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Engenharia

Continuum Modelling and Numerical Simulation of Hexagonal Close-Packed Materials

Dissertation presented to the Faculty of Engineering, University of Porto, as a requirement to obtain the PhD degree in Mechanical Engineering, carried out under the supervision of Professor Francisco Manuel Andrade Pires, Assistant Professor, Faculty of Engineering, University of Porto, Professor Abel Dias dos Santos, Assistant Professor, Faculty of Engineering, University of Porto, Professor Augusto Duarte Campos Barata da Rocha, Associate Professor, Faculty of Engineering, University of Porto.

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Abstract

Hexagonal Close-Packed (HCP) metals exhibit a very unconventional mechanical behaviour compared with other metallic materials. This is due to the limited number of slip systems and twinning mechanisms. A very pronounced strength asymmetry between yielding in tension and compression is observed at low strain levels and cold rolled sheets also show a strong planar anisotropy. Therefore, it is extremely challenging to develop accurate constitutive models and robust numerical integration algorithms to predict the behaviour of these materials. This has been a major motivation of this thesis.

Within the present work, a constitutive model proposed by Cazacu and Barlat is numerically treated and implemented within an implicit time integration scheme. The return mapping algorithm is based on the Closest Point Projection Method (CPPM) combined with an inexact line search strategy. Three interpolation methods are employed to overcome numerical instabilities and to enlarge the bowl of convergence.

To be able to handle regions of high rate-sensitivity to rate independent conditions, an elastic-viscoplastic model is developed by extending the formulation with the viscoplastic formulation proposed by Perzyna. The stability and accuracy of the numerical implementation is illustrated for high strain rates, which are characterized by very high power-exponents, under different stress states.

The identification of material parameters is addressed for an AZ31 alloy. Five optimization strategies are used to identify the parameters of the material constitutive model proposed by Cazacu and co-workers, which employs the principal values of the linearly transformed stress deviator. The ability to avoid local minima and obtain the global optimization value is assessed and discussed.

An extended yield criterion that can describe both the anisotropy of the material and the yielding asymmetry between tension and compression is also proposed. The yield function extends the Barlat 89 criterion to capture the behaviour of orthotropic HCP metal sheets under plane stress. The proposed criterion can capture very well the complex behaviour of HCP metal sheets with a limited number of material parameters.

The prediction of springback effects in sheet metal forming operations is extremely important. Therefore, a generic analytical solution is proposed for arbitrary elasto-plastic materials when subjected to external forces. Attention is also dedicated to the numerical simulation of springback. Four different yield criteria are used and two kinds of hardening law are considered.

Due to the fact that rate effects on HCP materials not only affect the magnitude of the yield surface but also its shape, an extended elasto-viscoplastic constitutive model is proposed. The yield surface of the model is obtained through a bilinear interpolation method and the numerical integration algorithm, which includes two stages, has been developed. The merit of the approach is analysed and discussed.

The efficiency and robustness of all the developments is illustrated by numerical examples, which are compared with results obtained by other authors.

Resumo

Os metais com uma estrutura hexagonal compacta apresentam um comportamento mecânico muito pouco convencional. Isto fica a dever-se ao número limitado de sistemas de deslizamento e a mecanismos de maclagem. Uma assimetria pronunciada é observada entre a tensão de escoamento à tração e compressão para baixos níveis de deformação. Simultaneamente, as chapas laminadas a frio evidenciam uma forte anisotropia planar. Deste modo, é particularmente desafiante o desenvolvimento de modelos constitutivos precisos e algoritmos de integração numérica robustos capazes de prever o comportamento deste tipo de materiais. Tal foi, precisamente, a motivação e o desígnio fundamental que estiveram na base dos trabalhos desta tese.

No presente trabalho, um modelo constitutivo proposto por Cazacu e Barlat é numericamente tratado e implementado dentro de um esquema de integração implícito. O algoritmo de integração das tensões baseia-se no método da Projeção do Ponto Mais Próximo, combinado com uma estratégia de pesquisa local. Três métodos de interpolação são utilizados com o objetivo de superar as instabilidades numéricas e aumentar o domínio de convergência.

Para poder lidar com regiões de elevada e baixa taxa de deformação, um modelo elasto - viscoplástico foi desenvolvido efetuando a extensão da formulação viscoplástica proposta por Perzyna. A estabilidade e a precisão da implementação numérica é ilustrada para altas taxas de deformação, que se caracterizam por expoentes elevados, sob diferentes estados de tensão.

A identificação dos parâmetros do material é executada para uma liga de alumínio AZ31. Cinco estratégias de otimização são usadas para identificar os parâmetros do modelo constitutivo do material que foi proposto por Cazacu e colaboradores, que utiliza os valores principais do tensor das tensões de desvio após transformação linear. A capacidade para serem evitados mínimos locais e obter o valor de otimização global é avaliada e discutida.

É igualmente proposto um critério de cedência, capaz de descrever simultaneamente a anisotropia do material e a assimetria do escoamento à tração e à compressão. A função de cedência estende o critério Barlat'89 para descrever o comportamento de chapas de metal ortotrópico com estrutura hexagonal compacta sob um estado plano de tensão. O critério proposto é capaz de captar com exatidão o comportamento complexo de chapas de metal com estrutura hexagonal compacta recorrendo a um número limitado de parâmetros.

A previsão de efeitos de retorno elástico em operações de conformação plástica de chapa é extremamente importante. Portanto, uma solução analítica genérica é proposta para materiais elasto - plástico arbitrários quando sujeitos a forças externas. A simulação numérica do efeito de retorno elástico é também alvo de atenção. Quatro

critérios de cedência distintos são utilizados e dois tipos de leis de endurecimento são considerados.

Uma vez que o efeito da taxa de deformação para materiais com estrutura hexagonal compacta afeta não só a magnitude da superfície de cedência, mas também a sua forma, é proposto um modelo constitutivo elasto - viscoplástico alargado. A superfície de cedência do modelo é calculada através de um método de interpolação bilinear e foi desenvolvido um algoritmo de integração numérica que inclui duas fases. O mérito da abordagem proposta é analisado e discutido.

A eficiência e robustez de todos os desenvolvimentos é ilustrada através de exemplos numéricos. Por seu turno, estes são comparados com resultados numéricos e experimentais obtidos por outros autores.

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Contents

Abstract	I
Resumo	III
Acknowledgement.....	V
List of Figures	XV
List of Tables.....	XIX
List of Abbreviations and Acronyms	XXI
List of Symbols and Operations	XXIII
Chapter 1 Introduction	1
1.1 Background.....	1
1.2 Outline of this thesis	4
1.3 References	9
Chapter 2 Hexagonal Close-Packed Materials: Physical Aspects and Constitutive Modeling.....	13
2.1 Review of the crystal microstructure and plastic behavior of HCP materials.....	13
2.1.1 The crystallography of HCP metals	13
2.1.2 Plastic behavior and deformation mechanism of HCP materials	15
2.2 Review of continuum phenomenological constitutive models.....	21
2.2.1 Yield criteria	22
2.2.2 Hardening rule	35
2.2.3 Flow rule.....	38
2.3 References	39
Chapter 3 Continuum Mechanics and Finite Element Method	43
3.1 Kinematics of deformation	43
3.1.1 Material and Spatial Description of Motion	43
3.1.2 The deformation gradient and polar decomposition.....	44
3.1.3 Strain measures.....	45
3.1.4 Stress measures.....	46

3.2 Kinetics of deformation	46
3.2.1 Conservation of mass	47
3.2.2 Momentum balance	47
3.2.3 The first principle of thermodynamics	48
3.2.4 The second principle of thermodynamics.....	48
3.2.5 The Clausius-Duhem inequality	49
3.2.6 Constitutive theory	49
3.3 Finite Element Formulation.....	51
3.3.1 Spatial discretization.....	52
3.3.2 Temporal discretization	52
3.3.3 Newton-Raphson method	53
3.4 References	54
Chapter 4 The Application of Line Search Strategies in Closest Point Projection Method (CPPM) to Cazacu 04 Model	55
4.1 Introduction	55
4.2 Formulation of constitutive model	56
4.2.1 Cazacu 04 yield criterion.....	56
4.2.2 Associated flow rule	57
4.2.3 Hardening rule	57
4.3 Integration algorithm for Cazacu 04 model	58
4.3.1 State variable update.....	58
4.3.2 Consistent tangent modular matrix.....	64
4.4 Line search strategy	66
4.4.1 Inexact conditions.....	67
4.4.2 Line search direction	68
4.4.3 Interpolation	68
4.5 Results and discussion.....	74
4.5.1 Single element validation	74
4.5.2 Comparisons among different integration algorithms	77
4.6 Iso-error maps.....	89

4.7 Benchmark test-Cup deep drawing.....	96
4.8 Conclusions	100
4.9 References	101
Chapter 5 An Elasto-Viscoplastic Model for Hexagonal Close Packed Materials: Formulation and Numerical Implementation.....	105
5.1 Introduction	105
5.2 Formulation of elasto-viscoplastic model.....	105
5.2.1 Cazacu 04 yield criterion.....	107
5.3 Stress integration of elasto-viscoplastic model	108
5.4 Numerical examples	110
5.4.1 Single element tests	111
5.4.2 3D iso-error maps of elasto-viscoplastic model	116
5.5 Conclusions	123
5.6 References	123
Chapter 6 Material Parameters Identification of Phenomenological Constitutive Equations by Deterministic and Evolutionary Approach	125
6.1 Introduction	125
6.2 Constitutive theory	126
6.3 Identification procedure and optimization methods	127
6.3.1 Classical gradient method.....	128
6.3.2 Genetic Algorithm (GA) method.....	129
6.3.3 Particle Swarm Optimization (PSO) method	131
6.3.4 Hybrid method of evolutionary optimization and gradient optimization	133
6.4 Results and discuss of identification	134
6.4.1 Yield surface of equivalent plastic strain ($\bar{\epsilon}^p=0.1$).....	134
6.4.2 Yield surface of equivalent plastic strain ($\bar{\epsilon}^p=0.01$).....	136
6.4 Conclusions	140
6.5 References	140
Chapter 7 An Extended Yield Criterion for Orthotropic Metallic Sheets under Plane Stress	143
7.1 Introduction	143

7.2 Fundamental review on anisotropic yield functions.....	145
7.2.1 Hill '48 yield criterion	146
7.2.2 Barlat '89 model	146
7.2.3 Cazacu '06 model	146
7.2.4 New proposed constitutive model	147
7.3 The assessment of the new proposed model.....	148
7.4 Identification procedure for AZ31 Alloy.....	150
7.4.1 Determination of the yield stress and the anisotropy parameters at different directions	151
7.4.2 Calibration procedure	153
7.5 Results and discussion of calibrated yield surfaces.....	154
7.6 Integration algorithm under plane stress	160
7.6.1 Stress and state variable updates	160
7.6.2 Elasto-plastic tangent operator	166
7.7 Numerical examples	168
7.7.1 Single element tests	169
7.7.2 Cup drawing test.....	171
7.8 Conclusions	173
7.9 References	174
Chapter 8 Analytical and Numerical Analysis of Springback for Hexagonal Close-Packed Materials	179
8.1 Introduction	179
8.2 Analytical theory of springback	180
8.2.1 Strain calculation	182
8.2.2 Positioning the elastic and plastic region	183
8.2.3 Force and moment calculation.....	185
8.2.4 Unloading and springback phase.....	186
8.3 Analytical algorithm description	188
8.4 Validation of the proposed analytical method.....	190
8.5 Results and discussion of analytical method	193

8.5.1 The influence of curvature on the neutral surface position, tensile yield critical position, and compressive yield critical position	193
8.5.2 The influence of curvature on the force distribution	196
8.5.3 The influence of curvature on the moment distribution	198
8.5.4 The comparison of the strain distribution between symmetrical and unsymmetrical material	200
8.5.5 Influence of strain rate on the bending	203
8.5.6 Influence of thickness on the bending	206
8.5.7 Influence of in-plane tension on springback.....	209
8.6 Numerical simulation of springback.....	213
8.7 Fundamental review of constitutive model	214
8.7.1 Yield criteria review	214
8.7.2 Hardening law	216
8.8 Implementation of the constitutive model	217
8.9 Numerical results on the unconstrained cylindrical bending simulation.....	219
8.9.1 Analysis of the plastic deformation	220
8.9.2 Effect of Number of Integration Point (NIP) through thickness	221
8.9.3 Influence of yield criteria on springback.....	223
8.10 Conclusions	225
8.11 References	226
Chapter 9 An Extended Elasto-Viscoplastic Model: Formulation and Numerical Implementation.....	229
9.1 Introduction	229
9.2 Determine the flow curves at different strain rates.....	231
9.2.1 Identification of the material parameters of flow rules	233
9.2.2 Tensile flow curve	233
9.2.3 Compressive flow curves	236
9.3 Constitutive formulation of elasto-viscoplastic model.....	237
9.3.1 Kinematic and flow rule for elasto-viscoplasticity.....	238
9.3.2 Cazacu 06 yield formulation	238
9.4 Parameter identification of constitutive model.....	239

9.5 Implementation of extended elasto-viscoplastic model.....	241
9.6 Numerical verification and examples	250
9.6.1 Single element test.....	250
9.6.2 Square box drawing.....	253
9.7 Conclusions	255
9.8 References	256
Chapter 10 Conclusions and Future Work	259
10.1 Conclusions	259
10.2 Suggestions for future work	264
Appendix A Equivalent Plastic Strain Rate of Cazacu’04 Model.....	267
Appendix B Proof of the Convexity of the Proposed Yield Function	269
Appendix C Derivatives of Cazacu 04 Constitutive Model	273
Appendix D Derivatives of New Constitutive Model.....	275
Appendix E Derivatives of Barlat 89 model.....	279
Appendix F Derivatives of Cazacu 06 Yield Criterion	283

List of Figures

Figure 2.1: Hexagonal Close-Packed crystal structures.	13
Figure 2.2: a) The primitive hexagonal unit cell illustrating the axes $\bar{a}_1 = \bar{a}_2 \neq \bar{c}$ and corresponding angles $\bar{\alpha} = \bar{\beta} = 90^\circ$, $\bar{\gamma} = 120^\circ$; b) the Hexagonal Close-Packed structures; the thick solid line in b) delineate the primitive hexagonal unit cell	14
Figure 2.3: Slip deformation mechanisms.	16
Figure 2.4: Frequently observed deformation modes in HCP metals: Basal $\langle \mathbf{a} \rangle$, prismatic $\langle \mathbf{a} \rangle$, and pyramidal $\langle \mathbf{a} \rangle$ slip system in HCP materials.....	16
Figure 2.5: Critical Resolved Shear Stress (CRSS) is plotted against temperature for basal slip, prismatic slip, pyramidal slip, and twinning in magnesium.	18
Figure 2.6: Crystallographic elements of twinning.	19
Figure 2.7: Twinning modes for HCP materials.	21
Figure 2.8: The schematic diagram of yield surface.	23
Figure 3.1: Configurations of a deformable body.	44
Figure 3.2: Schematic diagram of the Cauchy stress.....	47
Figure 3.3: Problem domain and boundaries.....	52
Figure 4.1: Graphical interpretation of stress return corresponding to the Closest Point Project Method (CPPM).	59
Figure 4.2: Single element test: (a) uniaxial tensile test in rolling direction; (b) uniaxial tensile test in transverse direction; (c) biaxial tensile test; (d) uniaxial compressive test in rolling direction; (e) uniaxial compressive test in transverse direction; (f) biaxial compressive test.....	75
Figure 4.3: Stress-strain curves at different stress states.	76
Figure 4.4: The yield surface evolution of pure titanium.	77
Figure 4.5: Schematic diagram of representative stress state.	78
Figure 4.6: Iterative number contours obtained from different integration schemes under different stress states at $\bar{\epsilon}^p = 0.6$	86
Figure 4.7: Iterative number contours obtained from different integration schemes under different stress states at $\bar{\epsilon}^p = 0.4$	89
Figure 4.8: Representative stress state at yield surface A ((Equivalent plastic strain $\bar{\epsilon}^p = 0.1$)) and B ((equivalent plastic strain $\bar{\epsilon}^p = 0.4$))......	91

Figure 4.9: Iso-error maps when $\hat{\mathbf{L}}$ is equal to identity tensor and $\hat{c}$ vanishes.	92
Figure 4.10: Iso-error maps of yield surface at equivalent plastic strain $\bar{\epsilon}^p = 0.0$	93
Figure 4.11: Iso error maps of yield surface at equivalent plastic strain $\bar{\epsilon}^p = 0.6$	95
Figure 4.12: Schematic view of circular cup deep drawing.	96
Figure 4.13: Cup drawing test finite element model.	97
Figure 4.14: Equivalent plastic strain contour.	98
Figure 4.15: Equivalent stress contour after drawing.	98
Figure 4.16: The plot of the punch displacement in the drawing direction vs. the prescribed edge path.	99
Figure 5.1: Uniaxial tensile tests.	111
Figure 5.2: Uniaxial tensile stress-strain curves.	113
Figure 5.3: Uniaxial compressive test.	114
Figure 5.4: Uniaxial compressive stress-strain curves.	115
Figure 5.5: Schematic diagram of stress states on the yield surface.	116
Figure 5.6: Iso-error maps under uniaxial tensile stress state with $\hat{\mu}\ \boldsymbol{\epsilon}\ = 1.0$	118
Figure 5.7: Iso-error maps under uniaxial-tensile test with $\hat{\mu}\ \boldsymbol{\epsilon}\ = 1000$	120
Figure 5.8: Iso-error maps under uniaxial compressive stress state with $\hat{\mu}\ \boldsymbol{\epsilon}\ = 1$	121
Figure 5.9: Iso-error maps under uniaxial compressive stress state with $\hat{\mu}\ \boldsymbol{\epsilon}\ = 1000$	122
Figure 6.1: Flow chart of GA algorithm.	130
Figure 6.2: Flow chart of PSO algorithm.	132
Figure 6.3: Stress-strain curve of AZ31 alloy at room temperature.	133
Figure 6.4: The error comparison among different optimization methods.	135
Figure 6.5: Yield locus of $\bar{\epsilon}^p=0.1$ in the $\sigma_{11} - \sigma_{22}$ space determined by different optimized methods.	136
Figure 6.6: The final error comparison among different optimization methods.	137
Figure 6.7: Yield locus of $\bar{\epsilon}^p=0.01$ in the $\sigma_{11} - \sigma_{22}$ space determined by different optimized methods.	138
Figure 6.8: Error value contours.	139
Figure 7.1: Yield surface of $M=2$	148
Figure 7.2: Yield surface of $M=4$	149
Figure 7.3: Yield surface of $M=6$	150
Figure 7.4: Yield surface evolution—Hill 48 yield criterion.	154
Figure 7.5: Yield surface evolution—Barlat 89 yield criterion.	155

Figure 7.6: Yield surface evolution—Cazacu 06 yield criterion with $\tilde{n}=2$	156
Figure 7.7: Yield surface evolution—Cazacu 06 yield criterion with $\tilde{n}=4$	156
Figure 7.8: Yield surface evolution—the proposed model.	157
Figure 7.9: Schematic illustration of the yield surface decomposition by deformation mechanism.	158
Figure 7.10: Uniaxial yield stress predicted by different yield criteria versus experimental data.....	159
Figure 7.11: Single element tests.....	170
Figure 7.12: Stress-strain curves.	170
Figure 7.13: Schematic section view of cup test drawing.	171
Figure 7.14: Finite element model of cup drawing test.	172
Figure 7.15: Equivalent plastic strain contour.....	172
Figure 7.16: Equivalent Stress contour.....	172
Figure 7.17: Displacement along the cup edge after stamping.....	173
Figure 8.1: A schematic diagram of bending in the longitudinal section.....	181
Figure 8.2: The strain and stress distribution diagram of bending in the cross section.....	182
Figure 8.3: Distributions of stress and strain within a beam before and after application of a moment sufficiently large to cause plastic deformation.....	188
Figure 8.4: Flow chart of analytical program for the bending process.	189
Figure 8.5: Comparison of moment evolution obtained from the theoretical method and the proposed analytical method.	192
Figure 8.6: Comparison of the strain distribution in the cross section between analytical solution, experiments and numerical results.....	192
Figure 8.7: Position distribution for symmetrical and unsymmetrical material.	194
Figure 8.8: Force distributions for the symmetrical material and the unsymmetrical material.	197
Figure 8.9: Bending moment distribution for the symmetrical material and the unsymmetrical material.	199
Figure 8.10: Strain distribution inside the sheet cross section.	201
Figure 8.11: Circumferential stress distribution inside the sheet cross section.....	202
Figure 8.12: Position vs bending curvature with different strain rates.	203
Figure 8.13: Force vs bending curvature with different strain rates.....	204
Figure 8.14: Moment vs bending curvature with different strain rates.	205
Figure 8.15: Springback factor ζ vs bending curvature with different strain rates.	206

Figure 8.16: Position vs bending curvature with different thicknesses.	207
Figure 8.17: Force vs bending curvature with different thicknesses.	207
Figure 8.18: Moment vs bending curvature with different thicknesses.	208
Figure 8.19: Springback factor ζ vs bending curvature with different thicknesses.	209
Figure 8.20: The schematic diagram of bending under external stretch tension.	210
Figure 8.21: Position vs bending curvature with different in-plane tension forces.	210
Figure 8.22: Moment vs bending curvature with different in-plane tension forces.	212
Figure 8.23: Springback factor ζ vs bending curvature with different in-plane tension forces.	213
Figure 8.24: The geometric model of springback test for simulation.	219
Figure 8.25: The strain contour in the top and bottom of the plate.	220
Figure 8.26: The springback angle simulated by von Mises with different NIP through thickness.	221
Figure 8.27: The springback angle simulated by the proposed yield model with different NIP through thickness.	222
Figure 8.28: The comparison of two integration methods.	223
Figure 8.29: Springback back simulated by different yield criteria.	224
Figure 9.1: Microstructures of AZ31B magnesium alloy sheet under different strain rates	230
Figure 9.2: Stress-strain relations for AZ31 under tensile test at RD direction.	235
Figure 9.3: Error comparisons among different flow laws.	236
Figure 9.4: Stress-strain relations for AZ31 under compressive test at RD direction.	237
Figure 9.5: Yield surface evolutions with different strain rates at plastic strain of 0.01.	240
Figure 9.6: Yield surface evolution with different strain rates at plastic strain of 0.03.	240
Figure 9.7: Yield surface evolutions with different strain rates at plastic strain of 0.06.	241
Figure 9.8: Schematic diagram of bilinear interpolation.	247
Figure 9.9: Finite element model of square box.	253
Figure 9.10: Contours of equivalent plastic strain.	254
Figure 9.11: Punch force-displacement curves.	255

List of Tables

Table 2.1: Summary of Slip system for HCP materials.	17
Table 4.1: Material parameter of Titanium	74
Table 4.2: Material parameters used in the simulation.....	74
Table 4.3: Comparisons of Iterative number at $\bar{\epsilon}^p = 0$, $\bar{\epsilon}^p = 0.6$	80
Table 4.4 Comparisons of Iterative number at $\bar{\epsilon}^p = 0.2$, $\bar{\epsilon}^p = 0.4$	82
Table 5.1: Material properties for pure titanium.	110
Table 5.2: Anisotropy coefficient values of pure titanium corresponding to the yield surface evolution.....	110
Table 6.1: Experimental values of the yield stresses and r -values under different loading conditions.	134
Table 6.2: The material parameters of the yield surface ($\bar{\epsilon}^p=0.1$) determined by different optimized methods.....	135
Table 6.3: The material parameters of the yield surface ($\bar{\epsilon}^p=0.01$) determined by different optimized methods.....	137
Table 7.1: Material parameters of AZ31B at quasi-static condition.	168
Table 7.2: Anisotropy coefficient values of AZ31 corresponding to the proposed yield surface evolution.....	169
Table 8.1: Material parameters at different strain rates for tensile and compressive tests.	190
Table 9.1: Material parameter of AZ31B.	250
Table 9.2: Residual norm of one-iterative return mapping algorithm for biaxial tensile test.....	251
Table 9.3: Residual norm of two-iterative return mapping algorithm for biaxial tensile test.....	251
Table 9.4: Residual norm of one iterative return mapping algorithm for biaxial compressive test.....	252
Table 9.5: Residual norm of two-iterative return mapping algorithm for biaxial compressive test.....	252

List of Abbreviations and Acronyms

The following table describes the significance of various abbreviations and acronyms used throughout the thesis.

Abbreviation	Meaning
AOF	Aggregate Objective Function
BCC	Body Centered Cubic
CP-FEM	Crystal Plasticity Finite Element Method
CPPM	Closest Point Projection Method
CRSS	Critical Resolved Shear Stress
DD	Orientation of 45° to the rolling direction of the sheet
DSD	Directed Search Domain
EBSD	Electron Back Scatter Diffraction
FCC	Face Centered Cubic
GA	Genetic Algorithm
Grad	Gradient optimization
HCP	Hexagonal Close Packed
IBVP	Initial Boundary Value Problem
IPE	Isotropic Plasticity Equivalent
K-B	Karafillis and Boyce
LQ	Line Search with Quadratic Interpolation
LQ3	Line Search with Three Point Quadratic Interpolation
LC	Line Search with Cubic Interpolation
N-R	Newton-Raphson method
NIP	Number of Integration Point
NBI	Normal Boundary Intersection
ODF	Orientation Distribution Function
PSO	Particle Swarm Optimization
PTR	Twin Reorientation
RD	Rolling Direction
SD	Strength Differential

SPO	Successive Pareto Optimization
TD	Transverse Direction
VFT	Volume Fraction Transfer
VPSC	Visco-Plastic Self-Consistent
Yld89	Barlat and Lian (1989) yield criterion
Yld91	Barlat 91 yield criterion
Yld 94	Barlat 94 yield criterion
Yld 97	Barlat 97 yield criterion
Yld2000	Barlat 2000 yield criterion

List of Symbols and Operations

The specific meaning of some important symbols and operations is listed below.

$\bar{a}, \bar{c}$	Lattice lengths of HCP structure
$\langle \mathbf{a} \rangle, \langle \mathbf{c} + \mathbf{a} \rangle$	Slip directions
$a_{BL}, c_{BL}, h_{BL}, p_{BL}, M_{BL}$	Barlat-Lian yield criterion parameters
$a_1^{Yld94}, a_2^{Yld94}, a_3^{Yld94}, m_{Yld94}$	Yld94 yield criterion parameters
$a_{Hill90}, b_{Hill90}, m_{Hill90}$	Hill 90 yield criterion parameters
$a_1, a_2, a_3, a_4, a_5, a_6, \hat{c}$	Cazacu 04 yield criterion parameters
a, h, b, c and f	New proposed yield criterion parameters
$\mathbf{b}$	Body force
$\mathbf{B}^g$	Global discrete symmetric gradient matrix
c_{dr}	Drucker criterion parameter
c_{KB}, m_{KB}	Karafillis and Boyce 93 yield criterion parameters
$C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44},$ $C_{55}, C_{66}, \tilde{k}$	Cazacu 06 yield criterion parameters
$\bar{C}_{11}, \bar{C}_{12}, \bar{C}_{21}, \bar{C}_{22}, \bar{C}_{66}, \bar{C}_{11}'', \bar{C}_{12}'',$ $\bar{C}_{21}'', \bar{C}_{22}''$ and $\bar{C}_{66}''$	Yld2000–2d yield criterion parameters
$\mathbf{D}$	Consistent tangent modulus
$\mathbf{D}^e$	Elastic tangent modular matrix
$\mathbf{D}^{ep}$	Elastoplastic tangent modular matrix
$\mathbf{D}^{vp}$	Elasto-viscoplastic tangent modular matrix
E	Young's modulus
$\bar{e}$	Distance between neutral surface and mid-plane
$\mathbf{F}$	Deformation gradient
$\mathbf{F}^e$	Elastic deformation gradient
$\mathbf{F}^p$	Plastic deformation gradient
$\mathbf{f}^{ext}$	Global external force vector
$\mathbf{f}^{int}$	Global internal force vector
$f_{Hill79}, g_{Hill79}, h_{Hill79}, a_{Hill79},$ $b_{Hill79}, c_{Hill79}, m_{Hill79}$	Hill 79 yield criterion parameters

$F_{Hill}, G_{Hill}, H_{Hill},$ $L_{Hill}, M_{Hill}, N_{Hill}$	Hill48 criterion anisotropic parameters
G	shear modulus
$\bar{g}$	Merit function
H	Hardening modulus
$\mathbf{I}$	Second order identity tensor
J	Determinant of the deformation gradient
J_2, J_3	Second and third invariants of deviatoric stress tensor
J_2^0, J_3^0	Second and third invariant of deviatoric transformed stress
$K_{T1}, K_{T2}, K'_{T2},$ $\eta_{T1}, \eta_{T2}, \eta'_{T2}, \gamma_{shear},$	Deformation twinning crystal parameters
$\mathbf{K}$	Global tangent stiffness matrix
K	Bulk modulus
K_S	Springback ratio
$\mathbf{L}_{KB}$	IPE tensor for K-B 93 yield criterion
$\mathbf{L}_{YLD}$	IPE stress tensor for Yld criteria
$\hat{\mathbf{L}}$	Transformed tensor of Cazacu 04 yield criterion
$\tilde{\mathbf{L}}$	Transformed tensor of Cazacu 06 yield criterion
m_{Yld91}	Yld91 yield criterion parameters
$\mathbf{N}$	Flow vector
$N_i^{(e)}$	Shape function associated with node i
$\mathbf{N}^g$	Global interpolation matrix
$\hat{\mathbf{N}}$	Unit normal vector to the yield surface
$\mathbf{P}$	First Piolla-Kirchhoff stress tensor
$\bar{\mathbf{p}}$	Line search direction
$\mathbf{R}$	Global finite element residual force vector
$\mathbf{R}, \mathbf{U}, \mathbf{V}$	Rotation, Right stretch and Left stretch tensor
r_0, r_{45}, r_{90}	Lankford anisotropic coefficients
$\bar{\mathbf{r}}$	Residual of constitutive equations sets
s	Entropy
$\mathbf{t}$	Surface traction

$\bar{t}$	Thickness
$\hat{T}$	Unit tangent vector to the yield surface
u	Displacement
α	internal variables
$\bar{\alpha}, \bar{\beta}, \bar{\gamma}$	Hexagonal unit cell parameters
$\bar{\alpha}_k$	Line search step length
β	Backstress tensor in kinematic hardening
$\dot{\gamma}$	Plastic multiplier
ϵ	Strain
ϵ^e	Elastic strain
ϵ^p	Plastic strain
ϵ^{vp}	Viscoplastic strain
$\bar{\epsilon}^p$	Equivalent plastic strain
$\hat{\epsilon}, \hat{\mu}$	Perzyna viscoplastic parameters
ζ	Springback factor
η	Relative stress tensor in kinematic hardening plasticity models
θ	Temperature
$\tilde{\theta}$	Lode angle
$\hat{\theta}$	Angle to the rolling direction
$\bar{\theta}$	Bending angle
λ, μ	Lamé coefficients
ν	Poisson ratio
ρ	Material density
$\bar{\rho}_m$	radius of curvature in the mid-plane
$\bar{\rho}$	Radius of curvature in the neutral surface plane
σ	Six components of the stress tensor
σ_Y	Yield stress obtained from a simple test
σ_{eq}	Equivalent stress
$\sigma_0, \sigma_{45}, \sigma_{90}, \sigma_b$	Yield stresses along the anisotropic axes
Φ	Yield function
Ψ	Plastic potential

ψ	Helmholtz specific free energy
ψ^e	Elastic component of helmholtz specific free energy
ψ^p	Plastic component of helmholtz specific free energy
$\mathbb{I}$	Four-order symmetric identity tensor
$\det(\cdot)$	Determinant of $(\cdot)$
$\text{div}x(\cdot)$	Divergence of $(\cdot)$
$\text{tr}(\cdot)$	Trace of $(\cdot)$
$\nabla(\cdot)$	Gradient of $(\cdot)$
$\frac{\partial}{\partial x}(\cdot)$	Derivative of $(\cdot)$ with respect to x
$(\cdot)^T$	The transpose of $(\cdot)$
$\ \boldsymbol{r}\ $	Euclidean norm of tensors and vectors
$\boldsymbol{u} \otimes \boldsymbol{v}$	Tensor product

Chapter 1

Introduction

1.1 Background

Currently there is a strong green awareness over sustainability, resources, energy and environment impacts. Concerning CO₂ emissions, transportation represents 22% of total emissions in world [1] or 28% in USA [2], from which road transports make up the vast majority of emissions from this sector.

In an effort to improve fuel efficiency for energy saving, an eventually competent way for reducing the climate-changing CO₂ emissions is provided by the application of lightweight materials that reduce the overall weight of vehicles in defense, aerospace, and automotive industries applications, *etc.* Although composites, particularly fiber-reinforced ones, are currently heavily investigated and developed for such purposes, lightweight metals, such as titanium alloys and magnesium alloys, remain as the most frequently chosen materials, partly due to their low density and high specific strength [3]. The United States Automotive Materials Partnership (USAMP) estimates that vehicle weight can be reduced by 290 *lbs* (131 kg) using magnesium alloys instead of steel or aluminum alloys [4].

Rather than time-consuming experiments, numerical techniques provide fast and cost-effective means to analyze and optimize the mechanical performance of products, having seen an increasing trend in various steps of sheet forming processes, from design to testing, over the last three decades [5-10]. The accuracy of metal forming simulation depends not only on the forming conditions (friction, tool geometry, *etc.*), but also on the choice of the constitutive model material and its numerical implementation into finite element programs. This consequently requires an in-depth knowledge of material constitutive models to describe the material behaviour, and methods to implement these into user-friendly numerical tools.

For polycrystalline materials, anisotropic metal plasticity has been a hot topic for many decades, and numerous constitutive models have been proposed based on two distinct approaches. The first approach, which is referred as crystal plasticity, describes the material constitutive behavior at the microstructural level. The polycrystalline behavior is described based on the behavior of each individual crystal. The theory of crystal

plasticity was developed to study material's heterogeneous plastic deformation based on modeling the slip of dislocations on discrete slip systems within the crystal. The method of crystal plasticity works well for solving problems of heterogeneous mechanical behavior, and it was extensively developed to study heterogeneous plastic deformation, lattice rotation and texture evolution when metals are subjected to large deformation. Therefore, this approach is usually conducted to understand the micromechanisms involved and to capture the macro mechanical behaviour. Although some studies [11-15] employed crystal plasticity to solve practical related problems met in manufacturing processes, like metal rolling and forming, simulation and analysis of metal forming operations using direct implementation of crystal plasticity approaches is still limited, due to the higher computational cost of the analysis. In the second approach, called phenomenological approach, the average behavior of all grains directly determines the global material behavior and the analysis are performed at the macroscopic level.

A large number of phenomenological yield functions [16-20] have been developed for plastic anisotropy of metals with cubic crystal structure after the original development of Hill (1948) [21]. However, the above mentioned constitutive models cannot be used directly to capture the behavior of materials with Hexagonal Close-Packed (HCP) crystal structure. Constitutive modelling of HCP materials, like magnesium alloys, is a rather challenging task because they exhibit a unique behaviour compared to cubic metals. Their HCP crystallographic structure promotes a very different mechanical behaviour when compared to other metals with Face Centered Cubic (FCC) and Body Centered Cubic (BCC) structures, such as steel, aluminium, *etc.* At room temperature, metals with HCP crystalline lattices have a reduced number of available slip systems, which makes the accommodation of arbitrary plastic deformation difficult. Therefore mechanical twinning goes along with dislocation slip during plastic deformation and plays an important role to accommodate the deformation. Mechanical twinning, unlike slip, is sensitive to the sign of the applied stress, *i.e.*, if a particular twin can be formed under a shear stress, it will not be formed by a shear stress of opposite direction. Because of the directionality of twinning, a very pronounced strength asymmetry between yielding in tension and compression, usually known as strength differential (SD) effect, is observed at room temperature. In addition, the cold rolled sheet also shows a strong anisotropy because of the evolution of crystallographic texture as a consequence of the rolling process. Hence, it is inevitable to study in detail the mechanical behaviour of HCP materials and to find a constitutive model which is able

to describe their unconventional stress-strain response. Owing to the unconventional yield surface evolution, the shape of the yield surface changes during deformation, which may lead to problems on the numerical implementation of the related constitutive model with the classical return mapping algorithm within an implicit finite element program. It is well-known that Newton's method can fail to converge if the initial values are too far from the solutions due to the change on the shape of the yield surface. Thus, the numerical strategy implemented for these kinds of yield locus evolution should be investigated.

The dynamic behavior of HCP materials must also be determined to support their introduction into vehicle body structures and high speed forming process. Even for the conventional sheet metal forming process, strain rate effects occur during practical forming process, although they are usually neglected in many simulations. When the deformation is accelerated, the strain rate effect becomes significant. Therefore, it is essential to study and be able to predict the strain rate behavior of HCP materials. A large number of research works [22-26] have been carried out on the experimental investigation of the mechanical behavior of materials under different strain rates and presented the stress-strain response with mathematical flow equations. However, for materials with HCP crystal structure, the above method may not be sufficient to capture the mechanical behavior at different strains, due to the fact that deformation twinning is remarkably affected by the deformation strain rate [23, 24]. As a consequence, it is necessary to model the constitutive behavior and develop new numerical algorithms for different strain rates.

The elastic recovery of HCP materials, commonly known as springback, is also an important phenomenon in sheet metal forming operations. Due to the low elastic modulus of these materials, the prediction of springback can be critical in many industries, where the final shape has to comply with strict dimensions. It has been one of the major concerns in sheet metal forming for decades. Finite element simulation of sheet metal forming is a well-established tool which is used in industrial practice to evaluate geometrical defects caused by elastic springback. A reliable numerical procedure can eliminate the need for experimental try-outs, and hence drastically reduce the lead time and manufacturing costs. The low accuracy of springback prediction is attributed to the lack of understanding of this phenomenon and to the use of oversimplified models that describe, for example, the material behavior or contact conditions during the deformation. Since materials with HCP crystal structure

demonstrate unconventional mechanical behavior, it is necessary to develop and apply accurate constitutive models to improve the prediction of springback behavior in HCP materials.

1.2 Outline of this thesis

The ambition of this dissertation is to contribute to the advancement of fundamental knowledge in the field of modeling and numerical simulation of plastic anisotropy and strength differential effects in HCP metals, to the development of strain rate dependent viscoplastic material model, and to the analysis of the springback behavior in HCP metals. This thesis is divided into ten chapters. Each chapter, except Chapter 1 and Chapter 10, addresses one specific topic. In the majority of the situations, the chapters are self-contained as they address one topic at a time and have already been prepared for publication. The output of this work is to offer enhanced simulation tools to be used for the sheet metal forming of HCP metals.

Chapter 2

This chapter briefly reviews the underlying deformation mechanisms and the mechanical behavior of HCP materials. It is divided into two sections. Section 2.1 presents the crystal structures of HCP materials. The deformation features of slip systems and deformation twinning in HCP metals are introduced. Section 2.2 presents a review of the most well-known phenomenological material constitutive models. Macroscopic plasticity models are discussed including isotropic and orthotropic yield criteria that describe the onset of the plastic behavior, plastic flow rule, and hardening laws which model subsequent plastic deformation.

Chapter 3

In this chapter, the basic concepts of standard continuum mechanics and thermodynamics are summarized for the constitutive model developments addressed in the following chapters. This chapter also recalls briefly the principles of quasi-static infinitesimal finite element methods.

Chapter 4

In this chapter, the Cazacu 04 constitutive model is used to describe the mechanical behavior for titanium alloys. The formulation of numerical algorithms with the Closest Point Projection Method (CPPM) for Cazacu 04 model under the usual assumption of convexity and associativity is addressed. The accuracy of this implicit integration

algorithm for Cazacu 04 model is assessed with iso-error maps at different stress states and accumulated plastic strain levels. Since the yield surface of titanium evolves from a nearly elliptical to a triangular-like shape with the increase of the accumulated plastic strain, the maximum curvature of the whole yield surface increases. This leads to the increase of nonlinearity of the set of constitutive equations. Aiming to expand the radius of convergence of Newton-Raphson scheme and increase the robustness of the CPPM algorithm, this chapter introduces inexact line search strategies into the classical CPPM implicit scheme. In the line search procedure, an iterative step size for update is determined by minimizing a prescribed objective function, such as the square of L2-norm of residual. The Goldstein's conditions are used to determine whether the iterative step is terminated avoiding excessive iterations with small steps. Three interpolation methods for step-length update are investigated, namely quadratic, three points quadratic and cubic interpolation method. Results indicate that the line search method with CPPM scheme is very useful not only to expand the convergence region, but also to improve the computational efficiency, particularly for the line search method with a cubic interpolation.

Chapter 5

In order to include viscoplastic mechanical behavior, in Chapter 5, an extension of the Cazacu 04 elasto-plastic anisotropic model that includes the effect of strain-rate is proposed. The Perzyna overstress method is used to model rate effects in the inviscid (elasto-plastic) model. A fully implicit elastic predictor return mapping method, coupled with line search, is implemented within an implicit quasi-static finite element environment. The closed form of the associated consistent tangent operator is presented. The resulting implementation can deal with small and large power-law exponents. The uniaxial tensile and uniaxial compressive tests are performed to evaluate the asymmetric mechanical behavior at different normalized strain rates. Finally, iso-error maps are used to estimate the accuracy and stability of the model at different strain rates and stress states.

Chapter 6

Due to the large number of material parameters that need to be determined simultaneously for the Cazacu 06 constitutive model, the identification strategy needs to be carefully selected in order to avoid local minima in the solution. This Chapter presents five optimization methodologies to calibrate the material parameters for an

AZ31 alloy. In particular, the gradient method, a Genetic Algorithm (GA), a Particle Swarm Optimization (PSO) method, a GA combined with the gradient method, and the PSO combined with gradient method [27-29]. All the above algorithms are based on the use of an error minimization function for computing the anisotropic coefficients, which allows to consider the constraints of the solution. The comparison of these strategies is presented and discussed. These five optimization strategies are applied to the identify material parameters of Cazacu 06 model at two typical accumulated plastic strain levels, where the shape of the yield surface is, respectively, elliptical and triangular-like. The final errors between theoretical and experimental values are compared. The yield locus of these two levels obtained from the above five identified strategies are also drawn. Finally, the error contour nearby the global optimized result is drawn to better understand the selection of identification strategy.

Chapter 7

An extended new yield criterion under plane-stress conditions for HCP sheet metal forming operations is proposed in Chapter 7. This yield function is able to simultaneously capture the tension/compression strength asymmetry effect and the anisotropic behavior of HCP materials. This yield criterion is developed by introducing an additional asymmetrical parameter into the anisotropic Barlat 89 yield criteria. The additional material parameter c represents the ratio between the yield stress in tension and the yield stress in compression. All material parameters are calibrated for a typical magnesium alloy AZ31 with the hybrid Particle Swarming Optimization (PSO) algorithm mentioned in Chapter 6. The proposed model is assessed and compared with several commonly used anisotropic yield criteria, such as Hill *et al.*(1948) [19], Barlat *et al.* (2003) [16] and Cazacu *et al.* (2006) [17]. The evolutions of the yield surfaces calibrated by the above yield criteria are drawn. Two benchmark tests are used to validate the proposed model. The comparison between the experimental results and finite element simulations indicates that the proposed constitutive model provides very accurate predictions for the material response under multi-axial loading. The model is also able to describe very well strength differential effects and the anisotropic mechanical behavior. The deep drawing of a cylindrical cup, which is a well-established benchmark test, is presented to further verify the planar anisotropic behaviour through the prediction of the earing profile.

Chapter 8

In order to compensate for the material elastic recovery under sheet metal forming operations, it is necessary to quantitatively predict springback. This chapter analyses the springback behavior of HCP materials by analytical methods and numerical methods, respectively. In the first part of this chapter, the mechanics of plane strain bending in combination with axial tension of an elastic-plastic sheet material is considered. A generic analytical solution for arbitrary elasto-plastic material is proposed under different external tensile conditions. The analytical solution has been implemented computationally and is available by running a user developed program. The calculated strain distribution through thickness is also compared with experimental results. The results showed that the proposed analytical model predicts the main trends of the springback behavior of magnesium alloy sheets reasonably well considering the simplicity of the analytical approach. In addition, the generic analytical evolution of deformation region, including the neutral surface position, the tensile yield critical position, the compressive yield critical position, bending force, and bending moment–curvature relationships for various deformation stages is presented and analyzed. Sensitivity tests are also conducted for the effect of an applied tensile force, strain rate, and sheet thickness on the springback of asymmetric metals. Besides, the conventional symmetric constitutive equations are compared with the asymmetric constitutive equations from our model.

Although the use of analytical models is advantageous because of their simplicity, the application of these models is limited to simple geometries. The amount of springback also depends upon many process variables such as friction, temperature, variations in the thickness and mechanical properties of the incoming sheet material. Moreover, complex strain histories and highly nonlinear deformation of the material during the forming process increase the difficulty of predicting springback. Therefore, the most widely used approach for predicting springback is to carry out computer simulations that rely on accurate material constitutive models. In the second part of this chapter, the effect of hardening laws and yield function types on spring-back is investigated for the AZ31 magnesium alloy. Four different yield criteria of different complexity are evaluated in the present investigation: von Mises, Hill, Barlat, and Cazacu 06 criteria. Two hardening laws are considered. One is the classical isotropic hardening; and the other employs the separated stress-strain response under tension and compression as hardening law. Since Cazacu 06 is able to capture the SD effect, only the other three

yield criteria were used in conjunction with separated hardening laws. All constitutive models were implemented in a fully-implicit finite element scheme by developing user material subroutines for ABAQUS commercial software. Numerical integration through the thickness of shell elements creates an additional source of error in the springback prediction [30], particularly when the material undergoes elastic plastic deformations. All traditional integration rules require a significant number of integration points to capture a nonlinear stress profile, especially for materials with high asymmetrical mechanical behavior. Therefore, in this chapter, the Number of Integration Points (NIP) used in conjunction with the asymmetrical constitutive model is investigated to understand the influence of NIP on the springback response. Finally, the springback of an unconstrained cylindrical bending test is simulated for all the constitutive models. From the results of these simulations, some conclusions regarding the influence of the constitutive model for springback simulations are drawn.

Chapter 9

The flow behavior of the AZ31B magnesium alloy for extended range of strain rates is studied in this chapter. Aiming to describe the flow behavior under tensile stress states with different strain rates, three commonly used hardening laws are extended based on Perzyna's theory of viscoplasticity, which are Ludwik, Swift and Voce. By comparison of the error value between the fitted true stress and experimental stress at each true strain, the extended Voce strain rate sensitivity formulation is able to provide a considerable agreement with experimental curves. In order to describe the mechanical behavior under compressive stress states, a new mathematical model is studied. The related material parameters are obtained by fitting the proposed equation with the corresponding experimental data. The gradient method has been used to obtain the global optimal fitting parameters. All these experimental results are used to identify the material parameters in Cazacu 06 model at different strain rates. The yield loci at several levels of accumulated plastic strains are drawn. The material coefficients inside the yield function, as well as the size of the elastic domain, are considered to be functions of both the accumulated plastic strain and strain rate. A bilinear interpolation technique is introduced to determine the evolution of the yield surface based on the results from experimental results. The identified constitutive model was implemented within the implicit finite element code ABAQUS and used to simulate the benchmark test-square cup drawing subjected to different loading conditions. The analysis of the

results is helpful to understand the behavior of HCP metal sheets in high speed impact or forming process technology.

Chapter 10

In this chapter, the main topics addressed in the thesis are briefly summarized and the conclusions of this work are drawn from the results obtained in the preceding chapters. Furthermore, this thesis also gives further recommendations for future research.

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Chapter 2

Hexagonal Close-Packed Materials: Physical Aspects and Constitutive Modeling

If I have been able to see further, it was only because I stood on the shoulders of giants.
-----Isaac Newton

The whole chapter has been divided into two main sections. In the first section, some basic knowledge of crystal structure and deformation mechanisms of Hexagonal Close-Packed (HCP) structured metals was reviewed. In the second section, the phenomenological constitutive model was discussed. Several commonly used yield criteria in metal forming were summarized. The plastic flow rule and hardening rule were also discussed. The review of all above section will be given to provide some background to this research project.

2.1 Review of the crystal microstructure and plastic behavior of HCP materials

2.1.1 The crystallography of HCP metals

Three relatively simple crystal structures are found for most of the common metals: Face-Centered Cubic (FCC), Body Centered Cubic (BCC) and Hexagonal Close-Packed (HCP) [1]. The unit cell of the HCP structure is a hexagonal prism with atoms at its corners and atoms at the centers of its upper and lower faces.

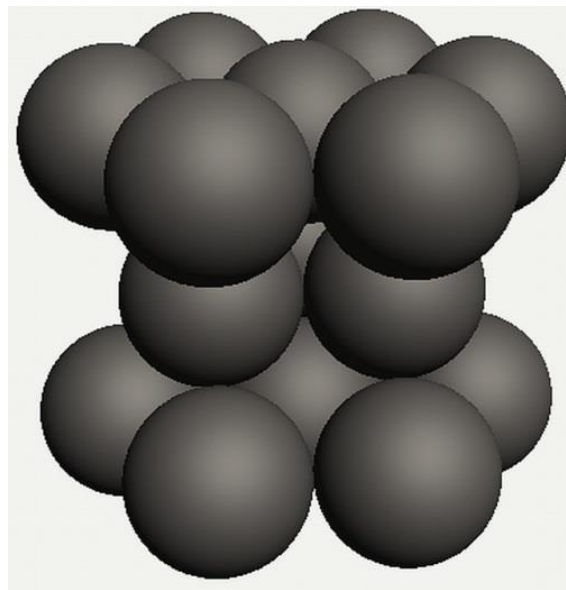


Figure 2.1: Hexagonal Close-Packed crystal structures.

There are also three atoms in a triangular configuration in a plane at half the height of the unit cell; each of these atoms is located directly above the center of an equilateral triangle of atoms in the basal plane, as shown in Figure 2.1. Since each of the corner atoms in the hexagon is shared by six adjacent cells and each of the face atoms is shared by two cells, the total number of atoms per unit cell in the HCP structure is six.

A comprehensive review of crystallography in HCP materials has been carried out by Partridge [2]. The primitive hexagonal unit cell (Figure 2.2 (a)) has axes $\bar{a}_1 = \bar{a}_2 \neq \bar{c}$ and corresponding angle $\bar{\alpha} = \bar{\beta} = 90^\circ, \bar{\gamma} = 120^\circ$; The symmetry of the hexagonal lattice can be illustrated by the hexagonal prism which can be constructed from three primitive hexagonal unit cells, such as shown in Figure 2.2 (b). In the HCP structure the atoms are stacked in a sequence of ABABAB..., as demonstrated in Figure 2.2 b.

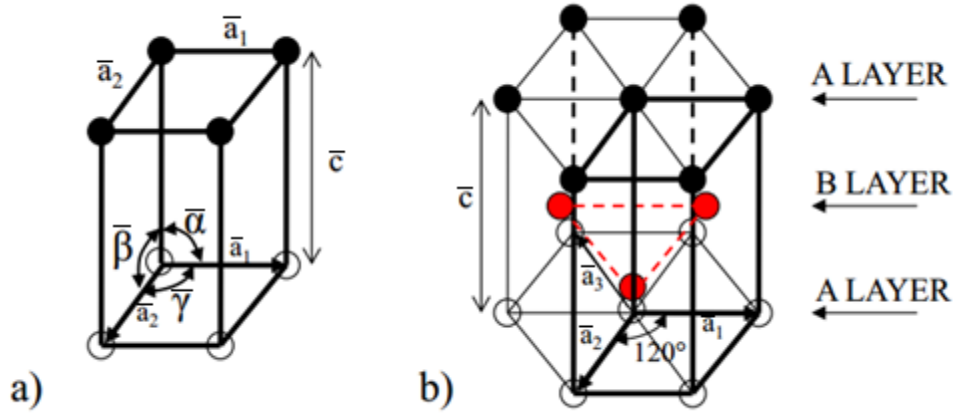


Figure 2.2: a) The primitive hexagonal unit cell illustrating the axes $\bar{a}_1 = \bar{a}_2 \neq \bar{c}$ and corresponding angles $\bar{\alpha} = \bar{\beta} = 90^\circ, \bar{\gamma} = 120^\circ$; b) the Hexagonal Close-Packed structures; the thick solid line in b) delineate the primitive hexagonal unit cell [3].

An ideal close packing of spherical atoms in this sequence generates a structure that is characterized in terms of lattice dimensions with a $\bar{c}/\bar{a}$ ratio of 1.632, where a and c represent, respectively, the short and long unit cell dimensions of Figure 2.2 a. None of these metals in their pure form has an ideal $\bar{c}/\bar{a}$ ratio. HCP metals are usually classified into two groups: those with $\bar{c}/\bar{a} > 1.633$ and those with $\bar{c}/\bar{a} < 1.633$. The former group contains only cadmium and zinc, the latter includes Mg, Ti and Zr *etc.* Magnesium and cobalt have a $\bar{c}/\bar{a}$ ratio within 1% of the ideal. However, the coordination number and the atomic packing factor for the HCP crystal structure are the same as for FCC: 12 and 0.74, respectively.

In the case of HCP metals, important crystallographic planes and directions of the hexagonal lattice can be described with a four index Miller-Bravais notation related to a coordinate system of three basal vectors $\bar{a}_i$ and the longitudinal axis $\bar{c}$.

2.1.2 Plastic behavior and deformation mechanism of HCP materials

Deformation mechanisms of metal can be categorized as crystallographic slip, deformation twinning and grain boundary sliding. Grain boundary sliding is negligible, except for a specific deformation condition such as superplasticity. Therefore, crystallographic slip and deformation twinning are two fundamental deformation mechanisms accommodating plastic deformation. These two deformation modes compete against each other when solid metals deform plastically under applied stresses and at a temperature where individual atoms are not mobile [2-4].

According to von Mises [5], five independent slip systems are needed to accommodate an arbitrary homogenous plastic deformation. Slip often dominates for high symmetry cubic crystal whereas deformation twinning is important for metals with lower symmetry HCP metals, where the five independent slip systems necessary for arbitrary shape change are not easily activated at room temperature. For instance, in FCC metals, the twelve $\{111\}\langle 110 \rangle$ slip systems provide five independent systems and satisfy this condition. However, the situation in HCP metals is much more complex owing to the following reasons: first, the deformation of HCP metals typically involves both slip and twinning, each slip and twinning system having different Critical Resolved Shear Stresses (CRSS); second, different families of slip systems and twin systems become activated depending on texture, loading, temperature and grain size. In order to know further, these two deformation mechanism are summarized in the next subsection.

a) Slip in HCP Crystals

Slip is one of the most important plastic deformation mechanisms in crystalline materials. It involves the motion of dislocations on a specific crystallographic plane along a specific crystallographic direction. The planes on which slip occurs are called slip planes, and the directions of the shear are the slip directions, which can be seen in Figure 2.3.

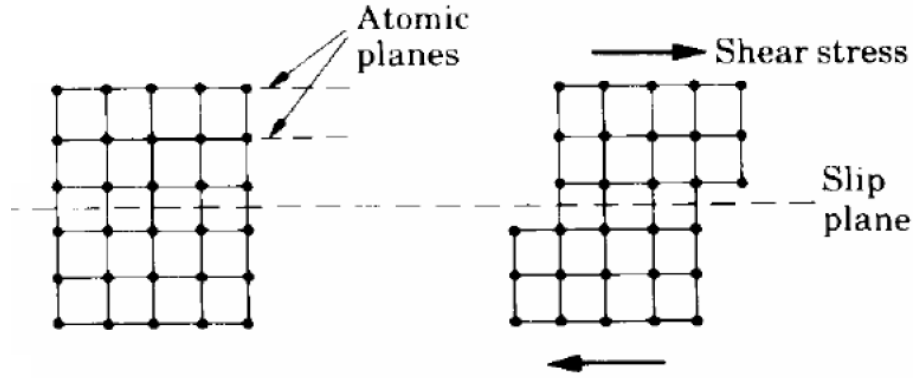


Figure 2.3: Slip deformation mechanisms [6].

These are crystallographic planes and directions that are characteristic of the crystal structure. Often, slip planes are the close packed planes and slip directions in the slip planes are the close packed directions. The magnitude and the direction of shear displacement on a particular plane are given by the Burgers vector [6].

For metals with Hexagonal Closed Pack (HCP) structure, the close-packed $\{0002\}$ basal planes and close-packed directions $\langle 11\bar{2}0 \rangle$ forms slip systems. At low temperatures, the three dominant sets of planes which contain this close-packed slip direction are: (i) the $\{0002\}$ basal plane, (ii) the three $\{10\bar{1}0\}$ prismatic planes, and (iii) the six $\{10\bar{1}1\}$ pyramidal planes.

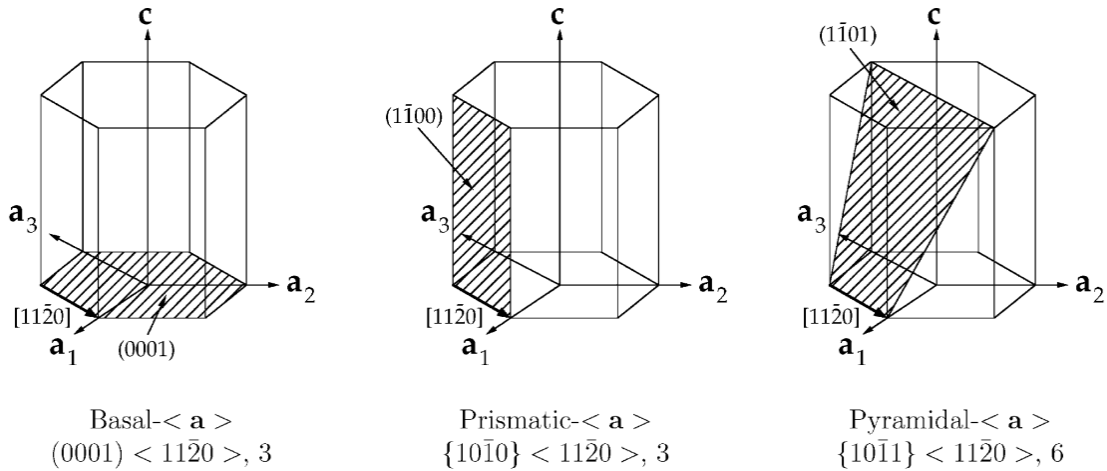


Figure 2.4: Frequently observed deformation modes in HCP metals: Basal $\langle \mathbf{a} \rangle$, prismatic $\langle \mathbf{a} \rangle$, and pyramidal $\langle \mathbf{a} \rangle$ slip system in HCP materials [7].

Crystallographic slip in HCP single crystals is commonly observed to occur on the basal- $\langle \mathbf{a} \rangle$ or prismatic- $\langle \mathbf{a} \rangle$ systems. The activation of pyramidal slip systems in polycrystalline aggregates occurs primarily due to the large stresses generated in grain-

boundary regions because of the misorientation between neighboring grains. Higher order slip systems on the pyramidal planes $\{10\bar{1}1\}$ in $\langle 11\bar{2}0 \rangle$ ($\langle \mathbf{a} \rangle$ type) or pyramidal $\{11\bar{2}2\}$ in $\langle \bar{1}123 \rangle$ directions ($\langle \mathbf{c} + \mathbf{a} \rangle$ type) which offers five independent slip systems on its own are present but are usually not favored as they have high critical resolved shear stress and are only reported to operate at higher temperatures [7]. As such, it is commonly believed that the improved ductility and formability at higher temperatures for magnesium alloy is the result of activation of higher order non-basal slip systems.

A non-exhaustive list of deformation modes frequently observed in HCP metals are presented in Figure 2.4. The total summary Slip system for HCP materials are listed in the Table 2.1. All the crystallographic planes and directions of the hexagonal lattice are described with a four index Miller-Bravais notation.

Table 2.1: Summary of Slip system for HCP materials.

Burgers vector	Slip plane	Slip direction	# of slip systems	
			Total	independent
$\langle \mathbf{a} \rangle$	basal $\{0001\}$	$\langle 11\bar{2}0 \rangle$	3	2
$\langle \mathbf{a} \rangle$	prismatic $\{10\bar{1}0\}$	$\langle 11\bar{2}0 \rangle$	3	2
$\langle \mathbf{a} \rangle$	pyramidal $\{10\bar{1}1\}$	$\langle 11\bar{2}0 \rangle$	6	4
$\langle \mathbf{c} + \mathbf{a} \rangle$	pyramidal $\{11\bar{2}2\}$	$\langle \bar{1}123 \rangle$	6	5

A slip system is only active if the resolved shear stress is greater than the Critical Resolved Shear Stress (CRSS) required for slip. Figure 2.5 presents the CRSS for basal slip, prismatic slip, pyramidal slip, and twinning as a function of temperature [7, 8]. At room temperature, only basal slip is active, which provides only two independent slip systems. The other slip systems must be provided by prismatic slip, pyramidal slip, or twinning. Prismatic slip and pyramidal slip require a much larger CRSS than basal slip, which makes them largely inactive at room temperature. Generally, there are too few active slip systems at room temperature for magnesium alloys to exhibit significant ductility. As temperature increases, the CRSSs for prismatic and pyramidal slip decrease significantly, and these additional slip systems become active. The increase in the number of active slip systems with increasing temperature results in a corresponding increase in the ductility of magnesium alloys.

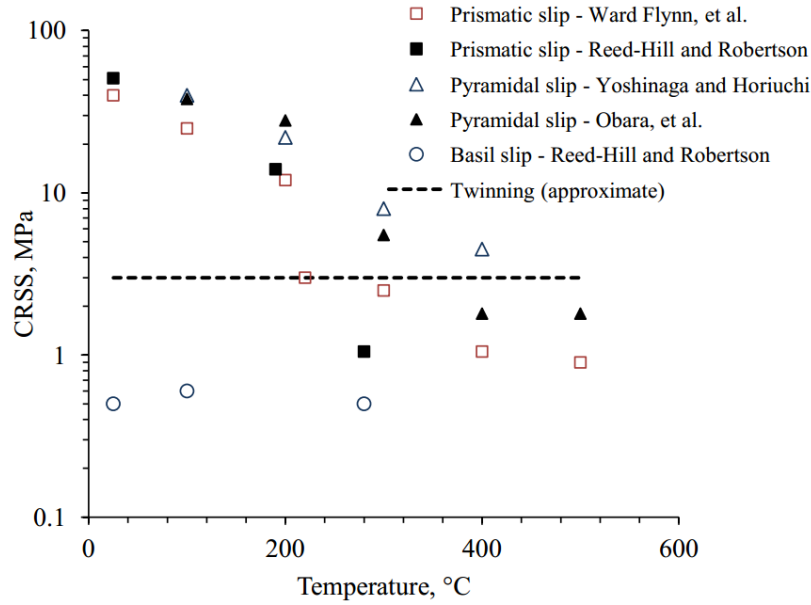


Figure 2.5: Critical Resolved Shear Stress (CRSS) is plotted against temperature for basal slip, prismatic slip, pyramidal slip, and twinning in magnesium [9].

b) Crystallography of twinning in HCP metals

Mechanical twinning is defined via a plane only, the twin plane. In contrast to slip, deformation by twinning is a homogeneous shear evenly distributed over a three-dimensional region with sharp boundaries. When mechanical deformation is created by twinning, the lattice structure of the material changes. The atoms move only a fraction of an interatomic space and this leads to a rearrangement of the lattice structure. Twinning is observed as wide bands under the microscope. These wide bands cannot be removed by polishing. The crystal structure of resulting twins is identical with that of parent matrix, but with a different orientation [10]. The total shear deformation from twinning is small and limited. A twinning mode is activated only under tension or compression (not both), depending upon the atomic structure. This is known as polarization. Deformation twins are typically in the shape of thin plates, as a result of the energetic barriers to the formation of twins [10-12].

The occurrence of slip is more common than twinning in metals with highly symmetric crystal structure, like cubic metals, where many symmetrically equivalent slip systems with low CRSS are available. The occurrence of twinning in metals is favored by the lack of slip activity such as deformation at low temperature or high strain rate. Twinning in magnesium alloy results in the reorientation of a volume of crystal planes so that dislocations may be placed in a favorable orientation with respect to the stress

direction so that slip can be continued which otherwise is immobile. Deformation twinning, however, is especially important for metals with low-symmetry crystal structures, like HCP metals, in which five independent slip systems necessary for arbitrary shape change are not easily activated at low temperatures or at high speeds. Therefore, ductility and formability of HCP metals and alloys are significantly influenced by the profuseness of operable twinning modes [11, 12].

Deformation twins are formed by a homogeneous simple shear of matrix lattice. Figure 2.6 illustrates the change from a sphere into an ellipsoid by the twinning shear, where the sphere is sheared along crystallographic direction η_{T1} with shear magnitude γ_{shear} . Only two material planes in the sphere remain undistorted after twinning namely, the K_{T1} (first undistorted plane) and K_{T2} (second undistorted but rotated plane) planes from convention. These two planes inclined at an angle θ_{shear} to each other. The magnitude of the twinning shear (γ_{shear}) is given by:

$$\gamma_{shear} = 2\cot(\theta_{shear}). \quad (2.1)$$

Plane K_{T1} , also called twinning plane, contains the crystallographic shear direction η_{T1} and experiences no displacement during twinning. However, plane K_{T2} is rotated to K'_{T2} by twinning. Plane K_{T2} intersects the plane of shear in direction η_{T2} and η'_{T2} before and after twinning, respectively.

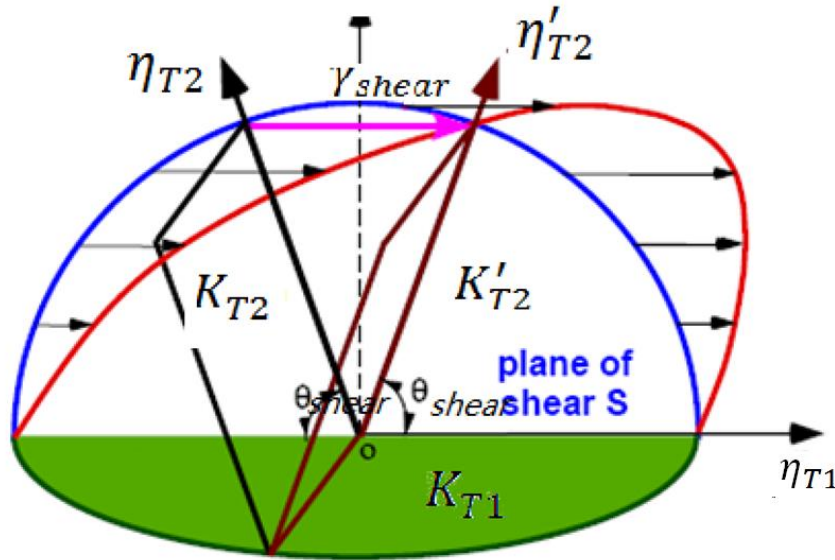


Figure 2.6: Crystallographic elements of twinning.

Four twinning elements, K_{T1} , K_{T2} , η_{T1} and η_{T2} are usually used to quote a twin, but only two of them, K_{T1} and η_{T2} , K_{T2} or η_{T1} are independent. Twin and matrix lattices are associated either by a reflection in certain plane like K_{T1} or by a rotation of 180° about

certain axis like η_{T1} . When twinning is incomplete, and for distance large compared with the twin lamella, the dimensional changes may be modified for directions lying in the quadrant bounded by K_{T2} and the plane normal to K_{T1} and η_{T1} . This effect is important when the twinning shear is large.

The most common deformation twinning mode in most HCP metals is of the $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$ type, which indicates that $\{10\bar{1}2\}$ is the twinning plane K_{T1} , $[10\bar{1}1]$ is the corresponding crystallographic shear direction η_{T1} . A schematic of these twinning mechanisms is shown in Figure 2.7. The direction and the magnitude of twinning shear in HCP metals depend on $\bar{c}/a$ ratios, packing densities, interplanar spacings, and stacking fault energies. The conjugate twinning plane K_{T2} and direction η_{T2} are crystallographic ally equivalent of K_{T1} and η_{T1} . When a “compressive” twin, which shortens $\bar{c}$ -axes, is activated, the twinning shear direction for $\{10\bar{1}2\}$ twin lies along $\langle 10\bar{1}1 \rangle$ if $\bar{c}/a > \sqrt{3}$ (as for Zn and Cd), when a “tensile” twin operates, the twinning shear direction for $\{10\bar{1}2\}$ twin lies along $\langle 10\bar{1}\bar{1} \rangle$ if $\bar{c}/a < \sqrt{3}$ (as for Mg, Zr and Ti) (Stoloff and Gensamer, 1963). The dependence of twinning shear strain (γ_{shear}) on the $\bar{c}/a$ ratio due to twinning on the $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$ twin system can be expressed by:

$$\gamma_{shear} = \frac{3 - (\bar{c}/a)^2}{\sqrt{3}\bar{c}/a}. \quad (2.2)$$

Only the stress which can drag the $\bar{c}$ -axis back to the ideal ratio could activate the twinning. For instance, $\{10\bar{1}2\}\langle 10\bar{1}1 \rangle$ twinning system in Mg or its alloys can only be activated when the $\bar{c}$ axis is elongated under certain stress state, such as tension in $\bar{c}$ direction or compression parallel to the basal plane. Opposite stress states that shorten the c axis cannot activate this twinning. This is called the polar nature of deformation twinning. Twinning can cause large lattice rotations, although the strain achieved is small. The angle of rotation and the shear strain of this deformation process depend on the activated twinning system. Thus, even at low deformations, twinning causes the basal poles to orient rapidly towards the normal direction; that is, parallel to the compressive force.

A significant consequence of the polarization of deformation twinning is that strongly textured Mg alloys can exhibit a tension/compression asymmetry, where the yield strength of the same sample condition is different if exposed to opposite strain fields, such as uniaxial tension versus compression. The yield asymmetry in Mg alloy can be quite remarkable; this asymmetry is also a function of the deformation temperature.

Considering that extension and contraction twins have different twinning stresses, the transition between slip and twin dominated flow will occur at different temperatures affecting the yield strength and the resulting yield asymmetry.

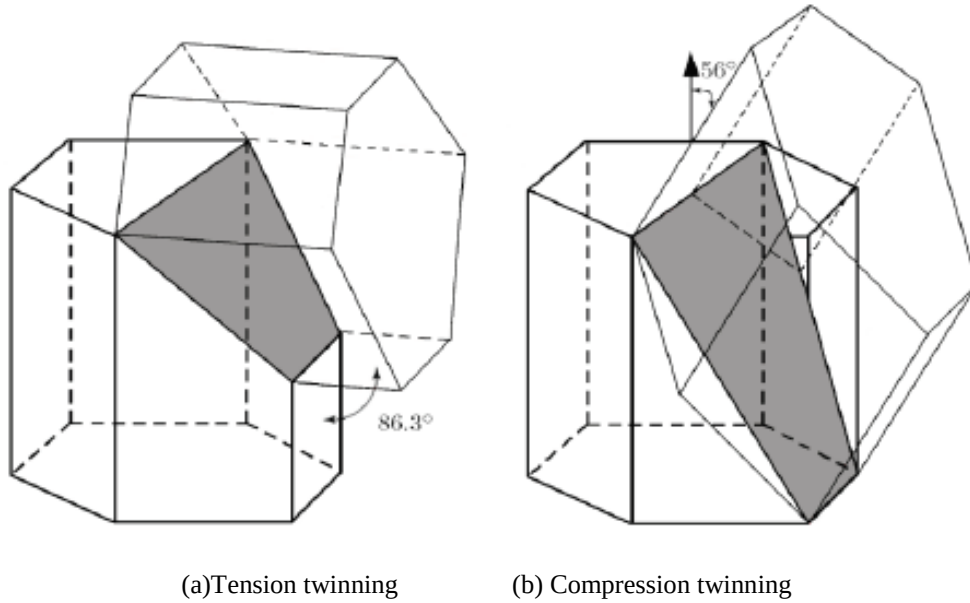


Figure 2.7: Twinning modes for HCP materials.

2.2 Review of continuum phenomenological constitutive models

In general, a material model is used to describe the stress-strain behavior of a material. It can be divided into two classes. The first model class is a physically-based model type representing a mechanical behavior. This kind of model is used to represent both microscopic phenomena, such as microstructure evolution, and macroscopic mechanical behavior, such as creep and relaxation, on a material scale, where the material mechanical behavior has been modeled by analyzing and measuring the dislocation density using transmission electron microscopy *etc*; then the microstructure size and the macroscopic stresses can be predicted [10, 12, 13]. The second model class is a phenomenological-based model based on the results of mechanical tests on a material. This material model is used to predict the mechanical behavior of a material and the model can be further used to predict the stress-strain behavior of a mechanical structure using, typically, a finite element simulation. The second type of model will be the focus of this study.

For sufficiently small values of stress and strain, a metal will reassume its original shape upon unloading. When loaded beyond this reversible (elastic) range, the specimen will not reassume its original shape upon unloading, but will exhibit a permanent (plastic) deformation. In this section the focus is on the conditions under which a material passes

from the elastic to the plastic state. A continuum phenomenological plasticity model is commonly used to describe the plastic behavior of a material in a general stress state. A complete phenomenological plasticity model is comprised of a yield surface expressing a relationship between the stress components at the moment when plastic ‘yielding’ occurs, the flow rule expressing the relationship between the components of the strain-rate, and a hardening law rule describing the evolution of the initial yield stress during the forming process. To establish the incremental equations for elastic-plastic hardening materials, several conditions are necessary to ensure an appropriate representation of plastic flow: continuity, uniqueness, irreversibility, consistency, and the flow rule. All these conditions, as well as a yield function, are used to determine a general stress-strain relation for plastic deformation.

2.2.1 Yield criteria

A yield surface is defined as the closed hyper surface which encloses the elastic region in the six-dimensional stress space. The shape of the yield surface depends on the entire deformation path from the reference state [14, 15]. Basically the yield function may be defined in two different ways: either by assuming that plastic yield begins when some physical quantity (energy, stress, *etc.*) attains a critical value or by approximating experimental data by an analytical function. The yield surfaces for actual materials are mainly smooth, but may have or develop pointed pyramidal or conical vertices. Physical theories of plasticity imply the formation of a corner or vertex at the loading point on the yield surface [15]. Experimental evidence, on the other hand, suggests that, while relatively high curvature at the loading point is often observed, sharp corners are seldom seen. Experiments also indicate that yield surfaces for metals are convex in Cauchy stress space, if elastic response within the yield surface is linear and unaffected by plastic flow [15, 16].

Phenomenological yield surfaces, although nothing more than a curve fit, have a number of advantages over yield surfaces obtained with polycrystalline theories. First of all, it usually have a simpler mathematical form, and easy to understand and manipulate. The physical quantities that are involved are measurable at the macroscopic level, thus, there is no need for averaging. Moreover, phenomenological yield surfaces do not require complex inputs, like the Orientation Distribution Function (ODF). Generally, all the required data are obtainable from a small number of standard tests like uniaxial tension and biaxial extension. From the experimental data, the surface can be identified

with very little computational cost. Another remarkable advantage is that phenomenological yield surfaces can also account for sources of anisotropy other than texture. Texture, in fact, is not the only origin of anisotropy in a material: elongated or flattened inclusions, voids and grain boundaries contribute as well.

In case of a multiaxial stress, the most general form of this function may be written

$$\Phi(\boldsymbol{\sigma}, \sigma_Y) = 0, \quad (2.3)$$

where σ_Y is the yield stress obtained from a simple test (tension, compression or shearing). $\boldsymbol{\sigma}$ are the six components of the stress tensor expressed in the reference frame of interests. Usually in the analysis of sheet metal forming all out of plane components of the stress vector are assumed to be equal to zero and the yield criterion is defined in the plane stress space. The yield surface reduces to a curve in the plane of the principal stresses σ_1 and σ_2 . The schematic diagram of the yield surface in plane stress space can be seen in Figure 2.8. All the points inside the surface ($\Phi < 0$) are related to an elastic state. The points on the surface ($\Phi = 0$) are related to a plastic state.

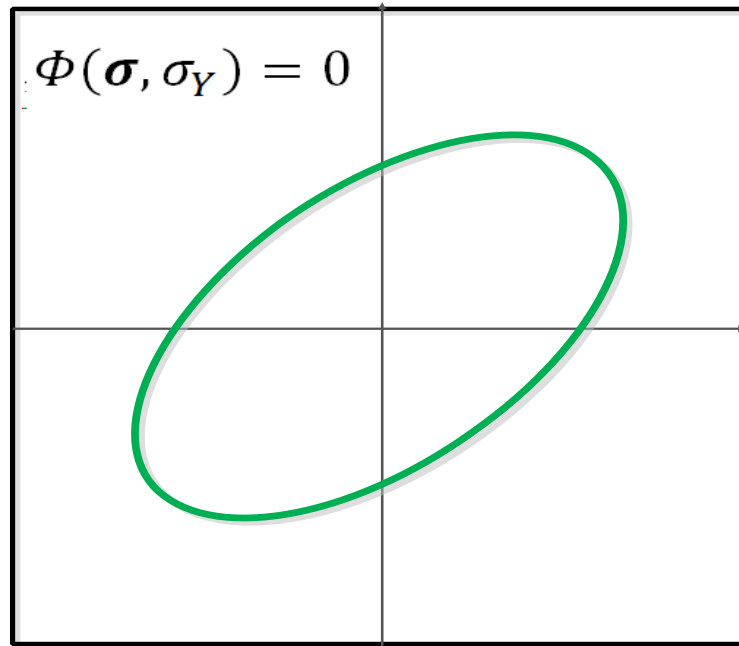


Figure 2.8: The schematic diagram of yield surface.

Numerous phenomenological yield criteria were proposed in last decades [12, 17, 18]. It can be generally categorized into two kinds, dependent on the material properties: isotropic and anisotropic yield criteria. The next subsection will summarize briefly the most popular and widely used yield criteria.

a) Isotropic yield criteria

If material is homogenous (it has the same properties throughout its solid body), then material can be considered as isotropic (its properties are independent of direction). Here two classical isotropic yield criteria are introduced.

von Mises yield criterion

This criterion was proposed independently by Huber [19] and von Mises [20] and further developed by Hencky [21]. It is based on the observation that a hydrostatic pressure cannot cause plastic yielding on the material. Thus, the conclusion that only the elastic energy of distortion influences the transition from elastic to a plastic state comes naturally. For simplicity, in the following text the criterion will often be referred to as the von Mises criterion. This criterion can be written as

$$\sigma_{eq} = \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)}{2}}. \quad (2.4)$$

This criterion gives better predictions for most polycrystalline metals than Tresca criteria. Moreover, it is more suitable in numerical models, since sharp corners in the yield surface are absent.

Drucker yield criterion

In order to represent the experimental data located between Tresca and von Mises yield surfaces, Drucker [22] proposed the following criterion:

$$\sigma_{eq} = J_2^3 - c_{dr} J_3^2, \quad (2.5)$$

where J_2 and J_3 are the second and third invariants of the stress tensor, respectively, and c_{dr} is a constant, which defines the curvature of the yield function.

$$J_2 = \frac{1}{2} s_{ij} s_{ij}, \quad (2.6)$$

$$J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}. \quad (2.7)$$

Here, s_{ij} are components of the stress deviator, which can be calculated by

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} I. \quad (2.8)$$

In order to ensure that the yield surface is convex, c_{dr} is limited to a given numerical range, $c_{dr} \in [-27/8, 2.25]$.

This yield criterion can be used to describe strength differential effect between tension and compression state.

b) Anisotropic yield criteria

Due to different and complex phenomena occurring during the rolling process, the flat rolled products such as sheet metals experience various degrees of anisotropy. As a consequence, the flow behaviors characterized by the Tresca and von Mises functions only provide a rough and frequently incorrect, estimate of the plastic properties. Therefore, use of appropriate anisotropic plasticity models becomes crucial to describe the mechanical behavior and ensure reasonable accuracy in finite element predictions. Normally rolled sheets exhibit orthotropic symmetry, that is, they have three orthogonal symmetry planes. These planes are identified with the direction of their unit normal: the rolling, the thickness, and the transverse direction. Yield surfaces that have orthotropic symmetry can be easily constructed by fixing the coordinate axes along the directions of symmetry. Several anisotropic yield functions have been proposed for sheet metal forming analysis in order to correlate with the experimental behavior as accurately as possible. However, for the sake of brevity the most frequently-used and only symmetric pressure insensitive anisotropic yield functions for sheet metals are reviewed in this subsection.

Hill 48 criterion

When a metal is subjected to large plastic deformations, the grain sizes and grain orientations are changed in the direction of deformation. As a result, the plastic yield behavior of the material shows directional dependency. Metals that have been processed by rolling, for example, will have characteristic material directions. The form of anisotropy exhibited by rolled sheets is such that the material properties are symmetric in about three mutually orthogonal planes. The lines of intersection of these planes form an orthogonal set of axes known as the principal axes of anisotropy. The axes are (a) in the rolling direction, (b) normal to the sheet, (c) in the plane of the sheet but normal to rolling direction. This form of anisotropy is called orthotropy. To predict the anisotropic yield behavior more accurately, Hill [23, 24] developed several yield criteria for anisotropic plastic deformations. Hill (1948) proposed a yield condition to the quantitative treatment of plastic anisotropy. It is a generalization of the von Mises criterion. The yield criterion is expressed by a quadratic function of the following type

$$\sigma_{eq} = [F_{Hill}(\sigma_{22} - \sigma_{33})^2 + G_{Hill}(\sigma_{33} - \sigma_{11})^2 + H_{Hill}(\sigma_{11} - \sigma_{22})^2 + 2L_{Hill}\sigma_{23}^2 + 2M_{Hill}\sigma_{13}^2 + 2N_{Hill}\sigma_{12}^2]^{1/2}, \quad (2.9)$$

where, F_{Hill} , G_{Hill} , H_{Hill} , L_{Hill} , M_{Hill} , N_{Hill} are material parameters. In the case of isotropy

$$L_{Hill} = M_{Hill} = N_{Hill} = 3F_{Hill} = 3G_{Hill} = 3H_{Hill}. \quad (2.10)$$

This criterion is restricted to materials with orthotropic physical symmetry, *i.e.*, to those with three mutually orthogonal symmetry planes. These can generally be inferred from the symmetry of the strain path employed to produce the anisotropy. The constants can be calculated with three tensile tests. The relative simplicity of Hill's yield surface makes it the most widely used anisotropic yield function, despite its many shortcomings: first of all, the surface is quadratic, thus the yield locus will always appear as an ellipse, no matter which section is taken, while the experiments show that the real yield locus is not an ellipse, at least in the plane-stress plane. Hill's surface has also another limitation: when Lankford coefficient $r < 1$, Hill's criterion implies that the ratio of the yield stress in biaxial tension to the yield stress in tension σ_u/σ_Y must be less than one, and conversely when $r < 1$, $\sigma_u/\sigma_Y > 1$. There are some materials in which this is not true. The F_{Hill} , G_{Hill} , H_{Hill} , L_{Hill} , M_{Hill} , and N_{Hill} parameters characterize the current state of anisotropy, and can be determined with uniaxial tension and shear test experimental data. Material coefficients of Hill'48 are given as follows

$$G_{Hill} + H_{Hill} = \frac{1}{X_{Hill}^2}; \quad (2.11)$$

$$F_{Hill} + H_{Hill} = \frac{1}{Y_{Hill}^2}; \quad (2.12)$$

$$G_{Hill} + F_{Hill} = \frac{1}{Z_{Hill}^2}; \quad (2.13)$$

$$2L_{Hill} = \frac{1}{R_{Hill}^2}; \quad (2.14)$$

$$2M_{Hill} = \frac{1}{S_{Hill}^2}; \quad (2.15)$$

$$2N_{Hill} = \frac{1}{T_{Hill}^2}. \quad (2.16)$$

where X_{Hill} , Y_{Hill} and Z_{Hill} are the tensile yield stresses in the principal directions of anisotropy and R_{Hill} , S_{Hill} and T_{Hill} the yield stresses in shear with respect to these axes. When simulating the sheet metal forming process, plane stress is assumed ignoring the stress in the thickness direction due to its insignificance compared with the other two orthogonal directions. Also assuming that the principal directions of the stress tensor are coincident with the anisotropic axes ($\sigma_{xx} = \sigma_1$, $\sigma_{yy} = \sigma_2$, $\sigma_{xy} = 0$), Hill's 48 yield criteria can be written as a dependence of the principal stress in the following form

$$\sigma_{eq}^2 = \sigma_1^2 - \frac{2r_0}{1+r_0}\sigma_1\sigma_2 + \frac{r_0(1+r_{90})}{r_{90}(1+r_0)}\sigma_2^2. \quad (2.17)$$

This yield criterion has the advantage that its basic assumption is easy to understand and only require a small number of mechanical parameters to determine the yields function. The parameters included in the yield functions have a direct physical meaning.

Hill 79 yield criterion

Following the work of Woodthorpe and Pearce [25] on the anomalous behavior of aluminum, Hill in 1979 [26] put forward a new non quadratic yield function similar to that of Hosford yield criteria for orthotropic metals. This function was expressed in its general form as well as in four special cases. If the directions of the principal stresses coincide with axes of anisotropy, the criterion is written as

$$\begin{aligned} \sigma_{eq}^{m_{Hill79}} = & f_{Hill79}|\sigma_2 - \sigma_3|^{m_{Hill79}} + g_{Hill79}|\sigma_3 - \sigma_1|^{m_{Hill79}} \\ & + h_{Hill79}|\sigma_1 - \sigma_2|^{m_{Hill79}} + a_{Hill79}|2\sigma_1 - \sigma_2 - \sigma_3|^{m_{Hill79}} \\ & + b_{Hill79}|2\sigma_2 - \sigma_1 - \sigma_3|^{m_{Hill79}} + c_{Hill79}|2\sigma_3 - \sigma_2 - \sigma_1|^{m_{Hill79}}. \end{aligned} \quad (2.18)$$

Here the σ_i are principal components and m is an exponent which can be a non integer. Seven parameters have to be determined experimentally, i.e. a_{Hill79} , b_{Hill79} , c_{Hill79} , f_{Hill79} , g_{Hill79} , h_{Hill79} , m_{Hill79} . As noted by Hosford [27], this general form recognizes the possibility of planar anisotropy. However, it cannot be used for loading conditions which involve shear relative to 1, 2 and 3 principal axes of anisotropy. If in-plane isotropy is assumed, then the criterion is valid, i.e., shear stress terms are not necessary since the 1 and 2 axes may be oriented in any direction in the sheet, m_{Hill79} (with $m_{Hill79} > 1$, integer or non-integer) is a material coefficient and is determined by matching the effective stress-strain curves for uniaxial and biaxial tests.

This yield criteria can be used to describe the woodthrope-Pearce anomalous behavior of materials, but it cannot describe the behavior of materials having $r_0/r_{90} \neq 1$ and $\sigma_0/\sigma_{90} \neq 1$. Due to the non-integer value of the exponent m_{Hill79} , it requires special numerical procedures even for the solution of quite simple cases.

Hill 90 yield criterion

The Hill 48 yield criterion can deal with the anisotropic metal, but it cannot represent “anomalous” behavior. This quadratic yield criterion cannot describe the plastic behavior of some materials such as aluminum alloys. This yield criterion proposed by Hill in 1979 [26] can only be used when the directions of the principal stresses are

coincident with the orthotropic axes. The proposed Hill 90 yield criteria can describe yield behavior of material with a general coordinate system [28]. Based on the Mohr's circle, Hill (1990) developed another non-quadratic yield function that is also restricted to plane-stress conditions:

$$2\sigma_{eq}^{m_{Hill90}} = |\sigma_{11} + \sigma_{22}|^{m_{Hill90}} + \left(\frac{\sigma_b}{\tau}\right)^{m_{Hill90}} |(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2|^{\frac{m_{Hill90}}{2}} + |\sigma_{11}^2 + \sigma_{22}^2 + 2\sigma_{12}^2|^{\frac{m_{Hill90}}{2}-1} \{-2a_{Hill90}(\sigma_{11}^2 - \sigma_{22}^2) + b_{Hill90}(\sigma_{11} - \sigma_{22})^2\}, \quad (2.19)$$

where τ is the shear yield stress, σ_b is the yield stress in equibiaxial tension, , and two parameters a_{Hill90} , and b_{Hill90} are defined as

$$a_{Hill90} = \frac{1}{4} \left[\left(\frac{2\sigma_b}{\sigma_0} \right)^{m_{Hill90}} - \left(\frac{\sigma_b}{\sigma_0} \right)^{m_{Hill90}} \right], \quad (2.20)$$

and

$$b_{Hill90} = \frac{1}{2} \left[\left(\frac{2\sigma_b}{\sigma_0} \right)^{m_{Hill90}} + \left(\frac{\sigma_b}{\sigma_0} \right)^{m_{Hill90}} \right] - \left(\frac{2\sigma_b}{\sigma_{45}} \right)^{m_{Hill90}}. \quad (2.21)$$

Being the value of the m_{Hill90} exponent as

$$m_{Hill90} = \frac{\ln[2(r_{45}+1)]}{\ln\left(\frac{2\sigma_b}{\sigma_{45}}\right)}, \quad (2.22)$$

where σ_0 and σ_{90} are the yield stresses along the anisotropic axes, and m_{Hill90} is the non-quadratic yield exponent. The yield exponent for this formulation will vary for different materials and is loosely coupled to the r value. In general, a material with a $r > 1$ will have $m_{Hill90} > 2$ whereas for $r < 1$, $1 < m_{Hill90} < 2$ will hold.

This yield criterion allows describing both the first order anomalous behavior ($r < 1, \sigma_b > \sigma_u$) and the second order anomalous behavior ($r_0 < r_{90}, \sigma_0 < \sigma_{90}$), and is able to describe very well the variation of the anisotropy coefficient and of the uniaxial yield stress in the plane of the sheet. However, the formulation of this model is not user-friendly and need large CPU time when used in the numerical simulation of sheet metal forming process.

K-B 93 yield criterion

Karafillis and Boyce (1993) [29] constructed a non-quadratic anisotropic yield criterion (so-called $K - B$) by mixing two isotropic yield functions

$$2\sigma_{eq}^{m_{KB}} = (1 - c_{KB})(|S_1 - S_2|^{m_{KB}} + |S_2 - S_3|^{m_{KB}} + |S_3 - S_1|^{m_{KB}}) + c_{KB} \frac{3^{m_{KB}}}{2^{m_{KB}-1}+1} (|S_1|^{m_{KB}} + |S_2|^{m_{KB}} + |S_3|^{m_{KB}}), \quad (2.23)$$

where c_{KB} is a material parameter ($0 \leq c \leq 1$), and S_1, S_2 and S_3 are the principal values of the Isotropic Plasticity Equivalent (IPE) tensor ($\mathbf{L}_{KB}$) that can be found in

Karafillis and Boyce (1993). In this formulation, tensor $\mathbf{L}_{KB}$ can describe different kinds of material anisotropy (triclinic, monoclinic, orthotropic, *etc.*). For plane stress conditions these values can be calculated as

$$S_{1,2} = \frac{S_{xx}+S_{yy}}{2} \pm \sqrt{\left(\frac{S_{xx}-S_{yy}}{2}\right)^2 + S_{xy}^2}, \text{ and } S_3 = S_{zz}. \quad (2.24)$$

If $c_{KB} = 0$ and the $\mathbf{L}_{KB}$ component coefficients are considered isotropic, then this yield function represents the Tresca yield criterion when $m_{KB} \rightarrow \infty$, and the von Mises yield criterion when $m_{KB} = 2$, and of course for orthotropic coefficients the Hill '48 yield function is restored. A good advantage of the criteria was that it uses only a uniaxial tensile test for identifying the material parameters. From a mathematical point of view the method proposed by Karafillis and Boyce was both elegant and rigorous. The disadvantage was the complicated format and the complex identification procedure of the parameters.

Barlat 89 yield criterion

Barlat and Lian (1989) [30] proposed a non-quadratic yield function (so-called *Yld89*) to model the behavior of orthotropic metallic sheets (typically rolled materials) under plane stress. This model is restricted to plane-stress conditions. The corresponding yield function, written at the outset exclusively in terms of in-plane components of the stress tensor, reads

$$2\sigma_{eq}^{M_{BL}} = a_{BL}|K_1 + K_2|^{M_{BL}} + a_{BL}|K_1 - K_2|^{M_{BL}} + c_{BL}|2K_2|^{M_{BL}}, \quad (2.25)$$

where K_1, K_2 is described as

$$K_1 = \frac{\sigma_{xx}+h_{BL}\sigma_{yy}}{2}; K_2 = \sqrt{\left(\frac{\sigma_{xx}-h_{BL}\sigma_{yy}}{2}\right)^2 + (b_{BL})^2\sigma_{xy}^2}. \quad (2.26)$$

The formulation is expressed in an x, y, z coordinate system, not necessary coincident with the principal directions. The included constants a_{BL} , h_{BL} , and b_{BL} are calculated from relations based on the measured r -values at rolling, transverse and 45° orientations. The materials parameters a_{BL} , c_{BL} , h_{BL} and b_{BL} can be determined using the Lankford coefficients, r_0 , r_{45} , and r_{90} , based upon crystalline plasticity considerations. For face centered cubic (FCC) materials, Barlat determined that a value of $M_{BL} = 8$ is appropriate, while for body centered cubic (BCC) materials a value of $M_{BL} = 6$ is appropriate based on crystalline plasticity calculations. a_{BL} and h_{BL} is described as

$$a_{BL} = 2 - c_{BL} = 2 - 2\sqrt{\frac{r_0}{1+r_0}\frac{r_{90}}{1+r_{90}}}; \quad (2.27)$$

$$h_{BL} = \sqrt{\frac{r_0}{1+r_0} \frac{1+r_{90}}{r_{90}}}. \quad (2.28)$$

The proposed yield criterion has great flexibility, and includes all planar stress components as well as the planar anisotropy. With correct exponent M_{BL} , and using the equibiaxial stress, a better agreement with yield locus predicted by the Bishop-Hill theory is achieved. The reduced numbers of the mechanical parameters make this criterion relatively easy for identification. The coefficients inside this model have not a direct and intuitive physical significance. This model only can be used in the plane stress state.

Yld91, Yld 94 and Yld97 yield criteria

A six component non-quadratic yield function (so-called *Yld91*) without restrictions to any stress state was developed by Barlat *et al.* [31] for orthotropic materials based on the isotropic Hosford criteria, which uses a linear transformation of the stress tensor

$$2\sigma_{eq}^{m_{Yld91}} = |S_1 - S_2|^{m_{Yld91}} + |S_2 - S_3|^{m_{Yld91}} + |S_3 - S_1|^{m_{Yld91}}, \quad (2.29)$$

where S_i are the principal values of an isotropic plasticity equivalent (IPE) stress tensor defined by

$$\mathbf{S} = \mathbf{L}_{YLD} \boldsymbol{\sigma}, \quad (2.30)$$

in which $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{L}_{YLD}$ is a symmetric and traceless fourth-ranked tensor. Tensor $\mathbf{L}_{YLD}$ introduces the material anisotropy into the formulations and its definition can be found in Barlat *et al.* [31].

In order to improve the performance of the above criterion, Barlat *et al.* [32] proposed the following yield criterion (so-called *Yld94*) to capture the experimental and polycrystalline yield surface shapes for alloys with very large amounts of cold reduction

$$2(\sigma_{eq}^{Yld94})^{m_{Yld94}} = a_1^{Yld94} |S_1 - S_2|^{m_{Yld94}} + a_2^{Yld94} |S_2 - S_3|^{m_{Yld94}} + a_3^{Yld94} |S_3 - S_1|^{m_{Yld94}}, \quad (2.31)$$

where S_i are the principal values of the same IPE tensor ($\mathbf{L}_{YLD}$) as the K-B yield function.

Coefficients a_k^{Yld94} are further parameters to describe anisotropy and are defined as [32]

$$a_k^{Yld94} = a_x p_{1k}^2 + a_y p_{2k}^2 + a_z p_{3k}^2, \quad (2.32)$$

where p_{ik} are the components of the transformation matrix $\mathbf{p}^{Yld}$ between the principal axes of anisotropy to the principal axes of $\mathbf{S}$ and a_x , a_y , and a_z are material coefficients. To overcome some limitations of *Yld94*, Barlat *et al.* [33] further proposed an improved

yield criterion (so-called *Yld97*) for plane stress cases. They defined another set of parameters β_j that is chosen to represent the angle between the anisotropy axes and the direction associated with S_1 or S_3 ($S_1 \geq S_2 \geq S_3$), whichever has the highest absolute value. Then the new coefficients a_k^{Yld94} are defined as functions of β_j , p_{ik} and other material coefficients. The formulations are rather complex and are not explained here; interested readers are referred to the paper [33].

