



Statistical properties for systems with weak invariant manifolds

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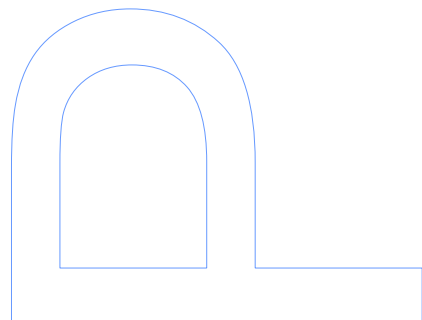
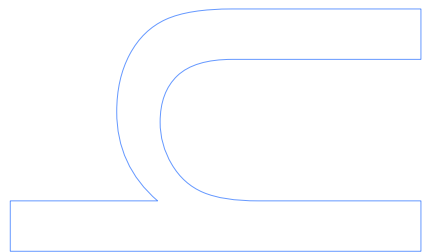
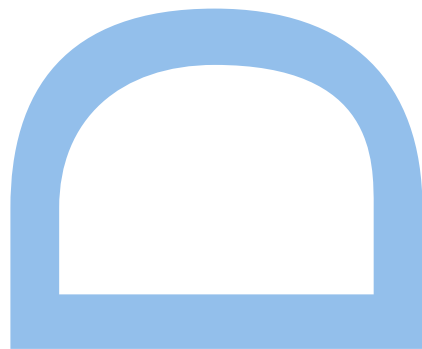
Statistical properties for systems with weak invariant manifolds

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Resumo

Nesta tese consideramos um sistema dinâmico $f : M \rightarrow M$, onde M é uma variedade Riemanniana e f é um difeomorfismo. Supõe-se que o sistema dinâmico tem uma estrutura Gibbs-Markov-Young, que consiste num conjunto de referência Λ com uma estrutura de produto hiperbólico que satisfaz certas propriedades. As propriedades que assumimos são a existência de uma partição de Markov $\Lambda_1, \Lambda_2, \dots$ de Λ , contração polinomial em folhas estáveis, contração polinomial para trás em folhas instáveis, uma propriedade de distorção limitada e uma certa regularidade da folheação estável.

Os objetivos principais desta tese consistem em provar resultados que estabelecem um controlo do decaimento de correlações e dos grandes desvios, bem como apresentar um exemplo de um sistema dinâmico que tem a estrutura Gibbs-Markov-Young descrita acima. Para podermos apresentar estes teoremas, precisamos primeiro de introduzir o conceito de tempo de retorno a Λ . Estes resultados fornecem controlos polinomiais do decaimento de correlações e dos grandes desvios, baseados no controlo polinomial da medida de Lebesgue da cauda do tempo de retorno.

Ferramentas essenciais para provar os teoremas principais são uma torre de Young, bem como uma torre quociente e uma torre produto, obtidas a partir da torre de Young. Recordamos alguns resultados sobre estas torres e provamos outros a partir das nossas propriedades.

Finalmente, apresentamos um exemplo de um sistema dinâmico definido no toro e provamos que este verifica todas as propriedades da estrutura Gibbs-Markov-Young considerada.

Abstract

In this thesis we consider a discrete dynamical system $f : M \rightarrow M$, where M is a Riemannian manifold and f is a diffeomorphism. We assume that the dynamical system has a Gibbs-Markov-Young structure, which consists of a reference set Λ with a hyperbolic product structure that satisfies certain properties. The properties assumed here are the existence of a Markov partition $\Lambda_1, \Lambda_2, \dots$ of Λ , polynomial contraction on stable leaves, polynomial backwards contraction on unstable leaves, a bounded distortion property and a certain regularity of the stable foliation.

The main goals of this thesis are to prove results establishing a control on the decay of correlations and large deviations, as well as presenting an example of a dynamical system satisfying the Gibbs-Markov-Young structure described above. In order to state these theorems, first we need to introduce the concept of return time to Λ . These results give polynomial controls on the decay of correlations and large deviations based on a polynomial control on the Lebesgue measure of the tail of the return time.

Essential tools to prove the main theorems are a Young tower, as well as a quotient tower and a tower product obtained from the Young tower. We recall some results about these towers and prove some others based on our properties.

Finally, we present an example of a dynamical system defined on the torus and we prove that it verifies all the properties of the Gibbs-Markov-Young structure that we considered.

Introduction

In this thesis we consider a discrete dynamical system $f : M \rightarrow M$, where M is a manifold and f is a diffeomorphism. If μ is an *invariant probability measure*, we say that f is *mixing* with respect to this measure if, for any measurable sets A and B , we have

$$\mu(f^{-n}(A) \cap B) - \mu(A)\mu(B) \xrightarrow[n]{} 0.$$

This setting will allow us to study some statistical properties of the dynamical system.

Given observables $\varphi, \psi : M \rightarrow \mathbb{R}$, their *correlation* is given by

$$\mathcal{C}_n(\varphi, \psi, \mu) = \left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Note that saying that f is mixing is equivalent to assuming that, for all measurable sets A and B , the correlation of the characteristic functions of A and B converges to zero. For sufficiently regular observables φ and ψ and suitable assumptions on the dynamical system, it is possible to obtain a control on the rate of decay of their correlation.

A special case that is important to study is dynamical systems with *SRB measures*, also known as *physical measures*. These measures were introduced by Bowen, Ruelle and Sinai [18, 7, 16] and their importance can be understood by recalling Birkhoff's Ergodic Theorem. It states that, if μ is an invariant probability measure, then, for μ almost every $x \in M$ and all continuous $\varphi : M \rightarrow \mathbb{R}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \xrightarrow[n]{} \int \varphi d\mu.$$

However, this theorem does not give any guarantee on the Lebesgue measure of the set where the above statement holds. SRB measures have the important property that this set has positive Lebesgue measure. In this thesis we will be working with a dynamic system with a SRB measure.

Given an observable $\phi : M \rightarrow \mathbb{R}$, the study of *large deviations* consists on obtaining a control on how much $\phi_n = \sum_{i=0}^{n-1} \phi \circ f^i$ deviates from the mean $\bar{\phi} = \int \phi d\mu$. More precisely, the aim is to study the asymptotic behaviour of

$$\mu \left\{ \left| \frac{1}{n} \phi_n - \bar{\phi} \right| > \varepsilon \right\}.$$

Several authors have studied rates of decay of correlations in different contexts. Bowen, Ruelle and Sinai [18, 7, 16] obtained exponential decay of correlations for uniformly hyperbolic diffeomorphisms. Later, some classes of non-uniformly hyperbolic diffeomorphisms were considered. First, Young [21] proved an exponential rate for the decay of correlations assuming there exists a reference set $\Lambda \subseteq M$ with a *hyperbolic product structure* and, among other properties, an exponential contraction along stable leaves and exponential backward contraction on unstable leaves. Later, Alves and Pinheiro [2] weakened these assumptions, removing the backward contraction but still imposing an exponential contraction along stable leaves. In that paper, they proved exponential or polynomial decay of correlations, depending on different hypothesis that we will explain later. In this thesis, we consider a dynamical system with similar properties as in [21, 2]. However, we only assume polynomial contraction on stable leaves and backward polynomial contraction on unstable leaves, consequently obtaining a polynomial decay of correlations. In addition, Young also obtained, in [22], a control on the rate of decay of correlations for non-invertible dynamical systems and, together with Benedicks in [6], for Hénon maps.

Many authors have proved results on large deviations for uniformly hyperbolic dynamical systems, some of which can be found in [15, 9, 10, 20, 19]. Later, Araújo and Pacifico, in [5], studied large deviations for certain classes of non-uniformly expanding maps and partially hyperbolic non-uniformly expanding diffeomorphisms. In [4], Araújo extended these results to a more general case. Melbourne and Nicol, in [14], obtained a control on large deviations for non-uniformly hyperbolic systems that verify certain properties, including exponential contraction on stable leaves and exponential backward contraction on unstable leaves. In [12], Melbourne obtained a slightly better result for large deviations. We obtain a similar result as in [14] and [12], but with weaker conditions on the dynamical system.

In this thesis we make extensive use of the framework developed by Young in [21, 22] and also used in [2]. The important tools that we use are a *Gibbs-Markov-Young structure*

defined in a reference set Λ , the *return time* to that reference set and a *Young tower* over the same set, this last one having been introduced by Young in [21].

The Gibbs-Markov-Young structure consists of a reference set Λ with a hyperbolic product structure that satisfies certain properties. The properties considered in each article may vary, but the ones we consider are a Markov partition $\Lambda_1, \Lambda_2, \dots$ of Λ , polynomial contraction on stable leaves, polynomial backwards contraction on unstable leaves, a bounded distortion property and a certain regularity of the stable foliation.

The Markov partition allows us to define a return map that, in each Λ_i , is an iterate of f such that all the points of Λ_i return to Λ by this map. The return time function is, in each Λ_i , the number of iterates until this return happens.

A Young tower is a new dynamical system which is defined based on the original one and making use of the return time function. The way this tower is defined means that we can study this new dynamical system and transfer most of the information obtained to the original dynamical system. We will also see that it is sufficient to study the dynamical system in a quotient tower obtained by identifying the points in the same stable leaf. For technical reasons, we are also going to consider a tower product and a simultaneous return time function.

Using the tower structure, it is possible, under certain conditions, to obtain a relation between the measure of the tail of the return time function and both the decay of correlations and large deviations. Young, in [21], for systems with exponential behaviour in stable and unstable leaves, proved exponential decay of correlations when the measure of the tail of the return time decreases exponentially. In [22], Young also proved, for non-invertible systems, both polynomial and exponential decay of correlations based, respectively, on polynomial and exponential control on the tail of the return time. Alves and Pinheiro, in [2], extended the result of [21] to a more general case, obtaining, in addition to the exponential decay of correlations, a polynomial decay of correlations assuming a polynomial return time. As for the large deviations, Melbourne and Nicol, in [14], also obtained exponential and polynomial control of large deviations, with the corresponding hypothesis on the tail of the return time. In this thesis, as we are only assuming a polynomial behaviour in the stable and unstable leaves, we could not expect any exponential results. In fact, we obtain a polynomial control on both the decay of correlations and large deviations, from a polynomial control of the tail of the return time.

The goal of Chapter 1 is to present the two main results of this thesis, which give a control on the decay of correlations and large deviations for a certain class of dynamical systems. In Section 1.1 we present some basic concepts and introduce the Gibbs-Markov-Young structure. This structure consists of the existence of a reference set Λ and the assumption of certain properties, namely, a Markov partition for that set, polynomial contraction on stable leaves, polynomial backward contraction on unstable leaves, a bounded distortion property and a certain regularity of the stable foliation. In this section we also define the concept of return time, which will be useful throughout this thesis. In Section 1.2 we introduce the first of the main theorems which states that we can obtain polynomial decay of correlations from the condition of a polynomial return time to Λ . Section 1.3 states the other main result, which gives a polynomial control on large deviations as long as we have a polynomial return time to Λ .

Chapter 2 is concerned with the tower structure, a tool that will be essential in the proofs of the main theorems. In Section 2.1 we introduce the Young tower, and present a result on the diameter control of elements of a certain partition. In Section 2.2 we define a quotient tower obtained from the tower by identifying points in the same stable leaf and state a result that will be proved in Section 2.4. In Section 2.3 we introduce a tower product, necessary to prove some of the results contained in Section 2.4.

Chapter 3 is divided in two sections, each one dedicated to the proof of one of the main theorems. In section 3.2 we use the result proved in Section 2.4.

In Chapter ?? we present an example that is obtained by a perturbation of an Anosov diffeomorphism on the torus, creating a point where the function has derivative one in both the stable and unstable directions. We prove that this example satisfies the properties of the Gibbs-Markov-Young structure defined in Chapter 1 and that the return time function to a certain reference set has a polynomial behaviour. This implies, using the main results from Chapter 1, that, for the example, there is a polynomial control on the decay of correlations and large deviations.

In the appendix we present some basic definitions.

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Chapter 1

Statement of results

Let M be a finite dimensional Riemannian compact manifold, Leb be the Lebesgue measure on the Borel sets of M and consider a diffeomorphism $f : M \rightarrow M$. Given a submanifold γ of M , let Leb_γ denote the measure on γ induced by the restriction of the Riemannian form to γ .

In this chapter we introduce a Gibbs-Markov-Young structure, which consists of assuming that f satisfies some properties on the stable and unstable leaves. We also state our two main theorems, one about the decay of correlations and the other about large deviations.

1.1 Gibbs-Markov-Young structures

In this section we introduce a hyperbolic product structure on a subset $\Lambda \subseteq M$. We impose that Λ has a Markov partition and that f satisfies polynomial forward or backward contraction on stable or unstable leaves, respectively. Additionally, we require a bounded distortion property on unstable leaves and some regularity of the stable foliation. We would like to point out that we are not assuming any kind of exponential contraction, forwards or backwards.

We start by introducing the necessary concepts in order to define a hyperbolic product structure.

Definition 1.1. An embedded disk $\gamma \subseteq M$ is said to be an *unstable manifold* if, for every

$x, y \in \gamma$,

$$d(f^{-n}(x), f^{-n}(y)) \xrightarrow[n]{} 0.$$

Analogously, an embedded disk $\gamma \subseteq M$ is a *stable manifold* if, for every $x, y \in \gamma$,

$$d(f^n(x), f^n(y)) \xrightarrow[n]{} 0.$$

Definition 1.2. We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable manifolds* if there is a compact set K^s , a unit disk D^u of some \mathbb{R}^n and a map $\phi^u : K^s \times D^u \rightarrow M$ such that:

- (a) $\gamma^u = \phi^u(\{x\} \times D^u)$ is an unstable manifold;
- (b) ϕ^u maps $K^s \times D^u$ homeomorphically onto its image;
- (c) $x \rightarrow \phi^u|_{\{x\} \times D^u}$ defines a continuous map from K^s into $\text{Emb}^1(D^u, M)$, where $\text{Emb}^1(D^u, M)$ denotes the space of C^1 embeddings from D^u into M .

Continuous families of C^1 stable manifolds are defined similarly.

We can now define a hyperbolic product structure in a subset Λ of M .

Definition 1.3. We say that $\Lambda \subseteq M$ has a *hyperbolic product structure* if there exists a continuous family of stable manifolds $\Gamma^s = \{\gamma^s\}$ and a continuous family of unstable manifolds $\Gamma^u = \{\gamma^u\}$ such that:

- (a) $\Lambda = (\bigcup \gamma^s) \cap (\bigcup \gamma^u)$;
- (b) $\dim \gamma^s + \dim \gamma^u = \dim M$;
- (c) each γ^s intersects each γ^u in exactly one point;
- (d) stable and unstable manifolds are transversal with angles bounded away from 0.

From now on, we consider $\Lambda \subseteq M$ to have a hyperbolic product structure, with Γ^s and Γ^u as their defining families.

Definition 1.4. A subset $\Lambda_1 \subseteq \Lambda$ is called an *s-subset* if Λ_1 also has a hyperbolic product structure and its defining families Γ_1^s and Γ_1^u can be chosen with $\Gamma_1^s \subseteq \Gamma^s$ and $\Gamma_1^u = \Gamma^u$. A subset $\Lambda_2 \subseteq \Lambda$ is called a *u-subset* if Λ_2 also has a hyperbolic product structure and its defining families Γ_2^s and Γ_2^u can be chosen with $\Gamma_2^s = \Gamma^s$ and $\Gamma_2^u \subseteq \Gamma^u$.

Given $x \in \Lambda$, denote by $\gamma^*(x)$ the element of Γ^* containing x , for $*$ $\in \{s, u\}$. For each $n \geq 1$ denote by $(f^n)^u$ the restriction of the map f^n to γ^u -disks, and by $\det D(f^n)^u$ the Jacobian of $(f^n)^u$.

Definition 1.5. Let Λ have a hyperbolic product structure. We say that Λ has a *Gibbs-Markov-Young (GMY) structure* if the properties (P₀)-(P₅) listed below hold.

(P₀) *Lebesgue detectable*

There exists an unstable manifold $\gamma \in \Gamma^u$ such that $\text{Leb}_\gamma(\Lambda \cap \gamma) > 0$.

(P₁) *Markov partition*

There are pairwise disjoint s -subsets $\Lambda_1, \Lambda_2, \dots \subseteq \Lambda$ such that:

- (a) $\text{Leb}_\gamma((\Lambda \setminus \bigcup_{i=1}^\infty \Lambda_i) \cap \gamma^u) = 0$ on each $\gamma^u \in \Gamma^u$;
- (b) for each $i \in \mathbb{N}$ there exists a $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is an u -subset and, for all $x \in \Lambda_i$,

$$f^{R_i}(\gamma^s(x)) \subseteq \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supseteq \gamma^u(f^{R_i}(x)).$$

For the remaining properties we assume that $C > 0$, $\alpha > 1$ and $0 < \beta < 1$ are constants depending only on f and Λ .

(P₂) *Polynomial contraction on stable leaves*

$$\forall y \in \gamma^s(x) \quad \forall n \in \mathbb{N} \quad d(f^n(x), f^n(y)) \leq \frac{C}{n^\alpha} d(x, y).$$

(P₃) *Backward polynomial contraction on unstable leaves*

$$\forall y \in \gamma^u(x) \quad \forall n \in \mathbb{N} \quad d(f^{-n}(x), f^{-n}(y)) \leq \frac{C}{n^\alpha} d(x, y).$$

We introduce a *return time function* $R : \Lambda \rightarrow \mathbb{N}$ and a *return function* $f^R : \Lambda \rightarrow \Lambda$ defined for each $i \in \mathbb{N}$ as

$$R|_{\Lambda_i} = R_i \quad \text{and} \quad f^R|_{\Lambda_i} = f^{R_i}|_{\Lambda_i}.$$

For $x, y \in \Lambda$, let the *separation time* $s(x, y)$ be defined as

$$s(x, y) = \min \{n \in \mathbb{N}_0 : (f^R)^n(x) \text{ and } (f^R)^n(y) \text{ are in distinct } \Lambda_i\}.$$

(P₄) *Bounded distortion*

For $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$

$$\log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} \leq C\beta^{s(f^R(x), f^R(y))}.$$

(P₅) *Regularity of the stable foliation*

For each $\gamma, \gamma' \in \Gamma^u$, defining

$$\begin{aligned} \Theta_{\gamma', \gamma} : \gamma' \cap \Lambda &\rightarrow \gamma \cap \Lambda \\ x &\mapsto \gamma^s(x) \cap \gamma, \end{aligned}$$

then

(a) Θ is absolutely continuous and

$$\frac{d(\Theta_* \text{Leb}_{\gamma'})}{d \text{Leb}_{\gamma}}(x) = \prod_{n=0}^{\infty} \frac{\det Df^u(f^n(x))}{\det Df^u(f^n(\Theta^{-1}(x)))};$$

(b) denoting

$$u(x) = \frac{d(\Theta_* \text{Leb}_{\gamma'})}{d \text{Leb}_{\gamma}}(x),$$

we have

$$\forall x, y \in \gamma' \cap \Lambda \quad \log \frac{u(x)}{u(y)} \leq C\beta^{s(x, y)}.$$

The properties of f that we present here are related to similar properties defined in [21] and [2]. The main difference here is that we only assume polynomial contraction on stable leaves as opposed to the exponential contraction in those two articles. We will now go into details over what is different about each property.

Property (P₁), about the Markov partition, is the same as in [2] and is an improvement of the corresponding property in [21].

Properties (P₂) and (P₃), polynomial contraction on stable leaves and backwards polynomial contraction on unstable leaves, are an improvement over [21], where exponential contraction is assumed. In [2], there is no backwards contraction assumed. However, that article also imposes exponential contraction on stable leaves.

Properties (P₄) and (P₅) are the same as properties (P₄) and (P₃) in [2]. Our properties (P₄) and (P₅) are different from the ones in [21]. However, as remarked in [2], these properties can be deduced from (P₄) and (P₅) of [21].

1.2 Decay of correlations

In this section we present one of the main results of this thesis, which establishes the decay of correlations.

Definition 1.6. An f -invariant probability measure μ is called a *Sinai-Ruelle-Bowen measure*, or *SRB measure*, if the Lyapunov exponents of f are nonzero μ almost everywhere and the conditional measures on local unstable manifolds are absolutely continuous with respect to the Lebesgue measures on these manifolds.

It was proved in [21, Theorem 1] that if f has a hyperbolic structure Λ such that R is integrable with respect to Leb_γ , for some $\gamma \in \Gamma^u$, then f has some SRB measure μ .

Given $0 < \eta \leq 1$, we define the *space of η -Hölder continuous functions*

$$H_\eta = \{\varphi : M \rightarrow \mathbb{R} : \exists C > 0 \forall x, y \in M \quad |\varphi(x) - \varphi(y)| \leq Cd(x, y)^\eta\}$$

with the seminorm

$$|\varphi|_\eta = \inf \{C > 0 : \forall x, y \in M \quad |\varphi(x) - \varphi(y)| \leq Cd(x, y)^\eta\}.$$

In the special case $\eta = 1$, these functions are called Lipschitz. The space H_η is a Banach space if we consider in H_η the norm

$$\|\cdot\|_\eta = \|\cdot\|_\infty + |\cdot|_\eta.$$

Definition 1.7. Given $n \in \mathbb{N}$, we define the *correlation of observables* $\varphi, \psi \in H_\eta$ as

$$\mathcal{C}_n(\varphi, \psi, \mu) = \left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

The proof of the following Theorem can be found in Section 3.1.

Theorem A. Suppose that f admits a GMY structure Λ with $\gcd\{R_i\} = 1$ for which there are $\gamma \in \Gamma^u$, $\zeta > 1$ and $C_1 > 0$ such that

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C_1}{n^\zeta}.$$

Then, given $\varphi, \psi \in H_\eta$, there exists $C_2 > 0$ such that for every $n \geq 1$

$$\mathcal{C}_n(\varphi, \psi, \mu) \leq C_2 \max \left\{ \frac{1}{n^{\zeta-1}}, \frac{1}{n^{\alpha\eta}} \right\},$$

where $\alpha > 0$ is the constant in (P_2) and (P_3) .

1.3 Large deviations

In this section we present our second main theorem, which establishes a control on large deviations.

