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# SECOND-ORDER NECESSARY CONDITIONS OF OPTIMALITY FOR ABNORMAL SOLUTIONS OF NONLINEAR PROBLEMS WITH EQUALITY AND INEQUALITY CONSTRAINTS

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Abstract: Second-order necessary conditions for an abnormal local minimizer of nonlinear optimization problem with equality and inequality constraints are presented and discussed. These are the best possible optimality conditions that can be obtained for this class of problems in that the associated set of Lagrange multipliers is the smallest possible.

Keywords: Mathematical Programming, Second-order Necessary Optimality Conditions, Abnormal Points

## 1. INTRODUCTION

We consider the following optimization problem

$$\text{Minimize } f(x)$$
$$F_1(x) = 0$$
$$F_2(x) \leq 0$$

where $f : X \to R^1$, $F_1 : X \to R^{k_1}$, and $F_2 : X \to R^{k_2}$ are given mappings, $X$ is a linear space, $R^k$ denotes the $k$-dimensional arithmetical space, and $k_1$ and $k_2$ are fixed. The non-positivity of a vector means that all its coordinates are non-positive. We shall assume that all functions $f$, $F_1$ and $F_2$ are smooth in the sense specified below.

In this article, we present and discuss second-order necessary conditions of optimality for an abnormal local minimizer of problem $(\mathcal{P})$, which improve the ones presented earlier in (Arutyunov, 1996; Arutyunov, 2000).

For the sake of illustration, let us consider the following particular instance of the problem $(\mathcal{P})$ featuring only equality type constraints:

$$f(x) \to \min, \quad F_1(x) = 0, \tag{1}$$

where the space $X$ is finite-dimensional, and $f$ and $F_1$ are twice continuously differentiable. Let $x_0$ be a solution of problem (1). Two cases may arise.

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Firstly, let us assume that $\mathrm{im}\,\frac{\partial F_1}{\partial x}(x_0) = R^{k_1}$ (here, im denotes the range of an operator), i.e., $x_0$ is a normal point. Then, the well-known first- and second-order necessary conditions of optimality hold, see (V.M. Alekseev, 1987). Denote by $\mathcal{L}_1$ the Lagrange function defined by

$$\mathcal{L}_1(x,\lambda) = \lambda^0 f(x) + \langle \lambda^1, F_1(x)\rangle.$$

These conditions guarantee the existence of a nonzero Lagrangian multiplier $\lambda = (\lambda^0, \lambda^1)$, with $\lambda^0 \geq 0$, such that

$$\frac{\partial \mathcal{L}_1}{\partial x}(x_0,\lambda) = 0,$$

and its second order derivative $\frac{\partial^2 \mathcal{L}_1}{\partial x^2}(x_0,\lambda)$ is non-negative definite on the linear subspace $\ker \frac{\partial F_1}{\partial x}(x_0)$. Here, and in what follows, $\langle\cdot,\cdot\rangle$ denotes the scalar product. Note that, in this case, $\lambda^0 > 0$, and $\ker \frac{\partial F_1}{\partial x}(x_0)$ is equal to the tangent subspace to the set $\{x : F_1(x) = 0\}$ at the point $x_0$.

Now, let us assume $x_0$ to be abnormal, i.e.,

$$\mathrm{im}\,\frac{\partial F_1}{\partial x}(x_0) \neq R^{k_1}.$$

The following simple example illustrates that the second-order necessary conditions stated above do not hold in general. Indeed, let us consider the following minimization problem

$$(E1)\begin{cases} & \langle a, x\rangle \to \min \\ \text{subject to} & x_1x_2 = 0, \\ & x_1^2 - x_2^2 = 0, \end{cases}$$

where $x = (x_1, x_2) \in R^2$, and $a \in R^2$ is any given nonzero vector. Here, the point $x = 0$ is the unique solution and it is abnormal. However, there is no Lagrange multiplier $\lambda$ such that $\frac{\partial^2 \mathcal{L}_1}{\partial x^2}(0,\lambda) \geq 0$.

