

Dynamic Characterization of Mechanical Components by Inverse Methods: Application to Bolted Joints

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Application to Bolted Joints

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Resumo

Na presente dissertação, componentes mecânicos denominados por juntas aparafusadas são analisados por métodos inversos, realizando-se assim uma caracterização dinâmica da referida junta de ligação. A caracterização dinâmica da junta de ligação realiza-se através de técnicas de desacoplamento que permitem obter as características de uma parte da viga sabendo-se o comportamento dinâmico de toda a viga e das restantes partes que serão desacopladas.

De forma a realizar o desacoplamento, será considerada uma técnica de acoplamento de impedâncias, através da qual provêm as correspondentes técnicas de desacoplamento. As funções de resposta em frequência puras serão obtidas através do Método de Elementos Finitos, sendo posteriormente adicionado ruído de forma a simular dados experimentais. Para realizar a validação das técnicas implementadas, diferentes exemplos provenientes de obras científicas serão utilizados de forma a comprovar resultados.

Com o objetivo de aumentar a eficácia destas técnicas de desacoplamento, introduzir-se-ão procedimentos experimentais tais como cancelamento de massas, análise de gamas de frequência e estimativa de funções de resposta em frequência não medidas. A performance e o impacto destes procedimentos serão avaliados e tidos em conta ao longo do trabalho.

Posteriormente, serão adoptadas três vigas com configurações diferentes de forma a realizar uma análise experimental das mesmas. A aquisição experimental da informação da vigas será realizada de duas formas distintas, medindo todas as *FRFs* em todos os pontos e realizando uma análise modal experimental através de uma linha ou coluna, a partir da qual se obtém o modelo modal da estrutura global e as *FRFs* são obtidas por sintetização.

As *FRFs* dos graus de liberdade de translação serão obtidas experimentalmente, enquanto que as *FRFs* correspondentes às rotações serão obtidas através de uma formulação que utiliza dados experimentais e teóricos provenientes do MEF. Uma vez realizada a caracterização dinâmica de toda a viga, as técnicas de desacoplamento abordadas anteriormente do ponto de vista teórico, serão implementadas com dados experimentais contaminados com ruído.

Abstract

In this dissertation, mechanical components called bolted joints are analysed through inverse methods, carrying out a dynamic characterization of the referred joint. This dynamic characterization makes use of uncoupling techniques which enable a dynamic identification of one part of the beam, knowing *a priori* the dynamic behaviour of the whole beam and the other substructures which will be uncoupled.

In order to carry out the uncoupling of a bolted joint, an impedance coupling technique will be taken into account, whereby the corresponding uncoupling techniques are formulated. The Frequency-response functions will be generated through the Finite Element Method, and then some noise will be introduced to simulate experimental data. To perform the validation of these techniques which were implemented, different examples from scientific works will be done in order to compare the similarity of the results.

Some experimental procedures were implemented with the purpose of enhance the effectiveness of these uncoupling techniques, such as a mass cancelation, an analysis of frequency ranges and an estimative of unmeasured *FRFs*. The performance and the impact of these procedures will be evaluated.

Then, an experimental analysis of three beams with different configurations will be carried out. The experimental work will follow two different ways to obtain the dynamic information of the beam. Firstly, the *FRFs* will be measured in all translation degrees-of-freedom, in order to obtain the experimental matrix. Secondly, several modal analyses will be carried out using a row or a column of the main matrix, from which all the *FRFs* will be synthesised.

The *FRFs* of the translation degrees-of-freedom are obtained experimentally, whereas the *FRFs* corresponding to the rotations are obtained through a formulation which uses experimental and theoretical data from the *MEF*. Once carried out the dynamic characterization of the whole beam, the uncoupling techniques, approached previously under the theoretical point of view, will be analysed with experimental data contaminated with noise.

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Nomenclature

Abbreviations

FRF	Frequency-Response Function
DOF	Degree-of-freedom
MAC	Modal Assurance Criterium
LAC	Local Amplitude Criterium
i	Internal coordinate
j	Joint coordinate
k	Internal joint coordinate

Matrix Variables

B	Deformation matrix
L	Matrix of the differential operators
M	Mass matrix
K	Stiffness matrix
$\bar{K}$	Complex stiffness matrix
N	Shape functions matrix
$(H)^+$	Pseudo-Inverse
$Z(\omega)$	Dynamic Stiffness
$\tilde{Z}(\omega)$	Condensated Dynamic Stiffness
$H(\omega)$	FRF matrix
H_{ij}^X	FRF matrix of structure X with coordinates i and j
Φ	Modal matrix

Scalar Variables

E	Young's modulus
I	Inertia Moment
ρ	Specific mass
A	Area of a surface
l	Length of a beam
ν	Poisson coefficient
η_r	Damping coefficient associated to a natural frequency
v	Weight parameter of the algorithm of frequency analysis
ω_i	Frequency of excitation
r_i	Amplification of matrix condition
R_i	Approximation of r_i

Vector Variables

$H(\omega)$	Frequency-Response Function
f_i^c	Load applied in substructure C in coordinate i
x_i^c	Displacement of coordinate i in substructure i
$H_{pq}(\omega)$	FRF of a structure excited in point q and response measured in point p
$\tilde{H}_{pq}(\omega)$	FRF with noise of a structure excited in point q and response measured in point p
$H_A(\omega)$	FRF obtained from a numerical model
$H_X(\omega)$	FRF obtained from a experimental analysis
$\{\Psi_A\}_i$	Modal vector of mode i of the numerical model
$\{\Psi_X\}_j$	Modal vector of mode j experimentally measured

Chapter 1

Introduction

The conception and development of mechanical projects involves always joints to carry out the connection between different sub-structures. The two main joints types widely used in mechanical projects divide into bolted and adhesive joints.

The bolted joints are the most used in mechanical construction, whereas adhesives have been improving their mechanical behavior along last decades although their application field is still a little limited. Normally, bolted joints provide a stronger mechanical connection, supporting high loads with good performances, being still the most used currently in main structures. On the other hand, adhesives joints have an application field more restrict than bolted joints, these have to be built with a determined size and thickness depending on the place where the connection joint will be applied. Therefore, the adhesive joints have their great applications in secondary substructures where there could be lower loads and the work conditions are not so demanding.

A bolted joint could be regarded as a discontinuity in a structure which introduces high stress concentrations, changing the structure features and its behavior. These connection joints could have a big influence in the system natural frequencies and in the vibration modes, causing non-linear responses which are undesirable and too complicated to analyse and understand. The non-linearities in the joint interface are directly related to force-displacement relations. The added flexibility introduced by the joints heavily affects the structure behaviour and when subjected to dynamic loading, most of the energy is dissipated in the joints.

The major source of damping in bolted joints comes from friction. When different substructures are assembled, we get a system which has higher damping values in comparison with individual members, and as this increase of damping is not caused by the members, some energy must be dissipated in the contact areas between the members when exposed to dynamic load.

Due to these problems introduced by connection joints, there is the necessity of develop new methodologies to build structural models which could characterize physically these type of joints and make easier and predictable the comprehension of their behaviour.

Nowadays, the analysis performed in the project of mechanical components are done using the *Finite Element Method* (MEF), which provides statics and dynamics analysis. This method makes estimations of natural frequencies and vibration modes, being possible the modeling of components with various shapes and sizes.

The finite element method has an important role in the project of mechanical systems, serving as a prediction tool of dimensioning errors and high stress concentration zones in an anticipated working phase, performing a better knowledge about the prototype before its construction. Unfortunately, this method does not enable one approach about the dispersion or uncertainty of bolted joints which is important to understand once these joints have a great impact on the main structures behavior, namely, modifying the natural frequencies and the vibration modes, making impossible the acquisition of information of the physical properties of global structure. In this work, this method has a vast importance, once all the theoretical solutions are based on the assumptions of such method.

Three innovative methodologies will be implemented in this work in order to improve the analysis of uncoupling techniques which will be approached throughout this work:

- Mass Cancellation;
- Estimative of Unmeasured *FRFs*.
- Analysis of Frequency Ranges.

These methodologies have an high impact and importance in the coherence and accuracy of results once without means such these, the final results would be even worse. The mass cancellation process allows to take into account the influence of the accelerometer's masses in the response behaviour, such as in response amplitude and natural frequencies. The estimative of unmeasured *FRFs* provides the achievement of responses which can not be experimentally measured. Indeed, this process has an high importance once experimentally most of *FRFs* involving rotational degrees-of-freedom are difficult or even impossible to measure or analyse. The analysis of frequency ranges enables the knowledge of the frequencies where the noise level is lower and consequently a better linear regression could be performed to determine the corresponding dynamic stiffness.

1.1 Motivation

This dissertation comes from another works done in the same area which promoted and raised problems and issues related to analysis of bolted joints once those researchers were not able to model these elements. For instance, a separated modal analysis of two substructures provides reliable results with modal shapes well defined and exact natural frequencies. When these two substructures are coupled using bolted joints to perform their connection, if another modal analyse is done, the results show an high discrepancy and lack of accuracy. Therefore, bolted joints are the cause of all this discrepancy and their behaviour needs to be studied and analysed to enable a better knowledge about their interference in the global system behaviour.

Therefore, this dissertation consists on a deep analysis of several uncoupling techniques and an attempt to introduce any modification or innovation which gives better results than those that have been achieved until the time.

1.2 Objectives

The analysis of bolted joints has recognized problems which appear when these identification methods are implemented. The main objective of this work is to analyse, develop, apply and discuss identification techniques of bolted joints. Initially, in this work all the data necessary to test and validate the uncoupling techniques will be generated by the *Finite Element Method*, and then the global structure data will be replaced by experimental data which include noise and measurement errors.

The introduction of experimental measured data requires the comprehension of problems related with experimental methods and devices used to process these data. These problems focus on questions related with numerical calculation, difficulties in the experimental data measurement and in the comparison between numerical and experimental results.

Throughout this work some techniques will be approached in order to analyse the different data obtained:

- Experimental Analysis Techniques;
- Numerical methods, implementation of calculation programmes;
- Coupling and Uncoupling Techniques;
- New Experimental Methodologies;
- Experimental Modal Analysis.

The joint dynamic properties will be identified with uncoupling techniques. Four different uncoupling techniques will be considered with the purpose to evaluate and verify which provides better results when exposed to high noise levels and other experimental errors.

In addition, this work is carried out with the purpose of finding solutions to problems related with numerical uncertainties adjacent to matrix operations such inversions and differences. Furthermore, when noise is introduced, this problem is still more visible than without noise and numerical discrepancy increases a lot. This problem is intrinsic to experimental devices and to the process of experimental data acquisition.

Then, an experimental work will be performed taking into account beams with bolted joints incorporated. One of most important objectives of this work is to apply all theoretical formulations to experimental data, replacing thus the origin of these data. Firstly, the global *FRF* matrix will be measured to achieve the beam dynamic characterization, and then we will measure a

row or a column in order to perform an experimental modal analysis whereby all the desired *FRFs* are synthesized in any points.

1.3 Thesis Structure

This dissertation will be essentially divided into two main parts, a theoretical part where all the formulations will be approached and implemented using a calculation software, *MATLAB*, and then, an experimental part in which some specimens will be analysed with the experimental means available.

In the first chapter, *Introduction*, a global contextualization of this dissertation will be carried out, such as a review of all work already done in the application field of this dissertation, as well as a detailed description of the different subjects which will be approached along the corresponding work.

In the second chapter, an extensive theoretical part will be presented with some numerical examples which will provide important information about the uncoupling techniques approached, trying to predict their behavior under experimental environments. This part of the work has an high importance once all the work developed here will be taken into account in the experimental measurements.

In the third chapter, some methodologies will be introduced to improve the uncoupling techniques behaviour and analyse the corresponding problems which appear when these techniques are implemented.

In the fourth chapter, some experimental measurements will be carried out using some aluminum specimens, previously studied and analysed. There will be considered three different aluminum specimens, a beam without any joint, a beam with a bolted joint and another beam which only has the length of the section considered to the bolted joints in previous beams. Here, the global *FRF* matrix is obtained carrying out measurements in all degrees-of-freedom.

In the fifth chapter, several experimental modal analysis will be carried out in order to evaluate another way to identify the bolted joint, measuring only a row or a column of the global *FRF* matrix.

Lastly, the conclusions of this dissertation will be presented, as well as some suggestions for future works in the field of interest.

1.4 State of the Art - Review

Over the last few years, lots of research in this scientific field has been done and some developments have been achieved. As said previously, this dissertation comes from another works

performed in the same area and the results of that researching will be taken into account in this work.

According to the different fields of this work, in each of the following sections is presented a summary of the work already published.

1.4.1 Bolted Joints

Bolted joints are one of the most common elements in project and mechanical construction performing several functions which have an high importance in mechanical systems. This system is essentially composed by a bolt and a nut which connect two different parts of a mechanical system. A bolted joint holds two or more parts together to form an assembly in a mechanical structure. Both elements are thread and therefore the surface of these components undergoes complex loads which cause nonlinear behaviour, normally very complicated or even impossible to analyse.

In terms of dimensioning, bolted joints are a surprisingly complicated matter, having many factors that affect the results when the assembly of a joint is carried out. The design of most joints is supported by experience and usually over dimensioned. The joint members are stronger and the bolts larger than would be in the case if we could control assembly and predict behavior more accurately.

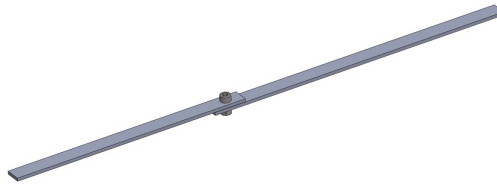


Figure 1.1: Beam with a bolted joint.

Mayer and Gaul [1] presented a new approach to model contact interfaces of joints using the finite element method with segment-to-segment contact elements like thin layer or zero thickness elements. This approach provides the simulation of multi-degree-of-freedom systems with structural joints and a preliminary knowledge of the modal damping properties.

A deep research using bolted joints was done by Batista [2]. In this work, coupling and uncoupling are performed with the purpose of identify the dynamic features of a bolted joint.

1.4.2 Coupling / Uncoupling Techniques

The coupling and uncoupling techniques have an important role in the project of mechanical components, allowing the knowledge of dynamic properties of different components and

subcomponents. Over the years many actors have been doing research in this field and in consequence these techniques have improved and their performance provides better results with much more accuracy.

An approach to this subject was performed by Nóbrega [3], who had studied two different modeling techniques for obtaining the dynamic model of a structure. A transfer gearbox and its components were analysed by experimental modal analysis, having shown good correlation with the experimental results. Two dynamic coupling techniques were developed and implemented, using a discrete system model. These two techniques did not provide satisfactory results.

An impedance uncoupling technique based on the classic coupling expression was approached by Maia et al. [4], in which there is no need to measure *FRFs* at the joint. In this article the problem related with difference and inversion of matrices is analysed, in order to understand its influence in the final results, principally when noise is introduced in results obtained numerically; this problem varies along the frequency range.

Jalali et al. [5] studied the behaviour of nonlinear joints identifying the corresponding parameters by force-state mapping from time-domain acceleration records in response to single frequency excitation close to the first natural frequency.

Kim et al. [6] investigated a new modeling technique with bolted joints, considering four different types of finite element models. The finite element models analysed took into account the pretension effect and the contact behaviour between the flanges to be joined. A large marine engine was analysed by this methodology.

Celtic and Boltezar [7] presented a methodology to establish a model of a joint from the substructures and assembly *FRFs* data, considering not only the translation degrees-of-freedom, but also the rotations ones. The method was validated experimentally and numerically through the analysis of a substructure.

The same authors [8] approached the influence of the coordinate reduction on the identification of a joint from measured frequency-response function (*FRF*). The theoretical model was numerically validated taking into account the effect of added noise.

A new methodology to identify joint dynamic properties with partially measured frequency response functions was proposed by Wang et al. [9]. Furthermore, another methodology was presented for estimate the unmeasured *FRFs* to perform the identification process. A numerical example was performed to analyse the effectiveness of this method.

Batista [2] performed a dynamic characterization of bolted joints. In this work different coupling and uncoupling techniques were considered, as well as methods to improve the experimental *FRFs* data such as mass cancellation and estimative of unmeasured frequency response functions. Several numerical examples were taken into account to analyse the performance of those methods.

Two new joint identification techniques were investigated by Mehrpouya et al. [10]. An inverse receptance coupling (*IRC*) method and a point-mass joint model were introduced, considering the joint as lumped mass, damping and stiffness matrices. The dynamic properties of

the joint were investigated using both methods through a finite element simulation and experimental tests.

1.4.3 Correlation Criteria

The comparison between numerical and experimental results requires criteria to quantify the similarity or difference between results in order to provide an idea of the results accuracy. Several correlation indices have been proposed along the years to compare similar *FRFs*.

Grafe [11] proposed a correlation criterion to compare two similar *FRFs*, the Local Amplitude Criterium, *LAC*, defined as:

$$LAC(\omega, i, j) = \frac{2 | H_{Xij}(\omega)^* H_{Aij}(\omega) |}{(H_{Xij}(\omega)^* H_{Xij}(\omega)) + (H_{Aij}(\omega)^* H_{Aij}(\omega))} \quad (1.1)$$

The indices i and j are the response and excitation coordinates, respectively. H_{Xij} is the *FRF* obtained experimentally, whereas H_{Aij} is the *FRF* obtained numerically, for each frequency ω . The apostrophe $*$ corresponds to the complex conjugate. This correlation criterion varies between 0 and 1 for each frequency ω . If $LAC(\omega, i, j)$ has values close to 1, H_{Xij} is closely related to H_{Aij} . The average result along the frequency range could be determined, providing an unique value for the correlation between both *FRFs*:

$$\overline{LAC_{ij}} = \frac{1}{N} \sum_{k=1}^N LAC_{ij}(\omega_k) \quad (1.2)$$

Another correlation criterium will be used in this work to compare the natural shapes obtained in a modal analysis with experimental data. Allemang and Brown [12] proposed the *Modal Assurance Criterion*, *MAC*, which is given by the following expression:

$$MAC(i, j) = \frac{|\{\Psi_A\}_i^H \{\Psi_X\}_j|^2}{(\{\Psi_A\}_i^H \{\Psi_A\}_i)(\{\Psi_X\}_j^H \{\Psi_X\}_j)} \quad (1.3)$$

The vector $\{\Psi_A\}_i$ corresponds to the mode i of the numerical model, whereas $\{\Psi_X\}_j$ represents the mode j experimentally identified and the superscript H is the hermitian operator. The correlation varies between 0 to 1.

1.4.4 Estimative of unmeasured Frequency-Response Functions

The measurement of some *FRFs* owns experimental limitations which affect the corresponding experimental procedures, and posteriorly the numerical analyses and data processing and

treatment. For instance, there are some points where the measurement could not be carried out due to any reasons and some *FRFs* types like the rotation ones are complicated or even impossible to determine once there are not still devices capable of perform these measurements. Therefore, there is the need of generate and find valid solutions to these unavailable *FRFs*.

Some authors have done during the last years some work in this subject and interesting formulations have been achieved. However, these formulations still possess an high uncertainty associated, mainly experimental and numerical errors.

Maia and Batista [13] proposed a new methodology for the estimation of frequency response functions which for some reason cannot be measured. This method is based on the classic coupling technique and the expressions obtained come from the coupling standard equation. The estimation of *FRFs* involving rotational degrees-of-freedom is now possible, generally impossible or difficult to measure and predict.

Wang et al. [9] presented a new methodology to identify joint dynamic properties and joint structures with partially measured frequency response functions (*FRFs*). The unmeasured *FRFs* of an unconstrained structure are obtained through modal calibration, using partially measured *FRFs* of the assembled structure calculated through the finite elements method.

Silva et al. [14] developed a technique for the cancellation of mass-loading effects. This process uses an uncoupling technique to perform the mass cancellation associated with the use of transducers which imply changes in the measured *FRF*.

Chapter 2

Coupling / Uncoupling

2.1 Introduction

The coupling and uncoupling of structures and corresponding substructures possess an high importance in several fields of mechanical engineering, permitting the knowledge of some sub-components' features from determined components through the analysis of the system's *FRFs*. In this chapter, different coupling and uncoupling techniques of structures are considered and analysed, and along this work these techniques will be tested through analyses and comparison of problems with results from articles and other works done in order to validate the implemented formulations.

The techniques of dynamic coupling divide essentially into three main groups:

- Impedance Coupling;
- Modal Coupling;
- Spatial Coupling.

The impedance coupling techniques or *FRF* techniques use the frequency responses function of subcomponents, obtaining as result the *FRFs* of the global structure.

The modal coupling techniques make use of the modal model properties of several subcomponents, and therefore this a problem of eigen values which needs a great number of coordinates in order to build a useful model. The order of the coupled system does not depend on the number of coordinates used necessarily, but on the number of modes used in each component.

The spatial coupling techniques use matrices which define the substructure properties to analyse, namely mass, stiffness and damping matrices. Figure 2.1 relates the three types of dynamic coupling. In fact, all the types enable the acquisition of dynamic properties of a structure with different ways. Generally, the spatial coupling is associated with a numerical way whereas the

impedance coupling relates to an experimental way; having the information about the spatial model or the impedance model, it is possible to obtain the modal model and consequently the dynamic properties of a structure.

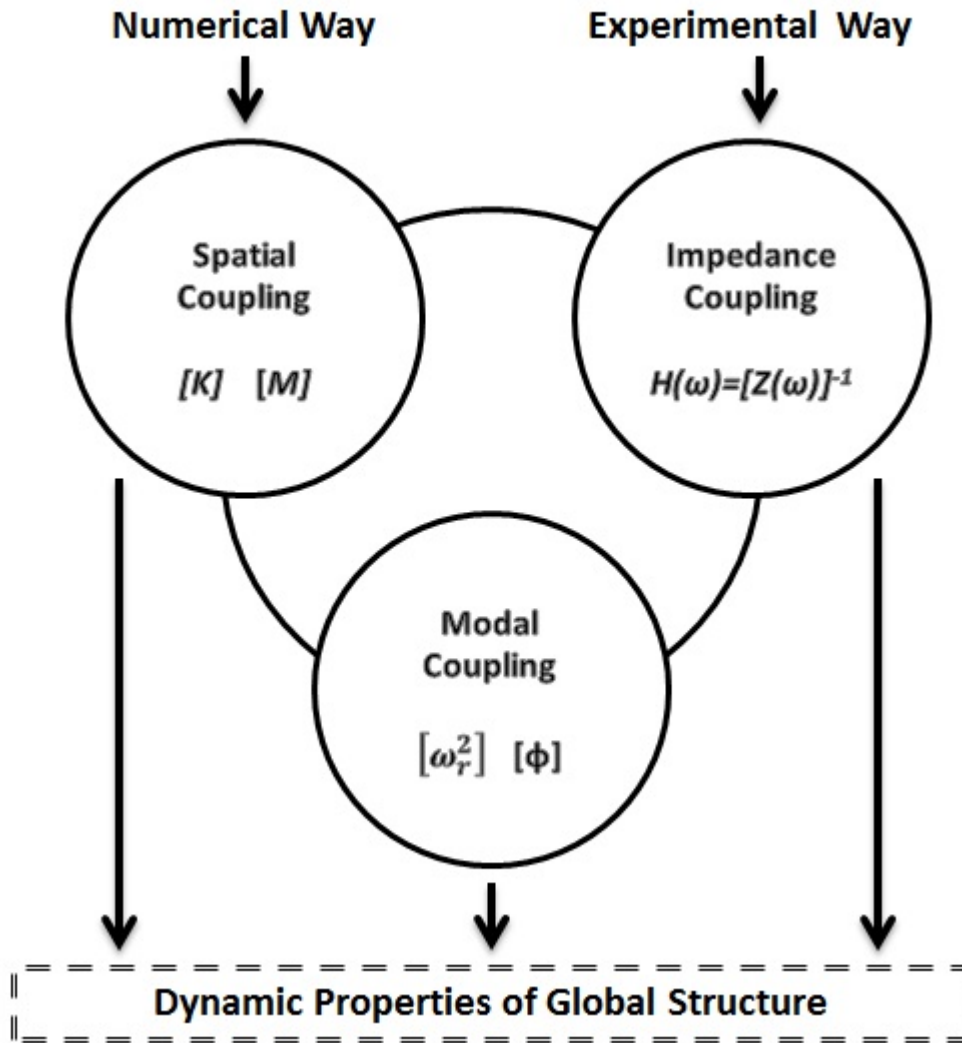


Figure 2.1: Coupling techniques.

In this work, several impedance coupling techniques will be approached owing to the facility associated with experimental and theoretical acquisition of *FRFs* and corresponding data treatment using a calculation software, *MATLAB*.

The coupling process of subcomponents is normally a process more stable than the substructures uncoupling, principally due to matrix operations carried out in this process which interfere somewhat in the results obtained. Associated to these numerical problems is the noise introduced in experimental measurements by devices used to perform the measurements. Unfortunately, it is completely impossible to avoid the noise introduced by experimental measurements and eventually the only option to reduce noise levels comes by a technological improvement of such devices.

2.2 Coupling Techniques

A coupling technique involves a connection between two structures in specific coordinates, assuming that the dynamic properties of both structures involved are known and well defined. The coupling of substructures A and B (Figure 2.2) can be performed with two different coupling techniques:

- Classic Coupling Technique;
- Alternative Coupling Technique.

The theoretical assumptions of these two techniques will be analysed in the following sections of this chapter, as well as the numerical expressions which define both methods.

2.2.1 Classic Coupling Technique

The classic coupling technique takes into account three different types of coordinates, as presented in figure 2.2. These coordinates are characterized as:

- coordinates i , corresponding only to substructure A;
- coordinates j , corresponding to the connection joint between substructures A and B;
- coordinates k , corresponding only to substructure B.

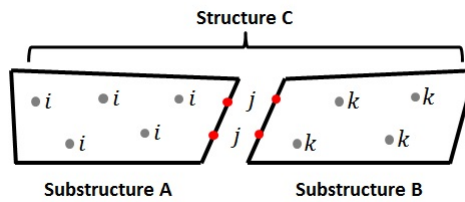


Figure 2.2: Coordinates to carry out the classic coupling.

This method assumes that the coupling of two substructures, A and B, is rigidly connected through the coupling coordinates j , whereby the efforts are transmitted [2]. In this work, substructure B corresponds to the bolted joint whereas structure A represents the other two parts of the beam without physical connection.

The assumptions of this technique focus on the following equations of force equilibrium and on the conditions of displacements compatibility [13]:

$$\begin{cases} f_j^A + f_j^B = f_j^C \\ x_j^A = x_j^B = x_j^C \end{cases} \quad (2.1)$$

The accelerance matrix H which relate the structure accelerations with the dynamic loads could be defined in the following way:

$$\bar{A}(\omega) = HF(\omega) \quad (2.2)$$

The accelerance matrices of substructures A and B and structure C will be:

$$H^C = \begin{bmatrix} H_{ii}^C & H_{ij}^C & H_{ik}^C \\ H_{ji}^C & H_{jj}^C & H_{jk}^C \\ H_{ki}^C & H_{kj}^C & H_{kk}^C \end{bmatrix} \quad H^A = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad H^B = \begin{bmatrix} H_{jj}^B & H_{jk}^B \\ H_{kj}^B & H_{kk}^B \end{bmatrix}$$

Relating the forces equilibrium equations and the displacements compatibility conditions with the accelerance definition, the accelerance matrix H^C is obtained.

$$H^C = \left[\begin{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{bmatrix} H_{jj}^B & H_{jk}^B \\ H_{kj}^B & H_{kk}^B \end{bmatrix}^{-1} \end{bmatrix} \right]^{-1} \quad (2.3)$$

The inversion process of the three matrices requires an high computational effort and furthermore, there is a high probability of having bad-conditioned matrices near the resonances and anti-resonance picks, which generate enormous numerical errors in their inverses. Therefore, the following matrix manipulation is performed:

$$(H^C)^{-1} = \left[\begin{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & \begin{bmatrix} H_{jj}^B & H_{jk}^B \\ H_{kj}^B & H_{kk}^B \end{bmatrix}^{-1} \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \right] \quad (2.4)$$

Simplifying the second member of equation 2.4:

$$(H^C)^{-1} = (H_{\#}^A)^{-1} + (H_{\#}^B)^{-1} - I_{\#} \quad (2.5)$$

Multiplying in equation 2.5 at left by $(H_{\#}^A)^{-1}H_{\#}^A$ and at right by $(H_{\#}^B)^{-1}H_{\#}^B$, comes:

$$(H^C)^{-1} = (H_{\#}^A)^{-1} + (H_{\#}^A + H_{\#}^B - H_{\#}^A I_{\#} H_{\#}^B)(H_{\#}^A)^{-1} \quad (2.6)$$

Thus, H^C could be calculated with one inversion only:

$$H^C = H_{\#}^B (H_{\#}^A + H_{\#}^B - H_{\#}^A I_{\#} H_{\#}^B)^{-1} H_{\#}^A \quad (2.7)$$

Calculating the content inside the parenthesis:

$$H^C = \begin{bmatrix} I & 0 & 0 \\ 0 & \begin{bmatrix} H_{jj}^B & H_{jk}^B \\ H_{kj}^B & H_{kk}^B \end{bmatrix} \\ 0 & \begin{bmatrix} H_{jj}^B & H_{jk}^B \\ H_{kj}^B & H_{kk}^B \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & H_{ij}^A & 0 \\ 0 & H_{jj} & H_{jk}^B \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} & 0 \\ 0 & 0 & I \end{bmatrix} \quad (2.8)$$

$$H_{jj} = H_{jj}^A + H_{jj}^B \quad (2.9)$$

The inversion of the second matrix in the second member can be written in the form:

$$\begin{bmatrix} I & H_{ij}^A & 0 \\ 0 & H_{jj} & H_{jk}^B \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -H_{ij}^A(H_{jj})^{-1} & H_{ij}^A(H_{jj})^{-1}H_{jk}^B \\ 0 & (H_{jj})^{-1} & -(H_{jj})^{-1}H_{jk}^B \\ 0 & 0 & I \end{bmatrix} \quad (2.10)$$

Replacing the equation 2.10 in equation 2.8 and carrying out the multiplication of the three matrices of the second member, gives:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C & H_{ik}^C \\ H_{ji}^C & H_{jj}^C & H_{jk}^C \\ H_{ki}^C & H_{kj}^C & H_{kk}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A - H_{ij}^A(H_{jj})^{-1}H_{ji}^A & H_{ij}^A - H_{ij}^A(H_{jj})^{-1}H_{jj}^A & H_{ij}^A(H_{jj})^{-1}H_{jk}^B \\ H_{ji}^B(H_{jj})^{-1}H_{ji}^A & H_{jj}^B(H_{jj})^{-1}H_{jj}^A & H_{jk}^B - H_{jj}^B(H_{jj})^{-1}H_{jk}^B \\ H_{kj}^B(H_{jj})^{-1}H_{ji}^A & H_{kj}^B(H_{jj})^{-1}H_{jj}^A & H_{kk}^B - H_{kj}^B(H_{jj})^{-1}H_{jk}^B \end{bmatrix} \quad (2.11)$$

Simplifying the expression 2.11,

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A & 0 \\ H_{ji}^A & H_{jj}^A & 0 \\ 0 & 0 & H_{kk}^B \end{bmatrix} - \begin{bmatrix} H_{ij}^A(H_{jj})^{-1}H_{ji}^A(H_{jj})^{-1} & H_{ij}^A(H_{jj})^{-1}H_{jj}^A & -H_{ij}^A(H_{jj})^{-1}H_{jk}^B \\ H_{ji}^A - H_{jj}^B(H_{jj})^{-1}H_{ji}^A & H_{jj}^A - H_{jj}^B(H_{jj})^{-1}H_{jj}^A & H_{jj}^B(H_{jj})^{-1}H_{jk}^B - H_{jk}^B \\ -H_{kj}^B(H_{jj})^{-1}H_{ji}^A & -H_{kj}^B(H_{jj})^{-1}H_{jj}^A & H_{kj}^B(H_{jj})^{-1}H_{jk}^B \end{bmatrix} \quad (2.12)$$

The elements of the second line of the second matrix are constituted by differences of matrices, so it is possible to simplify these three elements:

$$\begin{aligned} H_{ji}^A - H_{jj}^B(H_{jj}^A + H_{jj}^B)^{-1}H_{ji}^A &= (I - H_{jj}^B(H_{jj}^A + H_{jj}^B)^{-1})H_{ji}^A = \\ &= ((H_{jj}^A + H_{jj}^B)(H_{jj}^A + H_{jj}^B)^{-1} - H_{jj}^B(H_{jj}^A + H_{jj}^B)^{-1})H_{ji}^A = \\ &= ((H_{jj}^A + H_{jj}^B - H_{jj}^B)(H_{jj}^A + H_{jj}^B)^{-1})H_{ji}^A = H_{jj}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{ji}^A \end{aligned}$$

With a similar process, we can also obtain the following results to the other elements of second line:

$$H_{ji}^A - H_{jj}^B(H_{jj})^{-1}H_{ji}^A = H_{jj}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{jj}^A \quad (2.13)$$

$$H_{jj}^B(H_{jj})^{-1}H_{jk}^B - H_{jk}^B = -H_{jj}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{jk}^B \quad (2.14)$$

Replacing 2.13 and 2.14 in 2.11 results:

$$H^C = \begin{bmatrix} H_{ii}^A & H_{ij}^A & 0 \\ H_{ji}^A & H_{jj}^A & 0 \\ 0 & 0 & H_{kk}^B \end{bmatrix} - \begin{bmatrix} H_{ij}^A(H_{jj})^{-1}H_{ji}^A & H_{ij}^A(H_{jj})^{-1}H_{jj}^A & -H_{ij}^A(H_{jj})^{-1}H_{jk}^B \\ H_{jj}^A(H_{jj})^{-1}H_{ji}^A & H_{jj}^A(H_{jj})^{-1}H_{jj}^A & -H_{jj}^A(H_{jj})^{-1}H_{jk}^B \\ -H_{kj}^B(H_{jj})^{-1}H_{ji}^A & -H_{kj}^B(H_{jj})^{-1}H_{jj}^A & H_{kj}^B(H_{jj})^{-1}H_{jk}^B \end{bmatrix} \quad (2.15)$$

Simplifying the expression 2.15 in the following form:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C & H_{ik}^C \\ H_{ji}^C & H_{jj}^C & H_{jk}^C \\ H_{ki}^C & H_{kj}^C & H_{kk}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A & 0 \\ H_{ji}^A & H_{jj}^A & 0 \\ 0 & 0 & H_{kk}^B \end{bmatrix} - \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \\ -H_{kj}^B \end{bmatrix} (H_{jj}^A + H_{jj}^B)^{-1} \begin{bmatrix} H_{ji}^A & H_{jj}^A & -H_{jk}^B \end{bmatrix} \quad (2.16)$$

All these mathematical manipulations were performed with the objective of reduce the computational effort, making less matrices inversions and reducing the errors introduced by these operations. In expression 2.16 there is just one matrix inversion, restricting thus possible numerical problems.

2.2.2 Alternative Coupling Technique

The alternative coupling technique considers only the intern coordinates i of substructure A and the joint coordinates j of substructure B. In many situations, the intern coordinates of substructure B does not have interest or sometimes these coordinates have a difficult access.

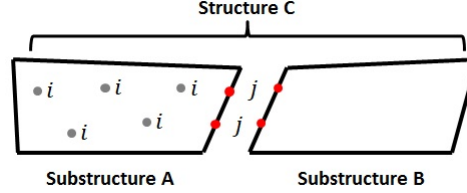


Figure 2.3: Coordinates to the realisation of alternative coupling

The dynamic stiffness of structure C could be written as the sum of the dynamic stiffness of structure A with the dynamic stiffness of substructure B:

$$Z^C = Z^A + Z^B \quad (2.17)$$

Collocating on evidence the matrix Z^A in the second member of equation 2.17:

$$Z^C = Z^A(I + H^A Z^B) \quad (2.18)$$

Inverting both members, it gives:

$$H^C = (I + H^A Z^B)^{-1} H^A \quad (2.19)$$

In the form of sub-matrices with the coordinates i and j :

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \left[\begin{bmatrix} I_{ii} & 0 \\ 0 & I_{jj} \end{bmatrix} + \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Z_{jj}^B \end{bmatrix} \right]^{-1} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.20)$$

Simplifying the previous equation:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} I_{ii} & H_{ij}^A Z_{jj}^B \\ 0 & I_{jj} + H_{jj}^A Z_{jj}^B \end{bmatrix}^{-1} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.21)$$

Developing equation 2.21 results:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A - H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{ij}^A - H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \\ (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \end{bmatrix} \quad (2.22)$$

Putting on evidence the matrix H^A :

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \\ H_{ji}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{jj}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \end{bmatrix}$$

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \\ H_{ji}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{jj}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \end{bmatrix} \quad (2.23)$$

The elements of the second line of the second matrix of the second member are constituted by differences of matrices, so it is possible to simplify these two elements:

$$\begin{aligned} H_{ji}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A &= \\ (I_{jj} - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1}) H_{ji}^A &= \\ = ((I_{jj} + H_{jj}^A Z_{jj}^B)(I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1}) H_{ji}^A &= \\ = ((I_{jj} + H_{jj}^A Z_{jj}^B - I_{jj})(I_{jj} + H_{jj}^A Z_{jj}^B)^{-1}) H_{ji}^A &= \\ = H_{jj}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & \end{aligned} \quad (2.24)$$

From the second member, following a similar process, it gives:

$$H_{jj}^A - (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A = H_{jj}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \quad (2.25)$$

Replacing the equations 2.24 and 2.25 in equation 2.23, it yields:

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{ij}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \\ H_{ji}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A & H_{ji}^A Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \end{bmatrix} \quad (2.26)$$

Reorganizing the previous expression 2.26:

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Z_{jj}^B (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} \end{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.27)$$

Simplifying the element which contains the dynamic stiffness of B:

$$Z^B(I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} = ((I_{jj} + H_{jj}^A Z_{jj}^B)H_{jj}^B)^{-1} = (H_{jj}^A + H_{jj}^B)^{-1} \quad (2.28)$$

Replacing the equation 2.28 in equation 2.27, gives:

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (H_{jj}^A + H_{jj}^B)^{-1} \end{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.29)$$

Lastly, simplifying the expression 2.29 it comes:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} (H_{jj}^A + H_{jj}^B)^{-1} \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.30)$$

The equation 2.30 represents the final expression obtained to the coupling between two sub-structures A and B. This expression provides an advantage once it is only necessary one matrix inversion to achieve the dynamic characterization of structure C.

2.3 Uncoupling Techniques

The uncoupling process of substructures possesses always an high complexity and difficulty when a joint analysis is required. The inverse process of coupling provides the achievement of the joint dynamic features through the coupling general equation (2.30).

According to equation 2.30, it is possible to develop three different expressions to determine the joint dynamic stiffness using different coordinates. Another formulation will be exposed, Wang formulation in order to compare with the previous expressions. Along this work will be considered the following four formulations:

- Formulation 1: without making use of the joint coordinates;
- Formulation 2: making only use of joint coordinates;
- Formulation 3: making only use of joint coordinates i and j ;
- Wang Formulation.

2.3.1 Formulation 1

For some reasons, sometimes it is not possible to use the joint coordinates j [2]. These coordinates could be inaccessible due to their location or other technical issue and their experimental measurement could not be performed. The corresponding measurement technique will not use the joint coordinates.

Therefore, the term of equation 2.30 to take into account is H_{ii}^C :

$$H_{ii}^C = H_{ii}^A - H_{ij}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{ji}^A \quad (2.31)$$

Organizing the expression 2.31:

$$H_{ii}^A - H_{ii}^C = H_{ij}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{ji}^A \quad (2.32)$$

Multiplying at left by $(H_{ij}^A)^{-1}$, and at right by $(H_{ji}^A)^{-1}$:

$$(H_{ij}^A)^{-1}(H_{ii}^A - H_{ii}^C)(H_{ji}^A)^{-1} = (H_{jj}^A + H_{jj}^B)^{-1} \quad (2.33)$$

$$H_{jj}^A + H_{jj}^B = (H_{ji}^A)(H_{ii}^A - H_{ii}^C)^{-1}(H_{ij}^A) \quad (2.34)$$

$$H_{jj}^B = H_{ji}^A (H_{ii}^A - H_{ii}^C)^{-1} H_{ij}^A - H_{jj}^A \quad (2.35)$$

This operation can only be applied if the number of coordinates i is equal to the number of coordinates j , once just with this condition the inversion operations are possible. In other words, the matrices must be square.

On the other hand, if the number of coordinates are different, i.e. $i > j$, H_{jj}^B is obtained multiplying at left by W_{ji} and at right by W_{ij} :

$$W_{ji} H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A W_{ij} = W_{ji} (H_{ii}^A - H_{ii}^C) W_{ij} \quad (2.36)$$

Rearranging the equation 2.36:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (W_{ji} H_{ij}^A)^{-1} W_{ji} (H_{ii}^A - H_{ii}^C) W_{ij} (H_{ji}^A W_{ij})^{-1} \quad (2.37)$$

Making $W_{ji} = H_{ji}^A$ and $W_{ij} = H_{ij}^A$:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (H_{ji}^A H_{ij}^A)^{-1} H_{ji}^A (H_{ii}^A - H_{ii}^C) H_{ij}^A (H_{ji}^A H_{ij}^A)^{-1} \quad (2.38)$$

The expression 2.38 simplifies itself considering $(H_{ji}^A H_{ij}^A)^{-1} H_{ji}^A$ as the pseudo-inverse of H_{ij}^A , and $H_{ij}^A (H_{ji}^A H_{ij}^A)^{-1}$ as the pseudo-inverse of H_{ji}^A . Replacing, respectively, by $(H_{ij}^A)^+$ and $(H_{ji}^A)^+$, in the expression 2.38 results:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (H_{ij}^A)^+ (H_{ii}^A - H_{ii}^C) (H_{ji}^A)^+ \quad (2.39)$$

Manipulating the expression 2.38:

$$H_{jj}^A + H_{jj}^B = H_{ij}^A (H_{ii}^A - H_{ii}^C)^{-1} H_{ji}^A \quad (2.40)$$

It results,

$$H_{jj}^B = H_{ij}^A (H_{ii}^A - H_{ii}^C)^{-1} H_{ji}^A - H_{jj}^A \quad (2.41)$$

Lastly, the operation could be more effective doing the inversion of both members:

$$H_{jj}^B = H_{ji}^A H_{ij}^A (H_{ji}^A (H_{ii}^A - H_{ii}^C) H_{ij}^A)^{-1} H_{ji}^A H_{ij}^A - H_{jj}^A \quad (2.42)$$

2.3.2 Formulation 2

Because of some experimental circumstances, sometimes just the joint coordinates are available to carry out such measurements [2]. Therefore, the term of equation 2.30 to take into account is H_{jj}^C :

$$H_{jj}^C = H_{jj}^A - H_{jj}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{jj}^A \quad (2.43)$$

Reorganizing the expression 2.43:

$$H_{jj}^A(H_{jj}^A + H_{jj}^B)^{-1}H_{jj}^A = H_{jj}^A - H_{jj}^C \quad (2.44)$$

Multiplying at left by $(H_{jj}^A)^{-1}$:

$$(H_{jj}^A + H_{jj}^B)^{-1}H_{jj}^A = (H_{jj}^A)^{-1}(H_{jj}^A - H_{jj}^C) \quad (2.45)$$

$$(H_{jj}^A + H_{jj}^B)^{-1}H_{jj}^A = I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C \quad (2.46)$$

Inverting the expression 2.46:

$$(H_{jj}^A)^{-1}(H_{jj}^A + H_{jj}^B) = (I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)^{-1} \quad (2.47)$$

Reorganizing the expression 2.47 in order to obtain a better computational solution:

$$H_{jj}^B = H_{jj}^A(I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)^{-1} - H_{jj}^A \quad (2.48)$$

The equation 2.48 could be simplified to achieve a simpler expression:

$$H_{jj}^B = H_{jj}^A[(I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)^{-1} - I_{jj}] \quad (2.49)$$

Continuing the simplification,

$$H_{jj}^B = H_{jj}^A[(I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)^{-1} - (I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)^{-1}(I_{jj} - (H_{jj}^A)^{-1}H_{jj}^C)] \quad (2.50)$$

$$H_{jj}^B = H_{jj}^A (I_{jj} - (H_{jj}^A)^{-1} H_{jj}^C)^{-1} [I_{jj} - (I_{jj} - (H_{jj}^A)^{-1} H_{jj}^C)] \quad (2.51)$$

$$H_{jj}^B = H_{jj}^A (I_{jj} - (H_{jj}^A)^{-1} H_{jj}^C)^{-1} (H_{jj}^A)^{-1} H_{jj}^C \quad (2.52)$$

$$H_{jj}^B = H_{jj}^A [H_{jj}^A (I_{jj} - (H_{jj}^A)^{-1} H_{jj}^C)]^{-1} H_{jj}^C \quad (2.53)$$

$$H_{jj}^B = H_{jj}^A (H_{jj}^A - H_{jj}^C)^{-1} H_{jj}^C \quad (2.54)$$

The final expression is given by:

$$H_{jj}^B = (I_{jj} - H_{jj}^C (H_{jj}^A)^{-1})^{-1} H_{jj}^C \quad (2.55)$$

Considering the coupling process, we can take on the same equations of force equilibrium and conditions of displacements compatibility, according to expression 2.56:

$$\begin{cases} f_j^A + f_j^B = f_j^C \\ x_j^A = x_j^B = x_j^C \end{cases} \quad (2.56)$$

To the substructure A, we have the following conditions:

$$\begin{cases} f_i^C = f_i^A \\ x_i^C = x_i^A \end{cases} \quad (2.57)$$

The dynamic stiffness matrix is given by:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix}^{-1} \begin{Bmatrix} x_i^C \\ x_j^C \end{Bmatrix} = \begin{Bmatrix} f_i^C \\ f_j^C \end{Bmatrix} \quad (2.58)$$

The condensed dynamic stiffness matrix of substructure A is given by:

$$\begin{bmatrix} \tilde{Z}_{ii}^A & \tilde{Z}_{ij}^C \\ \tilde{Z}_{ji}^C & \tilde{Z}_{jj}^C \end{bmatrix} \begin{Bmatrix} x_i^A \\ x_j^A \end{Bmatrix} = \begin{Bmatrix} f_i^A \\ f_j^A \end{Bmatrix} \quad (2.59)$$

If each member of equations 2.58 and 2.59 were subtracted, we have:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix}^{-1} \begin{Bmatrix} x_i^C \\ x_j^C \end{Bmatrix} - \begin{bmatrix} \tilde{Z}_{ii}^A & \tilde{Z}_{ij}^C \\ \tilde{Z}_{ji}^C & \tilde{Z}_{jj}^C \end{bmatrix} \begin{Bmatrix} x_i^A \\ x_j^A \end{Bmatrix} = \begin{Bmatrix} f_i^C \\ f_j^C \end{Bmatrix} - \begin{Bmatrix} f_i^A \\ f_j^A \end{Bmatrix} \quad (2.60)$$

Manipulating the equation in order to simplify:

$$\left(\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix}^{-1} - \begin{bmatrix} \tilde{Z}_{ii}^A & \tilde{Z}_{ij}^C \\ \tilde{Z}_{ji}^C & \tilde{Z}_{jj}^C \end{bmatrix} \right) \begin{Bmatrix} x_i^C \\ x_j^C \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_j^B \end{Bmatrix} \quad (2.61)$$

Multiplying at left by $[H^C]$:

$$\left(\begin{bmatrix} I_{ii} & 0 \\ 0 & I_{jj} \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix}^{-1} \begin{bmatrix} \tilde{Z}_{ii}^A & \tilde{Z}_{ij}^C \\ \tilde{Z}_{ji}^C & \tilde{Z}_{jj}^C \end{bmatrix} \right) \begin{Bmatrix} x_i^C \\ x_j^C \end{Bmatrix} = \begin{bmatrix} H_{ij}^C \\ H_{jj}^C \end{bmatrix} \{f_j^B\} \quad (2.62)$$

Developing previous equation,

$$\begin{bmatrix} I_{ii} - H_{ii}^C \tilde{Z}_{ii}^A - H_{ij}^C \tilde{Z}_{ji}^A & -H_{ii}^C \tilde{Z}_{ij}^A - H_{jj}^C \tilde{Z}_{ij}^A \\ -H_{ji}^C \tilde{Z}_{ii}^A - H_{jj}^C \tilde{Z}_{ji}^A & I_{jj} - H_{ji}^C \tilde{Z}_{ij}^A - H_{jj}^C \tilde{Z}_{jj}^A \end{bmatrix} \begin{Bmatrix} x_i^C \\ x_j^C \end{Bmatrix} = \begin{bmatrix} H_{ij}^C \\ H_{jj}^C \end{bmatrix} \{f_j^B\} \quad (2.63)$$

If it isn't necessary to measure on coordinates i , it comes:

$$\left[I_{jj} - H_{jj}^C \tilde{Z}_{jj}^A \right] \{x_j^B\} = H_{jj}^C \{f_j^B\} \quad (2.64)$$

As $Z_{jj}^A = (H_{jj}^A)^{-1}$, we can write:

$$\{x_j^B\} = (I_{jj} - H_{jj}^C (H_{jj}^A)^{-1})^{-1} H_{jj}^C \{f_j^B\} \quad (2.65)$$

Then, we obtain a similar expression to 2.55:

$$H_{jj}^B = (I_{jj} - H_{jj}^C (H_{jj}^A)^{-1})^{-1} H_{jj}^C \quad (2.66)$$

Alternatively, this formulation could be written in a simpler form, from the expression 2.44:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (H_{jj}^A)^{-1} (H_{jj}^A - H_{jj}^C) (H_{jj}^A)^{-1} \quad (2.67)$$

Manipulating the equation 2.67:

$$H_{jj}^B = H_{jj}^A (H_{jj}^A - H_{jj}^C)^{-1} H_{jj}^A - H_{jj}^A \quad (2.68)$$

Finally,

$$H_{jj}^B = (H_{jj}^A (H_{jj}^A - H_{jj}^C)^{-1} - I_{jj}) H_{jj}^A \quad (2.69)$$

2.3.3 Formulation 3

Following the expression 2.30, another way to determine H_{jj}^B is possible, making only use of joint coordinates i and j [2]. Therefore, the term of equation 2.30 to take into account is H_{ij}^C :

$$H_{ij}^C = H_{ij}^A - H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A \quad (2.70)$$

Organizing the expression 2.70 results:

$$H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A = H_{ij}^A - H_{ij}^C \quad (2.71)$$

Multiplying at left by W_{ij} :

$$W_{ji} H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A = W_{ji} (H_{ij}^A - H_{ij}^C) \quad (2.72)$$

Doing $W_{ji} = H_{ji}^A$:

$$H_{ji}^A H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A = H_{ji}^A (H_{ij}^A - H_{ij}^C) \quad (2.73)$$

Reorganizing the equation 2.73, it comes:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (H_{ji}^A H_{ij}^A)^{-1} H_{ji}^A (H_{ij}^A - H_{ij}^C) (H_{jj}^A)^{-1} \quad (2.74)$$

$$H_{jj}^A + H_{jj}^B = H_{jj}^A (H_{ji}^A (H_{ij}^A - H_{ij}^C))^{-1} H_{ji}^A H_{jj}^A \quad (2.75)$$

Lastly, the following expression is obtained:

$$H_{jj}^B = H_{jj}^A (H_{ji}^A (H_{ij}^A - H_{ij}^C))^{-1} H_{ji}^A H_{jj}^A - H_{jj}^A \quad (2.76)$$

2.3.4 Wang Formulation

Wang et al. [9] set up the classic uncoupling expression, using the coupling forces equilibrium performed by Ren and Beards [15]:

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (H_{jj}^A + H_{jj}^B)^{-1} \end{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.77)$$

Simplifying the expression 2.77:

$$\begin{bmatrix} H_{ij}^A H_{jj}^{-1} H_{ji}^A & H_{ij}^A H_{jj}^{-1} H_{jj}^A \\ H_{jj}^A H_{jj}^{-1} H_{ji}^A & H_{jj}^A H_{jj}^{-1} H_{jj}^A \end{bmatrix} = \begin{bmatrix} H_{ii}^A - H_{ii}^C & H_{ij}^A - H_{ij}^C \\ H_{ji}^A - H_{ji}^C & H_{jj}^A - H_{jj}^C \end{bmatrix} \quad (2.78)$$

where $H_{jj} = H_{jj}^A + H_{jj}^B$. Adding the following matrix to both members:

$$A = \begin{bmatrix} H_{ii}^A H_{ii}^A & H_{ii}^A H_{ij}^A \\ H_{ji}^A H_{ii}^A & H_{ji}^A H_{ij}^A \end{bmatrix}$$

Then, it comes:

$$\begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ij}^A H_{jj}^{-1} H_{ji}^A & H_{ii}^A H_{ij}^A + H_{ij}^A H_{jj}^{-1} H_{jj}^A \\ H_{ji}^A H_{ii}^A + H_{jj}^A H_{jj}^{-1} H_{ji}^A & H_{ji}^A H_{ij}^A + H_{jj}^A H_{jj}^{-1} H_{jj}^A \end{bmatrix} = \begin{bmatrix} H_{ii}^A - H_{ii}^C & H_{ij}^A - H_{ij}^C \\ H_{ji}^A - H_{ji}^C & H_{jj}^A - H_{jj}^C \end{bmatrix} + A \quad (2.79)$$

Isolating H_{jj} :

$$\begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} I_{ii} & 0 \\ 0 & H_{jj}^{-1} \end{bmatrix} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} = \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix} \quad (2.80)$$

Inverting the above matrix:

$$\begin{bmatrix} I_{ii} & 0 \\ 0 & H_{jj} \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix}^{-1} \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (2.81)$$

Using the expression $H_{jj} = H_{jj}^A + H_{jj}^B$, it is possible to determine H_{jj}^B :

$$H_{jj}^B = \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix}^{-1} \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} - H_{jj}^A \quad (2.82)$$

2.4 Review of Formulations

At this point of this work, a short summary can be done in order to clarify the four formulations introduced along this chapter:

Formulation 1

$$H_{jj}^B = H_{ji}^A H_{ij}^A (H_{ji}^A (H_{ii}^A - H_{ii}^C) H_{ij}^A)^{-1} H_{ji}^A H_{ij}^A - H_{jj}^A$$

Formulation 2

$$H_{jj}^B = (H_{jj}^A (H_{jj}^A - H_{jj}^C)^{-1} - I_{jj}) H_{jj}^A$$

Formulation 3

$$H_{jj}^B = H_{jj}^A (H_{ji}^A (H_{ij}^A - H_{ij}^C))^{-1} H_{ji}^A H_{ij}^A - H_{jj}^A$$

Wang Formulation

$$H_{jj}^B = \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix}^{-1} \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} - H_{jj}^A$$

2.5 Numerical Examples

Throughout this work, several uncoupling techniques have been regarded that differ numerically and consequently the results obtained are different. In order to test and validate those techniques, some numerical examples will be considered with the aim of understand which is the best technique to use experimentally. For each example, the four techniques exposed above will be analysed, introducing noise in the global system's *FRF* to simulate experimental data. Some formulations will have a better behaviour than other when exposed to the noise introduced by experimental measured data. This different behaviour depends on operations with matrix inversions which affect the matrix condition and consequently the results obtained. The matrix condition will be analysed along the frequency range to understand its dependence with resonance and anti-resonance picks.