Definition 1.8. If μ is an ergodic probability measure and $\varepsilon > 0$, the *large deviation* at time n of the time average of the observable ϕ from its spatial average is given by

$$LD(\phi, \varepsilon, n, \mu) = \mu \left\{ \left| \frac{1}{n} \sum_{i=1}^{n-1} \phi \circ f^i - \int \phi d\mu \right| > \varepsilon \right\}.$$

Theorem B. Suppose that f admits a GMY structure Λ with $\gcd\{R_i\} = 1$ for which there are $\gamma \in \Gamma^u$, $\zeta > 1$ and $C_1 > 0$ such that

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C_1}{n^\zeta}.$$

Then there are $\eta_0 > 0$ and $\zeta_0 = \zeta_0(\eta_0) > 1$ such that for all $\eta > \eta_0$, $1 < \zeta < \zeta_0$, $\varepsilon > 0$, $p > \max\{1, \zeta - 1\}$ and $\phi \in \mathcal{H}_\eta$, there exists $C_2 > 0$ such that for every $n \geq 1$

$$LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p}} \frac{1}{n^{\zeta-1}}.$$

This theorem will be proved in Section 3.2.

Chapter 2

Tower maps

In this chapter we are going to define a tower structure originally introduced by Young in [21]. Following Young, we will also define a quotient tower and a tower product. We will recall some necessary results and obtain improved versions of others.

2.1 Tower structure

Consider the set $\bigcup_{n \geq 0} f^n(\Lambda)$ and observe that it is preserved by f . In this section we introduce an extension of the dynamical system f restricted to this set, called a *tower extension* of f . We also prove a lemma that gives a control on the diameter of the elements of a certain partition of the tower.

We define a *tower* by

$$\Delta = \{(x, l) : x \in \Lambda \text{ and } 0 \leq l < R(x)\}$$

and a *tower map* $F : \Delta \rightarrow \Delta$ as

$$F(x, l) = \begin{cases} (x, l + 1) & \text{if } l + 1 < R(x), \\ (f^R(x), 0) & \text{if } l + 1 = R(x). \end{cases}$$

The set

$$\Delta_l = \{(x, l) \in \Delta\}$$

is called the l -th level of the tower. There is a natural identification between Δ_0 , the 0-th level of the tower, and Λ . So, we will make no distinction between them. Under this identification we easily conclude from the definitions that $F^R = f^R$ for each $x \in \Delta_0$. The l -th level of the tower is a copy of the set $\{R > l\} \subseteq \Delta_0$.

Let \mathcal{P} be a partition of Δ_0 into subsets $\Delta_{0,i}$ with $\Delta_{0,i} = \Lambda_i$ for $i \in \mathbb{N}$. We can now define a partition on each level of the tower, Δ_l , by defining its elements as

$$\Delta_{l,i} = \{(x, l) \in \Delta_l : x \in \Delta_{0,i}\}.$$

So, the set $\mathcal{Q} = \{\Delta_{l,i}\}_{l,i}$ is a partition of Δ . We introduce a sequence of partitions (\mathcal{Q}_n) of Δ defined as follows

$$\mathcal{Q}_0 = \mathcal{Q} \quad \text{and} \quad \mathcal{Q}_n = \bigvee_{i=0}^n F^{-i} \mathcal{Q} \quad \text{for } n \in \mathbb{N}. \quad (2.1)$$

For each point $x \in \Delta$, let $Q_n(x)$ be the element of \mathcal{Q}_n that contains that point.

Let us define a projection map

$$\begin{aligned} \pi : \quad \Delta &\rightarrow \bigcup_{n=0}^{\infty} f^n(\Delta_0) \\ (x, l) &\mapsto f^l(x) \end{aligned} \quad (2.2)$$

and observe that $f \circ \pi = \pi \circ F$.

Next, we will establish a polynomial upper bound on the diameter of the elements of the tower partition, which will be useful later. In [2, Lemma 3.2] an exponential control is obtained from stronger hypothesis.

Lemma 2.1. *There exists $C > 0$ such that, for all $k \in \mathbb{N}$ and $Q \in \mathcal{Q}_{2k}$,*

$$\text{diam}(\pi F^k(Q)) \leq \frac{C}{k^\alpha}.$$

Proof. Take $k > 0$ and $Q \in \mathcal{Q}_{2k}$. Fixing $x, y \in Q$, there exists $z = \gamma^u(x) \cap \gamma^s(y)$, by the hyperbolic structure of the dynamical system.

Choosing l such that $Q \subseteq \Delta_l$, then $y_0 = F^{-l}(y)$ and $z_0 = F^{-l}(z)$ are both in Δ_0 and

are in the same stable leaf. So, using (P₂),

$$\begin{aligned} d(\pi F^k(y), \pi F^k(z)) &= d(\pi F^{k+l}(y_0), \pi F^{k+l}(z_0)) \\ &= d(f^{k+l}(\pi y_0), f^{k+l}(\pi z_0)) \\ &\leq \frac{C}{(k+l)^\alpha} d(\pi y_0, \pi z_0) \leq \frac{C_1}{k^\alpha}, \end{aligned}$$

because M is compact.

The points $x_0 = F^{-l}(x)$ and $z_0 = F^{-l}(z)$ are both in Δ_0 and are in the same unstable leaf. So,

$$d(\pi F^k(x), \pi F^k(z)) = d(\pi F^{k+l}(x_0), \pi F^{k+l}(z_0)) = d(f^{k+l}(\pi x_0), f^{k+l}(\pi z_0)).$$

Since $x, z \in Q \cap \Delta_l$ and $Q \in \mathcal{Q}_{2k}$, and because of the tower structure, each pair of points $F^{-i}(x)$ and $F^{-i}(z)$, for $i = 0, \dots, l$, belongs to the same element of \mathcal{Q} . Then $x_0, z_0 \in Q'$, for some $Q' \in \mathcal{Q}_{2k+l}$, which implies that $F^{2k+l}(x_0), F^{2k+l}(z_0) \in \Delta_{l',i'}$, for some $l', i' \in \mathbb{N}$. Therefore, there exists $j \in \mathbb{N}_0$ such that $F^{2k+l+j}(x_0), F^{2k+l+j}(z_0) \in \Delta_0$ and so $f^{2k+l+j}(\pi x_0), f^{2k+l+j}(\pi z_0) \in \Lambda$. Then, using (P₃) and the compactness of M ,

$$\begin{aligned} d(f^{k+l}(\pi x_0), f^{k+l}(\pi z_0)) &= d(f^{-k-j}(f^{2k+l+j}(\pi x_0)), f^{-k-j}(f^{2k+l+j}(\pi z_0))) \\ &\leq \frac{C}{(k+j)^\alpha} d(f^{2k+l+j}(\pi x_0), f^{2k+l+j}(\pi z_0)) \leq \frac{C_1}{k^\alpha}. \end{aligned}$$

We can now conclude that

$$d(\pi F^k(x), \pi F^k(y)) \leq d(\pi F^k(y), \pi F^k(z)) + d(\pi F^k(x), \pi F^k(z)) \leq \frac{2C_1}{k^\alpha}.$$

□

2.2 Quotient dynamics

We will now introduce a quotient tower, obtained from the tower by identifying points in the same stable leaf. We are also going to present an improved version of a theorem, originally proved by Young in [22], which gives a control on the decay of correlations for functions defined in the quotient tower based on the measure of the tail of the return time.

Let \sim be the equivalence relation defined on Λ by $x \sim y$ if $y \in \gamma^s(x)$. Consider $\bar{\Lambda} = \Lambda/\sim$ and the *quotient tower* $\bar{\Delta}$, whose levels are $\bar{\Delta}_l = \Delta_l/\sim$ and set $\bar{\Delta}_{l,i} = \Delta_{l,i}/\sim$. Since the

tower map F takes γ^s -leaves to γ^s -leaves, we can define $\bar{F} : \bar{\Delta} \rightarrow \bar{\Delta}$ as the function obtained from F by this identification. We introduce a partition of $\bar{\Delta}$, $\bar{\mathcal{Q}} = \{\bar{\Delta}_{l,i}\}_{l,i}$ and a sequence of partitions $(\bar{\mathcal{Q}}_n)$ of $\bar{\Delta}$, defined analogously to (2.1), as follows

$$\bar{\mathcal{Q}}_0 = \bar{\mathcal{Q}} \quad \text{and} \quad \bar{\mathcal{Q}}_n = \bigvee_{i=0}^n \bar{F}^{-i} \bar{\mathcal{Q}} \quad \text{for } n \in \mathbb{N}.$$

Since R is constant on each stable leaf and f^R takes γ^s -leaves to γ^s -leaves, then the definitions of the *return time* $\bar{R} : \bar{\Delta}_0 \rightarrow \mathbb{N}$ and the *separation time* $\bar{s} : \bar{\Delta}_0 \times \bar{\Delta}_0 \rightarrow \mathbb{N}$ are naturally induced by the corresponding definitions in Δ_0 .

We extend the *separation time* \bar{s} to $\bar{\Delta} \times \bar{\Delta}$ in the following way:

- if x and y belong to the same $\bar{\Delta}_{l,i}$, take $\bar{s}(x, y) = \bar{s}(x_0, y_0)$, where x_0, y_0 are the corresponding elements of $\bar{\Delta}_{0,i}$;
- otherwise, take $\bar{s}(x, y) = 0$.

We now present an auxiliary result whose proof can be found in [2, Lemma 3.4].

Lemma 2.2. *There exists a constant $C_F > 0$ such that, given $k \in \mathbb{N}$ and $x, y \in \bar{\Delta}$ belonging to the same element of $\bar{\mathcal{Q}}_{k-1}$, we have*

$$\left| \frac{J\bar{F}^k(x)}{J\bar{F}^k(y)} - 1 \right| \leq C_F \beta^{\bar{s}(\bar{F}^k(x), \bar{F}^k(y))}.$$

We will define a measure \bar{m} on the quotient tower. To do that, we first need to define measures m_γ on each $\gamma \cap \Lambda$, $\gamma \in \Gamma^u$. Fix $\hat{\gamma} \in \Gamma^u$ and, for any given $\gamma \in \Gamma^u$ and $x \in \gamma \cap \Lambda$, let \hat{x} be the point in $\gamma^s(x) \cap \hat{\gamma}$. Define

$$\hat{u}(x) = \prod_{n=0}^{\infty} \frac{\det Df^u(f^n(x))}{\det Df^u(f^n(\hat{x}))}$$

and note that \hat{u} satisfies (P₅)-(b). For each $\gamma \in \Gamma^u$, define m_γ as the measure in γ such that

$$\frac{dm_\gamma}{d\text{Leb}_\gamma} = \hat{u} 1_{\gamma \cap \Lambda}.$$

We are going to see that, if $\Theta = \Theta_{\gamma, \gamma'}$ is as defined in (P₅), then

$$\Theta_* m_\gamma = m_{\gamma'}. \quad (2.3)$$

We will show this by verifying that the density of both measures with respect to $\text{Leb}_{\gamma'}$ coincide. From (P₅)-(a) we have

$$\frac{\widehat{u}(x')}{\widehat{u}(x)} = \prod_{n=0}^{\infty} \left(\frac{\det Df^u(f^n(x'))}{\det Df^u(f^n(\widehat{x}))} \cdot \frac{\det Df^u(f^n(\widehat{x}))}{\det Df^u(f^n(x))} \right) = \prod_{n=0}^{\infty} \frac{\det Df^u(f^n(x'))}{\det Df^u(f^n(x))} = \frac{d\Theta_* \text{Leb}_{\gamma}}{d\text{Leb}_{\gamma'}}(x'),$$

and so,

$$\frac{d\Theta_* m_{\gamma}}{d\text{Leb}_{\gamma'}}(x') = \widehat{u}(x) \frac{d\Theta_* \text{Leb}_{\gamma}}{d\text{Leb}_{\gamma'}}(x') = \widehat{u}(x') = \frac{d\Theta_* m_{\gamma'}}{d\text{Leb}_{\gamma'}}(x'),$$

proving what we wanted.

Define a measure m on Λ as the measure whose conditional measures on $\gamma \cap \Lambda$ for $\gamma \in \Gamma^u$ are the measures m_{γ} . We define a measure in Δ , also denoted by m , by letting $m|_{\Delta_l}$ be induced by the natural identification of Δ_l and a subset of Λ . Finally, since (2.3) holds, we can define a measure \bar{m} on $\bar{\Delta}$ whose representative on each $\gamma \in \Gamma^u$ is the measure m_{γ} defined above.

Definition 2.3. Given $0 < \beta < 1$, we define

$$\begin{aligned} \mathcal{F}_{\beta} &= \left\{ \varphi : \bar{\Delta} \rightarrow \mathbb{R} : \exists C_{\varphi} > 0 \forall x, y \in \bar{\Delta} \quad |\varphi(x) - \varphi(y)| \leq C_{\varphi} \beta^{\bar{s}(x,y)} \right\}, \\ \mathcal{F}_{\beta}^{+} &= \left\{ \varphi \in \mathcal{F}_{\beta} : \exists C_{\varphi} > 0 \text{ such that on each } \bar{\Delta}_{l,i}, \text{ either } \varphi \equiv 0 \text{ or} \right. \\ &\quad \left. \varphi > 0 \text{ and for all } x, y \in \bar{\Delta}_{l,i} \quad \left| \frac{\varphi(x)}{\varphi(y)} - 1 \right| \leq C_{\varphi} \beta^{\bar{s}(x,y)} \right\}. \end{aligned}$$

From now on, we denote by C_{φ} both the infimum of the constant in the definition of \mathcal{F}_{β} and of \mathcal{F}_{β}^{+} with respect to φ . We also denote by \mathcal{F}_{β} and \mathcal{F}_{β}^{+} the analogous sets defined for functions with domain M or Δ .

Definition 2.4. Given $\theta > 0$, we define

$$\begin{aligned} \mathcal{G}_{\theta} &= \left\{ \varphi : \bar{\Delta} \rightarrow \mathbb{R} : \exists c_{\varphi} > 0 \forall x, y \in \bar{\Delta} \quad |\varphi(x) - \varphi(y)| \leq \frac{c_{\varphi}}{\max\{\bar{s}(x,y), 1\}^{\theta}} \right\}, \\ \mathcal{G}_{\theta}^{+} &= \left\{ \varphi \in \mathcal{G}_{\theta} : \exists c_{\varphi} > 0 \text{ such that on each } \bar{\Delta}_{l,i}, \text{ either } \varphi \equiv 0 \text{ or} \right. \\ &\quad \left. \varphi > 0 \text{ and for all } x, y \in \bar{\Delta}_{l,i} \quad \left| \frac{\varphi(x)}{\varphi(y)} - 1 \right| \leq \frac{c_{\varphi}}{\max\{\bar{s}(x,y), 1\}^{\theta}} \right\}. \end{aligned}$$

As above, we denote by c_{φ} both the infimum of the constant in the definition of \mathcal{G}_{θ} and of \mathcal{G}_{θ}^{+} with respect to φ . The sets \mathcal{G}_{θ} and \mathcal{G}_{θ}^{+} also represent the analogous sets defined for functions with domain M or Δ .

Now, we state a theorem that will be useful throughout this chapter and whose proof can be found in [22, Lemma 2] and in [21, Theorem 1].

Theorem 2.5. *Assume that \bar{R} is integrable with respect to \bar{m} . Then*

1. \bar{F} has a unique invariant probability measure $\bar{\nu}$ equivalent to \bar{m} ;
2. $d\bar{\nu}/d\bar{m} \in \mathcal{F}_\beta^+$ and is bounded from below by a positive constant;
3. $(\bar{F}, \bar{\nu})$ is mixing.

The next theorem is similar to [22, Theorem 3] and [2, Theorem 3.6]. Note that we only assume that $\varphi \in \mathcal{G}_\theta^+$ instead of \mathcal{F}_β^+ , which forces us to impose some extra assumptions. However, if $\varphi \in \mathcal{F}_\beta^+$ we obtain the original result.

Theorem 2.6. *Let K be as defined in (2.4), assume that $\theta > 2e^K$ and $1 < \zeta < \frac{\theta}{e^K} - 1$. Take $\varphi \in \mathcal{G}_\theta^+$ and let $\bar{\lambda}$ be the measure whose density with respect to \bar{m} is φ . Given $C > 0$, there exists $C' > 0$, depending only on c_φ , such that*

$$\bar{m}\{\bar{R} > n\} \leq \frac{C}{n^\zeta} \quad \Rightarrow \quad |\bar{F}_*^n \bar{\lambda} - \bar{\nu}| \leq \frac{C'}{n^{\zeta-1}}.$$

Remember that we defined the correlation of observables φ, ψ as

$$\mathcal{C}_n(\psi, \varphi, \bar{\nu}) = \left| \int (\psi \circ \bar{F}^n) \varphi d\bar{\nu} - \int \psi d\bar{\nu} \int \varphi d\bar{\nu} \right|.$$

The following corollary will be important for the proof of Theorem B.

Corollary 2.7. *Let K be as defined in (2.4), assume that $\theta > 2e^K$ and $1 < \zeta < \frac{\theta}{e^K} - 1$. Take $\varphi \in \mathcal{G}_\theta$ and $\psi \in L^\infty$. Given $C > 0$, there exists $C' > 0$ depending only on c_φ and $\|\psi\|_\infty$, such that*

$$\bar{m}\{\bar{R} > n\} \leq \frac{C}{n^\zeta} \quad \Rightarrow \quad \mathcal{C}_n(\psi, \varphi, \bar{\nu}) \leq \frac{C'}{n^{\zeta-1}}.$$

2.3 Tower product

In this section we define the tower product structure, adapting several concepts introduced before to this new setting. We also state some auxiliary results that will be useful later.

From now on and until the end of this chapter we will simplify the notations by removing all bars.

Let λ and λ' be probability measures in Δ whose densities with respect to m are in \mathcal{G}_θ^+ and denote

$$\varphi = \frac{d\lambda}{d\text{Leb}} \quad \text{and} \quad \varphi' = \frac{d\lambda'}{d\text{Leb}}.$$

Consider the function

$$\begin{aligned} F \times F : \Delta \times \Delta &\rightarrow \Delta \times \Delta \\ (x, y) &\mapsto (F(x), F(y)) \end{aligned}$$

and the measure $P = \lambda \times \lambda'$ in $\Delta \times \Delta$. Let $\pi, \pi' : \Delta \times \Delta \rightarrow \Delta$ be the projections on the first and second coordinates, respectively. It can be easily verified that $F^n \circ \pi = \pi \circ (F \times F)^n$, for all $n \in \mathbb{N}$.

Remember the partition $\mathcal{Q} = \{\Delta_{l,i}\}$ of Δ and consider the partition $\mathcal{Q} \times \mathcal{Q}$ of $\Delta \times \Delta$. We observe that each element of $\mathcal{Q} \times \mathcal{Q}$ is sent bijectively by $F \times F$ onto a union of elements of $\mathcal{Q} \times \mathcal{Q}$. For $n \in \mathbb{N}$, we define

$$(\mathcal{Q} \times \mathcal{Q})_n = \bigvee_{i=0}^{n-1} (F \times F)^{-i}(\mathcal{Q} \times \mathcal{Q})$$

and denote by $(\mathcal{Q} \times \mathcal{Q})_n(x, x')$ the element of $(\mathcal{Q} \times \mathcal{Q})_n$ that contains the pair (x, x') of $\Delta \times \Delta$.

Define $\widehat{R} : \Delta \rightarrow \mathbb{N}$ as

$$\widehat{R}(x) = \min\{n \in \mathbb{N}_0 : F^n(x) \in \Delta_0\}.$$

Note that $\widehat{R}|_{\Delta_0} = R|_{\Delta_0}$.

As (F, ν) is mixing and $\frac{d\nu}{dm} \in L^\infty$, then there exists $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that, for all $n \geq n_0$, we have $m(F^{-n}(\Delta_0) \cap \Delta_0) \geq \delta_0$. Consider the sequence of *stopping times* $0 \equiv \tau_0 < \tau_1 < \dots$, defined in $\Delta \times \Delta$, as

$$\begin{aligned} \tau_1(x, x') &= n_0 + \widehat{R}(F^{n_0}x) \\ \tau_2(x, x') &= \tau_1 + n_0 + \widehat{R}(F^{\tau_1}x') \\ \tau_3(x, x') &= \tau_2 + n_0 + \widehat{R}(F^{\tau_2}x) \\ \tau_4(x, x') &= \tau_3 + n_0 + \widehat{R}(F^{\tau_3}x') \\ &\vdots \end{aligned}$$

Observe that $\tau_{i+1} - \tau_i \geq n_0$ for all $i \in \mathbb{N}$.

We introduce now the *simultaneous return time* $T : \Delta \times \Delta \rightarrow \mathbb{N}$ as

$$T(x, x') = \min_{i \geq 2} \{ \tau_i : (F^{\tau_i} x, F^{\tau_i} x') \in \Delta_0 \times \Delta_0 \}.$$

Note that, as (F, ν) is mixing, then $(F \times F, \nu \times \nu)$ is ergodic. So T is well defined $m \times m$ a.e.. We define a sequence of partitions of $\Delta \times \Delta$, $\xi_1 < \xi_2 < \dots$ as follows:

- $\xi_1(x, x') = (F^{-\tau_1(x)+1} \mathcal{Q})(x) \times \Delta$. The elements of ξ_1 are of the form $\Gamma = A \times \Delta$, where $\tau_1|_{A \times \Delta}$ is constant and A is sent bijectively to Δ_0 by F^{τ_1} ;
- for i even, ξ_i is the refinement of ξ_{i-1} obtained by partitioning $\Gamma \in \xi_{i-1}$ in the x' direction into sets $\tilde{\Gamma}$ such that $\tau_i|_{\tilde{\Gamma}}$ is constant and $\pi'(\tilde{\Gamma})$ is sent bijectively to Δ_0 by F^{τ_i} ;
- for i odd, $i > 1$, we do the same as in the previous point replacing the x' direction by the x direction and π' by π .

For convenience we define $\xi_0 = \{\Delta \times \Delta\}$. Note that

- $\forall i \in \mathbb{N} \forall n \leq i$ τ_n is ξ_i -measurable;
- $\forall i \in \mathbb{N}$ $\{T = \tau_i\}$ and $\{T > \tau_i\}$ are ξ_{i+1} -measurable.