The computed yield surfaces are also in good agreement with those predicted by the Bishop-Hill theory and experiments. A very good agreement between theory and experiment has also been found for the distribution of the uniaxial yield stresses and anisotropy coefficients in the plane of sheet. However, a large number of parameters ensure a good flexibility of the criteria; it also implies a large number of mechanical tests.

Yld2000–2d yield criterion

In order to remove the disadvantages of the *Yld94* and *Yld 97* yield criteria, but aiming to preserve their flexibility, Barlat [34] proposed a new model particularized for plane stress (2D). The *Yld2000* anisotropic yield function presented by Barlat *et al.* [34] is another accurate yield function for aluminum alloy sheets. The Barlat anisotropic yield functions are derived based on the approach of linear transformations of a stress tensor. The anisotropic yield function for plane stress plasticity can be expressed in general form:

$$2\sigma_{eq}^{m_{Yld2000}} = \bar{\phi}' + \bar{\phi}'', \quad (2.33)$$

where $m_{Yld2000}$ is an exponent determined based on the crystallographic structure of the material, and

$$\bar{\phi}' = |\bar{X}'_1 + \bar{X}'_2|^{m_{Yld2000}}, \quad (2.34)$$

$$\bar{\phi}'' = |2\bar{X}''_2 + \bar{X}''_1|^{m_{Yld2000}} + |2\bar{X}''_1 + \bar{X}''_2|^{m_{Yld2000}}, \quad (2.35)$$

$\bar{X}'_i$ and $\bar{X}''_i$, $i = 1, 2$, are the principal values of two linear transformations on the stress deviator. The linear transformations on the stress deviator are

$$\begin{bmatrix} \bar{X}'_{11} \\ \bar{X}'_{22} \\ \bar{X}'_{12} \end{bmatrix} = \begin{bmatrix} \bar{C}'_{11} & \bar{C}'_{12} & 0 \\ \bar{C}'_{21} & \bar{C}'_{22} & 0 \\ 0 & 0 & \bar{C}'_{66} \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{bmatrix}; \quad (2.36)$$

$$\begin{bmatrix} \bar{X}_{11}'' \\ \bar{X}_{22}'' \\ \bar{X}_{12}'' \end{bmatrix} = \begin{bmatrix} \bar{C}_{11}'' & \bar{C}_{12}'' & 0 \\ \bar{C}_{21}'' & \bar{C}_{22}'' & 0 \\ 0 & 0 & \bar{C}_{66}'' \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{bmatrix}, \quad (2.37)$$

where s_{ij} are the deviatoric stresses, $\bar{C}_{ij}'$ and $\bar{C}_{ij}''$ are material constants. Due to the fact that eight coefficients are incorporated in the linear transformations, we need eight material characteristics for evaluating them. We can calibrate the material constants $\bar{C}_{11}'$, $\bar{C}_{22}'$, $\bar{C}_{66}'$, $\bar{C}_{11}''$, $\bar{C}_{12}''$, $\bar{C}_{21}''$, $\bar{C}_{22}''$ and $\bar{C}_{66}''$ from the yield stresses σ_0 , σ_{45} , σ_{90} , and σ_b and the anisotropy parameters r_0 , r_{45} and r_{90} . Due to the complexity of the formulation, this criterion is not user-friendly. Some parameters could be evaluated by crystal plasticity models [34].

Cazacu 04 yield criterion

To describe both the asymmetry between yield in tension and compression and the anisotropy due to deformation twinning, the Drucker model [22] was extended to orthotropy by a fourth-order linear transformation operator ($\hat{\mathbf{L}}$) by Cazacu *et al* [35, 36]. Following this theory, the original Cauchy stress $\{\boldsymbol{\sigma}\}$ is transformed to a transformed tensor $\{\boldsymbol{\Sigma}\}$, which is defined as

$$\Sigma_{ij} = \hat{L}_{ijkl} \sigma_{kl}. \quad (2.38)$$

The tensor $\hat{\mathbf{L}}$ satisfies: (a) the symmetry conditions: $\hat{L}_{ijkl} = \hat{L}_{jikl} = \hat{L}_{jilk} = \hat{L}_{klij}$ ($i, j, k, l = 1, \dots, 3$), (b) the requirement of invariance with respect to the symmetry group of the material, and (c) $\hat{L}_{1k} + \hat{L}_{2k} + \hat{L}_{3k} = 0$ for $k=1, 2$ and 3 , which ensures that $\boldsymbol{\Sigma}$ is traceless and consequently yielding is independent of hydrostatic pressure. So the $\hat{\mathbf{L}}$ can be represented by

$$\hat{\mathbf{L}} = \begin{bmatrix} \frac{a_2+a_3}{3} & -\frac{a_3}{3} & -\frac{a_2}{3} & 0 & 0 & 0 \\ -\frac{a_3}{3} & \frac{a_1+a_3}{3} & -\frac{a_1}{3} & 0 & 0 & 0 \\ -\frac{a_2}{3} & -\frac{a_1}{3} & \frac{a_1+a_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}. \quad (2.39)$$

The Cazacu 04 anisotropic yield criteria can be expressed as

$$\sigma_{eq} = \hat{A} (J_2^{0/2} - \hat{c} J_3^0)^{1/3}, \quad (2.40)$$

where J_2^0 , J_3^0 is the second invariant of deviator transformed stress, the third invariant of deviator transformed stress, respectively, which can be obtained as

$$J_2^0 = \frac{1}{2} \text{tr}(\mathbf{\Sigma}^2); J_3^0 = \frac{1}{3} \text{tr}(\mathbf{\Sigma}^3). \quad (2.41)$$

$\hat{A}$ is a constant defined as

$$\hat{A} = 3[(a_2^2 + a_3^2 + a_2 a_3)^{\frac{3}{2}} - \hat{c}(a_2 + a_3)a_2 a_3]^{-1/3}, \quad (2.42)$$

and c is a material parameter, which can be expressed in terms of the yield in uniaxial tension, σ_T , and the yield in uniaxial compression, σ_c , as

$$\hat{c} = \frac{3\sqrt{3}(\sigma_T^3 - \sigma_c^3)}{2(\sigma_T^3 - \sigma_c^3)}. \quad (2.43)$$

To ensure convexity, the material constant $\hat{c}$ is limited to the range $[-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{4}]$.

When $\hat{c} = 0$ the proposed criterion reduces to the von Mises yield criterion. For any $\hat{c} \neq 0$, the yield function is homogeneous of degree three in stress and represents a “triangle” with rounded corners.

This model can be used to capture the asymmetrical mechanical behavior and anisotropic properties for titanium alloy. However, this criterion is not so sufficient flexible to describe the shape of yield surface, which neither triangular nor elliptical.

Cazacu 06 yield criterion

In order to capture the SD effect and anisotropic mechanical behavior with more flexibility, Plunkett *et al* [10] proposed a new general formulation with principal value of the deviatoric stress. It can be expressed by

$$\sigma_{eq}^{\tilde{n}} = (|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}}, \quad (2.44)$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are the principal values of transform stress $\mathbf{\Sigma}$. $\tilde{k}$ is the introduced material parameter.

$$\tilde{k} = \frac{1 - \tilde{H}}{1 + \tilde{H}}, \quad (2.45)$$

$$\tilde{H}\left(\frac{\sigma_t}{\sigma_c}\right) = \left[\frac{-2^{\tilde{n}} + 2\left(\frac{\sigma_t}{\sigma_c}\right)^{\tilde{n}}}{2 - 2^{\tilde{n}}\left(\frac{\sigma_t}{\sigma_c}\right)^{\tilde{n}}} \right]^{1/\tilde{n}}. \quad (2.46)$$

For $\tilde{k} \in [-1, 1]$ and any integer $\tilde{n} \geq 1$, the anisotropic yield function is convex in the variables $\Sigma_1, \Sigma_2, \Sigma_3$ (principal transformed stresses).

The transformed tensor $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} = \tilde{\mathbf{C}}\mathbf{S}. \quad (2.47)$$

where $\mathbf{S}$ is the original deviatoric stress tensor; $\tilde{\mathbf{C}}$ is the transformed tensor.

The only restrictions imposed on the tensor $\tilde{\mathbf{C}}$ are: (i) to satisfy the major and minor symmetries and (ii) to be invariant with respect to the orthotropy group. Thus, for 3-D stress conditions the orthotropic criterion involves 9 independent anisotropy coefficients; it reduces to the von Mises isotropic yield criterion for $\tilde{\mathbf{C}}$ equal to the 4th order identity tensor. It is worth noting that although the transformed tensor is not deviatoric, the orthotropic criterion is insensitive to hydrostatic pressure and thus the condition of plastic incompressibility is satisfied.

Let (x, y, z) be the reference frame associated with orthotropy. In the case of a sheet, x , y and z represent the rolling, transverse, and the normal directions. Relative to the orthotropy axes (x, y, z) , the tensor $\tilde{\mathbf{C}}$ is represent by

$$\tilde{\mathbf{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (2.48)$$

So the transformed stress can be obtained by

$$\boldsymbol{\Sigma} = \tilde{\mathbf{C}}\mathbf{S} = \tilde{\mathbf{C}}\tilde{\mathbf{A}}\boldsymbol{\sigma} = \tilde{\mathbf{L}}\boldsymbol{\sigma}, \quad (2.49)$$

where the matrix $\tilde{\mathbf{A}}$ is used to transform the stress to deviatoric stress, which can be written by

$$\tilde{\mathbf{A}} = \frac{2}{3} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \quad (2.50)$$

So the transformed matrix $\tilde{\mathbf{L}}$ can be obtained by

$$\tilde{\mathbf{L}} = \begin{bmatrix} \frac{(2C_{11}-C_{12}-C_{13})}{3} & \frac{(-C_{11}+2C_{12}-C_{13})}{3} & \frac{(-C_{11}-C_{12}+2C_{13})}{3} & 0 & 0 & 0 \\ \frac{(2C_{12}-C_{22}-C_{23})}{3} & \frac{(-C_{12}+2C_{22}-C_{23})}{3} & \frac{(-C_{12}-C_{22}+2C_{23})}{3} & 0 & 0 & 0 \\ \frac{(2C_{13}-C_{23}-C_{33})}{3} & \frac{(-C_{13}+2C_{23}-C_{33})}{3} & \frac{(-C_{13}-C_{23}+2C_{33})}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix}. \quad (2.51)$$

Let

$$\tilde{\phi}_1 = \frac{(2C_{11}-C_{12}-C_{13})}{3}, \tilde{\phi}_2 = \frac{(2C_{12}-C_{22}-C_{23})}{3}, \tilde{\phi}_3 = \frac{(2C_{13}-C_{23}-C_{33})}{3}, \quad (2.52)$$

$$\tilde{\varphi}_1 = \frac{(-C_{11}+2C_{12}-C_{13})}{3}, \tilde{\varphi}_2 = \frac{(-C_{12}+2C_{22}-C_{23})}{3}, \tilde{\varphi}_3 = \frac{(-C_{13}+2C_{23}-C_{33})}{3}, \quad (2.53)$$

$$\tilde{\psi}_1 = \frac{(-C_{11}-C_{12}+2C_{13})}{3}, \tilde{\psi}_2 = \frac{(-C_{12}-C_{22}+2C_{23})}{3}, \tilde{\psi}_3 = \frac{(-C_{13}-C_{23}+2C_{33})}{3}. \quad (2.54)$$

So the transformed stress can be simplified to note by

$$\Sigma = \begin{bmatrix} \tilde{\phi}_1\sigma_{11} + \tilde{\phi}_1\sigma_{22} + \tilde{\psi}_1\sigma_{33} \\ \tilde{\phi}_2\sigma_{11} + \tilde{\phi}_2\sigma_{22} + \tilde{\psi}_2\sigma_{33} \\ \tilde{\phi}_3\sigma_{11} + \tilde{\phi}_3\sigma_{22} + \tilde{\psi}_3\sigma_{33} \\ C_{44}\sigma_{12} \\ C_{55}\sigma_{23} \\ C_{66}\sigma_{13} \end{bmatrix}. \quad (2.55)$$

The most important advantage of this yield criterion consists in its capability to provide an accurate description of the tension/compression behavior specific to the magnesium and titanium alloys.

A brief conclusion about the application of above mentioned yield criteria is drawn as follows:

For isotropic material, the von Mises criterion is preferred. However, in the particular case of plane stress, the Tresca criteria can also be applied. For anisotropic materials, the present state-of-art is somewhat confusing since most of the above-described yield criteria are still being used. The choice of yield criteria will depend on experimental restrictions, and no whether biaxial yield stress needs to be measured. Current commercial FEM software frequently uses Hill 48 and Hill 90 yield criteria. Hill 48 yield criterion can be used obtain a simple approximation of the anisotropic behavior of sheet metal except aluminum alloys. Hill 90, Barlat 96 and K-B 93 yield criteria can be used to obtain more accurate models of the plastic behavior of anisotropic sheet metals, especially for aluminum alloys. Cazacu 04 and Cazacu 06 recently are widely used to describe the material with SD effect, particularly for HCP crystalline structure materials.

2.2.2 Hardening rule

During the plastic deformation process, the shape of the material changes and shows increased amount of strength properties, hence called as strain hardening. Strain hardening is an intrinsic hardening mechanism caused by the interactions of dislocations with one another resulting in an increased stress for continued dislocation motion and the characteristic parabolic nature of the plastic region in a typical polycrystalline stress-

strain curve. In plasticity, the hardening rule is used to describe the material behavior during the plastic deformation. Cyclic loading is a common type of loading in sheet metal forming. Therefore, in this section, two simplest approaches: Isotropic hardening and Kinematic hardening will be introduced [37-39].

a) Isotropic hardening

Yield loci of a hardening material are usually displayed as iso-contours of the plastic equivalent strain or plastic work called as iso-strain or iso-work, respectively, which is expressed as the area under the stress-plastic strain curve. In order to describe the evolution of the loading surface, the hardening effect of the material can be modeled by introducing internal variables. Different representation of common functions can be used to provide a best fit the experimental stress-strain curves of different materials. Generally, a scalar internal variable and a scalar associate force are suitable for modeling the material hardening

$$\alpha = \bar{\varepsilon}^p, \quad (2.56)$$

as an internal variable the accumulated equivalent plastic strain is applied. The definition of the accumulated equivalent plastic strain can be written as:

$$\frac{d\bar{\varepsilon}^p}{d\lambda} = \frac{\sigma : \dot{\varepsilon}^p}{\sigma_{eq}}. \quad (2.57)$$

The equivalent plastic strain $\bar{\varepsilon}^p$ is obtained by assuming the equivalence of the stress power given by the product of the equivalent quantities and the contraction of the tensorial quantities on two indices.

Thus for a given yield criterion and flow rule, hardening behavior in any process can be predicted from the knowledge of the function $\sigma_Y(\bar{\varepsilon}^p)$, and this function may, in principle, be determined from a simple standard test (such as uniaxial tension test).

So many functions have been proposed for the amount of expansion of the yield surface as a function of plastic strain. For the isotropic hardening, the yield function equation is written as:

$$\Phi(\sigma, \bar{\varepsilon}^p) = \sigma_{eq}(\sigma) - \sigma_Y(\bar{\varepsilon}^p) = 0. \quad (2.58)$$

In a mechanical model it is preferable to use a simple empirical law that approximates the data, such as power law, Swift law, Voce hardening law, etc.

Essentially, isotropic hardening in rate-independent plasticity corresponds to a local expansion of the yield surface. This results in a global uniform expansion of yield surface, without shape change. If the yield stress in tension increases due to hardening

the compression yield stress grows the same amount even though you might not have been loading the specimen in compression. It neglects the anisotropic effect on the subsequent yielding induced by deformation like Bauschinger effect which is observed experimentally.

b) Kinematic hardening

In the case of monotonically increasing loading, it is often reasonable to assume that any hardening that occurs is isotropic. If material elements undergo non-monotonous deformations, such isotropic hardening might not be so effective, even though deformations are approximately proportional. When sheet parts are removed from tools after forming, material elements experience elastic unloading and springback. During this reverse loading, material elements usually demonstrate the Bauschinger effect and the plastic behavior of material (as a part of cyclic deformation) is different in many aspects like yielding, hardening, and even elastic recovery. It is difficult to determine the hardening behavior or the expansion (contraction), translation and distortion of the yield surface quantitatively without knowledge of the stress strain relation for plastic deformation. In classical kinematic hardening models, the yield surface translates freely in any direction but is not permitted to rotate. In the modeling of cyclic deformation of materials, so-called cyclic plasticity, kinematic hardening must be considered because it is able to model different situations in loading and reverse loading conditions more realistically [38, 40]. Besides the Bauschinger effect, Ratcheting is another fact that is imperative to the definition of isotropic and kinematic hardening. Ratcheting is the accumulation of plastic strain, cycle-by-cycle, for stress amplitude with a non-zero mean stress. As loading is repeated, each consecutive hysteresis loop will displace forward in a pertinent rate due to the failure of complete closure of each loop. Ratcheting is generally considered to be dominated by kinematic hardening.

In order to reproduce the Bauschinger effect, the yield surface equation combined with kinematic hardening generally can be written as:

$$\Phi(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \sigma_{eq}(\boldsymbol{\eta}(\boldsymbol{\sigma}, \boldsymbol{\beta})) - \sigma_Y = 0, \quad (2.59)$$

where

$$\boldsymbol{\eta}(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \boldsymbol{\sigma} - \boldsymbol{\beta}, \quad (2.60)$$

is the relative stress tensor, defined as the difference between the stress $\boldsymbol{\sigma}$ and backstress tensor $\boldsymbol{\beta}$. The back stress tensor is the thermodynamical force associated with kinematic hardening. It represents the translation of the yield surface and determines the location

of the center of the yield surface in the space of stresses. σ_Y represents the initial yield strength.

A number of kinematic hardening models have been proposed to describe elasto-plastic mechanical behavior under cyclic loading conditions, such as Prager's linear kinematic hardening model, Ziegler's linear kinematic model, and Armstrong–Frederick hardening model *etc* [18, 40].

c) *Mixed isotropic/kinematic hardening*

Rather than purely isotropic or purely kinematic hardening, real-life materials show in general a combination of both; that is, under plastic straining, the yield surface expands/shrinks and translates simultaneously in stress space. Thus, more realistic plasticity models can be obtained by combining the above laws for isotropic and kinematic hardening.

2.2.3 Flow rule

Once the material is loaded beyond plastic limit the material will yield and plastic strain will increase. The material deformation behavior in plastic region is described by the plastic strain [18, 40]. The flow rule relates the change of plastic strain tensor to the change of stress state. The complete characterization of the general plasticity model requires the definition of the evolution laws for the internal variables associated with the dissipative phenomena.

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma} \mathbf{N}, \quad (2.61)$$

where $\mathbf{N}$ is the flow tensor, which can be written as

$$\mathbf{N} = \mathbf{N}(\boldsymbol{\sigma}, \bar{\boldsymbol{\varepsilon}}^p), \quad (2.62)$$

$\dot{\gamma}$ is the introduced plastic multiplier, which should abide by the complementarity equation.

In the formulation of multidimensional elasto-plasticity models, it is often convenient to define the flow rule in terms of a flow (or plastic) potential. If the plastic potential Ψ is differentiable, the evolution of the plastic strain is assumed to be proportional to the gradient of the plastic potential Ψ in stress space

$$\mathbf{N}(\boldsymbol{\sigma}, \bar{\boldsymbol{\varepsilon}}^p) = \frac{\partial \Psi}{\partial \boldsymbol{\sigma}}, \quad (2.63)$$

where Ψ is the plastic potential which is a function of the stress tensor and internal variables.

In general case, the plastic strain rate increment tensor $\dot{\epsilon}^p$ is independent of the yield function, namely,

$$\Psi \neq \Phi, \quad (2.64)$$

which is called as non-associative plasticity.

In this case, the plastic strain increment is not normal to the yield surface. In other words, the yield and plastic potential functions respectively describe the elastic limit and plastic strain rate direction independently.

For materials that obey either Drucker's or Ilyushin's postulate [41], given by the appropriate stress or strain cycle inequalities, it follows that the plastic strain increment must be co-directional with the outward normal to a locally smooth yield surface, while at a vertex it must lie within or on the cone of limiting outward normal. It means that the yield function is equal to the plastic potential that is known as associative plasticity

$$\Psi = \Phi. \quad (2.65)$$

The associative plastic theory is frequently used to describe deformation flow behavior and plays an important role in the guarantee of stability. The non-associative plastic flow theory is applied in some materials which dissipate energy by friction or with internal-microscopic frictional effects, such as geomaterials like rocks and soils.

The requirement of differentiability of the flow potential is, however, too restrictive and many practical plasticity models are based on the use of a non-differentiable Ψ . In such cases, the function Ψ is called pseudo-potential or generalized potential and the formulation of the evolution laws for internal variables can be dealt with by introducing the concept of subdifferential sets, which generalizes the classical definition of derivative.

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Chapter 3

Continuum Mechanics and Finite Element Method

What we need is not the will to believe but the will to find out.
-----Bertrand Russell

In this chapter, the basic concepts of continuum mechanics, laws of thermodynamics and constitutive theory of continuous media are briefly reviewed. This chapter is separated into three sections: the first section contains the fundamental concepts of deformation kinematics, various strain stress measures from the continuum mechanics point of view; in the second section, several laws of thermodynamics are presented, these principles are used to construct the constitutive equation to describe an objective physical phenomenon, the last section reviews briefly the formulation of quasi-static finite element method.

It is important to remark that the contents of this chapter are summarized from many established publications [1-3]. The entire notations used in this chapter are referred to the book [3].

3.1 Kinematics of deformation

Kinematics is the study of motion without regards to the forces responsible for that motion. The equations of motion are used to characterize the movement of an object. In this section, the theory related to the description of kinematics of deformation is summarized, where the concepts of motion and deformation are addressed.

3.1.1 Material and Spatial Description of Motion

Two descriptions may be used to characterize the motion of a solid: material or spatial descriptions. The material description, also called Lagrangian description, characterizes the deformation of the solid with reference to the undeformed configuration of the solid. Alternatively, all fields inherent with the motion of a solid may be described using a spatial description or Eulerian description. Assumed that the locations of material points of a three-dimension body in its reference configuration are denoted by vectors $\mathbf{X}$, the general motion of a continuum can be described by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad (3.1)$$

where, $\mathbf{x}$ denotes the location in deformed configuration at time t , which is continuously differentiable with respect to $\mathbf{X}$.

The components of $\mathbf{X}$ are the material coordinates of the particle, while those of $\mathbf{x}$ are the spatial coordinates.

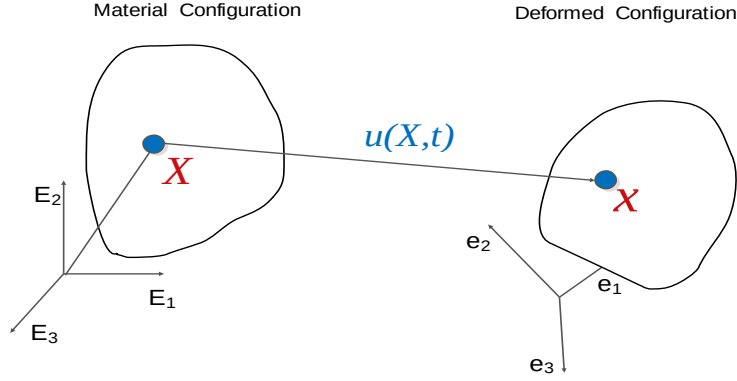


Figure 3.1: Configurations of a deformable body.

3.1.2 The deformation gradient and polar decomposition

Deformation at a material point $\mathbf{X}$ of a body is characterized by the changes of distances between any pair of material points within the small neighborhood of $\mathbf{X}$. An infinitesimal material element $d\mathbf{X}$ at the reference configuration is transformed through motion as

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \quad (3.2)$$

where, $d\mathbf{x}$ represents the transformed material element at deformed configuration.

The tensor $\mathbf{F}$ is called the deformation gradient at $\mathbf{X}$, which can also be written as

$$\mathbf{F} = \mathbf{x} \otimes \nabla^0 = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad (3.3)$$

in the deformed configuration at time t , and the gradient operator ∇^0 is defined with respect to material coordinates.

Using the previous definition together with the displacement motion, after a straightforward algebra manipulation, the equation (3.3) can be re-written in the following form:

$$\mathbf{F}\{\mathbf{X}, t\} = \mathbf{I} + \nabla_p \mathbf{u}, \quad (3.4)$$

where, $\mathbf{I}$ denotes the second order identity tensor. The deformation gradient is a second-order two point tensor and in general is not symmetric. It plays a key role in describing the local deformation in the vicinity of a particle.

In the spatial description, the deformation gradient is given by the following expression:

$$\mathbf{F}\{\mathbf{x}, t\} = [\mathbf{I} - \nabla_{\mathbf{x}}\mathbf{u}]^{-1}, \quad (3.5)$$

The deformation gradient $\mathbf{F}$ represents an affine mapping of the neighborhood of a material particle from the reference to deformed configuration. To further explain the deformation at any material point, by the polar decomposition theorem, the deformation gradient $\mathbf{F}$ can be decomposed into the product of a proper orthogonal tensor and a positive-definite symmetric tensor, such that

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}. \quad (3.6)$$

Each of these tensors is important in the development of strain measures. The symmetric tensor $\mathbf{U}$ is the right stretch tensor, with a basis in the reference configuration. $\mathbf{V}$ is the left stretch tensor, which is an object in the current configuration, and $\mathbf{R}$ is the rotation tensor, which is a local rotation tensor, connecting both configurations. Evidently,

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T, \quad (3.7)$$

where, the term $\mathbf{R}^T$ represents the transposed of the rotation tensor, so that $\mathbf{V}$ and $\mathbf{U}$ share the same eigenvalues.

3.1.3 Strain measures

From the description in the section 3.1.2, within an infinitesimal neighborhood of a generic material particle p , pure rotations can be distinguished from pure stretching by means of the polar decomposition of the deformation gradient. Under the action of pure rotations, the distances between particles remain fixed. Its reference configuration is a rigid deformation. Under stretching, the region surrounding p is strained, which can be characterized by $\mathbf{U}$ or $\mathbf{V}$, the distance change between material particles. Owing to that there are virtually infinite possibilities to measure strain, the choice of a strain measure remains arbitrary, which is dictated by mathematical and physical convenience. A well-known family of Lagrangian strain tensors, which is based on the Lagrangian triad, is defined by:

$$\mathbf{E}^{(m)} = \begin{cases} \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), & m \neq 0, \\ \ln[\mathbf{U}], & m = 0 \end{cases}, \quad (3.8)$$

where, m is a real number and $\ln[\mathbf{U}]$ denotes the tensor logarithm of the right stretch tensor $\mathbf{U}$.

Similarly, based on the left stretch tensor, the Eulerian counterpart of the Lagrangian family of strain measures above is defined by:

$$\boldsymbol{\varepsilon}^{(m)} = \begin{cases} \frac{1}{m}(\mathbf{V}^m - \mathbf{I}), & m \neq 0 \\ \ln[\mathbf{V}], & m = 0 \end{cases}. \quad (3.9)$$

3.1.4 Stress measures

The stress in a continuum solid is a consequence of the surface forces which are transferred within it. These surface forces may be mathematically quantified through the Cauchy axiom.

It is possible to prove that the Cauchy traction depends linearly on the normal unit vector $\mathbf{n}$ to the considered surface, seen in Figure 3.2. If this relation is satisfied then there is a second order tensor, $\boldsymbol{\sigma}$, such that the Cauchy force is given by:

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}, \quad (3.10)$$

where, $\mathbf{t}$ is the surface traction.

The second-order tensor is commonly as Cauchy stress tensor and it is characterized for being symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \quad (3.11)$$

while the Cauchy stress tensor characterize the stress, in a generic point of a continuum solid, with regard to the deformed configuration of the solid, in some circumstances, it is convenient to define it considering the undeformed configuration. Thus, the material version of the Cauchy stress tensor is known as the First Piolla-Kirchhoff stress tensor $\mathbf{P}$ and is mathematically defined as

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad (3.12)$$

where, $J = \det [\mathbf{F}]$ is the determinant of the deformation gradient.

3.2 Kinetics of deformation

Development of proper constitutive equations for a specific material and determination of the evolution laws for assumed internal state variables characterizing the material's behavior are the main challenges of the modern constitutive modeling. This can be effectively achieved, so far, through the thermodynamic principles by enforcing the balancing laws, the conservation of mass, the conservation of linear and angular momentums, and the first and second laws of thermodynamics. This section reviews some basic concepts of mechanics and thermodynamics of continuum body.

In order to state the fundamental laws of thermodynamics, it is necessary to introduce the scalar fields θ , e , s and r defined over body Ω which represent, respectively, the temperature, specific internal energy, specific entropy and the density of heat

production. In addition, $\mathbf{b}$ and $\mathbf{q}$ will denote the vector fields corresponding, respectively, to the body force (force per unit volume in the deformed configuration) and heat flux.

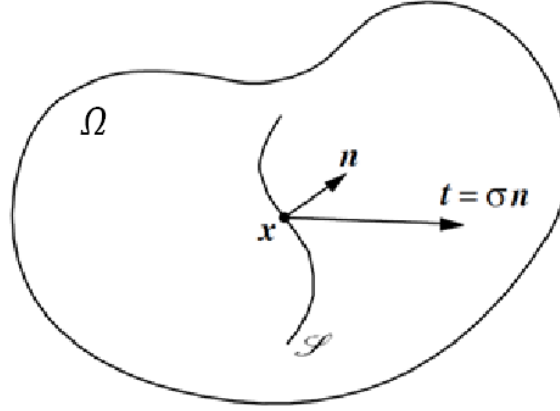


Figure 3.2: Schematic diagram of the Cauchy stress[3].

3.2.1 Conservation of mass

The principle of conservation of mass states that mass is neither created nor destroyed within a system. Even though we can have terms which represent sinks or sources of mass, these are mathematical constructs that allow us to transfer mass to or from the system surroundings. Therefore, the postulate of conservation of mass requires that

$$\dot{\rho} + \rho \operatorname{div}_x \dot{\mathbf{u}} = 0. \quad (3.13)$$

3.2.2 Momentum balance

The conservation of momentum principle is in fact the generalization of Newton's second law of motion for continuum mechanics. It can be stated as follows: the rate of change of the total momentum of any given set of particles equals the vectorial sum of all the forces acting on this set of particles. The balance of momentum for Ω can be expressed by the following partial differential equation with boundary condition:

$$\operatorname{div}_x \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \varphi(\Omega); \quad (3.14)$$

$$\mathbf{t} = \boldsymbol{\sigma} \mathbf{n} \quad \text{in } \varphi(\partial\Omega). \quad (3.15)$$

where $\mathbf{n}$ is the outward unit vector normal to the deformed boundary $\varphi(\partial\Omega)$ of Ω and $\mathbf{t}$ is the applied boundary traction vector field on $\varphi(\partial\Omega)$.

The above momentum balance equations are formulated in the spatial (deformed) configuration. Equivalently, they may be expressed in the reference (or material) configuration of Ω in terms of the first Piola–Kirchhoff stress tensor as

$$\operatorname{div}_p \mathbf{P} + \bar{\mathbf{b}} = \bar{\rho} \ddot{\mathbf{u}} \quad \text{in } \Omega; \quad (3.16)$$

$$\bar{\mathbf{t}} = \mathbf{P}\mathbf{m} \quad \text{in } \partial\Omega, \quad (3.17)$$

where $\text{div}_p \mathbf{P}$ represents the material divergence, $\bar{\mathbf{b}}$ is the reference body force measured per unit volume in the reference configuration, which can be written as

$$\bar{\mathbf{b}} = J\mathbf{b}; \quad (3.18)$$

$\bar{\rho}$ is the reference density (mass per unit volume in the reference configuration), which can be calculated by

$$\bar{\rho} = J\rho; \quad (3.19)$$

$\bar{\mathbf{t}}$ is the reference boundary traction (boundary force per unit reference area) and $\mathbf{m}$ is the outward normal to the boundary of Ω in its reference configuration.

3.2.3 The first principle of thermodynamics

The first principle of thermodynamics states that the time rate of change of the total energy of a continuum body is equal to the sum of the work done by the surroundings on the body and the heat added to the body. The total energy of a particle is the sum of the kinetic and potential energy. Before stating this principle, it is convenient to introduce the product

$$\boldsymbol{\sigma}:\mathbf{D},$$

which represents the stress power per unit volume in the deformed configuration of a body. The first principle of thermodynamics is mathematically expressed by the equation:

$$\rho\dot{e} = \boldsymbol{\sigma}:\mathbf{D} + \rho r - \text{div}_x \mathbf{q}, \quad (3.20)$$

which means that the rate of internal energy per unit deformed volume must equal the sum of the stress power and heat production per unit deformed volume minus the spatial divergence of the heat flux.

3.2.4 The second principle of thermodynamics

The first principle of thermodynamics is a statement of the energy balance, which applies regardless of the direction in which the energy conversion between work and heat is assumed to occur. The second principle of thermodynamics imposes restrictions on possible directions of thermodynamic processes. Clausius invented the thermodynamic potential - the entropy - to describe this uni-directional and irreversible degradation of energy. Formulated in terms of entropy, the second principle of thermodynamics says that whenever some form of energy is transformed into heat, the

global entropy increases. It can at best stay constant for reversible processes but can never decrease. It is expressed by means of the inequality:

$$\rho \dot{s} + \operatorname{div}_x \left[\frac{\mathbf{q}}{\theta} \right] - \frac{\rho r}{\theta} \geq 0. \quad (3.21)$$

3.2.5 The Clausius-Duhem inequality

By combination of the first and the first and second principles states above, we obtain the classic Clausius-Duhem (C-D) inequality by:

$$\rho \dot{s} + \operatorname{div}_x \left[\frac{\mathbf{q}}{\theta} \right] - \frac{1}{\theta} (\rho \dot{e} - \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div}_x \mathbf{q}) \geq 0. \quad (3.22)$$

As a macroscopic theory, irreversible thermodynamics does not give any explanation on the origin of entropy. Similarly to the case of plastic strains, the manipulation of entropy and other thermodynamic potentials will rely on postulated functions, valid over finite domains and containing coefficients to be determined by experiments.

The introduction of the specific free energy ψ (also known as the Helmholtz free energy per unit mass), defined by:

$$\psi = e - \theta s, \quad (3.23)$$

together with the relation:

$$\operatorname{div}_x \left[\frac{\mathbf{q}}{\theta} \right] = \frac{1}{\theta} \operatorname{div}_x \mathbf{q} - \frac{1}{\theta^2} \mathbf{q} \nabla_x \theta, \quad (3.24)$$

into the above fundamental inequality results in the Clausius-Duhem inequality:

$$\boldsymbol{\sigma} : \mathbf{D} - \rho (\dot{\psi} + s \dot{\theta}) - \frac{1}{\theta} \mathbf{q} \cdot \mathbf{g} \geq 0. \quad (3.25)$$

we have defined $\mathbf{g} = \nabla_x \theta$. Equivalently, the Clausius-Duhem inequality can be expressed in terms of dissipation per unit reference volume as:

$$\boldsymbol{\tau} : \mathbf{D} - \bar{\rho} (\dot{\psi} + s \dot{\theta}) - \frac{J}{\theta} \mathbf{q} \cdot \mathbf{g} \geq 0. \quad (3.26)$$

3.2.6 Constitutive theory

The constitutive model consists of a set of mathematical equations that relate the thermodynamic and kinematic quantities for a specific material within some finite range of applicability. When combined, the balance laws and the constitutive model form a complete set of equations that can theoretically be solved to determine the response of the system to imposed boundary conditions. In this section, the use of internal variables to formulate constitutive models of dissipative materials is addressed.

The principle of dissipation states that for any admissible process, the material model must satisfy the Clausius-Duhem (CD) inequality. An effective alternative to describe the dissipative constitutive behavior is the adoption of the so-called thermodynamics with internal variables. The thermodynamic state depends only on the instantaneous value of the state variables and not on their past history. This hypothesis is intimately connected with the assumption of the existence of a (fictitious) state of thermodynamic equilibrium known as the local accompanying state described by the current value of the state variables. Consider that at any time, the thermodynamic state at a point is defined by the set of state variables, as follows:

$$\{\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}\},$$

where, the terms $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and $\boldsymbol{\alpha}$ are the instantaneous values of the total strain, plastic strain, and the term $\boldsymbol{\alpha}$ represents the set of internal variables containing, in general, entities of scalar, vectorial and tensorial nature associated with dissipative mechanisms. Following the above hypothesis, the specific free energy potential, which plays a crucial role in the derivation of the model and provides the constitutive law for stress, is assumed to have the form:

$$\psi = \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}). \quad (3.27)$$

It is usual to assume that the free energy can be split as

$$\psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \boldsymbol{\alpha}) = \psi^e(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) + \psi^p(\boldsymbol{\alpha}), \quad (3.28)$$

into a sum of an elastic contribution, ψ^e , whose dependence upon strains and internal variables appears only through the elastic strain, and a contribution due to hardening, ψ^p . Following the above expression for the free energy, and using the chain rule to expand the Helmholtz free energy in terms of the independent variables gives

$$\dot{\psi} = \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e + \frac{\partial \psi^p}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}} = \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^p + \frac{\partial \psi^p}{\partial \boldsymbol{\alpha}} : \dot{\boldsymbol{\alpha}}. \quad (3.29)$$

Now, replacing Equation (3.29) in Equation (3.26), we obtain:

$$\left(\boldsymbol{\sigma} - \bar{\rho} \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e} \right) : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \boldsymbol{A} * \dot{\boldsymbol{\alpha}} \geq 0, \quad (3.30)$$

where

$$\boldsymbol{A} \equiv \bar{\rho} \frac{\partial \psi^p}{\partial \boldsymbol{\alpha}}, \quad (3.31)$$

is the hardening thermodynamical force and we note that $\boldsymbol{\sigma}$ is the thermodynamical stress associated with the plastic strain while the symbol $*$ indicates the appropriate product between $\boldsymbol{A}$ and $\dot{\boldsymbol{\alpha}}$. The above inequality implies a general elastic law of the form

$$\boldsymbol{\sigma} = \bar{\rho} \frac{\partial \psi^e}{\partial \boldsymbol{\varepsilon}^e}. \quad (3.32)$$

So that the requirement of non-negative dissipation can be reduced to

$$Y^p(\boldsymbol{\sigma}, \mathbf{A}, \dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}) \geq 0, \quad (3.33)$$

where the function Y^p , defined by

$$Y^p(\boldsymbol{\sigma}, \mathbf{A}, \dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\alpha}}) \equiv \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p - \mathbf{A} * \dot{\boldsymbol{\alpha}}, \quad (3.34)$$

is called the plastic dissipation function.

3.3 Finite Element Formulation

Any problem within the context of solids mechanics can be posed as an Initial Boundary Value Problem (IBVP). The solution of the IBVP delivers the prediction of how a given solid will mechanically behave when subjected to certain boundary conditions. Within the scope of this thesis, only quasi-static problems will be addressed, hence, any inertial effects will be neglected. The exact solution of such a problem requires that both force and moment equilibrium be maintained at all times over any arbitrary volume of the body. Thus the IBVP can be stated as follows:

Given the body force vector $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ and surface traction vector $\mathbf{t}^* = \mathbf{t}^*(\mathbf{x}, t)$, where $\mathbf{x}$ is the position vector, find the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, such that:

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) - \mathbf{b} = \mathbf{0} \text{ in } \Omega, \quad (3.35)$$

subjected to the boundary conditions

$$\mathbf{u} = \mathbf{u}^* \text{ on } \Gamma_u, \quad (3.36)$$

$$\mathbf{t}^* = \boldsymbol{\sigma} \cdot \mathbf{n} \text{ on } \Gamma_\sigma, \quad (3.37)$$

where ∇ is the vector gradient operator, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{u}^*$ is the prescribed displacement vector, $\mathbf{t}^*$ is the prescribed surface traction vector, $\mathbf{n}$ is the outward surface unit normal, Ω is the problem domain which occupies $\Omega = \mathbb{R}^3$, with boundary such that $\overline{\Gamma_u} \cup \overline{\Gamma_\sigma} \equiv \Gamma$ and $\overline{\Gamma_u} \cap \overline{\Gamma_\sigma} \equiv \Phi$, see Figure 3.3.

The preceding formulation can be written in a variational form, yielding the so called principle of virtual work.

$$\int_{\Omega} \boldsymbol{\sigma} : \nabla \eta d\Omega = \int_{\Omega} \mathbf{b} \cdot \eta d\Omega + \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \eta d\Gamma_\sigma. \quad (3.38)$$

To solve this IBVP numerically one of the general methods is the displacement-based approach method which is used in this work. Basically, this numerical method requires the application of the following three fundamental steps: spatial discretization and

temporal discretization, and solve the system of equations by using an iterative numerical method.

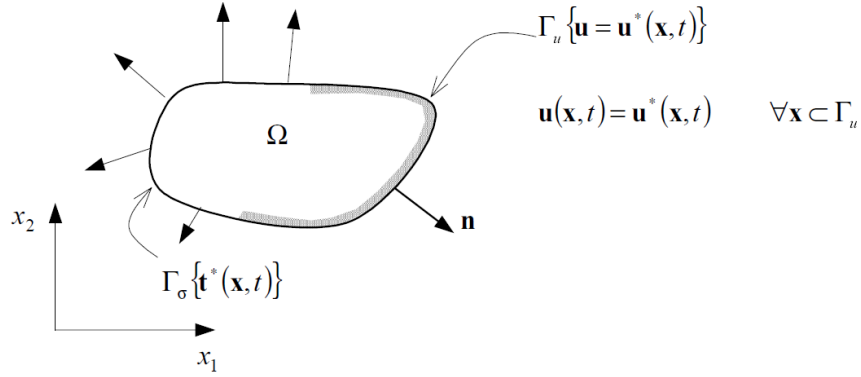


Figure 3.3: Problem domain and boundaries.

3.3.1 Spatial discretization

With a typical finite element framework, the field variables are discretised through the so-called interpolation or shape functions. In the case of displacement-based finite elements, the interpolated field variables are the displacements. For a given element e , the interpolation is assumed to be

$$\mathbf{u}(\mathbf{x}) \equiv \sum_{i=1}^{n_{node}} \mathbf{N}_i^{(e)}(\mathbf{x}) \mathbf{u}_i, \quad (3.39)$$

where $\mathbf{N}_i^{(e)}(\mathbf{x})$ is the shape function associated with node i , and n_{node} is the number of nodes of the element.

For the whole discrete global domain, the procedure is exactly the same, thus the global interpolation function can also be set as

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}^g(\mathbf{x}) \mathbf{u}, \quad (3.40)$$

where $\mathbf{N}^g$ is the global interpolation matrix, $\mathbf{u}$ denotes the global vector of nodal displacements.

In addition, the global discrete symmetric gradient matrix, $\mathbf{B}^g(\mathbf{x})$ can be defined as

$$\mathbf{B}^g(\mathbf{x}) = \frac{\partial \mathbf{N}^g(\mathbf{x})}{\partial \mathbf{x}}. \quad (3.41)$$

3.3.2 Temporal discretization

Besides a spatial discretization, the modeling of materials that have some kind of history dependence (e.g. strain-path or strain-rate dependent materials) also requires proper time discretization regardless whether they take strain rate effects into account or

not. Basically, the procedure for time discretization consists in dividing the overall time interval $[t_n, t_{n+1}]$ into $n + 1$ steps and, for each time step, equilibrium must be satisfied.

Given the set α_n of internal variables at t_n , the strain tensor ϵ_{n+1} at time t_{n+1} must determine the stress σ_{n+1} uniquely through the integration algorithm. Such an algorithm defines an (approximate) incremental constitutive function, $\hat{\sigma}$, for the stress tensor:

$$\sigma_{n+1} = \hat{\sigma}(\alpha_n, \epsilon_{n+1}), \quad (3.42)$$

whose outcome, σ_{n+1} , is expected to converge to the exact solution of the actual evolution problem as the strain increments are reduced.

The integration algorithm also defines a similar incremental constitutive function for the internal variables of the model:

$$\alpha_{n+1} = \hat{\alpha}(\alpha_n, \epsilon_{n+1}). \quad (3.43)$$

3.3.3 Newton-Raphson method

After spatial and temporal discretization, the incremental finite element equilibrium equation can be written as

$$\mathbf{R}(\mathbf{u}_{n+1}) = \mathbf{f}^{int}(\mathbf{u}_{n+1}) - \mathbf{f}^{ext} = \mathbf{0}, \quad (3.44)$$

where $\mathbf{f}^{int}$ and $\mathbf{f}^{ext}$ are, respectively, the internal and external force vectors, defined by

$$\mathbf{f}^{int} = \int_{\Omega} \mathbf{B}^T \hat{\sigma}(\alpha_n, \epsilon_{n+1}) d\Omega, \quad (3.45)$$

$$\mathbf{f}^{ext} = \int_{\Omega} \mathbf{N}^T \mathbf{b} d\Omega + \int_{\Gamma_{\sigma}} \mathbf{N}^T \mathbf{t}^* d\Gamma_{\sigma}, \quad (3.46)$$

and $\mathbf{R}$ is the residual error.

In order to solve the equilibrium equation introduced in the previous section, a robust and efficient numerical method is required. The most popular and widely used by the computational mechanics community is the Newton-Raphson.

If we perform a Taylor series expansion of the residual about an initial solution, $\mathbf{u}^{(i)}$, we obtain

$$\mathbf{R}^{(i+1)} = \mathbf{R}^{(i)} + \left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right] \Delta \mathbf{u}^{(i+1)} + \mathcal{O}(\{\Delta \mathbf{u}^{(i+1)}\}^2). \quad (3.47)$$

Neglecting high order terms, we obtain

$$\mathbf{R}^{(i+1)} = \mathbf{R}^{(i)} + \left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right] \Delta \mathbf{u}^{(i+1)}. \quad (3.48)$$

At the intersection of the $\mathbf{u}$ -axis, $\mathbf{R}^{(i+1)} = \mathbf{0}$, rearranging the equation gives

$$\mathbf{u}^{(i+1)} - \mathbf{u}^{(i)} = \Delta \mathbf{u}^{(i+1)} = - \left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right]^{-1} \mathbf{R}^{(i)}. \quad (3.49)$$

To start up the Newton–Raphson iterations, we need an initial guess, $\mathbf{u}_{n+1}^{(0)}$. The initial guess is usually taken as the converged (equilibrium) displacement vector at the end of the previous increment. The Newton-Raphson iterations are repeated until after some iteration (N), the following convergence criterion is satisfied:

$$\frac{|\mathbf{R}^{(N)}|}{|\mathbf{f}_{n+1}^{ext}|} \leq \epsilon_{tol}, \quad (3.50)$$

where ϵ_{tol} is a sufficiently small specified equilibrium convergence tolerance. The corresponding displacement vector, $\mathbf{u}_{n+1}^{(N)}$, is then accepted as sufficiently small to the solution of the incremental equilibrium equation.

Finally, we can get the global tangent stiffness matrix, $[\mathbf{K}]$, which is defined by

$$\mathbf{K} = \frac{\partial \mathbf{R}}{\partial \mathbf{u}}, \quad (3.51)$$

and this matrix is constructed in practice by assembling the element tangent stiffness matrices defined as

$$\mathbf{K} = \int \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega, \quad (3.52)$$

where $\mathbf{D}$ is the consistent tangent matrix. The matrix form of the fourth-order consistent tangent operator

$$\mathbf{D} \equiv \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}_{n+1}}. \quad (3.53)$$

The consistent tangent operator is the derivative of the incremental constitutive function $\hat{\boldsymbol{\sigma}}$. This generally implicit function is typically defined by some numerical algorithm for integration of the rate constitutive equations of the model.

3.4 References

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Chapter 4

The Application of Line Search Strategies in Closest Point Projection Method (CPPM) to Cazacu'04 Model

4.1 Introduction

Metals with HCP crystallographic structure promotes a very different mechanical behaviour when compared to other metals with Face Centered Cubic (FCC) and Body Centered Cubic (BCC) structures, such as steel, aluminium, *etc.* At room temperature, few slip systems require the activation of twinning to accommodate the deformation at room temperature. Therefore, a very pronounced strength differential (SD) effect is observed owing to the polar nature of twinning. Shear occurs only in a single direction rather than opposite directions. In addition, they always exhibit strong crystallographic texture, particularly for the cold rolling sheet, where the basal planes of the grains are aligned with the sheet, leading to a pronounced anisotropy. Owing to the unconventional yield surface evolution, the shape of the yield surface changes during deformation, which may lead to problems on the numerical implementation of the related constitutive model with the classical return mapping algorithm within an implicit finite element program.

In this chapter, Cazacu'04 model and the relevant constitutive equations are briefly outlined, followed by a closest point projection integration algorithm for Cazacu'04 model. The accuracy of this implicit integration algorithm for Cazacu'04 model is assessed with iso-error maps at different stress states and equivalent plastic strain level. Due to the yield surface of titanium change from the near elliptical to triangular-like shape, the curvature of the yield surface increases in some stress states, leading to the rise of the nonlinearity of the yield equation. In spite of the Newton-Raphson iterative scheme being quite attainable and commonly used, operating an asymptotic quadratic rate of convergence, it fails to converge for the strong nonlinearity leaded by the high curvature of the yield surface. This is particularly evident in the large ranges of the initial stress states that are far from the solution. Hence, three inexact line search methods are proposed and implemented to overcome numerical instabilities. The results indicate that the line search method with CPPM

scheme is useful not merely to expand the convergence region but also to improve the computational efficiency.

4.2 Formulation of constitutive model

4.2.1 Cazacu'04 yield criterion

Cazacu and Barlat [1-3] have extended Drucker's isotropic orthotropic yield criterion with a fourth-order linear transformation operator ($\hat{\mathbf{L}}$). They have also constructed generalizations of the deviatoric stress invariants. Following this theory, the original Cauchy stress $\{\boldsymbol{\sigma}\}$ is converted to a transformed tensor $\{\boldsymbol{\Sigma}\}$, which is defined as

$$\Sigma_{ij} = \hat{L}_{ijkl}\sigma_{kl}. \quad (4.1)$$

The fourth tensor $\hat{\mathbf{L}}$ is determined by both the material symmetry and mechanical anisotropy. For orthotropic materials (such as metal sheets), there are only six non-zero components in matrix $\hat{\mathbf{L}}$, meaning that the fourth order linear operator $\hat{\mathbf{L}}$ can be represented by:

$$\hat{\mathbf{L}} = \begin{bmatrix} \frac{a_2+a_3}{3} & -\frac{a_3}{3} & -\frac{a_2}{3} & 0 & 0 & 0 \\ -\frac{a_3}{3} & \frac{a_1+a_3}{3} & -\frac{a_1}{3} & 0 & 0 & 0 \\ -\frac{a_2}{3} & -\frac{a_1}{3} & \frac{a_1+a_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}. \quad (4.2)$$

The Cazacu'04 anisotropic yield criteria can be expressed as:

$$\Phi = \hat{A}(J_2^{0/2} - \hat{c}J_3^0)^{1/3} - \sigma_Y, \quad (4.3)$$

where J_2^0 is the second invariant of deviatoric transformed stress, and J_3^0 is the third invariant of deviatoric transformed stress, respectively, which can be obtained as:

$$J_2^0 = \frac{1}{2}tr(\boldsymbol{\Sigma}^2), \quad (4.4)$$

$$J_3^0 = \frac{1}{3}tr(\boldsymbol{\Sigma}^3). \quad (4.5)$$

The constant $\hat{A}$ is defined as:

$$\hat{A} = 3[(a_2^2 + a_3^2 + a_2a_3)^{\frac{3}{2}} - \hat{c}(a_2 + a_3)a_2a_3]^{-1/3}, \quad (4.6)$$

where $\hat{c}$ is a material parameter, which can be expressed in terms of the yield in uniaxial tension, σ_T , and the yield in uniaxial compression, σ_c , as:

$$\hat{c} = \frac{3\sqrt{3}(\sigma_T^3 - \sigma_c^3)}{2(\sigma_T^3 - \sigma_c^3)}. \quad (4.7)$$

To ensure the convexity of the yield function, the material constant $\hat{c}$ is limited to the range $[-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{4}]$.

4.2.2 Associated flow rule

The flow rule establishes a relation between the stress and the plastic strain increments (or rates). Regarding the present Cazacu'04 model, an associated plastic flow rule is adopted. Namely, the associated plasticity uses the yield surface function $\Phi(\boldsymbol{\sigma}, \bar{\varepsilon}^p)$ as a plastic dissipation potential. The plastic strain rate is assumed to be outward normal to the yield surface in the stress space according to:

$$\dot{\boldsymbol{\varepsilon}}_{ij}^p = \dot{\gamma} \frac{\partial \Phi}{\partial \sigma_{ij}}, \quad (4.8)$$

where $\dot{\gamma}$ is the plastic strain rate multiplier, Φ is the yield potential.

Being $\Phi < 0$, the sets of stress states are defined where the material behaves elastically in the sense that $\dot{\gamma} = 0$. Besides, being $\Phi = 0$, the stress is restricted to lie on the yield surface, and γ is determined implicitly, according to the Kuhn-Tucker conditions:

$$\gamma \dot{\Phi} = 0, \quad \gamma \geq 0, \quad \dot{\Phi} \leq 0. \quad (4.9)$$

An equivalent plastic strain rate $\dot{\bar{\varepsilon}}^p$, which is the plastic work rate conjugated to σ_{eq} , is defined such that:

$$\Delta W = \sigma_{ij} \Delta \varepsilon_{ij}^p = \sigma_{eq} \Delta \bar{\varepsilon}^p, \quad (4.10)$$

where σ_{eq} is the effective stress, while $\Delta \bar{\varepsilon}^p$ is the equivalent plastic strain increment.

In conclusion, it is possible to obtain a straightforward relation between $\dot{\gamma}$ and $\dot{\bar{\varepsilon}}^p$ by (the calculated details can be examined in Appendix A):

$$\Delta \bar{\varepsilon}^p = \frac{\boldsymbol{\sigma} : \frac{\partial \bar{\sigma}_{eq}}{\partial \boldsymbol{\sigma}}}{\bar{\sigma}_{eq}} \Delta \gamma = \Delta \gamma. \quad (4.11)$$

4.2.3 Hardening rule

In the rate-independent case, the phenomenon of hardening describes the changes in yield stress that results from plastic straining. Flow stress σ_Y represents the size of the yield function during deformation. The formulation of an adequate equation, describing changes in the flow stress of the material, will depend on the deformation

conditions. To achieve it, a power hardening law was used; the equation can be seen as follows:

$$\sigma_Y(\bar{\varepsilon}^p) = \sigma_{h0} + K_h(\bar{\varepsilon}^p)^{n_h}, \quad (4.12)$$

where σ_{h0} (initial yield strength), K_h (strength hardening coefficient), n_h (strain-hardening exponent) are material constants. $\bar{\varepsilon}^p$ is the effective plastic strain.

Box 4.1: Computation of the transfer matrix $\hat{\mathbf{L}}$ and hardening.

Given the accumulated plastic strain $\bar{\varepsilon}^p$

- (i) Compute the transformed matrix $\hat{\mathbf{L}}$ and material constant $\hat{c}$

$$a_i = a_i(\bar{\varepsilon}^p), \quad (i = 1, \dots, 6),$$

$$\hat{c} = \hat{c}(\bar{\varepsilon}^p) \quad (\hat{c} \in [-\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{4}]),$$

$$\hat{\mathbf{L}} = \begin{bmatrix} \frac{a_2+a_3}{3} & -\frac{a_3}{3} & -\frac{a_2}{3} & 0 & 0 & 0 \\ -\frac{a_3}{3} & \frac{a_1+a_3}{3} & -\frac{a_1}{3} & 0 & 0 & 0 \\ -\frac{a_2}{3} & -\frac{a_1}{3} & \frac{a_1+a_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix},$$

and the derivative of matrix $\hat{\mathbf{L}}$ to equivalent plastic strain $\bar{\varepsilon}^p$

$$D\hat{\mathbf{L}} = \frac{\partial \hat{\mathbf{L}}}{\partial \bar{\varepsilon}^p}.$$

- (ii) Hardening law

$$\sigma_Y = \sigma_Y(\bar{\varepsilon}^p),$$

$$H_Y = \frac{\partial \sigma_Y(\bar{\varepsilon}^p)}{\partial \bar{\varepsilon}^p}.$$

- (iii) Exit

4.3 Integration algorithm for Cazacu'04 model

4.3.1 State variable update

The basic assumption of the return-mapping scheme with small strains is that the strain increment is additively decomposed into elastic and plastic parts:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p. \quad (4.13)$$

Regarding numerical computation, it is convenient to express the stress and the strain increment tensors, used in the stress-strain relations, in a six-component column

matrix format. In this matrix, the shear strain increments are multiplied with a factor two in order to obtain tensor consistency. Isotropic elastic behavior is assumed until yielding is reached. The elastic response of the material is characterized by the generalized Hooke's law that relates stress and elastic strain increments linearly through the elastic constitutive tensor $\mathbf{D}^e$. For an isotropic elastic material, the relation between the stresses and the strains is given by:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} 2\mu + \lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu + \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu + \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & \mu & \mu \\ 0 & 0 & 0 & \mu & \mu & \mu \\ 0 & 0 & 0 & \mu & \mu & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} = \mathbf{D}^e \boldsymbol{\varepsilon}^e, \quad (4.14)$$

where the shear modulus μ and Lamé's constant λ can be expressed in terms of the Young's modulus E and the Poisson's ratio ν as $\mu = E/2(1 + \nu)$ and $\lambda = \nu E/(1 + \nu)(1 - 2\nu)$, respectively.

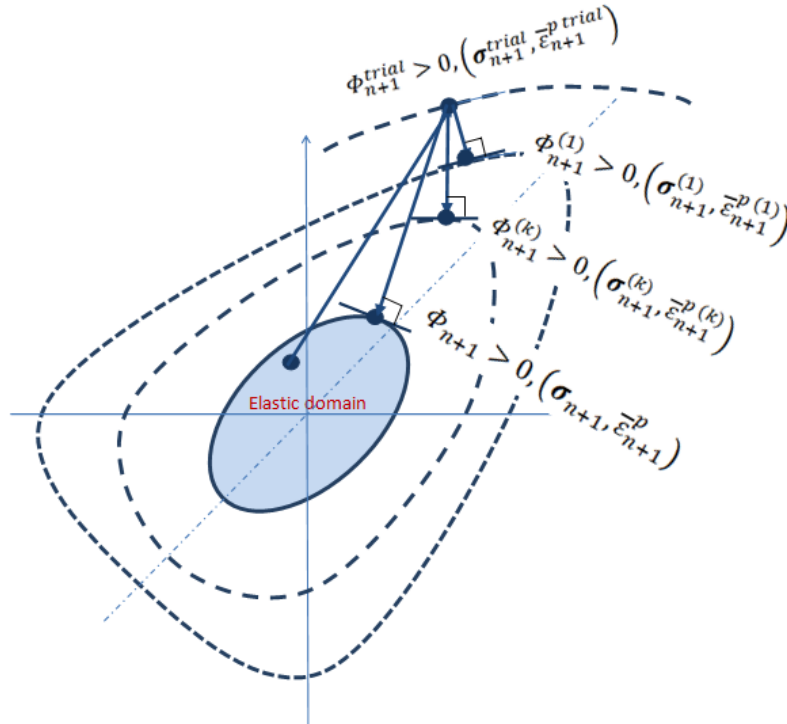


Figure 4.1: Graphical interpretation of stress return corresponding to the Closest Point Project Method (CPPM).

There exist a number of algorithms and techniques to update both the stresses and plastic strains, such as the forward Euler scheme, a generalized trapezoidal or mid-point algorithm, or the backward Euler (return mapping) scheme [4-6]. For the purpose of this work, the implicit, return mapping algorithm was selected. This

integration is carried out locally at each Gauss point in typical finite-element implementations. At each integration point, a time discretization is introduced. To ensure it, let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the time interval $[0, T]$, while letting $\Delta t_{n+1} = t_{n+1} - t_n$ be the step size in the n -th time step. At time interval $[t_n, t_{n+1}]$, all variables $\{\boldsymbol{\varepsilon}_n^e, \bar{\boldsymbol{\varepsilon}}_n^p, \boldsymbol{\sigma}_n\}$ at time t_n are assumed to be known, given the strain history from t_n up to another time t_{n+1} . In addition, a return mapping algorithm has to be used to update the stress for a finite increment of loading. The implicit stress integration for each time increment step can be divided into two steps. First, the elastic equations are integrated with total strain increments to obtain an elastic predictor. Zero plastic strain increment is assumed (elastic predictor step). Then, from an estimation of the total strain increment, an elastic trial stress is calculated. If this trial stress satisfies the Kunh-Tucker conditions, it is not only assumed to be the actual stress, but also that the material is in the elastic state. If not, it is accepted that the material is in the plastic domain.

The second step is related to the plastic correction: the elastically predicted stresses (the trial stress) are projected back onto the yield stress in the direction of the closest projection by correcting, iteratively, the plastic strain increments. This is named Closest Point Projection Method (CPPM). All stresses, strains and state variable are represented in the local material coordination. In addition, it is also convenient to use a co-rotational coordinate system in which the basic system rotates with the material. The overview of these implicit stress update scheme can be seen in Box 4.2. Stress return map corresponding to CPPM is presented in Figure 4.1.

Regarding the detailed update scheme, it is possible to describe it by firstly assuming that the input discrete strain increments are all in elastic regime for a given n -th time step, and that trial elastic strain and trial stress can be determined as:

$$\boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}} = \boldsymbol{\varepsilon}_n^e + \Delta \boldsymbol{\varepsilon}, \quad (4.15)$$

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{D}^e \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}, \quad (4.16)$$

where $\boldsymbol{\sigma}_{n+1}^{trial}$ is the initial stress for the $n + 1$ time increment step; $\boldsymbol{\varepsilon}_n^e$ is the elastic strain tensor from last equilibrium state; $\mathbf{D}^e$ is the fourth order elastic modulus matrix. The trial internal variable is equal to the equivalent plastic strain in the last equilibrium state, which could be written as:

$$\bar{\boldsymbol{\varepsilon}}_{n+1}^{p \text{ trial}} = \bar{\boldsymbol{\varepsilon}}_n^p. \quad (4.17)$$

If the following yield condition is satisfied with the trial state value of:

$$\sigma_{eq}(\boldsymbol{\sigma}_{n+1}^{trial}, \bar{\epsilon}_{n+1}^{p\ trial}) - \sigma_Y(\bar{\epsilon}_{n+1}^{p\ trial}) < 0, \quad (4.18)$$

the deformation of the material at the step $n + 1$ occurs in elastic region. The entire updated variable for the current increment step is equal to the trial state. It could be written as:

$$\boldsymbol{\epsilon}_{n+1}^e = \boldsymbol{\epsilon}_{n+1}^{e\ trial}, \quad (4.19)$$

$$\bar{\epsilon}_{n+1}^p = \bar{\epsilon}_{n+1}^{p\ trial} = \bar{\epsilon}_n^p, \quad (4.20)$$

$$\boldsymbol{\epsilon}_{n+1}^p = \boldsymbol{\epsilon}_n^p, \quad (4.21)$$

$$\boldsymbol{\sigma}_{n+1}^e = \boldsymbol{\sigma}_{n+1}^{e\ trial}. \quad (4.22)$$

In case the inequality Equation (4.18) is violated, the deformation moves into the elasto-plastic domain, and the trial elastic stress is taken as an initial value to solve the plastic corrector problem. The plastic strain increments are obtained to a degree that the following new trial stress stays on the new yield surface, which can be written as:

$$\boldsymbol{\epsilon}_{n+1}^e = \boldsymbol{\epsilon}_{n+1}^{e\ trial} - \Delta\boldsymbol{\epsilon}^p = \boldsymbol{\epsilon}_{n+1}^{e\ trial} - \Delta\gamma\mathbf{N}_{n+1}. \quad (4.23)$$

In the $n + 1$ step, the trial stress should be finally returned back to the yield surface. The incremental effective plastic strain, $\Delta\bar{\epsilon}_{n+1}^p$, satisfying the equilibrium state, allowing the following yielding condition to hold as:

$$\sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\epsilon}_n^p + \Delta\bar{\epsilon}_{n+1}^p) - \sigma_Y(\bar{\epsilon}_n^p + \Delta\bar{\epsilon}_{n+1}^p) = 0. \quad (4.24)$$

In parallel to what was deducted in the last section, the internal variable could be calculated by:

$$\bar{\epsilon}_{n+1}^p = \bar{\epsilon}_n^p + \Delta\gamma. \quad (4.25)$$

The previously presented Equations (4.23)-(4.25) are a set of non-linear equations. The upper boundary condition for this non-linear boundary problem is given by the initial trial stress. Respecting the lower boundary, it is determined from the equilibrium state variables from last time increment step.

For the purpose of this study, Newton-Raphson's iteration method is adopted to solve the non-linear equations set at each time step, which is referred as Newton-Raphson's closest point projection algorithm. In order to derive such an iterative loop, $\bar{\mathbf{r}}$ residuals are defined to represent the differences between the current constitutive variables and the backward-Euler constitutive variables. This can be presented as follows:

$$\begin{cases} \mathbf{r}_\sigma = \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\,trial} + \Delta\gamma \mathbf{N}_{n+1} \\ r_{\bar{\varepsilon}_p} = \bar{\varepsilon}_{n+1}^p - \bar{\varepsilon}_n^p - \Delta\gamma \\ r_\phi = \sigma_{eq}(\boldsymbol{\sigma}, \bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}_{n+1}^p) - \sigma_Y(\bar{\varepsilon}_n^p + \Delta\bar{\varepsilon}_{n+1}^p) \end{cases}. \quad (4.26)$$

These three residual equations are combined to:

$$\bar{\mathbf{r}} = \begin{bmatrix} \mathbf{r}_\sigma \\ r_{\bar{\varepsilon}_p} \\ r_\phi \end{bmatrix} = 0. \quad (4.27)$$

The truncated Taylor expansion can be applied to the Equation (4.26) that could be then it written as:

$$\begin{cases} \mathbf{r}_\sigma^{(k+1)} = \mathbf{r}_\sigma^{(k)} + \mathbf{D}^{e-1} d\boldsymbol{\sigma} + \Delta\gamma \frac{\partial N}{\partial \boldsymbol{\sigma}} d\boldsymbol{\sigma} + \Delta\gamma \frac{\partial N}{\partial \bar{\varepsilon}^p} d\bar{\varepsilon}^p + N d\Delta\gamma \\ r_{\bar{\varepsilon}_p}^{(k+1)} = r_{\bar{\varepsilon}_p}^{(k)} + d\bar{\varepsilon}^p - d\Delta\gamma \\ r_\phi^{(k+1)} = r_\phi^{(k)} + \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^p} d\bar{\varepsilon}^p - \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^p} d\bar{\varepsilon}^p. \end{cases}. \quad (4.28)$$

For ease of use in solving the state variables, let $\boldsymbol{\chi} = \begin{bmatrix} \boldsymbol{\sigma}_{n+1} \\ \bar{\varepsilon}_{p\,n+1} \\ \Delta\gamma_{n+1} \end{bmatrix}$.

The updated residual vector can be given as:

$$\bar{\mathbf{r}}_{n+1}^{k+1} = \bar{\mathbf{r}}_{n+1}^k + \theta \partial_{\boldsymbol{\chi}} \bar{\mathbf{r}}_{n+1}^{k+1} \cdot d\boldsymbol{\chi} = 0. \quad (4.29)$$

Being k the local iteration number, θ ($0 \leq \theta \leq 1$) is an interpolation parameter. At this point, the interpolation parameter θ is equal to unity. Hence, the above Taylor series expansion equation set can be rewrite in commonly used matrix format and presented as:

$$\begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial N}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial N}{\partial \bar{\varepsilon}^p} & N \\ 0 & 1 & -1 \\ \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} & \frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^p} - \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^p} & 0 \end{bmatrix} \begin{bmatrix} d\boldsymbol{\sigma} \\ d\bar{\varepsilon}^p \\ d\Delta\gamma \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_\sigma^{(k)} \\ -r_{\bar{\varepsilon}_p}^{(k)} \\ -r_\phi^{(k)} \end{bmatrix}. \quad (4.30)$$

$$\text{Define } \bar{\mathbf{D}}_{ep} = \begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial N}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial N}{\partial \bar{\varepsilon}^p} & N \\ 0 & 1 & -1 \\ \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} & \frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^p} - \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^p} & 0 \end{bmatrix},$$

This allows the incremental state variables to be calculated by:

$$\begin{bmatrix} d\boldsymbol{\sigma}^{(k+1)} \\ d\bar{\varepsilon}^{p(k+1)} \\ d\Delta\gamma^{(k+1)} \end{bmatrix} = \bar{\mathbf{D}}_{ep}^{-1} * \begin{bmatrix} -\mathbf{r}_\sigma^{(k)} \\ -r_{\bar{\varepsilon}_p}^{(k)} \\ -r_\phi^{(k)} \end{bmatrix}. \quad (4.31)$$

All the stress, internal variable and plastic multiplier can be all obtained in each iterative loop by:

$$\begin{bmatrix} \boldsymbol{\sigma}^{(k+1)} \\ \bar{\varepsilon}^p(k+1) \\ \Delta\gamma^{(k+1)} \end{bmatrix} = \begin{bmatrix} d\boldsymbol{\sigma}^{(k+1)} \\ d\bar{\varepsilon}^p(k+1) \\ d\Delta\gamma^{(k+1)} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\sigma}^{(k)} \\ \bar{\varepsilon}^p(k) \\ \Delta\gamma^{(k)} \end{bmatrix}. \quad (4.32)$$

The iterative process is stopped as soon as convergence has been achieved within the prescribed convergence criteria, namely, whenever the norm of residual $\bar{\mathbf{r}}$ is less than a small tolerance value.

Box 4.2: Fully implicit elastic predictor/return mapping algorithm for the Cazacu'04 model.

- (i) Elastic predictor. Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n , evaluate the elastic trial state:

$$\boldsymbol{\varepsilon}_{n+1}^{e\ trial} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon}; \bar{\varepsilon}_{n+1}^{p\ trial} = \bar{\varepsilon}_n^p;$$

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{D}^e \boldsymbol{\varepsilon}_{n+1}^{e\ trial}.$$

For getting all the coefficient parameters in transform matrix $\hat{\mathbf{L}}$, yield stress and hardening ---- Go to Box 4.1.

$$\boldsymbol{\Sigma}_{n+1}^{trial} = \hat{\mathbf{L}} \boldsymbol{\sigma}_{n+1}^{trial};$$

$$J_{2\ n+1}^{0\ trial} = \frac{1}{2} tr(\boldsymbol{\Sigma}_{n+1}^{trial^2});$$

$$J_{3\ n+1}^{0\ trial} = \frac{1}{3} tr(\boldsymbol{\Sigma}_{n+1}^{trial^3});$$

$$\sigma_{eq\ n+1}^{trial} = \hat{A}[(J_{2\ n+1}^{0\ trial})^{\frac{3}{2}} - \hat{c}J_{3\ n+1}^{0\ trial}]^{1/3}.$$

- (ii) Check Plastic admissibility

$$\text{If } \Phi = \sigma_{eq\ n+1}^{trial} - \sigma_Y(\bar{\varepsilon}_{n+1}^{p\ trial}) \leq 0;$$

Then set $()_{n+1} = ()_{n+1}^{trial}$ and Exit

- (iii) Return mapping. Solve the system of eight equations using the Newton-Raphson iterative method

$$\left\{ \begin{array}{l} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\ trial} + \Delta\gamma \mathbf{N}_{n+1} \\ \bar{\varepsilon}_{n+1}^p - \bar{\varepsilon}_n^p - \Delta\gamma \\ \Phi = \hat{A}(J_2^0(\boldsymbol{\sigma}, \bar{\varepsilon}_{n+1}^p))^{\frac{3}{2}} - \hat{c}(\bar{\varepsilon}_{n+1}^p)J_3^0(\boldsymbol{\sigma}, \bar{\varepsilon}_{n+1}^p)^{1/3} - \sigma_Y(\bar{\varepsilon}_{n+1}^p) \end{array} \right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix};$$

where,

$$\mathbf{N}_{n+1} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}},$$

For $\boldsymbol{\varepsilon}_{n+1}^e, \bar{\varepsilon}_{n+1}^p$ and $\Delta\gamma$, with

$$\boldsymbol{\sigma}_{n+1} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^e.$$

- (iv) Exit

4.3.2 Consistent tangent modular matrix

On the global level of finite element formulation, a consistent tangent modular matrix at the material points is required. The local stiffness is then assembled to get the global stiffness of the structure. Consistent tangent modular matrices significantly improve the convergence characteristics of the overall equilibrium iterations if a Newton-Raphson scheme is used [5, 7].

This consistent tangent modulus is defined by the following relationship:

$$\mathbf{D} \equiv \frac{\partial \hat{\boldsymbol{\sigma}}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}}. \quad (4.33)$$

(a) The elastic tangent modular matrix

If the stress is inside the elastic domain, or on the yield surface, and elastic unloading is assumed to occur, the tangent modulus $\mathbf{D}$ are consistent with any of the integration algorithms previously given by:

$$\mathbf{D} = \mathbf{D}^e \equiv \bar{\rho} \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon}^e{}^2}. \quad (4.34)$$

Here, we assume the elastic response is isotropic, such that $\mathbf{D}^e$ has the form:

$$\mathbf{D}^e = \lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \mathbb{I}, \quad (4.35)$$

where $\mathbf{I}$ is the second order identity tensor, $\mathbb{I}$ is the four-order symmetric identity tensor, λ and μ are the Lamé constants and shear modulus, and δ_{ij} is the Kronecker delta.

(b) The elastoplastic tangent modular matrix

If the stress occurs on the yield surface and it is also assumed that further plastic loading is going to occur, the tangent operator is, in such cases, called the elastoplastic consistent tangent and is denoted as $\mathbf{D}^{ep}$.

As previously mentioned in regard to the stress update integration algorithm, three equations should be met after Newton-Raphson iterative. The corresponding return mapping equations are rewritten as follows for the derivation of consistent tangent modular matrices conveniently:

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}} + \Delta\gamma \mathbf{N}_{n+1} \\ \bar{\boldsymbol{\varepsilon}}_{n+1}^p - \bar{\boldsymbol{\varepsilon}}_n^p - \Delta\gamma \\ \Phi(\boldsymbol{\sigma}_{n+1}, \bar{\boldsymbol{\varepsilon}}_{n+1}^p) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (4.36)$$

It is possible to identify, as basic unknowns of the system of algebraic Equations set (4.36), the updated elastic strain $\boldsymbol{\varepsilon}_{n+1}^e$; the updated set of internal variables $\bar{\boldsymbol{\varepsilon}}_{n+1}^p$; and,

lastly, the incremental plastic multiplier, $\Delta\gamma$. The corresponding updated stress tensor delivered by the aforementioned returned map is evaluated from the standard potential form as:

$$\boldsymbol{\sigma}_{n+1} = \bar{\rho} \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e} \Big|_{n+1}. \quad (4.37)$$

Clearly, by changing the elastic trial strain, $\boldsymbol{\varepsilon}_{n+1}^{e\,trial}$, that takes part in the return map equations, the solution $\{\boldsymbol{\varepsilon}_{n+1}^e, \alpha_{n+1}, \Delta\gamma\}$ will change accordingly and so will the updated stress $\boldsymbol{\sigma}_{n+1}$. In this way, the algebraic system of equations defines an implicit function for the stress tensor at t_{n+1} . The elastoplastic consistent tangent operator,

$$\mathbf{D}^{ep} \equiv \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\,trial}}, \quad (4.38)$$

i.e., the derivative of the implicit function for stress, places the linear tangent relationship between $\boldsymbol{\varepsilon}_{n+1}^{e\,trial}$ and $\boldsymbol{\sigma}_{n+1}$. Thus, the first step in its derivation is to linearize the return mapping equations (having the elastic trial strain – the system input – also as a variable). Straightforward differentiation of the general return-mapping equations yields the following linearized form:

$$\left\{ \begin{array}{l} d\boldsymbol{\varepsilon}^e + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} d\boldsymbol{\sigma} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^p} \cdot d\bar{\varepsilon}^p + d\Delta\gamma \mathbf{N} \\ d\bar{\varepsilon}^p - d\Delta\gamma \\ \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial \Phi}{\partial \bar{\varepsilon}^p} \cdot d\bar{\varepsilon}^p \end{array} \right\} = \left\{ \begin{array}{l} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{array} \right\}. \quad (4.39)$$

When rewritten it in the format of matrix product they are presented as:

$$\left[\begin{array}{ccc} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^p} & \mathbf{N} \\ 0 & 1 & -1 \\ \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} & \frac{\partial \Phi}{\partial \bar{\varepsilon}^p} & 0 \end{array} \right] \begin{bmatrix} d\boldsymbol{\sigma} \\ d\bar{\varepsilon}^p \\ d\Delta\gamma \end{bmatrix} = \begin{bmatrix} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{bmatrix}. \quad (4.40)$$

Finally, by inverting the above linear relation, we obtain the following:

$$\begin{bmatrix} d\boldsymbol{\sigma} \\ d\bar{\varepsilon}^p \\ d\Delta\gamma \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{bmatrix}. \quad (4.41)$$

At this stage, a symbolic inverted matrix is written to describe the matrix component.

Obviously, the elastoplastic consistent tangent modulus is:

$$\mathbf{D}^{ep} \equiv \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\,trial}} = \mathbf{D}_{11}. \quad (4.42)$$

After series mathematic deductions, the elastoplastic consistent tangent can be calculated as follows:

$$\mathbf{D}^{ep} = \left\{ [\mathbf{D}^e]^{-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} - \frac{\mathbf{N} \otimes (\mathbf{N} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\epsilon}^p})}{\frac{\partial \Phi}{\partial \bar{\epsilon}^p}} \right\}^{-1}. \quad (4.43)$$

4.4 Line search strategy

In case the initial trial state is not close to a solution, the previously mentioned Newton iteration cannot guarantee that the convergence is obtained. In such cases, line search algorithm and trust-region method are both two commonly used approaches to find a minimum $\bar{\chi}$ of an objective function $\bar{g}: \mathbb{R}^n \rightarrow \mathbb{R}$.

In order to improve convergence, Armero and Pérez-Foguet [8, 9], as well as Jeremić [10] all have used a line search technique for the local iterations. A closed-form return formula with line search algorithm in principal stress was employed for the Mohr–Coulomb failure surface, as described by Borja *et al.* [11]; Larsson and Runesson [12]; and Clausen *et al.* [13]. DeBorst [14] gave explicit return expressions using Koiter’s method, along with a region indicator, to deal with the singular yield points of the Mohr–Coulomb model. Following, Crisfield [15] employed a similar approach involving a two-vector return procedure. Also addressing this issue, Huang and Griffiths [16] compared five previously published algorithms for return mapping to Mohr–Coulomb in a finite-element analysis of bearing capacity.

In the present paper, we set the nonlinear equations as:

$$\bar{\mathbf{r}}(\boldsymbol{\chi}) = \begin{Bmatrix} \boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_{n+1}^{trial} + \Delta\gamma \mathbf{D}^e \mathbf{N}_{n+1} \\ \bar{\epsilon}_{n+1}^p - \bar{\epsilon}_n^p - \Delta\gamma \\ \sigma_{eq} - \sigma_Y \end{Bmatrix}, \quad (4.44)$$

where, $\bar{\mathbf{r}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector function, $\boldsymbol{\chi}$ is

$$\boldsymbol{\chi} = \{\boldsymbol{\sigma}, \bar{\epsilon}^p, \Delta\gamma\}. \quad (4.45)$$

Aiming to solve the nonlinear equations with line search method, a scalar-valued merit function should be defined, indicating whether a new iterate is better or worse than the current one. In this study, we chose a simple and widely used merit function, which can be written as:

$$\bar{g}(\boldsymbol{\chi}) = \frac{1}{2} \bar{\mathbf{r}}(\boldsymbol{\chi}) \cdot \bar{\mathbf{r}}(\boldsymbol{\chi}). \quad (4.46)$$

The factor 1/2 is introduced for convenience. All of the line search procedures require, before anything else, to find a suitable descent direction $\bar{\mathbf{p}}$ along which the objective function $\bar{g}(\boldsymbol{\chi})$ will be reduced. This can be calculated by various methods [7], such as gradient descent, Newton’s method, Quasi-Newton’s method, *etc.*

Following, an initial estimate $\bar{\alpha}_0$ is determined and a step size is computed, satisfying certain conditions by iterative method to decide how far χ should move along that direction $\bar{p}$. The step size can be determined either by an exactly or inexactly method. The basic iterative structure can be presented as follows:

- (a) Select a starting point χ_0 .
- (b) Determine the search direction $\bar{p}(k)$.
- (c) Find the value of $\bar{\alpha}_k$ to minimize $\bar{g}(\chi_k + \bar{\alpha}\bar{p}_k)$ with respect to $\bar{\alpha}$.
- (d) Set $\chi_{k+1} = \chi_k + \bar{\alpha}_k\bar{p}_k$.

At this point, step-length $\bar{\alpha}_k$ should be at the range of $[0,1]$. It should also be mentioned that $k = 0,1, \dots$ denotes the line search iteration number.

4.4.1 Inexact conditions

To obtain the exact minimized value of the Equation (4.46), it requires expensive computation costs, particularly for solving higher dimensional minimum problems. Thus, whenever computing the step length $\bar{\alpha}_k$, a trade off should be taken into account. If by one hand, it is desirable to choose $\bar{\alpha}_k$, so that a substantial reduction of $\bar{g}$ is possible, on the other the extended length of time required to accomplish it should also be addressed. Fortunately, inexact line searches with sufficient degree of descent are capable of guarantying convergence. It can terminate the iterative in finite iterates.

In the matter of inexact line search methods, it is possible to identify several of them as suitable to determine step-length $\bar{\alpha}_k$. This topic has been already studied and discussed by numerous mathematical researchers such as Goldstein [17], Armijo [18], Wolfe [19], Powell [20, 21], Dennis and Schnabel [7]. Among the different inexact line search methods, the backtracking line search has been highlighted by researchers as a very simple and quite effective one.

Given the aforementioned conditions, the Goldstein condition is a mostly applicable and useful termination condition. If the step length $\bar{\alpha}$ is chosen, new point is $\chi + \bar{\alpha}\bar{p}$. Selecting $\bar{\alpha} \in [0,1]$ and requiring $\bar{\alpha}$ satisfies the following inequality:

$$\bar{g}(\chi_k + \bar{\alpha}\bar{p}_k) \leq \bar{g}(\chi_k) + \bar{c}_1\bar{\alpha}\nabla\bar{g}_k^T\bar{p}_k. \quad (4.47)$$

We designed the procedure to be “efficient” in the sense that it computes the derivative $\nabla\bar{g}(\chi)$ as few times as possible.

where $0 < \bar{c}_1 < 1$. $\bar{c}_1$ is usually chosen to quite small. In this thesis, we set $\bar{c}_1 = 10^{-4}$.

In accordance to the previously described, we assume that $\bar{\mathbf{p}}$ is a descent direction and that $\bar{g}$ is bounded below along the direction $\bar{\mathbf{p}}$. The line search algorithm terminates with $\bar{\alpha}_*$ set to a step length that satisfies the Goldstein's conditions.

4.4.2 Line search direction

Each iteration of a line search method computes a search direction $\bar{\mathbf{p}}_k$ and then determines how far to move along that direction. The new variables can be updated by:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \bar{\alpha}_k \bar{\mathbf{p}}_k. \quad (4.48)$$

where, the positive scalar $\bar{\alpha}_k$ is called the step length. The success of a line search method relies on the effective choices of the direction $\bar{\mathbf{p}}_k$ to ensure that function $\bar{g}$ can be reduced along this direction. Specifically, it requires the satisfaction of the condition:

$$\bar{\mathbf{p}}_k^T \nabla \bar{g}_k < 0. \quad (4.49)$$

Generally, it can be written in the following form:

$$\bar{\mathbf{p}}_k = -\bar{\mathbf{B}}_k^{-1} \nabla \bar{g}_k, \quad (4.50)$$

where $\bar{\mathbf{B}}_k$ is a symmetric and nonsingular matrix. In the steepest descent method, $\bar{\mathbf{B}}_k$ is simply the identity matrix $\mathbf{I}$, while in Newton's method, $\bar{\mathbf{B}}_k$ is the exact Hessian $\nabla^2 \bar{g}(\mathbf{x}_k)$. In quasi-Newton methods, $\bar{\mathbf{B}}_k$ is an approximation to the Hessian matrix that is updated at every iteration by means of a low-rank formula.

4.4.3 Interpolation

Derivative-free line search algorithms include golden section and Fibonacci search [20, 21]. All of these sequential search methods typically store three trial points that determine an interval containing a one-dimensional minimizer. Golden and Fibonacci searches differ in the way trial step lengths are generated. However, these methods narrow the range slowly in the backtracking procedures. The polynomial interpolation method may be qualified as a good way to find a value of $\bar{\alpha}$ that satisfies the sufficient decrease inexact stopping criterion.

Assuming that the initial guess $\bar{\alpha}_0$ is given, if we have:

$$\bar{g}(\bar{\alpha}_0) \leq \bar{g}(0) + \bar{c}_1 \bar{\alpha}_0 \bar{g}'(0). \quad (4.51)$$

We are able to sustain that the step length satisfies the condition, and to terminate the line search. Otherwise, we know that the interval $[0, \bar{\alpha}_0]$ contains acceptable step

lengths. Polynomial interpolation methods are generally the most effective in terms of efficiency when the function to be minimized is continuous. These methods approximate a number of points with a univariant polynomial whose minimum can be calculated easily. In this study, three polynomial interpolation methods are selected, which are quadratic interpolation, three point quadratic (quadratic 3) interpolations, and cubic interpolation.

(a) Quadratic interpolation method

In order to have a quadratic fit of the merit function, the coefficients of the function are evaluated by making three measurements. The quadratic equation can be written as:

$$\bar{g}_q(\bar{\alpha}) = \tilde{a}\bar{\alpha}^2 + \bar{g}_q'(0)\bar{\alpha} + \bar{g}_q(0). \quad (4.52)$$

In order to compute the unknown coefficient $\tilde{a}$, the following may be selected:

$$\bar{g}_q(\bar{\alpha}_{k-1}) = \tilde{a}\bar{\alpha}_{k-1}^2 + \bar{g}_q'(0)\bar{\alpha}_{k-1} + \bar{g}_q(0). \quad (4.53)$$

Therefore, we have:

$$\tilde{a} = \frac{\bar{g}_q(\bar{\alpha}_{k-1}) - \bar{g}_q(0) - \bar{\alpha}_{k-1}\bar{g}_q'(0)}{\bar{\alpha}_{k-1}^2}. \quad (4.54)$$

Following, the quadratic approximated equation can be written as:

$$\bar{g}_q(\bar{\alpha}_k) = \left(\frac{\bar{g}_q(\bar{\alpha}_{k-1}) - \bar{g}_q(0) - \bar{\alpha}_{k-1}\bar{g}_q'(0)}{\bar{\alpha}_{k-1}^2} \right) \bar{\alpha}_k^2 + \bar{g}_q'(0)\bar{\alpha}_k + \bar{g}_q(0). \quad (4.55)$$

The minimum point of $\bar{g}_q$ can be estimated by finding the point where the derivative of $\bar{g}_q$ vanishes. Thus setting:

$$\bar{g}_q'(\bar{\alpha}_k) = \bar{g}_q'(0) + 2 \left(\frac{\bar{g}_q(\bar{\alpha}_{k-1}) - \bar{g}_q(0) - \bar{\alpha}_{k-1}\bar{g}_q'(0)}{\bar{\alpha}_{k-1}^2} \right) \bar{\alpha}_k = 0. \quad (4.56)$$

The stationary point of the above equation is obtained by setting the first derivative equal to zero to give:

$$\bar{\alpha}_k = \frac{-\bar{g}_q'(0)\bar{\alpha}_{k-1}^2}{2 [\bar{g}_q(\bar{\alpha}_{k-1}) - \bar{g}_q(0) - \bar{g}_q'(0)\bar{\alpha}_{k-1}]}. \quad (4.57)$$

(b) Three point quadratic (Quadratic 3) interpolation method

Quadratic 3 interpolation also uses the same quadratic approximation to fit the data, which can be written in a generic form as:

$$\bar{g}_{q3}(\bar{\alpha}) = \tilde{a}\bar{\alpha}^2 + \tilde{b}\bar{\alpha} + \tilde{c}. \quad (4.58)$$

Various simplifications in the solution of these equations can be reached whenever particular characteristics of the points are used. For example, the first point can generally be taken as $\bar{\alpha} = 0$. Other simplifications can be achieved when the points are evenly spaced. A general problem formula is defined as follows.

Given three unevenly spaced point $[\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3]$ and their associated function values $[\bar{g}_{q3}(\bar{\alpha}_1), \bar{g}_{q3}(\bar{\alpha}_2), \bar{g}_{q3}(\bar{\alpha}_3)]$, the values of $\tilde{a}$ and $\tilde{b}$ are determined by measurements at three points to determine $\bar{g}_{q3}(\bar{\alpha}_1)$, $\bar{g}_{q3}(\bar{\alpha}_2)$ and $\bar{g}_{q3}(\bar{\alpha}_3)$, respectively. Hence, Equation (4.58) can be written as:

$$\begin{cases} \bar{g}(\alpha_1) = \tilde{a}\bar{\alpha}_1^2 + \tilde{b}\bar{\alpha}_1 + \tilde{c} \\ \bar{g}(\alpha_2) = \tilde{a}\bar{\alpha}_2^2 + \tilde{b}\bar{\alpha}_2 + \tilde{c} \\ \bar{g}(\alpha_3) = \tilde{a}\bar{\alpha}_3^2 + \tilde{b}\bar{\alpha}_3 + \tilde{c} \end{cases} \quad (4.59)$$

The aforementioned equation presents are three linear algebraic equations in $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. In matrix form they can be written as:

$$\begin{bmatrix} \bar{\alpha}_1^2 & \bar{\alpha}_1 & 1 \\ \bar{\alpha}_2^2 & \bar{\alpha}_2 & 1 \\ \bar{\alpha}_3^2 & \bar{\alpha}_3 & 1 \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \end{bmatrix} = \begin{bmatrix} \bar{g}_{q3}(\alpha_1) \\ \bar{g}_{q3}(\alpha_2) \\ \bar{g}_{q3}(\alpha_3) \end{bmatrix}. \quad (4.60)$$

The minimum point of three parameter interpolation merit function $\bar{g}_{q3}$, as well as the point where the derivative of $\bar{g}_{q3}$ vanishes, can both be estimated the same way as the quadratic interpolation method. Thus setting:

$$\bar{g}_{q3}'(\alpha) = 2\tilde{a}\bar{\alpha} + \tilde{b} = 0. \quad (4.61)$$

This set of Equations (4.60) can be solved for $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$. The results are substituted into Equation (4.61) to compute $\bar{\alpha}^*$, the stationary point of the quadratic approximation to the merit function. The result is as follows:

$$\bar{\alpha}^* = \frac{\bar{g}_{q3}(\bar{\alpha}_1)(\bar{\alpha}_2^2 - \bar{\alpha}_3^2) + \bar{g}_{q3}(\bar{\alpha}_2)(\bar{\alpha}_3^2 - \bar{\alpha}_1^2) + \bar{g}_{q3}(\bar{\alpha}_3)(\bar{\alpha}_1^2 - \bar{\alpha}_2^2)}{2[\bar{g}_{q3}(\bar{\alpha}_1)(\bar{\alpha}_2 - \bar{\alpha}_3) + \bar{g}_{q3}(\bar{\alpha}_2)(\bar{\alpha}_3 - \bar{\alpha}_1) + \bar{g}_{q3}(\bar{\alpha}_3)(\bar{\alpha}_1 - \bar{\alpha}_2)]}. \quad (4.62)$$

Owing to that the step length $\bar{\alpha}$ are in the range $[0,1]$, so the value of $\bar{\alpha}_1$ could be set as 0; meanwhile, $\bar{\alpha}_2$, and $\bar{\alpha}_3$ can be set as the previous interpolate value $\bar{\alpha}_{k-1}, \bar{\alpha}_{k-2}$.

Thus, the new interpolated value $\bar{\alpha}_k$ can be computed as:

$$\bar{\alpha}_k = -\frac{1}{2} \frac{\bar{\alpha}_{k-1}^2 [\bar{g}_{q3}(\bar{\alpha}_{k-2}) - \bar{g}_{q3}(0)] - \bar{\alpha}_{k-2}^2 [\bar{g}_{q3}(\bar{\alpha}_{k-1}) - \bar{g}_{q3}(0)]}{\bar{\alpha}_{k-2} [\bar{g}_{q3}(\bar{\alpha}_{k-1}) - \bar{g}_{q3}(0)] - \bar{\alpha}_{k-1} [\bar{g}_{q3}(\bar{\alpha}_{k-2}) - \bar{g}_{q3}(0)]}. \quad (4.63)$$

The procedure requires selecting three initial points and computing the optimum of the quadratic approximation $\bar{\alpha}_k$, by Equation (4.57). Typically, the procedure continues using $\bar{\alpha}_k$ as one of the next three points to repeat the application of Equation (4.63). The point with the smallest value of $\bar{g}_{q3}$ is discarded. The procedure is repeated until a tolerance on the dependent variable is reached.

(c) Cubic interpolation method

For the cubic interpolation method, it is required to start a quadratic approximation $\bar{g}_q(\bar{\alpha})$ to $\bar{g}$, by interpolating the three pieces of information available— $\bar{g}(0)$, $\bar{g}'(0)$, and $\bar{g}(\bar{\alpha}_0)$, thus obtaining:

$$\bar{g}_q(\bar{\alpha}) = \left(\frac{\bar{g}(\bar{\alpha}_0) - \bar{g}(0) - \bar{\alpha}_0 \bar{g}'(0)}{\bar{\alpha}_0^2} \right) \bar{\alpha}^2 + \bar{\alpha} \bar{g}'(0) + \bar{g}(0). \quad (4.64)$$

(Note that this function is constructed so that it satisfies the interpolation conditions $\bar{g}_q(0) = \bar{g}(0)$, $\bar{g}'_q(0) = \bar{g}'(0)$, and $\bar{g}_q(\bar{\alpha}_0) = \bar{g}(\bar{\alpha}_0)$.)

