Optimal Control Problems for Path Planning of AUV using Simplified Models

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Abstract

Here we propose a simplified model for the path planning of an Autonomous Under Vehicle (AUV) in an horizontal plane when ocean currents are considered. The model includes kinetic equations and a simple dynamic equation. Our problem of interest is a minimum time problem with state constraints where the control appears linearly. This problem is solved numerically using the direct method. We extract various test from the Maximum Principle that are then used to test the numerical solution. In contrast to many other literature we apply the Maximum Principle as defined in [19].

Keywords: Optimal Control, Singular arc, Bang-bang control, Maximum Principle.
0.1 Introduction

Optimal control problems for autonomous vehicles have long proved to be a useful tool for robotics and, in particular, to determine references trajectories to execute certain tasks. On the other hand, they have been also of interest to illustrate and test many many theoretical concepts of optimal control (see [5], for example). The path planning of autonomous vehicles has been the focus of consideration attention in the last decade (see for example, [6] and reference within). In practice, computational optimization use optimal control to determine reference trajectories to be followed by the vehicle to accomplished its mission. Various models have been proposed for such simulations, with different degree of accuracy. Solving numerically optimal control problems based on accurate models for AUV’s may be a hard, if not impossible, task given their complexity. Moreover, validation of the numerical solution may turn to be a enormous problems. It is however well accepted that for the task of path planning simple models capturing the main characteristics of the vehicle are good enough (see [18]).

Here we propose a simplified model to determine the path of an Autonomous Underwater Vehicle (AUV) on a horizontal plane to go from one point to a target set \(T\) in the minimum time when currents are taken into account. We consider the motion using a simplified point mass model and we couple the kinetic equations of the motion with another differential equation depicting the dynamics of the vehicle. This equation involves the velocity and the thruster’s force. Noteworthy, we also impose constraints on the velocity which, in control terms, reduces to the inclusion of state constraints. We solve the problem numerically using a direct approach: we first discretize the problem and them, using A Mathematical Programming Language (AMPL) as the interface to the optimization solver Interior-Point optimization solver (IPOPT). Our numerical solution is validated using various tests provided by the maximum principle. Our approach differs from other literature since it is based on the Maximum Principle as stated in [19] where measures are associated with the multipliers of the state constraints. In this respect, we also bring to our discussion the concept of degeneracy of the Maximum Principle, a crucial aspect in Optimal Control.

**Notation:** If \(g \in \mathbb{R}^m\), the inequality \(g \leq 0\) is interpreted component-wise. Also, \(|\cdot|\) is the Euclidean norm or the induced matrix norm on \(\mathbb{R}^{p \times q}\). The closed unit ball centred at the origin is denoted by \(\mathbb{B}\) whereas \(\mathbb{B}\) denotes the open unit ball, regardless of the dimension of the underlying space. For any set \(A \subset \mathbb{R}^p\), \(\text{int} A\) denote the interior and convex hull of \(\mathcal{C}\), respectively. For any closed set \(S \subset \mathbb{R}^p\) the distance of a point \(y \in \mathbb{R}^p\) to the set \(A\) is defined as

\[d_S(y) := \inf \{|y - s| : s \in S\}.\]

For a function \(h : [a, b] \rightarrow \mathbb{R}^p\), we say that \(h \in W^{1,1}([a, b]; \mathbb{R}^p)\) if and only if \(h\) is absolutely continuous and that \(h \in L^1([a, b]; \mathbb{R}^p)\) iff \(h\) is integrable. The norm of \(L^1([a, b]; \mathbb{R}^p)\) is denoted by \(\|\cdot\|_1\) and the norm of \(L^\infty([a, b]; \mathbb{R}^p)\) is \(\|\cdot\|_\infty\).

Let \(C^*(\mathcal{C}; \mathbb{R})\) be the dual space of the continuous functions defined from \([a, b]\) to \(\mathbb{R}\), denoted by \(C([a, b]; \mathbb{R})\), with supremum norm. The norm of \(L^1([a, b]; \mathbb{R}^p)\) is denoted by \(\|\mu\|_{TV}\). The set of elements in \(C^*(\mathcal{C}; \mathbb{R})\) which take nonnegative values on nonnegative valued functions in \(C([a, b]; \mathbb{R})\) is here denoted by \(C^\circ([a, b]; \mathbb{R})\). For \(\mu \in C^\circ([a, b]; \mathbb{R})\), \(\|\mu\|_{TV} = \int_{[a, b]} \mu(dt)\).

We use concepts from nonsmooth analysis. They are well known so we refrain from stating them here but we refer the reader to [3], [4], [19], [17] and [13] for more information. Concerning nonsmooth analysis we use the following notation: \(N^S_S(x^*)\) is the limiting normal cone to the set \(S\) at \(x^*\) (also known as Mordukhovich normal cone), \(N^\circ_S(x^*)\) is the Clarke normal cone to \(S\) at \(x^*\), \(\partial^L f(x^*)\) is limiting subdifferential or Mordukhovich subdifferential of \(f\) at \(x^*\) and \(\partial^C f(x^*)\) is (Clarke) subdifferential of \(f\) at \(x^*\). If \(f\) is Lipschitz continuous near \(x^*\), the convex hull of the limiting subdifferential, co \(\partial^L f(x^*) = \partial^C f(x^*)\).

