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P H D   T H E S I S

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A Class of Equivalent Problems
Related to the
Riemann Hypothesis

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To My Parents

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Resumo

Durante um estudo da função zeta de Riemann, analizando alguns gráficos com ela relaçionadis e procurando relações entre a hipótese de Riemann e o tamanho (valor absoluto) da função-zeta, observamos uma relação interessante entre esses valores, nomeadamente, que na faixa $0 < \sigma < 1/2$ com $|t| \geq 6.5$, se tem (onde, como é usual, $s = \sigma + it$):

$$|\zeta(1-s)| \leq |\zeta(s)|$$

Mais tarde, apercebemo-nos que este resultado tinha já sido demonstrado por Dixon-Schoenfeld e Spira na década de 1960. No entanto, a nossa demonstração é diferente e tem a vantagem de envolver, em vez da fórmula assimptótica de Stirling, algumas desigualdades relacionadas com um produto infinito de π e a função Γ de Euler. O resultado principal do primeiro capítulo é, pois, que

$$|\zeta(1-s)| \leq |\zeta(s)|, \qquad \text{for} \quad 0 < \sigma < \frac{1}{2},$$

onde a igualdade ocorre só se $\zeta(s) = 0$.

No segundo capítulo dá-se um refinamento de estimativas de algumas funções relacionadas com a distribuição dos números primos, tais como as funções ψ e ϑ de Chebyshev, usando uma nova região livre de zeros e o cálculo de novos zeros da função zeta, obtidas por Kadiri e Gourdon, respectivamente.

No terceiro capítulo introduzimos e investigamos algumas sequências relacionadas com um teorema de Robin, que afirma que, a hipótese de Riemann é equivalente à desigualdade $\sigma(n) < e^\gamma n \log \log n$ para todos $n > 5040$, onde $\sigma(n)$ é a soma dos divisores de n e γ é a constante de Euler. Com base nesta desigualdade, introduzimos uma sequência de números, que apelidamos de extremamente abundantes, e mostramos que a hipótese de Riemann é verdadeira se e só se existe uma infinidade destes números. Investigamos ainda algumas das suas propriedades e a estrutura dos números extremamente abundantes, assim como algumas propriedades dos os números superabundantes e colossalmente abundantes. Finalmente apresentamos dois outros conjuntos de números, relacionados com os números extremamente abundantes, que mostram o quanto subtil é a hipótese de Riemann.

Abstract

While studying the Riemann zeta-function, observing some graphs related to it and looking for some relation between the Riemann hypothesis and the absolute value of the Riemann zeta function, we noted an interesting relationship between those values, namely, that in the strip $0 < \sigma < 1/2$ with $|t| \geq 6.5$, one has (as usual $s = \sigma + it$):

$$|\zeta(1 - s)| \leq |\zeta(s)|$$

Later we found that this result had been proved by Dixon-Schoenfeld and Spira in 1960's. Nevertheless, our proof was different and has the advantages of involving, instead of Stirling's asymptotic formula, some inequalities related to an infinite product for π and Euler's Γ-function. The main result of the first chapter is, thus, that

$$|\zeta(1 - s)| \leq |\zeta(s)|, \qquad \text{for} \quad 0 < \sigma < \frac{1}{2}, \tag{0.1}$$

where the equality takes place only if $\zeta(s) = 0$.

In the second chapter, we give an improvement for estimates of some functions related to the distribution of primes, such as Chebyshev's ψ and ϑ functions, using some new zero-free region and computations of new zeros of the zeta-function, obtained by Kadiri and Gourdon respectively.

In the third chapter, we introduce and investigate some sequences related to Robin's theorem, which states that, the Riemann hypothesis is equivalent to the inequality $\sigma(n) < e^{\gamma} n \log \log n$ for all $n > 5040$, where γ is Euler's constant. Inspired by this inequality, we introduce a sequence of numbers, that we call extremely abundant, and show that the Riemann hypothesis is true if and only if there are infinitely many of these numbers. Moreover, we investigate some of their properties and structure, as well as some properties of superabundant and colossally abundant numbers. Finally we introduce two other sets of numbers, related to extremely abundant numbers, that show how subtle the Riemann hypothesis is.

Contents

List of Tables

Notations

$f(x) = \Omega_+(g(x))$ $f(x) > Cg(x)$ for a suitable constant $C > 0$ and a sequence
$x = x_n$ s.t. $x_n \to \infty$

$f(x) = \Omega_-(g(x))$ $f(x) < -Cg(x)$ for a suitable constant $C > 0$ and a sequence
$x = x_n$ s.t. $x_n \to \infty$

$f(x) = \Omega_\pm(g(x))$ Both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold

$f(x) = \Omega(g(x))$ $|f(x)| = \Omega_+(g(x))$

$f(x) = O(g(x))$ $|f(x)| \leq A|g(x)|$ for some constant A and all values of $x > x_0$
for some x_0

$f(x) = o(g(x))$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$

$f \sim g$ $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

$\pi(x)$ the number of prime numbers $p \leq x$

$\Pi(x)$ Riemann prime counting function; i.e.
$\sum_{p^m \leq x} \frac{1}{m} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$

$\mathrm{li}(x)$ $\lim_{\varepsilon \to 0} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right)$

$\mathrm{Li}(x)$ $\int_2^x \frac{dt}{\log t}$

$\lfloor x \rfloor$ the integral part of x

$\sigma(n)$ sum of divisors of a positive integer n

$d(n)$ number of divisors of a positive integer n

$\phi(n)$ Euler's totient function: the number of positive integers
not exceeding n which are relatively prime to n

$\vartheta(x)$ Chebyshev's first function: $\sum_{p \leq x} \log p$

$\psi(x)$ Chebyshev's second function: $\sum_{p^m \leq x} \log p$

$\Psi(n)$ Dedekind's arithmetical function: $n \prod_{p|n}(1 + 1/p)$

SA Set of superabundant numbers, also abbreviation for superabundant

CA Set of colossally abundant numbers, also abbreviation for colossally abundant

XA Set of extremely abundant numbers, also abbreviation for extremely abundant

Chapter 0

Introduction

Contents

0.1   A Note About the Riemann Hypothesis

If I were to awaken after having slept for a thousand years, my first question would be: has the Riemann hypothesis been proven?

Attributed to David Hilbert ([6, p. 5])

As with any big problem, trying to solve such problems, even if one could not solve them, they shed light on the other parts of the life. The main motivation of this thesis, is to study a class of problems equivalent to the Riemann hypothesis (RH).

Among mathematicians many of them believe that the RH and its generalization is probably the most important problem in mathematics. This problem was the 8-th problem of Hilbert's famous list of 23 problems for the century in 1900 Paris Conference of the International Congress of Mathematicians.

The RH is connected to many other branches of mathematics, for instance it has a close relation to the prime numbers - building blocks of integers - and their distribution. Riemann(1826-1866) in his 8-page paper - the only paper he wrote in number theory - formulated the problem by saying:

... One finds in fact about this many real roots (of zeta function) within these bounds and it is very likely that all of the roots are real. One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation(B. Riemann: [23], p. 301).

In other words, Riemann conjectured that the real part of all non-trivial zeros of the zeta function is 1/2. The RH has been verified numerically up to 10^{13} zeros with no exception. The table below demonstrates the history of numerical verification of the zeros:

Year	Number of zeros	Computed by
1859 (approx)	1 (or 3)	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1041	E. C. Titchmarsh
1953	1104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,500,000	J. B. Rosser, et al.
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, et al.
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, et al.
2001	10,000,000,000	J. van de Lune(unpublished)
2004	900,000,000,000	S. Wedeniwski
2004	10,000,000,000,000	X. Gourdon

Table 1: Numerical verification of zeros of the zeta function ([6], p. 39)

Let $s = \sigma + it$ ($\sigma,\ t \in \mathbb{R}$) be a complex number. Let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \tag{0.1}$$

2

If $s = 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

is the harmonic series, which is divergent and so is when $\Re(s) \leq 1$. It converges whenever $\Re(s) > 1$, since

$$\left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq 1 + \int_1^{\infty} \frac{dt}{t^\sigma} = \frac{\sigma}{\sigma - 1}. \tag{0.2}$$

One of the most important relations which makes a connection between the prime numbers and the zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1}, \qquad (\sigma > 1). \tag{0.3}$$

This is called Euler's identity (or Euler product formula), and it had been used by Euler with positive *integer* values of s [27], and by Chebyshev with s as a *real* variable. Riemann then introduced the idea of treating s as a *complex* variable and studying the series on the left side by the methods of the theory of analytic functions (cf. [23], p. 7).

In 1859, Riemann proved that $\zeta(s)$ can be continued analytically to an analytic function over the whole complex plane, with the exception of $s = 1$.

One of the important relations in the theory of the zeta function $\zeta(s)$ is the functional equation which states some symmetry of this function. From Euler's product formula, $\zeta(s)$ has no zeros for $\Re(s) > 1$. From the functional equation $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$, and the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (critical strip) and are symmetric about the vertical line $\Re(s) = \frac{1}{2}$ (critical line) and the real axis $\Im(s) = 0$. Hardy in [34] proved that there are infinitely many zeros of the zeta function on the critical line. More details about the Riemann zeta function is discussed in the first chapter. We finish this section by a quote from E. Bombieri:

> *Even a single exception to Riemann's conjecture would have enormously strange consequences for the distribution of prime numbers.... If the RH turns out to be false, there will be huge oscillations in the distribution of primes. In an orchestra, that would be like one loud instrument that drowns out the others an aesthetically distasteful situation [5].*

0.2   Equivalent Statements to RH

As we indicated before, RH has a variety of equivalent statements in mathematics. Here we list some of number-theoretic ones ([6, Ch. 5], see also [16]).

1. Let $\pi(x)$ be the number of primes less than or equal to a real number x, and

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}.$$

   The assertion that

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\log x)$$

   is equivalent to the RH.

2. The second Chebyshev function $\psi(x)$ is the logarithm of the least common multiple of the integers from 1 to x.

   L. Schoenfeld [70] showed that if RH is true then

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2 x, \qquad (x > 73.2).$$

   Von Koch [77] proved that the RH is equivalent to the "best possible" bound for the error of the prime number theorem. Therefore that the RH is equivalent to [16]

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2 x, \qquad (x > 73.2).$$

   The following equivalence is due to Robin [64], which is the basis for the chapter 3 of the thesis:

3. The RH is equivalent to inequality

$$\sigma(n) < e^\gamma n \log\log n, \qquad (n > 5040),$$

   where γ is Euler's constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) \approx 0.577\,215\,664 \tag{0.4}$$

   and

$$\sigma(n) = \sum_{d|n} d \tag{0.5}$$

   is the sum of divisors of n.

0.3 Brief Description of the Chapters

- The first chapter consists of two sections. In the first section, we give the definition of the Riemann zeta function and list some of its known behaviors such as its extension, functional equation, symmetric location of zeros, etc. In the second section, over a curious studying of the Riemann zeta function and looking for some problem related to the RH, we found an interesting problem about the size of it. It is stated as a main theorem of this section. The corresponding theorem below has been proved independently by Spira ([74]) and Dixon and Schoenfeld [19] in 1960's. Our proof has an advantage, because it involves elementary calculus, new elementary inequalities and known formulas for some functions and constants. Moreover, we present a relation between the size of the Riemann zeta function and the RH. More precisely,

  Theorem. *Let $s = \sigma + it$, where $|t| \geq 12$. Then*

  $$|\zeta(1-s)| \leq |\zeta(s)|, \qquad for \quad 0 < \sigma < \frac{1}{2}, \tag{0.6}$$

  where the equality takes place only if $\zeta(s) = 0$.

  As a corollary, one can prove that the strict inequality in (0.6) is equivalent to the RH. Besides, we prove the following result related to the partial derivative of $|\zeta(s)|^2$ with respect to the real part of s

  Proposition. *If*

  $$\frac{\partial}{\partial \sigma}|\zeta(s)|^2 < 0, \qquad for \quad (0 < \sigma < \frac{1}{2}, \ |t| > 6.5), \tag{A}$$

  then the RH is true.

  We conclude the first chapter by the converse of the previous proposition as

  Conjecture. *The condition (A) is also necessary for the validity of the RH.*

- The second chapter is devoted to an improvement of some results due to Rosser and Schoenfeld (R-S) ([68], [70]) related to the distribution of prime numbers in the sense of Chebyshev's ψ function. To do this we use new zero free region [48] and recently calculated zeros of $\zeta(s)$ function [31]. For instance, we prove the following results.

(i) Let
$$\varepsilon_0(x) = \sqrt{8/\pi}X^{1/2}e^{-X}.$$

Then
$$|\psi(x) - x| < x\varepsilon_0(x), \qquad (x \geq 3)$$

and
$$|\vartheta(x) - x| < x\varepsilon_0(x), \qquad (x \geq 3).$$

(ii) Let T_0 be defined as in (2.23) and satisfy $T_0 \geq D$, where $2 \leq D \leq A$ and A is defined in 2.13. Let m be a positive integer and let $\delta > 0$. Then
$$|\psi(x) - x| < \varepsilon_0^* x, \qquad (x \geq e^b),$$

where
$$\varepsilon_0^* = \Omega_1^* e^{-b/2} + \Omega_3^* + \frac{m}{2}\delta + e^{-b}\log 2\pi,$$

where Ω_1^* and Ω_3^* are defined in 2.24 and 2.25.

For more details we refer to Theorem 2.20. We note that some of the results were mentioned by Dusart [21, p. 5]. However, our computed values in the tables at the end of the thesis are different from Dusart's. our method is similar to Rosser and Schoenfeld (see details in [68], [70]). Then we infer some estimates for certain functions of distribution of primes. Also we establish a sufficient condition for the RH. We conclude this chapter by giving the estimates for the certain product over primes which is like a dual for Mertens' third theorem.

- The third chapter is based on Robin's inequality (3.2) and his equivalence to the RH. Investigating the sum of divisors function and Robin's inequality, we noticed that the first integer n which violates this inequality, if exists, should have the property that $\sigma(n)/n > \sigma(m)/m$ for $m < n$. These numbers are called *superabundant numbers*. Akbary and Friggstad [2] proved that this is indeed the case. However, continuing to investigate Robin's inequality and modifying the Robin's inequality, we extracted and introduced a new subsequence of positive integers, which possibly can give a progress to the truth of the RH. Namely, in some way, it is a translation of Robin's criterion to a different aspect using the Gronwall theorem. But in our opinion these numbers have their own interest. We call this sequence "*extremely abundant numbers*" (in short XA numbers) and present some of their properties. For instance, we prove

(i) If there is any counterexample to Robin's inequality (3.2), then the least one is an XA number.

(ii) The RH is true if and only if $\#XA = \infty$.

(iii) If $n \in XA$, then $p(n) < \log n$.

Besides, we present a list of properties for the well known sequences of superabundant and colossally abundant and extremely abundant numbers. Finally, we state certain numerical results about superabundant and extremely abundant numbers, worth to be mentioned as well. Finally we demonstrate the delicacy of RH by defining a subset of superabundant which is is defined in a particular way (that is also a superset of extremely abundant number) and giving the proof for infinitude of the cardinality of this subset.

0.4 Own papers

– Nazardonyavi, Sadegh; Yakubovich, Semyon, *Another proof of Spira's inequality and its application to the Riemann hypothesis*, Journal of Mathematical Inequalities (7) No. 2 (2013), 167-174.

– Nazardonyavi, Sadegh; Yakubovich, Semyon, *Extremely abundant numbers and the Riemann hypothesis*, (submitted)

Chapter 1

Riemann Zeta-Function, Its size in Critical Strip and the RH

Contents

1.1 Preliminaries About the Riemann Zeta Function

1.1.1 Complex-Valued Functions

An *entire* function is a function which is analytic in the whole plane. For example polynomials, e^z, $\sin z$, and $\cos z$ are entire. If z_0 is an isolated singular point of f, then the Laurent series representation for $f(z)$ in a punctured disk $0 < |z - z_0| < R_2$ is:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

If in the principal part of f at z_0 (i.e., the second series above) $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$ $(m \geq 1)$, then the isolated singular point z_0 is called a pole of order m. If $m = 1$ then z_0 is a simple pole and b_1 as the residue of f at z_0. A meromorphic function is a function which is analytic except for poles(cf. [8, pp. 73, 231, 241, 291]). For example Riemann ζ function is a meromorphic in the hole plane except at $s = 1$ with residue 1.

1.1.2 Representations of Entire Functions

Assume that $f(z)$ is an entire function and has m zeros at the origin, and $a_1, a_2, \ldots, a_N$ are non-zero zeros of f. We can write (see [1])

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right).$$

If there are infinitely many zeros, then

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

This representation is valid if the infinite product converges uniformly on every compact set. We formulate here the famous Weierstrass theorem ([1, p. 194]).

Theorem 1.1 ([1], p. 195). *There exists an entire function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}},$$

where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and $g(z)$ is an entire function.

The product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{a_n}\right)^h}$$

converges and represent an entire function provided that the series $\sum 1/|a_n|^{h+1}$ converges. Assume that h is the smallest integer for which this series converges (see [1, p. 196]).

1.1.3 Gamma Function

The zeros of $\sin \pi z$ are the integers $z = \pm n$. Since $\sum 1/n$ diverges and $\sum 1/n^2$ converges, we must take $h = 1$. Then

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{1.1}$$

The function $\Gamma(z)$ is called Euler's gamma function ([1, p. 199]). It has the representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

or equivalently

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1 + 1/n)^z}{1 + z/n}. \tag{1.2}$$

It satisfies the following equations

$$\Gamma(z + 1) = z\Gamma(z),$$

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \tag{1.3}$$

and

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2}), \qquad \text{(Legendre's duplication formula)}. \tag{1.4}$$

The function $\Gamma(z)$ is a meromorphic function with poles at $z = 0, -1, -2, \ldots$ and has no zeros (cf. [1, pp. 197-200]).

1.1.4 The Riemann Zeta Function

The Riemann zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad (\Re(s) > 1).$$

Riemann introduced the notation $s = \sigma + it$ in his paper to denote a complex number.

The following theorem is a useful tool in number theory to compute series and it is called Abel's identity.

Theorem 1.2 ([44], p. 18). *Let $\lambda_1, \lambda_2, \ldots$ be a real sequence which increases (in the wide sense) and has the limit infinity, and let*

$$C(x) = \sum_{\lambda_n \leq x} c_n,$$

where the c_n may be real or complex, and the notation indicates a summation over the (finite) set of positive integers n for which $\lambda_n \leq x$. Then, if $X \geq \lambda_1$ and $\phi(x)$ has a continuous derivative, we have

$$\sum_{\lambda_n \leq X} c_n \phi(\lambda_n) = C(X)\phi(X) - \int_{\lambda_1}^{X} C(x)\phi'(x)dx. \qquad (1.5)$$

If, further, $C(X)\phi(X) \to 0$ as $X \to \infty$, then

$$\sum_{1}^{\infty} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} C(x)\phi'(x)dx,$$

provided that either side is convergent.

Using this theorem, we can extend the domain of definition of the ζ function to the left side of $\sigma = 1$. By Theorem 1.2, with $\lambda_n = n$, $c_n = 1$, $\phi(x) = x^{-s}$,

$$\sum_{n \leq X} \frac{1}{n^s} = s\int_1^X \frac{\lfloor x \rfloor}{x^{s+1}}dx + \frac{\lfloor X \rfloor}{X^s}, \qquad (X \geq 1).$$

where $\lfloor x \rfloor$ is the largest integer not greater than x. Writing $\lfloor x \rfloor = x - \{x\}$, so that $0 \leq \{x\} < 1$, we obtain

$$\sum_{n \leq X} \frac{1}{n^s} = \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int_1^X \frac{\{x\}}{x^{s+1}}dx + \frac{1}{X^{s-1}} - \frac{\{X\}}{X^s}.$$

Since$|1/X^{s-1}| = 1/X^{\sigma-1}$ and $|\{X\}/X^s| < 1/X^\sigma$, we deduce, making $X \to \infty$

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx, \qquad (\sigma > 1).$$

The integral in the right-hand side of the latter equation is convergent for $\sigma > 0$. So that this equation gives an analytic continuation of $\zeta(s)$ over the half-plane $\sigma > 0$ (cf. [44, p. 26]).

1.1.5 The Functional Equation

Riemann in his 1859 paper established the functional equation and used it to construct the analytic continuation of $\zeta(s)$ beyond the region $\sigma > 1$.

Theorem 1.3 ([76], Th. 2.1). *The function $\zeta(s)$ is regular (i.e., analytic) for all values of s except $s = 1$, where there is a simple pole with residue 1. It satisfies the functional equation*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s)\zeta(1-s). \tag{1.6}$$

Functional equation (1.6) may be written in the form

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.7}$$

with
$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s).$$

By (1.3) and (1.4) we get

$$
\begin{aligned}
\chi(s) &= 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s) \\
&= 2^s \pi^{s-1} \frac{\pi}{\Gamma(s/2)\Gamma(1-s/2)} \Gamma(1-s) \\
&= 2^s \pi^{s-1} \frac{\pi}{\Gamma(s/2)\Gamma(1-s/2)} \pi^{-1/2} 2^{-s} \Gamma\left(\frac{1}{2} - \frac{s}{2}\right) \Gamma\left(1 - \frac{s}{2}\right) \\
&= \pi^{s-1/2} \frac{\Gamma(1/2 - s/2)}{\Gamma(s/2)},
\end{aligned}
\tag{1.8}
$$

and by substitution
$$\chi(s)\chi(1-s) = 1.$$

1.1.6 Values of $\zeta(s)$ in the Integers

The Bernoulli numbers B_n can be defined by the generating function (cf. [28, p. 41])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

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From this one gets

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_{2n+1} = 0, \quad (n \in \mathbb{Z}^+),$$

and in general are given by the double sum (see [40], [30], [78])

$$B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n, \qquad (n = 0, 1, 2, \ldots).$$

It is known that

$$\zeta(-n) = \frac{(-1)^n}{n+1} B_{n+1}, \qquad n = 0, 1, 2, \ldots.$$

Since $B_{2n+1} = 0$ for $n \in \mathbb{Z}^+$, then $\zeta(-2n) = 0$ for $n \in \mathbb{Z}^+$.

