FACULDADE DE ECONOMIA DO PORTO

ESSAYS ON AMBIGUITY ABOUT STOCHASTIC VARIANCE

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TESE DE DOUTORAMENTO EM ECONOMIA

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Victor

Amigo que foste
com grinaldas
de flores marinhas

Além da terra
das pedras
das claras e translúcidas
gotas de chuva
do feno dos campos
do sorriso das raparigas
do vermelho das papoilas
das frases que são só fantasia

Numa névoa de longínquo olhar
Amigo que foste
não te digo adeus
digo até breve.

- Fernando Rente -

A todos os Meus: os que partiram,
os que me confortam com a sua presença,
os que ficarem quando eu partir.

Nota Biográfica

Gonçalo Manuel Albuquerque Pereira Oliveira de Faria nasceu a 9 de Novembro de 1977, no Porto.

Em Julho de 2000 terminou a licenciatura em Economia, pela Faculdade de Economia da Universidade do Porto, com média final de 15 valores. Em Julho de 2004 completou o programa de formação em avaliação e gestão de activos do CFA Institute (Charlottesville, USA), sendo desde então Chartered Financial Analyst (CFA) e membro do CFA Institute.

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Em Setembro de 2005 iniciou o programa de Doutoramento em Economia pela Faculdade de Economia da Universidade do Porto. Completou em Junho de 2008 a componente escolar do programa de Doutoramento com média final de 17 valores, tendo então iniciado os trabalhos da tese.

Apresentou os artigos que constituem a presente tese em seminários e *workshops* de doutoramento nas Universidades do Porto, do Minho e de Vigo e em encontros científicos: 3rd Economic Theory Workshop (Vigo); 1st World Finance Conference (Viana do Castelo); 6th World Congress of the Bachelier Finance Society (Toronto); 6th Portuguese Finance Network Conference (Ponta Delgada) e na 37th European Finance Association Meeting (Frankfurt).

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Sumário

Em 1921, Frank Knight realçou que a incerteza dos agentes económicos não é susceptível de ser totalmente descrita por uma distribuição de probabilidade. Isto motivou a distinção entre *risco* e *ambiguidade*, enquanto incerteza "probabilística" e "não probabilística", respectivamente.

Esta tese pretende ser um contributo para a literatura sobre decisões de carteira, "pricing" de activos em equilíbrio geral e de opções Europeias, considerando ambiguidade sobre o processo estocástico da variância. Tanto quanto é do nosso conhecimento, é a primeira vez que tal sucede e, por isso, esse é o principal contributo desta tese.

No capítulo 2, consideramos o modelo de decisões de carteira de Chacko e Viceira (2005), introduzindo ambiguidade sobre o valor esperado da variância através do método "multiple priors" de Gilboa e Schmeidler (1989). Assume-se que o investidor tem preferências recursivas (Duffie e Epstein (1992b)), com especificação de Kreps e Porteus (1978).

No capítulo 3, o investidor representativo tem uma função utilidade CRRA e é ambíguo sobre o processo estocástico da variância. É adoptado o método "multiplier preferences robust control" para obter a expressão analítica da decisão óptima de carteira.

No capítulo 4, desenvolvemos uma versão do modelo de equilíbrio geral de Cox et al. (1985a) considerando ambiguidade, com um processo produtivo e duas variáveis estado correlacionadas, em que apenas uma delas tem correlação com o crescimento do produto. A ambiguidade é sobre o processo estocástico dessa variável estado, sendo introduzida através do método "constraint preferences robust control".

No capítulo 5, generalizamos o modelo de avaliação de opções de Heston (1993) obtendo uma fórmula para o preço de uma opção de compra Europeia quando (i) a variância do retorno do activo subjacente é estocástica e (ii) existe ambiguidade sobre o seu processo estocástico. A especificação do preço da incerteza da variância é motivada pelo modelo de equilíbrio desenvolvido no capítulo 4.

Summary

In 1921, Frank Knight pointed out that uncertainty faced by economic agents may not be totally described by a probability distribution. This motivated the distinction between *risk* and *ambiguity*, as "probabilized" and "non-probabilized" uncertainty, respectively.

This thesis aims to contribute to the literature on portfolio choice, general equilibrium asset pricing and European option pricing, by considering ambiguity about the stochastic process of the variance. To the best of our knowledge, this is the first attempt to do so and therefore this is the major contribution of the present thesis.

In chapter 2, we extend the model of Chacko and Viceira (2005) for portfolio choice, introducing ambiguity about the expected value of variance through the multiple priors approach of Gilboa and Schmeidler (1989). The representative investor is assumed to have recursive preferences, as defined by Duffie and Epstein (1992b), with Kreps and Porteus (1978) specification.

In chapter 3, a representative investor with CRRA utility function over terminal wealth is assumed to be ambiguous about the model describing variance dynamics. A"multiplier preferences" robust control approach is adopted to obtain a closed-form optimal portfolio rule.

In chapter 4, we develop an ambiguous version of the Cox et al. (1985a) general equilibrium model, with a single production process and two correlated state variables, with only one of the state variables being correlated with the economy's output rate. Ambiguity is about the stochastic process of that state variable and is introduced through a "constraint preferences" robust control approach.

In chapter 5, we extend Heston's (1993) option pricing model in order to obtain a closed-form solution for the price of a European call option when (i) the underlying asset return's variance is stochastic and (ii) there exists ambiguity about the variance stochastic process. The variance uncertainty price specification used is motivated by the equilibrium model developed in chapter 4, although the option pricing formula is deducted through a no-arbitrage approach.

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Chapter 1

Introduction

"We perceive the world before we react to it, and we react not to what we perceive, but always to what we infer."

Knight (1921)

Decision theory under uncertainty has been posting, during the last two decades, relevant theoretical and experimental advances. Those advances have their conceptual root in the early work of Knight (1921), who pointed out that uncertainty faced by economic agents may not be totally described by a probability distribution. At that time, also Keynes (1921, 1936) highlighted the importance of "animal spirits" when investors are unable to estimate reliable probabilities.

This motivated the distinction between "probabilized" and "non-probabilized" uncertainty. As stated in Etner et al. (2009), risk "*represents situations in which information is available, in the form of probability distributions",* with the economic agent choosing among probability distributions defined on some outcome space (lotteries). Conversely, ambiguity corresponds to a situation where there is no information about probabilities.[1]

Ellsberg (1961), in a seminal experimental exercise, obtained evidence supporting the relevance of this conceptual distinction. That evidence became known in the literature as the "Ellsberg Paradox", and motivated further empirical works (see, for e.g., Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey).

As a simple variation of Ellsberg's (1961) experiments, imagine that there are two urns, each of them with 100 balls that can be either red or blue. In urn A (ambiguous urn), the breakdown between red and blue balls is not known. In urn B (risky urn), it is known that there are 50 red balls and 50 blue balls. Placed in a choice situation, it is found that the typical agent chooses to drawn a ball from urn B, i.e., has a strict preference for betting on the risky urn. This is because in that case the chance of winning (50%) is "safe" and well understood. However this implies a paradox: the implied "subjective" probabilities of getting either a red or a blue ball from urn A are both lower than 50%, meaning that standard additive probability can not represent Ellsberg's evidence about agents' behavior in such uncertain context.

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years has been ignoring ambiguity, reducing all uncertainty to risk. It is built up either on von Neumann and Morgenstern's (1944) expected utility theory, where the probabilities of the possible states of nature are known, or on Savage's (1954) subjective expected utility theory, where although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of the expected utility according to some subjective probability beliefs.

However, the lack of intuitive appeal of treating decision situations with no available

[1] In the literature, sometimes the term "uncertainty" refers exclusively to ambiguity, with the distinction being therefore between uncertainty and risk. The expression "Knightian uncertainty" is also common. We will use the term ambiguity alongside risk, which taken together, constitute the uncertainty faced by economic agents.

information in the same way as situations in which information is precise, coupled with empirical results such as those obtained in Ellsberg (1961), motivated the efforts during the last twenty years in generalizing Savage's (1954) subjective expected utility theory in order to incorporate ambiguity considerations through axiomatically founded ways. Those efforts resulted in four major types of models: Choquet Expected Utility, Maxmin Expected Utility, Robust Control and Smooth Ambiguity Aversion.[2]

The Choquet expected utility model of Schmeidler (1989) was the first axiomatically founded model, where beliefs under uncertainty are represented by capacities. Axiomatically, the major difference relatively to the subjective expected utility is the replacement of the "independence axiom" (also referred in the literature as the "sure thing principle") by the "comonotonic independence axiom". This type of model offers high flexibility in modeling attitudes towards ambiguity, but has the drawback of the difficulty in giving an intuitive interpretation for representing beliefs through capacities.

The next generation of models extended Savage's expected utility by substituting "precise" subjective beliefs (a single probability distribution) by "imprecise" subjective beliefs (set of probability distributions). This is the "Multiple Priors" family of models, of which the most well known is the Maxmin expected utility model of Gilboa and Schmeidler (1989). The axiomatization in this model, as in Schmeidler (1989), is done under the Anscombe and Aumann (1963) set up, with the major novelty being the replacement of the "independence axiom" by the "certainty independence axiom" and the "uncertainty aversion axiom". The latter can be read as follows: take two indifferent acts f and g for the agent, and a coefficient $\alpha \in [0,1]$. Then, the agent prefers $\alpha f + (1-\alpha) g$ to f (or g). This axiom captures the rationale behind the explanation of Ellsberg experimental results: intuitively it sets that randomization between indifferent acts can be valuable because it can hedge ambiguity.

The central issue is that, unlike the "independence axiom" in the expected utility theory and in the subjective expected utility theory, both the "comonotonic independence axiom" in Schmeidler (1989) and the "certainty independence axiom" and the "uncertainty aversion axiom" in Gilboa and Schmeidler (1989) do not exclude that hedging possibility.

The relationship between Schmeidler (1989) Choquet expected utility model and Gilboa and Schmeidler (1989) Maxmin expected utility model is that, under the assumption of uncertainty aversion, the set of priors of the latter is the core of the capacity in the former.[3]

In the setting of Gilboa and Schmeidler (1989), the ambiguity averse agent takes into

[2]As pointed out by Etner et al. (2009), the Wald maxmin criterion, later generalized by Arrow and Hurwicz (1972), can be seen as the ancestor of this literature on non-Bayesian decision criteria.

[3]From Gilboa and Schmeidler (1989): "*The core of ν, by definition, consists of all finitely additive probability measures that majorize ν pointwise (i.e. event-wise)*", where ν is a capacity.

account the minimum expected utility over all set of priors. That min operator and the set of priors define the attitude of the agent with respect to ambiguity. However, the issue of how the set of priors is determined is left open.

A more recent stream of models apply robust control techniques. These type of models were introduced in economics by Hansen and Sargent (1995), building on work done by engineers on robust control, and are associated with a model uncertainty assumption. The latter can be broadly described as the agent having a benchmark model but, not being totally sure about it, considers "statistically close" alternative models. Under this robust control approach, two main settings have been used in the ambiguity related literature: the "constraint preferences" and the "multiplier preferences".[4] Under the "constraint preferences" (e.g. in Hansen et al. (2006) and Gagliardini et al. (2009)), the set of priors is built under a constraint on the dimension of allowable perturbations from the benchmark model. Conversely, under the "multiplier preferences" type of model (e.g. in Maenhout (2006)), robust preferences are constructed by penalizing deviations from the benchmark model. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different, as highlighted by Trojani and Vanini (2004).

The relationship between the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001a), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006a).

More recently, the smooth ambiguity aversion utility theory has been developed on the back of the seminal work of Klibanoff et al. (2005). This setup is claimed to distinguish ambiguity from ambiguity aversion, allowing for smooth indifference curves and therefore, avoiding the infinite ambiguity aversion implied, for example, in Gilboa and Schmeidler (1989) approach. However, regarding this stream of models, there is still a debate in the literature about their axiomatic foundations (see Epstein (2010) and Klibanoff et al. (2009a) for a recent exchange on this), implying that Gilboa and Schmeidler's (1989) ambiguity aversion spirit (taking into account the minimum expected utility over all set of priors) continues to be the most used in the literature.

However, the implied decision criteria in Gilboa and Schmeidler (1989) may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. That is, there may exist intuitive bounds to limit that set of priors in each problem. This is why the criteria of Gilboa and Schmeidler (1989) is not so extreme as, for example, the Wald maxmin criteria. As Epstein and Schneider (2010)

[4] We adopt this terminology from Hansen and Sargent (2006).

claimed: *"Ultimately, the only way to argue that the model is extreme is to demonstrate extreme behavioral implications of the axioms, something that has not been done"*.

Further extensions of those models have been made in order to adapt them to dynamic problems. From a conceptual point of view, the main issue is how updated preferences are related with decisions taken ex-ante. Related with this, there are two central properties: dynamic consistency and consequentialism (see, for e.g., Etner et al. (2009) and Dominiak et al. (2009) for a good exposition on this). Dynamic consistency means that ex-ante contingent decisions are "respected" by updated preferences. Consequentialism means that only states of nature that are still possible matter for updated preferences. When dynamic consistency and consequentialism hold simultaneously, then preferences are of expected utility form and beliefs are updated according to the Bayes rule. However, as shown in Ghirardato (2002), going beyond expected utility models (violations of the "sure thing principle") imply dynamic inconsistencies when consequentialism is satisfied.

Notwithstanding, the literature on ambiguity in dynamic contexts has not yet reached a consensus about which of these rationality properties is more plausible to assume. Some studies drop consequentialism retaining the dynamic consistency property of ambiguous preferences, as in Eichberger and Kelsey (1996) and Hanany and Klibanoff (2007), while others do the opposite, as in Gilboa and Schmeidler (1993), Pires (2002) and Eichberger et al. (2007). In Sarin and Wakker (1998), Epstein and Schneider (2003) and Eichberger et al. (2005) it is shown that it is possible to maintain both properties under an ambiguity context, at expense of imposing restrictions on the domains of acts and conditioning events over which preferences are settled. Moreover, an interesting result is that, in one of the few experimental studies running a dynamic Ellsberg urn experiment, Dominiak et al. (2009) conclude that ambiguity averse agents more frequently decide with consequentialism than with dynamic consistency.

In the literature there are models on updating rules for ambiguous preferences, under non-recursive settings, as for e.g. in Gilboa and Schmeidler (1993), Hanany and Klibanoff (2007) and Hanany and Klibanoff (2009). There is also a stream of models that operates under recursive settings within the Choquet expected utility theory framework (e.g. Nishimura and Ozaki (2003) and Eichberger et al. (2005)), the Maxmin expected utility theory framework (e.g. Chen and Epstein (2002) and Epstein and Schneider (2003)) and the smooth ambiguity aversion theory framework (e.g. Klibanoff et al. (2009b)).

From this set of models, the recursive multiple priors setting (e.g. Chen and Epstein (2002) and Epstein and Schneider (2003)) is probably the one that has been gaining higher visibility. This setting lies in the "dynamic consistency axiom", which implies the rectangularity property of the set of priors and therefore the dynamic consistency of ambiguous

preferences. Rectangularity is the property that allows to update every prior through a Bayes rule.[5]

Regarding time consistency under the robust control approach, Hansen and Sargent (2006) claim that a type of time-consistency needed to study recursive competitive equilibrium and dynamic economic problems is guarantied both for the "multiplier preferences" setting (as shown in Maccheroni et al. (2006a) with axiomatic foundations given in Wang (2003) and Maccheroni et al. (2006b)) and for the "constraint preferences" setting (shown by the authors themselves through a dynamic game setting).[6]

Still regarding the dynamic consistency issue under robust control, as highlighted by Trojani and Vanini (2004), the key difference versus the recursive multiple priors setting, the most common dynamic approach in the ambiguity literature alongside the robust control, is that while the recursive multiple priors setting lies in the "dynamic consistency axiom", robust control settings are focused in formulating optimal dynamic robust rules where rectangularity is not a crucial property "per se" for the formulation of the set of priors of the model. However, if this set of multiple priors under robust control optimization satisfies the rectangularity property, then the implied robust preferences can be described through a representation as the recursive multiple priors of Chen and Epstein (2002) and Epstein and Schneider (2003). Trojani and Vanini (2004) explain this can happen under the "constraint preferences" approach within the robust control settings.

Additionally, some work has been done in order to incorporate learning effects within an ambiguity context (for e.g., in Wang (2003), Epstein and Schneider (2007, 2008), Miao (2009) and Epstein and Seo (2009a,b)). Although some reasons have been posted in the literature to ignore the effect of learning under an ambiguity context, such as its effect being small (Hansen et al. (2002)), that after some time not much can be learned (Chen and Epstein (2002)) and that ambiguity may be generated by hard-to-interpret signals (Epstein and Schneider (2010)), it is definitely a relevant research topic for the coming years.

Finance is one of the research areas within economic theory where ambiguity has been gaining more visibility. Generally speaking, ambiguity (aversion) has been introduced through different formal settings, while the source of ambiguity is typically related with either the expected return of the risky assets or the stochastic processes of their return.

Regarding the optimal portfolio choice, a general conclusion within the literature is that ambiguity aversion decreases the optimal allocation to the risky asset. In intertemporal

[5]See Epstein and Schneider (2003) for more details about this property.

[6]Regarding the dynamic consistency of preferences under the robust "constraint preferences" there has been some debate in the literature, namely in Chen and Epstein (2002) and Epstein and Schneider (2003).

problems, ambiguity aversion increases investor's intertemporal hedging demand and, in some settings, it even implies that a logarithmic utility investor hedges intertemporaly against potential adverse changes in the investment opportunity set.[7] Ahn et al. (2009) find empirical support for the relevance of studying ambiguity under the portfolio choice problem (about 2/3 of agents under their experience show a positive degree of ambiguity aversion). Bossaerts et al. (2010) conclude that ambiguity aversion can be observed in competitive markets and that it matters for portfolio choices and for asset prices. Examples of works studying optimal portfolio choice under ambiguity within a static (Multiple Priors) framework are Gollier (2006) and Garlappi et al. (2007) and within a dynamic setting are Chen et al. (2009) (Multiple Priors) and Maenhout (2004, 2006) (Robust Control).

Additionally, under ambiguity, non-participation in markets can be optimal for a relevant domain of analysis and some inertia on portfolio decisions emerges (Dow and Werlang (1992), Cao et al. (2005) and Ui (2009)). Moreover, with multiple assets, ambiguity gives rise to selective participation in the markets: when ambiguity is not independent across those assets then diversification can bring benefits, because some assets hedge ambiguity in others. The same doesn't happen when ambiguity between assets is independent. Uppal and Wang (2003), considering multiple assets with different ambiguity levels, also concluded that when overall ambiguity about the joint distribution of returns is high, small differences in ambiguity about the marginal return distribution leads to a significantly under-diversified portfolio (relatively to the standard mean-variance portfolio). Those "qualitative" features of investor's behavior, as the non-participation, selective and limited participation in the market and portfolio inertia, are hard (if not impossible) to rationalize under the standard Savage expected utility.

Regarding asset pricing, in equilibrium, an ambiguous agent requires a corresponding ambiguity premium on uncertain assets. This has been used to help explain some well known puzzles in the literature, namely the equity premium and interest rate puzzles (e.g., Epstein and Wang (1994, 1995), Chen and Epstein (2002), Trojani and Vanini (2002), Epstein and Miao (2003), Maenhout (2004, 2006), Ju and Miao (2008) and Barillas et al. (2009)). Leippold et al. (2008) show that ambiguity aversion, coupled with learning, can explain the excess volatility puzzle. Ambiguity has also been applied to help explain the "home bias" puzzle, as in Epstein and Miao (2003), Uppal and Wang (2003) and Benigno and Nisticò (2009): the intuition is that agents may feel less ambiguous about domestic assets versus foreign assets. Under the robust control approach, equilibrium asset pricing was studied, for e.g., by Hansen and Sargent (2001b), Anderson et al. (2003), Trojani and

[7] When uncertainty is exclusively risk, myopic behavior (i.e. zero intertemporal hedging demand) is always optimal for the logarithmic utility investor.

Vanini (2004) and Liu et al. (2005). In Gagliardini et al. (2009), a robust control approach is used to study the equilibrium interest rate term structure with ambiguity.

Regarding the option literature, almost all of the (few) works with ambiguity considerations are related to the exercise strategies of American options. In Miao and Wang (2009) a real options problem is studied, adopting a recursive multiple-priors utility model, with the source of ambiguity being the state process which influences the continuation and termination payoffs associated with the option. It is concluded that ambiguity lowers the option value. In Kast et al. (2010) and Jaimungal (2010), a similar problem regarding real options is studied by using Choquet-Brownian motions and robust control approach, respectively. In Riedel (2009), a general theory of optimal stopping under time-consistent ambiguity in discrete time is developed, and one of the applications is to American options. The source of ambiguity is the payoff of exercising the option, being used the recursive multiple-priors setting of Epstein and Schneider (2003). This work is extended in Riedel (2010) for a continuous-time environment. Chudjakow and Vorbrink (2009) use Riedel (2009) framework to study the exercise strategies for several American exotic options under ambiguity. In Liu et al. (2005) ambiguity aversion about rare events in the economy's endowment is considered, generating a correspondent equilibrium premium for rare-events, with the authors concluding that this ambiguity can help explain the well documented "smirk" pattern of option-implied volatility observed for options across moneyness levels.

Other applications are appearing in the literature, using models with either a single asset or multiple ambiguous assets and with a representative agent or heterogeneous agents, in order to study many different problems in Finance. A comprehensive survey on this is provided by Epstein and Schneider (2010).

In such context, this thesis aims to contribute to the literature on portfolio choice, general equilibrium asset pricing and European option pricing, by considering ambiguity about the stochastic process of the variance.[8] To the best of our knowledge, this is the first attempt to do so and therefore this is the major contribution of the present thesis. Throughout the thesis the reference to the source of ambiguity is sometimes abbreviated to "ambiguity about the stochastic variance" although it is always "ambiguity about the stochastic process of the variance". In fact, under diffusion models (as those that will be considered) the quadratic variation of an observed process is observable. Consequently, variance (its absolute value) is observable and therefore cannot exist ambiguity about it.[9]

[8] A note on terminology. In the literature, the term "volatility" sometimes is used as "variance" and other time as "standard deviation". In this thesis, the terminology used is volatility as "standard deviation", i.e., the square root of variance. Naturally, being ambiguous about variance implies being ambiguous about volatility.

[9] We thank Frank Riedel for this insight.