Furthermore, some examples from articles and Ph.D Thesis will be considered to validate the algorithms adopted and the finite element programmes used to extract the *FRF* of the structure and its substructures.

2.5.1 Numerical Example 1

In this example an uncoupling of a joint is performed, substructure B, from the global structure C [2]. The beam has the same section along its length, with different lengths for each substructure. In this example three substructures were considered, A1, A2 and B, forming the global structure C, while substructures A1 and A2 are considered as an only substructure A, for the sake of calculation. All the substructures are represented in figure 2.4.

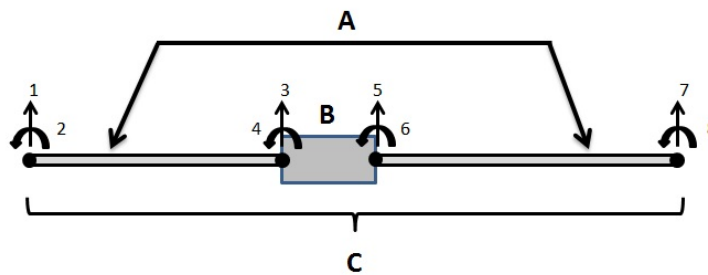


Figure 2.4: Coupling of substructures A and B, forming the structure C.

The analysis of this example was carried out using Euler-Bernoulli finite elements, each element with four degrees of freedom. For each substructure were considered ten Euler-Bernoulli elements. The *FRFs* of substructures A and B, and structure C, are calculated separately, from the FE spatial model by using a direct frequency analysis.

In this case it will be assumed that is not possible to perform measurements in the joint coordinates. Therefore, it should be used the formulation 1 to determine the joint dynamic characteristics:

$$H_{jj}^B = H_{ji}^A (H_{ii}^A - H_{ii}^C)^{-1} H_{ij}^A - H_{jj}^A$$

Table 2.1: Beam characteristics of numerical example 1.

Beam	Length	Width	Thickness	ν	E	ρ
A_1	270 mm	30 mm	5 mm	0.3	194 GPa	7562 Kg/m ³
B	200 mm	30 mm	5 mm	0.3	194 GPa	7562 Kg/m ³
A_2	370 mm	30 mm	5 mm	0.3	194 GPa	7562 Kg/m ³

To simulate the errors which are common in the acquisition of experimental data, the global structure's *FRFs* were contaminated with noise to evaluate the method stability. The introduction of noise is done according to the following expression, depending on the amplitude:

$$H_{pq}(\omega_k) = H_{pq}(\omega_k) + H_{pq}(\omega_k) \frac{\gamma}{50} (\text{rand}(1) - 0.5) \quad (2.83)$$

Where γ is the noise level in percentage, whereas $\text{rand}(1)$ corresponds to an uniform random distribution with values between 0 and 1. In this example 1 % is considered as noise level.

Two different analysis were done, one without introduction of noise and other considering noise. Without noise in the FRF matrix H_{ii}^C , all the *FRFs* curves obtained by this uncoupling method fit very well the exact solutions, showing a good stability. If some noise is introduced, the method becomes quite unstable, showing a great discrepancy when compared with the exact solution. The inversion operation $(H_{ii}^A - H_{ii}^C)$ could be considered as the major responsible for this instability. Furthermore, this matrix, which has to be inverted at each frequency, shows low values of matrix condition.

The results obtained for this example could be seen in figure 2.5. As we can see, when noise is introduced, the level of discrepancy increases and the *FRFs* with noise do not match very well with the exact solutions obtained by the *Finite Element Method*.

Another way to validate these results comes from an analysis over the natural frequencies, which are well visible in figure 2.5. Considering only the joint as beam, its natural frequencies could be obtained through the following theoretical expression [16]:

$$\omega_i = \frac{(\beta_i l)^2}{2\pi} \sqrt{\frac{EI}{\rho A l^4}} / Hz \quad (2.84)$$

Taking into account boundary conditions as free-free, the first value to $(\beta_i l)$ in expression 2.84 in which appears the first natural frequency is $(\beta_i l) = 4.7300$. The comparison between the experimental and theoretical results is presented in table 2.2.

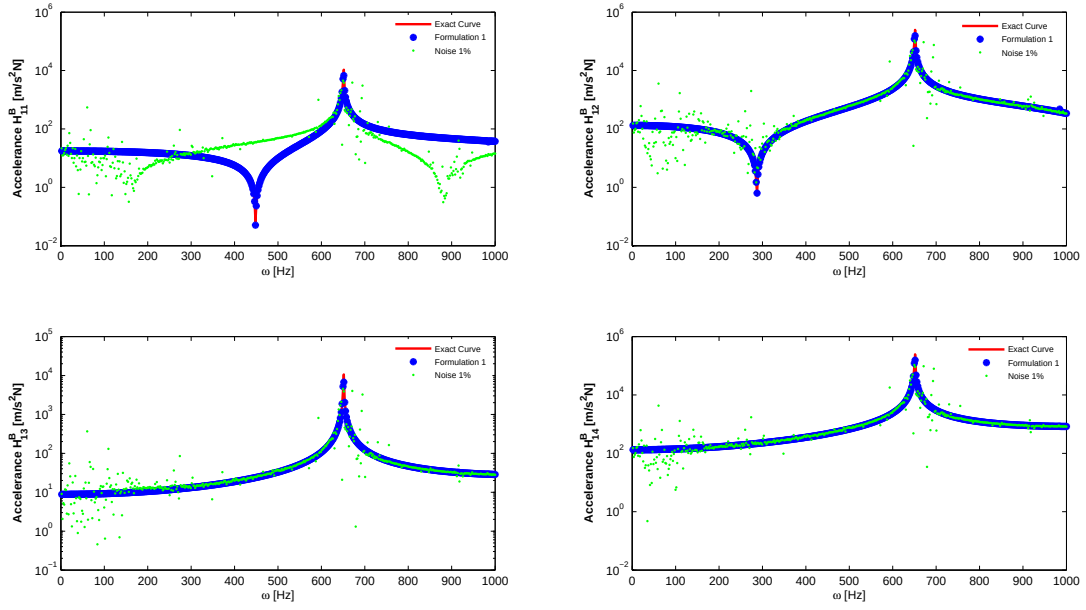


Figure 2.5: Uncoupling of substructure B from structure C.

Table 2.2: Analysis of natural frequencies.

<i>FRF</i>	Theoretical Value	Experimental Value	Error / %
H_{11}^B	$\omega_1 = 677.10Hz$	$\omega_1 = 651.70Hz$	$\varepsilon = 3.75$
H_{12}^B	$\omega_1 = 677.10Hz$	$\omega_1 = 651.70Hz$	$\varepsilon = 3.75$

Therefore, considering the results without the noise interference, it is possible notice that the natural frequencies are almost exact with a low error associated. This fact contributes to the validation of this numerical example.

2.5.2 Numerical Example 2

This numerical example was performed with the purpose of compare the following different uncoupling techniques:

- Formulation 1, without using joint coordinates;
- Formulation 2, using only joint coordinates;
- Formulation 3, using intern and joint coordinates;
- Wang Formulation.

2.5. NUMERICAL EXAMPLES

As in previous example, here the same number of substructures is considered with an identic arrangement. Figure 2.6 shows the number of degrees of freedom considered and their position along the beam. The joint section dimensions are different from the other two substructures in order to test the programm used to analyse beams with different sections. All the dimensions and physical properties of substructures are referred in table 2.3.

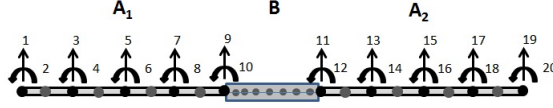


Figure 2.6: Coupling of substructures A and B, forming the structure C.

Table 2.3: Beam characteristics of numerical example 2.

Beam	Length	Width	Thickness	ν	E	ρ
A ₁	400 mm	25 mm	3 mm	0.3	210 GPa	7850 Kg/m ³
B	250 mm	25 mm	6 mm	0.3	210 GPa	7850 Kg/m ³
A ₂	400 mm	25 mm	3 mm	0.3	210 GPa	7850 Kg/m ³

The structure analysis based on a finite element approach, considering Euler-Bernoulli elements as in previous example. For each substructure were established eight Euler-Bernoulli elements, making a total of 24 elements for the global structure.

Figures 2.7 and 2.8 present the results obtained with the four uncoupling techniques. Analysing both graphics, almost all the formulations provide *FRFs* which fit very well the exact *FRF* and just formulation 1 shows discrepancy and instability in some frequency ranges. The instability of formulation 1 can be related with numerical operations which could be affecting the results.

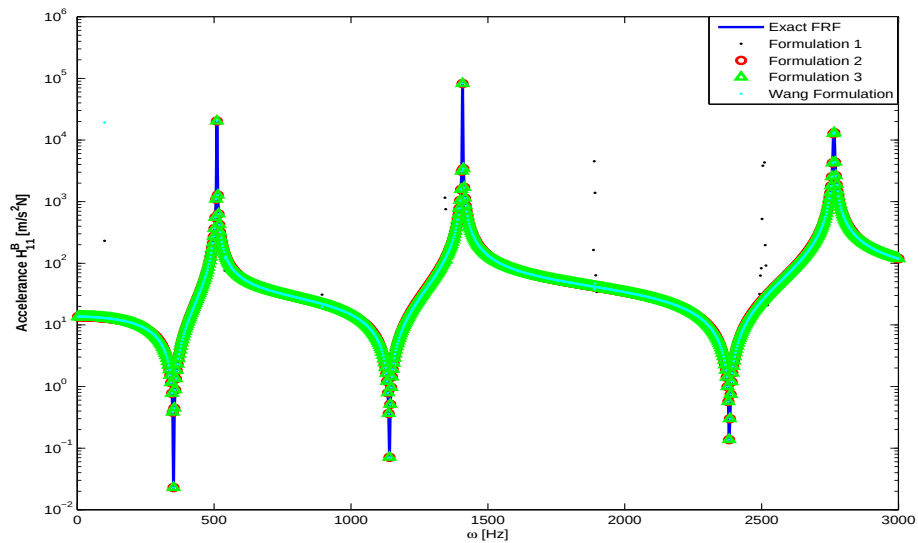


Figure 2.7: Accelerance H_{11}^B obtained with the uncoupling techniques.

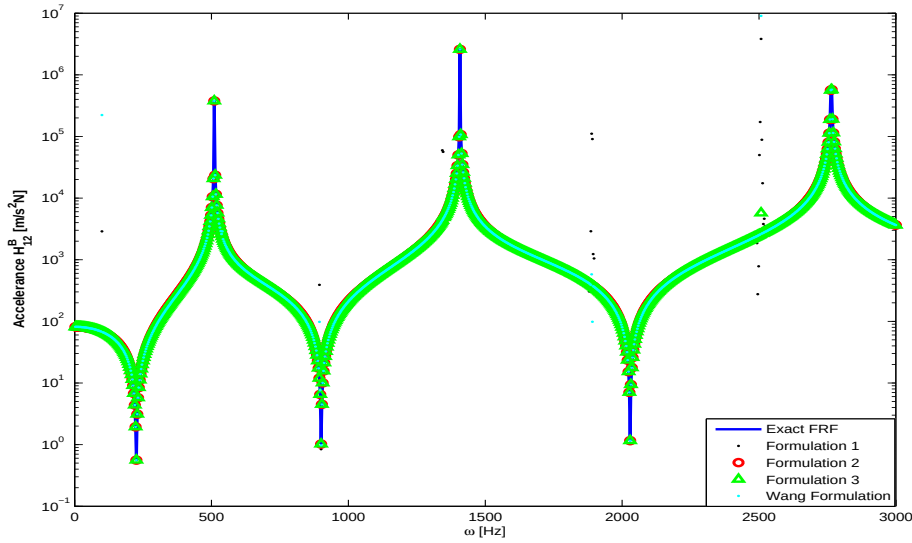


Figure 2.8: Accelerance H_{12}^B obtained with the uncoupling techniques.

Once again, a verification of the natural frequencies could be done taking into account the equation 2.84. As we can see in figures 2.7 and 2.8, all the formulations have coincident natural frequencies.

Table 2.4: Analysis of natural frequencies.

Natural Frequency	Theoretical Value / Hz	Experimental Value / Hz	Error / %
ω_1	510.38	510.3	0.016
ω_2	1406.92	1408.00	0.077
ω_3	2758.12	2765.00	0.249

Table 2.4 shows a good correlation between the theoretical and experimental results, and the error calculated is lower than 1 %.

Normally, experimental data owns always some errors caused by noise introduced during the corresponding measurements. To simulate these errors, a noise level of 1 % was introduced in previous *FRFs* obtained, considering noise which is dependant on the amplitude as in previous example. As we can see in figures 2.9 and 2.10, the noise introduced causes an high discrepancy and instability in results obtained, principally in resonance and anti-resonance zones. The first and third formulations show great instability along the frequency range and it is completely impossible to identify any natural frequencies as well as antiresonance picks. The second and Wang formulations provide much better results, despite the visible instability still present.

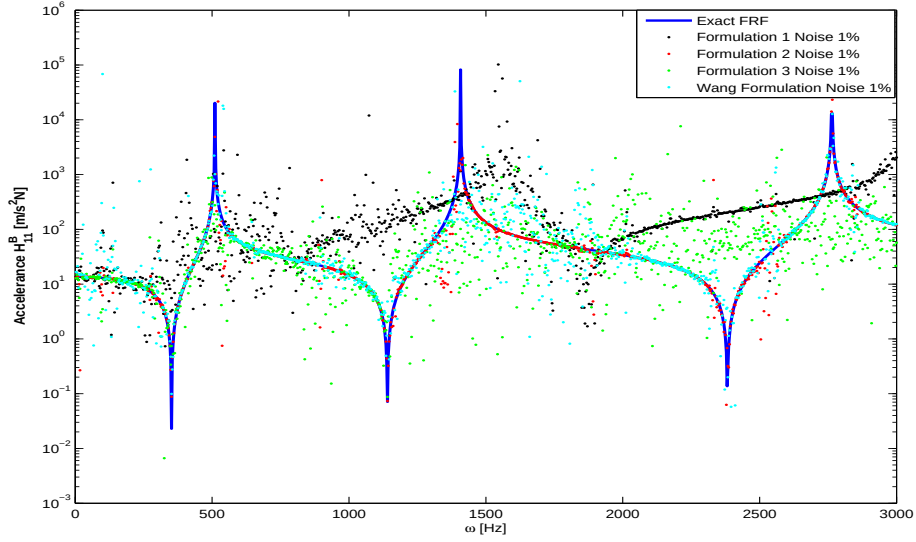


Figure 2.9: Accelerance H_{11}^B obtained with 1% of noise.

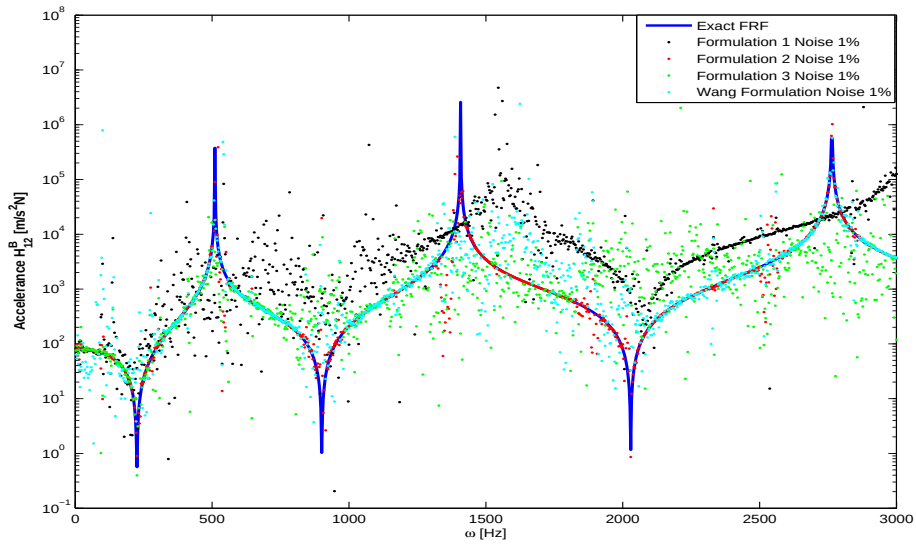


Figure 2.10: Accelerance H_{12}^B obtained with 1% of noise.

The comparison between results obtained with noise and the clean data was performed with a correlation criterion, LAC . Figure 2.11 shows this criterion applied to the four formulations for the joint degrees-of-freedom.

Observing figure 2.11, it is possible to conclude that formulation 2 possesses and provides better results once LAC values obtained are the highest, close to 1 which is the recommended. Relating to other formulations, only Wang formulation has sustainable LAC values whereas first and third formulations give lower values showing an high instability. Therefore, measurements in joint coordinates should be done whenever possible once formulation 2 provides

better results than other formulations.

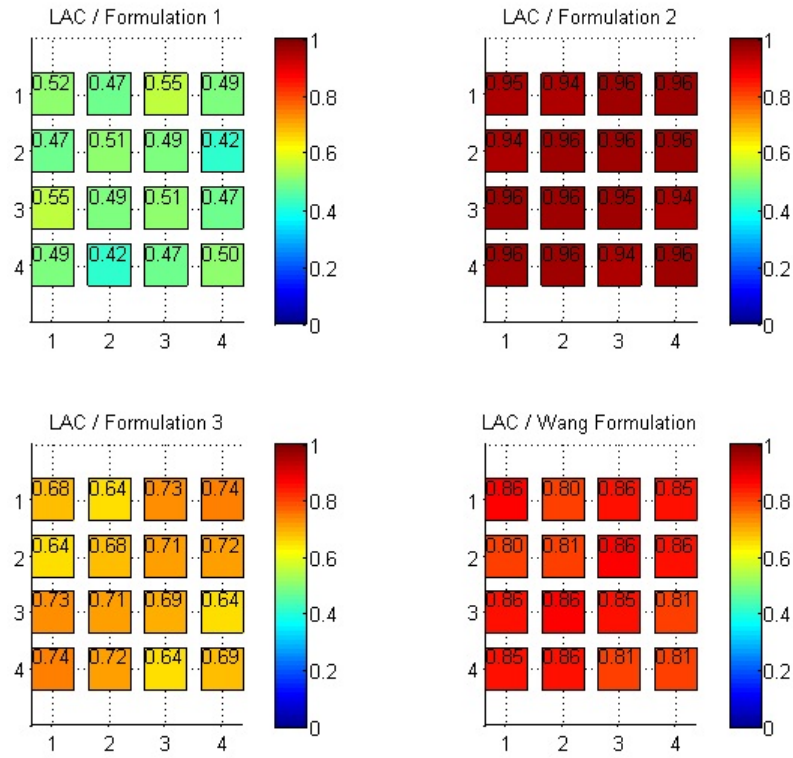


Figure 2.11: Average LAC of joint coordinates' FRFs.

2.5.3 Numerical Example 3

This numerical example was carried out to evaluate the same four uncoupling techniques approached in previous example, considering a different beam with a round section along its length [8]. The number of substructures is the same and Euler-Bernoulli finite elements were used. The number of elements and degrees-of-freedom is available in figure 2.12. The *FRFs* of both substructures were calculated separately considering eight Euler-Bernoulli elements for the main structure. For the sake of agreement and equality, only one finite element was taken into account to represent the joint model.

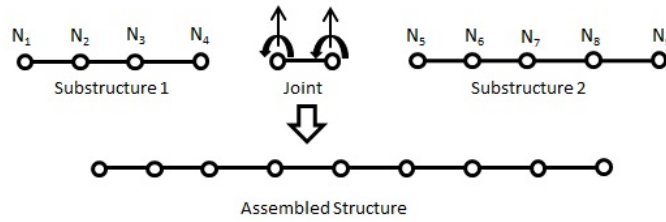


Figure 2.12: Coupling of substructures A and B, forming the structure C.

Here in this example, the joint has a geometry and properties equal to the other parts of the structure. The properties and characteristics of each Euler-Bernoulli element are presented in table 2.5.

Table 2.5: Geometry and material data of a beam finite element.

Length	$L = 0.1 \text{ m}$
Diameter	$d = 0.005 \text{ m}$
Young's modulus	$E = 2.07 \times 10^{11} \text{ N/m}^2$
Shear modulus	$G = 0.385.E$
Density	$\rho = 7547 \text{ Kg/m}^3$

The results obtained for the joint uncoupling are presented in figures 2.13 and 2.14. A detailed analysis of the results indicates that all techniques converge with exact curve at blue, even though first formulation already shows a low discrepancy in some frequencies. Table 2.6 provides a study about the natural frequency value, which isn't equal to the theoretical value calculated. Indeed, this fact is related to the number of elements considered to the joint, in this case just one. Thus, the natural frequencies depends also on the number of elements, decreasing the error with the raise of the number of elements taken into account.

The accelerance curves obtained are similar to results published in [8], which validates and confirms the results of this example. Although a good visible coherence, it is possible to check that there is an high level of discrepancy, and this fact could be related with the low number of finite elements considered to calculate the *FRFs*. Therefore, another factor affects the stability of these methods which are really sensitive to external conditions.

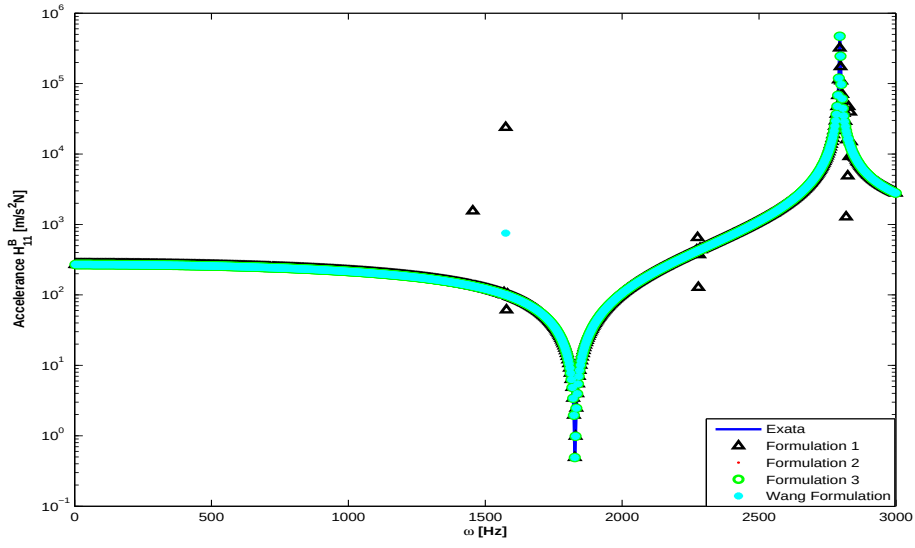
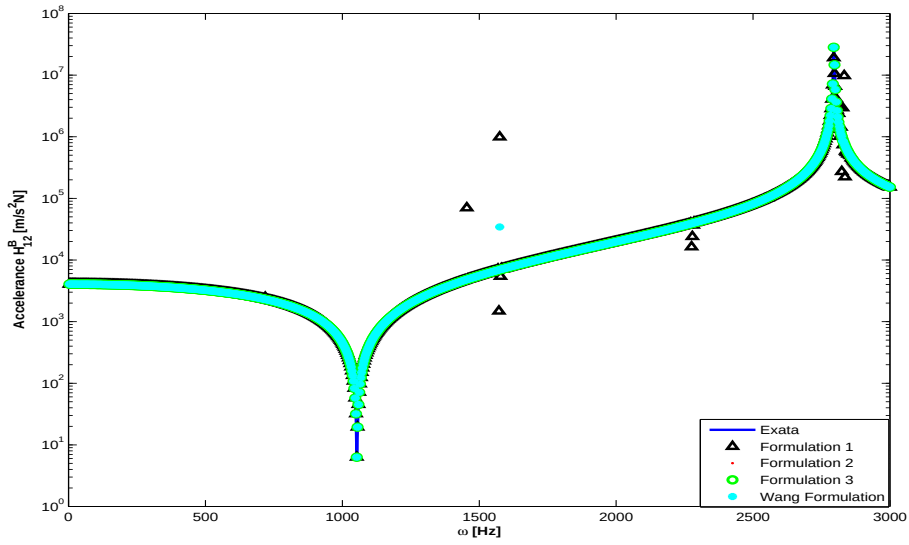

 Figure 2.13: Uncoupled acceleration H_{11}^B .

 Figure 2.14: Uncoupled acceleration H_{12}^B .

Table 2.6: Analysis of natural frequencies.

Natural Frequency	Theoretical Value / Hz	Experimental Value / Hz	Error / %
ω_1	2331.04	2795.00	19.89

A noise level of 1 % was introduced in the global system's *FRFs* to obtain the acceleration H_{11}^B . The noise introduced depends on the amplitude. Figure 2.15 presents the *FRFs* contaminated

2.5. NUMERICAL EXAMPLES

with noise where once again second and Wang formulation provide better results than other formulations.

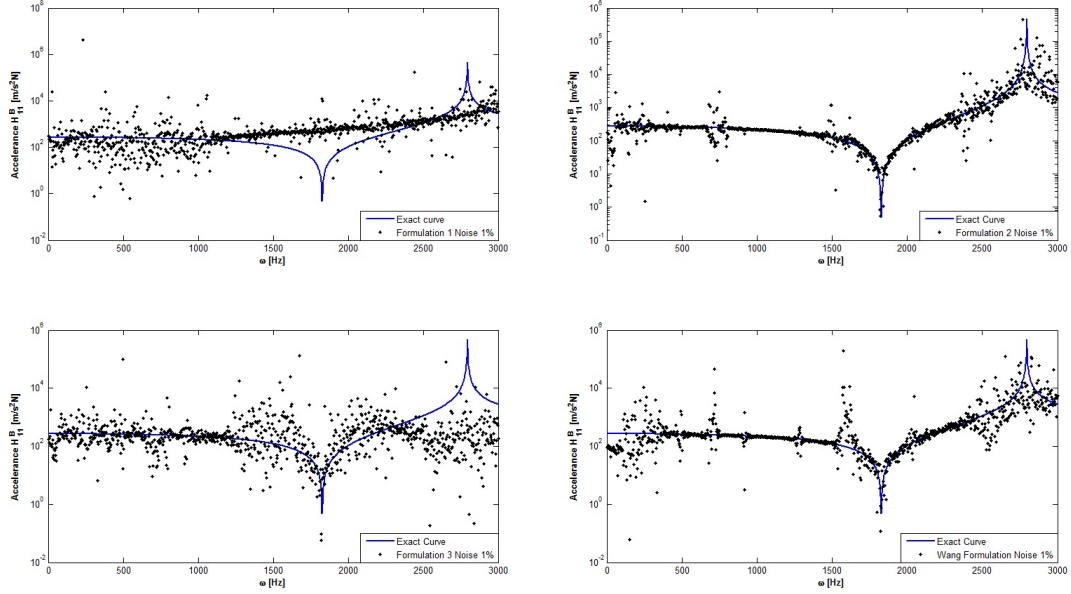


Figure 2.15: Uncoupled $FRFs H_{11}^B$ with 1 % of noise.

The contaminated $FRFs$ with noise were compared with the original ones using a correlation criterium, LAC . Figure 2.16 shows the mean LAC for the joint degrees-of-freedom; the second and Wang formulation give again better results than other formulations which reinforces that joint measurements should always be done whenever possible.

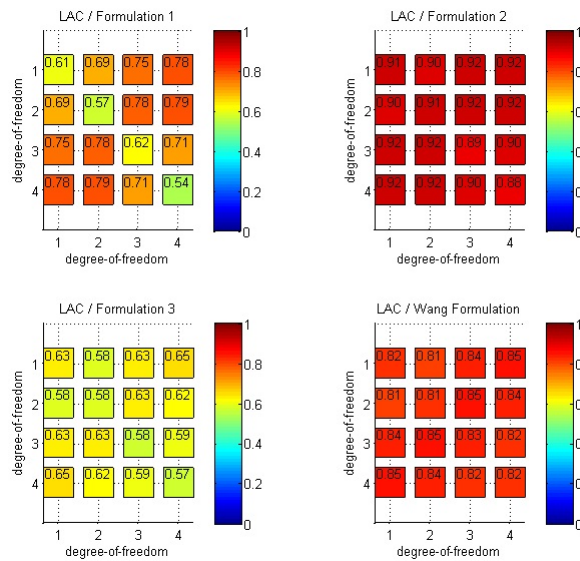


Figure 2.16: Average LAC of joint coordinates' $FRFs$.

2.5.4 Numerical Example 4

This example arises from the previous numerical example, but here damping is introduced in order to analyse its interference in uncoupling techniques [9]. The same four uncoupling techniques were considered and also were used Euler-Bernoulli elements. The dimensions and properties are exactly the same as seen in table 2.7 and in figure 2.17, respectively:

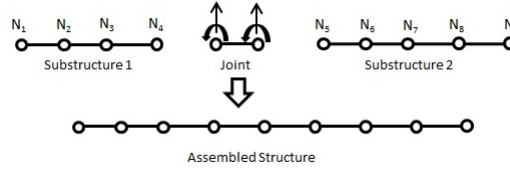


Figure 2.17: Coupling of substructures A and B, forming the structure C.

Table 2.7: Geometry and material data of a beam finite element.

Length	$L = 0.1m$
Diameter	$d = 0.005m$
Young's modulus	$E = 2.07 \times 10^{11} N/m^2$
Shear modulus	$G = 0.385.E$
Density	$\rho = 7547 Kg/m^3$
Overall Structural damping coefficient	$\eta = 0.001$

The damping introduced in this example is characterized as hysteretic or structural damping. Considering the following motion equation:

$$m\ddot{x} + \bar{k}x = F e^{j\omega t} \quad (2.85)$$

$$\bar{k} = k(1 + j\eta) \quad (2.86)$$

Substituting expression 2.86 in equation 2.85 gives:

$$m\ddot{x} + k(1 + j\eta)x = F e^{j\omega t} \quad (2.87)$$

Therefore, the stiffness matrix will have an imaginary part and damping will be proportional to displacement and in phase with velocity. As well as the stiffness matrix, the natural frequencies will also be complex:

$$\lambda_r = \omega_r^2(1 + j\eta_r) \quad (2.88)$$

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Figures 2.18 and 2.19 show the receptance functions, H_{11}^B and H_{12}^B , obtained in this numerical example. After a careful observation it is possible to understand that structural damping introduces some uncertainty in results comparing with previous example. The first and Wang formulation show some discrepancy in some frequency ranges whereas other formulations fit very well the exact solution.

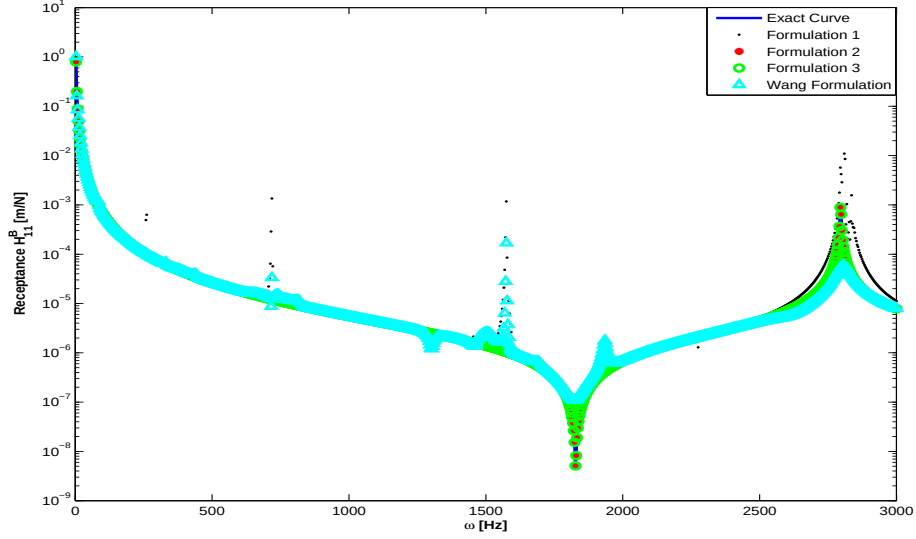


Figure 2.18: Results obtained with four different uncoupling techniques.

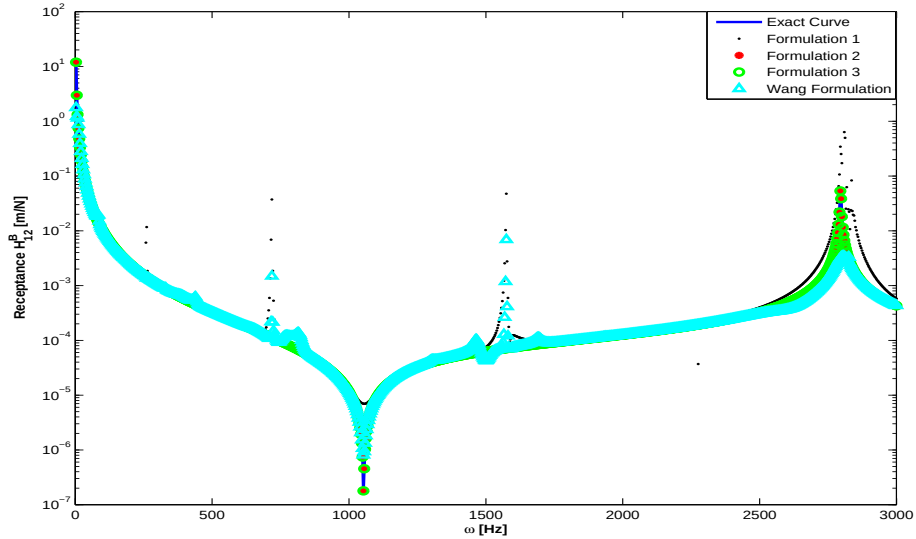


Figure 2.19: Results obtained with four different uncoupling techniques.

A noise level of 3 % was taken into account to evaluate the damping influence. Figure 2.20 presents the results obtained to the receptance H_{11}^B where it is possible to see again an huge uncertainty and discrepancy. Wang and second formulation give again better results than other

formulations, despite the high discrepancy.

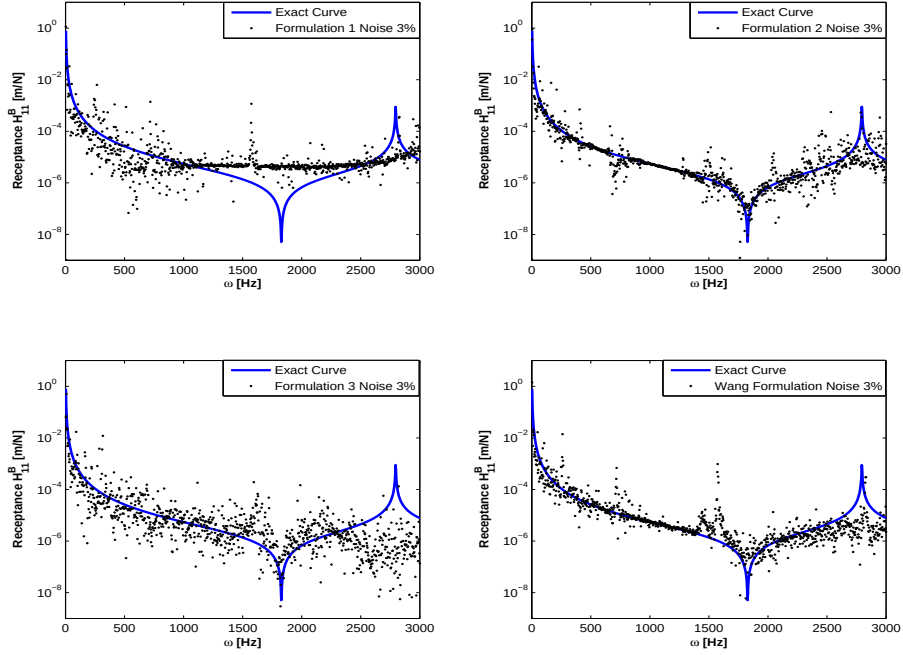


Figure 2.20: H_{11}^B with 3 % of noise.

In order to compare the results obtained with results presented in [9], the joint dynamic stiffness, D_{11}^B , was calculated applying the four formulations. Wang formulation shows similar results when compared to results found in [9], whereas second and third formulations show convergency and on the other hand, formulation 1 is showing an high discrepancy once again. The low number of elements considered could be affecting the accuracy of the results, as seen in previous numerical example.

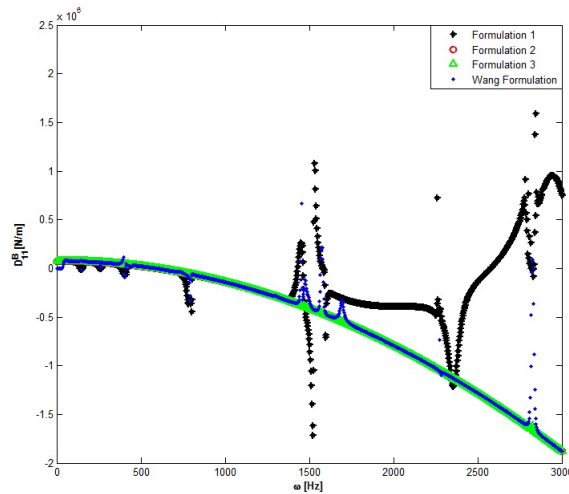


Figure 2.21: Dynamic stiffness, D_{11}^B .

2.5. NUMERICAL EXAMPLES

The correlation criterion (*mean LAC*) was used to compare *FRFs* obtained with noise and those without noise. Figure 2.22 shows the mean *LAC* values obtained for the different uncoupling techniques. Wang and second formulations present a good correspondence whereas the two other formulations have low *LAC* values indicating their weak experimental potential.

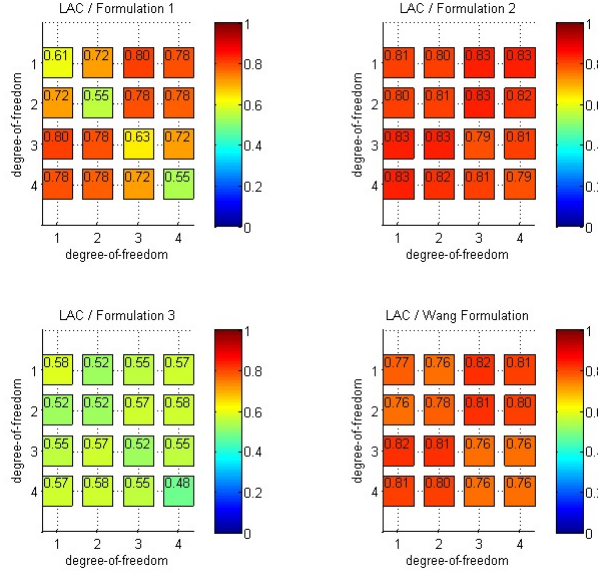


Figure 2.22: Average LAC of joint coordinates' *FRFs*.

2.5.5 Comparison of uncoupling formulations

The four uncoupling techniques taken into account in this dissertation have some differences that cause different results and behaviour under the same working conditions and experimental data. Therefore, a detailed understanding of these techniques is required in order to explain these experimental results.

The different behaviour of these techniques could be explained taking into account some numerical points, such as the number of inversions and differences involving matrices and in other field, the coordinates used in each formulation. Table 2.8 summarizes and organizes these particular aspects.

Table 2.8: Comparison of uncoupling formulations

	N.º of Inversions	N.º of Differences	Coordinates used
Formulation 1	1	2	only i
Formulation 2	1	2	only j
Formulation 3	1	2	i and j
Wang Formulation	1	1	i and j

The number of inversions and differences involving matrices affects directly the matrix con-

ditions which when exposed to experimental noise causes bad results. Thus, a detailed analysis of the expressions that perform the uncoupling is required to understand these different behaviours in order to evaluate these techniques.

Furthermore, the different coordinates used in each formulation have some influence in their behaviour. It has been said that the joint coordinates should be considered in experimental measurements whenever possible, once those provide more accurate results with more information about the corresponding structure.

Therefore, the formulations with better experimental behaviour will be those which have the low number of inversions and differences and use the largest number of coordinates, and rather joint coordinates. Indeed, these particular characteristics are intrinsic in the formulations which provide better results, the second and Wang formulations.

The different behaviour of these uncoupling formulations could be explained taking into account these particular characteristics which *a priori* wouldn't never affect and have so importance and impact in the uncoupling process.

Chapter 3

Experimental Data Analysis

3.1 Introduction

The uncoupling process can be considered as a very sensitive process in which every type of errors have a great impact in the final results. Consequently, some experimental techniques have been applied in order to improve the quality of the results and reduce the influence of such errors.

In this work, the following three techniques will be applied to the experimental results:

- Analysis of frequency ranges;
- Estimative of unmeasured *FRFs*;
- Mass Cancellation;

3.2 Analysis of frequency ranges

The uncoupling techniques approached along this work possess an high uncertainty associated and are too sensitive to small calculation errors, like noise automatically introduced in experimental measurements, which has a great impact in final results. These numerical problems come from matrix operations such inversions and differences that in presence of noise cause matrices badly conditioned and consequently a huge discrepancy. Batista [2] presented an algorithm that enables the identification of frequency ranges where the noise level is lower.

