OPTIMAL CONTROL PROBLEMS WITH NONREGULAR MIXED CONSTRAINTS [1]

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Abstract: In this paper we derive necessary conditions for a simple class of mixed constrained optimal control problems that do not satisfy known regularity conditions on the mixed constraints. Such problems are a particular class of nonregular problems. Notably the necessary conditions we derive are stated in terms of measures.

Keywords: Optimal control, mixed constraints, nonsmooth maximum principle.

1. INTRODUCTION

Consider the following mixed constrained optimal control problem:

$$
(P)\begin{cases}
\text{Minimize } l(x(0),x(1)) \\
\text{subject to} \\
\dot{x}(t) = f(t,x(t),u(t)) \quad \text{ae}\\
0 = b(t,x(t),u(t)) \quad \text{ae}\\
0 \geq g(t,x(t),u(t)) \quad \text{ae}\\
(x(0),x(1)) \in C
\end{cases}
$$

where $l:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $C\subset\mathbb{R}^n\times\mathbb{R}^n$ and

$$(f,b,g):[0,1]\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n\times\mathbb{R}^{m_b}\times\mathbb{R}^{m_g}.$$

For (P) define the scalar valued function

$$H(t,x,u,p,q,r) = \tag{1}$$
$$p\cdot f(t,x,u)+q\cdot b(t,x,u)+r\cdot g(t,x,u).$$

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Take $(\bar{x},\bar{u})$ to be a solution to (P). With respect to this process the set $\mathcal{I}_a(t)$ denotes the set of indexes of the *active constraints*, i.e.,

$$\mathcal{I}_a(t) = \tag{2}$$
$$\big\{i\in\{1,\dots,m_g\} \;\big|\; g_i(t,\bar{x}(t),\bar{u}(t))=0\big\},$$

and $q_a(t)$ denotes the cardinality of $\mathcal{I}_a(t)$. Also

$$\nabla_u g^{\mathcal{I}_a(t)}\big(t,\bar{x}(t),\bar{u}(t)\big)\in\mathbb{R}^{q_a(t)\times k}, \tag{3}$$

(if $q_a(t)=0$, then the latter holds vacuously) is the matrix we obtain after removing from $\nabla_u g(t,\bar{x}(t),\bar{u}(t))$ all the rows of index $i\notin\mathcal{I}_a(t)$.

Necessary conditions for smooth problem (P) are well known (see, for example, (Hestenes, 1966; Neustadt, 1976; Schwarzkopf, 1976; Stefani and Zezza, 1996; Milyutin and Osmolovskii, 1998; Dmitruk, 1993; Arutyunov, 2000)). Maximum principles have also been generalized to cover problems with nonsmooth data; see, for example, nonsmooth weak maximum principles, in the spirit of (Clarke, 1983), derived in (Pales and Zeidan, 1994; de Pinho and Ilchmann, 2002; de Pinho, 2003) and strong versions of the nonsmooth maximum principle in (de Pinho *et al.*, 2001) and (Devdariani and Ledyaev, 1996).

Roughly speaking, one may say that when the data of the problem under consideration is assumed to be *continuous with respect to t* necessary conditions have previously been derived under several regularity assumptions the more general being the full rank assumption

$$\det \Upsilon(t)\Upsilon(t)^T \neq 0 \quad \text{for all } t \in [0,1] \tag{4}$$

where

$$\Upsilon(t) = \begin{bmatrix} \nabla_u b(t, \bar{x}(t), \bar{u}(t)) \\ \nabla_u g^{\mathcal{I}_a(t)}(t, \bar{x}(t), \bar{u}(t)) \end{bmatrix}. \tag{5}$$

On the other hand, when the data is assumed to be *merely measurably with respect to t* (a standard assumption for nonsmooth problems), the aforementioned full rank condition (4) on matrix $\Upsilon(t)$ is replaced by the *uniformly full rank condition*

$$\det \Upsilon(t)\Upsilon(t)^T \geq K \quad \text{ae}, \tag{6}$$

where K is some positive constant. See (de Pinho, 2003) in this respect.