Define a sequence of *stopping times* in $\Delta \times \Delta$, $0 \equiv T_0 < T_1 < \dots$, as

$$T_1 = T \text{ and } T_n = T_{n-1} + T \circ (F \times F)^{T_{n-1}}, \text{ for } n \geq 2.$$

Consider the dynamical system $\hat{F} = (F \times F)^T : \Delta \times \Delta \rightarrow \Delta \times \Delta$. It is easy to verify that

$$\forall n \in \mathbb{N} \quad \hat{F}^n = (F \times F)^{T_n}.$$

Define a partition $\hat{\xi}_1$ of $\Delta \times \Delta$, composed by rectangles $\hat{\Gamma}$ such that $T|_{\hat{\Gamma}}$ is constant and $\hat{F} : \hat{\Gamma} \rightarrow \Delta_0 \times \Delta_0$ is bijective.

Define a sequence of partitions, $\hat{\xi}_2, \hat{\xi}_3, \dots$, by $\hat{\xi}_n = \hat{F}^{-(n-1)} \hat{\xi}_1$, for $n \geq 2$. Note that T_n is constant on each element of $\hat{\xi}_n$ and \hat{F}_n maps each element of $\hat{\xi}_n$ bijectively to $\Delta_0 \times \Delta_0$.

Consider the measure $m \times m$ for the dynamical system \widehat{F} and the Jacobian, $J\widehat{F}$, of \widehat{F} with respect to $m \times m$. Define a *separation time* $\widehat{s}: (\Delta \times \Delta) \times (\Delta \times \Delta) \rightarrow \mathbb{N}_0$ as

$$\widehat{s}(z, w) = \min \{n \in \mathbb{N}_0 : \widehat{F}^n z \text{ and } \widehat{F}^n w \text{ belong to different elements of } \widehat{\xi}_1\}.$$

Denoting

$$\Phi = \frac{dP}{d(m \times m)},$$

we observe that $\Phi(x, x') = \varphi(x)\varphi'(x')$. We may assume without loss of generality that $\varphi > 0$ and $\varphi' > 0$.

We are going to present some lemmas that will be needed later. The proof of the following one can be found in [22, Sublemma 3].

Lemma 2.8. *Let C_F be the constant defined in Lemma 2.2. For $z, w \in \Delta \times \Delta$ such that $\widehat{s}(z, w) \geq n$, for some $n \in \mathbb{N}$, we have*

$$\left| \log \frac{J\widehat{F}^n z}{J\widehat{F}^n w} \right| \leq 2C_F \beta^{\widehat{s}(\widehat{F}^n z, \widehat{F}^n w)}.$$

Lemma 2.9. *For all $z, w \in \Delta \times \Delta$, we have*

$$\left| \log \frac{\Phi(z)}{\Phi(w)} \right| \leq \frac{c_\Phi}{\widehat{s}(z, w)^\theta},$$

where $c_\Phi = c_\varphi + c_{\varphi'}$.

Proof. Let $z = (x, x')$ and $w = (y, y')$. Then, since $\log x \leq x - 1$ for $x \in \mathbb{R}^+$ and $\varphi, \varphi' \in \mathcal{G}_\theta^+$,

$$\begin{aligned} \left| \log \frac{\Phi(z)}{\Phi(w)} \right| &= \left| \log \frac{\varphi(x) \varphi'(x')}{\varphi(y) \varphi'(y')} \right| \leq \left| \log \frac{\varphi(x)}{\varphi(y)} \right| + \left| \log \frac{\varphi'(x')}{\varphi'(y')} \right| \\ &\leq \left| \frac{\varphi(x)}{\varphi(y)} - 1 \right| + \left| \frac{\varphi'(x')}{\varphi'(y')} - 1 \right| \\ &\leq c_\varphi \frac{1}{s(x, y)^\theta} + c_{\varphi'} \frac{1}{s(x', y')^\theta} \leq \frac{c_\varphi + c_{\varphi'}}{\widehat{s}(z, w)^\theta}. \end{aligned}$$

□

Lemma 2.10. *There exists a constant $C > 0$ depending only on c_φ and $c_{\varphi'}$, such that, for all $i \in \mathbb{N}$, $\Gamma \in \widehat{\xi}_i$, $z, w \in \Delta_0 \times \Delta_0$ and $Q = \widehat{F}_*^i(P|\Gamma)$, we have*

$$\left| \frac{dQ}{d\text{Leb}}(z) / \frac{dQ}{d\text{Leb}}(w) \right| \leq C.$$

Proof. Take $z_0, w_0 \in \Gamma$ such that $\widehat{F}^i(z_0) = z$ and $\widehat{F}^i(w_0) = w$. As $\widehat{s}(z_0, w_0) \geq i$, using Lemma 2.8 and Lemma 2.9, we get

$$\left| \frac{dQ}{d\text{Leb}}(z) \Big/ \frac{dQ}{d\text{Leb}}(w) \right| = \left| \frac{\Phi(z_0)}{J\widehat{F}^i(z_0)} \frac{J\widehat{F}^i(w_0)}{\Phi(w_0)} \right| = \frac{\Phi(z_0)}{\Phi(w_0)} \left| \frac{J\widehat{F}^i(w_0)}{J\widehat{F}^i(z_0)} \right| \leq e^{c_\Phi} e^{C_{\widehat{F}}}.$$

□

2.4 Probabilistic results

In what follows we will obtain the necessary results in order to prove Theorem 2.6 and Corollary 2.7, following the approach used in [22] and [2, Appendix A]. Recalling Lemma 2.8 we define $C_{\widehat{F}} = 2C_F$. Take

$$K > C_{\widehat{F}} + \frac{C_{\widehat{F}}}{1 - \beta} \quad (2.4)$$

and $\widehat{C} = K - C_{\widehat{F}}$. Observe that

$$\widehat{C} > \frac{C_{\widehat{F}}}{1 - \beta}.$$

Proposition 2.11. *There exists $\varepsilon_0 > 0$ such that, for all $i \geq 2$ and $\Gamma \in \xi_i$ with $T|_\Gamma > \tau_{i-1}$, we have*

$$P\{T = \tau_i \mid \Gamma\} \geq \varepsilon_0.$$

The constant ε_0 depends only on $c_\varphi, c_{\varphi'}$ and, if there exists $i \geq i_0(c_\varphi, c_{\varphi'})$ such that $i \geq i_0$, the dependence can be removed.

Proposition 2.12. *There exists $k_0 > 0$ such that, for all $i \in \mathbb{N}_0$, $\Gamma \in \xi_i$ and $n \in \mathbb{N}_0$,*

$$P\{\tau_{i+1} - \tau_i > n_0 + n \mid \Gamma\} \leq k_0 \text{Leb}\{\widehat{R} > n\}.$$

The constant k_0 depends only on $c_\varphi, c_{\varphi'}$ and, if there exists $i \geq i_0(c_\varphi, c_{\varphi'})$ such that $i \geq i_0$, the dependence can be removed.

The proofs of these two propositions follow the same steps of the proofs of (E1) and (E2) in [2, Subsections A.3.1 and A.3.2]. We only need to adapt the proof of Lemma A.2 of [2] to our case, which we do next.

Lemma 2.13. *There exists $C_0 = C_0(\varphi) > 0$ such that, for all $k \in \mathbb{N}$, $A \in \bigvee_{i=0}^{k-1} F^{-i}(\mathcal{Q})$ with $F^k(A) = \Delta_0$, $\mu = F_*^k(\lambda|A)$ and $x, y \in \Delta_0$, we have*

$$\left| \frac{d\mu}{d\text{Leb}}(x) / \frac{d\mu}{d\text{Leb}}(y) \right| \leq C_0.$$

The dependence of C_0 on c_φ may be removed if we assume that the number of visits $j \leq k$ of A to Δ_0 is bigger than a certain $j_0 = j_0(c_\varphi)$.

Proof. Given $x_0, y_0 \in A$ such that $F^k(x_0) = x$ and $F^k(y_0) = y$ then, as $\varphi \in \mathcal{G}_\theta^+$ and using Lemma 2.2,

$$\begin{aligned} \left| \frac{d\mu}{d\text{Leb}}(x) / \frac{d\mu}{d\text{Leb}}(y) \right| &= \left| \frac{\varphi(x_0)}{JF^k(x_0)} \frac{JF^k(y_0)}{\varphi(y_0)} \right| = \frac{\varphi(x_0)}{\varphi(y_0)} \left| \frac{JF^k(y_0)}{JF^k(x_0)} \right| \\ &\leq \left(1 + \frac{c_\varphi}{s(x_0, y_0)^\theta} \right) (1 + C_F \beta^{s(F^k(x_0), F^k(y_0))}) \\ &\leq \left(1 + \frac{c_\varphi}{j^\theta} \right) (1 + C_F) = C_0. \end{aligned}$$

□

This shows that Lemma A.2 of [2] is still valid in our case. So the proof of Proposition 2.11 is the same as the proof of (E1) in [2, Subsection A.3.1].

Now we state a proposition whose proof can be found in [2, Subsection A.2.1] and uses Propositions 2.11 and 2.12.

Proposition 2.14. *Let $C > 0$ and $\zeta > 1$ be such that $\text{Leb}\{R > n\} \leq Cn^{-\zeta}$. Then, there exists $C' > 0$ such that*

$$P\{T > n\} \leq \frac{C'}{n^{\zeta-1}}.$$

We want to define a sequence of densities $(\widehat{\Phi}_i)$ in $\Delta \times \Delta$ such that

- $\widehat{\Phi}_0 \geq \widehat{\Phi}_1 \geq \dots$;
- for all $i \in \mathbb{N}$ and $\widehat{\Gamma} \in \widehat{\xi}_i$,

$$\pi_* \widehat{F}_*^i((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma})) = \pi'_* \widehat{F}_*^i((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma})). \quad (2.5)$$

Let \widehat{C} be a constant such that

$$\widehat{C} > \frac{C_{\widehat{F}}}{1 - \beta}$$

and note that

$$K = \widehat{C} + C_{\widehat{F}}.$$

Take ζ as in Theorem 2.6. Noting that $1 < \zeta < \frac{\theta}{e^K} - 1$, we fix ρ such that

$$\zeta + 1 < \rho < \frac{\theta}{e^K}, \quad (2.6)$$

Take

$$\varepsilon_i = e^K \left(1 - \left(\frac{i-1}{i} \right)^\rho \right),$$

for $i \geq i_0$, where i_0 is such that $\varepsilon_{i_0} < 1$. Further restrictions on i_0 will be imposed during the proof of Lemma 2.16. Define, for $i < i_0$, $\widehat{\Phi}_i \equiv \Phi$ and, for $i \geq i_0$,

$$\widehat{\Phi}_i(z) = \left(\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \right) J\widehat{F}^i(z), \quad (2.7)$$

where $\widehat{\xi}_i(z)$ is the element of $\widehat{\xi}_i$ which contains z . It is easy to verify that the sequence $(\widehat{\Phi}_i(z))$ satisfies condition 2.5.

Lemma 2.15. *Assume that $\theta > e^K$. Then, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, we have*

$$\widehat{\Phi}_i \leq \left(\frac{i-1}{i} \right)^\rho \widehat{\Phi}_{i-1} \quad \text{in } \Delta \times \Delta.$$

Recall that, in the beginning of this section, we assumed that φ and φ' belong to \mathcal{G}_θ^+ . We want to prove that if θ is large enough then there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, we have

$$\widehat{\Phi}_i \leq \left(\frac{i-1}{i} \right)^\rho \widehat{\Phi}_{i-1} \quad \text{in } \Delta \times \Delta.$$

The next three lemmas are dedicating to proving this inequality.

For $z \in \Delta \times \Delta$, let

$$\widetilde{\Psi}_{i_0-1}(z) = \frac{\Phi}{J\widehat{F}^{i_0-1}(z)}$$

and, for $i \geq i_0$,

$$\Psi_i(z) = \frac{\widetilde{\Psi}_{i-1}(z)}{J\widehat{F}(\widehat{F}^{i-1})(z)},$$

$$\varepsilon_{i,z} = \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \Psi_i(w),$$

$$\widetilde{\Psi}_i(z) = \Psi_i(z) - \varepsilon_{i,z}.$$

Lemma 2.16. *Assume that $\theta > e^K$. Then, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$ and for all $z, w \in \Delta \times \Delta$ with $w \in \widehat{\xi}_i(z)$, we have*

$$\left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} \right| \leq \widehat{C}.$$

Proof. We divide this proof into several steps.

Step 1: By the definition of Ψ_i and Lemma 2.8,

$$\begin{aligned} \left| \log \frac{\Psi_i(z)}{\Psi_i(w)} \right| &\leq \left| \log \frac{\widetilde{\Psi}_{i-1}(z)}{\widetilde{\Psi}_{i-1}(w)} \right| + \left| \log \frac{J\widehat{F}(\widehat{F}^{i-1}w)}{J\widehat{F}(\widehat{F}^{i-1}z)} \right| \\ &\leq \left| \log \frac{\widetilde{\Psi}_{i-1}(z)}{\widetilde{\Psi}_{i-1}(w)} \right| + C_{\widehat{F}} \beta^{\widehat{F}^i z, \widehat{F}^i w}. \end{aligned}$$

Step 2: Setting $\widehat{\varepsilon}_i = \varepsilon_{i,z} = \varepsilon_{i,w}$, we get

$$\begin{aligned} \left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} - \log \frac{\Psi_i(z)}{\Psi_i(w)} \right| &= \left| \log \left(\frac{\Psi_i(z) - \widehat{\varepsilon}_i}{\Psi_i(z)} \frac{\Psi_i(w)}{\Psi_i(w) - \widehat{\varepsilon}_i} \right) \right| \\ &= \left| \log \left(1 + \frac{\widehat{\varepsilon}_i}{\Psi_i(w) - \widehat{\varepsilon}_i} \frac{\widehat{\varepsilon}_i}{\Psi_i(z)} \right) \right|. \end{aligned} \quad (2.8)$$

Noting that

$$\left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} - \log \frac{\Psi_i(z)}{\Psi_i(w)} \right| = \left| \log \frac{\widetilde{\Psi}_i(w)}{\widetilde{\Psi}_i(z)} - \log \frac{\Psi_i(w)}{\Psi_i(z)} \right|,$$

we may assume that $\Psi_i(w) \leq \Psi_i(z)$. Otherwise, we can swap the positions of z and w .

We can easily verify that, for all $0 < a \leq b < 1$, we have

$$\log \left(1 + \frac{b-a}{1-b} \right) \leq \frac{b}{1-b} \log \frac{b}{a}.$$

Taking $a = \frac{\widehat{\varepsilon}_i}{\Psi_i(z)}$ and $b = \frac{\widehat{\varepsilon}_i}{\Psi_i(w)}$ and recalling the definition of $\widehat{\varepsilon}_i$, we obtain

$$\left| \log \left(1 + \frac{\widehat{\varepsilon}_i}{\Psi_i(w) - \widehat{\varepsilon}_i} \frac{\widehat{\varepsilon}_i}{\Psi_i(z)} \right) \right| \leq \frac{\widehat{\varepsilon}_i}{1 - \widehat{\varepsilon}_i} \left| \log \frac{\widehat{\varepsilon}_i}{\Psi_i(z)} \right| \leq \frac{\varepsilon_i}{1 - \varepsilon_i} \left| \log \frac{\Psi_i(z)}{\Psi_i(w)} \right|. \quad (2.9)$$

Gathering the expressions (2.8) and (2.9), we obtain

$$\left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} - \log \frac{\Psi_i(z)}{\Psi_i(w)} \right| \leq \frac{\varepsilon_i}{1 - \varepsilon_i} \left| \log \frac{\Psi_i(z)}{\Psi_i(w)} \right|.$$

Denoting $\varepsilon'_i = \frac{\varepsilon_i}{1-\varepsilon_i}$, we conclude that

$$\left| \log \frac{\tilde{\Psi}_i(z)}{\tilde{\Psi}_i(w)} \right| \leq (1 + \varepsilon'_i) \left| \log \frac{\Psi_i(z)}{\Psi_i(w)} \right|.$$

Step 3: Note that

$$\Psi_{i_0}(z) = \frac{\tilde{\Psi}_{i_0-1}(z)}{J\hat{F}(\hat{F}^{i_0-1}(z))} = \frac{\Phi(z)}{J\hat{F}^{i_0-1}(z)J\hat{F}(\hat{F}^{i_0-1}(z))} = \frac{\Phi(z)}{J\hat{F}^{i_0}(z)},$$

and so, using step 2, Lemma 2.8 and Lemma 2.9,

$$\begin{aligned} \left| \log \frac{\tilde{\Psi}_{i_0}(z)}{\tilde{\Psi}_{i_0}(w)} \right| &\leq (1 + \varepsilon'_{i_0}) \left| \log \frac{\Psi_{i_0}(z)}{\Psi_{i_0}(w)} \right| \\ &\leq (1 + \varepsilon'_{i_0}) \left(\left| \log \frac{\Phi(z)}{\Phi(w)} \right| + \left| \log \frac{J\hat{F}^{i_0}(w)}{J\hat{F}^{i_0}(z)} \right| \right) \\ &\leq (1 + \varepsilon'_{i_0}) \left(\frac{c_\Phi}{\hat{s}(z, w)^\theta} + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^{i_0}z, \hat{F}^{i_0}w)} \right) \\ &= (1 + \varepsilon'_{i_0}) \left(\frac{c_\Phi}{(\hat{s}(\hat{F}^{i_0}z, \hat{F}^{i_0}w) + i_0)^\theta} + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^{i_0}z, \hat{F}^{i_0}w)} \right). \end{aligned} \quad (2.10)$$

Then

$$\left| \log \frac{\tilde{\Psi}_{i_0}(z)}{\tilde{\Psi}_{i_0}(w)} \right| \leq (1 + \varepsilon'_{i_0}) \left(\frac{c_\Phi}{i_0^\theta} + C_{\hat{F}} \right) \xrightarrow{i_0 \rightarrow \infty} C_{\hat{F}} < \hat{C}$$

and so we can choose i_0 sufficiently large such that

$$\left| \log \frac{\tilde{\Psi}_{i_0}(z)}{\tilde{\Psi}_{i_0}(w)} \right| \leq \hat{C},$$

obtaining the conclusion of the Lemma for $i = i_0$.

Step 4: Using steps 2 and 1, we obtain

$$\left| \log \frac{\tilde{\Psi}_i(z)}{\tilde{\Psi}_i(w)} \right| \leq (1 + \varepsilon'_i) \left(\left| \log \frac{\tilde{\Psi}_{i-1}(z)}{\tilde{\Psi}_{i-1}(w)} \right| + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w)} \right).$$

Step 5: Using the equality $\hat{s}(\hat{F}^{i-j}z, \hat{F}^{i-j}w) = \hat{s}(\hat{F}^i z, \hat{F}^i w) + j$ and the inequalities in steps

3 and 4, we get, for $i \geq i_0 + 1$,

$$\begin{aligned}
\left| \log \frac{\tilde{\Psi}_i(z)}{\tilde{\Psi}_i(w)} \right| &\leq (1 + \varepsilon'_i) \left((1 + \varepsilon'_{i-1}) \left(\left| \log \frac{\tilde{\Psi}_{i-2}(z)}{\tilde{\Psi}_{i-2}(w)} \right| + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w) + 1} \right) + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w)} \right) \\
&\leq \left| \log \frac{\tilde{\Psi}_{i_0}(z)}{\tilde{\Psi}_{i_0}(w)} \right| \prod_{j=i_0+1}^i (1 + \varepsilon'_j) + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w)} \left(\beta^{i-i_0-1} \prod_{j=i_0+1}^i (1 + \varepsilon'_j) \right. \\
&\quad \left. + \dots + \beta(1 + \varepsilon'_i)(1 + \varepsilon'_{i-1}) + (1 + \varepsilon'_i) \right) \\
&\leq \left(\frac{c_\Phi}{\hat{s}(z, w)^\rho} + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w) + i - i_0} \right) \prod_{j=i_0}^i (1 + \varepsilon'_j) \\
&\quad + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w)} \left(\beta^{i-i_0-1} \prod_{j=i_0+1}^i (1 + \varepsilon'_j) + \dots + (1 + \varepsilon'_i) \right) \\
&= \frac{c_\Phi}{(\hat{s}(\hat{F}^i z, \hat{F}^i w) + i)^\theta} \prod_{j=i_0}^i (1 + \varepsilon'_j) + C_{\hat{F}} \beta^{\hat{s}(\hat{F}^i z, \hat{F}^i w)} (1 + \varepsilon'_i) \sum_{k=i_0}^i \left(\prod_{j=k}^i (1 + \varepsilon'_j) \beta \right).
\end{aligned}$$

In the next two steps we will control the two terms of the previous expression.

