Mohammad Choubdar Soltan Ahmadi

# Mathematical Economics
# Sunspot equilibrium and
# International trade with tariffs

U. PORTO

FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Julho de 2014

Mohammad Choubdar Soltan Ahmadi

# Economia Matemática
# Equilíbrios Sunspot e
# Comércio Internacional com tarifas

*Tese submetida à Faculdade de Ciências da Universidade do Porto*
*para obtenção do grau de Doutor em Matemática*

Julho de 2014

U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO
DEPARTAMENTO DE MATEMÁTICA

# Acknowledgments

Firstly, I would like to thank my supervisor Professor Alberto Adrego Pinto for all the guidance and support he has provided me during my studies at the University of Porto.

I would also like to thank Prof. Diogo Pinheiro for all the collaboration and co-authorship during my PhD period.

I thank the financial support of LIAAD–INESC TEC through program PEst, Faculty of Sciences, University of Porto and Fundação para a Ciência e a Tecnologia (FCT). My research was supported by FCT - Fundação para a Ciência e Tecnologia grant with reference SFRH / BD / 51173 / 2010.

Last, but no means least, I would like to thank my beloved wife, Marjan, for her love and encouragement.

# Resumo

A presente tese está dividida em duas partes. Na primeira parte são estudadas falhas de coordenação de expectativas e a sua relação com a resiliência de equilíbrios do tipo sunspot sob perturbações aleatórias. A segunda parte da tese estuda os efeitos da imposição de tarifas no comércio internacional sob o ponto de vista da teoria dos jogos.

Desvios de trajetórias associadas a um equilíbrio de previsão perfeito podem ser interpretados como falhas de coordenação de expectativas por parte de agentes económicos. Nesta tese consideramos os dois casos limite em que tais desvios são, respectivamente, de pequena e grande amplitude. No que diz respeito ao limite no qual os desvios são de pequena amplitude, mostramos que estes geram um processo estocástico estacionário que está próximo ou da medida de probabilidade estacionária de um equilíbrio sunspot ou da medida de probabilidade associada a uma órbita periódica da dinâmica determinística subjacente. No que diz respeito a desvios de grande amplitude, obtemos condições sob as quais tais sistemas dinâmicos aleatórios admitem uma medida invariante absolutamente contínua e ergódica e obtemos um minorante positivo para o expoente de Lyapunov. Tais resultados são ilustrados no contexto de um modelo de equilíbrio geral com sobreposição de gerações.

Na segunda parte desta tese, estudamos um modelo de comércio internacional com incerteza nos custos de produção das firmas como um jogo estratégico nas tarifas dos governos. Para tal, consideramos duas empresas situadas em dois países e que vendem o mesmo bem homogéneo em ambos os países. Os governos de cada país decidem aplicar, ou não, tarifas sobre as importações dos bens produzidos no outro país. Calculamos o equilíbrio de Nash, o equilíbrio de Bayesian-Nash, e os correspondentes equilíbrios sociais ótimos para as funções de utilidade que são determinadas pelas quantidades económicas relevantes. Observamos que os equilíbrios de Nash não coincidem com os equilíbrios sociais ótimos o que gera as dificuldades mais pertinentes no comércio internacional, mas a nossa análise permite que estas dificuldades possam ser parcialmente resolvidas com recurso a acordos comerciais. Mostramos para o equilíbrio de Bayesian-Nash que o lucro esperado das empresas e que o bem-estar esperado dos países aumentam com a incerteza (variâncias) dos custos de produção de ambas as empresas.

# Abstract

This thesis consists of two parts. Part one studies the expectations coordination failures and sunspot equilibrium under effects of random perturbations. Part two studies the effects of tariffs in international trade from game theory point of view.

Deviations from a perfect foresight equilibrium path can be seen as coordination failures of expectations. In this thesis we consider the two limiting cases where such deviations are, respectively, small and large. For the limit case of small deviations, we show that these generate a stationary stochastic process which is close either to the stationary probability of a chaotic sunspot equilibrium or to a deterministic cycle. For the case of large deviations, we provide conditions under which an ergodic absolutely continuous invariant measure exists for such random dynamical systems and we obtain a positive lower bound for the corresponding Lyapunov exponent. We illustrate these results using an overlapping generations model.

In the second part, we study the international trade model with uncertainty on the production costs of the firms as a strategic game in the tariffs of the governments. We consider two firms located in different countries selling the same homogeneous good in both countries. Each government decides to impose or not a strategic tariff in the imports. We use the relevant economic quantities as the utilities of these strategic games and we compute the social optimum, the Nash equilibrium and the Bayesian-Nash equilibrium. We observe that the Nash equilibrium does not coincide with social optimum which is a main difficulty in international trade, but our analysis allow that these difficulties can be partially dealt with the use of trade agreements. For the Bayesian-Nash equilibrium, we show that the expected profit of the firms and the expected welfare of the countries increase with the variances of the production costs of both firms.

# Contents

# Chapter 1

# Introduction

In this thesis, we study two distinct models in Mathematical Economics. In chapter 2, we study expectations coordination failures and random perturbations of sunspot equilibrium in a macro economics model. In chapter 3, we study the effects of tariffs in a well-known international trade from a game theory point of view. Hence, for clarity of exposition, we separate the introduction in two parts.

## 1.1 Expectations Coordination Failures and Sunspot Equilibrium

Expectations coordination failures (ECF) is a concern in the recent literature of dynamics models with heterogeneous agents ([28], [50], [15], [29] and [4]). The heterogeneity in beliefs is a source of deviations from the traditional hypothesis of Rational Expectations (RE) and it is a challenge to find how far the actual dynamics under ECF is from the classical RE equilibrium.

The concept of sunspot equilibrium (SE) was firstly introduced to model the psychology of the market and the expectations of completely rational agents. In particular, Cass and Shell [13] defined the concept of sunspot equilibrium in the context of general equilibrium to study the influence of the agents expectations on market outcomes. More precisely, the notion of sunspot equilibria models the case where the market outcomes do not depend on the economy fundamentals, but rather on some external variable, representing the agents' beliefs about the future states of the world.

More recently, Lucas and Stokey [35] have argued that sunspots and contagion effects are sources of liquidity crises. They brought the argument of Cass and Shell [13] on expectation coordination to explain bank runs and consequently, the financial crises of 2008.

Sunspot equilibrium has been an active research area during the past decades and there exists an extensive literature in the subject nowadays. A good review for sunspot equilibria in economics models is the one by Chiappori and Guesnerie in [14]. In what concerns potential applications of the concept of sunspot equilibrium, this has been used to model subjects as distinct as bank-runs [40], lotteries [47, 48] and behavioral economics [19].

In sequential markets, given the present value of the state variable, a SE is a "zero" of the excess demand function realized as a transition function defining the conditional probability of reaching some given set of future state variable values. A SE is stationary if there exists a probability measure which is preserved under convolution with the transition function. From the economical point of view, this corresponds to saying that agents can move up and down in the earnings and wealth distribution while keeping the initial corresponding distribution invariant over time.

As we can see, the existence of SE requires that not only agents coordinate expectations, but also that they have to do it by using an extrinsic event. In spite of this seeming to be an extremely strong hypothesis, our result asserts that allowing for a lack of expectation coordination, the final stochastic dynamics are almost the same.

It is worth noting that despite the fact that there exist some learning rules that allow the convergence to the SE ([51] and [1]), they assume the knowledge of the agents of some particular features of the economy.

In section 2.1 of this thesis we will study how small deviations from a perfect foresight equilibrium representing possible expectations coordination failures lead the state variable to a stationary process. Using the results of dynamics of unimodal maps presented below, we prove that depending on the parameter values of the model the stationary process is close to one of two types of equilibria, namely, close to a deterministic cycle or close to the invariant measure of the global chaotic sunspot equilibrium (SE) presented in [1] (see [2]). That sort of SE arises in one-period looking forward economies where the backward perfect foresight map exhibits a chaotic behaviour and possesses a stationary probability distribution that is absolutely continuous with respect to the Lebesgue measure. Thus, small failures in coordination will produce stationary processes which are similar to either a chaotic SE or a deterministic cycle.

On the other hand, on Section 2.2 we consider the case of large random deviations from sunspot equilibria associated with unimodal backward perfect foresight maps. To proceed with our analysis, we start by considering how do “large” random perturbations change the asymptotic behaviour of a rather general class of unimodal maps. We provide appropriate conditions under which the random dynamical systems associated with the perturbed dynamics have an ergodic and absolutely continuous invariant measure with a positive lower bound for the corresponding Lyapunov exponent. Using such general results as a guideline, we move on to perform a numerical study of a family of random dynamical systems defined by random perturbations of backward perfect foresight maps known to yield sunspot equilibria for large subsets of parameter space. Contrary to what is done in Section 2.1, we do not consider the zero noise limit, but rather the case of non vanishing noise. We find evidence in support of the claim that random perturbations of such backward perfect foresight maps admit an absolutely continuous invariant measure with a positive Lyapunov exponent with smaller requirements in what concerns the size of the random perturbations when compared with the conditions required for the proof of our abstract results (see [3]).

The dynamics of unimodal maps play a central role in our analysis and it is a central topic in the modern dynamical system theory (see e.g. [16] for an overview). Namely, there has been great interest in the study of properties such as the existence and uniqueness of absolutely continuous invariant probability measures for this class of dynamical system (see [10, 31, 38]). A more recent topic of research concerns the stochastic stability of dynamical systems defined by unimodal maps under perturbations given by sequences of independent and identically distributed random variables. Results in this direction have been obtained by Benedicks and Young in [9] for the quadratic map family and with respect to the convergence induced by the weak$^*$-topology. Later on, Baladi and Viana proved in [7] the strong stochastic stability, i.e. with respect to the norm topology, for a wider class of unimodal maps. On the other hand, Lian and Stenlund [34] provide conditions under which a sufficiently large additive perturbation of multimodal maps of the circle yields a random dynamical system with an ergodic and absolutely continuous invariant with a positive lower bound for the corresponding Lyapunov exponent for almost every point of the circle. In section 2.2, we extended the results of Lian and Stenlund [34] to unimodal maps of the interval (see [3]).

## 1.2 International trade and game theory

In chapter 3 we will study the effects of tariffs in international trade from game theory point of view. There is a vast literature in international trade models with complete and incomplete information (see, for instance, [11, 12, 17, 18, 25, 33, 36, 20, 24, 23, 21, 22, 41, 46, 45]). Here, we consider a usual duopoly international trade model with complete and incomplete information, where there are two countries and a firm in each country that sells in its own country and exports to the other one (see [26]).

The international trade model has two stages: in the first stage, the governments simultaneously choose their tariff rates; and in the second stage, the firms observe the tariff rates and simultaneously choose their quantities for home consumption and for export.

The decision of the governments to impose or not tariffs can be interpreted as the actions of a game specified by the utilities considered for each country. The utilities (games) of the countries that we analysis are the relevant economic quantities of the international trade model for the consumers and firms. In particular, we consider the utilities given by the home quantities, the export quantities, the profit of the firms, the consumer surplus, the custom revenue and the welfare of the countries.

In section 3.1, we show for each one of the above utilities that there is a Nash equilibrium and a social optimum equilibrium (see [42] and [43]). Then, for each utility we classify the game in one of the following three typical classes depending of the values obtained for the Nash equilibrium and for the social optimum value: the social equilibrium (SE), where the social optimum coincides with the Nash equilibrium; the prisoner's dilemma (PD), where both utilities are bigger in the social optimum than in the Nash equilibrium; and the lose-win social strategies (LW), where one of the utilities is bigger in the social optimum and the other utility is bigger in the Nash equilibrium.

If the game is of prisoner's dilemma (PD) type, both governments can make a trade agreement such that they can improve their utilities by choosing the social tariffs. If the game is of the lose-win social strategies (LW) type, both governments can make a trade agreement such that both countries opt by the social tariffs but the wining country should compensate the loss of the other country. If the game is of the social equilibrium (SE) type, then there is a priori no need of a trade agreement.

In section 3.2, we introduce uncertainty effects on the production costs. We consider, as usual in the incomplete information literature, that each firm knows its own production cost but does not know the other firm's production cost.

We show that the expected profit of each firm increases with the variance of its production costs (see [44]). We prove that the expected welfare of each government increases with the variances of the production costs of both firms. Furthermore, the effect of the variance of the production costs of the foreign firm is lower than the effect of the variance of the production costs of the home firm in the increase of the welfare of the home government.

As before, the decision of the governments to impose or not a tariff can be interpreted as a game where the utilities are the expected welfares of the governments. In a fixed range of the exogenous parameters, we show that this game has similarities with the Prisoner's Dilemma game in the following sense (see [26]): The Nash equilibrium of the game consists of both governments to impose the tariffs obtained as the Bayesian-Nash equilibrium of the trade international model. However, the welfares of the governments are higher in the case where both governments do not impose tariffs than in the case where both governments choose to impose the Bayesian-Nash tariffs. Furthermore, if both governments do not impose tariffs then each government has an incentive to unilateral deviate from the zero tariff.

Further study is needed, for example, in the cases where the countries have different political, economical or social attitudes or different market sizes.

# Chapter 2

# Expectations Coordination Failures and Sunspot Equilibrium

Deviations from a perfect foresight equilibrium path can be seen as coordination failures of expectations. In this chapter we will consider the two limiting cases where such deviations are, respectively, small and large.

For the limit case of small deviations, we show that such these generate a stationary stochastic process which is close either to the stationary probability of a chaotic sunspot equilibrium or to a deterministic cycle. We illustrate the results using an overlapping generations model with parameter values satisfying the sufficient conditions of the theorems.

We then consider the opposite regime. More precisely, we study the influence that large random deviations have on a perfect foresight equilibrium path associated with an unimodal map. To obtain such description, we consider a large class of random dynamical systems obtained by additive random perturbations of unimodal maps of the interval by sequences of independent and identically distributed random variables. We provide conditions under which an ergodic absolutely continuous invariant measure exists for such random dynamical systems. Furthermore, under appropriate conditions we obtain a positive lower bound for the corresponding Lyapunov exponent for almost every point of the interval.

## 2.1 Stochastic Stability of Sunspot Equilibrium

In subsection 2.1.1, we briefly review the mathematical economics concepts and results that we use throughout this chapter. In subsection 2.1.2, we revisit the concept of stationary sunspot equilibria and the results of [1] on the existence of chaotic sunspots equilibria. Furthermore, we define the dynamics that small ECF generate in a one-step forward looking economic model and prove the main theorem relating that dynamics with the stationary measure of the chaotic sunspot equilibrium or with a cycle. In subsection 2.1.3 we illustrate our main result using an overlapping generations (OLG) model where the dynamics of ECF is close to either the stationary measure of the SE or the deterministic cycle depending on the relative risk aversion of the agents. In the final subsection 2.1.5 we present some well-known results on unimodal maps and small random perturbations of those systems that we used in the previous section.

### 2.1.1 Chaotic Sunspot Equilibrium and Expectations Coordination Failures

In this section we briefly review concepts and results that will be of use throughout this chapter.

#### Sunspot Equilibria

Let $X \subseteq \mathbb{R}^n$ be the state variable set, $\mathcal{B}(X)$ denote the Borel subsets of $X$ and $\mathcal{P}(X)$ be the set of probability measures on $\mathcal{B}(X)$. We assume that the equilibria of the economy are given by the zeroes of the function:

$$\tilde{Z} : X \times \mathcal{P}(X) \to \mathbb{R}^n \ ,$$

called *stochastic excess demand function.*

This nomenclature is due to the fact that in some economical models $\tilde{Z}(x, \mu)$ represents the excess demand function associated with a current value $x$ for the state variable and a probability measure $\mu$ for the future value of the state variable. If $\tilde{Z}(x, \mu) = 0$, the pair $(x, \mu)$ is called a *temporary equilibrium* for the economy described by $\tilde{Z}$. In this case it is also said that $\mu$ *rationalizes* $x$. We will assume that the following property holds for the stochastic excess demand function $\tilde{Z}$.

For each $x \in X$, define the set of measures $\rho(x)$ by

$$\rho(x) = \{\mu \in \mathcal{P}(X) | \tilde{Z}(x, \mu) = 0\} \ .$$

**CVR** (*Convex valuedness of rationalizing measures*) property: The stochastic excess demand function $\tilde{Z}$ has the CVR property if for every $x \in X$ and every family $\{\mu_i\}_{i=1}^k \in \rho(x)$, we have that

$$\sum_{i=1}^{k} \alpha_i \mu_i \in \rho(x) ,$$

where $\sum_{i=1}^k \alpha_i = 1$ and $\alpha_i \geq 0$, $i \in \{1, \ldots, k\}$.

Recall that a transition function defined on $X$ is a function $Q : X \times \mathcal{B}(X) \to [0,1]$ such that:

*i)* $Q(x, \cdot) \in \mathcal{P}(X)$ for every $x \in X$.

*ii)* $Q(\cdot, A) : X \to [0,1]$ is a measurable map for every $A \in \mathcal{B}(X)$

We are now ready to define what is meant by *sunspot equilibrium.* See [14, 1] for further details.

**Definition 2.1.** *A* sunspot equilibrium ***(SE)*** *is a pair* $(X_0, Q)$ *where* $X_0 \subset X$*, and* $Q : X_0 \times \mathcal{B}(X_0) \to [0,1]$ *is a transition function such that:*

i) *there exists* $x_0 \in X_0$ *such that* $Q(x_0, \cdot)$ *is not a Dirac measure.*

ii) $\tilde{Z}(x, Q(x, \cdot)) = 0$ *for every* $x \in X_0$.

A sunspot equilibrium has the property that for each value of the state variable $x \in X_0$, it is assigned a conditional probability measure $Q(x, \cdot)$ that rationalizes it, being truly stochastic for at least one $x_0 \in X_0$.

**Definition 2.2.** *A sunspot equilibrium* $(X_0, Q)$ *is said to be a* stationary sunspot equilibrium ***(SSE)*** *if there exists a probability measure* $\mu \in \mathcal{P}(X)$ *with support* $X_0$ *such that the equality*

$$\mu(A) = \int_{X_0} Q(x, A)\mu(dx)$$

*holds for every* $A \in \mathcal{B}(X_0)$.

We define the *deterministic excess demand function* $Z : X \times X \to \mathbb{R}^n$ to be given by

$$Z(x_0, x_1) := \tilde{Z}(x_0, \delta_{x_1}) ,$$

where $\delta_{x_1}$ denotes the Dirac measure supported at $x_1 \in X$.

If $x_0$ is the current value of the state variable and $x_1$ is the expectation for sure of its future value, the equation $Z(x_0, x_1) = 0$ is said to be a deterministic equilibrium condition and we say that $x_1$ deterministically rationalizes $x_0$.

**Definition 2.3.** *A* backward prefect foresight *(**bpf**) map is a function $\phi : X \to X$ such that $Z(\phi(x), x) = 0$ for all $x \in X$.*

If the sequence $\{x_t\}_{t\in\mathbb{Z}}$ is such that $x_t = \phi(x_{t+1})$ for all $t \in \mathbb{Z}$, then $Z(x_t, x_{t+1}) = Z(\phi(x_{t+1}), x_{t+1}) = 0$. This sequence is then a "bpf" equilibrium and occurs in economic models when the involved agents exactly predict the future equilibrium value of the state variable.

The following theorem, due to Araujo and Maldonado [1], guarantees the existence of SSE under general conditions on the corresponding bpf map and the stochastic excess demand function.

**Theorem 2.1** (Araujo–Maldonado)**.** *Let $\phi : X \to X$ be a bpf function associated with a stochastic excess demand function $\tilde{Z}$ with the CVR property. Assume that the following properties hold:*

*(i) There exists a partition $(A_i)_{i=1}^n$ of $X$ with non-empty interior such that $\phi : A_i \to \phi(A_i)$ is a diffeomorphism for all $i = 1, \ldots, n$ with inverse $\varphi_i$.*

*(ii) There exists a $\phi$-invariant measure $\mu \in \mathcal{P}(X)$ such that $\mu$ is absolutely continuous with respect to Lebesgue measure.*

*Then there exists a set $C$ with positive Lebesgue measure such that the transition function*

$$Q(x, \cdot) = \sum_{i=1}^{n} \frac{d(\mu \circ \varphi_i)}{d\mu}(x)\delta_{\varphi_i(x)}(.)$$

*is a SSE on the set $C$ with stationary measure $\mu$. This SSE will be called a **chaotic stationary sunspot equilibrium**.*

From now on, we will specialize to the case where the state space $X$ is a compact interval of $\mathbb{R}$ and the state variable follows the perfect foresight path given by the one-dimensional dynamical system

$$x_t = \phi(x_{t+1}) \ ,$$

where $\phi$ is the bpf map associated with some deterministic excess demand function $Z$.

### 2.1.2 Stochastic perturbations of unimodal maps

Let $X \subseteq \mathbb{R}$ be a compact interval, $\phi : X \to X$ be a backward perfect foresight map associated with a stochastic excess demand function with the CVR property and let $\{x_t\}_{t\in\mathbb{Z}}$ be a perfect foresight equilibrium for the model, i.e. the terms of the sequence $\{x_t\}_{t\in\mathbb{Z}}$ are such that $x_t = \phi(x_{t+1})$ for every $t \in \mathbb{Z}$. We say that a sequence $\{\tilde{x}_t\}_{t\in\mathbb{Z}}$ is a *path with expectation coordination failures* if $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t$ for all $t \in \mathbb{Z}$, where $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a discrete time stochastic process such that the random variables $\epsilon_t$, $t \in \mathbb{Z}$, are i.i.d. with distribution determined by some probability density function $\theta_\epsilon$ with support $[-\epsilon, \epsilon]$. Small failures in coordination give rise to a perturbed sequence which deviates slightly from the perfect foresight equilibrium sequence. The intuition behind this definition is the following: given a (possible misperceived) future state value $\tilde{x}_{t+1}$, the current value for the state variable under perfect foresight is $\phi(\tilde{x}_{t+1})$; however the heterogeneity of expectations introduces noise, leading to $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t$.

Our first goal is to show the stationarity of the process $\{\tilde{x}_t\}_{t\geq 0}$ and finally, to provide an analysis of the convergence of the stationary measure as $\epsilon \to 0$. Let us assume that the pair $(\phi, \theta_\epsilon)$ satisfies the following technical conditions:

(**H1**) $\phi \in C^3(X)$ is a unimodal map with non-flat critical point $x^*$ and negative Schwarzian derivative.

(**H2**) there exist constants $H_0 \geq 1$, $\gamma > 0$ and $0 < \alpha < \gamma/4$ such that for every $k \geq H_0$ the following inequalities hold:

- i) $|\phi^k(x^*) - x^*| \geq e^{-\alpha k}$;
- ii) $|D\phi^k(\phi(x^*))| \geq e^{\gamma k}$.

(**H3**) $\phi$ is topologically mixing in the dynamical interval bounded by $\phi(x^*)$ and $\phi^2(x^*)$.

(**H4**) For small enough $\epsilon > 0$, the probability density function $\theta_\epsilon : \mathbb{R} \to \mathbb{R}_0^+$ is such that

- (i) $\text{supp}(\theta_\epsilon) \subset \Omega_\epsilon = [-\epsilon, \epsilon]$,
- (ii) $\int_{\Omega_\epsilon} \theta_\epsilon(y)dy = 1$,
- (iii) $M = \sup_{\epsilon>0}(\epsilon \sup |\theta_\epsilon|) < \infty$,
- (iv) $J_\epsilon := \{t |\ \theta_\epsilon(t) > 0\}$ is an interval containing 0 and $\eta_\epsilon := \log(\theta_\epsilon|_{J_\epsilon})$ is a concave function.

The conditions above occur naturally in the one-dimensional dynamics literature. We provide further comments and motivation in subSection 2.1.5.

Under conditions (H1)-(H4), a remarkable theorem due to Baladi and Viana [7] guarantees that the map $\phi$ has an invariant measure with an integrable density, which is strongly stochastically stable. We will use such result to prove that small deviations from expectation coordination failures are stationary. Furthermore, the stationary probability distribution of the perturbed process representing paths with expectation coordination failures is close to that of the chaotic SE.

The transition function $P_\epsilon$ associated to the path $\{\tilde{x}_t\}_{t\in\mathbb{Z}}$ with expectation coordination failures is given by

$$P_\epsilon(x, A) := \int_{A-\{\phi(x)\}} \theta_\epsilon(t)dt = \int_A \theta_\epsilon\left(y - \phi(x)\right) dy \ ,$$

where $x \in X$ and $A \in \mathcal{B}(X)$. This represents the probability of the current (misperceived) value of the state variable being in $A$ given that the expected value for the future state is $x$. Thus, $P_\epsilon(x, A)$ can be seen as the backward conditional probability induced by the ECF process.

A Borel probability measure $\nu^\epsilon$ on $X$ is said to be *stationary* under the expectation coordination failures if

$$\nu^\epsilon(A) = \int P_\epsilon(x, A) d\nu^\epsilon(x) \tag{2.1}$$

for all $A \in \mathcal{B}(X)$. The probability $\nu^\epsilon$ is the distribution of the misperceived future values of the state variable that induces the same probability distribution for the current state variable values under the small ECF.

We now notice that whenever an ergodic $\phi$-invariant measure $\mu^0$ exists, we may use the bpf map to estimate the invariant distribution. Using the Birkhoff ergodic theorem, we have that for $\mu^0$-a.e. $x \in X$ the sequence of measures

$$\mu_N^0 := \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\phi^k(x)}$$

converges to $\mu^0$ in the weak topology. Therefore, using the backward perfect foresight trajectory, for $\mu^0$-almost every initial state $x$ we obtain the histograms associated with the measure $\mu_N^0$, as we will describe now.