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta(-2n) = 0, \qquad \zeta(1-2n) = -\frac{1}{2n} B_{2n}, \qquad n = 1, 2, \ldots.$$

Using functional equation (1.6) for zeta and the values of Gamma function in integers, one deduces the formula

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}, \qquad n \in \mathbb{Z}^+.$$

1.1.7 Zeros of the Riemann Zeta Function

From Euler's identity (0.3) we deduce that $\zeta(s)$ has no zeros for $\sigma > 1$. From the functional equation (1.6) we observe that $\zeta(s)$ has no zeros for $\sigma < 0$ except trivial zeros at $s = -2n$, ($n \in \mathbb{Z}^+$). Zeros, which lie inside the region $0 \leq \Re(s) \leq 1$ are called *non-trivial*. By the functional equation and the relation $\zeta(\bar{s}) = \overline{\zeta(s)}$ (reflection principle) one sees that non-trivial zeros are symmetric with respect to the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$. Hence, if ρ is a zero of ζ, then $\bar{\rho}$, $1-\rho$ and $1-\bar{\rho}$ are. Also we mentioned in the Introduction, Riemann made a conjecture that $\Re(\rho) = 1/2$ for all non-trivial zeros ρ. In 1896 Hadamard and de la Vallée Poussin proved independently that $\zeta(s) \neq 0$ on the line $\sigma = 1$ ([6, p. 16]).

Number of Zeros of $\zeta(s)$: Riemann-von Mangoldt Formula

Let $N(T)$, where $T > 0$, denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, for which $0 < \beta < 1$ and $0 < \gamma \leq T$. Let

$$\boxed{F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}} \tag{1.9}$$

and

$$R(T) = 0.137 \log T + 0.443 \log \log T + 1.588. \qquad (1.10)$$

Then the following theorem holds:

Theorem 1.4 ([66], Th. 19)**.** *For $T \geq 2$,*

$$|N(T) - F(T)| < R(T).$$

Zeros on the Critical Line $\sigma = 1/2$

Here we present a chronological review of some significant work which have been done about the zeros on the critical line.

- 1914: Hardy proved that there are infinitely many roots of $\zeta(s) = 0$ on the line $\Re(s) = 1/2$ ([34]).

- 1921: Hardy and Littlewood proved that the number of roots on the line segment from $1/2$ to $1/2 + iT$ is at least KT for some positive constant K and all sufficiently large T ([35]).

- 1942: Selberg proved that the number of such roots is at least $KT \log T$ for some positive constant K and all sufficiently large T ([71], [23, p. 226]).

- 1974: N. Levinson proved

$$N_0(T + U) - N_0(T) > C\{N(T + U) - N(T)\},$$

where $N_0(T)$ is the number of zeros of $\zeta(s)$ of the form $s = \frac{1}{2} + it$, $0 < t \leq T$ and $U = T(\log(T/2\pi))^{-10}$ and $C = 1/3$ ([53], [45]).

- 1979: D.R. Heath-Brown proved that the number of simple zeros is at least $KT \log T$ for some positive constant K and all sufficiently large T ([38]).

- 1989: Conrey proved that at least 40% of the zeros of the Riemann zeta function are on the critical line ([14], [15]).

- 2011: Bui et al. proved that at least 41.05% of the zeros of the Riemann zeta function are on the critical line ([9]).

1.1.8 Bounds of $\zeta(s)$

Bounds of $\zeta(\sigma + it)$ for $\sigma \geq 1$.

In the Introduction (see formula 0.2) it is shown that

$$|\zeta(s)| \leq \frac{\sigma}{\sigma - 1}.$$

However, this bound is much greater than $|\zeta(s)|$ when σ is close to 1. There are better upper bounds as the following theorems state. For instance,

Theorem 1.5 ([24], p. 46). *There exist absolute constants c_1, c_2, c_3, c_4, A and B such that for all $\sigma \geq 1$ and all $|t| \geq 8$, we have*

$$|\zeta(\sigma + it)| \leq c_1 \log |t|,$$

$$|\zeta'(\sigma + it)| \leq c_2 \log^2 |t|,$$

$$|\zeta(\sigma + it)|^{-1} \leq c_3 \log^A |t|,$$

$$|\zeta'/\zeta(\sigma + it)| \leq c_4 \log^B |t|.$$

One can choose in particular $c_1 = 4e$, $c_2 = 6e$, $c_3 = 16(6e)^7$, $c_4 = 16(6e)^8$, $A = 7$ and $B = 9$.

Size of $\zeta(\sigma + it)$ near to σ for $\sigma < 1$

For this case the following theorem is known.

Theorem 1.6 ([24], p. 44). *Let θ be a real number such that $0 < \theta < 1$. For all $\sigma \geq \theta$ and all $|t| \geq 1$*

$$|\zeta(\sigma + it)| \leq \frac{7}{4} \frac{|t|^{1-\theta}}{\theta(1-\theta)},$$

$$|\zeta'(\sigma + it)| \leq \frac{|t|^{1-\theta}}{\theta(1-\theta)} \left(\log |t| + \frac{1}{\theta} + \frac{5}{4} \right).$$

In the next section we will study the size (absolute value) of zeta function at points that are symmetric about the critical line and lie inside the critical strip.

1.2 On an Inequality for the Riemann Zeta Function in the Critical Strip

During the study of the Riemann zeta-function, observing its graphs and looking for some relation between the RH and the size of the Riemann zeta function, we

encountered an interesting problem for the estimation of its size in the critical strip, which is as follows:

$$|\zeta(1-s)| \leq |\zeta(s)|, \qquad (0 < \sigma < 1/2, \ |t| \geq 6.5).$$

Later we found that the same problem has been stated and proved independently by Spira [74] and Dixon-Schoenfeld [19] in 1960's. However, our method in the proof has some advantages, involving power inequalities related to an infinite product for π and formula (1.2) for Euler's gamma-function instead of the use of Stirling's asymptotic formula ([17], [50, p. 530]). Therefore, we state the main result of this section by the following

Theorem 1.7. *Let* $s = \sigma + it$, *and* $|t| \geq 12$. *Then*

$$|\zeta(1-s)| \leq |\zeta(s)|, \qquad for \quad 0 < \sigma < \frac{1}{2}, \tag{1.11}$$

where the equality takes place only if $\zeta(s) = 0$.

1.2.1 Auxiliary Lemmas

In order to prove Theorem 1.7, we will need some auxiliary elementary inequalities involving rational and logarithmic functions. Precisely, we have (see [54], §2)

$$\frac{1}{x+1} < \log\left(1 + \frac{1}{x}\right) < \frac{1}{x}, \qquad (x < -1, \ or \ x > 0), \tag{1.12}$$

$$\frac{1}{x+\frac{1}{2}} < \log\left(1 + \frac{1}{x}\right), \qquad x > 0, \tag{1.13}$$

$$\frac{x(2+x)}{2(1+x)} < \log(1+x) < \frac{2x}{2+x}, \qquad (-1 < x < 0). \tag{1.14}$$

$$\frac{2x}{2+x} < \log(1+x) < \frac{x(2+x)}{2(1+x)}, \qquad (x > 0). \tag{1.15}$$

Next we give some inequalities whose proofs are based on elementary calculus.

Lemma 1.8. *For any* $t \geq 1$

$$\left(1 + \frac{1}{tx+t-1}\right)^t \leq 1 + \frac{1}{x}, \qquad (x \leq -1, \ x > 0), \tag{1.16}$$

$$\left(1 + \frac{x}{t}\right)^t \leq 1 + \frac{2tx}{(1-t)x+2t}, \qquad (0 \leq x \leq 2). \tag{1.17}$$

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Finally, for $0 \leq a \leq 1$

$$\left(1 + \frac{1}{x}\right)^a \geq 1 + \frac{a}{x+1-a}, \qquad (x \leq -1, \; x > 0), \qquad (1.18)$$

where the equality holds only if $a = 0$, 1 or $x = -1$, and

$$\left(1 + \frac{1}{x}\right)^a \geq 1 + \frac{a}{x + \frac{1-a}{2}}, \qquad (x > 0), \qquad (1.19)$$

$$\left(1 + \frac{1}{x}\right)^a \leq 1 + \frac{a}{x + \frac{1-a}{2}}, \qquad (x \leq -1), \qquad (1.20)$$

where it becomes equality only if $a = 0$, 1.

Proof. In order to prove (1.16), let

$$f(t) = \left(1 + \frac{1}{tx + t - 1}\right)^t - \left(1 + \frac{1}{x}\right), \qquad (x \leq -1, \; x > 0).$$

Then its derivative is

$$f'(t) = \left(1 + \frac{1}{tx + t - 1}\right)^t \left(\log\left(1 + \frac{1}{tx + t - 1}\right) - \frac{1}{tx + t - 1}\right).$$

Calling inequality (1.12), it is easily seen that $f'(t) < 0$. Therefore $f(t)$ is a decreasing function and $f(1) = 0$. Hence $f(t) < 0$ for $t > 1$. To verify (1.17), observe that conditions $t \geq 1$ and $0 \leq x \leq 2$ imply the positiveness of both sides of the inequality, which is equivalent to

$$\left(1 + \frac{x}{t}\right)^t \left(1 + \frac{2tx}{(1-t)x + 2t}\right)^{-1} \leq 1, \qquad (0 \leq x \leq 2, \; t \geq 1).$$

Hence, denoting the left-hand side of the latter inequality by $g(x)$, one obtains

$$g'(x) = \frac{\left(1 + \frac{x}{t}\right)^t}{((1+t)x + 2t)^2}(1 - t^2)x^2 \leq 0, \qquad t \geq 1.$$

Since $g'(x) \leq 0$, then $g(x) \leq g(0) = 1$ for $x \geq 0$. The equality in (1.17) holds for $x = 0$ or $t = 1$. To prove (1.18), one replace $t = 1/a$ in (1.16). The proof of (1.19) and (1.20) is straightforward and similar, invoking with inequalities (1.15) and (1.14). $\qquad \square$

Lemma 1.9. *Let $0 < \sigma < 1/2$, $t \in \mathbb{R}$ and $x \geq (1 + \sqrt{3})/4$. Then*

$$\frac{(2x + 1 - \sigma)^2 + t^2}{(2x + \sigma)^2 + t^2} < \left\{\left(\frac{2x+1}{2x}\right)^2\right.$$

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$$\times \left(1 - \frac{(1+4x)((-1+\sigma)\sigma + t^2)}{(1+2x)^2((-1+\sigma)\sigma + t^2 + 4x^2)}\right)\Bigg\}^{1-2\sigma}. \quad (1.21)$$

If $t \geq 1/2$, one has

$$\frac{(1-\sigma)^2 + t^2}{\sigma^2 + t^2} < \left(1 + \frac{1}{(-1+\sigma)\sigma + t^2}\right)^{1-2\sigma}. \quad (1.22)$$

Finally, for $t \geq 12$, the following inequality holds

$$\left(\frac{(1-\sigma)^2 + t^2}{\sigma^2 + t^2}\right) \prod_{n=1}^{3} \frac{(2n+1-\sigma)^2 + t^2}{(2n+\sigma)^2 + t^2} < \left(\frac{1}{4} \prod_{n=1}^{3} \left(\frac{2n+1}{2n}\right)^2\right)^{1-2\sigma}. \quad (1.23)$$

Proof. Let $1 - 2\sigma = 1/y$. Then (1.21) is equivalent to

$$\left(1 + \frac{4(1+4x)}{y((-1/y+1+4x)^2 + 4t^2)}\right)^y < 1 + \frac{4(1+4x)y^2}{1 + (-1+4t^2+16x^2)y^2}. \quad (1.24)$$

It is not difficult to verify that

$$0 < \frac{4(1+4x)}{(-1/y+1+4x)^2 + 4t^2} \leq 2, \qquad \left(x \geq \frac{1+\sqrt{3}}{4}, \, t \in \mathbb{R}\right). \quad (1.25)$$

But relation (1.24) is just inequality (1.17) where

$$x := \frac{4(1+4x)}{(-1/y+1+4x)^2 + 4t^2}, \qquad t := y.$$

So (1.21) is proved. In the same manner one can establish (1.22). To prove (1.23) it is enough to verify the following inequality

$$\left(1 + \frac{1}{(-1+\sigma)\sigma + t^2}\right) \prod_{n=1}^{3} \left(1 - \frac{(1+4n)((-1+\sigma)\sigma + t^2)}{(1+2n)^2((-1+\sigma)\sigma + t^2 + 4n^2)}\right) < \frac{1}{4}.$$

Its left-hand side is increasing in σ and decreasing in t in the strip $]0, 1/2[\times]1/2, \infty[$. Therefore, one may put $\sigma = 1/2$ and $t = 12$ and see by straightforward computations that it is less than $1/4$. $\qquad \square$

1.2.2 Proof of the Main Result

Proof of Theorem 1.7. As we saw in §1.1.5, the functional equation (1.6) for the Riemann zeta-function can be written in the form (1.7).

Denoting $g(s) = \chi(s)^{-1}$, we have

$$g(s) = \pi^{\frac{1}{2}-s} \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2} - \frac{1}{2}s)},$$

and we will show that for $0 < \sigma < \frac{1}{2}$ and $t \geq 12$, $|g(\sigma + it)| < 1$.

In fact taking the infinite product (1.1) for the sine function and letting $z = \frac{1}{2}$, one arrives at the known Wallis' formula

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Moreover, the Gauss infinite product formula for the gamma function (1.2) yields

$$\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2} - \frac{1}{2}s)} = \frac{1-s}{s} \prod_{n=1}^{\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^{\frac{1}{2}-s} \left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}} \right).$$

Hence

$$
\begin{aligned}
g(s) &= \left(\frac{1-s}{s} \right) 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n)^2}{(2n-1)(2n+1)} \right)^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{1}{1+\frac{1}{n}} \right)^{\frac{1}{2}-s} \left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}} \right) \\
&= \left(\frac{1-s}{s} \right) 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n)^2 n}{(2n-1)(2n+1)(n+1)} \right)^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}} \\
&= \left(\frac{1-s}{s} \right) 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n)n}{(2n-1)(n+1)} \right)^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{\frac{1}{2}-s} \left(\frac{1+\frac{1-s}{2n}}{1+\frac{s}{2n}} \right) \\
&= \left(\frac{1-s}{s} \right) 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n)n}{(2n-1)(n+1)} \right)^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{\frac{1}{2}-s} \left(\frac{2n+1-s}{2n+s} \right) \\
&= \left(\frac{1-s}{s} \right) 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n+1)n}{(2n-1)(n+1)} \right)^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{1-2s} \left(\frac{2n+1-s}{2n+s} \right).
\end{aligned}
$$

Put

$$f(s) = 2^{\frac{1}{2}-s} \prod_{n=1}^{\infty} \left(\frac{(2n+1)n}{(2n-1)(n+1)} \right)^{\frac{1}{2}-s},$$

and

$$h(s) = h_1(s) h_2(s),$$

where

$$h_1(s) = \frac{1-s}{s}, \qquad h_2(s) = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{1-2s} \frac{2n+1-s}{2n+s}.$$

Plainly, for any N one has

$$\prod_{n=1}^{N} \frac{(2n+1)n}{(2n-1)(n+1)} = \frac{2N+1}{N+1},$$

and so

$$\prod_{n=1}^{\infty} \frac{(2n+1)n}{(2n-1)(n+1)} = 2.$$

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Hence
$$|f(s)| = 2^{1-2\sigma}.$$

Therefore, it is sufficient to show that for $0 < \sigma < \frac{1}{2}$ and $t \geq 12$
$$|h(s)| < 2^{2\sigma-1}. \tag{1.26}$$

Indeed, $|h_1(s)|$ is a decreasing function with respect to σ and t for $0 < \sigma < 1/2$ and $t > 0$. Meanwhile
$$|h_2(s)| = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{1-2\sigma} \left|\frac{2n+1-s}{2n+s}\right|, \tag{1.27}$$

is increasing with respect to σ in the strip $(\sigma, t) \in]0, 1/2[\times[1/2, \infty[$, and decreasing with respect to t in the strip $(\sigma, t) \in]0, 1/2[\times\mathbb{R}^+$.

Denoting by
$$h_{2,n}(\sigma, t) = \left(\frac{2n}{2n+1}\right)^{1-2\sigma} \left|\frac{2n+1-(\sigma+it)}{2n+(\sigma+it)}\right|,$$

the general term of the product and assuming for now
$$h_{2,n}(\sigma, t) < 1, \qquad (0 < \sigma < \frac{1}{2},\ t \geq 0), \tag{1.28}$$

we easily come out with the inequality
$$\prod_{n=1}^{N+1} h_{2,n}(\sigma, t) < \prod_{n=1}^{N} h_{2,n}(\sigma, t), \qquad (0 < \sigma < \frac{1}{2},\ t \geq 0).$$

To verify (1.28), we need to show that
$$\left(1 + \frac{1}{2n}\right)^{1-2\sigma} > \sqrt{\frac{(2n+1-\sigma)^2 + t^2}{(2n+\sigma)^2 + t^2}}, \qquad t \geq 0. \tag{1.29}$$

In fact,
$$\frac{(2n+1-\sigma)^2 + t^2}{(2n+\sigma)^2 + t^2} = 1 + \frac{(1-2\sigma)(4n+1)}{(2n+\sigma)^2 + t^2}. \tag{1.30}$$

Hence inequality (1.29) yields
$$\left(1 + \frac{1}{2n}\right)^{1-2\sigma} > \frac{2n+1-\sigma}{2n+\sigma} \geq \sqrt{\frac{(2n+1-\sigma)^2 + t^2}{(2n+\sigma)^2 + t^2}}. \tag{1.31}$$

However,
$$\frac{2n+1-\sigma}{2n+\sigma} = 1 + \frac{1-2\sigma}{2n+\sigma}.$$

So the first inequality in (1.31) follows immediately from (1.19), letting $x = 2n$ and $a = 1 - 2\sigma$. Thus we get inequality (1.28).

Further, we show that $\{h_{2,n}(\sigma,t)\}_{n=1}^{\infty}$ is an increasing sequence for any $(\sigma,t) \in$ $]0,1/2[\times\mathbb{R}$. To do this, we consider the function $H_2(y) = h_{2,y}(\sigma,t)$ and differentiate it with respect to y. Hence by straightforward calculations one derives

$$H_2'(y) = \frac{\dfrac{1-2\sigma}{y(2y+1)}\left(\dfrac{2y}{2y+1}\right)^{1-2\sigma}}{((2y+\sigma)^2+t^2)^2 \sqrt{\dfrac{(2y+1-\sigma)^2+t^2}{(2y+\sigma)^2+t^2}}}$$

$$\times\left\{(2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma)\right.$$

$$\left.+(1+6y(1+2y)-2(1-\sigma)\sigma)t^2+t^4\right\}.$$

Since

$$(2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma)+(1+6y(1+2y)-2(1-\sigma)\sigma)t^2+t^4$$

$$\geq (2y+1-\sigma)(1-\sigma)\sigma(2y+\sigma) > 0,$$

we find that the derivative is positive, and therefore $H_2(y)$ is increasing for $y > 0$. Now fixing $t \geq 1/2$ one justifies that $h_{2,n}(\sigma,t)$ is increasing by σ. Precisely,

$$\frac{\partial}{\partial\sigma}h_{2,n}(\sigma,t) = \left(\frac{2n}{2n+1}\right)^{1-2\sigma} \Big/ \left|\frac{2n+1-(\sigma+it)}{2n+(\sigma+it)}\right|$$

$$\times\left\{-(1+4n)(4n^2+2n+\sigma-\sigma^2+t^2)\right.$$

$$\left.+2((2n+1-\sigma)^2+t^2)((2n+\sigma)^2+t^2)\log(1+\frac{1}{2n})\right\}$$

and we achieve the goal, showing that the latter multiplier is positive. But this is true due to inequality (1.13), because it is greater than

$$\frac{-(1-2\sigma)^2(2n+1-\sigma)(2n+\sigma)+(8n(1+2n)+3-8(1-\sigma)\sigma)t^2+4t^4}{1+4n}$$

$$\geq \frac{1+(1-\sigma)\sigma(8n(1+2n)-3+4(1-\sigma)\sigma)}{1+4n} > 0, \qquad (0 < \sigma < 1/2,\ t \geq 1/2).$$

Returning to (1.27) we conclude that $|h_2(\sigma,t)|$ is increasing with respect to σ for $0 < \sigma < \frac{1}{2}$ and $t \geq 1/2$, and by (1.30) it is decreasing with respect to t for $0 < \sigma < \frac{1}{2}$ and $t > 0$.

Since

$$|h_N(s)| = |\frac{1-s}{s}|\prod_{n=1}^{N}\left(\frac{2n}{2n+1}\right)^{1-2\sigma}\left|\frac{2n+1-s}{2n+s}\right| \qquad (1.32)$$

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is decreasing by N, one has

$$|h(s)| \leq |h_N(s)|.$$

As $|h_N(s)|$ is decreasing by t, it is enough to show that

$$|h_N(s)| < 2^{2\sigma-1} \qquad \text{for} \qquad (t = 12, \text{and } N = 3)$$

and this has been established in (1.23). Moreover, since $\zeta(s)$ is reflexive with respect to the real axis, i.e., $\zeta(\bar{s}) = \overline{\zeta(s)}$, inequality (1.11) holds also for $t \leq -12$. Therefore, Theorem 1.7 is proved. $\qquad\square$

Remark 1.10. A computer simulation suggests that the main result is still valid for $t \in]6.5, 12[$ (See Figure 1.1). However, a direct proof by this approach is more complicated, because to achieve the goal we should increase the number N of terms in the product (1.32).
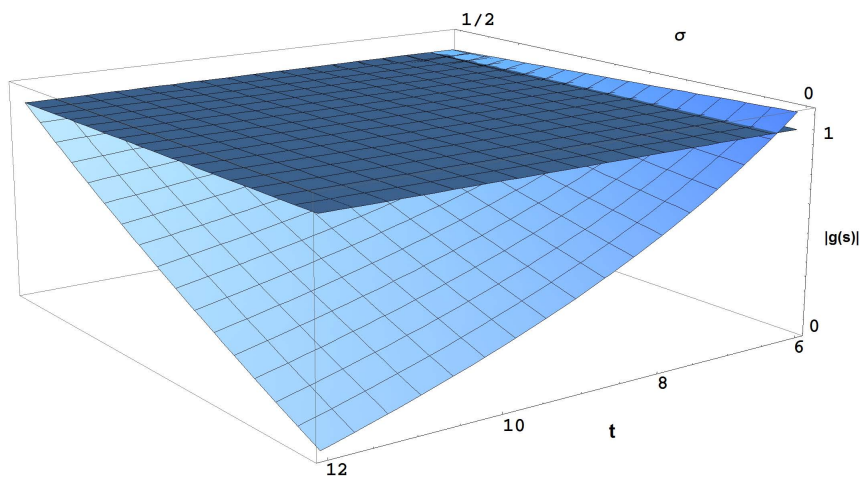
Figure 1.1: The graph of $|g(s)|$ for $6 < t < 12$.