Step 6: Recalling that $\varepsilon'_i = \frac{\varepsilon_i}{1 - \varepsilon_i}$ and $\varepsilon_i = e^K (1 - (\frac{i-1}{i})^\rho)$, it is easy to check that

$$\lim_i \frac{\varepsilon'_i}{\frac{1}{i}} = \lim_i \frac{\frac{\varepsilon_i}{1 - \varepsilon_i}}{\frac{1}{i}} = e^K \rho.$$

Remember that, in (2.6), we chose ρ such that $\theta > e^K \rho$. So, for i_0 sufficiently large and $i \geq i_0$, we have $\varepsilon'_i < \frac{\theta}{i}$. As $\log(1 + x) \leq x$ for $x > 0$, then

$$\log \prod_{j=i_0}^i (1 + \varepsilon'_j) = \sum_{j=i_0}^i \log(1 + \varepsilon'_j) \leq \sum_{j=i_0}^i \varepsilon'_j \leq \theta \sum_{j=i_0}^i \frac{1}{j} \leq \theta \log \frac{i}{i_0 - 1}.$$

So,

$$\prod_{j=i_0}^i (1 + \varepsilon'_j) \leq \left(\frac{i}{i_0 - 1} \right)^\theta$$

and

$$\frac{c_\Phi}{(\hat{s}(\hat{F}^i z, \hat{F}^i w) + i)^\theta} \prod_{j=i_0}^i (1 + \varepsilon'_j) \leq \frac{c_\Phi}{i^\theta} \left(\frac{i}{i_0 - 1} \right)^\theta = \frac{c_\Phi}{(i_0 - 1)^\theta}.$$

Step 7: We may choose i_0 sufficiently large such that $(1 + \varepsilon'_{i_0})\beta < 1$. Note that we will later impose additional restrictions on i_0 . So, recalling that (ε'_i) is a decreasing sequence

converging to zero, then, for all $i \geq i_0$,

$$\begin{aligned} C_{\widehat{F}} \beta^{\widehat{s}(\widehat{F}^i z, \widehat{F}^i w)} (1 + \varepsilon'_i) \sum_{k=i_0}^i \left(\prod_{j=k}^i (1 + \varepsilon'_j) \beta \right) &\leq C_{\widehat{F}} (1 + \varepsilon'_{i_0}) \sum_{k=0}^{\infty} ((1 + \varepsilon'_{i_0}) \beta)^k \\ &= \frac{C_{\widehat{F}} (1 + \varepsilon'_{i_0})}{1 - (1 + \varepsilon'_{i_0}) \beta}. \end{aligned}$$

Step 8: Replacing the conclusions of steps 6 and 7 on the expression in step 5, we obtain, for $i \geq i_0 + 1$,

$$\left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} \right| \leq \frac{c_{\Phi}}{(i_0 - 1)^{\theta}} + \frac{C_{\widehat{F}} (1 + \varepsilon'_{i_0})}{1 - (1 + \varepsilon'_{i_0}) \beta}.$$

As $\varepsilon'_{i_0} \xrightarrow{i_0 \rightarrow \infty} 0$, then

$$\frac{c_{\Phi}}{(i_0 - 1)^{\theta}} + \frac{C_{\widehat{F}} (1 + \varepsilon'_{i_0})}{1 - (1 + \varepsilon'_{i_0}) \beta} \xrightarrow{i_0 \rightarrow \infty} \frac{C_{\widehat{F}}}{1 - \beta}.$$

Observing that we chose $\widehat{C} > \frac{C_{\widehat{F}}}{1 - \beta}$, then there exists i_0 large enough such that, for $i \geq i_0 + 1$,

$$\left| \log \frac{\widetilde{\Psi}_i(z)}{\widetilde{\Psi}_i(w)} \right| \leq \widehat{C}.$$

Recalling that we proved the same result for $i = i_0$ in step 3, this concludes the proof. \square

Lemma 2.17. *Assume that $\theta > e^K$. Then, there exists $i_0 \in \mathbb{N}$ such that, for all $i \geq i_0$ and $\widehat{\Gamma} \in \widehat{\xi}_i$,*

$$\max_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \bigg/ \min_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \leq e^K.$$

Proof. Notice that, by the definitions, we have, for $i \geq i_0$,

$$\frac{\widehat{\Phi}_i}{J\widehat{F}^i(z)} = \frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \quad (2.11)$$

and

$$\widetilde{\Psi}_i(z) = \frac{\widetilde{\Psi}_{i-1}(z)}{J\widehat{F}(\widehat{F}^{i-1}(z))} - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widetilde{\Psi}_{i-1}(w)}{J\widehat{F}(\widehat{F}^{i-1}(w))}. \quad (2.12)$$

We will prove by induction that for all $z \in \Delta \times \Delta$ and all $i \geq i_0$ we have

$$\widetilde{\Psi}_{i-1}(z) = \frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^{i-1}(z)}, \quad (2.13)$$

which, since $J\widehat{F}^i(z) = J\widehat{F}(\widehat{F}^{i-1}(z))J\widehat{F}^{i-1}(z)$, is equivalent to

$$\frac{\widetilde{\Psi}_{i-1}(z)}{J\widehat{F}(\widehat{F}^{i-1}(z))} = \frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)}. \quad (2.14)$$

If $i = i_0$, then (2.13) is true by definition. Supposing now, by induction, that (2.14) is true, we will prove that it is also true replacing $i - 1$ by i . In fact, using (2.14) in (2.12) and remembering (2.11), we obtain

$$\widetilde{\Psi}_i(z) = \frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} = \frac{\widehat{\Phi}_i}{J\widehat{F}^i(z)},$$

which concludes the proof of (2.13). Using (2.13), we have

$$\frac{\widetilde{\Psi}_{i-1}(z)}{\widetilde{\Psi}_{i-1}(w)} = \frac{\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^{i-1}(z)}}{\frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^{i-1}(w)}} = \frac{\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} J\widehat{F}(\widehat{F}^{i-1}(z))}{\frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} J\widehat{F}(\widehat{F}^{i-1}(w))}$$

and so

$$\frac{\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)}}{\frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)}} = \frac{\widetilde{\Psi}_{i-1}(z)}{\widetilde{\Psi}_{i-1}(w)} \frac{J\widehat{F}(\widehat{F}^{i-1}(w))}{J\widehat{F}(\widehat{F}^{i-1}(z))}.$$

Since, by Lemma 2.16

$$\frac{\widetilde{\Psi}_{i-1}(z)}{\widetilde{\Psi}_{i-1}(w)} \leq e^{\widehat{C}},$$

and by Lemma 2.8

$$\frac{J\widehat{F}(\widehat{F}^{i-1}(w))}{J\widehat{F}(\widehat{F}^{i-1}(z))} \leq e^{C_{\widehat{F}}},$$

then

$$\max_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \bigg/ \min_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \leq e^{\widehat{C} + C_{\widehat{F}}} = e^K.$$

□

Lemma 2.18. *Assume that $\theta > e^K$. Then, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq i_0$, we have*

$$\widehat{\Phi}_i \leq \left(\frac{i-1}{i}\right)^\rho \widehat{\Phi}_{i-1} \quad \text{in } \Delta \times \Delta.$$

Proof. Observe that, for $i \geq i_0$ and $z \in \Delta \times \Delta$,

$$\begin{aligned}
\widehat{\Phi}_i(z) &\leq \left(\frac{i-1}{i}\right)^\rho \widehat{\Phi}_{i-1}(z) \\
\Leftrightarrow \widehat{\Phi}_{i-1}(z) - \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} J\widehat{F}^i(z) &\leq \widehat{\Phi}_{i-1}(z) - \left(1 - \left(\frac{i-1}{i}\right)^\rho\right) \widehat{\Phi}_{i-1}(z) \\
\Leftrightarrow \varepsilon_i \min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} &\geq \left(1 - \left(\frac{i-1}{i}\right)^\rho\right) \frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)} \\
\Leftrightarrow \varepsilon_i &\geq \left(1 - \left(\frac{i-1}{i}\right)^\rho\right) \frac{\frac{\widehat{\Phi}_{i-1}(z)}{J\widehat{F}^i(z)}}{\min_{w \in \widehat{\xi}_i(z)} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)}}.
\end{aligned}$$

Since, by Lemma 2.17, for all $\widehat{\Gamma} \in \widehat{\xi}_i$,

$$\max_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \bigg/ \min_{w \in \widehat{\Gamma}} \frac{\widehat{\Phi}_{i-1}(w)}{J\widehat{F}^i(w)} \leq e^K \quad (2.15)$$

the conclusion follows from our choice of ε_i . \square

Proposition 2.19. *Assume that $\theta > e^K$. Then there exists a constant $K_1 > 0$ such that, for all $n \in \mathbb{N}$,*

$$|F_*^n \lambda - F_*^n \lambda'| \leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} \frac{1}{i^\rho} P\{T_i \leq n < T_{i+1}\}.$$

The constant K_1 depends only on c_φ and $c_{\varphi'}$.

Proof. Given $n \in \mathbb{N}_0$, $z \in \Delta \times \Delta$ and recalling the definition of $\widehat{\Phi}_i$ given in (2.7), let Φ_0, Φ_1, \dots be defined as follows:

$$\Phi_n(z) = \widehat{\Phi}_i(z) \quad \text{for } T_i(z) \leq n < T_{i+1}(z). \quad (2.16)$$

We will prove that, for all $n \in \mathbb{N}$,

$$|F_*^n \lambda - F_*^n \lambda'| \leq 2 \int \Phi_n d(m \times m). \quad (2.17)$$

In fact, observing that $\Phi = \Phi_n + \sum_{k=1}^n (\Phi_{k-1} - \Phi_k)$, we have

$$\begin{aligned}
|F_*^n \lambda - F_*^n \lambda'| &= |\pi_*(F \times F)_*^n (\Phi(m \times m)) - \pi'_*(F \times F)_*^n (\Phi(m \times m))| \\
&= |\pi_*(F \times F)_*^n (\Phi_n(m \times m)) - \pi'_*(F \times F)_*^n (\Phi_n(m \times m))| \\
&\quad + \sum_{k=1}^n |(\pi - \pi')_*((F \times F)_*^n (\Phi_{k-1} - \Phi_k)(m \times m))|.
\end{aligned}$$

The first term in the last expression is bounded as follows

$$|\pi_*(F \times F)_*^n(\Phi_n(m \times m)) - \pi'_*(F \times F)_*^n(\Phi_n(m \times m))| \leq 2 \int \Phi_n d(m \times m).$$

We will now verify that the other terms vanish. Let $A_{k,i} = \{z \in \Delta \times \Delta : k = T_i(z)\}$ and $A_k = \bigcup A_{k,i}$. Note that each of the sets $A_{k,i}$ is a union of elements of $\Gamma \in \widehat{\xi}_i$ and $A_{k,i} \neq A_{k,j}$ for $i \neq j$. By (2.16) we have $\Phi_{k-1} - \Phi_k = \widehat{\Phi}_{i-1} - \widehat{\Phi}_i$ on $\Gamma \in \widehat{\xi}_i|A_{k,i}$ and $\Phi_{k-1} = \Phi_k$ on $\Delta \times \Delta \setminus A_k$. Given $k \in \mathbb{N}$ and remembering that, from (2.5),

$$\pi_* \widehat{F}_*^i((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma})) = \pi'_* \widehat{F}_*^i((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)((m \times m)|\widehat{\Gamma})),$$

we have

$$\begin{aligned} \pi_*(F \times F)_*^n(\Phi_{k-1} - \Phi_k)(m \times m) &= \sum_i \sum_{\Gamma \subseteq A_{k,i}} F_*^{n-k} \pi_*(F \times F)_*^{T_i} \left((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)(m \times m)|\Gamma \right) \\ &= \sum_i \sum_{\Gamma \subseteq A_{k,i}} F_*^{n-k} \pi'_*(F \times F)_*^{T_i} \left((\widehat{\Phi}_{i-1} - \widehat{\Phi}_i)(m \times m)|\Gamma \right) \\ &= \pi'_*(F \times F)_*^n(\Phi_{k-1} - \Phi_k)(m \times m). \end{aligned}$$

This completes the proof of (2.17). As a consequence, we have

$$\begin{aligned} |F_*^n \lambda - F_*^n \lambda'| &\leq 2 \int \Phi_n d(m \times m) \\ &= 2 \int_{\{T_{i_0} > n\}} \Phi_n d(m \times m) + 2 \sum_{i=i_0}^{\infty} \int_{\{T_i \leq n < T_{i+1}\}} \Phi_n d(m \times m). \end{aligned} \quad (2.18)$$

For the first term of this expression we have

$$\int_{\{T_{i_0} > n\}} \Phi_n d(m \times m) = \int_{\{T_{i_0} > n\}} \Phi d(m \times m) = P\{T_{i_0} > n\}$$

and for each of the others, using Lemma 2.18, we obtain

$$\begin{aligned} \int_{\{T_i \leq n < T_{i+1}\}} \Phi_n d(m \times m) &= \int_{\{T_i \leq n < T_{i+1}\}} \widehat{\Phi}_i d(m \times m) \\ &\leq \int_{\{T_i \leq n < T_{i+1}\}} \left(\frac{i_0 - 1}{i} \right)^\rho \Phi d(m \times m) \\ &= \left(\frac{i_0 - 1}{i} \right)^\rho P\{T_i \leq n < T_{i+1}\}. \end{aligned}$$

So, replacing the previous two expressions in (2.18), we get

$$|F_*^n \lambda - F_*^n \lambda'| \leq 2P\{T_{i_0} > n\} + 2(i_0 - 1)^\rho \sum_{i=i_0}^{\infty} \frac{1}{i^\rho} P\{T_i \leq n < T_{i+1}\}.$$

On the other hand,

$$\begin{aligned} P\{T_{i_0} > n\} &= P\{T > n\} + (i_0 - 1)^\rho \sum_{i=1}^{i_0-1} \frac{1}{(i_0 - 1)^\rho} P\{T_i \leq n < T_{i+1}\} \\ &\leq P\{T > n\} + (i_0 - 1)^\rho \sum_{i=1}^{i_0-1} \frac{1}{i^\rho} P\{T_i \leq n < T_{i+1}\}. \end{aligned}$$

Gathering the last two inequalities we conclude that

$$|F_*^n \lambda - F_*^n \lambda'| \leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} \frac{1}{i^\rho} P\{T_i \leq n < T_{i+1}\},$$

where K_1 depends only on i_0 . Fixing i_0 sufficiently large, from Lemma 2.18 we obtain the dependence of K_1 on φ and φ' . \square

The proof of the following proposition can be found in [2, Subsection A.3.4]. Note that it uses Lemma A.6 of the same article. While the proof of that lemma is not valid in our case, we obtained the same conclusion in Lemma 2.10.

Proposition 2.20. *There exists a constant $K_2 > 0$ such that, for $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$,*

$$P\{T_{i+1} - T_i > n\} \leq K_2(m \times m)\{T > n\}.$$

The constant K_2 depends only on c_φ and $c_{\varphi'}$.

We will now see that we can use Propositions 2.14, 2.19 and 2.20 to prove Theorem 2.6.

Proof of Theorem 2.6. Given $i \in \mathbb{N}$, we have

$$P\{T_i \leq n < T_{i+1}\} \leq \sum_{j=0}^i P\left\{T_{j+1} - T_j > \frac{n}{i+1}\right\}. \quad (2.19)$$

The last inequality is true because there exists $j \leq i$ such that $T_{j+1} - T_j > \frac{n}{i+1}$. In fact, if we assume the opposite, then $\frac{n}{i} i \geq \sum_{j=1}^i (T_{j+1} - T_j) = T_{i+1}$, which contradicts the assumption.

It follows, respectively from Proposition 2.19, (2.19) and Proposition 2.20 that

$$\begin{aligned}
|F_*^n \lambda - F_*^n \lambda'| &\leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} \frac{1}{i^\rho} P\{T_i \leq n < T_{i+1}\} \\
&\leq 2P\{T > n\} + K_1 \sum_{i=1}^{\infty} \frac{1}{i^\rho} \sum_{j=0}^i P\left\{T_{j+1} - T_j > \frac{n}{i+1}\right\} \\
&\leq 2P\{T > n\} + K_1 K_2 \sum_{i=1}^{\infty} \frac{1}{i^\rho} (i+1) (m \times m) \left\{T > \frac{n}{i+1}\right\}.
\end{aligned}$$

We know from Proposition 2.14 that $P\{T > n\} \leq C/n^{\zeta-1}$. So, taking $P = m \times m$ we obtain

$$\begin{aligned}
2P\{T > n\} + K_1 K_2 \sum_{i=1}^{\infty} \frac{1}{i^\rho} (i+1) (m \times m) \left\{T > \frac{n}{i+1}\right\} \\
\leq \frac{2C}{n^{\zeta-1}} + K_1 K_2 \sum_{i=1}^{\infty} \frac{i+1}{i^\rho} \left(\frac{i+1}{n}\right)^{\zeta-1} \\
\leq \left(2C + K_1 K_2 \sum_{i=1}^{\infty} \frac{(i+1)^\zeta}{i^\rho}\right) \frac{1}{n^{\zeta-1}}.
\end{aligned}$$

Since, in (2.6), we chose $\rho > \zeta + 1$, we obtain

$$|F_*^n \lambda - F_*^n \lambda'| \leq C' \frac{1}{n^{\zeta-1}}.$$

□

Proof of Corollary 2.7. Let $\rho = \frac{d\nu}{d\text{Leb}}$. Take $\tilde{\varphi} = b(\varphi + a)$, where $a \geq 0$ is such that $\tilde{\varphi}$ is bounded from below by a strictly positive constant and $b > 0$ is such that $\int \tilde{\varphi} \rho d\text{Leb} = 1$. Note that, since $\varphi \in \mathcal{G}_\theta$, then $\tilde{\varphi} \in \mathcal{G}_\theta^+$. In addition, as $\rho \in \mathcal{F}_\beta^+$ by Theorem 2.5 and $\mathcal{F}_\beta^+ \subseteq \mathcal{G}_\theta^+$, then $\tilde{\varphi} \rho \in \mathcal{G}_\theta^+$.

Let $P : L^2(\bar{\Delta}) \rightarrow L^2(\bar{\Delta})$ be the Perron-Frobenius operator associated with F , which is defined as follows:

$$\forall v, w \in L^2(\bar{\Delta}) \quad \int_{\bar{\Delta}} P(v) w d\bar{\nu} = \int_{\bar{\Delta}} v w \circ \bar{F} d\bar{\nu},$$

i.e., if μ is a signed measure and $\phi = \frac{d\mu}{d\text{Leb}}$, then $P(\phi) = \frac{d(F_*\mu)}{d\text{Leb}}$. So, if λ is the measure such

that $\frac{d\lambda}{d\text{Leb}} = \tilde{\varphi}\rho$, we have

$$\begin{aligned}
\mathcal{C}_n(\psi, \varphi, \nu) &= \left| \int (\psi \circ F^n) \varphi \, d\nu - \int \psi \, d\nu \int \varphi \, d\nu \right| \\
&= \frac{1}{b} \left| \int (\psi \circ F^n)(\tilde{\varphi}\rho) \, d\text{Leb} - \int \psi \rho \, d\text{Leb} \int \tilde{\varphi}\rho \, d\text{Leb} \right| \\
&= \frac{1}{b} \left| \int \psi P^n(\tilde{\varphi}\rho) \, d\text{Leb} - \int \psi \rho \, d\text{Leb} \right| \\
&\leq \frac{1}{b} \int |\psi| |P^n(\tilde{\varphi}\rho) - \rho| \, d\text{Leb} \\
&\leq \frac{1}{b} \|\psi\|_\infty |(F_*^n \lambda) - \nu|.
\end{aligned}$$

Since we already proved that $\frac{d\lambda}{d\text{Leb}} = \tilde{\varphi}\rho \in \mathcal{G}_\theta^+$, the conclusion follows from Theorem 2.6.

□

Chapter 3

Statistical properties

In this chapter, we will prove our two main theorems, stated in Sections 1.2 and 1.3.

3.1 Decay of correlations

This section is dedicated to the proof of Theorem A. We follow the approach of [21] and [2]. However, we have different hypothesis on the GMY structure.

Remember we want to prove that if f has a GMY structure, given $\zeta > 1$, $C_1 > 0$ and $\varphi, \psi \in H_\eta$, there is $C_2 > 0$ such that

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C_1}{n^\zeta} \quad \Rightarrow \quad \mathcal{C}_n(\varphi, \psi, \mu) \leq C_2 \max\left\{\frac{1}{n^{\zeta-1}}, \frac{1}{n^{\alpha\eta}}\right\},$$

where α is the exponent in (P₂) and (P₃) and C_2 depends only on C_1 , $|\varphi_\eta|$ and $|\psi_\eta|$.

We now present a theorem, which is an improved version of [22, Theorem 2], whose precise statement can be found in [2, Theorem 3.6] and proof in [2, Appendix A].

Theorem 3.1. *For $\varphi \in \mathcal{F}_\beta^+$ let $\bar{\lambda}$ be the measure whose density with respect to \bar{m} is φ .*

1. *If $\text{Leb}\{\bar{R} > n\} \leq Cn^{-\zeta}$, for some $C > 0$ and $\zeta > 1$, then there is $C' > 0$ such that*

$$|\bar{F}_*^n \bar{\lambda} - \bar{\nu}| \leq C' n^{-\zeta+1}.$$

2. *If $\text{Leb}\{\bar{R} > n\} \leq Ce^{-cn^\eta}$, for some $C, c > 0$ and $0 < \eta \leq 1$, then there is $C', c' > 0$ such that*

$$|\bar{F}_*^n \bar{\lambda} - \bar{\nu}| \leq C' e^{-c'n^\eta}.$$

Moreover, c' does not depend on φ and C' depends only on C_φ .

We will present three lemmas in which we keep reducing the problem of finding an upper bound for $\mathcal{C}_n(\varphi, \psi, \mu)$ to similar problems until we are left with finding an upper bound for the correlation of certain functions in $\bar{\Delta}$. Afterwards, we will show that we can use the above theorem to conclude the proof.

Remember that we defined $\mathcal{C}_n(\varphi, \psi, \mu) = \left| \int (\varphi \circ f^n) \psi d\mu - \int \varphi d\mu \int \psi d\mu \right|$ for φ, ψ belonging to H_η . We use analogous definitions and notation for functions in the tower Δ or in the quotient tower $\bar{\Delta}$ with respect to the corresponding measures.

It was shown in [22, Sections 2 and 4] that there exists a measure ν such that $\mu = \pi_* \nu$ and $\bar{\nu} = \bar{\pi}_* \nu$. For $\varphi, \psi \in H_\eta$, let $\tilde{\varphi} = \varphi \circ \pi$ and $\tilde{\psi} = \psi \circ \pi$.

First, note that

$$\begin{aligned} \int (\varphi \circ f^n) \psi d\mu &= \int (\varphi \circ f^n) \psi d(\pi_* \nu) = \int (\varphi \circ f^n \circ \pi) (\psi \circ \pi) d\nu \\ &= \int (\varphi \circ \pi \circ F^n) \tilde{\psi} d\nu = \int (\tilde{\varphi} \circ F^n) \tilde{\psi} d\nu \end{aligned}$$

and, arguing as above,

$$\int \varphi d\mu \int \psi d\mu = \int \tilde{\varphi} d\nu \int \tilde{\psi} d\nu.$$

So, $\mathcal{C}_n(\varphi, \psi, \mu) = \mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}, \nu)$.

