IMPULSIVE FEEDBACK CONTROL: A CONSTRUCTIVE APPROACH

Thesis by

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Dedicated to my dear Nelma
(Dedicado à minha querida Nelma)

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Abstract

There is an increasing number of control applications whose trajectories are better modeled by discontinuous functions. A general framework based on measure-driven differential inclusions is proposed. The discontinuities of the trajectories are an idealization of a strong actuation applied in a short period of time. This approach enables the definition of a trajectory linking the endpoints of a jump, which, in turn, enables important analytical tools. This model has been studied in the context of open loop control strategies and little effort has been given in what concerns feedback control laws. Thus, the impulsive Euler solution is developed in order to allow the definition of feedback control measures and, consequently, control the trajectory during the discontinuous period. This solution does not require any reparameterization technique and it is expressed in the original time frame. Using this solution, necessary and sufficient conditions for weak and strong invariance for impulsive control systems are provided. The results are constructive since they provide guidelines on how to compute control measures and selections in a feedback form. Applying the invariance results, optimality conditions for impulsive control systems expressed in terms of an Hamilton-Jacobi-Bellman equation are explored. The measure driven differential inclusion provides a formal framework where the control space is complete, guaranteeing the existence of the solution. Optimal feedback synthesis is possible due to the relationship between optimal control and invariance. Several practical applications are detailed where the relevance of this impulsive framework is demonstrated.

Resumo

A relevância de aplicações de controlo com trajectórias descontínuas tem aumentado. Assim, é proposto um modelo baseado em inclusões diferenciais controladas por medidas. As descontinuidades das trajectórias são idealizações da aplicação duma actuação forte num intervalo de tempo curto. Esta abordagem permite a definição duma trajectória ligando as extremidades do salto que, por sua vez, permite obter resultados analíticos importantes. Este modelo tem sido utilizado no contexto de estratégias de controlo em malha aberta e pouco esforço tem sido dado no que se refere a estratégias de controlo em malha fechada. Assim, é desenvolvida uma solução impulsional de Euler no sentido de permitir a definição da medida de controlo em malha fechada e, consequentemente, controlar a trajectória durante o seu período de salto discreto. Esta solução é definida em tempo original e não necessita de técnicas de reparameterização. Usando esta solução, são apresentadas condições necessárias e suficientes para invariância fraca e forte de sistemas de controlo impulsional. Os resultados são construtivos já que fornecem estratégias para o cálculo de medidas e selecções em malha fechada. Aplicando os resultados de invariância, é possível apresentar condições de optimalidade para sistemas de controlo impulsional sob a forma duma equação de Hamilton-Jacobi-Bellman. O modelo utilizado tem um espaço de controlo fechado o que garante a existência duma solução óptima. A síntese de controlo óptima é possível dada a relação entre o problema de controlo óptimo e invariância. Finalmente, são apresentadas aplicações onde é demonstrada a relevância do paradigma de controlo impulsional.

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Chapter 1

Introduction

1.1 Motivation

In many important engineering applications the automatic control plays a major role. The controller should compute the suitable actuation in order to achieve a desired performance. The engineering design requires a detailed analysis of the physical phenomena to determine the most suitable actuation. The inputs applied to a system influence their trajectories, which are directly related with its performance. The analysis of a control system has its foundations in the control theory, which, in turn, has its roots in formal objects of mathematics and computation.

Usually, the physical phenomena is modeled by formal mathematical objects while the implementation of a controller relies on computational techniques. There are several ways to model the behavior of a system. Among the most used are controlled differential equations, differential inclusions, discrete event systems and stochastic models. In controlled differential equations, for example, the output is modeled by a continuous function, controlled by another function. In stochastic models, the outputs does not only depend on the applied control but also on a probability distribution. These models are well suited when there is not enough information to provide a deterministic model and some uncertainty is present. In this thesis, we will focus in deterministic models based on differential inclusions and differential control equations.

There are essentially two possible control strategies. The open loop strategy uses the information given by a mathematical model to compute the control action in a given period of time. The control is computed before hand and is implemented as time evolves. On the other hand, we have the feedback control strategy where the control depends on the actual state of the system. The open loop strategy is well suited whenever there is a precise model and a demanding performance is required. The feedback approach does not require such a precise model and is designed to be robust in relation to model uncertainties. One of the most demanding objectives in control theory is to provide a feedback control law while ensuring the optimality of the decisions.

Most of the results developed in control theory are devoted to systems whose trajectories are modeled by continuous functions. However, the number of practical applications requiring discontinuous trajectories has been growing. Consider, for example, mechanical systems subject to impact collisions. The effect of the impact induces a fast change in the velocity of the system. This behavior appears in applications like robotic manipulators, vibro-impact mechanisms, walking (and, potentially, jumping) biped robots, juggling mechanical systems and micro-electromechanical systems (MEMS). In some economic systems the decisions (control) can lead some trajectories to have instantaneous jumps. Relevant examples are related to stock management, investment policies and exploration of renewable resources. Hybrid systems is another class of systems requiring discontinuous trajectories. Nowadays, virtually almost every control system has a computer or a digital communication device incorporated in the decision process. Clearly, this requires the composition of discrete and continuous systems, implying the existence of discontinuous trajectories.

The increasing interest on engineering applications requiring discontinuous trajectories and the lack of results on feedback control strategies for impulsive systems constitute the main motivation of this thesis.

1.2 Objectives

The objective of this thesis is to better understand systems whose trajectories may have discontinuities. We will address issues related with modeling, control and computational implementation.

We intent to develop a model where the behavior of the system during the jump is fully described. This will enable the computation of a feedback control strategy even in the discontinuous period. For this purpose, it will be necessary to consider a model with control measures and, additionally, a new solution concept should be developed in order to define feedback control measures. As mentioned before, the feedback control requires the assessment of the output trajectory and, in the jump period, we need an infinitely fast sampling scheme as well as an infinitely fast response speed. Off course, there is no such hardware in practice and we should provide guidelines on how to implement the sampling scheme in practice.

Applying the new solution concept, we intent to provide constructive results on invariance and optimal control for impulsive control systems. The results should provide precise guidelines on how to construct the control in order to achieve the desired performance (invariance, optimality).

We would like to cover a wide range of practical applications to demonstrate the relevance of the new solution concept in the analysis of impulsive control systems.

1.3 Contributions

Systems modeled by measure driven differential inclusions are usually analyzed in the context of open loop control and conditions for invariance, optimality and stability relies on reparameterization techniques [37, 10, 36, 33]. The reparameterization technique defines a new "time" scale to convert a measure driven differential inclusion into a differential inclusion without measures. However, there are many practical applications where the construction of a feedback control synthesis is required. For this purpose, we develop an extension of the Euler solution [14] for impulsive systems. The impulsive Euler solution is defined in the original time frame and does not require the usage of reparameterization techniques. Moreover, it allows the definition of the control measure in a feedback form, enabling the control of the discontinuities. These are the main novelties of the impulsive Euler solution in relation to the solution proposed in [45]. A solution concept expressed in the original time is essential to compute a control measure in a feedback form. Moreover, we show that the reparameterization technique distorts the vector fields of the system and does not synthesize the singular continuous component of the control measure. This is a drawback if the existence of a solution is required to prove, like in optimal control problems. The distortion effect is specially critical whenever the implementation of feedback control strategies is required.

We use the impulsive Euler solution to derive constructive results for problems on invariance. We present synthesis mechanisms such that the trajectories of a given impulsive system remain in a prescribed region. This control problem encounters many practical applications and is also an important analytical tool for other control problems like optimal control and stability. The results are constructive since they provide a synthesis mechanism for the control measure, allowing its application in computational systems.

Another contribution is the characterization of the solution of an optimal impulsive control problem with resource of an Hamilton-Jacobi-Bellman like equation. The usage of the impulsive Euler solution associated with the results on invariance allows the derivation of a constructive result, where optimal feedback selections and control measure are computed. In this case, instead of considering a predefined region in which invariance is studied, we have a set computed from the optimization problem data. Thus, the computation of feedback selections and a feedback control measure leading invariant this auxiliary set implies its optimality. This approach is presented in [14] in the context of conventional systems and we provide an extension for impulsive systems.

In the applications field, we provide the following contributions:

- We show the impulsive control framework is useful to model and control mechanical systems subject to impact collisions. There are many applications where such impact forces arise. Usually, the impacts are considered to be passive since the system in not controllable in the impact period. However, with the increasing capabilities of sensors and actuators, namely its sampling rate and response speed, it became feasible to control the system during the impact phase too. In this new paradigm, the impulsive framework turned an important tool to analyze this type of mechanical systems.

- The impulsive control framework is well suited to model and control the composition of dynamic systems. Examples of composed dynamic systems appear in many engineering applications like, for example, formations control and hybrid systems. In formations control, a network of autonomous vehicles is defined and the vehicles can interact to achieve multiple configurations. There is a continuous evolution mode, corresponding to the preservation of a given configuration, and an impulsive mode, where there is a change in the formation configuration.

Using the impulsive framework, we can better describe the trajectory of each vehicle during the reconfiguration period. Since the reconfiguration takes a very short period of time when compared with the continuous mode, we can assume the reconfiguration trajectory as being impulsive and associated with a control measure. Thus, we have the opportunity to integrate these two modes of operation in a single framework and synthesize controllers using the impulsive control model. In what concerns hybrid systems, they can be modeled by the impulsive control framework too [21]. The trajectory of a hybrid system evolves as a result of the interaction between continuous and discrete dynamics. This interaction reflects the compositional properties underlying the hybrid system model. On the other hand, an impulsive dynamic system can be regarded as a composition of multiple dynamic systems, and, therefore, is suited to model some classes of hybrid systems. The impulsive solution is not only amenable to a physical interpretation but also supports the analysis and control synthesis since a characterization of the trajectory during the jump is provided. One consequence of the usage of the impulsive Euler solution in the hybrid systems context is the possibility to overcome the so called Zeno behavior.

1.4 Organization of the thesis

In chapter 2 a model based on a measure driven differential inclusion is presented. The measure is interpreted as a control that can provoke a jump in the trajectory. Due to the properties of control measures. Then, the concepts of robust solution and impulsive Euler solution are introduced. The robust solution allows a practical interpretation for control measures and details how they can be suitably approximated by conventional controls. The impulsive Euler solution is defined in the original time frame and enables the definition of the control measure in a feedback form. Finally, some important properties of these solutions are provided.

In chapter 3 invariance results for the impulsive control system are provided. The impulsive Euler solution is used to compute feedback selections and feedback control measures such that the resulting trajectories have the invariance property (to remain inside a given closed set). Necessary and sufficient conditions are given and expressed in the original time frame.

In chapter 4 the notion of optimal impulsive control problem is introduced. Conditions characterizing verification and value functions are provided. Additionally, a constructive result where optimal feedback selections and control measures is presented.

Chapter 5 presents some practical applications of the impulsive control framework. The examples covered are impact dynamic systems, investment control, composition of dynamic systems, hybrid systems and formations of autonomous vehicles.

Finally, chapter 6 provides concluding remarks and possible future work.

Chapter 2

Impulsive solution concepts

The purpose of this chapter is to present the model of the impulsive control system. There are several alternatives but we use a model based on a measure driven differential inclusion, where the measure is interpreted as a control that can provoke a discontinuity in the trajectory. In some approaches presented in the literature, impulsive control systems have been modeled by only specifying the endpoint of a jump, not describing how it is conducted. In this chapter, we show how this approach can lead to situations where the solution in not defined, which is a consequence of the non completeness of the control space. On the other hand, the control measure belongs to a complete space. Besides the impulsive effect, the control measure can also originate an absolutely continuous or singular continuous trajectory. We will see that these properties are crucial for analysis and practical purposes. In measure driven differential inclusions, the jump is characterized by a trajectory joining the two endpoints. We introduce the concepts of robust solution and impulsive Euler solution for the measure driven differential inclusion. The robust solution allows a practical interpretation for the control measure and shows how we can suitably approximate it by a conventional (a measurable function) control. The impulsive Euler solution is defined in the original time frame and is suited for defining the control measure in a feedback form. We will present some important properties of these two solutions that will be fundamental for the subsequent chapters.

2.1 The model of the impulsive dynamic control system

The main object of this chapter is the following measure-driven controlled differential inclusion:

$$\begin{cases} dx(t) \in F(t, x(t))dt + G(t, x(t))d\mu(t), \ t \in [0, \infty) \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $F : [0, \infty) \times \mathbb{R}^n \hookrightarrow \mathbb{R}^n$, $G : [0, \infty) \times \mathbb{R}^n \hookrightarrow \mathbb{R}^{n \times q}$ are given multifunctions. The measure is such that $\mu \in C^*([0, +\infty); K)$, i.e., $\mu(A) \in K$ for any Borel set $A \subset [0, +\infty)$ and K being a positive convex pointed cone in $\mathbb{R}^q$. The space $C^*([0, +\infty); K)$ denotes the set in the dual space of continuous functions from $[0, +\infty)$ to $\mathbb{R}^q$ with values in K. The measure μ has three components, namely, the absolutely continuous μ_{ac}, the singular continuous μ_{sc} and the singular atomic component μ_{sa} (Lebesgue decomposition [42, 28]). In what follows B will denote the unit open ball whereas $\bar{B}$ will denote the closed unit ball in the Euclidean space.

The measure driven differential inclusion model subsumes the following measure driven control system:

$$dx(t) = f(t, x(t), u(t))dt + g(t, x(t), w(t))d\mu(t), \ t \in [0, +\infty), \tag{2.2}$$

where f and g are given functions, mapping $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{m_w} \to \mathbb{R}^{n \times q}$, respectively. The control functions u and w take values in some given subset $U \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^{m_w}$, respectively. By defining $F(t, x) := f(t, x, U)$ and $G(t, x) := g(t, x, W)$, we can apply the Filippov's lemma [14, 16] to show that there is an equivalence between the solutions of problem (2.1) and (2.2).

To avoid extra technicalities, we defined different control functions in f and g. However, if we have the same control variable in f and g ($u \equiv v$), then we should be careful in the way we choose selections for F and G when the measure is absolutely continuous, namely, the chosen selections should correspond to the same conventional control. In [35] this issue is detailed and optimality conditions for the measure driven control system are given. Hereafter, we will only consider the measure driven differential inclusion system. Here, $AC([0,+\infty);\mathbb{R}^n)$ means the space of absolutely continuous $\mathbb{R}^n$-valued functions on $[0,+\infty)$ and $BV^+([0,+\infty);\mathbb{R}^n)$ represents the space of $\mathbb{R}^n$-valued functions on $[0,+\infty)$ of bounded variation that are continuous from the right on $(0,+\infty)$. The measure $d\bar{\mu}$ denotes the total variation measure of $d\mu$ and is defined by

$$d\bar{\mu}(t) := \sum_{i=1}^{q} d\mu^i(t), \tag{2.3}$$

where μ^i represents each component of the vector-valued measure μ in the cone K.

This model is not only useful for analysis, but is also suitable to model practical phenomena, as we will have the opportunity to see in chapter 5.

There are also other well studied models for the impulsive system. We often encounter the impulse effect modeled through the specification of the endpoint of the jump, as follows:

$$\begin{cases} \frac{dx}{dt} = f(t,x), \ t \notin \tau_k \\ \Delta x = I_k(x), \ t = \tau_k. \end{cases} \tag{2.4}$$

The set $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset \mathbb{R}^+$ is an unbounded, closed, discrete set, representing the instants when the jumps occur. The function $I_k : \mathbb{R}^n \to \mathbb{R}^n$ represents the incremental change of the state at the time τ_k. In the argument of $I_k(x)$ we have the variable x^-, which represents the value of the state prior the jump. Models of this type can be found, with variations, in [5, 47, 9, 26, 27, 25]. However, we should mention that these models have several limitations face to the measure driven differential inclusion model, namely:

a) They do not admit the absolutely and singular continuous components;

b) They require hypotheses on minimum separation between atoms [25];

c) They require a finite number of atoms on all time interval or on its subintervals, in order to avoid the so called Zeno behavior (infinite discrete transitions in a finite time interval).

Under the practical point of view, it can be difficult to consider singular continuous measures but the formal framework has to consider mathematical objects that guarantee the development of analytical results. An important point is related with the completeness of the control space such that we can guarantee the existence of the solution. Imagine, for instance, we consider the model of equation 2.4. Then, we can use the classical optimization problem to show that the existence of the solution of the following example is not ensured:

$$\begin{aligned} &\min \int_{[0,1]} |x(t) - t| dt \\ &\text{s.t. } \dot{x} = 0,\ \Delta x = I_k,\ x(0) = 0, \end{aligned}$$

where I_k is assumed to be a controlled variable. We easily conclude that there is a minimizing sequence of controls converging to an absolutely or singular continuous component.

The issue of approximating ideal mathematical objects by simpler and more tractable ones is, under the computational point of view, very important but will have to be considered in an independent context.

2.2 Robust solution

At this point, we should note that the measure driven differential inclusion problem has an apparent ill-definition due to the discontinuity of x at the support of the singular atomic atoms. Thus, it is not clear a priori which value of x should be plugged in the argument of G when an atom of the measure is present (an atom can be interpreted as a δ-function). To overcome this ambiguity, we require the solution of the impulsive system of equation (2.1) to be defined in a way that it can be approximated by a conventional solution, where the measure is replaced by an absolutely continuous component $w_{ac}(t)dt$. This strategy has been used by several authors [11, 40, 36]. We adopt the solution presented in [33], which is a straightforward extension of [36], in order to encompass the unbounded interval $[0, +\infty)$.

Definition 2.2.1. *A trajectory x, with $x(0) = x_0$, is admissible for system of equation (2.1) if $x(t) = x_{ac}(t) + x_s(t)$ $\forall t \in [0, \infty)$, where*

$$\begin{cases} \dot{x}_{ac}(t) & \in & F(t, x(t)) + G(t, x(t))w_{ac}(t) \;\; L - a.e. \\ x_s(t) & = & \int_{[0,t]} g_{sc}(\tau)w_{sc}(\tau)d\bar{\mu}_{sc}(\tau) + \int_{[0,t]} g_{sa}(\tau)d\bar{\mu}_{sa}(\tau). \end{cases} \tag{2.5}$$

The variable w_{ac} represents the Lebesgue time derivative of μ_{ac} while w_{sc} is the Radon-Nikodym derivative of μ_{sc} with respect to $\bar{\mu}_{sc}$. The solution x_{ac} is an absolutely continuous function and the differential inclusion is to be interpreted in the selector sense, where the selections have to obey some regularity properties, namely, measurably and continuity on time and on the state, respectively. The function $g_{sc}(\cdot)$ is a $\bar{\mu}_{sc}$ measurable selection of $G(\cdot, x(\cdot))$ and $g_{sa}(\cdot)$ is a $\bar{\mu}_{sa}$ measurable selection of the multifunction

$$\tilde{G}(t, x(t^-); \mu(\{t\})) : [0, \infty) \times \mathbb{R}^n \times K \hookrightarrow \mathcal{P}(\mathbb{R}^n), \tag{2.6}$$

which specifies the reachable set of the singular dynamics at $(t, x(t^-))$ when the control measure has an atom $\mu\{t\}$.

To explain the meaning of multifunction $\tilde{G}$, we use the notion of graph completion of a time reparameterization function [36]. Hence, consider a possibly unbounded time interval T and a measure μ. Then, let us introduce a new time parameterization, which takes each time t to $\eta(t)$, defined as follows:

$$\eta(t) := t + \int_{[0,t]} d\bar{\mu}(\tau). \tag{2.7}$$

At the support of the singular atomic component t_a, the previous function is set-valued such that $\eta(t_a) = [\eta(t_a^-), \eta(t_a^+)]$. In this sense, the function η maps $[0, +\infty)$ into $\mathcal{P}([0, +\infty))$. To facilitate the notation, we define $M(\cdot) := \mathrm{col}(M^1(\cdot), \ldots, M^q(\cdot))$, where

$$M^i(t) = \int_{[0,t]} d\mu^i(\tau),$$

with $t > 0$, $M^i(0) = 0$ and $i = 1, \ldots, q$.

Definition 2.2.2. *The family of graph completions corresponding to the measure μ is the set of pairs (θ, γ), where $\theta : [0, +\infty) \to [0, +\infty)$ is the "inverse" of η in the sense that $\theta(s) = t$, $\forall s \in \eta(t)$, and $\gamma : \eta(t) \to \mathbb{R}^q$ is defined $\forall s \in \eta(t)$ by*

$$\gamma(s) := \begin{cases} M(\theta(s)) & , \text{ if } \bar{\mu}(\{t\}) = 0 \\ M(t^-) + \int_{\eta(t^-)}^{s} v(\sigma) d\sigma & , \text{ if } \bar{\mu}(\{t\}) > 0, \end{cases} \tag{2.8}$$

where $v(\cdot) : \eta(t) \to K$ satisfies

$$\sum_{i=1}^{q} v^i(s) = 1, \ \forall s \in \eta(t) \tag{2.9}$$

and

$$\int_{\eta(t)} v(s) ds = \mu(\{t\}). \tag{2.10}$$

Now that we have defined the graph completion, we are ready to define the multifunction $\tilde{G}(t, z; \alpha)$. When $|\alpha| = 0$, then $\tilde{G} = \{0\}$. Otherwise, if $|\alpha| > 0$, we have:

$$\begin{aligned} \tilde{G}(t, z; \alpha) := \Big\{ |\alpha|^{-1} \left[\xi(\eta(t)) - \xi(\eta(t^-))\right] : \dot{\xi}(s) \in G(t, \xi(s)) v(s) \ \eta\text{–a.e.}, \\ \xi(\eta(t^-)) = z, \ \xi \in AC([\eta(t^-), \eta(t)]; \mathbb{R}^n), \\ v(s) \in K \cap \bar{B}, \ \textstyle\int_{\eta(t^-)}^{\eta(t)} v(s) ds = \alpha \Big\} \end{aligned} \tag{2.11}$$

where $|\alpha| := \sum_{i=1}^{q} \alpha^i$. Note that the definition of multifunction $\tilde{G}$ allows to overcome the ambiguity mentioned before at the support of the singular atomic component. This is done by defining a singular dynamical system "explaining" the discontinuity of the trajectory x. In figure 2.1 is depicted this solution concept, where we can observe the singular trajectory evolving and connecting the two endpoints of a discontinuity of x. We say that (x, μ) is a feasible process for system (2.1) if $\mu \in C^*([0, \infty); K)$ and x is a robust solution of (2.1).

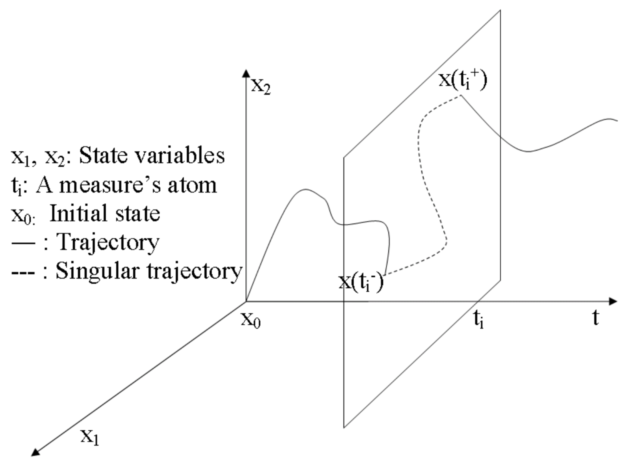

Figure 2.1: Solution concept sketch for impulsive systems

We remark that the emergence of the "additional control" v is due to the non-uniqueness of the integral of the matrix-valued selection $g \in G$ in relation to the measure μ_{sa}, in the absence of the commutativity property of the vector fields defined by its columns (Frobenius condition). In other words, the control v completely specifies how the measure components contribute to the total variation of $\bar{\mu}$. Without this additional control, it would be ambiguous how the jump on x was generated since more than one possibility would be possible due to the non commutativity property of G [12, 29]. This issue is shown in the next example, which is inspired in example 5 of [12].

Example 2.2.1. *Consider the following system:*

$$dx(t) = \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix} \begin{bmatrix} d\mu_1(t) \\ d\mu_2(t) \end{bmatrix},$$

with $x(0) = 0$ and $t \in [0,1]$. Observe that the columns of the previous matrix do not commute and the Lie bracket between them gives $[g_1, g_2] = (Dg_2)g_1 - (Dg_1)g_2 = [1,0]^T$. Assume that, at $t = 0$, we have a total variation singular atomic measure $\bar{\mu}(\{0\})$, whose total variation is given by $|\bar{\mu}(\{0\})| = 2$. Following the previous definitions, we can define $v(s)$ on $s \in [0,2]$. For instance, set $v(s) = [0,1]^T$ on $s \in [0,1)$ and $v(s) = [1,0]^T$ on $s \in [1,2]$. Clearly, $v(s)$ satisfies the constraints defined in equation (2.11) and $x(0^+)$ is given by:

$$x(0^+) = \int_0^1 \begin{bmatrix} 0 \\ x_1 \end{bmatrix} ds + \int_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

On the other hand, if, for instance, we set $v(s) = [1,0]^T$ on $s \in [0,2]$ then we obtain $x(0^+) = [2,0]^T$, which is different from the previous choice. Hence, for the same total variation measure, we can obtain different solutions in the singular phase. This ambiguity is resolved by specifying the extra control v.

In optimal control problems, we can use the non-uniqueness of the singular trajectory to select a solution giving the minimum cost [36]. Note that if the control measure μ is scalar-valued or the vector fields associated with the columns of the set-valued map G are commutative (see [12, 39, 40]), then the definition of the jump becomes significantly simpler due to the uniqueness of the solution.

For each time instant t in the support of μ_{sa}, the values of the trajectories are curves defined in $\mathbb{R}^n$, i.e.,

$$x^s(\cdot) := \{\xi : \eta(t) \to \mathbb{R}^n : \xi \text{ is defined as in definition of } \tilde{G}\}. \tag{2.12}$$

For instants t that are continuity points of the control measure μ, $x(\cdot)$ is, of course, a singleton. Whenever we refer to a trajectory x of the impulsive system, we mean an arc $x(t)$ and $x^s(s)$ defined in the suitable intervals. Note also that if we use the control formulation of equation (2.2), then the control variable w is a set-valued function. At the support of the singular atomic component, $w(t)$ corresponds to a set $w(s)$ parameterized by $s \in \eta(t)$.

Now, we present a proposition giving some properties of the reparameterization and graph completions presented before [33, 39, 40].

Proposition 2.2.1. *Let (θ, γ) be a family of graph completions of $\mu \in C^*([0, +\infty); K)$. Then,*

(i) *θ and γ are Lipschitz continuous, non-negative functions satisfying*

$$\dot{\theta}(s) + \sum_{i=1}^{q} \dot{\gamma}^i(s) = 1 \quad \mathcal{L} - a.e. \tag{2.13}$$

(ii) *For any Borel measure $\mu \in C^*([0, +\infty), K)$, integrable function $G : [0, +\infty) \to \mathbb{R}^{n \times q}$, Borel set $T \subset [0, +\infty)$ and $\mathbb{R}^q$ valued function $c : [0, +\infty) \to K \cap \bar{B}$ in L_1, we have*

$$\int_{\theta^{-1}(T)} G(\theta(s))\dot{\gamma}(s)ds = \int_T G(\tau)c(\tau)d\bar{\mu}(\tau). \tag{2.14}$$

(iii) *For any Lebesgue measurable function $f : [0, +\infty) \to \mathbb{R}^n$ and Borel set $S \subset [0, +\infty)$, $\theta(S)$ is also a Borel set and*

$$\int_S f(\theta(s))\dot{\theta}(s)ds = \int_{\theta(S)} f(\tau)d\tau. \tag{2.15}$$

The solutions of system of equation (2.1) are robust in the sense that the set of solutions has desirable closure properties with respect to perturbations of the driving measure μ and the initial state. The robustness property is essential if we need to interpret the measure μ_{sa} as an idealization of conventional controls taking large values in short periods. The following result [36, 32, 33] is a direct generalization of [39, Theorem 5.1]. This result enables the definition of an impulsive solution that can be approximated by a "conventional" trajectory.