Under mild conditions on the data, smooth weak maximum principles for (P) under the previous full rank conditions ((4) or (6)) assert the existence of an absolutely continuous function p, integrable functions q, r and a scalar $\lambda > 0$ such that

a. $\lambda + ||p||_\infty > 0$,

b. $\dot{p}(t) = H_x(t, \bar{x}(t), \bar{u}(t), p(t), q(t), r(t)) \quad \text{ae}$,

c. $0 = H_u(t, \bar{x}(t), \bar{u}(t), p(t), q(t), r(t)) \quad \text{ae}$,

d. $r(t) \cdot g(t, \bar{x}(t), \bar{u}(t)) = 0, \quad r(t) \leq 0 \quad \text{ae}$,

e. $(p(0), -p(1)) \in N_C(\bar{x}(0), \bar{x}(1)) + \lambda \nabla l(\bar{x}(0), \bar{x}(1))$,

f. There exists a $K_1 \in L^1$ such that $|(q(t), r(t))| \leq K_1(t)|p(t)| \quad \text{ae}$.

Here $N_C(\bar{x}(0), \bar{x}(1))$ denotes the *limiting normal cone* of the set C at $(\bar{x}(0), \bar{x}(1))$ (see for example (Vinter, 2000)).

It is of foremost importance to observe that there is no measures in the above conditions.

Problem (P) can still be "regular" even in situations where full rank conditions on $\Upsilon(t)$ are not satisfied. Indeed, the set of necessary conditions a–f for smooth problems or those given by (de Pinho, 2003, Theorem 3.1), for nonsmooth problems, still hold when full rank conditions are replaced by Mangasarian-Fromowitz constraints qualifications, also known as positively linear independence conditions on the mixed constraints (here we refer the reader to (Dmitruk, 1993)and (de Pinho, 2005)). To simplify the exposition we state here only a "uniform" version of such constraints qualifications.

HMF (de Pinho, 2005) There exist constants $K_1 > 0$, $K_2 > 0$ and integrable functions

$$h,\, a : [0,1] \to R^k$$

such that, for almost every $t \in [0,1]$, $|h(t)| = 1$ and

i. $a_i(t) \geq K_2 \quad$ ae for all $i \in \mathcal{I}_a(t)$,
ii. $\nabla_u \bar{g}(t) \cdot h(t) = a(t) \quad$ ae,
iii. $\nabla_u \bar{b}(t) \cdot h(t) = 0 \quad$ ae,
iv. $\det \nabla_u \bar{b}(t) \nabla_u \bar{b}(t)^T \geq K_1, \quad$ ae.

It is a simple matter to see that the uniform full rank (6) implies HMF whereas the opposite implication is not in general true.

In this paper we concentrate on necessary conditions for mixed constrained problems in the form of (P) when Mangasarian-Fromowitz constraints qualifications are not satisfied. Those are problems we call nonregular. Since derivation of necessary conditions for nonregular problems remains a largely unexplored subject (see (Dmitruk, 1993)), our aim is to give some insight on the necessary conditions for such problems.

We do not consider general nonregular problems. Rather we focus attention on a simple class of mixed constrained problems for which matrix $\Upsilon(t)$ looses rank. The problem of interest is a particular case of the following linear problem

$$(L) \quad \begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{subject to} \\ \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{a.e.} \\ \qquad 0 \geq Dx(t) + Eu(t) \quad \text{a.e. } t \\ (x(0), x(1)) \in C \end{cases}$$

where l and C are as defined before, A and B are matrix valued functions where $A(t) \in \mathcal{M}_{n\times n}$ and $B(t) \in \mathcal{M}_{n\times k}$, D and E are constant matrices such that $D \in \mathcal{M}_{m\times k}$ and $E \in \mathcal{M}_{m\times k}$, with $k \geq m$. Here $\mathcal{M}_{p\times q}$ denotes the set of real $p \times q$ matrices.

We show that when some hypotheses on matrix E are satisfied problems in the form of (L) are nonregular and additionally, they can be reduced to problems with pure state constraints. Such reduction permit the derivation necessary conditions for (L). Notably, such necessary conditions differ from those given by a–f above since they involve **measures**.