In order to address this issue, meaningful second order necessary conditions for problem $(\mathcal{P})$ were obtained without a priori normality assumptions imposed at the point $x_0$ in (Arutyunov, 2000). Next, we formulate these results from (Arutyunov, 2000). For this, consider the Lagrange function of problem $(\mathcal{P})$ $\mathcal{L} : X \times R^1 \times R^{k_1} \times R^{k_2} \to R^1$ defined by

$$\mathcal{L}(x,\lambda) = \lambda^0 f(x) + \langle \lambda^1, F_1(x)\rangle + \langle \lambda^2, F_2(x)\rangle,$$

$$\lambda = (\lambda^0, \lambda^1, \lambda^2),$$

$$\lambda^0 \in R^1,\ \lambda^1 \in R^{k_1},\ \lambda^2 \in R^{k_2}.$$

Let $x_0$ be a local minimizer for problem $(\mathcal{P})$, and the mappings $F_i$ and $f$ be twice continuously differentiable. For the sake of simplicity assume that $F_2(x_0) = 0$, and denote by $\Lambda(x_0)$ the set of all Lagrange multipliers $\lambda = (\lambda^0, \lambda^1, \lambda^2)$ satisfying the Lagrange multipliers rule at the point $x_0$:

$$\frac{\partial \mathcal{L}}{\partial x}(x_0,\lambda) = 0,$$

$$\lambda^0 \geq 0, \quad \lambda^2 \geq 0, \quad |\lambda| = 1.$$

Denote by $\Lambda_a(x_0)$ the set of all Lagrange multipliers $\lambda \in \Lambda(x_0)$ for which there exists a linear subspace $\Pi = \Pi(\lambda) \subseteq X$ satisfying

$$\Pi \subseteq \ker \frac{\partial F_1}{\partial x}(x_0) \cap \ker \frac{\partial F_2}{\partial x}(x_0),$$

$$\mathrm{codim}\,\Pi \leq k_1 + k_2,$$

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0,\lambda)[x,x] \geq 0, \forall x \in \Pi,$$

where codim means codimension of a linear subspace.

In (Arutyunov, 2000), it was proved that for any feasible descent direction for $(\mathcal{P})$, i.e., any vector $h \in X$ satisfying:

$$\frac{\partial F_1}{\partial x}(x_0)h = 0,$$

$$\frac{\partial F_2}{\partial x}(x_0)h \leq 0, \text{ and}$$

$$\langle \frac{\partial f}{\partial x}(x_0), h\rangle \leq 0,$$

there exists a Lagrange multiplier $\lambda \in \Lambda_a(x_0)$ (depending on $h$) such that

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0,\lambda)[h,h] \geq 0.$$

These necessary conditions constitute a natural generalization of the classical ones, (V.M. Alekseev, 1987), in the abnormal case. Note that the non-emptiness of the set $\Lambda_a(x_0)$ is in itself a significant necessary optimality condition.

With the help of the technique in (Mordukhovich, 2006), the above mentioned result in (Arutyunov, 2000) was afterwards generalized in (A.V. Arutyunov, 2006*b*) to a problem featuring more general set-inclusion constraints of the type $F(x) \in C$, where the set $C$ is assumed to be merely closed. On the other hand, the necessary optimality conditions for problem (1) with only equality type constraints were, under the additional assumption of abnormality of the point $x_0$, strengthened in (A.V. Arutyunov, 2006*a*). More specifically, in this reference, the following result was obtained:

If the local minimizer $x_0$ of problem (1) is abnormal, then, the set $\Lambda_r(x_0)$ in the necessary optimality conditions presented above can be replaced by the smaller set that contains all $\lambda \in \Lambda(x_0)$ such that $|\lambda| = 1$ and for which there exists a linear subspace $\Pi = \Pi(\lambda) \subseteq X$ satisfying:

$$\Pi \subseteq \ker \frac{\partial F_1}{\partial x}(x_0),$$

$$\mathrm{codim}\,\Pi \leq k_1 - 1,$$

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0, \forall x \in \Pi.$$

The main goal of this article is to present an extension of the above mentioned result to abnormal minimizers of the mathematical programming problem $(\mathcal{P})$ which is more general than the one in (1) due to the consideration of inequality type constraints. The approach to prove this result is based on a perturbation method developed in (Arutyunov, 2000) and on methods of real algebraic geometry, see (J. Bochnak, 1988).

