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Optimal investment timing using jump price processes: A real options approach*

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Abstract. In this work, we address an investment problem where investment can either be made immediately or postponed to a later time, in the hope that market conditions become more favourable. In our case, uncertainty is introduced through market price. When the investment is undertaken, a fixed sunk cost must be paid and a series of cash flows are to be received. Therefore, we are faced with an irreversible investment. Real options analysis provides an adequate framework for this type of problems by recognizing these two characteristics, uncertainty and irreversibility, explicitly. We have developed algorithmic solutions for this type of problems by modelling market prices by Markov jump processes.

1. Introduction.

The timing of market entry decisions is a central concern for business strategy. In a context of uncertainty and irreversibility the entry timing decision becomes even more important since by making a commitment we loose the option of waiting for a better opportunity.

The problem that we will discuss here is deciding the optimal timing for an irreversible investment. A firm has an investment opportunity in the exploration of a natural resource, such as oil. When the investment decision is made, the firm pays a sunk fixed cost and from then on it will start to receive the incomes of the exploration over a long period, considered infinite. This problem falls within real options problems, since at each point in time we have the option to invest or to wait for a more appropriate time and the investments discussed here typically involve real assets rather than financial assets.

The income received at each instant of time, the price of the resource, is considered to evolve according to a known stochastic dynamic model. Therefore, the expected total income will then be the integral over time of the expected discounted incomes received at each instant, i.e. the expected discounted prices of the resource integrated from the time of investment to infinity.

Usually, a simple static Net Present Value analysis of this problem is made. A comparison is made between the fixed cost of the investment and the total expected income. Questions like “Will the expected total income be greater then the investment cost?”, “If yes, what is the probability of that happening?”, and “How many years will it take for the investment to be paid?”, are typically answered by such analysis.

Here, we use a real options approach to this problem focusing on the optimal time for investment. Even if the answers to the previous questions are such that they would lead us to invest, the expected evolution of the resource price could be such that the conditions in the future will be even better. In this scenario, although a decision to invest now is a positive one, the decision to wait and invest in the future would be better, as long as the opportunity to invest remains available. In the real options literature this is known as the option to delay.

The question that we will address in this work will be the instant of time at which the best conditions for investment occurs, i.e. the optimal investment time. To answer this, instead of comparing the present expected total income just with the sunk cost of investment, we will have to compare the present expected total income with the expected total incomes of all future times and choose the maximum one. This problem, as we will see, falls within a category of problems known in the literature as “Optimal Stopping Problems”.

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We note that the solution we seek for is not a time value, but rather a policy. That is, a rule that specifies under which conditions we should enter the market. In fact, the results obtained state that the investment should be undertaken once the price reaches a certain threshold value. However, for a specific price value, it is possible to obtain an estimate for the waiting time till investment.

The evolution of the price will be modelled as a Markov jump process. The randomness of the process will be considered to occur not in a continuous way, but only at specific instants of time. This is particularly suitable when the main random factors that affect the price are point events, occurring at random times and with random intensity. There are several examples of price changes in natural resources caused by financial, political or environmental events. Oil prices are a typical example of a variable whose value is affected by abnormal news and their sudden variations are adequately modelled by jump processes. A possible application for the model and algorithms here developed is the decision to make an investment in the exploration of an oil field.

In this work, we discuss 3 solution methodologies, leading to 3 different algorithms, for the optimal stopping problem modelling an irreversible investment decision when the prices follow a Markov jump process.

For illustration purposes, though, we start by discussing a discrete time model.

2. Preliminaries.

Consider the Markov jump process $\left\{X_t; t \in \mathrm{IR}^+\right\}$ with transition function P_t, generator A, and state-space E (countable).

Let $\alpha \geq 0$ be a fixed number, the *α-potential of the function g* (bounded non-negative and defined on E) for the process X is the expected value of the total discounted return of the rewards g,given by

$$U^\alpha g(i) = E_i\left[\int_0^\infty e^{-\alpha t} g\left(X_t\right) dt\right],$$

and the matrix U^α can be computed as

$$U^\alpha(i,j) = \int_0^\infty e^{-\alpha t} P_t(i,j) dt,\ i,j \in E.$$

Proposition 2.1. *Let g be a bounded function and α >0 then the vector $u = U^\alpha \cdot \underline{g}$ is the unique solution of the system of linear equations*

$$\left(\alpha I - A\right) u = g.$$

Proposition 2.2. *For any stopping time T*

$$U^\alpha g(i) = E_i\left[\int_0^T e^{-\alpha t} g\left(X_T\right) dt\right] + E_i\left[e^{-\alpha T} U^\alpha g\left(X_T\right)\right].$$

For the proof of these propositions see [4].