1.2.3 An Application to the RH

Motivating our study of the size of $\zeta(s)$ and similar to [74], we apply the results of this chapter to the Riemann hypothesis. We have

Proposition 1.11. *The RH is true if and only if*

$$|\zeta(1-s)| < |\zeta(s)|, \qquad \text{for} \quad (0 < \sigma < \frac{1}{2}, \ |t| > 6.5).$$

As it is known [20], zeros of the derivative $\zeta'(s)$ of Riemann's zeta-function are connected with the behavior of zeros of $\zeta(s)$ itself. Indeed, Speiser's theorem [73] states that the RH is equivalent to $\zeta'(s)$ having no zeros on the left side of the critical line. Thus, one can get further tools to study RH, employing these properties.

We will formulate a sufficient condition for the RH to be true.

Proposition 1.12. *If*

$$\frac{\partial}{\partial\sigma}|\zeta(s)|^2 < 0, \qquad for \quad (0 < \sigma < \frac{1}{2}, \ |t| > 6.5), \qquad (A)$$

then the RH is true.

Proof. In fact, if the RH were not true, then by Speiser's theorem [73], there exists a number $s \in]0, 1/2[\times \mathbb{R}$, such that $\zeta'(s) = 0$. Hence $\frac{\partial}{\partial\sigma}|\zeta(s)|^2 = 0$.

$\square$

Finally in this chapter we conjecture the necessity of (A).

Conjecture. *The condition (A) is also necessary for the validity of the RH.*

Chapter 2

Chebyshev's Functions, Improved Bounds and the RH

Contents

2.1 Introduction and Preliminary Results

An integer greater than 1 is called prime if it is not a multiple of any smaller integers greater than 1. These numbers are important since they are the building blocks for integers. The fundamental theorem of arithmetic states that every integer greater than 1 is a product of prime numbers and this factorization is unique up to rearrangement. There are certain questions arising in the studying of prime numbers. For example: How many primes are there? How many primes are there less than a given number x? What is the distribution of prime numbers?

x	Count of primes $< x$	$\int \frac{dn}{\log n}$	Difference
500 000	41 556	41 606.4	50.4
1 000 000	78 501	78 627.5	126.5
1 500 000	114 112	114 263.1	151.1
2 000 000	148 883	149 054.8	171.8
2 500 000	183 016	183 245.0	229.0
3 000 000	216 745	216 970.6	225.6

Table 2.1: Prime counting function and logarithmic integral[23, p. 2]

How many primes are there which the difference is 2 (twin prime conjecture)? Every even integer greater than 2 can be expressed as the sum of two primes (Goldbach's conjecture)? Are there infinitely many Mersenne primes (primes of the form $2^n - 1$)? etc (see [62, Introduction]).

About 300 BC it was proved in Euclid's Elements that there are infinitely many prime numbers. Euler was the first one who discovered fundamental formula (0.3) as an analytic version of the fundamental theorem of arithmetic, and as a corollary it gives

$$\sum_p \frac{1}{p} = \infty.$$

Let $\pi(x)$ denote as usual the number of primes not exceeding x. In 1808 Legendre conjectured that

$$\pi(x) \sim \frac{x}{\log x - A(x)},$$

where $\lim_{x \to \infty} A(x) \approx 1.08366$.

It was conjectured by Gauss that $\pi(x)$ is asymptotically $\mathrm{Li}(x)$ (see [29]). Chebyshev ([12], [75]) proved the asymptotic estimate

$$(A_0 + o(1)) \frac{x}{\log x} \leq \pi(x) \leq \left(\frac{6}{5} A_0 + o(1) \right) \frac{x}{\log x} \qquad \text{as } x \to \infty,$$

with

$$A_0 = \log(2^{1/2} 3^{1/3} 5^{1/5} 30^{1/30}) \approx 0.92129. \qquad (2.1)$$

He also pointed out that

$$\liminf_{x \to \infty} \frac{\pi(x)}{x / \log x} \leq 1 \leq \limsup_{x \to \infty} \frac{\pi(x)}{x / \log x}.$$

Moreover, he proved the following result in a beautiful way

Theorem 2.1 ([12], p. 379). *For all $x > 1$*

$$\vartheta(x) < \frac{6}{5}A_0 x - A_0 x^{\frac{1}{2}} + \frac{5}{4\log 6}\log^2 x + \frac{5}{2}\log x + 2$$

$$\vartheta(x) > A_0 x - \frac{12}{5}A_0 x^{\frac{1}{2}} - \frac{5}{8\log 6}\log^2 x - \frac{15}{4}\log x - 3, \qquad (2.2)$$

where A_0 is defined in (2.1) and $\vartheta(x)$ is the first Chebyshev's function and it is defined by

$$\vartheta(x) = \sum_{p \leq x} \log p. \qquad (2.3)$$

As we mentioned in Chapter 1, in 1859 Riemann [63] started his paper with the fundamental formula of Euler (0.3). He defined the zeta function for complex numbers with real part σ greater than 1 and using analytic continuation to the whole complex plane except $s = 1$. In his paper he also gives an explicit formula which we will talk about later. Recall from §1.1.7 that J. Hadamard [33] and C. J. de la Vallée Poussin [18], independently and using methods of complex analysis, proved that there is no zeros on the line $\sigma = 1$. This fact implies the prime number theorem (PNT), i.e.,

$$\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1.$$

Finally in 1949, an elementary proof (without using complex analysis) of PNT was given by Selberg ([72]) and Erdős ([25]). The PNT can be expressed in different ways. Namely

Theorem 2.2 ([4], p. 79). *The following relations are logically equivalent:*

$$\lim_{x \to \infty} \frac{\pi(x)\log x}{x} = 1,$$

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1,$$

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1,$$

where

$$\psi(x) = \sum_{p^m \leq x} \log p,$$

The Riemann zeta function has a significant influence on the law of distribution of primes. Riemann introduced a tool which does this task and it is called explicit formula. This explicit formula gives a link between non-trivial zeros of the Riemann zeta function and Chebyshev's ψ function (see [45], [29]). Namely

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - \frac{1}{x^2}), \quad (x > 1, \ x \neq p^m), \qquad (2.4)$$

where

$$\sum_{\rho} \frac{x^{\rho}}{\rho} = \lim_{T \to \infty} \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho}, \qquad (\zeta(\rho) = 0,\ 0 < \Re\rho < 1).$$

and when $x = p^m$, then in the left-hand side of (2.4) put $\psi(x) - \frac{1}{2}\Lambda(x)$, where for any integer $n \geq 1$

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

The explicit expression (2.4) was proved by H. von Mangoldt in 1895.

As we see in the explicit formula (2.4), the size of the error term in PNT has a link to

$$\Theta = \sup\{\Re\rho : \ \zeta(\rho) = 0\}. \tag{2.5}$$

By the functional equation for zeta function we know that the non-trivial zeros are in the critical strip, symmetric about the line $\sigma = 1/2$ and therefore $1/2 \leq \Theta \leq 1$. Also $\Theta = 1/2$ if and only if RH is true ([44, p. 82]). Until now no upper bound $\Theta \leq 1 - \delta$ with $\delta > 0$ is known. As a first application of explicit formula (2.4), one has

Theorem 2.3 (cf. [44], Th. 30).

$$\psi(x) = x + O(x^{\Theta} \log^2 x),$$

$$\pi(x) = \mathrm{li}(x) + O(x^{\Theta} \log x).$$

2.1.1 Explicit Bounds for Distribution of Primes

Also, mathematicians have worked on the numerical verification of the RH and finding better zero-free region for the Riemann zeta function. The proof of Hadamard and de la Vallée Poussin (see [75], [6]) gives that all non-trivial zeros of zeta function lie in the region

$$\sigma \leq 1 - \frac{c}{\log^9(3 + |t|)}$$

for some $c > 0$. Later de la Vallée Poussin improved this result to

$$\sigma \leq 1 - \frac{c}{\log(3 + |t|)}.$$

Vinogradov and Korobov extended this zero free region and showed that

$$\sigma \leq 1 - \frac{c_0}{(\log(3 + |t|))^{2/3}(\log\log(3 + |t|))^{1/3}}.$$

Rosser and Schoenfeld are among the mathematicians who have made much efforts on determining zeros and an explicit zero free region of the Riemann zeta function. They give a explicit error term in prime number theorem employing the computation of the zeros on critical line and zero free region in 1975 and 1976 (see [68], [70]). More precisely, they determined that the first $3\,502\,500$ zeros lie on the critical line and proved

Theorem 2.4. *There is no zeros on the region*

$$\sigma \geq 1 - \frac{1}{R \log |t/17|}, \qquad R = 9.645\,908\,801. \tag{2.6}$$

Then they employed some estimates and deduced the explicit error term in the prime PNT given by

Theorem 2.5 ([68], Th. 2). *If* $\log x \geq 105$, *then*

$$|\psi(x) - x| < x\varepsilon(x),$$

where one may take either

$$\varepsilon(x) = 0.257634 \left\{ 1 + \frac{0.96642}{X} \right\} X^{3/4} e^{-X}, \qquad X = \sqrt{\log x/R},$$

where R is defined in (2.6), or simply by replacing $\sqrt{\log x/R}$

$$\varepsilon(x) = 0.110123 \left\{ 1 + \frac{3.0015}{\sqrt{\log x}} \right\} (\log x)^{3/8} e^{-\sqrt{(\log x)/R}}.$$

The above bounds were improved by Schoenfeld. Precisely

Theorem 2.6 (cf. [70], Th. 11). *Let $R = 9.645\,908\,801$. Then*

$$|\psi(x) - x| < x\varepsilon_0(x), \qquad (x \geq 17),$$

$$|\vartheta(x) - x| < x\varepsilon_0(x), \qquad (x \geq 101),$$

where

$$\varepsilon_0(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} e^{-X}, \qquad X = \sqrt{\log x/R}.$$

and R is defined in (2.6).

In 2010 Dusart [21] proved the explicit estimates for the functions of distribution of primes. Some of them will be employed in the sequel. Namely it has

Proposition 2.7 (cf. [21], Prop. 3.1, 3.2)**.** *The following estimates hold*

$$\psi(x) - \vartheta(x) < 1.00007\sqrt{x} + 1.78\sqrt[3]{x}, \qquad (x > 0),$$

$$\psi(x) - \vartheta(x) > 0.9999\sqrt{x}, \qquad\qquad (x \geq 121).$$

Theorem 2.8 ([21])**.** *One has*

$$|\vartheta(x) - x| < \eta_k \frac{x}{\log^k x}, \qquad (x \geq x_k),$$

with

k	0	1	1	2	2	2	2
η_k	1	1.2323	0.001	3.965	0.2	0.05	0.01
x_k	1	2	908994923	2	3594641	122568683	7713133853

and

k	3	3	3	3	4
η_k	20.83	10	1	0.78	1300
x_k	2	32321	89967803	158822621	2

In 1930 Hoheisel [41] proved that there is a number θ with $1 - 1/33000 = \theta_0 < \theta < 1$ such that for any $\varepsilon > 0$ the interval $(x, x + x^{\theta+\varepsilon})$ contains a prime for $x > x_0(\varepsilon)$. Heilbronn [39] showed that one may take $\theta_0 = 1 - 1/250$. Ingham [43] obtained that $\theta_0 = 5/8$. Montgomery [55] gets $\theta_0 = 3/5$. Later Huxley [42] gave $\theta_0 = 7/12$. Iwaniec and Jutila [46] gives $\theta = 13/23$. Later Iwaniec and Pintz [47] found that $\theta_0 = 23/42$. As we see all results above are implicit and they hold for $x \geq x(\theta)$. As we stated before, we concerned in this chapter with explicit bounds. Dusart in [22] and [21] gives the following estimates for the intervals that have at least one prime:

Lemma 2.9 ([22])**.** *For $k \geq 463$,*

$$p_{k+1} \leq p_k \left(1 + \frac{1}{2\log^2 p_k}\right).$$

where p_k is the k-th prime number.

Proposition ([21])**.** *For all $x \geq 396738$, there exists a prime p such that*

$$x < p \leq x \left(1 + \frac{1}{25\log^2 x}\right).$$

Theorem 2.10 ([21]).

$$\pi(x) \geq \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right), \qquad (x \geq 599),$$

$$\pi(x) \leq \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \qquad (x > 1).$$

Mertens' second theorem states that the asymptotic form of the harmonic series for the sum of reciprocal primes is given by (see [75, §8])

$$\lim_{x \to \infty}\left(\sum_{p \leq x} \frac{1}{p} - \log\log x - B\right) = 0.$$

where B is Meissel-Mertens constant and it is defined by (see [44, p. 23])

$$B = \gamma + \sum_p \left\{\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right\} \approx 0.261497212847643. \qquad (2.7)$$

One can derive the following formula for sum of reciprocal over primes by (see [67, p. 74])

$$\sum_{p \leq x} \frac{1}{p} - \log\log x - B = \frac{\vartheta(x) - x}{x \log x} - \int_x^\infty \{\vartheta(t) - t\}\left(\frac{1}{t^2 \log^2 t} + \frac{1}{t^2 \log t}\right) dt, \quad (2.8)$$

Therefore

$$\left|\sum_{p \leq x} \frac{1}{p} - \log\log x - B\right| \leq \frac{|\vartheta(x) - x|}{x \log x} + \int_x^\infty |\vartheta(t) - t|\left(\frac{1}{t^2 \log^2 t} + \frac{1}{t^2 \log t}\right) dt$$

$$< \frac{\eta_k}{\log^{k+1} x} + \int_x^\infty \frac{\eta_k}{\log^k t}\left(\frac{1}{t \log^2 t} + \frac{1}{t \log t}\right) dt$$

$$= \frac{\eta_k}{k \log^k x} + \left(1 + \frac{1}{k+1}\right)\frac{\eta_k}{\log^{k+1} x}. \qquad (2.9)$$

Taking $k = 2$ and $\eta = 0.2$ for $x \geq 359464$ and checking by computer for smaller values:

Theorem 2.11 ([21]). *We have*

$$\sum_{p \leq x} \frac{1}{p} - \log\log x - B \geq -\left(\frac{1}{10 \log^2 x} + \frac{4}{15 \log^3 x}\right), \qquad (x > 1),$$

$$\sum_{p \leq x} \frac{1}{p} - \log\log x - B \leq \frac{1}{10 \log^2 x} + \frac{4}{15 \log^3 x}, \qquad (x \geq 10372),$$

where B is Meissel-Mertens constant.

Taking the finite product in Euler product formula up to x

$$\lim_{x\to\infty}\left\{\log x\prod_{p\le x}\left(1-\frac{1}{p}\right)\right\}=e^{-\gamma}.$$

This result is known as Mertens' third theorem is (see [75, §8]; [37, p. 173]). Using the estimates in the previous theorem and taking exponential of both sides of the inequalities for $x\ge 3\,594\,641$ and checking by computer for smaller values:

Theorem 2.12 (cf. [21]). *We have*

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)<\frac{e^{-\gamma}}{\log x}\left(1+\frac{0.2}{\log^2 x}\right),\qquad (x>1),$$

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)>\frac{e^{-\gamma}}{\log x}\left(1-\frac{0.2}{\log^2 x}\right),\qquad (x\ge 2973),$$

$$\prod_{p\le x}\frac{p}{p-1}>e^{\gamma}\log x\left(1-\frac{0.2}{\log^2 x}\right),\qquad (x>1),$$

$$\prod_{p\le x}\frac{p}{p-1}<e^{\gamma}\log x\left(1+\frac{0.2}{\log^2 x}\right),\qquad (x\ge 2973).$$

2.1.2 Irregularities of the Distribution of Primes

As we observed in the previous section, there is a certain regularity in the distribution of primes. However, with these results one cannot guarantee too much. For instance, is it true that $\pi(x)<\mathrm{li}(x)$ for all x? It was Littlewood who proved the following interesting theorem which exhibit the irregularity in the distribution of prime numbers, from which one can deduce that (using Theorem 2.13 and relations (2.10) and (2.11) or Corollary 2.14 and formula (2.12)) $\pi(x)-\mathrm{li}(x)$ changes sign for infinitely many x.

Theorem 2.13 ([44], p. 100). *We have*

$$\psi(x)-x=\Omega_{\pm}(x^{1/2}\log\log\log x),\qquad as\ x\to\infty.$$

Let

$$\Pi(x)=\pi(x)+\sum_{m=2}^{\infty}\frac{1}{m}\pi(x^{\frac{1}{m}}). \qquad (2.10)$$

The following formula gives the relation between $\Pi(x)$ and $\psi(x)$

$$\Pi(x)=\sum_{2\le n\le x}\frac{\Lambda(n)}{\log n}=\frac{\psi(x)}{\log x}+\int_{2}^{x}\frac{\psi(t)}{t\log^2 t}dt. \qquad (2.11)$$

Combining Theorem 2.13 and Proposition 2.7 this irregularity is transfered to $\vartheta(x)$; that is

Corollary 2.14. *We have*

$$\vartheta(x) - x = \Omega_{\pm}(x^{1/2} \log \log \log x), \qquad \text{as} \quad x \to \infty.$$

The following formula makes a link between $pi(x)$ and $\vartheta(x)$

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} dt. \qquad (2.12)$$

2.2   A Sufficient Condition for the RH

Let

$$g(x) = e^{\gamma} \log \vartheta(x) \prod_{p \leq x} \left(1 - \frac{1}{p}\right).$$

Nicolas [56] proved that

(i) under RH, $g(x) < 1$ for $x \geq 2$;

(ii) if RH is false, then there exists a sequence of values x tends to $+\infty$ for which $g(x) < 1$, and there exists a sequence of values x tends to $+\infty$ for which $g(x) > 1$. If we denote by Θ (see 2.5) the upper bound of the real parts of the zeros of the Riemann zeta function, then, for any b, such that $1 - \Theta < b < 1/2$, we have

$$\log g(x) = \Omega_{\pm}(x^{-b}),$$

that is to say,

$$\limsup \ x^b \log g(x) > 0 \qquad \text{and} \qquad \liminf \ x^b \log g(x) < 0.$$

In the following proposition we prove a result similar to (ii) above for ψ function. Precisely

Proposition 2.15. *Let*

$$h(x) = e^{\gamma} \log \psi(x) \frac{\prod_{p \leq x}(1 - 1/p)}{\prod_{\frac{1}{2}\sqrt{x} < p \leq x}(1 - 1/p^2)}$$

If the RH is not true, then $h(x) < 1$ for infinitely many x and $h(x) > 1$ for infinitely many x.

Proof. Using Abel's identity (Theorem 1.2) and Theorem 2.8 we get

$$\log \prod_{p>x} \left(1 - \frac{1}{p^2}\right) < -\frac{1}{x \log x} + \frac{4}{x \log^2 x},$$

$$\log \prod_{p>x} \left(1 - \frac{1}{p^2}\right) > -\frac{1}{x \log x} - \frac{4}{x \log^2 x}.$$

Hence

$$\log \prod_{\frac{1}{2}\sqrt{x}<p\leq x} \left(1 - \frac{1}{p^2}\right) < -\frac{4}{\sqrt{x} \log x} + \frac{32}{\sqrt{x} \log^2 x},$$

$$\log \prod_{\frac{1}{2}\sqrt{x}<p\leq x} \left(1 - \frac{1}{p^2}\right) > -\frac{4}{\sqrt{x} \log x} - \frac{32}{\sqrt{x} \log^2 x}.$$

Note that

$$\psi(x) = \vartheta(x) + R(x), \qquad \text{where} \quad R(x) = \sum_{k=2}^{\infty} \vartheta(x^{1/k}).$$

Then applying Taylor's formula to $\log \log t$ one has

$$\log \log \psi(x) \leq \log \log \vartheta(x) + \frac{R(x)}{\vartheta(x) \log \vartheta(x)},$$

$$\log \log \psi(x) \geq \log \log \vartheta(x) + \frac{R(x)}{\vartheta(x) \log \vartheta(x)} - \frac{R(x)^2}{\vartheta(x)^2 \log \vartheta(x)},$$

where $R(x) \sim \sqrt{x}$ according to Theorem 2.2. Therefore

$$\log h(x) = \log g(x) + \left\{ \log \log \psi(x) - \log \log \vartheta(x) - \log \prod_{\sqrt{x}<p\leq x} \left(1 - \frac{1}{p^2}\right) \right\}$$

$$= \Omega_{\pm}(x^{-b}) + O\left(\frac{1}{\sqrt{x} \log x}\right) = \Omega_{\pm}(x^{-b}), \qquad (1 - \Theta < b < \frac{1}{2}).$$

$\square$

2.3 Improved Explicit Bounds for Chebyshev's Functions

2.3.1 Improved Explicit Bounds for Large Values of x

In this section we shall use the recent information about the number of zeros of the Riemann zeta function that are all on the critical line up to some height and the new zero free region. The RH has been verified until the 10^{13}-th zero by Gourdon

[31] (October 12th 2004). In 2005, Kadiri [48] gave an explicit zero free region for zeta function. Indeed, she proved the following lemma which is used in the proofs of the improved bounds in Theorems 2.19 and 2.20.

Lemma 2.16 ([48])**.** *The Riemann zeta-function $\zeta(s)$ with $s = \sigma + it$ does not vanish in the region*

$$\sigma \geq 1 - \frac{1}{R_0 \log |t|}, \qquad (|t| \geq 2, \ R_0 = 5.69693).$$

In other words, if $\rho = \beta + i\gamma$ is a zero of Riemann zeta function, then

$$\beta < 1 - \frac{1}{R_0 \log |\gamma|}, \qquad (|\gamma| \geq 2, \ R_0 = 5.69693).$$

Using these two recent statements we give a better explicit bound for the Chebyshev's functions. The methods of proofs of theorems in this section are essentially similar to those of Schoenfeld [70], therefore we consent to give just the sketch of the proofs.