All the formulations considered in this work have differences and inversions of matrices. Making a resume of all formulations regarded:

Formulation 1:

$$H_{jj}^B = H_{ji}^A H_{ij}^A (H_{ii}^A (H_{ii}^A - H_{ii}^C) H_{ij}^A)^{-1} H_{ji}^A H_{ij}^A - H_{jj}^A$$

Formulation 2:

$$H_{jj}^B = (H_{jj}^A (H_{jj}^A - H_{jj}^C)^{-1} - I_{jj}) H_{jj}^A$$

Formulation 3:

$$H_{jj}^B = H_{jj}^A (H_{ji}^A (H_{ij}^A - H_{ij}^C))^{-1} H_{ji}^A H_{ij}^A - H_{jj}^A$$

Wang Formulation:

$$H_{jj}^B = \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix}^{-1} \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} - H_{jj}^A$$

According to previous expressions, in each formulation the matrices which have to be inverted are the following:

$$A_1 = H_{ji}^A (H_{ii}^A - H_{ii}^C) H_{ij}^A$$

$$A_2 = H_{jj}^A - H_{jj}^C$$

$$A_3 = H_{ji}^A (H_{ij}^A - H_{ij}^C)$$

$$A_4 = \begin{bmatrix} H_{ii}^A H_{ii}^A + H_{ii}^A - H_{ii}^C & H_{ii}^A H_{ij}^A + H_{ij}^A - H_{ij}^C \\ H_{ji}^A H_{ii}^A + H_{ji}^A - H_{ji}^C & H_{ji}^A H_{ij}^A + H_{jj}^A - H_{jj}^C \end{bmatrix}$$

The condition of each matrix A_i will be calculated for each frequency ω in order to analyse the *FRF* behaviour and understand in which ranges the uncertainty and discrepancy are lower to provide better results. Normally, high values of matrix condition are associated with an high level of noise in a determined frequency range.

3.2.1 Algorithm

Once calculated the matrices condition along the frequency range, these data have to be processed and analysed. The function $r_i(\omega)$ relates the condition logarithm, making a magnification of the matrix condition value:

$$r_i(\omega) = \tau \ln(\text{cond}(A_i(\omega))) \quad (3.1)$$

3.2. ANALYSIS OF FREQUENCY RANGES

The parameter τ represents a magnification factor of the condition logarithm. To identify the ranges where this function is minimum was used a Matlab function that provides a regression along the frequency range:

$$R_i(\omega) = \text{csaps}(\omega, r_i(\omega), v) \quad (3.2)$$

where v is a weight parameter that varies between 0 to 1. If v is equal to zero the regression is linear, whereas if v is equal to 1 we have an interpolation that passes in all points along the frequency range.

To calculate the function minimums, the first and second derivative of $R_i(\omega)$ were analysed. It was established that the first derivative should be lower to a given parameter ξ and the second derivative should be positive:

$$\frac{dR_i}{d\omega} < \xi \bigwedge \frac{d^2R_i}{d\omega^2} > 0 \quad (3.3)$$

3.2.2 Numerical Example 5

This numerical example was carried out to implement the algorithm proposed in previous section. This algorithm provides a better comprehension and knowledge about the frequency ranges where the measurements should be done according to the assumptions of that algorithm. For substructures A_1 and A_2 were considered six Euler-Bernoulli finite elements, making a total of thirteen elements for the whole structure. Figure 3.1 shows the beam considered with the corresponding finite elements distribution and substructures.

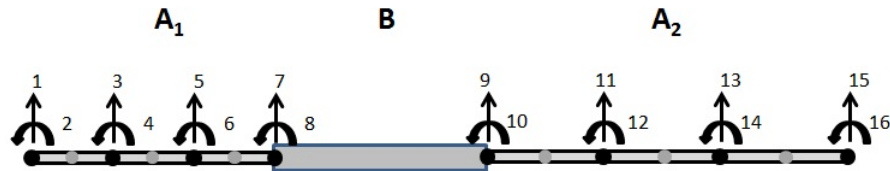


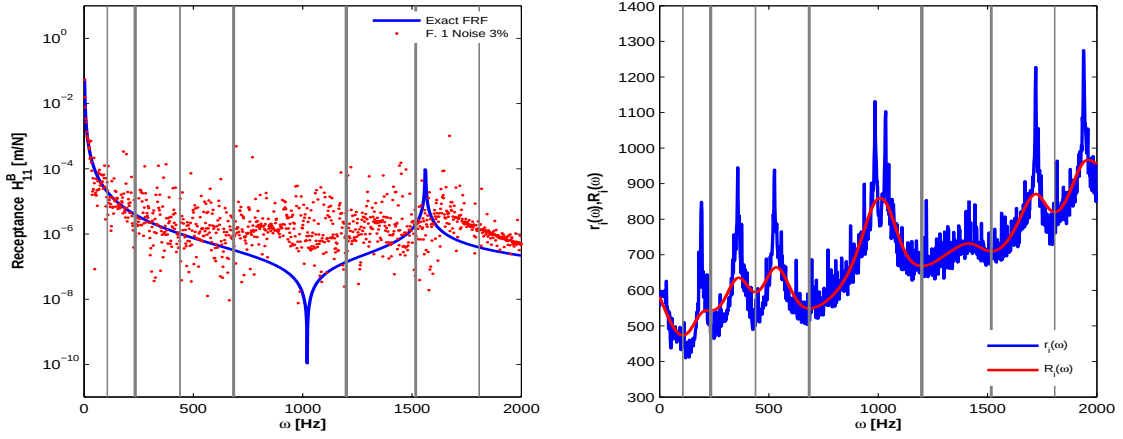
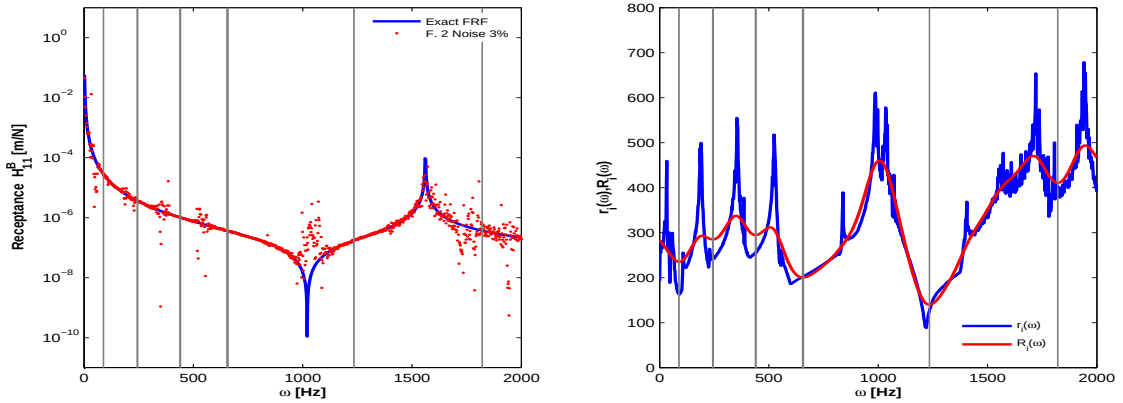
Figure 3.1: Coupling of substructures A and B, forming the structure C.

The beam's dimensions and characteristics are presented in table 3.1. Once again, in this example set up a noise level of 3 % dependent of the amplitude as considered in other examples.

Table 3.1: Characteristics and properties of the beam.

Beam	Length [mm]	Width [mm]	Thickness [mm]	E [GPa]	ρ [kg/m ³]
A1	270	30	5	194	7562
B	200	30	10	194	7562
A2	370	30	5	194	7562

The algorithm parameters set up as $\xi = 0.05$ and $v = 10^{-6}$ in order to perform the different analysis and find the frequency ranges where $R_i(\omega)$ has local minimums. The following figures show the results obtained for each formulation when this algorithm is applied. The grey bars correspond to the frequencies where $r_i(\omega)$ has local minimums.


 Figure 3.2: Receptance H_{11}^B , $r_i(\omega)$ and $R_i(\omega)$ of formulation 1.

 Figure 3.3: Receptance H_{11}^B , $r_i(\omega)$ and $R_i(\omega)$ of formulation 2.

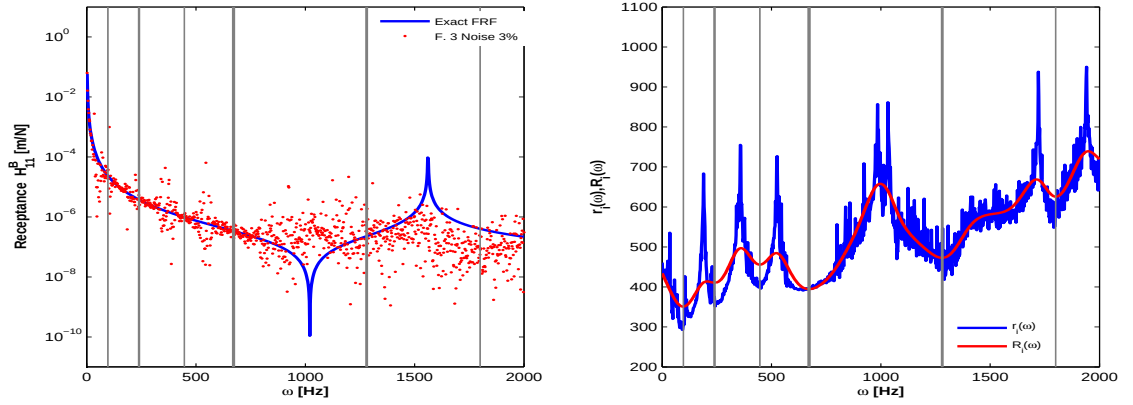


Figure 3.4: Receptance H_{11}^B , $r_i(\omega)$ and $R_i(\omega)$ of formulation 3.

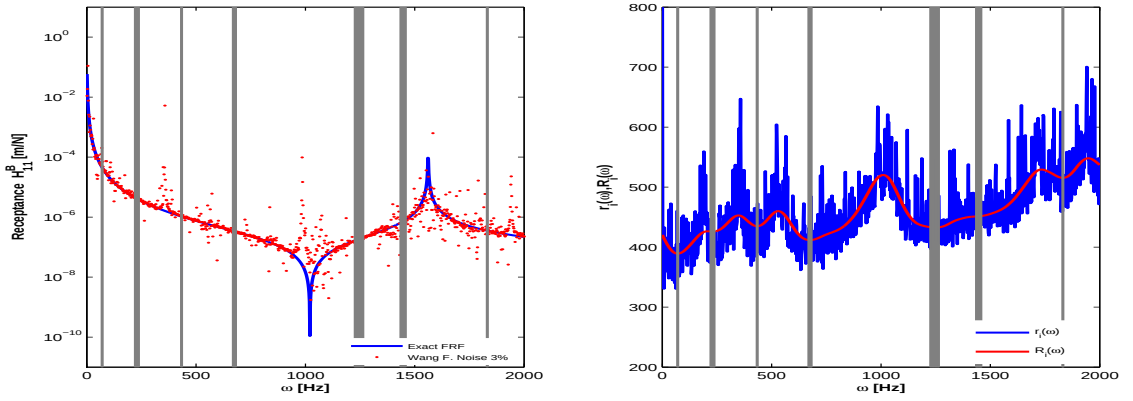


Figure 3.5: Receptance H_{11}^B , $r_i(\omega)$ and $R_i(\omega)$ of Wang formulation .

Through an analysis of previous figures, it is possible to get some conclusions. Wang and second formulations provide again better results than other formulations.

The first formulation (figure 3.2) shows an high discrepancy along the frequency range, being completely impossible to understand and fit the real *FRF*. The curve r_i reflects this fact with few frequencies where R_i is minimum.

The second formulation (figure 3.3) provides much better results. The level of uncertainty and discrepancy is lower than in previous formulation and noise doesn't affect so much the *FRF*. On the right side of figure 3.3 there are local minimums of R_i well visible and defined.

The third formulation (figure 3.4) doesn't provide good results and like in first formulation, it is possible to visualize an high discrepancy along the frequency range, although results are slightly better than in figure 3.2.

Wang formulation (figure 3.5) gives better results than last formulation. The noise effect is not too visible and there isn't a so huge discrepancy and uncertainty. On the right side of figure 3.5, it is possible to see some minimums of R_i . Despite the good results, formulation 2 provides

better results than Wang formulation.

Therefore, formulation 2 gives again better results and it reinforces that joints measurements should be carried out whenever possible. This algorithm serves as an interesting tool to choose the best frequency range to measure experimentally and its application gives information about matrix conditions for each frequency that is a complicated problem in this work.

3.2.2.1 Dynamic Stiffness Calculation

Once obtained the frequency ranges where the noise level is lower and the function R_i has local minimums according to the algorithm presented, these frequencies were selected in order to obtain the joint dynamic stiffness.

According to the results obtained in previous section, second and Wang formulations provide better results and consequently these two formulations were chosen to obtain the joint dynamic stiffness. Firstly, the joint dynamic stiffness was calculated at each frequency, $Z^B(\omega) = (H^B(\omega))^{-1}$, and then with the selected frequencies a quadratic regression was carried out using those points. Figures 3.6 and 3.7 show the corresponding results.

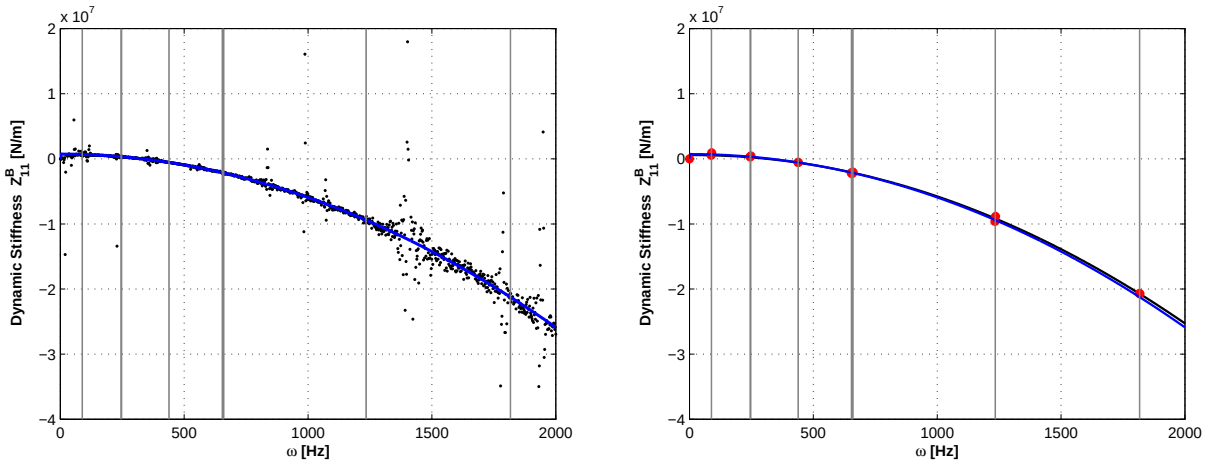


Figure 3.6: Z_{11}^B obtained with all frequencies and with the selected frequencies for formulation 2.

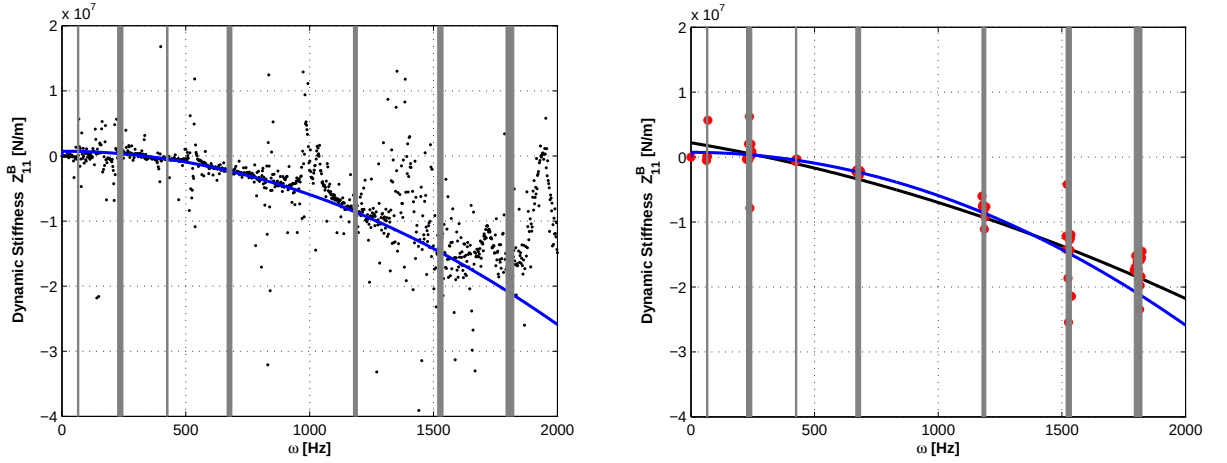


Figure 3.7: Z_{11}^B obtained with all frequencies and with the selected frequencies for Wang formulation.

The blue line corresponds to the exact dynamic stiffness whereas the black line is the dynamic stiffness obtained with the selected frequency ranges and the corresponding quadratic regression. Analysing the results obtained, formulation 2 provides again better results owing to the lower dispersion observed when noise is introduced in the samples.

Therefore, selecting the frequency ranges where the noise level is lower has a great importance and impact once we can obtain results much more accurate and close to the exact solution.

3.3 Estimative of Unmeasured FRF

An experimental analysis of a beam requires in most situations an estimative of *FRFs*, once some of those *FRFs* can not be measured due to experimental problems. Normally, in structures there are zones where is completely impossible to perform measurements owing to problems related to beam structure and geometry. Furthermore, measurements involving rotational degrees-of-freedom are difficult or even impossible to carry out and therefore, there is a need to find and develop formulations which allow the determination of these specific *FRFs*.

In [9] and [13], authors have approached this problem and new formulations have been developed. These formulations assume that part of the structure is modelled numerically, for instance using the finite element method, and the unavailable experimental *FRFs* are determined by expressions which allow the evaluation of these *FRFs*.

3.3.1 Theoretical Formulation

Maia and Batista [13] studied this problem and developed a new formulation to estimate unmeasured *FRFs* relating directly the dynamic stiffness of two substructures with the dynamic stiffness of the main structure, following a similar way as presented in section 2. The coordinates to take into account divide into two groups: experimentally measured coordinates, t , and coordinates r where the responses measurement is impossible to perform due to any reason, as shown in figure 3.8.

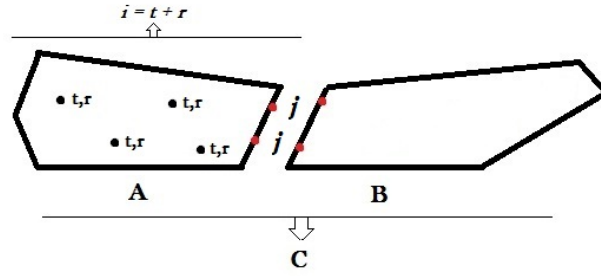


Figure 3.8: Coupling of substructures A and B, forming the structure C.

Using the alternative dynamic coupling expression to simplify, it comes:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ij}^A \\ H_{ji}^A \end{bmatrix} (H_{jj}^A + H_{jj}^B)^{-1} \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (3.4)$$

From equation 3.4 it is possible to write four equations, which possess information from the three structures:

$$H_{ii}^C = H_{ii}^A - H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{ji}^A \quad (3.5)$$

$$H_{ij}^C = H_{ij}^A - H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A \quad (3.6)$$

$$H_{ji}^C = H_{ji}^A - H_{jj}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{ji}^A \quad (3.7)$$

$$H_{jj}^C = H_{jj}^A - H_{jj}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A \quad (3.8)$$

Rearranging expression 3.5 and representing the pseudo-inverse $(H_{ij}^A)^+$,

$$(H_{ij}^A)^+ (H_{ii}^A - H_{ii}^C) = (H_{ii}^A - H_{ii}^C)^{-1} H_{ji}^A \quad (3.9)$$

Replacing in expression 3.7:

$$H_{ji}^C = H_{ji}^A - H_{jj}^A (H_{ij}^A)^+ (H_{jj}^A + H_{jj}^B) \quad (3.10)$$

Reorganizing the expression 3.7:

$$(H_{jj}^A)^{-1} (H_{ji}^A - H_{ji}^C) = (H_{jj}^A + H_{jj}^B)^{-1} H_{ji}^A \quad (3.11)$$

Replacing in expression 3.24:

$$H_{ii}^C = H_{ii}^A - H_{ij}^A (H_{jj}^A)^{-1} (H_{ji}^A + H_{ji}^C) \quad (3.12)$$

Reorganizing the expression 3.6:

$$(H_{ij}^A)^+ (H_{ij}^A - H_{ij}^C) = (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A \quad (3.13)$$

$$I_{ij} - (H_{ij}^A)^+ H_{ij}^C = (H_{jj}^A + H_{jj}^B)^{-1} H_{jj}^A \quad (3.14)$$

Replacing in expression 3.15:

$$H_{jj}^C = H_{jj}^A (H_{ij}^A)^+ H_{ij}^C \quad (3.15)$$

Summarizing, it is possible to determine with the following expressions the matrix H^C :

$$\begin{cases} H_{ji}^C = H_{ji}^A - H_{jj}^A (H_{ij}^A)^+ (H_{ii}^A - H_{ii}^C) \\ H_{ii}^C = H_{ii}^A - H_{ij}^A (H_{jj}^A)^{-1} (H_{ji}^A - H_{ji}^C) \\ H_{jj}^C = H_{jj}^A (H_{ij}^A)^+ H_{ij}^C \end{cases}$$

The coordinates i from structure C will be composed by coordinates t and r . Therefore, the new matrix H^C will suffer the following transformation:

$$H^C = \begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} \quad (3.16)$$

$$i = t + r \longrightarrow H^C = \begin{bmatrix} H_{tt}^C & H_{tr}^C & H_{tj}^C \\ H_{rt}^C & H_{rr}^C & H_{rj}^C \\ H_{jt}^C & H_{jr}^C & H_{jj}^C \end{bmatrix} \quad (3.17)$$

In the new matrix H^C only the sub matrix H_{tt}^C can be experimentally determined. Considering only coordinates i as translation coordinates, $i = t$:

$$H_{jt}^C = H_{jt}^A - H_{jj}^A (H_{tj}^A)^+ (H_{tt}^A - H_{tt}^C) \quad (3.18)$$

Relating translations with rotations, $i = r$ and $i = t$:

$$H_{rt}^C = H_{rt}^A - H_{rj}^A (H_{jj}^A)^{-1} (H_{jt}^A - H_{jt}^C) \quad (3.19)$$

$$i = t, i = r \longrightarrow H_{jr}^C = H_{jr}^A - H_{jj}^A (H_{tj}^A)^+ (H_{tr}^A - H_{tr}^C) \quad (3.20)$$

Considering only i coordinates as rotation coordinates, $i = r$:

$$H_{rr}^C = H_{rr}^A - H_{rj}^A (H_{jj}^A)^{-1} (H_{jr}^A - H_{jr}^C) \quad (3.21)$$

Lastly, considering coordinates i as translations, it gives:

$$i = t \longrightarrow H_{jj}^C = H_{jj}^A (H_{tj}^A)^+ H_{tj}^C \quad (3.22)$$

The matrix H^A is determined through the finite element method. Thus, starting from the responses known, all unknown FRFs can be estimated through a sequential process.

3.3.2 Alternative Formulation

All the FRF of matrix H^C could be determined through a simpler way. Resuming the equations 3.23, 3.24 and 3.25:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} (H_{jj}^A + H_{jj}^B)^{-1} \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (3.23)$$

$$H_{ii}^C = H_{ii}^A - H_{ij}^A (H_{jj}^A + H_{jj}^B)^{-1} H_{ji}^A \quad (3.24)$$

$$i = t + r \longrightarrow H^C = \begin{bmatrix} H_{tt}^C & H_{tr}^C & H_{tj}^C \\ H_{rt}^C & H_{rr}^C & H_{rj}^C \\ H_{jt}^C & H_{jr}^C & H_{jj}^C \end{bmatrix} \quad (3.25)$$

From equation 3.24, it is possible to write:

$$(H_{jj}^A + H_{jj}^B)^{-1} = (H_{ij}^A)^+ (H_{ii}^A - H_{ii}^C) (H_{ji}^A)^+ \quad (3.26)$$

Replacing the equation 3.26 in equation 3.4:

$$\begin{bmatrix} H_{ii}^C & H_{ij}^C \\ H_{ji}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{ii}^A & H_{ij}^A \\ H_{ji}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{ij}^A \\ H_{jj}^A \end{bmatrix} ((H_{ij}^A)^+ (H_{ii}^A - H_{ii}^C) (H_{ji}^A)^+) \begin{bmatrix} H_{ji}^A & H_{jj}^A \end{bmatrix} \quad (3.27)$$

If we use the coordinates i decomposition as presented in equation 3.27, the standard equation could be written as:

$$\begin{bmatrix} H_{tt}^C & H_{tr}^C & H_{tj}^C \\ H_{rt}^C & H_{rr}^C & H_{rj}^C \\ H_{jt}^C & H_{jr}^C & H_{jj}^C \end{bmatrix} = \begin{bmatrix} H_{tt}^A & H_{tr}^A & H_{tj}^A \\ H_{rt}^A & H_{rr}^A & H_{rj}^A \\ H_{jt}^A & H_{jr}^A & H_{jj}^A \end{bmatrix} - \begin{bmatrix} H_{tj}^A \\ H_{rj}^A \\ H_{jj}^A \end{bmatrix} ((H_{tj}^A)^+ (H_{tt}^A - H_{tt}^C) (H_{jt}^A)^+) \begin{bmatrix} H_{jt}^A & H_{jr}^A & H_{jj}^A \end{bmatrix} \quad (3.28)$$

3.3.3 Numerical Example 6

In order to evaluate the feasibility of the alternative formulation exposed in previous section, the following example was carried out. Figure 3.9 shows the approached structure and each substructure was divided into six Euler-Bernoulli finite elements with two degrees-of-freedom per node.

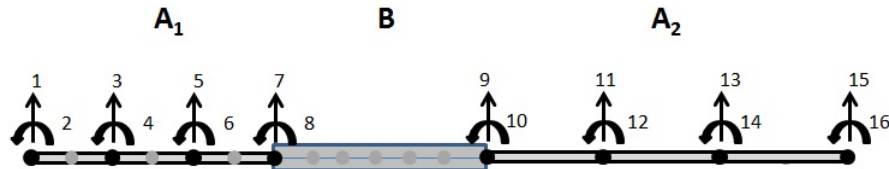


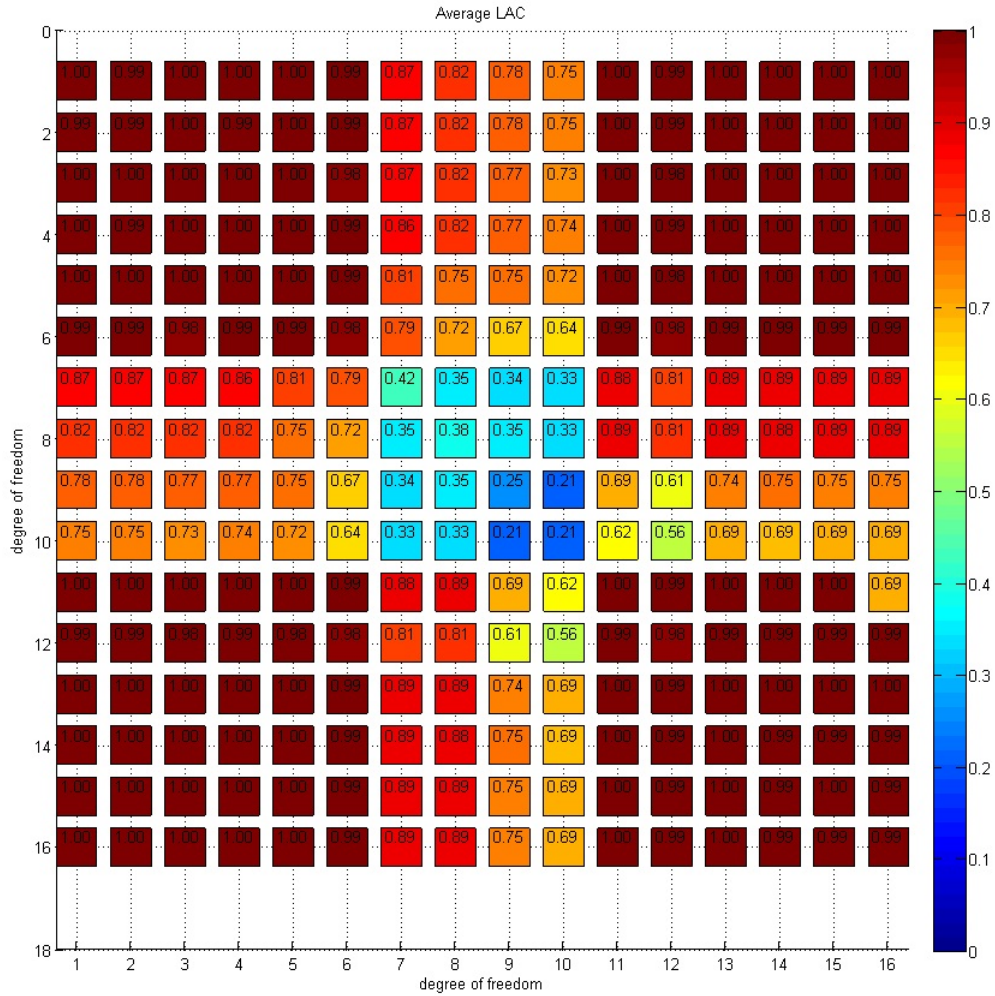
Figure 3.9: Coupling of substructures A and B, forming the structure C.

The characteristics and properties of the beam are referred in table 3.2. The translational degrees-of-freedom are considered as coordinates where is possible to perform measurements, coordinates t , whereas coordinates r represent either coordinate where measurements are impossible for any reason, in this specific case the rotational degrees-of-freedom.

Table 3.2: Characteristics and properties of the beam.

Beam	Length [mm]	Width [mm]	Thickness [mm]	E [GPa]	ρ [kg/m ³]
A1	270	30	5	194	7562
B	200	30	10	194	7562
A2	370	30	5	194	7562

The experimental errors were simulated through the introduction of noise according to expression 2.83. In the known *FRFs* were introduced 3 % of noise level.

Figure 3.10: Average *LAC*.

The comparison of results obtained through the alternative formulation with results from a finite element program was evaluated using the correlation criterium *LAC*, as illustrated in figure 3.10. Analysing the figure it is possible to conclude that all estimations involving joint coordinates have a lower *LAC* than other coordinates. Therefore, as seen in previous chapters the importance of carrying out measurements in joints coordinates has a considerable importance and impact in the final results, once their generation provides bad results.

Figure 3.11 represents some results obtained with this alternative formulation. The estimates of H_{24}^C and H_{36}^C indicate good coherence and accuracy, regardless of the noise introduced to simulate the experimental errors. The two other estimations involving joint coordinates have a worse accuracy when compared with two other curves. These results reinforce that experimentally joint coordinates should be measured when the location and the beam geometry allow the instrumentation of transducers.

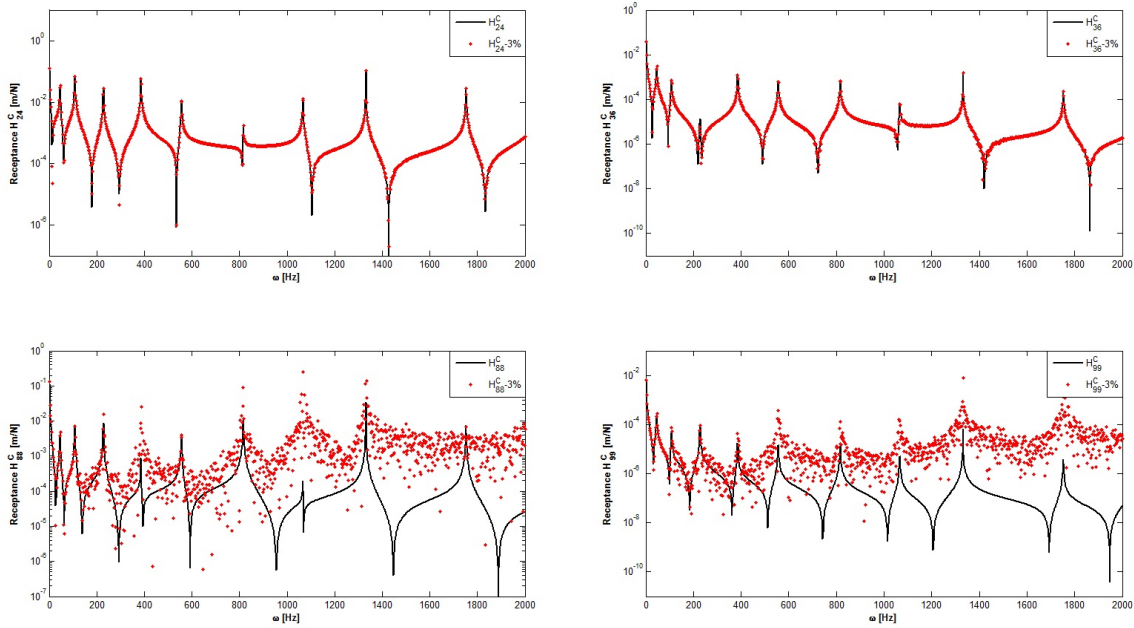


Figure 3.11: Estimated unmeasured FRFs.

3.4 Mass Cancellation

The use of transducers such as accelerometers and forces sensors induces changes in the measured *FRFs* due to the loading of the structure with extra masses (although sometimes it can be considered negligible). Normally, the experimental setup allows only for the measurement of *FRF* which includes the beam and the additional transducer's mass (the transducer behaves as rigid body in the frequency range of interest). Therefore, the mass or inertia cancellation should be done if one extra mass (normally one transducer) is added, and if this mass has a value close to a modal mass associated to a natural frequency, the frequency value will be lower than the real value.

Generally, the location of some resonances is affected by the transducers' mass, so a mass cancellation procedure is desirable in order to eliminate these effects. This procedure is also required if the transducer position is moved along the structure to perform the corresponding *FRF* measurements, as usually.

3.4.1 Theoretical Formulation

Silva et al. [14] suggested an algorithm to perform the mass cancelation based on the classic uncoupling technique. Following the equation 2.30, the necessary equations to the mass cancelation could be obtained:

$$H_{jj}^C = (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{jj}^A \quad (3.29)$$

$$H_{ji}^C = (I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} H_{ji}^A \quad (3.30)$$

In the previous expressions 3.29 and 3.30, H_{jj}^C and H_{ji}^C are the *FRFs* experimentally measured which will be cancelled, whereas H_{jj}^A and H_{ji}^A correspond to the substructure *FRFs* after mass cancelation, and Z_{jj}^B is the accelerometer dynamic stiffness.

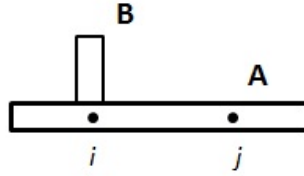


Figure 3.12: Coupling of structure A with accelerometer B.

Determining H_{jj}^A and H_{ji}^A through the following equality:

$$(I_{jj} + H_{jj}^A Z_{jj}^B)^{-1} = ((H_{jj}^B + H_{jj}^A) Z_{jj}^B)^{-1} = H_{jj}^B (H_{jj}^A + H_{jj}^B)^{-1} \quad (3.31)$$

Replacing above in expressions 3.29 and 3.30:

$$H_{jj}^C = H_{jj}^B (H_{jj}^B + H_{jj}^A)^{-1} H_{jj}^A \quad (3.32)$$

$$H_{ji}^C = H_{jj}^B (H_{jj}^B + H_{jj}^A)^{-1} H_{ji}^A \quad (3.33)$$

Rearranging the expression 3.32:

$$(H_{jj}^B)^{-1} H_{jj}^C = (H_{jj}^B + H_{jj}^A)^{-1} H_{jj}^A \quad (3.34)$$

Multiplying at left by $(H_{jj}^B + H_{jj}^A)$:

$$H_{jj}^C + H_{jj}^A (H_{jj}^B)^{-1} H_{jj}^C = H_{jj}^A \quad (3.35)$$

$$H_{jj}^C = H_{jj}^A (I_{jj} - (H_{jj}^B)^{-1} H_{jj}^C) \quad (3.36)$$

$$H_{jj}^A = H_{jj}^C (I_{jj} - (H_{jj}^B)^{-1} H_{jj}^C)^{-1} \quad (3.37)$$

If the accelerometer was considered as a rigid body, the accelerance comes:

$$H_{jj}^B = \frac{1}{m_b} \quad (3.38)$$

The classic mass cancelation expression of a direct *FRF* could be written as:

$$H_{jj}^A = H_{jj}^C (1 - m_b H_{jj}^C)^{-1} = \frac{H_{jj}^C}{1 - m_b H_{jj}^C} \quad (3.39)$$

Through equation 3.30:

$$(H_{jj}^B)^{-1} H_{ji}^C = (H_{jj}^B + H_{jj}^A)^{-1} H_{ji}^A \quad (3.40)$$

Multiplying at left by $(H_{jj}^B + H_{jj}^A)$:

$$H_{ji}^A = H_{ji}^C + H_{jj}^A (H_{jj}^B)^{-1} H_{ji}^C \quad (3.41)$$

$$H_{ji}^A = H_{ji}^C + m_b H_{jj}^A H_{ji}^C \quad (3.42)$$

$$H_{ji}^A = H_{ji}^C + \frac{m_b H_{jj}^C H_{ji}^C}{1 - m_b H_{jj}^C} \quad (3.43)$$

The classic mass cancelation expression of a transfer *FRF* could be written as:

$$H_{ji}^A = \frac{H_{ji}^C}{1 - m_b H_{jj}^C} \quad (3.44)$$

3.4.2 Numerical Example 7

This numerical example is performed with the aim of implement and analyse the formulations presented in the previous section related to the mass cancellation process. In order to implement those formulations a beam was considered with a determined finite element distribution, as presented in figure 3.13:

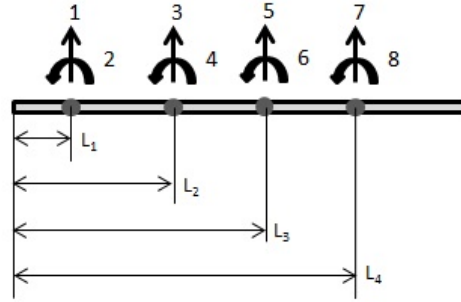


Figure 3.13: Finite element distribution.

The properties and characteristics of the beam are presented on table 3.3:

Table 3.3: Beam properties and characteristics.

L	500 mm	L_4	370 mm
L_1	5 mm	E	196 GPa
L_2	123.3 mm	ρ	7718.2 kg/m ³
L_3	246.7 mm	m_{ac}	4.8 g

The accelerometer will be collocated in the first node to analyse its influence in the beam behaviour. A finite element programme was used to perform the *FRFs* generation, firstly with the accelerometer mass and then without the accelerometer mass influence. Then, the mass cancellation formulations were implemented to compare the *FRFs* obtained by the formulations with the *FRFs* from the finite element program without the mass accelerometer influence. These two *FRFs* curves must be identic once the formulation objective is to cancel the accelerometer mass.

The accelerances corresponding to the translation degrees-of-freedom , H_{11} , H_{13} , H_{15} and H_{17} , were measured with the accelerometer in the first node and varying the excitation point along the beam.

Figure 3.14 shows the results obtained for this formulation. The black curve represents the *FRF* obtained with the instrumented accelerometer, whereas the red curve corresponds to the *FRF* calculated without the accelerometer influence and the blue points compose the *FRF* obtained by the mass cancellation process.

3.4. MASS CANCELATION

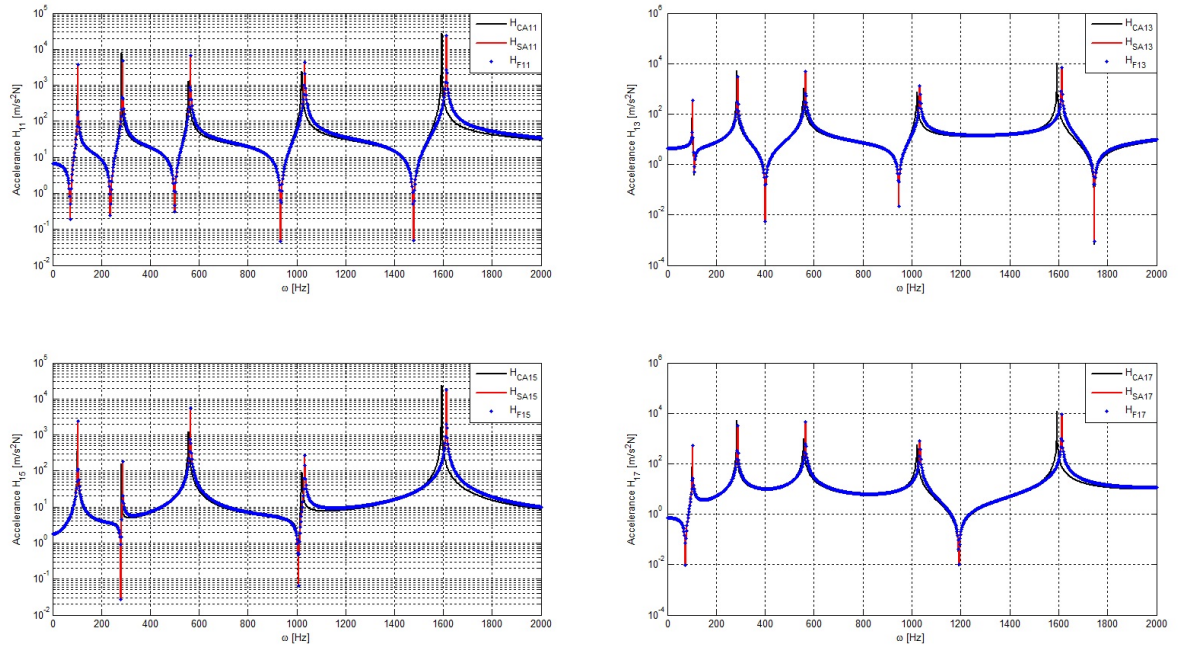


Figure 3.14: Results obtained by the mass cancellation formulations.

Through observation and analysis of figure 3.14, it is visible that blue points fit exactly the red curve which represents the response without the accelerometer mass contribution. This fact shows the great accuracy of the method. Furthermore, the natural frequencies of the *FRF* obtained by the mass cancellation process are lightly higher than the natural frequencies which have the influence of the accelerometer mass. Moreover, the mass cancellation process moves the natural frequencies to the right side once the accelerometer mass is removed and consequently the natural frequencies increase.

Chapter 4

Uncoupling with Experimental FRFs

4.1 Introduction

Throughout this work, several uncoupling techniques have been presented with the corresponding numerical examples which were tested and validated through theoretical expressions and other sources, [2], [8], [9].

Therefore, at this point of the work it is important to apply those uncoupling techniques to experimental cases in order to evaluate their performance under experimental data. The analysis of these techniques was carried out with aluminum specimens which hold bolted joints incorporated.

Figure 4.1 summarizes the treatment steps of experimental data acquired. Firstly, the experimental data acquired will be properly treated and organized. The following step is an estimative of unmeasured *FRFs* whereby the complete matrix H^C is obtained using numerical data from substructure A and the experimental data of the main structure.

Then, the four uncoupling techniques are applied in order to obtain the matrix H_{jj}^B which characterizes the joint. When the matrix H_{jj}^B is available, it is possible to achieve the joint dynamic stiffness carrying out the inverse of that matrix. Performing an alternative coupling between the joint information obtained through the uncoupling techniques and the matrix H^A obtained numerically, we achieve the global matrix H^C and consequently this matrix could be compared with the initial matrix H^C to evaluate the process effectiveness.

Three different specimens were considered in this experimental work. Firstly, the experimental work started with a simple beam without any joint (figure 4.2); the purpose of this arrangement is to analyse a beam for which there are theoretical and numerical solutions well defined in order to understand if the experimental setup is well tuned and totally prepared and capable of analyse the following beams with joints incorporated.

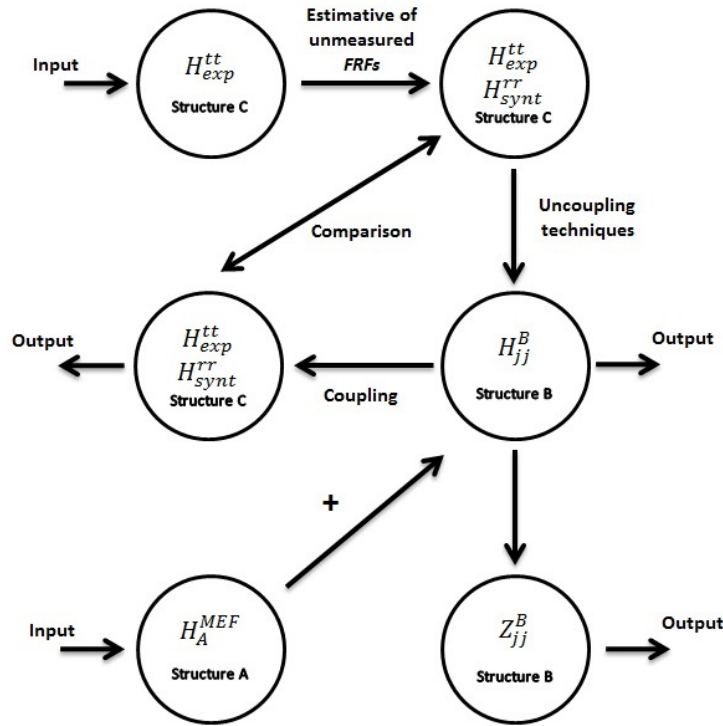


Figure 4.1: Experimental work diagram.

Figure 4.2: Specimen 1 (Beam 1).

Secondly, a beam with a bolted joint (figure 4.3) was analysed, using the setup previously prepared and tuned.



Figure 4.3: Specimen 2 (Beam 2).

Then, the bolted joint as considered in beam 2 (figure 4.4) was taken into account. Analysing this specimen, we want to measure directly the translation *FRFs* and then compare them with that obtained through the uncoupling techniques. The definition drawings of the three types of beams analysed are available in Appendix D.

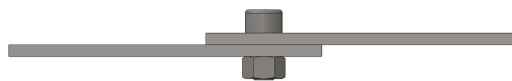


Figure 4.4: Specimen 3 (Bolted joint).

4.2. FINITE ELEMENT MESH

In this working phase, lots of experimental measurements will be carried out in the three specimens, using both the shaker and the hammer as excitation means. Table 4.1 summarizes all the experimental measurements, as well as the theoretical methodologies applied to each experimental data set.

Table 4.1: Experimental measurements carried out in each beam.

Methods	Beam 1		Beam 2		Beam 3	
	Excitation Mean		Excitation Mean		Excitation Mean	
	Hammer	Shaker	Hammer	Shaker	Hammer	Shaker
Measurements	X	X	X	X	X	-
Mass	-	-	X	X	-	-
Cancelation	-	-	X	-	-	-
Estimative of unmeasured FRFs	-	-	X	-	-	-
Uncoupling techniques	-	-	X	-	-	-
Analysis of frequencies ranges	-	-	X	-	-	-
New coupling	-	-	X	-	-	-

4.2 Finite Element Mesh

The experimental measurements need to be previously approached and studied in order to foresee the experimental results which we are looking for. Therefore, some theoretical studies were done at this step, such as a survey about the number of natural frequencies that could appear in a certain frequency range and the joint length once it is important to have at least one natural frequency in the frequency range experimentally taken into account.

The experimental frequency range taken into account depends on the capacities of the experimental devices which are available to perform these experimental tests. Thus, a frequency range between 0 to 800 Hz was considered with a resolution of 801 lines.

The finite element mesh considered is shown in figures 4.5 and 4.3, representing the degrees-of-freedom and measurement points, respectively. The nomenclature of the experimental *FRFs* is set up according to *DOF* distribution, for the sake of coherence with the numerical formulations. The position of each measurement point coincides exactly with the corresponding node of the finite element mesh, and therefore experimentally we just measure the translation *DOF* in a node. The experimental *FRFs* obtained will define the matrix H^C which corresponds to the global structure, whereas the theoretical *FRFs* which define the matrix H^A will be obtained through the finite element programme, *BAPMEF*.

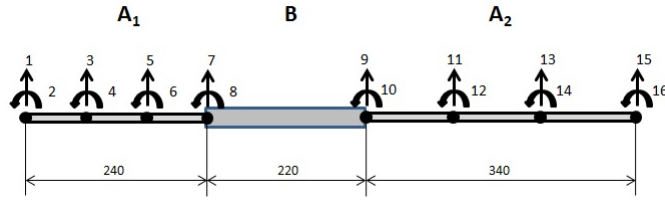


Figure 4.5: Degrees-of-freedom considered per substructure.