Essential in the forthcoming development is the generalization of a weak nonsmooth maximum principle recently established in (de Pinho *et al.*, 2005) for optimal control problems with pure state constraints.

Although our main result in this paper, Theorem 3.1 below, can be easily generalized to cover more general problems (for example, problems with nonlinear dynamics of the form $\dot{x}(t) = f(t, x(t), u(t))$) we opt to treat here the simpler problem (L). Indeed, the special structure of (L) allows us, at the same time, to simplify the exposition and to highlight the special features of nonregular problems.

This paper is organized in the following way. In the next section we give some essential definitions and, for completeness, we state a generalization of a weak nonsmooth maximum principle obtained in (de Pinho *et al.*, 2005) to cover problems where the pure state constraints can be vector-valued. In section 3, we state and discuss our main result, the proof of which is presented in the last section.

2. PRELIMINARIES

Here and throughout, B represents the closed unit ball centered at the origin and $|\cdot|$ the Euclidean norm or the induced matrix norm on $\mathbb{R}^{m\times k}$. The *Euclidean distance function* with respect to $A \subset \mathbb{R}^k$ is

$$d_A : \mathbb{R}^k \to \mathbb{R}, \qquad y \mapsto d_A(y) = \inf\{|y-x| : x \in A\}.$$

Here the linear space $W^{1,1}([0,1];\mathbb{R}^p)$ denotes the space of absolutely continuous functions, $L^1([0,1];\mathbb{R}^p)$ the space of integrable functions and $L^\infty([0,1];\mathbb{R}^p)$ the space of essentially bounded functions from $[0,1]$ to $\mathbb{R}^p$.

Crucial to all the analysis is the following result concerning the singular value decomposition of a matrix:

Theorem 2.1. (Horn and Jonhson, 1985, Theorem 7.3.5) If $E \in \mathcal{M}_{m\times k}$ and has rank m_e ($m_e \le \min\{m,k\}$), then there exist $V \in \mathcal{M}_{m\times m}$, $W^T \in \mathcal{M}_{k\times k}$, $\Sigma \in \mathcal{M}_{m\times k}$ and $\Lambda \in \mathcal{M}_{m_e\times m_e}$ such that

$$E = V\Sigma W^T,$$

where V and W are unitary,

$$\Sigma = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda = \operatorname{diag}\{\sigma_{ii}\}_{i=1}^{m_e}$$

with $\sigma_{ii} > 0$.

Recall that if a matrix U is unitary, then $U^TU = I$ and $U^{-1} = U^T$.

Consider now the following optimal control problem with vector valued pure state constraints:

$$(S)\quad \begin{cases} \text{Minimize } l(x(0), x(1)) \\ \text{subject to} \\ \qquad \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \\ \qquad\quad 0 \ge h(t, x(t)) \quad \text{for all } t \\ (x(0), x(1)) \in C \end{cases}$$

where l, $f$and C are as defined before and $h : [0,1]\times\mathbb{R}^n \to \mathbb{R}^{m_h}$, with $m_h \ge 1$.

Next we focus on (S) with possibly nonsmooth data (a standard procedure since the publication of the seminal book (Clarke, 1983)). Thus various concepts from nonsmooth analysis will be mentioned. Among those are the *limiting normal cone to* a set C, N_C, the *limiting subdifferential* of f, $\partial f(x)$, and the Clarke subdifferential, $\operatorname{co}\partial f$. The concepts of limiting normal cone and limiting subdifferential as well as the full calculus for these constructions in finite dimensions is described in, for example, (Rockafellar and Wets, 1998; Vinter, 2000; Mordukhovich, 2006).

In the case that the function f is Lipschitz continuous near x, the convex hull of the limiting subdifferential, $\operatorname{co}\partial f(x)$, coincides with the *Clarke subdifferential*, which may be defined directly (see (Clarke, 1983)).

