Adding flavor degrees of freedom to the AdS/CFT correspondence

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“We are ourselves only by the sum of our failures.”

Emil Cioran
In this thesis we study the addition of flavor degrees of freedom to the AdS/CFT correspondence through the introduction of a D7-brane to its original setup. Working on the gravity side of the “flavored” correspondence, we calculate numerically the meson spectra for two different arrangements of the $N$ D3-branes that source the background: the first consists of $N$ coincident D3 branes, the second corresponds to $N$ D3-branes evenly distributed on a circle.
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For those who want to learn.
Chapter 1

Preamble

I chose a thesis on AdS/CFT correspondence because this subject is like a zip file of modern theoretical physics. To understand it and work on it requires that one dominates a great variety of ideas: some well established concepts, such as quantum mechanics and general relativity, and some well informed proposals such as supersymmetry and string theory. Such high demand of theoretical and mathematical knowledge keeps me motivated and gives me the sense that I’m in the right path to understand the gears and bolts of the physical world.

Besides the personal point of view, AdS/CFT plays a very important role in the landscape of modern theoretical physics. The AdS/CFT statement provides a correspondence between two specific theories living in different dimensional spaces: a quantum field theory in 4 dimensions and a theory of gravity in 5 dimensions. This was a paradigm changer, allowing people to take different approaches to old problems - using AdS/CFT, previously intractable problems in QFT can be tackled through simple calculations in gravity theories.

But my words shouldn’t be taken for granted. One should let the numbers talk. Following the address

http://inspirehep.net/

searching for

\textit{cited:50->300000 year:1900->2014},

which is self-explanatory, and sorting the results by "times cited", one obtains, in first place, the paper [1] in which AdS/CFT correspondence was proposed for the first time. On the importance of AdS/CFT correspondence, I rest my case.
Chapter 1. Preamble

The original AdS/CFT correspondence involves two theories that are unrealistic in the sense that they are built upon principles which haven’t been experimentally confirmed. However, these theories share some aspects with more realistic and well confirmed theories. Using AdS/CFT correspondence to study the later theories requires that one deforms the original setup and comes up with some new correspondence. To the plethora of tweaks and extensions of AdS/CFT correspondence, one refers loosely as gauge-gravity correspondence. Formally this is the statement that gravitational theories in \((d + 1)\) dimensions can be equivalently described by a non-gravitational quantum field theories in \(d\) dimensions.

One of these tweaks has to do with the introduction of flavor degrees on freedom in the quantum field theory involved in the original AdS/CFT correspondence. This change is motivated by QCD and can hopefully help people to make some progress in understanding phenomena currently under the veil of nonperturbative dynamics. In this thesis we study this extension of AdS/CFT correspondence and calculate the meson spectra from the gravity side of the correspondence.

The organization of this document is the following: in chapter 2 we introduce the AdS/CFT correspondence in its original form, i.e., as Maldacena [1] proposed it; chapter 3 is devoted to a detailed exposition on the introduction of flavor degrees of freedom to the AdS/CFT correspondence, following the original work of Karch and Katz [2]; in chapter 4 we present an exercise based on the work of [3] in which we calculate meson spectra of a deformed arrangement of the flavored AdS/CFT correspondence.
Chapter 2

Introduction

This is a very long introduction. The objective of this introduction is to render this thesis with a self-contained linguistic map. Obviously, much of the details of the various subjects that are important to understand AdS/CFT won’t be presented here, but all concepts used throughout the text will be referred in this introduction, so that the reader can advance in the text without the feeling that there are some things that appear out of nowhere.

We’ll start by talking about gauge theories, then we’ll present some details on conformal field theories, specifying the discussion to two dimensions. Then, we move on to the very basics of supersymmetry and introduce some characteristics of $\mathcal{N} = 4$ SYM theory. After this, we talk about string theory and then supergravity. Before presenting the decoupling argument, that leads directly to AdS/CFT correspondence, we present some details on the Anti-de Sitter space. We finish this introduction with some further comments on the uses of the correspondence.

2.1 Gauge Theories

A gauge theory is a type of field theory in which the Lagrangian is invariant under a continuous group of local transformations. We are going to list some results and establish the jargon of these theories which play a fundamental role in the AdS/CFT correspondence.

2.1.1 $U(1)$ Model

We start with a simple case of gauge theory: the $U(1)$ model.
If we impose the invariance of the lagrangian of a complex field \( \mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi \) under the gauge transformation, we realize that the derivative of the field \( \partial_\mu \phi(x) \) doesn’t transform in such a way that leaves the lagrangian invariant. Because of that, one is forced to introduce an extra-field, the gauge field \( A_\mu(x) \), and to build the gauge covariant derivative \( D_\mu \phi(x) = (\partial_\mu + iA_\mu) \phi(x) \). This new derivative possesses pleasing transformation properties and by using it one can obtain a gauge invariant lagrangian for the complex scalar field. Another prime object in this very simple theory is the field strength tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) which is unaffected by gauge transformations. So, the introduction of gauge fields is a consequence of the imposition of gauge invariance on objects describing the dynamics of the field.

### 2.1.2 Non abelian gauge theories

The natural generalization is the imposition of gauge invariance for non-abelian groups. The interesting example of non-abelian groups in the AdS/CFT context is \( SU(N) \) - also known as Yang-Mills theories. In this case, the gauge fields are \( N \times N \) matrices \( (A_\mu)^a_b \) in which \( \mu \) is the coordinate index and \( a, b \) are matrix indices. In what follows, we indicate the generators of the gauge group as \((T^a)^i_j\) which satisfy the Lie algebra\(^1\) of \( SU(N) \)

\[
[T_a, T_b] = f_{ab}^c T^c. \tag{2.1}
\]

It is important to mention two different ways in which fields can transform under the influence of non-abelian groups:

Fields transforming in the fundamental representation (see footnote 1) of the gauge group are elements of an \( N \) dimensional vector space:

\[
\phi_i(x) \rightarrow (\exp (i\theta^a(x)T^a))^j_i \phi_j(x), \quad i, j = 1, 2, ..., N \tag{2.2}
\]

in which \( \theta^a(x) \) are infinitesimal parameters.

Fields transforming in the adjoint representation of the gauge group are aligned into the \( N^2 - 1 \) dimensional Lie algebra of \( SU(N) \),

\[
\phi^a_i \rightarrow (\exp (i\theta^b T^b))^k_i \phi^a_k (\exp (-i\theta^c T^c))^l_j. \tag{2.3}
\]

The non-abelian gauge fields \( A_\mu = A^a_\mu T^a \) give rise to a non-abelian field strength tensor in the adjoint representation

\(^1\)More details on Lie groups can be found in appendix D
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]  
(2.4)

which in terms of the generators can be written as

\[ F^a_{\mu\nu} = \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \right) T^a. \]  
(2.5)

and with this object one can form a gauge invariant action for the field strength by taking a trace over the \( i,j \) indices of the generators (\( g_{YM} \) is the gauge coupling)

\[ S[A] = -\frac{1}{g_{YM}^2} \int d^4x \text{Tr} \{ F^{\mu\nu} F_{\mu\nu} \} \]  
(2.6)

### 2.1.3 Large \( N \) expansion

We can rewrite the lagrangian for a \( SU(N) \) gauge theory as

\[ \mathcal{L} = -\frac{N}{\lambda} \text{Tr} \{ F^{\mu\nu} F_{\mu\nu} \} \]

where \( \lambda = g_{YM}^2 N \) is the ’t Hooft coupling. The large \( N \) expansion (or ’t Hooft expansion [4]) corresponds to keeping \( \lambda \) fixed and to perform an expansion of the amplitudes in powers of \( N \). Then, taking the limit \( N \to \infty \) at constant \( \lambda \), one obtains propagators of \( SU(N) \) that behave in the following way [5]

\[ \langle (A(x)_\mu)_{ij}^k (A(y)_\nu)_{kl}^h \rangle \sim \delta_i^l \delta_j^k \]  
(2.7)

and this suggests a double line notation (see figure 2.1) with which Feynman diagrams become networks of double lines, each one of these lines corresponding to one of the matrix indices \( i,j,k,l = 1,\ldots,N \).
Let us denote by the combination of $F$, $E$ and $V$ that appears in the exponent of $N$ in (43):
\[ F + E + V = \chi \]
which is nothing but the Euler characteristic of the triangulation, which only depends on the topology of the surface. If the diagram triangulates a surface with $h$ handles (an $h$-torus),
\[ \chi = 2 - 2h. \]
Thus the diagrams are weighed with a power of $N$ determined by the number of handles of the surface that they triangulate (see figure 7). The planar diagrams are those that can be drawn on a piece of paper without self-crossing. They correspond to $h = 0$ and their contribution is of the type $N^{2n}$, for some power $n$ which depends on the diagram considered.

Clearly, as the dependence on $N$ goes like $N^{2 - 2h}$ the diagrams with $h = 0$ are the dominant ones in the large $N$ expansion. For this reason the large $N$ limit is also called the planar limit of the gauge theory (see [18] for more details and examples).

\[ \text{Figure 7: Planar diagrams (left) can be drawn on a sphere, non-planar diagrams (right) have to be drawn on a torus. (Image obtained from [6])} \]

In such a framework, the propagators scale as $\lambda$ and the vertices as $\frac{N}{\lambda}$. Besides, the sum over indices in a trace contributes a factor of $N$ for each closed loop. So, one can characterize each Feynman diagram by the number of propagators, vertices and loops it has. If we refer to the number of vertices, propagators and loops as $V$, $E$ and $F$ respectively, we obtain
\[ \text{diagram } (V, E, F) \sim N^{V - F + E} \lambda^{E - V} = N^\chi \lambda^{E - V} \quad (2.8) \]
where $\chi = V - E + F = 2 - 2g$ is the Euler characteristic which is related to the genus $g$ of the closed oriented surface with $V$ vertices, $E$ edges and $F$ faces. It is important to note that the Euler characteristic only depends on the topology of the surface.

The previous relation between constituents of Feynman diagrams and characteristics of surfaces comes from the observation that, after assuming the double line notation, one can inscribe Feynman diagrams in closed surfaces (see figure 2.2) which will be organized in terms of their topology, being the only characteristic parameter the number of handles of each surface - the genus $g$. In this spirit, one defines planar diagrams as those that can be drawn on a piece of paper without crossing and, consequently, can be inscribed on a sphere ($g = 0$).

Having established the relation between Feynman diagrams and the genus of the surface in which it can be inscribed, one realizes that any physical quantity in a non-abelian gauge theory is given by a perturbative expansion of the type
\[ \sum_{g=0}^{\infty} N^{2 - 2g} \sum_{i=0}^{\infty} c_{g,i} \lambda^i = \sum_{g=0}^{\infty} N^{2 - 2g} f_g(g) \quad (2.9) \]
with \( f_g(\lambda) \) a polynomial depending on the ’t Hooft coupling. For large \( N \), the series is clearly dominated by surfaces of minimal genus in which one can only inscribe planar diagrams. This is the content of the large \( N \) expansion technique introduced by ’t Hooft.

There is however a further observation one can make. As we’ll see in section 2.4.4, in string perturbation theory there is an expansion of the type \( \mathcal{A} = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(\alpha')^2 \) for the amplitude \( \mathcal{A} \) of a specific process and one can establish a connection between string perturbation theory and non-abelian gauge perturbation theory if one makes the identification

\[
g_s \sim \frac{1}{N}. \tag{2.10}\n\]

### 2.2 Conformal field theory

This section deals with a subject that permeates string theory. In string theory one is interested in the internal degrees of freedom of fundamental strings, which are determined by its vibrational modes. The vibrational modes of a string can be studied by examining its worldsheet which is a two dimensional surface. In this process, it turns out that the vibrational modes of the string are described by a two-dimensional conformal field theory.

Having this in mind, this chapter will introduce the concept of CFT and present some characteristic results. We leave some hard-core details regarding the special case of 2 dimensional conformal field theory to appendix A.

#### 2.2.1 Definition

A conformal transformation is a change of coordinates \( \sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma) \) such that the metric changes by

\[
g_{\alpha\beta} \rightarrow \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{2.11}\n\]

A conformal field theory is a field theory which is invariant under these transformations.

If \( \epsilon \) is an infinitesimal conformal transformation, \( x^\mu \rightarrow x^\mu + \epsilon^\mu \), then it must obey the following equation

\[
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu\nu}. \tag{2.12}\n\]
where $d$ is the number of dimensions of the spacetime$^3$.

In the case of $d > 2$, we consider that the spacetime we’re dealing with has a Lorentzian signature: $g_{\mu \nu} = \eta_{\mu \nu}$. In this case (for $d > 2$) the solutions for such an equation are translations, Poincaré transformations, scale transformations and special conformal transformations. Altogether there are $\frac{1}{2}(d+2)(d+1)$ generators of the conformal group and the collection of these transformations forms the conformal group in $d$ dimensions, which is isomorphic to $SO(2,d)$. In this situation, this algebra generates all the conformal transformations except inversions; because of the inversion element, the group has two disconnected components.

For $d = 2$ we will use $g_{\mu \nu} = \delta_{\mu \nu}$\textsuperscript{4}, where $\delta_{\mu \nu}$ is the Euclidean metric. For this case, the equation defining the conformal transformations

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) g_{\mu \nu}$$ (2.13)

can be reduced to

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2$$ (2.14)

$$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1$$ (2.15)

which are the Cauchy-Riemann equations. Now, we can make a change of coordinates to the two dimensional complex plane $z, \bar{z} = x^1 \pm ix^2$ and write the infinitesimal transformation as $\epsilon = \epsilon^1 + i\epsilon^2$ and $\bar{\epsilon} = \epsilon^1 - i\epsilon^2$. So, the Cauchy-Riemann equations imply that

$$\partial_z \epsilon = 0$$ (2.16)

$$\partial_{\bar{z}} \epsilon = 0.$$ (2.17)

\textsuperscript{3}Invariance under these transformations can only hold if the theory has no preferred length scale.

\textsuperscript{4}In two dimensions this is not really a restriction on the theory since if the theory has conformal symmetry then one can use this symmetry to fix the metric to be flat.
Thus, in two dimensions, conformal transformations coincide with holomorphic and antiholomorphic coordinate transformations,

\[ z \mapsto f(z) \]  
\[ \bar{z} \mapsto \bar{f}(\bar{z}) \]  

which means that the conformal algebra in two dimensions is infinite dimensional. In infinitesimal form, the transformations read

\[ z \mapsto z' = z + \epsilon(z) \]  
\[ \bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) \].

### 2.2.2 Content of a conformal field theories

A conformal field theory, besides possessing conformal invariance, also satisfies the following properties:

- There is a set of field \( \{ A_i \} \) where the index \( i \) specifies the different fields. This set also contains all derivatives of each \( A_i \).
- There is a subset of fields \( \{ \phi_j \} \subseteq \{ A_i \} \), called quasi-primary fields which, under global conformal transformations \( x \mapsto x' \), transform according to

\[ \phi_j(x) \mapsto \left. \frac{\partial x'}{\partial x} \right| \Delta_j \phi_j(x') \]  

where \( \Delta_j \) is called the conformal dimension of the field \( \phi_j \) and \( d \) is the dimension of spacetime.
- The rest of the fields in \( \{ A_i \} \) that aren’t quasi-primary, can be expressed as linear combinations of the quasi-primary fields and their derivatives.

### 2.2.3 Stress-Energy Tensor

One important object in any field theory is the stress-energy tensor. From Noether’s theorem, one knows that for every continuous symmetry in a field theory, there is a
current $j_\mu$ which is conserved - $\partial^\mu j_\mu = 0$. In theories with a conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$, one has a conserved current which can be written as

$$j_\mu = T_{\mu\nu} \epsilon^\nu$$

where the tensor $T_{\mu\nu}$ is symmetric. This is the stress-energy tensor.

There’s a close relation between the trace of this tensor and the conformal nature of a given field theory: in a conformal field theory, the stress-energy tensor is traceless, that is, $T_{\mu\mu} = 0$. More details on this aspect can be found in appendix A.

### 2.2.4 Beta Function

A Lagrangian describing interacting quantum field theories possess numerical values that are used to measure the strength of interactions. These numerical values are known as couplings parameters.

The value of the coupling parameters depends on the energy scale of the physical process one is considering. The study of this dependence is made through the beta function $\beta(g)$ for a coupling parameter $g$. More specifically, if we denote the energy scale by $\mu$ the beta function is defined as

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}.$$  

(2.24)

This dependence of the coupling parameter on the energy scale is a fundamental characteristic of scale-dependence in quantum field theories. So, if one is in the presence of a scale-invariant theory, the beta function vanishes, i.e., the coupling parameters do not depend on the energy scale of the processes. A scale-independent quantum field theory is a conformal field theory. So, one concludes that, for a conformal field theory one must have

$$\beta = 0.$$  

(2.25)
2.3 Supersymmetry

In this section we’ll introduce some basic concepts of supersymmetry which will be useful to understand the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, which is the conformal field theory involved in the original AdS/CFT correspondence.

Poincaré symmetry is generated by the translations $\mathbb{R}^4$ and Lorentz transformations $SO(3,1)$, with generators $P_\mu$ and $L_{\mu\nu}$

$$[L_{\mu\nu}, P_\lambda] = -i (\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu) \quad (2.26)$$

$$[L_{\mu\nu}, L_{\lambda\rho}] = -i (\eta_{\mu\lambda} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\lambda} + \eta_{\nu\rho} L_{\mu\lambda} - \eta_{\nu\lambda} L_{\mu\rho}) \quad (2.27)$$

and supersymmetry enlarges this algebra by including supercharges $Q$: left-handed $Q^a_\alpha$ and right-handed $\bar{Q}^a_{\dot{\alpha}} = (Q^a_\alpha)^\dagger$ where $\alpha, \dot{\alpha} = 1, 2$ and $a$ counts the number of supersymmetries $a = 1, \ldots, N$.

The supercharges commute with the generators of translations and obey the algebra

$$\{Q^a_\alpha, \bar{Q}^b_{\dot{\beta}}\} = -2\sigma^a_{\alpha\dot{\beta}} P_\mu \delta^a_b, \quad \{Q^a_\alpha, Q^b_\beta\} = 2\varepsilon_{\alpha\beta} Z^{ab} \quad (2.28)$$

where $\sigma^a = (-I, \sigma^i)$ is a four vector os $2 \times 2$ matrices with the standard Pauli matrices $\sigma^i$ as their spatial entries and the operators $Z^{ab}$ is referred to as the central charges. Central charges commute with all Poincaré and supersymmetry generators $Q^a$ and obey, by symmetry reasons, to $Z^{ab} = -Z^{ba}$. Therefore, for $N = 1$ supersymmetry, we have $Z = 0$.

The supersymmetry algebra is invariant under global phase rotations of the supercharges into each other. This forms a $R$ symmetry group denoted $U(1)_R$. In addition, when $\mathcal{N} > 1$, the different supercharges may be rotated into one another under the unitary group $SU(N)_R$ which extends the $R$ symmetry.

In the $\mathcal{N} = 4$ theory one has $SU(4) \sim SO(6)$ $R$ symmetry, which is also the isometry group of the sphere in the $AdS_5 \times S^5$ background in which AdS/CFT correspondence will “take place”.
2.3.1 Field content of $\mathcal{N} = 4$ supersymmetric field theory

Representations of the supersymmetry algebra constitute the supersymmetry multiplets. Their components are spin 1 vector fields, spin 1/2 fermion fields and spin 0 scalar fields. In $\mathcal{N} = 4$ supersymmetry, we have maximal supersymmetry if the highest spin in a supersymmetric multiplet is 1. This implies that one cannot describe gravity with this theory, because the graviton is supposed to have spin 2 [7].

For any $\mathcal{N}$ with $1 \leq \mathcal{N} \leq 4$ we encounter one gauge multiplet, which is a multiplet transforming in the adjoint representation of the gauge group. For $\mathcal{N} = 4$ this is the only possible multiplet.

Lower symmetry $\mathcal{N} = 1$ and $\mathcal{N} = 2$ also admits matter multiplets. Here, we’ll only provide the names of such multiplets: the multiplet in the fundamental representation in $\mathcal{N} = 1$ supersymmetry is the chiral multiplet, and the multiplet in the fundamental representation in $\mathcal{N} = 2$ supersymmetry is the hypermultiplet.

The content of the $\mathcal{N} = 4$ multiplet is the following

- a vector field $A_\mu$ transforming in the 1 (singlet) of $SU(4)_R$;
- eight Weyl fermions $\lambda^a_\alpha$ and $\bar{\lambda}^a_\dot{\alpha}$, $a = 1, 2, 3, 4$, transforming in the 4 (fundamental) of $SU(4)_R$;
- six massless real scalars $X^i$, $i = 1, 2, \ldots, 6$, in the 6 (adjoint) of $SU(4)_R$.

The fields $A_\mu(x)$, $\lambda^a_\alpha(x)$, $\bar{\lambda}^a_\dot{\alpha}(x)$ and $X^i(x)$ of the supersymmetric multiplet presented before can be used to construct composite operators of $\mathcal{N} = 4$ SYM. Of central importance are single trace operators

$$\mathcal{O} = \text{Tr} \left\{ X^{i_1} X^{i_2} \ldots X^{i_n} \right\}. \quad (2.29)$$

2.3.2 Superfield nomenclature

Now, we’ll introduce the concept of superfield, which is a field that takes values in the superspace. The superspace is an extension of ordinary spacetime by the inclusion of additional fermionic coordinates. Points in superspace are identified by the coordinates

$$z^M = (x^\mu, \theta^a, \bar{\theta}_\dot{a}) \quad (2.30)$$
where $\theta$’s are anticommuting spinors. It was proposed [8] a function $\Phi(x, \theta, \bar{\theta})$ of the superspace coordinates, called superfield, which has a finite number of terms in its expansion in terms of $\theta$ and $\bar{\theta}$. To make things concrete, we’ll exhibit the expansion of a scalar superfield in terms of $\theta$ and $\bar{\theta}$ that will make explicit the appearance of the components of the supermultiplet (more details on [9]):

$$
\Phi(x, \theta, \bar{\theta}) = \phi(x) + \theta \psi(x) + \bar{\theta} \chi(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) + \theta \sigma^\mu \bar{\theta} V_\mu(x) + \theta \bar{\theta} \lambda(x) + \theta^2 \theta \alpha(x) + \bar{\theta}^2 \bar{\theta}^2 D(x)
$$

(2.31)

where $\phi, \psi, \bar{\chi}, M, N, V_\mu, \lambda, \alpha$ and $D$ are the component fields.

One can organize the previous multiplet in subsets of irreducible multiplets. Still dealing with the scalar superfield, one can construct an irreducible multiplet by demanding reality of $\Phi$:

$$
\Phi = \Phi^\ast
$$

(2.32)

and this is called a vector multiplet and is denoted by $V(x, \theta, \bar{\theta})$.

Another useful concept is the one of chiral $\Phi$ and antichiral $\bar{\Phi}$ superfields. These superfields also constitute an irreducible multiplet and they are defined as

$$
D_a \Phi = 0 \quad \text{and} \quad \bar{D}_a \Phi = 0
$$

(2.33)

where $D_a = \frac{\partial}{\partial \theta^a} + i \sigma^\mu_{a\dot{a}} \bar{\theta}^\dot{a} \partial_\mu$ and $\bar{D}_a = \frac{\partial}{\partial \bar{\theta}^a} + i \theta^a \sigma^\mu_{a\dot{a}} \partial_\mu$.

