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Capacity Bounds for Small-World and Dual Radio Networks

Departamento de Ciência de Computadores
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Aos meus pais, à minha irmã e à Kikas.
À Carla, o amor da minha vida.

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Abstract

Recent results from statistical physics show that large classes of complex networks, both man-made and of natural origin, are characterized by high clustering properties yet strikingly short path lengths between pairs of nodes. This class of networks are said to have a *small-world* topology. In the context of communication networks, *navigable* small-world topologies, i.e. those which admit efficient distributed routing algorithms, are deemed particularly effective, for example, in resource discovery tasks and peer-to-peer applications. Breaking with the traditional approach to small-world topologies that privileges graph parameters pertaining to connectivity, and intrigued by the fundamental limits of communication in networks that exploit this type of topology, in the first part of this thesis we investigate the *capacity* of these networks from the perspective of network information flow. Our contribution includes upper and lower bounds for the capacity of standard and navigable small-world models, and the somewhat surprising result that, with high probability, random rewiring does not alter the capacity of a small-world network.

Motivated by the proliferation of dual radio devices, we consider, in the second part of this thesis, communication networks in which the devices have two radio interfaces. With the goal of studying the performance gains in this networks when using the two radio interfaces in a combined manner, we define a wireless network model in which all devices have short-range transmission capability, but a subset of the nodes has a secondary long-range wireless interface. For the resulting class of random graph models, we present analytical bounds for both the connectivity and the max-flow min-cut capacity. The most striking conclusion to be drawn from our results is that the capacity of this class of networks grows quadratically with the fraction of dual radio devices, thus indicating that a small percentage of such devices is sufficient to improve significantly the capacity of the network.

Resumo

Grafos aleatórios do tipo Small-World e Power-Law têm sido utilizados como modelos para um número elevado de redes naturais e redes tecnológicas, porque capturam algumas das suas propriedades fundamentais. Em redes de comunicação, pensa-se que topologias Small-World navegáveis, i.e. aquelas que admitem algoritmos distribuídos de encaminhamento, sejam particularmente eficientes, por exemplo, em tarefas de descoberta de recursos e aplicações peer-to-peer. Apesar do potencial evidenciado por topologias deste tipo em redes de comunicação, a abordagem tradicional a redes Small-World privilegia parâmetros relacionados com a conectividade. Assim, torna-se crucial saber quais são os limites fundamentais de comunicação em redes que exploram este tipo de topologia. Com o objectivo de estudar esses limites, na primeira parte desta tese estudamos a *capacidade* destas redes do ponto de vista de fluxos de informação em redes. As nossas contribuições incluem limites superiores e inferiores para a capacidade de redes do tipo Small-World, incluindo um resultado surpreendente, que pode ter a seguinte interpretação: alterar aleatoriamente os extremos de algumas ligações não altera a capacidade da rede, com probabilidade convergente para 1.

Na segunda parte desta tese, motivados pela proliferação de aparelhos com duas interfaces de rádio, consideramos redes de comunicação em que os aparelhos são deste tipo. Com o objectivo de estudar os ganhos ao utilizar de uma forma combinada as duas interfaces de rádio, definimos um modelo para redes sem fios em que todos os aparelhos partilham uma tecnologia sem fios de curto alcance e alguns possuem uma segunda tecnologia sem fios, esta de longo alcance. Para a classe de grafos definida pelo modelo, apresentamos limites superiores e inferiores tanto para a probabilidade de uma instância do modelo ser conexa, como para a sua capacidade. A conclusão mais interessante a retirar dos nossos resultados é o facto de a capacidade desta classe de grafos crescer quadraticamente com a proporção de aparelhos que possuem as duas tecnologias sem fios, indicando assim que apenas uma pequena percentagem destes aparelhos é suficiente para melhorar significativamente a capacidade da rede.

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Chapter 1

Introduction

Although the capacity of networks (described by general weighted graphs) supporting multiple communicating parties is largely unknown, progress has recently been reported in several relevant instances of this problem. In the case where the network has one or more independent sources of information but only one sink, it is known that routing offers an optimal solution for transporting messages [LL04] — in this case the transmitted information behaves like *water in pipes* and the capacity can be obtained by classical network flow methods. Specifically, the capacity of the network follows from the well-known Ford-Fulkerson *max-flow min-cut* theorem [FF62], which asserts that the maximal amount of a flow (provided by the network) is equal to the capacity of a minimal cut, i.e. a nontrivial partition of the graph node set V into two parts such that the sum of the capacities of the edges connecting the two parts (the cut capacity) is minimum. In [BS06] it was shown that network flow methods also yield the capacity for networks with multiple *correlated* sources and one sink.

The case of general multicast networks, in which a single source broadcasts a number of messages to a set of sinks, is considered in [ACLY00], where it is shown that applying coding operations at intermediate nodes (i.e. *network coding*) is necessary to achieve the max-flow/min-cut bound of the network. In other words, if k messages are to be sent then the minimum cut between the source and each sink must be at least of size k. A converse proof for this problem, known as the *network information flow problem*, was provided by [Bor02], whereas linear network codes were proposed and discussed in [LYC03] and [KM03]. Max-flow min-cut capacity bounds for Erdös-Rényi graphs and random geometric graphs were presented in [RSW05].

Another problem in which network flow techniques have been found useful is that of finding the maximum stable throughput in certain networks. In this problem, posed by Gupta and Kumar in [GK00], it is sought to determine the maximum rate at which nodes can inject bits into a network, while keeping the system stable. This problem was reformulated in [PS05] as a multi-commodity flow problem, for which tight bounds were obtained using elementary counting techniques.

1.1 Small-World Networks

The first class of graphs of interest in this thesis is the class of small-world graphs, i.e. graphs with high clustering coefficients and small average path length. These graphs have sparked a fair amount of interest from the scientific community, mainly due to their ability to capture fundamental properties of relevant phenomena and structures in sociology, biology, statistical physics and man-made networks. Beyond well-known examples such as Milgram's "six degrees of separation" [Mil67] between any two people in the United States (over which some doubt has recently been casted [Kle02]) and the Hollywood graph with links between actors, small-world structures appear in such diverse networks as the U.S. electric power grid, the nervous system of the nematode worm *Caenorhabditis elegans* [AY92], food webs [WM00], telephone call graphs [ABW98], citation networks of scientists [New01], and, most strikingly, the World Wide Web [Bro00].

The term small-world graph was coined by Watts and Strogatz, who in their seminal paper [WS98] defined a class of models which interpolate between regular lattices and random Erdös-Rényi graphs by adding shortcuts or rewiring edges with a certain probability p. The most striking feature of these models is that for increasing values of p the average shortest-path length diminishes sharply, whereas the clustering coefficient remains practically constant during this transition.

Since the seminal work of [WS98], key properties of small-world networks, such as clustering coefficient, characteristic path length, and node degree distribution, have been studied by several authors (see e.g. [DM03] and references therein). The combination of strong local connectivity and long-range shortcut links renders small-world topologies potentially attractive in the context of communication networks, either to increase their capacity or simplify certain tasks. Recent examples include resource discovery in wireless networks [Hel03], design of heterogeneous networks [RKV04, DYT05], and peer-to-peer communications [MNW04].

When applying small-world principles to communication networks, we would not only like that short paths exist between any pairs of nodes, but also that such paths can easily be found using merely local information. In [Kle00] it was shown that this *navigability* property, which is key to the existence of effective distributed routing algorithms, is lacking in the small-world models of [WS98] and [NW99]. The alternative navigable model presented in [Kle00] consists of a grid to which shortcuts are added not uniformly but according to a harmonic distribution, such that the number of outgoing links per node is fixed and the link probability depends on the distance between the nodes. For this class of small-world networks a *greedy* routing algorithm, in which a message is sent through the outgoing link that takes it closest to the destination, was shown to be effective, thus opening the door towards a capacity-attaining solution.

Since small-world graphs were proposed as models for complex networks ([WS98], [NW99]), most contributions in the area of complex networks focus essentially on connectivity parameters such as the degree distribution, the clustering coefficient or the shortest path length between two nodes (see e.g. [Str01]). In spite of its undoubted relevance — particularly where communication networks are concerned — the *capacity* of small-world networks has, to the best of our knowledge, not yet been studied in any depth by the scientific community. One of the goals of this thesis is thus to provide a preliminary characterization of the capacity of small-world networks from the point of view of network information flow.

1.2 Dual Radio Networks

The second part of this thesis focuses on communication networks in which the devices have two radio interfaces - *dual radio networks*. The interest in these networks arises from the fact that wireless interfaces are standard commodities and communication devices with multiple radio interfaces appear in various products, thus leading to the natural question: can the aforementioned devices lead to substantial performance gains in wireless communication networks? Promising examples include [BAPW04], where multiple radios are used to provide better performance and greater functionality for users, and [PRW05], where it is shown that using radio hierarchies can reduce power consumption. In addition, [ABP+04] presents a link-layer protocol that works with multiple IEEE 802.11 radios and improves TCP throughput and latency. This growing interest in wireless systems with multiple radios (for example, a Bluetooth interface and an IEEE 802.11 wi-fi card) motivates us to study the impact of dual radio devices on the connectivity and capacity of wireless networks.

For classical single-radio networks, random geometric graphs provide a widely accepted model, whose connectivity is well understood. In [Pen99] Penrose shows a relationship between connectivity and minimum degree in terms of the value of the radio range. Gupta and Kumar derive in [GK98], the critical radio range for which the probability that the network is connected goes to one as the number of nodes goes to infinity. Ganesh and Xue [GX05] studied the connectivity and diameter of a class of networks similar to random geometric graphs, with the new feature of adding random shortcuts to the network, thus creating a *small-world* network.

1.3 Main Contributions

We provide a set of upper and lower bounds for the max-flow min-cut capacity of several classes of small-world networks, including navigable topologies, for which highly efficient distributed routing algorithms are known to exist and distributed network coding strategies are likely to be found. Our main contributions related to small-world networks, mainly from [CB06c] and [CB06a] (or [CB06b], for a complete set of results), are as follows:

- *Capacity Bounds on Small-World Networks with Added Shortcuts:* We prove a high concentration result which gives upper and lower bounds on the capacity of a small-world with shortcuts of probability p, thus describing the capacity growth due to the addition of random edges.

- *Rewiring does not alter the Capacity:* We construct assymptotically tight upper and lower bounds for the capacity of small-worlds with rewiring and prove that, with high probability, capacity will not change when the edges are altered in a random fashion.

- *Capacity Bounds for Kleinberg Networks:* We construct upper and lower bounds for the max-flow min-cut capacity of navigable small-world networks derived from a square lattice and illustrate how the choice of connectivity parameters affects communication.

- *Capacity Bounds for Navigable Small-World Networks on Ring Lattices:* Arguing that the corners present in the aforementioned Kleinberg networks introduce undesirable artefacts in the computation of the capacity, we define a navigable small world network based on a ring lattice, prove its navigability and derive a high-concentration result for the capacity of this instance.

Related to Dual Radio Networks, our main contributions, mainly from [CB07], are as follows:

- *Network Model:* We introduce a simple random graph model, the *Dual Radio Network* (DRN), where nodes with low-range radios are represented by a primary random geometric graph and the set of dual radio nodes with their additional long-range wireless links form a secondary random geometric graph.
- *Connectivity Bounds:* For this class of networks, we provide upper and lower bounds for the probability that an instance of a Dual Radio Network is connected;
- *Capacity Bounds:* Using a set of probabilistic tools, we derive upper and lower bounds for the max-flow min-cut capacity of this class of random networks.

1.4 Outline of the Thesis

The outline of the thesis is the following. In Chapter 2 we define two simple models for small-world networks and we present a set of tools (from random sampling in graphs) that allows us to study the max-flow min-cut capacity of randomized graphs in general. We use this set of tools to provide bounds on the max-flow min-cut capacity of the two models of interest in the chapter. The notion of *navigability* is presented in Chapter 3, where we define two navigable models for small-world networks and prove that this models are, indeed, navigable. Next, we provide upper and lower bounds on the max-flow min-cut capacity of these two classes of navigable small-world networks. In Chapter 4, we study networks in which some of the nodes have a second wireless technology, by defining a graph model for this networks, called *Dual Radio Networks*. We then provide bounds on the probability that a Dual Radio Network is connected, and we also provide bounds on their max-flow min-cut capacity. Chapter 5 concludes the thesis.