Proposition 2.2.2. *Consider multifunctions F and G with domain $[0,+\infty) \times \mathbb{R}^n$ satisfying the following conditions:*

(i) *$F(t,\cdot)$ and $G(\cdot,\cdot)$ have closed graphs and compact values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$, respectively;*

(ii) *F is a Lebesgue $\times$ Borel measurable and G is Borel measurable;*

(iii) *$F(t,x)$ and $G(t,x)$ are convex-valued for all (t,x).*

Consider $T > 0$ and take a sequence $\{x_{0,i}\}$ in $\mathbb{R}^n$ and a sequence $\{\mu_i\}$ in $C^*([0,T];K)$. Consider also elements $x_0 \in \mathbb{R}^n$ and $\mu \in C^*([0,T];K)$ such that, as $i \uparrow +\infty$, $x_{0,i} \to x_0$ and $\mu_i \to^* \mu$. Take also a sequence $\{x_i\}$ in $BV^+([0,T];K)$ such that $x_i(\cdot)$ is a solution to (2.1) in the sense of definition (2.2.1) with μ_i in place of μ. Suppose that the inclusion

$$\dot{y}(s) \in F(\theta(s), y(s))\dot{\theta}(s) + G(\theta(s), y(s))\dot{\gamma}(s) \tag{2.16}$$

is defined almost everywhere in $[0, T + \bar{\mu}([0,T])]$. For each i, assume the existence of $\beta(t) \in L^1$ and $c > 0$ such that $F(t, x_i(t)) \subset \beta(t)B$ a.e. and $G(t, x_i(t)) \subset cB$ for all t.

Then, there exist:

a) *A sequence of processes $(y_i, \theta_i, \gamma_i)$, solution of the differential inclusion (2.16) with $y_i(0) = x_{0,i}$ and $y_i(\cdot) \in AC([0, T + \bar{\mu}_i([0,T])]; \mathbb{R}^n)$;*

b) *A solution (y, θ, γ) of the differential inclusion (2.16) with initial condition $y(0) = x_0$ and $y(\cdot) \in AC([0, T + \bar{\mu}([0,T])]; \mathbb{R}^n)$;*

c) *A solution x of system (2.1) such that $x_i(t) = y_i(\eta_i(t))$ and $x(t) = y(\eta(t))$ for all $t \in (0,T]$.*

Furthermore, along a subsequence, one has the convergence $dx_i \to^* dx$ and $x_i(t) \to x(t)$ for all $t \in ([0,T] \backslash \mathcal{M}_\mu) \cup \{0,T\}$ (where $\mathcal{M}_\mu$ stands for the support of the atoms *of μ), and $y_i \to y$ strongly in $C([0, T + \bar{\mu}([0,T])]; \mathbb{R}^n)$. Here, $dx_i \to^* dx$ stands for the weak* convergence of the measure dx_i to dx.*

Before we conclude the presentation of the robust solution, we should refer that the measure was defined to be positive valued to avoid a possible ambiguous situation when the columns of G do not commute. In this case, we can apply singular atomic measures with opposite signals in a given at t. The two measures cancel each other but a jump can occur due to the noncommutativity property. Then, the "jump" cannot be explained and this is the main reason for not considering signed measures.

2.2.1 Issues on the reparameterization technique

The reparameterization technique presented before has its roots in the seminal work of Rishel [37] and was adopted by several other authors as in [39, 40, 36, 10, 45]. However, we remark here that this technique does not seem to be a proper tool when the measure has singular continuous components. In other words, the results that are derived from this technique are only valid if we assume that no singular continuous measures are present in a given problem. In some circumstances, this can be a drawback, namely, when the existence of the solutions is required (in optimal control problems for example).

Imagine, for instance, we have a measure supported on $[a, b]$ with $|\mu| > 0$ and having only the singular continuous component such that $|\mu| > 0$. An example of such measure is the Lebesgue-Stieljes measure associated with the Cantor function [28]. By definition, the function $\gamma(s)$ in this case is necessarily singular continuous, implying that $\dot{\gamma}(s) = 0$ $\mathcal{L}-$ a.e.. Hence, in the reparameterized dynamics we observe that the multifunction G does not contribute to the evolution of the trajectory, contradicting the robust solution. In this sense, the reparameterization technique does not synthesize the singular continuous component properly.

Even the seminal work of Rishel [37] has a similar problem. In this case, a control problem is studied where the function $g(t, u)$ does not depend on the state, as follows:

$$dx = f(t, x, u)dt + g(t, u)d\mu(t).$$

The reparameterization technique used in this case is slightly different (see equations (4)-(13) of [37]), which allows to overcome the difficulty presented before, but it remains another difficulty. This issue is shown in the following example.

Example 2.2.2. *Consider the following system:*

$$dx = 2dt + 1.d\mu_{sc}(t),\ t \in [0, 1],\ x(0) = 0 \tag{2.17}$$

such that $\eta(t) = t + \int_{[0,t]} d\mu_{sc}(t)$ *has the graphic shown in figure 2.2.*

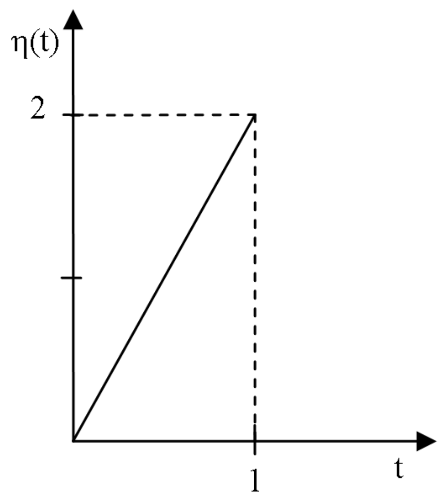

Figure 2.2: The $\eta(t)$ function

Using the definitions of Rishel, we can conclude that $\theta(s) = \frac{s}{2}$, *that* $\tilde{F}(s) = s - \theta(s) = \frac{s}{2}$ *and, consequently,* $v(s) = \frac{1}{2}$ *since* $\tilde{F}(s) = \int_0^s v(\tau)d\tau$. *Thus, the reparameterized version of this system is given by (equation (31) of [37]):*

$$\dot{y}(s) = 2(1 - v(s)) + 1.v(s) = \frac{3}{2},\ s \in [0, 2],$$

whose solution is given by:

$$y(s) = \frac{3}{2}s,\ s \in [0, 2].$$

On the other hand, projecting this in the original time frame, we obtain:

$$x(t) = y(\eta(t)) = \frac{3}{2}\left(t + \int_{[0,t]} d\mu_{sc}(t)\right).$$

However, we easily verify that $\dot{x}(t) = \frac{3}{2}$*, which contradicts the differential equation of the original problem (2.17). This happens because we have considered the function* $\theta(s)$ *as being Lipschitz continuous (stated in [37, 39, 40]), which lead us to an averaging among the absolutely continuous and singular continuous measures.*

The conclusion of this section constitutes one of the motivations to develop an Euler solution expressed in the original time frame. This is what is done in the next section.

2.3 The impulsive Euler solution

In this section, we present a solution for the impulsive control problem that does not require any regularity assumptions on the selections of F and G, namely continuity. This solution concept is based on the Euler solution already available for conventional systems [14] and provides an extension for the impulsive case. We will denominate it by impulsive Euler solution. Besides the non regularity assumption on the selections, the impulsive Euler solution will be derived in the original time frame and will address properly the case where the measure has singular continuous components. The main novelties relative to the results in [45, 46] are the computation of the solution in the original time frame and no reparameterization technique is required. These characteristics enable a better understanding of its meaning and, more importantly, give a formalism enabling the definition of the control measure in a feedback form. Consequently, the impulsive Euler solution will be an important tool to enable feedback synthesis mechanisms.

We assume the following hypotheses on the data:

(h1) For every $x \in \mathbb{R}^n$, $F(t,x)$ and $G(t,x)$ are nonempty compact convex sets.

(h2) $F(t,x)$ and $G(t,x)$ are upper semicontinuous.

(h3) Linear growth: there are positive constants γ and c such that for all (t,x)

$$\begin{aligned} v &\in F(t,x) \Longrightarrow \|v\| \leq \gamma\|x\| + c, \\ v &\in G(t,x) \Longrightarrow \|v\| \leq \gamma\|x\| + c. \end{aligned}$$

Recall that a multifunction F is upper semicontinuous at x if, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|y - x\| < \delta \Rightarrow F(y) \subset F(x) + \varepsilon B. \tag{2.18}$$

Clearly, the hypotheses (h2) and (h3) are analogous to continuity and linear growth when instead of having multifunctions we have just functions. In some of the results contained in this work, these hypotheses could be weakened and the conclusion would remain valid. However, we sacrifice stating minimal hypotheses to the goal of emphasizing the main ideas.

Since we are considering hypothesis $(h1)$, then there is no loss of generality if we consider multifunctions F and G independent of t. The case when the multifunctions depend on t measurably is more elaborated and will not be addressed here.

In order to simplify the presentation, we will address first the case where the measure is scalar valued and then we provide the natural extension for the vector-valued case. This methodology allows a better understanding of the main issues of the impulsive Euler solution. We briefly review the Euler solution for conventional systems (without control measures) for easy reference.

2.3.1 Conventional systems

For the sake of completeness, we review the Euler solution for conventional systems [14]. These results will be mentioned when we will present the impulsive Euler solution. Here, the system under study is the following:

$$\dot{x}(t) \in F(x(t)) \text{ a.e.}, \; t \in [a, b].$$

Let us consider any function $f(x) \in F(x)$ mapping $\mathbb{R}^n$ to $\mathbb{R}^n$ (without any regularity assumption). Then, consider the initial-value problem:

$$\dot{x}(t) = f(x(t)), \; x(a) = x_0, \tag{2.19}$$

The difficulty in finding a solution for (2.19) is that f is not, in general, continuous but is an arbitrary element of $F(x)$. A pragmatic approach is taken based upon the well known numerical solution procedure, which constructs piecewise-linear "approximate solutions" (Euler polygons) for partitions of the time interval. Letting the diameter of the partitions tend to zero, leads us to what we shall call an Euler solution of system (2.19).

Let $\pi_t = t_0, t_1, ..., t_N$ denote a partition of the interval $[a, b]$, where we shall always take $t_0 = a$ and $t_N = b$. On $[t_0, t_1]$, we consider the differential equation

$$\dot{x}(t) = f(x_0), \; x(t_0) = x_0,$$

which has constant right side. We denote by x_1 the resulting value $x(t_1)$. On the next subinterval $[t_1, t_2]$, we take

$$\dot{x}(t) = f(x_1), \; x(t_1) = x_1,$$

and so on, generating a piecewise-linear function $x_{\pi_t}(\cdot)$ on [a, b], whose "nodes" are given by (t_i, x_i) with $i = 0, 1, ..., N$. The diameter α_{π_t} of the partition π_t is defined by:

$$\alpha_{\pi_t} := \max\{t_i - t_{i-1} : 1 \leq i \leq N\}.$$

An Euler solution to the initial-value problem (2.19) means any arc x that is the uniform limit of Euler polygonal arcs $x_{\pi_{t,j}}$, corresponding to some sequence $\pi_{t,j}$ such that $\pi_{t,j} \to 0$. Here, by $\pi_{t,j} \to 0$, we mean that $\alpha_{\pi_{t,j}} \downarrow 0$. Clearly, as $\pi_{t,j} \to 0$, the number of partition points in $\pi_{t,j}$ must go to infinity. In theorem 4.1.7 of [14] is stated the existence of such limit as well as other important properties of this Euler solution. Despite the advantage in considering discontinuous feedback selections, there are some pathologies associated with the Euler solution. Whenever the function f is discontinuous, the solution may not be unique and it may fail to satisfy the condition $\dot{x}(t) = f(x(t))$.

2.3.2 Impulsive systems with scalar measures

In this section, we present the impulsive Euler solution of the following problem:

$$\begin{cases} dx(t) \in F(x(t))dt + G(x(t))d\mu(t), \ t \in [a,b] \\ x(a) = x_0, \end{cases} \tag{2.20}$$

with μ being positive and scalar valued. Assume we have selections $f \in F$ and $g \in G$, which are simply any functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, a scalar measure μ defined on $[a,b]$ with bounded total variation $|\mu([a,b])| \leq M < +\infty$. In particular, this constraint means that $\displaystyle\int_{[a,b]} d\mu(t) \leq M$. Thus, we obtain the following initial-value problem:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))d\mu(t), \ t \in [a,b] \\ x(a) = x_0. \end{cases} \tag{2.21}$$

In order to sample this system and obtain an Euler solution, we need first to define the following function:

$$\eta(t) := t + \int_{[a,t]} d\mu(\tau), \ t \in [a,b]. \tag{2.22}$$

Then, we define a partition on the range of function η as follows:

$$\pi = \{s_0, s_1, \cdots, s_N\}, \tag{2.23}$$

with $s_0 = \eta(a^-)$ and $s_N = \eta(b^+)$. The diameter of the partition π is defined by:

$$\alpha_\pi := \max\{s_i - s_{i-1} : 1 \leq i \leq N\}. \tag{2.24}$$

A uniform partition is not required but we assume, whenever the measure is singular atomic at a given instant t, there are node points precisely on $\eta(t^-)$ and on $\eta(t^+)$. Then, using the map of equation (2.22), we can compute the corresponding node points in the t-domain from the partition π. Let $t_0, t_1, \cdots, t_N$ be such points. For each partition point on the range of η there is only one corresponding point in the t-domain, since $\eta(t)$ is a strictly increasing function. If at a given instant t the measure is singular atomic, then there are partition nodes in the t-domain having the same value. In figure 2.3 is depicted a schematic view of this sampling scheme. At this point, we

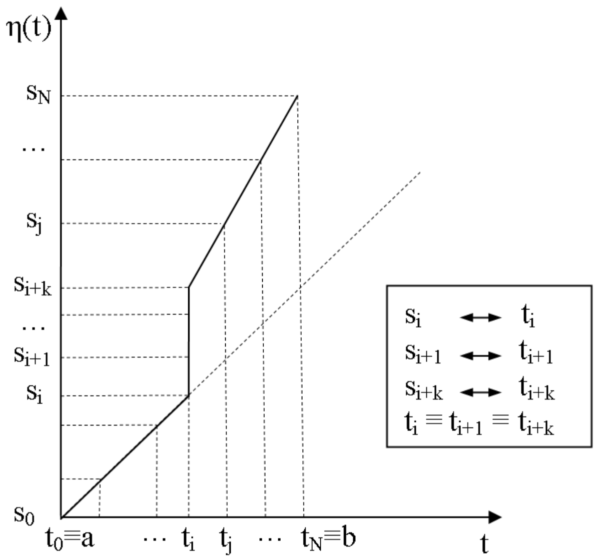

Figure 2.3: The sampling scheme of the impulsive control problem

should mention that there is a very precise practical meaning for this sampling scheme. The singular atomic component is an idealization for the case where $w_{ac}(t)$ takes large values during a short period of time. In practice, whenever we use a singular atomic control measure, we should apply the highest value possible for the variable $w_{ac}(t)$ and the fastest sampling scheme. Therefore, the impulsive sampling scheme constitutes a limiting situation of a practical implementation, enabling important analytical tools.

Whenever the measure is singular atomic at a given instant t_a, the partition should be performed on the set $[\eta(t_a^-), \eta(t_a^+)]$. Consequently, we should define a measure $d\mu_i(t)$ consistent with the fact that the singular atomic component of the measure is partitioned in segments defined by the partition of $\eta(t)$. Thus, for $i = 0, \cdots, N-1$, we define:

$$d\mu_i(t) := \begin{cases} d\mu(t) & \text{, with } t \in [t_i, t_{i+1}) \text{ if } t_{i+1} > t_i \\ (s_{i+1} - s_i)\delta_{t_i} & \text{, with } t = t_i \text{ if } t_{i+1} = t_i. \end{cases}$$

The function δ_{t_i} stands for the unitary Dirac function (a unitary singular atomic measure).

Remark 2.3.1. *If the multifunction F were dependent on t measurably, then some care must be taken on the choice of the partition points when the measure is absolutely continuous or singular continuous. The partition on the range of η should be such that the following condition holds for each node point i:*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{t_i - \varepsilon}^{t_i + \varepsilon} f(t, x(t))dt = f(t_i, x_i)$$

Now, we are in condition to present an Euler polygonal arc x_π, where the subscript π means that the Euler polygonal arc depends on a particular partition π made on the range of η. First, we consider the interval $t \in [t_0, t_1]$ and we define the following dynamical system:

$$dx(t) = f(x_0)dt + g(x_0)d\mu_0(t), \; x(a) = x_0.$$

We define the node point $x_1 := x(t_1)$. Note that the preceding dynamical system has a unique solution since the selections f and g are constant on $t \in [t_0, t_1]$. Also note that the interval $[t_0, t_1]$ can be a singleton, which means that the measure at t_0 has an atom. In this case, to compute the node point x_1, we apply a singular atomic measure with total variation equal to $s_1 - s_0$. Next, we proceed and we consider the interval $[t_1, t_2]$ and the following dynamical system:

$$dx(t) = f(x_1)dt + g(x_1)d\mu_1(t),\ x(t_1) = x_1.$$

The next node point will be defined as $x_2 := x(t_2)$. We proceed like this until we have covered all elements of the partition and an arc x_π is obtained on $[a, b]$. The diameter of the partition π is defined by:

$$\alpha_\pi := \max\{s_i - s_{i-1} : 1 \leq i \leq N\}.$$

An Euler solution to the initial-value problem (2.21) means any function x of bounded variation that is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding to some sequence $\pi_j \to 0$. In this context, $\pi_j \to 0$ means that $\alpha_{\pi_j} \downarrow 0$. Clearly, as the diameters of the partition π tends to zero, the number of node points tends to infinity. The following theorem presents some important properties of the Euler solution.

Theorem 2.3.1. *Suppose that, for positive constants γ, c and for all $x \in \mathbb{R}^n$, we have the linear growth conditions:*

$$\|f(x)\| \leq \gamma\|x\| + c,$$
$$\|g(x)\| \leq \gamma\|x\| + c,$$

where f, g are arbitrary selections of F, G, respectively. Moreover, assume also that we are given a measure μ such that $|\mu([a, b])| \leq M < +\infty$. Then,

a) *At least one Euler solution x exists for system (2.21) on $[a, b]$ and any Euler solution is of bounded variation.*

b) *Any Euler arc x on $[a, b]$ satisfies*

$$\|x(t) - x(a)\| \leq (t - a + |\mu([a, t])|)e^{\gamma(t-a+|\mu([a,t])|)}(\gamma\|x_0\| + c),\ a \leq t \leq b.$$

c) *If f and g are continuous selections, then any impulsive Euler solution x on $[a, b]$ is also a robust solution as presented in definition 2.2.1.*

Proof. Consider a partition π, as defined in equation (2.23), and the correspondent partition π_t in the t-domain. Consider also the correspondent Euler polygonal arc x_π and its nodes $x_0, x_1, \cdots, x_N$. Then, we can deduce the following relationship between consecutive node points:

$$\begin{aligned}\|x_{i+1} - x_i\| &= \left\|(t_{i+1} - t_i)f(x_i) + \int_{[t_i,t_{i+1}]} g(x_i)d\mu_i(t)\right\| \\ &\leq (t_{i+1} - t_i)(\gamma\|x_i\| + c) + (\gamma\|x_i\| + c)\int_{[t_i,t_{i+1}]} d\mu_i(t). \qquad (2.25)\end{aligned}$$

Note that t_{i+1} can be equal to t_i whenever the measure is singular atomic and, consequently, the interval $[t_i, t_{i+1}]$ can be a singleton such that $\displaystyle\int_{[t_i,t_{i+1}]} d\mu_i(t) = s_{i+1} - s_i$. Thus, with this estimative at hand, we deduce that:

$$\begin{aligned}\|x_{i+1} - x_0\| &\leq \|x_{i+1} - x_i\| + \|x_i - x_0\| \\ &\leq \left(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\mu_i(t)\right).(\gamma\|x_i\| + c) + \|x_i - x_0\| \\ &\leq \left(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\mu_i(t)\right).(\gamma\|x_i - x_0\| + \gamma\|x_0\| + c) + \|x_i - x_0\| \\ &= \left[\gamma.\left(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\mu_i(t)\right) + 1\right].\|x_i - x_0\| + \\ &\quad + \left(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\mu_i(t)\right)(\gamma\|x_0\| + c).\end{aligned}$$

Applying an exercise in induction (4.1.8 of [14]), we are led to the conclusion that:

$$\begin{aligned}\|x_i - x_0\| \leq \exp\left(\sum_{j=0}^{i-1}\gamma\left(t_{j+1} - t_j + \int_{[t_j,t_{j+1}]} d\mu_i\right)\right). \\ .\sum_{j=0}^{i-1}\left(t_{i+1} - t_i + \int_{[t_j,t_{j+1}]} d\mu_i\right).(\gamma\|x_0\| + c),\end{aligned}$$

which implies that $\|x_i - x_0\| \leq (b - a + M).e^{\gamma.(b-a+M)}.(\gamma\|x_0\| + c) := L$. Hence, we can conclude that all nodes of an Euler polygonal arc lie in a closed ball $x_i \in \bar{B}(x_0, L)$.

Since between node points the selections f and g are constant, then this property is also true for all values of x_π, which, in turn, implies that the Euler polygonal arc is a function of bounded variation. Now, let π_j be a sequence of partitions such that $\pi_j \downarrow 0$. The corresponding Euler polygonal arcs x_{π_j} all satisfy $\|x_{\pi_j} - x_0\|_\infty \leq L$ and they are functions of bounded variation. Therefore, by the Helly's selection principle [28] (a compactness theorem for sequences of functions of bounded variation and uniformly bounded), we can conclude that the sequence of Euler polygonal arcs has a limit, which is a function of bounded variation. This establishes part a) of the theorem.

The inequality of part b) is inherited by the limit of the subsequence x_{π_j}. We only need to identify t with b and M with $|\mu([a,t])|$.

Now we turn to prove part c) of the theorem. Let x_{π_j} denote a sequence of polygonal arcs converging uniformly to an Euler solution x. As shown before, the arcs x_{π_j} all lie in $\bar{B}(x_0; L)$. Since a continuous function on $\mathbb{R}^n$ is uniformly continuous on compact sets then, for any $\varepsilon > 0$, we can find $\delta > 0$, such that:

$$x, \tilde{x} \in \bar{B}(x_0; L),\ \|x - \tilde{x}\| < \delta \Rightarrow \begin{cases} \|f(x) - f(\tilde{x})\| < \varepsilon \\ \|g(x) - g(\tilde{x})\| < \varepsilon \end{cases}. \tag{2.26}$$

Consider any s not being a node point such that $s > \tilde{s}$ where $\tilde{s}$ represents the closest node point before t. Let t and $\tilde{t}$ be the corresponding node points in the t-domain. Then, let j be large enough so that the diameter of the partition π_j is such that:

$$\begin{aligned} \|x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})\| &\leq \left(t - \tilde{t} + \int_{[\tilde{t},t]} d\mu(t)\right)(\gamma\|x_{\pi_j}(\tilde{t})\| + c) \\ &\leq \alpha_{\pi_j}.(\gamma\|x_{\pi_j}(\tilde{t})\| + c) < \delta. \end{aligned}$$

The first inequality is due to equation (2.25) while the last inequality follows from equation (2.24) and the boundedness of $x_{\pi_j}(\tilde{t})$. When the Euler polygonal arc is a path joining the endpoints of a jump at $\tilde{t}$, t coincides with $\tilde{t}$ and in this case $x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})$

stands for $x_{\pi_j}(s) - x_{\pi_j}(\tilde{s})$. Thus, the following estimative can now be derived:

$$\begin{aligned}
& \|dx_{\pi_j}(t) - [f(x_{\pi_j}(t))dt + g(x_{\pi_j}(t))d\mu_{i_j}(t)]\| \\
= \ & \|f(x_{\pi_j}(\tilde{t}))dt + g(x_{\pi_j}(\tilde{t}))d\mu_{i_j}(t) - [f(x_{\pi_j}(t))dt + g(x_{\pi_j}(t))d\mu_{i_j}(t)]\| \\
= \ & \|(f(x_{\pi_j}(\tilde{t})) - f(x_{\pi_j}(t)))dt + (g(x_{\pi_j}(\tilde{t})) - g(x_{\pi_j}(t)))d\mu_{i_j}(t)\| \leq \varepsilon(dt + d\mu_{i_j}(t))
\end{aligned}$$

Consequently, for any $t \in [a, b]$, we have:

$$\begin{aligned}
& \|x_{\pi_j}(t) - x_{\pi_j}(a) - \int_{[a,t]} [f(x_{\pi_j}(\tau))d\tau + g(x_{\pi_j}(\tau))d\mu_{i_j}(\tau)]\| \\
= \ & \left\| \int_{[a,t]} \left\{ dx_\pi(\tau) - [f(x_{\pi_j}(\tau))d\tau + g(x_{\pi_j}(\tau))d\mu_{i_j}(\tau)] \right\} \right\| \leq \int_{[a,t]} \varepsilon(d\tau + d\mu_{i_j}(\tau)) \\
\leq \ & \varepsilon(b - a + M).
\end{aligned}$$

Letting $j \to \infty$, we obtain:

$$\left\| x(t) - x_0 - \int_{[a,t]} [f(x(\tau))dt + g(x(\tau))d\mu] \right\| \leq \varepsilon(b - a + M).$$

Since ε is arbitrary, we reach to the conclusion that

$$x(t) = x_0 + \int_{[a,t]} [f(x(\tau))dt + g(x(\tau))d\mu],$$

which gives the desired result. □

We have presented some important properties of the impulsive Euler solution. Now, we need to show that this solution is, in fact, a trajectory of system (2.20).