### 2.3.3 $\mathcal{N} = 4$ SYM Lagrangian

The best description for $\mathcal{N} = 4$ SYM would be in terms of $\mathcal{N} = 4$ superfields since most of the symmetries of the theory would be respected explicitly. However, such description wasn’t found yet and so, people write the action in terms of unconstrained $\mathcal{N} = 1$ superfields [10]. This can be done because the field content of the $\mathcal{N} = 4$ multiplet can be organized as an $\mathcal{N} = 1$ vector superfield $V$ and 3 chiral $\mathcal{N} = 1$ superfield $\Phi^i$. The action with explicit off-shell $\mathcal{N} = 1$ supersymmetry is
\[ S = \int d^6 z \text{Tr} \left( e^{-gV} \Phi_i e^{gV} \Phi_i \right) + \frac{1}{2g^2} \int d^6 z \text{Tr} W^\alpha W_\alpha \\
+ \frac{ig}{3!} \int d^6 z \varepsilon_{ijk} [\Phi^i, [\Phi^j, \Phi^k]] + \frac{ig}{3!} \int d^6 \bar{z} \varepsilon_{ijk} [\bar{\Phi}^i, [\bar{\Phi}^j, \bar{\Phi}^k]] \quad (2.34) \]

where \( W_\alpha = i d^2 \left( e^{-gV} \nabla_\alpha e^{gV} \right) \) is the gauge field strength, \( \nabla_\alpha \) (\( \bar{\nabla}_\dot{\alpha} \)) is the (anti)chiral covariant superspace derivative, \( \Phi_i \) (\( \bar{\Phi}_i \)) are covariant (anti)chiral fields\(^5\).

### 2.4 String Theory

In this section we’ll present the very basics of string theory. We’ll focus mainly on open strings and branes since it is the later object that will play the starring role in the AdS/CFT correspondence.

#### 2.4.1 Relativistic Closed String

In what follows\(^6\), we will work on \( D \)-dimensional Minkowski space \( \mathbb{R}^{1,D-1} \) and we’ll use the signature

\[ \eta_{\mu\nu} = \text{diag}(-1, +1, +1, \ldots, +1). \quad (2.38) \]

The basic constituent of string theory is an object extended along some characteristic distance \( l_s \). Such an object sweeps out a worldsheet in Minkowski space. We’ll parametrize this worldsheet by one timelike coordinate \( \tau \) and one spacelike coordinate \( \sigma \). The range of the spacelike coordinate is \( \sigma \in [0, 2\pi] \). One can define a map from the worldsheet to Minkowski space, \( X^\mu \) with \( \mu = 0, \ldots, D-1 \) which for closed strings must satisfy

\[ X^\mu (\sigma, \tau) = X^\mu (\sigma + 2\pi, \tau). \quad (2.39) \]

\(^5\)Some properties of these objects:

\[ \nabla_\alpha = e^{-V} D_\alpha e^V \]

\[ \bar{\nabla}_\dot{\alpha} = \bar{D}_{\dot{\alpha}} \]

\[ \nabla_\alpha \Phi = \nabla_\alpha \bar{\Phi} = 0 \]

\[^6\]This section deals mostly with bosonic string theory. Nevertheless, some remarks regarding superstring theory will made.
In direct analogy with the action of a relativistic point particle, the action of a relativistic string, known as the Nambu-Goto action, is

\[ S_{NG} = -T \int d^2 \sigma \sqrt{-\det \gamma} \]  

(2.40)

where \( \gamma \) is the pullback of the metric of the Minkowski space

\[ \gamma_{\alpha \beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu \nu} \]  

(2.41)

and \( T \) is the tension of the string and is related with its length by

\[ T = \frac{1}{2\pi \alpha'} \]  

(2.42)

\[ \alpha' = l_s^2. \]  

(2.43)

### 2.4.1.1 String Spectrum

Let’s analyse the spectrum of a single, free string\(^7\). The ground state is the vacuum state defined in appendix B

\[ |0; p\rangle. \]  

(2.44)

For this state, which has no oscillators excited, the mass formula gives

\[ M^2 = -\frac{1}{\alpha'} \frac{D - 2}{6} \]  

(2.45)

which is a negative mass-squared. This particles are called tachyons. The tachyon is a problem for the bosonic string - it means that the vacuum of the theory is unstable - which is removed by the introduction of supersymmetry. More on that later.

---

\(^7\)We took a big step from the end of the previous subsection to the beginning of the present one. Great part of the skipped details can be found in appendix B.

\(^8\)each operator \( \alpha^i \) and \( \tilde{\alpha}^j \) transform in the vector representation of \( SO(D - 2) \), which implies that the state \( \alpha^i_{-1} \alpha^j_{-1} |0, p\rangle \) transforms in \( SO(D - 2) \times SO(D - 2) \).
which correspond to $(D - 2)^2$ particle states\textsuperscript{9} with mass given by
\begin{equation}
M^2 = \frac{4}{\alpha'} \left( 1 - \frac{D - 2}{24} \right).
\end{equation}

As a general rule, Lorentz invariance implies that physical states form a representation of $SO(D - 1)$ for massive states and $SO(D - 2)$ for massless states\textsuperscript{10}. Since we have $(D - 2)^2$ states, it is not possible to put all these states in a representation of $SO(D - 1)$. So, if we want to preserve the $SO(1, D - 1)$ Lorentz symmetry of the theory, one has to consider that these states are described by a $SO(D - 2)$ representation, and so the states are massless. To guarantee the masslessness of the states, one has to put
\begin{equation}
D = 26.
\end{equation}

Having done this, one knows that the first excited states transform in the $24 \otimes 24$\textsuperscript{11} of $SO(24)$. The irreducible representations are
\begin{equation}
\text{traceless symmetric} \oplus \text{anti-symmetric} \oplus \text{trace}
\end{equation}
and to each of them one associates a massless field:
\begin{itemize}
  
  \item To the traceless symmetric mode one calls graviton and denotes it by $G_{\mu\nu}(X)$. This field transforms as a spin 2 particle under $SO(24)$. Due to general arguments [13], any theory of interacting massless spin 2 particles must be equivalent to general relativity. So, one should identify $G_{\mu\nu}(X)$ with the metric of spacetime.
  
  \item The anti-symmetric mode is denoted by $B_{\mu\nu}(X)$ and transforms under $SO(24)$ as an anti-symmetric second-rank tensor.
  
  \item The trace transforms as a scalar field. One denotes it by $\Phi(X)$ and calls it dilaton.
\end{itemize}

These three fields are common to all string theories.

\textsuperscript{9}because $i, j = 1, 2, \ldots, D - 2$

\textsuperscript{10}This has to do with Wigner’s classification of the Poincarè group. For more details, check [11] or [12].

\textsuperscript{11}The uses of this notation are explained in appendix D.
2.4.1.2 A touch of superstring theory

The details underpinning the association of string theory and supersymmetry won’t be presented here. However, it is important to mention the final result of such work. That’s what we’re going to do. The good reference for all the calculations is [12].

The principal difference between bosonic strings and superstrings is the addition of fermionic modes to the worldsheet - after doing it, the worldsheet theory becomes supersymmetric. Having done that, applying the quantization procedure described previously, one finds that

- The critical dimension of the superstring is \( D = 10 \).
- There is no tachyon in the spectrum\(^{12}\).
- The massless fields \( G_{\mu\nu}, B_{\mu\nu} \) and \( \Phi \) are all part of the spectrum of the superstring.

There are different ways of adding fermionic degrees of freedom to the bosonic string. To each different way corresponds a different superstring theory which has extra fields besides the massless, bosonic string ones: Type I, Type IIA, Type IIB, Heterotic \( SO(32) \) and Heterotic \( E_8 \times E_8 \). We’ll be mostly interested in Type IIB superstring theories, which has extra massless gauge fields that include a scalar, a 2-form and a 4-form.

2.4.2 Open Strings

Now, we move on to open strings. The study of the spectrum of these strings will bring new insights to the whole theory, particularly by shedding light on a new set of objects that play a fundamental role in the theory: D-branes. These are the objects to keep an eye on if one wants to understand where the AdS/CFT correspondence comes from.

2.4.2.1 Boundary Conditions and Branes

The spatial coordinate of the open string is parametrized by

\[ \sigma \in [0, \pi]. \] (2.50)

To describe the dynamics of an open string one uses, besides the Polyakov action (which in the following will be written in conformal gauge)

\(^{12}\)In the following subchapter, we’ll deal with open strings, which also have a tachyon in the spectrum. Supersymmetry will get rid of that too.
\[ S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X \]  

(2.51)

a set of boundary conditions that specify how the end points of the string move.

There are two types of boundary conditions:

- Neumann boundary conditions:
  \[ \partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0, \pi \]  
  (2.52)

- Dirichlet boundary conditions:
  \[ \delta X^\mu = 0 \quad \text{at } \sigma = 0, \pi \]  
  (2.53)

which means that the endpoints of the string lie at some constant position \( X^\mu = c^\mu \) in space.

These boundary conditions can be applied side by side. Let’s consider Neumann boundary conditions for some coordinates, and Dirichlet boundary conditions for others, i.e.

\[ \partial_\sigma X^a = 0 \quad \text{for } a = 0, \ldots, p \]
\[ X^I = c^I \quad \text{for } I = p + 1, \ldots, D - 1 \]

which fix the endpoints of the string to lie in a \((p + 1)\)-dimensional hypersurface in spacetime such that the \(SO(1,D - 1)\) Lorentz group is broken to

\[ SO(1,D - 1) \to SO(1,p) \times SO(D - p - 1). \]  

(2.54)

This hypersurface is called \(Dp\)-brane where the \(p\) stands for the number of spatial dimensions of the object. Branes are dynamical objects and play a fundamental role in string theory.

### 2.4.2.2 Open string spectrum

In what follows we will be focusing on type II string theories, since these are the ones we care if we want to understand AdS/CFT correspondence. Moreover, we will be assuming
that the theory with which we are dealing contains open and closed strings, as well as D-branes.

Similarly to closed strings, the ground state of an open string is a tachyon:

$$\alpha^i_n |0; p\rangle = 0 \quad n > 0 \quad (2.55)$$

with $$a = 1, \ldots, p - 1, p + 1, \ldots, D - 1$$ with mass

$$M^2 = -\frac{1}{\alpha'} \quad (2.56)$$

Now, the first excited states are massless and fall into two classes:

- Oscillators longitudinal to the brane

$$\alpha^a_{a-1} |0; p\rangle \quad a = 1, \ldots, p - 1; \quad (2.57)$$

notice that since the index $$a$$ lies within the brane, the state transforms under the group $$SO(1,p)$$. It is a spin 1 particle and so one identifies it with the quanta of a gauge field $$A_a$$, $$a = 0, \ldots, p$$;

- Oscillators transverse to the brane

$$\alpha^I_{p+1} |0; p\rangle \quad I = p + 1, \ldots, D - 1 \quad (2.58)$$

which are scalars under $$SO(1,p)$$ and transform as vectors under $$SO(D - p - 1)$$. These states can be thought of as arising from scalar fields $$\phi^I$$ living in the brane, which are fluctuations of the brane in the transverse directions.

### 2.4.2.3 Chan-Paton indices

Before moving on, one should pay close attention to the situation in which a string stretches between two branes.

Such string is described by

$$X^I = c^I + \left(\frac{d^I - c^I}{\pi}\right) \sigma + \text{Fourier modes} \quad (2.59)$$
and using the classical constraints\textsuperscript{13} one can obtain
its mass:

\[\partial^+ X \cdot \partial^+ X = \alpha'^2 p^2 + \frac{|\vec{d} - \vec{c}|^2}{4\pi^2} + \text{Fourier modes} = 0 \quad \text{(2.60)}\]

which can be translated to the classical mass condition

\[M^2 = \frac{|\vec{d} - \vec{c}|^2}{(2\pi\alpha')^2} + \text{Fourier modes.} \quad \text{(2.61)}\]

After quantization, the ground state of this string is only tachyonic if $|\vec{d} - \vec{c}|^2 < 4\pi^2\alpha'$.

One can generalize this analysis to the case of $N$ parallel branes. In that scenario, each endpoint of the string has $N$ possible places on which to end. To keep track of each string endpoint, one labels each one by the number $m, n = 1, \ldots, N$ of the brane in which it is placed. This label is called Chan-Paton index.

If the $N$ branes lie at the same spacetime position, each string has $N^2$ possibilities to arrange itself between two branes. Each string has the mass spectrum of an open string, which means that one is faced with $N^2$ different particles of each type. If one arranges the associated fields inside a $N \times N$ hermitian matrix, we end up with the open string tachyon $T^m_n$ and the massless fields

\[(\phi^I)^m_n, \quad (A_a)^m_n. \quad \text{(2.62)}\]

This way, the components of the matrix refer to the string the field came from; also, diagonal components refer to strings with both ends on the same brane.

\section*{2.4.2.4 Another touch from superstrings}

Open strings are also ingredients of superstring theories. Supersymmetry, however, restricts the possible types of stable branes each superstring theory can contain. Having said that, we simply mention that $Dp$-branes are stable in type IIA superstring theory if $p$ is even; if $p$ is odd, the branes are stable in type IIB superstring theory\textsuperscript{[14]}.

\textsuperscript{13}Classical constraints are mentioned in appendix B
2.4.3 Branes

Dp-branes are dynamic objects whose action is an extension of Nambu-Goto action, called Dirac action:

\[ S_{D_p} = -T_p \int d^{p+1}\xi \sqrt{-\det \gamma} \]  \hspace{1cm} (2.63)

where \( T_p = \frac{1}{(2\pi \alpha')^{p+1}} \) is the brane tension, \( \xi^a, a = 0, \ldots p \) are the worldvolume coordinates of the brane and \( \gamma_{ab} \) is the pullback of the spacetime metric onto the brane worldvolume

\[ \gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu}. \]  \hspace{1cm} (2.64)

For an infinite, flat Dp-brane, one can choose

\[ X^a = \xi^a \quad a = 0, \ldots p \]  \hspace{1cm} (2.65)

while the dynamical transverse coordinates are then identified with the fluctuations through

\[ X^I(\xi) = 2\pi \alpha' \phi^I(\xi) \quad I = p + 1, \ldots, D - 1. \]  \hspace{1cm} (2.66)

By recalling the states present in the open string spectrum, one can see that the action just presented is not enough to account for all the dynamics of the brane. The way it was presented, it only takes into account the transverse fluctuations of the brane leaving out the \( U(1) \) gauge field \( A_a \) which is on the brane itself.

2.4.3.1 Brane gauge fields

By analysing the consistency of the propagation of an open string in the presence of background gauge fields of the brane and imposing conformal invariance to the theory\(^\text{14}\) one obtains that the gauge field must satisfy

\[ \partial_b F_{ac} \left[ \frac{1}{1 - 4\pi \alpha' F^2} \right]^{ab} = 0 \]  \hspace{1cm} (2.67)

\(^\text{14}\)Detailed calculations can be found in [15]
where

\[ F_{ab}(X) = \frac{\partial A_b}{\partial X^a} - \frac{\partial A_a}{\partial X^b} = \partial_a A_b - \partial_b A_a. \] (2.68)

Now it is desirable to come up with an action whose equations of motion coincide with the condition just found. That action is the Born-Infeld action:

\[ S = -T_p \int d^{p+1} \xi \sqrt{-\det (\eta_{ab} + 2\pi \alpha' F_{ab})} \] (2.69)

where \( \xi \) are the worldvolume coordinates and \( T_p \) is the tension of the brane. Important remark: the Born-Infeld action arises from the one-loop beta function [16, p. 651].

Combining the action for the scalar fluctuations and the one for gauge fields on the brane, we obtain the Dirac-Born-Infeld (DBI) action

\[ S_{DBI} = -T_p \int d^{p+1} \xi \sqrt{-\det (\gamma_{ab} + 2\pi \alpha' F_{ab})} \] (2.70)

where, again, \( \gamma \) is the pullback of the spacetime metric onto the worldvolume \( \gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \eta_{\mu\nu} \).

Going beyond the scenario of low-energy dynamics of a brane in flat space, one can enquire about the behavior of the brane in the presence of a background created by closed string modes \( G_{\mu\nu}, B_{\mu\nu} \) and \( \Phi \). In such a situation, the dynamics is described by the following [17, p. 135]

\[ S_{DBI} = -T_p \int d^{p+1} \xi \exp \left(-\tilde{\Phi}\right) \sqrt{-\det (\gamma_{ab} + 2\pi \alpha' F_{ab} + B_{ab})} \] (2.71)

where

\[ \gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu} \] (2.72)

and

\[ B_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}; \] (2.73)

Note that the dilaton was decomposed into a constant and a varying piece: \( \Phi = \Phi_0 + \tilde{\Phi} \).
**U(N)-canny appearance of Yang-Mills** It was shown in [2.4.2](#) that the massless fields on $N$ coincident branes could be described by $N \times N$ Hermitian matrices, with the element of the matrix telling us which brane the end points terminate on. The gauge fields take the form

$$(A_a)_m^n$$

(2.74)

with $a = 0, \ldots, p$ and $m, n = 1, \ldots, N$. This gauge field describes a $U(N)$ gauge symmetry. So, the massless excitations of $N$ coincident branes are a $U(N)$ gauge field $(A_a)_m^n$, together with scalars $(\phi^I)_m^n$ which transform in the adjoint representation of the $U(N)$ gauge group. It was pointed out (in [2.4.2](#)) that the diagonal components $(\phi^I)_m^m$ have the interpretation of the transverse fluctuations of the $m^{th}$ brane.

We should look for the action that describes the interactions between these two fields. That task can only be accomplished in an incomplete way - obtaining its low-energy limit, corresponding to small field strengths. This can be shown [18] to be

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \text{Tr} \left( \frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} D_a \phi^I D^a \phi^I - \frac{1}{4} \sum_{I \neq J} \left[ \phi^I, \phi^J \right]^2 \right)$$

(2.75)

where, neglecting the matrix indices, $F_{ab} = \partial_a A_b - \partial_b A_a + i [A_a, A_b]$ and $D_a \phi^I = \partial_a \phi^I + i [A_a, \phi^I]$.

A flat, infinite $Dp$-brane breaks the Lorentz group of spacetime to

$$SO(1, D - 1) \rightarrow SO(1, p) \times SO(D - p - 1)$$

(2.76)

being the $SO(1, p)$ the Lorentz group of the D-brane worldvolume and $SO(D - p - 1)$ the global symmetry of the D-brane theory, rotating the scalar fields $\phi^I$.

When all branes are separated, the gauge group symmetry is broken from $U(N)$ to $U(1)^N$. As a consequence, strings that are stretched between two, non-coincident branes, acquire mass that is proportional to the separation of the branes [19]. More precisely

$$M_{\text{stretched}} = T |\vec{X}_n - \vec{X}_m|$$

(2.77)
where \( T = \frac{1}{2\pi\alpha'} \), \( \vec{X}_m \) is the position of the \( m^{th} \) brane, which is related to the scalar transverse fluctuations of the brane

\[
\vec{X}_m = 2\pi\alpha'\vec{\phi}_m
\]  

(2.78)

where \( \vec{\phi}_n \) is a vector composed of the diagonal elements \((\phi^I)_n\).

**Details of \( N \) D3-branes**  We are specially interested in D3-branes because they will play a fundamental role in the AdS/CFT correspondence. For that reason, it is important to point out some facts. The following facts are true for a type IIB superstring theory (check 2.4.2.4).

The low energy dynamics of a single D3 brane is described by \( \mathcal{N} = 4 \) SYM \( U(1) \) theory [20]. If \( N \) D3 branes are superimposed, the theory describing the low energy dynamics is \( \mathcal{N} = 4 \) SYM \( U(N) \). This theory splits into a \( U(1) \) part, which is free, and relates to the center of mass motions of the stack of D3-branes, and an \( SU(N) \) part, which is interacting, and relates to the relative motions of the branes.

**2.4.4 String Interactions**

In string theory, when one wants to calculate interactions, instead of calculating correlation functions such as

\[
\langle \phi(x_1)\ldots\phi(x_n) \rangle
\]

(2.79)

one can only consistently calculate the S-matrix, which corresponds to the previous correlation function on the limit \( x_i \to \infty \), \( \forall i = 1, 2, \ldots, n \).

Using the state-operator map (see appendix A), one knows that each state at infinity is equivalent to the insertion of an appropriate local operator on the worldsheet - check figure 2.4. To compute the S-matrix one uses conformal transformations to bring each of the operators at infinity to a finite distance. After this procedure, one ends up with a worldsheet with the topology of a sphere, dotted with local operators where used to be legs.

The calculation of the Polyakov path integral, requires the sum over all worldsheet metrics. However, when one is dealing with interactions, the sum over all worldsheet topologies is needed too[14]. Actually, it is the sum over topologies that will give rise to the perturbation expansion.
2.4.4.1 Summing over topologies

The way in which the topologies are accounted for in the expansion comes from a slight change in Polyakov action

\[ S_{\text{string}} = S_{\text{Polyakov}} + \lambda \chi \]  

in which \( \lambda \) is a real number and

\[ \chi = \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} R \]  

where \( R \) is the Ricci scalar and the integral is over the euclidean worldsheet. It turns out that \( \chi \) is a topological invariant and one can write it\(^{15}\) as

\[ \chi = 2 - 2h = 2(1 - g) \]  

where \( h \) refers to the number of handles on the worldsheet. Also, one refers to \( \chi \) as the \textit{Euler characteristic} and to \( g \) as the \textit{genus}.

Following this procedure, it is easy to see how one can take into account all possible worldsheet topologies: one sums over all possible values for the genus. So, the integral over worldsheets in weighted by

\[ \sum_{\text{topologies}} \sum_{\text{metrics}} e^{-S_{\text{string}}} \sim \sum_{g=0}^{\infty} e^{-2\lambda(1-g)} \int D\mathcal{X} Dg \ e^{-S_{\text{Polyakov}}}. \]  

\(^{15}\) according to Gauss-Bonnet theorem
Doing this, $e^\lambda$ plays the role of a string coupling. So, one has a good perturbation theory approximation if $e^\lambda \ll 1$. It is standard to define (check [14, p. 90])

$$g_s = e^\lambda$$

(2.84)

where $\lambda = \langle \phi \rangle$, i.e., it is the expectation value of the dilaton. Finally, one ends up with the sum over worldsheets becoming a sum over Riemann surfaces of increasing genus, with local operators inserted for the initial and final states. The Riemann surface of genus $g$ is weighted by

$$(g_s^2)^{g-1}.$$ (2.85)

### 2.4.4.2 Scattering amplitudes

To compute the scattering amplitude of $m$ states $\Lambda_i$, $i = 1, 2, ..., m$, one starts by assigning to each state a local operator $V_{\Lambda_i}$ with spacetime momenta $p_i$. The S-matrix element is then computed by evaluating the correlation function in the two-dimensional conformal field theory, with insertions of the operators:

$$A^{(m)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \frac{1}{\text{Vol}} \int D\sigma Dg e^{-S_{\text{Polyakov}}} \prod_{i=1}^m V_{\Lambda_i}(p_i).$$ (2.86)

This is all one wants to show about interacting strings. It is important to recall the comment made when we presented the large $N$ limit used in $SU(N)$ gauge theories.

