Unified Models of Dark Energy and Dark Matter

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“Error never leaves us, yet nevertheless a higher need constantly draws the aspiring mind silently towards the truth”

Johann Goethe
In this work we give a succinct review of the main ideas behind modern cosmology and introduce the concept of models that represent dark energy and dark matter by the same underlying fluid. The background evolution and general considerations of the generalized Chaplygin gas (GCG) are presented and the parameters of the model constrained using Supernovae Ia (SNIa) observations. Next, we deal with the evolution of cosmological perturbations of the GCG and find that the matter power spectrum of the model is inconsistent with the observations, unless we constrain the value of the parameter $\alpha$ close to zero. The importance of including baryons to account for the formation of structures is also discussed. By analysing the clustering properties of the GCG, we motivate the idea that it is essential to take in account the effects of non-linearities when considering unified dark energy models. Finally, we study a model that introduces a parameter $\epsilon$ to characterize the level of small non-linear clustering in these scenarios, using the GCG as a representative model. We conclude that the model is consistent for all values of $\alpha$, with the most recent SNIa observations, and avoids the late time oscillations of the matter power spectrum, providing that the level of non-linear clustering is sufficiently high.
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Chapter 1

Introduction

In 1916 Albert Einstein published [1] a new theory of the gravitational interaction which, unlike newtonian gravity, is fully consistent with Special Relativity. This theory, General Relativity (GR), ended with 250 years of newtonian hegemony. It changed radically the way we perceive space and time, and provided a solution to the problem of the precession of Mercury’s orbit that couldn’t be explained until then in newtonian terms. Also, Einstein was the first to calculate the correct value for the bending of starlight by the gravitational field of the Sun. In May 1919, Arthur Eddington and his collaborators confirmed Einstein results enhancing the status and providing empirical evidence for GR.

When Einstein applied his theory to the universe [2] in 1917, he assumed that it was static. In order to get a static solution he introduced a new term \( \Lambda \), named cosmological constant, that would work as a repulsive force. Meanwhile, Slipher’s observations of the spectrum of galaxies suggested that most of them were receding from the Earth. The combination of the Slipher’s results with Hubble’s measurements of galaxy distances [3] led to Hubble’s law in 1929 and to the conclusion that the universe is expanding. The motivation for a cosmological constant in Einstein’s equations had vanished and the static solution gave place to the evolving solutions developed by Friedmann [4] and Lemaître [5].

In the 1930’s Jan Oort and Fritz Zwicky, guided by astrophysical evidence, postulated the existence of a new kind of matter to account for “missing” mass in the astronomical objects, when comparing the dispersion velocities of stars in the Milky Way with the
observable stellar mass. The same phenomena was observed with galaxies in the Coma cluster. Luminous matter was not enough to account for the gravitational effects and dark matter (DM) became the mainstream answer for that problem.

All subsequent efforts led to the development of the Standard Cold Dark Matter (SCDM) model; a flat universe filled with only matter content (Einstein-de Sitter universe). Ordinary matter, DM and GR could explain all the observations with the required accuracy that had been achieved in the 1980’s. Later, from the observational data of Supernovae Type Ia (SN Ia), Riess et al. [6] and Perlmutter et al. [7] found that the expansion of the universe is accelerating. Assuming GR, this result required a new exotic energy, dubbed dark energy (DE). This led cosmologists to revive Einstein’s cosmological constant $\Lambda$ in order to account for the late time cosmic acceleration. It is the simplest candidate for DE and its energy density is constant in time and space. This new model for the universe (known as $\Lambda$CDM), proved to be consistent with a vast number of observations, making it the Standard Model of Cosmology.

So, over the last 100 years, modern cosmology has been built by this symbiosis between theory and observation. It is essential to improve the precision and reliability of cosmological observations, in order to probe the consistency of the cosmological theories. Despite of the good agreement of the $\Lambda$CDM scenario with the observations, the work on the theoretical front continues, exploring viable theoretical alternatives and to investigate more deeply their fundamental grounds. In this chapter we briefly introduce the foundations of the Standard Model of Cosmology, and also review some general concepts and formulations used in modern cosmology.

1.1 The Cosmological Principle

The Cosmological Principle states that, on large scales, the universe is homogeneous and isotropic at each instant of time (relaxing the temporal requisite we get the Perfect Cosmological Principle: a homogeneous and isotropic universe in space and time). By homogeneous we mean that the physical properties of the universe are the same everywhere. This can be in a way regarded as a philosophical statement following from the Copernican Principle, which assumes that Earth does not have a special location
in the universe, i.e. the part of the universe which we can observe is fairly representative. Isotropy implies that the large scale properties of the universe do not depend on the spatial direction. This assumption is supported by a vast number of observations and, in particular, by the latest results of the Planck mission, reporting deviations from isotropy in the Cosmic Microwave Background (CMB) radiation of a few parts in $10^{-5}$ [30].

1.2 Background Geometry

1.2.1 FLRW metric

The metric tensor defines the way we measure distances in a given system of coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu,$$  \hspace{1cm} (1.1)

where the Einstein summation convention over the set of indexed terms is assumed. Different coordinate systems can describe the same spacetime but some choices of coordinates may be more suitable than others, specially when some symmetry is known to be present. In our case we assume a homogeneous and isotropic universe. The general form of a metric that can describe an universe in expansion (or contraction) is given by the Friedmann-Lemaître-Robertson-Walker metric (FLRW)

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right].$$  \hspace{1cm} (1.2)

We choose units such that the speed of light is unity $c = 3 \times 10^8 \text{m s}^{-1} = 1$. Here $t$ is the cosmic time and the function $a(t)$ is the scale factor, which describes the expansion of the universe (we normalize its present value $a(t_0)$ to unity). $K$ is the spatial curvature and can be negative, zero or positive depending on whether the universe is open, flat or closed respectively. It is related with the scalar curvature (3 dimensional Ricci scalar) and, because the space is maximally symmetric, the curvature is the same in every point of the spacelike slices. If $K > 0$ there is a geometrical singularity at $r = r_K = 1/\sqrt{K}$ that can be removed with the coordinate transformation $r = r_K \sin\chi$ ($\chi \in [0, \pi]$) leading to a spatial part that represents a 3-dimensional hypersphere. The FLRW metric in
hyperspherical coordinates is given by

\[ ds^2 = -dt^2 + \frac{a^2(t)}{K} \left[ d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2 \right]. \tag{1.3} \]

The spatial coordinates \((r, \theta, \phi)\) are called comoving coordinates, as they remain a constant for a comoving observer (a free-falling observer for which the universe is homogeneous and isotropic). The conformal time is defined as

\[ d\eta = \frac{dt}{a(t)}, \tag{1.4} \]

so the complete comoving coordinate system is \((\eta, r, \theta, \phi)\).

### 1.2.2 Distances in Cosmology

Our observations generally rely on the information received in the form of photons, so it is crucial to understand the behaviour of light when considering the FLRW metric. From the geodesic equation it follows that the energy of massless particles goes as \(E \propto a^{-1}\). So, the wavelength of light expands as the universe expands, leading to a redshift of the light emitted by the observed objects. We can define it as

\[ 1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} \tag{1.5} \]

for a photon that leaves the source at a time \(t_{\text{em}}\) and is observed at \(t_{\text{obs}}\). The measured redshift in the present time \(t_0\) is related with the scale factor by

\[ a(t) = \frac{1}{1 + z}. \tag{1.6} \]

The concept of distance is embedded in the metric. Light travels along null geodesics, so the line interval is

\[ ds^2 = 0 = -dt^2 + a^2(t) \frac{dr^2}{1 - Kr^2} \tag{1.7} \]

where we can take \(\theta, \phi = 0\) assuming isotropy. This leads to the definition of proper distance

\[ d_P = a(t) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = \frac{a(t)}{\sqrt{-K}} \text{arcsinh} \left( \sqrt{-Kr} \right). \tag{1.8} \]
For later convenience, we define the function

$$f_K(x) = \frac{1}{\sqrt{-K}} \sinh \left( \sqrt{-K} x \right).$$  \hspace{1cm} (1.9)

The proper distance between two comoving objects grows proportionally to the scale factor. The recessional velocity (due to the expansion of the universe) is given by

$$v = \dot{d}_P = H d_P$$  \hspace{1cm} (1.10)

where $H = \dot{a}/a$ is the Hubble parameter (that describes the expansion rate of the universe) and the dot represents the derivative in order to the cosmic time ($\equiv d/dt$). The present Hubble constant is usually parameterized as

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1},$$  \hspace{1cm} (1.11)

where the estimated value of the parameter $h$ is $h = 0.673$ [30].

In cosmology it is usual to use observable quantities, such as the angular diameter and luminosity of objects (in addition to the redshift), to define distances. For objects with known size $s$ and an observed angular diameter $\theta(z)$, we introduce the angular diameter distance defined as $d_A(z) = s/\theta(z)$. From Eq. (1.2) we get the proper distance for a given $\theta$ as $s = a(t) r \theta(z)$. It follows that

$$d_A = a(t) r.$$  \hspace{1cm} (1.12)

In order to relate the radial coordinate with the redshift

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^z \frac{dz'}{H(z')} = \int_0^r \frac{dr'}{\sqrt{1-Kr'^2}},$$  \hspace{1cm} (1.13)

using the definition from Eq. (1.6). Solving for $r$ by Eq. (1.8) and Eq. (1.9) we get

$$r = f_K \left( \int_0^z \frac{dz'}{H(z')} \right).$$  \hspace{1cm} (1.14)
This enables us to rewrite Eq. (1.12) as
\[
d_A = \frac{1}{1+z} f_K \left( \int_0^z \frac{dz'}{H(z')} \right). \tag{1.15}
\]

One important standard ruler in cosmology comes from the CMB radiation. The predicted size of the sound horizon from the Baryonic Acoustic Oscillations (BAO) at the time of recombination, provides a scale that is measured through the acoustic peaks in the CMB. Their angular size gives a direct method of determining the spatial curvature of the universe. The recent data from the Planck Mission shows that the universe is practically flat \((K = 0)\).

Another important concept of distance in cosmology comes from the luminosity distance, defined as
\[
d_L \equiv \sqrt{\frac{L}{4\pi F}} \tag{1.16}
\]
where \(L = N_\gamma E_{em}/t_{em}\) is the absolute luminosity of an object and \(F = N_\gamma E_{obs}/(4\pi r^2 t_{obs})\) is the measured flux that is spreaded over a sphere. The number of photons \(N_\gamma\) is conserved but the observer takes a longer time \(t_{obs} = (1+z)t_{em}\) to receive the photons (from Eq. (1.5) and \(\lambda_{obs}/\lambda_{em} = t_{obs}/t_{em}\)). Also, their energy is redshifted so \(E_{em} = (1+z)E_{obs}\). It follows that
\[
d_L = (1+z) r = (1+z)^2 d_A. \tag{1.17}
\]

Objects with a known absolute luminosity are called standard candles, such as supernovae of type Ia (SNIa) [50]. We can get from the observed flux and redshift the corresponding luminosity distance \(d_L\). The measurements of distant SNIa led two research teams [6, 7] to first infer the current acceleration of the universe in the end of the 90’s.

1.3 General Relativity

The hypothesis for the kinematics and large-scale geometry (cosmography) is completely independent from the dynamical theory (cosmology). In order to get the dynamics of the universe one needs a theory which describes the gravitational interaction. Because
the universe is electrically neutral on large scales and the weak and strong interactions
have a very limited range, gravity is the dominant interaction on cosmic scales at the
present time. In GR gravity is a manifestation of the curvature of spacetime described
by a metric. The relation of the spacetime curvature with the distribution of matter
(expressed by the energy momentum-tensor $T_{\mu\nu}$) is given by the Einstein field equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $G_{\mu\nu}$ is the Einstein tensor given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$ (1.19)

The Ricci tensor $R_{\mu\nu}$ is expressed in terms of the Christoffel symbol

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha} - \Gamma_{\mu\alpha,\nu} + \Gamma_{\beta\alpha} \Gamma^\beta_{\mu\nu} - \Gamma_{\beta\nu} \Gamma^\beta_{\mu\alpha},$$ (1.20)

which in turn is obtained from the metric (and the inverse metric $g^{\mu\nu}$ where $g^{\mu\nu} g_{\nu\nu} = \delta^\mu_\alpha$)

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right).$$ (1.21)

The Ricci scalar is the contraction of the Ricci tensor $R = R^\mu_\mu = R_{\mu\nu} g^{\mu\nu}$.