Theorem 2.3.2. *Let f and g be any selections of F and G, respectively, and μ be a given measure with bounded total variation. If x is an impulsive Euler solution on $[a, b]$ of $dx(t) = f(x(t))dt + g(x(t))d\mu(t)$, $x(a) = x_0$, then x is a trajectory of the system:*

$$\begin{cases} dx(t) \in F(x(t))dt + G(x(t))d\mu(t), \ t \in [a, b] \\ x(a) = x_0. \end{cases} \tag{2.27}$$

Proof. Let any $s \in [\eta(a^-), \eta(b^+)]$ and let (s_i, x_i) represent any node point of an impulsive Euler polygonal arc before s. Consider also t and the node point t_i associated, respectively, with s and s_i in the t-domain. Defining $y_j(t) := x_i - x_{\pi_j}(t)$, then we have:

$$\begin{aligned} dx_{\pi_j}(t) = f(x_i)dt + g(x_i)d\mu_{i_j}(t) &\in F(x_i)dt + G(x_i)d\mu_{i_j}(t) \\ &= F(x_{\pi_j}(t) + y_j(t))dt + G(x_{\pi_j}(t) + y_j(t))d\mu_{i_j}(t). \end{aligned} \tag{2.28}$$

Then, by equation (2.25), the following estimate is obtained:

$$\|y_j(t)\| = \|x_{\pi_j}(t) - x_i\| \leq \left(t - t_i + \int_{[t_i,t]} d\mu_{i_j}(t)\right)(\gamma\|x_i\| + c) \leq \alpha_{\pi_j}.(\gamma\|x_i\| + c).$$

Since $\|x_i\|$ is bounded (see proof of theorem 2.3.1), we conclude that $y_j(t) \to 0$ as $\pi_j \downarrow 0$. The polygonal impulsive Euler arc x_{π_j} is a function of bounded variation and is uniformly bounded. Thus, by the Helly's selection principle, we conclude that x_{π_j} converges to a bounded variation function x. Therefore, we may suppose that $dx_{\pi_j} \to^* dx$. By construction, we conclude that $dx = v_F(t)dt + v_G(t)d\mu(t)$ where the functions $v_F(t)$ and $v_G(t)$ are originated by taking the limit of subsequences of $v_{F,j}(t) \in F(x_{\pi_j}(t) + y_j(t))$ and $v_{G,j}(t) \in G(x_{\pi_j}(t) + y_j(t))$. Then, applying a weak sequential compactness result (theorem 3.5.24 in [14]), we conclude that $v_F(t) \in F(x(t))$ and $v_G(t) \in G(x(t))$, which is precisely what we set to prove. □

2.3.3 Impulsive systems with vector-valued measures

Now, we devote attention to the case where the given measure μ is vector-valued such that $\mu \in K$, with K being a positive convex cone in $\mathbb{R}^q$:

$$\begin{cases} dx(t) \in F(x(t))dt + G(x(t))d\mu(t), \ t \in [a, b] \\ x(0) = x_0, \end{cases} \tag{2.29}$$

In this case, the multifunction G maps $\mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^{n\times q})$, while the selection g maps $\mathbb{R}^n \to \mathbb{R}^{n\times q}$. To define the impulsive Euler solution, we will use the definition of the total variation measure, defined in equation (2.3). In order to define the Euler nodes of this dynamical system, we introduce the following function:

$$\eta(t) := t + \int_{[a,t]} d\bar{\mu}(\tau),\ t \in [a,b]. \tag{2.30}$$

We define a partition on the range of function η:

$$\pi = \{s_0, s_1, \cdots, s_N\}, \tag{2.31}$$

where $s_0 = \eta(a^-)$ and $s_N = \eta(b^+)$. We do not require a uniform partition but, to simplify the presentation, whenever at least one of the components of the measure is singular atomic, we place node points precisely on $\eta(t^-)$ and on $\eta(t^+)$ (here t refers to the support of each atom). Having made this partition on the range of $\eta(t)$, we can compute the corresponding points in the t-domain. Let $t_0, t_1, \cdots, t_N$ be such points. For each node on the range of η there is only one corresponding point in t-domain, since $\eta(t)$ is a strictly increasing function. As a remark, note that if at a given instant t there is a singular atomic component, then there are partition nodes in the t-domain having the same value.

Whenever the measure has a singular atomic component at a given instant t_a, the sampling scheme makes a partition on $[\bar{\eta}(t_a^-), \bar{\eta}(t_a^+)]$. Thus, we need to define the measure $d\mu_i(t)$ for the vector-valued measure case, as was done for the scalar measure case. Thus, for $i = 0, \cdots, N-1$, we define:

$$d\mu_i(t) := \begin{cases} d\mu(t) & \text{, with } t \in [t_i, t_{i+1}) \text{ if } t_{i+1} > t_i \\ v(s)\delta_{t_i} & \text{, with } t = t_i \text{ if } t_{i+1} = t_i, \end{cases}$$

such that $\displaystyle\int_{s_i}^{s_{i+1}} \|v(s)\| ds = s_{i+1} - s_i$ and $v(s) \in K \cap \bar{B}$ on $s \in [s_i, s_{i+1})$.

Now we are in condition to present the Euler polygonal arc x_π. First, we consider the interval $t \in [t_0, t_1]$ and define the following dynamical system:

$$dx(t) = f(x_0)dt + g(x_0)d\mu_0(t),\ x(a) = x_0.$$

We define the node point $x_1 := x(t_1)$. Note that the preceding dynamical system has a unique solution since the selection f and g are set to be constant. Also note that the interval $[t_0, t_1]$ can be a singleton, which would mean that at least one component of the measure μ at t_0 has an atom. In this case, to compute the node point x_1, we would apply a singular atomic measure with total variation equal to $s_1 - s_0$. Since more than one component can be active, the next node point is computed with the help of an auxiliary variable v, which allows the complete definition of the singular trajectory (recall the need of the extra variable v in the robust solution). Since we are assuming the measure given in advance, the variable $v(s)$ is also specified before hand. Thus, the Euler node in this case is computed by $x_1 := x^s(s_1)$, subject to:

$$\frac{dx^s(s)}{ds} = g(x^s(s_0))v(s), \; s \in [s_0, s_1].$$

Next, we consider the interval $[t_1, t_2]$ and the following dynamical system:

$$dx(t) = f(x_1)dt + g(x_1)d\mu_1(t), \; x(t_1) = x_1.$$

The next node point will be defined by $x_2 := x(t_2)$ (the remarks on the singular trajectory apply as before). We proceed like this until we have covered all the elements of the partition and an Euler polygonal arc x_π is obtained on $[a, b]$.

An impulsive Euler solution of the initial-value problem means any function x of bounded variation that is the uniform limit of Euler polygonal arcs x_{π_j}, as the diameters of the partition π_j tends to zero. Clearly, as $\pi_j \downarrow 0$, the number of node points tends to infinity.

Now, with the impulsive Euler solution also defined for vector-valued measures, we can provide some properties of this solution as was done for the scalar measure case.

Theorem 2.3.3. *Suppose we are given selections f, g of F, G, respectively, such that,* *for positive constants γ and c and for all $x \in \mathbb{R}^n$, we have:*

$$\begin{aligned}\|f(x)\| &\leq \gamma\|x\| + c,\\ \|g_k(x)\| &\leq \gamma\|x\| + c,\ k = 1, \cdots, q.\end{aligned}$$

Moreover, assume also that we are given a measure $\mu \in K \subset \mathbb{R}^q$ such that $\|\bar{\mu}([a,b])\| \leq M < +\infty$. Then,

a) *At least one Euler solution x exists for system (2.29) on $[a,b]$ and any Euler solution is of bounded variation.*

b) *Any Euler arc x on $[a,b]$ satisfies*

$$\|x(t) - x(a)\| \leq (t - a + |\bar{\mu}([a,t])|)e^{\gamma(t-a+|\bar{\mu}([a,t])|)}(\gamma\|x_0\| + c),\ a \leq t \leq b.$$

c) *If f and g are continuous selections, then any impulsive Euler solution x on $[a,b]$ is also a robust solution as presented in definition 2.2.1.*

Proof. Consider a partition made in the range of equation (2.30). By the linear growth hypothesis on f and g_j, with $j = 1, \cdots, q$, and noting that $\left\|\int_{[t_i,t_{i+1}]} d\mu_i(t)\right\| \leq \int_{[t_i,t_{i+1}]} d\bar{\mu}_i(t)$, we obtain the following relationship between consecutive node points:

$$\begin{aligned}\|x_{i+1} - x_i\| &= \left\|(t_{i+1} - t_i)f(x_i) + \int_{[t_i,t_{i+1}]} g(x_i)d\mu_i(t)\right\|\\ &\leq \left(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\bar{\mu}_i(t)\right)(\gamma\|x_i\| + c).\end{aligned}$$

Then, we can apply the procedure presented in the proof of theorem 2.3.1 to conclude that parts a) and b) are valid.

Now we turn to prove part c) of the theorem. Let x_{π_j} denote a sequence of polygonal arcs converging uniformly to an Euler solution x. By condition b), the arcs x_{π_j} all lie in $\bar{B}(x_0; L)$, for some positive number$L$. Since a continuous function on $\mathbb{R}^n$

is uniformly continuous on compact sets then, for any $\varepsilon > 0$, we can find $\delta > 0$, such that:

$$x, \tilde{x} \in \bar{B}(x_0; L),\ \|x - \tilde{x}\| < \delta \Rightarrow \begin{cases} \|f(x) - f(\tilde{x})\| < \varepsilon \\ \|g_k(x) - g_k(\tilde{x})\| < \varepsilon,\ k = 1, \cdots, q. \end{cases} \tag{2.32}$$

Consider any s not being a node point such that $s > \tilde{s}$ where $\tilde{s}$ represents the closest node point before t. Let t and $\tilde{t}$ be the corresponding node points in the t-domain. Then, let j be large enough so that the diameter of the partition π_j is such that:

$$\begin{aligned} \|x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})\| &\leq \left(t - \tilde{t} + \int_{[\tilde{t},t]} d\bar{\mu}_{i_j}(t)\right)(\gamma\|x_{\pi_j}(\tilde{t})\| + c) \\ &\leq \alpha_{\pi_j}.(\gamma\|x_{\pi_j}(\tilde{t})\| + c) < \delta. \end{aligned}$$

When the Euler polygonal arc is a path joining the endpoints of a jump at $\tilde{t}$, t coincides with $\tilde{t}$ and, in this case, $x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})$ stands for $x_{\pi_j}(s) - x_{\pi_j}(\tilde{s})$. Thus, the following estimative can now be derived:

$$\begin{aligned} & \|dx_{\pi_j}(t) - [f(x_{\pi_j}(t))dt + g(x_{\pi_j}(t))d\mu_{i_j}(t)]\| \\ = \ & \|(f(x_{\pi_j}(\tilde{t})) - f(x_{\pi_j}(t)))dt + (g(x_{\pi_j}(\tilde{t})) - g(x_{\pi_j}(t)))d\mu_{i_j}(t)\| \leq \varepsilon(dt + d\bar{\mu}_{i_j}(t)) \end{aligned}$$

Consequently, for any $t \in [a, b]$, we have:

$$\begin{aligned} & \|x_{\pi_j}(t) - x_{\pi_j}(a) - \int_{[a,t]} [f(x_{\pi_j}(\tau))d\tau + g(x_{\pi_j}(\tau))d\mu(\tau)]\| \\ = \ & \left\| \int_{[a,t]} \{dx_\pi(\tau) - [f(x_{\pi_j}(\tau))d\tau + g(x_{\pi_j}(\tau))d\mu(\tau)]\} \right\| \leq \int_{[a,t]} \varepsilon(d\tau + d\bar{\mu}(\tau)) \\ \leq \ & \varepsilon(b - a + M). \end{aligned}$$

Letting $j \to \infty$, we obtain:

$$\left\| x(t) - x_0 - \int_{[a,t]} [f(x(\tau))dt + g(x(\tau))d\mu] \right\| \leq \varepsilon(b - a + M).$$

Since ε is arbitrary, we reach to the conclusion that

$$x(t) = x_0 + \int_{[a,t]} [f(x(\tau))dt + g(x(\tau))d\mu],$$

which gives the desired result. $\square$

The theorem 2.3.2 could also be extended for the case where the measure is vector-valued. With the Euler solution defined for this case and with the properties presented in theorem 2.3.3, the result follows immediately.

2.3.4 Signed scalar measures

Now, we make some comments on the case where the measure is signed and scalar. In this case, the robust solution should be adapted, namely, in the support of the singular atomic components. Hence, the function η must be redefined as follows:

$$\eta(t) := t + \left| \int_{[a,t]} d\mu(\tau) \right|,$$

and an additional control variable $v(s) \in \mathbb{R}$ should be considered at the atoms of the measure, with the following properties:

$$\int_{\eta(t^-)}^{\eta(t^+)} v(s) ds = \mu(\{t\})$$

and

$$|v(s)| = 1 \ \mathcal{L}\text{-a.e. on } s \in [\eta(t^-), \eta(t^+)].$$

The role of this extra control variable is to completely define the behavior of the singular arc. Consider, for example, that at a given instant t_a, we have a singular atomic measure such that $\left| \int_{\{t_a\}} |d\mu(\tau)| \right| = M$. If the variable v were not defined, it would not be clear if the total variation was due to a positive measure, to a negative measure or a a combination of both. Thus, the variable v serves precisely to eliminate this ambiguity.

The Euler polygonal arc is defined in a similar way as described in the vector-valued measure, with the necessary adaptations. Note that the measure is given in advance and, consequently, the variable v is also given on the atoms of the measure.

As done for the scalar unsigned measure case, we can present a result stating some properties about this Euler solution and also a result on compactness trajectories.

However, we skip its presentation since it is just an adaptation of the results presented in theorems 2.3.1, 2.3.2 and 2.3.3.

Chapter 3

Invariance for impulsive control systems

The aim of this chapter is to provide necessary and sufficient conditions of invariance for impulsive control systems. This work constitutes an extension of conventional systems [14, 4]. The objective of an invariance problem is to find selections of F, G and a control measure such that the resulting impulsive trajectory remains in a given set. We use the impulsive Euler solution to provide constructive results in the original time frame, enabling the synthesis of feedback selections and feedback control measures. This approach was introduced in [22].

Conditions for invariance have been presented in reparameterized time [34, 18]. However, these results are not constructive and have three drawbacks: firstly, the conditions are less intuitive since a time transformation is involved, secondly, the reparameterization technique does not synthesize the singular continuous component of the measure properly and, thirdly, the reparameterization induces a distortion on the vector fields due to the definition of a new time scale, disabling the construction of control measures in a feedback form. Below, we present a motivating example where the distortion effect occurs. We compute a relevant minimization, which will be useful through out this chapter, and we show that the results are different if we use the original or the reparameterized data. The impulsive Euler solution is an important tool to overcome these difficulties and allows the development of constructive results, using feedback synthesis mechanisms.

Example 3.0.1. *Consider the following control system:*

$$\dot{x} = f(x) + g(x).w_{ac}(t), \tag{3.1}$$

with $w_{ac} \in [0, M]$ *and* $x = (x_1, x_2)$. *For the purpose of the example, this control system suffices and it is not even necessary to include any control measure. Also, we can assume that the functions* f *and* g *are constant valued:*

$$f = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, g = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{3.2}$$

Using the reparameterization technique presented in chapter 2, this system can be described as:

$$\dot{y} = f(t)(1 - v(s)) + g(y)v(s), \tag{3.3}$$

where we identify $v(s)$ *with* $\dot{\gamma}(s)$. *Consider a set* S *defined as follows:*

$$S := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 1\}. \tag{3.4}$$

At any given point $x \notin S$, *consider the projection point* $p(x)$ *on* S. *The vector* $x - p \in N_S^P(x)$ *is given by:*

$$x - p = \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \text{ with } \xi < 0 \tag{3.5}$$

(see figure 3.1 for details). Then, we compute the following minimizations for a given $x \notin S$:

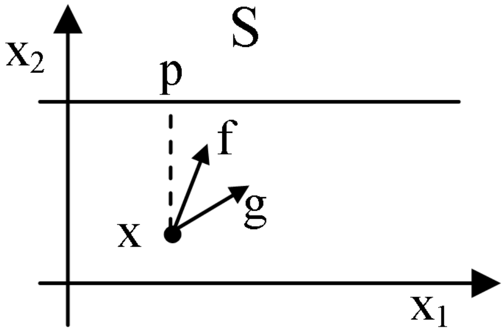

Figure 3.1: Illustration of set S

1. *Using the original time frame:*

$$\begin{aligned}\min_{w_{ac}\in[0,M]}\langle f+gw_{ac},x-p\rangle &= \min_{w_{ac}\in[0,M]}\left\langle \begin{bmatrix}1+2w_{ac}\\ 2+w_{ac}\end{bmatrix},\begin{bmatrix}0\\ \xi\end{bmatrix}\right\rangle\\ &= \min_{w_{ac}\in[0,M]}\xi(2+w_{ac}) \Rightarrow w_{ac}=M.\end{aligned}$$

2. *Using the reparameterized time:*

$$\begin{aligned}\min_{v\in\left[0,\frac{M}{1+M}\right]}\langle f(1-v)+gv,y-p\rangle &= \min_{v\in\left[0,\frac{M}{1+M}\right]}\left\langle \begin{bmatrix}1-v+2v\\ 2-2v+v\end{bmatrix},\begin{bmatrix}0\\ \xi\end{bmatrix}\right\rangle\\ &= \min_{v\in\left[0,\frac{M}{1+M}\right]}\xi(2-v) \Rightarrow v=0.\end{aligned}$$

Recall that $v(s)$ is identified with $\dot{\gamma}(s)$, which, by its definition, implies that it ranges from zero, when $w_{ac}=0$, to $\frac{M}{1+M}$, when $w_{ac}=M$. Hence, by equation (2.13), we deduce that $v\in[0,\frac{M}{1+M}]$. Clearly, $v=0$ corresponds to $w_{ac}=0$ in the original time frame, while $v=\frac{M}{1+M}$ corresponds to $w_{ac}=M$. In the last case, the function η is given by $\eta(t)=(1+M)t$ which, in turn, implies that $\theta(s)=\frac{s}{M+1}$. Consequently, $\dot{\theta}(s)=\frac{1}{1+M}$ and $v(s)=\dot{\gamma}(s)=1-\dot{\theta}(s)=\frac{M}{1+M}$.

From the previous minimizations, we observe that the usage of the reparameterized and original time frame systems does not lead to the same solution. This minimization *will be relevant in later developments, namely, in the definition of feedback control measures.*

3.1 Overview of results for conventional control systems

In this section, we give a brief overview of some definitions and results on nonsmooth analysis as well as some concepts about weak invariance of conventional systems (without control measures).

Many different topics in mathematics and its applications require to some extent a calculus for nondifferentiable functions and nonsmooth sets. The present topic, deriving constructive results on invariance, is another in which the best results necessitate nonsmooth analysis for their derivation. Here, we present relevant aspects that will be used later in this work. For a complete coverage of this issue, we refer to [16, 14, 13].

Let S be a closed subset of $\mathbb{R}^n$, and let $p \in S$. If $x \notin S$, and if one of the closest points to x in S is p, the vector $x - p$ (and any nonnegative scalar multiple of it) is said to be a proximal normal to S at p. The set of such vectors is the proximal normal cone to S at p, denoted $N_S^P(p)$. We set $N_S^P(p) = \{0\}$ if no x as described above exists.

Now, let $f : \mathbb{R}^n \to (-\infty, +\infty]$ be a lower semicontinuous, extended-valued function, and let x be a point such that $f(x)$ is finite. An element ζ of $\mathbb{R}^n$ is a proximal subgradient of f at x provided that for some $\sigma \geq 0$, and for all y in some neighborhood of x, we have:

$$f(y) - f(x) + \sigma|y - x|^2 \geq \langle \zeta, y - x \rangle.$$

The (possibly empty) set of proximal subgradients of f at x is denoted $\partial_P f(x)$.

The link between normals and subgradients is via the epigraph of f:

$$\text{epi}(f) := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}.$$

Then

$$\zeta \in \partial_P f(x) \Leftrightarrow (\zeta, -1) \in N^P_{epi(f)}(x, f(x)).$$

Another tool of nonsmooth analysis is the classical tangent cone to the set S at $x \in S$. The directional or Dini tangent cone (also called the contingent or Bouligand cone) is the set defined by:

$$T^B_S(x) := \left\{ \lim_{i \to +\infty} \frac{x_i - x}{t_i} : x_i \xrightarrow{S} x, t_i \downarrow 0 \right\}. \tag{3.6}$$

The Bouligand tangent cone can also be characterized in terms of the distance function $d_S(x)$ as follows:

$$T^B_S(x) := \{v \in \mathbb{R}^n : \lim_{h \to 0^+} \ \inf \frac{d_S(x + hv)}{h} = 0\}. \tag{3.7}$$

Recall that the function $d_S(x)$ is a map $\mathbb{R}^n \to \mathbb{R}$ and gives the minimum distance from x to the set S. The polar C^o of C is defined as follows

$$C^o := \{z \in X : \langle z, c \rangle \leq 0 \ \forall c \in C\}.$$

Now, we review some key ideas of weak invariance cast in the context of conventional control systems [14]:

$$\dot{x}(t) \in F(x), \ t \in [a, b]. \tag{3.8}$$

The results presented here are of constructive nature and provides a procedure to compute feedback selections. We do not require any regularity assumptions on the selections of F. For conventional systems, we define the Euler polygonal arc nodes by simply setting $\mu = 0$. In this sense, the impulsive Euler solution presented before is more general since it encompasses the conventional Euler solution [14].

We now introduce the proximal aiming concept that is useful in the process of deriving conditions for invariance. Thus, consider a given selection $f(x)$ of $F(x)$, possibly discontinuous, and the following dynamical system:

$$\dot{x}(t) = f(x(t)), \;\; x(a) = x_0. \tag{3.9}$$

We wish to verify if the value of the resulting Euler trajectory gets "closer" to a given closed set S, i.e. if $d_S(x(t)) \downarrow 0$. A sufficient condition for proximal aiming is given in the next proposition.

Proposition 3.1.1. *[14][Proposition 4.2.1] Let f satisfy the linear growth condition*

$$\|f(x)\| \leq \gamma\|x\| + c, \;\; \forall(t,x).$$

Let Ω be an open set containing $x(t)$ $\forall t \in [a,b]$, and suppose that every $(t,z) \in [a,b]\times\Omega$ satisfies the following "proximal aiming" condition: there exists $p \in proj_S(z)$ such that $\langle f(z), z-p\rangle \leq 0$. Then we have

$$d_S(x(t)) \leq d_S(x(a)) \;\forall t \in [a,b].$$

This proposition was proved in the setting of Euler solution. The Euler solutions for this conventional problem has important properties like Lipschitz continuity [14, Theorem 4.1.7].

As stated before, the good feature about using Euler arcs is that, if we have a constructive procedure for the selections of F (a feedback law), then the solution concept is well defined even if the chosen selections are non regular. This procedure can be used, for example, in weak invariance [14], a concept that we introduce now. The system (S,F) is called weakly invariant if, for all $x_0 \in S$, there exists a selection of F such that the resulting trajectory $x(t)$ on $[0,+\infty)$ satisfies

$$x(0) = x_0, \; x(t) \in S, \;\; \forall t \geq 0.$$

A sufficient condition is given in the next theorem. The prove is constructive and uses the Euler solution. The assumptions on F are the same as defined before in section 2.3.

Theorem 3.1.1. *(Weak invariance [14][Theorem 4.2.4]) Suppose that for every $x \in S$, we have*

$$\min_{v \in F(x)} \langle v, N_S^P(x) \rangle \leq 0. \tag{3.10}$$

Then (S, F) is weakly invariant.

Conditions for invariance were also presented in [4], where the issue was addressed with resource of other solution concept.

3.2 Proximal aiming

Proximal aiming for impulsive systems is an elemental tool for the derivation of invariance results. In this context, it is investigated whether the impulsive trajectories associated with given selections of F, G and given measure approach a predefined closed set. Sufficient conditions for this property are presented in the following proposition.

Proposition 3.2.1. *Let f, g satisfy the linear growth conditions*

$$\max\{\|f(x)\|, \|g(x)\|\} \leq \gamma \|x\| + c, \ \forall x \in \mathbb{R}^n$$

and let x be an impulsive Euler arc for the system $dx = f(x)dt + g(x)d\mu(t)$ on $t \in [a, b]$. Assume also that a measure μ is given such that $\int_{[a,b]} d\mu(t) \leq M < +\infty$. Let Ω be an open set such that $x(t) \in \Omega$, $\forall t \in [a, b]$ and $x^s(s) \in \Omega$, $\forall s \in [\eta(t_a^-), \eta(t_a^+)]$, where t_a is any point in the support of the singular atomic component and $x^s(\eta(t_a^-)) = x(t_a^-)$. Additionally, suppose that, for every $z \in \Omega$, there is a $p \in proj_S(z)$ such that:

$$\langle f(z) + g(z)w_{ac}(t), z - p \rangle \leq 0, \ \mu_{ac}\text{-a.e. on } [a, b],$$

$$\langle g(z), z - p \rangle \leq 0, \ \mu_{sa} \text{ and } \mu_{sc} \text{ a.e. on } [a, b],$$

where $w_{ac}(t)$ represents the Lebesgue time derivative of μ_{ac}. Then, we have $d_S(x(t)) \leq d_S(x(a^-))$ $\forall t \in [a,b]$ and, for every singular arc, $d_S(x^s(s)) \leq d_S(x^s(\eta(t_a^-)))$ $\forall s \in [\eta(t_a^-), \eta(t_a^+)]$.

Remark 3.2.1. *The conditions of the previous proposition can also be expressed in a compact form as $\langle f(z)dt + g(z)d\mu, z - p\rangle \leq 0$ μ-a.e. on $[a,b]$. In particular, the condition $\langle g(z), z-p\rangle \leq 0$, μ_{sa}-a.e. on $[a,b]$ means that the inequality remains true during a singular trajectory, on the support of the singular atomic measure.*

Proof. Let x_π be an impulsive Euler polygonal arc in the sequence converging uniformly to the Euler solution x. As before, consider the partition on the range of function η and the corresponding node points t_i and x_i, with $i = 0, 1, \cdots, N$, such that $x_0 = x(a^-)$. We can choose a set Ω such that the impulsive Euler arc x_π lies in it for all $t \in [a,b]$ (including the singular trajectory). For each node point we can compute its projection p_i on S such that $p_i \in \text{proj}_S(x_i)$. We know that $dx_\pi(t) = f(x_i)dt + g(x_i)d\mu_i(t)$ on $t \in [t_i, t_{i+1}]$ and, by hypothesis, that $\langle dx_\pi(t), x_i - p_i\rangle \leq 0$. Hence, the following estimative can be derived:

$$
\begin{aligned}
d_S^2(x_1) &\leq \|x_1 - p_0\|^2 \\
&= \|x_1 - x_0\|^2 + \|x_0 - p_0\|^2 + 2\langle x_1 - x_0, x_0 - p_0\rangle \\
&\leq \left(t_1 - t_0 + \int_{[t_0,t_1]} d\mu_0(t)\right)^2 (\gamma\|x_0\| + c)^2 + d_S^2(x_0) + 2\int_{[t_0,t_1]} \langle dx_\pi(t), x_0 - p_0\rangle \\
&\leq \left(t_1 - t_0 + \int_{[t_0,t_1]} d\mu_0(t)\right)^2 (\gamma\|x_0\| + c)^2 + d_S^2(x_0),
\end{aligned}
$$

where the first inequality is due to the fact that $p_0 \in S$, the second inequality is derived from equation (2.25), the definition of $d_S(x)$ and noting that $x_1 - x_0 = \int_{[t_0,t_1]} dx_\pi(t)$, and the third inequality is obtained by the hypothesis of the proposition plus the fact that $dx_\pi = f(x_0)dt + g(x_0)d\mu_0(t)$. We can follow the same procedure for all nodes of the Euler polygonal arc x_π, obtaining:

$$d_S^2(x_i) \leq \left(t_i - t_{i-1} + \int_{[t_{i-1},t_i]} d\mu_i(t)\right)^2 (\gamma\|x_{i-1}\| + c)^2 + d_S^2(x_{i-1}).$$

Note that, depending on the measure, it is possible that some time nodes t_i have the same value, which imply the singularity of the arc in that segment. With these estimates at hand, we can proceed in order to obtain:

$$\begin{aligned}
d_S^2(x_2) &\leq \left(t_2 - t_1 + \int_{[t_1,t_2]} d\mu_1(t)\right)^2 (\gamma\|x_1\| + c)^2 + d_S^2(x_1) \\
&\leq \left(t_2 - t_1 + \int_{[t_1,t_2]} d\mu_1(t)\right)^2 (\gamma\|x_1\| + c)^2 \\
&\quad + \left(t_1 - t_0 + \int_{[t_0,t_1]} d\mu_0(t)\right)^2 (\gamma\|x_0\| + c)^2 + d_S^2(x_0) \\
&\leq \left[(t_2 - t_1 + \int_{[t_1,t_2]} d\mu_!(t))^2 + (t_1 - t_0 + \int_{[t_0,t_1]} d\mu_0(t))^2\right] (\gamma L + c) + d_S^2(x_0).
\end{aligned}$$

Recall that, by theorem 2.3.1, x_i $\forall i$ has a finite upper bound L. Proceeding iteratively, we derive that:

$$\begin{aligned}
d_S^2(x_i) &\leq (\gamma L + c)^2 . \sum_{l=1}^{i} \left[t_l - t_{l-1} + \int_{[t_{l-1},t_l]} d\mu_{l-1}(t)\right]^2 + d_S^2(x_0) \\
&\leq (\gamma L + c)^2 . \alpha_\pi \sum_{l=1}^{i} \left[t_l - t_{l-1} + \int_{[t_{l-1},t_l]} d\mu_{l-1}(t)\right] + d_S^2(x_0) \\
&\leq (\gamma L + c)^2 . \alpha_\pi (b - a + M) + d_S^2(x_0).
\end{aligned}$$

If we consider the sequence x_{π_j} of polygonal arcs converging to x, then $\alpha_{\pi_j} \downarrow 0$. Thus, we deduce that

$$d_S(x(t)) \leq d_S(x(a^-)) \ \forall t \in [a, b]$$

and, when the trajectory is singular, that

$$d_S(x^s(s)) \leq d_S(x^s(\eta(t_a^-))) \ \forall s \in [\eta(t_a^-), \eta(t_a^+)],$$

where $x(\eta(t_a^-)) = x(t_a^-)$, as required. $\square$

3.3 Invariance for impulsive control systems

In this section, we use the impulsive Euler solution to present conditions for invariance. The proofs of these conditions are constructive in the sense that they provide a procedure to compute feedback selections and feedback control measures, such that the state trajectory have the desired property. This is possible due to the impulsive Euler solution since it allows discontinuous selections and measure dependence on the state variable.