It has been advocated in some literature (Cao et al. (2005), Garlappi et al. (2007), and Ui (2009)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, ambiguity about the stochastic process of the variance of the risky asset's return is studied because: (1) there is no "a priori" reason to assume that investors are not ambiguous about it; (2) we are able to find analytical solutions; (3) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world, as for e.g., option implied variance frequently differs both in level and dynamics from the statistical measure (Drechsler and Yaron (2008) and Todorov (2010)) and (4) of the variance relevance as an option pricing input. Moreover, for analytical tractability and focus of analysis, we will consider ambiguity exclusively about the stochastic variance.

The thesis includes five additional chapters, with four essays and a summary of research topics I believe to be interesting to explore in the future.

In chapters 2 and 3 we study the portfolio problem with ambiguity about the stochastic process of the variance under different settings, both regarding the specification of the representative investor preferences and the methodology used to introduce ambiguity aversion. In chapter 4, a general equilibrium asset pricing model with an ambiguous state variable is developed. In chapter 5, a closed-form solution is derived for the price of a European call option on an asset whose return's variance is stochastic and, simultaneously, there exists ambiguity about the variance process. We make use of the generic setting developed in chapter 4 in order to obtain the specification of the uncertainty price (risk and ambiguity) associated with the option's underlying asset return variance. The contribution of each of those essays is considered independently of the remaining ones.

In chapter 2, our aim is to understand what is the impact on the optimal dynamic consumption and portfolio rules from the ambiguity about the stochastic variance and if stochastic variance is relevant to portfolio choice. With this purpose, we extend the model of Chacko and Viceira (2005) for optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the expected value of the variance of the risky asset's return. Investor's preferences are described by the stochastic differential utility function introduced by Duffie and Epstein (1992b), with a specification representing

Kreps and Porteus (1978) preference setting. Formally, ambiguity aversion is introduced using Gilboa and Schmeidler (1989) Multiple Priors approach.

We find that, with investors being able to update their portfolio continuously (as a function of the instantaneous variance), ambiguity about the expected value of the variance has no impact. To shed some light on the case in which continuous portfolio updating is not possible, we also evaluate the effect of ambiguity when investors must use their expectation of future variance for their portfolio decision. In the latter scenario, demand for the risky asset can be decomposed into three components: myopic and hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand. Using long-run US data, Chacko and Viceira (2005) found that the intertemporal hedging demand is empirically small, suggesting a low impact of stochastic variance on portfolio choice. Using the same calibration, we find that ambiguity demand may be high, much more than intertemporal hedging demand. Therefore, we conclude that stochastic variance can be relevant for portfolio choice, not because of the variance risk, but because of investors' ambiguity about variance.

This chapter brings three major contributions. It is the first time that ambiguity aversion is introduced in a dynamic optimal portfolio choice setting with an explicit process for the stochastic variance. Secondly, it is the first time ambiguity aversion about the variance (process) of the risky asset return is considered in the portfolio choice problem. At last it suggests that, in some scenarios, ambiguity about the expected variance of the risky asset's return is important for the portfolio choice.

In chapter 3, the focus continues to be understanding the impact that ambiguity about the stochastic process of the variance has on the optimal dynamic portfolio choice. However, there are relevant differences in the preferences setting when compared with the approach in chapter 2. Firstly, it is assumed that the representative investor is ambiguous about the stochastic model describing the variance dynamics, instead of about one of its (unobserved) parameters. Additionally, ambiguity aversion is formally introduced through the robust control approach (multiplier preferences). Secondly, a CRRA utility function over terminal wealth is adopted, which strongly simplifies the analytical treatment of the problem. The setting of Maenhout (2006) for robust portfolio rules is closely followed.

We conclude that ambiguity about the variance process enhances the "risk-aversion" adjustment in the total asset demand, both in its myopic and hedging demand components. Moreover, it increases the hedging desire of the representative investor: a new hedging demand component emerges, and even the standard hedging demand is affected by ambiguity. As in chapter 2, the conclusion is that ambiguity about stochastic variance decreases the optimal demand of the risky asset. Additionally, under the empirically plausible case

of a negative correlation between shocks in the return of the risky asset and in its variance, the ambiguous investor always invests less in the risky asset than the myopic investor, and there is an horizon effect in the optimal trading strategy: the ambiguous investor will invest more in the risky asset the shorter is the horizon. As in chapter 2, a closed-form solution for the optimal portfolio rule is obtained, but in this case that rule embeds an instantaneous impact from the ambiguity about stochastic variance.

In our view, the major contribution of this chapter is that it is the first time that ambiguity aversion about the stochastic process of the variance of the risky asset return is considered, through a dynamic methodology, in a portfolio choice model. Moreover, it extends previous results obtained in the literature about the effect of ambiguity aversion on portfolio choice, particularly those in Maenhout (2006), for a different and relevant source of ambiguity in financial markets: the stochastic process of the variance.

In chapter 4, our focus is to study the channels through which ambiguity impacts equilibrium asset pricing and, related with this, obtain the specification of the price of risk and ambiguity in an intertemporal general equilibrium model of asset pricing. For that, we consider a version of the intertemporal general equilibrium model of Cox et al. (1985a) with a single production process and two correlated state variables, assuming that only one of the state variables has shocks that are correlated with those of the economy's output rate (this implies that only its changes should be hedged and its uncertainty priced). Ambiguity is about the stochastic process of that state variable and is formally introduced through a dynamic way, making use of the robust control approach (constraint preferences). The extension of the model of Cox et al. (1985b) made by Gagliardini et al. (2009) is closely followed.

We conclude that ambiguity about that state variable impacts asset pricing through two channels: the price of uncertainty associated with this state variable, and the interest rate. When ambiguity is considered, the equilibrium price of uncertainty and equilibrium interest rate can increase or decrease, depending on the concrete specification of the investment opportunity set considered and the gradient of the value function of the problem with respect to the ambiguous state variable.

The developed setting is generic and therefore can potentially consider any variable as the source of ambiguity. As an example, the developed generic setting is applied to the investment opportunity set used in Heston's (1993) stochastic volatility option pricing model, assuming that the option's underlying asset return variance is the ambiguous state variable. Through this example, we obtain a specification of the variance uncertainty price that can be used in an extension of Heston's (1993) model accommodating ambiguity aversion about the stochastic variance process of the option's underlying asset return.

This finding, alongside the development of the generic two-factor general equilibrium framework for asset pricing under ambiguity, which we see as a simple setting that can be used for many asset pricing problems, are the major contributions from this chapter.

In chapter 5, the major purpose is to obtain a closed-form solution for the price of a European call option when the variance of the underlying asset's return is stochastic and, there exists ambiguity about the stochastic process of the variance. Additionally, we want to study the impact on option pricing from ambiguity about the stochastic process that generates the variance of the option's underlying asset return.

For that we start from Heston's (1993) stochastic volatility option pricing model, where uncertainty is exclusively risk. The major novelty is that the variance uncertainty price (risk plus ambiguity) used to derive the closed-form solution of the European Call option accommodates an ambiguity component, becoming either a concave or convex function of the instantaneous variance (versus the linear specification in Heston (1993)). This new specification is theoretically motivated by the equilibrium model developed in chapter 4. However, the option pricing formula is obtained through a no-arbitrage approach and, consequently, we would obtain the same results by assuming exogenously the referred specification. In this sense, chapters 4 and 5 can be considered independent.

We conclude that if the variance ambiguity price is negative, ambiguity about stochastic variance implies higher option prices than when stochastic variance is considered without ambiguity (across all moneyness levels). The opposite happens if the variance ambiguity price is positive. Consistently, in the latter scenario, ambiguity aversion decreases the option's implied volatility, which helps to explain the variance premium puzzle.[10] In addition to this last feature, in our view this chapter brings two further relevant contributions: it is the first time that ambiguity aversion is considered within an option pricing problem with stochastic variance, and where the latter's process is the source of ambiguity, and more generally, it is obtained a closed-form solution for an extension to the model of Heston (1993) with a non-linear specification for the variance uncertainty price.

The focus in each of those essays is the deduction of the analytical solution of the developed model and its economic interpretation. Notwithstanding, in chapters 2 and 5, the model is simulated, using existing calibrations in the literature, in order to enhance the perception of obtained conclusions. All simulations are done by using Dynare (version 3.065) and MatLab (version 7.0.0.19920 R14).

Our method therefore comprehends the following stages:

[10]This puzzle refers to the fact that option-implied variance for a certain maturity tends to be higher than the conditional expectation of realized variance for that period of time (see, for e.g., Drechsler and Yaron (2008) and Todorov (2010)).

1. Set up the model:

   In this stage we choose a base model suitable to treat the research questions we are interested in: Chacko and Viceira (2005) in chapter 2; Maenhout (2006) in chapter 3; Cox et al. (1985a) and Gagliardini et al. (2009) in chapter 4; and Heston (1993) in chapter 5. We introduce ambiguity (aversion) about the stochastic process of the variance of the risky asset return through a specific way.

2. Analytical solutions:

   We deduct the analytical solutions of extended models with ambiguity about the stochastic process of the variance. We point out the differences and similarities between obtained analytical solutions and the ones in the base models so that the analytical impact, in each case, of ambiguity consideration is clearly identified. Additionally, we point out the economic meaning of the obtained solutions.

3. Model Simulation:

   In chapters 2 and 5, the model is simulated considering the same calibration of base models (Chacko and Viceira (2005) and Heston (1993), respectively), as it enhances comparative analysis between a scenario with and without ambiguity.

The mathematical proofs are presented in Appendices at the end of each chapter.

In chapter 6 are summarize some research topics, related with this thesis, that I foresee as interesting to be explored in the future.

Chapter 2

Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility[1]

[1]An early version of this chapter has been circulating as Faria et al. (2009). We are grateful to Alain Chateauneuf, Alejandro Balbás, Alexandre Lowenkron (discussant), Bengu Vuran (discussant), Frank Riedel, Helena Reis, Jerome Detemple (discussant), Mert Ural (discussant), Michiel Janssen (discussant), Paulo Pereira (discussant), and Raquel Gaspar (discussant) for useful comments and suggestions, and thank seminar participants at U.Minho, U.Vigo, the 5th Doctoral Workshop at U.Porto, the 3rd Economic Theory Workshop in Vigo, the 4th Luso-Brazilian Finance Meeting, the 1st World Finance Conference (Viana do Castelo), the 6th World Congress of the Bachelier Finance Society (Toronto), the 6th Portuguese Finance Network Conference (Ponta Delgada) and the 37th European Finance Association Meeting (Frankfurt).

2.1 Introduction

We study optimal dynamic consumption and portfolio choice, in the presence of stochastic variance, of an investor that is averse both to risk and ambiguity. In our setting, the variance of the risky asset's return is simultaneously the source of the risk and the ambiguity that are perceived by the investor.

There is a large literature on portfolio choice (see Campbell and Viceira (2002) for a survey), but few works study optimal dynamic portfolio choice with stochastic variance of the risky asset's return. Some examples are Kim and Omberg (1996), Chacko and Viceira (2005) and Liu (2007) under incomplete markets, and Schroder and Skiadas (1999) under complete markets. Schroder and Skiadas (2003) contains a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. In those papers, potential adverse changes in the investment opportunity set are associated with the stochastic variance of the risky asset's return, which therefore represents a source of risk to investors. This implies, from Merton (1973a), that stochastic variance is a key factor driving the investor's optimal intertemporal hedging demand. However, in those papers there is only risk, and no ambiguity.

Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and therefore for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a large body of empirical studies (surveyed, for e.g., in Camerer and Weber (1992) and Epstein and Schneider (2010)).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years ignored ambiguity, being based on the expected utility theory of von Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

However, gradually, ambiguity is being incorporated in decision theory. Two main approaches are being used: (i) the multiple priors (MP) approach, where the single probability measure of the standard expected and subjective utility models (precise beliefs) is replaced by a set of probabilities or priors (imprecise beliefs); (ii) the robust control (RC) approach, associated to an assumption of model uncertainty. The relationship between

the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001a), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006a).

Ahn et al. (2009) have found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also conclude that ambiguity aversion can be observed in competitive markets and that it matters for portfolio choices and for asset prices.

In studies of portfolio choice with ambiguity, Garlappi et al. (2007) and Gollier (2006) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.[2] The same conclusion was reached in a dynamic MP setting (e.g. Chen et al. (2009)) and in a dynamic RC model (e.g. Maenhout (2004, 2006)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, the source of ambiguity is the expected risky asset's return or the risky asset's return process.

We extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice with stochastic variance, by introducing ambiguity about the stochastic process of the variance, more concretely, about the expected value of the variance of the risky asset's return (an unobservable parameter). Motivation for this is provided by Chacko and Viceira themselves:

"*An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.*"

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2009)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

[2] Although the result in Gollier (2006) requires some restrictions on the set of priors and on the investor's attitude towards risk.

Nevertheless, we introduce ambiguity about the variance process of the risky asset's return because: (1) there is no "a priori" reason to assume that investors are not ambiguous about it, and in this case about one unobservable parameter of that process; (2) we are able to find analytical solutions; and (3) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world, as for e.g., option implied variance frequently differs both in level and dynamics from the statistical measure (Drechsler and Yaron (2008) and Todorov (2010)).

We consider ambiguity only about the variance process of the risky asset's return to keep a parsimonious model and to isolate its effects on the optimal dynamic consumption and portfolio choices. The reason to choose the expected value of variance as the source of ambiguity (besides analytical tractability) is that it seems more intuitive to assume that investors think more about that parameter of the variance process than about other parameters (like the variance of variance or the reversion parameter in the case of a mean-reverting variance process).

We introduce ambiguity through the MP approach, adopting the Maxmin expected utility framework of Gilboa and Schmeidler (1989), which enables us to introduce ambiguity aversion without changing significantly the setting of Chacko and Viceira (2005). The choice behavior is as if the investor has a set of possible values of the expected value of variance, and considers the worst possible of these values when evaluating each portfolio and consumption policy.

We find that, in our setting, the variance of the risky asset's return is always bad to the investor, in the sense that it decreases his utility. However, we conclude that ambiguity about variance has no impact whatsoever.

A crucial assumption of the model is that the investor is able to continuously update his portfolio as a function of the observed instantaneous variance, which means that the variance of the portfolio's return is always known by the investor. Without such continuous portfolio adjustment (which is not verified in reality, for e.g., due to information and transaction costs), the variance of the portfolio's return becomes uncertain, even if investors can observe the instantaneous variance. To shed some light on this scenario, we also consider an alternative case in which investors use their expectation about the future variance in their portfolio decision. We find that, in this case, the investor's ambiguity aversion about variance reduces the demand for the risky asset.

In this latter scenario, the demand for the risky asset can be decomposed into three components: myopic and intertemporal hedging demands, as in Chacko and Viceira (2005); and ambiguity demand, which is the novelty. Using the same calibration of Chacko and Viceira (2005), with long-run US data, we find that ambiguity demand seems to have a

relevant empirical dimension, much more than intertemporal hedging demand. We therefore conclude that, when investors use their expectation about future variance (and not the instantaneous variance) in their portfolio decision, the "ambiguity dimension" of the uncertainty about the stochastic variance is relevant, much more than its "risk dimension".

In our view, this chapter brings three major contributions: (1) it is the first time that ambiguity aversion is introduced in a dynamic optimal portfolio choice setting with an explicit process for the stochastic variance; (2) it is the first time that ambiguity aversion about the stochastic variance process is considered; (3) it suggests that, in some scenarios, ambiguity about the expected variance of the risky asset's return is important for portfolio choice.

The chapter is organized as follows. In section 2.2, we state the problem to be solved. In section 2.3, we present the analytical solution to the problem and the key results. In section 2.4, we show simulation outputs. In section 2.5, we conclude the chapter with some remarks.

2.2 Consumption and Portfolio Choice Problem

In section 2.2.1, we describe the investment opportunity set, which is the same as in the model of Chacko and Viceira (2005). In section 2.2.2, we disclose the preferences of the investor, extending the framework in Chacko and Viceira (2005) by introducing ambiguity about the expected value of the precision (which is the reciprocal of variance) of the risky asset's return. In section 2.2.3, we present the dynamic optimization problem to be solved by the investor.

2.2.1 Investment Opportunity Set

It is assumed that all wealth is allocated between a riskless asset with price B_t and a risky asset with price S_t. The instantaneous return of the riskless asset is described by:

$$\frac{dB_t}{B_t} = rdt\,, \tag{2.1}$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_S\,, \tag{2.2}$$

where μ is the expected return of the risky asset, W_S is a standard Brownian motion and y_t is the instantaneous precision of the risky asset's return process (the instantaneous variance is $v_t = \frac{1}{y_t}$). From (2.2), the expected excess return of the risky asset versus the riskless asset, $\mu - r$, is constant over time.

The precision, y_t, follows a mean-reverting, square-root process as used by Cox et al. (1985b):

$$dy_t = \kappa(\theta - y_t)\,dt + \sigma\sqrt{y_t}dW_y\,, \tag{2.3}$$

where the expected value of the precision is $E[y_t] = \theta$, the reversion parameter is $\kappa > 0$, and, thus, $Var[y_t] = \frac{\sigma^2\theta}{2\kappa}$. W_y is a standard Brownian motion. To guarantee standard integrability conditions, it is assumed that $2\kappa\theta > \sigma^2$, as in Cox et al. (1985b).

Applying Itô's Lemma to (2.3), a mean-reverting, square-root process for proportional changes in variance is obtained (Appendix 2.6.1) :

$$\frac{dv_t}{v_t} = \kappa_v(\theta_v - v_t)\,dt - \sigma\sqrt{v_t}dW_y\,, \tag{2.4}$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \kappa\left(\theta - \frac{\sigma^2}{\kappa}\right) = \frac{\kappa}{\theta_v}$.

Taking expectations of the second-order Taylor expansion of v_t around θ, the approximate unconditional mean of instantaneous variance is (Appendix 2.6.2):

$$E[v_t] \approx \frac{1}{\theta} + \frac{1}{2}\frac{\sigma^2}{\theta^2\kappa} = \frac{1}{\theta} + \frac{Var(y_t)}{\theta^3}\,. \tag{2.5}$$

As the expected return of the risky asset, μ, is assumed to be constant, (2.5) is also the unconditional variance of the risky asset's return. Chacko and Viceira (2005) perform a Monte Carlo simulation that validates this statement and the accuracy of the approximation (2.5), concluding that this approximation understates the true variance by 0.27%.

It is assumed that shocks to precision (W_y) are correlated with shocks to the return on the risky asset (W_S), with $dW_y dW_S = \rho dt$ and $\rho > 0$. From (2.4), this implies that the instantaneous correlation between proportional changes in variance and the risky asset's return is given by:

$$Corr_t\left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t}\right) = -Corr_t\left(dy_t, \frac{dS_t}{S_t}\right) = -\rho dt\,. \tag{2.6}$$

This investment opportunity setting incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect"

property (given by the negative correlation between returns and its variance), and the clustering property (as proportional changes in variance are higher when variance is high).

2.2.2 Investor Preferences

The investor faces ambiguity about the expected value of precision, θ (equivalently, from (2.5), about the expected value of the unconditional variance of the risky asset return, $E[v_t]$). Regarding the values of the remaining parameters of the investment opportunity set, it is assumed that there is no ambiguity.

Following Gilboa and Schmeidler (1989), we assume that the θ-ambiguity averse investor has a set of priors, given by the non-empty, closed and convex interval $\left[\underline{\theta}, \overline{\theta}\right]$, with $0 < \underline{\theta} \leq \theta \leq \overline{\theta}$, and makes his choice considering the minimal expected utility over all priors in the set. The higher the difference between $\underline{\theta}$ and $\overline{\theta}$, the higher the level of ambiguity.

Two comments about the set of priors $\left[\underline{\theta}, \overline{\theta}\right]$. First, it embeds any type of probability measure (and not just, for e.g., Dirac measures). As shown below, the value function of the investor's problem has a monotonic behavior relatively to the parameter θ and, under the framework of Gilboa and Schmeidler (1989), it results that one of the bounds (lower or upper) of the interval will be the solution of the ambiguity problem (if precision increases or decreases intertemporal utility, respectively). Second, no learning process about θ is considered. This would certainly enrich the analysis at expense of increased complexity. Moreover, there are some reasons that have been pointed out in the literature to ignore the effect of learning under an ambiguity context, such as its effect being small (Hansen et al. (2002)), that after some time not much can be learned (Chen and Epstein (2002)) and that ambiguity may be generated by hard-to-interpret signals (Epstein and Schneider (2010)).

The preferences of investors are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory in Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution by not constraining to be reciprocals of one another (as in standard additive intertemporal utility function).

The utility process that defines the SDU function is represented by:

$$J = E_t \left[\int_t^{\infty} f(C_s, J_s)\, ds \right], \tag{2.7}$$

where C_s represents current consumption and J_s is the continuation utility for the consumption flow C, at time $t = s$, with infinite time horizon. The function $f(C_s, J_s)$ is the normalized aggregator that generates J, which, as in Chacko and Viceira (2005), defines a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available. The normalized aggregator $f(C,J)$ is given by (Appendix 2.6.3):

$$f(C,J) = \frac{\beta}{1-\frac{1}{\psi}}(1-\gamma)J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right], \tag{2.8}$$

where $\gamma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the rate of time preference. With $\gamma = \frac{1}{\psi}$, (2.8) becomes the standard power utility representation.

When $\psi = 1$, the normalized aggregator $f(C,J)$ takes the form:

$$f(C,J) = \beta(1-\gamma)J\left\{ln(C) - \frac{1}{1-\gamma}ln[(1-\gamma)J]\right\}. \tag{2.9}$$

Two remarks on the preferences of the representative investor. The first regards the different dynamic nature of the two types of uncertainty - ambiguity and risk - in our setting. The attitude towards risk is settled through a recursive dynamic setting given by (2.7), where the dynamic consistency is guaranteed (Duffie and Epstein (1992a)). Conversely, ambiguity is introduced through the atemporal approach in Gilboa and Schmeidler (1989), assuming that the ambiguity problem has a "once-for-all" solution. This approach is justified by two main reasons: (i) it is a parsimonious way of extending the framework in Chacko and Viceira (2005) to incorporate θ-ambiguity;[3] (ii) although simple, it permits the study of the relevance of the dichotomy ambiguity-risk when assessing the impact of stochastic precision on the portfolio choice - we obtain an analytical solution that disentangles the ambiguity and risk components of investor's portfolio choice (section 2.3), enabling an elucidating empirical testing (section 2.4).