Some additional references on second-order necessary optimality conditions, where (R. Hettich, 1977) is a pioneer publication, can be found in (Arutyunov, 2000). We also single out the second-order necessary optimality conditions obtained in (Milyutin, 1981). Another approach to the first and second-order necessary optimality conditions for problems with inequality type of constraints for abnormal points is presented in (Avakov, 1989; Izmailov, 1999; A.F. Izmailov, 2001), as well as in the more recent articles (E.R. Avakov, 2006; E.R. Avakov, 2007*b*; E.R. Avakov, 2007*a*).

## 2. THE MAIN RESULT

In order to formulate the main result of this article, let us introduce some notation.

First, let us equip the linear space $X$ with the so called finite topology. Denote by $\mathcal{M}$ the set of all linear finite-dimensional subspaces $M \subseteq X$. A set is open in the finite topology if it has open intersection with every subspace $M \in \mathcal{M}$ (the openness of an intersection is meant in the sense of the unique separated vector topology of finite-dimensional space $M$). A local minimizer with respect to the finite topology is the weakest type of minimizers under consideration in optimization theory. For more details, see (Arutyunov, 2000). In what follows, by the term *local minimizer* we mean the local minimizer with respect to the finite topology.

Let vector $x_0 \in X$ be a local minimizer in problem $(\mathcal{P})$. We assume mappings $f$, $F_1$, and $F_2$ to be twice continuously differentiable in a neighborhood of $x_0$ with respect to the finite topology. This means that, for any subspace $M \in \mathcal{M}$ containing the point $x_0$, the restrictions of $f$, $F_1$, and $F_2$ to $M$ are twice continuously differentiable in some ($M$-dependent) neighborhood of vector $x_0$.

Therefore, there exist a linear functional $a : X \to R^1$, a bilinear form $q : X \times X \to R^1$, linear operators $A_i : X \to Y$, bilinear mappings $Q_i : X \times X \to Y$, with $i = 1, 2$, and, for $j = 0, 1, 2$, mappings $\alpha_j : X \to R^1$, such that, $\forall x \in X$,

$$f(x) = f(x_0) + \langle a, x - x_0\rangle + \frac{1}{2}q[x - x_0, x - x_0] + \alpha_0(x - x_0),$$

$$F_i(x) = F_i(x_0) + A_i(x - x_0) + \frac{1}{2}Q_i[x - x_0, x - x_0] + \alpha_i(x - x_0),$$

and, for an arbitrary $M \in \mathcal{M}$, such that $x \in M$, and

$$\frac{\alpha_j(x - x_0)}{\|x - x_0\|_M^2} \to 0, \quad \text{as } x \to x_0,$$

where $\|\cdot\|_M$ is a finite-dimensional norm in $M$.

In what follows, we denote $A_i$ by $F_i'(x_0) = \frac{\partial F}{\partial x}(x_0)$, $Q_i$ by $F_i''(x_0) = \frac{\partial^2 F_i}{\partial x^2}(x_0)$, respectively, the first- and second-order derivatives of $F_i$ and similarly for the derivatives of the function $f$ and the Lagrange function.

Consider the Lagrange function $\mathcal{L} : X \times R^1 \times R^{k_1} \times R^{k_2} \to R^1$ defined by

$$\mathcal{L}(x, \lambda) = \lambda^0 f(x) + \langle\lambda^1, F_1(x)\rangle + \langle\lambda^2, F_2(x)\rangle,$$
$$\lambda = (\lambda^0, \lambda^1, \lambda^2),$$
$$\lambda^0 \in R^1, \ \lambda^1 \in R^{k_1}, \ \lambda^2 \in R^{k_2}.$$

Denote by $\Lambda(x_0)$ the set of all $\lambda = (\lambda^0, \lambda^1, \lambda^2)$ that satisfy the Lagrange multipliers rule at the point $x_0$:

$$\frac{\partial \mathcal{L}}{\partial x}(x_0, \lambda) = 0, \tag{2}$$

$$\langle\lambda^2, F_2(x_0)\rangle = 0, \tag{3}$$

$$\lambda^0 \geq 0, \quad \lambda^2 \geq 0, \quad |\lambda| = 1. \tag{4}$$

By virtue of this rule (see (Arutyunov, 2000)), the set $\Lambda(x_0)$ is not empty. Elements $\lambda$ of this set are called Lagrange multipliers.

