FACULDADE DE ENGENHARIA DA UNIVERSIDADE DO PORTO

# Network Coding for Cooperation in Dynamic Wireless Networks

**Hana Khamfroush**

FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Programa Doutoral em Telecomunicações

Orientador: Doutor João Francisco Cordeiro de Oliveira Barros, Professor Associado com Agregação do Departamento de Engenharia Eletrotécnica e de Computadores da Faculdade de Engenharia da Universidade do Porto

Co-orientador: Doutor Daniel Enrique Lucani Roetter, Associate Professor in Department of Electronic Systems, Antennas, propagation and radio networking, The Faculty of Engineering and Science of Aalborg University

19 de Novembro de 2014

# Network Coding for Cooperation in Dynamic Wireless Networks

**Hana Khamfroush**

Programa Doutoral em Telecomunicações

Aprovado em provas públicas pelo Júri:

Presidente: Doutor Pedro Henrique Henriques Guedes de Oliveira, Por subdelegação do Diretor da FEUP

Arguente: Doutor Giuseppe Caire, Professor Catedrático, Department of Electrical Engineering, the Viterbi School of Engineering, USC/TU Berlin;

Arguente: Doutor Joel José Puga Coelho Rodrigues, Professor Auxiliar, Departamento de Informática, Universidade da Beira Interior;

Vogal: Doutor Alexandre Júlio Teixeira dos Santos, Professor Associado com Agregação, Departamento de Informática, Escola de Engenharia, Universidade do Minho (Membro da comissão científica do MAP-tele);

Vogal: Doutora Ana Cristina Costa Aguiar, Professora Auxiliar do Departamento de Engenharia Eletrotécnica da Universidade do Porto;

04 de Novembro de 2014

*To my devoted mother, and my dearest love, Sam.*

# Acknowledgments

Foremost, I would like to express my deepest gratitude and warmest thanks to my advisers, Prof. João Barros and Prof. Daniel Lucani, who have supported me throughout this journey with their patience and knowledge whilst allowing me to find my way and to learn how to do a productive research. Certainly, without your valuable help and contributions, this journey would not have been completed. Thank you so much João and Daniel!

At IT Porto, I was blessed with a friendly and cheerful group of fellow friends and colleagues who had a great impact on my work. Thank you all for the great moments we spent together and for your support during my PhD.

At Aalborg University, I had the privilege to meet and work with Prof. Frank Fitzek and his amazing team. I would not be able to do a big part of this work without their help and support.

I finish with expressing my deepest gratitude to my peerless family, who is the most basic source of my life energy.

To my dear mother who devoted her life to let us being in the place that we are standing now. Thank you mom for your unconditional support throughout all these years. I know you have given up many things for me to live my dreams. All I have is because of you.

To my love and my life passion, Sam, who was always there cheering me up and encouraging me when everything seems so dark and endless. We traversed this journey together, and without you I could never reach to the end. You are not only my love, you are my best friend and companion. Thank you so much for all of your continued support.

To my brother and my sister who always encourage me with their positive energy.

This work was supported by supported by the Fundação para a Ciençia e Tecnologia (Portuguese Foundation for Science and Technology) under grants SFRH/BD/72961/2010.

# Resumo

Com a crescente preocupação em minimizar o custo de transmissão de pacotes de aplicações sem fios multi-utilizador, a comunicação cooperativa por codificação de rede foi proposta e avaliada como uma poderosa tecnologia que pode proporcionar uma melhor qualidade de serviço em sistemas sem fios de próxima geração. Contribuições anteriores concentraram-se na avaliação de cenários pré-definidos de desempenho, em vez de procurar políticas ótimas que podem minimizar o custo total da transmissão fiável de pacotes. Nesta tese, estudamos a cooperação por codificação de rede para redes sem fios dinâmicas com o objetivo de encontrar políticas ótimas que minimizem o custo total da transmissão de pacotes e ajudem a projetar protocolos multicast ótimos para redes dinâmicas.

Começamos por analisar o ganho de protocolos de codificação de rede em termos de redução do número de nós ativos e a área ativa como indicativos do custo de transmissão de pacotes. Para este fim, um cenário de multicast em uma rede de malha é considerado. Procuramos encontrar um protocolo de codificação de rede que consiga o mesmo rendimento que na ausência de um protocolo de codificação de rede mas utilizando um menor número de nós ativos. Um novo protocolo de comunicação geográfico multicast sensível à codificação é proposto, sendo capaz de reduzir de forma significativa tanto a área de transmissão requerida como o número de nós activos quando comparado com multicast tradicional para o mesmo rendimento. É mostrado que o protocolo proposto tem complexidade linear em termos de número de receptores, em comparação com os protocolos tradicionais que têm complexidades não-lineares.

Em seguida, concentramo-nos em encontrar políticas cooperativas ideais de transmissão de pacotes codificados em rede que podem minimizar o custo de transmissão de pacotes fiável. Uma rede simples com uma fonte e dois receptores é considerada para responder à pergunta de qual o momento ideal para iniciar a cooperação entre os receptores a fim de minimizar o custo total da transmissão de pacotes. Modelamos este problema para os canais com perdas de pacotes variantes e invariantes no tempo através de um processo de decisão de Markov (MDP). As políticas de transmissão de pacotes selecionadas pelo MDP são analisados para extrair as regras ótimas de transmissão de pacotes. Propomos ainda quatro heurísticas poderosas que se mostra proporcionarem um desempenho próximo do ideal em ambientes sem fios dinâmicas. As nossas heurísticas podem ser utilizados com qualquer número de receptores. A implementação real das heurísticas propostas revela que são capazes de reduzir a média de tempo de conclusão por até $4.75x$ em comparação com simplesmente transmitindo pacotes codificados.

Finalmente, abordamos o impacto de um relé na redução do tempo de transmissão de pacotes de dados de uma fonte para dois receptores que usam codificação de rede e na presença de $X$ vizinhos ativos que partilham o mesmo canal. Mostramos que o uso de um relé na presença de vizinhos activas é benéfica mesmo se o canal entre relé e destino não

é melhor que o canal entre origem e destino.

# Abstract

With the growing concern on minimizing the cost of packet transmission in multi-user wireless applications, network coded cooperative communication has been proposed and evaluated as a powerful technology that can provide a better quality of service in the next generation wireless systems. Previous contributions have focused on performance evaluation of predefined scenarios rather than searching for optimal policies that can minimize the total cost of reliable packet transmission. In this thesis, we study network coded cooperation for dynamic wireless networks with the goal of finding optimal policies that can minimize the total cost of packet transmission and help us to design near optimal multi-cast protocols for dynamic networks.

We start by analysing the gain of network coding protocols in terms of reducing the number of active nodes and the active area as representatives of packet transmission cost. To this end, a multicast scenario in a mesh network is considered. We seek to find a network coding protocol that can achieve the same throughput as no network coding protocols while using less number of active nodes. A new multicast geographic coding-aware communication protocol is proposed which is capable of reducing significantly both the required transmission area and the number of active nodes when compared with traditional multi-casting while achieving the same throughput. The proposed protocol is shown to have linear complexity in terms of number of receivers, compared to the traditional protocols that have non-linear complexities.

We then focus on finding optimal network coded cooperative packet transmission policies that can minimize the cost of reliable packet transmission. A simple network with one source and two receivers is considered to answer the question of when is the optimal time of starting cooperation between receivers in order to minimize the total cost of packet transmission. We model this problem for time-invariant and time-varying erasure channels using a Markov decision process (MDP). Then, the packet transmission policies that are selected by the MDP are analysed to extract the optimal rules of packet transmission. We further propose four powerful heuristics that are shown to provide near-optimal performance in dynamic wireless environments. Our proposed heuristics could be used for any number of receivers. A real-world implementation of the proposed heuristics reveals that the proposed heuristics are able to reduce the mean of completion time by up to $4.75x$ compared with simply broadcasting coded packets.

Finally, we address the effect of a relay on reducing the transmission time of data packets from a source to two receivers using network coding and in the presence of $X$ active neighbours sharing the same channel. We show that using a relay in the presence of active neighbours is beneficial even if the channel from relay to destination is not better than the channel between source and destination.

# Contents

# List of Figures

# List of Tables

# Abbreviations

| | |
|---|---|
| BER | Bit Error Rate |
| BPSK | Binary Phase-shift keying |
| CDF | Cumulative Distribution Function |
| CT | Completion Time |
| D2D | Device To Device |
| GIF | Generalized Intermediate Feedback |
| GMF | Generalized Minimum Feedback |
| GS | Genie System |
| IF | Intermediate Feedback |
| I2V | Infrastructure-to-vehicle |
| LOS | Line-of-sight |
| MAC | Media Access Control |
| MDP | Markov Decision Process |
| MF | Minimum Feedback |
| NC | Network Coding |
| NC-CC | Network Coded Cooperative Communication |
| RLNC | Random Linear Network Coding |
| STD | Standard Deviation |
| TDMA | Time Division Multiple Access |
| VANET | Vehicular Ad hoc Network |

# Chapter 1

# Introduction

During the last decade, different deployments of wireless technology have been considered as promising approaches to provide high quality internet access for multiple users at the same time. Wireless mesh networks, mobile ad hoc networks (MANETs), and vehicular ad hoc networks (VANETs) are examples of these settings. In all of these examples, the dynamic nature of wireless communication channels on one side, and the time-varying topology of the network on the other side, result in problems such as unpredictable connections, variable packet transmission rates, and variable node densities. Therefore, designing appropriate routing protocols and reliable content distribution schemes with minimum cost for such unreliable environments are very challenging.

In this context, geographic routing techniques have been proposed for reliable transmission of data in dynamic environments. In geographic routing, a forwarding decision is made based on the positions of the current node, the destination node, and the candidate forwarding nodes between them. These techniques provide a natural solution for applications in which data is linked more tightly to specific locations than to specific terminals [1], [2]. A variety of comparative studies between topology-based routing techniques (e.g., AODV [3], DSR [4]) and geographic routing for wireless networks have identified the latter as a promising paradigm to improve performance in scenarios with positioning information and dynamic topology [5], [6].

Network coding (NC) [7] as a new emerging communication paradigm was also shown to improve the traditional data dissemination methods in terms of reliability and throughput. NC allows the nodes to use algebraic operations to mix packets they receive or generate instead of simply forward them. Recent works in [8–16] have revealed the benefits of NC for a variety of wireless communication scenarios.

On the other hand, cooperative communication that allows users in a network to hear and help the information transmission of each other was shown to considerably improve the communication performance in terms of bit error rate or outage probability. This is because short range transmission links between wireless users are typically faster, cheaper and more reliable than the links between the source and the users. Therefore, cooperation between devices is expected to substantially improve bandwidth and/or energy efficiency as well. Recently, cooperative communication protocols that use NC to improve reliability, efficiency, and security of the wireless networks have been proposed and extensively studied in the literature [17–31].

This thesis' research focus falls into the intersection of these three areas, i.e., geographic routing, network coding, and cooperative communications. Our goal is to investigate the optimal design of NC-based multicast protocols for dynamic wireless networks with time-varying channels. For a better understanding, Fig. 1.1 shows a schematic of the scenarios that are investigated in this thesis.

The first part of this thesis focuses on Scenario 1 of Fig. 1.1, where we investigate the design of a multi-hop geographic-based communication protocol that leverages NC to increase the throughput and the reliability of data transmission from a sender to multiple receivers. More precisely, we focus on developing network coding based routing protocols without looking at the dynamics of individual packet transmissions.

In the second part, we focus on Scenario 2 of Fig. 1.1, where the optimal design of packet transmission between each intermediate node and its neighbours that are seen as its local receivers is studied. As shown in Fig. 1.1, this scenario may exist as a part of the created paths in a wireless mesh network. Initially, the problem of cost minimization for packet transmission from a source to two receivers over time invariant erasure channels is investigated. Then, we extend our analysis for a network with $N$ receivers. Finally, we focus on a similar problem for time-varying erasure channels.

In the last part, Scenario 3 of Fig. 1.1 is considered, and we study the optimal use of a relay for reducing the transmission time of data packets from a source to two receivers using network coding and in the presence of multiple active neighbours. This scenario can happen in mesh networks with multiple active nodes.

Before discussing the framework of this research, we provide a brief overview on the background of the problems to show the position of our research.

## 1.1 Multipath Geographic routing

The design of efficient routing protocols for dynamically changing network topologies is a crucial part of building reliable and scalable ad hoc wireless or other similar mesh networks such as VANETs. If position information is available due to GPS or some kind of relative positioning techniques, a promising approach is given by geographic routing algorithms, where a forwarding decision is based on the positions of current, destination, and possible candidate nodes in the vicinity. The routing algorithms proposed in the past decade follow the traditional approach of topology-based routing; that is, information about paths are maintained (either reactively or pro-actively), and thus a topology

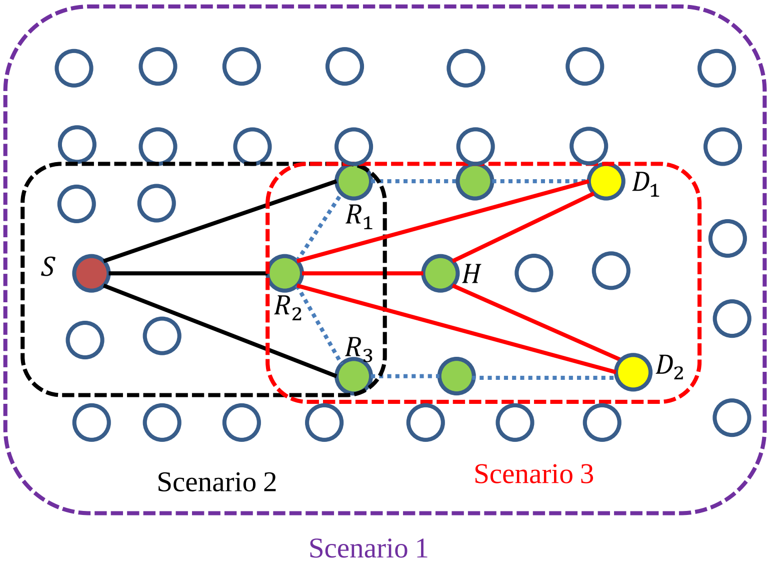


Figure 1.1: Schematic of the problems that are investigated in this thesis

change possibly requires distant nodes to change their routing table. This may generate a significant amount of traffic when the network topology changes frequently. In recent years, several novel geographic routing algorithms have been proposed that allow routers to be nearly stateless since packet forwarding is achieved by using information about the position of candidate nodes in the vicinity and the position of the destination node only. Information on the physical location might be determined by means of a global positioning technique like GPS, or relative positioning based on distance estimation on incoming signal strengths. There exists a broad literature on geographic routing algorithms, which could be divided into three basic sub-classes. First, greedy geographic algorithms which limit forwarding decisions based on the information about the location of the current forwarding node, its neighbours, and the final destination. Each intermediate node applies this greedy principle until the destination, is eventually reached. There are a lot of schemes based on greedy algorithms. The characteristics of greedy routing algorithms differ on the optimization criterion applied in each forwarding step.

A second sub-class of geographic routing algorithms is planar graph routing. In general, the geometric graph reflecting a wireless network is not planar. Thus, before planar graph routing can be performed, a planar sub-graph has to be extracted from the complete network graph. These algorithms are attractive for wireless ad hoc networks because they have been shown to scale better than other alternatives.

The third sub-class of geographic routing algorithms is based on partial flooding and keeping information about past routing tasks which, unlike greedy algorithms, can guarantee delivery based on memorization, e.g., [32]. However, these algorithms require an increased communication overhead and abandon the stateless property of single-path greedy routing.

Although simple, single path routing has inherent limitations in performance and reliability in practical settings. For example, the routing path failure may happen during data transmission because of collision, node dying out (no battery), node busy, or other accidents. Some applications require real time information and data, which means retransmission, is not possible. Node mobility may cause the existing point-to-point route invalid before another route must be chosen [33]. These kinds of problems motivate researchers to design a multipath routing scheme, which several paths is built from source to destinations. Traditionally, multipath routing protocols rely on link/node-disjoint paths to enhance the data throughput which requires complex algorithms to find, while with NC we do not need to search for node-disjoint paths as we can take advantage of combining packets and sending an efficient packet at the intersection nodes. We will use a similar approach to propose a geographical communication protocol that creates multiple paths with intersections and uses NC at the intersection nodes to increase the reliability and the throughput. Our proposed protocol differs from the classical geographic protocols in three ways: first, it has linear complexity of the path creation algorithm with respect to the number of destinations and the number of neighbouring nodes, while the traditional algorithms such as position-based multicast routing (PBM) [34] and geographic multicast routing (GMR) [35] have respectively, exponential and polynomial complexity. Second, our protocol achieves the capacity that is indicated by the max-flow min-cut theorem [36], and third, multiple paths created by our protocol may intersect with each other and the nodes are allowed to do NC.

## 1.2 Network Coding (NC)

In their seminal work [7], Ahlswede et al, proposed the concept of network coding, which allows the intermediate nodes to do some kind of coding and processing over received data packets and then, forward the packets. Network coding has shown to provide benefits in terms of throughput, robustness, complexity and security. The throughput benefit is achieved by using packet transmissions more efficiently, by communicating more information using less packet transmission. The classical example that is usually used to show the throughput benefit of NC is called "Butterfly network". Fig. 1.2 shows this example, where node $S$ wants to transmit two bits to two receivers $R_1, R_2$, over a network composed by links of unitary capacity. In the standard store-and-forward paradigm, the link between nodes 3 and 4 is a bottleneck, since only one bit can be forwarded by node 3, which leads to one of the receivers (either $R_1$ or $R_2$) obtaining incomplete information (see Fig. 1.2-(a)). In case of NC, node 3 is allowed to combine the two bits via a simple XOR operation, $a \bigoplus b$. The receivers can then combine the received data with this coded transmission in order to recover the missing information as shown in Fig. 1.2-(b). More

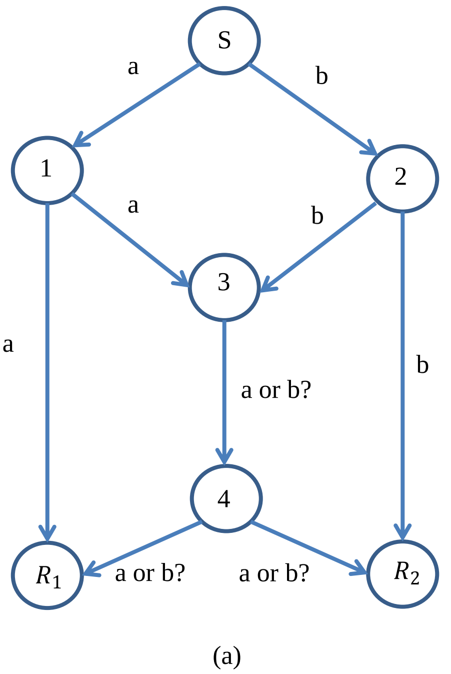


Figure 1.2: The Butterfly network example to explain the advantages of network coding over classical packet transmission (a) classical store-and-forward, (b) NC-based scheme.

precisely, since $R_1$ receives bit $a$ from node 1 and $a \bigoplus b$ from node 4, it can recover $b$ from this coded transmission. Similarly, $R_2$ can recover $a$ from $a \bigoplus b$ and $b$. Therefore, for NC-based scenario, we are able to transmit two bits by using one transmission from node 3, while in case of store-and-forward scheme, we have to do two transmissions. This means that NC doubled the throughput for this simple example. Although the butterfly example was first proposed for wired networks, but it was shown that the benefits of NC is not limited to wired networks and could be easily extended to wireless networks. Network coding can also increase robustness to packet losses. NC will solve the problem of packet loss using some packet redundancy such that even if only a subset of packets sent by the source is received at the sink, the original message can be recovered again. In terms of security, network coding may also provide some gains. Considering the butterfly example, if an adversary node receives only $a \bigoplus b$, it cannot decode the initial packets. So from this point of view, network coding can increase the security of the network. In this thesis, we mainly focus on the throughput gain and the complexity gain of NC.

## 1.3 Benefits of NC-Based Content Distribution

There exists many papers that show the benefits and the requirements of combining NC with existing content distribution methods in many wireless scenarios [37–40]. For example, Toledo et. al [37] proposed efficient multipath routing protocol for sensor networks by combining diffusion and network coding. The work in [38] has demonstrated that network coding may improve the overall performance of peer-to-peer content distribution. Authors in [39] investigate the implementation issues of NC for content-distribution in vehicular networks. They consider general resource constraints (e.g., CPU, disk, memory) besides bandwidth, that are likely to impact the encoding and storage management operations required by network coding. Ma et. al [40] evaluated the benefits of NC for peer-to-peer content distribution in terms of average downloading time at peers, total distribution time and system throughput.

Although many works have been done in this area, to the best of our knowledge, and at the time of conducting this research, there has been no work on using NC techniques in geographic-based routing algorithms. Considering the potential advantages of NC for content distribution in mesh networks, and the need to improve the throughput and the reliability of the traditional routing protocols in dynamic networks, we focus on the design of geographic NC-based routing protocols that could be used in mobile wireless mesh networks. Another reason for this focus is that the selection of minimum cost routing subgraph for multicast scenarios based on the traditional methods is very complex. While it has been shown that using network coding, corresponding problem will be converted to a linear optimization problem that has a low-complexity solution.

## 1.4 Network Coded Cooperative Communication (NC-CC)

Mainly, two general network coded cooperative scenarios have been investigated in the literature. First, scenarios with one/multiple sources transmitting data to multiple users with/without the help of relays and cooperative users attempting to receive packets from the source(s). Second, scenarios with multiple nodes working together to deliver their packets to a common destination. We call the first set of scenarios downstream cooperation and the second upstream cooperation. For downstream cooperation, the state of the art has considered evaluating the performance of network coding cooperation in terms of diversity multiplexing, and outage probability [17], as well as the application of network coding cooperation in (i) exploiting route selection strategies in multi-rate networks [18], (ii) proposing cluster-based routing protocols [19], (iii) improving user's perceived QoS in multimedia broadcast/multicast services (MBMS) [20], and (iv) session grouping and relay node selection [21–24]. For upstream cooperation, the literature has focused on (i) developing adaptive strategies [25] and constructing distributed network codes [26], (ii) evaluating performance of coded cooperation in a network with two cooperating users, in terms of cooperative diversity and outage probability [27–29], and maximal throughput [30], and (iii) determining implementation requirements and deploying cooperative strategies [31]. Despite of the extensive efforts to evaluate the performance of NC in cooperative scenarios, a more in-depth analysis of time-varying scenarios and, particularly, the design of optimal NC cooperative policies and protocols is missing in the literature. In fact, most of the previous works focus only on a predefined packet transmission policy and not on determining the optimal policy given protocol design considerations. We break from this trend by not assuming a transmission policy a priori and seeking an optimal policy to minimize the total cost of packet transmission in different scenarios.

## 1.5 Main contributions and thesis outline

In this thesis, we focus on the design of NC-based protocols for dynamic networks. First, we consider designing a geographic-based routing protocol that use NC to increase throughput and reliability. Then, we investigate how to disseminate data using local cooperation between nodes on a routing path to minimize the total cost of packet transmission. The main contributions of this thesis are as follows:

- *M-GeoCode: A Geographic Coding-Aware Communication Protocol:* We propose a geographic NC-based communication protocol for unicast and multicast sessions, M-GeoCode, which is capable of reducing significantly both the required transmission area and the number of active nodes when compared with classical multicasting while achieving the same throughput. M-GeoCode is using a modified directed diffusion policy to generate multiple paths within predefined geographic area. These paths may intersect each other at intermediate nodes, which use network coding to maximize the throughput. A comparison between M-GeoCode and the traditional geographic multi-cast communication schemes shows that M-GeoCode is able to achieve the same throughput while reducing the number of active nodes by a factor of 1.55 and the active area by a factor of 3.57, thus mitigating the interference and reducing the total energy consumption.

- *Optimal Network Coded Cooperative Communication over Time-Invariant Half-Duplex Channels:* We analyse the optimal design of NC-CC for a wireless network with one source, two receivers and half duplex erasure channels. The problem is modelled as a Markov decision process (MDP) and is solved for any field size, arbitrary number of packets, and arbitrary erasure probabilities of the channels. The proposed MDP solution results in an optimal transmission policy per time slot and we use it to design near-optimal heuristics for packet transmission in a network of one source and $N \geq 2$ receivers. We also present numerical results that illustrate the performance of the proposed heuristics under a variety of scenarios.

- *Real-World Implementation of the Proposed Network Coded Cooperative Communication Heuristics:* To complete our analysis, our proposed heuristics are implemented in a WiFi test-bed and compared with random linear network coding (RLNC) [9] broadcast in terms of completion time, total number of required transmissions, and percentage of delivered generations. Our measurements show that enabling cooperation only amongst pairs of devices can decrease the completion time and increase the reliability, compared to RLNC broadcast.

- *Optimal Network Coded Cooperative Communication over Time-Varying Half-Duplex Channels:* We investigate the optimal design of cooperative network-coded strategies for a three node wireless network with time-varying, half-duplex channels. The problem of cost minimization in this scenario is modelled using an MDP. Then, we analyse the performance of the heuristics that we proposed for the time-invariant channels model in scenarios with time-varying channels. To this end, two wireless channel models are used, namely, (a) an infrastructure-to-vehicle (I2V) in a highway scenario considering Rayleigh fading, and (b) real packet loss measurements for WiFi using Aalborg University's Raspberry Pi test-bed. We show that the proposed heuristics can also provide near-optimal performance in time-varying scenarios.

- *Optimal Network Coded Relay-Based Multicast Communication in the Presence of Active Neighbours:* We investigate the optimal use of a relay for reducing the transmission time of data packets to a single/multiple receivers using network coding and in the presence of active neighbours. The problem is formulated as an MDP and numerical results are provided comparing simple, close–to–optimal heuristics to the optimal scheme. Our results show that using a relay in the presence of active neighbours is beneficial even if the channel from relay to destinations is not better than the channel between source and destinations.

The remainder of this thesis is organized as follows. In Chapter 2, a network coded geographic-based communication protocol, M-GeoCode, is presented and its performance is compared to traditional multicast geographic routing protocols. The key idea of M-GeoCode is to select active nodes based on their geographic location and to leverage RLNC at the active nodes, to maximize the throughput within the active region. In Chapter 3, we focus on finding the optimal cooperative packet transmission policy that can be used in a simple network of one source and two or multiple receivers. We assume time-invariant erasure channels for this part of our analysis. Our results in this chapter, could be directly used to optimize the proposed geographic routing protocol in the future, by minimizing the cost of packet transmission in each hop of the multi-hop path. Four powerful heuristics are proposed that are shown to have near-optimal performance. We implemented the proposed heuristics in a WiFi test-bed consisting of Raspberry Pi nodes and the results of this implementation are presented in this chapter. In Chapter 4, we model a similar optimization problem for time-varying channels using an MDP. The performance of the heuristics that we proposed in Chapter 3, is evaluated for time-varying channels by using two scenarios, namely, (a) an I2V in a highway scenario considering Rayleigh fading, and (b) real packet loss measurements for WiFi using a Raspberry Pi test-bed. In Chapter 5, we address an effect that is typically overlooked in previous studies: the presence of active transmitting nodes in the neighbourhood of such devices, which is typical in wireless mesh networks. We investigate optimal transmission policy that minimizes the total cost of packet transmission in a crowded network with many active neighbours using the help of NC and a relay node. Two simple heuristics are proposed for relay-based multicast transmissions in the presence of active neighbours. Our numerical results show that the proposed heuristics have near-optimal performance. Finally, Chapter 6 presents the conclusions of this thesis and a discussion on possible directions for future work.

Parts of this work have been presented at the following international conferences:

- H. Khamfroush, D. Lucani, J. Barros, "GeoCode: A geographic coding-aware communication protocol", Proceedings of IEEE Conference on Intelligent Transportation Systems (ITSC), Washington DC, US, Oct 2011.

- H. Khamfroush, D. Lucani, J. Barros, "Minimizing The Completion Time Of A Wireless Cooperative Network Using Network Coding", Proceedings of IEEE International symposium on personal, Indoor and Mobile Radio Communications (PIMRC), London, UK, Sep 2013.

- H. Khamfroush, D. Lucani, J. Barros, "Network Coding for Wireless Cooperative Networks: Simple Rules, Near-optimal Delay", Proceedings of IEEE ICC, Workshop on Cooperative and Cognitive Mobile Networks, Sydney, Australia, June 2014.

- H. Khamfroush, P. Pahlevani, D. Lucani, M. Hundeboll, F. Fitzek, "On the Coded Packet Relay Network in the Presence of Neighbors: Benefits of Speaking in a Crowded Room", Proceedings of IEEE International Conference on Communications (ICC), Sydney, Australia, June 2014.

Part of this work has been accepted for publication in the following journals:

- H. Khamfroush, D. Lucani, P. Pahlevani, J. Barros, "On Optimal Policies for Network Coded Cooperation: Theory and Implementation", IEEE Journal of Selected Areas in Communications (IEEE JSAC).

- H. Khamfroush, D. Lucani, J. Barros, P. Pahlevani, "Network Coded Cooperation Over Time-Varying Channels", IEEE Transactions on Communications.

The last part of this work will be submitted to the following journal:

- P. Pahlevani, H. Khamfroush, D. E. Lucani, M. V. Pedersen, F. Fitzek, "On Opportunistic Network Coding for Local Optimization of Routing in Wireless Mesh Networks: Protocol Design and Performance Evaluation", to be submitted to IEEE Transactions on Communications.

# Chapter 2

# M-GeoCode: A Geographic Coding-Aware Communication Protocol

In this chapter, we present a new communication protocol called M-GeoCode that achieves a higher throughput for the same number of active nodes compared to node-disjoint multipath routing mechanisms. This means that, M-GeoCode requires a smaller transmission area and a smaller number of active nodes to achieve the same throughput compared to the traditional multi-path protocols. A comparison with traditional multipath routing algorithms, which deliver node-disjoint paths and non coding-aware solutions, reveals that M-GeoCode achieves the same throughput while reducing the number of active nodes by a factor of 1.55 and the active area by a factor of 3.57.

The remainder of this chapter is organized as follows. Section 2.1 presents related work. Our motivation and main contributions are presented in Section 2.2. Some useful definitions are stated in Section 2.3. Section 2.4, describes a formal problem statement. In Section 2.5, we propose our new multicast geographic coding-aware communication protocol (M-GeoCode). In Section 2.6, we analyse the benefits of M-GeoCode in terms of algorithm complexity, data delivery cost, and algorithm overhead in a grid network topology. Section 2.7, presents performance evaluation metrics that we use to compare M-GeoCode versus traditional protocols and the comparison results. Concluding remarks are presented in Section 2.8.

## 2.1 Related work

By leveraging advances in positioning systems (e.g. GPS) and wireless communication, geographic routing protocols for wireless networks are experiencing significant interest by the research community. Several geographic routing protocols for supporting unicast transmissions have been proposed in [1], [2], [41], [42]. To enhance the reliability and/or throughput of the data exchange amongst nodes in wireless networks, multipath routing algorithms relying on node-disjoint paths have been proposed [43]. This is particularly important when the network topology changes rapidly. By combining multi-path routing and geographical information, previous results have evidenced a higher reliability in data exchanges measured in terms of the packet delivery rate [44], [45]. These methods consider a source node sending repetitions of each packet through different, geographically-aware paths optimized for a specific parameter, e.g., path length. There are many scenarios of wireless sensor networks in which sensors are required to send the same report to several sinks whose positions are known in advance. In such scenarios, it is vital to count on an efficient multi-casting mechanism being able to alleviate the overall consumption of resources in the network [35]. Providing efficient multicast routing in wireless sensor networks poses special challenges compared to unicast data delivery. However, computing a minimal bandwidth consumption multicast tree in wireless multi-hop networks was proven to be NP-complete [46]. Therefore, heuristics are used in practical protocols. Previous works on geographic-based multi-casting in wireless networks include (but not limited to): position-based multicast routing (PBM) [34], scalable position-based multicast (SPBM) [47], differential destination multicast (DDM) [48], geographic multicast routing (GMR) [35] and hierarchical rendezvous point multicast (HRPM) [49]. It has been shown in [35] that GMR has less number of transmissions compared with PBM, SPBM and DDM. It is also proven that GMR has less amount of neighbor selection complexity compared with PBM, SPBM and DDM. HRPM is a hierarchical multi-casting protocol which efficiently reduces the byte overhead associated with each data packet by dividing a large group of multicast into multiple subgroups. HRPM improves the scalability of stateless location-based multicast with respect to the group size. It is mostly used when the number of destinations is large [49]. We use PBM and GMR as representatives of the existing geographic multicast protocols that have better performance than the others, and compare them with M-GeoCode.

- PBM protocol: It is a position-based multi-cast routing protocol that uses the geographic position of the nodes to make the forwarding decision. The main task of a forwarding node in PBM is to find a set of neighbours that should forward the packet next. These nodes are called the "next hop node". The current node will assign each destination of the packet to exactly one next hop node. Each next hop node then becomes forwarding node for this packet toward the assigned destinations. If the current node selects more than one next hop node, then the multicast packet is split. This maybe required to reach destinations which are located in different directions relative to the forwarding node. The main property of PBM is that each forwarding node autonomously decides how to forward the packet. This decision requires no global distribution structure such as a tree or a mesh. To determine the set of next hop nodes in PBM, a forwarding node minimizes the cost of forwarding that is defined as a function of two metrics: i) the number of neighbours that the packet is transmitted to, ii) the remaining distance to all destination.

- GMR protocol: Similar to PBM, GMR is also a localized geographic multicast routing protocol, where each node has to select a subset of its neighbours as relay nodes towards destinations. GMR optimizes the cost over progress ratio where the cost is equal to the number of neighbours selected for relaying and the progress is the overall reduction of the remaining distances to destinations. Such neighbour selection achieves a good trade-off between the bandwidth of the multicast tree and the effectiveness of the data distribution.

The work by Ahlswede et al. showed that in many multicast scenarios the optimal communication bandwidth can be achieved if and only if intermediate nodes in the network code information together. There is a broad literature on analysing NC benefits for routing in wireless networks [50–56]. For instance, MAC-independent opportunistic routing protocol, MORE, applied network coding to the opportunistic routing, and demonstrated its potential for unicast and multicast wireless mesh scenarios [53]. Using the simple idea of NC, MORE avoids node-coordination that was traditionally required by opportunistic routing protocols to have knowledge of which packets each node has received. Optimized multipath network coding protocol (OMNC) is an optimization based network coding protocol that controls the end-to-end transmission of coded packets in lossy wireless environments [54]. Katti et. al proposed COPE as an opportunistic approach to NC for wireless mesh networks, where routers mix packets from different sources to increase the information content of each transmission [55]. Authors in [56] proposed an NC-based geographical opportunistic routing protocol in which a number of opportunistic relays forward packets from a source to a destination and each relay tries to code maximum number of packets in each transmission.

To the best of our knowledge, there is no previous work on applying NC to design geographic multicast routing protocols. The only work that somehow deals with applying NC to geographical routing is the work presented by Tang et. al in [56], which only focused on a unicast session and needs a lot of control messages to keep track of the received packets by each node to make efficient coded packets. This increases the cost of the protocol and its complexity.

## 2.2 Motivation and Main Contributions

The fact that network coding allows nodes to transmit packets that result from joint encoding of multiple original information units, has led to have communication protocols that are easier to establish. The reason is that by using NC we are not limited to search for disjoint paths between the source and the destinations in a network graph and this simplifies the algorithms that are used to extract a routing sub-graph. Besides simplifying protocol design, NC also could bring other important benefits in terms of throughput, reliability, robustness and adaptability. Although there has been many works on showing the benefits of NC for many wireless scenarios, but there exist less efforts on designing NC-based protocols that can work in very dynamic/mobile scenarios such as VANETs. Ho et. al in [52] considered a distributed randomized network coding approach that enables efficient decentralized operation of multi-source multicast networks and showed that this approach provides substantial benefits over traditional routing methods in dynamically varying environments. In that paper, the authors just focused on showing the benefits of randomized NC for a predefined multicast spanning tree, and in fact, they are not trying to design a protocol.

As one of the relevant protocols for dynamic environments, geographic multicast routing protocols have shown to be promising. Thus, it may be beneficial to look at the possible ways of designing NC-based geographic routing protocols that could be used for multicast sessions. In this chapter, we are focusing on designing a network coded geographic multicast communication protocol that can outperform the performance of the traditional geographical multicast routing protocols.

Our main contributions in this chapter are as follows.

- **Protocol design:** We propose a new unicast and multicast geographic coding-aware communication protocol (M-GeoCode) that only allows the nodes inside a predefined geographic region (active region) to participate in the forwarding of the data packets. M-GeoCode is shown to achieve capacity, as indicated by the max-flow min-cut theorem [36], while allowing a distributed, message passing mechanism to determine the transmission sub-graph. A key difference to previous protocols lies in fact that M-GeoCode allows the intermediate nodes to use linear NC to increase throughput and robustness.

- **Mathematical analysis:** The performance of M-GeoCode is characterized in terms of the number of active nodes in a general network and a grid network model. The complexity of the neighbor selection algorithm of the proposed protocol is also computed and compared with traditional multicast algorithms. The results show that M-GeoCode has a linear complexity in terms of the number of destinations and the number of neighbouring nodes, in contrast to other traditional protocols like PBM and GMR who have exponential complexity and polynomial complexity, respectively.

- **Performance evaluation:** We provide simulation results that illustrate M-GeoCode's benefits over standard node-disjoint multi-path protocols namely Dijkstra, Greedy, and GMR in terms of the number of active nodes, the packet transmission area, and the throughput gains. Our results reveal that M-GeoCode is able to achieve the same throughput while reducing the active transmission area by a factor of 3 in unicast sessions and a factor of 3.57 in multicast sessions.

## 2.3 Definitions

Before stating the problem, we provide some useful definitions that are used in this chapter.

**Definition 1** (Min-cut)**.** In graph theory, a minimum cut of a graph is a cut (a partition of the vertices of a graph into two disjoint subsets that are joined by at least one edge) whose cut set has the smallest number of edges (unweighted case) or smallest sum of weights possible.

**Definition 2** (Rank of a Matrix)**.** The rank of a matrix $A$ is the size of the largest collection of linearly independent columns of $A$ (the column rank) or the size of the largest collection of linearly independent rows of $A$ (the row rank). For every matrix, the column rank is equal to the row rank

**Definition 3** (Max-flow min-cut)**.** Let $G = (V,E)$ be a directed graph representing a network with $s$ and $t$ being the source and the sink of $G$, respectively. The capacity of an edge is a mapping $c : E \rightarrow \mathbb{R}^+$, denoted by $c_{uv}$ or $c(u,v)$. It represents the maximum amount of flow that can pass through an edge. A flow is a mapping $f : E \rightarrow \mathbb{R}^+$ denoted by $f_{uv}$ or $f(u,v)$, subject to the following two constraints:
1. $f_{uv} \leq c_{uv}$ for each $(u,v) \in E$ (capacity constraint)
2. $\sum_{u:(u,v)\in E} f_{uv} = \sum_{u:(v,u)\in E} f_{vu}$ for each $v \in V \setminus \{s,t\}$.
The value of flow is defined by $|f| = \sum_{v\in V} f_{sv}$ , where $s$ is the source of $G$. It represents the amount of flow passing from the source to the destination. The maximum flow problem is to maximize $|f|$, that is, to route as much flow as possible from $s$ to $t$.
An $s$-$t$ cut $C = (S,T)$ is a partition of $V$ such that $s \in S$ and $t \in T$. The cut-set of $C$ is the set $\{(u,v) \in E \mid u \in S, v \in T\}$. Note that if the edges in the cut-set of $C$ are removed, $|f| = 0$. The capacity of an $s$-$t$ cut is defined by $c(S,T) = \sum_{(u,v)\in S\times T} c_{uv}$. The minimum $s$-$t$ cut problem is minimizing $c(S,T)$, that is, to determine $S$ and $T$ such that the capacity of the $S$-$T$ cut is minimal. The max-flow min-cut theorem states:
**The maximum value of an $s$-$t$ flow is equal to the minimum capacity over all $s$-$t$ cuts.**

## 2.4 Problem Statement

We start by describing the problem for a unicast session, then, we extend our problem statement for a multicast communication session.

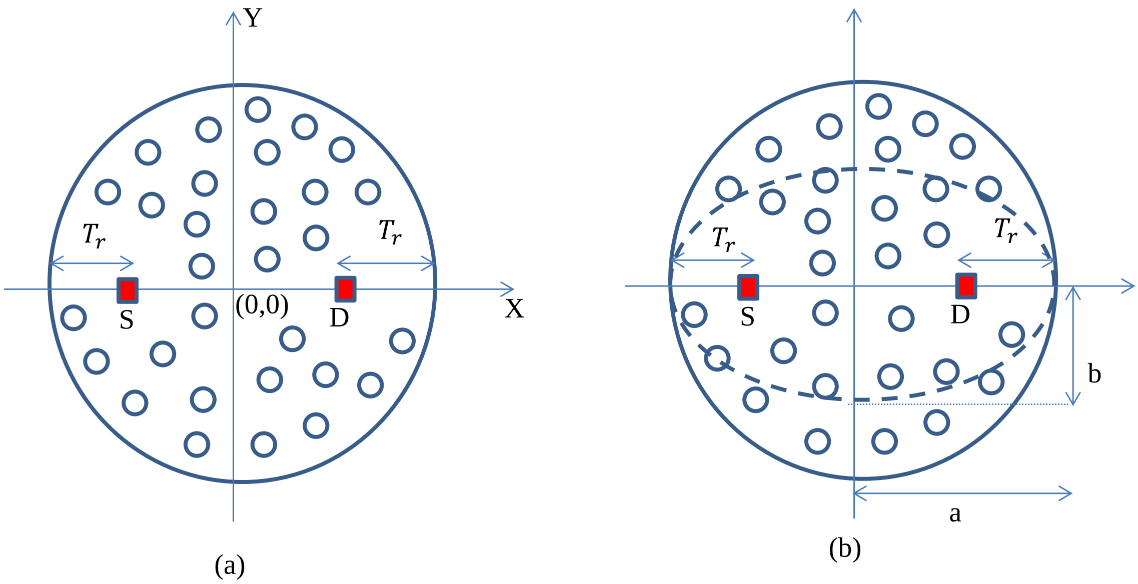


Figure 2.1: (a) Node deployment area, (b) active area

### 2.4.1 Unicast Session

We consider a wireless mesh network consisting of one source, one destination, and multiple intermediate nodes that are all located inside a circle area with radius $a$. We assume that all intermediate nodes are deployed according to a random uniform distribution and the transmission radius, $T_r$, is the same for all nodes. In our model, two nodes are connected if their euclidean distance is less than or equal to $T_r$. The center of circle is placed at the origin of a Cartesian coordinate system. We represent the network as an acyclic directed graph $G = (V,E)$, where $V$ is the set of nodes and $E$ is the set of edges. Edges are denoted by $e = (v,v') \in E$, in which $v = head(e)$ and $v' = tail(e)$. Moreover, we consider ideal links (delay-free, no losses). Without loss of generality, we assume that the source and destination are placed inside the circle area at $(-a+T_r,0)$ and $(a-T_r,0)$, respectively (See Fig. 2.1-(a) ). We define throughput, $\lambda$, as the total number of data packets, $N_t$, transmitted between source and destination in one time unit, i.e., $\lambda = N_t$. We assume that the capacity of each link is one bit per unit time. We define an ellipse inside the circle area as the "*active area*" and all the nodes inside the active area (except the source and the destination) as the "*active nodes*". We are aiming at proposing a NC-based protocol that can achieve the same throughput as non-NC based protocol by using less number of nodes. Therefore, we assume that for NC-based protocol only the nodes inside the ellipse are allowed to forward the packets, while for non-NC based protocols all nodes inside the circle can participate in the forwarding process. The source and the destination nodes constitute two focal points of the ellipse, as shown in Fig. 2.1(b). The semi-minor axis of the ellipse is denoted by $b$. We adopt random linear network coding (RLNC) framework [9] for our NC-based protocol. Meaning that every active node generates RLNC coded packets by linearly combining packets that are received from its neighbours or are available in its buffer. For example, an active node mixes $M$ packets, namely, $p_1, p_2, ..., p_M$ from the buffer and creates coded packets as a linear combination of these with some coding coefficients $\alpha_1, ..., \alpha_M$, i.e., $\sum_{i=1}^{M} \alpha_i p_i$. The coding coefficients are independently and randomly selected from a Galois field of size $q$, i.e., $GF(q)$.

Considering these assumptions, we are seeking the value of $b$ that provides the same or higher throughput for a NC-based protocol compared to a non-NC, node-disjoint multi-path protocol. This means that we seek $b$ that satisfies

$$\lambda_{G_{ellipse}} \geq \lambda_G \tag{2.1}$$

where $\lambda_{G_{ellipse}}$ represents the throughput achieved by the NC-based protocol for the sub-graph of the active area, i.e., nodes inside the ellipse, and $\lambda_G$ is the throughput achieved by a node-disjoint, multi-path geographic routing applied to the original network graph, $G$. This constitutes a proxy to determine the active area that provides the same throughput as using node-disjoint, multi-path routing over the entire network.

We define $R$ as the ratio between the ellipse area, $S_{ellipse}$, for $b$ satisfying Eq.(2.1) and the circle area, $S_{circle}$. So,

$$R = (S_{ellipse}/S_{circle}) = (\pi.ab)/(\pi.a^2) = b/a. \tag{2.2}$$

$R$ determines how much smaller the transmission area can be by using NC with respect to traditional node-disjoint multi-path algorithms.

### 2.4.2 Multicast Session

Let us assume that we have a more general case with one source and multiple destinations. We consider each destination separately and find the active region of that source-destination pair, $S$-$D_i$. Each pair has a min-cut value, $MC_{(S,D_i)}$, that is calculated according to Definition 1. For each specific source-destination pair, $S$-$D_i$, the active region is defined as an ellipse with two focal points at source $S$ and destination $D_i$. The $a$ parameter of each ellipse is defined as $a = (1/2)\ dis(S,D_i) + T_r$, which $dis(S,D_i)$ shows the Euclidean distance between source $S$ and destination $D_i$. We are seeking to find the $b$ parameter of each ellipse that satisfies Eq. (2.3).

$$MC_{(ellipse)} = MC_{(S,D_i)}, \tag{2.3}$$

where $MC_{(ellipse)}$ represents the min-cut of the sub-graph located inside the defined ellipse for pair $s - D_i$, and $MC_{(S,D_i)}$ represents the min-cut of the network graph for pair $S$-$D_i$. All nodes of the network graph that are located inside at least one of the defined ellipses are considered as "*active nodes*". Finally, the active region of M-GeoCode for the multicast scenario is equal to the union of the active regions for all existing pairs of $S-D_i$.

## 2.5 M-GeoCode: A Multicast Geographic Coding-aware Communication Protocol

Motivated by the problem we defined, a new multicast geographic coding-aware communication protocol, "M-GeoCode", is proposed. M-GeoCode relies on a modified directed diffusion policy [37] used to generate multiple paths within a predefined limited area, e.g., an ellipse, called active area. These paths may intersect each other at intermediate nodes which use NC to combine packets coming from different paths. The path creation phase of M-GeoCode is defined in a way that allows the source to determine the maximum achievable rate or capacity of the network. M-GeoCode is also proven to achieve the capacity of the active area. We propose two versions of M-GeoCode, that differ in the number of protocol steps and complexity of the control packets in the different steps. The first one, called "3-step M-GeoCode" includes three main phases: exploration phase, interest propagation phase, and packet transmission phase. The second version, "2-step M-GeoCode", includes two main phases: interest propagation phase and packet transmission phase. Although the 3-step version requires a larger set-up time, it also requires simpler control packets, which can be advantageous if the number of destinations is large. In the following sub-sections, we explain both versions in detail.