### 2.4.5 Effective String theory

The treatment made so far dealt with strings moving in a flat background. One can also analyse strings moving in curved backgrounds. The action in this case is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$ (2.87)

where $g_{\alpha\beta}$ is the worldsheet metric and $G_{\mu\nu}(X)$ is the spacetime metric. Writing the action in conformal gauge \(^{17}\) gives

\(^{16}\)According to the state-operator map (see appendix A)

\(^{17}\)choosing $g_{\alpha\beta} = e^{\phi} \eta_{\alpha\beta}$ with $\phi = 0$
Now, an important question arises: under what circumstances the previous action retains conformal invariance at a quantum level? We know that classically, this describes a conformally invariant theory, but at a quantum level, we’re not sure. To investigate that detail, we calculate a beta functional

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu}$$

(2.89)

so the requirement for the metric in the conformal gauge to be conformally invariant is that the target space must be Ricci flat: $R_{\mu\nu} = 0$, in other words, the background spacetime in which the string moves must obey the vacuum Einstein equations. This is an interesting result. However, it is a simplification. Besides the background metric, strings can interact with other fields, namely the antisymmetric tensor $B_{\mu\nu}$ and the dilaton $\Phi$.

### 2.4.5.1 $B_{\mu\nu}$ and $\Phi$

The way in which strings couple to $B_{\mu\nu}$ and $\Phi$ is described by

$$S = S_{\text{string}} + S_B + S_{\Phi} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + i B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} + \alpha' \Phi(X) R^{(2)} \right)$$

where $\epsilon^{\alpha\beta}$ is an antisymmetric 2-tensor normalized in the following way: $\sqrt{g} \epsilon^{12} = +1$ and $R^{(2)}$ is the two-dimensional Ricci scalar of the worldsheet. Writing this action makes one stumble upon a complicated issue: the coupling to the dilaton isn’t Weyl invariant. However, it is possible to restore Weyl invariance to the action by imposing that the stress tensor $T^a_\alpha$ vanishes (check A.3). What one obtains [21] is

$$\langle T^a_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{\mu\nu}(G) g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2\alpha'} \beta_{\mu\nu}(B) \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \beta(\Phi) R^{(2)}$$

(2.90)

---

18 Here we’re referring to a beta functional, instead of a beta function as in subsection 2.2.4. Since we’re dealing with functions that depend on the energy scale, it makes sense to define a beta functional as $\beta_{\mu\nu} \sim \mu \frac{\delta S_{\mu\nu}(X,\mu)}{\delta \mu}$, where, as usual, $\mu$ refers to the energy scale.
with the one-loop beta functionals

\[
\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\nu} H^\lambda
\]

(2.91)

\[
\beta_{\mu\nu}(B) = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu}
\]

(2.92)

\[
\beta(\Phi) = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}
\]

(2.93)

and \( H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \). If one desires to preserve Weyl invariance, one has to require that \( \beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0 \).

### 2.4.5.2 Low energy effective action

The equations \( \beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0 \) can be viewed as the equations of motion for the background in which the string propagates. It is possible to come up with an action in the 26th dimensional space that gives rise to the beta functionals as if they were the equations of motion. The action one is looking for is the low energy effective action of the bosonic string

\[
S \sim \int d^{26}X \sqrt{-G} e^{-2\Phi} \left( R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi \right)
\]

(2.94)

and after a different choice of frame, one can write

\[
S \sim \int d^{26}X \sqrt{-G} \left( \tilde{R} - \frac{1}{12} e^{-\frac{\Phi}{3}} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} \right).
\]

(2.95)

Here, it is important to note that the caveat ”low-energy” refers to the fact that we only worked with the one-loop beta functions.

### 2.4.5.3 Superstring low energy effective action

As presented in 2.4.1.2, it is possible to extend bosonic string theory for fermionic degrees of freedom using supersymmetry. Each type of superstring theory contains fields \( G_{\mu\nu} \), \( B_{\mu\nu} \) and \( \Phi \), common to the bosonic string, but also a collection of other massless fields. For each of these theories, the low-energy effective action describes the dynamics of these
fields in a 10-dimensional spacetime. In what follows, we focus on type IIB superstring theory because that is the one which is important to AdS/CFT correspondence.

The low energy effective action for type IIB superstring theory is

\[
S \sim \int d^{10}X \left[ \sqrt{-G} e^{-2\Phi} \left( R - \frac{1}{12} |H_3|^2 + 4 \partial_{\mu} \Phi \partial^{\mu} \Phi \right) + \sqrt{-G} \left( |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) + C_4 \wedge H_3 \wedge F_3 \right] + S_{\text{fermi}} \tag{2.96}
\]

where \( S_{\text{fermi}} \) describes the interactions of spacetime fermions. Also, one adopted the form notation: one denotes \( H_{\mu\nu\lambda} \) by \( H_3 \). In this notational spirit, one has the following fields: \( F_1 = dC_0 \), \( F_3 = dC_2 \), \( F_5 = dC_4 \), \( \tilde{F}_3 = F_3 - C_0 \wedge H_3 \) and \( \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \), where \( C_0 \) is a scalar, \( C_2 \) is a 2-form and \( C_4 \) is a 4-form. Besides this action, one needs a supplementary condition to correctly describe the low energy dynamics of type IIB theory: \( \tilde{F}_5 \) must be self-dual, i.e.

\[
\tilde{F}_5 = *\tilde{F}_5. \tag{2.97}
\]

### 2.5 A supergravity solution

Besides superstring theory, it is possible to take another path in the endeavour of finding a quantum theory of gravity. If we take supersymmetry and promote it to a local symmetry, one is forced to include gravity in the theory [22]. The gravity theory obtained in this way, called supergravity, is a quantum theory [23]. Specifically, if we promote the supersymmetry to a local symmetry in the standard model, one is forced to include in the theory the graviton and the gravitino.

We want to study solutions for the equations of motion of the fields involved in the effective action. We call \( p \)-branes to these solutions, where the \( p \) refers to the dimensionality of the extended object which will be the solution. As we said, supergravity can be obtained from a low-energy study of superstring theory. However, a much broader approach can be taken and here, that is what we’re going to do. We’ll start with a highly general action and then we’ll slowly specializing it so that in the end it coincides with the superstring effective low-energy theory. In the end, we will be interested in the metric coming out of this treatment.

We’ll be interested in extremal \( p \)-brane solutions which can be described by the metric
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\[ ds^2 = H^{- \frac{2}{p+1}} \left( -dt^2 + \sum_{i=1}^{p} (dx^i)^2 \right) + H^{\frac{2}{p-2}} \sum_{a=1}^{d} (dy^a)^2 \]  

(2.98)

where \( d \) refers to the number of transverse coordinates to the \( p \)-brane and

\[ H = 1 + \frac{Q}{(d-2)r^{d-2}} \]  

(2.99)

in which \( Q \) is the charge of the brane.

The specific case\(^\text{19}\) in which we interested is the one with \( p = 3 \) and \( D = 10 \). So, taking these values, we obtain from the previous equation

\[ H = 1 + \frac{4\pi g_s N l_s^4}{r^4} \]  

(2.100)

One defines the scaled variable

\[ U = \frac{r}{l_s^2} \]  

(2.101)

and consider the limit \( \alpha' = l_s^2 \to 0 \) and also \( r \to 0 \) in such a way that \( U \) becomes the meaningful variable

\[ H \approx \frac{4\pi g_s N}{U^4 l_s^4} \]  

(2.102)

\[ ds^2 = l_s^2 \left\{ \frac{U^2}{\sqrt{4\pi g_s N}} dz^2 + \sqrt{4\pi g_s N} \left( \frac{dU^2}{U^2} + d\Omega_5^2 \right) \right\} = \frac{U^2}{R^2} dz^2 + R^2 d\Omega_5^2 \]  

(2.103)

where

\[ dz^2 = -dt^2 + \sum_{i=1}^{3} (dx^i)^2 \]  

(2.104)

with \( \tilde{z} \) appropriately scaled version of coordinate \( z \). In the end, comparing this last equation to the equations in section 2.6, we see that, in this specific limit, we obtain the

\(^\text{19}\) A more interested reader may want to take a look at appendix C where we present a more general approach to supergravity solutions.
metric of $\text{AdS}_5 \times S^5$. When $l_s \to 0$ the metric has to be rescaled to get a finite result - by removing the $l_s^2$ overall factor. The radius parameter is given by

$$R^4 = 4\pi g_s N l_s^4. \quad (2.105)$$

### 2.5.1 $p$-branes and $Dp$-branes

Polchinski [24] realised that if one makes the identification between the extremal $p$-brane charge $Q$ and the tension of the corresponding $Dp$-brane

$$T_q = Q \quad (2.106)$$

the two objects are exactly the same. This identification is of extreme importance for the argumentation that leads to the AdS/CFT correspondence.

### 2.6 Anti-de Sitter space

Anti-de Sitter space is a maximally symmetric20 spacetime with constant negative curvature.

There are many different parametrizations of the metric, and it is important to understand how these metrics are related to each other.

Anti-de Sitter space is a solution to the Einstein equation in the vacuum with a negative cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \Lambda g_{\mu\nu} \implies \quad (2.107)$$

$$R = \frac{D}{2 - D} \Lambda \implies \quad (2.108)$$

$$R_{\mu\nu} = \frac{\Lambda}{2 - D} g_{\mu\nu}. \quad (2.109)$$

In addition, since it is maximally symmetric, the following condition applies [25, p. 27]

20Spacetime is maximally symmetric if it is both homogeneous and isotropic.
\[ R_{\mu\nu\rho\sigma} = \frac{R}{D(D-1)} (g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma}). \]  

We will consider a \( d + 1 \) dimensional Anti-de Sitter space. It can be represented as a hyperboloid of radius \( L \)

\[ X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = L^2 \]  

which is embedded in \( d + 2 \) dimensional flat space with metric

\[ ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2. \]  

These are called embedding coordinates.

To find the metric on the hyperboloid itself, one parametrizes the embedding coordinates \( X_i, i = 0, \ldots, d + 1 \) in the following way:

\[ X_0 = L \sec \rho \cos \tau \]  

\[ X_i = L \Omega_i \tan \rho, \quad i = 1, \ldots, d \]  

\[ X_{d+1} = L \sec \rho \sin \tau \]  

where \( 0 < \rho < \frac{\pi}{2}, -\pi < \tau < \pi, -1 < \Omega_i < 1 \) and \( \sum_{i=1}^{d} \Omega_i = 1 \). The coordinates \( \rho, \tau \) and \( \Omega_i \) represent the whole hyperboloid and they are called global coordinates. In these coordinates, the metric reads

\[ ds^2 = \frac{L^2}{\cos^2 \rho} \left( -d\tau^2 + d\rho^2 + \sin^2 \rho \sum_{i=1}^{d} d\Omega_i^2 \right). \]

This metric covers all of the \( d + 1 \) dimensional AdS space. The AdS boundary is at \( \rho = \frac{\pi}{2} \).

Another useful coordinate system describing AdS space is the so-called Poincaré coordinates. One starts by defining
\[ u = \frac{X_0 - X_d}{L^2} \quad \text{and} \quad v = \frac{X_0 + X_d}{L^2} \tag{2.117} \]

and

\[ x^i = \frac{X_i}{uL} \quad \text{and} \quad t = \frac{X_{d+1}}{uL}. \tag{2.118} \]

Then \( X_0 = \frac{L^2}{2}(u + v) \), \( X_d = \frac{L^2}{2}(v - u) \), \( X_i = uLx^i \) and \( X_{d+1} = uLt \). Plugging all this in the equation for the hyperboloid, one obtains an expression for \( v \) that depends on \( u \) and from that it is possible to write

\[ X_0 = \frac{1}{2u} \left( 1 + u^2 \left( L^2 - t^2 + \sum_{i=1}^{d-1} (x^i)^2 \right) \right) \tag{2.119} \]

\[ X_d = \frac{1}{2u} \left( 1 + u^2 \left( -L^2 - t^2 + \sum_{i=1}^{d-1} (x^i)^2 \right) \right) \tag{2.120} \]

\[ X_i = Rux^i, \quad i = 1, \ldots, d - 1 \tag{2.121} \]

\[ X_{d+1} = Rut. \tag{2.122} \]

This is nice because we have all \( X \) coordinates expressed in terms of \( u \) which allows us to write the metric in the form

\[ ds^2 = R^2 \left( \frac{du^2}{u^2} + u^2 (\eta_{\mu\nu} dx^\mu dx^\nu) \right) \tag{2.123} \]

Now, making another change of coordinates

\[ z = \frac{1}{u} \tag{2.124} \]

one obtains that the metric can be written as

\[ ds^2 = \frac{R^2}{z^2} \left( -dt^2 + dz^2 + \sum_{i=1}^{d-1} (x^i)^2 \right). \tag{2.125} \]
2.7 The decoupling argument - the realization of AdS/CFT correspondence

Finally, the star of the show: AdS/CFT correspondence will be presented now. We start by mentioning the two points of view one can take when analysing Dp-branes. These extended objects can be seen either as a locus where open strings can end (due to imposed boundary conditions) or as objects that emit closed strings. More specifically, in the latter case, the brane emits gravitons (closed strings) that deform the background, while in the former it is the place where open strings can end and whose interactions in a low energy regime are described by some effective theory.

In what follows, we are going to put everything that was referred so far together and make explicit the arguments that lead to AdS/CFT correspondence. We’ll work on a scenario of a propagating closed string, which will be our probe, in the presence of \( N \) D3-branes.

The argumentation will follow from the dual interpretation one can make of this scenario. On one hand, we can study the effect of the branes in the propagation of the closed string using the standard procedure of string interaction expansion, and perform sums of the form (check subsection 2.4.4)

On the other hand, we can focus on the effect of the gravitons, emitted by the branes, on the background space and then study the propagation of the free string in the correspondent deformed background.
Studying carefully these two points of view, one will be able to unveil the statement of AdS/CFT correspondence. The important aspect of taking these two approaches is to understand that, whatever description we find, the system being described is well defined, so the two descriptions have to be equivalent.

2.7.1 String interaction point of view (A)

Considering the system at low energies, i.e., energies lower than the string scale $1/l_s$, one knows that only massless string states can be excited and the effective action describing their interactions can be written as

$$S = S_{bulk} + S_{brane} + S_{int.}.$$  \hspace{1cm} (2.126)

One knows that closed string massless states are described by type IIB supergravity in ten dimensions (check sections 2.4 and 2.5) and the open string massless states generate $\mathcal{N} = 4$ SYM theory in 4 dimensions. $S_{bulk}$ and $S_{brane}$ correspond to the former and the latter theories. $S_{int}$ describes the interactions between brane modes and bulk modes.

One can schematically write the action of the bulk as

$$S_{bulk} \sim \frac{1}{2\kappa^{2}} \int \sqrt{g} R \sim \int (\partial h)^{2} + \kappa (\partial h)^{2} h + \cdots$$ \hspace{1cm} (2.127)

where we wrote $g = \eta + \kappa h$ with $\kappa$ the Newton constant. With this assumption, one is describing the propagation of free massless modes plus some interactions. Similarly, the action describing the interactions is proportional to positive powers of $\kappa$. To take the low energy limit it is convenient to keep the energy fixed and put $l_s \rightarrow 0 \iff \alpha' \rightarrow 0$ while keeping all the dimensionless parameters fixed. In this limit, the Newton constant goes to zero as well $\kappa \sim g_s \alpha' \rightarrow 0$, so that the interaction action vanishes. In addition, the brane action is reduced to a pure $\mathcal{N} = 4$ $U(N)$ gauge theory in 4 dimensions and the supergravity theory in the bulk becomes free because all interaction terms were proportional to $\kappa$ and dropped out.
So, from this point of view, in the low energy limit we end up with two decoupled systems:

\[(\mathcal{N} = 4 \text{ SYM in } 4\text{D}) \oplus (\text{SUGRA in } 10\text{D})\]  

(2.128)

It is important to stress that the limit of validity of the previous statement, the low energy limit, means that the 't Hooft coupling is very small: \(\lambda = g_s N \ll 1\). In this regime, when one starts to draw diagrams in the Yang-Mills theory of the brane, those with less number of loops are the ones that contribute the most (remember the discussion on subsection 2.1.3) and one obtains a very good description of the field theory.

### 2.7.2 Deformed background point of view (B)

If we think of the branes as objects that emit closed strings, it is possible to replace the effect of the brane by a suitable deformed background. The idea goes as follows: using the state operator map presented in appendix A one can replace the closed strings emitted by the brane (states) by operators inserted in the worldsheet of our propagating closed probe string.

Then, the usual sum over topologies performed in interacting string theory can be replaced by a sum of insertions in the worldsheet of the closed propagating string.

\[
\sum_{\text{topologies}} = \sum_{\text{insertions}}
\]

Qualitatively, this amounts to adding an extra term to the path integral, deforming the action describing the whole system,

\[
\sum_{\text{insertions}} \sim \exp \left( \int d^2 \sigma U \right)
\]  

(2.129)
which, after added to the original action, is equivalent to a deformation of the background. This is physically equivalent to say that a condensate of gravitons is changing the background. This descriptions touches the previously referred (in subsection 2.5.1) connection between supergravity solutions ($p$-branes) and D$p$-branes: one can find the effect of the stack of D$p$-branes on the background by solving the supergravity equations that describe $p$-branes.

So, the effect of having $p$-branes in the background (which is equivalent to the effect of having D$p$-branes and consider that they are emitting closed strings) is expressed by the metric (check section 2.5)

\[
 ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{\sqrt{1 + \frac{R^4}{r^4} dx_m dx^m}}, \quad \mu, \nu = 0, \ldots, 3; \ m,n = 4, \ldots, 9
\]  

(2.130)

where $r = x_m x^m$ measures the distance from the heavy object (the $p$-branes) and $R^4 = g_s N 4\pi \alpha'^2$.

Now, we explore different limits. Note that if $R \gg l_s \Leftrightarrow \lambda = g_s N \gg 1$. If we go very far from the brane, $r \gg R$, we obtain

\[
 ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \delta_{mn} dx^m dx^n
\]  

(2.131)

which is a flat ten dimensional metric. If we go close to the branes $r \ll R$

\[
 ds^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dx_m dx^m
\]  

(2.132)

and we can write $dx_m dx^m = dr^2 + r^2 d\Omega_5^2$ obtaining

\[
 ds^2 = \left[ \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2 \right] + L^2 d\Omega_5^2
\]  

(2.133)

which is the metric of $AdS_5 \times S^5$. So, we conclude that the presence of the branes\footnote{Let’s be diligent: here we can call, without abuse, brane to either $p$ or D$p$ branes.} has the effect of deforming the background to $AdS_5 \times S^5$ for $r \ll R$ (near the branes) and leaving the flat ten dimensional background intact for $r \gg R$ (far from the branes).

Having understood how the branes deform the background, we study the theoretical description of our scenario. We start by making explicit how time measurements close to the brane are related to time measurements by someone that is far away from it:
\[ \Delta t_{\text{close}} = \sqrt{\frac{r^2}{R^2}} \Delta t_{\infty}. \]  

(2.134)

The energy of a process taking place at position \( r \) will give different measurements for an observer sitting close to the brane and an observer far away from it. That difference is given by the expression

\[ E_{\text{close}} = \frac{R}{r} E_{\infty}. \]  

(2.135)

So, for an observer that is far away from the brane, any process that takes place near the brane, i.e., \( \frac{r_{\text{process}}}{R} \ll 1 \), is a low-energy one. Of particular interest are strings - for someone at infinity, strings near the brane are seen as low energy modes.

We used a supergravity description to obtain the effect of the branes on the 10 dimensional flat background. We know that supergravity can be obtained by studying the effective low energy effect of superstrings. So, one can positively argue that the deformations induced by the branes on the background, i.e., the supergravity effects, can be seen as a consequence of a huddle of low-energy strings that permeate the whole space.

So, the low energy, effective description of our scenario by an observer that sits far away from the brane includes two components:

- Supergravity in the bulk, i.e., in flat 10D spacetime
- anything close to the brane, including strings.

The final step of this analysis is to recover the fact that at low energies, it is possible to show that gravity decouples from any other system so, from this point of view one obtains, at low energy, two decoupled systems

\[ (\text{Superstring Theory in } AdS_5 \times S^5) \oplus (\text{SUGRA in 10D}). \]  

(2.136)

2.7.3 The statement

In the end we conclude that in the two equivalent descriptions (see figure 2.5) one has supergravity decoupled from some other theory, so, by “removing the common factor”, one can make the identification

\[ \text{one way of seeing this is to calculate the absorption cross-section of gravitons to the brane and note that the brane is very bad at absorbing them.} \]
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Figure 2.5: Schematic representation of the AdS/CFT duality. According to point of view (A), one is close to the brane and the relevant theory to the AdS/CFT correspondence is the one that governs the dynamics of the open strings in the brane. According to point of view (B), one is far away from the brane and the relevant description is the one which is encoded by supergravity generated by closed strings emitted by the brane.

\[(\text{Superstrings in } AdS_5 \times S^5) \equiv (\mathcal{N} = 4 \text{ SYM in } 4D)\] (2.137)

and this is the content of AdS/CFT duality, realized for the first time in [1].

2.7.4 Matching parameters

Taking into account what was shown in previous subsections, the parameters of both theories can be identified in the following way

\[4\pi g_s = g_{YM}^2 = \frac{\lambda}{N}, \quad R^2 = \sqrt{g_{YM}^2 N \alpha'} = \sqrt{\lambda} \alpha'\] (2.138)

remembering that \(\lambda\) is the ’t Hooft coupling, \(g_{YM}\) is the coupling of \(\mathcal{N} = 4\) SYM gauge theory living in the branes and \(\alpha' = l_s^2\).

The theories involved in the correspondence have validity regimes that can be conveniently expressed in conditions on the respective couplings. In the field theory side of the correspondence, one can trust the perturbative analysis in the Yang-Mills theory when the ’t Hooft coupling is small, so

\[\lambda \ll 1 \Rightarrow \lambda = g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^2} \ll 1.\] (2.139)

In the gravity side of the correspondence, the description becomes reliable when the radius of curvature \(R\) of the anti-de Sitter space and \(S^5\) becomes large compared to the string length
\[
\frac{R^4}{l_s^4} \gg 1 \Rightarrow \frac{R^4}{l_s^4} \sim g_\text{s} N \sim g_\text{s}^2 N = \lambda \gg 1.
\] (2.140)

From here, we see that the two theories involved in the conjecture coexist in opposite regimes: when one side is weakly coupled the other is strongly coupled and vice-versa.

### 2.8 The Gravity - Field Theory dictionary\(^{23}\)

In a little more general form than the one presented before, the AdS/CFT correspondence relates a 4d CFT to string theory on \(AdS_5 \times M\). If \(M\) is compact, the string theory is effectively five-dimensional. The \(AdS_5\) factor guarantees that the dual theory is conformal since its isometry group \(SO(2,4)\) is the same as the group of conformal transformations of a four-dimensional quantum field theory.

To define the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. We can rewrite the \(AdS_5\) metric (equation 2.125) with an euclidean signature (for later convenience)

\[
ds^2 = \frac{R^2}{z^2} \left[ dz^2 + \delta_{\mu \nu} dx^\mu dx^\nu \right].
\] (2.141)

From here, we can see that the metric has a conformal boundary at \(z \to 0\) isomorphic to Minkowski spacetime. The fields in \(AdS_5\) are the excitations of the string background: they contain the metric and other fields. In the validity limits of the correspondence, the interaction of all these fields is described by an effective action \(S_{AdS_5}(g_{\mu \nu}, A_\mu, \phi, ...).\)

Now, let’s suppose that we have a map between observables in the two theories. We can formulate a prescription to relate correlation functions in the CFT with scattering amplitudes in \(AdS_5\). In the CFT we can define the generating functional for correlation functions of a given operator \(O\) which is obtained by perturbing the lagrangian by a source term

\[
\mathcal{L}_{\text{CFT}} \rightarrow \mathcal{L}_{\text{CFT}} + h(x) O(x) \equiv \mathcal{L} + \mathcal{L}_h.
\] (2.142)

where \(h(x)\) is a source depending on 4 coordinates.

