An Algorithm for SDV Representation of 2D Behaviors

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***Abstract*— This paper deals with the characterization of 2D behaviors that are representable by means of special first order models, known as state/driving-variable (SDV) models. In previous work, [1], [2] we have shown how to identify SDV-representable behaviors using one of its full row rank representations. Here, we give a further refinement by showing that a 2D behavior is SDV-representable if and only if each of its kernel representations can be decomposed as a product of three 2D L-polynomial matrices: a zero right prime matrix, a cw-unital square matrix and a factor left prime matrix. Using that decomposition, we present a procedure to obtain SDV representations of a 2D behavior starting from any of its kernel representations.**

I. Introduction

In this paper we consider the class of 2D behaviors $\mathcal{B}$ that can be described as the solution set of a system of linear partial difference equations of the form

$$R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})w = 0$$

where

$$R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) = \sum_{(i,j)\in S} R_{ij}\sigma_1^i\sigma_2^j\,,$$

with $S \subset \mathbb{Z}^2$ finite, is a 2D Laurent-polynomial (L-polynomial) shift operator, σ_1 and σ_2 are the usual 2D shifts, (i.e. $\sigma_1 x(i,j) = x(i+1,j)$, $\sigma_2 x(i,j) = x(i,j+1)$, $\forall (i,j) \in \mathbb{Z}^2$ and $\forall x\colon \mathbb{Z}^2 \to \mathbb{R}^n$), and the system variable w is a vector valued signal whose components are not divided into inputs and outputs. For short, since $\mathcal{B} = \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$, we shall refer to such behaviors as kernel behaviors.

Note that $R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1})$ may be a higher order operator, since no limits are imposed on the degrees of monomials $\sigma_1^{k_1}\sigma_2^{k_2}$. The question that we investigate is the existence and construction of an alternative description which is first order, in the sense that at each point in the grid (i,j) is updated using the state and the driving-variable values at the two nearest neighbors $(i-1,j)$ and $(i,j-1)$. More concretely we are interested in representations of the form

$$\left\{\begin{array}{rcl} x &=& A(\sigma_1^{-1}, \sigma_2^{-1})x + B(\sigma_1^{-1}, \sigma_2^{-1})v \\ w &=& Cx + Dv\,, \end{array}\right. \tag{1}$$

where x and v are auxiliary variables, $A(s_1^{-1}, s_2^{-1}) = A_1 s_1^{-1} + A_2 s_2^{-1}$, $B(s_1^{-1}, s_2^{-1}) = B_1 s_1^{-1} + B_2 s_2^{-1}$ and A_1, A_2, B_1, B_2, C and D are real matrices of suitable dimensions. From the formal point of view, SDV models are very similar to the well-known Fornasini-Marchesini models, [3], but with the essential difference that here the role of the input is taken by the driving-variable, while the system variable w (which contains both inputs and outputs) plays the role of an output.

In previous work, [1], we have shown that SDV-representability is equivalent to the existence of a kernel representation that can be factored as a product of a left-prime 2D L-polynomial matrix by a square 2D L-polynomial matrix with 2D-proper inverse. This factorization condition emphasizes the existence a certain SDV-representable autonomous part of the behavior. Indeed, the kernel of the above mentioned square matrix is an SDV-representable autonomous part. Since the same behavior may have distinct autonomous parts with different representability properties, see [2], it seems suitable to present a SDV-representability characterization which does not rest on the properties of a certain autonomous part. Results in this direction have been obtained in [2]. Indeed, we have shown that a 2D behavior $\mathcal{B}$ is SDV-representable if and only if it has a full row rank kernel representation and, additionally, the gcd's of the maximal order minors of any full row rank kernel representation of $\mathcal{B}$ are unimodularly equivalent to a 2D L-polynomial with 2D-proper inverse. Here we reformulate this characterization in terms of the concept of cw-unital polynomials and obtain a further refinement. Furthermore, this is done using a different line of reasoning.

It is a well-known fact that every 2D L-polynomial matrix can be factored as a product of three matrices: a factor right prime matrix by a square non-singular matrix by a factor left prime matrix. We will show that SDV-representable behaviors are exactly the ones for which this decomposition of its kernel representations yields factorizations where the factor right matrix is indeed a zero right prime and the square matrix has cw-unital determinant. This result provides a way of deciding whether or not a kernel behavior is SDV-representable using any of its kernel representations.

Our approach to this representability problem allows to present a procedure for the construction of SDV-representations for a 2D behavior that can be performed starting from any of its kernel representations.