The second remark regards the preference for the timing of the resolution of risk. With the preference structure in Kreps and Porteus (1978), investors can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertem-

[3] A more elaborate way of doing it would be to introduce ambiguity through a recursive dynamic setting, guaranteeing the dynamic consistency property of ambiguous preferences. One way of doing this is through the recursive multiple priors utility specification of Chen and Epstein (2002), as it extends the SDU function of Duffie and Epstein (1992b) by replacing the single prior by a set of priors: the requirement would be to fulfill Theorem 2.1 in Chen and Epstein (2002) regarding the properties of the set of priors for θ and remaining conditions so that their Theorem 2.2 would be applicable.

poral utility function implies that investors are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between ψ and γ: if $\gamma > \frac{1}{\psi}$ $(<,=)$ investors have preference for early (late, indifferent) resolution of risk. Our specification (2.7) from Duffie and Epstein (1992a), being the continuous-time limit of Epstein and Zin (1989), inherits this property. However, on the contrary of other streams of literature with Epstein and Zin (1989) preferences, for e.g. the "long-run risk" literature (from the seminal work of Bansal and Yaron (2004)), we do not restrict the investor to have preference for early resolution of risk. Two main reasons support this decision: (i) as our model evolves in a long-run setting, the possibility of the "cost" becoming higher than the "benefit" of planning advantages brought by the early resolution of risk (Arai (1997)) should not be excluded and (ii) there is evidence that investors may have preference for late resolution of risk (Epstein and Zin (1991)).

2.2.3 Dynamic Optimization Problem

The dynamic optimization problem of the investor is to choose the consumption function, $C : [t_0, +\infty[\rightarrow \mathbb{R}_+$, and the fraction of wealth invested in the risky asset throughout time, $\pi : [t_0, +\infty[\rightarrow \mathbb{R}$, which maximize the θ-minimized expected utility (2.7), subject to the precision process (2.3) and to the intertemporal budget constraint (2.11):

$$\max_{\pi, C} \left\{ \min_{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]} E_{t_0} \left[\int_{t_0}^{\infty} f(C_s, J_s)\, ds \right] \right\} \tag{2.10}$$

$$s.t.$$

$$dX_t = [\pi_t (\mu - r) X_t + r X_t - C_t]\, dt + \pi_t \sqrt{\frac{1}{y_t}} X_t dW_S \,, \tag{2.11}$$

$$dy_t = \kappa \left(\hat{\theta} - y_t \right) dt + \sigma \sqrt{y_t} dW_y \,,$$

where C_t, X_t and π_t represent instantaneous consumption, wealth and fraction of wealth invested in the risky asset, respectively. Is is assumed that total wealth at the initial moment ($t_0 = 0$) is positive, i.e., $X_{t_0} > 0$.

Following Kamien and Schwartz (1991) (ch. 22), the Bellman equation of this problem

is:

$$0 = \max_{\pi,C} \min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} \left\{ f(C_s,J_s) + J_X(\pi_t(\mu-r)X_t + rX_t - C_t) + J_y\kappa\left(\hat{\theta}-y_t\right) + \right.$$
$$\left. + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \right\}, \tag{2.12}$$

where $f(C_s,J_s)$ is the normalized aggregator given in (2.8) and (2.9) for general values of ψ and for $\psi = 1$, respectively, and J_X, J_y, J_{XX}, J_{yy} and J_{Xy} are partial derivatives.

We will show that (2.16) is the exact solution of (2.12) when $\psi = 1$, and that (2.28) is an approximate solution valid for $\psi \neq 1$. This being true, the order of minimization and maximization in (2.12) can be changed by applying the Saddle Point Theorem (Fan (1953) and Sion (1958)), because: (i) the domain of $\hat{\theta}$ is compact; (ii) the argument is concave on π and on C (the expressions for J will be disclosed in the next two subsections);[4] and (iii) the argument is convex on $\hat{\theta}$.[5] We obtain:

$$0 = \min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} \max_{\pi,C} \left\{ f(C_s,J_s) + J_X(\pi_t(\mu-r)X_t + rX_t - C_t) + J_y\kappa\left(\hat{\theta}-y_t\right) + \right.$$
$$\left. + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \right\}, \tag{2.13}$$

which is the Bellman equation of the problem in which the order of the maximization and the minimization is changed. Therefore, the dynamic consumption-portfolio problem with stochastic precision faced by the investor that is both θ-ambiguity and risk averse can be rewritten as:

$$\min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} \left\{ \max_{\pi,C} E_{t_0}\left[\int_{t_0}^{\infty} f(C_s,J_s)\,ds\right]\right\}$$
$$s.t. \tag{2.14}$$
$$dX_t = [\pi_t(\mu-r)X_t + rX_t - C_t]\,dt + \pi_t\sqrt{\frac{1}{y_t}}X_t dW_S\,,$$

[4]There are two parcels that are linear on π and one that is quadratic but is associated with a minus sign (from $J_{XX} < 0$), therefore the argument is concave on π. Concavity on C also holds, as there is one parcel that is linear and, for the expressions for J given in the next two subsections, (2.8) and (2.9), $f(C_s,J_s)$ is concave on C.

[5]Note that J is not a function of $\hat{\theta}$, as it already considers the specific value that solves the minimization problem. This implies that the only relevant parcel of the argument of (2.12) to study convexity on $\hat{\theta}$ is $J_y\kappa\left(\hat{\theta}-y_t\right)$. The latter is linear, and therefore convex, on $\hat{\theta}$.

$$dy_t = \kappa\left(\hat{\theta} - y_t\right)dt + \sigma\sqrt{y_t}dW_y\,.$$

2.3 Optimal Consumption and Portfolio Rules

We solve the problem (2.14) in two steps. First, for each value of $\hat{\theta}$, we find the value function, $J_{t_0}(\hat{\theta})$, of the maximization problem, as in Chacko and Viceira (2005). Then, we minimize this value function with respect to $\hat{\theta}$ to obtain the value of $\hat{\theta}$ that is used by the ambiguity averse investor:

$$\theta^* = \underset{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}{argmin}\, J_{t_0}(\hat{\theta}).$$

The maximization problem, for each possible value of $\hat{\theta}$, is a stochastic continuous-time optimal control problem with two state variables, wealth (X_t) and precision of the risky asset's return (y_t), and two control variables, consumption (C_t) and fraction of wealth invested in the risky asset (π_t). This implies that the value function, $J_{t_0}(\hat{\theta})$, that solves the problem is a function of X_t and y_t. The corresponding Bellman equation is

$$\begin{aligned} 0 \;=\; & \max_{\pi,C}\Big\{ f\left(C_s,J_s\right) + J_X\left(\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right) + J_y\kappa\left(\hat{\theta} - y_t\right) + \\ & + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \Big\}\,. \end{aligned} \tag{2.15}$$

Chacko and Viceira (2005) found an exact solution of this maximization problem for $\psi = 1$, and an approximate solution for the general case of $\psi \neq 1$. We study θ-ambiguity aversion in both scenarios.

2.3.1 Exact solution $(\psi = 1)$

When the investor has unit elasticity of intertemporal substitution of consumption $(\psi = 1)$, the value function that solves (2.15), for any value of $\hat{\theta}$, is given by Chacko and Viceira (2005):

$$J\left(\hat{\theta},X_t,y_t\right) = exp\left\{Ay_t + B(\hat{\theta})\right\}\frac{X_t^{1-\gamma}}{1-\gamma}, \tag{2.16}$$

where A and $B(\hat{\theta})$ are given by

$$A = \frac{\gamma(1-\gamma)\left[\left(\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right) \pm \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}}\right]}{\sigma^2\left[\gamma(1-\rho^2)+\rho^2\right]}, \tag{2.17}$$

$$B(\hat{\theta}) = \frac{(1-\gamma)(\beta ln\beta + r - \beta)}{\beta} + \frac{\kappa\hat{\theta}}{\beta}A. \tag{2.18}$$

The sign of the square-root in A is "+" for $\gamma > 1$ and "-" for $\gamma \leqslant 1$ (Appendix 2.6.4).

This implies the following optimal consumption (C_t) and portfolio rules (π_t):

$$C_t = \beta X_t, \tag{2.19}$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \frac{\sigma\rho}{\gamma}Ay_t. \tag{2.20}$$

When $\psi = 1$, the optimal consumption rule is to consume a constant fraction of wealth throughout time (2.19). This means that the income and substitution effects on consumption that result from a change in the investment opportunity set are always exactly canceled out.

The optimal portfolio rule (2.20) has two components, well known from Merton (1973a): (i) mean-variance or myopic portfolio demand, $\frac{1}{\gamma}(\mu - r)\, y_t$; and (ii) intertemporal hedging demand, $\frac{\sigma\rho}{\gamma}Ay_t$. As highlighted in Chacko and Viceira (2005), the latter is zero (and therefore the portfolio myopic demand is optimal) when: the investor has unit coefficient relative risk aversion $(\gamma = 1)$; investment opportunities are constant $(\sigma = 0)$ or, being time-varying, it is not possible to use the risky asset to hedge against those changes $(\rho = 0)$. As both components of portfolio demand are linear functions of y_t, their ratio does not depend on y_t. From $E[y_t] = \theta$ and (2.20), the mean optimal allocation to the risky asset is:

$$\pi_\theta = \frac{1}{\gamma}(\mu - r)\,\theta + \frac{\sigma\rho}{\gamma}A\theta. \tag{2.21}$$

An ambiguity averse investor considers, from his set of priors, $\theta^* = \underset{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}{argmin}\, J_{t_0}(\hat{\theta})$, where:

$$J_{t_0}(\hat{\theta}) = exp\left\{Ay_{t_0} + B(\hat{\theta})\right\}\frac{X_{t_0}^{1-\gamma}}{1-\gamma}, \tag{2.22}$$

with y_{t_0} and X_{t_0} representing the instantaneous precision and the total wealth at the present moment $(t = t_0)$. From (2.22), the solution of the ambiguity problem will depend on the investor's risk preferences and on the characteristics of the investment opportunity set.

Proposition 1. *When* $\psi = 1$ *and* $\gamma \geqslant \omega$ *, where* $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$, *the solution of the ambiguity problem is:*[6]

$$\theta^* = \underline{\theta}\,.$$

Proof. *Appendix 2.6.5.*

From Proposition 1, we conclude that the domain of the solution to the θ-ambiguity problem depends on the combination of the level of investor's risk aversion (in Bliss and Panigirtzoglou (2004) is disclosed a review of estimated values of γ in the literature) and on the characterization of the investment opportunity set dynamics (represented by ω). In that domain $(\gamma \geqslant \omega)$, precision is always good, implying that $\theta^* = \underline{\theta}$.

Three further comments related to Proposition 1. First, although in our setting precision ends up being always good, the hypothesis of precision being bad should also be considered, as we did. This follows from the conclusion of Rothschild and Stiglitz (1970, 1971) that some risk averse investors may be better off with an increase of the variance (decrease of the precision) of returns.[7] If under the domain of analysis $(\gamma \geqslant \omega)$, there were scenarios where precision is bad, then the solution of the ambiguity problem would be $\theta^* = \overline{\theta}$.

Second, the approach of Gilboa and Schmeidler (1989) is sometimes criticized because it apparently implies extreme ambiguity aversion. However, the implied decision criteria may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. This is why Gilboa and Schmeidler's

[6]For $\gamma < \omega$, A in (2.17) is a complex number. From (2.16) this implies a complex value function J. As J is the optimized intertemporal utility, and utility functions are defined in the real space, the domain of the problem is restricted to values of parameters such that $\gamma \geqslant \omega$.

[7]In the finance literature, particularly on portfolio choice, the mean-variance (MV) approach, under which risk averse investors make their choices considering only the first two moments of the return distribution, has been extensively used. This despite the fact that the expected utility theory (EU) approach is more consistent. As a consequence, the risk-return trade-off is frequently and erroneously treated as the mean-variance trade-off. Although variance is probably the best scalar measure of risk, relevant probability distributions aren't exclusively characterized by their first two moments. At least their third and fourth moments - skewness and kurtosis - may be also informative and relevant. In Rothschild and Stiglitz (1970) this is demonstrated by introducing a richer definition of risk, based on the concept of a "mean-preserving spread". For the portfolio problem, in Rothschild and Stiglitz (1971) it is shown that an increase in the riskiness of the risky asset does not necessarily reduce its demand by risk-averse investors. For the MV and EU approaches to give the same results, it is necessary to assume the quadratic utility specification, which is both theoretically and empirically implausible, or to restrict the choice space in a way that the mean and variance contain all the relevant information (e.g. all distributions are Normal).

(1989) criteria is not so extreme as, for example, the Wald maxmin criteria. As stated by Epstein and Schneider (2010): *"Ultimately, the only way to argue that the model is extreme is to demonstrate extreme behavioral implications of the axioms, something that has not been done"*.[8]

Third, regarding the investor's preferences for the temporal resolution of risk, the domain of analysis $\gamma \geqslant \omega$ includes scenarios where the investor: has preference for late resolution of risk $(\omega \leqslant \gamma < 1)$; has preference for early resolution of risk $(\gamma > 1)$; is indifferent to that timing $(\gamma = 1)$. Only scenarios where the investor has a strong preference for late resolution of risk $(\gamma < \omega)$ are excluded.

Using Proposition 1, problem (2.14) becomes:

$$\max_{\pi,C} E_{t_0}\left[\int_{t_0}^{\infty} f(C_s, J_s)\, ds\right]_{|\hat{\theta}=\underline{\theta}}, \; with\, \gamma \geqslant \omega,\; \psi = 1,$$

subject to the same restrictions as before, but with $\hat{\theta} = \underline{\theta}$. The solution is given by equations (2.16)-(2.20), for a specific value of $\hat{\theta} = \underline{\theta}$ and a consistent set of remaining parameters values (such that $\gamma \geqslant \omega$).

It is immediate from (2.19) and (2.20) that both the optimal consumption and portfolio rules are not affected by ambiguity about θ, as neither depend on $\underline{\theta}$.[9] If the θ-ambiguity averse investor follows the instantaneous optimal portfolio policy (2.20), his mean optimal allocation to the risky asset is given by (2.21), the same as that of an investor who faces no ambiguity.

However, if the θ-ambiguity averse investor could not instantaneously adjust his portfolio following the observation of the instantaneous precision (as in (2.20)), then his expectation of the future precision would become relevant for the portfolio choice. Note that due to the existence of transaction costs and other market frictions, as well as due to human limitations, investors in the real world do not adjust their portfolios continuously. In this case, θ-ambiguity aversion becomes relevant as, from Proposition 1, the unconditional expectation of future precision differs between the ambiguity averse and the ambiguity neutral investor ($\underline{\theta}$ and θ, respectively) and, from Cox et al. (1985b), the instantaneous

[8]More recently, the smooth ambiguity aversion utility theory has been developed on the back of the seminal work of Klibanoff et al. (2005). It is claimed that this setup distinguishes ambiguity from ambiguity aversion and allows for smooth indifference curves, avoiding the infinite ambiguity aversion implied in Gilboa and Schmeidler (1989) approach. However, there is still a debate in the literature about the axiomatic foundations of this line of models (see Epstein (2010) and Klibanoff et al. (2009a) for a recent exchange on this). Also because of this, the approach of Gilboa and Schmeidler (1989) continues to be the most used in the literature.

[9]Note that (2.20) being a function of the instantaneous precision y_t, with dynamics described by the process (2.3), is indirectly impacted by the "true" θ parameter.

conditional expectation of precision depends on $\underline{\theta}$:

$$E_{(t-dt)}\left(y_t \mid y_{t-dt}\right) = \underline{\theta}\left(1-exp\left\{-\kappa dt\right\}\right)+y_{(t-dt)}exp\left\{-\kappa dt\right\}. \tag{2.23}$$

For an ambiguity neutral investor, the mean allocation to the risky asset continues to be given by (2.21). Proposition 2 elaborates on the portfolio choice of a θ-ambiguity averse investor.

Proposition 2. *When $\psi = 1$, $\gamma \geqslant \omega$, and the θ-ambiguity averse investor considers the expected precision of the risky asset return instead of the instantaneous precision, the mean allocation to the risky asset is:*

$$\pi_{\underline{\theta}} = \frac{1}{\gamma}(\mu - r)\,\underline{\theta} + \frac{\sigma\rho}{\gamma}A\underline{\theta}, \tag{2.24}$$

which can be decomposed into three components:

$$myopic\,demand = \frac{1}{\gamma}(\mu - r)\,\theta, \tag{2.25}$$

$$intertemporal\,hedging\,demand = \frac{\sigma\rho}{\gamma}A\theta, \tag{2.26}$$

$$ambiguity\,demand = \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A\right](\underline{\theta} - \theta). \tag{2.27}$$

In this "expectation-driven" scenario, the mean demand for the risky asset is therefore decomposed into three components: myopic and intertemporal hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand. The sum of (2.25) and (2.26) gives the mean optimal risky asset demand under no ambiguity (2.21) and under ambiguity when the investor is able to instantaneously change his asset allocation. By adding the ambiguity demand (2.27), one obtains the mean risky asset demand under ambiguity (2.24) when the investor cannot instantaneously adjust his portfolio.

The main conclusions about the impact on π_θ, in this "expectation-driven" scenario, of the existence of θ-ambiguity are stated in Proposition 3.

Proposition 3. *When $\gamma \geqslant \omega$ and $\psi = 1$, (i) the mean allocation to the risky asset is lower when investors are θ-ambiguity averse (ambiguity demand is always negative); (ii) a higher (lower) level of ambiguity (measured by $\theta - \underline{\theta}$) implies a smaller (greater) allocation to the risky asset; (iii) the intertemporal hedging demand for the risky asset is*

negative if $\gamma > 1$ and positive if $\omega \leqslant \gamma < 1$.

Proof: *Appendix 2.6.6.*

The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a setting where expected precision, and not the expected return, is the source of ambiguity.

However, note that from (2.27), if there were scenarios where precision is bad, implying $\theta^* = \overline{\theta}$, then ambiguity aversion would increase the investor's demand for the ambiguous risky asset. This is a result reached by Gollier (2006) in a setting with ambiguity about expected excess return, with constant precision and under the ambiguity aversion specification of Klibanoff et al. (2005), for certain choices of multiple priors for the risky asset's return.

The conclusion that an investor with $\gamma > 1$ has a negative intertemporal hedging demand, and the opposite when $\omega \leqslant \gamma < 1$, is consistent with the findings in Chacko and Viceira (2005) and applies to all scenarios and not only to the "expectation-driven" one. When risk aversion is low $(\omega \leqslant \gamma < 1)$, the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that $\rho > 0$). An investor with high risk-aversion $(\gamma > 1)$ is not willing to accept this trade-off.

2.3.2 Approximate solution $(\psi \neq 1)$

The general case in which the investor's elasticity of intertemporal substitution of consumption (ψ) may differ from unity is relevant because most empirical studies suggest that $\psi \neq 1$, either higher (Hansen and Singleton (1982), Attanasio and Weber (1989) and Guvenen (2001)) or lower (Hall (1988), Epstein and Zin (1991), Campbell (1999) and Vissing-Jorgensen (2002)). However, the Bellman equation (2.15) has no exact solution when $\psi \neq 1$. Chacko and Viceira (2005) found an approximate analytical solution for $\psi \neq 1$ that converges to the exact solution determined above when $\psi = 1$. This approximate solution may be interpreted as describing the behavior of an investor with bounded rationality.[10]

[10]Briefly, the approximation in Chacko and Viceira (2005) consists in a log-linear expansion of the consumption-wealth ratio around its unconditional mean used to linearize the Bellman equation. As referred by the authors, this approach can be view as a particular class of the perturbation methods of approximation presented in Judd (1998), where the approximation occurs around the particular point given by the unconditional mean of the log consumption-wealth ratio. For full details on this, please see Chacko and Viceira (2005) pp.1379.

The accuracy of the approximate solution is insufficient for $\gamma < 1$ and also for higher levels of γ when coupled with low values of ψ (Chacko and Viceira (2005), table 6). Therefore, we extend the analysis of the previous section to the general case of $\psi \neq 1$, but only for $\gamma > 1$.

The Bellman equation continues to be given by (2.15), for any $\hat{\theta}$ value, but now the normalized aggregator is given by (2.8). The value function that approximately solves (2.15) in this scenario is given by Chacko and Viceira (2005):

$$J\left(\hat{\theta},X_t,y_t\right) = exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_t + B_1(\hat{\theta})\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma}, \tag{2.28}$$

where A_1 and $B_1(\hat{\theta})$ are:

$$A_1 = \frac{\frac{\rho\sigma(\mu-r)(1-\gamma)}{\gamma} - (h_1+\kappa) + \sqrt{\left[(h_1+\kappa) - \frac{\rho\sigma(\mu-r)(1-\gamma)}{\gamma}\right]^2 - \frac{\sigma^2(\mu-r)^2(1-\gamma)\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2}}}{\frac{\sigma^2}{\gamma}\left(\frac{1-\gamma}{1-\psi}\right)\left[\gamma(1-\rho^2)+\rho^2\right]}, \tag{2.29}$$

$$B_1(\hat{\theta}) = \psi ln\beta + \frac{h_0 - \psi\beta - r(1-\psi)}{h_1} + \frac{\kappa\hat{\theta}}{h_1}A_1, \tag{2.30}$$

which implies the following optimal consumption (C_t) and portfolio rules (π_t):

$$C_t = \beta^{\psi} X_t exp\left\{-A_1 y_t - B_1(\hat{\theta})\right\}, \tag{2.31}$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)y_t + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1 y_t. \tag{2.32}$$

Two notes on expressions (2.29) and (2.30). Firstly, in (2.29), only the positive sign on the square root guarantees that this approximate solution converges to the exact solution when $\psi = 1$. Secondly, h_1 represents the unconditional mean of the consumption-wealth ratio around which the linear approximation of the Bellman equation (2.15) is made in order to obtain this approximate solution: $h_1 = exp\{\overline{c-x}\}$, with $c_t - x_t = ln\left(\frac{C_t}{X_t}\right)$. h_0 is defined as $h_0 = h_1(1 - ln(h_1))$.

Regarding the investor's optimal policies, the main difference between the set of equations (2.19)-(2.20) and (2.31)-(2.32) concerns the consumption policy. From (2.31), the optimal consumption-wealth ratio is not fixed (β when $\psi = 1$). It is a decreasing function of precision when $\psi > 1$ and an increasing function of precision when $\psi < 1$ (Appendix 2.6.8). This reflects the relative importance of the intertemporal income and substitution effects of precision on consumption: when $\psi > 1$, the intertemporal substitution effect dominates the income effect, and the contrary happens when $\psi < 1$. Regarding (2.32), its structure is the same as in (2.20): it has a myopic component and an intertemporal

hedging component that, for $\gamma > 1$, is always negative (Appendix 2.6.8). Chacko and Viceira (2005) show that for empirically plausible characterizations of the process for precision, expressions (2.20) and (2.32) are very close.

