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ON THE CONNECTIONS BETWEEN MATHEMATICAL PROGRAMMING AND DISCRETE OPTIMAL CONTROL

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ABSTRACT

This paper gives a short review of different theories and topics in the fields of Mathematical Programming and Discrete-time Optimal control Theory. Well-known and new results (for instance, the generalized Maximum Principle) are presented within a unified framework. Emphasis is placed in the trends of interplay. New research areas are also identified.

## 1. Introduction

This paper gives a brief account of different theories and topics in the fields of mathematical programming (section 2) and discrete optimal control theory (section 3). In particular we mention the recent contributions to discrete optimal control - the upper boundary approach. This work has deep roots in mathematical programming, a reason for making its introduction in section 2. Finally, in section 4, we will point out some connections between those fields. In general, the contribution of mathematical programming to the field of discrete optimal control is better recognized than the reverse case. The same happens with the development of the upper boundary approach.

This paper is a condensed version of research report [1], that can be obtained from the authors. In [1] a complete list of references are also presented. Moreover in [2] a survey of the discrete-time maximum principle is presented.

To provide a common basis for the discussion in the remainder of this paper, specific forms of the problems are now stated.

The mathematical programming (MP) problem is:

$$\begin{aligned} &\text{Maximize} && r(u) \\ &\text{subject to} && f(u) = \bar{x} \\ &&& u \in U \end{aligned}$$

where $r(.)$ is a real function defined on $E^n$, $f(.)$ is a function from $E^n$ into $E^m$, $u \in U \subset E^n$, and $\bar{x}$ is a given vector in $E^m$. Note that the stated MP problem is quite general. Inequality constraints can be transformed into equality constraints by adding slack variables. Moreover, we do not specify, for the moment, how the $U$ is defined.

The discrete optimal control (DOC) problem is:

$$\text{Maximize} \quad R = \sum_{i=0}^{N-1} r_i(x_i, u_i)$$

$$\begin{aligned} \text{subject to} \quad & x_{i+1} = f_i(x_i, u_i) \\ & u_i \in U_i \subset E^n, && i = 0,1,\ldots,N-1 \\ & x_i \in X_i \subset E^m, && i = 0,1,\ldots,N \\ & x_0 = \bar{x}_0 \end{aligned}$$

where $r_i(.,.)$ is a real function defined on $E^{m+n}$, $f_i(.,.)$ is a function from $E^{m+n}$ into $E^m$, for $i = 0,1,\ldots,N-1$, and $\bar{x}_0$ is a constant vector.

## 2. Mathematical Programming

### 2.1. Differential Optimization

Differentiability of the functions involved in the MP problem is required. In this way we have a basic distinction relative to the upper-boundary approach presented in section 2.2.

solved the unconstrained problem. This technique (generalized Everett's theorem) is quite attractive, because there are no conditions to verify in advance. Moreover the risk of wrong conclusions seems to be very small, because in that case no $u^* \in U$ maximizing $r(u)$ will be found, such that $f(u^*) = \bar{x}$.
The information obtained by this procedure can be used either to approximate the solution of the MP problem or to do a parametric study on the right-hand side. Although the MP problem may have a solution, the solution may not be obtained by maximizing the Lagrangian (relative to a specific type of supporting function). The situation can be such one that the desirable right-hand side cannot be generated, just because a support (of that type) to the upper boundary function, at the point $\bar{x}$, cannot be found - we have a gap.

The importance of considering linear supports is well known. Concavity of the upper boundary functions is a sufficient condition for the existence of supporting hyperplanes, and this property can be derived from special characteristics of the given $r, f$ and $U$ in the MP problem.

Saddle-point theory also provides conditions for the existence of a linear support. Its existence is guaranteed if a saddle-point to the Lagrangian linear in the multipliers exists.

## 3. Discrete Optimal Control

### 3.1 Maximum Principle

A point of contact between mathematical programming and discrete optimal control appeared along with the attempts, taken by both mathematical programming and control theories, to generate a parallel result of the continuous-time maximum principle for discrete optimal control.

