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Gibbs-Markov-Young structure with (stretched) exponential recurrence times for partially hyperbolic attractors

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To My Mother

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Resumo

No Capítulo 1 da tese estudamos conjuntos parcialmente hiperbólicos K sobre uma variedade Riemanniana M, cujo espaço tangente se decompõe da forma $T_K M = E^{cu} \oplus E^{ss}$, em que a direção centro-instável é não uniformemente expansora em algum disco instável local. Provamos que a existência de uma dinâmica induzida com decaimento (sub-)exponencial dos tempos de retorno pode ser deduzida da hipótese de decaimento (sub-)exponencial do tempo que os pontos típicos necessitam para alcançar algum comportamento uniformemente expansor na direção centro-instável. Usando um resultado de Young, obtemos Decaimento de Correlações (sub-)exponencial para estes atratores parcialmente hiperbólicos e, usando um resultado de Melbourne e Nicol, obtemos Grandes Desvios de ordem exponencial. O progresso principal deste trabalho reside na extensão de resultados de Alves, Dias, Gouëzel, Luzzatto e Pinheiro.

No Capítulo 2, como aplicação da estratégia geométrica da primeira parte, melhoramos um resultado de Alves-Freitas-Luzzatto-Vaienti em [4]. No sentido inverso aos resultados de Young usados na primeira parte da tese, em [4] os autores mostraram que, para sistemas não-invertíveis, o comportamento estocástico, tal como o decaimento de correlações com determinadas taxas, é suficiente para implicar a existência de uma estrutura GMY com as propriedades correspondentes. No caso (sub-)exponencial de [4], os autores colocaram uma hipótese adicional sobre a densidade da medida de probabilidade SRB, devido à estratégia global apresentada por Gouëzel em [15]. Com a estratégia local usada na primeira parte desta tese podemos remover esta hipótese adicional.

Abstract

In Chapter 1 of the thesis we study partially hyperbolic sets K on a Riemannian manifold M, whose tangent space splits as $T_K M = E^{cu} \oplus E^{ss}$, for which the center-unstable direction E^{cu} is non-uniformly expanding on some local unstable disk. We prove that the existence of an inducing scheme with (stretched) exponential decay of recurrence times can be deduced under the assumption of (stretched) exponential decay of the time that typical points need to achieve some uniform expanding behavior in the center-unstable direction. Using a result by Young we obtain (stretched) exponential Decay of Correlations for such partially hyperbolic attractors, and using a result by Melbourne and Nicol we obtain exponential Large Deviations. The main advantage is the extension of previous results by Alves, Dias, Gouëzel, Luzzatto and Pinheiro.

In Chapter 2, as an application of the geometrical strategy in the first part, we improve a result of Alves-Freitas-Luzzatto-Vaienti in [4]. In the contrary direction of Young's result applied in the first part, in [4] the authors showed that, in non-invertible systems, the stochastic-like behaviour such as decay of correlations at certain rates was sufficient to imply the existence of GMY structure with corresponding properties. In the (stretched) exponential case of [4], the authors stated an additional assumption on the density of the SRB measure because of the global strategy given by Gouëzel in [15]. Now, with the local strategy used in the first part, we move away that additional assumption.

Contents

Introduction

This work is about the ergodic theory of Dynamical Systems with hyperbolic properties in some sense. The main results are for discrete time systems. A classical approach in Dynamical Systems is to use particular geometrical structure to deduce statistical properties. In the late 60's and 70's, Sinai, Ruelle and Bowen brought Markov partitions and symbolic dynamics into the theory of uniformly hyperbolic systems; see [20, 9, 19]. Ruelle wrote: 'This allowed the powerful techniques and results of statistical mechanics to be applied into smooth dynamics' in [10, Preface]. To study the systems beyond uniformly hyperbolic, Young used Markov partition to build Young tower in [22, 23] for systems with nonuniform hyperbolicity, including Axiom A attractors, piecewise hyperbolic maps, billiards with convex scatterers, logistic maps, intermittent maps and Hénon-type attractors. Under these towers, Young studied some statistical properties of the non uniformly hyperbole systems, including the existence of SRB measures, exponential decay of correlation and the validity of the Central Limit Theorem for the SRB measure. Roughly speaking, a Markov structure is characterized by some selected region of the phase space that is divided into an at most countable number of subsets with associated *recurrence times*. Young called it 'horseshoe with infinitely many branches'. These structures have some properties which address to Gibbs states and for that reason they are nowadays sometimes referred to as Gibbs-Markov-Young (GMY) structures; see Definition 1.1.6.

In [11], Bonatti and Viana considered partially hyperbolic attractors with mostly contracting direction, i.e. the tangent bundle splitting as $E^{cs} \oplus E^u$, with the E^u direction uniformly expanding and the E^{cs} direction mostly contracting (negative Lyapunov exponents). They proved the existence of an SRB measure under those conditions. In [13], Castro showed the existence of GMY structure, thus obtaining statistical properties like exponential decay of correlations and the validity of the Central Limit Theorem. The Central Limit Theorem for these systems has also been obtained by Dolgopyat in [14].

However, as most of the richness of the dynamics in partially hyperbolic attractors appears in the unstable direction, the case $E^{cu} \oplus E^s$ (now with the stable direction being uniform and the unstable nonuniform) comprises more difficulties than the case $E^{cs} \oplus E^u$. The existence of SRB for some classes of partially hyperbolic attractors of the type $E^{cu} \oplus E^s$ has been proven by Alves, Bonatti and Viana in [2]. In [8], Alves and Pinherio obtained a GMY structure quite similar to that by Alves, Luzzatto and Pinheiro in [6] for non-uniformly expanding (NUE) systems. Given that the lack of expansion of the system at time n (hyperbolic times) is polynomially small, they got polynomial decay of recurrence times and thus polynomial decay of correlations. Their approach, originated from [22] for Axiom A attractors, has shown to be not efficient enough to estimate the tail of recurrence times for non-uniformly hyperbolic systems with exponential tail of hyperbolic times. This is due to the fact that at each step of their algorithmic construction just a definite fraction of hyperbolic times is used.

In [15], Gouëzel developed a new construction with much more efficient estimates for the tail of the recurrence times. As a starting point, Gouëzel used the fact that the attractor could be partitioned into finite number of sets with small size. That gave rise to more precise estimates than those in [6], yielding also the (streched) exponential case in the non-invertible endomorphism case. However, for important combinatorial reasons, Gouëzel's strategy could not be generalized to the partially hyperbolic setting $E^{cu} \oplus E^s$, in particular because the attractor is typically made of unstable leaves, which are not bounded in their intrinsic distance. Partially inspired by [15, 18], Alves, Dias and Luzzatto gave in [3] an improved *local* GMY structure, with much more efficiency than [6] in the use of hyperbolic times that made it possible to prove the integrability of recreance times under very general conditions.

The main goal of this thesis is Chapter 1, where we fill a gap in the theory of partially hyperbolic diffeomorphisms with centre unstable direction, where, after [8], GMY structures are only known with polynomial tail of recurrence times. From that we get (stretched) exponential Decay of Correlations and exponential Large Deviations for the systems under consideration, by Young, Alves, Pinheiro, Melbourne, and Nicol's related results in [22, 7, 17]. Our strategy is based in a mixture of techniques from [3] and [15] and we construct a GMY structure by a method similar to [3], where recurrence times were only proved to be integrable. To improve the efficiency of the algorithm in [8], our method has a main difference, namely, we keep track of all points with hyperbolic times at a given iterate and not just of a proportion of those points.

In Chapter 2 we consider non-invertible systems and remove an assumption in [4, Theorems A, B & C], where the density of an absolutely continuous invariant probability measure (acip) is supposed to be bounded from below on its support. We apply Young's result in Chapter 1: GMY structure with certain rates of mixing implies statistical properties, certain rates of decay of correlations for instance. Conversely, Alves, Freitas, Luzzatto and Vaienti showed in [4] that for non-uniformly expanding systems (positive Lypaunov exponents) the stochastic-like behavior such as decay of correlations at certain rates (polynomial, stretched exponential and exponential) for the acip was sufficient to imply the existence of an induced GMY structure; see [4, Theorems A, B & C].

Roughly speaking, the main steps in [4] are: first showing that, under certain circumstances, the decay of correlations implies large deviations, and then showing that large deviations implies the existence of GMY structures. In the second step, they used [6] in the polynomial case and [15] in the (stretched) exponential case. The main difference between these two works is that in [6] the authors perform a local construction while in [15] the construction is based on a global argument. In the local strategy, one immediately has that the acip has density bounded from below on the domain of the GMY structure. Contrarily, in the (stretched) exponential case in [4, Theorem A, B, C] the authors had to assume that the density of acip was bounded away from zero on its support. Now, with our local strategy in Chapter 1 we construct the GMY structure on a local unstable disk and so, the boundedness from below on the density of the acip can be removed also in the (stretched) stretched exponential cases.

We finally recall that Chapter 2 is just for non-invertible systems, and there is still an open question: can we get the parallel result in the partially hyperbolic attractors with non-uniformly expanding direction?

Chapter 1

(Stretched) exponential tail for partially hyperbolic attractors

1.1 Definitions and main results

In the beginning section we introduce the definition of Gibbs-Markov-Young structure in Subsection 1.1.1, state the partially hyperbolic attractor's setting in Subsection 1.1.2, then give our main theorem: under the assumption of tail of expansion, we construct a Gibbs-Markov-Young structure in partially hyperbolic attractors with a non-uniformly expanding central-unstable direction; moreover, the tail of recurrence times in the structure decays (stretched) exponentially fast. Further more, we discuss statistical properties of the systems such as decay of correlations and large deviations in Subsection 1.1.3.

1.1.1 Gibbs-Markov-Young structures

Here we recall the structures which have been introduced in [22]. Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M, Leb (Lebesgue measure) the normalized Riemannian volume on the Borel sets of M. Given a submanifold $\gamma \subset M$, and Leb_γ denotes the Lebesgue measure on γ induced by the restriction of the Riemannian structure to γ.

Definition 1.1.1. An embedded disk $\gamma \subset M$ is called an *unstable manifold* if for all $x, y \in \gamma$, $\exists 0 < \lambda < 1$, s.t.

$$\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \leq \lambda^n \quad \text{as } n \to \infty.$$

Similarly, γ is called a *stable manifold* if for all $x, y \in \gamma$

$$\mathrm{dist}(f^n(x), f^n(y)) < \lambda^n, \quad \text{as } n \to \infty.$$

Definition 1.1.2. Let Λ be a hyperbolic set, there exist $\epsilon, \delta > 0$, for any $x \in \Lambda$,

$$W^s_\delta(x) = \{y \in M : \mathrm{dist}(f^n(x), f^n(y)) \leq \delta, \quad \forall n \in \mathbb{N}_0\},$$

$$W^u_\delta(x) = \{y \in M : \mathrm{dist}(f^{-n}(x), f^{-n}(y)) \leq \delta, \quad \forall n \in \mathbb{N}_0\},$$

are called *local unstable manifold* and *local stable manifold* for x.

Definition 1.1.3. Given $n \geq 1$, let D^u be a unit disk in $\mathbb{R}^n$ and let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. A *continuous family of C^1 unstable manifolds* is a set Γ^u of unstable disks γ^u satisfying the following properties: there is a compact set K^s and a map $\Phi^u \colon K^s \times D^u \to M$ such that

1. $\gamma^u = \Phi^u(\{x\} \times D^u)$ is a local unstable manifold;
2. Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
3. $x \mapsto \Phi^u|(\{x\} \times D^u)$ is a continuous map from K^s to $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined analogously.

Definition 1.1.4. A subset $\Lambda \subset M$ has a *product structure* if, for some $n \geq 1$, there exist a continuous family of n-dimensional unstable manifolds $\Gamma^u = \cup\gamma^u$ and a continuous family of $(\dim(M) - n)$-dimensional stable manifolds $\Gamma^s = \cup\gamma^s$ such that

1. $\Lambda = \Gamma^u \cap \Gamma^s$;
2. each γ^s meets each γ^u in exactly one point, with the angle of γ^s and γ^u uniformly bounded away from zero.

Definition 1.1.5. Let $\Lambda \subset M$ have a product structure defined by families Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is an *s-subset* if Λ_0 has a hyperbolic product structure defined by families $\Gamma^s_0 \subset \Gamma^s$ and $\Gamma^u_0 = \Gamma^u$; *u-subsets* are defined similarly.

For $* = u, s$, given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, and let f^* denote the restriction of the map f to γ^*-disks and $|\det Df^*|$ denote the Jacobian of Df^*.

Definition 1.1.6. A set Λ with a product structure for which properties $(\mathbf{P_0})$-$(\mathbf{P_4})$ below hold will be called a *Gibbs-Markov-Young (GMY) structure*. From here on we assume that $C > 0$, $0 < \beta < 1$ and $0 < \zeta \le 1$ are constants depending only on f and Λ.

$(\mathbf{P_0})$ *Lebesgue detectable*: for every $\gamma \in \Gamma^u$, we have $\mathrm{Leb}_\gamma(\Lambda \cap \gamma) > 0$;

$(\mathbf{P_1})$ *Markov partition and recurrence times*: there are finitely or countably many pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

(a) for each $\gamma \in \Gamma^u$, $\mathrm{Leb}_\gamma\big((\Lambda \setminus \cup \Lambda_i) \cap \gamma\big) = 0$;

(b) for each $i \in \mathbb{N}$ there is integer $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

We define the *recurrence time* function $R\colon \cup_i \Lambda_i \to \mathbb{N}$ as $R|_{\Lambda_i} = R_i$. We call $f^{R_i} : \Lambda_i \to \Lambda$ the *induced map*.

$(\mathbf{P_2})$ *Uniform contraction on* Γ^s: for all $x \in \Lambda$, each $y \in \gamma^s(x)$ and $n \ge 1$

$$\mathrm{dist}(f^n(y), f^n(x)) \le C\beta^n.$$

$(\mathbf{P_3})$ *Backward contraction and bounded distortion on* Γ^u: for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, and $0 \le n < R_i$

(a) $$\mathrm{dist}(f^n(y), f^n(x)) \le C\beta^{R_i - n}\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y));$$

(b) $$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \le C\,\mathrm{dist}(f^{R_i}(x), f^{R_i}(y))^\zeta.$$

$(\mathbf{P_4})$ *Regularity of foliations*:

(a) *Convergence of* $D(f^i|\gamma^u)$: for all $y \in \gamma^s(x)$ and $n \ge 0$

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \le C\beta^n;$$

(b) *Absolutely continuity of* Γ^s: given $\gamma, \gamma' \in \Gamma^u$, define the holonomy map $\phi\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ as $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous with

$$\frac{d(\phi_* \mathrm{Leb}_\gamma)}{d\,\mathrm{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

(The notion of absolute continuity is precisely given in Subsection 1.3.7.) Under these conditions we say that $F = f^R : \Lambda \to \Lambda$ is an *induced GMY map*.

