Reach Set Formulation of a Model Predictive Control Scheme*

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1 Introduction

In this article, we introduce a new formulation of the Model Predictive Control (MPC), also designated by receding horizon control, scheme designed having in mind applications with hard real-time computational constraints.

Although the control scheme and results presented in this article are generic, the main motivation relies on the coordinated control of the motion of multiple Autonomous Underwater Vehicles (AUVs). Given the features exhibited by this class of control problems, notably

(a) modeling uncertainties, motion perturbations, environment variability, emergence of obstacles, etc., making the choice of receding horizon type of control schemes quite natural;

(b) performance optimization requirements subject to a number of very diverse type of constraints, for which the versatility of the optimal control paradigm is particularly well suited,

it is not surprising that a lot of work has been done proposing various types of MPC schemes, [8, 5, 6, 13], among others. The basic version consists in an iterative procedure involving the following ordered steps:

1. Initialization. Let t_0 be the current time, and set up the initial parameters or conditions specifying the initial state, prediction horizon and control horizon, respectively, x_0, T, and Δ, and, possibly other parameters.
2. Sample the state variable at time t_0.

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3. Compute the optimal control strategy, u^*, in the prediction optimal, i.e., $[t_0, t_0 + T]$, by solving the optimal control problem:

$$\begin{aligned}(P)\ \text{Minimize}\ & g(x(t_0+T)) + \int_{t_0}^{t_0+T} f_0(t, x(t), u(t))dt \\ \text{subject to}\ & \dot{x}(t) = f(t, x(t), u(t)),\ \ u(t) \in \Omega,\ \ \mathcal{L}\text{-a.e.} \\ & h(t, x(t)) \leq 0,\ \forall t,\ x(t_0+T) \in C_f,\end{aligned}$$

   where g is the endpoint cost functional, f_0 is the running cost integrand, f, and h represent, respectively, the vehicle dynamics, and the state constraints, C is a target set that may also be specified in order to ensure stability.

4. Apply the obtained optimal control during the current control horizon, $[t_0, t_0 + \Delta]$.

5. Slide time by Δ, i.e., $t_0 = t_0 + \Delta$, and adapt parameters and models as needed.

6. Goto step 2.

Excellent surveys have been considered in a number of seminal publications addressing a wide spectrum of important issues such as stability, sub-optimality, robustness, decentralized schemes, etc., e.g., [16, 17, 10, 6, 2, 7, 11, 13] and references cited therein.

Obviously, the price to pay for the long term optimization is the computationally intensive character of the control synthesis. In order to cope with this, the developments reported in [8, 22], which follow along the ones in [13], concern an MPC control scheme for the coordinated control of a formation of AUVs based on a linear quadratic optimal control problem. This formulation is particularly useful because it brings together the computational advantages of existing extremely efficient numerical solvers and the flexibility exhibited by the conventional optimal control problem formulation enabling the incorporation of a wide range of control and state constraints that arise in the considered class of problems.

As it was pointed out in [22], this scheme suits well the coordinated control of a formation of AUVs whose requirements typically involve the need of, [23]:

- Optimization of onboard resources.
- Flexibility of the control problem formulation.
- Feedback control synthesis enabling to deal with significant uncertainty.
- Amenability to decentralization [19, 20].

Unfortunately, these welcome features of the MPC control framework come with a very high computational complexity which is a prohibitively high price, particularly for applications involving a large number of vehicles subject to real-time requirements under very strict power constraints. The need to ensure the computational tractability of discrete time hybrid linear systems optimization, the modeling framework Mixed Logical Dynamical (MLD) for PieceWise Affine (PWA) dynamic control systems has been developed in [2]. Besides being a well posed framework to bridge the continuum time driven dynamics and the logical world, this modeling framework is also amenable for the tool HYSDEL, [12]. Affine controls of the bang-bang type are pre-computed off-line and then selected on-line. A similar idea is used in [7] to apply a MPC scheme to optimize the behavior of a power electronics system with extremely fast dynamics. In [13], a precious advantage is taken from available efficient linear quadratic solvers to address the real time constraints associated with the control of formations of aerial vehicles.

Here, we propose an MPC-like control that substantially reduces the on-line computational burden associated with the conventional MPC scheme. Its main features consist in:

- Replacing the optimization over the control space by another one over a local polyhedral approximation to the reachable set.
- Back-propagating the long term cost functional to the final time of the current control horizon, via the associated value function.
- Performing the optimization for the control synthesis over the control horizon which is much shorter than the control horizon considered in the usual MPC schemes.