### 2.5.1 3-step M-GeoCode

The three phases of 3-step M-GeoCode are as follows:

*Exploration phase:* The achievable rate of single source multicast scenario is characterized as the minimum of the min-cuts between the source and all nodes in its destination set, *Des*, i.e., $min\{min-cut(S,t);\forall t\in Des\}$ [57, 58]. The main goal of this phase is to explore the min-cut of each source-destination pair that can be used to determine the achievable rate of the graph at the end of the next step. This phase is started by sending exploration messages (EXP) from the source to all of its outgoing edges. Inspired by the work of Toledo et.al in [37] to determine the min-cut of a network graph, the content of each EXP message is defined as a unit vector in $\mathcal{I}^{\delta_O(S)}$ space, where $\delta_O(S)$ is the cardinality of the set of all edges that originate from source node $S$. The EXP messages generated at the source are linearly independent. For instance, if there are three edges originating from the source, then the EXP messages transmitted through those three links are $EXP_1=(1,0,0), EXP_2=(0,1,0), EXP_3=(0,0,1)$. Each intermediate node makes a new vector from the linear combination of all received vectors according to RLNC and forwards it toward all of its outgoing edges. This process is continued until all destinations receive the EXP packets. At the end of this phase, each destination is able to compute the min-cut between itself and the source from Theorem 1, but still the achievable rate (min-cut) of the network is unknown. The destinations are also able to characterize the size of the active area (ellipse) that they may use to transmit the interest packets in the interest propagation phase.

**Theorem 1.** *Given an acyclic directed graph, $G = (V, E)$ with a sender $S \in V$ and a set of $M$ destinations, $Des = \{D_j \mid j = 1, 2, 3, ..., M\} \subset V$, and diffusion mechanism of exploration phase of M-GeoCode, the min-cut (capacity) between source and each destination $D_j$ is calculated using*

$$rank\{EXP_i'\{e\} : tail(e) = D_j\} = MC_{(S,D_j)}, \forall i \in \mathcal{D}_{Tj}, \tag{2.4}$$

*Where $\mathcal{D}_{Tj}$ is the set of all edges terminating at the destination node $D_j$ and $EXP_i'$ is the i-th received exploration vector at destination $D_j$.*

*Proof.* The proof follows by substituting the sources (destination) in [37] with destinations (source) in M-GeoCode. □

*Interest propagation phase:* Once the exploration phase finishes, the interest propagation phase is started by sending the interest messages from each destination node to all of its outgoing edges located within a specified limited area, e.g., an ellipse. The size of the ellipse for each source-destination pair can be different from the others and it can be changed based on the calculated min-cut, total number of nodes and node's transmission radius. Fig. 2.2 shows the selected active areas for interest propagation in a network with 3 destinations. As we can see, the active area definition is such that the ellipses may intersect each other, i.e., there will be common nodes (solid fill nodes in Fig. 2.2). This is in contradiction to the traditional node-disjoint multicast protocols where usually avoid having common nodes between multiple paths.

Each interest packet contains 3 main components. Fig.2.3 shows the three components of each interest packet. The "Common nodes" component represents a parameter that changes while the interest packet propagates and its primary value is always zero. The "min-cut" component of the interest packet contains the value of the computed min-cut of the relevant *S-D* pair that was calculated from the exploration phase. Each interest packet also contains the location of the destination that is sending the packet and also the source location, $(S, D)$-*ID*, to determine the active area for that *S-D* pair. Node $i$ with coordinate $(x_i, y_i)$, forwards the received interest packet if and only if

$$(x_i/a)^2 + (y_i/b)^2 \leq 1, \tag{2.5}$$

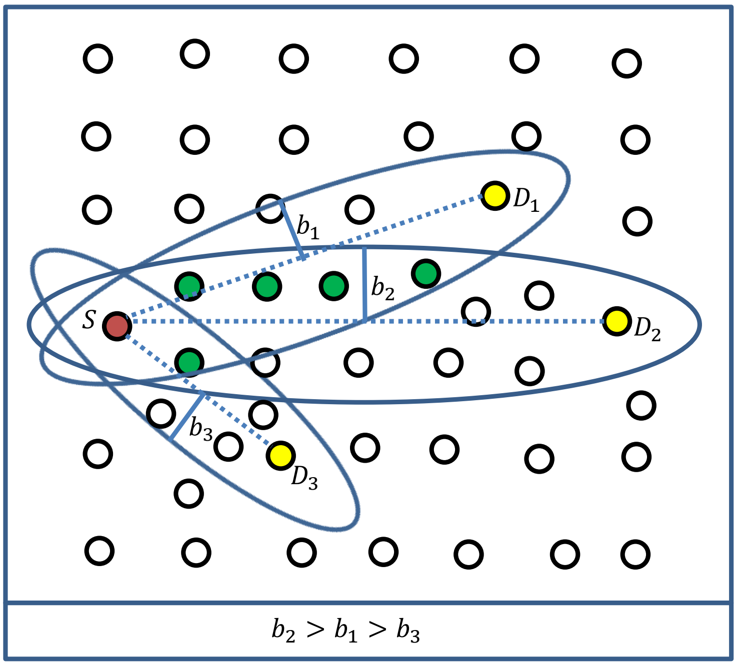


Figure 2.2: Limited areas of interest propagation in a multicast scenario with three destinations

is satisfied. The coordinate system is chosen in a way that the source and the destination lie at the focal points of the ellipse. $b$ represents the calculated value of the semi-minor axis of the ellipse for the received interest packet. $a$ is defined as we explained in Section 2.4. If an intermediate node receives more than one interest packet, it has to check Eq.(2.5) for all existing pairs of $(S,D)$-$ID$ that are received, to determine if it is an active node or not. If an intermediate node receives more than one interest packet and is an active node, it will make a new interest message from the received ones. The value of the "Common nodes" component for the new interest packet is equal to the number of different destination IDs that exist in the received interest messages minus one. Therefore, the "Common nodes" component ranges between zero and the total number of destination nodes minus one. The value of the "Min-cut" component for the new interest message is equal to the minimum value of the "Min-cut"'s of the received interest packets, because

| | | |
|---|---|---|
| Common nodes | Min-cut | (S,D)-ID |

Figure 2.3: Main components of interest packet in M-GeoCode

the maximum achievable rate of the network is equal to the minimum "min-cut" of the $S$-$D$ pairs. The $(S,D)$-$ID$ for the new interest message at each intermediate node, includes the union of $(S,D)$-$ID$ values of all received interest messages. The interest propagation phase continues until the source receives interest packets from all of its neighbours. Fig. 2.4 shows how the interest messages are transmitted along the network graph. In this example, there is one source and two destination nodes, $D_1, D_2$. The interest propagation phase is started by sending two different interest packets, namely, $(0,3,D_1)$ and $(0,2,D_2)$, respectively from $D_1$ and $D_2$ towards all of their outgoing edges located inside a predefined limited area. For simplicity, we replaced $(S,D_i)$-$ID$ with $D_i$. All of the intermediate nodes who receive the interest packets make a new interest packet and forward it toward their neighbours. In this example, node $f$ receives two interest packets, $(0,3,D_1)$ and $(1,2,(D_1,D_2))$, respectively from nodes $e$ and $i$. Since there are two different destination IDs, $D_1, D_2$, in the received packets, the value for the "Common nodes" parameter would be one. The new value of "Min-cut" will be the minimum of $(3,2)$ that is 2. The new set of "$(S,D)$-$ID$" will be $D_1, D_2$ that is the union of $D_1$ and $(D_1, D_2)$.

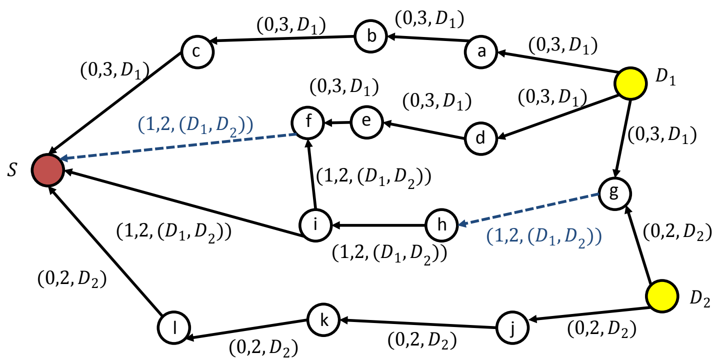


Figure 2.4: Interest propagation phase of 3-step M-GeoCode

*Packet transmission phase:* Once the source receives all interest packets, it calculates the achievable rate (maximum capacity) of the graph, $h$, from the "Min-cut" values of the received interest packets. The achievable rate of the multicast graph is equal to the minimum amount of the "Min-cut" values of the received interest packets. For instance, the maximum capacity of the multicast graph of Fig. 2.4 is $C_{max} = min(2,3) = 2$. This means that the source can send at most $h = 2$ symbols per unit time reliably. Therefore, the source chooses the appropriate number of the created paths between itself and destinations. The mechanism of the path selection is done based on maximizing the number of common-nodes and minimizing the total number of active nodes and path length. Therefore, the source chooses the paths related to the interest messages that have the maximum

"Common-nodes" and arrived at source earlier than the others. The process of path selection should be done such that there is enough number of paths, $h$, between each source and destination. Note that, although the path creation between source and destinations was done for each source-destination pair independently, the final decision about the selected paths is made considering all destinations together. In fact, one of our contributions is to propose a new neighbour selection scheme for multi-casting which is completely distributed and is able to create very efficient paths. Once the source has finished the path selection process, the data packets are transmitted through the selected paths from where the source received an interest message using RLNC.

### 2.5.2 2-step M-GeoCode

This version of M-GeoCode has two main phases and it can be used for small number of destinations, because the size of the interest packet increases as a function of number of destinations. There is no exploration phase in this version and the protocol is started by sending the interest messages from the destinations.

*Interest propagation phase:* The goal of this phase, is to provide enough information for the source to be able to estimate the achievable rate of the network graph and to create paths between destinations and the source. This phase is started by sending interest messages from each destination $D_j$ to all of its outgoing edges located within a predefined ellipse for that $S$-$D_j$ pair. The size of the ellipses is a metric that could be changed based on different parameters and it will be discussed in the future sub-sections. However, the size of the ellipses for different destinations are not necessarily the same and it depends on the location of the destinations (see Fig. 2.2). Similar to the exploration phase of "3-step M-GeoCode", the content of the interest messages created at each destination $D_j$ is independent unit vectors in $\mathcal{I}^{\delta_O(D_j)}$ space, where $\delta_O(D_j)$ is the cardinality of the set of all edges that originate from destination $D_j$. The interest messages are forwarded only inside the active area. Each destination node $D_j$ creates $k$ independent unit vectors, if it has $k$ outgoing edges. The ID (location) of destination is also added to the created interest message at each destination. Each intermediate node makes a new interest message from the received messages and sends it toward its active neighbours. If a node receives an interest message, while it is not located inside the active area, simply discards the packet. The active nodes are defined similar to 3-step M-GeoCode. Two different cases may happen, when a node wants to make a new interest message from the received ones:

- All received interest messages only include the same destination ID.
- The received interest messages include different destination IDs.

In the first case, a linear combination of the received unit vectors is created as a new interest message and the ID of destination is also added to the interest message. In the second case, only the received vectors that are coming from the same destination, are combined together. In this case, if there are vectors from distinct destinations, the intermediate node does not make a combination of them. It only adds them to the created interest message without any change. Fig. 2.5 shows an example of the interest propagation phase for 2-step M-GeoCode and we assumed that the graph only shows active nodes. It is seen that each interest message created at destination, includes two components, the first component is a unit vector and the second component is the ID of the destination who creates the interest message. Node $b$ in Fig. 2.5 receives two interest messages, $([0\ 1], D_1)$ and $([1\ 0], D_2)$ which have different destination IDs. Therefore, node $b$ only adds the contents of the received messages without doing any change and creates $([0\ 1], D_1, [1\ 0], D_2)$ as the new message to be sent. On the other hand, node $e$ receives two interest messages $([1\ 0], D_1)$ and $([0\ 1], D_1, [1\ 0], D_2)$. In this case, there are 2 vectors $[1\ 0], [0\ 1]$ both coming from destination $D_1$ and so, a new combination of them, $[1\ 1]$, is created. Therefore, the new created message is $([1\ 1], D_1, [1\ 0], D_2)$. At the end of the interest propagation phase, source is able to calculate the maximum achievable rate (min-cut) of the network using Theorem 2.

**Theorem 2.** *Given an acyclic directed graph, $G = (V, E)$, with a sender $S \in V$ and a set of $M$ destinations $Des = \{D_j \mid j = 1, 2, 3, ..., M\} \subset V$, and interest definition mechanism of 2-step M-GeoCode, the maximum achievable rate of the network or the multicast min-cut capacity is calculated from:*

$$\min\left\{\text{rank}\{g'_{ij}(e), : \text{tail}(e) = S\right\}, \forall i \in \mathcal{S}_{Tj}, \forall j \in \{1, 2, ..., M\} \tag{2.6}$$

*Where $\mathcal{S}_{Tj}$ is the set of all received vectors at source who have $D_j$ as destination ID and $g'_{ij}$ is the i-th received vector at source who is coming from destination $D_j$.*

*Proof.* The proof is an extended version of what presented in **Theorem 4** of [37] for the scenario with multiple source and one destination. The key difference of our framework with that work is the sub-graph (composed of active nodes) that we use instead of the original graph and the number of destinations sending the interest messages. Thus, we can consider the interest propagation phase of M-GeoCode as $M$ repetitions of the interest propagation phase in [37]. □

In the example of Fig. 2.5, the max achievable rate of this multicast scenario is 2. Since the rank of the three vectors $[1\ 0], [1\ 1], [0\ 1]$ coming from $D_1$, is 2 and the rank of vectors $[1\ 0], [1\ 1]$ coming from $D_2$, is also 2. Therefore, the minimum rank of the received vectors is equal to 2 and so, the achievable rate of the network is equal to 2.

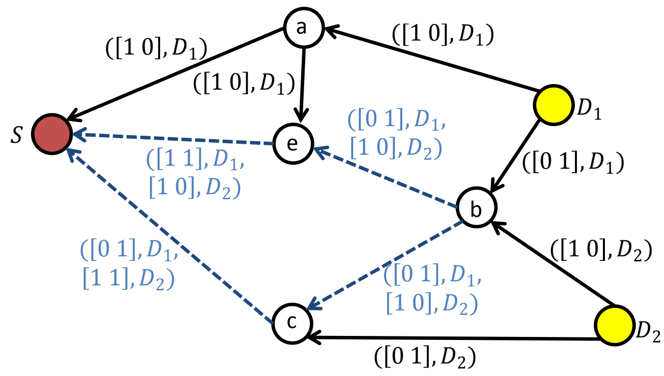


Figure 2.5: Interest propagation phase of 2-step M-GeoCode

*Packet transmission phase:* In this phase, the source determines which paths to activate. Although multiple objectives could be considered, we choose the paths that deliver the interest packets earlier than the others and the interest messages received from them include the maximum number of distinct destination IDs. This way, the number of active nodes decreases and the number of common nodes between multiple paths increases. For example, if the content of the received interest message includes $k$ distinct destination IDs, it means that the related path will end in $k$ distinct destinations.
The packet transmission phase is started by sending data packets through the selected paths. Intermediate nodes generate RLNC packets from the received packets.

## 2.6 Discussion

In this section, we analyse our proposed M-GoeCode protocol in terms of the complexity of path selection algorithm. We also evaluate its benefits against traditional node-disjoint multicast protocols in terms of the data delivery cost. To this end, we calculate the total number of nodes involved in the transmission process, the total number of transmissions, and the active area for unicast and multicast scenarios. An overhead calculation of the protocol is also presented.

### 2.6.1 Algorithm Complexity

We evaluate the worst case complexity of the neighbour selection algorithm for M-GeoCode and compare it with a geographic multicast routing protocol called GMR, in a multicast scenario. The results show that M-GeoCode outperforms GMR in terms of the neighbour selection complexity.

**Lemma 1.** *The complexity of the neighbour selection algorithm of M-GeoCode in the worst case is of $O(Tn)$, where $T$ is the number of destinations of the network and $n$ is the number of neighbours of the node currently multi-casting the message.*

*Proof.* In the worst case of M-GeoCode, an intermediate node may receive different interest packets from all $T$ destinations. Therefore, it has to determine it's active neighbours. In order to do that, if a neighbour node located at $(x,y)$ satisfies at least one of the $T$ equations in Eq. (2.7), then it will be considered as an active node.

$$(x/a_i)^2 + (y/b_i)^2 \leq 1 \quad \forall i \in \left\{1,2,...,T\right\} \tag{2.7}$$

Where $a_i$ and $b_i$ are respectively, the semi-major and semi-minor axes of the ellipse defined for the i-th $S$-$D_i$ pair. The coordinate system is chosen to ensure that the source $S$ and the destination $D_i$ lie at the focal points of the ellipse. This stage needs $T \times n$ comparisons to determine the active neighbour nodes. Thus, M-GeoCode has a complexity of $O(Tn)$ in the worst case scenario concluding our proof. □

*Remark* 1. GMR was proven to have a complexity of $O(Tn\, min(T,n)^3))$ [35]. Therefore, for all amounts of $n$ and $T$, $(T > n$ or $T \leq n)$, M-GeoCode has less complexity than GMR.

### 2.6.2 Data Delivery Cost of M-GeoCode

We analyse the data delivery cost, in terms of number of nodes involved in the transmission process, for both M-GeoCode protocol and node-disjoint multi-path protocols. We evaluate two different scenarios, unicast and multicast.

*Unicast Scenario:* For analytic tractability, we consider a very simple idealized setting: a square grid consisting of $N$ nodes deployed inside the circle (Fig. 2.6). The node transmission ranges are such that all nodes (except those on the sides of the square) can communicate with 8 neighbours on the grid. We focus on a simple unicast scenario, with the source placed along the left edge of the grid and the destination placed along the right edge (see Fig. 2.6). We assume that each packet transmission consumes a unit of energy. Thus, the total number of transmissions constitutes a proxy for the energy consumption of the network. In case of our protocol, we determine $b$ of the ellipse such that the min-cut of the sub-graph representing the active area is equal to the capacity of the network. In the following, we calculate the total cost of packet transmission for a simple node-disjoint multipath routing protocol as a proxy of the traditional geographic algorithms, and our M-GeoCode protocol. Note that according to the "max-flow min-cut" theorem, the maximum capacity of a network is equal to the min-cut of the graph of the network, that is equal to five in this example.

- Node-disjoint multipath routing protocols: The maximum number of node-disjoint paths in our example model is equal to the capacity of the network, i.e., 5 (See

Fig. 2.6). The number of links that are used to build five node-disjoint paths has two components: the number of diagonal links and non-diagonal links. In our example, 8 and $5(\sqrt{N}-1)-4$, respectively. The total number of transmissions for sending 5 packets is directly related to the number of active links and is given by $T_{x(disjoint-paths)} = 5(\sqrt{N}-1)+4$. The total number of nodes that are involved in the transmission process for the case of node-disjoint paths is determined by $N_{(disjoint-paths)} = 3\sqrt{N}+2(\sqrt{N}-2)$.

- NC-based multi-path protocols: Fig. 2.7 shows the minimum cost sub-graph of the grid network with min-cut equals to 5, that is obtained by M-GeoCode. The total number of transmissions to send 5 packets is given by two components: the number of active diagonal links, $2(\sqrt{N}-1)$, and active non-diagonal links, $3(\sqrt{N}-1)+4$. The total number of transmissions is $T_{x(M-GeoCode)} = 5(\sqrt{N}-1)+4$.

  The total number of nodes that are involved in the transmission process for M-GeoCode is calculated using $N_{(M-GeoCode)} = 3\sqrt{N}$

In our idealized scenario, the total number of transmissions for the same throughput is the same for node-disjoint multi-path and M-GeoCode. However, M-GeoCode involves less nodes to transmit those data packets, namely, $2(\sqrt{N}-2)$ less nodes. The ratio between nodes used by node-disjoint multi-path routing and M-GeoCode is

$$N_{(disjoint-paths)}/N_{(M-GeoCode)} = (3\sqrt{N}+2(\sqrt{N}-2))/3\sqrt{N}. \tag{2.8}$$

As $N$ goes to infinity, the ratio converges to 1.66. This means that M-GeoCode reduces the number of active nodes by a factor of 1.66 compared to node-disjoint multipath protocols while providing the same throughput.

*Multicast Scenario:* We consider a simple multicast scenario with one source and two destinations. A square grid consisting of $N$ nodes deployed inside the circle (Fig. 2.8), similar to the previous scenario is considered. The node transmission range is defined as the unicast example. We assume that the grid square center is placed at the origin of a Cartesian coordinate system. The coordinates of the source, $S$, and the two destination nodes, $D_1, D_2$, are defined as $(\frac{1-\sqrt{N}}{2}, 0), (\frac{\sqrt{N}-1}{2}, \frac{\sqrt{N}-3}{2}), (\frac{\sqrt{N}-1}{2}, \frac{3-\sqrt{N}}{2})$, respectively. The maximum achievable rate (min-cut) of the multicast scenario in this example is equal to five. It is easy to see that the maximum number of node-disjoint paths between $S$ and the two destinations is less than the capacity of the network and is equal to three. Fig. 2.8 shows the minimum cost node-disjoint multi-paths for this example. Therefore, to make a fair comparison between M-GeoCode and non-NC node-disjoint multi-path protocols, we define the active area of M-GeoCode such that the achieved min-cut is equal to three. Fig. 2.9 displays how to define the active area of M-GeoCode, $b_1$ (related to $S$-$D_1$ pair) and $b_2$ (related to $S$-$D_2$ pair) parameters, in order to achieve a min-cut of 3.

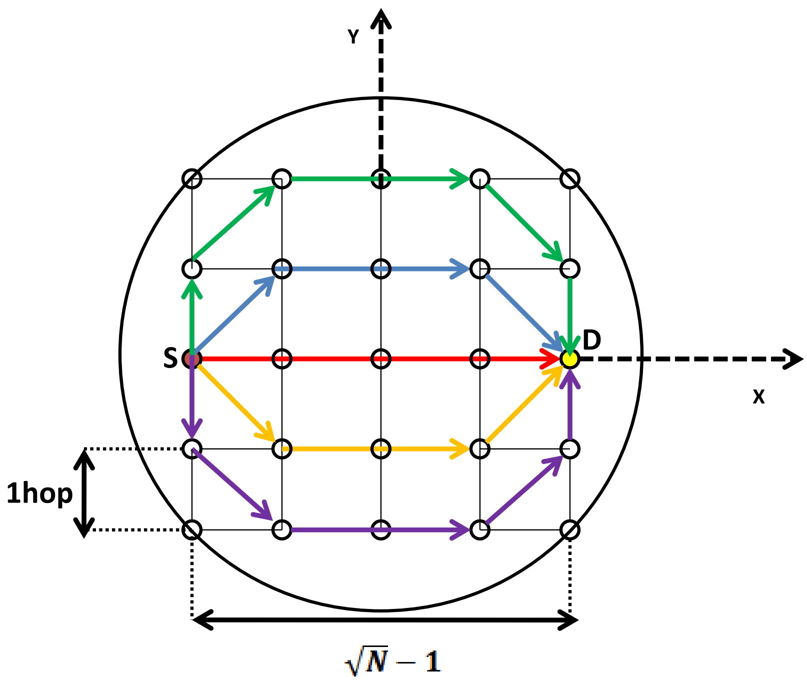


Figure 2.6: Example of square grid topology and node-disjoint paths(unicast scenario)

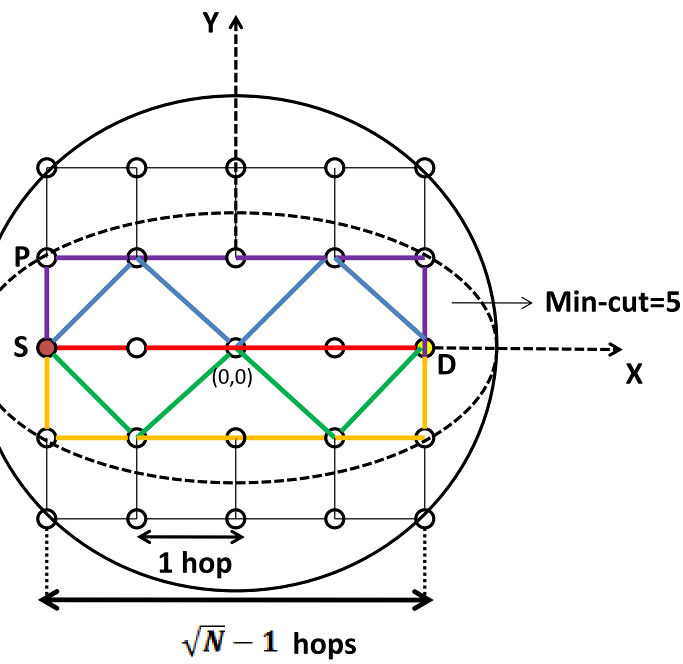


Figure 2.7: Selected sub-graph with minimum cost that achieves min-cut capacity(unicast scenario)

- Node-disjoint multipath routing protocols: In order to be consistent, we assume that $(\sqrt{N})$ has an odd value. Fig.2.8 shows the minimum cost sub-graph to create three possible node-disjoint paths between $S$ and $D_1$, $D_2$. The total number of active nodes in this case is determined by Eq.(2.9).

$$N_{(disjoint-paths)} = \begin{cases} 5\sqrt{N}-11 & for\sqrt{N}=5 \\ 5\sqrt{N}-10 & for\sqrt{N}=7 \\ 7\sqrt{N}-25 & for\sqrt{N}\geqslant 9 \end{cases} \tag{2.9}$$

- NC-based multi-path protocols: We applied M-GeoCode protocol to the grid network to achieve the min-cut equals to three. Fig.2.9 represents the selected minimum cost sub-graph by M-GeoCode and the active area. The active area in this case is composed of two ellipses with different sizes. We see that there are common nodes between the two ellipses that leads to decrease the total number of active nodes. The total number of active nodes that are involved in the transmission process is calculated as

$$N_{(M-GeoCode)} = (9\sqrt{N}-25)/2. \tag{2.10}$$

From Eq. (2.9) and Eq. (2.10), we see that the number of active nodes for M-GeoCode protocol is less than that for the traditional node-disjoint multi-path protocols. The ratio between the number of active nodes in two cases is calculated as

$$N_{(disjoint-paths)}/N_{(M-GeoCode)} = (7\sqrt{N}-25)/(4.5\sqrt{N}-12.5). \tag{2.11}$$

As $N$ goes to infinity, the ratio converges to 1.55. This means that for a multicast scenario with 2 destinations, M-GeoCode is able to reduce the number of active nodes by a factor of 1.55 while providing the same throughput. Another interesting point is that M-GeoCode can achieve the maximum capacity of the network by increasing the size of the active area, while in our example $(N = 25)$ it is not possible to create 5 node-disjoint paths between $S$ and $D_1, D_2$.

### 2.6.3 Overhead Analysis of M-GeoCode

M-GeoCode's interest propagation phase introduces some overheads in order to establish the paths from source to destinations. However, constraining the geographic region of transmitting the interest packets shall have the added value of reducing this overhead. For the unicast example of the grid network, it is straightforward to prove that the number of transmissions for the interest propagation phase in M-GeoCode is $9\sqrt{N}-7$, while for an equivalent directed diffusion mechanism involving all nodes in the grid network it is $2(\sqrt{N}-1)^2+2\sqrt{N}(\sqrt{N}-1)$. That is, M-GeoCode reduces the overhead by $O(\sqrt{N})$ with respect to the directed diffusion mechanism that involves all $N$ nodes and use a

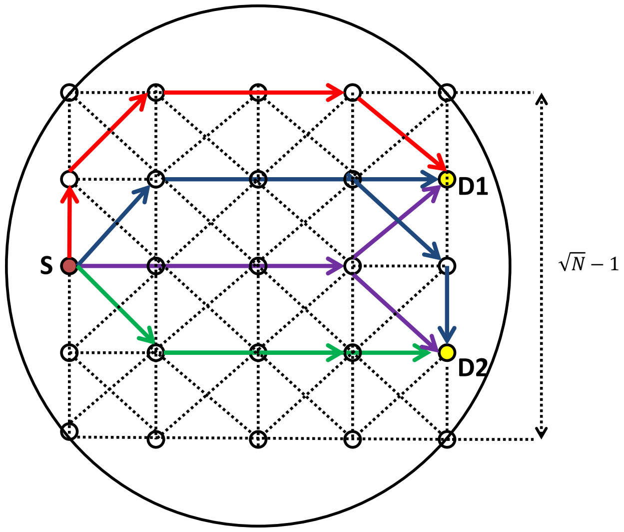


Figure 2.8: Example of square grid topology and node-disjoint paths (multicast scenario)

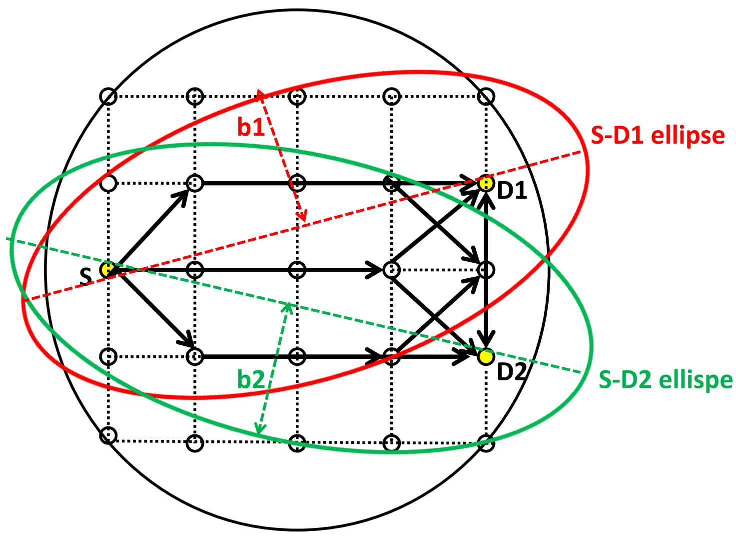


Figure 2.9: Selected sub-graph with minimum cost to achieve min-cut=3 using M-GeoCode (multicast scenario)

similar approach to create the paths. In our example, both mechanisms attain the network capacity.

### 2.6.4 Size of the Active Area

To have a better understanding of the size of the active area (ellipse), we characterize the effects of the possible effective parameters on it. Two different scenarios are considered as follows.

- *Distance between the source and the destination is a variable:* We consider a very simple idealized setting that was shown in Fig. 2.7. The square's center is placed at the origin of a Cartesian coordinate system. We assume that the grid is consisting of $N$ nodes and the coordinates of the source and the destination is always fixed at $S(-(\sqrt{N}-1)/2,0)$, $D((\sqrt{N}-1)/2,0)$, respectively. The transmission range of the nodes, $(T_r)$, is such that all nodes (except those on the sides of the square) can communicate with 8 neighbours on the grid. It is also assumed that the distance between two adjacent nodes is always equal to one hop and so the density of the nodes in the network is constant. Therefore, by increasing the number of nodes, $N$, the distance between $S$ and $D$ increases while the nodes density will remain constant.

**Lemma 2.** *In a grid network defined as Fig.2.7 with a constant value of node density, a lower bound of b to achieve the maximum capacity of network is calculated as*

$$b \geq \frac{1}{\sqrt{1-\frac{W^2}{(W+T_r)^2}}}, \tag{2.12}$$

*where* $W = \frac{\sqrt{N}-1}{2}$, *N represents the total number of network nodes, and* $T_r$ *is the transmission radius of nodes.*

*Proof.* The lower bound of $b$ is defined as the minimum value of $b$ which ensures that node $p$ is located over or inside the active area (ellipse). Because in this case, we can guarantee that the other active nodes are also located inside the ellipse. Therefore, the coordinate of $p(-(\sqrt{N}-1)/2,1)$ should satisfy

$$x^2/a^2 + y^2/b^2 \leq 1, \tag{2.13}$$

where $a = \frac{1}{2}distance(S,D) + T_r = \frac{\sqrt{N}-1}{2} + T_r$ and $b$ is the desired value. By substituting $x$ and $y$ with the coordinate of $p$, we have the following:

$$(-(\sqrt{N}-1)/2)^2/a^2 + 1^2/b^2 \leq 1, \tag{2.14}$$

which concludes the proof. □

We can see that for a constant value of $T_r$ and node density, the lower bound of $b$ is an increasing function with respect to $N$, meaning that by increasing the distance between $S$ and $D$, the size of the active area increases too.

- *Node density is a variable:* We consider a simple grid network with $N$ nodes as shown in Fig. 2.10. It is assumed that the distance between $S$ and $D$ is always equal to one, therefore, the density of the nodes increases by increasing the number of nodes. To find a lower bound of $b$, node $p$ should satisfy Eq. (2.13) similar to the previous section.

**Lemma 3.** *In a grid network scenario that the distance between the source and the destination is always fixed and the density of the nodes is a variable (see Fig.2.10), a lower bound of b to achieve the maximum capacity of the network is calculated as*

$$b \geq \frac{1}{2W.\sqrt{1-\frac{1}{4(\frac{1}{2}+T_r)^2}}}, \tag{2.15}$$

*where* $W = \frac{\sqrt{N}-1}{2}$.

*Proof.* The minimum value of $b$ is defined to ensure that node $p = (-\frac{1}{2}, \frac{1}{\sqrt{N}-1})$ is located over (or inside) the ellipse. Therefore, the coordinate of $p$ satisfies Eq. (2.13) which in this case $a = \frac{1}{2}distance(S,D) + T_r = \frac{1}{2} + T_r$. By substituting the coordinate of $p$ in Eq. (2.13) and simplifying, the lower bound of $b$ is attained as Eq. (2.15). □

The lower bound of $b$ in this case is a decreasing function of $W$, which means that by increasing the density of the nodes in a constant area, the size of the active area decreases.

Based on our observations, we can conclude that the size of the active area could be a function of the nodes density, distance between the source and the destinations, and $T_r$. We also see that the size of the active area is an increasing function of the distance between $S$ and $D$ and a decreasing function of the nodes density.

## 2.7 Performance Evaluation and Numerical Results

Before showing the simulation results of the M-GeoCode and the traditional geographic multicast protocols, we explain the comparison schemes and the metrics we use to evaluate the performance of M-GeoCode protocol.

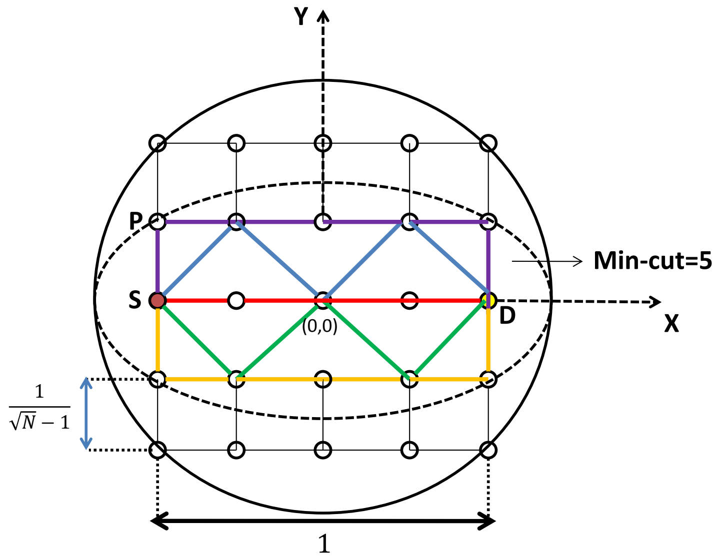


Figure 2.10: Minimum cost selected sub-graph to achieve min-cut=5 using M-GeoCode

### 2.7.1 Schemes

To make a fair comparison between the proposed M-GeoCode protocol and the traditional non-NC protocols, we use different schemes to create node-disjoint paths between source and destinations. Since finding the maximum number of node-disjoint paths in a graph is NP-hard [59], we focus on heuristics applied in practice. To calculate the possible maximum number of node-disjoint paths, we have to minimize the length of the created paths. We divided our analysis into two scenarios:

- *Unicast scenario:* In this case, we consider two node-disjoint protocols, namely a Dijkstra-based and a Greedy protocol. Both protocols use an iterative process that i) computes a path from source to destination using the network's graph, and ii) removes the nodes and the associated links from that path (except for $S,D$) and updates the network's graph. The process is repeated while $S$ and $D$ are connected. This way, we are creating node-disjoint path between the source and the destination.

  - Dijkstra-based protocol: Uses Dijkstra's algorithm [60] to determine the shortest-path between source and destination for step (i) of the iterative process.

    Dijkstra starts at the source, $S$, it grows a tree, $T$, that ultimately spans all vertices reachable from $S$. Vertices are added to $T$ in order of distance i.e., first $S$, then the vertex closest to $S$, then the next closest, and so on.

  - Greedy-based protocol: Uses a greedy algorithm [61] to create the path between source and destination for step (i) of the iterative process. Each node in the network sorts all of its neighbours based on their distances from the destination and chooses the one who has minimum distance as a next hop candidate. Node $S$ creates a link with its first next-hop candidate. That node then creates a link to its first next-hop candidate. This process continues until reaching $D$.

- *Multicast scenario:* Since we are interested in creating as much as possible number of paths between the source and the destinations, we use the protocol with less complexity and less amount of energy consumption. In this case, we use the idea of GMR to create the paths. Similar to the unicast case, an iterative process is used to build multiple paths between a source and destinations. We called the iterative process, GMR-based protocol. A minimum-cost tree using GMR protocol is created between the source and destinations in the first step. Then the nodes and the associated links are removed from the created path (except source and destinations) and the network's graph is updated. The process is repeated to create more paths until $S$ and destinations are connected. At each step, GMR selects a subset of nodes that is the best to propagate the message towards destinations. The selected subset is the one that reduces most the total distance to destinations per unit of cost, where the cost is the number of selected neighbours.

### 2.7.2 Comparison Metrics

We use the following metrics for the comparison between non-NC and NC protocols:

- *Throughput ratio:* We define the throughput ratio ($R_\lambda$) as

$$R_\lambda = \lambda_{(M-GeoCode)}/\lambda_{(Greedy)}, \tag{2.16}$$

  where $\lambda_{(M-GeoCode)}$ is the throughput of M-GeoCode and $\lambda_{(Greedy)}$ is the throughput of the greedy algorithm applied to the nodes located inside the circle area. To calculate the throughput achieved by non-NC and NC protocols in both unicast and multicast scenarios we use the following:

  - **Non-NC Protocols:** The number of node-disjoint paths selected by the protocol determines the achievable throughput.
  - **NC Protocols:** Min-cut of the sub-graph created by the protocol determines the achievable throughput similar to the work in [37]. This representation is following a very well-known theorem in NC stating: " linear network coding achieves the min-cut/max-flow bound for any multicast network with a single source and multiple destinations [7]".

- *Active nodes ratio:* In order to compare the number of active nodes, we define a metric called ”active node ratio”, $R_{active-nodes}$, that calculates the ratio between the number of active nodes of M-GeoCode ($N_{active-GeoCode}$) and the number of active nodes of the node-disjoint multipath protocol ($N_{active-node\ disjoint}$) to achieve the same throughput, i.e.,

$$R_{active-nodes} = \frac{N_{active-GeoCode}}{N_{active-node\ disjoint}} \tag{2.17}$$

- *Active area ratio:* This metric displays the area covered by the active nodes and is a proxy for network interference. According to interference definition in [62], we can conclude that the lower the active area, the lower the network interference. To do the comparison, we define

$$R_{active-area} = \frac{S_{active-GeoCode}}{S_{active-node\ disjoint}}, \tag{2.18}$$

where $S_{active-GeoCode}$ is the union of the transmission areas of the active nodes in the case of M-GeoCode and $S_{active-node\ disjoint}$ is the union of the transmission areas of the active nodes in the case of node-disjoint multipath protocols. To calculate the transmission areas of the active nodes for every active node, we define a circle centered at the active node with radius $T_r$. Then the total area of the created circles around the active nodes is computed as the union of the transmission areas. Therefore, the lower the value of $R_{active-area}$, the higher the efficiency of M-GeoCode in reducing network interference.

### 2.7.3 Simulation Results

We compare the performance of the traditional node-disjoint routing algorithms and M-GeoCode for unicast and multicast scenarios in terms of the metrics that we defined. In the case of unicast, we compare Dijkstra-based and Greedy-based algorithms, M-GeoCode, and the min-cut value of the original network graph. In case of multicast, we compare M-GeoCode and GMR-based node-disjoint multi-path algorithm. We implemented our network model in MATLAB.

- *Unicast Scenario:* We use the network model that was introduced at the beginning of this chapter in Fig. 2.1(b) with $a = 0.5$. We assume that the intermediate nodes deployed based on a random uniform distribution. We repeat our tests for 200 random deployments and the average throughput is determined by averaging over the 200 deployments.
  *Throughput gain of M-GeoCode:* Fig. 2.12 and 2.13 compare the throughput ($\lambda$) of different schemes for different values of $b$, $N = 700$, and $N = 1000$ nodes deployed inside the circle area. The transmission radius of the nodes is set to $T_r = 0.05$. Fig. 2.12, 2.13 show that for any $b$, the attainable throughput of NC (min-cut), $\lambda_{NC}$,

is larger than the throughput of the node-disjoint multipath algorithms, $\lambda_{non-NC}$. M-GeoCode is shown to achieve the same throughput as the min-cut of the network graph, i.e., M-GeoCode achieves capacity. We can see that for $N = 700$, the ratio between the throughput of M-GeoCode and the throughput of the node-disjoint protocols is changing from 1.2 to 1.59 which means that M-GeoCode is able to increase the throughput by a factor of 1.59. For the same experiment and $N = 1000$, M-GeoCode increases the value of throughput by a factor of 1.71. Another interesting observation is that the achievable throughput of both Dijkstra-based and Greedy-based protocols are very close to each other. This reveals that the number of possible node-disjoint paths is almost independent of the selected routing protocols.

We also use $R_\lambda$ criteria to make a comparison between M-GeoCode and the Greedy-based protocol in terms of throughput efficiency. For a specific value of $b$, we will calculate the throughput ratio, $R_\lambda$, for our two distributed algorithms, namely, M-GeoCode and the Greedy-based algorithm. Fig. 2.11, shows the comparison result for $N = 300$ and different values of $T_r$. Note that the value of $b$ that satisfies $R_\lambda = 1$ constitutes the point in which M-GeoCode provides the same throughput as the equivalent node-disjoint multipath algorithm. Therefore, Fig. 2.11 shows that by using M-GeoCode we are able to send at the same rate towards the destination while involving less nodes deployed in a smaller, constrained region. For $T_r = 0.15$, we observe that $R_\lambda = 1$ corresponds to M-GeoCode operating at $b = 0.16$. Meaning that the M-GeoCode reduces the required transmission area by $1/3$. Since we use random uniform distributed nodes, we can say that the M-GeoCode protocl requires 3 times less nodes to achieve the same throughput in average.

Fig. 2.11 also shows that there is a $b^*$ for each $N$ and $T_r$ such that every $b \geq b^*$ provides the maximum throughput for M-GeoCode. Thus, there is no incentive in using a $b > b^*$ from a throughput perspective.
*Transmission area reduction gain of M-GeoCode:* We use metric $R_{active-area}$ defined in Eq. (2.18) as the performance comparison metric. We use the Dijkstra-based algorithm to create node-disjoint paths. Fig. 2.14 illustrates the connection between the transmission radius ($T_r$) and $R_{active-area}$.

$R_{active-area}$ quantifies how much smaller the transmission area is by using M-GeoCode compared to the Dijkstra-based algorithm. Therefore, the smaller the $R_{active-area}$, the bigger the gain in terms of reducing the transmission area. We consider deployments with $N = 100, 300, 500$ nodes in the circle of Fig. 2.1(b).

We observe that for each $N$, there exists a $T_r$ that optimizes the reduction of the transmission area, i.e., minimizes $R_{active-area}$. Note that $R_{active-area}$ has values ranging

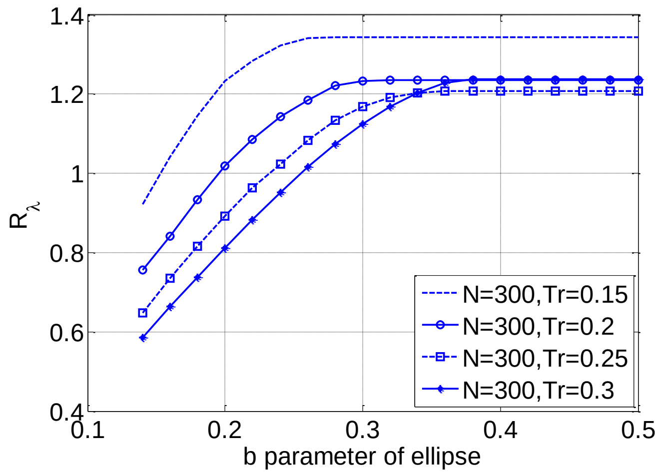


Figure 2.11: Comparison of M-GeoCode protocol and Greedy-based node-disjoint multi-path algorithm in terms of the throughput ratio ($R_\lambda$)

from 0.4 to 0.6, which means that M-GeoCode is able to decrease the transmission area by a factor of 2.5 while providing the same end-to-end throughput.

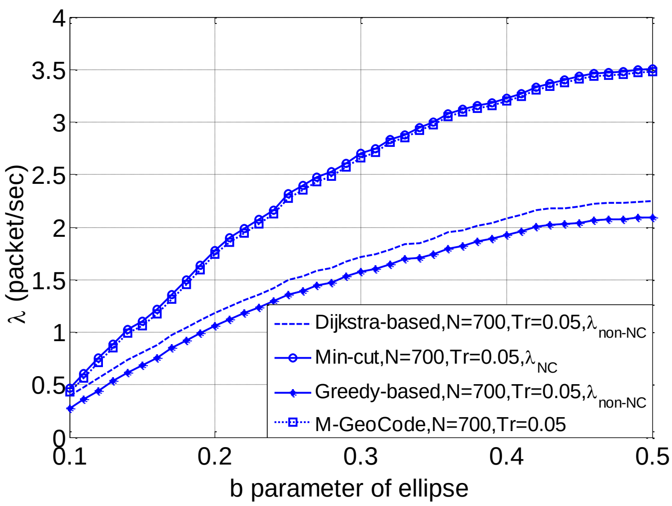


Figure 2.12: Throughput gain of M-GeoCode compared with node-disjoint path algorithms for different sizes of the ellipse and N=700

- *Multicast Scenario:* We assume that the network is composed of one source and 3 destinations. All nodes (except the source and the destinations) are deployed based

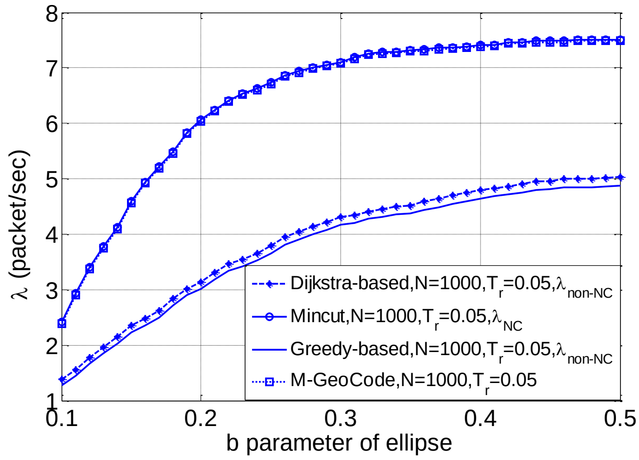


Figure 2.13: Throughput gain of M-GeoCode compared with node-disjoint path algorithms for different sizes of the ellipse and N=1000

on a random uniform distribution in a unit square area. We considered two different network models that are shown in Fig. 2.15 (a),(b). In the first model which we call it "line model", all three destinations are located over a line while in the second model, "triangle model", the destinations create a triangle. In both cases, the source is fixed at $S(0.3, 0.45)$. Our simulation results obtained by averaging over 100 random deployments of the nodes. Note that, although the time complexity of GMR is lower than the other geographic multicast algorithms like PBM, it is still time consuming to create a large number of node-disjoint paths using GMR algorithm. Therefore, we run our GMR-based protocol to achieve small number of node-disjoint paths (e.g., 3, 4, 5, 6) and then store the coordinates of the active nodes for each case. A similar process is repeated in the case of M-GeoCode to achieve the same throughput. The simulation results are divided to two main sections.