Einstein’s equations can be derived from an action integral, known as Einstein-Hilbert
action, defined as

$$S = \int d^4 x \sqrt{-g} \left( \mathcal{L}_G + \mathcal{L}_M \right).$$ (1.22)

where $g = \text{det} (g_{\mu\nu})$ and $\mathcal{L}_G = \frac{1}{16\pi G} R$. The second Lagrangean is related with the
energy-momentum tensor as

$$T_{\mu\nu} = -2 \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_M$$ (1.23)
1.4 Perfect Fluid Energy-Momentum Tensor

The 4-momentum \( p^\mu = m u^\mu \) (where \( m \) is the rest mass, \( u^\mu = dx^\mu/d\tau \) the 4-velocity and \( \tau \) the proper time) gives a complete description of the energy-momentum of a particle but is insufficient when dealing with extended systems like a fluid. Conceptually, the energy-momentum tensor \( T^{\mu\nu} \) (also called stress-energy tensor) describes the flux of the 4-momentum \( p^\mu \) across a surface of constant \( x^\nu \).

A perfect fluid can be written as

\[
T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}
\]  \hspace{1cm} \text{(1.24)}

where \( \rho \) is the energy density and \( p \) is the pressure. In the rest frame the 4-velocity is \( u^\mu = (-1,0,0,0) \) and Eq. (1.24) is diagonal, so a perfect fluid is completely described by a rest frame energy density \( \rho \) and an isotropic rest frame pressure \( p \)

\[
T^{\mu\nu}_r = \begin{pmatrix}
-\rho & 0 & 0 & 0 \\
0 & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix}
\]  \hspace{1cm} \text{(1.25)}

The description of the energy content as a perfect fluid is enough to characterize a wide variety of cosmological fluids.

1.5 Friedmann Equations

With the set up of the previous sections, we can build a model for the dynamics of the background universe. Considering the flat FLRW metric (in Cartesian coordinates)

\[
ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\]  \hspace{1cm} \text{(1.26)}

we get the non-vanishing Christoffel symbols
\[ \Gamma^0_{ij} = \delta_{ij} \dot{a} a, \quad (1.27) \]
\[ \Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij} H, \quad (1.28) \]

and the components of the Ricci tensor

\[ R_{00} = -3 \frac{\ddot{a}}{a}, \quad (1.29) \]
\[ R_{ij} = \delta_{ij} \left[ 2 \dot{a}^2 + a \ddot{a} \right]. \quad (1.30) \]

Finally, the Ricci scalar is

\[ R = -R_{00} + \frac{1}{a^2} R_{ii} = 6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right]. \quad (1.31) \]

The cosmological dynamics can be obtained by solving Eq. (1.18) in the presence of a source \( T^\mu_\nu = \text{diag}(-\rho, p, p, p) \). Taking the (00) and the (ii) component of the Einstein equation we get respectively

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G}{3} \rho, \quad (1.32) \]
\[ \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} = -8 \pi G p, \quad (1.33) \]

usually Eq. (1.32) is called the Friedmann equation. An useful relation that follows from Eq. (1.32) and Eq. (1.33) is the Raychaudhuri equation

\[ \frac{\dot{a}}{a} = -\frac{4 \pi G}{3} (\rho + 3p). \quad (1.34) \]

From the equation above, it is clear that a necessary condition for an accelerated expansion is \( p < -\rho/3 \). The conservation of the energy-momentum tensor follows from the Bianchi identities

\[ T^\mu_{\nu;\mu} = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0, \quad (1.35) \]
and this is usually written as

\[ \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0, \tag{1.36} \]

also called the continuity equation. To close the set of equations we need to define the functional dependence between the state variables. For that purpose we introduce the equation of state parameter

\[ w = \frac{p}{\rho}. \tag{1.37} \]

In the case of an barotropic fluid we have \( p = p(\rho) \) which simplifies the discussion; we will assume it for now.

If several components are present, the Friedmann equation (Eq. (1.32)) may be written as

\[ H^2 = \frac{8\pi G}{3} \sum_i \rho_i \tag{1.38} \]

On the other hand, Eq. (1.36) holds for each individual fluid if the different components interact minimally, so that the energy-momentum tensor is separately conserved.

For a given \( w \), Eq. (1.36) shows how the energy density evolves as the universe expands. Let us see, in a general way, how different fluids behave. The continuity equation may be rewritten as

\[ p = -\frac{d}{d\ln a} (a^3 \rho). \tag{1.39} \]

It follows that for \( p = 0 \) the energy in some volume remains constant. If \( p > 0 \) or \( p < 0 \) the total amount of energy respectively decreases or increases with time considering that the universe is expanding. The energy density for constant-\( w \) fluids evolves as

\[ \rho_i \propto a^{-3(1+w_i)}. \tag{1.40} \]

### 1.6 Universe Components

Having an equation that relates the scale factor and the energy density now we can understand how different kinds of perfect fluids determine the behaviour of the universe. The critical density is defined as the total energy content of an universe with \( K = 0 \)

\[ \rho_c = \frac{3}{8\pi G} H^2. \tag{1.41} \]
The density parameter is defined as

$$\Omega_i = \frac{\rho_i}{\rho_c}$$  \hspace{1cm} (1.42)

and the sum of all the present content (denoted by \((0)\)) leads to \(\Omega_{\text{tot}}^{(0)} = \sum \Omega_i^{(0)} = 1\). The question that follows is how the energy is distributed among the different components.

1.6.1 Standard Content

The relativistic species in the universe are composed mainly by photons and neutrinos\(^1\). Neutrinos can be considered relativistic particles provided that their masses are small. For a fluid representing the radiation component the equation of state is given by \(w = 1/3\) and the energy density scales as \(\rho \propto a^{-4}\). Most of the photons come from the CMB, and their energy density in the universe can be characterized by its present temperature. This results in a present density parameter for the radiation component which includes photons and neutrinos

$$\Omega_r^{(0)} = 9.3 \times 10^{-5}. \hspace{1cm} (1.43)$$

It turn out that the radiation contribution to the total energy density is irrelevant at the present time. In cosmology we refer to baryonic matter to account for all the nuclei and electrons in the universe. Although electrons are leptons, this terminology stuck among the cosmological community. Since nuclei are much more massive than electrons all the mass is virtually in the baryons. The fluid that describes this component is considered to be non-relativistic and pressureless, which gives \(w = 0\) and \(\rho \propto a^{-3}\). Unlike the relativistic case, the energy density of non-relativistic matter is not a function of the temperature alone, and need to be measured directly, by emission or absorption in the electromagnetic spectrum or by gravitational effects. A more “indirect” account for the present baryonic density comes from the Big Bang Nucleosynthesis (BNN) or from CMB anisotropies (acoustic peaks). All these estimates using different techniques agree at a

\(^1\)Cosmic neutrinos have not yet been observed, but the theoretical arguments in favour of their existence are well grounded.
good extent. From Planck Mission [24]

\[ \Omega_b^{(0)} = 0.0487 \] (1.44)

i.e. about 5% of the total energy content in the universe.

### 1.6.2 Dark Universe

Under the assumptions of isotropy and homogeneity on large scales, GR and well established physics out the gravitation domain, it is found that baryonic matter accounts for only about 5% of the total energy density of the universe (the other 95% being in a dark form).

The first piece of evidence that suggested something beyond the standard content comes from a matter component that would interact only gravitationally, and therefore would be unnoticed by direct measurements involving the interaction of matter and radiation. This kind of matter was hypothesized by Jan Oort in 1932 to account for the orbital velocities of stars in the Milky Way. Fritz Zwicky in 1933, while studying the orbital velocities of galaxies in clusters, found that the observations supported Oort idea. However, the best astrophysical evidence for dark matter comes from the Bullet Cluster [31, 32], where the gravitational lensing data is shown to be incompatible with a total matter distribution similar to that of luminous matter. Another indication comes from the CMB anisotropies and the distribution of galaxies in the universe which constrains the dark matter content in the universe to be

\[ \Omega_{DM}^{(0)} = 0.268. \] (1.45)

Since dark matter is described by a non-relativistic and pressureless fluid like the baryonic component, we define the total matter density as the sum of baryons and dark matter

\[ \Omega_m^{(0)} = 0.315. \] (1.46)

In 1998 the observations of Type Ia supernovae from two groups [6, 7] give direct evidence for the so called dark energy (DE). They concluded that the luminosity distance is larger for objects at high redshift, which indicates an universe dominated by this exotic
substance. This distant objects appear fainter than what we would expect in a matter dominated universe. A suitable fluid to describe DE must have negative pressure, in practice \( p < -\rho/3 \) to be the responsible for the current acceleration of the universe. There are other observations such as the CMB radiation, BAO [33] and large scale structures (LLS) [34] that lead to the same result

\[
\Omega_{DE}^{(0)} = 0.683. \tag{1.47}
\]

### 1.7 ΛCDM model

The provided density parameters are cosmological parameters of the so called ‘concordance model’, due to the fact that it is able to fit a vast number of independent observations, such as the existence and structure of the CMB, the distribution of galaxies, light elements abundances and the accelerated expansion of the universe. This model attempts to establish the main features of DM and DE.

It was already pointed that several observations indicate the existence of DM. First of all it is worth pointing that some of the DM is baryonic. But BBN and CMB observations restrict the value of baryonic content is such way, that we know with very high confidence that

\[
\Omega_{\text{lum}}^{(0)} < \Omega_{b}^{(0)} < \Omega_{m}^{(0)}. \tag{1.48}
\]

The inequality is consistent with what we would expect; all luminous matter is constituted by baryons and part of the total matter is not baryonic. The unknown nature of the nonbaryonic dark matter is called ‘dark matter problem’. Usually, the considered candidates for this component can be divided in hot dark matter (HDM), warm dark matter (WDM) and cold dark matter (CDM). This classification is based on the typical velocities of the particles. The effect that these different classes of DM have in large scale structure (LSS) formation make CDM the favourite one to describe the observations. In the meantime, there is no particle in the Standard Model of particle physics that is suitable as a CDM candidate.

The concept of DE, to account for the measurements from the luminosity distances to faraway supernovae, turns out to be of extreme importance in contemporary Cosmology. The current accelerated expansion can be achieved in different ways, but the simplest
one is the cosmological constant $\Lambda$, originally introduced by Einstein to get a static universe. This constant field with Lagrangean

$$\mathcal{L}_\Lambda = -\frac{\Lambda}{8\pi G}$$

(1.49)

can be introduced in Eq. (1.22). From this we get the Einstein equation where the cosmological constant is included

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}. \tag{1.50}$$

Working Eq. (1.50) we obtain the correspondent Friedmann equation

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i + \frac{\Lambda}{3}, \tag{1.51}$$

and the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3p_i) + \frac{\Lambda}{3}. \tag{1.52}$$

In the last equation it is clear that $\Lambda$ induces a repulsive force.

The energy momentum tensor for the cosmological constant describes a perfect fluid with

$$p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi G}, \tag{1.53}$$

leading to an equation of state where $w_\Lambda = -1$ (from Eq. (1.40) we get $\rho_\Lambda = \text{const.}$)

So, the Friedmann equation for the background evolution (where the radiation density can be ignored at late times) in the $\Lambda$CDM model is

$$H^2 = \frac{8\pi G}{3} \left[ \rho_b^{(0)} a^{-3} + \rho_{DM}^{(0)} a^{-3} + \rho_\Lambda^{(0)} \right], \tag{1.54}$$

or in terms of the density parameters

$$H^2 = H_0^2 \left[ \Omega_b^{(0)} a^{-3} + \Omega_{DM}^{(0)} a^{-3} + \Omega_\Lambda^{(0)} \right], \tag{1.55}$$

where $H_0^2 = 8\pi G \rho_c^{(0)}/3$. 

Although the ΛCDM model proved to be consistent with a large number of cosmological observations, the nature of dark matter and dark energy is not yet well established. The best candidate from the domain of particle physics to represent dark matter is a weakly interacting massive particle (WIMP) [43] which was not yet detected [44]. On the other hand, if we require the cosmological constant to explain the late-time accelerated expansion we have other problems. The natural interpretation of Λ as the vacuum energy leads to a discrepancy of many order of magnitude when compared with the observed value (the ’cosmological constant problem’) [63]. We also have the coincidence problem: why does the cosmological constant start dominating so close to the present (see Figure 1.1). These questions motivate the search of other models as viable alternatives to the standard model.