Let $\mathcal{A} = (A_i)_{i=1}^n$ be a partition of $X$. Define the map $P : X \to \mathcal{A}$ by $P(x) = A_i$ if and only if $x \in A_i$. Clearly, the map $P$ is well defined for every $x \in X$. Moreover, we are able to define $\mu_N^0(P(x))$ by

$$\mu_N^0(P(x)) = \frac{\sharp\{\phi^k(x) \in P(x) : \ k \in \{0, \cdots, N-1\}\}}{N} \ .$$

Assume for the time being that the measure $\mu^0$ is absolutely continuous with respect to Lebesgue measure $\lambda$. Denote its density by $\rho^0$. Then, for $\mu^0$-a.e. $x \in X$ we have that

$$\rho_N^{0,\mathcal{A}}(x) := \frac{\mu_N^0(P(x))}{\lambda(P(x))}$$

is an estimate of the density (Radon-Nikodym derivative) of the invariant measure $\mu^0$ and as $N \to \infty$ and the size of the partition converges to zero, we get the following convergence

$$\rho_N^{0,\mathcal{A}}(x) \to \rho^0(x) \ .$$

Let us now consider the path with expectation coordination failure $\{\tilde{x}_t\}_{t\in\mathbb{Z}}$, obtained from the perfect foresight equilibrium sequence $\{x_t\}_{t\in\mathbb{Z}}$ through perturbation by the stationary discrete-time process $\{\epsilon_t\}_{t\in\mathbb{Z}}$ described above, i.e. the terms of $\{\tilde{x}_t\}_{t\in\mathbb{Z}}$ are such that

$$\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \epsilon_t \ . \tag{2.2}$$

Assume that for every sufficiently small $\epsilon > 0$, the stochastic process defined by (2.2) has a unique ergodic absolutely continuous probability measure $\nu^\epsilon$ with density $\rho^\epsilon$. Thus, an estimate of the measure $\nu^\epsilon$ can be defined by the asymptotic distribution of the Birkhoff average of Dirac measures over the the path with expectations coordination failures $\{\tilde{x}_t\}_{t\in\mathbb{Z}}$ with $\tilde{x}_0 = x$, i.e. the measure

$$\nu_N^\epsilon := \frac{1}{N}\sum_{k=0}^{N-1} \delta_{\tilde{x}_{-k}} \tag{2.3}$$

converges to $\nu^\epsilon$ in the weak topology. Similarly to the deterministic dynamics, for every $x \in X$, every partition $\mathcal{A} = (A_i)_{i=1}^n$ of $X$ and every small $\epsilon > 0$, $\nu_N^\epsilon(P(x))$ is well defined by

$$\nu_N^\epsilon(P(x)) = \frac{\sharp\{(\phi+\epsilon_k)\circ\cdots\circ(\phi+\epsilon_1)(x) \in P(x):\ k \in \{0,\ldots,N-1\}\}}{N} \ .$$

For $\nu^\epsilon$-a.e. $x \in X$, we can estimate the density $\rho^\epsilon$ of the measure $\nu^\epsilon$ by

$$\rho_N^{\epsilon,\mathcal{A}}(x) := \frac{\nu_N^\epsilon(P(x))}{\lambda(P(x))} \ .$$

Note that $\rho_N^{\epsilon,\mathcal{A}}(x) \to \rho^\epsilon(x)$ as $N \to \infty$ and the size of the partition converges to zero.

The following theorem guarantees that the stationary measure associated to the path with expectations coordination failures (2.1) is close to the stationary measure under the chaotic sunspot equilibrium.

**Theorem 2.2.** *Assume that the stochastic excess demand function $\tilde{Z}$ satisfies the CVR property, the associated bpf map $\phi : X \to X$ satisfies conditions (H1)-(H3), and that the sequence of expectations coordination failures* (2.2) *is obtained by perturbing the bpf dynamics by a sequence of i.i.d. random perturbations $\{\epsilon_j\}_{j\in\mathbb{Z}}$ with density function satisfying (H4). Then, the following statements hold:*

*i) The approximate measure $\nu_N^\epsilon$ associated to the paths with expectations coordination failures converges to the measure $\mu^0$ of the deterministic dynamics $\phi$ in the weak*-topology, when $N \to \infty$ and $\epsilon \to 0$.*

*ii) The approximate density $\rho_N^{\epsilon,\mathcal{A}}$ associated to the paths with expectations coordination failures converges to the density $\rho^0$ of $\phi$ in the norm topology, when $N \to \infty$ , $\epsilon \to 0$ and for every sequence of partitions $\mathcal{A} = (A_i)_{i=1}^n$ such that $\max_i |A_i| \underset{n\to\infty}{\longrightarrow} 0$.*

*Proof.* We prove item $(i)$ first. Let $\epsilon > 0$ be small enough so that $\nu^\epsilon$ is the unique ergodic probability measure of the sequence of expectation coordination failures in (2.2). Furthermore, recall that for $\epsilon > 0$ small enough, $\nu^\epsilon$ is absolutely continuous with respect to the Lebesgue measure. By the definition of the measure $\nu_N^\epsilon$ given in (2.3) and ergodicity of $\nu^\epsilon$, we have that $\nu_N^\epsilon \underset{N\to\infty}{\longrightarrow} \nu^\epsilon$ in the weak*-topology. Under assumptions (H1)-(H3) and (H4), Baladi and Viana Theorem 2.4 ensures that $\nu^\epsilon \to \mu^0$ in the weak*-topology. Therefore, combining the two statements above, we obtain that $\nu_N^\epsilon \to \mu^0$ in the weak*-topology when $N \to \infty$ and $\epsilon \to 0$.

The proof of item $(ii)$ is similar. By ergodicity of $\nu^\epsilon$, we have that $\rho_N^{\epsilon,\mathcal{A}} \to \rho^\epsilon$ in $L^1(dx)$ when $N \to \infty$ and the size of the partition goes to zero. The result then follows by noting that Baladi and Viana Theorem 2.4 also ensures that $\rho^\epsilon \to \rho^0$ in $L^1(dx)$ as $\epsilon \to 0$. □

One of the key consequence of Theorem 2.2 is that under mild conditions, the approximate measures $\nu_N^\epsilon$ associated to the paths with expectations coordination failures approach the stationary measure $\mu^0$ of the SE given by Araujo and Maldonado [1] as $\epsilon \to 0$ and $N \to \infty$. We may call that process a *Rational revealing of beliefs*, i.e. as the number of iterations increases and providing that the size of the perturbations $\epsilon$ is small enough, the agents gain knowledge about the stationary measure $\mu^0$ from observing paths with expectation coordination failures.

Whenever the assumptions of Theorem 2.2 hold, then an absolutely continuous $\phi$-invariant measure $\mu^0$ exists, yielding a chaotic SE as in Araujo-Maldonado Theorem. In this case we say that $\mu^0$ is a *(full) spread of beliefs*. The other case is for a bpf map $\phi$ with an attracting cycle, in which case conditions (H2) and (H3) do not hold. We refer to such bpf maps as *ordered backward perfect foresight maps*. For such maps a unique $\phi$-invariant measure $\mu^0$ still exists. However, $\mu^0$ is no longer absolutely continuous with respect to the Lebesgue measure. Instead, $\mu^0$ is a convex linear combination of Dirac masses with support on the attracting cycle. In such case, we say that $\mu^0$ is a *(full) concentration of beliefs*. Due to hyperbolicity of the attracting cycle, the approximate measure $\mu_N^\epsilon$ associated with an ordered backward perfect foresight map defined similarly to (2.3), converges to a measure $\mu^0$ with a (full) concentration of beliefs. Therefore, the stationary process given by the small ECF converges either to the stationary measure of the chaotic SE or to a deterministic cycle.

### 2.1.3 An application: an overlapping generations model

In this section we will consider a two-period overlapping generations (OLG) model like the one introduced in [27] and provide an application for our main results. An analogous model was analysed by Azariadis and Guesnerie in [6] to prove the existence of cycles and sunspots with finite support. We show that depending on the parameter values of the model, small ECF will be either close to the stationary measure of the chaotic SE or close to a deterministic cycle.

Let us consider an OLG economy for a population with constant size and such that the proportions of young and old agents remain unchanged over time. We assume that the there exists a representative agent with preferences given by a separable utility function

$$U(c_t, c_{t+1}) = V_1(c_t) + V_2(c_{t+1}),$$

where $V_1$ and $V_2$ are the utilities representing the individual preferences with respect to the consumption plan in the first and second periods of their lives, and $c_t, c_{t+1}$ denote the corresponding consumption plan. We suppose also that one unit of the good is produced with one unit of the unique productive factor (labor) and let $l_1^*$ and $l_2^*$ denote the agents labor endowments in the first and second periods of their lives, respectively. Finally, we assume that there is a risk-free asset that can be purchased by the agents providing a gross return $z_t = 1 + r_t > 1$, where $r_t$ is the interest rate at period $t$, and that the dynamics of the money supply is defined by

$$M_{t+1} = M_t z_{t+1} \ ,$$

for some initial condition $M_0$.

We impose the following additional conditions on the utility function:

**(C1)** For each $\tau = 1, 2$ , $V_\tau$ is continuous on $[0, +\infty)$ and (at least) twice continuously differentiable on $(0, +\infty)$. Moreover, $V_\tau$ is strictly increasing and strictly concave and satisfies

$$\lim_{c_\tau \to +\infty} V_\tau'(c_\tau) = 0 \ , \qquad \lim_{c_\tau \to 0} V_\tau'(c_\tau) = +\infty \ , \qquad \bar{\Theta} := \frac{V_1'(l_1^*)}{V_2'(l_2^*)} < 1 \ .$$

(**C2**) The Arrow-Pratt relative degree of risk aversion of the old, given by

$$R_{V_2}(x) = -\frac{xV_2''(x)}{V_2'(x)} \ ,$$

is a non-decreasing function of $x$.

See [27] and the references therein for further comments and motivations for hypotheses (C1)-(C2).

We will now state the consumption-saving problem of the representative agent. Let $p_t$ and $p_{t+1}$ denote the prices of the unique good in the economy during the first and second periods of the individual's life. Note that while $p_t$ is known by the individual during the first stage of her life, her knowledge concerning $p_{t+1}$ consists of a probability distribution $\mu_{t+1}$ representing the likelihood of occurrence of particular values of $p_{t+1}$ and reflecting the individual's beliefs about the state of economy during the second period of her life. The agent must choose a consumption plan $c_t, c_{t+1}$ and the first period saving $m_t$ as the solution of the following optimization problem

$$\max_{\{c_t, c_{t+1}, m_t\}} \ V_1(c_t) + E_{\mu_{t+1}}\left[V_2(c_{t+1})\right] \tag{2.4}$$

subject to the budget constraints

$$p_t c_t + m_t = p_t l_1^*$$

$$p_{t+1} c_{t+1} = p_{t+1} l_2^* + z_t m_t \ .$$

Working out the first order condition for an interior solution of (2.4) leads to

$$-\frac{1}{p_t} V_1'\left(l_1^* - \frac{m_t}{p_t}\right) + E_{\mu_{t+1}}\left[\frac{z_t}{p_{t+1}} V_2'\left(l_2^* + \frac{z_t m_t}{p_{t+1}}\right)\right] = 0 \ . \tag{2.5}$$

**Monetary equilibrium condition and the bpf dynamics**

Under the monetary equilibrium condition $M_t = m$, and introducing the new variable

$$x_t = \frac{M_t}{p_t} \ ,$$

the first order condition (2.5), may be rewritten as

$$\nu_1(x_t) = E_{\mu_{t+1}}\left[\nu_2(x_{t+1})\right] \ , \tag{2.6}$$

where $\nu_1$ and $\nu_2$ are the the auxiliary functions defined by

$$\nu_1(x) = xV_1'(l_1^* - x) \ , \qquad \nu_2(x) = xV_2'(l_2^* + x) \ , \tag{2.7}$$

Note that condition (2.6) defines the following stochastic excess demand function

$$\tilde{Z}(x, \mu) = \nu_1(x) - E_\mu \left[\nu_2(x')\right]$$

and that $\tilde{Z}$ has the CVR property introduced in Section 2.1.1. The deterministic excess demand function associated with $\tilde{Z}$ is then given by

$$Z(x, x') = \tilde{Z}(x, \delta_{x'}) = \nu_1(x) - \nu_2(x') \ ,$$

defining the associated bpf map through the implicit relation

$$Z(\phi(x), x) = 0 \ .$$

Under condition (C1), it is possible to check that the map $\nu_1$ defined in (2.7) is strictly increasing, and thus invertible. Hence, we obtain that the backward perfect foresight map is of the form

$$\phi(x) = \nu_1^{-1}\left(\nu_2(x)\right) \ , \tag{2.8}$$

and the corresponding bpf dynamical system is determined by

$$x_t = \phi(x_{t+1}) = \nu_1^{-1}\left(\nu_2(x_{t+1})\right)$$

Grandmont [27] provided conditions under which the bpf map in (2.8) is unimodal with non-flat critical point and negative Schwarzian derivative, i.e. satisfies condition (H1) of Section 2.1.5. For the sake of completeness, we reproduce their criteria in the next lemma.

**Lemma 2.1.** *If conditions (C1) and (C2) hold and there exists $x_0 \geq 0$ such that $R_{V_2}(x_0) > 1$, then the bpf map $\phi$ is a unimodal map with $\phi(0) = 0$ and $\phi'(0) > 1$. Moreover, if $V_\tau \in C^3$ for $\tau = 1, 2$, $S\nu_1 \geq 0$ on $[0, l_1^*)$, $S\nu_2 < 0$ on $[0, \phi(x^*))$ and $\sup R_{V_2}(x) > 1$ then $\phi \in \mathcal{C}$.*

Now, let us consider the bpf map defined by (2.8) and parameterized by one of the parameters of the model (e.g. the relative risk aversion coefficient). In the Appendix we provide some technical features to be dealt with below. Thus, let $\phi_\lambda : X \to X$, $\lambda \in \Lambda \subset \mathbb{R}$, be the one-parameter family of bpf maps (2.8) with the property that the conditions of Lemma 2.1 hold for every $\lambda \in \Lambda$. Then, for every $\lambda \in \Lambda$, we have that

i) $\phi_\lambda$ is a $C^2$ unimodal maps of an interval $X$;

ii) $\phi_\lambda$ has a nondegenerate critical point $\bar{x}(\lambda)$;

iii) $\phi_\lambda$ has a repelling fixed point on the boundary of $X$.

We say that the one-parameter family $\phi_\lambda$, $\lambda \in \Lambda$, has a *Misiurewicz parameter with generic unfolding* if the following conditions hold

a) the map $(x, \lambda) \to \big(\phi_\lambda(x), D_x\phi_\lambda(x), D_x^2\phi_\lambda(x)\big)$ is $C^1$;

b) there exists a parameter value $\lambda_* \in \Lambda$ such that $\phi_{\lambda_*}$ is a Misiurewicz map;

c) the following transversality condition holds:

$$\lim_{n\to+\infty} \frac{D_\lambda \phi^n_{\lambda_*}(\bar{x}(\lambda_*))}{D_x\phi^{n-1}_{\lambda_*}(\phi_{\lambda_*}(\bar{x}(\lambda_*)))} \neq 0 \ .$$

Note that condition a) only depends on the choice of a sufficiently regular parametrization for such family of bpf maps. In what concerns conditions b) and c), we remark that in the case where $\phi_{\lambda_*}$ is a post-critically finite Misiurewicz map, i.e. $\phi_{\lambda_*}$ has no periodic attractors and some iterate $N$ of the critical point $\bar{x}(\lambda_*)$ reaches a repelling periodic point $P(\lambda_*)$, then condition c) is equivalent to the transversality of the curves $\lambda \mapsto \phi^N_\lambda(\bar{x}(\lambda))$ and $\lambda \mapsto P(\lambda)$.

The next result provides a set of conditions under which there exists a large set of bpf maps for which a strongly stochastically stable SSE exists.

**Theorem 2.3.** *Let $\phi_\lambda : X \to X$, $\lambda \in \Lambda \subset \mathbb{R}$, be a one-parameter family of bpf maps* (2.8) *with the property that the conditions of Lemma 2.1 hold for every $\lambda \in \Lambda$. Moreover, assume that $\phi_\lambda$ has a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding. Then, there exists a positive measure set $A \subset \Lambda$ having $\lambda_*$ as a density point such that for every $\lambda \in A$ there exists a SSE whose invariant measure $\mu_\lambda$ is an absolutely continuous SBR measure. Moreover, if the bpf dynamics are perturbed by an i.i.d process satisfying (H4), $\phi_\lambda$ is strongly stochastically stable.*

*Proof.* For one-parameter families of maps satisfying the conditions i), ii) and iii) above and having a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, there exists a positive measure set $A$ in the space of parameters with $\lambda^*$ as a density point and such that condition (H2) holds for every $\lambda \in A$ (see [16] for further details). As a consequence, for every $\lambda \in A$, we have that

1) $\phi_\lambda$ admits an acip $\mu_\lambda$, with a $L^p$ density for any $p < 2$;

2) $\mu_\lambda$ is a SBR measure;

3) $\phi_\lambda$ has positive Lyapunov exponent almost everywhere.

Hence, Theorem 2.1 ensures the existence of a SSE associated with each measure $\mu_\lambda \in A$. The strong stochastic stability of $\phi_\lambda$ follows from Theorem 2.4 by observing that condition (H3) holds for unimodal maps with negative Schwartzian derivative and an acip. □

We remark that there may be parameters outside of the set $A$ of Theorem 2.3 yielding a strongly stochastically stable SSE. In the alternative case where the bpf map $\phi$ is an ordered bpf map (see section 2.2), then $\phi$ has an attracting cycle leading a measure $\mu^0$ revealing a (full) concentration of beliefs.

**The case of constant relative risk aversion utilities**

Let us consider now the case of constant relative risk aversion utility functions of the form

$$V_\tau(c) = \frac{c^{1-\alpha_\tau}}{1-\alpha_\tau}; \quad \alpha_\tau > 0 \ , \qquad \tau = 1, 2 \ , \tag{2.9}$$

where $\alpha_1$ and $\alpha_2$ are, respectively, the relative degrees of risk aversion for the representative agent in the first and second periods of her life. It is possible to check that the bpf map associated with the choice of utilities (2.9) is such that $\phi \in \mathcal{C}$ if the relative risk aversion parameters are such that $\alpha_1 \in (0,1)$ and $\alpha_2 \in (2, +\infty)$ (see [1] for further details).

To illustrate the large abundance of strongly stochastically stable SSE for this particular family of utility functions, we numerically determine values of parameters $(\alpha_1, \alpha_2, l_1^*, l_2^*)$ under which the bpf map $\phi$ is a post-critically finite Misiurewicz map, i.e. $\phi$ has no periodic attractors and the critical orbit is pre-periodic to a repelling periodic orbit. Note that these are a subset of the set of Misiurewicz maps. To simplify the analysis, we fix the parameters $l_1^* = 3.51$ and $l_2^* = 0.55$ and work only on the two parameter space $(\alpha_1, \alpha_2) \in (0,1) \times (2, +\infty)$. We then numerically compute any intersections between the first $N$ iterates of the critical point and the periodic points up to some finite period $M$, excluding all the non-transverse intersections and all the intersections with attracting periodic points. Note that if the critical point is pre-periodic to a repelling periodic point, then there are no stable or neutral cycles, since for unimodal maps with negative Schwarzian these would attract the critical orbit. In Figure 2.1 it is possible to notice the different dynamical behaviours of the bpf map as the risk aversion parameter $\alpha_2$ increases from 2 to 7.5. For small values of $\alpha_2$, there exists a unique attracting fixed point of $\phi$. As $\alpha_2$ increases, periodic points of higher periods are generated by period-doubling bifurcations. All such maps $\phi$ are ordered bpf maps and lead to (full) concentration of beliefs, i.e. invariant measures supported on convex linear combination of Dirac measures. For some large enough values of $\alpha_2$ Misiurewicz maps can be found. See Figure 2.2 for the distribution of Misiurewicz maps in parameter space $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.29)$. As noted above, the values of parameters under which such maps occur are density points of positive measure sets where chaotic SSE exist and are robust with respect to sufficiently small stochastic perturbations. Such parameters are associated with invariant measures with a (full) spread of beliefs. Ultimately, one obtains a dichotomy between the subset of parameter space corresponding to (full) spread of beliefs and (full) concentration of beliefs.

To make the distinction between the behaviours associated with (full) spread of beliefs and (full) concentration of beliefs clear, we plot in Figures 2.3 and 2.4 some histograms associated with $10^5$ iterations of paths with coordination failures for two different set of parameters values. Figure 2.3 corresponds to a chaotic of parameters revealing a (full) spread of beliefs, i.e. a chaotic SE exists. Moreover, when perturbing the bpf map by a sequence of i.i.d. random variables satisfying assumption (H4), we see that the corresponding histograms are associated with an absolutely continuous measure. Instead, for the parameter values used to produce Figure 2.4, no chaotic SE exists. In this case, we observe a full concentration of beliefs with a stationary measure supported on a period two orbit.

### 2.1.4 Conclusions

We have considered small random perturbations of backward perfect foresight maps within a class of unimodal maps with non-flat critical point and negative Schwarzian that satisfy the Benedicks-Carleson conditions. In that setting we proved the stationarity of the time series generated by

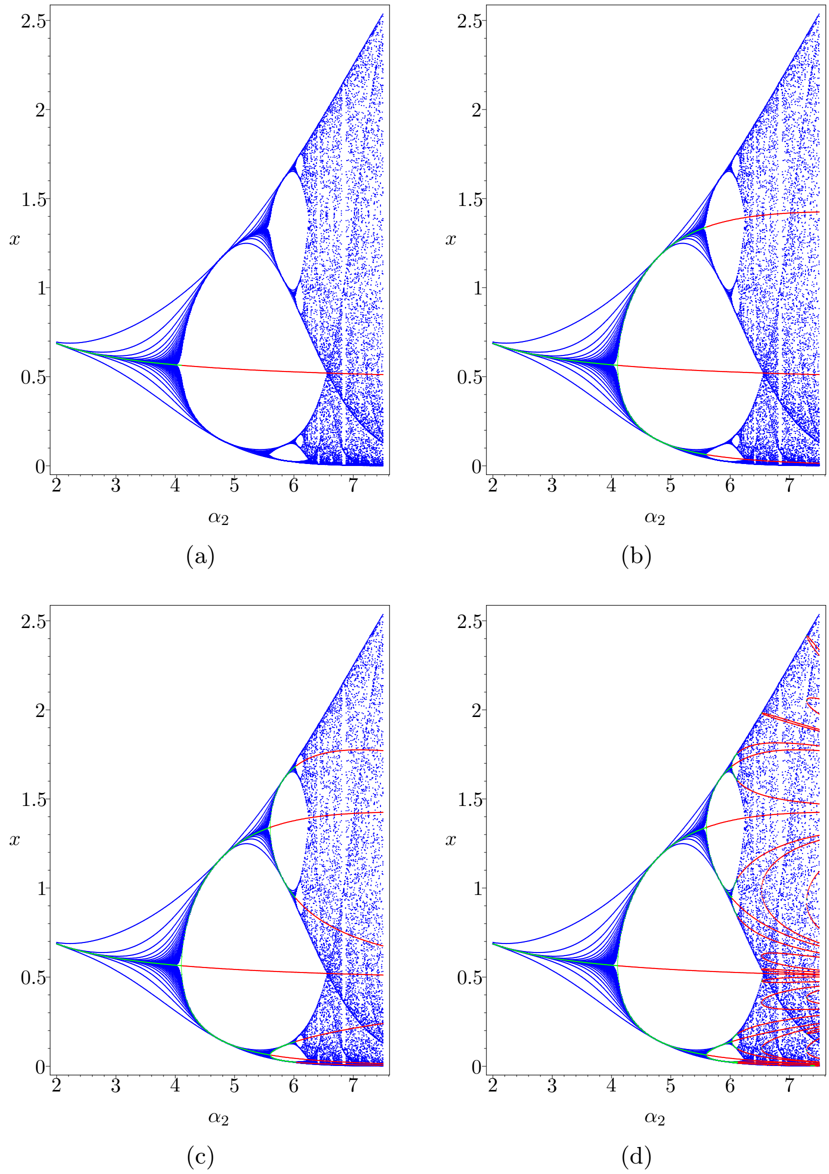


Figure 2.1: The first 100 iterates of critical point (in blue) and periodic points (up to period 8) of the bpf map $\phi$ in the $(\alpha_2, x)$ plane. We plot the stable periodic points in green and the unstable ones in red. Figure 2.1a contains only the fixed point of $\phi$, Figure 2.1b contains the fixed point and the period 2 orbit, Figure 2.1c contains periodic points of periods 1, 2 and 4, and finally, Figure 2.1d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $l_1^* = 3.51$, $l_2^* = 0.55$ and $\alpha_1 = 0.41$.

small expectations coordination failures as well as its closeness to the stationary measure of the

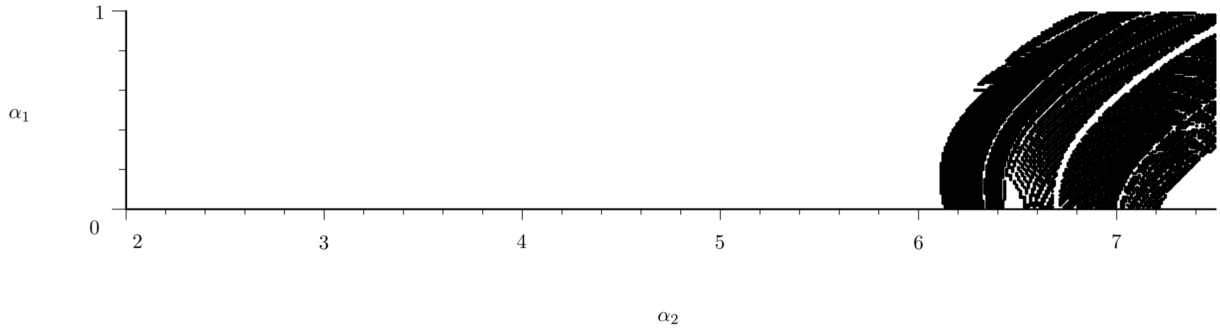


Figure 2.2: The distribution of Misiurewicz parameters for the family of bpf maps with $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.99)$ for fixed $l_1^* = 3.51$ and $l_2^* = 0.55$. These are obtained by considering intersections of the first 100 iterates of the critical point with unstable periodic points of periods 1, 2, 4 and 8.