Chapter 2

Small-World Networks

In this chapter, we provide the definitions for two simple classes of small-world networks, the Small-World Network with Shortcuts [NW99] and the Small-World Network with Rewiring [WS98], and study these classes in terms of their max-flow min-cut capacity.

2.1 Classes of Small-World Networks

In this section, we give rigorous definitions for the classes of small-world networks under consideration in this chapter. First, we require a precise notion of distance in a ring.

Definition 1. *Consider a set of n nodes connected by edges that form a ring (see Fig. 2.1, left plot). The ring distance between two nodes is defined as the minimum number of hops from one node to the other. If we number the nodes in clockwise direction, starting from any node, then the ring distance between nodes i and j is given by $d(i,j) = min\{|i-j|, n+i-j, n-|i-j|\}$.*

For simplicity, we refer to $d(i,j)$ as the *distance* between i and j. Next, we define a k-connected ring lattice, that serves as basis for some of the small-world models described next, can be defined as follows.

Definition 2. *A k-connected ring lattice (see Fig. 2.1) is a graph $L = (V_L, E_L)$ with nodes V_L and edges E_L, in which all nodes in V_L are placed on a ring and are connected to all the nodes within distance $\frac{k}{2}$.*

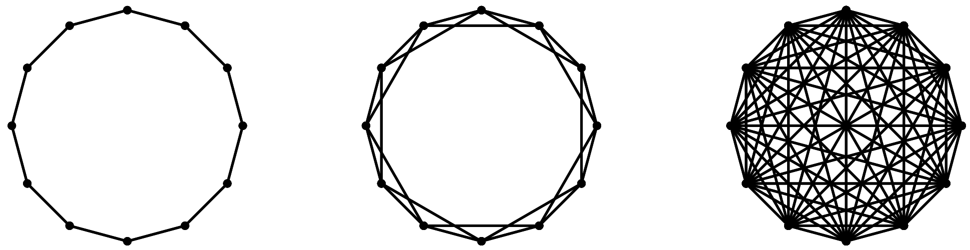

Figure 2.1: Illustration of a k-connected ring lattice.
From left to right $k = 2, 4, 12$.

Notice that in a k-connected ring lattice, all the nodes have degree k. We are now ready to define the small-world models of interest in this section.

Definition 3 (Small-World Network with Shortcuts [NW99], see Fig. 2.2)**.** *Consider a k-connected ring lattice $L = (V_L, E_L)$. To obtain a small-world network with shortcuts, we add to the ring lattice L each edge $e \notin E_L$ with probability p.*

Definition 4 (Small-World Network with Rewiring [WS98], see Fig. 2.3)**.** *To obtain a small-world network with rewiring, we use the following procedure. Consider a k-connected ring lattice $L = (V_L, E_L)$ and choose a node, say node u, and the edge that connects it to its nearest neighbor in a clockwise sense. With probability p, reconnect this edge to a node chosen uniformly at random over the set of nodes $\{v \in V_L : (u, v) \notin E_L\}$. Repeat this process by moving around the ring in clockwise direction, considering each node in turn until one lap is completed. Next, consider the edges that connect nodes to their second-nearest neighbors clockwise. As before, randomly rewire each of these edges with probability p, and continue this process, circulating around the ring and proceeding outward to more distant neighbors after each lap, until each edge in the original lattice has been considered once.*

2.2 Capacity Results for Small-World Networks

In Chapter 1, we argued that the max-flow min-cut capacity provides the fundamental limit of communication for various relevant network scenarios. Motivated by this observation, we will now use network flow methods and random sampling techniques in graphs to derive a set of bounds for the capacity of the small-world network models presented in the previous section.

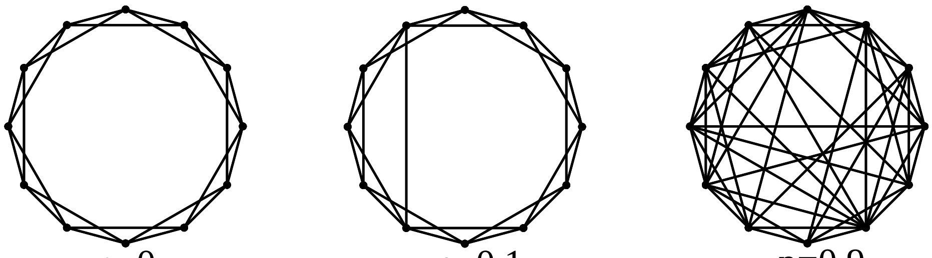

Figure 2.2: Small-world model with added shortcuts for different values of the adding probability p.

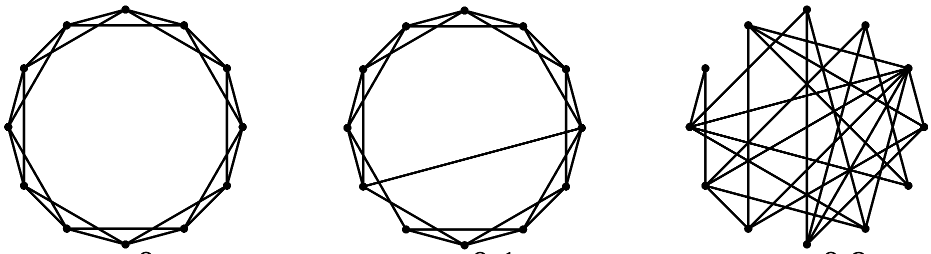

Figure 2.3: Small-world model with rewiring for different values of the rewiring probability p.

2.2.1 Preliminaries

We start by introducing some necessary mathematical tools. Let G be an undirected graph, representing a communication network, with edges of unitary weight[1].

In the spirit of the max-flow min-cut theorem of Ford and Fulkerson [FF62], we will refer to the global minimum cut of G as the max-flow min-cut capacity (or simply the *capacity*) of the graph.

Let G_s be the graph obtained by sampling on G, such that each edge e has sampling probability p_e. From G and G_s, we obtain G_w by assigning to each edge e the weight p_e, i.e. $w(e) = p_e, \forall e$. We denote the capacity of G_s and G_w by c_s and c_w, respectively.

It is helpful to view a cut in G_s as a sum of Bernoulli distributed random variables, whose outcome determines if an edge e connecting the two sides of the cut belongs to G_s or not. Thus, it is not difficult to see that the value of a cut in G_w is the expected

[1]For simplicity, in the rest of the thesis, we will assume that all the edges of the communication networks presented have unitary weight.

value of the same cut in G_s. The following theorem provides a characterization of how close a cut in G_s will be with respect to its expected value.

Theorem 1 (From [Kar94]). *Let* $\epsilon = \sqrt{2(d+2)\ln(n)/c_w}$. *Then, with probability* $1 - O(1/n^d)$, *every cut in* G_s *has value between* $(1-\epsilon)$ *and* $(1+\epsilon)$ *times its expected value.*

Notice that although d is a free parameter, there is a strict relationship between the value of d and the value of ϵ. In other words, the proximity to the expected value of the cut is intertwined with how close the probability is to one. *Theorem 1* yields also the following useful property.

Corollary 1. *Let* $\epsilon = \sqrt{2(d+2)\ln(n)/c_w}$. *Then, with high probability, the value of* c_s *lies between* $(1-\epsilon)c_w$ *and* $(1+\epsilon)c_w$.

Before using the previous random sampling results to determine bounds for the capacities of small-world models, we prove another useful lemma.

Lemma 1. *Let* $L = (V_L, E_L)$ *be a* k*-connected ring lattice and let* $G = (V_L, E)$ *be a fully connected graph (without self-loops), in which edges* $e \in E_L$ *have weight* $w_1 \geq 0$ *and edges* $f \notin E_L$ *have weight* $w_2 \geq 0$. *Then, the global minimum cut in* G *is* $kw_1 + (n-1-k)w_2$.

Proof. We start by splitting G into two subgraphs: a k-connected ring lattice L with weights w_1 and a graph F with nodes V_L and all remaining edges of weight w_2. Clearly, the value of a cut in G is the sum of the values of the same cut in L and in F. Moreover, both in L and in F, the global minimum cut is a cut in which one of the partitions consists of one node (any other partition increases the number of outgoing edges). Since each node in L has k edges of weight w_1 and each node in F has the remaining $n-1-k$ edges of weight w_2, the result follows. □

2.2.2 Capacity Bounds for Small-World Networks with Added Shortcuts

With the set of tools presented in the previous section, we are ready to state and prove our first main result.

Theorem 2. *With high probability, the value of the capacity of a small-world network with shortcuts lies between* $(1-\epsilon)c_w$ *and* $(1+\epsilon)c_w$, *with* $\epsilon = \sqrt{2(d+2)\ln(n)/c_w}$ *and* $c_w = k + (n-1-k)p$.

Proof. Let G_w be a fully connected graph with n nodes and with the edge weights (or equivalently, the sampling probabilities) defined as follows:

- The weight of the edges in the initial lattice of a small-world network with added shortcuts is one (because they are not removed);
- The weight of the remaining edges is p, (i.e. the probability that an edge is added).

Notice that G_w is a graph in the conditions of *Lemma 1*, with $w_1 = 1$ and $w_2 = p$. Therefore, the global minimum cut in G_w is $c_w = k + (n - 1 - k)p$, where k is the initial number of neighbors in the lattice. Using *Corollary 1*, the result follows. □

The obtained bounds are illustrated in Fig. 2.4. As one would expect, the capacity increases with p, i.e. as the number of added links become larger.

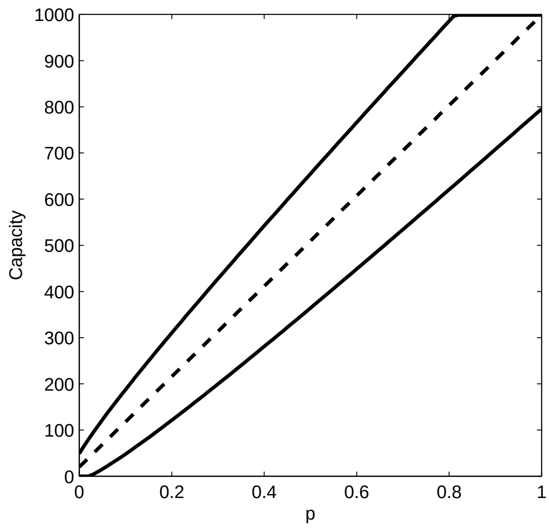

Figure 2.4: Bounds on the capacity of a small-world network with added shortcuts, for $n = 1000$, $k = 20$, and $d = 1$. The dashed line represents the expected value of the capacity, and the solid lines represent the bounds.

2.2.3 Capacity Bounds for Small-World Networks with Rewiring

In the previous class of small-world networks, edges were added to a k-connected ring lattice (with minimum cut k) and clearly the capacity could only grow with p. The next natural step is to ask what happens when edges are not added but rewired with probability p, as described in Section 2.1. Before presenting a theorem that answers this question, we will prove the following lemma.

Lemma 2. *Let G_w be a weighted fully connected graph whose weights correspond to the edge probabilities of a small-world network with rewiring, and let c_w be the global minimum cut in G_w. Then, $c_w \geq k$.*

Proof. We start with the initial lattice edges $(l, m) \in E_L$, and assign the weight $1 - p$ to their counterparts in G_w. In order to determine the weight of the non-initial edges that result from rewiring, consider the following events:

- $R(i, j)$: "Choose the edge $(i, j) \in E_L$ to be rewired";
- $C_i(j, l)$: "Rewire $(i, j) \in E_L$ to $(i, l) \notin E_L$".