Figure 1.1: Evolution of the density parameters Ω_0, Ω_{CDM} and Ω_Λ as a function of the redshift in the ΛCDM model.
Chapter 2

UDE Models - Background Cosmology

In the previous chapter we reviewed several hypothesis which are at the core of the Standard Cosmology: the homogeneity and isotropy of the universe on large scales (known as the Cosmological Principle, encoded in the FLRW metric), gravity as the dominant interaction and General Relativity as the correct description of the gravitational interaction. It is possible to obtain a solution to the Einstein’s equations (Friedmann equations) that encloses all these assumptions and provide a way to study the dynamics of the universe. We have found that, in order to explain the current observational data, baryonic matter and radiation are not enough, and other forms of energy that do not interact electromagnetically are required. The ΛCDM model assumes that this dark component is constituted by two different entities, where basically (cold) dark matter would be the responsible for the amount of structure observed at various scales, and dark energy would explain the late time acceleration of the universe. In spite of its success, there is still no clear direct detection of these dark forms\footnote{Some hints for the existence of dark matter particles come from cosmic rays experiments, but they are far from being conclusive. See ref.[21–24]} and their precise nature is still unknown. This is a weird fact on itself, considering that the universe would be constituted by $\sim 95\%$ of this dark substance.

But one should have in mind that all this follows from our initial assumptions, meaning that if we’re not making a correct initial description of the universe, these dark components may not exist at all. One of the possibilities is that GR would not hold on
large scales, or is somehow flawed. Some theories like $f(R)$ gravity [53], scalar-tensor theories [54] or MoND [55] (followed by TeVeS generalization [56]) try to substitute dark energy or/and dark matter by a modification of gravity [8]. Still, such theories are highly restricted from local gravity tests and observational constrains.

In this dissertation we work under the assumptions presented in Chapter 1, so we'll assume that dark components are real entities. The next question would be: How many of them? ΛCDM requires two: both dark energy and dark matter. But they do not have to be, a priori, independent entities. We will explore the possibilities of a unified scene of dark energy and dark matter (Unified Dark Energy - UDE, also known as Quartessence), an idea that sprung from the cosmological properties of the Chaplygin Gas [9].

2.1 Quartessence

General models dealing with the paradigm of unification of dark matter and dark energy are named Quartessence. They represent a special subclass of the k-essence models where the core idea is that the cosmic acceleration can be realized by the kinetic energy of a scalar field. The background dynamics for a general Unified Dark Energy (UDE) model can be easily obtained from Eqs. (1.32) and (1.33) [20]

$$\dot{\rho} = \frac{3}{4\pi G} H \frac{dH}{dt},$$

$$(2.1)$$

$$\dot{p} = -\frac{1}{4\pi G} \frac{d}{dt} \left[ \frac{\ddot{a}}{a} + \frac{1}{2} H^2 \right].$$

$$(2.2)$$

In the case of a barotropic fluid we have a sound speed of

$$c_s^2 = \frac{dp}{d\rho} = \frac{1}{3H} \frac{d}{dH} \left[ H^2 \left( q - \frac{1}{2} \right) \right],$$

$$(2.3)$$

where $q = -\frac{\ddot{a}}{aH^2}$ is the usual deceleration parameter. The sign of the sound speed squared is determined by the way $q$ is evolving. If evolves fast enough towards negative values then $c_s^2 > 0$; otherwise $c_s^2 < 0$. The evolution of $q$ is in turn linked to how fast the transition from dark matter to dark energy occurs for quartessence. If it is faster (slower) than ΛCDM, $c_s^2$ will be positive (negative). Therefore, the sign of $c_s^2$ is connected to the background dynamics.
2.2 The Chaplygin Gas

The first definite prototype of a quartessence model that was proposed is the Chaplygin gas model. We shall define the Chaplygin gas (CG) as a perfect fluid having the following equation of state

\[ p_{cg} = -\frac{A}{\rho_{cg}}, \tag{2.4} \]

where \( A \) is a positive constant. See [9, 17, 18] for a derivation of the above equation of state in the context of string theory. A straightforward generalization of Eq. (2.4) can be made [10]

\[ p_{cg} = -\frac{A}{\rho_{cg}^\alpha}, \tag{2.5} \]

with \( \alpha \) restricted to the interval \( 0 < \alpha \leq 1 \) and the state parameter given by \( w_{cg} = -\frac{A}{\rho_{cg}^\alpha+1} \). Negative values of \( \alpha \) can be assumed, but later we shall see that this would lead to instabilities associated with imaginary sound speeds.

Being a perfect fluid, the properties of the CG are completely specified once the equation of state is known. Let us see how the energy of the generalized Chaplygin gas (GCG) evolves with time in a homogeneous and isotropic universe. From Eq. (1.36) we get

\[ \rho_{cg}(a) = \left[ A + a^{-3(1+\alpha)} \left( \rho_{cg(0)}^{1+\alpha} - A \right) \right]^{\frac{1}{1+\alpha}}. \tag{2.6} \]

Eq. (2.6) can be rewritten as

\[ \rho_{cg}(a) = \rho_{cg(0)} \left[ \overline{A} + (1 - \overline{A}) a^{-3(1+\alpha)} \right]^{\frac{1}{1+\alpha}}, \tag{2.7} \]

where \( \overline{A} = A/\rho_{cg(0)}^{1+\alpha} \). From this expression it is clear that this fluid behaves as matter in the past \( (a \ll 1, \rho_{cg} \propto (1 - \overline{A})^{\frac{1}{1+\alpha}} a^{-3}) \) and as cosmological constant towards the future \( (a \gg 1, \rho_{cg} \propto \overline{A}^{\frac{1}{1+\alpha}}) \). If \( \overline{A} = 0 \) we recover the SCDM model. The evolution of the equation of state parameter of the GCG changes from a “matter state” \( (w_{cg} = 0) \) deep into the matter era to a “vacuum state” \( (w_{cg} = -1) \) as the universe expands (Figure 2.1). The energy density has a minimum of \( \rho_{min} = \rho_0 \overline{A}^{\frac{1}{1+\alpha}} = A^{\frac{1}{1+\alpha}} \), so the Chaplygin gas never dilutes completely. Notice that it is straightforward to incorporate a radiation dominated era for small values of the scale factor [15, 16].

The condition for an accelerating universe comes from Eq. (1.34) i.e \( \rho_{cg} + 3p_{cg} < 0 \) that gives for the scale factor
\[ a_{acc} > \left( \frac{1 - A}{2A} \right)^{\frac{1}{1+\alpha}}. \]  
(2.8)

For the Chaplygin gas, the sound speed will be proportional to the equation of state parameter

\[ c_{x, cg}^2 = \frac{dp}{d\rho} = -\alpha w_{cg}. \]  
(2.9)

and the deceleration parameter is

\[ q = \frac{1}{2} (1 + 3w_{cg}) \]  
(2.10)

From the previous analysis about the evolution of the energy density, we see that the sound speed changes from nearly zero to \( c_{x, cg}^2 = \alpha \) at later times (Figure 2.1). The constrain imposed from the beginning for \( \alpha \) limits \( c_{x, cg}^2 \leq 1 \), so that it is bounded by the speed of light.

![Figure 2.1](image-url)

**Figure 2.1:** The GCG state \( w = p/\rho \) and its square sound speed \( c_x^2 = dp/d\rho \) as a function of the scale factor. From this figure it is clear that the phase transition from the matter era \( (w = 0) \) to a cosmological constant era \( (w = -1) \) coincides with the transition to a non-null sound speed \( c_x^2 = -\alpha w \).
The non-null sound speed for the Chaplygin Gas is the conundrum of the model when linear perturbation theory is considered. Sound speed tell us how fast perturbations propagate across the fluid influencing the formation of structures. We will tackle this problem in Chapters 3 and 4.

2.3 Scalar Field implementation for GCG

Until now the description of the Chaplygin Gas was given in a hydrodynamic language, which is appropriate for a phenomenological description. It is interesting to describe the equation for the CG in terms of a scalar field [19]. Let us consider the action

\[ S_\phi = \int d^4x \sqrt{-g} \mathcal{L} (X, \phi) \tag{2.11} \]

with the canonical Lagrangian

\[ \mathcal{L} = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi), \tag{2.12} \]

where \( X = -\frac{1}{2} \nabla^\rho \phi \nabla_\rho \phi \) is the kinetic term and \( V(\phi) \) is some scalar potential. In a cosmological context, \( \phi \) is commonly called a “Quintessence” field. Its energy-momentum tensor may be obtained by varying the action in respect to the metric

\[ \delta S_\phi = \int d^4x \left[ \sqrt{-g} \left( -\frac{1}{2} \delta g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \delta \left( \sqrt{-g} \right) (X - V(\phi)) \right] \]

\[ = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[ -\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (X - V(\phi)) \right] \tag{2.13} \]

and therefore

\[ T_{\mu\nu}(\phi) = \nabla_\mu \phi \nabla_\nu \phi + (X - V(\phi)) g_{\mu\nu} \tag{2.14} \]

By making the the identifications

\[ u_\mu = \frac{\nabla_\mu \phi}{\sqrt{2X}} , \quad \rho = X + V(\phi) , \quad p = X - V(\phi) \tag{2.15} \]

we recover Eq. (1.24). So \( T_{\mu\nu}(\phi) \) can be written in the form of a perfect fluid when one considers a Quintessence field. Notice that the Lagrangian equals the pressure of the field \( \mathcal{L} = X - V = p \). The quantity \( \nabla_\mu \phi \) is required to be timelike since the 4-velocity
is a timelike vector. This procedure can be easily generalized to define a new class of scalars that enable us to find the correct Lagrangian for the GCG. Keeping a generic Lagrangian yields

\[ T_{\mu\nu} = \mathcal{L}(X,\phi) g_{\mu\nu} + \mathcal{L}(X,\phi)_X \nabla_\mu \phi \nabla_\nu \phi \quad (2.16) \]

and as before, the Lagrangian still plays the role of pressure. To explicitly rewrite Eq. (2.16) in a perfect fluid form we identify

\[ u_\mu = \frac{\nabla_\mu \phi}{\sqrt{2X}}, \quad \rho = 2X p, X - p \quad (2.17) \]

From this we conclude that if the pressure is a function of kinetic energy alone, so it is the energy density \((p = p(X) \rightarrow \rho = \rho(X))\). If the density is rewritten as a function of \(X = X(\rho)\) we get an explicit barotropic equation of state \(p = p(\rho)\). For instance if \(p \propto X^n\) the state parameter is \(w = \frac{1}{2n+1}\): \(n = 0\) corresponds to a cosmological constant \(\Lambda\), \(n = 1\) to a massless scalar field, \(n = 2\) to background radiation, and so on. In the \(n \rightarrow \infty\) limit we recover the case of pressureless non-relativistic matter \((w = 0)\).

Applying Eq. (2.5) to Eq. (2.17) delivers the following non-linear differential equation

\[ \rho_{cg} = 2X^\alpha \frac{A}{\rho_{cg} + T} \rho_{cg, X} + \frac{A}{\rho^\alpha} \quad (2.18) \]

The change of variable \(\xi = \frac{A}{\rho^\alpha} \) results in a simpler linear version

\[ 1 = -\frac{\alpha}{1+\alpha} 2X \xi_{,X} + \xi \quad (2.19) \]

with the solution \(\xi = 1 - (2X)^{\frac{1+\alpha}{2\alpha}}\). Finally, the Lagrangian that reproduces the GCG is

\[ \mathcal{L}(X) = p(X) = -\frac{A}{\rho_{cg}^\alpha} = -A^{\frac{1}{1+\alpha}} \xi(X)^{\frac{\alpha}{1+\alpha}} \quad (2.20) \]

and the restriction \(0 < 2X < 1\) ensures that \(p(X)\) has a non-null real value.

### 2.4 ΛCDM as a single fluid and GCG \(\alpha \rightarrow 0\) limit

In the first chapter the ΛCDM model was presented. Its usual interpretation relies on the assumption that dark energy and dark matter are two different entities played by the cosmological constant \(\Lambda\) and CDM respectively. This is mainly due to historical
reasons: DM was required in order to account for the observed dynamics of galaxies and clusters of galaxies well before the first evidence for the acceleration of the universe. Nevertheless, a single fluid interpretation is also possible. Considering two different fluids $\Lambda$ and CDM we have

\[ T_{\Lambda}^{\mu\nu} = p_{\Lambda} g^{\mu\nu}, \quad T_{\text{CDM}}^{\mu\nu} = \rho_{\text{CDM}} u^\mu u^\nu \]  

(2.21)

and we can rearrange them in a single energy-momentum tensor

\[ T^{\mu\nu} = \rho_{\text{CDM}} u^\mu u^\nu - p_{\Lambda} g^{\mu\nu} \]  

(2.22)

From the equation of state for the cosmological constant $p_{\Lambda} = -\rho_{\Lambda} = \text{const}$ it follows

\[ T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu} \]  

(2.23)

where we make the identifications $\rho = \rho_{\text{CDM}} + \rho_{\Lambda}$ and $p = p_{\Lambda}$. In this way we have both dark energy and dark matter as a single perfect fluid. This is actually a characteristic shared by the Chaplygin gas. If we take $\alpha = 0$ in Eq. (2.5) we get a model with density $\rho_{cg}$ and pressure $p_{cg} = -A = \text{const}$. The background dynamics (from Eqs. (1.32,1.33)) are identical for the $\Lambda$CDM as a single fluid and GCG with $\alpha = 0$ if we identify the total density and pressure [26]

\[ \rho_{cg} = \rho = \rho_{\text{CDM}} + \rho_{\Lambda} \]
\[ p_{cg} = p = p_{\Lambda} = -A \]  

(2.24)

and interpret $A$ in the Chaplygin gas as the vacuum energy density of the $\Lambda$CDM model. So, at the background level, the two models have the same dynamics. In fact this equivalence goes beyond the zero order. With the scalar field description of the GCG from the previous section, it can be shown that the $\alpha = 0$ GCG model is equivalent to $\Lambda$CDM [19] at all orders.