The results assumes the measure to be scalar valued and positive. The extension for vector-valued measures is presented later, with resource of the impulsive Euler solution for vector-valued measures. This approach allows a clearer presentation but without loss of generality.

3.3.1 Weak invariance

An impulsive control system is weakly invariant in relation to a given closed set if there exist impulsive Euler solutions such that they remain in this set. The topic of weak invariance encounters many practical control applications and serves also to study other problems like stability and optimality [14]. In the next chapter, we will have the opportunity to analyze an application in the context of optimal control in detail.

Definition 3.3.1. *The system (S, F, G) is said to be weakly invariant if for all $x_0 \in S$, there are selections of $F(x)$ and $G(x)$ and a measure μ such that the associated trajectory $x(t)$ on $t \in [0, +\infty)$ and $x^s(s)$ on $s \in [\eta(t_a^-), \eta(t_a^+)]$ is such that:*

$$x(0^-) = x_0, \ x(t) \in S \ \ \textit{and} \ x^s(s) \in S,$$

where $x(t)$ stands for the trajectory when the measure is non atomic while $x^s(s)$ stands for the singular arcs at t_a, which represents any point in the support of the singular *atomic component.*

We recall the definition of lower Hamiltonian h associated with a multifunction F, which maps $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ and is given by:

$$h_F(x,p) := \min_{v \in F(x)} \langle p, v \rangle.$$

Before we introduce the conditions characterizing weak invariance, we would like to remark that whenever the null vector belongs to G, then there is an equilibrium point associated with this selection if we apply a singular atomic measure. In this case the state and the time will not evolve and constitutes a type of indetermination. This reasoning opens new research topics related with equilibrium points for measure driven differential inclusions. The detailed study of this subject is, however, out of the scope of this thesis.

Theorem 3.3.1. *For every $x \in S$, suppose we have*

$$\min\{h_F(x,\zeta), h_G(x,\zeta)\} \leq 0, \ \forall\, \zeta \in N_S^P(x).$$

Then (S,F,G) is weakly invariant.

Proof. For each $x \in \mathbb{R}^n$ compute $p = p(x) \in \mathrm{proj}_S(x)$. Clearly, $x - p \in N_S^P(p)$. Then minimize the following functions:

$$v \to \langle v, x-p \rangle \text{ and } w \to \langle w, x-p \rangle,$$

with $v \in F(p)$ and $w \in G(p)$. We define the functions f_P and g_P by setting $f_P(x) = v$ and $g_P(x) = w$. If p_0 is any point in S, then we deduce:

$$\begin{aligned}
\|f_P(x)\| = \|v\| &\leq \gamma\|p\| + c \text{ (Since } F \text{ has linear growth)} \\
&\leq \gamma\|p - x\| + \gamma\|x\| + c \\
&= \gamma d_S(x) + \gamma\|x\| + c \\
&\leq \gamma\|x - p_0\| + \gamma\|x\| + c \\
&\leq 2\gamma\|x\| + \|\gamma p_0\| + c.
\end{aligned}$$

A similar bound can be derived for g_P. Hence, both f_P and g_P satisfy the linear growth condition.

By hypothesis, we have $\min\{\langle f_P(x), x-p\rangle, \langle g_P(x), x-p\rangle\} \leq 0$ for every $x \in \mathbb{R}^n$. Hence, we can compute a measure μ such that any impulsive Euler solution x of

$$dx = f_P(x)dt + g_P(x)d\mu(t), \tag{3.11}$$

on $t \in [0, +\infty)$ and $x(0) = x_0$, satisfies:

$$\langle f_P(x)dt + g_P(x)d\mu(t), x-p\rangle \leq 0 \;\; \mu\text{-a.e. on } t \in [0, +\infty). \tag{3.12}$$

The computation of such measure is state dependent and is computed as follows. Consider that we make a partition π of the interval $[0, 1]$, representing a portion of the range of $\eta(t)$. The measure is constructed at each node point i, with $i = 0, \cdots, N$. If we choose a singular atomic measure (only possible if $\langle g_P(x_i), x_i - p_i\rangle \leq 0$) then $t_{i+1} = t_i$ and $|\mu_i(\{t_i\})| = s_{i+1} - s_i$. Otherwise, if the option is an absolutely continuous or singular continuous measure (always possible except when $\langle f_P(x_i), x_i - p_i\rangle > 0$ and $\langle g_P(x_i), x_i - p_i\rangle = 0$), then we should choose μ_{ac} and μ_{sc} such that (3.12) is verified. We define $w_{ac}(t)$ to be constant along $[t_i, t_{i+1})$ and the measure μ_{sc} is selected in a way that the function $F_{sc}(t) = \int_{[t_i,t]} d\mu_{sc}(t)$ has constant increase rate k_{sc} on $t \in [t_i, t_{i+1})$ such that $F_{sc}(t_{i+1}) = k_{sc}(t_i)(t_{i+1} - t_i)$. In this case, the node point t_{i+1} is computed as follows:

$$\begin{aligned} s_{i+1} - s_i &= t_{i+1} - t_i + \int_{[t_i, t_{i+1}]} (w_{ac}(t)dt + d\mu_{sc}(t)) \\ &= (t_{i+1} - t_i)(1 + w_{ac}(t_i) + k_{sc}(t_i)) \Rightarrow \\ & \quad\; t_{i+1} = t_i + \frac{s_{i+1} - s_i}{1 + w_{ac}(t_i) + k_{sc}(t_i)}. \end{aligned} \tag{3.13}$$

Recall that s_i represents the i^{th} node of partition π. Clearly, x_π and η_π are computed simultaneously. The node points of function $\eta_\pi(t)$ are given by (t_i, s_i), with $i = 0, \cdots, N$ and $\eta_\pi(t) = \eta_\pi(t_i) + t - t_i + \displaystyle\int_{[t_i,t]} d\mu_{\pi,i}(t)$, $t \in [t_i, t_{i+1}]$, where $\eta_\pi(0^-) = 0$

and μ_π being the measure constructed as shown before. Now let π_j be a sequence of partitions such that $\pi_j \downarrow 0$. By construction, the corresponding functions η_{π_j} are of bounded variation and uniformly bounded. Hence, by the Helly's selection principle, we can conclude that there is a subsequence of the family $\{\eta_{\pi_j}\}$ whose limit is a function of bounded variation η. Thus, we conclude that there is a subsequence of $\{\mu_{\pi_j}\}$ such that $\mu_{\pi_j} \to^* \mu$. Simultaneously, as $\pi_j \downarrow 0$, we also obtain an impulsive Euler solution for system (3.11), associated with the measure computed before on $t \in [0, \tilde{t}_1]$, where $\tilde{t}_1 := \{t : \eta(t^+) = 1\}$. Then, we proceed and consider a partition π of the interval $[1, 2]$. Proceeding in a similar way as described before, we compute the measure μ on $t \in [\tilde{t}_1, \tilde{t}_2]$, with $\tilde{t}_2 := \{t : \eta(t^+) = 2\}$. Thus, considering the following intervals $[2, 3]$, $[3, 4]$, ..., we can compute the measure μ satisfying equation (3.12) for all $t \in [0, +\infty)$.

Now set $[a, b] = [0, \tilde{t}_1]$ and apply proposition 3.2.1 to conclude that the resulting impulsive Euler trajectory of $dx = f_P(x)dt + g_P(x)d\mu$, $x(0) = x_0$, on $t \in [0, \tilde{t}_1]$ necessarily lie in S. Note that, by construction, in this interval the total variation of the measure μ is bounded. Making a similar analysis for all the partitions of the type $[\tilde{t}_{k+1}, \tilde{t}_k]$, with $k = 1, \cdots$, we reach to the conclusion that the impulsive Euler solution lies in the set S for $t \in [0, +\infty)$.

At this point, we should note that neither f_P nor g_P are selections of F and G, respectively. Then, we cannot use theorem 2.3.2 to show that the impulsive Euler arc x is a trajectory of system (2.20). Hence, to show the desired result, we must define the multifunctions $F_S(x) := \text{co}\{F(p) : p \in \text{proj}_S(x)\}$ and $G_S(x) := \text{co}\{G(p) : p \in \text{proj}_S(x)\}$. Clearly, F_S and G_S are closed and convex and both satisfy $F_S(x) = F(x)$ and $G_S(x) = G(x)$ if $x \in S$. By construction, we conclude that f_P and g_P are selections of F_S and G_S and, by theorem 2.3.2, we conclude that the impulsive Euler solution x presented before is a trajectory of $dx \in F_S(x)dt + G_S(x)d\mu$, μ-a.e. on $[0, \tilde{t}_1]$. Since $x(t) \in S$ and $x^s(s) \in S$, we reach to the conclusion that x is also a trajectory of $dx \in F(x)dt + G(x)d\mu$, μ-a.e. on $[0, \tilde{t}_1]$, because $F_S = F$ and $G_S = G$ on S. Following the same procedure for all other intervals $[\tilde{t}_{k+1}, \tilde{t}_k]$, with $k = 1, \cdots,$ we conclude that x is, in fact, a trajectory of system (2.20) on $t \in [0, +\infty)$. $\square$

During the feedback construction of the control measure, it can happen that the structure of the problem requires a singular atomic measure with unbounded total variation. Whenever this happens, the trajectory evolves forever while time does not.

The sufficient conditions for weak invariance can be expressed in alternative forms and we can also show that they are necessary conditions too. We explore this issue in detail.

Theorem 3.3.2. *The following statements are equivalent:*

(a) $\exists\, \lambda \in [0,1]$ *such that* $\{\lambda F(x) + (1-\lambda)G(x)\} \cap T_S^B(x) \neq \emptyset\ \forall\ x \in S$.

(b) $\exists\, \lambda \in [0,1]$ *such that* $\{\lambda F(x) + (1-\lambda)G(x)\} \cap coT_S^B(x) \neq \emptyset\ \forall\ x \in S$.

(c) $\min\{h_F(x,\zeta), h_G(x,\zeta)\} \leq 0$, $\forall\ \zeta \in N_S^P(x)$.

(d) (S, F, G) *is weakly invariant.*

(e) $\forall\ x_0 \in S$, $\forall\ \varepsilon > 0$, $\exists\ \delta \in (0, \varepsilon)$ *such that*
$\{x(\delta) : dx(t) \in F(x(t))dt + G(x(t))d\mu(t),\ x(0) = x_0\} \cap S \neq \emptyset$, *whenever there are no atoms of* μ *supported on* $t \in [0, \delta]$ *and*
$\{x^s(\delta) : \dot{x}^s(s) \in G(x^s(s)),\ x^s(0) = x_0\} \cap S \neq \emptyset$, *whenever there is an atom of* μ *supported at* $t = 0$.

Proof. Condition (a) implies (b) since, clearly, $T_S^B(x) \subseteq coT_S^B(x)$. Now, we recall the following relationship [14, Exercise 2.7.1]:

$$coT_S^B(x) \subseteq [N_S^P(x)]^\circ := \{\tilde{v} : \langle \tilde{v}, \zeta \rangle \leq 0,\ \forall \zeta \in N_S^P(x)\}. \tag{3.14}$$

From condition (b), we know that, for every $x \in S$, there is a $\tilde{v}$ that belongs to $\lambda F(x) + (1-\lambda)G(x)$ and satisfies condition (3.14). Hence, there is a $\lambda \in [0,1]$, a $v \in F(x)$ and a $w \in G(x)$, such that $\langle v\lambda + w(1-\lambda), \zeta\rangle \leq 0, \forall\ \zeta \in N_S^P(x)$. Consequently, we obtain:

$$\lambda h_F(x,\zeta) + (1-\lambda)h_G(x,\zeta) \leq 0,\ \forall\, \zeta \in N_S^P(x).$$

Since λ is always nonnegative, the preceding equation implies precisely condition (c) of the theorem, which gives $(b) \Rightarrow (c)$. The fact that $(c) \Rightarrow (d)$ is a consequence of theorem 3.3.1. Condition (e) is an immediate consequence of the definition of weak invariance and, therefore, $(d) \Rightarrow (e)$. Hence, it remains to prove that $(e) \Rightarrow (a)$.

Consider first the case where there is an atom supported at $t = 0$. Then, for a sequence $\delta_i \downarrow 0$, there is a singular trajectory $x_i^s(s)$ on $s \in [0,\delta_i]$ such that $x^s(0) = x_0$ and $x_i(\delta_i) \in S$. From theorem 2.3.1 part b), we know that $\|x_i^s(\delta_i) - x_0\| \leq \delta_i e^{\gamma\delta_i}(\gamma\|x_0\| + c)$. Note that $e^{\gamma\delta_i} \to 1$ as $\delta_i \downarrow 0$. Hence, by taking a subsequence, we can assume that there is some v such that:

$$\frac{x_i^s(\delta_i) - x_0}{\delta_i} \to v.$$

By definition of Bouligand tangent cone T_S^B, we know that $v \in T_S^B(x)$. On the other hand, when there are not atoms of μ supported on $t \in [0,\delta_i]$, then, for a sequence $\delta_i \downarrow 0$, there are trajectories $x_i(t)$ on $t \in [0,\delta_i]$ with $x_i(0) = x_0$ such that $x_i(\delta_i) \in S$ (by condition (e)). Once again, using theorem 2.3.1 part b), we conclude that $\|x_i(\delta_i) - x_0\| \leq \bar{\delta}_i e^{\gamma\bar{\delta}_i}(\gamma\|x_0\| + c)$, with $\bar{\delta}_i$ being defined as $\bar{\delta}_i := \delta_i + \displaystyle\int_{[0,\delta_i]} d\mu(t)$. The quantity $\bar{\delta}_i$ represents the value of $\eta(\delta_i)$ associated with the measure μ and, clearly, if $\delta_i \downarrow 0$ then $\bar{\delta}_i \downarrow 0$. Thus, noting that $e^{\gamma\bar{\delta}_i} \to 1$, as $\bar{\delta}_i \downarrow 0$, we conclude, by taking a subsequence, that:

$$\frac{x_i(\delta_i) - x_0}{\bar{\delta}_i} \to v$$

for some v. Again, we know that $v \in T_S^B$. At this point, we only need to show that $v \in \{\lambda F(x) + (1-\lambda)G(x)\}$ for some $\lambda \in [0,1]$. For the case where we have an atom at $t = 0$, we have:

$$x_i^s(\delta_i) - x_0 = \int_0^{\delta_i} \dot{x}_i^s(s)ds, \tag{3.15}$$

with $\dot{x}_i(s) \in G(x_i(s))$ on $0 \leq s \leq \delta$. Otherwise, when no atoms exist at $t = 0$, we have that:

$$x_i(\delta_i) - x_0 = \int_{[0,\delta_i]} dx_i(t), \tag{3.16}$$

with $dx_i(t) \in F(x_i(t))dt + G(x_i(t))d\mu(t)$. For any given $\Delta > 0$ and i sufficiently large, both sets $\{x_i(t) : 0 \leq t \leq \delta_i\}$ and $\{x_i^s(s) : 0 \leq s \leq \delta_i\}$ lie in $x_0 + \Delta B$. Hence, given any $\varepsilon > 0$, taking Δ small enough will ensure, by the upper semicontinuity of F and G, that $x \in x_0 + \Delta B$ implies $F(x) \subset F(x_0) + \varepsilon B$ and $G(x) \subset G(x_0) + \varepsilon B$. Thus, considering equation (3.16), we can deduce that:

$$\begin{aligned} x_i(\delta_i) - x_0 \quad &\in \quad \int_{[0,\delta_i]} [(F(x_0) + \varepsilon B)dt + (G(x_0) + \varepsilon B)d\mu(t)] = \\ &= \quad \delta_i(F(x_0) + \varepsilon B) + (G(x_0) + \varepsilon B) \int_{[0,\delta_i]} d\mu(t). \end{aligned}$$

Dividing both terms in the preceding inclusion by $\bar{\delta}_i$, we obtain that (since both F and G are convex sets):

$$\frac{x_i(\delta_i) - x_0}{\bar{\delta}_i} \in \frac{\delta_i}{\delta_i + \int_{[0,\delta_i]} d\mu(t)}(F(x_0) + \varepsilon B) + \frac{\int_{[0,\delta_i]} d\mu(t)}{\delta_i + \int_{[0,\delta_i]} d\mu(t)}(G(x_0) + \varepsilon B).$$

Defining $\lambda_i := \dfrac{\delta_i}{\delta_i + \int_{[0,\delta_i]} d\mu(t)}$, we reach to the conclusion that :

$$\frac{x_i(\delta_i) - x_0}{\bar{\delta}_i} \in \lambda_i(F(x_0) + \varepsilon B) + (1 - \lambda_i)(G(x_0) + \varepsilon B).$$

The limit of λ_i as $i \uparrow \infty$ necessarily tends to a value $\lambda \in [0, 1]$, which leads us to:

$$v \in \lambda(F(x_0) + \varepsilon B) + (1 - \lambda)(G(x_0) + \varepsilon B).$$

Since ε is arbitrary, we obtain that $v \in \lambda F(x_0) + (1 - \lambda)G(x_0)$, as required. Now, considering equation (3.15), we can deduce that:

$$\frac{x_i^s(\delta_i) - x_0}{\delta_i} \in G(x_0) + \varepsilon B.$$

Thus, in the case where there is an atom at $t = 0$, we reach to the conclusion that $v \in G(x_0) + \varepsilon$, and, since ε is arbitrary, we have that $v \in G(x_0)$. Therefore, we have verified that $(b) \Rightarrow (a)$, which completes the proof. $\square$

In general, the feedback selections f_P and g_P defined in the proof of the theorem 3.3.1 are not selections of multifunctions F and G, except when $x \in S$. The next result presents an extension allowing the construction of feedback functions that are selections of F and G in the whole state space. In this case, we assume that the multifunctions F and G are Lipschitz continuous. A multifunction $F(x)$ is said to be Lipschitz continuous provided there exists a constant K such that $x_1, x_2 \in \mathbb{R}^n \Rightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}$

Theorem 3.3.3. *Let (S, F, G) be weakly invariant, where F and G are assumed to be locally Lipschitz. Then, there are feedback selections $\bar{f}_P \in F$, $\bar{g}_P \in G$ and a control measure μ, defined in subsets $[a, b] \subset [0, +\infty)$, with $|\mu([a, b])|$ being bounded, such that any impulsive Euler trajectory $x(t), x^s(s)$ of:*

$$dx = \bar{f}_P(x)dt + \bar{g}_P(x)d\mu(t),\ t \in [a, b],$$

with $x(a^-) \in S$, lies in the set S $\forall\, t \in [a, b]$ and $\forall s \in [\eta(t_a^-), \eta(t_a^+)]$.

Remark 3.3.1. *The existence of subsets $[a, b]$ such that $|\mu([a, b])|$ is bounded is ensured if we follow the construction mentioned in the proof of theorem 3.3.1. We only need to identify $[a, b]$ with $[\tilde{t}_k, \tilde{t}_{k+1}]$.*

Proof. Let f_P, g_P and μ be computed as in the proof of theorem 3.3.1. In general, when $x \notin S$, the selections $f_P(x)$ and $g_P(x)$ does not belong to $F(x)$ and $G(x)$, respectively. We define $\bar{f}_P(x)$ and $\bar{g}_P(x)$ to be the points in $F(x)$ and $G(x)$, respectively, closest to $f_P(x)$ and $g_P(x)$ such that they are selections of F and G. The measure μ is computed in the same manner as done in the proof of theorem 3.3.1.

Now, we need to show that, whenever $x_0 \in S$, any Euler solution $\bar{x}$ on $[a,b]$ generated by $d\bar{x} = \bar{f}_P(\bar{x})dt + \bar{g}_P(\bar{x})d\mu(t),\ x(a^-) = x_0$, is such that $\bar{x}(t) \in S\ \forall t \in [a,b]$ and $\bar{x}^s(s) \in S\ \forall s \in [\eta(t_a^-), \eta(t_a^+)]$. That $\bar{x}$ is a trajectory of system $d\bar{x} \in F(\bar{x})dt + G(\bar{x})d\mu(t),\ t \in [a,b]$ is an immediate consequence of theorem 2.3.2. From theorem 2.3.1, we know that there is a bound for $\bar{x}$ on $t \in [a,b]$. Hence, we may assume that $\|\bar{x}(t) - x_0\| < L$ and $\|\bar{x}^s(s) - x_0\| < L$. Let K_F and K_G be the Lipschitz ranks of F and G, respectively, on the set $x_0 + 2LB$. Thus, if $\|x - x_0\| < L$, then we obtain (recall that $d_S(x)$ is Lipschitz continuous with rank equal to one):

$$\|p - x_0\| \leq \|p - x\| + \|x - x_0\| = d_S(x) + \|x - x_0\| \leq 2\|x - x_0\| < 2L,$$

so that $p \in x_0 + 2LB$. Consequently, we deduce that:

$$\begin{aligned}
\langle \bar{f}_P(x)dt + \bar{g}_P(x)d\mu, x - p\rangle &= \left(\langle f_P(x), x - p\rangle + \langle \bar{f}_P(x) - f_P(x), x - p\rangle\right) dt + \\
&\quad \left(\langle g_P(x), x - p\rangle + \langle \bar{g}_P(x) - g_P(x), x - p\rangle\right) d\mu \\
&\leq \|\bar{f}_P(x) - f_P(x)\|.\|x - p\|dt + \|\bar{g}_P(x) - g_P(x)\|.\|x - p\|d\mu \\
&\leq K_F\|x - p\|^2 dt + K_G\|x - p\|^2 d\mu \\
&= K_F.d_S^2(x)dt + K_G.d_S^2(x)d\mu,
\end{aligned}$$

where the first inequality is obtained with resource of equation (3.12) and the second inequality is obtained by the Lipschitz continuity of F and G. With this estimative at hand, we can proceed in a similar manner as in the proof of proposition 3.2.1 and compute the following relationship between node points of an Euler polygonal arc $\bar{x}_\pi$, in the sequence converging to the Euler solution $\bar{x}$:

$$\begin{aligned}
d_S^2(\bar{x}_i) &\leq (\gamma L + c)^2 . \sum_{l=1}^{i} \left[t_l - t_{l-1} + \int_{[t_{l-1}, t_l]} d\mu_{l-1}(t) \right]^2 + d_S^2(\bar{x}_0) \\
&\quad +2 \sum_{l=1}^{i} \left[\int_{[t_{l-1}, t_l]} \big(K_F . d_S^2(\bar{x}_{l-1}) dt + K_G . d_S^2(\bar{x}_{l-1}) d\mu_{l-1} \big) \right] \\
&\leq (\gamma L + c)^2 . \alpha_\pi (b - a + M) + d_S^2(\bar{x}_0) \\
&\quad +2 \sum_{l=1}^{i} \left[\int_{[t_{l-1}, t_l]} \big(K_F . d_S^2(\bar{x}_{l-1}) dt + K_G . d_S^2(\bar{x}_{l-1}) d\mu_{l-1} \big) \right] .
\end{aligned}$$

If we consider the sequence $\bar{x}_{\pi_j}$ of polygonal arcs converging to $\bar{x}$, then $\alpha_{\pi_j} \downarrow 0$ and we can deduce that:

$$d_S^2(\bar{x}(t)) \leq d_S^2(\bar{x}(a^-)) + 2 \int_{[a,t]} (K_F . d_S^2(\bar{x}(t)) dt + K_G . d_S^2(\bar{x}(t)) d\mu), \ \forall t \in [a, b]$$

and, for every singular arc, that:

$$d_S^2(\bar{x}^s(s)) \leq d_S^2(\bar{x}^s(\eta(t^-))) + 2 \int_{\eta(t_a^-)}^{s} K_G . d_S^2(\bar{x}^s(s)) ds, \ \forall s \in [\eta(t_a^-), \eta(t_a^+)].$$

Here, t_a represents a point in the support of the singular atomic component with $\bar{x}^s(\eta(t^-)) = \bar{x}(t_a^-)$. Since the distance function is such that $d_S(x) \geq 0 \ \forall \ x \in \mathbb{R}^n$, we conclude from the preceding inequalities that $d_S(\bar{x}(t)) = 0$ and $d_S(\bar{x}^s(s)) = 0$ $\forall t \in [a, b]$ and $\forall s \in [\eta(t_a^-), \eta(t_a^+)]$, which, in turn, gives the desired result. □

The next corollary states that any trajectory of the impulsive system (2.20) can be generated by a feedback selection, as described in the proof of theorem 3.3.3.