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II. Preliminaries

In the following we consider discrete 2D systems $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B})$ that admit a kernel representation, i.e.,

$$\mathcal{B} = \ker R(\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}) \,.$$

The matrix $R(s_1, s_2, s_1^{-1}, s_2^{-1})$ is simply said to be a *representation* of Σ (and $\mathcal{B}$). We denote the set of all 2D systems $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B})$ with kernel representation by $\mathcal{L}^q$. Note that, a 2D system $\Sigma \in \mathcal{L}^q$ has infinitely many representations. In fact, R and $\tilde{R}$ are representations of Σ if and only if there exist L-polynomial matrices L and $\tilde{L}$, of suitable dimensions, such that $R = L\tilde{R}$ and $\tilde{R} = \tilde{L}R$, [4]. When two L-polynomial matrices represent the same behavior they are said to be *equivalent*. If L is unimodular, which means that L is invertible as a L-polynomial matrix, R and $\tilde{R}$ are said to be *unimodularly equivalent* (note that in this case $\tilde{L}$ is the inverse of L). For instance, two full row rank representations of the same behavior are unimodularly equivalent. Notice that not every behavior has a full row rank representation. $\mathcal{B}$ is said to be a *regular behavior* if it admits a full row rank representation.

Definition 1: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$. The system of equations

$$\left\{ \begin{array}{rcl} x & = & A(\sigma_1^{-1}, \sigma_2^{-1})x + B(\sigma_1^{-1}, \sigma_2^{-1})v \\ w & = & Cx + Dv \,, \end{array} \right. \tag{2}$$

where σ_1, σ_2 are the usual 2D shifts,

$$\begin{array}{l} A(s_1^{-1}, s_2^{-1}) = A_1 s_1^{-1} + A_2 s_2^{-1} \in \mathbb{R}^{n\times n}[s_1^{-1}, s_2^{-1}], \\ B(s_1^{-1}, s_2^{-1}) = B_1 s_1^{-1} + B_2 s_2^{-1} \in \mathbb{R}^{n\times m}[s_1^{-1}, s_2^{-1}], \\ C \in \mathbb{R}^{q\times n} \text{ and } D \in \mathbb{R}^{q\times m}, \end{array}$$

is called a *state/driving-variable representation* (SDV) of Σ (of $\mathcal{B}$) if

$$\mathcal{B} = \{w \colon \mathbb{Z}^2 \to \mathbb{R}^q |\ \exists\, x, v \text{ such that (2) holds}\}.$$

In this case $\mathcal{B}$ is said to be SDV-representable.

As mentioned in the Introduction, the question of the existence of such a representation has already been studied in [1], [2]. Let us recall some basic facts, already shown in that previous work, which are here our starting point.

It is a well-known fact that every 2D behavior is decomposable as a sum of its controllable part, $\mathcal{B}^c$, with an autonomous part, [5]. Taking this into account together with the fact that controllable behaviors are always SDV-representable, [4], it is simple to conclude the following, as shown in [1].

Proposition 1: Let $\mathcal{B}$ be a 2D kernel behavior. If $\mathcal{B}$ has an SDV-representable autonomous part, then $\mathcal{B}$ is SDV-representable.

Notice that, the reciprocal implication of the Proposition 1 holds, as shown in [1]. However, in this approach we need not use this implication and, naturally, it comes as a consequence of the results presented in the sequel (see Remark 2).

The SDV-representability of an autonomous behavior is characterized in [1], [2] in terms of the notion of 2D-properness. We say that a 2D rational function f is 2D-proper if $f = p/q$, where $p, q \in \mathbb{R}[s_1^{-1}, s_2^{-1}]$ and the zero-degree coefficient of q is nonzero. A 2D rational matrix will be called *2D-proper* if all its entries are 2D-proper rational functions. Recall that every 2D L-polynomial matrix $R(s_1, s_2, s_1^{-1}, s_2^{-1})$ can be written as

$$R(s_1, s_2, s_1^{-1}, s_2^{-1}) = \sum_{(i,j)\in S} R_{ij} s_1^i s_2^j \,,$$

where S is a finite subset of $\mathbb{Z}^2$ and R_{ij} is a nonzero constant matrix, for $(i, j) \in S$. The set S is the support of R, usually denoted by $\operatorname{supp}(R)$. Notice that, with this notation the definition of 2D-properness given above may be reformulated by saying that p and q have their supports in the third quarter plane and moreover the zero-degree coefficient of q is nonzero.

In fact, in [1] we have shown the following result.

Proposition 2: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ be an autonomous system. $\mathcal{B}$ is SDV-representable if and only if there exists a kernel representation of $\mathcal{B}$ with 2D-proper inverse.

Since every two square representations of an autonomous behavior are unimodularly equivalent we may state the next corollary.

Corollary 1: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$ be an autonomous system. $\mathcal{B}$ is SDV-representable if and only if $\mathcal{B}$ is regular and every square representation of $\mathcal{B}$ is unimodularly equivalent to a square L-polynomial matrix with 2D-proper inverse.

It is possible to identify a 2D square L-polynomial matrix unimodularly equivalent to a square L-polynomial matrix with 2D-proper inverse by its determinant. Indeed, the following has been proven in [2].

Lemma 1: A 2D L-polynomial square matrix M is unimodularly equivalent to a square L-polynomial matrix with 2D-proper inverse if and only if $\det M$ is unimodularly equivalent to a 2D L-polynomial with 2D-proper inverse.