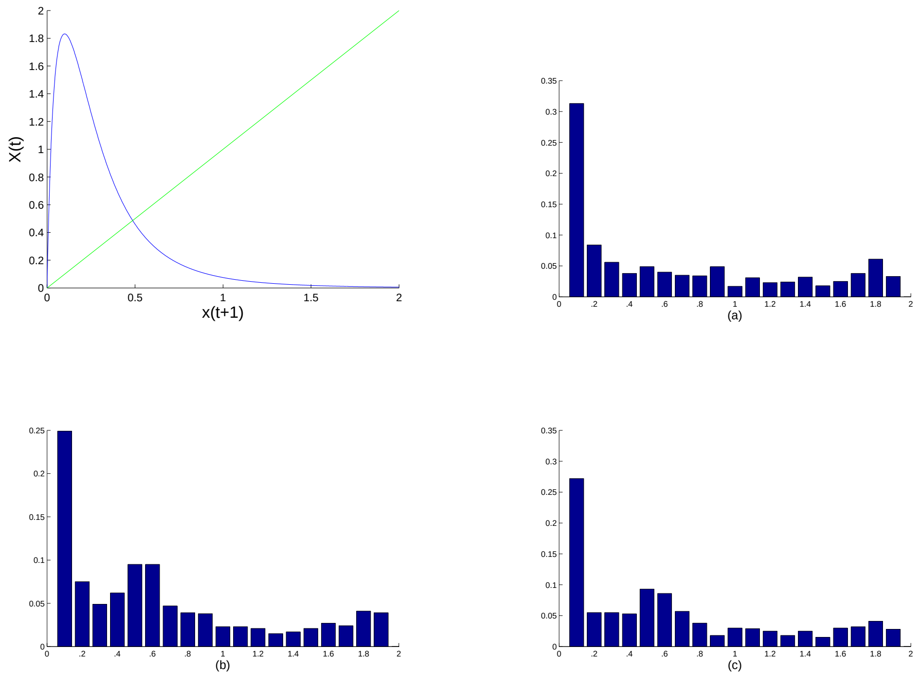


Figure 2.3: Map $\phi$ and the approximate densities of the dynamics of the bpf map $x_t = \phi(x_{t+1})$ with parameters $l_1^* = 3.5, l_2^* = 0.55, \alpha_1 = 0.2, \alpha_2 = 6.5$ under perturbations of maximum size $(a)\ \epsilon = 0.01,\ (b)\ \epsilon = 0.001,\ (c)\ \epsilon = 0,$ respectively.

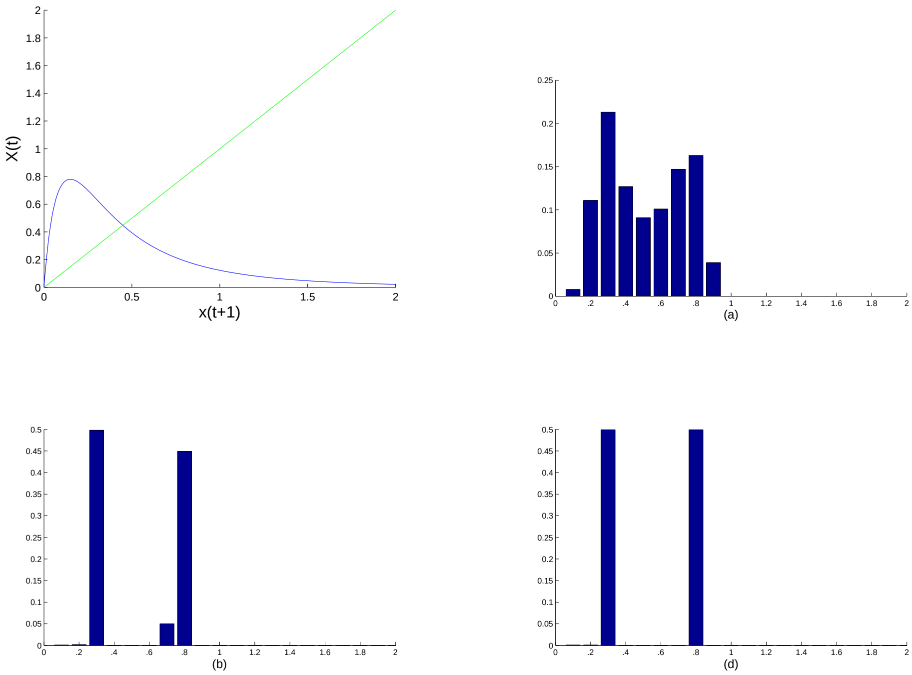


Figure 2.4: Map $\phi$ and the approximate densities of the dynamics of the bpf map $x_t = \phi(x_{t+1})$ with parameters $l_1^* = 3.5, l_2^* = 0.6, \alpha_1 = 0.2, \alpha_2 = 5$ under perturbations of maximum size $(a)\ \epsilon = 0.1,\ (b)\ \epsilon = 0.01,\ (d)\ \epsilon = 0$ respectively.

chaotic SE. As a consequence, small expectations coordination failures will converge to one of two equilibria: to the chaotic sunspot equilibrium defined by [1] or to a deterministic cycle. Finally, as an application we studied an overlapping generations model and showed that a large class of OLG models has unimodal bpf maps associated with its stochastic excess demand function. This family of unimodal bpf maps satisfies condition (H1) and for a subset of positive Lebesgue measure in parameter space, this family has an absolutely continuous invariant measure associated with the SSE.

### 2.1.5 Dynamics of unimodal maps

The aim of this subsection is to introduce the nomenclature and some of the key results from one-dimensional dynamics used throughout, as well as to provide appropriate motivation for such concepts and results.

**Deterministic unimodal maps**

Let $X = [a, b] \subset \mathbb{R}$ be a compact interval. A continuous map $\phi : X \to X$ is called *unimodal* if $\phi$ has a unique interior local maximum at $x^*$ such that $\phi$ is strictly increasing in $[a, x^*)$ and strictly decreasing in $(x^*, b]$. It is also common to require that $\phi(\partial X) \subset \partial X$, but we will not do so here. However, if such condition is required, we note that any endomorphism of a compact interval can be extended to a bigger interval so that the boundary of the larger interval is mapped into itself.

The local maximum of $\phi$ at $c$ will be called *non-flat* if there exists a $C^2$ local diffeomorphism $h$ such that $h(c) = 0$ and $\phi(x) = \phi(c) \pm |h(x)|^\alpha$, for some $\alpha \geq 2$. A notable example of a unimodal map is the quadratic family $f_\lambda : [0, 1] \to [0, 1]$ defined by $f_\lambda(x) = \lambda x(1 - x)$, for $\lambda \in (0, 4]$.

The *Schwarzian derivative* of a $C^3$ map $\phi$ is defined by

$$S\phi(x) = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2} \left( \frac{\phi''(x)}{\phi'(x)} \right)^2$$

for every $x \in X$ such that $\phi'(x) \neq 0$. Throughout this section, we will consider the following set of unimodal maps of the compact interval $X$:

$$\mathcal{C} = \left\{ \phi \in C^3(X) : \phi \text{ has a non-flat critical point and } S\phi < 0 \right\} .$$

Let $\mu$ be a $\phi$-invariant measure, i.e. $\mu(\phi^{-1}(A)) = \mu(A)$ for every measurable set $A \subset X$. We say that $\mu$ is a *SBR measure* (Sinai-Bowen-Ruelle measure) if there exists a set $B \subset X$ with positive Lebesgue measure such that

$$\frac{1}{N} \sum_{k=0}^{N-1} \delta_{\phi^k(x)} \underset{N \to \infty}{\longrightarrow} \mu$$

for all $x \in B$, where the limit is taken in the weak*-topology sense. There is a vast literature concerning the existence and uniqueness of absolutely continuous invariant measures for unimodal maps. In what follows, we provide only a quick overview of some results that will be useful in the sequel. See for instance [16, Chp. 5], as well as Blokh and Lyubich [10], Keller [31] and Nowicki and van Strien [38] for more details.

We start by a result due to Keller: if $\phi \in \mathcal{C}$ there exists a constant $\lambda_\phi \in \mathbb{R}$, called the *Lyapunov exponent*, such that

$$\limsup_{n \to \infty} \frac{1}{n} \log |D\phi^n(x)| = \lambda_\phi.$$

for almost all $x \in X$. Moreover, we have that $\lambda_\phi > 0$ if and only if $\phi$ has an absolutely continuous invariant probability measure (acip); and $\lambda_\phi < 0$ if and only if $\phi$ has a hyperbolic periodic attractor.

Benedicks and Carleson [8] provide the following sufficient conditions for the existence of an acip $\mu$ for a unimodal map $\phi$:

(**H1**) $\phi \in \mathcal{C}$;

(**H2**) there exist constants $H_0 \geq 1$, $\gamma > 0$ and $0 < \alpha < \gamma/4$ such that for every $k \geq H_0$ the following inequalities hold:

i) $|\phi^k(x^*) - x^*| \geq e^{-\alpha k}$;

ii) $|D\phi^k(\phi(x^*))| \geq e^{\gamma k}$.

(**H3**) $\phi$ is topologically mixing in the dynamical interval bounded by $\phi(x^*)$ and $\phi^2(x^*)$.

We notice that condition (H1) is very mild and that it is satisfied by a large family of unimodal maps. Before discussing condition (H2), we introduce a special class of unimodal maps admitting an acip under mild nonflatness conditions [37]. A unimodal map $\phi : X \to X$ is called a *Misiurewicz map* if it has no periodic attractors and if critical orbits do not accumulate on critical points. A *post-critically finite Misiurewicz map* is a Misiurewicz map for which the critical orbit is pre-periodic to a repelling periodic orbit. Indeed, note that item i) of condition (H2) above is trivially satisfied by Misiurewicz maps. Let us now provide the connection between item ii) of (H2) and Misiurewicz maps. Let $\phi_\lambda : X \to X,\ \lambda \in \Lambda \subseteq \mathbb{R}$, be a one-parameter family of unimodal maps $\phi \in \mathcal{C}$ passing through one Misiurewicz point with generic unfolding, i.e. the following conditions hold

a) the map $(x, \lambda) \to \big(\phi_\lambda(x), D_x\phi_\lambda(x), D_x^2\phi_\lambda(x)\big)$ is $C^1$;

b) there exists a parameter value $\lambda_* \in \Lambda$ such that $\phi_{\lambda_*}$ is a Misiurewicz map;

c) the following transversality condition holds:

$$\lim_{n\to+\infty} \frac{D_\lambda \phi^n_{\lambda_*}(\bar{x}(\lambda_*))}{D_x\phi^{n-1}_{\lambda_*}(\phi_{\lambda_*}(\bar{x}(\lambda_*)))} \neq 0 \ .$$

In spite of the set of Misiurewicz parameters having zero Lebesgue measure in the parameter space of the maps $\phi_\lambda$, there exists a positive measure set in that space with a Misiurewicz parameter as a density point and where conditions i) and ii) of (H2) are satisfied (see [49]) and therefore the existence of acips for our maps is guaranteed. Finally, we remark that assumption (H3) always holds for unimodal maps with negative Schwartzian derivative and an acip.

**Stochastic Perturbations of unimodal maps**

Consider the Markov chain $\chi^\epsilon$ generated by perturbations of the deterministic map $\phi : X \to X$ by a sequence of independent and identically distributed (i.i.d.) random variables $\{\epsilon_j\}_{j\geq 1}$ with density function $\theta_\epsilon$. Assume that the following condition holds:

(**H4**) For small enough $\epsilon > 0$, the probability density function $\theta_\epsilon : \mathbb{R} \to \mathbb{R}_0^+$ is such that

- (i) $\text{supp}(\theta_\epsilon) \subset \Omega_\epsilon = [-\epsilon, \epsilon]$,
- (ii) $\int_{\Omega_\epsilon} \theta_\epsilon(y)dy = 1$,
- (iii) $M = \sup_{\epsilon>0}(\epsilon \sup |\theta_\epsilon|) < \infty$,
- (iv) $J_\epsilon := \{t|\ \theta_\epsilon(t) > 0\}$ is an interval containing 0 and $\eta_\epsilon := \log(\theta_\epsilon|_{J_\epsilon})$ is a concave function.

Identify $\Omega_\epsilon^{\mathbb{N}} = [-\epsilon, \epsilon]^{\mathbb{N}}$ with $\{\Phi_\epsilon = (\phi + \epsilon_1, \phi + \epsilon_2, \cdots) : \ \epsilon_i \in [-\epsilon, \epsilon]\}$ and define the cocycle $\mathcal{F} : X \times \Omega_\epsilon^{\mathbb{N}} \to X \times \Omega_\epsilon^{\mathbb{N}}$ by

$$\mathcal{F}(x, \Phi_\epsilon) := (\phi(x) + \epsilon_1, \sigma(\Phi_\epsilon)) \ ,$$

where $\sigma : \Omega_\epsilon^{\mathbb{N}} \to \Omega_\epsilon^{\mathbb{N}}$ is the shift operator, i.e. $\sigma(a_1, a_2, \cdots) = (a_2, a_3, \cdots)$ for any sequence $(a_1, a_2, \cdots) \in \Omega_\epsilon^{\mathbb{N}}$. The cocycle $\mathcal{F}$ defines the random dynamics

$$\chi^\epsilon(x) = \phi(x) + \epsilon \ . \tag{2.10}$$

For every $x_0 \in X$, we define the $\epsilon$-random orbit of $x_0$ to be the sequence $\{\tilde{x}_k\}_{k\geq 0}$ given by

$$\tilde{x}_k := (\phi + \epsilon_k) \circ \cdots \circ (\phi + \epsilon_1)(x_0) \ , \ k \geq 1 \ .$$

The transition function $P_\epsilon$ associated with $\chi^\epsilon$ is given by

$$P_\epsilon(x, A) := \int_A \theta_\epsilon\, (y - \phi(x))\, dy \ ,$$

where $x \in X$ and $A \in \mathcal{B}(X)$. A Borel probability measure $\nu^\epsilon$ on $X$ is said to be *stationary* under the Markov chain $\chi^\epsilon$ if

$$\nu^\epsilon(A) = \int P_\epsilon(x, A) d\nu^\epsilon(x)$$

for all $A \in \mathcal{B}(X)$.

It is known that for every small enough $\epsilon > 0$, there exists a unique ergodic probability measure $\nu^\epsilon$ which is absolutely continuous with respect to the Lebesgue measure and stationary under $\chi^\epsilon$ (see [7] for more details).

Let $\phi \in \mathcal{C}$ be such that $\mu^0$ is its unique invariant absolutely continuous probability measure. Denote the density of $\mu^0$ by $\rho^0$. We say that $\phi$ is *strongly stochastically stable* under $\epsilon$-random perturbations if $\nu^\epsilon \to \mu^0$ in the norm topology as $\epsilon \to 0$, i.e.

$$\lim_{\epsilon \to 0} \left\| \frac{\mathrm{d}\mu^\epsilon}{\mathrm{d}\lambda} - \frac{\mathrm{d}\mu^0}{\mathrm{d}\lambda} \right\|_{L^1(\mathrm{d}\lambda)} = 0 \ .$$

Baladi and Viana [7] provide a set of sufficient conditions for the strong stochastic stability of unimodal maps. Let $\phi : X \to X$ be such that conditions (H1)-(H3) hold. Moreover, let $\chi^\epsilon$ be the Markov chain generated by random perturbations of $\phi$ by i.i.d. random variables satisfying $(H4)$.

**Theorem 2.4** (Baladi-Viana)**.** *Under conditions (H1)-(H4), $\phi$ has an invariant measure with an integrable density, which is strongly stochastically stable*

Taking into consideration the observation before assumption (H4), in this subsection concerning assumptions (H1)-(H3), we obtain that for one-parameter families of unimodal maps $\phi \in \mathcal{C}$ passing through one Misiurewicz point with generic unfolding, Baladi and Viana theorem applies to a set of positive measure in parameter space.

## 2.2 Large random perturbations of Sunspot Equilibrium

This section is devoted to study how large additive random perturbations influence the dynamics of unimodal maps in what concerns the existence of ergodic and absolutely continuous stationary measures for the corresponding random dynamical systems as well as positivity of the Lyapunov exponent. In subsection 2.2.1 we state the setup and necessary background. In subsection 2.2.2 we discuss the existence of a stochastic dynamical interval for the random dynamical system defined by the large perturbations of the deterministic dynamics. In subsection 2.2.3 we state conditions under which the random dynamical system admits a unique invariant measure which is both ergodic and equivalent to the Lebesgue measure. In subsection 2.2.4 we prove the existence of a positive lower bound for the Lyapunov exponent of a random dynamical system under appropriate conditions. Finally, in subsection 2.2.5 we illustrate the results of this section with the well-known overlapping generations model presented previously in subsection 2.1.3.

### 2.2.1 Setup and Background

Let $X = \mathbb{R}$ be the state variable set and let $\mathcal{B}(X)$ denote the Borel $\sigma$-algebra of $X$. Throughout this section we will assume that $\phi : X \to X$ is such that the following conditions hold:

Assumption (**A**):

1) $\phi$ is $C^2$ in supp($\phi$).

2) $\phi(0) = 0$.

3) $\phi'(0) > 1$.

4) There exists a unique $x^* > 0$ such that $\phi'(x^*) = 0$.

5) $\phi''(x^*) \neq 0$ and if there exists $\tilde{x}$ such that $\phi''(\tilde{x}) = 0$, then $\tilde{x} > x^*$.

6) $\lim_{x\to+\infty} \phi(x) = 0$.

7) $\lim_{x\to+\infty} \phi'(x) = 0$.

Assumption (A) ensures that we deal with a family of unimodal maps with a non-degenerate critical point $x^*$ and a repelling fixed point at zero. Indeed, we will consider two distinct cases herein. The case where supp $\phi$ is a compact subset of $X$ of the form $[0, b]$, with $b > 0$, and the case where supp $\phi = \mathbb{R}^+$. We also note that if there exists $\bar{x} > 0$ such that for every $x > \bar{x}$ we have $\phi(x) = 0$, then hypotheses 6) and 7) hold trivially.

Fix $\epsilon > 0$, let $\Omega_\epsilon = [-\epsilon, \epsilon]$ and denote by $\eta_\epsilon$ the uniform probability measure on $\Omega_\epsilon$. We will consider random dynamical systems (RDS) (see [5] for further details) obtained through perturbations of the (deterministic) dynamical system defined by a unimodal map $\phi$ satisfying assumption (A) by sequences of i.i.d. random variables with distribution $\eta_\epsilon$. Elements of such class of RDS are conveniently expressed through cocycles $\phi_\epsilon$ of the deterministic dynamical system $(X, \phi)$ over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \Omega_\epsilon^{\mathbb{N}}$, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$ and $\mathbb{P}$ is the product measure induced on $\Omega$ by the measure $\eta_\epsilon$ on $\Omega_\epsilon$. Thus, each state of the world $\omega \in \Omega$ corresponds to a sequence $\omega = (\omega_1, \omega_2, \cdots)$ and the cocycle $\phi_\epsilon : X \times \Omega \to X$ is such that

$$\phi_\epsilon(x, \omega) = \phi(x) + \pi_1(\sigma(\omega)) \ , \tag{2.11}$$

where $\sigma : \Omega \to \Omega$ is the shift map and $\pi_1 : \Omega \to \Omega_\epsilon$ is the projection onto the first component, i.e. for every sequence $(\omega_1, \omega_2, \omega_3, \cdots) \in \Omega$, we have that

$$\pi_1(\omega_1, \omega_2, \omega_3, \cdots) = \omega_1 \ .$$

We can associate with the cocycle $\phi_\epsilon$ above the skew-product $\Theta_\epsilon : X \times \Omega \to X \times \Omega$ given by

$$\Theta_\epsilon(x, \omega) = (\phi_\epsilon(x, \omega), \sigma(\omega)) \ .$$

Note that the skew-product above defines a dynamical system on $\Omega \times X$. Thus, each element $\omega \in \Omega$, can be seen as a sequence of i.i.d. $\eta_\epsilon$ random variables representing the additive perturbations to which $\phi$ is subject to in each iteration. In particular, the orbit of the RDS induced by $\phi_\epsilon$ associated with the sequence $\omega = \{\omega_n\}_{n \in \mathbb{N}}$ can be expressed as

$$\tilde{x}_n = \phi(\tilde{x}_{n-1}) + \omega_n \ , \tag{2.12}$$

where $\tilde{x}_0 = x_0 \in X$.

Recall that under assumption (A), the unimodal map $\phi$ has a non-empty *invariant (deterministic) dynamical interval* $\Lambda$, given by

$$\Lambda = [\phi^2(x^*), \phi(x^*)] \ .$$

Note that $\Lambda$ is the smallest invariant interval under $\phi$ that contains the critical point $x^*$. Note also that the dynamics of $\phi$ agree with those of the limiting case $\phi_0$, corresponding to zero size random perturbations. Similarly, we will say that $\Gamma \subseteq \operatorname{supp} \phi$ is a *stochastic dynamical interval* if the dynamical interval $\Lambda$ associated with $\phi$ is such that $\Lambda \subseteq \Gamma$ and, additionally, $\Gamma$ is the smallest $\mathbb{P}$-a.s. invariant interval under the RDS defined by $\phi_\epsilon$. In the sequel we will provide conditions guaranteeing the existence of such a stochastic dynamical interval (see Lemma 2.1).

A probability measure $\tilde{\mu}$ on $X \times \Omega$ is *invariant* under the random dynamical system $\phi_\epsilon$ if the following two conditions hold

1. $\Theta\tilde{\mu} = \tilde{\mu}$,

2. $\pi_\Omega \tilde{\mu} = \mathbb{P}$,

where $\pi_\Omega \tilde{\mu}$ denotes the marginal of $\tilde{\mu}$ on $(\Omega, \mathcal{F})$. Let $\psi_\epsilon : X \times \Omega_\epsilon \to X$ be the map given by

$$\psi_\epsilon(x, y) = \phi(x) + y \ .$$

A Borel probability measure $\mu$ on $X$ is *invariant for the random map* $\psi_\epsilon$ if

$$\mu(A) = \int_{\Omega_\epsilon} \mu\left(\psi_\epsilon^{-1}(A)\right) \mathrm{d}\eta_\epsilon(y) \ ,$$

for every Borel set $A \subseteq X$.

Let $A\Delta B$ denote the symmetric difference $(A \backslash B) \cup (B \backslash A)$. We say that an invariant measure $\mu$ on $X$ is *ergodic* if $\mu(A\Delta\psi_\epsilon^{-1}(A)) = 0$, for $\eta_\epsilon$ a.e. $y \in [-\epsilon, \epsilon]$, implies that $\mu(A)$ is either 0 or 1.

The *Lyapunov exponent* of the RDS induced by $\phi_\epsilon$ is given by

$$\lambda(\omega, x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|\phi'(\tilde{x}_k)| \ ,$$

where $\tilde{x}_k$ is a sequence of the form (2.12). By Birkhoff's ergodic theorem, the Lyapunov exponent $\lambda(\omega, x)$ exists $\mu \times \mathbb{P}$ a.s., provided $\mu$ is invariant for the random map $\psi_\epsilon$. Moreover, if $\mu$ is also ergodic, then $\lambda(\omega, x)$ is $\mu \times \mathbb{P}$ a.s. equal to the the constant

$$\lambda = \int_X \ln|\phi'(x)| \mathrm{d}\mu(x) \ .$$

In what follows, we will find mild conditions under which the RDS defined by the action of the cocycle $\phi_\epsilon$ has an ergodic invariant measure with a positive Lyapunov exponent for almost every point of its dynamical interval $\Gamma$.

### 2.2.2 Stochastic dynamical interval

In this section we will discuss the existence of a stochastic dynamical interval for the random dynamics defined by the cocycle $\phi_\epsilon$.

**Lemma 2.2.** *There exists $\epsilon_{\max} > 0$ such that for every $\epsilon \in [0, \epsilon_{\max}]$, the RDS induced by the cocycle $\phi_\epsilon$ has a stochastic dynamical interval $\Gamma \subset \mathbb{R}^+$.*

*Proof.* Since the dynamical interval associated with the unimodal map $\phi$ is given by $\Lambda = [\phi^2(x^*), \phi(x^*)]$, we obtain that the right most point of the stochastic dynamical interval $\Gamma$ is

$$x_R = \phi(x^*) + \epsilon \ .$$

We now note that $x_R$ is mapped into $\phi(\phi(x^*) + \epsilon)$ by the map $\phi$. Thus, a candidate for the left most point of $\Gamma$ turns out to be

$$x_L = \phi(\phi(x^*) + \epsilon) - \epsilon \ .$$

Hence, for the set

$$\Gamma = [\phi(\phi(x^*) + \epsilon) - \epsilon, \phi(x^*) + \epsilon]$$

to be *invariant* under the random dynamics defined by $\phi_\epsilon$, it is enough to check that

$$\phi(\phi(\phi(x^*) + \epsilon) - \epsilon) > \phi(\phi(x^*) + \epsilon) \ . \tag{2.13}$$

Note that if $\epsilon = 0$, the condition above reduces to $\phi^3(x^*) > \phi^2(x^*)$, which always holds since the deterministic dynamical interval $\Lambda = [\phi^2(x^*), \phi(x^*)]$ is invariant under the dynamics of $\phi$.

Since $\phi$ is a $C^2$ function (in supp $\phi$) by assumption (A), then by the implicit function theorem we obtain that there exists $\epsilon^* > 0$ such that

$$\phi(\phi(\phi(x^*) + \epsilon^*) - \epsilon^*) = \phi(\phi(x^*) + \epsilon^*) \ .$$

In particular, for every $0 < \epsilon < \epsilon^*$ we have that condition (2.13) holds.