Let f be a finite-valued function defined on E, the state-space of a Markov process X_t with semigroup P_t, and let $r \geq 0$.

The function f is said to be an *r-excessive function* for the process X_t if for all $x \in E$ and for all $t \geq 0$

$$f(x) \geq e^{-rt} P_t f(x),$$

or equivalently

$$rf(x) - Af(x) \geq 0.$$

A function which is 0-excessive is simply called excessive.

Theorem 2.3. *The value function V is the minimal r-excessive function that majorises g.*

See [18] chapter 3, for a proof.

Hence for a finite countable space E, V can be computed by linear programming as

$$\begin{array}{ll} \min & \sum_{j \in E} V(j) \\ \text{s.t.} & AV(x) - rV(x) \geq 0 \\ & V(x) - h(x) \geq 0 \\ \text{for all } x \in E, & \end{array}$$

where the second constraint is active (satisfied by equality) when x belongs to the stopping set and otherwise the first constraint is active.

Hence, these constraints can be written as the following Variational Inequalities

$$\max\{rV(x) - AV(x),\, h(x) - V(x)\} = 0, \;\; \forall x \in E.$$

3. Discrete Time Model.

The problem is to decide the best instant of time to invest in the extraction of a resource given its actual price and a stochastic model of the price evolution. It is assumed that the investment can be done immediately, once decided, and the corresponding income starts on the next time instant.

The prices are considered to evolve according to

$$p_{k+1} = (1 + w_k)p_k,$$

where w_k values are uncorrelated, belonging to a finite ordered set of values Ω={$\Omega_x, ..., \Omega_N$} distributed according to $F(\Omega_i)$=Prob($w_k \leq \Omega_i$) and corresponding mean $Ew_k = m$ and density $f(\Omega_i)$=Prob($w_k = \Omega_i$).

Our decision is the time instant to invest that maximises the expected discounted profit

$$\max_k \left\{ E\left[\sum_{i=k+1}^{\infty} (1+r)^{-i} p_i) - (1+r)^{-k} I \,|p_0, ..., p_k \right] \right\},$$

where I is the fixed cost of investment and r the interest rate.

Alternatively the decision at each time instant is to invest now or wait at least one more unit time

$$\max J_0(p_0)$$

where

$$J_k(p_k) = \max\left\{ \underset{\text{(invest)}}{E\left[\sum_{i=k+1}^{\infty} (1+r)^{-i} p_i) - (1+r)^{-k} I \right]}, \underset{\text{(wait)}}{E(J_{k+1}(p_{k+1}))} \right\},$$

or using the value function at current prices

$$V_K(p_k) = (1+r)^k J_k(p_k),$$

$$V_k(p_k) = \max\left\{ \underset{\text{(invest)}}{E\left[\sum_{i=1}^{\infty} (1+r)^{-i} p_i) - I \right]}, \underset{\text{(wait)}}{(1+r)^{-1} E(V_{k+1}(p_{k+1}))} \right\}.$$

For the infinite time horizon case, the case in which we are interested in, $V_k(p) = V(p)$ for all k, and so the value function V satisfies

$$V(p_k) = \max\left\{ \underset{\text{(invest)}}{E_{p_k}\left[\sum_{i=1}^{\infty} (1+r)^{-i} p_i) - I \right]}, \underset{\text{(wait)}}{(1+r)^{-1} E_{p_k}(V(p_{k+1}))} \right\},$$

which falls within the Optimal Stopping Problems.

3.1. Solution method.

Defining $h(p)$ as the expected return of investing now, i.e. the sum of the discounted incomes from now till infinity

$$h(p_k) = E\left(\sum_{i=1}^{\infty} (1+r)^{-i} p_{i+k} \right) - I = \sum_{i=1}^{\infty} \left(\frac{1+m}{1+r} \right)^{-i} p_k - I = \frac{1+m}{r-m} p_k - I, \qquad \text{if } r > m$$

and υ(p) as the expected return if we wait at least one unit time

$$\upsilon(p_k) = (1+r)^{-1}E(V_{k+1}(p_{k+1})).$$

At time k our decision will be to invest if $h(p_k) > \upsilon(p_k)$, to wait if $h(p_k) < \upsilon(p_k)$ and either decision is optimal if $h(p_k) = \upsilon(p_k)$. If in this last case we choose to invest, our decision rule will be to invest if and only if $h(p_k) \geq \upsilon(p_k)$, i.e.

$$\text{iff } \frac{1+m}{r-m}p_k - I \geq \upsilon(p_k),$$

$$\text{iff } p_k \geq \frac{r-m}{1+m}(\upsilon(p_k)+I),$$

$$\text{iff } p_k \geq p^* \text{ where } p^* \text{ satisfies } p^* = \frac{r-m}{1+m}(\upsilon(p^*)+I).$$

A graphical interpretation can be seen in Fig. 1.