From $E[y_t] = \theta$ and (2.32), the mean optimal allocation to the risky asset is

$$\pi_\theta = \frac{1}{\gamma}(\mu - r)\theta + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\theta. \tag{2.33}$$

An ambiguity averse investor considers, from his set of priors, $\theta^* = \underset{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}{argmin} J_{t_0}(\hat{\theta})$. The expression of $J_{t_0}(\hat{\theta})$ is now given by (2.28) at $t = t_0$:

$$J_{t_o}(\hat{\theta}) = exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_{t_0} + B_1(\hat{\theta})\right)\right\}\frac{X_{t_0}^{1-\gamma}}{1-\gamma}, \tag{2.34}$$

with y_{t_0} and X_{t_0} representing the instantaneous precision of the risky asset's return and the wealth at the present moment $(t = t_0)$. Proposition 4 states the conclusions on the ambiguity problem solution within this approximate formulation:

Proposition 4. *When $\psi \neq 1$ and $\gamma > 1$, the solution of the ambiguity problem is:*

$$\theta^* = \underline{\theta}.$$

Proof. *Appendix 2.6.7.*

Considering Proposition 4, the problem (2.14) to be solved becomes:

$$\underset{\pi,C}{max} E_{t_0}\left[\int_{t_0}^{\infty} f(C_s, J_s)\, ds\right]_{|\hat{\theta}=\underline{\theta}}, \; with\, \gamma > 1,\; \psi \neq 1,$$

subject to the same restrictions as above, but with $\hat{\theta} = \underline{\theta}$. The solution of this problem is given by expressions (2.28)-(2.32), with $\hat{\theta} = \underline{\theta}$.

In this case, the optimal consumption rule is affected by θ-ambiguity as expression (2.31) depends explicitly on $\hat{\theta}$: for each level of instantaneous precision, y_t, the consumption-wealth ratio is higher (lower) when investors are θ-ambiguity averse if and only if $\psi > 1$ $(\psi < 1)$ (Appendix 2.6.8).

Regarding the optimal portfolio rule, it happens the same thing as in the exact solution.

If the θ-ambiguity averse investor follows the optimal portfolio policy (2.32), his mean optimal allocation to the risky asset is given by (2.33), the same of an ambiguity neutral investor.

However, by analogy to the exposed for the exact solution, θ-ambiguity becomes relevant if the θ-ambiguity averse investor does not adjust continuously his portfolio for a given level of instantaneous precision (as in (2.32)). In this "expectation-driven" scenario, the investor's expectation of the risky asset return's precision becomes a key driver of the portfolio decision. In this scenario, while the mean allocation to the risky asset of an investor who faces no ambiguity continues to be given by (2.33), the choice of a θ-ambiguity averse investor is described by Proposition 5.

Proposition 5. *When $\psi \neq 1$, $\gamma > 1$, the mean consumption-wealth ratio and the mean allocation to the risky asset are given, respectively, by*

$$\left(\frac{C}{X}\right)_{\underline{\theta}} = c_{\underline{\theta}} = \beta^{\psi} exp\left\{-A_1\underline{\theta} - B_1(\underline{\theta})\right\}, \tag{2.35}$$

$$\pi_{\underline{\theta}} = \frac{1}{\gamma}(\mu - r)\,\underline{\theta} + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\underline{\theta}. \tag{2.36}$$

The mean allocation for the risky asset (2.36) can be decomposed into three components:

$$myopic\,demand = \frac{1}{\gamma}(\mu - r)\,\theta, \tag{2.37}$$

$$intertemporal\,hedging\,demand = \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\theta, \tag{2.38}$$

$$ambiguity\,demand = \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right](\underline{\theta} - \theta). \tag{2.39}$$

As in the exact solution, the risky asset demand in the "expectation-driven" scenario, is decomposed into three components: myopic and intertemporal hedging demands, as in Chacko and Viceira (2005), and ambiguity demand. The sum of (2.37) and (2.38) gives the mean optimal risky asset demand under no ambiguity (2.33) and under ambiguity with instantaneously portfolio updates. By adding the ambiguity demand (2.39), one obtains the mean risky asset demand with ambiguity, when the investor cannot continuously update his portfolio allocation, under the approximate solution (2.36).

The main conclusions about the impact, in this "expectation-driven" scenario, on the consumption-wealth ratio and on π_θ of the existence of θ-ambiguity are stated in Propo-

sition 6.

Proposition 6. *When $\gamma > 1$ and $\psi \neq 1$: (i) the mean allocation to the risky asset is lower when investors are θ-ambiguity averse (ambiguity demand is always negative); (ii) a higher (lower) level of ambiguity, measured by $(\theta - \underline{\theta})$, implies a smaller (greater) mean allocation to the risky asset; (iii) the intertemporal hedging demand is negative; (iv) the mean consumption-wealth ratio is higher (lower) when investors are θ-ambiguity averse if $\psi > 1$ ($\psi < 1$).*

Proof. *Appendix 2.6.8.*

2.4 Simulation

Chacko and Viceira (2005) found that, calibrating their model to long-run US data, the optimal intertemporal hedging demand is empirically small. This suggests that the "risk dimension" of stochastic variance is empirically not very relevant to the dynamic optimal portfolio decision. However, in their concluding remarks, they acknowledge that an important caveat of their analysis is that they have counterfactually assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have extended their model to account for ambiguity about one of the parameters of the precision process: the expected value of precision (inverse of variance). As a result, in a scenario where the ambiguity averse investor decides about his portfolio allocation considering his expectation of precision (and not the instantaneous precision), a third component of the demand for the risky asset appears. In addition to the "myopic demand" and the "intertemporal hedging demand", we obtain an "ambiguity demand" component. We denominated the referred scenario as "expectation-driven", and it is the one under which we run these simulations.

Using the calibration of Chacko and Viceira (2005), our simulation suggests that the ambiguity demand component of the allocation to the risky asset has a relevant empirical dimension, much higher than that of the intertemporal hedging demand. Stochastic variance may therefore have a significantly higher impact on the portfolio choice than what is suggested by the results of Chacko and Viceira (2005).

The reference parameter values used in the simulation are those estimated in Chacko and Viceira (2005) based on monthly excess stock returns on the CRSP value-weighted

portfolio over the T-Bill rate from January 1926 through December 2000:

$$\begin{aligned} \mu - r &= 0.0811, \\ \kappa &= 0.3374, \\ \theta &= 27.9345, \\ \sigma &= 0.6503, \\ \rho &= 0.5241, \\ r &= 0.015, \\ \beta &= 0.06. \end{aligned} \tag{2.40}$$

It is important to note that (2.5) is equivalent to (Appendix 2.6.9):

$$\theta \approx \frac{2\kappa + \sqrt{4\kappa^2 + 8\kappa\sigma^2 E[v_t]}}{4\kappa E[v_t]}. \tag{2.41}$$

Expression (2.41) sets the relation between θ and the expected variance of the risky asset's return, for a given pair of values for κ and σ. This relation is important for simulation purposes, because $\underline{\theta}$ is set for specified variance values[11], subject to the restriction that $2\kappa\theta > \sigma^2$ (section 2.2.1). Note that, for a given pair of values for κ and σ, and from (2.5):

$$\theta \in \left[\underline{\theta}, \overline{\theta}\right] \Longrightarrow E[v_t] \in \left[E[v_t]_{\theta=\overline{\theta}}, E[v_t]_{\theta=\underline{\theta}}\right]. \tag{2.42}$$

2.4.1 Simulation Results

Implications for the mean allocation to the risky asset from the introduction of θ-ambiguity are exemplified in Table 2.1, being consistent with conclusions in Proposition 3 and 6. The first column presents the results for the scenario without θ-ambiguity, in which the expected annual standard deviation of the risky asset's return is approximately equal to 19.1%. The other three columns represent scenarios with θ-ambiguity, in which the upper bound of the interval for the expected annual standard deviation values is increased to 20%, 25% and 30%. From (2.42), those three values are $\sqrt{E[v_t]_{\theta=\underline{\theta}}}$ values, i.e, they correspond (in each scenario) to the upper bound of intervals for $E[v_t]$ "built" by the θ-ambiguity averse investor. From (2.41) we obtain the related $\underline{\theta}$ values: this illustrates how the set of priors in the present analysis is part of the representation of the problem and how "intuitive" are its bounds, and therefore, why each of the scenarios analysed do not

[11] From (2.41), it is equivalent to consider a value for θ or $E[v_t]$. However, we believe it is more natural for investors to think in terms of $E[v_t]$ than θ. For the set of parameters in (2.40), the implied expected standard deviation of returns is 19.1314%.

correspond to implausible "extreme" scenarios of ambiguity.[12]

Simulations are run for different levels of risk aversion ($\gamma = 0.75$, 1, 2, 4, 20, 40) both using the exact solution ($\psi = 1$) and the approximate solution ($\psi = 1/0.75$ and $\psi = 1/1.5$). In panel A we show the mean allocation to the risky asset (percentage). In panels B and C, the intertemporal hedging demand and the ambiguity demand are shown as a percentage of the myopic demand.

Table 2.1: Mean allocation to the risky asset, intertemporal hedging demand and ambiguity demand.

	Expected Annual Standard Deviation of Risky Asset Returns											
	19,1314%			20%			25%			30%		
implied θ	θ = 27.935			$\underline{\theta}$ = 25.612			$\underline{\theta}$ = 16.604			$\underline{\theta}$ = 11.706		
ψ	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5
	A - Mean allocation to risky asset (%)											
γ												
0.75	-	305,66	-	-	280,24	-	-	181,68	-	-	128,09	-
1.00	-	226,55	-	-	207,71	-	-	134,66	-	-	94,94	-
2.00	111,38	111,37	111,37	102,12	102,11	102,11	66,22	66,20	66,18	46,69	46,67	46,65
4.00	55,26	55,24	55,21	50,67	50,64	50,62	32,85	32,83	32,81	23,16	23,15	23,13
20.00	10,99	10,98	10,97	10,08	10,07	10,05	6,53	6,52	6,52	4,60	4,60	4,59
40.00	5,49	5,48	5,48	5,03	5,03	5,02	3,26	3,26	3,26	2,30	2,30	2,30
	B - Ratio of hedging demand over myopic demand (%)											
γ												
0.75	-	1,19	-	-	1,19	-	-	1,19	-	-	1,19	-
1.00	-	0,00	-	-	0,00	-	-	0,00	-	-	0,00	-
2.00	-1,68	-1,68	-1,68	-1,67	-1,68	-1,68	-1,65	-1,68	-1,70	-1,64	-1,68	-1,71
4.00	-2,43	-2,47	-2,52	-2,43	-2,47	-2,52	-2,41	-2,47	-2,54	-2,40	-2,47	-2,55
20.00	-3,00	-3,09	-3,19	-2,99	-3,09	-3,19	-2,99	-3,09	-3,20	-2,99	-3,09	-3,20
40.00	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28
	C - Ratio of Ambiguity demand over myopic demand (%)											
γ												
0.75	-	0,00	-	-	-8,41	-	-	-41,04	-	-	-58,79	-
1.00	-	0,00	-	-	-8,32	-	-	-40,56	-	-	-58,10	-
2.00	0,00	0,00	0,00	-8,18	-8,18	-8,18	-39,89	-39,88	-39,87	-57,14	-57,12	-57,10
4.00	0,00	0,00	0,00	-8,11	-8,11	-8,11	-39,58	-39,56	-39,53	-56,70	-56,66	-56,62
20.00	0,00	0,00	0,00	-8,07	-8,06	-8,05	-39,35	-39,31	-39,27	-56,36	-56,30	-56,24
40.00	0,00	0,00	0,00	-8,06	-8,05	-8,04	-39,32	-39,28	-39,23	-56,32	-56,26	-56,19

Note 1 **- Panel A:** $E[\pi_t(y_t)] = \pi_\theta \times 100$; **Panel B**: for $\psi = 1$, (ratio of (2.26) over (2.25))$\times$100; for $\psi \neq 1$, (ratio of (2.38) over (2.37))$\times$100; **Panel C**: for $\psi = 1$, (ratio of (2.27) over (2.25))$\times$100; for $\psi \neq 1$, (ratio of (2.39) over (2.37))$\times$100.

Note 2 **-** Simulation results for $\gamma \leqslant 1$, $\psi \neq 1$ are not disclosed due to lower approximation accuracy.

Note 3 **-** $\omega = 0.14 < 0.75$: domain of Proposition 1 guarantied.

Let us give an example based on panel A of Table 2.1. Consider a risk-averse investor, with $\gamma = 2$ and $\psi = 1$, that is θ-ambiguity neutral. His mean optimal allocation to the risky asset corresponds to 111.4% of his wealth. If the investor becomes ambiguous about θ, under the "expectation-driven" scenario disclosed in previous sections, with the implied upper bound of the interval for annual expected standard deviation (from (2.42)) being equal to 25%, his mean allocation to the risky asset declines to 66,20% of his wealth.

Panel B reports the estimates of the intertemporal hedging demand, measured as a ratio of myopic demand. The overall results with θ-ambiguity are similar to the ones in the

[12]Implementation using Dynare version 3.065 and MatLab version 7.0.0.19920 R14.

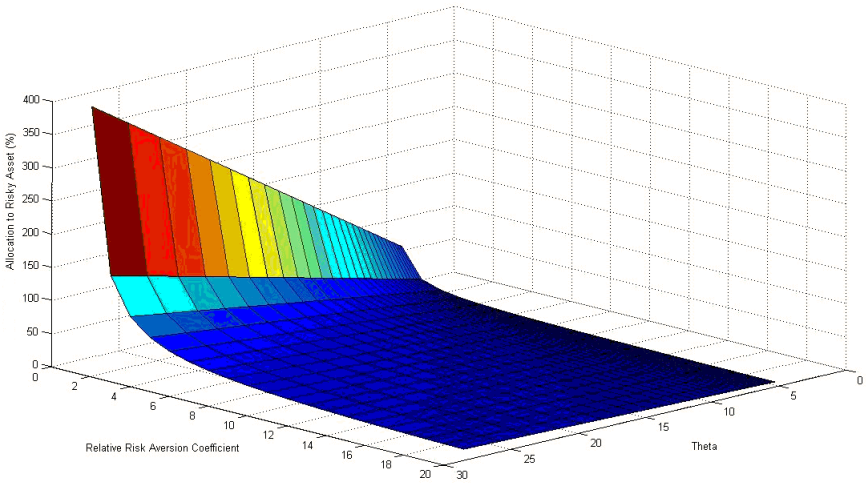
scenario without ambiguity (the same for $\psi = 1$ and very close in the other cases). The main results are: (i) this ratio does not vary with the precision of the return; (ii) intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma > 1$; (iii) intertemporal hedging demand is small - even for a very strongly risk averse investor ($\gamma = 40$).

Panel C presents the estimated ratios of ambiguity demand versus myopic demand. It shows that ambiguity demand is always negative, with an importance that increases with the level of ambiguity and decreases with the level of risk aversion, showing much less sensitivity to γ.

Results for the exact and the approximate solution are very close, denoting low sensitivity to the elasticity of intertemporal substitution (ψ).

In panel A, it is shown that, the demand for the risky asset is decreasing with the risk aversion, γ, and with the level of ambiguity, $\theta - \underline{\theta}$. This is graphically highlighted in Figure 2.1.

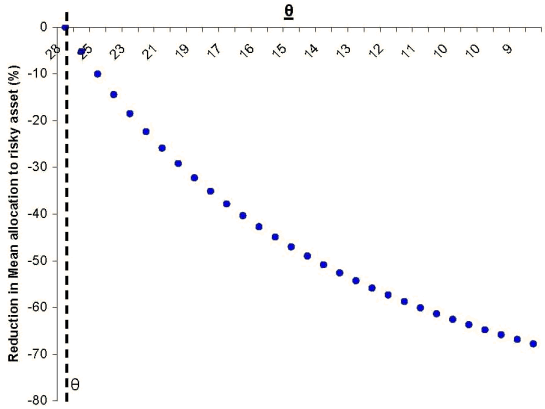
Figure 2.1: Mean allocation to the risky asset as a function of risk aversion and ambiguity level.



Note **-** Simulation run with $\psi = 1$.

It is clear that, under our setting and using calibration (2.40), the allocation to risky asset reacts strongly to changes in $\underline{\theta}$. Figure 2.2 further highlights this for a larger set of $\underline{\theta}$ values within a reasonable range of expected annual standard deviation of returns (from 19,1% to 35%).

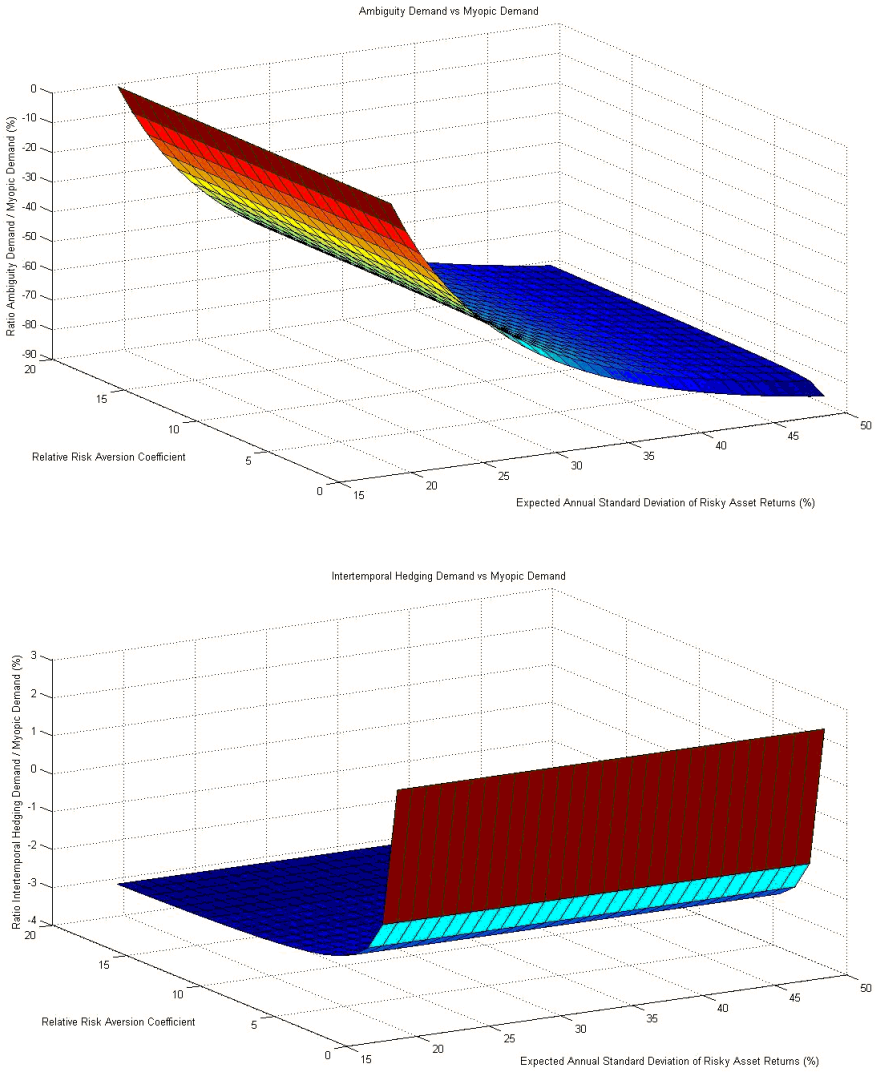
Figure 2.2: Mean allocation to the risky asset as a function of expected precision.



Note **-** Simulation run with $\psi = 1$.

Those results suggest that the ambiguity effect on the demand of the risky asset is empirically significant: even for a low level of ambiguity (second column in Table 2.1), where expected annual standard deviation is slightly adjusted from 19,1% (assumed to be the true value from calibration (2.40)) to 20%, the ambiguity effect corresponds to 8% of the investors myopic demand. This is further illustrated in Figure 2.3 for a wider range of values of expected annual standard deviation of returns and of γ. Figure 2.3 also highlights the fact of ambiguity demand being empirically much more relevant than intertemporal hedging demand (note the difference of scale in the vertical axis).

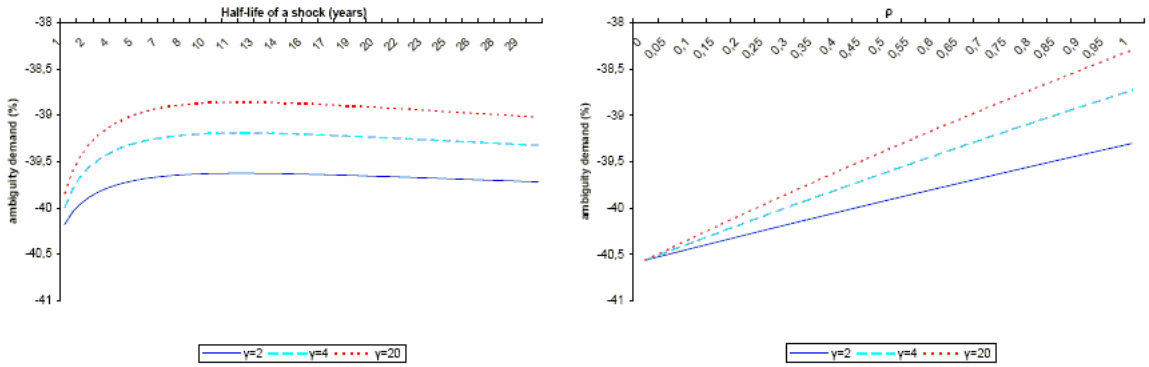
Figure 2.3: Ambiguity demand and intertemporal hedging demand *vs* myopic demand.



Note **-** Simulations run with $\psi = 1$.

In Figure 2.4 is illustrated the sensitivity of ambiguity demand to variations of two other parameters of the precision process (2.3): the reversion parameter (κ), which determines the persistence of shocks to precision, and the instantaneous correlation between shocks to precision and to risky asset return (ρ).

Figure 2.4: Effect on the ambiguity demand of changes in κ and ρ, with $\gamma = 2$, 4 and 20.



Note 1 **-** Simulations run with $\psi = 1$.

Note 2 **-** Half-life of a shock (years) to precision: $\ln(2)/\kappa$.

Note 3 **-** Ambiguity demand (%) measured as a ratio of myopic demand.