*Active node reduction gain of M-GeoCode:* We use $R_{active-nodes}$ as a comparison metric. We repeat our test for different number of nodes $N$, transmission radius $T_r$, throughputs $\lambda$, and different models of destination deployments (line and triangle models). The results are shown in Fig. 2.16, 2.17, 2.18. In all cases, M-GeoCode uses less nodes to achieve the same throughput as the GMR-based protocol. Therefore, the overall energy consumption and the interference are also reduced. We see that for a fixed value of $T_r$ and a predefined position of source and destinations, if the density of nodes increases, the gain of M-GeoCode increases too. For instance, to achieve $\lambda = 3$ in a network with $T_r = 0.15$, $N = 200$ and using the line model for destinations, $R_{active-nodes} = 0.8691$ while in case of $N = 300$, it is equal to 0.7902

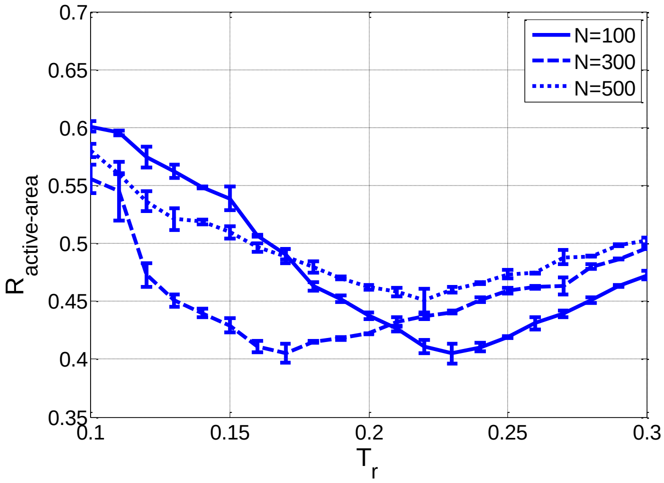


Figure 2.14: Comparison of the ratio between the active area of M-GeoCode and Dijkstra-based algorithm to achieve the smae throughput, for variable *N*

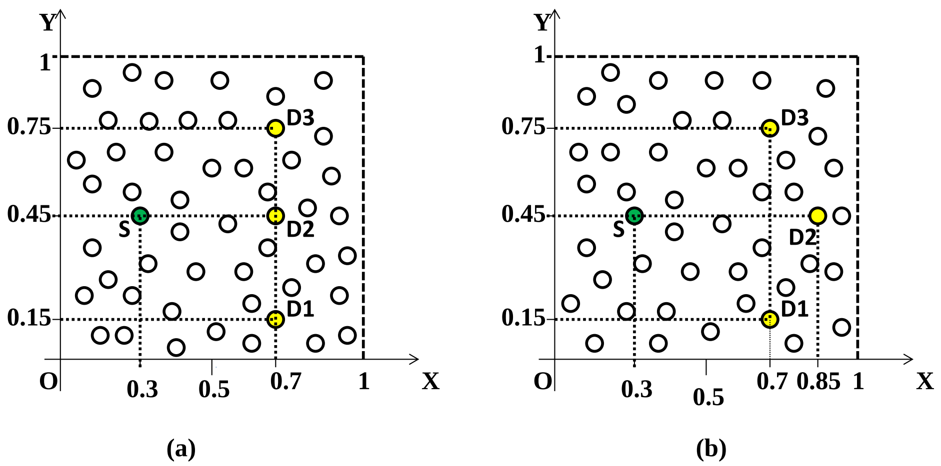


Figure 2.15: Multicast model with 3 different destinations (a) line model, (b) triangle model

(see Fig. 2.16). Therefore, for a fixed network model, by changing the number of nodes from 200 to 300, the gain of M-GeoCode in terms of active node ratio increase up to 11%.

For a fixed value of $N$ and $T_r$, and a determined throughput, it is seen that M-GeoCode works better for a line model compared to the other model. This is reasonable, because in line model the active areas (ellipses) of $S-D$ pairs have more intersections with each other than the triangle model and so, the M-GeoCode uses less number of nodes to create the paths. Therefore, the gain of the M-GeoCode may vary depending on the geographic positions of the source and the destinations (Fig. 2.17).

Fig. 2.18 shows that, for a fixed $N$, $T_r$, and destination model, by increasing the throughput, $R_{active-nodes}$ decreases. Therefore, the gain of the M-GeoCode increases by increasing the throughput value. For example, for the line model, $N = 300$, and $T_r = 0.15$, by increasing the throughput from 3 to 6, $R_{active-nodes}$ decreases from 0.7902 to 0.7. It means that to achieve $\lambda = 3$, M-GeoCode uses 79% of the nodes that GMR-based protocol uses, while in the case of $\lambda = 6$ M-GeoCode uses only 70% of the nodes that GMR-based protocol uses.

As it was shown in the case of unicast, there is an optimum value of $T_r$ for each specified scenario that can optimize the gain of M-GeoCode protocol. It can be seen from Fig. 2.16 that for $N = 300$, $T_r = 0.3$ maximizes the gain of M-GeoCode while in case of $N = 200$, the optimum value of $T_r$ is 0.25.

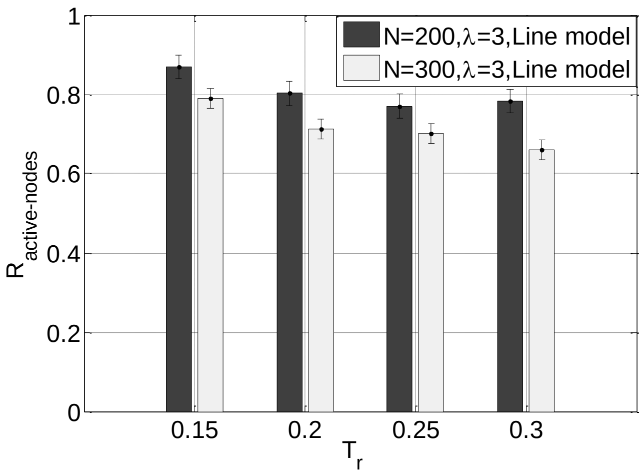


Figure 2.16: Active node ratio comparison for different values of $N$ and $Tr$

*Active area reduction gain of M-GeoCode:* We compare the distribution area of the active nodes for both GMR and M-GeoCode protocols, using $R_{active-area}$ metric. Similar to the previous tests, we repeated the simulations for different values

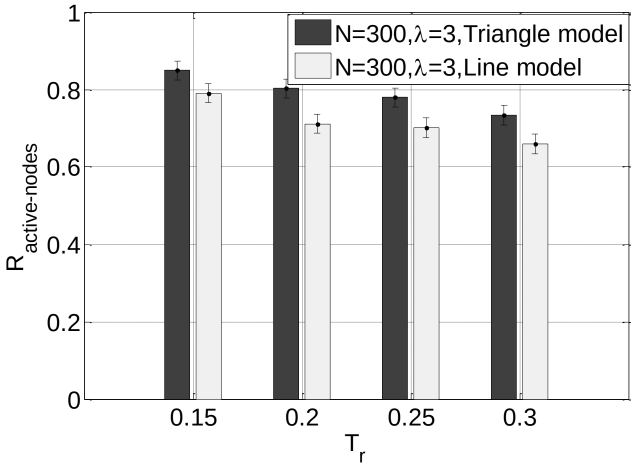


Figure 2.17: Active node ratio comparison for different destination deployment models

of $T_r$, $N$, deployment models and $\lambda$. The results of the simulations are shown in Fig. 2.19, 2.20, 2.21. In all cases, we see that there is an incredible gain of using M-GeoCode compared with GMR-based protocol in terms of the transmission area of the active nodes. This means that, M-GeoCode uses a smaller area than the GMR-based protocol to achieve the same throughput. Fig. 2.19 shows that for a specific node deployment model and specific value of $T_r$, if the density of the nodes increases, the gain of the M-GeoCode in terms of transmission area ratio increases too. For instance, if $N = 200$ the minimum value of $R_{active-area}$ (maximum gain of M-GeoCode) is equal to 0.48 while in case of $N = 300$, it is equal to 0.28. This states that for $N = 200$, M-GeoCode uses 48% of the transmission area that GMR-based protocol uses while for the same scenario and $N = 300$, M-GeoCode uses only 28.1% of the transmission area of GMR-based protocol.
Fig. 2.20 shows the comparison result for $N = 300$, $T_r = 0.15$, $\lambda = 3$ and two different models of destination deployments. We see that for a constant value of $T_r$ and $N$, the gain of M-GeoCode in terms of the reduction of the transmission area for line model is more than that for triangle model. For example, for $T_r = 0.25$, $N = 300$, and line model, M-GeoCode requires only 32% of the transmission area that GMR-based protocol needs, while for triangle model, M-GeoCode uses 39% of the transmission area of GMR-based protocol.
We also evaluated the effect of throughput on $R_{active-area}$ in Fig. 2.21. Our simulation results state that for fixed values of $N$, $T_r$ and destination model, the gain of M-GeoCode increases by increasing the amount of desired throughput. This is reasonable, because if the number of node-disjoint paths created by the GMR-based protocol increases, the transmission area of the active nodes increases too to avoid the intersection between the paths, while in case of M-GeoCode the size of the el-

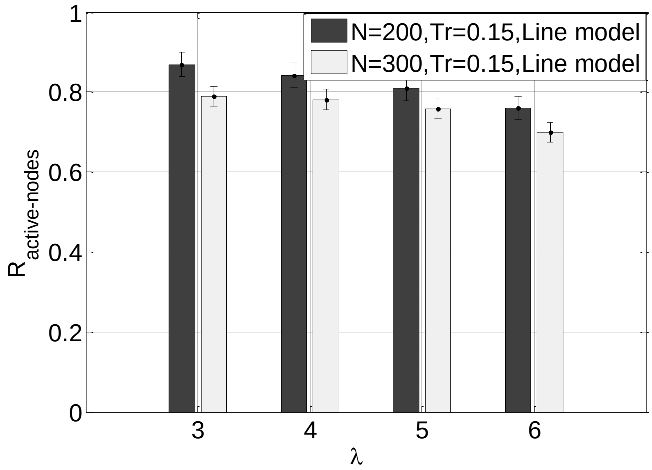


Figure 2.18: Active node ratio comparison for different values of $\lambda$ and $N$

lipses does not change too much. For instance, for $N = 300$, $T_r = 0.15$, and line model of destinations, $R_{active-area}$ is changing from 0.38 to 0.28, by increasing the amount of throughput from $\lambda = 3$ to $\lambda = 6$. Meaning that the percentage of the transmission area that is used by M-GeoCode decreases by 10% compared to the GMR-based protocol.

## 2.8 Concluding Remarks

In this chapter, we proposed a multicast geographic coding-aware communication protocol, M-GeoCode. The core idea of M-GeoCode is based on the reduction of the transmission area and the number of active nodes using NC, geographic information, and directed diffusion mechanisms. Through the analysis of a wide range of settings in our simulations, we show that M-GeoCode is able to achieve the capacity of the network graph and to require a significantly smaller active area with respect to the traditional node-disjoint multipath algorithms (without NC) while achieving the same throughput. Our results state that M-GeoCode is able to achieve the same throughput as traditional node-disjoint multipath algorithms while reduces the number of active nodes by a factor of 1.55. In terms of active nodes dispersion, our simulation results state that M-GeoCode is able to reduce the active area by up to a factor of 3.57 compared with the traditional multicast protocols. This means that M-GeoCode has the potential of reducing the total energy consumption and the interference. Although we have relied on an ellipse to determine the active area of M-GeoCode, any other geometric shape could be considered. M-GeoCode is promising in different applications of wireless networks such as vehicular networks, where geographic

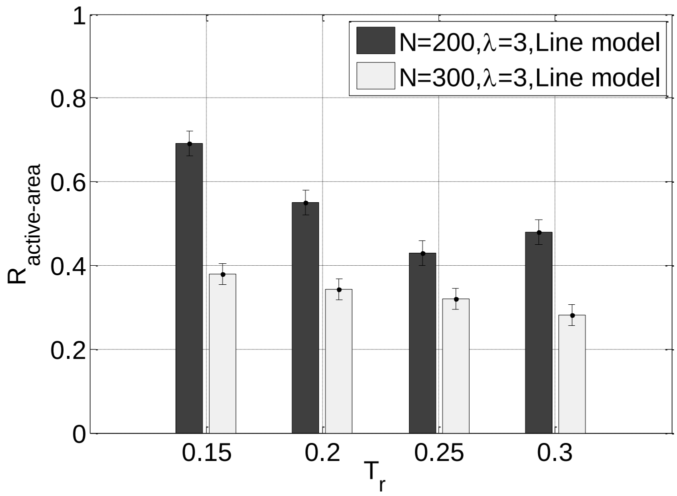


Figure 2.19: Active area ratio comparison for different values of $N$ and $Tr$

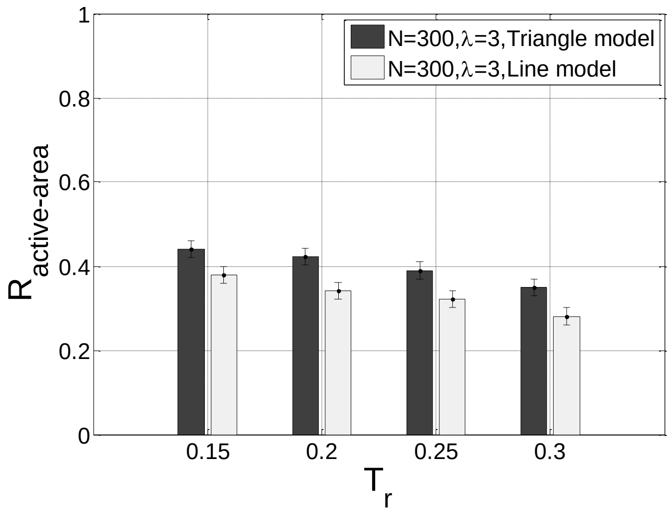


Figure 2.20: Active area ratio comparison for different destination deployment models

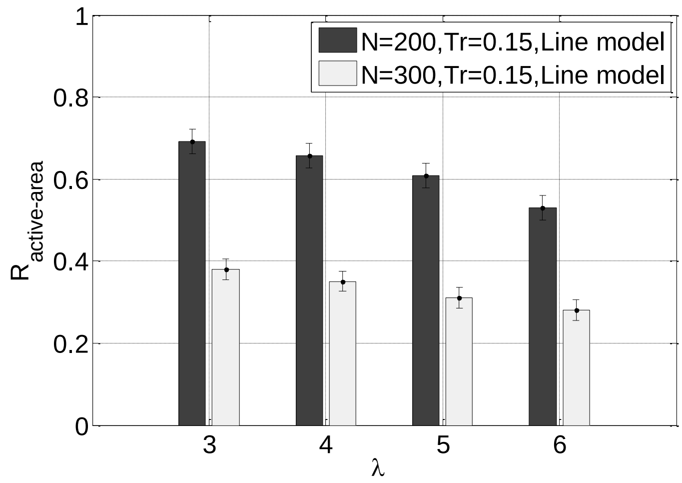


Figure 2.21: Active area ratio comparison for different values of $\lambda$ and $N$

information is available, specially in applications where the relevance of the data is linked to a geographical location of the nodes, and not the nodes per se.

# Chapter 3

# Optimal Network Coded Cooperation over Time-Invariant Erasure Channels

In Chapter 2, we looked at the problem of transmitting packets in a mesh network with one source and multiple destinations from a general perspective. In fact, we tried to design a protocol that minimizes the total cost of packet transmission by minimizing the number of active nodes, considering lossless channels. The dynamics and optimal policies for the transmission of data packets over the created paths was not considered in Chapter 2. The latter is the main focus of the current Chapter. More specifically, we characterize the performance of data transmission over real-world conditions, e.g., considering losses. We start by theoretical analysis for a simple network with one source and two receivers, where optimal packet transmission policies are obtained using an MDP model. Later, we extend our analysis for a general network with $N$ receivers. Four powerful packet transmission heuristics are also proposed, and tested in a real-world test-bed. The remainder of this chapter is organized as follows. Section 3.1 presents the related work. Our motivation and main contributions are stated in Section 3.2. In Section 3.3, first we describe the problem, and then MDP model of the problem is presented. In Section 3.4, we define three scenarios that are used to compare the performance of the optimal NC-based solution with no-NC based scenarios. Section 3.5 presents the results of comparison between the completion time of transmitting $M$ packets from a source to two receivers, using different NC-based and no-NC based schemes. In Section 3.6, we analyse the effect of different parameters of the network on the MDP solution and extract optimal packet transmission rules that help us to propose useful heuristics. In Section 3.7, two sets of packet transmission heuristics for a simple network of one source and two receivers are presented. We explain how to generalize the proposed heuristics for a network of one source and $N$ receivers in Section 3.8. Numerical results which compare the performance of the proposed heuristics, RLNC broadcasting, and the optimal MDP solution are presented in Section 3.9. The results of real-world implementation of the proposed heuristics are presented in Section 3.10. Finally, this chapter is concluded in Section 3.11.

## 3.1 Related Work

With the growing concern on minimizing the cost of packet transmission in wireless networks with multiple users, e.g., multi-media multicast services, finding optimal/near-optimal packet transmission policies that are efficient in cost while maintaining reliability has become critical. To this end, cooperative communication protocols that use network coding (NC) [7] to improve reliability, efficiency, and security of the network have been proposed and extensively studied in the literature [17–23], [27–29]. We will refer to the network coded cooperative communication as NC-CC. Most of the previous work in this area could be fit into one or both of the following categories: (a) performance analysis of NC-CC for a predefined scenario [17], [20–23], [27–29], or (b) protocol design and optimization of a NC-CC scenario in terms of a specific metric [18], [19], [24], [26], [30]. For example, [17], [21–23] evaluate the performance of different relay-based NC-CC in terms of bandwidth, outage probability, and achievable rate and [20] evaluates the performance of a non-relay based NC-CC scenario that uses NC only in short-range links in terms of number of transmitted packets. Authors in [27–29] provide some bounds on the bit error rate (BER), and outage probability of upstream NC-CC scenarios where multiple nodes working together to deliver their packets to a common destination. On the other hand, [18] looked at NC-CC from an optimization perspective and provided a theoretical formulation to calculate the maximal throughput of unicast traffic that can be achieved with cooperative network coding in multi-rate wireless networks. Authors in [18], also proposed a routing protocol based on the optimization result.
A cluster-based cooperative coding protocol was proposed in [19] which also optimizes the number of relay nodes per cluster to trade-off between performance and overhead. The optimization of session grouping, relay selection, and diversity order of the system for different NC-CC scenarios have been studied in [24], [26] and near-optimal solutions to these problems have been proposed. [25], [30], [63] focus on optimal design of NC-CC for upstream scenarios and try to develop adaptive strategies, or design optimal codes. Authors in [31] take a step forward and take a look at implementation requirements of some of the relay-based cooperative strategies. Previous work on downstream scenarios [17], [20], [21–23], where one/multiple sources transmit data to multiple users, focused mostly on relay-based NC-CC, while the performance and optimal design of non-relay NC-CC is not well understood.

## 3.2 Motivation and Main Contributions

Despite of the extensive efforts to evaluate the performance of NC in cooperative scenarios, a mathematical analysis on the design of optimal NC-CC policies in a non-relay based network that may be an infrastructure for the multi-hop routing protocols is missing in the literature. To fill this gap, we start by modelling the problem of minimizing the cost of transmitting $M$ packets from a source to two receivers over half-duplex erasure channels as a Markov decision process (MDP) and solve it for any field size, arbitrary erasure probability of channels, and $M$. We use this evaluation to propose two simple heuristics for the two receiver case that could be applied to the proposed M-GeoCode protocol and other NC-based protocols in order to minimize the total cost of packet transmission. We then present a generalized version of the heuristics for a network with one source and $N \geq 2$ receivers that are divided into $N/2$ clusters of two receivers that could be used in multi-user applications, such as video streaming. We focus on clusters of two receivers for two reasons. First, it is very likely to find a pair of nodes or users that want to cooperate to receive common data from a common source. Our results show that even this level of co-operation can provide a significant gain in completion time and reliability. Second, there are already efforts in wireless cellular technologies, such as device-to-device (D2D) [64] communications as a technology component for LTE-A, which would facilitate the implementation of our proposed heuristics in the near future. Our main contributions in this chapter are as follows.

- **Mathematical analysis:** we propose an MDP model to minimize the cost of transmitting $M$ packets from a source to two receivers over half-duplex erasure channels.
- **Proposal of near-optimal heuristics:** inspired by the optimal solution obtained by the MDP, we propose four powerful heuristics that could be used in a network of one source and $N$ receivers. We also provide a closed form expression for the completion time of the proposed heuristics that is shown to match our real-world implementation results.
- **Numerical results:** we evaluate the performance of the MDP solution and the proposed heuristics for different field size, packet erasure probabilities, number of receivers, and number of packets as well as the effect of spatial separation (or lack of it) between cooperation clusters. Our analytical results show that the proposed heuristics have better performance compared to RLNC broadcast in terms of completion time and reliable delivered packets.
- **Real-world implementation:** we implement our proposed heuristics in a wireless network coding test-bed implemented on Raspberry Pi's at Aalborg University. Our measurements show that the proposed heuristics are able to reduce the mean of completion time by up to $4.75x$ compared with RLNC broadcasting.

## 3.3 System Model

In this section, first, we provide a formal statement of the problem that we are going to solve, then, we explain how we mathematically model this problem.

### 3.3.1 Problem Statement

We consider a network consisting of one source, $S$, and $N$ receivers $R_1,\ldots,R_N$ as shown in Fig. 3.1. The source's goal is to transmit $M$ packets to the $N$ receivers. We assume that the receivers can also share data packets between them using unicast transmissions. We consider independent erasure channels for each receiver with $\varepsilon_i$ being the erasure probability of channel between $S$ and $R_i$, and $\varepsilon_{R_iR_j}$ being the erasure probability for transmissions from $R_i$ to $R_j$. The system is time-slotted and only one transmission is performed per time slot. When chosen to transmit, the source or one of the receivers generate random linear network coding (RLNC) packets [9]. The source generates a coded packet from the $M$ packets in its buffer at the time of transmission. The source uses packets $p_1, p_2, \ldots, p_M$ to create a linear combination with some coding coefficients $\alpha_1,\ldots,\alpha_M$, i.e., $\sum_{i=1}^{M}\alpha_i p_i$, randomly selected from the $q$ elements of a Galois field, i.e., $GF(q)$. For no-NC cases, we will explain how a packet is transmitted in the following sections.

Our goal is to find NC-based packet transmission policy that can minimize the total cost of finishing the transmission of $M$ packets from $S$ to $R_1,R_2,\ldots,R_N$. We divide $N$ receivers into $N/2$ clusters such that cluster $C_i$ includes two receivers $R_{2i-1},R_{2i}$ as shown in Fig. 3.1-(b). For now, we only focus on one of the clusters and therefore, our problem could be seen as optimization of the cost of packet transmission from $S$ to two receivers $R_1,R_2$ of cluster $C_1$ as shown in Fig. 3.2. Later, we will show how to generalize our optimal packet transmission policy for $N>2$ receivers.

For the network with one source and two receivers, there are five possible ways of packet transmission: (i) broadcast from $S$ to $R_1,R_2$, (ii) unicast from $S$ to $R_1$, (iii) unicast from $S$ to $R_2$, (iv) unicast from $R_1$ to $R_2$, (v) unicast from $R_2$ to $R_1$. Since we are transmitting coded packets, each receiver needs to collect $M$ degree of freedom (dof) to be able to decode the original packets, where a dof is defined as following.

**Definition 4** (Degree of freedom (dof)). A node has $k$ degrees of freedom if the dimension of its knowledge space is $k$.

**Definition 5** (Knowledge space). The knowledge space of a receiver $R_i$ at a given time $t$ is defined as the linear span of the linear combinations of the packets $P_1,P_2,\ldots,P_M$ received by $R_i$ until time $t$.

Each transmission from a sender (either source or one of the receivers) adds a dof to the set of packets received by a receiver if and only if the channel does not erase it

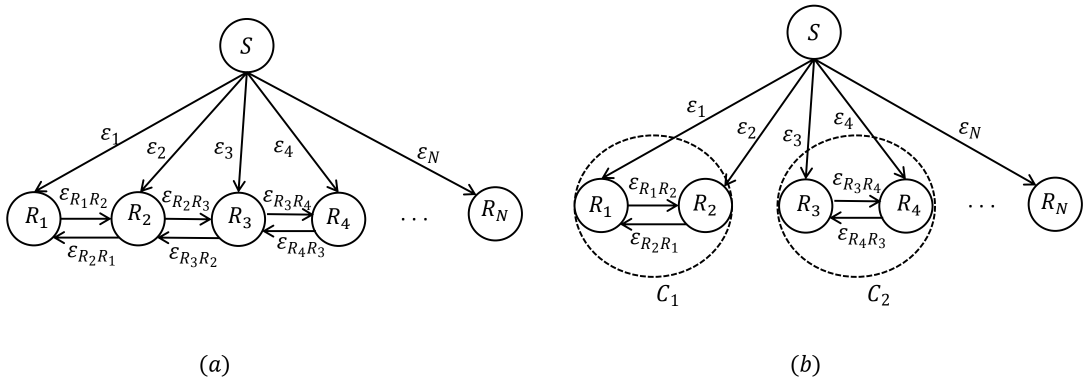


Figure 3.1: (a) Network model for $N$ receivers, (b) clustering of the receivers, $C_i$ shows $i$-th cluster.

and the sent linear combination is linearly independent of all previously received linear combinations. The cost of each transmission is defined as a function of different parameters depending on the application, e.g., if the application is not delay tolerant, then the completion time of packet transmission should be the cost we aim to minimize. In order to differentiate between the cost of packet transmission for unicast and broadcast actions, one unit of cost is assigned to each unicast transmission and the cost of each broadcast transmission is defined as $\beta \geq 1$ units.

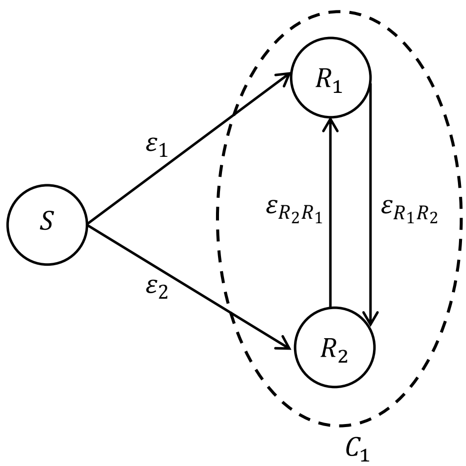


Figure 3.2: Closer look at one cluster as the network model for the MDP analysis

### 3.3.2 MDP Model of the Problem

We can formulate the problem of cost minimization as an MDP problem. An MDP is a discrete time stochastic control process. At each time step, the process is in some state $s$,

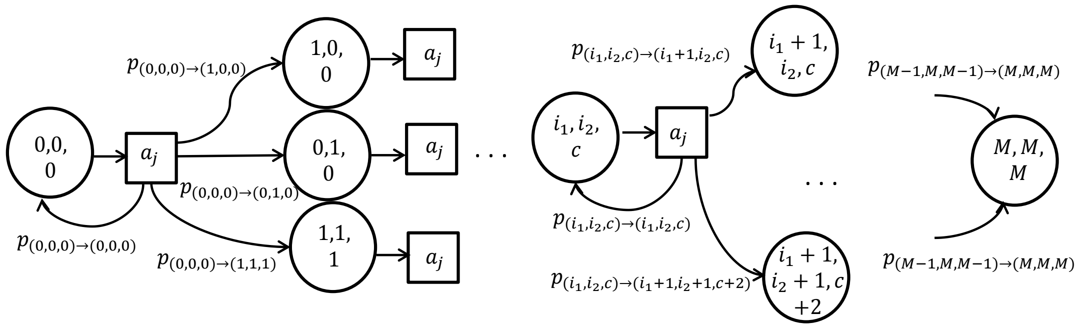


Figure 3.3: Schematic of the MDP model; $a_j$ represents the selected action

and the decision maker may choose any action $a$ that is available in state $s$. The process responds at the next time step by randomly moving into a new state $s^{'}$, and giving the decision maker a corresponding reward $R(s,a,s^{'})$ or adding a corresponding cost $C(s,a,s^{'})$. The probability that the process moves into its new state $s^{'}$ is influenced by the chosen action. Specifically, it is given by the state transition probability $P_{s\to s^{'}}$. Thus, the next state $s^{'}$ depends on the current state $s$ and the decision maker's action $a$. But given $s$ and $a$, it is conditionally independent of all previous states and actions; in other words, the state transitions of an MDP possess the Markov property. Markov property refers to the memoryless property of a stochastic process. Looking at the cost minimization problem for the network defined in Fig. 3.2, we conclude that it could be very well described by an MDP.

To solve the optimization problem and determine the optimal policy for minimizing the cost, we assume a Genie system (GS), in the sense that each node in the network has perfect knowledge of the system state. We drop this assumption for our practical schemes. At each time step, the process is in a state $s$, and the system may choose an action $a_j$ that is possible in state $s$. The action chosen will determine the probability to transition into a new state. In the following, we specify the state, possible actions, transition probabilities, and the cost function of our MDP model for the network defined in Fig. 3.2. A schematic of the proposed MDP model is shown in Fig. 3.3, where each circle represents a state of network, $a_j$ represents the selected action by the MDP, and the arrows represent the transitions between different states. $P_{x\to y}$ represents the transition probability for a transition from state $x$ to state $y$.
*State Definition:* Each state $s$ is defined as $s=(i_1,i_2,c)$, where $i_k$ is the number of dof at receiver $R_k$, and $c$ represents the dimension of the common knowledge between $R_1$ and $R_2$. As a simple example, assume that we are aiming at transmitting 3 packets $P_1,P_2,P_3$ from $S$ to $R_1,R_2$ and the set of packets received by $R_1$ and $R_2$ until now are respectively, $\{P_1+P_2,P_3\}$ and $\{P_1+P_2+P_3\}$. The state of the network in this case is shown as $(2,1,1)$,

because the dofs of the received packets by $R_1$ and $R_2$ are respectively, 2 and 1 and the common knowledge between receivers is $P_1+P_2+P_3$ which has a dimension of 1. It is clear that $c$ cannot be greater than the minimum of $i_1, i_2$ and therefore, only the states that satisfy $c \leq min(i_1, i_2)$ are meaningful. The maximum dof at each receiver is reached when $M$ linear independent packets are received. When all nodes receive $M$ dof, they all share the same knowledge. Thus, there exists an absorbing state defined as $s_{abs} = (M,M,M)$. A state of a Markov chain is called absorbing if it has a transition probability $p_{s_{abs}\to s_{abs}} = 1$, which implies that the process never changes state once it reaches state $s_{abs}$.

*Possible Actions:* We define actions $a_1, a_2, ..., a_5$, that cover all possible ways of transmitting a packet in the network of Fig. 3.2. Action $a_1$ is defined as broadcast from $S$ to $R_1, R_2$. Actions $a_2, a_3$ define the unicast transmissions from $S$ to $R_1$ and $R_2$, respectively. Actions $a_4, a_5$ define cooperation between receivers. More precisely, $a_4$ defines unicast transmission from $R_1$ to $R_2$ and $a_5$ defines unicast transmission from $R_2$ to $R_1$. Action $a_6$ is defined as "do not transmit", to allow the system to stop transmissions after $R_1$ and $R_2$ have both received the $M$ packets.

*Transition Probabilities:* We explain the transition probabilities for our MDP model assuming that $q$ has an arbitrary value. The possible states to which state $(i_1, i_2, c)$ can transit to with non-zero probability depends on the action chosen and the total knowledge ($\mathcal{K}$) that is available to both receivers at the time of observation. $\mathcal{K}$ indicates the dof of the union of the received packets by the two receivers and is calculated as $\mathcal{K} = i_1 + i_2 - c$. In the following, we refer to $I_{(x\in X)}$ as an indicator function that is equal to one when $x \in X$ and zero otherwise. For simplicity, $I_{(i'_1=i_1+k_1, i'_2=i_2+k_2, c'=c+k_3)}$ is denoted by $I_{(k_1,k_2,k_3)}$ and $\bar{\varepsilon}_i = 1-\varepsilon_i$. We also define $F_a(X) = (1-q^{-M+X}), F_b(X,Y,Z) = \frac{(1-q^{-M+X})(q^{-M+Y}-q^{-M+Z})}{(1-q^{-M+Z})}$, and $F_c(X,Y,Z) = \frac{(1-q^{-M+X})(1-q^{-M+Y})}{(1-q^{-M+Z})}$ to do more simplification. The non-zero transition probabilities for the six actions are summarized as follows.

**Action $a_1$ (Broadcast):** When the source broadcasts, it creates different possible state transitions depending on the erasure probability of the channels and also the packets that have been previously received by each receiver. A detailed explanation of how we calculate these probabilities is found in Theorem 1 of [65]. Here we explain key cases.

**Definition 6** (Innovative packet)**.** We say a packet is innovative to a receiver if it is linearly independent from the packets previously received by that receiver.

On the one hand, if source sends a packet that is innovative to both $R_1, R_2$ and is received without erasure at $R_1, R_2$, we may have different transitions depending on the packets previously received by the two receivers. If the packet sent by $S$ is also innovative to the union of the received packets by the two receivers, then $i_1, i_2, c$ are all increased by one since both $R_1, R_2$ have received the same packet that is innovative to both of them. If the packet sent by $S$ is not innovative to the union of the received packets by the two receivers, then $c$ is increased by two while $i_1, i_2$ are increased by one. Let us illustrate this with an example. Assuming that $M=3$ and the sets of packets received by $R_1,R_2$ at the time of observation are $\{P_1,P_3\}$ and $\{P_2+P_3\}$, respectively. The network state is then $s(2,1,0)$ and the union of the received packets by the two receivers is shown as $\{P_1,P_3,P_2+P_3\}$. Now assume that source broadcasts a new coded packet $P_1+P_2+P_3$, which adds one dof to both $R_1$ and $R_2$ but not to the union of the packets received by both of them. However, the dimension of the common knowledge is increased by two and the system then transits to a new state $s'=(3,2,2)$.

On the other hand, if source sends a packet that is innovative to $R_1$ and non-innovative to $R_2$, and is received without erasure at $R_1$, then both $i_1$ and $c$ are increased by one while $i_2$ is not changed. This is because the packet received by $R_1$ already exits in $R_2$ and therefore, it increases the dimension of the common knowledge by one. The probability that the packet sent by source is innovative to receiver $R_k$ is calculated by $1-q^{-M+i_k}$, where $i_k$ is the dof of the packets that previously have been received by $R_k$. The following equations show the transition probabilities for action $a_1$.

$$\begin{aligned}
&P_{(i_1,i_2,c)\to(i_1',i_2',c')} = \Big[\varepsilon_1\varepsilon_2+\varepsilon_1\bar{\varepsilon}_2\, q^{-M+i_2}+\bar{\varepsilon}_1\varepsilon_2\, q^{-M+i_1}+\bar{\varepsilon}_1\bar{\varepsilon}_2\times \\
&\big(q^{-M+i_2}-F_b(i_1,i_2,c)\big)\Big]\times I_{(0,0,0)}+\varepsilon_1\bar{\varepsilon}_2F_a(\mathcal{K})\times I_{(0,1,0)}+ \\
&\bar{\varepsilon}_1\varepsilon_2F_a(\mathcal{K})\times I_{(1,0,0)}+\bar{\varepsilon}_1\bar{\varepsilon}_2F_a(\mathcal{K})\times I_{(1,1,1)}+ \\
&\Big[\bar{\varepsilon}_2\, F_b(i_2,i_1,c)+\varepsilon_1\bar{\varepsilon}_2\, \big(F_c(i_1,i_2,c)-F_a(\mathcal{K})\big)\Big]\times I_{(0,1,1)}+ \\
&\Big[\bar{\varepsilon}_1F_b(i_1,i_2,c)+\varepsilon_2\bar{\varepsilon}_1\, \big(F_c(i_1,i_2,c)-F_a(\mathcal{K})\big)\Big]\times I_{(1,0,1)}+ \\
&\bar{\varepsilon}_1\bar{\varepsilon}_2\, \big[F_c(i_1,i_2,c)-F_a(\mathcal{K})\big]\times I_{(1,1,2)}.
\end{aligned} \tag{3.1}$$

**Action $a_2$ (unicast from $S$ to $R_1$):** This is a special case of broadcasting, $a_1$ when $\varepsilon_2=1$. This means that packets going through that link are erased and cannot be received by $R_2$. Therefore,

$$\begin{aligned}
&P_{(i_1,i_2,c)\to(i_1',i_2',c')} = [\varepsilon_1+\bar{\varepsilon}_1\, q^{-M+i_1}]\times I_{(0,0,0)}+\bar{\varepsilon}_1F_a(\mathcal{K})\times \\
&I_{(1,0,0)}+\bar{\varepsilon}_1\big[F_a(i_1)-F_a(\mathcal{K})\big]\times I_{(1,0,1)}.
\end{aligned} \tag{3.2}$$

**Action $a_3$ (unicast from $S$ to $R_2$):** This is a special case of broadcasting, $a_1$ when $\varepsilon_1=1$. Therefore,

$$\begin{aligned}
&P_{(i_1,i_2,c)\to(i_1',i_2',c')} = [\varepsilon_2+\bar{\varepsilon}_2\, q^{-M+i_2}]\times I_{(0,0,0)}+\bar{\varepsilon}_2\, F_a(\mathcal{K})\times \\
&I_{(0,1,0)}+\bar{\varepsilon}_2\, \big[F_a(i_2)-F_a(\mathcal{K})\big]\times I_{(0,1,1)}.
\end{aligned} \tag{3.3}$$

**Action $a_4$ (unicast from $R_1$ to $R_2$):** If $i_1=c$, the probability of sending an innovative packet from $R_1$ to $R_2$ is zero, because the coded packets received by $R_2$ span the dimensions in $R_1$. Thus, any linear combination of these coded packets will be non-innovative to the set of packets received by $R_2$. If $i_1 > c$ and $i_2 < M$, the probability that the packet sent by $R_1$ is innovative to $R_2$ is $1 - q^{-i_1+c}$. Thus,

- If $i_1 > c$ and $i_2 < M$, then $P_{(i_1,i_2,c)\to(i_1',i_2',c')} = \left[\varepsilon_{R_1R_2} + \varepsilon_{\bar{R_1}R_2}\frac{q^c}{q^{i_1}}\right] \times I_{(0,0,0)} + \varepsilon_{\bar{R_1}R_2}(1 - \frac{q^c}{q^{i_1}}) \times I_{(0,1,1)}$.
- If $i_2 = M$ or $i_1 = c$, then $P_{(i_1,i_2,c)\to(i_1',i_2',c')} = I_{(0,0,0)}$.

**Action $a_5$ (unicast from $R_2$ to $R_1$):** This case is symmetric to the case of action $a_4$. Therefore, the transition probabilities are summarized as:

- If $i_2 > c$ and $i_1 < M$, then $P_{(i_1,i_2,c)\to(i_1',i_2',c')} = \left[\varepsilon_{R_2R_1} + \varepsilon_{\bar{R_2}R_1}\frac{q^c}{q^{i_2}}\right] \times I_{(0,0,0)} + \varepsilon_{\bar{R_2}R_1}(1 - \frac{q^c}{q^{i_2}}) \times I_{(1,0,1)}$.
- If $i_1 = M$ or $i_2 = c$, then $P_{(i_1,i_2,c)\to(i_1',i_2',c')} = I_{(0,0,0)}$.

**Action $a_6$ (do not transmit):** For all values of $i_1, i_2, c$, $P_{(i_1,i_2,c)\to(i_1',i_2',c')} = I_{(0,0,0)}$.

*Cost Function:* We define a general, parametric cost function that could be applied to various wireless network technologies considering that each unicast transmission costs one unit and broadcast transmissions cost $\beta$ units. This allows the formulation to capture different metrics, such as, completion time, energy consumption, a combination of time and energy. For example, if we think of energy as the cost metric, the cost of each broadcast transmission could be calculated as $E_{tx} + 2E_{rx}$, and the cost of each unicast transmission is $E_{tx} + E_{rx}$, where $E_{tx}$ is the energy consumed by transmitter, and $E_{rx}$ is the energy consumed by each receiver. By defining the cost of unicast as one unit of cost, the cost of broadcast is $\beta = \frac{E_{tx}+2E_{rx}}{E_{tx}+E_{rx}}$, which is greater than one. This leads to

$$C(s,a_j,s') = \begin{cases} 1, & \forall s \in S_T \mid s \neq (M,M,M),\ j = 2,...,5 \\ \beta, & \forall s \in S_T \mid s \neq (M,M,M),\ j = 1 \\ \mathcal{D}, & \text{if } s = (M,M,M),\ j = 1,...,5 \\ \mathcal{D}, & \forall s \in S_T \mid s \neq (M,M,M),\ j = 6 \\ 0, & \text{if } s = (M,M,M),\ j = 6, \end{cases} \tag{3.4}$$

where $C(s,a_j,s')$ is the cost of transition from state $s$ to state $s'$ by choosing action $a_j$ and $S_T$ is the set of all meaningful states. $\mathcal{D}$ is an arbitrary large number that is much greater than $\beta$. By defining $\mathcal{D} >> \beta$, we make sure that the MDP does not choose any one of the actions $a_1, a_2, ..., a_5$ if the system is in the absorbing state $s_{abs}$ and it chooses action $a_6$ that has the minimum cost. Thus, no additional cost is incurred once the absorbing state is reached.

*Optimization Algorithm:* We use the value iteration algorithm (Bellman equations) [66] to solve the optimization. A value function is defined as $V_t : S_T \to R^+$ for iteration $t$ that associates to each state $s \in S_T$ a lower bound on the minimal total cost to be paid starting from that state. We summarize the steps to find an optimal policy as

$$V_{t+1}(s) \leftarrow \min_{a_j} E(C(s,a_j,s') + \zeta V_t(s')), \tag{3.5}$$

where $V_0(s) = 0$, $\forall s \in S_T$. $E(X)$ is the expected value of random variable $X$, and $C(s,a_j,s')$ is the cost of transition from state $s$ to state $s'$ using action $a_j$. This algorithm iterates until $\max_s |V_{t+1}(s) - V_t(s)| < \delta$ is satisfied. $\zeta \in (0,1]$ is called discount factor and is used to make sure that the equation converges when $t$ goes to infinity. $\delta$ shows the degree of optimality of the result and has a very small positive value. We set $\delta = 0.0001$ for our analysis. The algorithm chooses an action that minimizes the cost of transition into a new state, $C(s,a_j,s')$ and the total cost that we pay starting from the new state, $V_k(s^{'})$.

## 3.4 Comparison Schemes

In order to compare the performance of no-NC and NC in a real system (non-GS) with the calculated minimum completion time for a GS, we define three scenarios: (i) NC with full feedback, where NC is allowed to use, but a GS is not available. In this scheme, the system state is known in the cost of sending some feedback messages, (ii) No-NC and no-feedback, where NC is not allowed and there is not any feedback on the received packets by the receivers, and (iii) No-NC with full feedback, where NC is not allowed, but full feedback is provided by the receivers.

To make a fair comparison between the optimal solution that we found for a GS, and these scenarios in terms of the total packet transmission cost, we calculate the optimal performance of each scenario by using an MDP model.

### 3.4.1 NC with Full-Feedback (NC, Full-Feedback)

In this scenario, we assume that a GS is not available, but for each packet transmission by a sender, a feedback message is sent out by the receiver. If the action is broadcast, we assume that both $R_1, R_2$ are sending out a distinct feedback message, which contains the ID of the packet they received (or not). Therefore, the cost of broadcast is always more than the cost of unicast because in case of broadcast two feedback messages are sent. Except the cost function, all other components of the MDP model for this scenario is the same as the MDP model we defined in Section 3.3. We add the cost of sending full feedback (feedback per transmission) as a percentage of time slot to the cost of packet transmission and modify the general cost function as

$$C(s,a_j,s') = \begin{cases} 1+\alpha, & \forall s \in S_T \mid s \neq (M,M,M),\ j=2,\dots,5 \\ \beta+2\alpha, & \forall s \in S_T \mid s \neq (M,M,M),\ j=1 \\ \mathcal{D}, & \text{if } s=(M,M,M),\ j=1,\dots,5 \\ \mathcal{D}, & \forall s \in S_T \mid s \neq (M,M,M),\ j=6 \\ 0, & \text{if } s=(M,M,M),\ j=6, \end{cases} \tag{3.6}$$

where $\alpha$ represents the feedback cost. It is assumed that the feedback is overheard by all nodes in the network and therefore, the state information of the network and the packets that were received by $R_1,R_2$ are known to all nodes at each time slot.