Figure 4.6: Measurement points considered per substructure.

Each substructure A_i divides into eighteen Euler-Bernoulli finite elements. A study about the impact of the number of finite elements per substructure in the uncoupling techniques was carried out. In other words, a study about the mesh effect was done and it has been found out a number of finite elements from which there is no interference in the final results; the number of elements is close to the value considered. The properties and dimensions of each substructure are given on table 4.2. To the structure which corresponds to the joint, we just consider the length for the sake of coherence, once it has incorporated the bolt and the nut which change the beam properties in that area.

Table 4.2: Characteristics and properties of the experimental beam.

Beam	Length [mm]	Width [mm]	Thickness [mm]	E [GPa]	ρ [kg/m ³]
A1	240	25	5	70	2700
B	220	-	-	-	-
A2	340	25	5	70	2700

The *FRFs* of both substructures were calculated through the finite element programme taking into account the assumptions presented previously. Nowadays, the experimental devices available just enable measurements of translation degrees-of-freedom and the corresponding *FRFs* of rotation have to be estimated considering the finite element results and the experimental results. The same methodology exposed in section 3.3 will be adopted in this case and then with all *FRFs* available, the joint uncoupling will be carried out using the four uncoupling techniques presented.

A previous study about modal shapes was carried out in order to evaluate the position of the two suspension ropes. The two suspension ropes were placed on the position of the vibration nodes of a beam, considering the nodes of the first modal shape for a free-free beam. So, the wire ropes were placed at 179.2 mm of each extremity of the beam. Then, a beam with a concentrated mass, to take into account the influence of the bolted joint, was simulated in order to study the nodes position of each modal shape. Table 4.3 shows the results obtained for each mode.

Table 4.3: Nodes Position per Mode.

Nodes Position per Mode / [mm]	
Mode 1	288 - 512
Mode 2	224 - 400 - 576
Mode 3	176 - 320 - 464 - 624
Mode 4	144 - 272 - 400 - 528 - 640

Analysing table 4.3, there is no node identified in this survey that coincides exactly with the position of suspension ropes. Thus, this fact avoids some experimental problems that could exist if one node of a mode coincides with the suspension ropes or would be in its neighborhood.

4.3 Experimental Analysis of Beam 1

As previously stated, Beam 1 will serve as a way of comparison between the results obtained with a shaker and a hammer, and also as a validation and adjustment of the experimental setup. Thus, some measurements were performed in this beam using a hammer and a shaker. Using the shaker in a fixed excitation node, the responses were measured in all nodes changing the corresponding position of the accelerometer; using the hammer to excite all the nodes, the response was measured in a fixed node varying the corresponding excitation node. All the results obtained in this experimental analysis are available in Appendix B.

4.3.1 Using the Hammer 5800SL

Figure 4.8 shows the results obtained with a hammer as excitation mean. As we can see, the experimental *FRF* fits in a good way the theoretical *FRF* obtained by the finite element method. The behaviour of both curves is similar, although there is some discrepancy in certain frequency ranges. In the frequency range between 0 to 50 Hz, we can see some discrepancy due to the influence of extra factors, such as the suspension ropes.

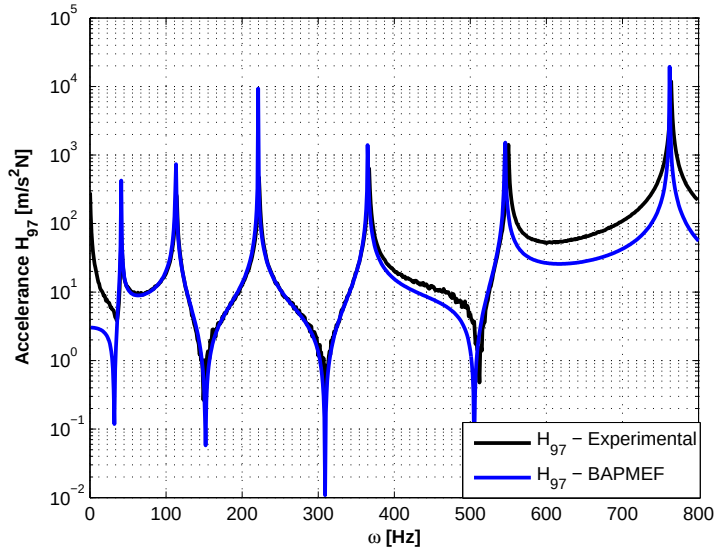
Figure 4.7: Accelerance H_{97} .

Table 4.4 shows a study about the natural frequencies. The experimental values obtained are really near the theoretical values calculated, which indicates an excellent accuracy obtained with this excitation mean.

Table 4.4: Natural frequencies obtained using a hammer.

Natural Frequency	Theoretical Value	Experimental Value	Error / %
ω_1	40.889	41.000	0.272
ω_2	112.715	113.000	0.253
ω_3	220.966	221.000	0.015
ω_4	365.270	367.000	0.474
ω_5	545.643	550.000	0.799
ω_6	762.102	764.000	0.249

4.3.2 Using the Shaker LDS210 (Force transducer 8200)

The results obtained with shaker are presented in figure 4.8. Analysing this figure, we can see that the experimental curve fits very well the theoretical curve, despite some discrepancy present in the highest frequencies that could be related with the drive rod which introduces a natural frequency near the highest frequencies. Here again, it is possible to see some discrepancy in the first frequencies owing to external factors as stated in previous section.

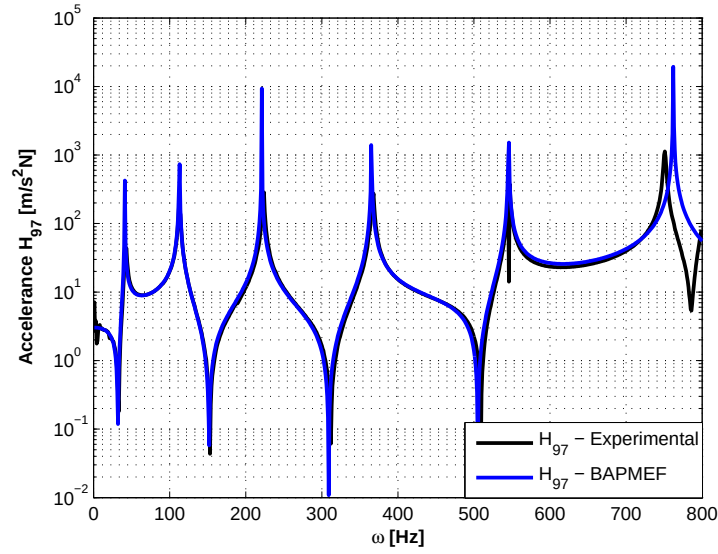


Figure 4.8: Accelerance H_{97} .

Table 4.5 gives a study about natural frequencies. The results obtained experimentally are really similar to the theoretical values, despite an higher error in some natural frequencies.

Table 4.5: Natural frequencies obtained using a shaker.

Natural Frequency	Theoretical Value	Experimental Value	Error / %
ω_1	40.889	42.000	2.717
ω_2	112.715	111.000	1.522
ω_3	220.966	221.000	0.015
ω_4	365.270	366.000	0.199
ω_5	545.643	543.000	0.484
ω_6	762.102	756.000	0.801

4.4 Evaluation of the torque effect

The torque effect contributes somewhat to the behaviour of frequency-response functions. Thus, an experimental study was performed with four different torque values, 5 N.m, 10 N.m, 15 N.m and 20 N.m. A beam with a bolted joint (figure 4.3) was taken into account and a hammer was used to excite the beam, and the response was measured with an accelerometer. Using a dynamometric key, the measurements with four different torques were performed.

The beam was suspended with two nylon ropes in order to simulate free-free boundary conditions. These ropes were collocated specifically in the vibrations nodes of the first vibration mode to get a better analysis without the interference of the other modal modes. Figure 4.9 shows the results obtained.

Analysing figure 4.9 it is possible to understand that the torque affects lightly the *FRFs*' behaviour and just in the sixth natural frequency is visible some discrepancy between the curves. Thus, torque affects the *FRFs*' behaviour and its interference could be considered null to certain values. Figure 4.13 provides a zoom over the sixth natural frequency zone.

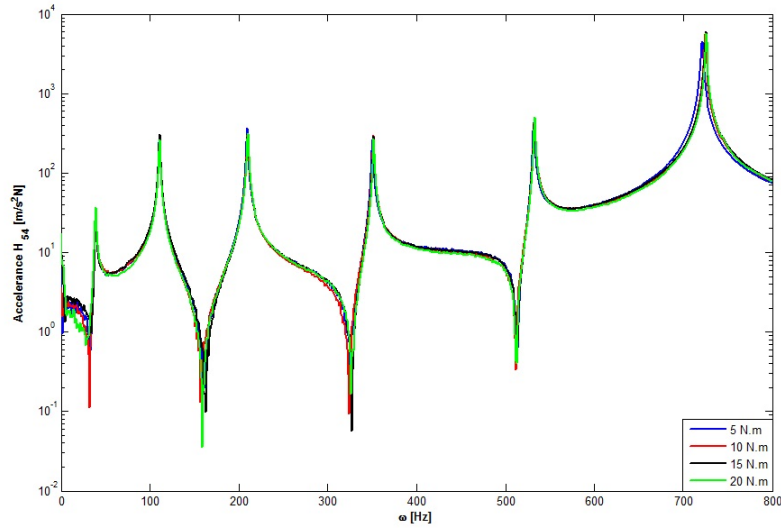


Figure 4.9: Experimental results with different values of torque.

Indeed, the torque value has some interference and it becomes constant from values around 15 N.m. In the experimental measurements a torque of 20 N.m was taken into account in order to ensure that there is no interference of this factor in the final results.

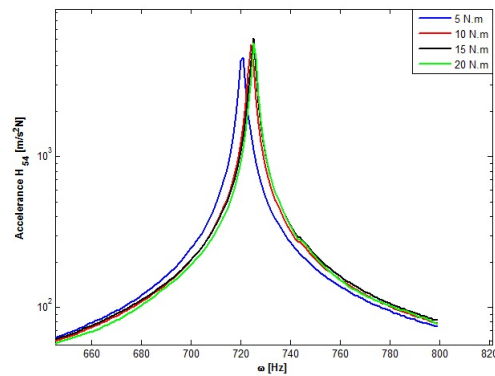


Figure 4.10: Zoom taken from the sixth natural frequency.

4.5 Experimental Analysis of Beam 2

This experimental analysis was carried out with the purpose of identify a joint using the beam 2 (figure 4.3) which holds a bolted joint in its structure. It took into account the same number of measurement points with an identic disposition as considered in previous experimental analysis. Once again, the measurements used as excitation means a hammer and a shaker.

Furthermore, in this experimental analysis the four uncoupling techniques were implemented in order to obtain the joint dynamic behaviour. The effects of transducers' mass were removed according to a mass cancelation process. Therefore, once the experimental *FRFs* obtained, the *FRFs* which involve rotational degrees-of-freedom were generated considering the algorithm proposed in section 3.3. Then, with the matrix H^C completely available, the four uncoupling techniques were implemented. All the results obtained in this experimental analysis are available in Appendix B.

4.5.1 Using the Hammer 5800SL

The frequency response functions were measured in all degrees-of-freedom, exciting all points and measuring the response in the corresponding points. The necessity of this process comes from the fact that we need the complete matrix H^C , at each frequency ω , to perform the joint uncoupling with the four techniques.

Figure 4.11 shows some results obtained in this experimental analysis. We can see a good coherence and accuracy both in direct and in transfer *FRFs*.

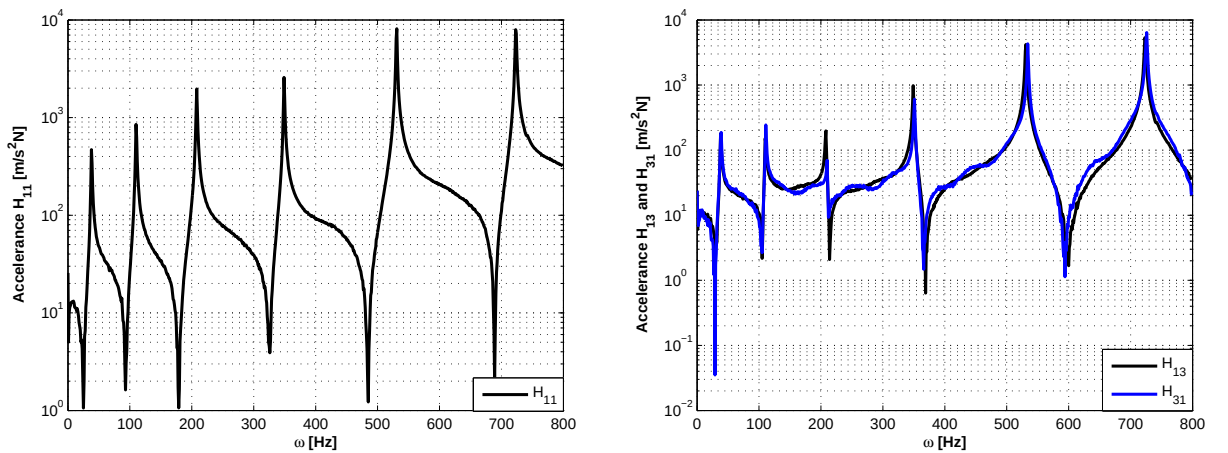


Figure 4.11: Accelerances H_{11} and H_{13} .

For the sake of coherence and accuracy, all the experimental *FRFs* were subjected to a mass

cancelation process once the accelerometer mass influences the structure behaviour. Although the accelerometer has a low mass, 1 g, it is important to eliminate this effect to avoid any incoherence. Therefore, this methodology was applied to each matrix row obtained experimentally following a similar process as in section 3.4.

In order to evaluate the compatibility and correspondence between the transfer *FRFs*, the correlation criterion *LAC* was applied to the measured *FRFs* and then to the cancelled ones. Figure 4.12 shows the results of the application of this indicator, where is possible to see a slight increase of the mean value in most transfer *FRFs*. For the sake of coherence, the main diagonal has always the maximum value, 1, once those represent direct *FRFs*.

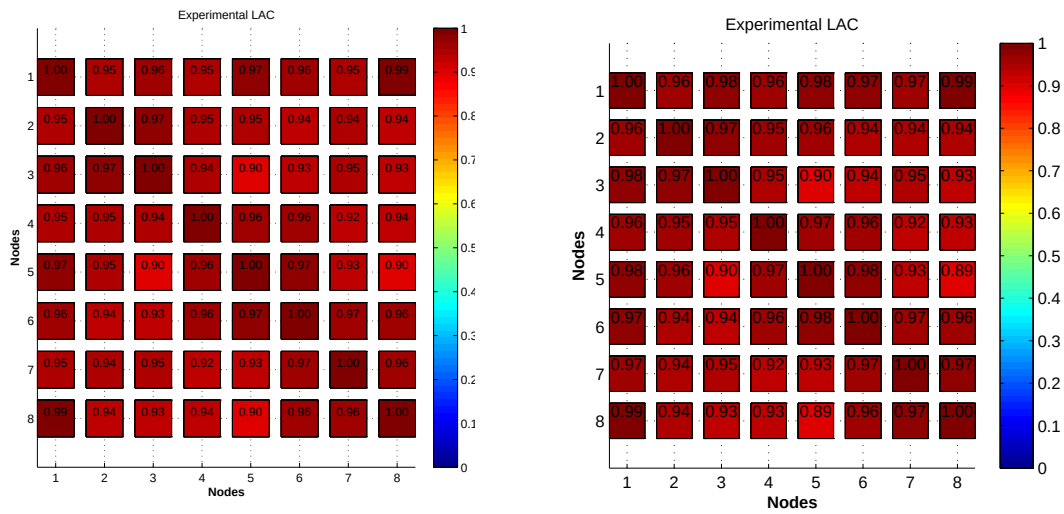


Figure 4.12: Experimental *LAC* between transfer *FRFs* before and after of mass cancelation process, respectively.

Once obtained and treated all the experimental *FRFs*, the frequency-response functions related with rotational degrees-of-freedom were generated. The results obtained showed a good accuracy, despite the noise usually present in experimental data. In fact, these results overcome the initial expectations, once worse results were expected according to numerical example previously approached.

In this specific case, extra mass is introduced in the structure by the beam overlap in the joint section, and also by the bolt with the corresponding nut used to link the two substructures. Therefore, if we introduce any concentrated mass in the structure, we will expect lower natural frequencies. In order to simulate this fact, a beam with the joint length was analysed, considering a concentrated mass which will represent the influence of these extra masses. It had been considered to these extra masses, a global value of 40 g.

Thus, the uncoupling techniques were applied taking into account the experimental data previously acquired and the rotational *FRFs* generated. Figure 4.13 shows the accelerance H_{11}^B obtained through the four uncoupling techniques.

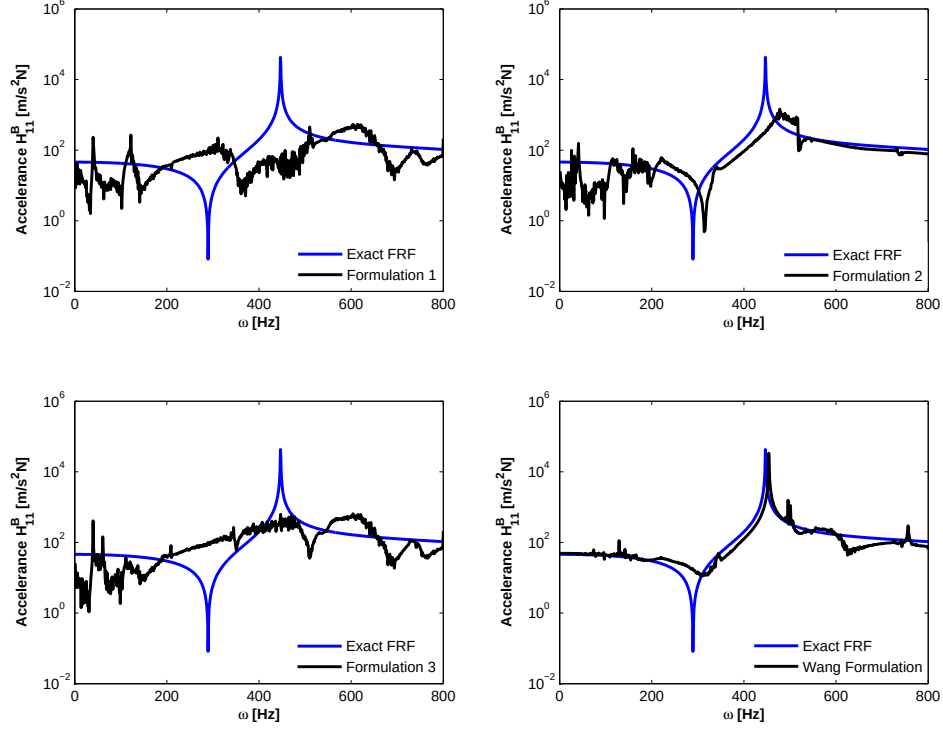


Figure 4.13: Accelerance H_{11}^B obtained by uncoupling from experimental results.

The theoretical examples approached in previous section of this work suggested that there are two formulations which provide better results, the second and Wang formulations. In fact, analysing figure 4.13, these two formulations give better results. The two other formulations, the first and the third, provided bad results as it was expected taking into account the theoretical examples.

Indeed, the results obtained were much better than previously expected. The *FRF* obtained with Wang formulation almost fits the exact curve, and it is visible the resonance and antiresonance zones with a natural frequency well defined. From the theoretical examples, it was expected a better behaviour of formulation 2 once this had produced excellent results in previous applications. Therefore, the uncoupling techniques and other methods used were implemented with success and the results obtained verify these facts.

• Dynamic Stiffness Calculation

Having the accelerance matrix H_{jj}^B available, the joint dynamic stiffness can be calculated. The final results are presented in figure 4.14. The dynamic stiffness calculated with all frequencies takes the following shape because of noise which affects its behaviour along the frequency range and consequently the dynamic stiffness becomes completely different. According to numerical examples carried out in previous sections, it was expected a dynamic stiffness equivalent but the noise naturally added in experimental measurements causes this behaviour.

Analysing both figures, the second formulation provides better results than Wang formulation, despite the low accuracy as we can see.

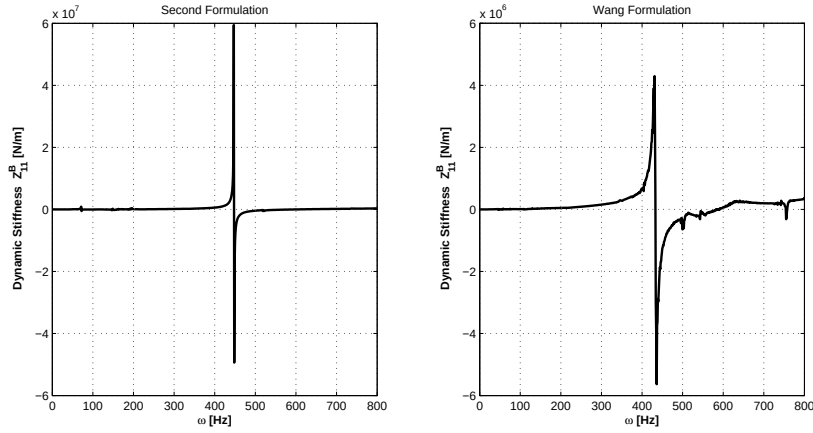


Figure 4.14: Dynamic stiffness Z_{11}^B according to second and Wang formulations.

- Coupling using matrix H_{jj}^B :

Then, the inverse of the uncoupling process, the coupling, was carried out using the joint FRFs, H_{jj}^B , obtained through the uncoupling process which took in account the experimental data. The alternative coupling technique, which doesn't use coordinates k , was implemented once we just have available the joint coordinates. The matrix H_{jj}^B of second and Wang formulations were selected to carry out the respective coupling, once these formulations gave the best results. Figure 4.15 shows the results obtained.

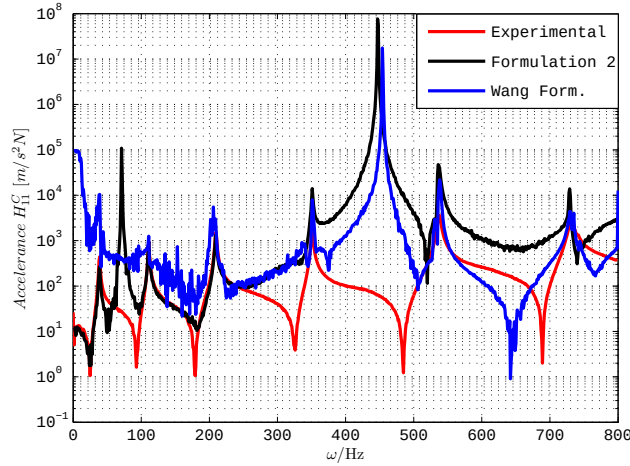


Figure 4.15: Accelerances H_{11}^C according to alternative coupling technique.

Analysing the results obtained, we can see that there is an huge discrepancy between the experimental FRF and the FRF obtained from the coupling procedure. This discrepancy and uncertainty was expected, once these experimental data contain noise and have been numerically manipulated during the processes of uncoupling and coupling.

In spite of the great discrepancy, it is possible to identify some natural frequencies taking into account the matrix H_{jj}^B obtained with second and Wang formulations. Although there is a certain correspondence between the natural frequencies of theoretical and experimental curves, in the *FRF* obtained with the coupling process appears a natural frequency well defined (close to 450 Hz) that isn't indicated in any experimental curve. This fact could be related with any joint dynamic characteristic that is unknown and belongs to the joint.

4.5.2 Using the Shaker LDS210 (Force transducer 8200)

This experimental analysis was carried out to compare the results obtained with hammer in previous analysis, and to evaluate the shaker performance as an excitation mean. In this experimental analysis, the drive rod choice had collocated some problems due to its length, because it was introducing an extra undesirable natural frequency. This drive rod was replaced for another more robust, but it had introduced extra mass in the system due to the new force transducer. Here again, the *FRFs* were measured in all degrees-of-freedom, once we need the whole matrix H^C , at each excitation frequency ω . Figure 4.16 shows some results of this experimental analysis.

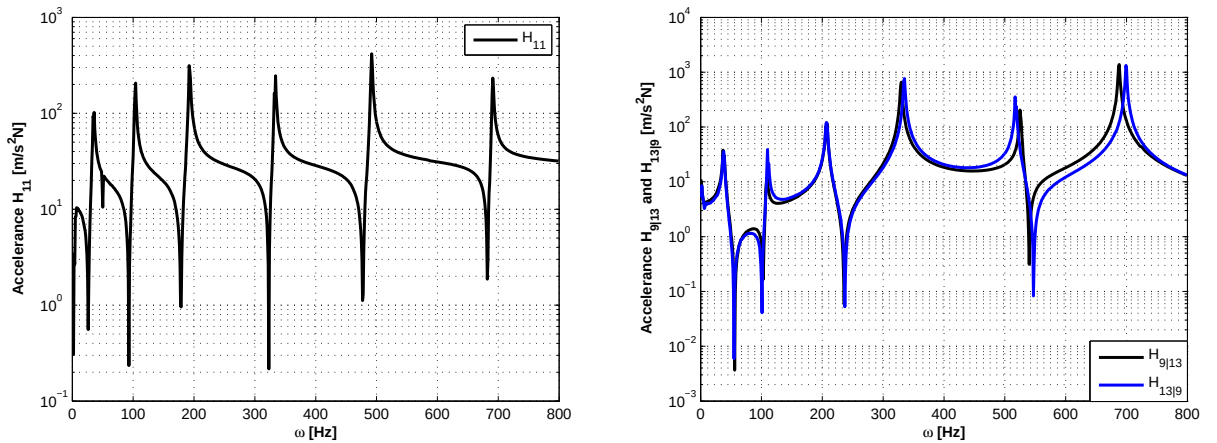
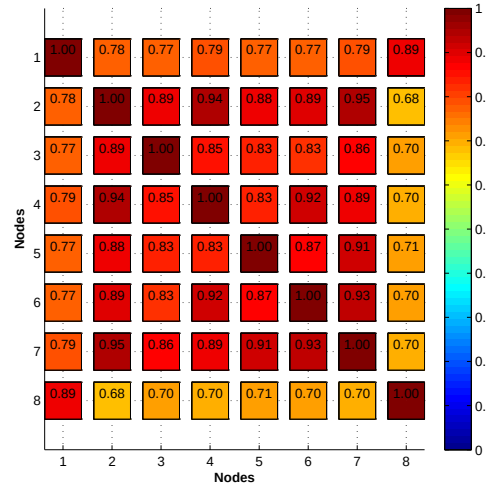


Figure 4.16: Accelerances $H_{1|1}$ and $H_{9|13}$.

As we can see in previous figure, there is a difference in transfer *FRFs* $H_{5|13}$ and $H_{13|5}$ which must have a similar dynamic behavior with coincident curves. The correlation criterium *LAC* was applied to the transfer *FRFs* in order to study this fact. Figure 4.17 reflects this fact with low values for correlation criterium *LAC*.


 Figure 4.17: Experimental *LAC* between transfer *FRFs*.

The main diagonal of the *LAC* matrix has always the highest value 1 once we are comparing the same *FRF*, for the sake of coherence. When the same criterium was applied in the experimental data obtained with the hammer, the lowest value was 0.90 and in this concrete case we have values close to 0.7 which indicates that there is something wrong in this analysis. In fact, this detail has a valid physical explanation which is related to the interference of the force sensor which measures the excitation (figure 4.18 [17]). According to the force sensor catalogue, the active mass which affects directly the beam behaviour is close to 18 g.

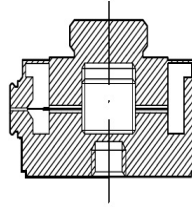


Figure 4.18: Force sensor used to measure the excitation (Force transducer 8203).

Considering again the example of the transfer *FRFs*, $H_{5|13}$ and $H_{13|5}$, the experimental setup used in each case is different once in first case we have influence of the force transducer in degree-of-freedom 13, and in second case the force transducer is placed on the degree-of-freedom 5. Therefore, the mass distribution in both cases is different due to the introduction of the transducer mass influence in different points and consequently the *FRFs* obtained are different in terms of natural frequencies and magnitude.

The influence of this mass needs to be cancelled in order to achieve *FRFs* without this effect. The mass cancelation algorithm approached in section 3.4 was implemented to this experimental data. To the force sensor mass, it was considered a mass of 18 g plus the washer mass, that does the coupling between the force sensor and the beam, 1.42 g which does a total of 19.420 g. Figure 4.19 shows a direct and a transfer *FRF* that were cancelled.

4.5. EXPERIMENTAL ANALYSIS OF BEAM 2

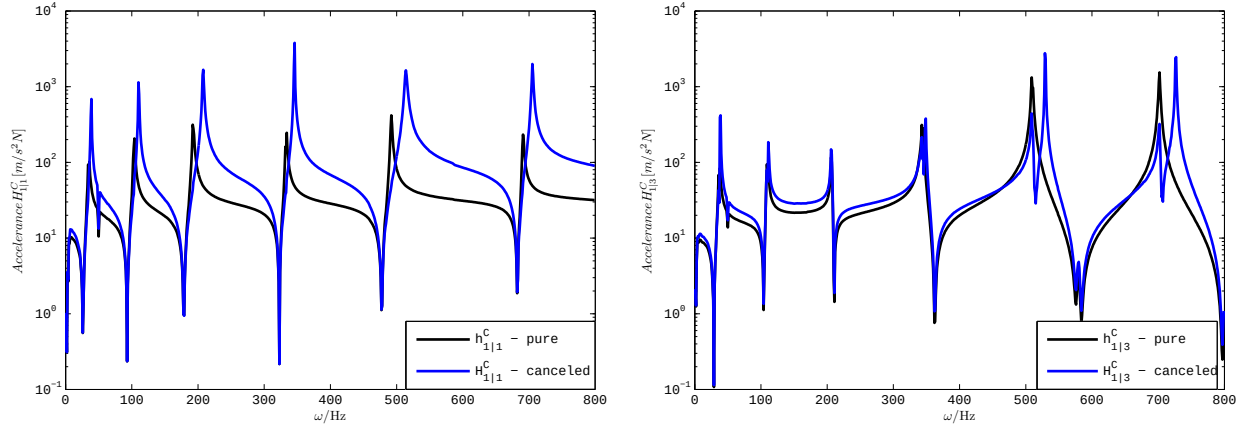


Figure 4.19: Accelerances $H^C_{1|1}$ and $H^C_{1|3}$.

Analysing the results of the mass cancellation process, the direct *FRF* gives coherent results whereas in the transfer *FRF* there is the introduction of new natural frequencies where must exist just one natural frequency. The only explanation to this fact comes from the accelerometer mass, 1 g, that we have considered irrelevant and negligible to this process due to its low mass. This fact can be easily clarified once the accelerometer mass position varies while the measurements are carried out in a column with the force sensor placed always on the same position. For instance, to cancel the *FRF* $H_{2|7}$ we need the direct *FRF* $H_{7|7}$, and accelerometer has been moved from node 2 to 7. Therefore, the system under analysis is not the same once the accelerometer mass had moved.

A numerical simulation was performed in order to understand the problem and identify its source. Thereunto, a beam as presented in figure 4.20 was taken into account, and its characteristics are available in table 4.6.

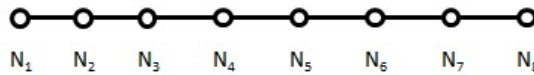


Figure 4.20: Finite element mesh considered in this example.

Table 4.6: Geometry and material data of a beam finite element.

Length	$L = 0.1 \text{ m}$
Width	$b = 0.025 \text{ m}$
Height	$h = 5\text{E}^{-3} \text{ m}$
Young's modulus	$E = 70 \text{ GPa}$
Density	$\rho = 2700 \text{ Kg/m}^3$

Making use of the finite element programme, two measurements were performed to obtain a direct and a transfer *FRF*. To measure the *FRF* $H_{7|7}$, it was considered a mass of 20 g in node

7 to represent the force transducer effect and a mass of 1 g in the same point to represent the accelerometer effect. To the other $FRF H_{2|7}$, it was considered the force sensor mass in the same point and the accelerometer mass was moved to the node 2. Indeed, this corresponds exactly to the experimental procedure taken into account. Figure 4.21 shows the results obtained in this simulation.

This simulation represents exactly the experimental results and in fact, the accelerometer mass has some interference in the structure and can not be negligible in this particular case. The big point is that we may have always a system with the same mass distribution, either we want to cancel the accelerometer mass or the force sensor mass. Indeed, if we want to cancel the $FRF H_{ij}$ which has the accelerometer mass placed on node i , we need the direct $FRF H_{jj}$ which now has the accelerometer mass placed on node j . If we want to cancel the force sensor mass, the same fact occurs oppositely.

Indeed, this problem has a justified explanation and recently some authors have been studying this phenomenon, and for instance Si et al. [18] give some solutions to solve or avoid this kind of problems.

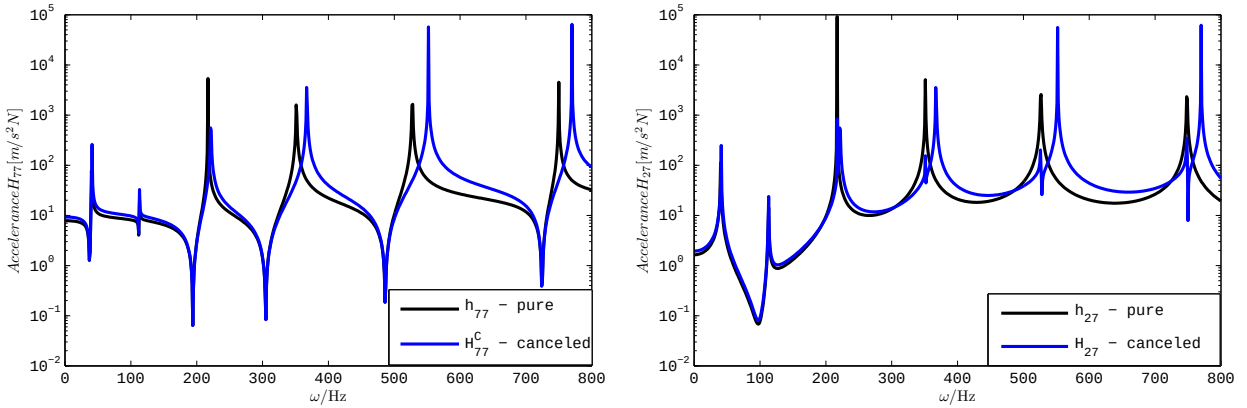


Figure 4.21: Accelerances $H_{7|7}$ and $H_{2|7}$.

4.6 Experimental analysis of the bolted joint

Lastly, the bolted joint was analysed in order to experimentally characterize the bolted joint included in previous specimen. The excitation of the joint was done with a hammer, whereas the response was measured with the accelerometer. For the sake of clarity, in this experimental analysis we just measure the translation degrees-of-freedom, once the rotational ones aren't available due to reasons already stated in this work. Figures 4.22 shows a direct and a transfer FRF .

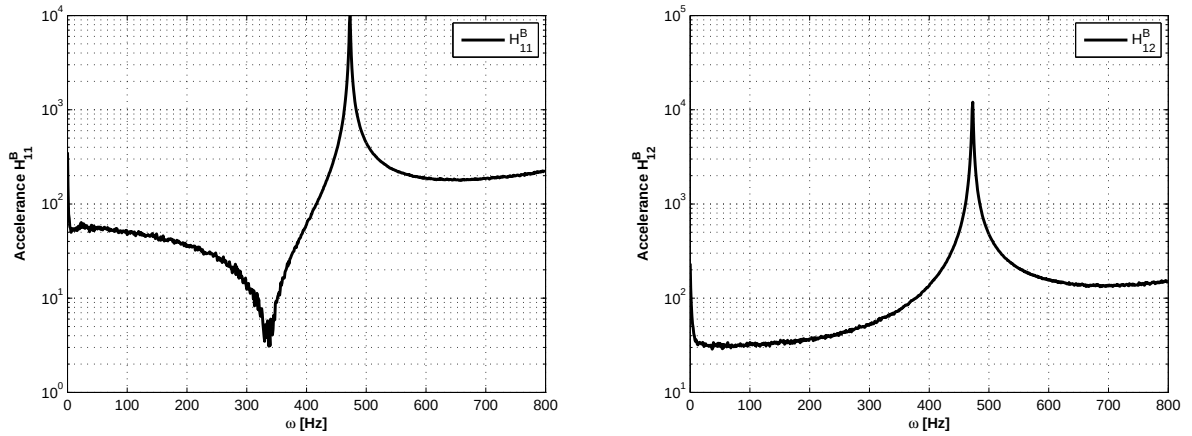


Figure 4.22: Accelerances $H_{1|1}$ and $H_{1|2}$.

Indeed, the behaviour of both *FRFs* is similar to the uncoupling techniques results, having a resonance placed on the same zone. The resonance found is lightly higher than that which was theoretically considered, once in fact the bolted joint and its overlap do not work entirely as a concentrated mass, but in a different way with a mass distribution more homogeneous. Even though, the *FRFs* are really similar and therefore this another point which validates the work carried out.

Figure 4.23 shows the experimental setup of the bolted joint where it is possible to visualize the accelerometer 27A11 normally used to measure the response in this work.



Figure 4.23: Experimental setup of the bolted joint.

4.7 Discussion of results

The experimental analysis of beam 1 provided good results, although there are some differences between the two excitation means of the beam. The shaker provides results that fit very well the exact solution and the natural frequencies are near the exact values. On the other

hand, the hammer provides values for the natural frequencies much more close, but the curve doesn't fit so very well the theoretical solution.

Therefore, exciting the beam with a shaker provides better results than with an hammer, even though an higher error in natural frequencies which in most cases does not interfere so much in the results.

The torque applied to the bolted joint affects slightly the *FRFs'* behaviour along the frequency range. If the torque applied is low, any undesirable phenomenons such as slipping and trepidation between the two beam parts will occur, causing uncertainty and lack of accuracy in the dynamic responses acquired. Actually, this effect must be canceled in order to avoid even more uncertainty in this type of analysis. The value from what there is not influence of this extra factor was detected and considered to every bolted joints used in this work to carry out experimental measurements.

The analysis of beam 2 provided a comparison between two excitation means as an application to perform the joint uncoupling. The hammer gave much better results than the shaker due to some technical reasons. The experimental analysis using the hammer demonstrated an excellent practical applicability, once the accelerometer mass almost hadn't affected the beam behaviour but its cancelation was performed for the sake of coherence, and the natural frequencies showed a good accuracy. The joint uncoupling exceeded the results expected with two formulations providing good results, once it was expected an huge discrepancy and uncertainty caused by the noise which is automatically inserted in experimental data. The estimative of unmeasured *FRFs* gave an important contribute to these results, once they need to be estimated. The coupling between experimental and theoretical data revealed some uncertainty, although it was possible to identify some natural frequencies.

On the other hand, some experimental problems appeared when the shaker was used as excitation mean. Firstly, the drive rod was introducing an extra natural frequency in the samples due to its interference. Different drive rod lengths were experimented without success and another drive road was considered. The high mass of the new force transducer that we had to use due to the new drive road changed the beam behaviour and consequently there were some differences in transfer *FRFs*. Then, a mass cancelation methodology was implemented without success and the reasons to this problem were properly explained with a theoretical formulation. This problem allows to understand that the accelerometer low mass could have some influence and thus its cancelation was carried out in the previous experimental analysis with the hammer. Because of the worst quality of the results, the uncoupling techniques and other weren't applied to this data set.

Lastly, the bolted joint was analysed and good results were achieved. Indeed, the *FRFs* obtained are really similar to that obtained with the uncoupling techniques. Furthermore, the natural frequency identified is close to what was expected, even though a bit higher than the theoretical one. This fact shows that the overlap and bolt/nut introduce a special mass distribution, once it is not exactly the same as if we considered a concentrated mass in the bolt position.

Chapter 5

Uncoupling with Experimental Modal Analysis

5.1 Introduction

Another way to study a beam with a bolted joint involves the execution of an experimental modal analysis. To perform this analysis, the programme *CADA-PC* was used to process the experimental data. The main advantage of this technique is the direct determination of the coefficients related to the system modal properties which are usually the parameters sought [19]. Another advantage of this process is that we just need to measure a row or a line of the *FRF* matrix to perform the corresponding modal identification, instead of measure the *FRFs* in all points as was done in all experimental analysis in last chapter.

In other words, a modal analysis gives the modal matrix $[\phi]$ with the corresponding vibration modes, as well as the natural frequencies and the damping coefficients for each vibration mode. Having all these data available, we can synthesise any *FRF* in either points of the structure which we desire.

Basically, the modal analysis process consists on a theoretical curve fitting of the adopted model to the experimental curves, in order to obtain the parameters of theoretical curves which better fit the experimental curves obtained. These parameters are usually called modal parameters which describe the system under analysis. To obtain these modal parameters, the programme *CADA-PC*, which works with the experimental setup, was sought.

The software *CADA-PC* implements a multi degree-of-freedom technique for parameter estimation. It uses a time domain algorithm to identify poles (frequency and damping values), called the *Least Square Complex Exponential* method. In a second phase, residues are identified with the *Least Square Frequency Domain* technique [20].

The synthesising of each *FRF* is performed with an expansion in simple fractions (Equation 5.1). This expression takes into account a viscous damping expressed by the coefficient ξ and

the influence of the different modes, carrying out a sweep in frequency.

$$H_{jk}(\omega) = \sum_{r=1}^n \frac{\phi_{jr}\phi_{kr}}{(\omega_r^2 - \omega^2) + 2j\xi_r\omega_r\omega} \quad (5.1)$$

The implementation of this type of analysis has some advantageous points, principally a lower level of noise in the samples and an improvement of accuracy on the dynamic behaviour of corresponding *FRFs* which are calculated according to the expansion in simple fractions. Therefore, we are looking for better results than that we have got in previous experimental analysis.

The experimental analysis made use of the same beam used in previous experimental analysis (beam 2) with the same finite element mesh in order to keep the correspondence of the final results for a later discussion. The acquisition of a row or a column of the *FRF* matrix was carried out with the same excitation means, hammer and shaker.

In this chapter, the data treatment will follow a similar sequence as presented by diagram 4.1. The big difference is that the *FRFs* of the translation *DOFs* will be synthesised taking into account the modal analysis outputs, the matrix modal and the corresponding natural frequencies identified and damping coefficients. From this point, all the procedure sequence is the same as done in previous experimental works. Table 5.1 summarizes all the experimental modal analyses done and the formulations implemented in each specific case.

Table 5.1: Experimental modal analysis carried out in beam 2.

		Mass Cancelation	Estimative of unmeasured FRFs	Uncoupling Techniques	Dynamic stiffness	Coupling
Shaker	Column 8	-	X	X	X	X
	Laser vibrometer (x2)	X	X	X	X	X
	Dummy Masses (x2)	X	X	X	X	X
Hammer	Row 2	-	X	X	X	X
	Row 4	-	X	X	X	X
	Row 5	-	X	X	X	X

5.2 Modal Analysis with Shaker LDS210

This experimental modal analysis used the shaker as excitation mean, using the force transducer 8200 to measure the excitation. Firstly, an experimental modal analysis will be carried out with pure data, using the eighth column. Due to problems related with transducer's masses which had an huge interference in previous data treatment, here we will try to avoid these problems taking into account an experimental setup with some modifications. Thereunto, a beam with dummy masses will be analysed, as well as a laser vibrometer will be used to measure the response, once this transducer does not introduce extra mass in the system, which is a great advantage. Thus, it will be possible to compare different analyses which have the transducers influence and other where different mass cancelation processes are implemented.

5.2.1 Modal Analysis with column 8

The shaker was placed on an extremity of the beam, and the response was measured in all degrees-of-freedom. Therefore, the main structure has the influence of the force sensor mass, as well as the accelerometer mass which changes its positions according to the measurement point required. Using the identification software *CADA-PC*, the corresponding modal identification gives the modal parameters, the damping coefficients and natural frequencies (table 5.2). The frequency-response functions of the translation degrees-of-freedom were calculated according to expression 5.1 presented above.

Table 5.2: Modal Parameters of the beam identification process.

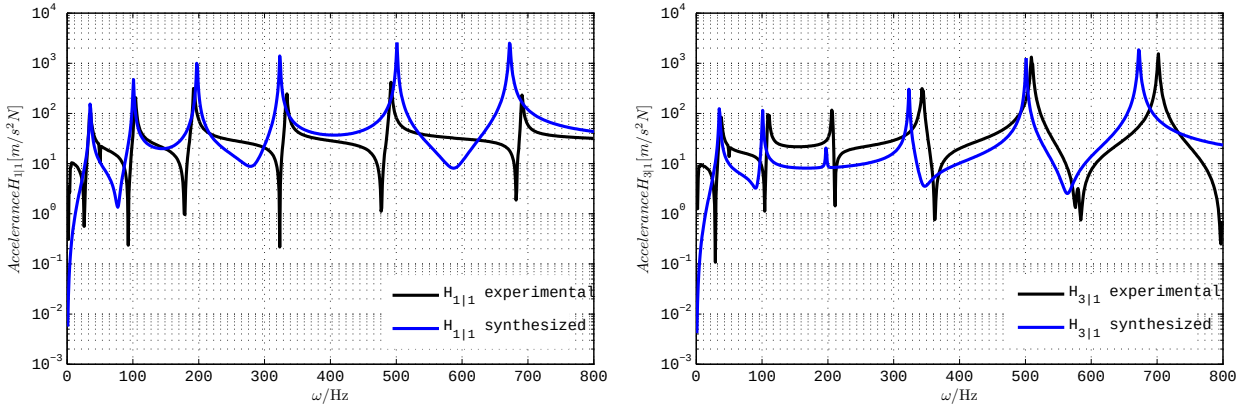
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	35.331	1.790	8
Mode 2	100.850	0.572	8
Mode 3	197.382	0.301	8
Mode 4	323.269	0.197	8
Mode 5	501.015	0.129	8
Mode 6	672.425	0.141	8

Table 5.3 shows the values of the correlation criterium *MAC*, which relates the identified mode shapes obtained in this modal analysis.

Table 5.3: MAC Criterion.

Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	35.331	100	1.3	8.9	13.9	6.5	4.4
2	100.850	1.3	100	19.3	10.1	6.6	2.9
3	197.382	8.9	19.3	100	9.9	2.0	2.2
4	323.269	13.9	10.1	9.9	100	5.8	0.8
5	501.015	6.5	6.6	2.0	5.8	100	10.4
6	672.425	4.4	2.9	2.2	0.8	10.4	100

Figure 5.1 shows two synthesised *FRFs*, $H_{1|1}^C$ and $H_{3|1}^C$. The synthesized *FRF* matches in a good way the experimental *FRF*, despite a certain discrepancy in some frequency ranges. As we can see, in the synthesized *FRF* there is no interference in the first frequencies (0-100 Hz) which normally is related to boundary conditions, for instance the influence of suspension ropes, in the acquisition of an experimental *FRF*. Therefore, we can conclude that the modal analysis was carried out with success, although there are some differences in natural frequencies, as well as some lack of coherence in resonance and anti-resonance zones. Indeed, these differences are the consequence of the different position of the force transducer, because in the experimental curve it is placed on DOF 1, whereas in the synthesised *FRF* it is placed on DOF 15 (measurement point 8), causing different mass distributions that explain this difference in terms of natural frequencies.


 Figure 5.1: Accelerances $H_{1|1}^C$ and $H_{3|1}^C$.

• Joint Uncoupling

The modal analysis process provided all the information about the beam dynamic behaviour that we need to obtain the necessary *FRFs* which are required to apply the uncoupling algorithms. Having this information, the *FRFs* were calculated in all degrees-of-freedom taking into account the extracted modal parameters.

In this numerical approach we are looking for results much more accurate, because the influence of extra factors such as noise and boundary conditions is reduced or even annulled.

For the sake of coherence, it is important to notice that these modal parameters still have the influence of the transducer's mass used to perform the experimental measurements.

Having the translation *FRFs* available, the rotation ones were estimated according to the formulation used in previous numerical examples. Once obtained the matrix H^C , the uncoupling techniques were implemented in order to obtain the joint dynamic characterization. Figure 5.2 shows the results obtained with the four uncoupling techniques.