Notice that $\mathcal{P}(R(i, j)) = p, \forall (i, j) \in E_L$.
Let i and j be two non-initially connected nodes. The notation $i \leftrightarrow j$ denotes the event that the nodes i and j are connected. We have that

$$\begin{aligned}\mathcal{P}(i \leftrightarrow j) &= \mathcal{P}\left([\cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))] \cup [\cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))]\right) \\ &\overset{(a)}{=} \mathcal{P}\left(\cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))\right) + \mathcal{P}\left(\cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))\right) \\ &\quad -\mathcal{P}\left([\cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))] \cap [\cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))]\right)\end{aligned}$$

where (a) follows from the fact that, for any two events A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B)$, and take $A = \cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))$ and $B = \cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))$.

Because we do not consider multiple edges, we have that the events $R(i, i+x) \cap C_i(i+x, j)$ and $R(j, j+y) \cap C_j(j+y, i)$ are mutually exclusive, $\forall x, y$, and, for each i, j, the same is true for the pair of events $R(i, i+x) \cap C_i(i+x, j)$ and $R(i, i+y) \cap C_i(i+y, j)$, and for the pair $R(j, j+x) \cap C_j(j+x, i)$ and $R(j, j+y) \cap C_j(j+y, i)$. Therefore,

$$\mathcal{P}\left([\cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))] \cap [\cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))]\right) = 0,$$

$$\mathcal{P}\left(\cup_{x=1}^{k/2}(R(i, i+x) \cap C_i(i+x, j))\right) = \sum_{x=1}^{k/2} \mathcal{P}(R(i, i+x) \cap C_i(i+x, j))$$

and

$$\mathcal{P}\left(\cup_{x=1}^{k/2}(R(j, j+x) \cap C_j(j+x, i))\right) = \sum_{x=1}^{k/2} \mathcal{P}(R(j, j+x) \cap C_j(j+x, i)).$$

Thus

$$\begin{aligned}\mathcal{P}(i \leftrightarrow j) &= \sum_{x=1}^{k/2} \left(\mathcal{P}(R(i,i+x) \cap C_i(i+x,j)) + \mathcal{P}(R(j,j+x) \cap C_j(j+x,i))\right) \\ &\overset{(a)}{=} \sum_{x=1}^{k/2} [\mathcal{P}(C_i(i+x,j)|R(i,i+x))\mathcal{P}(R(i,i+x)) \\ &\quad +\mathcal{P}(C_j(j+x,i)|R(j,j+x))\mathcal{P}(R(j,j+x))] \\ &\overset{(b)}{=} p \cdot \left(\sum_{x=1}^{k/2}(\mathcal{P}(C_i(i+x,j)|R(i,i+x)) + \mathcal{P}(C_j(j+x,i)|R(j,j+x)))\right)\end{aligned}$$

where (a) follows from the fact that, for any two events A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(B|A)\mathcal{P}(A)$, and (b) follows from substituting $\mathcal{P}(R(i,i+x))$ and $\mathcal{P}(R(j,j+x))$ by p.

We have $\mathcal{P}(C_i(i+x,j)|R(i,i+x)) = \frac{1}{m}$, where m is the number of possible new connections from node i when we rewire the edge $(i, i{+}x)$. It is possible that, regardless of whether or not some rewiring has previously occurred, none of the other nodes chose to rewire to node i. In this case, $m = n-k-1$. Notice that this is the highest it can get, therefore $m \leq n-k-1$. Thus, we have

$$\mathcal{P}(C_i(i+x,j)|R(i,i+x)) \geq \frac{1}{n-k-1}.$$

Analogously, $\mathcal{P}(C_j(j+x,i)|R(j,j+x))) \geq \frac{1}{n-k-1}$. Therefore,

$$\mathcal{P}(i \leftrightarrow j) \geq p \cdot \left(\sum_{x=1}^{k/2} \frac{2}{n-k-1}\right) = \frac{pk}{n-k-1}.$$

Consider a fully connected weighted graph F with the weights defined as follows: all the edges $(i,j) \notin E_L$ have the weight $\frac{pk}{n-k-1}$, and all the others edges $(i,j) \in E_L$ have the weight $1-p$. Notice that F is a graph satisfying the conditions of Lemma 1, with $w_1 = 1-p$ and $w_2 = \frac{pk}{n-k-1}$. Therefore, because there are k initial edges and $n-k-1$ non-initial edges in each node,

$$c_F = k(1-p) + (n-k-1)\frac{pk}{n-k-1} = k.$$

Notice that, in this situation, all the weights in F are a lower bound of the weights in G_w. Therefore, a cut in F is a lower bound for the corresponding cut in G_w. Then, the global minimum cut in F is a lower bound for all the cuts in G_w, in particular, for c_w: $c_w \geq c_F = k$. □

With this lemma, we are now ready to state and prove our next result.

Theorem 3 (Rewiring does not alter capacity.). *With high probability, the capacity of a small-world network with rewiring has a value in the interval* $[(1-\epsilon)k, k]$ *with* $\epsilon = \sqrt{2(d+2)ln(n)/k}$.

Proof. Based on *Lemma 2* and *Corollary 1*, we have that, with high probability, $c_s \geq (1-\epsilon_w)k$, with $\epsilon_w = \sqrt{2(d+2)ln(n)/c_w}$. Now, from the fact that $c_w \geq k$, we have that $\epsilon = \sqrt{2(d+2)ln(n)/k} \geq \epsilon_w$. Then, $(1-\epsilon_w)k \geq (1-\epsilon)k$, and the first part of the result follows.

Next, we prove by contradiction that $c_s \leq k$. Suppose that the proposition $c_s > k$ is true. Let c_i be the cut in which one of the partitions consists of node i, $i = 1, ..., n$. Because c_s is the global minimum cut in G_s, we have that $c_i > k$, $\forall i = 1, ..., n$. Notice that c_i is the degree of node i. Then, because in the k-connected ring lattice all nodes have degree k and all nodes in G_s have degree greater than k (because $c_i > k, \forall i$), we have that the number of edges in G_s must be greater than the number of edges in the k-connected ring lattice. But this is clearly absurd, because in the construction of G_s, we do not add new edges to the k-connected ring lattice, we just rewire some of the existent edges. The contradiction arises from the initial assumption $c_s > k$, thus $c_s \leq k$. □

2.3 Summary

We studied the max-flow min-cut capacity of two classes of small-world networks. Using classical network flow arguments and concentration results from random sampling in graphs, we provided bounds for the capacity of two standard models for small-world networks. In particular, we presented a tight result for small-world networks with rewiring, which permits the following interpretation: *With high probability, rewiring does not alter the capacity of the network.* This observation is not obvious, because we can easily find ways to rewire the ring lattice in order to obtain, for instance, a *bottleneck*. But, according to the previous results, such instances occur with very low probability.

Chapter 3

Navigable Small-World Networks

As we argue in Chapter 1, when considering small-world networks as communication networks, an important aspect is the ability to find short paths between any pairs of nodes, using only local information. This property guarantees that efficient distributed routing algorithms exists. Kleinberg, in his seminal work [Kle00], proved that this *navigability* property is lacking in the models of Watts and Strogatz, and introduced a new model (*Definition 5*). Motivated by the relevance of the navigability property, in this chapter, we study two small-world models that exhibit this navigability property, stating the results that prove that they are, indeed, navigable, and presenting bounds on their max-flow min-cut capacity.

3.1 Classes of Navigable Small-World Networks

In this section, we provide rigorous definitions for the two models of interest in the rest of this chapter.

Definition 5 (Kleinberg Network [Kle00], see Fig. 3.1)**.** *We begin from a two-dimensional grid and a set of nodes that are identified with the set of lattice points in a* $n \times n$ *square,* $\{(x, y) : x \in \{1, 2, ..., n\}, y \in \{1, 2, ..., n\}\}$*, and we define the lattice distance between two nodes* (x_1, x_2) *and* (y_1, y_2) *to be the number of lattice steps (or hops) separating them:* $d(x, y) = |y_1 - x_1| + |y_2 - x_2|$*. For a constant* $h \geq 1$*,* $\forall u_1, u_2 \in \{1, ..., n\}$*, the node* (u_1, u_2) *is connected to every other node within lattice distance* h *(we denote the set of this initial edges as* E_L*). For universal constants* $q \geq 0$ *and* $r \geq 0$*, we also construct edges between* u *and* q *other nodes using random trials; the* i^{th} *edge from* u *has endpoint* v *with probability proportional to* $d(u, v)^{-r}$*. To ensure* *a valid probability distribution, consider the set of nodes that are not connected with* u *in the initial lattice,* $N_u = \{w : d(u, w) > h\}$, *and divide* $d(u, v)^{-r}$ *by the appropriate normalizing constant* $s(u) = \sum_{w \in N_u} [d(u, w)]^{-r}$.

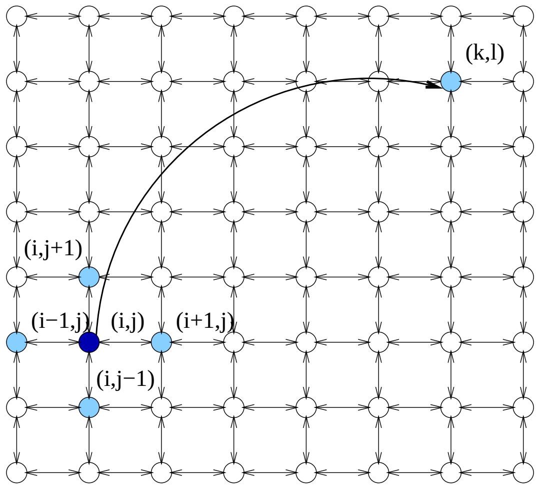

Figure 3.1: Kleinberg network, for $h = q = 1$. Lightly shaded circles represent the nodes that are directly connected to node (i, j), i.e. the four direct neighbors of (i, j) and one additional node (k, l) connected by a shortcut.

In the next section, we will see that this model exhibits unexpected effects related to the corners of the chosen base lattice. Motivated by this observation, we construct a somewhat different model, which uses a ring lattice but still keeps the key relationship between shortcut probability and node distance that assures the navigability of the model.

Definition 6 (Navigable Small-World Ring)**.** *Consider a* k*-connected ring lattice. For universal constants* $q \geq 0$ *and* $r \geq 0$, $\forall i$, *we add new edges from node* i *to* q *other nodes randomly: each added edge has an endpoint* j *with probability proportional to* $d(i, j)^{-r}$. *To ensure a valid probability distribution, consider* $N_i = \left\{j : d(i, j) > \frac{k}{2}\right\}$ *and divide* $d(i, j)^{-r}$ *by the appropriate normalizing constant* $s_i = \sum_{j \in N_i} d(i, j)^{-r}$.

3.2 Results on Navigability

In this section, we give an insight on the notion of navigability and we state the results of [Kle00] that prove that his model is navigable. Next, we use similar techniques to those in [Kle00] to prove that a navigable small-world ring is, indeed, navigable.

In his work, Kleinberg uses the notion of *decentralized routing algorithms* to study the navigability of his model.

Definition 7 (From [Kle00]). *Consider a graph G with an underlying metric δ_G. A decentralized routing algorithm in G is an algorithm with the goal of sending a message from a source to a destination, with the knowledge, at each step, of the underlying metric, the position of the destination, and the contacts of the current message holder and of all the nodes seen so far.*

Definition 8 (From [Kle00]). *A greedy decentralized routing algorithm is a decentralized routing algorithm operating greedily: at each step, it sends the message to the contact of the current message holder that is closest (in the sense of the underlying metric) to destination.*

The meaning of navigability is the following: we say that a network is *navigable* if it admits an efficient decentralized routing algorithm.

In [Kle00], Kleinberg proved that the models presented by Watts and Strogatz do not admit efficient decentralized routing algorithms, in constrast with his model:

Theorem 4 (From [Kle00]). *For $r = 2$, there is a constant α_2, independent of n, such that the expected delivery time of a greedy decentralized routing algorithm in a Kleinberg network is at most $\alpha_2 \cdot \log^2(n)$.*

Theorem 5 (From [Kle00]).