As an immediate consequence of Corollary 1 and Lemma 1 it is possible to state the next corollary.

Corollary 2: An autonomous system $\mathcal{B}$ is SDV-representable if an only if $\mathcal{B}$ is regular and every square representation is such that its determinant is unimodularly equivalent to a 2D L-polynomial with 2D-proper inverse.

III. Representable autonomous behaviors

Let us consider the componentwise order (cw-order) in $\mathbb{Z}^2$, [6], given by

$$\begin{array}{c}(m_1, m_2) \leq_{\text{cw}} (n_1, n_2) \\ \text{if and only if} \\ m_1 \leq n_1 \ \wedge m_2 \leq n_2 \quad .\end{array} \tag{3}$$

Notice that $\leq_{\text{cw}}$ is a partial order of $\mathbb{Z}^2$. So, as expected, given a L-polynomial matrix R it may not exist $(d_1, d_2) \in \text{supp}\,(R)$ such that $(d_1, d_2) \leq_{\text{cw}} (i, j), \forall (i, j) \in \text{supp}\,(R)$. This observation gives rise to the following definitions:

Definition 2:

(i) A L-polynomial $p \in \mathbb{R}[s_1, s_2, s_1^{-1}, s_2^{-1}]$ is cw-unital if p has the form

$$p = p_{d_1 d_2} s_1^{d_1} s_2^{d_2} + \sum_{(i,j) <_{\text{cw}} (d_1, d_2)} p_{ij} s_1^i s_2^j \ ,$$

where $p_{d_1 d_2}$ is nonzero, [6].

(ii) A square L-polynomial matrix $R \in \mathbb{R}^{g \times g}[s_1, s_2, s_1^{-1}, s_2^{-1}]$ is cw-unital if R has the form

$$R = R_{d_1 d_2} s_1^{d_1} s_2^{d_2} + \sum_{(i,j) <_{\text{cw}} (d_1, d_2)} R_{ij} s_1^i s_2^j \ ,$$

where $R_{d_1 d_2}$ is invertible.

Remark 1: The previous definitions may be rewritten in the following manner. Let S be a finite subset of $\mathbb{Z}^2$ and denote by $sup_{cw}(S)$ the supremum of S with respect to the componentwise order defined by (3). It is clear that,

(i) A L-polynomial p is cw-unital if and only if $sup_{cw}(\text{supp}\,(p)) \in \text{supp}\,(p)$;

(ii) A L-polynomial square matrix R is cw-unital if and only if $sup_{cw}(\text{supp}\,(R)) = (d_1, d_2) \in \text{supp}\,(R)$ and the correspondent coefficient $R_{d_1 d_2}$ is invertible.

■

In fact, every cw-unital L-polynomial is unimodularly equivalent to a L-polynomial with proper inverse and vice-versa.

Lemma 2: A L-polynomial p is cw-unital if and only if p is unimodularly equivalent to a polynomial with proper inverse.

Proof: p is cw-unital if and only if

$$p = p_{d_1 d_2} s_1^{d_1} s_2^{d_2} + \sum_{(i,j) <_{\text{cw}} (d_1, d_2)} p_{ij} s_1^i s_2^j \ ,$$

where $p_{d_1 d_2}$ is nonzero, *i.e.*, if and only if

$$s_1^{-d_1} s_2^{-d_2} p = p_{d_1 d_2} + \sum_{(i,j) <_{\text{cw}} (0,0)} p_{ij} s_1^i s_2^j \ ,$$

where $p_{d_1 d_2}$ is nonzero, which means that p is unimodularly equivalent to a polynomial with proper inverse. ■

Notice that not every matrix that is unimodularly equivalent to a matrix with proper inverse is cw-unital. However, it is easy to show the following lemma (for the less obvious equivalence, (ii) $\Leftrightarrow$ (iii), see [1]).

Lemma 3: Let R be a square 2D L-polynomial matrix. The following statements are equivalent:

(i) $\det R$ is cw-unital;

(ii) R is unimodularly equivalent to a 2D L-polynomial matrix with 2D proper inverse;

(iii) R is unimodularly equivalent to a 2D L-polynomial matrix with support in the third quarter plane and invertible independent term.

(iv) R is unimodularly equivalent to a cw- unital 2D L-polynomial matrix

Considering Lemma 2, Corollary 2 may be restated in the following nicer way.

Proposition 3: An autonomous system $\mathcal{B}$ is SDV-representable if an only if $\mathcal{B}$ is regular and every square representation is such that its determinant is cw-unital.

IV. General representability

In this section we consider systems $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$, that are not necessarily autonomous, and show how to obtain the SDV-representability characterization correspondent to the one given in Proposition 3. This is made based on a factorization that emphasizes the controllable part of the behavior.

The following lemma is basically a reformulation of Lemma A.1 in [7] and this was pointed out to us by E. Zerz.