Analysing the results obtained, we can notice that there is an improvement on the level of accuracy in some formulations comparing to previous experimental examples. Once again, the second and Wang formulations provided better results than other formulations. The behaviour of first and third formulations was expected with an huge discrepancy and uncertainty. The second formulation gives good results, where is well visible an anti-resonance and a resonance pick, even though an anti-resonance better defined than the resonance. The Wang formulation gives an excellent result with a resonance very well defined, whereas the anti-resonance zone has some discrepancy and uncertainty.

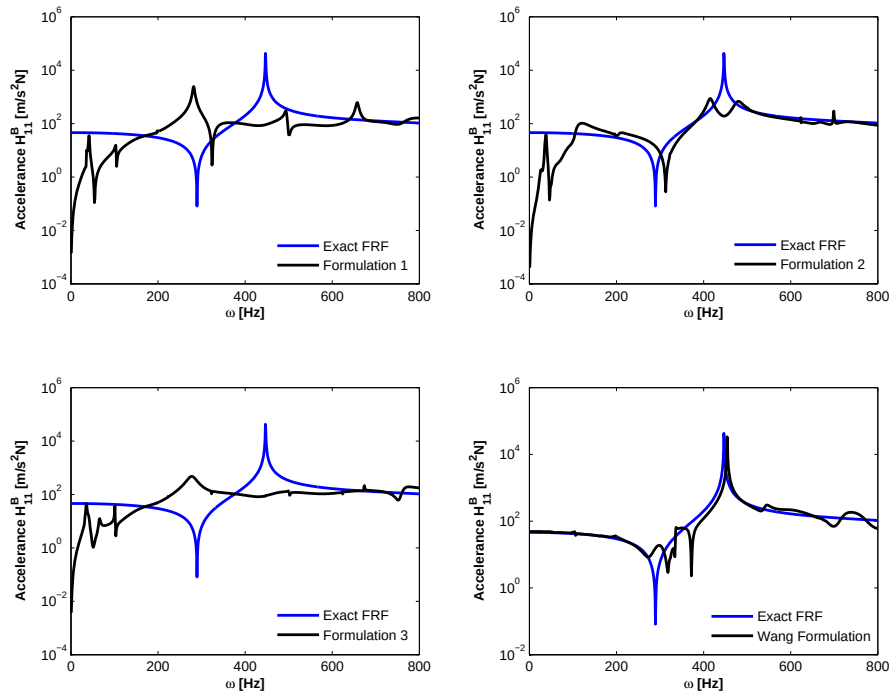


Figure 5.2: Accelerance H_{11}^B of the uncoupled joint.

- **Dynamic Stiffness**

If we have the joint *FRFs* available, the corresponding dynamic stiffness can be calculated. The formulations with better results, second and Wang, were selected (figure 5.3). The dynamic stiffness obtained takes the expected shape when there are extra factors (introduction of noise) which contaminate the experimental samples, although these results have a lower level of noise as said previously.

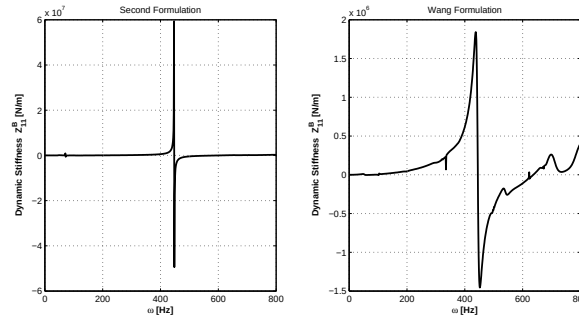


Figure 5.3: Dynamic Stiffness Z_{11}^B of second and Wang formulations.

- Coupling with joint *FRFs*

Once obtained the joint *FRF* matrix through the uncoupling process, the inverse process was executed in order to compare the results with the *FRFs* experimentally measured. The alternative coupling technique made use of matrix H_{jj}^B from second and Wang formulations because both provided good results. This process will introduce even more numerical uncertainty in the samples once new matrix inversions have to be performed. Figure 5.4 shows the results obtained.

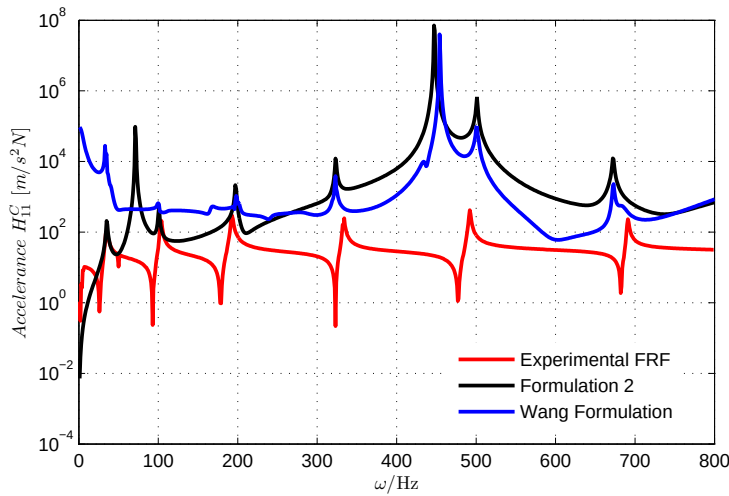


Figure 5.4: Accelerance H_{11}^C according to alternative coupling technique.

In terms of natural frequencies, there is a good correspondence between the *FRFs* obtained and the synthesized ones by coupling, even though the appearance of a spurious natural frequency close to 450 Hz which has appeared in other experimental examples without a concrete explanation. In terms of magnitude, there are differences in some frequency ranges, principally in the range where appears the new unknown natural frequency.

In these examples, second formulation provides better results than Wang formulation, mainly in first frequencies, but this formulation introduces a spurious natural frequency at 80 Hz that doesn't appear in Wang formulation.

5.2.2 Modal Analysis using the Laser Vibrometer and Shaker LDS201

The previous modal analysis, which used the eighth column, has the influence of the force sensor and accelerometer mass that cause a different mass distribution. In previous chapter, the numerical example that used the shaker as excitation mean gave bad results when a mass cancelation process was required to eliminate the transducers influence, once that process wasn't coherent due to the accelerometer position. Therefore, the influence of this transducers must be canceled and in this section we will propose a different experimental setup capable of consider these effects. To measure the response, a laser vibrometer will be used once it doesn't introduce extra mass in the system like the accelerometer normally introduces, and thus we just have to cancel the force sensor mass, becoming the process much more feasible. Two different ways will be followed, varying the order between the mass cancelation process and the modal analysis. Table 5.4 presents the two different procedures.

Table 5.4: Different procedures for this experimental setup.

Shaker LDS 210 + Laser Vibrometer			
First Procedure		Second Procedure	
1°	Experimental modal analysis	1°	Mass cancelation (force sensor)
2°	Synthesising	2°	Experimental modal analysis
3°	Mass cancelation (force sensor)	3°	Synthesising
4°	Uncoupling	4°	Uncoupling

Figure 5.5 shows this experimental setup, where it is possible to see the shaker as well as the laser vibrometer.



Figure 5.5: Experimental setup.

5.2.2.1 First Procedure: Synthesising + Mass Cancelation

In the first procedure, we measured the experimental *FRFs* exciting the beam in an extremity (measurement point 8) and measuring the response in all points. Then, a modal analysis was carried out with these experimental data. Table 5.5 shows the results of this modal analysis.

Table 5.5: Modal Parameters of the beam modal identification.

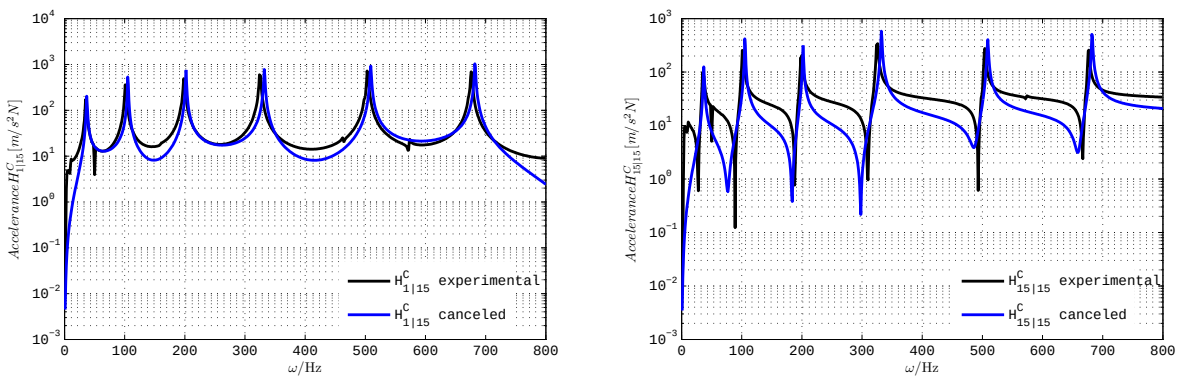
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	35.607	1.499	8
Mode 2	101.518	0.565	8
Mode 3	198.765	0.313	8
Mode 4	324.907	0.206	8
Mode 5	503.691	0.126	8
Mode 6	676.206	0.119	8

Table 5.6 has a representation of the correlation criterion *MAC* that relates the identified mode shapes. It is possible to conclude that this modal analysis was carried out with success, once this criterium gives consistent results.

Table 5.6: MAC Criterion.

Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	35.607	100	3.5	18.2	14.7	8.7	0.2
2	101.518	3.5	100	17.8	14.5	3.7	1.8
3	198.765	18.2	17.8	100	5.0	15.2	10.4
4	324.907	14.7	14.5	5.0	100	10.0	13.1
5	503.691	8.7	3.7	15.2	10.0	100	8.4
6	676.206	0.2	1.8	10.4	13.1	8.4	100

After, having all the *FRFs* synthesised, a mass cancellation process was implemented in order to remove the force sensor influence from the main structure. To the force sensor and corresponding washer that does the connection, a total mass of 19.42 g was taken into account. Figure 5.6 shows two samples of the mass cancellation process, where we can see an increment of the natural frequencies between the experimental curve and the canceled one. This is a direct cause of the mass cancellation, once mass is taken away from the structure and the natural frequencies lightly increase.


 Figure 5.6: Accelerances $H_{1|15}^C$ and $H_{15|15}^C$ experimental and canceled.

- Joint Uncoupling

Having all the experimental *FRFs* canceled, the rotational *FRFs* were generated in order to carry out the joint uncoupling. Figure 5.7 gives the results of this process. Here again, the same two formulations give the better results whereas the two other show an high level of uncertainty and discrepancy. However, the second and Wang formulations present some discrepancy in certain frequency ranges, but it is still possible to visualize and understand the resonances and anti-resonances zones.

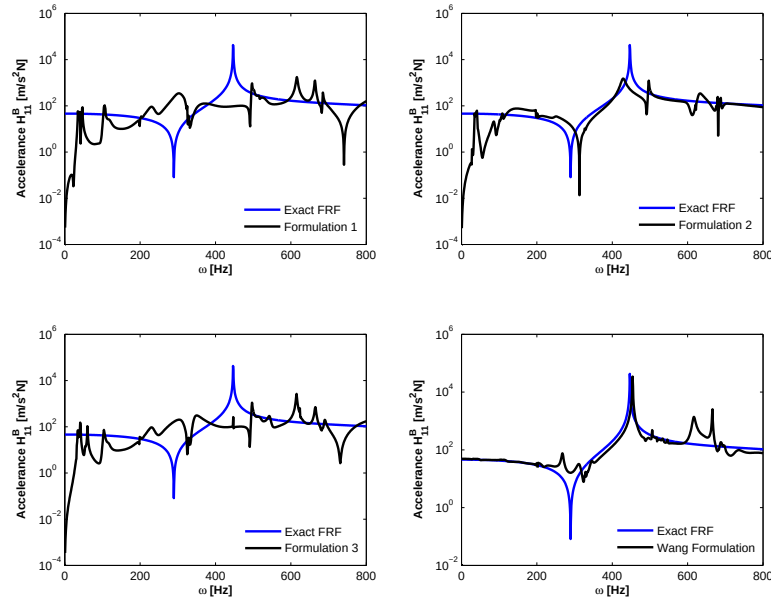


Figure 5.7: Accelerance H_{11}^B of the uncoupled joint.

- Dynamic Stiffness

Having the joint dynamic identification done, the dynamic stiffness could be calculated. Figure 5.21 presents the results obtained, and second formulation provides a curve better defined whereas Wang formulation carries the uncertainty present in the joint uncoupling.

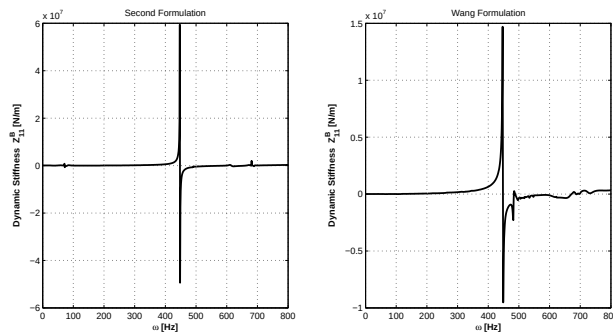


Figure 5.8: Dynamic stiffness Z_{11}^B of the uncoupled joint.

- Coupling with joint *FRFs*

The results of second and Wang formulations were selected to carry out the coupling between the joint and two substructures. In this specific case, we are trying to compare these results with the synthesised *FRFs* obtained in the beginning. Figure 5.9 shows the results of this coupling, where there is an huge discrepancy, being complicated to identify any natural frequency. Basically, just in the first natural frequency there is some correspondence between the exact *FRF* and the two formulations, and Wang formulation holds an higher discrepancy than second formulation.

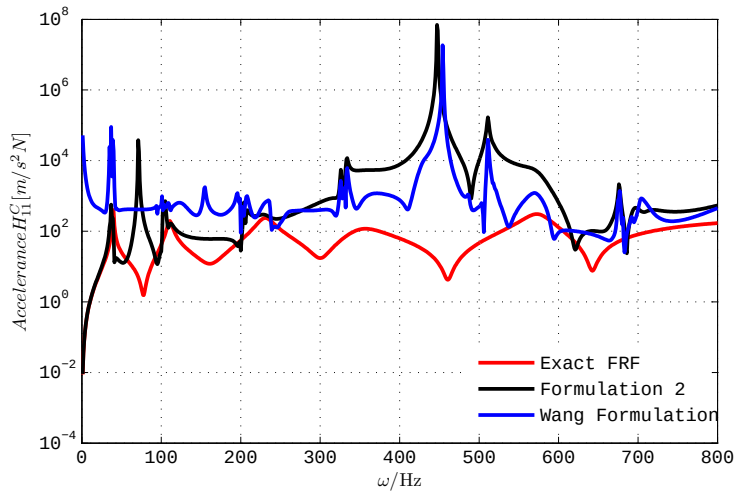


Figure 5.9: Accelerance H_{11}^C according to alternative coupling technique.

5.2.2.2 Second Procedure: Mass Cancellation + Synthesising

Here, the cancellation of the force transducer mass was done first and then we performed an experimental modal analysis with the canceled *FRFs*, the inverse process of previous analysis which might be similar. Carrying out first the mass cancellation, we are supposedly introducing even more numerical uncertainty in this data set. Table 5.7 gives the results of this modal analysis, in which the natural frequencies are slightly higher than previous modal analysis, that reveals the effect of the mass cancellation process.

Table 5.7: Modal Parameters of the beam modal identification.

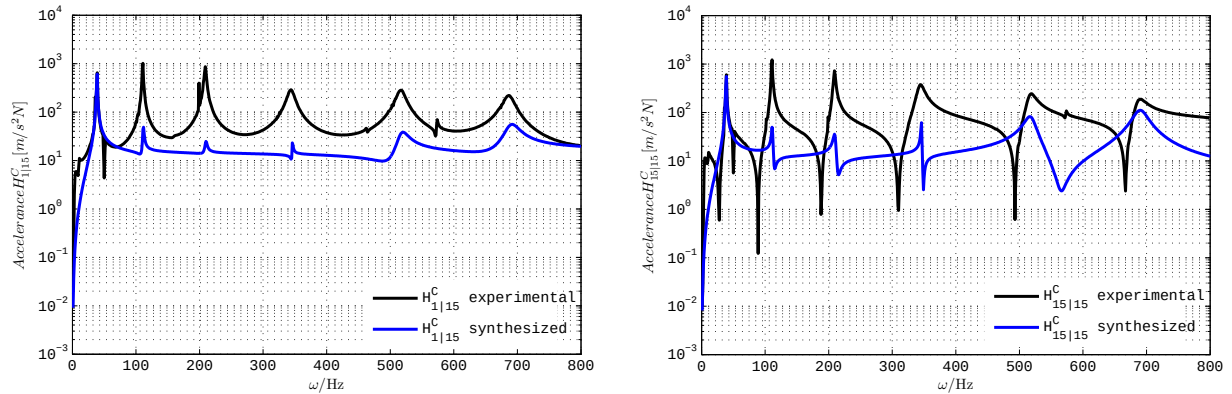
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	38.617	0.526	8
Mode 2	111.435	0.780	8
Mode 3	209.871	0.893	8
Mode 4	346.013	0.186	8
Mode 5	517.813	1.678	8
Mode 6	690.070	1.655	8

Table 5.8 gives the correlation criterion MAC that relates the identified mode shapes. Carefully analysing the MAC table, we could presuppose that this modal analysis isn't completely coherent, once it is possible to see high values out of the main diagonal.

Table 5.8: MAC Criterion.

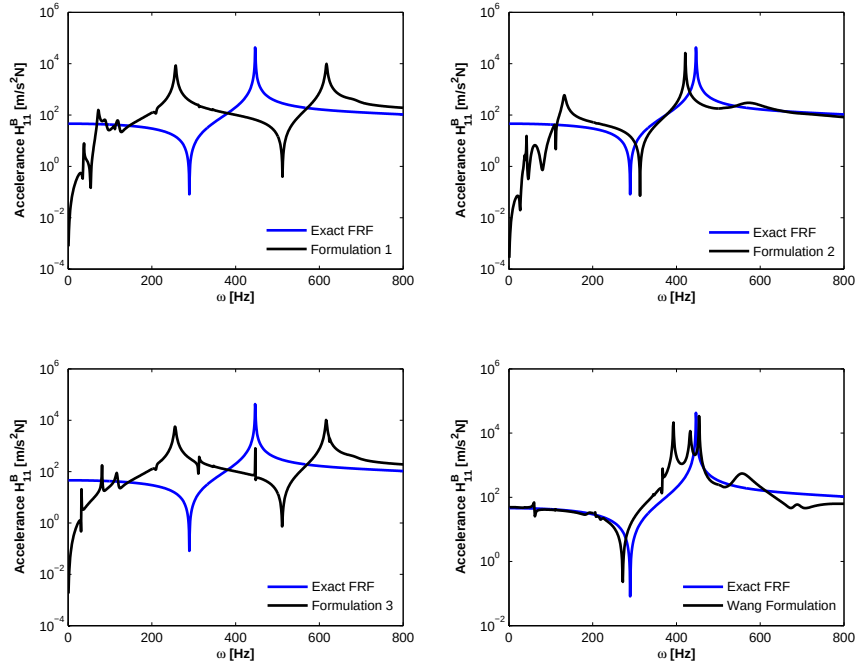
Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	38.617	100	2.3	23.0	11.5	8.7	2.8
2	111.435	2.3	100	72.3	61.8	3.7	31.9
3	209.871	23.0	72.3	100	81.8	15.2	40.5
4	346.013	11.5	61.8	81.4	100	10.0	72.4
5	517.813	2.2	33.1	42.1	77.1	100	98.6
6	690.070	2.8	31.9	40.5	13.1	72.4	100

Then, having the modal matrix available and the corresponding parameters, the necessary $FRFs$ were synthesised. Figure 5.10 supplies two samples of this synthesising and a comparison between the experimental and synthesised FRF . Indeed, there is a big difference between the two curves, which compromises the effectiveness of this modal analysis as said previously. The introduction of one more mathematical operation in the experimental data set could have increased the noise level.

Figure 5.10: Accelerances $H_{1|15}^C$ and $H_{15|15}^C$.

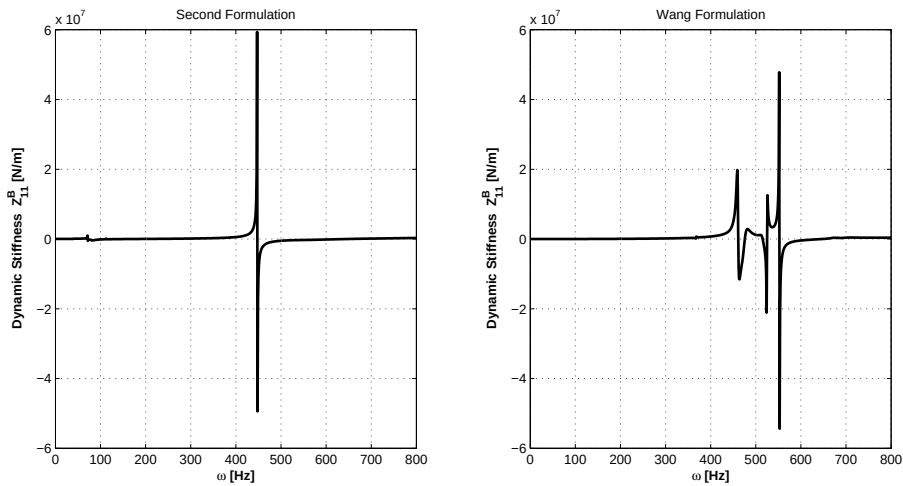
• Joint Uncoupling

Then, the rotational $FRFs$ were generated as usual and the global matrix H^C was set up to carry out the joint uncoupling. Figure 5.11 provides the results of the four uncoupling techniques, where once again the same formulations give the best results. However, second formulation identifies better the resonance and anti-resonance zones, whereas Wang formulation introduces two picks close to the natural frequency position.


 Figure 5.11: Accelerance H_{11}^B of the uncoupled joint.

- Dynamic Stiffness

Figure 5.12 gives the dynamic stiffness calculated for the second and Wang formulations. The dynamic stiffness of Wang formulation shows some uncertainty that is directly related with the uncertainty in the resonance zone of the uncoupled *FRF*. In spite of the good concordance, second formulation takes the usual shape when we are in presence of contaminated data.


 Figure 5.12: Accelerance H_{11}^B of the uncoupled joint.

- **Coupling with joint FRFs**

Therefore, if we have the joint dynamic characterization available, we could couple this information with numerical matrix H^A and obtain again the dynamic characterization of the whole beam. Figure 5.13 gives the results of this coupling, where it is possible to visualize an high uncertainty. Firstly, the synthesised FRF H_{11}^C is really poor as a consequence of this modal analysis, and the two other curves give again bad results. However, the first natural frequency could be easily identified in both formulations.

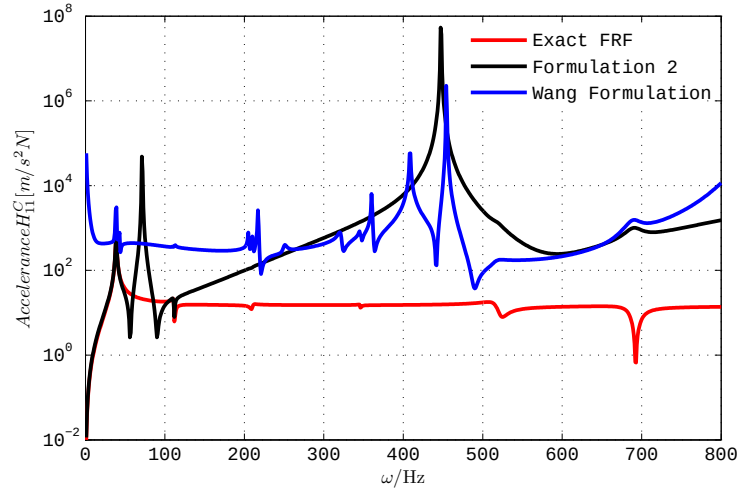


Figure 5.13: Accelerance H_{11}^C according to alternative coupling technique.

5.2.3 Modal Analysis with dummy masses

Another methodology to reduce the transducers influence is presented in this section, considering dummy masses with the same accelerometer mass, in order to simulate its influence in all measurement points and have an identic mass distribution in all measurements. Thus, the force sensor will be placed on an extremity (point 8), while whenever the accelerometer position is changed, in its position is collocated a dummy mass, whereas in the new measurement point the dummy mass is replaced by the accelerometer, having all the measurement points an identic distribution of mass. For the sake of coherence, we have considered dummy masses with 0.970 g (three M3 nuts), to compensate the accelerometer mass of 0.978 g.

Here again, two different procedures will be followed as in previous section, doing in first procedure the synthesising and mass cancelation respectively, whereas in second procedure the order of these processes is changed. Theoretically, these two procedures may give identic results, being thus reversible. Table 5.9 summarizes the sequence of each procedure.

Table 5.9: Different procedures for this experimental setup with dummy masses.

Shaker LDS210 + Accelerometer 27A11 + Dummy masses			
First Procedure		Second Procedure	
1°	Experimental modal analysis	1°	Mass cancelation (force sensor)
2°	Synthesising	2°	Experimental modal analysis
3°	Mass cancelation (force sensor)	3°	Synthesising
4°	Uncoupling	4°	Uncoupling

Figure 5.14 shows the experimental setup with the dummy masses in each measurement point.

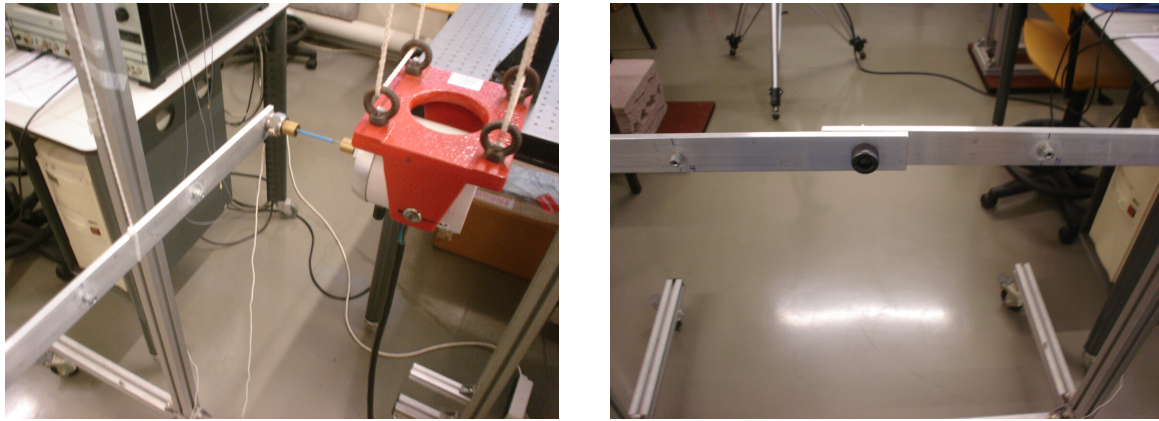


Figure 5.14: Experimental setup with dummy masses.

5.2.3.1 First Procedure: Synthesising + Mass Cancelation

In this first procedure, the response was measured in all points with the shaker placed on an extremity (point 8), having all the points a dummy mass to simulate the accelerometer. Then, an experimental modal analysis was carried out and the final results are shown in table 5.10. Considering dummy masses, concentrated masses are introduced in the system and consequently the natural frequencies decrease and table 5.10 verifies this fact.

Table 5.10: Modal Parameters of the beam modal identification.

	Frequency / [Hz]	ξ [%]	N.° of DOFs
Mode 1	35.607	1.499	8
Mode 2	101.518	0.565	8
Mode 3	198.765	0.313	8
Mode 4	324.907	0.206	8
Mode 5	494.049	0.128	8
Mode 6	663.045	0.134	8

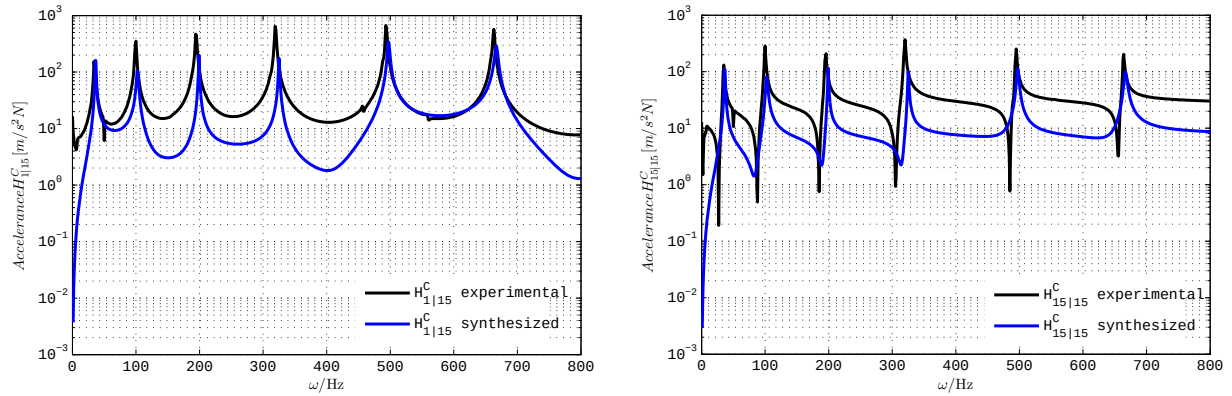
Table 5.11 gives a representation of the correlation criterion *MAC* that relates the identified

mode shapes, where it is possible to understand that there is good correspondence between the different modes.

Table 5.11: MAC Criterion.

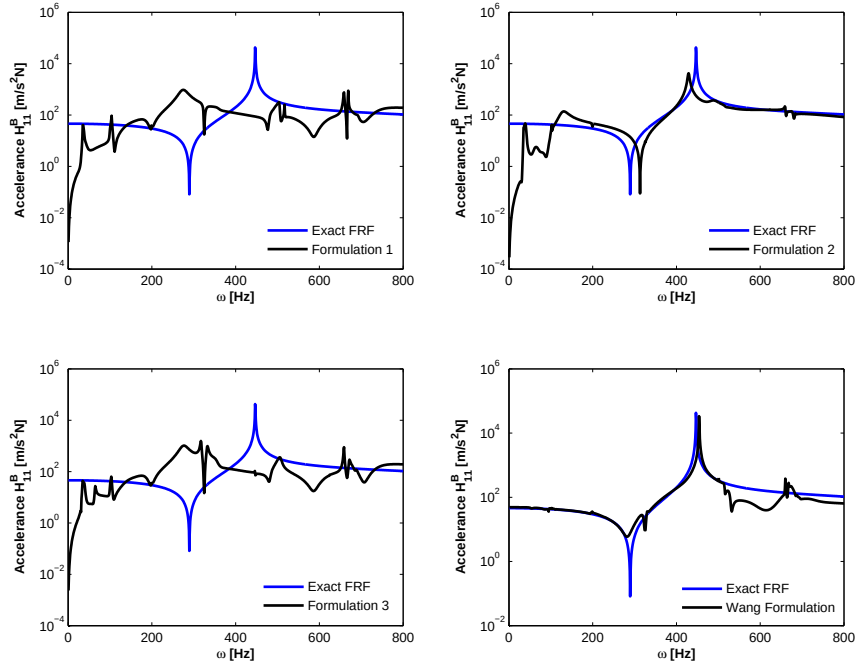
Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	35.607	100	1.9	16.5	15.5	4.9	0.8
2	101.518	1.9	100	9.3	15	4.9	1.8
3	198.765	16.5	9.3	100	1.3	7.0	7.8
4	324.907	15.5	15.0	1.3	100	4.7	8.0
5	494.049	4.9	4.9	7.0	4.7	100	5.2
6	663.045	0.8	1.6	7.8	8.0	5.2	100

Then, all the necessary *FRFs* were synthesised and implemented a mass cancelation process, considering only the force sensor mass to cancelation effects. Therefore, the main structure will have the influence of seven dummy masses and the accelerometer mass. Figure 5.15 gives the results of this process, comparing the experimental with synthesised and canceled *FRF*. As we can see, the mass cancelation process increases lightly the natural frequencies. The synthesised *FRF* shows good coherence in terms of natural frequencies, but the anti-resonances aren't coherent with the experimental one.

Figure 5.15: Accelerances $H_{1|15}^C$ and $H_{15|15}^C$.

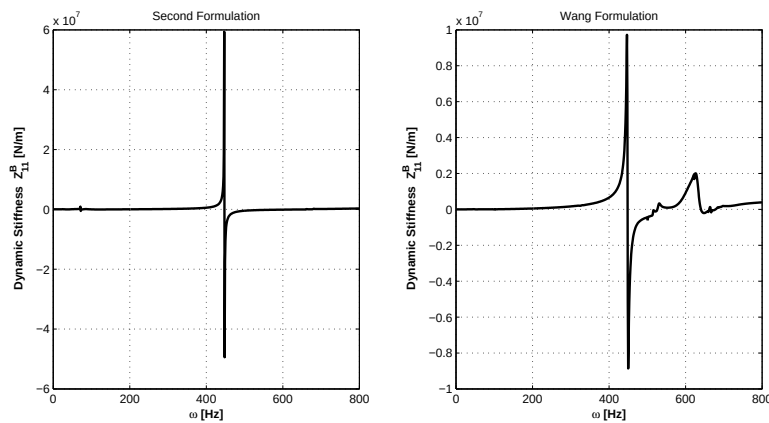
• Joint Uncoupling

Then, the rotational *FRFs* were generated to have the necessary information available to carry out the joint uncoupling. Figure 5.16 gives the results of this uncoupling process, and once again second and Wang formulations provide the best results. In both formulations, it is possible to identify the resonance and anti-resonance zones, demonstrating however some discrepancy in certain frequency ranges.


 Figure 5.16: Accelerance H_{11}^B of the uncoupled joint.

- **Dynamic Stiffness**

Inverting the matrix H_{jj}^B , the dynamic stiffness Z_{jj}^B can be immediately calculated. Figure 5.21 gives the results of the two formulations, where second formulation provides a curve more accurate than Wang formulation, reflecting the discrepancy present in the uncoupling results. The dynamic stiffness takes again the characteristic behaviour when the data are contaminated with noise.


 Figure 5.17: Dynamic stiffness Z_{11}^B of the uncoupled joint.

- **Coupling with joint FRFs**

Once obtained the joint dynamic identification, a coupling between this information and

the structure H^A can be performed. Figure 5.18 gives the results of this coupling for the second and wang formulations. Here again, there is an huge discrepancy and incoherence between the synthesised $FRFs$ and the two other curves, but it is possible to identify four natural frequencies up to 400 Hz. Indeed, in this case is well visible the influence of the successive numerical manipulations conditioned by the noise introduced, making impossible the achievement of coherent results that theoretically would give identical $FRFs$ that were taken into account in the beginning of the cycle.

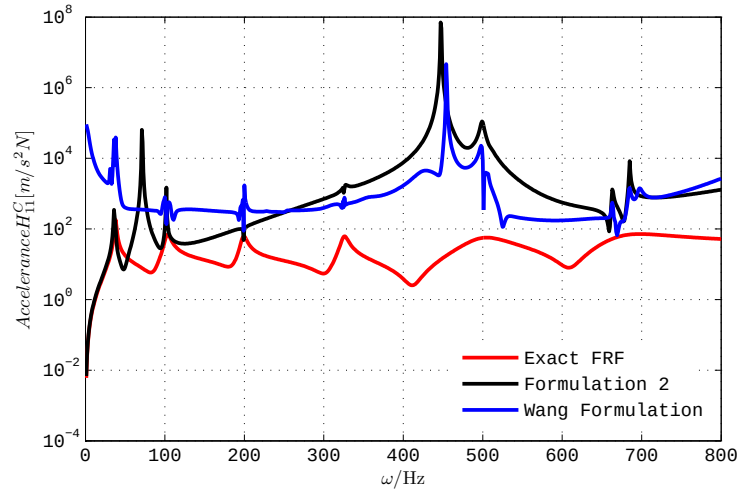


Figure 5.18: Accelerance H_{11}^C according to alternative coupling technique.

5.2.3.2 Second Procedure: Mass Cancellation + Synthesising

In this case, the cancellation of the force sensor mass was done first, and then performed the experimental modal analysis. For the sake of clarity, in this modal analysis the dummy masses are considered in the main structure, and no cancellation is done for this masses. The experimental setup is exactly the same, but the experimental data are treated before the modal analysis. Table 5.12 presents the results of this modal analysis. The natural frequencies are slightly higher than in previous modal analysis, once the mass cancellation had already been implemented.

Table 5.12: Modal Parameters of the beam modal identification.

	Frequency / [Hz]	ξ [%]	N.° of DOFs
Mode 1	37.779	1.16	8
Mode 2	107.477	0.238	8
Mode 3	203.601	0.153	8
Mode 4	335.535	0.154	8
Mode 5	506.300	0.076	8
Mode 6	674.711	0.130	8

Table 5.13 gives the correlation criterion MAC that relates the identified mode shapes. Analysing

the MAC table, we can expect good results from this modal analysis once there is a good coherence in the final results.

Table 5.13: MAC Criterion.

Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	37.779	100	0.2	27.0	0.7	10.2	1.7
2	107.477	0.2	100	2.0	20.9	0.0	6.4
3	203.601	27.0	2.0	100	0.1	17.5	1.0
4	335.535	0.7	20.9	0.1	100	0.3	19.5
5	506.300	10.2	0.0	17.5	0.3	100	0.5
6	674.711	1.7	6.4	1.0	19.5	0.5	100

Having the modal matrix and corresponding modal parameters available, the *FRFs* were synthesised in the required points. Figure 5.19 gives two samples, a transfer and a direct *FRF*, which compares the experimental curve already canceled and the synthesised *FRF*. Indeed, there is an excellent coherence between the two curves, once the synthesised *FRF* fits very well the experimental one both in resonance and anti-resonance zones.

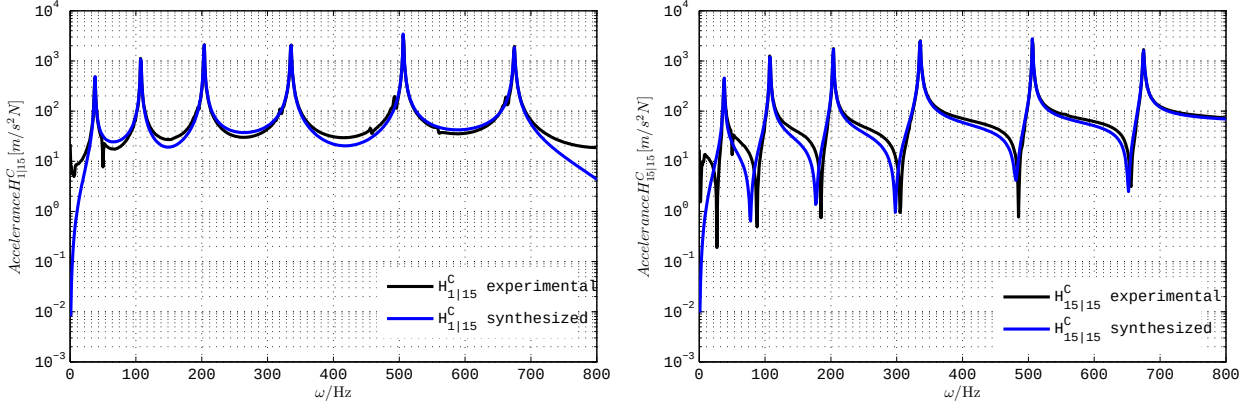


Figure 5.19: Accelerances $H_{1|15}^C$ and $H_{15|15}^C$.

• Joint Uncoupling

Then, the rotational *FRFs* were generated through the synthesised *FRFs* and the numerical matrix H^A . Having the necessary information available, the uncoupling techniques were implemented in order to achieve the joint dynamic characterization. Here again, the formulations which gave the best results are the same, although there are an higher uncertainty if we compare with the previous modal analysis, once the mass cancelation process done before the modal analysis could have contaminated even more the experimental data. In second and Wang formulations, there are more than one resonance identified and the level of uncertainty really increases comparing to previous modal analysis.

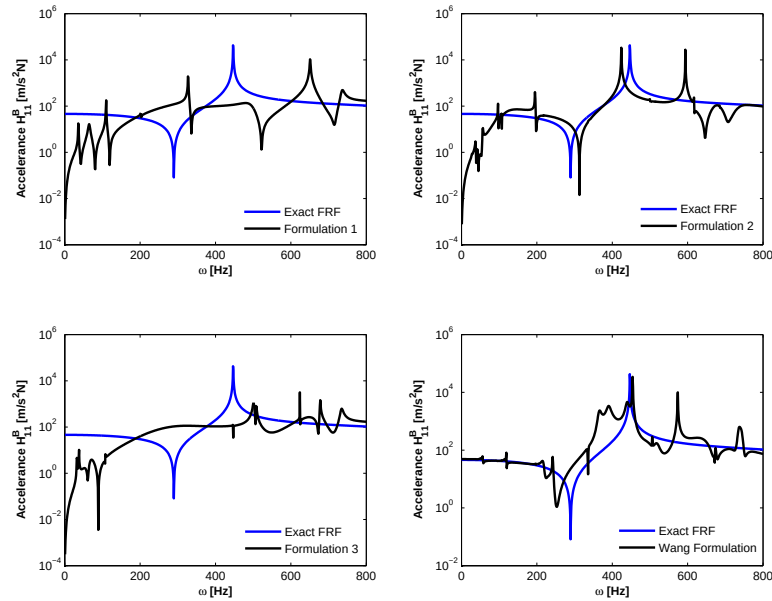


Figure 5.20: Accelerance H_{11}^B of the uncoupled joint.

- **Dynamic Stiffness**

Once obtained the joint dynamic characterization, the dynamic stiffness is calculated inverting the matrix H_{jj}^B . Figure 5.12 gives the dynamic stiffness Z_{11}^B according to second and Wang formulations. Second formulation provides good results despite the uncertainty present above, whereas Wang formulation reflects the high uncertainty at the right of the joint natural frequency.

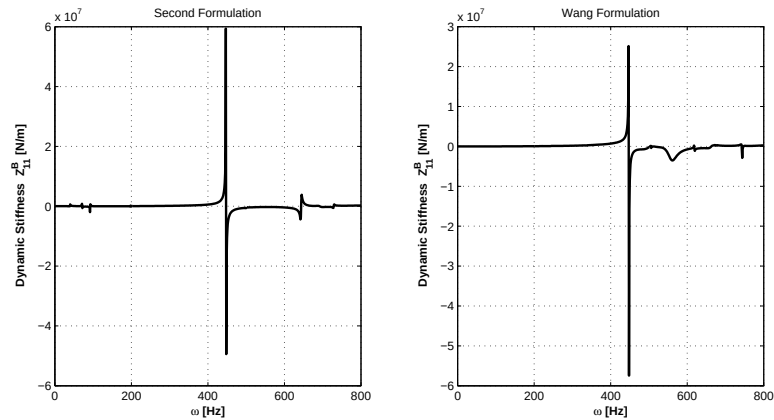


Figure 5.21: Dynamic stiffness Z_{11}^B of the uncoupled joint.

- **Coupling with joint FRFs**

If the dynamic characterization of the bolted joint is available, a new coupling can be performed and then compare these results with the initial FRFs of the main structure. Figure 5.22

provides the results of this coupling, and the discrepancy verified during the uncoupling process affects these results. This discrepancy is a direct consequence of several mathematical operations and of the noise that is generally introduced during experimental measurements. Analysing the results, in both formulations is complicated to fit the exact FRF , although the second formulation gives better results.

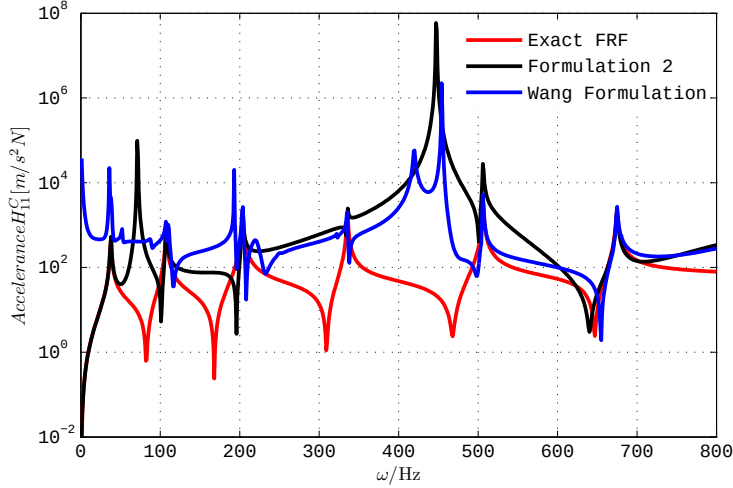


Figure 5.22: Accelerance H_{11}^C according to alternative coupling technique.

5.3 Modal Analysis with hammer 5800SL

This experimental modal analysis used the hammer as excitation mean. The data acquired during the previous experimental work with the hammer, in which we had measured the entire matrix H^C were used to perform these modal analysis. Once we have the entire matrix available, three different rows will be selected from that matrix and then a modal analysis will be performed for each matrix row. For each row, the accelerometer is always placed on the same position whereas the hammer varies its position. Thus, in each row we have always the same system with an identic mass distribution.

5.3.1 Modal Analysis using the second row

In this modal analysis, the accelerometer was placed on the second measurement point, whereas the hammer excited the other measurement points to obtain the translation $FRFs$. The modal identification process provided the modal parameters, which are available in table 5.14.

Table 5.14: Modal Parameters of the beam identification process (row 2).

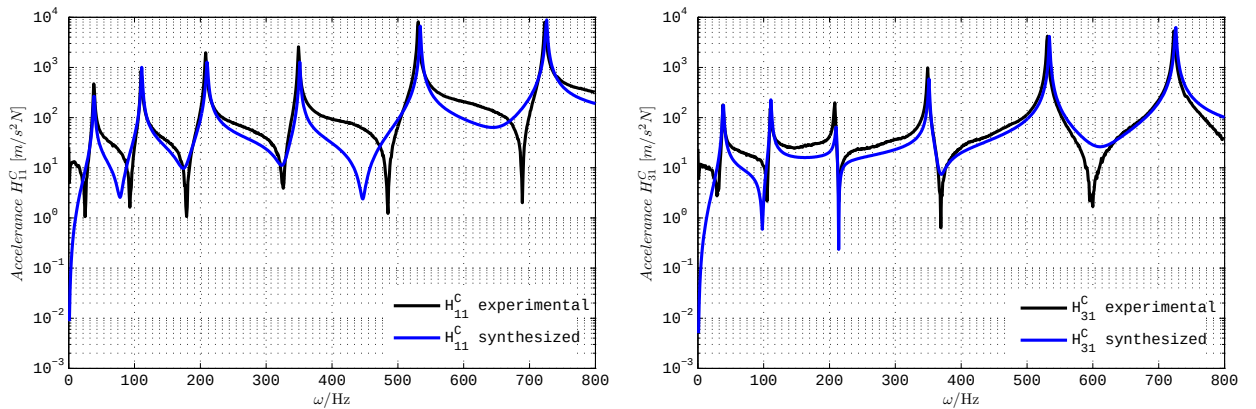
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	38.484	1.593	8
Mode 2	111.060	0.593	8
Mode 3	210.193	0.449	8
Mode 4	350.983	0.195	8
Mode 5	534.058	0.128	8
Mode 6	725.716	0.124	8

Table 5.15 has a representation of the correlation criterion *MAC* that relates the identified mode shapes.

Table 5.15: MAC Criterion (row 2).

Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	38.484	100	3.0	27.2	0.0	9.7	8.3
2	111.060	3.0	100	2.2	16.7	1.1	15.9
3	210.193	27.2	2.2	100	0.5	24.5	0.1
4	350.983	0	16.7	0.5	100	0.4	23.0
5	534.058	69.7	1.1	24.5	0.4	100	0.0
6	725.716	8.3	15.9	0.1	23.0	0.0	100

Once obtained the modal matrix and the corresponding parameters, either *FRF* could be synthesised in any point. Figure 5.23 compares the experimental *FRFs* with the synthesised ones. As we can see, there is a good correspondence in terms of natural frequencies and magnitude, although there are some differences in the anti-resonances zones.

Figure 5.23: Accelerances H_{11}^C and H_{31}^C (row 2).