### 3.4.2 No-NC and No-Feedback (No-NC, No-Feedback)

In this scenario, NC is not used and it is assumed that when a node transmits, it chooses a packet from its buffer at random to send it to the other nodes. The possible actions and the cost function are similar to the NC scenario defined in Section 3.3, but the state definition and the transition probabilities are changed to account for this random packet selection. In the following, we explain how to define these two components of the MDP model.
*State Definition:* Each state is defined by three elements $s(i_1,i_2,c)$, where $i_1$ and $i_2$ are the number of distinct packets received by receivers $R_1$ and $R_2$, respectively and $c$ is the number of packets that were received by both $R_1,R_2$.
*Transition Probabilities:* We may send a packet that was already received by both receivers and therefore the state of the network does not change even if the packet was received successfully. Similar to the NC-based scenario, we define the total knowledge of the two receivers as $\mathcal{K} = i_1 + i_2 - c$. Clearly, the transition probabilities also depend on the number of packets received by each receiver, their total knowledge and the probability that the sent packet is erased by the channels or not. For example, if $\mathcal{K} < M$, the probability of sending a packet that is new to both $R_1,R_2$ is equal to the probability that the selected packet was not received by any one of the receivers before, i.e., $\frac{M-\mathcal{K}}{M}$. On the other hand, the probability that the packet that is sent is new to $R_1$ but not to $R_2$, is equal to the probability that the selected packet is not in $c$ but it is in $i_2$, therefore it is calculated as $\frac{i_2-c}{M}$. We can summarize all possible non-zero transition probabilities as follows.
**Action $a_1$ (broadcast):**

- If $\mathcal{K} < M$

$$P_{(i_1,i_2,c)\to(i_1',i_2',c')} =$$

$$\begin{cases} \varepsilon_1\varepsilon_2+\varepsilon_1\bar{\varepsilon}_2\frac{i_2}{M}+\varepsilon_2\bar{\varepsilon}_1\frac{i_1}{M}+ & \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\frac{c}{M}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2,\ c^{'}=c \\ \bar{\varepsilon}_1\varepsilon_2\frac{M-\mathcal{K}}{M}, & \text{if } i_1^{'}=i_1+1,\ i_2^{'}=i_2,\ c^{'}=c \\ \bar{\varepsilon}_2\varepsilon_1\frac{M-\mathcal{K}}{M}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2+1,\ c^{'}=c \\ \bar{\varepsilon}_1\frac{i_2-c}{M}, & \text{if } i_1^{'}=i_1+1,\ i_2^{'}=i_2,\ c^{'}=c+1 \\ \bar{\varepsilon}_2\frac{i_1-c}{M}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2+1,\ c^{'}=c+1 \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\frac{M-\mathcal{K}}{M}, & \text{if } i_1^{'}=i_1+1,\ i_2^{'}=i_2+1,\ c^{'}=c+1. \end{cases} \tag{3.7}$$

- If $\mathcal{K}=M$

$$P_{(i_1,i_2,c)\to(i_1^{'},i_2^{'},c^{'})}=$$

$$\begin{cases} \varepsilon_1\varepsilon_2+\varepsilon_1\bar{\varepsilon}_2\frac{i_2}{M}+\varepsilon_2\bar{\varepsilon}_1\frac{i_1}{M}+ & \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\frac{c}{M}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2,\ c^{'}=c \\ \bar{\varepsilon}_1\frac{i_2-c}{M}, & \text{if } i_1^{'}=i_1+1,\ i_2^{'}=i_2,\ c^{'}=c+1 \\ \bar{\varepsilon}_2\frac{i_1-c}{M}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2+1,\ c^{'}=c+1. \end{cases} \tag{3.8}$$

- If $\mathcal{K}=M$, $i_1=M$, $i_2=M$, then $P_{(i_1,i_2,c)\to(i_1,i_2,c)}=1$.

**Action $a_2$ (unicast from $S$ to $R_1$):** it is seen as a special case of broadcasting by substituting $\varepsilon_2=1$.
**Action $a_3$ (unicast from $S$ to $R_2$):** it is symmetric to $a_2$.
**Action $a_4$ (unicast from $R_1$ to $R_2$):** In this case, $R_1$ may send a new packet if and only if $i_1>c$ and $i_2<M$. If these conditions are satisfied, then the probability that the packet sent by $R_1$ is new to $R_2$ is equal to the probability that the selected packet by $R_1$ is not in the common packets received by both receivers, therefore, it is equal to $\frac{i_1-c}{i_1}$. Therefore, the transition probabilities for action $a_4$ is calculated as

- If $i_2<M$, $i_1>c$

$$P_{(i_1,i_2,c)\to(i_1^{'},i_2^{'},c^{'})}=$$

$$\begin{cases} \varepsilon_{R_1R_2}+\varepsilon_{\bar{R_1}R_2}\frac{c}{i_1}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2,\ c^{'}=c \\ \varepsilon_{\bar{R_1}R_2}\frac{i_1-c}{i_1}, & \text{if } i_1^{'}=i_1,\ i_2^{'}=i_2+1,\ c^{'}=c+1. \end{cases} \tag{3.9}$$

- If $i_2=M$ or $i_1=c$, then $P_{(i_1,i_2,c)\to(i_1,i_2,c)}=1$.

**Action $a_5$ (unicast from $R_2$ to $R_1$):** it is symmetric to $a_4$.
**Action $a_6$ (do not transmit):** For all values of $i_1,i_2,c$

$$P_{(i_1,i_2,c)\to(i_1,i_2,c)} = 1. \tag{3.10}$$

### 3.4.3 No-NC with Full-Feedback (No-NC, Full-Feedback)

In this case, we assume that NC is not allowed, but the ID of the packets that were received by $R_1, R_2$ is known to the system through sending feedback. Except the transition probability, all other components of the MDP model in this scenario would be the same as the previous scenario in Section 3.4.2. By knowing the ID of the packets that were received by $R_1, R_2$ in a no-NC scenario, the sender is able to choose an optimum packet to send compared to the previous case where the sender has to choose a packet randomly from its buffer. Therefore, we propose a heuristic for when coding is not possible in the presence of full feedback to optimize the completion time. The key idea of the proposed heuristic is to do broadcast for $\mathcal{K} < M$ and unicast when $\mathcal{K} = M$. The system sends those packets that are new at least to one of the receivers. Assuming that the network is not in any of the following three states $(M,M,M), (M,i_2,c), (i_1,M,c)$, two different cases may happen:

$\mathcal{K} < M$**:**

1. If the action is any one of $a_1$, $a_2$, $a_3$: the sender sends a packet that is new to both receivers.
2. If the action is $a_4$ and $i_1 > c$: $R_1$ selects a packet that is new to $R_2$ and sends it.
3. If the action is $a_4$ and $i_1 = c$: $R_1$ cannot send a new packet to $R_2$ and so it selects a packet randomly to send.
4. If the action is $a_5$: the case is symmetric to $a_4$.

$\mathcal{K} = M$**:**

1. If the action is $a_1$: the packet is selected randomly.
2. If the action is $a_2$ or $a_3$: the sender chooses a packet that is new for the receiver and sends it.
3. If the action is $a_4$ and $i_1 > c$: $R_1$ selects a packet that is new to $R_2$ and sends it.
4. If the action is $a_4$ and $i_1 = c$: $R_1$ cannot send a new packet to $R_2$ and chooses a packet randomly.
5. If the action is $a_5$: the case is symmetric to $a_4$.

By using this heuristic, the calculation of the transition probabilities is straightforward. For example, assuming that the action is broadcast, and $\mathcal{K} < M$, then the non-zero transition probabilities are as follows.

$$P_{(i_1,i_2,c)\to(i_1^{'},i_2^{'},c^{'})} =$$

$$\begin{cases} \varepsilon_1\varepsilon_2, & \text{if } i_1^{'} = i_1,\ i_2^{'} = i_2,\ c^{'} = c \\ \bar{\varepsilon}_1\varepsilon_2, & \text{if } i_1^{'} = i_1+1,\ i_2^{'} = i_2,\ c^{'} = c \\ \bar{\varepsilon}_2\varepsilon_1, & \text{if } i_1^{'} = i_1,\ i_2^{'} = i_2+1,\ c^{'} = c \\ \bar{\varepsilon}_1\bar{\varepsilon}_2, & \text{if } i_1^{'} = i_1+1,\ i_2^{'} = i_2+1,\ c^{'} = c+1. \end{cases} \tag{3.11}$$

## 3.5 Comparison of the MDP solution for NC and no-NC scenarios

For the network of Fig. 3.2, we compare the expected completion time of transmitting $M$ packets from $S$ to $R_1, R_2$ calculated by the optimal MDP solutions, for the four scenarios that we explained, namely, (i) Optimal solution proposed for a GS, (ii) NC with Full-Feedback, (iii) No-NC and No-Feedback, and (iv) No-NC with Full-Feedback. In this part of our analysis, we assume a symmetric channel between receivers, meaning that $\varepsilon_{R_1R_2} = \varepsilon_{R_2R_1}$.

*Optimal solution for GS (no-Feedback):* In a GS, we have perfect knowledge of the system state, but not the information about the ID of the packets that were received by each receiver. For no-NC scenarios, this is equal to the case that no-feedback is available (Section 3.4.2). Therefore, we compare the completion time of a Genie system for NC and the no-NC and no-feedback scenario. We set $\varepsilon_2 = 0.8, \varepsilon_{R_1R_2} = 0.1$ and change the erasure probability of the link between $S, R_1$, $(\varepsilon_1)$. Fig. 3.4 shows the comparison between NC and no-NC. We see that NC is able to decrease the expected completion time by a factor of 2.9 compared with the no-NC case. It is also shown that by increasing the erasure probability of a channel, the completion time of no-NC increases exponentially whereas in the case of NC it increases linearly. This means that NC can be used in a dynamic network in which the erasure probabilities are changing fast.

*Full-Feedback assumption:* We compare the performance of no-NC with full-feedback scenario with the optimal NC solution for GS. Fig.3.5 shows the expected completion time for different values of feedback cost $(\alpha = \frac{1}{3}, \frac{1}{2}, 1)$. If the cost of feedback is small (e.g. $\alpha = \frac{1}{3}$), the expected completion time in case of no-NC decreases by a factor of 2 compared with the case of no-NC and no-feedback, but it is still 1.5 times bigger than the optimal completion time of NC and GS. If the cost of feedback is large and equal to packet transmission cost ($\alpha = 1$), the expected completion time of no-NC could be bigger than what we have for no-NC and no-feedback. This means that in case of no-NC, if the cost of sending feedback is too high, then full feedback is not warranted to increase the

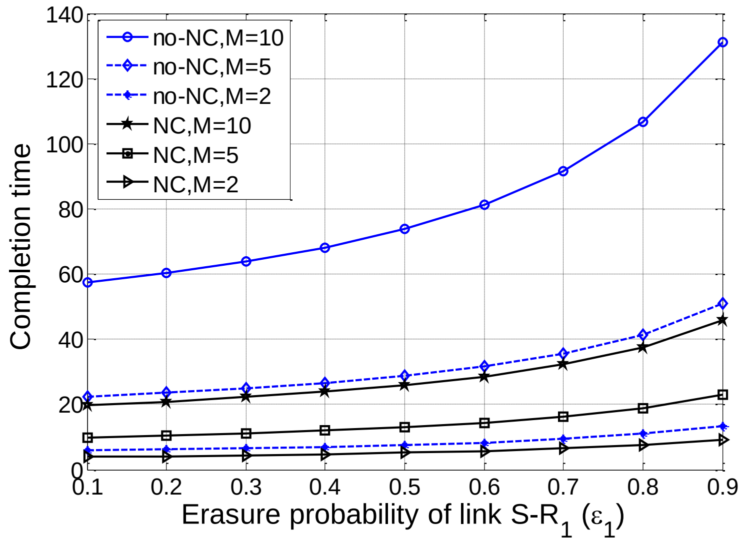


Figure 3.4: Completion time of a GS for different values of $\varepsilon_1$ and NC and no-NC scenarios

probability of sending new packets to receivers. It is seen that there is an upper bound for the cost of feedback $\alpha$ depending on $M$. This helps us to decide whether to send or not to send the feedback to improve the completion time.

*All broadcast (no-MDP):* In order to show the benefit of having cooperation between receivers, we compare the optimal solution selected by the MDP with the case where no decision making process is done and the source always broadcasts data packets toward $R_1$ and $R_2$ until both of them get all $M$ packets successfully. A GS is assumed. Note that in case of no-NC and no-MDP, the source is randomly selecting a packet from its buffer and broadcast it toward destinations. Fig. 3.6 shows the comparison of the MDP and no-MDP approaches for NC and no-NC scenarios and two sets of channels erasure probabilities, in terms of expected completion time. It is shown that for NC scenario, the MDP approach decreases the expected completion time by a factor of 2.13 compared with the no-MDP approach. It is also seen that the MDP approach for NC decreases the expected completion time by a factor of 6.1 compared with the no-NC no-MDP approach.

*Different cost for broadcast and unicast ($\beta > 1$):* We assume that the cost of broadcast is more than the cost of unicast, so, $\beta > 1$. Fig. 3.7, illustrates the expected completion cost for different values of $\beta$. It is seen that there is an upper bound, $\beta_{UB}$, in which the expected completion time remains constant for the values greater than that. This means that if $\beta \geq \beta_{UB}$, the MDP algorithm chooses to do unicast and broadcast is not used. We can see that for $\beta > 1$, NC still is able to reduce the expected completion time by up to 3 times compared with non-NC, which means that even for larger cost of broadcast compared with unicast, NC outperforms non-NC scenarios in terms of completion time.

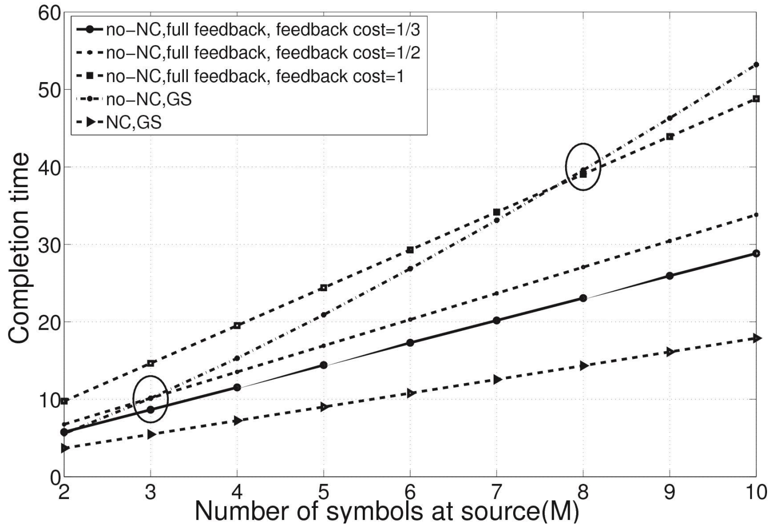


Figure 3.5: Completion time for different cost of feedback, $\varepsilon_1 = 0.2, \varepsilon_2 = 0.5, \varepsilon_{R_1R_2} = 0.3$

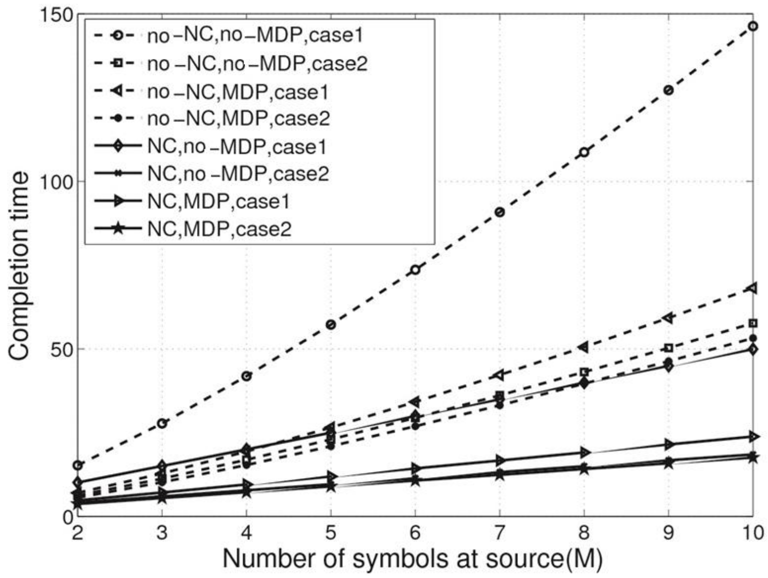


Figure 3.6: Completion time for MDP and non-MDP, case1: $\varepsilon_1 = 0.4, \varepsilon_2 = 0.8, \varepsilon_{R_1R_2} = 0.1$, case2: $\varepsilon_1 = 0.4, \varepsilon_2 = 0.4, \varepsilon_{R_1R_2} = 0.2$.

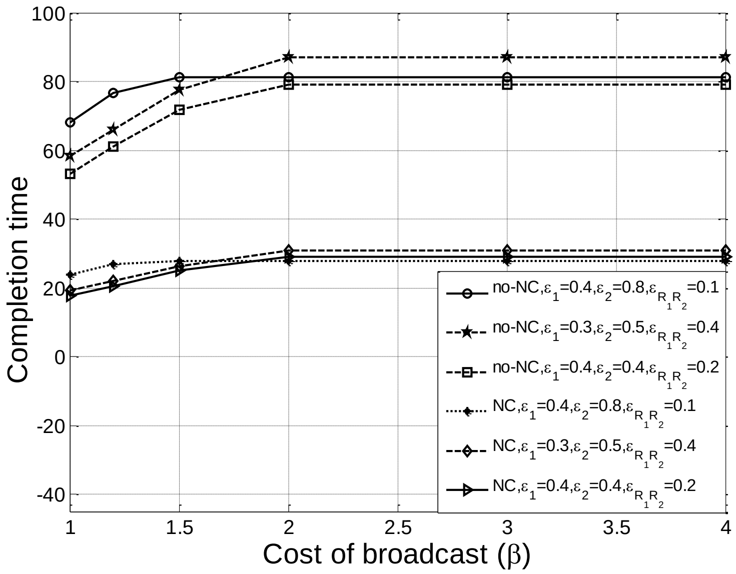


Figure 3.7: Completion time for different values of $\beta$

## 3.6 Analysis of the MDP Solution and Optimal Policy Extraction

Now that we have shown the gain of the optimal NC scheme over no-NC schemes, we focus on the MDP solution for NC scenario in GS, and analyse the impact of different parameters of the network on the MDP solution. Our goal is to extract general rules from the MDP solution that can be used to propose useful heuristics for packet transmission in a network of one source and $N \geq 2$ receivers.

For the network of Fig. 3.2, we evaluate the distribution of the selected actions by the MDP for a GS under different network conditions. Unless explicitly stated, we assume that $q$ is large enough so that any RLNC packet received from the source is independent from previously received packets with very high probability. We further assume that the channel between receivers is a symmetric channel with erasure probability $\varepsilon_{R_1R_2}$. We analyse the relationship between the selected actions by the MDP and different characteristics of the network, e.g., the erasure probabilities, $\beta$, $M$, and $q$. We calculate the percentage of usage for each action $a_j$ as $percentage\ of\ usage = (N_{a_j}/N_S) \times 100\%$, where $N_{a_j}$ is the number of states that MDP chooses to do action $a_j$ and $N_S$ is the total number of meaningful states. We neglect the role of action $a_6$, which is selected only once at the end of the transmission. We also investigate the distribution of the selected actions for different instants of time to understand the time-evolution of the actions.

*Effect of the erasure probabilities:* We assume that $M = 20, \beta = 1$, $\varepsilon_1 = 0.9, \varepsilon_2 = 0.7$, and $\varepsilon_{R_1R_2}$ is changing to evaluate the effect of erasure probabilities of the channels on the actions selected by the MDP. Fig. 3.8 shows the distribution of the selected actions by the MDP at the time that packet transmission is completed. The x-axis of the graph shows

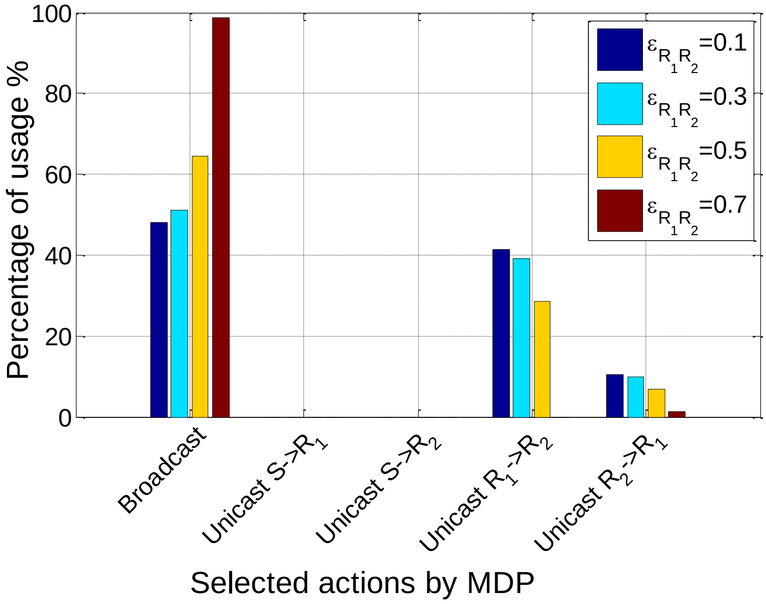


Figure 3.8: Distribution of the selected actions for $\beta = 1, \varepsilon_1 = 0.9, \varepsilon_2 = 0.7, M = 20$ and variable $\varepsilon_{R_1R_2}$

five possible actions, $a_1 - a_5$, and the y-axis shows the percentage of usage of each action by the MDP. By changing the erasure probabilities of the channels, the actions selected by the MDP change too. For instance, if $\varepsilon_{R_1R_2} = 0.1$, the MDP chooses to do broadcast only in 48% of the states and in the remaining 52% the two receivers are cooperating to help each other, while for $\varepsilon_{R_1R_2} = 0.7$, the MDP chooses to do broadcast in 98% of the states. Therefore, broadcast is shown to be used with high probability by increasing the erasure probability of the channel between receivers.

*Effect of M:* We assume that $\beta = 1, \varepsilon_1 = 0.8, \varepsilon_2 = 0.6, \varepsilon_{R_1R_2} = 0.3$ and $M$ is variable. Fig. 3.9 shows the distribution of the actions selected by the MDP at the time that packet transmission is completed. We can see that the set of selected actions by the MDP does not change by changing the value of $M$, although the percentage of the usage of each action may be changed. This means that we may be able to extract a general decision making policy to select the appropriate actions independent of the number of packets. It is also seen that by increasing $M$ from 10 to 25, the percentage of usage of broadcast is decreased from 75% to 60%. Meaning that larger $M$, may lead to higher probability of choosing cooperation between receivers by the MDP.

*Effect of* $(\beta)$*:* To see the effect of $\beta$ on actions selected by the MDP, the network characteristics are set at $\varepsilon_1 = 0.4, \varepsilon_2 = 0.8, \varepsilon_{R_1R_2} = 0.1, M = 10$ and $\beta$ is variable. Fig. 3.10 shows the results. We see that for $\beta \geq 1.4$, the MDP does not select broadcast and for $\beta = 1.2$, it chooses broadcast only in 50% of the states. This means that there are many cases that broadcasting leads to extra cost of packet transmission. We have done a similar test for

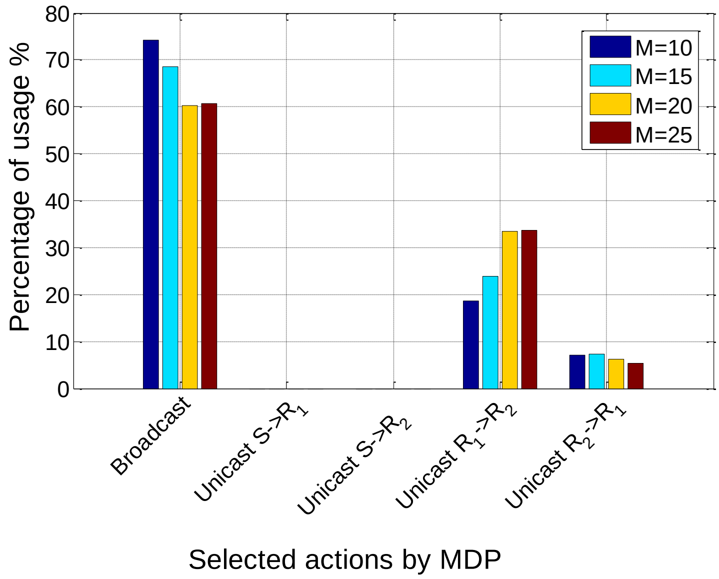


Figure 3.9: Distribution of the selected actions for $\beta = 1, \varepsilon_1 = 0.8, \varepsilon_2 = 0.6, \varepsilon_{R_1R_2} = 0.3$ and variable $M$

different scenarios and concluded that if $\frac{\beta}{(1-\varepsilon_1)+(1-\varepsilon_2)} \leq \frac{1}{1-\varepsilon_1}$ and $\frac{\beta}{(1-\varepsilon_1)+(1-\varepsilon_2)} \leq \frac{1}{1-\varepsilon_2}$, the MDP solution chooses broadcast in some cases, otherwise no broadcast action is selected. $\frac{\beta}{(1-\varepsilon_1)+(1-\varepsilon_2)}$ represents the cost of successful reception of one packet at the receiver side, i.e., either $R_1$ or $R_2$, per broadcast transmission. $\frac{1}{1-\varepsilon_k}$ represents the cost of successful transmission of one packet from the source to the receiver $R_k$ by using unicast transmission over the link between $S, R_k$.

*Time-evolution of selected actions:* We define a new parameter, $\gamma$, that shows the percentage of $M$ packets that were received by at least one of the receivers at the time of observation. Therefore, if the system is in state $s(i_1, i_2, c)$, then $\gamma$ is defined as $\gamma = (\mathcal{K}/M) \times 100\%$, where $\mathcal{K} = i_1 + i_2 - c$. We evaluate the distribution of the selected actions by the MDP for different values of $\gamma$, i.e., $0\% < \gamma \leq 100\%$. We split the case of $\gamma = 100\%$ into two sub-cases:

- If $\mathcal{K} = M$ but $i_1 \neq M, i_2 \neq M$ that is shown as $\gamma = 100\%$ in our numerical results.
- If $i_1 = M$ or $i_2 = M$ that are shown as $i_1 = M$ and $i_2 = M$, respectively.

The network characteristics are fixed at $\varepsilon_1 = 0.8, \varepsilon_2 = 0.6, \varepsilon_{R_1R_2} = 0.3$, $\beta = 1$, $M = 20$ and Fig. 3.11 shows the percentage of usage of each action for different values of $\gamma$. We can see that for $\gamma = 100\%$ (case (i)), the MDP chooses to do unicast from $R_1$ to $R_2$. This means that, when the total knowledge of the receivers is equal to $M$, while none of the receivers has $M$ dof, the source can stop transmitting and the two receivers can exchange their data. For $i_1 = M$ or $i_2 = M$ (case (ii)), the selected action would be cooperation between receivers in 100% of the states for this specific scenario. Meaning that the action

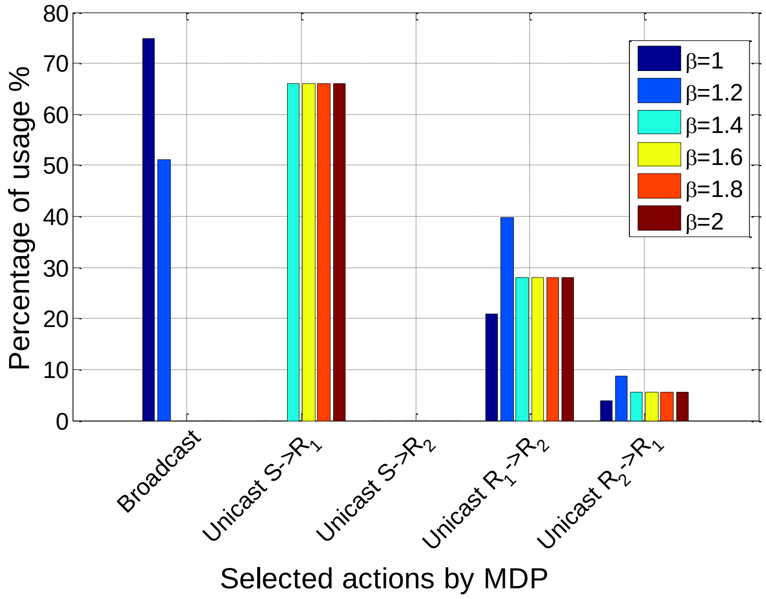


Figure 3.10: Distribution of the selected actions for $M = 10, \varepsilon_1 = 0.4, \varepsilon_2 = 0.8, \varepsilon_{R_1R_2} = 0.1$ and variable $\beta$

selected remains constant until the end of the packet transmission. These results are valid assuming that the erasure probabilities of the channels remain constant during packet transmission process.

*Effect of field size (q):* we analyse the effect of field size by computing three metrics: (i) percentage of cooperation between receivers that is calculated as the percentage of usage of actions $a_4, a_5$, (ii) expected completion time of transmitting $M$ packets, and (iii) the value of $\gamma$ at the time of starting cooperation between receivers. Fig. 3.12(a)-(c) show the results of these experiments assuming that $\varepsilon_1 = 0.5, \varepsilon_2 = 0.7$, $M = 10$, and $\varepsilon_{R_1R_2}$ is changing from 0.1 to 0.7. The field size is varied from $GF(2)$ to $GF(2^{16})$.

Fig. 3.12-(a) shows that depending on the erasure probability of the channels, the MDP solution may or may not be changed by changing $q$. For example, in case of $\varepsilon_{R_1R_2} \leq 0.2$, increasing the field size leads to an increase in the percentage of cooperation between receivers, while for $\varepsilon_{R_1R_2} > 0.2$, the percentage of cooperation remains constant for any value of $q$. Fig. 3.12-(b) shows that an increase of the field size from $GF(2)$ to $GF(2^4)$, can decrease the completion time by up to 16%, while for the field size greater than $GF(2^4)$, the completion time remains constant. Fig. 3.12-(c) shows the values of $\gamma$ at the time that the MDP chooses cooperation for the first time for different values of $q$. This states the relationship between $q$ and minimum knowledge that the receivers should have before starting cooperation. It is seen that for $q < 2^8$, cooperation between receivers can be started only when $\mathcal{K} = M$, i.e., $\gamma = 100\%$, while for $q \geq 2^8$, the cooperation could be started earlier if $\varepsilon_{R_1R_2}$ is small, i.e., $\varepsilon_{R_1R_2} < 0.3$. As an example, for $\varepsilon_{R_1R_2} < 0.3$, and

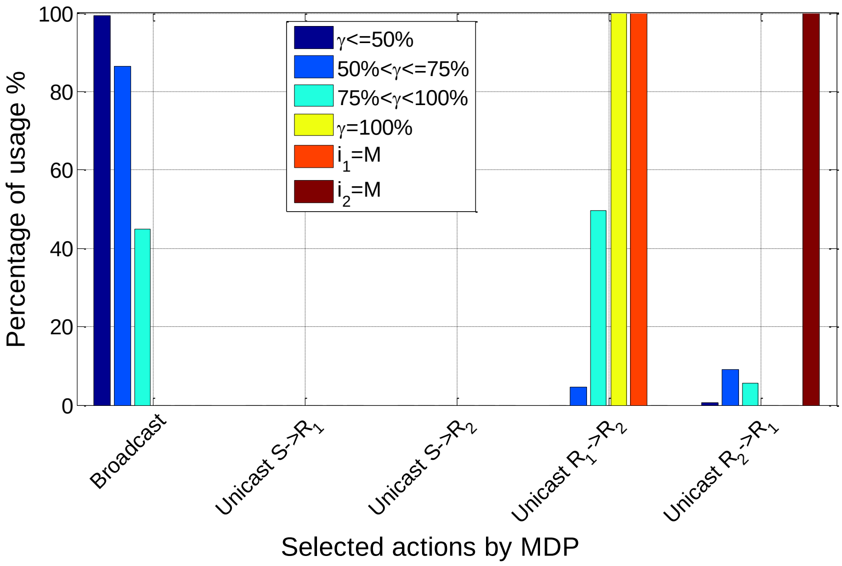


Figure 3.11: Distribution of the selected actions for $M = 20, \varepsilon_1 = 0.8, \varepsilon_2 = 0.6, \varepsilon_{R_1R_2} = 0.3$, $\beta = 1$

$GF(2^{16})$, cooperation is started at $\gamma = 70\%$, while for $\varepsilon_{R_1R_2} \geq 0.3$ cooperation starts at $\gamma = 100\%$. Therefore, by starting cooperation a bit later than what the MDP calculates, we may be able to provide a general solution that works well for any value of field size and erasure probability.

*Some intuitions behind the MDP solution:* Inspired by the actions that the MDP chooses in each state depending on the network characteristics, we obtained some general rules for the optimal packet transmission. There are four critical points where the MDP may need to select a new action: (a) at the beginning of packet transmission, (b) when $\mathcal{K} = M$ while none of the receivers has $M$ dof, (c) when $i_1 = M, i_2 \neq M$, (d) when $i_2 = M, i_1 \neq M$. Based on the behaviour of the MDP at these critical points, we define a cost metric, $C(a_j)$, that is a function of erasure probabilities of the channels and calculates the cost of successful transmission of one packet from a sender (either $S$ or one of the receivers) to a receiver by using action $a_j$ and it can help us to select the appropriate action for each point.

$$C(a_j) = \left[\frac{\beta}{(1-\varepsilon_1)+(1-\varepsilon_2)}\right] \times I_{(a_j=a_1)} + \frac{1}{1-\varepsilon_1} \times I_{(a_j=a_2)} + \frac{1}{1-\varepsilon_2} \times I_{(a_j=a_3)} + \frac{1}{1-\varepsilon_{R_1R_2}} \times I_{(a_j=a_4)} + \frac{1}{1-\varepsilon_{R_2R_1}} \times I_{(a_j=a_5)}, \tag{3.12}$$

where $I_{x \in X}$ represents the indicator function, $\beta$ is the cost of one broadcast transmission as defined in Section 3.3, $\varepsilon_k$ represents the erasure probability of the channel between $S$ and receiver $R_k$, and $\varepsilon_{R_iR_j}$ is the erasure probability of the channel between receivers $R_i, R_j$, when $R_i$ is transmitting toward $R_j$. According to our analysis of the MDP solution,

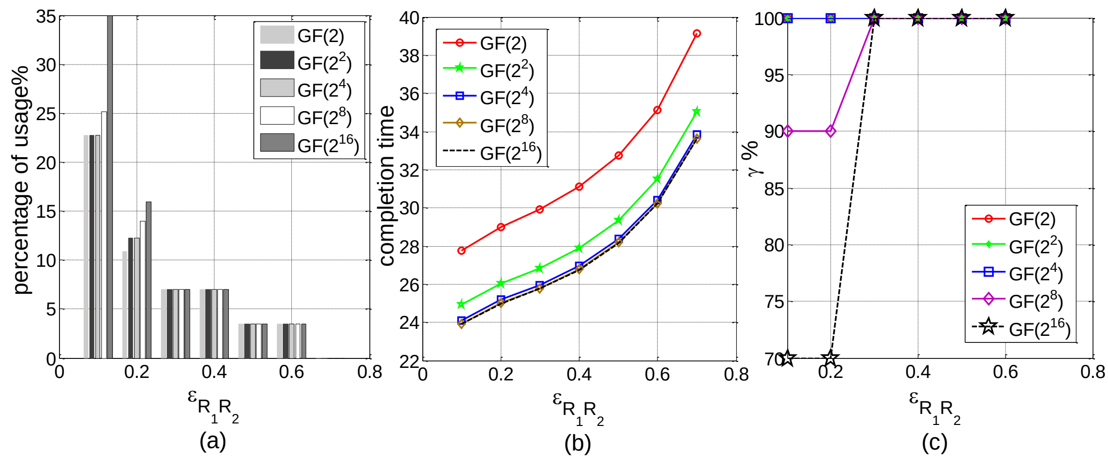


Figure 3.12: Comparison between the MDP solution for variable field size, for $M = 10, N = 2, \varepsilon_1 = 0.5, \varepsilon_2 = 0.7$ (a) percentage of cooperation, (b) completion time, (c) $\gamma$ at time of cooperation

we extracted the following transmission rules for the four critical points.

- At critical point (a), the MDP chooses one of the actions $a_1, a_2, a_3$ that minimizes $C(a_j)$.
- At critical point (b), the MDP chooses one of the actions $a_1, a_2, ..., a_5$ that minimizes $C(a_j)$.
- At critical point (c), the MDP chooses one of the actions $a_3, a_4$ that minimizes $C(a_j)$.
- At critical point (d), the MDP chooses one of the actions $a_2, a_5$ that minimizes $C(a_j)$.
- if $i_1 = i_2 = M$, the MDP chooses action $a_6$ and the transmission process is completed.

## 3.7 Proposed Heuristics

Since maintaining accurate system state information after each transmission is prohibitive due to the large signalling overhead required, we now focus on heuristics that drop the GS assumption and focus on limited signalling. Inspired by the packet transmission rules obtained from the MDP solution for a GS, we propose two heuristics, that use only a limited number of feedback packets from each receiver per generation of $M$ packets. Note that the proposed heuristics are initially designed for a network with one source and two receivers, but afterwards, we will generalize them for a network with $N$ receivers. In the following, we refer to $a^*$ as the selected action and $a'$ as the set of solutions that represent the minimum cost actions for a given subset of actions.

### 3.7.1 Minimum-Feedback (MF)

In this heuristic, a receiver can start sending recoded packets to its neighbour only when it receives $M$ dof and is able to decode the original packets. A receiver sends feedback only if $M$ dof are received successfully. The packet transmission process is divided into two phases. First, the source selects one of the three actions $a_1, a_2, a_3$ that has the minimum cost to start the transmission with. To this end, it computes

$$a' = \arg \min_{a_j \in \{a_1, a_2, a_3\}} C(a_j), \tag{3.13}$$

where $C(a)$ is calculated from Eq. (5.2) and depends on the erasure probabilities of the channels. Source starts sending RLNC packets using $a^* = a'$ and continues sending. If $a' = \{a_1, a_2\}$, then $a^* = a_1$, i.e., broadcast is favoured over unicast when they represent the same cost. Second, upon receiving feedback from either $R_1$ or $R_2$, the second phase is started by choosing a new action based on the following rules:

- If the received feedback has come from $R_1$, the new selected action is calculated as:

$$a^* = \arg \min_{a_j \in \{a_3, a_4\}} C(a_j)$$

- If the received feedback has come from $R_2$, the new selected action is calculated as:

$$a^* = \arg \min_{a_j \in \{a_2, a_5\}} C(a_j)$$

- If the received feedback has come from both $R_1, R_2$, the packet transmission is stopped.

### 3.7.2 Intermediate-Feedback (IF)

Considering the rules we extracted from the MDP solution, we propose IF heuristic which allows receivers to start exchanging recoded packets among each other when the total knowledge of the received packets by the two receivers is equal to M, i.e., $\mathcal{K} = M$. Therefore, cooperation between receivers may be started earlier compared to the MF heuristic. The IF heuristic uses feedback not only when a receiver gets $M$ dof successfully, but also when $\mathcal{K} = M$. Sending feedback whenever $\mathcal{K} = M$ needs some level of signalling among receivers to exchange the information about the number of packets received by two receivers. To minimize the number of signalling, we calculate the expected number of RLNC packets, $\hat{M}$, that the source has to broadcast to expect that the total number of packets received by two receivers is equal to $M$. Under our high field size assumption for RLNC, this leads to have $\mathcal{K} = M$. We calculate $\hat{M}$ as $\hat{M} = M \times \frac{1}{1-\varepsilon_1\varepsilon_2}$, where $1-\varepsilon_1\varepsilon_2$ is the probability of adding one degree of freedom to the set of packets received by two receivers per broadcast transmission. Now each receiver can calculate the expected number of packets it needs to receive such that the expected total number of packets received by the two receivers at that time is equal to $M$. For $R_1$, this means $ms_1 = \hat{M} \times (1-\varepsilon_1)$ and for $R_2$, this means $ms_2 = \hat{M} \times (1-\varepsilon_2)$. At the beginning of packet transmission, $R_1, R_2$ calculate $ms_1, ms_2$, respectively, as the milestone they need to reach before starting exchanging data with each other. We can summarize the process of packet transmission in two phases. Phase 1, is similar to the first phase of the MF. Clearly, if the selected action for phase 1 is not broadcast, the previous scheme of sending feedback can be used without the need for signalling among receivers. Because in that case, only one of the receivers is receiving packets in phase 1. If the selected action in phase 1 is broadcast, the first receiver to reach its milestone sends a feedback stating the dimensions they possess. For example, after getting $ms_1$ packets at $R_1$, it will signal to both $R_2$ and $S$ that it has reached that specific milestone and sends extra information in terms of the dimensions it has. To reduce the size of the feedback packets, we can use the compressed format of feedback that was proposed in [67] to convey the required information instead of sending all dimensions explicitly. Then, the other receiver (in this case $R_2$) can determine if $R_1$ and $R_2$ have enough dimensions jointly. If they do, then $R_2$ sends a feedback packet that informs $S$ to stop phase 1 of the process and informs $R_1$ that they can begin the exchange process. Otherwise, $R_2$ will wait until getting what is missing and then feedback is sent. This way, the number of signalling packets sent is small, although the scheme is suboptimal as $R_1$ will have accumulated additional coded packets by that time. Instead of waiting to get the missing dofs, $R_2$ can send a signal to $R_1$ telling the number of the coded packets that it needs to reach the milestone. By having this information, both $R_1, R_2$ can recalculate a new milestone. Once the first has reached the new milestone, it will feedback as before. This approach trades-off accuracy with the use of feedback. A similar process occurs if $R_2$ is the first to reach the milestone.

Phase 2 is started upon receiving feedback that acknowledges one of the three cases, $i_1 = M$ or $i_2 = M$ or $\mathcal{K} = M$. The decision on the new action is made based on what the feedback acknowledges and the cost of actions.

- If the feedback acknowledges $\mathcal{K} = M$, while $i_1 \neq M$, $i_2 \neq M$, then the source calculates $a' = \arg\min_{a\in\{a_1,a_2,a_3,a_4,a_5\}} C(a)$ and the new action would be $a^* = a^{'}$. This means that the selected action would be one of the five actions $a_1 - a_5$ that has the minimum cost as defined in Eq. (5.2). Note that if $a' = \{a_4, a_5\}$, then $a^* = a_4$ for the case of $i_1 > i_2$ and $a^* = a_5$ otherwise.

- If the feedback acknowledges $i_1 = M$, then the selected action would be $a^* = \arg\min_{a\in\{a_3,a_4\}} C(a)$.

- If the feedback acknowledges $i_2 = M$, then the selected action would be $a^* = \arg\min_{a\in\{a_2,a_5\}} C(a)$.
- If the feedback acknowledges $i_1 = i_2 = M$, then $a^* = a_6$ and the packet transmission is stopped.

We emphasize that for both MF and IF heuristics, the received dofs determine when to start cooperation, while the selected action depends on the erasure probabilities of channels. In highly dynamic environments, a receiver can send a wake up message to ask source to restart transmission if it does not receive anything from its neighbour after a given waiting time, $T_{wait}$.

*Remark* 2. *Less feedback events:* we can avoid sending feedback when $\mathcal{K} = M$, instead each receiver that reaches its milestone starts transmitting RLNC packets toward the other receiver, without sending a feedback. Similar to the MF heuristic, a feedback is sent only when one of the receivers gets $M$ dof. The only difference with the MF heuristic is that in this scheme cooperation between receivers start earlier. This way, we can decrease the cost of signalling by sending only one feedback per receiver per generation, but the total number of transmissions may increase, since the source is not stopped when $\mathcal{K} = M$.

## 3.8 Generalization of the Proposed Heuristics for N Receivers

Given the complexity of characterizing the MDP model of a network with more than two receivers, we extend the IF and MF heuristics for a network with $N > 2$ receivers as in Fig. 3.1. We do not claim optimality of these heuristics for $N > 2$ receivers, but show that they can significantly improve the performance of the RLNC broadcast in practical scenarios. The channels between receivers are assumed to be symmetric with $\varepsilon_{R_iR_j}$ showing the erasure probability of the channel between $R_i$ and $R_j$.

We propose four generalized versions of the IF and MF heuristics depending on the application requirements. All versions share as common step the organization of the set of $N$ receivers into $\lceil N/2 \rceil$ clusters. The source can determine which nodes are in which cluster based on the geographic position of the receivers or the erasure probabilities of the channels. The clustering problem is an interesting issue, but one that goes beyond the scope of this thesis. Finally, we define the total knowledge of a cluster as the combined dof of the two receivers of the cluster.

### 3.8.1 *Generalized Minimum-Feedback (GMF) heuristics*

We propose two versions of the MF for a general network with $N$ receivers: (a) *Generalized MF Right-Away Heuristics (GMF RA)*, (b) *Generalized MF Wait Heuristics (GMF*

*WAIT)*. Similar to the MF, in case of the GMF heuristics, a feedback is sent whenever a receiver gets $M$ dof.

*1) GMF RA:* this heuristic includes two phases. First, the source starts broadcasting RLNC packets and continues sending until it receives feedback from at least one receiver of each cluster. This way, we ensure that when source stops transmitting, each cluster has collected enough dof to complete the transmission. Second, each receiver that collects $M$ dof, sends a feedback to the source and simultaneously starts sending RLNC packets to its neighbour inside the same cluster. The packets sent by receivers are built by recoding the coded packets that exist in their buffer. Packet transmissions inside the clusters continues until all receivers collect $M$ dof.

*2) GMF WAIT:* this is similar to the GMF RA heuristics. The key difference is that GMF WAIT does not start cooperation between receivers as soon as one of the receivers gets $M$ dof. Instead, a receiver that collects $M$ dof, sends a feedback to the source and waits to receive an ACK from it. Upon receiving an ACK from the source, the receiver starts sending recoded RLNC packets to its neighbour in the same cluster. The ACK is sent by the source when it receives feedback from at least one receiver of each cluster. The source stops transmission after sending this ACK.

### 3.8.2 *Generalized Intermediate-Feedback Heuristics (GIF)*

Two versions of the IF heuristics are proposed for a general network with $N$ receivers: (a) *Generalized IF Right-Away Heuristics (GIF RA)*, (b) *Generalized IF Wait Heuristics (GIF WAIT)*. Similar to the IF heuristics and for every cluster, each receiver defines its milestone as defined in Section 3.7.2, based on the expected number of linearly independent packets that it has to receive from the source to expect that the total knowledge of the cluster is equal to $M$.

*1) GIF RA:* this heuristic has two main phases. First, the source starts broadcasting RLNC packets and continues transmitting until it receives completion feedback from at least one receiver of each cluster. Second, when a receiver reaches its milestone, it starts sending recoded RLNC packets to its neighbour in the same cluster. Therefore, cooperation between receivers can start while source is still transmitting. A completion feedback is sent by a receiver whenever it has $M$ dof.

*2) GIF WAIT:* this is similar to the RA case. First, the source starts broadcasting RLNC packets and continues transmitting until it receives the milestone feedback from all receivers. At this moment, source broadcasts an ACK to tell the receivers to start exchanging packets among each other. In the second phase, a receiver that reaches its milestone, sends a milestone feedback to the source and waits to get an ACK from the source to start transmitting toward its neighbor.

Since cooperation between the receivers is started by stopping the source before each cluster has guaranteed reception of all dof, GIF considers an additional common step to wakeup the source for additional transmissions, if needed. If a receiver gets $A$ consecutive non-innovative packets from its neighbour, it sends a wake-up message to the source and the source restarts broadcasting RLNC packets. After the source wakes-up, it continues transmitting until it receives the completion feedback from at least one receiver of each cluster. $A = 5$ in our implementation.

### 3.8.3 Total cost of packet transmission for the GIF WAIT heuristics

We calculate the total number of packet transmission for the GIF WAIT as a proxy to characterize the GIF heuristics. Small changes are needed to derive similar results for GMF. To simplify our analysis, we assume that the source does not need to wakeup after it stops transmitting in the first phase. We compute the cost of packet transmission for the GIF WAIT as $T_{GW} = t_1 + t_2$, where $t_1$ represents the expected time that is required to have the total knowledge of $M$ at all $\lceil N/2 \rceil$ clusters (cost of the first phase), and $t_2$ represents the expected time that is required to exchange packets inside the clusters (cost of second phase).