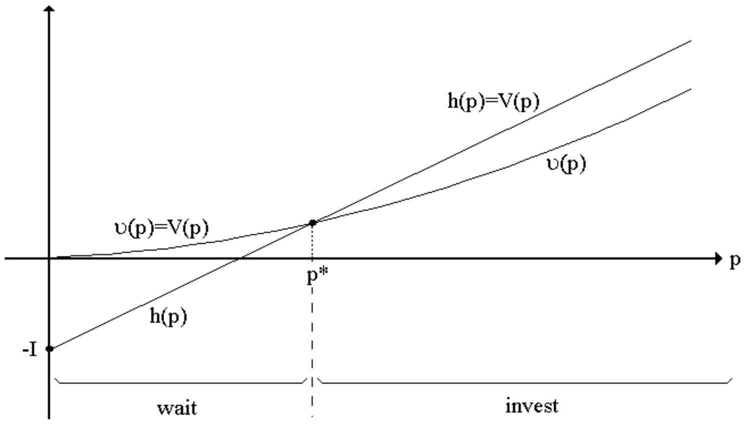

Figure 1: Optimal stopping graphical interpretation.

Note that for p greater than p^* (inside the stopping region), $h(p)$ is greater than $\upsilon(p)$ because we are in fact loosing an opportunity if we do not make the correct decision even if it is only for one instant of time. The value function V coincides with $\upsilon(p)$ for p less or equal to p^* and coincides with $h(p)$ for p greater or equal to p^*.

To achieve the solution it remains only to develop $\upsilon(p)$.

3.1.1. Determination of υ(p). We know that $\upsilon(p_k)$ satisfies

$\upsilon(0) = 0,$
$\upsilon(p*) = h(p*),$
υ(p)=(1+r)$^{-1}$E { V [(1+w)p] }=(1+r)$^{-1}$ E{max{ h[(1+w)p],υ[(1+w)p]}}.

In order to develop it further we will consider separately the cases when (1+w)$p > p*$ and when (1+w)$p \leq p*$.

a) Case (1+w)$p > p* \Leftrightarrow w > \frac{p*}{p} - 1$

Let $N1$ be such that $\Omega_{N1} = \min\{\Omega_i \in \Omega : \Omega_i > \frac{p*}{p} - 1\}$, i.e. $N1$ is the index of the first term in Ω that satisfies the condition of being in this case.

$$\upsilon^a(p) = (1+r)^{-1}E\{h[(1+w)p]|w > \Omega_{N1}\} =$$

$$= (1+r)^{-1}\left(\frac{1+m}{r-m}E[(1+w)p|w>\Omega_{N1}]-I\right)=$$

$$= (1+r)^{-1}\left(\frac{1+m}{r-m}\frac{\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)}{\sum_{i=N1}^{N}f(\Omega_i)}p-I\right).$$

And the probability of being in this case is

$$\sum_{i=N1}^{N} f(\Omega_i) = 1-\text{Prob}\left(w\leq\frac{p*}{p}-1\right)=1-F\left(\frac{p*}{p}-1\right).$$

b) Case $(1+w)p\leq p*\Leftrightarrow w\leq\frac{p*}{p}-1$

Let $N2$ be such that $\Omega_{N2}=max\left\{\Omega_i\in\Omega:\Omega_i\leq\frac{p*}{p}-1\right\}$. i.e. $N2$ is the index of the last term in Ω that satisfies the condition of being in this case.

$$\upsilon^b(p)=(1+r)^{-1}E\{\upsilon[(1+w)p]|w\leq\Omega_{N1}\}=$$

$$=(1+r)^{-1}\frac{\sum_{i=1}^{N2}\nu\left[(1+\Omega_i)p\right]f(\Omega_i)}{\sum_{i=1}^{N2}f(\Omega_i)}.$$

And the probability of being in this case is

$$\sum_{i=1}^{N2} f(\Omega_i)=\text{Prob}\left(w\leq\frac{p*}{p}-1\right)=F\left(\frac{p*}{p}-1\right).$$

Finally, using the Bayes rule, $\upsilon(p)$ is given by

$$(1+r)\upsilon(p)=\left[1-F\left(\frac{p*}{p}-1\right)\right]\upsilon^a(p)+F\left(\frac{p*}{p}-1\right)\upsilon^b(p),$$

$$(1+r)\upsilon(p)=\frac{I+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p-I\left[1-F\left(\frac{p*}{p}-1\right)\right]+\sum_{i=1}^{N2}\upsilon\left[(1+\Omega_i)p\right]f(\Omega_i).$$

3.1.2. Determination of an estimate for the Optimal “waiting” time. Recalling that the original problem was

$$\max_k\left\{E\left[\sum_{i=k+1}^{\infty}(1+r)^{-i}p_i)-(1+r)^{-k}I|p_0,...,p_k\right]\right\}.$$