\(^{23}\)A summary of the original work of [26] and [27]
2.8.1 Example: Scalar Field

Let’s consider that the operator $\mathcal{O}$ is associated with the scalar field $\hat{h}$ is $AdS_5$, which, for simplicity, we assume to be a scalar. To work out these details, let’s assume an even more general setting in which we calculate the behaviour of a scalar field in $AdS_{d+1}$.

One starts with the action of a field $\hat{h}(z, x)$ in $AdS_{d+1}$:

$$S_{AdS} \sim \int d^{d+1}x \sqrt{G} \left[ \left( \partial \hat{h} \right)^2 + m^2 \hat{h}^2 \right]. \tag{2.143}$$

The equation of motion can be written as

$$z^{d+1} \partial_z \left( z^{1-d} \partial_z \hat{h} \right) + z^2 \partial^\mu \partial_\mu \hat{h} - m^2 R^2 \hat{h} = 0. \tag{2.144}$$

Then we perform a Fourier transform of $\hat{h}$ in the $x^\mu$ coordinates

$$\hat{h}(z, x^\mu) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x} f_k(z) \tag{2.145}$$

and the equation of motion reads

$$z^{d+1} \partial_z \left( z^{1-d} \partial_z f_k \right) - k^2 z^2 f_k - m^2 R^2 f_k = 0. \tag{2.146}$$

To solve this equation near the boundary ($z \to 0$), one puts $f_k \sim z^\beta$ for some exponent $\beta$ and keep the leading terms near $z = 0$. One finds that

$$\beta (\beta - d) - m^2 R^2 = 0 \tag{2.147}$$

with solutions

$$\beta = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - m^2 R^2} \tag{2.148}$$

So, near the boundary, the function $f_k$ behaves as

$$f_k(z) \approx A(k) z^{d-\Delta} + B(k) z^\Delta \tag{2.149}$$

where $\Delta$ is given by
\[ \Delta = \frac{d}{2} + \nu, \quad \nu = \sqrt{\frac{d^2}{4} + m^2 R^2}. \] (2.150)

Performing the inverse Fourier transform we obtain

\[ \hat{h}(z, x) \approx A(x) z^{d-\Delta} + B(x) z^{\Delta}, \quad z \to 0. \] (2.151)

\( \Delta \) is real if \( \nu \in \mathbb{R} \), which happens if the mass \( m \) satisfies the Breitenlohner-Freedman (BF) bound [28, 29]

\[ m^2 \geq -\left( \frac{d}{2R} \right)^2. \] (2.152)

This means that \( m^2 \) can be negative but it must satisfy the BF bound. Assuming that the bound is satisfied, we proceed, noting that

\[ d - \Delta \leq \Delta \Leftrightarrow \nu = 2\Delta - d \geq 0 \] (2.153)

which is satisfied if the mass is above the BF bound. Then the term \( z^{d-\Delta} \) is the dominant one in the limit \( z \to 0 \). Taking the boundary in \( z = \epsilon \) and focusing on the dominant term, one can write

\[ \hat{h}(z = \epsilon, x) \approx \epsilon^{d-\Delta} A(x). \] (2.154)

Because \( d - \Delta \leq 0 \) if \( m^2 > 0 \), the dominant term is divergent as one approaches the boundary \( z = \epsilon \to 0 \). To identify the CFT source \( h(x) \) from the boundary value of the field \( \hat{h}(z, x) \) one needs to remove the divergences of the latter. To do it, one takes out the divergent multiplicative factor from the previous equation and identify \( h(x) \) with \( A(x) \), which can be equivalently written as

\[ h(x) = \lim_{z \to 0} z^{\Delta-d} \hat{h}(z, x). \] (2.155)

To interpret the meaning of \( \Delta \), one looks at the boundary action. If \( \mathcal{O} \) is the operator dual to \( \hat{h} \), the action is given by

\[ S_{\text{boundary}} \sim \int d^d x \sqrt{G_{z=\epsilon}} \hat{h}(\epsilon, x) \mathcal{O}(\epsilon, x). \] (2.156)
where \( G_{z=\epsilon} = (\frac{L}{\epsilon})^{2d} \) is the determinant of the induced metric at the boundary \( z = \epsilon \).

Plugging \( \hat{h}(\epsilon, x) = \epsilon^{d-\Delta} h(x) \), we obtain

\[
S_{\text{boundary}} \sim L^d \int d^d x h(x) \epsilon^{-\Delta} \mathcal{O}(\epsilon, x).
\] (2.157)

To make \( S_{\text{boundary}} \) finite and independent of \( \epsilon \) as \( \epsilon \to 0 \) one has to require that

\[
\mathcal{O}(\epsilon, x) = \epsilon^{\Delta} \mathcal{O}(0, x) = \epsilon^{\Delta} \mathcal{O}(x).
\] (2.158)

But this requirement is a scale transformation in the CFT. Thus, \( \Delta \) must be interpreted as the scaling dimension of the dual operator \( \mathcal{O} \).

### 2.8.1.1 The CFT operators

Focusing on \( d = 4 \), the dimension/mass relation 2.150 reads

\[
\Delta = 2 + \sqrt{4 + m^2 R^2}.
\] (2.159)

From now on, we’ll use the results of appendix E on the mass of scalar fields on AdS\(_5\). There, we saw that there’s a tower of massive fields which originate from Kaluza-Klein reduction on the \( S^5 \) with masses given by

\[
m_l^2 = \frac{l(l+4)}{R^2}, \quad l = 0, 1, 2, \ldots
\] (2.160)

So, for the massless \( l = 0 \) case, we have \( \Delta = 4 \). Thus, the CFT dual operator should be a scalar operator with dimension 4. The only CFT candidate with these characteristics is the “glueball operator”

\[
\mathcal{O} = \text{Tr} [F_{\mu\nu} F^{\mu\nu}] .
\] (2.161)

For higher \( l \) order of the Kaluza-Klein modes, the dimensions are

\[
\Delta_l = 2 + \sqrt{4 + l(l+4)} = 4 + l
\] (2.162)
and in this case the dual operator should transform under the corresponding representation of $SO(6)$, the symmetry group of $S^5$, which in this case means that it has to transform as a symmetric tensor with $l$ indices. Such operators have the form

$$O_{i_1,\ldots,i_l} = \text{Tr} \left[ \phi_{(i_1,\ldots,i_l)} F_{\mu\nu} F^{\mu\nu} \right],$$

with $\phi_{(i_1,\ldots,i_l)}$ being the traceless symmetric product of $l$ scalar fields $\phi_i$ of $\mathcal{N} = 4$ SYM.

### 2.8.2 Higher spin fields

The results previously presented for the scalar field can be generalized to any antisymmetric tensor $A_{\mu_1\ldots\mu_p}$ with $p$ indices. In this case, the dimension of the dual operator is the largest root of the quadratic equation

$$(\Delta - p)(\Delta + p - d) = m^2 R^2.$$  \hfill (2.164)

### 2.8.3 Correlation functions

In field theory, correlators of the form

$$\langle O(x_1) \cdots O(x_n) \rangle$$  \hfill (2.165)

can be calculated from a generating functional.

The generating functional is just

$$Z_{QFT}[h] = \left\langle \exp \left[ \int \mathcal{L}_h \right] \right\rangle_{QFT}$$  \hfill (2.166)

and the correlators can be calculated using the formula (see for instance [30])

$$\left\langle \prod_i O(x_i) \right\rangle = \prod_i \frac{\delta}{\delta h(x_i)} \log Z_{QFT}[h = 0].$$  \hfill (2.167)

Considering a bulk field $\hat{h}(z,x)$ in $AdS$, let $h_0(x)$ be its boundary value

$$h_0(x) = \hat{h}(z = 0, x).$$  \hfill (2.168)
Chapter 2. Introduction

The field $h_0(x)$ is related to a source of some dual operator $O$ in the QFT. The source isn’t the value of $\hat{h}(z, x)$ at $z = 0$ but the limit

$$\lim_{z \to 0} z^{\Delta - d} \hat{h}(z, x) = h(x).$$  \hspace{1cm} (2.169)

Then the AdS/CFT recipe for the generating functional is $[26, 27]$

$$Z_{QFT}[h_0(x)] = \left\langle \exp \left[ \int h_0(x) O(x) \right] \right\rangle_{QFT} = Z_{gravity}\left[\hat{h}(z, x) \to h_0(x) = \hat{h}(z = 0, x)\right]$$  \hspace{1cm} (2.170)

where $Z_{gravity}\left[\hat{h}(z, x) \to h_0(x) = \hat{h}(z = 0, x)\right]$ is the path integral in the gravity theory over all functions which have the value $h_0(x)$ at the boundary of $AdS$\(^{24}\)

$$Z_{gravity}\left[\hat{h}(z, x) \to h_0(x) = \hat{h}(z = 0, x)\right] = \sum_{\hat{h}(z, x) \to h_0(x)} \exp\left(S_{gravity}\right).$$  \hspace{1cm} (2.171)

2.9 Small geometric detour

In quantum field theories, the possible vacua are usually labelled by the vacuum expectation values of scalar fields. These vacuum expectation values can take any value for which the potential function is a minimum. When the potential function has continuous families of global minima, the space of vacua for the quantum field theory is often called the moduli space.

In the original AdS/CFT setup, the gauge theory involved was presented at the origin of moduli space. However, it is also desirable to extend the investigations away from the origin of moduli space. Giving expectation values to certain scalar fields moves one onto the Coulomb Branch - a space of maximally supersymmetric but non-conformal vacua.

$\mathcal{N} = 4$ $SU(N)$ SYM includes the scalar fields $\Phi_i$ ($i = 1 \ldots 6$) which transform in the vector representation of the $SO(6)$ $R$ symmetry group, and the adjoint representation of $SU(N)$. The Coulomb branch corresponds to giving these fields expectation values subject to the condition $[\Phi_i, \Phi_j] = 0$. Upon diagonalizing the fields, the moduli space is parametrized by the $6N$ eigenvalues $y_i^{(a)} (a = 1 \ldots N)$

\(^{24}\)here we’ll write the action in the euclidean signature.
Tracelessness reduces the number of independent eigenvalues to \(6(N - 1)\). At generic points of this branch, the gauge symmetry is broken to \(U(1)^{N - 1}\).

One can, alternatively, parametrize the moduli space in terms of gauge invariant operators.

Type IIB supergravity solutions for systems of D3-branes admit the form

\[
ds^2 = H_{D3}^{-1/2} \left( -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + H_{D3}^{1/2} \sum_{i=1}^{6} dy_i^2
\]

for any harmonic function \(H_{D3}(\vec{y})\). A general harmonic function has the integral representation

\[
H_{D3}(\vec{y}) = 1 + R^4 \int d^6y' \frac{\sigma_{D3}(\vec{y}')}{|\vec{y} - \vec{y}'|^4}
\]

where \(R^4 = 4\pi g_s \alpha'^2 N_{D3}\) and the distribution function \(\sigma_{D3}\) is normalized as

\[
\int d^6y' \sigma_{D3}(\vec{y}') = N_{D3}
\]

so that \(N_{D3}\) is the total number of D3-branes. Solutions representing multiple D3-branes at distinct points are referred to as multi-center solutions. Such supergravity solutions are written in terms of a harmonic function which has sources at the positions of the D3-branes.

It is apparent that the space of multi-center solutions corresponds to the space of possible distribution functions one can write down. The distribution

\[
\sigma_{D3}(\vec{y}) = \sum_{a=1}^{N} \delta^{(6)} \left( \vec{y} - \vec{y}^{(a)} \right)
\]

corresponds to D3-branes located at the discrete points \(\vec{y}^{(a)}\). These points are naturally identified with the eigenvalues that result from the diagonalization of the scalar fields of \(\mathcal{N} = 4\) SYM.
So, we see that multi-center solutions are in one-to-one correspondence with points on the Coulomb branch of the gauge theory. This fact suggests that the Coulomb branch of the gauge theory is equivalent to the space of multi-center solutions in the supergravity theory.

2.10 Beyond AdS/CFT

However spectacular and very ingenious, the AdS/CFT correspondence is built upon non-realistic field theories. Since $\mathcal{N} = 4$ SYM is a $SU(N)$ gauge theory, it’s natural to think that, following the path opened by AdS/CFT correspondence, it would be possible to study QCD, an $SU(3)$ Yang Mills theory, based on the SYM. However, SYM can’t even be thought of as a toy model of QCD. Here are some reasons for that: QCD confines while SYM is not confining, QCD is not supersymmetric while SYM is maximally supersymmetric, QCD has quarks while SYM has adjoint matter, QCD has $N = 3$ while SYM has $N \to \infty$.

Even though the scenario looks quite helpless, it is possible to tweak the bulk theory, the gravitational side of the duality, to obtain a field theory that is closer to QCD [31–33]. In the following chapter, we will use one of those tweaks and calculate the meson spectra of such deformed AdS/CFT setup.
Chapter 3

Adding Flavor to AdS/CFT

In QCD quarks are fermions transforming in the fundamental representation of the $SU(3)$ gauge group. The theory present in the correspondence, $\mathcal{N} = 4$ SYM, describes supersymmetric gluons in the adjoint representation. Adding degrees of freedom in the fundamental representation to $\mathcal{N} = 4$ SYM requires a further generalization of the AdS/CFT correspondence.

3.1 Adding more branes to AdS/CFT

One possible generalization of AdS/CFT correspondence is realized by the introduction of additional branes to the $\text{AdS}_5 \times S^5$ background. The dual field theories of these setups are conformal field theories with a spacetime defect$^1$. These defect conformal field theories involve fields which are confined to a lower-dimensional subspace of the original four-dimensional spacetime.

This generalization can be realized for D3/Dp brane intersections with $p = 2, 5, 7^2$. The near-horizon geometry is again $\text{AdS}_5 \times S^5$ with the new $Dp$ branes wrapping an $\text{AdS}_{k+2} \times S^k$ submanifold, where $(k + 1) = (p + 1)/2$ is the dimension of the intersection which agrees with the dimension of the defect. The massless open string degrees of freedom of the D3/Dp intersection correspond to a $\mathcal{N} = 4$ SYM multiplet (generated by 3-3 strings) coupled to a fundamental hypermultiplet (3-p and p-3 strings) localized.

$^1$One refers to defect Conformal Field Theories when one talks about theories whose amount of supersymmetry was reduced but nevertheless preserve conformal invariance.

$^2$One is excluding $p = 1$ and $p = 9$ because the former gives rise to point sources of magnetic charge for the gauge fields and we stumble upon the monopole problem, which is something we don’t want to talk about (more details on the subject can be found in [34]) and the latter, when intersected with D3 branes, is a non-supersymmetric configuration and we’re not going to talk about it. Moreover, all the possible $p$’s are odd because we’re dealing with a Type IIB superstring theory (check subsection 2.4.2 where we say that type IIB string theory only contains odd $p$ Dp-branes.).
at the \((k + 1)\)-dimensional intersection. The p-p strings were left out of the discussion because we’ll be working in a limit in which they decouple from the other strings. The decoupling is achieved by making \(N \to \infty\) while keeping the ‘t Hooft coupling \(\lambda \equiv g_{YM}^2 N = 4\pi g_s N\) fixed. This is the usual ‘t Hooft limit for the gauge theory describing the \(N\) D3 branes. The ‘t Hooft coupling for the \(N_f\) orthogonal Dp-branes is

\[
\lambda' = 2(2\pi)^{p-2} g_s l_p^{p-3} N_f = \lambda (2\pi l_s)^{p-3} N_f / N
\]

which vanishes in the above limit if \(N_f\) is kept fixed. This implies that the \(SU(N_f)\) gauge theory on the Dp-branes (generated by p-p strings) decouples and the group \(SU(N_f)\) becomes the flavor symmetry of \(N_f\) flavors. So, for \(\lambda \ll 1\) the appropriate description of this system is given by a four-dimensional \(\mathcal{N} = 4\) \(SU(N)\) gauge theory coupled to \(N_f\) hypermultiplets at a k+1 dimensional defect.

On the other hand, for \(\lambda \gg 1\), one may replace the \(N\) D3-branes by the geometry \(AdS_5 \times S^5\), according to the usual AdS/CFT correspondence. The Dp-branes may be treated as a probe of the \(AdS_5 \times S^5\) geometry. Comparing the tension of both stacks of branes

\[
T_{D_p} = \frac{\nu}{(2\pi l_s)^{p-3}} T_{D_3} \quad (\nu = N_f / N),
\]

we see that the tension \(T_{D_p}\) (and thus the backreaction of the Dp-branes) can be neglected in the probe limit \(\nu \to 0\) keeping \(\nu / l_p^{p-3} \ll 1\).

To conclude, one can say that the AdS/CFT correspondence acts twice in the background with Dp-branes embedded. This means that closed strings on \(AdS_5\) should be dual to \(\mathcal{N} = 4\) \(SU(N)\) SYM theory on \(\mathbb{R}^{1,3}\), while open string modes on the Dp-brane should be dual to the fundamental hypermultiplet on the \(\mathbb{R}^{1,k}\) defect.

### 3.1.1 Adding D7 branes

Of all the possible brane intersections adduced above, the D3/D7 intersection is special since fundamental fields are allowed to propagate in all four spacetime dimensions. This opens up the possibility for studying flavor in supersymmetric extensions of QCD. It is possible to introduce mass for the fundamental matter by separating the D7-branes from the D3-branes. The dual description involves D7 probe branes on which the induced metric is only asymptotically \(AdS_5 \times S^3\). In this case there is a discrete spectrum of
mesons. This spectrum has been computed exactly at large 't Hooft coupling in [3], and we will be describing it in the following sections.

3.2 D3/D7 Intersection

As referred in the previous section, one is interested in representing degrees of freedom in the fundamental representation of the gauge group. In the original setup of AdS/CFT correspondence, one has degrees of freedom in the adjoint representation of the gauge group that are generated by the open strings with both ends on the $N$ D3 branes. Adding $N_f$ D7 branes, the strings between the D3 and the D7 generate degrees of freedom in the fundamental representation, making the corresponding field theory more like QCD.

3.2.1 The new degrees of freedom

Karch and Katz proposed[2] the study of the system with intersecting D3 and D7 branes (check figure 3.1). Their proposal consists of adding a stack of $N_f$ D7 branes to the original AdS/CFT setup. These D7 branes will be called ”flavor branes” and they will be positioned in the following way

\[
\begin{array}{ccccccccccc}
\text{space} & \rightarrow & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{coordinates} & \rightarrow & x^0 & x^1 & x^2 & x^3 & Y^1 & Y^2 & Y^3 & Y^4 & Y^5 & Y^6 \\
\text{D3} & & \times & \times & \times & \times & & & & & & \\
\text{D7} & & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

(3.3)

It is very important that the number of D7 branes is much smaller than the number of D3 branes if we want to safely work on the probe approximation. As we saw in section 2.5, the placement of D3 branes in the 10 dimensional flat space ”curves” the space near the branes - the space reacts to the energy possessed by the branes, rearranging itself, near the branes, into a $AdS_5 \times S^5$ space. If we don’t want to perturb this ”near horizon” geometry with the introduction of the D7 branes, we must assure that their effect on the geometry of the space is negligible compared to the effect of the D3 branes and that is achieved by imposing the condition $N_f \ll N$. 
3.2.2 Consequences of the new brane on the original setup...

3.2.2.1 ...from the Field theory point of view

The field theory corresponding to this brane setup ($N$ $D3$ branes plus $N_f$ $D7$ branes) is a $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory which, in addition to the degrees of freedom of $\mathcal{N} = 4$ SYM, contains $N_f$ hypermultiplets in the fundamental representation of the gauge group.

This field theory appears as follows: the $\mathcal{N} = 4$ SYM multiplet is generated by massless open string modes on the $D3$ branes (3–3 strings) whereas the $\mathcal{N} = 2$ hypermultiplet descends from the strings between $D3$ and $D7$ branes (3-7 strings).

Besides these two possibilities, one also obtains degrees of freedom that come from strings with both ends on the stack of $N_f$ $D7$ branes (7-7 strings). These degrees of freedom happen to be negligible from this point of view since they decouple from the 3-3 and 3-7 degrees of freedom. As we discussed, this decoupling is achieved by taking the usual large-$N$ limit while keeping the four-dimensional ’t Hooft coupling $\lambda = g^2_{YM} N = g_s N$ and $N_f$ fixed. The eight dimensional ’t Hooft coupling $\lambda'$ for the $N_f$ $D7$ branes is $\lambda' = \lambda (2\pi l_s)^4 N_f / N$ which vanishes in the low energy $\alpha' \to 0$ (i.e. $l_s \to 0$) limit. The 7-7 strings therefore do not interact with the other strings anymore, and the $SU(N_f)$ gauge group on the $D7$ branes plays the role of a global flavor group in the four-dimensional theory.

3.2.2.2 ...from the Gravity point of view

On the gravity side of the duality, the $\mathcal{N} = 4$ SYM degrees of freedom are described by supergravity on $AdS_5 \times S^5$ with added degrees of freedom corresponding to the $D7$ brane probe within the 10 dimensional curved spacetime. Strings with both ends on the flavor brane are in the adjoint representation of $SU(N_f)$ symmetry of the quarks. These states are described by fluctuations of the brane in the background geometry as we saw in subsection 2.4.3. Small oscillations of the brane are dual to the gauge theory mesons [35].

So, by putting these new branes in the original AdS/CFT setup, one obtains a new correspondence. In addition to the original AdS/CFT correspondence, gauge invariant field theory operators involving fundamental fields are mapped to the fluctuations of the $D7$ brane probe inside $AdS_5 \times S^5$. 
3.2.3 Massive degrees of freedom

The D3/D7 brane intersection has an $SO(4) \times SO(2)$ isometry in the directions transverse to the D3 branes. The $SO(4)$ rotates $Y^1, Y^2, Y^3, Y^4$ while the $SO(2)$ group acts on $Y^5, Y^6$. By separating the D3 branes from the D7 branes in the 89-direction by a distance $L$ explicitly breaks the $SO(2)$ group. One knows, from section 2.4 that if one has a string stretching from one brane to another, the string acquires mass, whose value is proportional to the distance between the branes. So, if the new “flavor branes” are separated from the D3 branes in some direction transverse to both branes, then the minimum length string between the two branes has non-zero energy and hence the fundamental fields are massive. The mass of these fundamental degrees of freedom is $m_q = L/2\pi\alpha'$. 

3.3 Mesons from field theory

As we saw in a previous section, the field theory in the worldvolume of the intersection of $N$ D3 branes with $N_f$ D7 branes is a $\mathcal{N} = 4$ $SU(N)$ SYM theory with supersymmetry broken to $\mathcal{N} = 2$ by $N_f$ hypermultiplets in the fundamental representation of $SU(N)$.