**Theorem 3.** *For the network of Fig. 3.1-(b), the expected number of time slots required to get $M$ dof in all $N/2$ clusters with GIF WAIT is calculated as*

$$t_1 = M + \sum_{t=M}^{\infty} \Big[1 - \prod_{i=1}^{\frac{N}{2}} \Big( \sum_{\tau=M}^{t} \binom{\tau - 1}{M - 1} \varepsilon_{C_i}^{(\tau - M)} \bar{\varepsilon}_{C_i}^{M} \Big) \Big], \tag{3.14}$$

*where $\bar{\varepsilon}_{C_i} = 1 - \varepsilon_{C_i}$ and $\varepsilon_{C_i} = \varepsilon_{R_1C_i} \times \varepsilon_{R_2C_i}$. $\varepsilon_{R_1C_i}$ and $\varepsilon_{R_2C_i}$ represent the erasure probability of the link between $S$ and the first and second receiver of cluster $C_i$, respectively.*

*Proof.* Each cluster is seen as a virtual node. The first phase of GIF WAIT broadcasts RLNC packets from the source to $N/2$ virtual receivers. Under the independent channel assumption, the equivalent erasure probability of the link between the source and cluster $C_i$ is $\varepsilon_{C_i} = \varepsilon_{R_1C_i} \times \varepsilon_{R_2C_i}$, as the goal is to convey $M$ dof to the virtual node. Using the result in [68] for broadcasting $M$ packets with RLNC to $N/2$ receivers each with loss $\varepsilon_{C_i}$ concludes the proof. □

**Theorem 4.** *Assuming that each cluster in Fig. 3.1-(b) has $M$ dof after phase 1 of the GIF WAIT, that $\varepsilon_{R_1R_2k}$ is the erasure probability of the link between the two receivers of cluster $C_k$, $s_j$ is the state of a cluster as defined for the MDP model in Section 3.3.2, and $s_j(1,3)$ is the third element, $c$, of state $s_j$. Also consider that $S_{possible}$ is the set of all states that satisfy $i_1 + i_2 - c = M$, that $P_{1j}^{(t)}$ is the probability that the network transits from state $s_1 = (0,0,0)$ to state $s_j$ in $t$ transmissions, and that $P$ is the transition probability* *matrix of the action $a_1$ defined in Section 3.3.2. Then, the expected number of time slots to complete the transmission of M packets to N receivers by exchanging packets between receivers of the clusters in GIF WAIT is*

$$t_2 = \sum_{k=1}^{\frac{N}{2}} \sum_{t=M}^{\infty} \sum_{j=1}^{|S_{possible}|} \left[ \frac{M - s_j(1,3)}{1 - \varepsilon_{R_1R_2k}} \times \frac{P_{1j}^{(t)}}{\sum_{v=1}^{|S_{possible}|} P_{1v}^{(t)}} \right] \times$$

$$\left[ \prod_{i=1}^{\frac{N}{2}} \left( \sum_{\tau=M}^{t} \binom{\tau-1}{M-1} \varepsilon_{C_i}^{(\tau-M)} \bar{\varepsilon_{C_i}}^{M} \right) - \prod_{l=1}^{\frac{N}{2}} \left( \sum_{\tau=M}^{t-1} \binom{\tau-1}{M-1} \varepsilon_{C_l}^{(\tau-M)} \bar{\varepsilon_{C_l}}^{M} \right) \right], \tag{3.15}$$

*where $P_{ij}^{(n)}$ shows the ij-th entry of the matrix $P^n$.*

*Proof.* Under the assumption of only one transmission per time slot, $t_2$ is

$$t_2 = \sum_{k=1}^{\frac{N}{2}} E(N_{tx,C_k} \mid \mathcal{K}_C = M) = \sum_{k=1}^{\frac{N}{2}} \sum_{t=M}^{\infty} E(N_{tx,C_k} \mid \mathcal{K}_C = M, N_{tx,S} = t) \times p(N_{tx,S} = t), \tag{3.16}$$

where $E(X)$ shows the expected value of random variable $X$. $N_{tx,C_k}$ represents the number of transmissions that is needed inside cluster $C_k$, and $\mathcal{K}_C = M$ means that the total dof of all clusters is equal to $M$. $N_{tx,S}$ shows the total number of broadcast transmissions performed by the source before having $\mathcal{K}_C = M$. On the other hand,

$$E(N_{tx,C_k} \mid \mathcal{K}_C = M, N_{tx,S} = t) = \sum_{j=1}^{|S_{possible}|} E\left(N_{tx,C_k} \mid \mathcal{K}_C = M, N_{tx,S} = t, state(C_k(t) = s_j)\right)$$

$$\times p(state(C_k(t) = s_j)) = \sum_{j=1}^{|S_{possible}|} \frac{M - s_j(1,3)}{1 - \varepsilon_{R_1R_2k}} \times \frac{P_{1j}^{(t)}}{\sum_{v=1}^{|S_{possible}|} P_{1v}^{(t)}}, \tag{3.17}$$

where $p(state(C_k(t) = s_j))$ represents the probability that cluster $C_k$ is in state $s_j$ at time $t$. $|S_{possible}|$ represents the size of set $S_{possible}$. The term $P_{1j}^{(t)} / \left( \sum_{v=1}^{|S_{possible}|} P_{1v}^{(t)} \right)$ calculates the probability of being in state $s_j$ at time $t$ according to Chapman-Kolmogorov theorem [69]. To calculate $p(N_{tx,S} = t)$, we define $p(N_{tx,S} = t) = p(Y = t)$, where $Y = \max_{i \in \{1,..,N\}} X_i$ and $X_i$ is defined as the number of transmissions required before cluster $C_i$ has $M$ dof and it is a Pascal random variable. Thus, $Y$ is the maximum of $N/2$ independent Pascal random variables and

$$p(Y = t) = p(Y \leq t) - p(Y \leq t-1) = p(\max_{i \in \{1,..,\frac{N}{2}\}} X_i \leq t) - p(\max_{i \in \{1,..,\frac{N}{2}\}} X_i \leq t-1) =$$

$$\prod_{i=1}^{\frac{N}{2}} \left( \sum_{\tau=M}^{t} \binom{\tau-1}{M-1} \varepsilon_{C_i}^{(\tau-M)} \bar{\varepsilon_{C_i}}^{M} \right) - \prod_{l=1}^{\frac{N}{2}} \left( \sum_{\tau=M}^{t-1} \binom{\tau-1}{M-1} \varepsilon_{C_l}^{(\tau-M)} \bar{\varepsilon_{C_l}}^{M} \right). \tag{3.18}$$

By substituting Eq. (3.17), (3.18) into Eq. (3.16), the proof is completed. □

*Remark* 3. *Spatial separation between clusters:* if the clusters are spatially separated from each other or different channels are available at the same time, then simultaneous transmissions could be performed by different clusters in every time slot. Therefore, the total cost of packet transmissions would be $t_1$ plus the maximum number of transmissions inside the clusters.

$$T_{GW} = \quad t_1 + \max_{k\in 1,2,\ldots,\frac{N}{2}} \left( \sum_{t=M}^{\infty} \sum_{j=1}^{|S_{possible}|} \left[ \frac{M - s_j(1,3)}{1-\varepsilon_{R_1R_2k}} \times \frac{P_{1j}^{(t)}}{\sum_{j=1}^{|S_{possible}|} P_{1j}^{(t)}} \right] \times \right.$$
$$\left. \left[ \prod_{i=1}^{\frac{N}{2}} \Big( \sum_{\tau=M}^{t} \binom{\tau-1}{M-1} \varepsilon_{C_i}^{(\tau-M)} \bar{\varepsilon_{C_i}}^{M} \Big) - \prod_{l=1}^{\frac{N}{2}} \Big( \sum_{\tau=M}^{t-1} \binom{\tau-1}{M-1} \varepsilon_{C_l}{}^{(\tau-M)} \bar{\varepsilon_{C_l}}^{M} \Big) \right] \right).$$

Finally, if the cost of broadcast is $\beta$ times bigger than the cost of unicast, the total cost of packet transmission is calculated as: $T_{GW} = (\beta \times t_1) + t_2$.

## 3.9 Numerical Results

In this section, we provide numerical results to compare the performance of the proposed heuristics versus RLNC broadcasting and the optimal MDP solution in terms of the total cost of transmitting $M$ packets to $N$ receivers. Symmetric channels and high field size are assumed and the cost of feedback is neglected. The effect of feedback cost is incorporated in our real-world implementation results.

### 3.9.1 Performance evaluation of the IF and MF heuristics for $N = 2$

We compare the performance of the MF and IF heuristics, RLNC broadcast, and the optimal MDP solution in terms of the total cost of packet transmission to see the degree of optimality of the proposed heuristics. Fig. 3.13-(a) shows the result for $M = 10$ and $N = 2$ receivers for the network defined in Fig. 3.2. Network characteristics are set at $\varepsilon_1 = 0.6, \varepsilon_{R_1R_2} = 0.4, \beta = 1$ and $\varepsilon_2$ is variable. It is seen that the performance of the MF and IF heuristics are matched to the optimal performance of the MDP. We also see that the proposed heuristics are able to decrease the total cost of packet transmission up to a factor of 2.7 with respect to the RLNC broadcast. Although by increasing $\varepsilon_2$ the total cost of packet transmission increases for all four schemes, these changes are much faster in the case of RLNC broadcasting compared with the proposed heuristics. Meaning that the proposed heuristics provide a more stable performance compared with the RLNC broadcast. A similar experiment is performed for $\beta = 1.2$, $\varepsilon_1 = \varepsilon_2 = 0.8$, and $\varepsilon_{R_1R_2}$ is varied. Fig. 3.13-(b) shows the result. We see that the MF and IF heuristics

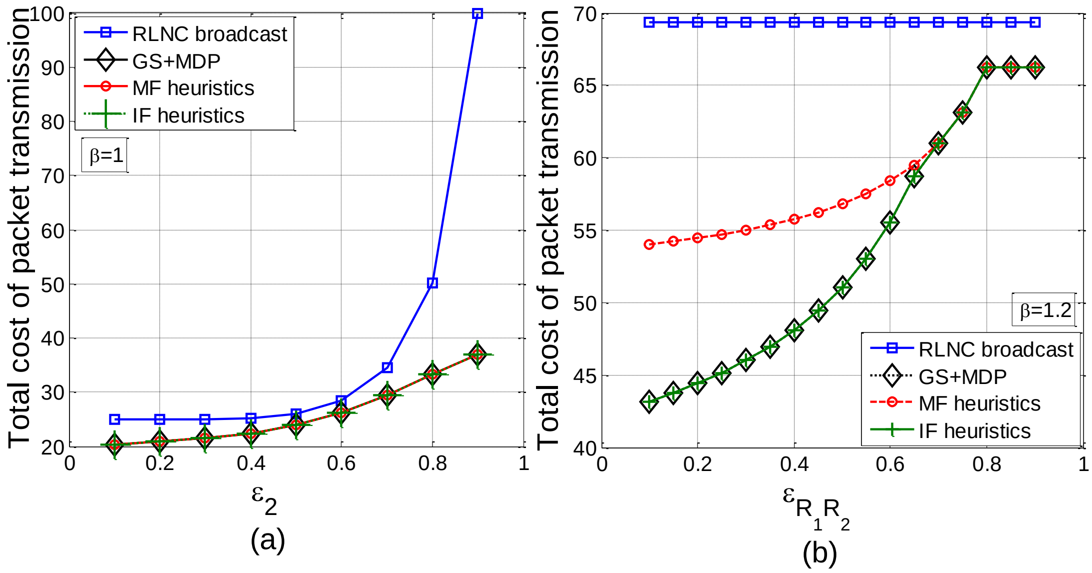


Figure 3.13: Comparison in terms of total cost of packet transmission for $M = 10$ (a) $\varepsilon_1 = 0.6, \varepsilon_{R_1R_2} = 0.4, \beta = 1$, and $\varepsilon_2$ is variable (b) $\varepsilon_1 = 0.8, \varepsilon_2 = 0.8, \beta = 1.2$, and $\varepsilon_{R_1R_2}$ is variable.

provide a gain of respectively, 1.3 and 1.6 compared with the RLNC broadcast. It is also seen that for any value of $\varepsilon_{R_1R_2}$, the performance of the IF heuristics matches the optimal performance of the MDP, while the MF heuristics can achieve the optimal performance only if $\varepsilon_{R_1R_2} \geq 0.7$. The reason is that for $\varepsilon_{R_1R_2} > 0.7$, the selected action by the MDP is changed when a receiver gets $M$ dof and this corresponds to the behaviour of the MF, while this is not the case for $\varepsilon_{R_1R_2} \leq 0.7$. Our observations in this section, state that by sending only two feedback packets (one per receiver) at the right time, we are able to obtain the optimal performance of the MDP solution.

### 3.9.2 Performance evaluation of the GIF WAIT heuristics for $N \geq 2$

Considering the network defined in Fig. 3.1-(b), we evaluate network performance for two network configurations: (i) no spatial separation between clusters, (ii) spatial separation between clusters. We define the gain of the GIF WAIT heuristics as $Gain = T_{Broadcast}/T_{GW}$, where $T_{Broadcast}$ is the total cost of transmitting $M$ packets using the RLNC broadcast and $T_{GW}$ represents the total cost of packet transmission using the GIF WAIT as calculated in Section 3.8.3.

*Effect of the erasure probabilities:* we evaluate the gain of the GIF WAIT for the two configurations, and $M = 10$, $\beta = 1$, and different erasure probabilities. Fig. 3.14-(a) shows the results for $N = 6$, $\varepsilon_1 = 0.6, \varepsilon_2 = 0.85, \varepsilon_3 = 0.2, \varepsilon_4 = 0.8, \varepsilon_5 = 0.4, \varepsilon_6 = 0.3, \varepsilon_{R_5R_6} = 0.6,$

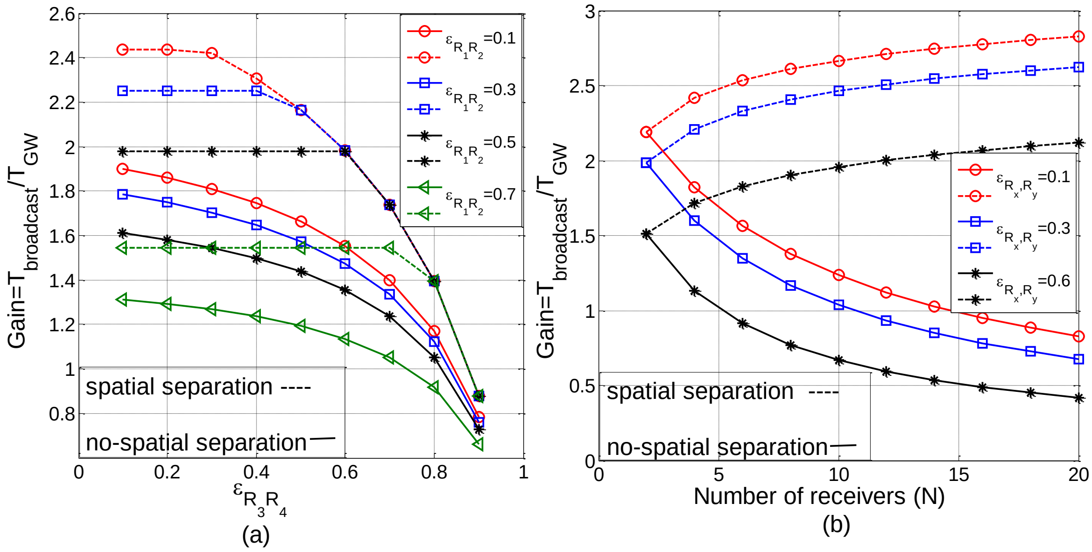


Figure 3.14: Effect of erasure probability and $N$ on the GIF WAIT gain for $M = 10$, a) $\varepsilon_1 = 0.6, \varepsilon_2 = 0.85, \varepsilon_3 = 0.2, \varepsilon_4 = 0.8, \varepsilon_5 = 0.4, \varepsilon_6 = 0.3, \varepsilon_{R_5R_6} = 0.6, N = 6$, (b) $\varepsilon_x = 0.4, \varepsilon_y = 0.8$ and symmetric clusters

and changing $\varepsilon_{R_1R_2}, \varepsilon_{R_3R_4}$. We can see that the gain of the GIF WAIT for the second configuration (spatial separation) is always larger than that of the non-spatial separation configuration. For example, for $\varepsilon_{R_3R_4} = 0.1$, the gain of the GIF WAIT for the first configuration is 1.9, while it is 2.43 for the second configuration. This is because of concurrent transmissions inside different clusters due to spatial separation between clusters. We also see that by increasing $\varepsilon_{R_3R_4}$, the gain of the GIF WAIT for the second configuration remains constant up to a specific point. This states that for the spatial separation between clusters, cluster with the worst channel between receivers has maximum effect on the total cost of packet transmission.

*Effect of N:* we evaluate the gain of the GIF WAIT heuristics for different number of receivers, $N$, and the two configurations. It is assumed that each cluster $C_i$ includes two receivers $R_{2i-1}, R_{2i}$. For the sake of simplicity, we assume that the clusters are symmetric, meaning that $\varepsilon_1 = \varepsilon_3 = \ldots = \varepsilon_{N-1} = \varepsilon_x$, and $\varepsilon_2 = \varepsilon_4 = \ldots = \varepsilon_N = \varepsilon_y$, and $\varepsilon_{R_1R_2} = \varepsilon_{R_3R_4} = \ldots = \varepsilon_{R_{N-1}R_N} = \varepsilon_{R_xR_y}$. Fig. 3.14-(b) shows the gain of the GIF WAIT with respect to RLNC broadcast for $2 \leq N \leq 20, M = 10, \varepsilon_x = 0.4, \varepsilon_y = 0.8$. Interestingly, by increasing $N$ for a spatial separation configuration, the gain of the GIF WAIT increases too. Meaning that the proposed heuristics can outperform the RLNC broadcast for any number of receivers, if the clusters have enough spatial separation. For instance, a gain of 2.8 is achieved compared with the RLNC broadcast for $N = 20$ and spatial separation. Also in case of non-spatial separation and for $\varepsilon_{R_xR_y} < 0.6$ and $N < 10$, GIF WAIT still provides up to $2.25x$ gains compared to the RLNC broadcast. For $N > 15$ and the first configuration, the

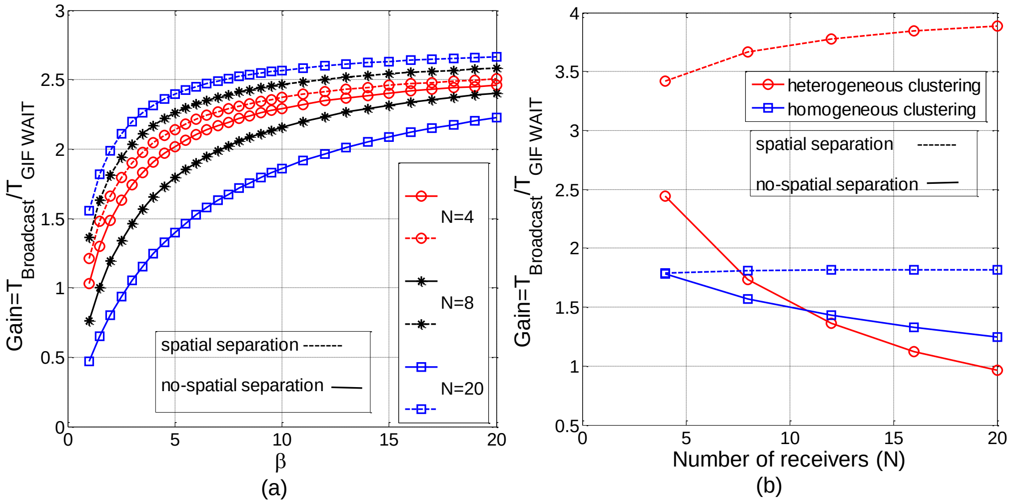


Figure 3.15: Gain of GIF vs. RLNC broadcast; effect of $\beta$ and different clustering methods a) $\varepsilon_x = 0.6, \varepsilon_y = 0.8, \varepsilon_z = 0.7, \varepsilon_w = 0.3, \varepsilon_{R_xR_y} = 0.7, \varepsilon_{R_zR_w} = 0.4, M = 10$, (b) $\varepsilon_x = 0.5, \varepsilon_y = 0.9, \varepsilon_{R_xR_y} = 0.4, M = 10$

RLNC broadcast has a better performance compared with the GIF WAIT. This is because of large number of transmissions required to be done inside each cluster to complete the packet transmission task. Thus, the use of our heuristics may depend on the network and channel conditions as well as geographic positions of the receivers.

*Effect of $\beta$:* we evaluate the effect of different costs of broadcast and unicast communication on the performance of the proposed heuristics. To this end, the ratio between broadcast cost and unicast cost $(\beta)$ is varied from 1 to 20. This could be interpreted as different packet transmission rates due to different modulations for unicast/broadcast transmissions. A network of $N$ receivers is considered and the receivers are divided into $N/2$ clusters such that cluster $C_i$ includes receivers $R_{2i-1}, R_{2i}$. The erasure probabilities are set to $\varepsilon_1 = \varepsilon_5 = \ldots = \varepsilon_x = 0.6, \varepsilon_2 = \varepsilon_6 = \ldots = \varepsilon_y = 0.8, \varepsilon_3 = \varepsilon_7 = \ldots = \varepsilon_z = 0.7, \varepsilon_4 = \varepsilon_8 = \ldots = \varepsilon_w = 0.3, \varepsilon_{R_xR_y} = 0.7, \varepsilon_{R_zR_w} = 0.4$. Fig. 3.15-(a) shows the gain of the GIF WAIT heuristics for $N = 4, 8, 20$, different values of $\beta$, and the two configurations. It is seen that by increasing $\beta$, the gain of the GIF WAIT for the two configurations increases too. For example, in case of no-spatial separation and $N = 20$, the gain of the GIF WAIT varies from 0.5 to 2.2 by changing $\beta$ from 1 to 20. This states that for the applications with different modulation schemes for unicast and broadcast, the GIF WAIT can provide some gains even for no spatial-separation and large $N$. For the case of spatial separation, the GIF WAIT always provides a gain greater than one for any value of $N, \beta$.

*Effect of different clustering schemes:* we study different ways of clustering of the receivers to see its effect on the gain of the GIF WAIT heuristic. It is assumed that the erasure probability of the channels between the source and each receiver, $\varepsilon_i$, can accept one of the two values $0.5, 0.9$ and the erasure probability of the channels between the receivers are fixed at $\varepsilon_{R_iR_j} = 0.4$. For simplicity, we assume that for any value of $N$, there are $N/2$ receivers with $\varepsilon_i = 0.5$ and $N/2$ receivers with $\varepsilon_i = 0.9$. We cluster the receivers in two ways: a) homogeneous clustering, where the receivers with similar links to the source are clustered into the same cluster, b) heterogeneous clustering, where the receivers with different links to the source are clustered into the same cluster. Fig. 3.15-(b) compares the performance of the proposed heuristics for the two schemes of clustering and configurations with spatial and no-spatial separation. We assume that $M = 10$ and $N$ is variable. In case of non-spatial separation and $N \leq 10$, we see that heterogeneous clustering provides a better performance than homogeneous clustering. For $N > 10$, the performance of homogeneous clustering is better than the performance of heterogeneous clustering. In case of the spatial separation between clusters, heterogeneous clustering always provides a better performance than homogeneous clustering. The reason is that the cluster that has the maximum number of transmissions between the receivers plays a dominant role in the total completion time of the network. On the other hand, by having a cluster of heterogeneous receivers, the required time to start cooperation between receivers is decreased compared to the homogeneous clustering, where the start of cooperation is delayed. Therefore, the total completion time is decreased too.

## 3.10 Real-World Implementation Results

In order to see the performance of the proposed heuristics and their respective gains in the real-world applications, we use a wireless network coding test-bed implemented on Raspberry Pi's of Model B at Aalborg University (AAU). A C++ library, called KODO library [70], is used to provide RLNC for the Raspberry Pis. A Wi-Fi dongle of type TP–Link (TL–WN722N) was used for every node to create a wireless network among Raspberry nodes. We will use UDP protocol for all kind of communications in the network. A detailed description on the test-bed that we used is available in [71]. Our measurement set-up consists of $N+1$ Raspberry Pi nodes, one is performing as source and the remaining $N$ nodes are performing as receivers. All nodes are placed inside the campus building. We hard coded the clustering process of the proposed heuristics, therefore we define which nodes belong to which clusters before starting packet transmission. Two different deployments of nodes, namely near-field and far-field deployments, are considered to evaluate the gain of the proposed heuristics under different network conditions. Completion time of packet transmission, $CT$, and total number of transmissions per packet, $T_{X_{PP}}$, are measured as performance evaluation metrics. $CT$ is calculated as $CT = t_{ack} - t_0$, where $t_0$ is the local time at the source at the beginning of packet transmission and $t_{ack}$ represents the local time at the source when it receives completion feedback from all receivers. Therefore, the calculated completion time incorporates the cost of feedback messages as well. $T_{X_{PP}} = \frac{N_{T_X}}{M}$, is a proxy of energy consumption and is calculated as total number of data packets sent by source or exchanged between receivers, $N_{T_x}$, divided by number of packets transmitted successfully, $M$. In the following, we provide the results of our measurements for the two deployments.

### 3.10.1 First set-up: synthetic losses

In this set-up, devices are placed inside a University lab at close distance with limited losses. We introduce synthetic losses for each channel of the network. Our goal is to characterize the parameter space and compare to our theoretical analysis. We consider symmetric channels between receivers, i.e., $\varepsilon_{R_iR_j} = \varepsilon_{R_jR_i}$, and $M = 100$ packets of 100 B. Data generation is set to 10 KBps unless stated otherwise. The following results show the average over 1000 trials at different times of day. A trial consists of transmitting one generation of $M$ packets.

*Effect of erasure probability:* Let us consider $N = 4$ receivers divided into two clusters, $C_1 = \{R_1, R_2\}$ and $C_2 = \{R_3, R_4\}$, for $\varepsilon_1 = 0.6, \varepsilon_2 = 0.8, \varepsilon_3 = 0.2, \varepsilon_4 = 0.5$, and the values of $\varepsilon_{R_1R_2} = \varepsilon_{R_3R_4}$ are varied from 0.1 to 0.6. Fig. 3.16 shows the measured $T_{X_{PP}}, CT$ for the proposed heuristics and RLNC broadcast and $q = 2^8$. We compare these measurements to the theoretical analysis of $T_{X_{PP}}$ for the GIF WAIT heuristics and RLNC broadcast from Section 3.8.3. Fig. 3.16-(a) shows that our measurements are close to the theoretical results. The small gap between them could be explained in part by the interference and correlation between transmissions in the real-world measurements. Fig. 3.16 also demonstrate that our heuristics outperform RLNC broadcast in terms of both $T_{X_{PP}}$ and $CT$ for any value of $\varepsilon_{R_1R_2}, \varepsilon_{R_3R_4}$. Although the gain of the proposed heuristics is decreased by increasing $\varepsilon_{R_1R_2}$, they can still provide at least 30% gain in $T_{X_{PP}}$ and 45% gain in $CT$ in the worst case. Fig. 3.16 also shows that GMF WAIT minimizes $T_{X_{PP}}$, while the GIF RA has the minimum $CT$. This is because in GIF RA, the receivers start exchanging data between each other earlier than the other schemes. Therefore, the completion time is reduced while the number of transmissions increases. More specifically, GMF WAIT provides up to 1.71 fold gain in $T_{X_{PP}}$ and GIF RA provides up to 2.27 fold gain in $CT$ with respect to the RLNC broadcast. We also compare the performance of the proposed heuristics under different field size assumption. Fig. 3.17 shows the results of the same experiment for $q = 2$ and $q = 2^8$. A similar trend as shown for $q = 2^8$ in Fig. 3.16, is seen for the case of small field size. We also see that the difference between coding with $q = 2$ and $q = 2^8$ in case of GMF WAIT and GIF WAIT heuristics is much less than that of the GMF RA and GIF RA. The reason is that in the case of WAIT heuristics, cooperation between receivers is started late compared with the RA heuristics, and this leads to have a smaller probability of sending non-innovative packets, and therefore, a smaller number of transmissions per packet. For the remaining tests, we only focus on the completion time for the case of $q = 2^8$.

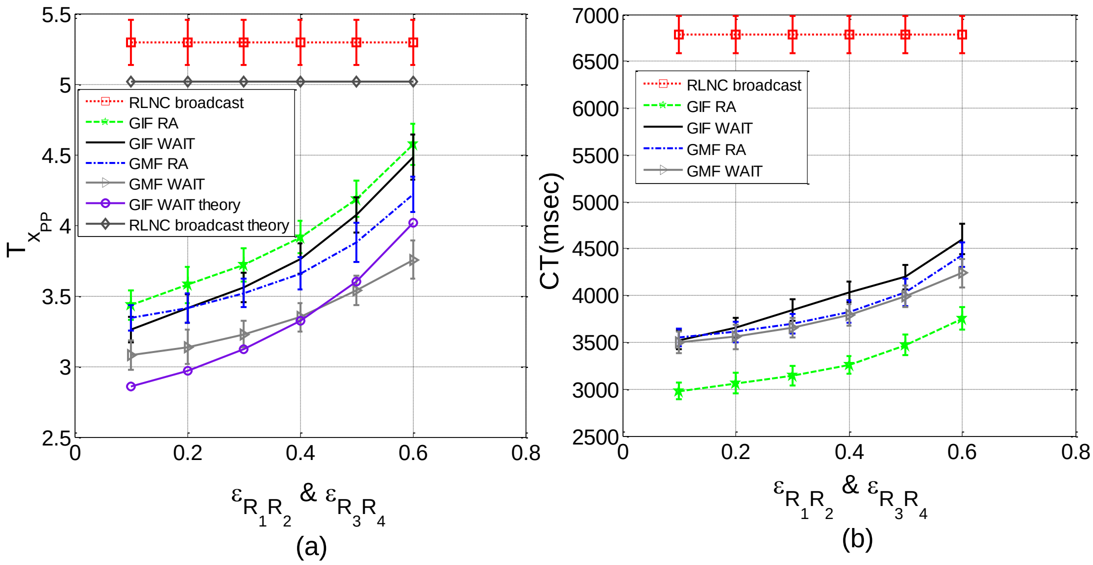


Figure 3.16: Comparison between the proposed heuristics and RLNC broadcast for $N = 4, \varepsilon_1 = 0.6, \varepsilon_2 = 0.8, \varepsilon_3 = 0.2, \varepsilon_4 = 0.5$ (a) number of transmissions per packet, (b) completion time.

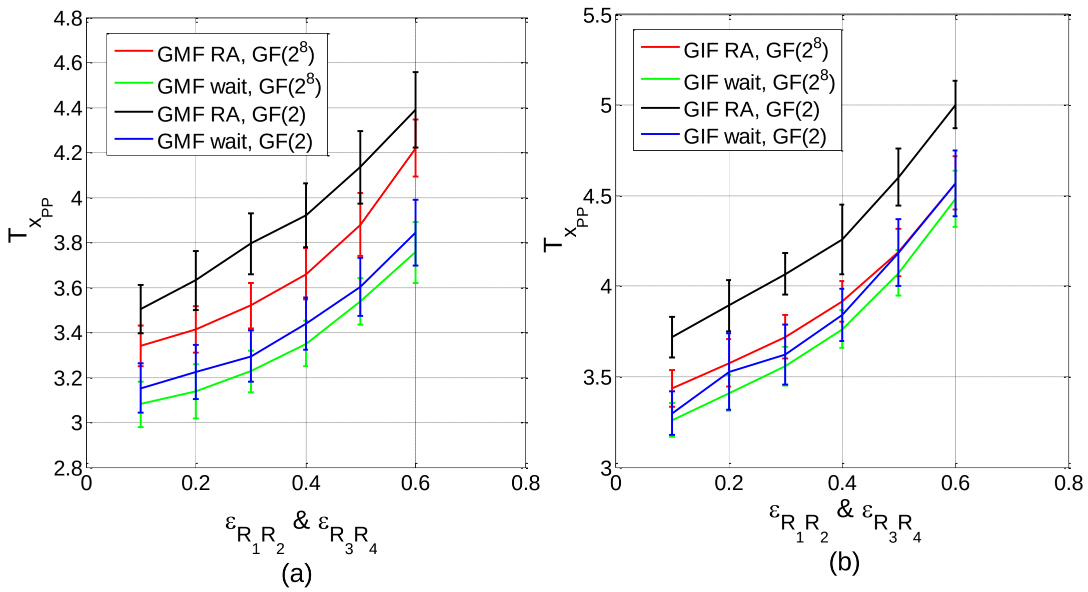


Figure 3.17: Comparison between performance of the proposed heuristics for $q = 2$ and $q = 2^8$, $N = 4, \varepsilon_1 = 0.6, \varepsilon_2 = 0.8, \varepsilon_3 = 0.2, \varepsilon_4 = 0.5$ (a) GMF heuristics, (b) GIF heuristics.

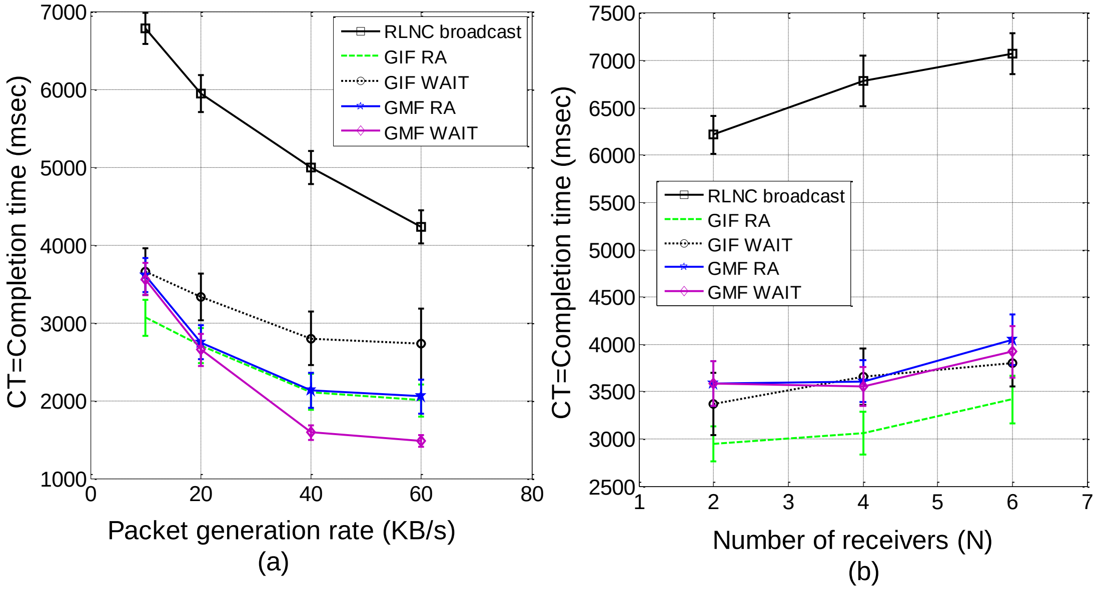


Figure 3.18: Comparison between the proposed heuristics and RLNC broadcast, (a) $N = 4$, and packet generation rate is varied, (b) $N$ is varied.

*Effect of packet generation rate:* The network is set as before except that $\varepsilon_{R_1R_2} = \varepsilon_{R_3R_4} = 0.2$. Fig. 3.18-(a) shows the completion time of the proposed heuristics and the RLNC broadcast. For high rate of generating packets, GMF WAIT has a better performance compared to other heuristics. For example, for generation rate 60 KBps, GMF WAIT provides a gain of 2.85x compared to RLNC broadcast. This means that for the higher rates of data generation, the best choice consists in postponing cooperation until the source stops transmitting packets.

*Effect of N:* The experiment is performed for $N = 2,4,6$ receivers divided into $N/2$ clusters such that receivers $R_{2i}, R_{2i-1}$ are members of cluster $C_i$, for $i = 1,2,3$. The erasure probabilities are set to $\varepsilon_1 = \varepsilon_5 = 0.6, \varepsilon_2 = \varepsilon_6 = 0.8, \varepsilon_3 = 0.2, \varepsilon_4 = 0.5, \varepsilon_{R_1R_2} = \varepsilon_{R_3R_4} = \varepsilon_{R_5R_6} = 0.2$. Fig. 3.18-(b) shows the average $CT$ for different $N$ and supports that our heuristics provide better performance than RLNC broadcast, with GIF RA providing the best results, over the studied $N$ range. For $N = 6$, GIF RA decreases $CT$ by 51% compared with the RLNC broadcast, while GIF WAIT, GMF WAIT, and GMF RA provide a $46\%, 44\%$, and 42% reduction in $CT$, respectively.

### 3.10.2 Second set-up: far field deployment

After showing the gain of the proposed heuristics for a fixed value of packet loss, we now study their performance under real-life, time-varying channel conditions. We build a network of 5 Raspberry Pis, one source and 4 receivers. All five nodes are fixed during our tests but the channels between them suffer variations over time. Fig. 3.19-(a) shows the deployment of nodes. The receivers are divided into two clusters, $C_1 : \{R_1, R_2\}$ and $C_2 : \{R_3, R_4\}$. Each cluster is located inside a separate room and the source is located inside a room in another building as shown in Fig. 3.19-(a). Receivers $R_1, R_3$ are located by the window and have line-of-sight (LOS) to the source, and receivers $R_2, R_4$ do not have LOS to the source but each has LOS to its neighbour. Three schemes were implemented: (a) RLNC broadcast, (b) GMF RA, and (c) GMF WAIT. We measure the completion time of transmitting $M = 100$ packets of size 100B to the four receivers by using each one of these schemes. Packet generation rate is fixed at 20 KBps. In order to confine the duration of each experiment, the maximum number of transmissions from the source is set to 10000 transmissions, i.e., the source stops transmitting packets after 10000 transmissions even if some receivers have not completed all $M$ packets. This only discards cases with extremely high packet losses, e.g., more than 99% of packets lost from the source. To calculate the mean and standard deviation (STD), we prune out these incomplete tests and calculate over the completed tests. We also calculate the percentage of incomplete tests in our 2000 trials as a representative metric of reliability. To incorporate the effect of interference from other devices in the network, we repeated our tests over different times during the day.

Table. 3.1 shows the mean and the STD of the completion time as well as the percentage of incomplete tests. These results show that GMF RA and GMF WAIT are able to decrease the mean of completion time compared with the RLNC broadcast by up to $4.75x$ and $4.25x$, respectively. It also shows that the percentage of incomplete trials for the RLNC broadcast is 12%, while in case of the two heuristics is zero. Fig. 3.19-(b) shows the distribution of the $CT$ for the three schemes. Clearly, our heuristics have a distribution concentrated around their mean, while RLNC broadcast has a large deviation. Thus, our heuristics not only reduce the completion time, but provide a more predictable and reliable solution compared to RLNC broadcast. Interestingly, GMF WAIT is more predictable than GMF RA, even when its mean $CT$ is worse.

Table 3.1: Far-field deployment results

| Result | RLNC Broadcast | GMF RA | GMF WAIT |
|---|---|---|---|
| Mean of completion time (msec) | 8.2782e+03 | 1.7459e+03 | 1.9477e+03 |
| STD of completion time (msec) | 1.2233e+04 | 807.1879 | 583.1516 |
| Percentage of incomplete cases after 10000 transmissions from source | 12% | 0% | 0% |

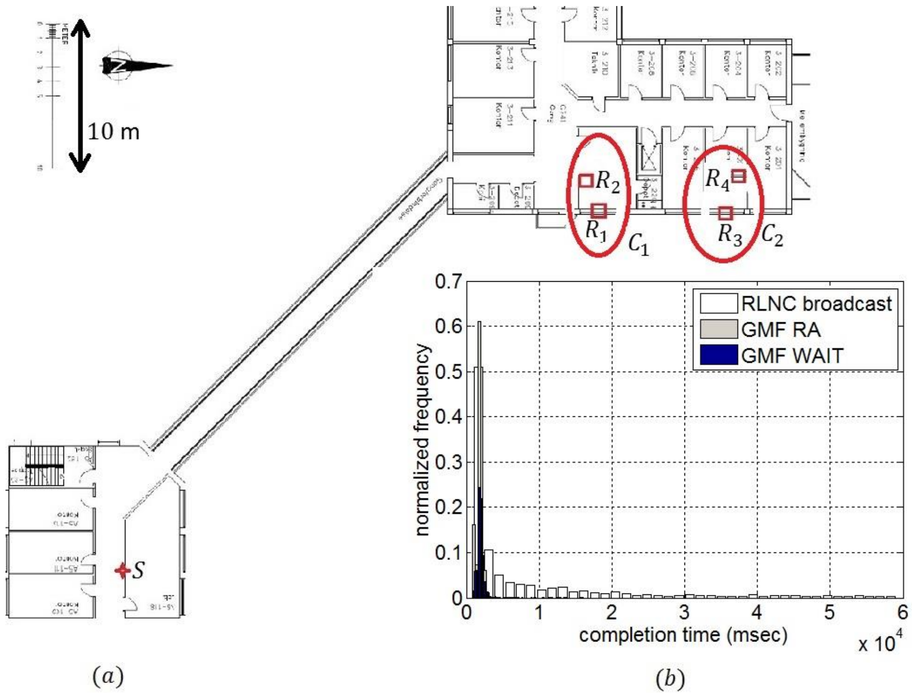


Figure 3.19: (a) Deployment of 5 Raspberry Pi nodes for the far-field analysis, (b) Distribution of completion time for one source, four receivers, and far-field deployment.

## 3.11 Concluding Remarks

In this chapter, we investigated optimal designs of cooperative network coded communications over half duplex channels, which is a key to design efficient packet transmission protocols with minimum cost. We determined the optimal time to start network coded cooperation between receivers of a non-relay based network that leads to minimize the total cost of packet transmission. We deeply analysed the case of one source and two receivers for any field size, arbitrary number of packets, and arbitrary erasure probabilities by modelling the optimization problem as an MDP. We also proposed simple, yet powerful heuristics that have been shown to be beneficial for a general case of one source and $N$ receivers. For more than two receivers, the proposed heuristics considered a clustering approach to map the problem of packet transmission to $N$ receivers into a simpler problem of packet transmission to the two receivers of each cluster. Therefore, we were able to reuse some of the transmission rules obtained by the MDP model for one source and two receivers. We implemented the proposed heuristics in a wireless NC test-bed. Our real-world measurements showed that the proposed heuristics can outperform the RLNC broadcast in terms of completion time by up to $4.75x$, while increasing the packet transmission reliability by up to $12\%$.

# Chapter 4

# Optimal Network Coded Cooperation over Time-Varying Erasure Channels

In Chapter 3, we have seen how to design an optimal network coded cooperative communication protocol to be used in networks with time-invariant erasure channels. We have shown that our proposed heuristics can provide a better performance compared to RLNC broadcast scheme in static real-world scenarios. However, we did not investigate their performance in very dynamic wireless networks where the quality of channels may change over time drastically due to node mobilities. This kind of network models could be very useful to improve the performance of routing algorithms for wireless networks with mobile nodes, where the channels between the nodes are changing over time due to nodes movements. For instance, the performance of the geographic routing that is a relevant strategy for VANETs can be improved by minimizing the cost of packet transmission over the created paths.

Although there has been some work in the literature that show the benefits of network coded cooperative communications on improving the performance of wireless networks, a very limited fraction of them have consider time varying channel conditions. To the best of our knowledge, we are the first who provide an in-depth analysis of the optimal solution for the total cost minimization problem in a time-varying scenario. More precisely, we focus on the problem of minimizing the total cost required to complete the transmission of $M$ packets to two moving receivers from a static source, assuming independent erasure channels between nodes. First, we model the problem using an MDP, and then we evaluate the performance of the heuristics that we proposed in Chapter 3 for time-varying channels. We consider two relevant practical scenarios to model the channels: (a) infrastructure-to-vehicle (I2V) communication in a highway scenario, and (b) a WiFi scenario using a Raspberry Pi test-bed at Aalborg University.

The remainder of this chapter is organized as follows. Section 4.1 presents our main contributions. In Section 4.2, first, we state the problem that we are going to solve, and then, we present the MDP model of the problem. Section 4.3 defines two time-varying scenarios that we use for the evaluation of the proposed heuristics. In Section 4.4, we provide the results of performance evaluation for the proposed heuristics and the MDP solution. Section 4.5 presents the concluding remarks of this chapter.

## 4.1 Main Contributions

Our main contributions in this chapter are as follows:

- **Mathematical analysis of a multi-user network for time-varying channels:** we solve the problem of minimizing the completion cost of transmitting $M$ packets from a common source to two receivers for time-varying channels scenario by using an MDP model. Since the number of meaningful states for the MDP model increases by the number of packets we are transmitting, we consider a finite time horizon and a finite number of packets to minimize the complexity of the MDP model. Given this complexity, we use our MDP solution as a way to evaluate the performance of our proposed heuristics in time-varying scenarios for small and moderate $M$.

- **Performance evaluation of the proposed heuristics for time-varying channels:** two relevant time-varying setups are considered for evaluating performance of our heuristics. We show that the IF and MF heuristics can outperform the performance of RLNC broadcasting in terms of completion time by a factor of 2 and 1.75, respectively. We also define a metric called percentage of reliability that shows the percentage of completed transmissions of $M$ packets in $T$ time slots and it could be considered as a proxy to measure the maximum $M$ that can be transmitted reliably. Our results show that the IF heuristic is able to increase the reliability of packet transmission in a finite time horizon by a factor of four with respect to the RLNC broadcasting.

## 4.2 System Model

### 4.2.1 Problem Statement

Similar to Chapter 3, we consider a network with one source, $S$, and two receivers, $R_1, R_2$, as shown in Fig. 4.1. The only difference with the problem defined in Chapter 3 is that in this chapter, independent time-varying erasure channels are considered for each receiver, with $\varepsilon_i(t)$ representing the erasure probability of channel $i$ at time $t$, while the erasure probabilities of the channels in the previous chapter were fixed over time. For simplicity, we assume symmetric channels, later we will show the validity of our analysis for the asymmetric channels as well. The following assumptions are made:

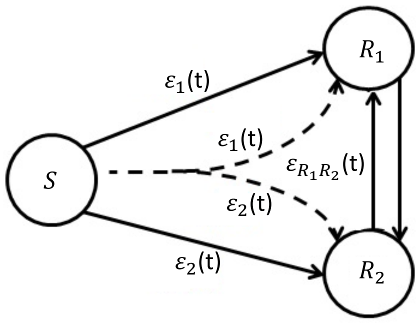


Figure 4.1: Network model; solid lines represent unicast, dotted lines represent broadcast, and $\varepsilon_i(t)$ represents erasure probability at time $t$.

- The receivers can share their knowledge with each other using unicast transmissions over the available channel between them or receive data directly from the source.
- A time-slotted system is considered and one transmission is allowed per time slot.
- When chosen to transmit, the source or one of the receivers generate RLNC coded packets as defined in Chapter 3.
- It is assumed that $q$ is large enough so that any RLNC packet received from the source is independent from previously received packets with very high probability.

Our goal is to find an optimal/near-optimal transmission policy that can minimize the total cost of successful transmission of $M$ packets from the source to the two receivers over time-varying channels.