Recall that $N(T)$, $F(T)$ and $R(T)$ are defined as in §1.1.7. Choose A such that $F(A) = 10^{13}$. Then

$$A = 2\,445\,999\,556\,030.342\,362\,641, \tag{2.13}$$
$$\log A = 28.525\,474\,972. \tag{2.14}$$

We also applied the following lemmas in the proofs of theorems for the order of $\psi(x) - x$ (i.e., Theorems 2.19 and 2.20). In the next lemma we use the same method as in [65], letting $r = 29$ instead of $r = 8$ and get

Lemma 2.17. *Let $\rho = \beta + i\gamma$ denote the non-trivial zero of the Riemann zeta function. Then*

$$\sum_{\rho} \frac{1}{|\gamma^3|} < 0.00146435, \qquad \sum_{\rho} \frac{1}{\gamma^4} < 7.43617 \cdot 10^{-5}, \qquad \sum_{\rho} \frac{1}{|\gamma^5|} < 4.46243 \cdot 10^{-6},$$

$$\sum_{\rho} \frac{1}{\gamma^6} < 2.88348 \cdot 10^{-7}, \qquad \sum_{\rho} \frac{1}{|\gamma^7|} < 1.93507 \cdot 10^{-8}.$$

Define for $x \geq 1$

$$X = \sqrt{\frac{\log x}{R_0}}, \qquad R_0 = 5.69693. \tag{2.15}$$

Also for positive ν, positive integer m, and non-negative reals T_1 and T_2, define

$$R_m(\nu) = \{(1+\nu)^{m+1} + 1\}^m, \tag{2.16}$$

$$S_1(m,\nu) = 2 \sum_{\substack{\beta \leq 1/2 \\ 0 < \gamma \leq T_1}} \frac{2 + m\nu}{2|\rho|}, \tag{2.17}$$

$$S_2(m,\nu) = 2 \sum_{\substack{\beta \leq 1/2 \\ \gamma > T_1}} \frac{R_m(\nu)}{\nu^m |\rho(\rho+1)\cdots(\rho+m)|}, \tag{2.18}$$

$$S_3(m,\nu) = 2 \sum_{\substack{\beta > 1/2 \\ 0 < \gamma \leq T_2}} \frac{(2+m\nu)\exp(-X^2/\log\gamma)}{2|\rho|}, \tag{2.19}$$

$$S_4(m,\nu) = 2 \sum_{\substack{\beta > 1/2 \\ \gamma > T_2}} \frac{R_m(\nu)\exp(-X^2/\log\gamma)}{\nu^m |\rho(\rho+1)\cdots(\rho+m)|}, \tag{2.20}$$

and

$$\phi_m(y) = \frac{e^{-X^2/\log y}}{y^{m+1}},$$

$$q(y) = \frac{0.137\log y + 0.443}{y \log y \log(y/2\pi)}. \tag{2.21}$$

The following lemma is the basis for finding estimates for Chebyshev's function in the proof of Rosser and Scoenfeld [68], [70].

Lemma 2.18 ([68], Lemma 8). *Let T_1 and T_2 be non-negative real numbers. Let m be a positive integer. Let $x > 1$ and $0 < \delta < (x-1)/(xm)$. Then*

$$\frac{1}{x}\left|\psi(x) - \left\{x - \log 2\pi - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right)\right\}\right| \tag{2.22}$$

$$\leq \frac{1}{\sqrt{x}}\{S_1(m,\delta) + S_2(m,\delta)\} + S_3(m,\delta) + S_4(m,\delta) + \frac{m\delta}{2}.$$

Therefore, to compute the estimates for error term we need only to minimize the terms in the right-hand side of (2.22).

Note that if we replace 17 with 1 in the proof of Theorem 2.6 and adjust the terms in its proof when it is necessary, we could prove in a similar manner the following theorem

Theorem 2.19. *Let*

$$\varepsilon_0(x) = \sqrt{8/\pi}X^{1/2}e^{-X}.$$

Then

$$|\psi(x) - x| < x\varepsilon_0(x), \qquad (x \geq 3)$$

and

$$|\vartheta(x) - x| < x\varepsilon_0(x), \qquad (x \geq 3).$$

As you may observe the error term in Theorem 2.6 has the coefficient $\sqrt{8/(17\pi)}$ which is smaller than $\sqrt{8/\pi}$ in the next theorem and therefore gives a better bound. However, as we use a better zero-free region, we will get a better bound when $x \geq e^{255}$.

2.3.2   Improved Explicit Bounds for Moderate Values of x

As in Theorem 2.19, the role of A the verified height of RH (defined in (2.13)) was not vigorous, but it has more efficient role in estimating of the Chebyshev's function for moderate values of x using the next theorem. Let

$$T_0 = \frac{1}{\delta}\left(\frac{2R_m(\delta)}{2 + m\delta}\right)^{1/m}, \tag{2.23}$$

$$G(D) = \sum_{0 < \gamma \leq D} \frac{1}{(\gamma^2 + 1/4)^{1/2}} - \frac{1}{4\pi}\left\{\left(\log\frac{D}{2\pi} - 1\right)^2 + 1\right\}$$
$$+ \frac{1}{D}\left\{0.137\log D + 0.443\left(\log\log D + \frac{1}{\log D}\right) + 2.6 - N(D)\right\},$$

and

$$C(D) = 4\pi\left(0.137 + \frac{0.443}{\log D}\right).$$

Theorem 2.20 ([70], Lemma 9*). *Let T_0 be defined as above and satisfy $T_0 \geq D$, where $2 \leq D \leq A$. Let m be a positive integer and let $\delta > 0$. Then*

$$S_1(m, \delta) + S_2(m, \delta) < \Omega_1^*,$$

where

$$\Omega_1^* = \frac{2 + m\delta}{4\pi}\left\{\left(\log\frac{T_0}{2\pi} + \frac{1}{m}\right)^2 + 4\pi G(D) + \frac{1}{m^2} - \frac{mC(D)}{(m+1)T_0}\right\} \tag{2.24}$$

and $G(D)$ and $C(D)$ are defined as above. Moreover, if

$$\Omega_3^* = \frac{1}{2\pi}h_3(T_2) + e_3(T_2), \qquad T_2 \geq A, \tag{2.25}$$

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where

$$h_3(T) = \frac{2 + m\delta}{2} \int_A^T \phi_0(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{\delta^m} \int_T^\infty \phi_m(y) \log \frac{y}{2\pi} dy$$

and

$$e_3(T) = q(T) \left\{ -\frac{2 + m\delta}{2} \int_A^T \phi_0(y) \log \frac{y}{2\pi} dy + \frac{R_m(\delta)}{\delta^m} \int_T^\infty \phi_m(y) \log \frac{y}{2\pi} dy \right\}$$
$$+ R(T)\phi_0(T)\{2 + m\delta + 2\frac{R_m(\delta)}{(\delta T)^m}\},$$

then

$$|\psi(x) - x| < \varepsilon_0^* x, \qquad (x \geq e^b),$$

where

$$\varepsilon_0^* = \Omega_1^* e^{-b/2} + \Omega_3^* + \frac{m}{2}\delta + e^{-b} \log 2\pi. \qquad (2.26)$$

Table 3.2 is made from the above Theorem.

2.4 Improved Explicit Bounds and Distribution of Primes

Using Theorem 2.20 we can get a little better estimate for the first Chebyshev's function (2.3).

Proposition 2.21. *Let $x_k \geq 8 \cdot 10^{11}$. Then*

$$|\vartheta(x) - x| < \eta_k' \frac{x}{\log^k x}, \qquad (x \geq x_k),$$

where

k	0	1	2	3	4
η_k'	0.00002945957104	0.00082486799	0.0230963037	0.6466965035	1230

Proof. Let $e^b \leq x < e^{b+1}$. Appealing to Proposition 2.7, we treat $\vartheta(x) - x$ in the following way

$$\vartheta(x) - x = \vartheta(x) - \psi(x) + \psi(x) - x$$
$$< -0.9999\sqrt{x} + x\varepsilon_0^*$$
$$= \left(-0.9999\frac{\log^k x}{\sqrt{x}} + \varepsilon_0^* \log^k x \right) \frac{x}{\log^k x}.$$

In the same manner we find

$$\vartheta(x) - x = \vartheta(x) - \psi(x) + \psi(x) - x$$
$$> -1.00007\sqrt{x} - 1.78\sqrt[3]{x} - x\varepsilon_0^*$$
$$= \left(-1.00007\frac{\log^k x}{\sqrt{x}} - 1.78\frac{\log^k x}{\sqrt[3]{x^2}} - \varepsilon_0^* \log^k x\right)\frac{x}{\log^k x}.$$

To estimate η_k', it is enough to choose $x = e^{b+1}$ in each parenthesis. For instance, to estimate η_1' in the interval $[8 \cdot 10^{11}, e^{28})$, we have $\varepsilon_0^* = 0.0000284888$ (see computations just before Table 3.2 at the end of thesis), and

$$\vartheta(x) - x < \left(-0.9999\frac{28}{\sqrt{e^{28}}} + 0.0000284888(28)\right)\frac{x}{\log x} < 0.000774406\frac{x}{\log x},$$

$$\vartheta(x) - x > \left(-1.00007\frac{28}{\sqrt{e^{28}}} - 1.78\frac{28}{\sqrt[3]{e^{2\cdot28}}} - 0.0000284888(28)\right)\frac{x}{\log x}$$
$$> -0.00082486799\frac{x}{\log x}.$$

Continuing this process for all intervals $[e^b, e^{b+1})$ where $b = 28, 29, \ldots$ up to $x = e^{5200}$, we get the desired results. $\qquad\square$

Remark 2.22. The number 5200 in Table 3.2 is chosen as the last number, since for $x \geq e^{5204}$ we obtain $\varepsilon_0 < \varepsilon_0^*$, therefore we can apply then Theorem 2.19.

Applying the previous proposition, we obtain the estimates for the function $\pi(x)$.

Proposition 2.23. *Let* $x \geq 8 \cdot 10^{11}$. *Then*

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.0796}{\log x}\right),$$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.2703}{\log^2 x}\right).$$

Proof. By Abel's identity (Theorem 1.2)

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(y)}{y \log^2 y}dy$$
$$< \frac{x}{\log x}\left(1 + \frac{\eta_k'}{\log^k x}\right) + \int_2^x \frac{1}{\log^2 y}\left(1 + \frac{\eta_k'}{\log^k y}\right)dy. \qquad (2.27)$$

We are looking for inequality of this type:

$$\pi(x) < A_2(x), \qquad (x \geq 8 \cdot 10^{11}),$$

where
$$A_2(x) = \frac{x}{\log x}\left(1 + \frac{c}{\log x}\right),$$
and c is a constant which will be determined in the following.

Let $A_1(x)$ be the right-hand side of (2.27). Therefore we must have $A_1(x) < A_2(x)$ for $x \geq 8 \cdot 10^{11}$. To have this inequality it is enough to have $A_1(x_0) \leq A_2(x_0)$ with $x_0 = 8 \cdot 10^{11}$ and $A_1'(x) < A_2'(x)$ for $x \geq x_0$. Indeed,
$$A_1'(x) = \frac{1}{\log x} + \frac{2\eta_k' x}{\log^{k+1} x} - \frac{\eta_k'(-1 + x + kx)}{\log^{k+2} x}$$
and
$$A_2'(x) = \frac{1}{\log x} + \frac{-1+c}{\log^2 x} - \frac{2c}{\log^3 x}.$$
We apply the case η_1' in Proposition 2.21, and get for $x \geq 8 \cdot 10^{11}$
$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1.0796}{\log x}\right)$$
or if we let
$$A_2(x) = \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{c'}{\log x}\right),$$
by a similar method we arrive at
$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{2.2703}{\log^2 x}\right).$$

$\square$

Note that if the values of the function $\text{li}(x)$ can be calculated in some way, we could use the following formula
$$\text{li}(x) - \text{li}(2) = \int_2^x \frac{1}{\log y}dy = \left[\frac{y}{\log y} + \frac{y}{\log^2 y} + \frac{2!y}{\log^3 y} + \frac{3!y}{\log^4 y} + \cdots + \frac{j!y}{\log^{j+1} y}\right]_2^x$$
$$+ (j+1)! \int_2^x \frac{1}{\log^{j+2} y}dy, \qquad (j = 0, 1, \ldots)$$
to compute the integral in (2.27) instead of the method of differential calculus which we applied above.

In the next proposition we will determine the length of intervals which contain at least one prime.

Proposition 2.24. *For all $x \geq 492\,227$, there exist at least one prime p such that*
$$x < p \leq x\left(1 + \frac{0.0297139}{\log^2 x}\right).$$

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Proof. Since the first Chebyshev's function has a jump of the size $\log p$ on a prime p, for having a prime in the interval $[x, y)$ it is enough to find y such that $\vartheta(y) - \vartheta(x) > 0$. Assume $y = x\left(1 + \frac{\alpha_k}{\log^k x}\right)$ where $k = 1, 2, 3, 4$ and α_k is a constant. Hence,

$$\vartheta(y) - \vartheta(x) > y\left(1 - \frac{\eta'_k}{\log^k y}\right) - x\left(1 + \frac{\eta'_k}{\log^k x}\right)$$
$$> \frac{x}{\log^k x}\left\{\alpha_k\left(1 - \frac{\eta'_k}{\log^k x}\right) - 2\eta'_k\right\}.$$

If

$$\alpha_k > \frac{2\eta'_k}{1 - \eta'_k/\log^k x},$$

then we get the desired condition. Now according to [70], p. 355

$$p_{n+1} - p_n \leq 652, \qquad \text{for all } p_n \leq 2.686 \cdot 10^{12}.$$

On the other hand, $\varepsilon_0^* = 0.0000170896$ (defined 2.26) for $x \geq x_0 = 2.686 \cdot 10^{12}$. From here we have

$$|\vartheta(x) - x| < 0.0148566\frac{x}{\log^2 x}, \qquad (x \geq x_0).$$

Therefore,

$$\alpha_2 > \frac{2(0.0148567)}{1 - (0.0148567)/\log^2 x_0} \approx 0.0297139.$$

For $5\,254\,433 \leq x < 2.686 \cdot 10^{12}$ we note that

$$0.0297139\frac{x}{\log^2 x} > 652.$$

For $492\,227 \leq x < 5\,254\,433$ we check it by computer. $\qquad\square$

2.5 Explicit Estimates for $\prod_{p \leq x}(1 + 1/p)$

In this subsection we give bounds for $\prod_{p \leq x}(1 + 1/p)$.

First we determine some values for which we encounter later. Let

$$S(x) = \sum_{p > x}\left\{\log\left(1 + \frac{1}{p}\right) - \frac{1}{p}\right\} = \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{p > x}\frac{1}{p^n}.$$

Hence,

$$\sum_{p > x}\left(\frac{1}{2p^2} - \frac{1}{3p^3}\right) < -S(x) < \sum_{p > x}\frac{1}{2p^2}.$$

Using Abel's identity (Theorem 1.2) and estimates for $\vartheta(x)$ in Proposition 2.8 one obtains

$$\sum_{p>x} \frac{1}{2p^2} < \frac{1}{x \log x}$$

and

$$\sum_{p>x} \left(\frac{1}{2p^2} - \frac{1}{3p^3} \right) > \frac{1}{2x \log x} - \frac{5}{x \log^2 x}.$$

From

$$\prod_p \left(1 + \frac{1}{p} \right) = \prod_p \left(1 - \frac{1}{p^2} \right) \Big/ \prod_p \left(1 - \frac{1}{p} \right)$$

and definition of B in (2.7) we have

$$\sum_p \left\{ \log \left(1 + \frac{1}{p} \right) - \frac{1}{p} \right\} = \log \frac{6}{\pi^2} + \gamma - B. \tag{2.28}$$

Therefore,

$$\sum_{p \le x} \frac{1}{p} - B = \sum_{p \le x} \log \left(1 + \frac{1}{p} \right) + \sum_{p > x} \left\{ \log \left(1 + \frac{1}{p} \right) - \frac{1}{p} \right\} - \log \frac{6}{\pi^2} - \gamma.$$

Now by (2.9)

$$\left| \sum_{p \le x} \log \left(1 + \frac{1}{p} \right) + S(x) - \log \frac{6}{\pi^2} - \gamma - \log \log x \right| < C_k(x), \tag{2.29}$$

where $C_k(x)$ is the right-hand side of (2.9). Expanding terms inside absolute value (2.29), we get

$$\sum_{p \le x} \log \left(1 + \frac{1}{p} \right) < \log \frac{6}{\pi^2} + \gamma + \log \log x + C_k(x) - S(x),$$

$$\sum_{p \le x} \log \left(1 + \frac{1}{p} \right) > \log \frac{6}{\pi^2} + \gamma + \log \log x - C_k(x) - S(x).$$

We take exponential of both sides in each inequality, and using the estimate

$$1 + t < e^t < \frac{1}{1-t}, \qquad (t < 1) \tag{2.30}$$

and noting that $-S(x)$ is very small compared to the difference between the two sides of latter estimate when t replace with $C_k(x) < 1$, so that it is negligible. Thus we arrive at

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Proposition 2.25. *We have*

$$\prod_{p \le x} \left(1 + \frac{1}{p} \right) < \frac{6e^{\gamma}}{\pi^2} \frac{1}{1 - C_k(x)} \log x, \tag{2.31}$$

$$\prod_{p \le x} \left(1 + \frac{1}{p} \right) > \frac{6e^{\gamma}}{\pi^2} \{ 1 - C_k(x) \} \log x. \tag{2.32}$$

for all $x \ge x_k$ where x_k depends on η_k.

We can treat the proof of the Proposition 2.25 in a different way. In this method we do not use the estimates for $\prod_p (1 - 1/p^2)$, $\prod_p (1 - 1/p)$ or $\prod_{p \le x} (1 - 1/p)$.

Recall that for $t > 0$ we have the inequality (cf. 1.15)

$$\frac{1}{t + 1/2} < \log \left(1 + \frac{1}{t} \right) < \frac{1}{2} \left(\frac{1}{t} + \frac{1}{t+1} \right).$$

Let

$$\sum_{p \le x} \left\{ \log \left(1 + \frac{1}{p} \right) - \frac{1}{2} \left(\frac{1}{p} + \frac{1}{p+1} \right) \right\} = -a_x,$$

$$\sum_{p \le x} \left\{ \log \left(1 + \frac{1}{p} \right) - \frac{1}{p + 1/2} \right\} = b_x.$$

It is clear that

$$a_x + b_x = \frac{1}{2} \sum_{p \le x} \left(\frac{1}{p} - \frac{2}{p + 1/2} + \frac{1}{p+1} \right).$$

Therefore,

$$\log \prod_{p \le x} \left(1 + \frac{1}{p} \right) = \sum_{p \le x} \log \left(1 + \frac{1}{p} \right) = \frac{1}{2} \sum_{p \le x} \left(\frac{1}{p} + \frac{1}{p+1} \right) - a_x$$

$$= \sum_{p \le x} \frac{1}{p} - \frac{1}{2} \sum_{p \le x} \frac{1}{p(p+1)} - a_x$$

$$< \log \log x + B + C_k(x) - \frac{1}{2} \sum_{p \le x} \frac{1}{p(p+1)} - a_x,$$

$$\log \prod_{p \le x} \left(1 + \frac{1}{p} \right) = \sum_{p \le x} \log \left(1 + \frac{1}{p} \right) = \sum_{p \le x} \frac{1}{p + 1/2} + b_x$$

$$= \sum_{p \le x} \frac{1}{p} - \frac{1}{2} \sum_{p \le x} \frac{1}{p(p + 1/2)} + b_x$$

$$> \log \log x + B - C_k(x) - \frac{1}{2} \sum_{p \le x} \frac{1}{p(p + 1/2)} + b_x.$$

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Taking exponential of both side in each inequality and using (2.30) we get the bounds in the proposition.[1]

Corollary 2.26. *We have*

$$\prod_{x<p\leq y}\left(1+\frac{1}{p}\right) < \frac{\log y}{\log x}\left\{\frac{1}{1-C_k(x)-C_k(y)}\right\}, \qquad (x \geq x_k)$$

and

$$\prod_{x<p\leq y}\left(1+\frac{1}{p}\right) > \frac{\log y}{\log x}\left\{1-C_k(x)-C_k(y)\right\}, \qquad (x \geq x_k),$$

where x_k depends on η_k.

Note that in this corollary, for simplicity, we used the estimates of first method in the proof of Proposition 2.25.

---

[1]Continuing the second method we arrive at

$$\log \prod_{x<p\leq y}\left(1+\frac{1}{p}\right) < \log\log y - \log\log x + C_k(x) + C_k(y) - \frac{1}{2}\sum_{x<p\leq y}\frac{1}{p(p+1)} - (a_y - a_x)$$

and

$$\log \prod_{x<p\leq y}\left(1+\frac{1}{p}\right) > \log\log y - \log\log x - C_k(x) - C_k(y) - \frac{1}{2}\sum_{x<p\leq y}\frac{1}{p(p+1/2)} + (b_y - b_x),$$

which are slightly better than the bounds in Corollary 2.26.

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Chapter 3

Extremely Abundant Numbers and the RH

Contents

3.1 Introduction and Background

There are several equivalent statements to the RH. Some of them are related to the asymptotic behavior of arithmetic functions. In this chapter we will work on Robin's equivalence that is related to the behavior of sum of divisors $\sigma(n)$ and its ratio to n which can be expressed as (see 0.5)

$$\frac{\sigma(n)}{n} = \prod_{p|n} \frac{1 - 1/p_k^{\alpha_k+1}}{1 - 1/p}, \qquad \text{where } n = \prod_{k=1}^{r} p_k^{\alpha_k}. \tag{3.1}$$

Indeed, Robin [64] proved that

Theorem 3.1. *The RH is equivalent to inequality*

$$\frac{\sigma(n)}{n} < e^\gamma \log \log n, \qquad (n > 5040), \tag{3.2}$$

where γ is Euler's constant.