A few observations are in order. First, the reachable set and the value function can be estimated off line by taking into account all the (state and control) constraints as a function of the initial state. Second, the fact that we consider polyhedral approximations to the reachable sets allow us to deal with optimization problems subject to inequality affine constraints, and, thus to deal with efficient optimizations solvers such as the one considered in [22]. Third, the difficulties inherent to the linearization of the control system dynamics, either by first order approximation with its associated control issues, or by lower level feedback control deeply compromising the overall optimization, are avoided. Fourth, uncertainties and perturbations can be dealt with by considering mini-max optimization schemes where the synthesized control optimizes the worst case due to the their potential effect.

This article is organized as follows. In the next section, the basic reach set based MPC scheme is presented and its equivalence to the conventional scheme justified. Based on the results of the previous sections, a multi intermediate step reach set algorithm is presented in the fifth section and its implementation is discussed. The article is closed with some brief conclusions.

2 The basic Reach Set Based MPC Scheme

In this section, we formulate the reachable set based MPC scheme in the context of approximating a long (possibly infinite) time horizon optimization problem by a sequence of sliding shorter time horizon sub-problems initialized with the current sampled state.

First, we consider the optimization of a dynamic control system over a very long time horizon $[0, T]$ (which, with some adaptations, could well be infinite) as follows.

$$(P_{T_f})\ \text{Minimize}\ g(x(T_f)) + \int_0^{T_f} l(t, x(t), u(t))dt$$
$$\text{subject to}\ \dot{x}(t) = f(t, x(t), u(t)),\ \ u(t) \in \Omega,\ \ \mathcal{L} - a.e.$$
$$x(T_f) \in C_f,\ x(0)\ \text{is given.}$$

Note that, for infinite horizon, we may consider trajectories converging asymptotically to some equilibrium point in the state space. Now, let $T < T_f$. Then, by the Principle of Optimality, [15], the following equivalent finite horizon optimal control problem can be formulated:

$$(P_T)\ \text{Minimize}\ V(t{+}T, x(t{+}T)) + \int_t^{t+T} l(\tau, x(\tau), u(\tau))d\tau$$
$$\text{subject to}\ \dot{x}(\tau) = f(\tau, x(\tau), u(\tau)),\ \ u(t) \in \Omega,\ \ \mathcal{L} - a.e. \tag{1}$$

with $x(t)$ given. Here,

$$V(t, z) := \min_{u\in\mathcal{U}, \xi\in C_f} \{g(\xi) + \int_t^{T_f} l(\tau, x(\tau), u(\tau))d\tau : x(T_f){=}\xi,$$
$$x(t){=}z,\ \dot{x}(\tau){=}f(\tau, x(\tau), u(\tau)),\ \mathcal{L}\text{-a.e.}\}$$

The value function can be propagated by solving the Hamilton-Jacobi-Bellman equation, [1], which, for problems with null running cost and finite horizon, may be represented by:

$$\frac{\partial}{\partial t}V(t,x) + \min_{u\in\Omega}\left\{\langle\frac{\partial}{\partial x}V(t,x), f(t,x,u)\rangle + l(t,x,u)\right\} = 0,\ \ V(T_f, x(T_f)) = g(x(T_f)).$$

In general, the value function is, at most, merely continuous, which means that the partial derivatives have to be understood in some generalized sense, and the solution concept has to be cast in a nonsmooth context, [1]. Moreover, there are a number of results characterizing the interplay between value functions and the forward and backward Reach sets, [15].

There are a number of software packages to solve the Hamilton-Jacobi-Bellman equation numerically and thus compute the value function, see, for example, [21, 18, 3]. Once computed, the value function can be stored in a look-up table and invoked to determine the next optimal control at any point (t, x) in time and phase space. The value function might have to be updated when there are changes in the environment or in the system that affect the formulation of the underlying optimal control problem .

Now, in order to derive the Reach Set formulation of this MPC scheme, let us define the forward reach set at time t, from the state x_0 and time $t_0 \le t$, [**?**], by

$$\mathcal{R}_f(t; t_0, x_0) := \{x(t) : \dot{x} = f(t, x, u),\ u \in \mathcal{U},\ x(t_0) = x_0\}.$$

Notice that $\mathcal{R}_f(t; t_0, C_0) = \bigcup_{x_0\in C_0} \mathcal{R}_f(t; t_0, x_0)$. In a similar way, for some $t \le t_1$, the backward reach set is given by $\mathcal{R}_b(t; t_1, C_1) = \{z \in \boldsymbol{R}^n : \mathcal{R}_f(t_1; t, z) \cap C_1 \ne \emptyset\}$.