The new trial value $\bar{\alpha}_1$ is defined as the minimum of this quadratic, that is, we obtain:

$$\bar{\alpha}_1 = - \frac{\bar{g}'(0) \bar{\alpha}_0^2}{2[\bar{g}(\bar{\alpha}_0) - \bar{g}(0) - \bar{g}'(0) \bar{\alpha}_0]}. \quad (4.65)$$

If the sufficient decrease condition inequity Equation (4.47) is satisfied at $\bar{\alpha}_1$, the search is considered complete. Otherwise, a cubic function is constructed, interpolating the four pieces of information $\bar{g}(0)$, $\bar{g}'(0)$, $\bar{g}(\bar{\alpha}_0)$, and $\bar{g}(\bar{\alpha}_1)$, obtaining:

$$\bar{g}_c(\bar{\alpha}) = \tilde{a} \bar{\alpha}^3 + \tilde{b} \bar{\alpha}^2 + \bar{\alpha} \bar{g}'(0) + \bar{g}(0). \quad (4.66)$$

If settled the following:

$$\bar{g}_c(\bar{\alpha}_{k-1}) = \tilde{a} \bar{\alpha}_{k-1}^3 + \tilde{b} \bar{\alpha}_{k-1}^2 + \bar{\alpha}_{k-1} \bar{g}'(0) + \bar{g}(0), \quad (4.67)$$

$$\bar{g}_c(\bar{\alpha}_{k-2}) = \tilde{a} \bar{\alpha}_{k-2}^3 + \tilde{b} \bar{\alpha}_{k-2}^2 + \bar{\alpha}_{k-2} \bar{g}'(0) + \bar{g}(0), \quad (4.68)$$

After arrangement, it is obtained:

$$\begin{bmatrix} \bar{\alpha}_{k-1}^3 & \bar{\alpha}_{k-1}^2 \\ \bar{\alpha}_{k-2}^3 & \bar{\alpha}_{k-2}^2 \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} \bar{g}_c(\bar{\alpha}_{k-1}) - \bar{g}(0) - \bar{g}'(0) \bar{\alpha}_{k-1} \\ \bar{g}_c(\bar{\alpha}_{k-2}) - \bar{g}(0) - \bar{g}'(0) \bar{\alpha}_{k-2} \end{bmatrix}. \quad (4.69)$$

Subsequently, the coefficient $\tilde{a}$ and $\tilde{b}$ can be obtained by:

$$\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \frac{\begin{bmatrix} \bar{\alpha}_{k-2}^2 & -\bar{\alpha}_{k-1}^2 \\ -\bar{\alpha}_{k-2}^3 & \bar{\alpha}_{k-1}^3 \end{bmatrix} \begin{bmatrix} \bar{g}_c(\bar{\alpha}_{k-1}) - \bar{g}(0) - \bar{g}'(0) \bar{\alpha}_{k-1} \\ \bar{g}_c(\bar{\alpha}_{k-2}) - \bar{g}(0) - \bar{g}'(0) \bar{\alpha}_{k-2} \end{bmatrix}}{\bar{\alpha}_{k-1}^2 \bar{\alpha}_{k-2}^2 (\bar{\alpha}_{k-1} - \bar{\alpha}_{k-2})}. \quad (4.70)$$

By differentiating $\bar{g}_c(\bar{\alpha})$, the stationary point of the cubic approximation to the merit function is:

$$\bar{g}_c'(\bar{\alpha}_k) = 3\tilde{a}\bar{\alpha}_k^2 + 2\tilde{b}\bar{\alpha}_k + \bar{g}'(0). \quad (4.71)$$

Lastly, the result is:

$$\bar{\alpha}_k = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 3\tilde{a}\bar{g}'(0)}}{3\tilde{a}}. \quad (4.72)$$

If necessary, this process may be repeated, using a cubic interpolate of $\bar{g}(0)$, $\bar{g}'(0)$ and the two most recent values of $\bar{g}$, until the step length $\bar{\alpha}$ satisfies the stopping criterion.

Box 4.3: Line search algorithm.

1. Input data: $\chi^{(k)} := \{\sigma^{(k)}, \bar{\varepsilon}^{p(k)}, \Delta\gamma^{(k)}\}$, compute $\bar{\mathbf{r}}(\chi^{(k)})$ and $\bar{\mathbf{p}}(\chi^{(k)})$:

$$\bar{\mathbf{r}}^{(k)} := \bar{\mathbf{r}}(\chi^{(k)}) := \begin{Bmatrix} \sigma^{(k)} - \sigma_{n+1}^{trial} + \Delta\gamma \mathbf{D}^e \mathbf{N}^{(k)} \\ \bar{\varepsilon}^{p(k)} - \bar{\varepsilon}_n^p - \Delta\gamma^{(k)} \\ \sigma_{eq}(\sigma^{(k)}, \bar{\varepsilon}^{p(k)}) - \sigma_Y(\bar{\varepsilon}^{p(k)}) \end{Bmatrix},$$

$$\bar{\mathbf{p}}^{(k)} := \bar{\mathbf{p}}(\chi^{(k)}) := -(\bar{\mathbf{J}}^{(k)})^{-1} \bar{\mathbf{r}}^{(k)},$$

where,

$$\bar{\mathbf{J}}^{(k)} := \nabla \bar{\mathbf{r}}(\chi^{(k)}) = \begin{bmatrix} \mathbb{I} + \Delta\gamma \frac{\partial \mathbf{N}^{(k)}}{\partial \sigma} & \Delta\gamma \frac{\partial \mathbf{N}^{(k)}}{\partial \bar{\varepsilon}^p} & \mathbf{N}^{(k)} \\ \mathbf{0} & 1 & -1 \\ \mathbf{N}^{(k)} & \frac{\partial \sigma_{eq}(\sigma^{(k)}, \bar{\varepsilon}^{p(k)})}{\partial \bar{\varepsilon}^p} - \frac{\partial \sigma_Y(\bar{\varepsilon}^{p(k)})}{\partial \bar{\varepsilon}^p} & 0 \end{bmatrix},$$

$$\mathbf{N}^{(k)} := \frac{\partial \sigma_{eq}(\sigma^{(k)}, \bar{\varepsilon}^{p(k)})}{\partial \sigma}.$$

2. Initialize: set $j = 1$, $\bar{\alpha}_{(1)}^{(k)} = 1$ and

$$\bar{g}^{(k)} := \bar{\mathbf{r}}^{(k)} \cdot \bar{\mathbf{r}}^{(k)} / 2,$$

$$\bar{g}'^{(k)} := \bar{\mathbf{r}}^{(k)} \nabla \bar{\mathbf{r}}(\chi^{(k)}) = \bar{\mathbf{J}}^{(k)} \bar{\mathbf{r}}^{(k)}.$$

3. Compute the new unknowns, residuals and merit function:

$$\chi_{(j)}^{(k+1)} := \chi_{(j)}^{(k)} + \bar{\alpha}_{(j)}^{(k)} \bar{\mathbf{p}}^{(k)},$$

$$\bar{\mathbf{r}}_{(j)}^{(k+1)} := \bar{\mathbf{r}}(\chi_{(j)}^{(k+1)}),$$

$$\bar{g}_{(j)}^{(k+1)} := \bar{\mathbf{r}}_{(j)}^{(k+1)} \cdot \bar{\mathbf{r}}_{(j)}^{(k+1)} / 2.$$

4. Check Goldstein's condition:

If $\bar{g}_{(j)}^{(k+1)} \leq \bar{g}^{(k)} + \bar{c}_1 \bar{\alpha}_{(j)}^{(k)} \bar{g}'^{(k)T} \bar{p}^{(k)}$ Then

$\chi^{(k+1)} = \chi_{(j)}^{(k+1)}$ and Exit.

5. Check for maximum number of curve fittings:

If $j = j_{max}$ Then notify, set $\chi^{(k+1)} = \chi_{(j)}^{(k+1)}$ and Exit.

6. Polynomial interpolation to obtain new value of step length $\bar{\alpha}_{(j+1)}^{(k)}$:

a. Quadratic interpolation method:

$$\bar{\alpha}_{(j+1)}^{(k)} = \frac{-\bar{g}_q'(0) (\bar{\alpha}_{(j)}^{(k)})^2}{2 [\bar{g}_q(\bar{\alpha}_{(j)}^{(k)}) - \bar{g}_q(0) - \bar{g}_q'(0) \bar{\alpha}_{(j)}^{(k)}]}.$$

b. Quadratic 3 interpolation method:

$$\bar{\alpha}_{(j+1)}^{(k)} = -\frac{1}{2} \frac{(\bar{\alpha}_{(j)}^{(k)})^2 [\bar{g}_{q3}(\bar{\alpha}_{(j-1)}^{(k)}) - \bar{g}_{q3}(0)] - (\bar{\alpha}_{(j-1)}^{(k)})^2 [\bar{g}_{q3}(\bar{\alpha}_{(j)}^{(k)}) - \bar{g}_{q3}(0)]}{\bar{\alpha}_{(j-1)}^{(k)} [\bar{g}_{q3}(\bar{\alpha}_{(j)}^{(k)}) - \bar{g}_{q3}(0)] - \bar{\alpha}_{(j)}^{(k)} [\bar{g}_{q3}(\bar{\alpha}_{(j-1)}^{(k)}) - \bar{g}_{q3}(0)]}.$$

c. Cubic interpolation method

$$\bar{\alpha}_{(j+1)}^{(k)} = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 3\tilde{a}\bar{g}'(0)}}{3\tilde{a}},$$

where,

$$\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \frac{\begin{bmatrix} (\bar{\alpha}_{(j-1)}^{(k)})^2 & -(\bar{\alpha}_{(j)}^{(k)})^2 \\ -(\bar{\alpha}_{(j-1)}^{(k)})^3 & (\bar{\alpha}_{(j)}^{(k)})^3 \end{bmatrix} \begin{bmatrix} \bar{g}_c(\bar{\alpha}_{(j)}^{(k)}) - \bar{g}(0) - \bar{g}'(0)\bar{\alpha}_{(j)}^{(k)} \\ \bar{g}_c(\bar{\alpha}_{(j-1)}^{(k)}) - \bar{g}(0) - \bar{g}'(0)\bar{\alpha}_{(j-1)}^{(k)} \end{bmatrix}}{(\bar{\alpha}_{(j)}^{(k)})^2 (\bar{\alpha}_{(j-1)}^{(k)})^2 (\bar{\alpha}_{(j)}^{(k)} - \bar{\alpha}_{(j-1)}^{(k)})}.$$

In order to avoid small step length, the final length could be calculated by:

$$\bar{\alpha}_{(j+1)}^{(k)} := \text{MAX} \{ \tilde{\eta} \bar{\alpha}_{(j)}^{(k)}, \bar{\alpha}_{(j+1)}^{(k)} \}.$$

7. Set $j \leftarrow j + 1$ and Go to 3.

Remark: it is worth to notice that if any $\bar{\alpha}_k$ is either too close to its predecessor $\bar{\alpha}_{k-1}$, or else too much smaller than $\bar{\alpha}_{k-1}$, it will cause the oscillation of the iterative loops. In order to ensure reasonable progress on each iteration, and that the final $\bar{\alpha}$ is not too small, the final step length can be calculated by:

$$\bar{\alpha}_k := \text{MAX} \{ \tilde{\eta} \bar{\alpha}_{k-1}, \bar{\alpha}_k \}. \quad (4.73)$$

The detailed line search algorithm is presented in Box 4.3. The parameter $\tilde{\eta} = 0.1$ is considered in the numerical simulations presented in this thesis.

4.5 Results and discussion

4.5.1 Single element validation

The constitutive model described above was implemented in an implicit finite element environment. In this sub-section, the verification of the constitutive model implementation was undertaken using a series of single eight-noded brick element test cases. In order to reflect the asymmetric mechanical behavior, and to show the mechanical response dependent on the state of stress, seven representative test cases, as illustrated in Figure 4.2, were considered.

Table 4.1: Material parameter of Titanium [22].

	ρ (g/cm^3)	E (GPa) Modulus	ν Poisson's ratio	$\sigma_Y(\bar{\epsilon}^p) = \sigma_{h0} + K_h(\bar{\epsilon}^p)^{n_h}$		
				σ_{h0} (MPa)	K_h (MPa)	n_h
Titanium	4.51	120	0.361	208.0	260.536	0.532

Table 4.2: Material parameters used in the simulation [23].

$\bar{\epsilon}^p$	a_1	a_2	a_3	a_4	a_5	a_6	$\hat{c}$
0.0	0.5454	0.5010	1.09	0.7246	-0.8675	-0.8675	-0.2168
0.025	0.5231	0.4745	0.9034	0.7309	0.7202	0.7202	-0.2198
0.05	0.6694	0.5585	1.1030	0.9138	0.9381	0.9381	-0.2291
0.075	0.6960	0.5969	1.1270	0.9838	0.9716	0.9716	-0.2607
0.1	0.5356	0.4768	0.8603	0.7761	0.7714	0.7714	-0.2754
0.2	0.0610	0.0576	0.0869	0.0870	0.0794	0.0794	-0.5908
0.4	0.0632	0.0620	0.0788	0.0816	0.0801	0.0801	-1.0330
0.5	0.9547	0.9570	1.2140	1.1810	1.1760	1.1760	-1.1480

The single element with one integration point is subjected to prescribed displacements, simulating different stress states. The initial element dimension is set to 10 mm. The geometry and the boundary conditions applied to this sample at different stress states, namely

(a) uniaxial tensile along rolling direction;

- (b) uniaxial tensile along transverse direction;
- (c) biaxial tensile test;
- (d) uniaxial compressive along rolling direction;
- (e) uniaxial compressive along transverse direction;
- (f) biaxial compressive.

Respecting the totality of examples presented in this chapter, all the residual norm tolerance in the local stress update loops are set to $TOL = 10^{-6}$. All the material parameters of Cazacu'04 model used in the following simulations are withdrawn from reference from the thesis [23] and listed in Table 4.1 and Table 4.2.

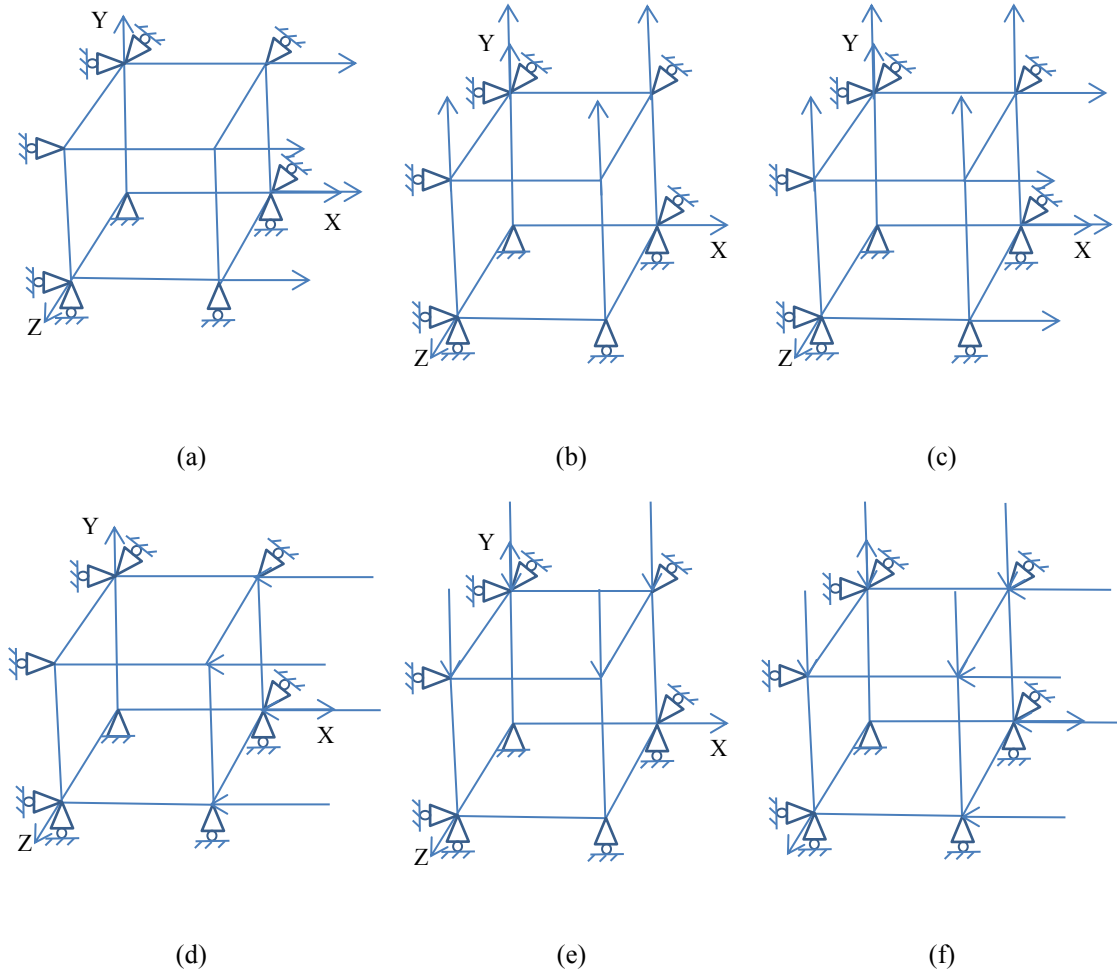


Figure 4.2: Single element test: (a) uniaxial tensile test in rolling direction; (b) uniaxial tensile test in transverse direction; (c) biaxial tensile test; (d) uniaxial compressive test in rolling direction; (e) uniaxial compressive test in transverse direction; (f) biaxial compressive test.

As far as Figure 4.3 is concerned, stress-strain curves at different stress states are presented. From this, it can be seen that the results obtained by simulation under uniaxial tensile test in the rolling direction match very well with the experimental data; meanwhile, under uniaxial compressive tensile test, the obtained results by

simulation present a slight deviation from the experimental results.

The flow stress-strain curve for uniaxial tension tests demonstrates a normal concave-down shape; on the contrary, the stress-strain curve under uniaxial compressive tests presents a concave-up shape whenever the plastic strain is less than 0.4. After this value, the flow curve also changes to the same concave-down shape as in uniaxial tension test. By the comparison of the stress-strain under the compressive and tensile tests, it is quite evident that the strength in the compressive stress state is larger than in the tensile one. Besides, the strength differential gap increases with the augmentation of plastic deformation. A pronounced strength differential (SD) effect between tension stress and compression stress state is detected. This effect is also notoriously observable in both the biaxial tensile stress and biaxial compressive stress states. Nevertheless, the difference between the rolling and transverse directions is very small, whether it takes place in uniaxial tensile stress state or uniaxial compressive stress state.

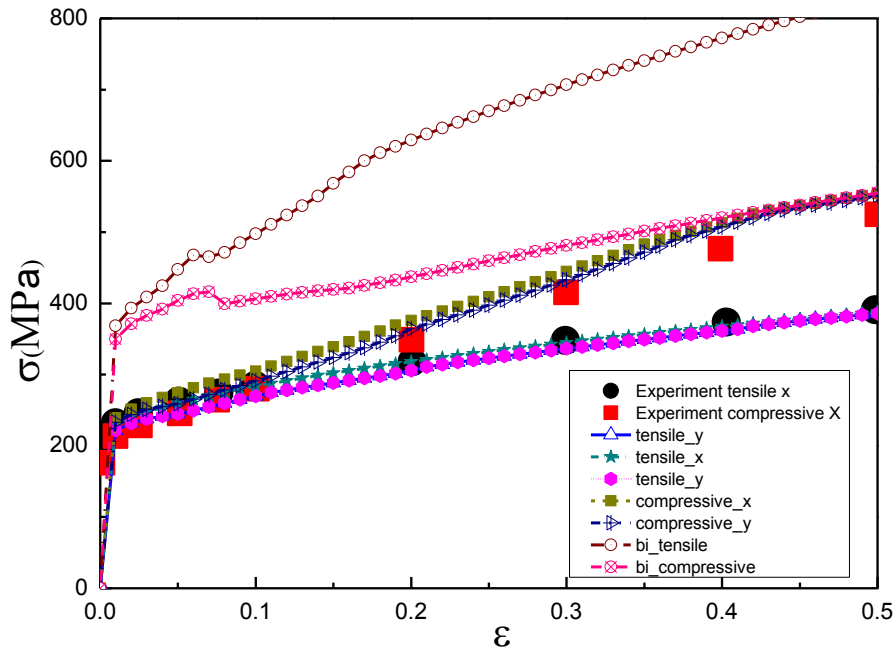


Figure 4.3: Stress-strain curves at different stress states.

The yield surface for pure titanium, at different levels of accumulative effective plastic strain, is displayed in Figure 4.4. It is clearly identified that the initial yield surface resembles an elliptical shape, and with the increase of the accumulative

effective plastic strain; the shape of yield surface evolves from the elliptical-like shape to the triangular-like shape. This shape's evolution of yield surface not only reflects the difference of deformation mechanism at different stress state (the changes from the slip dominated deformation to twinning dominated plastic deformation), but also changes the curvature of the yield locus, which has a great influence on the implemented numerical algorithms, particularly, the convergence of Newton-Raphson iterative in the stress-update subroutine.

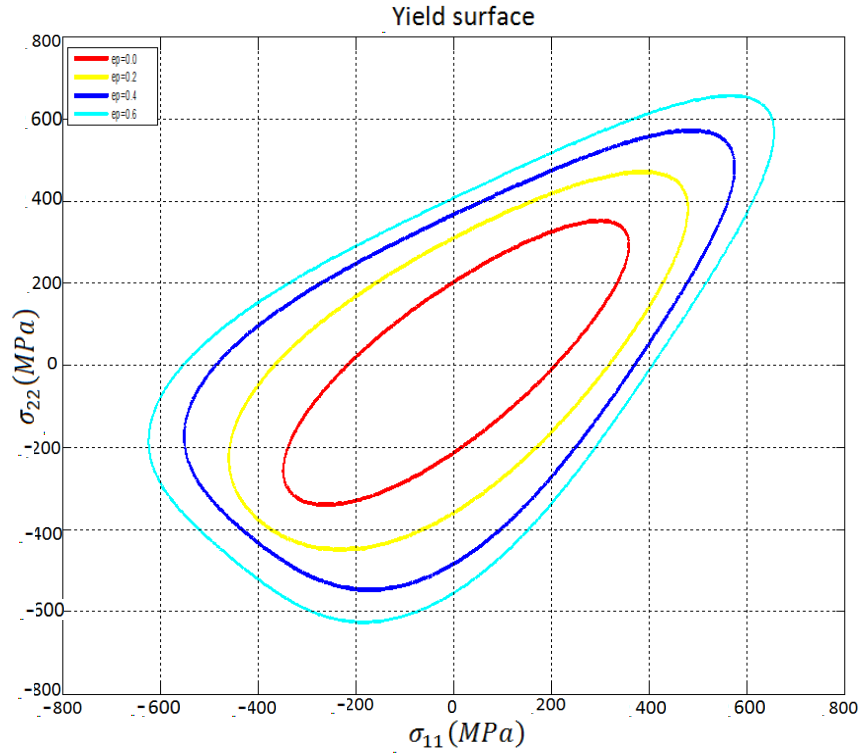


Figure 4.4: The yield surface evolution of pure titanium.

4.5.2 Comparisons among different integration algorithms

(a) Comparisons of maximum iterative numbers under different stress states

As analyzed in the aforementioned section, due to the alternation of deformation mechanism, the HCP crystal structure metal demonstrates pronounced anisotropic properties and SD effect; this may reflect in the yield locus's evolution during the deformation. This means that the shape's evolution in the yield locus directly results in the curvature change. Using the standard Newton-Raphson closest point projection scheme, it is possible to determine the asymptotic quadratic rate of convergence for nonlinear system of algebraic equations. However, it requires that initial estimates should be close to the final solution. Consequently, this originates difficulties in

obtaining the convergence in the stress-updated subroutine due to shape's change in the yield surface.

In order to investigate the numerical convergence properties, in this section, both standard Newton-Raphson's closest-point projection iterative scheme (N-R CPPM), and Newton-Raphson's iterative scheme with line search algorithm are used in stress-updated subroutine. For the line search algorithm, three kinds of interpolation methods were used, which is quadratic interpolation, three point quadratic (quadratic 3) interpolation, and cubic interpolation.

Owing to the anisotropic properties and SD effect, eight points on the yield surface are selected, representing a wide range of possible states' of stress. These points, labeled A, B, C, D, E, F, G, H in Figure 4.5, correspond, respectively, to uniaxial tensile in rolling direction, biaxial tensile, uniaxial tensile in transverse direction, pure shear stress, uniaxial compressive in rolling direction, biaxial compressive, uniaxial compressive in transverse direction, and pure shear stress.

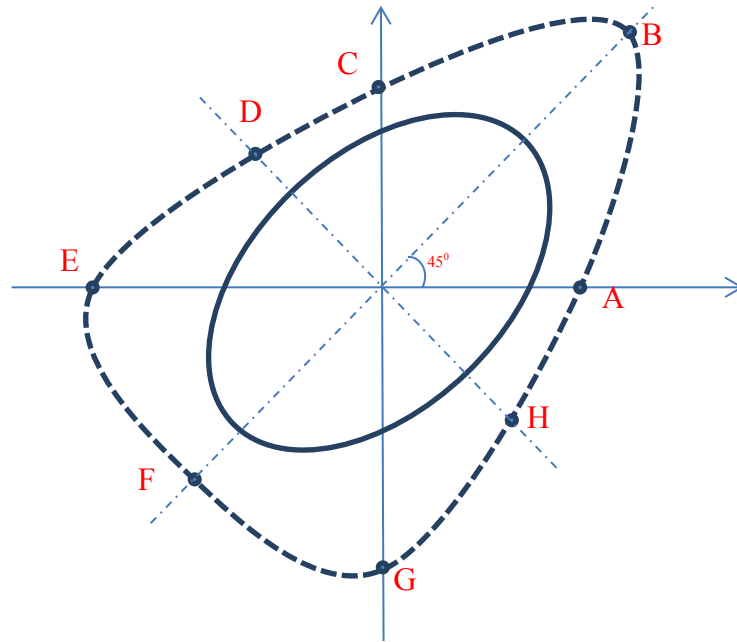


Figure 4.5: Schematic diagram of representative stress state.

Considering the influence of plastic flow direction at each representative stress state, unit normal direction $\hat{\mathbf{N}}$ and tangent direction $\hat{\mathbf{T}}$ of yield surface are chosen to construct the total strain increment. As a result, the total strain increment is assigned in the form:

$$\Delta\epsilon(a_1, a_2) = \hat{a}_1 \|\Delta\epsilon\| \hat{\mathbf{N}} + \hat{a}_2 \|\Delta\epsilon\| \hat{\mathbf{T}}, \quad (4.74)$$

where $\hat{a}_1$ and $\hat{a}_2$ are positive scalars.

In order to study the convergence during the shape's change of yield surface, four typical effective plastic strain levels: $\bar{\varepsilon}^p = 0$, $\bar{\varepsilon}^p = 0.2$, $\bar{\varepsilon}^p = 0.4$ and $\bar{\varepsilon}^p = 0.6$, are chosen, in representation of the initial yield surface (elliptical-like yield surface), critical yield surface - in which the shape changes from elliptical-like to triangular-like shape - , triangular-like evolving surface, and fixed triangular-like shape yield surface.

Meanwhile, two total strain increments, $\|\Delta\boldsymbol{\varepsilon}\| = 0.005$ and $\|\Delta\boldsymbol{\varepsilon}\| = 0.01$, are used to study the influence of strain increment on the convergence for different iterative algorithms. The convergence iterative numbers in the aforementioned representative stress states, obtained from different algorithms, are summarized in the Table 4.3 and Table 4.4. In these tables, "NC" denotes non-convergence. The iterative number filled in Table 4.3 and Table 4.4 is the maximum iterative number of all plastic directions under each representative stress state. Therefore, Table 4.3 and Table 4.4 merely present an overview comparison of the different integration schemes. Details are explained and discussed in Section 4.5.2 (b).

Examining Table 4.3, it is possible to observe that when the effective plastic strain reaches $\bar{\varepsilon}^p = 0$ (initial yield surface), the Newton-Raphson's line search with algorithm presents the same iterative numbers, in convergence with the standard Newton-Raphson scheme. As a matter of fact, in such a case, the line search algorithm is not activated. This means that all the interpolated methods generate the same results.

It is not difficult to understand that the initial yield surface is an elliptical-like shape, similar to the von-Mises yield criterion, or Hill'48 yield criterion shapes. The curvature of the yield surface is not big, and when the small strain increment is loaded, the shape's change of the yield surface is very slightly, keeping its elliptical-like shape. By this, it is possible to sustain that the standard Newton-Raphson's scheme is enough to obtain the quadratic convergence rate.

Confronting the results at different strain increments, it is clearly perceived that when the strain increment increases from $\|\Delta\boldsymbol{\varepsilon}\| = 0.005$ to $\|\Delta\boldsymbol{\varepsilon}\| = 0.01$, the required maximum iterative numbers to be convergent have remarkably increase; moreover, when the Newton-Raphson's with line search algorithm is used the maximum iterative number has a slightly decrease at the uniaxial tensile stress state in transverse

direction (point C) and biaxial tensile state (D). This is an indication that the line search algorithm is activated. In comparison with the iterative number, it is accurate to say that the aforementioned three interpolation methods presented the same iterative number.

As a result, it could be summarized that whenever the yield surface displays an elliptical-like shape, and the shape does not present significant changes during its following evolution, N-R with line search algorithm produces similar results to the standard N-R algorithm; besides using standard N-R algorithm it is possible to obtain the quadratic convergence.

Table 4.3: Comparisons of Iterative number at $\bar{\varepsilon}^p = 0$, $\bar{\varepsilon}^p = 0.6$.

	$\bar{\varepsilon}^p$	$\ \Delta \boldsymbol{\varepsilon}\ $	TX (A)	TY (C)	BT (B)	CX (E)	CY (G)	BC (F)	Shear (D)	Shear (H)
Standard N-R	0	0.005	12	12	11	10	10	10	12	12
		0.01	21	23	19	18	18	16	19	20
	0.6	0.005	NC	NC	NC	NC	NC	NC	NC	NC
		0.01	NC	NC	NC	NC	NC	NC	NC	NC
N-R+LQ	0	0.005	12	12	11	10	10	10	12	12
		0.01	21	20	18	18	18	16	19	20
	0.6	0.005	15	16	13	10	10	15	16	16
		0.01	16	16	13	13	13	13	16	16
N-R +LQ3	0	0.005	12	12	11	10	10	10	12	12
		0.01	21	20	18	18	18	16	19	20
	0.6	0.005	15	16	13	10	10	15	16	16
		0.01	16	16	13	13	13	13	16	16
N-R+LC	0	0.005	12	12	11	10	10	10	12	12
		0.01	21	20	18	18	18	16	19	20
	0.6	0.005	15	16	13	10	10	15	16	16
		0.01	16	16	13	13	13	13	16	16

Note: Standard N-R denotes Newton-Raphson's closest-point projection iterative scheme; N-R+LQ denotes Newton-Raphson's iterative scheme with line search algorithm, where the quadratic interpolation was used; N-R +LQ3 represents

Newton-Raphson's iterative scheme with line search algorithm, where the three point quadratic interpolation was used; N-R+LC represents Newton-Raphson's iterative scheme with line search algorithm, where the cubic interpolation was used.

TX: represents the uniaxial tensile in rolling direction

TY: represents uniaxial tensile in transverse direction;

BT: represents biaxial tensile;

CX: represents uniaxial compressive in rolling direction;

CY: represents uniaxial compressive in transverse direction;

BC: represents biaxial compressive.

When effective plastic strain $\bar{\epsilon}^p$ is equal to 0.6, the yield surface has already changed into the triangular-like shape, and its shape will not be changed again, not even if the material continues undergoing the deformation. That is because the calibrated material parameters in Table 4.2 only can supply data with the accumulated plastic strain less than $\bar{\epsilon}^p = 0.5$, so the following deformation keep using the same material parameters for transformation matrix as $\bar{\epsilon}^p = 0.5$.

From Table 4.3, it can be observed that when effective plastic strain $\bar{\epsilon}^p = 0.6$, even if small strain increment $\|\Delta\epsilon\| = 0.005$ is used, the standard Newton-Raphson cannot obtain convergence at all the stress states. Nevertheless, N-R with line search algorithm only requires a few iterative numbers to obtain the convergence, slightly more than the iterative number at the same stress state of the initial yield surface ($\bar{\epsilon}^p = 0$).

When the strain increment increases into double value, $\|\Delta\epsilon\| = 0.01$, the maximum iterative number of N-R with line search algorithm does not show the evidence of a significant increase, with almost the same iterative number as small strain increment $\|\Delta\epsilon\| = 0.005$. Regarding both the fixed elliptical-like shape yield surface (the initial yield surface) and the fixed triangular-like shape yield surface ($\bar{\epsilon}^p = 0.6$) without shape change during the following deformation load, standard N-R algorithm is able to reach a high level of quadratic convergence for the fixed elliptical-like shape yield surface (the initial yield surface). In spite of this, and respecting the fixed triangular-like shape yield surface ($\bar{\epsilon}^p = 0.6$), N-R algorithm does not prove able to obtain convergence. Although N-R with line search does not demonstrate a substantially higher convergence improvement than standard N-R for the fixed elliptical-like shape yield surface (the initial yield surface), on the other hand it can remarkably improve

the convergence for the fixed triangular-like shape yield surface ($\bar{\varepsilon}^p = 0.6$). With respect to fixed yield surfaces, no matter if fixed elliptical-like shape yield surface or fixed triangular-like shape yield surface ones, interpolation methods do not present different iterative numbers, nor an improvement on the iterative convergence.

Table 4.4 Comparisons of Iterative number at $\bar{\varepsilon}^p = 0.2$, $\bar{\varepsilon}^p = 0.4$.

	$\bar{\varepsilon}^p$	$\ \Delta\boldsymbol{\varepsilon}\ $	TX (A)	TY (C)	BT (B)	CX (E)	CY (G)	BC (F)	Shear (D)	Shear (H)
Standard N-R	0.2	0.005	NC	62	43	21	20	NC	35	33
		0.01	NC	NC	NC	NC	NC	NC	NC	NC
	0.4	0.005	NC	NC	NC	NC	NC	NC	NC	NC
		0.01	NC	NC	NC	NC	NC	NC	NC	NC
N-R+LQ	0.2	0.005	NC	NC	NC	15	16	NC	NC	NC
		0.01	NC	NC	NC	15	17	NC	NC	NC
	0.4	0.005	15	15	14	10	11	18	16	16
		0.01	16	16	14	13	13	18	15	14
N- R+LQ3	0.2	0.005	NC	NC	NC	15	16	NC	NC	NC
		0.01	NC	NC	NC	15	17	NC	NC	NC
	0.4	0.005	15	15	14	10	11	18	16	16
		0.01	16	16	14	13	13	18	15	14
N-R+LC	0.2	0.005	NC	NC	NC	11	16	NC	NC	NC
		0.01	NC	NC	NC	13	16	NC	NC	NC
	0.4	0.005	10	11	8	8	8	14	10	10
		0.01	10	10	11	10	10	13	11	11

In Table 4.4, it is presented the maximum iterative number of all plastic flow direction for the yield surface at effective plastic strain $\bar{\varepsilon}^p = 0.2$, while $\bar{\varepsilon}^p = 0.4$. It is also possible to observe that for the yield surface of effective plastic strain $\bar{\varepsilon}^p = 0.2$, when total strain increments $\|\Delta\boldsymbol{\varepsilon}\| = 0.005$, the standard N-R iterative scheme is unable to obtain convergence at uniaxial tensile stress state (point A) and biaxial compressive stress state (point F) for all plastic direction. Respecting the other representative stress state, it requires a large iterative number to be in convergence compared with the

initial yield surface; when total strain increment increases into $\|\Delta\boldsymbol{\varepsilon}\| = 0.01$, it proves not able to obtain convergence. This can also be seen in the N-R with line search iterative scheme. The details from the comparisons between N-R scheme and N-R with line search iterative scheme will be discussed further, in the next subsection of this study in order to understand the influence of the line search and interpolation method on convergence.

These nonconvergent results occur because the yield surface of effective plastic strain $\bar{\varepsilon}^p = 0.2$ is a critical yield one. From this surface, with the increase of hardening during the subsequent deformation, the shape of yield surface changes significantly and abruptly, from its elliptical-like shape to a triangular-like one. This causes a major alteration in the curvature of the yield surface. Once a large increment is loaded, there is a chance that the initial estimation in the plastic corrected step may largely deviate from the final solution, thus originating the nonlinear equation sets difficult to converge.

Concerning the yield surface of effective plastic strain $\bar{\varepsilon}^p = 0.4$, the standard N-R iterative scheme is also unable to obtain convergence for all plastic flow direction and at all representative stress states, as the fixed yield surface of effective plastic strain being $\bar{\varepsilon}^p = 0.6$. Nevertheless, N-R line search is not only able of obtaining convergence for all plastic directions and at all representative stress states, as it also requires some iterative number to obtain convergence. This is explained by the evolved characteristics of the yield surface. Although the shape of the yield surface continues triangularization with the external load, the basic shape has already been settled, therefore the continuous loading only turns the shape more triangular. This implies that the curvature does not suffer an abrupt change. Instead, it is a smooth shape's evolution. It is possible to conclude that the N-R with line search iterative scheme can be effective to improve the convergence as the fixed triangular-like yield surface of effective plastic strain is of $\bar{\varepsilon}^p = 0.6$. By comparison of different interpolation methods in the N-R with line search iterative scheme, it is verifiable that both the quadratic and the three parameter quadratic (quadratic 3) interpolations present the same results, and that the maximum iterative number at each representative stress state is equal. The cubic interpolation methods present a much smaller maximum iterative number than both the aforementioned interpolation methods. Hence, it is possible to argue that the N-R with cubic interpolated line search iterative

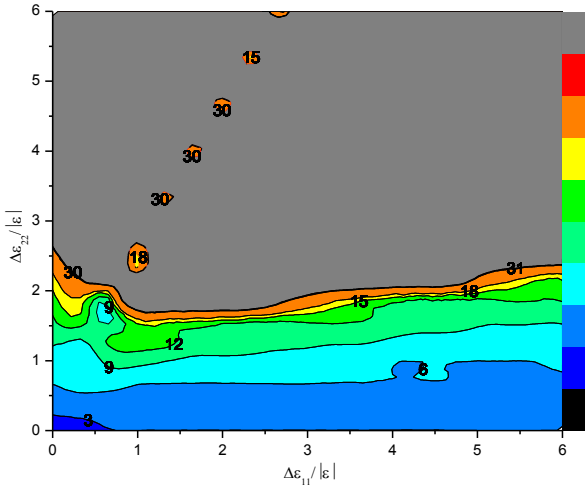
scheme can remarkably improve the convergence, even in large total strain increment loading; besides, the maximum iterative number presents a mere slight increase. We then conclude, accordingly to what was previously demonstrated, that the N-R with cubic interpolated line search iterative scheme is a very robust one.

(b) Comparison of iterative number contours under different stress states

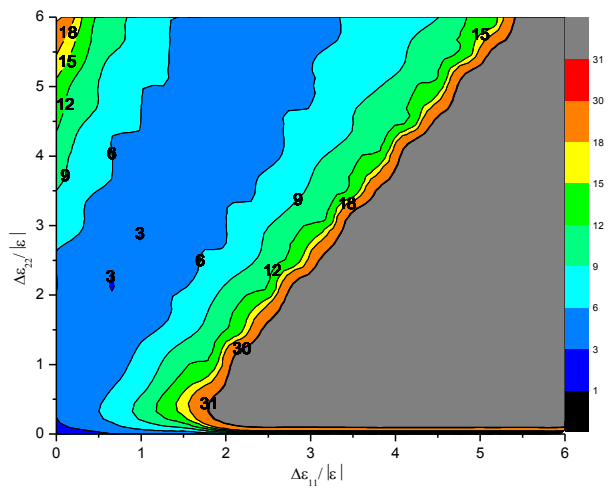
Even though the above overview discussion has already elucidated some improvements of the N-R line search iterative scheme, that is not enough for the demonstration of details. In fact, none of the iterative schemes is able to obtain the convergence for the yield surface of effective plastic strain $\bar{\epsilon}^p = 0.2$ at any given strain increments. This is illustrative of the fact, that all iterative schemes have some places of nonconvergence. It is not possible to demonstrate the specific influence of line search and interpolation on the convergence properties during the iterative plastic-corrected steps. In this sub-section, some typical iterative number contours, at different representative stress states, iterative schemes and total strain increments will be presented.

Although it is possible to use the N-R model for capturing any anisotropic mechanical response, the titanium alloy studied in this paper shows only some slightly anisotropic properties, which can be seen in the analysis of single element validation in the sub-section 4.5.1. Therefore, the properties in uniaxial tensile and compressive stress state in rolling direction are respectively similar to the uniaxial tensile and compressive stress state in transverse direction. This is quite easy to verify if compared with the maximum iterative number in Table 4.3 and Table 4.4.

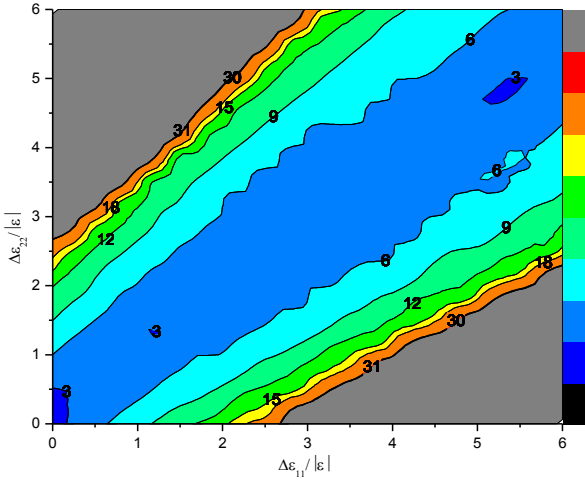
As a result, in this subsection, four representative stress states will be investigated, namely uniaxial tensile in rolling direction, uniaxial compressive in rolling direction, biaxial tensile, biaxial compressive. As analysed in the previous sub-section, regarding the fixed elliptical-like shape yield surface (the initial yield surface, which is also the yield surface of effective plastic strain $\bar{\epsilon}^p = 0.0$), all the iterative schemes demonstrate the same maximum iterative number, together with the fact that the line search algorithm does not activate. Thus, the iterative number contour of N-R combined with line search algorithm is the same as the standard N-R scheme.



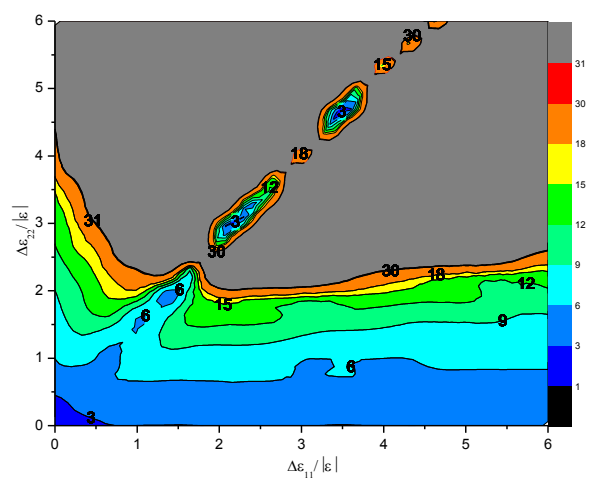
(a) Tensile X (N-R)



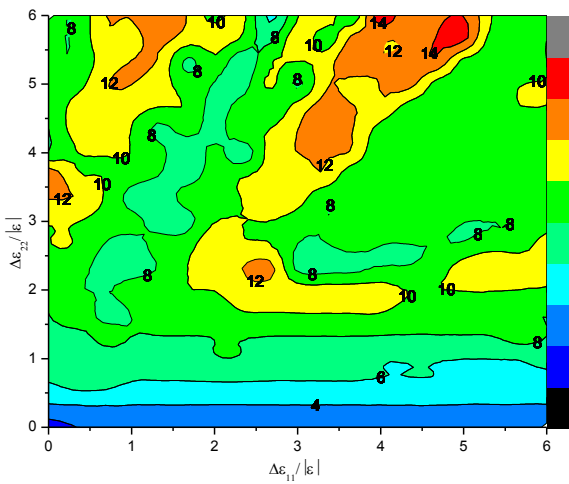
(b) Biaxial tensile (N-R)



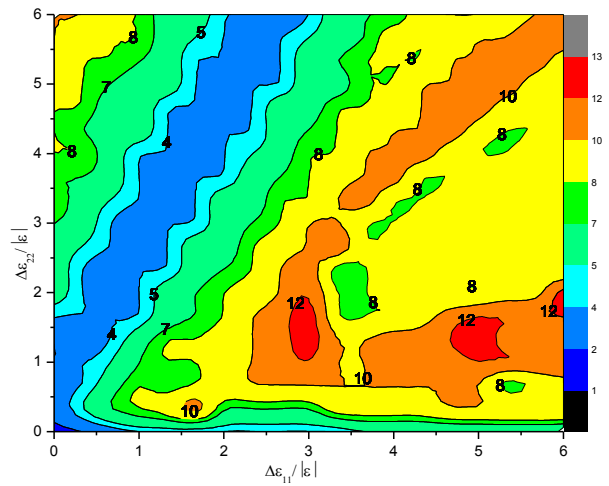
(c) Compressive-X (N-R)



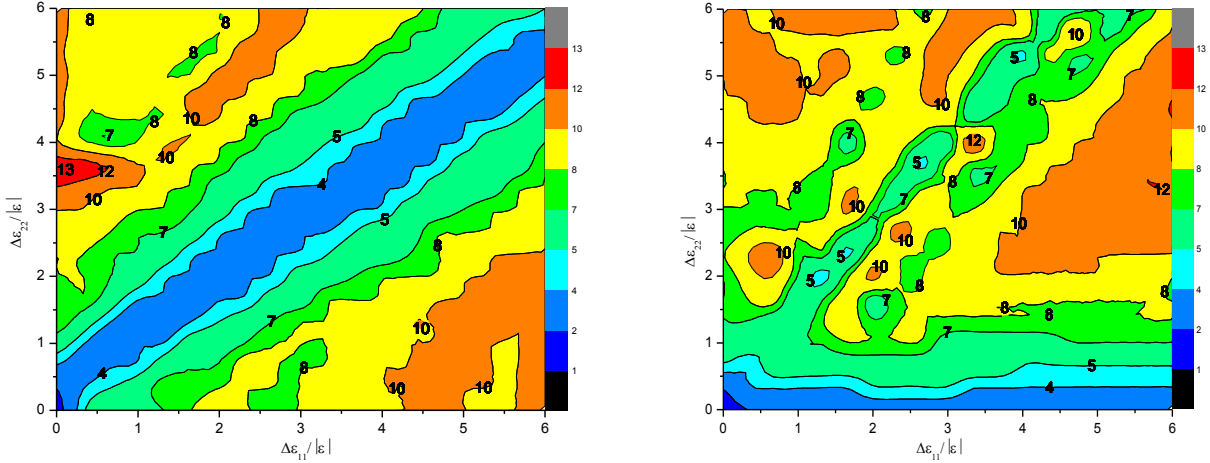
(d) Biaxial compressive (N-R)



(e) Tensile X (N-R +LQ)



(f) Biaxial tensile (N-R +LQ)



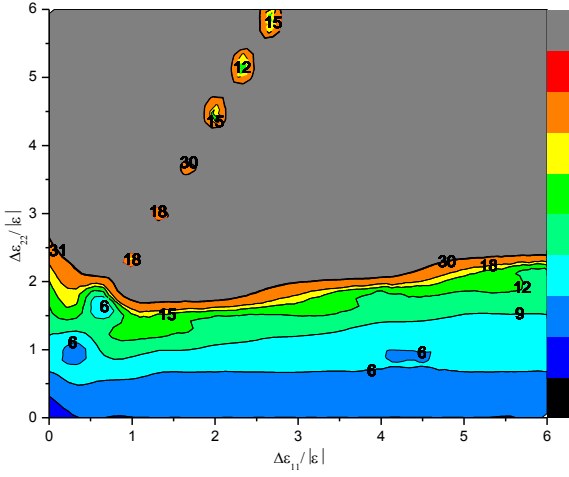
(g) Compressive-X (N-R +LQ)

(h) Biaxial compressive (N-R +LQ)

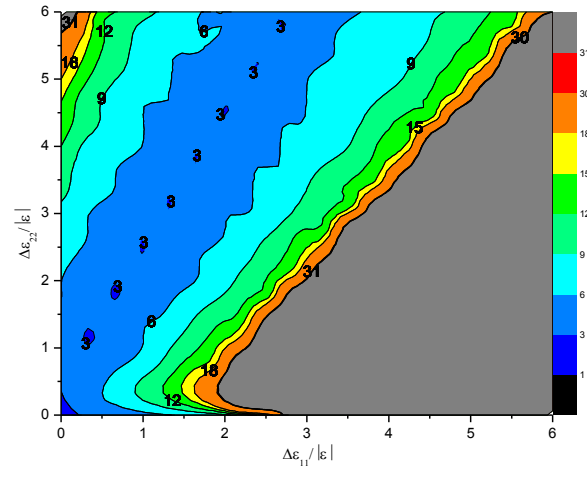
Figure 4.6: Iterative number contours obtained from different integration schemes under different stress states at $\bar{\epsilon}^p = 0.6$.

In order to describe the influence of line search algorithm on the convergence, two representative yield surfaces are chosen. One is the fixed triangular-like yield surface, where the effective plastic strain is of $\bar{\epsilon}^p = 0.6$; the other is the yield surface at effective plastic strain $\bar{\epsilon}^p = 0.4$, where the shape of yield surface will evolve in the following deformation. The total strain increment $\|\Delta \epsilon\|$ is set as 0.01.

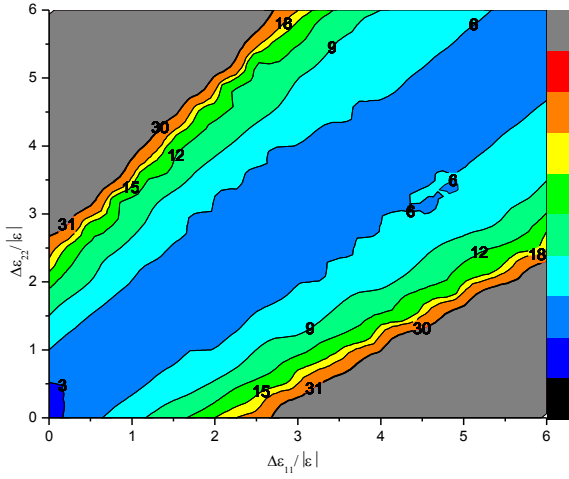
Figure 4.6 presents iterative number contours under different stress states at $\bar{\epsilon}^p = 0.6$. From there, it is clear seen that N-R scheme although cannot make all strain increment domains converge, some parts are still easy to converge. For example, In the case of tensile X (see in Figure 4.6 (a)), if all strain increment is loaded along the rolling direction, the N-R scheme is very effective to make problems converge. By comparison of iterative number contours between tensile and compressive tests, it is clearly seen that the contour is remarkable different. This is also indicates the asymmetry of the yield surface. Figure 4.6 (e), (f), (g), (h) present the iterative number contours under different stress states obtained by N-R combined line search algorithm. From there, it can be seen that in the entire strain increment domain, the nonlinear stress updated equations sets can get convergence. The line search algorithm can be effective to improve the convergence in the area, where the problems cannot converge with the N-R scheme.



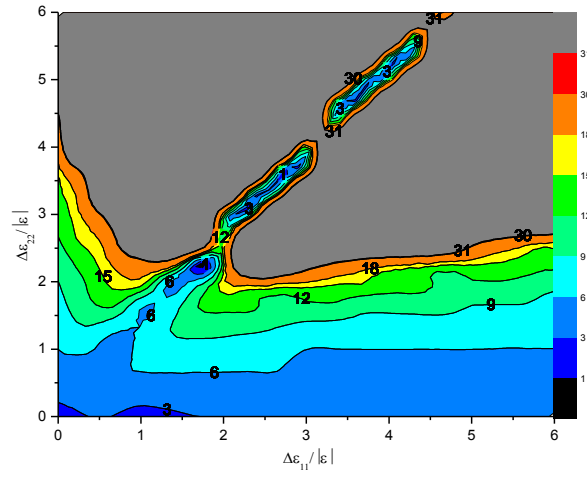
(a) Tensile X (N-R)



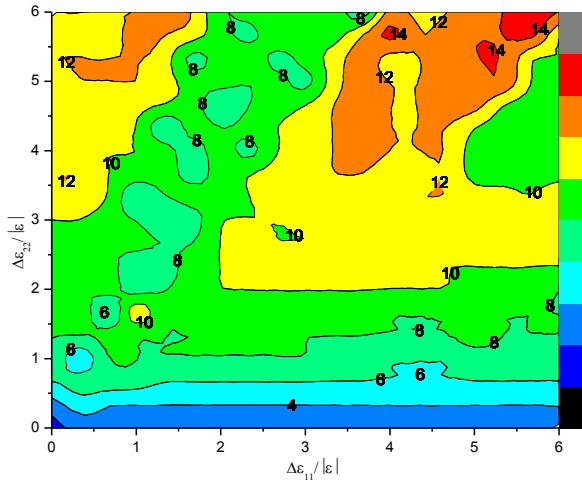
(b) Biaxial tensile (N-R)



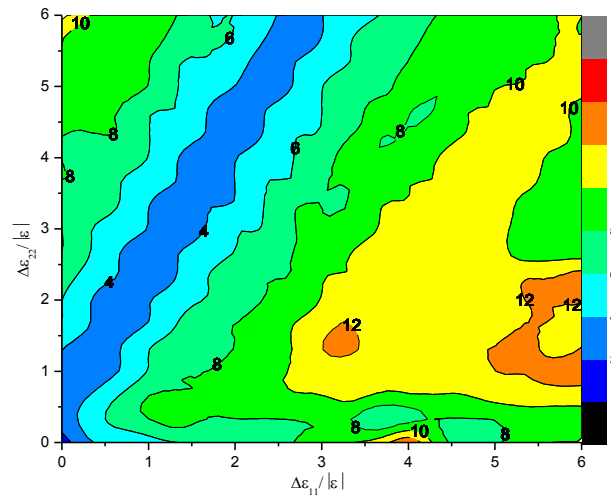
(c) Compressive-X (N-R)



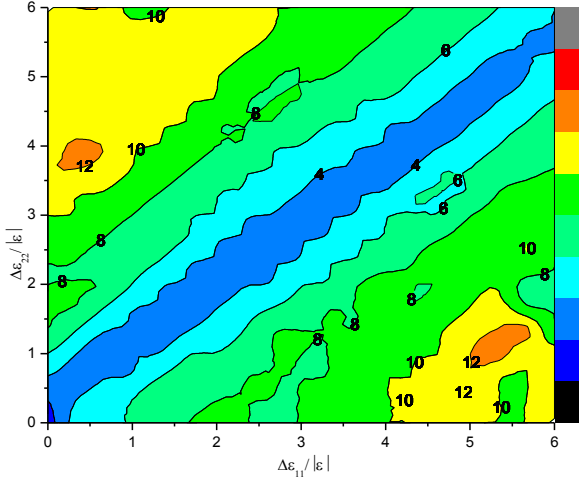
(d) Biaxial compressive (N-R)



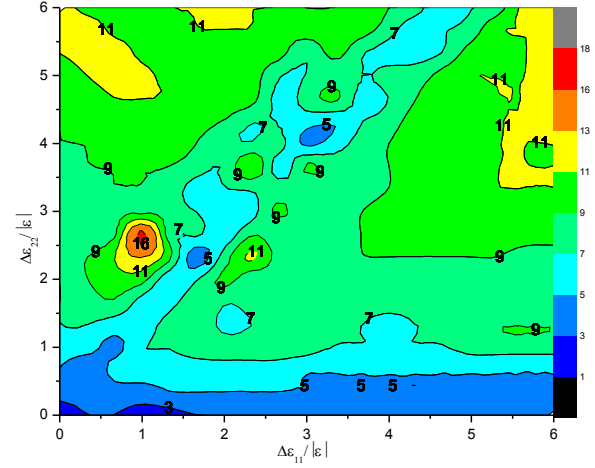
(e) Tensile X (N-R+LQ)



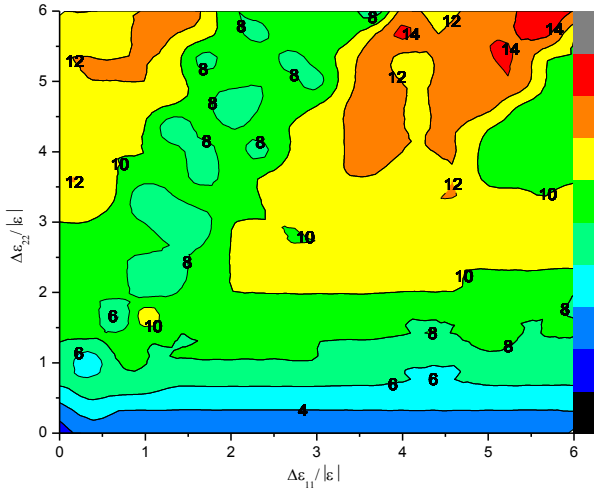
(f) Biaxial tensile (N-R+LQ)



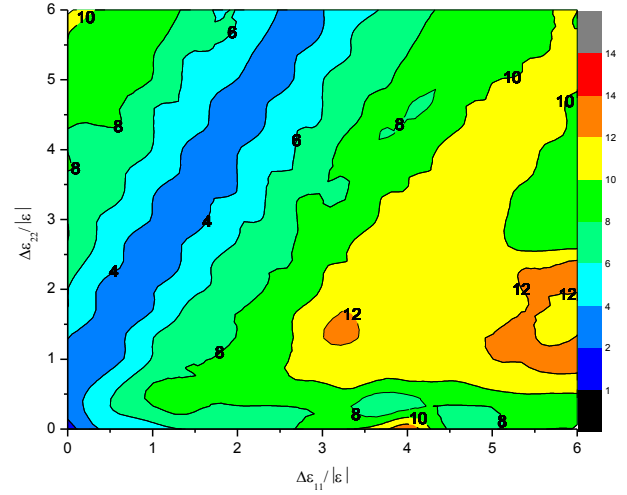
(g) Compressive-X (N-R+LQ)



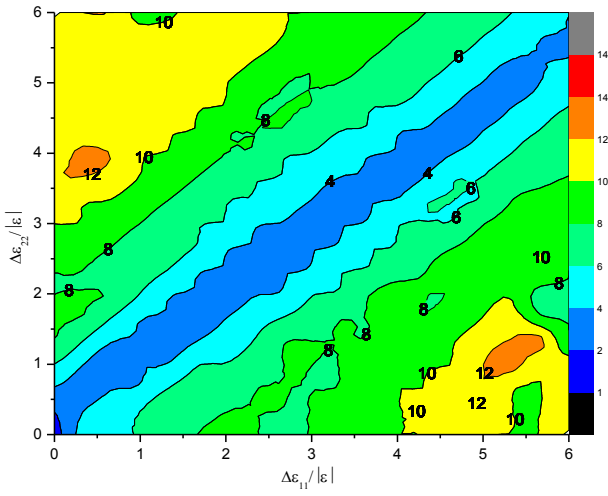
(h) Biaxial compressive (N-R+LQ)



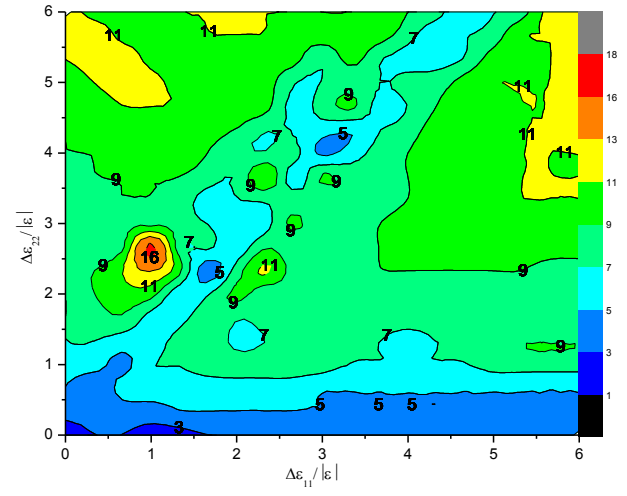
(i) Tensile X (N-R+LQ3)



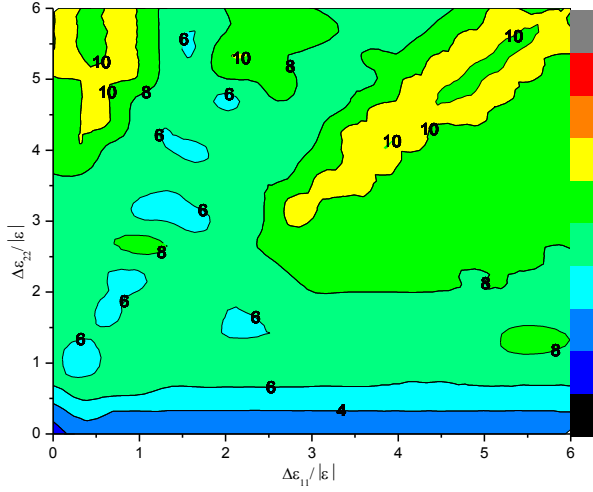
(j) Biaxial tensile (N-R+LQ3)



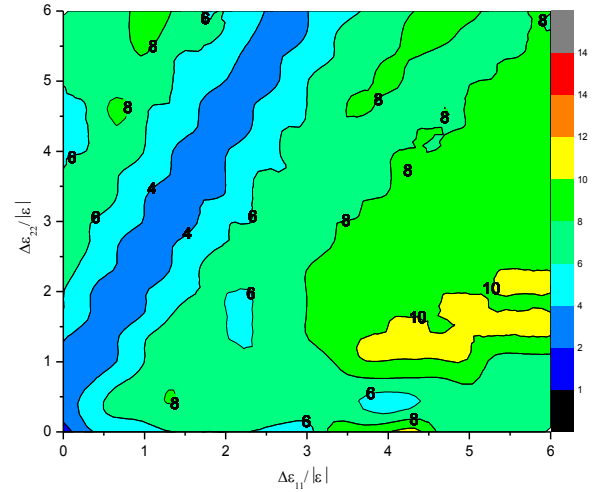
(k) Compressive-X (N-R+LQ3)



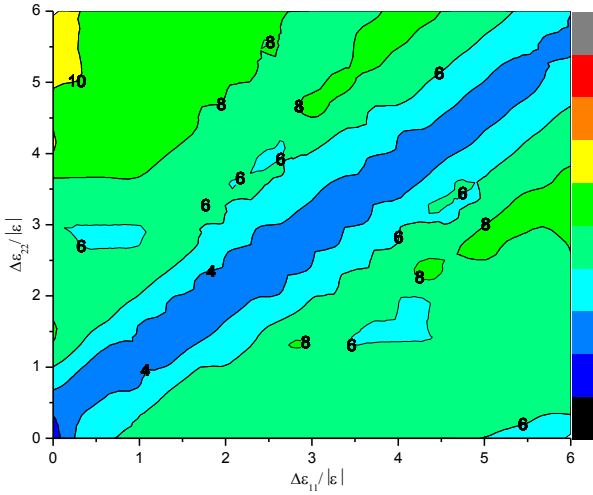
(l) Biaxial compressive (N-R+LQ3)



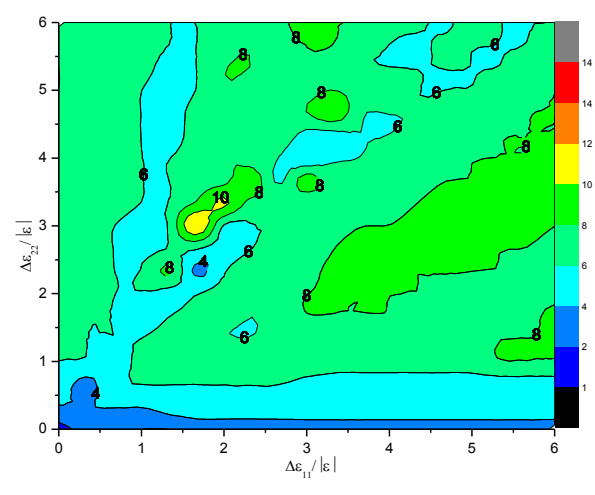
(m) Tensile X (N-R+LC)



(n) Biaxial tensile (N-R+LC)



(o) Compressive-X (N-R+LC)



(p) Biaxial compressive (N-R+LC)

Figure 4.7: Iterative number contours obtained from different integration schemes under different stress states at $\bar{\varepsilon}^p = 0.4$.

Figure 4.7 presents iterative number contours for different integration scheme under different stress states at $\bar{\varepsilon}^p = 0.4$. From the comparisons, it can be seen that all the line search algorithms can improve the convergence. N-R with the 3 point quadratic (quadratic 3) interpolated line search algorithm has the same effectiveness to convergence as the quadratic interpolated line search algorithm. However, the N-R with the cubic interpolated line search algorithm can reduce the iterative number and further improve the convergence.

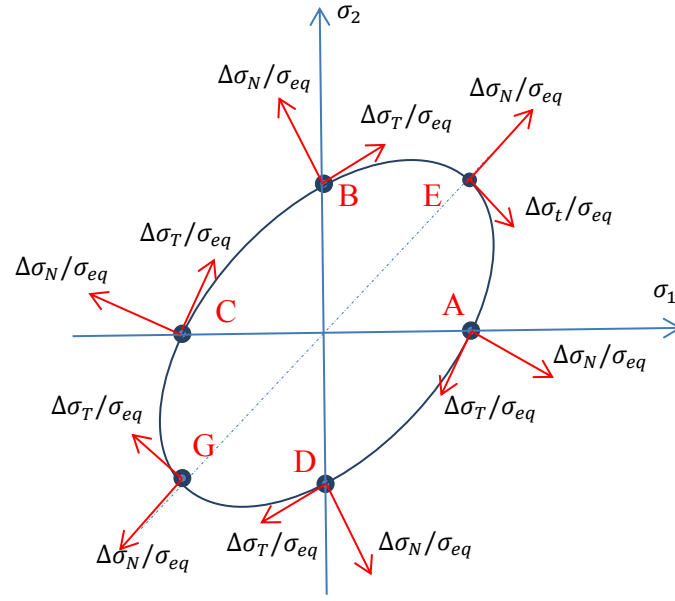
4.6 Iso-error maps

In this section, a numerical assessment of the accuracy and stability of the small strain

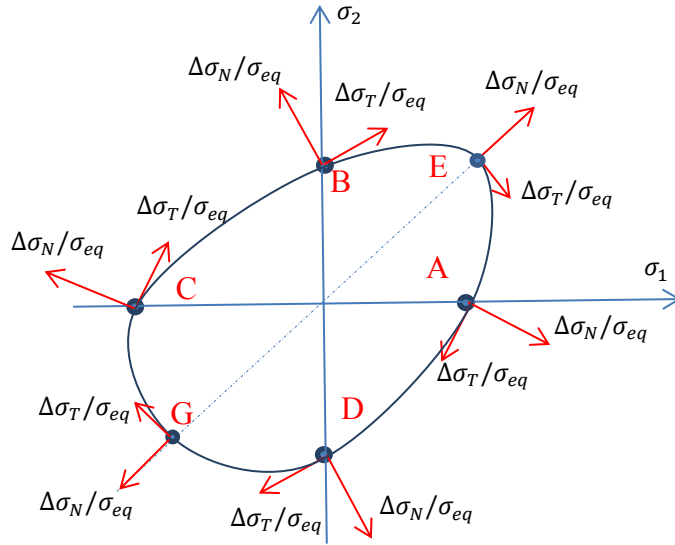
part of the integration algorithm for the Cazacu'04 constitutive model is discussed. Iso-error maps have proved to be an efficient and most reliable tool for testing numerically the accuracy of integration algorithms in elasto-plasticity. Besides, they are also used to solve some strain controlled problems by the closest-point projection algorithm [5, 24].

Owing simultaneously to the anisotropy properties and to the shape's change of yield surface, it is possible to verify that with the evolution of equivalent plastic strain during the plastic deformation, not only different stress states generate different error, but also that the same stress states, at different plastic deformation, have different iso-error maps. According to the yield surface evolution presented in Figure 4.4, the yield surface at the equivalent plastic strain of 0.4 changes from an elliptical-like surface, at the small deformation, to a triangular-like one. In order to reflect the influence of the yield surface change on the iso-error maps, two yield surfaces were selected as object of study. The first is the initial yield surface (the equivalent plastic strain $\bar{\epsilon}^p = 0.0$), identified as surface A hereafter; the second one is the yield surface of the equivalent plastic strain of 0.4, identified as surface B. The iso-error map of the initial yield surface is obtained starting from a value $\hat{\sigma}_0$ of the stress lying on the boundary of the initial elastic domain and computing the associated strain $\hat{\epsilon}_0 = \mathbf{D}^{e-1} \hat{\sigma}_0$. Regarding the iso-error map of the surface B, the stress magnitude of each stress state should be computed according to the Cazacu'04 yield criterion, allowing the associated strain to be calculated afterwards by the same method as the initial yield surface.

In order to show the error difference at different stress states, six points on those two representative yield surfaces are selected, representing a wide range of possible stress states. These points, labeled A, B, C, D, E, G in Figure 4.8, correspond, respectively, to uniaxial tensile in rolling direction, biaxial tensile, uniaxial tensile in transverse direction, pure shear stress, uniaxial compressive in rolling direction, biaxial compressive, uniaxial compressive in transverse direction, and, lastly, to pure shear stress.



Yield surface A ($\bar{\epsilon}^p=0.0$)



Yield surface B ($\bar{\epsilon}^p=0.4$)

Figure 4.8: Representative stress state at yield surface A ((Equivalent plastic strain $\bar{\epsilon}^p = 0.1$)) and B ((equivalent plastic strain $\bar{\epsilon}^p = 0.4$)).

To draw the iso-error maps, for each selected representative point on the yield surface, a sequence of specified normalized strain increments was considered. Afterward, the stresses, corresponding to the states of strain, in accordance to this methodology, are computed by applying the closest point projection algorithm. A sequence of strain

increments is applied in order to correspond to specified normalized elastic trial stress increments of the form:

$$\Delta\sigma^{trial} = \frac{\Delta\sigma_T}{\sigma_{eq}} \hat{T}_p + \frac{\Delta\sigma_N}{\sigma_{eq}} \hat{N}_p, \quad (4.75)$$

where, $\hat{N}_p$ and $\hat{T}_p$ are, respectively, the unit (in Euclidean norm) projected normal and tangent vectors on the $\sigma_{11} - \sigma_{22}$ stress plane, and σ_{eq} is the equivalent stress.

Applying increments of trial stress in the tangential and normal direction to the yield surface, the error is evaluated by:

$$ERROR(\%) = \frac{\sqrt{(\sigma_{Num} - \sigma_{EXACT}) : (\sigma_{Num} - \sigma_{EXACT})}}{\sqrt{\sigma_{EXACT} : \sigma_{EXACT}}}, \quad (4.76)$$

where, σ_{Num} is numerical solution, σ_{EXACT} is the exact solution. Here, due to the lack of an analytical solution, σ_{EXACT} is assumed to be the stress obtained by subincrementation of each stress increment into 1000 steps.

By varying the prescribed increment sizes $\Delta\sigma_T$ and $\Delta\sigma_N$, respectively associated with the tangential and normal directions to the yield surface, the related iso-error map can be drawn.

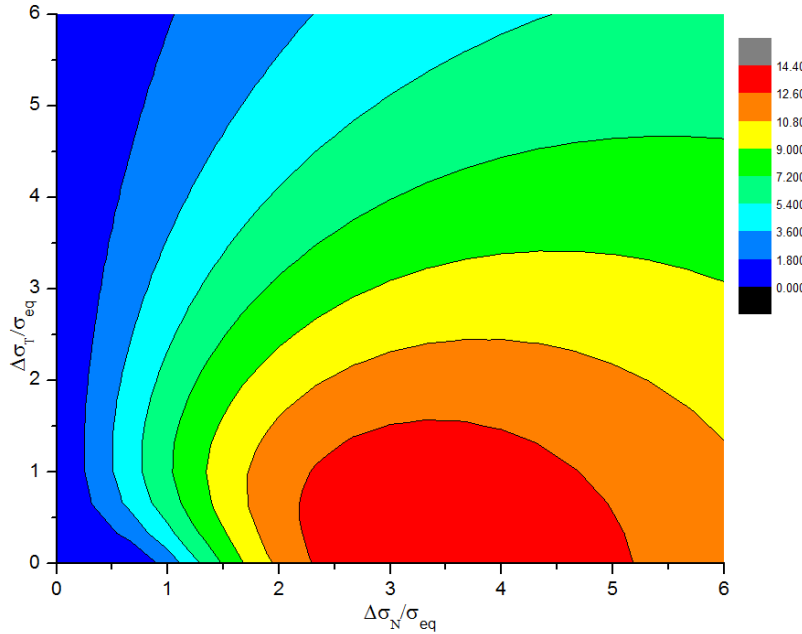


Figure 4.9: Iso-error maps when $\hat{\mathbf{L}}$ is equal to identity tensor and $\hat{c}$ vanishes.

If all of a_1 , a_2 , a_3 , a_4 , a_5 , and a_6 are set equal to one, and $\hat{c}$ equal to zero, the transformed tensor $\hat{\mathbf{L}}$, is equal to the identity tensor, originating the degradation of the current Cazacu'04 constitutive model into the von Mises model. Figure 4.9 presents the contour of the equivalent plastic strain that is obtained by the reduced Cazacu'04

model. The iso-error map for this limiting scenario is depicted and it is quite secure to conclude that the results are in complete agreement with the von Mises iso-error map.

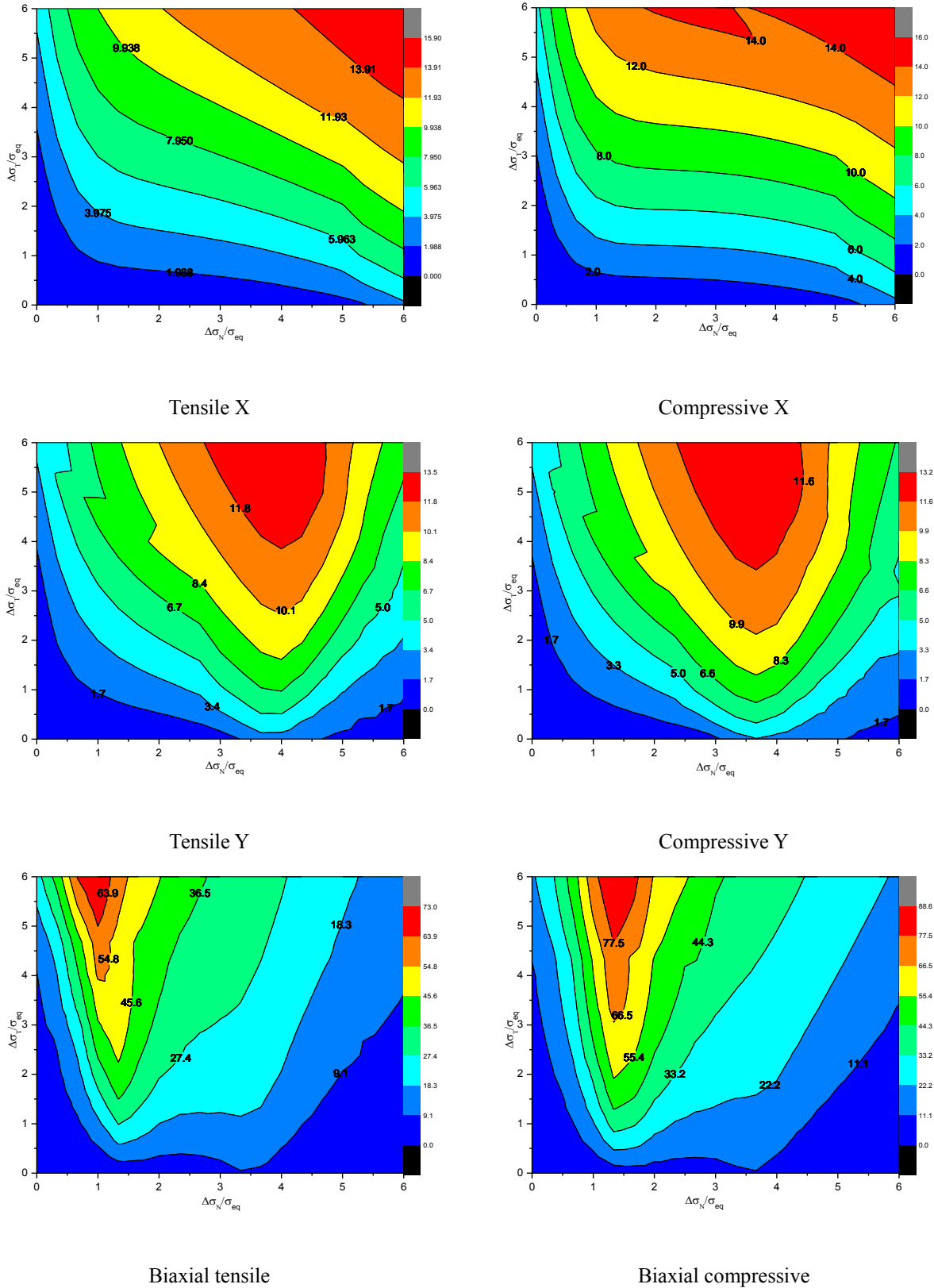


Figure 4.10: Iso-error maps of yield surface at equivalent plastic strain $\bar{\epsilon}^p = 0.0$.

Figure 4.10 depicts the iso-error maps of the yield surface of effective plastic strain $\bar{\varepsilon}^p = 0.0$ at six representative points. From there, it is possible to observe that the iso-error maps in the tension stress state are very similar to the corresponding compressive stress state. This also reflects that, in this specific context, the material does not present the asymmetrical tension/compression difference (SD effect). This can be easily explained by the yield surface shape. When the effective plastic strain is $\bar{\varepsilon}^p = 0.0$, the yield surface still keeps its classical elliptical-like shape. The iso-error maps in point A (uniaxial tensile in rolling direction) displays an eminent difference from the corresponding point on the yield surface of von-Mises criterion, which have already been shown in Figure 4.9.

By comparison of the iso-error maps between the point A (uniaxial tensile stress state in rolling direction) and the point B (uniaxial tensile stress state in transverse direction) in Figure 4.10, it is outstandingly perceived that the iso-error contours are different; moreover, the maximum iso-error value of the point A located in a different place from the point B, besides being somewhat larger than the point B. Regarding the biaxial tension stress state, the maximum value of iso-error moves close to the original point, and its value is much larger than the uniaxial tension state. It also indicates that large increment loading would generate a larger deviation from the exact value.

As far as Figure 4.11 is concerned, it displays the iso-error maps of the yield surface of effective plastic strain as $\bar{\varepsilon}^p = 0.6$ at six representative stress states. From there, it is clearly observable that the iso-error contour of the tension stress state is definitely different from the compression stress state, which is justified by the strength differential effect. Compared with the yield surface of effective plastic strain of $\bar{\varepsilon}^p = 0.0$, the iso-error maps also present a significant deal of change. It also reflects the evolution of the yield surface from $\bar{\varepsilon}^p = 0.0$ to $\bar{\varepsilon}^p = 0.6$. Comparing the iso-error maps between the uniaxial tensile stress state with the uniaxial compressive stress state, both of them in rolling direction it is clear that not only the place of maximum iso-error makes a change, moving close to original point, but also that the such a value is almost twice than the uniaxial tensile stress state in rolling direction. Although iso-error maps of stress states, along the transverse direction, are also capable of change, the place of maximum value of the compression stress state also moves close to original, when compared with the tension stress state. In fact, the

maximum value does not present a notorious increase. Similar to the iso-error map of the yield surface of equivalent plastic strain $\bar{\varepsilon}^p = 0.0$, the maximum value of the iso-error map of biaxial stress state is much larger than the uniaxial stress state. The biaxial compression stress state has a larger value than the biaxial tension stress state.

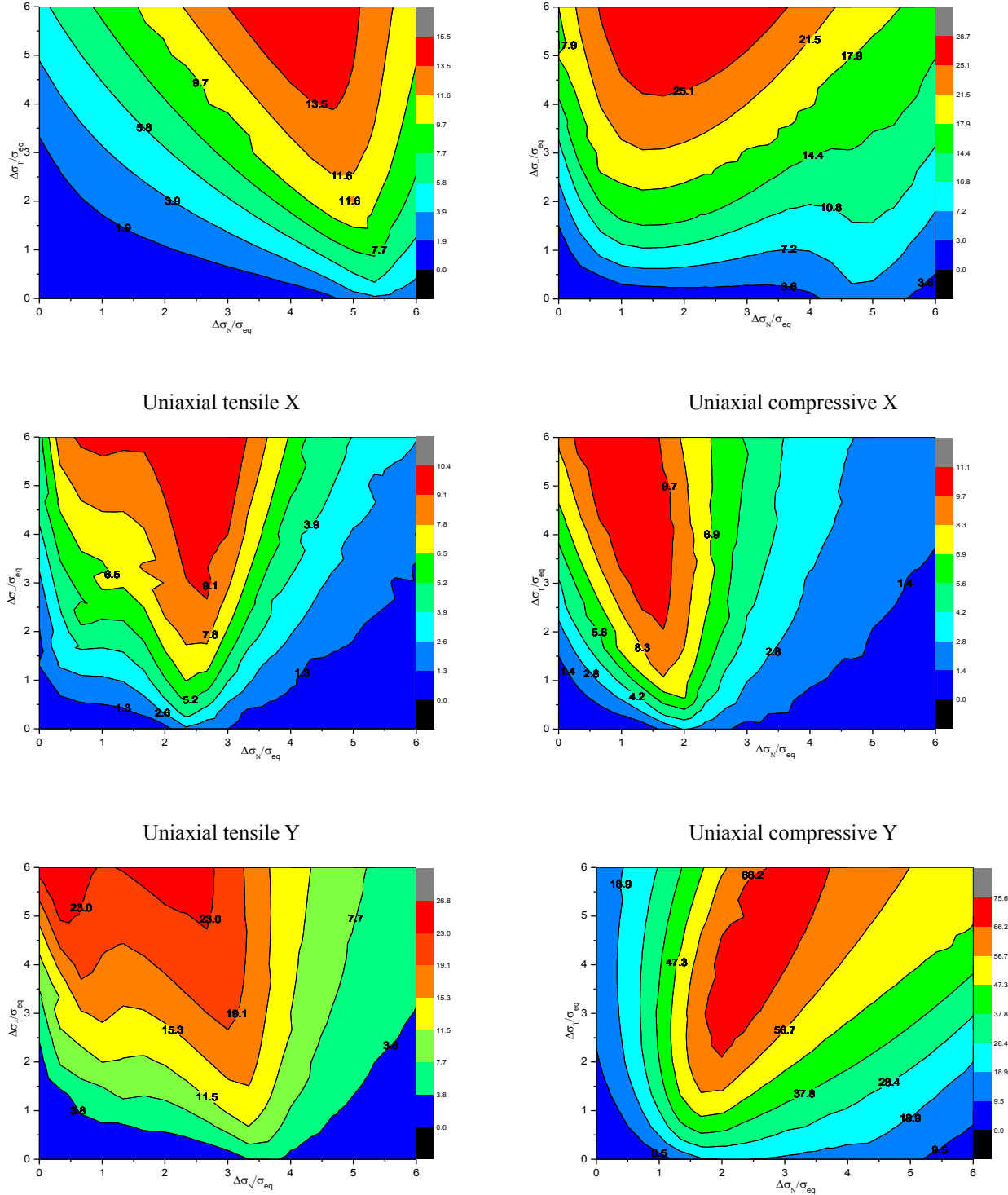


Figure 4.11: Iso error maps of yield surface at equivalent plastic strain $\bar{\varepsilon}^p = 0.6$.

4.7 Benchmark test-Cup deep drawing

Deep drawing is one of the most widely used processes for forming sheet metal parts in the automotive industry [25]. It is also a popular process in the assessment of formability in sheet metals. In this section, the Newton-Raphson with line-search cubic interpolation iterative scheme is applied to solve the nonlinear equation set in a complicated benchmark test. A validation of the presented approach is discussed, based on performed cylindrical cup deep drawing simulations. The dimension of all parts was listed as follows:

Punch diameter: $D_p = 97.46$ mm

Punch profile radius: $r_p = 12.70$ mm

Die opening diameter: $D_d = 101.48$ mm

Die profile radius: $r_d = 12.70$ mm

Blank radius: $D_b = 158.76$ mm

The initial thickness of the blank is 1.0 mm. Its schematic diagrams are presented in Figure 4.12.

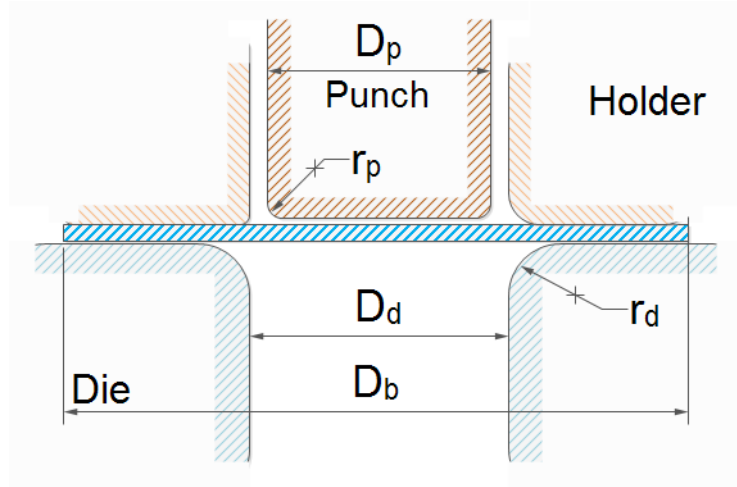
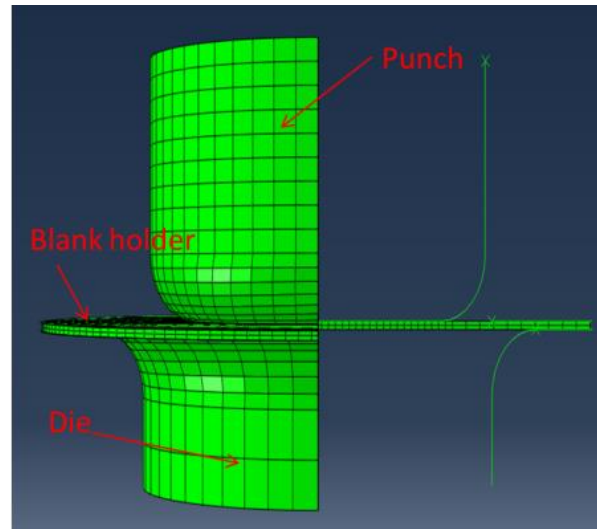


Figure 4.12: Schematic view of circular cup deep drawing.

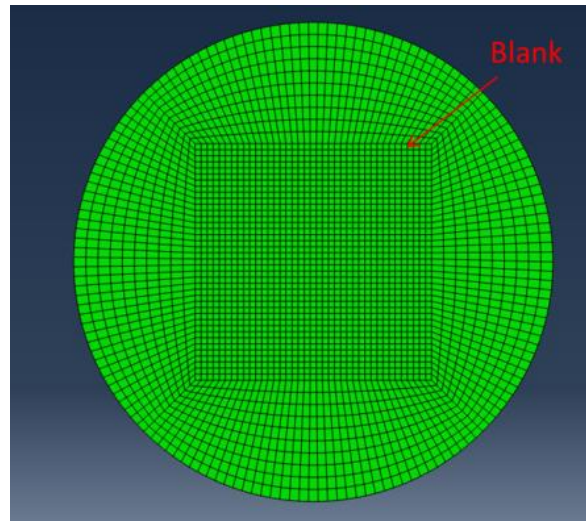
The finite element model is presented in Figure 4.13. The left side of Figure 4.13 (a) demonstrates half of finite element model; the right side shows the cross section of the half cup deep drawing model. The punch, die and blank holder are considered as rigid surfaces in the simulation, meshed with linear quadratic rigid elements (R3D4). The circular blank is meshed with 6162 reduced integrated eight-noded brick elements (C3D8R), which can be seen in Figure 4.13 (b). The displacement of the punch is set to be large enough, allowing pulling the whole blank into the die.

In order to avoid the occurrence of wrinkling during drawing, the constant blank

holder force of 30 kN is applied to the blank. The friction is considered with the Coulomb model, the coefficient of friction being 0.01 for all contact surfaces. The material parameters and hardening parameters, as presented in both Table 4.1 and Table 4.2, are also used in the current benchmark cup drawing model. Besides, an automatic time incrementation is used in the finite element simulation, where the initial and maximal time increment sizes are specified. In this automatic time incrementation, the time increment is reduced if either the global equilibrium iteration or the local iteration of the algorithm fails to converge [26].



(a) Finite element model



(b) Blank mesh

Figure 4.13: Cup drawing test finite element model.

The final shape and the equivalent plastic strain contour after the drawing operation are presented in Figure 4.14. From this sideview of the contour, it is possible to observe, firstly, a pronounced earing effect, and secondly that the maximum plastic

deformation primarily occurs at the neighboring place of the earing. This occurs because whenever a cylindrical cup deep drawing is performed on circular blanks it causes an uneven cup-rim (i.e. a number of high points called ears and an equal number of low points known as troughs), as a result of directional properties or anisotropy of the blank. The effect of anisotropy in deep drawing is generally viewed based on the number of ears, their location with respect to the rolling direction and, lastly, to their amplitude. In this simulation, four ears occur after deep drawing. This result also validates that Cazacu'04 constitutive model can be used to describe the anisotropic mechanical behavior between the rolling and transverse directions.

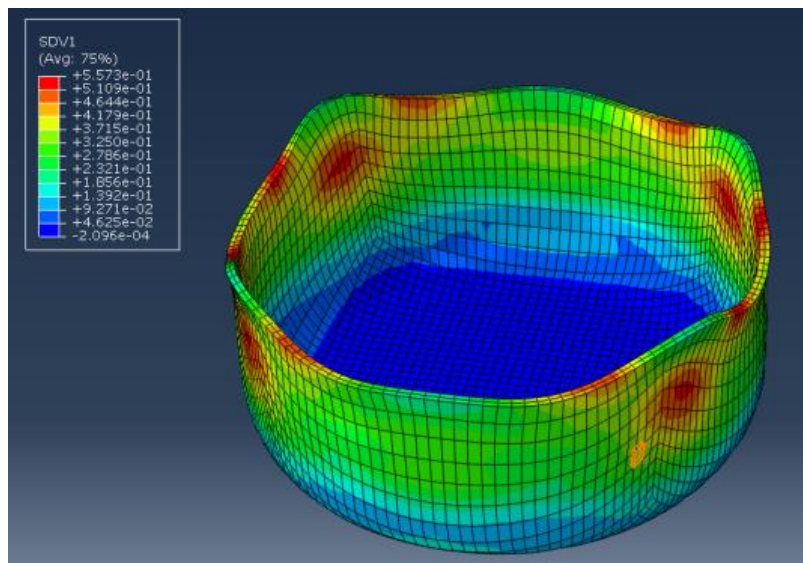


Figure 4.14: Equivalent plastic strain contour.

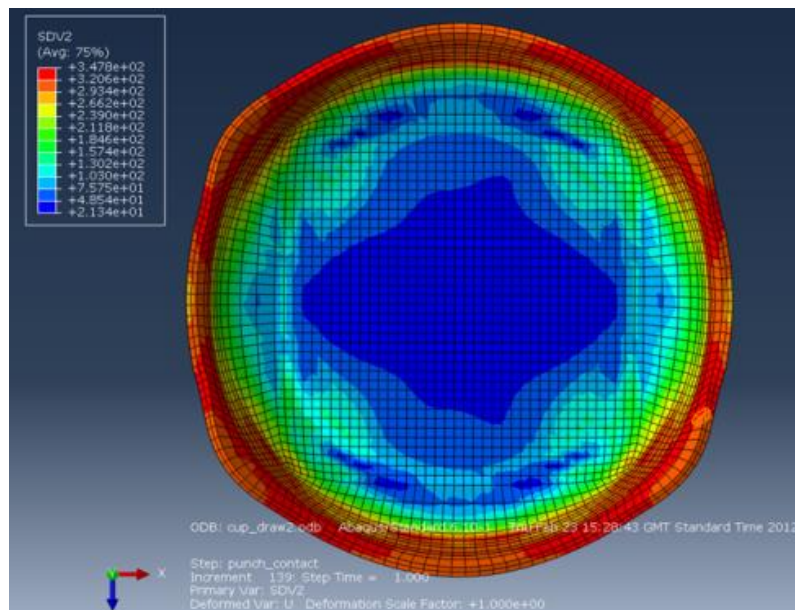
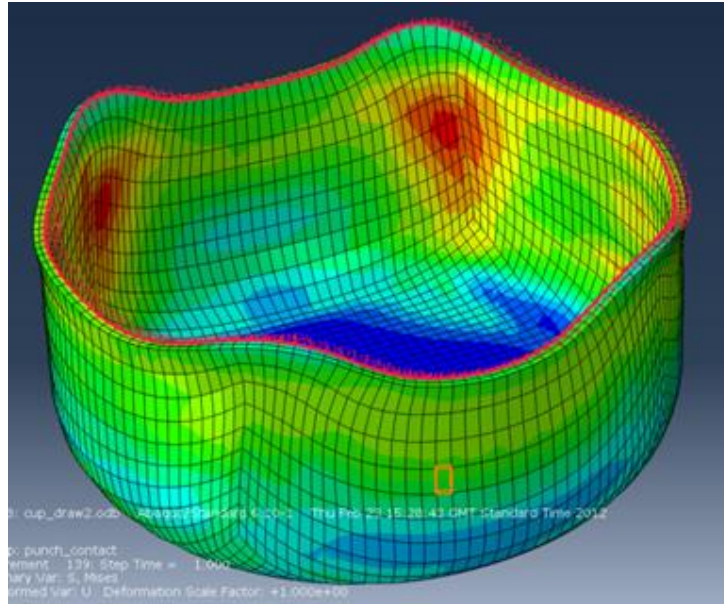
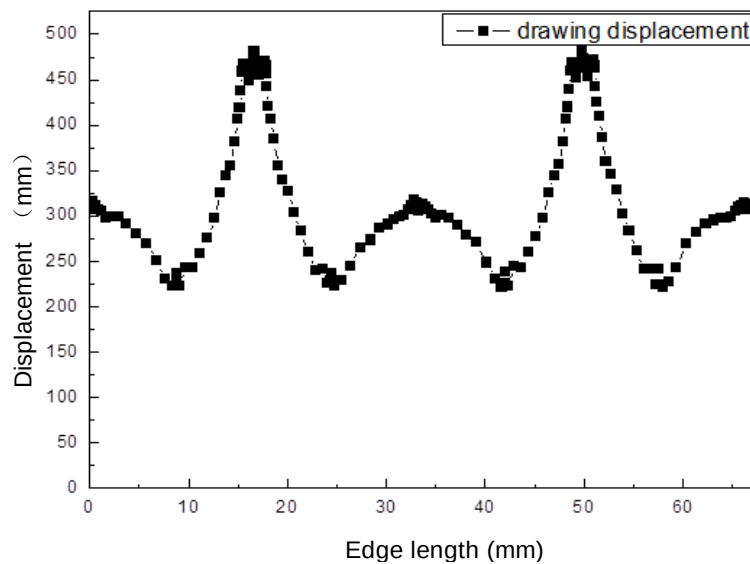


Figure 4.15: Equivalent stress contour after drawing.

The top view of the equivalent effective stress contour is presented by the Figure 4.15. Taking it under consideration, it is quite clear to understand that the stress distribution in the bottom of the drawing cup is not isotropic; The contour in transverse direction is different from the rolling direction. The maximum stress is concentrated on the edge side of the drawing cup. It is also possible to clearly see that, at room temperature, the earing profile exhibits four troughs in both RD and TD and ears at 45° with similar amplitude in their respective directions.



(a) Edge path



(b) Displacement vs edge length

Figure 4.16: The plot of the punch displacement in the drawing direction vs. the prescribed edge path.

The prescribed edge path is demonstrated by Figure 4.16, together with the punch displacement along the deep drawing direction. From this figure, it is possible to further examine the ear distribution. The troughs locate considerably close to 45° against the rolling direction, while the earrings occur close to the RD and TD direction.

4.8 Conclusions

In the present paper, the Cazacu'04 constitutive model was described and used to model the mechanical behavior of pure titanium.

Several fully implicit integration strategies, based on the Closest Point Projection Method (CPPM), were implemented within an implicit environment. One of these is the standard Newton-Raphson Closest Point Projection Method (CPPM), being the other three CPPM with line search algorithm by using different interpolation methods. As expected, the standard Newton-Raphson (N-R) proved to be convergent, when the level of accumulated effective plastic strain was low and the yield surface had a elliptical shape, regardless of the initial stress state (uniaxial stress state, biaxial stress state, shear stress state, *etc.*). In this case, the maximum number of iterations was around 9-10. When the yield surface presents an elliptical-like shape, and the shape does not present a substantial change during its following evolution, N-R with line search algorithm presented similar results to the standard N-R algorithm; in this case, the N-R algorithm, is able to obtain the quadratic convergence

In turn, as far as accumulative effective plastic strain as $\bar{\epsilon}^p = 0.6$ is concerned, the yield surface changed more significantly and into a triangular-like shape. The standard N-R method is unable to obtain convergence. The N-R with line search requires a small iterative number to obtain the convergence, however, all the interpolated methods give the same results.