### 2.5 UDE Background Tests

In order to confront the theoretical predictions with the observations we got to rely in a statistical analysis of the problem. Given our model and the observations, we want to determine the best parameters that fit the results. Here we will test the background
dynamics of the GCG plus a baryonic component given by the equation

\[ \frac{H^2}{H_0^2} = \Omega_b^{(0)} a^{-3} + \Omega_{cg}^{(0)} \left[ \bar{A} + (1 - \bar{A}) a^{-3} \right] \frac{1}{1 + \alpha} \]  \hspace{1cm} (2.25)

and compare the distance modulus available from Union 2.1 data set [35]. The distance modulus is defined as the difference between the apparent and absolute magnitude

\[ \mu = m - M = 5 \log_{10} (d_L) + 25 \]  \hspace{1cm} (2.26)

and assuming that the universe is flat we get the luminosity distance from Eq. (1.17)

\[ d_L = (1 + z) \int_0^z \frac{dz'}{H(z')} \]  \hspace{1cm} (2.27)

Since SNIa are standard candles and the redshift is known, this makes the distance modulus a suitable quantity to test the background dynamics of cosmological models.

### 2.5.1 Likelihood Function

Let us assume that a model \( M(p) \) is able to predict some observable quantity \( \mu_M \) given the \( l \) parameters \( p_1, \ldots, p_l \) of the theory. If we have a set of several observations \( \{ \mu_i \} \), we want to obtain the probability of \( M(p) \) delivering the right result when confronted with the data \( \{ \mu_i \} \), i.e. \( P(M(p)|\{\mu_i\}) \). In our particular case, we are dealing with supernovae observations, and it can safely be assumed that they are independent measurements, so

\[ P(\{\mu_i\}|M(p)) = P(\mu_1|M(p)) \ldots P(\mu_n|M(p)) \]  \hspace{1cm} (2.28)

We assume that a single observation will be the signal plus some noise component given by a Gaussian distribution. So, if we have just one data point, the probability of getting \( \mu_i \) given the theory is

\[ P(\mu_i|M(p)) \propto \exp \left( -\frac{(\mu_i - \mu_M)^2}{2\sigma_i^2} \right) \]  \hspace{1cm} (2.29)

where the distribution as zero mean and variance \( \sigma_i^2 \). The last expression is known as likelihood function. But we want to know what are the theoretical parameters given the available data. From Bayes’ theorem we have the following rule for the conditional
probabilities: \( P(A \cap B) = P(B|A) P(A) = P(A|B) P(B) \). This delivers the result

\[
P(M(p)|\mu_i) = \frac{P(M(p))}{P(\mu_i)} P(\mu_i|M(p))
\]

(2.30)

The denominator can be seen as a constant independent of the model; when integrating the probability \( P(M(p), \mu_i) \) over all values of the parameters we must get 1. The first term in the numerator is called the model’s prior. Taking a conservative approach we assume a uniform prior.

From Eq. (2.28) we get

\[
P(M(p) | \{\mu_i\}) \propto \exp \left( -\sum_i \frac{(\mu_i - \mu_M)^2}{2\sigma_i^2} \right).
\]

(2.31)

Usually the sum is denoted as \( \chi^2 \) and the likelihood is simply expressed as \( L = \exp(-\chi^2/2) \). The best-fit values are the ones where the likelihood function is bigger; then we can obtain the confidence regions for the model parameters computing the likelihood at every point \( p = (p_1, \ldots, p_l) \).

### 2.5.2 Results

We have plotted the confidence regions for the GCG model to constrain the parameters \( \bar{A} \) and \( \alpha \) in a \( 100 \times 100 \) grid using supernova data. The Hubble constant was marginalized in order to eliminate the uncertainty in this parameter i.e. \( \mathcal{L}(\bar{A}, \alpha) = \sum_i \mathcal{L}(\bar{A}, \alpha, h_i) \) for the interval \([0.665, 0.688]\). The \( \chi^2 \) fitting comes from

\[
\chi^2 = (\bar{\mu}_M - \bar{\mu})^T C^{-1} (\bar{\mu}_M - \bar{\mu})
\]

(2.32)

where \( \bar{\mu}_M \) is the distance modulus given by Eq. (2.26) for the GCG and \( \bar{\mu} \) and \( C \) are respectively the distance modulus and covariance matrix of the Union 2.1 data set [35]. Since the measurements are independent, the covariance matrix is diagonal. This data set consists of 580 distance modulus of supernova at small and large redshifts. We see that the confidence region in the parameter space (Figure (2.2)) restricts \( \bar{A} \) to the interval \( 0.69 < \bar{A} < 0.77 \), while \( \alpha \) can take the values \( 0 \leq \alpha \lesssim 0.4 \). Notice that we can fix the value of \( \bar{A} \) for the GCG model in order to match the densities of CDM for \( a \ll 1 \).
and of the cosmological constant when $\alpha \gg 1$

$$\bar{A} = \left(\frac{1 - \Omega_m^{(0)}}{1 - \Omega_b^{(0)}}\right)^{1+\alpha}.$$  \hspace{1cm} (2.33)

So, for $\alpha = 0$, $\Omega_m^{(0)} = 0.315$ and $\Omega_b^{(0)} = 0.0487$ we get $\bar{A} \sim 0.72$. These values are well inside the confidence region, which is not surprising, since the GCG is completely equivalent to the $\Lambda$CDM model when $\alpha = 0$.

Although $\alpha$ is not so highly restricted as $\bar{A}$, this likelihood analysis rules out the value $\alpha = 1$, associated to the original proposal of the Chaplygin Gas. Previous results using supernova data (see for instance [12]) did not confine this parameter and all range of values were possible. Largely, this was due to the overestimation of the Hubble parameter ($h \simeq 0.72$) when compared with the latest results from the Planck mission ($h = 0.673$), and also due to a higher uncertainty of the value. Because this analysis is only sensitive to the background evolution, we expect a higher restriction to the value

![Figure 2.2: Confidence region for the GCG parameters ($\bar{A}, \alpha$). We assume an universe with GCG and brayons with a present density $\Omega_b^{(0)} = 0.0487$ such that $\Omega_{cg} + \Omega_b = 1$. The Hubble constant was marginalized for the interval [0.665, 0.688]](image-url)
of $\mathfrak{A}$, since it is related with the energy density of the GCG model and therefore, is the responsible for the background dynamics.
Chapter 3

Linear Perturbation Theory for UDE

Up to this point we discussed the consequences of a homogeneous and isotropic universe described by the FLRW metric. Although it is an adequate picture for large scales, our universe is more complex and far richer; it contains inhomogeneous structures such as stars, galaxies and clusters of galaxies. So, in order to take into account the inhomogeneities present in our universe, we present some basic concepts of linear perturbation theory, where small perturbations around the ‘unperturbed’ background are considered (for a detailed account see for instance [13, 36–38]).

Perturbation theory is usually used in the context of cosmology to describe the formation of cosmic structures assuming that the early universe was in a nearly uniform state; a valid assumption given the very small CMB anisotropies. In the process, non-linear structures evolve from small initial perturbations due to self-gravity. These primordial perturbations (the ‘seeds’ of cosmic structure) are predicted by the inflation scenario, and have so far agreed very well with the observations.

3.1 The perturbed metric

To get the perturbed relativistic equations we must consider perturbations to the FLRW metric
\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \]  

where \( g^{(0)}_{\mu\nu} \) stands for the background metric (zero-th order) and \( \delta g_{\mu\nu} \) is the perturbed part. We assume that these deviations can grow due to gravitational instability and form cosmic structures. Let us write the background metric directly in terms of the conformal time

\[ ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left( -d\eta^2 + \delta_{ij} dx^i dx^j \right). \]  

We will also make use of the conformal Hubble parameter

\[ \mathcal{H} = \frac{1}{a} \frac{da}{d\eta} = aH. \]  

The most general linear perturbation around the FLRW metric when decomposed into its irreducible parts reads as

\[ ds^2 = a^2(\eta) \left\{ -(1 + 2\psi) d\eta^2 + 2w_i d\eta dx^i + [(1 - 2\phi) \delta_{ij} + 2h_{ij}] dx^i dx^j \right\}, \]  

where \( \psi \) and \( \phi \) are spatial scalars, \( w_i \) is a 3-vector and \( h_{ij} \) is a symmetric and traceless \( (\delta_{ij} h_{ij} = 0) \) second order tensor. Physically, tensor modes correspond to gravity waves, vectors modes are associated to rotational velocity perturbations and scalars are related to the density perturbations. At first order perturbations these modes decouple completely and can be treated separately. The 3-vector \( \mathbf{w} \) can be decomposed into longitudinal and transverse parts

\[ \mathbf{w} = \mathbf{w}_\parallel + \mathbf{w}_\perp \]  

such that \( \nabla \times \mathbf{w}_\parallel = 0 \) and \( \nabla \cdot \mathbf{w}_\perp = 0 \). Given that the curl of a gradient is always zero, the longitudinal part can be written as the gradient of a scalar \( \mathbf{w}_\parallel = \nabla w_s \). On the other hand the transverse part we can write it as \( \mathbf{w}_\perp = \nabla \times \mathbf{w}_v \). It is clear that the longitudinal part has one degree of freedom and the transverse part two degrees of freedom. By definition \( \mathbf{w}_\parallel \) represents a scalar perturbation and \( \mathbf{w}_\perp \) represents a vector perturbation. Analogously, we can decompose \( h_{ij} \) as

\[ h = h_\parallel + h_\perp + h_T. \]
The longitudinal part can be derived from a scalar function so it will have just one degree of freedom
\[
h_{ij||} = \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) h_s, \tag{3.7}
\]
while \( h_\perp \) and \( h_T \) carry two degrees each. Here, we shall study the growth of cosmic structures and, consequently, only scalar perturbations shall be considered. So, when we make reference to the quantities \( u_i \) and \( h_{ij} \) we are just assuming the correspondent scalar part.

### 3.2 Perturbed Field Equations

In order to derive the first-order Einstein equations we decompose the Einstein tensor \( G_\mu^\nu \) (Eq. (1.19)) and the energy-momentum tensor \( T_\mu^\nu \) (Eq. (1.25)) into background and perturbed parts
\[
G_\mu^\nu = G_\mu^\nu(0) + \delta G_\mu^\nu, \quad T_\mu^\nu = T_\mu^\nu(0) + \delta T_\mu^\nu. \tag{3.8}
\]
which results in the Einstein equations for the background
\[
G_\mu^\nu(0) = 8\pi G T_\mu^\nu(0), \tag{3.9}
\]
and for the perturbations
\[
\delta G_\mu^\nu = 8\pi G \delta T_\mu^\nu. \tag{3.10}
\]
The perturbed Einstein tensor is
\[
\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} (\delta g_{\mu\nu} R + g_{\mu\nu} \delta R) \Rightarrow \delta G_\mu^\nu = \delta g^{\mu\alpha} G_{\alpha\nu} + g^{\mu\alpha} \delta G_{\alpha\nu}, \tag{3.11}
\]
and going all the way through the perturbed Ricci scalar \( \delta R \), Ricci tensor \( \delta R_{\mu\nu} \) and Christoffel symbols \( \delta \Gamma^\mu_{\nu\lambda} \) we get everything in terms of the perturbed metric
\[
\delta R = \delta g^{\mu\alpha} R_{\alpha\mu} + g^{\mu\alpha} \delta R_{\alpha\mu}, \tag{3.12}
\]
\[
\delta R_{\mu\nu} = \delta \Gamma^\alpha_{\mu\nu,\alpha} - \delta \Gamma^\alpha_{\mu\alpha,\nu} + \delta \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\nu} \delta \Gamma^\beta_{\alpha\beta} \delta G_{\alpha\nu}, \tag{3.13}
\]
\[
\delta \Gamma^\mu_{\nu\lambda} = \frac{1}{2} \delta g^{\mu\alpha} (g_{\alpha\nu,\lambda} + g_{\alpha\lambda,\nu} - g_{\nu\lambda,\alpha}) + \frac{1}{2} g^{\mu\alpha} (\delta g_{\alpha\nu,\lambda} + \delta g_{\alpha\lambda,\nu} - \delta g_{\nu\lambda,\alpha}). \tag{3.14}
\]
From the condition $g_{\nu\alpha}g^{\alpha\mu} = \delta^\mu_\nu$ it follows that, at first order (drooping the quadratic terms of the perturbed quantities)

$$\delta g^{\mu\nu} = -\delta g_{\alpha\beta} g^{(0)\alpha\mu} g^{(0)\beta\nu}. \quad (3.15)$$