Given $n \in \mathbb{N}$, fix a positive integer $k < n/2$. Let $\bar{\varphi}_k$ be a discretization of $\tilde{\varphi}$ defined on Δ as

$$\bar{\varphi}_k|_Q = \inf\{\tilde{\varphi} \circ F^k(x) : x \in Q\}, \quad \text{for } Q \in \mathcal{Q}_{2k}.$$

The next lemma proves that we only need to obtain the conclusion for $\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu)$.

Lemma 3.2. *For $\varphi, \psi \in H_\eta$, let $\tilde{\varphi}$, $\tilde{\psi}$ and $\bar{\varphi}_k$ be defined as above. Then*

$$|\mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu)| \leq \frac{C_2}{k^{\alpha\eta}},$$

where C_2 is a constant depending only on $|\varphi|_\eta$ and on $\|\psi\|_\infty$.

Proof. Notice that, since ν is F -invariant,

$$\mathcal{C}_{n-k}(\tilde{\varphi} \circ F^k, \tilde{\psi}, \nu) = \left| \int (\tilde{\varphi} \circ F^k \circ F^{n-k}) \tilde{\psi} d\nu - \int \tilde{\varphi} \circ F^k d\nu \int \tilde{\psi} d\nu \right| = \mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}, \nu). \quad (3.1)$$

Using the fact that φ is Hölder continuous and Lemma 2.1, we observe that for $Q \in \mathcal{Q}_{2k}$ and all $x, y \in Q$,

$$\begin{aligned} |\tilde{\varphi} \circ F^k(x) - \tilde{\varphi} \circ F^k(y)| &= |\varphi \circ \pi \circ F^k(x) - \varphi \circ \pi \circ F^k(y)| \\ &\leq |\varphi|_\eta d(\pi F^k(x), \pi F^k(y))^\eta \\ &\leq |\varphi|_\eta \text{diam}(\pi F^k(Q))^\eta \leq |\varphi|_\eta \left(\frac{C}{k^\alpha}\right)^\eta, \end{aligned}$$

which implies that, for any $x \in Q$,

$$|\tilde{\varphi} \circ F^k(x) - \bar{\varphi}_k(x)| = |\varphi \circ \pi \circ F^k(x) - \inf\{\varphi \circ \pi \circ F^k(y) : y \in Q\}| \leq |\varphi|_\eta \left(\frac{C}{k^\alpha}\right)^\eta. \quad (3.2)$$

Applying (3.1), (3.2) and the F -invariance of ν we obtain

$$\begin{aligned} |\mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu)| &= |\mathcal{C}_{n-k}(\tilde{\varphi} \circ F^k, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu)| \\ &= \left| \int (\tilde{\varphi} \circ F^k \circ F^{n-k}) \tilde{\psi} d\nu - \int \tilde{\varphi} \circ F^k d\nu \int \tilde{\psi} d\nu \right. \\ &\quad \left. - \int (\bar{\varphi}_k \circ F^{n-k}) \tilde{\psi} d\nu - \int \bar{\varphi}_k d\nu \int \tilde{\psi} d\nu \right| \\ &\leq \left| \int (\tilde{\varphi} \circ F^k - \bar{\varphi}_k) \circ F^{n-k} \tilde{\psi} d\nu \right| + \left| \int (\tilde{\varphi} \circ F^k - \bar{\varphi}_k) d\nu \int \tilde{\psi} d\nu \right| \\ &\leq \|\psi\|_\infty \left(\int |(\tilde{\varphi} \circ F^k - \bar{\varphi}_k) \circ F^{n-k}| d\nu + \int |\tilde{\varphi} \circ F^k - \bar{\varphi}_k| d\nu \right) \\ &\leq 2\|\psi\|_\infty |\varphi|_\eta \left(\frac{C}{k^\alpha}\right)^\eta. \end{aligned}$$

We only need to take $C_2 = 2\|\psi\|_\infty |\varphi|_\eta C^\eta$. □

Define $\bar{\psi}_k$ in a similar way to $\bar{\varphi}_k$. Denote by $\bar{\psi}_k \nu$ the signed measure whose density with respect to ν is $\bar{\psi}_k$ and by $\tilde{\psi}_k$ the density of $F_*^k \bar{\psi}_k \nu$ with respect to ν .

Note that, if ν is a signed measure, then the *total variation* of ν is defined as $|\nu| = \nu^+ + \nu^-$.

Now, we will see that it is sufficient to obtain the upper bound for $\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k, \nu)$.

Lemma 3.3. *For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$, $\tilde{\psi}$ and $\tilde{\psi}_k$ be defined as before. Then*

$$|\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k, \nu)| \leq \frac{C_3}{k^{\alpha\eta}},$$

where C_3 is a constant depending only on $|\varphi|_\infty$ and on $\|\psi\|_\eta$.

Proof. Observe that, since $\|\bar{\varphi}_k\|_\infty \leq \|\varphi\|_\infty$,

$$\begin{aligned}
& |\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k, \nu)| \\
&= \left| \int (\bar{\varphi}_k \circ F^{n-k}) \tilde{\psi} d\nu - \int \bar{\varphi}_k d\nu \int \tilde{\psi} d\nu \right. \\
&\quad \left. - \int (\bar{\varphi}_k \circ F^{n-k}) \tilde{\psi}_k d\nu - \int \bar{\varphi}_k d\nu \int \tilde{\psi}_k d\nu \right| \\
&\leq \int |\bar{\varphi}_k \circ F^{n-k}| |\tilde{\psi} - \tilde{\psi}_k| d\nu + \int |\bar{\varphi}_k| d\nu \int |\tilde{\psi} - \tilde{\psi}_k| d\nu \\
&\leq 2\|\varphi\|_\infty \int |\tilde{\psi} - \tilde{\psi}_k| d\nu.
\end{aligned} \tag{3.3}$$

Note that

$$F_*^k((\tilde{\psi} \circ F^k)\nu) = \tilde{\psi}\nu$$

and so, by the definition of $\tilde{\psi}_k$, we have

$$\begin{aligned}
\int |\tilde{\psi} - \tilde{\psi}_k| d\nu &= |\tilde{\psi}\nu - \tilde{\psi}_k\nu| \\
&= |F_*^k((\tilde{\psi} \circ F^k)\nu) - F_*^k(\tilde{\psi}_k\nu)| \\
&\leq |(\tilde{\psi} \circ F^k - \tilde{\psi}_k)\nu| \\
&= \int |\tilde{\psi} \circ F^k - \tilde{\psi}_k| d\nu.
\end{aligned} \tag{3.4}$$

Using Lemma 2.1, (3.3), (3.4) and the same argument as in (3.2) we get

$$|\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}, \nu) - \mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k, \nu)| \leq 2\|\varphi\|_\infty \int |\tilde{\psi} \circ F^k - \tilde{\psi}_k| d\nu \leq 2\|\varphi\|_\infty |\psi|_\eta \left(\frac{C_1}{k^\alpha}\right)^\eta.$$

To conclude, we just need to take $C_3 = 2\|\varphi\|_\infty |\psi|_\eta C_1^\eta$. \square

We will now show that we only need to prove the result for $\mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu})$.

Lemma 3.4. *For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$, $\tilde{\psi}_k$ and $\bar{\psi}_k$ be defined as before. Then*

$$\mathcal{C}_{n-k}(\bar{\varphi}_k, \tilde{\psi}_k, \nu) = \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}).$$

Proof. We have, by definition of $\tilde{\psi}_k$,

$$\int (\bar{\varphi}_k \circ F^{n-k}) \tilde{\psi}_k d\nu = \int \bar{\varphi}_k \circ F^{n-k} d(\tilde{\psi}_k\nu) = \int \bar{\varphi}_k d(F_*^{n-k}(\tilde{\psi}_k\nu)) = \int \bar{\varphi}_k d(F_*^n(\bar{\psi}_k\nu)).$$

Since $\bar{\varphi}_k$ is constant on γ^s leaves, then

$$\int \bar{\varphi}_k d(F_*^n(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d(\bar{\pi} F_*^n(\bar{\psi}_k \nu))$$

and, as F and \bar{F} are semi-conjugated by $\bar{\pi}$,

$$\int \bar{\varphi}_k d(\bar{\pi} F_*^n(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d(\bar{F}_*^n(\bar{\psi}_k \bar{\nu})) = \int (\bar{\varphi}_k \circ \bar{F}^n) \bar{\psi}_k d\bar{\nu}.$$

So, we have proved that

$$\int (\bar{\varphi}_k \circ F^{n-k}) \tilde{\psi}_k d\nu = \int (\bar{\varphi}_k \circ \bar{F}_*^n) \bar{\psi}_k d\bar{\nu}. \quad (3.5)$$

Additionally, as $\bar{\varphi}_k$ is constant on γ^s leaves, and using the definition of $\tilde{\psi}_k$ and the \bar{F} -invariance of $\bar{\nu}$,

$$\int \bar{\varphi}_k d\nu \int \tilde{\psi}_k d\nu = \int \bar{\varphi}_k d\bar{\nu} \int d(F_*^k(\bar{\psi}_k \nu)) = \int \bar{\varphi}_k d\bar{\nu} \int \bar{\psi}_k d\bar{\nu}. \quad (3.6)$$

Gathering (3.5) and (3.6), we obtain the conclusion. \square

Without loss of generality we may assume that $\bar{\psi}_k$ is not the null function. We define

$$b_k = \frac{1}{\int (\bar{\psi}_k + 2\|\bar{\psi}_k\|_\infty) d\bar{\nu}} \quad \text{and} \quad \hat{\psi}_k = b_k(\bar{\psi}_k + 2\|\bar{\psi}_k\|_\infty).$$

Since $\|\bar{\psi}_k\|_\infty \leq \bar{\psi}_k + 2\|\bar{\psi}_k\|_\infty \leq 3\|\bar{\psi}_k\|_\infty$, then

$$\frac{1}{3\|\bar{\psi}_k\|_\infty} \leq b_k \leq \frac{1}{\|\bar{\psi}_k\|_\infty}.$$

Defining $\bar{\rho} = \frac{d\bar{\nu}}{d\bar{m}}$, we have, by the definition of b_k ,

$$\int \hat{\psi}_k \bar{\rho} d\bar{m} = 1.$$

Since $\bar{\psi}_k$ is constant on elements of \mathcal{Q}_{2k} , so is $\hat{\psi}_k$. Denote by $\hat{\lambda}_k$ the probability measure on $\bar{\Delta}$ whose density with respect to \bar{m} is $\hat{\psi}_k \bar{\rho}$.

Lemma 3.5. *For $\varphi, \psi \in H_\eta$, let $\bar{\varphi}_k$ and $\bar{\psi}_k$ be defined as before. Then*

$$\mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}) \leq C_4 |\bar{F}_*^{n-2k} \bar{\lambda}_k - \nu|,$$

where C_4 is a constant depending only on $|\varphi|_\infty$ and on $\|\psi\|_\infty$.

Proof. Notice that, by the definition of $\widehat{\psi}_k$ and the \bar{F} -invariance of $\bar{\nu}$,

$$\begin{aligned} \int (\bar{\varphi}_k \circ \bar{F}^n) \bar{\psi}_k d\bar{\nu} &= \int (\bar{\varphi}_k \circ \bar{F}^n) \left(\frac{1}{b_k} \widehat{\psi}_k - 2\|\bar{\psi}_k\|_\infty \right) d\bar{\nu} \\ &= \frac{1}{b_k} \int (\bar{\varphi}_k \circ \bar{F}^n) \widehat{\psi}_k d\bar{\nu} - 2\|\bar{\psi}_k\|_\infty \int \bar{\varphi}_k \circ \bar{F}^n d\bar{\nu} \\ &= \frac{1}{b_k} \int (\bar{\varphi}_k \circ \bar{F}^n) \widehat{\psi}_k d\bar{\nu} - 2\|\bar{\psi}_k\|_\infty \int \bar{\varphi}_k d\bar{\nu} \end{aligned}$$

and, similarly,

$$\int \bar{\varphi}_k d\bar{\nu} \int \bar{\psi}_k d\bar{\nu} = \frac{1}{b_k} \int \bar{\varphi}_k d\bar{\nu} \int \widehat{\psi}_k d\bar{\nu} - 2\|\bar{\psi}_k\|_\infty \int \bar{\varphi}_k d\bar{\nu}.$$

Then, using the last two equalities and the definitions of $\bar{\rho}$ and $\widehat{\lambda}_k$, we obtain

$$\begin{aligned} \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}) &= \left| \int (\bar{\varphi}_k \circ \bar{F}^n) \bar{\psi}_k d\bar{\nu} - \int \bar{\varphi}_k d\bar{\nu} \int \bar{\psi}_k d\bar{\nu} \right| \\ &= \frac{1}{b_k} \left| \int (\bar{\varphi}_k \circ \bar{F}^n) \widehat{\psi}_k d\bar{\nu} - \int \bar{\varphi}_k d\bar{\nu} \int \widehat{\psi}_k d\bar{\nu} \right| \\ &= \frac{1}{b_k} \left| \int (\bar{\varphi}_k \circ \bar{F}^n) \widehat{\psi}_k \bar{\rho} d\bar{m} - \int \bar{\varphi}_k \bar{\rho} d\bar{m} \int \widehat{\psi}_k \bar{\rho} d\bar{m} \right| \\ &= \frac{1}{b_k} \left| \int \bar{\varphi}_k \frac{d(\bar{F}_*^n \widehat{\lambda}_k)}{d\bar{m}} d\bar{m} - \int \bar{\varphi}_k \bar{\rho} d\bar{m} \right| \\ &\leq \frac{1}{b_k} \int |\bar{\varphi}_k| \left| \frac{d(\bar{F}_*^n \widehat{\lambda}_k)}{d\bar{m}} - \bar{\rho} \right| d\bar{m}. \end{aligned} \tag{3.7}$$

Setting $\bar{\lambda}_k = \bar{F}_*^{2k} \widehat{\lambda}_k$ and since $k < n/2$, we have

$$\frac{d}{d\bar{m}} \bar{F}_*^n \widehat{\lambda}_k = \frac{d}{d\bar{m}} \bar{F}_*^{n-2k} \bar{\lambda}_k,$$

and so, using (3.7), we have

$$\begin{aligned} \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}) &\leq \frac{1}{b_k} \int |\bar{\varphi}_k| \left| \frac{d(\bar{F}_*^n \widehat{\lambda}_k)}{d\bar{m}} - \bar{\rho} \right| d\bar{m} \\ &= \frac{1}{b_k} \int |\bar{\varphi}_k| \left| \frac{d(\bar{F}_*^{n-2k} \bar{\lambda}_k)}{d\bar{m}} - \frac{d\bar{\nu}}{d\bar{m}} \right| d\bar{m} \\ &\leq \frac{1}{b_k} \|\bar{\varphi}_k\|_\infty |\bar{F}_*^{n-2k} \bar{\lambda}_k - \bar{\nu}|. \end{aligned}$$

Since $\frac{1}{b_k} \leq 3\|\bar{\psi}_k\|_\infty$, we get

$$\begin{aligned} \mathcal{C}_n(\bar{\varphi}_k, \bar{\psi}_k, \bar{\nu}) &\leq \frac{1}{b_k} \|\bar{\varphi}_k\|_\infty |\bar{F}_*^{n-2k} \bar{\lambda}_k - \bar{\nu}| \\ &\leq 3\|\bar{\psi}_k\|_\infty \|\bar{\varphi}_k\|_\infty |\bar{F}_*^{n-2k} \bar{\lambda}_k - \bar{\nu}| \\ &\leq 3\|\psi\|_\infty \|\varphi\|_\infty |\bar{F}_*^{n-2k} \bar{\lambda}_k - \bar{\nu}|. \end{aligned}$$

We just need to take $C_4 = 3\|\psi\|_\infty\|\varphi\|_\infty$. □

Gathering everything that was proved in the previous lemmas, we get

$$\begin{aligned}\mathcal{C}_n(\varphi, \psi, \mu) &= \mathcal{C}_n(\tilde{\varphi}, \tilde{\psi}, \mu) \\ &\leq \mathcal{C}_{n-k}(\tilde{\varphi}_k, \tilde{\psi}, \mu) + \frac{C_2}{k^{\alpha\eta}} \\ &\leq \mathcal{C}_{n-k}(\tilde{\varphi}_k, \tilde{\psi}_k, \nu) + \frac{C_3 + C_2}{k^{\alpha\eta}} \\ &\leq C_4|\bar{F}_*^{n-2k}\bar{\lambda}_k - \nu| + \frac{C_3 + C_2}{k^{\alpha\eta}}.\end{aligned}$$

Let ϕ_k be the density of the measure $\bar{\lambda}_k$ with respect to \bar{m} . The next lemma, whose proof is given in [2, Lemma 4.1], implies that $\phi_k \in \mathcal{F}_\beta^+$.

Lemma 3.6. *There is $C > 0$, not depending on ϕ_k , such that*

$$|\phi_k(\bar{x}) - \phi_k(\bar{y})| \leq C\beta^{\bar{s}(\bar{x}, \bar{y})}, \quad \forall \bar{x}, \bar{y} \in \bar{\Delta}.$$

Lemma 3.6 and the inequality $\mathcal{C}_n(\varphi, \psi, \mu) \leq C_4|\bar{F}_*^{n-2k}\bar{\lambda}_k - \nu| + \frac{C_3+C_2}{k^{\alpha\eta}}$, proved before, allow us to use Theorem 3.1 to obtain

$$\mathcal{C}_n(\varphi, \psi, \mu) \leq C_4 \frac{C'}{(n-2k)^{\zeta-1}} + \frac{C_3 + C_2}{k^{\alpha\eta}} \leq C \max\left\{\frac{1}{n^{\zeta-1}}, \frac{1}{n^{\alpha\eta}}\right\}.$$

This concludes the proof of Theorem A.

3.2 Large deviations

In this section we will prove Theorem B. It is based on the proofs in [14] and [12], although our assumptions are different.

Recall that we want to prove that, if f has a GMY structure Λ , then there are $\eta_0 > 0$ and $\zeta_0 > 1$ such that for all $\eta > \eta_0$, $1 < \zeta < \zeta_0$, $\varepsilon > 0$, $p > \max\{1, \zeta - 1\}$, $C_1 > 0$ and $\phi \in \mathcal{H}_\eta$, there exists $C_2 > 0$ such that

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C_1}{n^\zeta} \quad \Rightarrow \quad LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p}n^{\zeta-1}}.$$

where C_2 depends only on C_1 , p , $|\psi|_\eta$ and c_ϕ .

Remark 3.7. If $\zeta \geq \frac{\alpha\eta-1}{e^{C_1}} - 1$, we obtain the conclusion of the above theorem replacing ζ by any κ such that $1 < \kappa < \frac{\alpha\eta-1}{e^{C_1}} - 1$.

The proof of Theorem B uses the construction of a function $\psi \in \mathcal{G}_\theta(\bar{\Delta})$, which will be done in Proposition 3.10, for $\theta = \alpha\eta - 1$. However, if $\psi \in \mathcal{F}_\gamma(\bar{\Delta})$ with $0 < \gamma < 1$, then $\psi \in \mathcal{G}_\theta(\bar{\Delta})$ for all $\theta > 0$. So, in this case, Theorem B is valid for any $\zeta > 1$, choosing θ arbitrarily large.

Lemma 3.8. *There exists a constant $C_3 > 0$ such that, for all $x, y \in \gamma^u$ with $s(x, y) \neq 0$ and all $0 \leq k < R$, we have*

$$d(f^k x, f^k y) \leq \frac{C_3}{s(x, y)^\alpha}.$$

Proof. Let $n \in \mathbb{N}$ be such that $s(x, y) = n$. Then, using (P₃), we get

$$\begin{aligned} d(f^k x, f^k y) &= d(f^{k-R_n}(f^{R_n} x), f^{k-R_n}(f^{R_n} y)) \leq \frac{C'}{(R_n - k)^\alpha} d(f^{R_n} x, f^{R_n} y) \\ &\leq \frac{C'}{(R_n - k)^\alpha} \text{diam}(M) \leq \frac{C_3}{(R_n - k)^\alpha}. \end{aligned}$$

Since $R - k \geq 1$, then $R_n - k \geq n$, and so

$$d(f^k x, f^k y) \leq \frac{C_3}{(s(x, y))^\alpha}.$$

□

Definition 3.9. We say that a function $\psi : \Delta \rightarrow \mathbb{R}$ depends only on future coordinates if, given $x, y \in \Delta$ with $y \in \gamma_s(x)$, then $\psi(x) = \psi(y)$.

We will now present a result that characterizes an Hölder function in M with the help of a function in the quotient tower with a certain regularity. It is an adaptation to the polynomial case of [13, Lemma 3.2] which proves an analogous result for the exponential case.

Proposition 3.10. *Let f has a GMY structure Λ and $\phi : M \rightarrow \mathbb{R}$ be a function belonging to \mathcal{H}_η for $\eta > 1/\alpha$. Then there exist functions $\chi, \psi : \Delta \rightarrow \mathbb{R}$ such that:*

1. $\chi \in L^\infty(\Delta)$ and $\|\chi\|_\infty$ depends only on $|\phi|_\eta$;
2. $\phi \circ \pi = \psi + \chi - \chi \circ F$;

3. ψ depends only on future coordinates;

4. the function $\psi : \bar{\Delta} \rightarrow \mathbb{R}$ belongs to \mathcal{G}_θ , for $\theta = \alpha\eta - 1$. More precisely, there exists $c_\psi > 0$, depending only on $|\phi|_\eta$, such that for all $p, q \in \bar{\Delta}$,

$$|\psi(p) - \psi(q)| \leq \frac{c_\psi}{\max\{\bar{s}(p, q), 1\}^\theta}.$$

Proof. Let us fix $\gamma^u \in \Gamma^u$. Given $p = (x, l) \in \Delta$, let $\hat{p} = (\hat{x}, l)$, where \hat{x} is the unique point in $\gamma^s(x) \cap \gamma^u$ and define

$$\chi(p) = \sum_{j=0}^{\infty} (\phi\pi F^j(p) - \phi\pi F^j(\hat{p})).$$

Observing that $\pi \circ F^j(p) = f^j \circ \pi(p) = f^{j+l}(x)$ and using (P₂), we have

$$\begin{aligned} |\chi(p)| &\leq \sum_{j=0}^{\infty} \left| \phi\pi F^j(p) - \phi\pi F^j(\hat{p}) \right| \\ &\leq \sum_{j=0}^{\infty} |\phi|_\eta d(f^{j+l}(x), f^{j+l}(\hat{x}))^\eta \\ &\leq |\phi|_\eta C^\eta \sum_{j=0}^{\infty} \frac{1}{j^{\alpha\eta}} = C' |\phi|_\eta, \end{aligned}$$

since $\alpha\eta > 1$. So, 1. is verified.