1.1.2 Partially hyperbolic attractors

Here we recall the definition of partially hyperbolic attractors with mostly expanding center-unstable direction and then we state the main theorem, Theorem A. This result extends the polynomial estimates in [8, Theorem A] to the (stretched) exponential case.

Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. A set $K \subset M$ is said to be invariant if $f(K) = K$.

Definition 1.1.7. A compact invariant subset $K \subset M$ has a *dominated splitting*, if there exists a continuous Df-invariant splitting $T_K M = E^{cs} \oplus E^{cu}$ and $0 < \lambda < 1$ such that (for some choice of Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for all } x \in K. \tag{1.1}$$

We call E^{cs} the *center-stable bundle* and E^{cu} the *center-unstable bundle.*

Definition 1.1.8. A compact invariant set $K \subset M$ is called *partially hyperbolic*, if it has a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$ for which E^{cs} is *uniformly contracting* or E^{cu} is *uniformly expanding*, i.e. there is $0 < \lambda < 1$ such that (for some choice of a Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1} \leq \lambda, \quad \text{for all } x \in K.$$

In this work we consider partially hyperbolic sets of the same type of those considered in [2], for which the center-stable direction is uniformly contracting and the central-unstable direction is non-uniformly expanding. To emphasize that, we shall write E^s instead of E^{cs}.

Definition 1.1.9. Given $b > 0$, we say that f is *non-uniformly expanding* at a point $x \in K$ in the central-unstable direction, if

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -b. \tag{NUE}$$

If f satisfies (NUE) at $x \in K$, then the *expansion time* function at x

$$\mathcal{E}(x) = \min\left\{ N \geq 1 \colon \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{f^i(x)}\| < -b, \quad \forall n \geq N \right\} \tag{1.2}$$

is defined and finite.

$\{\mathcal{E} > n\}$ is the set of points which, up to time n, have not yet achieved exponential growth of the derivative along orbits. We call $\{\mathcal{E} > n\}$ *the tail of hyperbolic times* (at time n).

We remark that if condition (NUE) holds for every point in a subset with positive Lebesgue measure of a forward invariant set $\tilde{K} \subset M$, then $K = \cap_{n \geq 0} f^n(\tilde{K})$ contains some local unstable disk D for which condition (NUE) is satisfied Leb_D almost everywhere; see [8, Theorem A].

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism with $K \subset M$ an invariant transitive partially hyperbolic set. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$ and $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then there exists $\Lambda \subset K$ with a GMY structure. Moreover, there exists $d > 0$ such that $\mathrm{Leb}_\gamma\{R > n\} = \mathcal{O}(e^{-dn^\tau})$ for any $\gamma \in \Gamma^u$.*

The proof of this result will be given in Subsection 1.3.

Under the assumptions of Theorem A, the set Λ coincides with Γ^u, but there are other possibilities, e.g. Λ is a Cantor set for the Hénon attractors in [12].

In Subsection 1.4 we present an open class of diffeomorphisms for which $K = M$ is partially hyperbolic and satisfies the assumptions of Theorem A. The transitivity of the diffeomorphisms in that class was proved in [21].

1.1.3 Statistical properties

A good way of describing the dynamical behavior of chaotic dynamical systems is through invariant probability measures and, in our context, a special role is played by SRB measures.

Definition 1.1.10. An f-invariant probability measure μ on the Borel sets of M is called an *Sinai-Ruelle-Bowen (SRB) measure* if f has no zero Lypaunov exponents μ almost everywhere and the conditional measures of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on these manifolds.

It is well known that SRB measures are *physical measures*: for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \tag{1.3}$$

SRB measures for partially hyperbolic diffeomorphisms whose central direction is non-uniformly expanding were already obtained in [2]. Under the assumptions of Theorem A, we also get the existence of such measures by means of [22, Theorem 1].

Definition 1.1.11. We define the *correlation functions* of observables $\varphi, \psi\colon M \to \mathbb{R}$ with respect to a measure μ as

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \right|, \quad n \geq 0.$$

Sometimes it is possible to obtain specific rates for which $\mathrm{Cor}_\mu(\varphi, \psi)$ decays to 0 as n tends to infinity, at least for certain classes of observables with some regularity. See that if we take the observables as characteristic functions of Borel sets, we get the classical definition of *mixing*.

The next corollary follows from Theorem A together with [7, Theorem B]; see also [7, Remark 2.4]. Though in [7] the decay of correlations depends on some backward decay rates in the unstable direction, in our case we clearly have exponential backward contraction along that direction. So the next result is indeed an extension of [8, Corollary B] to the (stretched) exponential case.

Corollary B (Decay of Correlations)**.** *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$ and $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then some power f^k has an SRB measure μ and there is $d > 0$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^{kn}) = \mathcal{O}(e^{-dn^\tau})$ for Hölder continuous $\varphi\colon M \to \mathbb{R}$, and $\psi \in L^\infty(\mu)$.*

If the recurrence times associated to the elements of the GMY structure given by Theorem A are relatively prime, i.e. $\gcd\{R_i\} = 1$, then the same conclusion holds with respect to f, i.e. for $k = 1$.

Definition 1.1.12. Given an observable $\varphi : M \to \mathbb{R}$, a probability measure μ and a small constant $\epsilon > 0$, we define the *large deviation* at time n of the time average of φ from the spatial average as

$$\mathrm{LD}_\mu(\varphi, \epsilon, n) = \mu\left(\left\{ x \in M : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi\, d\mu \right| > \epsilon \right\}\right).$$

By Theorem A and [17, Theorem 4.1], we also deduce a result of large deviations for the SRB measure μ of f.

Corollary C (Large Deviations)**.** *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and $c > 0$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn})$. Then there is $d > 0$ such that for any Hölder continuous $\varphi : M \to \mathbb{R}$ and any $\epsilon > 0$ we have $\mathrm{LD}_\mu(\varphi, \epsilon, n) = \mathcal{O}(e^{-dn})$.*

In Corollary C we do not need to take any power of f; see the considerations in [17, Subsection 2.2]. It remains an interesting open question to know whether we have a similar result in the stretched exponential case; this depends only on a stretched exponential version of [17, Theorem 4.1].

Further statistical properties, as the Central Limit Theorem or an Almost Sure Invariant Principle, which have already been obtained in [8], could still be deduced form Theorem A.

1.2 Preliminary results

In this section we discuss the bounded distortion property at hyperbolic times (firstly appeared in [5]) for iterations of f over disks which are tangent to a center-unstable cone filed. The material here is mainly from [2].

Firstly we give the precise definition of center-unstable cone field. We denote the continuous extensions of E^s and E^{cu} to some neighborhood U of K by $\tilde{E}^s$ and $\tilde{E}^{cu}$. The extensions are not necessarily invariant under Df. Notice the set U will be necessary in the GMY construction; see Subsection 1.3.5. These extensions may not be invariant under Df.

Definition 1.2.1. Given $0 < a < 1$, the *center-unstable cone field $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ of width a* is defined by

$$C_a^{cu}(x) = \left\{v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_1\| \leq a\|v_2\|\right\};$$

the *stable cone field $C_a^s = (C_a^s(x))_{x\in U}$ of width a* is defined similarly,

$$C_a^s(x) = \left\{v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_2\| \leq a\|v_1\|\right\}.$$

We notice that the dominated splitting property still holds for the extension. Up to slightly increasing $\lambda < 1$, we fix $a > 0$ and U small enough so that the domination condition (1.1) still holds for any point $x \in U \cap f^{-1}(U)$ and every tangent vector $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$:

$$\|Df(x)v^s\| \cdot \|Df^{-1}(f(x))v^{cu}\| \leq \lambda\|v^s\|\,\|v^{cu}\|.$$

The center-unstable cone field is forward invariant

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x)), \quad \text{any } x \in K,$$

and this holds for any $x \in U \cap f^{-1}(U)$ by continuity.

The cu-direction tangent bundle of the iterates of a C^2 submanifold are Hölder continuous as long as they do not leave U, with uniform Hölder constants. We only need the existence of a dominated splitting $E^{cs} \oplus E^{cu}$.

Definition 1.2.2. An embedded C^1 submanifold $L \subset U$ is *tangent* to the centre-unstable cone field, if $T_xL \subset C_a^{cu}(x)$, at every point $x \in L$.

If a submanifold L satisfies Definition 1.2.2, then $f(L)$ is also tangent to the centre-unstable cone field by the domination property as far as $f(L)$ is in U.

The tangent bundle TL is said to be *Hölder continuous*, if the sections $x \to T_xL$ of the Grassmannian bundles over L are Hölder continuous.

For a subset T_xL and a vector $v \in TM$, let $\operatorname{dist}(v, T_xL) = \min_{u \in T_xL} \|v - u\|$, which means $\operatorname{dist}(v, T_xL)$ is the length of the distance between v and its orthogonal projection of T_xL. Taken $x, y \in L$ for subbundles T_xL and T_yL, we define

$$\operatorname{dist}(T_xL, T_yL) = \max\left\{ \max_{v \in T_xL, \|v\|=1} \operatorname{dist}(v, T_yL), \max_{w \in T_yL, \|w\|=1} \operatorname{dist}(w, T_xL) \right\}.$$

Definition 1.2.3. For constants $C > 0$ and $\zeta \in (0, 1]$, the tangent bundle TL is said to be $(C, \zeta)-$Hölder continuous, if

$$\operatorname{dist}(T_xL, T_yL) \le C \operatorname{dist}_L(x, y)^\zeta \quad \text{for all } y \in B(x, \varepsilon) \cap L \textit{ and } x \in U.$$

Here $\operatorname{dist}_L(x, y)$ is the length of geodesic along L connecting x and y. Given a C^1 submanifold $L \subset U$, we define

$$\kappa(L) = \inf\{C > 0 : TL \text{ is } (C, \zeta)\text{-Hölder}\}.$$

The next result on the Hölder control of the tangent direction is all we need. See its proof in [2, Corollary 2.4].

Proposition 1.2.4. *Given $C_1 > 0$ such that for any C^1 submanifold $L \subset U$ tangent to C_a^{cu},*

1. *there is $n_0 \geq 1$, then $\kappa(f^n(L)) \leq C_1$ for every $n \geq n_0$ and $f^k(L) \subset U$ for all $0 \leq k \leq n$;*

2. *if $\kappa(L) \leq C_1$, then $\kappa(f^n(L)) \leq C_1$ for all $n \geq 1$ and $f^k(L) \subset U$ for all $0 \leq k \leq n$;*

3. *if L, n are as in last item, then we have the functions*

$$J_k : f^k(L) \ni x \mapsto \log|\det\big(Df \mid T_x f^k(L)\big)|, \quad 0 \leq k \leq n,$$

are (C', ζ)-Hölder continuous with $C' > 0$ depending only on C_1 and f.

This proposition would be useful in proving Item (3) of Lemma 1.2.9, i.e the bounded distortion estimates at hyperbolic times in next subsection.

We can derive uniform expansion and bounded distortion from NUE assumption in the centre-unstable direction, with the definition below. Here we do not need the full strength of partially hyperbolic, we only consider the cu-direction has condition (NUE).

Definition 1.2.5. Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \qquad \text{for all } 1 \leq k \leq n.$$

For $n \geq 1$, we define

$$H_n(\sigma) = \{x \in K \colon \; n \text{ is a } \sigma\text{-hyperbolic time for } x \;\}.$$

Remark 1.2.6. Given $0 < \sigma < 1$ and $x \in H_n(\sigma)$, we obtain

$$\|Df^{-k} \mid E^{cu}_{f^n(x)}\| \leq \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \tag{1.4}$$

which means $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for $1 \leq k \leq n$.

The next result states the existence of σ-hyperbolic times for points satisfying Definition 1.2.5 and gives indeed the positive frequency for such points. Its proof can be found in [2, Lemma 3.1, Corollary 3.2].

Proposition 1.2.7. *There exist $0 < \theta \leq 1$ and $0 < \sigma < 1$ such that for every $x \in K$ with $\mathcal{E}(x) \leq n$ there exist σ-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq n$ for x with $l \geq \theta n$.*

In the sequel, we consider a fixed σ and simply write H_n instead of $H_n(\sigma)$.

Remark 1.2.8. If $a > 0$ and $\delta_1 > 0$ are sufficiently small such that the δ_1-neighborhood of K is contained in U, we get by continuity

$$\|Df^{-1}(f(y))v\| \leq \frac{1}{\sqrt{\sigma}}\|Df^{-1}|E^{cu}_{f(x)}\|\,\|v\|, \tag{1.5}$$

whenever $x \in K$, $\operatorname{dist}(y, x) \leq \delta_1$, and any $v \in C_a^{cu}(y)$.

For a given disk $\Delta \subset M$, we denote the distance between $x, y \in \Delta$ by $\operatorname{dist}_\Delta(x, y)$, measured along Δ. Items (1)-(3) in the next result have been proved in [2, Lemma 5.2 & Corollary 5.3], and Item (4) is a consequence of Item (2).

Lemma 1.2.9. *Let $0 < \delta < \delta_1$, $n_0 \geq 1$ and $0 < \zeta \leq 1$. Let $\Delta \subset U$ be a C^1 disk of radius δ tangent to the centre-unstable cone field with $\kappa(\Delta) \leq C_1$ and $x \in \Delta \cap K$. There exists $C_2 > 1$ such that if $n \geq n_0$ and $x \in H_n$, then there exists a neighborhood $V_n(x)$ of x and $V_n(x) \subset \Delta$ so that:*

1. *f^n maps $V_n(x)$ diffeomorphically onto a centre-unstable ball $B(f^n(x), \delta_1)$;*
2. *for every $1 \leq k \leq n$ and $y, z \in V_n(x)$,*

$$\operatorname{dist}_{f^{n-k}(V_n(x))}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \operatorname{dist}_{f^n(V_n(x))}(f^n(y), f^n(z));$$

3. *for all $y, z \in V_n(x)$*

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \leq C_2 \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))^\zeta,$$

and for any Borel sets $Y, Z \subset V_n(x)$,

$$\frac{1}{C_2}\frac{\operatorname{Leb}(Y)}{\operatorname{Leb}(Z)} \leq \frac{\operatorname{Leb}(f^n(Y))}{\operatorname{Leb}(f^n(Z))} \leq C_2\frac{\operatorname{Leb}(Y)}{\operatorname{Leb}(Z)};$$

4. *$V_n(x) \subset B(x, \delta_1\sigma^{n/2})$.*

The sets $V_n(x)$ will be called *hyperbolic pre-balls*, and their images $B(f^n(x), \delta_1)$ called *hyperbolic balls*. Item (3) gives the bounded distortion at hyperbolic times.

1.3 The GMY structure

In this section we prove the existence of the product structure in the attractor. We essentially describe the geometrical and dynamical nature. This process has three steps. Firstly we prove the existence of a centre-unstable disk Δ whose subsets return to a neighborhood of Δ under forward iterations and the image projects along stable leaves covering Δ completely. Secondly, we define a partition on Δ by these subsets. This construction is inspired essentially by [8, Section 3] and [3, Section 3 & 4]. That is, we improve the structure construction of [3] for NUE systems; see Subsection 1.3.2. Then we generalize the structure to the partially hyperbolic attractor setting as in [8]. Notice that we improve the product structure in [8] in Subsection 1.3.5. Finally we show that the set with a product structure satisfies Definition 1.1.6.