For our purposes it is enough to assume that the perturbed fluid can also be treated as a perfect fluid, which implies that $\delta T_{ij} = 0$ if $i \neq j$. So, restating Eq. (1.25)

$$T^\mu_\nu = (\rho + p) u^\mu u_\nu + p \delta^\mu_\nu \quad (3.16)$$

we have

$$\delta T^\mu_\nu = (\delta \rho + \delta p) u^\mu u_\nu + (\rho + p) \left( u^\mu \delta u_\nu + u_\nu \delta u^\mu \right) + \delta p \delta^\mu_\nu \quad (3.17)$$

Now we need to evaluate the velocity perturbations. The fluid velocity can be written as $u^\mu = u^0(1, v^i)$ where $v_i = v^i = dx^i/d\eta = adx^i/dt$ is the coordinate 3-velocity and $u^0 = d\eta/dt$. From the normalization condition for the 4-velocity $g_{\mu\nu} u^\mu u^\nu = -1$ we obtain

$$u^0 = \frac{1}{a \sqrt{1 - v^2}} \left[ 1 - \psi - w_i v^i + \phi v^2 - h_{ij} v^i v^j \right] \quad (3.18)$$

If the fluid is non-relativistic the quadratic terms can be neglected. Keeping only linear terms of the perturbed quantities leads to

$$u^\mu = \left[ \frac{1}{a} \left(1 - \psi\right), \frac{v^i}{a} \right] \quad (3.19)$$

$$u_\mu = g_{\mu\nu} u^\nu = [-a (1 + \psi), a (v_i + w_i)] \quad (3.20)$$

The 4-velocity in the background is $u^\mu(0) = (a^{-1}, 0, 0, 0)$ so, from $u^\mu = u^\mu(0) + \delta u^\mu$ we have

$$\delta u^\mu = \left( -\frac{\psi}{a}, u^i \right), \quad \delta u_\mu = (-a \psi, u_i) \quad (3.21)$$

which results in the total energy-momentum tensor

$$T^0_0 = -(\rho + \delta \rho), \quad T^i_0 = -(\rho + p) v^i$$

$$T^0_i = (\rho + p) (v_i + w_i), \quad T^i_j = (p + \delta p) \delta^i_j \quad (3.22)$$
For later convenience let us introduce the notation for the perturbed quantities. The density contrast and the velocity divergence are defined as

\[ \delta = \delta (x) = \frac{\rho (x) - \bar{\rho}}{\bar{\rho}} \quad (3.23) \]

\[ \theta = \nabla_i v^i \quad (3.24) \]

where \( \bar{\rho} \) is the spatial average of the energy density. Also, as it was defined for the background quantities, \( w = p/\rho \) is the equation of state and \( c_s^2 = \delta p/\delta \rho \) is the sound velocity. We are assuming a barotropic fluid so that, even when perturbed, \( p \) depends only on \( \rho \) alone.

### 3.3 Statistics of Scalar Perturbations

It is not expected that the theory for the density perturbations presented in the previous section would be able to predict \( \delta (x) \) at some particular location \( x \). We got to rely on a statistical statement to compare the theory with observations. For that purpose, we will explore some basic statistical properties enforced on the density contrast. A more complete account of the application of statistical physics in cosmology can be found in [25].

Let us assume that the density contrast (Eq. (3.23)) is a random field. From the definition it follows that it has zero mean \( \langle \delta (x) \rangle = 0 \). Statistical homogeneity requires that the statistical properties of the translated field are the same as the original field. That is, it is possible to divide the universe into sufficiently large regions, such that in each region of volume \( V = L^3 \) the mass distribution is the same on average, and the statistical properties are similar. So this volume represents a fair sample of the universe \( (L \gg l_s \text{ such that } l_s \text{ identify the maximum scale where significant structure still exists}) \). This results in a 2-point correlation function that only depends on the distance \( r \) between them

\[ \xi (r) = \langle \delta (x) \delta (x + r) \rangle = \frac{1}{V} \int \delta (x) \delta (x + r) \, d^3x. \quad (3.25) \]
The angle brackets denote the average over the normalization volume $V$. The decomposition of the field in Fourier components is

$$\delta(x) = \frac{V}{(2\pi)^3} \int e^{ik \cdot x} \delta_k d^3k,$$

(3.26)

$$\delta_k = \frac{1}{V} \int e^{-ik \cdot x} \delta(x) d^3x,$$

(3.27)

and because the density contrast is a real field, the condition $\delta_k^* = \delta_{-k}$ has to be satisfied (notice that the numerical prefactors vary substantially in the literature). Assuming spatial isotropy, the correlation function only depends on the modulus $r = |r|$, so we can integrate the angular part and rewrite Eq. (3.25) as

$$\xi(r) = \frac{V}{(2\pi)^3} \int |\delta_k|^2 e^{-ikr \cos \theta} d^3k = \frac{1}{2\pi^2} \int P(k) \frac{\sin(kr)k^2}{kr} dk.$$  

(3.28)

The quantity $P(k) = V |\delta_k|^2$ is usually called the power spectrum of the field and depends only on $k = |k|$. It has dimensions of length$^3$, but one can also express the power spectrum as a dimensionless function

$$\triangle(k) = \frac{k^3 P(k)}{2\pi^2}.$$  

(3.29)

The growth of cosmic structures due to their own gravity can be described using scalar perturbations. The power spectrum plays a central role in cosmology due to its ability to describe the level of clustering in the linear and mildly non-linear regime. The non-linear scale is roughly set by $\triangle(k_{nl}) \simeq 1$. Inflation theory predicts a power law for the primordial power spectrum $P_i(k) = Ak^{n_s}$, where $n_s$ is the spectral index and $A$ is the amplitude of the power spectrum. While its shape can be known, the overall amplitude is not specified by the current models of inflation, and it has to be fixed by the observations. The usual way is through the observed mass fluctuation amplitude at the present day on $8 \, h^{-1}\text{Mpc}$, a typical scale of galaxy clusters. The root mean square of the density fluctuations is (remember that $\langle \delta(x) \rangle = 0$)

$$\sigma^2 = \langle \delta^2(x) \rangle = \frac{1}{V} \int \delta^2(x) d^3x = \frac{V}{(2\pi)^3} \int d^3k |\delta_k|^2 = \frac{V}{2\pi^2} \int_0^\infty |\delta_k|^2 k^2 dk.$$  

(3.30)

The power spectrum completely characterizes the density fluctuations, when one assumes Gaussian statistics. We expect $P(k)$ to become small on large scales, but for small scales
this does not necessarily happen and the power spectrum may not converge. The usual
way to deal with this unwanted feature of the density field is to perform a cut-off at
a given scale, where we consider that the structures have highly non-linear properties
inside that region. For that purpose, we define a spherical top-hat window function

\[
W(x) = \begin{cases} 
\frac{1}{V}, & x < R \\
0, & x > R 
\end{cases}
\]  (3.31)

This function selects particles inside a spherical volume \( V \), and in this way structures
smaller than the smoothing scale are wiped out. The Fourier transform of the Window
function is

\[
W(k, R) = \int W(x) e^{-i k \cdot x} d^3x = \frac{3 \left( \sin(kR) - kR \cos(kR) \right)}{(kR)^3}
\]  (3.32)

so we are able to write a convergent version of Eq. (3.30)

\[
\sigma_R^2 = \frac{1}{2\pi^2} \int_0^\infty P(k) W^2(k, R) k^2 dk < \sigma^2.
\]  (3.33)

From Eq. (3.33) we can fix the amplitude of the power spectrum once \( \sigma_R^2 \) is known, the
usual choice being \( R = 8 h^{-1}\text{Mpc} \).

The observed distribution of matter that we are able to detect is predominantly from
late times. To relate the power spectrum at these times with the primordial power
spectrum predicted by the inflationary model, we introduce the fitting form of Bardeen,
Bond, Keiser and Szalay (1986) or BBKS Transfer Function [62]

\[
T(k) = \frac{\ln \left( 1 + 2.34 \left( \frac{k}{\Gamma} \right) \right)}{2.34 \left( \frac{k}{\Gamma} \right)} \left[ 1 + 3.89 \left( \frac{k}{\Gamma} \right) + \left( 16.1 \left( \frac{k}{\Gamma} \right) \right)^2 + \left( 5.46 \left( \frac{k}{\Gamma} \right) \right)^3 + \left( 6.71 \left( \frac{k}{\Gamma} \right) \right)^4 \right]^{-\frac{1}{2}}.
\]  (3.34)

The wave number \( k \) is given in units of \( h \text{Mpc}^{-1} \) and the coefficient \( \Gamma = \Omega^{(0)}_{\text{m}} h \) is called
the shape parameter. The shape parameter takes in account the stagnation period of
CDM perturbations for scales inside the Hubble radius, due to Meszaros effect, during
the radiation era. This is due to the fact that, even if CDM does not couple to photons,
the rate of the background expansion during the radiation era prevents the gravitational
collapse. However, scales larger than the Hubble radius at matter-radiation equality do not experience this effect as they become smaller than the Hubble radius already in the matter era. So, the transfer function enable us to process the primordial power spectrum from the radiation era to the late times using linear theory. The spectrum is preserved for large scales but the shape is changed on small scales. When considering baryons, we have to be ware of the fact that they are strongly coupled to photons until recombination. One must apply the empirical correction by Suguyama [48] to take this effect in account
\[
\Gamma_b = \Gamma_{\exp} \left[ -\Omega_b^{(0)} \left( 1 + \frac{\sqrt{2}}{\Omega_m^{(0)}} \right) \right].
\] (3.35)

Finally, we get the following expression for the power spectrum
\[
P(k) = A k^n s T^2(k) |\delta(k)|^2,
\] (3.36)

where the normalization constant is obtained from the condition in Eq. (3.33).

### 3.4 Fixing the Gauge

In the first chapter we have defined comoving coordinates in the background universe \( g_{\mu\nu}^{(0)} \), such that observers expanding with the universe remain at fixed coordinates. In the presence of perturbations \( \delta g_{\mu\nu} \) we do not have a preferred coordinate system. An essential requirement is that in the limit of zero perturbation those coordinates are reduced to the conformal coordinates. This choice of coordinates is called a gauge, and there are problems in which it is more convenient to work in some specific gauge, either for analytical or numerical simplification.

See for instance references [41, 42] for a gauge-invariant treatment of cosmological perturbations.

#### 3.4.1 Conformal Newtonian Gauge

This is in fact the most intuitive gauge that we can choose because it provides a direct link between the variables in relativistic perturbations and the Newtonian theory of small perturbations when we take the Newtonian gravity limit. The perturbed metric
is obtained making \( w_i = h_{ij} = 0 \) in Eq. (3.4). Once the expansion of the universe is neglected \( (a = 1) \) the metric describes a weak gravitational field. Here the observers are attached to the points in the unperturbed frame and will detect a velocity field of particles falling into the clumps of matter, measuring a gravitational potential. The metric perturbations \( \phi \) and \( \psi \) are known as the Bardeen potentials. The function \( \phi \) is also called Newtonian potential since it becomes equal to the Newtonian potential perturbation in appropriated limit; \( \psi \) is the Newtonian curvature perturbation and determines the curvature of the spatial section at \( t = \text{const.} \) which are flat in the unperturbed universe.

From Eq. (3.11) we get the components of the perturbed Einstein tensor

\[
\delta G^0_0 = \frac{2}{a^2} \left[ 3\mathcal{H} \left( \mathcal{H} \psi - \phi' \right) + \nabla^2 \phi \right],
\]

\[
\delta G^0_i = \frac{2}{a^2} \left( \phi' - \mathcal{H} \psi \right)_{ij},
\]

\[
\delta G^i_j = \frac{2}{a^2} \left[ \left( \mathcal{H}^2 + 2\mathcal{H}' \right) \psi + \mathcal{H} \psi' - \phi'' - 2\mathcal{H} \phi' - 2\mathcal{H} \phi \right] \delta^i_j + \frac{1}{a^2} \left[ \nabla^2 \left( \phi + \psi \right) \delta^i_j - \left( \phi + \psi \right)_{ij} \right].
\]

The prime stands for \( ' = \frac{d}{d\eta} \), the symbol \( | \) represents the covariant derivative with the 3-spatial metric and \( \nabla^2 \phi = \phi_{,\alpha}^{\alpha} \).