Until a certain level of half-life of the persistence of a shock to precision (around 7 years), the more persistent is the shock in precision, the lower is the absolute value of the ambiguity demand. Above that level, this effect disappears. The intuition for this can be that the higher is the persistence of shocks in precision, the larger is the period of time that it stays away from its expected value θ, and, therefore, the less relevant is this parameter and the ambiguity about it.

The higher is the instantaneous correlation between shocks to precision and to risky asset returns, the lower is the absolute value of the ambiguity demand. This means that the easier it is to hedge variations in precision, the less relevant is the ambiguity about its expected value. However, Figure 4 also suggests that both effects are small.

Table 2.2 shows the impacts on the mean consumption-wealth ratio (Panel A) and on the long term expected return on wealth (Panel B) from the introduction of θ-ambiguity. Panel A shows that: (i) in the case of exact solution $(\psi = 1)$, the consumption-wealth ratio is constant and equal to β as in (2.19); (ii) in the case of approximate solution $(\psi \neq 1)$, when $\psi > 1$ $(\psi < 1)$ the consumption-wealth ratio is higher (lower) when there is θ-ambiguity, in consistency with Proposition 6, and the same happens with the level of risk aversion. Figure 2.5 illustrate well this conclusion. Moreover, both Table 2.2 and Figure 2.5 show that changes in the consumption-wealth ratio from changes in the level of ambiguity and risk aversion are of small empirical dimension.

Table 2.2: Consumption-wealth ratio and long-term expected return on wealth.

	Expected Annual Standard Deviation of Risky Asset Returns											
	19,1314%			20%			25%			30%		
implied θ	θ = 27.935			θ = 25.612			θ = 16.604			θ = 11.706		
ψ	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5
	A - Consumption-wealth ratio (%)											
γ												
0.75	-	6,00	-	-	6,00	-	-	6,00	-	-	6,00	-
1.00	-	6,00	-	-	6,00	-	-	6,00	-	-	6,00	-
2.00	6,02	6,00	5,98	6,15	6,00	5,85	6,62	6,00	5,38	6,88	6,00	5,12
4.00	6,77	6,00	5,23	6,83	6,00	5,16	7,07	6,00	4,93	7,20	6,00	4,80
20.00	7,36	6,00	4,64	7,37	6,00	4,63	7,41	6,00	4,58	7,44	6,00	4,56
40.00	7,43	6,00	4,57	7,43	6,00	4,57	7,46	6,00	4,54	7,47	6,00	4,53
	B - Long-term expected return on wealth (%)											
γ												
0.75	-	26,29	-	-	24,23	-	-	16,23	-	-	11,89	-
1.00	-	19,87	-	-	18,35	-	-	12,42	-	-	9,20	-
2.00	10,53	10,53	10,53	9,78	9,78	9,78	6,87	6,87	6,87	5,29	5,29	5,28
4.00	5,98	5,98	5,98	5,61	5,61	5,61	4,16	4,16	4,16	3,38	3,38	3,38
20.00	2,39	2,39	2,39	2,32	2,32	2,32	2,03	2,03	2,03	1,87	1,87	1,87
40.00	1,95	1,94	1,94	1,91	1,91	1,91	1,76	1,76	1,76	1,69	1,69	1,69

Note 1 **- Panel A**: $c_t/x_t = exp\{E[c_t - x_t] \times 100\}$; **Panel B**: $(\pi_\theta(\mu - r) + r) \times 100$

Note 2 **-** Simulation results for $\gamma \leqslant 1$, $\psi \neq 1$ are not disclosed due to lower approximation accuracy.

Note 3 **-** $\omega = 0.14 < 0.75$: domain of Proposition 1 guarantied.

In Panel B, it is possible to see that the long-term expected return on wealth, measured by $(\pi_\theta(\mu - r) + r)$, of an investor that is both risk averse and θ-ambiguity averse is a decreasing function of both risk aversion and the level of ambiguity, which was expectable considering results from Panel A of Table 2.1. This is further illustrated in Figure 2.6.

Figure 2.5: Consumption-wealth ratio as a function of expected precision.

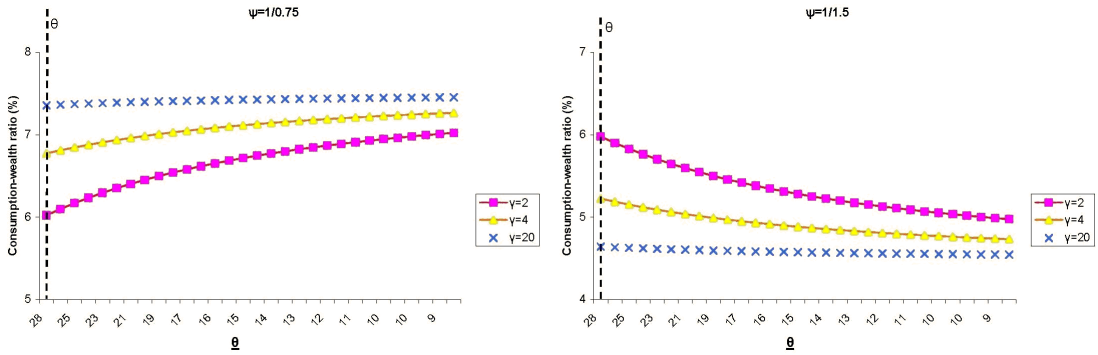
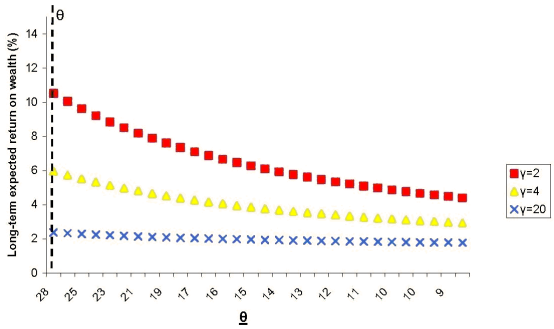


Figure 2.6: Long-term expected return on wealth as a function of expected precision and risk aversion.



2.5 Concluding Remarks

We presented an extension of the model of Chacko and Viceira (2005) for optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the expected precision (inverse of variance) of the risky asset's return (parameter θ). In our setting, precision of the risky asset return is therefore simultaneously the source of risk and ambiguity perceived by the risk averse and θ-ambiguity averse investor.

Long-horizon investors with recursive preferences, as defined by Duffie and Epstein (1992a) with Kreps and Porteus' (1978) specification, have two assets to invest in, a risk-free asset and a risky asset. The precision of the risky asset return is stochastic and the investor is ambiguous about its expected value (unobserved parameter), with ambiguity aversion in the spirit of Gilboa and Schmeidler's (1989) Maxmin expected utility model. Not knowing the true value of precision, investors consider the worst possible value in a given interval.

A preliminary conclusion is that precision is always good, in the sense that it increases the utility attainable by the investor. This implies that the solution of the ambiguity problem is given by the lower bound $(\underline{\theta})$ of the assumed interval for possible θ values.

The main conclusions concern the impact on optimal dynamic policies from ambiguity about the expected value of precision. Regarding the optimal consumption policy, when the intertemporal elasticity of consumption (ψ) is different from one, the optimal consumption-wealth decision is always affected by the θ-ambiguity: consumption is higher when investors are ambiguity averse compared to when they are ambiguity neutral if $\psi > 1$ and the contrary when $\psi < 1$. When $\psi = 1$, the mean optimal consumption-wealth ratio does not depend on the level of precision, as found by Chacko and Viceira (2005).

Regarding the optimal demand for the risky asset, we conclude that ambiguity about θ has no impact. This result should be viewed having in mind that the model assumes that investors can adjust their portfolios continuously, as a function of the instantaneously observed precision. In the real world, investors cannot update their portfolios continuously, for reasons that are not easy to model (such as transaction costs or human limitations). Incorporating such market frictions in the model is an interesting task for future research.

To provide an additional perspective in the context of the present model, we also studied an "expectation-driven" scenario in which case investors use their expectations of future precision instead of the instantaneous precision. In this scenario, the mean allocation to the risky asset decreases with ambiguity, i.e., ambiguity demand is negative.

The last conclusion of this chapter respects to the empirical relevance of ambiguity demand, as determined in this latter scenario. Chacko and Viceira (2005) concluded that the variance of the risky asset's return generates a small intertemporal hedging demand, suggesting low relevance of the stochastic variance (precision) in the dynamic portfolio decision. Using the same calibration, our simulation suggests that the ambiguity demand component of the risky asset demand is relevant and has a much greater empirical dimension than that of intertemporal hedging demand. This indicates that stochastic variance (precision) may have a much higher impact on investors portfolio decision than found in Chacko and Viceira (2005). Recovering the Knightian conceptual dichotomy of ambiguity versus risk (Knight (1921)), we conclude that under those circumstances the "ambiguity dimension" of variance seems to be much more relevant than its "risk dimension" for dynamic optimal portfolio decisions.

Appendix

2.6 Mathematical Proofs

2.6.1 Equation (2.4)

We start by stating Itô's lemma. Let L_t be a variable that follows the Itô process:

$$dL_t = \varsigma\left(L_t, t\right) dt + \nu\left(L_t, t\right) dBM_t\,, \tag{2.43}$$

where BM_t is a standard Brownian motion and ς and ν are functions of L_t and t. Let $g\left(t, L_t\right)$ be a continuous differentiable function as regards its two arguments. Then,

$$Y = \left\{Y_t = g\left(t, L_t\right)\right\},$$

is a stochastic variable with its stochastic differential given by:

$$dY_t = \frac{\partial g}{\partial t}\left(t, L_t\right) dt + \frac{\partial g}{\partial L}\left(t, L_t\right) dL_t + \frac{1}{2}\frac{\partial^2 g}{\partial L^2}\left(t, L_t\right)\left(dL_t\right)^2.$$

Applying the rule $dt^2 = dt\,dBM_t = 0$ and $\left(dBM_t\right)^2 = dt$, then:

$$dY_t = \left(\frac{\partial g}{\partial t}\left(t, L_t\right) + \frac{\partial g}{\partial L}\left(t, L_t\right)\varsigma + \frac{1}{2}\frac{\partial^2 g}{\partial L^2}\left(t, L_t\right)\nu^2\right) dt + \frac{\partial g}{\partial L}\left(t, L_t\right)\nu dBM_t\,, \tag{2.44}$$

i.e., the variable Y_t also follows an Itô Process.

In our setting, considering $v_t = g\left(y_t\right) = \frac{1}{y_t}$, then $g_y = -\frac{1}{y_t^2}$ and $g_{yy} = \frac{2}{y_t^3}$, where g_y and g_{yy} are the first and second derivatives of $g\left(\cdot\right)$ in order to y_t. Moreover, making the parallel with (2.43), in our setting we have:

$$\varsigma = \kappa\left(\theta - y_t\right),$$

$$\nu = \sigma\sqrt{y_t}.$$

Using these inputs and applying (2.44) to (2.3), we find that the stochastic process v_t is given by:

$$\frac{dv_t}{v_t} = \left(\kappa + v_t\sigma^2 - v_t\kappa\theta\right) dt - \sigma\sqrt{v_t}dW_y\,. \tag{2.45}$$

Defining $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \frac{\kappa}{\theta_v}$ (which imply that $\kappa + v_t\sigma^2 - v_t\kappa\theta = \kappa_v\left(\theta_v - v_t\right)$), we find (2.4).

□

2.6.2 Equation (2.5)

Applying the second-order Taylor expansion of $v_t = \frac{1}{y_t}$ around θ:

$$v_t \approx \frac{1}{\theta} - \frac{1}{\theta^2}(y_t - \theta) + \frac{1}{\theta^3}(y_t - \theta)^2 . \tag{2.46}$$

Taking expectations:

$$E[v_t] \approx \frac{1}{\theta} - \frac{1}{\theta^2}(E[y_t] - \theta) + \frac{1}{\theta^3} E\left[(y_t - \theta)^2\right] . \tag{2.47}$$

From Cox et al. (1985b) it is known that $E[y_t] = \theta$ and $E\left[(y_t - \theta)^2\right] = Var[y_t] = \frac{\sigma^2\theta}{2\kappa}$. Substituting those results in (2.47), we obtain (2.5).

□

2.6.3 Stochastic Differential Utility

Consumption processes C are chosen from the space D of square-integrable progressively measurable processes valued in $\mathscr{C} = \mathbb{R}_+$ (see Duffie and Epstein (1992a) footnote 2 for further details). Duffie and Epstein (1992a) define the stochastic differential utility (SDU) function $U : D \to \mathbb{R}$ by two primitive functions, $\bar{f} : \mathscr{C} \times \mathbb{R} \to \mathbb{R}$ and $\bar{A} : \mathbb{R} \to \mathbb{R}$. The function $\bar{A}(\cdot)$ is called variance multiplier, as it applies a penalty (or reward) as a multiple of the utility "volatility". For deterministic consumption process, $\bar{A}(\cdot)$ is therefore irrelevant and only $\bar{f}(\cdot)$ matters.

As stated in Duffie and Epstein (1992a) if, for each consumption process C, there exists a well-defined utility process $\bar{J}$, then the SDU function U is defined by $U(C) = \bar{J}_0$, the initial value of that utility process. The pair $\left(\bar{f}, \bar{A}\right)$ generating $\bar{J}$ is called an aggregator.

Two aggregators $\left(\bar{f}, \bar{A}\right)$ and (f, A) are said to be ordinally equivalent if they generate ordinally equivalent utility functions, i.e. represent the same preference ordering of consumption processes. In Duffie and Epstein (1992a) it is presented a method through which for any aggregator $\left(\bar{f}, \bar{A}\right)$ an ordinally equivalent aggregator (f, A), with variance multiplier $A(\cdot)$ equal to zero, is obtained. (f, A) is called the normalized aggregator of $\left(\bar{f}, \bar{A}\right)$ and generates the utility process J given by:

$$J_t^T = E_t\left[\int_t^T f(C_s, J_s)\, ds\right] . \tag{2.48}$$

Note how (2.48) is close to (2.7). The difference between the expressions is the infinite time horizon in the later which is also considered in Duffie and Epstein (1992a) with $J_t \equiv \lim_{T\to\infty} J_t^T$.

The method to obtain (f,A) from $\left(\bar{f},\bar{A}\right)$, consists in a change of variables through a twice continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and with $\varphi(0) = 0$. φ satisfies the differential equation $\varphi''(x) = \bar{A}(x)\,\varphi'(x)$, which implies:

$$\varphi(J) = \delta_2 + \delta_1 \int_{J_0}^{J} exp\left[\int_{J_0}^{u} \bar{A}(x)\,dx\right] du, \tag{2.49}$$

where J_0 is arbitrary, δ_2 and δ_1 are constants, with $\delta_1 > 0$, defined so that $\varphi(0) = 0$. Using φ, the relationship between the two ordinally equivalent aggregators $\left(\bar{f},\bar{A}\right)$ and (f,A) is given by:

$$\begin{aligned} \bar{f}(C,z) &= \frac{f(C,\varphi(z))}{\varphi'(z)}, \quad (C,z) \in \mathscr{C} \times \mathbb{R}, \\ \bar{A}(x) &= \varphi'(x)A[\varphi(x)] + \frac{\varphi''(x)}{\varphi'(x)}. \end{aligned} \tag{2.50}$$

In this chapter, following Chacko and Viceira (2005), the utility process J generated by the normalized aggregator $f(C_s,J_s)$ is set to define a SDU function ordinally equivalent to Kreps and Porteus (1978) utility specification. The aggregator $\left(\bar{f},\bar{A}\right)$ for this particular utility function is defined in Duffie and Epstein (1992a) as:

$$\bar{f}(C,J) = \frac{\beta}{\xi}\frac{C^{\xi} - J^{\xi}}{J^{\xi-1}}, \qquad \bar{A}(J) = \frac{\alpha - 1}{J}, \tag{2.51}$$

with $\mathscr{C} = \mathbb{R}_+$, $0 \neq \xi \leq 1$, $0 \leq \beta$, $0 \neq \alpha \leq 1$, and $J > 0$.

Deduction of expression (2.8)

Replacing (2.51), $\bar{A}(x) = \frac{\alpha-1}{x}$, in (2.49):

$$\begin{aligned}
\varphi(J) &= \delta_2 + \delta_1 \int_{J_0}^{J} exp\left[\int_{J_0}^{u} \frac{\alpha-1}{x} dx\right] du \\
&= \delta_2 + \frac{\delta_1}{J_0} \int_{J_0}^{J} u^{\alpha-1} du \\
&= \delta_2 + \delta_1 \left(\frac{J^\alpha}{\alpha}\right). \qquad (2.52)
\end{aligned}$$

To have $\varphi(0) = 0$, we set $\delta_2 = 0$. Assuming $\delta_1 = 1$, possible as in Duffie and Epstein (1992a) the only requirement is $\delta_1 > 0$, expression (2.52) yields $\varphi(J) = \frac{J^\alpha}{\alpha}$ and:

$$\varphi'(J) = J^{\alpha-1}. \qquad (2.53)$$

Introducing (2.51) and (2.53) in (2.50), gives:

$$\frac{\beta}{\xi} \frac{C^\xi - J^\xi}{J^{(\xi-\alpha)}} = f(C, \varphi(J)). \qquad (2.54)$$

From $\varphi(J) = \frac{J^\alpha}{\alpha}$ we get $(\alpha\varphi(J))^{\frac{1}{\alpha}} = J$.

Introducing this result in (2.54):

$$\begin{aligned}
\frac{\beta}{\xi} \frac{C^\xi - (\alpha J)^{\frac{\xi}{\alpha}}}{(\alpha J)^{\frac{\xi-\alpha}{\alpha}}} &= f(C,J) \\
\frac{\beta}{\xi} \alpha J \left(\left(\frac{C}{(\alpha J)^{\frac{1}{\alpha}}}\right)^\xi - 1\right) &= f(C,J). \qquad (2.55)
\end{aligned}$$

Expression (2.8) follows simply by changing notation: $\xi = 1 - \frac{1}{\psi}$ and $\alpha = 1 - \gamma$.

□

2.6.4 Sign of the square-root in (2.17)

Regarding (2.17), an issue to be addressed is the sign of the square root. With $\psi = 1$, as $\gamma \to 1$, the utility representation (2.9) converges to the log-utility representation. The optimal portfolio rule (2.20) in the special case of log-utility $(\gamma = \psi = 1)$ is well-known

(Merton (1969, 1971, 1973a)):

$$\pi_t = (\mu - r)\, y_t\,,$$

i.e., the intertemporal hedging demand component disappears (if $\psi = \gamma = 1$, then $A = B = 0$). It is therefore necessary to guarantee that with $\psi = 1$, $\lim_{\gamma\to 1} A = 0$[13] as the limit of (2.20) as $\gamma \to 1$ is given by:

$$\lim_{\gamma\to 1}\pi_t = (\mu - r)\, y_t + \left(\lim_{\gamma\to 1} A\right)\rho\sigma y_t\,.$$

From (2.17), $\lim_{\gamma\to 1} A$ is:

$$\lim_{\gamma\to 1} A = \frac{(\beta+\kappa) \pm \lim_{\gamma\to 1}(1-\gamma)\gamma\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}}}{\sigma^2}. \tag{2.56}$$

If $\gamma \to 1^+$, i.e., $\gamma > 1$, then $(1-\gamma) < 0$ and the discriminant of the square root in (2.56) is always > 0. By assumption, $(\beta + \kappa) > 0$, which implies that, in order to have $\lim_{\gamma\to 1^+} A = 0$, the "$+$" sign must be considered.

The same rational implies that when $\gamma < 1$, the "$-$" sign of the square root guarantees $\lim_{\gamma\to 1^-} A = 0$ (it can be easily shown that the discriminant of the square root in (2.56) is positive as γ approaches 1 from below).

When $\gamma = 1$, only the "-" sign of the square root gives $A = 0$, as:

$$A\,|_{\gamma=1} = \frac{(\beta+\kappa) \pm (\beta+\kappa)}{\sigma^2}.$$

□

2.6.5 Proof of Proposition 1

The domain of analysis is set so that A in (2.17) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$\left(\frac{\rho\sigma\,(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 \geqslant \frac{\sigma^2\,(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\,(1-\gamma)}. \tag{2.57}$$

[13] From (2.18) $\lim_{\gamma\to 1} A = 0 \Longrightarrow \lim_{\gamma\to 1} B = 0$.

For $\gamma > 1$ it is straightforward to conclude that $\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 > \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}$, and therefore (2.57) is always true.
For $\gamma < 1$, (2.57) is true as long as:

$$\begin{aligned}
\frac{\gamma}{1-\gamma} &\geqslant \frac{\sigma^2(\mu-r)^2}{(\beta+\kappa)^2} + \frac{2\rho\sigma(\mu-r)}{(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \omega,
\end{aligned}$$

by making $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}$. Note that $\omega < 1$ as $(\beta+\kappa)^2 > 0$.

The domain of analysis is therefore $\gamma \geqslant \omega$.

From (2.17)-(2.18) and (2.22): $\frac{dJ_{t_o}}{d\theta} = \frac{\kappa exp\left\{Ay_{t_0}+B\right\}X_{t_o}^{1-\gamma}}{\beta}\frac{A}{1-\gamma}$. This implies that:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = Sign\,\frac{A}{1-\gamma}, \tag{2.58}$$

as $\frac{\kappa exp\left\{Ay_{t_0}+B\right\}X_{t_o}^{1-\gamma}}{\beta} > 0$. Further development of (2.58) gives:

$$\begin{aligned}
Sign\left(\frac{dJ_{t_o}}{d\theta}\right) &= Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right. \\
&\left.\pm\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma(1-\rho^2)+\rho^2\right]}{\gamma^2(1-\gamma)}}\right\}
\end{aligned} \tag{2.59}$$

since $\frac{\gamma}{\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]} > 0$ as, by assumption, $\gamma > 0$ and $0 < \rho < 1$. Recovering conclusions regarding the sign of the square root in Appendix 2.6.4, one gets:
For $\gamma > 1$:

$$\begin{aligned}
Sign\left(\frac{dJ_{t_o}}{d\theta}\right) &= Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right. \\
&\left.+\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma(1-\rho^2)+\rho^2\right]}{\gamma^2(1-\gamma)}}\right\}.
\end{aligned}$$

Although $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} < 0$, when $\gamma > 1$ it results:

$$\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}} > \left|\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right|,$$

implying that $Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = "+"|_{\gamma>1}$.