1. *Let $0 \leq r < 2$. There is a constant α_r, depending on p, q, r, but independent of n, such that the expected delivery time of any decentralized routing algorithm in a Kleinberg network is at least $\alpha_r \cdot n^{(2-r)/3}$.*

2. *Let $r > 2$. There is a constant α_r, depending on p, q, r, but independent of n, so that the expected delivery time of any decentralized routing algorithm in a Kleinberg network is at least $\alpha_r \cdot n^{(r-2)/(r-1)}$.*

Theorem 4 shows that, in fact, a Kleinberg network is navigable, while *Theorem 5* shows that the models from Watts and Strogatz are not navigable, because this corresponds to the case when we consider uniformly chosen shortcuts, therefore corresponding to $r = 0$.

The next theorem shows that a navigable small-world ring is indeed navigable, in the sense that the expected delivery time of a decentralized routing algorithm is logarithmic. The proof is essentially based on the proof of *Theorem 4* presented by Kleinberg.

Theorem 6. *For $r = 1$, the expected delivery time of a greedy decentralized routing algorithm in a navigable small-world ring is at most $\frac{\ln^2(2n)}{\ln(2)}$.*

Proof. First, we need to show that $\sum_{u \in N_v} d(u,v)^{-1}$ is uniformly bounded. For even n, it is not difficult to see that there is a single node that maximizes the distance to node 1; that node is node $\frac{n}{2}+1$, and we have that $d\left(1, \frac{n}{2}+1\right) = \frac{n}{2}$. For distances $d < \frac{n}{2}$, there are two nodes at distance d to node 1. Therefore, if n is even, we have that

$$\sum_{u \in N_v} d(u,v)^{-1} = \left(\frac{n}{2}\right)^{-1} + 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} i^{-1} \leq 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}} i^{-1}.$$

When n is odd, it is also easy to see that there are two nodes that maximize the distance to node 1: nodes $\frac{n+1}{2}$ and $\frac{n+3}{2}$, with the maximum distance being $\frac{n-1}{2}$. Therefore, if n is odd, we have that

$$\sum_{u \in N_v} d(u,v)^{-1} = 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-1}{2}} i^{-1}.$$

Therefore, we have that $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{u \in N_v} d(u,v)^{-1} &\leq 2 \cdot \sum_{i=\frac{k}{2}+1}^{\left\lfloor \frac{n}{2} \right\rfloor} i^{-1} \\
&\leq 2 \cdot \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} i^{-1} \\
&\leq 2 + 2\ln\left(\frac{n}{2}\right) \\
&\leq 2\ln(2n)
\end{aligned}$$

For $j > 0$, we say that the decentralized routing algorithm is in *phase* j if the distance between the current message holder and the destination is d such that $2^j < d \leq 2^{j+1}$. We say that the algorithm is in phase 0 if the distance between the current message holder and the destination is at most 2. Because the maximum distance in the ring-lattice is at most $\frac{n}{2}$, we have that $j \leq \log\left(\frac{n}{2}\right)$.

Now, suppose that we are in phase j and the current message holder is node u. The task is to determine the probability of phase j ending in this step. Let B_j be the set of nodes within lattice distance 2^j of the destination. Phase j ending in this step means that u chooses a long-range contact $v \in B_j$. Each node $v \in B_j$ has probability of being chosen as long-range contact of u at least

$$\frac{(2^j)^{-1}}{\sum_{v \in N_u} d(u,v)^{-1}} \geq \frac{1}{2^{j+1} \cdot \ln(2n)}.$$

We have that the number of nodes in B_j, denoted by $|B_j|$, satisfies

$$|B_j| = 1 + 2 \cdot \sum_{i=1}^{2^j} i \geq 2^{2j}.$$

Therefore, with A denoting the event "Phase j ends in this step", we have that

$$\mathcal{P}(A) \geq \frac{2^{2j}}{2^{j+1} \cdot \ln(2n)} = \frac{2^{j-1}}{\ln(2n)} \geq \frac{1}{\ln(2n)}.$$

Let N_j be the number of steps spent in phase j. Now, we must compute the expected value of N_j. Notice that the maximum number of steps spent in phase j is the number of nodes at distance of the destination d such that $2^j < d \leq 2^{j+1}$, which is

$$\begin{aligned}
m &= 2 \cdot \sum_{i=2^j}^{2^{j+1}-1} i \\
&= 2 \cdot \left(\sum_{i=1}^{2^{j+1}-1} i - \sum_{i=1}^{2^j-1} i \right) \\
&= 2^{j+1}(2^{j+1}-1) - 2^j(2^j-1) \\
&\leq 2^{2j+2}.
\end{aligned}$$

Therefore, the expected value of N_j satisfies the following:

$$\begin{aligned}
E(N_j) &= \sum_{i=1}^{m} \mathcal{P}(N_j \geq i) \\
&\leq \sum_{i=1}^{2^{2j+2}} \mathcal{P}(N_j \geq i) \\
&\leq \sum_{i=1}^{2^{2j+2}} \left(1 - \frac{1}{\ln(2n)}\right)^{i-1} \\
&\leq \sum_{i=1}^{\infty} \left(1 - \frac{1}{\ln(2n)}\right)^{i-1} \\
&= \ln(2n)
\end{aligned}$$

Now, denoting by N the total number of steps implemented by the algorithm, we have that

$$N = \sum_{i=0}^{\log\left(\frac{n}{2}\right)} N_j.$$

Therefore, by linearity of the expected value, we have that

$$\begin{aligned}
E(N) &= \sum_{i=0}^{\log\left(\frac{n}{2}\right)} E(N_j) \\
&\leq \left(1 + \log\left(\frac{n}{2}\right)\right) \cdot \ln(2n) \\
&= \left(\log(2) + \log\left(\frac{n}{2}\right)\right) \cdot \ln(2n) \\
&= \log(n) \cdot \ln(2n) \\
&= \frac{\ln(n) \cdot \ln(2n)}{\ln(2)} \\
&\leq \frac{\ln^2(2n)}{\ln(2)}
\end{aligned}$$

□

3.3 Capacity Results for Navigable Small-World Networks

In this section, we study the max-flow min-cut capacity of the navigable models for small-world networks defined in Section 3.1.

3.3.1 Capacity Bounds for Kleinberg Networks

Before proceeding with the bounds for the capacity of Kleinberg networks, we require an algorithm to calculate the normalizing constants $s(x,y) = \sum_{(i,j)\in N_{(x,y)}} [d((x,y),(i,j))]^{-r}$ for $x, y \in \{1, ...n\}$. For this purpose, notice that the previous sum can be written as

$$s(x,y) = \sum_{(i,j)\neq(x,y)} [d((i,j),(x,y))]^{-r} - \sum_{(i,j)\notin N_{(x,y)}} [d((i,j),(x,y))]^{-r}.$$

Clearly, the first term can be easily calculated. Thus, the challenging task is to present an algorithm that deals with the calculation of $\sum_{(i,j)\notin N_{(x,y)}} [d((i,j),(x,y))]^{-r}$. The

Table 3.1: Algorithm for computing normalizing constants

Algorithm 1.

$z = [0]_{n\times n}$
$for\ i = 0 : min\{h, n - y\}$
    $for\ j = 0 : min\{h - i, n - x\}$
        $z(x + j, y + i) = (i + j)^{-r}$
    $for\ j = 1 : min\{h - i, x - 1\}$
        $z(x - j, y + i) = (i + j)^{-r}$
$for\ i = 1 : min\{h, y - 1\}$
    $for\ j = 0 : min\{h - i, n - x\}$
        $z(x + j, y - i) = (i + j)^{-r}$
    $for\ j = 1 : min\{h-i, h-(m_1-i), x-1\}$
        $z(x - j, y - i) = (i + j)^{-r}$
$z(x, y) = 0$
$z = \sum_{i=1}^{n} \sum_{j=1}^{n} z(i, j)$
$s(x, y) = \sum_{(i,j)\neq(x,y)} (|i - x| + |j - y|)^{-r} - z$

nodes $(i, j) \notin N_{(x,y)}$ are the nodes initially connected to node (x, y), i.e., the nodes at a distance $t \leq h$ from node (x, y). It is not difficult to see that the nodes at a distance t from node (x, y) are the nodes in the square line formed by the nodes $(x - t, y)$, $(x + t, y)$, $(x, y + t)$ and $(x, y - t)$. Then, we could just look at nodes in the square formed by the nodes $(x - h, y)$, $(x + h, y)$, $(x, y + h)$ and $(x, y - h)$ and sum all the corresponding distances to node (x, y). A corner effect occurs when this square lies outside the base lattice. Assume that we start by calculating the distances to the nodes in line $y + i$, with $i \geq 0$.

To avoid calculating extra distances (i.e., distances of nodes that are out of the grid), we must make sure that this line verifies $y + i \leq n$ and also $y + i \leq h$. For this reason, i must vary according to $i \in \{0 \ldots \min\{h, n - y\}\}$. Now, in each line $y + i$, we first look at the nodes in the right side of (x, y), i.e., we calculate the distances of the nodes $(x + j, y + i)$, with $j \geq 0$. Now, notice that in the line y, we have h points on the right side of (x, y) that are in the square (regardless of whether they are in the grid). Because the distance is the minimum number of steps in the grid, we have that in line $y + i$ there are $h - i$ points at the right side of (x, y) that are inside the square. As a consequence, j must vary according to $j \in \{0 \ldots \min\{h-i, n-x\}\}$. Now, when looking at the nodes on the left side (i.e., the nodes $(x - j, y + i)$, with $i \geq 1$), the idea is the same, the only difference is that, in this case, the variation for j is $j \in \{1 \ldots \min\{h-i, x-1\}\}$. Then, we proceed analogously for the lines below (x, y), i.e., the lines $y - i$, with $i \in \{1 \ldots \min\{h, y-1\}\}$. This algorithm is summarized in Table 3.1. The matrix z is a buffer for the distances, i.e., $z(u_1, u_2) = d((x, y), (u_1, u_2))$. We impose $z(x, y) = 0$, because $d((x, y), (x, y))^{-r}$ is also calculated in this procedure.

The following quantities will be instrumental towards characterizing the capacity:

$$
\begin{aligned}
M &= \max\left\{\frac{h(h+3)}{2} + q, (1-\epsilon)c_w\right\} \\
\epsilon &= \sqrt{2(d+2)\ln(n^2)/c_w} \\
c_w &= \frac{h(h+3)}{2} + \sum_{x=1}^{h+1} \sum_{y=h+2-x}^{n} f(x, y) + \sum_{x=h+2}^{n} \sum_{y=1}^{n} f(x, y) \qquad (3.1) \\
f(x, y) &= q \cdot (g_{(x,y)}(1, 1) + g_{(1,1)}(x, y)) \\
g_{(x,y)}(a, b) &= \left(1 - \frac{(x+y-2)^{-r}}{s(a, b)}\right)^{q-1} \cdot \frac{(x+y-2)^{-r}}{s(a, b)} \\
s(1, 1) &= \sum_{i=h+1}^{n-1} (i+1) \cdot i^{-r} + \sum_{i=0}^{n-2} (n-1-i) \cdot (n+i)^{-r}.
\end{aligned}
$$

Recall that $s(x, y)$ can be calculated using *Algorithm 1.* The proof of the capacity will rely heavily on the following lemma:

Lemma 3. *Let G_w be the weighted graph associated with a Kleinberg network, and c_w be the global minimum cut in G_w. Then, for $h < n - 1$, c_w is given by (3.1).*

Proof. All the edges $e \in E_L$ have weight 1 (because they are never removed), all nodes in G_w have degree $n^2 - 1$, and the weights of these edges depend only on the distance between the nodes they connect. Therefore, the global minimum cut in G_w must be a cut in which one of the partitions consists of a single node. Because the weight of an edge in G_w decreases with the distance between the nodes that it connects, the global minimum cut in G_w must be a cut in which one of the partitions consists of a single node that maximizes the distance to other nodes. Therefore, because a corner node has more nodes at a greater distance than the other nodes and has also a smaller number of nodes to which it is connected, c_w must be a cut in which one of the partitions consists of a corner node: $(1, 1), (1, n), (n, 1)$ or (n, n).