Throughout this chapter, as Robin used in [64], we let

$$f(n) = \frac{\sigma(n)}{n \log \log n}. \tag{3.3}$$

First we present a historical overview of the works which have been done on two closely related arithmetic functions $\frac{\sigma(n)}{n}$ and $\frac{n}{\phi(n)}$, where $\phi(n)$ is Euler's totient function, which is defined as the number of positive integers not exceeding n that are relatively prime to n.

In 1913, Gronwall [32] in his study of asymptotic maximal size for $\sigma(n)$, found that the order of $\sigma(n)$ is always "very nearly n" (see [36], Th. 323), proving

Theorem 3.2 ([32]). *Let f be defined by (3.3). Then*

$$\limsup_{n \to \infty} f(n) = e^\gamma. \tag{3.4}$$

Ramanujan in his unpublished manuscript [60] proved that if N is a generalized superior highly composite number, i.e., a number of CA which we introduce in the next section, then under the RH

$$\liminf_{N \to \infty} \left(\frac{\sigma(N)}{N} - e^\gamma \log \log N \right) \sqrt{\log N} \geq - e^\gamma (2\sqrt{2} + \gamma - \log 4\pi) \approx -1.558,$$

and

$$\limsup_{N \to \infty} \left(\frac{\sigma(N)}{N} - e^\gamma \log \log N \right) \sqrt{\log N} \leq - e^\gamma (2\sqrt{2} - 4 - \gamma + \log 4\pi) \approx -1.393.$$

Robin [64] demonstrated that

$$f(n) \leq e^\gamma + \frac{0.648214}{(\log \log n)^2}, \qquad (n \geq 3), \tag{3.5}$$

where $0.648214 \approx (\frac{7}{3} - e^\gamma \log \log 12) \log \log 12$ and the left-hand side of (3.5) attains its maximum at $n = 12$. In the same spirit, Lagarias [51] proved that the RH is equivalent to the inequality

$$\sigma(n) \leq e^{H_n} \log H_n + H_n, \qquad (n \geq 1),$$

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where $H_n = \sum_{j=1}^{n} 1/j$ and it is called the n-th harmonic number.

Investigating upper and lower bounds of arithmetic functions, Landau ([52], pp. 216–219) obtained the following limits:

$$\liminf_{n\to\infty} \frac{\varphi(n) \log\log n}{n} = e^{-\gamma}\ , \qquad \limsup_{n\to\infty} \frac{\varphi(n)}{n} = 1.$$

It is known that the totient function can be expressed as a product over the distinct prime divisors of n (see for instance [4, Th. 2.4]); i.e.,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{3.6}$$

Furthermore, Nicolas ([56], [57]) proved that, if the RH is true, then we have for all $k \geq 2$,

$$\frac{N_k}{\varphi(N_k) \log\log N_k} > e^{\gamma}, \tag{3.7}$$

where $N_k = \prod_{j=1}^{k} p_j$ and p_j is the j-th prime. On the other hand, if the RH is false, then there are infinitely many k for which (3.7) is true, and infinitely many k for which (3.7) is false.

Compared to numbers N_k which are the smallest integers that maximize $n/\varphi(n)$, there are integers which play the same role for $\sigma(n)/n$ and they are called superabundant numbers. In other words, n is a *superabundant number* if ([3], see also [60])

$$\frac{\sigma(n)}{n} > \frac{\sigma(m)}{m} \qquad \text{for all } m < n. \tag{3.8}$$

Briggs [7] describes a computational study of the successive maxima of the relative sum-of-divisors function $\sigma(n)/n$. He also studies computationally the density of these numbers. Wójtowicz [79] showed that the values of f are close to 0 on a set of asymptotic density 1. Another study on Robin's inequality (3.2) can be found in [13] in which Choie et al. showed that RH holds true if and only if every natural number divisible by a fifth power greater than 1 satisfies Robin's inequality.

In 2009, Akbary and Friggstad [2] established the following interesting theorem which enables to limit our attention to a narrow sequence of positive integers to find a probable counterexample to (3.2).

Theorem 3.3 ([2], Th. 3). *If there is any counterexample to Robin's inequality, then the least such counterexample is a superabundant number.*

Unfortunately, to our knowledge, there is no known algorithm to produce su-perabundant numbers. On the other hand, if $Q(x)$ denotes the number of super-abundants not exceeding x, then (see [3])

$$Q(x) > c \frac{\log x \log \log x}{(\log \log \log x)^2},$$

and in [26] it was even proved that for every $\delta < 5/48$

$$Q(x) > (\log x)^{1+\delta}, \qquad (x > x_0).$$

As a natural question in this direction, it is interesting to determine the least number, if exists, that violates inequality (3.2) which belongs to a thinner sequence of positive integers, and study its properties. Following this demand, we introduce a new sequence of numbers and call its elements *extremely abundant numbers*. We will present in the sequel some of their properties. Surprisingly enough, we will prove that the least number, if any, should be an extremely abundant number. Therefore, we will establish another criterion, which is equivalent to the RH.

Before starting the definition and results we mention a recent paper by Caveney et al. [10]. They defined a positive integer n as an extraordinary number, if n is composite and $f(n) \geq f(kn)$ for all

$$k \in \mathbb{N} \cup \{1/p : \ p \text{ is a prime factor of } n\}.$$

Under these conditions, they showed that the smallest extraordinary number is $n = 4$. Then they proved that the RH is true, if and only if, 4 is the only extraordinary number. For more properties of these numbers and comparison with superabundant and colossally abundant numbers we refer to [11].

3.2 Extremely Abundant Numbers: Definition and Motivations

In this section we define a new subsequence of superabundant numbers which will be contributed to the RH. Namely, our contribution and motivation of this definition will be to give Theorems 3.6 and 3.7 below.

Definition 3.4. A positive integer n is an *extremely abundant* number, if either $n = 10080$ or $n > 10080$ and

$$\forall m \quad \text{s.t.} \quad 10080 \leq m < n, \qquad f(m) < f(n). \tag{3.9}$$

where $f(n)$ is given by (3.3).

Here 10080 has been chosen as the smallest superabundant number greater than 5040. In Table 3.1 we list the first 20 extremely abundant numbers. To find them we used the list of superabundant numbers provided in [58] or [49].

We call a positive integer n (cf. [3] and [60])

(i) *colossally abundant*, if for some $\varepsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\varepsilon}} > \frac{\sigma(m)}{m^{1+\varepsilon}}, \qquad (m < n) \quad \text{and} \quad \frac{\sigma(n)}{n^{1+\varepsilon}} \geq \frac{\sigma(m)}{m^{1+\varepsilon}}, \quad (m > n); \qquad (3.10)$$

(ii) *highly composite*, if $d(n) > d(m)$ for all $m < n$, where $d(n) = \sum_{d|n} 1$ is the number of divisors of n;

(iii) *generalized superior highly composite*, if there is a positive number ε such that

$$\frac{\sigma_{-s}(n)}{n^{\varepsilon}} \geq \frac{\sigma_{-s}(m)}{m^{\varepsilon}}, \qquad (m < n) \qquad \text{and} \qquad \frac{\sigma_{-s}(n)}{n^{\varepsilon}} > \frac{\sigma_{-s}(m)}{m^{\varepsilon}}, \qquad (m > n),$$

where $\sigma_{-s}(n) = \sum_{d|n} d^{-s}$.

It was Ramanujan who initiated the study of these classes of numbers in an unpublished part of his 1915 work on highly composite numbers ([59], [60], [61]). More precisely, he defined rather general classes of these numbers. For instance, he defined generalized highly composite numbers, containing as a subset superabundant numbers ([59], §59), and he introduced the generalized superior highly composite numbers, including as a particular case colossally abundant numbers. For more details about these numbers see [3], [26] and [60].

We denote the following sets of integers by

$$SA = \{n : \quad n \text{ is superabundant}\},$$
$$CA = \{n : \quad n \text{ is colossally abundant}\},$$
$$XA = \{n : \quad n \text{ is extremely abundant}\}.$$

We also use SA, CA and XA as abbreviations of the corresponding sets. Clearly, $XA \neq CA$ (see Table 3.1). Indeed, we shall prove that infinitely many numbers of CA are not in XA and that, if RH holds, then infinitely many numbers of XA are in CA.

As an elementary result from the definition of XA numbers we get

Proposition 3.5. *The inclusion $XA \subset SA$ holds.*

Proof. First, $10080 \in SA$. Further, if $n > 10080$ and $n \in XA$, then for $10080 \leq m < n$ we have

$$\frac{\sigma(n)}{n} = f(n) \log \log n > f(m) \log \log m = \frac{\sigma(m)}{m}.$$

In particular, for $m = 10080$, we get

$$\frac{\sigma(n)}{n} > \frac{\sigma(10080)}{10080}.$$

So that, for $m < 10080$, we have

$$\frac{\sigma(n)}{n} > \frac{\sigma(10080)}{10080} > \frac{\sigma(m)}{m},$$

since $10080 \in SA$. Therefore, n belongs to SA. $\qquad\square$

Next, motivating our construction of XA numbers, we will establish the first interesting result of the chapter.

Theorem 3.6. *If there is any counterexample to Robin's inequality (3.2), then the least one is an XA number.*

Proof. By doing some computer calculations we observe that there is no counterexample to Robin's inequality (3.2) for $5040 < n \leq 10080$. Now let $n > 10080$ be the least counterexample to inequality (3.2). For m satisfying $10080 \leq m < n$ we have

$$f(m) < e^\gamma \leq f(n).$$

Therefore, n is an XA. $\qquad\square$

As we mentioned in Introduction, we will prove an equivalent criterion to the RH whose proof is based on Robin's inequality (3.2) and Theorem 3.2. Let $\#A$ denote the cardinal number of a set A. This result has its own interest which will be discussed in §3.5.

Theorem 3.7. *The RH is true if and only if $\#XA = \infty$.*

Proof. Sufficiency. Assume that RH is not true. Then from Theorem 3.6, $f(m) \geq e^\gamma$ for some $m \geq 10080$. From Theorem 3.2, $M = \sup_{n \geq 10080} f(n)$ is finite and hence there exists n_0 such that $f(n_0) = M \geq e^\gamma$ (if $M = e^\gamma$ then set $n_0 = m$). An integer $n > n_0$ satisfies $f(n) \leq M = f(n_0)$ and n can not be in XA so that $\#XA \leq n_0$.

Necessity. On the other hand, if RH is true, then inequality (3.2) is true. If $\#XA$ is finite, then there exists an m such that for every $n > m$, $f(n) \le f(m)$. Then

$$\limsup_{n\to\infty} f(n) \le f(m) < e^\gamma,$$

which is a contradiction to Theorem 3.2. □

There are some primes which cannot be the largest prime factors of any XA number. For example, referring to Table 3.1, suggests that there is no XA number with the largest prime factor $p(n) = 149$ (one can prove this using Proposition 3.14). Do there exist infinitely many such primes?

3.3 Auxiliary Lemmas

Before we state several properties of SA, CA and XA numbers, we give the following lemmas which will be needed in the sequel. We note that inequality (1.12) or by changing variable $x = 1/t$

$$\frac{t}{1+t} < \log(1+t) < t, \qquad (t > 0), \tag{3.11}$$

will be employed frequently.

Lemma 3.8. *Let a, b be positive constants and $x, y \in \mathbb{R}^+$ for which*

$$\log x > a,$$

and

$$x\left(1 - \frac{a}{\log x}\right) < y < x\left(1 + \frac{b}{\log x}\right).$$

Then

$$y\left(1 - \frac{c}{\log y}\right) < x < y\left(1 + \frac{d}{\log y}\right),$$

where

$$c \ge b\left(1 - \frac{b - \frac{b}{\log x}}{\log x + b}\right), \qquad d \ge a\left(1 + \frac{a + \frac{b}{\log x}}{\log x - a}\right).$$

Proof. Dividing by x, inverting both sides and multiplying by y, we get

$$\frac{y}{1 + b/\log x} < x < \frac{y}{1 - a/\log x}.$$

We are looking for constants c and d such that

$$1 - \frac{c}{\log y} < \frac{1}{1 + b/\log x},$$

or equivalently

$$c > (\log y)\frac{b}{\log x + b},$$

and

$$\frac{1}{1 - a/\log x} < 1 + \frac{d}{\log y},$$

or equivalently

$$d > (\log y)\frac{a}{\log x - a}.$$

First we determine c. Since

$$y < x\left(1 + \frac{b}{\log x}\right),$$

then

$$\log y < \log x + \log\left(1 + \frac{b}{\log x}\right) < \log x + \frac{b}{\log x}.$$

So that if

$$
\begin{aligned}
c &> \left(\log x + \frac{b}{\log x}\right)\frac{b}{\log x + b} \\
&= b\left(\log x + b - b + \frac{b}{\log x}\right)\frac{1}{\log x + b} \\
&= b\left(1 - \frac{b - \frac{b}{\log x}}{\log x + b}\right),
\end{aligned}
$$

then

$$c > \log y\frac{b}{\log x + b},$$

and hence

$$x > y\left(1 - \frac{c}{\log y}\right).$$

Similarly, if

$$
\begin{aligned}
d &> \left(\log x + \frac{b}{\log x}\right)\frac{a}{\log x - a} \\
&= \left(\log x - a + a + \frac{b}{\log x}\right)\frac{a}{\log x - a} \\
&= a\left(1 + \frac{a + \frac{b}{\log x}}{\log x - a}\right),
\end{aligned}
$$

then

$$d > \log y\frac{a}{\log x - a},$$

and therefore

$$x < y\left(1 + \frac{d}{\log y}\right).$$

$\square$

Similarly one can show

Lemma 3.9. *Let a, b be positive constants and $x, y \in \mathbb{R}^+$ for which*

$$\log^2 x > a,$$

and

$$x \left(1 - \frac{a}{\log^2 x} \right) < y < x \left(1 + \frac{b}{\log^2 x} \right),$$

Then

$$y \left(1 - \frac{c}{\log^2 y} \right) < x < y \left(1 + \frac{d}{\log^2 y} \right),$$

where

$$c \geq b \left(1 - \frac{b - \frac{2b}{\log x} - \frac{b^2}{\log^4 x}}{\log^2 x + b} \right), \qquad d \geq a \left(1 + \frac{a + \frac{2b}{\log x} + \frac{b^2}{\log^4 x}}{\log^2 x - a} \right).$$

By elementary differential calculus one proves also

Lemma 3.10. *Let $h(x) = \log \log x$. Then*

$$g(y) = \frac{yh(y) - xh(x)}{(y - x)h(x)}, \qquad (y > x > e).$$

is increasing. In particular, if $c > 1$ and $e < x < y < cx$, we have $g(y) < g(cx)$.

We will need in the sequel the following inequality

$$\frac{1}{c-1} \left(c \frac{\log \log cx}{\log \log x} - 1 \right) < 1 + \frac{c}{c-1} \frac{\log c}{\log x \log \log x}, \qquad (x > e, \ c > 1). \quad (3.12)$$

Indeed

$$
\begin{aligned}
\frac{1}{c-1} \left(c \frac{\log \log cx}{\log \log x} - 1 \right) &= \frac{1}{c-1} \left(c \frac{\log \log cx - \log \log x}{\log \log x} + c - 1 \right) \\
&= \frac{1}{c-1} \left(\frac{c}{\log \log x} \log \left(1 + \frac{\log c}{\log x} \right) + c - 1 \right) \\
&< \frac{1}{c-1} \left(\frac{c}{\log \log x} \left(\frac{\log c}{\log x} \right) + c - 1 \right) \\
&= 1 + \frac{c}{c-1} \frac{\log c}{\log x \log \log x}.
\end{aligned}
$$

Lemma 3.11. *Let $x \geq 11$. Then, for $y > x$ the following inequality holds*

$$\frac{\log \log y}{\log \log x} < \frac{\sqrt{y}}{\sqrt{x}}.$$

Recall that the prime number theorem is equivalent to

$$\psi(x) \sim x, \qquad (3.13)$$

where $\psi(x)$ is Chebyshev's function (Theorem 2.2; see also [36], Th. 434; [44], Th. 3, 12). The following result is a corollary of 2.1 which we will use in the sequel. Of course we could use the explicit bounds which we got in the previous chapter, but for historical point of view (due to Chebyshev) we use the following corollary in one of our results.

Corollary 3.12. *We have*

$$\vartheta(x) > \frac{\log 2}{2} x, \qquad (x \geq 3).$$

Proof. First we prove that the right-hand side of (2.2) is greater than $\frac{\log 2}{2} x$ in $[x_0, \infty)$ for some x_0. Let

$$g(x) = A_0 x - \frac{12}{5} A_0 x^{\frac{1}{2}} - \frac{5}{8 \log 6} \log^2 x - \frac{15}{4} \log x - 3 - \frac{\log 2}{2} x,$$

where A_0 is defined in (2.1). Then

$$g'(x) = (A_0 - \frac{\log 2}{2}) - \frac{6}{5} A_0 x^{-\frac{1}{2}} - \frac{5}{4 \log 6} \frac{\log x}{x} - \frac{15}{4x}.$$

As $\log x < \frac{3}{4} x^{\frac{1}{2}}$ and $x^{\frac{1}{2}} < x$ for $x > 1$,

$$g'(x) > (A_0 - \frac{\log 2}{2}) - \left(\frac{6}{5} A_0 + \frac{15}{16 \log 6} + \frac{15}{4} \right) x^{-\frac{1}{2}}.$$

So, for

$$x > x_0 = \left(\frac{6}{5} A_0 + \frac{15}{16 \log 6} + \frac{15}{4} \right)^2 / (A_0 - \frac{\log 2}{2})^2 \approx 87.591$$

we have $g'(x) > 0$; i.e., $g(x)$ is increasing. Also $g(x_0) > 0$ and therefore $g(x) > 0$, for $x \geq x_0$. Now for $3 \leq x < x_0$, verify $\vartheta(x) > \frac{\log 2}{2} x$ using direct computation. $\square$

3.4 Some Properties of SA, CA and XA Numbers

This section is divided into three subsections, for which we will exhibit several properties of SA, CA and XA numbers, respectively. In the following, when there is no ambiguity, we simply denote by p the largest prime factor of n.

3.4.1 SA Numbers

As the starting point, we show that for any real positive $x \geq 1$, there is at least one SA number in the interval $[x, 2x)$. In other words

Proposition 3.13. *Let $n < n'$ be two consecutive SA numbers. Then*

$$\frac{n'}{n} \leq 2.$$

Proof. Let $n = \prod_{q=2}^{p} q^{k_q}$. We compare n with $2n$. In fact

$$\frac{\sigma(2n)/(2n)}{\sigma(n)/n} = \frac{2^{k_2+2} - 1}{2^{k_2+2} - 2} > 1.$$

Hence, $n' \leq 2n$. $\qquad\square$

Alaoglu and Erdős [3] proved that If $n = 2^{k_2} \cdots p^{k_p}$ is a superabundant number then $k_2 \geq \ldots \geq k_p$ and the exponent of greatest prime factor of n is 1 except $n = 4,\ 36$.

Proposition 3.14 ([3], Th. 2)**.** *Let q and r be prime factors of $n \in SA$ such that $q < r$ and*

$$\beta := \left\lfloor \frac{k_q \log q}{\log r} \right\rfloor,$$

where k_q is the exponent of q. Then k_r (the exponent of r) has one of the three values : $\beta - 1,\ \beta + 1,\ \beta$.

As we observe, the above proposition determines the exponent of each prime factor of a SA number with error of at most 1 in terms of smaller prime factor of that number. In the next theorem we give a lower bound for the exponent k_q related to the largest prime factor of n.

Theorem 3.15. *Let $n \in SA$ and $2 \leq q \leq p$ (where p is the greatest prime factor of n) be a prime factor of n. Then*

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq k_q.$$

Proof. If $q = p$, it is trivial. Let $q < p$ and $k_q = k$ and suppose that $k \leq [\log p / \log q] - 1$. Hence

$$q^{k+1} < p. \tag{3.14}$$

Now we compare values of $\sigma(s)/s$, taking $s = n$ and $s = m = nq^{k+1}/p$. Since $\sigma(s)/s$ is multiplicative, we restrict our attention to the factors q and p. But n is SA and $m < n$, then

$$1 < \frac{\sigma(n)/n}{\sigma(m)/m} = \frac{q^{2k+2} - q^{k+1}}{q^{2k+2} - 1}\left(1 + \frac{1}{p}\right) = \frac{1}{1 + 1/q^{k+1}}\left(1 + \frac{1}{p}\right).$$

Consequently, $p < q^{k+1}$, which contradicts (3.14). $\qquad\square$

The following proposition gives the asymptotic relation between prime factors of SA numbers.

Proposition 3.16 ([3], p. 453). *Let δ denote*

$$\delta = \frac{(\log\log p)^2}{\log p \log q}, \qquad (q^{1-\theta} < \log p),$$

$$\delta = \frac{\log p}{q^{1-\theta}\log q}, \qquad (q^{1-\theta} > \log p),$$

where $\theta \geq 5/8$ is the number which was discussed just before Lemma 2.9. Then

$$\log\frac{q^{k+1} - 1}{q^{k+1} - q} > \frac{\log q}{\log p}\log\left(1 + \frac{1}{p}\right)\{1 + O(\delta)\}, \qquad (3.15)$$

$$\log\frac{q^{k+2} - 1}{q^{k+2} - q} < \frac{\log q}{\log p}\log\left(1 + \frac{1}{p}\right)\{1 + O(\delta)\}. \qquad (3.16)$$

Corollary 3.17. *Let $n \in SA$ and $2 \leq q \leq p$ (where p is the greatest prime factor of n) be a fixed prime factor of n. Then there exist two positive constants c and c' (depending on q) such that*

$$cp\frac{\log p}{\log q} < q^{k_q} < c'p\frac{\log p}{\log q}.$$

Proof. By inequality (3.11)

$$\log\frac{q^{k+1} - 1}{q^{k+1} - q} = \log\left(1 + \frac{q-1}{q^{k+1} - q}\right) < \frac{q-1}{q^{k+1} - q} \leq \frac{1}{q^k}$$

and (3.15), there exists a $c' > 0$ such that

$$q^k < c'\frac{p\log p}{\log q}.$$

On the other hand, again from inequality (3.11)

$$\log\frac{q^{k+2} - 1}{q^{k+2} - q} = \log\left(1 + \frac{q-1}{q^{k+2} - q}\right) > \frac{q-1}{q^{k+2} - 1} > \frac{1}{2q^{k+1}}$$

and (3.16), there exists a $c > 0$ such that

$$q^k > c\frac{p\log p}{\log q}.$$

$\qquad\square$

Corollary 3.18. *For large enough SA number* $n = 2^k \cdots p$

$$p < 2^{k-1}. \tag{3.17}$$

Corollary 3.19. *Let* $n = 2^k \cdots p$ *be a SA number. Then for large enough* n

$$\left\lfloor \frac{k \log 2}{\log p} \right\rfloor = 1.$$

Proof. By Corollary 3.17 for $q = 2$ we have

$$\log(c\, p \frac{\log p}{\log 2}) < k \log 2 < \log(c'\, p \frac{\log p}{\log 2}).$$

Hence, for large enough p

$$1 < 1 + \frac{\log(cp \log p / \log 2)}{\log p} < \frac{k \log 2}{\log p} < 1 + \frac{\log(c'p \log p / \log 2)}{\log p} < 2.$$

Therefore,

$$\left\lfloor \frac{k \log 2}{\log p} \right\rfloor = 1.$$

$\square$

Remark 3.20. In [3] it was proved that $q^{k_q} < 2^{k_2+2}$ and, in p. 455 it was remarked that for large SA n, $q^{k_q} < 2^{k_2}$ for $q > 11$.