In the case that at the present time k, the decision obtained is to wait, we may wish to know (given the present data $p_0,...,p_k$) how much time we should wait in order to invest. The answer is given by

$$\tau=\min\{t\geq 0:E[h(p_{k+t})|p_0,...,p_k]\geq E[\upsilon(p_{k+t})|p_k]\}=$$

$$=\min\{t\geq 0:E_{pk}[h(p_{k+t})]\geq \quad E_{pk}[\upsilon(p_{k+t})]\},$$

which, as we have seen, is equivalent to

$$\tau=min\{t\geq 0:E[p_{k+t}\quad|p_0,...,p_k]\geq p^*\}.$$

As $E(p_{k+t}|p_0,...,p_k)=E(p_{k+t}|p_k)$=(1+$m)^t p_k$, we get

$$\tau=min\{t\geq 0:(1+m)^t p_k\quad\geq p^*\},$$

and so $\tau=\min\left\{t\geq 0:t\geq\frac{\ln(p^*/p_k)}{\ln(1+m)}\right\}$.

Hence for each value of p_k, the corresponding stopping time can be directly determined as $\tau=\frac{\ln(p^*/p_k)}{\ln(1+m)}$.

3.2. Algorithm. An algorithm to compute the solution to this problem could be the following:

```
1.  Iteration index K=1
2.  Set initial guess for p*^1
3.  Initialise υ(p) as straight lines
      for p =0...p*^1
```

$$\upsilon(p) = \frac{h(p^{*1})}{p^{*1}}p,$$

```
      for p = p*^1...P_max
```

$$\upsilon(p) = h(p).$$

```
4.  Update estimate of υ(p)
      for p=0..P_max
```

$$\upsilon(p) = (1+r)^{-1}\left\{\frac{1+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p - I\left[1-F\left(\frac{p*}{p}-I\right)\right] + \sum_{i=1}^{N2}\upsilon\left[(1+\Omega_i)p\right]f(\Omega_i)\right\}.$$

```
5.  Stop condition
      If max_p |υ(p) − υ_OLD(p)| < ε then STOP.
6.  Update estimate of p*
```

$$p*^{k+1} = \min\left\{p : \upsilon(p) = h(p)\right\}.$$

```
7.  k=k+1; GOTO 4.
```

3.3. Special Case. In the special case where the prices are monotonically increasing (i.e. Prob(w >0)=1) we have that $F\left(\frac{p*}{p}-1\right) = 0$ for $p = p$*, and so the expression for $\upsilon(p)$ simplifies to

$$\upsilon(p^*) = (1+r)^{-1}\left\{\frac{1+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p^* - I\right\},$$

and as $\upsilon(p^*) = h(p^*)$ and $\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)$=1+$m$,

$$(1+r)^{-1}\left[\frac{(1+m)^2}{r-m}p^* - I\right] = \frac{1+m}{r-m}p^* - I.$$

Finally, after some algebra we get

$$p^* = \frac{rI}{1+m}.$$

So p^*can be determined explicitly in closed form for this special case. In the general case this can be used as an initial guess for p^*in an algorithm for a more general case.

3.4. Reformulating as a finite state Markov chain model. As the evolution of the prices considered before is a process with independent increments, it satisfies the Markov property.

In order to get a finite state-space (the previous case had not a finite state-space unless the (1+Ω_ι)'s were multiples of each other) we will first apply logarithms to both sides of equation (1) getting

$$\log P_{k+1} = \log P_k + \log\left(1+w_k\right).$$

Defining

$$X_k = \log P_k,$$

we have now a countable state-space for the process $\{X_k,\, k \in IN\}$. If we now set upper and lower bounds for the state-space, $X_{\min}$ and $X_{\max}$, defining $x_1 = X_{\min}$ and $x_n = X_{\max}$, we get a finite state-space $X = \{x_1, x_2, ..., x_n\}$. Naturally, by clipping the state-space in this way, we will get a different process, but in a real problem, the majority of the possible prices will certainly be within a bounded interval with high probability and so the difference will not be significant.

In such case, the new gain function $h(x)$ would be defined as

$$h(x) = E\left[\sum_{i=1}^{\infty} (1+r)^{-i} e^{X_i}\right] - I,$$

which is the $(1+r)$-potential of the exponential function minus the fixed cost of investment I.

Defining

$$\begin{array}{l} \underline{h} = \left[h\left(x_1\right), ..., h\left(x_n\right)\right]^T, \\ \underline{f} = \left[\exp\left(x_1\right), ..., \exp\left(x_n\right)\right]^T, \\ \underline{I} = I * \left[1, 1, ..., 1\right]^T, \end{array}$$

and I_n as the identity matrix of dimension n.