Lemma 4: Let $R_1 \in \mathbb{R}^{g_1 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}]$ and $R_2 \in \mathbb{R}^{g_2 \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}]$. Then $\ker R_1 + \ker R_2 = \ker L$, where L is a least common left factor of R_1 and R_2.

Proof: Let $L \in \mathbb{R}^{g \times q}[s_1, s_2, s_1^{-1}, s_2^{-1}]$ be a matrix such that

$$\ker R_1 + \ker R_2 = \ker L \, .$$

Note that such a L-polynomial matrix exists, see for instance [8]. Thus, we shall show that L is a least common left factor of R_1 and R_2. In fact, $w \in \ker L$ iff there exist $w_1 \in \ker R_1$ and $w_2 \in \ker R_2$ such that $w = w_1 + w_2$. That is, $w \in \ker L$ iff there exist $w_1 \in \ker R_1$ and $w_2 \in \ker R_2$ such that

$$\begin{bmatrix} I & I \\ R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} w \ . \tag{4}$$

So, (4) is a representation of $\ker L$ with auxiliary variables w_1 and w_2. Then, there exist L-polynomial matrices F_1 and F_2 such that the L-polynomial matrix

$$\begin{bmatrix} L & -F_1 & -F_2 \end{bmatrix}$$

is a minimal left annihilator of

$$\begin{bmatrix} I & I \\ R_1 & 0 \\ 0 & R_2 \end{bmatrix} .$$

That is equivalent to say that $L = F_1 R_1$, $L = F_2 R_2$ and if

$$\begin{bmatrix} \bar{L} & -\bar{F}_1 & -\bar{F}_2 \end{bmatrix} \begin{bmatrix} I & I \\ R_1 & 0 \\ 0 & R_2 \end{bmatrix} = 0$$

then

$$\begin{bmatrix} \bar{L} & -\bar{F}_1 & -\bar{F}_2 \end{bmatrix} = T \begin{bmatrix} L & -F_1 & -F_2 \end{bmatrix}$$

for some L-polynomial matrix T. Therefore, L is a least common left factor of R_1 and R_2. ■

It is a well-known fact that a full row rank L-polynomial matrix may be factored as a product of a square matrix by factor left prime one. In the following lemma we establish a relation between the determinant of such a square matrix and a representation of a certain autonomous part of the behavior.

Lemma 5: If $\mathcal{B} = \ker(\Delta H_c)$, where Δ is a nonsingular square 2D L-polynomial and H_c is a factor left prime one, then there exists an autonomous part of $\mathcal{B}$, $\widehat{\mathcal{B}}^a$, such that $\widehat{\mathcal{B}}^a = \ker R^a$, where $\det R^a = \det(\Delta).d(s_1)$, with $d(s_1)$ a polynomial in s_1.

Proof: First of all notice that H_c is a representation of the controllable part of $\mathcal{B}$, see for instance [5]. Let $H = \Delta H_c$. Since H_c is factor left prime, there exist C_1 and C_2 such that

$$\det \begin{bmatrix} H_c \\ C_i \end{bmatrix} = d_i(s_i),\ i = 1, 2\,,$$

where $d_1(s_1)$ and $d_2(s_2)$ are nonzero polynomials in s_1 and s_2, respectively, [9].

Notice that

$$\ker H \cap \ker C_1 = \ker \left(\begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_c \\ C_1 \end{bmatrix} \right).$$

Let $R^a := \begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_c \\ C_1 \end{bmatrix}$. We next show that

$$\widehat{\mathcal{B}}^a = \ker R^a$$

is an autonomous part of $\mathcal{B}$. By Lemma 4

$$\ker H_c + \ker R^a = \ker L\,, \tag{5}$$

where L is a $l.c.l.f.(H_c, R^a)$. In order to determine a $l.c.l.f.(H_c, R^a)$, we identify the multiples of H_c and R^a as being the matrices of the form

$$L_0 H_c,\ \text{for some matrix } L_0\,,$$

and

$$L_1 \Delta H_c + L_2 C_1,\ \text{for some matrices } L_1, L_2\,,$$

respectively. Therefore the common left multiples of H_c and R^a are matrices of the previous forms for which L_0, L_1, L_2 are such that

$$L_0 H_c = L_1 \Delta H_c + L_2 C_1\,,$$

that is,

$$\begin{bmatrix} L_0 - L_1\Delta & -L_2 \end{bmatrix} \begin{bmatrix} H_c \\ C_1 \end{bmatrix} = 0\,.$$

By construction $\begin{bmatrix} H_c \\ C_1 \end{bmatrix}$ is full row rank. Hence, $L_0 = L_1 \Delta$, and $L_2 = 0$, and the common left multiples of H_c and R^a are of the form $L_1 \Delta H_c$. Therefore, $L = \Delta H_c$ is a $l.c.l.f.(H_c, R^a)$. Since $\ker H_c$ is the controllable part of $\mathcal{B}$, from equation (5) we conclude that $\ker R^a$ is an autonomous part of $\mathcal{B}$. Furthermore, by construction, R^a is square and $\det R^a = \det(\Delta).d(s_1)$, with $d(s_1)$ a polynomial in s_1. ■

Next we give a characterization of SDV-representable 2D behaviors. Clearly, Proposition 3 is contained in the following one.