The result follows from taking

$$\epsilon_{\max} = \min\{\phi^2(x^*), b - \phi(x^*), \epsilon^*\} \ ,$$

where $b - \phi(x^*)$ is identified with $+\infty$ in the case where supp $\phi = \mathbb{R}^+$. □

**Remark 2.1.** *In the case where $\epsilon > \epsilon_{\max}$, there may be orbits of the RDS induced by the cocycle $\phi_\epsilon$ that leave the stochastic dynamical interval $\Gamma$ of the previous lemma with strictly positive probability. Such orbits are captured by the point $x = 0$ in the boundary of supp $\phi$ with positive probability. Moreover, if $\phi$ has an (absolutely continuous) SRB measure with support $\Lambda = [\phi^2(x^*), \phi(x^*)]$, then for Lebesgue almost every point $x_0 \in \Lambda$, the orbit of $x_0$ by the RDS induced by $\phi_\epsilon$ is captured by $x = 0$ with full probability.*

### 2.2.3 Ergodic measure

For a given $\epsilon > 0$, denote the closed $\epsilon$-neighborhood of a set $A \in \mathcal{B}(X)$ by $B_\epsilon(A) = \{x \in X : \mathrm{dist}(x, A) \le \epsilon\}$. From this point onwards, we will assume that $\epsilon > 0$ is such that a stochastic dynamic interval $\Gamma$ exists for the RDS defined by $\phi_\epsilon$. For any $x \in \Gamma$ and $A \in \mathcal{B}(\Gamma)$, let

$$p(x, A) = \frac{1}{2\epsilon} m(A \cap B_\epsilon(\phi(x))) \tag{2.14}$$

be the transition probability representing the likelihood of a point $x \in \Gamma$ to be mapped onto a set $A \in \mathcal{B}(\Gamma)$ by the random dynamical system induced by $\phi_\epsilon$. We call $p(x, A)$ the *transition kernel*. A probability measure $\mu \in \mathcal{P}(\Gamma)$ is said to be *stationary* under the Markov process associated with $\phi_\epsilon$ if

$$\mu(A) = \int_\Gamma p(x, A)\, \mathrm{d}\mu(x) \tag{2.15}$$

for every $A \in \mathcal{B}(\Gamma)$. We note that a stationary measure as defined above is invariant in average over randomly perturbed orbits. In particular, a stationary measure for the Markov chain induced by the dynamics of $\phi_\epsilon$ is an invariant measure for the corresponding random map $\psi_\epsilon$. Additionally, a $\phi_\epsilon$ invariant measure $\tilde{\mu} = \mu \times \mathbb{P}$ is ergodic if and only if $\mu$ is ergodic (see the monographs [5, 32] for further details).

**Lemma 2.3.** *Let $\phi$ be such that assumption (A) holds, then for every $0 < \epsilon < \epsilon_{\max}$ the RDS induced by $\phi_\epsilon$ admits a stationary measure $\mu$ that is absolutely continuous with respect to the Lebesgue measure.*

*Proof.* Since for every $x \in \Gamma$ and every $A \in \mathcal{B}(\Gamma)$ the transition kernel $p(x, A)$ is a continuous function of both variables and the map $x \mapsto p(x, A)$ is continuous, a stationary measure $\mu$ is guaranteed to exist [32]. Moreover, any stationary measure $\mu$ is absolutely continuous with respect to the Lebesgue measure $m$, $\mu << m$, since for any $\epsilon > 0$ we have that

$$\mu(A) \leq \max_{x\in\Gamma} p(x, A) \leq \frac{1}{2\epsilon} m(A) \ .$$

Therefore, there exists a density function $\rho$ such that

$$\mu(A) = \int_A \rho(x) \ \mathrm{d}m(x) \ , \tag{2.16}$$

concluding the proof. □

The following theorem provides conditions guaranteeing the existence and uniqueness of an invariant ergodic measure $\mu$ on $\Gamma$ for the RDS induced by $\phi_\epsilon$. Its proof uses the strategy developed in [34] by Lian and Stenlund, extending it to the setup under consideration here. Before providing its statement, let us introduce the following notation:

$$S_{\phi,K} = \{x \in \Lambda : |\phi'(x)| < K\} \ ,$$

where $\Lambda$ denotes the (deterministic) dynamical interval of $\phi$.

**Theorem 2.5.** *Let $\phi$ be such that assumption (A) holds and let $\epsilon_{\max}$ be as given in Lemma 2.2. If $m(S_{\phi,2}) < 2\epsilon_{\max}$, then for every $\epsilon$ such that $m(S_{\phi,2})/2 < \epsilon < \epsilon_{\max}$, the random dynamical system induced by $\phi_\epsilon$ admits a unique invariant measure which is both ergodic and equivalent to the Lebesgue measure $m$ on $\Gamma$.*

*Proof.* According to Lemma 2.3, we know that there exists a stationary measure with support equal to $\Gamma$ for the $\epsilon$-perturbed system defined by $\phi_\epsilon$. Moreover, such measure is invariant and absolutely continuous with respect to the Lebesgue measure on $\Gamma$. Also, if there exists an ergodic stationary measure which is equivalent to the Lebesgue measure, it is unique. Hence, we only need to prove that the absolutely continuous invariant measure $\mu$ is equivalent to $m$ and ergodic.

For Lebesgue almost every point $x \in \mathrm{supp}\mu$, since $\mathrm{d}\mu = \rho \ \mathrm{d}m$ is stationary, we have that

$$\mu(\mathrm{supp}\mu) = \int_\Gamma p(x, \mathrm{supp}\mu) \ \mathrm{d}\mu(x) = \int_{\mathrm{supp}\mu} p(x, \mathrm{supp}\mu)\rho(x) \ \mathrm{d}m(x) \ .$$

Moreover, identity (2.16) implies that

$$\mu(\mathrm{supp}\mu) = \int_{\mathrm{supp}\mu} \rho(x) \ \mathrm{d}m(x) \ .$$

Hence, we obtain that

$$\int_{\text{supp}\mu} (1 - p(x, \text{supp}\mu))\rho(x)\ \mathrm{d}m(x) = 0$$

and thus

$$p(x, \text{supp}\mu) = 1\ .$$

Since $\mu << m$, we get that $m(\text{supp}\mu) > 0$. To prove ergodicity, for any set $A \in \mathcal{B}(\Gamma)$ invariant mod $\mu$ with $\mu(A) > 0$ we need to check that $\mu(A) = 1$. Since $\mu << m$, it is sufficient to show that $m(A) = m(\Gamma)$. Since supp$\mu$ is an invariant subset of $\Gamma$ mod $\mu$, we assume without loss of generality that $A \subset \text{supp}\mu$.

The rest of this proof is divided into the following two steps:

(1) For every $k \geq 0$, consider the sequence $I_{k+1} = B_\epsilon(\phi(I_k))$, where $I_0 = B_\epsilon(\phi(x))$ and $x \in A$. Then, we have $m(I_k \backslash A) = 0$.

(2) There exist $n > 0$ and $\epsilon_0 > 0$ such that for every $\epsilon_0 < \epsilon < \epsilon_{\text{max}}$ and every $k > n$, we have $I_k = \Gamma$

Hence, by proving the two statements above, we conclude that an ergodic measure exists for the RDS defined by $\phi_\epsilon$. Moreover, we also obtain that the probability measure $\mu$ is equivalent to the Lebesgue measure $m$ on $\Gamma$.

1. By considering $A \subset \text{supp}\mu$, the statements $\mu$-a.e. and $m$-a.e. are equivalent. Since $A$ is assumed to be invariant, then for a.e. $x \in A$ we have that $p(x, A) = 1$. Pick such an $x$. Thus, by defining the interval $I_0 = B_\epsilon(\phi(x))$ we have $m(I_0 \backslash A) = 0$, because

$$p(x, A) = \frac{1}{2\epsilon} m\left(A \cap B_\epsilon(\phi(x))\right) = 1\ ,$$

by invariance of $A$. Also, since $A \subset \text{supp}\mu$ and $m(A \cap I_0) = 2\epsilon$, by absolute continuity of $\mu$ with respect to $m$ we conclude that $\mu(A \cap I_0) > 0$. By invariance of $A$, we also have that $p(y, A) = 1$ for a.e. $y \in A \cap I_0$. Denote the set of such $y$ by $\tilde{I}_0$. Then $I_0 = \tilde{I}_0 \cup N_0$ for some $m$-null set $N_0$. Also for any $k \geq 1$, we define $I_k = B_\epsilon(\phi(I_{k-1}))$ and

$$\tilde{I}_k = \{y \in I_k \cap A :\ p(y, A) = 1\} \subset A$$

inductively. Then by induction we obtain that $I_k = \tilde{I}_k \cup N_k$ for some $m$-null set $N_k$. Note also that $I_k$ contains a countable dense set, which we will denote by $\hat{I}_k$. Since $I_k$ and $I_{k-1} = \tilde{I}_{k-1} \cup N_{i-1}$

are closed intervals and by induction $m(N_{k-1}) = 0$, then

$$I_k = \bigcup_{y \in \hat{I}_{k-1}} B_\epsilon(\phi(y)) \cup \partial I_k \ ,$$

where $\partial I_k$ denote the boundary of $I_k$. Also, for each $y \in \tilde{I}_{k-1}$ we have $p(y, A) = 1$, which from (2.14) implies

$$m(B_\epsilon(\phi(y)) \backslash A) = 0 \ .$$

Since $\hat{I}_{k-1}$ is countable, we conclude that $m(I_k \backslash A) = 0$. Moreover, we also obtain that $m(I_k \backslash \tilde{I}_k) = 0$ by invariance of $A$.

2. Note that for any $K > 2$, if $\epsilon$ is such that $Km(S_{\phi,K})/4 < \epsilon < \epsilon_{\max}$, then for any interval $I \subset \Gamma$, we have that

$$\begin{aligned} m\left(B_\epsilon(\phi(I))\right) &\geq \min\{m(\Gamma), 2\epsilon + m\left(\phi(I \cap (S_{\phi,K})^c)\right)\} \\ &\geq \min\{m(\Gamma), 2\epsilon + \frac{K}{2}\, m\left(I \cap (S_{\phi,K})^c\right)\} \\ &\geq \min\{m(\Gamma), 2\epsilon + \frac{K}{2}\left(m(I) - m(S_{\phi,K})\right)\} \\ &\geq \min\{m(\Gamma), \frac{K}{2}\, m(I)\} \ . \end{aligned}$$

The above estimate shows that the interval $I_k$ grows exponentially with $k$ until it covers $\Gamma$. Thus, there exists a positive number $n$ such that for $k \geq n$, we have $I_k = \Gamma$.

Hence, $\mu$ is an ergodic measure and we have $m(\mathrm{supp}\mu) = m(\Gamma)$ which implies that the ergodic probability measure $\mu$ and the Lebesgue measure $m$ are equivalent. □

### 2.2.4 Positive Lyapunov exponent

Let $g : X \to X$ be the map defined by

$$g(x) = L\phi(x) \ , \tag{2.17}$$

where $L \geq 1$ and $\phi$ satisfies assumption (A). From this point onwards we will use the same notation for the RDS defined by the unimodal map $g$ as we have used in Section 2.2.1 for the RDS associated with the unimodal map $\phi$.

Note that for every $K \geq 1$, the map $g$ is uniformly expanding on the complement of the set

$$S_{g,K} = \{x \in \Lambda_g : \ |g'(x)| \leq K\} \ ,$$

where $\Lambda_g$ denotes the deterministic dynamical interval associated with the unimodal map $g$. For every $x \in S_{g,K}$, since $-K/L \leq \phi'(x) \leq K/L$, there exists a sufficiently large $L$ such that the set $S_{g,K}$ is equal to the union of at most two disjoint intervals, i.e.

$$S_{g,K} = J_{g,K} \dot{\sqcup}\, T_{g,K} \ ,$$

where $J_{g,K}$ is the (non-empty) connected component of $S_{g,K}$ containing the critical point $x^*$ and $T_{g,K}$ is the (possibly empty) connected component of $S_{g,K}$ not containing the critical point (occurring, for instance in the case where supp $\phi$ is not bounded).

Let $m$ be the Lebesgue measure. For a sufficiently large $L$, the length of the interval $J_{g,K}$ can be estimated using a Taylor series expansion around the critical point $x^*$. Hence, we obtain that the length of the interval $J_{g,K}$ satisfies:

$$m(J_{g,K}) = \frac{2K}{|g''(x^*)|} + O((K/L)^2) \ . \tag{2.18}$$

We are now going to describe how to compute the length of the tail set $T_{g,K}$. We start by noting that whenever $\phi$ is a concave map we have that $T_{g,K}$ is the empty set and thus

$$m(T_{g,K}) = 0 \ .$$

On the other hand, if $\phi$ is not a concave map, then by item 5 in Assumption (A), there exists $\tilde{x} > x^*$ such that $\phi''(x) < 0$ for every $x < \tilde{x}$ and $\phi''(x) \geq 0$ for every $x > \tilde{x}$. Let $\Gamma_g$ be the stochastic dynamical interval associated with the unimodal map $g$ and denote by

$$x_g^+ = g(x^*) + \epsilon$$

the right most point of $\Gamma_g$. Let us define $x_{g,K}^-$ as the unique solution of

$$g'(x_{g,K}^-) = -K$$

such that $x_{g,K}^- > \tilde{x}$. If $x_{g,K}^- > x_g^+$, then we again have that $T_{g,K}$ is the empty set and $m(T_{g,K}) = 0$. If instead $x_{g,K}^- < x_g^+$, we get that

$$T_{g,K} = [x_{g,K}^-, x_g^+]$$

and thus

$$m(T_{g,K}) = x_g^+ - x_{g,K}^- \ .$$

Moreover, note that the maps $g(x)$ and $\phi(x)$ differ only by the multiplicative factor $L \geq 1$. Hence, assumption (A) holds for the map $g(x)$ whenever it holds for the map $\phi(x)$. Let $\epsilon_{\max}(L)$ be the value defined in Lemma 2.2 when applied to the map $g$ in (2.17) for a fixed value $L \geq 1$. Thus, according to Theorem 2.5, if $m(S_{g,2}) < 2\epsilon_{\max}(L)$ then for every $m(S_{g,2})/2 < \epsilon < \epsilon_{\max}(L)$, the $\epsilon$-perturbed RDS defined by the auxiliary map $g(x)$ defined in (2.17) admits a unique ergodic invariant measure.

Let $\mathcal{E}_g$ be the set of all values $L \geq 1$ such that the RDS associated with the unimodal map $g$ has a non-empty stochastic dynamical interval $\Gamma_g \subseteq \mathbb{R}^+$. For each $L \in \mathcal{E}_g$ and $K \in (1,2)$, define

$$\epsilon_0(L,K) = \frac{K}{\inf_{J_{g,1}} |g''|} + \frac{x_g^+ - x_{g,K}^-}{2} + \frac{1}{\ln K}\left(\frac{1}{\inf_{J_{g,1}} |g''|} + \frac{1}{\inf_{T_{g,1}} |g''|}\right) .$$

The following theorem provides conditions ensuring positivity of the Lyapunov exponent for the RDS defined by the cocycle $g_\epsilon$.

**Theorem 2.6.** *Let $\phi$ be such that assumption (A) holds and let $g$ be as given in* (2.17). *For every $L \in \mathcal{E}_g$ and $K \in (1,2)$ such that $\epsilon_0(L,K) < \epsilon_{\max}(L)$ and every $\epsilon$ such that $\epsilon_0(L,K) < \epsilon < \epsilon_{\max}(L)$, the RDS defined by the cocycle $g_\epsilon$ has a positive lower bound $\beta$ for its Lyapunov exponent $\lambda$, given by*

$$\beta = \left(1 - \frac{K}{\epsilon \ \inf_{J_{g,1}} |g''|} - \frac{x_g^+ - x_{g,K}^-}{2\epsilon}\right) \ln K - \frac{1}{\epsilon \ \inf_{J_{g,1}} |g''|} - \frac{1}{2\epsilon \ \inf_{T_{g,1}} |g''|} .$$

*Proof.* The Lyapunov exponent of the RDS defined by $g_\epsilon$ is equal to the constant

$$\lambda = \int_X \ln |g'(x)| \mathrm{d}\mu(x) .$$

To obtain a lower estimate for the Lyapunov exponent $\lambda$, we need to start by bounding the invariant density $\rho$ from above. The transition kernel $p(x, \cdot)$ in (2.14) is a Borel probability measure such that

$$p(x, A) = \frac{1}{2\epsilon} m(A \cap B_\epsilon(g(x))) = \int_A \psi(x,y) \mathrm{d}m(y) ,$$

where the density function $\psi(x, \cdot)$ is given by the Radon-Nikodym derivative

$$\psi(x,y) = \frac{dp(x,\cdot)}{dm}|_y = \frac{1}{2\epsilon} 1_{B_\epsilon(g(x))}(y)$$

and $1_A(y)$ denotes the indicator function of the set $A$. By substituting $A = B_\delta(x_0)$ in (2.15) for the stationary Borel measure $\mu$ on $\Gamma_g$, we get

$$\mu(B_\delta(x_0)) = \int_{\Gamma_g} p(x, B_\delta(x_0)) \ \mathrm{d}\mu(x) . \qquad (2.19)$$

Since $\mu$ is absolutely continuous with density $\rho$, we obtain that

$$\lim_{\delta\to 0^+} \frac{\mu(B_\delta(x_0))}{m(B_\delta(x_0))} = \rho(x_0) \ .$$

Dividing (2.19) by $m(B_\delta(x_0))$ and taking the limit when $\delta \to 0^+$, we get that

$$\rho(x_0) = \int_{\Gamma_g} \psi(x, x_0) \mathrm{d}\mu(x) \ . \tag{2.20}$$

Thus, we obtain that for every $x \in \Gamma$ the following inequality holds:

$$\rho(x) \leq \frac{1}{2\epsilon} \ , \tag{2.21}$$

i.e. the density $\rho$ has a finite upper bound.

For the second part of the proof, take $K > 1$ and note that

$$\begin{aligned} \mu(J_{g,K}) &= \int_{\Gamma_g} p(x, J_{g,K}) \mathrm{d}\mu(x) \\ &= \int_{\Gamma_g} p(x, J_{g,K}) \rho(x) \mathrm{d}m(x) \end{aligned}$$

Thus, by definition of the transition kernel $p(x, A)$, we have that

$$\begin{aligned} \mu(J_{g,K}) &\leq m(J_{g,K}) \sup_x \rho(x) \\ &\leq \frac{2K}{|g''(x^*)|} \frac{1}{2\epsilon} \\ &\leq \frac{K}{\epsilon \ \inf_{J_{g,K}} |g''(x)|} \ . \end{aligned}$$

Similarly, based on the remark preceding the statement of Theorem 2.6 we have that

$$\begin{aligned} \mu(T_{g,K}) &\leq m(T_{g,K}) \sup_x \rho(x) \\ &= \frac{x_g^+ - x_{g,K}^-}{2\epsilon} \ . \end{aligned}$$

Moreover, using integration by parts and the change of variables $r = |g'(x)|$, we conclude that there exist constants $C_1$ and $C_2$ defined by

$$C_1 = -\frac{2}{\inf_{J_{g,1}} |g''(x)|} \qquad \text{and} \qquad C_2 = -\frac{1}{\inf_{T_{g,1}} |g''(x)|}$$

and such that

$$\int_{I_{g,1}} \ln|g'(x)|\mathrm{d}m(x) \geq C_1$$

and

$$\int_{T_{g,1}} \ln|g'(x)|\mathrm{d}m(x) \geq C_2 \ .$$

To find a lower bound for $\lambda$, fix a number $1 < K < 2$ and take $L$ sufficiently large such that

$$S_{g,K} = J_{g,K} \dot{\sqcup}\, T_{g,K} \ .$$

Note also that $J_{g,1} \subseteq J_{g,K}$. Thus, we obtain that

$$\begin{aligned}
\lambda &\geq \int_{(S_{g,K})^c} \ln|g'|\mathrm{d}\mu + \int_{J_{g,1}} \ln|g'|\mathrm{d}\mu + \int_{T_{g,1}} \ln|g'|\mathrm{d}\mu \\
&\geq \big(\mu(\Lambda_g) - \mu(J_{g,K}) - \mu(T_{g,K})\big)\ln K \\
&\quad + \sup_{J_{g,1}} \rho(x) \int_{J_{g,1}} \ln|g'|\mathrm{d}m + \sup_{T_{g,1}} \rho(x) \int_{T_{g,1}} \ln|g'|\mathrm{d}m \ .
\end{aligned}$$

Therefore, combining the inequality (2.21) and identity (2.18) with the previous inequality, we obtain

$$\lambda \geq \left(1 - \frac{K}{\epsilon \ \inf_{J_{g,1}}|g''|} - \frac{x_g^+ - x_{g,K}^-}{2\epsilon}\right)\ln K - \frac{1}{\epsilon \ \inf_{J_{g,1}}|g''|} - \frac{1}{2\epsilon \ \inf_{T_{g,1}}|g''|} \ . \tag{2.22}$$

Hence, we obtain that if $\epsilon_0(L,K)$ is given by

$$\epsilon_0(L,K) = \frac{K}{\inf_{J_{g,1}}|g''|} + \frac{x_g^+ - x_{g,K}^-}{2} + \frac{1}{\ln K}\left(\frac{1}{\inf_{J_{g,1}}|g''|} + \frac{1}{\inf_{T_{g,1}}|g''|}\right) \ ,$$

if $L$ is sufficiently large and $\epsilon$ is such that $\epsilon_0(L,K) < \epsilon < \epsilon_{\max}$, the $\epsilon$-perturbed RDS defined by the auxiliary map $g$ has a positive Lyapunov exponent for a.e. $x \in \Gamma_g$ with lower bound given by the right hand side of (2.22). □

**Corollary 2.1.** *Let $\phi$ be such that assumption (A) holds and let $\epsilon_{\max}$ be as given in Lemma 2.2. Suppose that for fixed $1 < K < 2$ we have that*

$$\phi'(\phi(x^*) + \epsilon_{\max}) < -K \tag{2.23}$$

*and let*

$$\epsilon_0 := \left(K + \frac{1}{\ln K}\right)\frac{1}{\inf_{J_{\phi,1}}|\phi''|} < \epsilon_{\max} \ .$$

*Then for every $\epsilon$ such that $\epsilon_0 < \epsilon < \epsilon_{\max}$ the RDS defined by $\phi_\epsilon$ has a positive Lyapunov exponent.*

*Proof.* Inequality (2.23) ensures that $T_{\phi,K} = \emptyset$ for every fixed $1 < K < 2$. Thus, from the proof of Theorem 2.6, we get

$$\epsilon_0 = \left(K + \frac{1}{\ln K}\right) \frac{1}{\inf_{J_{\phi,1}} |\phi''|} \ .$$

Hence, if $\epsilon_0 < \epsilon < \epsilon_{\max}$, the RDS defined by the map $\phi$ has a positive Lyapunov exponent for a.e. $x \in \Gamma$. □

The following example exhibits one relevant family of unimodal maps to which Theorem 2.6 applies.

**Example 2.1.** *Consider the following two-parameter family of quadratic maps:*

$$\phi_{\mu,a} = \frac{4\mu}{a^2} x(a-x) \ .$$

*We are interested in the following regime of parameters: $a > 0$ large and $\mu$ close to, but larger than, $a/2$. It is easy to see that the critical point of the unimodal maps $\phi_{\mu,a}$ is $x^* = a/2$. Moreover, computing the first two iterates of the critical point, we obtain that the deterministic dynamical interval associated with this family of maps is*

$$\Lambda_{\phi_{\mu,a}} = \left[\frac{4\mu^2}{a^2}(a-\mu), \mu\right] \ ,$$

*which has length equal to*

$$m\left(\Lambda_{\phi_{\mu,a}}\right) = \frac{4\mu}{a^2}\left(\mu - \frac{a}{2}\right)^2 \ .$$

*Additionally, following the characterization obtained for $\epsilon^*$ in Lemma 2.2, we get that*

$$\epsilon^* = \frac{a}{\mu}\left(\mu - \frac{a}{2}\right) \ .$$

*Hence, we conclude that*

$$\epsilon_{max} = \min\left\{a - \mu, \frac{4\mu^2}{a^2}(a-\mu), \frac{a}{\mu}\left(\mu - \frac{a}{2}\right)\right\} \ .$$

*It is clear that by making $a$ larger, and consequently, taking larger values of $\mu > a/2$, the bound $\epsilon_{max}$ can be made as large as desired. Furthermore, it is also possible to check that for a fixed length of the deterministic dynamical interval of $\phi_{\mu,a}$, the bound $\epsilon_{max}$ can also be made arbitrarily large, thus fulfilling the conditions of Theorem 2.5, guaranteeing that the RDS obtained by additive random perturbations of the unimodal map $\phi_{\mu,a}$ has an ergodic invariant measure which is equivalent to the Lebesgue measure.*

*Notice that for the family of maps $\phi_{\mu,a}$ under consideration now, the set tail set $T_{\phi_{\mu,a},K}$ is empty for every choice of $K \in (1,2)$. Hence, recalling the definition of $\epsilon_0$ given in Corollary 2.6, we obtain*

$$\epsilon_0 = \frac{a^2}{8\mu}\left(K + \ln(K)\right)$$

*Comparing $\epsilon_0$ with $\epsilon_{max}$, we obtain that $\epsilon_0 < \epsilon_{max}$ provided the following inequality holds:*

$$\mu > \frac{a}{2} + \frac{a}{8}\left(K + \frac{1}{\ln(K)}\right) .$$

*Thus, provided the inequality above is satisfied, then for every $\epsilon \in (\epsilon_0, \epsilon_{max})$ we obtain that the RDS obtained by additive perturbations of $\phi_{\mu,a}$ with maximum size $\epsilon$ has a positive Lyapunov exponent.*

*Even though this example is based in a quadratic map (with support on the compact interval $[0,a]$), it is an easy task to construct an example of a map whose support is the positive half line, by changing $\phi_{\mu,a}$ outside of its dynamical interval in such a way that the conditions in assumption (A) are preserved.*

### 2.2.5 An application to an overlapping generations model

In this section we will consider a two-period overlapping generations (OLG) model like the one introduced in [1] and discuss a potential application for our main results. We provide numerical evidence showing that the bpf map $\phi$ for this OLG model always has a positive Lyapunov exponent for almost all points under a sufficiently large stochastic perturbation.