1.3.1 The reference disk

Let D be a local unstable disk as in the assumption of Theorem A. Given δ_1 as in Remark 1.2.8, we take $0 < \delta_s < \delta_1/2$ such that points in K have local stable manifolds of radius δ_s. In particular, these local stable leaves are contained in U; recall (1.5).

Definition 1.3.1. Given a disk $\Delta \subset D$, we define the *cylinder* over Δ

$$\mathcal{C}(\Delta) = \bigcup_{x \in \Delta} W^s_{\delta_s}(x),$$

and consider π be the projection from $\mathcal{C}(\Delta)$ onto Δ along local stable leaves. We say that a center-unstable disk γ^u *u-crosses* $\mathcal{C}(\Delta)$ if

$$\pi(\gamma^u \cap \mathcal{C}(\Delta)) = \Delta.$$

From Lemma 1.2.9 we know that if $\Delta \subset U$ is a small C^1 disk tangent to the centre-unstable cone field with $\kappa(\Delta) \leq C_1$ and $x \in \Delta \cap K$, then for each $x \in H_n$, there is a hyperbolic pre-ball which is sent by f^n diffeomorphically onto the ball $B(f^n(x), \delta_1)$. For technical reasons (see Lemma 1.3.9) we shall take $\delta_1' \ll \delta_1$ and consider $V_n'(x)$ the part of $V_n(x)$ which is sent by f^n onto $B(f^n(x), \delta_1')$. The sets $V_n'(x)$ will also be called hyperbolic pre-balls.

The next lemma follows immediately from [8, Lemma 3.1 & 3.2].

Lemma 1.3.2. *There are $\delta_0 > 0$, a point $p \in D$ and $N_0 \geq 1$ such that for each hyperbolic pre-ball $V_n'(x)$ there is $0 \leq m \leq N_0$ for which $f^{n+m}(V_n'(x))$ u-crosses $\mathcal{C}(\Delta_0)$, where $\Delta_0 = B(p, \delta_0) \subset D$.*

From here on we fix the two center-unstable disks centered at p

$$\Delta_0^0 = \Delta_0 = B(p, \delta_0) \quad \text{and} \quad \Delta_0^1 = B(p, 2\delta_0),$$

and the corresponding cylinders

$$\mathcal{C}_0^i = \bigcup_{x \in \Delta_0^i} W_{\delta_s}^s(x), \qquad \text{for } i = 0, 1. \tag{1.6}$$

The projection along stable leaves is denoted by π. We have

$$\pi(\mathcal{C}_0^i) = \Delta_0^i, \quad \text{for } i = 0, 1.$$

Remark 1.3.3. We assume that each disk γ^u u-crossing $\mathcal{C}_0^i$ $(i = 0, 1)$ is a disk centered at a point of $W_{\delta_s}^s(p)$ and with the same radius of Δ_0^i. We ignore the difference of radius caused by the height of the cylinder and the angles of the two dominated splitting bundles. Let the top and bottom components of $\partial \mathcal{C}_0^1$ be denotes by $\partial^u \mathcal{C}_0^1$, i.e. the set of points $z \in \partial \mathcal{C}_0^1$ such that $z \in \partial W_{\delta_s}^s(x)$ for some $x \in \Delta_0$. By the domination property, we may take $\delta_0 > 0$ small enough so that any centre-unstable disk γ^u which is contained in $\mathcal{C}_0^1$ and intersecting $W_{\delta_s/2}^s(p)$ does not reach $\partial^u \mathcal{C}_0^1$.

Given a hyperbolic pre-ball $V_n'(x)$, for $0 \leq m \leq N_0$ as in the conclusion of Lemma 1.3.2, we define

$$\omega_{n,m}^{i,x} = (f|_{V_n'(x)}^{n+m})^{-1}(f^{n+m}(\Delta_0^i) \cap \mathcal{C}_0^i), \quad i = 0, 1. \tag{1.7}$$

The sets of the type $\omega_{n,m}^{0,x}$, with $x \in H_n \cap \Delta_0$, are the natural candidates to be in the partition $\mathcal{P}$. In the sequel, sometimes we omit m, i and x in the notation $\omega_{n,m}^{i,x}$ and simply use ω_n to denote some element at step n.

For $k \geq n$, set the *annulus* around the element $\omega_n = \omega_{n,m}^{0,x}$

$$A_k(\omega_n) = \{y \in V_n(x) : 0 \leq \mathrm{dist}_D(f^{R(\omega_n)}(y), \Delta_0) \leq \delta_0 \sigma^{\frac{k-n}{2}}\}. \tag{1.8}$$

Obviously

$$A_n(\omega_n) \cup \omega_n = \omega_{n,m}^{1,x}.$$

1.3.2 Partition on the reference disk

In this subsection we describe an algorithm of a (Leb_D mod 0) partition $\mathcal{P}$ of Δ_0. The algorithm is similar to the one in [3], but in the present context of a diffeomorphism, each element of the partition will return to another u-leaf which *u-crosses* $\mathcal{C}_0^0$. Along the process we shall introduce sequences of objects (Δ_n), (Ω_n), (A_n) and (S_n). For each n, Δ_n is the set of points left in the reference disk up to time n, Ω_n is the union of elements of the partition at step n, and A_n is the union of rings around the chosen elements at time n. The set S_n (*satellite*) contains the components which could have been chosen for the partition but are too close to already chosen elements. Compared to [3], now S_n gathers more points which is the crucial step of the (stretched) exponential estimates of the tail of recurrence time in Subsection 1.3.4. More precise notation will be shown along the constructing process.

First step of induction

Given $n_0 \in \mathbb{N}$, we only consider the dynamics after time n_0. Remember $\Delta_0^c = D \setminus \Delta_0$. By the third assertion of [3, Lemma 3.7], there is a finite set of points $I_{n_0} = \{z_1, \ldots, z_{N_{n_0}}\} \in H_{n_0} \cap \Delta_0$ such that

$$H_{n_0} \cap \Delta_0 \subset V'_{n_0}(z_1) \cup \cdots \cup V'_{n_0}(z_{N_{n_0}}).$$

Consider a maximal family of pairwise disjoint sets of type (1.7) contained in Δ_0

$$\{\omega^{1,x_0}_{n_0,m_0}, \omega^{1,x_1}_{n_0,m_1}, \ldots, \omega^{1,x_{k_{n_0}}}_{n_0,m_{k_{n_0}}}\},$$

and denote

$$\Omega_{n_0} = \{\omega^{0,x_0}_{n_0,m_0}, \omega^{0,x_1}_{n_0,m_1}, \ldots, \omega^{0,x_{k_{n_0}}}_{n_0,m_{k_{n_0}}}\}.$$

The subsets in Ω_{n_0} are the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm. We obtain the *recurrence time* $R(\omega^{0,x_i}_{n_0,m_i}) = n_0 + m_i$ with $0 \le i \le k_{n_0}$. Recalling (1.8), we define

$$A_{n_0}(\Omega_{n_0}) = \bigcup_{\omega \in \Omega_{n_0}} A_{n_0}(\omega).$$

We need to keep track of the sets $\{\omega^{1,z}_{n_0,m} : z \in I_{n_0}, 0 \le m \le N_0\}$ which overlap $\Omega_{n_0} \cup A_{n_0}(\Omega_{n_0})$ or Δ_0^c. Given $\omega \in \Omega_{n_0}$, for each $0 \le m \le N_0$, we define

$$I^m_{n_0}(\omega) = \left\{x \in I_{n_0} : \omega^{1,x}_{n_0,m} \cap (\omega \cup A_{n_0}(\omega)) \ne \emptyset\right\},$$

and the n_0-*satellite* around ω

$$S_{n_0}(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I_{n_0}^m(\omega)} V'_{n_0}(x) \cap (\Delta_0 \setminus \omega), \tag{1.9}$$

We write

$$S_{n_0}(\Delta_0) = \bigcup_{\omega \in \Omega_{n_0}} S_{n_0}(\omega).$$

Similarly, we define the n_0-satellite associated to $\Delta_0^c = D \setminus \Delta_0$

$$S_{n_0}(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\omega_{n_0,m}^{1,x} \cap \Delta_0^c \neq \emptyset} V'_{n_0}(x) \cap \Delta_0, \quad x \in I_{n_0}.$$

We will show in the general step, the volume of $S_{n_0}(\Delta_0^c)$ is exponentially small. The 'global' n_0-satellite is

$$S_{n_0} = \bigcup_{\omega \in \Omega_{n_0}} S_{n_0}(\omega) \cup S_{n_0}(\Delta_0^c).$$

The remaining points at step n_0 are

$$\Delta_{n_0} = \Delta_0 \setminus \Omega_{n_0}.$$

Clearly,

$$H_{n_0} \cap \Delta_0 \subset S_{n_0} \cup \Omega_{n_0}.$$

General step of induction

The general step of the construction follows the ideas above with minor modifications. As before, there is a finite set of points $I_n = \{z_1, \dots, z_{N_n}\} \in H_n \cap \Delta_0$ such that

$$H_n \cap \Delta_0 \subset V'_n(z_1) \cup \dots \cup V'_n(z_{N_n}).$$

Assume that the sets Ω_i, Δ_i and S_i are defined for each $i \leq n-1$. Assuming

$$\Omega_\ell = \{\omega_{\ell,m_0}^{0,x_0}, \omega_{\ell,m_1}^{0,x_1}, \dots, \omega_{\ell,m_{k_\ell}}^{0,x_{k_\ell}}\}$$

for $n_0 \leq \ell \leq n-1$, we let

$$A_n(\Omega_\ell) = \cup_{\omega \in \Omega_\ell} A_n(\omega).$$

Now we consider a maximal family of pairwise disjoint sets of type (1.7) contained in Δ_{n-1}

$$\{\omega_{n,m_0}^{1,x_0}, \omega_{n,m_1}^{1,x_1}, \dots, \omega_{n,m_{k_n}}^{1,x_{k_n}}\}$$

satisfying

$$\omega_{n,m}^{1,x_i} \cap \left(\cup_{\ell=n_0}^{n-1} \{A_n(\Omega_\ell) \cup \Omega_\ell\}\right) = \emptyset, \quad i = 1, \dots, k_n,$$

and define

$$\Omega_n = \{\omega_{n,m_0}^{0,x_0}, \omega_{n,m_1}^{0,x_1}, \dots, \omega_{n,m_{k_n}}^{0,x_{k_n}}\}.$$

The subsets in Ω_n are the elements of the partition $\mathcal{P}$ constructed in the n-step of the algorithm. Set the *recurrence time* $R(x) = n + m_i$ for each $x \in \omega_{n,m_i}^{0,x_i}$ with $0 \le i \le \ell_n$. Given $\omega \in \Omega_{n_0} \cup \dots \cup \Omega_n$ and $0 \le m \le N_0$, we set

$$I_n^m(\omega) = \left\{x \in I_n : \omega_{n,m}^{1,x} \cap (\omega \cup A_n(\omega)) \neq \emptyset\right\},$$

define

$$S_n(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I_n^m(\omega)} V_n'(x) \cap (\Delta_0 \setminus \omega) \tag{1.10}$$

and

$$S_n(\Delta_0) = \bigcup_{\omega \in \Omega_{n_0} \cup \dots \cup \Omega_n} S_n(\omega).$$

Similarly, the *n-satellite* associated to Δ_0^c is

$$S_n(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\omega_{n,m}^{1,x} \cap \Delta_0^c \neq \emptyset} V_n'(x) \cap \Delta_0, \quad x \in I_n.$$

Remark 1.3.4. We have an observation that the volume of $S_n(\Delta_0^c)$ decays exponentially. Actually, it follows from the definition of $S_n(\Delta_0^c)$ and Lemma 1.2.9 that

$$S_n(\Delta_0^c) \subset \{x \in \Delta_0 : \operatorname{dist}_D(x, \partial\Delta_0) \le 2\delta_0 \sigma^{n/2}\}.$$

Thus, there exists $\rho > 0$ such that $\mathrm{Leb}_D(S_n(\Delta_0^c)) \le \rho\sigma^{n/2}$.

Finally we define the 'global' *n-satellite* associate to $\Omega_{n_0} \cup \dots \cup \Omega_n \cup \Delta_0^c$

$$S_n = S_n(\Delta_0) \cup S_n(\Delta_0^c),$$

and

$$\Delta_n = \Delta_0 \setminus \bigcup_{i=n_0}^{n} \Omega_i.$$

We clearly have

$$H_n \cap \Delta_0 \subset S_n \cup \bigcup_{i=n_0}^{n} \Omega_i. \tag{1.11}$$

1.3.3 Estimates on the satellites

For the sake of notational simplicity, we shall avoid the superscript 0 in the sets $\omega_{n,m}^{0,x}$. The next lemma shows that, given n and m, the conditional volume of the union of $\omega_{n,m}^{x}$ which intersects one chosen element is proportional to the conditional volume of this element. The proportion constant is uniformly summable with respect to n.

Though we consider here the case of partially hyperbolic attractor, and also the construction is modified a bit, the proofs of the next two lemmas are still essentially the same as [3, Lemmas 4.4 & 4.5].