Proposition 4: Let $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathcal{B}) \in \mathcal{L}^q$. $\mathcal{B}$ is SDV-representable if and only if $\mathcal{B}$ is regular and every full row rank kernel representation R of $\mathcal{B}$ is such that $R = \Delta H_c$, where H_c is factor left prime and Δ is square with cw-unital determinant.

Proof: Assume that $\mathcal{B}$ is SDV-representable. That is, there exists a representation of $\mathcal{B}$ of the following form

$$\begin{cases} x &= A(\sigma_1^{-1}, \sigma_2^{-1})x + B(\sigma_1^{-1}, \sigma_2^{-1})v \\ w &= Cx + Dv \end{cases}.$$

So, $\mathcal{B}$ is described by an latent variable representation as follows, [4], [10],

$$\begin{bmatrix} 0 \\ I \end{bmatrix} w = \begin{bmatrix} I - A(\sigma_1^{-1}, \sigma_2^{-1}) & -B(\sigma_1^{-1}, \sigma_2^{-1}) \\ C & D \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}.$$

In order to eliminate the latent variables $\mathrm{col}(x, v)$, let $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ be a minimal left annihilator of

$$\begin{bmatrix} I - A(\sigma_1^{-1}, \sigma_2^{-1}) & -B(\sigma_1^{-1}, \sigma_2^{-1}) \\ C & D \end{bmatrix}.$$

Thus, $\mathcal{B} = \ker L_2$.

On the other hand, $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ is a left annihilator of

$$\begin{bmatrix} I - A(\sigma_1^{-1}, \sigma_2^{-1}) \\ C \end{bmatrix}.$$

Therefore, there exists a L-polynomial matrix F such that

$$\begin{bmatrix} L_1 & L_2 \end{bmatrix} = F \begin{bmatrix} M_1 & M_2 \end{bmatrix}$$

where $\begin{bmatrix} M_1 & M_2 \end{bmatrix}$ is a minimal left annihilator of $\begin{bmatrix} I - A(\sigma_1^{-1}, \sigma_2^{-1}) \\ C \end{bmatrix}$. Since $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ is factor left prime, F is also factor left prime. Note also that, by construction, M_2 is a square matrix and its determinant is a divisor of $\det(I - A(\sigma_1^{-1}, \sigma_2^{-1}))$. Since $\det(I - A(\sigma_1^{-1}, \sigma_2^{-1}))$ is a cw-unital L-polynomial and every divisor of a cw-unital L-polynomial is also a cw-unital L-polynomial (this is a consequence of [6, p. 116]), we conclude that $\det M_2$ is cw-unital. Thus

$$L_2 = F M_2$$

where F is factor left prime and $\det M_2$ is cw-unital. Moreover, it is possible to prove from this that L_2 has the desired factorization. In fact, let R_c be a representation of

the controllable part of $\mathcal{B}$. Since $\mathcal{B}^c \subset \mathcal{B}$, there exists Δ such that $\Delta R_c = F M_2$, that is,

$$\begin{bmatrix} -F & \Delta \end{bmatrix} \begin{bmatrix} M_2 \\ R_c \end{bmatrix} = 0$$

But F is factor left prime and consequently $\begin{bmatrix} -F & \Delta \end{bmatrix}$ is also left factor prime. So, $\begin{bmatrix} -F & \Delta \end{bmatrix}$ is a minimal left annihilator of $\begin{bmatrix} M_2 \\ R_c \end{bmatrix}$. Therefore, the determinant of the non-singular matrix Δ must divide $\det M_2$ and hence be cw-unital. Thus, we have shown the existence of a full row rank representation for $\mathcal{B}$ of the form ΔR_c, where Δ is square with cw-unital determinant and R_c is factor left prime. Furthermore, it is easy to see that any other full row rank representation of the same behavior also admits a similar decomposition. Indeed, if R is a full row rank representation of $\mathcal{B}$, there exists an unimodular matrix U such that $R = U M_2 L_2$ and it is quite trivial to check that $\det U M_2$ is cw-unital.

Conversely, let us suppose that $\mathcal{B} = \ker(\Delta H_c)$, where H_c is left factor prime and Δ is square with cw-unital determinant. According to Lemma 5, there exists an autonomous part of $\mathcal{B}$, $\mathcal{B}^a$, such that $\mathcal{B}^a = \ker R^a$, where $\det R^a = \det(\Delta).d(s_1)$, with $d(s_1)$ a polynomial in s_1. Since $\det \Delta$ is cw-unital it is easy to check that $\det R^a$ is also cw-unital. ■