Let us consider an OLG economy for a population with constant size and such that the proportions of young and old agents remain unchanged over time. We assume that the there exists a representative agent with preferences given by a separable utility function

$$U(c_t, c_{t+1}) = V_1(c_t) + V_2(c_{t+1}) ,$$

where $V_1$ and $V_2$ are the utilities representing the individual preferences concerning consumption in the first and second periods of their life, and $c_t, c_{t+1}$ denote the corresponding consumption in each period. We suppose also that one unit of the good is produced with one unit of the unique productive factor (labor) and let $l_1^*$ and $l_2^*$ denote the agents labor endowments in the first and second periods of their life, respectively. Finally, we assume that there is a risk-free asset that can be purchased by the agents providing a gross return $z_t = 1 + r_t > 1$, where $r_t$ is the interest rate at period $t$, and that the dynamics of the money supply are determined by

$$M_{t+1} = M_t z_{t+1} \ ,$$

for some initial condition $M_0$.

At this point, let us recall some basic notations from the literature (see [1], [14] for further details). A map

$$\tilde{Z} : X \times \mathcal{P}(X) \to \mathbb{R}^n \ ,$$

is called *stochastic excess demand function* if an equilibrium condition is given by the zero set of this map. We can then define a *deterministic excess demand function* $Z : X \times X \to \mathbb{R}^n$ as being given by

$$Z(x_0, x_1) := \tilde{Z}(x_0, \delta_{x_1}) \ ,$$

where $\delta_{x_1}$ denotes the Dirac measure supported at $x_1 \in X$. If $x_0$ is the current value of the state variable and $x_1$ is the expectation for sure of its future value, the equation $Z(x_0, x_1) = 0$ is said to be a deterministic equilibrium condition. A *backward prefect foresight* (**bpf**) map is a function $\phi : X \to X$ such that $Z(\phi(x), x) = 0$ for all $x \in X$.

We impose the following additional conditions on the utility function $V_\tau$, $\tau = 1, 2$ describing the agents preferences.

(**C1**) For each $\tau = 1, 2$ , $V_\tau$ is continuous on $[0, +\infty)$ and twice continuously differentiable on $(0, +\infty)$. Moreover, $V_\tau$ is strictly increasing and strictly concave and satisfies

$$\lim_{c_\tau \to +\infty} V'_\tau(c_\tau) = 0 \ , \qquad \lim_{c_\tau \to 0} V'_\tau(c_\tau) = +\infty \ , \qquad \bar{\Theta} := \frac{V'_1(l^*_1)}{V'_2(l^*_2)} < 1 \ .$$

(**C2**) The Arrow-Pratt relative degree of risk aversion of the old, given by

$$R_{V_2}(x) = -\frac{x V''_2(x)}{V'_2(x)} \ ,$$

is a non-decreasing function of $x$.

See [27] and the references therein for further comments and motivation for the hypothesis (C1)-(C2), guaranteeing that the bpf map $\phi$ is unimodal.

We will now introduce a consumption-savings problem for the representative agent. Let $p_t$ and $p_{t+1}$ denote the prices of the unique good in the economy during the first and second periods of the individual's life. Note that while $p_t$ is known by the individual during the first stage of her life, her knowledge concerning $p_{t+1}$ consists of a probability distribution $\mu_{t+1}$ representing the likelihood of occurrence of particular values of $p_{t+1}$ and reflecting the individual's beliefs about the state of economy during the second period of her life. The agent must choose a consumption plan $c_t, c_{t+1}$ and the first period savings $m_t$ as a solution to the following optimization problem

$$\max_{\{c_t, c_{t+1}, m_t\}} V_1(c_t) + E_{\mu_{t+1}}\left[V_2(c_{t+1})\right] \tag{2.24}$$

subject to the budget constraints

$$p_t c_t + m_t = p_t l_1^*$$
$$p_{t+1} c_{t+1} = p_{t+1} l_2^* + z_t m_t \ .$$

Working out the first order condition for an interior solution of (2.24) leads to

$$-\frac{1}{p_t} V_1'\left(l_1^* - \frac{m_t}{p_t}\right) + E_{\mu_{t+1}}\left[\frac{z_t}{p_{t+1}} V_2'\left(l_2^* + \frac{z_t m_t}{p_{t+1}}\right)\right] = 0 \ . \tag{2.25}$$

Under the monetary equilibrium condition $M_t = m_t$, and introducing the new variable

$$x_t = \frac{M_t}{p_t} \ ,$$

the first order condition (2.25) may be rewritten as

$$\nu_1(x_t) = E_{\mu_{t+1}}\left[\nu_2(x_{t+1})\right] \ , \tag{2.26}$$

where $\nu_1$ and $\nu_2$ are the the auxiliary functions defined by

$$\nu_1(x) = x V_1'(l_1^* - x) \ , \qquad \nu_2(x) = x V_2'(l_2^* + x) \ . \tag{2.27}$$

The equilibrium condition (2.26) defines the stochastic excess demand function

$$\tilde{Z}(x, \mu) = \nu_1(x) - E_\mu\left[\nu_2(x')\right] \ .$$

The deterministic excess demand function associated with $\tilde{Z}$ is then given by

$$Z(x, x') = \tilde{Z}(x, \delta_{x'}) = \nu_1(x) - \nu_2(x') \ ,$$

defining the associated bpf map through the implicit relation

$$Z(\phi(x), x) = 0 \ .$$

Under condition (C1), it is possible to check that the map $\nu_1$ defined in (2.27) is strictly increasing, and thus invertible. Hence, we obtain that the backward perfect foresight map is of the form

$$\phi(x) = \nu_1^{-1}\left(\nu_2(x)\right) \ , \tag{2.28}$$

and the corresponding bpf dynamical system is determined by

$$x_t = \phi(x_{t+1}) = \nu_1^{-1}\left(\nu_2(x_{t+1})\right) \ .$$

This bpf map represents the dynamics of the prices of the unique good in the economy. If conditions $(C1)$ and $(C2)$ hold and there exists $x_0 \geq 0$ such that $R_{V_2}(x_0) > 1$, then in [2] it is proven that the bpf map $\phi$ is a unimodal map with $\phi(0) = 0$ and $\phi'(0) > 1$.

Now, as a particular example, we consider the constant relative risk aversion (CRRA) utility function

$$V_\tau(c) = \frac{c^{1-\alpha_\tau}}{1-\alpha_\tau} \ , \qquad \tau = 1, 2 \ , \tag{2.29}$$

where $\alpha_1 \in (0,1)$ and $\alpha_2 \in (2, +\infty)$ are, respectively, the relative degrees of risk aversion for the representative agent in the first and second periods of her life. Thus, the deterministic excess demand function is given by

$$Z(x, x') = \frac{x}{(l_1^* - x)^{\alpha_1}} + \frac{x'}{(l_2^* + x')^{\alpha_2}} \tag{2.30}$$

and we obtain the bpf map $\phi$ from solving the implicit equation $Z(\phi(x), x) = 0$.

Depending on the values of the parameter in the constant relative risk aversion (CRRA) utility functions (2.29), the associated bpf map $\phi(x)$ defined by (2.28) has an attracting periodic orbit and the dynamics has a negative Lyapunov exponent or the bpf map $\phi$ has an absolutely continuous invariant probability measure and the dynamics has a positive Lyapunov exponent. In the following corollary, we provide conditions under which the $\epsilon$-perturbed RDS $\phi_\epsilon$ has a positive Lyapunov exponent.

**Corollary 2.2.** *Assume that conditions $(C1)$ and $(C2)$ hold and let $\epsilon_{\max}$ and $\epsilon_0$ be the quantities given in Lemma 2.2 and Theorem 2.6, respectively. Then, if $\epsilon_0 < \epsilon_{\max}$, the RDS defined by the bpf map $\phi$ associated with the deterministic excess demand function in (2.30) has a positive Lyapunov exponent whenever the random perturbations have size $\epsilon$ such that $\epsilon \in (\epsilon_0, \epsilon_{\max})$.*

| $\epsilon$ | 0 | 0.005 | 0.05 | 0.1 | 0.14 |
|---|---|---|---|---|---|
| LE | $-0.5901$ | $-0.4041$ | 0.0087 | 0.4818 | 0.7772 |

Table 2.1: Lyapunov exponent of some different $\epsilon$-perturbed RDS $\phi_\epsilon$.

*Proof.* Follows from Theorem 2.6 by noting that Conditions (C1) and (C2) together with the choice of risk aversion parameters $\alpha_1 \in (0,1)$ and $\alpha_2 \in (2,+\infty)$ guarantee the criteria listed in Assumption (A). □

We should remark that the estimates obtained in Theorem 2.6 and Corollary 2.1 are not sharp. Those results can be improved in two directions. On the one hand, the lower bound obtained for the Lyapunov exponent can be made smaller. On the other hand, the range of values of $\epsilon$ to which those results apply is rather restrictive. Indeed, we believe that the Lyapunov exponent is still positive for random perturbations of size $\epsilon$ smaller than $\epsilon_0$. To further support these statements, we will now provide some numerical results.

We fix the parameters $l_1^* = 3.5, l_2^* = 0.6, \alpha_1 = 0.2, \alpha_2 = 5$ in the deterministic excess demand function (2.30). Due to the form of (2.30) and the definition of the bpf map in (2.28), it is possible to check that the critical point of $\phi$ is given by

$$x^* = \frac{l_2^*}{\alpha_2 - 1} \ .$$

For a plot of the bpf map for this choice of parameters, see Figure 2.5.

Using numerical simulation it is also possible to obtain the estimate

$$\epsilon_{\max} \approx 0.147\ldots$$

for the maximum value of $\epsilon$ for which a stochastic dynamical interval still exists for the RDS determined by the cocycle defined by the bpf map $\phi$.

We then estimate the Lyapunov exponent for such RDS using standard numerical methods for one-dimensional maps ([30, 39]) for varing sizes of the perturbation size $\epsilon$. We provide a summary of the outcomes of some of these numerical experiments in the table 2.1.

We remark that the Lyapunov exponent turns from negative to positive when $\epsilon \approx 0.045\ldots$. Additionally, in Figures 2.6 we plot the histograms of the RDS orbits used to compute the Lyapunov exponents in Table 2.1. Note that for values of $\epsilon$ very close to zero, the histograms

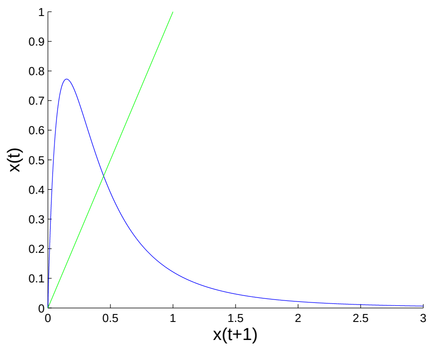


Figure 2.5: The bpf map $\phi$ for fixed values of parameters $l_1^* = 3.5, l_2^* = 0.6, \alpha_1 = 0.2, \alpha_2 = 5$.

look as if the system as a singular measure (in this case having support on a period two orbit). However, for large values of $\epsilon$, these histograms are already good approximations for the densities associated with absolutely continuous stationary measures, whose existence is guaranteed by Theorem 2.5 for sufficiently large $\epsilon$.

### 2.2.6 Conclusions

We have studied the influence of a large additive deviations on a perfect foresight equilibrium path associated with an unimodal map, where the large deviations are modeled through a sequence of independent and identically distributed random variables. To move on with such study, we have obtained general conditions under which an ergodic absolutely continuous invariant measure exists for random dynamical systems defined by i.i.d. additive random perturbations of unimodal maps. Furthermore, under appropriate conditions, we have obtained a positive lower bound for the corresponding Lyapunov exponent for almost every point of the interval.

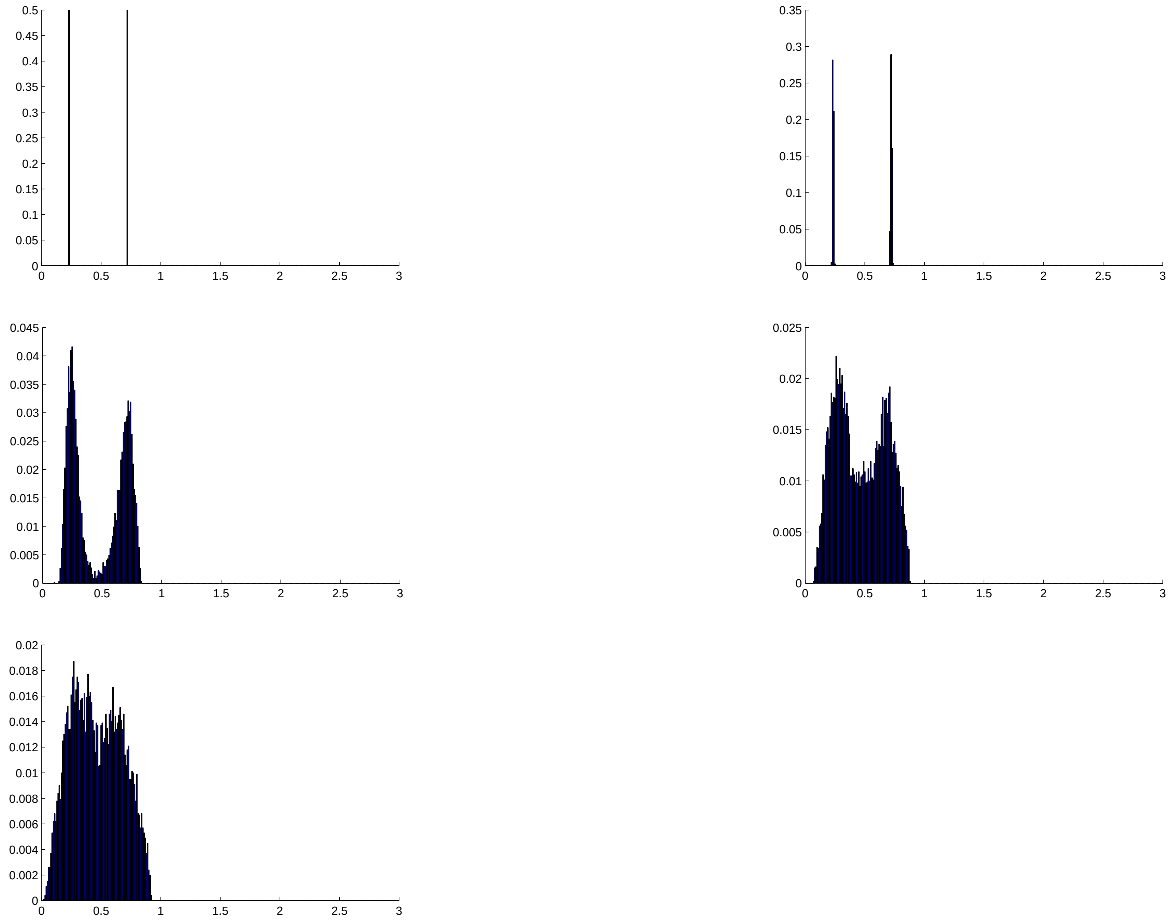


Figure 2.6: Histograms for the distribution of iterations of the bpf map $\phi$ subject to perturbations of maximum size $\epsilon = 0$ (top left), $\epsilon = 0.005$ (top right), $\epsilon = 0.05$ (second row left), $\epsilon = 0.1$ (second row right) and $\epsilon = 0.14$ (bottom).

# Chapter 3

# International trade and game theory

We consider two firms located in different countries selling the same homogeneous good in both countries. In each country there is a tariff on imports of the good produced in the other country. We study the international trade model as a strategic game in the tariffs of the governments. We use the relevant economic quantities as the utilities of these strategic games and we compare the Nash equilibrium with the social optimum equilibrium. The lack of coincidence of these equilibria is a main difficulty in international trade that can be partially dealt with the use of trade agreements.

Then we add uncertainty on the production costs of the firms and we compute the Bayesian-Nash equilibrium. We show that the expected profit of the firms and the expected welfare of the countries increase with the variances of the production costs of both firms. When the production costs are similar enough, we show that this international trade model is like the Prisoner's Dilemma (see [26]) in the sense that the Bayesian-Nash equilibrium consists in both firms imposing tariffs but if both governments do not impose tariffs then both countries will get a higher welfare.

## 3.1 Strategic tariffs in international duopoly game

In this section we study an international trade model consisting of a game with two stages. For the second stage of the game we find the Nash equilibrium for the firms in terms of the home quantities and the export quantities that maximize their profits in a competitive market. Then, using the Nash equilibrium for the firms, we find for the tariffs imposed by the governments the

Nash equilibrium and the social equilibrium using different utilities related with the different economic agents involved in the international trade model.

In subsection 3.1.1 we explain the three typical game outcomes that can arise from comparing the Nash equilibrium with the social equilibrium. In subsection 3.1.2 we define the international duopoly model that we use in this chapter. In subsection 3.1.3 we determine the Nash equilibrium of the second subgame in the case of complete information. In subsection 3.1.4 we find the Nash tariffs and the social tariffs for different utilities related with the different economic agents involved in the international trade model. In subsection 3.1.5 we compute the Nash welfare and the social welfare and we find which of the three typical games occurs depending upon the initial production costs. Finally, in subsection 3.1.6 we do the static analysis of the relevant economic quantities of the international trade model by comparing the perfect Nash equilibrium with the social optimum equilibrium for the welfare of the countries.

### 3.1.1 Strategic tariffs

In this subsection, we introduce the most relevant game theoretical concepts that we will use in the other sections to understand the strategic behaviour of firms, consumers and governments of the countries.

Let $u_i(t_i, t_j)$ and $u_j(t_i, t_j)$ be two relevant economic quantities of the countries $X_i$ and $X_j$ depending only upon the tariffs $t_i$ and $t_j$ imposed by the governments of the two countries. For instance, for every pair of tariffs $(t_i, t_j)$, the functions $u_i(t_i, t_j)$ and $u_j(t_i, t_j)$ can be the profit of the firms or the consumer surplus at the competitive Nash equilibrium for the quantities produced by the firms.

We are going to interpret $u_i(t_i, t_j)$ and $u_j(t_i, t_j)$ as the utilities of a game where the players are the governments of the countries and their actions are the tariffs $(t_i, t_j)$.

The quantity $t_i^{BR}(t_j) \equiv t_i^{BR}(t_j; u)$ is the *best response* of the country $X_i$ for the utility $u_i$, if for all tariffs $t_i$,

$$u_i(t_i^{BR}(t_j), t_j) \geq u_i(t_i, t_j) \ .$$

A pair of tariffs $(t_i^N, t_j^N) \equiv (t_i^N(u), t_j^N(u))$ is a *Nash equilibrium* or a *global strategic optimum*, if for all tariffs $t_i$

$$u_i(t_i^N, t_j^N) \geq u_i(t_i, t_j^N)$$

and for all tariffs $t_j$

$$u_j(t_i^N, t_j^N) \geq u_i(t_i^N, t_j) \ .$$

In other words, a pair of tariffs $(t_i^N, t_j^N)$ is a Nash equilibrium, if

$$t_i^N = t_i^{BR}(t_j^N) \qquad \text{and} \qquad t_j^N = t_j^{BR}(t_i^N) \ .$$

A pair of tariffs $(t_i^P, t_j^P) \equiv (t_i^P(W), t_j^P(W))$ is a *Pareto optimum*, if there is no pair $(t_i, t_j)$ of tariffs such that

$$u_i(t_i, t_j) \geq u_i(t_i^P, t_j^P) \qquad \text{for all} \ \ i, j \in \{1, 2\},$$

and at least one utility $u_i, \ i \in \{1, 2\}$ gets a better payoff with $(t_i, t_j)$ than with $(t_i^P, t_j^P)$, i.e.

$$u_i(t_i, t_j) > u_i(t_i^P, t_j^P) \ .$$

The *social utility* $u_S$ is

$$u_S(t_i, t_j) = u_i(t_i, t_j) + u_j(t_i, t_j) \ .$$

The quantity $t_i^{SR}(t_j) \equiv t_i^{SR}(t_j; u)$ is the *social best response*, if for all tariffs $t_i$

$$u_S(t_i^{SR}(t_j), t_j) \geq u_S(t_i, t_j) \ .$$

A pair of tariffs $(t_i^S, t_j^S) \equiv (t_i^S(u), t_j^S(u))$ is a *social optimum*, if for all tariffs $t_i$

$$u_S(t_i^S, t_j^S) \geq u_S(t_i, t_j^S) \ ,$$

and for all tariffs $t_j$

$$u_S(t_i^S, t_j^S) \geq u_S(t_i^S, t_j) \ .$$

In other words, a pair of tariffs $(t_i^S, t_j^S)$ is a social optimum, if

$$t_i^S = t_i^{SR}(t_j^S) \qquad \text{and} \qquad t_j^S = t_j^{SR}(t_i^S) \ .$$

We observe that a social optimum is a Pareto optimum. For games with a unique Nash equilibrium, we describe the three typical games outcomes when we compare the social optimum with the Nash equilibrium.

(**SE**) **Social equilibrium**: When the social optimum coincides with the Nash equilibrium

$$(t_i^S, t_j^S) = (t_i^N, t_j^N)$$

and the social optimum is the only Pareto optimum. In this case, the individualist Nash choice of the tariffs by the governments leads to a social equilibrium. Hence, a priori there is no need of a trade agreement between the two governments of the two countries.

(**PD**) **Prisoner's dilemma**: When the social optimum $(t_i^S, t_j^S)$ is different from the Nash equilibrium

$$t_i^S \neq t_i^N \qquad \text{or} \qquad t_j^S \neq t_j^N$$

and both utilities are bigger in the social optimum than in the Nash equilibrium,

$$u_i(t_i^S, t_j^S) > u_i(t_i^N, t_j^N) \qquad \text{and} \qquad u_j(t_i^S, t_j^S) > u_j(t_i^N, t_j^N) \ .$$

In this case, the game is like the Prisoner's dilemma, where the Nash strategy leads to a lower outcome for both countries than if they would agree among therein (through a trade agreement) in opting for the social optimum.

(**LW**) **Lose-win social strategies**: When the social optimum $(t_i^S, t_j^S)$ is different from the Nash equilibrium

$$t_i^S \neq t_i^N \qquad \text{or} \qquad t_j^S \neq t_j^N$$

and one of the utilities is bigger in the social optimum and the other utility is bigger in the Nash equilibrium,

$$u_i(t_i^S, t_j^S) < u_i(t_i^N, t_j^N) \qquad \text{and} \qquad u_j(t_i^S, t_j^S) > u_j(t_i^N, t_j^N) \ .$$

In this case, the governments can implement an external mechanism (trade agreement) that will force them to opt for the social optimum in such a way that the country with the advantage in its utility compensates the loss in the utility of the other country and can also give some extra benefit to persuade the other country to implement the social equilibrium.

### 3.1.2 International duopoly model

In this subsection, we introduce the relevant economic quantities of the international duopoly model.

The international duopoly model is a game with two stages (subgames). In the first stage, both governments choose simultaneously their Nash or social tariffs for a utility given by a relevant economic quantity; and, in the second stage, the firms choose simultaneously their home and export quantities to maximize competitively their profits.

The *home consumption* $h_i$ is the quantity produced by the firm $F_i$ and consumed in its own country $X_i$. The *export* $e_i$ is the quantity produced by the firm $F_i$ and consumed in the country $X_j$ of the other firm $F_j$, where $i, j \in \{1, 2\}$ with $i \neq j$. The *tariff rate* $t_i$ is determined by the government of country $X_i$ on the import quantity $e_j$. The *total quantity* $q_i$ produced by firm $F_i$ is

$$q_i \equiv q_i(h_i, e_i) = h_i + e_i \ .$$

The *aggregate quantity* $Q_i$ sold on the market in the country $X_i$ is

$$Q_i \equiv Q_i(h_i, e_j) = h_i + e_j \ .$$

The *inverse demand* $p_i$ in the country $X_i$ is

$$p_i \equiv p_i(h_i, e_j) = \alpha - Q_i \ ,$$

where $\alpha \geq 0$ is the *demand intercept*. The *payoff* $\pi_i$ of firm $F_i$ is

$$\pi_i \equiv \pi_i(h_i, e_i, h_j, e_j; t_i, t_j) = (p_i - c_i)h_i + (p_j - c_i)e_i - t_j e_i \ ,$$

where $c_i \geq 0$ is the firm $F_i$'s *unitary production cost*. The *custom revenue* $CR_i$ of the country $X_i$ is given by

$$CR_i \equiv CR_i(e_j; t_i) = t_i e_j \ .$$

The *consumer surplus* $CS_i$ in the country $X_i$ is given by

$$CS_i \equiv CS_i(h_i, e_j) = \frac{1}{2} Q_i^2 \ .$$

The *welfare* $W_i$ of the country $X_i$ is

$$W_i \equiv W_i(h_i, e_i, h_j, e_j; t_i, t_j) = CR_i + CS_i + \pi_i \ .$$

### 3.1.3 Second stage Nash equilibrium

In this subsection, we give an original presentation of the Nash equilibrium of the second subgame in the case of complete information, i.e. when both firms have full information on their and others utility functions.