Corollary 3.3.1. *A given impulsive arc $\bar{x}$ defined on $t \in [a, b]$ is a trajectory of system (2.20) if and only if there are feedback selections f, g of F, G (possibly not autonomous) and a feedback measure μ such that $\bar{x}$ is an impulsive Euler solution for the initial-value problem $dx = f(t, x)dt + g(t, x)d\mu(t), x(a^-) = \bar{x}(a^-)$.*

Proof. The "if" part of the proof is given by theorem 2.3.2. Then, we only need to prove that for any given impulsive trajectory $\bar{x}$ there is always an impulsive Euler solution generating it. For that purpose let us define the closed set T:

$$T := \{(t, \bar{x}(t)) \cup (t, \bar{x}^s(s)) : \ a \leq t \leq b, \ s \in [\eta(t_a^-), \eta(t_a^+)]\},$$

with t_a being a point in the support of the singular atomic measure. By construction, this set is weakly invariant associated with the augmented system:

$$\begin{bmatrix} dt \\ dx \end{bmatrix} \in \begin{bmatrix} \{1\} \\ F(x) \end{bmatrix} d\tau + \begin{bmatrix} \{0\} \\ G(x) \end{bmatrix} d\mu(\tau),$$

where t is viewed as another state variable. By theorem 3.3.3, we can conclude that there are feedback selections of the form $(1, f(t,x))$ and $(0, g(t,x))$ and a measure μ such that any Euler solution of system $dx = f(t,x)dt + g(t,x)d\mu, \ x(a) = \bar{x}(a)$, satisfies $(t, x(t)) \cup (t, x^s(s)) \in T$ on $t \in [a,b]$ and on $s \in [\eta(t_a^-, \eta(t_a^+))]$. But this implies that $x(t) = \bar{x}(t)$ and $x^s(s) = \bar{x}^s(s)$, which means that $\bar{x}$ can always be generated by an impulsive Euler solution. $\square$

From theorem 3.3.1 and 3.3.3 we deduce the following algorithm to compute feedback control measures and selections such that the resulting impulsive Euler trajectories is invariant in relation to a closed set S:

Algorithm 3.3.1. *Impulsive feedback control for Weak Invariance:*

1. *For each $x \in R^n$, compute $p := p(x) \in proj_S(x)$.*

2. *For each $x \in R^n$, minimize the functions: $v \to \langle v, x - p\rangle$ and $w \to \langle w, x - p\rangle$, with $v \in F(p)$ and $w \in G(p)$. Set $f_P(x) = v$ and $g_P(x) = w$.*

3. *Compute a measure μ such that $\langle f_P(x), x - p\rangle dt + \langle g_P(x), x - p\rangle d\mu \leq 0$ μ-a.e. on $t \in [0, 1]$, as detailed in theorem 3.3.1.*

4. *Choose* $\bar{f}_P(x)$ *and* $\bar{g}_P(x)$ *to be the points in* $F(x)$ *and* $G(x)$ *closest to* $f_P(x)$ *and* $g_P(x)$*, respectively.*

5. *Compute the impulsive Euler solution of* $dx = \bar{f}_P(x)dt + \bar{g}_P(x)d\mu(t)$ *with* $x(0) \in S$ *on* $t \in [0, +\infty)$*. The impulsive Euler solution computed this way will be always inside the set* S*.*

3.3.2 Strong invariance

Until now, we were concerned in finding at least one trajectory of the impulsive control system such that it remained inside a given closed set. Now, in the context of strong invariance, we require that all impulsive trajectories remain inside a given closed set.

Definition 3.3.2. *The system* (S, F, G) *is called strongly invariant if every trajectory of* $dx \in F(x)dt + G(x)d\mu(t)$*, on* $t \in [0, +\infty)$ *and* $x(0^-) \in S$*, is such that* $x(t) \in S$ *for all* $t \in [0, +\infty)$ *and* $x^s(s) \in S$ *for* $s \in [\eta(t_a^-), \eta(t_a^+)]$*, where* t_a *is a point in the support of the singular atomic component.*

Before proceed and present necessary and sufficient conditions for strong invariance, we recall the definition of upper Hamiltonian H_F associated with a multifunction F:

$$H_F(x,p) := \max_{v \in F(x)} \langle p, v \rangle.$$

Theorem 3.3.4. *Let the multifunctions* F *and* G *be locally Lipschitz.* (S, F, G) *is strongly invariant if and only if:*

$$\max\{H_F(x,\zeta), H_G(x,\zeta)\} \leq 0, \; \forall\, \zeta \in N_S^P(x), \; \forall x \in S. \tag{3.17}$$

Proof. We start by proving that the condition is sufficient. Let $\bar{x}$ be any trajectory of the impulsive system $dx \in F(x)dt + G(x)d\mu(t)$ on $[a, b]$. From corollary 3.3.1, we already know that there are feedback selections f, g of F, G and a feedback measure

μ such that $\bar{x}$ is an Euler solution of the initial value problem $dx = f(t,x)dt + g(t,x)d\mu(t)$, $\bar{x}(a) = x_0$. From theorem 2.3.1, we also know that $\|\bar{x}(t) - x_0\| < L$ and $\|\bar{x}^s(s) - x_0\| < L$ (we are assuming an interval $[a,b]$ such that $|\mu([a,b])|$ is bounded). If x lies in the set $x_0 + LB$ and $p \in \text{proj}_S(x)$, then we can conclude that:

$$\|p - x_0\| \le \|p - x\| + \|x - x_0\| = d_S(x) + \|x - x_0\| \le 2\|x - x_0\|,$$

so that $p \in x_0 + 2LB$. Let K_F and K_G be the Lipschitz constants of F and G on $x_0 + 2L$, respectively, and consider any $x \in x_0 + 2LB$ and $p \in \text{proj}_S(x)$. Clearly, we have $x - p \in N_S^P(p)$. Since $f(t,x) \in F(x)$ and $g(t,x) \in G(x)$, then there exist $v \in F(p)$ and $w \in F(p)$ such that

$$\|v - f(t,x)\| \le K_F\|p - x\| = K_F d_S(x) \text{ and } \|w - g(t,x)\| \le K_G d_S(x).$$

By hypothesis, we know that $\langle v, x-p\rangle \le 0$ and $\langle w, x-p\rangle \le 0$, which implies:

$$\begin{aligned}
&\langle f(t,x)dt + g(t,x)d\mu, x-p\rangle \\
= \;& \left(\langle v, x-p\rangle + \langle f(t,x) - v, x-p\rangle\right) dt + \left(\langle w, x-p\rangle + \langle g(t,x) - w, x-p\rangle\right) d\mu \\
\le \;& \|f(t,x) - v\|.\|x-p\|dt + \|g(t,x) - w\|.\|x-p\|d\mu \\
\le \;& K_F\|x-p\|^2 dt + K_G\|x-p\|^2 d\mu = K_F.d_S^2(x)dt + K_G.d_S^2(x)d\mu.
\end{aligned}$$

As noted in the proof of theorem 3.3.3, this last equation implies that:

$$d_S^2(\bar{x}(t)) \le d_S^2(\bar{x}(a^-)) + 2\int_{[a,t]} (K_F.d_S^2(\bar{x}(t))dt + K_G.d_S^2(\bar{x}(t))d\mu) \;\forall t \in [a,b]$$

and, for every singular arc, that:

$$d_S^2(\bar{x}^s(s)) \le d_S^2(\bar{x}^s(\eta(t^-))) + \int_{\eta(t_a^-)}^{s} K_G.d_S^2(\bar{x}^s(s))ds \;\forall s \in [\eta(t_a^-), \eta(t_a^+)],$$

By hypothesis, we know that $d_S^2(\bar{x}(a^-)) = 0$. Again, following the same argument as in the proof of theorem 3.3.3, we deduce that $\bar{x}(t) \in S$ for $t \in [a,b]$ and $\bar{x}^s(s) \in S$

for $s \in [\eta(t_a^-), \eta(t_a^+)]$. This, together with the arbitrary choice of $\bar{x}$, leads us to the conclusion that (S, F, G) is strongly invariant.

Now, we turn to show that the condition is also necessary. Thus, consider any $\tilde{x} \in S$ and let $\tilde{v} \in F(\tilde{x})$ and $\tilde{w} \in G(\tilde{x})$ be given. Set $\tilde{F}(x) = \{\tilde{f}(x)\}$ and $\tilde{G}(x) = \{\tilde{g}(x)\}$ where $\tilde{f}(x)$ and $\tilde{g}(x)$ are the closest points in $F(x)$ and $G(x)$ to $\tilde{v}$ and $\tilde{w}$, respectively. Note that $\tilde{f}(\tilde{x}) = \tilde{v}$ and $\tilde{g}(\tilde{x}) = \tilde{w}$. Moreover, $\tilde{f}$ and $\tilde{g}$ are continuous selections of F, G (exercise 4.3.3 b) in [14]). Hence, $\tilde{F}$ and $\tilde{G}$ satisfies the same hypotheses as F and G (except Lipschitz continuity). This means that (S, F, G) being strongly invariant implies that $(S, \tilde{F}, \tilde{G})$ is also strongly invariant and, consequently, weakly invariant. Therefore, by theorem 3.3.1, which does not require Lipschitz continuity, we conclude that $\min\{h_{\tilde{F}}(\tilde{x}, \tilde{\zeta}), h_{\tilde{G}}(\tilde{x}, \tilde{\zeta})\} \leq 0,\ \forall\, \tilde{\zeta} \in N_S^P(\tilde{x})$, which, by the definition of $\tilde{F}$ and $\tilde{G}$, is the same as $\min\{\langle \tilde{v}, \tilde{\zeta}\rangle, \langle \tilde{w}, \tilde{\zeta}\rangle\} \leq 0,\ \forall\, \tilde{\zeta} \in N_S^P(\tilde{x})$. Since by construction $\tilde{v}$ and $\tilde{w}$ are arbitrary, we conclude that condition of equation (3.17) holds. $\square$

3.3.3 Extension to the vector-valued measure case

We have presented the invariance results under the assumption that the measure is scalar-valued. This was useful to focus on the essential features of the results and to turn the presentation lighter in terms of notation. However, there is no loss of generality in this assumption. With the impulsive Euler solution and its properties, the task of extending the invariance results for this case is possible. The major issue is related with the non-uniqueness of the solution in the absence of the commutativity assumption at the atoms of the measure. In this sense, the auxiliary variable v should also be specified at the atoms of the measure.

Theorem 3.3.5. *For every $x \in S$, suppose we have*

$$\min\{h_F(x, \zeta), h_{G_1}(x, \zeta), \cdots, h_{G_q}(x, \zeta)\} \leq 0,\ \forall\, \zeta \in N_S^P(x),$$

where G_k, $k = 1, \cdots, q$ represents the k^{th} column of multifunction G. Then (S, F, G) is weakly invariant.

Proof. Here, we only focus on the differences relatively to the proof of theorem 3.3.1. We construct functions f_P, $g_{P,k}$, $k = 1, \cdots, q$ in a similar way as done in theorem 3.3.1. Then, we only need to compute a vector-valued measure such that:

$$\langle f_P(x)dt + g_{P,1}(x)d\mu^1(t) + \cdots + g_{P,q}(x)d\mu^q(t), x - p\rangle \leq 0 \tag{3.18}$$

μ-a.e. on $t \in [0, +\infty)$. The construction of such measure follows the same lines as in the proof of theorem 3.3.1 but, here, we should construct the function η associated with the total variation measure. Hence, if we opt by a singular atomic component of the measure, we should specify what components contribute to the total variation of the measure by specifying the quantity $v(s)$ such that $\sum_{k=1}^{q} v^k(s) = 1$. Whenever we choose an absolutely or singular continuous component, we compute the node points t_i (the partition in the original time frame) as follows:

$$t_{i+1} = t_i + \frac{s_{i+1} - s_i}{1 + \sum_{k=1}^{q} w_{ac}^k(t_i) + \sum_{k=1}^{q} k_{sc}^k(t_i)}.$$

Recall that s_i represents the node points in the range of function η associated with the total variation measure. The rest of the proof is a straightforward extension of theorem 3.3.1. □

3.4 Example

We consider a pendulum controlled by changing its radius r [12]. In figure 3.2 is sketched the mechanical system. The state variables are the displacement angle θ and the angular velocity ω. The kinetic energy of this system is given by:

$$T(r, \theta, \omega, \dot{r}) = \frac{M}{2}(\dot{r}^2 + r^2\dot{\theta}^2), \tag{3.19}$$

where M represents the mass of the pendulum. We are assuming the mass of the bar as being negligible. On the other hand, the potential energy is given by:

$$V(r, \theta) = -Mgr\cos(\theta). \tag{3.20}$$

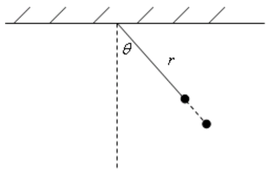

Figure 3.2: Pendulum controlled by changing its length

The radius is controlled by the control variable $m \in [0, +\infty)$ such that $\dot{r}(t) = m(t)$. Consequently, the radius can be controlled very fast in relation to the dynamic response of the pendulum. The dynamic of variable θ is obtained by computing the Lagrangian $L = T - V$ and applying the Euler-Lagrangian method:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}. \tag{3.21}$$

From the previous equation, we may deduce:

$$2Mr\dot{\theta}\dot{r} + Mr^2\ddot{\theta} = -Mgr\sin(\theta). \tag{3.22}$$

Then, we derive that:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g\sin(\theta)}{r} - \frac{2\omega}{r}m \\ \dot{r} &= m. \end{aligned} \tag{3.23}$$

For simplicity, we assume that $\sin(\theta) \simeq \theta$ and $g = 1$. If we consider the ideal case when the radius can be controlled by a measure μ, then we have the following model:

$$\begin{aligned} \dot{\theta} &= \omega \\ d\omega &= -\frac{\theta}{r}dt - \frac{2\omega}{r}d\mu(t) \\ dr &= d\mu(t). \end{aligned} \tag{3.24}$$

The measure is assumed to be positive valued and the initial condition for r is always equal to one. This means that the radius can only be increased, and, consequently r is always greater or equal to one.

Next, we consider the problem of verifying if this dynamic system is weakly invariant in relation to a given set. To address this problem, we will use theorem 3.3.1. Then, after verifying the weak invariance property, we should compute feedback selection and a feedback measure, leading the system weak invariant. Again, for this purpose, we will make use of the constructive proof of theorem 3.3.1 together with theorem 3.3.3. In the context of this example we define $x = (\theta, \omega, r)$. Let the set S be defined as in figure 4.1, where the lateral extremities are segments of ellipses with equation:

$$\frac{\theta^2}{2} + \omega^2 = 1. \tag{3.25}$$

We assume no constraints on $r(t)$.

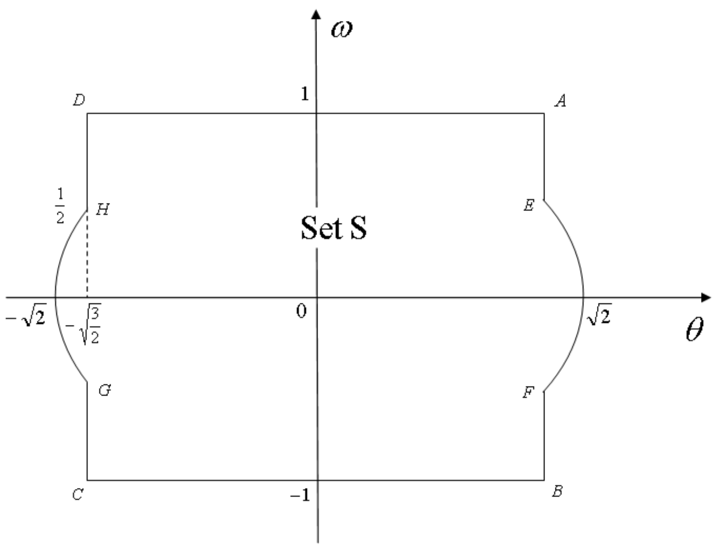

Figure 3.3: The set S

We start by showing that the system is weakly invariant. In first place, we need to compute the proximal normal cone $N_S^P(x)$ and then verify if the condition of theorem

3.3.1 is satisfied for all $x \in S$. If so, the system will be weakly invariant. The proximal normal cone is computed for each $x \in S$, as follows:

case a) For all interior points of S we clearly have $N_S^P(x) = 0$;

case b) If $\omega = 1$ or $\omega = -1$ and $|\theta| < \sqrt{\frac{3}{2}}$ then $N_S^P(x) = \lambda(0,1,0)$, with $\lambda > 0$ or $\lambda < 0$, respectively;

case c) If $\theta = \sqrt{\frac{3}{2}}$ or $\theta = -\sqrt{\frac{3}{2}}$ and $\frac{1}{2} < |\omega| < 1$ then $N_S^P(x) = \lambda(1,0,0)$, with $\lambda > 0$ or $\lambda < 0$, respectively;

case d) If $x = A$ or $x = C$ then $N_S^P(x) = \lambda(\gamma, 1-\gamma, 0)$, with $\gamma \in [0,1]$ and $\lambda > 0$ or $\lambda < 0$, respectively;

case e) If $x = B$ or $x = D$ then $N_S^P(x) = \lambda(\gamma, -1+\gamma, 0)$, with $\gamma \in [0,1]$ and $\lambda > 0$ or $\lambda < 0$, respectively;

case f) It x is over the segments of ellipse represented in figure 4.1 then the proximal normal cone coincides with the classical normal vector, which is given by:

$$N_S^P(x) = \lambda(\theta, 2\omega, 0),\ \lambda > 0; \tag{3.26}$$

case g At points E, F, G and H the proximal normal cone reduces trivially to the set $\{0\}$ (there are no points outside S such that the closest point is E, F, G or H).

Now, we need to verify if the condition

$$\min\{h_F(x,\zeta), h_G(x,\zeta)\} \leq 0,\ \forall\, \zeta \in N_S^P(x) \tag{3.27}$$

is satisfied in order to conclude about the weak invariance of (S, F, G) (theorem 3.3.1). In this example, the multifunctions F and G are single-valued and are given by:

$$F(x) = \left\{[1, -\frac{\theta}{r}, 0]^T\right\} \text{ and } G(x) = \left\{[0, -\frac{2\omega}{r}, 1]^T\right\}.$$

For the cases $a)$ and $g)$ we know that $N_S^P = 0$ and the condition (3.27) is trivially satisfied. For the case $b)$ we have that:

$$h_G(x, N_S^P) = \langle (0, -\frac{2\omega}{r}, 1), \lambda(0,1,0) \rangle = 0.$$

On the other hand, the lower Hamiltonian associated with F is given by:

$$h_F(x, N_S^P) = \langle (1, -\frac{\theta}{r}, 0), \lambda(0,1,0) \rangle = -\lambda \frac{\theta}{r}.$$

By a straightforward analysis, we conclude that when $\theta < 0$ and $\omega = 1$ or $\theta > 0$ and $\omega = -1$ we have $h_F(x, N_S^P) > 0$. For the other cases, the lower Hamiltonian associated with F is less or equal to zero. Hence, for this case the condition (3.27) is also satisfied. Making a similar analysis for the other cases we could conclude that condition (3.27) is always satisfied and, consequently, (S, F, G) is weakly invariant.

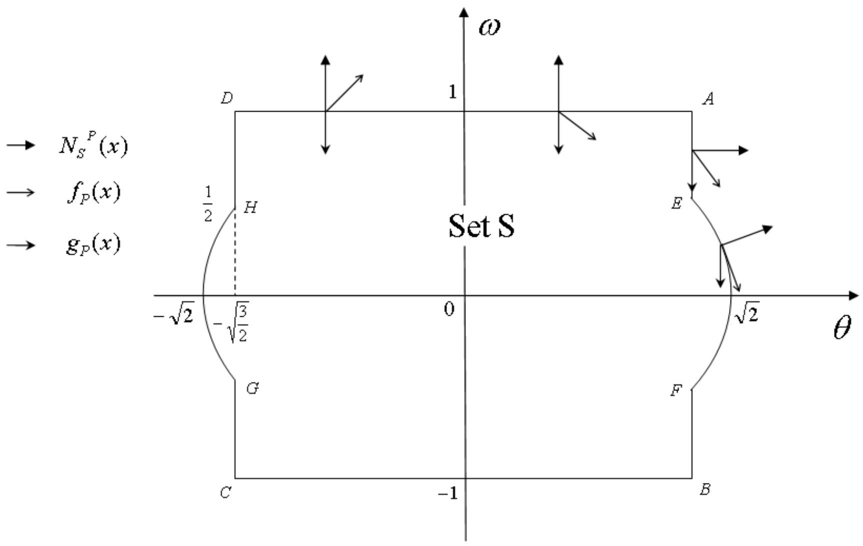

Figure 3.4: Representation of some proximal normal cones together with functions f_P and g_P

Now that we already know this system is weakly invariant, we are able to compute the feedback control following the algorithm 3.3.1.

- In first place we should compute the projection in the set S of any point in $x \in \mathbb{R}^3$. Clearly, if $x \in S$ the projection coincides if x. For the other points we simply compute the point in S giving the shortest distance to x.

- In this problem the multifunctions F and G are single valued. Thus, $f_P(x)$ and $g_P(x)$ are simply given by $f_P(x) = \left[1, -\frac{p_\theta}{p_r}, 0\right]^T$ and $g_P(x) = \left[0, -\frac{2p_\omega}{p_r}, 1\right]^T$, where p_θ, p_ω and p_r represent the components of the projection vector $p(\theta, \omega, r)$ on set S.

- We should compute a measure μ such that $\langle f_P(x), x-p\rangle dt + \langle g_P(x), x-p\rangle d\mu \leq 0$ μ-a.e., as detailed in theorem 3.3.1. Clearly, if $x \in S$ we can apply any control measure since the previous condition holds trivially. For any $x \in \mathbb{R}^3$ such that its projection lies within the point D (depicted in figure 3.4) and $(0, 1, r)$ $\forall r$, then the measure should nonzero, since $\langle f_P(x), x - p\rangle > 0$, and can either be singular atomic, absolutely continuous or singular continuous. If the measure is singular atomic, we clearly have that $\langle g_P(x), x - p\rangle \leq 0$ (see figure 3.4). If we choose the measure to be absolutely continuous, then w_{ac} should be set as:

$$w_{ac} \geq \frac{\langle f_P(x), x - p\rangle}{\langle g_P(x), x - p\rangle}.$$

  For $x \in \mathbb{R}^3$ such that its projection in S is in the segment within points A and E, depicted in figure 3.4, then the only option is to apply a singular atomic measure since $\langle f_P, x - p\rangle > 0$ and $\langle g_P, x - p\rangle = 0$. For the remaining points in $\mathbb{R}^3$, we should proceed in a similar manner and, this way, we obtain a complete feedback synthesis for the control measure.

- The computation of functions $\bar{f}_P$ and $\bar{g}_P$, which are defined in theorem 3.3.3, is a simple task since multifunctions F and G are single-valued. Hence, $\bar{f}_P(x) = \left[1, -\frac{\theta}{r}, 0\right]^T$ and $\bar{g}_P(x) = \left[0, -\frac{2\omega}{r}, 1\right]^T$.

- Finally, for any given initial condition $x(0) \in S$, we compute the impulsive Euler solution of $dx = \bar{f}_P(x)dt + \bar{g}_P(x)d\mu(t)$, $t \in [0, +\infty)$. This solution leads the system invariant, as required.

The preceding procedure gave us a complete feedback synthesis both for selections and for the control measure. For illustration purposes, we sketch in figure 3.5 some Euler polygonal trajectories of $dx = f_P(x)dt + g_P(x)d\mu$, whenever the initial conditions are given by $x(0^-) = (\sqrt{\frac{3}{2}}, 1, 1)$ and $x(0^-) = \left(-\sqrt{\frac{3}{2}}, \frac{3}{4}, 1\right)$.

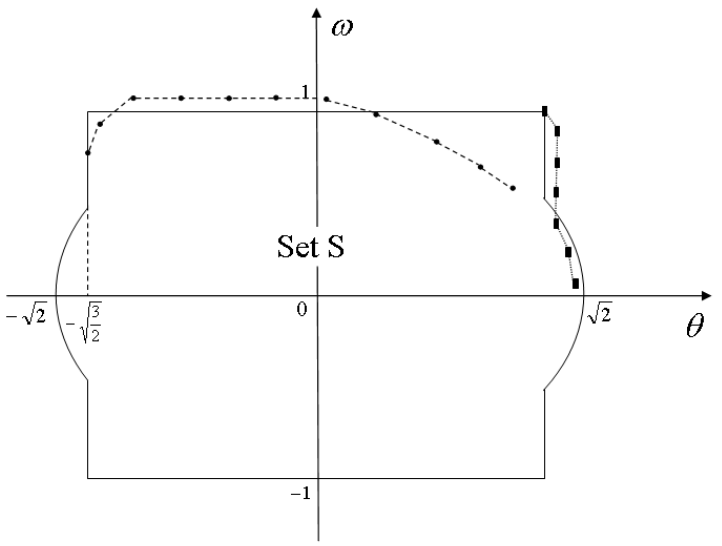

Figure 3.5: Representation of some Euler arcs with two different initial conditions

Note that small portions of the impulsive Euler polygonal arc can lie outside the set S. This is due to a known limitation of the impulsive Euler solution, resulting from the fact that the control measure and the feedback selections are arbitrary when $x \in S$. However, as we take smaller diameters of partition π on the range of η the resulting arc gets closer to the set S. The impulsive Euler solution is obtained as $\pi \downarrow 0$ and, in the limit, the trajectory will be within the set S.

At this point, we should refer that if a square like set S was chosen the system

(S, F, G) would be invariant but, in some circumstances, the measure should be singular atomic with infinite bounded variation. Imagine, for instance, the initial condition is such that $x(0^-) \equiv A$ (see figure 3.4). As we observed in the feedback synthesis construction, the measure should be set to be singular atomic. In this case, the singular dynamics is given by:

$$\dot{\omega}^s = -\frac{2\omega^s}{r^s}, \tag{3.28}$$

$$\dot{r^s} = 1, \tag{3.29}$$

whose solution is:

$$\omega^s(s) = \frac{\omega^s(0)}{s+1},\ r^s(s) = s + 1. \tag{3.30}$$

Since no conventional trajectory lets the system invariant, then the only solution is to apply a singular atomic measure. However, $w(s) \to 0$ as $s \to +\infty$. Hence, a singular atomic measure with unbounded total variation is required. Formally, this situation means that the trajectory keeps involving while time is froze forever.

In practice, we do not have control measures available. The measure driven control systems represent an idealization which is well suited for analysis purposes. In this sense, whenever the control synthesis mechanism requires a singular atomic measure, in practice, this means that we should use the highest value possible for the variable $w_{ac}(t)$. Consequently, we never have a situation where the time is froze but, instead, a control mechanism demanding a fast sampling scheme and high control values. Clearly, the trajectory approximating the ideal impulsive solution may lie outside the set S. However, we know that as we tend to the ideal situation, where we have measures as controls, the trajectories tend to the set S. There is an analogy of this practical meaning for conventional systems as well. Recall that, in conventional systems, we need to sample infinitely fast to get the trajectory inside the set S. Clearly, sampling infinitely fast is unreachable in practice. In this sense, the conventional Euler solution is also a useful analysis tool, corresponding to a limit situation of a practical implementation.

Chapter 4

The Hamilton-Jacobi equation for impulsive control systems

The purpose of this chapter is to study the following optimal control problem:

$$(P) \quad \min l(x(b)) \text{ subject to}$$
$$dx(t) \in F(x(t))dt + G(x(t))d\mu(t),\ t \in [a,b],\ x(a) = x_0,$$

where the function l is assumed to be continuous and, without loss of generality, we only consider scalar measures. Multifunctions F and G satisfy hypothesis $(h1)$, $(h2)$ and $(h3)$ presented in section 2.3 and, additionally, they are assumed to be Lipschitz continuous. An extension for vector-valued measures would be straightforward using the impulsive Euler solution presented in chapter 2.

Instead of determining the optimal decisions from a fixed state of the system (open loop approach), we are interested in deriving the optimal control synthesis, which allows the determination of the optimal decisions from any state of the system (feedback approach). This is the approach suggested by the seminal work of Richard Bellman.

We provide conditions characterizing verification functions and value functions associated with problem (P). A verification function allows to check if a putative minimizer is, in fact, a solution of problem (P), while the value function gives the optimal value of the problem for given initial conditions. Suppose that, instead of having problem (P), we have problem $P(\tau,\alpha)$, whose initial conditions are given by $x(\tau^-) = \alpha$, with $(\tau,\alpha) \in [a,b] \times \mathbb{R}^n$. The value function $V(\tau,\alpha)$ associated with problem $P(\tau,\alpha)$ is the infimum of $l(x(b))$ when the initial conditions are given by (τ,α). The superscript $(-)$ in τ means that we can apply a singular atomic control measure and provoke a singular trajectory at τ. We are not considering any constraint on the total variation of the measure, implying that the value function of problem (P) is independent of the total variation of the measure. A different approach was taken in [30, 7] where the total variation of the measure has to satisfy a constraint. For this problem it is known that the value function $V(t,x)$ is continuous both in t and x [7, 20].