Remark 2: From Proposition 4, it follows that if $\mathcal{B}$ is SDV-representable then $\mathcal{B} = \ker \Delta H_c$, where Δ is cw-unital and H_c is factor left prime. Thus, as shown in the proof of the same proposition, it is possible to exhibit an autonomous part

$$\mathcal{B}^a = \ker \begin{bmatrix} \Delta H_c \\ C \end{bmatrix}$$

such that

$$\det \begin{bmatrix} \Delta H_c \\ C \end{bmatrix} = \det(\Delta) d(s_1) \ ,$$

where $d(s_1)$ is polynomial in s_1 only. Therefore, $\mathcal{B}^a$ is an SDV-representable autonomous part of $\mathcal{B}$. Notice that, this type of autonomous part construction was proposed in [7].■

As already mentioned, every full row rank 2D L-polynomial matrix can be factored as a product of a square non-singular matrix by a factor left prime one. It follows directly from the previous proposition that SDV-decomposable behavior are exactly those where, in every such a decomposition, the square matrix has cw-unital determinant.

Corollary 3: If $\mathcal{B} = \ker(\Delta F)$, where Δ is a non-singular square matrix and F is factor left prime, then $\mathcal{B}$ is SDV-representable if and only if $\det \Delta$ is cw-unital.

Notice that, according to the previous corollary, if one has a full row rank representation of behavior, in order to check its SDV-representability one may simply calculate a maximal left factor of such representation and then see whether or not its determinant is cw-unital. However, even when a behavior is regular, there are representations without full row rank. Next we shall see how such representations can be characterized for SDV-representable behaviors.

Clearly, even with the full row rank condition is dropped, every 2D L-polynomial matrix can be factored as a product of three matrices: a factor right prime matrix by a square non-singular matrix by a left factor left prime. We will show that SDV-representable behaviors are exactly the ones for which the above mentioned decomposition yields a factorization where the factor right prime matrix is in fact zero right prime and the square matrix has cw-unital determinant.

Proposition 5: If $\mathcal{B} = \ker(P \Delta F)$, where P is factor right prime, Δ is a non-singular square matrix and F is factor left prime, then $\mathcal{B}$ is SDV-representable if and only if P is zero right prime and Δ has cw-unital determinant.

Proof: Suppose that $R = P \Delta F$, where P is zero right prime, Δ is cw-unital matrix and F is factor left prime, is a representation of $\mathcal{B}$. Since P is zero right prime there exists N such that $U = \begin{bmatrix} P & N \end{bmatrix}$ is unimodular and

$$R = \begin{bmatrix} P & N \end{bmatrix} \begin{bmatrix} \Delta F \\ 0 \end{bmatrix} .$$

Considering $U^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$,

$$U_1 R = \Delta F \ .$$

So, ΔF is a representation of $\mathcal{B}$. Therefore $\mathcal{B}$ is SDV-representable, by Proposition 4.

Reciprocally, let us suppose that $\mathcal{B}$ is SDV-representable and that $R = \ker(P \Delta F)$, where P, $g \times r$, is factor right prime, Δ, $r \times r$, is non-singular and F, $r \times q$, is factor left prime is a representation of $\mathcal{B}$. Since $\mathcal{B}$ is SDV-representable there exists a full row rank representation $\tilde{R}$ (with rank r) such that

$$\tilde{R} = M R_c$$

where M is a $r \times r$ matrix with cw-unital determinant and R_c, $r \times q$, factor left prime representation of $\mathcal{B}^c$, the controllable part of $\mathcal{B}$. On the other hand, F is also a representation of $\mathcal{B}^c$, so there exists U, unimodular, such that $F = U R_c$. Thus,

$$R = P \bar{\Delta} R_c, \text{ where } \bar{\Delta} = U \Delta \ .$$

Because R and $\tilde{R}$ are both representations of $\mathcal{B}$, there exist L-polynomial matrices L and $\tilde{L}$ such that

$$R = L \tilde{R} \tag{6}$$

$$\tilde{R} = \tilde{L} R \tag{7}$$

Thus, from (7), $M R_c = \tilde{L} P \bar{\Delta} R_c$, that is, (because R_c is full row rank),

$$M = \tilde{L} P \bar{\Delta} \tag{8}$$

Also, from (6), $P \bar{\Delta} R_c = L M R_c$ and consequently

$$P \bar{\Delta} = L M \ . \tag{9}$$

This, together with (8), yields

$$P \bar{\Delta} = L \tilde{L} P \bar{\Delta} \ .$$

Therefore, $P = L(\tilde{L}P)$. As P is factor right prime and $\tilde{L}P$ is a square factor of P, $\tilde{L}P$ must be unimodular. Hence P is zero right prime. Moreover, recalling that $\det M$ is cw-unital, from (8), $\det(\bar{\Delta})$ is cw-unital. Consequently, also $\det(\Delta)$ is cw-unital. ■

V. The Representation Construction

In the previous section, we have shown that a behavior is SDV-representable if and only if any of its representations admits a factorization into a product of a zero right prime matrix by a square matrix with cw-unital determinant by a factor left prime matrix. In this section, we propose a method to construct an SDV-represention for a behavior starting from one of its representations using the referred factorization.