Respecting the effective plastic strain of $\bar{\epsilon}^p = 0.4$, the cubic interpolation methods presented a much smaller maximum iterative number when compared to both the aforementioned interpolation methods.

It is possible, then, to sustain that all coupled N-R with line search scheme, particularly, N-R with cubic interpolated line search scheme were able to reduce the number of iterations significantly.

Subsequently, iso-error maps have been drawn for two types of yield surface at six representative stress states. One of those was an elliptical-like shape, the other a

triangular-like one. It was verified that the stability at tensile and compressive stress states had a pronounced difference, and also that large increments could not be employed, due to the loss of convergence promoted by the model.

Finally, a benchmark test-cup drawing was performed to validate the model. This made it possible to conclude that the simulated results demonstrated how the model captured the earing phenomenon after drawing the pure titanium sheet, in agreement with the experimental results.

4.9 References

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Chapter 5

An Elasto-Viscoplastic Model for Hexagonal Close Packed Materials: Formulation and Numerical Implementation

5.1 Introduction

Titanium alloys are extensively used in defense applications, as well as aerospace and automotive industries. In practical engineering problems, materials show the rate-dependent behavior even at the room temperature. It is therefore necessary to derive the constitutive model based on a rate dependent elasto-viscoplasticity model. In the recent decades, novel viscoplastic formulations were proposed, which resemble the mathematical framework of classical elastoplastic flow theory [1-3]. In view of these advances, rate independent plasticity extends to rate dependent material behavior. Well-established elastoplastic strategies may be used to integrate the viscoplastic rate equations based on the algorithmic tangent operator of elastoplasticity [4]. In this Chapter, an elasto-viscoplastic model for Hexagonal Close Packed (HCP) materials is formulated. The Perzyna overstress viscoplastic model is coupled with Cazacu 04 yield criterion here and the resulting constitutive model is implemented within an implicit quasi-static finite element environment. In the numerical stress integration, a primal Closest Point Projection Method (CPPM) integration algorithm and a closed form consistent tangent operator are derived. Iso-error maps are used to illustrate the accuracy and stability of the algorithm at different strain rates.

5.2 Formulation of elasto-viscoplastic model

In the infinitesimal strain domain, similar to the classical plastic model, the total strain rate, $\dot{\boldsymbol{\epsilon}}$, in an elasto-viscoplastic material point can be additively decomposed into an elastic component, $\dot{\boldsymbol{\epsilon}}^e$, and a viscoplastic component, $\dot{\boldsymbol{\epsilon}}^{vp}$:

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^{vp}, \quad (5.1)$$

where the superimposed dot denotes the time derivative. The stress rate, $\dot{\boldsymbol{\sigma}}$, is related to the strain rate via the constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^e : \dot{\boldsymbol{\epsilon}}^e, \quad (5.2)$$

with $\mathbf{D}^e$ the fourth-order tensor containing the tangential elastic stiffness moduli. Furthermore, the viscoplastic strain rate evolves via a flow rule,

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \dot{\gamma} \mathbf{N}. \quad (5.3)$$

The scalar, $\dot{\gamma}$, is a non-negative parameter, known as the consistency parameter [5]. The vector $\mathbf{N}$ is the viscoplastic flow vector.

Over the last decades, numerous viscoplastic models [6-8] have been proposed. Perzyna's overstress theory provides a unified approach to analyze a wide range of engineering problems being widely used to capture rate-dependent effects in solid materials, such as Luders bands and the Portevin-Le Chatelier effect in metals, shear banding and creep in geomaterials, as well as to analyze localization and bifurcation properties. The main feature of this model is that the inviscid yield function used for describing the viscoplastic strain can become larger than zero, an effect that is known as overstress. Viscoplasticity can be modeled by direct incorporation of the time-dependency in a yield function which, together with the consistency parameter, obeys the classical Kuhn-Tucker relations.

In Perzyna's model, the extension of traditional elastoplasticity assumes that the ratio of the over-stress function and the viscosity defines the viscoplastic multiplier, the consistency parameter being expressed by

$$\dot{\gamma} = \frac{\langle \hat{\Phi}(\sigma_{eq}) \rangle}{\hat{\mu}}, \quad (5.4)$$

where $\hat{\mu}$ is the viscosity parameter, $\hat{\Phi}$ represents the overstress function that depends on the rate-independent yield surface σ_{eq} and “ $\langle \cdot \rangle$ ” are the McCauley brackets, such that:

$$\langle \hat{\Phi}(\sigma_{eq}) \rangle = \begin{cases} \hat{\Phi}(\sigma_{eq}), & \text{if } \hat{\Phi}(\sigma_{eq}) \geq 0 \\ 0, & \text{if } \hat{\Phi}(\sigma_{eq}) < 0 \end{cases} \quad (5.5)$$

The overstress function $\hat{\Phi}$ must fulfill the following two conditions:

- (i) $\hat{\Phi}(\sigma_{eq})$ is continuous in $[0, \infty)$,
- (ii) $\hat{\Phi}(\sigma_{eq})$ is convex in $[0, \infty)$.

So that a rate-independent elastoplastic model is recovered if $\hat{\mu} \rightarrow 0$, which represents the value at which the viscosity is negligible. Here, in this chapter, a widely used expression for overstress function $\hat{\Phi}$ is employed:

$$\hat{\Phi}(\sigma_{eq}) = \left(\frac{\sigma_{eq}}{\sigma_Y} \right)^{1/\hat{\varepsilon}} - 1, \quad (5.6)$$

where $\hat{\epsilon}$ is a calibration material parameter, used to normalize the overstress function, chosen as the initial yield stress. It may also be used as the hardening stress or the combination of both the hardening and the yield stresses.

5.2.1 Cazacu 04 yield criterion

The yield function proposed by Cazacu *et al.* [9] was obtained by extending the orthotropic Drucker's isotropic yield criterion with constructed generalizations of the invariants of the deviatoric stress. This model has been formulated in detail in Chapter 4 and will be summarized here for completeness.

Cazacu 04 anisotropic yield function can be expressed by

$$\sigma_{eq} = \hat{A}(J_2^{0\frac{3}{2}} - \hat{c}J_3^0)^{1/3}, \quad (5.7)$$

where, J_2^0 , is the second invariant of the deviatoric transformed stress, and J_3^0 , is the third invariant of deviatoric transformed stress, respectively, which can be obtained as

$$J_2^0 = \frac{1}{2}tr(\boldsymbol{\Sigma}^2), \quad (5.8)$$

$$J_3^0 = \frac{1}{3}tr(\boldsymbol{\Sigma}^3). \quad (5.9)$$

where, transformed tensor $\{\boldsymbol{\Sigma}\}$ is defined as

$$\Sigma_{ij} = \hat{L}_{ijkl}\sigma_{kl}. \quad (5.10)$$

The fourth-order linear transformation operator ($\hat{\mathbf{L}}$) can be represented in 6×6 matrix format as

$$\hat{\mathbf{L}} = \begin{bmatrix} \frac{a_2 + a_3}{3} & -\frac{a_3}{3} & -\frac{a_2}{3} & 0 & 0 & 0 \\ -\frac{a_3}{3} & \frac{a_1 + a_3}{3} & -\frac{a_1}{3} & 0 & 0 & 0 \\ -\frac{a_2}{3} & -\frac{a_1}{3} & \frac{a_1 + a_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{bmatrix}. \quad (5.11)$$

The constant $\hat{A}$ is defined as

$$\hat{A} = 3[(a_2^2 + a_3^2 + a_2a_3)^{\frac{3}{2}} - \hat{c}(a_2 + a_3)a_2a_3]^{-\frac{1}{3}}, \quad (5.12)$$

where $\hat{c}$ is a material parameter.

5.3 Stress integration of elasto-viscoplastic model

In this section, the evolution of the different quantities from time t_n to time $t_{n+1} = t_n + \Delta t$ at each integration point will be treated with CPPM stress integration algorithm. Within a typical time interval $[t_n, t_{n+1}]$, it is assumed that all variables $\{\boldsymbol{\sigma}_n, \boldsymbol{\varepsilon}_n^e, \boldsymbol{\varepsilon}_n^{vp}, \bar{\varepsilon}_n^{vp}\}$ at time t_n are known, and the stress together with the updated variables, which characterize the inelastic response of the material, are pursued for given strain increment $\Delta \boldsymbol{\varepsilon}$. The numerical integration of elasto-viscoplastic constitutive equations is typically carried out by means of the so-called elastic predictor return mapping schemes. In order to integrate the governing equations in a similar means to generalized plasticity, the overstress Perzyna model is chosen. The strain rate is assumed to be computed by

$$\dot{\bar{\varepsilon}}^{vp} = \Delta \bar{\varepsilon}^{vp} / \Delta t. \quad (5.13)$$

A fully implicit elastic predictor return mapping method is implemented within an implicit quasi-static finite element environment, having the format given below.

(i) Elastic predictor

The material is assumed to behave purely elastically at this current time interval, the state variables at t_n , the total strain increment $\Delta \boldsymbol{\varepsilon}$ is given. Thus, the elastic trial state can be evaluated as

$$\boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}} = \boldsymbol{\varepsilon}_n^e + \Delta \boldsymbol{\varepsilon}; \quad (5.14)$$

$$\bar{\varepsilon}_{n+1}^{vp \text{ trial}} = \bar{\varepsilon}_n^{vp}; \quad (5.15)$$

$$\boldsymbol{p}_{n+1}^{trial} = K \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}; \quad (5.16)$$

$$\boldsymbol{s}_{n+1}^{trial} = 2G \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}; \quad (5.17)$$

$$\boldsymbol{\sigma}_{n+1}^{trial} = \boldsymbol{s}_{n+1}^{trial} + \boldsymbol{p}_{n+1}^{trial} \boldsymbol{I}; \quad (5.18)$$

$$\boldsymbol{\Sigma}_{n+1}^{trial} = \hat{\mathbf{L}} \boldsymbol{\sigma}_{n+1}^{trial}; \quad (5.19)$$

$$J_{2 \text{ } n+1}^{0 \text{ trial}} = \frac{1}{2} \text{tr}(\boldsymbol{\Sigma}_{n+1}^{trial^2}); \quad (5.20)$$

$$J_{3 \text{ } n+1}^{0 \text{ trial}} = \frac{1}{3} \text{tr}(\boldsymbol{\Sigma}_{n+1}^{trial^3}); \quad (5.21)$$

$$\sigma_{eq \text{ } n+1}^{trial} = \hat{A}[(J_{2 \text{ } n+1}^{0 \text{ trial}})^{\frac{3}{2}} - \hat{c} J_{3 \text{ } n+1}^{0 \text{ trial}}]^{1/3}. \quad (5.22)$$

(ii) Check plastic admissibility

If the yield function $\Phi = \hat{A}[(J_{2 \text{ } n+1}^{0 \text{ trial}})^{\frac{3}{2}} - \hat{c} J_{3 \text{ } n+1}^{0 \text{ trial}}]^{1/3} - \sigma_Y(\bar{\varepsilon}_{n+1}^{vp \text{ trial}}) \leq 0$, then the

process is indeed within elastic at the current interval. All the variables to be updated at time t_{n+1} are assigned the corresponding values at the previous trial stage. Otherwise, it goes to a viscoplastic return-mapping stage to obtain all the required variables at time t_{n+1} .

(iii) Return mapping

As previously mentioned, the consistency parameter in Perzyna's model is expressed by

$$\dot{\gamma} = \frac{\langle \hat{\Phi}(\sigma_{eq}) \rangle}{\hat{\mu}} = \frac{1}{\hat{\mu}} \left(\left(\frac{\sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp})}{\sigma_Y(\bar{\varepsilon}_{n+1}^{vp})} \right)^{1/\hat{\varepsilon}} - 1 \right). \quad (5.23)$$

In order to integrate into the similar nonlinear system of equations to elasto-plastic model, the above equation can be rewritten as

$$\sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) - \sigma_{Y,n+1} \left(1 + \frac{\hat{\mu} \Delta \gamma_{n+1}}{\Delta t} \right)^{\hat{\varepsilon}} = 0, \quad (5.24)$$

where $\Delta \gamma$ represents the incremental multiplier, which is equal to $\dot{\gamma} \Delta t$.

Combined with general viscoplastic nonlinear system of equations for viscoplastic model, the new equations system could be written as

$$\left\{ \begin{array}{l} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e,trial} + \Delta t \hat{\mathbf{G}}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) \\ \bar{\varepsilon}_{n+1}^{vp} - \bar{\varepsilon}_n^{vp} + \Delta t \hat{f}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) \\ \sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) - \sigma_{Y,n+1} \left(1 + \frac{\hat{\mu} \Delta \gamma_{n+1}}{\Delta t} \right)^{\hat{\varepsilon}} \end{array} \right\} = \left\{ \begin{array}{l} \mathbf{0} \\ 0 \\ 0 \end{array} \right\}, \quad (5.25)$$

where, the constitutive function $\hat{\mathbf{G}}$ and $\hat{f}$ can be calculated by

$$\hat{\mathbf{G}}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) = \dot{\gamma} \mathbf{N}_{n+1}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}), \quad (5.26)$$

$$\hat{f}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) = -\dot{\gamma}, \quad (5.27)$$

$\mathbf{N}$ is the flow vector following the terminology of the rate-independent theory, for the associated constitutive model; the plastic strain rate is assumed to be normal to the yield surface and, thus, the flow vector can be written as

$$\mathbf{N}_{n+1} = \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}}. \quad (5.28)$$

From this moment on it is possible to the same algorithm to solve the above system of eight equations as elasto-plastic model. In order to enhance the convergence of the iterative process, a line search method is adopted. The process is stopped as soon as convergence is achieved within some prescribed convergence norm. The solution is found for $\boldsymbol{\varepsilon}_{n+1}^e$, $\bar{\varepsilon}_{n+1}^{vp}$ and $\Delta \gamma$. The stress tensor can be obtained by

$$\boldsymbol{\sigma}_{n+1} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^e. \quad (5.29)$$

In order to obtain quadratic convergence in the solution of an initial boundary value problem within an implicit finite element environment, it is required to assemble the tangent operator consistent with the general algorithm with the tangent stiffness matrix. The fourth-order tensor can be obtained by computing the derivative of the updated stress tensor $\boldsymbol{\sigma}_{n+1}$ with respect to the final strain $\boldsymbol{\varepsilon}_{n+1}$. When the stress state lies within the elastic domain and no viscoplastic flow occurs, the tangent operator is the elastic tangent operator $\mathbf{D}^e$. Otherwise, the elasto-viscoplastic consistent tangent operator has to be derived by consistently linearizing the viscoplastic return-mapping algorithm under viscoplastic flow. The closed form for the viscoplastic tangent operator can be expressed by

$$\mathbf{D}^{vp} = \left\{ \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial N}{\partial \boldsymbol{\sigma}} - \frac{N \otimes (N + \Delta\gamma \frac{\partial N}{\partial \boldsymbol{\varepsilon}^{vp}})}{\left(\frac{\partial \sigma_{eq}}{\partial \boldsymbol{\varepsilon}^{vp}} - \frac{\partial \sigma_Y}{\partial \boldsymbol{\varepsilon}^{vp}} \left(1 + \frac{\hat{\mu} \Delta\gamma}{\Delta t} \right)^{\hat{\varepsilon}} - \hat{\varepsilon} \sigma_Y \left(1 + \frac{\hat{\mu} \Delta\gamma}{\Delta t} \right)^{\hat{\varepsilon}-1} \left(\frac{\hat{\mu}}{\Delta t} \right) \right)} \right\}^{-1}. \quad (5.30)$$

It is important to remark that this tangent operator is not symmetric.

5.4 Numerical examples

In this section, some numerical examples are presented to verify the implementation of the proposed elasto-viscoplastic constitutive model.

Table 5.1: Material properties for pure titanium [10].

	ρ (g/cm ³)	E , Modulus (GPa)	ν , Poisson's Ratio
Titanium	4.51	120	0.361

Table 5.2: Anisotropy coefficient values of pure titanium corresponding to the yield surface evolution [10].

Strain ($\bar{\varepsilon}^p$)	a_1	a_2	a_3	a_4	a_5	a_6	$\hat{c}$
0.0	0.5454	0.501	1.09	0.7246	-0.8675	-0.8675	-0.2168
0.025	0.5231	0.4745	0.9034	0.7309	0.7202	0.7202	-0.2198
0.05	0.6694	0.5585	1.103	0.9138	0.9381	0.9381	-0.2291
0.075	0.6960	0.5969	1.127	0.9838	0.9716	0.9716	-0.2607
0.1	0.5356	0.4768	0.8603	0.7761	0.7714	0.7714	-0.2754
0.2	0.061	0.0576	0.0869	0.087	0.0794	0.0794	-0.5908
0.4	0.0632	0.062	0.0788	0.0816	0.0801	0.0801	-1.0330
0.5	0.9547	0.957	1.2140	1.181	1.176	1.176	-1.1480

The anisotropy coefficient values of pure titanium corresponding to the Cazacu 04 constitutive model are listed in Table 5.2.

The numerical examples are divided into the second subsections with different objectives. In the first set of numerical examples, the uniaxial tensile and uniaxial-compressive tests are performed respectively to evaluate the asymmetric mechanical behavior at different strain rates; in the second set of numerical examples, several iso-error maps are presented to estimate the accuracy of the CPPM stress integration scheme at different strain rates. The material constants, employed in all examples, are listed in Table 5.1.

5.4.1 Single element tests

In order to verify the implementation of the elasto-viscoplastic constitutive model in the FE code, single element simulations were carried out for both plates under two different uniaxial stress conditions.

(a) Uniaxial tensile test

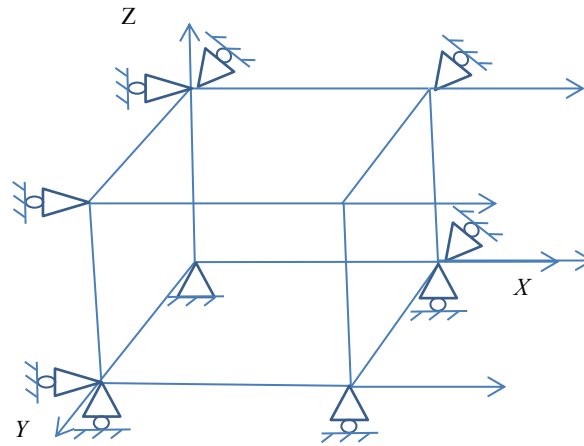
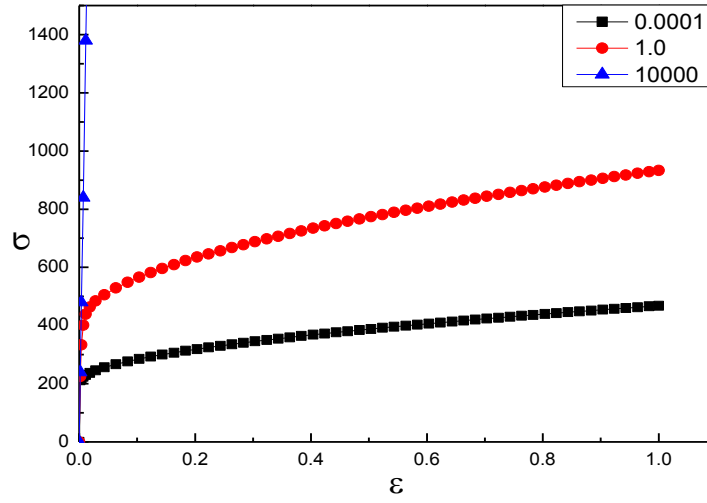


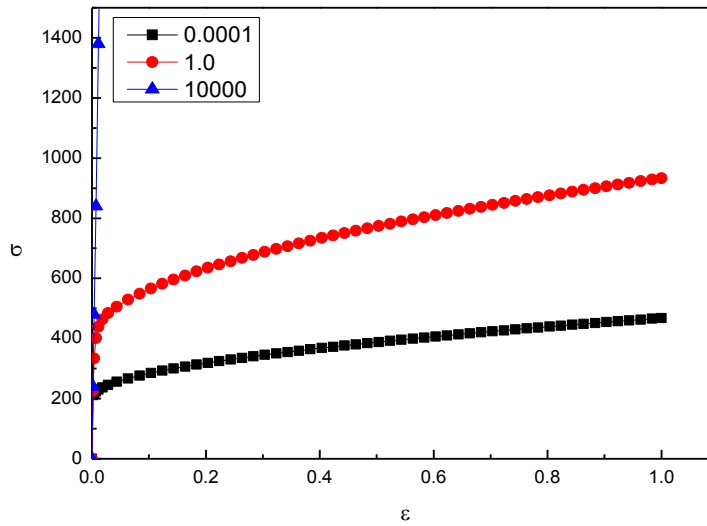
Figure 5.1: Uniaxial tensile tests.

In this example, a uniaxial stress state is applied to a single eight-node brick element (with one integration point). The length of the element is $10 \times 10 \times 10 \text{ mm}^3$. This example is conducted to demonstrate the ability of the finite element formulation to capture strain rate sensitivity under uniaxial tension stress state. Four nodes on one face of the element were restrained, the element is stretched with a prescribed constant (in time) velocity, $\hat{v}$, along the rolling direction (x direction). The geometry and the boundary

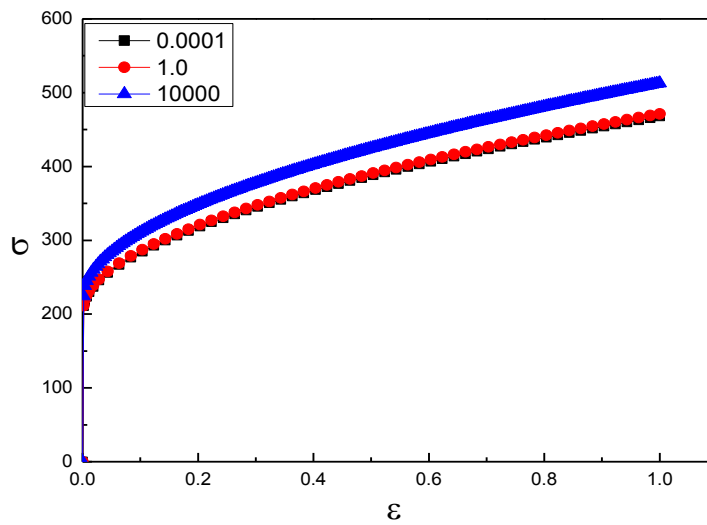
conditions applied to the single element for the uniaxial tensile test in x direction are presented in Figure 5.1.



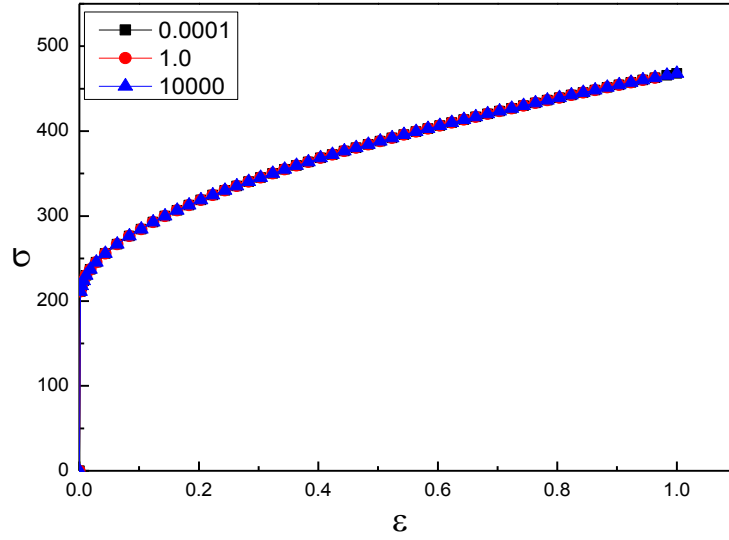
(a) $\hat{\varepsilon} = 10^1$



(b) $\hat{\varepsilon} = 10^0$



(c) $\hat{\varepsilon} = 10^{-2}$

(d) $\hat{\epsilon} = 10^{-6}$ Figure 5.2: Uniaxial tensile stress-strain curves ($\hat{\epsilon}$ -power law exponent; see in Equation (5.23)).

The normalized stretching rate can be conveniently defined for the unit cell by $\hat{\nu}^* = \hat{\mu} \hat{\nu}$ and the simulation is carried out for three different values of $\hat{\nu}^*$

$$\hat{\nu}^* = 10^{-4}, 10^0, 10^4. \quad (5.31)$$

This choice covers from very slow to very fast strain rates and it is meant to demonstrate the robustness of the integration algorithm over a wide range of strain rates. In order to show the effect of the rate-sensitivity parameter on the behavior of the model, in the uniaxial-tensile stress state, four values of $\hat{\epsilon}$ are considered

$$\hat{\epsilon} = 10^1, 10^0, 10^{-2}, 10^{-6}. \quad (5.32)$$

A series of stress-strain curves for each power-law exponent $\hat{\epsilon} = 10^1, 10^0, 10^{-2}$ and 10^{-6} are drawn from the numerical uniaxial tensile test, which are presented, respectively, in Figure 5.2 (a), (b), (c), (d). It can be seen that when the power-law exponent $\hat{\epsilon}$ is larger than 1.0, the material shows a significant strain rate sensitivity. The mechanical response increases remarkably with the increase of strain rate, when $\hat{\epsilon} = 10^1$ and $\hat{\epsilon} = 10^0$. In addition, when $\hat{\nu}^* = 10^4$, the material does not yield during the deformation and demonstrates a very high stiffness. When the power-law exponent $\hat{\epsilon} = 10^{-2}$, although the material still presents viscoplastic mechanical behavior, the influence of the strain rate on the response decreases. Particularly, when $\hat{\nu}^* = 10^{-4}$ and $\hat{\nu}^* = 10^0$, the mechanical response almost coincides. This behavior becomes more obvious when the power-law exponent $\hat{\epsilon} = 10^{-6}$. In this case, all the stress-strain curves coincide and the viscoplastic model is effectively reduced to the Cazacu 04 elasto-plastic model.

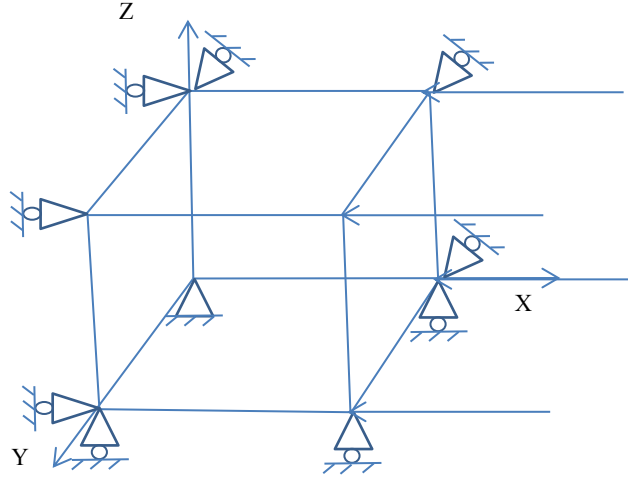
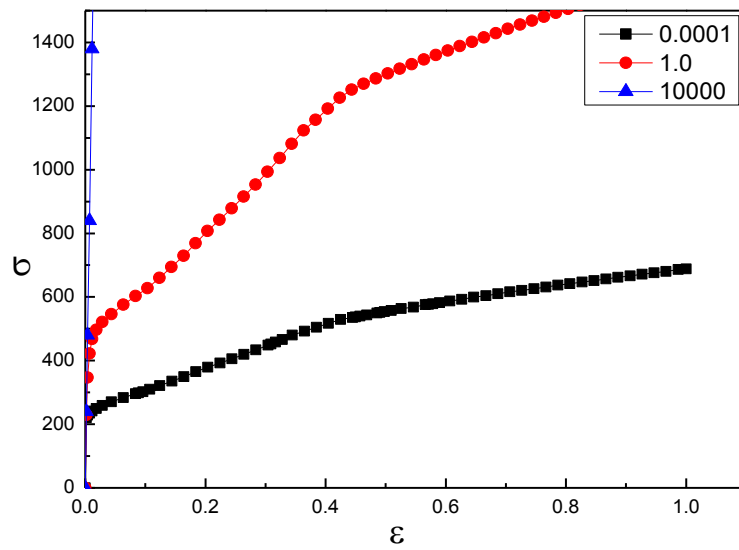
(b) Uniaxial compressive test

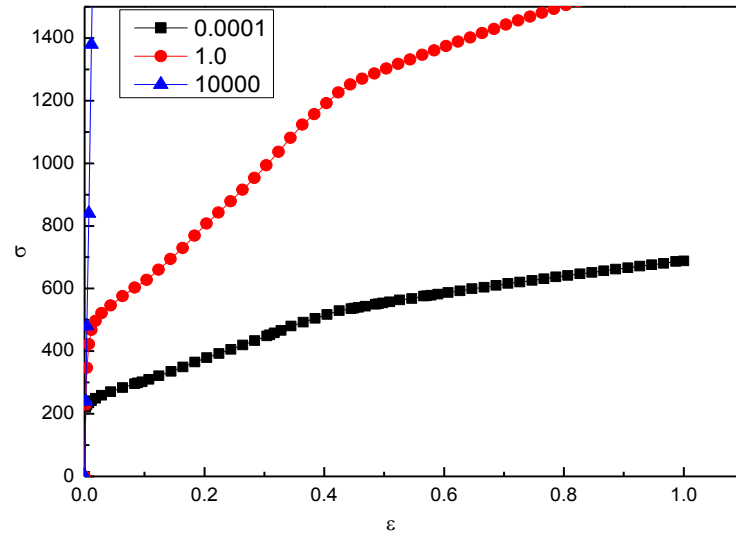
Figure 5.3: Uniaxial compressive test.

Owing to the different mechanical behavior of titanium alloy under compressive stress states and the tensile stress states, a uniaxial compressive benchmark test is used to study the stress integration algorithm of the viscoplastic Cazacu 04 model at different strain rates. Similar to the tensile test, a single eight-node brick element (see Figure 5.3) is loaded with a prescribed displacement. The simulation is also carried out for three different values of $\hat{v}^*$

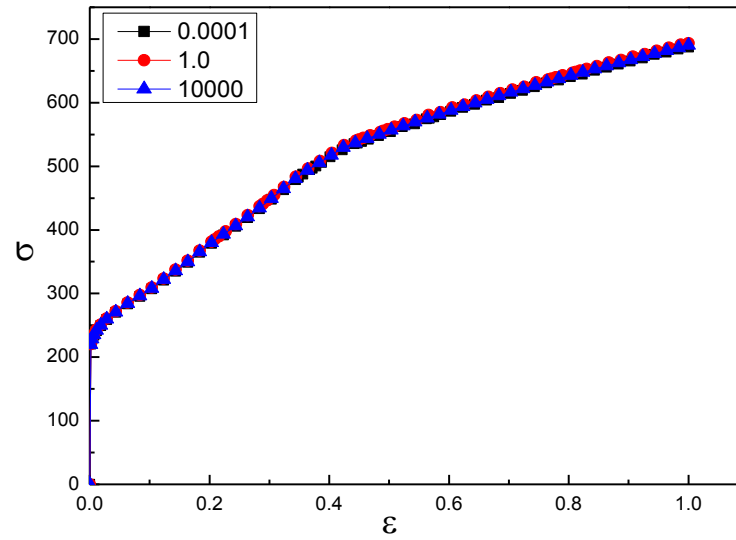
$$\hat{v}^* = 10^{-4}, 10^0, 10^4, \quad (5.33)$$

and four values of $\hat{\epsilon} = 10^1, 10^0, 10^{-2}, 10^{-6}$ are also studied [7].

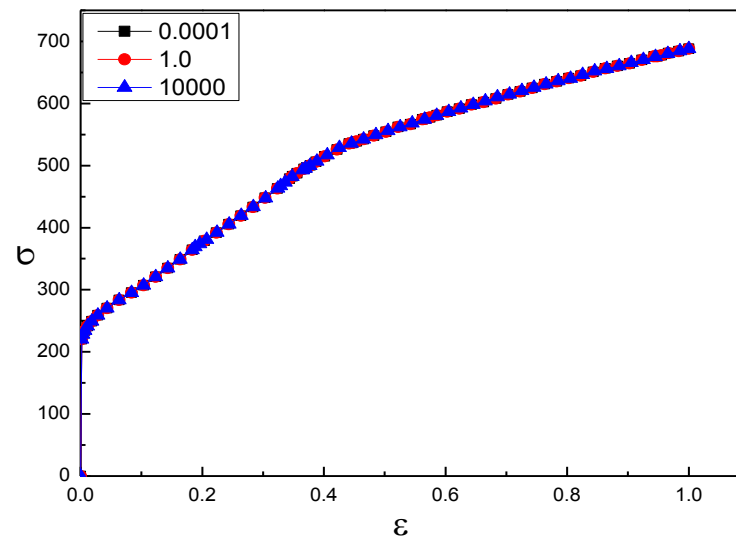
(a) $\hat{\epsilon} = 10^1$



(b) $\hat{\epsilon} = 10^0$



(c) $\hat{\epsilon} = 10^{-2}$



(d) $\hat{\epsilon} = 10^{-6}$

Figure 5.4: Uniaxial compressive stress-strain curves.

Stress-strain curves for the power-law exponent $\dot{\epsilon} = 10^1, 10^0, 10^{-2}$ and 10^{-6} are also drawn from this numerical uniaxial compressive test, which are shown, respectively, in Figure 5.4 (a), (b), (c), (d). From these, we can observe that the mechanical response under compressive stress states also presents strong strain rate sensitivity for high power-law exponent. By comparison with the stress-strain curve under uniaxial tensile stress states, it can be concluded that the strain rate sensitivity in compression is higher than the tensile stress state, which can be concluded from the Figure 5.4 (c). In the compression test, the stress-strain curves coincide for all the strain rates when the power-law exponent $\dot{\epsilon} = 10^{-2}$, however, in the scenario of tensile loading the mechanical response at high strain rate of 10^4 is higher for low ($\dot{\nu}^* = 10^{-4}$) and medium ($\dot{\nu}^* = 10^0$) strain rates.

5.4.2 3D iso-error maps of elasto-viscoplastic model

In order to assess the accuracy of the stress integration algorithm for elasto-viscoplastic model, 3D iso-error maps are drawn by standard numerical testing procedure under realistic finite time/strain steps [11]. Using the three-dimensional implementation of the model, we start from a stress point at time t_n , $\boldsymbol{\sigma}_n$, lying on the yield surface, and a sequence of strain increments is applied corresponding to specified normalized elastic trial stress increments of the form

$$\Delta\boldsymbol{\sigma}^{trial} = \frac{\Delta\sigma_T}{\sigma_{eq}}\hat{\mathbf{T}} + \frac{\Delta\sigma_N}{\sigma_{eq}}\hat{\mathbf{N}}, \quad (5.34)$$

where $\hat{\mathbf{N}}$ and $\hat{\mathbf{T}}$ are, respectively, the unit (in Euclidean norm) normal and tangent vectors to the yield surface and σ_{eq} is the equivalent stress.

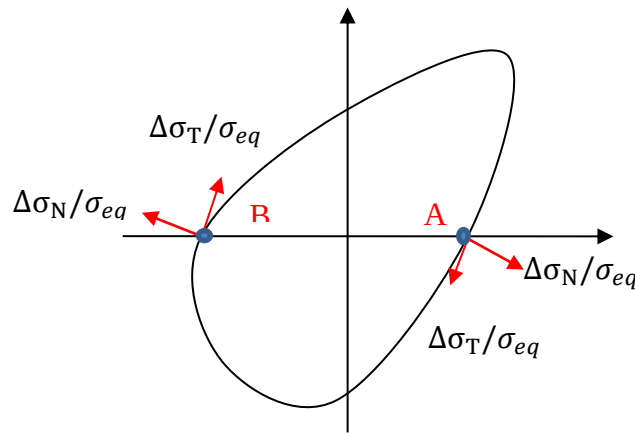


Figure 5.5: Schematic diagram of stress states on the yield surface.

Applying increments of trial stress in the tangential and normal direction to the yield surface, the error is evaluated by

$$ERROR(\%) = \frac{\sqrt{(\sigma_{Num} - \sigma_{EXACT}) : (\sigma_{Num} - \sigma_{EXACT})}}{\sqrt{\sigma_{EXACT} : \sigma_{EXACT}}}, \quad (5.35)$$

where, σ_{Num} is the numerical solution, and σ_{EXACT} is the exact solution. Here, due to the lack of an analytical solution, σ_{EXACT} is assumed to be the stress obtained by sub-incrementation of each stress increment into 1000 steps.

In order to evaluate the accuracy properties and remarkable robustness at different strain rates, several iso-error maps are drawn at low and high strain rates with the non-dimensional factor:

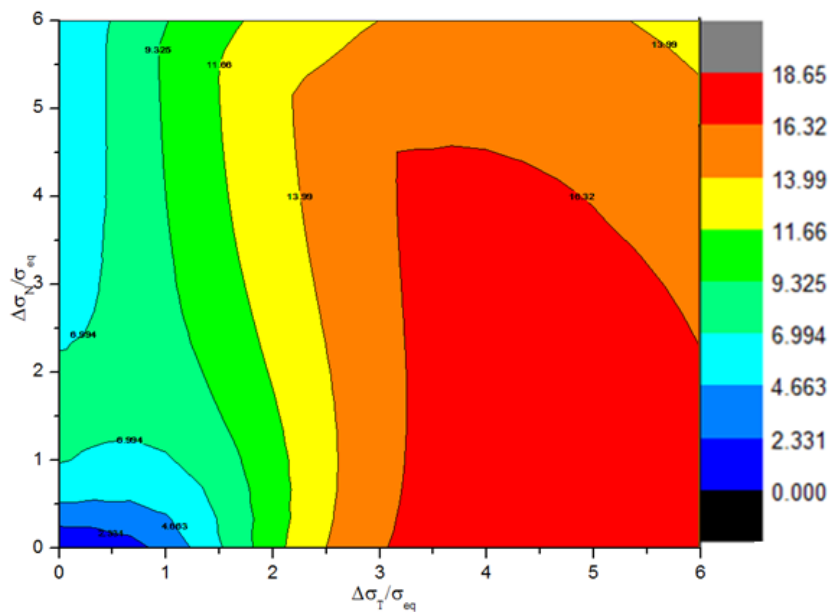
$$\hat{\mu} \|\dot{\epsilon}\|$$

Set, respectively, to 1 and 1000. In order to investigate the influence of power-law exponent, three values chosen: $10^0, 10^{-1}, 0$.

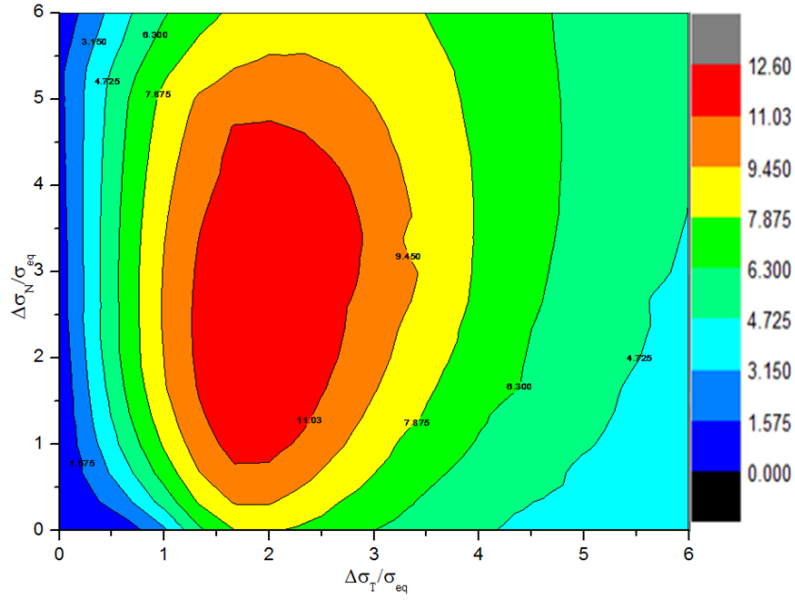
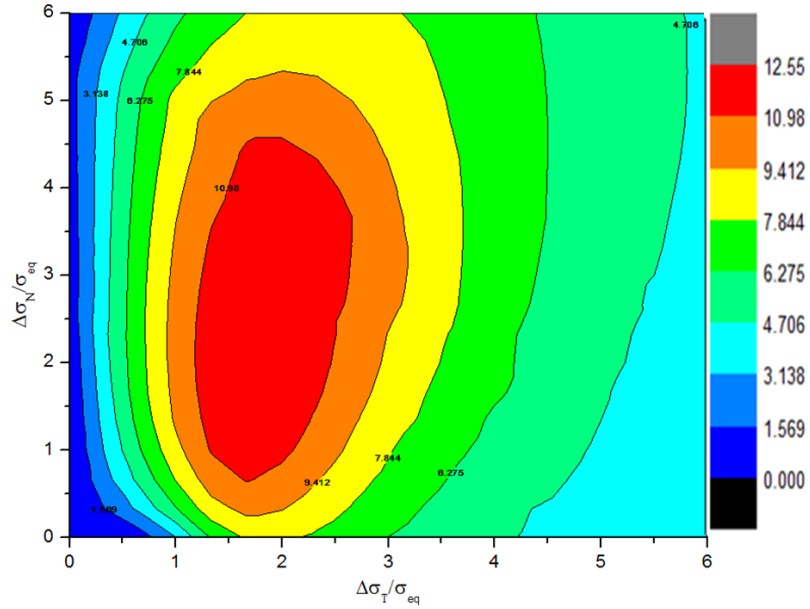
In order to analyze the iso-error map under different stress states, we select two typical stress states to study due to the asymmetrical strength effect of pure titanium, which can be seen in Figure 5.5. One is the point A under uniaxial tensile stress state; the other is point B under uniaxial compressive stress state.

(a) Iso-error maps under uniaxial tensile stress state

Figure 5.6 and Figure 5.7 show the iso-error maps under uniaxial tensile stress state obtained at the aforementioned low and high strain rates for different power-law exponents, $\hat{\epsilon}$.

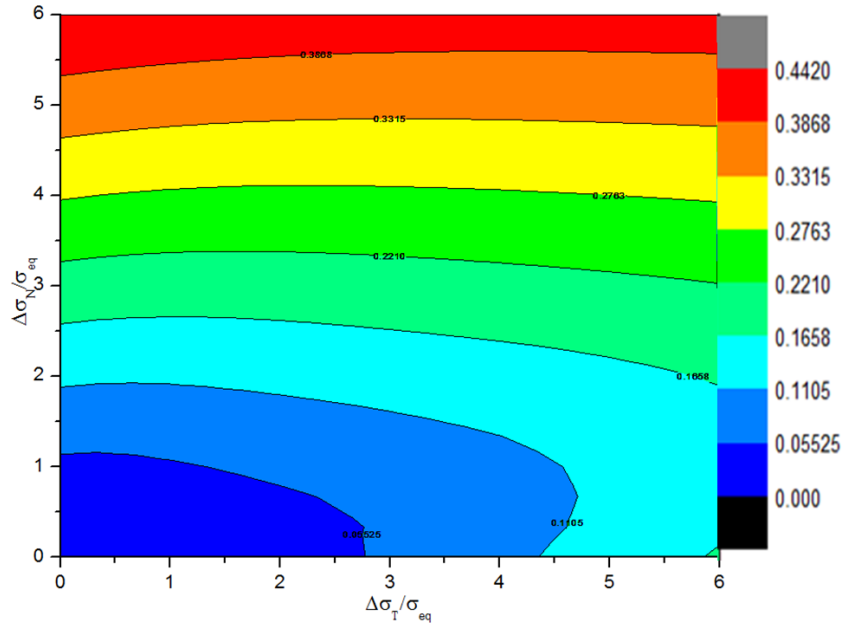
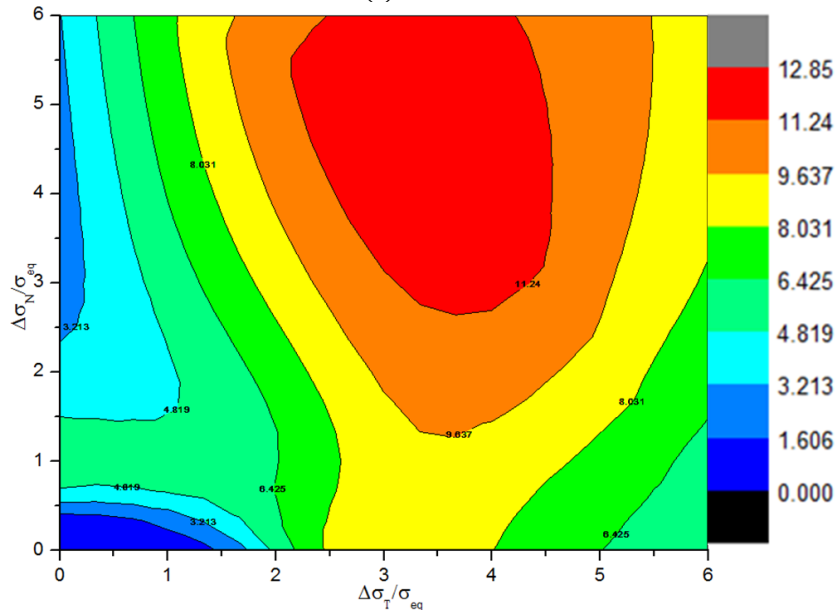


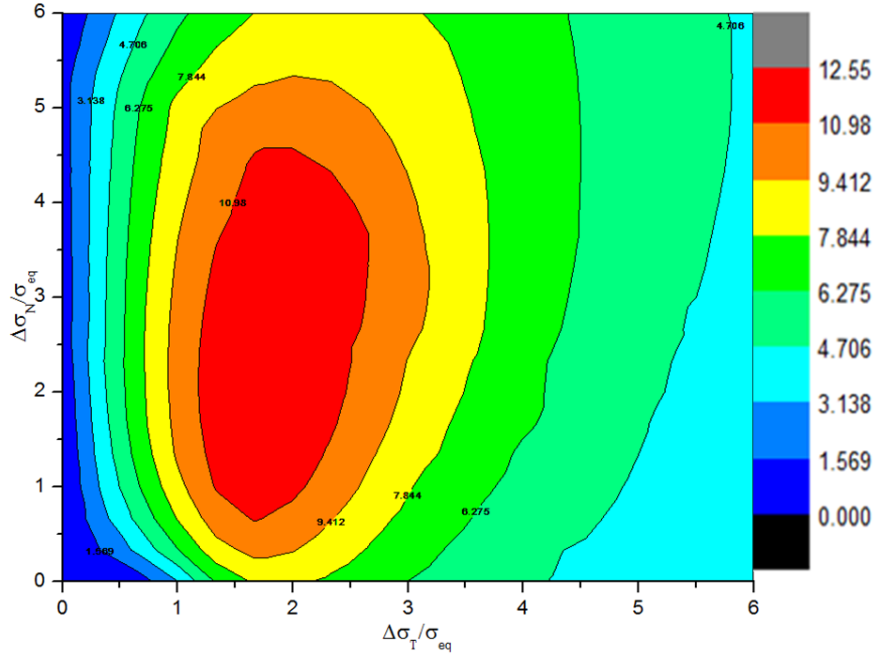
(a) $\hat{\epsilon} = 10^0$

(b) $\hat{\epsilon} = 10^{-1}$ (c) $\hat{\epsilon} = 0$ (rate-independent)Figure 5.6: Iso-error maps under uniaxial tensile stress state with $\hat{\mu}||\dot{\epsilon}|| = 1.0$.

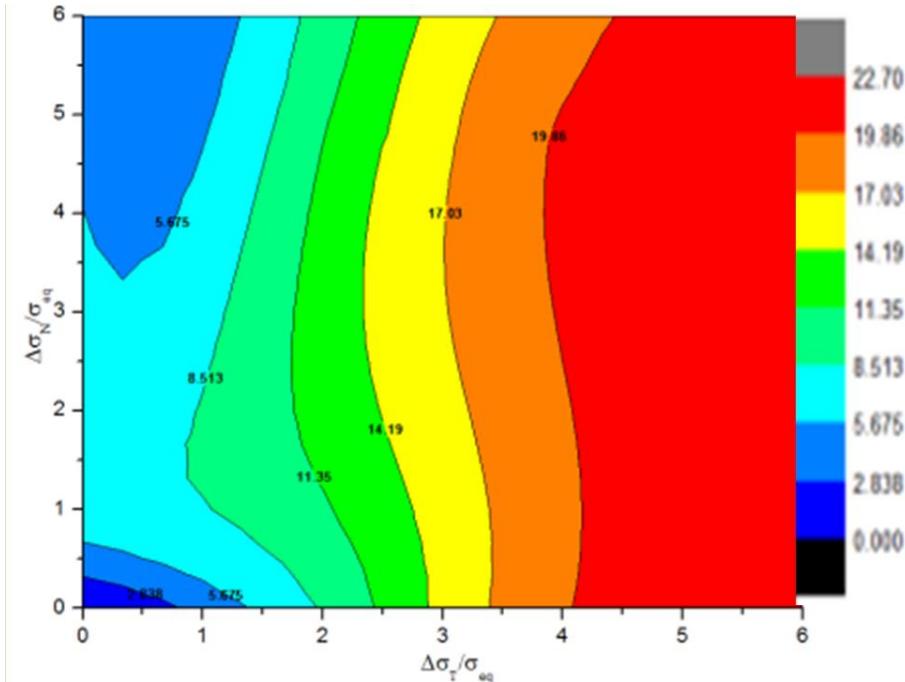
For the power-law exponent $\hat{\epsilon} = 0$, the standard rate-independent Cazacu 04 elastoplastic model is recovered. The resulting iso-error map is, in this case, identical to the rate-independent map. For the small power-law exponent, $\hat{\epsilon} = 0.1$, the iso-error map is almost the same as the rate-independent one at the low strain rate (see in Figure 5.6 (b)). However, the iso-error map changes significantly at the high strain rate (see in Figure 5.7 (b)). This also occurs for the power-law exponent $\hat{\epsilon} = 10^0$. It can also be seen that the maximum value of iso-error maps for high power-law exponent is higher than in the

rate-independent map; nevertheless, it decreases sharply at high strain rate. The larger the power-law exponent $\hat{\epsilon}$, the more strain rate sensitivity of the iso-error value. From the maximum iso-error value, it is possible to conclude that the current stress integration algorithm is remarkable robust and can effectively handle any power law exponent $\hat{\epsilon}$ from small (corresponding to high rate-sensitivity) to extremely large values (corresponding to effectively rate-independent conditions) under low as well as high strain rates.

(a) $\hat{\epsilon} = 10^0$ (b) $\hat{\epsilon} = 10^{-1}$

(c) $\hat{\epsilon} = 0$ (rate-independent)Figure 5.7: Iso-error maps under uniaxial-tensile test with $\hat{\mu}||\dot{\epsilon}|| = 1000$.**(b) Iso-erro maps under uniaxial compressive stress state**

The iso-error maps under a uniaxial compressive stress state obtained at low and high strain rates for different power-law exponents ($\hat{\epsilon}$), are presented in Figure 5.8 and Figure 5.9. The same stress integration algorithms and tolerances are applied to draw iso-error maps at uniaxial-compressive stress state.

(a) $\hat{\epsilon} = 10^0$

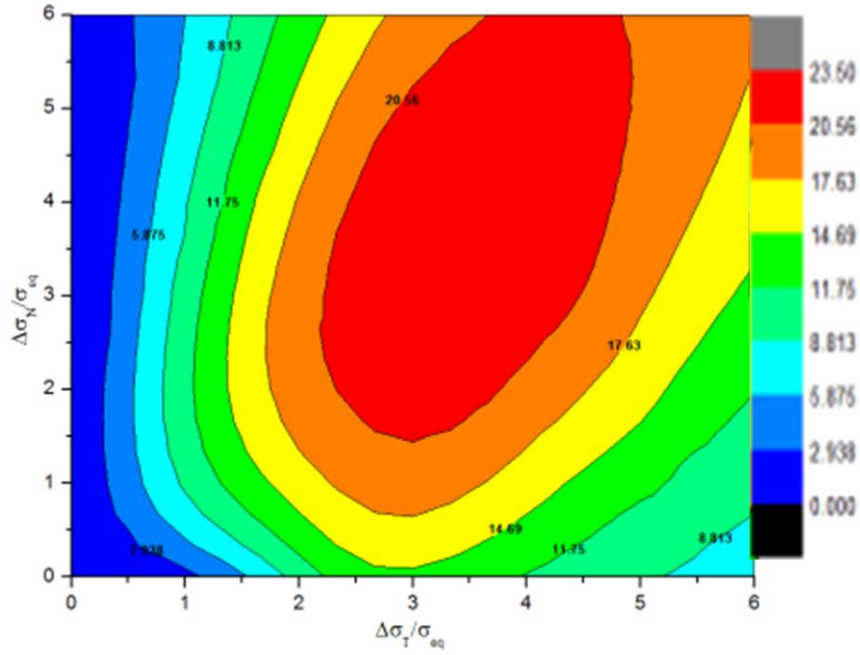
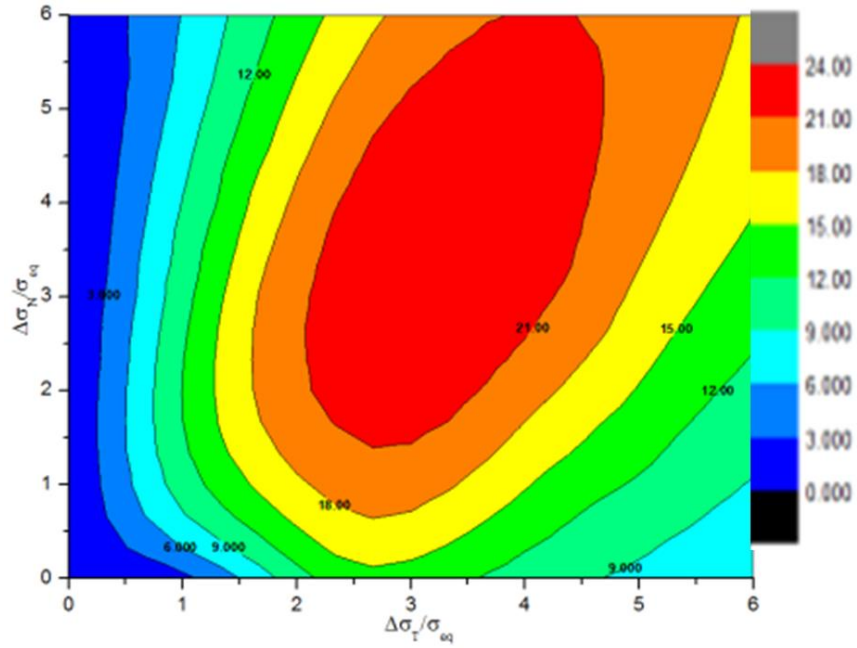
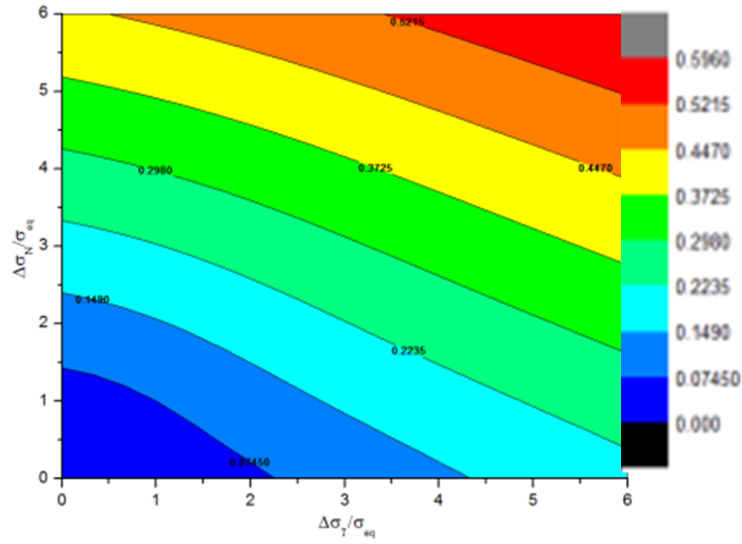
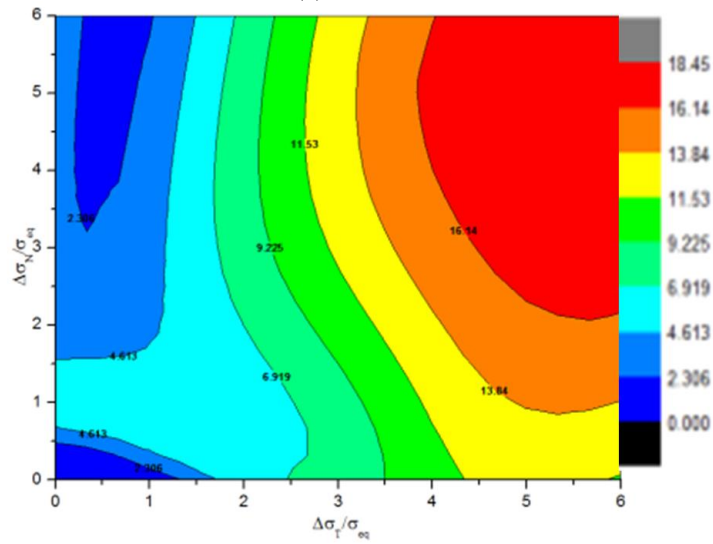

 (b) $\hat{\epsilon} = 10^{-1}$

 (c) $\hat{\epsilon} = 0$ (rate-independent)

 Figure 5.8: Iso-error maps under uniaxial compressive stress state with $\hat{\mu}\|\dot{\epsilon}\| = 1$.

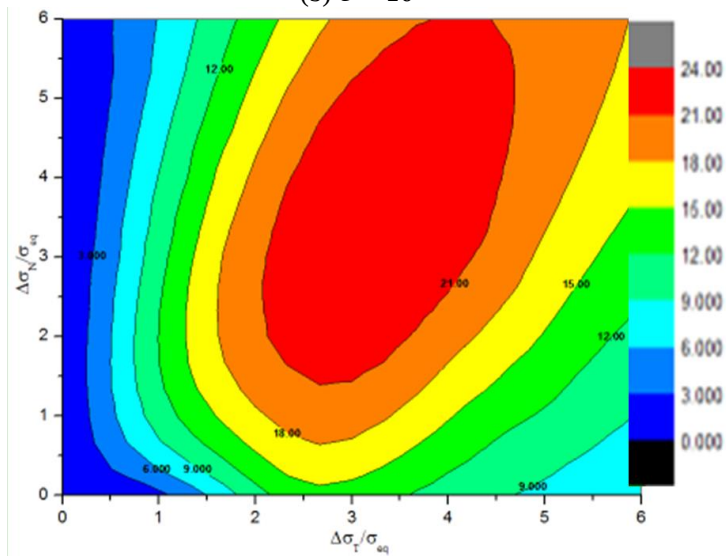
From Figure 5.8 and Figure 5.9, the same behavior for the strain rate sensitivity can be observed in the uniaxial compressive stress state. From the comparison between the iso-error maps under uniaxial compressive stress state and under uniaxial tensile stress state, it is clear that the iso-error maps are completely different, which illustrates the importance of the strength differential (SD) effect.



(a) $\hat{\epsilon} = 10^0$



(b) $\hat{\epsilon} = 10^{-1}$



(c) $\hat{\epsilon} = 0$ (rate-independent)

Figure 5.9: Iso-error maps under uniaxial compressive stress state with $\hat{\mu}\|\dot{\epsilon}\| = 1000$.

5.5 Conclusions

In this work, an elastic-viscoplastic constitutive model coupled with Cazacu 04 yield function is proposed to simulate the mechanical behavior of metals with an HCP crystal structure at different deformation strain rates. The CPPM algorithm is implemented within an implicit quasi-static finite element environment. Two numerical examples show that the integration algorithm can be effectively used at high strain rate even for very high power-exponent. It is also shown that the stress evolution under compressive stress states is higher than under tensile stress states and presents a higher strain rate sensitivity.

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Chapter 6

Material Parameters Identification of Phenomenological Constitutive Equations by Deterministic and Evolutionary Approach

6.1 Introduction

In the sheet metal forming, numerical simulation plays a more and more important role to design and optimize the processing and tool geometry. Numerical simulation has already been proved as efficient and useful to decrease the cost of experimental investigations. These numerical simulations utilize and rely on work material constitutive models to simulate deformation conditions that take place in the real sheet metal forming process. Therefore, an accurate and reliable constitutive model is required to describe the plasticity behavior of metals, particularly for materials with Hexagonal Close Packed (HCP) crystal structure, since their HCP crystallographic structure promotes a very different mechanical behavior when compared to other metals with Face-Centered Cubic (FCC) and Body-Centered Cubic (BCC) structures, such as steel, aluminum, *etc.* [1, 2]. Classical yield criterion and less material parameters included are not sufficient. The flexible yield loci are generally necessary to introduce a large number of material parameters. Higher complexity of material constitutive relations would require more elaborate and robust identification techniques. The more material parameters are found in the constitutive model, the more difficult is such identification. It not only requires more typical mechanical experiment data to calibrate, but also is easy to trap into a local minima value during the optimized process.

There exist many strategies to find a solution to the optimization problem, namely constructing a single Aggregate Objective Function (AOF), the Normal Boundary Intersection (NBI), Successive Pareto Optimization (SPO), and Directed Search Domain (DSD) methods, evolutionary algorithms, *etc.* [3, 4].

In this Chapter, the Cazacu 06 model is summarized in the second section, which includes eight material parameters to be identified under the plane stress condition. The third section introduces the objective function used in the identification procedure and several typical optimized methods, which are gradient method, genetic algorithm, and particle swarm optimization. In the fourth section, a comparison of all these optimized

methods to identify material parameters in the Cazacu 06 constitutive model is addressed, proposing the hybrid optimization method between the evolutionary optimized method and the classical gradient method, besides, the calibrated results are presented and discussed. Finally, the main conclusions are drawn in the last section.

6.2 Constitutive theory

To describe both the asymmetry between tension and compression and the anisotropy observed in HCP metal sheets, Cazacu *et al* [1] introduced a general and rigorous method which is based on the theory of representation of tensor functions. A fourth order linear transformation is multiplied by the stress deviator $\mathbf{S}$ to extend an isotropic yield function to an orthotropic one, so that the transformed tensor Σ can be defined as:

$$\Sigma = \tilde{\mathcal{C}}\mathbf{S}. \quad (6.1)$$

where $\tilde{\mathcal{C}}$ is a 4th order tensor, which includes nine independent anisotropy coefficients for three dimensional stress conditions. Let (x, y, z) be the reference frame associated with orthotropy. In the case of a sheet, x , y and z represent the rolling, transverse, and the normal directions. Relative to the orthotropy axes (x, y, z) , the tensor $\tilde{\mathcal{C}}$ is represented by

$$\tilde{\mathcal{C}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (6.2)$$

So the transformed stress can be obtained by

$$\Sigma = \begin{bmatrix} \tilde{\phi}_1\sigma_{11} + \tilde{\phi}_2\sigma_{22} + \tilde{\psi}_1\sigma_{33} \\ \tilde{\phi}_2\sigma_{11} + \tilde{\phi}_2\sigma_{22} + \tilde{\psi}_2\sigma_{33} \\ \tilde{\phi}_3\sigma_{11} + \tilde{\phi}_3\sigma_{22} + \tilde{\psi}_3\sigma_{33} \\ C_{44}\sigma_{12} \\ C_{55}\sigma_{23} \\ C_{66}\sigma_{13} \end{bmatrix}, \quad (6.3)$$

where,

$$\tilde{\phi}_1 = \frac{(2C_{11}-C_{12}-C_{13})}{3}; \quad \tilde{\phi}_2 = \frac{(2C_{12}-C_{22}-C_{23})}{3}; \quad \tilde{\phi}_3 = \frac{(2C_{13}-C_{23}-C_{33})}{3}, \quad (6.4)$$

$$\tilde{\phi}_1 = \frac{(2C_{12}-C_{11}-C_{13})}{3}; \quad \tilde{\phi}_2 = \frac{(2C_{22}-C_{12}-C_{23})}{3}; \quad \tilde{\phi}_3 = \frac{(2C_{23}-C_{13}-C_{33})}{3}, \quad (6.5)$$

$$\tilde{\psi}_1 = \frac{(2C_{13}-C_{11}-C_{12})}{3}; \quad \tilde{\psi}_2 = \frac{(2C_{23}-C_{12}-C_{22})}{3}; \quad \tilde{\psi}_3 = \frac{(2C_{33}-C_{13}-C_{23})}{3}. \quad (6.6)$$

To describe the strength differential effect between tension and compression, a material parameter $\tilde{k}$ is introduced by Cazacu *et al* [1] and the orthotropic criterion can be expressed by

$$\sigma_{eq} = \left[(|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}} \right]^{1/\tilde{n}}, \quad (6.7)$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are the principal values of Σ . In order to ensure the convexity of the yield function, the introduced parameter, $\tilde{k}$, should be within the range of $\tilde{k} \in [-1, 1]$. When the transformed matrix $\tilde{\mathcal{C}}$ is equal to the identity matrix, the proposed formulation can reproduce, as a limiting case, the von Mises yield criterion.

6.3 Identification procedure and optimization methods

In sheet metal forming it is a common practice to assume that the sheet is approximately subjected to a plane stress loading during the entire processing [5]. Thus, the yield function may be considered as a function of the in-plane stress $\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{21}$. So anisotropy material parameters C_{55} and C_{66} are not necessary to calibrate. To identify the remained material parameters in the Cazacu 06 constitutive model, a set of experiments should be performed. The most widely experimental used tests are uniaxial tensile at different directions. In the current model, in order to calibrate material parameters related to the SD effect in the Cazacu 06 model, a series of compressive test should also be performed. So material parameters $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}$ can be calibrated from the above experimental tests. The required input variables in the identification procedure can be seen as follows:

- Three directional yield stresses obtained from uniaxial tensile tests at specimens having an orientation of $0^\circ, 45^\circ$ and 90° to the original rolling direction of the sheet. The associated yield stresses are denoted here as $\sigma_0^T, \sigma_{45}^T, \sigma_{90}^T$.
- Three directional yield stresses obtained from uniaxial compressive tests at specimens having an orientation of $0^\circ, 45^\circ$ and 90° to the original rolling direction of the sheet. The associated yield stresses are denoted here as $\sigma_0^C, \sigma_{45}^C, \sigma_{90}^C$.
- Three r -values corresponding to $\sigma_0^T, \sigma_{45}^T, \sigma_{90}^T$, which are denoted as $r_0^T, r_{45}^T, r_{90}^T$.

In many industrial applications, empirical or physical models involving some unknown parameters are used for design or simulation purposes. The identification of the model parameters is obtained by using either Trial-and-Error method or an optimization method being the act of obtaining the best result under the given circumstances. Trial-and-Error method is suitable when the model is simple and the number of unknown

parameters is reasonable. Optimization method is, however, used when the model is complex and a huge number of unknown parameters are under consideration. The Trial-and-Error method is not sufficient to handle the constitutive model with numerous material parameters. Therefore, it is necessary to find the suitable and efficient optimization method for the identification of these models.

The identification of the constitutive models parameters can be formulated as an optimization task. Material parameters in the constitutive model are interdependent and are difficult to quantify. Therefore, a mathematical optimization model should be built to allow this identification. Generally, the optimization problem corresponding to the parameters' identification of experimental values can be formulated as follows: determine n material parameters $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}$ of a given constitutive law by minimizing the summation of the errors squares. The errors are the differences between the measured experimental data and computed results at the given calibrated points of experimental domain. In this study, the measured experimental data includes stress value and r value at different stress state at a certain deformation level. In order to avoid the order effect difference between stress and r value, each item is normalized, and the final objective function (Error function) could be written as

$$Error(\underline{x}) = \sum_i w_i \left(\frac{\sigma_i^T}{\sigma_i^{Tth}} - 1 \right)^2 + \sum_j w_j \left(\frac{\sigma_j^C}{\sigma_j^{Cth}} - 1 \right)^2 + \sum_k w_k \left(\frac{r_k^T}{r_k^{th}} - 1 \right)^2, \quad (6.8)$$

where i is the number of experimental uniaxial tensile yield stresses, j is the number of experimental uniaxial compressive yield stress, k is the number of experimental tensile Lankford coefficients, while w_i, w_j, w_k are weights given to the respective experimental values. In this paper, all of these weights coefficient are fixed to 1.0. $\sigma_i^{Tth}, \sigma_j^{Cth}$ and r_k^{th} are the theoretical stress values corresponding, respectively, to each of the related experimental value. The summary of the computational method could be seen in [6]. $\underline{x}$ is a vector, which represents the calibrated material parameters $C_{11}, C_{22}, C_{33}, C_{12}, C_{13}, C_{23}, C_{44}$ and $\tilde{k}$.

6.3.1 Classical gradient method

The gradient-based algorithms usually converge quickly in the vicinity of the solution, and are therefore very interesting in terms of rapidity. However, they have some limitations, being strongly dependent on user skills. If the initial guess for the set of parameters is not carefully chosen, the iterative method can lead to convergence difficulties. In the gradient method, it is required to calculate the derivatives of objective

function with respect to design variables. However, the objective function used to identify material parameters is always substantially complicated; the derivatives are not easy to calculate by hand. In this paper, finite differencing approach is used to calculate the approximation derivatives based on the Tayloy's theorem. Finite perturbations in the variable values of χ are used to examine the resulting differences in the function values. The approximation of the derivative can be calculated by taking ratios of the function difference to variable difference. The general steps of gradient method can be written as follows [7].

- (a) Choose a starting point χ_0 .
- (b) Determine the search direction $\bar{p}^{(k)}$, here the direction is calculated by the approximated finite differencing method.
- (c) Set $\chi^{(k+1)} = \chi^{(k)} + \bar{\alpha}^k \bar{p}^{(k)}$, and $k = k + 1$.
- (d) Repeat until the stop criteria is met.

Remark: the $\bar{\alpha}^k$ value in (c) is calculated by Line search algorithm.

6.3.2 Genetic Algorithm (GA) method

Genetic algorithm is derivative-free stochastic search optimization methods inspired by the natural process of evolution, adapted to solve optimization problems and are able to deal with difficult, big sized, high cardinality and multi-modal problems.

The genetic algorithm (GA) is a selective random evolutionary search algorithm designed to achieve an optimum solution as proposed by Holland [8, 9]. The initial solutions (population) are usually randomly generated in the range of prescribed limits. Genetic algorithms code the candidate solutions of an optimization algorithm as a string of characters usually binary digits, which are usually called chromosomes. Each chromosome represents a point in the search space. Each point of the population is associated with the fitness function value. This fitness function measures the robustness of each solution. The fitness-dependent technique is used to select the parents for the next population from the current one, setting them in the mating pool, which is called selection. In this process, a probabilistic selection operator ensures the 'fittest' individuals the highest probability to produce offspring.

After the selection, two more basic genetic operators, cross-over, and mutations are used to formulate the offspring generation. The next generation of solutions is obtained by a crossover technique from the individuals in the mating pool. Two parents generate two offspring by passing on parts of their chromosomes. The method exchanges the bits

of the chromosome's bitstream beginning at a single random point and stores them in the offspring's chromosomes. In order to ensure genetic diversity, a mutation operator is used. Mutation is a background operator that produced some random changes in various chromosomes. Random mutation of newly generated offspring induces variability in the population preventing the premature convergence. It plays an important role not only replacing the genes lost during the evolutionary process in a new form, but also producing new genes; this procedure is referred to as an elitism strategy. After evolution, the fitness value of all altered individuals is evaluated and the next selection step starts by checking the stopping criterion. This algorithm is repeated until the stop criteria are reached [9, 10]. One iteration of the algorithm is referred to as a generation. The simplified flowchart of GA algorithm can be seen in Figure 6.1.

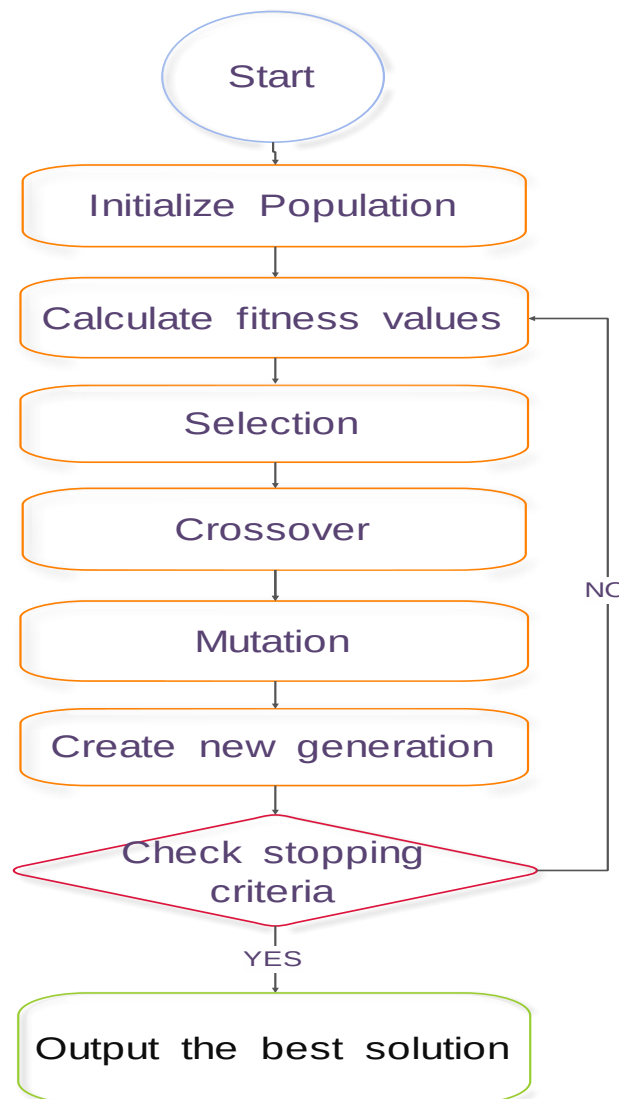


Figure 6.1: Flow chart of GA algorithm.

Remark: the fitness function (criterion) used in the GA algorithm is the same objective function as gradient optimization method, which is used to indicate how good or bad a candidate solution is.

6.3.3 Particle Swarm Optimization (PSO) method

PSO is a robust stochastic optimization technique based on the movement and intelligence of swarms, which is also a kind of evolutionary computational techniques. It is motivated from the simulation of social behavior of animals such as bird flocking, fish schooling, and swarm. Mathematical description of basic PSO is briefly summarized [11, 12].

PSO is initialized with a population of random solutions and this initial population evolves over generations to find optima. Each particle is represented by a position and a velocity in the n -dimensional problem space. The velocity enables each particle to move through the problem space instead of dying and mutating the mechanisms of genetic algorithms. All the position and velocity should be constraint in the pre-defined ranges. Their fitness values are decided by objective function for the position of each particle. Each individual is treated as a volume-less particle. Further, each particle has a memory and hence is capable of remembering the best position in the search space it visited. At each iteration, modification of the position of a particle is performed by using its previous position information and its current velocity. Each particle knows its best position (personal best) so far and the best position achieved by the group (group best) among all personal bests. These principles can be formulated as:

$$\begin{cases} \bar{v}_i^{k+1} = \bar{w}\bar{v}_i^k + \bar{c}_1 rand_1(\mathbf{pbest}_i - \bar{x}_i^k) + \bar{c}_2 rand_2(\mathbf{gbest}_i - \bar{x}_i^k) \\ \bar{x}_i^{k+1} = \bar{x}_i^k + \bar{v}_i^{k+1} \end{cases}, \quad (6.9)$$

where, $\bar{v}_i^k$: velocity of agent i at iteration k

$\bar{x}_i^k$: current position of agent i at iteration k

$\mathbf{pbest}_i$: personal best of agent i

$\mathbf{gbest}_i$: best position in group

$rand$: random number between 0 and 1

$\bar{w}$: weighting function

$\bar{c}_j$: weighting factor $j = 1, 2$

The weighting function $\bar{w}$, which is also called inertia weight, is employed to control the impact the previous history of velocities on the current velocity, thereby influencing the trade-off between global (wide-ranging) and local exploration abilities of the “flying

points”. A larger inertia weight facilitate global exploration (search new area) while a smaller inertia weight tends to facilitate local exploration to fine-tune the current search area. The inclusion of random variable endows the PSO with the ability of stochastic searching. The weighting factors, $\bar{c}_j$, compromise the inevitable trade-off between exploration and exploitation. After updating, the new vector $\bar{v}_i^{k+1}$ should be checked and secured within a pre-specified range to avoid violent random walking. The updated position should also be checked and limited to the allowed range.

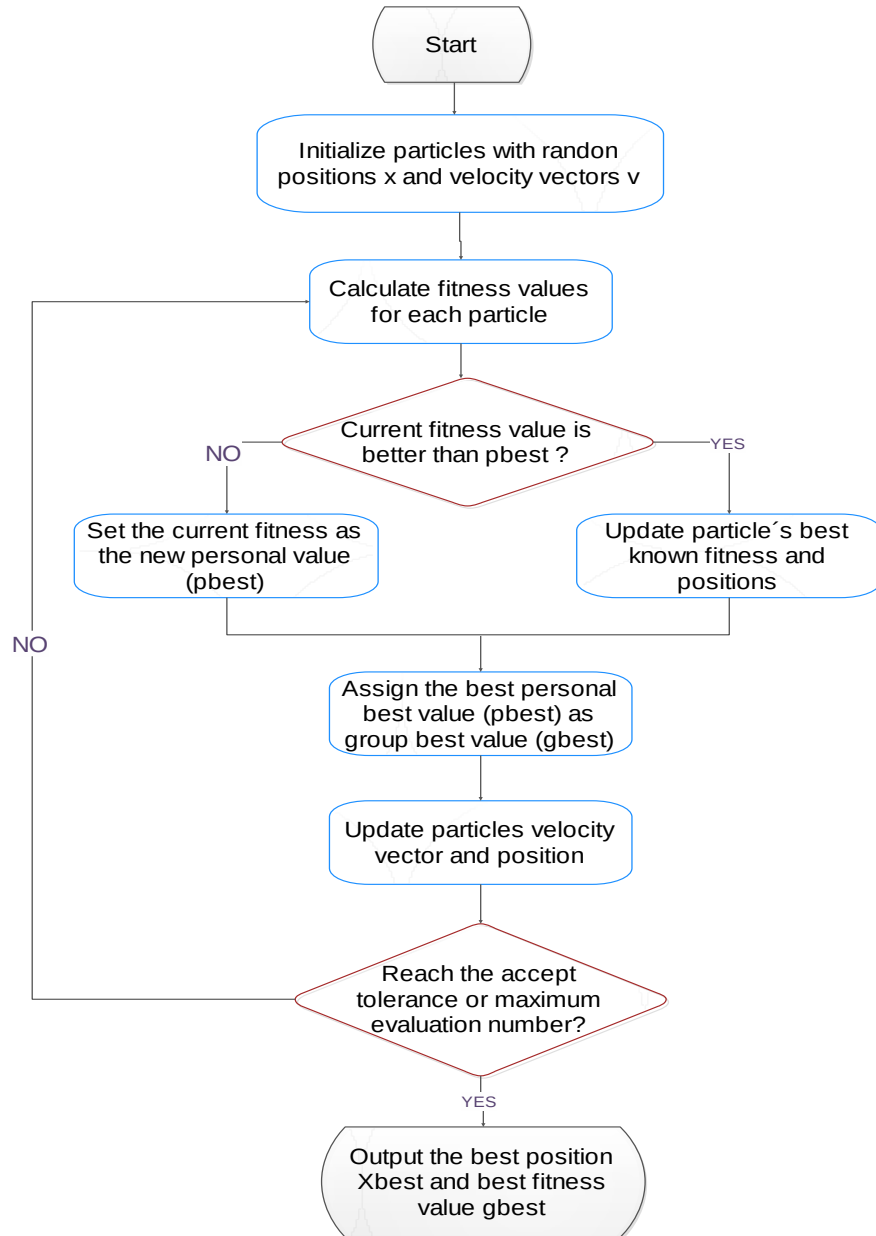


Figure 6.2: Flow chart of PSO algorithm.

The above iterative procedure continues until the occurrence of the stop condition: whenever the PSO executes the maximum number of iterations or the minimum error

requirement is reached. The simplified flow chart of PSO algorithm can be observed in Figure 6.2.

6.3.4 Hybrid method of evolutionary optimization and gradient optimization

Owing to what was mentioned above, the classical gradient method is strongly dependent on the initial value; it is easy trapped into local minima once a wrong starting value is chosen. Stochastic evolutionary methods can overcome local minima; however, they frequently are not effective with respect to convergence.

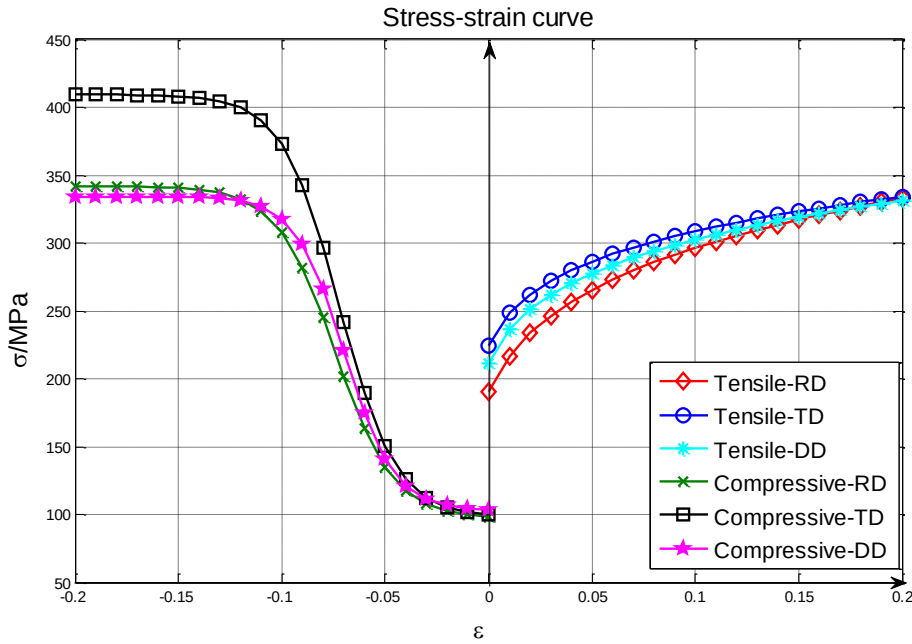


Figure 6.3: Stress-strain curve of AZ31 alloy at room temperature.

In order to avoid this problem, a hybrid scheme was developed. The hybrid algorithm was obtained using the evolutionary algorithm first, to reach a point near the global minimum, and then a gradient method was performed to obtain the minima. In this paper, the gradient method using the initial value obtained from the GA method is denoted as GA+Grad. If PSO method is used to generate initial value for the subsequent gradient methods, this hybrid method is named as PSO+Grad.

In this contribution, the proposed optimization methods are applied to calibrate the material parameters for AZ31 alloy. The experiment tests were performed by Khan *et al.* [13]. All the stress-strain curves are shown in Figure 6.3. Owing to that the yield surface of AZ31 alloy has two typical yield locus shapes, which evolves from the triangle-like shape to elliptic-like shape during the deformation; two representative points are chosen to study the material parameters calibrations with different optimized

methods. One is an elliptical-like shape which is correspondence to the $\bar{\varepsilon}^p=0.1$. The other is a triangular-like shape which is correspondence to the $\bar{\varepsilon}^p=0.01$. The input variables for calibration in these two representative points can be seen in Table 6.1.

Owing to the convex constraint of the Cazacu 06 constitutive model, the material parameter $\tilde{k}$ is fixed in the range of $[-1.0, 1.0]$. The remaining material parameters ($C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}$) are set in the range of $[-2.0, 2.0]$.

Table 6.1: Experimental values of the yield stresses and r -values under different loading conditions.

$\bar{\varepsilon}^p$	σ_0^T	σ_{90}^T	σ_{45}^T	σ_0^C	σ_{90}^C	σ_{45}^C	r_0^T	r_{90}^T	r_{45}^T
0.0	216.84	248.27	236.35	99.80	101.41	104.56	0.3063	0.6580	0.5475
0.1	296.27	308.68	302.59	308.08	372.66	317.38	1.5962	3.0259	2.3269

6.4 Results and discuss of identification

In this section, the gradient methods and GA algorithm are performed by MATLAB optimization TOOLBOX. PSO method is developed by user developed code.

6.4.1 Yield surface of equivalent plastic strain ($\bar{\varepsilon}^p=0.1$)

For the classical gradient optimized method, the yield surface starts evolution from a classical elliptical yield locus (von-Mises yield criteria) during the optimization. Therefore, the starting point ($\underline{x}_0$) of the gradient optimization method can be chosen as $[1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0]$. In this situation, the Cazacu 06 model is reduced to the von Mises model. The calibrated material parameters are presented in Table 6.2.

The error value comparison of the yield surface calibration of $\bar{\varepsilon}^p=0.1$ among different optimization methods is shown in Figure 6.4. From there, it can be concluded that classical gradient method and PSO+Grad methods obtain a similar error value, which is 0.00601. It means that the point $[1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1]$ is a good starting point for gradient method. It also means that for the elliptical-like shape yield surface, the calibration procedure can be set starting from the yield surface of von-Mises. Meanwhile, the final error value obtained from GA and GA+Grad is a little big larger than the gradient method and PSO+Grad. It means that the hybrid optimization (GA+Grad) is trapped into local minima after performing gradient optimization, where the starting trial values are obtained from the GA algorithm. However, the PSO method can jumps out of local trap and obtain a good initial value, which is close to the final global optimization value.

Therefore, the hybrid PSO+Grad optimization can obtain the same calibrated material parameters and yield surface as the classical gradient optimization method.

Table 6.2: The material parameters of the yield surface ($\bar{\varepsilon}^p=0.1$) determined by different optimized methods.

	Gradient	GA	PSO	GA+Grad	PSO+Grad
C_{11}	0.7922	1.1584	0.4336	1.1622	0.4624
C_{22}	1.1066	0.9996	0.7791	0.9936	0.7768
C_{33}	0.8679	0.0419	0.5728	0.05163	0.5381
C_{12}	-0.01188	0.1949	-0.3588	0.1891	-0.3417
C_{13}	-0.003839	0.8528	-0.3314	0.8546	-0.3337
C_{23}	0.249	0.8863	-0.0743	0.8829	-0.08086
C_{44}	0.9959	0.9177	0.9871	0.9295	0.9959
$\tilde{k}$	-0.08515	0.4342	-0.1067	0.4227	-0.08515
Error	0.00601	0.008819	0.0073	0.008575	0.0060096

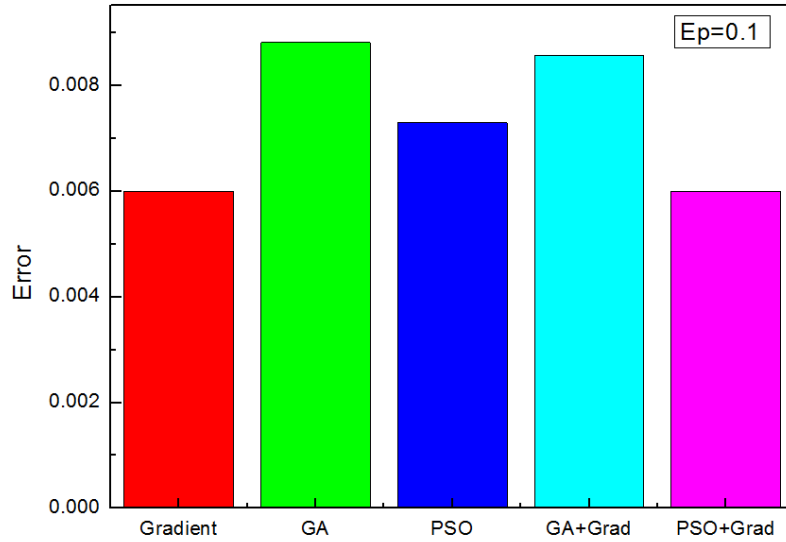


Figure 6.4: The error comparison among different optimization methods.

Figure 6.5 shows the yield surface of $\bar{\varepsilon}^p=0.1$ obtained from different strategies. From there, it can be concluded that the yield locus obtained from the classical gradient method, PSO, and PSO+Grad method is very close, the shape looks like elliptical. Nevertheless, the yield surface obtained from GA and GA+Grad is a triangular-like shape. Although the error value gap among these optimization methods is not large,

from the shape of the yield surface it is easy to conclude that once the optimization is trapped into an unsuitable local value, the shape definitely changes.

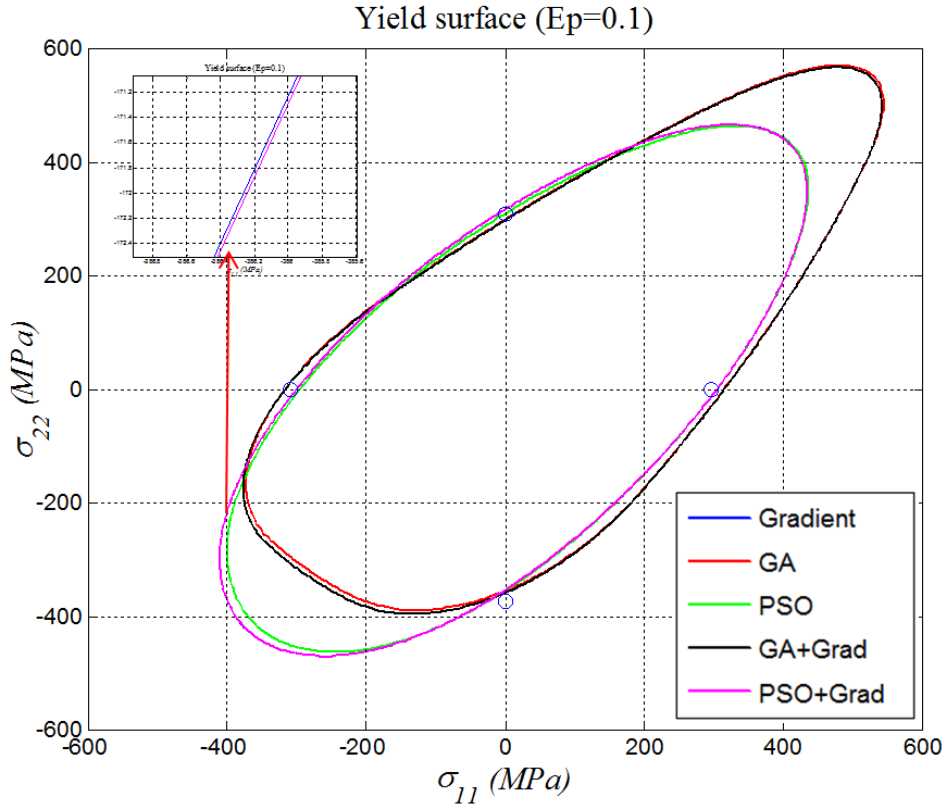


Figure 6.5: Yield locus of $\bar{\epsilon}^p=0.1$ in the $\sigma_{11} - \sigma_{22}$ space determined by different optimized methods.

6.4.2 Yield surface of equivalent plastic strain ($\bar{\epsilon}^p=0.01$)

For the yield surface of equivalent plastic strain equal to 0.01, the same starting point is used for the classical gradient method as yield surface of equivalent plastic strain $\bar{\epsilon}^p=0.1$. The calibrated material parameters obtained from hybrid optimized methods are presented in Table 6.3.

Figure 6.6 displays the error value comparison among different optimization methods. From there, it is obviously seen that the classical gradient method has a large value compared with other optimized methods. It indicates that the classical gradient method only gets a local optimization. It also means that the starting point $[1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0]$ is not a good choice in this situation. A hybrid optimization strategy is necessary to determine the starting point for the classical gradient method. Although the error value obtained by GA algorithm is half of the gradient method, it is still trapped in local optimization. PSO algorithm is able to provide a good initial value for the subsequent gradient methods, so the PSO+Grad algorithm obtains the least error value.

Table 6.3: The material parameters of the yield surface ($\bar{\epsilon}^p=0.01$) determined by different optimized methods.

	Gradient	GA	PSO	GA+Grad	PSO+Grad
C_{11}	1.3806	0.7133	-0.2704	0.7224	-0.2013
C_{22}	1.2963	0.8708	0.1117	0.8642	0.07752
C_{33}	0.1478	1.1857	-0.2396	1.1197	-0.2043
C_{12}	1.7787	1.7054	-1.1754	1.6975	-1.2433
C_{13}	-0.971	-0.05562	0.9854	-0.05378	0.9598
C_{23}	-0.6322	0.03232	0.8074	0.0310	0.8308
C_{44}	2.267E-5	0.9119	1.0450	0.8895	1.0771
$\tilde{k}$	0.5107	0.9103	-0.6159	1.00019	-0.5908
Error	0.08992	0.04198	0.0085	0.03524	0.001703

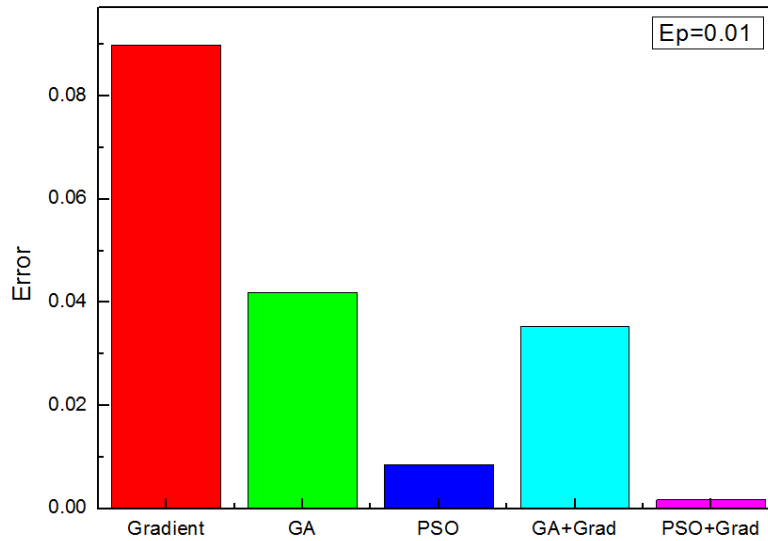


Figure 6.6: The final error comparison among different optimization methods.

Figure 6.7 presents the yield surface of $\bar{\epsilon}^p=0.01$ obtained from different optimization methods. From there, it can be concluded that the yield locus obtained from the classical gradient method keeps an elliptical shape, but the long axis changes from the biaxial stress state into the shear stress state. It is obviously trapped into local minima. The shape is not reasonable for the real yield surface. The GA and GA+Grad have already got the triangle-like shape, but it is not in the global optimization value. PSO, and PSO+Grad method are very close, the shape looks like triangle.