In $\mathcal{N} = 1$ superspace language the $\mathcal{N} = 2$ hypermultiplets are described by $N_f$ chiral superfields $Q^r, \tilde{Q}_r$, $r = 1, \ldots, N_f$. The Lagrangian describing the theory is...
\[ \mathcal{L} = \int d^4 \theta \left[ \text{Tr} \left( e^{-gV} \bar{\Phi} e^{gV} \Phi^i \right) + \text{tr} \left( Q e^{gV} Q + \bar{Q} e^{-gV} \bar{Q} \right) \right] + \frac{1}{2g^2} \int d^2 \theta \text{Tr} (W^a W_a) \\
+ i \int d^2 \theta \left[ g \text{Tr} \left( \Phi^1 [\Phi^2, \Phi^3] \right) + g \text{tr} \left( \bar{Q} \Phi^1 Q \right) + m \text{tr} \left( \bar{Q} Q \right) \right] + \text{h.c.} \]

(3.4)

where trace Tr is over color indices and tr is over flavor ones and \( m \) is the mass of the hypermultiplets (here, we’re considering that they all have the same mass). This action is \( \mathcal{N} = 2 \) supersymmetric with \((W, \Phi_1)\) realizing a \( \mathcal{N} = 2 \) vector multiplet and \((\Phi_2, \Phi_3)\) an adjoint matter hypermultiplet. We write \( \Phi_1 = X^4 + iX^5 \), \( \Phi_2 = X^6 + iX^7 \) and \( \Phi_3 = X^8 + iX^9 \).

This theory has a \( SU(2)_{\Phi} \times SU(2)_R \) global invariance corresponding to a symmetry that exchanges \( (\Phi_2, \Phi_3) \) and to the \( \mathcal{N} = 2 \) R-symmetry, respectively. For \( m = 0 \), there is a \( U(1) \) R-symmetry under which \((Q^r, \bar{Q}^r)\) and \((\Phi_2, \Phi_3)\) are neutral, whereas \( \Phi_1 \) has charge 2 and \( W_a \) has charge 1 [36, 37]. The theory also possesses a \( U(1) \) baryonic symmetry under which only \((Q^r, \bar{Q}^r)\) are charged \((1, -1)\).

For \( m = 0 \) and in the large \( N \) limit with \( N_f \) fixed, the theory is superconformal - the \( \beta \) function for the ’t Hooft coupling \( \lambda = g^2 N \) is proportional to \( \lambda^2 N_f / N \) [36, 38] and vanishes for \( N_f / N \to 0 \). For mass \( m \neq 0 \) conformal symmetry is explicitly broken.

### 3.3.1 Meson operators

Mesons are quark-antiquark bound states. The \( \mathcal{N} = 2 \) worldvolume field theory of the D3/D7 brane intersection contains the scalar meson operators

\[ \mathcal{M}^A_{\alpha} = \bar{\psi}_i \sigma^A_{ij} X^l \psi_j + \bar{q}^m X^l X^l q^m, \quad i, m = 1, 2 \]

(3.5)

where \( X^l_i \) denotes the vector \((X^8, X^9)\) and \( \sigma^A = (\sigma^1, \sigma^2) \) is a doublet of Pauli matrices. Both \( X^l_i \) and \( \sigma^A \) transform in the 2 of \( U(1)_R \); \( q^m \) and \( \psi_i \) are components of the fundamental hypermultiplets \( Q \) and \( \bar{Q} \); \( X^l \) denotes the symmetric, traceless operator insertion \( X^{(i_1 \cdots i_l)} \) of \( l \) adjoint scalars \( X^i \) \((i = 4, 5, 6, 7)\) which transform in the fundamental representation of \( SO(4) \approx SU(2)_\Phi \times SU(2)_R \). The conformal dimension of this operator is \( \Delta = 3 + l \). The spectrum of these operators and also of the ones corresponding to vector mesons was found in [3] and we’ll describe it in the following.
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### 3.4 Analytical Meson Spectra coming from the D3/D7 intersection

In what follows, we will be reproducing the basic equations that describe the analytical calculation of the meson spectrum\(^3\) as presented in [3]. A more detailed analysis of the calculations will be presented in the next chapter.

#### 3.4.1 The model

We start by presenting the model. From the AdS/CFT original setup, one knows that the \(N = 4\) SYM theory is dual to type IIB string theory on \(AdS_5 \times S^5\) with metric

\[
ds^2 = \frac{r^2}{R^2}ds^2 \left( E^{(1,3)} \right) + \frac{R^2}{r^2}d\vec{Y} \cdot d\vec{Y} 
\]

(3.6)

with \(r = |\vec{Y}|\) and \(R^2 = \sqrt{4\pi g_s N'\alpha'}\) where \(Y^i, i = 1, \ldots, 6\) parametrize the 456789-space (check equation 3.3).

After adding a D7-brane to the original setup and separating it from the D3-branes in the 89-plane at \(|\vec{Y}| = L\), the induced metric on the D7-brane worldvolume is

\[
ds^2 = \frac{\rho^2 + L^2}{R^2}ds^2 \left( E^{(1,3)} \right) + \frac{R^2}{\rho^2 + L^2}d\rho^2 + \frac{R^2 \rho^2}{\rho^2 + L^2}d\Omega_3^2 
\]

(3.7)

where \(\rho^2 = r^2 - L^2\) and \(\Omega_3\) are the spherical coordinates in the 4567-space (check figure 3.2).

#### 3.4.2 The spectrum

To calculate the spectrum of scalar and vector mesons, which are dual to open string excitations of the \(D7\)-brane, the DBI action is analysed

\[
S_{D7} = -T_7 \int d^8 \xi \sqrt{-\det \left( P \left[ G \right]_{ab} + 2\pi \alpha' F_{ab} \right) + \frac{(2\pi \alpha')^2}{2}T_7 \int P \left[ C^{(4)} \right] \wedge F \wedge F} 
\]

(3.8)

\(^3\)This kind of calculation can be generalized to setups in which \(Dk\) probe branes \((k = p, p+2, p+4\) and \(p < 5\)) are added to a background of \(N Dp\)-branes - see [39]
where $G_{ab}$ is the bulk metric in equation 3.6, $C^{(4)}$ is the relevant part of the Ramond-Ramond potential\footnote{D-branes are objects that possess charge and, consequently, they are sources of potentials (just like electric charge is a source of the electric potential). $C^{(4)}$ is a differential form which describes the potential sourced by the D-brane charge. Check \cite{19}.} appearing in the Wess-Zumino term, $T_7 = \left[ (2\pi)^7 g_s \alpha' \right]^{-1}$ is the $D7$-brane tension and $P$ denotes the pullback of a bulk field to the brane’s worldvolume.

### 3.4.2.1 Fluctuations of the scalar fields

To study scalar fluctuation, one can safely ignore the effect of gauge fields. So, one obtains the following equation of motion for the radial direction

\[ \partial^2 \varphi + 3 \frac{3}{\varrho} \partial \varphi + \left( \frac{\bar{M}^2}{(1 + \varrho^2)^2} - \frac{l(l+2)}{\varrho^2} \right) \varphi = 0 \quad (3.9) \]

with

\[ \varrho = \frac{\rho}{L}, \quad \bar{M}^2 = -\frac{k^2 R^4}{L^2} \quad (3.10) \]

whose solution is given by

\[ \phi(\rho) = -\frac{\rho^l}{(\rho^2 + L^2)^{n+l+1}} F \left( -\left( n + l + 1 \right), -n; l + 2; \frac{-\rho^2 / L^2}{\rho^2 + L^2} \right) \quad (3.11) \]

in which $F(a, b; c; y)$ is the standard hypergeometric function. The properties of hypergeometric functions \cite{40} can be used to extract a spectrum for the parameter $\bar{M}$. By

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{D7-brane.png}
\caption{The placement of the D7 brane.}
\end{figure}
demanding that the hypergeometric function series terminates in such a way that its highest order term is suppressed by the factor $\rho^{l-2\alpha}$, one obtains

$$\bar{M}^2 = 4(n + l + 1)(n + l + 2).$$ (3.12)

The mass values $M$ for scalar mesons can be obtained using the fact that

$$M^2 = -k^2$$ (3.13)

which implies

$$M = \frac{L}{R^2} \bar{M}. \quad (3.14)$$

**Conformal dimension of dual operators**  One can make a similar analysis to the one which was presented in subsection 2.8.1 to the free scalar field. In this case, since the lagrangian (check equation 3.4 of [3] or equation 4.4) isn’t canonically normalized, the conformal dimension is obtained in a different way. In this case, at infinity, the normalizable modes behave as $\rho^{-\Delta + p}$ and the non-normalizable ones as $\rho^{\Delta - 4 + p}$, for some value of $p$. So, after obtaining the exponents through the analysis of the equation of motion - let’s call $\alpha$ to the non-normalizable exponent and $\beta$ to the normalizable exponent - the conformal dimension can be obtained by

$$2\Delta - 4 = \alpha - \beta. \quad (3.15)$$

Applying this recipe to the scalar case, one obtains

$$\Delta = l + 3 \quad (3.16)$$

where $l$ is an arbitrary value of the “quantum angular number” coming from the scalar spherical harmonics of the $S^3$.

### 3.4.2.2 Fluctuations of the gauge fields

The equations of motion for the gauge field on the D7-branes are

$$\partial_a \left( \sqrt{-\det G} F^{ab} \right) - \frac{4\rho (\rho^2 + L^2)}{R^4} \epsilon^{ijk} \partial_j A_k = 0 \quad (3.17)$$
where $\epsilon^{ijk}$ is a tensor density and $G$ is the induced metric on the D7-brane. From now on, an index convention will be established in order to make calculations more comfortable. The indices $a, b, c, \ldots$ will run over all the D7 coordinates; Latin indices $i, j, k, \ldots$ will denote the coordinates on the $S^3$ (unit radius) and Greek letters $\mu, \nu, \ldots$ will refer to directions parallel to the D3-brane with the exception of $\rho$ which will always refer to the coordinate $\rho = \sqrt{(Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2}$.

To solve the previous equation one expands the $A_\mu$ and $A_\rho$ in scalar spherical harmonics on $S^3$ and the $A_i$ in vector spherical harmonics. Since there are three classes of vector spherical harmonics, one finds three different types of modes, each one corresponding to a different mass spectrum and conformal dimension. (In what follows, we’ll denote the 3-sphere vector spherical harmonics by $Y^l_i$ and the scalar ones as $Y_l$.)

**Type I**

\[ A_\mu = 0, \quad A_\rho = 0, \quad A_i = \phi_I^+ (\rho) e^{i k \cdot x} Y^l (S^3) \quad (3.18) \]

and the mass spectrum is

\[ \bar{M}^2_{I,+} = 4(n + l + 2)(n + l + 3), \quad n \geq 0, l \geq 1; \quad (3.19) \]

\[ \bar{M}^2_{I,-} = 4(n + l)(n + l + 1), \quad n \geq 0, l \geq 1. \quad (3.20) \]

The conformal dimensions of the respective dual operators are

\[ \Delta_{I,+} = l + 5; \quad (3.21) \]

\[ \Delta_{I,-} = l + 1; \quad (3.22) \]

**Type II**

\[ A_\mu = \zeta_\mu \phi_{II}(\rho) e^{i k \cdot x} Y^l (S^3), \quad k \cdot \zeta = 0, \quad A_\rho = 0, \quad A_i = 0 \quad (3.23) \]

and the mass spectrum is

\[ \bar{M}^2_{II} = 4(n + l + 1)(n + l + 2), \quad n \geq 0, l \geq 0. \quad (3.24) \]

The conformal dimension of the dual operator is
\[ \Delta_{II} = l + 3. \] (3.25)

**Type III**

\[ A_\mu = 0, \quad A_\rho = \phi_{III}(\rho)e^{ik \cdot x} Y^l(S^3), \quad A_i = \tilde{\phi}_{III}(\rho)e^{ik \cdot x} \nabla_i Y^l(S^3) \] (3.26)

and the mass spectrum is

\[ \bar{M}_{III}^2 = 4(n + l + 1)(n + l + 2), \quad n \geq 0, l \geq 1. \] (3.27)

The conformal dimension of the dual operator is

\[ \Delta_{III} = l + 3. \] (3.28)
Chapter 4

Meson spectra in deformed background

In this chapter we start by reproducing numerically the meson spectra obtained analytically in [3]. Then we use the same numerical techniques to calculate the meson spectra that emerges from the fluctuations of a D7-brane placed in a background generated by a non-trivial distribution of D3-branes.

4.1 Numerical solution

4.1.1 Scalar fields

Starting from the action for a D7-brane, ignoring the gauge fields, one writes

\[ S = \mu_7 \int d^8 \xi \sqrt{-\det (P^a [G]_{ab})}. \] (4.1)

To calculate the equations of motion explicitly, one perturbs the metric around a stable profile of the D7 brane. This stable profile is expressed by the following parametrization

\[ x \mapsto X(x) \]
\[ (\mathbb{M}^4, Y^1, \ldots, Y^4) \mapsto (\mathbb{M}^4, Y^1, \ldots, Y^4, 0, L) \]

which means that the brane is parametrized by the identity map to the 10 dimensional space, and sits at constant values for the two remaining coordinates. Note that \( L = 0 \)
Chapter 4. Meson spectra in deformed background

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corresponds to the scenario in which the D7-brane coincides with the D3-branes. The fluctuations are triggered in the $Y^5$ and $Y^6$ directions and are parametrized by scalar functions $\phi, \chi$ that depend on all the coordinates of the D7-brane

$$Y^5 = 0 + 2\pi\alpha'\chi, \quad Y^6 = L + 2\pi\alpha'\phi$$ (4.2)

where the fluctuations are normalized with the factor $2\pi\alpha'$. Having done this, the parametrization of the D7-brane becomes

$$x \mapsto X(x)$$

$$(M^4, Y^1, \ldots, Y^4) \mapsto (M^4, Y^1, \ldots, Y^4, 0 + 2\pi\alpha'\chi, L + 2\pi\alpha'\phi).$$

Now one calculates the pullback of the metric

$$P \left[ G \right]_{ab} = G_{ab} \left( 1 + G^{ab} \left( 2\pi\alpha' \right)^2 \frac{R^2}{r^2} \left( \partial_a\chi \partial_b\chi + \partial_a\phi \partial_b\phi \right) \right), \quad a, b = 1, \ldots 8$$ (4.3)

where $R$ is the radius of the AdS space and $r = \sum_{i=1}^{6} (Y^i)^2$. We also approximate the square root in the action such that the lagrangian can be written as

$$\mathcal{L} = \sqrt{-\det (P \left[ G \right]_{ab})} \approx \sqrt{-\det (G_{ab})} \left( 1 + \frac{1}{2} \left( 2\pi\alpha' \right)^2 \frac{R^2}{r^2} G^{cd} \left( \partial_c\chi \partial_d\chi + \partial_c\phi \partial_d\phi \right) \right).$$ (4.4)

Having obtained this, the equation of motion for each scalar fluctuation is

$$\partial_a \left( \frac{\sqrt{-\det G_{ab}G_{ab}\partial_b\Phi}}{r^2} \right) = 0$$ (4.5)

where $\Phi$ stands for any of the scalar fluctuations. Keeping this notation in mind, we proceed and after some manipulations one obtains

$$\frac{R^4}{(\rho^2 + L^2)^2} \partial^\mu \partial_\mu \Phi + \frac{1}{\rho^2} \partial_\rho \left( \rho^3 \partial_\rho \Phi \right) + \frac{1}{\rho^2} \frac{1}{\sqrt{\det G}} \left( \sqrt{\det G} \partial_\iota \Phi \right) = 0$$ (4.6)

where one used the index notation introduced in the previous chapter - $\mu$ for space-time coordinates ($\mu = 1, \ldots, 4$), $i$ for the $S^3$ coordinates and $\rho$ stands for the coordinate itself
\[(\rho^2 = (Y^1)^2 + \cdots + (Y^4)^2)\] and \(\tilde{G}\) represents the \(S^3\) metric. The previous equation can be written as

\[
\frac{R^4}{(\rho^2 + L^2)^2} \partial^\mu \partial_\mu \Phi + \frac{1}{\rho^2} \partial_\rho (\rho^2 \partial_\rho \Phi) + \frac{1}{\rho^2} \nabla^i \nabla_i \Phi = 0 \tag{4.7}
\]

where \(\nabla^i\) stands for the covariant derivative in the \(S^3\).

Considering the symmetry of the \(S^3\) one can perform a separation of variables

\[
\Phi = \varphi(\rho) \exp(ik \cdot x) Y^l(S^3) \tag{4.8}
\]

where \(Y^l(S^3)\) are the scalar spherical harmonics in the \(S^3\), which satisfy \(\nabla^i \nabla_i Y^l = -l(l + 2) Y^l\). Using this separation of variables on the equation of motion, one ends up with

\[
\partial^2_\rho \varphi + \frac{3}{\rho} \partial_\rho \varphi + \left(\frac{-k^2 R^4}{(\rho^2 + L^2)^2} - \frac{l(l + 2)}{\rho^2}\right) \varphi = 0 \tag{4.9}
\]

which, after the change of variables

\[
\rho = \frac{\varrho}{L}, \quad \bar{M}^2 = -\frac{k^2 R^4}{L^2} \tag{4.10}
\]

becomes

\[
\partial^2_{\varrho} \varphi + \frac{3}{\varrho} \partial_\varrho \varphi + \left(\frac{\bar{M}^2}{(1 + \varrho^2)^2} - \frac{l(l + 2)}{\varrho^2}\right) \varphi = 0. \tag{4.11}
\]

### 4.1.1.1 Numerical vs. Analytical results (scalar fields)

The numerical results are obtained using the shooting method. We start by studying the equation of motion near the branes, i.e., for the case \(\rho \to 0\) in order to obtain boundary conditions from where we can "shoot". Having those conditions, we’re all set to solve numerically the equation of motion. The solution depends on the mass parameter \(\bar{M}\). By studying the dependence of the solution on this parameter, we’re able to obtain the spectrum. Specifically, we solve the equation of motion for \(\rho \to \infty\) and we extract the values of \(\bar{M}\) for which the solution is normalizable. This is a general procedure that was applied for all the following cases.
We present the results of the application of this method to the scalar field equation of motion in table 4.1. We show only the first 3 cases, which agree very well with the analytical results; this gives us confidence that the numerical method we’re using is trustable.

### 4.1.2 Gauge Fields

In this case, we’ll be dealing with the full DBI action, which we’ll write as

$$S_{DBI} = S_{BI} + S_{WZ}$$  \hspace{1cm} (4.12)

in which

$$S_{BI} = -\mu_7 \int d^8 \xi \sqrt{-\det (P [G]_{ab} + 2\pi\alpha' F_{ab})}$$  \hspace{1cm} (4.13)

and

$$L_{WZ} = \frac{(2\pi\alpha')^2}{2} \mu_7 \int P \left[ C^{(4)} \right] \wedge F \wedge F.$$  \hspace{1cm} (4.14)

From $S_{BI}$ we obtain the first part of the equation of motion for the gauge fields

$$\partial_\mu \left( \sqrt{-\det G} F^{\mu a} \right) = 0$$  \hspace{1cm} (4.15)

in which the indices $\mu, a$ do not follow the geometric convention introduced before. From $S_{WZ}$ we obtain

$$-\frac{4\rho (\rho^2 + L^2)}{R^4} \varepsilon^{\gamma\sigma} \partial_\gamma A_\sigma = 0$$  \hspace{1cm} (4.16)
in which $b$ has to be an index of a $S^3$ coordinate, otherwise the Levi-Civita is identically zero. The complete equation of motion is

$$\partial_\mu \left( \sqrt{-\det G_{\mu\nu}} \right) - \frac{4\rho^2 + L^2}{R^4} \varepsilon^{\nu\gamma\sigma} \partial_\gamma A_\sigma = 0. \quad (4.17)$$

From here on, we'll be using the index convention introduced in subsection 3.4.2.2.

### 4.1.2.1 Type I

In this case we have

$$A_\mu = 0; \quad A_\rho = 0; \quad A_i = \phi^\pm (\rho) e^{ik \cdot x} Y^I_l (S^3) \quad (4.18)$$

and we see that the equations for $A_\mu$ and $A_\rho$ are identically satisfied. Having done this, we fix $b = i$ in equation 4.17 and one obtains

$$\partial_\mu \partial_\mu A_i + \frac{1}{\rho} \partial_\rho \left( \rho^2 + L^2 \right) A_i + \frac{4\rho^2 + L^2}{R^4 \rho^2} \left( \nabla_j \nabla^j A_i - R^A_i A_j \right) = 0. \quad (4.19)$$

Using the separation of variables of equation 4.18 and making the usual change of variables (check equation 4.10), we obtain

$$\frac{1}{\rho} \partial_\rho \left( \rho (1 + \varrho)^2 \partial_\rho \phi^\pm_1 \right) + \bar{M}^2 \phi^\pm_1 - (l + 1)^2 \frac{(1 + \varrho^2)^2}{g^2} \phi^\pm_1 + 4 (l + 1) (1 + \varrho^2) \phi^\pm_1 = 0. \quad (4.20)$$

### 4.1.2.2 Type II

Type II gauge fields are written as

$$A_\mu = \zeta_\mu \phi^I_l (\rho) e^{ik \cdot x} Y^l (S^3), \quad k \cdot \zeta = 0, \quad A_\rho = 0, \quad A_i = 0 \quad (4.21)$$

and the only component with non-trivial equation of motion is $\mu$, which is
\[ \left( \frac{R^2}{\rho^2 + L^2} \right)^2 \partial_\rho \partial_\nu A_\mu + \frac{1}{\rho^2} \partial_\rho \left( \rho^3 \partial_\nu A_\mu \right) + \frac{1}{\rho^2} \nabla^i \nabla_i A_\mu = 0 \quad (4.22) \]

which, after using the separation of variables and the usual change of coordinates can be written as

\[ \tilde{M}^2 \left( \frac{1}{\varrho + 1} \right)^2 \phi_{II} + \frac{1}{\varrho} \partial_\varrho \left( \varrho^3 \partial_\varrho \phi_{II} \right) - l(l + 2) \frac{1}{\varrho^2} \phi_{II} = 0 \quad (4.23) \]

### 4.1.2.3 Type III

Finally, type III gauge fields

\[ A_\mu = 0, \quad A_\rho = \phi_{III}(\rho) e^{ik \cdot \hat{Y}}(S^3), \quad A_i = \tilde{\phi}_{III}(\rho) e^{ik \cdot \nabla_i} Y_i(S^3) \quad (4.24) \]

have an equation of motion with a more complex derivation. Following [3] we know that the equation of motion for \( b = \rho \) and \( b = i \), being \( i \) any of the \( S^3 \) coordinates, are equivalent. So, for the sake of simplicity we are going to deal only with the former.