For (P) or (S) a process is a pair (x,u) comprising a function $x \in W^{1,1}([0,1];\mathbb{R}^n)$ and a measurable function $u : [0,1] \to \mathbb{R}^k$ satisfying the constraints. Take a process $(\bar{x}(t), \bar{u}(t))$ to (S) or (P) and a parameter $\varepsilon > 0$. Define the set

$$T_\varepsilon(t) = \{(x,u) : x \in \bar{x}(t) + \varepsilon B,\ u \in \bar{u}(t) + \varepsilon B\}.$$

Let $\bar{f}(t)$, $\bar{b}(t)$, $\bar{g}(t)$ etc, denote the corresponding function evaluated at $(t, \bar{x}(t), \bar{u}(t))$.

A process $(\bar{x}, \bar{u})$ is a *weak minimizer* to (P) or (S) if there exists some $\varepsilon > 0$, such that it minimizes the cost over all processes (x,u) which satisfy

$$(x(t), u(t)) \in T_\varepsilon(t) \quad \text{a.e.}$$

We now focus on (S). The following hypotheses, which make reference to a parameter $\varepsilon > 0$, are imposed:

H1. The function $t \to f(t,x,u)$ is Lebesgue measurable for each pair (x,u) and there exists a function K_f in $L^1([0,1];\mathbb{R})$ such that

$$\left|f(t,x,u) - f(t,x',u')\right| \le K(t)\,|(x,u) - (x',u')|$$

for $(x,u), (x',u') \in T_\varepsilon(t)$ ae.

H2. The endpoint constraint set C is closed and l is locally Lipschitz in a neighborhood of $(\bar{x}(0), \bar{x}(1))$.

H3. For $x \in \bar{x}(t) + \varepsilon B$ the function $t \to h(t,x)$ is continuous and there exists a scalar $K_h > 0$ such that $x \to h(t,x)$ is Lipschitz of rank K_h for all $t \in [0,1]$.

For (S) define the function

$$H_S(t,x,p,u) = p \cdot f(t,x,u).$$

The following theorem is a straightforward generalization of Unmaximized Hamiltonian type conditions for optimal problems with pure state constraints, proved in (de Pinho *et al.*, 2005). The proof consists on considering a scalar functional

$$h^+(t,x) = \max\{h_j(t,x) : j \in \{1,\ldots,m_h\}\}$$

following the approach in (Vinter, 2000).

Theorem 2.2. Let $(\bar{x},\bar{u})$ be a weak minimizer to problem (S). Assume that H1–H3 are satisfied for some parameter ε. Then there exists an absolutely continuous function $p : [0,1] \mapsto \mathbb{R}^n$, integrable function $\gamma_j : [0,1] \to \mathbb{R}^n$, for $j = 1,\ldots,m_h$, nonnegative Radon measures $\mu_j \in C^*([0,1],\mathbb{R})$, $j = 1,\ldots,m_h$, and a scalar $\lambda \geq 0$ such that

i. $\displaystyle\sum_{j=1}^{m_h} \mu_j\{[0,1]\} + ||p||_\infty + \lambda > 0,$

ii. $(-\dot{p}(t),0) \in \operatorname{co}\partial_{x,u}H(t,\bar{x}(t),\pi(t),\bar{u}(t))$ a.e.

iii. $(p(0),-\pi(1)) \in N_C(\bar{x}(0),\bar{x}(1)) + \lambda\partial l(\bar{x}(0),\bar{x}(1)),$

where

$$\operatorname{supp}\{\mu_j\} \subset \{t \in [0,1] : h_j(t,\bar{x}(t)) = 0\},$$

π is defined as

$$\pi(t) = \begin{cases} p(t) + \displaystyle\int_{[0,t)} \sum_{j=1}^{m_h} \gamma_i(\tau)\mu_j(d\tau), & t \in [0,1) \\ p(t) + \displaystyle\int_{[0,1]} \sum_{j=1}^{m_h} \gamma_i(\tau)\mu_j(d\tau), & t = 1. \end{cases}$$

and

$$\gamma_j(t) \in \partial_x^{>} h_j(t,\bar{x}(t))\ \mu_j\text{-a.e.}$$

with

$$\partial_x^{>} h_j(t,x) = \operatorname{co}\Big\{\xi : \exists (t_i,x_i) \xrightarrow{h} (t,x) : h_j(t_i,x_i) > 0 \quad \forall i,\ \nabla_x h_j(t_i,x_i) \to \xi\Big\}.$$

3. MAIN RESULT

We shall invoke the following set of hypotheses on (L):

AH1. For almost every $t \in [0,1]$ we have

$$\mathcal{I}_a(t) = \{1,\ldots,m\}.$$

AH2. The matrix E has rank m_e where $1 \geq m_e < m$. Writing $E = \begin{bmatrix} E_e \\ E_s \end{bmatrix}$, then $E_e \in \mathcal{M}_{m_e\times k}$ is of full rank and there exists a matrix $Q \in \mathcal{M}_{(m-m_e)\times m_e}$ such that $E_s = -QE_e$.