Consider c_w as the cut in which one of the partitions consists of node $(1, 1)$. Let $w(u, v)$ be the weight of the edge connecting the nodes u and v. This way, $c_w = \sum_{u \neq (1,1)} w((1, 1), u)$. Now, we must count how many edges connecting node $(1, 1)$ are in E_L, therefore, having weight 1. For this, we define an auxiliary way to numerate diagonals: $\{(1,1)\}$ is the diagonal 0, $\{(1,2),(2,1)\}$ is diagonal 1, and so on (see Figure 3.2).

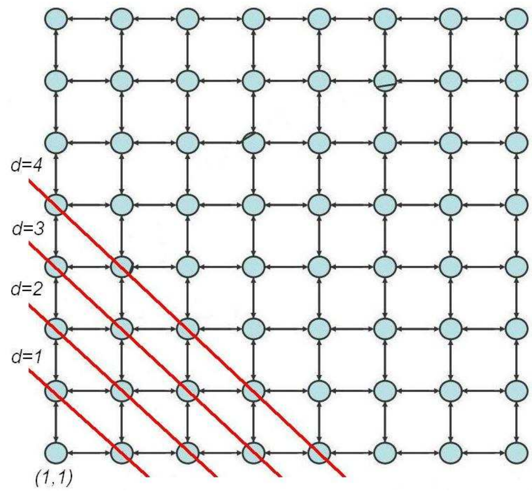

Figure 3.2: Numeration of the diagonals of a square lattice.

It is not difficult to see that the nodes in the i^{th} diagonal have a distance i to node $(1,1)$, $i \in \{1,...,2(n-1)\}$. Now, for $i \leq n-1$, there are $i+1$ nodes in the i^{th} diagonal. Then, because we are considering $h < n-1$, there are $\sum_{i=1}^{h} i+1 = h(h+3)/2$ nodes initially connected to node $(1,1)$. Thus, there are $h(h+3)/2$ edges with weight 1. Therefore, we have that:

$$c_w = \frac{h(h+3)}{2} + \sum_{x=1}^{h+1} \sum_{y=h+2-x}^{n} w((1,1),(x,y)) + \sum_{x=h+2}^{n} \sum_{y=1}^{n} w((1,1),(x,y)).$$

Next, we determine the weights, $w(u,v)$. Consider two nodes that are not initially connected, $u=(u_1,u_2)$ and $v=(v_1,v_2)$, and the edge (u,v). This edge can be added in two different trials: one for node u and another one for node v. Because we do not consider multiple edges, these can be viewed as two mutually exclusive trials. Therefore, the weight of this edge is the sum of the probabilities of adding this edge when considering node u and when considering node v. Let us focus on node u. The trial "add edge (u,v)" follows a Binomial distribution, with q Bernoulli distributed random variables, with success probability

$$a_u(v) = \frac{d(u,v)^{-r}}{s(u)} = \frac{(|u_1-v_1|+|u_2-v_2|)^{-r}}{s(u)}.$$

Therefore, the probability of adding the edge (u, v), when considering node u, is $q \cdot (1 - a_u(v))^{q-1} \cdot a_u(v)$. Therefore, the weight of the edge $((u_1, u_2), (v_1, v_2))$ is

$$w(u, v) = q \cdot (1 - a_u(v))^{q-1} \cdot a_u(v) + q \cdot (1 - a_v(u))^{q-1} \cdot a_v(u).$$

As we have seen, the global minimum cut in G_w is the cut in which one of the partitions consists of node $(1, 1)$. We have that, if (x, y) is a node of the grid, $x \geq 1$ and $y \geq 1$. Then, $d((1, 1), (x, y)) = |x - 1| + |y - 1| = x + y - 2$. Therefore, $a_{(1,1)}(x, y) = \frac{x+y-2}{s(1,1)}$ and $a_{(x,y)}(1, 1) = \frac{x+y-2}{s(x,y)}$. Now, observing that we can calculate $s(1, 1)$ as

$$s(1, 1) = \sum_{i=h+1}^{n-1} (i + 1) \cdot i^{-r} + \sum_{i=0}^{n-2} (n - 1 - i) \cdot (n + i)^{-r}$$

and using expression (3.1) for c_w, the result follows. □

We are now ready to state our main result in this section.

Theorem 7. *For $h < n - 1$ the capacity of a Kleinberg small-world network graph lies, with high probability, in the interval $[M, (1 + \epsilon)c_w]$.*

Proof. Using *Lemma 3* and *Corollary 1*, we have that, with high probability, $c_s \in [(1 - \epsilon)c_w, (1 + \epsilon)c_w]$. A tighter lower bound can be obtained for c_s as follows. Each node has a number of initial edges, determined by h, and q additional shortcut edges. The nodes with less initial edges are obviously the corner nodes, which exhibit $\frac{h(h+3)}{2}$ initial connections. Therefore, we have that $c_s \geq \frac{h(h+3)}{2} + q$, and the result follows. □

The bounds of *Theorem 7* are illustrated in Fig. 3.3.

3.3.2 Capacity Bounds for Navigable Small-World Rings

As we have seen, Kleinberg's model exhibits corner's effects in terms of capacity. With the goal of overcoming this undesired feature, we defined a new class of small-world networks, the navigable small-world ring (see *Definition 6* in Section 3.1). Now we study the capacity of this class of networks by proving the following result:

Theorem 8. *With high probability, the capacity of the navigable small-world ring has a value in the interval $[\max\{k, (1 - \epsilon)c_w)\}, (1 + \epsilon)c_w)]$, with $\epsilon = \sqrt{2(d + 2)\ln(n)/c_w}$ and*

$$c_w = k + 2^{rq+1} s^{-q} q(1 + a_n)(n - a_n)^{-r} \left(2^{-r} s - (n - a_n)^{-r}\right)^{q-1} + 4qs^{-q} \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-a_n}{2}-1} i^{-r} \left(s - i^{-r}\right)^{q-1}$$

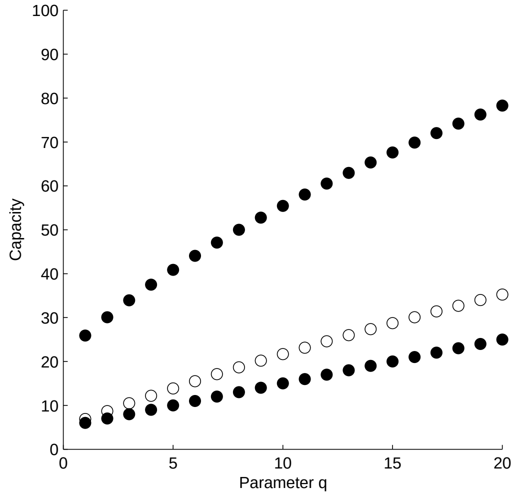

Figure 3.3: Bounds for the capacity of Kleinberg small-world network for $n = 80$ (i.e. 1600 nodes), $h = 2$, $r = 2$ e $d = 1$, and different values of the shortcut parameter q. The white circles represent the expected value of the capacity and the black circles represent the bounds computed according to *Theorem 7*.

with $s = (1 + a_n) \cdot \left(\frac{n-a_n}{2}\right)^{-r} + 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-a_n}{2}-1} i^{-r}$, *where* $a_n = \frac{1-(-1)^n}{2}$.

Proof. Consider the fully connected graph $G_w = (V_L, E)$ associated to the navigable small-world graph. The task is to determine the weights of the edges of G_w. The edges $e \in E_L$ have weight 1, because we never remove them. Now, notice that the ring distance between two nodes does not depend on which node is numbered first. It is therefore correct to state that all the nodes have the same number of nodes at a distance d. Therefore, we have that the normalizing constants are equal, for all nodes: $s_i = s_j, \forall i, j$. Let $s = s_i$. We also have that the weight of each edge only depends on the distance between the nodes that it connects. Therefore, it is sufficient to determine the weights of the edges of a single node, say node 1.

First, we must compute the normalizing constant s. We must distinguish between two different situations: even n or odd n. If n is even, it is not difficult to see that there is a single node that maximizes the distance to node 1. That node is node $\frac{n}{2}+1$, and we have that $d\left(1, \frac{n}{2}+1\right) = \frac{n}{2}$. For distances $d < \frac{n}{2}$, there are two nodes at distance d to node 1. Therefore, if n is even, we have that

$$s = \left(\frac{n}{2}\right)^{-r} + 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n}{2}-1} i^{-r}.$$

When n is odd, it is also easy to see that there are two nodes that maximize the distance to node 1: nodes $\frac{n+1}{2}$ and $\frac{n+3}{2}$, with the maximum distance being $\frac{n-1}{2}$. Therefore, if n is odd, we have that

$$s = 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-1}{2}} i^{-r}.$$

Now, notice that $a_n = \frac{1-(-1)^n}{2}$ is equal to 0 if n is even, and it is equal to 1 if n is odd. Therefore, $\forall n$,

$$s = (1 + a_n) \cdot \left(\frac{n - a_n}{2}\right)^{-r} + 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-a_n}{2}-1} i^{-r}.$$

Consider a node that is not initially connected to node 1, say node i. The edge $(1, i)$ can be added in two different trials: one for node 1 and another for node i. Because we do not consider multiple edges, these two trials are mutually exclusive. Therefore, the weight of the edge $(1, i)$ is the sum of the probabilities of adding this edge when looking at node 1 and when looking at node i. Because the normalizing constant is the same for all nodes, these two probabilities are equal. Consequently, let us focus on node 1. The trial "add edge $(1, i)$" follows a Binomial distribution, with q Bernoulli distributed random variables and with success probability $p = \frac{d(1,i)^{-r}}{s}$. Therefore, the probability of adding edge $(1, i)$ when considering node 1 is $qp \cdot (1 - p)^{q-1}$. Therefore, the weight of the edge $(1, i)$ is

$$w(1, i) = 2q \cdot \frac{d(1, i)^{-r}}{s} \cdot \left(1 - \frac{d(1, i)^{-r}}{s}\right)^{q-1}.$$

We have seen that all the nodes have the same number of nodes at a distance d. We also have that all the edges in the ring lattice have unitary weight. Based on these two observations and the fact that G_w is a fully connected graph, it is clear that the global minimum cut in G_w, denoted c_w, is a cut in which one of the partitions consists of a single node, say node 1. Thus, we may write

$$\begin{aligned} c_w &= k + \sum_{i \in N_1} w(1, i) \\ &= k + 2(1 + a_n)q\frac{\left(\frac{n-a_n}{2}\right)^{-r}}{s}\left(1 - \frac{\left(\frac{n-a_n}{2}\right)^{-r}}{s}\right)^{q-1} + 2 \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-a_n}{2}-1} 2q\frac{i^{-r}}{s}\left(1 - \frac{i^{-r}}{s}\right)^{q-1} \\ &= k + 2^{rq+1}s^{-q}q(1+a_n)(n-a_n)^{-r}\left(2^{-r}s - (n-a_n)^{-r}\right)^{q-1} + 4qs^{-q} \cdot \sum_{i=\frac{k}{2}+1}^{\frac{n-a_n}{2}-1} i^{-r}\left(s-i^{-r}\right)^{q-1} \end{aligned}$$

Now, using *Corollary 1* and noticing that, because we only add new edges to the initial k-connected ring lattice and this lattice has capacity k, the capacity can only be greater than k, we obtain the desired bounds. □

The result is illustrated in Fig. 3.4.