0.2 Minimum Time problems with state constraints and controls appearing linearly

In this section we deduce necessary conditions for a minimum time problem with vector valued state constraints using techniques provided in [19]. To maintain some of its generality we consider some data Lipschitz continuous.
Central to all this work is the following general minimum time optimal control problem with state constraints: 

\[
\begin{align*}
(M) & \quad \text{Minimize } t_f \\
& \quad \text{subject to } \\
& \quad \dot{y}(t) = f(y(t)) + g(y(t))u(t) \quad \text{a.e. } t \in [0, t_f] \\
& \quad u(t) \in \Omega \quad \text{a.e. } t \in [0, t_f] \\
& \quad h(y(t)) \leq 0 \quad \text{for all } t \in [0, t_f] \\
& \quad (y(0), y(t_f)) \in \{y_0\} \times E,
\end{align*}
\]

Here \(t_f\) is a choice variable to be determined, \(f : \mathbb{R}^n \to \mathbb{R}^n\), \(h : \mathbb{R}^n \to \mathbb{R}^K\) are vector functions, \(g : \mathbb{R}^n \to \mathbb{R}^{m \times n}\) is a matrix function, \(\Omega \subset \mathbb{R}^n\) and \(E \subset \mathbb{R}^n\) are closed sets and \(y_0 \in \mathbb{R}^n\). We denote the \(k\) components of \(h\) by \(h_i\), \(i = 1, \ldots, k\).

A feasible process for \((M)\) is a triple \((t_f, y, u)\), satisfying the constraints of the problem with \(t_f > 0\), where \(u\) is a measurable control functions and \(y\), the state variable, is an absolutely continuous function. For any \(t_f > 0\), we identify a function \(\bar{y} : [0, t_f] \to \mathbb{R}^n\) with its extension \(y_e\) to all \([0, +\infty[\) by constant extrapolation of end values to the right: for example, if \(\bar{y} \in \mathbb{R}^n\) and \(t > t_f\), then \(|\bar{y} - y_e(t)| := |\bar{y} - y(t_f)|\).

In this way, given \(t_f^1, t_f^2 > 0\) and two absolutely continuous functions \(y : [a, t_f^1] \to \mathbb{R}^n\) and \(\bar{y} : [0, t_f^2] \to \mathbb{R}^n\) we define  

\[
\|y - \bar{y}\|_{L^\infty} := \|y_e - \bar{y}_e\|_{L^\infty},
\]

where \(y_e\) and \(\bar{y}_e\) are the extensions of \(y\) and \(\bar{y}\). We say that \((\bar{t}_f, \bar{y}, \bar{u})\) is a strong local minimizer for \((M)\) if there exists a \(\varepsilon > 0\) such that \(t_f \leq \bar{t}_f\) over all feasible processes \((t_f, y, u)\) of \((M)\) satisfying  

\[
|t_f - \bar{t}_f| + \|y - \bar{y}\|_{L^\infty} \leq \varepsilon.
\]

As it is customary in the literature (see, for example, \([19]\)) necessary conditions of optimality for \((M)\) can be derived reformulating the free time problem into a problem \((R)\) with fixed end time \([a, \bar{t}_f]\) to which known necessary conditions are then applied. To illustrate such procedure, we need to assume that the following hypotheses, which make reference to the process \((\bar{t}_f, \bar{y}, \bar{u})\) and parameter \(\varepsilon\), hold:

\begin{enumerate}
  \item [(H1)] The set \(E\) is closed and \(\Omega \subset \mathbb{R}^n\) is a compact set.
  \item [(H2)] There exist \(\delta > 0\), \(K_f > 0\) and \(K_g > 0\) such that
  \[
  |f(y) - f(y')| \leq K_f|y - y'|, \quad |g(y) - g(y')| \leq K_g|y - y'|
  \]
  for all \(y, y' \in \bar{y}(t) + \delta B\) a.e. \(t \in [0, \bar{t}_f]\).
  \item [(H3)] The function \(h\) is continuously differentiable\(^2\).
\end{enumerate}

Now let us consider the fixed time problem  

\[
(R) \quad \begin{align*}
& \quad \text{Minimize } \tau(\bar{t}_f) \\
& \quad \text{subject to } \\
& \quad \dot{\tau}(s) = \gamma(s) \quad \text{a.e. } s \in [0, \bar{t}_f], \\
& \quad \dot{\bar{z}}(s) = \gamma(s)f(z(s)) + \gamma(s)g(z(s))a(s) \quad \text{a.e. } s \in [0, \bar{t}_f], \\
& \quad \gamma(s), a(s) \in \left[\frac{1}{2}, \frac{3}{2}\right] \times \Omega \quad \text{a.e. } s \in [0, \bar{t}_f] \\
& \quad h(z(s)) \leq 0 \quad \text{for all } s \in [0, \bar{t}_f] \\
& \quad \tau(0), \bar{z}(0), z(\bar{t}_f) \in \{0\} \times \{\bar{y}_0\} \times E, \\
& \quad |\tau(\bar{t}_f) - \bar{t}_f| \leq \varepsilon.
\end{align*}
\]