Let $i, j \in \{1, 2\}$ with $i \neq j$. Define

$$T_i \equiv T_i(c_i, c_j) = (\alpha + c_i - 2c_j)/2 \ ,$$
$$T_j \equiv T_j(c_i, c_j) = (\alpha + c_j - 2c_i)/2 \ .$$

**Assumption (A1)**: For all $i \in \{1, 2\}$, $T_i > 0$ and

$$0 \leq t_i \leq T_i \ .$$

By assumption (A1), we obtain that

$$\alpha - c_i = \frac{2(T_i + 2T_j)}{3} > 0 \ ;$$

and if $c_i = c_j = c$ then

$$T_i = T_j = \frac{\alpha - c}{2} > 0 \ .$$

The *best response* $(h_i^{BR}(e_j), e_i^{BR}(h_j; t_j))$ of the firm $F_i$ is the solution of

$$(h_i^{BR}(e_j), e_i^{BR}(h_j; t_j)) = \arg\max_{(h_i, e_i)} \pi_i(h_i, e_i, h_j, e_j; t_i, t_j).$$

Hence,

$$\begin{cases} h_i^{BR}(e_j) = \frac{\alpha - e_j - c_i}{2} \\ \\ e_i^{BR}(h_j; t_j) = \frac{\alpha - h_j - c_i - t_j}{2} \end{cases} .$$

The *Nash equilibrium* $(h_i^N(t_i), e_i^N(t_j); h_j^N(t_j), e_j^N(t_i))$ is the solution of

$$\begin{cases} (h_i^N(t_i), e_i^N(t_j)) = (h_i^{BR}(e_j^N(t_i)), e_i^{BR}(h_j^N(t_j); t_j)) \\ \\ (h_j^N(t_j), e_j^N(t_i)) = (h_j^{BR}(e_i^N(t_j)), e_j^{BR}(h_i^N(t_i); t_i)) \end{cases} .$$

Under assumption (A1), for every $t_i \in [0, T_i]$ and every $t_j \in [0, T_j]$, the home $h_i^N(t_i)$ and export $e_i^N(t_j)$ quantities for the firms at the Nash equilibrium (see [26]) are

$$h_i^N(t_i) \equiv h_i^N(c_i, c_j; t_i) = \frac{2T_j + t_i}{3} \ ,$$

$$e_i^N(t_j) \equiv e_i^N(c_i, c_j; t_j) = \frac{2(T_j - t_j)}{3} \ .$$

We observe that the export quantity $e_i^N(t_j)$ is positive if, and only if, assumption (A1) holds.

### 3.1.4 Strategic games

In this subsection, we will analyse the advantages and disadvantages of the use of tariffs for the firms, the consumers and the governments of the countries. To do it, we will use the relevant economic quantities as utilities $u_i(t_i, t_j)$ and $u_j(t_i, t_j)$ of a game where the players are the governments of the countries and their actions are the tariffs $(t_i, t_j)$. For each pair of utilities that we will consider, we will find which of the three typical games occurs: social equilibrium (SE), prisoner's dilemma (PD), or lose-win social strategies (LW).

**Tariff effects in produced quantities and prices**

The home quantity $h_i^N(t_i)$ increases with the tariff $t_i$, and so

$$t_i^{BR}(t_j; h) = T_i \qquad \text{and} \qquad (t_i^N(h), t_j^N(h)) = (T_i, T_j) \ .$$

The social utility $h_S^N(t_i, t_j)$ is

$$h_S^N(t_i, t_j) = \frac{2(T_i + T_j) + t_i + t_j}{3}$$

and so

$$t_i^{SR}(t_j; h) = T_i \qquad \text{and} \qquad (t_i^S(h), t_j^S(h)) = (T_i, T_j) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(h) = t_i^N(h) = T_i \ .$$

Therefore, the game, with utility $u_i = h_i$, is of the type **SE**.

The export quantity $e_i^N(t_j)$ decreases with the tariff $t_j$, but does not depend upon the tariff $t_i$. Hence, every tariff $t_i$ is a best response and so every pair of tariffs is a Nash equilibrium

$$t_i^{BR}(t_j; e) \in [0, T_i] \qquad \text{and} \qquad (t_i^N(e), t_j^N(e)) \in [0, T_i] \times [0, T_j] \ .$$

The social utility $e_S^N(t_i, t_j)$ is

$$e_S^N(t_i, t_j) = \frac{2(T_i + T_j) - 2(t_i + t_j)}{3}$$

and so

$$t_i^{SR}(t_j; e) = 0 \qquad \text{and} \qquad (t_i^S(e), t_j^S(e)) = (0,0) \ .$$

Hence, there is a unique social optimum, that is the unique Pareto optimum

$$t_i^S(e) = 0 \ .$$

The total quantity $q_i^N(t_i, t_j)$ produced by firm $F_i$ is

$$q_i^N(t_i, t_j) \equiv q_i^N(c_i, c_j; t_i, t_j) = \frac{1}{3}(4T_j + t_i - 2t_j)$$

and so the total quantity $q_i^N(t_i, t_j)$ increases with $t_i$ and decreases with $t_j$. Thus, there is a unique Nash equilibrium

$$t_i^{BR}(t_j; q) = T_i \qquad \text{and} \qquad (t_i^N(q), t_j^N(q)) = (T_i, T_j) \ .$$

For every pair of tariffs $(t_i, t_j)$, with $t_i \geq t_j$, we have

$$q_i^N(t_i - t_j, 0) > q_i^N(t_i, t_j)$$

and

$$q_j^N(t_i - t_j, 0) > q_j^N(t_i, t_j) \ .$$

Hence, a pair of tariffs $(t_i, t_j)$ is a Pareto optimum with respect to the utility $q_i^N(t_i, t_j)$ if, and only if,

$$(t_i, t_j) \in \mathcal{P}(q) \ ,$$

where

$$\mathcal{P}(q) = \{(t_i, t_j) \in [0, t_i] \times [0, t_j] : t_i = 0 \vee t_j = 0\} \ .$$

The social utility $q_S^N(t_i, t_j)$ is

$$q_S^N(t_i, t_j) = \frac{4(T_i + T_j) - (t_i + t_j)}{3}$$

and so

$$t_i^{SR}(t_j; q) = 0 \qquad \text{and} \qquad (t_i^S(q), t_j^S(q)) = (0,0) \ .$$

Hence, there is a unique social optimum but it does not coincide with the Nash equilibrium

$$t_i^S(q) \neq t_i^N(q)$$

and

$$q_i^N(t_i^N, t_j^N) < q_i^N(t_i^S, t_j^S) \qquad \text{and} \qquad q_j^N(t_i^N, t_j^N) < q_j^N(t_i^S, t_j^S) \ .$$

Therefore, the game, with utility $u_i = q_i$, is of the type **PD**.

The aggregate quantity $Q_i^N(t_i)$ in the market of country $X_i$ is

$$Q_i^N(t_i) \equiv Q_i^N(c_i, c_j; t_i) = \frac{2(T_i + T_j) - t_i}{3}$$

and so

$$t_i^{BR}(t_j; Q) = 0 \qquad \text{and} \qquad (t_i^N(Q), t_j^N(Q)) = (0, 0) \ .$$

The social utility $Q_S^N(t_i, t_j)$ is

$$Q_S^N(t_i, t_j) = q_S^N(t_i, t_j)$$

and so

$$t_i^{SR}(t_j; Q) = 0 \qquad \text{and} \qquad (t_i^S(Q), t_j^S(Q)) = (0, 0) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(Q) = t_i^N(Q) = 0 \ .$$

Therefore, the game, with utility $u_i = Q_i$, is of the type **SE**.

The inverse demand function $p_i^N(t_i)$ of the firm $F_i$ is

$$p_i^N(t_i) \equiv p_i^N(c_i, c_j; t_i) = \frac{\alpha + c_i + c_j + t_i}{3}$$

and so

$$t_i^{BR}(t_j; p) = T_i \qquad \text{and} \qquad (t_i^N(p), t_j^N(p)) = (T_i, T_j) \ .$$

The social utility $p_S^N(t_i, t_j)$ is

$$p_S^N(t_i, t_j) = \frac{2(\alpha + c_i + c_j) + t_i + t_j}{3}$$

and so

$$t_i^{SR}(t_j; p) = T_i \qquad \text{and} \qquad (t_i^S(p), t_j^S(p)) = (T_i, T_j) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(p) = t_i^N(p) = T_i \ .$$

Therefore, the game, with utility $u_i = p_i^N$, is of the type **SE**.

**Consumers savings effects from using tariffs**

The consumers savings are measured by the consumer surplus. The consumer surplus $CS_i^N(t_i)$ is

$$CS_i^N(t_i) \equiv CS_i^N(c_i, c_j; t_i) = \frac{\big(2(T_i + T_j) - t_i\big)^2}{18} \ ,$$

and so

$$t_i^{BR}(t_j; CS) = 0 \qquad \text{and} \qquad (t_i^N(CS), t_j^N(CS)) = (0, 0) \ .$$

The social utility $CS_S^N(t_i, t_j)$ is

$$CS_S^N(t_i, t_j) = \frac{\big(2(T_i + T_j) - t_i\big)^2}{18} + \frac{\big(2(T_i + T_j) - t_j\big)^2}{18}$$

and so

$$t_i^{SR}(t_j; CS) = 0 \qquad \text{and} \qquad (t_i^S(CS), t_j^S(CS)) = (0, 0) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(CS) = t_i^N(CS) = 0 \ .$$

Therefore, the game, with utility $u_i = CS_i$, is of the type **SE**.

**Firms profits effects from using tariffs**

The profit $\pi_i^N(t_i, t_j)$ of the firm $F_i$ is

$$\pi_i^N(t_i, t_j) \equiv \pi_i^N(c_i, c_j; t_i, t_j) = \frac{1}{9}[(2T_j + t_i)^2 + 4(T_j - t_j)^2] \ .$$

Thus, the profit $\pi_i^N(t_i, t_j)$ increases with $t_i$ and decreases with $t_j$, and so

$$t_i^{BR}(t_j, \pi) = T_i \qquad \text{and} \qquad (t_i^N(\pi), t_j^N(\pi)) = (T_i, T_j) \ .$$

The social utility $\pi_S^N(t_i, t_j)$ is

$$\pi_S^N(t_i, t_j) = \frac{1}{9}[(2T_j + t_i)^2 + (2T_i + t_j)^2 + 4(T_i - t_i)^2 + 4(T_j - t_j)^2] \ .$$

Hence,

$$\frac{\partial \pi_S^N}{\partial t_i} = \frac{4(T_j - 2T_i) + 10t_i}{9} \ .$$

Noting that

$$\frac{\partial^2 \pi_S^N}{\partial t_i^2} = \frac{10}{9} > 0$$

we obtain that the local maxima of $\pi_S^N$ is attained at the boundary points of the admissible tariffs

$$t_i^{SR}(t_j;\pi) \in \{0, T_i\} \ .$$

Since,

$$\pi_S^N(T_i, t_j) - \pi_S^N(0, t_j) = \frac{T_i}{9}(4T_j - 3T_i)$$

there are two possible cases:

*Case I:* $9T_j < 12T_i < 16T_j$. We have

$$t_i^{SR}(t_j;\pi) = T_i \qquad \text{and} \qquad t_j^{SR}(t_i;\pi) = T_j \ .$$

Thus,

$$(t_i^S(\pi), t_j^S(\pi)) = (T_i, T_j) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(\pi) = t_i^N(\pi) = T_i \ .$$

Therefore, the game, with utility $u_i = \pi_i^N$, is of the type **SE**.

*Case II:* $0 < 4T_j \leq 3T_i$ (similarly, $0 < 4T_i \leq 3T_j$). We have

$$t_i^{SR}(t_j;\pi) = 0 \qquad \text{and} \qquad t_j^{SR}(t_i;\pi) = T_j \ .$$

Therefore,

$$(t_i^S(\pi), t_j^S(\pi)) = (0, T_j) \ .$$

Hence, there is a unique social optimum but it does not coincide with the Nash equilibrium

$$t_i^N(\pi) \neq t_i^S(\pi) \qquad \text{and} \qquad t_j^N(\pi) = t_j^S(\pi) = T_j \ .$$

Furthermore,

$$\pi_j^N(t_i^N, t_j^N) < \pi_j^N(t_i^S, t_j^S) \qquad \text{and} \qquad \pi_i^N(t_i^N, t_j^N) > \pi_i^N(t_i^S, t_j^S) \ .$$

Therefore, the game, with utility $u_i = \pi_i^N$, is of the type **LW**.

**Governments direct gains from using tariffs**

The direct gain in using tariffs, by the governments, are given by the custom revenues. The custom revenue $CR_i^N(t_i)$ is

$$CR_i^N(t_i) \equiv CR_i^N(c_i, c_j; t_i) = \frac{2t_i(T_i - t_i)}{3} \ .$$

By assumption (A1), $CR_i^N(t_i) > 0$. The custom revenue increases with the tariff $t_i \in [0, T_i/2]$, and it decreases with the tariff $t_i \in [T_i/2, T_i]$,

$$CR_i^N(0) = CR_i^N(T_i) \leq CR_i^N(t_i) \leq CR_i^N\left(\frac{T_i}{2}\right) = \frac{T_i^2}{6}$$

and so

$$t_i^{BR}(t_j, CR) = \frac{T_i}{2} \qquad \text{and} \qquad (t_i^N(CR), t_j^N(CR)) = \left(\frac{T_i}{2}, \frac{T_j}{2}\right) \ .$$

The social utility $CR_S^N(t_i, t_j)$ is

$$CR_S^N(t_i, t_j) = \frac{2t_i(T_i - t_i)}{3} + \frac{2t_j(T_j - t_j)}{3}$$

and so

$$t_i^{SR}(t_j; CR) = \frac{T_i}{2} \qquad \text{and} \qquad \left(t_i^S(CR), t_j^S(CR)\right) = \left(\frac{T_i}{2}, \frac{T_j}{2}\right) \ .$$

Hence, there is a unique social optimum (that is the unique Pareto optimum) and coincides with the Nash equilibrium

$$t_i^S(CR) = t_i^N(CR) = \frac{T_i}{2} \ .$$

Therefore, the game, with utility $u_i = CR_i^N$, is of the type **SE**.

### 3.1.5 Nash and social welfares

In this subsection, we will find which of the three typical games occurs depending upon the production costs: social equilibrium (SE), prisoner's dilemma (PD), or lose-win social strategies (LW).

Recall that the welfare $W_i^N(t_i, t_j)$ of the country $X_i$ is

$$\begin{aligned} W_i^N(t_i, t_j) &= \pi_i^N(t_i, t_j) + CR_i^N(t_i) + CS_i^N(t_i) \\ &= \frac{1}{9}\left[10T_j^2 + 2T_i^2 + 4T_iT_j + (4T_i + 2T_j)t_i - 8T_jt_j + 4t_j^2 - \frac{9t_i^2}{2}\right] \ . \end{aligned}$$

We have that

$$\frac{\partial W_i^N}{\partial t_i} = \frac{4T_i + 2T_j}{9} - t_i \ .$$

Therefore, the maximum point of the polynomial $W_i^N(t_i, t_j)$ is

$$A_{W,i} = \frac{2(T_j + 2T_i)}{9} \ .$$

Noting that $A_{W,i} < T_i$ is equivalent to $2T_j < 5T_i$, we get that the best response is

$$t_i^{BR}(t_j, W) = \begin{cases} A_{W,i} \ , & \text{if } T_j < \frac{5T_i}{2} \ ; \\ \\ T_i \ , & \text{otherwise} \ . \end{cases}$$

The social utility $W_S^N(t_i, t_j)$ is

$$W_S^N(t_i, t_j) = W_i^N(t_i, t_j) + W_j^N(t_i, t_j) \ .$$

Hence, we have that

$$\frac{\partial W_S^N}{\partial t_i} = \frac{2T_j - 4T_i - t_i}{9} \ .$$

Let

$$B_{W_S,i} = 2(T_j - 2T_i) \ .$$

Noting that $0 < B_{W_S,i} < T_i$ is equivalent to $2T_i < T_j < 5T_i/2$, we get that the social best response is

$$t_i^{SR}(t_j; W) = \begin{cases} 0 \ , & \text{if } T_j \leq 2T_i \ ; \\ \\ B_{W_S,i} \ , & \text{if } 2T_i < T_j < \frac{5T_i}{2} \ ; \\ \\ T_i \ , & \text{if } T_j \geq \frac{5T_i}{2} \ . \end{cases}$$

Hence, we have three possible cases:

*Case I:* $T_i \leq 2T_j$ and $T_j \leq 2T_i$ . The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (A_{W,i}, A_{W,j}) \ .$$

The social equilibrium is

$$(t_i^S(W), t_j^S(W)) = (0, 0) \ .$$

In case I, there are two possible subcases:

*(a)* $0.63\ldots T_j < T_i < 1.57\ldots T_j$ and $0.63\ldots T_i < T_j < 1.57\ldots T_i$. Then,

$$W_i^N(t_i^S(W), t_j^S(W)) > W_i^N(t_i^N(W), t_j^N(W))$$

and

$$W_j^N(t_i^S(W), t_j^S(W)) > W_j^N(t_i^N(W), t_j^N(W)) \ .$$

Hence, there is a unique social equilibrium but it does not coincide with the Nash equilibrium. Furthermore, the game is of the type **PD**.

*(b)* $1.57\ldots T_j < T_i \leq 2T_j$ *(case (c)* $0.5T_j \leq T_i < 0.63\ldots T_j$ *is similar)*, then

$$W_i^N(t_i^S(W), t_j^S(W)) < W_i^N(t_i^N(W), t_j^N(W))$$

and

$$W_j^N(t_i^S(W), t_j^S(W)) > W_j^N(t_i^N(W), t_j^N(W)) \ .$$

Hence, there is a unique social equilibrium but it does not coincide with the Nash equilibrium. Furthermore, the game is of the type **LW**.

*Case II:* $2T_i < T_j < 5T_i/2$ (similarly, $2T_i/5 < T_j < T_i/2$). The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (A_{W,i}, A_{W,j}) \ .$$

The social equilibrium is

$$(t_i^S(W), t_j^S(W)) = (B_{W_S,i}, 0) \ .$$

Since

$$B_{W_S,i} < A_{W,i}$$

then,

$$t_i^S(W) \neq t_i^N(W) \qquad \text{and} \qquad t_j^S(W) \neq t_j^N(W) \ .$$

Moreover,

$$W_i^N(t_i^S(W), t_j^S(W)) > W_i^N(t_i^N(W), t_j^N(W))$$

and

$$W_j^N(t_i^S(W), t_j^S(W)) < W_j^N(t_i^N(W), t_j^N(W)) \ .$$

Hence, there is a unique social equilibrium but it does not coincide with the Nash equilibrium. Furthermore, the game is of the type **LW**.

*Case III:* $5T_i \leq 2T_j$ (similarly, $5T_j \leq 2T_i$). The Nash equilibrium is

$$(t_i^N(W), t_j^N(W)) = (T_i, A_{W,j}) \ .$$

The social equilibrium is

$$(t_i^S(W), t_j^S(W)) = (T_i, 0) \ .$$

Thus,

$$W_i^N(t_i^S(W), t_j^S(W)) > W_i^N(t_i^N(W), t_j^N(W))$$

and

$$W_j^N(t_i^S(W), t_j^S(W)) < W_j^N(t_i^N(W), t_j^N(W)) \ .$$

Hence, there is a unique social equilibrium but it does not coincide with the Nash equilibrium. Furthermore, the game is of the type **LW**.

### 3.1.6 Static analysis

In this subsection, we compare the values of the relevant economic quantities of the international trade model computed in the case of the perfect Nash subgame equilibrium with the ones computed in the social optimum.

For every pair of tariffs $(t_i, t_j) \in [0, T_i] \times [0, T_j]$, we assume that in the second subgame the firms choose the Nash equilibrium

$$(h_i^N(t_i), e_i^N(t_j), h_j^N(t_j), e_j^N(t_i))$$

computed in the previous section.

The *perfect Nash subgame equilibrium* consists in the governments to choose the pair of tariffs $(t_i^N, t_j^N)$ that maximize the competitive welfare of the governments, i.e. the Nash equilibrium for the first stage game

$$(t_i^N, t_j^N) = (t_i^N(W), t_j^N(W)) \ ,$$

where $(t_i^N(W), t_j^N(W))$ was computed in the previous section.

The *social optimum* consists in both governments to choose the pair of tariffs $(t_i^S, t_j^S)$ that maximize the joint welfare of the countries, i.e. the social optimum for the first stage game

$$(t_i^S, t_j^S) = (t_i^S(W), t_j^S(W)) \ ,$$

where $(t_i^S(W), t_j^S(W))$ was computed in the previous section.

Therefore, we have three possible cases:

*Case I:* $T_i \leq 2T_j$ and $T_j \leq 2T_i$ . The Nash tariffs for the first stage game are

$$(t_i^N, t_j^N) = \big(2(T_j + 2T_i)/9, 2(T_i + 2T_j)/9\big) \ .$$

The social tariffs for the first stage game are

$$(t_i^S, t_j^S) = (0, 0) \ .$$

The home quantity $h_i^P$ at the subgame Nash perfect equilibrium is

$$h_i^P = \frac{4(5T_j + T_i)}{27} \ .$$

The home quantity $h_i^S$ at the social optimum is

$$h_i^S = \frac{2T_j}{3} \ .$$

Hence,

$$h_i^P - h_i^S = \frac{4T_i + 2T_j}{27} > 0.$$

The export quantity $e_i^P$ at the subgame Nash perfect equilibrium is

$$e_i^P = \frac{2(5T_j - 2T_i)}{27} \ .$$

The export quantity $e_i^S$ at the social optimum is

$$e_i^S = \frac{2T_j}{3} \ .$$

Hence,

$$e_i^P - e_i^S = -\frac{4T_i + 18T_j}{27} < 0.$$

The total quantity $q_i^P$ produced by firm $F_i$ at the subgame Nash perfect equilibrium is

$$q_i^P = \frac{30T_j}{27} \ .$$

The total quantity $q_i^S$ at the social optimum is

$$q_i^S = \frac{4T_j}{3} \ .$$

Hence,

$$q_i^P - q_i^S = -\frac{2T_j}{9} < 0 \ .$$

The aggregate quantity $Q_i^P$ in the market of country $X_i$ at the subgame Nash perfect equilibrium is

$$Q_i^P = \frac{2(7T_i + 8T_j)}{27} \ .$$

The aggregate quantity $Q_i^S$ at the social optimum is

$$Q_i^S = \frac{2T_i + 2T_j}{3} \ .$$

Hence,

$$Q_i^P - Q_i^S = -\frac{4T_i + 2T_j}{27} < 0 \ .$$

The inverse demand function $p_i^P$ of the firm $F_i$ at the subgame Nash perfect equilibrium is

$$p_i^P = \alpha - \frac{2(7T_i + 8T_j)}{27} \ .$$

The inverse demand function $p_i^S$ at the social optimum is

$$p_i^S = \alpha - \frac{2T_i + 2T_j}{3} \ .$$

Hence,

$$p_i^P - p_i^S = Q_i^S - Q_i^P > 0.$$

Recall that $A_{W,i} = 2(T_i + 2T_j)/9$. The profit $\pi_i^P$ of the firm $F_i$ at the subgame Nash perfect equilibrium is

$$\pi_i^P = \frac{1}{9}\left[(2T_j + A_{W,i})^2 + 4(T_j - A_{W,j})^2\right] \ .$$

The profit $\pi_i^S$ of the firm $F_i$ at the social optimum is

$$\pi_i^S = \frac{8T_j^2}{9} \ .$$

Hence,

$$\pi_i^P - \pi_i^S = \frac{4}{3^6}(8T_i^2 + 20T_iT_j - 37T_j^2) \ .$$

Let $K$ be such that $T_i = KT_j$. Recall that in this case $0.5 \leq K \leq 2$. Hence,

$$\pi_i^P - \pi_i^S = \frac{4T_j^2}{3^6}(8K^2 + 20K - 37) \ .$$

Thus, there are two possible subcases:

*(a)* $0.81\ldots \leq K \leq 1.23\ldots$, then

$$\pi_i^P - \pi_i^S < 0 \qquad \text{and} \qquad \pi_j^P - \pi_j^S < 0 \ .$$

*(b)* $1.24\ldots \leq K \leq 2$ *(case (c)* $0.5 \leq K \leq 0.8\ldots$ *is similar)*, then

$$\pi_i^P - \pi_i^S > 0 \qquad \text{and} \qquad \pi_j^P - \pi_j^S < 0 \ .$$

The custom revenue $CR_i^P$ at the subgame Nash perfect equilibrium is

$$CR_i^P = \frac{2^2}{3^5}(T_j + 2T_i)(5T_i - 2T_j) \ .$$

The custom revenue $CR_i^S$ at the social optimum is

$$CR_i^S = 0 \ .$$

Hence,

$$CR_i^P - CR_i^S = CR_i^P > 0 \ .$$

The consumer surplus $CS_i^P$ at the subgame Nash perfect equilibrium is

$$CS_i^P = \frac{2(7T_i + 8T_j)^2}{3^6} \ .$$

The consumer surplus $CS_i^S$ at the social optimum is

$$CS_i^S = \frac{2(T_i + T_j)^2}{9} \ .$$

Hence,

$$CS_i^P - CS_i^S = \frac{-64T_i^2 - 100T_iT_j - 34T_j^2}{3^6} < 0 \ .$$

The welfare $W_i^P$ of the country $X_i$ at the subgame Nash perfect equilibrium is

$$W_i^P = \pi_i^P + CR_i^P + CS_i^P \ .$$

The welfare $W_i^S$ of the country $X_i$ at the social optimum is

$$W_i^S = \pi_i^S + CR_i^S + CS_i^S \ .$$

Hence,

$$W_i^P - W_i^S = \frac{2}{3^6}(44T_i^2 - 4T_iT_j - 103T_j^2) \ .$$

Let $K$ be such that $T_i = KT_j$. Hence,

$$W_i^P - W_i^S = \frac{2T_j^2}{3^6}(44K^2 - 4K - 103) \ .$$

Thus, there are two possible subcases:

*(a)* $0.64\ldots \leq K \leq 1.57\ldots$, then

$$W_i^P - W_i^S < 0 \qquad \text{and} \qquad W_j^P - W_j^S < 0 \ .$$

For instance, when $T_i = T_j$.