Lemma 1.3.5. *(1) There exists $C_3 > 0$ such that, for any $n \geq n_0$, $0 \leq m \leq N_0$, and finitely many points $\{x_1, \dots, x_N\} \in I_n$ satisfying $\omega_{n,m}^{x_i} = \omega_{n,m}^{x_1}$ $(1 \leq i \leq N)$, we have*

$$\mathrm{Leb}_D\left(\bigcup_{i=1}^{N} V_n'(x_i)\right) \leq C_3 \,\mathrm{Leb}_D(\omega_{n,m}^{x_1}).$$

(2) There exists $C_4 > 0$ such that for $k \geq n_0$, $\omega \in \Omega_k$ and $0 \leq m \leq N_0$, given any $n \geq k$, we obtain

$$\mathrm{Leb}_D\left(\bigcup_{x\in I_n^m(\omega)} \omega_{n,m}^{x}\right) \leq C_4 \sigma^{\frac{n-k}{2}}\,\mathrm{Leb}_D(\omega).$$

Proposition 1.3.6. *There exists $C_5 > 0$ such that for any $\omega \in \Omega_k$ and $n \geq k$, we have*

$$\mathrm{Leb}_D(S_n(\omega)) \leq C_5 \sigma^{\frac{n-k}{2}}\,\mathrm{Leb}_D(\omega).$$

Proof. Consider now $k \geq n_0$ and $n \geq k$. Fix $\omega \in \Omega_k$ and consider $S_n(\omega)$ the n-satellite associated to it. By definition of $S_n(\omega)$ and Lemma 1.3.5 Item (1) we have

$$\begin{aligned}\mathrm{Leb}_D(S_n(\omega)) &\leq \sum_{m=0}^{N_0}\sum_{x\in I_n^m(\omega)} \mathrm{Leb}_D\left(V_n'(x)\cap(\Delta_0\setminus\omega)\right) + \mathrm{Leb}_D(V_k'(\omega)\setminus\omega)\\ &\leq C_3\sum_{m=0}^{N_0}\mathrm{Leb}_D\left(\bigcup_{x\in I_n^m(\omega)}\omega_{n,m}^{x}\right) + C_3\,\mathrm{Leb}_D(\omega).\end{aligned}$$

In this last step we have used the obvious fact that for fixed n, m the sets of the form $\omega_{n,m}^{x}$ with $x \in I_n^m(\omega)$ are pairwise disjoint. Thus, by Lemma 1.3.5 Item (2),

$$\mathrm{Leb}_D(S_n(\omega)) \leq C_3(C_4(N_0+1)+1)\sigma^{\frac{n-k}{2}}\,\mathrm{Leb}_D(\omega).$$

Take $C_5 = C_3(C_4(N_0+1)+1)$, then we finish the proof. □

Definition 1.3.7. Given $k \geq n_0$ and $\omega_{k,m}^x \in \Omega_k$, for some $x \in \Delta_0$ and $0 \leq m \leq N_0$, we define for $n \geq k$

$$B_n^k(x) = S_n(\omega_{k,m}^x) \cup \omega_{k,m}^x \quad \text{and} \quad t(B_n^k(x)) = k.$$

Notice that k and n are both hyperbolic times for points in Δ_0. The set $\omega_{k,m}^x$ will be called the *core* of $B_n^k(x)$ and denoted as $C(B_n^k(x))$.

The next result follows immediately from Proposition 1.3.6.

Corollary 1.3.8. *For all $n \geq k$ and x, we have*

$$\mathrm{Leb}_D(B_n^k(x)) \leq (C_5 + 1)\,\mathrm{Leb}_D(C(B_n^k(x))).$$

The dependence of δ_1' on δ_1 becomes clear in the next lemma.

Lemma 1.3.9. *If $n_0 \leq k \leq k'$, $n \geq k$, $n' \geq k'$ and $B_n^k(x) \cap B_{n'}^{k'}(y) \neq \emptyset$, then*

$$C(B_n^k(x)) \cup C(B_{n'}^{k'}(y)) \subset V_k(x).$$

Proof. Since k and $n \geq k$ are hyperbolic times, by the second assertion of Lemma 1.2.9

$$\mathrm{diam}_{f^k(D)}(f^k(B_n^k(x))) \leq 2\delta_1' + 4\delta_1'\sigma^{\frac{n-k}{2}}.$$

Use again the second assertion of Lemma 1.2.9, and we finally have

$$\mathrm{diam}_D(B_n^k(x)) \leq (2\delta_1' + 4\delta_1'\sigma^{\frac{n-k}{2}})\sigma^{\frac{k}{2}} \leq 6\delta_1'\sigma^{\frac{k}{2}}.$$

Similarly

$$\mathrm{diam}_D(B_{n'}^{k'}(y)) \leq 6\delta_1'\sigma^{\frac{k'}{2}}.$$

Now observe that it is enough to obtain the conclusion of the lemma for $n = k$ and $n' = k'$. By the computation above, we have

$$\mathrm{diam}_D(B_n^n(x)) \leq 6\delta_1'\sigma^{n/2} \quad \text{and} \quad \mathrm{diam}_D(B_{n'}^{n'}(y)) \leq 6\delta_1'\sigma^{n'/2} \leq 6\delta_1'\sigma^{n/2}.$$

Then we have

$$\mathrm{dist}_{f^n(D)}(f^n(x), \partial f^n(V_n'(y))) \leq 7\delta_1' \ll \delta_1,$$

so $f^n(V_n'(y)) \subset B(f^n(x), \delta_1)$. We build a set $W_n'(y) = f^{-n}(f^n(V_n'(y))) \cap V_n(x)$. By the definition of V_n, f^n is an isomorphism between $W_n'(y)$ and $f^n(V_n'(y))$. But also f^n is an isomorphism between $V_n'(y)$ and $f^n(V_n'(y))$. By the uniqueness in Lemma 1.2.9, $V_n'(y) = W_n'(y)$. In particular, $V_n'(y) \subset V_n(x)$. And so $C(B_{n'}^{k'}(y)) \subset V_n(x)$. Then

$$C(B_n^k(x)) \cup C(B_{n'}^{k'}(y)) \subset V_n(x).$$

The result follows immediately. $\square$

Lemma 1.3.10. *There exists $P \geq N_0$ such that for all $n_0 \leq t_1 \leq t_2$,*

$$B_{t_2+P}^{t_2}(y) \cap B_{t_2+P}^{t_1}(x) = \emptyset.$$

Proof. Suppose, on the contrary, that we have $B_{t_2+P}^{t_2}(y) \cap B_{t_2+P}^{t_1}(x) \neq \emptyset$ for all $P \geq N_0$. Take a point z in the intersection. Then, let $R_1 = R(C(B_{t_2+P}^{t_1}(x)))$, and recall that $t_2 + P$ is a hyperbolic time in the definitions of $B_{t_2+P}^{t_1}(x)$ and $B_{t_2+P}^{t_2}(y)$. By the second assertion of Lemma 1.2.9, we obtain

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B_{t_2+P}^{t_1}(x)))\right) \leq 2\delta_1'\sigma^{\frac{t_2+P-R_1}{2}};$$

and also

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq 2\delta_1'\sigma^{\frac{t_2+P-R_1}{2}}.$$

Hence,

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B_{t_2+P}^{t_1}(x))), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq 4\delta_1'\sigma^{\frac{t_2+P-R_1}{2}}.$$

Let P large enough such that $4\delta_1'\sigma^{P/2} < \delta_0\sigma^{N_0/2}$, and we have

$$\operatorname{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B_{t_2+P}^{t_1}(x))), f^{R_1}(C(B_{t_2+P}^{t_2}(y)))\right) \leq \delta_0\sigma^{\frac{t_2-t_1}{2}}$$

which means $C(B_{t_2+P}^{t_2}(y)) \subset A_{t_2}(C(B_{t_1+P}^{t_1}(x)))$. This is a contradiction. $\square$

1.3.4 Tail of recurrence times

Though our constructions are very different from [15], our approach on the estimates below is inspired in [15, Subsection 3.2]. Given a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$, $c > 0$, we assume $\operatorname{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Recalling Remark 1.3.4, there exists a constant $\rho > 0$ such that for all $n \in \mathbb{N}$

$$\operatorname{Leb}_D\{x \in D \mid \operatorname{dist}_D(x, \partial\Delta_0) \leq 2\delta_0\sigma^{\frac{n}{2}}\} \leq \rho\sigma^{\frac{n}{2}}. \tag{1.12}$$

Recall that Δ_n is the complement at time n, and θ is defined in Proposition 1.2.7.

We will show $\mathrm{Leb}_D(\Delta_n)$ decays (stretched) exponentially.

Recall that $\mathrm{Leb}_D\{\mathcal{E} > n\}$ is (stretched) exponentially small and $\mathrm{Leb}_D(\{x \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq 2\delta_0\sigma^{\frac{\theta n}{4}}\}$ decays exponentially as in (1.12). Take $x \in \Delta_n$ which does not belong either to $\{\mathcal{E} > n\} \cap D$ or to $\{x \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq 2\delta_0\sigma^{\frac{\theta n}{4}}\}$. By Proposition 1.2.7, for n large, x has at least θn hyperbolic times between 1 and n, then x has at least $\frac{\theta n}{2}$ hyperbolic times between $\frac{\theta n}{2}$ and n. We order them as $\frac{\theta n}{2} \leq t_1 < \cdots < t_k \leq n$, then $x \in H_{t_i} \cap \Delta_0$ for $1 \leq i \leq k$.

We know from the construction (see (1.11) in Subsection 1.3.2)

$$H_{t_i} \cap \Delta_0 \subset S_{t_i} \cup \bigcup_{j=n_0}^{t_i} \Omega_j, \quad \text{for } 1 \leq i \leq k.$$

If $x \notin S_{t_i}$, then $x \in \cup_{j=n_0}^{t_i} \Omega_j$ which means $x \notin \Delta_n$. A contradiction. So we get $x \in S_{t_i}$. As $x \in \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) > 2\delta_0\sigma^{\frac{\theta n}{4}}\}$, we have

$$x \in H_{t_i} \cap \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) > 2\delta_0\sigma^{t_i/2}\}, \quad \text{for } 1 \leq i \leq k.$$

Recalling Remark 1.3.4, we obtain $x \notin S_{t_i}(\Delta_0^c)$. Then we obtain

$$x \in S_{t_i}(\Delta_0), \quad \text{for } i = 1, \ldots, k.$$

We simply take $k = \frac{\theta n}{2}$. Thus, x belongs to the set

$$Z\left(\frac{\theta n}{2}, n\right) := \left\{x \mid \exists t_1 < \ldots < t_{\frac{\theta n}{2}} \leq n, x \in \bigcap_{i=1}^{\frac{\theta n}{2}} S_{t_i}(\Delta_0)\right\} \cap \Delta_n.$$

So we have

$$\Delta_n \subset \{x \in \Delta_0 \mid \mathcal{E} > n\} \cup \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq 2\delta_0\sigma^{\frac{\theta n}{4}}\} \cup Z(\theta n/2, n).$$

Since the second set has exponentially small measure by (1.12), it remains to see that the measure of $Z(\theta n/2, n)$ decays exponentially fast. This follows from Proposition 1.3.11 below. Observe that if there exists $d > 0$ such that

$$\mathrm{Leb}_D(\Delta_n) \leq \mathcal{O}(e^{-dn^\tau}),$$

for any large integer k, we have $\mathcal{R}_k = \{R > k\} \subset \Delta_{k-N_0}$, and so

$$\mathrm{Leb}_D\{R > k\} \leq \mathrm{Leb}_D(\Delta_{k-N_0}) = \mathcal{O}(e^{-d(k-N_0)^\tau}) = \mathcal{O}(e^{-dk^\tau}).$$

The next proposition shows that the set of points contained in finite satellite sets and have not been chosen yet has a measure which decays exponentially.

Proposition 1.3.11. *Set, for integers k, N,*

$$Z(k,N) \;=\; \left\{x \mid \exists t_1 < \ldots < t_k \le N, x \in \bigcap_{i=1}^{k} S_{t_i}(\Delta_0) \cap \Delta_N\right\}.$$

There exist $D_5 > 0$ and $\lambda_5 < 1$ such that, for all N and $1 \le k \le N$,

$$\mathrm{Leb}_D(Z(k,N)) \le D_5 \lambda_5^k \,\mathrm{Leb}_D(\Delta_0).$$

For the proof of this result we need several lemmas that we prove in the sequel. We fix some integer $P' \ge P$ (see P in Lemma 1.3.10) whose value will be made precise in the proof of Proposition 1.3.11. In Lemma 1.3.12, 1.3.13 and 1.3.14 we simply denote $B_i = B^{t_i}_{t_i+m_i}(x)$ for some t_i, x, and $m_i \le P'$.

Lemma 1.3.12. *Set $E \in \mathbf{N}$, and*

$$\begin{aligned} Z_1(k,B_0) \;=\; &\Big\{x \,|\, \exists B_1', B_1, \ldots, B_r', B_r, \textit{so that } \forall 1 \le i \le r,\ t_{i-1} \le t_i' \le t_i - E, B_i \not\subseteq B_i', \\ &\sum_{i=1}^{r} \left[\frac{t_i - t_i'}{E}\right] \ge k \textit{ and } x \in \bigcap_{i=0}^{r} B_i \cap \bigcap_{i=1}^{r} B_i'\Big\}. \end{aligned}$$

There is D_1 (independent of E, P'), for all k and B_0,

$$\mathrm{Leb}_D(Z_1(k,B_0)) \le D_1 (D_1 \sigma^{E/2})^k \,\mathrm{Leb}_D(C(B_0)).$$

Proof. Choose $D_1 > 0$ large enough such that

$$\frac{1}{1 - D_1^{-1}} (\rho^2 C_2^3) \le D_1 \quad \text{and} \quad C_5 + 1 \le D_1.$$

We will prove the assertion by induction on $k \ge 0$. When $k = 0$, recall Corollary 1.3.8 and we obtain

$$\mathrm{Leb}_D(Z_1(0,B_0)) \le \mathrm{Leb}_D(B_0) \le (C_5+1)\,\mathrm{Leb}_D(C(B_0)) \le D_1 \,\mathrm{Leb}_D(C(B_0)). \tag{1.13}$$

When $k \ge 1$, by decomposition, we have

$$Z_1(k,B_0) \subset \bigcup_{t=1}^{k} \bigcup_{B_1' \cap B_0 \ne \emptyset} \bigcup_{\substack{B_1' \cap B_1 \ne \emptyset, B_1 \not\subseteq B_1', \\ \left[\frac{t_1 - t_1'}{E}\right] \ge t}} Z_1(k-t, B_1).$$

Let $n = t_1 - t_1'$. Fix some B_1', and take one from all the possible B_1's. We still call it B_1. It is contained in a ring of size $\sigma^{\frac{n}{2}}$ around B_1'. More precisely, setting $p = t_1'$ and defining $Q_1' = f^p(B_1')$, $Q_1 = f^p(C(B_1'))$, we will show that

$$f^p(B_1) \subset \mathcal{C} := \{y \mid \mathrm{dist}_{f^p(D)}(y, \partial Q_1') \leq 6\delta_1' \sigma^{\frac{n}{2}}\}. \tag{1.14}$$

Since B_1 contains a point of $\partial B_1'$, $f^p(B_1)$ contains a point of $\partial Q_1'$. We obtain

$$\mathrm{diam}_{f^p(D)}\, f^p(B_1) \leq \sigma^{\frac{n}{2}} \mathrm{diam}_{f^{p+n}(D)}\, f^{p+n}(B_1) \leq 6\delta_1' \sigma^{\frac{n}{2}}.$$

Then we get (1.14). By (1.12), there is ρ satisfying

$$\mathrm{Leb}_{f^p(D)}(\mathcal{C}) \leq \rho \sigma^{\frac{n}{2}}\, \mathrm{Leb}_{f^p(D)}(Q_1').$$

Hence

$$\mathrm{Leb}_{f^p(D)}(f^p(B_1)) \leq \rho \sigma^{\frac{n}{2}}\, \mathrm{Leb}_{f^p(D)}(Q_1').$$

Since $C_5 + 1 \leq D_1$,

$$\mathrm{Leb}_D(B_1) \leq D_1\, \mathrm{Leb}_D(C(B_1)).$$

By the bounded distortion constant C_2, we have

$$\mathrm{Leb}_{f^p(D)}(Q_1') \leq C_2 D_1\, \mathrm{Leb}_{f^p(D)}(Q_1).$$

Then obviously,

$$\mathrm{Leb}_{f^p(D)}(f^p(B_1)) \leq C_2 D_1 \rho \sigma^{\frac{n}{2}}\, \mathrm{Leb}_{f^p(D)}(Q_1). \tag{1.15}$$