The process \((\dot{\tau}(s) = s, \ddot{z} = \bar{y}, \bar{a} = \bar{u}, \ddot{\gamma} = 1)\) is a strong local minimizer for \((R)\). To see this, and seeking a contradiction, suppose that there exists a feasible process \((\tau, z, a, \gamma)\) for \((R)\) such that

\[
\|\tau - \bar{\tau}\|_{L^\infty} + \|z - \bar{z}\|_{L^\infty} \leq 2\varepsilon.
\]

\(^1\)For non autonomous problems, the same can be done when the data is Lipschitz continuous with respect to time.

\(^2\)We remark that the forthcoming analysis holds if Lipschitz continuity of \(h\) is imposed instead of (H3). However, for our propose, (H3) is enough and it will somewhat simplify the exposition.
where $0 < \tau(\bar{t}_f) \leq \bar{t}_f$. Define $\phi : [0, \bar{t}_f] \to [0, t_f]$ so that

$$\tau(\bar{t}_f) = t_f$$

and $\phi(s) = \int_0^s 1 \, ds$.

Then $\phi$ is a Lipschitz continuous and strictly increasing function with Lipschitz continuous inverse. It is a simple matter to see that the functions $y(t) = z \circ \phi^{-1}(t)$ and $u(t) = a \circ \phi^{-1}(t)$ (observe that $y : [0, t_f] \to \mathbb{R}^n$ and $u : [0, t_f] \to \mathbb{R}^m$) satisfy

$$y(0) = y_0, \quad t_f = \tau(\bar{t}_f), \quad y(t_f) = z(\bar{t}_f), \quad |\tau(\bar{t}_f) - \bar{t}_f| \leq \varepsilon,$$

$$\dot{y}(t) = f(y(t)) + g(y(t))u(t) \text{ a.e. } t \in [0, t_f],$$

$$u(t) \in A \text{ a.e. } t \in [0, t_f],$$

$$h(y(t)) \leq 0 \text{ a.e. } t \in [0, t_f]$$

and

$$|t_f - \bar{t}_f| + \|y - \bar{y}\|_{L^\infty} \leq 2\varepsilon.$$

Thus $(\tau, y, u, \gamma)$ is a feasible process for $(M)$ with $t_f \leq \bar{t}_f$ contradicting the optimality of $(\bar{\tau}, \bar{y}, \bar{u}, \bar{\gamma})$ and proving our claim.

Assume that the data of our problem $(M)$ satisfy (H1) and (H2). Following the approach in [19], Chapter 8 (now with the aforementioned state constraints), we apply the nonsmooth Maximum Principle given by Theorem 9.3.1, Multiple State Constraint version, in [19] (page 331) for $(\bar{R})$ with reference to the strong local minimizer $(\bar{\tau}, \bar{y}, \bar{u}, \bar{\gamma})$. We then deduce the existence of an absolutely continuous function $p : [0, t_f] \to \mathbb{R}^n$, $\lambda \geq 0$ and $\mu_i \in C^0([0, t_f]), i = 1, \ldots, k$ such that $\text{supp}\{\mu_i\} \subset \{s \in [0, t_f] : h_i(y(s))\}$ and

(i) $\langle p, \mu_1, \mu_2, \ldots, \mu_k, \lambda \rangle \neq (0, 0, \ldots, 0, 0),$

(ii) $-p(s) \in \partial F q(s) \cdot (f(y(s)) + g(y(s))u(s)) \text{ a.e. } s \in [0, \bar{t}_f],$

(iii) $q(s) \cdot (f(y(s)) + g(y(s))u(s)) = \max_{u \in A} q(t) \cdot (f(y(t)) + g(y(t))u) \text{ a.e. } s \in [0, t_f],$

(iv) $\lambda = -q(s) \cdot (f(y(t)) + g(y(s))u(s)) \text{ a.e. } s \in [0, \bar{t}_f],$

(v) $-q(\bar{t}_f) \in N_E^d(y(\bar{t}_f)),$

where

$$q(s) := \begin{cases} p(s) + \int_{[0, s]} \sum_{i=1}^k \nabla h_i(y(\sigma))\mu_i(\sigma) & \text{for } s < \bar{t}_f, \\ p(\bar{t}_f) + \int_{[0, \bar{t}_f]} \sum_{i=1}^k \nabla h_i(y(t))\mu_i(ds) & \text{for } s = \bar{t}_f. \end{cases}$$

Define the Hamiltonian for $(M)$ to be

$$H(y, p, u) := p \cdot (f(y) + g(y)u)$$

and rewriting the (i)–(v) in terms of the data of the original problem $(M)$ we deduce the following theorem