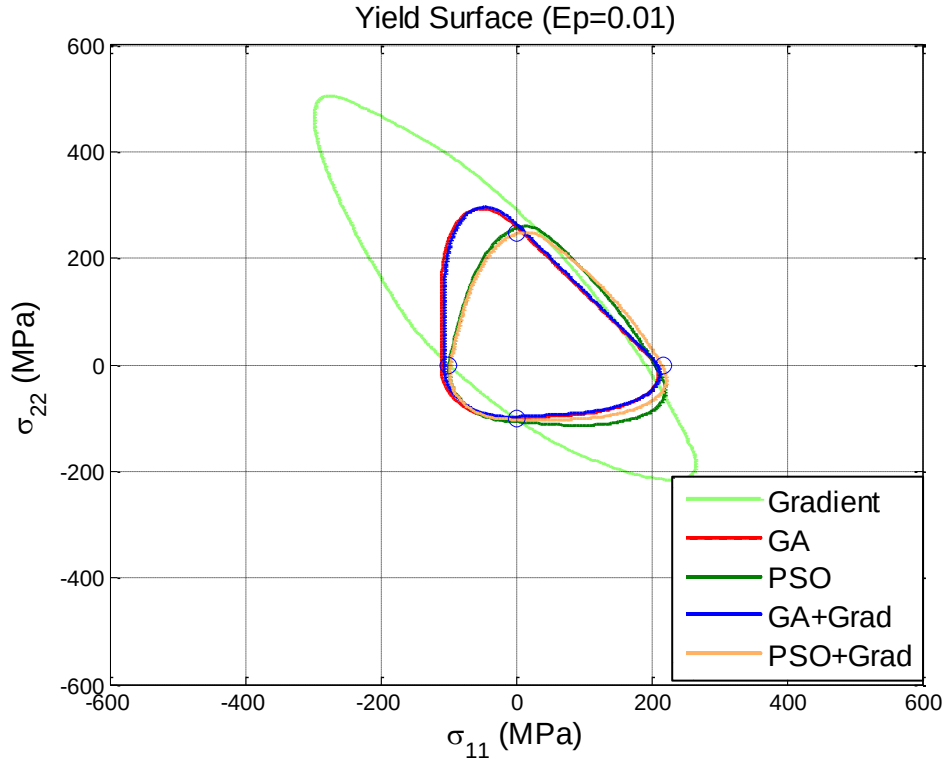
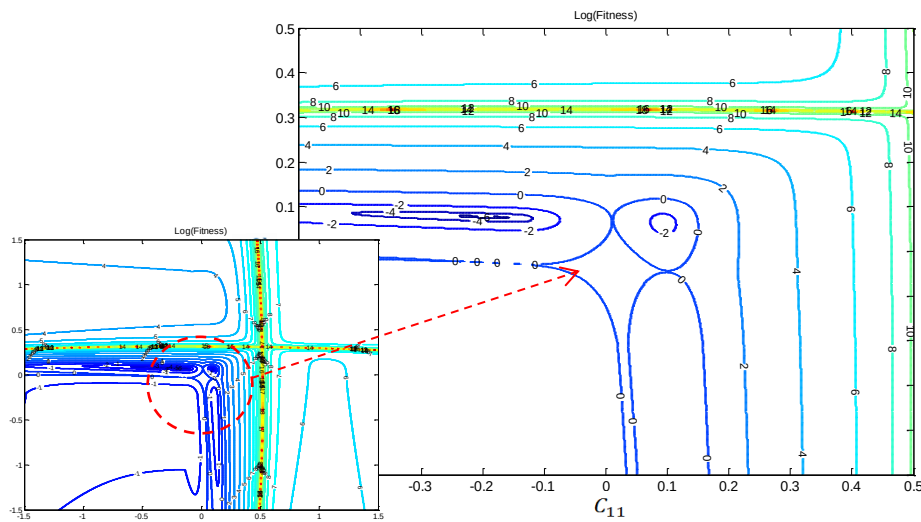
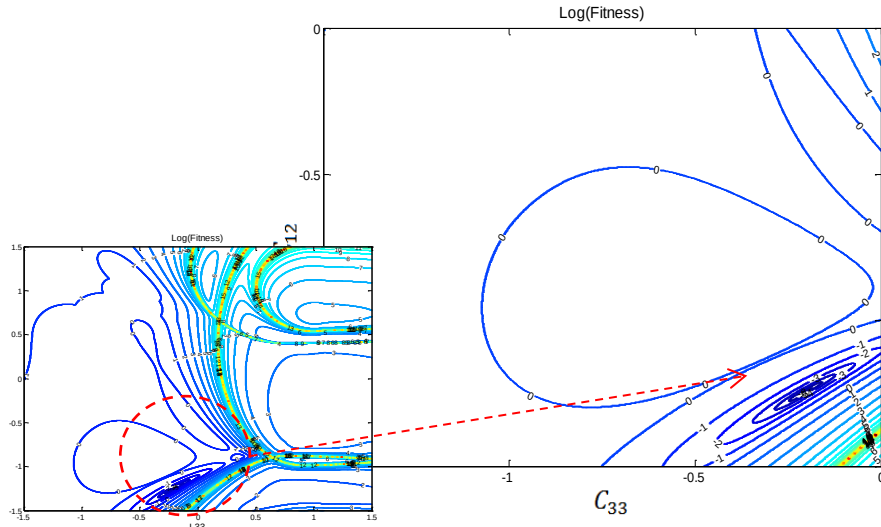


Figure 6.7: Yield locus of $\bar{\epsilon}^p=0.01$ in the $\sigma_{11} - \sigma_{22}$ space determined by different optimized methods.

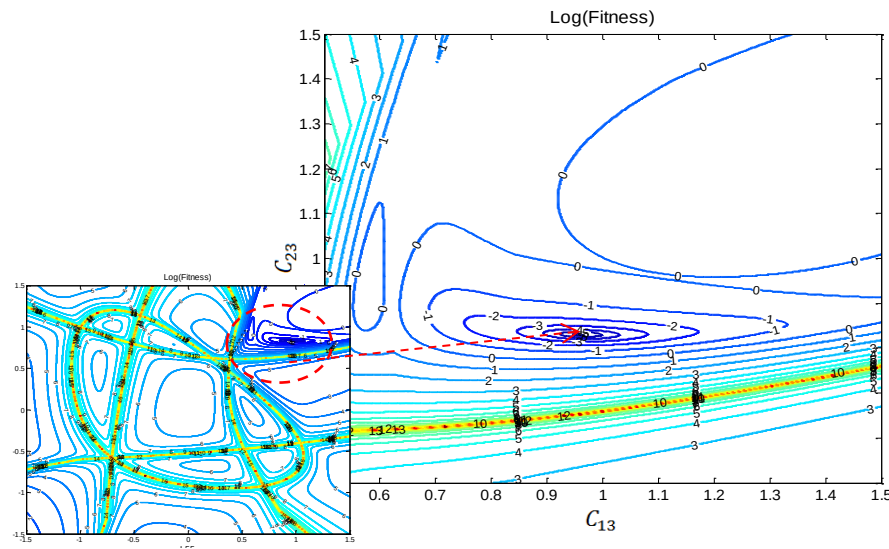
In order to explain clearly, several error value distributions around the global minima are drawn. Owing to that there exist eight variables, but it is not possible to draw an eight dimensional contour. Therefore, we just list several mutual variables error distributions contours, which could be observed in Figure 6.8. From there, it is easy to see that the error value distribution near the global minima is very complicated. There exists numerous local minima value. Once the wrong initial value is chosen, it is very easy to trap into local minima and generate the inaccuracy yield surface.



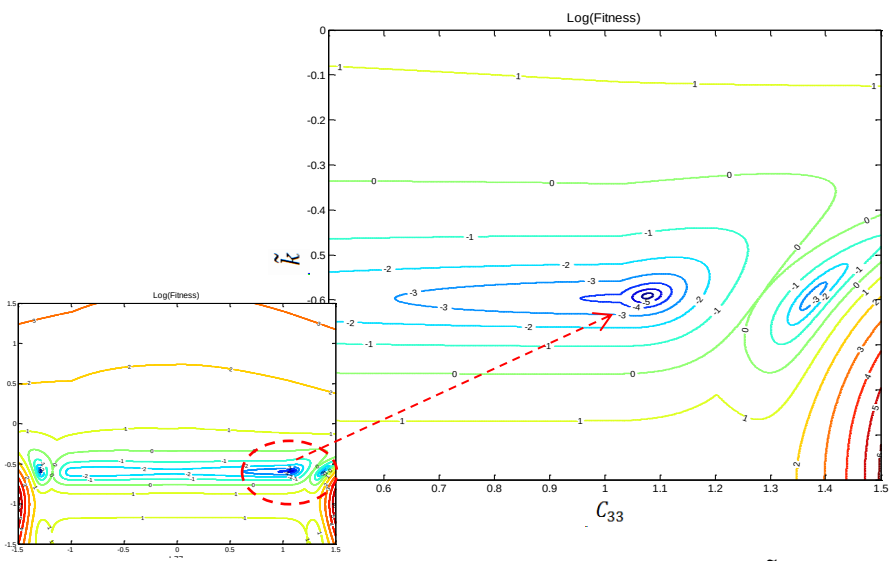
Error Contours of variable C_{11} and C_{22}



Error Contours of variable C_{33} and C_{12}



Error Contours of variable C_{13} and C_{23}



Error Contours of variable C_{33} and $\tilde{k}$

Figure 6.8: Error value contours.

6.4 Conclusions

In this work, five optimized strategies were used to calibrate the material parameters of Cazacu 06 constitutive model for typical HCP crystal structured material- AZ31 alloy. The classical gradient method is effective to calibrate the yield surface at large strain levels. At that stage, the yield surface of AZ31 alloy is elliptical-like. The initial value of the constitutive model can be chosen as $[1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0]$. At the low strain levels, the yield surface demonstrates triangular-like shape. The above initial value for the gradient optimization method is easy trapped into local minima, and evolutionary hybrid methods are used. By comparison of GA and PSO, PSO algorithm can escape the local minima trap and obtain the global optimization value. The optimized yield surface is more accurate.

6.5 References

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Chapter 7

An Extended Yield Criterion for Orthotropic Metallic Sheets under Plane Stress

7.1 Introduction

Sheet metal forming simulations rely on an accurate description of the anisotropic yielding behavior. In recent years, numerous studies [1-4] have been conducted using the elasto-viscoplastic Crystal Plasticity Finite Element Method (CP-FEM) in order to understand the micro-mechanisms involved and to capture the macro mechanical behavior. Some efforts have been carried out to model the plastic behavior of Hexagonal Close-Packed (HCP) polycrystals using crystal plasticity and continuum theories. Because of the complex material behavior, crystal plasticity has been more popular in modeling HCP metals. Van Houtte [5] first proposed an approach for modeling twinning in a statistical manner. Tome *et al.* [6] developed the Predominant Twin Reorientation (PTR) and Volume Fraction Transfer (VFT) schemes and implemented them into their Visco-Plastic Self-Consistent (VPSC) model to represent twinning activity. In the PTR scheme, twinning reorientation occurs only on the slip system with the highest activity until the grain is exhausted; while in the VFT scheme, all twinning reorientations are considered through a volume fraction transfer from the old orientation cell to the new orientation cell in the Euler space when twinning occurs. Besides, the PTR method has been later used to study the deformation mechanisms in magnesium alloys [7, 8]. Kalidindi [1] developed a modified VFT scheme in which the lattice orientation of the twinned region is predefined. Staroselsky *et al.* [9] used the probabilistic lattice reorientation condition suggested by Van Houtte [5] to model the room-temperature behavior of the magnesium alloy AZ31B. In more recent works, the models [2, 3, 10, 11] account for more complex phenomena such as geometrically necessary dislocations, Hall-Petch effect, and slip-twinning interaction, double twinning. Wang *et al.* [4, 12] developed a comprehensive model for describing the twinning/untwining behavior occurring during cyclic loading for magnesium alloys. The physically-based crystal plasticity models are suitable for studying the deformation mechanisms. However, a full 3D forming simulation using crystal plasticity model may be impractical because of large computation time. Furthermore, its high computational

cost limits its application to real sheet metal forming. Therefore, phenomenological yield surface functions are still a good choice in metal forming finite element simulations.

Continuum (non-crystal plasticity) approaches have also been tried to represent the behavior of HCP metals. Although numerous yield criteria [13-16] have been proposed for cubic materials, all of them fail to capture this unconventional mechanical behavior. Cazacu and coworkers (Cazacu and Barlat, 2004 [14], Cazacu *et al.*, 2006 [17] and Plunkett *et al.*, 2008 [18]) developed a series of yield criteria for describing anisotropy and yield differential effects of HCP metals. Cazacu and Barlat (2004) [14] used the generalized second and third invariants of the stress deviator, whereas Cazacu *et al.* (2006) [17] and Plunkett *et al.* (2008) [18] used the principal values of the linearly transformed stress deviator. Barnett *et al.* [19] proposed an analytical model for one-dimensional twinning dominated flow in magnesium. Kim *et al.* [20] used Cazacu's yield function (Cazacu *et al.*, 2006 [17]) and different hardening equations depending on the strain paths. Cazacu 06 model predicts a high curvature of the yield function in loading direction, while the opposite region of the yield function is rather flat. The modified Cazacu model was adopted by numerous researchers [21-23] by introducing multi-transformed matrix to capture the mechanical behavior. Soare *et al.* [24] recently discussed four, six and eighth-order polynomial yield functions, proposing a scheme of material parameter identification based on the numerical optimization technique which includes the constraints to satisfy the convexity of the yield loci at several representative cross sections of the yield locus. This approach is, however, sophisticated for the optimization; in fact, a substantially more material data in addition to r_0 , r_{45} , r_{90} , σ_0 , σ_{45} , σ_{90} and σ_b . As for the convexity, their proof presents no major practical problems, but theoretically still it is not perfect.

Lee *et al.* [20, 25] used a modified Drucker-Prager model for the initial yield surface and the two-surface hardening model where the gap between the two functions can be controlled to describe the unique hardening behavior. The strength differential (SD) effect as well as the initial anisotropy can be captured in a relatively straightforward manner by existing models. Lee *et al.* [20] presented a one dimensional analytical model for predicting springback of magnesium alloy sheets. Li *et al.* [26] proposed the constitutive model TWINLAW that uses different kinematic hardening rules depending on the dominating deformation mechanisms (slip, twinning, and untwining), while keeping track of the amount of grains that can be twinned. Barlat *et al.* [27], on their

turn, proposed a homogeneous anisotropic hardening rule to describe plastic behavior during strain path changes.

The description of anisotropy based on the non-associated flow rule, where two different functions for the yield criterion and the plastic potential are used, has been addressed by some studies [28, 29]. In the present chapter, our framework is the description of anisotropy with the yield function based on the associated flow rule.

Although some of yield criteria mentioned before can describe yielding of material with good accuracy, this entire model introduces numerous material parameters, and lead to the problem of the identification and a great problem for the real application. In this chapter, a new anisotropic constitutive model under plane stress condition is proposed. The related material parameters are identified by the hybrid Particle Swarming optimization (PSO) algorithm. The comparison of the fitted yield surface between the new proposed model and several commonly used yield criteria are also addressed. Finally, the proposed constitutive model and several classical anisotropic yield functions are implemented by using a primal Closest Point Projection Method (CPPM) within an implicit quasi-static finite element environment and applied to the deep drawing of a cylindrical cup.

7.2 Fundamental review on anisotropic yield functions

Plasticity theory deals with yielding of materials under complex stress states. It is used to decide whether or not a material will yield under a stress state and to determine the shape change that will occur if it does yield. It is commonly described by a mathematical function of the stress states that will cause yielding or plastic flow.

The yield criterion, as a function of all state variables, can be written in a generic form as:

$$\Phi(\boldsymbol{\sigma}, \sigma_Y) = \sigma_{eq} - \sigma_Y = 0. \quad (7.1)$$

For the modelling of mechanical behavior of HCP materials, it is essential to select an appropriated yield criterion. Three anisotropic yield functions, (Hill 48 [30], Barlat 89 [31], and Cazacu 06 [17]), and a new proposed yield criterion were chosen to evaluate the mechanical performance of AZ31 alloy. The first two yield functions were selected due to the simplicity of their numerical formulas and the small number of experimental data required for calibration. The Cazacu 06 yield criterion requires more material data to identify the anisotropy coefficients and thus is more flexible than the previous two.

7.2.1 Hill 48 yield criterion

Hill's (1948) quadratic yield function [30, 32] may be noted as one of the first and most comprehensive yield criteria proposed as a generalization of the Huber-Mises-Hencky yield function criterion, which has been widely used to evaluate the plastic properties of anisotropic metals. The yield function associated with the Hill criterion can be cast in the following form:

$$\sigma_{eq} = [F_{Hill}(\sigma_{22} - \sigma_{33})^2 + G_{Hill}(\sigma_{33} - \sigma_{11})^2 + H_{Hill}(\sigma_{11} - \sigma_{22})^2 + 2L_{Hill}\sigma_{23}^2 + 2M_{Hill}\sigma_{13}^2 + 2N_{Hill}\sigma_{12}^2]^{1/2}, \quad (7.2)$$

where, F_{Hill} , G_{Hill} , H_{Hill} , L_{Hill} , M_{Hill} , N_{Hill} are material parameters, which characterize the state of anisotropy and to be determined experimentally.

7.2.2 Barlat 89 model

Barlat *et al* [31] proposed a more general anisotropic yield criterion based on the Hosford's criterion by expressing it in an x, y, z coordinate system, not necessarily coincident with the principal directions. The corresponding yield function, written at the outset exclusively in terms of in-plane components of the stress tensor, can be written as

$$\sigma_{Barlat} = \left(\frac{a_{BL}|K_1 + K_2|^{M_{BL}} + a_{BL}|K_1 - K_2|^{M_{BL}} + (2 - a_{BL})|2K_2|^{M_{BL}}}{2} \right)^{\frac{1}{M_{BL}}}, \quad (7.3)$$

where

$$K_1 = \frac{\sigma_{11} + h_{BL}\sigma_{22}}{2}, \quad (7.4)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2}\right)^2 + (b_{BL})^2\sigma_{12}^2}, \quad (7.5)$$

where M_{BL} , a_{BL} , h_{BL} , and b_{BL} are material constants. The yield function is convex if

$$M_{BL} > 1, a_{BL}, h_{BL}, b_{BL} > 0, \text{ and } a_{BL} < 2.$$

7.2.3 Cazacu 06 model

Cazacu 06 anisotropic yield function can be expressed by

$$\sigma_{Cazacu} = \left[(|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}} \right]^{1/\tilde{n}}, \quad (7.6)$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are the principal values of Σ . The transformed tensor Σ can be defined as:

$$\Sigma = \tilde{\mathbf{L}}\sigma, \quad (7.7)$$

where $\tilde{\mathbf{L}}$ is a 4th order tensor, which includes nine independent anisotropy coefficients for three dimensional stress conditions, which is expressed by

$$\tilde{\mathbf{L}} = \begin{bmatrix} \frac{(2C_{11}-C_{12}-C_{13})}{3} & \frac{(-C_{11}+2C_{12}-C_{13})}{3} & \frac{(-C_{11}-C_{12}+2C_{13})}{3} & 0 & 0 & 0 \\ \frac{(2C_{12}-C_{22}-C_{23})}{3} & \frac{(-C_{12}+2C_{22}-C_{23})}{3} & \frac{(-C_{12}-C_{22}+2C_{23})}{3} & 0 & 0 & 0 \\ \frac{(2C_{13}-C_{23}-C_{33})}{3} & \frac{(-C_{13}+2C_{23}-C_{33})}{3} & \frac{(-C_{13}-C_{23}+2C_{33})}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (7.8)$$

In order to ensure the convexity of yield function, the introduced parameter, $\tilde{k}$, should be within the range of $\tilde{k} \in [-1, 1]$.

7.2.4 New proposed constitutive model

The Barlat 89 model is simple and highly adequate to capture the anisotropic mechanical behaviour. However, it cannot capture the SD effect between tension and compression stress state. In order to overcome this insufficiency without the change of major form of the yield model, some improving tasks are required. In the present study, we extend further the Barlat 89 model by adding the term of the absolute value of the principal stress, and the extended yield criterion can be seen as

$$\sigma_{eq} = \left(\frac{a|(K_1 + K_2) - c|K_1 + K_2|^M + a|(K_1 - K_2) - c|K_1 - K_2|^M}{f|2K_2|^M} \right)^{1/M}, \quad (7.9)$$

where,

$$K_1 = \frac{\sigma_{11} + h\sigma_{22}}{2}, \quad (7.10)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h\sigma_{22}}{2}\right)^2 + b^2\sigma_{12}^2}, \quad (7.11)$$

and a , h , b , c and f are material parameters. Parameters a , h , f and b reflect the anisotropic mechanical behaviour. Parameter c defines the asymmetrical strength between tension and compression stress state. When c is equal to zero, the proposed yield criteria can be reduced to the original Barlat 89 model. In this contribution, the parameter f was calculated by

$$f = 2 - a. \quad (7.12)$$

Finally, it is important to remark that the yield function is convex when

$$M > 1, a, b, f, h > 0.$$

The mathematical prove of the convexity for this new proposed yield criteria is deduced in appendix B.

Since the effective stress σ_{eq} is the first order homogeneous function in stresses, from the work equivalence principle it follows that the law of evolution for the effective plastic strain (associated with σ_{eq}) reduces to $\dot{\bar{\epsilon}}^p = \dot{\gamma}$. The loading-unloading conditions can be expressed in Kuhn-Tucker form as Equation (7.13)

$$\Phi \leq 0, \dot{\gamma} \geq 0, \dot{\gamma}\Phi = 0. \quad (7.13)$$

The phenomenon of hardening describes the changes in yield stress that result from plastic straining and the flow stress, σ_Y , represents the size of the yield function during deformation.

7.3 The assessment of the new proposed model

In this section, we will discuss the yield surfaces of the proposed anisotropic criterion for different material parameters. In order to evaluate the influence of the new introduced parameter c on the yield surface, a set of c value are chosen, which is -0.5, 0.0, and 0.5. For the original Barlat 89 model, the power index M_{BL} has a great influence on the yield surface; thus, in this section, the influence of power index M on the proposed model is also discussed.

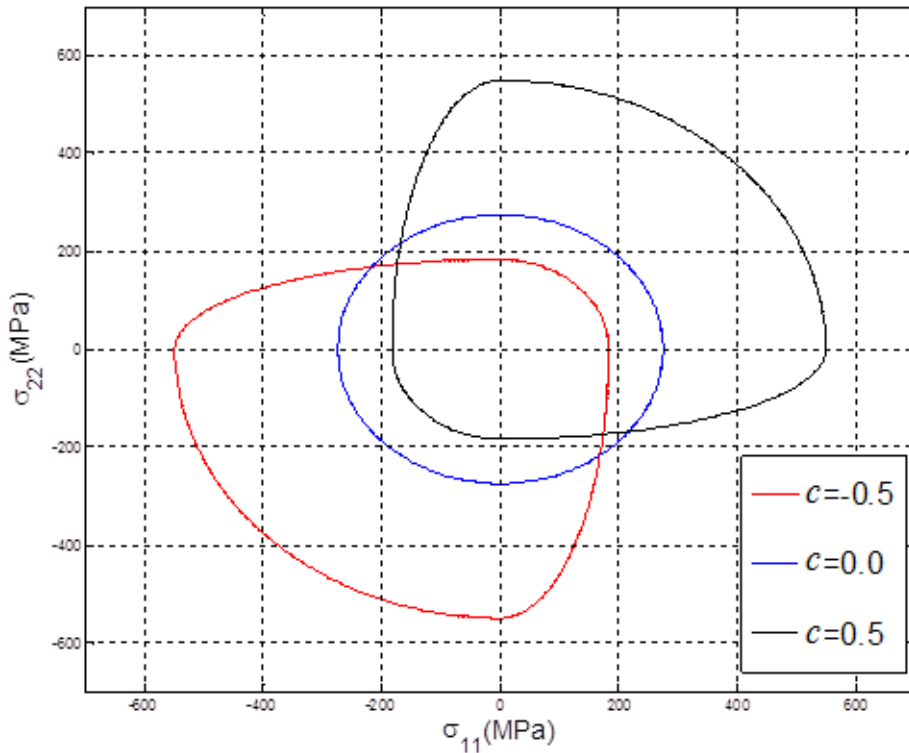


Figure 7.1: Yield surface of $M = 2$.

Figure 7.1 shows the yield surface alternation at different c values with the power index $M = 2$. It can be seen that when the c value is equal to zero, the yield surface presents a symmetrical ellipse. When the c value is equal to -0.5 , the yield surface in the first quadrant is shrunk, meanwhile, the yield surface in the third quadrant is expanded. The size of the yield surface is also expanded. The c value can be effective to reflect the tension-compression asymmetry. It makes the yield surface change from the elliptical shape to triangular-like shape. From the comparison between the yield surface of $c = 0.0$ and $c = -0.5$ in these two quadrants, the alternation in the third quadrant of the yield surface of $c = -0.5$ presents a larger changes than the first quadrant. It is also seen that the turn corner in the first quadrant of the yield surface of $c = -0.5$ becomes sharper than the yield surface of $c = 0.0$; besides, the yield surface of $c = -0.5$ becomes smoother than the yield surface of $c = 0.0$ in the third quadrant. When the c value is equal to 0.5 , the situation is opposite.

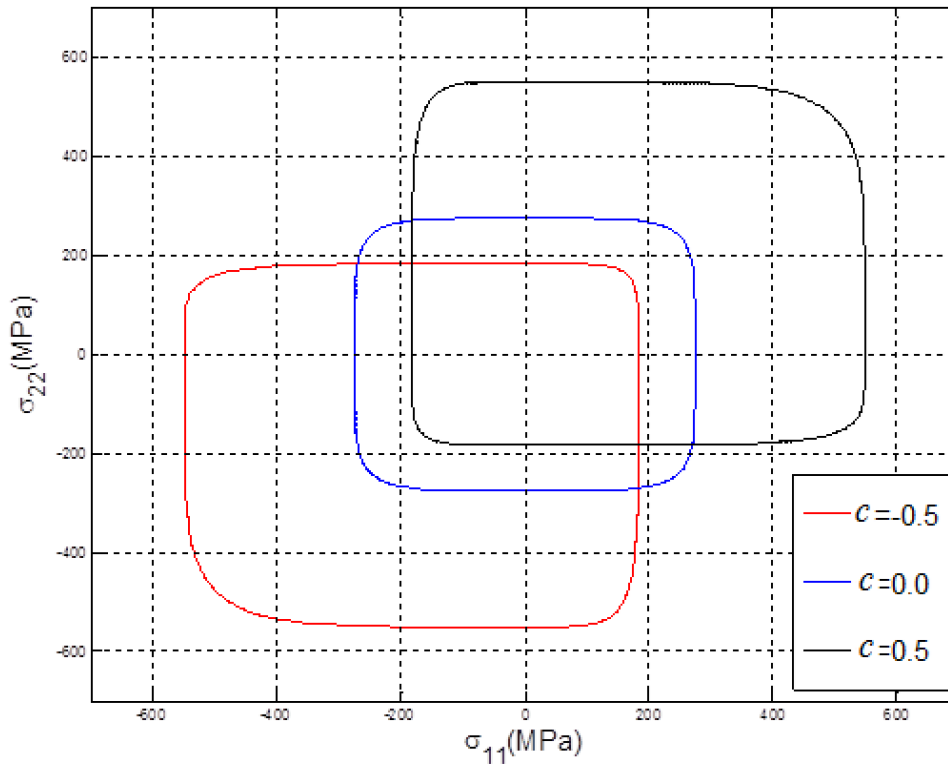


Figure 7.2: Yield surface of $M = 4$.

Figure 7.2 shows the yield surface alternation at different c values with the power index $M = 4$. It is observable that when the c value is equal to zero, the yield surface is symmetrical rectangular-like shape. Besides, it is subjected to a similar evolution law to the yield surface of the power index $M = 2$ when the c value changes. When the c value is negative, the corner in the first quadrant becomes sharper than the yield surface of

$c = 0.0$, while the yield surface in the third quadrant become smoother than the yield surface of $c = 0.0$. Although it still causes a great impact on the size of the yield surface, the influence of the same c value on the shape alternation of yield surface become smaller, and the yield surface of $c = -0.5$ still keeps a rectangular-like shape. This also can be clearly seen in Figure 7.3, which demonstrates the yield surface alternation of power index $M = 6$.

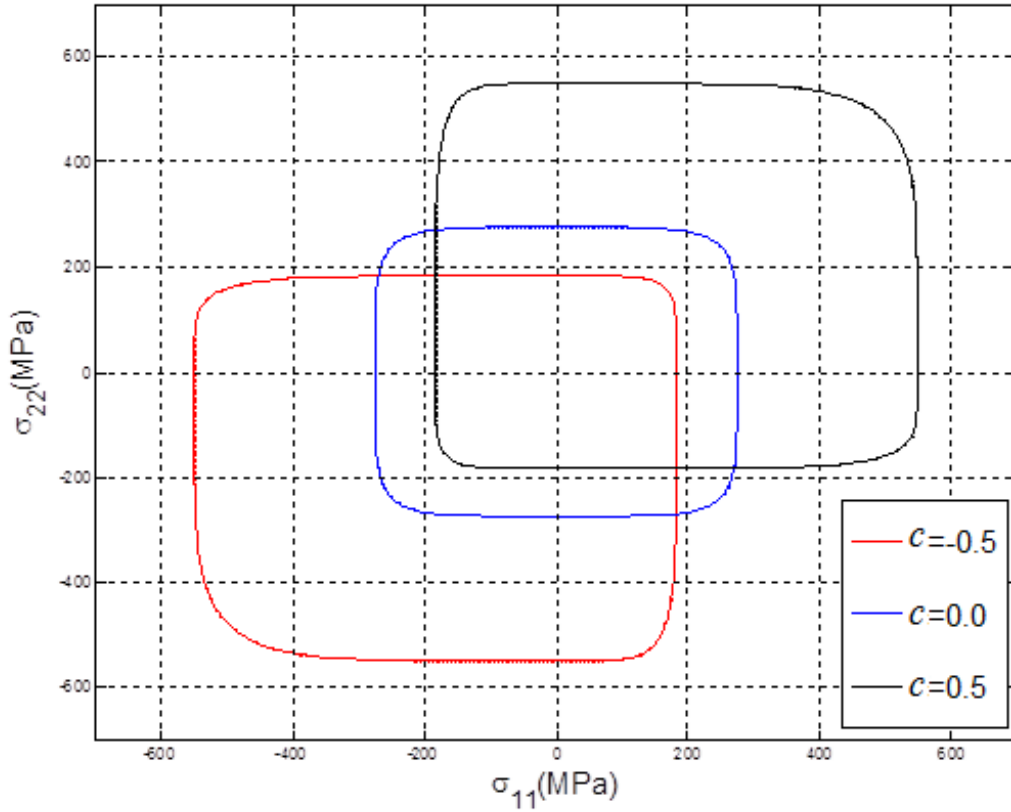


Figure 7.3: Yield surface of $M = 6$.

It can be concluded that the introduced material parameter c not only change the size of yield surface to capture the asymmetrical strength effect between the tension and compression, but also affect the shape of yield surface.

7.4 Identification procedure for AZ31 Alloy

In the following sections, we envision applications to sheet forming. It is common practice to assume that the sheet is subjected to a plane stress loading during the entire processing. Their anisotropy, mainly due to the rolling process, can be reasonably approximated within the orthotropic symmetry class. We assume an orthogonal material frame assigned to the sheet with the x -axis along the rolling direction, the y -axis along the transverse direction in the sheet's plane, and the z -axis in the thickness direction. We

let σ denotes the stress tensor, and $\sigma_x, \sigma_y, \sigma_{xy}$, its components with respect to the material frame.

To identify material parameters in the above mentioned constitutive models, a set of experiments should be performed. The most widely experimental tests used are uniaxial tensile at different directions. In the current model, in order to calibrate material parameters related to the SD effect, a series of compressive test also should be performed. The required input variables can be seen as follows:

- Three directional yield stresses obtained from uniaxial tensile tests at specimens having an orientation of $0^\circ, 45^\circ$ and 90° to the original rolling direction of the sheet. The associated yield stresses are denoted here as $\sigma_0^T, \sigma_{45}^T, \sigma_{90}^T$.
- Three directional yield stresses obtained from uniaxial compressive tests at specimens having an orientation of $0^\circ, 45^\circ$ and 90° to the original rolling direction of the sheet. The associated yield stresses are denoted here as $\sigma_0^C, \sigma_{45}^C, \sigma_{90}^C$.
- Three r -values corresponding to $\sigma_0^T, \sigma_{45}^T, \sigma_{90}^T$, which are denoted as $r_0^T, r_{45}^T, r_{90}^T$.

7.4.1 Determination of the yield stress and the anisotropy parameters at different directions

Consider uniaxial loading in the plane (x, y) of the sheet along a direction at angle $\hat{\theta}$ with the rolling direction and denoted by $\sigma_{\hat{\theta}}^T$ and $\sigma_{\hat{\theta}}^C$, the uniaxial tensile and uniaxial compressive yield stresses, respectively. From the tensor transformation rules, the stress components in a uniaxial test specimen with orientation angle $\hat{\theta}$ with respect to the original rolling direction are given as

$$\sigma_{xx} = \sigma_{\hat{\theta}} \cdot \cos^2 \hat{\theta}, \quad (7.14)$$

$$\sigma_{yy} = \sigma_{\hat{\theta}} \cdot \sin^2 \hat{\theta}, \quad (7.15)$$

$$\sigma_{xy} = \sigma_{yx} = \sigma_{\hat{\theta}} \cdot \cos \hat{\theta} \cdot \sin \hat{\theta}, \quad (7.16)$$

where $\sigma_{\hat{\theta}}$ is the yield stress under uniaxial load. When uniaxial tensile test is performed, the $\sigma_{\hat{\theta}}$ is a positive value and denoted as $\sigma_{\hat{\theta}}^T$. If the specimen is under the uniaxial compressive test, the $\sigma_{\hat{\theta}}$ is a negative value, which can be denoted as $-\sigma_{\hat{\theta}}^C$. Therefore, the yield stress at different direction under uniaxial tensile test can be given by

$$\sigma_{\hat{\theta}}^T = \frac{\sigma_{eq}}{\left(a \left| (x_1^T + x_2^T) - c |x_1^T + x_2^T| \right|^M + a \left| (x_1^T - x_2^T) - c |x_1^T - x_2^T| \right|^M + f |2x_2^T|^M \right)^{\frac{1}{M}}}, \quad (7.17)$$

where

$$X_1^T = \frac{(1+h)+(1-h)\cos 2\hat{\theta}}{4}, \quad (7.18)$$

$$X_2^T = \frac{\sqrt{2(h^2+2h+1-4b^2)\cos 4\hat{\theta}+8(1-h^2)\cos 2\hat{\theta}+2(3h^2-2h+2+4b^2)}}{8}. \quad (7.19)$$

The yield stress under uniaxial compressive test can be calculated as follows.

$$\sigma_{\hat{\theta}}^C = \frac{\sigma_{eq}}{\left(a|(X_1^C+X_2^C)-c|X_1^C+X_2^C|^M+a|(X_1^C-X_2^C)-c|X_1^C-X_2^C|^M+f|2X_2^C|^M\right)^{\frac{1}{M}}}, \quad (7.20)$$

where

$$X_1^C = -\frac{(1+h)+(1-h)\cos 2\hat{\theta}}{4}, \quad (7.21)$$

$$X_2^C = \frac{\sqrt{2(h^2+2h+1-4b^2)\cos 4\hat{\theta}+8(1-h^2)\cos 2\hat{\theta}+2(3h^2-2h+2+4b^2)}}{8}. \quad (7.22)$$

The anisotropy parameter $r_{\hat{\theta}}$ is an important parameter to calibrate the yield surface in the sheet metal forming. It is defined as follows:

$$r_{\hat{\theta}} = -\frac{d\varepsilon_w}{(d\varepsilon_w+d\varepsilon_l)}. \quad (7.23)$$

Owing to that the plastic deformation flows along the normal direction of the associated yield surface, and the plastic deformation rate can be written as

$$\dot{\varepsilon}_{xx} = \dot{\gamma} \frac{\partial \Phi}{\partial \sigma_{xx}}, \quad (7.24)$$

$$\dot{\varepsilon}_{yy} = \dot{\gamma} \frac{\partial \Phi}{\partial \sigma_{yy}}, \quad (7.25)$$

$$\dot{\varepsilon}_{xy} + \dot{\varepsilon}_{yx} = \dot{\gamma} \frac{\partial \Phi}{\partial \sigma_{xy}}, \quad (7.26)$$

where $\dot{\gamma}$ is a multiplier factor.

After substitution, the plastic anisotropy is found as follows. For uniaxial tension performed along an arbitrary orientation $\hat{\theta}$ measured counterclockwise from the x -axis, the ratio of the strain rates $r_{\hat{\theta}}$ is defined by:

$$r_{\hat{\theta}} = \frac{\partial \Phi / \partial \sigma_{90+\hat{\theta}}}{\partial \Phi / \partial \sigma_{zz}} = -\frac{\frac{\partial \Phi}{\partial \sigma_{90+\hat{\theta}}}}{\left(\frac{\partial \Phi}{\partial \sigma_{90+\hat{\theta}}} + \frac{\partial \Phi}{\partial \sigma_{\hat{\theta}}}\right)}, \quad (7.27)$$

where

$$\frac{\partial \Phi}{\partial \sigma_{\hat{\theta}}} = \frac{\partial \Phi}{\partial \sigma_{xx}} \cos^2 \hat{\theta} + \frac{\partial \Phi}{\partial \sigma_{yy}} \sin^2 \hat{\theta} + 2 \frac{\partial \Phi}{\partial \sigma_{xy}} \sin \hat{\theta} \cos \hat{\theta}, \quad (7.28)$$

and

$$\frac{\partial \Phi}{\partial \sigma_{90+\hat{\theta}}} = \frac{\partial \Phi}{\partial \sigma_{xx}} \sin^2 \hat{\theta} + \frac{\partial \Phi}{\partial \sigma_{yy}} \cos^2 \hat{\theta} - 2 \frac{\partial \Phi}{\partial \sigma_{xy}} \sin \hat{\theta} \cos \hat{\theta}, \quad (7.29)$$

which leads to

$$r_{\hat{\theta}} = - \frac{\frac{\partial \Phi}{\partial \sigma_{xx}} \sin^2 \hat{\theta} + \frac{\partial \Phi}{\partial \sigma_{yy}} \cos^2 \hat{\theta} - 2 \frac{\partial \Phi}{\partial \sigma_{xy}} \cos \hat{\theta} \sin \hat{\theta}}{\frac{\partial \Phi}{\partial \sigma_{xx}} + \frac{\partial \Phi}{\partial \sigma_{yy}}}, \quad (7.30)$$

where $\hat{\theta} = 0^\circ, 45^\circ$ and 90° , and Φ is the yield function.

$$r_0 = \frac{fh}{[a(1-c)^M + f] - fh}; \quad (7.31)$$

$$r_{90} = \frac{f}{[a(1-c)^M + f]h - f}. \quad (7.32)$$

7.4.2 Calibration procedure

The identification of the constitutive models parameters can be formulated as an optimization task. Material parameters in the constitutive model are interdependent and are difficult to quantify. Therefore, a mathematical optimization model should be built. Generally, the optimization problem corresponding to the parameters identification of experimental values can be formulated as follows: determine material parameters a, b, h, c of a given constitutive law so as to minimize the summation the squares of the errors of differences between the measured experimental data and computed results at the given calibrated points of experimental domain. In this study, the measured experimental data include stress values and r values at different stress states at a certain deformation level. In order to avoid the order effect difference between stress and r value, each item is normalized; consequently, the final objective function (Error function) could be written as

$$Error(\underline{x}) = \sum_i w_i \left(\frac{\sigma_i^T}{\sigma_i^{Tth}} - 1 \right)^2 + \sum_j w_j \left(\frac{\sigma_j^C}{\sigma_j^{Cth}} - 1 \right)^2 + \sum_k w_k \left(\frac{r_k^T}{r_k^{th}} - 1 \right)^2, \quad (7.33)$$

where i is the number of experimental tensile yield stresses, j is the number of experimental compressive yield stress, k is the number of experimental tensile Lankford coefficients, while w_i, w_j, w_k are weights given to the respective experimental values. In this chapter, all of these weights coefficient are fixed to 1.0. The scalars $\sigma_i^{Tth}, \sigma_j^{Cth}$ and r_k^{th} are the theoretical stress values corresponding to the related experimental value respectively. Owing to that the classical gradient method is strongly dependent on the initial value; it is easy trapped into local minima once a wrong starting value is chosen. Stochastic evolutionary methods can overcome local minima; however, they very often are not effective with respect to convergence. In order to avoid this problem, a hybrid scheme was developed. The hybrid algorithm was obtained using first the evolutionary Particle Swarming optimization (PSO) algorithm to obtain a point near the global

minimum, and then a gradient method was performed to obtain the minima. The calibration detail could be seen in Chapter 6.

7.5 Results and discussion of calibrated yield surfaces

Akhtar S. Khan *et al.* [33], reported results of an experimental investigation of the monotonic mechanical response of a commercial AZ31 magnesium alloy sheet in tension, compression, and simple shear at large strains for different strain rates and temperatures. The characterization of the in-plane anisotropy of the sheet was done by performing uniaxial tensile and compression tests at the rolling direction (RD), at 45° to rolling direction (DD), and at the transverse direction. The material displays significant asymmetric yield and hardening behavior.

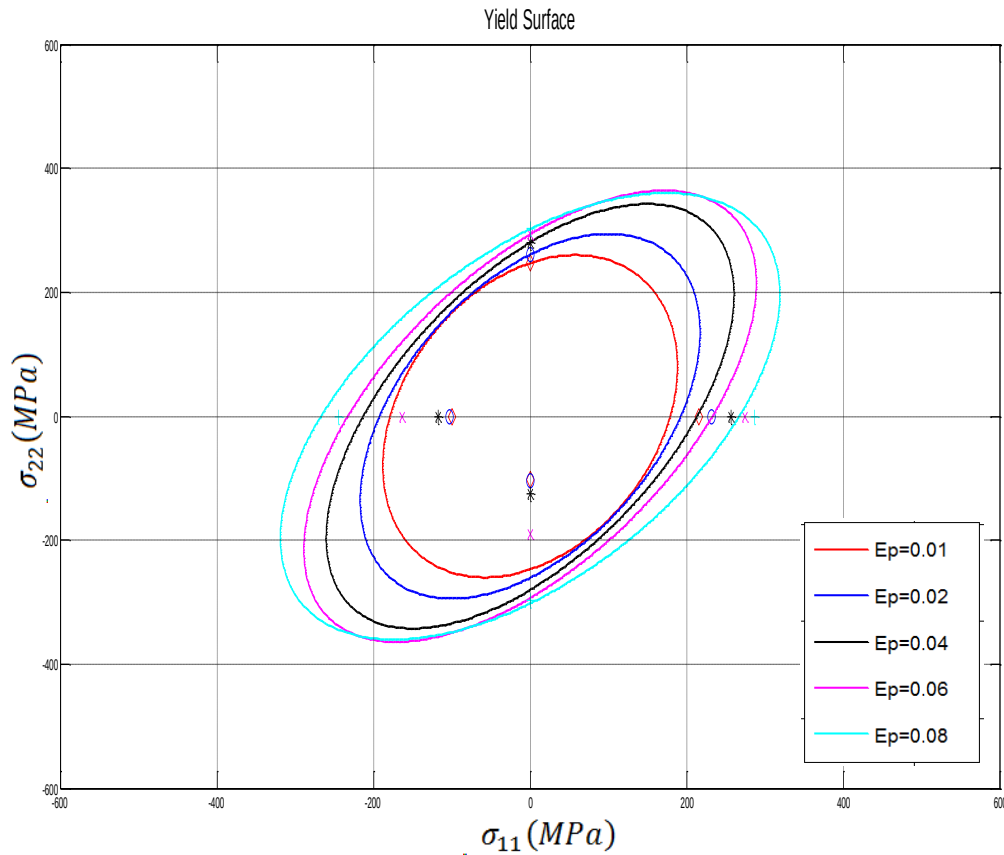


Figure 7.4: Yield surface evolution—Hill 48 yield criterion.

In this study, all the experimental data used to identify the material parameters for the constitutive model is extracted from Akhtar *et al.* work [33]. These data are applied to calibrate the above mentioned yield criteria. In order to evaluate the related evolution during deformation, the yield surface will be drawn in five deformation stage, the plastic strain of which is 0.01, 0.02, 0.04, 0.06, and 0.08.

Figure 7.4 shows the yield surface evolution calibrated by Hill 48 yield criterion. Owing to that the anisotropic parameter r value is not constant, since it evolves during the continuous deformation. The material parameters in Hill 48 criterion are calibrated for these five equivalent plastic strain levels respectively. From there, it is clearly seen that the yield surface cannot pass through all experimental points. It exists a large difference gap between experimental data and yield surface under compressive stress state. It means that Hill yield criterion cannot fit the experimental results.

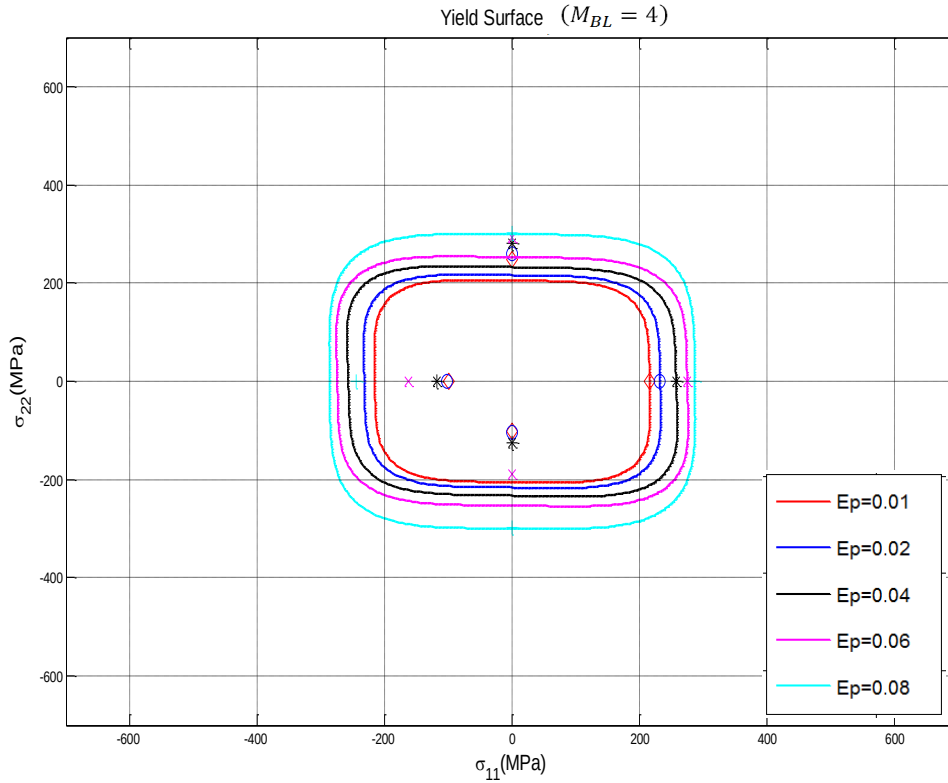


Figure 7.5: Yield surface evolution—Barlat 89 yield criterion.

Figure 7.5 shows the yield surface evolution calibrated by Barlat 89 yield criterion with power index $M_{BL} = 4$ for these five equivalent plastic strain levels. From there, it is clearly seen that the yield surface passes through all experimental points under the uniaxial tensile stress state at both rolling and transverse direction. It indicates that the Barlat 89 model is better to characterize the anisotropic mechanical behavior than Hill 48 yield criterion. However, this model still keep a large difference gap between experimental data and yield surface under compressive stress state.

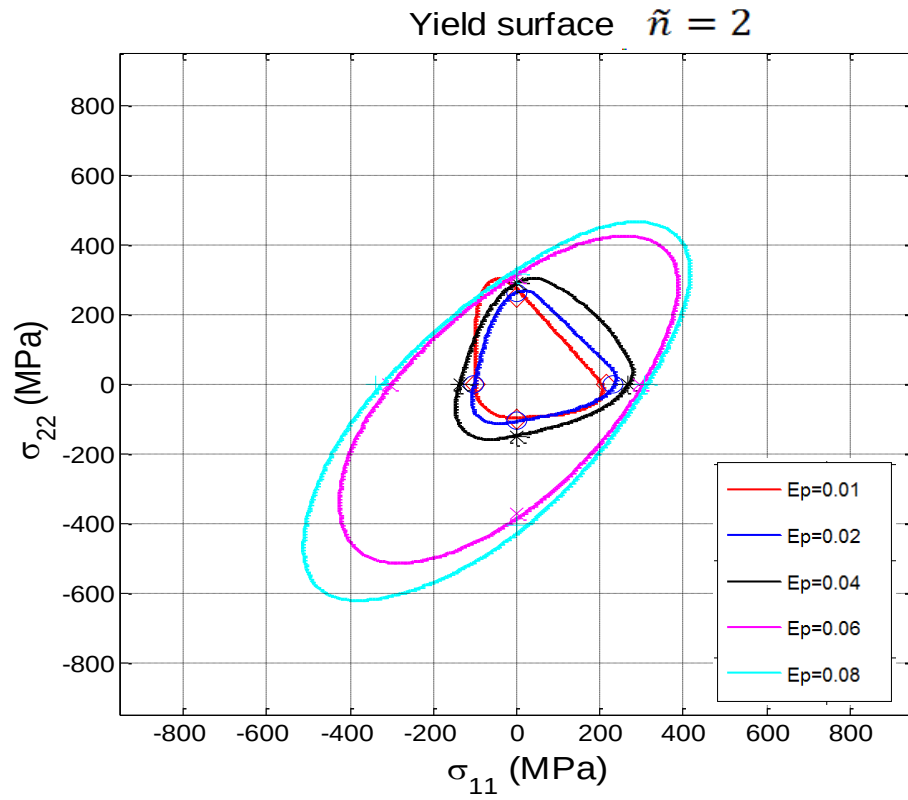


Figure 7.6: Yield surface evolution—Cazacu 06 yield criterion with $\tilde{n}=2$.

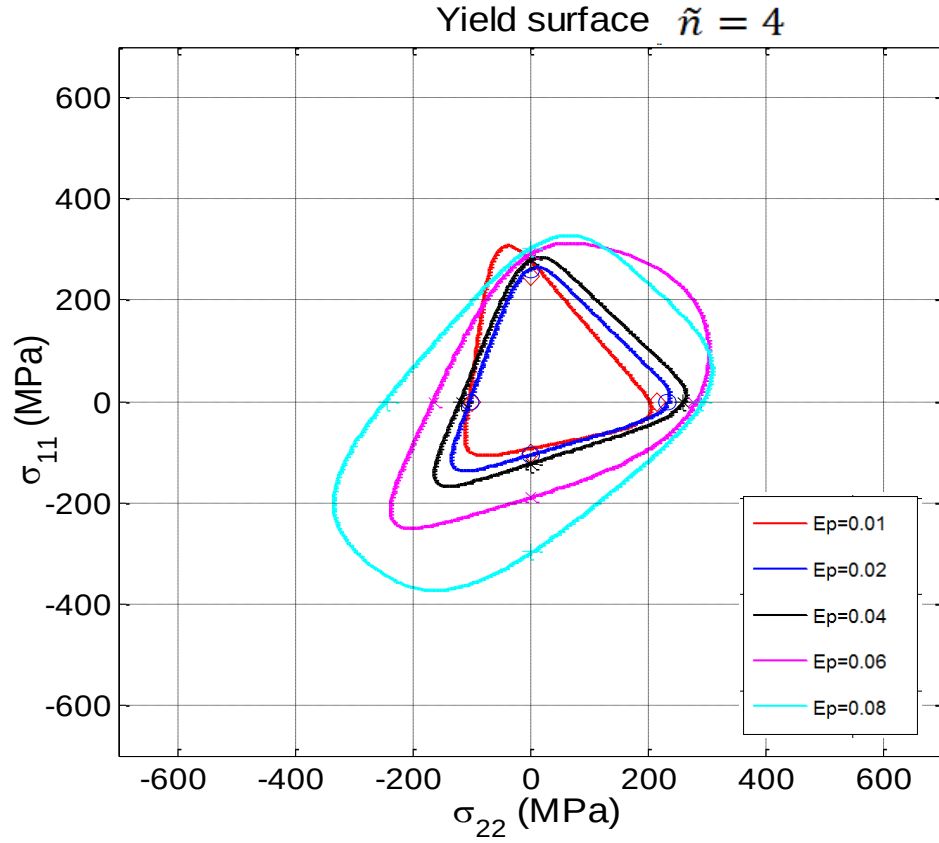


Figure 7.7: Yield surface evolution—Cazacu 06 yield criterion with $\tilde{n}=4$.

Cazacu 06 is a flexible 3-Dimensional yield criterion, which not only can capture the anisotropic mechanical behaviour, but also reflect the strength difference (SD) effect. Under the plane stress condition, the yield function may be considered as a function of the in-plane stress σ_{11} , σ_{22} , σ_{12} , σ_{21} . Therefore, anisotropy material parameters C_{55} and C_{66} is not necessary to calibrate. This model has already been used to calibrate the material with Hexagonal Close-Packed (HCP) structure, such as pure titanium, titanium alloy, among others. Several researchers investigate the calibrated yield surface for magnesium with different power indexes. In this study, and in order to establish a comparison later on, two power indexes are adopted, one is the power index $\tilde{n} = 2$, and the other one is the power index $\tilde{n} = 4$. The yield surface evolution of Cazacu 06 constitutive model with these two power index during deformation are presented in Figure 7.6 and Figure 7.7, respectively.

From there, it is clear seen that the yield surface calibrated by Cazacu 06 can pass through all given experimental points. However, at the small deformation, the yield surface is almost closed to a triangular-like shape. It gives an underestimated value for the biaxial tensile stress state. This also occurs in the large power index ($\tilde{n} = 4$).

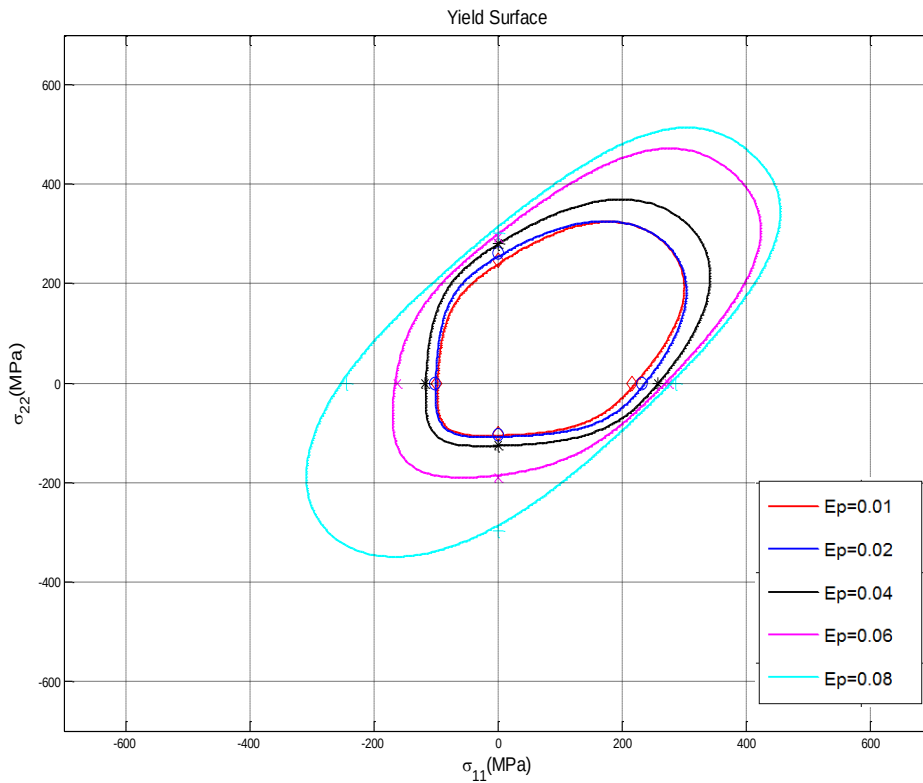


Figure 7.8: Yield surface evolution—the proposed model.

Figure 7.8 demonstrates the yield surface evolution calibrated by the proposed yield criteria with the power index $M = 4$. From there, it can be seen that the calibrated yield surface also pass through all the given experimental points. The yield surface evolves from the pentagon-like shape to the elliptical shape with the increase of the plastic deformation. These results can be explained by the Figure 7.9.

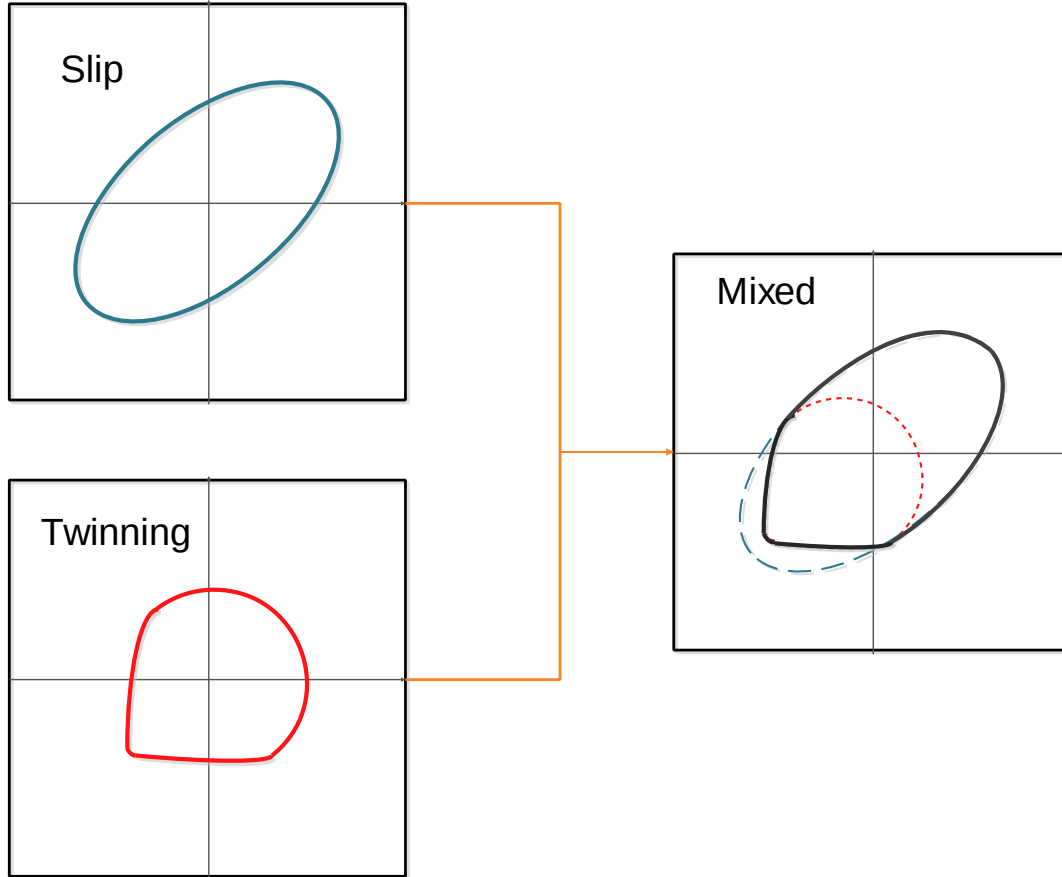


Figure 7.9: Schematic illustration of the yield surface decomposition by deformation mechanism.

As commonly known, the yield surface is strongly dependent on the deformation mechanism. The yield surface dominated by the slip deformation mechanism always demonstrates the elliptical-like shape. The yield surface dominated by deformation twinning shows a triangular-like shape. For the magnesium alloy, it demonstrates complex mechanical behavior. Numerous experiments demonstrate that magnesium alloy led to plastic deformation under tensile stress state by the slip mechanism, however, under the compressive stress state, it is dominated by deformation twinning. Therefore, the ideal yield surface should be the combined surface of these two typical yield surface shapes. In the Figure 7.9, the blue elliptical yield surface is used to denote the slip dominated mode, the red sector yield surface denotes the twinning dominated mode. In view of the above deformation mechanism analysis, the mixed surface consists

of two parts, one being the elliptical yield surface in the positive side, the other being the sector like shape in the negative side, which is denoted by the black solid line. The yield surface calibrated by the proposed yield criteria presents high similarities with the analyzed schematically shape. The results also point out that this proposed yield criteria is more accurate than the other yield criteria.

In order to establish a comparison of these models, besides the comparison of the yield locus geometry, the description of the planar distribution for uniaxial yield stress must also be addressed. The planar distribution of the uniaxial yield stresses predicted by the models is presented in Figure 7.10. As one may notice, both of the predicted results from the proposed yield criterion and Cazacu 06 yield criterion are close to the experimental data. Meanwhile, the predictions of the Hill 48 and Barlat 89 are in poorer agreement with the experimental data, due to the fact that only the uniaxial yield stress corresponding to the rolling direction is used by their identification procedure. Hence, it is clearly seen that the proposed yield criterion is also flexible enough to describe the anisotropic yield criteria for both tensile and compressive loadings. Compared with the Cazacu 06 model, the proposed yield criterion requires less material parameters to be identified.

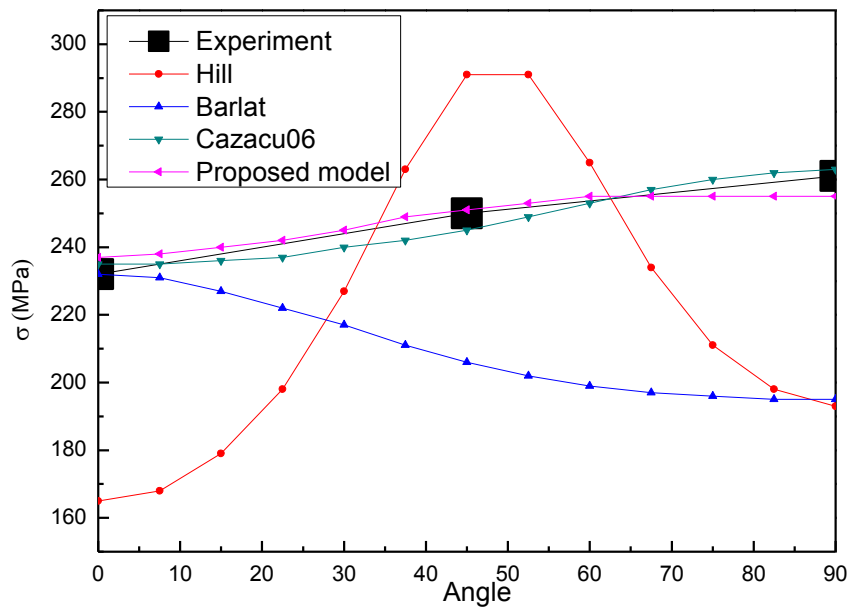


Figure 7.10: Uniaxial yield stress predicted by different yield criteria versus experimental data.

7.6 Integration algorithm under plane stress

7.6.1 Stress and state variable updates

In sheet metal forming, the plane stress assumption is typically introduced in the analysis of bodies in which one of the dimensions – the thickness – is much smaller than the others and is subjected to loads that generate dominant stresses perpendicular to the thickness direction. In order to incorporate the constitutive models described in section 7.2 into an elasto-plastic FE analysis for sheet metal forming, user material (UMAT) subroutines under plane stress state were developed and employed in conjunction with the FE code ABAQUS/Standard. As the three dimensional constitutive model, the subroutine under plane stress is also mainly composed of three parts, which deal with: (1) the updating of stresses and state-dependent variables, (2) the detection of elastic-plastic transition, and (3) the calculation of the tangent modulus. Three-dimensional radial return algorithm can be easily modified for the plane strain problem, but not for problems with additional constraints on stresses, such as plane stress condition. For implicit integration algorithm, three main general approaches may be adopted to deal with the plane stress elastoplastic evolution problem. The first one is direct inclusion of the plane stress constraint into the three dimensional elastic predictor and plastic corrector algorithm equations at each Gauss point. This method requires a nested Newton return mapping iteration for plane stress enforcement; the second method consists in the use the standard three dimensional return mapping at the Gauss point level with plane stress condition added as a structural constraint at the global structural level; the third method is the use of plane stress projected constitutive equations. The plane stress constraint is enforced at the Gauss point level. In this section, the implementation of the third method is summarized for the general case.

Within the infinitesimal elasto-plastic finite element context, the elastic strains are usually much smaller than the plastic strains, and hence an additive decomposition of the total strain rate $\dot{\boldsymbol{\epsilon}}$ into an elastic part $\dot{\boldsymbol{\epsilon}}^e$ and a plastic part $\dot{\boldsymbol{\epsilon}}^p$ is usually considered. Thus, the constitutive equations can be written in a rate form as

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p. \quad (7.34)$$

For the plane-stress state, such as in the case of shell or membranes elements formulations, the strain component normal to the element in-plane local directions(x, y), that is, ϵ_{33} , is not defined from the kinematics but by the restriction $\sigma_{33} = 0$. However, when a shell element is used, ABAQUS generates the transverse stress components (σ_{13}

and σ_{23}) outside the UMAT based on Mindline assumption with elasticity theory. Thus, it is not necessary to include the out-of-plane items during the elasto-plastic stress integration procedure. Isotropic elastic behavior is assumed until yielding is reached. Therefore, the elastic stress-strain relationship is given by the Hooke's law, which can be written as

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}^e : \dot{\boldsymbol{\varepsilon}}^e, \quad (7.35)$$

where, $\mathbf{D}^e$ denotes the fourth order elasticity tensor, which for the plane-stress condition is written as:

$$\mathbf{D}^e = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \quad (7.36)$$

where E is the elastic modulus and ν is Poisson's ratio. Stress and strain tensors ($\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$) are stored in vector form for the sake of simplicity and efficiency:

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \quad (7.37)$$

and

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12}(= 2\varepsilon_{12}) \end{Bmatrix}. \quad (7.38)$$

The shear strain increments are multiplied with a factor two in order to obtain tensor consistency.

It should be noted that the strain components ε_{33} , ε_{33}^e , ε_{33}^p do not enter in the formulation explicitly. The elastic strain ε_{33}^e can be calculated by the static assumption of ($\sigma_{33} = 0$) that defines constraints for the resulting boundary-value problem, which are incorporated in advance in the representation of the tensors in the constitutive equations, which can be given by

$$\varepsilon_{33}^e = -\frac{\nu}{1-\nu}(\varepsilon_{11}^e + \varepsilon_{22}^e) = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}). \quad (7.39)$$

On the other hand, based on the assumption of constancy of volume the through thickness plastic strain ε_{33}^p can be expressed as

$$\varepsilon_{33}^p = -(\varepsilon_{11}^p + \varepsilon_{22}^p). \quad (7.40)$$

Therefore, the total strain component ε_{33} can be finally given by

$$\varepsilon_{33} = \left(-\frac{\nu}{(\varepsilon_{11} + \varepsilon_{22})} - (1 - 2\nu)(\varepsilon_{11}^p + \varepsilon_{22}^p) \right) / (1 - \nu). \quad (7.41)$$

For a shell element, the thickness is updated based on Equation (7.41) after the stress integration procedure. So, the equilibrium equation is still solved under a rigid-normal assumption of a shell element (no thickness change).

The incremental solution of the initial boundary-value problem for elasto-plasticity problems requires a discrete time integration of the constitutive equations of the model over a time increment. The stress and updated variables, which characterize the inelastic response of the material, are pursued for a given strain increment and the previous values of the internal variables. The numerical integration of elasto-plasticity constitutive equations is typically carried out by means of the so-called elastic predictor return mapping schemes, which is called Closest Point Projection Method (CPPM). In the algorithm, a trial stress is made to relax to the closest point on the yield surface. A summary of the implicit CPPM, developed in this work, is presented in Box 7.1.

Box 7.1: Fully implicit closest point projection method (CPPM).

(i) Elastic predictor. Given $\Delta \boldsymbol{\varepsilon}$ and the state variables at t_n , evaluate the elastic trial state:

$$\begin{aligned}\boldsymbol{\varepsilon}_{n+1}^{e\,trial} &= \boldsymbol{\varepsilon}_n^e + \Delta \boldsymbol{\varepsilon}; \bar{\boldsymbol{\varepsilon}}_{n+1}^p = \bar{\boldsymbol{\varepsilon}}_n^p \\ P_{n+1}^{trial} &= K \boldsymbol{\varepsilon}_{n+1}^{e\,trial}; \mathbf{S}_{n+1}^{trial} = 2G \boldsymbol{\varepsilon}_{n+1}^{e\,trial}; \\ \boldsymbol{\sigma}_{n+1}^{trial} &= \mathbf{S}_{n+1}^{trial} + P_{n+1}^{trial} \mathbf{I}.\end{aligned}$$

Hill 48 yield criterion:

$$\sigma_{eq\,n+1}^{trial} = \left[F_{Hill} (\sigma_{22}^{trial} - \sigma_{33}^{trial})^2 + G_{Hill} (\sigma_{33}^{trial} - \sigma_{11}^{trial})^2 + H_{Hill} (\sigma_{11}^{trial} - \sigma_{22}^{trial})^2 + 2N_{Hill} (\sigma_{12}^{trial})^2 \right]^{1/2};$$

Barlat 89 yield criterion:

$$\begin{aligned}K_1 &= \frac{\sigma_{11}^{trial} + h_{BL} \sigma_{22}^{trial}}{2}; K_2 = \sqrt{\left(\frac{\sigma_{11}^{trial} - h_{BL} \sigma_{22}^{trial}}{2} \right)^2 + (b_{BL})^2 (\sigma_{12}^{trial})^2}, \\ \sigma_{eq\,n+1}^{trial} &= \left(\frac{a_{BL} |K_1^{trial} + K_2^{trial}|^{M_{BL}} + a_{BL} |K_1^{trial} - K_2^{trial}|^{M_{BL}} + (2 - a_{BL}) |2K_2^{trial}|^{M_{BL}}}{2} \right)^{1/M_{BL}};\end{aligned}$$

Cazacu 06 yield criterion:

$$\begin{aligned}\boldsymbol{\Sigma}_{n+1}^{trial} &= \tilde{\mathbf{L}} \boldsymbol{\sigma}_{n+1}^{trial}; \\ J_{2\,n+1}^{0\,trial} &= \frac{1}{2} tr(\boldsymbol{\Sigma}_{n+1}^{trial^2}); J_{3\,n+1}^{0\,trial} = \frac{1}{3} tr(\boldsymbol{\Sigma}_{n+1}^{trial^3}); \\ \cos(3\tilde{\theta}) &= \frac{J_{3\,n+1}^{0\,trial}}{J_{2\,n+1}^{0\,trial}} \left(\frac{3}{J_{2\,n+1}^{0\,trial}} \right)^{3/2};\end{aligned}$$

$$\Sigma_1^{trial} = 2\cos(\tilde{\theta})\sqrt{\frac{J_{2n+1}^{0trial}}{3}} + \frac{p_{n+1}^{trial}}{3};$$

$$\Sigma_2^{trial} = 2\cos\left(\tilde{\theta} - \frac{2\pi}{3}\right)\sqrt{\frac{J_{2n+1}^{0trial}}{3}} + \frac{p_{n+1}^{trial}}{3},$$

$$\Sigma_3^{trial} = 2\cos\left(\tilde{\theta} + \frac{2\pi}{3}\right)\sqrt{\frac{J_{2n+1}^{0trial}}{3}} + \frac{p_{n+1}^{trial}}{3};$$

$$\sigma_{eq\ n+1}^{trial} = \left[(|\Sigma_1^{trial}| - \tilde{k}\Sigma_1^{trial})^{\tilde{n}} + (|\Sigma_2^{trial}| - \tilde{k}\Sigma_2^{trial})^{\tilde{n}} + (|\Sigma_3^{trial}| - \tilde{k}\Sigma_3^{trial})^{\tilde{n}} \right]^{\frac{1}{\tilde{n}}};$$

New proposed yield criterion:

$$K_1 = \frac{\sigma_{11}^{trial} + h\sigma_{22}^{trial}}{2}; K_2 = \sqrt{\left(\frac{\sigma_{11}^{trial} - h\sigma_{22}^{trial}}{2}\right)^2 + b^2(\sigma_{12}^{trial})^2},$$

$$\sigma_{eq\ n+1}^{trial} = \left(a \left| (K_1^{trial} + K_2^{trial}) - c|K_1^{trial} + K_2^{trial}| \right|^M + a \left| (K_1^{trial} - K_2^{trial}) - c|K_1^{trial} - K_2^{trial}| \right|^M + f|2K_2^{trial}|^M \right)^{1/M};$$

(ii) Check Plastic admissibility

$$\text{If } \Phi = \sigma_{eq\ n+1}^{trial} - \sigma_Y(\bar{\varepsilon}_{n+1}^{p\ trial}) \leq 0$$

Then set $(\)_{n+1} = (\)_{n+1}^{trial}$ and Exit

(iii) Return mapping. Solve the system of five equations using the Newton-Raphson iterative method

$$\begin{cases} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\ trial} + \Delta\gamma \mathbf{N}_{n+1} \\ \bar{\varepsilon}_{n+1}^p - \bar{\varepsilon}_n^p - \Delta\gamma \\ \Phi = \sigma_{eq} - \sigma_Y \end{cases} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

where

$$\mathbf{N}_{n+1} = \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}}$$

The solution is found for $\boldsymbol{\varepsilon}_{n+1}^e$, $\bar{\varepsilon}_{n+1}^p$ and $\Delta\gamma$. The stress tensor can be obtained by

$$\boldsymbol{\sigma}_{n+1} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^e$$

(iv) Exit

This integration is carried out locally at each Gauss point in typical finite-element implementations. At each integration point, a time discretization or to be more precise a semi- discretization is introduced. At time interval $[t_n, t_{n+1}]$, all variables $\{\boldsymbol{\varepsilon}_n^e, \bar{\varepsilon}_n^p, \boldsymbol{\sigma}_n\}$ at time t_n are assumed to be known, given the strain history from t_n up to another time t_{n+1} , and a return mapping algorithm has to be used to update the stress for a finite increment of loading. The implicit stress integration for each time increment step can be

divided into two steps. First, the elastic equations are integrated with total strain increments to obtain an elastic predictor. Zero plastic strain increment is assumed (elastic predictor step). From an estimate of the total strain increment, an elastic trial stress is calculated. If this trial stress satisfies the Kunh-Tucker conditions it is assumed to be the actual stress and the material is in the elastic state. If not, the material is in the plastic domain. Second step is the plastic corrector step: the elastically predicted stresses (the trial stress) is projected back onto the yield stress in the direction of the closest projection by correcting, iteratively, the plastic strain increments. All stresses, strains, state variables are represented in the local material coordination. Also, it is convenient to use a co-rotational coordinate system in which the basic system rotates with the material. The overview of these implicit stress update scheme is presented in Box 7.1.

The closest point projection method (CPPM) requires solving a system of non-linear equation questions, such as robustness and rate of convergence of the underlying algorithms. The detailed update scheme is described as follows.

By assuming that the input discrete strain increments are all in elastic regime for a given n -th time step, a trial elastic strain and trial stress can be determined as:

$$\boldsymbol{\varepsilon}_{n+1}^{e\,trial} = \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon}, \quad (7.42)$$

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{D}^e \boldsymbol{\varepsilon}_{n+1}^{e\,trial}, \quad (7.43)$$

where $\boldsymbol{\sigma}_{n+1}^{trial}$ is the initial stress for the $n + 1$ time increment step; $\boldsymbol{\varepsilon}_n^e$ is the elastic strain tensor from last equilibrium state; $\mathbf{D}^e$ is the fourth order elastic modulus matrix under plane stress. As regard to the unknown $(\boldsymbol{\varepsilon}_{33}^e)_{n+1}^{trial}$, one starting point can be the previously converged out-of-plane elastic strain, which can be set as

$$\boldsymbol{\varepsilon}_{33}^{e\,trial} = (\boldsymbol{\varepsilon}_{33}^e)_n. \quad (7.44)$$

The trial internal variable is equal to the equivalent plastic strain in the last equilibrium state, which can be written as

$$\bar{\varepsilon}_{n+1}^{p\,trial} = \bar{\varepsilon}_n^p, \quad (7.45)$$

If the yield condition is satisfied with the trial state value

$$\sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^p) - \sigma_Y(\bar{\varepsilon}_{n+1}^p) < 0, \quad (7.46)$$

the deformation of the material at the step $n + 1$ is in elastic region. The entire updated variable for the current increment step is equal to the trial state. It could be written as

$$\boldsymbol{\varepsilon}_{n+1}^e = \boldsymbol{\varepsilon}_{n+1}^{e\,trial}, \quad (7.47)$$

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_{n+1}^{p\,trial} = \bar{\varepsilon}_n^p, \quad (7.48)$$

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p, \quad (7.49)$$

$$\boldsymbol{\sigma}_{n+1}^e = \boldsymbol{\sigma}_{n+1}^{e\,trial}, \quad (7.50)$$

If the inequality Equation (7.46) is violated, the deformation goes into the elasto-plastic domain, and the trial elastic stress is taken as an initial value for the solution of the plastic corrector problem. The plastic strain increments are obtained such that the following new trial stress stays on the new yield surface:

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1}^{trial} - \mathbf{D}^e \Delta \boldsymbol{\varepsilon}^p, \quad (7.51)$$

In the $n + 1$ step, the trial stress should be finally returned back to the yield surface. The incremental effective plastic strain, $\Delta \bar{\varepsilon}_{n+1}^p$, satisfies the equilibrium state. Hence, the following yielding condition holds

$$\sigma_{eq}(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_n^p + \Delta \bar{\varepsilon}^p) - \sigma_Y(\bar{\varepsilon}_n^p + \Delta \bar{\varepsilon}^p) = 0. \quad (7.52)$$

As deduced in the last section, it is possible to calculate the internal variable by

$$\bar{\varepsilon}_{n+1}^p = \bar{\varepsilon}_n^p + \Delta \gamma. \quad (7.53)$$

Equations (7.51)-(7.53) are a set of non-linear equations for the three unknowns $(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^p, \Delta \gamma_{n+1})$. It can be written as

$$\bar{\mathbf{r}}(\boldsymbol{\chi}) = 0, \quad (7.54)$$

where

$$\bar{\mathbf{r}} = \begin{Bmatrix} \mathbf{r}_\sigma \\ r_{\bar{\varepsilon}^p} \\ r_{\Delta \gamma} \end{Bmatrix}; \quad \boldsymbol{\chi} = \begin{Bmatrix} \boldsymbol{\sigma}_{n+1} \\ \bar{\varepsilon}_{n+1}^p \\ \Delta \gamma_{n+1} \end{Bmatrix}, \quad (7.55)$$

denote the residual and solution vectors, respectively.

The upper boundary condition for this non-linear boundary problem is given by the initial trial stress while the lower boundary is determined from the equilibrium state variables from last time increment step. The system of three nonlinear algebraic equations can be solved by using different iteration procedures. Generally, the Newton-Raphson iterative algorithm is used to solve the equation set. However, if the material constant M in the proposed yield criteria is of large value, the convergence bowl of the above Newton-Raphson scheme can be dramatically reduced. Such ill-conditioned behaviour stems from the high curvature alteration for large M in the neighbourhood of points in stress space. Under such conditions, small changes in stress on the yield surface result in large changes in flow vector direction, characterizing a set of stiff evolution equations. Therefore, at this study, a line search algorithm is adopted to

stabilize the Newton-Raphson scheme for the range of strain increments expected to be present in practical finite element computations, which is referred to as LNR-CPPM algorithm. In order to derive such an iterative loop, $\bar{\mathbf{r}}$ residuals are defined to represent the differences between the current constitutive variables and the backward-Euler constitutive variables. The iteration procedure for this method is characterized by

$$\boldsymbol{\chi}^{(k+1)} = \boldsymbol{\chi}^{(k)} + \bar{\alpha}^{(k+1)} \Delta \boldsymbol{\chi}^{(k+1)}, \quad (7.56)$$

With

$$\Delta \boldsymbol{\chi}^{(k+1)} = -\left(\bar{\mathbf{J}}^{(k)}\right)^{-1} \bar{\mathbf{r}}^{(k)}, \quad (7.57)$$

where $\bar{\mathbf{J}}$ denotes the Jacobian matrix of the algebraic system, which can be given by

$$\bar{\mathbf{J}} = \begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\epsilon}^p} & \mathbf{N} \\ 0 & 1 & -1 \\ \mathbf{N} & -1 & 0 \end{bmatrix}, \quad (7.58)$$

where $\mathbf{N}_{n+1} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}$, $\Delta\gamma$ is the multiplier.

$\bar{\alpha}$ is the step-length, which can be computed by line search algorithm.

The iterative process is stopped as soon as convergence is achieved within the prescribed convergence criteria, namely, when the norm of residual is less than a small tolerance value. The detail of CPPM algorithm combined with line search strategy can be seen in Chapter 4.

7.6.2 Elasto-plastic tangent operator

In modern computational plasticity, it is now recognized that, in order to achieve the asymptotically quadratic rate of force-balance convergence, it is theoretically possible to use the global Newton-Raphson force balance iterations, material tangent operators that are consistent with the implemented (discrete) form of the constitutive models. This leads to so-called consistent tangent operators which generally differ quite significantly from the continuum elastoplastic tangent moduli which can be derived from the rate form of the constitutive equations and the plastic consistency condition.

If the stress is inside the elastic domain or if it is on the yield surface and elastic unloading is assumed to occur, the tangent modulus $\mathbf{D}$ consistent with any of the integration algorithms has the same form as the fourth order elasticity tensor under plane stress condition, which is given by

$$\mathbf{D} = \mathbf{D}^e \quad (7.59)$$

If materials violate the yield function, a elastoplastic consistent tangent $\mathbf{D}^{ep}$ should be defined.

As previously mentioned in the stress update integration algorithm, three equations should be met after Newton-Raphson iterative. The corresponding return mapping equations are rewritten as follows for the derivation of consistent tangent modular matrices conveniently.

$$\begin{cases} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\,trial} + \Delta\gamma \mathbf{N}_{n+1} \\ \bar{\varepsilon}_{n+1}^p - \bar{\varepsilon}_n^p - \Delta\gamma \\ \Phi(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^p) \end{cases} = \begin{cases} \mathbf{0} \\ 0 \\ 0 \end{cases}. \quad (7.60)$$

Thus, the first step in its derivation is to linearize the return mapping equations (having the elastic trial strain – the system input – also as a variable). Straightforward differentiation of the general return-mapping equations yields the following linearized form:

$$\begin{cases} d\boldsymbol{\varepsilon}^e + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^p} \cdot d\bar{\varepsilon}^p + d\Delta\gamma \mathbf{N} \\ d\bar{\varepsilon}^p - d\Delta\gamma \\ \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial \Phi}{\partial \bar{\varepsilon}^p} \cdot d\bar{\varepsilon}^p \end{cases} = \begin{cases} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{cases}. \quad (7.61)$$

Rewriting Equation set (7.61) in the format of matrix product, we obtain

$$\begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^p} & \mathbf{N} \\ 0 & 1 & -1 \\ \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} & \frac{\partial \Phi}{\partial \bar{\varepsilon}^p} & 0 \end{bmatrix} \begin{bmatrix} d\boldsymbol{\sigma} \\ d\bar{\varepsilon}^p \\ d\Delta\gamma \end{bmatrix} = \begin{bmatrix} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{bmatrix}. \quad (7.62)$$

Finally, by inverting the Equation set (7.62), we obtain

$$\begin{bmatrix} d\boldsymbol{\sigma} \\ d\bar{\varepsilon}^p \\ d\Delta\gamma \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} d\boldsymbol{\varepsilon}^{e\,trial} \\ 0 \\ 0 \end{bmatrix}. \quad (7.63)$$

In the previous system, the inverted matrix is expressed in symbolic form.

Obviously, the elastoplastic consistent tangent modulus are:

$$\mathbf{D}^{ep} \equiv \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e\,trial}} = \mathbf{D}_{11}. \quad (7.64)$$

After a series of mathematic deductions, the elastoplastic consistent tangent can be calculated by

$$\mathbf{D}^{ep} = \left\{ [\mathbf{D}^e]^{-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} - \frac{\mathbf{N} \otimes (\mathbf{N} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^p})}{\frac{\partial \Phi}{\partial \bar{\varepsilon}^p}} \right\}^{-1}. \quad (7.65)$$

It is important to remark that this tangent operator is not symmetric.

It should be remarked also that for most shell elements, the default shear stiffness values are not calculated when the material mechanical behavior is computed by a user-defined

subroutine. The transverse shear stiffness should be specified as the initial, linear elastic stiffness of the shell in response to pure transverse shear strains in the element formulation. The transverse shear stiffness is computed by matching the shear response for the case of the shell bending about one axis to a three-dimensional solid element. For a homogeneous shell made of a linear, orthotropic elastic material, the transverse shear stiffness terms are given by:

$$K_{11}^{ts} = \frac{5}{6} G_{13} t; \quad (7.66)$$

$$K_{22}^{ts} = \frac{5}{6} G_{23} t, \quad (7.67)$$

and

$$K_{12}^{ts} = K_{21}^{ts} = 0.0, \quad (7.68)$$

where, G_{13} , G_{23} are the material's shear moduli in the out-of-plane direction. The number 5/6 is the shear correction coefficient that results from matching the transverse shear energy to that for a three-dimensional structure in pure bending.

7.7 Numerical examples

In order to validate the prediction capability of the developed elasto-plastic formulation, two types of numerical examples were considered: single element test, and the circular cup drawing. In all the numerical simulations discussed hereafter, the Voce hardening law was chosen to describe the effective plastic stress-strain hardening relationship, which can be expressed by

$$\sigma_Y = A_V + B_V (1 - e^{-C_V \bar{\epsilon}^p}), \quad (7.69)$$

where A_V , B_V , C_V are the material parameters identified by the related uniaxial tensile test. The scalar $\bar{\epsilon}^p$ denotes the equivalent plastic strain. Material parameters are presented in Table 7.1.

Table 7.1: Material parameters of AZ31B at quasi-static condition.

	ρ (g/cm ³)	E Modulus (GPa)	ν Poisson's Ratio	$\sigma_Y = A_V + B_V(1 - e^{-C_V \bar{\epsilon}^p})$,		
				A_V	B_V	C_V
AZ31	1.77	45	0.35	195.78	117.57	18.46

Owing to that the shape of yield surface changes during the deformation, the anisotropic coefficients are not a constant during deformation, which evolves with the accumulated plastic strain. The anisotropic coefficients at several fixed accumulated plastic strain

levels were calibrated by the hybrid evolutionary optimization method. The yield surface evolution during iterative process can be calculated by a linear interpolation scheme between a discrete numbers of yield surfaces corresponding to fixed levels of accumulated plastic strain. A sufficient number of points were used to ensure that the hardening curve was recreated with enough accuracy. The anisotropy coefficients at the prescribed strain level are presented in Table 7.2.

Table 7.2: Anisotropy coefficient values of AZ31 corresponding to the proposed yield surface evolution.

$\bar{\epsilon}^p$	a	h	b	c
0.01	0.647	0.925	0.929	1.91
0.02	0.739	0.931	0.939	1.94
0.04	0.581	0.925	0.954	1.98
0.06	0.118	0.897	0.944	2.24

7.7.1 Single element tests

One of the element tests is demonstrated the robustness of the numerical scheme of the UMAT. In this section, a simple benchmark test is presented to verify the implementation of the proposed constitutive model within an implicit quasi-static finite element environment.

A uniaxial stress state is applied to a single four-node shell element (with one integration point). The size of the element is 10x10 mm. In order to evaluate the asymmetric mechanical behavior, the uniaxial tensile and uniaxial-compressive tests are performed. The material constants, which are employed in the test, are listed in Table 7.1. The anisotropy coefficient values of magnesium alloy AZ31 corresponding to the yield surface evolution are identified with the proposed description in the previous section, which can be seen in Table 7.2.

Figure 7.11 shows the schematic with boundary conditions and load conditions for a single element model. This particular test was done with the four-node doubly curved reduced integration plane stress elements (ABAQUS element type S4R) with hourglass control. The thickness is set to 1 mm. The transverse shear stiffness is computed by matching the shear response for the case of the shell bending about one axis to a three-dimensional solid element, as mentioned in Equation (7.66)-(7.68).

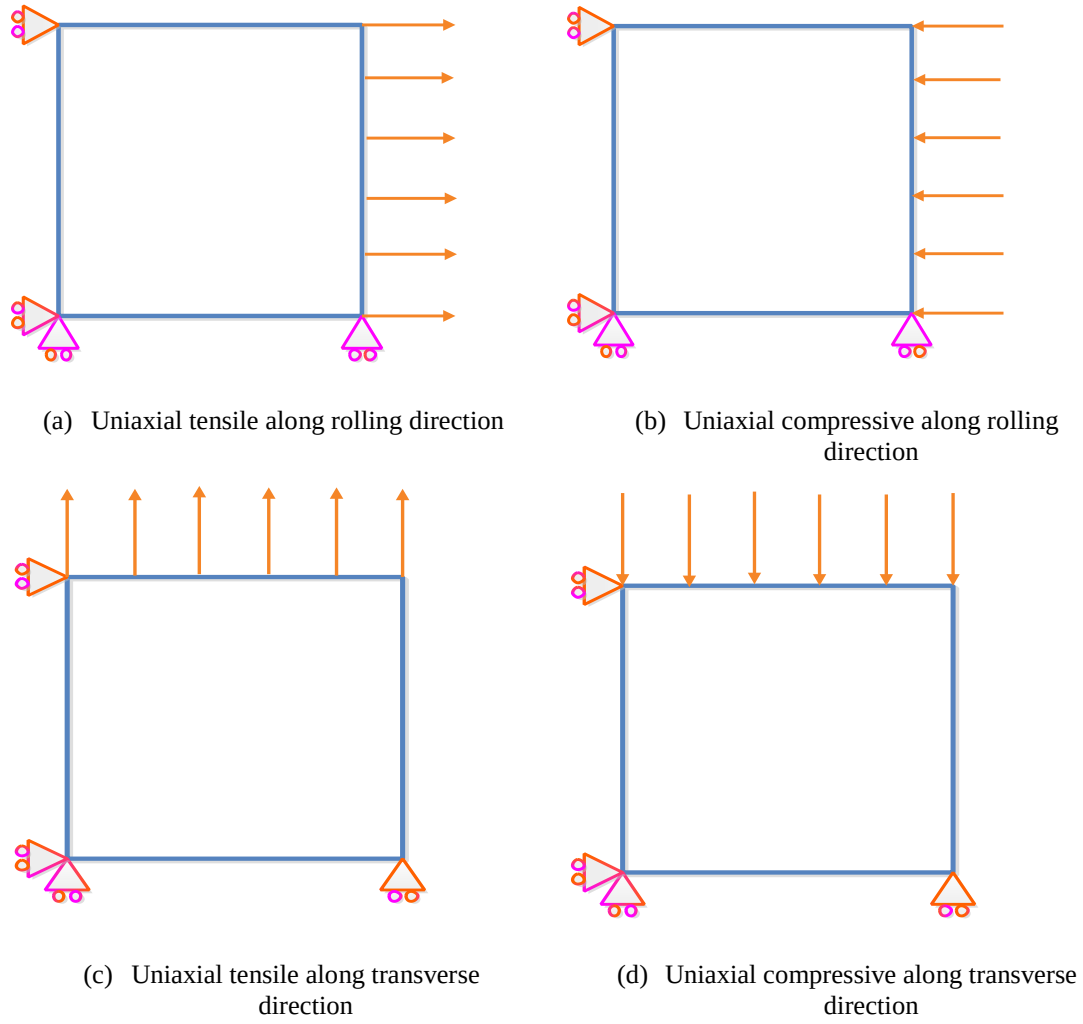


Figure 7.11: Single element tests

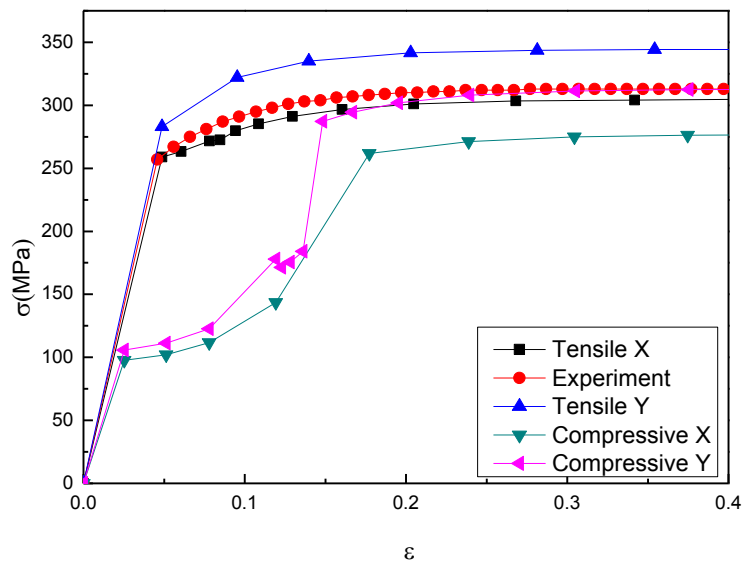


Figure 7.12: Stress-strain curves.

Figure 7.12 shows the stress strain curve obtained from the numerical simulation with the proposed yield criterion. It is clearly seen that there is a strength gap between tension and compression stress state in the small deformation. The material has a different mechanical response in the rolling direction and transverse direction. The stress-strain curve obtained from the proposed approach presents a better agreement with experimental value.

7.7.2 Cup drawing test

In this subsection, the application of the new proposed constitutive model is extended to relatively complex model – a cylindrical cup drawing test.

Deep drawing with circular punch and blank is a popular process in the formability assessment of sheet metals and it verifies the planar anisotropic behaviour through the prediction of the earing profile.

This test was simulated for AZ31 alloy sheet based on the proposed yield function, assuming isotropic hardening. The schematic views of the cup drawing process analyzed in this chapter are presented in Figure 7.13. The dimension of all parts can be seen in Chapter 4.

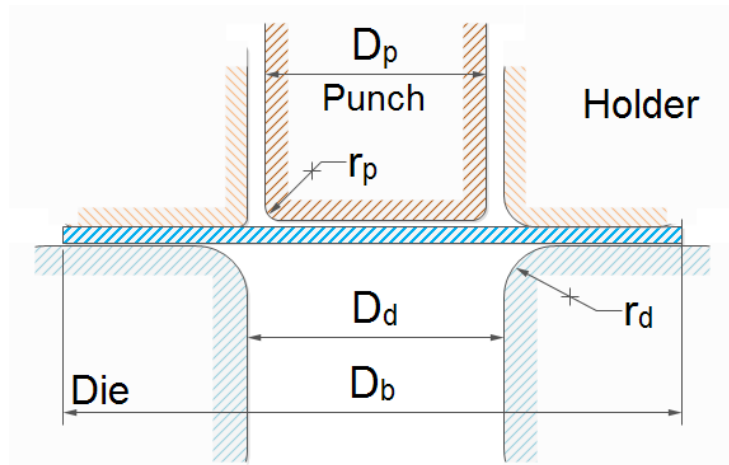


Figure 7.13: Schematic section view of cup test drawing.

The simulations were performed with the implicit code ABAQUS/Standard together with developed user material subroutine UMAT. The full finite element model is demonstrated in Figure 7.14. The tooling is described by discrete rigid surfaces, and meshed with linear quadratic rigid elements (R3D4). The circular blank is meshed with 1045 reduced integrated four-noded shell element (S4R). In order to avoid the occurrence of wrinkling during drawing, the constant blank holder force of 25 kN is applied to the blank. A standard surface-to-surface contact is defined between the punch

and the specimen, where wherein the friction behaviour is described by the Coulomb's law. The coefficient of friction is equal to 0.01.

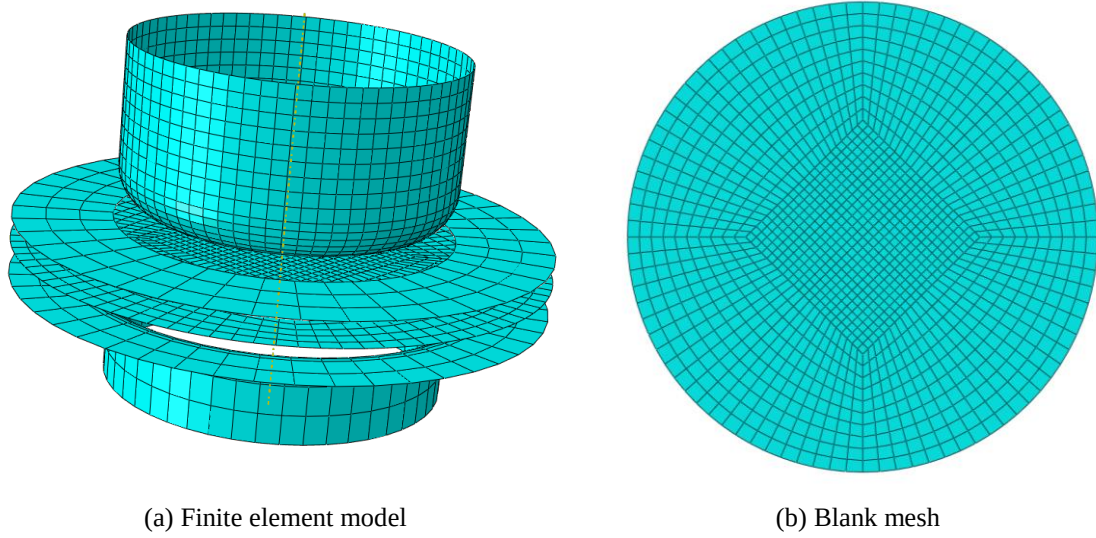


Figure 7.14: Finite element model of cup drawing test.