### 4.2.2 The MDP model of the problem

Similar to the case of time-invariant channels model, to determine the optimal policy for minimizing the cost, we assume a GS, where each node in the network has perfect knowledge of the system state and also the erasure probabilities of the channels per time slot. Thus, the MDP policy has all required information to choose an optimal action per time slot. Note that optimality does not refer to the method we use to solve the MDP, but to the underlying assumptions of the model. In the following, we specify the state, possible actions, transition probabilities, and optimization algorithm in our model.
*State definition:* each state $s$ is defined as $s = (i_1, i_2, c, t)$, where $i_k$ is the number of degrees of freedom for the received packets at receiver $R_k$, and $c$ represents the dimension of the common knowledge between $R_1$ and $R_2$. $t$ represents the instance of time that the observation is made and could have an integer value in $\{0, 1, ..., T\}$, where $T$ is a finite time horizon that is defined large enough so that the transmission of $M$ packets can be finished before $T$ is expired. Since the complexity of the MDP model depends on the number of

meaningful states, we will look at the network for a finite horizon of time to reduce the complexity. As a simple example, assume that we are aiming at transmitting 4 packets $\{P_1, P_2, P_3, P_4\}$ from a source to two receivers $R_1, R_2$ and the set of packets received by $R_1$ and $R_2$ at time $t$ are respectively, $\{P_1, P_1 + P_2, P_3 + P_4\}$ and $\{P_3, P_4, P_1 + P_2 + P_4\}$. The state of network in this case is shown as $s = (3,3,2,t)$, because the *dof* of the received packets by $R_1$ and $R_2$ are 3 and the common knowledge between receivers is shown as $\{P_1 + P_2, P_3 + P_4\}$ that has a dimension of 2.
We call a state *a meaningful state* if and only if the elements of the state satisfy all of the following three conditions: I) $t \geq i_1 + i_2 - c$, II) $c \leq min(i_1, i_2)$, III) $i_1 + i_2 - c \leq M$. Condition (I) indicates the minimum number of time slots that is required to have $i_1$ *dof* at $R_1$ and $i_2$ *dof* at $R_2$. Condition (II) states that the dimension of the common knowledge between two receivers cannot be greater than the minimum of the knowledge received by the two receivers, and condition (III) states that the total knowledge received by both receivers cannot be greater than $M$. In the previous example, the set of states defined as $s = (3,3,2,t), \forall t \in \{4,5,...,T\}$ are the meaningful states. The set of absorbing states of our MDP model is defined as $s_{abs} = (M,M,M,t)$, $\forall t \in \{M, M+1,...,T\}$, meaning that if the network is in one of these states it cannot leave the state and a self-transition is performed for every selected action. In other words, the packet transmission process is completed when the network is in an absorbing state.
*Possible Actions* $(a_j)$*:* similar to Chapter 3, we define five actions, $a_1 - a_5$ that cover all possible ways of packet transmission in the network of Fig.4.1, assuming fixed transmission rate and the same modulation for all nodes. We may be able to define more actions if we are allowing for various slot sizes for different modulations and coding schemes, but these are beyond the scope of this work. Action $a_1$ is defined as broadcast from $S$ to $R_1, R_2$. Actions $a_2, a_3$ define the unicast transmissions from $S$ to $R_1$ and $R_2$, respectively. Actions $a_4, a_5$ define cooperation between receivers. More precisely, $a_4$ defines unicast transmission of RLNC packets from $R_1$ to $R_2$ and $a_5$ defines unicast transmission of RLNC packets from $R_2$ to $R_1$. To allow the system to stop transmission after $R_1$ and $R_2$ have both received the $M$ packets, action $a_6$ is defined as “do not transmit".
*Transition Probabilities:* the possible states to which state $(i_1, i_2, c, t)$ can transit to with non-zero probability depends on the action chosen and the total knowledge ($\mathcal{K} = i_1 + i_2 - c$) that is available to both receivers at time $t$. $\mathcal{K}$ indicates the *dof* of the union of the received packets by both receivers. We refer to $p_{x \to y}$ as the probability of transition from state $x$ to state $y$. Note that there are two cases where the state of the network does not change: 1) the packet is not received correctly (is erased by the channel), and 2) the packet is received correctly but it is non-innovative, i.e., the received packet is not linearly independent from previously received packets.

Similar to the previous chapter, we refer to $I_{(x \in X)}$ as the indicator function. For simplicity, $I_{(i_1' = i_1 + k_1, i_2' = i_2 + k_2, c' = c + k_3, t' = t + k_4)}$ is denoted by $I_{(k_1, k_2, k_3, k_4)}$ and $\bar{\varepsilon}_i(t) = 1 - \varepsilon_i(t)$. The

non-zero transition probabilities for the six actions are summarized as follows.
**Action $a_1$ (Broadcast):** when the source broadcasts, it creates different possible state transitions. These can be obtained by combinatorial arguments and we explain the more surprising cases.
On the one hand, assuming that the packet is received without erasure at $R_1, R_2$ and depending on the total knowledge ($\mathcal{K}$) that is available to both. If $\mathcal{K} < M$ and the packet is not erased by any one of the channels, then the dimension of the common knowledge between $R_1, R_2$ is increased by one since both $R_1, R_2$ have received the same packet that is innovative to both of them. If $\mathcal{K} = M$ while none of the receivers has $M$ *dof*, and the packet is not erased, the dimension of the common knowledge between $R_1, R_2$ is increased by two. Let us illustrate this with an example. Assuming that $M = 3$ and the sets of packets received by $R_1, R_2$ until time $t$ are $\{P_1, P_3\}$ and $\{P_2 + P_3\}$, respectively. The network state is then $s = (2, 1, 0, t)$. Now assume that source broadcasts a new coded packet $P_1 + P_2 + P_3$, which adds one *dof* to both $R_1$ and $R_2$. However, the dimension of the common knowledge is increased by two and the system then transits to a new state $s' = (3, 2, 2, t+1)$. The time instance, $t$, is increased by one, since the transition between the two states is performed in one time slot.
On the other hand, if only one of the receivers has $M$ *dof* and the other one has less than $M$ *dof*, then any new coded packet sent by source that is not erased by the channels adds one *dof* to the receiver with *dof* $< M$ and also increases the dimension of the common knowledge by one. This is because the receiver with $M$ *dof* already has enough number of linear independent coded packets to decode the original packets and therefore, any new coded packet transmitted by source is non-innovative to the set of packets received by that receiver. We now summarize all possible transitions with non-zero probabilities for source broadcasting as

- If $\mathcal{K} < M$, $i_1 < M$, $i_2 < M$, then

$$P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = \varepsilon_1(t)\varepsilon_2(t)I_{(0,0,0,1)} + \varepsilon_1(t)\bar{\varepsilon}_2(t)I_{(0,1,0,1)} + \bar{\varepsilon}_1(t)\varepsilon_2(t)I_{(1,0,0,1)} + \bar{\varepsilon}_1(t)\bar{\varepsilon}_2(t)I_{(1,1,1,1)}. \tag{4.1}$$

- If $\mathcal{K} = M$, $i_1 < M$, $i_2 < M$, then

$$P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = \varepsilon_1(t)\varepsilon_2(t)I_{(0,0,0,1)} + \varepsilon_1(t)\bar{\varepsilon}_2(t)I_{(0,1,1,1)} + \bar{\varepsilon}_1(t)\varepsilon_2(t)I_{(1,0,1,1)} + \bar{\varepsilon}_1(t)\bar{\varepsilon}_2(t)I_{(1,1,2,1)}. \tag{4.2}$$

- If $\mathcal{K} = M$, $i_1 = M$, $i_2 \neq M$, then

$$P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = \varepsilon_2(t)I_{(0,0,0,1)} + \bar{\varepsilon}_2(t)I_{(0,1,1,1)}. \tag{4.3}$$

- If $\mathcal{K} = M$, $i_1 \neq M$, $i_2 = M$, then

$$P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = \varepsilon_1(t)I_{(0,0,0,1)} + \bar{\varepsilon}_1(t)I_{(1,0,1,1)}. \tag{4.4}$$

- If $\mathcal{K} = i_1 = i_2 = M$, then $P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = I_{(0,0,0,0)}$.

**Action $a_2$ (unicast from $S$ to $R_1$):** this action could be seen as a special case of broadcasting, $a_1$, in the sense that the erasure probability of the link between $S$ and $R_2$ is one, i.e., $\varepsilon_2 = 1$.
**Action $a_3$ (unicast from $S$ to $R_2$):** this action could be seen as a special case of broadcasting, $a_1$, in the sense that the erasure probability of the link between $S$ and $R_1$ is one, i.e., $\varepsilon_1 = 1$.
**Action $a_4$ (unicast from $R_1$ to $R_2$):** On the one hand, if the number of *dof* at $R_1$ is equal to the common knowledge of $R_1, R_2$, then the coded packets created by $R_1$ cannot add a *dof* to $R_2$. On the other hand, if the number of dof at $R_1$ is greater than the common knowledge of $R_1, R_2$, then any coded packet created by $R_1$ adds one *dof* to the set of received packets by $R_2$ under our high field size assumption. Thus,

- If $i_2 < M$, $i_1 > c$, then

$$P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = \varepsilon_3(t)I_{(0,0,0,1)} + \bar{\varepsilon}_3(t)I_{(0,1,1,1)}. \tag{4.5}$$

- If $i_2 < M$, $i_1 = c$ or $i_2 = M$, $i_1 \neq M$, then $P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = I_{(0,0,0,1)}$.
- If $i_1 = i_2 = c = M$, then $P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = I_{(0,0,0,0)}$.

**Action $a_5$ (unicast from $R_2$ to $R_1$):** it is similar to $a_4$, replacing $i_1$ by $i_2$ and vice versa.
**Action $a_6$ (do not transmit):** if the system is not in an absorbing state, the time instance is increased by one while the *dof* of the receivers does not change, therefore, $P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = I_{(0,0,0,1)}$. If the system is in an absorbing state, the probability of transition into a new state is zero, i.e., $P_{(i_1,i_2,c,t)\to(i_1',i_2',c',t')} = I_{(0,0,0,0)}$.

For a better understanding, we show a schematic of the possible transitions among states in Fig. 4.2. Assuming that the network state at time $t_0$ is $s_i = (i_1, i_2, c, t_0)$, there are 7 possible transitions depending on the action selected by the MDP as shown in Fig. 4.2. The possible states to which state $s_i(i_1, i_2, c, t_0)$ may transit to could be summarized as: $s_{i+1} = (i_1, i_2, c, t_0 + 1), s_{i+2} = (i_1 + 1, i_2, c, t_0 + 1), s_{i+3} = (i_1, i_2 + 1, c, t_0 + 1), s_{i+4} = (i_1 +$

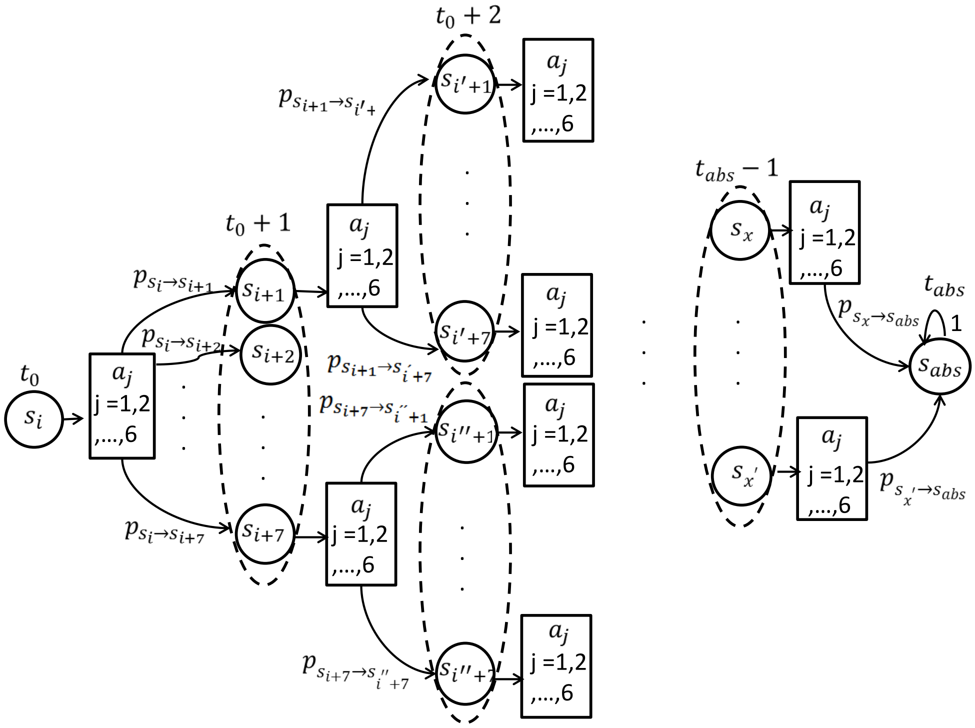

Figure 4.2: schematic of the MDP model; $a_j$ is the selected action, and $s_i$ represents the state of network at time $t_0$.

$1, i_2, c+1, t_0+1), s_{i+5} = (i_1, i_2+1, c+1, t_0+1), s_{i+6} = (i_1+1, i_2+1, c+1, t_0+1), s_{i+7} = (i_1+1, i_2+1, c+2, t_0+1)$. For non-absorbing states, the probability of having a self transition by using any action is zero since the time is always increasing by one and therefore, the network state is changing. The transitions are continued until the network is reached to an absorbing state, $s_{abs}$. A cost function and an optimization algorithm is defined in the following to complete the MDP model.

*Cost Function:* the cost function is defined similar to the cost function defined for time-invariant channels model in Section. 3.3.2. The only difference is the way we define the meaningful and the absorbing states. Therefore, the cost of each transition from state $s$ to state $s'$ by choosing action $a_j$ is defined as

$$C(s, a_j, s') = \begin{cases} 1, & \forall s \in S_T \mid s \neq (M, M, M, t),\ j = 2, \ldots, 5 \\ \beta, & \forall s \in S_T \mid s \neq (M, M, M, t),\ j = 1 \\ \mathcal{D}, & \text{if } s = (M, M, M, t),\ j = 1, \ldots, 5 \\ \mathcal{D}, & \forall s \in S_T \mid s \neq (M, M, M, t),\ j = 6 \\ 0, & \text{if } s = (M, M, M, t),\ j = 6, \end{cases} \tag{4.6}$$

where $C(s, a_j, s')$ is the cost of transition from state $s$ to state $s'$ by choosing action $a_j$ and $S_T$ is the set of all meaningful states. $\beta$ is the cost of one broadcast transmission and in

general, it could be greater than the cost of a unicast transmission. $\mathcal{D}$ is an arbitrary large number that is much greater than $\beta$. We define $\mathcal{D} >> \beta$ to ensure that the MDP does not choose any one of the actions $a_1, a_2, ..., a_5$ if the system is in an absorbing state.
*Optimization Algorithm:* we use the value iteration algorithm (Bellman equations) as defined in Section. 3.3.2 to solve the optimization problem and to minimize the total cost of the transmission of $M$ packets. Note that, the iterative algorithm chooses an action that minimizes the cost of transition into a new state, $C(s, a_j, s')$, and the total cost that we pay starting from that state, $V_k(s')$. Thus, the selected action by the algorithm takes into account both the current state and the future state of channels, and makes an optimal decision for each time slot.

*Remark* 4. We can model the case of two unicasts from source to two receivers, using a similar MDP model. Assume that the two receivers are interested in two different sets of packets (flows) each of size $M$, and mixing packets from different flows is allowed. The packet transmission problem in this case could be seen as a special case of the multicast problem that we discussed. The only difference is that each receiver has to receive at least $2M$ dof (instead of $M$ in previous case) to decode all original packets of the two flows. Therefore, except the absorbing state which is $s_{abs}(2M, 2M, 2M, t)$, all other elements of the MDP model remain the same.

## 4.3 Time-Varying Scenarios Used for Heuristics Evaluation

We define two set-ups to analyse the performance of the proposed heuristics in a time-varying scenario in terms of throughput and reliability. As a relevant example of time-varying environments, we consider an I2V communication in a highway scenario as our first set-up, which reflects the gain of our heuristics in a highly dynamic environment. The second set-up is a three-node wireless test-bed that reflects the performance of our proposed heuristics in a less dynamic environment. Assuming that we are aiming at transmitting $M$ packets in $T$ time slots, the percentage of reliability is calculated as $P_{reliability}\% = \frac{N_{completed}}{N_{experiment}} \times 100$, where $N_{completed}$ is the number of experiments which the packet transmission is completed successfully and $N_{experiment}$ is the total number of experiments. We prune out the cases where packet transmission is not completed and calculate the mean of completion time only for the cases where transmission is completed successfully.

### 4.3.1 First network set-up (I2V)

Considering the network defined in Fig. 4.3-(a), we want to calculate the cost of transmitting $M$ packets from a fixed access point (AP) representing a source (S), to two moving

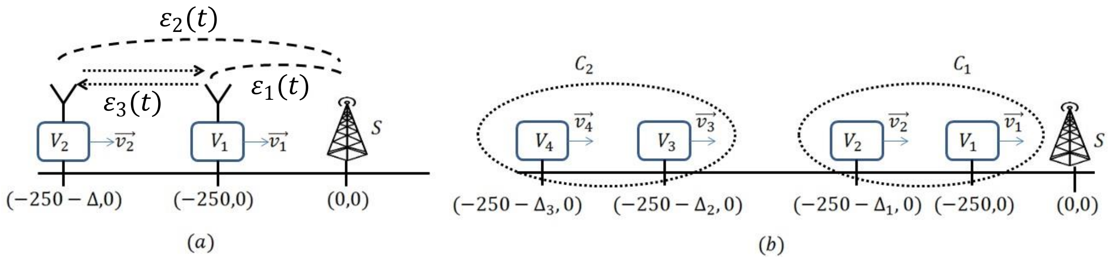


Figure 4.3: Network model for first setup; $\varepsilon_i(t)$ is the erasure probability of channel $i$ at time $t$, (a) the case of two receivers, (b) the case of four receivers divided into two clusters $C_1, C_2$.

vehicles $V_1, V_2$. We use a realistic model of vehicle distances, and their respective travelling speeds, based on stereoscopic aerial photography as proposed in [72]. Authors of [72] show that the distribution of inter-vehicle spacing can be well fitted with an exponential probability distribution with mean 51.58 $m$ and the speed distribution of vehicles is well approximated by a normal probability distribution with mean 106.98 $km/h$ and standard deviation 21.09 $km/h$. We assume that, both vehicles are moving in the same direction and $\vec{v_1}, \vec{v_2}$ show their speeds that are selected randomly and according to the normal distribution. The initial coordinates of the AP and the two vehicles are defined according to Fig. 4.3-(a). $\Delta$ represents the distance between two vehicles that is selected randomly according to the exponential distribution. A time-slotted system is considered with one transmission per time slot. We use a Rayleigh fading channel model and BPSK modulation. According to [73] the average bit error rate (BER) of BPSK modulation is

$$BER = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_b}{1+\bar{\gamma}_b}}\right), \tag{4.7}$$

where $\bar{\gamma}_b$ represents the average SNR per bit that is calculated as $\bar{\gamma}_b = E_b/N_0$. $E_b$ is energy per bit and $N_0$ is the noise power. Since the vehicles are moving, the packet loss/erasure probability changes over time. To find a relationship between the distance of transmitter-receiver and the BER, we assume that $\gamma_b$ is known at a reference distance $(d_{ref})$ and is shown as $\gamma_{ref}$. By using the log-distance path-loss model [74], we can find a relationship between the path loss $(P_L)$, the average received power, and the distance between transmitter and receiver. In other words, $P_L = P_{T_x(dBm)} - P_{R_x(dBm)} = P_{L0} + 10\alpha log(\frac{d}{d_{ref}}) + X_g$, where $P_{L0}$ is the path-loss at a reference distance $(d_{ref})$ in dB, $\alpha$ is the path loss exponent, and $X_g$ is a random variable reflecting the attenuation caused by fading in dB. By calculating the expected value of path loss for an arbitrary distance $d$, we will have $P_L = \frac{P_{Tx}}{P_{Rx}} = k \times (\frac{d}{d_{ref}})^{\alpha}$, where $k$ is a constant value. Assuming that the transmission power is constant for all nodes, the received power by a receiver at arbitrary distance $d$ from the transmitter is calculated as $P_R = P_{R_0}(\frac{d_{ref}}{d})^{\alpha}$, where $P_{R_0}$ is the received power by the receiver at $d_{ref}$ and $P_R$ is the received power by the receiver at distance $d$. Therefore, the average SNR per bit ($\bar{\gamma}_b$) for a pair of transmitter-receiver with distance $d$ is calculated as $\bar{\gamma}_b = \gamma_{ref} \times (\frac{d_{ref}}{d})^{\alpha}$. We set $\gamma_{ref} = 25dB, d_{ref} = 250m, \alpha = 2$ for our numerical analysis. By substituting $\bar{\gamma}_b$ into Eq. (4.7), we can calculate the BER for a pair of transmitter-receiver with distance $d$. We define the packet loss probability of the channel as $1-(1-BER)^n$, which matches the erasure probability of the channel, where $n$ is the packet size in bit and is fixed at $n = 4$ $Kbit$ for our numerical analysis.

### 4.3.2 Second network set-up (Raspberry pi test-bed)

We use a wireless network coding test-bed implemented on Raspberry Pi's in Aalborg University to measure packet losses over time. The goal is to collect statistics about the channel loss as a function of time and to use these measurements to compare the performance of our heuristics versus RLNC broadcasting. All three nodes are fixed during our tests and they are located inside two buildings as shown in Fig. 4.4-(a), but the channels suffer wide variations over time. Two receivers are located inside the same room and the source is located inside a room in another building as shown in Fig. 4.4-(a). $R_1$ has line-of-sight (LOS) to the source, while $R_2$ does not have LOS to the source but there is LOS between $R_1, R_2$. To ensure that our measurements keep track of the correlation

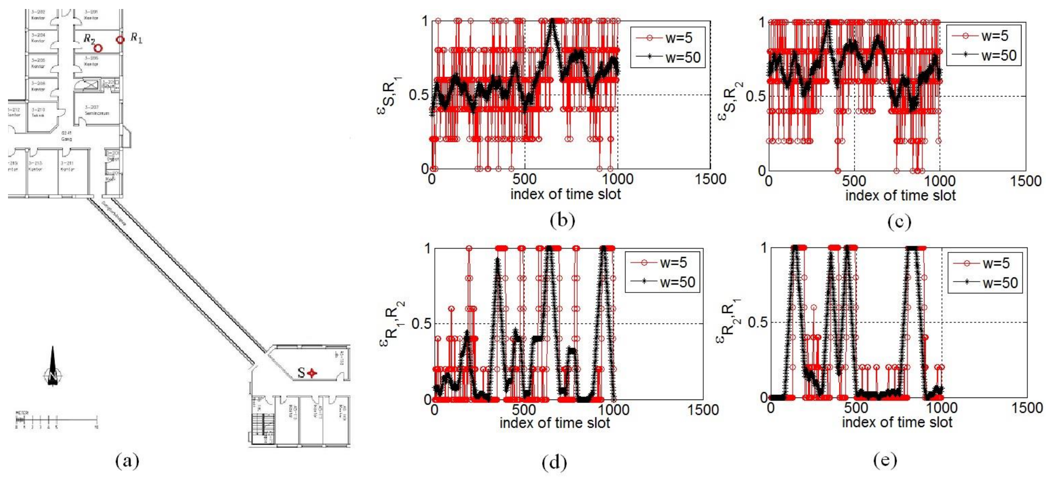


Figure 4.4: (a) Deployment of nodes in the wireless test-bed, (b) erasure prob. between $S, R_1$, (c) erasure prob. between $s, R_2$, (d) erasure prob. between $R_1, R_2$, (e) erasure prob. between $R_2, R_1$.

between erasure probability of the channels, a node transmits for $10s$ while the others record the packet losses. After the $10s$, another node transmits and so on. Each sender broadcasts packets of 1KB every $0.1s$. We record the sequence number of the received packets by each receiver. A moving average filter with window size $w$ is applied to the collected data set to model the erasure probability of the channels. We do not make any assumption about having symmetric channels between receivers, and instead we use the real measurement for the loss probability of the channel between nodes. Therefore, we may have different costs for actions $a_4, a_5$. Fig. 4.4 (b)-(e) show the erasure probabilities obtained by our measurements for $T = 1000$, $w = 5$, and $w = 50$.

## 4.4 Heuristics Evaluation for Time-Varying Environments

In Chapter 3, we proposed the MF and IF heuristics and showed that they have a near-optimal performance in a network with erasure channels that are not changing over time. In this section, we try to evaluate the performance of the proposed MF and IF heuristics under the assumption of time-varying channels and for the two proposed set-ups. In fact, we want to see if the proposed heuristics could be used for time-varying environments as well.

Our analysis is divided into two parts. First, an analysis for small number of packets, where we compare the performance of the proposed heuristics with the performance of the optimal MDP solution obtained in Section 4.2.2 in terms of expected completion cost. This is to validate our claim about near-optimality of the proposed heuristics for time-varying environment. Second, an analysis for large number of packets, where we use mean-field analysis to show the gain of the proposed heuristics compared with RLNC broadcast in terms of expected completion cost and reliability.

### 4.4.1 First Set-up: Performance evaluation for small $M$

Assuming the network in Fig. 4.3-(a), we compare the performance of the MDP, and the proposed heuristics in terms of expected completion time for $M = 5, \beta = 1, T = 100$. A time slot is $0.1s$. We repeat the same experiment for 200 different random pairs of $\Delta, \overrightarrow{v_1}, \overrightarrow{v_2}$ that are selected randomly according to the model we explained before. We calculate $X = \frac{CT_{Heu}}{CT_{MDP}}$ as the ratio between the expected completion times of packet transmission by using a heuristic and the MDP solution. Fig. 4.5-(a) shows the Cumulative Distribution Function (CDF) of $X$ for 200 random tests. It is seen that both heuristics can perform close to the optimal. For example, in case of the IF and for $X = 1.25$, the calculated CDF is 0.95. This means the probability that the completion time by using the IF is within $1dB$ of the optimal completion time is 0.95. Also in case of the MF, the probability of being within $1.3dB$ of the optimal solution is 0.95. These observations state that the proposed heuristics can also provide a close-to-optimal performance in highly dynamic environment.

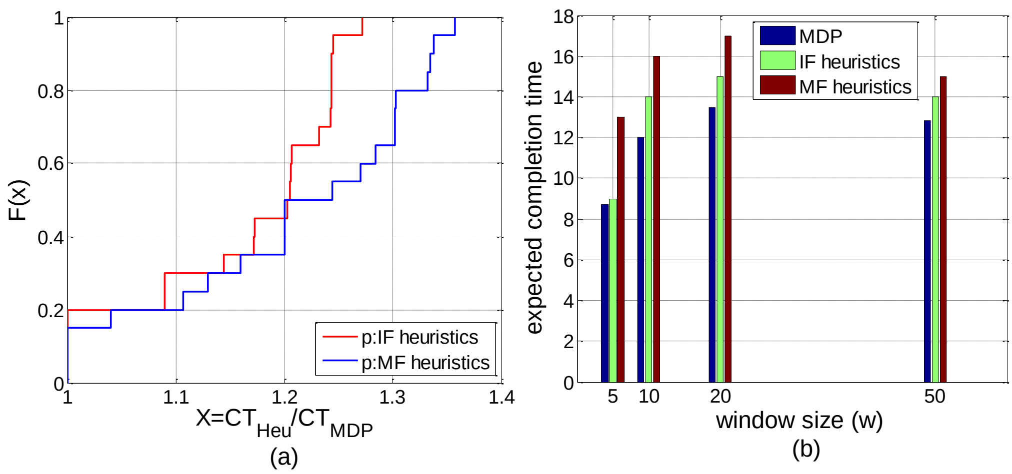


Figure 4.5: (a) CDF of the ratio between completion time of the proposed heuristics and the MDP for the first set-up, $M = 5$, (b) completion time comparison for the second set-up, $M = 5$.

### 4.4.2 First Set-up: Performance evaluation for large $M$

Considering the network defined in Fig. 4.3-(a), we compare the performance of the IF, the MF, and RLNC broadcasting for $50 \leq M \leq 3500$ and $\beta \geq 1$ in terms of the total cost of packet transmission and the reliability. The results are shown in Fig. 4.6. We use $T = 5000$ time slots, where each time slot is $4ms$. We repeat the same experiment for 1000 randomly selected pairs of vehicle speeds and inter-vehicle distances and our results show the average of these samples. Note that by increasing the generation size up to 3500, we just want to see where the performance of our heuristics breaks, and obviously we are not advocating for using generations of this size in practice. The reason is that the overhead per packet used to transmit the coding coefficients increases dramatically, e.g., 3500 bytes for GF($2^8$) according to [75], and the computational complexity would be high. Therefore in practice, we divide a large generation into smaller chunks of packets and use our co-operative approach for each chunk of data. For $M = 50, \beta = 1$, the IF and MF heuristics, respectively, reduce the completion time by a factor of 2 and 1.75, compared to RLNC broadcasting (see Fig.4.6-(a)). Although for larger number of packets, e.g., $M \geq 750$, the gain of heuristics is reduced, the IF heuristic still provides a gain in terms of the percent-age of reliability. The reason of having less gain in case of larger generation size is that the period of time that we are running using a channel estimate that we have made at the beginning of the transmission of the generation is increased. Since our heuristics make decision without knowing the future state of the channels and only based on their current state, the larger the generation size becomes, the less accurate the estimate becomes as time progresses. Ultimately, this results in a lower gain. Fig.4.6-(b) shows that in case of the IF, the percentage of reliability for transmitting 2500 packets is 63%, while in case of RLNC broadcasting it is only 16%. Therefore, in this case, the IF increases the per-

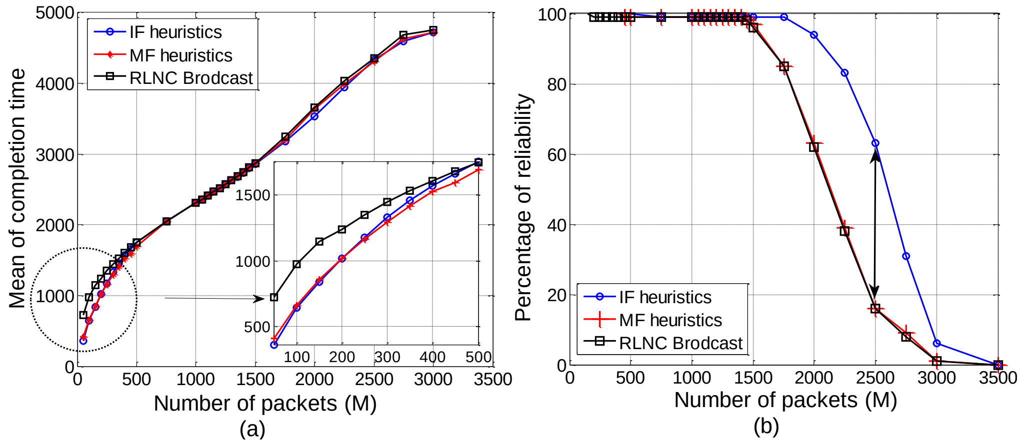


Figure 4.6: Comparison between IF, MF, and RLNC broadcasting for time-varying scenario, $\beta = 1$, and first set-up; a) completion time comparison, b) reliability comparison.

centage of reliability by a factor of 3.98 with respect to RLNC broadcasting. For $\beta > 1$, we calculate the gain of heuristics with respect to RLNC broadcasting (see Fig. 4.7). The network characteristics are defined as we defined for $\beta = 1$. Fig. 4.7 shows that the MF achieves less gain compared with the IF, while the stability of the gain achieved by the MF is more compared to that of the IF by increasing the values of $\beta$. This is because the packet transmission between receivers in MF heuristics is started later than IF heuristics, so in case of the MF having a lower cost for the transmission between receivers has a smaller impact on the total cost compared with the IF. For example, if $M = 50$ and $\beta$ is changed from 1 to 5, the gain of the IF increases from 1.75 to 2.25, while the gain of the MF is changed from 1.65 to 1.8.

We also evaluated the gain of the generalized versions of heuristics, namely GMF RA and GIF RA heuristics as defined in Section. 3.8, for the network defined in Fig. 4.3-(b), where four receivers are divided into two clusters, $C_1, C_2$. Note that the GMF WAIT and GIF WAIT heuristics may not provide very well performance in highly dynamic environments due to the time which the receivers are waiting to start cooperation. Because in this case, the decision that we made about the start time of cooperation at the beginning of packet transmission may not be valid anymore, since the quality of the channels between receivers may change drastically over time.

The initial positions of the source and four vehicles are shown in Fig. 4.3-(b). $\Delta_1, \Delta_2, \Delta_3$ are three randomly selected inter vehicle distances. The speed of each vehicle is selected randomly as we explained before. A time horizon of $T = 10000$ time slots is considered, where each time slot is $0.01 sec$. Fig. 4.8 shows the results of this experiment for 1000 random pairs of vehicle speeds and inter-vehicle distances. We see that the GIF RA and GMF RA heuristics are able to reduce the completion time by a factor of respectively, $1.41, 1.26$ with respect to RLNC broadcast. Fig. 4.8-(b) illustrates that the reliability of

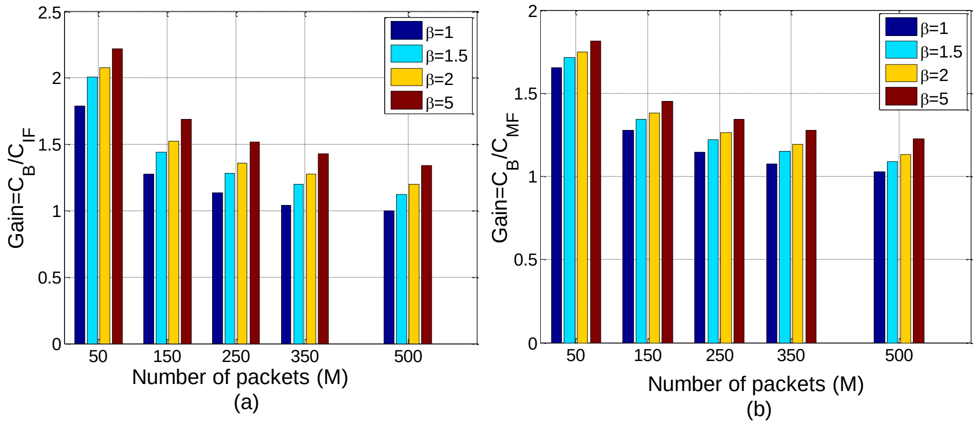


Figure 4.7: Gain of the IF and MF heuristics w.r.t RLNC broadcasting for time-varying scenario, first set-up and varying $\beta$; a) gain of IF heuristic, b) gain of MF heuristic.

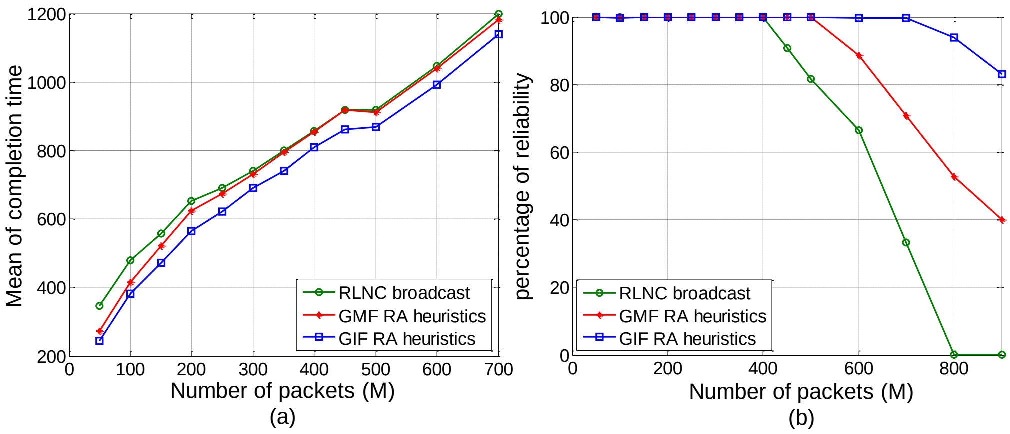


Figure 4.8: Comparison between MF, IF and RLNC broadcast in terms of (a) completion time and (b) reliability for $\beta = 1$, $N = 4$ receivers and the first set-up.

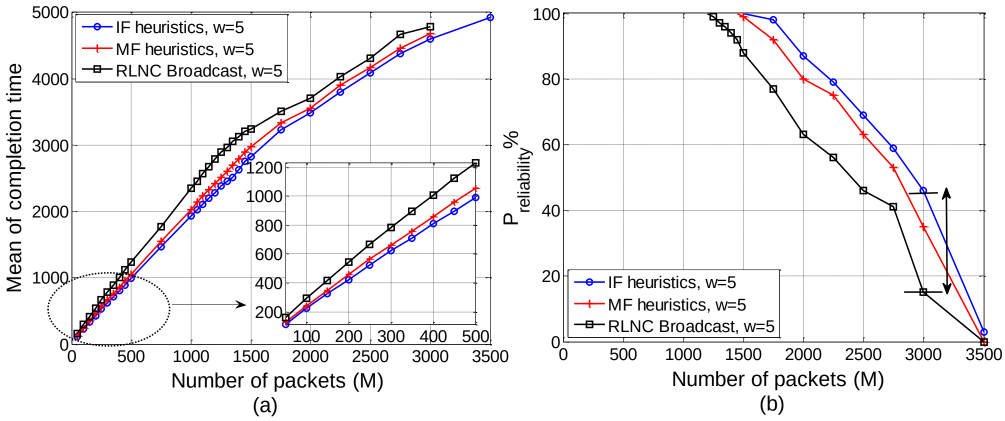


Figure 4.9: Comparison between IF, MF, and RLNC broadcasting for time-varying scenario, second set-up, and $w = 5$; a) completion time comparison, b) reliability comparison.

transmitting 800 packets in 10000 time slots for RLNC broadcast is zero, while in case of GIF RA and GMF RA, it is respectively, $93\%, 53\%$.

### 4.4.3 Second Set-up: Performance evaluation for small $M$

Considering the second set-up, we compare the performance of the MDP solution and the heuristics for $M = 5$. Fig. 4.5-(b), shows the completion time for the optimal MDP solution and the proposed heuristics. We see that the performance of the IF is very close to the performance of the MDP solution for all values of $w$. In case of the MF, its performance is getting closer to the optimal MDP solution performance by increasing the window size. Meaning that the MF has better performance in less dynamic environments.

### 4.4.4 Second Set-up: Performance evaluation for large $M$

Fig. 4.9 shows the mean of completion time and the reliability of the two heuristics and RLNC broadcasting for the second setup. we assume $w = 5$ and $\beta = 1$. For $M = 3000$, the percentage of reliability of the IF is 46%, while for RLNC broadcasting it is only 15%. Meaning that the IF increases the percentage of reliability by a factor of 3 compared with RLNC broadcasting. For smaller $M$, we see that both IF and MF heuristics decrease the expected completion time with respect to RLNC broadcasting. For example, for $M = 50$ the expected completion time by using the IF, the MF and the RLNC broadcasting are respectively, $115.22, 127.01$, and $161.71$. This leads to have a 40% gain for the IF and a 27% gain for the MF in terms of the expected completion time.

## 4.5 Concluding Remarks

In this chapter, the problem of minimizing the total cost of transmitting $M$ packets from a source to two receivers has been solved for a time-varying wireless network by taking advantage of network-coded cooperation between receivers. We modelled the problem as an MDP problem. Then we evaluated the performance of the MF and IF heuristics that were proposed in Chapter 3, in two time-varying network set-ups representing practical time-varying environments, namely, (i) I2V communication in a highway, and (ii) real packet loss measurements for WiFi using Aalborg University's Raspberry Pi test-bed. We have shown that both heuristics can have close-to-optimal performance. A comparison with RLNC broadcasting reveals that the proposed heuristics are able to decrease the completion time by a factor of 2 and increase the percentage of reliability by a factor of 3.98 in time-varying environments. Although our analysis has been done for a network of three nodes, the results of this analysis have significant impact on improving the design of routing protocols in dynamic wireless networks. In fact, the proposed heuristics could be applied to the traditional multi-hop protocols to improve their performance in terms of the cost of packet transmission per single hop.

# Chapter 5

# Optimal Network Coded Relay-Based Multi-casting in the Presence of Active Neighbours

In Chapters 3, 4, we focused on the optimal design of network coded cooperative communications in a network consisting of one source, two receivers, and no helper/relay nodes. Although in that model, a receiver can act as a relay as well, but we neglected the impact of relay nodes in improving the performance of a single transmission link which is important in designing effective multi-hop routing protocols. Thus, in this chapter, we focus on the impact of relay nodes, and study the optimal use of a relay for reducing the transmission time of data packets from a source to one/multiple receivers using network coding. More importantly, we address an effect that is typically overlooked in previous studies: the presence of active transmitting nodes in the neighbourhood of such devices, which is typical in wireless mesh networks.

The remainder of this chapter is organized as follows. Section 5.1 presents the related work and our motivation. Our main contributions are listed in Section 5.2. Section 5.3 states the problem. Section 5.4 presents the MDP model of the problem for both unicast and multicast scenarios. In Section 5.5, we define schemes that are used to compare the performance of the relay approaches and non-relay approaches. Section 5.6 shows the results of performance comparison between the relay-based schemes, non-relay based schemes, and the optimal MDP solution for different scenarios. Finally, Section 5.7 presents the concluding remarks.

## 5.1 Related Work and Motivation

The broadcast nature of wireless channels, which allows potentially all nodes in the transmission range to receive the packets, has opened a series of potential advantages and challenges in the use of the transmission medium in wireless networks. In fact, exploiting relay nodes to improve performance of a single transmission link has been the focus of research under different contexts, but particularly at the physical (PHY) layer, for several decades. The advent of network coding offers a key mechanism to exploit the benefits of a relay with packet–level interactions, instead of tailored PHY layer mechanisms, by providing a richer, controllable and throughput optimal alternative to simply repeating the same data packet from the relay. The use of random linear network coding (RLNC) allows the system to improve performance requiring minimal if any coordination between relay and source. Nodes need only combine data packets linearly in a finite field using coding coefficients drawn uniformly at random from the elements of the field. Recent results focused on the coded erasure relay channel, e.g., [76], [77] have studied both performance benefits as well as where and how much to code in a simple network. [78], [79] investigate the problem of relaying from a physical layer perspective for multiple users and multiple relays. Taking a step further, PlayNCool [80], [81] provided more practical mechanisms for exploiting relays in a wireless mesh network to reinforce links chosen by an underlying routing mechanism. This contrasted with previous approaches, e.g., [82], [83], which focused on defining their own routing scheme. Another interesting feature of [80], [81] is the potential increase in performance due to neighbouring nodes. Inspired by the flow analysis and simulations in [80], [81], we focus on determining the optimal transmission policy to send $M$ data packets from a source, $S$, to two receivers $R_1, R_2$ with the help of a helper/relay, $H$, and in the presence of $X$ active neighbours sharing the same channel. To the best of our knowledge, this is the first in-depth analytical work looking at this problem.

## 5.2 Main Contributions

Seeking to understand the effect of neighbouring nodes on the performance of the packet erasure relay channel in the presence of network coding, we make the following contributions in this chapter:

- **Mathematical Analysis**: we model the problem as a Markov Decision Process (MDP). The cost of packet transmission is defined as the number of time slots that is used to send packets plus the number of time slots that the sender needs to wait in order to have a time slot allocated to it. For simplicity, we assume a dynamic TDMA medium access control (MAC), although random access can also be modelled with our approach albeit with additional complexity. First, we model the problem for the case of one source, one helper, and one receiver. Then, we provide the MDP model for a multicast scenario with two receivers.

- **Proposal of Heuristics:** we propose an extension of the PlayNCool protocol for a multicast scenario that is shown to achieve close-to-optimal performance in the presence of multiple active neighbours.

- **Numerical Results and Comparison to Heuristics:** we calculate the expected completion time for different scenarios, e.g, different number of neighbours, different number of packets, different erasure probabilities of the links between source, relay, and receivers. These results show two key and counter-intuitive results. First, that the judicious use of a relay can provide gains of up to $3.5x$ with respect to the use of the direct link. Second, that the operating region where the relay provides benefits can be significantly extended with respect to the result in [76] when the coded relay network is in the presence of active neighbours. Finally, a comparison between the optimal results obtained by the MDP and the simulation results of PlayNCool is provided showing that PlayNCool provides a close–to–optimal solution for many unicast scenarios. We also show that a simple extension of the PlayNCool protocol can provide a close-to-optimal performance in a multicast scenario.

## 5.3 Problem Statement

We consider a network that consists of one source, $S$, one helper (relay), $H$, and two receivers ($R_1, R_2$), in the presence of $X$ neighbours, that also use the same channel to transmit data packets (See Fig. 5.1-(a)). A time-slotted system is assumed with only one transmission per time slot and no collisions. We assume a genie and fair time division multiple access (TDMA) medium access control, i.e., a TDMA scheme that allows for immediate dynamic allocation of resources based on the users requirements. We model losses between $S$, $H$, $R_1$, $R_2$ as independent, time invariant erasure channels, where there is some probability of losing each transmitted packet. The probability of packet loss is given by $\varepsilon_i$, for the links from $S$ to $R_i$, $\varepsilon_{HR_i}$ for the links from $H$ to receiver $R_i$, and $\varepsilon_{SH}$, for the link from $S$ to $H$. The source is assumed to have $M$ data packets to transmit, namely, packets $p_1, p_2, ..., p_M$. When transmitting, the source and the helper send linear combinations of the contents of their buffer following the rules of RLNC. Similar to Section 3.3.2, RLNC packets at the source are generated by linear combinations of the $M$ original packets using randomly chosen coding coefficients $\alpha_{1,k}, ..., \alpha_{M,k}$ to create the $k$-th coded packet, i.e., $\sum_{i=1}^{M} \alpha_{i,k} p_i$. The helper is recoding the packets of its buffer. The coding coefficients are selected independently and randomly from a Galois field of size $q$, i.e., $GF(q)$, using a uniform distribution over the elements of the field. It is assumed that $q$ is large enough so that any RLNC packet received from the source is independent from previously received packets with very high probability. However, this is not the case

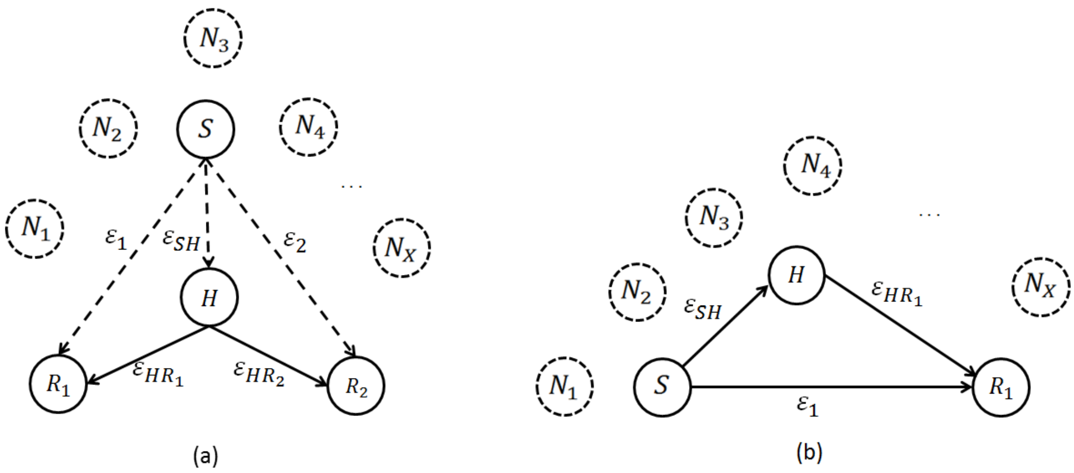


Figure 5.1: A coded packet relay network with neighbors. All nodes are in transmission range of each other and share a single transmission channel, (a) multicast scenario, (b) unicast scenario

for transmissions between $H$ and $R_1, R_2$ because they may share common linear combinations. For now, we only focus on a unicast scenario with one receiver as a special case of the multicast scenario. Fig. 5.1-(b) shows the unicast model of the problem, where a source wants to transmit $M$ data packets to a receiver, $R_1$, with the help of a helper node, $H$. Later, we will show how to extend the MDP model for multicast scenario.