The link between a proximal Hamilton-Jacobi-Bellman equation and the value function V is not obtained by studying a partial differential equation directly but, instead, by using results derived in the context of invariance. This reasoning was applied for "conventional" systems in [14]. For this purpose, we need to present the functional counterpart of invariance of impulsive system, which is precisely what is addressed in the section 4.2.

For the sake of completeness and to situate our contributions, we provide, in the next section, a survey of the theory developed for the nonsmooth theory of Hamilton-Jacobi equation in the context of conventional systems (for a complete overview see [16, 14, 44]).

4.1 Survey of nonsmooth theory of Hamilton-Jacobi equation

Consider the following optimal control problem:

$$\begin{aligned}&(P_c) \quad \min l(x(b)) \text{ subject to}\\&\dot{x}(t) \in F(x(t)),\ t \in [a,b],\ x(a) = x_0,\end{aligned}$$

where the multifunction F satisfies the hypotheses $(h1)$, $(h2)$ and $(h3)$ presented in section 2.3 and, additionally, is assumed to be a Lipschitz continuous. The function l is assumed to be continuous. Note that if we allow the cost function to take the value $+\infty$, then in this formulation it is implicit the endpoint constraint $x(b) \in C$, where $C := \{x \in \mathbb{R}^n : l(x) < +\infty\}$.

Dynamic programming relates the solutions of this problem with the solutions of the Hamilton-Jacobi equation:

$$\begin{aligned}\varphi_t(t,x) + h(x, \varphi_x(t,x)) = 0,&\ \text{for all } (t,x) \in D\\ \varphi(b,x) = l(x),&\ \text{for all } x \in D_1\ ,\end{aligned}$$

where D and D_1 are given subsets of $[a,b] \times \mathbb{R}^n$ and $\mathbb{R}^n$, respectively. The definition of function h is present in section 3.3.1.

The link between the optimal control problem (P_c) and the Hamilton-Jacobi equation is the value function $V : [a,b] \to \mathbb{R}$. For this conventional optimal control problem, the value function $V(t,x)$ is defined, for each $(t,x) \in [a,b] \times \mathbb{R}^n$, to be the infimum cost of the problem:

$$\begin{aligned}&(P_{t,x}) \quad \min l(y(b)) \text{ subject to}\\&\dot{y}(s) \in F(y(s)),\ s \in [t,b],\ y(t) = x.\end{aligned}$$

Then, the value function can be written as $V(t,x) = \inf(P_{t,x})$. From the classical theory, we know that if V is a C^1 function then V is a solution of the Hamilton-Jacobi equation when $D = (a, b) \times \mathbb{R}^n$ and $D_1 = \mathbb{R}^n$. Clearly, $V(a, x_0)$ gives us the minimum cost of problem (P_c), but, in some circumstances, it enables the determination of minimizers too. Thus, let us assume that V is a C^1 function and let define, for each (t, x), the following function:

$$\chi(t, x) := \{v \in F(x) : V_x(t, x).v = h(x, V_x(t, x))\}.$$

If we assume that this function is nonempty and single-valued, we may also suppose that

$$\dot{y}(s) = \chi(s, y(s)), \ y(t) = x$$

has an absolutely continuous solution on $[t, b]$. Then, from the following calculation

$$\begin{aligned} V(t, x) &= V(b, y(b)) - \int_t^b \frac{d}{ds} V(s, y(s)) ds \\ &= V(b, y(b)) - \int_t^b [V_t(s, y(s)) + V_x(s, y(s)).\chi(s, y(s))] ds \\ &= V(b, y(b)) - \int_t^b [V_t(s, y(s)) + h(y(s), V_x(s, y(s)))] ds \\ &= V(b, y(b)) - 0 = l(y(b)). \end{aligned}$$

We can conclude that the trajectory y has cost equal to $V(t, x)$, which shows its optimality in relation to problem $(P_{t,x})$. Note that if we set $(t, x) = (a, x_0)$, then we obtain the optimal solution of (P_c). This approach has the advantage of providing a feedback optimal control law. Thus, dynamic programming establishes the link between the optimal control problem and a partial differential equation.

In classical theory, the value function started to a have a different role, namely, providing sufficient conditions of optimality to test if a given putative minimizer is in fact a solution of problem (P_c). The usage of the Hamilton-Jacobi equation for this purpose is called the Carathéodory method. However, confirming that a arc is a minimizer can be achieved in more favorable circumstances by constructing verification functions.

A natural question arising is to know whether the Hamilton-Jacobi has a unique solution that coincides with the value function. This issue is not trivial since in many optimal control problems the value function is not differentiable. In this sense, it was necessary to develop new solution concepts for the Hamilton-Jacobi equation. The solution concept should be defined in a way that the value function (possibly nondifferentiable) is the unique solution of the Hamilton-Jacobi equation.

Viscosity solutions, developed by M. Crandall and P. L. Lions, met this requirement whenever the value function is continuous. A continuous function φ is said to be a continuous viscosity solution of

$$\varphi_t(t,x) + h(x, \varphi_x(t,x)) = 0, \quad \text{for all } (t,x) \in (a,b) \times \mathbb{R}^n$$

if the following two conditions are satisfied:

- For any point $(t,x) \in (a,b) \times \mathbb{R}^n$ and any C^1 function $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $(t',x') \to \varphi(t',x') - w(t',x')$ has a local minimum at (t,x) we have

$$\nabla_t w(t,x) + h(x, \nabla_x w(t,x)) \leq 0;$$

- For any point $(t,x) \in (a,b) \times \mathbb{R}^n$ and any C^1 function $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $(t',x') \to \varphi(t',x') - w(t',x')$ has a local maximum at (t,x), we have

$$\nabla_t w(t,x) + h(x, \nabla_x w(t,x)) \geq 0.$$

Methods in the analysis of nonlinear partial differential equations (viscosity techniques) established that the value function is the unique continuous function that is solution of the Hamilton-Jacobi equation in the viscosity sense.

The proximal solution to the Hamilton-Jacobi equation is another solution concept with the required properties. With this solution concept, the value function is only required to be lower semicontinuous, which is an important characteristic if we wish to consider terminal constraints in the optimal control problem. We do not explore this type of problem as it is out of the scope of our objectives for this chapter. Thus, a continuous function $\varphi : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ is said to be a proximal solution of the Hamilton-Jacobi equation if, at each point $(t,x) \in (a,b) \times \mathbb{R}^n$ such that $\partial_P \varphi(t,x)$ is nonempty, we have:

$$\theta + h(x,\zeta) = 0 \text{ for all } (\theta,\zeta) \in \partial_P \varphi(t,x).$$

The link between proximal solutions of the Hamilton-Jacobi equation and the value function V is not obtained by studying the partial differential equation directly but by using results derived in the context of invariance. This is addressed in [14] and is the path we chose to extend dynamic programming for impulsive systems. Consequently, we can apply, in this optimization context, the results on invariance developed before for impulsive systems. For that purpose, we need to present the functional counterpart of invariance, which is precisely what is performed in the next section.

Before concluding, we should mention that there are other solution concepts that allow the study of the Hamilton-Jacobi equation. For example, in [43], problems with measurable dynamics in relation to the time are studied and, in [23, 24], the contingent solution in which the cost function can take values in $\mathbb{R} \times \{+\infty\}$ is addressed. In [38], it is provided a constructive procedure to obtain optimal controls in a feedback form with resource of piecewise constant functions, which can be viewed as a variation of the Euler solution of [14].

4.2 Monotonicity properties

Before we present the main results of this chapter, we first introduce the functional counterpart of invariance for impulsive systems. These monotonicity properties are the extension for the impulsive problem of the conventional results presented in [14]. These concepts will play a major role in the upcoming proofs.

4.2.1 Weakly decreasing systems

In this section, we provide the functional counterpart of weak invariance. Consider a lower semicontinuous function φ mapping $\mathbb{R}^n$ to $(-\infty, +\infty]$. The system (φ, F, G) is said to be **weakly decreasing** if, for any $\alpha \in \mathbb{R}^n$, there exists an impulsive trajectory $x(t)$ on $t \in [0, +\infty)$ and $x^s(s)$ on $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp } \mu_{sa}$, with $x(0^-) = \alpha$, satisfying:

$$\varphi(x(t)) \leq \varphi(x(0^-)) := \varphi(\alpha) \text{ and } \varphi(x^s(s)) \leq \varphi(x(0^-)) := \varphi(\alpha).$$

The concepts of weak invariance and weak decreasing systems are related as follows:

Lemma 4.2.1. (φ, F, G) *is weakly decreasing iff* $(epi\varphi, F \times \{0\}, G \times \{0\})$ *is weakly invariant.*

Proof. Suppose first that (φ, F, G) is weakly decreasing. Thus, there exists an impulsive trajectory such that $\varphi(x(t)) \leq \varphi(\alpha)$ $\forall$ $t \in [0, +\infty)$ and $\varphi(x^s(s)) \leq \varphi(\alpha)$ $\forall$ $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp } \mu_{sa}$. Consider a new added state variable r with $dr(t) = 0$ and choose $r(0^-) = r_0$ such that $\varphi(\alpha) \leq r_0$. Then, by construction, $r(t) = r_0$ $\forall$ $t \in [0, +\infty)$ and $r^s(s) = r_0$ $\forall$ $s \in [\eta(t_a^-), \eta(t_a^+)]$. Hence, we conclude that:

$$(x(t), r(t)) \in \text{epi } \varphi := \{(\bar{x}, \bar{r}) \in \text{dom } \varphi \times \mathbb{R} : \varphi(\bar{x}) \leq \bar{r}\} \text{ and } (x^s(s), r^s(s)) \in \text{epi } \varphi,$$

for $t \in [0, +\infty)$ and for $s \in [\eta(t_a^-), \eta(t_a^+)]$, which implies that $(\text{epi}\varphi, F \times \{0\}, G \times \{0\})$ is weakly invariant.

For the converse, consider that $(\text{epi}\varphi, F \times \{0\}, G \times \{0\})$ is weakly invariant. Then, by definition, there exists a trajectory x, r such that $(x(t), r(t)) \in \text{epi}\varphi(x) := \{(x, r) : \varphi(x) \leq r\}$ and $(x^s(s), r^s(s)) \in \text{epi}\varphi(x)$. Then, we derive that:

$$\varphi(x(t)) \leq r(t); \forall\, t \in [0, +\infty) \text{ and } \varphi(x^s(s)) \leq r^s(s)\ \forall s \in [\eta(t_a^-), \eta(t_a^+)].$$

Since $dr = 0$ we have that $r(t) = r(a) = r_0$ and $r(s) = r_0$. Hence, we can conclude that $\varphi(x(t)) \leq r_0$ $\forall$ $t \in [0, +\infty)$ and $\varphi(x^s(s)) \leq r_0$ $\forall$ $s \in [\eta(t_a^-), \eta(t_a^+)]$, which means that (φ, F, G) is weakly decreasing. $\square$

Now, we are in condition to provide necessary and sufficient conditions for weakly decreasing systems.

Theorem 4.2.1. *(φ, F, G) is weakly decreasing iff for every $x \in \mathbb{R}^n$*

$$\min\{h_F(x,\zeta), h_G(x,\zeta)\} \leq 0, \ \forall\, \zeta \in \partial_P\varphi(x). \tag{4.1}$$

Proof. Suppose first that (φ, F, G) is weakly decreasing. By lemma 4.2.1, we know that this implies that $(\text{epi}\varphi, F\times\{0\}, G\times\{0\})$ is weakly invariant. Then, by theorem 3.3.1 this implies that for any vector $(\zeta,\lambda) \in N^P_{\text{epi}\varphi}(x,r)$, where $(x,r) \in \text{epi}\ \varphi$, we have:

$$\min\left\{\min_{v\in F(x)}\langle(\zeta,\lambda),(v,0)\rangle,\ \min_{w\in G(x)}\langle(\zeta,\lambda),(w,0)\rangle\right\} \leq 0.$$

If $\zeta \in \partial_P\varphi(x)$, then $(\zeta,-1) \in N^P_{\text{epi}\varphi}(x,\varphi(x))$ and we conclude that:

$$\min\left\{\min_{v\in F(x)}\langle\zeta,v\rangle,\ \min_{w\in G(x)}\langle\zeta,w\rangle\right\} \leq 0,$$

which is precisely what we need to prove.

Now, we prove the sufficiency of the condition. By lemma 4.2.1, it suffices to prove that $(\text{epi}\varphi, F\times\{0\}, G\times\{0\})$ is weakly invariant. Consequently, we only need to find, for any $(\zeta,\lambda) \in N^P_{\text{epi}\varphi}(x,r)$, elements $v \in F(x)$ and $w \in G(x)$ such that $\min\{\langle v,\zeta\rangle,\langle w,\zeta\rangle\} \leq 0$. From nonsmooth analysis, we know that $\lambda \leq 0$ and that $(\zeta,\lambda) \in N^P_{\text{epi}\ \varphi}(x,\varphi(x))$ (exercise 1.2.1 d) in [14]). If $\lambda < 0$, we have $(\zeta/(-\lambda),-1) \in N^P_{\text{epi}\varphi}(x,r)$, which implies $-\zeta/\lambda \in \partial_P\varphi(x)$. By hypothesis, we can ensure the existence of v and w satisfying $\min\{\langle v,-\zeta/\lambda\rangle,\langle w,-\zeta/\lambda\} \leq 0$. Since $\lambda < 0$, then we can conclude that $\min\{\langle v,\zeta\rangle,\langle w,\zeta\} \leq 0$. Now, we only have to consider the case when $\lambda = 0$. Then, we have $(\zeta,0) \in N^P_{\text{epi}\ \varphi}(x,\varphi(x))$. From exercise 1.11.23 in [14], we can deduce the existence of sequences $(\zeta_i,-\varepsilon_i)$, with $\varepsilon_i > 0$, and $(x_i,\varphi(x_i))$ such that:

$$(\zeta_i,-\varepsilon_i) \to (\zeta,0),\ (\zeta_i,-\varepsilon_i) \in N^P_{\text{epi}\ \varphi}(x_i,\varphi(x_i)),\ (x_i,\varphi(x_i)) \to (x,\varphi(x)).$$

From the case when $\lambda < 0$, we know that there exist $v_i \in F(x_i), w_i \in G(x_i)$ satisfying $\min\{\langle v_i, \zeta_i\rangle, \langle w_i, \zeta_i\} \leq 0$. Since F, G are locally bounded, then the sequences $\{v_i\}$ and $\{w_i\}$ are bounded. Extracting a subsequence (we do not relabel), we can suppose that v_i and w_i converge to $v \in F(x)$ and $w \in G(x)$, respectively, due to the upper semicontinuity of F and G. Hence, we can also conclude that $\min\{\langle v, \zeta\rangle, \langle w, \zeta\} \leq 0$, for $\lambda = 0$, which implies that $(\text{epi}\varphi, F \times \{0\}, G \times \{0\})$ is weakly invariant and, therefore, (φ, F, G) is weakly decreasing. □

4.2.2 Strongly decreasing systems

Now we devote attention to strongly decreasing systems which are the functional counterpart of strong invariance.

Consider a lower semicontinuous function φ mapping $\mathbb{R}^n$ to $(-\infty, +\infty]$. The system (φ, F, G) is said to be **strongly decreasing** if, for any $\alpha \in \mathbb{R}^n$, all impulsive trajectories $x(t)$ on $t \in [0, +\infty)$ and $x^s(s)$ on $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp}\,\mu_{sa}$, with $x(0^-) = \alpha$, satisfy:

$$\varphi(x(t)) \leq \varphi(\alpha)\ \forall t \in [0, +\infty) \text{ and } \varphi(x^s(s)) \leq \varphi(\alpha)\ \forall s \in [\eta(t_a^-), \eta(t_a^+)].$$

Note that (φ, F, G) is strongly decreasing if and only if all trajectories x are such that the functions

$$t \to \varphi(x(t)) \text{ and } s \to \varphi(x^s(s)) \tag{4.2}$$

are decreasing on $t \in [0, +\infty)$ and on $s \in [\eta(t_a^-), \eta(t_a^+)]$.

Theorem 4.2.2. *Let the multifunctions F and G be Lipschitz continuous. Then, (φ, F, G) is strongly decreasing iff, for every $x \in \mathbb{R}^n$,*

$$\max\{H_F(x, \zeta), H_G(x, \zeta)\} \leq 0, \zeta \in \partial_P\varphi(x). \tag{4.3}$$

The proof is omitted since there is no additional technical considerations relatively to the proof of theorem 4.2.1. Here, instead of invoking theorem 3.3.1 we use theorem 3.3.4. Note that we require Lipschitz continuity for F, G as in theorem 3.3.4.

4.2.3 Strongly increasing systems

In this section, we introduce the concept of strongly increasing systems.

Consider a lower semicontinuous function φ mapping $\mathbb{R}^n$ to $(-\infty, +\infty]$. The system (φ, F, G) is said to be **strongly increasing** if, for any $\alpha \in \mathbb{R}^n$, all impulsive trajectories $x(t)$ on $t \in [0, +\infty)$ and $x^s(s)$ on $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp } \mu_{sa}$, with $x(0^-) = \alpha$, satisfy:

$$\varphi(x(t)) \leq \varphi(x(b^+)) \ \forall\, t \in [a, b] \text{ and } \varphi(x^s(s)) \leq \varphi(x(b^+)) \ \forall\, s \in [\eta(t_a^-), \eta(t_a^+)].$$

As usual, the superscript $+$ on b means that if there is an atom at b we should consider the endpoint of the jump. If there is no jump at $t = b$, then $x(b^+)$ coincides with $x(b)$. As noted before, (φ, F, G) is strongly increasing on $[a, b]$ if and only if the functions

$$t \to \varphi(x(t)) \text{ and } s \to \varphi(x^s(s)) \tag{4.4}$$

are increasing for all trajectories x on $t \in [a, b]$ and on $s \in [\eta(t_a^-), \eta(t_a^+)]$.

The following result gives conditions characterizing strongly increasing systems.

Theorem 4.2.3. *Let the multifunctions F and G be Lipschitz continuous. Then, (φ, F, G) is strongly increasing iff for every $x \in \mathbb{R}^n$*

$$\min\{h_F(x, \zeta), h_G(x, \zeta)\} \geq 0, \zeta \in \partial_P \varphi(x). \tag{4.5}$$

Proof. Let x be a trajectory of $dx \in F(x)dt + G(x)d\mu(t)$ on $t \in [a, b]$. Then, define $y(t) := x(b+a-t)$ for $t \in [a, b]$ and $y^s(s) := x^s(\eta(t_a^+)+\eta(t_a^-)-s)$ for $s \in [\eta(t_a^-), \eta(t_a^+)]$.

By construction we have that:

$$\begin{aligned} dy(t) = -dx(b+a-t) &\in -F(x(b+a-t))dt - G(x(b+a-t))d\mu(t) \\ &= -F(y(t))dt - G(y(t))d\mu(t) \; \mu - a.e.. \end{aligned}$$

There is a one-to-one correspondence between x and y, so that $\varphi(t, x(t))$ and $\varphi(t, x^s(s))$ are increasing functions for all x if and only if $\varphi(t, y(t))$ and $\varphi(t, y^s(s))$ are decreasing functions for all y. Hence, showing that the original system (φ, F, G) is strongly increasing on $[a, b]$ is the same as showing the auxiliary system $(\varphi, -F, -G)$ is strongly decreasing on $t \in [a, b]$. Applying theorem 4.2.2 to the auxiliary system we deduce that

$$\max\{H_{(-F)}(y, \zeta), H_{(-G)}(y, \zeta)\} \leq 0, \zeta \in \partial_P \varphi(x),$$

for every $y \in \mathbb{R}^n$, that implies

$$\max\{-h_F(y, \zeta), -h_G(y, \zeta)\} \leq 0, \zeta \in \partial_P \varphi(x).$$

In turn, this last inequality leads us to the conclusion that

$$\min\{h_F(y, \zeta), h_G(y, \zeta)\} \geq 0, \zeta \in \partial_P \varphi(x).$$

Due to the previous mentioned equivalence we reach to the desired result. □

4.2.4 The nonautonomous case

Now we consider the case when the function φ depends also on the variable t. Let $\varphi(t, x)$ be a lower semicontinuous function mapping $(a, b) \times \mathbb{R}^n$ to $(-\infty, +\infty]$. The system (φ, F, G) is said to be **weakly decreasing** on $(a, b) \times \mathbb{R}^n$ provided that, for any interval $[\bar{a}, \bar{b}]$ contained in (a, b), for any $\alpha \in \mathbb{R}^n$ and $\tau \in [\bar{a}, \bar{b}]$, there exists an impulsive trajectory x on $t \in [\tau, \bar{b}]$ and $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp } \mu_{sa}$, such that:

$$\varphi(t, x(t)) \leq \varphi(\tau, \alpha) \text{ and } \varphi(t, x^s(s)) \leq \varphi(\tau, \alpha).$$

The system (φ, F, G) is said to be **strongly increasing** on $(a, b) \times \mathbb{R}^n$ if any impulsive trajectory x $\forall t \in [\bar{a}, \bar{b}]$ and $\forall s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp } \mu_{sa}$ satisfies:

$$\varphi(t, x(t)) \leq \varphi(\bar{b}, x(\bar{b}^+)) \text{ and } \varphi(t, x^s(s)) \leq \varphi(\bar{b}, x(\bar{b}^+)).$$

We define the function φ in an open set to avoid some pathological functions to appear when the domain of definition is a closed set.

Proposition 4.2.1. *Let $\varphi(t, x)$ be a lower semicontinuous function on $(a, b) \times \mathbb{R}^n$.*

(a) *(φ, F, G) is weakly decreasing on $(a, b) \times \mathbb{R}^n$ if and only if*

$$\min\{\theta + h_F(x, \zeta), h_G(x, \zeta)\} \leq 0, \ \forall (\theta, \zeta) \in \partial_P \varphi(t, x) \ \forall (t, x) \in (a, b) \times \mathbb{R}^n. \quad (4.6)$$

(b) *(φ, F, G) is strongly increasing on $(a, b) \times \mathbb{R}^n$ if and only if*

$$\min\{\theta + h_F(x, \zeta), h_G(x, \zeta)\} \geq 0, \ \forall (\theta, \zeta) \in \partial_P \varphi(t, x) \ \forall (t, x) \in (a, b) \times \mathbb{R}^n. \quad (4.7)$$

To prove this proposition, we should consider an augmented system $\{1\} \times F$ and $\{0\} \times G$, where t is viewed as another state variable. This, together with theorem 4.2.1, leads us to part a) of the proposition. On the other hand, part b) is deduced with resource of a reverse time technique, where a strongly decreasing system is transformed in a strongly increasing system. Then, applying theorem 3.3.4 in the context of strongly increasing systems, in a similar way as done in theorem 4.2.1, leads us to part b) of the proposition.

4.3 Verification functions

Imagine we have a feasible arc $\bar{x}(t), \bar{x}^s(s)$, which we suspect it is an optimal solution of problem (P). A verification function is useful to confirm that $\bar{x}(t), \bar{x}^s(s)$ is in fact an optimal solution. The following result provides conditions characterizing verification functions.

Proposition 4.3.1. *Let $\bar{x}(t), \bar{x}^s(s)$ be a feasible arc for problem (P) and suppose there is a continuous function $\varphi(t,x)$ on $[a,b] \times \mathbb{R}^n$ satisfying:*

$$\min\{\theta + h_F(x,\zeta), h_G(x,\zeta)\} \geq 0,\ \forall(\theta,\zeta) \in \partial_P\varphi(t,x),\ \forall(t,x) \in (a,b) \times \mathbb{R}^n, \tag{4.8}$$

$$h_G(x,\zeta) \geq 0,\ \forall\zeta \in \partial_P\varphi(t,\cdot)(x),\ \forall(t,x) \in \{\{a\} \cup \{b\}\} \times \mathbb{R}^n, \tag{4.9}$$

$$\varphi(b,\cdot) \leq l(\cdot), \tag{4.10}$$

$$\varphi(a,x_0) = l(\bar{x}(b^+)) \tag{4.11}$$

Then $\bar{x}, \bar{x}^s$ solves problem (P) and its value is given by $\varphi(a,x_0)$.

Proof. Condition (4.8) of the theorem implies, by proposition 4.2.1, that (φ, F, G) is strongly increasing on $(a,b) \times \mathbb{R}^n$. Moreover, condition (4.9) implies that at $t = a$ and $t = b$ the singular dynamics is also strongly increasing on $s \in [\eta(a^-), \eta(a^+)]$ and on $s \in [\eta(b^-), \eta(b^+)]$, respectively. Then, consider any feasible arc x, x^s for problem (P). We can deduce:

$$\begin{aligned} l(x(b^+)) &\geq \varphi(b, x(b^+)) \\ &\geq \varphi(b, x(b^-)) \geq \varphi(a, x(a^+)) \geq \varphi(a, x(a^-)) = \varphi(a, x_0) = l(\bar{x}(b^+)), \end{aligned}$$

where the first, the second, the third and the fourth inequalities are given by conditions (4.10), (4.9), (4.8) and (4.9), respectively. The preceding inequality shows that $\bar{x}, \bar{x}^s$ is in fact a minimizer of problem (P). ☐

Now, consider the value function $V(\tau,\alpha)$ associated to problem $P(\tau,\alpha)$. Appealing to the principle of optimality, we know that $t \to V(t, x(t))$ and $s \to V(t, x^s(s))$ are increasing functions on $t \in [a,b]$ and on $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in \text{supp}\mu_{sa}$ for every feasible arc of problem (P). Hence, we conclude that (V, F, G) is a strongly increasing system implying and, by proposition 4.2.1, we conclude that $V(t,x)$ satisfies condition (4.8). That V satisfies condition (4.9) is an immediate consequence of the fact that the singular dynamics are strongly increasing. By definition, the value function $V(t,x)$

also satisfies conditions (4.10) and (4.11). From this discussion, we conclude that at least one verification function exists, being the value function one such function. However, it would be interesting to have a full characterization of function V in terms of a Hamilton-Jacobi-Bellman like function. This is precisely what is presented in the next result.

Theorem 4.3.1. *The unique continuous function $\varphi : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\min\{\theta + h_F(x, \zeta), h_G(x, \zeta)\} = 0,\ \forall(\theta, \zeta) \in \partial_P\varphi(t, x),\ \forall(t, x) \in (a, b) \times \mathbb{R}^n, \tag{4.12}$$

$$h_G(x, \zeta) = 0,\ \forall\zeta \in \partial_P\varphi(t, \cdot)(x),\ \forall(t, x) \in \{\{a\} \cup \{b\}\} \times \mathbb{R}^n \tag{4.13}$$

$$\varphi(b, x) = \psi(x) := \inf\{l(\tilde{x}(s)) :\ \dot{\tilde{x}}(s) \in G(\tilde{x}(s)),\ \tilde{x}(0) = x\}\ \forall x \in \mathbb{R}^n \tag{4.14}$$

is the value function associated with problem (P).