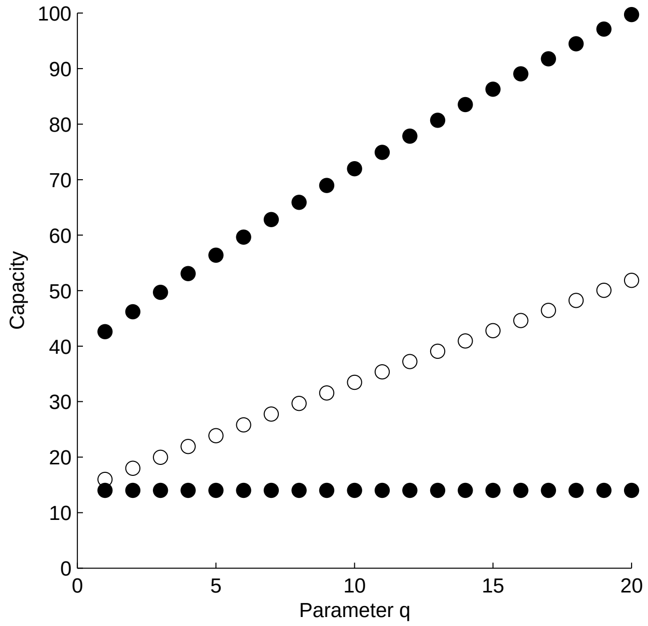

Figure 3.4: Bounds for the capacity of a navigable small-world ring for $n = 1600$, $k = 14$, $r = 1$ e $d = 1$, and different values of the shortcut parameter q. The white circles represent the expected value of the capacity and the black circles represent the bounds computed according to *Theorem 8*.

3.4 Summary

We presented two navigable small-world network models and, using similar definitions to those in [Kle00], we provide an insight on the notion of navigability. We state the results that prove that the two classes of networks studied in this chapter are indeed navigable.

Using the set of tools used in Section 2.2 in Chapter 2, we provide upper and lower bounds on the max-flow min-cut capacity of Kleinberg Networks and of Navigable Small-World Rings.

Chapter 4

Dual Radio Networks

As discussed in Section 1.2, there is a growing interest in communication networks in which the devices have multiple radio interfaces, due to the fact that it is natural to ask if there are significant performance gains when using these multiple wireless interfaces in a combined manner. In this chapter, we will study communication networks with two wireless technologies - *dual radio networks*.

To be able to study the aforementioned class of networks analytically, we provide a rigorous definition of our model for dual radio networks. Next, we study this class of networks in terms of the probability that a dual radio network is connected, and we also provide upper and lower bounds on the min-cut max-flow capacity of a dual radio network.

4.1 Problem Statement

In this section, we give a rigorous definition for the class of networks under consideration in the rest of the chapter.

Definition 9. *A Dual Radio Network (DRN) is a graph $G(n, p, r_s, r_L) = (V, E)$ constructed by the following procedure. Assign n nodes uniformly at random in the set T, where T is the torus obtained by identifying the opposite sides of the box $[0, 1]^2$, and define V as the set of these n nodes. For a parameter r_S, each pair of nodes (a, b), with $a, b \in V$, is connected if their Euclidian distance verifies $d(a, b) \leq r_S$, and let E_S be the set of edges created in this step. Now, for a parameter p, define the set V_L by the following: for node i, $i \in V_L$ with probability p, and repeat this procedure*

$\forall i \in V$. For a parameter r_L, each pair of nodes (a, b), $a, b \in V_L$ is connected if their Euclidian distance satisfies $d(a, b) \leq r_L$, and let E_L be the set of edges created in this step. Finally, the set of edges of a DRN is defined by $E = E_S \cup E_L$.

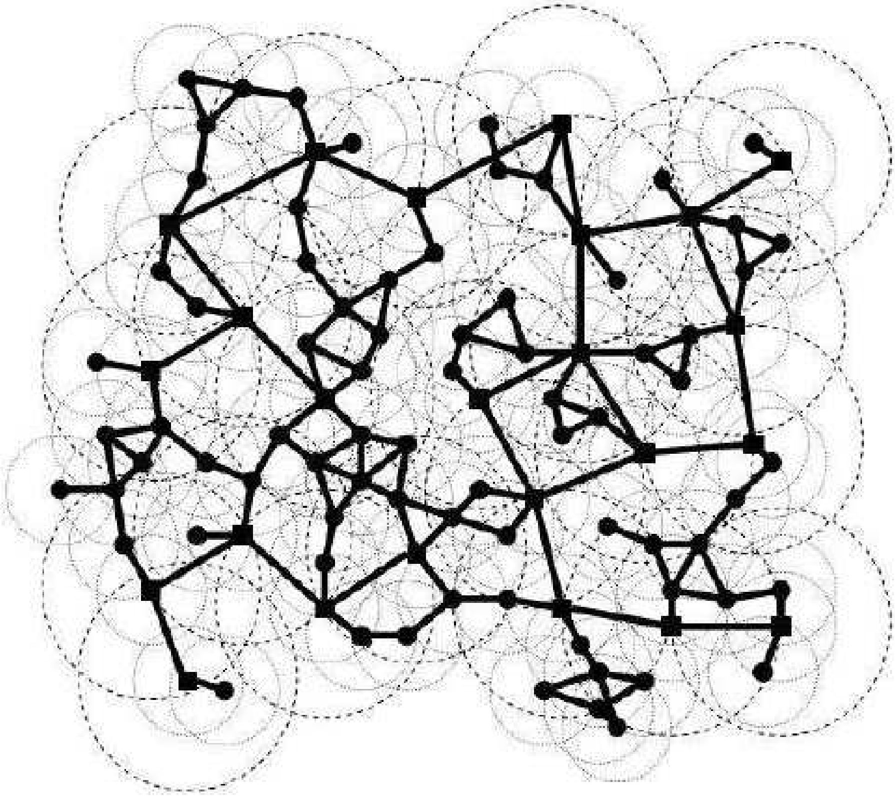

Figure 4.1: Illustration of Dual Radio Networks.
The square nodes represent the devices with two wireless technologies, and the circular nodes represent the nodes with only one wireless technology. The small and large circumferences represent the coverage area of the short-range and long-range wireless interfaces, respectively.

Fig. 4.1 provides an illustration of Dual Radio Networks. In the definition above, notice that, for two nodes $a, b \in V$ such that $r_S < d(a, b) \leq r_L$, they are connected only if both are elements of the set V_L. In terms of the wireless systems that this class of networks pretends to model, this is a realistic feature, since devices with the higher-level wireless technology can only communicate using this technology with devices that also have the higher-level wireless technology.

In the rest of the chapter, we study this class of networks in terms of connectivity and capacity. We say that a network is *connected* if for each pair of nodes there exists a path connecting them. As in Chapter 2 and Chapter 3, we will refer to the global minimum cut of a graph as the max-flow min-cut capacity (or simply the *capacity*) of the graph.

4.2 Results on the connectivity of a Dual Radio Network

In this section, we study the connectivity of the class of networks introduced in Section 4.1, providing an upper and a lower bound on the probability of an instance of a Dual Radio Network being connected.

Lemma 4. *For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that there is no isolated node in $G\left(n, p, r_S, r_L\right)$ satisfies:*

$$\mathcal{P}\{\textit{no isolated node}\} \leq 1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}.$$

Proof. First, we calculate the probability that a node $\mathbf{Y}$ is connected to node $\mathbf{X}$, given the position of $\mathbf{X}$. This probability is given by

$$\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y} | \mathbf{X}) = \mathcal{P}\left(\{d\left(\mathbf{X}, \mathbf{Y}\right) \leq r_S\} \cup \left(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{d\left(\mathbf{X}, \mathbf{Y}\right) \leq r_L\}\right) \middle| \, \mathbf{X}\right).$$

Using the notation $\mathcal{P}\left(A|\mathbf{X}\right) = \mathcal{P}_{\mathbf{X}}(A)$ and $d(\mathbf{X}, \mathbf{Y}) = D$, we have the following:

$$\begin{aligned}
\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) &\overset{(a)}{=} \mathcal{P}_{\mathbf{X}}\left(D \leq r_S\right) + \mathcal{P}_{\mathbf{X}}\left(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}\right) \\
&\quad - \mathcal{P}_{\mathbf{X}}\left(\{D \leq r_S\} \cap \{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}\right) \\
&\overset{(b)}{=} \mathcal{P}_{\mathbf{X}}\left(D \leq r_S\right) + \mathcal{P}_{\mathbf{X}}\left(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}\right) \\
&\quad - \mathcal{P}_{\mathbf{X}}\left(\{D \leq r_S\} \cap \{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\}\right)
\end{aligned}$$

where (a) follows from the fact that for any two events A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B)$, and (b) is justified by noticing that $D \leq r_S \Rightarrow D \leq r_L$, thus $\{D \leq r_S\} \cap \{D \leq r_L\} = \{D \leq r_S\}$.

The events $\{D \leq r_L\}$ and $\{\mathbf{X} \in V_L\}$ are independent, and the same is true for the events $\{D \leq r_L\}$ and $\{\mathbf{Y} \in V_L\}$. Because the set of nodes V_L is formed by nodes selected at random and in an independent fashion, we have that the events $\{\mathbf{X} \in V_L\}$ and $\{\mathbf{Y} \in V_L\}$ are independent. Therefore:

$$\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}) = \mathcal{P}_{\mathbf{X}}(\mathbf{X} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(\mathbf{Y} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(D \leq r_L).$$

Using analogous arguments, we have that

$$\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \le r_S\}) = \mathcal{P}_{\mathbf{X}}(\mathbf{X} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(\mathbf{Y} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(D \le r_S).$$

Noticing that the events $\{\mathbf{X} \in V_L\}$ and $\{\mathbf{Y} \in V_L\}$ are independent of the position of $\mathbf{X}$, we have that $\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) = \mathcal{P}_{\mathbf{X}}(D \le r_S) + \mathcal{P}(\mathbf{X} \in V_L) \cdot \mathcal{P}(\mathbf{Y} \in V_L) \cdot (\mathcal{P}_{\mathbf{X}}(D \le r_L) - \mathcal{P}_{\mathbf{X}}(D \le r_S))$.

Because the set where the nodes are placed is a torus, we have that $\mathcal{P}_{\mathbf{X}}(D \le \rho) = \pi\rho^2$, with $\rho \le 1/\sqrt{\pi}$. Noticing that $\mathcal{P}(\mathbf{X} \in V_L) = \mathcal{P}(\mathbf{Y} \in V_L) = p$, we have that:

$$\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2).$$

Now, to compute the probability that a node at a position $\mathbf{X}$ is isolated, we argue that the events $\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}$, conditioning on the fact that the position of node $\mathbf{X}$ is given (say $\mathbf{X} = (x_1, x_2) = \mathbf{x}$), are mutually independent. Without loss of generality, consider the case $\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} \leftrightarrow \mathbf{Y}_2, \ldots, \mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}, \mathbf{X} = \mathbf{x})$. We have that

$$\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} \leftrightarrow \mathbf{Y}_2, \ldots, \mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}, \mathbf{X} = \mathbf{x}) = \mathcal{P}(\mathbf{Y}_1 \leftrightarrow \mathbf{x} | \mathbf{Y}_2 \leftrightarrow \mathbf{x}, \ldots, \mathbf{Y}_{n-1} \leftrightarrow \mathbf{x}),$$

where we exploited the fact that the position of $\mathbf{X}$ is fixed. Now, notice that none of the events $\{\mathbf{Y}_2 \leftrightarrow \mathbf{x}\}, \ldots, \{\mathbf{Y}_{n-1} \leftrightarrow \mathbf{x}\}$ affects the event $\{\mathbf{Y}_1 \leftrightarrow \mathbf{x}\}$, because we do not have information about the existence of connection between $\mathbf{Y}_1$ and any of the $\mathbf{Y}_i$. Therefore,

$$\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} \leftrightarrow \mathbf{Y}_2, \ldots, \mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}, \mathbf{X} = \mathbf{x}) = \mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} = \mathbf{x}).$$

Since we can use similar arguments for different subsets of the collection

$$\{\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}\},$$

we have that the events $\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}$ are mutually independent, conditioned on the fact that the position of node $\mathbf{X}$ is given.