The cores $C(B_1)$ of those possible B_1's are pairwise disjoint by construction. And importantly, the possible cores $C(B_1)$ must be all contained in $V_p(x_1')$, and $C(B_1') = \omega_{t_1'}^{x_1'}$ by Lemma 1.3.9. We know that f^p is a diffeomorphism on $V_p(x_1')$. So $f^p(C(B_1)) \subset f^p(V_p(x_1'))$. As

$$f^p(C(B_1)) \subset f^p(B_1) \subset \mathcal{C},$$

then

$$\sum_{B_1' \cap B_1 \neq \emptyset, \left[\frac{t_1 - t_1'}{E}\right] \geq t} \mathrm{Leb}_{f^p(D)}(f^p(C(B_1))) \leq \mathrm{Leb}_{f^p(D)}\, \mathcal{C}.$$

Remember that $Q_1 = f^p(C(B_1'))$. By (1.15) we get

$$\sum_{B_1' \cap B_1 \neq \emptyset, \left[\frac{t_1 - t_1'}{E}\right] \geq t} \mathrm{Leb}_{f^p(D)}(f^p(C(B_1))) \;\leq\; C_2 D_1 \rho \sigma^{\frac{n}{2}}\, \mathrm{Leb}_{f^p(D)}(Q_1).$$

Now, using the bounded distortion constant C_2, we obtain

$$\sum_{B'_1 \cap B_1 \neq \emptyset, \left[\frac{t_1 - t'_1}{E}\right] \geq t} \operatorname{Leb}_D(C(B_1)) \;\; \leq \;\; D_1 C_2^2 \rho \sigma^{\frac{Et}{2}} \operatorname{Leb}_D(C(B'_1)). \tag{1.16}$$

After that, write $q = t_0$ and $C(B_0) = \omega^x_{q,m}$. The possible sets $C(B'_1)$'s are pairwise disjoint by construction, and included in $V_q(x)$ by Lemma 1.3.9. Indeed, f^q is a diffeomorphism on $V_q(x)$ and its distortion is bounded by C_2. Since $f^q(V_q(x)) = B(f^q(x), \delta_1)$, we suppose that $\operatorname{Leb}_{f^q(D)}(B(f^q(x), \delta_1)) \leq \rho \operatorname{Leb}_{f^q(D)}(f^q(C(B_0)))$. By bounded distortion, obtaining by

$$\operatorname{Leb}_D(V_q(x)) \leq C_2 \operatorname{Leb}_D(C(B_0)) \frac{\operatorname{Leb}_{f^q(D)}(B(f^q(x), \delta_1))}{\operatorname{Leb}_{f^q(D)}(f^q(C(B_0)))},$$

we have

$$\sum_{B'_1 \cap B_0 \neq \emptyset} \operatorname{Leb}_D(C(B'_1)) \leq \rho C_2 \operatorname{Leb}_D(C(B_0)).$$

Finally, the induction assumption gives

$$\begin{aligned}
\operatorname{Leb}_D(Z_1(k, B_0)) \;\; &\leq \;\; \sum_{t=1}^{k} \sum_{B'_1 \cap B_0 \neq \emptyset} \sum_{\left[\frac{t_1 - t'_1}{E}\right] \geq t} \operatorname{Leb}_D(Z_1(k-t, B_1)) \\
&\leq \;\; \sum_{t=1}^{k} \sum_{B'_1 \cap B_0 \neq \emptyset} \sum_{\left[\frac{t_1 - t'_1}{E}\right] \geq t} D_1 (D_1 \sigma^{E/2})^{k-t} \operatorname{Leb}_D(C(B_1)) \\
&\leq \;\; \sum_{t=1}^{k} \sum_{B'_1 \cap B_0 \neq \emptyset} D_1 (D_1 \sigma^{\frac{E}{2}})^{k-t} D_1 C_2^2 \rho \sigma^{\frac{Nt}{2}} \operatorname{Leb}_D(C(B'_1)) \\
&\leq \;\; D_1 (D_1 \sigma^{\frac{E}{2}})^{k} D_1 C_2^3 \rho^2 \sum_{t=1}^{k} (D_1)^{-t} \operatorname{Leb}_D(C(B_0)).
\end{aligned}$$

By the definition of D_1, we have $D_1 \rho^2 C_2^3 (\sum_{t=1}^{k} (D_1)^{-t}) \leq 1$. Then we get

$$\operatorname{Leb}_D(Z_1(k, B_0)) \leq D_1 (D_1 \sigma^{\frac{E}{2}})^k \operatorname{Leb}_D(C(B_0)),$$

which ends the proof. □

Lemma 1.3.13. *Set*

$$Z_2(k, N) = \{x \mid \exists B_1 \supsetneq B_2 \ldots \supsetneq B_k \, with \, t_1 < \cdots < t_k \leq N \, and \, x \in B_1 \cap \ldots \cap B_k \cap \Delta_N\}.$$

Then there exists $\lambda_2 < 1$ such that for all $N \geq 1$ and $1 \leq k \leq N$,

$$\mathrm{Leb}_D(Z_2(k,N)) \leq \lambda_2^k \,\mathrm{Leb}_D(\Delta_0).$$

Proof. We assume N is fixed in this proof, so $Z_2(k) := Z_2(k,N)$. We will prove that the conclusion of the lemma holds for $\lambda_2 = \frac{D_1}{D_1+1}$. By Corollary 1.3.8 and $C_5 + 1 \leq D_1$, for each possible B, we get

$$\mathrm{Leb}_D(B) \quad \leq \quad D_1 \,\mathrm{Leb}_D(C(B)). \tag{1.17}$$

We define $\mathcal{Q}_1$ as a maximal class of sets B with $t(B) \leq N$ and not contained in any other B's. Consider $\mathcal{Q}_2 \subset \mathcal{Q}_1^c$ as the class of sets B with $t(B) \leq N$ which are included in elements of $\mathcal{Q}_1$. Next we define $\mathcal{Q}_3 \subset \mathcal{Q}_2^c$ as the class of sets B with $t(B) \leq N$ which are included in elements of $\mathcal{Q}_2$. We proceed inductively. Notice that this process must stop in a finite number of steps because we always take $t(B) \leq N$. We say that an element in $\mathcal{Q}_i$ has *rank i*.

Let now

$$G_k = \bigcup_{i=1}^{k} \bigcup_{B \in \mathcal{Q}_k} C(B),$$

and

$$\tilde{Z}_2(k) = \left(\bigcup_{B \in \mathcal{Q}_k} B \right) \setminus G_k.$$

Now we prove that $Z_2(k) \subset \tilde{Z}_2(k)$. Given $x \in Z_2(k)$, we have $x \in B_1 \cap \ldots \cap B_k \cap \Delta_N$ with $B_1 \supsetneq B_2 \ldots \supsetneq B_k$ and $t(B_k) \leq N$. We clearly have that B_k is of rank $r \geq k$. Take $B_1' \supsetneq B_2' \ldots \supsetneq B_{r-1}' \supsetneq B_r'$ a sequence with $B_i' \in \mathcal{Q}_i$ and $B_r' = B_k$. In particular, $x \in B_i'$ for $i = 1, \ldots, k$, and so $x \in \bigcup_{B \in \mathcal{Q}_k} B$. On the other hand, since $x \in \Delta_N$ and $G_k \cap \Delta_N = \emptyset$, we get $x \notin G_k$. So $x \in \tilde{Z}_2(k)$.

Now we deduce the relation between $\mathrm{Leb}_D(\tilde{Z}_2(k+1))$ and $\mathrm{Leb}_D(\tilde{Z}_2(k))$, in such a way that we may estimate $\mathrm{Leb}_D(\tilde{Z}_2(k))$. Take $B \in \mathcal{Q}_{k+1}$. Let B' be an element of rank k containing B. As the cores are pairwise disjoint by nature, $C(B) \cap G_k = \emptyset$. We obtain $C(B) \subset B' \setminus G_k \subset \tilde{Z}_2(k)$. By definition $C(B) \subset G_{k+1}$, thus $C(B) \cap \tilde{Z}_2(k+1) = \emptyset$. This means that $C(B) \subset \tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1)$. Finally, by (1.17),

$$\mathrm{Leb}_D(\tilde{Z}_2(k+1)) \quad \leq \quad \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(B)$$

$$\leq D_1 \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(C(B))$$
$$\leq D_1 \,\mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1))$$

since the $C(B)$ are pairwise disjoint. Then, we obtain

$$\begin{aligned}(D_1+1)\,\mathrm{Leb}_D(\tilde{Z}_2(k+1)) &\leq D_1\,\mathrm{Leb}_D(\tilde{Z}_2(k+1)) + D_1\,\mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1)) \\ &= D_1\,\mathrm{Leb}_D(\tilde{Z}_2(k)).\end{aligned}$$

It yields $\mathrm{Leb}_D(\tilde{Z}_2(k)) \leq \left(\frac{D_1}{D_1+1}\right)^k \mathrm{Leb}_D(\Delta_0)$ by induction. Since $Z_2(k) \subset \tilde{Z}_2(k)$, the same inequality holds for $Z_2(k)$. This ends the proof. □

The results of Lemma 1.3.12 and Lemma 1.3.13 are enough for us to assert next lemma:

Lemma 1.3.14. *Set*

$$Z_3(k,N) = \left\{ x \mid \exists t_1 < \ldots < t_k \leq N, x \in S_{t_1+m_1}(\Omega_{t_1}) \cap \ldots \cap S_{t_k+m_k}(\Omega_{t_k}) \cap \Delta_N \right\},$$

where $m_1, \ldots, m_q < P'$. There are constants $D_3 > 0$ and $\lambda_3 < 1$ (both independent of P') such that, for all N and $1 \leq k \leq N$,

$$\mathrm{Leb}_D(Z_3(k,N)) \leq D_3 \lambda_3^k \,\mathrm{Leb}_D(\Delta_0).$$

Proof. Choose E large enough s.t. $D_1 \sigma^{E/2} < 1$ (recall Lemma 1.3.12). Let us write $N = rE + s$ with $s < E$. Given an arbitrary $x \in Z_3(k,N)$, then there exist instants $t_1 < \ldots < t_k$ as in the definition of $Z_3(k,N)$. For $0 \leq u < r$, take from each interval $[uE, (u+1)E)$ the first appeared $t_i \in \{t_1, \ldots, t_k\}$ (if there is at least one). Denote the got subsequence t_i's by $t_{1'} < \cdots < t_{k'}$. Since $t_1 < \cdots < t_k \leq N$, we can see $k' \geq [\frac{k}{E}]$, which means $Ek' + E \geq k$. Keeping only the instants with odd indexes, we get a sequence of instants $u_1 < \ldots < u_\ell$ with $2\ell \geq k'$, and necessarily $\ell \geq \frac{k-E}{2E}$. Moreover, we have $u_{i+1} - u_i \geq E$ for $1 \leq i \leq \ell$ by construction.

Now, according to our construction process, we know that associated to each instant u_i there must be some set B_i such that $x \in B_i$, for $1 \leq i \leq \ell$. Define

$$I = \{1 \leq i \leq \ell, B_i \subset B_1 \cap \cdots \cap B_{i-1}\} \quad \text{and} \quad J = [1,\ell] \setminus I.$$

If $\#I \geq \ell/2$, we keep only the elements with indexes in I. Recalling Z_2 in Lemma 1.3.13, we have $x \in Z_2(\ell/2, N)$. Then $Z_2(\ell/2, N)$ has an exponentially small measure in ℓ (then in

k). Otherwise, if $\#I \leq \ell/2$, then $\#J \geq \ell/2$. Let $j_0 = \sup J$ and $i_0 = \inf\{i < j_0, B_{j_0} \not\subseteq B_i\}$. Let $j_1 = \sup\{j \leq i_0, j \in J\}$, $i_1 = \inf\{i < j_1, B_{j_1} \not\subseteq B_i\}$, and continue the process. The process must necessarily stop at some step i_n. Then $J \subset \cup_{s=0}^n (i_s, j_s]$ by construction. We obtain $\sum_{s=0}^n (j_s - i_s) \geq \#J \geq \ell/2$, which shows that

$$\sum_{s=0}^{n} \left[\frac{t(B_{j_s}) - t(B_{i_s})}{E}\right] = \sum_{s=0}^{n} \left[\frac{u_{j_s} - u_{i_s}}{E}\right] \geq \ell/2,$$

since $|u_j - u_i| \geq E(j-i)$ by the process. Hence $x \in Z_1(\ell/2, B_{i_n})$ with the sequence $B_{i_n}, B_{i_n}, B_{j_n}, \ldots, B_{i_0}, B_{j_0}$. As the cores are pairwise disjoint by nature, we use the estimate of Lemma 1.3.12 and, summing over all the possible B'_{i_n}s, we get

$$\mathrm{Leb}_D(Z_3(k,N)) \leq D_3 \lambda_3^k \,\mathrm{Leb}_D(\Delta_0).$$

□

Lemma 1.3.15. *Given $B_1 = B_{t_1}^{t_1}(x_1)$, we let*

$$\begin{aligned} Z_4(n_1,\ldots,n_k,B_1) \quad = \quad & \Big\{x \mid \exists\, t_2,\ldots,t_k \textit{ with } t_1 < \ldots < t_k \textit{ and } x_2,\ldots,x_k, \\ & \textit{s.t. } x \in \bigcap_{i=1}^{k} B_{t_i+n_i}^{t_i}(x_i) \cap \Delta_N \Big\}. \end{aligned}$$

Then, there is $D_4 > 0$ (independent of $B_1, n_1, \ldots, n_k$) such that for $n_1, \ldots, n_k > P$,

$$\mathrm{Leb}_D(Z_4(n_1,\ldots,n_k,B_1)) \leq D_4(D_4\sigma^{n_1/2})\ldots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_D(C(B_1)).$$

Proof. The proof is by induction on k. Taking $D_4 > {C_5}^{1/2}$ (recall C_5 in Proposition 1.3.6), we get the result immediately when $k = 1$. Now suppose $k > 1$. Let $x \in Z_4(n_1,\ldots,n_k,B_1)$. There exists $B_2 = B_{t_2}^{t_2}(x_2)$ constructed at an instant $t_2 > t_1$, and $x \in Z_4(n_2,\ldots,n_k,B_2)$. Suppose

$$\mathrm{Leb}_D(Z_4(n_2,\ldots,n_k,B_2)) \leq D_4(D_4\sigma^{n_2/2})\ldots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_D(C(B_2)).$$