**Theorem 0.2.1** Let $(\bar{t}_f, \bar{y}, \bar{u})$ be a strong local minimizer for $(M)$. Assume that the assumptions (H1)–(H4) are satisfied. Then there exist an absolutely continuous function $p : [0, \bar{t}_f] \to \mathbb{R}^n$ and $\mu_i \in C^0([0, \bar{t}_f]), i = 1, \ldots, k$ such that

$$\text{supp}\{\mu_i\} \subset \{s \in [0, \bar{t}_f] : h_i(y(s))\}$$
and

\begin{align}
    (a) & \quad (p, \mu_1, \mu_2, \ldots, \mu_k, \lambda) \neq (0, 0, \ldots, 0), \\
    (b) & \quad -\dot{p}(t) \in \partial_C^\circ q(t) \cdot \{ f(\dot{y}(t)) + g(\dot{y}(t))u(t) \} \text{ a.e. } t \in [0, \bar{t}_f], \tag{4} \\
    (c) & \quad q(t) \cdot (f(\dot{y}(t)) + g(\dot{y}(t))u(t)) = \max_{u \in A} q(t) \cdot (f(\dot{y}(t)) + g(\dot{y}(t))u(t)) \text{ a.e. } t \in [0, \bar{t}_f], \tag{5} \\
    (d) & \quad q(t) \cdot (f(\dot{y}(t)) + g(\dot{y}(t))u(t)) = \lambda \text{ a.e. } t \in [0, \bar{t}_f], \tag{6} \\
    (d) & \quad -q(T) \in N_E^L(\dot{y}(\bar{t}_f)), \tag{7} \\
\end{align}

where \( q \) is as defined in \[2\].

### 0.3 AUV Problem

We consider the problem of determining the path of an Autonomous Underwater Vehicle (AUV) on a horizontal plane to go from one point to a target set \( T \) in the minimum time when currents are taken into account. We focus our attention to the movement of the AUV in the horizontal plane and describe the motion using a simplified point mass model. For information on more realistic models of underwater vehicles we refer the reader to \([6]\), for example.

We consider the following simplified kinematic model (see, for example, \([18]\) and references therein):

\[
\begin{align*}
\dot{x}(t) &= u(t) \cos(\phi(t)) + v_x, \\
\dot{y}(t) &= u(t) \sin(\phi(t)) + v_y, \\
\dot{\phi}(t) &= r(t),
\end{align*}
\]

where \((x, y)\) denotes the position of the vehicle on the horizontal plane of constant depth, while \( \phi \) represents its orientation, \( u \) is the velocity of the vehicle, \( r \) the angular velocity, and \( v = (v_x, v_y) \) the current velocity, which might depend on the position on the horizontal plane. The above equations are as in the well known Zermelo’s problem. However, we couple these equations with a simplified dynamics equation of the form:

\[
\dot{u}(t) = f(t) - Ku(t)|u(t)|, \tag{9}
\]

where \( u \), the surge velocity of the vehicle, is a state and the thruster’s force \( f \) is an additional control. The term \(-Ku(t)|u(t)|\) depicts the quadratic drag force \((21)\) and, throughout this paper, we consider \( K = 1 \). We assume that the velocity is limited

\[
u(t) \in [0, 2]. \tag{10}\]

Clearly, the velocity is not negative and so the term \(-Ku(t)|u(t)|\) in \((9)\) can be written simply by \( Ku^2(t)\).

As in Zermelo’s problem we consider that the velocity of ocean currents is known. For simplicity of the analysis, we assume that the velocity of the currents has components merely on the \( x \) but depending on the \( y \) position: \( v(t) = (0, \text{tanh}(y(t)), 0) \).

To reflect the fact that the power of the thruster is limited and to bound the heading rate (making the model more realistic), we impose that the control variables \((f, r)\) take values in a given control set

\[
(f, r) \in [-5, 5] \times [-\pi, \pi]. \tag{11}
\]

Our aim is to determine the minimum time \( t_f \) needed to drive the vehicle from the point \((x_0, y_0) = (40, -2)\) to the target set

\[
T = \{(x, y) : x^2 + y^2 \leq 0.05\}. \tag{12}
\]

The initial and final configurations of the vehicle are \((x_0, y_0, \phi_0, u_0) = (40, -2, \pi, 0)\) and \((x_f, y_f, \phi_f, u_f) = (x_1, y_1, \pi, 0)\), where \((x_1, y_1) \in T\). Putting all together and considering \( h(x, y, \phi, u) = (h_1(x, y, \phi, u), h_2(x, y, \phi, u)) \),
with \( h_1(x,y,\phi,u) = u - 2 \) and \( h_2(x,y,\phi,u) = -u \), we now have the optimal control problem:

\[
\begin{align*}
(P) & \begin{cases}
\text{Minimize } t_f \\
\text{subject to }
\begin{align*}
\dot{x}(t) &= u(t) \cos(\phi(t)) + 0.8 \tanh(y(t)) \quad \text{a.e. } t \in [0,t_f], \\
\dot{y}(t) &= u(t) \sin(\phi(t)) \quad \text{a.e. } t \in [0,t_f], \\
\dot{\phi}(t) &= r(t) \quad \text{a.e. } t \in [0,t_f], \\
\dot{u}(t) &= f(t) - u(t).|u(t)| \quad \text{a.e. } t \in [0,t_f],
\end{align*}
\end{cases}
\end{align*}
\]

where \( (\bar{x},\bar{y},\bar{\phi},\bar{u}) \), \( (\bar{f},\bar{r}) \), and \( \mu = (\mu_x,\mu_y,\mu_\phi,\mu_u) \) are the extremal point for the data of \( (\bar{x},\bar{y},\bar{\phi},\bar{u}) \), \( (\bar{f},\bar{r}) \), and \( \mu = (\mu_x,\mu_y,\mu_\phi,\mu_u) \) are the extremal point for the data of \( (\bar{x},\bar{y},\bar{\phi},\bar{u}) \) and \( (\bar{f},\bar{r}) \) is the state variable, \( u \) is the control variable and \( p = (p_x,p_y,p_\phi,p_u) \) is the adjoint multiplier. It is then a simple matter to see that \( (P) \) is in the form of \( (M) \) when

\[
f(y) = \begin{bmatrix}
  u(t) \cos(\phi(t)) + 0.8 \tanh(y(t)) \\
  u(t) \sin(\phi(t)) \\
  0 \\
  -u(t).|u(t)|
\end{bmatrix}, \quad g(y) = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  1 \\
\end{bmatrix}.
\]

The Hamiltonian function for \( (P) \) is then

\[
H(y,p,u) = p_x u \cos(\phi) + 0.8 p_x \tanh(y) + p_y u \sin(\phi) + p_\phi r + p_u (f - u^2).
\]

Assume that \( (\bar{f}, \bar{y}) = (\bar{x}, \bar{y}, \bar{\phi}, \bar{u}), \bar{u} = (\bar{f}, \bar{r}) \) is a strong local solution of \( (P) \). Since the data of \( (P) \) satisfies the conditions under which Theorem 1.2.1 holds, we deduce that there exist absolutely continuous function \( p \), Borel regular measures \( \mu_1, \mu_2 \) and \( \lambda \geq 0 \), not all 0, such that

\[
\begin{align*}
\dot{p}_x(t) &= 0, \\
\dot{p}_y(t) &= -0.8 p_x(t) \frac{1}{\cosh^2(y(t))}, \\
\dot{p}_\phi(t) &= p_x(t) \bar{u}(t) \sin(\bar{\phi}(t)) - p_y(t) \bar{u}(t) \cos(\bar{\phi}(t)), \\
\dot{p}_u(t) &= -p_x(t) \bar{u}(t) \cos(\bar{\phi}(t)) - p_y(t) \bar{u}(t) \sin(\bar{\phi}(t)) + 2q_u(t) \bar{u}(t), \\
\dot{p}_r(t) + q_u(t) \bar{f}(t) &= \max_{(f,r) \in [-5,5] \times [-\pi,\pi]} p_x(t) r + q_u(t) f,
\end{align*}
\]

(13)

(14)

(15)

(16)

(17)

together with

\[
\lambda = p_x(t) \bar{u} \cos(\bar{\phi}(t)) + 0.8 p_x(t) \tanh(\bar{y}(t)) + p_y(t) \bar{u}(t) \sin(\bar{\phi}(t)) + p_r(t) \bar{r}(t) + q_u(t) (\bar{f}(t) - \bar{u}^2(t)), \text{ a.e. } t \in [0,\bar{f}],
\]

(18)

\[
(-p_x(\bar{f}),-p_y(\bar{f})) \in N_T^L(\bar{x}(\bar{f}),\bar{y}(\bar{f})),
\]

(19)

where

\[
q_u(t) := \begin{cases}
  p_u(t) + \int_{[0,t]} \mu_1(ds) - \int_{[0,t]} \mu_2(ds) & \text{for } t < \bar{f}, \\
  p_u(\bar{f}) + \int_{[0,\bar{f}]} \mu_1(ds) - \int_{[0,\bar{f}]} \mu_2(ds) & \text{for } t = \bar{f}.
\end{cases}
\]

(19)

Next we extract information about the optimal solution from the above conditions. Before proceeding it is worth to recall the following facts:

**F1:** Since our final state \( y(\bar{f}) \) is constrained to take values in \( T \times \{(\pi,0)\} \), we cannot, a priori, assume that \( \lambda = 1 \).
F2: The left-continuous function of bounded variation \( \nu(t) = \int_{[0,t)} \mu_1(\,d\tau) \) can be further decomposed uniquely as (see [10])

\[
\nu(t) = \nu^a(t) + \nu^r(t) + \nu^i(t),
\]

where \( \nu^a(t) = \int_0^t \dot{\nu}(s) \, ds \) is an absolutely continuous function, \( \nu^r(t) = \int_{[0,t)} \mu_2^r(\,d\tau) \) is a continuous but not absolutely continuous function and \( \nu^i \) is a pure jump function with at most a countable number of jumps. Let \( \tau_i \) denote the points of discontinuity of \( \nu \).