The probability that a node at a position $\mathbf{X}$ is isolated is given by

$$\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \text{ is isolated}\}) = \mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \not\leftrightarrow \mathbf{Y}_1\} \cap \{\mathbf{X} \not\leftrightarrow \mathbf{Y}_2\} \cap ... \cap \{\mathbf{X} \not\leftrightarrow \mathbf{Y}_{n-1}\}).$$

Because the events $\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, ..., \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}$ are mutually independent, conditioned on the position of node $\mathbf{X}$, the same is true for the events $\{\mathbf{X} \not\leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \not\leftrightarrow \mathbf{Y}_2\}, ..., \{\mathbf{X} \not\leftrightarrow \mathbf{Y}_{n-1}\}$.

Thus,

$$\mathcal{P}_{\mathbf{X}}(\{\mathbf{X}\text{ is isolated}\}) = \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}.$$

Therefore, the probability of a node being isolated is given by:

$$\begin{aligned}\mathcal{P}(\{\text{a node is isolated}\}) &= \int_0^1 \int_0^1 \mathcal{P}_{\mathbf{X}}(\{\mathbf{X}\text{ is isolated}\})\mathrm{d}\mathbf{x} \\ &= \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}. \end{aligned} \tag{4.1}$$

Now, let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ represent the nodes of the graph. We have that:

$$\begin{aligned}\mathcal{P}(\{\text{no isolated node}\}) &= \mathcal{P}(\{\mathbf{X}_1 \text{ is not isolated}\} \cap \cdots \cap \{\mathbf{X}_n \text{ is not isolated}\}) \\ &= 1 - \mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \cdots \cup \{\mathbf{X}_n \text{ is isolated}\}).\end{aligned}$$

Now, notice that

$$\mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \cdots \cup \{\mathbf{X}_n \text{ is isolated}\}) \geq \mathcal{P}(\mathbf{X}_1 \text{ is isolated}).$$

Therefore, by (4.1), we have that

$$\mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \cdots \cup \{\mathbf{X}_n \text{ is isolated}\}) \geq \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}$$

and the result follows. □

After determining the probability of having an isolated node in a DRN, we calculate a bound on the probability that a DRN is disconnected.

Lemma 5. *For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that $G(n, p, r_S, r_L)$ is disconnected, $P_d(n, p, r_S, r_L)$, satisfies:*

$$P_d(n, p, r_S, r_L) \leq \frac{1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^n}{\pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)} - 1.$$

Proof. For $k > 1$, select a node from $G(k, p, r_S, r_L)$, say node k. To $G(k, p, r_S, r_L)$ be disconnected, or node k is isolated, or the subgraph obtained by removing node k and all its edges (which can be viewed as $G(k-1, p, r_S, r_L)$) is disconnected. Thus, we have that $\{G(k, p, r_S, r_L) \text{ is disconnected}\} = \{G(k-1, p, r_S, r_L) \text{ is disconnected}\} \cup \{\text{node } k \text{ is isolated}\}$. Therefore:

$$P_d(k, p, r_S, r_L) \leq \mathcal{P}(\text{node } k \text{ is isolated in } G(k, p, r_S, r_L)) + P_d(k-1, p, r_S, r_L).$$

After recursion, we have that:

$$\begin{aligned}
P_d(n,p,r_S,r_L) &\leq \mathcal{P}(a\,node\,is\,isolated\,in\,G(2,p,r_S,r_L)) \\
&\quad + \sum_{k=3}^{n} \mathcal{P}(node\,k\,is\,isolated\,in\,G(k,p,r_S,r_L)) \\
&\stackrel{(a)}{\leq} 1-\pi r_S^2 - \pi p^2(r_L^2-r_S^2) + \sum_{k=3}^{n}\left(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2)\right)^{k-1} \\
&\leq 1-\pi r_S^2 - \pi p^2(r_L^2-r_S^2) + \sum_{k=2}^{n-1}\left(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2)\right)^{k} \\
&\leq \sum_{k=1}^{n-1}\left(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2)\right)^{k}.
\end{aligned}$$

where (a) follows from (4.1).

Because $\sum_{k=1}^{n} a^k = \frac{a-a^{n+1}}{1-a}$, we have that

$$P_d(n,p,r_S,r_L) \leq \frac{1-\pi r_S^2-\pi p^2(r_L^2-r_S^2)-(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2))^n}{\pi r_S^2+\pi p^2(r_L^2-r_S^2)}$$

and the result follows. □

Using the previous two lemmas, we are able to state our main result in terms of connectivity.

Theorem 9. *For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that $G(n,p,r_S,r_L)$ is connected, $P_c(n,p,r_S,r_L)$, satisfies:*

$$P_c(n,p,r_S,r_L) \geq \max\{2-\frac{1-(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2))^n}{\pi r_S^2+\pi p^2(r_L^2-r_S^2)},0\}$$

and

$$P_c(n,p,r_S,r_L) \leq 1-\left(1-\pi r_S^2-\pi p^2(r_L^2-r_S^2)\right)^{n-1}.$$

Proof. It is easy to see that:

$$P_c(n,p,r_S,r_L) \leq \mathcal{P}\left(\{no\,isolated\,node\,in\,G(n,p,r_S,r_L)\}\right).$$

Thus, using *Lemma 4*, we have the upper bound for $P_c(n,p,r_S,r_L)$. Noticing that $P_c(n,p,r_S,r_L) = 1-P_d(n,p,r_S,r_L)$, using *Lemma 5* and taking the maximum between the lower bound obtained and zero (because a probability is always lower bounded by zero), the result follows. □

4.3 Capacity Results for Dual Radio Networks

In this section, we study the max-flow min-cut capacity of a Dual Radio Network. It is important to notice that, for this goal, is not possible to use the set of tools used in Chapter 2 and Chapter 3 to derive bounds on the capacity of the networks of interest there. This happens due to the fact that, when considering a Dual Radio Network, we do not have a base network (or metric) to start with, because the position of the nodes is also random. Therefore, we need to use a different approach, which is described next, to study the capacity of a Dual Radio Network.

We consider a multiple-source multiple-terminal transmission on a DRN with n nodes, denoting by $\{s_1, \ldots, s_\alpha\}$ the set of the α sources and by $\{t_1, \ldots, t_\beta\}$ the set of the β terminals. Let i and j be two nodes of a DRN. C_{ij} is the capacity of the edge (i, j), defined by $C_{ij} = 1$ if nodes i and j are connected, and $C_{ij} = 0$ otherwise. This means that $C_{ij} = 1$, if $d(i, j) \leq r_S$ or $i \in V_L \wedge j \in V_L \wedge d(i, j) \leq r_L$, and $C_{ij} = 0$ otherwise. Notice that $E\{C_{ij}\} = \mathcal{P}(i \leftrightarrow j)$ and, as we have seen in Section 4.2:

$$\mathcal{P}(i \leftrightarrow j) = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2),$$

with $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, which we assume in the following. Let $\mu = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)$. The techniques used for proving the following results are similar to those used in [RSW05].

Before stating and proving the results on the capacity of Dual Radio Networks, we present a useful inequality.

Lemma 6 (Hoeffding's inequality, from [Hoe63]). *For $X_1, X_2, \ldots, X_m$ independent random variables with $\mathcal{P}(X_i \in [a_i, b_i]) = 1$, $\forall i \in \{1, 2, \ldots, m\}$, if we define $S = X_1 + X_2 + \cdots + X_m$, then:*

$$\mathcal{P}(S - E(S) \geq mt) \leq \exp\left(-\frac{2m^2t^2}{\sum_{i=1}^{m} (b_i - a_i)^2} \right).$$

First, we determine an upper bound on the probability that the capacity of a cut does not take a value much greater than its expected value.

Lemma 7. *Let G be a random instance of a DRN, and consider a single-source single-terminal transmission (i.e. $\alpha = \beta = 1$). Let N be the number of relay nodes, i.e. $N = n - 2$. Let C_k be the capacity of a cut in G in which one of the partitions consists* *of k nodes and the source. For $\epsilon > 0$ and $N \geq 2$,*

$$\mathcal{P}(C_k \leq (1-\epsilon)E\{C_k\}) \leq e^{-(N+1+k(N-k))\mu^2\epsilon^2/N^2}.$$

Proof. We have that:

$$\mathcal{P}(C_k \leq (1-\epsilon)E(C_k)) = \mathcal{P}(-C_k - E(-C_k) \geq \epsilon E(C_k)). \tag{4.2}$$

To compute the desired upper bound, we shall use the Hoeffding's inequality (*Lemma 6*). More precisely, we shall use this inequality for $m = 1$. First, notice that C_k is upper bounded by the value of a similar cut in the complete graph, i.e.

$$C_k \leq (k+1)(N-k+1) = N+1+k(N-k).$$

Therefore, we have that $C_k \in [0, N+1+k(N-k)]$. Thus, applying Hoeffding's inequality in (4.2), we have that

$$\mathcal{P}(C_k \leq (1-\epsilon)E(C_k)) \leq \exp\left(-\frac{2\epsilon^2(E(C_k))^2}{(N+1+k(N-k))^2}\right). \tag{4.3}$$

Now, notice that C_k is the sum of $N+1+k(N-k)$ random variables of the form C_{ij}, with $C_{ij} = 1$, if $i \leftrightarrow j$ and $C_{ij} = 0$, if $i \not\leftrightarrow j$, i.e. i is not connected to j. Therefore, for each of these random variables, we have that $E(C_{ij}) = \mathcal{P}(i \leftrightarrow j) = \mu$. Thus:

$$E(C_k) = (N+1+k(N-k))\mu.$$

Now, notice that $N+1+k(N-k) \leq 2N^2$, for $N \geq 2$, thus $\frac{1}{N+1+k(N-k)} \geq \frac{1}{2N^2}$, for $N \geq 2$. Therefore

$$\exp\left(-\frac{2\epsilon^2(E(C_k))^2}{(N+1+k(N-k))^2}\right) \leq \exp\left(-\frac{(N+1+k(N-k))\mu^2\epsilon^2}{N^2}\right).$$

Thus, by (4.3), the result follows. □

Using the previous result, we obtain an useful inequality.

Corollary 2. *Let C_k and N be as defined in Lemma 7 and let A_k be the event $\{C_k < (1-\epsilon)E\{C_k\}\}$. Then:*

$$\mathcal{P}(\cup_k A_k) \leq 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N.$$

Proof. By *Lemma 7*, we have that $\mathcal{P}(A_k) \leq e^{-(N+1+k(N-k)\mu^2)\epsilon^2/N^2}$, which also provides:

$$\mathcal{P}(A_k) \leq e^{-(N+k(N-k))\mu^2\epsilon^2/N^2}.$$

Notice that, for each $k \in \{0, ..., N\}$, there are $\binom{N}{k}$ cuts in which one of the partitions consists on k nodes and the source. Therefore:

$$\begin{aligned} \mathcal{P}(\cup_k A_k) &\leq \sum_{k=0}^{N} \binom{N}{k} \mathcal{P}(A_k) \\ &\leq \sum_{k=0}^{N} \binom{N}{k} e^{-(N+k(N-k))\mu^2\epsilon^2/N^2} \end{aligned}$$

Let $\beta = e^{-\mu^2\epsilon^2/N}$. Then:

$$\begin{aligned} \mathcal{P}(\cup_k A_k) &\leq \beta \sum_{k=0}^{N} \binom{N}{k} \beta^{N\frac{k}{N}(1-\frac{k}{N})} \\ &= \beta \left(\sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N}{k} \beta^{N\frac{k}{N}(1-\frac{k}{N})} + \sum_{k=\lfloor N/2 \rfloor+1}^{N} \binom{N}{k} \beta^{N\frac{k}{N}(1-\frac{k}{N})} \right). \end{aligned}$$

Notice that, when $\frac{k}{N} \in [0, 1/2]$,

$$\frac{k}{N}(1 - \frac{k}{N}) \geq \frac{k}{2N},$$

and when $\frac{k}{N} \in [1/2, 1]$,

$$\frac{k}{N}(1 - \frac{k}{N}) \geq \frac{N-k}{2N}.$$

Therefore:

$$\begin{aligned} \mathcal{P}(\cup_k A_k) &\leq \beta \left(\sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N}{k} \beta^{N\frac{k}{2N}} + \sum_{k=\lfloor N/2 \rfloor+1}^{N} \binom{N}{k} \beta^{N\frac{1}{2}(1-\frac{k}{N})} \right) \\ &\leq \beta \left(\sum_{k=0}^{N} \binom{N}{k} \left(\beta^{\frac{1}{2}}\right)^k + \sum_{k=0}^{N} \binom{N}{k} \left(\beta^{\frac{1}{2}}\right)^{N-k} \right) \\ &\overset{(a)}{\leq} 2\beta(1+\sqrt{\beta})^N \\ &\overset{(b)}{\leq} 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N \end{aligned}$$

where (a) follows from the fact that $(x+y)^m = \sum_{k=0}^{m} \binom{m}{k} x^k y^{m-k}$, thus

$$\sum_{k=0}^{N} \left(\sqrt{\beta}\right)^k = (1+\sqrt{\beta})^N = \sum_{k=0}^{N} \left(\sqrt{\beta}\right)^{N-k},$$

and (b) follows from substituting β by $e^{-\mu^2\epsilon^2/N}$. □

Now, using *Corollary 2*, we obtain the first result related to the capacity of a DRN, which is valid for the single-source single-terminal transmission problem.