- Joint uncoupling

Having the matrix H^C completely available, the uncoupling techniques can be applied to obtain the joint dynamic characterization (figure 5.24). Here again, second and Wang formulations provided better results whereas the other formulations showed a huge discrepancy as expected. In terms of resonances, both formulations identify very well the natural frequency, and the anti-resonance just is distinguishable in second formulation. Therefore, in this case second formulation behaves better than Wang formulation.

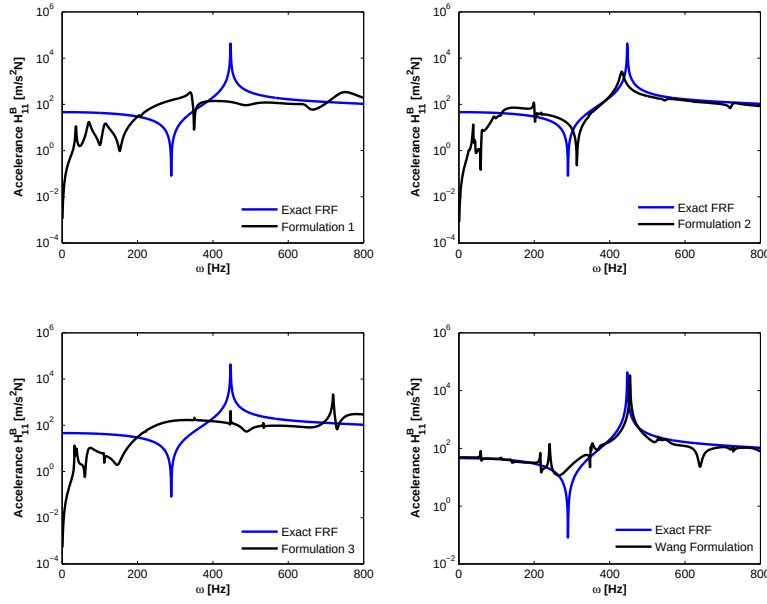


Figure 5.24: Accelerance H_{11}^B of the uncoupled joint (row 2).

- Dynamic Stiffness

Figure 5.25 gives the dynamic stiffness according to second and wang formulations once these formulations gave coherent results. Both curves are well defined, reflecting the impact of noise which normally influences the dynamic stiffness behaviour.

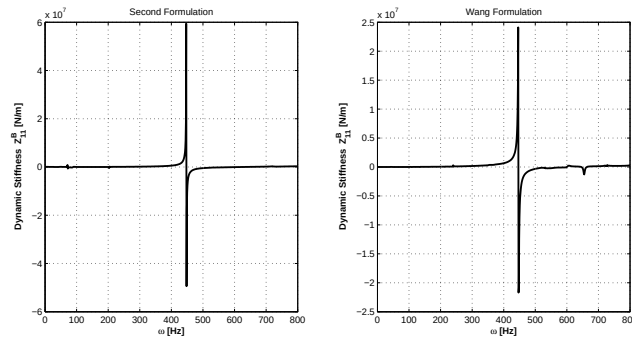


Figure 5.25: Dynamic stiffness Z_{11}^B of the uncoupled joint.

- Coupling with joint FRFs

Then, the results obtained through the second and Wang formulations were selected to carry out the coupling between these data and the theoretical matrix H^A . Figure 5.26 gives one sample of this coupling where there is correspondence in some natural frequencies, although there is a new natural frequency that appears again close to 450 Hz.

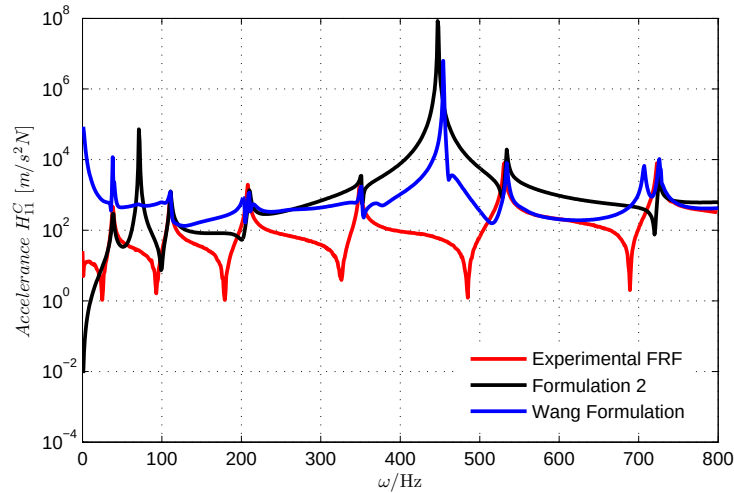


Figure 5.26: Accelerance H_{11}^C according to alternative coupling technique.

5.3.2 Modal Analysis of forth row

This modal analysis used the forth row of matrix H^C , with the accelerometer placed on a joint coordinate, the node 4. Having the accelerometer placed on the forth node, the *FRFs* were measured in every nodes changing the hammer position. Table 5.16 gives the results of this modal analysis.

Table 5.16: Modal Parameters of the beam identification process (row 4).

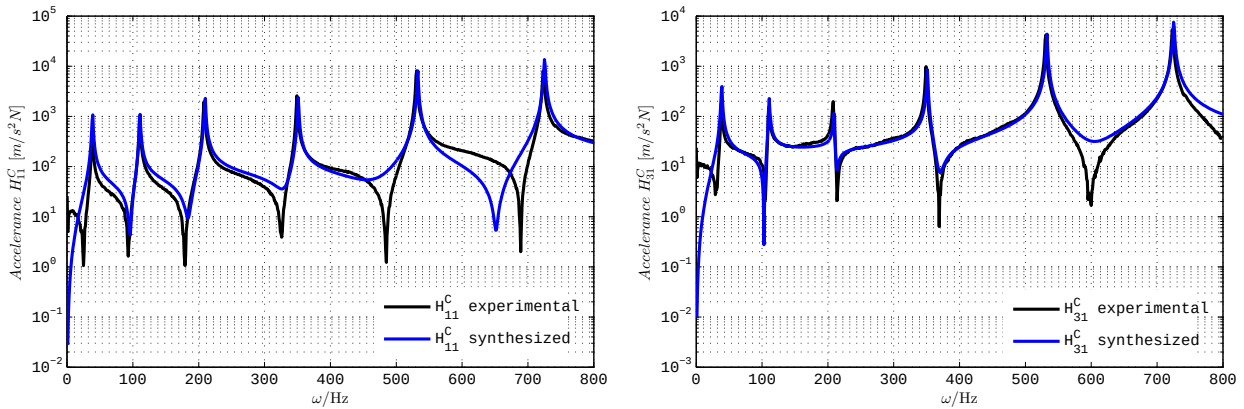
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	38.649	1.597	8
Mode 2	110.751	0.595	8
Mode 3	209.611	0.312	8
Mode 4	351.136	0.185	8
Mode 5	532.682	0.120	8
Mode 6	725.282	0.115	8

Table 5.17 gives a representation of the correlation criterion *MAC* that relates the identified mode shapes. Analysing the *MAC* table, there are some high values out of the main diagonal, despite the good coherence of the synthesised *FRFs* in comparison with experimental ones.

Table 5.17: MAC Criterion (row 4).

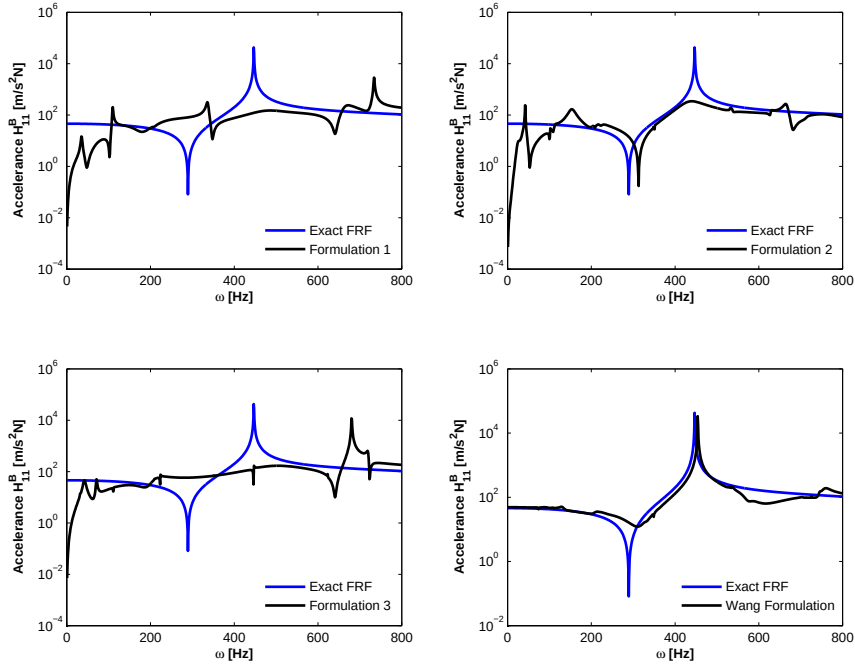
Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	38.64	100	0.4	37.6	0.9	23.3	1.9
2	110.75	0.4	100	1.0	23.4	0.1	18.3
3	209.61	37.6	1.0	100	0.6	26.3	0.1
4	351.14	0.9	23.4	0.6	100	0.2	24.0
5	532.68	23.3	0.1	26.3	0.2	100	1.3
6	725.28	1.9	18.3	0.1	24.0	1.3	100

Once obtained the modal matrix and corresponding parameters, the *FRFs* were synthesised in every points in order to perform the joint uncoupling. Figure 5.27 gives a comparison between a direct and a transfer *FRF* synthesised and experimentally measured. This synthesising gave really good results, once the synthesised *FRF* almost fits completely the experimental curve, both the direct and the transfer *FRF*. Even the anti-resonances are almost exact, having only some discrepancies in the highest frequencies.


 Figure 5.27: Accelerances H_{11}^C and H_{31}^C (row 4).

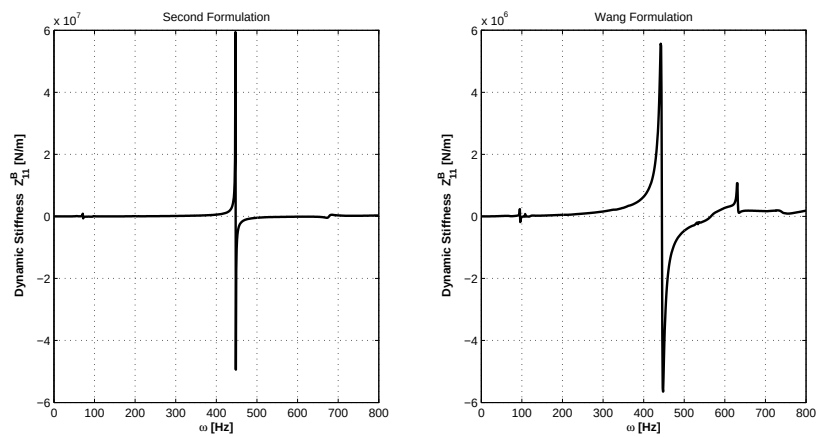
• Joint uncoupling

Then, the rotational *FRFs* were generated from the synthesised *FRFs* according to the modal model. Having all the information of the main structure available, the joint uncoupling was carried out. Figure 5.28 shows the results of the four formulations. Here again, the second and Wang formulations give the best results. The second formulation has an anti-resonance well defined, whereas Wang formulation provides a resonance pick well defined and much less uncertainty than the second formulation in first frequencies.

Figure 5.28: Accelerance H_{11}^B of the uncoupled joint (row 4).

- **Dynamic Stiffness**

Once obtained the joint dynamic characterization, it is possible to obtain the dynamic stiffness inverting directly the matrix H_{jj}^B . Figure 5.29 shows the dynamic stiffness of the formulations with the best results. The presence of noise in experimental data causes again the typical shape of these curves.

Figure 5.29: Dynamic stiffness Z_{11}^B according to second and wang formulations.

- **Coupling with joint FRFs**

Then, a new coupling between the experimental matrix H_{jj}^B and matrix H^A was carried

out to obtain a dynamic characterization of the whole structure. Figure 5.30 shows the FRF H_{11}^C where we can see a good correspondence in most frequencies for both formulations, and appears again an undesirable natural frequency close to 450 Hz that causes an high discrepancy in terms of magnitude in that frequency range.

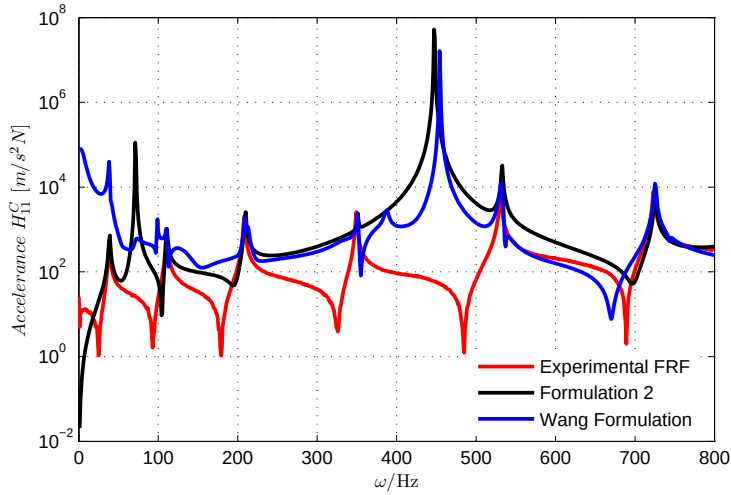


Figure 5.30: Accelerance H_{11}^C according to alternative coupling technique.

5.3.3 Modal Analysis of fifth row

Another modal analysis was carried out, this time using the fifth row of matrix H^C . This modal analysis corresponds to measure the response in measurement point five whereas the excitation is measured in all points. The modal parameters achieved are available in table 5.18.

Table 5.18: Modal Parameters of the beam identification process (row 5).

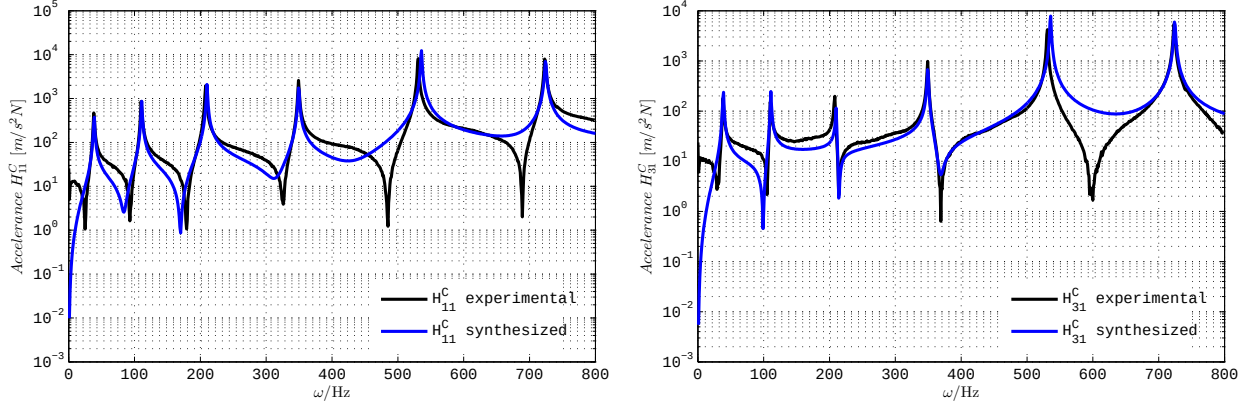
	Frequency / [Hz]	ξ [%]	N.º of DOFs
Mode 1	38.620	1.403	8
Mode 2	111.900	0.535	8
Mode 3	210.081	0.303	8
Mode 4	349.378	0.194	8
Mode 5	535.666	0.116	8
Mode 6	724.347	0.121	8

Table 5.19 provides the correlation criterion MAC which relates the identified mode shapes. This modal analysis seems to have given better results than the previous analysis, once the values out of the main diagonal aren't so high.

Table 5.19: MAC Criterion (row 5).

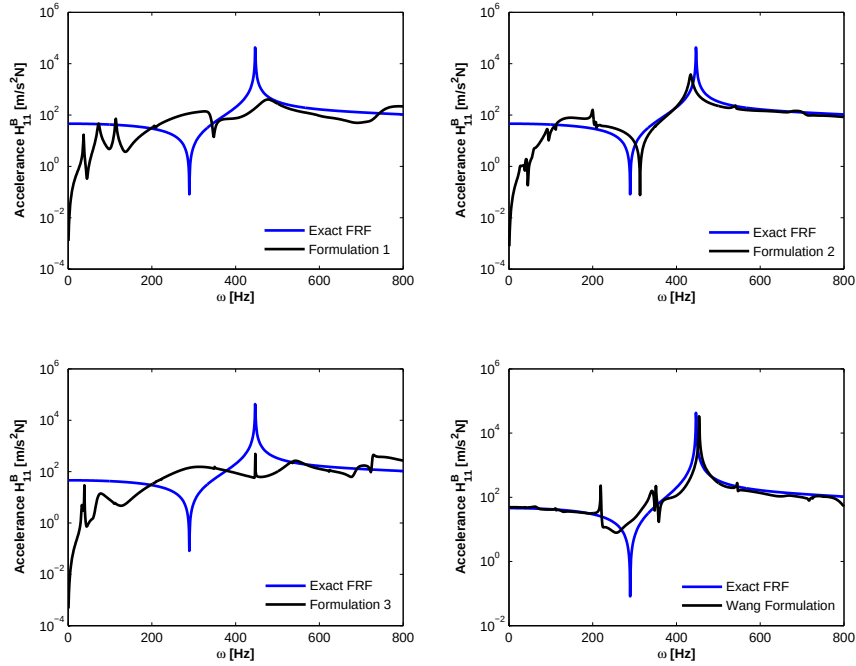
Mode	Frequency [Hz]	Mode 1 [%]	Mode 2 [%]	Mode 3 [%]	Mode 4 [%]	Mode 5 [%]	Mode 6 [%]
1	38.620	100	4.2	32.9	0.1	10.8	6.7
2	111.900	4.2	100	0.3	20.1	1.5	8.6
3	210.081	32.9	0.3	100	0.5	19.9	1.1
4	349.378	0.1	20.1	0.5	100	0.4	17.7
5	535.666	10.8	1.5	19.9	0.4	100	0.2
6	724.347	6.7	8.6	1.1	17.7	0.2	100

Figure 5.31 shows two synthesised *FRFs* according to the modal analysis results. The natural frequencies coincide exactly with the experimental curve, which indicates that there is a good correspondence between the *FRFs* of this matrix row. Furthermore, the discrepancy normally present in the first frequencies doesn't appear in synthesized *FRFs*, providing a good contribute to the performance of the following uncoupling techniques. In terms of anti-resonances, there is some similarity between both curves but the last anti-resonance is not very defined as previous ones.

Figure 5.31: Accelerances H_{11}^C and H_{31}^C (row 5).

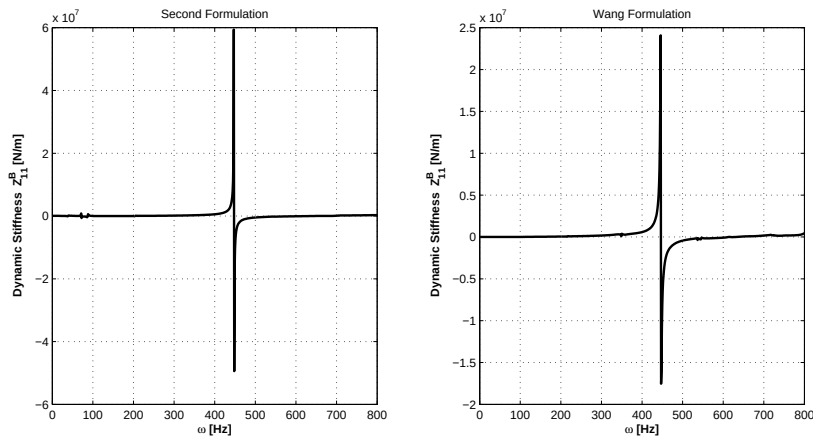
• Joint uncoupling

Then, the four uncoupling techniques were applied to obtain the joint dynamic characterization. Once again, the two formulations which gave better results are the second and Wang formulations. The second formulation provides indeed an excellent result, once the resonance and anti-resonance are very well defined and only is possible to find out some discrepancy in first frequencies. On the other hand, Wang formulation identifies very well the beam natural frequency whereas the anti-resonance isn't visible with some discrepancy in that zone. The first and third formulations gave poor results, as theoretically expected according to the numerical examples approached.


 Figure 5.32: Accelerance H_{11}^B of the uncoupled joint (row 5).

- **Dynamic Stiffness**

Figure 5.33 gives the joint dynamic stiffness of the two formulations which gave the best results. Once again, the dynamic stiffness takes the usual shape when experimental data is taken into account.


 Figure 5.33: Dynamic Stiffness Z_{11}^B of the uncoupled joint.

- **Coupling with joint FRFs**

A new coupling was performed taking into account the best results which the uncoupling techniques provided. Indeed, there is correspondence in some natural frequencies, as shown

in figure 5.34. The two formulations have a similar behaviour along the frequency, introducing again a new undesirable natural frequency close to 450 Hz. In terms of magnitude, the two formulations fit in a good way the experimental curve, excepting in the new natural frequency introduced.

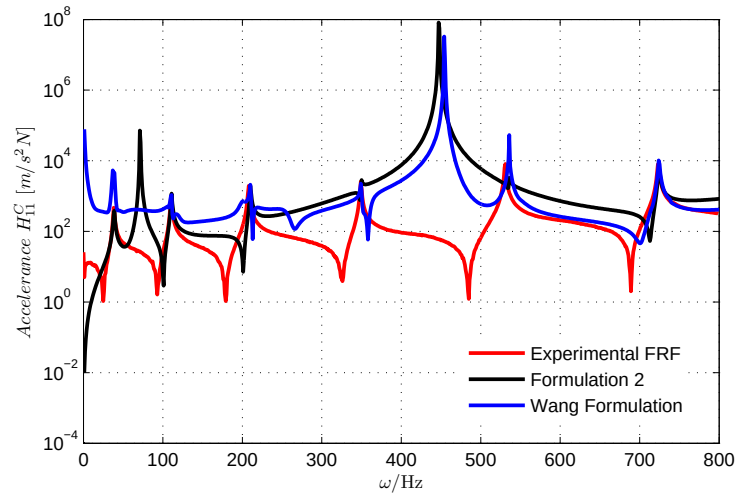


Figure 5.34: Accelerance H_{11}^C according to alternative coupling technique.

5.3.3.1 Evaluation of unmeasured FRFs

The algorithm presented in section 3.3 allows an estimative of unmeasured *FRFs* which are required to carry out the joint uncoupling. Thereunto, the experimental modal analysis that uses the fifth row was taken into account, in order to compare the estimated *FRFs* with *FRFs* provided by the *MEF* programme, *BAPMEF*. Here, we are considering transfer and direct *FRFs* that involve rotations, once these correspond to the *DOFs* where is impossible to do experimental measurements. The degrees-of-freedom of the extremities were considered in this study to evaluate the performance of the algorithm in this specific experimental example. The theoretical example uses a beam with the same dimensions, with a concentrated mass of 40 g in the joint position to simulate its interference. Figure 5.35 gives the comparison of four *FRFs* that involve rotations and translation *DOFs*.

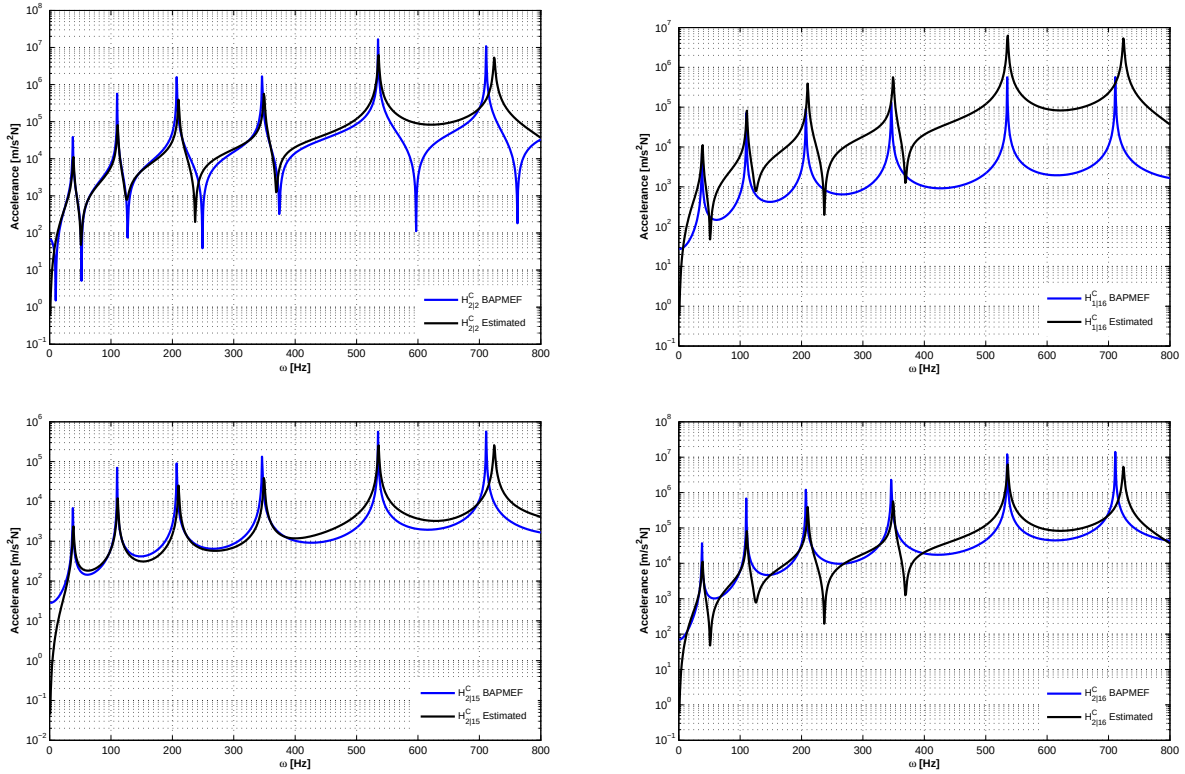


Figure 5.35: Comparison of estimated *FRFs*.

As we can see, there is indeed a good similarity between the theoretical and estimated *FRFs* in most cases, despite the *FRF* $H^C_{1|16}$ where there isn't correspondence in terms of amplitude and behaviour. However, despite some incoherence in any *FRF*, this algorithm provides excellent results, much better than expected when compared to the numerical example. Here again, we can point out that the natural frequencies of the theoretical *FRFs* are slightly lower than the natural frequencies of the estimated *FRFs*, meaning that the mass distribution of the bolted joint isn't exactly the same as considering a bolted jointed, as previously discussed in this dissertation.

Therefore, the implementation of this algorithm was carried out with success, despite some differences in any *FRFs* which have a direct interference in the results of the joint uncoupling. This study validates experimentally this algorithm that had provided an indispensable tool along this dissertation.

5.3.3.2 Analysis of frequency ranges

Up to now, in this work several experimental examples were carried out and different results were achieved. The algorithm that allows the selection of frequencies where the noise level is lower was applied in this example, once this provided a better approximation to the dynamic stiffness. Figures 5.36 and 5.37 show the results obtained with this algorithm. The blue curve represents the exact dynamic stiffness, whereas the black one represents the approximated function.

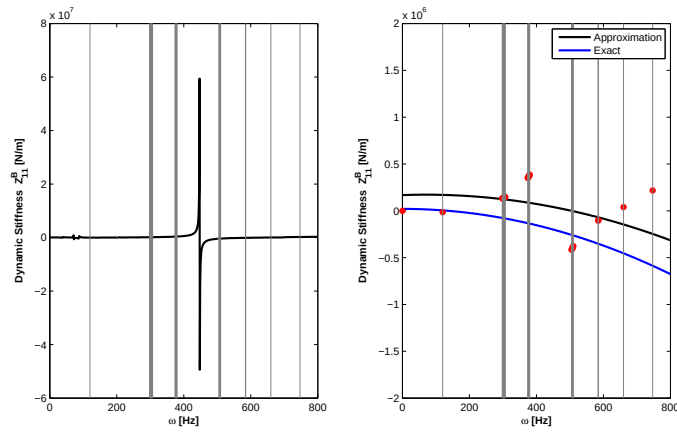


Figure 5.36: Dynamic Stiffness Z_{11}^B obtained with all frequencies and with the selected frequencies, according to second formulation.

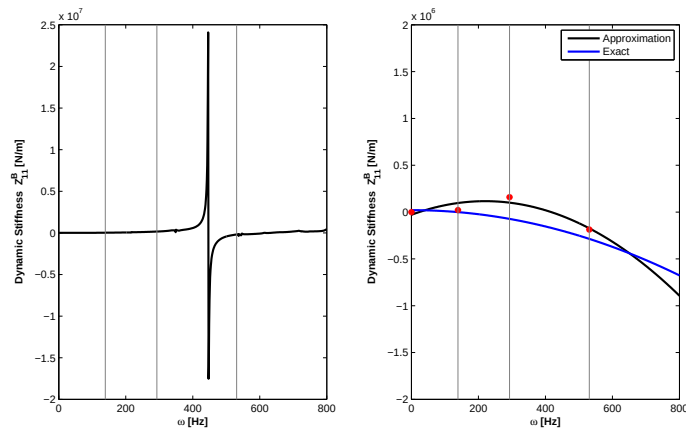


Figure 5.37: Dynamic Stiffness Z_{11}^B obtained with all frequencies and with the selected frequencies, according to Wang formulation.

The input parameters of the algorithm that selects the frequencies are available in table 5.20. The parameters were adjusted for each formulation in order to obtain a good approximation function.

Table 5.20: Input parameters of the algorithm.

	τ	v	ξ
Second formulation	30	40E^{-6}	0.03
Wang formulation	30	60E^{-6}	0.02

Analysing the results obtained, we have achieved a good approximation function for each formulation, which validates this algorithm. The exact curve corresponds to a beam with the same joint lenght, but it has no concentrated mass that would simulate the presence of the nut and respective bolt. Therefore, the curve is not completely equivalent in physical terms, but this represents indeed the behaviour of the joint dynamic stiffness.

5.4 Discussion of results

The experimental modal analyses carried out and presented in this chapter revealed a good feasibility and effectiveness in terms of final results, as well as an alternative to the experimental procedure applied in previous chapter where is required the measurements of all *FRFs*.

Firstly, the experimental modal analyses used the shaker as excitation mean and different ways of measuring the response. The force sensor introduced extra mass on the system and consequently mass cancelation procedures were required.

The modal analysis which used the eighth column of the experimental matrix H_{tt}^C pointed out the important influence of the transducers' mass. The synthesised *FRFs*, using the modal parameters of this column, showed a big difference in the natural frequencies values, principally when the *FRFs* of other columns were compared with the experimental ones. Indeed, this difference is directly related with the influence of the force sensor mass which can not be ignored. Despite the finding of this important fact, the results of the algorithm usually applied showed good coherence in the joint uncoupling. This fact helps to better understand why in previous chapter the measurements that used the shaker weren't successful, due to the different force transducer position in each matrix column.

In order to cancel the force sensor mass and become the structure as consistent as possible, two experimental alternatives were presented, using a laser vibrometer to measure the response and collocating dummy masses in all points to simulate the accelerometer influence. In these two alternatives, we were looking for eliminate the accelerometer influence and become the mass cancelation algorithm more effective, due to the problems related with the accelerometer influence properly explained earlier, and then cancel numerically the force sensor mass.

The laser vibrometer provided a good experimental efficiency, once this transducer doesn't introduce extra mass in the system. The cancelation of the force sensor mass was made in two

different ways and the moment of this mass cancelation process had a great impact in the results. Performing the mass cancelation process before the modal analysis, we are introducing even more numerical uncertainty in the samples and consequently the quality of the results decrease. In the modal analysis, the identification of natural frequencies became complicated and consequently the MAC table reflected the poor results of this modal analysis. The inverse process, doing first the synthesising of the *FRFs* and then the mass cancelation revealed a better effectiveness, once the mass cancelation is done with data much less contaminated with noise and there is correspondence between the natural frequencies. The second way showed therefore a better effectiveness, but these two processes would give theoretically the same results and here is well reflected the influence of one more mathematical operation. In terms of results, the second way provided better results, although we could visualize some discrepancy in the uncoupling results comparing to previous experimental results.

The use of dummy masses showed a good availability and effectiveness once it was possible to simulate the accelerometer influence in all measurement points. Indeed, the mass distribution of the beam became different once basically eight concentrated masses were introduced in the structure and consequently it was verified that the natural frequencies decreased lightly. In this experimental setup, the mass cancelation process followed the same way, before and after of the modal analysis. The impact of the mass cancelation process wasn't so perceptible, but it still had some influence. The results of the first way, doing first the modal analysis, gave better results once we were canceling data much less contaminated.

Taking into account both ways, the use of dummy masses provided better results than the use of the laser vibrometer, but we were expecting better results with the laser vibrometer. The way to measure the response could have influenced the results, once there is no explanation to the bad results obtained with the laser vibrometer. Theoretically, the laser vibrometer would be the best procedure once just the force sensor mass is introduced in the system, and then the force sensor is canceled, supplying the information of the real beam.

Another experimental modal analyses were carried out, this time using the hammer as excitation mean and the accelerometer to measure the response. Three different matrix rows were used in order to compare the results. In this specific case, the influence of accelerometer mass was ignored and evaluating the final results, this condition had't almost any effect in the main structure. Therefore, in each row we are analysing always a system with the same mass distribution, whereas if another row is selected, just the accelerometer position is changed.

Both modal analyses of the three rows provided good results with a good correspondence and coherence. The synthesised *FRFs* fitted very well the experimental ones, showing an exact coincidence in terms of natural frequencies whereas the anti-resonances zones were generally correspondent, although there were some discrepant cases. The forth row provided an excellent approximation to the uncoupled joint, using the Wang formulation, with a resonance and anti-resonance very well defined. On the other hand, the fifth row gave an excellent result to the uncoupled joint using the second formulation. The coupling between the uncoupled joint and the other beam substructure gave generally better results than the experimental setup using the shaker, being possible to achieve correspondence in some natural frequencies. However, the comparison between the results of this coupling and the initial *FRFs* revealed that the mathematical operations have an huge impact and influence in the experimental data quality. In many examples, the *FRFs* obtained with the new coupling process presented a new natural frequency close to 450 Hz, which indeed has an equivalent value to the joint natural frequency

and can there can be some relationship between this two facts once this spurious natural frequency of the main structure was unexpected and absent.

The comparison of estimated *FRFs* with the corresponding theoretical homonym, showed the high effectiveness of this algorithm that allowed the achievement of interesting results. In the four examples studied, only one had evidenced some discrepancy, despite a good similarity with the theoretical curve. The algorithm of selection of frequency ranges allowed a coherent quadratic approximation to the theoretical dynamic stiffness, which proves the correct implementation of this algorithm.

Chapter 6

Conclusion

6.1 Conclusions

6.1.1 Theoretical Work

This dissertation presents a deep study about a dynamic characterization of mechanical components, in this case bolted joints, using uncoupling techniques to perform that dynamic identification. Indeed, these techniques are really sensitive to any kind of error or other external influence and experimental techniques have to be applied in order to enhance the performance of that uncoupling techniques.

The uncoupling techniques showed a good performance and accuracy when exposed to clean data, but when contaminated with noise their behaviour changed a lot and consequently a formulation differentiation was made. The introduction of noise affects directly the matrix condition, and this is the main factor which affects their performance, as well as matrix inversions and differences which contributes even more to this fact. Associated to this important detail, the type of coordinates used in each formulation has an important role in the accuracy of these methods. An analysis was carried out taking into account these details, and therefore whenever possible we should measure in joint coordinates, as well as give more importance to that formulations which have a low mathematical manipulation effort. Between the formulations approached, the second and Wang formulations provided always better results once these techniques fulfill all the conditions which less affect their mathematical performance.

The experimental techniques considered to enhance the performance of the uncoupling techniques revealed a great utility, once it was shown that they can help a lot in the analysis of experimental data which normally is polluted with noise. The estimative of unmeasured *FRFs* was in fact an important tool to perform the uncoupling, because without rotational *FRFs* these uncoupling techniques couldn't be applied. The analysis of frequency ranges allowed a better approximation to the dynamic stiffness, which when exposed to contaminated data takes a strange shape. The mass cancelation process confirmed that the transducers used in experimental simulations have an important impact in the structure dynamic behaviour which can

not be ignored as we found out in the experimental work.

The Euler-Bernoulli finite elements used showed a good applicability and feasibility. During the elaboration of the different numerical examples, it was stated that the number of elements of each substructure influences directly the accuracy of the uncoupling techniques. Increasing the number of elements, the accuracy of those techniques also improves to a certain number from what there is no need to use more elements.

The correlation criterion *LAC* provided the comparison between theoretical and contaminated *FRFs*, being an important tool that made easier the evaluation on the uncoupling techniques effectiveness, once all of them have given different results.

These theoretical formulations were validated with scientific work such as articles and other which verified the results and gave motivation to an experimental work based on this previous theoretical work carried out.

6.1.2 Experimental Work

The experimental work focused basically on two different ways of achieve the global matrix H^C , measuring all the *FRFs* and through a row or a column, perform a modal analysis that provide the information of the whole structure.

The uncoupling techniques had a similar behaviour to what were obtained in the theoretical examples, proving that the numerical noise introduced has some similarity to the reality. Here again, the formulations that use joint coordinates provided better results, which reinforces again that joint measurements must be carried out.

The measurements of all *FRFs* with the hammer as excitation mean gave coherent results. The mass cancelation process increased slightly the mean *LAC* between the transfer *FRFs*, once the accelerometer mass is really low. Unfortunately, the measurements with the shaker gave bad results principally due to the influence of the transducers' mass which cause different mass distributions in each column and consequently large differences in the transfer *FRFs* as stated by indicator *LAC*.

In this dissertation, the mass cancelation had an important role and this process was indeed much more sensible and complicated than initially expected. Actually, the influence of transducers' masses has an extraordinary importance and a strong impact in the structures behavior. The knowledge and discovery of this important interference led us to adopt different experimental setups to avoid or decrease these type of errors. The use of dummy masses and a laser vibrometer were two solutions presented in this work that eliminated this problems and became the mass cancelation completely effective in numerical terms, but the results weren't what we were really looking for.

The experimental modal analyses were carried with success, providing good results when the hammer was used as excitation mean. In relation to the experimental work that measured all the *FRFs*, this process has important advantage once the noise level in this case is much

lower than in other experimental way. Consequently, if we have a lower level of noise, the results will be much better because the noise has a strong impact in mathematical operations as seen along this work. Therefore, this experimental process is much more advantageous than the other one, and is in fact another working way in this area that gives better results. Furthermore, considering n as the number of measurement points, a modal analysis is much less time consuming than the first experimental procedure once in a modal analysis are required only n measurements whereas in first procedure are required n^2 measurements. The execution of an experimental modal analysis appears as an alternative with excellent results comparing to the usual experimental work that uses all the *FRFs*.

In terms of excitation means, in this work the hammer provided often better results than the shaker. The use of the shaker introduced extra mass that caused the problems properly documented in this work, despite the good results obtained in terms of adjustment and coherence in the resonance and anti-resonance zones. Actually, the source of this problem is in the drive rod used with the shaker, once the force transducer had a mass that could not be ignored. On the other hand, the hammer gave always results with exact natural frequencies, despite the lack of accuracy in certain frequency ranges. The use of the shaker is more practical in terms of data acquisition, whereas using the hammer, the excitation is done manually.

Both correlation criterions used in the experimental work, *LAC* and *MAC*, showed a great applicability and effectiveness when was required the comparison of results. The criterion *LAC* provided the comprehension of some problems, such as the differences between *FRFs* that were causing strange results, as well as the influence of concentrated introduced in the system by transducers. The correlation criterion *MAC* had evaluated all the experimental modal analysis, relating the different modal shapes identified in each analysis and giving an idea about the analysis effectiveness.

The dissertation now ending contributed to deepen these subjects and understand the real boundaries of this knowledge field. These matters are principally influenced by noise that influences all the mathematical operations and consequently all procedures, as well as small experimental interferences that sometimes have an important impact in the results and must be avoided and eliminated whenever possible.

6.2 Future Works

The uncoupling of mechanical components from their main structures is an extensive subject with lots of constraints that affect the performance of these methods. Indeed, these methodologies are really sensitive and vulnerable to extra factors as we have seen along this dissertation, and there is an urgent necessity of develop new techniques capable of reduce or even eliminate the source of some errors which have an high effect. Therefore, this scientific field requires much more research and a long way has still to be made to achieve better results.

The technological progress could give an important help to these uncoupling techniques, once if the experimental devices improved their capacities, the high noise levels usually present in experimental data would decrease and consequently the level of uncertainty would not affect so much these techniques. This is an important point because noise causes bad conditioned

matrices which have a direct impact in mathematical operations such as inversions and differences of matrices.

The influence of transducers' masses appears as a subject which is still not very well studied, and in this work they caused some problems for what were presented some solutions. These details influence directly the structure's natural frequencies once extra mass is added, and consequently the results hold with their interference. Indeed, an experimental measurement has these problems and therefore new methodologies need to be developed in order to remove the influence of that masses. Some work have already been done in this area ([18]), but new methodologies more feasible and practical are required.

Another matter that need to be deeply approached is the joint mass distribution, because the beam overlap in conjunction with the bolt provides a different mass distribution, as well as dynamic behaviours which depend on mass distribution. The contact between the bolt and the two beam parts could introduce any non-linear behaviour even as other unknown phenomena. In this work, it was point out that considering a beam with a concentrated mass, which would represent the extra mass introduced, is not exactly the same once the beam natural frequency differs a little from the value that the uncoupling techniques give.

A matter that could be even more approached is the dynamic stiffness. In this work, we have calculated the dynamic stiffness in each experimental analysis, as well as from the theoretical point of view and a algorithm of identification of frequencies was applied. However, the real value of this physical quantity isn't still well developed and approached. A depth study about this concept could provide extra information and knowledge about the real capacity and coverage of this matter.

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Appendix A

Euler-Bernoulli Finite Elements

Throughout this work, Euler-Bernoulli elements have been used in the examples approached. In such element there are two degrees of freedom at a node in its local coordinate system, in which one is by deflection in the y direction, ν , and other is by rotation in the $x-y$ plane, θ [21]. Thus, each Euler-Bernoulli element has a total of four degrees-of-freedom, as shown in figure A.1.

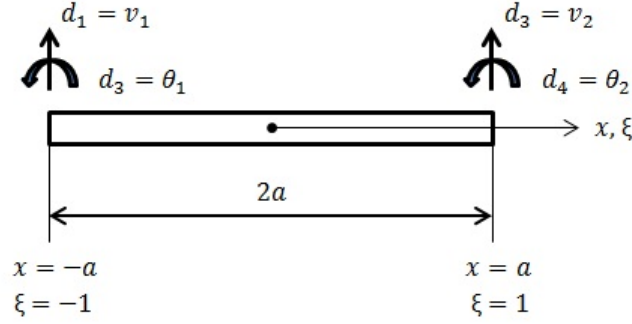


Figure A.1: Euler-Bernoulli finite element.

The relationship between the natural coordinate system and the local coordinate system can be expressed as:

$$\xi = \frac{x}{a} \quad (\text{A.1})$$

To obtain the four shape functions in the natural coordinates, the displacement in an element is assumed in the form of third order polynomial of ξ which contains the corresponding four unknowns constants:

$$\nu(\xi) = \alpha_0 + \alpha_1 \nu + \alpha_2 \nu^2 + \alpha_3 \nu^3 \quad (\text{A.2})$$

In equation A.2, α_i represents each unknown constant. The above expression can be represented in the following matrix form:

$$\nu(\xi) = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad (\text{A.3})$$

Or,

$$\nu(\xi) = p^T(\xi) \alpha \quad (\text{A.4})$$

Where $\mathbf{p}$ is the vector of basis functions and α is the vector of coefficients. The rotation can be obtained from equations A.1 and A.2:

$$\theta = \frac{d\nu}{dx} = \frac{d\nu}{d\xi} \frac{d\xi}{dx} = \frac{1}{a}(\alpha_0 + 2\alpha_2\xi + 3\alpha_3\xi^2) \quad (\text{A.5})$$

The four unknowns constants can be determined by the following four boundary conditions:

$$\left. \begin{array}{l} x = -a \quad \nu(-1) = \nu_1 \\ \xi = -1 \quad \frac{d\nu}{dx} = \theta_1 \end{array} \right| \left. \begin{array}{l} x = a \quad \nu(1) = \nu_2 \\ \xi = 1 \quad \frac{d\nu}{dx} = \theta_2 \end{array} \right.$$

Then, the application of these four conditions yields the following expression:

$$\begin{Bmatrix} \nu_1 \\ \theta_1 \\ \nu_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & \frac{1}{a} & -\frac{2}{a} & \frac{3}{a} \\ 1 & 1 & 1 & 1 \\ 0 & \frac{1}{a} & \frac{2}{a} & \frac{3}{a} \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad (\text{A.6})$$

Or,

$$d_e = A_e \alpha \quad (\text{A.7})$$

Solving the previous expression A.7, in order to α :

$$\alpha = A_e^{-1} d_e \quad (\text{A.8})$$

Where

$$A_e^{-1} = \frac{1}{4} \begin{bmatrix} 2 & a & 2 & -a \\ -3 & -a & 3 & -a \\ 0 & -a & 0 & a \\ 1 & a & -1 & a \end{bmatrix} \quad (\text{A.9})$$

Thus, replacing equation A.8 in equation A.4, it comes:

$$\nu = N(\xi) d_e \quad (\text{A.10})$$

The matrix N is given as:

$$N(\xi) = P A_e^{-1} = [N_1(\xi) \quad N_2(\xi) \quad N_3(\xi) \quad N_4(\xi)] \quad (\text{A.11})$$

The corresponding shapes functions are the following:

$$\begin{cases} N_1(\xi) = \frac{1}{4}(2 - 3\xi + \xi^3) \\ N_2(\xi) = \frac{1}{4}a(1 - \xi - \xi^2 + \xi^3) \\ N_3(\xi) = \frac{1}{4}(2 + 3\xi - \xi^3) \\ N_4(\xi) = \frac{a}{4}(-1 - \xi + \xi^2 + \xi^3) \end{cases}$$

- **Strain Matrix**

Once obtained the four shape functions, the strain matrix can now be derived. Therefore, the strain matrix relates the strain and the deflection, as:

$$\epsilon_{xx} = B d_e \quad (\text{A.12})$$

The strain matrix **B** is given by:

$$B = -y L N = -y \frac{d^2}{dx^2} N = -\frac{y}{a^2} \frac{d^2}{d\xi^2} N = -\frac{y}{a^2} N'' \quad (\text{A.13})$$

Where N'' is the matrix of the second derivatives of the shape functions,

$$N'' = [N_1'' \quad N_2'' \quad N_3'' \quad N_4''] \quad (\text{A.14})$$

- **Stiffness Matrix**

Having the strain matrix defined, the stiffness matrix for an element can be easily obtained, according to the following expression:

$$\begin{aligned} K_e &= \int_V B^T c B dV = E \int_A y^2 dA \int_{-a}^a \left(\frac{d^2}{dx^2} N \right)^T \left(\frac{d^2}{dx^2} N \right) dx = \\ &= EI_z \int_{-1}^1 \frac{1}{a^4} \left[\frac{d^2}{d\xi^2} N \right]^T \left[\frac{d^2}{d\xi^2} N \right] a d\xi = \frac{EI_z}{a^3} \int_{-1}^1 N''^T N'' d\xi \end{aligned} \quad (\text{A.15})$$

$$I_z = \int_A y^2 dA \quad (\text{A.16})$$

Equation A.16 represents the moment of inertia of a cross section of a section with respect to the z axis. Replacing equation A.14 into A.16 and evaluating the corresponding integrals, it gives:

$$K_e = \frac{EI_z}{2a^3} \begin{bmatrix} 3 & 3a & -3 & 3a \\ 3a & 4a^2 & -3a & 2a^2 \\ -3 & -3a & 3 & -3a \\ 3a & 2a^2 & -3a & 4a^2 \end{bmatrix} \quad (\text{A.17})$$

- **Mass Matrix**

The mass matrix can be obtained through the following integral:

$$m_e = \int_V \rho N^T N dV = \rho \int_A y^2 dA \int_{-a}^a N^T N dx = \rho A \int_{-1}^1 N^T N a d\xi \quad (\text{A.18})$$

Then, evaluating the integral in expression A.18, it leads to:

$$m_e = \frac{\rho A a}{105} \begin{bmatrix} 78 & 22a & 27 & -13a \\ 22a & 8a^2 & 13a & -6a^2 \\ 27 & 13a & 78 & -22a \\ -13a & -6a^2 & -22a & 8a^2 \end{bmatrix} \quad (\text{A.19})$$

- **Force Vector**

Lastly, the force vector of an element can be obtained. Supposing that an element is loaded by an external distributed force f_y along the x-axis, and two concentrated forces f_{s1} and f_{s2} , and two concentrated moments m_{s1} and m_{s2} , at nodes 1 and 2, respectively; the overall nodal force vector becomes:

$$\begin{aligned} f_e &= \int_V N^T f_b dV + \int_{S_f} N^T f_s dS_f = \\ &= f_y a \int_{-1}^1 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} d\xi + \begin{Bmatrix} f_{s1} \\ m_{s1} \\ f_{s2} \\ m_{s2} \end{Bmatrix} = \begin{Bmatrix} f_y a + f_{s1} \\ f_y a^3/3 + m_{s1} \\ f_y a + f_{s2} \\ -f_y a^2/3 + m_{s2} \end{Bmatrix} \end{aligned} \quad (\text{A.20})$$

- **Frequency Response Function**

Along this work, all the frequency response functions (*FRFs*) have been calculated taking into account the spatial model of a beam, defined by its mass, damping and stiffness matrices. Therefore, the finite element program used, BAPMEF, provides first the matrices which define the spatial model and then the desirable *FRFs* are calculated in the selected degrees-of-freedom.