Defining $\psi = \phi \circ \pi - \chi + \chi \circ F$, 2. is verified and, as

$$\begin{aligned} \psi(p) &= \phi\pi(p) - \sum_{j=0}^{\infty} \phi\pi F^j(p) + \sum_{j=0}^{\infty} \phi\pi F^j(\hat{p}) + \sum_{j=0}^{\infty} \phi\pi F^{j+1}(p) - \sum_{j=0}^{\infty} \phi\pi F^j(\hat{F}p) \\ &= \sum_{j=0}^{\infty} (\phi\pi F^j(\hat{p}) - \phi\pi F^j(\hat{F}p)), \end{aligned}$$

ψ depends only on future coordinates. So, 3. is proved.

We are left to prove 4. Let $n \in \mathbb{N}$ and $p, q \in \Delta$. Then

$$\begin{aligned} |\psi(p) - \psi(q)| &\leq \sum_{j=0}^n |\phi\pi F^j(\hat{p}) - \phi\pi F^j(\hat{q})| + \sum_{j=0}^{n-1} |\phi\pi F^j(\hat{F}p) - \phi\pi F^j(\hat{F}q)| \\ &\quad + \sum_{j=n+1}^{\infty} |\phi\pi F^j(\hat{p}) - \phi\pi F^{j-1}(\hat{F}p)| + \sum_{j=n+1}^{\infty} |\phi\pi F^j(\hat{q}) - \phi\pi F^{j-1}(\hat{F}q)|. \quad (3.8) \end{aligned}$$

Since the choice of n is arbitrary we can assume that $s(p, q) \approx 2n$. This will mean that there will be no separation during the calculations of the first two terms.

We will consider separately each term of the right-hand side of (3.8). We start with the third term. When $p \neq (x, R(x) - 1)$ then $F\widehat{p} = \widehat{Fp}$. If $p = (x, R(x) - 1)$ then $F\widehat{p} = (f^R\widehat{x}, 0)$ and $\widehat{Fp} = (\widehat{f^Rx}, 0)$ and so $\pi F^j\widehat{p} = f^{j-1}f^R\widehat{x}$ and $\pi F^{j-1}\widehat{Fp} = f^{j-1}\widehat{f^Rx}$. But $f^R\widehat{x}$ and $\widehat{f^Rx}$ belong to the same stable leaf, and then, using (P₂),

$$\begin{aligned} |\phi\pi F^j(\widehat{p}) - \phi\pi F^{j-1}(\widehat{Fp})| &= |\phi f^{j-1}f^R(\widehat{x}) - \phi f^{j-1}(\widehat{f^Rx})| \\ &\leq |\phi|_\eta d(f^{j-1}f^R(\widehat{x}), f^{j-1}(\widehat{f^Rx}))^\eta \leq |\phi|_\eta \frac{1}{(j-1)^{\alpha\eta}} = |\phi|_\eta \frac{1}{(j-1)^{\theta+1}} \end{aligned}$$

and then, recalling that $s(p, q) \approx 2n$,

$$\begin{aligned} \sum_{j=n+1}^{\infty} |\phi\pi F^j\widehat{p} - \phi\pi F^{j-1}\widehat{Fp}| &\leq |\phi|_\eta \sum_{j=n+1}^{\infty} \frac{1}{(j-1)^{\theta+1}} \\ &\leq C' |\phi|_\eta \frac{1}{n^\theta} \approx 2^\theta C' |\phi|_\eta \frac{1}{s(p, q)^\theta}. \end{aligned} \quad (3.9)$$

The calculations for the fourth term of the right-hand side of (3.8) are similar.

Let us consider now the first term. Recall that there is no separation during the calculations. Let $p = (x, l)$ and $q = (y, l)$. Then

$$\pi F^j(\widehat{p}) = f^{j+l}(\widehat{x}) = f^L f^{R(x)^J}(\widehat{x}), \quad \text{where } J \leq j \text{ and } L < R((f^R)^J(\widehat{x})),$$

and

$$\pi F^j(\widehat{q}) = f^{j+l}(\widehat{y}) = f^L f^{R(x)^J}(\widehat{y}), \quad \text{where } J \leq j \text{ and } L < R((f^R)^J(\widehat{y})).$$

Then, since $\phi \in \mathcal{H}_\eta$ and using the calculations above and Lemma 3.8,

$$\begin{aligned} |\phi\pi F^j(\widehat{p}) - \phi\pi F^j(\widehat{q})| &\leq |\phi|_\eta d(\pi F^j(\widehat{p}), \pi F^j(\widehat{q}))^\eta = |\phi|_\eta d(f^L f^{R(x)^J}(\widehat{x}), f^L f^{R(x)^J}(\widehat{y}))^\eta \\ &\leq C_3 |\phi|_\eta \frac{1}{s(f^{R(x)^J}(\widehat{x}), f^{R(x)^J}(\widehat{y}))^{\alpha\eta}} = C_3 |\phi|_\eta \frac{1}{(s(\widehat{x}, \widehat{y}) - J)^{\theta+1}} \\ &\leq C_3 |\phi|_\eta \frac{1}{(s(\widehat{x}, \widehat{y}) - j)^{\theta+1}} \approx C_3 |\phi|_\eta \frac{1}{(2n - j)^{\theta+1}}. \end{aligned}$$

So, we have

$$\begin{aligned} \sum_{j=0}^n |\phi\pi F^j(\widehat{p}) - \phi\pi F^j(\widehat{q})| &\leq C_3 |\phi|_\eta \sum_{j=0}^n \frac{1}{(2n - j)^{\theta+1}} \\ &\leq C'' |\phi|_\eta \frac{1}{n^\theta} \approx 2^\theta C'' |\phi|_\eta \frac{1}{s(p, q)^\theta}. \end{aligned} \quad (3.10)$$

The calculations for the second term of the right-hand side of (3.8) are analogous.

From (3.9) and (3.10), we obtain

$$|\psi(p) - \psi(q)| \leq \frac{c_\psi}{s(p, q)^\theta},$$

where c_ψ depends only on $|\phi|_\eta$. □

A sequence of σ -algebras $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ is said to form a *filtration* if, for all $i \in \mathbb{N}$, $\mathcal{F}_i \subset \mathcal{F}_{i+1}$. A sequence of random variables $\{X_i\}_{i \in \mathbb{N}}$ is *adapted to a filtration* $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$, if, for all $i \in \mathbb{N}$, X_i is measurable with respect to \mathcal{F}_i . A sequence of random variables $\{X_i\}_{i \in \mathbb{N}}$ is said to be a sequence of *martingale differences* with respect to a filtration $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ if it is adapted to the filtration and, for all $i \in \mathbb{N}$,

$$E(X_1) = 0, \quad E(X_{i+1}|\mathcal{F}_i) = 0$$

Before proceeding, we present an auxiliary result that can be found in [17, Theorem 2.5].

Theorem 3.11. *Let $\{X_i\}$ be a sequence of L^2 random variables with filtration \mathcal{G}_i . Let $p \geq 1$ and define*

$$b_{i,n} = \max_{i \leq k \leq n} \|X_i \sum_{j=i}^k E(X_j|\mathcal{G}_i)\|_p. \quad (3.11)$$

Then

$$E|X_1 + \cdots + X_n|^{2p} \leq (4p \sum_{i=1}^n b_{i,n})^p.$$

We recall Markov Inequality, which states that given a measure space (X, \mathcal{A}, μ) , a function $f \in L^1(x, \mu)$ and $\varepsilon > 0$ then

$$\mu\{|f| > \varepsilon\} \leq \frac{1}{\varepsilon} \int_X |f| d\mu.$$

Given $\psi : \bar{\Delta} \rightarrow \mathbb{R}$, we define $\psi_n = \sum_{i=1}^{n-1} \psi \circ \bar{F}^i$.

In the next proposition we prove that, in the quotient tower, a control on the decay of correlations implies a control on large deviations. This proof is based on [12, Theorem 1.2].

Proposition 3.12. *Let $\zeta > 0$ and $\psi \in \mathcal{G}_\theta(\bar{\Delta})$, for some $\theta > 0$. Suppose there exists $C_4 > 0$ such that, for all $w \in L^\infty(\bar{\Delta})$ and all $n \geq n_0$ we have*

$$C_n(w, \psi, \bar{\nu}) \leq \frac{C_4}{n^\zeta},$$

where C_4 depends only on c_ψ and $\|\psi\|_\infty$. Then, for $\varepsilon > 0$ and $p > \max\{1, \zeta\}$,

$$LD(\psi, \varepsilon, n, \bar{\nu}) \leq \frac{C_5}{\varepsilon^{2p} n^\zeta},$$

where $C_5 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$.

Proof. We may assume, without loss of generality, that $\int \psi d\bar{\nu} = 0$. By Markov's inequality, we have

$$\bar{\nu}\left\{\left|\frac{1}{n}\psi_n\right| > \varepsilon\right\} = \bar{\nu}\left\{\left|\frac{1}{n}\psi_n\right|^{2p} > \varepsilon^{2p}\right\} \leq \frac{1}{\varepsilon^{2p}} \int_{\bar{\Delta}} \left|\frac{1}{n}\psi_n\right|^{2p} d\bar{\nu} = \|\psi_n\|_{2p}^{2p} \frac{1}{\varepsilon^{2p}} \frac{1}{n^{2p}}$$

and so we only need to prove that

$$\|\psi_n\|_{2p}^{2p} \leq C_5 n^{2p-\zeta}, \quad (3.12)$$

where $C_5 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$.

Recall that we defined $P : L^2(\bar{\Delta}) \rightarrow L^2(\bar{\Delta})$, the Perron-Frobenius operator associated with F , as follows:

$$\forall v, w \in L^2(\bar{\Delta}) \quad \int_{\bar{\Delta}} P(v) w d\bar{\nu} = \int_{\bar{\Delta}} v w \circ \bar{F} d\bar{\nu}.$$

By this definition and the hypothesis, we have, for all $w \in L^\infty(\bar{\Delta})$,

$$\left| \int_{\bar{\Delta}} P^n(\psi) w d\bar{\nu} \right| = \left| \int_{\bar{\Delta}} \psi w \circ \bar{F}^n d\bar{\nu} \right| = C_n(w, \psi, \bar{\nu}) \leq \frac{C_4}{n^\zeta}. \quad (3.13)$$

Choosing $w = \text{sgn } P^n(\psi)$ in (3.13) we get

$$\|P^n(\psi)\|_1 = \int_{\bar{\Delta}} P^n(\psi) \text{sgn}(P^n\psi) d\bar{\nu} \leq \frac{C_4}{n^\zeta}.$$

Note that, from now on, C_4 depends only on c_ψ as $\|\text{sgn } P^n(\psi)\|_\infty = 1$.

Since $\|P^n(\psi)\|_\infty \leq \|\psi\|_\infty$ we have

$$\begin{aligned} \|P^n(\psi)\|_p &= \left(\int_{\bar{\Delta}} |P^n(\psi)|^{p-1} |P^n(\psi)| d\bar{\nu} \right)^{\frac{1}{p}} \\ &\leq \|P^n(\psi)\|_\infty^{1-\frac{1}{p}} \|P^n(\psi)\|_1^{\frac{1}{p}} \leq \|\psi\|_\infty^{1-\frac{1}{p}} \frac{(C_4)^{\frac{1}{p}}}{n^{\frac{\zeta}{p}}} = \frac{C'}{n^{\frac{\zeta}{p}}}, \end{aligned} \quad (3.14)$$

where $C' = (C_4)^{1/p} \|\psi\|_\infty^{1-1/p}$ depends only on p , c_ψ and $\|\psi\|_\infty$.

Define

$$\chi_k = \sum_{n=1}^k P^n(\psi) \quad \text{and} \quad \varphi_k = \psi - \chi_k \circ F + \chi_k - P^k(\psi). \quad (3.15)$$

Observe that, from (3.14), $\chi_k, \varphi_k \in L^p(\bar{\Delta})$,

$$\|\chi_k\|_p \leq \sum_{n=1}^k \|P^n(\psi)\|_p \leq C' \sum_{n=1}^k \frac{1}{n^{\frac{\zeta}{p}}} \leq C' \frac{k^{1-\frac{\zeta}{p}}}{1-\frac{\zeta}{p}} \quad (3.16)$$

and

$$\|\varphi_k\|_p \leq \|\psi\|_p + 2\|\chi_k\|_p + \|P^k(\psi)\|_p \leq 4C' \frac{k^{1-\frac{\zeta}{p}}}{1-\frac{\zeta}{p}} \quad (3.17)$$

for k sufficiently large.

We are going to prove that $P(\varphi_k) = 0$. In fact, given $w \in L^2(\bar{\Delta})$, we have, since $\bar{\nu}$ is F -invariant,

$$\begin{aligned} \int_{\bar{\Delta}} P(\chi_k)w \, d\bar{\nu} - \int_{\bar{\Delta}} P(\chi_k \circ F)w \, d\bar{\nu} &= \int_{\bar{\Delta}} \chi_k w \circ F \, d\bar{\nu} - \int_{\bar{\Delta}} \chi_k \circ F w \circ F \, d\bar{\nu} \\ &= \int_{\bar{\Delta}} \chi_k w \circ F \, d\bar{\nu} - \int_{\bar{\Delta}} \chi_k w \, d\bar{\nu} \\ &= \sum_{n=1}^k \int_{\bar{\Delta}} P^n(\psi) w \circ F \, d\bar{\nu} - \sum_{n=1}^k \int_{\bar{\Delta}} P^n(\psi) w \, d\bar{\nu} \\ &= \sum_{n=1}^k \int_{\bar{\Delta}} \psi(w \circ F^{n+1} - w \circ F^n) \, d\bar{\nu} \\ &= \int_{\bar{\Delta}} \psi(w \circ F^{k+1} - w \circ F) \, d\bar{\nu}. \end{aligned}$$

On the other hand,

$$\int_{\bar{\Delta}} P(\psi)w - P^{k+1}(\psi)w \, d\nu = \int_{\bar{\Delta}} \psi(w \circ F - w \circ F^{k+1}) \, d\nu.$$

So, $P(\varphi_k) = P(\psi) - P(\chi_k \circ F) + P(\chi_k) - P^{k+1}(\psi) = 0$.

The operator P is the adjoint operator of $U : L^2(\bar{\Delta}, \bar{\nu}) \rightarrow L^2(\bar{\Delta}, \bar{\nu})$ defined by $U(v) = v \circ F$. Besides, $P \circ U = I$, where I is the identity operator, and $U \circ P = E(\cdot | F^{-1}\mathcal{M})$, where \mathcal{M} is the underlying σ -algebra. So, $E(\varphi_k | F^{-1}\mathcal{M}) = 0$ and $E(\varphi_k \circ F^j | F^{-(n+1)}\mathcal{M}) = 0$. Then, $\{\varphi_k \circ F^n : n \in \mathbb{N}_0\}$ is a sequence of reverse martingale differences. Passing to the

natural extension (see [17]), $\{\varphi_k \circ F^n : n \in \mathbb{N}_0\}$ is a sequence of martingale differences with respect to a filtration $\{\mathcal{G}_n : n \in \mathbb{N}_0\}$.

Defining $X_j = \psi \circ F^j$ in Theorem 3.11, we have

$$b_{i,n} = \max_{i \leq l \leq n} \left\| \psi \circ F^i \sum_{j=i}^l E(\psi \circ F^j | \mathcal{G}_i) \right\|_p \leq \|\psi\|_\infty \max_{i \leq l \leq n} \left\| \sum_{j=i}^l E(\psi \circ F^j | \mathcal{G}_i) \right\|_p$$

and, by that theorem, we obtain

$$\|\psi_n\|_{2p}^{2p} \leq (4p \sum_{i=1}^n b_{i,n})^p. \quad (3.18)$$

Recalling the definition of φ_k in (3.15), we have

$$\sum_{j=i}^l E(\psi \circ F^j | \mathcal{G}_i) = \varphi_k \circ F^i + E(\chi_k \circ F^{l+1} | \mathcal{G}_i) - E(\chi_k \circ F^i | \mathcal{G}_i) + \sum_{j=i}^l E(P^k(\psi) \circ F^j | \mathcal{G}_i),$$

and so, using (3.14), (3.16) and (3.17), we obtain

$$\left\| \sum_{j=i}^l E(\psi \circ F^j | \mathcal{G}_i) \right\|_p \leq \|\varphi_k\|_p + 2\|\chi_k\|_p + n\|P^k(\psi)\|_p \leq C' \left(6 \frac{k^{1-\frac{\zeta}{p}}}{1-\frac{\zeta}{p}} + \frac{n}{k^{\frac{\zeta}{p}}} \right).$$

Then,

$$b_{i,n} \leq C'' \left(6 \frac{k^{1-\frac{\zeta}{p}}}{1-\frac{\zeta}{p}} + \frac{n}{k^{\frac{\zeta}{p}}} \right),$$

where C'' depends only on p , c_ψ and $\|\psi\|_\infty$. Then, recalling (3.18) and choosing $k = n$, we conclude that

$$\|\psi_n\|_{2p}^{2p} \leq (4p \sum_{i=1}^n b_{i,n})^p \leq C_5 n^{2p-\zeta},$$

where C_5 depends only on p , c_ψ and $\|\psi\|_\infty$. □

Proposition 3.13. *Suppose that f has a GMY structure Λ and $\phi : M \rightarrow \mathbb{R}$ is a function belonging to \mathcal{H}_η . Assume that there exist $\psi \in \mathcal{G}_\theta$, for some $\theta > 0$, and $\chi \in L^\infty(\Delta)$ such that $\phi \circ \pi = \psi + \chi - \chi \circ \bar{F}$ where ψ depends only on future coordinates. Fixing $\zeta > 0$, assume that, for all $w \in L^\infty(\bar{\Delta})$ and all $n \geq n_0$ there exists $C_4 > 0$, depending only on c_ψ and $\|w\|_\infty$, such that*

$$C_n(w, \psi, \bar{\nu}) \leq \frac{C_4}{n^\zeta}.$$

Then, for $\varepsilon > 0$ and $p > \max\{1, \zeta\}$,

$$LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p} n^\zeta},$$

where $C_2 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$.

Proof. We may assume, without loss of generality, that $\int \phi d\mu = 0$.

By assumption, we can write $\phi \circ \pi = \psi + \chi - \chi \circ \bar{F}$ where $\psi \in \mathcal{G}_\theta$, for some $\theta > 0$, $\chi \in L^\infty(\Delta)$ and ψ depends only on future coordinates. By Proposition 3.12 we have

$$\bar{\nu}\left\{\left|\frac{1}{n}\psi_n\right| > \varepsilon\right\} \leq \frac{C_5}{\varepsilon^{2p} n^\zeta}, \quad (3.19)$$

where $C_5 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$.

Note that

$$\mu\left\{\left|\frac{1}{n}\phi_n(x)\right| > \varepsilon\right\} = \nu\left\{\left|\frac{1}{n}\phi_n(\pi y)\right| > \varepsilon\right\} \quad (3.20)$$

and

$$\begin{aligned} \phi_n \circ \pi &= \sum_{k=0}^{n-1} \phi \circ f^k \circ \pi = \sum_{k=0}^{n-1} \phi \circ \pi \circ F^k \\ &= \sum_{k=0}^{n-1} \psi \circ F^k + \sum_{k=0}^{n-1} \chi \circ F^k - \sum_{k=0}^{n-1} \chi \circ F^{k+1} \\ &= \psi_n + \chi - \chi \circ F^n. \end{aligned}$$

Let $y \in \Delta$ be such that $\frac{1}{n}|\psi_n(y) + \chi(y) - \chi(F^n y)| > \varepsilon$. Then $\frac{1}{n}|\psi_n(y)| + \frac{2}{n}\|\chi\|_\infty > \varepsilon$ and so

$$\left\{\frac{1}{n}|\phi_n \circ \pi| > \varepsilon\right\} = \left\{\frac{1}{n}|\psi_n + \chi - \chi \circ F^n| > \varepsilon\right\} \subseteq \left\{\frac{1}{n}|\psi_n| > \varepsilon - \frac{2}{n}\|\chi\|_\infty\right\}.$$

From (3.19), (3.20) and the last inclusion we get, for a sufficiently large n_0 and $n \geq n_0$,

$$\mu\left\{\left|\frac{1}{n}\phi_n(x)\right| > \varepsilon\right\} \leq \bar{\nu}\left\{\frac{1}{n}|\psi_n| > \varepsilon - \frac{2}{n}\|\chi\|_\infty\right\} \leq \frac{C_2}{\varepsilon^{2p} n^\zeta}, \quad (3.21)$$

where $C_2 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$. \square

Proof of Theorem B. Note that to obtain the conclusion we only need to verify the assumptions of Proposition 3.13.

$\psi \in \mathcal{G}_\theta(\Delta)$, for $\theta = \alpha\eta - 1$, and $\chi \in L^\infty(\Delta)$ such that $\phi \circ \pi = \psi + \chi - \chi \circ \bar{F}$, where ψ depends only on future coordinates and c_ψ depends only on $|\phi|_\eta$. So, we may apply Corollary 2.7, obtaining, for all $w \in L^\infty(\bar{\Delta})$,

$$C_n(w, \psi, \nu) \leq \frac{C_4}{n^{\zeta-1}},$$

where C_4 depends only on c_ψ and $\|w\|_\infty$. Consequently, using Proposition 3.13,

$$LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p} n^{\zeta-1}}$$

where $C_2 > 0$ depends only on p , c_ψ and $\|\psi\|_\infty$ and so, only on p , $|\phi|_\eta$ and $\|\psi\|_\infty$. As $\|\psi\|_\infty \leq \|\phi\|_\infty + 2\|\chi\|_\infty$ and, by Proposition 3.10, $\|\chi\|_\infty$ depends only on c_ϕ , we conclude that C_2 depends only on p , $|\phi|_\eta$ and c_ϕ . \square

Chapter 4

An example

Here we give an example of a diffeomorphism f of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with a GMY structure Λ having polynomial decay for the Lebesgue measure of the tail of the return time. As a consequence, we deduce that f satisfies the results on polynomial decay of correlations and large deviations from Section 1.2 and 1.3.