In order to facilitate the presentation and without any loss of generality, we proceed with a standard change of variable to eliminate the running cost. Let

$\tilde{V}(t,\tilde{x}) = V(t,x) + y$ where $\tilde{x} = (x,y)$, being $\dot{y} = l(t,x,u)$, with $y(0) = 0$. Without relabeling (i.e., $x = \tilde{x}$, and $V = \tilde{V}$), the optimal control problem (P_T) can be expressed in terms of reach sets and the value function as follows:

$$(P_T) \text{ Minimize } V(t+T, x(t+T)) \text{ subject to } x(t+T) \in \mathcal{R}_f(t+T; t, x(t)).$$

Let T be the prediction horizon, Δ, the control horizon, and t the current time. Then, the MPC scheme can be formulated as follows:

1. Initialization: $t = t_0$, $x(t_0)$
2. Solve (P_T) over $[t, t+T]$ to obtain u^*
3. Apply u^* during $[t, t+\Delta]$
4. Sample x at $t+\Delta$ to obtain $\bar{x} = x(t+\Delta)$
5. Slide time, i.e., $t = t+\Delta$, and goto 2.

Under mild assumptions on the data and, by using the fact that the value function is continuous, it has been shown that this scheme is robust. Stability is proved by showing that there exists an uniform neighborhood along the reference trajectory for which the value function satisfies a Lyapunov inequality in a generalized sense. From the continuity of the value function, we obtain sub-optimality estimates in both global and local senses as a consequence of the asymptotic performance convergence of this MPC-like control scheme.

In order to overcome the difficulties inherent to the Reach Set complexity, a scheme to compute both inner and outer polyhedral approximations was derived in [9]. The advantage of this scheme with respect to other approaches, [14], relies on the fact that it yields affine constraints in the optimization problems to be solved on-line.

3 The Multistep Reach Set Based MPC Scheme

Now, we introduce a variant of the previous Reach Set based scheme that ensures robustness to short term perturbations. If changes in the dynamics are detected or more accurate characterizations of uncertainties and perturbations are available, then, in order to ensure the feasibility of the control, a new estimate of the local reach set for which a new approximation will have to be computed is required. Moreover, the scheme introduced in the previous section suffers from a major weakness. If sampling fails (this is a reasonable event in the networked control of multiple vehicles context), then the scheme described in the previous section does not provide recently computed controls, and the only possibility is to proceed with totally simulated data. To overcome this weakness, a multi intermediate step reach set based scheme is proposed to enable a richer trade-off between complexity, robustness and sub-optimality which can be adjusted to the available on-board resources. This scheme consists in (i) considering a certain sub-optimality for each optimization step, and in (ii) organizing this step in a number of intermediate steps for which only feasibility is required.

1. Initialization. Take $T = N_T\Delta$, N, $\tilde{N}$.

2. Optimization step. Let $z_0^* = x(t)$ and compute $\mathcal{R}_f^N(t+T; t, x(t))$.
3. Compute $z_{N_T}^* = \underset{z \in \mathcal{R}_f^N(t+T;t,z_0^*)}{\arg\min} \{V(t+T, z)\}$.
4. Intermediate control steps. For $i = 1$ to N_T, let $t_i = t + i\Delta$, and compute:
   (i) $\mathcal{R}_f^N(t_i; t_{i-1}, z_{i-1}^*)$, and $\mathcal{R}_b^{\bar{N}}(t_i; t+T, z_{N_T}^*)$,
   (ii) $z_i^* \in \mathcal{R}_f^N(t_i; t_{i-1}, z_{i-1}^*) \cap \mathcal{R}_b^{\bar{N}}(t_i; t+T, z_{N_T}^*)$,
   (iii) u_i^* driving the state from z_{i-1}^* to z_i^* on $[t_{i-1}, t_i]$.
5. Let $u^* = u_1^*$ and apply u^* during $[t, t+\Delta]$.
6. Sampling. Obtain x at $t+\Delta$ to obtain $x(t+\Delta)$ and slide time, i.e., $t = t+\Delta$.
7. If $x(t)$ is close to z_i^*, then goto 4, otherwise goto 2.

Again we observe that the forward and backward reach sets required here can be pre-computed, stored on-board and recruited whenever necessary by consulting a look-up table. Intermediate steps in the optimization step ensures robustness to perturbations without having to perform heavy optimization computations. These will take place only when the deviation from the optimizing reference becomes too large.

4 Conclusions

The novel MPC scheme based on reach sets and value functions proposed here enables the transfer of a significant run-time computational effort underlying the conventional schemes to an off-line stage. In fact, this new scheme requires only a cyclic short-horizon optimization of a back propagated value function on the state space constrained by the intersection of a polyhedral approximation to the reach set and the free spaced dictated by the external constraints. Although, several important formal properties have been presented, there are a wide number of issues that remain to be exploited, which will be useful to guide future numerical experimentation. The networked control of a formation of Autonomous Underwater Vehicles served as a motivation and provided the requirements for the proposed general control schemes.

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