*(b)* $1.58\ldots \leq K \leq 2$ *(case (c)* $0.5 \leq K \leq 0.63\ldots$ *is similar)*, then

$$W_i^P - W_i^S > 0 \qquad \text{and} \qquad W_j^P - W_j^S < 0 \ .$$

*Case II:* $2T_i < T_j < 5T_i/2$ ($0.4T_i < T_j < 0.5T_i$ is similar). The Nash tariffs for the first stage game are

$$(t_i^N, t_j^N) = \big(2(T_j + 2T_i)/9, 2(T_i + 2T_j)/9\big) \ .$$

The social tariffs for the first stage game are

$$(t_i^S, t_j^S) = (B_{W_S,i}, 0) \ .$$

The home quantity $h_i^P$ at the subgame Nash perfect equilibrium is

$$h_i^P = \frac{4(5T_j + T_i)}{27} \ .$$

The home quantities $h_i^S$ and $h_i^S$ at the social optimum are

$$h_i^S = \frac{4(T_j - T_i)}{3} \qquad \text{and} \qquad h_j^S = \frac{2T_i}{3} \ .$$

Hence,

$$h_i^P - h_i^S = \frac{8(5T_i - 2T_j)}{27} \geq 0 \qquad \text{and} \qquad h_j^P - h_j^S = \frac{2T_i + 4T_j}{27} > 0 \ .$$

The export quantity $e_i^P$ at the subgame Nash perfect equilibrium is

$$e_i^P = \frac{2(5T_j - 2T_i)}{27} \ .$$

The export quantities $e_i^S$ and $e_j^S$ at the social optimum are

$$e_i^S = \frac{2T_j}{3} \qquad \text{and} \qquad e_j^S = \frac{10T_i - 4T_j}{3} \ .$$

Hence,

$$e_i^P - e_i^S = -\frac{4T_i + 8T_j}{27} < 0 \qquad \text{and} \qquad e_j^P - e_j^S = \frac{16(2T_j - 5T_j)}{27} < 0 \ .$$

The total quantity $q_i^P$ produced by firm $F_i$ at the subgame Nash perfect equilibrium is

$$q_i^P = \frac{10T_j}{9} \ .$$

The total quantities $q_i^S$ and $q_j^S$ at the social optimum are

$$q_i^S = \frac{6T_j - 4T_i}{3} \qquad \text{and} \qquad q_j^S = \frac{4(3T_i - T_j)}{3} \ .$$

Hence,

$$q_i^P - q_i^S = \frac{12(3T_i - 2T_j)}{27} < 0$$

and

$$q_j^P - q_j^S = \frac{2(6T_j - 13T_i)}{9} \ ,$$

1) if $12T_i < 6T_j < 13T_i$, then $q_j^P - q_j^S < 0$;

2) if $13T_i \leq 6T_j < 15T_i$, then $q_j^P - q_j^S \geq 0$.

The aggregate quantity $Q_i^P$ in the market of country $X_i$ at the subgame Nash perfect equilibrium is

$$Q_i^P = \frac{2(7T_i + 8T_j)}{27} \ .$$

The aggregate quantities $Q_i^S$ and $Q_i^S$ at the social optimum are

$$Q_i^S = 2T_i \qquad \text{and} \qquad Q_j^S = \frac{2T_i + 2T_j}{3} \ .$$

Hence,

$$Q_i^P - Q_i^S = \frac{8(2T_j - 5T_i)}{27} < 0 \qquad \text{and} \qquad Q_i^P - Q_i^S = -\frac{2T_i + 4T_j}{27} < 0 \ .$$

The inverse demand function $p_i^P$ of the firm $F_i$ at the subgame Nash perfect equilibrium is

$$p_i^P = \alpha - \frac{2(7T_i + 8T_j)}{27} \ .$$

The inverse demand functions $p_i^S$ and $p_j^S$ at the social optimum are

$$p_i^S = \alpha - 2T_i \qquad \text{and} \qquad p_j^S = \alpha - \frac{2T_i + 2T_j}{3} \ .$$

Hence,

$$p_i^P - p_i^S = Q_i^S - Q_i^P > 0 \qquad \text{and} \qquad p_j^P - p_j^S = Q_j^S - Q_j^P > 0.$$

Recall that $A_{W,i} = 2(T_j + 2T_i)/9$. The profit $\pi_i^P$ of the firm $F_i$ at the subgame Nash perfect equilibrium is

$$\pi_i^P = \frac{1}{9}\left[(2T_j + A_{W,i})^2 + 4(T_j - A_{W,j})^2\right] \ .$$

The profits $\pi_i^S$ and $\pi_j^S$ of the firms $F_i$ and $F_j$ at the social optimum are

$$\pi_i^S = \frac{1}{9}\left[16T_i^2 + 20T_j^2 - 32T_iT_j\right]$$

and

$$\pi_j^S = \frac{1}{9}\left[104T_i^2 + 16T_j^2 - 80T_iT_j\right] \ .$$

Hence,

$$\pi_i^P - \pi_i^S = \frac{16}{3^6}(-70T_j^2 + 167T_iT_j - 79T_i^2)$$

and

$$\pi_j^P - \pi_j^S = \frac{4}{3^6}(-316T_j^2 + 1640T_iT_j - 1981T_i^2) \ .$$

Let $K$ be such that $T_j = KT_i$ then, in this case $2 < K < 2.5$. Therefore,

$$\pi_i^P - \pi_i^S = \frac{16T_i^2}{3^6}(-70K^2 + 167K - 79) < 0$$

and

$$\pi_j^P - \pi_j^S = \frac{4T_i^2}{3^6}(-316K^2 + 1640K - 1981) < 0 \ .$$

The custom revenue $CR_i^P$ at the subgame Nash perfect equilibrium is

$$CR_i^P = \frac{2^2}{3^5}(T_j + 2T_i)(5T_i - 2T_j) \ .$$

The custom revenues $CR_i^S$ and $CR_j^S$ at the social optimum are

$$CR_i^S = \frac{4}{3}\left[9T_iT_j - 10T_i^2 - 2T_j^2\right] \qquad \text{and} \qquad CR_j^S = 0 \ .$$

Hence,

$$CR_i^P - CR_i^S = \frac{16}{3^5}\ (40T_j^2 - 182T_iT_j + 205T_i^2)$$

Let $K$ be such that $T_j = KT_i$ then

$$CR_i^P - CR_i^S = \frac{16T_i^2}{3^5}\,(40K^2 - 182K + 205)\ .$$

In this case $2 < K < 2.5$. Therefore,

1) if $2 < K < 2.05\ldots$, then $CR_i^P - CR_i^S > 0$ ;

2) if $2.05\ldots \leq k < 2.5$, then $CR_i^P - CR_i^S < 0$

and

$$CR_j^P - CR_j^S = CR_j^P > 0\ .$$

The consumer surplus $CS_i^P$ at the subgame Nash perfect equilibrium is

$$CS_i^P = \frac{2(7T_i + 8T_j)^2}{3^6}\ .$$

The consumer surpluses $CS_i^S$ and $CS_j^S$ at the social optimum are

$$CS_i^S = 2T_i^2 \qquad \text{and} \qquad CS_j^S = \frac{2(T_i + T_j)^2}{9}\ .$$

Hence,

$$CS_i^P - CS_i^S = \frac{2}{3^6}\,(64T_j^2 + 112T_iT_j - 1409T_i^2)$$

Let $K$ be such that $T_j = KT_i$ then in this case $2 < K < 2.5$. Therefore,

$$CS_i^P - CS_i^S = \frac{2T_i^2}{3^6}\,(64K^2 + 112K - 1409) < 0$$

and

$$CS_j^P - CS_j^S = \frac{-64T_j^2 - 100T_iT_j - 34T_i^2}{3^6} < 0\ .$$

The welfare $W_i^P$ of the country $X_i$ at the subgame Nash perfect equilibrium is

$$W_i^P = \pi_i^P + CR_i^P + CS_i^P\ .$$

The welfare $W_i^S$ of the country $X_i$ at the social optimum is

$$W_i^S = \pi_i^S + CR_i^S + CS_i^S\ .$$

Hence,

$$W_i^P - W_i^S = \frac{2}{3^6}\,(464T_j^2 - 2920T_iT_j + 2879T_i^2)$$

and

$$W_j^P - W_j^S = \frac{2}{3^6}\left(-604T_j^2 + 3236T_iT_j - 3991T_i^2\right) .$$

Let $K$ be such that $T_j = KT_i$. Hence,

$$W_i^P - W_i^S = \frac{2T_i^2}{3^6}\left(464K^2 - 2920K + 2879\right) < 0$$

and

$$W_j^P - W_j^S = \frac{2T_i^2}{3^6}\left(-604K^2 + 3236K - 3991\right) > 0 .$$

*Case III:* $5T_i \leq 2T_j$ ($5T_j \leq 2T_i$ is similar). The Nash tariffs for the first stage game are

$$(t_i^N, t_j^N) = \left(T_i, 2(T_i + 2T_j)/9\right) .$$

The social tariffs for the first stage game are

$$(t_i^S, t_j^S) = (T_i, 0) .$$

The home $h_i^P$ and $h_j^P$ quantities at the perfect Nash equilibrium are

$$h_i^P = \frac{2T_j + T_i}{3}$$

and

$$h_j^P = \frac{4(5T_i + T_j)}{27} .$$

The home $h_i^P$ and $h_j^P$ quantities at the social optimum are

$$h_i^S = h_i^P = \frac{2T_j + T_i}{3}$$

and

$$h_j^S = \frac{2T_i}{3} .$$

Hence,

$$h_i^P - h_i^S = 0$$

and

$$h_j^P - h_j^S = \frac{2(T_i + 2T_j)}{27} > 0 .$$

The export $e_i^P$ and $h_j^P$ quantities at the perfect Nash equilibrium are

$$e_i^P = \frac{2(5T_j - 2T_i)}{27}$$

and

$$e_j^P = 0 \ .$$

The export $e_i^S$ and $h_j^S$ quantities at the social optimum are

$$e_i^S = \frac{2T_j}{3}$$

and

$$e_j^S = 0 \ .$$

Hence,

$$e_i^P - e_i^S = -\frac{4(T_i + 2T_j)}{27} < 0$$

and

$$e_j^P - e_j^S = 0 \ .$$

The total quantities $q_i^P$ and $q_j^P$ respectively, produced by the firms $F_i$ and $F_j$ at the perfect Nash equilibrium are

$$q_i^P = \frac{28T_j + 5T_i}{27}$$

and

$$q_j^P = h_j^P \ .$$

The total quantities $q_i^S$ and $q_j^S$ respectively, produced by the firms $F_i$ and $F_j$ at the social optimum are

$$q_i^S = \frac{4T_j + T_i}{3}$$

and

$$q_j^S = h_j^S = \frac{2T_i}{3} \ .$$

Hence,

$$q_i^P - q_i^S = -\frac{4(T_i + 2T_j)}{27} < 0$$

and

$$q_j^P - q_j^S = \frac{2(T_i + 2T_j)}{27} > 0 \ .$$

The aggregate quantities $Q_i^P$ and $Q_j^P$ respectively, in the market of countries $X_i$ and $X_j$ at the perfect Nash equilibrium are

$$Q_i^P = h_i^P = \frac{2T_j + T_i}{3}$$

and

$$Q_j^P = \frac{2(8T_i + 7T_j)}{27} \ .$$

The aggregate quantities $Q_i^S$ and $Q_j^S$ respectively, produced by the firms $F_i$ and $F_j$ at the social optimum are

$$Q_i^S = Q_i^P = h_i^P$$

and

$$Q_j^S = \frac{2(T_i + T_j)}{3} \ .$$

Hence,

$$Q_i^P - Q_i^S = 0$$

and

$$Q_j^P - Q_j^S = -\frac{2(T_i + 2T_j)}{27} < 0 \ .$$

The inverse demand functions $p_i^P$ and $p_j^P$ respectively, of the firms $F_i$ and $F_j$ at the perfect Nash equilibrium are

$$p_i^P = \alpha - \frac{2T_j + T_i}{3}$$

and

$$p_j^P = \alpha - \frac{2(8T_i + 7T_j)}{27} \ .$$

The inverse demand functions $p_i^P$ and $p_j^P$ respectively, of the firms $F_i$ and $F_j$ at the social optimum are

$$p_i^S = p_i^P = \alpha - \frac{2T_j + T_i}{3}$$

and

$$p_j^S = \alpha - \frac{2(T_i + T_j)}{3} \ .$$

Hence,

$$p_i^P - p_i^S = 0$$

and

$$p_j^P - p_j^S = Q_j^S - Q_j^P = \frac{2(T_i + 2T_j)}{27} > 0 \ .$$

The profits $\pi_i^P$ and $\pi_j^P$ of the firms $F_i$ and $F_j$ at the perfect Nash equilibrium respectively, are

$$\pi_i^P = \frac{1}{9}[(2T_j + T_i)^2 + 4(T_j - A_{W,j})^2]$$

and

$$\pi_j^P = \frac{1}{9}(2T_i + A_{W,j})^2 \ .$$

The profits $\pi_i^S$ and $\pi_j^S$ of the firms $F_i$ and $F_j$ at the social optimum respectively, are

$$\pi_i^S = \frac{1}{9}[(2T_j + T_i)^2 + 4T_j^2]$$

and

$$\pi_j^S = \frac{4}{9}\ T_i^2 \ .$$

Hence,

$$\pi_i^P - \pi_i^S = \frac{4}{81}(T_i^2 - 5T_iT_j - 14T_j^2) < 0$$

and

$$\pi_j^P - \pi_j^S = \frac{1}{9}(4T_iA_{W,j} + A_{W,j}^2) > 0 \ .$$

The custom revenues $CR_i^P$ and $CR_j^P$ at the perfect Nash equilibrium respectively, are

$$CR_i^P(t_i) = 0$$

and

$$CR_j^P = \frac{2A_{W,j}(T_j - A_{W,j})}{3} \ .$$

The custom revenues $CR_i^P$ and $CR_j^P$ at the social optimum respectively, are

$$CR_i^S = 0$$

and

$$CR_j^S = 0 \ .$$

Hence,

$$CR_i^P - CR_i^S = 0$$

and

$$CR_j^P - CR_j^S = CR_j^P = \frac{2A_{W,j}(T_j - A_{W,j})}{3} > 0 \ .$$

The consumer surpluses $CS_i^P$ and $CS_j^P$ at the perfect Nash equilibrium respectively, are

$$CS_i^P = \frac{(T_i + 2T_j)^2}{18}$$

and

$$CS_j^P = \frac{2(8T_i + 7T_j)^2}{27^2} \ .$$

The consumer surpluses $CS_i^S$ and $CS_j^S$ at the social optimum respectively, are

$$CS_i^S = CS_i^P = \frac{(T_i + 2T_j)^2}{18}$$

and

$$CS_j^S = \frac{2(T_i + T_j)^2}{9} \ .$$

Hence,

$$CS_i^P - CS_i^S = 0$$

and

$$CS_j^P - CS_j^S = \frac{2}{3^6}(-17T_i^2 - 32T_j^2 - 50T_iT_j) < 0 \ .$$

The welfares $W_i^P$ and $W_j^P$ of the countries $X_i$ and $X_j$ at the perfect Nash equilibrium, respectively, are

$$W_i^P = \pi_i^P + CS_i^P$$

and

$$W_j^P = \pi_j^P + CR_j^P + CS_j^P \ .$$

The welfares $W_i^S$ and $W_j^S$ of the countries $X_i$ and $X_j$ at the social optimum respectively, are

$$W_i^S = \pi_i^S + CS_i^S$$

and

$$W_j^P = \pi_j^S + CR_j^S + CS_j^S \ .$$

Hence,

$$W_i^P - W_i^S = \pi_i^P - \pi_i^S = \frac{4}{81} \ (T_i^2 - 5T_iT_j - 14T_j^2) < 0$$

and

$$\begin{aligned} W_j^P - W_j^S &= (\pi_j^P - \pi_j^S) + (CR_j^P - CR_j^S) + (CS_j^P - CS_j^S) \\ &= \frac{2}{3^4} \ (T_i + 2T_j)^2 > 0 \ . \end{aligned}$$

### 3.1.7 Conclusions

For every pair of tariffs $(t_i, t_j)$, we found the Nash equilibrium for the second subgame, i.e. the home and export quantities such that the firms maximize strategically their profits. Then, using the Nash equilibrium for the home and export quantities, we found the tariffs that lead to a Nash equilibria or to a social equilibria for different utilities given by the relevant economic quantities. We observed that the Nash equilibria and the social optimum tariffs for the home quantities are the same and equal with the maximal tariff. For the export quantities all tariffs lead to a Nash equilibrium but only the $(0, 0)$ tariffs are a social optimum. These different behaviour in the home and export quantities lead us to do a full analysis of the tariffs economical impact in all the relevant economic quantities.

| Export quantities $(e_i, e_j)$ of the firms | | |
|---|---|---|
| Nash tariffs | Social tariffs | Game type |
| $(t_i, t_j)$ with $t_i > 0$ and $t_j > 0$ | $(0, 0)$ | PD |
| $(0, 0)$ | $(0, 0)$ | SE |
| $(t_i, 0)$ with $t_i > 0$ | $(0, 0)$ | LW |
| $(0, t_j)$ with $t_j > 0$ | $(0, 0)$ | LW |

For the total quantities produced by the firms we found that the Nash tariffs are the maximal tariffs and the social tariffs are the zero tariffs and, so, the game is of prisoner's dilemma type. For the aggregate quantities, prices, custom revenues and consumer surpluses we found that the Nash tariffs coincide with the social tariffs. However, for the aggregate quantities and the consumer surpluses the tariffs are zero; for the custom revenues are half of the maximal tariffs; and for the prices, like for the home quantities, are the maximal tariffs.

| | SE game | | | | | PD game |
|---|---|---|---|---|---|---|
| Econ. quantity | $h_i$ | $Q_i$ | $p_i$ | $CR_i$ | $CS_i$ | $q_i$ |
| Nash tariff | $T_i$ | 0 | $T_i$ | $T_i/2$ | 0 | $T_i$ |
| Social tariff | $T_i$ | 0 | $T_i$ | $T_i/2$ | 0 | 0 |

For the profits of the firms we found that the Nash tariffs are the maximal tariffs and the social tariffs can be zero or the maximal tariffs. If $T_i$ and $T_j$ are similar the Nash and social tariffs are equal and the game is of the SE type; if $T_i$ and $T_j$ are too different the game is of the LW type.

| Profits $(\pi_i, \pi_j)$ of the firms | | | |
|---|---|---|---|
| Condition | Nash tariffs | Social tariffs | Game type |
| If $0 < T_j < 3T_i/4$ | $(T_i, T_j)$ | $(0, T_j)$ | LW |
| If $3T_i/4 < T_j < 4T_i/3$ | $(T_i, T_j)$ | $(T_i, T_j)$ | SE |
| If $T_j < 4T_i/3$ | $(T_i, T_j)$ | $(T_i, 0)$ | LW |

For the welfares of the countries we found two types of Nash tariffs, three types of social tariffs, and two types of games. We observe that for $(0, 0)$ social tariffs and $(A_{W,i}, A_{W,j})$ Nash tariffs, the game is of type PD if the maximal tariffs $T_i$ and $T_j$ are similar. However, the game is of type LW if the maximal tariffs $T_i$ and $T_j$ are too different. In the LW type game, the country with lower maximal tariff has a gain in its welfare and the country with higher maximal tariff has a lost in its welfare when we compare their welfares at the Nash equilibrium with their welfares at the social optimum.

| Welfares $(W_i, W_j)$ of the countries | | | |
|---|---|---|---|
| Condition | Nash tariffs | Social tariffs | Game type |
| If $0 < T_j \leq 2T_i/5$ | $(A_{W,i}, T_j)$ | $(0, T_j)$ | LW |
| If $2T_i/5 < T_j < T_i/2$ | $(A_{W,i}, A_{W,j})$ | $(0, B_{W_S,j})$ | LW |
| If $T_i/2 \leq T_j < 0.63\ldots T_i$ | $(A_{W,i}, A_{W,j})$ | $(0, 0)$ | LW |
| If $0.63\ldots T_i < T_j < 1.57\ldots T_i$ | $(A_{W,i}, A_{W,j})$ | $(0, 0)$ | PD |
| If $1.57\ldots T_i < T_j \leq 2T_i$ | $(A_{W,i}, A_{W,j})$ | $(0, 0)$ | LW |
| If $2T_i < T_j < 5T_i/2$ | $(A_{W,i}, A_{W,j})$ | $(B_{W_S,i}, 0)$ | LW |
| If $5T_i/2 \leq T_j$ | $(T_i, A_{W,j})$ | $(T_i, 0)$ | LW |

## 3.2 Uncertainty costs on an international duopoly with tariffs

In this section we study the effects of uncertainty in the initial production costs on the expected profits of the firms and on the expected welfares of the countries.

In subsection 3.2.1 we compute the Bayesian-Nash equilibrium of the home and export quantities for the competing firms. In subsection 3.2.2, we find the Bayesian-Nash tariffs that maximize the welfares of the countries. Then, we show that the decision of the governments to impose or not a tariff can be interpreted as a game that it is similar to the Prisoner's Dilemma (see [26]).

### 3.2.1 Costs uncertainty

In this subsection, for every pair of tariffs, we compute the home and export quantities practiced by both firms at the Bayesian-Nash equilibrium for the second stage game. The static analysis for the relevant economic quantities is done along the section and it comes straightforward from the explicit formulas obtained.

We suppose that each firm has two different technologies $L$ and $H$ and uses one of them according to a certain probability distribution. The use of one or the other technology affects the unitary production cost

$$c_i : \{L, H\} \to \mathbb{R}_0^+ \ ,$$

where $c_{i,L} < c_{i,H}$ for $i \in \{1, 2\}$. For $k \in \{L, H\}$ and $i \in \{1, 2\}$, let $\theta_{i,k}$ be the probability of the firm $F_i$ to use technology $k$. Hence, $\theta_{i,H} \geq 0$, $\theta_{i,L} \geq 0$ and $\theta_{i,H} + \theta_{i,L} = 1$.

Let $i, j \in \{1, 2\}$, with $i \neq j$. Given a random variable

$$X_{i,j} : \{L, H\}^2 \to \mathbb{R} \ ,$$

the expectation $E_i(X_{i,j}) : \{L, H\} \to \mathbb{R}$, with respect to the technology of country $X_i$, is the random variable

$$E_i(X_{i,j})(k_j) = \sum_{k_i \in \{H,L\}} \theta_{i,k_i} X_{i,j}(k_i, k_j)$$

and the expectation $E_j(X_{i,j}) : \{L, H\} \to \mathbb{R}$, with respect to the technology of country $X_j$, is the random variable

$$E_j(X_{i,j})(k_i) = \sum_{k_j \in \{H,L\}} \theta_{j,k_j} X_{i,j}(k_i, k_j) \ .$$

Furthermore, the expectation $E(X)$, with respect to the technologies of both countries, is

$$\begin{aligned} E(X) &= E_i(E_j(X_{i,j})) = E_j(E_i(X_{i,j})) \\ &= \sum_{k_i \in \{H,L\}} \sum_{k_j \in \{H,L\}} \theta_{i,k_i} \theta_{j,k_j} X_{i,j}(k_i, k_j) \in \mathbb{R} \ . \end{aligned}$$

The home quantity $h_i$ and the export quantity $e_i$ of firm $F_i$ are random variables

$$h_i : \{L, H\} \to \mathbb{R}_0^+ \qquad \text{and} \qquad e_i : \{L, H\} \to \mathbb{R}_0^+ \ .$$

The *ex-ante profit* $\pi_i^A : \{L, H\} \to \mathbb{R}_0^+$ of firm $F_i$ is

$$\pi_i^A(h_i, e_i, h_j, e_j; t_j)(k_i) = E_j\Big(\pi_i(h_i, e_i, h_j, e_j; t_j)\Big)(k_i)$$

$$= \sum_{k_j \in \{H,L\}} \pi_i(h_i(k_i), e_i(k_i), h_j(k_j), e_j(k_j); t_j) \ .$$

The *expected cost* $E_i$ of the firm $F_i$ is

$$E_i \equiv E(c_i) = \sum_{k \in \{H,L\}} \theta_{i,k} c_i(k) \in \mathbb{R}_0^+ \ .$$

The expected squared cost $E_i^2$ of the firm $F_i$ is

$$E_i^2 \equiv E(c_i^2) = \sum_{k \in \{H,L\}} \theta_{i,k} (c_i(k))^2 \in \mathbb{R}_0^+ \ .$$

The *variance cost* $V_i$ of the firm $F_i$ is

$$V_i \equiv V(c_i) = E(c_i^2) - \big(E(c_i)\big)^2 \in \mathbb{R}_0^+ \ .$$

The *cost difference* $\Delta_i : \{L, H\} \to \mathbb{R}$ of Firm $F_i$ is

$$\Delta_i(k_i) = c_i(k_i) - E_i \ .$$

Let

$$T_i \equiv T_i(c_i, c_j) = (\alpha + c_i - 2c_j)/2 \ .$$

The *complete maximal tariff* $T_i^E$ of the government of state $X_i$ is

$$T_i^E \equiv T_i(E_i, E_j) = \frac{\alpha + E(c_i) - 2E(c_j)}{2} \ .$$

The *incomplete maximal tariff* $\bar{T}_i$ of the government of state $X_i$ is

$$\bar{T}_i = T_i^E - \frac{3}{4}\Delta_i(H) \ .$$

**Assumption 1**: For all $i, j \in \{1, 2\}$ with $i \neq j$, we have $\bar{T}_i > 0$ and

$$0 \leq t_i \leq \bar{T}_i \ .$$

The Bayesian-Nash equilibrium of the second stage game is determined by the home quantities and the export quantities that maximize the ex-ante profit of both firms.