F3: A state constraint \( h(y(t)) \leq 0 \) has a boundary interval \([t_0^b, t_1^b]\) if \( h(y(t)) = 0 \forall t \in [t_0^b, t_1^b] \) and the points \( t_0^b \) and \( t_1^b \) are called junction points if \( h(y(t)) < 0 \) for \( t \) in neighbourhoods of those point, on the left of \( t_0^b \) and on the right of \( t_1^b \). In this case \( t_0^b \) is called an entry point and \( t_1^b \) an exit point, it has a contact point if there exist \( \sigma_1 \) and \( \delta > 0 \) such that \( h(y(\sigma_1)) = 0 \) and \( h(y(t)) < 0 \) for all \( t \in ([\sigma_1 - \delta, \sigma_1] \cup [\sigma_1, \sigma_1 + \delta]) \cap [0, t_f] \) and, finally, it has an interior interval \([t_0^b, t_1^b] \) if \( h(y(t)) < 0 \) for all \( t \in [t_0^b, t_1^b] \).

We are now in position to turn to our problem. Problem \( \langle P \rangle \) has two state inequality constraints \( h_1(u) = u - 2 \) and \( h_2(u) = -u \) that are never simultaneously active. Taking into account the physical meaning of the problem it is to be expected that \( h_2 \) will be active only at the extreme points \( t = 0 \) and \( t = t_f \) (since \( u(0) = u(t_f) \)) and, consequently, it is reasonable to expect that there is no boundary interval for \( h_2 \). The same, however, cannot be said about \( h_1 \).

A word of caution in this regard is called for. Indeed, the fact that \( h_2 \) is active at \( t = 0 \), because \( u(0) = 0 \), could undermine the applicability of the necessary condition (a)–(d) of Theorem 0.2.1 since it is well known that when the initial state is on the boundary of a state constraint, the maximum principle may fail to provide any information about the solution, i.e., the maximum principle may be degenerate. In such situations, non-degenerate forms of the maximum principle have been established in the literature under different constraints qualifications (see, for example, [8], [2], [11], [13] and, more recently, [9], and references within). However, it is a simple matter to see that the inward pointing velocity conditions (see in [9]) holds. Indeed, for a constant control \( u(t) = (\pi, f) \), where \( f \in [1, 5] \) (for example) we have

\[
\nabla h_2(\bar{y}(0)) \cdot (f(\bar{y}(0)) + g(\bar{y}(0))u(t)) = -f < 5.
\]

Since the control appears linearly in the Hamiltonian, we know that the optimal solution of our problem is a concatenation of bang and singular arcs. We say that \([t_0^s, t_1^s]\) is a singular interval for the control component \( r \) if \( \bar{r}(t) \in [-\pi, \pi] \) for \( t \in [t_0^s, t_1^s] \) and that \([t_0^b, t_1^b]\) is a singular interval for the control component \( f \) if \( \bar{f}(t) \in [-5, 5] \) for \( t \in [t_0^b, t_1^b] \).

Let us define the switching function as

\[
\psi(y, p(t)) = \begin{bmatrix} \psi_f(y, p(t)) \\ \psi_r(y, p(t)) \end{bmatrix} = \begin{bmatrix} q_u(t) \\ \mu(t) \end{bmatrix}.
\]

Evaluating \( \psi \) along the optimal trajectory \( y \) and for a certain \( p \), \( \psi \) depends on \( t \). Thus we write merely \( \psi(t) \) when the dependence on \( y \) and \( p \) is clearly understood. From [17], we get the following control laws:

\[
\bar{f}(t) = \begin{cases} -5, & \text{if } \psi_f(t) > 0 \\ \bar{f}^\text{sing}(t), & \text{if } \psi_f(t) = 0 \\ 5, & \text{if } \psi_f(t) < 0 \end{cases}, \quad \bar{r}(t) = \begin{cases} -\pi, & \text{if } \psi_r(t) > 0 \\ \bar{r}^\text{sing}(t), & \text{if } \psi_r(t) = 0 \\ \pi, & \text{if } \psi_r(t) < 0 \end{cases}
\]

In the above, both \( f^\text{sing} \) and \( r^\text{sing} \), the values of the singular controls, represent values in \([-5, 5] \) and \([-\pi, \pi]\).