There exists P given by Lemma 1.3.10, such that $B_{t_2+P}^{t_2}(x_2) \cap B_{t_2+P}^{t_1}(x_1) = \emptyset$. But for all $1 \leq i \leq k$, we have $x \in B_{t_i+n_i}^{t_i}(x_i)$. So, $t_1 + n_1 < t_2 + P$, i.e. $t_2 - t_1 > n_1 - P$. By the uniform expansion at hyperbolic times, we get

$$\mathrm{diam}_{f^{t_1}(D)}(f^{t_1}(B_2)) \leq \sigma^{\frac{t_2-t_1}{2}}\,\mathrm{diam}_{f^{t_2}(D)}(f^{t_2}(B_2)) \leq 6\delta_1'\sigma^{\frac{n_1-P}{2}}.$$

On the other hand, setting $Q = f^{t_1}(C(B_1))$, we have $\mathrm{dist}_{f^{t_1}(D)}(f^{t_1}(x), \partial Q) \le 2\delta_1' \sigma^{\frac{n_1}{2}}$ when $x \in B^{t_1}_{t_1+n_1}(x_1) \cap B_2$. Then, taking $D_4 \ge 2\delta_1' + 6\delta_1'\sigma^{-P}$, we have

$$f^{t_1}(B_2) \subset \mathcal{C} := \{y \mid \mathrm{dist}_{f^{t_1}(D)}(y, \partial Q) \le D_4 \sigma^{\frac{n_1}{2}}\}.$$

By induction and bounded distortion, we get

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_2, \dots, n_k, B_2))) \le C_2 D_4 (D_4 \sigma^{n_2/2}) \dots (D_4 \sigma^{n_k/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2))).$$

The possible cores $C(B_2)$'s are pairwise disjoint by nature and contained in $V_{t_1}(x_1)$ by Lemma 1.3.9. The sets $f^{t_1}(C(B_2))$ are still pairwise disjoint, since f^{t_1} is injective on $V_{t_1}(x_1)$. So they are all contained in the annulus $\mathcal{C}$. We have

$$\begin{aligned}\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_1, \dots, n_k, B_1))) &\le \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_2, \dots, n_k, B_2))) \\ &\le C_2 D_4 (D_4 \sigma^{n_2/2}) \dots (D_4 \sigma^{n_k/2}) \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2))) \\ &\le C_2 D_4 (D_4 \sigma^{n_2/2}) \dots (D_4 \sigma^{n_k/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}).\end{aligned}$$

By (1.14) and (1.15), we similarly get $\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}) \le C_2 D_1 \rho \sigma^{n_1/2} \,\mathrm{Leb}_{f^{t_1}(D)}(Q)$ in which $Q = f^{t_1}(C(B_1))$. Hence,

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_1, \dots, n_k, B_1))) \le C_2^2 D_1 D_4 \rho \sigma^{n_1/2} (D_4 \sigma^{n_2/2}) \dots (D_4 \sigma^{n_k/2}) \,\mathrm{Leb}_{f^{t_1}(D)}(Q).$$

By the bounded distortion constant C_2 of the map f^{t_1} on $V_{t_1}(x_1)$, we get

$$\mathrm{Leb}_D(Z_4(n_1, \dots, n_k, B_1)) \le C_2^3 D_1 \rho (D_4 \sigma^{n_1/2})(D_4 \sigma^{n_2/2}) \dots (D_4 \sigma^{n_k/2}) \,\mathrm{Leb}_D(C(B_1)).$$

Taking $D_4 \ge C_2^3 D_1 \rho$, we finish the proof. □

Now we are ready to complete the proof of the metric estimates.

Proof of Proposition 1.3.11. Take $P' \ge P$ (recall P in Lemma 1.3.10) so that

$$\sigma^{1/2} + D_3 \sigma^{P'/2} < 1.$$

Let $x \in Z(k, N)$, consider all the instants u_i for which x is in some $S_{u_i+n_i}(\omega^y_{u_i,m})$ with $n_i \ge P'$, ordered so that $u_1 < \dots < u_p$. Then $x \in Z_4(n_1, \dots, n_p, B_1)$ for some B_1. If $\sum_{i=1}^p n_i \ge k/2$, we finish the proof. Otherwise, $\sum_{i=1}^p n_i < k/2$, and $p < k/2P'$. Let

$v_1 < \ldots < v_q$ be other instants for which $x \in S_{v_i+m_i}(\omega^z_{v_i,\tilde{m}})$, for times $m_1, \ldots, m_q < P'$. We have $p + q \geq k$, then $q \geq \frac{(2P'-1)k}{2P'} \geq \frac{k}{2P'}$, where $P' > 1$. This shows $P'q \geq \frac{k}{2}$.

Thus we have

$$Z(k, N) \subset \bigcup_{B_1} \bigcup_{\substack{n_1,\ldots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} Z_4(n_1, \ldots, n_p, B_1) \cup Z_3\left(\frac{k}{2P'}, N\right).$$

By Lemma 1.3.14 and Lemma 1.3.15, we obtain

$$\mathrm{Leb}_D(Z(k, N)) \leq \sum_{B_1} \sum_{\substack{n_1,\ldots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} D_3(D_3\sigma^{n_1/2}) \ldots (D_3\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_1)) + D_3\lambda_3^{\frac{k}{2P'}} \,\mathrm{Leb}_D(\Delta_0).$$

We have $\sum_{B_1} \mathrm{Leb}_D(C(B_1)) \leq \mathrm{Leb}_D(\Delta_0) < \infty$, because the cores $C(B_1)$ are pairwise disjoint. What is left is to show that

$$\sum_{\substack{n_1,\ldots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} (D_3\sigma^{n_1/2}) \ldots (D_3\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_1))$$

is exponentially small. Let us adopt

$$\sum_n \sum_{\substack{n_1,\ldots,n_p \geq P', \\ \sum n_i = n}} (D_3\sigma^{n_1/2}) \ldots (D_3\sigma^{n_p/2}) z^n = \sum_{p=1}^{\infty} \left(D_3 \sum_{n=P'}^{\infty} \sigma^{n/2} z^n \right)^p = \frac{D_3\sigma^{P'/2} z^{P'}}{1 - \sigma^{1/2} z - D_3\sigma^{P'/2} z^{P'}}.$$

Under the hypothesis $\sigma^{1/2} + D_3\sigma^{P'/2} < 1$, the function above has no extreme pole in the unit disk's neighbourhood in $\mathbb{C}$. Thus its coefficients decay exponentially fast. There are constants $D_6 > 0$ and $\lambda_6 < 1$ such that

$$\sum_{\substack{n_1,\ldots,n_p \geq P', \\ \sum n_i = n}} (D_3\sigma^{n_1/2}) \ldots (D_3\sigma^{n_p/2}) \leq D_6\lambda_6^n.$$

Then we sum over $n \geq k/2$ and B_1 to obtain constants $D_5 > 0$, $\lambda_5 < 1$ such that

$$\mathrm{Leb}_D(Z(k, N)) \leq D_5\lambda_5^k \,\mathrm{Leb}_D(\Delta_0).$$

$\square$

1.3.5 Product structure

Consider the center-unstable disk $\Delta_0 \subset D$ and the partition $\mathcal{P}$ of Δ_0 (Leb_D mod 0) defined in Subsection 1.3.2. We define

$$\Gamma^s = \left\{ W^s_{\delta_s}(x) : \, x \in \Delta_0 \right\}.$$

And we define the family of unstable leaves Γ^u as the set of all local unstable leaves intersecting $\mathcal{C}^0$ (recall equation (1.6) in Subsection 1.3.2) which u-cross Δ_0. Clearly Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. It is necessary to prove that Γ^u is compact. By the domination property and Ascoli-Arzela Theorem, any limit leaf Δ_∞ of leaves in Γ^u is a u-disk and u-crossing Δ_0, at the same time it is contained in $\mathcal{C}^0$ since $\mathcal{C}^0$ is closed. As the definition of Γ^u, we can see $\Delta_\infty \in \Gamma^u$. So Γ^u is compact.

Relatively, the *s-subsets* are as the following: we define $\mathcal{C}(\omega)$ as the cylinder made by the stable leaves passing through the points in ω, i.e.

$$\mathcal{C}(\omega) = \bigcup_{x \in \omega} W^s_{\delta_s}(x).$$

The pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots$ are the sets $\{\mathcal{C}(\omega) \cap \Gamma^u\}_{\omega \in \mathcal{P}}$.

Then we should check that $f^{R_i}(\Lambda_i)$ is *u-subset*. Given an element $\omega \in \mathcal{P}$, by construction there is some $R(\omega) \in \mathbb{N}$ such that $f^{R(\omega)}(\omega)$ is a center-unstable disk u-crossing $\mathcal{C}^0$. Since each γ^u is a copy of Δ_0 but with a different center, and very important that, $\Gamma^u \cap \mathcal{C}(\omega) \in \cup_{x \in \omega} W^s_{\delta_s}(x)$. Since by construction $f^{R(\omega)}(\omega)$ intersects $W^s_{\delta_s/4}(p)$, then according to the choice of δ_0 and the invariance of the stable foliation, we have that each element of $f^{R(\omega)}(\mathcal{C}(\omega) \cap \Gamma^u)$ must u-cross $\mathcal{C}^0$ and is contained in the $\lambda^{R(\omega)}\delta_s$ height neighborhood of $f^{R(\omega)}(\omega)$. Ignore the difference caused by the angle. We can say it is contained in $\mathcal{C}^0$. So, that is a *u-subset*.

In the sequel, the *product structure* $\Lambda = \Gamma^u \cap \Gamma^s$ will be proven as a *GMY structure*. Observe that the set Λ coincides with the union of the leaves in Γ^u. We can diminish it so that we say $\Lambda \subset K$ as the assertion of Theorem A. Properties $(\mathbf{P_0})$ until $(\mathbf{P_2})$ are satisfied by nature. In the following we prove $(\mathbf{P_3})$. The proof of $(\mathbf{P_4})$ is a repeat of that in [8, Subsection 3.5].

1.3.6 Uniform expansion and bounded distortion

Here we prove property $(\mathbf{P_3})$(a).

Lemma 1.3.16. *There is $C > 0$ such that, given $\omega \in \mathcal{P}$ and $\gamma \in \Gamma^u$, we have for all $1 \le k \le R(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$*

$$\operatorname{dist}_{f^{R(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)-k}(x), f^{R(\omega)-k}(y)) \le C\sigma^{k/2} \operatorname{dist}_{f^{R(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)}(x), f^{R(\omega)}(y)).$$

Proof. Let ω be an element of partition $\mathcal{P}$ constructed in the Subsection 1.3.2. So there are a point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \le n(\omega) \le R(\omega)$. Since we take $\delta_s, \delta_0 < \delta_1/2$, by (1.5), $n(\omega)$ is a $\sqrt{\sigma}$-hyperbolic time for every point in $\mathcal{C}(\omega) \cap \gamma$. Recalling (1.4), we obtain that for all $1 \le k \le n(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$

$$\operatorname{dist}_{f^{n(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)-k}(x), f^{n(\omega)-k}(y)) \le \sigma^{k/2} \operatorname{dist}_{f^{n(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)}(x), f^{n(\omega)}(y)).$$

Considering $R(\omega) - n(\omega) \le N_0$, we take C depending only on N_0 and the derivative of f, then we get the result. □

Property $(\mathbf{P_3})$(b) follows from Proposition 1.2.4 together with Lemma 1.3.16 as in [2, Proposition 2.8]. We state it here for the completeness.

Lemma 1.3.17. *There is $\bar{C} > 0$ such that, for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, we have*

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \le \bar{C} \operatorname{dist}(f^{R_i}(x), f^{R_i}(y))^\zeta.$$

Proof. For $0 \le k < R_i$ and $y \in \gamma^u(x) \in \Gamma^u$, we denote $J_k(y) = \log|\det Df^u(f^k(y))|$ as in the last item of Prop. 1.2.4. Then,

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} = \sum_{k=0}^{R_i-1}(J_k(x) - J_k(y)) \le \sum_{k=0}^{R_i-1} L_1 \operatorname{dist}_D(f^k(x), f^k(y))^\zeta.$$

By Prop. 1.2.4, the sum of $\operatorname{dist}_D(f^k(x), f^k(y))^\zeta$ over $0 \le k \le R_i$ is bounded by

$$\operatorname{dist}_D(f^{R_i}(x), f^{R_i}(y))^\zeta/(1 - \sigma^{\zeta/2}).$$

Take $\bar{C} = L_1(1 - \sigma^{\zeta/2})$, then we have the result. □

1.3.7 Regularity of the foliations

$(\mathbf{P_4})$ has been proved in [8]. This is a standard result for uniformly hyperbolic attractors, and we adapt the classical ideas to our partially hyperbolic setting. $(\mathbf{P_4})$(a) follows from the next result whose proof may be found in [8, Corollary 3.8].

Proposition 1.3.18. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \geq 0$*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n.$$

For $(\mathbf{P_4})$(b), we need some useful notions. We say that $\phi : N \to G$, where N and G are submanifolds of M, is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$, called the *Jacobian* of ϕ, such that

$$\mathrm{Leb}_G(\phi(A)) = \int_A J d\,\mathrm{Leb}_N .$$

Finally, property $(\mathbf{P_4})$(b) follows from the next result whose proof is given in [8, Proposition 3.9].

Proposition 1.3.19. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi\colon \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

We deduce from Proposition 1.3.18 that this infinite product converges uniformly.

1.4 Application

Here we present a open robust class of partially hyperbolic diffeomorphisms (or, more generally, diffeomorphisms with a dominated splitting) whose centre-unstable direction is non-uniformly expanding at Lebesgue almost everywhere in M. The example was introduced in [2, Appendix] as the following: assume $K = M$, through deformation of a uniformly hyperbolic map by isotopy inside some small region, we can prove the new map satisfies the condition (NUE) in the cu-direction. Then we prove $\mathrm{Leb}_D\{\mathcal{E} > n\}$ is exponentially small. The following is a sketch of the main steps.

We consider a linear Anosov diffeomorphism f_0 on the d-dimensional torus $M = T^d$, $d \geq 2$. Thus we have the hyperbolic splitting $TM = E^u \oplus E^s$. Let $V \subset M$ be some small compact domain, such that $f_0|_V$ is injective. Let $\pi : \mathbb{R}^d \to T^d$ be the canonical projection, there exist unit open cubes K^0, K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $f_0(V) \subset \pi(K^1)$. We obtain f in a sufficiently small C^1-neighborhood of f_0, and f satisfies the assumptions of Theorem A. Let f be a diffeomorphism on T^d such that:

(1) f has invariant cone fields C^{cu} and C^s which are with small width $\alpha > 0$ and contain, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism f_0;

(2) f^{cu} is *volume expanding everywhere*: there is $\sigma_1 > 0$ such that $|\det(Df|T_xD^{cu})| > \sigma_1$ for any $x \in M$ and any disk D^{cu} through x tangent to the center-unstable cone field C^{cu};

(3) f is C^1-close to f_0 in the compliment of V, so that f^{cu} is *expanding outside* V: there is $\sigma_2 < 1$ satisfying $\|(Df|T_xD^{cu})^{-1}\| < \sigma_2$ for $x \in M \setminus V$ and any disks D^{cu} tangent to C^{cu};

(4) f^{cu} is *not too contracting* on V: there is small $\delta_0 > 0$ satisfying $\|(Df|T_xD^{cu})^{-1}\| < 1 + \delta_0$ for any $x \in V$ and any disks D^{cu} tangent to C^{cu}.