For the network of Fig. 5.1-(b), if the relay can help the source by transmitting coded packets, when is this beneficial? If the erasure probability of the link between $S, R_1$, $\varepsilon_1$, is larger than the erasure probability of the link between $H, R_1$, $\varepsilon_{HR_1}$, it is clear that it is beneficial to ask for help. If $\varepsilon_1$ is lower than $\varepsilon_{HR_1}$, the potential benefits are not as clear. In fact, [76] showed that $\varepsilon_1 < \varepsilon_{HR_1}$ for the specific case of no neighbours ($X = 0$, in our case) is optimally solved without a relay. This means that in an isolated environment with no interference the relay should not be used as it is stealing wireless resources from the source. However, the use of a relay may become beneficial in the presence of neighbours (interferers) in the environment. Although the relay may be using resources that could be allocated to the source, it is inherently providing a larger share among all nodes if the MAC distributes resources equally among the nodes. The heuristics proposed in [80] suggest that this improvement is possible, but the gap between the heuristics and the optimal policy is not addressed. Having these questions in mind, we are interested in finding a packet transmission policy that can minimize the total cost of finishing the transmission of $M$ packets from $S$ to $R_1$ with/without the help of a relay and in the presence of $X$ active neighbours. The cost is defined as the number of active neighbours that use the same channel to transmit plus the number of time slots that we use to transmit packets toward receiver.

## 5.4 MDP Model of Problem

We model the problem as an MDP. First, we provide the MDP model for unicast scenario, and then, we explain how to extend this model for a multicast scenario. For determining the optimal policy in both cases, we assume that we have a Genie system (GS) in which the state information of the network is available per time slot and thus, it can help us to choose the best action.

### 5.4.1 MDP Model for the Unicast Scenario

In the following, we specify the state, possible actions, and transition probabilities of the MDP model for the network defined in Fig. 5.1-(b).

*State Definition:* Using the definition of dof in Definition 4, each state is defined by a triplet $s(i_1, i_H, c)$, where $i_1$ is the number of dof of the received packets at receiver $R_1$, and $i_H$ is the number of dof of the received packets at the helper, $H$. $c$ is the number of dof of $H$ and $R_1$ combined, i.e., the dimension of the common knowledge between $H$ and $R_1$. We define a single absorbing state in this case as being composed by a set of states of the form $s_{abs} = (M, i_H, c)$, where $i_H$ can change from zero to $M$. Similar to the MDP model defined in Section 3.3.2, the meaningful states should satisfy $c \leq min(i_1, i_H)$.

*Possible Actions:* We define actions $a_1, a_2, a_3, a_4$ as all possible ways of transmitting a packet in the network of Fig. 5.1-(b) as follows. In our analysis, we assume that the cost of one unicast communication is equal to the cost of a broadcast. Therefore, we do not consider the unicast actions from $S$ to $H$ or $R_1$ as a separate action.

- Action $a_1$: broadcast from $S$ to $H, R_1$.
- Action $a_2$: unicast from $H$ to $R_1$.
- Action $a_3$: first, broadcast from $S$ to $H, R_1$, then unicast from $H$ to $R_1$ in two consecutive time slots.
- Action $a_4$: do not transmit.

*Transition Probabilities:* The possible states to which state $(i_1, i_H, c)$ can transit to with non–zero probability depends on the action that we choose and also the total knowledge ($\mathcal{K} = i_1 + i_H - c$) that is available to both helper and receiver at time $t$. Similar to our notation in the previous chapters, $I_{x \in X}$ is an indicator function. $I_{(i_1' = i_1 + k_1, i_H' = i_H + k_2, c' = c + k_3)}$ is denoted by $I_{(k_1, k_2, k_3)}$ and $\bar{\varepsilon}_i = 1 - \varepsilon_i$. In order to calculate the transition probabilities between different states, we should note that there are two cases where the state of the network does not change, 1) the packet is not received correctly (is erased by the channel), 2) the packet is received correctly but it is not innovative to the set of received packets at receiver. The non–zero transition probabilities for the 4 possible actions are summarized as follows:

**Action** $a_1$ **(source broadcast):** This action could be seen as a special instant of the broadcast action in Section 3.3.2. The only difference is the absorbing states and that here we assume large value for the field size and the effect of small field size is neglected. Therefore, the transition probabilities could be obtained in similar way with small changes.

When the source is broadcasting, there are different possible state transitions. We will explain the more surprising cases, while the rest can be obtained via combinatorial arguments. On the one hand, assuming that the packet is received without erasure at $H$ and $R_1$ and depending on the total knowledge that is available to both. If the total knowledge is less than $M$ and the packet is not erased by any one of the channels, then the common knowledge between $H, R_1$ is increased by one since both $H, R_1$ have received the same packet that is innovative to both of them. If the total knowledge is equal to $M$ and the packet is not erased, the common knowledge between $H, R_1$ is increased by two. Let us illustrate this with an example. Assuming that $M = 3$ and the set of packets received by $R_1$ and $H$ until now is $P_1, P_3$ and $P_2 + P_3$, respectively. The network state is then $s = (2,1,0)$. Now assume that source broadcasts $P_1 + P_2 + P_3$, which adds one dof to $R_1$ and $H$. However, the common knowledge is increased by two and the system then transits to a new state $s' = (3,2,2)$. On the other hand, if the relay has $M$ dof, then any new coded packet sent by the source adds one dof to the receiver and increases the common knowledge by one. This is because $H$ already has all dof needed to decode the original packets and the common knowledge simply equal to the knowledge at $R_1$. We now summarize all possible transitions with non–zero probabilities for source broadcasting as

- If $\mathcal{K} < M$, $i_1 < M$, $i_H < M$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0)} + \bar{\varepsilon_1}\varepsilon_{SH}\, I_{(1,0,0)} + \varepsilon_1\bar{\varepsilon_{SH}}\, I_{(0,1,0)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,1,1)}.$$

- If $\mathcal{K} = M$, $i_1 < M$, $i_H < M$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0)} + \bar{\varepsilon_1}\varepsilon_{SH}\, I_{(1,0,1)} + \varepsilon_1\bar{\varepsilon_{SH}}\, I_{(0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,1,2)}.$$

- If $\mathcal{K} = M$, $i_1 \neq M$, $i_H = M$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\, I_{(0,0,0)} + \bar{\varepsilon_1}\, I_{(1,0,1)}.$$

- If $i_1 = M$, then $P_{(i_1,i_H,c)\to(i_1',i_H',c')} = I_{(0,0,0)}$.

**Action** $a_2$ **(unicast from** $H$ **to** $R_1$**):** If the number of dof at $H$ is equal to the common knowledge of $H, R_1$, the helper cannot send a packet to $R_1$ that adds one dof to it. On the other hand, if the number of dof at $H$ is greater than the common knowledge, then the packet sent by $H$ adds one dof to the set of received packets by $R_1$ under our high field size assumption. We summarize the transition probabilities as

- If $i_1 < M$, $i_H > c$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_{HR_1}\, I_{(0,0,0)} + \varepsilon_{\bar{H}R_1}\, I_{(1,0,1)}.$$

- If $i_1 < M$, $i_H = c$ or $i_1 = M$, then $P_{(i_1,i_H,c)\to(i_1',i_H',c')} = I_{(0,0,0)}$

**Action $a_3$ (first broadcast, then unicast from $H$ to $R_1$):** This action includes two consecutive phases and constitutes a combination of $a_1$ and $a_2$ occurring in the same transmission round. Starting by state $s$, first we use broadcast to transit to a new state $\hat{s}$ with probability $p_{s\to\hat{s}}$ and then assuming that the system is in state $\hat{s}$, we calculate the transition probability of transition from $\hat{s}$ to $s'$ using action $a_2$ as $p_{\hat{s}\to s'}$. Therefore, the transition probability of going from state $s$ to state $s'$ using action $a_3$ is calculated as $p_{s\to s'} = p_{s\to\hat{s}} \times p_{\hat{s}\to s'}$. Using combinatorial arguments, the transitions are as follows.

- If $\mathcal{K} < M$, $c < i_H < M$, $i_1 < M-1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0)} + \varepsilon_1\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\, I_{(1,0,1)} +$$
$$\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{HR_1}\, I_{(0,1,0)} + \left[\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{\bar{H}R_1} + \bar{\varepsilon}_1\varepsilon_{\bar{S}H}\varepsilon_{HR_1}\right]\times$$
$$I_{(1,1,1)} + \bar{\varepsilon}_1\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,0,0)} + \bar{\varepsilon}_1\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\, I_{(2,0,1)} +$$
$$\bar{\varepsilon}_1\varepsilon_{\bar{S}H}\varepsilon_{\bar{H}R_1}\, I_{(2,1,2)}.$$

- If $\mathcal{K} < M$, $c < i_H < M$, $i_1 = M-1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0)} + \varepsilon_1\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\, I_{(1,0,1)} +$$
$$\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{HR_1}\, I_{(0,1,0)} + \left[\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{\bar{H}R_1} + \bar{\varepsilon}_1\varepsilon_{\bar{S}H}\right]\times$$
$$I_{(1,1,1)} + \varepsilon_{SH}\bar{\varepsilon}_1\, I_{(1,0,0)}.$$

- If $\mathcal{K} < M$, $i_H < M$, $i_1 < M-1$, $i_H = c$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0)} + \varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{HR_1}\, I_{(0,1,0)} +$$
$$\left[\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{\bar{H}R_1} + \bar{\varepsilon}_1\varepsilon_{\bar{S}H}\right] I_{(1,1,1)} + \bar{\varepsilon}_1\varepsilon_{SH}\, I_{(1,0,0)}.$$

- If $\mathcal{K} < M$, $i_H < M$, $i_1 = M-1$, $i_H = c$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0)} + \varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{HR_1}\, I_{(0,1,0)} +$$
$$\left[\varepsilon_1\varepsilon_{\bar{S}H}\varepsilon_{\bar{H}R_1} + \bar{\varepsilon}_1\varepsilon_{\bar{S}H}\right] \times I_{(1,1,1)} + \bar{\varepsilon}_1\varepsilon_{SH} I_{(1,0,0)}.$$

- If $\mathcal{K} = M$, $c+1 < i_H < M$, $i_1 < M-1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0)} + \left[\varepsilon_1\varepsilon_{SH}\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\varepsilon_{SH}\varepsilon_{HR_1}\right] I_{(1,0,1)} + \bar{\varepsilon_1}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\, I_{(2,0,2)} + \varepsilon_1\varepsilon_{\bar{SH}}\varepsilon_{HR_1} \times I_{(0,1,1)} + \left[\varepsilon_1\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\right] I_{(1,1,2)} + \bar{\varepsilon_1}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\, I_{(2,1,3)}.$$

- If $\mathcal{K} = M$, $i_H = M$, $i_1 < M-1$, $i_H > c+1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{HR_1}\, I_{(0,0,0)} + \left[\varepsilon_1\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\varepsilon_{HR_1}\right] I_{(1,0,1)} + \bar{\varepsilon_1}\varepsilon_{\bar{HR}_1}\, I_{(2,0,2)}.$$

- If $i_1 = M$, then $P_{(i_1,i_H,c)\to(i_1',i_H',c')} = I_{(0,0,0)}$.
- If $\mathcal{K} = M$, $i_H = M$, $i_1 = M-1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{HR_1}\, I_{(0,0,0)} + \left[\varepsilon_1\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\right] I_{(1,0,1)}.$$

- If $\mathcal{K} = M$, $i_H = c+1$, $i_1 = M-1$:

$$P_{(i_1,i_H,c)\to(i_1',i_H',c')} = \varepsilon_1\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0)} + [\varepsilon_1\varepsilon_{SH}\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\varepsilon_{SH}]\, I_{(1,0,1)} + \varepsilon_1\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\, I_{(0,1,1)} + \left[\varepsilon_1\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1} + \bar{\varepsilon_1}\varepsilon_{\bar{SH}}\right] I_{(1,1,2)}.$$

**Action $a_4$ (do not transmit):** in this case, the state of the system is not changed, and therefore, $P_{(i_1,i_H,c)\to(i_1',i_H',c')} = I_{(0,0,0)}$.

*Cost Function:* It is assumed that one transmission is done per time slot. Therefore, every time the source or the helper transmit a packet, they have to wait for $X$ time slots to get one time slot assigned for them to transmit their packets again. If both $S, H$ transmit in two consecutive time slots, then the number of time slots that is used is $X+2$ in that transmission round. On the other hand, if only one transmits the number of slots in a round is $X+1$. Fig. 5.2 shows the cost of actions $a_1, a_2, a_3$. This leads to

$$C(s,a_j,s') = \begin{cases} X+1, & \forall s \in S \mid s \neq (M,i_H,c), \\ & \quad j \in 1,2 \\ (X+2), & \forall s \in S \mid s \neq (M,i_H,c), j=3 \\ \mathcal{D}, & for\ s = (M,i_H,c), j \in 1,2,3, \\ \mathcal{D}, & \forall s \in S \mid s \neq (M,i_H,c), j=4, \\ 0, & \text{if } s = (M,i_H,c),\ j=4, \end{cases} \tag{5.1}$$

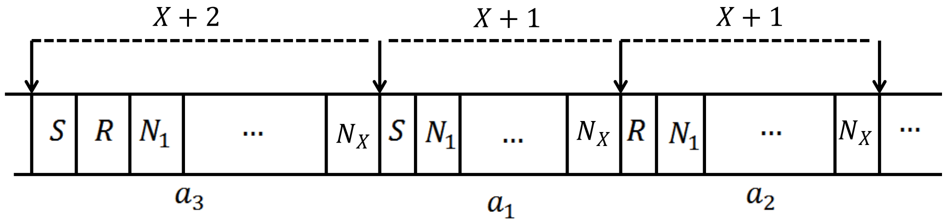


Figure 5.2: Cost (required time slots) of three key actions

where $C(s, a_j, s')$ is the cost of transition from state $s$ to state $s'$ by choosing action $a_j$ and $S$ is the set of all possible states. $\mathcal{D}$ is an arbitrary large number that is much greater than $X$. By defining $\mathcal{D} >> X$, we make sure that the MDP does not choose any one of the actions $a_1, a_2, a_3$ if the system is in the absorbing states, $s(M, i_H, c)$, and it chooses action $a_4$ that has the minimum cost. This leads to stopping the process at the absorbing states.

*Optimization Algorithm:* We use the value iteration algorithm as defined in Section 3.3.2, to solve the optimization problem and to minimize the total cost of the transmission of $M$ packets.

### 5.4.2 MDP Model for the Multicast Scenario

In this section, we explain how to model the problem of optimal packet transmission for the network of Fig. 5.1-(a), where a source wants to transmit $M$ packets to two receivers $R_1, R_2$.

*State Definition:* Each state is defined by five elements $s(i_1, i_2, i_H, c_1, c_2)$, where $i_1, i_2, i_H$ respectively, represent the number of dof of the received packets by $R_1, R_2, H$. $c_1, c_2$ represent the dimension of the common knowledge between $R_1, H$ and $R_2, H$, respectively. Clearly, the meaningful states should satisfy the following conditions: 1) $i_1 + i_H - c_1 \leq M$, 2) $i_2 + i_H - c_2 \leq M$, 3) $c_1 \leq i_1$, 4) $c_1 \leq i_H$, 5) $c_2 \leq i_2$, 6) $c_2 \leq i_H$. Since the goal of multicast transmission is to deliver $M$ packets to both receivers, the absorbing state in this case is shown as $s_{abs} = (M, M, i_H, c_1, c_2)$, where $c_1, c_2, i_H$ can change from zero to $M$ and $c_1 \leq i_H, c_2 \leq i_H$.

*Possible actions:* Using a similar argument as unicast scenario, the possible ways of packet transmission in the network of Fig. 5.1-(a) represent possible actions. Therefore, four actions are defined as follows:

- Action $a_1$: broadcast from $S$ to $R_1, R_2, H$
- Action $a_2$: broadcast from $H$ to $R_1, R_2$
- Action $a_3$: first, broadcast from $S$ to $R_1, R_2, H$, then broadcast from $H$ to $R_1, R_2$, in two consecutive time slots
- Action $a_4$: no transmission

*Transition probabilities:* The possible states to which state $(i_1, i_2, i_H, c_1, c_2)$ can transit to with non–zero probability depends on the action that we choose and also the total knowledge that is available to any pair of relay and receiver at time $t$. The total knowledge of pair $(H, R_x)$ is defined as $\mathcal{K}_x = i_x + i_H - c_x$. For simplicity, we use a similar notation as the previous model. Therefore, $I_{x\in X}$ is used as an indicator function, and $I_{(i_1'=i_1+k_1, i_2'=i_2+k_2, i_H'=i_H+k_3, c_1'=c_1+k_4, c_2'=c_2+k_5)}$ is denoted by $I_{(k_1,k_2,k_3,k_4,k_5)}$, and $\bar{\varepsilon}_i = 1 - \varepsilon_i$. Note that in two main cases, the state of the network does not change: I) none of $R_1, R_2, H$ have received the packet correctly, i.e., the packet was erased, II) the packet that is received by receivers or the helper is not innovative to the set of previously received packets. In the following, we summarize the non-zero transition probabilities for all possible actions.

**Action $a_1$ (Broadcast from $S$ to $R_1, R_2, H$):**

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M$ and $i_1 < M, i_2 < M, i_H < M$:

$$
\begin{aligned}
&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\
&\varepsilon_1\varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,0,0)} \\
&+\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(0,1,0,0,0)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(0,0,1,0,0)} \\
&+\bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(1,0,1,1,0)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(0,1,1,0,1)} \\
&+\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(1,1,1,1,1)}.
\end{aligned}
$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M$ and $i_1 < M, i_2 < M, i_H < M$:

$$
\begin{aligned}
&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\
&\varepsilon_1\varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,1,0)} \\
&+\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(0,1,0,0,0)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(0,0,1,1,0)} \\
&+\bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(1,0,1,2,0)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(0,1,1,1,1)} \\
&+\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(1,1,0,1,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(1,1,1,2,1)}.
\end{aligned}
$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M$ and $i_1 < M, i_2 < M, i_H < M$ :

$$
\begin{aligned}
&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\
&\varepsilon_1\varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,0,0)} \\
&+\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(0,0,1,0,1)} \\
&+\bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{SH}\, I_{(1,0,1,1,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(0,1,1,0,2)} \\
&+\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(1,1,0,0,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{SH}\, I_{(1,1,1,1,2)}.
\end{aligned}
$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_1 < M, i_2 < M, i_H < M$ :

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,1,0)} \\ +\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\bar{\varepsilon_{SH}}\, I_{(0,0,1,1,1)} \\ +\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\, I_{(1,0,1,2,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(0,1,1,1,2)} \\ +\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M$ and $i_1 = M$ and $\mathcal{K}_2 < M, i_2 < M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon_2}\varepsilon_{SH}\, I_{(0,1,0,0,0)} \\ +\varepsilon_2\bar{\varepsilon_{SH}}\, I_{(0,0,1,1,0)} + \bar{\varepsilon_2}\bar{\varepsilon_{SH}} I_{(0,1,1,1,1)}.$$

- If $\mathcal{K}_2 = M$ and $i_2 = M$ and $\mathcal{K}_1 < M, i_1 < M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_{SH}\, I_{(1,0,0,0,0)} \\ +\varepsilon_1\bar{\varepsilon_{SH}}\, I_{(0,0,1,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,0,1,1,1)}.$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_1 = M$ and $i_2 < M, i_H < M$ :

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon_2}\varepsilon_{SH}\, I_{(0,1,0,0,1)} \\ +\varepsilon_2\bar{\varepsilon_{SH}}\, I_{(0,0,1,1,1)} + \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(0,1,1,1,2)}.$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_2 = M$ and $i_1 < M, i_H < M$ :

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_{SH}\, I_{(0,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_{SH}\, I_{(1,0,0,1,0)} \\ +\varepsilon_1\bar{\varepsilon_{SH}}\, I_{(0,0,1,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,0,1,2,1)}.$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_H = M$ and $i_1 < M, i_2 < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\, I_{(0,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2 I_{(1,0,0,1,0)} \\ +\varepsilon_1\bar{\varepsilon_2} I_{(0,1,0,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2} I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_1, i_H = M$ and $i_2 < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_2 I_{(0,0,0,0,0)} + \bar{\varepsilon_2} I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = \mathcal{K}_2 = M$ and $i_2 = M, i_H = M$ and $i_1 < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_1\ I_{(0,0,0,0,0)} + \bar{\varepsilon_1} I_{(1,0,0,1,0)}.$$

- If $i_1 = i_2 = M$ then $P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = I_{(0,0,0,0,0)}$

**Action $a_2$ (Broadcast from $H$ to $R_1, R_2$):** Since the helper recode the encoded packets that it has in its buffer, the probability of sending an innovative packet from helper depends on the dof received by the helper, and the common knowledge between helper and each receiver. More precisely, if $i_H$ is equal to $c_1$, any packet that the helper creates would be a linear combination of the packets that already exist in the buffer of receiver $R_1$. Therefore, $H$ cannot send an innovative packet to $R_1$. A similar argument is made for the case of $i_H = c_2$. If $i_H > c1, i_H > c_2, i_1 < M, i_2 < M$, then under our high field size assumption, any packet created by the helper is innovative to both $R_1, R_2$ with very high probability.

- If $i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_{HR_1}\varepsilon_{HR_2}\ I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_1}\varepsilon_{HR_2} I_{(1,0,0,1,0)} + \varepsilon_{HR_1}\varepsilon_{\bar{H}R_2} I_{(0,1,0,0,1)} + \varepsilon_{\bar{H}R_1}\varepsilon_{\bar{H}R_2}\ I_{(1,1,0,1,1)}.$$

- If $i_1 < M, i_2 < M, i_H < M$ and $i_H = c_2, i_H > c_1$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_{HR_1}\ I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_1} I_{(1,0,0,1,0)}.$$

- If $i_1 < M, i_2 < M, i_H < M$ and $i_H = c_1, i_H > c_2$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_{HR_2}\ I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_2} I_{(0,1,0,0,1)}.$$

- If $i_H = M$ and $i_1 < M, i_2 < M$

$$\begin{aligned}&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_1}\varepsilon_{\bar{H}R_2} I_{(1,1,0,1,1)}, \\ &+\varepsilon_{\bar{H}R_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \varepsilon_{HR_1}\varepsilon_{\bar{H}R_2}\, I_{(0,1,0,0,1)}.\end{aligned}$$

- If $i_1 = M, i_2 < M, i_H < M$ and $i_H > c_2$

$$\begin{aligned}&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_2} I_{(0,1,0,0,1)}.\end{aligned}$$

- If $i_2 = M, i_1 < M, i_H < M$ and $i_H > c_1$

$$\begin{aligned}&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\varepsilon_{HR_1}\, I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_1} I_{(1,0,0,1,0)}.\end{aligned}$$

- If $i_1 = i_2 = M$ then $P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = I_{(0,0,0,0,0)}$.
- If $i_1, i_2, i_H < M$ and $i_H = c_1 = c_2$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = I_{(0,0,0,0,0)}.$$

- If $i_1 < M, i_H < M$ and $i_2 = M$ and $i_H = c_1$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = I_{(0,0,0,0,0)}.$$

- If $i_1 < M$ and $i_2 = i_H = M$

$$\begin{aligned}&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\varepsilon_{HR_1}\, I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_1}\, I_{(1,0,0,1,0)}.\end{aligned}$$

- If $i_1 = M$ and $i_2 < M, i_H < M$ and $i_H = c_2$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = I_{(0,0,0,0,0)}.$$

- If $i_1 = i_H = M$ and $i_2 < M$

$$\begin{aligned}&P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_{\bar{H}R_2}\, I_{(0,1,0,0,1)}.\end{aligned}$$

- If $i_1 = i_2 = M$

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = I_{(0,0,0,0,0)}.$$

**Action $a_3$ (Broadcast from $S$ to $R_1, R_2, H$, then broadcast from $H$ to $R_1, R_2$, consecutively):** this action could be seen as a combination of two previous actions. Therefore, we only explain some of the possible transitions, and the rest can be obtained using combinatorial arguments. We can classify possible transitions based on the total knowledge that is available to the pairs of helper and receivers, i.e., $\mathcal{K}_1, \mathcal{K}_2$, the dof of the packets received by each receiver, and the relationship between the dof of the packets received by the helper and $c_1, c_2$. There are 12 different combinations of these conditions.

1. If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$
2. If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$
3. If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M$
4. If $\mathcal{K}_1 = \mathcal{T}_2 = M, i_1 < M, i_2 < M, i_H < M$
5. If $i_1 = M, \mathcal{K}_2 < M, i_2 < M, i_H < M$
6. If $i_2 = M, \mathcal{K}_1 < M, i_1 < M, i_H < M$
7. If $i_1 = M, \mathcal{K}_2 = M, i_2 < M, i_H < M$
8. If $i_2 = M, \mathcal{K}_1 = M, i_1 < M, i_H < M$
9. If $i_R = M, \mathcal{K}_1 = \mathcal{K}_2 = M, i_1 < M, i_2 < M$
10. If $i_1 = i_H = \mathcal{K}_1 = \mathcal{K}_2 = M, i_2 < M$
11. If $i_2 = i_H = \mathcal{K}_1 = \mathcal{K}_2 = M, i_1 < M$
12. If $i_1 = i_2 = M$

We describe all non-zero transition probabilities of this action in Appendix A. Here, we provide one example for each category which can help to understand how the transition probabilities of this action is calculated.

- *Case (1):* In this case, if the packet sent by $S$ is received without erasure at $R_1, R_2, H$, and the packet sent by $H$ is received without erasure at $R_1, R_2$, and $i_H$ is greater than $c_1, c_2$, then the common knowledge of each pair $(H, R_i)$ is increased by two and $i_1, i_2$ are also increased by two at the end of this transaction. Let us illustrate this case with an example. Assuming that $M = 5$ and the set of packets received by $R_1, R_2, H$ until now are $R_1 : \{P_1 + P_2\}$, $R_2 : \{P_2 + P_3\}$, and $H : \{P_1 + P_2 + P_3\}$. Therefore, the network state is $s(1,1,1,0,0)$. Now assume that the source broadcasts $P_4+P_5$ which definitely, adds one dof to the set of received packets by each one of $R_1,R_2,H$, and increases the common knowledge of each pair $(H,R_i)$ by one. Now the set of packets in the buffer of helper is $H:\{P_1+P_2+P_3,P_4+P_5\}$. Assume that $H$ broadcasts recoded packet $P_1+P_2+P_3+P_4+P_5$ that adds one dof to each one of $R_1,R_2$, and one dof to $c_1,c_2$. Therefore, the system transits to state $s'(i_1+2,i_2+2,i_H+1,c_1+2,c_2+2)$ after these two consecutive broadcast transmissions.

- *Case (2):* If $i_H$ is greater than $c_1+1$ and $c_2$, then the common knowledge of pairs $(H,R_1)$ and $(H,R_2)$ could be at most increased by three and two, respectively depending on the packet reception situation at helper and receivers. For example, assume that source wants to transmit 5 packets, namely, $P_1,P_2,P_3,P_4,P_5$ and the set of the packets received by $R_1,R_2,R$ until now are $R_1:\{P_1,P_3+P_4\}$, $R_2:\{P_2\}$, and $H:\{P_2,P_4,P_5\}$, which satisfy the conditions of case 2. The network state is shown as $s(2,1,3,0,1)$. Now, assume that $S$ broadcasts $P_1+P_2+P_3+P_4+P_5$, which adds one dof to $R_1$, $R_2$, $H$, but it adds two dof to the common knowledge of pair $H,R_1$, and only one dof to the common knowledge of pair $H,R_2$. If $H$ broadcasts $P_1+P_2+P_3$ to $R_1,R_2$, it adds one dof to both $R_1,R_2$ and one dof to the common knowledge of each pair. Therefore the new state of the network would be $s'(4,3,4,3,3)$.
- *Case (3):* This case is symmetric to case (2).
- *Case (4):* If $i_H>c_1+1,i_H>c_2+1$ and $i_1<M-1,i_2<M-1,i_H<M-1$, then the maximum dof that could be added to $R_1,R_2$ is 2 which is corresponds to the case that both receivers receive packets sent by $S,H$ without erasure, while the dof of the common knowledge of the helper and each receiver may be increased by up to three. To understand this case, let us give an example. Assume that $M=5$ and the set of packets received by $R_1,R_2,H$ until now is $R_1:\{P_1,P_2\},R_2:\{P_1+P_2,P_2+P_3\},H:\{P_3,P_4,P_5\}$. Therefore, $\mathcal{K}_1=\mathcal{K}_1=M$ and the network state is $s(2,2,3,0,0)$. Now, assume that $S$ broadcasts $P_1+P_2+P_5$ that adds one dof to $R_1,R_2,H$, and two dof to the common knowledge of both pairs of $H,R_i$. Then, if $H$ broadcasts $P_4+P_5$, it adds one dof to $R_1,R_2$ and one dof to the common knowledge of each pair $H,R_i$. Therefore, the new state of the network would be $s'(4,4,4,3,3)$. As we can see, by having two consecutive broadcast transmissions from $S$ and $H$, the common knowledge of the helper and a receiver may be increased by three.
- *Case (5):* In this case, $R_1$ already has $M$ dof and decoded the original packets. Therefore, $i_1$ cannot change, while the common knowledge between $H,R_1$ may be at most increased by one if the packet broadcast by $S$ is received by $H$. Because any packet sent by the source adds one dof to $H$, and one dof to $c_1$. If $i_H>c_2,i_2<M-1,i_H<M-1$ or $i_H=M-1,i_2<M-1$, then the dof of the

received packets by $R_2$ and also the common knowledge between $H,R_2$ could be increased by two if the packets sent by $S,H$ are received by $R_2$ without erasure. For example, if $M=5$, and the set of packets received by $R_2,H$ until now is $R_2\{P_1+P_2,P_3+P_4\},H\{P_1+P_3,P_2+P_5\}$, respectively, and $R_1$ has received all dof. The network state is $s(5,2,2,2,0)$. Now, assume that $S$ broadcasts $P_1+P_4+P_5$ that is received without erasure at $R_2,H$ and therefore, adds one dof to each and also to the common knowledge of them. The received packet at $H$ also adds one dof to the common knowledge of $H,R_1$, because $R_1$ has all original packets and the packet sent by $S$ already exists in $R_1$. Then, if $H$ broadcasts $P_1+P_2+P_3+P_5$, it adds one dof to $R_2$ and also to the common knowledge of $H,R_2$. Therefore, the new state of the network is $s'(5,4,3,3,2)$.

- *Case (6):* This case is symmetric to case (5).

- *Case (7):* In this case, $R_1$ already collected $M$ dof and $H,R_2$ jointly have $M$ dof. Therefore, if the packet sent by $S$ is received without erasure at $R_2,H$, it adds one dof to $R_2,H$ and two dof to the common knowledge of $R_2,H$. Also if the dof of the packets received by $H$ after source transmits is greater than the common knowledge between $H,R_2$, then the packet sent by $H$ can add one dof to $R_2$ and also one dof to the common knowledge of $H,R_2$.

- *Case (8):* It is symmetric to case (7).

- *Case (9):* In this case, $H$ already has $M$ dof and therefore, the packet sent by $S$ can add one $dof$ to the common knowledge of each pair of $H,R_i$ and one dof to the packets received by a receiver. Therefore, in this scenario, the transition with maximum number of added dofs would be from state $s(i_1,i_2,i_R,c_1,c_2)$ to state $s'(i_1+2,i_2+2,i_R,c_1+2,c_2+2)$ depending on the reception of the packets at receivers.

- *Case (10):* In this case, both $H,R_1$ already collected $M$ dof, therefore, only the dof of $R_2$ and its common knowledge with $H$ may change. If the dof of $R_2$ is less than $M-1$ and the packets sent by $S,R$ are received without erasure at $R_2$, then the dof of $R_2$ is increased by two. Also the common knowledge between $H,R_2$ is increased by two at the end of packet transmission. This is because the packet sent by $S$ already exists in $H$ and therefore, it adds one dof to the common knowledge of $H,R_2$ and the packet sent by $H$ also adds one dof. Therefore, the the transition with maximum number of added dofs would be from state $s(i_1,i_2,i_R,c_1,c_2)$ to state $s'(i_1,i_2+2,i_R,c_1,c_2+2)$ depending on the reception of the packets at $R_2$.

- *Case (11):* This is symmetric to case (10).

- *Case (12):* In this case, both receivers already received $M$ dof and therefore, the network is in the absorbing state and it can not leave its current state.

Note that we explained the extreme cases with maximum number of added dofs. The remaining cases could be extracted from the previous examples with combinatorial arguments.

*Cost Function:* We assume that one transmission is performed in one time slot. Using a similar argument as we did for the unicast scenario, the cost of transition from state $s$ to state $s'$, $C(s,a_j,s')$, by choosing action $a_j$ in a multicast scenario is defined as:

$$C(s,a_j,s') = \begin{cases} X+1, & \forall s \in S \mid s \neq (M,M,i_H,c_1,c_2), \\ & \qquad j \in 1,2 \\ (X+2), & \forall s \in S \mid s \neq (M,M,i_H,c_1,c_2), \\ & \qquad j = 3 \\ \mathcal{D}, & for\ s = (M,M,i_H,c_1,c_2), \\ & \qquad j \in 1,2,3, \\ \mathcal{D}, & \forall s \in S \mid s \neq (M,M,i_H,c_1,c_2), \\ & \qquad j = 4, \\ 0, & \text{if } s = (M,M,i_H,c_1,c_2),\ j = 4, \end{cases} \tag{5.2}$$

where $S$ is the set of all meaningful states, and $\mathcal{D}$ is an arbitrary large number that is much greater than $X$. By defining $\mathcal{D} >> X$, we are pushing the MDP to choose action $a_4$ at the absorbing states that has the minimum cost and leads to stop the process at the absorbing states.

*Optimization Algorithm:* Similar to unicast, we use the value iteration algorithm to solve the optimization problem.

## 5.5 Comparison Schemes

Now that we have the optimal relay-based solution provided by the MDP, we can compare the performance of the relay approaches, with the performance of a non–relay approach in the presence of active neighbours and using RLNC. In the following, we explain the schemes that we use for unicast and multicast scenarios.

### 5.5.1 Schemes used for Unicast

The following three schemes are compared.

1. *MDP*: The MDP scheme is the optimal solution to the problem that we have discussed before and is computed as discussed in Section 5.4.

2. *PlayNCool:* The PlayNCool scheme uses a simple heuristic to transmit packets opportunistically. The source starts broadcasting until the helper receives a reasonable number of dof (but not enough to decode) before it starts to send. When the helper starts sending, it will also listen to transmissions from the source to gather additional dofs. Both helper and source transmit RLNC packets until the receiver receives enough dof to decode. The number of broadcast transmissions before helper starts sending, $r$, depends on the erasure probabilities of the channels. This means that the helper makes decisions based only on knowledge of its own state and channel statistics, but not on the receiver state. If the helper is close to source and far from receiver, i.e., $(1-\varepsilon_{SH}) \times \varepsilon_1 > 1-\varepsilon_{HR_1}$, $r$ is calculated as $r = \frac{1}{(1-\varepsilon_{SH})\varepsilon_1}$. If the helper is closer to receiver, i.e., $(1-\varepsilon_{SH}) \times \varepsilon_1 \leq 1-\varepsilon_{HR_1}$, the number of transmissions before helper starts sending is calculated as [80]:

$$r = \frac{-M.C(\varepsilon_{SH}, \varepsilon_1, \varepsilon_{HR_1})}{D(\varepsilon_{SH}, \varepsilon_1, \varepsilon_{HR_1}) - (1-\varepsilon_1).C(\varepsilon_{SH}, \varepsilon_1, \varepsilon_{HR_1})}, \tag{5.3}$$

where $C(\varepsilon_{SH}, \varepsilon_1, \varepsilon_{HR_1}) = (-1+\varepsilon_{HR_1}+\varepsilon_1-\varepsilon_{SH}.\varepsilon_1)$ and $D(\varepsilon_{SH}, \varepsilon_1, \varepsilon_{HR_1}) = (2-\varepsilon_1-\varepsilon_{HR_1}).(\varepsilon_1-\varepsilon_{SH}.\varepsilon_1)$. Therefore, the helper starts transmission when it receives $(1-\varepsilon_{SH}) \times r$ packets.

3. *Non-relay approach:* In this scheme, the helper is not used, and basically source is transmitting RLNC packets to the receiver until it gets $M$ dof.

### 5.5.2 Schemes used for Multicast

In case of multicast scenario, the optimal MDP solution is compared with two extended versions of PlayNCool heuristic that was originally proposed for unicast sessions. Our proposed heuristics use a similar concept to transmit packets with the help of relay toward both receivers. We explain the two versions of PlayNCool that could be used for a multicast session.

1. *Max-PlayNCool heuristic:* Considering each pair of helper-receiver, $(H, R_i)$, we calculate $r_i$, as the required number of broadcast transmissions from $S$ before the helper starts sending to receiver $R_i$, as it was calculated by PlayNCool. The helper starts broadcasting coded packets when it receives $(1-\varepsilon_{SH}) \times r_{max}$ packets, where $r_{max} = max(r_1, r_2)$.

2. *Min-PlayNCool heuristic:* This is similar to the previous version, and the only difference is that the helper starts transmitting after getting $(1-\varepsilon_{SH}) \times r_{min}$ packets, where $r_{min} = min(r_1, r_2)$. Therefore, in this case, the helper is activated earlier compared to the Max-PlayNCool heuristic.

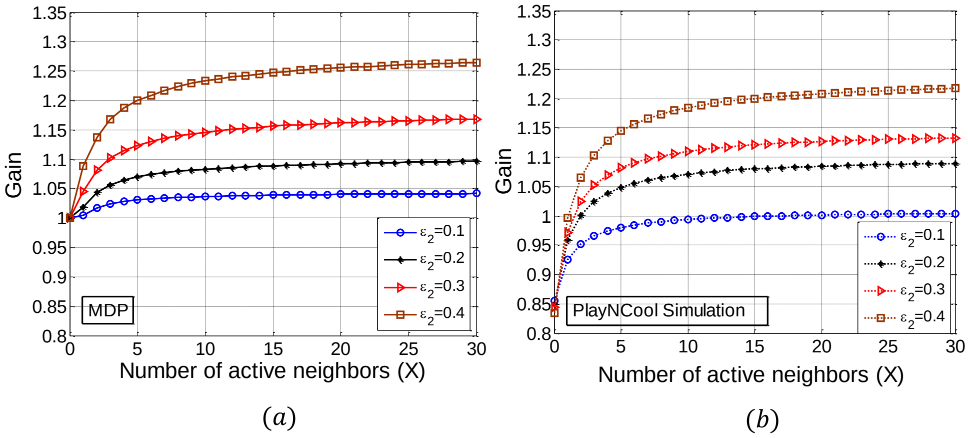


Figure 5.3: Comparison between MDP, and PlayNCool simulation for $\varepsilon_{SH} = 0.2, \varepsilon_{HR_1} = 0.8, M = 10$ and different number of active neighbours, (a) MDP, (b) PlayNCool

3. *RLNC Broadcast:* In this scheme, helper is not used, and the source simply broadcasts RLNC packets until both $R_1, R_2$ receive $M$ dof.

## 5.6 Numerical Results

In this section, we compare the performance of non-relay based schemes, PlayNCool heuristic, and its extended versions with the optimal MDP solution. We use the C++ KODO library [70] to simulate the PlayNCool, Max-PlayNCool, and Min-PlayNCool heuristics.

### 5.6.1 Unicast Scenario

We consider three scenarios to analyse the effect of different parameters of the network on the gain of coded packet relay networks: a) $M, X$ are fixed while erasure probability of channels are varied, b) erasure probability of channels and $M$ are fixed while $X$ is varied, and c) erasure probability of channels and $X$ are fixed while $M$ is varied. The gain in the presence of $X$ active neighbours is defined as the completion time of sending $M$ packets from $S$ to $R_1$ without helper and using non-relay approach, $(CT_{WH})$, divided by the completion time of a helper approach $(CT_H)$ that is calculated by simulation or the MDP solution:

$$Gain = \frac{CT_{WH}}{CT_H}. \tag{5.4}$$

*The effect of erasure probabilities:* We investigate different scenarios to validate our claim that a *crowded room* (i.e., active neighbours) allows the relay to provide additional benefits. First, we consider the case where the erasure probability of the channel between

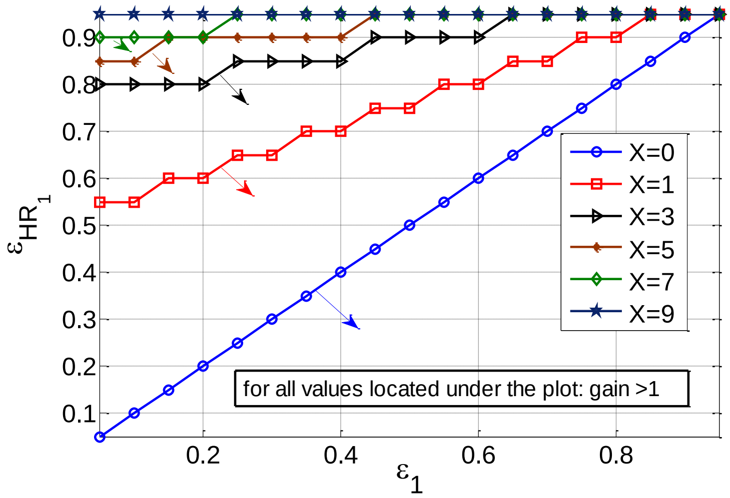


Figure 5.4: The map of possible area of getting benefit of using relay for $\varepsilon_{SH} = 0.2, M = 10$ and different values of $\varepsilon_1, \varepsilon_{HR_1}, X$: pairs of $(\varepsilon_1, \varepsilon_{HR_1})$ under the curve of $X$ provide gain$> 1$, i.e., there is a gain of using the relay

$H$ and $R_1$ is more than the erasure probability of channel between $S$ and $R_1$, which was shown in [76] to require no relay to achieve optimal performance (no other active nodes). Fig. 5.3 illustrates that the use of the relay can be beneficial if there are active neighbour nodes in the system. This corresponds to cases with a gain larger than 1. The MDP solution demonstrates that even a small number of neighbours is sufficient to make the use of a relay attractive.

Fig. 5.3 also shows that PlayNCool does not provide a good solution for this region until there is a large number of active nodes, suggesting that improvements are needed in the heuristics of [80, 81]. However, when enough neighbour nodes are active, the performance of PlayNCool comes closer to the performance of the optimal MDP solution. Fig. 5.3 shows that even a poor link between $H$ and $R_1$ ($\varepsilon_{HR_1} = 0.8$ in this case) can help in decreasing the time to complete the transmission of $M = 10$ packets by around 40 %.

In order to have a better understanding of the effect of neighbour nodes in the usefulness of a relay, we illustrate the operating region where the relay provides benefits. This useful operating region for the erasure probabilities of the links between $S, R_1$ ($\varepsilon_1$) and $H, R_1$ ($\varepsilon_{HR_1}$) is defined for each $X$ value as the area under the curve (pointed by an arrow) in Fig. 5.4. In other words, the relay provides gains for pairs of $(\varepsilon_1, \varepsilon_{HR_1})$ that are located under the curve for each $X$. The curves were calculated using the MDP solution for $X = 0,1,3,5,7,9$ and different pairs $(\varepsilon_1, \varepsilon_{HR_1})$. Fig. 5.4 for the case of $X = 0$, which is the same as having no neighbours in the network, confirms the result in [76]. That is, if $\varepsilon_1 < \varepsilon_{HR_1}$ there is no gain of using relay. By increasing the number of active neighbours, we increase the region where we get benefits of using a relay. Even a single neighbour, i.e., $X = 1$, provides a significant increase in the useful operating region. For $X = 9$, es-

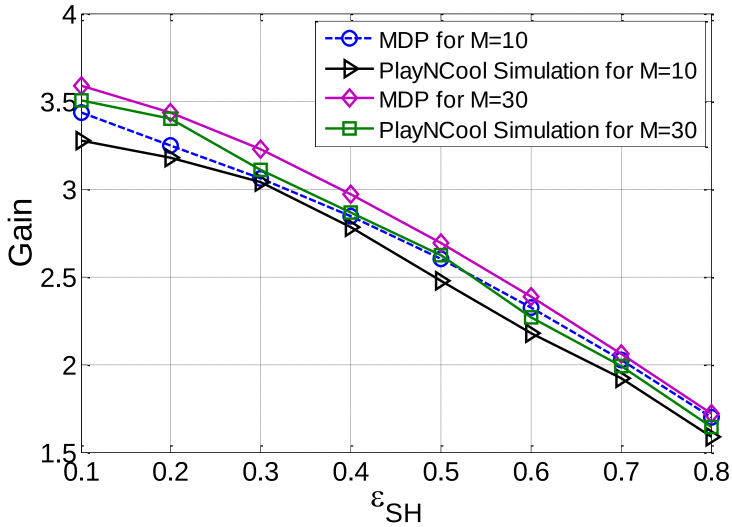


Figure 5.5: Gains of MDP and PlayNCool simulation for $\varepsilon_2 = 0.8, \varepsilon_3 = 0.3$, $X = 5$, and different values of $\varepsilon_1$ and $M$

sentially any pair $(\varepsilon_1, \varepsilon_{HR_1})$ benefits of using a relay, as shown in Fig. 5.4. Finer grained results can be computed using a larger number of points, but the key result still holds: the presence of neighbours makes the relay useful in a wider range of channel conditions.

Second, we consider the case where the link between $H, R_1$ is better than the link between $S, R_1$. We assume that $\varepsilon_1 = 0.8$ and there are $X = 5$ active neighbours in the network. Fig. 5.5 shows a similar experiment for the case where $\varepsilon_{HR_1} = 0.3$ and $\varepsilon_{SH}$ is changed for both $M = 10$ and $M = 30$ packets. Fig. 5.5 shows that by increasing the erasure probability of the channel between $H, R_1$, the gain of relay approaches decreases but it is still greater than one. This means that even if the channel between $H$ and $R_1$ is not substantially better than the one between $S$ and $R_1$, the presence of active neighbours makes the use of a relay (helper) beneficial to speed up the packet transmission process. Also, Fig. 5.5 shows that by increasing the value of $M$, the gap between the gain calculated by the MDP and the simulation is decreased. This is explained because PlayNCool assumes that $H$ is always sending innovative packets to $R_1$, while this is not always true as we have shown in the MDP analysis. By increasing the number of packets, the probability of sending innovative packets increases and therefore, the performance of PlayNCool is closer to the MDP solution.