Preparing for the validation of numerical results, we know consider a situation that suggests itself from the physical meaning of our problem. Let us then suppose that there exists no boundary interval for \( h_2 \) and there exist only one boundary interval for \( h_1 \), \([t_0^b, t_1^b] \subset [0, t_f] \), where \( 0 < t_0^b < t_1^b < t_f \) is the first point in \([0, t_f]\) where \( h_1(y(t)) = 0 \) and \( t_0^b < t_1^b < t_f \) is the last point where \( h_1(y(t)) = 0 \). We also assume that there is not other contact points for both \( h_1 \) and \( h_2 \) besides those mentioned above in the whole interval of interest \([0, t_f]\). Under such assumption we can deduce that

(1) for all \( t \in [t_0^b, t_1^b] \) we have \( \bar{u}(t) = 2 \). Consequently \( \bar{u}(t) = 0 \) and \( \bar{f}(t) = 4 \) for \( t \in [t_0^b, t_1^b] \). Thus \([t_0^b, t_1^b] \subset [t_0^b, t_1^b] \) and since \( \bar{f} \) is singular, we have \( q_u(t) = 0 \) for all \( t \in [t_0^b, t_1^b] \). Recall that in this situation we have\(^\text{3}\)

\[
q_u(t) = p_u(t) \quad \text{for } t \in [0, t_0^b],
\]

\(^\text{3}\)The fact that the measure \( \mu_1 \) is assumed 0 here is because of the nondegeneracy of our maximum principle.
and
\[ 0 = p_u(t) + \int_{[0,t]} \mu_1(d\sigma) \text{ for } t \in [t_0^b, t_1^f]. \]  
(21)

(2) Using the notation introduced in F2, we have
\[ \nu^j(\tau_i) = \mu_1(\{\tau_i\}) = \nu(\tau_i^+) - \nu(\tau_i^-) \]
and \( \nu^j(t) = 0 \) if \( t \neq \tau_i \). Since \( \nu(t) = 0 \) for \( t < t_0^b \) and \( q_u(t) = 0 \) for \( t \in [t_0^b, t_1^f] \) we have two possibilities:
(i) either \( \tau_i = t_0^b \) and then \( \mu_1(\{\tau_i\}) \neq 0 \) and \( 0 = p_u(t) + \mu_1(\{\tau_i\}) + \nu^a(t), \)
(ii) or, \( t_0^b \) is not a point of discontinuity of \( \nu \) and then we have \( \mu_1(\{\tau_i\}) = 0 \) and \( p_u(t) = \nu^a(t). \)

(3) Doing a similar analysis at \( t_1^f \) we conclude from the above that, then at \( t_0^b \) and at \( t_1^f \) we have
\[ q_u(t_0^b) - q_u(t_0^b^-) = \mu_1(\{t_0^b\}), \quad q_u(t_1^f) - q_u(t_1^f^-) = \mu_1(\{t_1^f\}). \]  
(22)

If \( t_0^b \) (or \( t_1^f \)) is is not a discontinuity point of \( q_u \), then \( \mu_1(\{t_0^b\}) = 0 \) \( (\mu_1(\{t_1^f\}) = 0). \)

(4) Moreover, \( q_u \) has no jumps inside the boundary interval and \( \nu^a(t) = 0 \) if \( t \in [0, t_0^b] \cup [t_1^f, t_f]. \)

(5) We can then write
\[ p_u(t) = \begin{cases} 
q_u(t) & \text{if } t \in [0, t_0^b], \\
q_u(t_0^b^-) - \nu^a(t) & \text{if } t \in [t_0^b, t_1^f], \\
q_u(t) + q_u(t_1^f^-) + q_u(t_1^f) & \text{if } t \in [t_1^f, t_f]. 
\end{cases} \]  
(23)

(6) We now turn to the control \( r \). Again, based on the physical meaning of \( r \), it is not unreasonable to expect that \( r \) takes singular values except at small neighbourhoods of 0 and \( t_f \) and when \( u(t) \neq 0 \). We should then expect that the existence of a singular interval \([t_0^b, t_1^f]\), with \( 0 < t_0^b < t_1^f < t_f \) and we expect \([t_0^b, t_1^f] \subset [t_0^b, t_1^f]. \)

Seeking a closed form for \( r^{\text{sing}} \) we set \( \psi_r(t) = 0 \) for \( t \in [t_0^b, t_1^f]. \) We have
\[ \frac{d}{dt} \psi_r(t) = \dot{\psi}_r(t) = p_r(t) \hat{u}(t) \sin(\phi(t)) - p_y(t) \hat{u}(t) \cos(\phi(t)) = 0, \quad t \in [t_0^b, t_1^f]. \]

Using this equality, the fact that \( \hat{u}(t) \neq 0 \) and imposing that \( \frac{d^2}{dt^2} \psi_r = 0 \) we get
\[ r(p_r(t) \cos(\phi(t)) + p_y(t) \sin(\phi(t))) - \dot{\psi}_r(t) \cos(\phi(t)) = 0. \]

If
\[ p_r(t) \cos(\phi(t)) + p_y(t) \sin(\phi(t)) \neq 0 \text{ for } t \in [t_0^b, t_1^f], \]  
(24)

then the generalized Legendre-Clebsch condition \( \frac{\partial}{\partial r} \left( \frac{d^2}{dt^2} \psi_r \right) \neq 0 \) holds allowing us to deduce that
\[ r^{\text{sing}}(y(t)) = -0.8 \cos^2(\phi(t)) \frac{1}{\cosh^2(\hat{y}(t))}. \]  
(25)

The expression \( [25] \) holds provided that \( \cos(\phi(t)) \neq 0 \) for \( t \in [t_0^b, t_1^f]. \) We now turn to \( f \). We already know that the control \( f \) is singular in the boundary interval if this interval exists. The question is if there exists a singular interval outside or containing the boundary interval and what value would \( f \) take there. However, an analysis for \( \sigma_f \) analogous to the one in \( 6 \) is not possible since we have \( \frac{\partial}{\partial f} \left( \frac{d^2}{dt^2} \psi_f \right) = q_u(t) = 0 \) and so it does not provide a test for optimality of this arc.