Proof. From the previous discussion, we already know that V satisfies half of conditions (4.12) and (4.13) (the $\geq$ part). Condition (4.14) holds trivially. Hence, we only have to show that:

$$\min\{\theta + h_F(x, \zeta), h_G(x, \zeta)\} \leq 0,\ \forall(\theta, \zeta) \in \partial_P V(t, x),\ \forall(t, x) \in (a, b) \times \mathbb{R}^n \tag{4.15}$$

$$h_G(x, \zeta) \leq 0,\ \forall\zeta \in \partial_P V(t, \cdot)(x),\ \forall(t, x) \in \{\{a\} \cup \{b\}\} \times \mathbb{R}^n \tag{4.16}$$

To show these inequalities, we appeal again to the principle of optimality to state that an optimal arc $\bar{x}, \bar{x}^s$ for problem (P) makes the mappings $t \to V(t, \bar{x}(t))$ and $s \to V(t, \bar{x}^s(s))$ constant on $t \in [a, b]$ and on $s \in [\eta(t_a^-), \eta(t_a^+)]$. Hence, we can conclude that (V, F, G) is weakly decreasing on (a, b) which, by proposition 4.2.1, implies the inequality (4.15). The condition (4.16) is just a consequence of the fact that if the optimal arc $\bar{x}, \bar{x}^s$ has atoms at $t = a$ and $t = b$ then the mappings $s \to V(a, \bar{x}^s(s))$ and $s \to V(b, \bar{x}^s(s))$ are constant along the singular trajectory.

To conclude the proof, we now need to show that V is, in fact, the unique function satisfying the conditions of the theorem. Let us assume that there is another function

φ satisfying the conditions of the theorem. Let (τ, α) be given with $a \leq \tau \leq b$. By the conditions of the theorem and using proposition 4.2.1, we deduce that (φ, F, G) is weakly decreasing and, consequently, there is a trajectory x, x^s on $[\tau, b]$ with $x(\tau^-) = \alpha$ such that:

$$\begin{aligned} \varphi(t, x(t)) &\leq \varphi(\tau, \alpha)\ \forall t \in [\tau, b), \\ \varphi(t, x^s(s)) &\leq \varphi(\tau, \alpha)\ \forall s \in [\eta(t_a^-), \eta(t_a^+)]. \end{aligned}$$

Hence, we can deduce that:

$$V(\tau, \alpha) \leq V(b, x(b^+)) = \psi(x(b^+)) = \varphi(b, x(b^+)) \leq \varphi(b, x(b^-)) \leq \varphi(\tau, \alpha).$$

Note the prominence of the continuity of function φ in the last inequality, where $\varphi(b, x(b^-))$ is to be interpreted as $\lim_{t \uparrow b} \varphi(t, x(t))$. Consequently, we obtain that $V \leq \varphi$. On the other hand, we also know that the system (φ, F, G) is strongly increasing and, consequently, for a given optimal trajectory $\bar{x}$ with $\bar{x}(\tau^-) = \alpha$, we know that:

$$V(\tau, \alpha) = l(\bar{x}(b^+)) = \varphi(b, \bar{x}(b^+)) \geq \varphi(b, \bar{x}(b^-)) \geq \varphi(\tau, \alpha),$$

which leads us to the conclusion that $V \leq \varphi$. Therefore, the value function V is the unique solution satisfying the conditions of the theorem. $\square$

The previous result gives conditions characterizing the value function associated with problem (P). Now, an interesting question would be to know if we can compute the optimal solution from the value function. The answer to this question is given in the following result.

Theorem 4.3.2. *There are feedback selections and a control measure such that the resulting Euler solution is an optimal solution of problem (P).*

Proof. Consider the set

$$S := \{(t,x) \in [a,b] \times \mathbb{R}^n : V(t,x) = V(a,x_0)\}.$$

Since the value function is continuous, we conclude that the set S is closed. By the principle of optimality, we already know that there is a trajectory $\bar{x}, \bar{x}^s$ making the mappings $t \to V(t, \bar{x}(t))$ and $s \to V(t, \bar{x}^s(s))$ constant on $t \in [a,b]$ and on $s \in [\eta(t_a^-), \eta(t_a^+)]$ $\forall t_a \in$ supp μ_{sa}. Consequently, we conclude that the augmented system is such that $(S, \bar{F}, \bar{G})$ is weakly invariant on $t \in [a,b]$ ($\bar{F} := \{1\} \times F$ and $\bar{G} := \{0\} \times G$). Then, using the constructive result of theorem 3.3.3, we can derive feedback selections of $\bar{F}$ and $\bar{G}$ and a feedback control measure μ such that the system $(S, \bar{F}, \bar{G})$ become invariant. These selections and control measure produce an impulsive Euler trajectory that makes the value function constant along the trajectory. In turn, by the principle of optimality, this implies the optimality of the trajectory and, consequently, of the feedback selections and of the feedback control measure. $\square$

The previous result is not only of existence type but it is also constructive in the sense that it provides a method to compute optimal feedback selections and optimal feedback control measures. The algorithm is as follows:

Algorithm 4.3.1. *Optimal feedback control for impulsive systems.*

1. *Consider the augmented system $\tilde{F} := \{1\} \times F$, $\tilde{G} := \{0\} \times G$ and the set S computed from the value function as described in theorem 4.3.2.*

2. *For each $(t,x) \in [0,\infty) \times R^n$, compute $(p_t, p_x) := p(t,x) \in proj_S(t,x)$.*

3. *For each $(t,x) \in [0,\infty) \times R^n$, minimize the functions: $v \to \langle (1,v), (t,x) - p \rangle$ and $w \to \langle (0,w), (t,x) - p \rangle$, with $v \in F(p_x)$ and $w \in G(p_x)$. Set $f_P(t,x) = [1,v]^T$ and $g_P(t,x) = [0,w]^T$.*

4. *Compute a measure* μ *such that* $[\langle f_P(t,x),(t,x)-p\rangle]\,dt+\langle g_P(t,x),(t,x)-p\rangle d\mu \leq 0$ μ*-a.e. on* $t\in[0,1]$*, as detailed in theorem 3.3.1.*

5. *Choose* $\bar{f}_P(t,x)$ *and* $\bar{g}_P(t,x)$ *to be the points in* $\{1\}\times F(x)$ *and* $\{0\}\times G(x)$ *closest to* $f_P(t,x)$ *and* $g_P(t,x)$*, respectively.*

6. *Compute the impulsive Euler solution of* $[dt,dx]^T=\bar{f}_P(t,x)d\tau+\bar{g}_P(t,x)d\mu(\tau)$ *on* $\tau\in[0,1]$*. The component* x *of the augmented state will be the optimal solution of problem (P).*

4.4 Example

In this example, we intent to illustrate the usage of the algorithm presented before. Thus, we consider the following optimal impulsive control problem:

$$\begin{aligned}
&\min|x(1)|\\
&\text{s.t. } dx\in F(x)dt+d\mu(t)\\
&\text{with } F(x)=[-1,1];\ \mu\geq 0;\\
&x(0^-)=0;(t,x)\in[0,1]\times\mathbb{R}.
\end{aligned}$$

First, we compute the value function $V(t,x)$ associated with this problem. For this purpose we need to consider the auxiliary problem $(P(\tau,\alpha))$ with $\tau\in[0,1]$ and $\alpha\in\mathbb{R}$.

$$\text{If } \alpha\geq 1-\tau \Rightarrow \begin{cases} f^*(t,x)=-1\\ \mu^*(\{t\})=0,\end{cases} \tag{4.17}$$

where $*$ represents the optimality of the selection and of the control measure on $t\in[\tau,1]$. The resulting trajectory when applying this control is:

$$x^*(t)=\alpha-(t-\tau), \tag{4.18}$$

which means that $|x(1)| = \alpha - 1 + \tau$. If $\alpha \leq 1 - \tau$, then it is always possible to find a selection and a control measure such that $x(1)$ reaches its global minimum, which is equal zero. Note also that, for given initial conditions, the optimal control measure has always bounded total variation. From the previous discussion, we conclude that the value function for this problem is given by:

$$V(t, x) = \max\{0, x - 1 + t\} \tag{4.19}$$

The invariant set S is given by $S := \{(t, x) \in [0, 1] \times \mathbb{R} : \max\{0, x - 1 + t\} = V(0, 0) = 0\} \equiv \{(t, x) : \ x - 1 + t \leq 0, \ t \in [0, 1]\}$. In figure 4.1 this set is depicted.

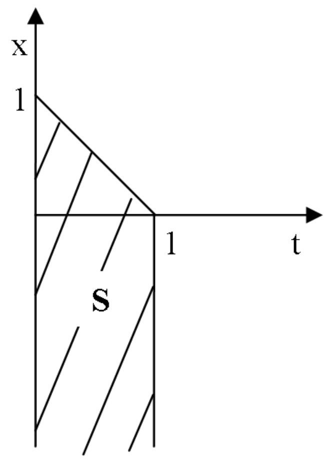

Figure 4.1: Invariant set of the optimal impulsive control problem

Now, we devote attention to point 2 and 3 of the algorithm. Whenever $x \in S$ the control measure and the selection is arbitrary. Since $G(x) = \{1\}$ then, clearly, $g_P(x) = [0, 1]^T \ \forall (t, x) \in [0, \infty) \times \mathbb{R}$. Now we turn to the computation of $f_P(t, x) \in \{1\} \times F(x)$. To clarify the presentation we divide the state space in zones as depicted in figure 4.2. Thus, depending on which zone (t, x) is, we compute f_P as follows:

- For $t < 0$ and $x < 1$ (zone A), we have $(t, x) - p = \lambda[-1, 0]^T$ with $\lambda > 0$ (recall that p represents the projection of (t, x) on S). Then, we compute the minimum:

$$\min_{v\in[-1,1]} \left\langle \begin{bmatrix} 1 \\ v \end{bmatrix}, \begin{bmatrix} -\lambda \\ 0 \end{bmatrix} \right\rangle = -\lambda \; \forall v \in [-1,1].$$

Thus, we set $f_P(t,x) = [1,v]^T$ with $v \in [-1,1]$.

- Whenever

$$x \geq \begin{cases} 1, & \text{if } t \leq 0 \\ 1+t, & \text{if } t > 0 \end{cases}$$

(zone B) then $p = [0,1]^T$ and $(t,x) - p = [t, x-1]^T$. Then, for every (t,x) as defined before, we have:

$$\min_{v\in[-1,1]} \left\langle \begin{bmatrix} 1 \\ v \end{bmatrix}, \begin{bmatrix} t \\ x-1 \end{bmatrix} \right\rangle = \min_{v\in[-1,1]} t + v(x-1).$$

Since $x - 1 \geq 0$ and $x - 1 \geq t$, we conclude that the minimum is less or equal to zero and is obtained when $v = -1$. Consequently, we set the feedback selection $f_P(t,x) = [1,-1]^T$.

- For $t \in (0,+\infty)$ and $x > |t-1|$ (zone C), then $(t,x) - p = \lambda[1,1]^T$, with $\lambda > 0$, and, consequently, we obtain:

$$\min_{v\in[-1,1]} \left\langle \begin{bmatrix} 1 \\ v \end{bmatrix}, \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = 0 \text{ for } v = -1.$$

Again, we set $f_P(t,x) = [1,-1]^T$.

- If $t > 1$ and $x < 0$ (zone E), we have $(t,x) - p = \lambda[1,0]^T$, $\lambda > 0$. For this case we have:

$$\min_{v\in[-1,1]} \left\langle \begin{bmatrix} 1 \\ v \end{bmatrix}, \begin{bmatrix} \lambda \\ 0 \end{bmatrix} \right\rangle = \lambda > 0 \; \forall \, v \in [-1,1].$$

We set $f_P(t,x) = [1,v]^T$, $\forall \, v \in [-1,1]$.

- If $t > 1$ and $0 \leq x \leq t-1$ (zone D), we have $p = [1,0]^T$. Then, we obtain

$$\min_{v\in[-1,1]} \left\langle \begin{bmatrix} 1 \\ v \end{bmatrix}, \begin{bmatrix} t-1 \\ x \end{bmatrix} \right\rangle > 0,$$

with $v = -1$. We set $f_P(t,x) = [1,-1]^T$.

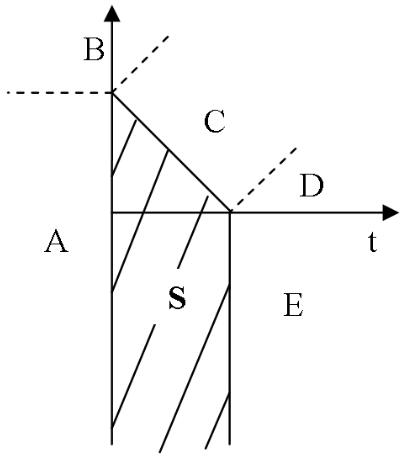

Figure 4.2: State space division

At this point, we devote attention to point 4 of the algorithm. Thus, we need to choose the measure μ such that:

$$\langle f_P(t,x)dt + g_P(t,x)d\mu, (t,x) - p\rangle \leq 0. \tag{4.20}$$

Whenever $(t,x) \in S$, the measure is arbitrary since, clearly, $(t,x) - p = 0$. If $(t,x) \in A$, the measure is also arbitrary since $\langle \bar{g}_P, (t,x) - p\rangle = 0$ and $\langle f_P, (t,x) - p\rangle = -\lambda$ (the measure does not affect the signal of (4.20)). Whenever $(t,x) \in B$, then $\langle f_P, (t,x) - p\rangle \leq 0$ and $\langle g_P, (t,x) - p\rangle > 0$, which means that if we set $\mu = 0$ then the equation (4.20) remains valid. If $(t,x) \in C$, then, necessarily, we should set $\mu = 0$ since $\langle f_P, (t,x) - p\rangle = 0$ and $\langle g_P, (t,x) - p\rangle > 0$ (any contribution of g_P will make (4.20) greater to zero). If $(t,x) \in E$, we have $\langle g_P, (t,x) - p\rangle = 0$ and $\langle f_P, (t,x) - p\rangle > 0$. This shows that the measure μ has to be singular atomic in order to keep the inequality (4.20) less or equal to zero. When $(t,x) \in D$, we conclude that $p(t,x) = (1,0)$. This corresponds to the final optimal point. At this point, the set S is no longer invariant but this does not bring any additional difficulty since the final time and global optimal have been reached.

In this example, the computation of function $\bar{f}_P$ and $\bar{g}_P$ is a simple task since both multifunctions F and G do not depend on the state, meaning that $\bar{f}_P \equiv f_P$ and $\bar{g}_P \equiv g_P$. This corresponds to point 5 of the algorithm.

The optimal solution for problem (P) is computed with resource of the Euler solution as described in point 6 of the algorithm.

Chapter 5

Applications

The aim of this chapter is to present some practical applications where the impulsive control framework and its solution concept play a major role in terms of modeling, control and computational implementation. We cover some engineering systems for which it is relevant to consider discontinuous trajectories. We use the impulsive Euler solution and the feedback control measure to address these applications in a way that the jump can be controlled. Usually, discontinuities represent an idealization of very fast variations of the trajectories in a short period of time. As we have seen before, a measure can be approximated by a measurable function, enabling a practical implementation. We will provide examples on impact dynamic systems and on investment control. We will also see that the impulsive control formulation can be used for the composition of dynamic systems and, in particular, is well suited to model hybrid systems and to control formations of autonomous vehicles. Finally, we will provide an extension of the invariance concept. There are several practical applications where the singular arc does not need to satisfy the invariance property and a quasi-invariant system will be introduced.

5.1 Impact dynamics

In this section, we show the impulsive control framework is useful to model mechanical systems subject to impact collisions. There are numerous examples where impact forces arise: applications on robotic manipulators, vibro-impact mechanisms, walking (and, potentially, jumping) biped robots, juggling mechanical systems and micro-electromechanical systems (MEMS). In [8] and in the references therein, a detailed description of these applications is provided.

Usually, the impacts are considered passive since the behavior of the state during this stage cannot be controlled. However, with the increasing capabilities of sensors and actuators, in terms of sampling rate and response speed, it has become feasible to control the system during the impact phase as well. In this new paradigm, the impulsive framework turned an important tool for modeling and controller synthesis purposes.

The impact period takes a very short period of time when compared with the normal operation of the system. For this reason, we can ideally describe the impact dynamics by a singular trajectory associated with a singular atomic measure. Hence, the multifunction $F(x)$ models the dynamical system in the non-impact period while, on the other hand, $G(x)$ models the impact dynamics. Despite we describe the trajectory in the impact period, the impulsive model assumes the state transition to be instantaneous (singular) face to the non-impact evolution. This approach is clearly a useful idealization in some practical applications. The results on invariance can be applied in impact mechanics if we control the application of the impact force.

In order to get more insight about impact dynamics, we model a bouncing ball subject to the gravity and to the ground's reaction force. In the literature [9, 26] we often find models not specifying the trajectory during the impact and only the final velocity after the impact is detailed. However, the impulsive framework allows also the complete definition of the trajectory during the impact. The bouncing ball system has two modes: when the ball is only subject to the gravity force and when it is also subject to the ground's reaction force. In the first mode, we have the following dynamical model:

$$\begin{aligned} \dot{y}(t) &= v(t) \\ \dot{v}(t) &= -g. \end{aligned}$$

Here, y denotes the vertical position of the ball while v represents its velocity. For simplicity, we assume a unitary mass. The impact is modeled by a linear under damped spring:

$$\begin{aligned} \dot{y}(t) &= v(t) \\ \dot{v}(t) &= -g - ky(t) - cv(t), \end{aligned}$$

where k and c represent the spring and damp coefficients, respectively. The complete model of this system is given as follows:

$$\begin{aligned} \dot{y}(t) &= v(t) \\ \dot{v}(t) &= -g + (-ky(t) - cv(t))m(t), \end{aligned}$$

with

$$m(t) = \begin{cases} 0, & \text{if } y > 0 \\ 1, & \text{if } y \leq 0 \end{cases}.$$

Taking a closer look to this model, we conclude that whenever $m(t) = 1$, which corresponds to the case where the reaction force is active, we have that

$$\left\| \int_A [v, (-ky(t) - cv(t))]^T \, dt \right\| >> \left\| \int_A [v, -g]^T \, dt \right\|,$$

where the set A represents the interval when the reaction force is active. The last equation means the impact dynamics are much faster than the "free flight" dynamics and we can assume an ideal situation where we consider the impact dynamics to evolve instantaneously. Thus, when $y \leq 0$, we assume the trajectory is singular and we neglect the influence of $-gdt$. When the trajectory is singular, the time does not evolve and the trajectory is parameterized by $s \in A$. Moreover, the effect of $-gdt$ is negligible. Formally, the ideal situation is achieved in the limit $m^i(t)dt \rightarrow^* d\mu_{sa}(t)$, when $y \leq 0$. We model the complete system in a compact form, as follows:

$$\begin{aligned} dy &= vdt + vd\mu(t) && (5.1)\\ dv &= -gdt + (-ky - cv)d\mu(t), && (5.2) \end{aligned}$$

where the measure has only the singular atomic component and its support corresponds to the instants when the ball hits the ground:

$$\text{supp } \mu = \{t : y(t) \leq 0\}.$$

The norm of the measure is such that the endpoint of the jump should coincide with $y(\eta(t_a^+)) = 0$. Thus, the impulse evolves as long as the ball is in contact with the ground. The singular dynamics is given by:

$$\begin{aligned} \dot{y}(s) &= v(s)\\ \dot{v}(s) &= -ky(s) - cv(s),\ s \in [0, |\mu\{t_a\}|]. \end{aligned}$$

By a straightforward analysis, we conclude $|\mu\{t_a\}|$ has to be $\dfrac{2\pi}{\sqrt{(4k - c^2)}} = \dfrac{\pi}{\omega_d}$ (ω_d stands for the damped natural frequency). Clearly, the interval where the parameter s belongs models the period the ball is in contact with the floor. Figure 5.1 depicts some trajectories of this impulsive system.

Since we are not assuming any friction during the non-impact phase and the singular dynamics are given by an under damped model, we can conclude that there are not accumulation points (the Zeno behavior) as described in [9, 26]. In fact, $v(t^+) \rightarrow 0$ as $t \uparrow +\infty$. Even if we do not consider the singular dynamics modeled

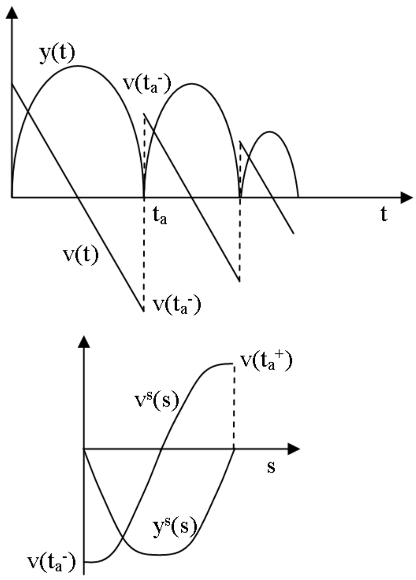

Figure 5.1: Trajectory of the bouncing ball during the conventional and singular phases

by an under damped spring model, we can also conclude that the measure driven differential inclusion allows to solve some well known drawbacks like Zeno behavior. Thus, lets imagine a constant singular dynamics is defined:

$$\begin{aligned} dy &= vdt \\ dv &= -gdt + d\mu_{sa}(t). \end{aligned} \tag{5.3}$$

Again, the singular atomic measure is supported whenever $y(t) \leq 0$ and its total variation is such that the final singular velocity $v(t^+)$ have value $-\alpha v(t^-)$ (α is some scalar value such that $0 < \alpha \leq 1$). In this sense, the measure is computed in a feedback form. Observe this measure driven differential equation models exactly the same effect as in [9, 26]. The advantage is that we can apply the impulsive Euler solution to overcome the so called Zeno behavior and define the solution even after the accumulation points. Imagine, for instance, the initial point is given by

$(x_0, v_0) = (0, 0)$ and we have some partition π in the range of the function $\eta(t)$. Whenever $y_i \leq 0$, we apply a singular atomic measure with total variation equal to $s_{i+1} - s_i$. Otherwise, we set $\mu = 0$ and $t_{i+1} = t_i + (s_{i+1} - s_i)$. Using the partition π, we can define an Euler polygonal arc x_π, v_π on $[0, +\infty)$. Clearly, as we take the limit $\pi_j \downarrow 0$, we conclude that $x_{\pi_j}, v_{\pi_j} \to (0, 0)$ and, consequently, $(0, 0)$ is an impulsive Euler solution of system (5.3) when $(x_0, v_0) = (0, 0)$.

The main merit of the impulsive framework resides in the possibility to derive mathematical models for the impact and no-impact periods in a single framework. Until now we have not seen any control action during the impact. However, we can extend the previous example to the cases where we can control the support of the measure and the trajectory during the impact. Consider the familiar table tennis game where the player, by rotating his wrist, can control the spin of the ball during the impact. Once again, we can model this situation into two separate models and then place them in a single impulsive control framework. Let us consider, for instance, a simple cinematic model for the control of the spin:

$$\dot{\theta} = u, \ u \in [-1, 1].$$

Consider also the coefficients k and c being dependent on the spin of the ball during the impact. Thus, we obtain the following complete model:

$$dy = vdt + vd\mu(t) \tag{5.4}$$

$$dv = -gdt + (-k(\theta)y - c(\theta)v)d\mu(t) \tag{5.5}$$

$$d\theta = ud\mu(t). \tag{5.6}$$

Now, there are two controls available. The measure itself and the spin control variable u. The first control allows deciding when to activate the impact while the second enables the control of the singular trajectory. In this example, we assume that μ has only the singular atomic component.

The impulsive formulation constitutes a modeling advantage since we can separate the system behavior into two distinct models. In this sense, we avoid unnecessary complexity since we only consider the relevant features of each mode. Moreover, in terms of controller synthesis, we have available analysis tools enabling the controller's synthesis (recall the procedures presented in chapters 3 and 4).

Taking control actions during the impact requires great sensing and actuation capabilities in terms of sampling rate and response speed, respectively. If such hardware is available, then we can apply the control feedback laws derived for the impulsive framework. Whenever we detect or provoke a impact force, we implement a fast sampling scheme and apply the feedback control derived for the singular dynamics. On the other hand, when we are in the conventional dynamics, we apply a slower sampling scheme and apply the conventional dynamics controller. In fact, properties of invariance and optimality are only guaranteed to hold in the ideal limit, where we have an "infinity fast" sampling scheme. Thus, in practice, we should show that the duration of the impact is in fact negligible face to the conventional evolution and the performance is not affected by the approximation. Hence, a robustness result should be derived to validate the practical application of the feedback control.

5.2 Investment control

There are several applications in economic theory where the models allow jumps in the controlled variables. Classical examples are investment control, stock management and natural resource exploration. We will illustrate the relevance of our impulsive control model in this context by modeling the problem on resource economics presented in [17]. In this problem, the optimal exploration of a fishery population is studied. Investments on capital can be done instantaneously. The population dynamics of the fishery resource is modeled by the dynamical system:

$$\dot{x}(t) = F(x(t)) - qE(t)x(t),\ x(0) = x_0,$$

where $x(t)$ is the population biomass at time t, $F(x)$ is the natural growth function, q is the catchability coefficient (constant) and $E(t)$ is fishing effort at time t. Regarding the biological growth function $F(x)$, The following condition is assumed:

$$F(0) = F(\bar{x}) = 0,\ F(x) > 0,\ \ddot{F}(x) < 0 \text{ for } 0 < x < \bar{x}.$$

The variables $x(t)$ and $E(t)$ are subject to the following constraints:

$$x(t) > 0 \text{ and } 0 \leq E(t) \leq E_{max} = K(t),$$

where E_{max} is maximum effort capacity and $K(t)$ is the amount of capital invested in the fishery at time t. We shall think of $K(t)$ as representing the number of fishing vessels available to the fishery. The previous control constraint asserts that the maximum effort capacity equals the number of vessels available, and that the actual level of effort employed at any time cannot exceed E_{max}. Possible adjustments to the level of capital are modeled by the measure driven dynamical system:

$$dK(t) = d\mu(t) - \gamma K(t)dt,\ K(0) = K_0,\ d\mu(t) \geq 0, \tag{5.7}$$

where the measure control $d\mu(t)$ models the gross investment rate at time t and γ is the rate of depreciation (constant). The capital stock $K(t)$ are subject to the constraint $K(t) \geq 0$. In terms of modeling, equation (5.7) constitutes the main novelty face to the model presented in [17], since the gross investment in modelled by a measure while in [17] is used a control function that can take the value $+\infty$.

Our control objective is to maximize the discounted net cash flow in a period T, subject to the dynamical constraints presented before. The objective function is defined as:

$$J = \int_0^T e^{-\delta t}[pqE(t)x(t) - cE(t)]dt - \int_{[0,T]} \pi d\mu(t),$$

where δ is the instantaneous rate of discount (constant), p is the price of landed fish (constant), c is the operating cost per unit effort (constant) and π is price (purchase or replacement) of capital (constant). The problem we face is that of determining the optimal effort and investment policies $E(t)$, $d\mu(t)$, leading to the maximization of the previous objective.

The fact that the jump of the state variable K is provoked by a singular atomic control measure enables another modeling and control synthesis features, namely, the possibility to have a singular dynamic system associated with a control measure and a formal definition of a feedback control measure. In this problem, the singular dynamics is trivial since it is constant and equal to one. However, this well studied investment problem is sufficient to show the role of the impulsive Euler solution and the ability to define the control measure in a feedback form. Thus, we can re-use the synthesis presented in [17] to show that the following control synthesis is optimal:

- Whenever the state is in the region R_3 (see figure 5.2), we should apply a singular atomic control measure. In the context of the impulsive Euler solution, the singular atomic measure is constructed with resource of small total variation singular atomic measures. Depending on the state variable, this allows to determine if we should continue to increase the measure's total variation or not. Whenever a small portion of an impulsive Euler polygonal arc enters in the region R_2, we should stop to increase the total variation of the measure and apply the control strategy relative to the region R_2.

- If the state variable is in the region R_2, then the control $E(t)$ is set to be equal to $K(t)$ while the control measure is set to zero.