For example, if $f_1 : T^d \to T^d$ is a diffeomorphism satisfying Item (1), (2), (4) above and coinciding with f_0 outside V, then any f in a C^1 neighborhood of f_1 satisfies all the conditions (1)-(4). The C^1 open classes of transitive non-Anosov diffeomorphisms given in [11, Sec. 6], and also other robust examples from [16], are constructed in this way and they satisfy: both these diffeomorphisms and their inverse satisfy condition (1)-(4) above.

Then we say that any f satisfying (1)-(4) is non-uniformly expanding along cu-direction on a full Lebesgue set of points in M.

Let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of T^d into small subsets such that f is injective on B_j, for $1 \le j \le p+1$. There exist open cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ such that

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1).$$

Let $\mathcal{F}_0^u$ be the unstable foliation of f_0, and $\mathcal{F}_j = f^j(\mathcal{F}_0^u)$ for all $j \ge 0$. By Item (1), each $\mathcal{F}_j$ is a foliation of T^d tangent to the centre-unstable cone field C^{cu}. For any subset E of a leaf of $\mathcal{F}_j$, $j \ge 0$, we denote $\mathrm{Leb}_j(E)$ the Lebesgue measure of E inside the leaf.

Let us fix any small disk D_0 contained in a leaf of $\mathcal{F}_0$. We still have the same arguments as in [2, Lemma A.1]:

Lemma 1.4.1. *[2, Lemma A.1] Let $B_1, \ldots, B_p, B_{p+1} = V$ be an arbitrary partition of M such that f is injective on B_j, for $1 \le j \le p+1$. There exist $\theta > 0$ such that, the orbit of Lebesgue almost every $x \in D_0$ spends a fraction θ of the time in $B_1 \cup \ldots \cup B_p$:*

$$\#\{0 \le j < n : f^j(x) \in B_1 \cup \ldots \cup B_p\} \ge \theta n$$

for every large n.

We conclude that Leb_{D_0}-almost every point $x \in D_0$ spends a positive fraction θ of time outside domain V. Then by Item (3) and (4) above, there exists $c_0 > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df \mid E^{cu}_{f^j(x)})^{-1}\| \le -c_0$$

for Leb_{D_0}-almost every point $x \in D_0$. Since D_0 was an arbitrary disk intersect foliations $\mathcal{F}^s_0$ transversely, and the strong stable foliation is absolutely continuous, we say f is non-uniformly expanding along E^{cu}, at Lebesgue almost everywhere in $M = T^d$.

Moreover, by the proof of Lemma 1.4.1, the induced Lebesgue measure of the set

$$\{x \in D_0 : \|D(f^{cu})^j(x)^{-1}\| > e^{-c_0 j} \text{ for some } j \ge n\}$$

is exponentially small, which means there exists a constant $c > 0$,

$$\mathrm{Leb}_{D_0}\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn}).$$

Chapter 2

Decay of correlations implies GMY structures

Here we focus on non-invertible systems. The material is mainly from [4] and Chapter 1.

2.1 Definitions and main results

We suppose M is a compact Riemannian manifold (possible with boundary) of dimension $d \geq 1$ and Leb a normalized Riemannian volume form on M that we call *Lebesgue measure*. Let $f : M \to M$ be a continuous map which is a C^{1+} local diffeomorphism in M but outside of the zero Lebesgue measure *critical set* $\mathcal{C}$.

2.1.1 Preliminaries

We state the definition of Gibbs-Markov-Young structure, then introduce the definition of expanding measure and decay of correlations with respect to the measure.

Definition 2.1.1. ([4, Def.1.1]) The map f admits a *Gibbs-Markov-Young induced map* if there is a ball $\Delta \subset M$, a countable partition $\mathcal{P}$ (Leb mod 0) of Δ into topological balls U with smooth boundaries, and a recurrence time function $R : \Delta \to \mathbb{N}$ constant on elements of $\mathcal{P}$ satisfying the following properties:

i) Markov: for each $U \in \mathcal{P}$, $R = R(U)$, $f^R : U \to \Delta$ is a C^{1+} diffeomorphism. Thus the induced map $F : \Delta \to \Delta$ given by $F(x) = f^{R(x)}(x)$ is defined almost everywhere and satisfies the classical Markov property.

ii) Uniformly expansivity: there is $0 < \kappa < 1$, s.t. for a.e. $x \in \Delta$, $\|DF(x)^{-1}\| \leq \kappa$. In particular the *separation time* $s(x, y)$ which denotes the maximum integer such that $F^i(x)$ and $F^i(y)$ belong to the same element of the partition $\mathcal{P}$ for all $i \leq s(x, y)$, is defined and finite for almost all $x, y \in \Delta$.

iii) Bounded distortion: there is $C > 0$, s.t. for any pair of points $x, y \in \Delta$ with $1 \leq s(x, y) < \infty$,

$$\left|1 - \frac{\det DF(x)}{\det DF(y)}\right| \leq C\kappa^{-s(F(x),F(y))}.$$

In the sequel we denote Gibbs-Markov-Young by GMY. We call F an induced GMY map. The set Δ on which the above conditions (i)-(iii) hold is called a *GMY structure.*

The *tail set* of the recurrence time function at time n is defined as

$$\mathcal{R}_n = \{x \in \Delta : R(x) > n\}$$

of points whose recurrence time is larger than n.

Definition 2.1.2 (Invariant measure)**.** Let $(M, \mathscr{B})$ be a measurable space and let $f : M \to M$ be a measurable function. A measure μ on $(M, \mathscr{B})$ is said to be invariant under f if, for every measurable set $B \in \mathscr{B}$,

$$\mu(f^{-1}(B)) = \mu(B).$$

Definition 2.1.3 (Expanding measure)**.** An invariant probability measure μ is called *expanding* if all of its Lyapunov exponents are positive, i.e. for μ almost every x and every $v \in T_xM \setminus \{0\}$,

$$\lambda(x, v) := \liminf_{n\to\infty} \frac{1}{n} \log \|Df^{-n}(x)v\|^{-1} > 0. \tag{2.1}$$

Moreover, we say μ is *regularly expanding* if μ is expanding and $\log \|Df^{-1}\| \in L^1(\mu)$.

In [3], Alves, Dias and Luzzatto show that for a large class of maps including multidimensional maps with 'non-degenerate' critical sets (see Definition 2.1.7), a C^{1+} local diffeomorphism f admits a GMY induced map if and only if f admits an ergodic regularly expanding acip measure.

Definition 2.1.4 (Decay of correlations)**.** Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces of real valued measurable functions defined on M. The *correlation* of non-zero functions $\varphi \in \mathcal{B}_1$ and $\psi \in \mathcal{B}_2$ with respect to a measure μ is defined as

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) := \frac{1}{\|\varphi\|_{\mathcal{B}_1}\|\psi\|_{\mathcal{B}_2}}\left|\int \varphi(\psi \circ f^n)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu\right|.$$

For every $\varphi \in \mathcal{B}_1$ and every $\psi \in \mathcal{B}_2$, if

$$\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \to 0, \qquad \text{as } n \to \infty,$$

then we say that we have *decay of correlations* with respect of μ for the observables in $\mathcal{B}_1$ against observables in $\mathcal{B}_2$. Here we take $\mathcal{B}_1 = \mathcal{H}_\alpha$ the space of Hölder continuous functions with Hölder constant $\alpha > 0$, and $\mathcal{B}_2 = L^p$ for $p = 1$ or ∞.

2.1.2 Motivation

As an application of the local strategy of the previous chapter, when we only consider non-invertible maps, the main goal of the second part is to improve the result of Item (2) of Theorem A, B, C in [4], more precisely, we remove the additional assumption in [4, Theorem A, B, C, 3.1, 4.2, Proposition 4.1] that the density of acip is supposed to be bounded from below on the support of acip in the (stretched) exponential case. [4] showed that in some non-invertible systems, the stochastic-like behaviour such as decay of correlations at certain rates (polynomial, sub-exponential, exponential) was sufficient to imply the existence of an induced GMY map with the corresponding properties. At this point the geometry is both necessary and sufficient for the statistical properties of the dynamical systems.

Roughly speaking, starting from rates of mixing, the authors got the estimate of the tail of recurrence times via large deviations in [4]. From large deviation to recurrence times, there was a crucial lemma ([4, Lemma 3.2]) requiring a set A on which the density of acip $d\mu/d\,\mathrm{Leb} > a$ for some $a > 0$. They proved that the tail of expansion times on A decayed at certain rates, then the tail of recurrence times decayed at certain rates; this is the main result in [6, 15]. The difference is: in [6] Alves, Luzzatto and Pinheiro used a local strategy and obtained the polynomial rates, while in [15] Gouëzel used a global strategy and got the (stretched) exponential rates. As shown in [3] there exists a ball $\Delta_0 \subset \mathrm{supp}(\mu)$ centered at a point p whose preimages are dense in the support of μ, such that the density of acip is bounded from below on Δ_0. By the local structure in [6], taking $A = \Delta_0$ in the polynomial case ([4, Theorem A, Item (1)]), they deduced the existence of GMY structure in [4, Theorem 3.1]. Differently, in the (stretched) exponential case ([4, Theorem A, Item (2)]), by the global structure given by [15], they chose $A = \mathrm{supp}(\mu)$ and assumed the density to be bounded from below on A. Then they also got the existence of GMY structure in [4, Theorem 4.2]. That is how the density assumption comes.

By the new improvement in the first chapter, where the local strategy is based on a ball Δ_0 as in [6], we apply $A = \Delta_0$ for Item (2) of [4, Theorem A]. Therefore we get rid of the additional assumption of $d\mu/d\,\mathrm{Leb}$ on $\mathrm{supp}(\mu)$. We consider two cases of both local diffeomorphism and maps with critical sets. The technique used to construct the local structure in Subsection 1.3 for diffeomorphisms with (stretched) exponential tail can easily be adapted to the endomorphism case. The presence of critical sets in the second case is overcomed by the non-degenerate condition (Definition 2.1.7); refer to Theorem 2.3.8. See more precisely in Subsection 2.3.

2.1.3 Local diffeomorphisms

We start by the setting of C^{1+} local diffeomorphism.

The proofs of the following theorems follow from [4, Theorem A & B] with the subset $A = \Delta_0$ in Lemma 2.3.3 which comes from the *local* approach in Subsection 1.3. We state the main proofs in Subsection 2.3 for completeness.

Theorem D. *Let $f : M \to M$ be a C^{1+} local diffeomorphism and $\alpha > 0$, and f admits an ergodic regularly expanding acip μ. If there are constants $c > 0$, $0 < \tau \leq 1$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathcal{O}(e^{-cn^\tau})$ for every $\varphi \in \mathcal{H}_\alpha$ and $\psi \in L^\infty(\mu)$, then there is a GMY induced map such that $\mathrm{Leb}(\mathcal{R}_n) = \mathcal{O}(e^{-dn^{\tau'}})$ for $d > 0$ and $\tau' = \tau/(\tau+2)$.*

Theorem E. *Let $f : M \to M$ be a C^{1+} local diffeomorphism, and f admits an ergodic regularly expanding acip μ. Suppose that there is $\xi(n)$ with $\sum_{n=0}^\infty \xi(n) < \infty$ such that we have $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) \leq \xi(n)$ for all $\varphi \in \mathcal{H}_\alpha$ and $\psi \in L^1(\mu)$. Then there is a GMY induced map with $\mathrm{Leb}(\mathcal{R}_n) = \mathcal{O}(e^{-dn})$ for some $d > 0$.*

Remark 2.1.5. Notice that exponential decay of correlations against L^∞ observables is a particular case which satisfies the above assumptions.

Combining Theorem D with Young [23, Theorem 3], we know the decay of correlations is stretched exponential if and only if there exists a GMY induced map with stretched exponential tail of recurrence times. we know from Young [23, Theorem 3] that exponential decay of the recurrence times implies exponential decay of correlations against L^∞. An 'if and only if' statement in exponential case could be obtained either by relaxing the assumptions on the decay of correlations against L^1 functions in Theorem E, or by showing that the assumption of Theorem E is true whenever its result holds. This is still an open question.

2.1.4 Maps with critical sets

There are many examples which may fail to be local diffeomorphisms due to the presence of critical points and/or singular points, and we denote the union of all these point as the *critical set*; see Definition 2.1.6. Since the partition structure of GMY induced maps allows in some sense to avoid bad regions of the phase space, most results of local diffeomorphisms in last subsection will be applied here successfully, under some additional mild assumptions (non-degenerate) on critical set such that possible accumulation of the images or preimages of the critical set do not further affect the existence of GMY structure and the decay of recurrence times.

Definition 2.1.6. The union of critical/singular points is called the *critical set*, in which we say x is a *critical point* if $Df(x)$ is not invertible (where $\| \det Df \| = 0$); x is a *singular point* if $Df(x)$ does not exist or $\|Df\| = \infty$ (including the case in which f is discontinuous at x).

Definition 2.1.7. The critical set $\mathcal{C}$ is called *non-degeneracy* if there exist constants $B > 1$ and $d > 0$ such that for any $\epsilon > 0$ the following conditions hold:

$$\text{(C0)} \qquad \mathrm{Leb}(\{x : \mathrm{dist}(x, \mathcal{C}) \leq \epsilon\}) \leq B\epsilon^d \text{ (in particular } \mathrm{Leb}(\mathcal{C}) = 0);$$

and there is $\beta > 0$ such that for any $x \in M \setminus \mathcal{C}$ we obtain

$$\text{(C1)} \qquad B^{-1} \,\mathrm{dist}(x, \mathcal{C})^{\beta} \leq \|Df(x)\| \leq B \,\mathrm{dist}(x, \mathcal{C})^{-\beta};$$

Moreover, for all $x, y \in M \setminus \mathcal{C}$:

$$\text{(C2)} \qquad \left|\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\| \,\right| \leq B \left|\log(\mathrm{dist}(x, \mathcal{C})) - \log(\mathrm{dist}(y, \mathcal{C}))\,\right|;$$

$$\text{(C3)} \qquad \left|\log |\det Df(x)| - \log |\det Df(y)|\,\right| \leq B \left|\log(\mathrm{dist}(x, \mathcal{C})) - \log(\mathrm{dist}(y, \mathcal{C}))\,\right|.$$

Remark 2.1.8. The conditions (C2) and (C3) above deduce the corresponding conditions applied in [2, 6, 15].

With the non-degenerate assumption, the results in Subsec. 2.1.3 in the stretched exponential case still hold; see Theorem F. We will give the proof of Theorem F in Subsection 2.3. Notice that we do not have a parallel result of Theorem E for exponential case, we will explain the reason in Subsection 2.3.2.