The perturbed part of the energy-momentum tensor for this gauge follows from Eq. (3.22)

\[
\delta T^0_0 = -\delta \rho,
\]

\[
\delta T^0_i = -\delta T^i_0 = \rho (1 + w) v^i,
\]

\[
\delta T^1_1 = \delta T^2_2 = \delta T^3_3 = c_s^2 \delta \rho.
\]

The components of Eq. (3.10) are

\[
3\mathcal{H} \left( \mathcal{H} \psi - \phi' \right) + \nabla^2 \phi = -4\pi G a^2 \delta \rho,
\]

\[
\nabla^2 \left( \phi' - \mathcal{H} \psi \right) = 4\pi G a^2 (1 + w) \rho \theta,
\]

\[
\psi = -\phi,
\]
As we saw, the perturbation quantities can be written in terms of a Fourier expansion where they are the sum of plane waves. Since the equations are linear each plane wave obeys the same equations with a different comoving wavenumber \( k \). From Eq. (3.26) we can rewrite the previous equations for each Fourier mode \( k \). Every perturbation variable reduces to a Fourier amplitude and the Laplacian operator is replaced by \( -k^2 \), i.e \( \nabla^2 \rightarrow -k^2 \). Also, since the equations are linear, the terms \( e^{i k \cdot x} \) and pre-factors can be dropped. So

\[
\begin{align*}
\phi'' + 2\mathcal{H}\phi' - \mathcal{H}\psi' - \left( \mathcal{H}^2 + 2\mathcal{H}' \right) \psi &= -4\pi G a^2 c_s^2 \delta \rho. \\
\end{align*}
\] (3.46)

Combining Eqs. (3.47), (3.48) and employing the relation for the conformal Hubble parameter

\[
\mathcal{H}' = -\frac{1}{2} (1 + 3w) \mathcal{H}^2
\] (3.50)

we get the relativistic Poisson equation

\[
k^2 \phi = 4\pi G a^2 \delta \rho + 3\mathcal{H} \left( \mathcal{H} \psi - \phi' \right),
\] (3.47)

\[
-k^2 \left( \phi' - \mathcal{H} \psi \right) = 4\pi G a^2 (1 + w) \rho \theta,
\] (3.48)

\[
\phi'' + 2\mathcal{H}\phi' - \mathcal{H}\psi - \left( \mathcal{H}^2 + 2\mathcal{H}' \right) \psi = -4\pi G a^2 c_s^2.
\] (3.49)

3.4.2 Synchronous Gauge

The synchronous gauge was first introduced by Lifshitz in 1946 [39] and is defined by the conditions \( \psi = w_i = 0 \). With this gauge choice the conformal time \( \eta \) coincide with the proper time and observers following geodesics do not change their spacial coordinates (they only move along \( \eta \)-threads), so the coordinates in synchronous gauge

\[
\begin{align*}
\delta_k'' + \mathcal{H} \left( 1 + 3c_s^2 - 6w \right) \delta_k' - \left[ \frac{3}{2} \mathcal{H} \left( 1 - 6c_s^2 - 3w^2 + 8w \right) - c_s^2 k^2 \right] \delta_k = 0
\end{align*}
\] (3.52)
are Lagrangian coordinates and the observers are attached to the perturbed particles. This characteristic implies that large density perturbations will deform the coordinate lines giving rise to caustic formation (singularities). Since this problem is only noticeable when perturbations grow large enough, the synchronous gauge can be safely used in the linear regime \( |\delta \rho / \rho| \ll 1 \). There are also some subtleties concerning the physical interpretation of perturbations due to the fact that the gauge is not entirely fixed on scales larger than the Hubble horizon (see [13]).

Here the perturbation equations for several interacting (minimally coupled) fluids will be given in the synchronous gauge. From the conservation of the energy-momentum tensor \( T^\mu_{\nu;\mu} = T^\mu_{\nu;\mu} - \Gamma^\alpha_{\nu\beta} T^\beta_{\alpha} + \Gamma^\alpha_{\beta\alpha} T^\beta_{\nu} = 0 \) we have for the component \( \nu = 0 \)

\[
(\delta \rho)' + 3H(\delta \rho + \delta p) = -(\rho + p) \left( \theta + 3\phi' \right),
\]

known as the perturbed continuity equation. For the \( \nu = i \) component we get the relativistic analogue of the Euler equation in the Newtonian context

\[
\theta' + H(1 - 3w) \theta + \nabla^2 \left( c_s^2 s_k + 6h \right) - 6H\phi' = 0.
\]

We have again the correspondent equations in Fourier space

\[
\delta_k' + 3H \left( c_s^2 - w \right) \delta_k + (1 + w) \left( \theta_k - 3\phi_k' \right) = 0,
\]

\[
\theta_k' + H \left( 1 - 3c_s^2 \right) \theta_k - \frac{c_s^2 k^2}{1 + w} \delta_k = 0.
\]

Together with the conditions from the Einstein equations, we close the system of equations describing the scalar perturbations for \( n \) interacting fluids. The relevant components from Eq. (3.10) are the (00)

\[
\frac{1}{3} \nabla^2 (h_s + 6\phi) - 6H\phi' = 8\pi G a^2 \rho \sum_i \delta_i,
\]

and (ii) component

\[
3\phi'' + 6H\phi' - \nabla^2 \left( \phi + \frac{1}{6} h_s \right) = 12\pi G \sum_i \delta p_i.
\]
Here the sum goes over all \( n \) fluid components in the model. Combining both equations one finds that
\[
\phi'' + \mathcal{H}\phi' - \frac{1}{2}\mathcal{H}^2 \sum_i (1 + 3c_{s,i}^2) \Omega_i \delta_i = 0. \tag{3.59}
\]

### 3.5 GCG Linear Evolution

After delivering in the last section the theory beyond the linear perturbations we continue the discussion of the GCG, and the restrictions to the parameters of the model when compared with the observations. The conclusions presented in the paper titled as “The End of Unified Dark Matter” [11] by Sandvick et al. were quite alarming for any attempt to unify dark energy and dark matter. Here it is considered a universe filled with Chaplygin Gas, which leads to the background relation
\[
H^2 = \frac{8\pi G}{3} \rho_{cg}. \tag{3.60}
\]

The equation for the density perturbations (in the Newtonian gauge) for a given mode \( k \), consider scales well inside the Hubble horizon (i.e. \( k \gg H_0 \)). So, from Eq. (3.52) we make the identification \( \delta^*_k \equiv \delta_{cg,k} \) leading to
\[
\delta_{cg,k}'' + (2 + 3c_{s}^2 - 6w) H \delta_{cg,k}' - \left[ \frac{3}{2} \mathcal{H}^2 \left( 1 - 6c_{s}^2 - 3w^2 + 8w \right) - \left( \frac{c_{s}k}{a} \right)^2 \right] \delta_{cg,k} = 0. \tag{3.61}
\]

To solve Eq. (3.61), a usual procedure is to change the independent variable in order to decrease its numerical complexity. Here we use the logarithm scale factor \( x = ln (a) \) for that purpose, so \( \dot{x} = \frac{d}{dx} = \frac{d}{dln(a)} \)
\[
\ddot{\delta}_{cg,k} + \left[ 2 + \xi - 6w + 3c_{s}^2 \right] \dot{\delta}_{cg,k} - \left[ \frac{3}{2} \left( 1 - 6c_{s}^2 - 3w^2 + 8w \right) - c_{s}^2 \left( \frac{k}{aH} \right)^2 \right] \delta_{cg,k} = 0, \tag{3.62}
\]
and the new variable defined as \( \xi = -\frac{3}{2} \left( 1 + w \right) \). Sound speed is \( c_{s}^2 = \frac{\partial p}{\partial \rho} = -\alpha w \) as usual. The state parameter is given by
\[
w = \left[ - \left( 1 + \frac{1}{\mathcal{A}} a^{-3(1+\alpha)} \right) \right]^{-1} \tag{3.63}
\]
and since \( w \leq 0 \) from Eq. (3.63) the sound speed \( c_{s}^2 \) is positive for \( \alpha > 0 \) and negative for \( \alpha < 0 \). We consider small values for \( \alpha \) and fix \( \mathcal{A} = 1 - \Omega_m^{(0)} \).
The equation is solved starting with a set of normalized conditions at \( a = 0.01 \) and evolved up to the present, using a standard Runge-Kutta method. Here we make the analysis with the latest Planck results for the cosmological parameters. Namely, we take for the present matter density \( \Omega_m^{(0)} = 0.315 \) and \( \sigma_8 = 0.83 \) to normalize the amplitude of the power spectrum. We also made similar analysis using the perturbation equations in the synchronous gauge. The final result in the two different gauges coincide, so this provides a consistency check of the numerical calculations.

Deep in the matter era, the sound speed is much smaller than unity and only starts to grow around the end of it. It is possible to estimate what is the condition that controls the behaviour of the density perturbations. Taking only scales smaller than the Hubble radius (well inside the horizon \( k \gg H \)) the evolution of a perturbation is described as a

\[
P(k) \sim k^{-4/3} \left( \frac{k}{a_H} \right)^n
\]

Figure 3.1: The matter power spectrum for perturbations in the GCG model where we have used the latest results from Planck mission for the cosmological parameters. The data points represent the LRG power spectrum from SDSS DR7 [40]
wave using linear theory

\[ \delta_{cg,k} + \left( \frac{c_s k}{aH} \right)^2 \delta_{cg,k} = 0. \]  

(3.64)

When \( c_s^2 > 0 \), we have an oscillatory solution and for \( c_s^2 < 0 \) the solution is composed by a growing mode and a decaying mode (which eventually vanishes). This provides a simple interpretation about the effect that the signal of the sound speed has on the behaviour of the density perturbations. Ignoring just the drag term in Eq. (3.61) we have

\[ |c_s^2| \lesssim \frac{3}{2} \left( \frac{aH}{k} \right)^2 \]  

(3.65)

as a condition for perturbations that can grow via the gravitational instability. So, when \( c_s^2 \gtrsim \frac{3}{2} \left( \frac{aH}{k} \right)^2 \) the perturbations show oscillations; that is equivalent to say that the physical wavelength \( \lambda = (2\pi/k) a \) is bigger than the Jeans length defined as \( \lambda_J = |c_s| \sqrt{\pi/G\rho_{cg}} \) by Eq. (3.60). The density contrast oscillate as an acoustical wave and inhomogeneities do not grow (pressure support). For \( c_s^2 < 0 \) the condition Eq. (3.65) is verified, so collapsing regions and voids grow exponentially. This effect of exponential instability is more noticeable on smaller scales, that leads to the first approximation in Eq. (3.64).

In Ref. [11] they concluded that only models with a small \( \alpha \) (about \( |\alpha| \lesssim 10^{-5} \)) could fit the observed matter power spectrum, so the behaviour of the GCG would be close to the ΛCDM model. Taking in account the latest observational data does not change the previous conclusions for the GCG (see Figure 3.1).

3.6 Baryons + GCG Model

Such dramatic result from Sandvick and his collaborators is partially due to the fact that baryons have been neglected in their analysis. The transition in the quartessence background from the “CDM” state to a cosmological constant is accompanied by a large sound speed (unless \( \alpha \approx 0 \)) which damps the perturbations for \( \alpha > 0 \) or exponentially enhances them for \( \alpha < 0 \).

Because baryons have a low sound speed they are quite important when one wants to study the consequences of the model on the formation of large scale structures. Indeed, this is what was done in the paper [12] by Beça et al. To study the perturbations of the
GCG minimally coupled with baryons we use Eqs. (3.55), (3.56) and (3.59). Since we will treat baryons as a pressureless non-relativistic fluid with null background speed we have $\theta_b = \nabla_i v^i = 0$ and $c_{s,b}^2 = 0$. From Eq. (3.56) we get $\dot{\delta}_b = 3 \dot{\phi}$. Analogously to the previous section, we change the independent variable from $\eta$ to $\ln(a)$

\begin{align*}
\delta''_b + (2 + \xi) \delta'_b - \frac{3}{2} \left[ \Omega_b \delta_b + (1 - 3 \alpha w_{cg}) \Omega_{cg} \delta_{cg} \right] = 0 \\
\delta'_c + (1 + w_{cg}) \left[ \frac{\theta_{cg}}{H} - \delta'_b \right] - 3 w_{cg} (1 + \alpha) \delta_{cg} = 0 \\
\theta'_c + (1 + 3 \alpha w_{cg}) \theta_{cg} + \frac{\alpha w_{cg} k^2}{H(1 + w_{cg})} \delta_{cg} = 0
\end{align*}

where $\xi = H'/H$ and the prime stands for $' = d/d\ln(a)$.