AH3. The matrix valued function $t \to (A(t),B(t))$ is integrable.

Observe that in AH1 we assume that *all the constraints are active*. It follows that $E = E^{\mathcal{I}_a(t)}$. Hypothesis AH2 asserts that E is **not** of full rank. Since we also assume the existence of a matrix Q such that $E_s = -QE_e$ it follows that HMF is not satisfied. Thus, when the data of (L) satisfies AH1–AH2, (L) is a *nonregular* problem. In this respect it is important to observe that the rank of E is assumed to be greater or equal than 1.

Taking into account the partition of matrix E in AH2, write $D = \begin{bmatrix} D_e \\ D_s \end{bmatrix}$ and $B(t) = [B_e(t)\,|\,B_s(t)]$, where $D_e \in \mathcal{M}_{m_e\times n}$, $B_e(t) \in \mathcal{M}_{n\times m_e}$, $D_s \in \mathcal{M}_{(m-m_e)\times n}$ and $B_s(t) \in \mathcal{M}_{n\times(k-m_e)}$.

In what follows

$$\hat{D} = QD_e + D_s.$$

Theorem 3.1. Let $(\bar{x},\bar{u})$ be a weak minimizer to (L). If hypotheses H2, AH1, AH2 and AH3 are satisfied, then there exist an absolutely continuous function $p : [0,1] \mapsto \mathbb{R}^n$, integrable function $r : [0,1] \to \mathbb{R}^{m_e}$, nonnegative Radon measures $\mu_j \in C^*([0,1],\mathbb{R})$, with $j = 1,\ldots,m-m_e$, and a scalar $\lambda \geq 0$ such that

i. $\displaystyle\sum_{j=1}^{m-m_e} \mu_j\{[0,1]\} + ||p||_\infty + \lambda > 0,$

ii. $-\dot{p}(t) = \pi(t)\cdot A(t) + r(t)\cdot D_e$ a.e.

iii. $0 = \pi(t)\cdot B(t) + r(t)\cdot E_e$ a.e.

iv. $(p(0),-\pi(1)) \in \lambda\partial l(\bar{x}(0),\bar{x}(1)) + N_C(\bar{x}(0),\bar{x}(1)),$

where

$$\operatorname{supp}\{\mu_j\} \subset \Big\{t \in [0,1] : \hat{d}_j\bar{x}(t) = 0\Big\}, \tag{7}$$

and π is defined as

$$\pi(t) = \begin{cases} p(t) + \displaystyle\int_{[0,t)} \sum_{j=1}^{m-m_e} \hat{d}_j\mu_j(d\tau), & t \in [0,1) \\ p(t) + \displaystyle\int_{[0,1]} \sum_{j=1}^{m-m_e} \hat{d}_j\mu_j(d\tau), & t = 1. \end{cases} \tag{8}$$

with $\hat{d}_j$ being the j-th row of $\hat{D}$.

A remarkable difference between the conclusions of Theorem 3.1 and the necessary conditions given by a–f above is that r, the multiplier associated with mixed constraints, is now defined by (iii) in terms of the bounded variation function π and not in terms of the absolutely continuous function p.

Although AH1–AH2 are strong assumptions on the data of (L) they nevertheless are of interest since they permit the illustration of some special features of nonregular problems in a simple and straightforward way. We emphasize the fact that necessary conditions analogous as those given by the above theorem can be derived when AH1-AH2 are replaced by weaker assumptions. Theorem 3.1 holds when, for example, when the constraints are active in subinterval of $[0,1]$ and not in the whole interval. However the proof of this Theorem would be much more demanding. Also, and as stated in the Introduction, Theorem 3.1 can easily be generalized to cover problems with nonsmooth dynamics of the form $\dot{x}(t) = f(t, x(t), u(t))$.