Although the maximum principle caused no big difficulties when applied to the optimization of continuous systems it became apparent the infeasibility to obtain, for the optimization of discrete-time systems, as strong results as in the continuous-time case. In fact, an equivalent formulation in discrete-time is, in general, neither necessary nor a sufficient condition for optimality.

the directional convexity property - which may be translated i terms of existence of supporting hyperplanes to the upper boundary functions (at each stage) at the optimal state. The generalized maximum principle offers, a priori, more chances by allowing more general supporting functions and, therefore, generalized Hamiltonian functions. The generalized maximum principle basically affirms that if the DOC problem has an optimal solution then supports to the upper-boundary functions exist so that the respective Hamiltonian (at every stage) is maximized by the optimal solution.

It is possible to distinguish in the necessary conditions of the maximum principle (on the linear or generalized version) a "maximum principle" part and an adjoint equations part. Different assumptions may be taken for each part, still keeping the validity of the main results. However the "maximum principle" part without the adjoint equations part has no practical relevance unless there is a way of choosing the right supporting surfaces to the upper boundary functions. This is in straight relation with the possibility of the upper boundary approach to deal with nonsmooth problems.

Halkin's principle of optimal evolution stems naturally from the upper boundary approach. Some work has been done to connect, in some way, this principle with Bellman's principle of optimality, in the spirit of the current approach. Therefore we can see dynamic programming (as applied to discrete optimal control) naturally placed within the upper boundary approach.

## 4. Mathematical Programming and Discrete Optimal Control

Along with the discussion around the different techniques to cope with the MP problem and the DOC problem, made in the previous sections, we have already touched the subject of the connections between mathematical programming and discrete optimal control. The equivalence of the MP and DOC problems immediately suggests the possibility of contacts and of an interplay between those theories. Moreover, we can expect that existing connections translate, in some way, the inner life of each of those fields - we mean the relations relative to the different approaches developed a) within mathematical programming and b) within discrete optimal control.

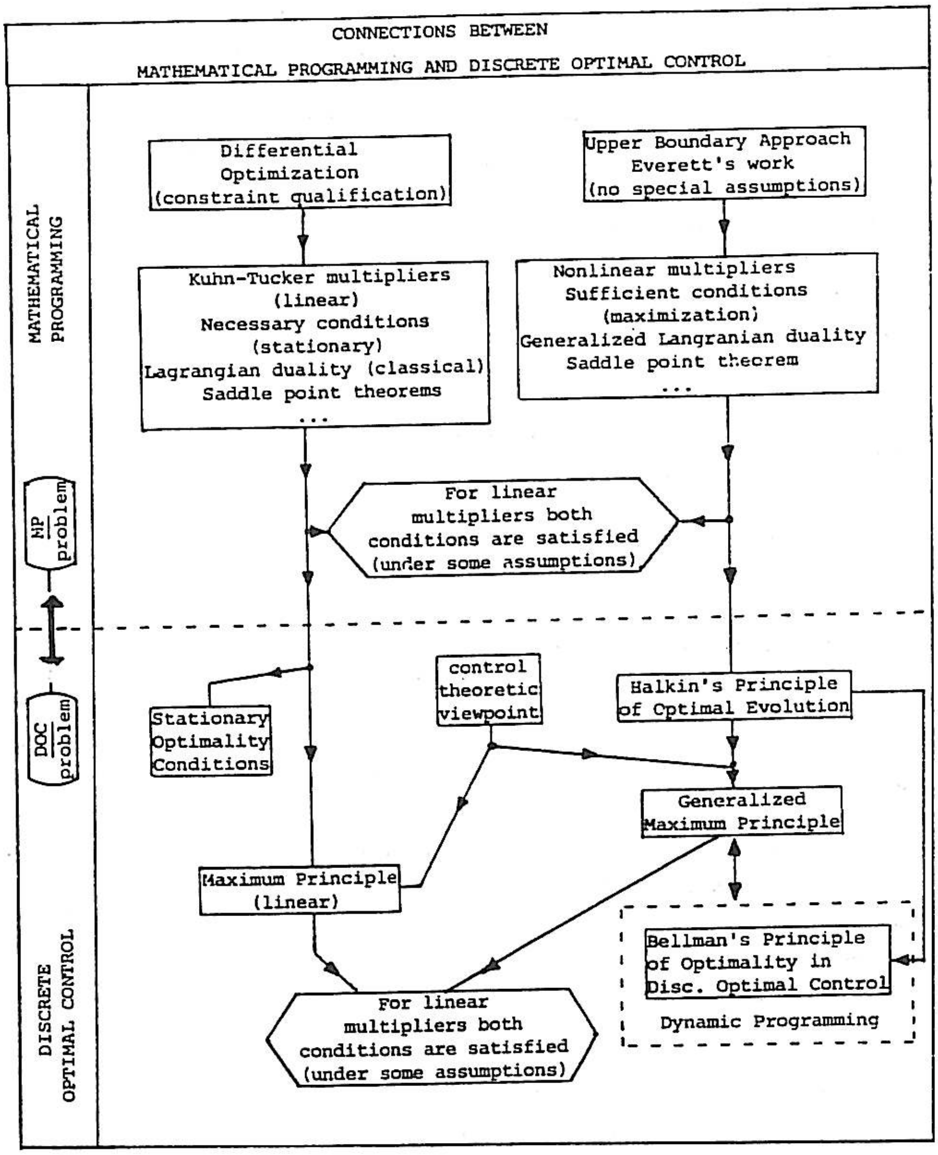
conditions arising from the different types of approaches, referred in section 3, give the same information.