By Proposition 2.1

$$\underline{h} = \left(I_n - (1+r)Q\right)\underline{f} - \underline{I}.$$

Now the problem,

$$V\left(X_0\right) = \max_k E\left[(1+r)^{-k}\, h\left(X_k\right)\right],$$

would be solved simply for this finite state-space case, by calculating V for all $x \in X$, such that V satisfies

$$V(x) = \max\left\{h(x), TV(x)\right\},$$

where

$$TV\left(X_k\right) = (1+r)^{-1} EV\left(X_{k+1}\right) = (1+r)^{-1} \sum_{x_j \in X} Q\left(X_k, x_j\right) V\left(x_j\right).$$

Solution method 1. This function V can be computed iteratively as

$$\begin{cases} V_0(x) = h(x) \\ V_{m+1}(x) = \max\left\{h(x),\, TV_m(x)\right\}, \end{cases}$$

and $V_m \to V$ as $m \to \infty$.

Solution method 2. It is known, from the optimal stopping literature that V, the value function, is the minimum $(1+r)$-excessive function that majorises h. Hence, it can be computed by linear programming as

$$\begin{array}{ll} \min & \sum_{x \in X} V(x) \\ s.t. & V(x) \geq h(x) \\ & V(x) \geq (1+r)^{-1} TV(x) \\ & V(x) \geq 0. \end{array}$$

These alternative solution methods will be adapted and implemented for the case of the Markov jump processes, as is developed in the next section.

4. Markov Jump Model.

In this section, we consider the price to be modelled as a Markov jump process. Such process is able to capture price jumps that are induced by the occurrence of rare events typically related to the arrival of new information. This information might be of different nature: technological (inovations)[11, 9, 12], competition (new product or competitor entry)[10, 2], political (wars, expropriation, change in legislation) [5, 19], natural conditions (natural hazards, hurricanes). More recently Martzoukos and Trigiorgious [16] have considered various types of rare events simultaneously.

Consider the resource price p_t to evolve according to

$$p_t = e^{X_t},$$

where X_t is a piecewise constant Markov process, taking values in a finite ordered set $X = \{x_1, x_2, ..., x_n\}$, having a jump rate λ and with the post-jump location defined by the transition probabilities

$Q(i,j) = \text{Prob}\,(X_T = x_j | X_T = x_i)$ in which $Q(i,i) = 0$ for i=0...N.

Our problem is to decide the time instant to invest that maximises the expected discounted profit

$$\rho(x) = \sup_{\tau \geq 0} E_x \left[e^{-r\tau} h(X_\tau) \right] ,$$

where h is the expected income at the time of investment.

For the Markov jump process $\{X_t\}$ considered above, we have the following transition function.

- For a constant jump rate λ the transition function is given by

$$P_t = \sum_{n=0}^{\infty} Q^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \ \ , \, t \geq 0 \, .$$

- For a jump rate λ dependent on the state, the transition function is given by

$$P_t(i,j) = e^{-\lambda(i)t}\delta_{ij} + \int_0^t \lambda(i) e^{-\lambda(i)s} \sum_{k \in E} Q(i,k) P_{t-s}(k,j) ds,$$

where $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

The generator A for this process is given by

$$A(i,j) = \begin{cases} -\lambda(i) & \text{if } \ i = j \\ -\lambda(i)Q(i,j) & \text{if } i \neq j \end{cases}$$

or in matrix notation $A = \Lambda(Q - I)$ where $\Lambda = \text{diag}(\lambda(1), \lambda(1), \ldots, \lambda(N))$.

4.1. Solution Method. In order to determine the optimal stopping policy, the first thing to do is to compute the gain function h, the expected income at time of investment.

Defining

$$f(X_t) = \exp(X_t)$$

and $V(x)$ as the r-potential of f

$$V(x) = U^r f(x) = E_x \left[\int_0^\infty e^{-rt} f(X_t) dt \right] .$$

The vector $\underline{V}$=[$V(x_1)$, $V(x_2)$,..., $V(x_N)$]^T can be computed as the unique solution of the system of equations

$$(rI_N - A)\underline{V} = \underline{f}$$

where $\underline{f}$=[$f(x_1)$, $f(x_2)$,..., $f(x_N)$]^T and I_N is the identity matrix. Hence, the vector $\underline{h}$=[$h(x_1)$, $h(x_2)$,..., $h(x_N)$]^T is given by

$$h = V - I,$$

where $\underline{I}$=I*[1,1,...,1]^T.

Now to compute the value function we will develop, for the Markov jump processes case, the three methods referred to previously.

4.1.1. Solution Method 1. This method follows closely the first method described in the previous section and it is based on determining an approximation of the function ρ.