As we can see from Figure 3.2 the growth of the GCG density perturbations (here with $\alpha = 1$) follow the baryonic perturbations until the GCG background transitions to a cosmological constant.

**Figure 3.2:** Evolution of perturbations for two scales $k = 0.1 \, h^{-1}\text{Mpc}$ and $k = 0.01 \, h^{-1}\text{Mpc}$ in a model of baryons and GCG ($\alpha = 1$) interacting minimally. The amplitude of the GCG perturbations start to decay when the transition in the background to a cosmological constant occurs.
cosmological constant, which happens at a different scale factor for different perturbation scales, and oscillates before vanishing completely. Then, only baryons can carry over gravitational clustering due to their low sound speed. In fact, the value of $\alpha$ is not restricted by the LSS data. Still, they concluded that even if baryonic matter is taken in account to explain the normal growth of inhomogeneities, the combination of the LSS results with the SNIa observations restricted the parameter space, and the Chaplygin Gas would be forced to behave very closely to $\Lambda$CDM model.

### 3.7 UDE in Jeopardy

From the last discussions about the ability of the GCG to deal with the observations, as far as linear theory is concerned, it is clear that the model parameters are highly constrained. To have structure formation one needs low pressure effects during the matter era, but also a negative pressure to account for the late-time cosmic acceleration. This by itself, does not reveal any fundamental flaw for UDE models. After all, GCG with $\alpha = 0$ is gravitationally indistinguishable from $\Lambda$CDM. So any attempts grounded in the frame of General Relativity, to dismiss these theories that unify dark matter and dark energy leave the core of the quartessence idea untouched; and by *lex parsimoniae* alone, we even would had to choose the single fluid interpretation instead of two components of unknown nature. Some proposals to avoid the problem of large structure formation in the Chaplygin gas can be found at [66–68]. But in order to fix the problem they need to introduce non-adiabatic contributions (silent quartessence) or modify the equation of state. Furthermore, the constrains obtained assume that linear theory is a valid approximation for large scales. But it turns out that we must be careful about this assumption in these unified scenarios. In the next chapter we will discuss how interesting new features may also appear on non-linear scales, and their importance in the context of perfect fluid UDE models.
Chapter 4

UDE - Non-linear Dynamics

So far we have tested GCG as a representative model for the more general UDE models. We examined that the background evolution with $\alpha \lesssim 0.4$ is consistent with the supernovae Ia observations in a flat universe. Other phenomenological tests such as gravitational lensing [29] or high precision CMB radiation data [27, 28] had also been successfully confronted within the GCG model. But the critical failure for a pure CG comes when one wants to explain the formation of cosmic structures. The model was not able to reproduce the matter power spectrum since large sound velocities at late times force $\alpha$ to be close to zero, in order to be consistent with the latest observations. See [51] for a recent analysis combining CMB + SN Ia + LSS(m). Although this kind of heuristic to explain the formation of structures is a usual procedure in cosmology (where we assume a background and then small perturbations are considered), there are some idiosyncrasies concerning the exotic equation of state of the Chaplygin gas. In this chapter we will argue why this treatment cannot be applied in a straightforward manner to UDE models, and how the idea behind the non-linear clustering in the GCG provides a way to be completely consistent with the latest observations, without the strong restrictions in $\alpha$ verified in previous analysis.

4.1 Backreaction effect in cosmology

The real universe can only be regarded as homogeneous and isotropic at sufficiently large scales. On smaller scales it exhibits hierarchical structure such as galaxies and clusters.
So, the question if an inhomogeneous universe evolve on average like a homogeneous solution is not new [65]. This comes as a natural question due to the non-linear nature of GR.

The perturbative treatment presented in the previous chapter comes to aid due to our inability to solve the Einstein equations with all generality. Gravitational field equations are highly non-linear and we can solve them only in a few high symmetry situations (see for instance [45]). However, for a perturbative expansion we need first to define a background i.e. a manifold that could be representative of the average universe. But this process of averaging in General Relativity is a bit hazy, because in the process we also average complex non-linear interactions. The effect of non-linearities on the average expansion is known as backreaction. There are some hints suggesting that these non-linear terms are important still on cosmological scales [46, 47]. It turns out that small scale clustering has an important effect on the large scale evolution, and in particular this effect turns to be of uttermost importance in UDE scenarios.

4.2 The Chaplygin Gas on Small Scales

As we have presented in Section 2, the dynamics of an universe where the Cosmological Principle and General Relativity holds on large scales are partially described by Eqs. (1.32) and (1.33). Because these assumptions are only valid for the average universe, the functions that characterize the fluids are also average quantities \( \langle p \rangle \) and \( \langle \rho \rangle \). Furthermore, in order to fully specify the dynamics, one needs the equation of state relating the pressure and density by the parameter \( w \). In its turn the equation of state \( p = w \rho \) is a local relation. So, if we assume the perturbative decomposition [19]

\[
p = \langle p \rangle + \delta p + \ldots
\]

\[
\rho = \langle \rho \rangle + \delta \rho + \ldots
\]

we clearly see that the average pressure and density are not necessarily the same as the local ones. For the GCG this feature is evident since

\[
\langle p \rangle = -A \langle \rho^{-\alpha} \rangle \neq -A \langle p \rangle^{-\alpha}
\]
When one considers the ΛCDM model, linear theory is an excellent tool to understand the deviations from a homogeneous universe. Perturbations are predicted by inflationary models and their evolution up to the present provides an explanation for the observed large scale structures. Dark matter becomes highly clustered during the matter era and this turns out as not being relevant for the average background pressure, because CDM is considered to be a pressureless fluid. So dark energy evolves normally and is not affected by inhomogeneities in the dark matter distribution. With $\alpha = 0$ in Eq. (4.3) we get $\langle p \rangle = -A$, which has the same form as the local equation of state for the cosmological constant $p = -A$.

![Figure 4.1: Linear evolution of the mass dispersion in the baryon component (solid lines) and Chaplygin gas (dot-dashed lines) for different scale factors. At early times both components evolve in the same way. When pressure effects arise, the density perturbations of the Chaplygin gas stops growing and only the baryons can contribute for the formation of structures at late times (although at a slower pace).](image)

On the other hand, when we are dealing with unification models, dark energy and dark matter are represented by the same underlying fluid. So the discrepancy that arises between the local and global equation of state means that the average background will
be affected by the inhomogeneities in the quartessence distribution. The mass dispersion of the density fluctuations for the model studied in section 3.6 is depicted in Figure 4.1 were we assume $\alpha = 1$. At early times, the Chaplygin gas behaves like matter, so it evolves in same way as the baryonic component. While the baryon fluctuations can keep growing, the Chaplygin gas becomes non-linear on small scales very early in the matter era. This collapsed fraction of the Chaplygin gas decouple from the background, and do not contribute any more for the overall pressure, since it starts behaving like dark matter. Therefore, these regions stay in a high energy density state and null pressure, and the transition for a dark energy stage (which requires lower density) never happens [69].

4.3 Non-linear Chaplygin Gas Cosmologies

4.3.1 Background evolution

Now we will study the backreaction effect of small scale non-linearities using the GCG as a representative family of UDE models, following closely the work done in [14]. We shall assume that the distribution of the Chaplygin Gas in a large comoving volume $V$ of the universe is composed of two types of regions: collapsed regions where the local density is much higher than the average density $\rho_{cg,+} \gg \langle \rho_{cg} \rangle$ and underdense regions with local density smaller than the average $\rho_{cg,-} < \langle \rho_{cg} \rangle$. Most of the universe is filled by these underdense regions so $V_- \sim V$. We shall define a parameter $\epsilon$ as the average fraction of the total Chaplygin gas energy $E$ which is incorporated into collapsed objects $E_+$, in an attempt to quantify the level of small scale clustering

$$\epsilon = \frac{E_+}{E}$$

(4.4)

and $0 \leq \epsilon < 1$. The contribution of the collapsed and underdense regions to the average universe is respectively

$$\rho_+ = \frac{E_+}{V} = \epsilon \langle \rho_{cg} \rangle,$$

(4.5)

$$\rho_- = \frac{E_-}{V} = \frac{E - E_+}{V} = (1 - \epsilon) \langle \rho_{cg} \rangle.$$  

(4.6)

Although we are considering the effect of the small scale non-linearities in the GCG component, the Friedmann equations (Eqs. (1.38) and (1.36)) remain valid. This comes
from the fact that the universe retain its geometrical properties on average, so the FLRW metric remain as a valid assumption. In our case, the backreaction effect is not due to geometrical properties but about the influence of small scale clustering on the CG equation of state. Also, the CG EoS still has the same functional form (Eq. (2.4)) but now the pressure contribution is due to the underdense regions (since we assume $p_+ = 0$)

$$w_{cg} = \frac{p_-}{\langle \rho_{cg} \rangle} = \frac{\rho_-}{\langle \rho_{cg} \rangle} \rho_- = (1 - \epsilon) w_-.$$  

(4.7)

Here $w_- = p_- / \rho_-$ plays the role of the effective DE equation of state parameter. To study the evolution of the parameter $\epsilon$ we assume a simple model where $E_+$ remains fixed. Given that $E \propto \langle \rho_{cg} \rangle a^3$ one has

$$\epsilon = \frac{E_+}{\langle \rho_{cg} \rangle} a^3.$$  

(4.8)

At early times ($a \ll 1$) the Chaplygin gas behaves like dark matter so the energy density evolves as $\langle \rho_{cg} \rangle \propto a^{-3}$. It follows that the collapsed fraction of the CG $\epsilon$ is constant deep in the matter era and do not contribute to the overall pressure.

Along with the GCG, the evolution of the baryonic component is also taken in account. So the Friedmann equation reads

$$H^2 = \frac{8\pi G}{3} \left[ \rho_b + \langle \rho_{cg} \rangle \right].$$  

(4.9)

We shall assume the latest Planck results to fix the present time (when $\Omega_b^{(0)} = 0.0487$) and ensure that at recombination this GCG model is fully consistent with the Planck CMB constrains.

The evolution of the parameter $\epsilon$ with the scale factor is obtained by solving the equation for the GCG energy density (Eq. (1.36))

$$a \langle \rho_{cg} \rangle \frac{d \langle \rho_{cg} \rangle}{da} + 3(1 + w_{cg}) = 0,$$  

(4.10)
Figure 4.2: Evolution of the parameter $\epsilon$ with the scale factor $a$. Here we show four different models. With the values $\alpha = 0$ and $\epsilon_i = \left[1 + \frac{Aa_3}{1 - A}\right]^{-1}$, we get the $\Lambda$CDM case, which is in turn influenced by the small scale clustering from the relation

$$w_{cg} = (1 - \epsilon)w_+ = -(1 - \epsilon)\frac{A}{\left[(1 - \epsilon)\langle\rho_{cg}\rangle\right]^{\alpha+1}}. \quad (4.11)$$

Now the evolution of the average energy density of the CG takes into account that the non-linear clustering occurs during the matter era and that such a process is irreversible. Once gravitational collapse occurs in these regions, they do not contribute any more for the pressure $p_{cg}$. We solved the differential equation numerically for different initial fractions $\epsilon_i$ and the results are shown in Figure 4.2. The evolution of $\epsilon = E_+/E$ shows that $\epsilon \to \epsilon_i$ for $a \ll 1$ and evolves rapidly towards zero for $a \gg 1$. Actually for $\alpha = 0$ the value of $\epsilon_i$ is not relevant for the evolution of the average density (from Eq. (4.11)) and the dark energy component is not affected by the inhomogeneities in the dark matter sector. This independence of the evolution of $\langle\rho_{cg}\rangle$ from the non-linear clustering was discussed in section 4.2 since with $\alpha = 0$ we get $w = -A/\rho_{cg}$, which is completely equivalent to $\Lambda$CDM model.
**Figure 4.3:** Evolution of the equation of state parameter $w_\gamma$ with the scale factor for the models considered in Figure 4.2. The models smoothly interpolate between a dark matter and dark energy state.