Denote by $I = I(x_0)$ the set of all indices $i \in \{1, .., k_2\}$ such that $F_2^i(x_0) = 0$. ($F_s^i(x)$ are the coordinates of the vector $F_s(x)$, $s = 1, 2$). For an integer nonnegative number $r$, we denote by $\Lambda_r(x_0)$ the set of vectors $\lambda \in \Lambda(x_0)$ such that there exists a linear subspace

$$\Pi = \Pi(\lambda) \subseteq \ker \frac{\partial F_1}{\partial x}(x_0) \bigcap \Big(\bigcap_{i \in I} \ker \frac{\partial F_2^i}{\partial x}(x_0)\Big)$$

satisfying:

$$\operatorname{codim} \Pi \leq r,$$
$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0 \ \ \forall x \in \Pi.$$

Consider the cone of critical directions at point $x_0$:

$$\mathcal{K}(x_0) = \Big\{x \in X : \Big\langle\frac{\partial f}{\partial x}(x_0), x\Big\rangle \leq 0, \ \frac{\partial F_1}{\partial x}(x_0)x = 0, \ \Big\langle\frac{\partial F_2^i}{\partial x}(x_0), x\Big\rangle \leq 0, \ i \in I\Big\}.$$

Put $k = k_1 + |I(x_0)|$, where $|I|$ denotes the number of elements in the set $I$.

We shall say that a point $x_0$ is *abnormal*, if $k > 0$ and the vectors $\frac{\partial F_1^j}{\partial x}(x_0)$, $j = 1,..,k_1$, $\frac{\partial F_2^i}{\partial x}(x_0)$, $i \in I$ are linearly dependent.

**Theorem 2.1** *Let the point $x_0$ be a local minimizer for problem $(\mathcal{P})$. Assume that $x_0$ is an abnormal point.*

*Then,*

$$\Lambda_{k-1}(x_0) \neq \emptyset$$

*and the following inequality holds for any vector $h \in \mathcal{K}(x_0)$*

$$\max_{\lambda \in \Lambda_{k-1}(x_0)} \frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[h, h] \geq 0. \qquad (5)$$

## 3. DISCUSSION

This theorem only deals with abnormal minimizers. However if a minimizer is normal then second-order necessary conditions are well known (see for example (Arutyunov, 2000; V.M. Alekseev, 1987), and also our introduction) that we refer to by *classical second-order necessary conditions.* Classical second-order necessary conditions do not hold for abnormal minimizers as it was clearly illustrated with the example $(E1)$ in the introduction.

So, the following question naturally arises:

*When do classical second-order necessary conditions still follow from our theorem?*

*Or, in an equivalent way, when is it possible to use an universal Lagrange multiplier in (5), thus omitting the maximum operation?*

Some answers follow below.

The simplest application of Theorem 2.1 concerns the case $k = 1$. Indeed, the theorem states that if $x_0$ is an abnormal minimizer of problem $\mathcal{P}$ and $k = 1$, then there exists a Lagrange multiplier $\lambda$ such that $\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0$, $\forall x \in X$. Hence, in spite of the abnormality, the classical second-order necessary optimality conditions hold.

A less trivial application concerns the case of abnormal problems when $k = 2$ and Mangasarian-Fromovitz constrained qualification $(MFCQ)$ holds at an abnormal minimizer $x_0$.