Figure 7.15 and Figure 7.16 show the equivalent plastic strain contour and equivalent stress contour after the drawing operation, respectively. From there, it can be clearly seen that the material in the transverse direction has larger plastic deformation than in the rolling direction. It might not be easy to see the ear phenomenon; however, an uneven cup rim and non-circular distribution in the equivalent stress contour can still be seen, which indicates that the material has an anisotropic mechanical.

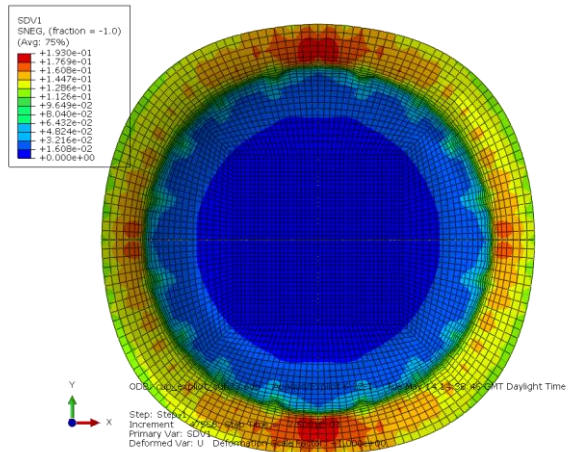


Figure 7.15: Equivalent plastic strain contour.

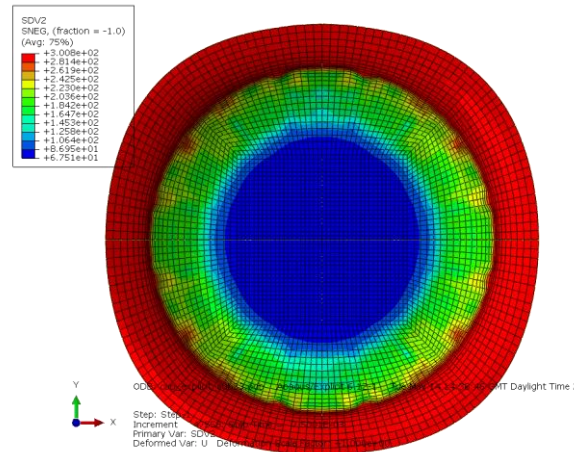
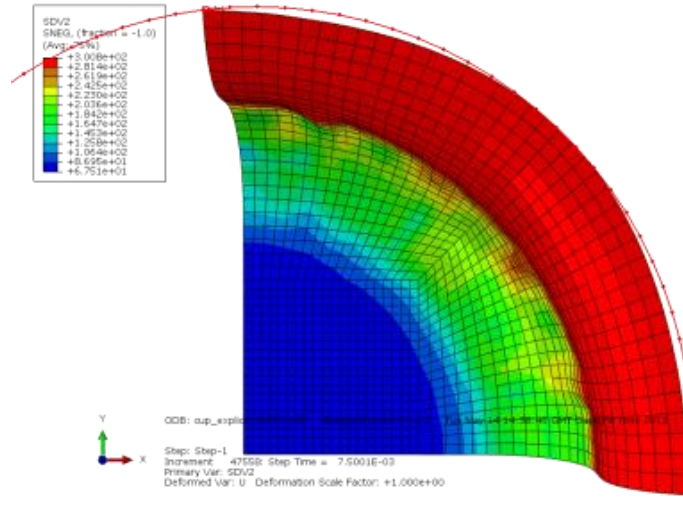


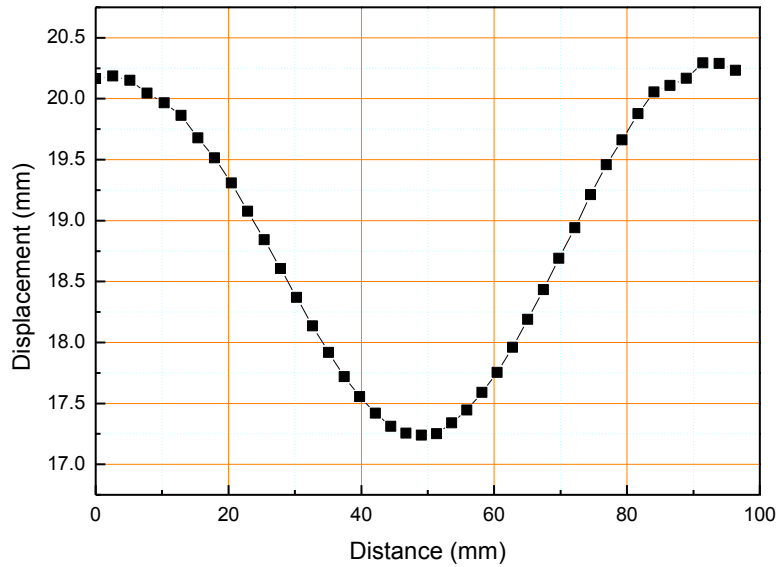
Figure 7.16: Equivalent Stress contour.

The anisotropic behavior can be better enhanced by extracting nodal displacements at the outer flange of cylindrical cup for one quarter of its geometry, as seen in Figure 7.17. It is possible to observe that the smallest displacement is located in the node at the angle

45° with respect to the rolling direction. The drawing displacement in the rolling direction and transverse direction doesn't presents remarkable difference.



(a) Selected node path



(b) Displacement distribution of the selected node path
Figure 7.17: Displacement along the cup edge after stamping.

7.8 Conclusions

In this paper, the new constitutive model is proposed by extending the original Barlat 89 yield criteria to capture anisotropic and SD effect simultaneously. The influence of the introduced material parameter in the new proposed model was assessed. It can be concluded that with the increase of the power index M in the proposed model, the yield surface becomes more rectangular. The introduced parameter c can enhance

the difference between the tension and compression stress state, making it smoother in one side and sharper on the other one. However, this effect is not as obvious when the power index value M is big. After that it is also applied to model the mechanical behavior of magnesium alloy AZ31. By comparison with other commonly used models, such as Hill 48 yield criteria, Barlat 89 model, and Cazacu 06, it proves that the new proposed one presents accurate results. A high level of agreement was found between theoretical and experimental results; hence a substantially high accurate description for the mechanical behaviour of AZ31 alloy is possible, requiring less material parameters than Cazacu 06 model. Finally, a fully implicit integration algorithm based on CPPM algorithm coupled with line search algorithm was implemented within an implicit environment. A simple test was used to validate the model and the simulated results are able to demonstrate that the new proposed model can be effective to reflect the asymmetrical mechanical behavior and anisotropic properties in the real application. A benchmark test-cup drawing was performed to show the good consistency of the implementation, and the simulated results were able to show that the model captures the AZ31B earing phenomenon after drawing, which has a good agreement with experimental evidence.

7.9 References

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Chapter 8

Analytical and Numerical Analysis of Springback for Hexagonal Close-Packed Materials

8.1 Introduction

Springback is a recovery phenomenon needed to be taken into account when processing a variety of metallic parts in automobiles, aircrafts, building products and domestic appliances. Once a deformed sheet-metal part is removed from the dies in which it was formed, the elastic component of strain is recovered, especially where bending, bending-unbending, and reverse bending are performed. The elastic recovery is accompanied by strain and, consequently, it causes the final shape of the part to change. This additional deformation of sheet metal parts after the loading is removed is called springback, which is one of the most critical behavior in sheet metal forming, and a considerable amount of time is needed to compensate for this behavior. Springback is a complex physical phenomenon, which is mainly governed by the stress state obtained at the end of a deformation.

Several methods are available for the prediction of springback [1-4]. These methods can be categorized into two types: analytical methods, on one hand, and the numerical methods with finite element analysis, on the other hand. Some researchers also proposed semi-analytical approaches [5-7]. The usefulness of numerical methods is apparent in their ability to treat complex tool shapes and realistic nonlinear material behavior. However, despite a relatively simplified methodology, analytical methods for predicting the springback phenomenon of sheet materials have the advantage that the process and material effects on the springback can be qualitatively and quickly estimated during the design process. Although not generally applicable, these models are powerful tools that can be used to quickly visualise dependencies between some material and process parameters and the amount of springback. Early analytical solutions for springback of plane stress, pure bending with large radius of curvature (radius of curvature $\bar{\rho}$ =thickness $\bar{t}$) were established for elastic-perfectly plastic material response [8], and were extended to plane-strain pure bending [9], and initially curved cases [10]. Further developments include solutions for plane-strain bending with superimposed tension for arbitrary $\bar{\rho} = \bar{t}$ [11-14] and for plane-stress bending of narrow

strips. This classical analytical approaches are based on pure bending theory without an applied tensile force, and the springback is assumed to consist of elastic unbending on the removal of the applied moment [9, 15]. Later, the effect of the tensile force superimposed on the sheet bending process was considered for perfect plastic [16], rigid power law [17, 18], elastic linear hardening [19], and elastic nonlinear power-law materials [20].

Accurate prediction of sheet springback by finite element method also depends strongly on the material hardening rule [21, 22]. The commonly used plasticity model describes symmetric material behavior, *i.e.*, the stress–strain responses are assumed to be approximately the same both in tension and compression. Obviously, these can not be used to capture the mechanical behavior of magnesium alloy sheets with a strong in-plane texture since these materials have revealed that strong asymmetries exist in both the initial yield stress and subsequent flow stress in continuous tension and compression. Therefore, without a suitable description of the magnesium alloy's asymmetric stress–strain responses, an accurate estimation of the springback in magnesium alloy sheets may not be possible. Although Lee *et al* [23–25] considered the asymmetrical effect and derived analytical bending moment–curvature relationships for various deformation stages, the hardening curve has already been simplified. The derived equation is only available for the linear hardening material.

From section 8.2 to section 8.5 of this chapter, an extended analytical method for arbitrary material is proposed; besides the mechanics of plane strain bending in combination with tension of an elastic-plastic sheet material are considered. Furthermore, the generic analytical deformation region evolution, bending force and bending moment–curvature relationships for various deformation stages are discussed. Sensitivity tests were also conducted for the effect of an applied tensile force, sheet thickness on the springback of asymmetric metals, comparing the conventional symmetric constitutive equations with the asymmetric constitutive equations from our model. From section 8.6 to section 8.9, finite element method is used to simulate the springback. Several constitutive models are implemented and applied to study the benchmark cylindrical bending test.

8.2 Analytical theory of springback

To understand the unusual phenomenon of springback in magnesium alloy sheets, especially the reverse effect of the constrained tensile force, a simple but analytical

bending theory with applied tensile force is extended in this chapter to the asymmetric materials.

Bending a sheet material along a straight line is considered. Owing to that the bend radius is always more than three or four times the sheet thickness, so it could be assumed that a plane strain condition exists in the plane perpendicular to the bending line and that cross-sections remain plane and normal to the mid-surface after the deformation (Kirchhoff hypothesis). A two-dimensional representation of the considered problem is shown in Figure 8.1. During bending, a radial direction $\bar{\rho}$ and a circumferential direction $\bar{\theta}$ can be distinguished. Because of the Kirchhoff hypothesis these directions are the principal strain directions. A unit width of a continuous sheet in which a cylindrical bent region of radius of curvature $\bar{\rho}$ is flanked by flat sheet. The bend angle is $\bar{\theta}$, and a moment per unit width $\bar{M}$, and a tension (force per unit width) $\hat{T}$ are applied.

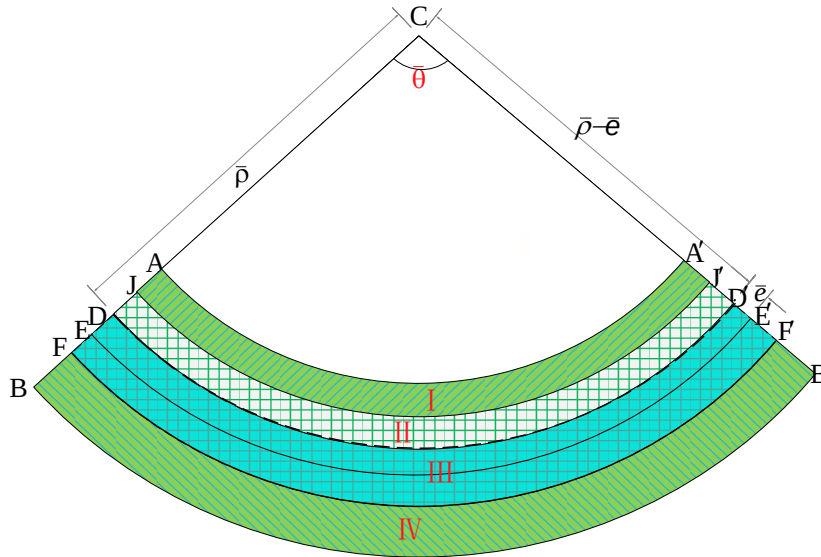


Figure 8.1: A schematic diagram of bending in the longitudinal section.

In this schematic diagram, any cross section perpendicular to the axis of the member remains plane, and the plane of the section passes through point C. The line $A - A'$ along which the upper face of the member intersects the plane of the couples will have a constant curvature. During the bending process, the line $A - A'$, which is originally a straight line, will be transformed into a arc of the circle of center C. Line $D - D'$ is the neutral axis, in which the longitudinal normal strain ε_θ and stress σ_θ are zero, Line $E - E'$ is the center line. Line $J - J'$ is the critical line, where the material initiates the

compressive yield. Line $F - F'$ is the other critical line, where the material initiates the tensile yield.

These characteristic line separated the longitudinal section of the sheet into four regions: the region $A - A' - J' - J$ (also called I region) is the compressive plastic region; the region $J - J' - D' - D$ (also called II region) is the compressive elastic region; the region $D - D' - F' - F$ (also called III region) is the tensile elastic region; lastly, the region $F - F' - B' - B$ (also called IV region) is the tensile plastic region.

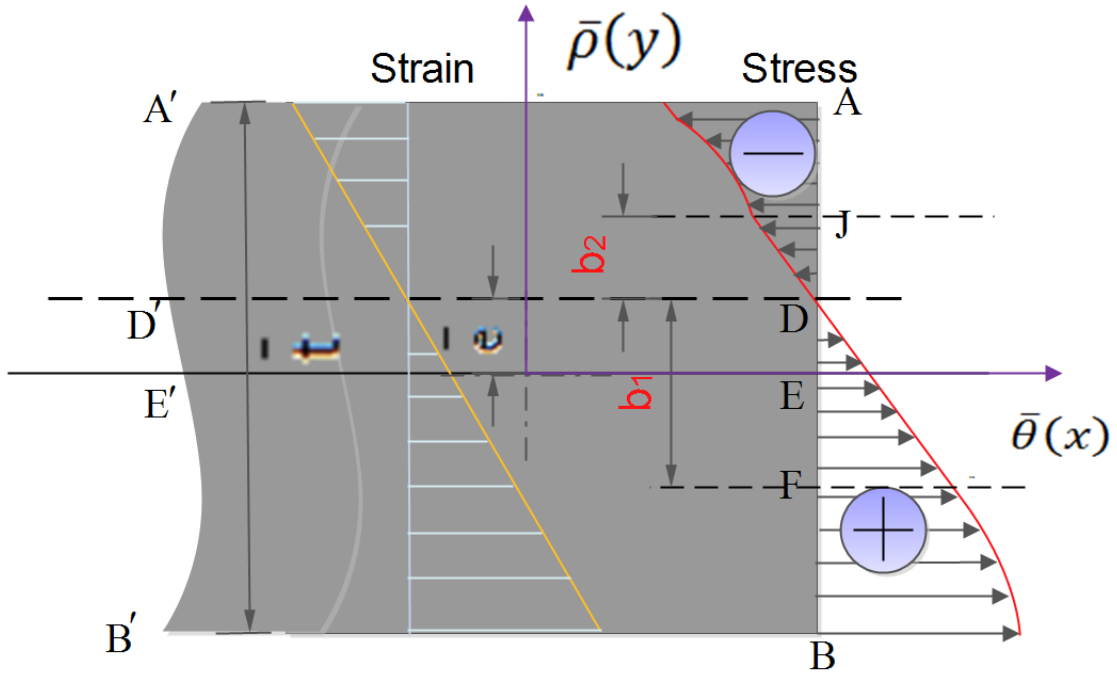


Figure 8.2: The strain and stress distribution diagram of bending in the cross section.

8.2.1 Strain calculation

In order to perform the mechanic analysis for the bending process, the strain and stress distribution diagram in the cross section is drawn in Figure 8.2.

A line segment of initial length L_0 , situated at the distance y above the mid-surface, after bending to an angle $\bar{\theta}$ with a radius of curvature $\bar{\rho}$, will deform to a length L . The radius of curvature in the mid-plane is defined as $\bar{\rho}_m = \bar{R} + \bar{t}/2$, where $\bar{t}$ is the sheet thickness, $\bar{R}$ is the curvature radius of internal surface (AA').

The line segment's length L at position y can be expressed as:

$$L = (\bar{\rho}_m - y)\bar{\theta}. \quad (8.1)$$

The initial length L_0 is equal to

$$L_0 = \bar{\rho}\bar{\theta}, \quad (8.2)$$

where $\bar{\rho}$ is the radius of curvature in the neutral surface plane.

It is assumed that the distance between the neutral surface plane and the mid-plane is $\bar{e}$, so the radius of the curvature in the neutral surface plane can be calculated by

$$\bar{\rho} = \bar{\rho}_m - \bar{e}. \quad (8.3)$$

Therefore, the circumferential true strain is given by:

$$\varepsilon_\theta = \ln\left(\frac{L}{L_0}\right) = \ln\left(\frac{(\bar{\rho}_m - y)\bar{\theta}}{\bar{\rho}\bar{\theta}}\right) = \ln\left(\frac{\bar{\rho} + \bar{e} - y}{\bar{\rho}}\right). \quad (8.4)$$

In the case of large $\bar{R}$, the difference between the true strain and the engineering strain is negligible, and the engineering strain is easier to handle. Thus, The membrane and bending engineering strains can be written as follows:

$$\varepsilon_\theta = \frac{\Delta L}{L_0}. \quad (8.5)$$

Finally, the total engineering circumferential strain becomes:

$$\varepsilon_\theta = \frac{\bar{e} - y}{\bar{\rho}}. \quad (8.6)$$

Because of the requirement that transverse sections remain plane, identical deformations will occur in all planes parallel to the plane of symmetry.

8.2.2 Positioning the elastic and plastic region

As long as the norm circumferential stress σ_θ does not exceed the initial yield strength σ_Y^0 , Hooke's law applies, and the stress distribution across the section is linear. Otherwise, the plastic yield can be initiated either from the compressive region (or above the pseudo-neutral surface) or tensile region (or below the pseudo-neutral surface), depending on the magnitude of the applied tensile force and ratio between the tensile and compressive yield stresses. Let b_1 and b_2 be variables that determine the critical position of yield points in elastic regions where the material is in tension or compression stress state (see Figure 8.2), respectively. These variables are defined in relation to the neutral line which is situated at $y_{nu} = \bar{e}$. Therefore, the coordinate of the critical yield points in tension is:

$$y_T^0 = \bar{e} - b_1, \quad (8.7)$$

where, the subscript T denotes tension.

The variable b_1 can be calculated by using the initial yield stress in the uniaxial tensile stress

$$b_1 = \bar{\rho} \frac{\sigma_T^0}{E}. \quad (8.8)$$

where σ_T^0 is the initial yield strength in the uniaxial tensile test.

The coordinate of the critical yield points in compression can be given by

$$y_C^0 = \bar{e} + b_2, \quad (8.9)$$

where the subscript C denotes compression.

Similary, the variable b_2 can be calculated by using the initial yield stress in the uniaxial compressive stress

$$b_2 = \bar{\rho} \frac{\sigma_C^0}{E}, \quad (8.10)$$

where σ_C^0 is the initial yield strength in the uniaxial compressive test.

The total circumferential strain in the region of plastic deformations can be defined as the sum of two parts – elastic strain and the plastic strain due to the material workhardening:

$$\varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p. \quad (8.11)$$

Rewriting the above equation, the strain due to the material workhardening can be derived:

$$\varepsilon_\theta^p = \varepsilon_\theta - \varepsilon_\theta^e. \quad (8.12)$$

When the position of materials $y_T^0 < y < y_C^0$, the member locates in the elastic region. No plastic deformation occurs, $\varepsilon_\theta^p = 0$, so the elastic strain ε_θ^e is equal to the total strain ε_θ .

In the outer regions of the sheet, the stress will be determined by the hardening law, although the strain continues to increase linearly with distance from the neutral axis, as in the elastic case. The outer region can be separated by the stress state into two sub-regions, tension and compression regions.

When the position of materials $y > y_C^0$, the member located in the compression region. The plastic strain ε_θ^p can be given by

$$\varepsilon_\theta^p = \varepsilon_\theta - \varepsilon_\theta^e = -\frac{\bar{e} - y}{\bar{\rho}} - \frac{\sigma_Y^c(\varepsilon_\theta^p)}{E}, \quad (8.13)$$

where, $\sigma_Y^C(\varepsilon_\theta^p)$ is the hardening law in uniaxial compressive test, which is a function of hardening plastic strain.

When the position of materials $y < y_T^0$, the member is located in the tension region. Similarly to compression region, the ε_θ^p can be given by

$$\varepsilon_\theta^p = \varepsilon_\theta - \varepsilon_\theta^e = \frac{\bar{e} - y}{\bar{\rho}} - \frac{\sigma_Y^T(\bar{\varepsilon}^p)}{E}. \quad (8.14)$$

Owing to that hardening law is dependent of the equivalent plastic strain $\bar{\varepsilon}^p$, so it is required to use a iterative method to calculate the equivalent plastic strain at each given position y .

According to the hardening law, the circumferential stress in the plastic part of the material can be written as:

$$\sigma = \begin{cases} \sigma_Y^T(\bar{\varepsilon}^p) & (\sigma > \sigma_T^0) \\ E\varepsilon & (\sigma_C^0 \leq \sigma \leq \sigma_T^0), \\ \sigma_Y^C(\bar{\varepsilon}^p) & (\sigma < \sigma_C^0) \end{cases} \quad (8.15)$$

where σ_Y^T is the flow stress in the uniaxial tensile test, σ_Y^C is the flow stress in the uniaxial compressive test. σ_T^0 is the initial yield stress in uniaxial tensile test, σ_C^0 is the initial yield stress in compressive test.

8.2.3 Force and moment calculation

The plastic strain is negative in the compression region; therefore its absolute value is used in the power law to calculate the circumferential stress. The forces and bending moments acting on the sheet per unit length can then be found from:

$$\bar{T} = \int_{-\bar{t}/2}^{\bar{t}/2} \sigma_\theta dy, \quad (8.16)$$

$$\bar{M} = \int_{-\bar{t}/2}^{\bar{t}/2} \sigma_\theta y dy. \quad (8.17)$$

The tensile force can be split into three components:

$$\bar{T} = T^e + T_T^p + T_C^p. \quad (8.18)$$

In this equation T^e is the force caused by the elastic stress, while T_T^p and T_C^p are the tensile and the compressive forces caused by the plastic stresses. The contribution of the elastic and the plastic stresses to the total tension will be:

$$T^e = \int_{y_T^0}^{y_0^C} \sigma_\theta^e dy = \int_{y_T^0}^{y_0^C} E \frac{\bar{e} - y}{\bar{\rho}} dy = \frac{E}{2\bar{\rho}} [(\bar{e} - y_T^0)^2 - (\bar{e} - y_0^C)^2], \quad (8.19)$$

$$T_T^p = \int_{-\bar{t}/2}^{y_0^T} \sigma_Y^T(y) dy, \quad (8.20)$$

$$T_C^p = \int_{y_0^C}^{\bar{t}/2} -\sigma_Y^C(y) dy. \quad (8.21)$$

Bending moments are produced by transverse loads applied to beams. The bending moment acting on a section of the beam, due to an applied transverse force, is given by the product of the applied force and its distance from that section. It is balanced by the internal moment arising from the stresses generated. This is given by a summation of all of the internal moments acting on individual elements within the section. These are given by the force acting on the element (stress times area of element) multiplied by its distance from the neutral axis, y .

Balancing the external and internal moments during the bending, the bending moment can be written in the form

$$\bar{M} = \int_{-\bar{t}/2}^{\bar{t}/2} \sigma_\theta y dy. \quad (8.22)$$

The total moment per unit width acting about the mid-plane can be described as a sum of three components:

$$\bar{M} = M^{tol} = M^e + M_T^p + M_C^p, \quad (8.23)$$

where M^e is the elastic part, while M_T^p and M_C^p are the plastic parts of the total bending moment in the region of tension and compression. These components can be found as follows:

$$\begin{aligned} M^e &= \int_{y_0^T}^{y_0^C} \sigma_\theta^e y dy = \int_{y_0^T}^{y_0^C} E \frac{e - y}{\bar{\rho}} y dy \\ &= \frac{E}{\bar{\rho}} \left[\frac{\bar{e}}{2} ((y_0^C)^2 - (y_0^T)^2) - \frac{1}{3} ((y_0^C)^3 - (y_0^T)^3) \right], \end{aligned} \quad (8.24)$$

$$M_T^p = \int_{-\bar{t}/2}^{y_0^T} \sigma_Y^T(y) y dy, \quad (8.25)$$

$$M_C^p = \int_{y_0^C}^{\bar{t}/2} -\sigma_Y^C(y) y dy. \quad (8.26)$$

8.2.4 Unloading and springback phase

After bending the sheet to the radius $\bar{R}$, a moment $\bar{M}$ remains. If external loads are removed, this bending moment is released and the sheet will spring back to a different shape to reach a new equilibrium. The magnitude of the stresses will decrease and the

amount of shape change (i.e. change in curvature) can be related to the applied bending moment. The change in internal stresses due to elastic unloading reads:

$$\Delta\sigma_\theta = E\Delta\varepsilon_\theta. \quad (8.27)$$

with

$$\Delta\varepsilon_\theta = \frac{y}{\bar{\rho}} - \frac{y}{\bar{\rho}'} = \Delta\left(\frac{1}{\bar{\rho}}\right)y, \quad (8.28)$$

where $\bar{\rho}'$ is the radius of curvature after unloading. The change in internal stresses causes a change in bending moment, $\Delta\bar{M}$:

$$\Delta\bar{M} = \int_{-\bar{t}/2}^{\bar{t}/2} y\Delta\sigma_\theta dy = \int_{-\bar{t}/2}^{\bar{t}/2} yE\Delta\left(\frac{1}{\bar{\rho}}\right)y dy = \frac{\bar{t}^3}{12}E\Delta\left(\frac{1}{\bar{\rho}}\right). \quad (8.29)$$

The removal of external loads results in $\Delta\bar{M} = -\bar{M}$. Thus the change in curvature is related to the applied bending moment via:

$$\frac{\bar{t}^3}{12}E\Delta\left(\frac{1}{\bar{\rho}}\right) = -\bar{M}, \quad (8.30)$$

$$\Delta\left(\frac{1}{\bar{\rho}}\right) = -\frac{12\bar{M}}{E\bar{t}^3}. \quad (8.31)$$

The change in curvature leads to a change in bending angle. It can be determined from the arc length L_0 of the bend which remains constant after bending and during unloading, hence:

$$L_0 = \bar{\theta}\bar{\rho} \rightarrow \bar{\theta} = L_0\frac{1}{\bar{\rho}}, \quad (8.32)$$

An expression for the change in angle $\Delta\theta$ can be obtained by differentiation of the above equation to the curvature:

$$\Delta\bar{\theta} = L_0\Delta\left(\frac{1}{\bar{\rho}}\right) = \bar{\rho}\bar{\theta}\Delta\left(\frac{1}{\bar{\rho}}\right), \quad (8.33)$$

where $\Delta\bar{\theta}$ can then be calculated from:

$$\Delta\bar{\theta} = -\frac{12\bar{M}}{E\bar{t}^3}\bar{\rho}\bar{\theta}. \quad (8.34)$$

In order to establish a comparison of the springback effect, a springback factor ζ is defined by

$$\zeta = -\frac{12\bar{M}}{E\bar{t}^3}\bar{\rho}. \quad (8.35)$$

In bending, a parameter called the springback ratio is commonly defined as

$$Ks = \frac{\bar{\theta}_1}{\bar{\theta}_0}. \quad (8.36)$$

Therefore, the relationship between the springback factor ζ and the springback ratio Ks can be built as

$$Ks = \zeta + 1. \quad (8.37)$$

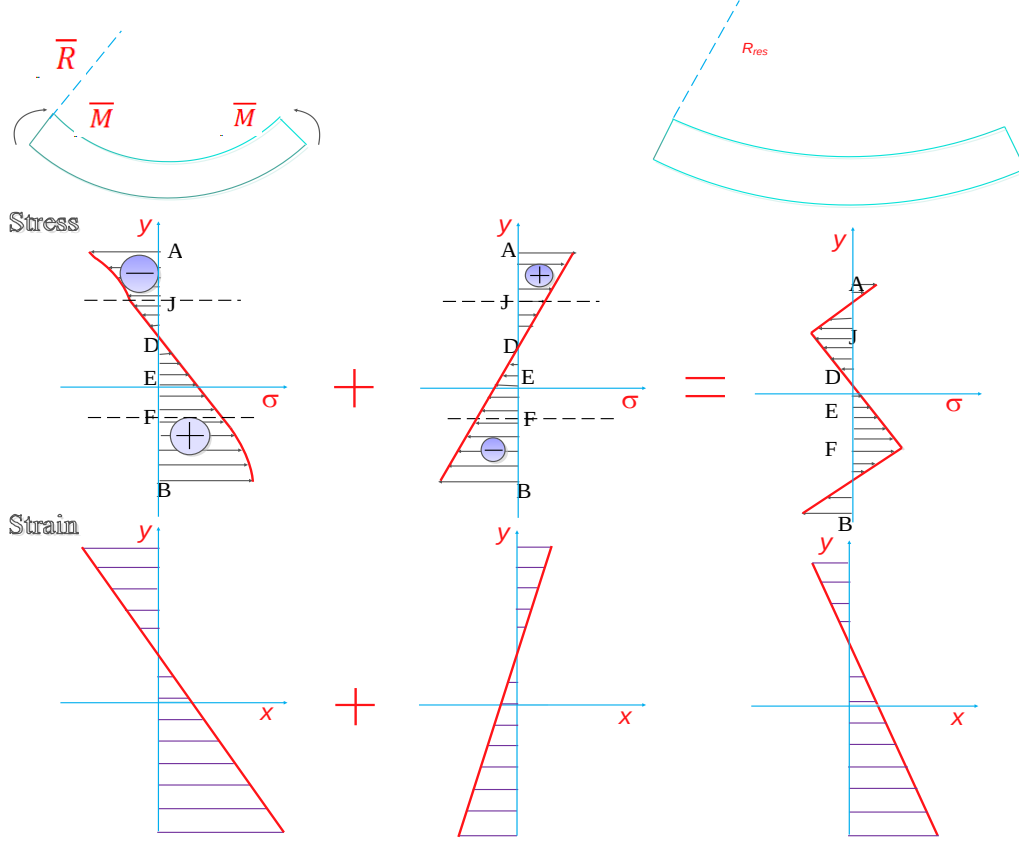


Figure 8.3: Distributions of stress and strain within a beam before and after application of a moment sufficiently large to cause plastic deformation.

8.3 Analytical algorithm description

Due to the asymmetry mechanical behavior, the neutral plane is not coincident with the mid-plane. During the bending process, the neutral plane alternates. The eccentric distance $\bar{e}$ from the mid-plane is initially unknown. It is necessary to determinate the location of the neutral plane before calculate the bending moment. After that, the entire variables like curvature radius of the neutral plane, and strain at the cross section can be calculated. Owing to that the circumferential force is equal to zero (or the external force if the external tension force applied), so it is necessary to compute the eccentric distance $\bar{e}$ based on an iterative method, which is called global iterative process. During the

global iterative process, the stress at each position y is calculated by a local iterative if the material at that point is located in the plastic deformation region, otherwise, it can be calculated directly by elastic Hooke's law. After that, in order to calculate the total force in the circumferential direction, an integration of stress along the radius direction should be done. The global iterative process is repeated until the circumferential force vanishes (the absolute value is less than the prescribed tolerance) or equal to the external tension force. The entire flow chart of the algorithm is shown in Figure 8.4.

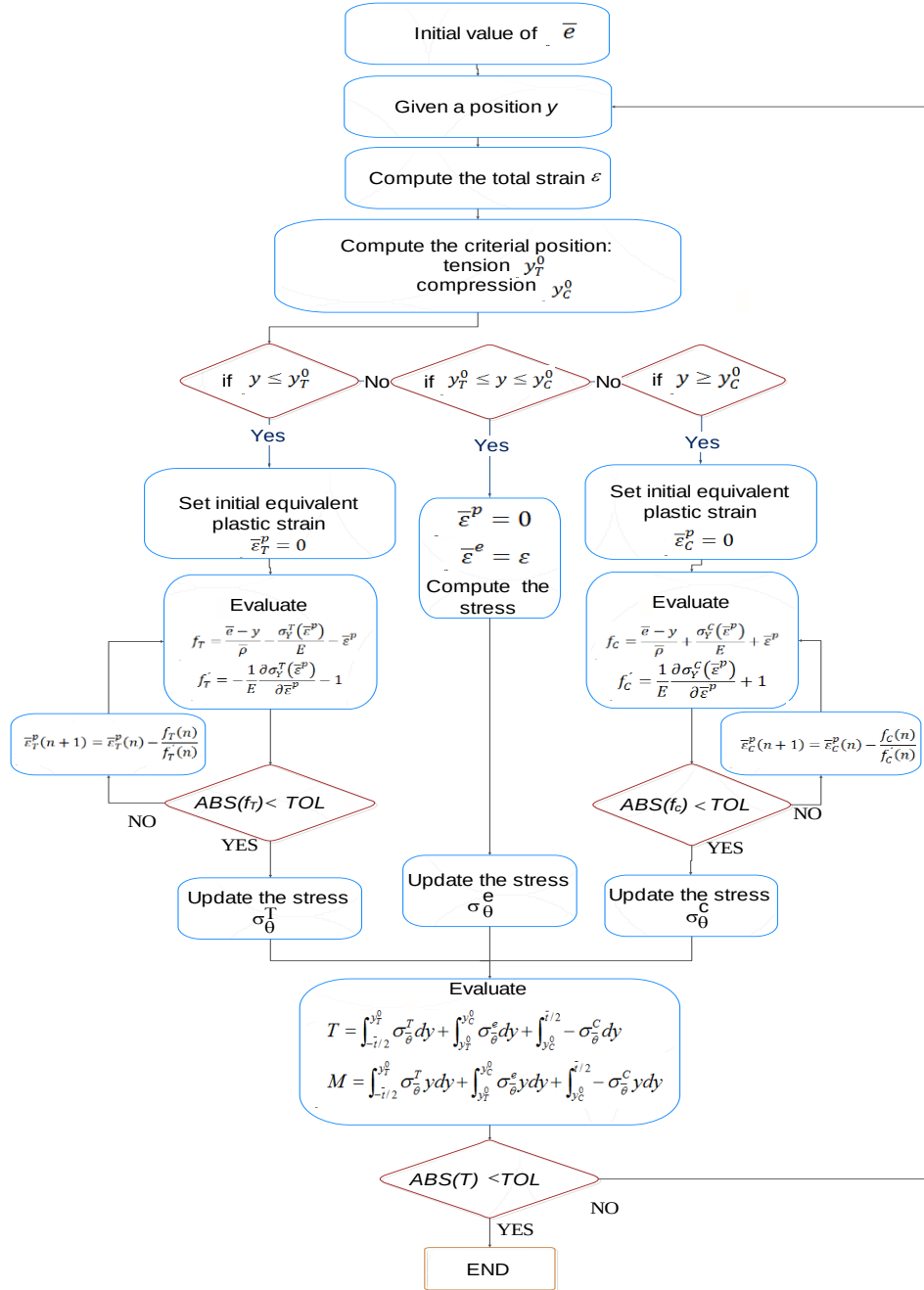


Figure 8.4: Flow chart of analytical program for the bending process.

The selected material in this chapter is magnesium alloy AZ31. The material Young's modulus is 45 GPa, Poisson's ratio is 0.35.

The compressive curve can be fitted by a new proposed equation, which can be written as

$$\sigma_c = \sigma_{c0} + K_c * \left[1 + \tanh[Q_c(\bar{\varepsilon}^p + \varepsilon_{c0})] \right], \quad (8.38)$$

where, σ_{c0} , K_c , Q_c , ε_{c0} are material parameters.

For the uniaxial tensile curve, the classical Voce hardening equation is adopted here, which can be written by

$$\sigma_T = A_V + B_V \left(1 - e^{-C_V \bar{\varepsilon}^p} \right), \quad (8.39)$$

where A_V , B_V , C_V are material parameters.

All the above material parameters are calibrated by classical gradient method based on the MATLAB toolbox. All the fitted material parameters at different strain rate are listed in Table 8.1.

Table 8.1: Material parameters at different strain rates for tensile and compressive tests.

	Strain rate	Compressive				Tensile		
		$\sigma_c = \sigma_{c0} + K_c * \left[1 + \tanh[Q_c(\bar{\varepsilon}^p + \varepsilon_{c0})] \right]$				$\sigma_T = A_V + B_V \left(1 - e^{-C_V \bar{\varepsilon}^p} \right)$		
		σ_{c0}	K_c	Q_c	ε_{c0}	A_V	B_V	C_V
RD	1.0	97.21	122.22	35.40	-0.074	195.78	117.57	18.46
	0.01	86.48	124.24	31.59	-0.076	176.26	133.11	18.74
	0.0001	96.18	112.82	36.50	-0.078	169.30	132.14	18.02

8.4 Validation of the proposed analytical method

In order to validate the proposed analytical method and programming code, two simple bending processes are chosen to make comparisons with theoretical values. One of them is the elastic bending process; the other is perfectly-plastic bending process.

For members made of elastic material, assuming the member to have a rectangular cross section of unit width and thickness $\bar{t}$, the theoretical moment can be computed by the following equation

$$\bar{M} = \frac{1}{12\bar{\rho}_m} E \bar{t}^3. \quad (8.40)$$

The evolution relation of the elastic moment against the bending curvature of the sheet for the elastic bending process can be drawn, which is presented in Figure 8.5 (a).

For members made of symmetric elastic, perfectly-plastic materials, the theoretical moment can be calculated by

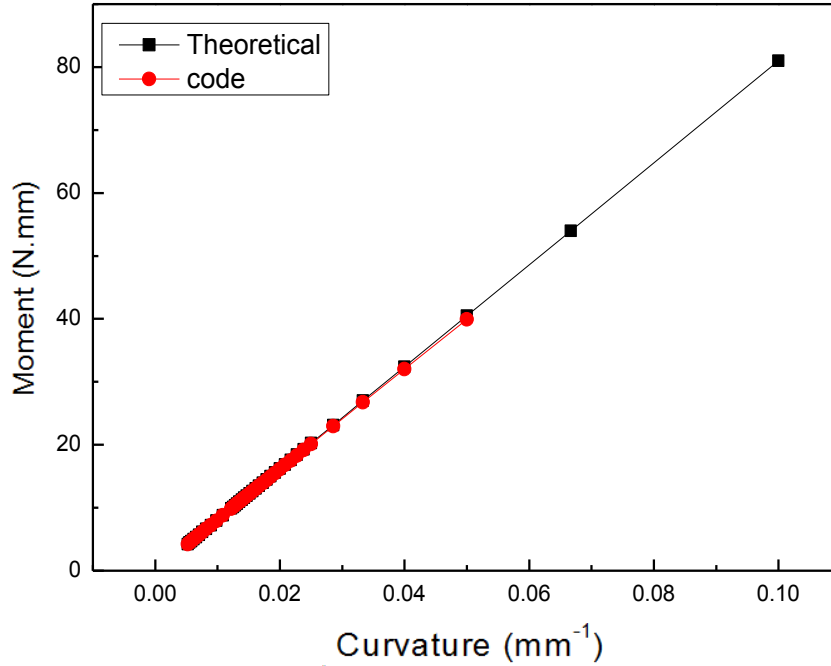
$$\bar{M} = \frac{3}{2} \bar{M}_Y \left(1 - \frac{4}{3} \frac{y_Y^2}{\bar{t}^2} \right), \quad (8.41)$$

where, $\bar{M}_Y$ is the maximum elastic moment for the rectangular beam with unit width, which is given by $\bar{M}_Y = \bar{t}^2 \sigma_Y^T / 6$;

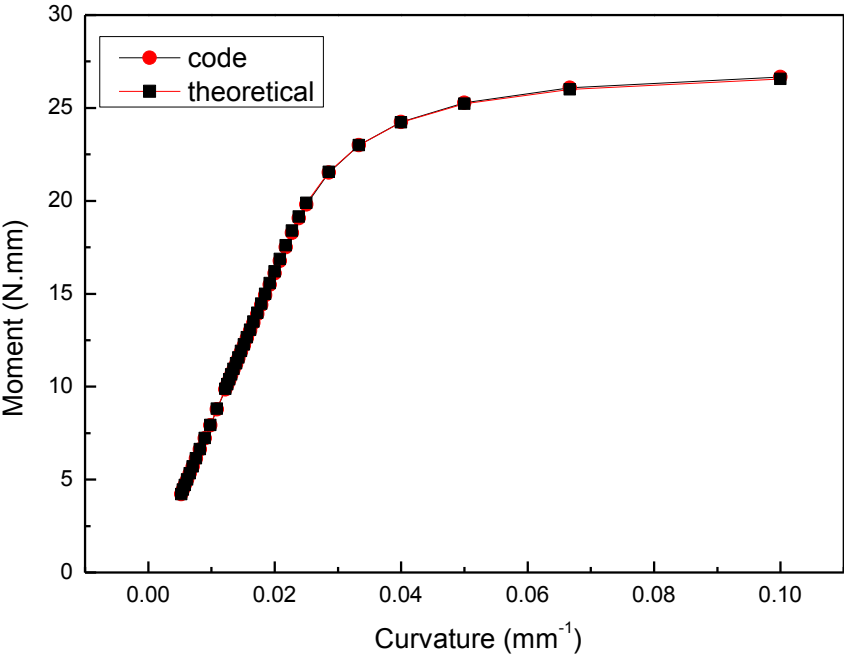
y_Y represents the critical position, which is equal to $\bar{\rho}_m \sigma_Y^T / E$.

The evolution relation between total moment and the bending curvature of the sheet for the symmetric elastic, perfectly-plastic bending can be drawn, which is demonstrated in Figure 8.5 (b).

In Figure 8.5 (a) and (b), the theoretical moment evolution against the curvature is drawn with black curve; the calculated value obtained from the user developed code is drawn with red curve. From the comparisons, it is clearly seen that the proposed analytical method and user developed code is able to give a good prediction.



(a) Elastic bending



(b) Elastic, perfectly-plastic bending

Figure 8.5: Comparison of moment evolution obtained from the theoretical method and the proposed analytical method.

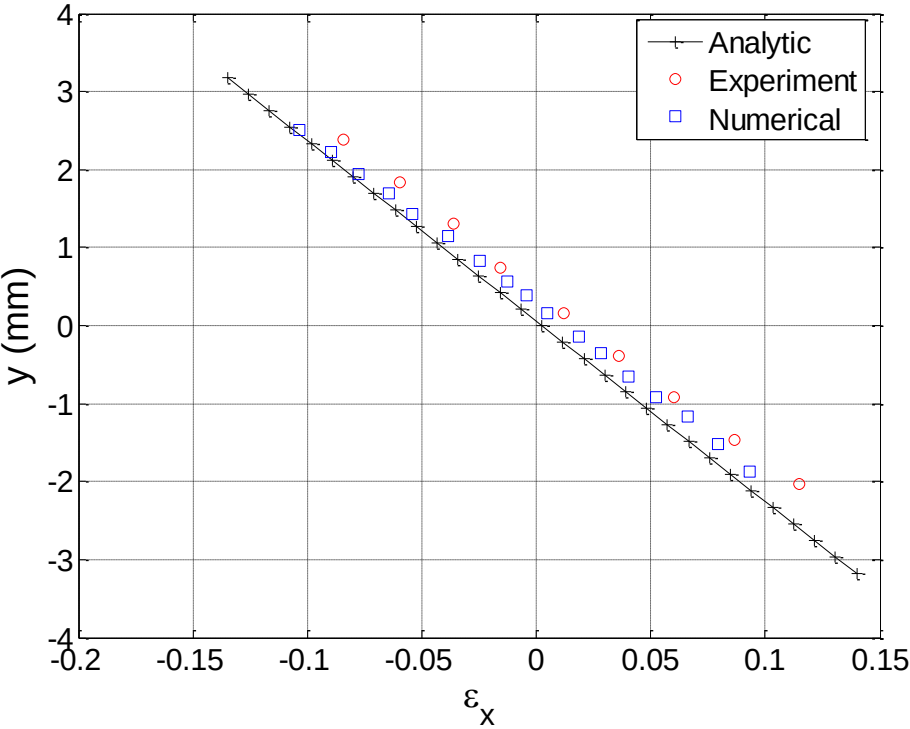


Figure 8.6: Comparison of the strain distribution in the cross section between analytical solution, experiments and numerical results.

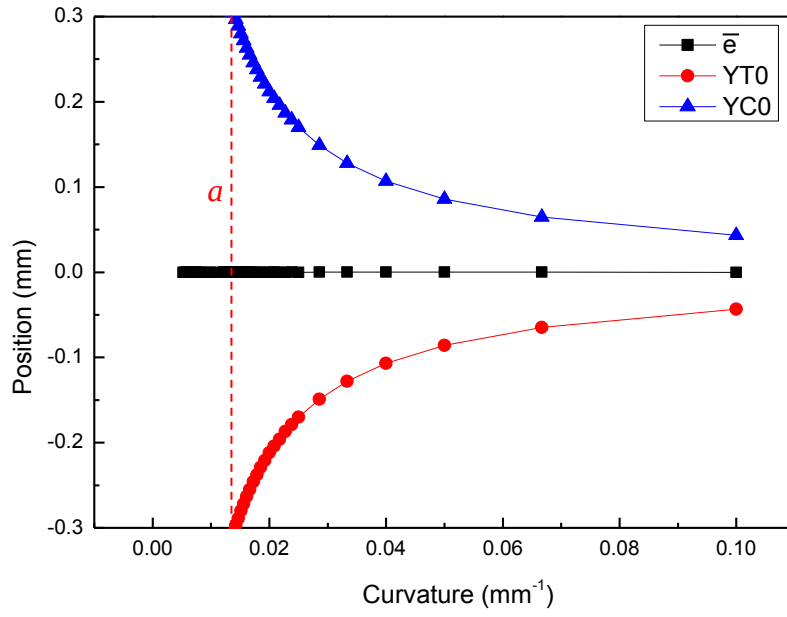
In order to further validate the proposed analytical method and the user developed code, a four point bending test was performed on beams of high purity α titanium material by Nixon *et al* [26]. The proposed analytical method is used to compute the strain distribution along the thickness. All the experimental data and numerical strain distribution in the beam cross section is extracted from literature [26]. Figure 8.6 demonstrates the circumferential strain distribution through the thickness obtained from the numerical simulation by FEM, analytical computation and experiments, respectively. From the comparison of Figure 8.6, it can be observed that the theoretical value is almost coincident with the calculated value by the proposed analytical method and the developed code. Therefore, it further validates the accuracy of both the proposed analytical method and the implemented programming code.

8.5 Results and discussion of analytical method

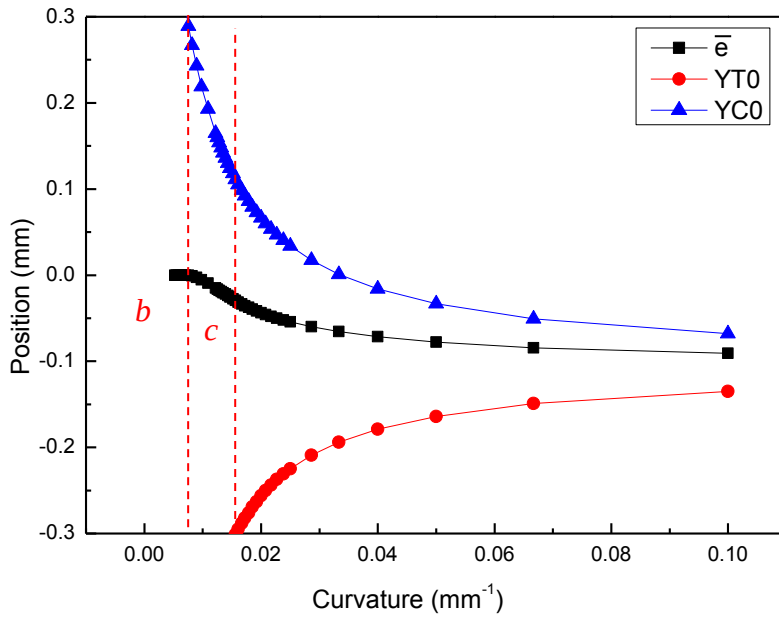
After validating the accuracy of the proposed method and code, it is used to investigate the influence factors during bending process and make comparisons between symmetrical and unsymmetrical materials. The used unsymmetrical material in this section is Magnesium alloy AZ31. Except the subsection 8.5.6, the thickness of the sheet in the other subsection is set as 0.6 mm. The material parameters used to describe the mechanical behavior under uniaxial tensile and uniaxial compressive test can be seen in Table 8.1. In order to make a comparison, the symmetrical material uses the same stress-strain curve as the unsymmetrical material under uniaxial tension stress state. All the bending curvature in this section represents the inner surface curvature of the sheet. Owing to the Kirchhoff hypothesis, the curvature radius should be ten times greater than the sheet thickness. In this section, the minimum curvature radius is set as 10 mm.

8.5.1 The influence of curvature on the neutral surface position, tensile yield critical position, and compressive yield critical position

Figure 8.7 (a) shows the neutral surface position, the tensile yield critical position and the compressive critical position inside the sheet cross section for the symmetrical material and unsymmetrical materials, respectively. The surface at position of 0.3 is the inner surface; the surface at position of -0.3 is the outside surface.



(a) Symmetrical material



(b) Unsymmetrical material

Figure 8.7: Position distribution for symmetrical and unsymmetrical material.

In the Figure 8.7 (a), with the increasing bending curvature, the neutral surface position evolution curve is drawn by square black line, which is denoted by $\bar{e}$. The initial tensile yield position (also called the tensile yield critical position) is denoted by $YT0$. The initial compressive yield position (also called the compressive yield critical position) is

denoted by $YC0$. From there, it can be seen that the neutral surface is always coincident with the mid-surface of the sheet, which is independent on the bending curvature. As the curvature increases (the curvature radius decreases), the tensile yield critical position and the compressive yield critical position move closer and closer to the neutral plane (here is also the mid-plane). It also means that the tensile and compressive plastic region inside the section of the sheet gradually expands with the increasing bending curvature. This occurs because with the increase of the bending curvature, the material inside the section plane is subjected to increasing deformation; moreover the total strain gradually increases. From the Figure 8.7 (a), it is also possible to conclude that these two critical positions are symmetrical with respect to the mid-plane. It should be noted that a dotted red line a is drawn to separate the figure into two parts. The critical yield position only exists in the right region; this means that in the left side, the strain distribution inside the section of the sheet is not sufficient to initiate yield from both tensile and compressive plastic regions; besides the stress-strain through the thickness is purely elastic, and there exists a critical bending curvature, where the material in the surface starts to plastic yield. This change does not affect the neutral position, which still locates in the mid-plane. That is because the material has isotropic elastic behavior and the same stress-strain flow behavior under the tensile and compressive stress states hence the material in the inner and outside surfaces yields simultaneously.

In contrast to the symmetrical material, Figure 8.7 (b) demonstrates a remarkable different evolution curves for unsymmetrical material. It can be seen that the neutral position is not a straight line during the bending process. With the increasing bending curvature, the neutral position moves gradually toward the outside surface. Compared with the symmetrical material, the tensile yield critical position and the compressive yield critical position are no longer symmetrical and both of them move toward the outside surface. The compressive plastic deformation inside the section is much larger than the symmetrical material; meanwhile, the tensile plastic deformation reduces substantially when compared with the symmetrical material. Not only the plastic deformation region alternates, but the elastic region also changes. From the figure, we can observe that the distance between neutral position and the compressive yield critical position is smaller than the distance between neutral position and the tensile yield critical position. This means that although these three typical position evolution curves move towards the outside surface, it does not mean that the material in elastic deformation region is symmetrical with respect to the neutral surface. Actually, the

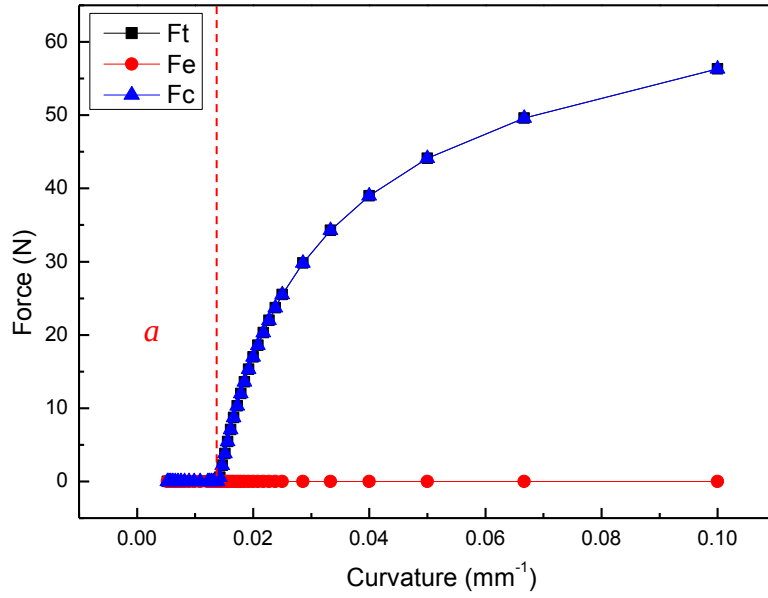
material is subjected to a higher tensile elastic deformation than compressive deformation. From the neutral surface position evolution curve, it is easy to see that there exist three stages with the increasing bending curvature. In the left side of red dotted line b , the value of the neutral position is zero, which means that it is coincident with mid-plane, because that there is no material inside the entire cross section yields, all the material suffers elastic deformation. Owing to that the Young's modulus E in the tensile and compressive stress state is the same, so the deformation is symmetrical with respect to the mid-plane. In the second stage between line b and line c , the neutral position starts to move towards the outside surface. This occurs because the material close to the inner surface reaches plastic deformation; meanwhile, the material close to the outside surface still presents elastic deformation, and owing to that the strength difference between compressive and tensile is very large, with the increasing curvature, the neutral surface position moves quickly toward the outside surface. Once the material enters into the third stage (the right side of line c), although the neutral surface position still moves toward to the outside surface, the changes are not as large as the second stage; this is because the strength gap between tensile and compressive is reduced.

8.5.2 The influence of curvature on the force distribution

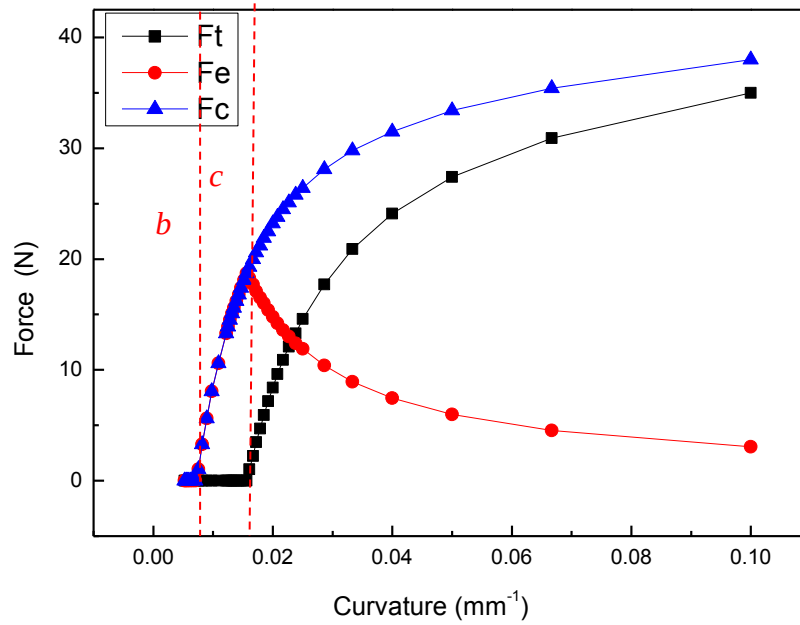
In order to analyze the bending process further, the force distribution evolution in elastic region, compressive plastic region, and tensile plastic region are drawn with respect to the curvature. Figure 8.8 (a), (b) shows the force-curvature curve for symmetrical and unsymmetrical materials, respectively.

In the Figure 8.8, F_t denotes the value of the force value in the tensile plastic deformation region, F_e denotes the force value in the elastic deformation region, and F_c denotes the force value in the compressive deformation region. From the Figure 8.8(a), it is obviously seen that the total force in the elastic region is always equal to zero, the forces value in the compressive plastic deformation region are coincident with the force value in the tensile plastic deformation region in the left side of red dotted line- a the force is also equal to zero, once the curvature surpasses a critical value, the force increase with the increasing the curvature. This is easy to be interpreted by the position distribution analysis. In the left side of the red dotted line a , all the material is in elastic region, and the elastic deformation is symmetrical with respect to the neutral plane (mid-plane), so all the force value is equal to zero. However, in the right side of the red dotted line a , with the increasing curvature, the plastic deformation region expands

gradually, so the force value in the compressive and tensile plastic region increases gradually.



(a) Symmetrical material



(b) Unsymmetrical material

Figure 8.8: Force distributions for the symmetrical material and the unsymmetrical material.

With regard to the unsymmetrical material, the force distribution has a remarkable difference from the symmetrical material, which can be seen in Figure 8.8 (b). In the left

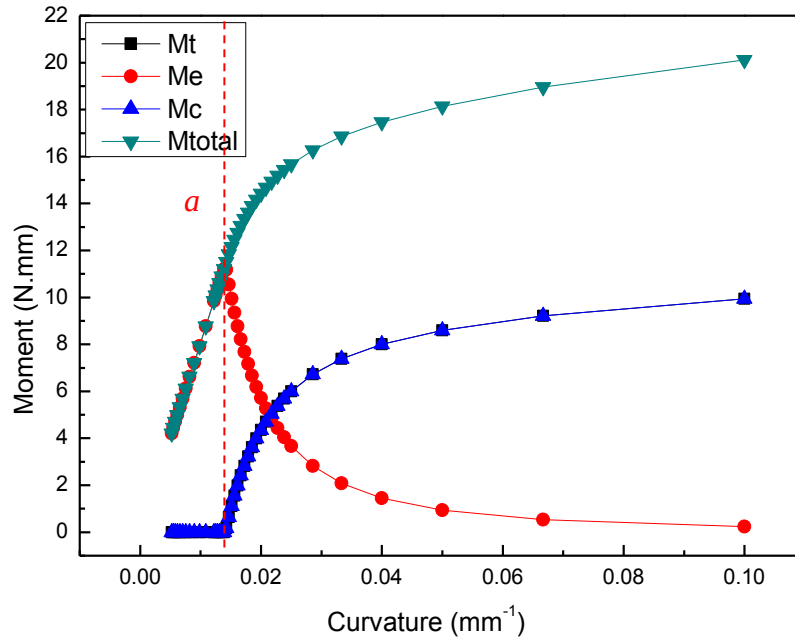
side of line b , all the material is in the elastic region, and this elastic region is symmetrical with respect to the neutral plane (mid-plane), so all the force is equal to zero. In the domain between lines b and c , the force in tensile plastic region is still equal to zero, however, the force value in the compressive plastic and elastic region increases with the increasing curvature. That is because the compressive plastic region enlarges gradually with the increasing of the curvature, so the force in compressive plastic region increases. Regarding the force in elastic region, that is due to the neutral surface position moving toward the outside surface, and the tensile elastic and compressive elastic region being no longer symmetrical. In the right side of line c , with the increasing curvature, the compressive plastic region continues to expand, so the force in compressive plastic region continues to increase, but the material close to the outside surface enters into plastic deformation and expands gradually with increasing curvature and therefore the force in the tensile plastic region increases. With the unsymmetrical expansion of the tensile plastic region and the compressive plastic region, the force in elastic region also changes; the force in elastic region reduces with the increasing curvature, which is also clearly observed by making a comparison with the distance between the yield critical position and neutral position for compressive and tensile plastic deformation.

8.5.3 The influence of curvature on the moment distribution

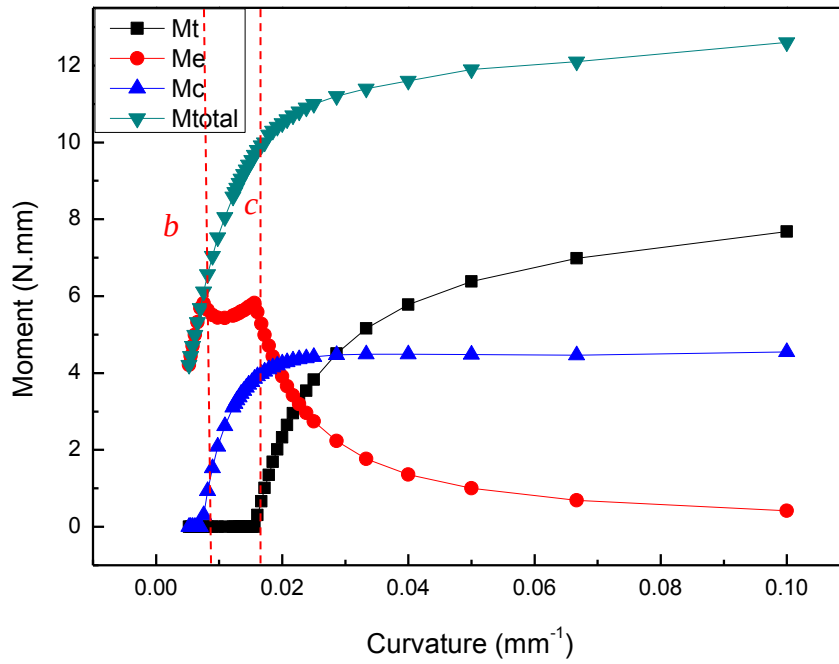
With regard to the bending process, the bending moment is an important item to be further investigated. In order to analyze the distribution of the bending moment clearly, four moment values are drawn with respect to the bending curvature, as shown in Figure 8.9. These four moment values include the moment in the tensile plastic region, in the elastic region, in the compressive region, and the total bending moment, denoted by M_t , M_e , M_c , M_{total} , respectively.

Figure 8.9 (a) shows the moment-bending curvature for symmetrical material. In the left side of line a , the material is only subjected to elastic deformation, so the moment in the tensile plastic region and the compressive plastic region is equal to zero. The total bending moment should be equal to the moment in the elastic region. In the right side of the line a , the region where the material enters into plastic deformation expands gradually with the increasing bending curvature, owing to the symmetry with respect to the mid-plane; the moment value in tensile plastic region is also coincident with the

compressive plastic region. Meanwhile, the elastic region inside the section decreases substantially, therefore, the moment caused by the elastic deformation decreases sharply.



(a) Symmetrical material



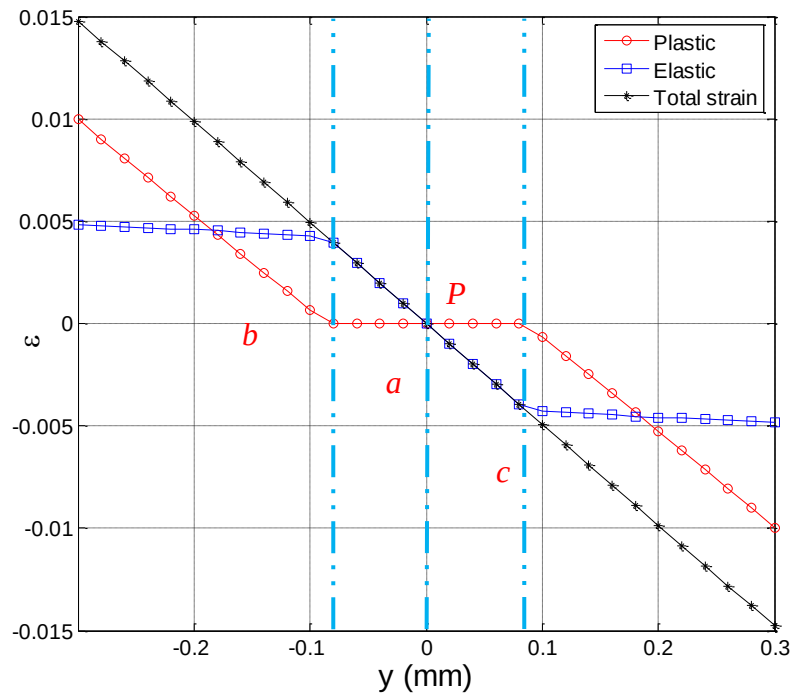
(b) Unsymmetrical material

Figure 8.9: Bending moment distribution for the symmetrical material and the unsymmetrical material.

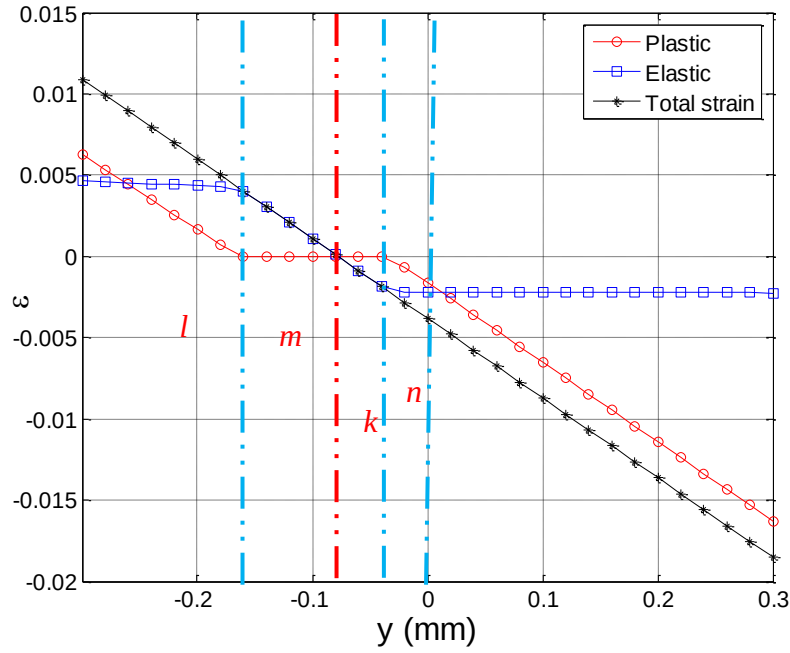
For the unsymmetrical material (seen in Figure 8.9 (b)), in the left side of line b , the material only suffers elastic deformation, so the moment distribution is the same as the symmetrical material. The moment in the tensile plastic region is not coincident with the compressive plastic region. In the domain between the line b and the line c , the material close to the inner surface yields, and the compressive plastic region expands. The moment in the compressive plastic region increases with the increasing bending curvature. The elastic region not only reduces, but also the neutral surface position moves toward the outside surface, so the moment in the elastic region changes. The change, however, is complicated, which is dependent on the above two effects. After the material close to the outside surface enters into yield, the elastic region sharply decreases, so the moment in elastic region decreases quickly. The total bending moment continues increase owing to the increasing the moment in the tensile plastic region.

8.5.4 The comparison of the strain distribution between symmetrical and unsymmetrical material

In order to obtain more information inside the cross section the sheet, it is necessary to understand the elastic strains and plastic strains, as well as the total strain distribution through the thickness. To clearly demonstrate the difference between symmetrical and unsymmetrical material, both of these two materials are bended to the same curvature ($\bar{R} = 20$ mm). The strain distributions are presented in Figure 8.10.



(a) Symmetrical material



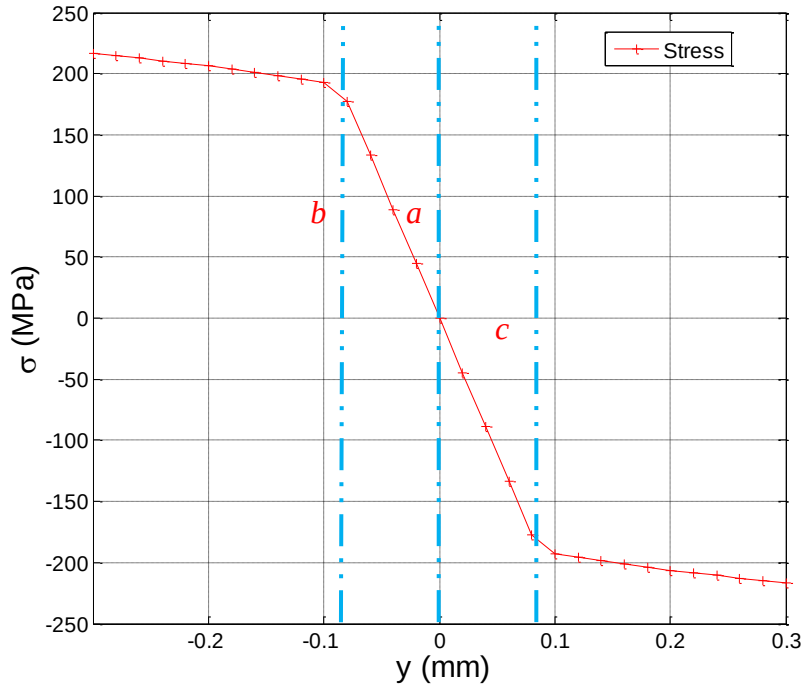
(b) Unsymmetrical material

Figure 8.10: Strain distribution inside the sheet cross section.

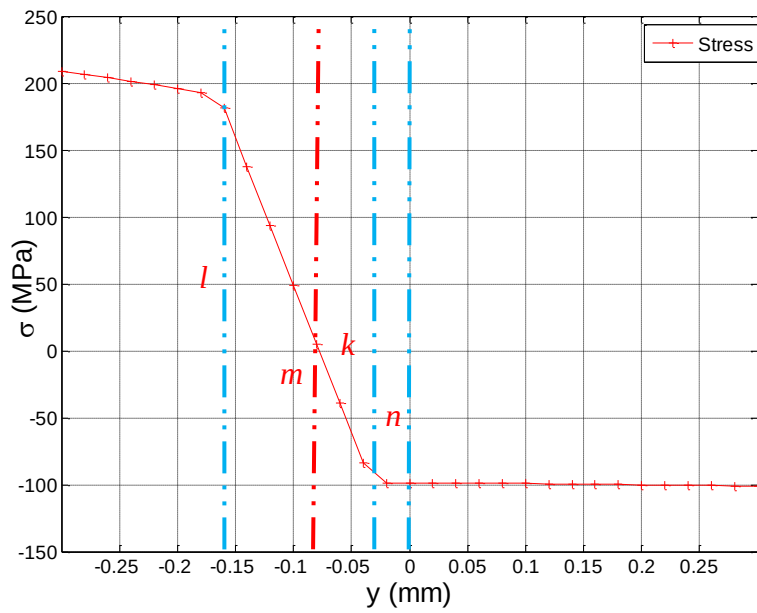
The horizontal axis represents the position through thickness; the vertical axis represents the strain value. From the Figure 8.11 (a), it can be clearly seen that the neutral plane is coincident with the mid-plane, and all the strain distributions which include elastic, plastic and total strain are symmetrical with respect to the point P . The material between line b and line c suffers elastic deformation, so the elastic strain distribution curve is coincident with the total strain distribution curve. The materials where locates in the left side of line b and in the right side of line c is subjected to plastic deformation. The furthest the material is from the region between line b and line c (also far from the mid-plane), the larger plastic deformation it suffers. Owing to the hardening effect, in this stage, the elastic strain still has a slightly increase.

Figure 8.10 (b) shows the strain distribution of unsymmetrical material through the thickness in the cross section the sheet. There are four blue center dotted lines drawn in the figure. Line l is the critical line between tensile elastic deformation and tensile plastic deformation. Line m denotes the position of the neutral surface. Line k denotes the critical position where the material initiates the compressive plastic deformation. Line n represents the mid-plane position. From there, it is easy to see that the strain distribution is no longer symmetrical. The distance between line l and line m is larger than the distance between line m and line k . It means there is a higher number of

material subjected to tensile elastic deformation than compressive elastic deformation. The maximum value of the elastic strain in the compressive deformation is much smaller than in the tensile deformation region. However, the maximum value of the plastic strain and the total strain in the compressive deformation is larger than the tensile deformation region.



(a) Symmetrical material



(b) Unsymmetrical material

Figure 8.11: Circumferential stress distribution inside the sheet cross section.

The circumferential stress distributions inside the cross section of the sheet are presented in Figure 8.11 for symmetrical and unsymmetrical material, respectively. From there, it can be concluded that the stress distribution is symmetrical between the tensile and compressive stress state. The Figure 8.11 (b) describes the asymmetrical hardening between tensile and compressive stress state.

8.5.5 Influence of strain rate on the bending

As it is known, stamping velocity has a great influence on the bending process, particularly, on the final springback angle. In order to study the influence of the stamping velocity, the strain rate effect should be discussed. In this study, three kinds of strain rate are investigated: 1 s^{-1} , 0.01 s^{-1} , and 0.0001 s^{-1} . In this section, three aspects are used to analyze the influence of strain rate, which are the position, force and bending moment.

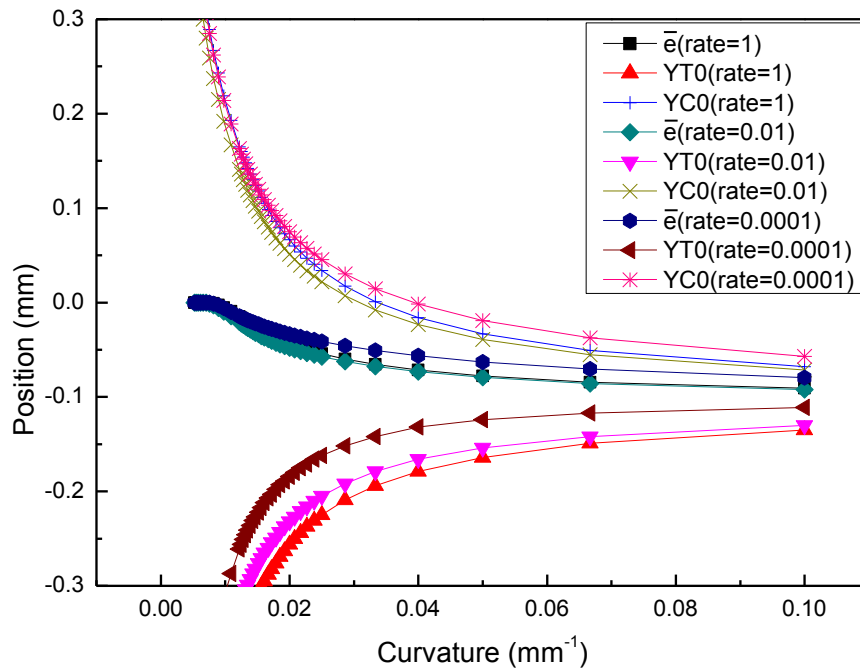


Figure 8.12: Position vs bending curvature with different strain rates.

Figure 8.12 shows the position distribution evolution against bending curvature with different strain rates. It can be seen that with the increasing strain rate, all the neutral surface position, the tensile yield critical position and the compressive yield critical position move gradually down (it is also close to the outside surface). By comparison, we can see that all these three position distributions in quasi-static (0.0001 s^{-1}) is

remarkably different from the other two dynamical states. From the moving distance, it can be seen that when the strain rate changes, the change of neutral surface position is smaller, however, the compressive yield critical position, the tensile yield critical position change much larger. It also reflects that with the increasing strain rate, the tensile plastic region reduces substantially; however, the compressive plastic region increases only slightly generating the expansion of the tensile elastic region.

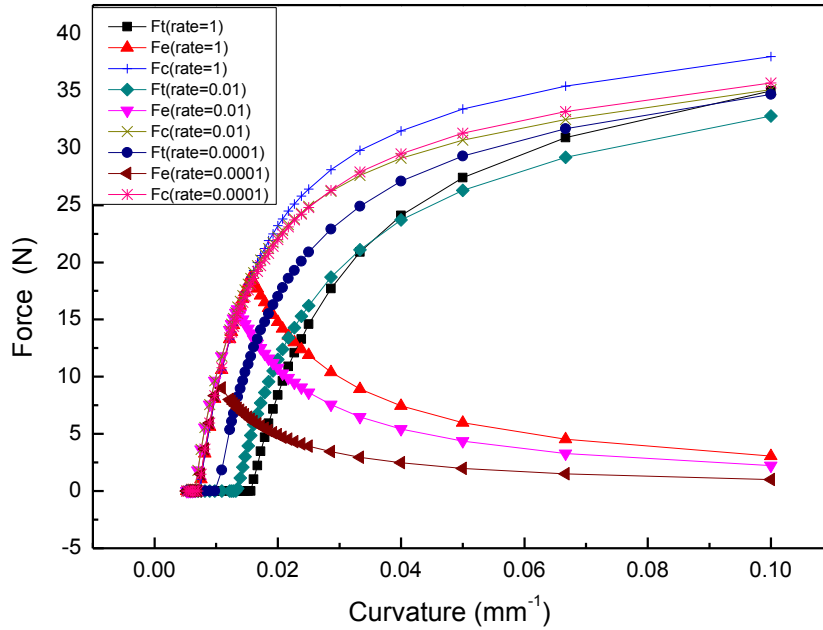


Figure 8.13: Force vs bending curvature with different strain rates.

Figure 8.13 shows the force-bending curvature with different strain rates. From there, by comparison among the force in elastic region, compressive plastic region, and tensile plastic region, it is obviously seen that the strain rate effect remarkably affects the elastic region. Not only the force in elastic region is changed, the elastic turned point is also obviously different with different strain rates. That is because the tensile yield critical position is affected by the strain rate, the higher the strain rate, the larger the critical curvature value occurs in the outside surface. In contrast, the critical curvature value that occurs in the inner surface does not significantly change with the strain rate alternation.

Once the position distribution and force distribution with respect to the strain rate in different strain rates conditions are known, the moment distribution with respect to the strain rate can be used to comprehensively analyze the influence of the strain rate.

Figure 8.14 demonstrates the moment distribution in different strain rates conditions. The moment distribution has similar rules to the force distribution. The difference is that the moment in the compressive plastic region does not increase with the increasing bending curvature as soon as the curvature is larger than a critical value. However, the moment in tensile plastic region always increases with the bending curvature increasing. With regard to the moment in the elastic region, the differences with different strain rate are obvious for the small bending curvature, with the bending curvature increasing, the difference gradually reduces, the moment value approaches to zero. The total bending moment keeps increasing with the augmentation of the bending curvature, in elastic domain, and it is coincident with the moment in the elastic region.

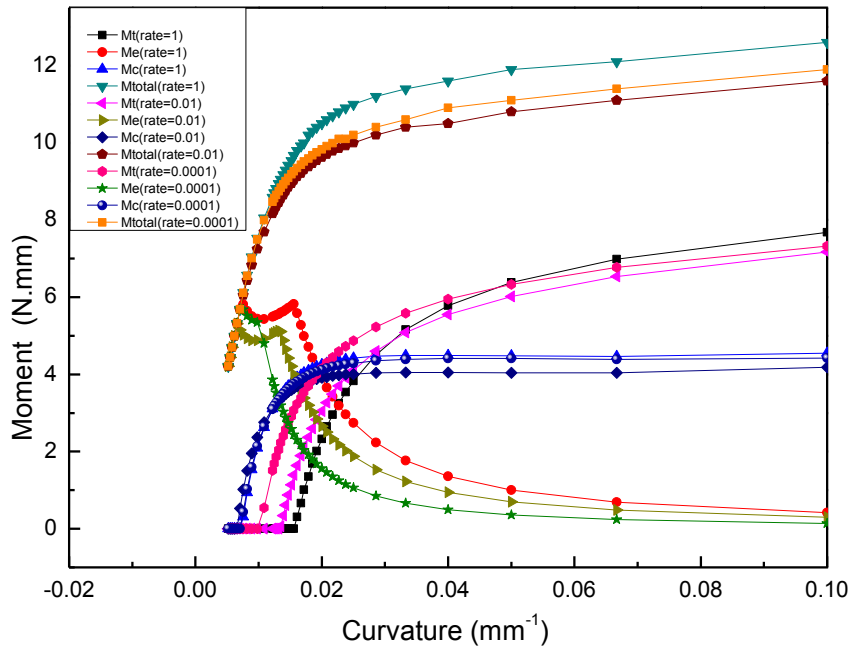


Figure 8.14: Moment vs bending curvature with different strain rates.

Figure 8.15 shows the springback factor ζ evolution when the bending curvature changes with different strain rates. From there, it is obvious to see that when the bending curvatures increase, the value of the springback factor ζ gradually decreases. In the left side of the line a , the springback factor ζ is equal to unit. It means that the angle change during the springback stage is equal to the bending angle obtained from the previous stamping step. In other words, this is an elastic deformation, all the deformation is recovered after the removal of the external force. With the increase of the bending curvature, more and more plastic deformation occurs. The deformation that

can be recovered is reduced; therefore, the angle change ratio $\Delta\bar{\theta}/\bar{\theta}$ reduces. From the comparisons among different strain rates, the springback factor does not alternate significantly.

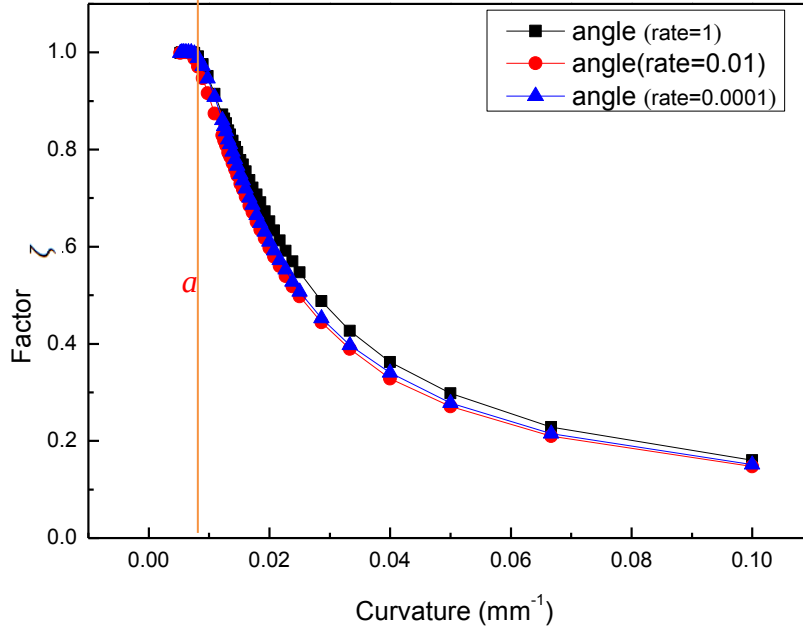


Figure 8.15: Springback factor ζ vs bending curvature with different strain rates.

8.5.6 Influence of thickness on the bending

During bending process, the thickness of the sheet is also a key factor to take under consideration. Sheets with different thicknesses affect the stress distribution inside the section. In this study, the influence of the sheet thickness on the bending is analyzed by establishing a comparison of two sheets. The thickness of these two sheets is 0.6 mm, 1.0 mm, respectively.

Figure 8.16 shows the neutral surface position, the tensile yield critical position, and the compressive yield critical position evolution for the sheet with the thickness of 0.6 mm and 1 mm, when the bending curvature alternates. From there, it is easily seen that the gap of all the above three position between these two sheets increases with the increasing bending curvature.

By comparing the evolution curve of the neutral surface position with different thicknesses, it is observable that the neutral surface position clearly moves toward the outside surface. It also means that the material subjected to compressive deformation outstandingly increases with the increasing thickness. Inherently, the elastic region does

not change significantly, and both the tensile elastic and compressive elastic regions almost keep the same weight.

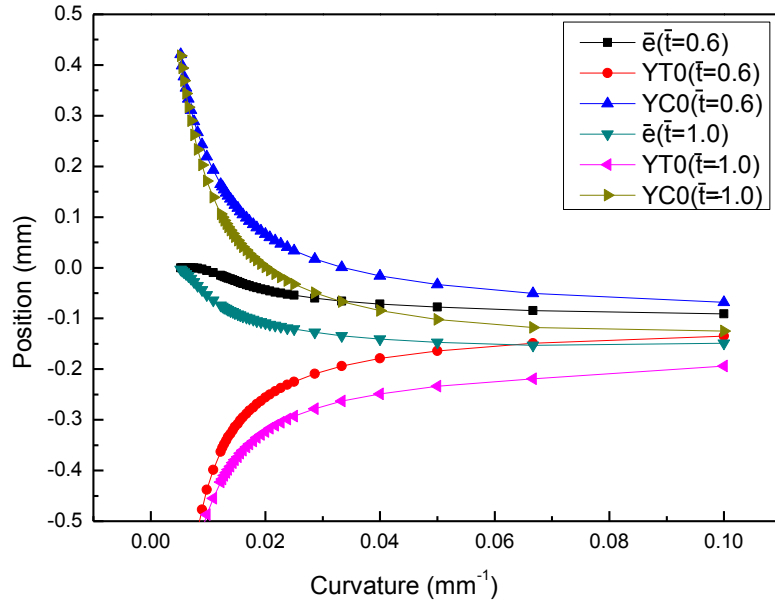


Figure 8.16: Position vs bending curvature with different thicknesses.

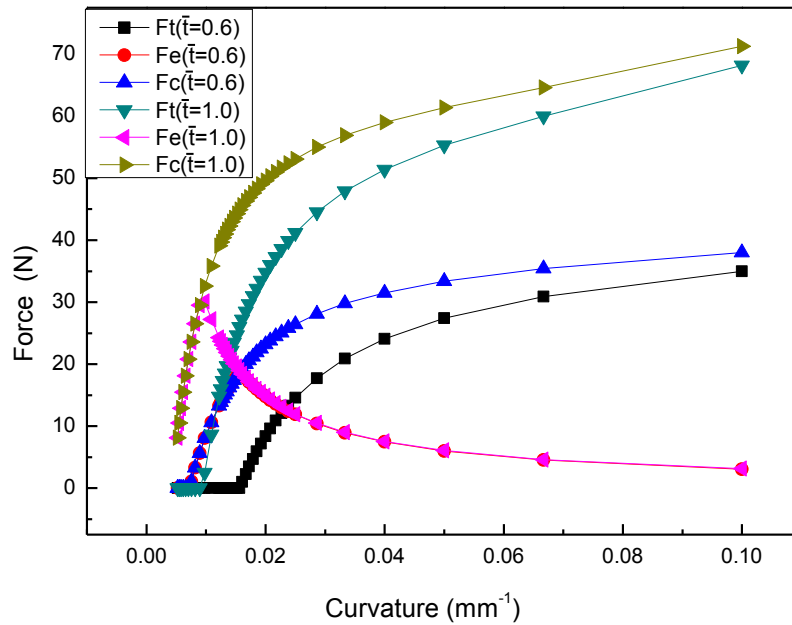


Figure 8.17: Force vs bending curvature with different thicknesses.

Figure 8.17 demonstrates the force evolution curve with respect to the bending curvature for the sheets with different thicknesses. From there, it can be seen that the force evolution curve in the elastic region, with respect to the bending curvature for

different thicknesses, is almost coincident. It means that the force in the elastic region is independent on the sheet thickness. The critical curvature values where the material in the inner surface starts to yield reduces when the thickness of the sheet increases, which is called the starting compressive yield curvature. The starting tensile yield curvature also reduces when the thickness changes from 0.6 mm to 1mm, and the change of the starting tensile yield curvature is larger than the starting compressive yield curvature. With the increasing of the bending curvature, the force gap of these two sheets gradually decreases.

Finally, the moment evolution with respect to the bending curvature for different thickness of sheet is described in Figure 8.18. From there, it is clearly observed that the total moment for the thick sheet (thickness of 1mm) is much larger than the sheet with the 0.6 mm thickness. The moment in the elastic region is almost coincident when the material in the outside surface starts to the tensile yield stress state; before that, the moment in elastic region with different thicknesses has a large gap. The moment in the tensile plastic region and compressive plastic region have a similar evolution rule to the force evolution with respect to the bending curvature.

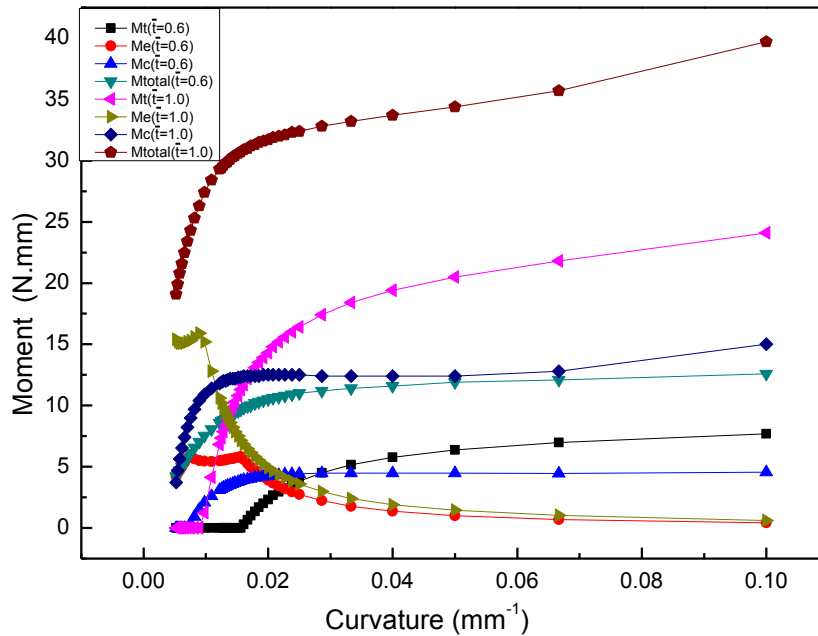


Figure 8.18: Moment vs bending curvature with different thicknesses.

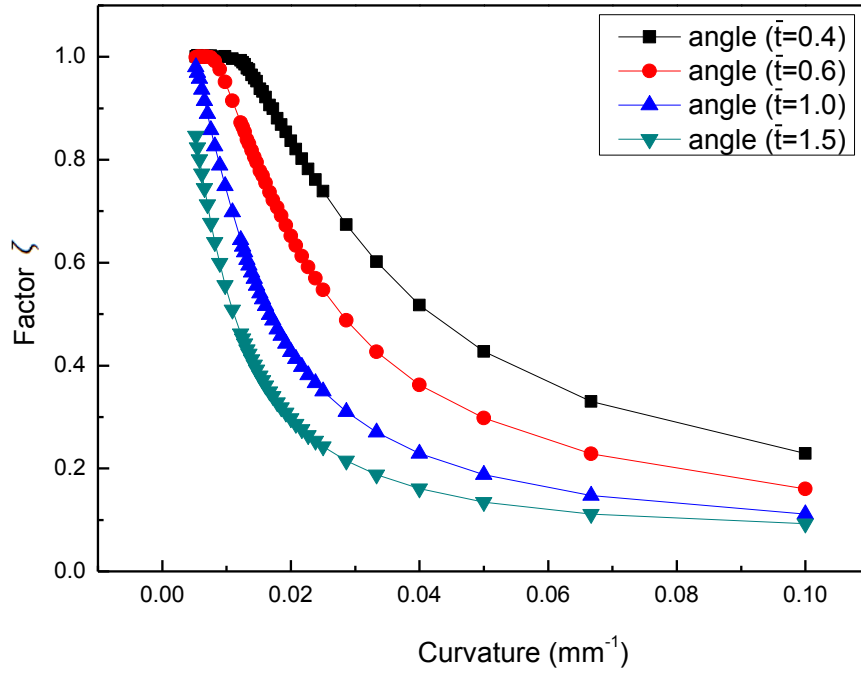


Figure 8.19: Springback factor ζ vs bending curvature with different thicknesses.

Figure 8.19 shows the springback factor ζ evolution when the bending curvature changes with different sheet thicknesses. From there, it can be seen that the sheet thickness has a great influence on the springback factor ζ . With the increasing sheet thickness, the elastic region shrinks. At the same bending curvature, the springback factor ζ decreases with the increasing sheet thickness. It is also seen that for the thicker sheet ($\bar{t} = 1.5$), the springback factor ζ reduces sharply when the bending curvature is in the range of small value, after that even the bending curvature continues increasing, the springback factor ζ slowly decreases.

8.5.7 Influence of in-plane tension on springback

During the sheet metal forming, bending holder and draw bead are always used to control the metal flow. This generates an in-plane tension stress, which affects the final forming. The schematic diagram can be seen in Figure 8.20. The symbol $\hat{T}$ denotes the in-plane force of the sheet. The bend angle is $\bar{\theta}$, and a moment per unit width $\bar{M}$, as well as a tension (force per unit width) $\hat{T}$ are applied.

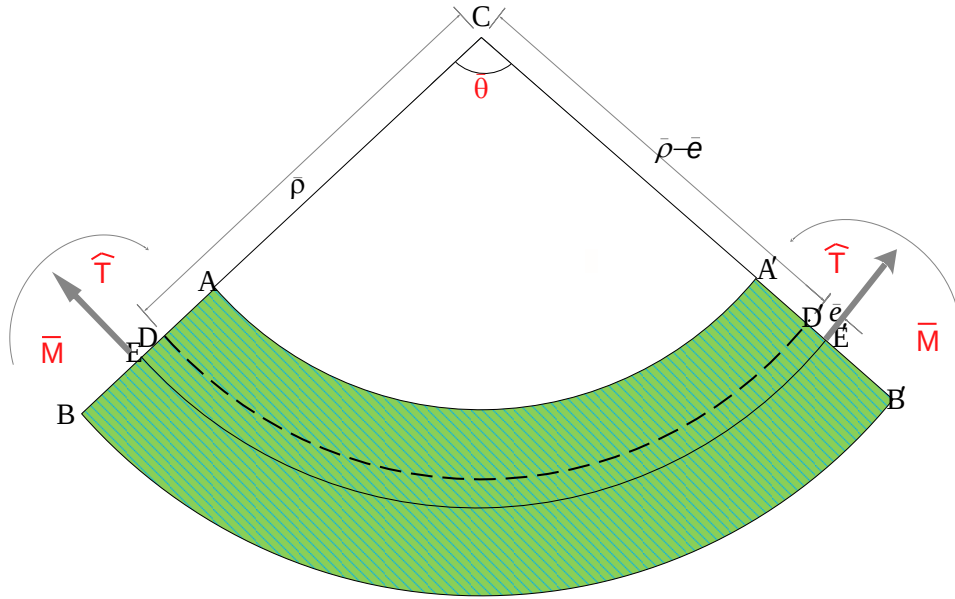


Figure 8.20: The schematic diagram of bending under external stretch tension.

From the analysis of the above sections, the moment distribution in different region has a highly similar evolution rule with respect to the corresponding bending curvature distribution. Therefore only the position distribution evolution and the moment distribution evolution with respect to the bending curvature for different in-plane tension force are discussed.