Equation A.21 gives the how in which a *FRF* is calculated at each frequency ω , involving an inversion at each frequency, what becomes computationally the process too demanding. The matrix K corresponds to the stiffness matrix of the global beam or structure, whereas M corresponds to the mass matrix of the global structure. For the sake of simplicity, all the expressions

presented above are formulated to a single element, whereas K and M are the final product of the assembly of all elementary matrices.

$$H(\omega) = (K - \omega^2 M)^{-1} \quad (\text{A.21})$$

To overcome the necessity of invert the matrix at each frequency, the programme *BAPMEF*, calculates a *FRF* according to expression A.22. This way corresponds to a system of equations, where $\{H\}_j$ is the *FRF* that will be calculated, whereas $\{F\}_j$ is the force vector composed by a unitary load in excitation *DOF* and zeros in the other components.

$$[K - \omega^2 M] \{H\}_j = \{F\}_j \quad (\text{A.22})$$

Here, we obtain a column of the *FRFs* matrix corresponding to the excitation at degree-of-freedom j and response for all the degrees-of-freedom, whereas according to expression A.21 we obtain the entire *FRF* matrix at each frequency ω .

Appendix B

Experimental results

B.1 Experimental FRFs

B.1.1 Beam 1

B.1.1.1 Results obtained with Hammer 5800SL / Accelerometer 27A11

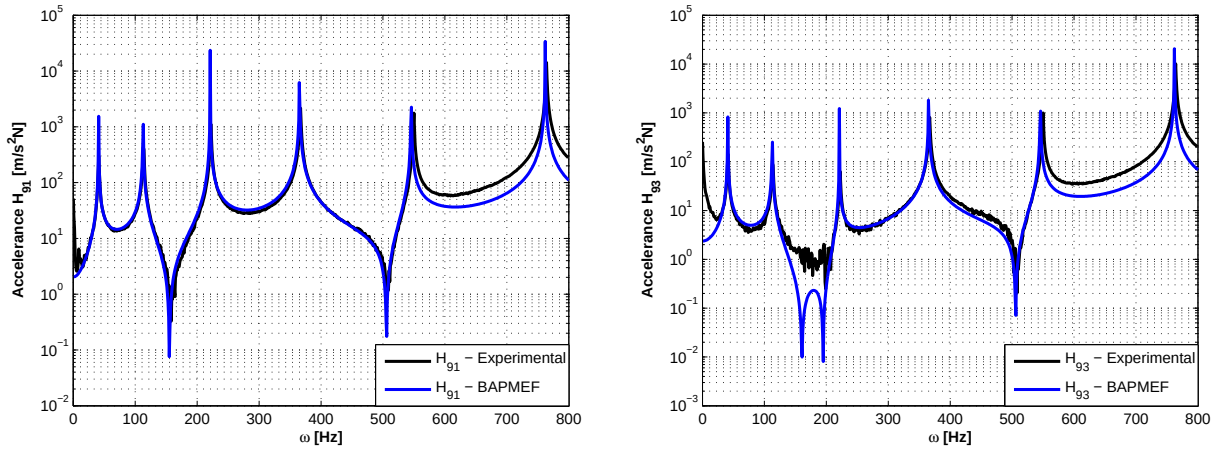


Figure B.1: Accelerances H_{91} and H_{93} .

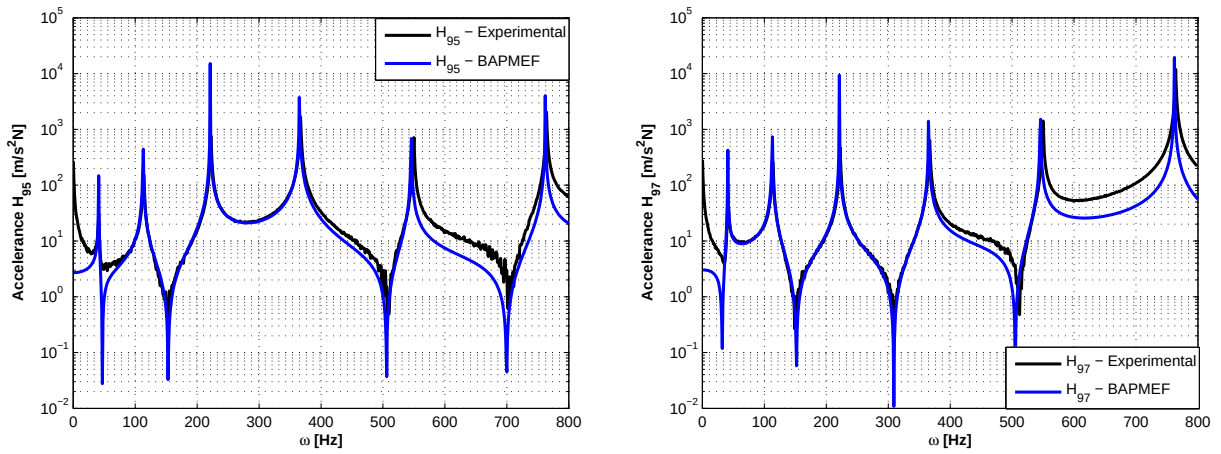
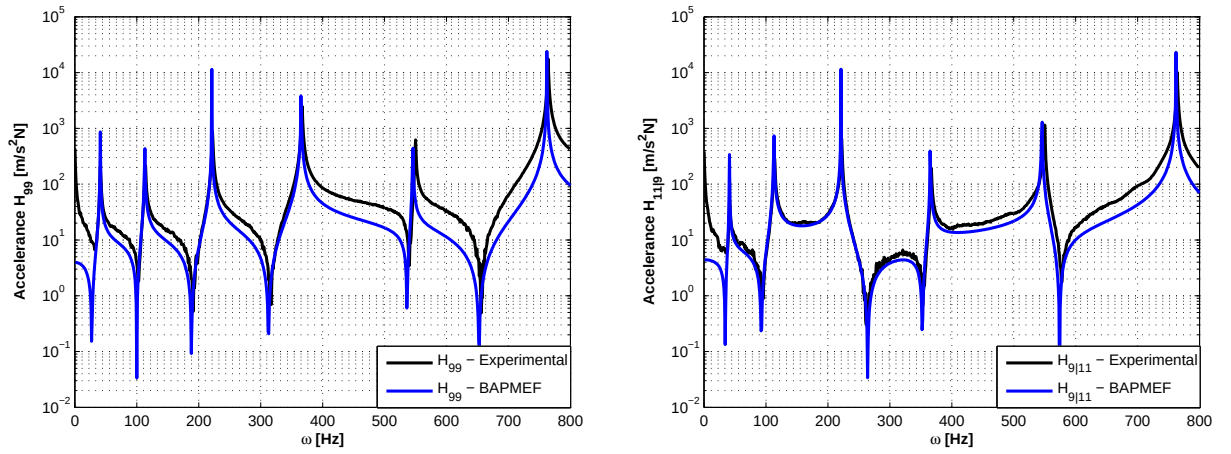
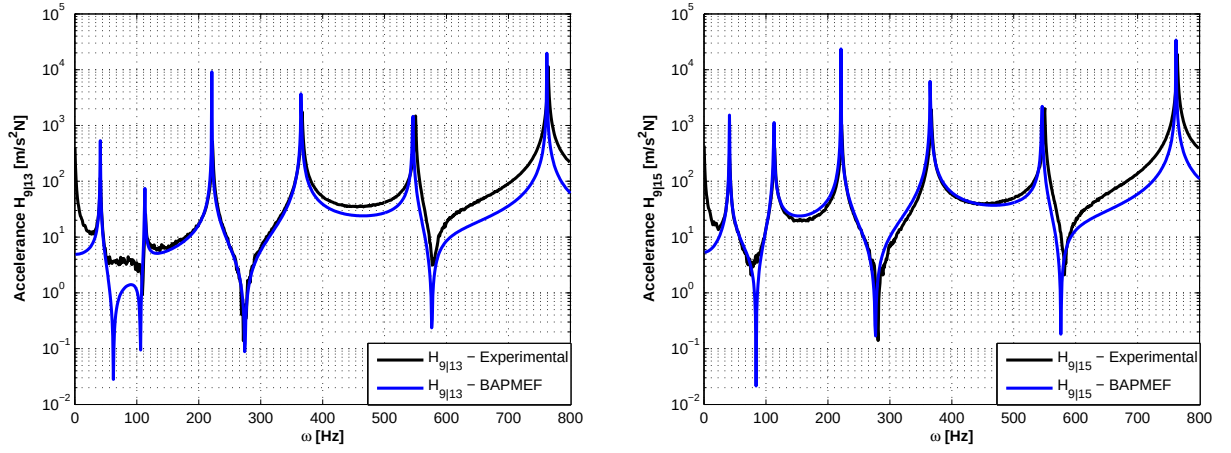


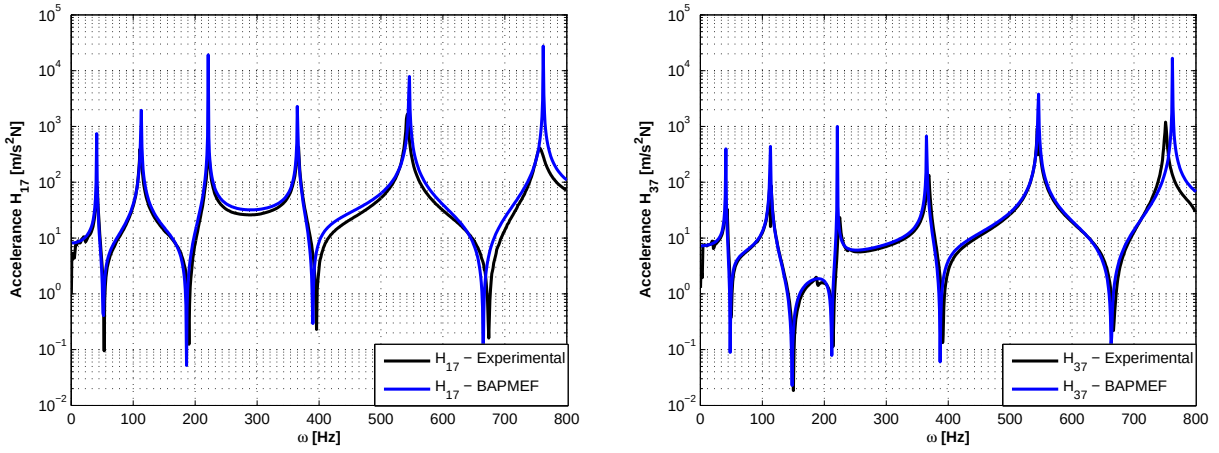
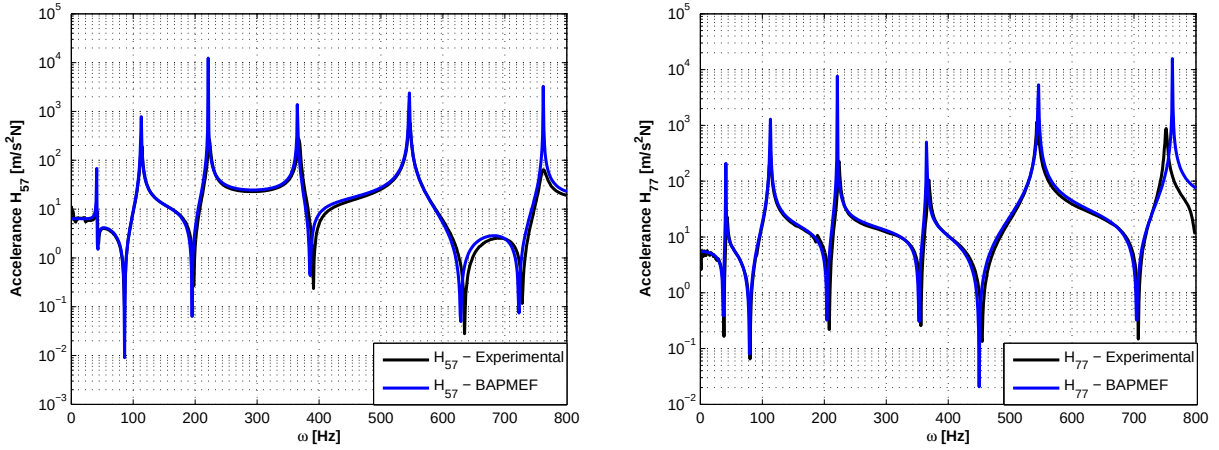
Figure B.2: Accelerances H_{95} and H_{97} .

Figure B.3: Accelerances H_{99} and $H_{9|11}$.Figure B.4: Accelerances $H_{9|13}$ and $H_{9|15}$.

This experimental test was carried out with an impact hammer 5800SL. The response was measured on node five, corresponding to the degrees-of-freedom nine and ten, and all the measurement points were excited with the impact hammer. Only the translation degrees-of-freedom have been measured.

B.1.1.2 Results obtained with Shaker LDS201 / Accelerometer 27A11

This experimental test was carried out with the shaker LDS201. The excitation was taken into account in measurement point 4, having the shaker instrumented in this node. Then, the response was measured in all nodes along the beam. Once again, only the translation degrees-of-freedom have been measured.

Figure B.5: Accelerances H_{17} and H_{37} .Figure B.6: Accelerances H_{57} and H_{77} .

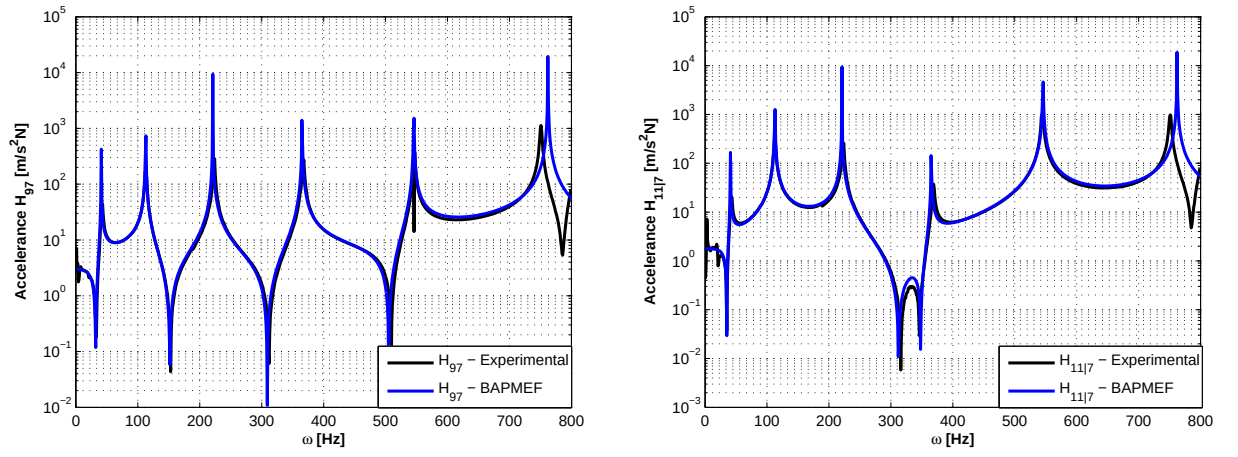


Figure B.7: Accelerances H_{97} and $H_{11|7}$.

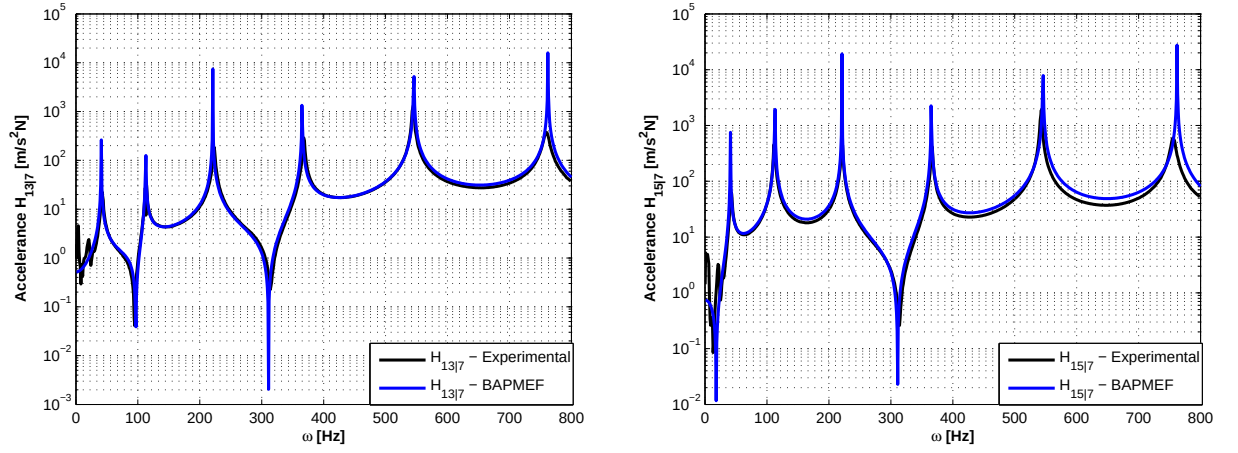


Figure B.8: Accelerances $H_{13|7}$ and $H_{15|7}$.

B.1.2 Beam 2

B.1.2.1 Results obtained with Hammer 5800SL / Accelerometer 27A11

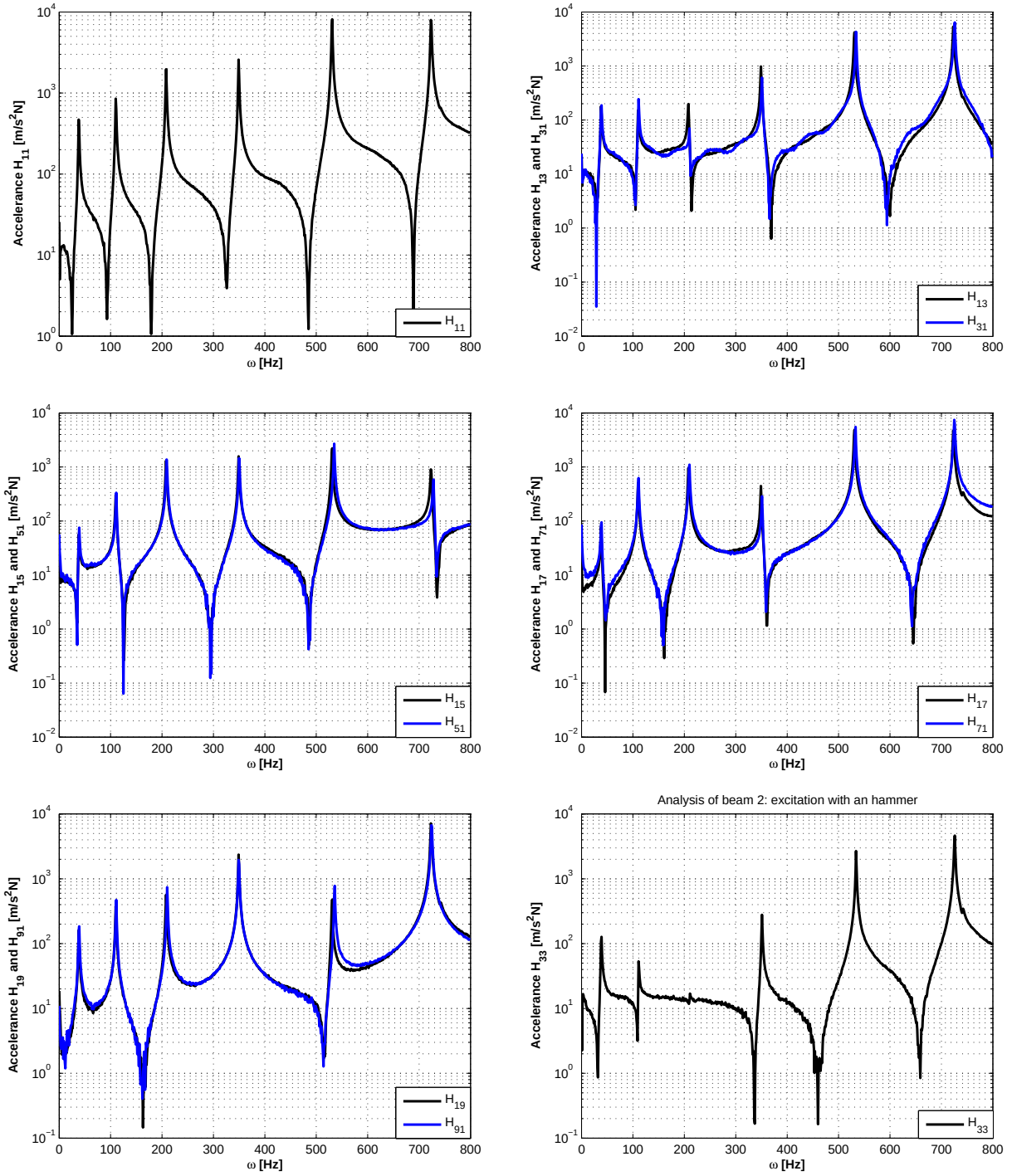


Figure B.9: Accelerances of experimental analysis of beam 2 (1).

B.1. EXPERIMENTAL FRFS

This experimental analysis used the beam 2 as experimental specimen. The excitation was made with hammer 27A11 and the response measured with accelerometer 27A11. The FRFs were acquired in all measurement points, in order to obtain the whole experimental matrix H_{tt}^C .

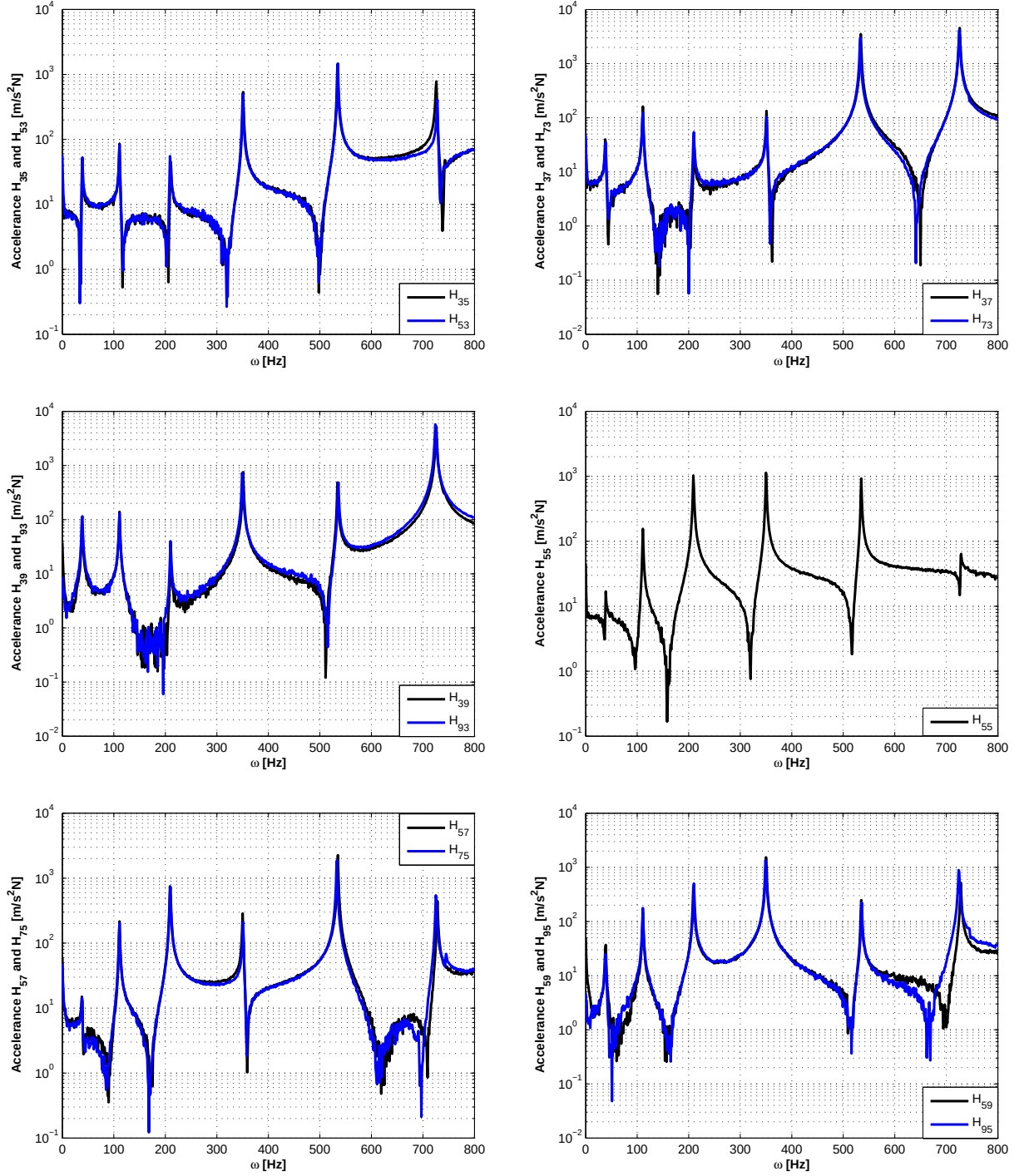


Figure B.10: Accelerances of experimental analysis of beam 2 (2).

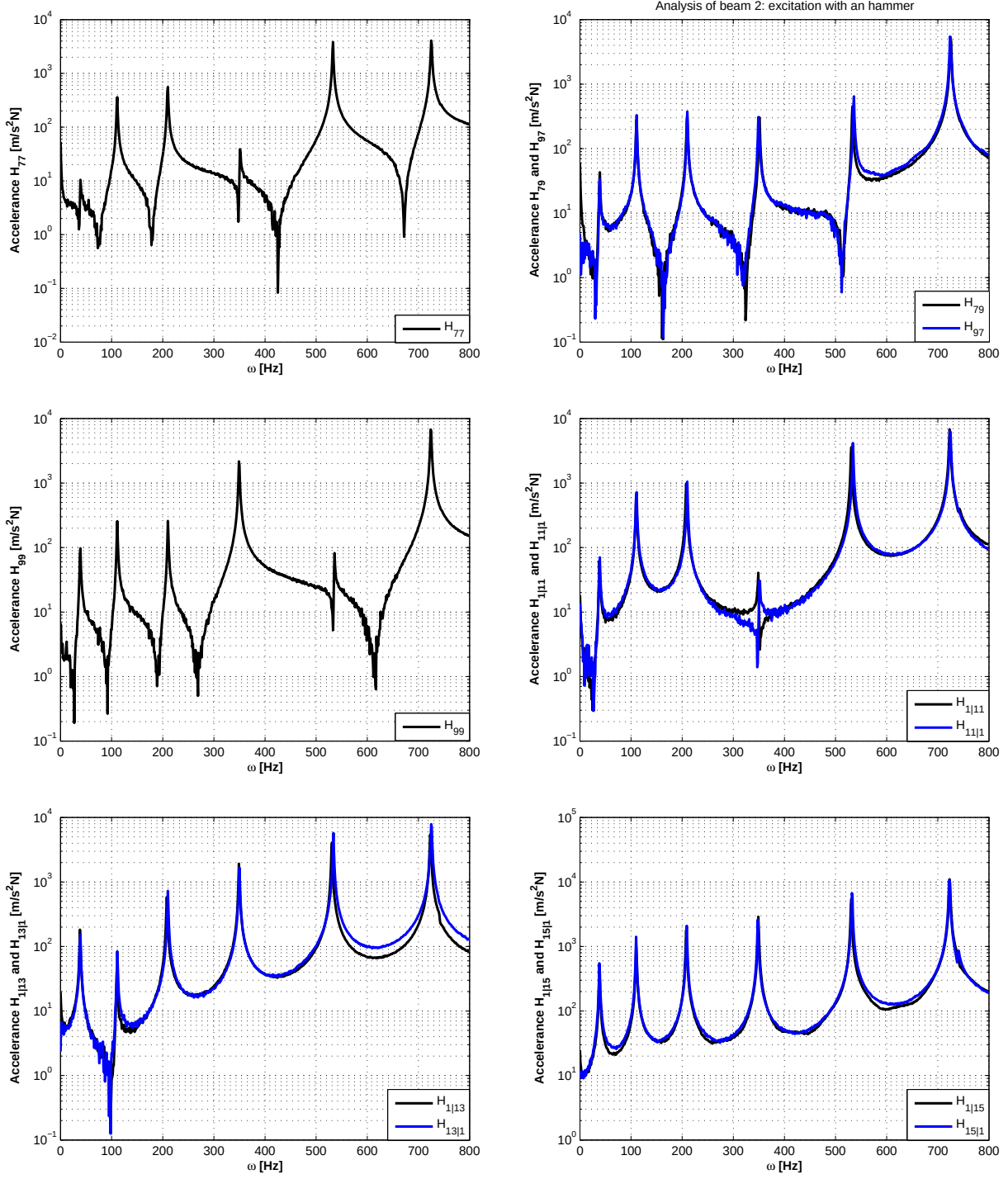


Figure B.11: Accelerances of experimental analysis of beam 2 (3).

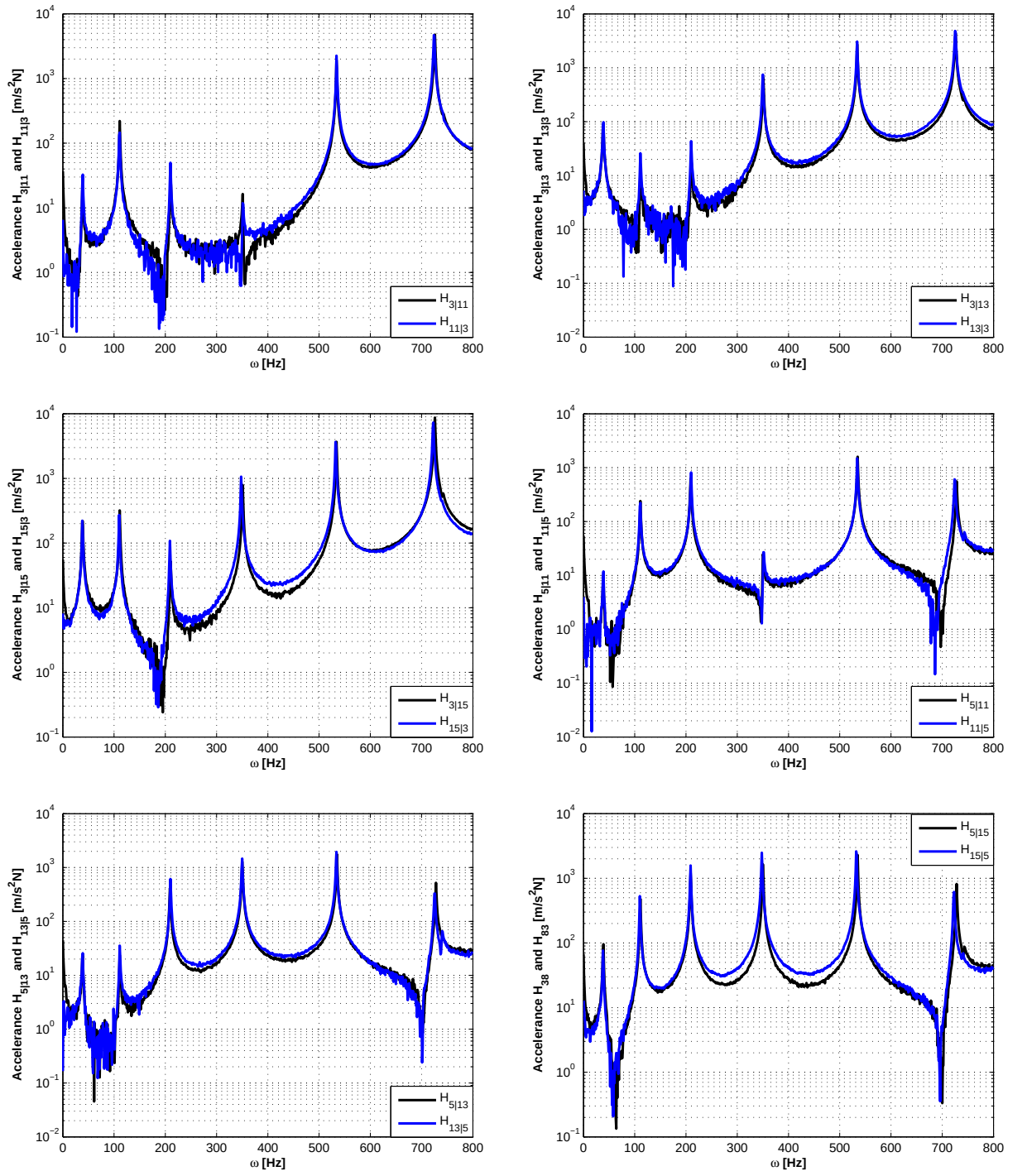


Figure B.12: Accelerances of experimental analysis of beam 2 (4).

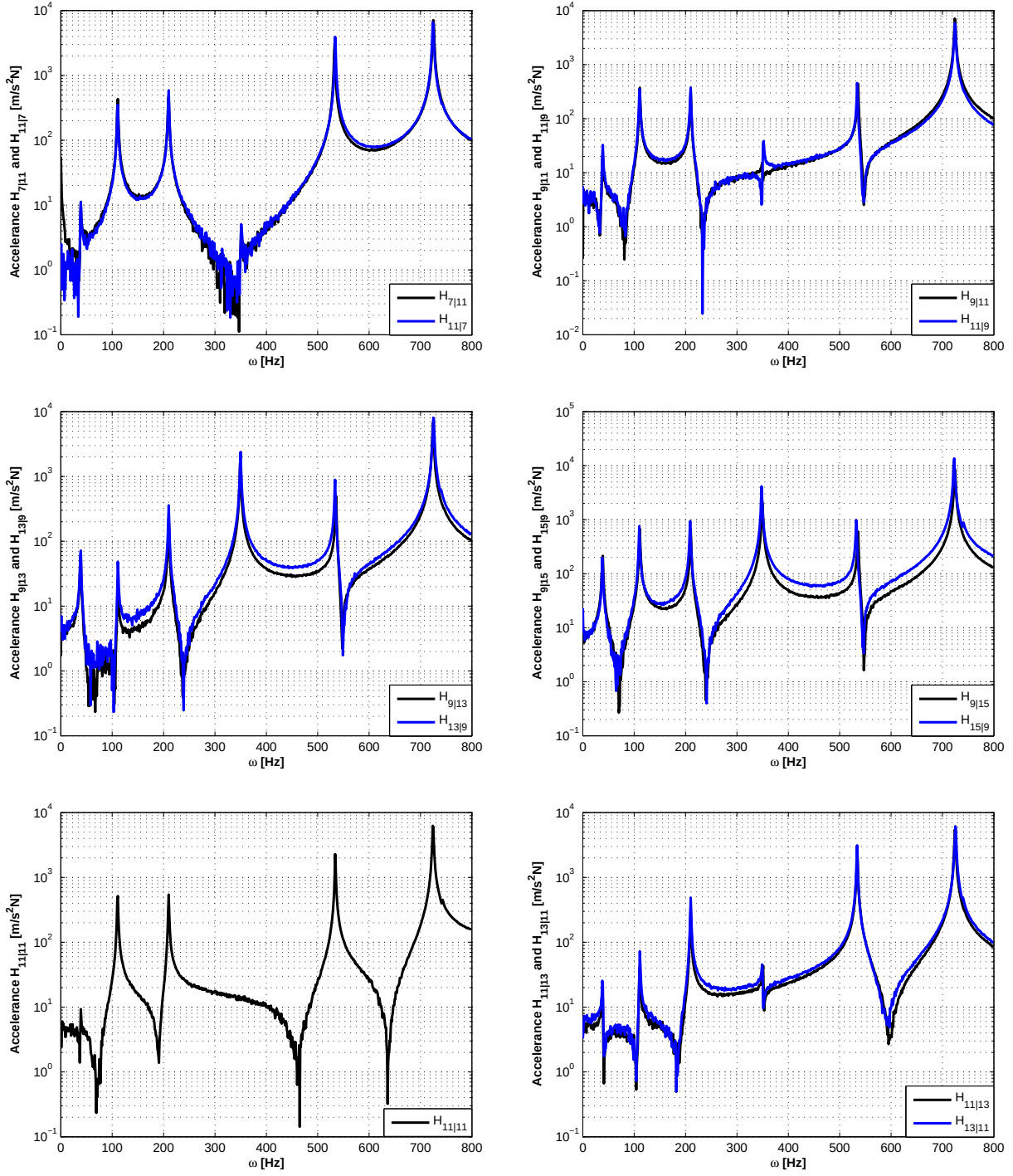


Figure B.13: Accelerances of experimental analysis of beam 2 (5).

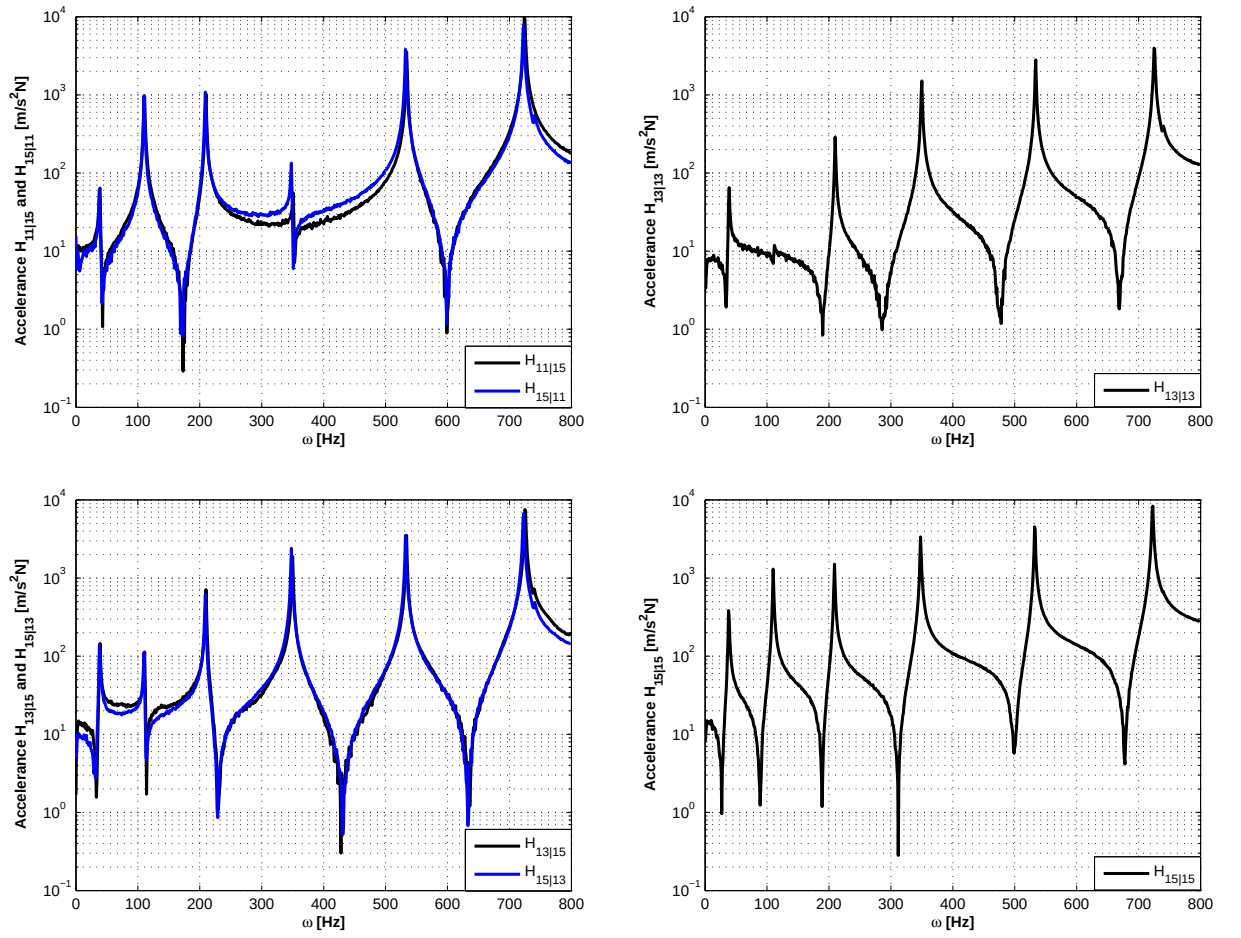


Figure B.14: Accelerances of experimental analysis of beam 2 (6).

B.1.2.2 Results obtained with Shaker LDS201 / Accelerometer 27A11

This experimental analysis used the beam 2 as experimental specimen. The excitation was made with shaker LDS210 and the response measured with accelerometer 27A11. The FRFs were acquired in all measurement points to obtain the whole experimental matrix H_{tt}^C .

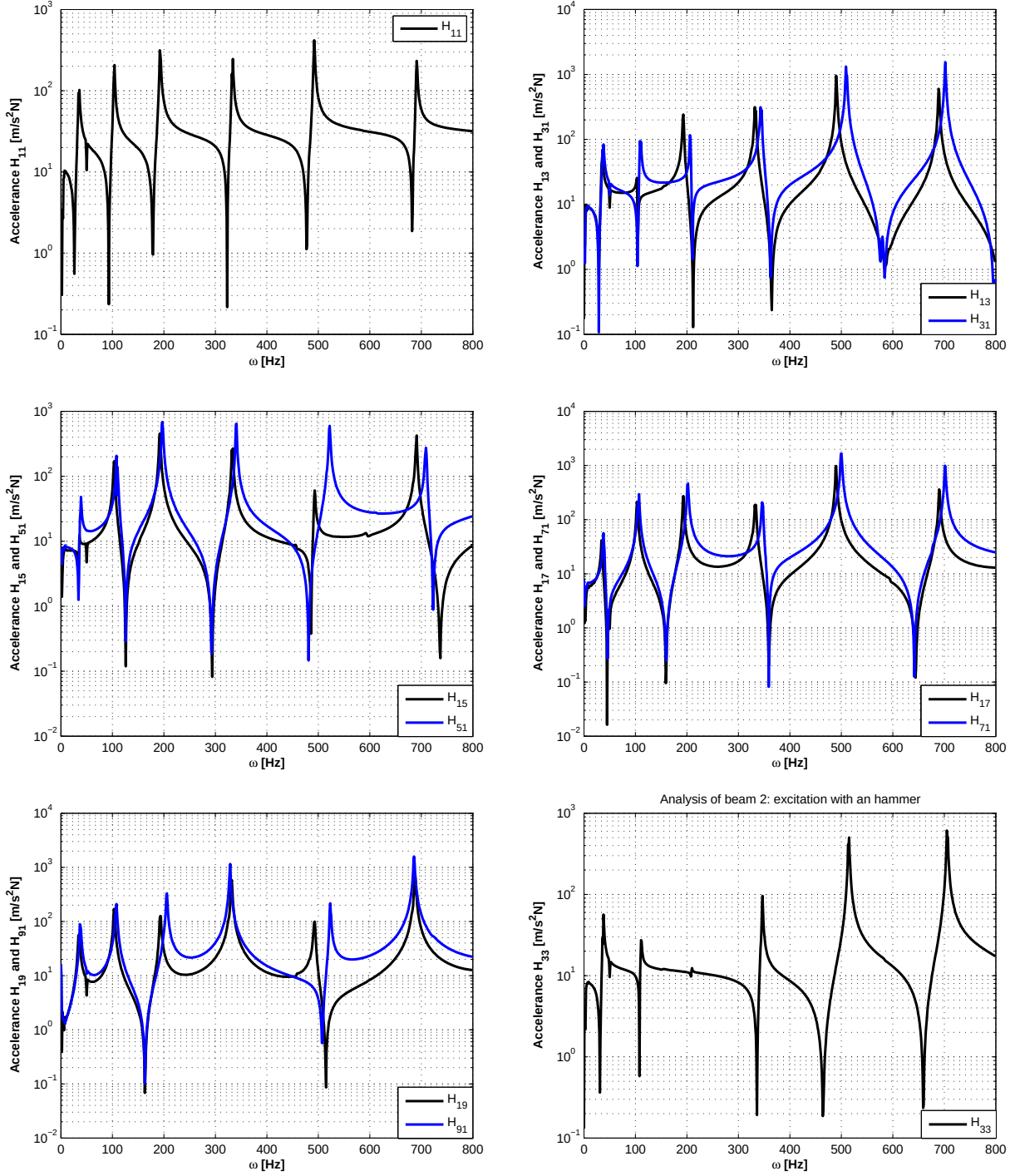


Figure B.15: Accelerances of experimental analysis of beam 2 (1).

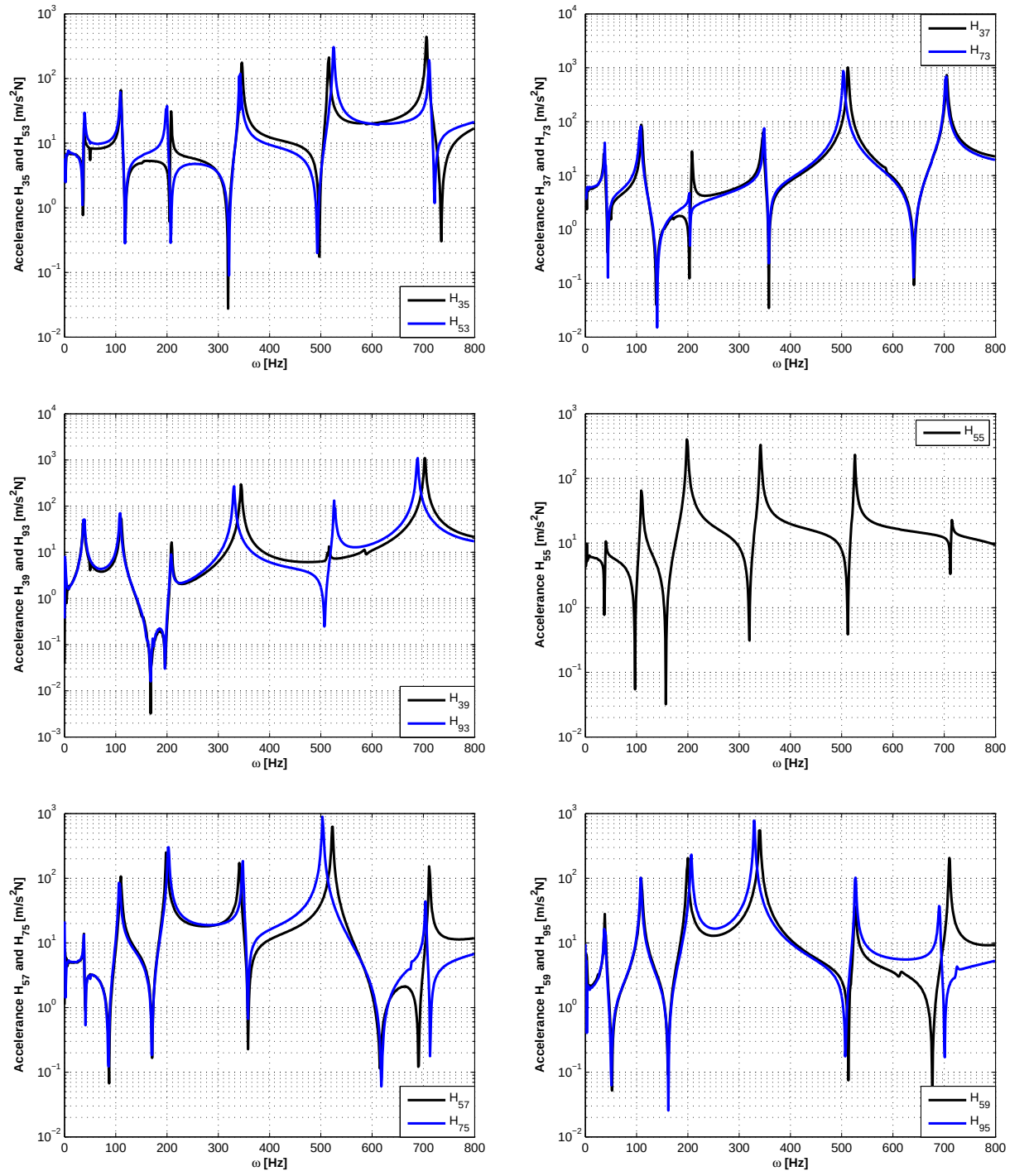


Figure B.16: Accelerances of experimental analysis of beam 2 (2).