Defining
$$\nu_\delta(x_t) = e^{-r\delta}\rho(x_{t+\delta}), \tag{1}$$
as the expected return if we wait a small time δ, we have the obvious result
$$\lim_{\delta\downarrow 0}\nu_\delta(x) \to \rho(x).$$

This means that, for very small δ, $\nu_\delta(x)$ will give us a good estimate of $\rho o x)$, and so an "almost optimal" solution would be given by

Invest

iff $h(X_t) \geq \nu_\delta(X_t)$,

iff $X_t \geq X^*_\delta$,

where X^*_δ is given by
$$X^*_\delta = \inf\{x \in X : h(x) \geq \nu_\delta(x)\}.$$

Otherwise, wait for a time τ_δ, given by
$$\tau_\delta = \inf\{t \geq 0 : h(X_t) \geq \nu_\delta(X_t)\}.$$

Now we will develop $\nu_\delta(x)$, knowing that it is given by
$$\nu_\delta(X_t) = e^{-r\delta}\rho(X_t) = e^{-r\delta}E\max\{h(X_{t+\delta}), \nu_\delta(X_{t+\delta})\}.$$

Similar to what we have done for the discrete time case, we will consider separately the two cases when a) $X_{t+\delta} \geq X^*_\delta$ and when b) $X_{t+\delta} < X^*_\delta$.

a) Case $X_{t+\delta} \geq X^*_\delta$

We will start by defining P_a, the probability of being in this case as
$$P_a = Prob(X_{t+\delta} \geq X^*_\delta \quad |X_t = x_i) = \sum_{j=1}^{N} p(i,j,\delta) I_{x_j \geq X^*}.$$

But as the Poisson Process for small intervals [t,t+δ[can be given by
$$\text{Prob}(N_\delta = m) = \begin{cases} \lambda\delta + o(\delta) \text{ if } m = 1 \\ o(\delta) \text{ if } m > 1 \\ 1 - \lambda\delta + o(\delta) \text{ if } m = 0\,, \end{cases} \tag{2}$$
P_a can be more simply denoted as
$$P_a = \lambda\delta \sum_{j=1}^{N} q_{ij} I_{x_j \geq X^*} + (1{-}\lambda\delta) I_{x_j \geq X^*}. \tag{3}$$

And so
$$\nu^a_\delta(X_t) = e^{-r\delta}E\max\{h(X_{t+\delta}) | X_{t+\delta} \geq X^*_\delta\} =$$
$$= e^{-r\delta}\frac{\lambda\delta \sum\limits_{j=1}^{N} h(x_j) q_{ij} I_{x_j \geq X^*} + h(x_i)(1{-}\lambda\delta) I_{x_i \geq X^*}}{P_a}.$$

b) Case $X_{t+\delta} < X^*_\delta$

Defining P_b the probability of being in this case as,

P_b=Prob($X_{t+\delta} < X^*_\delta \quad | X_t = x_i$)=

$$= \lambda\delta \sum_{j=1}^{N} q_{ij} I_{x_j < X^*} + (1 - \lambda\delta) I_{x_j < X^*}, \tag{4}$$

we have

$$\nu^b_\delta(X_t) = e^{-r\delta} E \max \{\nu(X_{t+\delta}) | X_{t+\delta} < X^*_\delta\} =$$

$$= e^{-r\delta} \frac{\lambda\delta \sum_{j=1}^{N} \nu(x_j) q_{ij} I_{x_j < X^*} + \nu(x_i)(1 - \lambda\delta) I_{x_i < X^*}}{P_b}$$

Finally $\nu_\delta(x)$ is then

$$\nu_\delta(x) = \mathrm{g}\nu^a_\delta(X_t) P_a + \nu^b_\delta(X_t) P_b =$$

$$= e^{-r\delta} \left\{ \lambda\delta \sum_{j=1}^{N} q_{ij} [h(x_j) I_{x_j \geq X^*} + \nu(x_j) I_{x_j < X^*}] + (1 - \lambda\delta)[h(x_i) I_{x_i \geq X^*} + \nu(x_i) I_{x_i < X^*}] \right\}. \tag{5}$$

Having a recursive equation for $\nu_\delta(x)$ we will only need now a fixed point to start with.

Choosing this point to be the largest value in X, say x_N, we get

$$\nu_\delta(x_N) = h(x_N).$$

If this is not true for this point, then it is also not true for any other, and then we would have the trivial solution "never to invest".

Algorithm.