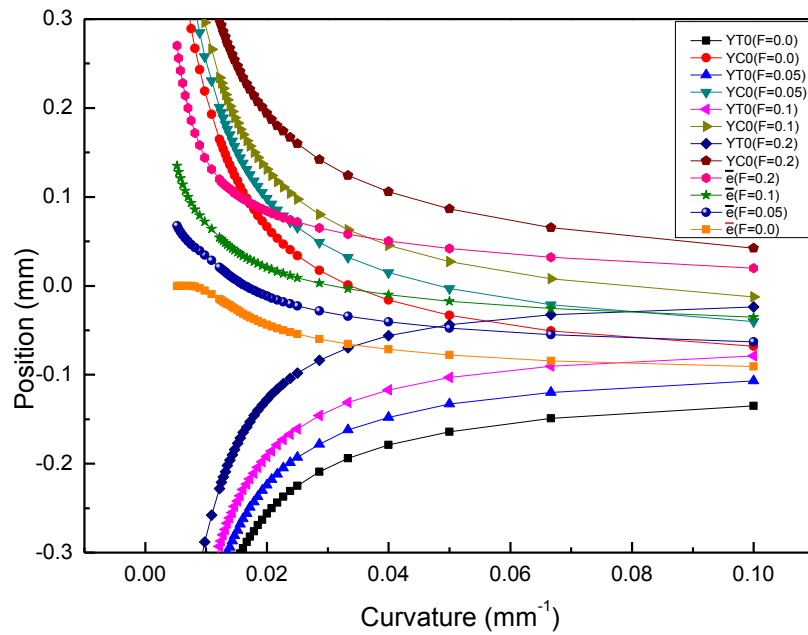
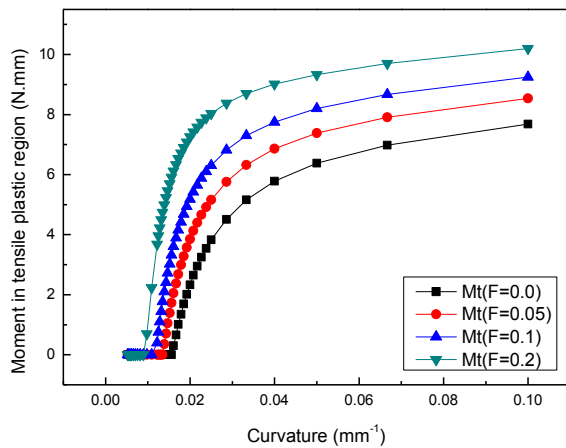


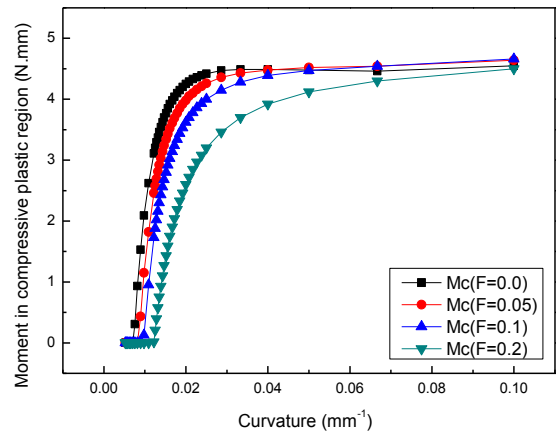
Figure 8.21: Position vs bending curvature with different in-plane tension forces.

The external in-plane force is denoted by a factor F . This factor F is defined in this section as the ratio between external in-plane force and the initial tensile yield strength. Figure 8.21 demonstrates the neutral surface position, the tensile yield critical position, and the compressive yield critical position evolution with respect to the bending curvature for different external in-plane tensile forces. From there, it can be concluded that with the increase of the external in-plane tensile force, all these three characteristic critical positions move toward inner surface. If compared with the tensile yield critical position and the compressive yield critical position, the neutral surface moves much more toward the inner surface of the sheet at the same external force. The pure elastic region reduces gradually even in the small bending curvature. Besides that, another interesting thing is also observed: with the increase of the bending curvature, the moved distance of the above three critical positions toward inner surface increases.

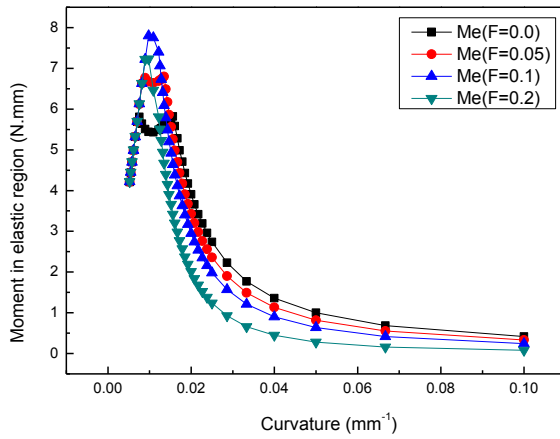
Figure 8.22 shows the moment distributions with different in-plane tension forces. From the comparison between the moment distribution in the tensile plastic region and in the compressive plastic region, it can be seen that the moment in the tensile plastic region increases with the increasing external tensile forces, and the larger the bending curvature is, the larger value the moment is. However, in the compressive plastic region, there is a mid-range of curvature, where the external force has a great influence on the moment. When the bending curvature surpasses this special range and continues to increase, the moment does not change substantially, but as a whole the moment in the compressive plastic region reduces when the external tensile force increases.



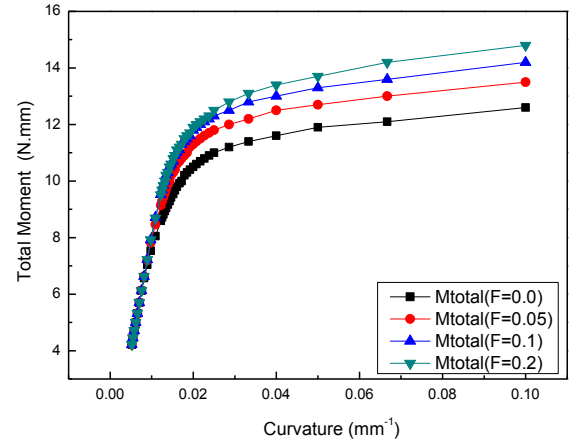
(a) Moment in the tensile plastic region



(b) Moment in the compressive plastic region



(c) Moment in the elastic region



(d) Total bending moment

Figure 8.22: Moment vs bending curvature with different in-plane tension forces.

Regarding the moment evolution curve in the elastic region with respect to the bending curvature, Figure 8.22 (c) demonstrates that with the increasing external force, the transition from the low bending curvature stage to the high curvature stage becomes more and more sharp. In this transition range of the bending curvature, the moment value alternates in an elaborated way. For the small external tensile force, there exists a concave down transition curvature range from the small curvature to high curvature, and showing that the higher the external force applies the higher the moment in the elastic region. When the large external tensile force applies, that concave transition vanishes, and is replaced by a sharp peak. In this case, the moment reduces when the external tensile force increases. In the high curvature range, the moment in the elastic region reduces steeply into almost zero, and the more external force it applies, the quicker the moment reduces. It is also notable that the total moment curve increases as the tensile force in Figure 8.22 (d).

Figure 8.23 shows the springback factor ζ evolution when the bending curvature changes with different in-plane tension forces. From there, it can be seen that the external in-plane tension force does not alternate the shape of the springback factor ζ evolution curve with the bending curvature. At the same bending curvature, the springback factor ζ has a slight increase with the increasing external in-plane forces. In a whole, the external in-plane stress has no great influence on the springback effect.

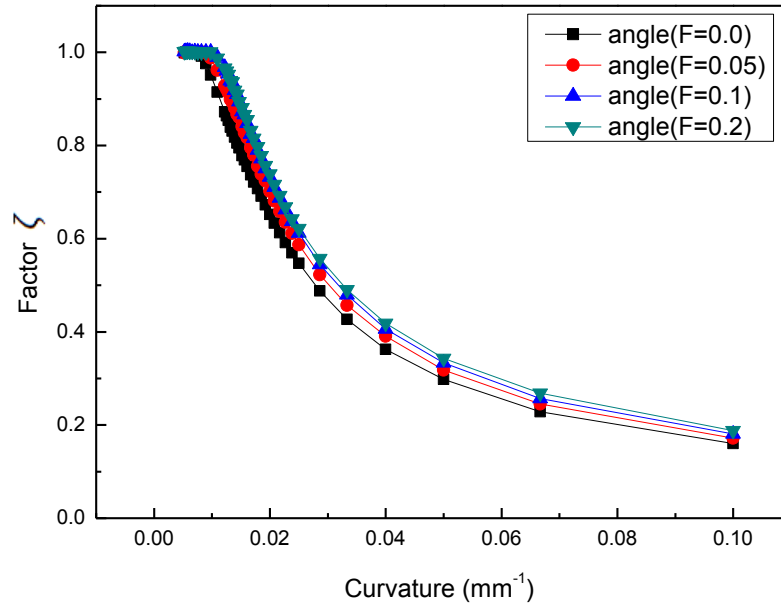


Figure 8.23: Springback factor ζ vs bending curvature with different in-plane tension forces.

8.6 Numerical simulation of springback

Nowadays, numerical simulations about springback prediction based on finite element methods replace time-consuming trial-and-error methods in tooling design [1, 22]. To be able to efficiently use finite element software to predict springback in sheet metal forming, springback needs to be considered as a complex physical phenomenon. The accuracy of springback calculation is strongly dependent on precise evaluation of the stress during forming. The accuracy of sheet metal forming and springback simulation depends not only on the forming conditions (contact, friction, tool and binder geometry, *etc.*), but also on the choice of the material constitutive model and its numerical implementation into finite element programs. Accurate modeling of springback in sheet metal forming requires an accurate understanding of this phenomenon. Some researchers have proposed various practical anisotropic yield criteria and successfully applied them in commercial sheet metal forming simulation software for materials with BCC or FCC crystal structure [27, 28]. However, for the HCP material, not only the anisotropy characteristic of the sheet metal, that plays a very important role in improving the realistic simulation precision, but also the asymmetrical effect, are both crucial factors affecting the prediction of springback. In the following sections, several anisotropic constitutive models with an asymmetrical hardening law are implemented in the implicit integration scheme, and applied to model the unconstrained cylindrical

bending test of Numisheet 2002 benchmarks. Besides, several critical factors that affect the simulation of springback in sheet metal forming are also investigated.

8.7 Fundamental review of constitutive model

In a phenomenological model, the mathematical description of the plastic behaviour uses three major concepts: the yield criterion, the hardening law and the flow rule.

The yield condition establishes a convex surface in the stress space, which bounds the elastic region. It can be expressed as a function of the so-called “state variables”, for an isotropic hardening law, by the general expression

$$\Phi(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = \sigma_{eq} - \sigma_Y(\bar{\varepsilon}^p), \quad (8.42)$$

where σ_Y is a scalar function representing the stress–strain hardening behavior, and defining the size of the yield surface. Also, σ_{eq} is the equivalent value of the stress tensor $\boldsymbol{\sigma}$, defined by the yield function.

8.7.1 Yield criteria review

A brief description of yield functions was addressed in detail in Chapter 2 and Chapter 7. This section presents several yield functions under plane stress condition.

(a) von Mises yield criterion

The von Mises criterion is a commonly used isotropic yield criterion. This criterion can be written as

$$\sigma_{Mises} = \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11})^2 + 6\sigma_{12}^2}{2}}. \quad (8.43)$$

(a) Hill yield criterion

The quadratic Hill yield criterion is widely selected to capture the anisotropic mechanical properties, which can be written under plane stress condition as

$$\sigma_{Hill} = [F_{Hill}(\sigma_{22})^2 + G_{Hill}(\sigma_{11})^2 + H_{Hill}(\sigma_{11} - \sigma_{22})^2 + 2N_{Hill}\sigma_{12}^2]^{1/2}, \quad (8.44)$$

where, F_{Hill} , G_{Hill} , H_{Hill} , L_{Hill} , M_{Hill} , N_{Hill} are material parameters.

(b) Barlat yield criterion

In order to include the shear stresses into yield function, Barlat *et al* [29] proposed a more general anisotropic yield criterion based on the Hosford’s criterion by expressing

it in an x, y, z coordinate system, not necessarily coincident with the principal directions. The formulation can be written as

$$\sigma_{Barlat} = \left(\frac{a_{BL}|K_1 + K_2|^{M_{BL}} + a_{BL}|K_1 - K_2|^{M_{BL}} + (2 - a_{BL})|2K_2|^{M_{BL}}}{2} \right)^{\frac{1}{M_{BL}}}. \quad (8.45)$$

The coefficients K_1 and K_2 are given by

$$K_1 = \frac{\sigma_{11} + h_{BL}\sigma_{22}}{2}, \quad (8.46)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2}\right)^2 + (b_{BL})^2\sigma_{12}^2}, \quad (8.47)$$

and a_{BL} , h_{BL} , and b_{BL} are material parameters.

(c) Cazacu 06 anisotropic yield criterion

To describe both the asymmetry between tension and compression, not disregarding the anisotropy observed in HCP metal sheets, Cazacu and Barlat [30] introduced a general and rigorous method which is based on the theory of tensor functions representation. A material parameter $\tilde{k}$ is introduced and the proposed orthotropic criterion can be expressed by

$$\sigma_{cazacu}^{\tilde{n}} = (|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}}, \quad (8.48)$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are the principal values of Σ . The transformed tensor Σ can be defined as:

$$\Sigma = \tilde{\mathbf{L}}\sigma, \quad (8.49)$$

where $\tilde{\mathbf{L}}$ is a 4th order tensor, which includes 7 independent anisotropy coefficients for plane stress conditions. The tensor $\tilde{\mathbf{L}}$ is represent by

$$\tilde{\mathbf{L}} = \begin{bmatrix} \frac{(2C_{11}-C_{12}-C_{13})}{3} & \frac{(-C_{11}+2C_{12}-C_{13})}{3} & \frac{(-C_{11}-C_{12}+2C_{13})}{3} & 0 \\ \frac{(2C_{12}-C_{22}-C_{23})}{3} & \frac{(-C_{12}+2C_{22}-C_{23})}{3} & \frac{(-C_{12}-C_{22}+2C_{23})}{3} & 0 \\ \frac{(2C_{13}-C_{23}-C_{33})}{3} & \frac{(-C_{13}+2C_{23}-C_{33})}{3} & \frac{(-C_{13}-C_{23}+2C_{33})}{3} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix}. \quad (8.50)$$

In order to ensure the convexity of yield function, the introduced parameter, $\tilde{k}$, should be within the range of $\tilde{k} \in [-1, 1]$. When the transformed matrix $\tilde{\mathbf{L}}$ is equal to the identity matrix, the proposed formulation can reproduce, as a limiting case, the von Mises yield criterion.

(d) New proposed yield criterion

In the chapter 7, we extent further the Barlat 89 model by adding the term of the absolute value of the principal stress, the extended yield criterion can be seen as

$$\sigma_{eq} = \left(\frac{a|(K_1 + K_2) - c|K_1 + K_2||^M + a|(K_1 - K_2) - c|K_1 - K_2||^M}{f|2K_2|^M} \right)^{1/M}, \quad (8.51)$$

where,

$$K_1 = \frac{\sigma_{11} + h\sigma_{22}}{2}, \quad (8.52)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h\sigma_{22}}{2}\right)^2 + b^2\sigma_{12}^2}, \quad (8.53)$$

and a , h , b , c and f are material parameters. Parameters a , h , f and b reflect the anisotropic mechanical behavior. Parameter c defines the asymmetrical strength between tension and compression stress state. In this contribution, the parameter f was calculated by

$$f = 2 - a. \quad (8.54)$$

The loading-unloading conditions can be expressed in Kuhn-Tuckner form as Equation (8.55).

$$\Phi \leq 0, \dot{\gamma} \geq 0, \dot{\gamma}\Phi = 0. \quad (8.55)$$

The phenomenon of hardening describes the changes in yield stress that result from plastic straining and the flow stress, σ_Y , represents the size of the yield function during deformation.

8.7.2 Hardening law

For processes in which the sheet metal is subjected to relatively linear loading paths (such as hemispherical punch stretching or cup-drawing), the simple assumption of isotropic hardening may provide accurate springback predictions.

An appropriate equation describing changes in the flow stress of the material depends on deformation conditions. The AZ31 magnesium alloy has asymmetric property, which means it has different strain and stress curves for tension and compression process. Based on this, different hardening law is proposed for the simulation. The same separated hardening laws are chosen as the analytical method to describe the uniaxial true stress-strain curve response in both tension and compression stress states. All the mechanical parameters can be identified by the related experimental test, which can be seen in Table 8.1.

8.8 Implementation of the constitutive model

In this section, the numerical solution strategy adopted in this work is summarized. The incremental solution of the initial boundary-value problem for elasto-plasticity problems requires a discrete time integration of the constitutive equations of the model over a time increment, from a known state at time t_n to the unknown state at t_{n+1} , given the total strain increment $\Delta\boldsymbol{\varepsilon}$. Therefore, the discretization of the constitutive equations within a generic pseudo-time interval $[t_n, t_{n+1}]$ is performed for all above mention constitutive models.

The numerical integration of elasto-plasticity constitutive equations is typically carried out by means of the so-called elastic predictor return mapping schemes. A fully implicit elastic predictor return mapping algorithm, which is called Closest Point Projection Method (CPPM), is implemented within an implicit quasi-static finite element environment. In the algorithm, a trial stress is made to relax to the closest point on the yield surface.

The elastic predictor step of the algorithm in the present case is clearly identical to that of the plane stress projected in von Mises model and, therefore, requires no further consideration. However, the return mapping step requires the solution of the algebraic system of equations.

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}^{e\,trial} + \Delta\gamma \cdot \mathbf{N}_{n+1} \\ \bar{\varepsilon}_{n+1}^p - \bar{\varepsilon}_n^p - \Delta\gamma \\ \boldsymbol{\Phi}(\boldsymbol{\sigma}(\boldsymbol{\varepsilon}_{n+1}^e), \sigma_Y(\bar{\varepsilon}_{n+1}^p)) \end{Bmatrix} := \begin{Bmatrix} \mathbf{0} \\ 0 \\ 0 \end{Bmatrix}, \quad (8.56)$$

where $\boldsymbol{\varepsilon}_{n+1}^e, \bar{\varepsilon}_{n+1}^p, \Delta\gamma$ are the unknowns and $\mathbf{N}_{n+1}$ is the flow vector.

The procedures of the elastic predictor step are described in details in Box 8.1.

Notice that, for the yield criteria (von Mises yield criterion, Hill 48 yield criterion) with a separated hardening law, the hydrostatic pressure P is used as the criteria to select the uniaxial tensile or uniaxial compressive stress-strain response curve as the hardening law in the constitutive model. For the Cazacu 06 yield criterion and the new proposed yield criterion in Chapter 7, only the uniaxial tensile stress-strain curve is considered as the hardening law, due to that the yield function can describe the asymmetrical mechanical behavior.

Box 8.1: The implementation of Closest Point Projection Method (CPPM) for springback simulation

- (i) Elastic predictor. Given $\Delta \boldsymbol{\varepsilon}$ and the state variables at t_n , evaluate the elastic trial state:

$$\begin{aligned}\boldsymbol{\varepsilon}_{n+1}^{e\ trial} &= \boldsymbol{\varepsilon}_n^e + \Delta \boldsymbol{\varepsilon}; \bar{\boldsymbol{\varepsilon}}_{n+1}^{p\ trial} = \bar{\boldsymbol{\varepsilon}}_n^p; \\ p_{n+1}^{trial} &= K \boldsymbol{\varepsilon}_{n+1}^{e\ trial}; \mathbf{S}_{n+1}^{trial} = 2G \boldsymbol{\varepsilon}_{n+1}^{e\ trial}; \\ \boldsymbol{\sigma}_{n+1}^{trial} &= \mathbf{S}_{n+1}^{trial} + p_{n+1}^{trial} \mathbf{I}; \\ \sigma_{eq\ n+1}^{trial} &= F(\boldsymbol{\sigma}_{n+1}^{trial}).\end{aligned}$$

- (ii) Check Plastic admissibility

For the yield criteria (von Mises, Hill '48) with a separated asymmetrical hardening model

If $p_{n+1}^{trial} > 0$ Then

$$\Phi = \sigma_{eq\ n+1}^{trial} - \sigma_Y^T(\bar{\boldsymbol{\varepsilon}}_{n+1}^{p\ trial});$$

Else

$$\Phi = \sigma_{eq\ n+1}^{trial} - \sigma_Y^C(\bar{\boldsymbol{\varepsilon}}_{n+1}^{p\ trial}).$$

End If

For the other yield criteria (Cazacu 06, New proposed yield criterion) with an isotropic hardening model

$$\Phi = \sigma_{eq\ n+1}^{trial} - \sigma_Y^T(\bar{\boldsymbol{\varepsilon}}_{n+1}^{p\ trial}).$$

If $\Phi \leq 0$

Then set $(\)_{n+1} = (\)_{n+1}^{trial}$ and Exit

- (iii) Return mapping. Solve the system of five equations using the Newton-Raphson iterative method

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\ trial} + \Delta \gamma \mathbf{N}_{n+1} \\ \bar{\boldsymbol{\varepsilon}}_{n+1}^p - \bar{\boldsymbol{\varepsilon}}_n^p - \Delta \gamma \\ \Phi = \sigma_{eq} - \sigma_Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

where

$$\mathbf{N}_{n+1} = \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}},$$

The solution is found for $\boldsymbol{\varepsilon}_{n+1}^e$, $\bar{\boldsymbol{\varepsilon}}_{n+1}^p$ and $\Delta \gamma$. The stress tensor can be obtained by

$$\boldsymbol{\sigma}_{n+1} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^e.$$

- (iv) Exit

8.9 Numerical results on the unconstrained cylindrical bending simulation

The unconstrained cylindrical bending and springback analysis from Numisheet 2002 international benchmark is chosen to test the solid-shell element performance in large bending deformation and springback analysis. Kim and Hyung Jong [31] completed the stamping experiments and provided the experiment parameters and test results. This example has been frequently taken by many researchers because it involves the simple die geometry but complex contact condition. Besides the accuracy of the element model can be conveniently evaluated. The FE model of unconstrained cylindrical bending is presented in Figure 8.24. The cylindrical punch and die are considered as rigid surfaces in the simulation. The rectangular blank is meshed with the reduced integrated four-noded shell element (S4R). The initial blank is a 120 mm×30 mm×0.6 mm square plate. The material used in the simulation is AZ31, and the material parameters are identified in Table 8.1. The whole process is separated into two steps. The first step, called forming step, consists in the cylindrical punch moving down to a specific position by setting the prescribed punch stroke, which is called stamping step. The second step is the springback step, where all the contact information among punch, blank and die are removed. After the forming process, the effect of dies on blank deformation is not considered during the springback prediction. The springback is considered as a process of residual stress unloading. The total process is conducted by a standard implicit scheme in the finite element software ABAQUS/standard solver.

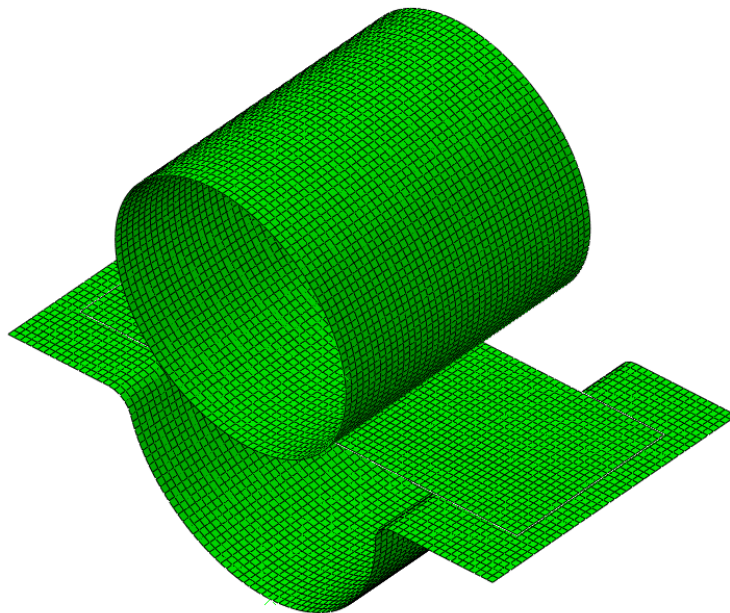


Figure 8.24: The geometric model of springback test for simulation.

The friction behavior was modeled using the Coulomb friction law. The friction coefficient μ_f between the blank and the punch was assumed to be the same as the friction coefficient μ_f between the blank and a die of 0.1.

The following set of boundary conditions has been applied: a reference point is specified on each rigid surface. For the reference point on the die surface all degrees of freedom (i.e. displacement in horizontal and vertical directions and rotation) have been suppressed. For the reference point on the punch surface, the displacement component in horizontal direction and the rotation are suppressed, while the displacement of the punch in vertical direction is specified. In the analysis of sheet metal forming it is common practice to assume that the elastic modulus remains constant.

There are several critical factors that influence the simulation of springback in sheet metal forming. A detailed study of numerical issues associated with springback prediction was carried out by Li *et al.* [22]. However, the parameters affecting the numerical simulation of springback are specific to each problem. Particularly, the material has a strong strength differential (SD) effect. Therefore, a detailed investigation is carried out to evaluate the effect of the integration point, yield criteria.

8.9.1 Analysis of the plastic deformation

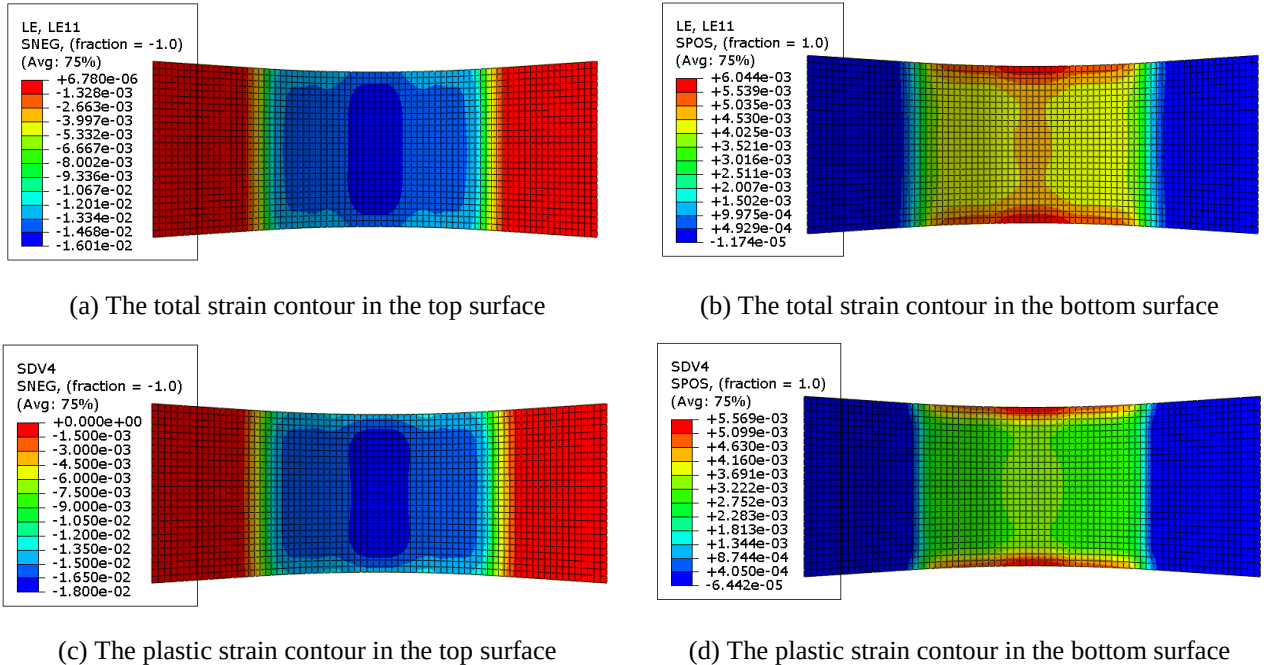


Figure 8.25: The strain contour in the top and bottom of the plate.

Figure 8.25 shows the strain distribution in the top and bottom surface after stamping process. Figure 8.25 (a) and Figure 8.25 (b) demonstrates the total strain contour on the

surface of the plate. Figure 8.25 (c) and Figure 8.25 (d) demonstrate the equivalent plastic strain contour on the surface of the plate. Here, the surface of “*Fraction* = -1 ” denotes the top surface, also denoting the surface that suffers compressive longitudinal stress. The surface of “*Fraction* = 1 ” denotes the surface that suffers tensile longitudinal stress. By comparison of the (a) and (b) in Figure 8.25, it can be seen that the magnitude of maximum strain in the top surface is much larger than in the bottom surface. It indicates that the neutral surface of the plate locates close to the bottom surface. In each transverse section, it is found that the material in the center of the top surface have a larger strain than the marginal part. However, the material on the bottom surface has an opposite distribution, while the material located in the margin of the edge has a larger strain. This rule also occurs in the plastic strain contour.

8.9.2 Effect of Number of Integration Point (NIP) through thickness

Most shell elements require numerical integration of stress through the thickness to obtain bending moments and tensile forces. This introduces numerical error in the results. Some parameters may affect the local strain/stress significantly, such as Number of Integration Points (NIP) through thickness.

Figure 8.26 shows the simulation results by the von Mises yield criterion using S4R with a variable number of integration points along the shell thickness. It can be observed that the springback angle increases with the increase of the integration point number through thickness. After the integration point is larger than 13, the springback angle gets to a stable value.

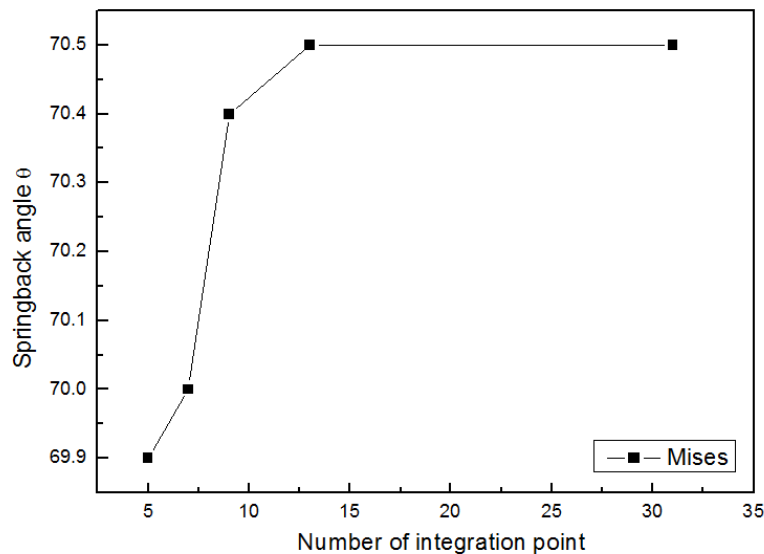


Figure 8.26: The springback angle simulated by von Mises with different NIP through thickness.

By comparison of the springback angle between the value of 5 integration point through thickness and the stable value, it is possible to find that the absolute magnitude gap is 0.6° , and the relative gap error is 0.85%. Many integration points are required to obtain springback results within 1% of the limiting case (with sufficient integration points that springback angle no longer changes with additional points). It means that five integration points could be enough to obtain good results. Taking under consideration the computational cost, five integration points through the thickness is the best choice for the symmetrical material.

Figure 8.27 presents the springback angle evolution as simulated by the proposed yield criteria with a variable number of integration points through thickness. From there, it is found that similarly to the results obtained by von Mises yield criteria, the springback angle increases with the increment of the integration point's number though thickness. However, it is seen that the absolute magnitude value between 5 integration point and the stable value is up to 2.60, the relative error is equal to 6.13%. This gap value is much larger than the results obtained by von Mises. It indicates that the influence of integration points on the asymmetrical material is much larger than the symmetrical material. That is due to the fact that the neutral surface is not coincident with the mid-plane of the plate, the strain though thickness is asymmetrical.

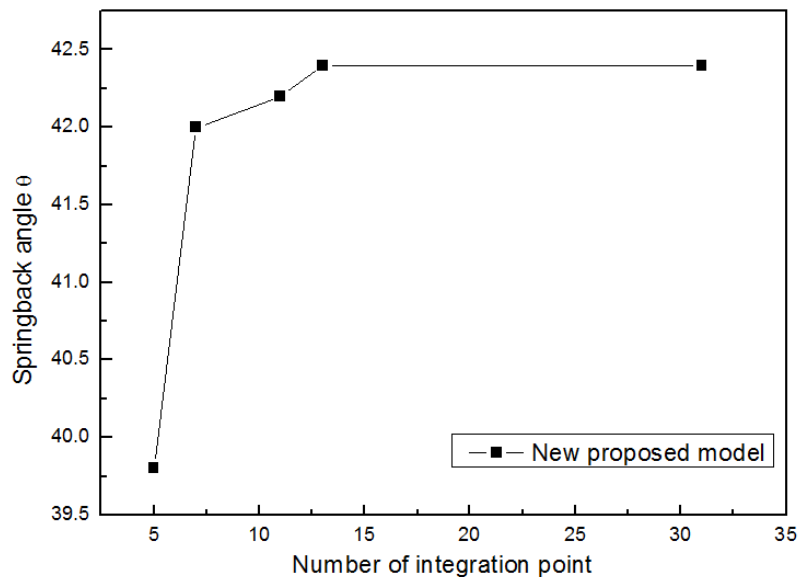


Figure 8.27: The springback angle simulated by the proposed yield model with different NIP through thickness.

As analyzed in the Section 8.5, for the magnesium alloy, the neutral plane moves toward the outside surface that is subjected to the tensile stress state. It results in that the

distance between neutral plane and outer surface becomes small, and the total stress distributed in these zone narrows. In order to obtain accurate results, it is necessary to increase the integration point number for the asymmetrical material.

A variety of techniques can be used for numerical integration. At their simplest, most rely on fitting an approximate function through a number of points at which the integrand is calculated, also integrating analytically the approximating function. ABAQUS supply two integration methods for the shell element in their library. One is Simpson integration method, the other is Gauss integration method. The Simpson integration method is the default option for the shell element in the ABAQUS. The simulated results obtained from different integration algorithm through thickness are presented in Figure 8.28. From there, it can be seen that the results obtained from the Gauss integration method have a narrower gap than the results obtained from the Simpson integration method, besides requiring a less number of integration points through thickness to get the stable springback value when compared with Simpson integration method. Therefore, it can be concluded that the Gauss integration point is a better choice in this particular case.

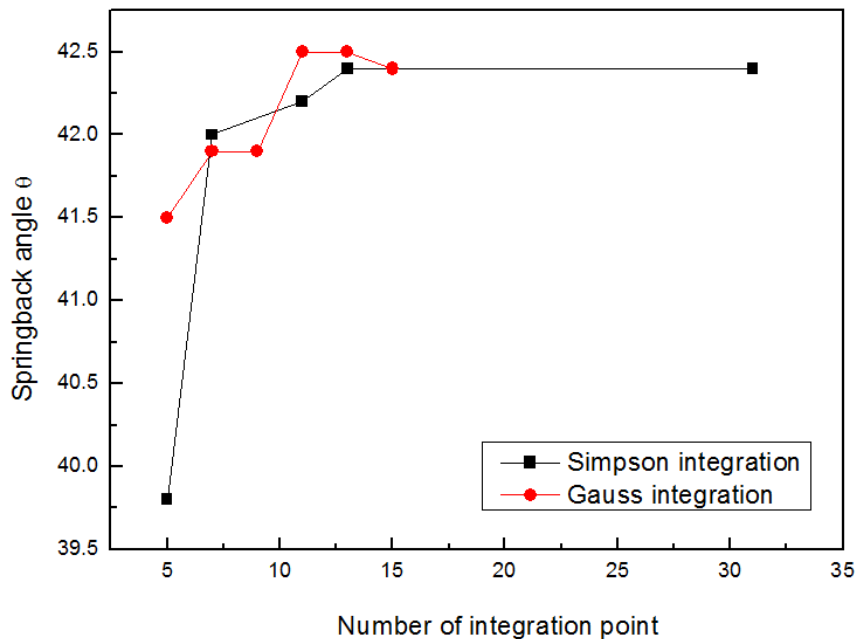


Figure 8.28: The comparison of two integration methods.

8.9.3 Influence of yield criteria on springback

The yield function used in the finite element simulations of the forming operations is very important in the view of prediction springback, since they represent a convex yield stress surface in the stress field, which limits the elastic range of the materials. In order

to study the influence of the yield criteria on the springback, several yield criteria, like von Mises, Hill 48, Barlat 89 and Cazacu 06 yield criterion are implemented in the present study. In order to describe the asymmetrical hardening, several authors also use a simple method to modify classical anisotropic constitutive models by adopting different stress-strain responses for the tension and compression stress state, respectively.

In this study, the von Mises and Hill 48 criteria are also implemented by the above separated mechanical response. In order to avoid the influence of the number of integration point, total plate is meshed with shell element S4R, where 13 integration points are used through thickness.

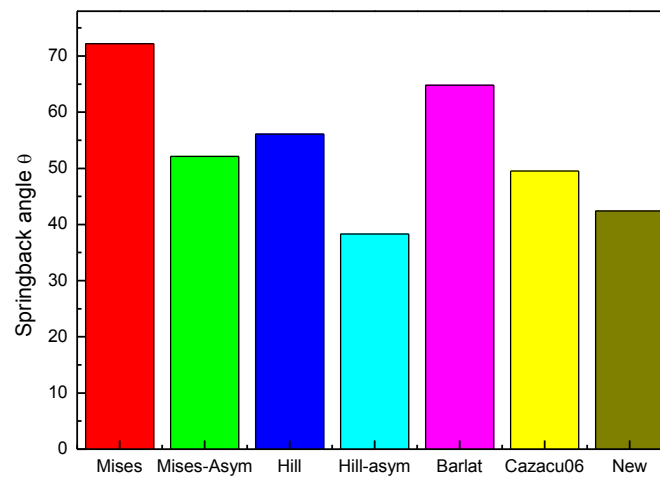


Figure 8.29: Springback back simulated by different yield criteria.

Figure 8.29 demonstrates the springback angle simulated by the above different yield criteria. Mises-Asym denotes that the results are obtained by adopting the von Mises yield criterion with a separated isotropic hardening under tension and compression stress state, respectively. Hill-asym means Hill yield criterion with a separated isotropic hardening law. From there it is clearly seen that all the yield criteria which cannot reflect the asymmetrical strength effect, like classical von Mises. Hill and Barlat yield criteria is larger than the other yield criteria which consider the SD effect, also having a great deviation from the analytic results. It means that the SD effect has a great influence on the springback. By the comparison of the remaining results obtained by the simulation and analytic criteria, it is clear that the result obtained by Cazacu 06 yield

criteria is close to the prediction by analytic methods. Hill-asym model can also give a good prediction in this simple benchmark case.

8.10 Conclusions

In this chapter, two types of methods (analytical and numerical method) were used to study the springback behavior of Magnesium alloy. Therefore, the chapter was divided into two parts.

In the first part, a generic analytical method was proposed. This model can not only be used to the prediction of the symmetrical model, but also the unsymmetrical model. This model is available for arbitrary mechanical hardening behavior. The verification and validation of this analytical model was performed by experimental and simulation analyses. After that, it was applied to analyze the mechanics of the AZ31 plate during bending process. The influence of the sheet thickness, bending curvature, strain rate and in-plane external tension stress on the neutral surface distribution, the distribution of the force and the moment in elastic, compressive plastic, tension plastic of the plate section were investigated, respectively.

- For the bending of unsymmetrical material, there exist four typical regions: compression plastic region, compression elastic region, tension elastic region and tension plastic region. These four regions are not symmetrical; the neutral surface is not coincided with the mid-plane. With the increasing bending curvature, the compression region expanded. The neutral surface also moved toward outside surface.
- By the comparison of springback factor of plates with different thickness, the springback factor presented a significant increase when the plate thickness augmented.
- With an applied external in-plane tension force, the bending moment increased with the increasing external tension force. The incremental moment gap of each external tension force augmented with the increasing bending curvature.

In the second part, a series of constitutive models were implemented by user-defined material subroutines into the commercial nonlinear finite element code ABAQUS. The CPPM numerical algorithm was used to integrate the stress and other state variables over each time increment. All these constitutive models can be categorized into two types. One is the conventional yield criteria coupled with pure isotropic hardening model. These constitutive models itself can only reflect the anisotropic properties. The

other model can capture anisotropic and SD effect simultaneously. By comparison of the springback angle of the unconstrained cylindrical bending test using the above anisotropic models, the model considering the SD effect was able to obtain a more accurate prediction than the previous model. The elaborated model takes into account the anisotropic behavior of materials and asymmetrical strength at the same time. Then several simulations with varied integration number through thickness were performed. It was concluded that for the symmetrical material, 5 integration points through the thickness are sufficient to obtain a convergence solution for springback, and the asymmetrical model is more sensitive of NIP through thickness than the symmetrical material. Therefore, it is necessary to employ a higher number of integration points through thickness.

8.11 References

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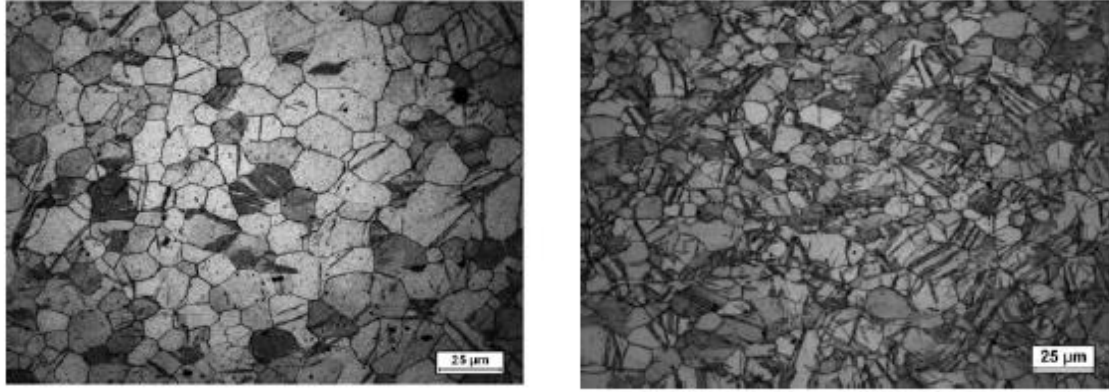
Chapter 9

An Extended Elasto-Viscoplastic Model: Formulation and Numerical Implementation

9.1 Introduction

Even though large numbers of studies on the plastic deformation behavior of magnesium (Mg) alloys have been carried out, most of them have their focus on the materials response under quasi-static loading conditions [1-5]. However, in some structural applications, components must be designed to operate over a broad range of strain rates and temperatures. For instance, the high strain rate behavior is of great interest to automotive, aerospace and/or defense industries since wide range of critical components must have the proper mechanical properties to work under severe loading conditions, such as crash or impact. Furthermore, in most metalworking processes, materials undergo large amounts of strain at different strain rates, such as electromagnetic forming. Strain hardening and strain-rate sensitivity, determining the formability of magnesium alloys, have been identified as important factors to take under consideration in the effort of making this material compete with other structural materials for practical applications. Therefore, it is inevitable to study the mechanical behavior of Mg alloy at dynamic strain rates. Kocks and Mecking [6, 7] observed that strain hardening rates are different for different deformation modes. They attributed this behavior to differences in the texture evolution behavior. In Mg alloys, the variations in the strain hardening behavior can also be ascribed to the influence of the initial texture on slip and twinning. Ulcacia *et al* [8-10] carried out a series of experiments and investigated the mechanical behavior at high strain rates of AZ31 magnesium alloy sheet, as well as the influence of the strain rate and temperature on the deformation mechanisms by means of Electron Back Scatter Diffraction (EBSD) and neutron diffraction. It is shown which amount of mechanical twinning occurs at a similar plastic work under different strain rates. It is concluded that the twinning plays an important role on high strain rate deformation of this alloy. The microstructure of AZ31 is presented in Figure 9.1. Although twinning cannot dominate plasticity at large strains, since the shear strains are modest even at large twinned volume fractions, deformation twinning can give rise to a radical reorientation of the volume fraction of the crystal that

has twinned. This leads to substantial texture modification. Subsequent plasticity by slip in a twin-modified texture then progresses quite differently due to the anisotropic nature of the deformation mechanisms. These amounts of occurred mechanical twinning undoubtedly alter the yield asymmetrical mechanical behaviors at high strain rates from the quasi-static conditions owing to their polarity.



(a) $T=20\text{ }^{\circ}\text{C}$, $\dot{\epsilon} = 10^{-3}\text{ s}^{-1}$, $\epsilon \approx 0.2$

(b) $T=20\text{ }^{\circ}\text{C}$, $\dot{\epsilon} = 10^3\text{ s}^{-1}$, $\epsilon \approx 0.28$

Figure 9.1: Microstructures of AZ31B magnesium alloy sheet under different strain rates [9].

In order to be able to accurately simulate some high impact processes, including metal forming, crashworthiness, and high velocity impact problems, it is necessary to understand the material behavior at different deformation conditions. This entails the need to develop the constitutive behavior that describes the material behavior at various deformation conditions. In recent years, numerous studies [11, 12] have been conducted with the elasto-viscoplastic Crystal Plasticity Finite Element Method (CP-FEM) in order to understand the micro-mechanisms involved and capture the macro mechanical behavior. But its high computational cost limits its application to real sheet metal forming. Phenomenological yield surface functions are still a good choice in engineering practice. The phenomenological models are actually designed for simplicity of computational implementation for static and dynamic analysis as they are relatively easy to calibrate with a minimum set of experimental data in the form of stress–strain curves at different strain rates and temperatures. Although numerous computational models with the existing yield criteria [6, 13, 14] such as Hill from 1948 and 1990, Hosford, Barlat, or Barlat and Lian, Karafillis *et al* have been proposed to describe temperature and strain-rate effects on the inelastic response, all of them can give rise to robust predictive methods for simulating the anisotropic high strain-rate behavior of Body Centered Cubic (BCC) and Face Centered Cubic (FCC) polycrystals. However, they fail to capture this unconventional mechanical behavior of HCP crystal structure

materials. Cazacu and Plunkett *et al.* [15] used the Perzyna's theory of viscoplasticity with the Cazacu 06 yield for describing the high strain-rate and large deformation behavior of HCP polycrystals. Although the Cazacu 06 model used the transformed principal stress with a fourth-order linear transformation operator on the Cauchy stress tensor, to simultaneously account for the initial plastic anisotropy and strength differential (SD) effect, it is also able to reflect both the anisotropic and yield asymmetrical mechanical behavior. In spite of this, it still fails to describe the rate-dependent yield surface, i.e. the influence of mechanical twinning occurred at high strain rate on the yield surface. It cannot only describe the uniformly expansion of the material's yield loci as the strain rates is increased to higher values.

Therefore, the objectives of this chapter are the development of rate-dependent material constitutive models that integrate a nonlinear elasto-viscoplastic model and their implementation into an implicit nonlinear finite element scheme. The proposed model employs an extended Cazacu 06 yield surface which accounted for the influence of strain rate on material response based on Perzyna's theory of viscoplasticity. The viscoplastic strain is modelled using Perzyna's viscoplasticity theory. The outline of this paper is as follows: in section 9.2, the flow curves at different strain rates are fitted by the Perzyna viscoplastic model. The material parameters inside this model are identified by the least square optimization method by using the experimental data at different strain rates. In section 9.3, the Cazacu 06 yield criterion is briefly described. In section 9.4, the material parameters in the viscoplastic constitutive equations are modeled as a function of strain rate. Section 9.5 implements the extended elasto-viscoplastic model using a primal Closest Point Projection Method (CPPM) within an implicit finite element environment. The numerical algorithms and computational implementation of the constitutive model in the FE code ABAQUS are presented in detail. In section 9.6, several numerical examples are performed in order to validate and test the formulated constitutive model.

9.2 Determine the flow curves at different strain rates

Khan *et al* [16] carried out standard quasi-static and dynamic tensile or compressive test to study the mechanical response and texture evolution of AZ31 at large strains for different strain rates. The true stress-strain data at room temperature under different strain rates in literature [16] is extracted in this study to characterize the relevant rate-dependent mechanical behavior and on this basis to determine material parameters of

the flow curve. In order to obtain the effective true stress–true plastic strain curves, the plastic strain is calculated from the true stress–true strain data by removing the elastic strain, which can be determined by

$$\varepsilon_{\theta}^p = \varepsilon_{\theta} - \frac{\sigma}{E}, \quad (9.1)$$

where E is Young's modulus and ν is Poisson's ratio, and σ is the true stress.

ε_{θ}^p is the plastic strain, ε_{θ} is the total true strain.

The term flow stress is used to describe the stress necessary to continue deformation at any stage of plastic strain. Mathematical descriptions of true stress–strain curves are needed in engineering analyses that involve plastic deformation, such as predicting energy absorption in automobile crashes, designing of dies for stamping parts, analyzing the stresses around cracks, *etc.* This mathematical equation usually is a function of strain with many empirical constants by fitting the experimental test data. There are two approaches to determine the constitutive behavior relation of AZ31 magnesium alloy. The first is describing the behavior in terms of microstructure, deformation conditions and various materials constants related to the properties of the material. The second approach solely describes the behavior in terms of the deformation conditions and various material constants. It is also named engineering models as they are more common in engineering applications. Usually, these empirical models have little physics in them and usually the constants cannot be related to any physical process or phenomenon. In spite of this inherent disadvantage, these models are extensively used in FEM codes to model plasticity as they require low computational resources and provide excellent fits for a given deformation condition. Owing to that the AZ31 magnesium alloy has asymmetric mechanical behavior, which means it has different strain and stress curves for tension and compression process, and different flow laws should be chosen to describe the uniaxial true stress-strain curve response in tension and compression stress state respectively. Dynamic flow curve modeling should take into account parameters such as strain, strain rate in order to accurately capture the strain rate sensitivity behaviour of materials. A constitutive relation needs to define the dependence of flow stress on coupled effect of strain hardening, and strain rate. In order to integrate the influence of strain rate into the usual flow equation at room temperature, the flow equation is expressed as multiplicative terms, as follows:

$$\sigma = \sigma(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p) = \sigma_Y(\bar{\varepsilon}^p) \cdot \tilde{g}(\dot{\bar{\varepsilon}}^p), \quad (9.2)$$

where, $\sigma_Y(\bar{\varepsilon}^p)$ represents the usual flow law part, which is independent on strain rate. $\tilde{g}(\dot{\varepsilon}^p)$ is a function of strain rate. A choice of different constitutive equations is available to describe strain rate dependency, such as piecewise linear plasticity, plastic kinematic hardening, *etc.* In this contribution, Perzyna's overstress function is adopted to govern the time dependent straining of the material. Therefore, strain rate part can be written as

$$\tilde{g}(\dot{\varepsilon}^p) = \left(1 + \mu \dot{\varepsilon}^p\right)^{\hat{\varepsilon}}. \quad (9.3)$$

9.2.1 Identification of the material parameters of flow rules

The calibration of the flow law is performed using an optimization procedure, regardless of its complexity. To do so, we begin by defining an error parameter to quantify the error between a simulated curve and the corresponding experimental data. The error in stress between simulation and experiment is measured at several discrete plastic strain values. Each error value is divided by the measured experimental stress value at that plastic strain level. The quadratic values of these normalized error values are then averaged as follows:

$$Error = \frac{1}{num} \sum_{i=1}^{num} \left(\frac{\bar{\sigma}_i^{sim}}{\bar{\sigma}^{exp}} - 1 \right)^2, \quad (9.4)$$

where the subscript *num* defines the number of experimental tests used to measure stress points. The subscripts *sim*, *exp* refers respectively to the simulated values by proposed flow equations, and experimental values.

To this end a least-squares error functional is considered as an identification criterion in order to minimize the distance of the simulated data to the experimental data. A general framework and technical details for the minimization of the least-squares error functional is presented elsewhere, see [17, 18].

9.2.2 Tensile flow curve

As regarding the strain hardening part, considerable effort has been directed toward developing empirical relations that describe the hardening behavior of polycrystalline materials. Different hardening laws have been proposed by several authors, such as Ludwik [19], Swift [20], and Voce [21]. These hardening laws have been widely used to describe the work-hardening behavior of materials. In order to accurately describe the flow behavior of the materials at different strain rates, the true stress (σ) - plastic strain

$(\bar{\varepsilon}^p)$ data along the rolling direction is fitted to three modified flow models by integrating the Perzyna's overstress function into the above mentioned work hardening models- modified Ludwik law, modified Swift law and modified Voce law.

(i) Modified Ludwik law

$$\sigma_{lT} = [\sigma_{lo} + A_l(\bar{\varepsilon}^p)^{n_l}] \left(1 + \hat{\mu}_l \dot{\bar{\varepsilon}}^p\right)^{\hat{\varepsilon}_l}, \quad (9.5)$$

where: σ_{lo} is the initial strength (MPa), A_l is material parameter, n_l is the strain-hardening exponent, $\hat{\varepsilon}_l$ is the strain-rate sensitivity index, $\hat{\mu}_l$ is the viscosity-related parameter.

(ii) Modified Swift law

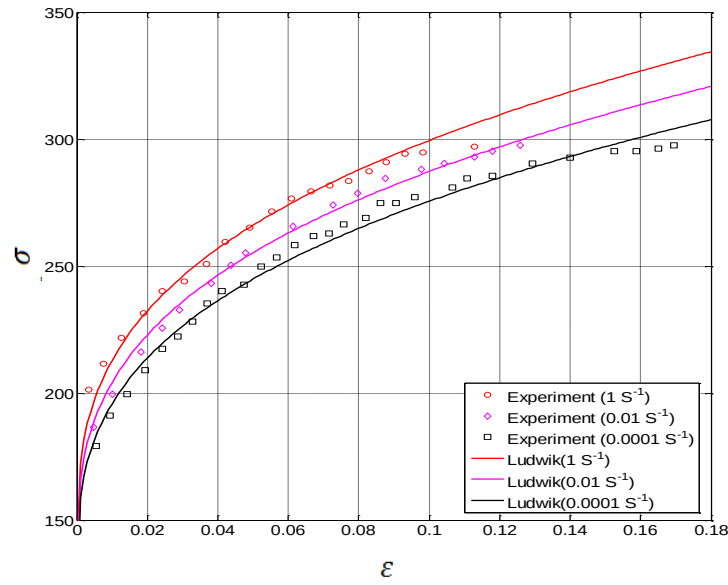
$$\sigma_{sT} = K_s(\varepsilon_{so} + \bar{\varepsilon}^p)^{n_s} \left(1 + \hat{\mu}_s \dot{\bar{\varepsilon}}^p\right)^{\hat{\varepsilon}_s}, \quad (9.6)$$

where: K_s is the strength coefficient (MPa), ε_{so} is the pre-strain, n_s is the strain-hardening exponent, $\hat{\varepsilon}_s$ is the strain-rate sensitivity index, $\hat{\mu}_s$ is the viscosity-related parameter.

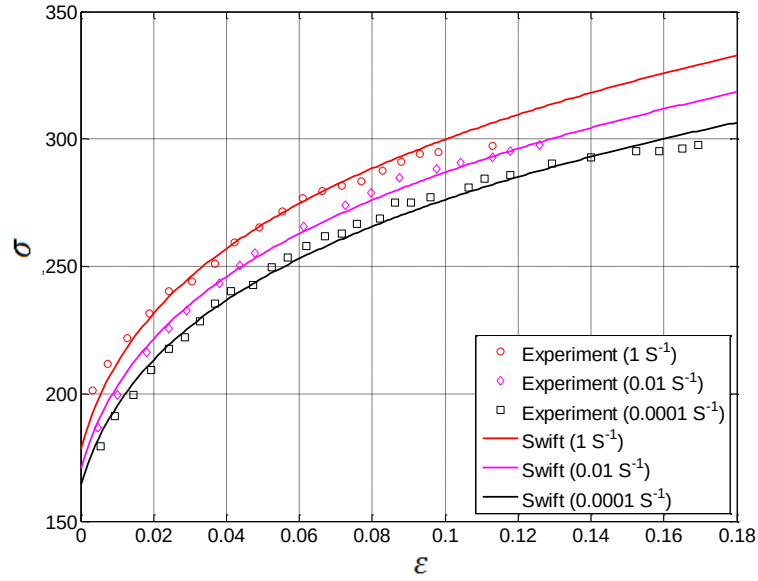
(iii) Modified Voce law

$$\sigma_{vT} = \left[A_v + B_v \left(1 - e^{-C_v \bar{\varepsilon}^p}\right)\right] \left(1 + \hat{\mu} \dot{\bar{\varepsilon}}^p\right)^{\hat{\varepsilon}}, \quad (9.7)$$

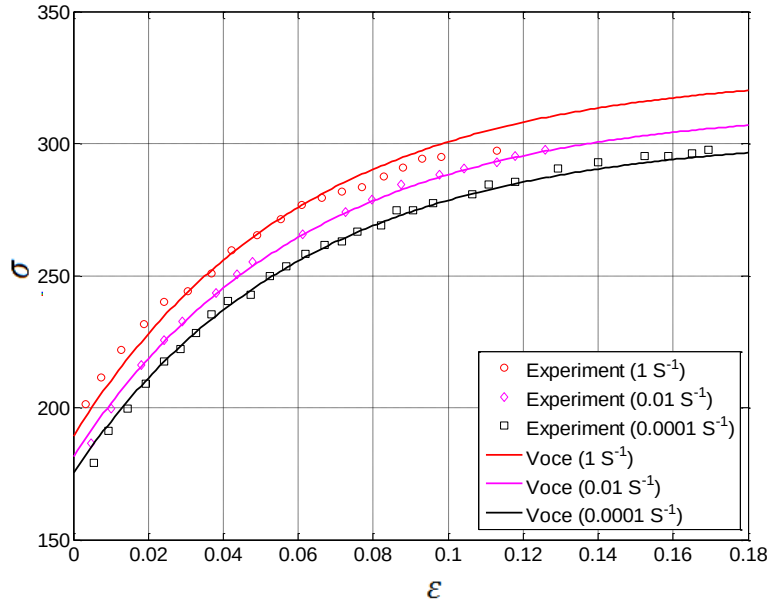
where: A_v, B_v, C_v is the material parameters, $\hat{\varepsilon}$ is the strain-rate sensitivity index, $\hat{\mu}$ is the viscosity-related parameter.



(a) Modified Ludwik flow law



(b) Modified Swift flow law



(c) Modified Voce flow law

Figure 9.2: Stress-strain relations for AZ31 under tensile test at RD direction.

It must be noted that the reason that a proper flow function generally gives appropriate results for applications involving monotonic loading is that the functions presented in Equation (9.5) to Equation (9.7) are capable of providing a suitable fit to the uniaxial tensile stress-strain curve for different orientations with respect to the rolling direction. After material parameters identification using a least-squares method, three flow laws (Modified Ludwik flow law, Modified Swift flow law, Modified Voce flow law) predictions for the three different strain rates illustrated in Figure 9.2 have been obtained. It can be seen that significant increase of stress response is observed with the

increase of strain rate. All the models are more or less capable of describing the trends at various strain rates. However, by the comparison of errors among different flow laws in Figure 9.3, it is clearly seen that the modified Voce model has least error value and meet a better accordance between experiments and simulations than the other two models.

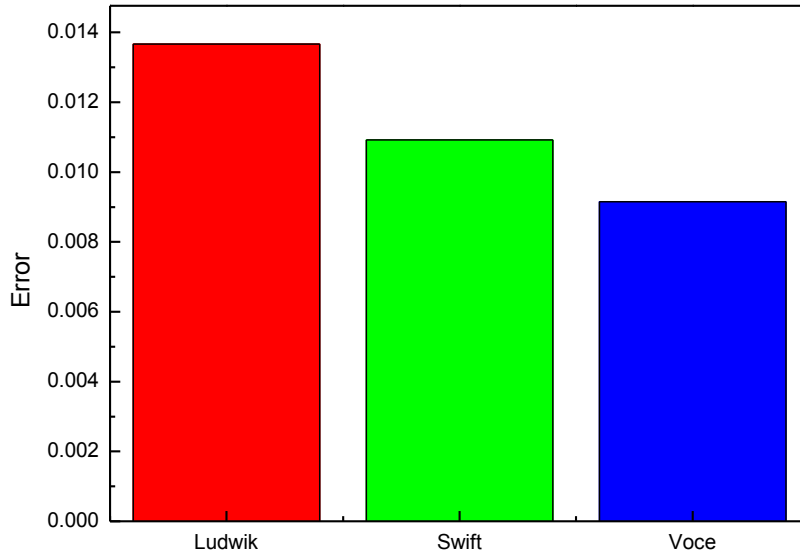


Figure 9.3: Error comparisons among different flow laws.

9.2.3 Compressive flow curves

Owing to that the magnesium alloy demonstrates an asymmetrical mechanical behavior between tension and compression test, the stress-strain curve under compression test exhibits concave start-up phase caused by the occurrence of massive twinning, which is obviously different from the stress-strain curve under tension test. Thus, the mathematical description of the characteristic of stress-strain curves under compression test is available to use the same flow equation as tensile test. It is obvious that it should be a complex mathematical function to describe the flow behavior under compression test.

After the analysis of physical essence and curve shape of compressive flow curve, a new flow law is proposed to capture the atypical stress-strain response, which can be written as

$$\sigma_{VC} = \left[\sigma_{VC0} + K_{VC} * \left[1 + \tanh[Q_{VC}(\bar{\epsilon}^p + \epsilon_{VC0})] \right] \right] \left(1 + \hat{\mu}_{VC} \bar{\epsilon}^p \right)^{\hat{\epsilon}_{VC}}, \quad (9.8)$$

where σ_{vc0} , K_{vc} , Q_{vc} , ε_{vc0} is material parameters identified by the related compressive test, $\hat{\varepsilon}_{vc}$ is the strain-rate sensitivity index, $\hat{\mu}_{vc}$ is the viscosity-related parameter.

For evaluation of parameters of the function, the same calibration procedure as tensile test was used, with the utilization of the gradient optimization method. A special attention was dedicated to the initial estimates of the parameters of the function. The progression of the calibrated functions, in comparison with experimental data, is demonstrated in Figure 9.4.

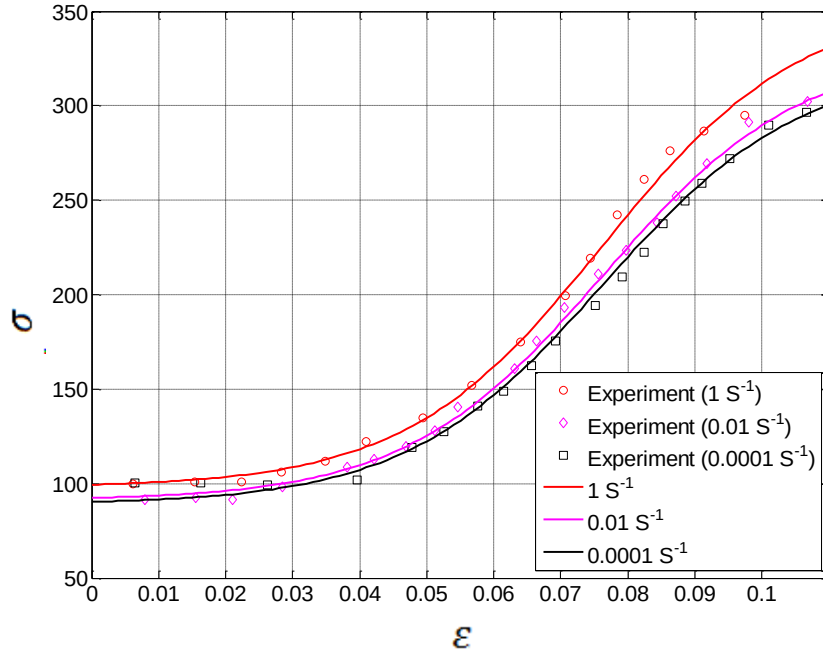


Figure 9.4: Stress-strain relations for AZ31 under compressive test at RD direction.

It is evident that the determined functions describe accurately the experimental data. By comparison with the stress strain response under tensile test, the tension-compression asymmetry is generally observed. These stress differences are consistent with the predominance of crystallographic slip in tension and $\{10\bar{1}2\}$ twinning followed by strain hardening in compression. Although the strain hardening rate also increases with increasing strain rate, the influence of strain rate on the flow stress under compressive test is clearly less than tensile test. This means that these dynamic mechanical properties of AZ31 magnesium sheet enhances further SD effect, and it may affect the evolution of yield surface during the deformation.

9.3 Constitutive formulation of elasto-viscoplastic model

Regarding viscoplastic model, material behavior there are basically three different approaches [22]: formulations with a yield surface, which demarcate an elastic domain,

formulations without yield surfaces and, lastly, formulations based on overstress. Generally, for the construction of the viscoplastic material model, the total stress is decomposed into two parts, namely a rate-independent equilibrium stress σ_{eq} and a rate-dependent overstress σ^v . However, due to the influence of the strain rate on the shape of yield surface of HCP structure materials, the equilibrium stress part of the constitutive model should also be strain rate dependent. The objective of this section is to develop a macroscopic anisotropic model that describes simultaneously the influence of strain-rate, and evolving texture on the inelastic response of metals with HCP crystalline structure.

9.3.1 Kinematic and flow rule for elasto-viscoplasticity

The constitutive equations are formulated within a geometrically linear theory for small strains. The strain-rate $\dot{\boldsymbol{\varepsilon}}$ can be decomposed additively into an elastic part $\dot{\boldsymbol{\varepsilon}}^e$ and a viscoplastic part $\dot{\boldsymbol{\varepsilon}}^{vp}$,

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{vp}. \quad (9.9)$$

According to the Perzyna's overstress approach, the viscoplastic strain can be written as

$$\dot{\boldsymbol{\varepsilon}}^{vp} = \gamma \langle \hat{\Phi} \rangle \mathbf{N}. \quad (9.10)$$

The evolution of the viscoplastic strain-rate is considered to be given by an overstress type law of the form:

$$\dot{\gamma} = \frac{1}{\hat{\mu}} \hat{\Phi}(\sigma_{eq}) = \begin{cases} \frac{1}{\hat{\mu}} \left[\left(\frac{\sigma_{eq}}{\sigma_Y} \right)^{1/\hat{\varepsilon}} - 1 \right], & \text{if } \hat{\Phi} \geq 0 \\ 0, & \text{if } \hat{\Phi} < 0 \end{cases}, \quad (9.11)$$

where, γ is a viscosity parameter, and $\hat{\varepsilon}$ is a strain-rate sensitivity constant.

σ_{eq} is the yield function, which is different from the quasi-static yield function. It can be expressed as the function of stress tensor, accumulated plastic strain and strain rate, $\mathbf{N}$ is a vector normal to the above yield function, which can be defined (in a rate form) as

$$\mathbf{N} = \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}}. \quad (9.12)$$

9.3.2 Cazacu 06 yield formulation

To describe the strength differential effect between tension and compression, a material parameter $\tilde{k}$ is introduced and the proposed orthotropic criterion by Cazacu and Plunkett *et al* [14]. This model has been presented in detail in Chapter 6, and will be summarized here for completeness. It can be expressed by

$$\sigma_{eq}^{\tilde{n}} = (|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}}, \quad (9.13)$$

where $\Sigma_1, \Sigma_2, \Sigma_3$ are the principal values of Σ . The transformed tensor Σ , which can be defined as:

$$\Sigma = \tilde{\mathbf{L}}\sigma, \quad (9.14)$$

where $\tilde{\mathbf{L}}$ is a fourth order tensor, which includes nine independent anisotropy coefficients for three dimensional stress conditions.

The transformed matrix $\tilde{\mathbf{L}}$ can be obtained by

$$\tilde{\mathbf{L}} = \begin{bmatrix} \frac{(2C_{11}-C_{12}-C_{13})}{3} & \frac{(-C_{11}+2C_{12}-C_{13})}{3} & \frac{(-C_{11}-C_{12}+2C_{13})}{3} & 0 & 0 & 0 \\ \frac{(2C_{12}-C_{22}-C_{23})}{3} & \frac{(-C_{12}+2C_{22}-C_{23})}{3} & \frac{(-C_{12}-C_{22}+2C_{23})}{3} & 0 & 0 & 0 \\ \frac{(2C_{13}-C_{23}-C_{33})}{3} & \frac{(-C_{13}+2C_{23}-C_{33})}{3} & \frac{(-C_{13}-C_{23}+2C_{33})}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (9.15)$$

When the transformed matrix $\tilde{\mathbf{L}}$ is equal to the identity matrix, the proposed formulation can reproduce, as a limiting case, the von Mises yield criteria. In order to ensure the convexity of yield function, the introduced parameter, $\tilde{k}$, should be within the range of $\tilde{k} \in [-1, 1]$.

9.4 Parameter identification of constitutive model

A similar identification procedure to the strain rate independent constitutive model in Chapter 6 is adopted. Experimental results from uniaxial tension and compression yield stresses in rolling direction, 45° to rolling direction, and transverse direction, and r -ratios in the rolling, and transverse to rolling directions for the uniaxial tensile case are used. The error function is also based in the summation of the squares of the errors defined from a comparison between the results obtained from the constitutive equations for a given set of parameters to be identified and the corresponding experimental values. The difference is that all the data should be selected at the same plastic strain and strain rate level. A hybrid evolutionary optimization method is done to minimize the comparative errors between numerical results, obtained from the constitutive equations at each accumulated plastic strain and strain rate. In this contribution, three accumulated strain levels are chosen, 0.03, 0.06, and 0.1. Three strain rate are adopted, 0.0001 s^{-1} , 0.01 s^{-1} , 1 s^{-1} .

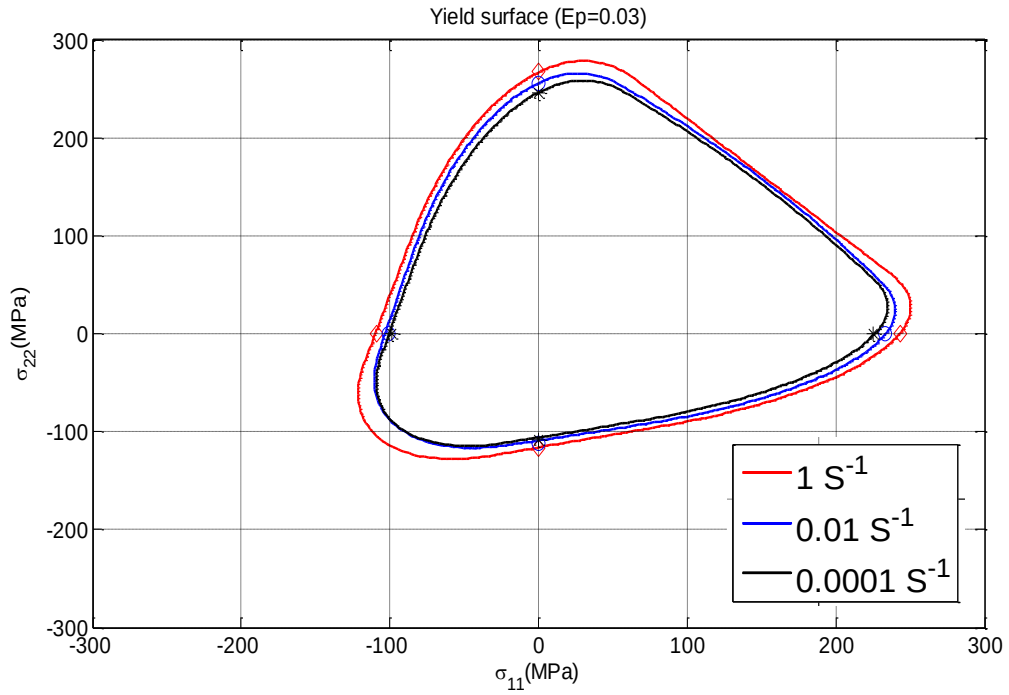


Figure 9.5: Yield surface evolutions with different strain rates at plastic strain of 0.01.

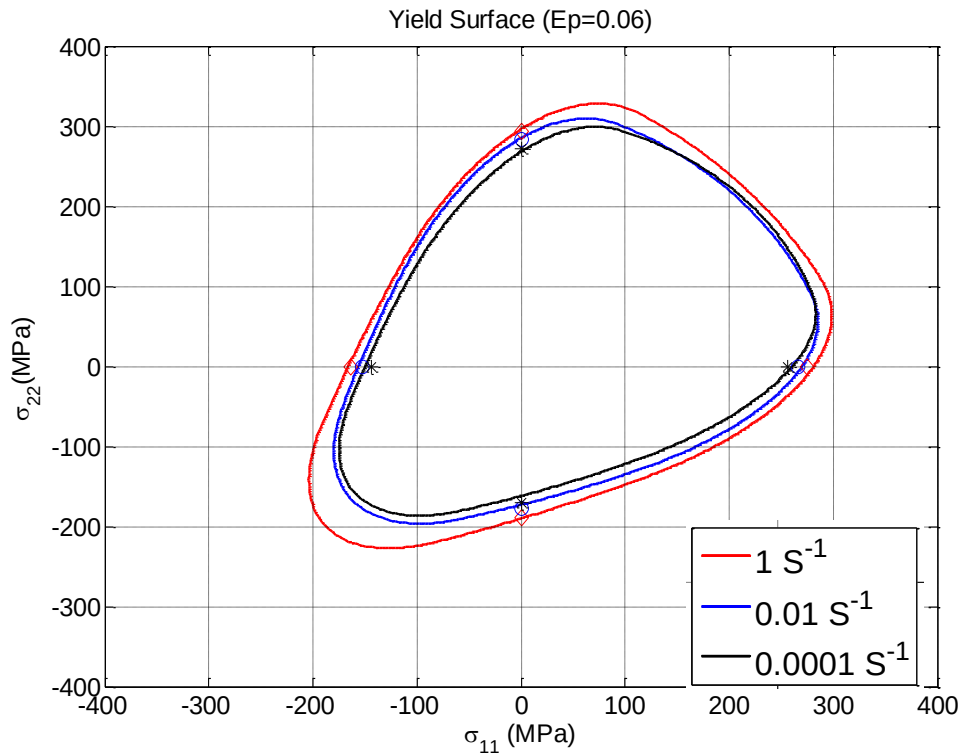


Figure 9.6: Yield surface evolution with different strain rates at plastic strain of 0.03.

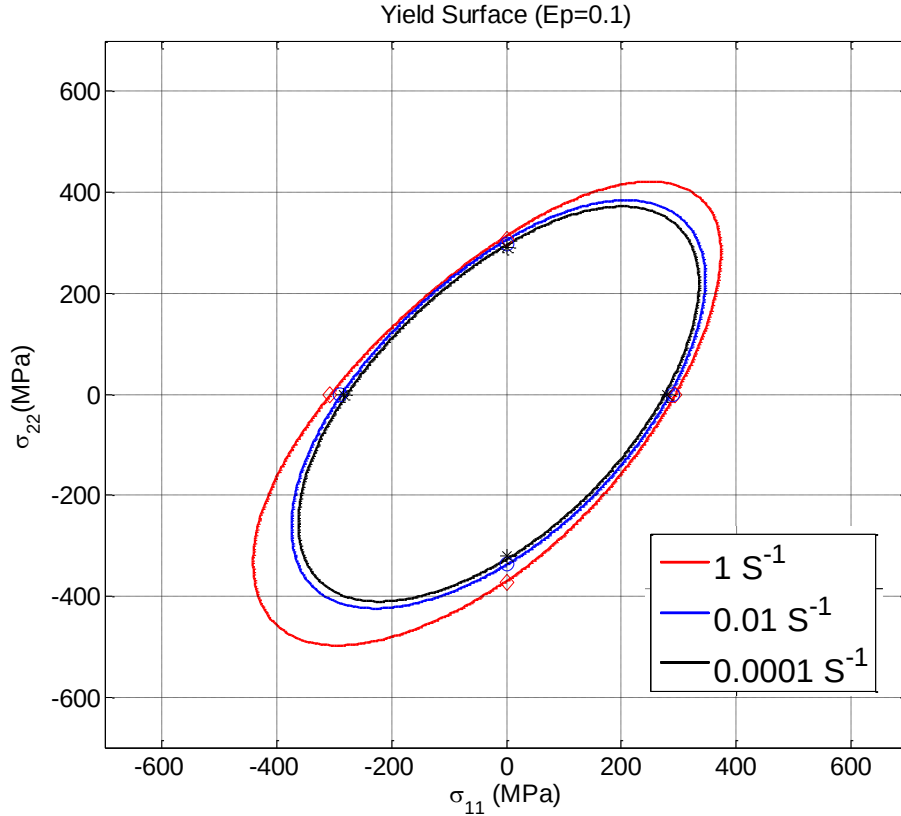


Figure 9.7: Yield surface evolutions with different strain rates at plastic strain of 0.06.

The comparison between predicted yield surfaces by the proposed strain rate dependent yield criterion and the corresponding experimental results at different strain rate, levels are presented in Figure 9.5 to Figure 9.7, respectively. From these figures, it can be seen that excellent correlation exists at all these experimental conditions. The extended Cazacu 06 model is available to properly capture the anisotropic stress response at different strain rate. From the comparison, it can be seen that with the increase of the strain rate, the yield surface presents a significant change, particularly for the large strain rate. The yield surface at different stress state expands at different deformation level. It indicates that the strain rate not only enhances the asymmetrical mechanical behavior, but it also affects the anisotropic properties. Thus, it is necessary to take the strain rate effect into account during the high speed deformation.

9.5 Implementation of extended elasto-viscoplastic model

To solve nonlinear boundary value, the finite element numerical scheme iteratively try out discrete displacements at the discretized material space until trial values ultimately satisfy the static principle of momentum or the equilibrium equation. The total process is discretized into small steps and variables such as stresses, strains are incrementally

integrated throughout the process. The solution is thus obtained for the chosen values of pseudo-time in the incremental sequence: $0 < t_1 < t_2 < \dots < t_n < t_{n+1} < \dots < T$. If the deformation is assumed to follow the proportional true strain path during the time step following the incremental deformation theory and also if the strain rate remains constant during the step, the strain rate is conveniently given by the strain increment as

$$\dot{\varepsilon} = \frac{\Delta \varepsilon}{\Delta t}, \quad (9.16)$$

where $\Delta \varepsilon$ is the true strain increment in the materially embedded coordinate system.

In displacement-based finite element formulations, stress updates take place at the Gauss points for a prescribed nodal displacement. In other words and as for most numerical schemes which are built to solve a constitutive equation, we suppose that the strain history path is known. We are then interested in the evolution of the other problem variables, in particular the stresses, the viscoplastic strain, and the total strain. The integration of the rate equations is a problem of evolution that can be regarded as follows. At incremental sequence n the total strain field and the plastic strain field as well as the hardening parameters are known:

$$\{\boldsymbol{\varepsilon}_n, \Delta \boldsymbol{\varepsilon}_{n+1}, \boldsymbol{\varepsilon}_n^{vp}, \bar{\varepsilon}^{vp}\}.$$

The elastic strain and stress fields are regarded as dependent variables which can always be obtained from the basic variables through the relations

$$\boldsymbol{\varepsilon}_n^e = \boldsymbol{\varepsilon}_n - \boldsymbol{\varepsilon}_n^{vp}. \quad (9.17)$$

The elastic part is assumed to be strain rate independent.

In viscoplasticity, all the stress values are admissible, since the consistency parameter of the classical plasticity is replaced by a parameter $\dot{\gamma}$, defined by the constitutive Equation (9.11). The discrete form of that equation can be written as

$$\Delta \gamma = \frac{\Delta t}{\hat{\mu}} \left[\left(\frac{\sigma_{eq}}{\sigma_Y} \right)^{1/\hat{\varepsilon}} - 1 \right]. \quad (9.18)$$

Implicit return mappings rely heavily on Newton-Raphson schemes to iteratively arrive at a solution. These schemes typically construct residual vectors $\bar{\mathbf{r}}$ as a function of the unknowns, which can be written as

$$\begin{cases} \mathbf{r}_\sigma = \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\,trial} + \Delta \gamma \mathbf{N}_{n+1} \\ r_{\bar{\varepsilon}^{vp}} = \bar{\varepsilon}_{n+1}^{vp} - \bar{\varepsilon}_n^{vp} - \Delta \gamma \\ r_\phi = \frac{\sigma_{eq}(\boldsymbol{\sigma}, \bar{\varepsilon}_{n+1}^{vp}, \dot{\bar{\varepsilon}}_{n+1}^{vp})}{\sigma_Y(\bar{\varepsilon}_{n+1}^{vp})} - \left(1 + \hat{\mu} \frac{\Delta \gamma}{\Delta t} \right)^{\hat{\varepsilon}} \end{cases} \quad (9.19)$$

with $\boldsymbol{\varepsilon}_{n+1}^e(\boldsymbol{\sigma}_{n+1})$, $\bar{\varepsilon}_{n+1}^{vp}$, $\Delta\gamma$ as the unknowns. Solution to the local system of generally nonlinear equations is achieved when $\bar{\mathbf{r}}(\boldsymbol{\chi}) = \mathbf{0}$ and the rate of convergence is intimately dependent on the consistent local tangent. A flow chart explicating the fully implicit return mapping algorithm is given in Box 9.1. The asymptotic quadratic rate of convergence of this iterative scheme makes this approach a very attractive one. However, it is limited to global convergence properties, and often leads often to a lack of convergence for large ranges of the initial trial state, particularly for the strong nonlinear evolution equations arisen from the high curvature of the yield function. It requires using line search algorithm to increase the efficiency of the Newton Raphson method. Owing to that the influence of the strain rate on the shape of yield surface, it results that the initial trial stress state and trial state variables may be too far from the solution. Besides the line search algorithm may not be enough to enhance convergence. In order to solve this problem, two iterative processes are adopted. In the first iterative, strain rate keeps as a constant, herein which is set as 0, after the first iterative process converges, the obtained results are used to be considered as the initial value for the second iterative process.

Box 9.1: Fully implicit CPPM of extended elasto-viscoplastic model .

- (i) Elastic predictor: Given $\Delta\boldsymbol{\varepsilon}$ and the state variables at t_n , evaluate the elastic trial state:

$$\begin{aligned}
 \boldsymbol{\varepsilon}_{n+1}^{e\,trial} &= \boldsymbol{\varepsilon}_n^e + \Delta\boldsymbol{\varepsilon} ; \bar{\varepsilon}_{n+1}^{vp\,trial} = \bar{\varepsilon}_n^{vp} ; \\
 p_{n+1}^{trial} &= K\varepsilon_{v\,n+1}^{e\,trial}, \mathbf{S}_{n+1}^{trial} = 2G\boldsymbol{\varepsilon}_{d\,n+1}^{e\,trial}, \\
 \boldsymbol{\sigma}_{n+1}^{trial} &= \mathbf{S}_{n+1}^{trial} + p_{n+1}^{trial}\mathbf{I}; \\
 \boldsymbol{\Sigma}_{n+1}^{trial} &= \hat{\mathbf{L}}\mathbf{S}_{n+1}^{trial}, \\
 J_{2\,n+1}^{0\,trial} &= \frac{1}{2}tr(\boldsymbol{\Sigma}_{n+1}^{trial^2}) ; J_{3\,n+1}^{0\,trial} = \frac{1}{3}tr(\boldsymbol{\Sigma}_{n+1}^{trial^3}); \\
 \cos(3\tilde{\theta}) &= \frac{J_{3\,n+1}^{0\,trial}}{2} \left(\frac{3}{J_{2\,n+1}^{0\,trial}} \right)^{3/2} ; \\
 \Sigma_1^{trial} &= 2\cos(\tilde{\theta}) \sqrt{\frac{J_{2\,n+1}^{0\,trial}}{3}} + \frac{p_{n+1}^{trial}}{3}; \\
 \Sigma_2^{trial} &= 2\cos\left(\tilde{\theta} - \frac{2\pi}{3}\right) \sqrt{\frac{J_{2\,n+1}^{0\,trial}}{3}} + \frac{p_{n+1}^{trial}}{3}, \\
 \Sigma_3^{trial} &= 2\cos\left(\tilde{\theta} + \frac{2\pi}{3}\right) \sqrt{\frac{J_{2\,n+1}^{0\,trial}}{3}} + \frac{p_{n+1}^{trial}}{3};
 \end{aligned}$$

$$\sigma_{eq\,n+1}^{trial} = \left[(|\Sigma_1^{trial}| - \tilde{k}\Sigma_1^{trial})^{\tilde{n}} + (|\Sigma_2^{trial}| - \tilde{k}\Sigma_2^{trial})^{\tilde{n}} + (|\Sigma_3^{trial}| - \tilde{k}\Sigma_3^{trial})^{\tilde{n}} \right]^{\frac{1}{\tilde{n}}};$$

(ii) Check plastic admissibility:

$$\text{If } \Phi = \sigma_{eq\,n+1}^{trial} - \sigma_Y(\bar{\varepsilon}_{n+1}^{vp\,trial}) \leq 0$$

Then set $(\cdot)_{n+1} = (\cdot)_{n+1}^{trial}$, and Exit.

(iii) Return mapping: Solve the system of eight equations using the Newton-Raphson iterative method (see Box 9.2)

$$\bar{\mathbf{r}}(\chi) = \begin{Bmatrix} \mathbf{r}_\sigma \\ r_{\bar{\varepsilon}^{vp}} \\ r_\Phi \end{Bmatrix} = 0 \quad \text{for} \quad \chi := \begin{Bmatrix} \boldsymbol{\sigma} \\ \bar{\varepsilon}^{vp} \\ \Delta\gamma \end{Bmatrix}$$

where

$$\begin{aligned} \mathbf{r}_\sigma &= \boldsymbol{\varepsilon}_{n+1}^e - \boldsymbol{\varepsilon}_{n+1}^{e\,trial} + \Delta\gamma \mathbf{N}_{n+1}, \\ r_{\bar{\varepsilon}^{vp}} &= \bar{\varepsilon}_{n+1}^{vp} - \bar{\varepsilon}_n^{vp} - \Delta\gamma, \\ r_\Phi &= \frac{\sigma_{eq}(\boldsymbol{\sigma}, \bar{\varepsilon}_{n+1}^{vp}, \dot{\bar{\varepsilon}}_{n+1}^{vp})}{\sigma_Y(\bar{\varepsilon}_{n+1}^{vp})} - \left(1 + \hat{\mu} \frac{\Delta\gamma}{\Delta t}\right)^{\hat{\varepsilon}}, \end{aligned}$$

The solution is found for $\boldsymbol{\varepsilon}_{n+1}^e$, $\bar{\varepsilon}_{n+1}^{vp}$ and $\Delta\gamma$. The stress tensor can be obtained by

$$\boldsymbol{\sigma}_{n+1} = \mathbf{D}^e : \boldsymbol{\varepsilon}_{n+1}^e.$$

(iv) Exit

Box 9.2: Implementation of Newton iteration process for the extended viscoplastic model.

(1) Input data: $\{\boldsymbol{\sigma}_{n+1}^{trial}, \bar{\varepsilon}_n^{vp}\}$ and $\Phi_{n+1}^{trial} > 0$.

(2) Initialization: set $i = 0$

$$\chi^{(0)} = \begin{Bmatrix} \boldsymbol{\sigma}_{n+1}^{trial} \\ \bar{\varepsilon}_n^{vp} \\ 0 \end{Bmatrix}.$$

(3) Compute the residual $\bar{\mathbf{r}}^{(i)}$ and the jacobian $\bar{\mathbf{J}}^{(i)}$

If $IFLAG = 0$ then

$$\begin{aligned} \dot{\bar{\varepsilon}}^{vp} &= 0; \\ \bar{\mathbf{r}}(\chi^{(i)}) &:= \begin{Bmatrix} \boldsymbol{\sigma}^{(i)} - \boldsymbol{\sigma}_{n+1}^{trial} + \Delta\gamma \mathbf{D}^e \mathbf{N}^{(i)} \\ \bar{\varepsilon}^{vp(i)} - \bar{\varepsilon}_n^{vp} - \Delta\gamma^{(i)} \\ \frac{\sigma_{eq}(\boldsymbol{\sigma}^{(i)}, \bar{\varepsilon}^{vp(i)})}{\sigma_Y(\bar{\varepsilon}^{vp(i)})} - \left(1 + \frac{\hat{\mu}\Delta\gamma^{(i)}}{\Delta t}\right)^{\hat{\varepsilon}} \end{Bmatrix}; \end{aligned}$$

$$\bar{\mathbf{J}}^{(i)} = \begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^{vp}} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \dot{\bar{\varepsilon}}^{vp}} \frac{1}{\Delta t} & \mathbf{N} \\ 0 & 1 & -1 \\ \mathbf{N}/\sigma_Y & \frac{1}{\sigma_Y} \frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^{vp}} - \frac{\sigma_{eq}}{\sigma_Y^2} \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^{vp}} & -\hat{\varepsilon} \left(1 + \frac{\hat{\mu} \Delta\gamma}{\Delta t} \right)^{\hat{\varepsilon}-1} \left(\frac{\hat{\mu}}{\Delta t} \right) \end{bmatrix};$$

$$\bar{\mathbf{p}}^{(i)} := \bar{\mathbf{p}}(\chi^{(i)}) := -(\bar{\mathbf{J}}^{(i)})^{-1} \bar{\mathbf{r}}^{(i)}.$$

Else

$$\dot{\bar{\varepsilon}}^{vp} = \Delta \bar{\varepsilon}^{vp(i)} / \Delta t;$$

$$\bar{\mathbf{r}}(\chi^{(i)}) := \begin{Bmatrix} \boldsymbol{\sigma}^{(i)} - \boldsymbol{\sigma}_{n+1}^{trial} + \Delta\gamma \mathbf{D}^e \mathbf{N}^{(i)} \\ \bar{\varepsilon}^{vp(i)} - \bar{\varepsilon}_n^{vp} - \Delta\gamma^{(i)} \\ \sigma_{eq}(\boldsymbol{\sigma}^{(i)}, \bar{\varepsilon}^{vp(i)}, \dot{\bar{\varepsilon}}^{vp}) / \sigma_Y(\bar{\varepsilon}^{vp(i)}) - (1 + \hat{\mu} \Delta\gamma^{(i)} / \Delta t)^{\hat{\varepsilon}} \end{Bmatrix};$$

$$\bar{\mathbf{J}}^{(i)} = \begin{bmatrix} \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} & \Delta\gamma \frac{\partial \mathbf{N}}{\partial \bar{\varepsilon}^{vp}} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \dot{\bar{\varepsilon}}^{vp}} \frac{1}{\Delta t} & \mathbf{N} \\ 0 & 1 & -1 \\ \mathbf{N}/\sigma_Y & \frac{\left(\frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^{vp}} + \frac{\partial \sigma_{eq}}{\partial \dot{\bar{\varepsilon}}^{vp}} \frac{1}{\Delta t} \right)}{\sigma_Y} - \frac{\sigma_{eq}}{\sigma_Y^2} \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}^{vp}} & -(1 + \hat{\mu} \Delta\gamma / \Delta t)^{\hat{\varepsilon}-1} (\hat{\mu} \hat{\varepsilon} / \Delta t) \end{bmatrix};$$

$$\bar{\mathbf{p}}^{(i)} := \bar{\mathbf{p}}(\chi^{(i)}) := -(\bar{\mathbf{J}}^{(i)})^{-1} \bar{\mathbf{r}}^{(i)}.$$

End if

(4) Apply the Line Search Scheme In Box 9.3.

Input: $\chi^{(i)}$, $\bar{\mathbf{r}}^{(i)}$, $\bar{\mathbf{p}}^{(i)}$, and *IFLAG*.

Output: $\chi^{(i+1)}$.

(5) If *IFLAG* = 0 then

Check convergence → Set the converge value as the initial value of new iterative

$$(\boldsymbol{\sigma}^{(0)}, \bar{\varepsilon}^{vp(0)}) \leftarrow (\boldsymbol{\sigma}^{(i+1)}, \bar{\varepsilon}^{vp(i+1)}), \text{ and } IFLAG = 1.$$

Go To (3).

else

Check convergence → Set $(\boldsymbol{\sigma}_{n+1}, \bar{\varepsilon}_{n+1}^{vp}) \leftarrow (\boldsymbol{\sigma}^{(i+1)}, \bar{\varepsilon}^{vp(i+1)})$, and Exit.

End if

(6) Set $j \leftarrow j + 1$ and Go To 3.

The application of a Newton scheme in the solution of the resulting non-linear algebraic equations is summarized in Box 9.2. All the equations are written in residual form and the Jacobian matrix utilized in the iterative process identified. The line search algorithm

is briefly summarized in Box 9.3. This procedure is iterated until the criterion $\|\bar{\mathbf{r}}\| < Tol$ is satisfied. Here, $\|\bar{\mathbf{r}}\|$ is the norm of the vector $\bar{\mathbf{r}}$ and Tol is a numerical tolerance. A tolerance value of 10^{-6} was used for Tol in this contribution.

Note that, when implementing the Newton-Raphson iterative procedures, a flag parameter (IFLAG) has been introduced allowing to only perform one iterative process when IFLAG =1 and two iterative processes when IFLAG =0. This allows the comparison between two iterative calculations, using exactly the same analysis computational tools.

(a) Bilinear interpolation

Due to that only a finite set of pre-strain levels and pre-strain rate levels are obtained by the material parameter identification procedure. Therefore, it is necessary to utilize a interpolation method to the material coefficients $\tilde{\mathbf{L}}(\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}^{vp})$; $\tilde{k}(\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}^{vp})$ and their corresponding derivative with respect to the strain and strain rate during the stress integration scheme.

In mathematics, bilinear interpolation is an extension of linear interpolation for interpolating functions of two variables (in this contribution, accumulated plastic strain $\bar{\varepsilon}^{vp}$ and strain rate $\dot{\bar{\varepsilon}}^{vp}$ on a regular 2D grid). Given the anisotropic coefficients in $\tilde{\mathbf{L}}$ matrix, and $\tilde{k}$ at a finite set of pre-strain levels ($\bar{\varepsilon}_1^{vp} < \bar{\varepsilon}_2^{vp} < \dots < \bar{\varepsilon}_N^{vp}$) and pre-strain rate level ($\dot{\bar{\varepsilon}}_1^{vp} < \dot{\bar{\varepsilon}}_2^{vp} < \dots < \dot{\bar{\varepsilon}}_M^{vp}$), the bilinear interpolation is to perform linear interpolation in accumulated plastic strain direction first, and then again in the strain rate direction. Although each step is linear in the sampled values and in the position, the interpolation as a whole is not linear but rather quadratic in the sample location. The detail can be seen as follows.

Assumed that given values at four nearby points $Q_{11} = (\bar{\varepsilon}_1^{vp}, \dot{\bar{\varepsilon}}_1^{vp})$, $Q_{12} = (\bar{\varepsilon}_1^{vp}, \dot{\bar{\varepsilon}}_2^{vp})$, $Q_{21} = (\bar{\varepsilon}_2^{vp}, \dot{\bar{\varepsilon}}_1^{vp})$, and $Q_{22} = (\bar{\varepsilon}_2^{vp}, \dot{\bar{\varepsilon}}_2^{vp})$, find desired value at point $P(\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}^{vp})$ see in Figure 9.8.

Firstly, a linear interpolation along the x -direction (plastic strain factor) is performed. The value of point $V_1 = (\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}_1^{vp})$ and point $V_2 = (\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}_2^{vp})$ can be calculated by

$$f(V_1) \approx \frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} f(Q_{11}) + \frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} f(Q_{21}); \quad (9.20)$$

$$f(V_2) \approx \frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} f(Q_{12}) + \frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} f(Q_{22}). \quad (9.21)$$

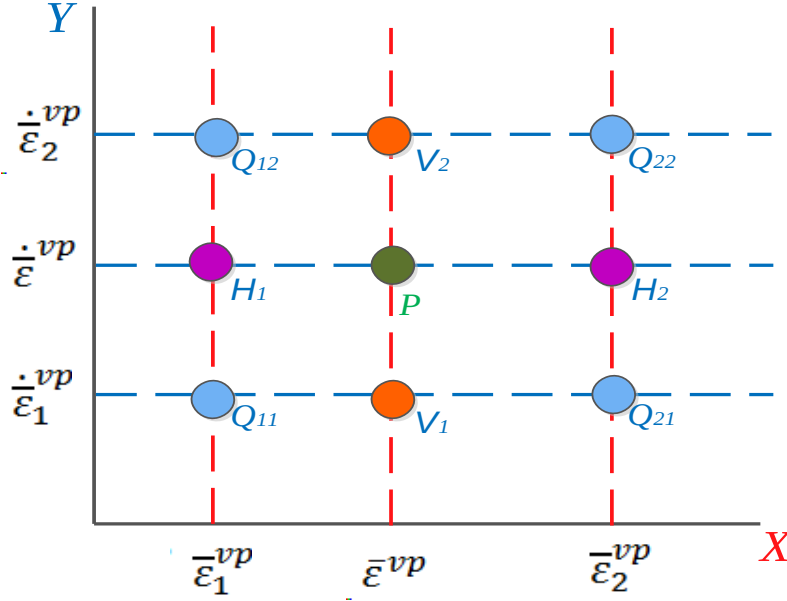


Figure 9.8: Schematic diagram of bilinear interpolation.