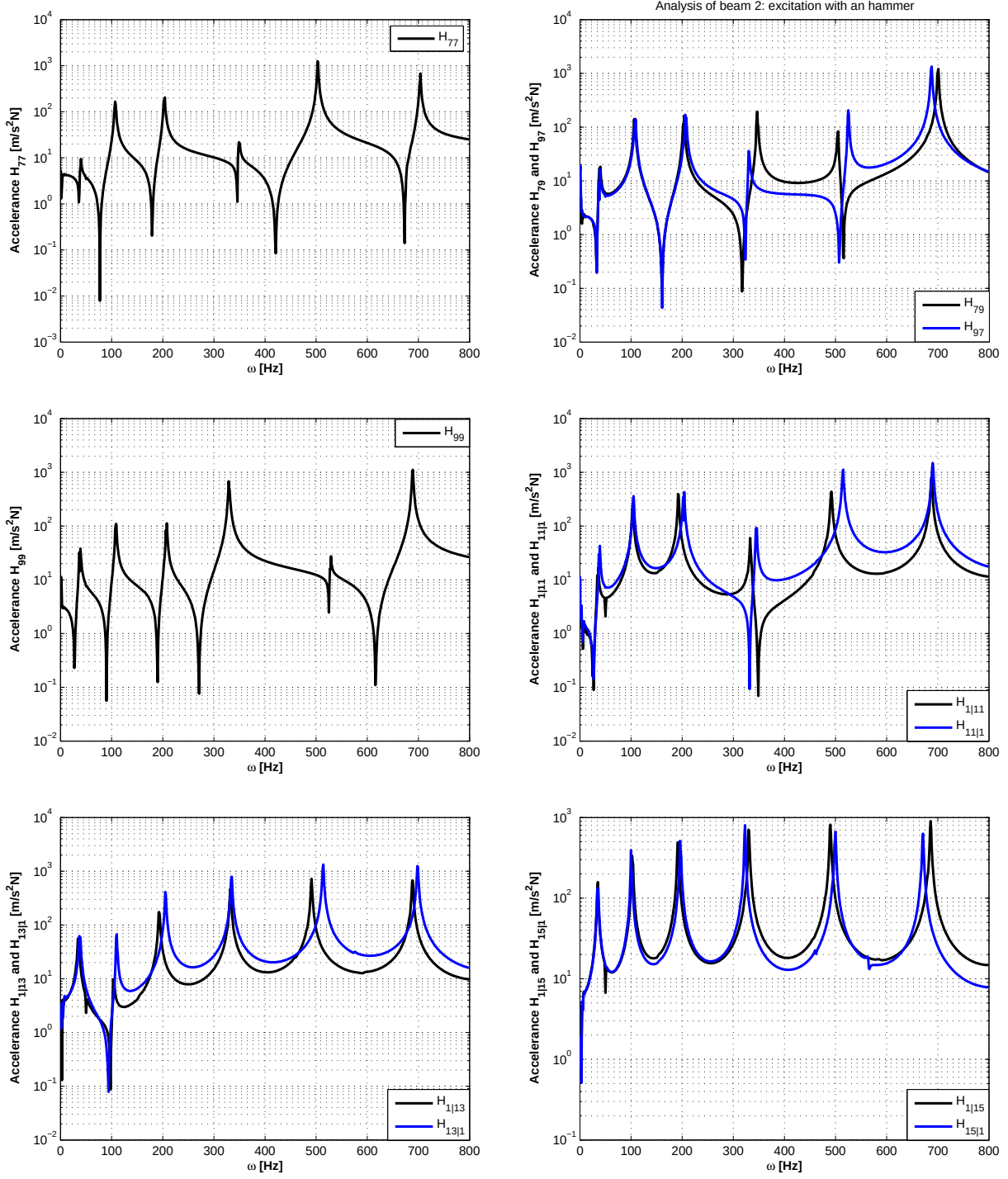


Figure B.17: Accelerances of experimental analysis of beam 2 (3).

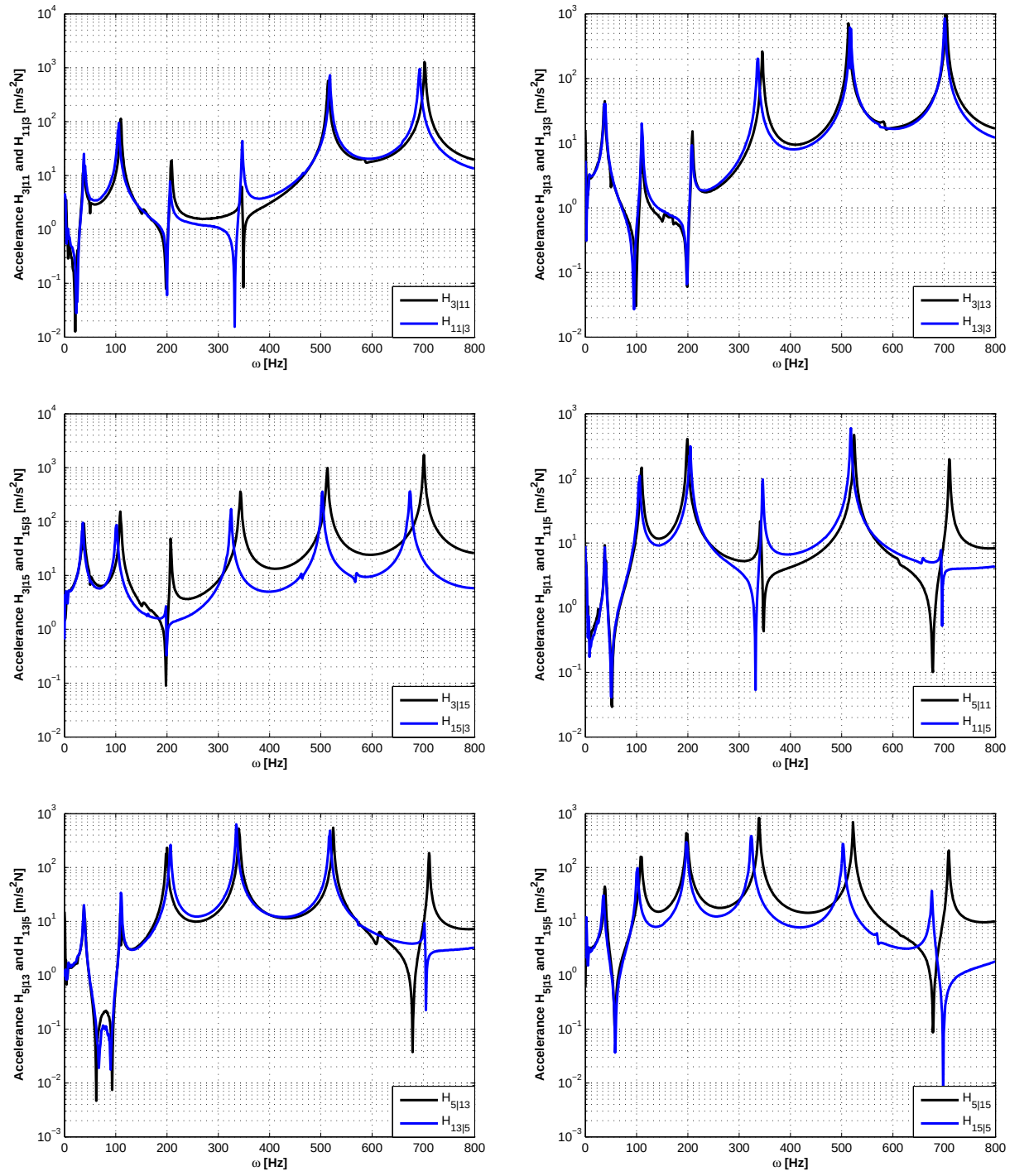


Figure B.18: Accelerances of experimental analysis of beam 2 (4).

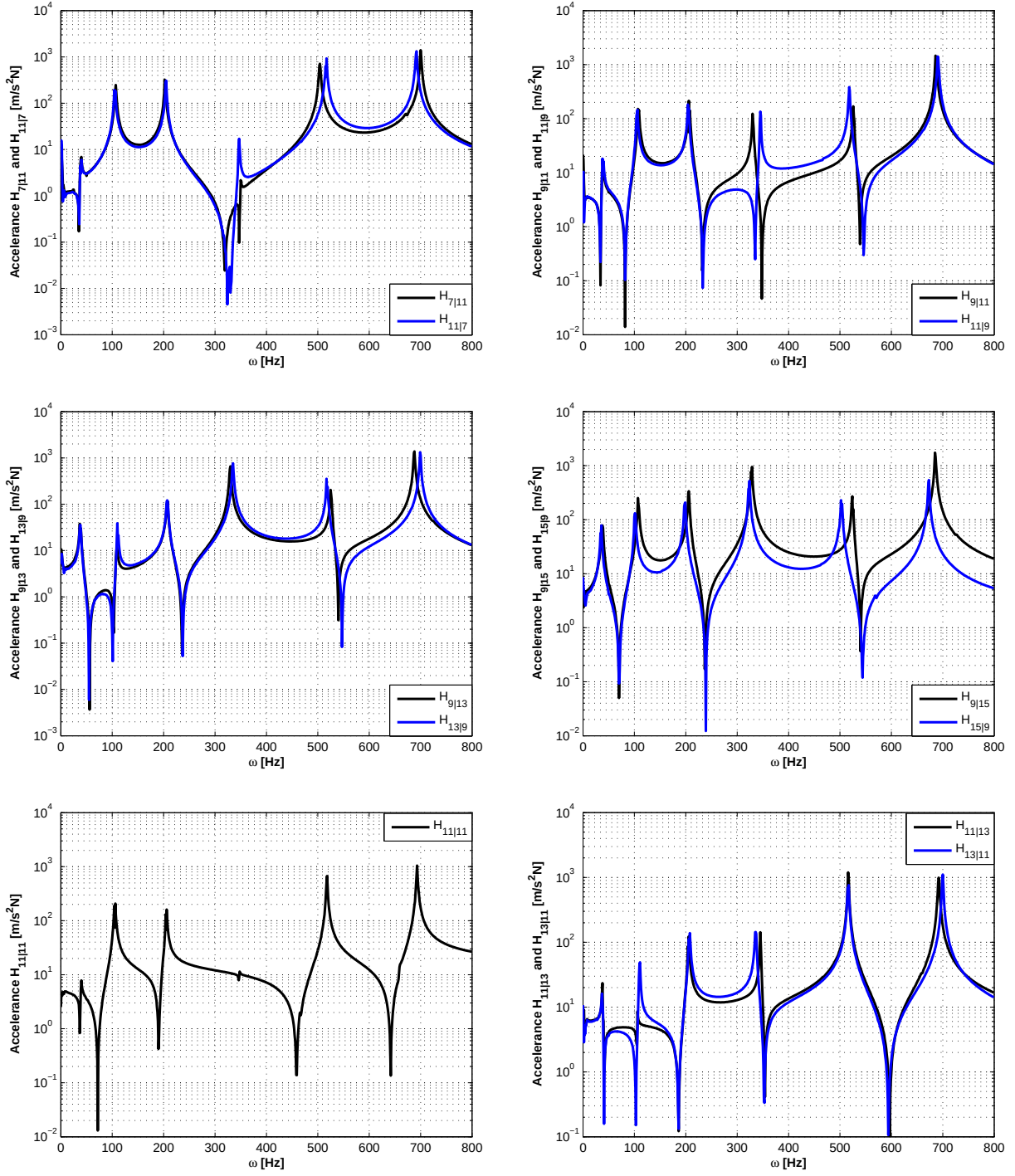


Figure B.19: Accelerances of experimental analysis of beam 2 (5).

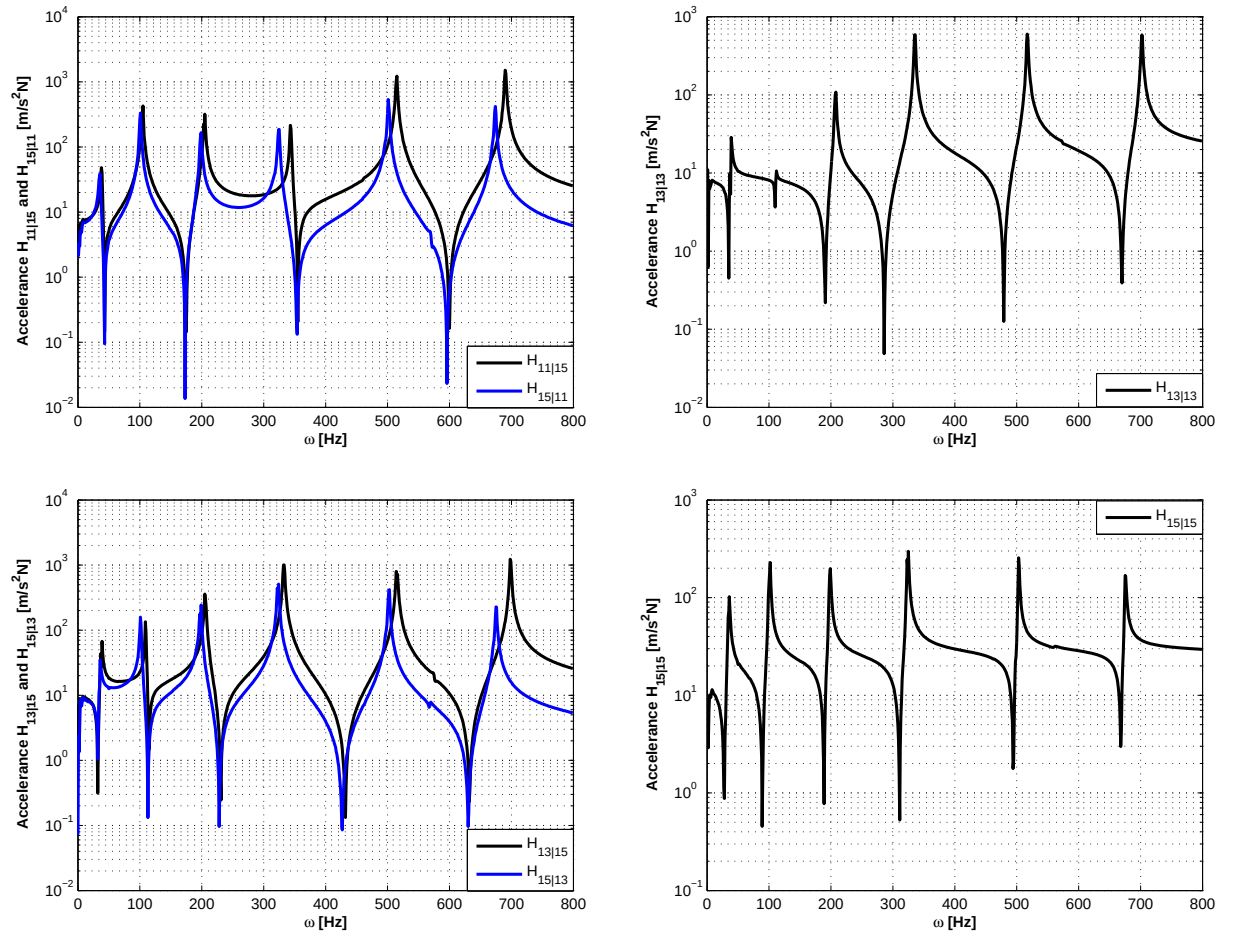


Figure B.20: Accelerances of experimental analysis of beam 2 (6).

B.1.2.3 Results obtained with shaker LDS201 / Laser Vibrometer

In this experimental analysis, the laser vibrometer was used to eliminate the accelerometer mass influence. Here, we had just measured a column in order to carry out a modal analysis. The excitation was done in measurement point 8 whereas the response was obtained in all measurement points.

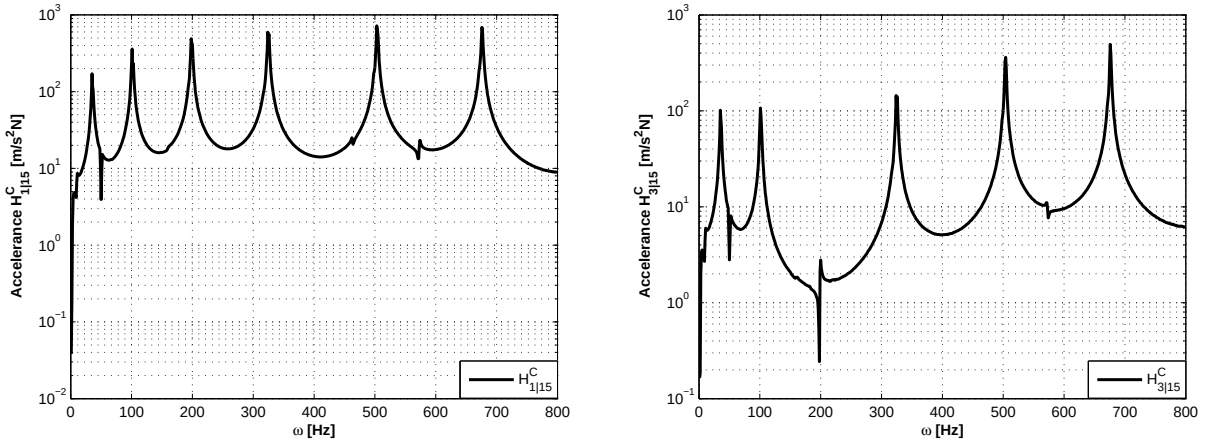


Figure B.21: Accelerances $H_{1|15}^C$ and $H_{3|15}^C$.

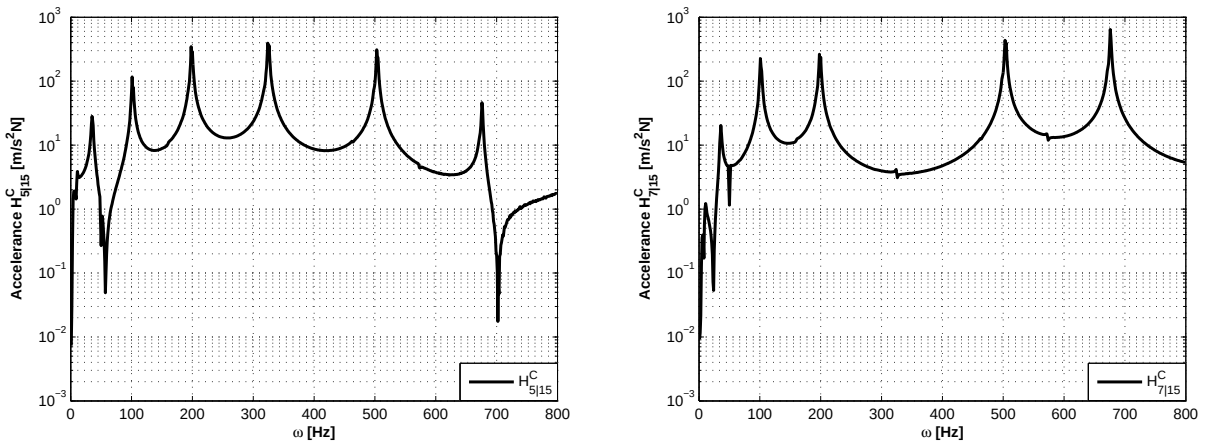


Figure B.22: Accelerances $H_{5|15}^C$ and $H_{7|15}^C$.

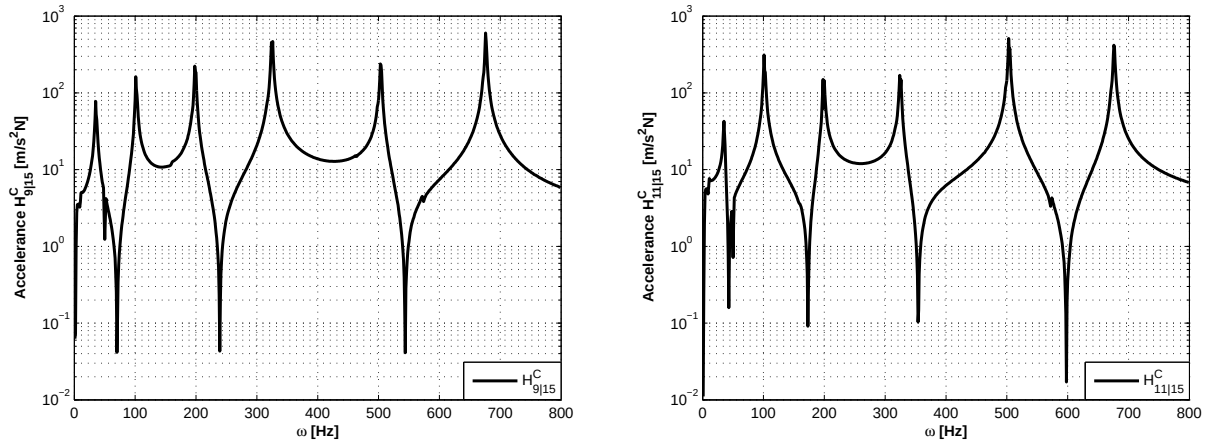


Figure B.23: Accelerances $H_{9|15}$ and $H_{11|15}$.

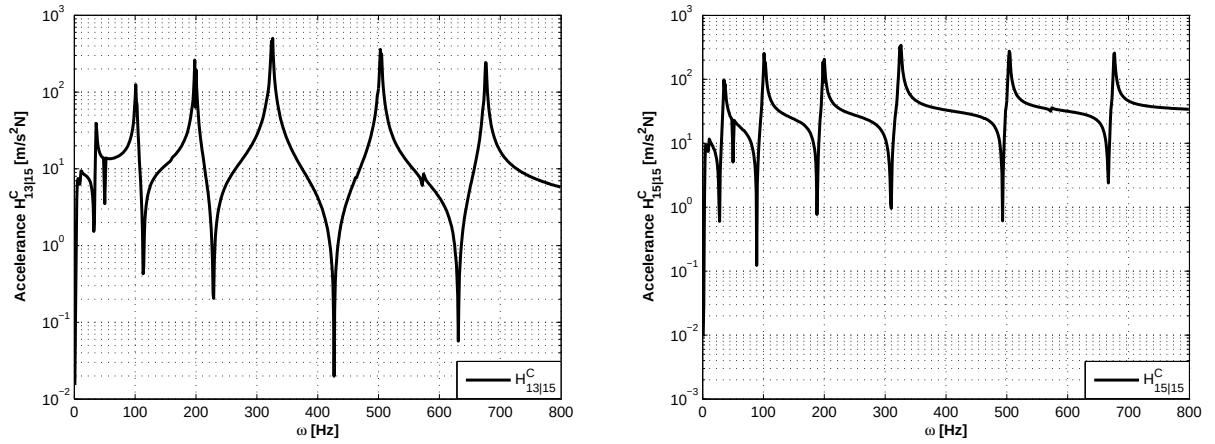
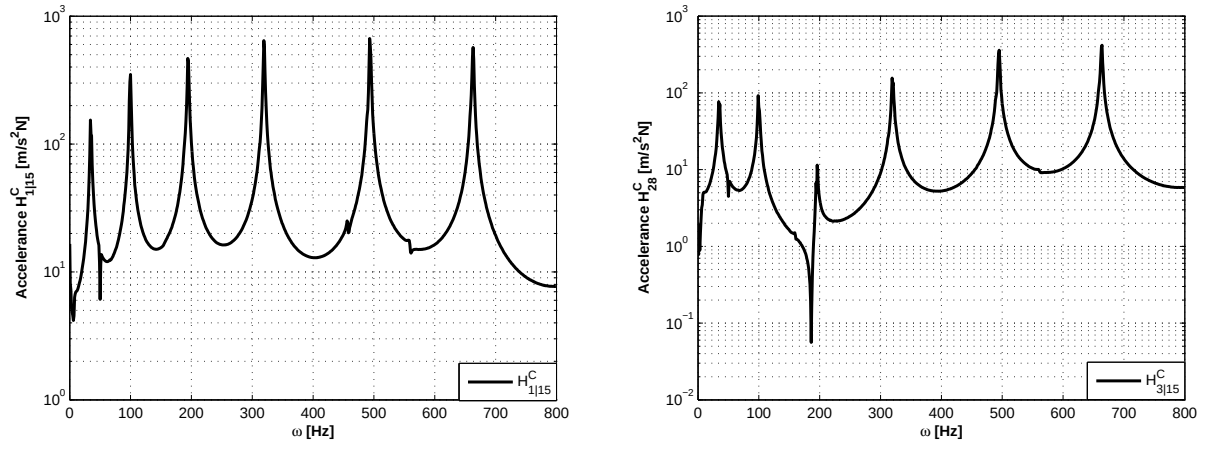
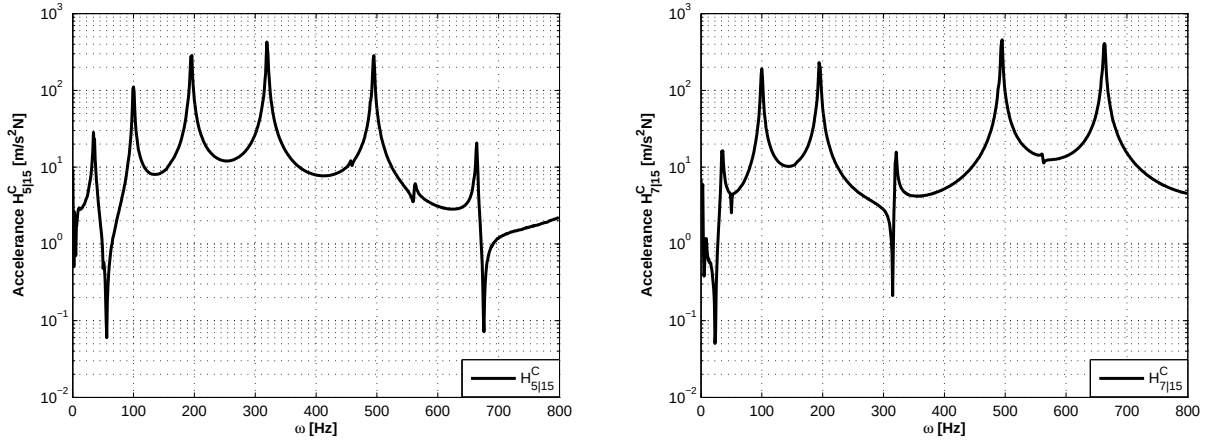
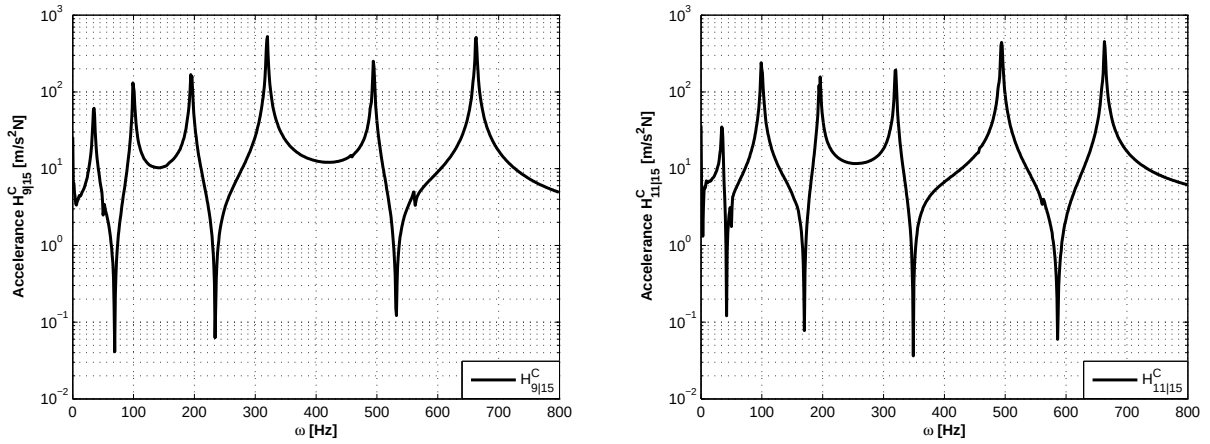


Figure B.24: Accelerances $H_{13|15}$ and $H_{15|15}$.

B.1.2.4 Results obtained with Shaker LDS201 / Accelerometer 27A11 and dummy masses

In this experimental analysis, dummy masses were considered to simulate the accelerometer mass in all measurement points. Here again, we had just measured a column in order to carry out a modal analysis. The excitation was done in measurement point 8 whereas the response was obtained in all measurement points using the accelerometer 27A11..

Figure B.25: Accelerances $H_{1|15}$ and $H_{3|15}$.Figure B.26: Accelerances $H_{5|15}$ and $H_{7|15}$.Figure B.27: Accelerances $H_{9|15}$ and $H_{11|15}$.

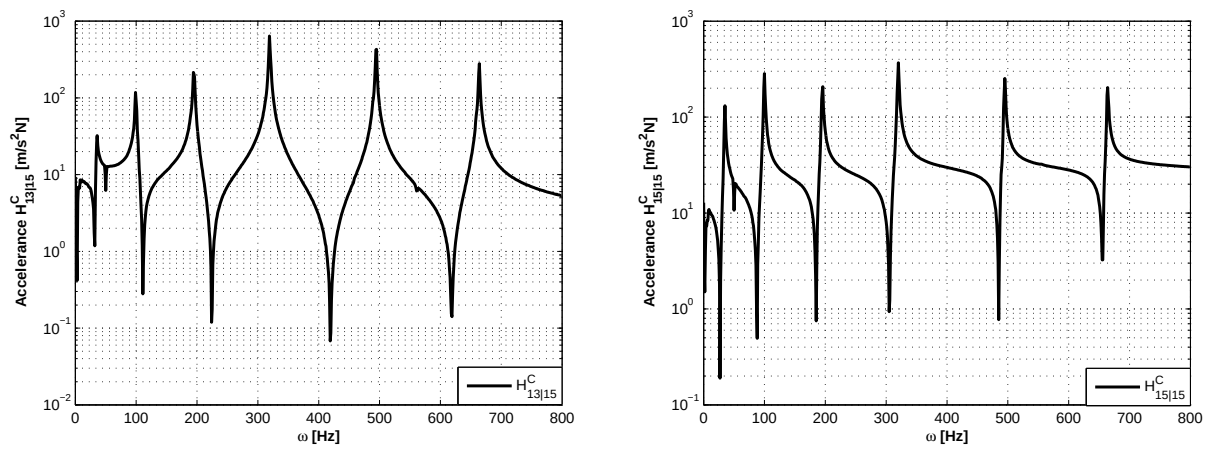


Figure B.28: Accelerances $H_{13|15}$ and $H_{15|15}$.

Appendix C

BAPMEF

Along this work, a finite element programme called *BAPMEF* was used to perform all the theoretical analysis of beams, as well as other subjects and so on. This programme makes use of a commercial software, *MATLAB*, where it is fully implemented. This programme provides three different types of analysis:

- Static Analysis;
- Modal Analysis;
- Frequency Analysis.

The static analysis provides a study about the displacements in each degree-of-freedom that a beam allows when exposed to a determined static load, as well as its deformed shape. The modal analysis gives the vibration modes of a beam with any boundary conditions and the corresponding natural frequency for each mode. A frequency analysis provides a frequency-response function in the corresponding point of response and excitation which are initially selected and defined according to the user.

This programme holds two different types of finite elements, Euler-Bernoulli and timoshenko elements. For each element the corresponding mass and stiffness are calculated and then the assembly is performed considering the position of each element in a global matrix of mass or stiffness. There is also the option to add hysterical damping taking into account an imaginary stiffness matrix.

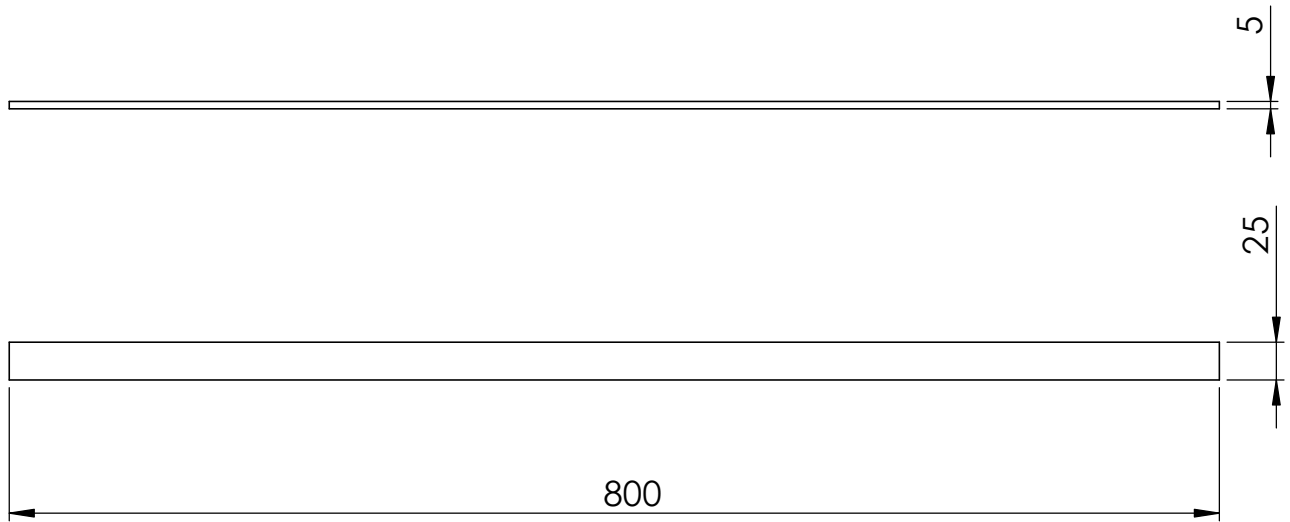
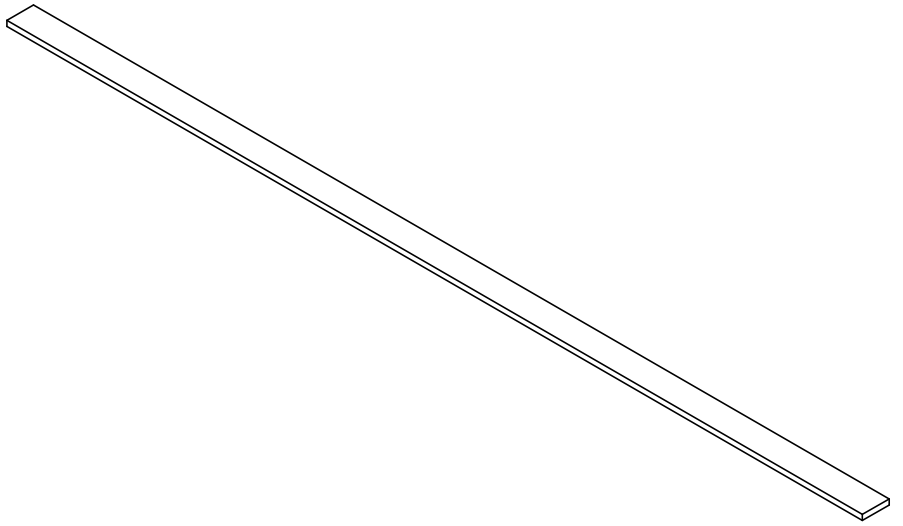
The achievement of frequency-response functions is done according to the spatial model, where are considered the mass and stiffness matrices at each excitation frequency. If there is damping in the structure, it is considered in the new imaginary stiffness matrix.

The influence of concentrated masses along the beam can also be analysed with this programme, simply by introducing in the programme the value of the corresponding mass and node where it is applied. This option was principally implemented to evaluate the influence of transducers in the process of mass cancelation.

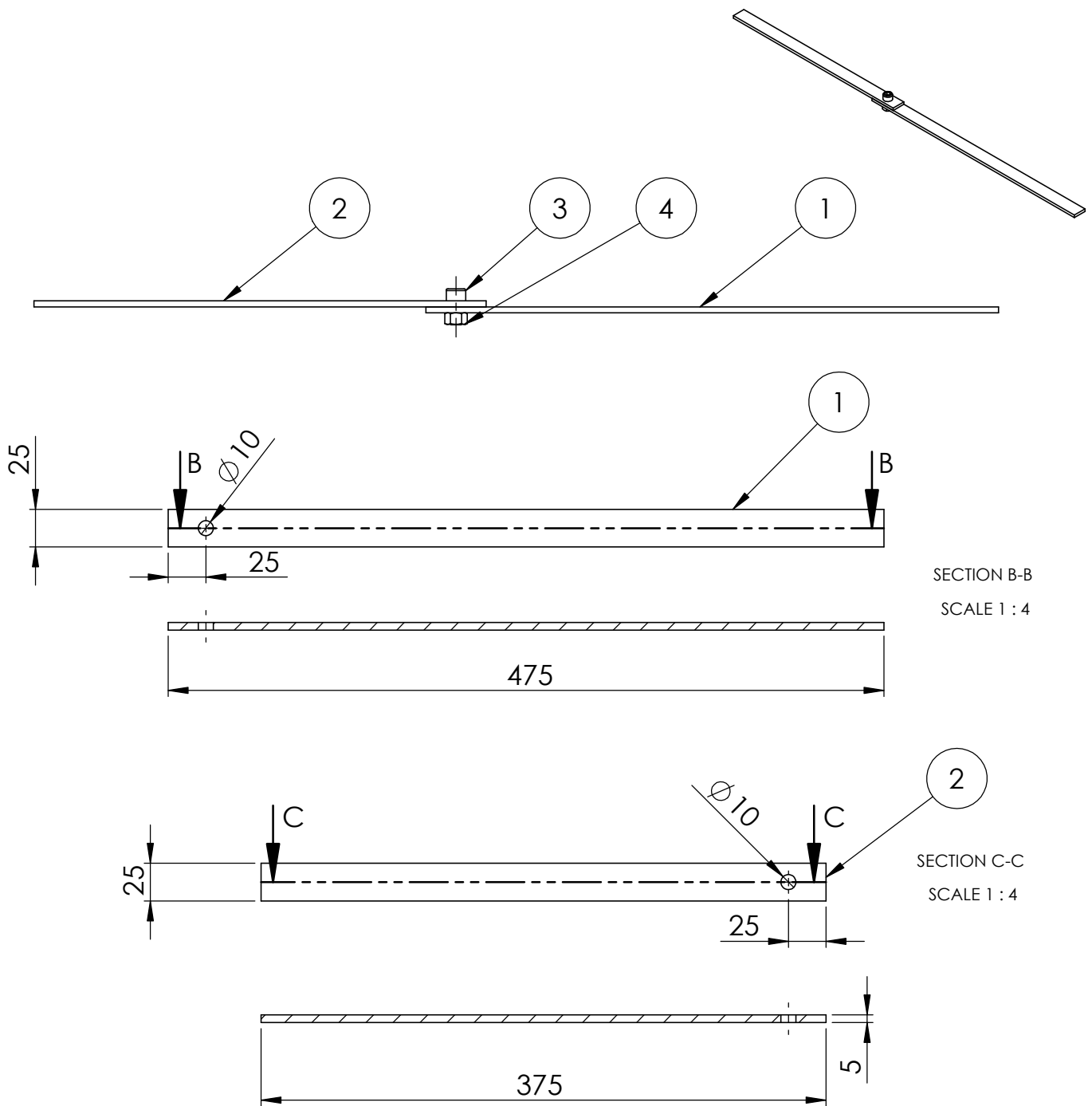
The inputs of the programme are the beam geometry, the physical properties of each element, the boundary conditions and the corresponding load distribution. Depending on the analysis chosen in the beginning, the outputs of the programme vary according to this option.

Appendix D

Definition Drawings

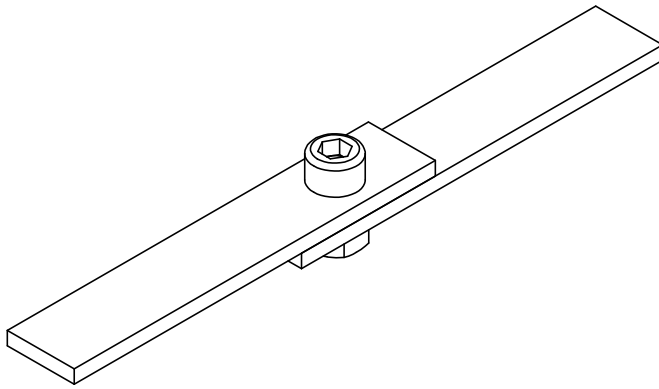


UNLESS OTHERWISE SPECIFIED: DIMENSIONS ARE IN MILLIMETERS SURFACE FINISH: TOLERANCES: LINEAR: ANGULAR:		FINISH:				DEBUR AND BREAK SHARP EDGES		DO NOT SCALE DRAWING		REVISION	
								Faculty of Engineering of University of Oporto			
	NAME	SIGNATURE	DATE					TITLE: Dynamic Characterization of Mechanical Components by Inverse Methods			
DRAWN	Diogo Pontes										
CHK'D											
APPV'D											
MFG											
Q.A					MATERIAL: Aluminium			DWG NO.		Beam 1	
SolidWorks Student License								A4			
Academic Use Only											
				WEIGHT:				SCALE:1:10		SHEET 1 OF 1	

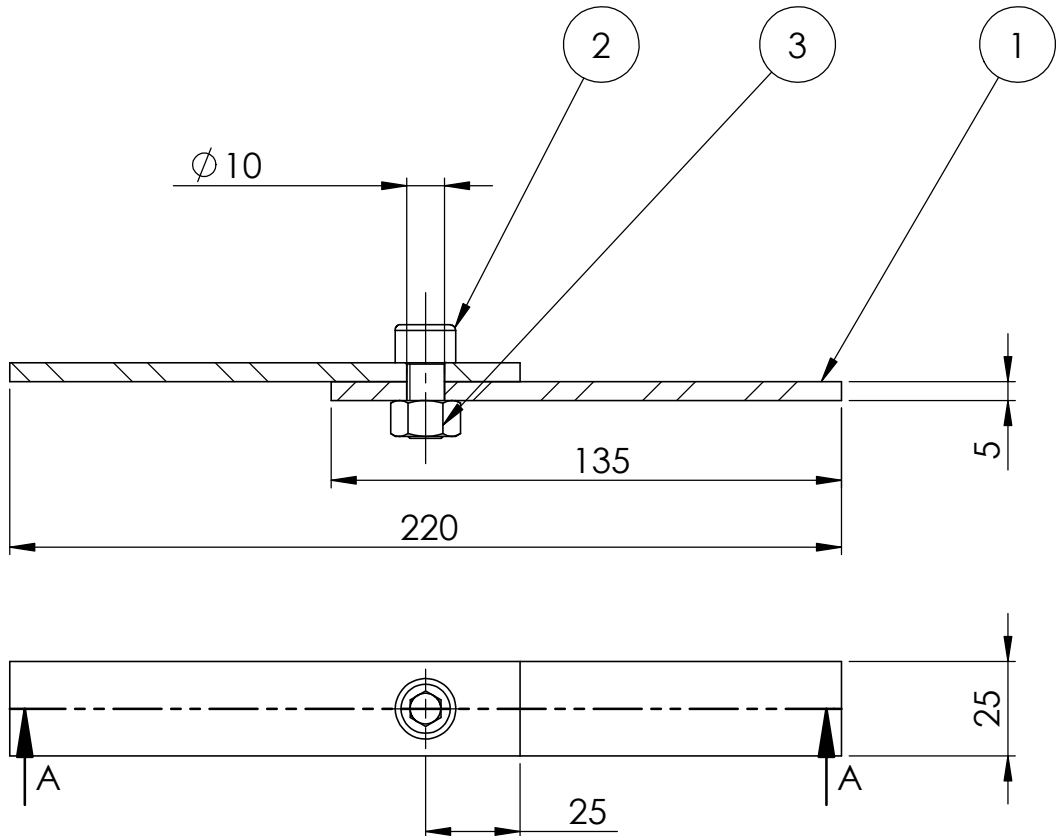


NO.	PART NUMBER	DESCRIPTION	QTY.
1	Beam475mm	Aluminum	1
2	Beam375mm	Aluminum	1
3	ISO 4762 M10 x 20 --- 20N		1
4	Hexagon Nut ISO - 4034 - M10 - N		1

UNLESS OTHERWISE SPECIFIED: DIMENSIONS ARE IN MILLIMETERS SURFACE FINISH: TOLERANCES: LINEAR: ANGULAR:		FINISH:		DEBUR AND BREAK SHARP EDGES		DO NOT SCALE DRAWING		REVISION	
						Faculty of Engineering of University of Oporto			
						Dynamic Characterization of Mechanical Components by Inverse Methods			
NAME		SIGNATURE		DATE		TITLE:			
DRAWN Diogo Pontes									
CHK'D									
APPV'D									
MFG									
Q.A						DWG NO.		A4	
SolidWorks Educational License Instructional Use Only						MATERIAL:		Beam 2	
WEIGHT:						SCALE:1:10		SHEET 1 OF 1	



SECTION A-A
SCALE 1 : 2



ITEM NO.	PART NUMBER	Material	QTY.
1	Beam L = 135 mm	Aluminium	2
2	ISO 4762 M10 x 20 --- 20N		1
3	Hexagon Nut ISO - 4034 - M10 - N		1

UNLESS OTHERWISE SPECIFIED: DIMENSIONS ARE IN MILLIMETERS SURFACE FINISH: TOLERANCES: LINEAR: ANGULAR:		FINISH:		DEBUR AND BREAK SHARP EDGES		DO NOT SCALE DRAWING		REVISION	
						Faculty of Engineering of University of Oporto			
NAME		SIGNATURE		DATE		TITLE: Dynamic Characterization of Mechanical Components by Inverse Methods			
DRAWN Diogo Pontes									
CHK'D									
APPV'D									
MFG									
Q.A				MATERIAL:		DWG NO.		Beam 3	
SolidWorks Student License Academic Use Only						WEIGHT:		SCALE:1:5	
								SHEET 1 OF 1	

A4

Appendix E

Experimental Equipment List

Spectral Analyser

Trademark / Model	Brüel&Kjaer / 2035
Input Models	2x3019 (25 kHz)
Output Models	1x3106 (25 kHz)

Power Amplifier PA25E

Trademark / Model	Ling Dynamic Systems (LDS) / PA25E
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Force transducer BK8200

Trademark / Model	Brüel&Kjaer / 8200 / 1777312
Sensitivity	3.88 pC/N
Mass	21.0 g

Electrodynamic shaker LDS201

Trademark / Model	Ling Dynamic Systems (LDS) / V401
Range	17.8 N / 5-13000 Hz
Total mass	1.81 Kg

Laser Vibrometer

Trademark / Model	Polytec / OFV 303
Sensitivity	0.025 ms ⁻¹ /V

Laser Vibrometer Controller

Trademark / Model	Polytec / OFV 3001
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APPENDIX E. EXPERIMENTAL EQUIPMENT LIST

Hammer 5800SL

Trademark / Model	DYTRAN / 5800SL
Sensitivity	106.40 mV/Lb.F

Force transducer 8203

Trademark / Model	Brüel & Kjaer / 8203
Sensitivity	3.3 pC/N

Accelerometer 27A11

Trademark / Model	ENDEVCO / 27A11
Sensitivity	9.859 mV/g
Mass	1 g