Having said that, we start with \( b = \mu \)

\[ \frac{1}{\rho} \partial_\rho \left( \rho^3 \partial_\varrho A_\rho \right) = - \frac{1}{\sqrt{- \det \hat{G}}} \partial_j \left( \sqrt{- \det \hat{G}} \hat{G}^{ij} \partial_i A_j \right) \quad (4.25) \]

which, after the separation of variables is written as

\[ \frac{1}{\rho} \partial_\rho \left( \rho^3 \phi_{III} \right) = l(l + 2) \tilde{\phi}_{III}. \quad (4.26) \]

For \( b = \rho \) we obtain the equation

\[ \partial_\mu \partial_\nu A_\rho + \left( \frac{\rho^2 + L^2}{R^2} \right) \frac{1}{\rho^2} \frac{1}{\sqrt{- \det \hat{G}}} \partial_i \left( \sqrt{- \det \hat{G}} \hat{G}^{ii} \{ \partial_i A_\rho - \partial_\rho A_i \} \right) = 0 \quad (4.27) \]

which after the application of the separation of variables reads

\[ - k^2 \phi_{III} - l(l + 2) \left( \frac{\rho^2 + L^2}{R^2} \right)^2 \frac{1}{\rho^2} \phi_{III} + l(l + 2) \left( \frac{\rho^2 + L^2}{R^2} \right)^2 \frac{1}{\rho^2} \partial_\rho \tilde{\phi}_{III} = 0. \quad (4.28) \]
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\[(l.n)\quad \tilde{M}_{l,+} = \sqrt{4(l + n + 2)(l + n + 3)} \quad \text{Numerical result}\]

\begin{align*}
(1, 0) & \quad \sqrt{48} \approx 6, 9282032... \quad 6, 9282 \\
(1, 1) & \quad \sqrt{80} \approx 8, 9442719... \quad 8, 94427 \\
(2, 0) & \quad \sqrt{80} \approx 8, 9442719... \quad 8, 94427
\end{align*}

Table 4.2: Numerical results for gauge fields type \(I^+. \) Remember that \(\tilde{M}_{l,+}^2 = 4(n + l + 2)(n + l + 3), \quad n \geq 0, l \geq 1.\)

\begin{align*}
(1, 0) & \quad \sqrt{8} \approx 2, 828427... \quad 3, 05685 \\
(1, 1) & \quad \sqrt{24} \approx 4, 898979... \quad 5, 15187 \\
(2, 0) & \quad \sqrt{24} \approx 4, 898979... \quad 4, 89918 \\
(2, 1) & \quad \sqrt{48} \approx 6, 9282032... \quad 6, 92871 \\
(3, 0) & \quad \sqrt{48} \approx 6, 9282032... \quad 6, 9282
\end{align*}

Table 4.3: Numerical results for gauge fields type \(I^-. \) Remember that \(\tilde{M}_{l,-}^2 = 4(n + l)(n + l + 1), \quad n \geq 0, l \geq 1.\)

One can get rid of \(\tilde{\phi}_{III}\) with the help of the equation of motion for the case \(b = \mu; \) doing it implies

\[\begin{align*}
-k^2 \phi_{III} - & \quad l(l + 2) \left( \frac{\rho^2 + L^2}{R^2} \right)^2 \frac{1}{\rho^2} \phi_{III} + \left( \frac{\rho^2 + L^2}{R^2} \right)^2 \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho^3 \phi_{III} \right) \right) = 0 \quad (4.29)
\end{align*}\]

and after changing the variables one obtains

\[\begin{align*}
\partial_{\rho} \left( \frac{1}{\rho} \partial_{\rho} \left( \rho^3 \phi_{III} \right) \right) - l(l + 2) \phi_{III} + \frac{\tilde{M}_{l,-}^2 \rho^2}{(1 + \rho^2)^2} \phi_{III} = 0. \quad (4.30)
\end{align*}\]

4.1.3 Numerical vs. Analytical results (gauge fields)

4.1.3.1 Type I

Type I gauge fields are separated in two types, one for each vector spherical harmonic \(\mathcal{Y}_{l_{1},+}^l\) and \(\mathcal{Y}_{l_{2},-}^l.\)

In table 4.2, we present the numerical results obtained for type \(I^+\) gauge fields.

The numerical results for type \(I^-\) are presented in table 4.3. We find a disagreement with the analytical results for \(l = 1.\) However, for \(l = 2\) the agreement is good. Some words on the former fact are presented in subsection 4.1.4.
Chapter 4. Meson spectra in deformed background

<table>
<thead>
<tr>
<th>$(l,n)$</th>
<th>$\hat{M}_{II} = \sqrt{4(l+n+1)(l+n+2)}$</th>
<th>Numerical result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,0)$</td>
<td>$\sqrt{8} \approx 2.828427...$</td>
<td>2.82853</td>
</tr>
<tr>
<td>$(0,1)$</td>
<td>$\sqrt{24} \approx 4.898979...$</td>
<td>4.89929</td>
</tr>
<tr>
<td>$(1,0)$</td>
<td>$\sqrt{24} \approx 4.898979...$</td>
<td>4.89898</td>
</tr>
</tbody>
</table>

Table 4.4: Numerical results for gauge fields type $II$. Remember that $\hat{M}_{II}^2 = 4(n + l + 1)(n + l + 2)$, $n \geq 0, l \geq 0$.

<table>
<thead>
<tr>
<th>$(l,n)$</th>
<th>$\hat{M}_{III} = \sqrt{4(l+n+1)(l+n+2)}$</th>
<th>Numerical result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,0)$</td>
<td>$\sqrt{24} \approx 4.898979...$</td>
<td>4.89898</td>
</tr>
<tr>
<td>$(1,1)$</td>
<td>$\sqrt{48} \approx 6.9282032...$</td>
<td>6.9282</td>
</tr>
<tr>
<td>$(2,0)$</td>
<td>$\sqrt{48} \approx 6.9282032...$</td>
<td>6.9282</td>
</tr>
</tbody>
</table>

Table 4.5: Numerical results for gauge fields type $III$. Remember that $\hat{M}_{III}^2 = 4(m + l + 1)(m + l + 2)$, $n \geq 0, l \geq 1$.

4.1.3.2 Type II and Type III

For type II and type III we obtained good agreement with the analytical results. Table 4.4 shows the numerical results for the case of type II gauge fields and table 4.5 are for the case of type III gauge fields.

4.1.4 Comments

Comparing our calculation with the ones from [3] we found two inconsistencies:

- The equation of motion for type III gauge fields that we obtained (check equation 4.30) doesn’t totally agree with the one presented on the paper;
- The numerical calculations of the spectra of type I- gauge fields (check table 4.3) don’t agree with the analytical results for $l = 1$.

The first point is easy to explain. The equation that we obtained gives good numerical results compared to the analytical results of the paper, which suggests that the difference mentioned above is a consequence of a typo. The second point can be explained either by some deficiency of our calculation method, which doesn’t seem plausible to assume since we obtained good results for all the other cases, or by the assumption of another typo. The latter case is, obviously, the simplest and most naive suggestion. One could go into more detail, exploring the representations of the gauge fields, trying to find a more profound justification for this second point. We’re not going into that.
4.2 Meson spectroscopy in a deformed background

Having tested the numerical methods to solve the equations of motion for the fluctuations of the D7-brane, we approach the same problem but in a different background. In what follows, we will be calculating the meson spectrum in a deformed background. This deformation corresponds to an $S^3$ distribution of D3-branes with radius $\rho = r_0$ in which the coordinates describing the position of the branes are $Y^1, Y^2, Y^3, Y^4$ such that $\rho = \sum_{i=1}^4 Y^i$. This configuration describes an $\mathcal{N} = 4$ SYM field theory with vacuum expectation values for all six scalar fields. More details on the background can be found in [41, 42].

With this setup, the metric is given by

$$ds^2 = H^{-1/2}ds^2 \left( \mathbb{E}^{(1,3)} \right) + H^{1/2}d\vec{Y} \cdot d\vec{Y} \quad (4.31)$$

with (check section 2.9)

$$H(\vec{Y}) = 1 + R^4 \int d^6Y' \frac{\sigma^{(S^3)}(\vec{Y}')}{|\vec{y} - \vec{Y}'|^4} \quad (4.32)$$

where $\sigma^{(\text{disc})}(\vec{Y})$ is the distribution function obtained after taking the large $N$ limit and $R^4 = 4\pi g_s N\alpha'$. Besides, if one takes the near-horizon limit, the harmonic function can be written as [41]

$$H = \frac{2R^4}{\sqrt{(r^2 + r_0^2)^2 - 4r_0^2\rho^2} \left( r^2 + r_0^2 + \sqrt{(r^2 + r_0^2)^2 - 4r_0^2\rho^2} \right)}. \quad (4.33)$$

where

$$r^2 = (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 + (Y^5)^2 + (Y^6)^2 \quad (4.34)$$

and

$$\rho^2 = (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 \quad (4.35)$$
4.3 Equations of Motion and Numerical results in the deformed background

To obtain the meson spectra, one has to perturb the D7-brane around a stable position and minimize the action describing those perturbations. In this case, we place the D7-brane the same way it was placed in [3]. So, choosing the same configuration as before, the D7-brane will be positioned at $Y^5 = 0$ and $Y^6 = L$.

4.3.1 Scalar field

Again, the scalar fields equations of motion are obtained from fluctuations around the equilibrium position of the D7-brane

$$Y^5 = 0 + 2\pi\alpha' \chi, \quad Y^6 = L + 2\pi\alpha' \phi. \quad (4.36)$$

Using these perturbations on the harmonic function of equation 4.33 and expanding it to second order in $\alpha'$ we obtain

$$H = \frac{2R^4}{\sqrt{(\rho^2 + L^2 + r_0^2)^2 - 4r_0^2 \rho^2}} \quad (4.37)$$

and the equations of motion for the scalar field read

$$\partial_a \left( H^{1/2} \rho^3 \sqrt{-\det \tilde{G}_{ab} G^{ab} \partial_b \Phi} \right) = 0 \quad (4.38)$$

where $\tilde{G}_{ab}$ is the determinant of the metric of $S^3$. This equation can be rewritten as

$$H \partial^\mu \partial_\mu \Phi + \frac{1}{\rho^2} \partial_\rho \left( \rho^3 \partial_\rho \Phi \right) + \frac{1}{\rho^2} \nabla^i \nabla_i \Phi = 0 \quad (4.39)$$

in which we are following the notation of the previous chapter. Putting in the separation of variables

$$\Phi = \varphi(\rho)e^{ik \cdot x} \gamma^l \left( S^3 \right) \quad (4.40)$$
we obtain
\[ \partial_{\rho}^2 \varphi + \frac{3}{\rho} \partial_\rho \varphi + \left( -H k^2 - \frac{l(l+2)}{\rho^2} \right) \varphi = 0 \] (4.41)

and with the following relabelling
\[ \varrho = \frac{\rho}{R} \] (4.42)
\[ \tilde{L} = \frac{L}{R} \] (4.43)
\[ \tilde{r}_0 = \frac{r_0}{R} \] (4.44)
\[ \mathcal{M} = -k^2 R^2 \] (4.45)

one can write
\[ \partial_{\varrho}^2 \varphi + \frac{2}{\varrho} \partial_{\varrho} \varphi + \left( \mathcal{M}^2 \tilde{H} - \frac{l(l+2)}{\varrho^2} \right) \varphi = 0. \] (4.46)

where we defined
\[ \tilde{H} = \frac{2}{\sqrt{\left( \varrho^2 + \tilde{L}^2 + \tilde{r}_0^2 \right)^2 - 4\tilde{r}_0^2 \varrho^2 \left( \varrho^2 + \tilde{L}^2 + \tilde{r}_0^2 \right) + \sqrt{\left( \varrho^2 + \tilde{L}^2 + \tilde{r}_0^2 \right)^2 - 4\tilde{r}_0^2 \varrho^2}}} \] (4.47)

Before moving on we make a small comment regarding the parameter \( \mathcal{M} \) just defined and the Myers’ mass \( \bar{M} = -\frac{k^2 R^2}{L} \). The former can be rewritten as
\[ \mathcal{M} = \frac{L}{R} \bar{M} \Leftrightarrow \mathcal{M} = \bar{L} \bar{M}. \] (4.48)

Taking into account that the harmonic function 4.33 will reduce to the harmonic function used in [3] if \( r_0 = 0 \), one can infer that the plot of \( \mathcal{M} \) against \( \tilde{L} \) in the case \( r_0 = 0 \), will be a straight line passing in the origin with the corresponding Myers’ mass as the slope.
In what follows, we study how the radius of the distribution of D3-branes affects the mass of the mesons. For the first massive state allowed for scalar mesons \((l = 0, n = 0)\), the difference between having \(r_0 = 0\) and \(r_0 \neq 0\) can be checked in figure 4.1.

### 4.3.2 Gauge Fields

Following the reasoning of subsection 4.1.2, we obtain that the equations of motion for the gauge fields are

\[
\partial_a \left( \sqrt{-\det G^{ab}} \right) - \partial_\rho [H]^{-1} \varepsilon^{b\gamma\sigma} \partial_{\gamma} A_{\sigma} = 0 \tag{4.49}
\]

which can be expanded

\[
\partial_\mu (\partial_\mu A_i) + \frac{1}{\rho} \partial_\rho (\rho H^{-1} \partial_\rho A_i) + \frac{1}{\rho^2 H} \left( \nabla_j \nabla^j A_i - R^j_i A_j \right) - \frac{1}{\rho} \partial_\rho [H]^{-1} \varepsilon_{ijk} \partial_j A_k = 0 \tag{4.50}
\]

#### 4.3.2.1 Type I

After the separation of variables for the type I gauge fields (equation 4.18), one obtains

\[
- k^2 \Phi_I^\pm + \frac{1}{\rho} \partial_\rho (\rho H^{-1} \partial_\rho \Phi_I^\pm) - \frac{1}{\rho^2 H} (l + 1)^2 \Phi_I^\pm + \frac{1}{\rho} \partial_\rho [H]^{-1} (l + 1) \Phi_I^\pm = 0. \tag{4.51}
\]

After the change of variables mentioned before, the equation of motion can be written as
Chapter 4. Meson spectra in deformed background

Figure 4.2: Behavior of $\mathcal{M}(l = 1, n = 0)$ for gauge mesons type I+ with respect to $\tilde{L}$ with finite and null $r_0$.

Figure 4.3: Behavior of $\mathcal{M}(l = 1, n = 0)$ for gauge mesons type I- with respect to $\tilde{L}$ with finite and null $r_0$.

$$\mathcal{M}^2 \Phi_{\pm}^I + \frac{1}{\varrho} \partial_{\varrho} \left( \varrho \tilde{H}^{-1} \partial_{\varrho} \Phi_{\pm}^I \right) - \frac{1}{\varrho^2 \tilde{H}} (l + 1)^2 \Phi_{\pm}^I \mp \frac{1}{\varrho} \partial_{\varrho} \left[ \tilde{H} \right]^{-1} (l + 1) \Phi_{\pm}^I = 0. \quad (4.52)$$

In figure 4.2 and 4.3 is presented the variation of some mass values with the effect of a finite radius on the the values of the mesons masses.

4.3.2.2 Type II

For type II gauge fields, equation 4.49 can be written as

$$H \partial_{\varrho} (\partial_{\varrho} A_\mu) + \frac{1}{\varrho^3} \partial_{\varrho} (\varrho^3 \partial_{\varrho} A_\mu) + \frac{1}{\varrho^2} \nabla^i \nabla_i A_\mu = 0 \quad (4.53)$$

and using the separation of variables, we obtain

$$-H k^2 \Phi_{II} + \frac{1}{\varrho^3} \partial_{\varrho} (\varrho^3 \partial_{\varrho} \Phi_{II}) - \frac{1}{\varrho^2} (l + 2) \Phi_{II} = 0. \quad (4.54)$$
Using the usual change of variables, the equation can be rewritten as

$$M^2 H \Phi_{II} + \frac{1}{\bar{g}^3} \partial_{\bar{g}} (\bar{g}^3 \partial_{\bar{g}} \Phi_{II}) - \frac{1}{\bar{g}^2} l(l+2) \Phi_{II} = 0. \quad (4.55)$$

Figure 4.4 shows the results of the calculations.

### 4.3.2.3 Type III

For type III gauge fields, similarly to the original case from [3], we obtain for $b = \mu$

$$- \partial_\rho (\rho^3 \partial_\mu A_\rho) - \frac{\rho}{\sqrt{-\det \tilde{G}}} \partial_i \left( \sqrt{-\det \tilde{G}} \tilde{G}^{ii} \partial_\mu A_i \right) = 0 \quad (4.56)$$

and for the case $b = \rho$, which will be equivalent to the case $b = i$, we obtain

$$\partial_\rho \partial_\mu A_\rho + \frac{1}{\rho^2} [H]^{-1} \frac{1}{\sqrt{-\det \tilde{G}}} \partial_i \left( \sqrt{-\det \tilde{G}} \tilde{G}^{ii} \{ \partial_i A_\rho - \partial_\rho A_i \} \right) = 0. \quad (4.57)$$

Using the separation of variables, we obtain for $b = \mu$

$$\frac{1}{\rho} \partial_\rho (\rho^3 \Phi_{III}) = l(l+2) \tilde{\Phi}_{III}. \quad (4.58)$$

and for the case $b = \rho$

$$- k^2 \rho^2 H \Phi_{III} - l(l+2) \Phi_{III} + \partial_\rho \left( l(l+2) \tilde{\Phi}_{III} \right) = 0. \quad (4.59)$$

Using the former equation one obtains the following from the latter
Figure 4.5: Behavior of $\mathcal{M}(l = 1, n = 0)$ for gauge mesons type III with respect to $\tilde{L}$ with finite and null $r_0$.

\[ \partial_{\rho} \left( \frac{1}{\rho} \partial_{\rho} (\rho^3 \Phi_{III}) \right) - l(l + 2)\Phi_{III} - k^2 \rho^2 H \Phi_{III} = 0. \] (4.60)

Changing coordinates in the usual way we obtain

\[ \partial_{\varrho} \left( \frac{1}{\varrho} \partial_{\varrho} (\varrho^3 \Phi_{III}) \right) - l(l + 2)\Phi_{III} + \mathcal{M}^2 \varrho^2 \bar{H} \Phi_{III} = 0. \] (4.61)

Figure 4.5 shows the results of the numerical calculations.

4.3.3 The limiting case $\tilde{L} = 0$

Due to numerical difficulties, we weren’t able to explore the limit $\tilde{L} \to 0$ for the case $\tilde{r}_0 \neq 0$ properly. In order to understand what happens in this limiting case we study the analytical details of the equations, following the approach employed in [3]. So, we put $\tilde{L} = 0$ in the harmonic function $\bar{H}$

\[ \bar{H} = \frac{2}{\sqrt{(\tilde{r}_0^2 - \varrho^2)^2 \left( \tilde{r}_0^2 + \varrho^2 + \sqrt{(\tilde{r}_0^2 - \varrho^2)^2} \right)}} \] (4.62)

and we solve analytically, using hypergeometric functions, the equations of motion previously presented. By using the properties of these functions, one can obtain analytical formulas for the mass of mesons when $\tilde{L} = 0$ and $\tilde{r}_0 \neq 0$. Table 4.6 summarizes the results. Note that the mass spectra presented depend on two integers $l$ and $n$, both of them having the same range of validity as in [3].

One can see that the plots in figures 4.1, 4.2, 4.4 and 4.5, for the case $\tilde{r}_0 = 1$ will go to zero in the limit $\tilde{L} \to 0$ because one is plotting the mass value $\mathcal{M}$ for $n = 0$. 
Chapter 4. *Meson spectra in deformed background*

<table>
<thead>
<tr>
<th>Field</th>
<th>Mass Spectrum when $L = 0$</th>
<th>Validity range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>$M = 2\tilde{r}_0\sqrt{n(1+l+n)}$</td>
<td>$n \geq 0, l \geq 0$</td>
</tr>
<tr>
<td>Gauge type I+</td>
<td>$M = 2\tilde{r}_0\sqrt{n(2+l+n)}$</td>
<td>$n \geq 0, l \geq 1$</td>
</tr>
<tr>
<td>Gauge type I-</td>
<td>$M = 2\tilde{r}_0\sqrt{(n+1)(1+l+n)}$</td>
<td>$n \geq 0, l \geq 1$</td>
</tr>
<tr>
<td>Gauge type II</td>
<td>$M = 2\tilde{r}_0\sqrt{n(1+l+n)}$</td>
<td>$n \geq 0, l \geq 0$</td>
</tr>
<tr>
<td>Gauge type III</td>
<td>$M = 2\tilde{r}_0\sqrt{n(1+l+n)}$</td>
<td>$n \geq 0, l \geq 1$</td>
</tr>
</tbody>
</table>

Table 4.6: Analytical mass spectra when the system has $\tilde{L} = 0$. The integers $(n, l)$ have the same range of validity as in [3].

For gauge fields type I-, one sees, again, inconsistencies between analytical and numerical results. Analytically, one finds that these fields will always have non-zero mass as long as $\tilde{r}_0 \neq 0$, however, one doesn’t verify that numerically, as can be seen in figure 4.3. In this figure, one sees that mass values behaving in a similar manner to the other fields, going to zero as $\tilde{L}$ approaches 0.

To verify the analytical results for all the other case, one can study the behavior of $M$ with $n = 1$. Due to lack of time, we only present the case for type III gauge fields in figure 4.6 and we put together the $n = 0$ and $n = 1$ in figure 4.7 for the sake of appreciation.

4.3.3.1 Comments

The mass spectra for $\tilde{L} = 0$ and $\tilde{r}_0 \neq 0$ The key difference between having the D3-branes on top of each other and distributing them in a non-trivial way is the effect that the radius of the D3-brane distribution has on the mass of the fields. Having the D3-branes on top of each other, as in [3], one ends up with a conformal field theory when the D7-brane is in the origin of coordinates, i.e., $L = 0 \Leftrightarrow \tilde{L} = 0$. This means that we end up with null masses for the fields as can be seen by the formula.
Using a non-trivial distribution of D3-branes, parametrized by the factor \( \tilde{r}_0 \), one breaks the conformal symmetry of the theory with \( L = 0 \Leftrightarrow \tilde{L} = 0 \) by giving vacuum expectation values to the scalar fields and moving away from the origin of the Coulomb branch. As previously shown, for \( \tilde{r}_0 \neq 0 \) one obtains finite values for the mass of the fields even when \( L = 0 \Leftrightarrow \tilde{L} = 0 \) (see table 4.6).

### The pathological case of gauge fields type I-

The calculations for gauge fields type I- show inconsistencies. The effect of varying \( \tilde{L} \) on the mass of the field (see figure 4.3) for the \( n = 0, l = 1 \) case is similar to the other fields, however, the analytical result for the limiting case \( \tilde{L} = 0 \) is special for type I-. Based on that result, one would expect to see a finite mass, even for \( n = 0 \) and that isn’t verified. One could argue that the case \( l = 1 \) shouldn’t be trusted based on the inconsistencies pointed out in section 4.1.4, however, even the case \( l = 2, n = 0 \) doesn’t show the behavior expected from analytical results. This is intriguing and we don’t know how to explain it. A deeper understanding of the appearance of this field in the theory might give some hints on this pathological behavior. Because we lack the time to do it, we’re not going further than this.
Chapter 5

Conclusion

The AdS/CFT correspondence is truly a remarkable tool. Not only demands for the finest uses of modern theoretical ideas, it also paved the way for a new approach on the relation between theories that include gravity and theories that do not include it. The plethora of extensions and deformations that germinated from it reveals its versatility and puts it in a very comfortable position among the putative ideas for underlying principles of the physical world.

Even though the flavored extension presented in this thesis is one of the simplest, it captures the spirit of the deformation endeavour. The paradigm of altering the field theory through the introduction of new branes is clearly exposed and the appearance of new degrees of freedom is well understood. Also, the possibility of adding new branes and changing the configuration of the D3 branes that source the background in this “brane engineering” process, allows one to explore the ways in which the details of the gravity side influence the degrees of freedom of the field theory and deepen the comprehension on the nature of the AdS/CFT correspondence.

The calculation of the meson spectra in the ”flavored” correspondence is, before anything else, a proof of concept on the way one can obtain legitimate, useful information of the field theory working on the dual gravitation theory. The fact that we calculated both scalar and gauge field fluctuations made the exercise even more interesting and stimulating. Also, and since these calculations were made for two different configurations of D3-branes, it allows one to take a closer look on the impact of the gravitational side of the correspondence on the field theory objects. Specifically, the case of a $S^3$ distribution of D3-branes makes it clear how a characteristic of the distribution, namely the radius of the distribution, influences the masses of the mesons in the field theory.
For the reasons adduced above, similar and more complex explorations of AdS/CFT will continue to be studied and hopefully that effort will have generous fruits that will provide us with more hints for this fascinating puzzle called Nature.
Appendix A

Conformal Field Theory in 2 dimensions

In this appendix we present some results on conformal field theory in 2 dimensions that are specially useful to understand the interaction of strings.