Theorem F. *Let $f : M \to M$ be a C^{1+} local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. Suppose that f admits an ergodic regularly expanding acip μ with $\frac{d\mu}{d\,\mathrm{Leb}} \in L^p(\mathrm{Leb})$ for some $p > 1$, if there are $c > 0$, $0 < \tau \leq 1$ such that $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathcal{O}(e^{-cn^\tau})$ for every $\varphi \in \mathcal{H}_\alpha$ and $\psi \in L^\infty(\mu)$, then for any $\gamma > 0$ there exists a GMY induced map such that $\mathrm{Leb}(\mathcal{R}_n) = \mathcal{O}(e^{-dn^{\tau'-\gamma}})$ for some $d > 0$ and $\tau' = \tau/(3\tau + 6)$.*

Thus, in the very general setting of maps with critical sets we obtain a converse to Young's results: the rates of decay of correlations is stretched exponential if and only if there exists a GMY induced map with stretched exponential tail of recurrence times.

In the last two sections we prove Theorem D, E and F. The arguments are inspired from [4, Section 2,3,4,5].

2.2 Decay of correlations implies large deviations

A crucial role in our arguments is large deviation. To prove the main theorems, we firstly show that the rate of decay of correlations deduces certain estimates of large deviations. In this subsection, we do not need any Riemannian structure on M.

Definition 2.2.1 (Large deviations)**.** Given a probability measure μ and a small constant $\epsilon > 0$, we define the *large deviation* at time n of the time average of an observable $\varphi : M \to \mathbb{R}$ from the spatial average as

$$\mathrm{LD}_\mu(\varphi, \epsilon, n) := \mu\left(\left\{x \in M : \left|\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) - \int \varphi\, d\mu\right| > \epsilon\right\}\right).$$

For local diffeomorphism, decay of correlations imply large deviations as [4, Theorem D & Theorem E]; see (2.2) in next subsection. The proofs are in [4, Subsection 2.2]. For maps with critical sets, refer to the parallel results in [4, Proposition 4.1]. We do not repeat the complete results in this subsection.

The main tools in the proof are Perron-Frobenius and Koopman operators. We apply Azuma-Hoeffding Inequality[1] (refer to [4, Theorem A.1, Appendix A]) on large deviations for a sequence of martingale difference. It yields the explicit expressions for the constants in Theorem D, E and F.

[1] Azuma-Hoeffding: Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of martingale differences. If there is $a > 0$ such that $\|X_i\|_\infty < a$ for all $1 \leq i \leq n$, then for all $b \in \mathbb{R}$ we have $\mu\left(\sum_{i=1}^n X_i \geq nb\right) \leq e^{-n\frac{b^2}{2a^2}}$.

2.3 Large deviations imply GMY structures

In this section, we prove Theorem D, E and F. Here Lemma 2.3.3 plays an important role to show that when the large deviation for some Hölder continuous function ϕ is (sub-)exponentially small, the tail of expansion $\mathrm{Leb}(\{\mathcal{E} > n\} \cap \Delta_0)$ decays (sub-)exponentially. Then we obtain an induced GMY structure with the tail of recurrence times (sub-)exponentially small by Theorem A in Chapter 1.

2.3.1 For local diffeomorphisms

In this subsection we prove Theorem D and E. We let

$$\phi(x) := \log \|Df(x)^{-1}\|,$$

and then notice that the function ϕ is Hölder continuous when f is a C^{1+} local diffeomorphisms (see Subsection 2.1.3). Recalling the assumptions of Theorems D and E and the results in Subsection 2.2 (which we omitted for briefness): there are $c > 0$, $0 < \tau \leq 1$, such that $\mathrm{Cor}_\mu(\phi, \psi \circ f^n) = \mathcal{O}(e^{-cn^\tau})$ for $\phi \in \mathcal{H}$ and every $\psi \in L^\infty(\mu)$, then there is $c' = c'(c, \phi, \epsilon) > 0$ such that for any $\epsilon > 0$ small enough we have

$$\mathrm{LD}_\mu(\phi, \epsilon, n) = \mathcal{O}(e^{-c' n^{\frac{\tau}{\tau+2}}}). \tag{2.2}$$

The next theorem is similar with [4, Theorem 3.1], now we move away that condition which supposes $d\mu/d\,\mathrm{Leb}$ is uniformly bounded away from zero on its support. The only difference is we apply $A = \Delta_0$ in Lemma 2.3.3. In [4, Lemma 3.2] $A = \mathrm{supp}(\mu)$ because the authors applied the global strategy in [15] by Gouëzel, then they needed the condition $\frac{d\mu}{d\,\mathrm{Leb}}|_{\mathrm{supp}(\mu)} > a$ for some $a > 0$. Now we take Δ_0 instead of $\mathrm{supp}(\mu)$ by the local strategy in Subsection 1.3. We take advantage of the result in [3] (see [3, Subsec. 4.4]) that there exists a local disk $\Delta_0 \subset \mathrm{supp}(\mu)$ centered at a point p whose preimages are dense in the support of μ, such that $\frac{d\mu}{d\,\mathrm{Leb}}|_{\Delta_0} > a$ for some constant $a > 0$.

Theorem D and E then follow directly from:

Theorem 2.3.1. *Let f be a C^{1+} local diffeomorphism with an ergodic regularly expanding acip μ, if there exist $c > 0$, $0 < \tau \leq 1$ such that for small $\epsilon > 0$ we have $\mathrm{LD}_\mu(\phi, \epsilon, n) = \mathcal{O}(e^{-cn^\tau})$, then there exists a GMY induced map with $\mathrm{Leb}(\mathcal{R}_n) = \mathcal{O}(e^{-dn^\tau})$, for some $d > 0$.*

Remark 2.3.2. Notice that $\tau = 1$ is in the exponential case, and the large deviation rates are not necessarily uniform in ϵ.

To prove this, we state a general result which will also be applied in the other case (maps with critical sets); see Subsection 2.3.2. Given an arbitrary $\varphi \in L^1(\mu)$, we define

$$N_\epsilon(x) := \min\left\{N : \left|\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) - \int \varphi d\mu\right| \leq \epsilon, \forall n \geq N\right\}. \tag{2.3}$$

The next lemma is very important when we prove Theorem 2.3.1; see its proof in [4, Lemma 3.2].

Lemma 2.3.3. *Let $A \subset M$ be a subset on which $d\mu/d\,\mathrm{Leb} > a$ for some $a > 0$. Given $\varphi \in L^1(\mu)$ and $\epsilon > 0$ there exists $\xi : \mathbb{N} \to \mathbb{R}^+$ such that $\mathrm{LD}_\mu(\varphi, \epsilon, n) \leq \xi(n)$. Then for every $n \geq 1$ we have*

$$\mathrm{Leb}(\{N_\epsilon > n\} \cap A) \leq \frac{1}{a}\sum_{\ell \geq n}\xi(\ell).$$

Proof of Theorem 2.3.1. Since μ is regularly expanding, for μ-almost everywhere we have

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\phi(f^j(x)) = \int \phi d\mu = \lambda < 0$$

by Birkhoff's Ergodic Theorem.

Then the expansion time

$$\mathcal{E}(x) := \min\left\{N : \frac{1}{n}\sum_{i=0}^{n-1}\phi(f^j(x)) \leq \lambda/2, \ \forall n \geq N\right\}$$

is well defined μ-almost everywhere in M. Recalling (2.3), we take $\varphi = \phi$ and $\epsilon = \lambda/2$ to obtain

$$\{\mathcal{E} > n\} \cap A \subseteq \{N_\epsilon > n\} \cap A.$$

By [22, Lemma 2] and [3, Subsec. 4.4], there is a local unstable disk $\Delta_0 \subset \mathrm{supp}(\mu)$ centered at a point p whose finitely many preimages are dense in $\mathrm{supp}(\mu)$, and $\frac{d\mu}{d\,\mathrm{Leb}}|_{\Delta_0}$ is bounded from below. We take $A = \Delta_0$ in Lemma 2.3.3. By Theorem A in Subsection 1.1.2, there exists a GMY structure, and $\mathrm{Leb}(\mathcal{R}_n)$ decays (sub-)exponentially when $\mathrm{Leb}(\{\mathcal{E} > n\} \cap \Delta_0)$ is (sub-)exponentially small. $\square$

Remark 2.3.4. In Gouëzel's result [15, Theorem 3.1], the induced GMY maps are constructed in a global sense when $\{\mathcal{E} > n\}$ decays (sub-)exponentially fast and the tail of recurrence times has the same rates of decay. [4, Theorem 3.1] thus concluded by a more global assumption : $d\mu/d\,\mathrm{Leb}$ is uniformly bounded from below on $\mathrm{supp}(\mu)$.

For the (sub-)exponential case we take advantage of our result in Chapter 1. When $A = \Delta_0$, we have that $\mathrm{Leb}(\{\mathcal{E} > n\} \cap \Delta_0)$ is (sub-)exponentially small, then there exists a GMY structure with (sub-)exponential tail of recurrence times by Theorem A in Chapter 1.

2.3.2 For maps with critical sets

We consider maps with critical sets $\mathcal{C}$ (see Subsection 2.1.4) and prove Theorem F. The strategy is similar with the one applied in Subsection 2.3.1, we have the construction in parallel with Theorem A in Chapter 1; see Theorem 2.3.8. Now since the function $\log \|Df^{-1}\|$ is not necessarily Hölder continuous, we cannot apply directly the result of Subsection 2.2 which give bounds on the large deviation rates. Moreover, we also need to consider another function $-\log \mathrm{dist}(x, \mathcal{C})$ which is also not Hölder continuous. Let $\mathrm{d}(x, \mathcal{C}) := \mathrm{dist}(x, \mathcal{C})$, and we define

$$\phi_1(x) = \log \|Df^{-1}\| \quad \text{and} \quad \phi_2(x) = \phi_2^{(\delta)}(x) = \begin{cases} -\log \mathrm{d}(x, \mathcal{C}), & \text{if } \mathrm{d}(x, \mathcal{C}) < \delta; \\ \frac{\log \delta}{\delta}(\mathrm{d}(x, \mathcal{C}) - 2\delta), & \text{if } \delta \leq \mathrm{d}(x, \mathcal{C}) < 2\delta; \\ 0, & \text{if } \mathrm{d}(x, \mathcal{C}) \geq 2\delta, \end{cases}$$

The constant $\delta > 0$ is sufficiently small and to be fixed in the proof of [4, Theorem 4.2]. We have $\phi_1, \phi_2 \in L^1(\mu)$, see [4, Lemma 4.3]. We want to get some estimates of large deviations as (2.2); refer to [4, Proposition 4.1].

Since we have the estimates for Hölder continuous functions, we obtain large deviation estimates for functions $\log \|Df^{-1}\|$ and $-\log \mathrm{d}(x, \mathcal{C})$ by an approximation argument, although they are not Hölder continuous. The strategy in [4, Sec. 5] is to approximate ϕ_1 and ϕ_2 by 'truncated' functions which are Hölder continuous. For $c > 0$ and $0 < \tau \leq 1$, when we assume $\mathrm{Cor}_\mu(\varphi, \psi \circ f^n) = \mathcal{O}(e^{-cn^\tau})$ for every $\varphi \in \mathcal{H}$ and $\psi \in L^\infty(\mu)$ as before, there exist $c' > 0$ for any $\gamma > 0$ and $\epsilon > 0$ sufficiently small, we have

$$\mathrm{LD}_\mu(\phi_i, \epsilon, n) = \mathcal{O}(e^{-c' n^{\frac{\tau}{3\tau+6} - \gamma}}) \quad \text{for } i = 1, 2.$$

At this point we lose the exponential estimates such that we are not able to prove a parallel version of Theorem E for maps with critical sets.

The proof of Theorem 2.3.5 is similar with the proof of [4, Theorem 4.2], which needs Lemma 2.3.3 again. We omit the proof. The only difference is that [4, Theorem 4.2] uses Gouëzel's result [15, Theorem 3.1] which gives a global structure while we have a endormorphism version of Theorem A which gives a local structure; see Theorem 2.3.8.

That is why we do not need to assume $d\mu/d\,\mathrm{Leb}$ is bounded from below on $\mathrm{supp}(\mu)$; also refer to Theorem 2.3.1 and Remark 2.3.4.

Theorem 2.3.5. *Let $f : M \to M$ be a C^{1+} local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. Suppose that f admits an ergodic regularly expanding acip μ with $d\mu/d\,\mathrm{Leb} \in L^p(\mathrm{Leb})$ for some $p > 1$. Then $\phi_i \in L^1(\mu)$ for $i = 1, 2$. If there are constants $c > 0$ and $0 < \tau \leq 1$ such that for small $\epsilon > 0$ we have $\mathrm{LD}_\mu(\phi_i, \epsilon, n) = \mathcal{O}(e^{-cn^\tau})$ for $i = 1, 2$, then there exist a GMY induced map with $\mathrm{Leb}(\mathcal{R}_n) = \mathcal{O}(e^{-dn^\tau})$ for some $d > 0$.*

Remark 2.3.6. Notice that $\tau = 1$ is in the exponential case, and the large deviation rates are not necessarily uniform in ϵ.

Now there is only one step which is missing. Recalling the proof of Theorem 2.3.1 and Remark 2.3.4, we apply Theorem A in Chapter 1. Since we do not consider partially hyperbolic attractors admitting critical sets in Theorem A, now we need a endormorphism version of Theorem A which admits critical sets. Let us briefly recall Gouëzel's result in [15].

Theorem 2.3.7 ([15, Theorem 3.1]). *Let $f : M \to M$ be a transitive C^{1+} local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. If there exist $c > 0$ and $0 < \tau \leq 1$ such that*

$$\mathrm{Leb}(\Gamma_n) \leq \mathcal{O}(e^{-cn^\tau}),$$

then M admits a Gibbs-Markov-Young structure; moreover, there is $d > 0$,

$$\mathrm{Leb}\{R > n\} \leq \mathcal{O}(e^{-dn^\tau}).$$

The following theorem is the endormorphism case of Theorem A, we omit the proof since it is a repeat of Subsection 1.3. The existence of critical sets is overcomed by the non-degenerate condition in Definition 2.1.7.

Theorem 2.3.8. *Let $f : M \to M$ be a transitive C^{1+} local diffeomorphism outside a non-degenerate critical set $\mathcal{C}$. If there exist $c > 0$ and $0 < \tau \leq 1$ such that*

$$\mathrm{Leb}(\Gamma_n) \leq \mathcal{O}(e^{-cn^\tau}),$$

then there exists a local unstable disk $\Delta \subset M$ which admits a Gibbs-Markov-Young structure; moreover, there is $d > 0$,

$$\mathrm{Leb}\{R > n\} \leq \mathcal{O}(e^{-dn^\tau}).$$

Remark 2.3.9 (Closing Remark)**.** We finally recall that Chapter 2 is just for non-invertible systems, and there is still an open question: can we get the parallel result in the partially hyperbolic attractors with non-uniformly expanding direction?

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