Retaking, more closely the initial subject about the connections between mathematical programming and discrete optimal control, we observe that the search for necessary conditions of optimality, resulting in the formulation of the maximum principle (linear or generalized version) permitted us to suppose that the optimal values of the states $x_i$ and $x_{i+1}$ (see the statement of the DOC problem) are known, for any i, therefore obtaining static optimization problems (interrelated) relative to the control variables $u_i$ . Mathematical programming becomes, then, applicable; the upper boundary approach to discrete optimal control is a clear example of that, in the way that it is a natural extension of concepts and methods used in static optimization theory (see section 2.2). Saddle-point theorems, results of duality theory in mathematical programming have their own role and interpretation in discrete optimal control. For instence, in the case of the maximum principle (linear) it is clear a relation between the separated application (stage by stage) of that principle and duality theory. Maximization of the Hamiltonian, for given $x_i$ and $x_{i+1}$ , corresponds to determining a dual function (regardless constant parcels). Dual variables appear as slopes of supporting hyperplanes (tangent) to the upper boundary sets.

The maximum principles, by dividing a problem in a series of interrelated ones, to which mathematical programming may be applied, give a good insight to the DOC problem. This profundity in analysis is an advantage of these principles relatively to dynamic programming. Adjoint equations, relating multipliers, are the connections among the interrelated problems mentioned.

We have already mentioned, in section 3.1., that the maximum principle (linear) may also be obtained only by mathematical programming theory, - not considering techniques using control theory.

In conclusion we observe that we have different tools to cope with the MP problem and the DOC problem. Multipliers in linear or non-linear forms, appearing in different contexts, always played an essential role. However classes of MP problems and DOC problems were identified for which a correspondence (equality in some cases) could be found among the different types of multipliers.


CONNECTIONS BETWEEN
MATHEMATICAL PROGRAMMING AND DISCRETE OPTIMAL CONTROL
MATHEMATICAL PROGRAMMING
Differential Optimization (constraint qualification)
Upper Boundary Approach Everett's work (no special assumptions)
Kuhn-Tucker multipliers (linear) Necessary conditions (stationary) Lagrangian duality (classical) Saddle point theorems ...
Nonlinear multipliers Sufficient conditions (maximization) Generalized Langranian duality Saddle point theorem ...
MP problem
For linear multipliers both conditions are satisfied (under some assumptions)
DOC problem
Stationary Optimality Conditions
control theoretic viewpoint
Halkin's Principle of Optimal Evolution
Generalized Maximum Principle
Maximum Principle (linear)
Bellman's Principle of Optimality in Disc. Optimal Control
Dynamic Programming
For linear multipliers both conditions are satisfied (under some assumptions)
DISCRETE OPTIMAL CONTROL