**Figure 4.4:** Deceleration parameter $q = \frac{1}{2} (\Omega_{\gamma} (1 + 3w) + \Omega_b)$ for the models considered in Figure 4.2. For the considered models, the transition from a dark matter to a dark energy state is faster than in the $\Lambda$CDM case ($\alpha = 0$). This background behaviour delivers a positive sound speed when the transition occurs.
For the effective DE EoS parameter $w_-$ its value smoothly interpolates from $w_- = 0$ when matter dominates, into $w_- = -1$ at later times entering the dark energy era (except for the case $\alpha = 0$, the $\Lambda$CDM limit of the Chaplygin Gas). The time at which this transition occurs is controlled by the parameter $\epsilon_i$. When the amount of small scale clustering is increased, the transition from a CDM behaviour to a cosmological constant occurs at larger and larger redshifts. Taking in account that the case $\epsilon_i = 0$ yields the usual GCG for which the scale factor at the transition is given by Eq. (2.8). For the other models, during the $w_- = 0$ phase, the evolution of the energy density evolves as pressureless matter

$$\rho_-(t) = (1 - \epsilon) \rho(0) (1 - \epsilon_i) a^{-3}. \quad (4.12)$$

So one finds that the $a_{tr}$ between the two phases is roughly proportional to $(1 - \epsilon_i)^{1/3}$

$$a_{tr} = (1 - \epsilon_i)^{1/3} \left[ \frac{1 - \bar{A}}{2\bar{A}} \right]^{1/(1+\alpha)}. \quad (4.13)$$

The deceleration parameter (Figure 4.4) is in this case given by

$$q = \frac{1}{2} (\Omega_{cg} (1 + 3w_{cg}) + \Omega_b) \quad (4.14)$$

Deep in the matter era the models have a null sound speed. Since the transition for a dark energy state is faster than in the $\Lambda$CDM case, the admitted models have a positive sound speed (see section 2.1).

### 4.3.2 Observations

The values for the parameters $\alpha$ and $\epsilon_i$ can be constrained using supernova data, in a similar way as it was done in section 2.5.2. In this case, we fix the parameter $\bar{A} = 0.72$ and the Hubble constant was marginalized (see Figure 4.5). By including the non-linear clustering effect, all the range $0 \leq \alpha \leq 1$ is consistent with the supernova constraints. Otherwise, if $\epsilon_i = 0$ the Chaplygin gas model ($\alpha = 1$) is ruled out. Also, from [14] it is shown that the angular diameter distance to the last scattering surface ($z \approx 1100$) and the sound horizon for Non-linear Chaplygin Gas models can be compatible with current
observational constrains, as long as the level of non-linear clustering is high enough ($\epsilon_i > 0.9$).

### 4.4 Results and Comments

#### 4.4.1 Evolution of density perturbations

We have checked that at the background level this model is consistent with the current observational constrains. The ordinary GCG model revealed late time oscillations (or exponential growth if $\alpha < 0$) of the matter power spectrum, when the non-null sound speed starts to emerge after the transition from $w_{cg} = 0$ to $w_{cg} = -1$, spoiling the formation of structures. This behaviour leads to a substantial restriction of the parameter space, and in the end GCG does not detach from the ΛCDM model.
Now we will study the evolution of the density perturbations in the context of the non-linear Chaplygin gas model. We will have density perturbations from underdense regions and collapsed regions interacting gravitationally. So we must solve the system of Eqs. (3.55, 3.56, 3.59) in a similarly way to the case of section 3.4, where baryons were included to the Chaplygin gas. Here the non-linear clustering parameter determines the fraction of collapsed Chaplygin gas $\Omega_+ = \epsilon$, and the density parameter of the underdense regions $\Omega_- = 1 - \epsilon$. The collapsed regions behave like dust ($p_+ = 0$) and we will include the baryonic perturbations in the $+$ component as a matter of simplification, so $\delta_+ = \delta_b$ and now $\Omega_+ \sim \epsilon_i$ at early times. Notice that the non-linear evolution of the Chaplygin gas is taken in account for the evolution of the background quantities. The identification of clustered and underdense regions for the density perturbations enable us to use the linear equations in similar manner. The full set of equations describing the evolution of density perturbations is

\[ \begin{align*}
\delta_+ &, \quad \alpha = 0.0 \\
\delta_- &, \quad \alpha = 0.0 \\
\delta_+ &, \quad \epsilon_i = 0.5 \\
\delta_- & \\
\delta_+ &, \quad \epsilon_i = 0.9 \\
\delta_- 
\end{align*} \]

Figure 4.6: The evolution of perturbations of collapsed regions $\delta_+$ and underdense regions $\delta_-$ for the scale $k = 0.1h\,\text{Mpc}^{-1}$. For $\alpha = 0$ we recover the $\Lambda$CDM case. The underdense density fluctuations start to decay sooner for models with higher initial non-linear clustering $\epsilon_i$. If $\epsilon_i \to 1$ the non-linear CG perturbations match the density contrast of the $\Lambda$CDM model.
\[
\begin{aligned}
    &\delta''_+ + (2 + \xi) \delta'_+ - \frac{3}{2} [\Omega_+ \delta_+ + (1 - 3\alpha w_-) \Omega_- \delta_-] = 0 \\
    &\delta'_- + (1 + w_-) \left[ \frac{\theta}{aH} - \delta'_+ \right] - 3w_- (1 + \alpha) \delta_- = 0 \\
    &\theta'_- + (1 + 3\alpha w_-) \theta_- + \frac{\alpha w_- k^2}{aH(1 + w_-)} \delta_- = 0
\end{aligned}
\]

(4.15)

Deep in the matter era \( w_- = 0 \) and \( \xi = -3(1 + w)/2 = -3/2 \) so we can simplify Eq. (4.15)

\[
\begin{aligned}
    &\delta''_+ + \frac{1}{2} \delta'_+ - \frac{3}{2} (\Omega_+ \delta_+ + \Omega_- \delta_-) = 0 \\
    &\delta'_- = \delta'_+ \\
    &\theta'_- = \theta_-
\end{aligned}
\]

(4.16)

This gives the standard result for the DM perturbations with a growing mode \( \delta_+ \propto a \)

and a decaying mode \( \delta_- \propto a^{-3/2} \) which implies that \( \delta'_{-,+} \propto a \). We have used normalized initial conditions \([\delta_+, \delta'_+ \delta_-, \theta_-]_{a=0.01} = [1, 1, 1, 0]\) to evolve our system of equations

\[\text{Figure 4.7: The evolution of } D(a)/a \text{ with the redshift } z.\] During the matter era the quantity \( D(a)/a \) remains constant. The decrease of the gravitational potential at late times, due to the cosmic acceleration, leads to the variation of \( D(a)/a \). With a sufficiently high level of non-linear clustering, the non-linear Chaplygin model approaches the behaviour of the \( \Lambda \)CDM model for all values of \( \alpha \).
from $a = 0.01$ until today. Figure 4.4 illustrates typical solutions. We see that the density perturbations of the two regions for all $\epsilon_i$ grow in unison, and then decouple when the sound velocity prevents the growth of perturbations of the underdense regions. This happens at different times on different scales, when the comoving sound horizon times the comoving wave number ($\sim \frac{\omega k}{aH}$) becomes of order unity. So, the density contrast $\delta_-$ oscillates rapidly around zero which makes that contribution negligible. From the first equation from (4.16) we get

$$\delta_+'' + \frac{1}{2} \delta_+'' + \frac{1}{2} \frac{3}{2} \epsilon \delta_+ = 0,$$

(4.17)

For modes that have this behaviour during the matter era, the non-linear clustering parameter can be assumed constant $\epsilon = \epsilon_i$, so the growing mode solution is $\delta_+ \propto a^{1 - \frac{3(1 - \epsilon_i)}{2}}$. Also, as $\epsilon_i \to 1$, the evolution of $\delta_+$ approaches the behaviour of DM perturbations in the $\Lambda$CDM model.

One can also study the growth function for the collapsed regions, defined as

$$D(a) = \frac{\delta_+ (a)}{\delta_+ (a_i)},$$

(4.18)

During the matter dominated era we have a constant evolution of the quantity $D(a)/a = 1$. This value decays when the universe starts the cosmic expansion and the function $\epsilon$ transitions towards zero. Since this variation of $D(a)$ is related with the variation of the gravitational potential after the matter era, models with a sufficiently high level of non-linear clustering (which approach the $\Lambda$CDM model for any value of $\alpha$) would be consistent with the integrated Sachs-Wolf (ISW) effect in the CMB temperature anisotropies [57, 58]. Notice that models with lower $\epsilon_i$ start to deviate from the $\Lambda$CDM model for lower redshifts than models with higher clustering parameter. This is due to the fact that the transition for a dark energy state happens at later times (see Figure 4.4), and only then the sound speed become non null. Nonetheless, models with higher $\epsilon_i$ have a smoother transition and globally approach the $\Lambda$CDM profile for $\epsilon_i \to 1$.

### 4.4.2 Power Spectrum

We have assumed a primordial spectrum $k^{n_s}$ and processed it using the transfer function given by Eq. (3.34). The power spectrum is normalized by Eq. (3.33). It should be
noted that the plot of the Figure 4.8 was drawn by interpolating several points of the power spectrum. As suspected from the analysis of the density perturbations, when the effects of non-linear clustering of the Chaplygin gas are considered, the oscillations of the matter power spectrum (see Figure 3.1 for the comparison) are absent. Also, when $\epsilon_i \rightarrow 1$, the shape gets close to the $\Lambda$CDM result no matter what the value of $\alpha$ is. The non-linear properties of the CG provides a way to be consistent with the LSS observations. For $\epsilon_i = 0$ and considering the density fluctuation $\delta_-$ we recover the standard GCG model, which is depicted in Figure 4.8 for $\alpha = 10^{-5}$. From construction, we expect that the density fluctuations from clustered regions ($\delta_+$) are the responsible for the formation of structures. Nevertheless, the underdense regions of the Chaplygin gas behave like matter ($w_- \approx 0$) for low scale factors (deep in the matter era). This

![Figure 4.8: Power Spectrum for the ΛCDM model ($\alpha = 0$). The two models considering $\delta_+$ with $\alpha = 1$ and initial clustering parameter of $\epsilon_i = 0.5, 0.9$, do not show the characteristic oscillations of the original CG for small scales (here with $\epsilon_i = 0$, $\alpha = 10^{-5}$ and assuming the density contrast of the underdense regions $\delta_-$). The power spectrum for a model with high initial non-linear clustering ($\epsilon_i \rightarrow 1$) and any value of $\alpha$ gets closer to the ΛCDM result.](image-url)
means that we have matter in a low density environment that remains unclustered, until the CG transition for a dark energy state \( w_\gamma = -1 \). Taking in account the total density fluctuation

\[
\delta = \Omega_+ \delta_+ + \Omega_- \delta_-
\]  

(4.19)

gives a natural interpretation of the ordinary Chaplygin gas (by identifying \( \Omega_+ = 0 \) we have \( \delta = \delta_- \)) as a low density matter fluid during the matter era. The high sound speed in this component that emerges when the fluid starts to behave as dark energy is avoided when \( \epsilon_i \to 1 \), thus rendering the models consistent with the observations \(^1\).

### 4.5 Conclusions

In the review of the background dynamics of the GCG model from Chapter 2, we have presented the constrains to the parameters \( \alpha \) and \( A \) from SNIa observations. Unlike previous results where \( \alpha \) could assume all values in the interval \([0, 1]\), the parameter space is constrained to \( 0 \leq \alpha \lesssim 0.4 \) and \( 0.69 \lesssim A \lesssim 0.77 \). The value \( \alpha = 1 \), that corresponds to the original proposal for the Chaplygin Gas, is excluded if the latest results from \([30]\) for the cosmological parameters are assumed. When theory of perturbations is taken in account any non null value for \( \alpha \) is practically excluded, since the Chaplygin Gas density perturbations manifest late time oscillations when it starts to behave as a cosmological constant. We need \( \alpha \lesssim 10^{-5} \) in order to reproduce a power spectrum in agreement with the data from LSS observations. Although baryons are of critical importance for the formation of LSS, considering a universe composed of GCG and baryons do not significantly alleviates the previous constrains. Since the GCG model is completely equivalent to the \( \Lambda \)CDM model for \( \alpha = 0 \), the data does not show any preference between this attempt to interpret DE and DM as a single fluid and the standard model. However, the traditional way to understand the formation of structures can not be safely applied to UDE models. Collapsed regions and voids can have a potential effect on the behaviour of the average universe through backreaction effects. By assuming a model where we add a new parameter characterizing the level of small scale clustering, one can account for the backreaction effects of collapsed and underdense regions.

\(^1\)Available in http://lambda.gsfc.nasa.gov/toolbox/lrgdr/
background evolution of this model is consistent with SNIa data for all values of $\alpha$, considering that the non-linear clustering of the GCG is high enough. The analysis of the evolution of density perturbations show that the late time oscillations can be avoided if the clustering parameter is close to unity, and independently of the value of $\alpha$. This behaviour of the perturbations results in a matter power spectrum consistent with the LSS observations. In addition to the ability of this model to have a background dynamics and evolution of density fluctuations in agreement with the latest observations, the introduced parameter $\epsilon$ has also a straightforward physical interpretation. Although this work was focused on the GCG model, it is expected that the main results hold in general for UDE models, where the considered quartessence candidate is a perfect fluid.
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