For $\omega \leqslant \gamma < 1$:

$$\begin{aligned} Sign\left(\frac{dJ_{t_o}}{d\theta}\right) &= Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right. . \\ &\left. -\sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}}\right\} \end{aligned}$$

For $\gamma < 1$, one gets $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} > 0$ if $\gamma > \frac{\rho\sigma(\mu-r)}{(\beta+\kappa)+\rho\sigma(\mu-r)}$.
As $\frac{\rho\sigma(\mu-r)}{(\beta+\kappa)+\rho\sigma(\mu-r)} < \omega$ it results that $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} > 0\,|_{\omega\leqslant\gamma<1}$. This implies:

$$\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} > \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}}$$

when $\omega \leqslant \gamma < 1$, and consequently $Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = "+"|_{\omega\leqslant\gamma<1}$.

At last, by applying l'Hôpital rule to the calculation of $\lim_{\gamma\to 1}\left(\frac{dJ_{t_o}}{d\theta}\right)$ one obtains $\lim_{\gamma\to 1}\left(\frac{dJ_{t_o}}{d\theta}\right) = \frac{\sigma^2(\mu-r)^2}{2(\beta+\kappa)}$, which is > 0. Thus overall:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = "+"|_{\gamma\geqslant\omega}\,.$$

Following the spirit of Gilboa and Schmeidler (1989) Maxmin expected utility model, as an increase in the expected precision of the risky asset return increases utility, then the lowest possible value for the expected precision is considered, i.e., $\theta^* = \underline{\theta}$.

□

2.6.6 Proof of Proposition 3

Proof of (i): $\underline{\theta} < \theta \Rightarrow \pi_{\underline{\theta}} < \pi_{\theta}$.

The difference between the mean allocation to the risky asset when ambiguity is considered $\left(\pi_{\underline{\theta}}\right)$ versus when there is no ambiguity (π_{θ}), is given by (2.27):

$$\pi_{\underline{\theta}} - \pi_{\theta} \quad = \quad \left(\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A\right)(\underline{\theta} - \theta)\,.$$

We need only to show that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ in the domain under analysis $(\gamma \geqslant \omega)$. For $\gamma > 1$, it is algebraically easy to show that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$, by substituting the expression for A (2.17) and considering the positive sign of its square root in line with conclusions in Appendix 2.6.4.

For $\gamma = 1$, it results that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A = (\mu - r) > 0$, by substituting the expression for A (2.17) and considering the negative sign of its square root in line with conclusions in Appendix 2.6.4.

For $\omega \leqslant \gamma < 1$, from (2.58) and conclusions in Appendix 2.6.5, it is immediate to conclude that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$, as $A > 0$.

Since $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ and $\underline{\theta} < \theta$ (when $\underline{\theta} = \theta$ there is no ambiguity), it is immediate to conclude that $\left(\pi_{\underline{\theta}} - \pi_{\theta}\right) < 0$ and that ambiguity demand (2.27) is always negative.

□

Proof of (ii): *$\pi_{\underline{\theta}}$ is decreasing with$(\theta - \underline{\theta})$.*

The relation between the mean allocation to the risky asset when there exists ambiguity $(\pi_{\underline{\theta}})$ and changes in the level of ambiguity, measured by $(\theta - \underline{\theta})$, is:

$$\frac{d\pi_{\underline{\theta}}}{d(\theta - \underline{\theta})} = -\left(\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A\right).$$

As $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ (see previous proof of (i)), it results that an increased level of ambiguity implies a reduction of the mean allocation to the risky asset.

□

Proof of (iii): *Sign of the intertemporal hedging demand.*

From (2.26), and since $\sigma, \rho, \gamma, \theta > 0$:

$$Sign(HD) = Sign(A).$$

From (2.58) and conclusions in Appendix 2.6.5:

$$\gamma \geqslant \omega : \quad \begin{cases} \gamma > 1 : & Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = "+" \Rightarrow Sign\,(A) = "-" \\ \\ \omega \leqslant \gamma < 1 : & Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = "+" \Rightarrow Sign\,(A) = "+" \end{cases}$$

When $\gamma = 1$, intertemporal hedging demand is null because $A = 0\,|_{\gamma=1}$ (see Appendix 2.6.4).

□

2.6.7 Proof of Proposition 4

From (2.29)-(2.30) and (2.34):

$$\frac{dJ_{t_0}}{d\theta} = -\frac{1}{1-\psi}\frac{A_1\kappa}{h_1} exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)(A_1 y_{t_0} + B_1)\right\} X_{t_0}^{1-\gamma}.$$

As $exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)(A_1 y_{t_0} + B_1)\right\} > 0$ and $X_{t_0}^{1-\gamma}$, κ, $h_1 > 0$:

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = Sign\left(-\frac{1}{1-\psi}A_1\right). \tag{2.60}$$

Expression (2.29) for A_1 is further simplified to:

$$A_1 = \frac{(1-\psi)\left[b + \sqrt{b^2+u}\right]}{(1-\gamma)\,e}, \; with \tag{2.61}$$

$$\begin{aligned} b &= \rho\sigma\,(\mu - r)\,(1-\gamma) - \gamma(h_1 + \kappa), \\ u &= -\sigma^2\,(\mu - r)^2\,(1-\gamma)\left[\gamma\left(1-\rho^2\right) + \rho^2\right], \\ e &= \sigma^2\left[\gamma\left(1-\rho^2\right) + \rho^2\right], \; always > 0 \; as \; 0 < \rho < 1, \gamma > 1. \end{aligned}$$

Substituting (2.61) into (2.60), for $\gamma > 1$ one obtains:

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = Sign\left(b+\sqrt{b^2+u}\right).$$

As $\sqrt{b^2+u} > b$, because $b < 0$ and $u > 0$ when $\gamma > 1$, then

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = \text{"}+\text{"},$$

which, following the same rational as in Appendix 2.6.5, implies $\theta^* = \underline{\theta}$.

□

2.6.8 Proof of Proposition 6

Proof of (i): $\underline{\theta} < \theta \Rightarrow \pi_{\underline{\theta}} < \pi_\theta$.

The difference between the mean allocation to the risky asset under ambiguity aversion $\left(\pi_{\underline{\theta}}\right)$ versus under ambiguity neutrality (π_θ), is given by (2.39):

$$\pi_{\underline{\theta}} - \pi_\theta = \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right](\underline{\theta} - \theta). \tag{2.62}$$

Substituting expression (2.29) for A_1 into $\left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right]$, after some algebra the latter expression is simplified to:

$$\frac{(\mu-r)\sigma^2\gamma}{\gamma\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]} + \frac{\sigma\rho\left[\gamma(h_1+\kappa)-\sqrt{\left[(h_1+\kappa)\gamma-\rho\sigma(\mu-r)(1-\gamma)\right]^2-\sigma^2(\mu-r)^2(1-\gamma)\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}\right]}{\gamma\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]},$$

which is always positive as both parcels are positive.

Going back to (2.62), and considering that $\left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right] > 0$ and $\underline{\theta} < \theta$, it is immediate to conclude that $\pi_{\underline{\theta}} - \pi_\theta < 0$, and that ambiguity demand is always negative.

□

Proof of (ii): *$\pi_{\underline{\theta}}$ is decreasing with$(\theta - \underline{\theta})$.*

The relation between the mean allocation to the risky asset under ambiguity aversion and changes in the level of ambiguity, measured by $(\theta - \underline{\theta})$, is given by:

$$\frac{d\pi_{\underline{\theta}}}{d(\theta - \underline{\theta})} = -\left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma - 1)}{(1-\psi)}A_1\right].$$

As $\left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right] > 0$ (see previous proof of (i)), it results that an increased level of ambiguity implies a reduction of the mean allocation to the risky asset.

□

Proof of (iii): *Sign of intertemporal hedging demand* (HD).

From (2.38) and considering that $\sigma, \rho, \theta > 0$ and $\gamma > 1$, it results that the sign of intertemporal hedging demand (HD) is:

$$Sign(HD) \quad = \quad Sign\left(\frac{A_1}{1-\psi}\right).$$

From (2.60) and conclusions on Appendix 2.6.7 it is known that $\frac{A_1}{1-\psi} < 0$.

□

Proof of (iv): *$c_{\underline{\theta}}$ is higher (lower) under θ-ambiguity aversion if $\psi > 1$ (< 1).*

Under the "expectation-driven" scenario, from (2.35) and (2.29)-(2.30):

$$\begin{aligned} \frac{dc_{\underline{\theta}}}{d\underline{\theta}} &= -A_1\left(1 + \frac{\kappa}{h_1}\right)\beta^{\psi} exp\{-(A_1\underline{\theta} + B_1(\underline{\theta}))\} \\ \Rightarrow \quad Sign\left(\frac{dc_{\underline{\theta}}}{d\underline{\theta}}\right) &= Sign(-A_1), \end{aligned} \tag{2.63}$$

as $\left(1 + \frac{\kappa}{h_1}\right)\beta^{\psi} exp\{-(A_1\underline{\theta} + B_1(\underline{\theta}))\} > 0$. From (2.61), for $\gamma > 1$:

$$\begin{aligned} Sign(-A_1) &= Sign\left[(1-\psi)\left(\frac{b + \sqrt{b^2 + u}}{(\gamma - 1)e}\right)\right] \\ \Leftrightarrow Sign(-A_1) &= Sign(1-\psi), \end{aligned}$$

as $\frac{b+\sqrt{b^2+u}}{(\gamma-1)e} > 0$ (see Appendix 2.6.7), and therefore:

$$\psi < 1 \quad : \quad \frac{dc_\theta}{d\theta} > 0\,,$$

$$\psi > 1 \quad : \quad \frac{dc_\theta}{d\theta} < 0\,.$$

Consequently, when $\psi < 1$, θ-ambiguity aversion decreases the optimal consumption wealth ratio ($\underline{\theta} < \theta$). The contrary happens when $\psi > 1$. The same conclusion is obtained when studying $\frac{dc_t}{dy_t}$, with c_t given by (2.31), as $Sign\left(\frac{dc_t}{dy_t}\right) = Sign\left(-A_1\right)$.

□

2.6.9 Expression (2.41)

From (2.5):

$$E\left[v_t\right]2\theta^2\kappa - 2\theta\kappa - \sigma^2 \approx 0\,,$$

which is a quadratic equation for θ with roots:

$$\theta \approx \frac{2\kappa \pm \sqrt{4\kappa^2 + 8\kappa\sigma^2 E\left[v_t\right]}}{4\kappa E\left[v_t\right]}\,.$$

As $\theta > 0$, $2\kappa < \sqrt{4\kappa^2 + 8\kappa\sigma^2 E\left[v_t\right]}$ and $4\kappa E\left[v_t\right] > 0$, only the positive sign of the square root is relevant.

□

Chapter 3

Robust Portfolio Choice with Ambiguity about Stochastic Volatility[1]

[1]An early version of this chapter has been circulating as Faria and Correia-da Silva (2010c). We are grateful to Alejandro Balbás, Cláudia Ribeiro and Frank Riedel for useful comments and suggestions.

3.1 Introduction

We study optimal dynamic portfolio choice, in the presence of stochastic variance, of an investor that is averse both to risk and ambiguity. In our setting, the variance of the risky asset's return is simultaneously the source of the risk and the ambiguity that are perceived by the investor.

In a previous work (Faria et al. (2009)), a similar problem has been studied, but with two major differences regarding the ambiguity about the stochastic variance process. First, here it is assumed that the representative investor is ambiguous about the stochastic model describing the variance dynamics, while in Faria et al. (2009) the source of ambiguity was an unobservable parameter (expected value) of the variance stochastic process. Secondly, here we follow a dynamic robust control approach, while in Faria et al. (2009) we employed a static Multiple Priors approach.

There is a large literature on portfolio choice (see Campbell and Viceira (2002) for a survey), but few works study optimal dynamic portfolio choice with stochastic variance of the risky asset's return. Some examples are Kim and Omberg (1996), Chacko and Viceira (2005) and Liu (2007) under incomplete markets, and Schroder and Skiadas (1999) under complete markets. Schroder and Skiadas (2003) contains a general closed-form solution for the consumption-portfolio problem, which includes the other models as special cases. In those papers, potential adverse changes in the investment opportunity set are associated with the stochastic variance of the risky asset's return, which therefore represents a source of risk to investors. This implies, from Merton (1973a), that stochastic variance is a key factor driving the investor's optimal intertemporal hedging demand. However, in those papers there is only risk, and no ambiguity.

The distinction between risk and ambiguity was firstly pointed out by Knight (1921) and latter supported by the empirical experiments of Ellsberg (1961) and others (see, for e.g, Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey). The reason for this distinction is that economic agents may not be able to completely describe the uncertainty that they face by using a single probability distribution. Risk refers to uncertainty that can be represented by a probability distribution, while ambiguity refers to uncertainty that cannot. This distinction has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. That is why a rapidly growing literature on asset pricing and portfolio choice under ambiguity is emerging. This literature was comprehensively surveyed by Epstein and Schneider (2010).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years ignored ambiguity, being based on the expected utility theory of von

Neumann and Morgenstern (1944), where the probabilities of the possible states of nature are known, and on the subjective expected utility theory of Savage (1954), where although probabilities are not necessarily known, the choice behavior of an agent coincides with the maximization of expected utility according to some subjective probability beliefs.

However, gradually, ambiguity is being incorporated in decision theory (an extensive review on decision theory under ambiguity has been carried out by Etner et al. (2009)). Two main approaches are being used: the robust control (RC) approach, associated to an assumption of model uncertainty, and the multiple priors (MP) approach, where the single probability measure of the standard expected and subjective utility models is replaced by a set of probabilities or priors. The relationship between these two types of models has been discussed in the literature, namely in Hansen and Sargent (2001a), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006a).

Ahn et al. (2009) have found empirical support for the relevance of studying the portfolio choice problem under ambiguity (about 2/3 of agents in their experience showed a positive degree of ambiguity aversion). Bossaerts et al. (2010) also conclude that ambiguity aversion can be observed in competitive markets and that it matters for portfolio choices and for asset prices.

In studies of portfolio choice with ambiguity, Garlappi et al. (2007) and Gollier (2006) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.[2] The same conclusion was reached in a dynamic MP setting (e.g. Chen et al. (2009)) and in a dynamic RC model (e.g. Maenhout (2004, 2006)). The implications of ambiguity aversion for portfolio diversification have also been studied (Uppal and Wang (2003)). In all these works, the source of ambiguity is the expected risky asset's return or the risky asset's return process.

In this chapter, we introduce ambiguity about the risky asset's return variance process. More concretely, it is assumed that the representative investor is ambiguous about the stochastic model describing the variance dynamics.

It has been advocated in the literature (Cao et al. (2005), Garlappi et al. (2007) and Ui (2009)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and

[2] Although the result in Gollier (2006) requires some restrictions on the set of priors and on the investor's attitude towards risk.

higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity about the variance process of the risky asset's return because: (1) there is no "a priori" reason to assume that investors are not ambiguous about it; (2) we are able to find analytical solutions; and (3) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world, as for e.g., option implied variance frequently differs both in level and dynamics from the statistical measure (Drechsler and Yaron (2008) and Todorov (2010)). It is only considered ambiguity about the variance (process) of the risky asset's return in order to keep a parsimonious model and to isolate its effects on the optimal dynamic portfolio rule.

We follow closely the setting of Maenhout (2006) for robust portfolio rules. The portfolio-choice problem of an ambiguity averse investor with stochastic investment opportunities is solved, with a closed-form solution being obtained.

We find out that ambiguity about the stochastic process of variance impacts the optimal demand for the risky asset in two ways. Firstly, it enhances the "risk-aversion" adjustment in the total asset demand, both in the myopic and the hedging demand components. Secondly, it increases the hedging desire of the representative investor (even a logarithmic utility investor hedges intertemporaly). A new hedging demand component emerges exclusively due to ambiguity consideration, but even the standard hedging demand is affected by ambiguity. Those effects of ambiguity into optimal dynamic portfolio allocation can be interpreted as an increase in the effective risk aversion of the representative investor, as in Maenhout (2006).

It is also found that if there is a negative correlation between shocks in the return of the risky asset and in its variance, which is empirically plausible (at least for equity markets), then the investor with ambiguity aversion always invests less in the risky asset than the "myopic" investor. The hedging demand components are therefore always negative. Moreover, in this scenario, there is an horizon effect in the optimal trading strategy, exclusively driven by the hedging demand: the robust investor invests more in the risky asset if the horizon is shorter.

In our view, the major contribution of this chapter is to introduce, for the first time, ambiguity aversion about the stochastic variance process of the risky asset return through a dynamic approach, in the optimal portfolio choice problem. Moreover, it extends previous results obtained in the literature about the effect of ambiguity aversion on portfolio choice, particularly those in Maenhout (2006), for a different and relevant source of ambiguity in financial markets: the stochastic process of variance.

The chapter is organized as follows. In section 3.2, the intuition and mechanism of the

robustness approach towards model uncertainty followed in this chapter is explained. It is a technical descriptive section, in line with that in Anderson et al. (2003) and Maenhout (2006). In section 3.3, the preferences of the representative investor and the investment opportunity set are described, and consistently, the dynamic portfolio optimization problem to be solved by the investor is formulated. The solution of the optimization problem is presented in section 3.4. Some concluding remarks are made in section 3.5.

3.2 Model Uncertainty and the Robustness Approach

It is assumed that the representative investor is ambiguous about the model describing his investment opportunity set. The formal way of treating this ambiguity is through robust control techniques, firstly introduced in economics by Hansen and Sargent (1995). The representative investor considers a base model as his reference belief but, being ambiguous about it, considers a family of alternative models around it. Those alternative models are statistically difficult to distinguish from the benchmark model. This statistical distance between alternative and reference models is formally measured by the derivative of relative entropy. Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative investor chooses from the set of alternative models, the one that corresponds to the worst case scenario, i.e., the one associated with the lowest expected utility.

Under the robust control approach, two main settings have been used in the ambiguity related literature: the "constraint preferences" and the "multiplier preferences".[3] Under the "constraint preferences" (e.g. in Hansen et al. (2006)), there is a constraint on the magnitude of the allowable perturbations from the benchmark model. Conversely, under the "multiplier preferences" type of model (e.g. in Maenhout (2006)), preferences for robustness are constructed by penalizing deviations from the benchmark model. Although both settings are related, through the Lagrange Multiplier Theorem (Hansen and Sargent (2006)), they end up being structurally very different, as highlighted by Trojani and Vanini (2004).

In this chapter, the "multiplier preferences" type of model is adopted. Following the framework developed by Anderson et al. (2003), the introduction of robustness is formally translated in obtaining a modified Bellman equation. This Bellman equation is derived by considering a changed differential generator, reflecting the existence of model uncertainty, and an entropy penalty, reflecting the representative investor's preference for robustness.

[3] We adopt this terminology from Hansen and Sargent (2006).

As referred by Hansen and Sargent (2006), the "multiplier preferences" approach is dynamically consistent (see Maccheroni et al. (2006a)) and has been axiomatically founded by Wang (2003) and Maccheroni et al. (2006b).

However, still regarding this dynamic consistency issue under robust control, Trojani and Vanini (2004) highlight a key difference versus the recursive multiple priors setting (Chen and Epstein (2002), Epstein and Schneider (2003)), the most common dynamic approach in the ambiguity literature alongside the robust control: while the recursive multiple priors setting lies in the "dynamic consistency axiom", which implies the rectangularity property of the set of priors and therefore the dynamic consistency of ambiguous preferences,[4] robust control settings are focused in formulating optimal dynamic robust rules where rectangularity is not a crucial property "per se" for the formulation of the set of priors of the model (see Trojani and Vanini (2004) for more details about this).

As in Maenhout (2006), it is assumed that the representative investor derives utility exclusively through terminal wealth, which is analytically useful for the study of dynamic portfolio choice under stochastic investment opportunities. In this case, the standard Bellman equation of the problem is simply given by:

$$0 \;=\; \sup_{\alpha} \mathscr{D}^{(\alpha)} V(x)\,, \tag{3.1}$$

where α represents the fraction of wealth allocated to the portfolio (the control variable), $\mathscr{D}$ is the differential generator (Dynkin generator), $V(x)$ is the value function and x is the vector of state variables.

Formally, the differential generator $\mathscr{D}f$ (for a generic function f) is the infinitesimal generator of the family of conditional expectation operators $\{T_s : s \in \mathbb{R}_+\}$:

$$\mathscr{D}f \equiv \lim_{s\to 0} \frac{T_s(f) - f}{s}\,, \; with\; T_s(f)(y) = E\left[f(x_s) \mid x_0 = y\right].$$

As explained in Anderson et al. (2003), the expectation operator T_s and corresponding differential generator $\mathscr{D}$ in (3.1) reflect a specific underlying model for the vector of state variables x. As stated above, the investor with preference for robustness takes this model as the "reference model", but being ambiguous about its specification, considers alternative models. These alternative models, each indexed by a strictly positive function w, are represented by T_s^w, defined as:

$$T_s^w(f) \;\equiv\; \frac{T_s(wf)}{T_s(w)}. \tag{3.2}$$

[4]Rectangularity is the property that allows updating every prior under the recursive multiple priors utility through a Bayes rule. See Epstein and Schneider (2003) for details about this property.

Associated with this distorted expectation operator $T_s^w(f)$, emerges a changed differential generator to be used in the Bellman equation (3.1):

$$\mathscr{D}^w f \equiv \lim_{s \to 0} \frac{T_s^w(f) - f}{s}. \tag{3.3}$$

This differential generator (3.3) represents the uncertainty felt by the representative investor regarding the model specification. The next step is to formally introduce his preference for robustness, or equivalently, his aversion towards ambiguity.

Broadly speaking, in the spirit of Gilboa and Schmeidler (1989), this is achieved with the investor choosing the worst alternative model, w^*, i.e., the one that generates the lower expected utility. However, in order to guarantee that model uncertainty is meaningful, the set of models to be considered by the investor has to be statistically close to the reference model so that the investor has difficulty in distinguishing them. The formal way to measure the "distance" between alternative models is the derivative of relative entropy, denoted by $I'(w)$ and given by:

$$I'(w) \equiv \lim_{s \to 0} \frac{I_s(w)}{s}, \tag{3.4}$$

where $I_s(w)$ represents the relative entropy. Intuitively, $I'(w)$ represents a penalty (entropy penalty) introduced in the intertemporal optimization problem incurred when investor selects an adverse alternative model and departs from the reference model.