A.1 Two dimensional metric and field transformations

The two dimensional metric

\[ ds^2 = (dx^1)^2 + (dx^2)^2 \]  \hspace{1cm} (A.1)

can be rewritten in terms of the coordinates \( z = x^1 + ix^2 \) and \( \bar{z} = x^1 - ix^2 \)

\[ ds^2 = dz d\bar{z}. \]  \hspace{1cm} (A.2)

Under a general coordinate transformation, \( z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z}) \), we have that the metric transforms as\(^1\)

\[ ds^2 \mapsto \frac{df}{dz} \frac{d\bar{f}}{d\bar{z}} ds^2. \]  \hspace{1cm} (A.3)

A field \( \Phi(z, \bar{z}) \) that transforms under a general coordinate change\(^2\) as

\(^1\)because \( dz \mapsto d(f(z)) = \frac{df}{dz} dz \) and \( d\bar{z} \mapsto d(\bar{f}(\bar{z})) = \frac{d\bar{f}}{d\bar{z}} d\bar{z} \)

\(^2z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})\)
Appendix A. Conformal Field Theory in 2 dimensions

\[ \Phi(z, \bar{z}) \mapsto \left( \frac{df}{dz} \right)^h \left( \frac{d\bar{f}}{d\bar{z}} \right)^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})), \quad (A.4) \]

is called primary field. The constants \((h, \bar{h})\) are called the conformal weights of the primary field. Those fields that do not transform as previously shown are called secondary or descendant fields. Primary fields are automatically quasi-primary while descendants may or may not be quasi-primary.

Under an infinitesimal transformation \(f(z) = z + \epsilon(z), \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})\) we have that

\[ \left( \frac{\partial f}{\partial z} \right)^h = (1 + \partial_z \epsilon(z))^h = 1 + h\partial_z \epsilon(z) + \mathcal{O}(\epsilon^2), \quad (A.5) \]

\[ \left( \frac{\partial \bar{f}}{\partial \bar{z}} \right)^{\bar{h}} = (1 + \partial_{\bar{z}} \bar{\epsilon}(\bar{z}))^{\bar{h}} = 1 + \bar{h}\partial_{\bar{z}} \bar{\epsilon}(\bar{z}) + \mathcal{O}(\epsilon^2) \quad (A.6) \]

and so, under infinitesimal transformation, a primary field transforms as following

\[ \Phi(z, \bar{z}) \mapsto \Phi(z, \bar{z}) + \left[ (h\partial_z \epsilon(z) + \epsilon(z)\partial_z) + (\bar{h}\partial_{\bar{z}} \bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}}) \right] \Phi(z, \bar{z}) + \mathcal{O}(\epsilon^2, \bar{\epsilon}^2, \epsilon\bar{\epsilon}). \quad (A.7) \]

So, putting this tidier, one can write that, up to first order in \(\epsilon\) and \(\bar{\epsilon}\), we have

\[ \delta_{\epsilon, \bar{\epsilon}} \Phi(z, \bar{z}) = \left[ (h\partial_z \epsilon(z) + \epsilon(z)\partial_z) + (\bar{h}\partial_{\bar{z}} \bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z})\partial_{\bar{z}}) \right] \Phi(z, \bar{z}). \quad (A.8) \]

A.2 Radial Quantization

We continue in a flat two dimensional Euclidean surface with coordinates labeled by \(\sigma^0\) for time and \(\sigma^1\) for space. The metric is given by

\[ ds^2 = (d\sigma^0)^2 + (d\sigma^1)^2. \quad (A.9) \]

To eliminate infrared divergences, one take \(\sigma^1\) periodic, \(\sigma^1 \in [0, 2\pi]\) which makes the space topologically equivalent to an infinitely long cylinder: \(\mathbb{R} \times S^1\).

Moving on, one can define Wick rotated light-cone coordinates for the Euclidean surface
\[ \zeta = \sigma^0 + i\sigma^1 \quad \text{(A.10)} \]

\[ \bar{\zeta} = \sigma^0 - i\sigma^1 \quad \text{(A.11)} \]

and the metric reads

\[ ds^2 = d\zeta d\bar{\zeta}. \quad \text{(A.12)} \]

Going a little bit further, one can map the infinitely long cylinder to the complex plane, with coordinates denoted by \( z \), through the map

\[ \zeta \mapsto z = \exp \zeta = \exp \left( \sigma^0 + i\sigma^1 \right) \quad \text{(A.13)} \]

\[ \bar{\zeta} \mapsto \bar{z} = \exp \bar{\zeta} = \exp \left( \sigma^0 - i\sigma^1 \right) \quad \text{(A.14)} \]

which makes the infinite past stand at the point \( z = 0 \) and the infinite future at \( z = \infty \).

To the process of quantizing a theory on a manifold whose geometry is given by the complex plane it is called radial quantization.

From now on, we will consider \( z \) and \( \bar{z} \) as independent coordinates. This will allow us to use all the machinery of complex analysis and whenever we want to get back to the real world, we only need to remember that \( \bar{z} = z^* \).

### A.3 Conserved Currents and Symmetry Generators

According to Noether’s theorem, if a \( d + 1 \) dimensional quantum theory has an exact symmetry then, associated to that symmetry, there is a conserved current \( j^\mu \). For a theory which is invariant under an infinitesimal coordinate transformation \( x^\mu \mapsto x^\mu + \delta x^\mu = x^\mu + \epsilon^\mu \), the corresponding conserved current is given by

\[ j_\mu = T_{\mu\nu} \epsilon^\nu, \quad \text{(A.15)} \]

where \( T_{\mu\nu} \) is the stress-energy tensor. Furthermore, there is also a conserved charge \( Q \) defined by
\[ Q = \int_{\partial M} j^\theta d^d x \]  
\hspace{1cm} (A.16)

where \( \partial M \) is the \( d \) dimensional manifold constructed by taking a fixed time slice of the \( d + 1 \) dimensional spacetime manifold in which the theory is defined. From Stoke’s theorem, we obtain that

\[ \frac{d}{d\tau} Q = 0. \]  
\hspace{1cm} (A.17)

The conserved \( Q \) generates the symmetry, i.e., if one has a field \( A \) that is described by a theory with a specific symmetry then, under this symmetry the variation of \( A \) is given by

\[ \delta \epsilon A = \epsilon [Q, A]. \]  
\hspace{1cm} (A.18)

### A.3.1 The stress-energy tensor is traceless

The particular case of dilatations provides an excellent insight into the structure of the stress-energy tensor of a conformal field theory. For an infinitesimal dilatation, one has

\[ \epsilon^\mu = \lambda x^\mu \]  
\hspace{1cm} (A.19)

and so, the current \( j^\mu \) corresponding to this transformation is given by

\[ j^\mu = \lambda T^\mu_\nu x^\nu. \]  
\hspace{1cm} (A.20)

If we’re dealing with a conformally invariant theory, we are, in particular, dealing with a theory that is invariant under the action of dilatations. So, the current is conserved and we should have \( \partial^\mu j_\mu = 0 \). But this implies that

\[ \partial^\mu j_\mu = \partial^\mu (\lambda T^\mu_\nu x^\nu) = 0. \]  
\hspace{1cm} (A.21)

Remembering that \( \partial^\nu T^\mu_\nu = 0 \), one can write

\[ \partial^\mu (\lambda T^\mu_\nu x^\nu) = \lambda T^\mu_\nu \partial^\mu x^\nu = \lambda T^\mu_\nu \delta^\mu_\nu = \lambda T^\mu_\mu \Rightarrow \]  
\hspace{1cm} (A.22)
and we conclude that the stress-energy tensor is traceless.

A.4 Metric and Stress-Energy tensor

Before continuing, let’s pause to analyse our $\mathbb{C}^2$ plane $(z, \bar{z})$. Remembering that in light-cone coordinates the metric is

$$ds^2 = d\zeta d\bar{\zeta}$$

and that one can write $\zeta = \ln z$, $\bar{\zeta} = \ln \bar{z}$, the metric can be recast as

$$ds^2 = \frac{1}{|z|^2} dz d\bar{z}.$$  \hfill (A.25)

Now, the scaling factor of $\frac{1}{|z|^2}$ can be removed via a conformal transformation, and since we are working on the assumption that the field theory in analysis is conformal, the whole set of results will be invariant under such transformation. So, one can, without loss of generality, take the metric of the complex plane to be

$$ds^2 = dz d\bar{z}$$

and from here one can infer the components of the metric

$$g_{zz} = g_{\bar{z}\bar{z}} = 0$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}.$$

With this knowledge, one can obtain the components of the stress-energy tensor

$$T_{zz} = \frac{1}{4} (T_{00} - 2iT_{10} - T_{11}),$$  \hfill (A.29)
Appendix A. Conformal Field Theory in 2 dimensions

\[ T_{zz} = \frac{1}{4} (T_{00} + 2iT_{10} - T_{11}), \tag{A.30} \]

\[ T_{\bar{z}z} = T_{z\bar{z}} = \frac{1}{4} (T_{00} + T_{11}) = \frac{1}{4} T^\mu_{\mu}. \tag{A.31} \]

By translational invariance we have that \( \partial^\mu T_{\mu\nu} = 0 \) which implies that

\[ \partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0, \tag{A.32} \]

\[ \partial_\bar{z} T_{\bar{z}z} + \partial_z T_{zz} = 0. \tag{A.33} \]

Because the stress-energy tensor is traceless, we know that \( T_{00} + T_{11} = 0 \), which implies that \( T_{\bar{z}z} = T_{z\bar{z}} = 0 \) and from the previous set of equations one obtains

\[ \partial_{\bar{z}} T_{\bar{z}z} = 0, \tag{A.34} \]

\[ \partial_z T_{zz} = 0. \tag{A.35} \]

From these last two equations one can see that \( T_{\bar{z}z} \) is a holomorphic function that only depends on \( \bar{z} \) and \( T_{z\bar{z}} \) is an anti-holomorphic function that only depends on \( z \).

Using these facts, we will write \( T(z) \equiv T_{zz}(z) \) and \( \tilde{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}) \). We see, then, that the stress-energy tensor factorizes into holomorphic and anti-holomorphic pieces.

A.5 Symmetries and Currents

Consider the generator of a general coordinate transformation in terms of complex coordinates

\[ \delta z = \epsilon(z), \tag{A.36} \]

\[ \delta \bar{z} = \bar{\epsilon}(\bar{z}). \tag{A.37} \]
where \( \epsilon(z) \) is a holomorphic function and \( \bar{\epsilon}(\bar{z}) \) is an anti-holomorphic function. The corresponding charge for this transformation is given, in terms of complex coordinates, by

\[
Q = \frac{1}{2\pi i} \oint_C \left( dzT(z) \epsilon(z) + d\bar{z}\bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \right),
\]

where the contour \( C \) is over a circle in the complex plane whose radius corresponds with a value of \( \sigma^0 \) for the time slice of the cylinder.

Having this, the variation of a field \( \Phi(w, \bar{w}) \) with respect to the above transformation is given by the “equal time” commutator of the field with the charge \( Q \)

\[
\delta_{\epsilon} \Phi(w, \bar{w}) \equiv [Q, \Phi(w, \bar{w})]
\]

and so

\[
\delta_{\epsilon} \Phi(w, \bar{w}) = \frac{1}{2\pi i} \left[ \oint dzT(z) \epsilon(z) \Phi(w, \bar{w}) \right] + \left[ \oint d\bar{z}\bar{T}(\bar{z}) \bar{\epsilon}(\bar{z}) \Phi(w, \bar{w}) \right].
\] (A.40)

Taking into account that time "flows" radially in the complex plane, one can see that if we let \( T(z) \) act first and then \( \Phi(w, \bar{w}) \), the product is well defined only if \(|w| < |z|\). To make this precise in the expressions, one defines the radial ordering, that for two arbitrary operators \( A(z) \) and \( B(w) \) reads

\[
R[A(z)B(w)] = \begin{cases} 
A(z)B(w) & \text{if } |w| < |z| \\
B(w)A(z) & \text{if } |z| < |w| 
\end{cases}.
\] (A.41)

Using this new tool, and considering that \( \Phi(w, \bar{w}) \) is a primary operator, it is possible to obtain

\[
R[T(z)\Phi(w, \bar{w})] = \frac{\hbar}{(z - w)^2} \Phi(w, \bar{w}) + \frac{1}{z - w} \partial_w \Phi(w, \bar{w}) + \text{regular terms} \quad \text{(A.42)}
\]

\[
R[\bar{T}(\bar{z})\Phi(w, \bar{w})] = \frac{\bar{\hbar}}{(\bar{z} - \bar{w})^2} \Phi(w, \bar{w}) + \frac{1}{\bar{z} - \bar{w}} \partial_{\bar{w}} \Phi(w, \bar{w}) + \text{regular terms} \quad \text{(A.43)}
\]
where \((h, \bar{h})\) are the conformal weights of the primary field \(\Phi (w, \bar{w})\) and the "regular terms" correspond to terms that have zeroth order poles.

The weights express how operators transform under rotations and scaling transformations. Considering that, one calls the spin \(s\) to

\[
\begin{align*}
  s &= h - \bar{h} \quad (A.44)
\end{align*}
\]

and scaling dimension \(\Delta\) to

\[
\begin{align*}
  \Delta &= h + \bar{h}. \quad (A.45)
\end{align*}
\]

### A.6 Charges of the Conformal Symmetry Current

The current corresponding to conformal symmetry is given by

\[
J(z) = T(z)\varepsilon(z) \quad (A.46)
\]

where \(\varepsilon(z)\) is some holomorphic function. Similarly, we have that the anti-holomorphic part of the current \(\bar{J}(\bar{z}) = \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z})\) where \(\bar{\varepsilon}(\bar{z})\) is some anti-holomorphic function. We’ll focus on the holomorphic part.

Since \(\varepsilon(z)\) is holomorphic, one can expand it in Laurent series. For \(z\) close to zero, one expect it to take the form of \(z^{n+1}, \ n \in \mathbb{Z}\), and so we get an infinite set of currents \(J^n(z) = T(z)z^{n+1}\) corresponding to each value of \(n\). For each of these currents one can write the conserved charge, which we denote by \(L_n\), as

\[
L_n = \frac{1}{2\pi i} \oint_C dz T(z)z^{n+1} \quad (A.47)
\]

where \(C\) is a closed contour which encloses the origin \(z = 0\). One can formally invert the previous relation, using Cauchy’s theorem, to obtain a Laurent series of \(T(z)\)

\[
T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2}L_n. \quad (A.48)
\]
When one quantizes a theory, the conserved charges of some symmetry become the generators of the transformations. So, we see that the operator $L_n$ generates the conformal transformation that is described by $\delta z = z^{n+1}$.

The algebra of these commutators has the same structure of the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}$$

where $c$ is called the central charge.

### A.6.1 Representation Theory of Virasoro Algebra

We will only be interested in unitary representations, where a representation of the Virasoro algebra is called unitary if all generators $L_n$ are realized as operators acting in a Hilbert space, along with the condition that $L_n^\dagger = L_{-n}$. The maximal set of generators which commute with all of the generators of the algebra is the central charge $c$ and $L_0$. So, each state inside the representation will be labeled by the eigenvalues of $L_0$ and $c$, which we’ll denote, respectively, by $h$ and $c$. So, the states will be denoted by $|h, c\rangle$.

Combining the previous facts with the commutation relations, one obtains

$$L_0L_n|h, c\rangle = (h - n)L_n|h, c\rangle$$

from which one can conclude that $L_{n>0}$ decreases $h$ by $n$ and $L_{n<0}$ increases $h$ by $n$. So, if we start with a state $|h, c\rangle$ with the lowest value possible for $h$, one can generate an infinite number of other states by acting with the generators $L_{n<0}$. The state with the lowest possible value of $h$ is called primary state and the set of states that can be obtained from it by acting with $L_{n<0}$ is known as descendant states.

### A.7 The Vacuum State

The vacuum of the theory can be defined by the condition that it respects the maximum number of symmetries, i.e., it should be annihilated by the maximum number of conserved charges, which can be translated by the condition

$$L_n|0\rangle = 0, \quad \forall n \in \mathbb{Z}$$

Consider also that, if $h$ takes its lowest value possible, $|h, c\rangle$ will be annihilated by all $L_{n>0}$.
but this is not possible because of the central charge. Having said that, the maximal symmetry one can impose on the vacuum is

\[ L_n |0\rangle = 0, \quad \forall n > 0. \]  

(A.52)

### A.8 State-Operator Map

There is a relation between primary states and primary conformal fields. Consider a primary conformal field \( \Phi(z) \) with weight \( h \).

Now, one defines a state \(|h,c\rangle\) by

\[ |h,c\rangle = \Phi(0) |0\rangle \equiv |\Phi\rangle \]  

(A.53)

i.e., one creates this state by acting on the vacuum with the primary conformal field, just like it is done in QFT by acting on the vacuum with the modes of the fields. This state turns out to be a primary state, i.e., it satisfies

\[ L_0 |\Phi\rangle = h |\Phi\rangle \]  

(A.54)

and

\[ L_n |\Phi\rangle = 0, \quad n > 0. \]  

(A.55)

It can also be proven\([43]\) that the same correspondence exists for descendant fields and descendant states, i.e., by acting on the vacuum with a descendant field, one obtains a descendant state whose eigenvalue associated to the generator \( L_0 \) is the same as the weight of the respective field.

This nice correspondence is known as the state-operator map and the relevant definition can be written as

\[ |\Phi\rangle = \lim_{z \to 0} \Phi(z) |0\rangle \]  

(A.56)

\[ ^4\text{Obviously, we are being lazy here. } \Phi \text{ also depends on } \bar{z} \text{ (remember that we are considering that } z \text{ and } \bar{z} \text{ are independent variables) and also encodes the weight } h. \text{ However, since we will only deal with the holomorphic sector of the Virasoro algebra (the one we denote by } L_n), \text{ we decide to make explicit only the holomorphic part of the field. Have any problems with this? Deal with it, life is hard!} \]
Appendix B

String Theory

In this appendix we present general details on the quantization of the bosonic string.

B.1 Polyakov action

There is another way of writing the action of a string that is, classically, equivalent to the Nambu-Goto action. It is called Polyakov action and eliminates the square root of the pullback metric at the expense of introducing another field $g_{\alpha\beta}$

$$S_{\text{Polyakov}} = -\frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{-\det g_{\alpha\beta}} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$ (B.1)

It is of utmost importance to understand the symmetries of this action because they will play a crucial role in what’s following. The symmetries of the Polyakov action are

- Poincaré invariance

$$X^\mu \rightarrow \Lambda^\mu_\nu X^\nu + c^\mu;$$ (B.2)

- Reparametrization invariance\(^1\). One may redefine the worldsheet coordinates $\sigma^{\alpha} \rightarrow \tilde{\sigma}^{\alpha}(\sigma)$ which imply the following transformation on the fields

$$X^\mu(\sigma) \rightarrow \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma);$$ (B.3)

$$g_{\alpha\beta}(\sigma) \rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^d}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma);$$ (B.4)

\(^1\)also known as invariance under diffeomorphisms.
• Weyl invariance, under which the fields $X^\mu$ do not change

$$X^\mu(\sigma) \rightarrow X^\mu(\sigma)$$ (B.5)

but the $g_{\alpha\beta}$ transforms as

$$g_{\alpha\beta}(\sigma) \rightarrow \Omega^2(\sigma) g_{\alpha\beta}(\sigma) = \tilde{g}_{\alpha\beta}(\tilde{\sigma}),$$ (B.6)

which means that two metrics related by a Weyl transformation are to be considered as the same physical state.

As referred, the Polyakov action is obtained by introducing an extra field on the Nambu-Goto action. This step has a price: when one is trying to describe the motion of the string, its equations of motion aren’t enough - one also has to take into account constraints coming from the equation of motion of the field $g_{\alpha\beta}$.

### B.2 Equations of motion and constraints

Using the parametrization invariance of the Polyakov action allows one to make a wise choice for the metric

$$g_{\alpha\beta} = e^{2\phi} \eta_{\alpha\beta}$$ (B.7)

to which one refers to as conformal gauge. One can still use the freedom of Weyl transformations and set $\phi = 0$ which leaves the action with the following form

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X$$ (B.8)

and the equations of motion are simply

$$\partial_\alpha \partial^\alpha X^\mu = 0.$$ (B.9)

Besides the equations of motion for the string, one also has constraints that come from the fact that one has fixed the extra field $\eta_{\alpha\beta}$ on the action. The choice for the metric has implications on the freedom of the string equation of motion. These constraints are
\[ \partial_0 X \cdot \partial_1 X = 0 \quad (B.10) \]

\[ \frac{1}{2} \left( (\partial_0 X)^2 + (\partial_1 X)^2 \right) = 0. \quad (B.11) \]

**B.3 Mode expansion**

Taking a big step forward, we are going to write the most general solution for the equations of motion of the string

\[ X^\mu(\sigma, \tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \quad (B.12) \]

for arbitrary functions \( X^\mu_L \) and \( X^\mu_R \), with \( \sigma^\pm = \tau \pm \sigma \). These solutions must obey the periodicity conditions

\[ X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau). \quad (B.13) \]

The most general, periodic solution can be expanded in Fourier modes,

\[ X^\mu_L(\sigma^+) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma^+} \quad (B.14) \]

\[ X^\mu_R(\sigma^-) = \frac{1}{2} x^\mu + \frac{1}{2} \alpha' p^\mu \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-} \quad (B.15) \]

where \( x^\mu \) and \( p^\mu \) are the position and momentum of the center of mass of the string and \( \tilde{\alpha}_n^\mu, \alpha_n^\mu \) are the Fourier modes. Note that the requirement that \( X^\mu \) is real implies that

\[ \alpha_n^\mu = (\alpha_{-n}^\mu)^* \quad \tilde{\alpha}_n^\mu = (\tilde{\alpha}_{-n}^\mu)^*. \quad (B.16) \]

Taking into account the constraints and applying them to the Fourier expanded fields \( X^\mu \) we obtain that

\[ \sum_m \alpha_{n-m} \cdot \alpha_m = \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m = 0, \quad n \in \mathbb{Z} \quad (B.17) \]
where the $n = 0$ modes are defined as

$$\alpha^\mu_0 \equiv \sqrt{\alpha'} p^\mu \equiv \tilde{\alpha}^\mu_0.$$  \hfill (B.18)

Bearing in mind that one can write the rest mass of a particle as

$$p_\mu p^\mu = -M^2$$  \hfill (B.19)

it is possible to obtain the effective mass of the string in terms of the excited oscillator modes

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}$$  \hfill (B.20)

which is known as level matching condition.

### B.4 Closed String Quantization

In what follows, we will present a quantization procedure that is called Lightcone Quantization. With this approach, one immediately obtains the physical degrees of freedom by taking into account the constraints of the classical theory (equations B.10 and B.11) while one is quantizing the theory.