**Theorem 3.1.** *Let* $(t_i, t_j) \in [0, \bar{T}_i] \times [0, \bar{T}_j]$. *Under assumption 1, the Bayesian-Nash equilibrium of the second stage game for the home consumption* $h_i^B(t_i) : \{L, H\} \to \mathbb{R}_0^+$ *is*

$$h_i^B(k_i; t_i) = \frac{1}{3}\big(2T_j^E + t_i\big) - \frac{1}{2}\Delta_i(k_i) \ ;$$

*and for the export quantity* $e_i^B(t_j) : \{L, H\} \to \mathbb{R}_0^+$ *is*

$$e_i^B(k_i; t_j) = \frac{2}{3}\big(T_j^E - t_j\big) - \frac{1}{2}\Delta_i(k_i) \ ,$$

*for* $i, j \in \{1, 2\}$ *with* $j \neq i$.

From Theorem 3.1, we obtain the following expected economic quantities. The expected home quantity is

$$E\big(h_i^B(t_i)\big) = \frac{2T_j^E + t_i}{3} \ .$$

Hence, the expected Bayesian-Nash home quantity $E\big(h_i^B(t_i)\big)$ increases with the tariff $t_i$. The expected export quantity is

$$E\big(e_i^B(t_j)\big) = \frac{2}{3}\big(T_j^E - t_j\big)$$

Hence, the expected Bayesian-Nash export quantity $E\big(e_i^B(t_j)\big)$ decreases with the tariff $t_j$ but does not depend upon the tariff $t_i$. The Bayesian-Nash total quantity

$$q_i^B(t_i, t_j) : \{L, H\} \to \mathbb{R}_0^+$$

produced by the firm $F_i$ is

$$q_i^B(k_i; t_i, t_j) = h_i^B(t_i) + e_i^B(t_j) = \frac{4T_i^E + t_i - 2t_j}{3} - \Delta_i(k_i) \ .$$

Thus, the expected Bayesian-Nash total quantity is

$$E(q_i^B(t_i,t_j)) = \frac{4T_i^E + t_i - 2t_j}{3} \ .$$

Hence, $q_i^B(t_i,t_j)(k_i)$ and $E(q_i^B(t_i,t_j))$ increase with $t_i$ and decrease with $t_j$. The Bayesian-Nash aggregate quantity

$$Q_i(t_i) : \{L,H\}^2 \to \mathbb{R}_0^+$$

in the market of the country $X_i$ is

$$Q_i^B(k_i,k_j;t_i) = \frac{2(T_i^E + T_j^E) - t_i}{3} - \frac{\Delta_i(k_i) + \Delta_j(k_j)}{2} \ .$$

Thus, the expected Bayesian-Nash aggregate quantity is

$$E(Q_i^B(t_i)) = \frac{2(T_i^E + T_j^E) - t_i}{3} \ .$$

Hence, the expected Bayesian-Nash aggregate quantity $E(Q_i^B(t_i))$ decreases with tariff $t_i$. The Bayesian-Nash inverse demand function

$$p_i^B(t_i) : \{L,H\}^2 \to \mathbb{R}$$

is

$$\begin{aligned} p_i^B(k_i,k_j;t_i) &\equiv p_i^B(c_i(k_i), c_j(k_j), E_i, E_j; t_i) \\ &= \alpha - \frac{2(T_i^E + T_j^E) - t_i}{3} + \frac{\Delta_i(k_i) + \Delta_j(k_j)}{2} \ . \end{aligned}$$

Thus, the expected Bayesian-Nash inverse demand function is

$$E(p_i^B(t_i)) = \alpha - \frac{2(T_i^E + T_j^E) - t_i}{3} \ .$$

The expected Bayesian-Nash inverse demand function increase with tariff $t_i$. The ex-post Bayesian-Nash profit

$$\pi_i^B(t_i,t_j) : \{L,H\}^2 \to \mathbb{R}$$

is

$$\pi_i^B(k_i,k_j;t_i,t_j) \equiv \pi_i^B(c_i(k_i), c_j(k_j), E_i, E_j; t_i, t_j)$$

$$= \big(p_i^B(k_i;t_i) - c_i(k_i)\big) h_i^B(k_i;t_i) + \big(p_j^B(k_j;t_j) - c_i(k_i) - t_j\big) e_i^B(k_k;t_j)$$

$$= \tfrac{1}{9}\big[(2T_j^E + t_i)^2 + 4(T_j^E - t_j)^2\big] + \tfrac{4T_j^E + t_i - 2t_j}{6}\ \Delta_j - \tfrac{1}{2}\Delta_i\Delta_j + \tfrac{1}{2}\Delta_i^2 \ .$$

Furthermore,

$$\frac{\partial \pi_i^B}{\partial t_i} = \frac{2}{9}(2T_j^E + t_i) + \frac{\Delta_j}{6} \ .$$

Noting that

$$\frac{\partial^2 \pi_i^B}{\partial t_i^2} > 0 \ ,$$

we obtain that the local maxima of $\pi_i^B$ is attained at the boundary points of the admissible tariffs

$$t_i^{BR}(t_j; \pi_i^B(k_i, k_j)) \in \{0, \bar{T}_i\} \ .$$

The expected Bayesian-Nash profit is

$$E(\pi_i^B(t_i, t_j)) = \frac{(2T_j^E + t_i)^2 + 4(T_j^E - t_j)^2}{9} + \frac{V_i}{2} \ .$$

Hence, the expected Bayesian-Nash profit $E(\pi_i^B(t_i, t_j))$ increases with tariff $t_i$ and decrease with tariff $t_j$. The Bayesian-Nash custom revenue

$$CR_i^B(t_i) : \{L, H\} \to \mathbb{R}$$

is

$$CR_i^B(k_j; t_i) = \frac{2t_i(T_i^E - t_i)}{3} - \frac{t_i}{2} \ \Delta_j(k_j) \ .$$

Furthermore,

$$\frac{\partial CR_i}{\partial t_i} = \frac{2(T_i^E - 2t_i)}{3} - \frac{\Delta_j(k_j)}{2} \ ,$$

and

$$\frac{\partial^2 CR_i}{\partial t_i^2} < 0 \ .$$

Hence, the Bayesian-Nash custom revenue $CR_i^B(k_j; t_i)$ increases with tariff $t_i \in [0, T_i^E/2 - 3/8\Delta_j(k_j)]$ and decreases with tariff $t_i \in [T_i^E/2 - 3/8\Delta_j(k_j), \bar{T}_i^E]$. The expected Bayesian-Nash custom revenue is

$$E(CR_i^B(t_i)) = \frac{2t_i(T_i^E - t_i)}{3} \ .$$

The Bayesian-Nash consumer surplus

$$CS_i^B(t_i) : \{L, H\}^2 \to \mathbb{R}$$

is

$$\begin{aligned} CS_i^B(k_i, k_j; t_i) &= \frac{1}{18}(2T_i^E + 2T_j^E - t_i)^2 + (\frac{t_i}{6} - \frac{T_i^E + T_j^E}{3})(\Delta_i + \Delta_j) \\ &+ \frac{\Delta_i \Delta_j}{4} + \frac{1}{8}(\Delta_i^2 + \Delta_j^2) \end{aligned}$$

Furthermore,

$$\frac{\partial CS_i}{\partial t_i} = \frac{t_i - 2(T_i^E + T_j^E)}{9} + \frac{\Delta_i + \Delta_j}{6} < 0 \ ,$$

and

$$\frac{\partial^2 CS_i}{\partial t_i^2} > 0 \ .$$

Hence, the Bayesian-Nash consumer surplus $CS_i^B(k_i, k_j; t_i)$ decreases with tariff $t_i \in [0, \bar{T}_i^E]$. The expected Bayesian-Nash consumer surplus is

$$E(CS_i(t_i)) = \frac{(2T_i^E + 2T_j^E - t_i)^2}{18} + \frac{V_i + V_j}{8} \ .$$

*Proof of Theorem 3.1.* Following [26], for $i, j \in \{L, H\}$ with $i \neq j$, the Bayesian-Nash home quantities

$$h_i^B : \{L, H\} \to \mathbb{R}_0^+ \qquad \text{and} \qquad h_j^B : \{L, H\} \to \mathbb{R}_0^+$$

and the Bayesian-Nash export quantities

$$e_i^B : \{L, H\} \to \mathbb{R}_0^+ \qquad \text{and} \qquad e_j^B : \{L, H\} \to \mathbb{R}_0^+$$

are the solutions of the maximization problem

$$\begin{aligned}
&\max\nolimits_{(h_i,e_i)} \pi_i^A(h_i, e_i, h_j, e_j; t_j) \\
&\quad = \max\nolimits_{(h_i,e_i)} \textstyle\sum_{k_j \in \{H,L\}} \theta_{j,k_j} \Big[\big(\alpha - h_i(k_i) - e_j(k_j) - c_i(k_i)\big)h_i(k_i) \\
&\qquad + \big(\alpha - h_j(k_j) - e_i(k_i) - c_i(k_i) - t_j\big)e_i(k_i)\Big] \\
&\quad = \max\nolimits_{(h_i,e_i)} \big(\alpha - h_i(k_i) - c_i(k_i) - \textstyle\sum_{k_j \in \{H,L\}} \theta_{j,k_j} e_j(k_j)\big)h_i(k_i) \\
&\qquad + \big(\alpha - e_i(k_i) - c_i(k_i) - t_j - \textstyle\sum_{k_j \in \{H,L\}} h_j(k_j)\big)e_i(k_i) \ ,
\end{aligned}$$

and of the maximization problem

$$\begin{aligned}
&\max\nolimits_{(h_j,e_j)} \pi_j^A(h_i, e_i, h_j, e_j; t_i) \\
&\quad = \max\nolimits_{(h_j,e_j)} \big(\alpha - h_j(k_j)n - c_j(k_j) - \textstyle\sum_{k_i \in \{H,L\}} \theta_{i,k_i} e_i(k_i)\big)h_j(k_j) \\
&\qquad + \big(\alpha - e_j(k_j) - c_j(k_j) - t_i - \textstyle\sum_{k_i \in \{H,L\}} \theta_{i,k_i} h_i(k_i)\big)e_j(k_j) \ .
\end{aligned}$$

Therefore, these optimization problems are equivalent to the following two independent optimization problems for each market: In the market of the country $X_i$,

$$\max\nolimits_{h_i(k_i)} h_i(k_i)\big(\alpha - h_i(k_i) - c_i(k_i) - \textstyle\sum_{k_j \in \{H,L\}} \theta_{j,k_j} e_j(k_j))\big)$$

$$\max\nolimits_{e_j(k_j)} e_j(k_j)(\alpha - e_j(k_j) - c_j(k_j) - t_i - \textstyle\sum_{k_i \in \{H,L\}} \theta_{i,k_i} h_i(k_i)) \ ;$$

and in the market of the country $X_j$,

$$\max_{h_j(k_j)} h_j(k_j)\big(\alpha - h_j(k_j) - c_j(k_j) - \textstyle\sum_{k_i\in\{H,L\}} \theta_{i,k_i} e_i(k_i))\big)$$

$$\max_{e_i(k_i)} e_i(k_i)(\alpha - e_i(k_i) - c_i(k_i) - t_j - \textstyle\sum_{k_j\in\{H,L\}} h_j(k_j)) \ .$$

Hence, in the market of the country $X_i$, by the first order condition, we obtain

$$h_i(k_i) = (\alpha - c_i(k_i) - \sum_{k_j\in\{H,L\}} \theta_{j,k_j} e_j(k_j))/2$$

$$e_j(k_j) = (\alpha - c_j(k_j) - t_i - \sum_{k_i\in\{H,L\}} \theta_{i,k_i} h_i(k_i))/2 \ .$$

Therefore, by solving the system of four equations with four unknowns $h_i(L), h_i(H), e_j(L)$ and $e_j(H)$, we obtain

$$h_i^B(k_i) = (2T_j^E + t_i)/3 - \Delta_i(k_i)/2$$

$$e_j^B(k_j) = 2(T_i^E - t_i)/3 - \Delta_j(k_j)/2 \ .$$

Similarly, in the market of the country $X_j$, we obtain

$$h_j(k_j) = (\alpha - c_j(k_j) - \sum_{k_i\in\{H,L\}} \theta_{i,k_i} e_i(k_i))/2$$

$$e_i(k_i) = (\alpha - c_i(k_i) - t_j - \sum_{k_j\in\{H,L\}} h_j(k_j))/2 \ .$$

Therefore, by solving the system of four equations with four unknowns $h_j(L), h_j(H), e_i(L)$ and $e_i(H)$, we obtain

$$h_j^B(k_j) = (2T_i^E + t_j)/3 - \Delta_j(k_j)/2$$

$$e_i^B(k_i) = 2(T_j^E - t_j)/3 - \Delta_i(k_i)/2 \ .$$

□

### 3.2.2 Welfare and the Prisioner's dilemma for tariffs

In this subsection, we find the subgame perfect equilibrium that it is characterized by the the Bayesian-Nash tariffs that maximize the welfare of the countries using the Bayesian-Nash home and export quantities found in the previous section. Then, we show that the decision of the governments to impose or not a tariff can be interpreted as a game that it is like the Prisoner's Dilemma.

The ex-post Bayesian-Nash welfare $W_i^B(k_i, k_j; t_i, t_j)$ of the country $X_i$ is

$$W_i^B(k_i, k_j; t_i, t_j) = \pi_i^B(k_i, k_j; t_i, t_j) + CR_i^B(k_i; t_i) + CS_i^B(k_i, k_j; t_i) \ .$$

The expected Bayesian-Nash welfare $E(W_i^B(t_i, t_j))$ of the country $X_i$ is

$$\begin{aligned} E(W_i^B(t_i, t_j)) &= \frac{2}{9}[(T_i^E + T_j^E)^2 + 4(T_j^E)^2 + (2T_i^E + T_j^E)t_i - 4T_j^E t_j)] \\ &\quad - \frac{t_i^2}{2} + \frac{4t_j^2}{9} + \frac{1}{8}(5V_i + V_j) \ . \end{aligned}$$

Hence,

$$\frac{\partial E(W_i^B)}{\partial t_i} = \frac{2}{9}(2T_i^E + T_j^E) - t_i \ .$$

and

$$\frac{\partial^2 E(W_i^B)}{\partial t_i^2} = -1 < 0 \ .$$

The subgame perfect equilibrium consists in finding the tariffs that maximize the expected Bayesian-Nash welfare of both countries. Hence, if $2T_j^E < 5T_i^E$ and $2T_i^E < 5T_j^E$ then, the Bayesian-Nash tariffs $(t_i^B, t_j^B)$ are

$$(t_i^B, t_j^B) = \left(\frac{2}{9}(2T_i^E + T_j^E) \ , \ \frac{2}{9}(2T_j^E + T_i^E)\right) \ .$$

**Theorem 3.2.** *If $T_i^E/T_j^E \in [0.64, 1.57]$, then for the expected Bayesian-Nash welfares of the both counties we have*

$$E\left(W_i^B(t_i^B, 0)\right) > E\left(W_i^B(0, 0)\right) > E\left(W_i^B(t_i^B, t_j^B)\right) > E\left(W_i^B(0, t_j^B)\right) \ .$$

Therefore, the inequalities obtained in Theorem 3.2 for the expected welfares of both countries show that the decision of the governments to impose or not a tariff can be interpreted as a game that it is like the Prisoner's Dilemma.

*Proof.* We have that

$$\begin{aligned} E\left(W_i^B(t_i^B, 0)\right) - E\left(W_i^B(0, 0)\right) &= E\left(W_i^B(t_i^B, t_j^B)\right) - E\left(W_i^B(0, t_j^B)\right) \\ &= \frac{(t_i^B)^2}{2} > 0 \ . \end{aligned}$$

Since $7T_j^E > 2T_i^E$, we get

$$E\left(W_i^B(0,0)\right) - E\left(W_i^B(0,t_j^B)\right) = \frac{4}{9}\, t_j^B(2T_j^E - t_j^B) = \frac{8}{81}\, t_j^B(7T_j^E - 2T_i^E) > 0\ .$$

Furthermore, we have that

$$E\left(W_i^B(0,0)\right) - E\left(W_i^B(t_i^B,t_j^B)\right) = \frac{2}{3^6}\left(103(T_j^E)^2 + 4T_i^E T_j^E - 44(T_i^E)^2\right)\ .$$

Letting $K$ be such that $T_j^E = KT_i^E$, we get

$$E\left(W_i^B(0,0)\right) - E\left(W_i^B(t_i^B,t_j^B)\right) = \frac{2(T_i^E)^2}{3^6}\left(103K^2 + 4K - 44\right)\ .$$

Hence, if $0.64 \le K \le 1.57$, then

$$E\left(W_i^B(0,0)\right) - E\left(W_i^B(t_i^B,t_j^B)\right) > 0\ .$$

Therefore,

$$E\left(W_i^B(t_i^B,0)\right) > E\left(W_i^B(0,0)\right) > E\left(W_i^B(t_i^B,t_j^B)\right) > E\left(W_i^B(0,t_j^B)\right)\ .$$

□

### 3.2.3 Conclusions

We proved that the expected profit of each firm increases with the variance of its production costs. We showed that the expected welfare of each government increases with the variances of both production costs, being the effect of the variance of the production costs of the foreign firm smaller than the effect of the variance of the production costs of the home firm.

We showed that the decision of the governments to impose or not a tariff can be interpreted as a game where the utilities are the expected welfares of the governments. We show that this game is like the Prisoner's Dilemma because the welfares of the countries are higher in the case where both governments do not impose tariffs than in the case where both governments decide to impose the Bayesian-Nash tariffs.

For future research, it will be interesting a) to allow the intercept demands of both countries not to be the same, since the countries can have markets with different dimensions; b) to consider that both governments can choose export subsidies or production subsidies; c) to check the robustness of the prisoner's dilemma in these and other extensions.

# Bibliography


[1] A. Araujo and W. Maldonado, *Ergodic chaos, learning and sunspot equilibrium*, Econom. Theory **15** (2000), no. 1, 163–184.

[2] A.P. Araujo, M. Choubdar, W. L. Maldonado, D. Pinheiro, and A. A. Pinto, *Stationary sunspot equilibrium as limit of expectations coordination failures*, X **X** (2014), no. X.

[3] ———, *Large random perturbations of sunspot equilibrium*, X **X** (201X), no. X.

[4] J. Arifovic, J. Jiang, and Y. Xu, *Experimental evidence of bank runs as pure coordination failures*, Journal of Economic Dynamics & Control **Forthcoming** (2013).

[5] L. Arnold, *Random dynamical systems*, Springer, 1998.

[6] C. Azariadis and R. Guesnerie, *Sunspots and cycles*, The Review of Economic Studies **53** (1986), no. 5, 725–737.

[7] V. Baladi and M. Viana, *Strong stochastic stability and rate of mixing for unimodal maps*, Ann. Sci. Éc. Norm. Supér. **29** (1994), no. 4, 483–517.

[8] M. Benedicks and L. Carleson, *The dynamics of the hénon map*, Ann. of Math. **133** (1991), no. 1, 73–169.

[9] M. Benedicks and L.-S. Young, *Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps*, Ergodic Theory Dynam. Systems **12** (1992), no. 1, 13–37.

[10] A. Blokh and M. Yu. Lyubich, *Measure and dimension of solenoidal attractors of one dimensional dynamical systems*, Comm. Math. Phys. **127** (1990), no. 3, 573–583.

[11] J. A. Brander, *Intra-industry trade in identical commodities*, Journal of International Economics **11** (1981), 1–14.

[12] Geanakoplos J. D. Bulow, J. I. and P. D. Klemperer, *Multi-market oligopoly: Strategic substitutes and complements*, Journal of Political Economy **93** (1985), 488–511.

[13] D. Cass and K. Shell, *Do sunspots matter?*, Journal of Political Economy **91** (1983), no. 2, 193–227.

[14] P. A. Chiappori and R. Guesnerie, *Sunspot equilibria in sequential markets models*, Handbook of Mathematical Economics, vol. 4, Elsevier, 1991, pp. 1683–1762.

[15] A. Citanna, H. Crés, J. Dréze, P. J.-J. Herings, and A. Villanacci, *Continua of under-employment equilibria reflecting coordination failures, also at walrasian prices*, Journal of Mathematical Economics **36** (2001), 169–200.

[16] W. de Melo and S. van Strien, *One-dimensional dynamics*, Springer, 1993.

[17] A. Dixit and G. Grossman, *Targed export promotion with several oligopolistic industries*, Journal of International Economics **21** (1986), 233–249.

[18] G. Eaton and G. Grossman, *Optimal trade and industrial policy under oligopoly*, Quarterly Journal of Economics CI (1984), 383–406.

[19] D. Fehr, F. Heinemann, and A. Llorente-Saguer, *The power of sunspots: An experimental analysis*, SFB 649 Discussion Paper, Humboldt University, 2011.

[20] F.A. Ferreira, *Applications of mathematics to industrial organization*, Ph.D Thesis, Universidade do Porto, Portugal, 2007.

[21] Ferreira F. Ferreira, F. A. and A.A. Pinto, *Bayesian price leadership, chapter in tas, k. et al. (eds.): Mathematical methods in engineering, 371–379*, Springer, Dordrecht, 2007.

[22] ——, *Unknown costs in a duopoly with differentiated products, chapter in tas, k. et al. (eds.): Mathematical methods in engineering, 359–369*, Springer, Dordrecht, 2007.

[23] ———, *Flexibility in stackelberg leadership, chapter in j. a. tenreiro machado, bela patkai and imre j. rudas (eds.): Intelligent engineering systems and computational cybernetics, 399–405*, Springer Science+Business Media B.V., 2008.

[24] Ferreira F. Ferreira M. Ferreira, F. A. and A.A. Pinto, *Quantity competition in a differentiated duopoly, chapter in j. a. tenreiro machado, bela patkai and imre j. rudas (eds.): Intelligent engineering systems and computational cybernetics, 365–374*, Springer Science+Business Media B.V., 2008.

[25] E. O'N. Fisher and C. A. Wilson, *Price competition between two international firms facing tariffs*, International Journal of Industrial Organization **13** (1995), 67–87.

[26] R. Gibbons, *A primer in game theory*, Pearson Prentice Hall, Harlow, 1992.

[27] J.-M. Grandmont, *Stabilizing competitive business cycles*, Journal of Economic Theory **40** (1986), 57–76.

[28] R. Guesnerie, *Successes and failures in coordinating expectations*, European Economic Review **37** (1993), 243–268.

[29] ———, *Expectational coordination failures and market outcomes' volatility*, Paris School of Economics Working Paper **22** (2011).

[30] S. M. Hammel, J. A. Yorke, and Grebogi C., *Do numerical orbits of chaotic dynamical processes represent true orbits?*, JOURNAL OF COMPLEXITY **3** (1987), no. 3, 136–145.

[31] G. Keller, *Exponents, attractors and hopf decompositions for interval maps*, Ergodic Theory Dynam. Systems **10** (1990), no. 4, 717–744.

[32] Y. Kifer, *Ergodic theory of random transformations*, Birkhauser Boston, MA, 1986.

[33] K. Krishna, *Trade restrictions as facilitating practices*, Journal of International Economics **26** (1989), 251–270.

[34] Z. Lian and M. Stenlund, *Positive lyapunov exponent by a random perturbation*, Dynamical Systems **27** (2012), no. 2, 239–252.

[35] R. Lucas and N. Stokey, *Liquidity crises*, Policy Paper **11** (2011), no. 3.

[36] J. McMillan, *Game theory in international economics*, Harwood Academic Publishers, Chur, Switzerland, 1986.

[37] M. Misiurewicz, *Absolutely continuous measures for certain maps of an interval*, Publ. Math. Inst. Hautes Études Sci. **53** (1981), no. 1, 17–51.

[38] T. Nowicki and S. van Strien, *Invariant measures exist under a summability condition for unimodal maps*, Invent. Math. **105** (1991), no. 1, 123–136.

[39] H. E. Nusse and J. A. Yorke, *Is every approximate trajectory of some process near an exact trajectory of a nearby process?*, Comm. Math. Phys. **114** (1988), no. 3, 363–379.

[40] J. Peck and K. Shell, *Equilibrium bank runs*, Journal of Political Economy **111** (2003), no. 1, 103–123.

[41] A. A. Pinto, *Game theory and duopoly models*, Interdisciplinary Applied Mathematics Series. Springer Verlag. Accepted, 2009.

[42] A. A. Pinto and M. Choubdar, *Strategic tariffs in international trade*, X **X** (2014).

[43] Faria E. Pinto, A. A. and M. Choubdar, *Nash and social welfare impact in international trade*, X **X** (2014).

[44] Faria E. Ferreira F. A. Pinto, A. A. and M. Choubdar, *Uncertainty costs on an international duopoly with tariffs*, X **X** (2014).

[45] Oliveira B. M. Ferreira F. A. Pinto, A. A. and F. Ferreira, *Investing to survive in a duopoly model, chapter in j. a. tenreiro machado, bela patkai and imre j. rudas (eds.): Intelligent engineering systems and computational cybernetics, 407–414*, Springer Science+Business Media B.V., 2009.

[46] ———, *Stochasticity favoring the effects of the r&d strategies of the firms, chapter in j. a. tenreiro machado, bela patkai and imre j. rudas (eds.): Intelligent engineering systems and computational cybernetics, 415–423*, Springer Science+Business Media B.V., 2009.

[47] E. C. Prescott and K. Shell, *Introduction to sunspots and lotteries*, J. Econom. Theory **107** (2002), no. 1, 1–10.

[48] E. C. Prescott and R. M. Townsend, *Pareto optima and competitive equilibria with adverse selection and moral hazard*, Econometrica **52** (1984), no. 1, 21–45.

[49] D. Sands, *Misiurewicz maps are rare*, Comm. Math. Phys. **197** (1998), 109–129.

[50] F. Vega-Redondo, *Shaping long-run expectations in problems of coordination*, European Journal of Political Economy **13** (1997), 783–806.

[51] M. Woodford, *Learning to believe in sunspots*, Econometrica **58** (1990), 277–307.