The relative entropy $I_s(w)$ is the expectation of the log Radon-Nikodym derivative of the alternative model's measure with respect to the reference model's measure. That expectation is build up under the alternative model w, being given by:

$$I_s(w) \equiv E^w \left[log \frac{w}{T_s(w)} \right].$$

Technically, for the Radon-Nikodym derivative to exist, and therefore to be possible the use of relative entropy criteria, it is necessary that alternative models are absolutely continuous with respect to the reference model. Note that the lower is the entropy of an alternative model, the harder it is for the investor to statistically differentiate it from the reference model.

With the differential generator (3.3) and the entropy penalty (3.4), the Bellman equation

for the investor with preference for robustness is obtained and given by:

$$0 = \sup_{\alpha} \inf_{w} \left\{ \mathscr{D}^{w,(\alpha)} V(x) + \frac{1}{\psi} I'(w) \right\}, \psi \geqslant 0. \tag{3.5}$$

Note that, as in Anderson et al. (2003) and Maenhout (2006), the entropy penalty $I'(w)$ in (3.5) is weighted by a non-negative parameter ψ, which measures the preference for robustness ($\psi = 0$ corresponds to the expected-utility scenario). The higher is ψ, the higher the preference for robustness of the investor or, by other words, the less is his confidence in the reference model, which motivates the consideration of alternative models with higher relative entropy. This parameter is assumed to be constant and state-independent in Anderson et al. (2003). However, in Maenhout (2004, 2006), ψ is state-dependent, denoted by $\psi(x) > 0$, in order to guarantee an "homothetic robustness".[5] In this chapter, the latter approach is followed.

As will be presented in section 3.3.1, the state vector x in our setting follows a diffusion. Anderson et al. (2003) show that, in this case, expressions for both the differential generator $\mathscr{D}^w V$ and for the entropy penalty $I'(w)$ simplify significantly. Concretely, assuming that vector x follows a diffusion driven by a standard Brownian Motion B_t with the diffusion coefficient matrix $\Lambda(x_t)$:

$$dx_t = \mu(x_t)\,dt + \Lambda(x_t)\,dB_t, \tag{3.6}$$

defining the variance-covariance matrix of x_t as $\Sigma(x_t) \equiv \Lambda(x_t)\Lambda(x_t)^\top$ and considering the logarithmic derivative of w, $u(x) \equiv \frac{\partial \ln w(x)}{\partial x}$, expressions for $\mathscr{D}^w V$ and $I'(w)$ can be rewritten as:

$$\mathscr{D}^w V = \mathscr{D} V + u(x)^\top \Sigma(x) \left(\frac{\partial V}{\partial x} \right), \tag{3.7}$$

$$I'(w) = \frac{1}{2} u(x)^\top \Sigma(x)\, u(x). \tag{3.8}$$

The assumption of an Itô process for the state variables, x_t, as in (3.6) combined with the use of relative entropy, implies that investor's ambiguity about the model specification ends up being exclusively related with the drift functions of the state variables processes.

[5] In Maenhout (2004), pp.960, are pointed out some reasons to support the homotheticity assumption: "*Although economics exhibit growth, rates of return are stationary. Second, when the scale of the state variables matters, natural unit invariance of optimal decisions disappears and calibrations have to take this into account. Finally, homotheticity facilitates aggregation and the construction of a representative agent*". Preserving this homotheticity property guarantees that (Maenhout (2004), pp.961) "*...robustness will no longer wear off as wealth rises*".

This is suggested by (3.7) and is technically founded in the Cameron-Martin-Girsanov Theorem which states that, through a change of probability measure, a given Itô process can always be transformed into another one with a different drift function. The requirement is that probability measures are absolutely continuous, which guarantees that they are equivalent (i.e. define the same possible events). In the present case, the absolute continuity property is imposed by the use of relative entropy, as previously stated.

Overall, on the back of (3.7), alternative models for x_t considered by the investor with preference for robustness include an added drift component versus the reference model (3.6):

$$dx_t = \mu(x_t)\,dt + \Lambda(x_t)\left[\Lambda(x_t)^\top u(x_t)\,dt + dB_t\right]. \tag{3.9}$$

3.3 Dynamic Portfolio Problem

3.3.1 Investment Opportunity Set

The investment opportunity set in this chapter is the same as in Faria et al. (2009). It is assumed that wealth can be allocated between a riskless asset with price R_t and a risky asset with price S_t. The instantaneous return of the riskless asset is described by:

$$\frac{dR_t}{R_t} = rdt\,, \tag{3.10}$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{X_t}}dB_{1t}\,, \tag{3.11}$$

where μ is the expected return of the risky asset, B_{1t} is a standard Brownian motion and X_t is the instantaneous precision of the risky asset's return (the instantaneous variance is $v_t = \frac{1}{X_t}$). From (3.11), the expected excess return of the risky asset versus the riskless asset, $\mu - r$, is constant over time.

The precision, X_t, is assumed to follow a mean-reverting, square-root process as in Cox et al. (1985b):

$$dX_t = \lambda\left(\overline{X} - X_t\right)dt + \sigma_X\sqrt{X_t}dB_{Xt}\,, \tag{3.12}$$

where $\sigma_X > 0$, the expected value of the precision is $E[X_t] = \overline{X}$, the reversion parameter is $\lambda > 0$ and $Var(X_t) = \frac{\overline{X}\sigma_X^2}{2\lambda}$. To guarantee standard integrability conditions, it is assumed that $2\lambda\overline{X} > \sigma_X^2$ as in Cox et al. (1985b). B_{Xt} is also a standard Brownian motion.

It is assumed that shocks to precision are correlated with shocks to the return of the risky asset, with $dB_{Xt}dB_{1t} = \rho dt$ and $\rho \in [-1, 1]$. Given this instantaneous correlation between B_{Xt} and B_{1t}, and considering a Brownian motion B_{2t} independent of B_{1t}, dB_{Xt} can be expressed as:

$$dB_{Xt} = \rho dB_{1t} + \sqrt{1-\rho^2} dB_{2t}\,. \tag{3.13}$$

The reason for considering the precision of the risky asset return, instead of its variance, is exclusively due to analytical reasons (as, for e.g., in Chacko and Viceira (2005)). However, considering that one is the reciprocal of the other, it is straightforward to obtain the dynamics for the instantaneous variance and its properties. Applying Itô's Lemma to (3.12), a mean-reverting, square-root process for proportional changes in variance is obtained (Faria et al. (2009)):

$$\frac{dv_t}{v_t} = \lambda_v(\overline{v} - v_t)\,dt - \sigma_X\sqrt{v_t}dB_{Xt}\,, \tag{3.14}$$

where $\overline{v} = \left(\overline{X} - \frac{\sigma_X^2}{\lambda}\right)^{-1}$ and $\lambda_v = \lambda\left(\overline{X} - \frac{\sigma_X^2}{\lambda}\right) = \frac{\lambda}{\overline{v}}$.

Taking expectations of the second-order Taylor expansion of v_t around $\overline{X}$, the approximate unconditional mean of instantaneous variance is (Faria et al. (2009)):

$$E[v_t] \approx \frac{1}{\overline{X}} + \frac{1}{2}\frac{\sigma_X^2}{\overline{X}^2\lambda} = \frac{1}{\overline{X}} + \frac{Var(X_t)}{\overline{X}^3}\,. \tag{3.15}$$

As the expected return of the risky asset, μ, is assumed to be constant, (3.15) is also the expected unconditional variance of the risky asset's return.

From (3.14), this implies that the instantaneous correlation between proportional changes in variance and the risky asset's return is given by:

$$Corr_t\left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t}\right) = -Corr_t\left(dX_t, \frac{dS_t}{S_t}\right) = -\rho dt\,. \tag{3.16}$$

This investment opportunity setting incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect" property in case of $\rho > 0$ (given by the negative correlation between returns and its variance), and the clustering property (proportional changes in variance are higher when variance is high).

As wealth is invested between the riskless asset and the risky asset, and the trading strat-

egy α_t represents the fraction of wealth invested in the risky asset, the instantaneous wealth, W_t, dynamics is given by (Appendix 3.6.1):

$$dW_t = [\alpha_t(\mu - r) + r] W_t dt + \alpha_t W_t \sqrt{\frac{1}{X_t}} dB_{1t}. \tag{3.17}$$

3.3.2 Investor Preferences

The investor faces ambiguity about the stochastic process (3.12) for the precision of the risky asset return. As explained in section 3.2, the investor with preference for robustness considers the process (3.12) as his reference belief but, being ambiguous about it, considers a set of statistically close alternative models around it. Being ambiguity averse, investor will chose the alternative model that generates the lower expected utility.

From section 3.3.1, as the precision of the risky asset return follows a diffusion process, then alternative models for X_t considered by the investor include an added drift component versus the reference model (3.12). This will be analytically deducted in the following section.

Additionally, it is assumed that investor starts with an endowment of wealth, W_0, and maximizes his utility from terminal wealth, W_T, by defining a self-financing optimal dynamic trading strategy during $t \in [0, T]$. The control variable is therefore α_t, the instantaneous fraction of wealth allocated to the risky asset.

The representative investor's utility is given by:

$$U(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}, \tag{3.18}$$

where $\gamma > 1$ represents the coefficient of relative risk aversion. The limit of $U(W_T)$ for $\gamma = 1$ is given by $ln(W_T)$. As in Maenhout (2006), it is assumed that $\gamma \geqslant 1$ for three major reasons: (i) it guarantees that solutions for the optimization problem are well-behaved[6]; (ii) as in the spirit of Gilboa and Schmeidler (1989), the investor adopts a conservative approach by choosing the alternative model that delivers the lowest expected utility, it seems more intuitive to consider relatively high risk averse investors and (iii) most empirical studies estimate that $\gamma > 1$, even if there are exceptions (see Bliss and Panigirtzoglou (2004), table 7).

By defining utility over terminal wealth, the intertemporal consumption-savings decision

[6]In Kim and Omberg (1996) the expected utility solutions are studied in detail for $\gamma < 1$. Kim and Omberg (1996) characterize well-behaved and "nirvana" solutions (where investor's expected utility approaches infinity).

(considered in Faria et al. (2009)) is ignored. Despite the economic relevance of that decision, this enables the construction of a much simpler setting where, nonetheless, robust control continues to make sense, as highlighted by Maenhout (2006).

3.3.3 Bellman Equation

The dynamic optimization problem of the representative investor is to maximize his expected terminal utility (3.18), under the worst alternative model for the dynamics of state variables:

$$\begin{aligned} & \underset{\alpha}{sup}\left\{\underset{u}{inf}\, E_t^u\left[U\left(W_T\right)\right]\right\} \\ & s.t. \\ dW_t &= \left[\alpha_t\left(\mu-r\right)+r\right]W_t dt+\alpha_t W_t\sqrt{\frac{1}{X_t}}dB_{1t}\,, \\ dX_t &= \lambda\left(\overline{X}-X_t\right)dt+\sigma_X\sqrt{X_t}dB_{Xt}^u\,, \end{aligned}$$

where E_t^u and B_{Xt}^u denote the expectation operator and the Brownian motion in the stochastic process of precision under the alternative measure u.

This Maxmin problem has two state variables, the instantaneous wealth, W_t, and the instantaneous precision, X_t, and two control variables, the instantaneous fraction of wealth invested in the risky asset, α_t, and the alternative model for the precision dynamics, u. The value function will therefore be a function of W_t, X_t and t, being denoted as $V\left(W,X,t\right)$.

In order to obtain the Bellman equation corresponding to this optimization problem, we follow closely the notation of section 3.2. As the state variables, X_t and W_t, follow diffusion processes, expressions (3.7) and (3.8) from Anderson et al. (2003) for the differential generator and for the entropy penalty, respectively, apply.

We start with the differential generator. In our setting, and making use of (3.13), the diffusion coefficient matrix Λ is given by:

$$\Lambda \equiv \begin{bmatrix} \alpha W\sqrt{\frac{1}{X}} & 0 \\ \\ \sigma_X\sqrt{X}\rho & \sigma_X\sqrt{X}\sqrt{1-\rho^2} \end{bmatrix},$$

which, considering that the variance-covariance matrix is given by $\Sigma\equiv\Lambda\Lambda^\top$, implies that:

$$\Sigma \equiv \begin{bmatrix} \frac{\alpha^2W^2}{X} & \alpha W\sigma_X\rho \\ \\ \sigma_X\rho\,\alpha W & \sigma_X^2X \end{bmatrix}.$$

Next, denote $\partial V \equiv \begin{bmatrix} V_W & V_X \end{bmatrix}^\top$ as the column vector of partial derivatives of the value function V with respect to the state variables. The Hessian matrix is given by:

$$\partial^2 V \equiv \begin{bmatrix} V_{WW} & V_{WX} \\ V_{XW} & V_{XX} \end{bmatrix},$$

and, therefore:

$$\Sigma\partial^2 V \equiv \begin{bmatrix} \left(\frac{\alpha^2 W^2}{X} V_{WW} + \alpha W \sigma_X \rho V_{XW}\right) & \left(\frac{\alpha^2 W^2}{X} V_{WX} + \alpha W \sigma_X \rho V_{XX}\right) \\ \left(\sigma_X \rho \alpha W V_{WW} + \sigma_X^2 X V_{XW}\right) & \left(\sigma_X \rho \alpha W V_{WX} + \sigma_X^2 X V_{XX}\right) \end{bmatrix},$$

with $trace\left(\Sigma\partial^2 V\right) = \frac{\alpha^2 W^2}{X} V_{WW} + 2\sigma_X \rho \alpha W V_{WX} + \sigma_X^2 X V_{XX}$.

As is well known (for e.g., Kamien and Schwartz (1991), ch. 22) the standard differential generator $\mathscr{D}^\alpha V$ is given by:

$$\mathscr{D}^\alpha V = V_W \left[\alpha\left(\mu - r\right) + r\right] W + V_X \left[\lambda\left(\overline{X} - X\right)\right] + V_t + \frac{1}{2} trace\left(\Sigma\partial^2 V\right),$$

which, substituting the expression for $trace\left(\Sigma\partial^2 V\right)$ becomes,

$$\begin{aligned} \mathscr{D}^\alpha V &= V_W \left[\alpha\left(\mu - r\right) + r\right] W + V_X \left[\lambda\left(\overline{X} - X\right)\right] + V_t + \frac{1}{2}\frac{\alpha^2 W^2}{X} V_{WW} \\ &\quad + \sigma_X \rho \alpha W V_{WX} + \frac{1}{2}\sigma_X^2 X V_{XX}. \end{aligned}$$

To obtain the differential generator given in (3.7), we only have to calculate the second parcel $u^\top \Sigma\partial V$:

$$u^\top \Sigma\partial V \equiv \begin{bmatrix} u_W & u_X \end{bmatrix} \begin{bmatrix} \frac{\alpha^2 W^2}{X} & \alpha W \sigma_X \rho \\ \sigma_X \rho \alpha W & \sigma_X^2 X \end{bmatrix} \begin{bmatrix} V_W \\ V_X \end{bmatrix},$$

$$\Leftrightarrow u^\top \Sigma\partial V \equiv u_W \frac{\alpha^2 W^2}{X} V_W + u_X \alpha W \sigma_X \rho V_W + u_W \sigma_X \rho \alpha W V_X + u_X \sigma_X^2 X V_X.$$

From (3.7), the differential generator $\left[\mathscr{D}^\alpha V + u^\top \Sigma\partial V\right]$ in the present setting is therefore

given by:

$$V_W\left[\alpha\left(\mu-r\right)+r\right]W+V_X\left[\lambda\left(\overline{X}-X\right)\right]+V_t+\frac{1}{2}\frac{\alpha^2W^2}{X}V_{WW}+\sigma_X\rho\alpha WV_{WX}$$
$$+\frac{1}{2}\sigma_X^2XV_{XX}+u_W\frac{\alpha^2W^2}{X}V_W+u_X\alpha W\sigma_X\rho V_W+u_W\sigma_X\rho\alpha WV_X+u_X\sigma_X^2XV_X\,.$$

The expression of the entropy penalty, $\left[\frac{1}{2}u^\top\Sigma u\right]$ in (3.8), in our setting is, therefore:

$$\begin{aligned}
&\frac{1}{2}\begin{bmatrix} u_W & u_X \end{bmatrix}\begin{bmatrix} \frac{\alpha^2W^2}{X} & \alpha W\sigma_X\rho \\ \sigma_X\rho\alpha W & \sigma_X^2X \end{bmatrix}\begin{bmatrix} u_W \\ u_X \end{bmatrix} \\
=\;&\frac{1}{2}\left[\left(u_W\right)^2\frac{\alpha^2W^2}{X}+2u_Wu_X\sigma_X\rho\alpha W+\left(u_X\right)^2\sigma_X^2X\right].
\end{aligned}$$

From (3.5), given the obtained expressions for the differential generator and the entropy penalty, the Bellman equation for the representative investor with preference for robustness is

$$\begin{aligned}
0 &= \sup_{\alpha}\inf_{u}\left\{\mathscr{D}^\alpha V+u^\top\Sigma\partial V+\frac{1}{2\psi}u^\top\Sigma u\right\} \\
\Leftrightarrow\; 0 &= \sup_{\alpha}\inf_{u}\left\{V_W\left[\alpha\left(\mu-r\right)+r\right]W+V_X\left[\lambda\left(\overline{X}-X\right)\right]+V_t+\frac{1}{2}\frac{\alpha^2W^2}{X}V_{WW}\right. \\
&\quad+\sigma_X\rho\alpha WV_{WX}+\frac{1}{2}\sigma_X^2XV_{XX}+u_W\frac{\alpha^2W^2}{X}V_W+u_X\alpha W\sigma_X\rho V_W+u_W\sigma_X\rho\alpha WV_X \\
&\quad\left.+u_X\sigma_X^2XV_X+\frac{1}{2\psi}\left[\left(u_W\right)^2\frac{\alpha^2W^2}{X}+2u_Wu_X\sigma_X\rho\alpha W+\left(u_X\right)^2\sigma_X^2X\right]\right\}, \qquad (3.19)
\end{aligned}$$

subject to the boundary condition:

$$V\left(W,X,T\right) = \frac{W_T^{1-\gamma}}{1-\gamma}. \qquad (3.20)$$

In Maenhout (2004), it is argued that ψ should be state-dependent so that the CRRA value function preserves homotheticity property. This is kept in Maenhout (2006), and there are other examples in the literature with this state-dependent specification of ψ (e.g. Uppal and Wang (2003) and Liu et al. (2005)). We also use a state-dependent ψ with the specification:

$$\psi\left(W,X,t\right) = \frac{\theta}{\left(1-\gamma\right)V\left(W,X,t\right)}, \qquad (3.21)$$

where the parameter $\theta \geqslant 0$ is a robustness preference parameter ($\theta = 0$ corresponds to standard expected utility maximization).

3.4 Optimal Dynamic Robust Portfolio Rule

As in Maenhout (2006), we start by solving for the optimal vector u (alternative model). Considering (3.19), the first order conditions with respect to u are (Appendix 3.6.2):

$$\begin{bmatrix} u_W^* \\ \\ u_X^* \end{bmatrix} = \begin{bmatrix} -\psi V_W \\ \\ -\psi V_X \end{bmatrix}. \tag{3.22}$$

Substituting this result in the Bellman equation (3.19), we obtain (Appendix 3.6.3):

$$\begin{aligned} 0 &= \sup_{\alpha} \left\{ V_W \left[\alpha(\mu - r) + r\right] W + V_X \left[\lambda \left(\overline{X} - X\right)\right] + V_t + \frac{1}{2}\frac{\alpha^2 W^2}{X} V_{WW} + \sigma_X \rho \alpha W V_{WX} \right. \\ &\quad \left. + \frac{1}{2}\sigma_X^2 X V_{XX} - \frac{\psi}{2}\left[\frac{\alpha^2 W^2}{X} V_W^2 + 2\alpha W \sigma_X \rho V_X V_W + V_X^2 \sigma_X^2 X\right]\right\}. \end{aligned} \tag{3.23}$$

Solving this equation, we obtain the optimal portfolio rule (Appendix 3.6.3):

$$\begin{aligned} \alpha^* &= \frac{-V_W}{\left(V_{WW} - \psi V_W^2\right) W}(\mu - r) X + \frac{-V_{WX}}{\left(V_{WW} - \psi V_W^2\right) W} \rho \sigma_X X \\ &\quad + \frac{V_W V_X \psi}{\left(V_{WW} - \psi V_W^2\right) W} \rho \sigma_X X. \end{aligned} \tag{3.24}$$

The first two components in (3.24) correspond to the standard ones in the intertemporal optimal portfolio decision (from Merton (1973a)): the first term is the myopic demand and the second term is the standard hedging demand that vanishes when: (i) the investment opportunities are constant and therefore there is no need for hedging ($\sigma_X = 0$); (ii) the investment opportunities change but it is not possible to use the risky asset to hedge against those changes ($\rho = 0$); (iii) there is no preference for hedging ($V_{WX} = 0$). Note, however, that in both terms the risk aversion adjustment factor differs from the standard ones, $\frac{-V_W}{V_{WW}W}$ and $\frac{-V_{WX}}{(V_{WW})W}$, respectively, due to the existence of preference for robustness (through ψ).

The last term in (3.24) is the major novelty versus the standard results (without ambiguity). This is a very similar term to the standard hedging demand, except that if the investor is not interested in hedging ($V_{WX} = 0$), this term does not disappear, as it depends on $V_W V_X$ and not on V_{WX}. This term results from the concern about the quadratic

variation in the partial derivatives of the value function, visible in (3.23), that is present only when investor has preference for robustness ($\psi > 0$). Preference for robustness generates, therefore, an additional component of the hedging demand.

The value function that solves the Bellman equation (3.23) with the specification of ψ given by (3.21) and subject to the boundary condition (3.20) is given by:

$$V(W,X,t) = \frac{W^{1-\gamma}}{1-\gamma} e^{[(1-\gamma)(A_t+B_tX)]}, \tag{3.25}$$

with A_t and B_t given by (Appendix 3.6.4):

$$A_t = r(T-t) - \frac{2\beta_0\lambda\overline{X}}{\beta_1+\eta}(T-t) + \frac{4\beta_0\lambda\overline{X}}{\eta^2-\beta_1^2} ln\left[\frac{(\beta_1+\eta)-(\beta_1-\eta)e^{(T-t)\eta}}{2\eta}\right], \tag{3.26}$$

$$B_t = \frac{2\beta_0\left(1-e^{-\eta(T-t)}\right)}{2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)}, \tag{3.27}$$

where:

$$\begin{aligned}
\beta_0 &= \frac{(\mu-r)^2}{2(\gamma+\theta)}, \\
\beta_1 &= \frac{(\mu-r)(1-\gamma-\theta)\rho\sigma_X}{\gamma+\theta} - \lambda, \\
\beta_2 &= \frac{\sigma_X^2}{2}(1-\gamma-\theta)\left[1+\frac{(1-\gamma-\theta)\rho^2}{\gamma+\theta}\right], \\
\eta &= \sqrt{\beta_1^2-4\beta_0\beta_2}.
\end{aligned} \tag{3.28}$$

Given this explicit analytical solution of the Bellman equation (3.23), the corresponding explicit expression for the optimal portfolio rule in (3.24) is given in Proposition 1.