As before, one uses the freedom endowed by symmetries to fix the worldsheet metric to

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$  \hfill (B.21)

which still leaves us with some gauge freedom. Particularly, any coordinate transformation $\sigma \rightarrow \tilde{\sigma}(\sigma)$ which changes the metric by

$$\eta_{\alpha\beta} \rightarrow \Omega^2(\sigma)\eta_{\alpha\beta}$$  \hfill (B.22)

can be undone by a Weyl transformation. So we can choose to use lightcone coordinates

$$\sigma^\pm = \tau \pm \sigma$$  \hfill (B.23)

such that the flat metric on the worldsheet takes the form
\[ ds^2 = -d\sigma^+ d\sigma^- \] (B.24)

With this choice of coordinates, any transformation of the form

\[ \sigma^+ \to \tilde{\sigma}^+(\sigma^+), \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-) \] (B.25)

simply multiplies the flat metric by a factor.

Now, one wants to gauge fix the remaining reparametrization invariance. To do that, one adopts spacetime lightcone coordinates

\[ X^\pm = \sqrt{\frac{1}{2}} (X^0 \pm X^{D-1}) \] (B.26)

and so, the spacetime Minkowski metric reads

\[ ds^2 = -2dX^+ dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \] (B.27)

With all these redefinitions, the solution for the equation of motion of \( X^+ \) reads

\[ X^+ = X^+_L (\sigma^+) + X^+_R (\sigma^-) \] (B.28)

and one can use the freedom coming from reparametrization invariance to choose coordinates such that

\[ X^+_L = \frac{1}{2} x^+ + \frac{1}{2} \alpha' p^+ \sigma^+, \quad X^+_R = \frac{1}{2} x^+ + \frac{1}{2} \alpha' p^+ \sigma^- \] (B.29)

and so, the lightcone gauge can be compactly expressed in

\[ X^+ = x^+ + \alpha' p^+ \tau \] (B.30)

where \( x^+ \) and \( p^+ \) are constants. Now, one looks at the solution for \( X^- (\sigma^+, \sigma^-) = X^-_L (\sigma^+) + X^-_R (\sigma^-) \) that one can write in the terms of the modes

\(^2\text{as long as } p^+ \neq 0 \text{ one can always change } x^+ \text{ by a shift in } \tau.\)
\begin{equation}
X_L^-(\sigma^+) = \frac{1}{2} x^- + \frac{1}{2} \alpha^' p^- \sigma^+ + i \sqrt{\frac{\alpha^'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-i n \sigma^+} \tag{B.31}
\end{equation}

\begin{equation}
X_R^-(\sigma^-) = \frac{1}{2} x^- + \frac{1}{2} \alpha^' p^- \sigma^- + i \sqrt{\frac{\alpha^'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-i n \sigma^-} \tag{B.32}
\end{equation}

where \(x^-\) is an undetermined integration constant and \(p^-, \alpha_n^- \) and \(\tilde{\alpha}_n^-\) are fixed by constraints that come from the fact that the theory possesses Weyl invariance. By imposing these conditions on the mode expansion version of \(X^- (\sigma^+, \sigma^-)\) it is possible to reconstruct the level matching condition

\begin{equation}
M^2 = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha^'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i n \alpha^i_n = \frac{4}{\alpha^'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_n^i \tilde{\alpha}_n^i \tag{B.33}
\end{equation}

but notice the difference: the sum is over oscillators \(\alpha^i\) and \(\tilde{\alpha}^i\) with \(i = 1, \ldots, D - 2\).

One will refer to these as transverse modes.

### B.5 Quantization itself

Now, one imposes the commutation relations

\begin{equation}
[x^i, p^j] = i \delta^{ij}, \quad [x^-, p^+] = -i, \quad [x^+, p^-] = -i \tag{B.34}
\end{equation}

\begin{equation}
[\alpha^i_n, \alpha^j_{m}] = [\tilde{\alpha}^i_n, \tilde{\alpha}^j_{m}] = n \delta^{ij} \delta_{n+m,0}. \tag{B.35}
\end{equation}

One uses states that are eigenstates of \(p^\mu\), with \(\mu = 0, \ldots, D\) and the Fock space grows from the vacuum state \(|0; p\rangle\)

\begin{equation}
p^\mu |0; p\rangle = p^\mu |0; p\rangle, \quad \alpha^i_n |0; p\rangle = \tilde{\alpha}^i_n |0; p\rangle = 0 \text{ for } n > 0 \tag{B.36}
\end{equation}

by acting on it with the creation operators \(\alpha^i_{-n}\) and \(\tilde{\alpha}^i_{-n}\) with \(n > 0\) and \(i = 1, \ldots, D - 2\).

To take care of the ordering ambiguity of operators that one faces in the process of quantization, a constant\(^3\) is added to the mass relation\(^4\)

---

\(^3\)the normal-ordering constant

\(^4\)For more details, check [14]
\[ M^2 = \frac{4}{\alpha'} \left( \sum_{i=1}^{D-2} \sum_{n>0} \alpha_i^i_n \alpha^i_n - a \right) = \frac{4}{\alpha'} \left( \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_i^i_n \tilde{\alpha}^i_n - a \right). \]  (B.37)

Defining

\[ N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha_i^i_n, \quad \tilde{N} = \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_i^i_n \]  (B.38)

one ends up with

\[ M^2 = \frac{4}{\alpha'} (N - a) = \frac{4}{\alpha'} (\tilde{N} - a). \]  (B.39)

It can be proven that\(^5\)

\[ a = \frac{D - 2}{4}. \]  (B.40)

\(^5\)Check [14]
Appendix C

Supergravity Solutions

In this appendix, we give details regarding the general $p$-brane solution of supergravity.

We’ll start by considering the generic supergravity action (in the Einstein Frame) [44]

$$S_s = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \sum_n \frac{1}{n!} e^{a_n \phi} F_n^2 \right) + S_{\text{Fermi}}. \quad (C.1)$$

$S_{\text{Fermi}}$ represents the fermionic terms. $\phi$ is the dilaton and $F_n$ are n-form field strengths. The Newton constant in $D$ dimensions is related to Einstein’s constant as $16\pi G_D = 2\kappa_D^2$ and $a_n = -\frac{1}{2}(n - 5)^1$. The n-form is defined as $F_n = \frac{1}{n!} F_{\mu_1 \ldots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$.

We write $D = (p+1)+d$ where $d$ is the number of dimensions transverse to the $p$-brane.

The equations of motion for the general action presented before are

$$R^\mu_\nu = \frac{1}{2} \partial^\mu \phi \partial_\nu \phi + \frac{1}{2n!} e^{a \phi} \left( n F^{\mu \xi_2 \cdots \xi_n} F_{\nu \xi_2 \cdots \xi_n} - \frac{n-1}{D-2} \delta^\mu_\nu F_n^2 \right) \quad (C.2)$$

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} \partial^\nu \phi g^\mu \nu \right) = \frac{a}{2n!} F_n^2 \quad (C.3)$$

$$\partial_\mu \left( \sqrt{g} e^{a \phi} F^{\mu \xi_2 \cdots \xi_n} \right) = 0 \quad (C.4)$$

where it was considered the case with $F_n \neq 0$ only for one value of $n$.

One plugs in the ansatz

\[\text{[one writes } a \text{ for } a_n]\]
Appendix C. Supergravity Solutions

\[ ds^2 = g_{\mu\nu} dz^\mu dz^\nu = sB^2 dt^2 + C^2 \sum_{i=1}^p (dx^i)^2 + F^2 dr^2 + G^2 r^2 d\Omega_{d-1}^2 \]  \hspace{1cm} (C.5)

which is a diagonal metric with components being functions of the transverse distance coordinate \( r^2 = \sum_{a=1}^d (y^a)^2 \) only. It’s important to note that all coefficients \( B, C, F, G \) should tend to 1 in the limit \( r \to \infty \) so that one gets back a flat metric.

The solutions, in the Einstein frame, are \[B = f^{\frac{1}{2}} H^{-\frac{d-2}{2}}, \quad C = H^{-\frac{d-2}{2}}, \quad F = f^{-\frac{1}{2}} H^{\frac{p+1}{2}}, \quad G = H^{\frac{p+1}{2}}, \quad e^\phi H^{a\frac{D-2}{2}} \]  \hspace{1cm} (C.6)

where

\[ H = 1 + \left( \frac{h}{r} \right)^{d-2}, \quad f = 1 - \left( \frac{r_0}{r} \right)^{d-2} \]  \hspace{1cm} (C.7)

\[ \Delta = (p + 1)(d - 2) + \frac{1}{2} a^2 (D - 2) \]  \hspace{1cm} (C.8)

\[ h^{2(d-2)} + r_0^{d-2} h^{d-2} = \frac{\Delta Q^2}{2(d-2)(D-2)} \]  \hspace{1cm} (C.9)

so that the metric reads

\[ ds^2 = H^{-2\frac{d-2}{2}} \left( -f dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + H^{2\frac{p+1}{2}} \left( f^{-1} dr^2 + r^2 (d\Omega_{d-1})^2 \right) . \]  \hspace{1cm} (C.10)

The constant \( Q \) is the charge of the brane.

The solutions with \( r_0 = 0 \) are known as extremal solutions. We will only consider these because it is possible to relate them to \( Dp \)-branes\[24\] and that’s the main point.

The extremal solution corresponds to the brane being in the ground state in a quantum description \[44\]. In this case, the solution simplifies to

\[ f = 1 \]  \hspace{1cm} (C.11)

\[ H = 1 + \frac{Q}{(d-2)r^{d-2}} \]  \hspace{1cm} (C.12)
\[ ds^2 = H^{-\frac{2}{p+1}} \left( -dt^2 + \sum_{i=1}^{p} (dx^i)^2 \right) + H^{\frac{2}{p-2}} \sum_{a=1}^{d} (dy^a)^2 \] (C.13)

where
\[ \sum_{a=1}^{d} (dy^a)^2 = dr^2 + r^2 (d\Omega_{d-1})^2. \] (C.14)
Appendix D

Representations

In this appendix we want to introduce the basic concepts of Lie group representation theory. We use the Lorentz group as an example to make the concepts concrete.

D.1 Lie Groups Representations

A Lie group is a group in which each element $g$ depends in a continuous and differentiable way on a set of real parameters $\theta^a$, $a = 1, \ldots, N$. We’ll refer to a generic element of a Lie group as $g(\theta)$ and to the identity as $g(0) = e$.

A linear representation $R$ of a group is an operation that assigns to a generic, abstract element $g$ of a group a linear operator $D_R(g)$ defined on a linear space $g \mapsto D_R(g)$ (D.1)

with the properties $D_R(e) = I$ and $D_R(g_1)D_R(g_2) = D_R(g_1g_2)$ in which $I$ is the identity element of the space of linear operators. The space in which the operators act is the basis of the representation. The dimension of the representation is defined as the dimension $n$ of the base space. If one writes a generic element of the base space as $(\phi^1, \ldots, \phi^n)$, a group element $g$ induces a transformation of the vector space

$$\phi^i \rightarrow [D_R(g)]^i_j \phi^j$$  (D.2)

A representation $R$ is called reducible if the action of any $D_R(g)$ on the vectors in the subspace gives another vector of the subspace - the invariant subspace. A representation
with no invariant subspaces is irreducible. A representation is completely reducible if, for all elements \( g \), the matrices of \( D_R(g) \) can be written in block diagonal form - that means the basis of the representation can be organised in several invariant subspaces, i.e., a direct sum of irreducible representations. Two representations \( R \) and \( R' \) are equivalent if there is a matrix \( S \) independent of \( g \), such that for all \( g \) we have \( D_R(g) = S^{-1} D_{R'}(g) S \). In general, when one changes the representation the explicit form and dimension of the matrices \( D_R(g) \) change, but there is an important property of the Lie group that is independent of representation: the Lie algebra.

By the assumption of smoothness, for \( \theta^a \) infinitesimal, we have in the neighbourhood of identity that

\[
D_R(\theta) \approx 1 + i\theta_a T^a_R \quad (D.3)
\]

where \( T^a_R = -i \frac{\partial D_R}{\partial \theta_a} (\theta = 0) \). \( T^a_R \) are known as the generators of the group in the representation \( R \). It can be shown \cite[p. 102]{45} that the generic group element \( g(\theta) \) can always be represented by

\[
D_R(g(\theta)) = \exp (i\theta_a T^a_R) . \quad (D.4)
\]

From consistency considerations of the group structure one obtains

\[
\left[T^a, T^b \right] = i f^a_{\, bc} T^c \quad (D.5)
\]

which is the Lie algebra of the group. \( f^a_{\, bc} \) are known as structure constants and they are independent of the representation.

Casimir Operators are operators constructed with the generators \( T^a \) which commute with all the \( T^a \). In each irreducible representation, the Casimir operators are proportional to the identity matrix and the proportionality constant labels the representation. For instance, in the angular momentum algebra \( [J^i, J^j] = i \varepsilon^{ijk} J^k \), \( i, j = x, y, z \), the Casimir is \( J^2 \). In an irreducible representation, \( J^2 \) is equal to \( j(j+1) \) times the identity matrix.

\(^1\)The factor \( i \) in the definition is chosen so that, if in the representation \( R \) the generators are hermitian, then the matrices \( D_R(g) \) are unitary; in that case \( R \) is a unitary representation.


D.2 The Lorentz Group

The Lorentz group is the group of all linear coordinate transformations $x^\mu \mapsto \Lambda^\mu_\nu x^\nu$ which leaves invariant the quantity $\eta_{\mu\nu} x^\mu x^\nu = -t^2 + x^2 + y^2 + z^2$.

The group of transformations of a space with coordinates $(y_1, \ldots, y_m, x_1, \ldots, x_n)$ which leaves invariant the quadratic form $-(y_1^2 + \cdots + y_m^2) + (x_1^2 + \cdots + x_n^2)$ is called the orthogonal group $O(m,n)$. The Lorentz group is $O(1,3)$.

The condition that $\Lambda$ must satisfy in order to leave invariant $\eta_{\mu\nu} x^\mu x^\nu = -t^2 + x^2 + y^2 + z^2$ is

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\rho\sigma} x^\rho x^\sigma$$

which can be translated into a condition on the determinant of the matrix

$$(\det \Lambda)^2 = 1 \iff \det \Lambda = \pm 1.$$ (D.7)

Transformations with $\det \Lambda = +1$ and $\Lambda^0_0 \geq 1$ are called proper orthochronous Lorentz transformations\(^2\); they are a subgroup of $O(1,3)$ denoted by $SO(1,3)$.

Considering an infinitesimal transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$$ (D.8)

and using the fact that

$$\eta_{\rho\sigma} = \eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma$$ (D.9)

one can conclude that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$ (D.10)

which allows for 6 independent elements to characterize the group.

\(^2\)One could also choose $\det \Lambda = -1$ and $\Lambda^0_0 \leq -1$, being such transformations called improper non-orthochronous. They are discrete transformations, unlike proper orthochronous which are continuous, and they are related to time and parity transformations [46, vol. 1, p.58] [47, p. 16].
D.2.1 The Lorentz algebra

6 independent parameters $\omega_{\mu\nu}$ imply 6 generators $J^{\mu\nu}$ with $J^{\mu\nu} = -J^{\nu\mu}$. So, a generic element $\Lambda$ of the Lorentz group can be written as

$$\Lambda = \exp \left( -\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu} \right). \quad (D.11)$$

By definition, a set of objects $\phi^i$, $i = 1, \ldots, n$ transform in a representation $R$ of dimension $n$ of the Lorentz group if, under Lorentz transformation it is verified the following:

$$\phi^i \mapsto \left[ \exp \left( -\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}_R \right) \right]^{i\, j} \phi^j \quad (D.12)$$

where $J^{\mu\nu}_R$ are the Lorentz generators in the representation $R$ and take the form of $n \times n$ matrices. Under an infinitesimal transformation with infinitesimal parameters $\omega_{\mu\nu}$, the variation of $\phi^i$ is

$$\delta \phi^i = -\frac{i}{2} \omega_{\mu\nu} \left[ J^{\mu\nu}_R \right]^{i\, j} \phi^j. \quad (D.13)$$

This is important because all physical quantities can be classified according to their transformation properties under the Lorentz group:

- a scalar is invariant under Lorentz transformations;
- contravariant four-vectors transforms as $V^\mu \rightarrow \Lambda^\mu_{\mu'} V^{\nu'}$;
- a covariant four-vector transforms as $V_\mu \rightarrow \Lambda^\nu_{\nu'} V^\mu$;
- a tensor transforms as $T^{\mu\nu} \rightarrow \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} T^{\mu'\nu'}$.

The explicit form of the generators $\left[ J^{\mu\nu}_R \right]^{i\, j}$ as $n \times n$ matrices depends on the particular representation that one is considering

- **scalar**: index $i$ takes only one value - it is a one dimensional representation and $\left[ J^{\mu\nu}_R \right]^{i\, j}$ is a $1 \times 1$ matrix. By definition, on a scalar a Lorentz transformation is the identity and $\delta \phi = 0$ which implies that $J^{\mu\nu} = 0$. It’s the trivial representation;
- **4-vector**: $i, j$ indices in $\left[ J^{\mu\nu}_R \right]^{i\, j}$ are Lorentz indices, so $\left[ J^{\mu\nu}_R \right]^{i\, j}_\sigma$ is a $4 \times 4$ matrix\(^3\)

\(^3\)This can be seen by observing that under an infinitesimal Lorentz transformation, the 4-vector variation is $\delta V^\mu = \omega^\mu_{\nu'} V^{\nu'}$ which can be rewritten as $\delta V^\rho = -\frac{i}{2} \omega_{\mu\nu} \left[ J^{\mu\nu}_R \right]^{i\, j}_\sigma V^\sigma$. 

\(\omega_{\mu\nu} \) is the \(2\times2\) matrix.
Appendix D. Representations

\[ [J^\mu\nu]_\rho = i (\eta^{\mu\rho} \delta^\nu_\sigma - \eta^{\mu\sigma} \delta^\nu_\rho) \]  
(D.14)

such that

\[ \delta V^\rho = -i/2 \omega_{\mu\nu} [J^\mu\nu]_\rho V^\sigma. \]  
(D.15)

This representation is irreducible because a generic Lorentz transformation mixes all indices so there are no invariant subspaces. Using the previous expression, one can obtain the Lie algebra of $SO(1, 3)$

\[ [J^\mu\nu, J^\rho\sigma] = i (\eta^{\nu\rho} J^\mu\sigma - \eta^{\mu\rho} J^\nu\sigma - \eta^{\mu\sigma} J^\nu\rho + \eta^{\nu\sigma} J^\mu\rho). \]  
(D.16)

### D.2.1.1 Tensor representation

The logical next step is to treat tensor representations. The new aspect in this case is the fact that tensor representations are reducible, i.e., one can always slice the tensor into parts that transform independently of each other.

If the tensor is symmetric it will still be symmetric after Lorentz transformation (the same for an anti-symmetric tensor). A tensor with two indices has 16 components: six of them (from the top triangle) correspond to the antisymmetric part $A^{\mu\nu} = 1/2 (T^{\mu\nu} - T^{\nu\mu})$ and ten of them to the symmetric part $S^{\mu\nu} = 1/2 (T^{\mu\nu} + T^{\nu\mu})$. Besides, the trace of a symmetric tensor is invariant under Lorentz transformations$^4$. So we can separate the symmetric tensors into two classes: the ones without trace and the ones with trace $S$.

Because the trace is invariant, each value of the trace identifies its tensor, so there is a one dimensional scalar representation for the set of tensors with trace.

### D.2.1.2 The fundamental representation

An irreducible representation is denoted by its dimensionality, written in boldface:

- scalar: 1
- 4-vector: 4

$^4$Let’s identify the trace of the symmetric part of the tensor as $S = \eta_{\mu\nu} S^{\mu\nu}$. To see that it is invariant, let’s calculate the trace $S'$ of the symmetric part of the transformed tensor $S'^{\mu\nu}$

\[ S' = \eta_{\mu\nu} S'^{\mu\nu} = \eta_{\mu\nu} A_\mu^\mu A_\nu^\nu S^{\mu\nu} = \eta_{\mu\sigma} S^{\mu\sigma} = S. \]  
(D.17)
Appendix D. Representations

- antisymmetric tensor: 6
- traceless symmetric tensor: 9

The tensor representation is a tensor product of two 4-vector representations\(^5\), which means that each of the two indices of \(T^{\mu\nu}\) transform independently as a 4-vector index - the 4-vector representation is called the fundamental representation of \(SO(1,3)\). Above, we concluded that the rank two tensor representation can be decomposed in a direct sum of the 1, 6 and 9. Summing all this information up, one writes:

\[
4 \otimes 4 = 1 \oplus 6 \oplus 9. \tag{D.18}
\]

D.2.1.3 The adjoint representation

As well as the fundamental representation, this representation also plays an important role. The adjoint representation has the same dimension as the number of generators. We can use the same type of indices \(a, b, c\) for labelling the generator and its matrix elements

\[
[T^a_{ADJ}]_c^b = -if_{c}^{ab}
\]

where \(f_{c}^{ab}\) are the structure constants. This way, the Lie algebra condition is satisfied

\[
\left[ T^a_{ADJ}, T^b_{ADJ} \right] = if_{c}^{ab}T^c_{ADJ}. \tag{D.20}
\]

For the Lorentz group, the adjoint representation has dimension six, so it is given by the antisymmetric tensor \(A^{\mu\nu}\).

\(^5\)The most general irreducible tensor representation of the Lorentz group is found starting from a generic tensor with an arbitrary number of indices, removing all traces and then symmetrizing or antisymmetrizing over all pairs of indices.
Appendix E

Kaluza-Klein reduction in $S^5$ for a scalar field

In this appendix we present the basic ideas behind the process of Kaluza Klein reduction using the simple example of a scalar field in a gravity theory on $AdS_5 \times S^5$.

Any gravity field $\phi(x, \Omega)$ on $AdS_5 \times S^5$ can be reduced to a tower of fields on $AdS_5 \times S^5$ by expanding it in terms of the harmonics on $S^5$

$$\phi(x, \Omega) = \sum_l \phi_l(x) \mathcal{Y}_l(\Omega) \quad (E.1)$$

with $x$ representing the coordinates of $AdS_5$, $\Omega$ the coordinates of $S^5$ and $\mathcal{Y}_l(\Omega)$ spherical harmonics on $S^5$.

To make it concrete, let’s take the example of a massless scalar field in $AdS_5 \times S^5$ whose equation of motion is the Klein-Gordon in ten dimensions

$$\nabla^2 \phi = 0. \quad (E.2)$$

Using the fact that the metric factorizes into a $AdS_5$ and a $S^5$ part, one can write

$$\nabla^2 = \nabla_{AdS_5}^2 + \nabla_{S^5}^2 \quad (E.3)$$

where the “squared angular momentum” operator in $SO(6)$, $\nabla_{S^5}^2$, acts on the spherical harmonics as
Appendix E. Kaluza-Klein reduction in $S^5$ for a scalar field

\[ \nabla^2_{S^5} \mathcal{Y}_l (\Omega) = - \frac{l(l+4)}{R^2} \mathcal{Y}_l (\Omega), \quad l = 0, 1, 2, \ldots \tag{E.4} \]

where $R$ is the radius of the $S^5$.

So, the reduced $AdS_5$ fields $\phi_l$ satisfy a massive Klein-Gordon equation

\[ \nabla^2_{AdS^5} \phi_l = m_l^2 \phi_l, \quad m_l^2 = \frac{l(l+4)}{R^2}. \tag{E.5} \]

One ends up with a tower of massive fields $\phi_l$, with a particular set of masses, which originate from a single massless field after Kaluza-Klein reduction on the $S^5$. 
Bibliography