Nondegeneracy of the necessary conditions given by Theorem 3.1 shall be treated in future work.

4. PROOF OF MAIN RESULT

Now we prove Theorem 3.1. We start by proving that (L) can be associated with a problem in the form of (S). In this respect AH2 is crucial.

Consider the full rank matrix E_e. Theorem 2.1 asserts the existence of unitary matrices V_e and W_e, of real positive numbers σ_{ii}, $i \in \{1,\dots,m_e\}$ such that

$$E_e = V_e \,[\Lambda_e \,|\, 0]\, W_e^T,$$

where

$$\Lambda = \text{diag}\,\{\sigma_{ii}\}_{i=1}^{m_e}.$$

Take any vector $u \in \mathbb{R}^k$ and define $v = W_e^T u \in R^k$. Write

$$v = \begin{bmatrix} v_e \\ v_s \end{bmatrix},$$

where $v_s \in \mathbb{R}^{k-m_e}$. Likewise write $\bar{v}(t) = W_e^T \bar{u}(t)$.

Set also

$$\begin{aligned} \left[\hat{B}_e(t) \,|\, \hat{B}_s(t)\right] &= B(t)W_e \\ \hat{A}(t) &= A(t) - \hat{B}_e(t)\Lambda_e^{-1} V_e^T D_e, \\ \hat{D} &= QD_e + D_s. \end{aligned}$$

Proposition 4.1. If $(\bar{x}, \bar{u})$ is a weak minimizer to (L), then $(\bar{x}, \bar{v}_s)$, is a weak minimizer for

$$(L_{aux}) \quad \begin{cases} \text{Minimize } l(x(1)) \\ \text{subject to} \\ \dot{x}(t) = \hat{A}(t)x(t) + \hat{B}_s(t)v_s(t) & \text{ae} \\ 0 \geq \hat{D}x(t) & \text{for all } t \\ (x(0), x(1)) \in C \end{cases}$$

Proof. It is a simple matter to see that if $(\bar{x}, \bar{u})$ solves (L), then $(\bar{x}, \bar{v}_e)$ is admissible to (L_{aux}). Indeed, we have

$$D\bar{x}(t) + E\bar{u}(t) = 0 \iff \begin{cases} D_e\bar{x}(t) + E_e\bar{u}(t) = 0 \\ D_s\bar{x}(t) + E_s\bar{u}(t) = 0 \end{cases}$$

for almost every $t \in [0,1]$. Premultiply the first equation of the above system by Q. This, together with AH2, yields

$$QD_e\bar{x}(t) = -QE_e\bar{u}(t) = E_s(t)\bar{u}(t).$$

Thus

$$\begin{cases} D_e\bar{x}(t) + E_e\bar{u}(t) = 0 & \text{a.e.} \\ \hat{D}\bar{x}(t) = 0 & \text{a.e.} \end{cases}$$

But $\hat{D}$ is a constant matrix and $\bar{x}$ is absolutely continuous. So the equality $\hat{D}\bar{x}(t) = 0$ holds for all $t \in [0,1]$.

Now consider $\bar{v}(t) = W_e^T \bar{u}(t)$. Observe that

$$\begin{aligned} &D_e\bar{x}(t) + E_e\bar{u}(t) \\ &= D_e\bar{x}(t) + V_e\,[\Lambda_e\,|0]\,W_e^T\bar{u}(t) \\ &= D_e\bar{x}(t) + V_e\,[\Lambda_e\,|0]\,\bar{v}(t) \\ &= 0 \end{aligned}$$