We start with an orientation preserving C^2 Anosov diffeomorphism f_0 of \mathbb{T}^2 and we consider a finite Markov partition W_0, \dots, W_d for f_0 such that the fixed point $(0, 0)$ belongs to the interior of W_0 . Considering the hyperbolic decomposition into stable and unstable sub-bundles $TM = E^s \oplus E^u$, we assume that there is $0 < \lambda < 1$ such that

$$\|Df|_{E^s}\| < \lambda \quad \text{and} \quad \|Df^{-1}|_{E^u}\| < \lambda.$$

We assume moreover that the transition matrix A of f_0 is *aperiodic*, i.e. some power of A having all entries strictly positive. By a suitable change of coordinates we can suppose that $f_0(a, b) = (\phi_0(a), \psi_0(b))$ for all $(a, b) \in W_0$, the local stable manifold of $(0, 0)$ is $\{a = 0\}$, the local unstable manifold of $(0, 0)$ is $\{b = 0\}$ and both ϕ_0 and ψ_0 are orientation preserving.

Now we consider $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, a perturbation of f_0 that coincides with f_0 out of W_0 and $f(a, b) = (\phi(a), \psi(b))$ for $(a, b) \in W_0$. For definiteness we assume that $W_0 = [a'_0, a_0] \times [b'_0, b_0]$ and consider $V_0 = [\phi_0^{-1}(a'_0), \phi_0^{-1}(a_0)] \times [\psi_0(b'_0), \psi_0(b_0)]$. Observe that V_0 is a neighborhood of $(0, 0)$ strictly contained in W_0 . We assume that for some $0 < \theta < 1$ we have

$$\phi(a) = a(1 + a^\theta) \quad \text{and} \quad \psi(b) = \phi^{-1}(b) \quad \forall (a, b) \in V_0,$$

and assume that ϕ and ψ coincide respectively with ϕ_0 and ψ_0 in $W_0 \setminus V_0$. Note that ϕ

is the so-called intermittent map of the type considered in [22, Section 6.2] and $(0, 0)$ is a fixed point of f with $\phi'(0) = 1 = \psi'(0)$.

Observe that as we have not modified the geometric structure of f_0 , then the set W_1 is completely foliated by a set Γ^s of stable leaves and a set Γ^u of unstable leaves. Our goal now is to prove that f satisfies the properties (P₁)-(P₅) on the set $\Lambda = W_1$ (any other $W_i \neq W_0$ would be fine) and that we have recurrence times with polynomial decay to some unstable leaf on W_1 , thus being in the conditions of Theorems A and B. As a consequence we obtain the following results.

Theorem 4.1. *Let f be as above and take $\varphi, \psi \in H_\eta$. Then, f has a physical measure μ and there exists $C_2 > 0$ such that for every $n \geq 1$,*

1. *if $\eta > \frac{1}{\theta+1}$, then $\mathcal{C}_n(\varphi, \psi, \mu) \leq \frac{C_2}{n^{1/\theta}}$;*
2. *if $\eta \leq \frac{1}{\theta+1}$, then $\mathcal{C}_n(\varphi, \psi, \mu) \leq \frac{C_2}{n^{(1+1/\theta)\eta}}$.*

Theorem 4.2. *Let f be as above. There are $\eta_0 > 0$ and $\zeta_0 = \zeta_0(\eta_0) > 1$ such that for all $\eta > \eta_0$, $1 < \zeta < \zeta_0$, $\varepsilon > 0$, $p > 1/\theta$ and $\phi \in \mathcal{H}_\eta$, there exists $C_2 > 0$ such that for every $n \geq 1$*

$$LD(\phi, \varepsilon, n, \mu) \leq \frac{C_2}{\varepsilon^{2p}} \frac{1}{n^{1/\theta}}.$$

We consider the sequences $(a_n)_n$ and $(a'_n)_n$ defined recursively for $n \geq 1$ as

$$a_n = \phi^{-1}(a_{n-1}) \quad \text{and} \quad a'_n = \phi^{-1}(a'_{n-1}).$$

For all $n \geq 0$, set

$$J_n = [a_{n+1}, a_n] \times [b'_0, b_0] \quad \text{and} \quad J'_n = [a'_n, a'_{n+1}] \times [b'_0, b_0].$$

Observe that these sets form a (lebesgue mod 0) partition of W_0 . Setting for $i = 1, \dots, k$ and $n \geq 0$

$$\widehat{R}|_{W_i} = 1, \quad \widehat{R}|_{J_n} = n + 1 \quad \text{and} \quad \widehat{R}|_{J'_n} = n + 1,$$

define

$$\widehat{R}_1 = \widehat{R} - 1 + n_0, \quad \widehat{R}_i = \widehat{R}_{i-1} + (\widehat{R} - 1) \circ f^{\widehat{R}_{i-1}} + n_0 \quad \text{for } i \geq 2$$

and, for $x \in W_1$, let $R(x)$ be equal to the smallest \widehat{R}_i such that $f^{\widehat{R}_i}(x) \in W_1$. Note that as we are assuming the transition matrix of f_0 (and thus of f) with respect to the partition W_0, \dots, W_k to be aperiodic, then R is well defined.

4.1 Invariant manifolds

Here we prove that the manifolds in Γ^s and Γ^u satisfy (P₂) and (P₃). We start by proving some useful estimates about the map ϕ . It follows from the results in the beginning of [22, Section 6.2] that $(a_n)_n$ and $(a'_n)_n$ have the same asymptotics of the sequence $1/n^{1/\theta}$. In particular, there is $C > 0$ such that for all $n \geq 1$ we have

$$\Delta a_n := a_n - a_{n+1} \leq \frac{C}{n^{1+1/\theta}}, \quad (4.1)$$

and a similar estimate holds for $(a'_n)_n$. For the sake of notational simplicity we shall consider $\tau = 1/\theta$.

Lemma 4.3. *There exists $C > 0$, such that for all $n \geq 0$ and all $x \in [a_{n+1}, a_n]$, we have*

$$|(\phi^n)'(x)| \geq Cn^{\tau+1}.$$

Proof. By the definition of a_n , we have

$$|\phi^n(a_n) - \phi^n(a_{n+1})| = |a_0 - a_1|$$

and so, using the Mean Value Theorem and (4.1), we get, for some $\xi \in [a_{n+1}, a_n]$,

$$|(\phi_u^n)'(\xi_k)| = \frac{\Delta a_0}{\Delta a_n} \geq Cn^{\tau+1}.$$

Using the previous lemma for $a = \xi$ and any $b \in [a_{n+1}, a_n]$, we obtain the same conclusion for any point in $[a_{n+1}, a_n]$, concluding the proof. \square

To simplify notation, we write f'_u to mean the derivative of f in the unstable direction.

Proposition 4.4. *There exists $C > 0$ such that*

(a) *for all $n \in \mathbb{N}$ and $x, y \in \gamma^u \in \Gamma^u$ we have*

$$d(f^{-n}(x), f^{-n}(y)) \leq \frac{C}{n^{\tau+1}} d(x, y);$$

(b) *for all $n \in \mathbb{N}$ and $x, y \in \gamma^s \in \Gamma^s$ we have*

$$d(f^n(x), f^n(y)) \leq \frac{C}{n^{\tau+1}} d(x, y).$$

Proof. We shall prove (a). The proof of (b) follows similar arguments.

Consider $x, y \in \gamma^u$ and let $n \in \mathbb{N}$. We first assume that the orbits of x and y visit W_0 exactly at the same moments. Then, it is enough to prove that there is $C > 0$ such that for a given point $z \in T^2$ we have

$$|(f_u^{-n})'(z)| \leq \frac{C}{n^{\tau+1}}.$$

Let $K = \{j \in \mathbb{N}, 0 \leq j \leq n : f^{-j}(x) \in W_0\}$ and $M = \{0, \dots, n\} \setminus K$. The set K can be written as $K = \cup_{i=1}^k K_i$, where

$$K_i = \{-k_i, \dots, -k_i + p_i \mid -k_i - 1 \notin K, \quad -k_i + p_i + 1 \notin K\}.$$

Analogously, we write $M = \cup_{i=1}^m M_i$, where

$$M_i = \{-m_i, \dots, -m_i + q_i \mid -m_i - 1 \notin M, \quad -m_i + q_i + 1 \notin M\}.$$

Considering $P = \sum_{i=1}^k p_i$ and $Q = \sum_{i=1}^m q_i$, we have $P + Q = n$.

Note that, since $k_i + p_i \in K$ and $k_i + p_i + 1 \notin K$, then $f^{k_i + p_i}(x) \in J_0$. Since we assumed that the orbits of x and y visit W_0 exactly at the same moments, then $f^{k_i + p_i}(y) \in J_0$. Observe that f coincides with ϕ in $K \cap \gamma^u$. Using the Mean Value Theorem and Lemma 4.3 we get, for some $\xi \in J_0$,

$$d(f^{-k_i}(x), f^{-k_i}(y)) \leq (\phi^{-p_i})'(\xi) d(f^{-k_i + p_i}(x), f^{-k_i + p_i}(y)) \leq \frac{C}{p_i^{\tau+1}} d(f^{-k_i + p_i}(x), f^{-k_i + p_i}(y)). \quad (4.2)$$

For the iterates $m \in M$ we have $f^{-m}(x) \notin W_0$, and so the behavior of $(f_u)'$ is the same of the unperturbed Anosov case. In particular, there is exponential backward contraction: there is $\lambda > 1$ such that

$$d(f^{-m}(x), f^{-m}(y)) \leq \lambda d(f^{-(m-1)}(x), f^{-(m-1)}(y)). \quad (4.3)$$

Gathering (4.2) and (4.3), we obtain, for any $n \in \mathbb{N}$,

$$d(f^{-n}(x), f^{-n}(y)) \leq \lambda^Q \prod_{i=1}^k \frac{C}{p_i^{\tau+1}} d(x, y).$$

Now it is enough to prove that

$$\prod_{i=1}^k \frac{C}{p_i^{\tau+1}} \leq \frac{C}{P^{\tau+1}}. \quad (4.4)$$

We have for each i

$$\frac{C}{p_i^{\tau+1}} = \left(\frac{p_i}{C^{\frac{1}{\tau+1}}} \right)^{-\tau-1}.$$

With no loss of generality, we may assume that each $p_i/C^{\frac{1}{\tau+1}} \geq 2$. Actually, if this were not the case we would have the p_i 's uniformly bounded, meaning that the corresponding p_i iterates would be uniformly bounded away from the stable leaf of $(0,0)$. In particular, there would be some $0 < \lambda_0 < 1$ such that $|(f_u^{-1})'| \leq \lambda_0$ and this case could be treated as the case of the previous case with λ_0 playing the role of λ .

Let us now prove (4.4) under the assumption that $p_i/C^{\frac{1}{\tau+1}} \geq 2$ for each $1 \leq i \leq k$. This in particular implies that

$$\prod_{i=1}^k \frac{p_i}{C^{\frac{1}{\tau+1}}} \geq \sum_{i=1}^k \frac{p_i}{C^{\frac{1}{\tau+1}}}.$$

Using this we get

$$\prod_{i=1}^k \frac{C}{p_i^{\tau+1}} = \left(\prod_{i=1}^k \frac{C^{\frac{1}{\tau+1}}}{p_i} \right)^{\tau+1} \leq \left(\sum_{i=1}^k \frac{C^{\frac{1}{\tau+1}}}{p_i} \right)^{\tau+1} = \left(\frac{C^{\frac{1}{\tau+1}}}{P} \right)^{\tau+1} = \frac{C}{P^{\tau+1}},$$

thus proving (4.4).

Let us finally consider the case where the orbits of x and y do not visit W_0 at the same moments. Assume that there is $j \leq n$ such that $f^j(x) \in J \cup J'$ and $f^j(y) \notin J \cup J'$. Choosing the size of the rectangle W_1 sufficiently small (and thus the length of $\gamma^u(x)$), we may assure that we necessarily have $f^j(x)$ (uniformly) bounded away from $\gamma^s(0,0)$. In particular, there is some λ_0 such that $|f'_u| \geq \lambda_0$, and so we may repeat the calculations above we λ_0 playing the role of λ . \square

4.2 Bounded distortion

Here we prove the bounded distortion property (P₄).

The following lemma is proved in [22, Lemma 5].

Lemma 4.5. *There exists $C > 0$ such that, for all $i, n \in \mathbb{N}$ with $i \leq n$, and for all $a, b \in [a_{n+1}, a_n]$,*

$$\log \frac{(\phi^i)'(a)}{(\phi^i)'(b)} \leq C \frac{|\phi^i(a) - \phi^i(b)|}{\Delta a_{n-i}} \leq C.$$

Lemma 4.6. *There exists $C > 0$ and $0 < \beta < 1$ such that for all $x, y \in \gamma^u \in \Gamma^u$ we have $d(x, y) \leq C\beta^{s(x,y)}$.*

Proof. We will start by showing that there exists $0 < \beta < 1$ such that, for $x, y \in \Lambda \cap \gamma^u$ with $s(x, y) \neq 0$, we have $d(x, y) \leq \beta d(f^R(x), f^R(y))$. In fact, since $f^R(x), f^R(y) \notin W_0$ and f behaves like an Anosov diffeomorphism outside W_0 , then

$$d(f^{R-1}(x), f^{R-1}(y)) \leq \beta d(f^R(x), f^R(y)) \quad \text{for some } 0 < \beta < 1$$

and so $d(x, y) \leq \beta d(f^R(x), f^R(y))$.

Applying this inequality successively, we obtain

$$d(x, y) \leq \beta d(f^R(x), f^R(y)) \leq \dots \leq \beta^s d((f^R)^s(x), (f^R)^s(y)) \leq C\beta^{s(x,y)},$$

where C is the diameter of M . □

Proposition 4.7. *For $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$,*

$$\left| \log \frac{(f_u^R)'(x)}{(f_u^R)'(y)} \right| \leq C\beta^{s(f^R(x), f^R(y))}.$$

Proof. Let $\gamma \in \Gamma^u$ and $x, y \in \Lambda \cap \gamma$. We have

$$\left| \log \frac{(f_u^R)'(x)}{(f_u^R)'(y)} \right| \leq \sum_{j=0}^{R-1} |\log f'_u(f^j x) - \log f'_u(f^j y)|. \quad (4.5)$$

As in Proposition 4.4, without loss of generality we may assume that the orbits of x and y visit W_0 exactly at the same moments. Let $K = \{j \in \mathbb{N}_0, 0 \leq j \leq R-1 : f^j(x) \in W_0\}$ and $M = \{0, \dots, R\} \setminus K$. The set K can be written as $K = \cup_{i=1}^k K_i$, where

$$K_i = \{k_i, \dots, k_i + p_i \mid k_i - 1 \notin K, \quad k_i + p_i + 1 \notin K\}.$$

Analogously we can write $M = \cup_{i=1}^m M_i$, where

$$M_i = \{m_i, \dots, m_i + q_i \mid m_i - 1 \notin M, \quad m_i + q_i + 1 \notin M\}.$$

We will consider now the terms of the right hand side of (4.5) which belong to some K_i . Note that, since $k_i + p_i \in K$ and $k_i + p_i + 1 \notin K$, then $f^{k_i+p_i} \in J_0$. From Lemma 4.5,

$$\sum_{j=0}^{p_i-1} |\log f'_u(f^{k_i+j}x) - \log f'_u(f^{k_i+j}y)| \leq C_1 d(f^{k_i+p_i}x, f^{k_i+p_i}y).$$

So, adding the term $j = p_i$, we obtain

$$\sum_{j=0}^{p_i} |\log f'_u(f^{k_i+j}x) - \log f'_u(f^{k_i+j}y)| \leq Cd(f^{k_i+p_i}x, f^{k_i+p_i}y),$$

because there exists $\xi \in J_0$ such that

$$|\log f'_u(f^{k_i+p_i}x) - \log f'_u(f^{k_i+p_i}y)| = \left| \frac{f''_u(\xi)}{f'_u(\xi)} \right| d(f^{k_i+p_i}x, f^{k_i+p_i}y).$$

Let us now consider the terms belonging to some M_i . Since f is of class C^2 outside W_0 , using the Mean Value Theorem and the fact that f is uniformly expanding on unstable leaves, we have, for $x \in M_i$,

$$\begin{aligned} \sum_{j=0}^{q_i} |\log f'_u(f^{m_i+j}x) - \log f'_u(f^{m_i+j}y)| &\leq C_3 \sum_{j=0}^{q_i} d(f^{m_i+j}x, f^{m_i+j}y) \\ &\leq C_3 \sum_{j=0}^{q_i} \beta^{q_i-j+1} d(f^{m_i+q_i}x, f^{m_i+q_i}y) \leq Cd(f^{m_i+q_i}x, f^{m_i+q_i}y). \end{aligned}$$

Gathering the conclusions we obtained for K and M , using Proposition 4.4-(a),

$$\begin{aligned} \sum_{p \in K \cup L \cup M} |\log f'_u(f^p x) - \log f'_u(f^p y)| &\leq C \left(\sum_{i=0}^k d(f^{k_i+p_i}x, f^{k_i+p_i}y) + \sum_{i=0}^m d(f^{m_i+q_i}x, f^{m_i+q_i}y) \right) \\ &\leq C \left(\sum_{i=0}^k \frac{1}{(R-k_i-p_i)^{\tau+1}} + \sum_{i=0}^m \frac{1}{(R-m_i-q_i)^{\tau+1}} \right) d(f^R x, f^R y) \\ &\leq C' d((f^R)x, (f^R)y), \end{aligned}$$

and so,

$$\left| \log \frac{(f_u^R)'(x)}{(f_u^R)'(y)} \right| \leq C' d((f^R)x, (f^R)y). \quad (4.6)$$

Applying Lemma 4.6, we have $d((f^R)x, (f^R)y) \leq C_2 \beta^{s(f^R(x), f^R(y))}$ for some $C_2 > 0$, thus concluding the proof. \square

4.3 Regularity of the stable foliation

To prove property (P₅)-(a), we follow the ideas in [3, Section 3.5]. The proof of the following lemma can be found in [11, Theorem 3.3.].

Lemma 4.8. *Let N and P be manifolds, where P has finite volume, and, for every $n \in \mathbb{N}$, let $\Theta_n : N \rightarrow P$ be an absolutely continuous map with Jacobian J_n . If we assume that*

- (a) Θ_n converges uniformly to an injective continuous map $\Theta : N \rightarrow P$,
- (b) J_n converges uniformly to an integrable continuous map $J : N \rightarrow \mathbb{R}$,

then Θ is absolutely continuous with Jacobian J .

Until the end of this section we denote $\Theta = \Theta_{\gamma', \gamma}(x)$ to simplify the notation. The next lemma can be found in [3, Lemma 3.11] and it is a consequence of [11, Lemma 3.8].

Lemma 4.9. *Given $\gamma, \gamma' \in \Gamma^u$ and $\Theta : \gamma' \rightarrow \gamma$, then, for every $n \in \mathbb{N}$, there exists an absolutely continuous function $\pi_n : f^n(\gamma') \rightarrow f^n(\gamma)$ with Jacobian G_n such that*

- (a) $\lim_{n \rightarrow \infty} \sup_{x \in \gamma} \{d_{f^n(\gamma')} (f^n(x), f^n(\Theta(x)))\} = 0;$
- (b) $\lim_{n \rightarrow \infty} \sup_{x \in f^n(\gamma)} \{|1 - G_n(x)|\} = 0.$

Lemma 4.10. *There exists $C > 0$ such that for all $x, y \in \gamma^s \in \Gamma^s$ and $n \in \mathbb{N}$ we have*

$$\log \prod_{i=n}^{\infty} \frac{\det Df(f^i(x))}{\det Df(f^i(y))} \leq \frac{C}{n^\tau}.$$

Proof. Note that

$$\log \prod_{i=n}^{\infty} \frac{\det Df(f^i(x))}{\det Df(f^i(y))} \leq \sum_{i=n}^{\infty} |\log (\det Df(f^i(x))) - \log (\det Df(f^i(y)))|.$$

We now need to control each term of the above sum. We divide this in three cases.

Assume first that $f^i(x), f^i(y) \in W_0$. Since $f^i(y) \in \gamma^s(f^i(x))$ and f has a product form in W_0 , then $|\log (\det Df(f^i(x))) - \log (\det Df(f^i(y)))| = 0$.

Assume now that $f^i(x), f^i(y) \notin W_0$. As f behaves like an Anosov diffeomorphism outside W_0 , then $\log \det Df$ is Lipschitz. So, using the polynomial contraction on stable leaves, we get

$$|\log (\det Df(f^i(x))) - \log (\det Df(f^i(y)))| \leq C_1 d(f^i(x), f^i(y)) \leq \frac{C_2}{i^{\tau+1}}.$$

Finally, for $f^i(x) \in W_0$ and $f^i(y) \notin W_0$, choose the point z in the same stable leaf as $f^i(x)$ such that z is in the boundary of W_0 and between $f^i(x)$ and $f^i(y)$. Then, applying the first case to $f^i(x)$ and z , and the second case to z and $f^i(y)$, we obtain the conclusion.