Let $\mathcal{B} = \ker R$ and consider a factorization of R into the following form

$$R = P\Delta F\,,$$

where P, $g \times r$, is factor right prime, Δ, $r \times r$, is a non-singular square matrix and F, $r \times q$, is factor left prime. If P is not zero right prime or $\det(\Delta)$ is not cw-unital, then $\mathcal{B}$ is not SDV-representable, else $\mathcal{B}$ is SDV-representable and $\mathcal{B} = \ker(\Delta F)$. In this case, as shown in the proof of Lemma 5 , there exist F_1 such that

$$\begin{bmatrix} \Delta F \\ F_1 \end{bmatrix}$$

has cw-unital determinant. Moreover,

$$\begin{bmatrix} \Delta F \\ F_1 \end{bmatrix} w = \begin{bmatrix} 0 \\ I \end{bmatrix} v$$

is an latent variable representation of $\mathcal{B}$. Since

$$\det\left(\begin{bmatrix} \Delta F \\ F_1 \end{bmatrix}\right)$$

is cw-unital, there exists an unimodular matrix U such that

$$\bar{R} = U \begin{bmatrix} \Delta F \\ F_1 \end{bmatrix}$$

has its support in the third quarter plane and unit independent term. Considering $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, we have the following latent variable representation of $\mathcal{B}$

$$\bar{R}w = U_2 v \tag{10}$$

Notice that, without loss of generality, we may take U_2 such that its support also lies in the third quarter plane and zero independent term. In fact, if such does not happen, we multiply U_2 by $s_1^{-p_1}s_2^{-p_2}$, for suitable $p_1, p_2 \in \mathbb{N}$, and take an other driving variable $\bar{v} = \sigma_1^{p_1}\sigma_2^{p_2}v$. From (10) it is possible to construct an SDV-representation for $\mathcal{B}$ using some extra auxiliary variables (the states), performing a rather technical, but easy to follow, procedure, that we describe next.

Consider in (10) the L-polynomial matrices written in the following way

$$\begin{aligned} \bar{R} &= I + \bar{R}_{10}s_1^{-1} + \bar{R}_{01}s_2^{-1} + \dots \\ &\quad + \bar{R}_{k0}s_1^{-k} + \bar{R}_{k1}s_1^{-k}s_2^{-1} + \dots \bar{R}_{0k}s_2^{-k} \end{aligned} \tag{11}$$

$$\begin{aligned} U_2 &= \tilde{U}_{10}s_1^{-1} + \tilde{U}_{01}s_2^{-1} + \dots \\ &\quad + \tilde{U}_{l0}s_1^{-1} + \tilde{U}_{l1}s_1^{-l}s_2^{-1} + \dots \tilde{U}_{0l}s_2^{-l} \end{aligned} \tag{12}$$

where

$$k = \max_{(i,j)\in \text{supp }(\bar{R})} \{-i-j\}\,,$$

$$l = \max_{(i,j)\in \text{supp }(U_2)} \{-i-j\}\,,$$

$\bar{R}_{ij}$, for $i, j = 1\dots, k$, are $r \times r$ real matrices,

$\tilde{U}_{ij}$, for $i, j = 0\dots, l$, are $r \times (q-r)$ real matrices,

and the L-monomials are taken in the degree-lexicographic order.

Take w and define the following $\frac{k(k+1)}{2}$ auxiliary variables:

$$\begin{array}{rcl} \bar{x}_{00} &=& w \\ \left.\begin{array}{rcl} \bar{x}_{10} &=& \sigma_1^{-1}\bar{x}_{00} \\ \bar{x}_{01} &=& \sigma_2^{-1}\bar{x}_{00} \end{array}\right\} & 2 & \\ \vdots & & \\ \left.\begin{array}{rcl} \bar{x}_{i0} &=& \sigma_1^{-1}\bar{x}_{(i-1)0} \\ \bar{x}_{(i-1)1} &=& \sigma_1^{-1}\bar{x}_{(i-2)1} \\ \vdots & & \\ \bar{x}_{1(i-1)} &=& \sigma_1^{-1}\bar{x}_{0(i-1)} \\ \bar{x}_{0i} &=& \sigma_2^{-1}\bar{x}_{0(i-1)} \end{array}\right\} & i\,, & \text{for } i = 3, 4, \dots, k. \end{array}$$