After values of mid-interpolated points V_1 and V_2 are obtained, the value of the desired point P can be calculated by interpolating the above two mid-interpolated points in the y -direction (strain rate factor), which can be written as

$$f(P) \approx \frac{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} f(V_1) + \frac{\dot{\varepsilon}^{vp} - \dot{\varepsilon}_1^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} f(V_2). \quad (9.22)$$

After a series of mathematical operations, the desired estimate of $f(\bar{\varepsilon}^{vp}, \dot{\varepsilon}^{vp})$ can be written as

$$\begin{aligned} f(\bar{\varepsilon}^{vp}, \dot{\varepsilon}^{vp}) &\approx \left(\frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) f(Q_{11}) + \left(\frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) f(Q_{21}) + \\ &\quad \left(\frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}^{vp} - \dot{\varepsilon}_1^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) f(Q_{12}) + \left(\frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}^{vp} - \dot{\varepsilon}_1^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) f(Q_{22}). \end{aligned} \quad (9.23)$$

In order to obtain the derivative of the value at the desired point P with respect to the accumulated plastic strain and strain rate, which is required in the stress integration algorithm, the partial derivate to the plastic strain $\bar{\varepsilon}^{vp}$ could be calculated by

$$\begin{aligned} \frac{\partial f(\bar{\varepsilon}^{vp}, \dot{\varepsilon}^{vp})}{\partial \bar{\varepsilon}^{vp}} &\approx - \left(\frac{f(Q_{11})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) + \left(\frac{f(Q_{21})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) - \\ &\quad \left(\frac{f(Q_{12})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}^{vp} - \dot{\varepsilon}_1^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right) + \left(\frac{f(Q_{22})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{\dot{\varepsilon}^{vp} - \dot{\varepsilon}_1^{vp}}{\dot{\varepsilon}_2^{vp} - \dot{\varepsilon}_1^{vp}} \right). \end{aligned} \quad (9.24)$$

The partial derivate of the desired estimate to the strain rate $\dot{\bar{\varepsilon}}^{vp}$ can be computed by

$$\begin{aligned} \frac{\partial f(\bar{\varepsilon}^{vp}, \dot{\bar{\varepsilon}}^{vp})}{\partial \dot{\bar{\varepsilon}}^{vp}} \approx & - \left(\frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{f(Q_{11})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) - \left(\frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{f(Q_{21})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) + \\ & \left(\frac{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{f(Q_{12})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) + \left(\frac{\bar{\varepsilon}^{vp} - \bar{\varepsilon}_1^{vp}}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right) \left(\frac{f(Q_{22})}{\bar{\varepsilon}_2^{vp} - \bar{\varepsilon}_1^{vp}} \right). \end{aligned} \quad (9.25)$$

Box 9.3: A line search algorithm for the extended viscoplastic model.

(1) Input data: $\chi^{(i)}, \bar{\mathbf{r}}^{(i)}, \bar{\mathbf{p}}^{(i)}$.

(2) Initialize:

Set $j = 0, \bar{\alpha}_{(j)}^{(i)} = 1, \bar{g}^{(i)} = \bar{\mathbf{r}}^{(i)} : \bar{\mathbf{r}}^{(i)} / 2$ and $\bar{g}'^{(i)} := -2\bar{g}^{(i)}$.

(3) Compute the new unknowns, residual and merit function:

$$\chi_{(j)}^{(i+1)} = \chi^{(i)} + \bar{\alpha}_{(j)}^{(i)} \bar{\mathbf{p}}^{(i)};$$

$$\bar{\mathbf{r}}_{(j)}^{(i+1)} = \bar{\mathbf{r}}(\chi_{(j)}^{(i+1)});$$

$$\bar{g}_{(j)}^{(i+1)} = \bar{\mathbf{r}}_{(j)}^{(i+1)} : \bar{\mathbf{r}}_{(j)}^{(i+1)} / 2.$$

(4) Check Goldstein's condition

If $\bar{g}_{(j)}^{(i+1)} \leq \bar{g}^{(i)} + \bar{c}_1 \bar{\alpha}_{(j)}^{(i)} \bar{g}'^{(i)T} \bar{\mathbf{p}}^{(i)}$ Then Set $\chi^{(i+1)} = \chi_{(j)}^{(i+1)}$ and Exit.

If $j = j_{max}$ Then Set $\chi^{(i+1)} = \chi_{(j)}^{(i+1)}$ and Exit.

(5) Compute new value of Linea search parameter:

$$\bar{\alpha}_{(j+1)}^{(k)} = \frac{-\tilde{b} + \sqrt{\tilde{b}^2 - 3\tilde{a}\bar{g}'(0)}}{3\tilde{a}},$$

where,

$$\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \frac{\begin{bmatrix} (\bar{\alpha}_{(j-1)}^{(i)})^2 & -(\bar{\alpha}_{(j)}^{(i)})^2 \\ -(\bar{\alpha}_{(j-1)}^{(i)})^3 & (\bar{\alpha}_{(j)}^{(i)})^3 \end{bmatrix} \begin{bmatrix} \bar{g}_c(\bar{\alpha}_{(j)}^{(i)}) - \bar{g}(0) - \bar{g}'(0)\bar{\alpha}_{(j)}^{(i)} \\ \bar{g}_c(\bar{\alpha}_{(j-1)}^{(i)}) - \bar{g}(0) - \bar{g}'(0)\bar{\alpha}_{(j-1)}^{(i)} \end{bmatrix}}{(\bar{\alpha}_{(j)}^{(i)})^2 (\bar{\alpha}_{(j-1)}^{(i)})^2 (\bar{\alpha}_{(j)}^{(i)} - \bar{\alpha}_{(j-1)}^{(i)})}.$$

In order to avoid small step length in the iterative process, the final length could be calculated by

$$\bar{\alpha}_{(j+1)}^{(i)} := \text{MAX} \{ \tilde{\eta} \bar{\alpha}_{(j)}^{(i)}, \bar{\alpha}_{(j+1)}^{(i)} \}.$$

(6) Set $j \leftarrow j + 1$ and Go To (3).

(b) Consistent tangent matrix

In an implicit displacement based FE formulation, equilibrium in a time step can be obtained by means of the Newton-Raphson method. The viscoplastic consistent tangent modulus is needed for implicit finite element codes to preserve the quadratic rate of convergence in the solution of an initial boundary value problem by linearizing the nonlinear equilibrium equation at each iteration. Consistency in this context means that the formula for the tangential stiffness matrix is consistent with the underlying time integration scheme. The tangent operator consistent with the general algorithm is needed to assemble the tangent stiffness matrix.

To construct the consistent elasto-viscoplastic modulus, the change of the stress is evaluated by a special return mapping algorithm corresponding to an infinitesimal change of total strain increment is considered. The fourth-order tensor can be found by computing the derivative of the updated $\boldsymbol{\sigma}_{n+1}$ with respect to the final strain $\boldsymbol{\varepsilon}_{n+1}$. Analogously to the general procedure for the rate-independent case, when the stress state lies within the elastic domain and no viscoplastic flow is possible, the tangential stiffness matrix is equal to the elasticity matrix

$$\mathbf{D}^{vp} = \mathbf{D}^e. \quad (9.26)$$

Otherwise, the elasto-viscoplastic consistent tangent operator can be derived by consistently linearizing the consistency viscoplastic return-mapping algorithm under viscoplastic flow. Considering the time step $n + 1$, the viscoplastic strain increment is defined as

$$\dot{\boldsymbol{\varepsilon}}_{n+1}^{vp} = \Delta\gamma \mathbf{N}. \quad (9.27)$$

Consequently, the stress-strain relation and the internal variables read

$$\boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}} - \Delta\gamma \mathbf{N} = \mathbf{D}^{e-1} \boldsymbol{\sigma}. \quad (9.28)$$

The consistent tangent operator can be obtained by differentiating the above equation, which gives

$$\frac{\partial \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}}{\partial \boldsymbol{\sigma}} - \frac{\partial \Delta\gamma}{\partial \boldsymbol{\sigma}} \mathbf{N} - \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} = \mathbf{D}^{e-1}. \quad (9.29)$$

After rearranging equation 9.34 can be written as

$$\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}_{n+1}^{e \text{ trial}}} = \left\{ \mathbf{D}^{e-1} + \Delta\gamma \frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} + \frac{\partial \Delta\gamma}{\partial \boldsymbol{\sigma}} \mathbf{N} \right\}^{-1}. \quad (9.30)$$

The incremental multiplier derivative is obtained by differentiating the Equation (9.18) with respect to $\boldsymbol{\sigma}$. This gives

$$\frac{\partial \Delta \gamma}{\partial \sigma} = - \frac{\frac{1}{\sigma_Y} \left(N + \Delta \gamma \frac{\partial N}{\partial \bar{\epsilon}^{vp}} + \Delta \gamma \frac{\partial N}{\partial \bar{\epsilon}^{vp}} \frac{1}{\Delta t} \right)}{\frac{1}{\sigma_Y} \left(\frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^{vp}} + \frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^{vp}} \frac{1}{\Delta t} \right) - \frac{\sigma_{eq}}{\sigma_Y^2} \frac{\partial \sigma_Y}{\partial \bar{\epsilon}^{vp}} - \hat{\epsilon} \left(1 + \frac{\hat{\mu} \Delta \gamma}{\Delta t} \right)^{\hat{\epsilon}-1} \left(\frac{\hat{\mu}}{\Delta t} \right)}. \quad (9.31)$$

Then, with substitution of Equation (9.31) into Equation (9.30) we obtain the following closed form expression for the viscoplastic consistent tangent operator:

$$\mathbf{D}^{vp} = \left\{ \mathbf{D}^{e-1} + \Delta \gamma \frac{\partial N}{\partial \sigma} - \frac{\frac{N}{\sigma_Y} \otimes \left(N + \Delta \gamma \frac{\partial N}{\partial \bar{\epsilon}^{vp}} + \Delta \gamma \frac{\partial N}{\partial \bar{\epsilon}^{vp}} \frac{1}{\Delta t} \right)}{\frac{1}{\sigma_Y} \left(\frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^{vp}} + \frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^{vp}} \frac{1}{\Delta t} \right) - \frac{\sigma_{eq}}{\sigma_Y^2} \frac{\partial \sigma_Y}{\partial \bar{\epsilon}^{vp}} - \hat{\epsilon} \left(1 + \frac{\hat{\mu} \Delta \gamma}{\Delta t} \right)^{\hat{\epsilon}-1} \left(\frac{\hat{\mu}}{\Delta t} \right)} \right\}^{-1}. \quad (9.32)$$

It should be noted that the operator $\mathbf{D}^{vp}$ in the present case is not symmetric. When the yield function is independent on the strain rate, it is reduced to the general viscoplastic hardening constitutive consistent tangent matrix.

9.6 Numerical verification and examples

The stress update algorithm and the calculation of the viscoplastic consistent tangent moduli presented in the section 9.5 are implemented in the commercial finite element code ABAQUS/Standard by writing a user-defined material subroutine UMAT.

Table 9.1: Material parameter of AZ31B.

				Flow Curve				
	ρ (g/cm ³)	Modulus E (GPa)	Poisson's Ratio ν	$\sigma_{VT} = \left[A_V + B_V \left(1 - e^{-C_V \bar{\epsilon}^{vp}} \right) \right] \left(1 + \hat{\mu} \bar{\epsilon}^{vp} \right)^{\hat{\epsilon}}$				
				A_V (MPa)	B_V (MPa)	C_V	$\hat{\mu}$	$\hat{\epsilon}$
AZ31	1.77	45	0.35	169.30	132.14	18.02	0.009162	7287.65

In this section, several numerical benchmark tests are performed in order to validate the proposed elasto-viscoplastic model and assess the performance of the developed user subroutine. All the material parameters used in the benchmark numerical tests are presented in Table 9.1. The modified Voce flow law was used to describe the uniaxial true stress-strain curve response at rolling direction. The description of flow law can be seen in Equation (9.7).

9.6.1 Single element test

In order to validate the performance of return mapping algorithms, two simple single element tests are performed. One is the biaxial tensile test; the other one is the biaxial

compressive test. The single element is subjected to the prescribed displacements to simulate the different stress state, such as biaxial tensile and compressive tests. The initial element dimension is set equal to 10 mm. In all examples presented in this chapter, the residual norm is set to $ToL = 10^{-6}$.

(a) Biaxial tensile test

Table 9.2 and Table 9.3 list the evolution of residual for one Newton-Raphson iterative strategy and two Newton-Raphson iterative strategies return mapping algorithms under biaxial tensile test, respectively.

Table 9.2: Residual norm of one-iterative return mapping algorithm for biaxial tensile test.

<i>Iteration number</i>	<i>Residual</i>
1	49.72627
2	12.30161
3	0.227088
4	1.41E-03
5	1.11E-05
6	3.97E-12

Table 9.3: Residual norm of two-iterative return mapping algorithm for biaxial tensile test.

<i>Iteration number</i>	<i>Residual</i>
First iterative	
1	49.93169
2	5.269009
3	0.287509
4	4.22E-03
5	9.25E-05
6	2.23E-10
Second iterative	
1	2.23E-10

From the comparison, it can be seen that the one-iterative return mapping strategy requires six iterative to get convergence, meanwhile, the two-iterative return mapping strategy also requires six iterative to obtain the initial value for the second iterative

procedure, one more iterative to get convergence. This means that these two return mapping strategies have a similar effect to converge under biaxial tensile test.

(b) Biaxial compressive test

Owing to the different mechanical behavior of Magnesium alloy under compression stress state from the tension stress state, a biaxial compressive test is used to study these two above mentioned return mapping integration algorithms. The residual norms of these two strategies are listed in Table 9.4 and Table 9.5, respectively.

Table 9.4: Residual norm of one iterative return mapping algorithm for biaxial compressive test.

<i>Iteration number</i>	<i>Residual</i>
1	1.9125E-003
2	1.7244E-003
3	NC

Table 9.5: Residual norm of two-iterative return mapping algorithm for biaxial compressive test.

<i>Iteration number</i>	<i>Residual</i>
First iterative	
1	0.142657
2	0.102714
3	5.49E-03
4	1.76E-07
Second iterative	
1	3.08E-04
2	1.68E-10

Note: NC represents non convergence

From the Table 9.4, it can be seen that the iteration cannot be converged if the one-iterative return mapping algorithm is chosen. That is because the initial trial value is far from the solution. However, these two iterative return mapping algorithms only requires fourth iterative to obtain initial values for the second iterative process (see in Table 9.5). It makes the initial values closely approaches the solution. Thus, only two more iterative processes can get the final converge in the second iterative procedure. This indicates that the two-iterative integration algorithm can effectively improve the convergence.

9.6.2 Square box drawing

Deep drawing is a manufacturing process that is used extensively in the sheet metal forming into geometries that include cup or box like structures. Finite element simulations provide an effective mean to investigate the interaction between the process parameters and the material response. They provide useful information for fine-tuning the production processes. In this study the deep drawing process of square cups is modeled for the magnesium alloy AZ31B using FEA to validate further the proposed elasto-viscoplastic model and show the applicability of the proposed integration algorithm.

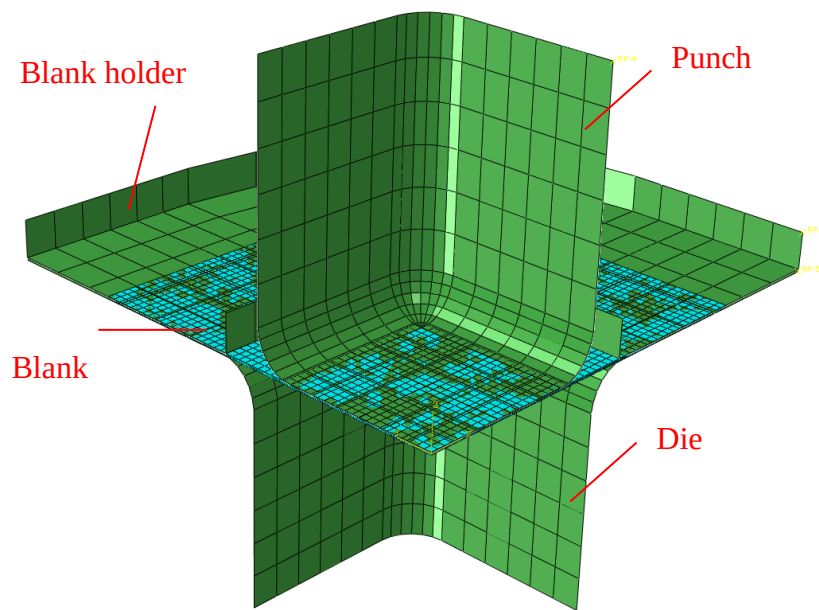


Figure 9.9: Finite element model of square box.

Finite element of the square cup drawing process is schematized in Figure 9.9. Given the symmetry of the problem, it is sufficient to model only a one- quarter sector of the box. However, for easier visualization we employ a one-quarter model. Symmetry boundary conditions are applied at the quarter edges of the blank. The punch and the blank holders are allowed to move only in the vertical direction. Allowing vertical motion of the blank holder accommodates changes in the blank thickness during forming. The movement of the punch is defined using a reference node. This reference node was also employed to obtain the drawing force during the simulation. The rigid die is a flat surface with a square hole 102.5 mm by 102.5 mm, rounded at the edges with a radius of 10 mm. The rigid square punch measures 100 mm by 100 mm and is rounded at the edges with the same 10 mm radius. The rigid blank holder can be considered a flat

plate, since the blank never comes close to its edges. We use four-node, three-dimensional rigid surface elements (type R3D4) to model the die, the punch, and the blank holder. The blank is initially square, 200 mm by 200 mm, and is 0.6 mm thick. The blank is modeled with eight-node, brick elements (type C3D8R).

The material of the blank is in contact with the face of the punch, the die and the holder. Contact interaction is considered between the blank and the punch with a friction coefficient of 0.25 and between the blank and the die with a friction coefficient of 0.125. Two different constitutive models are applied into the above square box drawing process. One is the strain rate dependent yield function model; the other is the strain rate independent yield function model.

(a) Equivalent plastic strain distribution

Figure 9.10 shows the contour of equivalent plastic strain obtained from rate dependent constitutive model and rate independent constitutive model, respectively.

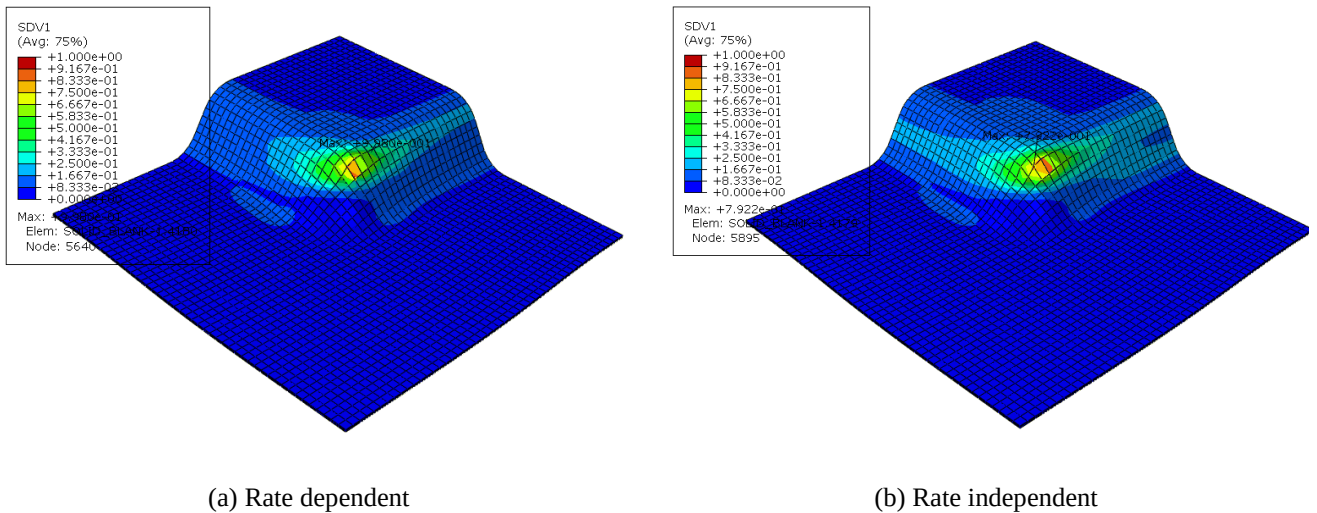


Figure 9.10: Contours of equivalent plastic strain.

From the comparison, it can be seen that the distribution of equivalent plastic strain is different. The plastic deformation of rate dependent yield function model is mainly localized in corner place; in contrast, the plastic deformation of rate independent yield function model distributes along the edge. The maximum value of the equivalent plastic strain for rate dependent model is larger than the strain rate independent model.

(b) Punch force

Figure 9.11 shows the force evolution reaction on the punch during the drawing process. From there, it is clearly seen that the punch force–displacement curve for the rate independent yield function model is lower than for rate dependent yield function model.

That is because during the deep drawing of square cups, the parts of the blank are subjected to different deformation rate under different deformation zones. It does not only affect the flow hardening law, but it also makes the yield surface different at different zones for the rate dependent model.

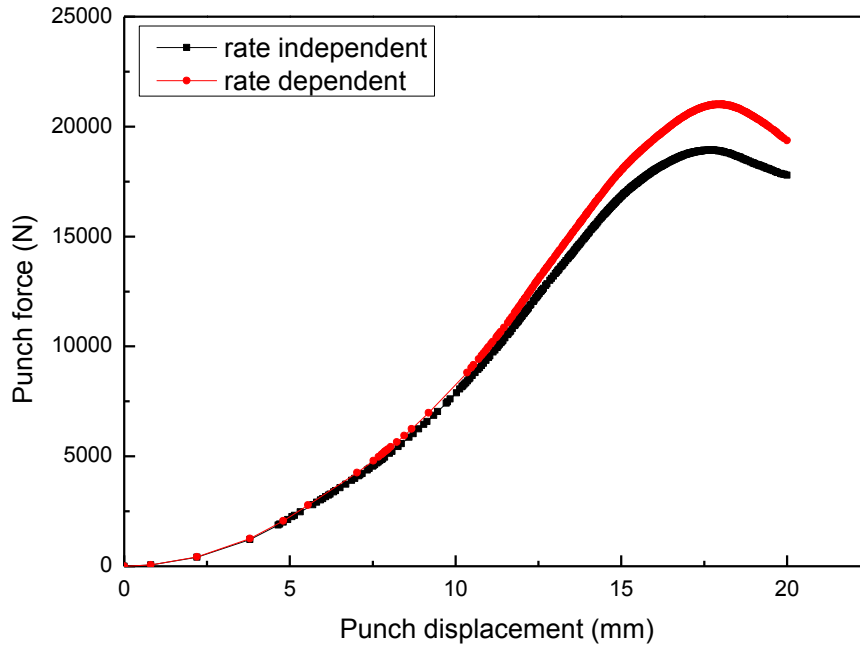


Figure 9.11: Punch force-displacement curves.

9.7 Conclusions

The work in this chapter describes an extended elasto-viscoplastic constitutive model that has been employed to capture the mechanical behavior of AZ31 alloy under different strain rates. Strain rate dependence of yield behavior of AZ31 alloy was investigated analytically. The material constants of the model have been calibrated as a function of equivalent plastic strain and strain rate, thus allowing the implementation of the corresponding constitutive equations in a UMAT subroutine of the ABAQUS platform. In order to enhance the convergence, two return mapping algorithms were adopted: one is the Newton-Raphson iterative combined with line search algorithm; the other is two Newton-Raphson iterative combined with line search algorithm. The validation of the UMAT has been carried out by several benchmark tests. The general conclusions of this study can be listed as follows:

- The extended flow law can describe the mechanical behavior of AZ31 alloy at each uniaxial stress state under different strain rates at room temperature.
- The extended elasto-viscoplastic model takes the influence of strain rate on the shape of yield surface and can capture the mechanical behavior on the multiaxial stress space. It describes effectively the evolution of the anisotropic, asymmetric effects with the strain rate. From the presented simulations of the square box deep drawing, it can be concluded that the shape of the yield locus has an important effect on the calculated punch force–displacement curve. This was demonstrated by the difference in the results obtained for the strain rate independent yield surface and strain rate dependent yield surface.
- Two-iterative return mapping algorithm can effectively improve the convergence compared with the classical one iterative return mapping algorithm.

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Chapter 10

Conclusions and Future Work

Accurate simulations of forming processes using the finite element method require reliable constitutive models to describe the mechanical behavior of materials. The study of this topic is of great importance and, in recent years, has been subjected to numerous studies regarding the formulation of mathematical models capable of describing the mechanical behavior in the most realistic way. These scientific developments bring to the industry, in general, competitive gains with regard to the development of more efficient and durable mechanical components as well as the improvement of manufacturing processes.

In this chapter, the implications and importance of the work presented in this thesis are discussed with the intention of accurately describing the mechanical quasi-static/dynamic behavior of Hexagonal Close-Packed (HCP) crystal structure and applying the various theoretical facets into practical metal forming process. The conclusions drawn from these studies and recommendations for further research are presented.

10.1 Conclusions

This study focuses on two categories of HCP crystal metals: titanium alloys and magnesium alloys. Detailed conclusions for the developments proposed in this thesis are given as follows.

Chapter 4

In this chapter, a brief summary of the formulation of Cazacu 04 constitutive model was presented and used to model the quasi-static mechanical behavior of a titanium alloy. Several fully implicit integration algorithms of the Cazacu 04 model based on the Closest Point Projection Method (CPPM) were implemented within an implicit environment. They can be classified into two types: the first is the standard Newton-Raphson Closest Point Projection Method (N-R CPPM) and the second is the Newton-Raphson Closest Point Projection Method combined with line search algorithms (L-CPPM). In particular, the second strategy was combined with three different interpolation algorithms, namely quadratic (LQ), three points quadratic (LQ3), and cubic (LC) interpolation. All these integration methods have been investigated under

different stress states and strain increments. It was concluded that whenever the level of accumulated plastic strain is low and the yield surface has an elliptical shape or an elliptical-like shape, regardless of the initial stress state (uniaxial stress state, biaxial stress state, *etc.*), and the shape does not present a substantial change during its following evolution, the standard Newton-Raphson (N-R) proved to be convergent. The L-CPPM presented similar results to the standard N-R algorithm. Nevertheless, when the yield surface changed more significantly into a triangular-like shape, the standard N-R method doesn't achieve convergence while the L-CPPM scheme was able to converge and reduce the number of iterations significantly. Through the comparison of the iterative number contours with different interpolated line search algorithms during the yield surface evolution, it is possible to conclude that the cubic interpolation method converges in less iterations when compared to the other two aforementioned interpolation methods. Therefore the line search strategy, particularly the cubic interpolated line search coupled CPPM integration algorithm is very effective, and the minimization of the objective functional is superior from the viewpoint of the computational cost.

Subsequently, iso-error maps have been drawn for two types of yield surface at six representative stress states. One of them departure from a yield surface with an elliptical-like shape, the other a triangular-like one. It was verified that the stability at tensile and compressive stress states had a pronounced difference, and also that large increments could not be employed, due to the loss of convergence promoted by the model.

Finally, a benchmark test-cup drawing was performed to validate the model. This allows to conclude that the model was able to capture the earing phenomenon in an effective manner after drawing the pure titanium sheet in agreement with the experimental results. The presented integration algorithm is suitable for the application to highly nonlinear problems.

Chapter 5

In order to simulate the dynamic mechanical behavior of metals with an HCP crystal structure at different strain rates, in this chapter, an elasto-viscoplastic constitutive model coupled with Cazacu 04 yield function is proposed. The model employed the CPPM coupled with the line search algorithm and was implemented within an implicit quasi-static finite element environment. Two numerical examples show that the

integration algorithm can be effectively used at high strain rates even for very high power-exponents. It is also shown that the stress evolution under compressive stress states is higher than tensile stress states and presents higher strain rate sensitivity.

Chapter 6

This chapter addresses the identification of material parameters of a phenomenological constitutive model for magnesium alloy. The aggregate-objective optimization technique has been developed to assess the optimal set of coefficients of the Cazacu06 ' yield criterion. The optimization problem with a multi-objective nature has been solved in a single run by combining each objective of the problem of assessing the optimal coefficients into the single objective function. Five optimization strategies were used to calibrate the material parameters of Cazacu 06 constitutive model for a typical HCP crystal structured material- AZ31 alloy. The classical gradient method is effective to calibrate the yield surface at small deformation. At that stage, the yield surface of AZ31 alloy has an elliptical-like shape. The initial value of material parameters of the constitutive model can be chosen as $[1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0]$. At large deformations, the precious initial value is easily trapped into local minima, and hybrid evolutionary methods are used. By comparison of hybrid GA and hybrid PSO optimized strategies, it can be concluded that hybrid PSO algorithm can escape the local minima trap and obtain the global optimization value, giving a better accuracy to the optimized yield surface.

Chapter 7

In this chapter, a yield criterion for numerical analyses of sheet forming processes was developed to model the mechanical behavior of HCP crystal structure material by extending the classical Barlat 89 model. The proposed yield function was expressed in terms of the stress components under plane stress condition. The influence of the introduced material parameter in the new model was assessed. It can be concluded that with the increase of the power index M in the proposed model, the yield surface becomes more rectangular. The parameter c can enhance the difference between tensile and compressive stress states, making the shape of the yield surface smoother on one side and sharper on the other side. However, this effect is not so noticeable when the power index value is small. The novel constitutive model was developed to account for the complexity of the yielding behavior of magnesium alloys. The convexity of the

proposed yield function was also proved. The material identification procedure proposed in chapter 6 was successfully applied to the AZ31 magnesium alloy. The evolution of the shape of the yield locus was also accommodated by expressing the model parameters as a function of the equivalent plastic strain. By comparison with other commonly used models such as Hill 48 yield criteria, Barlat 89 model, and Cazacu 06, it was possible to show that the new model has a close agreement with theoretical and experimental results, is able to capture the mechanical behavior of AZ31 alloy, and requires less material parameters than Cazacu 06 model. Finally, a fully implicit integration algorithm under plane stress condition based on the CPPM coupled with line search algorithm was implemented within an implicit environment. The calibrated model was employed in a simple test to validate the model, and the simulated results are able to show that the new proposed model can reflect the asymmetrical mechanical behavior and anisotropic properties in real applications. A benchmark test-cup drawing was performed to show the robustness of the implementation, and the simulated results show that the model captures the earing phenomenon after drawing the AZ31B sheet, in close agreement with the experimental results.

Chapter 8

To obtain reliable information from the FE analysis of springback, it is essential to fully understand the phenomenon and to carefully consider all sources of error of its prediction. In this chapter, two types of methods (analytical and numerical) were used to study the springback behavior of magnesium alloy. Therefore, the whole chapter can be divided into two parts.

The springback dependency on some material and process parameters was demonstrated with the help of a simple analytical model. In the first part, a generic analytical method was proposed. The presented analytical model can be used to predict the change of curvature in the elastic-plastic sheet in the case of plane strain bending under tension. This model not only can be used to predict a symmetrical model, but also an unsymmetrical model. It is available for arbitrary mechanical hardening behavior. The verification and validation of this analytical model was performed by experimental and simulation analyses. In particular, it is applied to analyze the mechanics of a AZ31 plate during bending process. The influence of the thickness, bending curvature, strain rate and in-plane external tensile stress on the neutral surface distribution, the distribution of

the force and the moment in elastic, compressive plastic, tensile plastic of the plate section, respectively.

- For the bending of an unsymmetrical material, there exist four typical regions, compressive plastic region, compressive elastic region, tensile elastic region and tensile plastic region. These four regions are not symmetrical, and the neutral surface is not coincided with the mid-plane. With the increasing bending curvature, the compressive region expands. The neutral surface also moves towards the outside surface.
- By analyzing the springback factor of plates with different thickness, it is possible to conclude that the springback factor has a significant increase when the plate thickness increases.
- Tension strongly affects the bending moment. With an applied external in-plane tensile force, the bending moment increases with the increasing external tensile force. The incremental moment gap of each external tensile force increases with the increasing bending curvature.

For a successful FE analysis of springback, it is important to prepare an adequate numerical model. Special attention must be given to choosing the appropriate constitutive law which must accurately represent the material behavior during deformation. In the second part, a series of constitutive models were implemented in user-defined subroutines into the commercial nonlinear finite element code ABAQUS. The CPPM algorithm was used to integrate the stress and other state variables over each time increment. All these constitutive models can be categorized into two types. The first is the conventional yield criteria coupled with pure isotropic hardening model. These models can only reflect the anisotropic properties. The second model can capture anisotropic and SD effects simultaneously. By comparison of the springback angle of the unconstraint cylindrical bending test using the above anisotropic models, the model considering the SD effect is able to obtain a more accurate prediction than the previous models. The elaborated model takes into account the anisotropic behavior of materials and asymmetrical strength at the same time. Several simulations with different Number of Integration Points (NIP) through thickness were performed. It was concluded that for the symmetrical material, five integration points through the thickness are sufficient to obtain a convergence solution for springback, and the asymmetrical model is more sensitive of the NIP through thickness than the symmetrical material. Therefore, it is necessary to employ more integration points through the thickness.

Chapter 9

The work in this chapter describes an extended elasto-viscoplastic constitutive model that has been employed to capture the mechanical behavior of AZ31 alloy under different strain rates. An equivalent plastic strain and strain rate dependent anisotropic material model was investigated for AZ31 alloy sheet. The material parameters of the extended elasto-viscoplastic model were established and calibrated as a function of equivalent plastic strain and strain rate, thus allowing the implementation of the corresponding constitutive equations in a UMAT subroutine of the ABAQUS platform. In order to enhance the convergence, two return mapping algorithms were adopted: one is the Newton-Raphson iterative combined with line search algorithm; the other is two Newton-Raphson iterative combined with line search algorithm. The validation of the UMAT has been carried out by several benchmark tests. The general conclusions of this study can be listed as follows:

- This chapter investigates the identification of the whole set of parameters of three strain rate dependent flow law using the optimized algorithm. The extended flow law can describe the mechanical behavior of AZ31 alloy at each uniaxial stress state under different strain rates at room temperature.
- The extended elasto-viscoplastic model takes the influence of strain rate on the shape of yield surface and can capture the mechanical behavior on the multiaxial stress space. It describes effectively the evolution of the anisotropic, asymmetric effects with the strain rate.
- Two-iterative return mapping algorithm can effectively improve the convergence compared with the classical one iterative return mapping algorithm. From the presented simulations of the square box deep drawing, it can be concluded that the shape of the yield locus has an important effect on the calculated punch force–displacement curve. This was demonstrated by the difference in the results obtained for the strain rate independent yield surface and strain rate dependent yield surface. This numerical example also indicates that the two iterative return mapping is more robust and stable.

10.2 Suggestions for future work

The deformation behavior of magnesium alloys at high strain rates (10^3 s^{-1}) and at elevated temperatures has not been thoroughly investigated yet. The influence of temperature, loading condition (tension-compression) and texture on the deformation

mechanisms and flow stress at high strain rates is still unknown. Future work should take into account various strain rates and temperature effects. The application of the developed constitutive models to other forming conditions as well as industrial examples should also be investigated in future works.

Furthermore, in order to improve the predictive ability of constitutive models for sheet metal processes undergoing cyclic loading (such as alternative tension-compression or bending unbending stresses in deep drawing processes including draw beads), the kinematic hardening effects associated with stress reversal should be taken into account in future research work.

Appendix A

Equivalent Plastic Strain Rate of Cazacu'04 Model

An equivalent plastic strain rate $\dot{\bar{\epsilon}}^p$ which is the plastic work rate conjugate to the equivalent stress σ_{eq} . The plastic work rate is defined such that:

$$\dot{W} = \sigma_{ij} \dot{\epsilon}_{ij}^p = \sigma_{eq} \dot{\bar{\epsilon}}^p. \quad (A.1)$$

According to the associated plastic flow rule, the plastic strain rate can be written as

$$\dot{\epsilon}_{ij}^p = \Delta\gamma \frac{\partial \sigma_{eq}}{\partial \sigma}. \quad (A.2)$$

Thus,

$$\sigma : \frac{\partial \sigma_{eq}}{\partial \sigma} = \sigma_{ij} \left[\frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial J_2^0}{\partial \Sigma_{kl}} - \hat{c} \frac{\partial J_3^0}{\partial \Sigma_{kl}} \right) \hat{L}_{klij} \right]. \quad (A.3)$$

After regrouping the equation, it can be given by:

$$\begin{aligned} \sigma : \frac{\partial \sigma_{eq}}{\partial \sigma} &= \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial J_2^0}{\partial \Sigma_{kl}} - \hat{c} \frac{\partial J_3^0}{\partial \Sigma_{kl}} \right) \hat{L}_{klij} \sigma_{ij} \\ &= \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} \Sigma_{kl} - \hat{c} \left(\Sigma_{km} \Sigma_{ml} - \frac{2}{3} \delta_{kl} J_2^0 \right) \right) \Sigma_{kl} \\ &= \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} \Sigma_{kl} \Sigma_{kl} - \hat{c} (\Sigma_{km} \Sigma_{ml} \Sigma_{kl} - \frac{2}{3} \delta_{kl} \Sigma_{kl} J_2^0) \right). \end{aligned} \quad (A.4)$$

According to the definition of J_2^0 , J_3^0 , σ_{eq} , regrouping the above equations and simplifying, it is rewritten as

$$\sigma : \frac{\partial \sigma_{eq}}{\partial \sigma} = \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} (2J_2^0) - \hat{c} 3J_3^0 \right) = \frac{3\hat{A}^3}{3\sigma_{eq}^2} \left(J_2^{0\frac{3}{2}} - \hat{c} J_3^0 \right) = \sigma_{eq}. \quad (A.5)$$

So

$$\frac{\sigma : \frac{\partial \sigma_{eq}}{\partial \sigma}}{\sigma_{eq}} = 1, \quad (A.6)$$

$$\Delta \bar{\epsilon}^p = \frac{\sigma : \frac{\partial \sigma_{eq}}{\partial \sigma}}{\sigma_{eq}} \Delta\gamma = \Delta\gamma. \quad (A.7)$$

Appendix B

Proof of the Convexity of the Proposed Yield Function

In order for a material to be a stable plastic material, one of the main requirements that yield surfaces must satisfy is convexity. The convexity of the yield surface is a consequence of the work assumption or can be derived as a consequence of the assumption of maximum rate of dissipation. It also guarantees a unique relationship between the stresses and plastic strain increments assuming an associated flow rule.

In mathematics, a real-valued function $\hat{f}(x)$ defined on an interval is called convex (or convex downward or concave upward) if the line segment between any two points on the graph of the function lies above the graph, in a Euclidean space (or more generally a vector space) of at least two dimensions. Equivalently, a function is convex if its epigraph (the set of points on or above the graph of the function) is a convex set. Well-known examples of convex functions are the quadratic function. $\hat{f}: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex if $\text{Dom } \hat{f}$ is convex and for all $x, y \in \text{dom } \hat{f}, \theta \in [0,1]$

$$\hat{f}(\theta x + (1 - \theta)y) \leq \theta \hat{f}(x) + (1 - \theta)\hat{f}(y). \quad (\text{B.1})$$

This also can be judged by evaluating its Hessian matrix.

For the plane stress state, owing to the third principal stress is equal to zero, therefore, there only exists two in-plane principal stress σ_1, σ_2 . So the Hessian matrix $\hat{\mathbf{H}}$ is the second derivative of the yield function, which can be written as

$$\hat{\mathbf{H}} = \begin{pmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{12} & \hat{H}_{22} \end{pmatrix}, \quad (\text{B.2})$$

where,

$$\hat{H}_{11} = \frac{\partial^2 \Phi}{\partial \sigma_{11}^2}, \quad (\text{B.3})$$

$$\hat{H}_{22} = \frac{\partial^2 \Phi}{\partial \sigma_{22}^2}, \quad (\text{B.4})$$

$$\hat{H}_{12} = \frac{\partial^2 \Phi}{\partial \sigma_{11} \partial \sigma_{22}}. \quad (\text{B.5})$$

If and only if $\hat{H}_{11} \geq 0$ ($\hat{H}_{22} \geq 0$) and $\hat{H}_{11}\hat{H}_{22} - \hat{H}_{12}^2 \geq 0$, the Hessian matrix is positive semi-definite. The related yield functions to be convex.

Let

$$\hat{F} = a[(K_1 + K_2) - c|K_1 + K_2|]^M + a[(K_1 - K_2) - c|K_1 - K_2|]^M + f|2K_2|^M. \quad (\text{B.6})$$

where,

$$K_1 = \frac{\sigma_{11} + h\sigma_{22}}{2}, \quad (\text{B.7})$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h\sigma_{22}}{2}\right)^2 + b^2\sigma_{12}^2}. \quad (\text{B.8})$$

and a, h, f, b, c , and M are material parameters.

Then

$$\hat{H}_{11} = \frac{\partial^2 \hat{F}}{\partial \sigma_{11}^2}, \quad (\text{B.9})$$

$$\hat{H}_{22} = \frac{\partial^2 \hat{F}}{\partial \sigma_{22}^2}, \quad (\text{B.10})$$

$$\hat{H}_{12} = \frac{\partial^2 \hat{F}}{\partial \sigma_{11} \partial \sigma_{22}}. \quad (\text{B.11})$$

Therefore

$$\hat{H}_{11}\hat{H}_{22} - \hat{H}_{12}^2 = \frac{\partial^2 \hat{F}}{\partial \sigma_{11}^2} \frac{\partial^2 \hat{F}}{\partial \sigma_{22}^2} - \left(\frac{\partial^2 \hat{F}}{\partial \sigma_{11} \partial \sigma_{22}} \right)^2. \quad (\text{B.12})$$

The derivative of $\hat{F}$ with respect to stress component can be calculated by

$$\frac{\partial \hat{F}}{\partial \sigma_{11}} = \frac{1}{2} \left[\frac{\partial \hat{F}}{\partial K_1} + \frac{\partial \hat{F}}{\partial K_2} \left(\frac{\sigma_{11} - h\sigma_{22}}{2K_2} \right) \right], \quad (\text{B.13})$$

$$\frac{\partial \hat{F}}{\partial \sigma_{22}} = \frac{h}{2} \left[\frac{\partial \hat{F}}{\partial K_1} - \frac{\partial \hat{F}}{\partial K_2} \left(\frac{\sigma_{11} - h\sigma_{22}}{2K_2} \right) \right], \quad (\text{B.14})$$

$$\frac{\partial^2 \hat{F}}{\partial \sigma_{11}^2} = \frac{1}{4} \left[\frac{\partial^2 \hat{F}}{\partial K_1^2} + \frac{\partial^2 \hat{F}}{\partial K_2^2} + \frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2} \left(\frac{\sigma_{11} - h\sigma_{22}}{K_2} \right) \right], \quad (\text{B.15})$$

$$\frac{\partial^2 \hat{F}}{\partial \sigma_{11} \partial \sigma_{22}} = \frac{h}{4} \left[\frac{\partial^2 \hat{F}}{\partial K_1^2} - \frac{\partial^2 \hat{F}}{\partial K_2^2} \right], \quad (\text{B.16})$$

$$\frac{\partial^2 \hat{F}}{\partial \sigma_{22}^2} = \frac{h^2}{4} \left[\frac{\partial^2 \hat{F}}{\partial K_1^2} + \frac{\partial^2 \hat{F}}{\partial K_2^2} - \frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2} \left(\frac{\sigma_{11} - h\sigma_{22}}{K_2} \right) \right]. \quad (\text{B.17})$$

After a series mathematical operation, we can obtain

$$\hat{H}_{11}\hat{H}_{22} - \hat{H}_{12}^2 = \frac{h^2}{4} \left[\frac{\partial^2 \hat{F}}{\partial K_1^2} \frac{\partial^2 \hat{F}}{\partial K_2^2} - \left(\frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2} \right)^2 \right]. \quad (\text{B.18})$$

If we consider the yield function $\hat{F}$ is the function of variables K_1 and K_2 , then

$$\frac{\partial^2 \hat{F}}{\partial K_1^2} \frac{\partial^2 \hat{F}}{\partial K_2^2} - \left(\frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2} \right)^2,$$

is the determinant of Hessian matrix of yield function $\hat{F}$ with respect to variables K_1 and K_2 , where,

$$\hat{H}'_{11} = \frac{\partial^2 \hat{F}}{\partial K_1^2}, \quad (\text{B.19})$$

$$\hat{H}'_{22} = \frac{\partial^2 \hat{F}}{\partial K_2^2}, \quad (\text{B.20})$$

$$\hat{H}'_{12} = \frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2}. \quad (\text{B.21})$$

This indicates that if the yield function $\hat{F}$ is convex to variables K_1 and K_2 , it is also convex to the stress components.

In order to simplify the scripts, let

$$\hat{P} = (K_1 + K_2) - c|K_1 + K_2|, \quad (\text{B.22})$$

$$\hat{Q} = (K_1 - K_2) - c|K_1 - K_2|. \quad (\text{B.23})$$

The derivative of $\hat{F}$ with respect to K_1 and K_2 can be calculated by

$$\frac{\partial \hat{F}}{\partial K_1} = aM\hat{P}^{M-1} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right) + aM\hat{Q}^{M-1} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right), \quad (\text{B.24})$$

$$\frac{\partial \hat{F}}{\partial K_2} = aM\hat{P}^{M-1} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right) - aM\hat{Q}^{M-1} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right) + 2fM|2K_2|^{M-1}, \quad (\text{B.25})$$

$$\frac{\partial^2 \hat{F}}{\partial K_1^2} = aM(M-1)\hat{P}^{M-2} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right)^2 + aM(M-1)\hat{Q}^{M-2} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right)^2, \quad (\text{B.26})$$

$$\frac{\partial^2 \hat{F}}{\partial K_2^2} = aM(M-1)\hat{P}^{M-2} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right)^2 + aM(M-1)\hat{Q}^{M-2} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right)^2 + 4fM(M-1)|2K_2|^{M-2}, \quad (\text{B.27})$$

$$\begin{aligned} \frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2} &= aM(M-1)\hat{P}^{M-2} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right)^2 \\ &\quad - aM(M-1)\hat{Q}^{M-2} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right)^2. \end{aligned} \quad (\text{B.28})$$

Thus,

$$\begin{aligned} \frac{\partial^2 \hat{F}}{\partial K_1^2} \frac{\partial^2 \hat{F}}{\partial K_2^2} - \left(\frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2}\right)^2 &= 4a^2M^2(M-1)^2\hat{P}^{M-2}\hat{Q}^{M-2} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right)^4 \\ &\quad + 4fM(M-1) \left[aM(M-1)\hat{P}^{M-2} \left(1 - c \frac{K_1+K_2}{|K_1+K_2|}\right)^2 + \right. \\ &\quad \left. aM(M-1)\hat{Q}^{M-2} \left(1 - c \frac{K_1-K_2}{|K_1-K_2|}\right)^2 \right] |2K_2|^{M-2}. \end{aligned} \quad (\text{B.29})$$

Therefore, when $M > 1$, $a, c, f > 0$, we can obtain that

$$\hat{H}'_{11}\hat{H}'_{22} - \hat{H}'_{12}\hat{H}'_{12} = \frac{\partial^2 \hat{F}}{\partial K_1^2} \frac{\partial^2 \hat{F}}{\partial K_2^2} - \left(\frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2}\right)^2 \geq 0. \quad (\text{B.30})$$

Then

$$\hat{H}_{11}\hat{H}_{22} - \hat{H}_{12}^2 = \frac{h^2}{4} \left[\frac{\partial^2 \hat{F}}{\partial K_1^2} \frac{\partial^2 \hat{F}}{\partial K_2^2} - \left(\frac{\partial^2 \hat{F}}{\partial K_1 \partial K_2}\right)^2 \right] \geq 0. \quad (\text{B.31})$$

It is concluded that the proposed yield function is convex function.

Appendix C

Derivatives of Cazacu'04 Constitutive Model

The yield function of Cazacu'04 model can be written as:

$$\sigma_{eq} = \hat{A}(J_2^{0\frac{3}{2}} - \hat{c}J_3^0)^{1/3}, \quad (C.1)$$

$$\boldsymbol{\Sigma} = \hat{\mathbf{L}}\boldsymbol{\sigma}, \quad (C.2)$$

$$J_2^0 = \frac{1}{2}tr(\boldsymbol{\Sigma}^2), \quad (C.3)$$

$$J_3^0 = \frac{1}{3}tr(\boldsymbol{\Sigma}^3), \quad (C.4)$$

$$\hat{A} = 3[(a_2^2 + a_3^2 + a_2a_3)^{\frac{3}{2}} - \hat{c}(a_2 + a_3)a_2a_3]^{-1/3}. \quad (C.5)$$

Derivatives of the yield function by the Cartesian components of the stress tensor are evaluated using the equivalent form of the yield function given as follows.

First derivative of J_2^0 with respect to stress tensor Σ_{ij} can be given by:

$$\frac{\partial J_2^0}{\partial \Sigma_{ij}} = S_{ij}. \quad (C.6)$$

First derivative of J_3^0 with respect to stress tensor Σ_{ij} can be calculated by

$$\frac{\partial J_3^0}{\partial \Sigma_{ij}} = S_{ik}S_{kj} - \frac{2}{3}\delta_{ij}J_2^0. \quad (C.7)$$

First derivative of the deviatoric transformed stress S_{ij} with respect to stress tensor Σ_{mn} , or second derivative of J_2^0 with respect to stress tensors Σ_{ij} and Σ_{mn} :

$$\frac{\partial^2 J_2^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}} = \delta_{mi}\delta_{nj} - \frac{1}{3}\delta_{ij}\delta_{mn}. \quad (C.8)$$

Second derivative of J_3^0 with respect to stress tensors Σ_{ij} and Σ_{mn} :

$$\frac{\partial^2 J_3^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}} = S_{ni}\delta_{jm} + S_{jm}\delta_{ni} - \frac{2}{3}S_{ij}\delta_{mn} - \frac{2}{3}S_{mn}\delta_{ij}. \quad (C.9)$$

Using the chain rule, the derivative of both the second and third invariants of deviatoric transformed stress, with respect to the Cauchy stress can be obtained by:

$$\frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} = \frac{\partial J_2^0}{\partial \boldsymbol{\Sigma}} \frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\sigma}}, \quad (C.10)$$

$$\frac{\partial J_3^0}{\partial \boldsymbol{\sigma}} = \frac{\partial J_3^0}{\partial \boldsymbol{\Sigma}} \frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\sigma}}. \quad (C.11)$$

Therefore, the derivative of yield function to the Cauchy stress is:

$$\mathbf{N} = \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \frac{\hat{A}^3}{3\sigma_{eq}} \left(\frac{3}{2}J_2^{0\frac{1}{2}} \frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} - \hat{c} \frac{\partial J_3^0}{\partial \boldsymbol{\sigma}} \right). \quad (C.12)$$

The derivative of coefficient $\hat{A}$ to equivalent plastic strain can be given by:

$$\begin{aligned} \frac{\partial \hat{A}}{\partial \bar{\varepsilon}_p} = & - \left[(a_2^2 + a_3^2 + a_2 a_3)^{\frac{3}{2}} - \hat{c} (a_2 + a_3) a_2 a_3 \right]^{-\frac{4}{3}} \\ & \left[\left(\frac{3}{2} (a_2^2 + a_3^2 + a_2 a_3)^{\frac{1}{2}} \right) \left(2a_2 \frac{\partial a_2}{\partial \bar{\varepsilon}_p} + 2a_3 \frac{\partial a_3}{\partial \bar{\varepsilon}_p} + a_2 \frac{\partial a_3}{\partial \bar{\varepsilon}_p} + a_3 \frac{\partial a_2}{\partial \bar{\varepsilon}_p} \right) - \right. \\ & \left. \frac{\partial \hat{c}}{\partial \bar{\varepsilon}_p} (a_2 + a_3) a_2 a_3 - 2a_2 a_3 \hat{c} \left(\frac{\partial a_2}{\partial \bar{\varepsilon}_p} + \frac{\partial a_3}{\partial \bar{\varepsilon}_p} \right) \right]. \end{aligned} \quad (C.13)$$

Owing to the dependence of the transformed matrix $\hat{\mathbf{L}}$ on the equivalent plastic strain, the derivative of the second and third invariants of the deviatoric transformed stress with respect to the equivalent plastic strain can be given by:

$$\frac{\partial J_2^0}{\partial \bar{\varepsilon}_p} = \frac{\partial J_2^0}{\partial \boldsymbol{\Sigma}} \frac{\partial \boldsymbol{\Sigma}}{\partial \bar{\varepsilon}_p}, \quad (C.14)$$

$$\frac{\partial J_3^0}{\partial \bar{\varepsilon}_p} = \frac{\partial J_3^0}{\partial \boldsymbol{\Sigma}} \frac{\partial \boldsymbol{\Sigma}}{\partial \bar{\varepsilon}_p}. \quad (C.15)$$

and the $\frac{\partial \boldsymbol{\Sigma}}{\partial \bar{\varepsilon}_p}$ can be calculated by:

$$\frac{\partial \boldsymbol{\Sigma}}{\partial \bar{\varepsilon}_p} = \frac{\partial \hat{\mathbf{L}}}{\partial \bar{\varepsilon}_p} \boldsymbol{\sigma} = \frac{\partial \hat{\mathbf{L}}}{\partial a_m} \frac{\partial a_m}{\partial \bar{\varepsilon}_p} \boldsymbol{\sigma}. \quad (C.16)$$

Finally, the derivative of the yield function to the equivalent plastic strain can be given by:

$$\frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}_p} = \frac{\partial \hat{A}}{\partial \bar{\varepsilon}_p} (J_2^{0\frac{3}{2}} - \hat{c} J_3^0)^{1/3} + \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial J_2^0}{\partial \bar{\varepsilon}_p} - \hat{c} \frac{\partial J_3^0}{\partial \bar{\varepsilon}_p} - \frac{\partial \hat{c}}{\partial \bar{\varepsilon}_p} J_3^0 \right). \quad (C.17)$$

Owing to that, the transformed matrix $\hat{\mathbf{L}}$ is independent on the stress tensor, so the second order derivative of J_3^0, J_2^0 to the Cauchy stress can be calculated by:

$$\frac{\partial^2 J_2^0}{\partial \boldsymbol{\sigma}^2} = \frac{\partial^2 J_2^0}{\partial \boldsymbol{\Sigma}^2} \frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\sigma}}, \quad (C.18)$$

$$\frac{\partial^2 J_3^0}{\partial \boldsymbol{\sigma}^2} = \frac{\partial^2 J_3^0}{\partial \boldsymbol{\Sigma}^2} \frac{\partial \boldsymbol{\Sigma}}{\partial \boldsymbol{\sigma}}. \quad (C.19)$$

So, the second order derivative of yield function to Cauchy stress can be obtained by:

$$\frac{\partial^2 \sigma_{eq}}{\partial \boldsymbol{\sigma}^2} = - \frac{2}{\sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} + \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{4} J_2^{0-\frac{1}{2}} \frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} + \frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial^2 J_2^0}{\partial \boldsymbol{\sigma}^2} - \hat{c} \frac{\partial^2 J_3^0}{\partial \boldsymbol{\sigma}^2} \right). \quad (C.20)$$

while the derivative of flow direction $\mathbf{N}$ to equivalent plastic strain can be obtained by :

$$\begin{aligned} \frac{\partial^2 \sigma_{eq}}{\partial \boldsymbol{\sigma} \partial \bar{\varepsilon}_p} = & \left(\frac{\hat{A}^2}{\sigma_{eq}^2} \frac{\partial \hat{A}}{\partial \bar{\varepsilon}_p} - \frac{2\hat{A}^3}{3\sigma_{eq}^3} \frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}_p} \right) \left(\frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} - \hat{c} \frac{\partial J_3^0}{\partial \boldsymbol{\sigma}} \right) + \frac{\hat{A}^3}{3\sigma_{eq}^2} \left(\frac{3}{4} J_2^{0-\frac{1}{2}} \frac{\partial J_2^0}{\partial \boldsymbol{\sigma}} \frac{\partial J_2^0}{\partial \bar{\varepsilon}_p} + \right. \\ & \left. \frac{3}{2} J_2^{0\frac{1}{2}} \frac{\partial^2 J_2^0}{\partial \boldsymbol{\sigma} \partial \bar{\varepsilon}_p} - \hat{c} \frac{\partial^2 J_3^0}{\partial \boldsymbol{\sigma} \partial \bar{\varepsilon}_p} - \frac{\partial \hat{c}}{\partial \bar{\varepsilon}_p} \right). \end{aligned} \quad (C.21)$$

Appendix D

Derivatives of New Constitutive Model

$$K_1 = \frac{\sigma_{11} + h\sigma_{22}}{2}, \quad (D.1)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h\sigma_{22}}{2}\right)^2 + b^2\sigma_{12}^2}. \quad (D.2)$$

The new proposed yield criteria is shown as follows

$$\sigma_{eq} = (a(\bar{\Sigma}_1 - c|\bar{\Sigma}_1|)^M + a(\bar{\Sigma}_2 - c|\bar{\Sigma}_2|)^M + f|2K_2|^M)^{1/M}, \quad (D.3)$$

where

$$\bar{\Sigma}_1 = K_1 + K_2, \quad (D.4)$$

$$\bar{\Sigma}_2 = K_1 - K_2. \quad (D.5)$$

The first derivative of K_1, K_2 with respect to stress componet

$$\frac{\partial K_1}{\partial \sigma} = \begin{bmatrix} 1/2 \\ h/2 \\ 0 \end{bmatrix}, \quad (D.6)$$

$$\frac{\partial K_2}{\partial \sigma} = \frac{1}{2K_2} \begin{bmatrix} (\sigma_{11} - h\sigma_{22})/2 \\ -h(\sigma_{11} - h\sigma_{22})/2 \\ 2b^2\sigma_{12} \end{bmatrix} = \frac{\tilde{B}}{2K_2}. \quad (D.7)$$

where

$$\tilde{B} = \begin{bmatrix} (\sigma_{11} - h\sigma_{22})/2 \\ -h(\sigma_{11} - h\sigma_{22})/2 \\ 2b^2\sigma_{12} \end{bmatrix}. \quad (D.8)$$

The derivative of K_1, K_2 with respect to the equivalent plastic strain

$$\frac{\partial K_1}{\partial \bar{\epsilon}^p} = \frac{\partial h}{\partial \bar{\epsilon}^p} \frac{\sigma_{22}}{2}, \quad (D.9)$$

$$\frac{\partial K_2}{\partial \bar{\epsilon}^p} = \frac{1}{2K_2} \left(2b\sigma_{12}^2 \frac{\partial b}{\partial \bar{\epsilon}^p} - \left(\frac{\sigma_{11} - h\sigma_{22}}{2} \right) \frac{\sigma_{22}}{2} \frac{\partial h}{\partial \bar{\epsilon}^p} \right). \quad (D.10)$$

The second derivative of K_1, K_2 with respect to stress component

$$\frac{\partial^2 K_1}{\partial^2 \sigma} = \mathbf{0}, \quad (D.11)$$

$$\frac{\partial^2 K_2}{\partial^2 \sigma} = -\frac{1}{K_2} \frac{\partial K_2}{\partial \sigma} \otimes \frac{\partial K_2}{\partial \sigma} + \frac{1}{2K_2} \frac{\partial \tilde{B}}{\partial \sigma}. \quad (D.12)$$

where

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} 1/2 & -h/2 & 0 \\ -h/2 & h^2/2 & 0 \\ 0 & 0 & 2b^2 \end{bmatrix}. \quad (\text{D.13})$$

The second derivative K_1, K_2 with respect to stress and equivalent plastic strain

$$\frac{\partial K_1^2}{\partial \boldsymbol{\sigma} \partial \bar{\varepsilon}^p} = \begin{bmatrix} 0 \\ \partial h / \partial \bar{\varepsilon}^p / 2 \\ 0 \end{bmatrix}, \quad (\text{D.14})$$

$$\frac{\partial K_2^2}{\partial \boldsymbol{\sigma} \partial \bar{\varepsilon}^p} = -\frac{1}{K_2} \frac{\partial K_2}{\partial \bar{\varepsilon}^p} \frac{\partial K_2}{\partial \boldsymbol{\sigma}} + \frac{1}{2K_2} \frac{\partial \tilde{\mathbf{B}}}{\partial \bar{\varepsilon}^p}. \quad (\text{D.15})$$

where

$$\frac{\partial \tilde{\mathbf{B}}}{\partial \bar{\varepsilon}^p} = \begin{bmatrix} -\frac{\partial h}{\partial \bar{\varepsilon}^p} \frac{\sigma_{22}}{2} \\ \frac{\partial h}{\partial \bar{\varepsilon}^p} \left(-\frac{\sigma_{11}}{2} + h\sigma_{22} \right) \\ 4b\sigma_{12} \frac{\partial b}{\partial \bar{\varepsilon}^p} \end{bmatrix}. \quad (\text{D.16})$$

In order to derivative the yield function to variable, here we set

$$\tilde{g}_1 = a |(K_1 + K_2) - c|K_1 + K_2||^M, \quad (\text{D.17})$$

$$\tilde{g}_2 = a |(K_1 - K_2) - c|K_1 - K_2||^M, \quad (\text{D.18})$$

$$\tilde{g}_3 = (2 - a)|2K_2|^M. \quad (\text{D.19})$$

So the yield function can be written as

$$\sigma_{eq} = (\tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3)^{1/M}. \quad (\text{D.20})$$

So, the first derivative of $\tilde{g}_1, \tilde{g}_2$ and $\tilde{g}_3$ with respect to stress can be given by

$$\frac{\partial \tilde{g}_1}{\partial \boldsymbol{\sigma}} = \frac{M\tilde{g}_1}{K_1 + K_2} \left(\frac{\partial K_1}{\partial \boldsymbol{\sigma}} + \frac{\partial K_2}{\partial \boldsymbol{\sigma}} \right), \quad (\text{D.21})$$

$$\frac{\partial \tilde{g}_2}{\partial \boldsymbol{\sigma}} = \frac{M\tilde{g}_2}{K_1 - K_2} \left(\frac{\partial K_1}{\partial \boldsymbol{\sigma}} - \frac{\partial K_2}{\partial \boldsymbol{\sigma}} \right), \quad (\text{D.22})$$

$$\frac{\partial \tilde{g}_3}{\partial \boldsymbol{\sigma}} = \frac{M\tilde{g}_3}{K_2} \frac{\partial K_2}{\partial \boldsymbol{\sigma}}. \quad (\text{D.23})$$

The first derivative of $\tilde{g}_1, \tilde{g}_2$ and $\tilde{g}_3$ with respect to equivalent plastic strain can be given by

$$\frac{\partial \sigma_{eq}}{\partial \bar{\varepsilon}^p} = \frac{1}{M} (\tilde{g}_1 + \tilde{g}_2 + \tilde{g}_3)^{\frac{1}{M}-1} \left(\frac{\partial \tilde{g}_1}{\partial \bar{\varepsilon}^p} + \frac{\partial \tilde{g}_2}{\partial \bar{\varepsilon}^p} + \frac{\partial \tilde{g}_3}{\partial \bar{\varepsilon}^p} \right). \quad (\text{D.24})$$

where

$$\frac{\partial \tilde{g}_1}{\partial \bar{\varepsilon}^p} = \frac{\tilde{g}_1}{a} \frac{\partial a}{\partial \bar{\varepsilon}^p} + \frac{M\tilde{g}_1}{K_1 + K_2} \left(\frac{\partial K_1}{\partial \bar{\varepsilon}^p} + \frac{\partial K_2}{\partial \bar{\varepsilon}^p} \right) - aM |(K_1 + K_2) - c|K_1 + K_2||^{M-1} \quad (\text{D.25})$$

$$|K_1 + K_2| \frac{\partial c}{\partial \bar{\varepsilon}^p},$$

$$\frac{\partial \tilde{g}_2}{\partial \bar{\varepsilon}^p} = \frac{\tilde{g}_2}{a} \frac{\partial a}{\partial \bar{\varepsilon}^p} + \frac{M\tilde{g}_2}{K_1 - K_2} \left(\frac{\partial K_1}{\partial \bar{\varepsilon}^p} - \frac{\partial K_2}{\partial \bar{\varepsilon}^p} \right) - aM |(K_1 - K_2) - c|K_1 - K_2||^{M-1} \quad (\text{D.26})$$

$$|K_1 - K_2| \frac{\partial c}{\partial \bar{\epsilon}^p},$$

$$\frac{\partial \tilde{g}_3}{\partial \bar{\epsilon}^p} = -\frac{\partial a}{\partial \bar{\epsilon}^p} |2K_2|^M + \frac{M \tilde{g}_3}{K_2} \frac{\partial K_2}{\partial \bar{\epsilon}^p}. \quad (D.27)$$

The second derivative of $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$ with respect to stress

$$\frac{\partial^2 \tilde{g}_1}{\partial \sigma^2} = \frac{M}{K_1 + K_2} \left(\frac{\partial K_1}{\partial \sigma} + \frac{\partial K_2}{\partial \sigma} \right) \otimes \frac{\partial \tilde{g}_1}{\partial \sigma} - \frac{M \tilde{g}_1}{(K_1 + K_2)^2} \frac{\partial(K_1 + K_2)}{\partial \sigma} \otimes \frac{\partial(K_1 + K_2)}{\partial \sigma} \\ + \frac{M \tilde{g}_1}{K_1 + K_2} \left(\frac{\partial K_1^2}{\partial^2 \sigma} + \frac{\partial K_2^2}{\partial^2 \sigma} \right), \quad (D.28)$$

$$\frac{\partial^2 \tilde{g}_2}{\partial \sigma^2} = \frac{M}{K_1 - K_2} \left(\frac{\partial K_1}{\partial \sigma} - \frac{\partial K_2}{\partial \sigma} \right) \otimes \frac{\partial \tilde{g}_2}{\partial \sigma} - \frac{M \tilde{g}_2}{(K_1 - K_2)^2} \frac{\partial(K_1 - K_2)}{\partial \sigma} \otimes \frac{\partial(K_1 - K_2)}{\partial \sigma} \\ + \frac{M \tilde{g}_2}{K_1 - K_2} \left(\frac{\partial K_1^2}{\partial^2 \sigma} - \frac{\partial K_2^2}{\partial^2 \sigma} \right), \quad (D.29)$$

$$\frac{\partial^2 \tilde{g}_3}{\partial \sigma^2} = \frac{M}{K_2} \frac{\partial K_2}{\partial \sigma} \otimes \frac{\partial \tilde{g}_3}{\partial \sigma} - \frac{M \tilde{g}_3}{K_2^2} \frac{\partial K_2}{\partial \sigma} \otimes \frac{\partial K_2}{\partial \sigma} + \frac{M \tilde{g}_3}{K_2} \frac{\partial K_2^2}{\partial^2 \sigma}. \quad (D.30)$$

The second derivative of $\tilde{g}_1$, $\tilde{g}_2$ and $\tilde{g}_3$ with respect to stress and equivalent plastic strain

$$\frac{\partial^2 \tilde{g}_1}{\partial \sigma \partial \bar{\epsilon}^p} = \frac{M}{K_1 + K_2} \frac{\partial \tilde{g}_1}{\partial \bar{\epsilon}^p} \left(\frac{\partial K_1}{\partial \sigma} + \frac{\partial K_2}{\partial \sigma} \right) - \frac{M \tilde{g}_1}{(K_1 + K_2)^2} \frac{\partial(K_1 + K_2)}{\partial \bar{\epsilon}^p} \frac{\partial(K_1 + K_2)}{\partial \sigma} \\ + \frac{M \tilde{g}_1}{K_1 + K_2} \left(\frac{\partial K_1^2}{\partial \sigma \partial \bar{\epsilon}^p} + \frac{\partial K_2^2}{\partial \sigma \partial \bar{\epsilon}^p} \right), \quad (D.31)$$

$$\frac{\partial^2 \tilde{g}_2}{\partial \sigma \partial \bar{\epsilon}^p} = \frac{M}{K_1 - K_2} \frac{\partial \tilde{g}_2}{\partial \bar{\epsilon}^p} \left(\frac{\partial K_1}{\partial \sigma} - \frac{\partial K_2}{\partial \sigma} \right) - \frac{M \tilde{g}_2}{(K_1 - K_2)^2} \frac{\partial(K_1 - K_2)}{\partial \bar{\epsilon}^p} \frac{\partial(K_1 - K_2)}{\partial \sigma} \\ + \frac{M \tilde{g}_2}{K_1 - K_2} \left(\frac{\partial K_1^2}{\partial \sigma \partial \bar{\epsilon}^p} - \frac{\partial K_2^2}{\partial \sigma \partial \bar{\epsilon}^p} \right), \quad (D.32)$$

$$\frac{\partial^2 \tilde{g}_3}{\partial \sigma \partial \bar{\epsilon}^p} = \frac{M}{K_2} \frac{\partial \tilde{g}_3}{\partial \bar{\epsilon}^p} \frac{\partial K_2}{\partial \sigma} - \frac{M \tilde{g}_3}{K_2^2} \frac{\partial K_2}{\partial \bar{\epsilon}^p} \frac{\partial K_2}{\partial \sigma} + \frac{M \tilde{g}_3}{K_2} \frac{\partial K_2^2}{\partial \sigma \partial \bar{\epsilon}^p}. \quad (D.33)$$

Finally, the derivative of the yield function with respect to stress can be given by

$$\frac{\partial \sigma_{eq}}{\partial \sigma} = \frac{\sigma_{eq}^{1-M}}{2M} \left(\frac{\partial \tilde{g}_1}{\partial \sigma} + \frac{\partial \tilde{g}_2}{\partial \sigma} + \frac{\partial \tilde{g}_3}{\partial \sigma} \right). \quad (D.34)$$

The derivative of the yield function with respect to equivalent plastic strain

$$\frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^p} = \frac{\sigma_{eq}^{1-M}}{2M} \left(\frac{\partial \tilde{g}_1}{\partial \bar{\epsilon}^p} + \frac{\partial \tilde{g}_2}{\partial \bar{\epsilon}^p} + \frac{\partial \tilde{g}_3}{\partial \bar{\epsilon}^p} \right). \quad (D.35)$$

The second derivative of the yield function with respect to stress

$$\frac{\partial^2 \sigma_{eq}}{\partial \sigma^2} = \frac{1-M}{\sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \sigma} \otimes \frac{\partial \sigma_{eq}}{\partial \sigma} + \frac{\sigma_{eq}^{1-M}}{2M} \left(\frac{\partial^2 \tilde{g}_1}{\partial \sigma^2} + \frac{\partial^2 \tilde{g}_2}{\partial \sigma^2} + \frac{\partial^2 \tilde{g}_3}{\partial \sigma^2} \right). \quad (D.36)$$

The derivative of the yield function with respect to stress and equivalent plastic strain

$$\frac{\partial^2 \sigma_{eq}}{\partial \sigma \partial \bar{\epsilon}^p} = \frac{1-M}{\sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}^p} \frac{\partial \sigma_{eq}}{\partial \sigma} + \frac{\sigma_{eq}^{1-M}}{2M} \left(\frac{\partial^2 \tilde{g}_1}{\partial \sigma \partial \bar{\epsilon}^p} + \frac{\partial^2 \tilde{g}_2}{\partial \sigma \partial \bar{\epsilon}^p} + \frac{\partial^2 \tilde{g}_3}{\partial \sigma \partial \bar{\epsilon}^p} \right). \quad (D.37)$$

Appendix E

Derivatives of Barlat'89 model

The formulation of Barlat'89 model is written as follows

$$\sigma_{Barlat} = \left(\frac{a_{BL}|K_1+K_2|^{M_{BL}} + a_{BL}|K_1-K_2|^{M_{BL}} + (2-a_{BL})|2K_2|^{M_{BL}}}{2} \right)^{1/M} \quad (E.1)$$

where K_1 and K_2 are given by

$$K_1 = \frac{\sigma_{11} + h_{BL}\sigma_{22}}{2}, \quad (E.2)$$

$$K_2 = \sqrt{\left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2}\right)^2 + (b_{BL})^2 \sigma_{12}^2}, \quad (E.3)$$

and M_{BL} , a_{BL} , h_{BL} , and b_{BL} are material parameters.

Let

$$f_{BL} = a_{BL}|K_1 + K_2|^{M_{BL}} + a_{BL}|K_1 - K_2|^{M_{BL}} + (2 - a_{BL})|2K_2|^{M_{BL}}. \quad (E.4)$$

Then

$$\sigma_{Barlat} = \left(\frac{f_{BL}}{2} \right)^{1/M_{BL}}. \quad (E.5)$$

The equation of the flow vector $\mathbf{N}$, defined below, requires the terms of the first derivative of the function f_{BL} depending on σ_{11} , σ_{22} and σ_{12} .

$$\mathbf{N} = \frac{1}{2M_{BL}} \left(\frac{f_{BL}}{2} \right)^{\frac{1-M_{BL}}{M_{BL}}} \frac{\partial f_{BL}}{\partial \boldsymbol{\sigma}}, \quad (E.6)$$

and

$$\frac{\partial \mathbf{N}}{\partial \boldsymbol{\sigma}} = \frac{1}{2M_{BL}} \left(\frac{f_{BL}}{2} \right)^{\frac{1-M_{BL}}{M_{BL}}} \frac{\partial^2 f_{BL}}{\partial \sigma^2} + \frac{1-M_{BL}}{4M_{BL}^2} \left(\frac{f_{BL}}{2} \right)^{\frac{1-2M_{BL}}{M_{BL}}} \frac{\partial f_{BL}}{\partial \sigma} \otimes \frac{\partial f_{BL}}{\partial \sigma}. \quad (E.7)$$

The first derivative of f_{BL} with respect to σ_{11} , σ_{22} and σ_{12} is:

$$\frac{\partial f_{BL}}{\partial \sigma_{11}} = \frac{M_{BL}}{2} \left[a_{BL} \left(1 + \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_1 + K_2|^{M_{BL}-1} + a_{BL} \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_1 - K_2|^{M_{BL}-1} + (2 - a_{BL}) 2^{M_{BL}} \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_2|^{M_{BL}-1} \right], \quad (E.8)$$

$$\frac{\partial f_{BL}}{\partial \sigma_{22}} = \frac{M_{BL}h_{BL}}{2} \left[a_{BL} \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_1 + K_2|^{M_{BL}-1} + a_{BL} \left(1 + \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_1 - K_2|^{M_{BL}-1} - (2 - a_{BL}) 2^{M_{BL}} \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_2|^{M_{BL}-1} \right], \quad (E.9)$$

$$\frac{\partial f_{BL}}{\partial \sigma_{12}} = \frac{M_{BL}(b_{BL})^2 \sigma_{12}}{K_2} [a_{BL}(K_1 + K_2)|K_1 + K_2|^{M_{BL}-2} - a_{BL}(K_1 - K_2)|K_1 - K_2|^{M_{BL}-2} + 2(2 - a_{BL})(2K_2)^{M_{BL}-1}]. \quad (E.10)$$

The second derivative of function f_{BL} with respect to σ_{11}

$$\begin{aligned} \frac{\partial^2 f_{BL}}{\partial \sigma_{11}^2} = & \frac{M_{BL}}{4} \left\{ a_{BL} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 + K_2|^{M_{BL}-1} + (M_{BL} - \right. \right. \\ & 1) \left(1 + \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 |K_1 + K_2|^{M_{BL}-2} \Big] + \\ & a_{BL} \left[-\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 - K_2|^{M_{BL}-1} + \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 (M_{BL} - \right. \\ & 1) |K_1 - K_2|^{M_{BL}-2} \Big] + (2 - a_{BL}) 2^{M_{BL}} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_2|^{M_{BL}-1} + \right. \\ & \left. \left. (M_{BL} - 1) \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 |K_2|^{M_{BL}-2} \right] \right\}. \end{aligned} \quad (E.11)$$

The second derivative of function f_{BL} with respect to σ_{22} :

$$\begin{aligned} \frac{\partial^2 f_{BL}}{\partial \sigma_{22}^2} = & \frac{M_{BL}(h_{BL})^2}{4} \left\{ a_{BL} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 + K_2|^{M_{BL}-1} + \right. \right. \\ & \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 (M_{BL} - 1) |K_1 + K_2|^{M_{BL}-2} \Big] + \\ & a_{BL} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 - K_2|^{M_{BL}-1} - \left(1 + \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 (M_{BL} - \right. \\ & 1) |K_1 - K_2|^{M_{BL}-2} \Big] + (2 - a_{BL}) 2^{M_{BL}} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_2|^{M_{BL}-1} + \right. \\ & \left. \left. (M_{BL} - 1) \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 |K_2|^{M_{BL}-2} \right] \right\}. \end{aligned} \quad (E.12)$$

The second derivative of function f_{BL} depending on σ_{12} is:

$$\begin{aligned} \frac{\partial^2 f_{BL}}{\partial \sigma_{12}^2} = & \frac{M_{BL}b_{BL}}{K_2} \left[\left(1 - \frac{\sigma_{12}^2(b_{BL})^2}{K_2^2} \right) \{ a_{BL} |K_1 + K_2|^{M_{BL}-1} - a_{BL} |K_1 - K_2|^{M_{BL}-1} \right. \right. \\ & \left. \left. - 2(2 - a_{BL})(2K_2)^{M_{BL}-1} \right\} + \frac{(b_{BL})^2 \sigma_{12}^2}{K_2} (M_{BL} - 1) \right. \\ & \left. [a_{BL} (|K_1 + K_2|^{M_{BL}-2} + |K_1 - K_2|^{M_{BL}-2}) + 2^{M_{BL}} (2 - a_{BL})(K_2)^{M_{BL}-2}] \right] \end{aligned} \quad (E.13)$$

The second derivative of function f_{BL} depending on σ_{11} and σ_{22} :

$$\begin{aligned} \frac{\partial^2 f_{BL}}{\partial \sigma_{11} \partial \sigma_{22}} = & \frac{M_{BL}h_{BL}}{4} \left\{ a_{BL} \left[-\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 + K_2|^{M_{BL}-1} + \right. \right. \\ & (M_{BL} - 1) \left(1 + \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) |K_1 + K_2|^{M_{BL}-2} \Big] + \\ & a_{BL} \left[\frac{1}{K_2} \left(1 - \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_1 - K_2|^{M_{BL}-1} + \left(1 - \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) \left(1 + \right. \\ & \left. \frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right) (M_{BL} - 1) |K_1 - K_2|^{M_{BL}-2} \Big] - (2 - a_{BL}) 2^{M_{BL}} \left[\frac{1}{K_2} \left(1 - \right. \right. \\ & \left. \left. \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 \right) |K_2|^{M_{BL}-1} + (M_{BL} - 1) \left(\frac{\sigma_{11} - h_{BL}\sigma_{22}}{2K_2} \right)^2 |K_2|^{M_{BL}-2} \right] \right\}. \end{aligned} \quad (E.14)$$

The second derivative of function f_{BL} depending on σ_{12} and σ_{11} :

$$\begin{aligned}
\frac{\partial^2 f_{BL}}{\partial \sigma_{12} \partial \sigma_{11}} = & \frac{M_{BL}(b_{BL})^2 \sigma_{12}}{2K_2} \left[\left(\frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2^2} \right) \left\{ a_{BL} \left(|K_1 - K_2|^{M_{BL}-1} \right) \right. \right. \\
& \left. \left. - 2(2 - a_{BL})(2K_2)^{M-1} \right\} \right. \\
& \left. + (M_{BL} - 1) \left[a_{BL} \left(\left(1 + \frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) |K_1 + K_2|^{M_{BL}-2} \right) \right. \right. \\
& \left. \left. - \left(1 - \frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) |K_1 - K_2|^{M_{BL}-2} \right) \right. \right. \\
& \left. \left. + 2^{M_{BL}} \left(\frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) (2 - a_{BL})(K_2)^{M_{BL}-2} \right] \right] .
\end{aligned} \tag{E.15}$$

The second derivative of function f_{BL} depending on σ_{12} and σ_{22} , we obtain the above expression:

$$\begin{aligned}
\frac{\partial^2 f_{BL}}{\partial \sigma_{12} \partial \sigma_{22}} = & \frac{M_{BL}(b_{BL})^2 h_{BL} \sigma_{12}}{2K_2} \left[\left(\frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2^2} \right) \left\{ a_{BL} \left(|K_1 + K_2|^{M_{BL}-1} \right) \right. \right. \\
& \left. \left. + 2(2 - a_{BL})(2K_2)^{M_{BL}-1} \right\} \right. \\
& \left. + (M_{BL} - 1) \left[a_{BL} \left(\left(1 - \frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) |K_1 + K_2|^{M_{BL}-2} \right) \right. \right. \\
& \left. \left. - \left(1 + \frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) |K_1 - K_2|^{M_{BL}-2} \right) \right. \right. \\
& \left. \left. - 2^{M_{BL}} \left(\frac{\sigma_{11} - h_{BL} \sigma_{22}}{2K_2} \right) (2 - a_{BL})(K_2)^{M_{BL}-2} \right] \right] .
\end{aligned} \tag{E.16}$$

Appendix F

Derivatives of Cazacu'06 Yield Criterion

The formula of Cazacu'06 is written as

$$\sigma_{eq} = \left[(|\Sigma_1| - \tilde{k}\Sigma_1)^{\tilde{n}} + (|\Sigma_2| - \tilde{k}\Sigma_2)^{\tilde{n}} + (|\Sigma_3| - \tilde{k}\Sigma_3)^{\tilde{n}} \right]^{1/\tilde{n}}, \quad (\text{F.1})$$

where

$$\Sigma_1 = 2\cos(\tilde{\theta})\sqrt{\frac{J_2^0}{3}} + \frac{I_1^0}{3}, \quad (\text{F.2})$$

$$\Sigma_2 = 2\cos\left(\tilde{\theta} - \frac{2\pi}{3}\right)\sqrt{\frac{J_2^0}{3}} + \frac{I_1^0}{3}, \quad (\text{F.3})$$

$$\Sigma_3 = 2\cos\left(\tilde{\theta} + \frac{2\pi}{3}\right)\sqrt{\frac{J_2^0}{3}} + \frac{I_1^0}{3}, \quad (\text{F.4})$$

and

$$J_2^0 = \frac{1}{2}\text{tr}(\Sigma^2), \quad (\text{F.5})$$

$$J_3^0 = \frac{1}{3}\text{tr}(\Sigma^3), \quad (\text{F.6})$$

$$I_1^0 = \text{tr}(\Sigma). \quad (\text{F.7})$$

The derivative of Lode angle $\tilde{\theta}$ to deviatoric stress invariant $J_2^0 J_3^0$

$$\frac{\partial \tilde{\theta}}{\partial J_2^0} = \frac{3\sqrt{3}}{4\sin 3\tilde{\theta}} \frac{J_3}{J_2^{0\ 5/2}}, \quad (\text{F.8})$$

$$\frac{\partial \tilde{\theta}}{\partial J_3^0} = -\frac{1}{6\sin 3\tilde{\theta}} \left(\frac{3}{J_2^0}\right)^{\frac{3}{2}}. \quad (\text{F.9})$$

So the derivative of principal stress to the second deviatoric stress invariant J_2^0

$$\frac{\partial \Sigma_1}{\partial J_2^0} = -\frac{3J_3^0 \sin \tilde{\theta}}{2J_2^{0\ 2} \sin 3\tilde{\theta}} + \frac{1}{3} \cos \tilde{\theta} \left(\frac{3}{J_2^0}\right)^{1/2}, \quad (\text{F.10})$$

$$\frac{\partial \Sigma_2}{\partial J_2^0} = -\frac{3J_3^0 \sin\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{2J_2^{0\ 2} \sin 3\tilde{\theta}} + \frac{1}{3} \cos\left(\tilde{\theta} - \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0}\right)^{1/2}, \quad (\text{F.11})$$

$$\frac{\partial \Sigma_3}{\partial J_2^0} = -\frac{3J_3^0 \sin\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{2J_2^{0\ 2} \sin 3\tilde{\theta}} + \frac{1}{3} \cos\left(\tilde{\theta} + \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0}\right)^{1/2}. \quad (\text{F.12})$$

and the derivative of principal stress to the third deviatoric stress invariant J_3^0

$$\frac{\partial \Sigma_1}{\partial J_3^0} = \frac{\sin \tilde{\theta}}{J_2^0 \sin 3\tilde{\theta}}, \quad (\text{F.13})$$

$$\frac{\partial \Sigma_2}{\partial J_3^0} = \frac{\sin\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{J_2^0 \sin 3\tilde{\theta}}, \quad (\text{F.14})$$

$$\frac{\partial \Sigma_3}{\partial J_3^0} = \frac{\sin\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{J_2^0 \sin 3\tilde{\theta}}. \quad (F.15)$$

The second derivative of principal stress to the deviatoric stress invariant J_2^0, J_3^0

$$\begin{aligned} \frac{\partial^2 \Sigma_1}{\partial J_2^0} &= 3J_3^0 J_2^0^{-3} \frac{\sin \tilde{\theta}}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos \tilde{\theta}}{\sin 3\tilde{\theta}} - \frac{3\sin \tilde{\theta} \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_2^0} \\ &\quad - \frac{1}{3} \sin \tilde{\theta} \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_2^0} - \frac{\sqrt{3}}{6} \cos \tilde{\theta} J_2^0^{-3/2}, \end{aligned} \quad (F.16)$$

$$\begin{aligned} \frac{\partial^2 \Sigma_1}{\partial J_2^0 \partial J_3^0} &= -\frac{3}{2} J_2^0^{-2} \frac{\sin \tilde{\theta}}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos \tilde{\theta}}{\sin 3\tilde{\theta}} - \frac{3\sin \tilde{\theta} \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_3^0} \\ &\quad - \frac{1}{3} \sin \tilde{\theta} \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_3^0}, \end{aligned} \quad (F.17)$$

$$\begin{aligned} \frac{\partial^2 \Sigma_2}{\partial J_2^0} &= 3J_3^0 J_2^0^{-3} \frac{\sin\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3\sin\left(\tilde{\theta} - \frac{2\pi}{3}\right) \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_2^0} \\ &\quad - \frac{1}{3} \sin\left(\tilde{\theta} - \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_2^0} - \frac{\sqrt{3}}{6} \cos\left(\tilde{\theta} - \frac{2\pi}{3}\right) J_2^0^{-3/2}, \end{aligned} \quad (F.18)$$

$$\begin{aligned} \frac{\partial^2 \Sigma_2}{\partial J_2^0 \partial J_3^0} &= -\frac{3}{2} J_2^0^{-2} \frac{\sin\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos\left(\tilde{\theta} - \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3\sin\left(\tilde{\theta} - \frac{2\pi}{3}\right) \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_3^0} \\ &\quad - \frac{1}{3} \sin\left(\tilde{\theta} - \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_3^0}, \end{aligned} \quad (F.19)$$

$$\begin{aligned} \frac{\partial^2 \Sigma_3}{\partial J_2^0} &= 3J_3^0 J_2^0^{-3} \frac{\sin\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3\sin\left(\tilde{\theta} + \frac{2\pi}{3}\right) \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_2^0} \\ &\quad - \frac{1}{3} \sin\left(\tilde{\theta} + \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_2^0} - \frac{\sqrt{3}}{6} \cos\left(\tilde{\theta} + \frac{2\pi}{3}\right) J_2^0^{-3/2}, \end{aligned} \quad (F.20)$$

$$\begin{aligned} \frac{\partial^2 \Sigma_3}{\partial J_2^0 \partial J_3^0} &= -\frac{3}{2} J_2^0^{-2} \frac{\sin\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3J_3^0}{2J_2^0{}^2} \left(\frac{\cos\left(\tilde{\theta} + \frac{2\pi}{3}\right)}{\sin 3\tilde{\theta}} - \frac{3\sin\left(\tilde{\theta} + \frac{2\pi}{3}\right) \cos 3\tilde{\theta}}{(\sin 3\tilde{\theta})^2} \right) \frac{\partial \tilde{\theta}}{\partial J_3^0} \\ &\quad - \frac{1}{3} \sin\left(\tilde{\theta} + \frac{2\pi}{3}\right) \left(\frac{3}{J_2^0} \right)^{\frac{1}{2}} \frac{\partial \tilde{\theta}}{\partial J_3^0}. \end{aligned} \quad (F.21)$$

The derivative of equivalent stress σ_{eq} to principal stress

$$\frac{\partial \sigma_{eq}}{\partial \Sigma_1} = \sigma_{eq}^{1-\tilde{n}} \frac{(|\Sigma_1| - k\Sigma_1)^{\tilde{n}}}{\Sigma_1}, \quad (F.22)$$

$$\frac{\partial \sigma_{eq}}{\partial \Sigma_2} = \sigma_{eq}^{1-\tilde{n}} \frac{(|\Sigma_2| - k\Sigma_2)^{\tilde{n}}}{\Sigma_2}, \quad (F.23)$$

$$\frac{\partial \sigma_{eq}}{\partial \Sigma_3} = \sigma_{eq}^{1-\tilde{n}} \frac{(|\Sigma_3| - k\Sigma_3)^{\tilde{n}}}{\Sigma_3}, \quad (F.24)$$

The second derivative of equivalent stress to principal stress

$$\frac{\partial^2 \sigma_{eq}}{\partial \Sigma_i \partial \Sigma_j} = (1 - \tilde{n}) \sigma_{eq}^{-\tilde{n}} \frac{\partial \sigma_{eq}}{\partial \Sigma_j} \frac{(|\Sigma_i| - k\Sigma_i)^{\tilde{n}}}{\Sigma_i} + \frac{(\tilde{n}-1) \sigma_{eq}^{1-\tilde{n}} (|\Sigma_i| - k\Sigma_i)^{\tilde{n}}}{\Sigma_i^2} \delta_{ij}. \quad (F.25)$$

The derivative of principal stress Σ_i to the transformed stress Σ_{jk}

$$\frac{\partial \Sigma_1}{\partial \Sigma_{ij}} = \frac{\delta_{ij}}{3} + \frac{\partial \Sigma_1}{\partial J_2^0} \frac{\partial J_2^0}{\partial \Sigma_{ij}} + \frac{\partial \Sigma_1}{\partial J_3^0} \frac{\partial J_3^0}{\partial \Sigma_{ij}}, \quad (\text{F.26})$$

$$\frac{\partial \Sigma_2}{\partial \Sigma_{ij}} = \frac{\delta_{ij}}{3} + \frac{\partial \Sigma_2}{\partial J_2^0} \frac{\partial J_2^0}{\partial \Sigma_{ij}} + \frac{\partial \Sigma_2}{\partial J_3^0} \frac{\partial J_3^0}{\partial \Sigma_{ij}}, \quad (\text{F.27})$$

$$\frac{\partial \Sigma_3}{\partial \Sigma_{ij}} = \frac{\delta_{ij}}{3} + \frac{\partial \Sigma_3}{\partial J_2^0} \frac{\partial J_2^0}{\partial \Sigma_{ij}} + \frac{\partial \Sigma_3}{\partial J_3^0} \frac{\partial J_3^0}{\partial \Sigma_{ij}}. \quad (\text{F.28})$$

The second derivative of principal stress Σ_i to the transformed stress Σ_{jk}

$$\begin{aligned} \frac{\partial^2 \Sigma_1}{\partial \Sigma_{ij} \partial \Sigma_{mn}} &= \frac{\partial^2 \Sigma_1}{\partial J_2^{0^2}} \frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} + \frac{\partial^2 \Sigma_1}{\partial J_2^0 \partial J_3^0} \left(\frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} \right) \\ &+ \frac{\partial^2 \Sigma_1}{\partial J_3^{0^2}} \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial \Sigma_1}{\partial J_2^0} \frac{\partial^2 J_2^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}} + \frac{\partial \Sigma_1}{\partial J_3^0} \frac{\partial^2 J_3^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}}, \end{aligned} \quad (\text{F.29})$$

$$\begin{aligned} \frac{\partial^2 \Sigma_2}{\partial \Sigma_{ij} \partial \Sigma_{mn}} &= \frac{\partial^2 \Sigma_2}{\partial J_2^{0^2}} \frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} + \frac{\partial^2 \Sigma_2}{\partial J_2^0 \partial J_3^0} \left(\frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} \right) \\ &+ \frac{\partial^2 \Sigma_2}{\partial J_3^{0^2}} \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial \Sigma_2}{\partial J_2^0} \frac{\partial^2 J_2^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}} + \frac{\partial \Sigma_2}{\partial J_3^0} \frac{\partial^2 J_3^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}}, \end{aligned} \quad (\text{F.30})$$

$$\begin{aligned} \frac{\partial^2 \Sigma_3}{\partial \Sigma_{ij} \partial \Sigma_{mn}} &= \frac{\partial^2 \Sigma_3}{\partial J_2^{0^2}} \frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} + \frac{\partial^2 \Sigma_3}{\partial J_2^0 \partial J_3^0} \left(\frac{\partial J_2^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_2^0}{\partial \Sigma_{mn}} \right) \\ &+ \frac{\partial^2 \Sigma_3}{\partial J_3^{0^2}} \frac{\partial J_3^0}{\partial \Sigma_{ij}} \frac{\partial J_3^0}{\partial \Sigma_{mn}} + \frac{\partial \Sigma_3}{\partial J_2^0} \frac{\partial^2 J_2^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}} + \frac{\partial \Sigma_3}{\partial J_3^0} \frac{\partial^2 J_3^0}{\partial \Sigma_{ij} \partial \Sigma_{mn}}. \end{aligned} \quad (\text{F.31})$$

The derivative of the transformed stress Σ_{kl} to the original stress σ_{mn}

$$\frac{\partial \Sigma_{kl}}{\partial \sigma_{mn}} = \tilde{\mathbf{L}}. \quad (\text{F.32})$$

So the derivative of the equivalent stress σ_{eq} to the original stress σ_{mn}

$$\frac{\partial \sigma_{eq}}{\partial \sigma_{mn}} = \frac{\partial \sigma_{eq}}{\partial \Sigma_i} \frac{\partial \Sigma_i}{\partial \Sigma_{jk}} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}}. \quad (\text{F.33})$$

The second derivative of the equivalent stress σ_{eq} to the stress σ_{mn}

$$\frac{\partial^2 \sigma_{eq}}{\partial \sigma_{mn} \partial \sigma_{pq}} = \frac{\partial^2 \sigma_{eq}}{\partial \Sigma_i \partial \Sigma_l} \frac{\partial \Sigma_l}{\partial \Sigma_{st}} \frac{\partial \Sigma_{st}}{\partial \sigma_{pq}} \frac{\partial \Sigma_i}{\partial \Sigma_{jk}} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}} + \frac{\partial \sigma_{eq}}{\partial \Sigma_i} \frac{\partial^2 \Sigma_i}{\partial \Sigma_{jk} \partial \Sigma_{st}} \frac{\partial \Sigma_{st}}{\partial \sigma_{pq}} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}}. \quad (\text{F.34})$$

The second derivative of the equivalent stress σ_{eq} to the stress σ_{mn} and equivalent plastic strain

$$\begin{aligned} \frac{\partial^2 \sigma_{eq}}{\partial \sigma_{mn} \partial \bar{\epsilon}_p} &= \frac{\partial^2 \sigma_{eq}}{\partial \Sigma_i \partial \Sigma_l} \frac{\partial \Sigma_l}{\partial \Sigma_{st}} \frac{\partial \Sigma_{st}}{\partial \bar{\epsilon}_p} \frac{\partial \Sigma_i}{\partial \Sigma_{jk}} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}} + \frac{\partial \sigma_{eq}}{\partial \Sigma_i} \frac{\partial^2 \Sigma_i}{\partial \Sigma_{jk} \partial \Sigma_{st}} \frac{\partial \Sigma_{st}}{\partial \bar{\epsilon}_p} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}} \\ &+ \frac{\partial \sigma_{eq}}{\partial \Sigma_i} \frac{\partial \Sigma_i}{\partial \Sigma_{jk}} \frac{\partial^2 \Sigma_{jk}}{\partial \sigma_{mn} \partial \bar{\epsilon}_p}. \end{aligned} \quad (\text{F.35})$$

The derivative of the equivalent stress σ_{eq} to the equivalent plastic strain

$$\frac{\partial \sigma_{eq}}{\partial \bar{\epsilon}_p} = \frac{\partial \sigma_{eq}}{\partial \Sigma_i} \frac{\partial \Sigma_i}{\partial \Sigma_{jk}} \frac{\partial \Sigma_{jk}}{\partial \sigma_{mn}}. \quad (\text{F.36})$$