Adding all the terms, we conclude that

$$\sum_{i=n}^{\infty} |\log(\det Df(f^i(x))) - \log(\det Df(f^i(y)))| \leq C_3 \sum_{i=n}^{\infty} \frac{1}{i^{\tau+1}} \leq \frac{C}{n^{\tau}}.$$

□

We define, for $n \in \mathbb{N}$, the map $\Theta_n : \gamma' \rightarrow \gamma$ as $\Theta_n = f^{-R_n} \circ \pi_{R_n} \circ f^{R_n}$. Note that Θ_n is absolutely continuous, its Jacobian is

$$J_n(x) = \frac{|\det(Df^{R_n})(x)|}{|\det(Df^{R_n})\Theta_n(x)|} G_{R_n}(f^{R_n}(x))$$

and the Jacobian of Θ is given by

$$J(x) = \frac{d(\Theta_* \text{Leb}_{\gamma'})}{d \text{Leb}_{\gamma'}}.$$

Proposition 4.11. *For $\gamma', \gamma \in \Gamma^u$, the function Θ is absolutely continuous and its Jacobian is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df(f^i(x))}{\det Df(f^i(\Theta(x)))}.$$

Note that Lemma 4.10 implies that the product in the above proposition is finite. The proof of this proposition is a direct consequence of the following lemma together with Lemma 4.8.

Lemma 4.12. *The functions Θ_n converge uniformly to Θ and their Jacobians J_n converge uniformly to J .*

Proof. Using (P₃), we have, for $x \in \gamma$,

$$d_{\gamma}(\Theta_n(x), \Theta(x)) = d_{\gamma}(f^{-R_n} \pi_{R_n} f^{R_n}(x), f^{-R_n} f^{R_n} \Theta(x)) \leq \frac{C}{(R_n)^{\tau+1}} d_{f^{R_n}(\gamma)}(\pi_{R_n} f^{R_n}(x), f^{R_n} \Theta(x))$$

and, since $R_n \xrightarrow[n]{\rightarrow} \infty$ and $d_{f^{R_n}(\gamma)}(\pi_{R_n} f^{R_n}(x), f^{R_n} \Theta(x))$ is bounded, then the uniform convergence follows.

We write

$$J_n(x) = \frac{|\det(Df^{R_n})(x)|}{|\det(Df^{R_n})\Theta(x)|} \frac{|\det(Df^{R_n})\Theta(x)|}{|\det(Df^{R_n})\Theta_n(x)|} G_{R_n}(f^{R_n}(x)),$$

By Lemma 4.9, $G_{R_n}(f^{R_n}(x))$ converges uniformly to one. To control the second factor note that, by (4.6) applied to the point $\Theta(x)$ and $\Theta_n(x)$, we have

$$\left| \log \frac{|\det(Df^{R_n})\Theta(x)|}{|\det(Df^{R_n})\Theta_n(x)|} \right| \leq C d_{f^{R_n}(\gamma')}(f^{R_n}(\Theta(x)), f^{R_n}(\Theta_n(x))).$$

So,

$$\frac{|\det(Df^{R_n})\Theta(x)|}{|\det(Df^{R_n})\Theta_n(x)|} \xrightarrow{n} 1.$$

We are left to prove that the first factor converges uniformly to J . Notice that

$$\log \frac{|\det(Df^{R_n})(x)|}{|\det(Df^{R_n})\Theta(x)|} = \sum_{i=0}^{R_n} \log \frac{\det Df(f^i(x))}{\det Df(f^i(\Theta(x)))}$$

and so

$$\log J(x) - \log \frac{|\det(Df^{R_n})(x)|}{|\det(Df^{R_n})\Theta(x)|} = \sum_{R_n+1}^{\infty} \log \frac{\det Df(f^i(x))}{\det Df(f^i(\Theta(x)))},$$

which converges uniformly to zero, by Lemma 4.10. \square

The next proposition proves (P₅)-(b).

Proposition 4.13. *For each $\gamma, \gamma' \in \Gamma^u$, the map Θ is absolutely continuous and denoting*

$$u(x) = \frac{d(\Theta_* \text{Leb}_{\gamma'})}{d \text{Leb}_{\gamma}}(x),$$

we have

$$\log \frac{u(x)}{u(y)} \leq C \beta^{s(x,y)}, \quad \forall x, y \in \gamma' \cap \Lambda.$$

Proof. It is known that (P₅)-(b) is satisfied by Anosov diffeomorphisms. But f is topologically conjugate to the Anosov diffeomorphism f_0 . Since the separation time is invariant by topological conjugacy, then so is (P₅)-(b). \square

4.4 Recurrence times

Our goal in this section is to prove that there exists $C > 0$ such that for all $\gamma \in \Gamma^u$ and $n \in \mathbb{N}$ we have

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C}{n^{\tau+1}}.$$

Since we have assumed the transition matrix of the initial Markov partition aperiodic, then there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $f^n(W_j)$ intersects W_k , for all j, k .

Lemma 4.14. *For $L \in \{W_1, \dots, W_d, J_0, J'_0\}$, there exists $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that, for all $n \geq n_0$ and $j \in \{1, \dots, d\}$, we have*

$$\text{Leb}_\gamma(f^{-n}(W_j) \cap L) \geq \delta_0.$$

Proof. Choosing n_0 as in above, we know that, for all $n \geq n_0$, we have $f^n(L)$ intersects W_k , for all k . Since, in addition, $f^n(L)$ must cross the entire length of the unstable direction of any W_k it intersects, then $f^n(L)$ crosses the entire length of the unstable direction of every W_k . Then

$$\frac{\text{Leb}_\gamma(f^{-n}(W_j) \cap L)}{\text{Leb}_\gamma(L)} = \frac{\int_{f^n(f^{-n}(W_j) \cap L)} (f_u^{-n})' d\text{Leb}_\gamma}{\int_{f^n(L)} (f_u^{-n})' d\text{Leb}_\gamma} = \frac{\int_{W_j} (f_u^{-n})' d\text{Leb}_\gamma}{\int_{\cup W_k} (f_u^{-n})' d\text{Leb}_\gamma}. \quad (4.7)$$

Let $R_0 = 0$. Choosing $k \in \mathbb{N}_0$ such that $R_k \leq n < R_{k+1}$, note that $(f_u^{n-R_k})'(x) \geq 1$ and so,

$$(f_u^n)'(x) = (f_u^{R_k})'(f^{n-R_k}(x))(f_u^{n-R_k})'(x) \geq (f_u^{R_k})'(f^{n-R_k}(x)).$$

Analogously, since $(f_u^{n-R_{k+1}})'(x) \leq 1$, then

$$(f_u^n)'(x) = (f_u^{R_{k+1}})'(f^{n-R_{k+1}}(x))(f_u^{n-R_{k+1}})'(x) \leq (f_u^{R_{k+1}})'(f^{n-R_{k+1}}(x)).$$

Consequently,

$$\frac{1}{(f_u^{R_{k+1}})'(f^{n-R_{k+1}}(y))} \leq (f_u^{-n})'(y) \leq \frac{1}{(f_u^{R_k})'(f^{n-R_k}(y))}. \quad (4.8)$$

Applying (P₄), there exists $C > 0$ such that, for all $m \in \mathbb{N}$,

$$e^{1/C} \leq \frac{(f_u^{R_m})'(z)}{(f_u^{R_m})'(w)} \leq e^C.$$

Fixing w_0 we have

$$(f_u^{R_k})'(f^{n-R_k}(y)) \geq e^{1/C} (f_u^{R_k})'(w_0) \quad \text{and} \quad (f_u^{R_{k+1}})'(f^{n-R_{k+1}}(y)) \leq e^C (f_u^{R_{k+1}})'(w_0).$$

Then, from (4.7), (4.8) and the previous inequalities, we obtain

$$\begin{aligned} \frac{\text{Leb}_\gamma(f^{-n}(W_j) \cap L)}{\text{Leb}_\gamma(L)} &\geq \frac{\int_{W_j} \frac{1}{(f_u^{R_{k+1}})'(f^{n-R_{k+1}})} d\text{Leb}_\gamma}{\int_{\cup W_k} \frac{1}{(f_u^{R_k})'(f^{n-R_k})} d\text{Leb}_\gamma} \geq \frac{\frac{1}{e^C (f_u^{R_{k+1}})'(w_0)} \text{Leb}_\gamma(W_j)}{\frac{1}{e^{1/C} (f_u^{R_k})'(w_0)} \text{Leb}_\gamma(\cup W_k)} \\ &= \frac{e^{1/C}}{e^C (f_u^{R_k})'(w_0)} \frac{(f_u^{R_k})'(w_0)}{(f_u^{R_k})'(f^{n-R_k}(w_0))} \frac{\text{Leb}_\gamma(W_j)}{\text{Leb}_\gamma(\cup W_k)} \geq \frac{e^{2/C}}{e^C (f_u^{R_k})'(w_0)} \frac{\text{Leb}_\gamma(W_j)}{\text{Leb}_\gamma(\cup W_k)}, \end{aligned}$$

using (P₄) in the last step. Finally,

$$\text{Leb}_\gamma(f^{-n}(W_j) \cap L) \geq \frac{e^{2/C} \text{Leb}_\gamma(L)}{e^C (f_u^{R_k})'(w_0) \text{Leb}_\gamma(\cup W_k)} \min_{j=1, \dots, d} \{\text{Leb}_\gamma(W_j)\} = \delta_0.$$

□

Define the σ -algebra

$$\mathcal{B}_i = \bigvee_{j=0}^{\widehat{R}_{i-1}} f^{-j} \mathcal{A}.$$

Lemma 4.15. *There exists $\varepsilon_0 > 0$ such that, for all $i \in \mathbb{N}$ and all $\omega \in \mathcal{B}_i$ with $R_{|\omega} > \widehat{R}_{i-1}$,*

$$\text{Leb}_\gamma \{R = \widehat{R}_i \mid \omega\} \geq \varepsilon_0.$$

Proof. Fix $i \in \mathbb{N}$ and let $\omega \in \mathcal{B}_i$ be such that $R_{|\omega} > \widehat{R}_{i-1}$. It follows from the definition of \mathcal{B}_i that $f^{\widehat{R}_{i-1}}\omega \in \mathcal{A}$. Set $n = \widehat{R}_{i-1} + (\widehat{R} - 1) \circ f^{\widehat{R}_{i-1}}$. If $f^{\widehat{R}_{i-1}}\omega = W_l$, for some $l \neq 0$, since $(\widehat{R}-1)W_l = W_l$, then $f^n W_l = W_l$. If $f^{\widehat{R}_{i-1}}\omega = J_l$, for some $l \in \mathbb{N}_0$, since $(\widehat{R}-1)J_l = J_0$, then $f^n J_l = J_0$ (analogously, $f^n J'_l = J'_0$). So, we proved that $f^n \omega = L \in \{W_1, \dots, W_d, J_0, J'_0\}$.

Calling $A = L \cap f^{-n_0} W_k$ and noting that $\widehat{R}_i(x) = n + n_0$, we have

$$\begin{aligned} B &= \{x \in \omega : R(x) = \widehat{R}_i(x)\} = \{x \in \omega : f^{n+n_0}(x) \in W_k\} \\ &= \{x \in f^{-n} L : x \in f^{-(n+n_0)} W_k\} = f^{-n}(A). \end{aligned}$$

From Lemma 4.14, we know that $\text{Leb}_\gamma(A) \geq \delta_0 > 0$. We are left to prove that $\text{Leb}_\gamma(f^{-n}(A)) \geq \varepsilon_0$. But, if we prove that $(f_u^{-n})'_{|A} \geq \delta_1 > 0$, then we get

$$\text{Leb}_\gamma(f^{-n}(A)) = \int_A (f_u^{-n})' d\text{Leb}_\gamma \geq \delta_1 \delta_0 = \varepsilon_0.$$

To prove that $(f_u^{-n})'|_A \geq \delta_1 > 0$, we only need to find an upper bound for $(f_u^n)'$ in B . If $z \in A$ then $z = f^n(x)$, for some $x \in B$ and $R(x) = \widehat{R}_i(x) = n + n_0$. So,

$$(f_u^n)'(x) = (f^{-n_0} \circ f_u^{R(x)})'(x) = (f_u^{-n_0})'(f^R(x)) (f_u^R)'(x).$$

Since n_0 is fixed and $(f_u^{-n_0})'$ is a continuous function with a compact domain, then $(f_u^{-n_0})'$ has an upper bound. So, we only need to control $(f_u^R)'$ in B . Using (4.6), there exists a constant $C > 0$ such that, for $x, y \in L$,

$$\left| \log \frac{(f_u^R)'(x)}{(f_u^R)'(y)} \right| \leq Cd(f^R(x), f^R(y))$$

and so

$$\left| \frac{(f_u^R)'(x)}{(f_u^R)'(y)} \right| \leq e^{C \text{diam}(M)}.$$

Fixing $y_0 \in L$, we get

$$|(f_u^R)'(x)| \leq e^{C \text{diam}(M)} |(f_u^R)'(y_0)| = C_1,$$

concluding the proof. \square

Lemma 4.16. *For all $i, n \in \mathbb{N}$ and all $\omega \in \mathcal{B}_i$,*

$$\text{Leb}_\gamma \{ \widehat{R}_{i+1} - \widehat{R}_i > n_0 + n \mid \omega \} \leq \text{Leb}_\gamma \{ \widehat{R} > n \}.$$

Proof. Let $A = \{x \in \omega : \widehat{R}_{i+1}(x) - \widehat{R}_i(x) > n_0 + n\}$. For $x \in A$ we have $(\widehat{R} - 1)(f^{\widehat{R}_i}(x)) = \widehat{R}_{i+1}(x) - \widehat{R}_i(x) - n_0 > n$. Then $f^{\widehat{R}_i}(A) \subseteq \bigcup_{k \geq n+2} (J_k \cup J'_k)$. So

$$A \subseteq f^{-\widehat{R}_i} \left(\bigcup_{k \geq n+2} (J_k \cup J'_k) \right) \subseteq f^{-\widehat{R}_i} \{ \widehat{R} > n \}$$

and, as $(f_u^{-\widehat{R}_i})' \leq 1$, then

$$\text{Leb}_\gamma(A) \leq \text{Leb}_\gamma (f^{-\widehat{R}_i} \{ \widehat{R} > n \}) = \int_{\{\widehat{R} > n\}} (f_u^{-\widehat{R}_i})' d\text{Leb}_\gamma \leq \text{Leb}_\gamma \{ \widehat{R} > n \}.$$

\square

In the proof of the next result we use ideas from [22, Section 4.1] and [2, Section A.2.1].

Proposition 4.17. *There exists $C > 0$ such that, for sufficiently large n ,*

$$\text{Leb}_\gamma\{R > n\} \leq \frac{C}{n^{\tau+1}}.$$

Proof. We start by noting that

$$\text{Leb}_\gamma\{\widehat{R} > n\} = \text{Leb}_\gamma\left(\bigcup_{i \geq n} (J_i \cup J'_i)\right) = \text{Leb}_\gamma([0, a_n] \cup [a'_n, 0]) \leq \frac{C}{n^{\tau+1}}. \quad (4.9)$$

Defining $\widehat{R}_0 = 0$, observe that $\text{Leb}_\gamma\{R > n\} = \text{(I)} + \text{(II)}$, where

$$\begin{aligned} \text{(I)} &= \sum_{i \leq \frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor} \text{Leb}_\gamma\{R > n; \widehat{R}_{i-1} \leq n < \widehat{R}_i\}, \\ \text{(II)} &= \text{Leb}_\gamma\left\{R > n; n \geq \widehat{R}_{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor}\right\}. \end{aligned}$$

First we will see that there exists $\varepsilon_0 > 0$ and $C > 0$, a constant depending on f , but not on n , such that

$$\text{(II)} \leq C(1 - \varepsilon_0)^{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor}.$$

In fact, taking $n \geq 4n_0$, and so $\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor \geq 2$, we have

$$\begin{aligned} \text{(II)} &= \text{Leb}_\gamma\left\{R > n; n \geq \widehat{R}_{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor}\right\} \leq \text{Leb}_\gamma\left\{R \geq \widehat{R}_{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor}\right\} \\ &= \text{Leb}_\gamma\{R > \widehat{R}_2\} \text{Leb}_\gamma\{R > \widehat{R}_3 \mid R > \widehat{R}_2\} \cdots \text{Leb}_\gamma\left\{R > \widehat{R}_{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor} \mid R > \widehat{R}_{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor - 1}\right\} \\ &\leq C(1 - \varepsilon_0)^{\frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor}, \quad \text{applying Lemma 4.15 to each factor.} \end{aligned} \quad (4.10)$$

We will now focus on (I). Let $k \geq 2n_0$. By (4.9),

$$\text{Leb}_\gamma\left\{\widehat{R} > \frac{n}{i} - n_0\right\} \leq \frac{C}{\left(\frac{n}{i} - n_0\right)^{\tau+1}} \leq C_1 \left(\frac{i}{n_0}\right)^{\tau+1}, \quad \forall i \leq \frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor. \quad (4.11)$$

Fixing i , we have

$$\begin{aligned} \text{Leb}_\gamma\left\{R > n \mid \widehat{R}_{i-1} \leq n < \widehat{R}_i\right\} &\leq \text{Leb}_\gamma\left\{R > \widehat{R}_{i-1}; n < \widehat{R}_i\right\} \\ &\leq \sum_{j=1}^i \text{Leb}_\gamma\left\{R > \widehat{R}_{i-1}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i}\right\}. \end{aligned} \quad (4.12)$$

The last inequality is true because there exists $j \leq i$ such that $\widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i}$. In fact, if we assume the opposite, then $\frac{n}{i} i \geq \sum_{j=1}^i (\widehat{R}_j - \widehat{R}_{j-1}) = \widehat{R}_i$, which contradicts the assumption.

We will now prove that each term of the sum (4.12) is less then or equal to $C(1-\varepsilon_0)^i \frac{i^{\tau+1}}{n^{\tau+1}}$. Considering first the case $i, j \geq 2$, define

$$\begin{aligned} a &= \text{Leb}_\gamma \{R > \widehat{R}_2\} \text{Leb}_\gamma \{R > \widehat{R}_3 \mid R > \widehat{R}_2\} \cdots \text{Leb}_\gamma \{R > \widehat{R}_{j-2} \mid R > \widehat{R}_{j-3}\}, \\ b &= \text{Leb}_\gamma \{R > \widehat{R}_{j-1}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i} \mid R > \widehat{R}_{j-2}\}, \\ c &= \text{Leb}_\gamma \{R > \widehat{R}_j \mid R > \widehat{R}_{j-1}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i}\} \cdots m\{R > \widehat{R}_{i-1} \mid R > \widehat{R}_{i-2}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i}\}, \end{aligned}$$

where if $j = 2$ or $j = 3$ we take $a = 1$ and if $j = i$ we take $c = 1$. Note that

$$\text{Leb}_\gamma \left\{ R > \widehat{R}_{i-1}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i} \right\} = a \cdot b \cdot c.$$

Applying Lemma 4.15 to each factor in a , we get $a \leq (1-\varepsilon_0)^{j-1}$. Each factor in c is of the form $\text{Leb}_\gamma \{R > \widehat{R}_k \mid R > \widehat{R}_{k-1}; \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i}\}$ with $j \leq k < i$. Using again Lemma 4.15, we conclude that $c \leq (1-\varepsilon_0)^{i-j}$. Using Lemma 4.16 and (4.11), we get

$$b \leq \text{Leb}_\gamma \left\{ \widehat{R}_j - \widehat{R}_{j-1} > \frac{n}{i} \mid R > \widehat{R}_{j-2} \right\} \leq \text{Leb}_\gamma \left\{ \widehat{R} > \frac{n}{i} - n_0 \right\} \leq C \left(\frac{i}{n} \right)^{\tau+1}.$$

Gathering all the estimates above we get

$$(I) \leq \sum_{i \leq \frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor} a \cdot b \cdot c \leq C \sum_{i \leq \frac{1}{2} \left\lfloor \frac{n}{n_0} \right\rfloor} (1-\varepsilon_0)^{i-1} \left(\frac{i}{n} \right)^{\tau+1} \leq \frac{C}{n^{\tau+1}} \sum_{i=1}^{\infty} (1-\varepsilon_0)^{i-1} i^{\tau+1} = \frac{C_1}{n^{\tau+1}}. \quad (4.13)$$

For the term $i = 1$ of (I), we have, by the definition of \widehat{R}_1 ,

$$\begin{aligned} \text{Leb}_\gamma \{R > \widehat{R}_0; \widehat{R}_0 < n < \widehat{R}_1\} &\leq \text{Leb}_\gamma \{\widehat{R}_1 > n\} = \text{Leb}_\gamma \{\widehat{R} > n - n_0 + 1\} \\ &= \text{Leb}_\gamma \left(\bigcup_{k \geq n - n_0 + 1} (J_k \cup J'_k) \right) = \text{Leb}_\gamma ([0, a_{n-n_0+1}] \cup [a'_{n-n_0+1}, 0]) \\ &\leq \frac{C}{(n - n_0 + 1)^{\tau+1}} \leq \frac{C_1}{n^{\tau+1}}, \end{aligned}$$

for any $n \geq n_1$, with n_1 sufficiently large. For $i \geq 2$ and $j = 1$, considering each term of the sum in (4.12),

$$\begin{aligned} \text{Leb}_\gamma \{R > \widehat{R}_{i-1}; \widehat{R}_1 - \widehat{R}_0 > \frac{n}{i}\} &\leq \text{Leb}_\gamma \left\{ \widehat{R}_1 - \widehat{R}_0 > \frac{n}{i} \right\} \text{Leb}_\gamma \{R > \widehat{R}_1 \mid \widehat{R}_1 - \widehat{R}_0 > \frac{n}{i}\} \\ &\quad \cdot \text{Leb}_\gamma \{R > \widehat{R}_2 \mid R > \widehat{R}_1; \widehat{R}_1 - \widehat{R}_0 > \frac{n}{i}\} \cdots \text{Leb}_\gamma \{R > \widehat{R}_{i-1} \mid R > \widehat{R}_{i-2}; \widehat{R}_1 - \widehat{R}_0 > \frac{n}{i}\} \\ &\leq C(1-\varepsilon_0)^{i-1}, \end{aligned}$$

arguing as we did to estimate c in the general case. Finally, from (4.10), (4.13) and the calculations for the small terms, we have, for sufficiently large n ,

$$(I) + (II) \leq \frac{C_1}{n^{\tau+1}} + C(1 - \varepsilon_0)^{\frac{1}{2} \left[\frac{n}{n_0} \right]} \leq \frac{C_2}{n^{\tau+1}}.$$

□

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