- Finally, if the state is in region R_1, then both $E(t)$ and $d\mu(t)$ should be zero.

Note that this feedback control strategy leads us to a chattering situation along the switching surface σ_1, above point B. This happens because the state oscillates among regions R_1 and R_2, leading to a consecutive change in the control strategy ($E(t) = 0$ and $E(t) = K(t)$). In the switching surface σ_2, a similar situation happens around the long term optimal point (x^*, K^*) where the state oscillates between region R_2 and region R_3. For each of these cases, the impulsive Euler solution allows to conclude that, in the limit, the solution coincides with the border marked in figure 5.2 with segments A, B and with $(x^*, 0), (x^*, K^*)$, respectively.

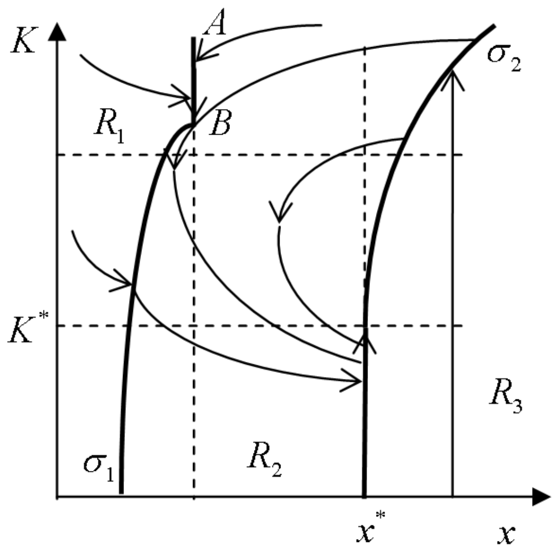

Figure 5.2: Control synthesis for the investment problem [17]

5.3 Composition of dynamic systems

The composition of dynamic systems can be addressed under the impulsive control framework. In this context, the state variable of a dynamical system is affected by different dynamical systems in a simultaneous form, in an alternate form or in a singular form. This composition can be modeled by the impulsive control framework as follows:

$$dx(t) \in F_1(x(t))d\mu_1(t) + \cdots + F_n(x(t))d\mu_n(t).$$

We assume there are n dynamical systems affecting the state variable. The measures $\mu_1, \cdots, \mu_n$ model the way each dynamical system contributes to the state evolution. Thus, if the measures are absolutely continuous in relation to the Lebesgue measure, then they weight the influence of each dynamical system in the state evolution. When a subset of measures is singular atomic, the only dynamic systems contributing to the state evolution are precisely those associated with the singular atomic measures. In this case, the evolution of the state trajectory is parameterized by s, which models the total variation of the measure. The trajectories are singular in relation to conventional time and the state $x(t)$ can have "jumps" in relation to the Lebesgue measure. A third situation happens when some measures are absolutely continuous while others are singular continuous in relation to the Lebesgue measure. In this case, the trajectory is continuous but some dynamical systems contribute singularly to its evolution.

The previous model encounters many practical applications, namely, whenever the composition between "fast" and "slow" dynamic systems is required. Examples are impact dynamics (explored before), formations control, failure detection and recovery and hybrid systems. Hybrid systems and formations control are further explored in the next subsections. In failure detection, whenever the system is working properly, a dynamical model associated with dt is used. Otherwise, a failure is detected, the control system applies a new sampling scheme and a new dynamical model associated with the measure $d\mu$. The control is designed to overcome the failure in a very short period of time.

5.3.1 Hybrid systems

Hybrid systems have been considered a convenient modeling framework to describe large classes of systems and extensively used in a wide range of applications, [1, 2, 3]. The trajectory of a hybrid system evolves as a result of the interaction between continuous and discrete dynamics. This interaction reflects the compositional properties underlying the hybrid system model. On the other hand, as shown before, an impulsive dynamic system can be regarded as a composition of multiple dynamic systems, and, hence, is suited to model the behavior of some classes of hybrid systems [21]. By exhibiting jumps in the trajectory, the impulsive control paradigm encompasses evolutions due to the interaction of continuous and discrete dynamics.

The relationship between hybrid and impulsive systems enables the usage of wealthy results available for impulsive systems. In [5], the relationship between differential impulsive inclusions and hybrid systems has been recognized in the context of reachability and viability theory. A distinguishing feature of this work and the one discussed here is precisely the way we describe the system behavior "during" the jump associated with the fact that jumps are provoked by control measures. Thus, instead of considering just a discrete transition in the state trajectory, in the impulsive formalism, we also have the process that leads to that transition.

We adopt the taxonomy for hybrid systems of [9] to show the impulsive systems formulation may model some classes of hybrid systems. We will give a formulation in terms of impulsive systems for the four classes of hybrid systems of that taxonomy: autonomous switching, autonomous impulses, controlled switching and controlled impulses. For each of these classes, we will give the same example as in [9] for comparison purposes.

Autonomous switching

In this class of hybrid systems, the vector field may switch during the system operation. The switching happens instantaneously and we model this situation by a predefined feedback control law. Thus, we parameterize the vector field by a state variable q:

$$\dot{x}(t) = f(x(t), u(t), q(t)), \quad t \geq 0. \tag{5.8}$$

Additionally, we assume that the vector field may also depend on the state and on the control u. The parameter q is assumed to be a state variable that may change in discrete times

$$dq(t) = g(t, x(t))d\mu(t). \tag{5.9}$$

The measure μ models the switching and should be defined in a feedback form since we are considering the autonomous case. One possible way to define such feedback control measure law is to specify a set where we wish the trajectories to remain. In the next example, we show how to use this approach to model autonomous switching systems.

Example 5.3.1. *In this example we model of dynamical system with hysteresis:*

$$\dot{x} = H(x) + u, \tag{5.10}$$

where the state x and the control $u \in U$ are real numbers. The function H is an hysteresis having a threshold of Δ and is depicted in figure 5.3 [9]. We can model this

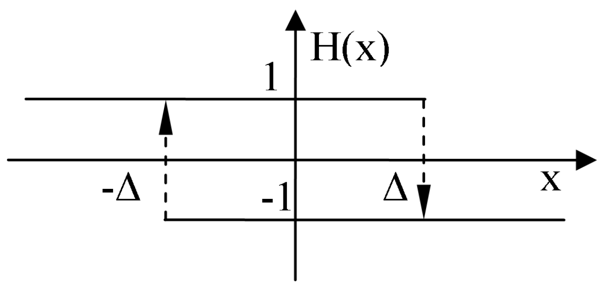

Figure 5.3: Hysteresis function

system using the impulsive formulation, as follows:

$$\begin{aligned} \dot{x} &= u + q, & (5.11)\\ dq &= -2\sigma(x)\mu_{sa}(dt), & (5.12) \end{aligned}$$

where σ is a suitable Lipschitz continuous approximation to the signal function, $q(0) \in \{-1, 1\}$ and the measure μ being characterized in a feedback form such that *the impulsive trajectories are invariant in relation to the set depicted in figure 5.4 by the solid line. The previous hysteresis system is sometimes modeled by an hybrid automata. Thus, from this example, we conclude that hybrid automata can be suitably modeled by composing dynamic systems using the impulsive framework.*

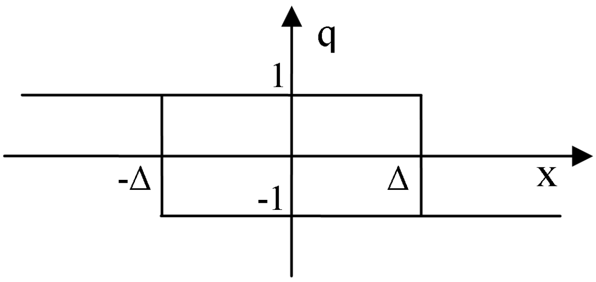

Figure 5.4: The invariant set for the hysteresis problem

Autonomous impulses

In this class of hybrid system, the state trajectory may exhibit a discontinuity when hitting prescribed regions of the state space. This systems can also be modeled as an impulsive system, as follows:

$$dx \in F(x)dt + G(x)d\mu(t), \tag{5.13}$$

Since the jumps in the trajectory happens autonomously, the measure should be defined in a feedback form. The multifunction F determines the system evolution in the continuous phase while the set-valued map G defines the trajectory in the jump period. The singular dynamics corresponds to a discrete evolution. In this formalism, the control of the jump is enabled by choosing appropriate selections of G. The possible discrete locations will correspond to points in the reachable set of the singular dynamics.

A typical example of this type of hybrid system was already presented in the context of impact dynamics. The bouncing ball is a typical example of autonomous jumps. We saw that the well know Zeno-behavior can be overcome using the impulsive Euler solution.

Controlled Switching

In controlled switching hybrid systems the vector field of equation (5.8) commutes not autonomously but in response to a control command. In other words, the feedback control measure is not predefined and the controller is free to take any action. In the impulsive framework, the number of possible states q in equation (5.9) may be infinite. However, this formalism may also be applied in situations where a finite number of vector fields is required. This is accomplished by introducing trajectory constraints, as it is illustrated in the next example.

Example 5.3.2. *Consider a simple satellite system where we can control its orientation by means of reaction jets:*

$$\dot{\omega}(t) = \tau v(t), \ \ t \geq 0. \tag{5.14}$$

Here, ω *is the angular speed,* τ *is the constant force applied by the jets and* $v \in \{-1, 0, 1\}$ *is a variable that controls the direction of the jets (reverse, off and forward). This hybrid system can be formulated as an impulsive control system as follows:*

$$d\omega(t) = \tau v(t) dt \tag{5.15}$$

$$dv(t) = d\mu_{sa}(t) \tag{5.16}$$

subject to $v(t) \in \{-1, 0, 1\}$*. In this example, the measure is only assumed to have the singular atomic component. In this example, the measure in not given in advance but is chosen by the controller. In this sense, we can control the composition of dynamical systems.*

Controlled impulses

Hybrid systems of controlled impulses class are similar to the autonomous impulses except that now we do not have a pre-specification of the measure. The state may jump whenever the controller activates the measure. We reinforce here that, by using the impulsive control formalism, we control the composition of different dynamic systems by the control measures.

Example 5.3.3. *Consider an inventory management model*

$$dx(t) = -a(t)dt + d\mu(t) \tag{5.17}$$

where x represents the stock level, the function a represents the stock consumption rate and the measure μ is the control variable that restores the stock level. By a suitable choice of the measure, the level stock can be controlled. Clearly, this impulsive model allows the composition of the dynamical system modeling the stock consumption and the dynamical system modeling the restoring stock level

5.3.2 Formations control

We have seen that the impulsive control framework is well suited to model the composition between "fast" and "slow" dynamic systems. This composition arises in many engineering applications, like the control of formations [6, 41, 19, 31]. In this application, we have a network of autonomous vehicles that may interact between each other in order to achieve multiple configurations. There are two modes of operation. The working mode is concerned with maintaining the actual configuration of the vehicles while the reconfiguration mode is concerned with the fast modification of the actual configuration.

Formations of autonomous systems encounter many applications nowadays. As an example, for Earth science applications, formation flying spacecraft can provide distributed sensing for gravitational field mapping, contemporaneous spatial sampling of atmospheric data, co-observations (i.e., near-simultaneous observations of the same science target by instruments on multiple platforms), and synthetic radio-frequency and radar apertures. In particular, it will become possible to expand science capabilities by deploying large numbers of low cost, Earth-orbiting miniaturized spacecraft. Also, in the field of underwater observations, there are many applications for formations control. Usually, autonomous underwater vehicles have limited room to place all necessary sensors. Thus, it is convenient to have a network of vehicles where each vehicle carries its own sensor. Then, the vehicles can share information in order to achieve a certain objective. An added advantage of this paradigm is that new formation members can be introduced to expand or upgrade the formation, or to replace a failed member.

In formations control, we can assume there is a continuous evolution mode, corresponding to the preservation of a given configuration, and an impulsive mode, where there is a change in the formation configuration. Usually, this problem is studied under the hybrid systems formulation. However, using the impulsive framework, we can better describe the trajectory of each vehicle during the reconfiguration period. Since the reconfiguration takes a very short period of time, when compared with the working mode, we can assume the reconfiguration trajectory as being singular and associated with a singular atomic measure. Hence, we can integrate these two modes of operation in a single impulsive framework and synthesize controllers using the approaches developed before.

In order to show the advantages of the impulsive control formulation, we start by a simple example where each element of the formation has a one-dimensional state controlled by two variables. One control variable has small amplitude and is well suited to preserve the configuration of the formation. The other control variable can take bigger values and is useful for fast reconfigurations. Due to the characteristics and applications of each controller and their applications, the second controller can be approximated by a control measure. Note that the reconfiguration phase should take the less time possible since, in general, in this period the system is not performing the main task for which it was designed. Hence, we consider a network of N vehicles with the following dynamics:

$$dx^i(t) \in U^i dt + d\mu^i,$$

with $i = 1, \cdots, N$ and $U^i \in [-\delta, \delta]$. For the working phase, we provide references x^i_{ref} for each vehicle. The role of the control (selection) $u^i \in U^i$ is to keep the variable $x^i(t)$ close to its reference while rejecting external disturbances. On the other hand, the role of the control measures μ^i is to enable fast reconfigurations. We consider three possibilities for the reconfiguration:

- The first approach is to assign individually new references for each vehicle (no information exchange). Since we are assuming positive valued measures then, necessarily, the new assigned references for each vehicle should be greater or equal to the previous reference. The new references are given along $t \in [0, +\infty)$. Hence, using the proximal aiming approach, a feedback control measure can be computed such that $x^i(t)$ reaches its reference value. If we always assume increasing references, then the existence of such control measure is assured.

- The second approach provides references that depend on each other. As an example, we define $x^i_{ref}(t_a) = \alpha$ and $x^j_{ref}(t_a) = x^i_{ref}(t_a) + \beta$, with $i \neq j$ and $i, j = 1, \cdots, N$. In this case, before the reconfiguration period, the subsystems should exchange information, namely, their new references assignments.

- The third approach is more involving since we require information exchange during the reconfiguration period. Let $x(t_a^-)$ be the positions of the vehicles before the reconfiguration. Now, we define the new references, not specifying the values individually for each vehicle, but defining a set in $\mathbb{R}^N$ where the elements x^i should be at t_a^+. In fact, this problem can be cast in the proximal aiming setting and we can compute the measure in feedback. For this purpose, we can use the results on proximal aiming and on invariance. Clearly, the reference set should be defined in a way that it is reachable by the singular dynamics. We already know, from the previous chapters, that a sufficient condition for this to happen is that for every $x \in \mathbb{R}^N$ there exists a control $v(x) \in K \cap \bar{B}$ such that $\langle v, x - p \rangle < 0$, where $p = \text{proj}_{S_{ref}}(x)$ and the control $v(x)$ represents the auxiliary singular control defined in the robust solution. For illustration purposes, let us consider $N = 2$ and a set S_{ref} being defined in the following form:

$$S_{ref} := \{(x_1, x_2) : \bar{x} - \varepsilon_1 \leq \frac{x_1 + x_2}{2} \leq \bar{x} + \varepsilon,\ (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 \leq \varepsilon_2\},$$

where $\bar{x}$ represents the new desired average value and ε_1 and ε_2 are given positive constants. In this particular case, the set S_{ref} models the objective to place the vehicles in a formation such that they are in a neighborhood of a new assigned mean value and the standard deviation is also within the specified limits. Imagine, for instance, the system configuration before the jump is $x(t_a^-) = (0, 0)$ and we require that $x(t_a^+) \in S_{ref}$. Then, we can construct the feedback control measure such that the singular trajectory reaches the set S_{ref}. We only need to choose, for each x, the singular control v giving the minimum $\min_{v \in K \cap \bar{B}} \langle v, x - p \rangle < 0$. In figure 5.5 is depicted a possible singular trajectory that drives the system from the initial point to the reference set. During the reconfiguration period, the agents should communicate their states in order to implement this algorithm.

Now that we introduced some concepts about formation control, we can provide a more complex model with more adherence to practical applications. We are going

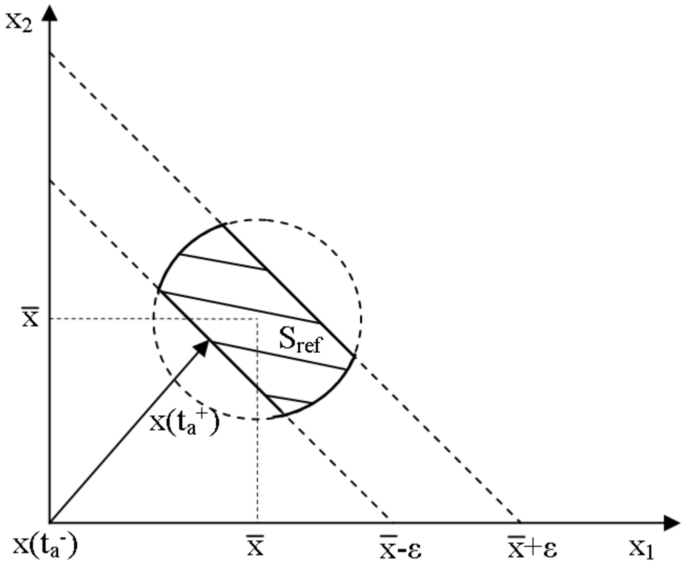

Figure 5.5: Reconfiguration of a formation control

to consider a formation of autonomous underwater vehicles. We will consider a simplified model for the vehicles to focus on some the relevant features of the impulsive framework. We will consider a 2-dimensional plane and unitary mass vehicles. The model of each vehicle is given as follows:

$$\begin{bmatrix} \dot{x}^i \\ \dot{v}_x^i \\ \dot{y}^i \\ \dot{v}_y^i \end{bmatrix} = f^i(X^i, U^i) := \begin{bmatrix} v_x^i \\ u^i \cos(\theta^i) - d.(v_x^i)^2 \\ v_y^i \\ u^i \sin(\theta^i) - d.(v_y^i)^2 \end{bmatrix},$$

where $i = 1, \cdots, N$, x^i, y^i represents the position of each vehicle within an inertial coordinate frame, and v_x^i, v_y^i represents the velocity expressed in this coordinate frame. The parameter d represents the drag coefficient and u^i is the thruster force applied in each vehicle. We assume that the angle of the vehicle in relation to the inertial frame is a control variable. The variable $X^i := (x^i, v_x^i, y^i, v_y^i)^T$ represents the 4-dimensional state variable while $U^i := (u^i, \theta^i)^T$ represents the control variable.

Now, consider the following formation control problem. During the non-reconfiguration mode, we wish to keep the velocities v_x^i and v_y^i and the angle θ^i constants and equal to their values after a reconfiguration. Lets represent these values by $\bar{v}_x^i$, $\bar{v}_y^i$ and $\bar{\theta}^i$. We can develop a linearization around these working points with the purpose to simplify the controller synthesis and to enhance its performance. The steady state thruster force $\bar{u}^i$ can be computed by the following relationship: $0 = \bar{u}\cos\bar{\theta} - d.(\bar{v}_x^i)^2$. Hence, the linear model around the working point is given by:

$$\begin{bmatrix} \dot{x}^i \\ \dot{v}_x^i \\ \dot{y}^i \\ \dot{v}_y^i \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2d & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2d \end{bmatrix} \cdot \begin{bmatrix} \delta x^i \\ \delta v_x^i \\ \delta y^i \\ \delta v_y^i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \cos\bar{\theta} & -\bar{u}^i \sin\bar{\theta} \\ 0 & 0 \\ \sin\bar{\theta} & \bar{u}^i \cos\bar{\theta} \end{bmatrix} \cdot \begin{bmatrix} \delta u^i \\ \delta\theta^i \end{bmatrix} \tag{5.18}$$

where $\delta x^i := x^i - \bar{x}^i$ (same notation for the other state variables), $\delta u^i := u^i - \bar{u}^i$ and $\delta\theta^i := \theta^i - \bar{\theta}^i$. On the other hand, in the reconfiguration phase, we need to use the complete nonlinear model since substantial motion is required. This phase is designed to take a short period of time when compared with the other mode. Therefore, we design the reconfiguration trajectory to be singular in relation to the non-reconfiguration mode. In this way, we can compose two distinct dynamic systems in a single framework, as follows:

$$dX^i(t) = (A^i \delta X^i(t) + B^i \delta U^i(t))dt + f(X^i(t), U^i(t))d\mu^i(t).$$

The measures μ^i are supported at the instants where a reconfiguration is needed. This allows to use distinct models for different modes of operation. The working phase is associated with the measure dt while the reconfiguration phase is associated with the measure $d\mu$. In the first mode, we can use classical linear controller synthesis to stabilize the system around the point $\bar{X}^i$. In the reconfiguration mode, we can use the proximal aiming methods presented before.

5.4 Quasi-Invariant systems

In many practical applications the measure is only allowed to be active if the state is in a given set. We have seen examples of this situation in hybrid systems and in impact dynamic systems. For example, in the bouncing ball system the measure is only active whenever the ball hits the ground. In hybrid systems, namely in autonomous impulses and switching, the transition is performed whenever the state lies in a predefined region. Hence, the notion of quasi-invariant systems intents to provide a characterization of weak invariance whenever the measure can only be active in a given closed set S_G.

Definition 5.4.1. *The system $(S_F \cup S_G, F, G)$ is quasi-weak invariant if there is at least one impulsive trajectory with $x(0) \in S_F \cup S_G$ such that $x(t), x^s(s) \in S_F \cup S_G$ and the control measure μ can only be active if the state variable is in the S_G.*

As an immediate consequence of theorem 3.3.2, we can conclude that a system $(S_F \cup S_G, F, G)$ is quasi-weak invariant if and only if the following conditions hold:

$$h_F(x, N^P_{S_F \cup S_G}(x)) \leq 0, \; \forall x \in S_F \backslash S_G \tag{5.19}$$

$$\min\{h_F(x,\zeta), h_G(x,\zeta)\} \leq 0, \; \forall \zeta \in N^P_{S_F \cup S_G}(x), \; \forall x \in S_G. \tag{5.20}$$

Chapter 6

Conclusion

6.1 Main achievements

The increasing interest on hybrid systems and systems having discontinuous trajectories compelled us to further develop systems modeled by measure driven differential inclusions. This class of systems have control measures that are responsible for the discontinuities of the trajectory. We chose this model since it provides a formal definition on how the jump is conducted and due to the completeness of the control space. These characteristics are important to better characterize the physical phenomena and for analysis purposes.

Hybrid systems have a compositional behavior between continuous dynamic systems and discrete event systems. This interaction requires a mathematical model in which the trajectories can be discontinuous. In this context, the composition occurs in a controlled way and, for this reason, the measure driven differential inclusion model is well suited to analyze hybrid systems. Due to the increasing sensing and actuator capabilities, in terms of sampling and response speed, it became possible to control fast transitions on the trajectories. To address this issue in a way that a feedback control law can be derived, we developed the impulsive Euler solution (chapter

2). This solution does not require any regularity assumptions on the selections of the measure driven differential inclusion, which allows the characterization of the solution even if the selections are discontinuous. Additionally, the impulsive Euler solution permits the construction of the control measure in a feedback form. We showed that instead of making a partition in the time axis (as done in the nonimpulsive model), we should define a partition in the range of a function depending on time and on the total variation of the measure. This scheme leads to an adaptive sampling in the sense that it depends on the actual value of the measure. Thus, even when the measure is singular atomic, we can define it in a feedback form: we construct the measure in small partitions of total variation and, depending on the state, we decide if we should increase its total variation or not.

The impulsive Euler solution is essential to develop constructive results on invariance and optimal control. The results are constructive since they provide synthesis mechanisms for the control. In other words, we are able to define a procedure to construct feedback selections and control measures such that the resulting trajectories have the desired properties: to remain inside a given closed set or to be optimal in relation to some cost function. The optimal control problem is studied with resource of the invariance results. In this case, the invariance set is not given in advance but is derived from the value function associated with the optimal control problem. The feedback selections and the control measure leading this set invariant are proved to be optimal.

The definition of the control measure in a feedback form requires a solution defined in the original time frame. We noted that the reparameterization technique, which has being used in the open loop control context, requires a new time scale and, for this reason, makes a distortion on the vector fields of the measure driven differential inclusion. This is one reason for which the reparameterization technique is not suited for a feedback control strategy. Moreover, we showed that, even in the open loop case, this technique does not synthesize the singular continuous component of the measure.

We presented a set of examples where was illustrated the relevance of the previous results in engineering applications. In the hybrid systems context, the so called Zeno behavior was overcome with resource of the impulsive Euler solution and relating the impulsive control framework with hybrid systems. We also observed how the compositional behavior of hybrid systems and complexity can be addressed using the impulsive control framework. In particular, we developed a complete example on formations control where these issues were addressed. Applications in impact dynamics and investment control were also explored and, again, the importance of the impulsive formulation was demonstrated.

6.2 Challenges

The impulsive framework models additional information related with the description of the singular trajectory. This brings an additional complexity and, consequently, there is a trade-off between the completeness of the model and the computational effort associated with the implementation of the control laws. Moreover, since a complete control space is required, we have to consider the singular continuous component of the measure whose practical meaning is not yet fully understood.

The control measure represents an idealization of a practical implementation. The singular atomic component, for instance, is an idealization of a measurable function taking large values in short periods of time. Hence, the provided analytical results have only an approximated implementation in practice. Thus, the robustness of the controllers should be addressed in order to understand how the difference between the ideal model and its practical implementation affects the derived results.

6.3 Future work

The purpose of this section is to present possible future developments following the concepts of this thesis.

The issue of extending the impulsive model for the case where the measure is signed and vector-valued is not yet addressed and possesses some interesting challenges. It is missing a solution at the support of the singular atomic measure whenever the columns of the multifunction associated with the measure do not commute. The challenge arises when there is a jump in the trajectory while the singular atomic atoms cancel each other. So there is not a justification (a control action) for a jump in the trajectory. Additionally, the measure driven control system (instead of measure driven differential inclusion) is not fully addressed and it would be valuable to define an impulsive Euler solution, as performed for conventional systems in [15].

The issue of stability was not addressed but the impulsive Euler solution can also be an important analytical tool to develop constructive results in this domain. In the literature, the issue of stability for measure driven systems has been studied with resource of the reparameterization technique [32, 33] and little effort has been given to constructive methods involving a feedback control strategy. The notion of an equilibrium point for impulsive systems needs clarification and is a enabling concept for the study of impulsive stability.

The inclusion of trajectory constraints in the optimal control problem brings another technical challenges, like the possibility of the value function being discontinuous.

As we already know, a feedback control law can be designed such that it is robust relative to model uncertainties and sensor errors. In this sense, it would be advantageous to make a formal analysis on robustness of the controllers synthesized before. In this domain, we should also consider that a measure is an idealization of a practical object. For this reason, the controllers should also be analyzed in order to conclude if the required properties remain valid under the approximating objects.

Under the computational point of view, it is clear how to implement the absolutely continuous and singular atomic components of the measure. However, the implementation of the singular continuous component is not immediately clear and further research will be necessary. We have seen the importance of having a complete control space for analysis purposes and, for this reason, the treatment of the singular continuous component plays a major role.

The development of efficient computational tools implementing the impulsive control synthesis is another important challenge. This is directly related with the added complexity of the impulsive model and there should be a reasonable trade-off between complexity and computation. In this vein, it would be necessary to develop effective numerical algorithms for the Hamilton-Jacobi equation for impulsive problems.

Finally, a major future challenge will be to explore further the connection between hybrid systems and impulsive systems. Hybrid systems have a very strong engineering motivation. On the other hand, there are important analytical results for impulsive systems. Thus, the relationship between these two subjects can be relevant to enhance the performance of several control applications by providing new analytical and computational methodologies.

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