1. Determine $h(x)$
2. Set initial guess for X^*
3. Initialise $\nu_\delta(x)$ as $\nu_\delta(x) = h(x)$
4. Update estimate of $\nu_\delta(x)$ for $i = 1, ..., N$

$$\begin{cases} \nu_\delta(X_N) = h(X_N) \\ \nu_\delta(x_i) = ...(equation\, 10) \end{cases}$$

5. Stop condition

   If $\max_i |\nu(x_i) - \nu_{old}(x_i) < \varepsilon|$ then stop
6. Update estimate of X^*

$$X^* = \min_i \{x_i : \nu(x_i) \leq h(x_i)\}$$

7. Goto 4.

4.1.2. Solution Method 2. The previous method can be supported by the known result that the value function is the minimal r-excessive function that majorises h; where the r-excessive function is defined as a function $f$satisfying:

$$f(x) \geq e^{-rt} P_t f(x).$$

We have also seen that an equivalent definition of the r-excessive function is a function $f$satisfying:

$$rf(x) - Af(x) \geq 0.$$

Where A is the generator of the process $\{X_t\}$ defined as $A = \Lambda\,(Q - I)$ where $\Lambda = \text{diag}\,(\lambda(1), \lambda(2), ...)$.

With this definition of r-excessive function we no longer have to consider any value δ for a discretization of t, because we are already using an infinitesimal operator - the generator A.

It follows the result of the *Variational Inequalities*

$$\max\{rV(x) - AV(x),\, h(x) - V(x)\} = 0,\ \ \forall x\ inE,$$

which can be formulated as a linear programming problem

$$\begin{array}{l} \min \sum\limits_{j \in E} V(j) \\ \text{s.t.} \quad AV(x) - rV(x) \geq 0 \\ \qquad V(x) - h(x) \geq 0 \\ \text{for all } x \in E, \end{array}$$

An implementation of this method in MATLAB was carried out, where a MATLAB built in function for solving the linear programming problem, was used.

4.1.3. Solution Method 3. Another particularity of this problem is that since r >0 and X_t is constant between jumps, the optimal solution will always be to invest in the instants of time immediately after the jump.

This is because if for a certain value X_t the decision is to invest, we should do so as soon as possible as we will only be loosing money by waiting.

This can be justified with an example.

Consider $X_t = x$ constant, then

$$h(x) = \int_0^\infty e^{-rt+x} dt - I = \frac{e^x}{r} - I,$$

and so

$$\rho(x) = \underset{\tau}{\text{Sup}} \left\{ e^{-r\tau} \left(\frac{e^x}{r} - I \right) \right\},$$

gives, obviously, τ=0 if r >0.

So, instead of considering a small time instant δ as in the previous method, we can consider T_n, the next jump time.

So defining

$$J_n(x_i) = \underset{\tau \leq T_n}{\text{Sup}}\ E_i\left[e^{-r\tau} h\left(X_\tau\right)\right],$$

it can also be given, as we have seen, by

$$J_n(x_i) = \max\Big\{ \underset{\text{(wait)}}{J_{n-1}(x_i)}, \underset{\text{(invest)}}{E_i\left[e^{-rT_n} h\left(X_{T_n}\right)\right]} \Big\}.$$

Since

$$E_i\left[e^{-rT_1} h\left(X_{T_1}\right)\right] = \int\limits_0^\infty \lambda e^{-\lambda t} dt \cdot \sum_{j=1}^N Q(i,j) h(x_j) = \frac{\lambda}{\lambda + r} \left(Q\underline{h}\right)_i,$$

$J_n(x_i)$ can be computed for i=1...N and for n=1,2,... as

$$\begin{cases} J_0(x_i) = h(x_i) \\ J_n(x_i) = \max\left\{J_{n-1}(x_i), \left(\frac{\lambda}{\lambda+r}\right)^n (Q^n \underline{h})_i\right\} \end{cases}.$$

Moreover, as λ is a bounded function

$$\lim_{n\to\infty} T_n = \infty,$$

and so

$$J_n(x) \underset{n\to\infty}{\longrightarrow} \underset{\tau<\infty}{\text{Sup}}\, E_x\left[e^{-r\varepsilon} h\left(X_\tau\right)\right] = \rho(x),$$

which is our aim to determine.

Algorithm.

1. Determine $\underline{h}$ (as in the previous methods)
2. n=0
   FOR i=1...N
   $$J_0(x_i) = h(x_i)$$
3. REPEAT
   δ_{max}=0
   FOR i=1...N
   $$J_{n+1}(x_i) = \max\left\{J_n(x_i), \left(\frac{\lambda}{\lambda+r}\right)^{n+1} \left(Q^{n+1}\underline{h}\right)_i\right\}$$
   $$-\delta = |J_{n+1}(x_i) - J_n(x_i)|$$
   $$\delta_{max} = \max\{\delta, \delta_{max}\}$$
   $n = n$+1
   UNTIL $\delta_{max} <\varepsilon$

4.2. Example and comparison of the algorithms. To illustrate the algorithms consider the following example:

Interest rate: r=1,
Fixed cost of inv.: I=10,
$X_{Max} = \ln(200)$,
$X_{Min} = \ln(2)$,
$X = \{X_{Min}, ..., X_{Max}\}$ with N=20 points equally spaced,
Jump rate λ=10.