Proposition 3.21 ([3], Th. 7). *If* $n = 2^k \cdots p \in SA$, *then*

$$p \sim \log n.$$

From Corollary 3.17 and Proposition 3.21 it follows that

Proposition 3.22. *For large enough* $n \in SA$

$$\log n < 2^{k_2}.$$

Proof. We use Remark 3.20, Theorem 2.10 and Corollary 3.18 to get

$$\begin{aligned}
\frac{\log n}{2^{k_2}} &= \frac{\sum \log q^{k_q}}{2^{k_2}} \\
&< \frac{5 \log 2^{k_2+2} + (\pi(p(n)) - 5) \log 2^{k_2})}{2^{k_2}} \\
&= \pi(p(n)) \frac{\log 2^{k_2}}{2^{k_2}} + \frac{10 \log 2}{2^{k_2}} \\
&< \frac{p(n)}{\log p(n)} \left(1 + \frac{1.2762}{\log p(n)} \right) \frac{\log 2^{k_2}}{2^{k_2}} + \frac{10 \log 2}{2^{k_2}} \\
&= \frac{p(n)}{2^{k_2}} \frac{\log 2^{k_2}}{\log p(n)} \left(1 + \frac{1.2762}{\log p(n)} \right) + \frac{10 \log 2}{2^{k_2}} \\
&< 1,
\end{aligned}$$

where $p(n) = p$ is the greatest prime factor of n. $\square$

Proposition 3.23. *Let* $n = 2^{k_2} \cdots q^{k_q} \cdots p \in SA$. *Then*

$$\psi(p) \leq \log n. \tag{3.18}$$

Moreover,

$$\lim_{n \to \infty} \frac{\psi(p)}{\log n} = 1. \tag{3.19}$$

Proof. In fact, by Theorem 3.15

$$\psi(p) = \sum_{q \leq p} \left\lfloor \frac{\log p}{\log q} \right\rfloor \log q \leq \sum_{q \leq p} k_q \log q = \log n.$$

In order to prove (3.19) we appeal to (3.13) and Proposition 3.21. □

Proposition 3.24 ([3], Lemma 4). *If* q *is the greatest prime of exponent* k, *and if* $q^{1-\theta} > \log p$ *(where* $\theta \geq 5/8$*), then all primes between* q *and* $q + q^\theta$ *have exponent* $k - 1$.

Remark 3.25. From the above proposition we observe that there is some n_0 such that for any superabundant number $n > n_0$ there exists a prime factor of n with exponent 2 and there exists a prime factor of n with exponent 3.

Proposition 3.26 ([3], Th. 4). *If* q *is either the greatest prime of exponent* k *or the least prime of exponent* $k - 1$, *and if* $q^{1-\theta} > \log p$, *then*

$$q^k = \frac{p \log p}{\log q} \left\{ 1 + O\left(\frac{\log p}{q^{1-\theta} \log q} \right) \right\}.$$

From Remark 3.25 and Proposition 3.26 we get

Corollary 3.27. *Let* x_k *(with* $k = 2, 3$*) denote the greatest prime factor of exponent* k *or the least prime of exponent* $k - 1$ *in decomposition of* $n \in SA$ *and* $x_k^{1-\theta} > \log p$. *Then for large enough* $n \in SA$

$$\sqrt{\frac{3}{2}p} < x_2 < \frac{3}{2}\sqrt{p},$$

and

$$\sqrt[3]{\frac{5}{2}p} < x_3 < \frac{3}{2}\sqrt[3]{p},$$

where p *is the greatest prime factor of* n.

Lemma 3.28. *For large enough* $n = 2^{k_2} \cdots q^{k_q} \cdots p \in SA$

$$\frac{\log n}{\vartheta(p)} < 1 + \frac{3}{2\sqrt{p}} \left(1 + \frac{4\eta_1}{\log p} \right),$$

where η_1 *is defined in (2.8) or (2.21).*

Proof. Let x_2 be the largest prime factor with exponent 2. From Corollary 3.27 for large enough $n \in SA$

$$\frac{\log n}{\vartheta(p)} - 1 = \frac{1}{\vartheta(p)} \left\{ \sum_{2 \leq q \leq x_3} (k_q - 1) \log q + \vartheta(x_2) - \vartheta(x_3) \right\}$$

$$< \frac{1}{\vartheta(p)} \left\{ \vartheta(x_2) + (k_2 - 2)\vartheta(x_3) \right\}$$

$$< \frac{1}{\vartheta(p)} \left\{ \vartheta(\frac{3}{2}\sqrt{p}) + \left(\frac{\log 2p \log p}{\log 2} - 2 \right) \vartheta(\frac{3}{2}\sqrt[3]{p}) \right\}$$

$$< \frac{1}{p(1 - \eta_1/\log p)} \frac{3\sqrt{p}}{2} \left\{ 1 + \frac{\eta_1}{\log \frac{3}{2}\sqrt{p}} + \frac{\log \frac{p \log p}{2}}{\log 2} \cdot \frac{\sqrt[3]{p}}{\sqrt{p}} \left(1 + \frac{\eta_1}{\log \frac{3}{2}\sqrt[3]{p}} \right) \right\}$$

$$< \frac{3}{2\sqrt{p}} \left(1 + \frac{4\eta_1}{\log p} \right), \qquad p > p_0 \quad \text{for some } p_0$$

where η_1 is that in Theorem 2.8. $\qquad \square$

In Proposition 3.21 it was proved that the $\log n$ is asymptotic to $p(n)$. In the next proposition we give better bounds for this approximation.

Proposition 3.29. *For $n = 2^k \cdots p \in SA$ we have*

$$\log n > p \left(1 - \frac{\eta_1}{\log p} \right)$$

and for large enough $n \in SA$

$$\log n < p \left(1 + \frac{2\eta_1}{\log p} \right), \tag{3.20}$$

where η_1 is defined in (2.8) or (2.21).

Proof. The first inequality holds by (3.18) and Theorem 2.8 or Proposition 2.21 (for $p > 8 \cdot 10^{11}$). Concerning the second inequality, we find

$$\frac{\log n}{p} = \frac{\log n}{\vartheta(p)} \frac{\vartheta(p)}{p}$$

$$< \left\{ 1 + \frac{3}{2\sqrt{p}} \left(1 + \frac{4\eta_1}{\log p} \right) \right\} \left(1 + \frac{\eta_1}{\log p} \right)$$

$$< \left(1 + \frac{2\eta_1}{\log p} \right), \qquad (p > p_0 \quad \text{for some } p_0).$$

$\qquad \square$

From Lemma 3.9 and Proposition 3.29, we conclude

Corollary 3.30. *For large enough* $n = 2^k \cdots p \in SA$, *we have*

$$\log n \left(1 - \frac{2\eta_1}{\log\log n} \right) < p < \log n \left(1 + \frac{2\eta_1}{\log\log n} \right),$$

where η_1 is defined in (2.8) or (2.21).

As we mentioned in the Introduction, two functions $\sigma(n)/n$ and $n/\phi(n)$ are close functions (see (3.1) and (3.6)). Here we will show how close they are for SA numbers. In §18.3 and §18.4 of [36], it is proved that

$$\frac{6}{\pi^2} < \frac{\sigma(n)\varphi(n)}{n^2} < 1,$$

and

$$\varliminf_{n\to\infty} \frac{\sigma(n)\varphi(n)}{n^2} = \frac{6}{\pi^2}, \qquad \varlimsup_{n\to\infty} \frac{\sigma(n)\varphi(n)}{n^2} = 1.$$

Proposition 3.31. *For $n = 2^{k_2} \cdots q^{k_q} \cdots p \in SA$, we have*

$$\frac{\sigma(n)}{n} > \{1 - \varepsilon(p)\} \frac{n}{\varphi(n)},$$

where

$$\varepsilon(p) = \frac{6}{\sqrt{p}\log p} \left(1 + \frac{1}{\log p} \right).$$

Proof. We show that

$$\frac{\sigma(n)}{n} \cdot \frac{\varphi(n)}{n} = \prod_{q \leq p} \left(1 - \frac{1}{q^{k_q+1}} \right) > 1 - \frac{6}{\sqrt{p}\log p} \left(1 + \frac{1}{\log p} \right). \qquad (3.21)$$

Hence, using logarithmic inequality (1.12) and Theorem 3.15 and Theorem 2.10, we obtain

$$\log \prod_{q \leq p} \left(1 - \frac{1}{q^{k_q+1}} \right) = \sum_{q \leq p} \log \left(1 - \frac{1}{q^{k_q+1}} \right) > -\sum_{q \leq p} \frac{1}{q^{k_q+1} - 1}$$

$$= -\sum_{q \leq x_2} \frac{1}{q^{k_q+1} - 1} - \sum_{x_2 < q \leq p} \frac{1}{q^2 - 1}$$

$$> -\sum_{q \leq x_2} \frac{1}{q^{\log p/\log q} - 1} - \frac{2\sqrt{2}}{\sqrt{p}\log p}$$

$$= -\sum_{q \leq x_2} \frac{1}{p - 1} - \frac{2\sqrt{2}}{\sqrt{p}\log p}$$

$$= -\frac{\pi(x_2)}{p - 1} - \frac{2\sqrt{2}}{\sqrt{p}\log p}$$

$$> -\frac{1}{p - 1} \frac{3\sqrt{p}}{2\log(3/2)\sqrt{p}} \left(1 + \frac{1.2762}{\log(3/2)\sqrt{p}} \right) - \frac{2\sqrt{2}}{\sqrt{p}\log p}$$

$$> -\frac{6}{\sqrt{p}\log p} \left(1 + \frac{1}{\log p} \right),$$

where x_2 is the greatest prime factor of exponent 2. Therefore, taking the exponential of both sides and using $e^{-x} > 1 - x$, we get (3.21). $\qquad\square$

Proposition 3.32. *Let* $n = 2^k \cdots p \in SA$. *Then*

$$\lim_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma.$$

More precisely,

$$\frac{\sigma(n)}{n \log \log n} < e^\gamma \left(1 + \frac{0.363945701}{(\log \log n)^2} \right), \qquad (n \geq 3)$$

and for large enough $n \in SA$

$$\frac{\sigma(n)}{n \log \log n} > e^\gamma \left(1 - \frac{1}{(\log \log n)^2} \right). \qquad (3.22)$$

Proof. The first inequality is exactly (3.5), where $0.363945701 \approx (0.6482136495)e^{-\gamma}$.

By using Proposition 3.31, Lemma 2.12 and Corollary 3.30, we get for large enough n

$$\begin{aligned}
\frac{\sigma(n)}{n} &> \{1 - \varepsilon(p)\} e^\gamma (\log p) \left(1 - \frac{0.2}{\log^2 p} \right) \\
&= e^\gamma (\log p) \left\{ 1 - \frac{6}{\sqrt{p} \log p} \left(1 + \frac{1}{\log p} \right) \right\} \left(1 - \frac{0.2}{\log^2 p} \right) \\
&> e^\gamma (\log \log n) \left(1 - \frac{1}{(\log \log n)^2} \right).
\end{aligned}$$

$\qquad\square$

In [57], it was proved that under RH

$$\psi(x) - \vartheta(x) \geq \sqrt{x}, \qquad (x \geq 121)$$

and

$$\psi(x) - \vartheta(x) - \sqrt{x} \leq 1.3327681611 \sqrt[3]{x}, \qquad (x \geq 1).$$

The following proposition is an application of Corollary 3.27 and Lemma 3.28.

Proposition 3.33. *For large enough* $n \in SA$ *we have*

$$\sqrt{p} < \log n - \vartheta(p) < \sqrt{3p}$$

Proof.

$$\log n - \vartheta(p) = \sum_{q \le x_3} (k_q - 1) \log q + \vartheta(x_2) - \vartheta(x_3)$$

$$\ge \vartheta(x_2) + \vartheta(x_3)$$

$$\ge \vartheta(\sqrt{3p/2}) + \vartheta(\sqrt[3]{5p/2})$$

$$> \sqrt{\frac{3}{2}p} \left(1 - \frac{\eta_1}{\log \sqrt{3p/2}} \right) + \sqrt[3]{\frac{5}{2}p} \left(1 - \frac{\eta_1}{\log \sqrt[3]{5p/2}} \right)$$

$$> \sqrt{p}, \qquad p > p_0 \quad \text{for some } p_0.$$

Hence the first inequality holds. The second inequality is a corollary of Lemma 3.28. □

Behavior of some functions in SA numbers

In this part we present some results on the behaviors of some special arithmetic functions when their arguments are SA numbers. Before we start the results we give a definition.

Definition 3.34. Let g be a real-valued function and $A = \{a_n\}_{n \in I}$ (where I is an ordered subset of natural numbers $\mathbb{N}$ by the usual order $<$) be an increasing sequence of integers. We say that g is an increasing (decreasing) function on A (or for $a_n \in A$), if $g(a_n) \le g(a_{n+1})$ (if $g(a_n) \ge g(a_{n+1})$) for all $n \in I$.

In this part we use a_n to denote an SA number.

Lemma 3.35. *Let*

$$g(n) = \sigma(n) - n.$$

Then g is increasing for $a_n \in SA$.

Proof. Let $a_n, a_{n+1} \in SA$. By definition of SA numbers (3.8)

$$\frac{\sigma(a_{n+1})}{a_{n+1}} > \frac{\sigma(a_n)}{a_n} > 1, \qquad (n > 1).$$

Therefore,

$$\frac{\sigma(a_{n+1})}{\sigma(a_n)} > \frac{a_{n+1}}{a_n} \Rightarrow \frac{\sigma(a_{n+1})}{\sigma(a_n)} - 1 > \frac{a_{n+1}}{a_n} - 1$$

$$\Rightarrow \sigma(a_n) \left(\frac{\sigma(a_{n+1})}{\sigma(a_n)} - 1 \right) > a_n \left(\frac{a_{n+1}}{a_n} - 1 \right)$$

$$\Rightarrow \sigma(a_{n+1}) - a_{n+1} > \sigma(a_n) - a_n.$$

□

Proposition 3.36. *Let*

$$g(n) = \frac{\sigma(n)^{\sigma(n)}}{n^n}.$$

Then g is increasing for $a_n \in SA$.

Proof. Indeed, definition of SA numbers (3.8) and Lemma 3.35 imply

$$
\begin{aligned}
\frac{\sigma(a_{n+1})^{\sigma(a_{n+1})}}{(a_{n+1})^{a_{n+1}}} &= \left(\frac{\sigma(a_{n+1})}{a_{n+1}}\right)^{\sigma(a_{n+1})} (a_{n+1})^{\sigma(a_{n+1})-a_{n+1}} \\
&> \left(\frac{\sigma(a_n)}{a_n}\right)^{\sigma(a_{n+1})} (a_{n+1})^{\sigma(a_n)-a_n} \\
&> \left(\frac{\sigma(a_n)}{a_n}\right)^{\sigma(a_n)} (a_n)^{\sigma(a_n)-a_n} \\
&= \frac{\sigma(a_n)^{\sigma(a_n)}}{(a_n)^{a_n}}.
\end{aligned}
$$

$\square$

Now we prove a stronger result.

Theorem 3.37. *Let*

$$g(n) = \sigma(n) - n \log \log n.$$

Then g is increasing for large enough $n \in SA$.

Proof. Let n, n' be two consecutive SA numbers. By Lemma 3.10, Proposition 3.13 and inequality (3.12), with $c = 2$, $x = n$, $y = n'$, we obtain

$$
\begin{aligned}
\frac{1}{n'/n - 1}\left(\frac{n'}{n}\frac{\log\log n'}{\log\log n} - 1\right) &\leq 2\frac{\log\log(2n)}{\log\log n} - 1 \\
&< 1 + 2\frac{\log 2}{\log n \log\log n} \\
&< \frac{\log\log 12}{\log\log 6}, \qquad (n \geq 24).
\end{aligned}
$$

This gives

$$\frac{n'}{n} - 1 > \frac{\log\log 6}{\log\log 12}\left(\frac{n'}{n}\frac{\log\log n'}{\log\log n} - 1\right). \tag{3.23}$$

By definition of SA numbers

$$\frac{\sigma(n')}{\sigma(n)} > \frac{n'}{n}.$$

Hence, via (3.23) we derive

$$\frac{\sigma(n')}{\sigma(n)} - 1 > \frac{n'}{n} - 1$$

$$= \frac{\log\log 6}{\log\log 12} \frac{n'\log\log n'}{n\log\log n} - \frac{\log\log 6}{\log\log 12} \frac{n'\log\log n'}{n\log\log n} + \frac{n'}{n} - 1$$

$$> \frac{\log\log 6}{\log\log 12} \left(\frac{n'\log\log n'}{n\log\log n} - 1 \right). \tag{3.24}$$

On the other hand since $\frac{\log\log 12}{\log\log 6} < 1.56077 < e^\gamma$, by Proposition 3.32, for large enough n

$$\sigma(n) > \frac{\log\log 12}{\log\log 6}(n\log\log n). \tag{3.25}$$

Multiplying both sides of (3.24) and (3.25), we get

$$\sigma(n') - \sigma(n) > n'\log\log n' - n\log\log n.$$

Therefore,

$$\sigma(n') - n'\log\log n' > \sigma(n) - n\log\log n.$$

$\square$

Proposition 3.38. *Let*

$$g(n) = \frac{\sigma(n)^{\sigma(n)}}{(n\log\log n)^{n\log\log n}}.$$

Then g is increasing for large enough $n \in SA$.

Proof. By Proposition 3.32 we have for large enough $n \in SA$

$$\sigma(n) > \frac{3}{2}n\log\log n. \tag{3.26}$$

We show that for two consecutive SA n, n'

$$\frac{\sigma(n')^{\sigma(n')}}{(n'\log\log n')^{n'\log\log n'}} > \frac{\sigma(n)^{\sigma(n)}}{(n\log\log n)^{n\log\log n}}.$$

Indeed,

$$\frac{\sigma(n')^{\sigma(n')}}{\sigma(n)^{\sigma(n)}} \frac{(n\log\log n)^{n\log\log n}}{(n'\log\log n')^{n'\log\log n'}} = \left(\frac{\sigma(n')}{\sigma(n)} \right)^{\sigma(n')} \left(\frac{n\log\log n}{n'\log\log n'} \right)^{n'\log\log n'} \tag{3.27}$$

$$\times \left\{ \frac{\sigma(n)^{\sigma(n')-\sigma(n)}}{(n\log\log n)^{n'\log\log n'-n\log\log n}} \right\}.$$

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By Theorem 3.37, the term inside $\{\}$ is greater than 1. Moreover,

$$\left(\frac{\sigma(n')}{\sigma(n)}\right)^{\sigma(n')}\left(\frac{n\log\log n}{n'\log\log n'}\right)^{n'\log\log n'} > \left(\frac{n'}{n}\right)^{\sigma(n')}\left(\frac{n\log\log n}{n'\log\log n'}\right)^{n'\log\log n'}$$
$$= \left(\frac{n'}{n}\right)^{\sigma(n')-n'\log\log n'}\left(\frac{\log\log n}{\log\log n'}\right)^{n'\log\log n'}.$$

However, due to (3.26) the right-hand side of the equality is greater than

$$\left(\frac{n'}{n}\right)^{\frac{1}{2}n'\log\log n'}\left(\frac{\log\log n}{\log\log n'}\right)^{n'\log\log n'}. \tag{3.28}$$

Finally appealing to Lemma 3.11 we conclude that (3.28) is greater than 1. $\square$

Proposition 3.39. *Let $A = \{a_n\}$ be a sequence for which any prime factor of a_n is a prime factor of a_{n+1}, and*

$$g(n) = \frac{n}{\varphi(n)},$$

Then g is increasing for $a_n \in A$.

Proof. If $p(a_{n+1}) = p(a_n)$, it is clear. Let $p(a_{n+1}) \geq p_{k+1} > p_k = p(a_n)$

$$\frac{a_{n+1}/\varphi(a_{n+1})}{a_n/\varphi(a_n)} \geq \frac{1}{1 - 1/p_{k+1}} > 1.$$

$\square$

Proposition 3.40. *Let $\{a_n\}_{n=1}^{\infty} \subset SA$ be such that $p(a_{n+1}) \geq p(a_n)$,*

$$g(n) = \frac{\sigma(n)}{\varphi(n)},$$

then g is increasing for $a_n \in \{a_n\}_{n=1}^{\infty} \subset SA$.

Proof. Using definition of SA number and the previous proposition we have

$$\frac{\sigma(a_{n+1})}{\varphi(a_{n+1})} = \frac{\sigma(a_{n+1})}{\varphi(a_{n+1})} \cdot \frac{a_{n+1}}{a_{n+1}} > \frac{\sigma(a_n)}{a_n}\frac{a_n}{\varphi(a_n)} = \frac{\sigma(a_n)}{\varphi(a_n)}.$$

$\square$

Let $\Psi(n)$ denote Dedekind's arithmetical function of n which is defined by

$$\Psi(n) = n\prod_{p|n}\left(1 + \frac{1}{p}\right), \qquad \Psi(1) = 1,$$

where the product is taken over all primes p dividing n.