Corollary 3. *Let $C_{min}(s_1 \to t_1)$ be the global minimum cut in an instance of DRN. Then:*

$$\mathcal{P}(C_{min}(s_1 \to t_1) \leq (1-\epsilon)(N+1)\mu) \leq 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N.$$

Proof. Let $\tilde{A}_k$ be the event $\{C_k < (1-\epsilon)E\{C_0\}\}$ and let A_k be the event $\{C_k < (1-\epsilon)E\{C_k\}\}$. We have that $E\{C_k\} = (N+1+k(N-k))\mu$. Therefore, $E\{C_k\} \geq E\{C_0\}$, $\forall k \in 0, ..., N$. Thus $\tilde{A}_k \subseteq A_k$, which implies that $\cup_k \tilde{A}_k \subseteq \cup_k A_k$. Therefore,

$$\begin{aligned} \mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)E\{C_0\}) &= \mathcal{P}(\cup_k \tilde{A}_k) \\ &\leq \mathcal{P}(\cup_k A_k). \end{aligned}$$

Using *Corollary 2* and noticing that $E\{C_0\} = (N+1)\mu$, the result follows. □

Now, we are ready to state our main result in terms of capacity of a DRN:

Theorem 10. *Let $C_{min}(\alpha, \beta)$ be the global minimum cut for a transmission with α sources and β terminals, in an instance of a DRN. Let $\epsilon = \sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$ with $d > 0$, and $\mu = \pi r_S^2 + \pi p^2(r_L^2 - r_S^2)$. Then:*

$$C_{min}(\alpha, \beta) > (1-\epsilon)(n-1)\mu$$

with probability $1 - O\left(\frac{\alpha\beta}{n^{2d}}\right)$, and

$$C_{min}(\alpha, \beta) < (1+\epsilon)\alpha(n-\alpha)\mu$$

with probability $1 - O\left(\frac{1}{n^{4nd}}\right)$.

Proof. Recall that, for a single-source single terminal transmission, $N = n-2$. Therefore, $\mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu) = \mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(N+1)\mu)$. Thus, replacing ϵ in *Corollary 3* by the expression $\sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}} = \sqrt{\frac{2Nd\ln N}{\mu^2}}$, we have that:

$$\begin{aligned} \mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu) &\leq 2e^{\frac{-2dN\mu^2 \ln N}{N\mu^2}} \cdot [1 + e^{\frac{-2dN\mu^2 \ln N}{2N\mu^2}}]^N \\ &\leq \frac{2}{N^{2d}} \cdot \left[1 + \frac{1}{N^d}\right]^N. \end{aligned}$$

We have that $(x+y)^N = \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}$, thus:

$$\left[1 + \frac{1}{N^d}\right]^N = \sum_{k=0}^{N} \binom{N}{k} \left(\frac{1}{N^d}\right)^k .$$

Therefore, we have that:

$$\begin{aligned}
\mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu) &\leq \frac{2}{N^{2d}} \cdot \sum_{k=0}^{N} \binom{N}{k} \left(\frac{1}{N^d}\right)^k \\
&\overset{(a)}{\leq} \frac{2}{N^{2d}} \cdot \sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k \\
&\overset{(b)}{\leq} \frac{2}{N^{2d} - N^{d+1}} \\
&\approx O\left(\frac{1}{N^{2d}}\right) \\
&= O\left(\frac{1}{n^{2d}}\right)
\end{aligned}$$

where:

- (a) follows from the fact that $\binom{N}{k} = \frac{N!}{(N-k)!k!} = \frac{N \times (N-1) \times \cdots \times (N-k+1)}{k!}$, thus $\binom{N}{k} \leq N \times (N-1) \times \cdots \times (N-k+1) \leq N^k$;
- (b) follows from the fact that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, therefore:

$$\sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k = \frac{1}{1 - N^{1-d}},$$

  which implies that $\frac{2}{N^{2d}} \cdot \sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k = \frac{2}{N^{2d} - N^{d+1}}$

Now, back to the multiple-source multiple-terminal transmission, we have that

$$\mathcal{P}(C_{\min}(\alpha, \beta) \leq (1-\epsilon)(n-1)\mu) = \mathcal{P}\left(\cup_{i=1}^{\alpha} \cup_{j=1}^{\beta} \{C_{\min}(s_i \to t_j) \leq (1-\epsilon)(n-1)\mu\}\right).$$

Therefore, by the union bound:

$$\mathcal{P}(C_{\min}(\alpha, \beta) \leq (1-\epsilon)(n-1)\mu) \leq \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \mathcal{P}\left(C_{\min}(s_i \to t_j) \leq (1-\epsilon)(n-1)\mu\right).$$

From the fact that, as we derive in *Corollary 3*, $\mathcal{P}\left(C_{\min}(s_i \to t_j) \leq (1-\epsilon)(n-1)\mu\right)$ does not depend on nodes i and j, we have that $\mathcal{P}(C_{\min}(\alpha,\beta) \leq (1-\epsilon)(n-1)\mu) \leq \alpha\beta\mathcal{P}\left(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu\right)$. Therefore, we have that:

$$\mathcal{P}(C_{\min}(\alpha,\beta) \geq (1-\epsilon)(n-1)\mu) \geq 1 - \alpha\beta \cdot \mathcal{P}\left(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu\right)$$

and, because we already proved that $\mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(n-1)\mu) = O\left(\frac{1}{n^{2d}}\right)$, the first part of the theorem follows.

Now, to compute the upper bound on $\mathcal{P}(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu)$, notice that, by definition, any cut (that contains in one partition the source nodes and in the other partition the terminal nodes) is greater or equal to $C_{\min}(\alpha,\beta)$. Thus, the value of the cut in which one of the partitions consists of source nodes only (denoted by $C^*(\alpha,\beta)$) is greater or equal to $C_{\min}(\alpha,\beta)$. This means that, if $C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu$, then $C^*(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu$. Therefore, because $\mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu\right) = \mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(N+\beta)\mu\right)$, we have that:

$$\mathcal{P}(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu) \leq \mathcal{P}(C^*(\alpha,\beta) \geq (1+\epsilon)\alpha(N+\beta)\mu),$$

which is equivalent to

$$\mathcal{P}(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu) \leq \mathcal{P}(C^*(\alpha,\beta) - \alpha(N+\beta)\mu \geq \epsilon\alpha(N+\beta)\mu).$$

Noticing that $C^*(\alpha,\beta) \in [0, \alpha(N+\beta)]$, $E(C^*(\alpha,\beta)) = \alpha(N+\beta)\mu$, and applying Hoeffding's inequality (*Lemma 6*), we have that

$$\mathcal{P}(C^*(\alpha,\beta) - \alpha(N+\beta)\mu \geq \epsilon\alpha(N+\beta)\mu) \leq \exp\left(-\frac{2\epsilon^2\alpha^2(N+\beta)^2\mu^2}{\alpha^2(N+\beta)^2}\right).$$

Therefore

$$\begin{aligned}\mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu\right) &\leq \exp\left(-2\epsilon^2\mu^2\right) \\ &\stackrel{(a)}{\leq} \frac{1}{N^{4Nd}}\end{aligned}$$

where (a) follows from the substitution of ϵ by $\sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$. Thus, we have that

$$\mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu\right) = \mathrm{O}\left(\frac{1}{n^{4nd}}\right)$$

and the result follows. □

This result shows that the capacity grows linearly with $\mu = \pi r_S^2 + \pi p^2(r_L^2 - r_S^2)$. Thus, the capacity for a multiple-source multiple-terminal transmission grows quadratically in function of the parameter p, which represents the percentage of nodes with two wireless technologies. Thus, this result shows that there is a significant benefit (in terms of capacity) by using dual-radio schemes in wireless systems.

Setting $\alpha = \beta = 1$ in *Theorem 10*, we obtain the following bounds for the capacity of a single-source single-terminal transmission:

Corollary 4. *Let C_{min} be the global minimum cut for a single-source single-terminal transmission in an instance of a DRN. Let $\epsilon = \sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$, and $\mu = \pi r_S^2 + \pi p^2(r_L^2 - r_S^2)$. Then:*

$$C_{min} > (1-\epsilon)(n-1)\mu$$

with probability $1 - O\left(\frac{1}{n^{2d}}\right)$, and

$$C_{min} < (1+\epsilon)(n-1)\mu$$

with probability $1 - O\left(\frac{1}{n^{4nd}}\right)$.

4.4 Summary

We defined a class of random geometric graphs that models a wireless network in which all devices share the same short-range radio capability, and some of them have a secondary long-range wireless interface. For this class of networks, we provided upper and lower bounds on the probability of its connectivity. We also provided bounds for the capacity of this class of networks, showing that the use of dual radio technologies can improve the capacity of the network. Specifically, we proved that the capacity of our model grows quadratically with the fraction of devices with two wireless interfaces, which shows that there is a significant benefit (in terms of capacity) by using dual-radio schemes in wireless systems.

Chapter 5

Conclusions

We studied the max-flow min-cut capacity of different classes of networks. Regarding small-world networks, we presented bounds on the capacity of four different models for small-worlds: two simple models and two navigable ones. This notion of *navigability*, which is important when considering communication networks, is defined and results on the navigability of the models for small-world networks are presented. Regarding capacity, we present a somehow surprising result. The bounds derived for small-world networks with rewiring permit the following interpretation: *With high probability, rewiring does not alter the capacity of the network*. This observation is not obvious, because we can easily find ways to rewire the ring lattice in order to obtain, for instance, a *bottleneck*. But, according to the previous results, such instances occur with very low probability.

In the second part of the thesis, we defined a class of random geometric graphs that models a wireless network in which all devices share the same short-range radio capability, and some of them have a secondary long-range wireless interface. For this class of networks, we provided upper and lower bounds on the probability of its connectivity. We also provided bounds for the capacity of this class of networks, showing that the use of dual radio technologies can improve the capacity of the network. Specifically, we showed that the capacity of our model grows quadratically with the fraction of devices with two wireless interfaces.

5.1 Future Work

Possible directions for future work in small-world networks include tighter capacity results, extensions to other classes of small-world networks (e.g. weighted models and those used in peer-to-peer networks [MNW04]), and understanding if and how small-world topologies can be exploited in the design of capacity-attaining network codes and distributed network coding algorithms. At a more conceptual level, we are intrigued by the possibility that the notion of capacity may help us answer a very central question: *why* small-world topologies are ubiquitous in real-world networks.

As part of our ongoing work in dual radio networks, we are analyzing the diameter and the clustering coefficient of dual radio networks and exploring their relationship with small-world networks.

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