### 0.4 Numerical Results

We now present the numerical solution of problem \((P)\) as defined in section 3 with
\[ T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 0.05\}. \]
We use the Implicit Euler Method to solve AUV’s problem with 10000 grid nodes. As mentioned in the Introduction, we use the Applied Modeling Programming Language, AMPL, as the interface with version 3.8.0 of IPOPT.

We obtain \( \tilde{t}_f = 14.944 \) as the minimum time. We provide here the computed values of the endpoints of the state \( x \) and \( y \) and the initial values of the multipliers.

**Numerical results for \( (x(0), y(0)) = (40, -2) \):**
- \( x(\tilde{t}_f) = 0.15625 \), \( y(\tilde{t}_f) = -0.15995 \), \( \tilde{t}_f = 14.944 \),
- \( p_x(0) = 0.35811 \), \( p_y(0) = 0.05657 \),
- \( p_u(0) = -0.000112 \), \( p_u(0) = -0.14208 \).

In Figure 1 we present the trajectory in the horizontal plane together with the graphs of both control, \( r \) and \( f \).

![Figure 1](image1.png)

Figure 1: Minimum time trajectory for the AUV on the left and the optimal controls on the right when \( (x(0), y(0)) = (40, -2) \).

As for the controls \( r \) and \( f \) they are both discontinuous and both are bang-singular-bang with two switching points. The control \( r \) has the first switching point \( t_0 \) very close to \( 0 \) and both \( r \) and \( f \) have the second switching time close to \( \tilde{t}_f \). The computed state variables are plotted in Figure 2.

![Figure 2](image2.png)

Figure 2: State variables for the AUV problem when \( (x(0), y(0)) = (40, -2) \).

The numericals show that the singular interval of \( \tilde{f} \) coincides with the boundary interval of the state constraint \( u - 2 \leq 0 \) as shown in Figure 3 where the graph of the state variable \( u \) is plotted together with the graph of the control \( f \).

Our analysis shows that the multiplier \( q_u \) must be 0 along the boundary arc. This is confirmed by the numerical values as shown in Figure 4 where we present the computed multipliers \( p_x, p_y, p_\phi \) and \( q_u \).

Since the computed solution tell us at \( \tilde{t}_f \) the vehicles reaches the boundary of \( T \) in the fourth quadrant, we deduce from (19) and (13) that we must have \( p_x(t) = p_x(\tilde{t}_f) > 0 \) and that \( p_y(\tilde{t}_f) < 0 \). This is confirmed by the computed values of the multipliers presented in Figure 4. A careful analysis of the computed values of \( \phi \), \( p_\phi \) and
The computed values of $u$ plotted together with those of $f$: the singular interval for $f$ coincides with the boundary interval for $u$. The values of the switching point for $f$ are approximately 0.64 and 14.62.

Computed multipliers for the case when $(x(0), y(0)) = (40, -2)$.

$p_x$ confirms that (24) holds. Confronting the numerical expression of the analytical value of the $r_{\text{sing}}$ in (25) with the computed value of $\bar{r}$, shown in 5, we get a match of these values. The numerics also show that the multiplier $\lambda$ is 1 as shown in the right side of figure 5 where the Hamiltonian is plotted verifying (apart from some residual numerical chattering) (18).

On the left the graph of the analytical value of $r$ in (25) and the numerical value of the optimal control $r$ are presenting showing that they coincide. The Hamiltonian is on the right allowing us to conclude that the numeric $\lambda$ is indeed 1.

With AMPL we have also access to the values of the bounded variations functions (see, in this respect, F2 in section 0.3)

$$
\nu(t) = \int_{[0,t]} \mu_1(\sigma) d\sigma, \quad \eta(t) = \int_{[0,t]} \mu_2(\sigma) d\sigma
$$

presented in figure 6. The function $\nu$ is indeed zero outside the boundary interval and that $\mu_1$ has one atom at the entry point of the boundary interval. Also, $\mu_2$ has a atom on the end point $\bar{t}$ and it is 0 elsewhere.

Finally, being the use of AMPL validated for problem $(P)$, figure 7 shows numerically obtained optimal trajectories for variants of this problem with different values of $(x(0), y(0))$. 
Figure 6: Graphs of the bounded variation functions (26) for the case when \((x(0), y(0)) = (40, -2)\).

Figure 7: Minimum time trajectory for the AUV for various values of \((x(0), y(0))\).

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