Proposition 3.41. *Let $\{a_n\}_{n=1}^\infty \subset SA$ be such that $p(a_{n+1}) \geq p(a_n)$,*

$$\frac{\sigma(n)^{\Psi(n)}}{n^n}.$$

then g is increasing for $a_n \in \{a_n\}_{n=1}^\infty \subset SA$.

Proof. Let $p(a_{n+1}) \geq p(a_n)$. Then

$$\Psi(a_{n+1}) - a_{n+1} > \Psi(a_n) - a_n.$$

So that

$$
\begin{aligned}
\frac{\sigma(a_{n+1})^{\Psi(a_{n+1})}}{(a_{n+1})^{a_{n+1}}} &= \left(\frac{\sigma(a_{n+1})}{a_{n+1}}\right)^{\Psi(a_{n+1})} a_{n+1}^{\Psi(a_{n+1})-a_{n+1}} > \left(\frac{\sigma(a_n)}{a_n}\right)^{\Psi(a_{n+1})} a_{n+1}^{\Psi(a_{n+1})-a_{n+1}} \\
&> \left(\frac{\sigma(a_n)}{a_n}\right)^{\Psi(a_n)} a_{n+1}^{\Psi(a_{n+1})-a_{n+1}} > \left(\frac{\sigma(a_n)}{a_n}\right)^{\Psi(a_n)} a_n^{\Psi(a_n)-a_n} \\
&= \frac{\sigma(a_n)^{\Psi(a_n)}}{(a_n)^{a_n}}.
\end{aligned}
$$

$\square$

3.4.2 CA Numbers

From the definition of CA numbers (3.10) it is easily seen that $CA \subset SA$. Here we describe the algorithm to produce CA numbers. For more details see [11], [3], [26], [64]. Let F be defined by

$$F(x,k) = \frac{\log(1 + 1/(x + \cdots + x^k))}{\log x}. \tag{3.29}$$

For $\varepsilon > 0$, we define x_1 to be the only number such that

$$F(x_1, 1) = \varepsilon,$$

and x_k (where $k > 1$)to be the only number such that

$$F(x_k, k) = \varepsilon.$$

Let

$$E_p = \{F(p, \alpha): \ \alpha \geq 1\},$$

and

$$E = \bigcup E_p = \{\varepsilon_1, \varepsilon_2, \ldots\}.$$

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If $\varepsilon \notin E$, then the function $\sigma(n)/n^{1+\varepsilon}$ attains its maximum at a single point N_ε whose prime decomposition is

$$N_\varepsilon = \prod p^{\alpha_p(\varepsilon)}, \qquad \alpha_p(\varepsilon) = \left\lfloor \frac{\log \frac{p^{1+\varepsilon}-1}{p^\varepsilon-1}}{\log p} \right\rfloor - 1 \qquad (3.30)$$

or if prefer

$$\alpha_p(\varepsilon) = \begin{cases} k, & x_{k+1} < p < x_k, \ k \geq 1; \\ 0, & p > x = x_1. \end{cases}$$

If $\varepsilon \in E$, then by theorem of six exponentials at most two x_k's are prime (see [11], [3], [26], [64]). Hence, there are either two or four CA numbers of parameter ε, is defined by

$$N_\varepsilon = \prod_{k=1}^{K} \prod_{\substack{p < x_k \\ \text{or} \\ p \leq x_k}} p. \qquad (3.31)$$

In fact formula (3.31) gives all possible values of a CA number for a parameter ε in or not in E.

If N is the largest CA number of parameter ε, then

$$F(p, 1) = \varepsilon \Rightarrow p(N) = p, \qquad (3.32)$$

where $p(N)$ is the largest prime factor of N.

It was proved by Robin ([64], Proposition 1) that the maximum order of the function f defined in (3.3) is attained by CA numbers. Using this fact, one has

Proposition 3.42. *Let $3 \leq N < n < N'$, where N and N' are two successive CA numbers. Then*

$$f(n) < \max\{f(N), f(N')\}. \qquad (3.33)$$

Proof. Robin [64, Prop. 1] proved the inequality

$$f(n) \leq \max\{f(N), f(N')\}.$$

But, in fact, due to the strict convexity of the function $t \mapsto \varepsilon t - \log \log t$, Robin's proof naturally extends to the strict inequality (3.33). $\qquad \square$

This fact shows, that if there is any counterexample to (3.2), then there exists at list one CA number which violates it.

Corollary 3.43. *Let $N < N'$ be two consecutive CA numbers. If there exists an XA number $n > 10080$ satisfying $N < n < N'$, then N' is also an XA.*

Proof. Let us set

$$X = \{m \in XA : \ N < m < N'\}.$$

By the assumption $n \in XA$, then we have $X \neq \emptyset$. Let $n' = \max X$. Since $n' \in XA$ and $n' > N$ then $f(n') > f(N)$. From inequality (3.33) we must have $f(n') < f(N')$. Hence $N' \in XA$. $\qquad\square$

Remark 3.44. In the case $N < n = 10080 < N'$, we have $N = 5040$, $N' = 55440$ and

$$f(N) \approx 1.790\,973\,367, \qquad f(n) \approx 1.755\,814\,339, \qquad f(N') \approx 1.751\,246\,515.$$

Hence inequality (3.33) is satisfied with $f(n) < f(N) = \max\{f(N), \ f(N')\}$.

Theorem 3.45. *If RH holds, then there exist infinitely many CA numbers that are also XA.*

Proof. If RH holds, then by Theorem 3.7, $\#XA = \infty$. Let n be in XA. Since $\#CA = \infty$ (see [3], [26]), there exist two successive CA numbers N, N' such that $N < n \leq N'$. If $N' = n$ then it is readily in XA, otherwise N' belongs to XA via Corollary 3.43. $\qquad\square$

It will be seen that there exist infinitely many CA numbers N for which the largest prime factor p is greater than $\log N$. For this purpose, we will use the following

Lemma 3.46 ([11], Lemma 3). *Let N be a CA number of parameter $\varepsilon < F(2,1) = \log(3/2)/\log 2$ and define $x = x(\varepsilon)$ by (3.29). Then*

(i) *for some constant $c > 0$*

$$\log N \leq \vartheta(x) + c\sqrt{x}.$$

(ii) *Moreover, if N is the largest CA number of parameter ε, then*

$$\vartheta(x) \leq \log N \leq \vartheta(x) + c\sqrt{x}.$$

The following lemma is a corollary of Littlewood oscillation for Chebyshev's ϑ function (Corollary 2.14).

Lemma 3.47 ([11]). *There exists a constant $c > 0$ such that for infinitely many primes p we have*

$$\vartheta(p) < p - c\sqrt{p}\log\log\log p, \tag{3.34}$$

and for infinitely many other primes p we have

$$\vartheta(p) > p + c\sqrt{p}\log\log\log p.$$

These results give

Theorem 3.48. *There are infinitely many CA numbers N_ε, such that $\log N_\varepsilon < p(N_\varepsilon)$.*

Proof. We choose p large enough as in (3.34) and N_ε the largest CA number of parameter

$$\varepsilon = F(p, 1).$$

Then, from (3.32), one has $p(N_\varepsilon) = p$. By Lemma 3.46(ii)

$$\log N_\varepsilon - \vartheta(p) < c\sqrt{p}, \qquad \text{(for some } c > 0\text{).}$$

On the other hand, by Lemma 3.47 there exists a constant $c' > 0$ such that

$$\vartheta(p) - p < -c'\sqrt{p}\log\log\log p, \qquad (c' > 0).$$

Hence

$$\log N_\varepsilon - p < \{c - c'\log\log\log p\}\sqrt{p} < 0,$$

and this is the desired result. $\square$

3.4.3 XA Numbers

Returning to XA numbers, here we present some of their properties and describe the structure of these numbers.

Theorem 3.49. *Let $n = 2^{k_2}\cdots p$ be an XA number. Then*

$$p < \log n.$$

Proof. For $n = 10080$ we have

$$p(10080) = 7 < 9.218 < \log(10080).$$

Let $n > 10080$ be an XA number and $m = n/p$. Then $m > 10080$, since for all primes p we have $\vartheta(p) > \frac{\log 2}{2}p > \frac{p}{3}$ (this follows from Corollary 3.12). Therefore,

for a number $n \in SA$ we have $\log n \geq \vartheta(p(n)) > p(n)/3$ and $m = n/p(n) > n/(3\log n) > 10080$ if $n \geq 400\,000$. For $n < 400\,000$ we can check by computation. Hence by Definition 3.4

$$1 + \frac{1}{p} = \frac{\sigma(n)/n}{\sigma(m)/m} > \frac{\log\log n}{\log\log m}.$$

So

$$1 + \frac{1}{p} > \frac{\log\log n}{\log\log m} \Rightarrow \frac{1}{p} > \frac{\log(1 + \log p/\log m)}{\log\log m}.$$

Using inequality (1.12) we have

$$\frac{1}{p} > \frac{\log p}{\log n \log\log m} > \frac{\log p}{\log n \log\log n} \Rightarrow p < \log n.$$

$\square$

We mention a similar result proved by Choie et al. ([13], Lemma 6.1)

Proposition 3.50. *Let $t \geq 2$ be fixed. Suppose that there exists a t-free integer exceeding 5040 that does not satisfy Robin's inequality. Let $n = 2^{k_2} \cdots p$ be the smallest such integer. Then $p < \log n$.*

In the previous section we showed that, if RH holds, then there exist infinitely many CA numbers that are also XA. Next theorem is a conclusion of Theorems 3.48 and 3.49 which is independent of RH.

Theorem 3.51. *There exist infinitely many CA numbers that are not XA.*

We know that by Definition 3.4, for $n \in XA$ the function $\sigma(n)/n$ is strictly increasing and $\phi(n)/n$ is decreasing. Next theorem compares the increase and decrease power by adding these two functions.

Theorem 3.52. *Let*

$$g(n) = \frac{\sigma(n) + \varphi(n)}{n}. \tag{3.35}$$

For two consecutive XA numbers $n = 2^k \cdots p$ and $n' = 2^{k'} \cdots p'$, if $p' \geq p$ and $\log(n'/n) > 1/(3\log p')$, then $g(n) < g(n')$ for large enough n, $n' \in XA$.

Proof. If the largest primes of n and n' are equal, it is clear. Let $p' = p_{k+1} > p_k = p$. If $n > 10080$ is XA, then (see (3.3))

$$f(n) > f(10080) > 1.75.$$

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Using inequality (1.12), Proposition 3.29, Lemma 2.12 and Lemma 2.9, we deduce for large enough n

$$
\frac{\sigma(n')}{n'} + \frac{\varphi(n')}{n'} - \frac{\sigma(n)}{n} - \frac{\varphi(n)}{n}
$$

$$
> \frac{\sigma(n)}{n} \frac{\log \log n' - \log \log n}{\log \log n} - \frac{1}{p_{k+1}} \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)
$$

$$
> 1.75 \log \frac{\log n'}{\log n} - \frac{1}{p_{k+1}} \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)
$$

$$
> 1.75 \frac{\log(n'/n)}{\log n'} - \frac{1}{p_{k+1}} \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)
$$

$$
> 1.75 \frac{\log(n'/n)}{\log n'} - \frac{1}{p_{k+1}} \frac{e^{-\gamma}}{\log p_k} \left(1 + \frac{0.2}{\log^2 p_k}\right)
$$

$$
> 1.75 \frac{\log(n'/n)}{p_{k+1}(1 + \frac{2\eta_1}{\log p_{k+1}})} - \frac{1}{p_{k+1}} \frac{e^{-\gamma}}{\log p_k} \left(1 + \frac{0.2}{\log^2 p_k}\right)
$$

$$
> \frac{1}{p_{k+1}} \left\{ \frac{1.75}{3 \log p_{k+1}(1 + \frac{2\eta_1}{\log p_{k+1}})} - \frac{e^{-\gamma}}{\log p_k} \left(1 + \frac{0.2}{\log^2 p_k}\right) \right\}
$$

$$
> 0.
$$

$\square$

Remark 3.53. We checked that (3.35) (without further assumptions in the theorem) is increasing up to 8150-th element of XA.

Structure of XA

We can describe the structure of XA numbers (for large enough ones). Next theorem will determine the exponents of the prime factors of an XA numbers (for large enough XA) with an error at most 1.

Theorem 3.54. *Let* $n = 2^{k_2} \cdots q^{k_q} \cdots p \in XA$, *and*

$$
\alpha_q(p) = \left\lfloor \log_q \left(1 + (q-1)\frac{p \log p}{q \log q}\right) \right\rfloor. \tag{3.36}
$$

Then for large enough $n \in XA$ *we have* $|k_q - \alpha_q(p)| \leq 1$.

Proof. Let $k_q = k$ and $k - \alpha_q(p) \geq 2$. Then

$$
q^k \geq q^{\alpha_q(p)+2} > q \left(1 + (q-1)\frac{p \log p}{q \log q}\right). \tag{3.37}
$$

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Now compare $f(n)$ with $f(m)$ where $m = n/q$. Since $n \in XA$ we must have

$$\frac{\sigma(n)/n}{\sigma(m)/m} = \frac{q^{k+1} - 1}{q^{k+1} - q} > \frac{\log\log n}{\log\log m},$$

or using inequality (1.12)

$$q^k < 1 + (q - 1)\frac{\log n \log\log m}{q \log q}. \tag{3.38}$$

From (3.37) and (3.38) we get

$$\log n \log\log m - qp \log p > q \log q,$$

and this is a contradiction with (3.20).

Now assume $k - \alpha_q(p) \le -2$. Then

$$\frac{q^{k+2} - 1}{q - 1} \le \frac{p \log p}{q \log q}.$$

Put $m = nq/p$. We show that under the assumption $k - \alpha_q(p) \le -2$ we have that $f(n) < f(m)$ or simply

$$\frac{\sigma(n)/n}{\sigma(m)/m} = \left(1 - \frac{q-1}{q^{k+2} - 1}\right)\left(1 + \frac{1}{p}\right) < 1 + \frac{\log p/q}{\log n \log\log m}.$$

It is enough to show that

$$\frac{1}{p} - \frac{q \log q}{p \log p} < \frac{\log p/q}{\log n \log\log m}. \tag{3.39}$$

If the left-hand side of (3.39) is negative, then clearly the inequality holds. Suppose that the left-hand side is positive. Then by (3.20) we have

$$\frac{\log n \log\log m}{p \log p} < 1 + \frac{(q-1)\log q}{\log p - q \log q}, \qquad (n > n_1).$$

Hence for $n > n_1$

$$\frac{\sigma(n)/n}{\sigma(m)/m} < 1 - \frac{q-1}{q^{k+2} - 1} + \frac{1}{p} < 1 + \frac{\log p/q}{\log n \log\log m} < \frac{\log\log n}{\log\log m},$$

which is a contradiction with the definition of $n \in XA$. $\qquad\square$

We conclude this subsection by the following interesting conjecture.

Conjecture. *Let $n = 2^{k_2} \cdots q^{k_q} \cdots p \in XA$ and $\alpha_q(p)$ is defined by (3.36). Then for all $n \in XA$ we have $|k_q - \alpha_q(p)| \le 1$.*

3.5 Delicacy of the RH

We already proved that under the RH the number of XA numbers are infinite. Here we present an interesting theorem which demonstrates the delicacy of the RH by showing the infinitude of some superset of XA numbers which is defined by an inequality which is quite close to that (i.e., (3.9)) in the definition of XA numbers, independent of RH.

Lemma 3.55. *If $m \geq 3$, then there exists $n > m$ such that*

$$\frac{\sigma(n)/n}{\sigma(m)/m} > 1 + \frac{\log n/m}{\log n \log \log m}. \tag{3.40}$$

Proof. Given $m \geq 3$. Then by (3.5)

$$\frac{\sigma(m)}{m} \leq \left(e^{\gamma} + \frac{0.648214}{(\log \log m)^2}\right) \log \log m, \tag{3.41}$$

Since

$$\frac{\log \log m}{\log \log m'} \left(1 + \frac{\log m'/m}{\log m' \log \log m}\right) < 1$$

and decreasing for $m' > m$ and tends to 0 as m' goes to infinity, then for some $m' > m$ we have

$$\frac{\log \log m}{\log \log m'} \left(1 + \frac{\log m'/m}{\log m' \log \log m}\right) \left(e^{\gamma} + \frac{0.648214}{(\log \log m)^2}\right) = e^{\gamma} - \varepsilon, \tag{3.42}$$

where $\varepsilon > 0$. Hence by Gronwall's theorem there is $n \geq m'$ such that

$$\frac{\sigma(n)}{n} > (e^{\gamma} - \varepsilon) \log \log n$$
$$= \frac{\log \log m}{\log \log m'} \left(1 + \frac{\log m'/m}{\log m' \log \log m}\right) \left(e^{\gamma} + \frac{0.648214}{(\log \log m)^2}\right) \log \log n$$
$$\geq \left(1 + \frac{\log n/m}{\log n \log \log m}\right) \frac{\sigma(m)}{m},$$

where the last inequality holds by (3.41) and (3.42). $\qquad\square$

Definition 3.56. Given $n_1 = 10080$. Let n_{k+1} to be the first integer greater than n_k such that

$$\frac{\sigma(n_{k+1})/n_{k+1}}{\sigma(n_k)/n_k} > 1 + \frac{\log n_{k+1}/n_k}{\log n_{k+1} \log \log n_k}, \qquad (k = 1, 2, \ldots).$$

We define X' to be the set of all $n_1, n_2, n_3, \ldots$.

$$XA \subset X' \subset S. \tag{3.43}$$

Now we are going to state the main theorem of this paper which is the second step towards the delicacy of the RH, i.e.,

Theorem 3.57. *The set X' has infinite number of elements.*

Proof. If the RH is true, then the set X' has infinite elements by (3.43). If RH is not true, then there exists $m_0 \geq 10080$ such that

$$\frac{\sigma(m_0)/m_0}{\sigma(m)/m} > \frac{\log \log m_0}{\log \log m}, \qquad \text{for all} \quad m \geq 10080.$$

By Lemma 3.44 there exists $m' > m_0$ such that m' satisfies the inequality

$$\frac{\sigma(m')/m'}{\sigma(m_0)/m_0} > 1 + \frac{\log m'/m_0}{\log m' \log \log m_0}.$$

Let n be the first number greater than m_0 which satisfies

$$\frac{\sigma(n)/n}{\sigma(m_0)/m_0} > 1 + \frac{\log n/m_0}{\log n \log \log m_0}.$$

Then $n \in X'$. $\qquad \square$

Lemma 3.58. *If $m \geq 3$, then there exists $n > m$ such that*

$$\frac{\sigma(n)/n}{\sigma(m)/m} > 1 + \frac{2 \log n/m}{(\log m + \log n) \log \log m}. \tag{3.44}$$

Proof. The proof is similar to that of Lemma 3.55. $\qquad \square$

Definition 3.59. Given $n_1 = 10080$. Let n_{k+1} to be the first integer greater than n_k such that

$$\frac{\sigma(n_{k+1})/n_{k+1}}{\sigma(n_k)/n_k} > 1 + \frac{2 \log n_{k+1}/n_k}{(\log n_k + \log n_{k+1}) \log \log n_k}, \qquad (k = 1, 2, \ldots).$$

We define X'' to be the set of all $n_1, \ n_2, \ n_3, \ldots$.

It is easily seen that $XA \subset X'' \subset X'$. In a similar method of the proof of Theorem 3.57 one can prove that

Theorem 3.60. *The set X'' has infinite number of elements.*

Note that

$$\#XA = 9240, \qquad \#X'' = 9279, \qquad \#X' = 9535.$$

up to the $300\,000^{th}$ element of S (we used the list of SA numbers tabulated in [58]) and

$$\#(X'' - XA) = 39, \qquad \#(X' - XA) = 295.$$

We list here the elements of $X''\backslash XA$ up to $s_{300\,000}$:

$$
\begin{aligned}
X''\backslash XA = \{ & s_{55}, s_{62}, s_{91}, s_{106}, s_{116}, s_{127}, s_{128}, s_{137}, s_{138}, s_{149}, s_{181}, s_{196}, s_{212}, s_{219}, \\
& s_{224}, s_{231}, s_{232}, s_{246}, s_{247}, s_{259}, s_{260}, s_{263}, s_{272}, s_{273}, s_{276}, s_{288}, s_{294}, \\
& s_{299}, s_{305}, s_{311}, s_{317}, s_{330}, s_{340}, s_{341}, s_{343}, s_{354}, s_{65343}, s_{271143}, s_{271151} \}
\end{aligned}
$$

We conclude this section by formulating another criterion for the RH (using Robin's theorem) with Chebyshev's ψ function.

Proposition 3.61. *The RH is true if and only if*

$$\frac{\sigma(\mathrm{lcm}(n))}{\mathrm{lcm}(n)} < e^{\gamma} \log \psi(n), \qquad (n \geq 11) \tag{3.45}$$

where $\mathrm{lcm}(n) = \mathrm{lcm}(1, 2, \ldots, n)$ *is the least common multiples of the first n positive integers.*

Proof. If the RH is true, then by Robin's theorem inequality (3.45) holds. On the other hand, if the RH is not true, then according to Proposition 2.15 and noting that

$$\frac{\sigma(\mathrm{lcm}(n))}{\mathrm{lcm}(n)} > \frac{\prod_{\frac{1}{2}\sqrt{n}<p\leq n}(1 - 1/p^2)}{\prod_{p\leq n}(1 - 1/p)}, \qquad (n \geq 121).$$

inequality (3.45) does not hold. $\square$

3.6 Numerical Experiments

In this section we give some numerical results mainly for the set of XA numbers up to its 13770-th element, which is less than $C_1 = s_{500\,000}$ (i.e., $500\,000$-th SA number) basing on the list provided by T. D. Noe [58]. We examined Property 3.62 to 3.65 and Remark 3.66 below for the corresponding XA numbers extracted from the list.

Property 3.62. *Let $n = 2^{k_2} \cdots q^{k_q} \cdots r^{k_r} \cdots p$ be an XA number, where $2 \leq q < r \leq p$. Then for $10080 < n \leq C_1$*

(i) $\log n < q^{k_q+1}$,

(ii) $r^{k_r} < q^{k_q+1} < r^{k_r+2}$,

(iii) $q^{k_q} < k_q p$,

(iv) $q^{k_q} \log q < \log n \log \log n < q^{k_q+2}$.