*The effect of number of active neighbours:* We assume that $\varepsilon_{SH} = 0.3, \varepsilon_1 = 0.8, \varepsilon_{HR_1} = 0.5$ and $M = 10$ for the network depicted in Fig. 5.1-(b). In order to see the effect of network traffic on the gain of the relay approaches, we change the number of active neighbours that are competing to access the same channel. Fig. 5.6 presents the gain of PlayNCool protocol with the gain of the optimal MDP solution for 0 to 29 neighbours. By

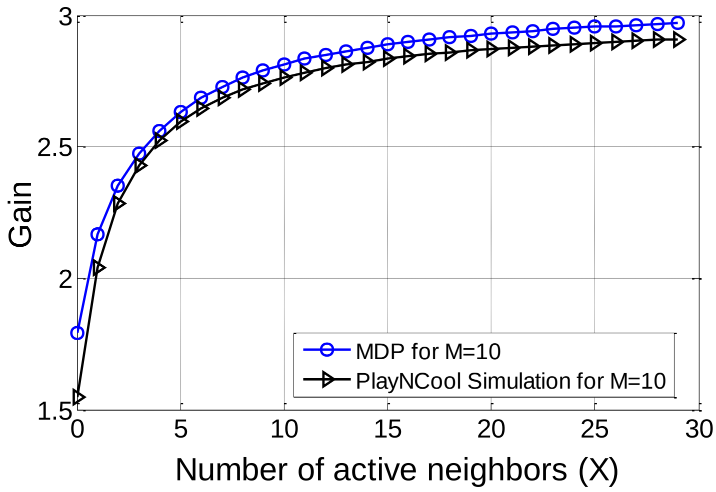


Figure 5.6: Gains of MDP and PlayNCool simulation for $\varepsilon_{SH} = 0.3, \varepsilon_1 = 0.8, \varepsilon_{HR_1} = 0.5, M = 10$ packets, and different number of active neighbours $(X)$

increasing the number of interfering nodes, the gain of using a relay approach increases. Fig. 5.6 shows that the gap between the PlayNCool heuristic and optimal MDP solution is below 10%, which is quite impressive since PlayNCool does not assume perfect knowledge of the system state.

*The effect of number of packets (M):* We assume that $\varepsilon_{SH} = 0.3, \varepsilon_1 = 0.8, \varepsilon_{HR_1} = 0.6$ and $X = 5$ active neighbours. We change the number of packets that are transmitted from $S$ to $R_1$. Fig. 5.7 compares the gains of PlayNCool and the MDP solution with respect to the non-relay approach (direct link) for $M$ changing from 5 to 30. By increasing the number of packets, the gain of both MDP and PlayNCool increases while their gap decreases.

### 5.6.2 Multicast Scenario

We compare the performance of the optimal MDP solution, the proposed heuristics, and RLNC broadcast in terms of completion time. Two scenarios are considered to evaluate the effect of different parameters on the gain of coded packet relay networks: (1) $M$, $X$ are fixed and the erasure probabilities are varied, (2) erasure probabilities and $M$ are fixed and $X$ is varied.

*The effect of erasure probabilities:* We assume that $M = 10, X = 10, \varepsilon_2 = 0.6, \varepsilon_{SH} = 0.4, \varepsilon_{HR_1} = 0.3, \varepsilon_{HR_2} = 0.4$ and the erasure probability between $S, R_1$ is varied for the network of Fig. 5.1-(a). Fig. 5.8 shows the completion time of transmitting 10 packets to two receivers $R_1, R_2$, using different schemes, namely, MDP, Max-PlayNCool, Min-PlayNCool, RLNC broadcast. It is seen that for all values of $\varepsilon_1$, using a coded relay-based approach outperforms RLNC broadcast in terms of completion time. Even for the case that $\varepsilon_1$ is less than $\varepsilon_{HR_1}$, using a relay in the presence of active neighbours provides

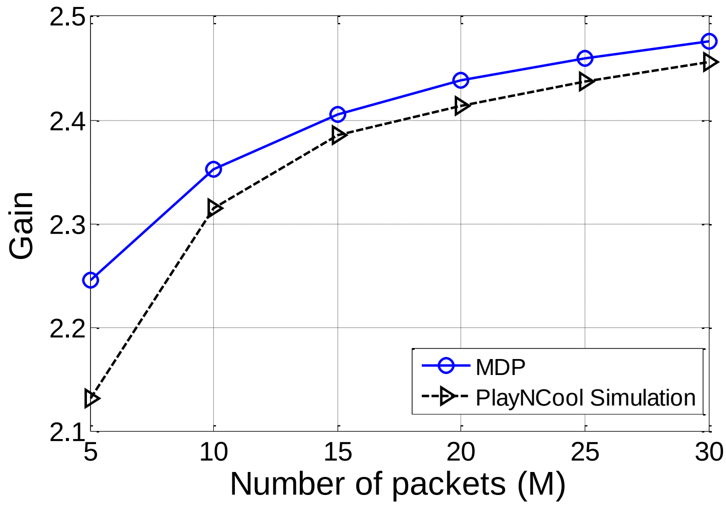


Figure 5.7: Gains of MDP and PlayNCool simulations for $\varepsilon_{SH} = 0.3, \varepsilon_1 = 0.8, \varepsilon_{HR_1} = 0.6, X = 5$ and different $M$

a gain of 1.68 compared with no-relay schemes. We also see that both Min-PlayNCool and Max-PlayNCool heuristics can provide a close-to-optimal performance for a network of one source, one helper, and two receivers. We also see that for this specific scenario, the Min-PlayNCool heuristic has a better performance compared with Max-PlayNCool heuristic. This somehow states that if the helper is activated earlier, it may lead to a better performance depending on the network characteristics.

*The effect of number of active neighbours:* We assume that $M = 10, \varepsilon_1 = \varepsilon_2 = 0.6, \varepsilon_{SH} = 0.4$ and $X$ is changing from 0 to 30. We also change the values of $\varepsilon_{HR_1}, \varepsilon_{HR_2}$. For simplicity, it is assumed that $\varepsilon_{HR_1} = \varepsilon_{HR_2}$. Fig. 5.9 shows the completion time for different schemes. It is seen that for $X = 0$, the gap between the completion time of the relay-based and non-relay based schemes is small, while by increasing the number of active neighbours from $X = 0$ to $X = 30$, this gap increases drastically. For example, in case of $X = 30$, using a relay-based approach can decrease the completion time by a factor of 1.95 compared to RLNC broadcast. Interestingly, we see that for $\varepsilon_{HR_1} = \varepsilon_{HR_2} = 0.8$, the MDP and the proposed heuristics can reduce the completion time by up to 1.33 times. This states that even for the case where the channels between the helper and the receivers are not better than the direct channels between $S$ and the receivers, using a relay could be still beneficial in the presence of active neighbours. Similar to the previous scenario, we see that for all values of erasure probabilities, the proposed heuristics can provide a close-to-optimal performance.

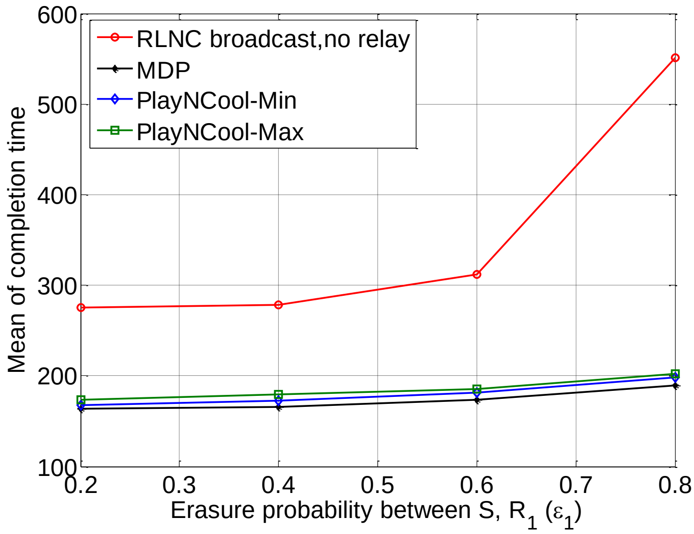


Figure 5.8: Comparison between the MDP, proposed heuristics, and RLNC broadcast in terms of completion time, $M = 10, X = 10, \varepsilon_2 = 0.6, \varepsilon_{SH} = 0.4, \varepsilon_{HR_1} = 0.3, \varepsilon_{HR_2} = 0.4$

## 5.7 Concluding Remarks

In this chapter, we determined the optimal policy to minimize the transmission time of $M$ packets from a source to two receivers in the presence of $X$ active neighbours by using RLNC and a relay approach. The problem has been modelled using an MDP, and it was solved using value iteration optimization algorithm for both unicast and multicast sessions. We compared the performance of the optimal MDP solution to that of the PlayNCool protocol proposed in [80], [81] in terms of the completion time for a transmission of $M$ packets for different unicast scenarios, e.g, different number of active neighbours, different number of packets, and different channel conditions. We also proposed two extended versions of PlayNCool protocol for multicast scenarios that were shown to provide close-to-optimal performance in the presence of active neighbours. We showed that using a relay in the presence of active neighbours is beneficial even if the channel from relay to receivers is not better than the channel between source and receivers. Our numerical results show that in systems with a fair medium access control mechanism (MAC), the use of a relay in a crowded medium brings forth considerable and unforeseen improvements, including up to 3.5$x$ gains in terms of throughput compared to using only the direct link in some of our examples, and a considerable extension of the operating region where using a relay is beneficial. These results have a great impact on designing locally optimized network coding protocols for wireless mesh networks.

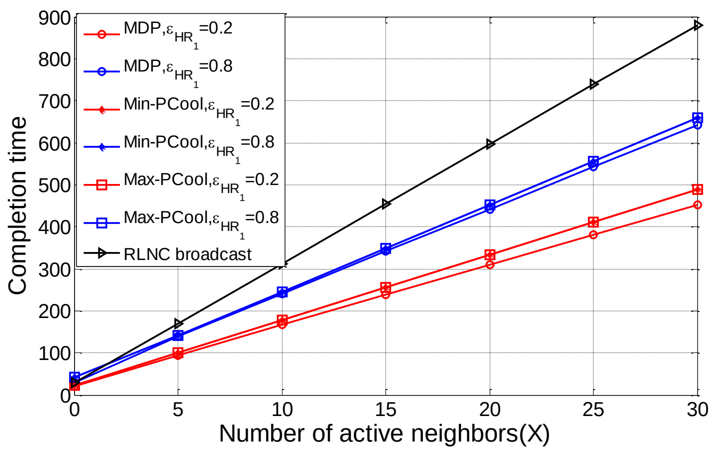


Figure 5.9: Comparison between the MDP, proposed heuristics, and RLNC broadcast in terms of completion time, $M = 10, \varepsilon_1 = \varepsilon_2 = 0.6, \varepsilon_{SH} = 0.4$

# Chapter 6

# Conclusions and Future Directions

## 6.1 Summary

Considering the large potential of network coding and cooperative communications for improving the performance of wireless networks, we focused on finding a comprehensive solution that can minimize the total cost of packet transmission in dynamic mesh networks. To this end, we have studied the protocol design problem in two levels. In the first level, we proposed a protocol that creates multiple paths between source and destinations with minimum number of active nodes, without looking at the dynamics of the active nodes. The proposed protocol combines the idea of network coding with geographic routing and proposed to be used in dynamic wireless mesh networks such as VANETs, where the geographic information of the nodes are available. Through theoretical analysis of the proposed protocol, we exhibited the role that network coding can have in reducing the number of active nodes, the active area of packet transmission, and the complexity of the path selection in unicast and multi-cast scenarios. Unlike the previous path selection algorithms for multicast scenarios, where multiple node-disjoint paths have to be created to achieve the capacity of the network, we have shown that there is no need to search for node-disjoint paths if network coding is used. It was also shown that network coding is able to achieve the maximum capacity of the network while providing a path selection algorithm with linear complexity compared to the supra-linear complexity that the traditional path selection algorithms have. Although simple, this idea can have a large impact on the design of next generation multicast protocols for wireless mesh networks.

In the second level, our focus was on the optimal policies for packet transmission over the created paths. We set out to design locally optimized network coded communication protocols by analysing a family of problems that may happen in an existing multi-hop routing protocol. Two simple scenarios have been analysed by using Markov Decision Process. First, a network of one source and two receivers has been studied where the source is aiming at transmitting $M$ packets to the two receivers over half-duplex erasure channels. Both time-varying and time-invariant channels have been studied. We showed using theory and real-world implementation, that sending recoded packets among receivers can have a significant impact on reducing the completion time, and increasing the throughput. This improvement comes at the cost of sending few feedback messages to start cooperation at the appropriate time. According to the optimal MDP solution, cooperation between receivers has to be started at the time that the two receivers jointly have enough coded packets to decode original packets or one of the receivers already decoded the original packets. For a general network with one source and $N$ receivers, we proposed extended versions of the heuristics by dividing $N$ receivers to $N/2$ clusters, and starting cooperation between two receivers of each cluster at the appropriate time. Our real-world implementation results state that the proposed heuristics have close-to-optimal performance. For a network with four receivers, it was shown that enabling cooperation amongst pairs of receivers can decrease the completion time by up to 4.75, while delivering 100% of the 10000 generations transmitted compared to RLNC broadcast delivering only 88% of them. Besides the benefits that our proposed heuristics might bring in designing optimal multicast routing protocols, they could be very useful in improving the performance of wireless applications in which multiple users are interested in receiving the same content from a common source, e.g., video streaming applications, D2D communication, and etc.

The second scenario consists of one source, one relay node, and two receivers with time-invariant erasure channels, where the source wants to transmit $M$ packets to both receivers, in the presence of multiple active neighbours sharing the same channel. Seeking to find optimal packet transmission policy for this scenario, we modelled the problem as a Markov Decision Process. Our results state that using a relay in the presence of active neighbours is beneficial even if the channel from relay to receivers is not better than the channel between source and receivers. We also showed that choosing the right moment to activate the relay node, can bring up to $3.5x$ gains in terms of throughput compared to using only the direct link in some of our examples. This is particularly relevant for dense mesh networks, where large number of nodes contend for the channel at the MAC layer. As for the right moment of activating the relay nodes, we proposed that a relay starts sending recoded packets when at least one pair of relay-receiver nodes has jointly received $M$ coded packets, i.e., relay together with one of the receivers have $M$ degree of freedom. The logic behind this policy, is that the relay should start sending recoded packets when the probability of sending innovative packets from the source to the pair of relay and one receiver is very low, i.e., the total knowledge of the relay and one receiver is $M$. This way, the receivers can benefit from the packets transmitted by the relay, and at the same time, the relay may receive innovative packets from the source. Therefore, in contrast to the similar coded relay-based protocols where a relay is always sending recoded packets, our optimal solution allows a system designer to determine the appropriate time that a relay should be activated, which in turn leads to send less number of non-innovative packets and decrease the time and the energy consumption.

## 6.2 Future Work

In Chapter 2, we proposed a network coded geographic communication protocol that was shown to decrease the transmission area and the number of active nodes using network coding. Although we tried to minimize the number of active nodes by directing all packet transmissions inside a limited geographic area, we did not apply optimal ways of transmitting packets to the created paths. Thus, one natural next step is to include in the proposed M-GeoCode protocol, the optimal network coded cooperative packet transmission policies that we proposed. Further work includes evaluating the effect of packet loss on the performance of M-GeoCode, and extending the proposed protocol for a heterogeneous network consisting of the nodes with different coding capacities. Other directions for further work includes proposing network coded geographic routing protocols for multi-layer multicast scenarios where different priorities are assigned to different layers of data, that could be translated as multi-view video streaming applications in VANETs.

In Chapters 3, 4, we solved the problem of optimal design of network coded cooperative communication for time-invariant and time-varying channels in a simple network of one source and two receivers. In our framework, we assumed multicast scenario in which both receivers are interested in the same content. Although this scenario has lots of applicability in many practical applications, such as cooperative download in dynamic wireless networks, considering other frameworks with more complex requirements, such as multiple unicast flows with inter-flow network coding, or index coding are still challenging and could be seen as potential next steps. Another interesting scenario in the context of network coded cooperative communication would be a network with multiple source and multiple receivers, where each receiver is interested in a specific flow transmitted by one of the sources. Further work in this area includes finding optimal code structures for cooperative multicast scenarios, and investigating cooperative network coded packet transmission with strict deadlines on packet delivery that may happen in the safety applications of VANETs or other control applications of wireless networks.

In Chapter 3, for the heuristics analysis, we assumed that the receivers have been clustered a priori, and we did not discussed the optimal methods of clustering the receivers. We only evaluated two methods of clustering, namely, heterogeneous and homogeneous clustering for the case that all clusters start the cooperation phase together. Our primary results state that heterogeneous clustering always provides a better performance than homogeneous clustering, if there is spatial separation between clusters. These results were presented for a specific scenario, therefore, a complete investigation on the effects of different methods of clustering could be conducted as a new line of research. Also one can investigate the effect of clusters with more than two receivers on the completion time of the proposed heuristics.

In Chapter 5, using a Markov decision process model, we determined the optimal policy to minimize the transmission time of $M$ packets from a source to two receivers in the presence of active neighbours. It was shown that the potential gain of a relay node is increased by increasing the number of active neighbours. We assumed identical coding and modulation schemes for the transmissions from source and relay in the network model that we analysed. There are different levels of complexity that can be added to the model we considered. For example, we can consider the effects of asymmetric coding and modulation schemes for transmission from source and relay, which can increase even more the usefulness of the relay, as well as more complex topologies, e.g., multi hop scenarios, and sharing of relay by multiple flows. Other relevant lines of research in this area are as follows: 1) extending our analysis for a network with multiple relay nodes assuming that every relay only maintain a portion of the coded packets, 2) investigating the overall effect of such a local enhancing on a multi-hop protocol, and 3) designing locally optimized multicast protocols for mesh networks. Finally, the physical model of the network including the probability of erasure, length of the time slots and range of transmission are very abstract in our model and can be made more realistic.

# Appendix A

# Transition probabilities for action $a_3$ of multicast scenario

In this appendix, we summarize all non-zero transition probabilities of action $a_3$ for the network defined in Fig. 5.1-(a). As we mentioned, this action is a combination of actions $a_1, a_2$, therefore, the transition probabilities are calculated using combinatorial arguments.

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{\bar{H}R_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\varepsilon_{\bar{H}R_2}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $i_H = c_1, i_R > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{\bar{H}R_2}\, I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $i_H = c_2, i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{\bar{H}R_1}\, I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $I_h = c_1 = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{SH}\, I_{(0,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,0,1,0)}$$
$$+\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(2,0,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H = c_1 = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,0,1,0)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{\bar{HR}_2}\, I_{(0,2,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\, I_{(1,1,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H = c_1 = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(0,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\, I_{(1,1,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(0,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \\ \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,2,0)}$$
$$+\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H = c_1, i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,2,0)}$$
$$+\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M-1$ and $i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M-1$ and $i_H \geq c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,0,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,1,1)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,0,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M-1$ and $i_H \geq c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,0,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H < M-1$ and $i_H \geq c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,0,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,0,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,0,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,1,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,0,2)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,0,1,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,0,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(2,1,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H = c_1 = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\, I_{(1,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\ I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\ I_{(2,1,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\ I_{(1,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_2}\ I_{(1,1,0,0,0)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{\bar{HR}_2}\ I_{(1,2,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\ I_{(1,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{SH}\ I_{(1,1,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\ I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\ I_{(2,1,1,2,1)} \\ + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\ I_{(1,2,1,1,2)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\ I_{(2,2,1,2,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{HR_2}\ I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_2}\ I_{(1,2,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_1}\, I_{(2,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H = c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_1}\, I_{(2,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_1}\, I_{(2,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_1}\varepsilon_{HR_2}\, I_{(2,1,1,2,1)} \\ + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_1}\varepsilon_{\overline{HR}_2}\, I_{(1,2,1,1,2)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_1}\varepsilon_{\overline{HR}_2}\, I_{(2,2,1,2,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{\overline{SH}}\varepsilon_{\overline{HR}_2}\, I_{(1,2,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 < M, i_1 = M-1, i_2 = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1+1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,0,2,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1+1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(2,0,0,2,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon}_{HR_2}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon}_1\varepsilon_2\varepsilon_{SH}\, I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,1,0)}$$
$$+\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(0,2,0,0,1)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,2,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon}_{HR_1}\, I_{(1,1,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,0)} + \varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\bar{\varepsilon}_{HR_1}\, I_{(1,1,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon}_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,1,0)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)}$$
$$+\varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(0,1,1,1,1)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,1,0)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)}$$
$$+\varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(0,1,1,1,1)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{SH}\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H > c_1+1, i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,3,0)} \\ +\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,3,0)} \\ +\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H < M-1$ and $i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,2,0)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M-1$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,1,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,2,1)} \\ +\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H < M-1$ and $i_H > c_1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,1,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H < M-1$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,1,1)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\ I_{(0,1,1,1,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\ I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\ I_{(0,1,1,1,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\ I_{(1,1,1,2,1)} \\ +\varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\ I_{(0,2,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\ I_{(1,2,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1+1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\ I_{(1,1,0,1,0)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\ I_{(2,1,0,2,0)} \\ +\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\ I_{(1,2,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\ I_{(2,2,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1+1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\ I_{(1,1,0,1,0)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\ I_{(2,1,0,2,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H > c_1+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\ I_{(1,1,0,1,0)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\ I_{(2,1,0,2,0)}.$$

- $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\ I_{(1,1,0,1,0)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_2}\ I_{(1,2,0,1,1)}.$$

- $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\ I_{(1,1,0,1,0)}.$$

- $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(1,1,0,1,0)}.$$

- $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1+1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,1,3,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1+1, i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,1,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H > c_1+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,1,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,1,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 < M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,1,3,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,0,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{\bar{HR}_2}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,2,0,0,2)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,2,0,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H = c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_2}\, I_{(0,2,0,0,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M-1$ and $i_H \geq c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)}$$
$$+\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,0,2)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,1,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)}$$
$$+\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,0,2)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,2,1)}$$
$$+\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,2,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H = c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,1,2,1)}$$
$$+\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,1,2,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H < M-1$ and $i_H \geq c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,0,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,0,3)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,3)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H < M-1$ and $i_H \geq c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,0,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,0,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,0,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,0,3)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,3)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,0,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,0,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H = c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,0,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(2,1,0,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,1,0,0,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,0,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(1,1,0,0,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1, i_H > c_2+1$:

$$\begin{aligned} &P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(2,1,1,2,2)} \\ &+\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(1,2,1,1,3)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(2,2,1,2,3)}. \end{aligned}$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H = c_1, i_H > c_2+1$:

$$\begin{aligned} &P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_2}\, I_{(1,2,1,1,3)}. \end{aligned}$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H > c_1$:

$$\begin{aligned} &P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\, I_{(2,1,1,2,2)}. \end{aligned}$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H = M-1$:

$$\begin{aligned} &P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\, I_{(2,1,1,2,2)}. \end{aligned}$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H = M-1$:

$$\begin{aligned} &P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ &\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(2,1,1,2,2)} \\ &+\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(1,2,1,1,3)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(2,2,1,2,3)}. \end{aligned}$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_2 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,3)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,1,3)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\, I_{(1,1,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M$ and $i_H > c_1 + 1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,0,0,2,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,1,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H < M$ and $i_H > c_1, i_H > c_2 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,0,0,2)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{\bar{HR}_1}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,1,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,1,2)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,1,1,1)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(0,1,1,1,2)} + \varepsilon_1\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H < M-1$ and $i_H > c_1+1, i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(2,0,1,3,1)} \\ +\bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(2,1,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{HR_2}\, I_{(2,0,1,3,1)} \\ +\bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_1}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_1}\varepsilon_{\bar{HR}_2}\, I_{(2,1,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{HR_2}\, I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\varepsilon_2\varepsilon_{\bar{SH}}\varepsilon_{\bar{HR}_2}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H < M-1$ and $i_H > c_1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,1,3)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H < M-1$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,1,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,1,1,2)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)}$$
$$+\varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(0,2,1,1,3)} + \varepsilon_1\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M$ and $i_H > c_1+1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,0,2,1)}$$
$$+\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,1,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,0,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M$ and $i_H > c_1+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_1}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\, I_{(2,1,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H < M$ and $i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} =$$
$$\bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(1,2,0,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\varepsilon_{SH}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_1+1, i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,1,3,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,1,3,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H < M-1$ and $i_H > c_1+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,1,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,1,1,3,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\varepsilon_{HR_2}\, I_{(2,1,1,3,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\bar{\varepsilon_{HR_2}}\, I_{(2,2,1,3,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H < M-1$ and $i_H > c_2+1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\, I_{(1,1,1,2,2)} + \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\, I_{(1,2,1,2,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_2}\bar{\varepsilon_{SH}}\ I_{(1,1,1,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\ I_{(0,0,0,0,0)} + \varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\ I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\varepsilon_{SH}\ I_{(0,0,0,0,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M-1, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\ I_{(0,1,0,0,0)} + \bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\ I_{(0,2,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M-1, i_H < M$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\varepsilon_{SH}\ I_{(0,1,0,0,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\varepsilon_{SH}\ I_{(0,1,0,0,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M, i_H < M-1$ and $i_H \geq c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\ I_{(0,0,1,1,0)} + \varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\ I_{(0,1,1,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\ I_{(0,0,1,1,0)} + \varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\ I_{(0,1,1,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M-1, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \varepsilon_{HR_2} \, I_{(0,1,1,1,1)} + \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \varepsilon_{\bar{HR}_2} \, I_{(0,2,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M-1, i_H < M-1$ and $i_H = c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \, I_{(0,1,1,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \varepsilon_{HR_2} \, I_{(0,1,1,1,1)} + \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \varepsilon_{\bar{HR}_2} \, I_{(0,2,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 < M, i_1 = M, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2 \varepsilon_{\bar{SH}} \, I_{(0,1,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1 \varepsilon_{SH} \varepsilon_{HR_1} \, I_{(0,0,0,0,0)} + \varepsilon_1 \varepsilon_{SH} \varepsilon_{\bar{HR}_1} \, I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1 \varepsilon_{SH} \, I_{(0,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1 \varepsilon_{SH} \varepsilon_{HR_1} \, I_{(1,0,0,0,0)} + \bar{\varepsilon}_1 \varepsilon_{SH} \varepsilon_{\bar{HR}_1} \, I_{(2,0,0,1,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1 \varepsilon_{SH} \, I_{(1,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\varepsilon_{SH}\, I_{(1,0,0,0,0)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H < M-1$ and $i_H \geq c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,0,1,0,1)} + \varepsilon_1\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,0,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(0,0,1,0,1)} + \varepsilon_1\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(1,0,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M-1$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,0,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M-1$ and $i_H = c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,0,1,1,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\, I_{(1,0,1,1,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\, I_{(2,0,1,2,1)}.$$

- If $\mathcal{K}_1 < M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\, I_{(1,0,1,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M, i_H < M$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\varepsilon_{SH}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_2\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\, I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M-1, i_H < M$ and $i_H > c_2 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\varepsilon_{SH}\varepsilon_{HR_2}\; I_{(0,1,0,0,1)} + \bar{\varepsilon_2}\varepsilon_{SH}\bar{\varepsilon_{HR_2}}\; I_{(0,2,0,0,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\varepsilon_{SH}\; I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M, i_H < M-1$ and $i_H > c_2$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\; I_{(0,0,1,1,1)} + \varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\; I_{(0,1,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_2\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\; I_{(0,0,1,1,1)} + \varepsilon_2\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\; I_{(0,1,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M-1, i_H < M-1$ and $i_H > c_2 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\; I_{(0,1,1,1,2)} + \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\; I_{(0,2,1,1,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M-1, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\varepsilon_{HR_2}\; I_{(0,1,1,1,2)} + \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_2}}\; I_{(0,2,1,1,3)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 = M-1, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \bar{\varepsilon_2}\bar{\varepsilon_{SH}}\; I_{(0,1,1,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H < M$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \varepsilon_1\varepsilon_{SH}\varepsilon_{HR_1}\; I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\; I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M$ and $i_H > c_1 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_{SH}\varepsilon_{HR_1}\ I_{(1,0,0,1,0)} + \bar{\varepsilon_1}\varepsilon_{SH}\bar{\varepsilon_{HR_1}}\ I_{(2,0,0,2,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\varepsilon_{SH}\ I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H < M-1$ and $i_H > c_1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\ I_{(0,0,1,1,1)} + \varepsilon_1\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\ I_{(1,0,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\ I_{(0,0,1,1,1)} + \varepsilon_1\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\ I_{(1,0,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H < M-1$ and $i_H > c_1 + 1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\ I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\ I_{(2,0,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H = M-1$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\varepsilon_{HR_1}\ I_{(1,0,1,2,1)} + \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\bar{\varepsilon_{HR_1}}\ I_{(2,0,1,3,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M, i_H < M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon_1}\bar{\varepsilon_{SH}}\ I_{(1,0,1,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_2\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,0,0,0,0)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} \\ +\varepsilon_1\varepsilon_2\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\varepsilon_2\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\varepsilon_2\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(2,0,0,2,0)} \\ +\bar{\varepsilon}_1\varepsilon_2\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,1,0,1,1)} + \bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(2,1,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\varepsilon_2\varepsilon_{HR_2}\, I_{(1,0,0,1,0)} + \bar{\varepsilon}_1\varepsilon_2\bar{\varepsilon}_{HR_2}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{HR_1}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\, I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 < M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\bar{\varepsilon}_2\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(0,1,0,0,1)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)} \\ +\varepsilon_1\bar{\varepsilon}_2\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(0,2,0,0,2)} + \varepsilon_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,2,0,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{HR_1}\, I_{(1,1,0,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\, I_{(2,1,0,2,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 < M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{HR_1}\varepsilon_{HR_2}\, I_{(1,1,0,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\varepsilon_{HR_2}\, I_{(2,1,0,2,1)} \\ +\bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(1,2,0,1,2)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_{HR_1}\bar{\varepsilon}_{HR_2}\, I_{(2,2,0,2,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 < M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\varepsilon_{HR_2}\ I_{(1,1,0,1,1)} + \bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon_{HR_2}}\ I_{(1,2,0,1,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\bar{\varepsilon}_2\ I_{(1,1,0,1,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_2\varepsilon_{HR_2}\ I_{(0,0,0,0,0)} + \varepsilon_2\bar{\varepsilon_{HR_2}}\ I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 < M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2\varepsilon_{HR_2}\ I_{(0,1,0,0,1)} + \bar{\varepsilon}_2\bar{\varepsilon_{HR_2}}\ I_{(0,2,0,0,2)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M, i_2 = M-1, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_2\ I_{(0,1,0,0,1)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M, i_2 = M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \varepsilon_1\varepsilon_{HR_1}\ I_{(0,0,0,0,0)} + \varepsilon_1\bar{\varepsilon_{HR_1}}\ I_{(1,0,0,1,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 < M-1, i_2 = M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\varepsilon_{HR_1}\ I_{(1,0,0,1,0)} + \bar{\varepsilon}_1\bar{\varepsilon_{HR_1}}\ I_{(2,0,0,2,0)}.$$

- If $\mathcal{K}_1 = M, \mathcal{K}_2 = M, i_1 = M-1, i_2 = M, i_H = M$:

$$P_{(i_1,i_2,i_H,c_1,c_2)\to(i_1',i_2',i_H',c_1',c_2')} = \\ \bar{\varepsilon}_1\ I_{(1,0,0,1,0)}.$$

- If $i_1 = M, i_2 = M$, then $P_{(i_1,i_2,i_H,c_1,c_2)\rightarrow(i_1^{'},i_2^{'},i_H^{'},c_1^{'},c_2^{'})} = I_{(0,0,0,0,0)}$.

# References

[1] L. Shu, Y. Zhang, T.Yang, Y. Wang, and M. Hauswirth. Geographic routing in wireless multimedia sensor networks. In *Proc. International Conference on Future Generation Communication and Networking*, pages 68–73, 2008.

[2] Y. Wang, T. Wu, W. Lee, and C. H. Ke. A novel geographic routing strategy over vanet. In *Proc. 24th IEEE Int. Conf. on Advanced Info. Networking and Applications*, pages 873–879, April 2010.

[3] C. Perkins, E. Belding-Royer, and S.Das. *RFC Editor*. 1st edition, 2003.

[4] D. Johnson, D. Maltz, and J. Broch. *Ad Hoc Networking*. Addison-Wesley Longman Publishing Co., Inc., 2001.

[5] G. Liu, B. Lee, B. Seet, C. Foh, K.Wong, and K. Lee. A routing strategy for metropolis vehicular communications. In *Proc. of Int. Conf. on Info. Networking (ICOIN)*, pages 134–143, Feb 2004.

[6] H. Fubler, M. Mauve, H. Hartenstein, M. Kasemann, and D. Vollmer. Location-based routing for vehicular ad-hoc networks. In *Proc. of ACM SIGMOBILE Mobile Computing and Comm. Review (MC2R)*, Oct 2003.

[7] R. Ahlswede, N. Cai, S. R. Li, and R. W. Yeung. Network information flow. *IEEE Trans. on Info. Theory*, Jul. 2000.

[8] C. Fragouli, J. Widmer, and J. Y. Le Boudec. A network coding approach to energy ecient broadcasting: From theory to practice. In *Proc. of IEEE INFOCOM*, Apr. 2006.

[9] T. Ho, M. Medard, R. Koetter, D.R. Karger, M. Eros, Jun Shi, and B. Leong. A random linear network coding approach to multicast. *IEEE Trans. on Info. Theory*, Oct. 2006.

[10] S. Katti, H. Rahul, Wenjun Hu, D. Katabi, M. Medard, and J. Crowcroft. Xors in the air: Practical wireless network coding. *IEEE/ACM Transactions on Networking*, Jun. 2008.

[11] R. Koetter and M. Medard. An algebraic approach to network coding. *IEEE/ACM Transactions on Networking*, 2003.

[12] Desmond S. Lun, Muriel Medard, Ralf Koetter, and Michelle Eros. On coding for reliable communication over packet networks. *Physical Communication*, 2008.

[13] Chi Zhang, Xiaoyan Zhu, and Yuguang Fang. On the improvement of scaling laws for large-scale manets with network coding. *IEEE Journal on selected areas in communications,* , 2009.

[14] C. Gkantsidis and P.R. Rodriguez. Network coding for large scale content distribution. In *Proc. of IEEE INFOCOM*, Mar. 2005.

[15] Yunnan Wu. Distributing layered content using network coding. In *Proc. of Sensor, Mesh and Ad Hoc Communications and Networks Workshops (SECON)*, 2008.

[16] B. Liu and Chuan Wu. Uusee: Large-scale operational on-demand streaming with random network coding. In *Proc. of INFOCOM*, 2010.

[17] C. Peng, Q. Zhang, M. Zhao, and Y. Yao. On the performance analysis of network-coded cooperation in wireless networks. In *Proc. of IEEE Int. Conf. on Comp. Comm. (INFOCOM)*, May 2007.

[18] J. Zhang and Q. Zhang. Cooperative network coding-aware routing for multi-rate wireless networks. In *Proc. of IEEE Int. Conf. on Comp. Comm. (INFOCOM)*, Apr. 2009.

[19] Z. J. Haas and T. Chen. Cluster-based cooperative communication with network coding in wireless networks. In *Proc. of IEEE MILCOM*, Nov. 2010.

[20] Q. Zhang, J. Heide, M. V. Pedersen, and F. H. P. Fitzek. Mbms with user cooperation and network coding. In *Proc. of IEEE Globecom*, Dec. 2011.

[21] S. Sharma, Y. shi, J. Liu, Y. T. Hu, and S. Kompella. Is network coding always good for cooperative communications? In *Proc. of IEEE INFOCOM*, Mar. 2010.

[22] S. Sharma, Y. shi, J. Liu, Y. T. Hu, S. Kompella, and S. F. Midkiff. Network Coding in Cooperative Communications: Friend or Foe? *IEEE Trans. on Mob. Comp.*, Jul. 2012.

[23] J. Du, M. Xiao, and M. Skoglund. Cooperative Network Coding Strategies for Wireless Relay Networks with Backhaul. *IEEE Trans. on Communications.*, Sep. 2011.

[24] S. Sharma, Y. Shi, Y. T. Hou, H. D. Sherali, and S. Kompella. Optimizing network-coded cooperative communications via joint session grouping and relay node. In *Proc. of IEEE INFOCOM*, Apr. 2011.

[25] X. Bao and J. Li. Matching code-on-graph with network-on-graph: adaptive network coding for wireless relay networks. In *Proc. of 43rd Annual Allerton Conf. Commun., Control Computing*, Sep. 2005.

[26] J. L. Robelatto, B. F. Uchoa-Filho, and Y. Li. Multi-User Cooperative Diversity through Network Coding Based on Classical Coding Theory. *IEEE Trans. on Signal Processing*, Feb. 2012.

[27] T. E. Hunter and A. Nosratinia. Cooperative diversity through coding. In *Proc. of IEEE ISIT*, Jul. 2002.

[28] T. Hunter and A. Nostratinia. Diversity through coded cooperation. *IEEE Trans. Wireless Commun.*, Feb. 2006.

[29] T. Hunter, S. Sanayei, and A. Nostratinia. Outage analysis of coded cooperation. *IEEE Trans. on Info. Theory*, Feb. 2006.

[30] L. Xiao, T. E. Fuja, J. Kliewer, and Costello D. J. A Network Coding Approach to Cooperative Diversity. *IEEE Trans. on Info. Theory*, Oct. 2007.

[31] M. Peng, C. Yang, Z. Zhao, and W. Wang. Cooperative Network Coding in Relay-Based IMT-Advanced Systems. *IEEE Comm. Magazine*, Apr. 2012.

[32] I. Stojmenovic and X. Tin. loop-Free Hybrid Single path/Flooding Routing Algorithms with Guaranteed Delivery for Wireless Networks. *IEEE Trans. Parallel and Distributed. Systems* , Oct. 2001.

[33] D. Saha, S. Toy, S. Bandyopadhyay, T. Ueda, and S. Tanaka. An adaptive framework for multipath routing via maximally zone-disjoint shortest paths in ad hoc wireless networks with directional antenna. In *Proc. of IEEE Globecom*, 2003.

[34] M. Mauve, H. Fubler, J. Widmer, and T. Lang. Position-based multicast routing for mobile ad-hoc networks. *ACM SIGMOBILE Mobile Computing and Comm. Review*, Jul. 2003.

[35] J. Sanchez, P. Ruiz, J. Liu, and I. Stojmenovic. Bandwidth-efficient geographic multicast routing protocol for wireless sensor networks. *IEEE Sensors*, May. 2007.

[36] B. Bollobas. *Modern Graph Theory*. Springer, 2000.

[37] A. Toledo and X. Wang. Efficient multipath in sensor networks using diffusion and network coding. In *Proc. of 40th IEEE CISS*, Mar. 2006.

[38] C. Gkantsidis, P. Rodriguez, G. Grilli, and M. Gerla. Network coding for large scale content distribution. In *Proc. of IEEE joint Conference of Computer and Communication Society (INFOCOM)*, 2005.

[39] S. Lee, U. Lee, K. Lee, and M. Gerla. Content distribution in vanets using network coding: The effect of disk i/o and processing o/h. In *Proc. of IEEE SECON*, 2008.

[40] G. Ma, Y. Xu, M. Lin, and Y. Xuan. A content distribution system based on sparse linear network coding. In *Proc. of Third Workshop on Network Coding (Netcod)*, 2007.

[41] H. Frey. Scalable geographic routing algorithms for wireless adhoc networks. In *Proc. of IEEE Network Comm. Society*, Jul. 2004.

[42] F. Kuhn, R. Wattenhofer, and A. Zollinger. An algorithmic approach to geographic routing in ad hoc and sensor networks. *IEEE/ACM Trans. on Networking*, Feb. 2008.

[43] J. Tsai and T. Moors. A review of multipath routing protocols: From wireless ad hoc to mesh networks. In *Proc. of ACoRN Early Career Researcher Workshop*, 2006.

[44] B. Karp and H. T. Kung. Gpsr: Greedy perimeter stateless routing for wireless networks. In *Proc. of ACM/IEEE Mobicom*, Aug. 2000.

[45] K. Selcuk Candan S. Wu. Power-aware single and multipath geographic routing in sensor networks. *Elsevier*.

[46] P.M. Ruiz and A.F. Gomez-Skarmeta. Approximating optimal multicast trees in wireless multihop networks. In *Proc. of IEEE Symposium on computers and communications*, Jun. 2005.

[47] M. Transier, H. Fubler, J. Widmer, M. Mauve, and W. Effelsberg. Scalable position-based multicast for mobile ad-hoc networks. In *Proc. of workshop on Broadband Wireless Multimedia*, 2004.

[48] Ji. Lusheng and M.S. Corson. Differential destination multicast-a manet multicast routing protocol for small groups. In *Proc. of 20th Annual Joint Conference of the IEEE Computer and Communications Societies*, 2001.

[49] S.M. Das, H. Pucha, and Y.C. Hu. Distributed hashing for scalable multicast in wireless ad hoc networks. *IEEE Trans. on Parallel and Distributed Systems*, Mar. 2008.

[50] D.S. Lun, N. Ratnakar, R. Koetter, M. Medard, E. Ahmed, and Hyunjoo Lee. Achieving minimum-cost multicast: a decentralized approach based on network coding. In *Proc. of IEEE Int. Conf. on Computer and Communications (INFOCOM)*, Mar. 2005.

[51] T. Ho, R. Koetter, M. Medard, D.R. Karger, and M. Effros. The benefits of coding over routing in a randomized setting. In *Proc. of IEEE Int. Symposium on Information Theory*, Jul. 2003.

[52] T. Ho, B. Leong, M. Medard, R. Koetter, Yu-Han Chang, and M. Effros. On the utility of network coding in dynamic environments. In *Proc. of Int. Workshop on Wireless Ad-Hoc Networks*, Jun. 2004.

[53] S. Chachulski, M. Jennings, S. Katti, and D. Katabi. Trading structure for randomness in wireless opportunistic routing. In *Proc. of ACM SIGCOMM*, 2007.

[54] Xinyu Zhang and Baochun Li. Optimized multipath network coding in lossy wireless networks. In *Proc. of Int. Conf. on Distributed Computing Systems*, Jun. 2008.

[55] S. Katti, D. Katabi, W. Hu, H. Rahul, and M. Medard. The importance of being opportunistic: Practical network coding for wireless environments. In *Proc. of Int. Conf. Allerton*, 2005.

[56] X. Tang and Q. Liu. Network coding based geographical opportunistic routing for ad hoc cognitive radio networks. In *Proc. of IEEE Globecom Workshop*, 2012.

[57] S.A. Aly, V. Kapoor, J. Meng, and A. Klappenecker. Bounds on the network coding capacity for wireless random networks. In *Information Theory and Applications Workshop*, Feb. 2007.

[58] S. Sengupta, Minghua Chen, P.A. Chou, and Jin Li. On optimality of routing for multi-source multicast communication scenarios with node uplink constraints. In *IEEE Int. Symposium on Information Theory (ISIT)*, Jul.2008.

[59] C.L. Li, S.T. McCormic, and D. Simchi-Levi. The complexity of finding two disjoint paths with min-max objective function. *Discrete Applied Mathematics*, pages 105–115, 1990.

[60] Y. Chao and W. Hongxia. Developed dijkstra shortest path search algorithm and simulation. In *Proc. IEEE ICCDA*, 2010.

[61] R. Flury, S.V. Pemmaraju, and R. Wattenhofer. Greedy routing with bounded stretch. In *Proc. IEEE Conf. on Computer Communications*, Jul. 2004.

[62] P. von Rickenbach, R. Wattenhofer, and A. Zollinger. Algorithmic models of interference in wireless ad hoc and sensor networks. *IEEE/ACM Trans. on Networking*, pages 172–185, Feb. 2009.

[63] M. Xiao and M. Skoglund. Multiple-user cooperative communications based on linear network coding. *IEEE Trans. on Comms.*, pages 172–185, Dec. 2010.

[64] 3gpp: 3rd generation partnership project. *technical specification group SA; feasibility study for proximity services (ProSe) (Release 12)*, Aug. 2012.

[65] M. Nistor, D. E. Lucani, T. T. V. Vinhoza, R. Costa, and J. Barros. On the delay distribution of random linear network coding. *IEEE Journal on Selected Areas in Comm. (JSAC)*, May. 2011.

[66] R. Bellman. A markovian decision process. *Jour. of Math. and Mech.*, 1957.

[67] D. Koutsonikolas, C. C. Wang, and Y. Charlie Hu. Efficient network-coding-based opportunistic routing through cumulative coded. *IEEE/ACM Trans. on Networking.*, Oct. 2011.

[68] A. Eryilmaz, A. Ozdaglar, M. Medard, and E. Ahmed. On the delay and throughput gains of coding in unreliable networks. *IEEE Trans. on Info. Theory*, Dec. 2008.

[69] James Norris. *Markov Chains*. Cambridge University Press, New York, NY, USA, 1997.

[70] M. V. Pedersen, J. Heide, and F. Fitzek. Kodo: An open and research oriented network coding library. *Lecture Notes in Computer Science*, pages 145–152, 2011.

[71] A. Paramanathan, P. Pahlevani, S. Thorsteinsson, M. Hundeboll, D. E. Lucani, and F. H. P. Fitzek. Sharing the pi: Testbed description and performance evaluation of network coding on the raspberry pi. In *Proc. IEEE VTS Vehicular Technology Conference (VTC Spring)*, May 2014.

[72] M. Boban, T. T. V. Vinhoza, M. Ferreira, J. Barros, and O. K. Tonguz. Impact of vehicles as obstacles in vehicular ad hoc networks. *IEEE JSAC*, Jan. 2011.

[73] Andrea Goldsmith. *Wireless Communications*. Cambridge University Press, New York, NY, USA, Aug. 2005.

[74] Theodore Rappaport. *Wireless communications principles and practices*. Prentice Hall, 2002.

[75] A. Paramanathan, M. V. Pedersen, D. Lucani, and F. H. P. Fitzek. Lean and mean: Network coding for commercial devices. *IEEE wireless comm.*, Oct. 2013.

[76] X. Shi, M. Medard, and D. E. Lucani. Whether and where to code in the wireless packet erasure relay channel. *IEEE Jou. of Sel. Areas in Comm.*, pages 1379–1389, Aug. 2013.

[77] G. Giacaglia, X. Shi, M. Kim, D. E. Lucani, and M. Medard. Systematic network coding with the aid of a full-duplex relay. In *Proc. IEEE Int. Comm. Conf. (ICC)*, Jun. 2013.

[78] Y. Wang, C. Hu, H. Liu, M. Peng, and W. Wang. Network coding in cooperative relay networks. In *Proc. IEEE PIMRC*, Sep. 2008.

[79] M. Xiao and M. Skoglund. Design of network codes for multiple-user multiple-relay wireless networks. In *Proc. ISIT*, Jul. 2009.

[80] P. Pahlevani, D. E. Lucani, M. V. Pederson, and F. H. P. Fitzek. Playncool: Opportunistic network coding for local optimization of routing in wireless mesh networks. In *Proc. Globecom Workshops*, Dec. 2013.

[81] P. Pahlevani, D. E. Lucani, M. V. Pederson, and F. H. P. Fitzek. Network coding to enhance standard routing protocols in wireless mesh networks. In *Proc. Globecom Workshops*, Dec. 2013.

[82] S. Chachulski, M. Jennings, S. Katti, and D. Katabi. More: A network coding approach to opportunistic routing. In *Proc. SIGCOMM,*, Aug. 2007.

[83] D. Koutsonikolas, C. C. Wang, and Y. Charlie Hu. Efficient network- coding-based opportunistic routing through cumulative coded. *IEEE/ACM Trans. on Networking*, Oct. 2011.