For the 1st Algorithm (corresponding to the solution method 1):

Error ε=1e-3.
δ=1e-4.

For the 3rd Algorithm:

Error ε=1e-8,

Results of the 1st algorithm

```
Xstar=X10=1.817
Iteration:1 Error:1.059
Xstar=X13=2.504
Iteration:5 Error:0.06069
Xstar=X15=3.102
Iteration:10 Error:0.0428
Xstar=X15=3.102
Iteration:15 Error:0.03091
Xstar=X15=3.102
Iteration:20 Error:0.02279
Xstar=X15=3.102
Iteration:25 Error:0.01707
Xstar=X15=3.102
Iteration:30 Error:0.01295
Xstar=X15=3.102
Iteration:35 Error:0.009921
Xstar=X15=3.102
Final Result:
Iteration:37 Error:0.0008942
Xstar=X15=3.102
```

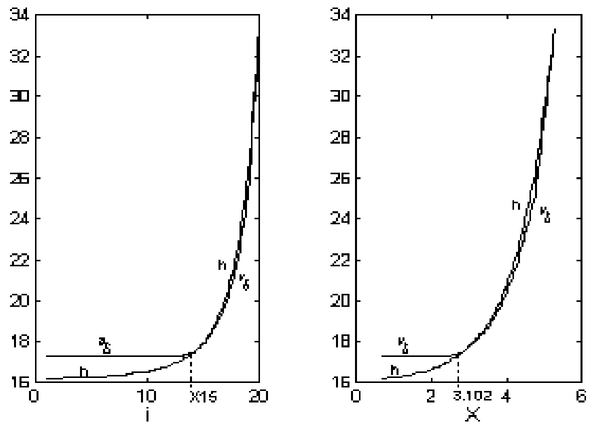

Figure 2: 1st algorithm results.

This algorithm converges very fast if $\delta << \varepsilon$. Moreover, reducing δs the difference between ν_δ and h for $X >= X^*$ is reduced. This difference is an effect of the discretisation. But if we reduce δ too much, we start having numerical problems (because the term that is multiplied by δ vanishes).

Results of the 2nd algorithm

```
Xstar=X15=3.102
```

This algorithm was the fastest to run for this example, but we have used a MATLAB built-in function to determine the optimal solution of the linear programming problem. Although the linear programming optimisation methods are usually very efficient (the Simplex algorithm, for example, usually converges in few iterations), this efficiency is not guaranteed for every instance of the problem.

Results of the 3rd algorithm

```
Iteration:0 Error:0.7273
Iteration:5 Error:0.03046
```

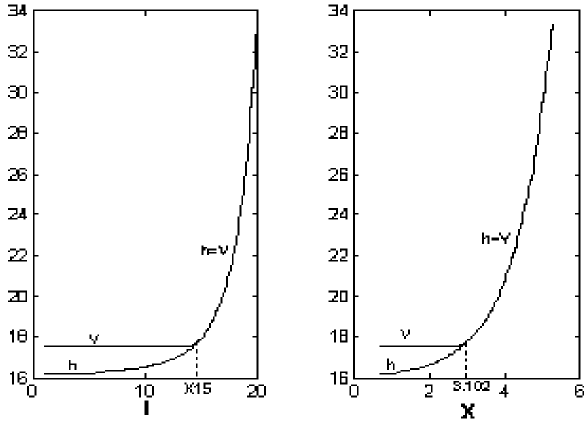

Figure 3: 2nd algorithm results.

```
Iteration:10 Error:0.002785
Iteration:15 Error:0.0002593
Iteration:20 Error:0.00002414
Iteration:25 Error:2.248e-006
Iteration:30 Error:2.093e-007
Iteration:35 Error:1.949e-008
Iteration:37 Error:7.539e-009
Xstar=X15=3.102
```

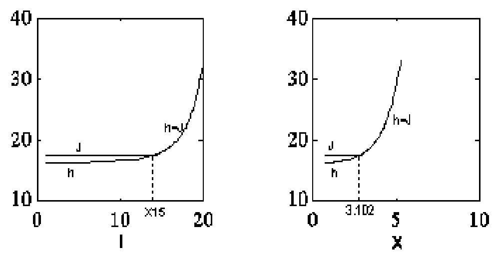

Figure 4: 3rd algorithm results.

As expected we have found the same solution as with the previous methods. This method is probably the one conceptually most simple, resulting in a more clear implementation. Therefore, it can be more easily expanded to cover more general Markov processes or additional problem features.

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