Property 3.63. *Let $n = 2^{k_2} \cdots x_k^k \cdots p$ be an XA number. Then*

$$\sqrt{p} < x_2 < \sqrt{2p}, \qquad for \ \ 10080 < n \leq C_1.$$

Property 3.64. *Let $n = 2^{\alpha_2} \cdots q^{\alpha_q} \cdots p$ and $n' = 2^{\beta_2} \cdots q^{\beta_q} \cdots p'$ be two consecutive XA numbers greater than 10080. Then for $10080 < n \leq C_1$*

$$\alpha_q - \beta_q \in \{-1, 0, 1\}, \qquad for \ all \ \ 2 \leq q \leq p'.$$

Property 3.65. *If m, n are XA and $m < n$, then for $10080 < n \leq C_1$*

(i) $p(m) \leq p(n)$,

(ii) $d(m) \leq d(n)$.

Remark 3.66. We note that Property 3.65 is not true for SA numbers. For example

$$s_{47} = (19\sharp)(3\sharp)^2 2, \qquad s_{48} = (17\sharp)(5\sharp)(3\sharp)2^3, \qquad p(s_{48}) = 17 < 19 = p(s_{47}).$$

and

$$s_{173} = (59\sharp)(7\sharp)(5\sharp)(3\sharp)^2 2^3, \qquad s_{174} = (61\sharp)(7\sharp)(3\sharp)^2 2^2, \qquad \frac{d(s_{173})}{d(s_{174})} = \frac{36}{35} > 1,$$

where s_k denotes k-th SA number.

Using Table of SA and CA numbers in [58] we have

$$\#\{n \in XA : n < C\} = 24\,875,$$
$$\#\{n \in CA : n < C\} = 21\,187,$$
$$\#\{n \in CA \cap XA : n < C\} = 20\,468,$$
$$\#\{n \in CA \setminus XA : n < C\} = 719,$$
$$\#\{n \in XA \setminus CA : n < C\} = 4407,$$

where $C = s_{1000,000}$.

The following properties have been checked up to C_2 (250,000-th element of SA numbers) and for 8150-th element of XA numbers in this domain.

	n	Type	$f(n)$	$p(n)$	$\log n$	k_2
1	$(7\sharp)(3\sharp)2^3 = 10080$	SA	1.75581	7	9.21831	5
2	$(113\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	CA	1.75718	113	126.444	8
3	$(127\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	CA	1.75737	127	131.288	8
4	$(131\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	CA	1.75764	131	136.163	8
5	$(137\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	CA	1.75778	137	141.083	8
6	$(139\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	CA	1.75821	139	146.018	8
7	$(139\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$	CA	1.75826	139	146.711	9
8	$(151\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^3$	SA	1.75831	151	156.039	8
9	$(151\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$	CA	1.75849	151	156.732	9
10	$(151\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.75860	151	158.678	9
11	$(157\sharp)(13\sharp)(5\sharp)(3\sharp)^2 2^4$	SA	1.75864	157	161.788	9
12	$(157\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^3$	SA	1.75866	157	163.041	8
13	$(157\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.75892	157	163.734	9
14	$(163\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.75914	163	168.828	9
15	$(163\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	SA	1.75918	163	171.661	9
16	$(167\sharp)(13\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.75943	167	173.946	9
17	$(167\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.75966	167	176.779	9
18	$(173\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.76006	173	181.933	9
19	$(179\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.76038	179	187.120	9
20	$(181\sharp)(17\sharp)(7\sharp)(3\sharp)^2 2^4$	CA	1.76089	181	192.318	9

Table 3.1: First 10 extremely abundant numbers ($p_k\sharp = \prod_{j=1}^{k} p_j$ is primorial of p)

Property 3.67. *If $n,\ n' \in SA$ are consecutive, then*

$$\frac{\sigma(n')/n'}{\sigma(n)/n} < 1 + \frac{1}{p'}, \qquad (n' < C_2).$$

Property 3.68. *If $n,\ n' \in XA$ are consecutive, then*

$$\frac{n'}{n} > 1 + c\,\frac{(\log\log n)^2}{\log n}, \qquad (0 < c \le 4,\ n' < C_2),$$

$$\frac{n'}{n} > 1 + c\,\frac{(\log\log n)^2}{\sqrt{\log n}}, \qquad (0 < c \le 0.195,\ n' < C_2).$$

Property 3.69. *If $n, n' \in XA$ are consecutive, then*

$$\frac{f(n')}{f(n)} < 1 + \frac{1}{p'}, \qquad (n' < C_2).$$

The number of distinct prime factors of a number n is denoted by $\omega(n)$ ([69]). From Property 3.68 we easily can get

$$g(n) = \frac{n}{\omega(n)}$$

is increasing for $n \in XA$, where $n < C_2$.

Some of the following functions are mentioned in [69].

Property 3.70. *The composition*

$$\sigma\left(n \left\lfloor \frac{\sigma(n)}{n} \right\rfloor\right)$$

is increasing for $n \in SA$, $n < C_2$.

Property 3.71. *Let g be*

$$(1) \ \frac{\sigma(n)^{\varphi(n)}}{n^n} \qquad\qquad (2) \ \frac{\Psi(n)^{\varphi(n)}}{n^n}.$$

Then, g is decreasing for $n \in SA$, $n < C_2$.

Property 3.72. *Let g be*

$$(1) \ \frac{\Psi(n)^{\sigma(n)}}{n^n} \qquad\qquad (2) \ \frac{\varphi(n)^{\sigma(n)}}{n^n}, \qquad (a_n > a_3)$$

$$(3) \ \frac{\varphi(n)^{\Psi(n)}}{n^n}, \qquad (a_n > a_3, \ p(a_{n+1}) \geq p(a_n)).$$

Then, g is increasing for $n \in SA$, $n < C_2$.

Property 3.73. *Let g be each of the following arithmetic functions:*

$$(1) \ \frac{\varphi(n)}{\varphi(\varphi(n))} \qquad\qquad (2) \ \frac{n}{\varphi(\varphi(n))}$$

$$(3) \ d(n)\omega(n) \qquad\qquad (4) \ \omega(\varphi(n)).$$

Then g is increasing for $n \in XA$, $n < C_2$.

Property 3.74. *The compositions*

$$(1) \ \varphi\left(n \left\lfloor \frac{\sigma(n)}{n} \right\rfloor\right) \qquad (2) \ \varphi\left(n \left\lfloor \frac{n}{\varphi(n)} \right\rfloor\right) \qquad (3) \ \varphi\left(n \left\lfloor \frac{\Psi(n)}{n} \right\rfloor\right)$$

are increasing for $n \in XA$, $n < C_2$.

Property 3.75. *Let $g(m) = \operatorname{lcm}(1, 2, \ldots, m)$. Let $n = 2^{k_2} \cdots p \in SA$, then*

$$f(n) > f(g(p)), \qquad (s_{49} < n < C_2).$$

Conclusion and Future Work

During this thesis we tried to work on a number theoretic problem equivalent to the Riemann hypothesis, i.e. Robin's criterion. Some of questions which may be answered are as follows:

- If the Riemann hypothesis is true, then

$$\frac{\partial}{\partial \sigma} |\zeta(s)|^2 < 0, \qquad \text{for} \quad (0 < \sigma < \frac{1}{2}, \ |t| > 6.5).$$

- Let $n = 2^{\alpha_2} \cdots q^{\alpha_q} \cdots p$ and $n' = 2^{\beta_2} \cdots q^{\beta_q} \cdots p'$ be two consecutive extremely abundant numbers greater than 10080. Then for $10080 < n$

$$\alpha_q - \beta_q \in \{-1, 0, 1\}, \qquad \text{for all} \ 2 \le q \le p'.$$

- If the Riemann hypothesis is true, then the previous statement is true.

- If m, n are extremely abundant and $m < n$, then for $10080 < n$

  (i) $p(m) \le p(n)$,

  (ii) $d(m) \le d(n)$.

- If $n, \ n' \in SA$ are consecutive, then

$$\frac{\sigma(n')/n'}{\sigma(n)/n} < 1 + \frac{1}{p'}.$$

- If $n, \ n' \in XA$ are consecutive, then

$$\frac{f(n')}{f(n)} < 1 + \frac{1}{p'}.$$

- The first integer which violates the Robin's inequality belongs to $XA \cap CA$.

- What is the relation between $\vartheta(x^r)$ and $\vartheta(x)^r$, where $r \in \mathbb{R}^+$?

Tables of Estimation for Chebyshev's Function

In Theorem 2.20 we take

$$D = 2500$$

For $b = \log(8 \cdot 10^{11}) \approx 27.4079$, we have $m = 1$, $\delta = 9 \cdot 10^{-6}$ and $\varepsilon_0^* = 2.84888 \cdot 10^{-5}$.

For $b = \log 10^{16} \approx 36.8414$, we have $m = 2$, $\delta = 5.24 \cdot 10^{-8}$ and $\varepsilon_0^* = 4.66629 \cdot 10^{-7}$.

In this case for $b \geq 5213$,

$$\varepsilon_0^* < \Omega_1^* e^{-b/2} + \Omega_3^* + m\delta/2 + \log(2\pi)e^{-b}$$

The values in the Tables 3.3, 3.4, 3.5 are computed

$$|\psi(x) - x| < \eta_k \frac{x}{\log^k x}, \qquad \left(e^{b_1} \leq x \leq x_2 = e^{b_2}\right),$$

where

$$\eta_k = \varepsilon_0^* \log^k x_2, \qquad \left(e^{b_1} \leq x \leq x_2 = e^{b_2}\right).$$

Table 3.2: $|\psi(x) - x| < x\varepsilon_0^*,$ $(x \geq e^b)$, for Theorem 2.20

b	m	δ	ε		b	m	δ	ε
18.42	1	4.78(-4)	1.14790(-3)		900	22	2.08(-12)	2.39881(-11)
18.43	1	4.76(-4)	1.14336(-3)		950	21	2.15(-12)	2.36469(-11)
18.44	1	4.74(-4)	1.13884(-3)		1000	21	2.12(-12)	2.33114(-11)
18.45	1	4.71(-4)	1.13434(-3)		1050	21	2.09(-12)	2.29819(-11)
18.5	1	4.61(-4)	1.11208(-3)		1100	20	2.16(-12)	2.26511(-11)
18.7	1	4.22(-4)	1.02723(-3)		1150	20	2.13(-12)	2.23185(-11)
19.0	1	3.70(-4)	9.11615(-4)		1200	20	2.09(-12)	2.19902(-11)
19.5	1	2.96(-4)	7.46453(-4)		1250	19	2.17(-12)	2.16664(-11)
20	1	2.37(-4)	6.10561(-4)		1300	19	2.13(-12)	2.13331(-11)
21	1	1.52(-4)	4.07253(-4)		1350	19	2.10(-12)	2.10050(-11)
22	1	9.68(-5)	2.70618(-4)		1400	19	2.07(-12)	2.06828(-11)
23	1	6.17(-5)	1.79207(-4)		1450	18	2.14(-12)	2.03552(-11)
24	1	3.93(-5)	1.18314(-4)		1500	18	2.11(-12)	2.00268(-11)
25	1	2.51(-5)	7.79224(-5)		1550	18	2.07(-12)	1.97045(-11)
26	1	1.61(-5)	5.12515(-5)		1600	17	2.15(-12)	1.93836(-11)
27	1	1.06(-5)	3.37385(-5)		1650	17	2.12(-12)	1.90541(-11)
28	1	7.22(-6)	2.23274(-5)		1700	17	2.08(-12)	1.87301(-11)
29	1	5.26(-6)	1.49727(-5)		1750	17	2.05(-12)	1.84126(-11)
30	2	1.26(-6)	9.41428(-6)		1800	16	2.13(-12)	1.80866(-11)
35	2	1.22(-7)	1.05471(-6)		1850	16	2.09(-12)	1.77616(-11)
40	3	7.81(-9)	1.16290(-7)		1900	16	2.05(-12)	1.74427(-11)
45	4	5.60(-10)	1.23408(-8)		1950	15	2.14(-12)	1.71251(-11)
50	7	3.45(-11)	1.30541(-9)		2000	15	2.10(-12)	1.67987(-11)
75	26	2.20(-12)	3.32667(-11)		2100	15	2.02(-12)	1.61646(-11)
100	26	2.18(-12)	3.25398(-11)		2200	14	2.07(-12)	1.55206(-11)
150	26	2.16(-12)	3.13387(-11)		2300	13	2.12(-12)	1.48944(-11)
200	26	2.13(-12)	3.03713(-11)		2400	13	2.04(-12)	1.42535(-11)
250	25	2.18(-12)	2.95752(-11)		2500	12	2.10(-12)	1.36270(-11)
300	25	2.15(-12)	2.88982(-11)		2600	12	2.00(-12)	1.29976(-11)
350	25	2.13(-12)	2.83142(-11)		2700	11	2.06(-12)	1.23732(-11)
400	25	2.10(-12)	2.78000(-11)		3000	10	1.92(-12)	1.05303(-11)
450	24	2.16(-12)	2.73267(-11)		3200	9	1.86(-12)	9.32308(-12)
500	24	2.13(-12)	2.68923(-11)		3500	7	1.89(-12)	7.53761(-12)
550	24	2.10(-12)	2.64897(-11)		3700	6	1.83(-12)	6.39612(-12)
600	23	2.16(-12)	2.61010(-11)		4000	5	1.60(-12)	4.78674(-12)
650	23	2.14(-12)	2.57273(-11)		4500	3	1.23(-12)	2.46504(-12)
700	23	2.11(-12)	2.53666(-11)		4700	2	1.20(-12)	1.77229(-12)
750	22	2.17(-12)	2.50149(-11)		5000	2	6.51(-13)	9.76476(-13)
800	22	2.14(-12)	2.46639(-11)		5100	2	5.34(-13)	8.00754(-13)
850	22	2.11(-12)	2.43220(-11)		5200	2	4.38(-13)	6.56727(-13)

Table 3.3: η_k for the case $|\psi(x) - x| < x\varepsilon_0^*$

b	η_1	η_2	η_3	η_4
18.42	0.0211558	0.389901	7.18587	132.436
18.43	0.0210836	0.388781	7.16912	132.199
18.44	0.0210116	0.387664	7.15241	131.962
18.45	0.0209853	0.388227	7.18220	132.871
18.5	0.0207960	0.388884	7.27214	135.989
18.7	0.0195173	0.370829	7.04574	133.869
19.0	0.0177765	0.346641	6.75951	131.81
19.5	0.0149291	0.298581	5.97162	119.432
20	0.0128218	0.269258	5.65441	118.743
21	0.00895956	0.19711	4.33643	95.4014
22	0.00622421	0.143157	3.29261	75.73
23	0.00430097	0.103223	2.47736	59.4567
24	0.00295785	0.0739463	1.84866	46.2165
25	0.00202598	0.0526755	1.36956	35.6087
26	0.00138379	0.0373624	1.00878	27.2372
27	0.000944678	0.026451	0.740628	20.7376
28	0.000647495	0.0187774	0.544543	15.7918
29	0.000449182	0.0134755	0.404264	12.1279
30	0.0003295	0.0115325	0.403637	14.1273
35	0.0000421884	0.00168754	0.0675015	2.70006
40	5.23306(-6)	0.000235488	0.0105969	0.476863
45	6.17042(-7)	0.0000308521	0.00154261	0.0771303
50	9.79061(-8)	7.34296(-6)	0.000550722	0.0413041
75	3.32667(-9)	3.32667(-7)	0.0000332667	0.00332667
100	4.88096(-9)	7.32145(-7)	0.000109822	0.0164733
150	6.26774(-9)	1.25355(-6)	0.00025071	0.0501419
200	7.59281(-9)	1.8982(-6)	0.000474551	0.118638
250	8.87255(-9)	2.66177(-6)	0.00079853	0.239559
300	1.01144(-8)	3.54003(-6)	0.00123901	0.433654
350	1.13257(-8)	4.53027(-6)	0.00181211	0.724843
400	1.251(-8)	5.62949(-6)	0.00253327	1.13997
450	1.36634(-8)	6.83168(-6)	0.00341584	1.70792
500	1.47907(-8)	8.13491(-6)	0.0044742	2.46081
550	1.58938(-8)	9.5363(-6)	0.00572178	3.43307
600	1.69656(-8)	0.0000110277	0.00716799	4.65919
650	1.80091(-8)	0.0000126064	0.00882445	6.17712
700	1.90249(-8)	0.0000142687	0.0107015	8.02615
750	2.0012(-8)	0.0000160096	0.0128076	10.2461
800	2.09643(-8)	0.0000178197	0.0151467	12.8747
850	2.18898(-8)	0.0000197008	0.0177307	15.9577

Table 3.4: η_k for the case $|\psi(x) - x| < x\varepsilon_0^*$

b	η_1	η_2	η_3	η_4
900	2.27887(-8)	0.0000216493	0.0205668	19.5385
950	2.36469(-8)	0.0000236469	0.0236469	23.6469
1000	2.4477(-8)	0.0000257009	0.0269859	28.3352
1050	2.52801(-8)	0.0000278081	0.0305889	33.6478
1100	2.60488(-8)	0.0000299561	0.0344495	39.617
1150	2.67822(-8)	0.0000321386	0.0385663	46.2796
1200	2.74877(-8)	0.0000343597	0.0429496	53.687
1250	2.81663(-8)	0.0000366162	0.0476011	61.8814
1300	2.87997(-8)	0.0000388796	0.0524875	70.8582
1350	2.94071(-8)	0.0000411699	0.0576378	80.693
1400	2.999(-8)	0.0000434855	0.063054	91.4283
1450	3.05328(-8)	0.0000457993	0.0686989	103.048
1500	3.10416(-8)	0.0000481145	0.0745774	115.595
1550	3.15272(-8)	0.0000504435	0.0807096	129.135
1600	3.1983(-8)	0.000052772	0.0870738	143.672
1650	3.2392(-8)	0.0000550663	0.0936128	159.142
1700	3.27777(-8)	0.000057361	0.100382	175.668
1750	3.31427(-8)	0.0000596569	0.107382	193.288
1800	3.34602(-8)	0.0000619014	0.114518	211.857
1850	3.3747(-8)	0.0000641193	0.121827	231.471
1900	3.40133(-8)	0.0000663259	0.129336	252.204
1950	3.42503(-8)	0.0000685005	0.137001	274.002
2000	3.52773(-8)	0.0000740823	0.155573	326.703
2100	3.55621(-8)	0.0000782366	0.172121	378.665
2200	3.56974(-8)	0.0000821041	0.188839	434.331
2300	3.57465(-8)	0.0000857917	0.2059	494.16
2400	3.56337(-8)	0.0000890842	0.22271	556.776
2500	3.54303(-8)	0.0000921187	0.239509	622.722
2600	3.50936(-8)	0.0000947527	0.255832	690.747
2700	3.71195(-8)	0.000111358	0.334075	1002.23
3000	3.3697(-8)	0.00010783	0.345057	1104.18
3200	3.26308(-8)	0.000114208	0.399727	1399.04
3500	2.8266(-8)	0.000105998	0.397491	1490.59
3750	2.4491(-8)	0.0000979639	0.391855	1567.42
4000	2.01043(-8)	0.0000844381	0.35464	1489.49
4200	1.69978(-8)	0.0000764902	0.344206	1548.93
4500	1.15857(-8)	0.0000544527	0.255928	1202.86
4700	8.86144(-9)	0.0000443072	0.221536	1107.68
5000	4.98003(-9)	0.0000253981	0.12953	660.606
5100	4.16392(-9)	0.0000216524	0.112592	585.48
5200	3.42352(-9)	0.0000178468	0.0930354	484.993

Table 3.5: η_k for the case $|\psi(x) - x| < x\varepsilon_0^*$

b	m	δ	ε	η_1	η_2	η_3	η_4
3800	6	1.67(-12)	5.86122(-12)	2.23312(-8)	0.000085082	0.324163	1235.06
3810	5	1.94(-12)	5.80739(-12)	2.21842(-8)	0.0000847437	0.323721	1236.61
3820	5	1.92(-12)	5.74859(-12)	2.20171(-8)	0.0000843255	0.322967	1236.96
3830	5	1.90(-12)	5.69039(-12)	2.18511(-8)	0.0000839082	0.322207	1237.28
3840	5	1.88(-12)	5.63277(-12)	2.16862(-8)	0.0000834918	0.321443	1237.56
3850	5	1.86(-12)	5.57575(-12)	2.15224(-8)	0.0000830764	0.320675	1237.8
3860	5	1.84(-12)	5.51930(-12)	2.13597(-8)	0.000082662	0.319902	1238.02
3870	5	1.82(-12)	5.46344(-12)	2.11982(-8)	0.0000822488	0.319126	1238.21
3880	5	1.80(-12)	5.40817(-12)	2.10378(-8)	0.0000818369	0.318346	1238.36
3890	5	1.78(-12)	5.35348(-12)	2.08786(-8)	0.0000814265	0.317563	1238.5
3900	5	1.77(-12)	5.29927(-12)	2.07201(-8)	0.0000810158	0.316772	1238.58
3910	5	1.75(-12)	5.24559(-12)	2.05627(-8)	0.0000806058	0.315975	1238.62
3920	5	1.73(-12)	5.19249(-12)	2.04065(-8)	0.0000801975	0.315176	1238.64
3930	5	1.71(-12)	5.13998(-12)	2.02515(-8)	0.000079791	0.314376	1238.64
3940	5	1.70(-12)	5.08798(-12)	2.00975(-8)	0.0000793852	0.313572	1238.61
3950	5	1.68(-12)	5.03642(-12)	1.99442(-8)	0.0000789791	0.312757	1238.52
3960	5	1.66(-12)	4.98545(-12)	1.97922(-8)	0.0000785752	0.311944	1238.42
3970	5	1.64(-12)	4.93509(-12)	1.96417(-8)	0.0000781739	0.311132	1238.31
3980	5	1.63(-12)	4.88504(-12)	1.94913(-8)	0.0000777704	0.310304	1238.11
3990	5	1.61(-12)	4.83561(-12)	1.93424(-8)	0.0000773697	0.309479	1237.92
4000	5	1.60(-12)	4.78674(-12)	1.91948(-8)	0.0000769713	0.308655	1237.71

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