Similarly, if $l > 1$, take v and define some extra $\frac{(l-1)(l+2)}{2}$ auxiliary variables:

$$\left.\begin{array}{rcl} \tilde{x}_{10} &=& \sigma_1^{-1}v \\ \tilde{x}_{01} &=& \sigma_2^{-1}v \end{array}\right\} \quad 2$$

$$\left.\begin{array}{rcl} \tilde{x}_{20} &=& \sigma_1^{-1}\tilde{x}_{10} \\ \tilde{x}_{21} &=& \sigma_1^{-1}\tilde{x}_{01} \\ \tilde{x}_{02} &=& \sigma_2^{-1}\tilde{x}_{01} \end{array}\right\} \; 3$$

$$\begin{array}{rcl} \vdots & & \\ \left.\begin{array}{rcl} \tilde{x}_{j0} &=& \sigma_1^{-1}\tilde{x}_{(j-1)0} \\ \tilde{x}_{(j-1)1} &=& \sigma_1^{-1}\tilde{x}_{(j-2)1} \\ \vdots & & \\ \tilde{x}_{1(j-1)} &=& \sigma_1^{-1}\tilde{x}_{0(j-1)} \\ \tilde{x}_{0j} &=& \sigma_2^{-1}\tilde{x}_{0(j-1)} \end{array}\right\} & j\,, & \text{for } j = 4, 5, \dots, l. \end{array}$$

Considering the state variables

$$x = \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix},$$

where

$$\bar{x} = \mathrm{col}(\bar{x}_{00}, \bar{x}_{10}, \bar{x}_{01}, \ldots, \bar{x}_{(k-1)0}, \ldots, \bar{x}_{1(k-2)}, \bar{x}_{0(k-1)})$$

$$\tilde{x} = \mathrm{col}(\tilde{x}_{10}, \tilde{x}_{01}, \ldots, \tilde{x}_{(l-1)0}, \ldots, \tilde{x}_{1(l-2)}, \tilde{x}_{0(l-1)}),$$

the following SDV representation of $\mathcal{B}$ is obtained (from (10), (11) and (12)):

$$\begin{cases} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v \\ w = \begin{bmatrix} I_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{x} \end{bmatrix} \end{cases}$$

where

$$B_1 = \left[\begin{array}{c} \tilde{U}_{10} s_1^{-1} + \tilde{U}_{01} s_2^{-1} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

$$B_2 = \left[\begin{array}{c} I_{q-r} s_1^{-1} \\ I_{q-r} s_2^{-1} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right]$$

and

$$A_{11} = \left[\begin{array}{ccccc|c} \bar{R}_1 & \cdots & \bar{R}_i & \cdots & \bar{R}_{k-1} & \bar{R}_k \\ \hline V_1 & & & & & 0 \\ & \ddots & & & \mathbf{0} & \vdots \\ & & V_i & & & \\ \mathbf{0} & & & \ddots & & \\ & & & & V_{k-1} & 0 \end{array}\right]$$

$$A_{12} = \left[\begin{array}{ccccc|c} \tilde{U}_2 & \cdots & \tilde{U}_j & \cdots & \tilde{U}_{l-1} & \tilde{U}_l \\ \hline 0 & & & & & 0 \\ & \ddots & & & \mathbf{0} & \vdots \\ & & 0 & & & \\ \mathbf{0} & & & \ddots & & \\ & & & & 0 & 0 \end{array}\right]$$

$$A_{22} = \left[\begin{array}{ccccc|c} 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 \\ \hline W_2 & & & & & 0 \\ & \ddots & & & \mathbf{0} & \vdots \\ & & W_j & & & \\ \mathbf{0} & & & \ddots & & \\ & & & & W_{l-1} & 0 \end{array}\right]$$

with

$$\bar{R}_i = -\left[\bar{R}_{i0} s_1^{-1} \middle| \cdots \middle| \bar{R}_{2(i-2)} s_1^{-1} \middle| \bar{R}_{1(i-1)} s_1^{-1} + \bar{R}_{0i} s_2^{-1}\right], \quad i = 1, \ldots, k$$

$$V_1 = \begin{bmatrix} s_1^{-1} I_q \\ s_2^{-1} I_q \end{bmatrix}$$

$$V_i = \left[\begin{array}{c|c} s_1^{-1} I_{iq} & 0 \\ \hline 0 & V_1 \end{array}\right], \quad i = 2, \ldots, k$$

$$\tilde{U}_j = \left[\tilde{U}_{j0} s_1^{-1} \middle| \cdots \middle| \tilde{U}_{2(j-2)} s_1^{-1} \middle| \tilde{U}_{1(j-1)} s_1^{-1} + \tilde{U}_{0j} s_2^{-1}\right], \quad j = 2, \ldots, l$$

$$W_j = \left[\begin{array}{c|c} s_1^{-1} I_{j(q-r)} & 0 \\ \hline 0 & W_1 \end{array}\right], \quad j = 2, \ldots, l.$$

Note that, if $l = 0, 1$ the variables $\tilde{x}$ are not considered and consequently the blocks A_{22} and B_2 do not appear.

VI. Final Remarks

In this paper we have proposed a strategy for testing if a 2D kernel behavior is SDV-representable. More concretely, we have shown that if a kernel representation of a behavior is decomposed as a product of a factor right prime matrix by a non-singular square matrix by a factor left prime matrix, then the behavior is SDV-representable if and only if the factor right prime matrix is indeed zero right prime and the square matrix has cw-unital determinant. This representability test improves a previous result obtained in [2], as it is independent from the chosen behavior representation. The obtained decomposition allows to set up an algorithm for the construction of 2D SVD representations.

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