Premultiply this last equation by V_e^T we deduce that

$$V_e^T D_e\bar{x}(t) + \Lambda_e\bar{v}_e(t) = 0 \quad \text{a.e.}$$

It follows that

$$\bar{v}_e(t) = -\Lambda_e^{-1} V_e^T D_e\bar{x}(t).$$

We also have

$$\begin{aligned} &A(t)\bar{x}(t) + B(t)\bar{u}(t) \\ &= A(t)\bar{x}(t) + B(t)W_e\bar{v}(t) \\ &= A(t)\bar{x}(t) + \hat{B}_e(t)\bar{v}_e(t) + \hat{B}_s(t)\bar{v}_s(t) \\ &= (A(t) - \hat{B}_e(t)\Lambda_e^{-1}V_e^T D_e)\bar{x}(t) + \hat{B}_s(t)\bar{v}_s(t). \end{aligned}$$

We conclude from the above that

$$\dot{\bar{x}}(t) = \hat{A}(t)\bar{x}(t) + \hat{B}_s(t)\bar{v}_s(t).$$

It follows that $(\bar{x}, \bar{v}_s)$ is admissible to (L_{aux}).

Suppose now that $(\tilde{x}, \tilde{v}_s)$ is an admissible solution to (L_{aux}) with lesser cost. Set

$$\tilde{v}_e(t) = -\Lambda_e^{-1} V_e^T D_e\tilde{x}(t), \qquad \tilde{v}(t) = \begin{bmatrix} \tilde{v}_e(t) \\ \tilde{v}_s(t) \end{bmatrix}$$

and define

$$\tilde{u}(t) = W_e\tilde{v}(t).$$

As it can easily be verified we have

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t) \quad \text{a.e.}$$

Also

$$\begin{aligned} &V_e^T D_e\tilde{x}(t) + \Lambda_e\tilde{v}_e(t) \\ &= V_e^T D_e\tilde{x}(t) + [\Lambda_e\,|\,0]\,\tilde{v}(t) \\ &= V_e^T D_e\tilde{x}(t) + [\Lambda_e\,|\,0]\,W_e^T\tilde{u}(t) \\ &= 0. \end{aligned}$$

Premultiplying this equation by V_e we get

$$D_e\tilde{x}(t) + E_e\tilde{u}(t) = 0.$$

Again premultiply this last equation by Q. Appealing to AH2 we deduce that

$$QD_e\tilde{x}(t) = E_s\tilde{u}(t).$$

Taking into account the definition of $\hat{D}$ we now deduce that

$$0 \geq \hat{D}\tilde{x}(t) = (QD_e + D_s)\tilde{x}(t) = D_s\tilde{x}(t) + E_s\tilde{u}(t).$$

We conclude that

$$\begin{aligned} D_e\tilde{x}(t) + E_e\tilde{u}(t) &= 0 \\ D_s\tilde{x}(t) + E_s\tilde{u}(t) &\leq 0 \end{aligned}$$

that is, $(\tilde{x}, \tilde{u})$ is an admissible solution to (L) with lesser cost, contradicting the optimality of $(\bar{x}, \bar{u})$. □

It is an easy task to verify that (L_{aux}) is an optimal control with pure state constraints satisfying the conditions under which Theorem 2.2 holds. Application of Theorem 2.2 asserts the existence of an absolutely continuous function $p : [0,1] \mapsto \mathbb{R}^n$, nonnegative Radon measures $\mu_j \in C^*([0,1], \mathbb{R})$, with $j = 1, \ldots, m - m_e$, and a scalar $\lambda \geq 0$ such that (i) and (7) of the Theorem holds with π as defined in (8). Moreover

$$\begin{aligned} -\dot{p}(t) &= \pi(t) \cdot \hat{A}(t) \\ &= \pi(t) \cdot A(t) - \pi(t) \cdot \hat{B}_e(t)\Lambda_e^{-1}V_e^T D_e \\ 0 &= \hat{B}_s(t) \cdot \pi(t) \end{aligned}$$

Define the vector valued function $r : [0,1] \to \mathbb{R}^{m_e}$ by

$$\begin{bmatrix} r(t) \\ 0 \end{bmatrix} = - \begin{bmatrix} V_e & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Lambda_e^{-1} & 0 \\ 0 & I \end{bmatrix} W_e^T B^T(t)\pi(t)$$

where $\begin{bmatrix} r(t) \\ 0 \end{bmatrix} \in \mathbb{R}^k$.

Then (ii) and (iii) follows. The proof is complete.

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