Then, there exists a Lagrange multiplier $\lambda$ such that

$$\frac{\partial^2 \mathcal{L}}{\partial x^2}(x_0, \lambda)[x, x] \geq 0, \quad \forall x \in \mathcal{K}(x_0).$$

Let us prove it. Indeed, since the case $k = 1$ was already considered, we can assume that all the constraints of the problem are active. In view of $(MFCQ)$ and of the abnormality, it follows that $k_1 = 0$, $k_2 = 2$ (i.e. only inequality type constraints are present), and vectors $\frac{\partial F_2^i}{\partial x}(x_0)$, $i = 1, 2$, are nonzero and co-directional. Consider two cases: $\frac{\partial f}{\partial x}(x_0) = 0$ and $\frac{\partial f}{\partial x}(x_0) \neq 0$. If $\frac{\partial f}{\partial x}(x_0) = 0$, then, by Lagrange principle, the set $\Lambda(x_0)$ is singleton with $\lambda_0 = 1$, $\lambda_2 = 0$ and our assertion is a trivial corollary of the condition (5). Assume that $\frac{\partial f}{\partial x}(x_0) \neq 0$.

Then, by virtue of the Lagrange principle, we have that $\lambda_0 \neq 0$ and $\frac{\partial f}{\partial x}(x_0) = -\alpha \frac{\partial F_2^1}{\partial x}(x_0)$, where $\alpha$ is some positive number. Therefore, $\mathcal{K}(x_0) = \ker \frac{\partial F_2}{\partial x}(x_0)$. Now, since the codimension of the kernel is exactly 1, our assertion follows directly from Theorem 2.1. Thus, once again, in spite of the abnormality, the classical second-order necessary optimality conditions hold.

In the case $k \geq 3$, it is not possible to assert whether classical second-order necessary conditions of optimality hold even when the $(MFCQ)$ is assumed (see example $(E4)$ below).

The fact that $\Lambda_0 = \emptyset$ in the example $(E1)$ presented in the introduction shows that Theorem 2.1 can not be improved in the following sense. If $k \geq 2$, then, in general, the set $\Lambda_{k-1}$ can not be replaced by the smaller set $\Lambda_{k-2}$. Note that, in the example mentioned above, there are only equality type constraints. In spite of the presence of inequality type constraints in the example $(E2)$ below, $\Lambda_{k-2} = \emptyset$.

$$(E2) \left\{ \begin{array}{rl} & x_1 \to \min \\ \text{subject to} & x_1 x_2 = 0, \\ & x_1^2 - x_2^2 \leq 0. \end{array} \right.$$

Here, the feasible set is the line $\{x : x_1 = 0\}$ and hence $x = 0$ is a minimizer. Here, we have $k = 2$ and $\Lambda_0 = \emptyset$.

In this example, equality and inequality type constraints are present. Now, let us provide an example featuring only inequality type constraints.

$$(E3) \left\{ \begin{array}{rl} & x_1 x_2 - x_2^2 \to \min \\ \text{subject to} & -x_1 x_2 \leq 0, \\ & x_2^2 - x_1^2 \leq 0. \end{array} \right.$$

It is a straightforward task to verify that $x = 0$ is a minimizer. However, once again $k = 2$ and $\Lambda_0 = \emptyset$. Note that, in this example, all the functions are quadratic.

A simple modification of example $(E3)$ shows that even $(MFCQ)$ does not allow us to replace $\Lambda_{k-1}$ by $\Lambda_{k-2}$. Indeed, consider problem

$$(E4)\left\{\begin{array}{rl} & -x_3 \to \min \\ \text{subject to} & x_3 + x_1x_2 - x_2^2 \leq 0, \\ & x_3 - x_1x_2 \leq 0, \\ & x_3 + x_2^2 - x_1^2 \leq 0, \end{array}\right.$$

where $k = 3$, $x = (x_1, x_2, x_3) \in R^3$. Since $(x_1, x_2) = 0$ is a solution to problem $(E3)$, then, for any admissible point $x$ of problem $(E4)$ we have $x_3 \leq 0$.

Therefore, $x = 0$ is also a solution to problem $(E4)$. Obviously, $(MFCQ)$ holds for this problem, and, also, $\Lambda_1 = \emptyset$.

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