Proposition 1. *From the value function (3.25), the optimal robust portfolio rule for an investor (with $\gamma \geqslant 1$) that is ambiguous about the risky asset's return precision stochastic process is:*

$$\alpha_t^* = \frac{1}{\gamma+\theta}[(\mu-r)+(1-\gamma-\theta)\sigma_X\rho B_t]X, \tag{3.29}$$

with B_t given by (3.27).

Proof. *Appendix 3.6.4 .*

Following previous comments to expression (3.24) and attending to the concrete portfolio rule (3.29), it is clear that robustness impacts in two ways the optimal demand for the risky asset. Firstly, it enhances the "risk-aversion" adjustment in the total asset demand, both in the myopic and the hedging demand components. Secondly, it increases the hedging desire of the representative investor.

Observing (3.29) with more detail, it is clear that in the myopic demand component, given by $\frac{(\mu-r)}{\gamma+\theta}X$ (which corresponds to the first term on the right-hand side in (3.24)), the risk aversion correction factor is reinforced by the θ parameter. Regarding the hedging demand component it has two components: the standard term, proportional to $1-\gamma$, given by $\frac{(1-\gamma)}{\gamma+\theta}\sigma_X \rho B_t X$ (corresponding to the second term on the right-hand side in (3.24)), and a new one, proportional to θ, given by $\frac{-\theta}{\gamma+\theta}\sigma_X \rho B_t X$ (corresponding to the third term on the right-hand side in (3.24)). The later is exclusively due to the preference for robustness, but note that even the standard hedging demand is affected by robustness, as θ appears explicitly and implicitly (through B_t).

As noted by Maenhout (2006), those effects of robustness on optimal dynamic portfolio allocation can be interpreted as an increase in the effective risk aversion of the representative investor (from γ to $\gamma+\theta$). Maenhout (2006) formalizes this idea in his Proposition 2, which can be replicated in our setting as θ appears additively with γ in (3.27) (due to specifications in (3.28)) and in (3.29):

Proposition 2. *Under the investment opportunity set described in section 3.3.1, an investor with homothetic preference for robustness* $\psi = \frac{\theta}{(1-\gamma)V(W,X,t)} > 0$ *and a CRRA utility function over terminal wealth* $U(W_T) = \frac{W_T^{1-\gamma}}{1-\gamma}$ *,with* $\gamma \geqslant 1$*, is observationally equivalent to an investor without preference for robustness and a CRRA utility function* $U(W_T) = \frac{W_T^{1-\gamma-\theta}}{1-\gamma-\theta}$.[7]

This proposition has, at least, two important implications. First, under the present "multiplier preferences" setting, the effect of robustness is exclusively quantitative, corresponding to an increase in the effective risk aversion, as qualitatively it does not change the standard conclusions in the literature regarding optimal dynamic portfolio choice. As

[7]The idea that the effects of robustness on optimal dynamic portfolio choice can be interpreted as an increase in the effective risk aversion of the representative investor has also been pointed out by Maenhout (2004). However, the setting in Maenhout (2004) is different of that of Maenhout (2006). The major difference is that Maenhout (2004) considers i.i.d. returns and the identified observational equivalence is between a robust CRRA investor (with risk aversion γ and preference for robustness θ) and a non-robust Duffie-Epstein investor (stochastic differential utility) with elasticity of intertemporal risk aversion γ^{-1} and risk aversion $\gamma+\theta$.

pointed out by Epstein and Schneider (2010), this contrasts with the Multiple Priors approach, which allows ambiguity to have "first-order" effects on portfolio choice and asset pricing, that is, it enables the expansion of the range of qualitative behavior that can be rationalized versus the standard expected utility theory.[8] However, it should be stressed that under the "constraint preferences" robust control approach it is also possible to generate some "first-order" effects on portfolio choice as shown, for e.g, by Trojani and Vanini (2004).

The second implication from Proposition 2, coupled with expression (3.29), is that when robustness is considered, the logarithmic utility investor ($\gamma = 1$) hedges intertemporaly (Appendix 3.6.5). The intuition for this is that as robustness increases the effective risk aversion, then the logarithmic investor with preference for robustness is more risk averse than the logarithmic investor without preference for robustness, and therefore hedges intertemporaly.

Finally, under the assumptions $\gamma \geqslant 1$ and $\rho > 0$, we obtain (Appendix 3.6.6):

$$\begin{aligned} B_t &> 0, \forall_{t<T}, \\ \frac{\partial B_t}{\partial t} &< 0, \forall_{t<T}. \end{aligned} \tag{3.30}$$

This implies that some qualitative effects from robustness found in Kim and Omberg (1996) and Maenhout (2006) are also obtained in the present setting. Concretely, results in (3.30) imply that if $\rho > 0$, i.e., if the correlation between shocks to the precision of the risky asset return and shocks to the return is positive (equivalently, if there is a negative correlation between shocks in the return of the risky asset and in its variance, which is empirically plausible, at least for equity markets), then, from (3.29), the robust investor with CRRA preferences ($\gamma \geqslant 1$) always invests less in the risky asset than the myopic investor. The hedging demand component is therefore always negative when $\rho > 0$. This is consistent with findings in some portfolio literature with stochastic variance in intertemporal settings without robustness considerations, like for e.g. in Chacko and Viceira (2005).

Moreover, considering (3.29), it is immediate to conclude that there is an horizon effect in the optimal trading strategy, exclusively driven by the hedging demand. If $\rho > 0$, as

[8]For example, as briefly exposed in the introduction of this thesis, on the contrary of what happens in standard expected utility theory, the multiple-priors approach can generate selective participation, optimal underdiversification and portfolio inertia at portfolios that hedge ambiguity. Also as pointed out by Epstein and Schneider (2010), under multiple-priors settings large non-zero average excess returns on an asset can emerge on the back of investor's lack of information regarding that asset, even if its payoffs do not covary much with marginal utility, contrasting with conclusions in the standard "risk-only" literature (e.g. Cox et al. (1985a), pp 376).

$\frac{\partial B_t}{\partial t} < 0$, then the robust investor will invest more in the risky asset when the horizon is shorter. The reasoning for this is that the shorter is the investment horizon, the smaller is the required hedging demand, implying that the closer is the risky asset demand to the myopic demand.

At last, expression (3.29) for the optimal portfolio rule has a relevant difference from the one obtained in Faria et al. (2009), where the source of ambiguity is also about the stochastic precision process, but emerging from one unobservable parameter of the precision process (its expected value $\overline{X}$). In Faria et al. (2009) it is concluded that, if ambiguity averse investor is able to update his portfolio continuously, as a function of the instantaneous precision, then that ambiguity has no effect. The impact only exists, and is found to be empirically relevant and in the same direction of the present setting of decreasing the optimal demand of the risky asset, if investor makes use of his expectation of future precision. Conversely, in the present setting, it is immediate to conclude from (3.29) that ambiguity aversion, or equivalently the preference for robustness, impacts the optimal instantaneous allocation to the risky asset.

3.5 Concluding Remarks

We have analyzed the optimal intertemporal portfolio problem of an investor that is ambiguous about the stochastic model describing the variance dynamics. In our setting, variance of the risky asset return is therefore simultaneously the source of risk and ambiguity perceived by the representative investor.

It is assumed that the investor has two assets to invest in, a risk-free asset and a risky asset. The investor starts with an endowment of wealth and maximizes his utility from terminal wealth by defining a self-financing optimal dynamic trading strategy on the risky asset. The risky asset return's variance is stochastic and the investor is ambiguous about its process. Robust control techniques (multiplier preferences) are applied to solve the portfolio problem: the ambiguous-investor considers a set of statistically close alternative models around his "reference-belief". Being ambiguity averse, under the spirit of Gilboa and Schmeidler (1989), the investor will chose the alternative model that generates the lower expected utility.

A closed-form robust portfolio rule is obtained. Ambiguity about the variance stochastic process enhances the "risk-aversion" adjustment in the total risky asset demand, both in its myopic and hedging components. It also increases the hedging desire of the representative investor. A new hedging demand component emerges exclusively due to ambiguity considerations, but even the standard hedging demand is affected by ambiguity. Those

effects of ambiguity into optimal dynamic portfolio allocation can be interpreted as an increase in the effective risk aversion of the representative investor.

It is also found that when there is a negative correlation between shocks in the return of the risky asset and in its variance (leverage effect), then the ambiguous investor always invests less in the risky asset than the myopic investor. The hedging demand components are therefore always negative. Moreover, under this scenario, there is an horizon effect in the optimal trading strategy, exclusively driven by the hedging demand: the robust investor will invest more in the risky asset the shorter is the horizon.

Comparing those conclusions with those reached in Faria et al. (2009), here we have obtained a dynamic optimal portfolio rule that is instantaneously impacted by the ambiguity about the stochastic variance process, in contrast with that obtained in Faria et al. (2009). Both works coincide that this ambiguity decreases the optimal demand of the risky asset, under a negative correlation between shocks in the return of the risky asset and in its variance.

We left open the calibration of the obtained model for optimal portfolio decision under ambiguity about the stochastic process of the variance. We see this as an interesting topic for future research. The model calibration requires the ambiguity preference parameter θ to be estimated through a well-founded way and, for that, the detection-error probabilities methodology provided by Anderson et al. (2003) could be applied. More concretely, the method for analytical calculation of the detection-error probabilities developed in Maenhout (2006) could be used, as it delivers a semi-closed form (up to a system of ODEs and a Fourier inversion).

Appendix

3.6 Mathematical Proofs

3.6.1 Equation (3.17)

Considering that in time t the fraction α_t of wealth is invested in the risky asset and the remaining $(1-\alpha_t)$ in the riskless asset, from (3.10) and (3.11), the instantaneous wealth W_t dynamics is given by:

$$\begin{aligned}
dW_t &= r(1-\alpha_t)W_t dt + \alpha_t W_t \frac{dS_t}{S_t}, \\
\Leftrightarrow dW_t &= r(1-\alpha_t)W_t dt + \alpha_t W_t \mu dt + \alpha_t W_t \sqrt{\frac{1}{X_t}} dB_{1t}, \\
\Leftrightarrow dW_t &= [\alpha_t(\mu - r) + r] W_t dt + \alpha_t W_t \sqrt{\frac{1}{X_t}} dB_{1t}.
\end{aligned}$$

□

3.6.2 System (3.22)

From the Bellman equation (3.19), the first order condition regarding u_w is:

$$\begin{aligned}
0 &= \frac{\alpha^2 W^2}{X} V_W + \sigma_X \rho \alpha W V_X + \frac{1}{\psi} u_W \frac{\alpha^2 W^2}{X} + \frac{1}{\psi} u_X \sigma_X \rho \alpha W, \\
\Leftrightarrow \frac{1}{\psi} u_W \frac{\alpha^2 W^2}{X} &= -\frac{\alpha^2 W^2}{X} V_W - \sigma_X \rho \alpha W V_X - \frac{1}{\psi} u_X \sigma_X \rho \alpha W, \\
\Leftrightarrow u_W &= -\psi V_W - \psi \frac{\sigma_X \rho \alpha W X}{\alpha^2 W^2} V_X - \frac{u_X \sigma_X \rho \alpha W X}{\alpha^2 W^2}, \\
\Leftrightarrow u_W &= -\psi V_W - \psi \frac{\sigma_X \rho X}{\alpha W} V_X - \frac{u_X \sigma_X \rho X}{\alpha W}. \qquad (3.31)
\end{aligned}$$

Similarly, the first order condition regarding u_X is:

$$\begin{aligned}
0 &= \sigma_X \rho \alpha W V_W + \sigma_X^2 X V_X + \frac{u_W \sigma_X \rho \alpha W}{\psi} + \frac{u_X \sigma_X^2 X}{\psi}, \\
\Leftrightarrow \frac{u_X \sigma_X^2 X}{\psi} &= -\sigma_X \rho \alpha W V_W - \sigma_X^2 X V_X - \frac{u_W \sigma_X \rho \alpha W}{\psi}, \\
\Leftrightarrow u_X &= -\psi \frac{\rho \alpha W V_W}{\sigma_X X} - \psi V_X - \frac{u_W \rho \alpha W}{\sigma_X X}. \qquad (3.32)
\end{aligned}$$

Denoting u_W^* and u_X^* those first order solutions and substituting into (3.31) the result in

(3.32) one obtains:

$$
\begin{aligned}
u_W^* &= -\psi V_W - \psi \frac{\sigma_X \rho X}{\alpha W} V_X - \left[-\psi \frac{\rho \alpha W V_W}{\sigma_X X} - \psi V_X - \frac{u_W^* \rho \alpha W}{\sigma_X X} \right] \frac{\sigma_X \rho X}{\alpha W}, \\
\Leftrightarrow u_W^* &= -\psi V_W - \psi \frac{\sigma_X \rho X}{\alpha W} V_X + \psi \rho^2 V_W + \psi V_X \frac{\sigma_X \rho X}{\alpha W} + u_W^* \rho^2, \\
\Leftrightarrow \left(1-\rho^2\right) u_W^* &= -\psi \left(1-\rho^2\right) V_W, \\
\Leftrightarrow u_W^* &= -\psi V_W.
\end{aligned}
$$

At last, substituting this result into (3.32) one gets:

$$
\begin{aligned}
u_X^* &= -\psi \frac{\rho \alpha W V_W}{\sigma_X X} - \psi V_X + \psi \frac{V_W \rho \alpha W}{\sigma_X X}, \\
\Leftrightarrow u_X^* &= -\psi V_X.
\end{aligned}
$$

It is immediate to conclude that $\frac{\partial u_W^*}{\partial u_W} = \frac{\partial u_X^*}{\partial u_X} = 0$, and therefore second order conditions are satisfied, confirming that both u_W^* and u_X^* are the optimal solution for the minimization component of the Bellman equation (3.19).

□

3.6.3 Bellman Equation (3.23) and Equation (3.24)

3.6.3.1 Bellman Equation (3.23)

Considering the first order conditions with respect to u (Appendix 3.6.2):

$$
\begin{bmatrix} u_W^* \\ \\ u_X^* \end{bmatrix} = \begin{bmatrix} -\psi V_W \\ \\ -\psi V_X \end{bmatrix},
$$

and substituting this result into the Bellman equation (3.19) it becomes:

$$
\begin{aligned}
0 &= \sup_{\alpha} \Big\{ V_W \left[\alpha\left(\mu - r\right) + r\right] W + V_X \left[\lambda\left(\overline{X} - X\right)\right] + V_t + \frac{1}{2}\frac{\alpha^2 W^2}{X} V_{WW} \\
&\quad + \sigma_X \rho \alpha W V_{WX} + \frac{1}{2}\sigma_X^2 X V_{XX} - \psi \frac{\alpha^2 W^2}{X} V_W^2 - \psi \alpha W \sigma_X \rho V_X V_W \\
&\quad - \psi \sigma_X \rho \alpha W V_W V_X - \psi \sigma_X^2 X V_X^2 \\
&\quad + \frac{1}{2\psi}\left[\psi^2 V_W^2 \frac{\alpha^2 W^2}{X} + 2\psi^2 V_W V_X \sigma_X \rho \alpha W + \psi^2 V_X^2 \sigma_X^2 X\right] \Big\} , \\
\Leftrightarrow 0 &= \sup_{\alpha} \Big\{ V_W \left[\alpha\left(\mu - r\right) + r\right] W + V_X \left[\lambda\left(\overline{X} - X\right)\right] + V_t + \frac{1}{2}\frac{\alpha^2 W^2}{X} V_{WW} \\
&\quad + \sigma_X \rho \alpha W V_{WX} + \frac{1}{2}\sigma_X^2 X V_{XX} - \psi \frac{\alpha^2 W^2}{X} V_W^2 - 2\psi \alpha W \sigma_X \rho V_X V_W \\
&\quad - \psi \sigma_X^2 X V_X^2 + \frac{\psi}{2} V_W^2 \frac{\alpha^2 W^2}{X} + \psi V_W V_X \sigma_X \rho \alpha W + \frac{\psi}{2} V_X^2 \sigma_X^2 X \Big\} , \\
\Leftrightarrow 0 &= \sup_{\alpha} \Big\{ V_W \left[\alpha\left(\mu - r\right) + r\right] W + V_X \left[\lambda\left(\overline{X} - X\right)\right] + V_t + \frac{1}{2}\frac{\alpha^2 W^2}{X} V_{WW} \\
&\quad + \sigma_X \rho \alpha W V_{WX} + \frac{1}{2}\sigma_X^2 X V_{XX} \\
&\quad - \frac{\psi}{2}\left[\frac{\alpha^2 W^2}{X} V_W^2 + 2\alpha W \sigma_X \rho V_X V_W + V_X^2 \sigma_X^2 X\right] \Big\} ,
\end{aligned}
$$

which is (3.23).

□

3.6.3.2 Equation (3.24)

From the Bellman equation (3.23), the first order condition regarding α is:

$$
\begin{aligned}
0 &= V_W \left(\mu - r\right) W + \frac{\alpha^* W^2}{X} V_{WW} + \rho \sigma_X W V_{WX} - \psi \frac{\alpha^* W^2}{X} V_W^2 \\
&\quad - \psi W \sigma_X \rho V_W V_X , \\
\Leftrightarrow \alpha^* \left(V_{WW} - \psi V_W^2\right) \frac{W^2}{X} &= -V_W \left(\mu - r\right) W - \rho \sigma_X W V_{WX} + \psi W \sigma_X \rho V_W V_X , \\
\Leftrightarrow \alpha^* &= \frac{-V_W}{\left(V_{WW} - \psi V_W^2\right) W} \left(\mu - r\right) X + \frac{-V_{WX}}{\left(V_{WW} - \psi V_W^2\right) W} \rho \sigma_X X \\
&\quad + \frac{\psi V_W V_X}{\left(V_{WW} - \psi V_W^2\right) W} \rho \sigma_X X .
\end{aligned}
$$

From this result it is immediate to conclude that $\frac{\partial \alpha^*}{\partial \alpha} = 0$ and therefore second order conditions are also verified confirming that α^* is the optimal portfolio rule.

$\square$

3.6.4 Portfolio Rule (3.29), Expressions (3.26) and (3.27)

3.6.4.1 Portfolio Rule (3.29)

From (3.25), the first and second order partial derivatives of the value function $V(W,X,t)$ are given by:

$$\begin{aligned}
V_W &= W^{-\gamma} e^{[(1-\gamma)(A_t+B_tX)]}, \\
V_{WW} &= -\gamma W^{(-1-\gamma)} e^{[(1-\gamma)(A_t+B_tX)]}, \\
V_{WX} &= W^{-\gamma}(1-\gamma) B_t e^{[(1-\gamma)(A_t+B_tX)]}, \\
V_X &= W^{(1-\gamma)} B_t e^{[(1-\gamma)(A_t+B_tX)]}, \\
V_{XX} &= W^{(1-\gamma)} B_t^2 (1-\gamma) e^{[(1-\gamma)(A_t+B_tX)]}, \\
V_t &= W^{(1-\gamma)} \left(\frac{\partial A_t}{\partial t} + \frac{\partial B_t}{\partial t} X \right) e^{[(1-\gamma)(A_t+B_tX)]},
\end{aligned} \tag{3.33}$$

and the ψ specification (3.21) is given by:

$$\psi = \frac{\theta}{W^{(1-\gamma)} e^{[(1-\gamma)(A_t+B_tX)]}}. \tag{3.34}$$

Substituting expressions (3.33) and (3.34) into (3.24) one obtains:

$$
\begin{aligned}
\alpha^* &= \frac{-V_W}{\left(V_{WW}-\psi V_W^2\right)W}(\mu-r)X+\frac{-V_{WX}}{\left(V_{WW}-\psi V_W^2\right)W}\rho\sigma_X X \\
&\quad +\frac{V_W V_X \psi}{\left(V_{WW}-\psi V_W^2\right)W}\rho\sigma_X X\,, \\
\Leftrightarrow \alpha^* &= \frac{-W^{-\gamma}e^{[(1-\gamma)(A_t+B_tX)]}}{\left(-\gamma W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}-\theta W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}\right)W}(\mu-r)X \\
&\quad +\frac{-W^{-\gamma}(1-\gamma)B_t e^{[(1-\gamma)(A_t+B_tX)]}\rho\sigma_X X}{\left(-\gamma W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}-\theta W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}\right)W} \\
&\quad +\frac{W^{(1-2\gamma)}e^{2[(1-\gamma)(A_t+B_tX)]}B_t\theta\rho\sigma_X X}{W^{(2-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}\left(-\gamma W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}-\theta W^{(-1-\gamma)}e^{[(1-\gamma)(A_t+B_tX)]}\right)}\,, \\
\Leftrightarrow \alpha^* &= \frac{-W^{-\gamma}e^{[(1-\gamma)(A_t+B_tX)]}(\mu-r)X-W^{-\gamma}(1-\gamma)B_t e^{[(1-\gamma)(A_t+B_tX)]}\rho\sigma_X X}{-W^{-\gamma}(\gamma+\theta)e^{[(1-\gamma)(A_t+B_tX)]}} \\
&\quad -\frac{e^{[(1-\gamma)(A_t+B_tX)]}B_t\theta\rho\sigma_X X}{(\gamma+\theta)e^{[(1-\gamma)(A_t+B_tX)]}}\,, \\
\Leftrightarrow \alpha^* &= \frac{1}{(\gamma+\theta)}\left[(\mu-r)+(1-\gamma-\theta)\sigma_X\rho B_t\right]X\,,
\end{aligned}
$$

which is (3.29).

□

3.6.4.2 Expressions (3.26) and (3.27)

Substituting the obtained optimal portfolio rule and expressions (3.33) and (3.34) into the Bellman equation (3.23) it results an equation with an independent term and other dependent on X. Collecting them, gives a system of first-order nonlinear ordinary differential equations:

$$
\begin{aligned}
\frac{\partial A_t}{\partial t} &= -r-\lambda\overline{X}B_t\,, \\
\frac{\partial B_t}{\partial t} &= -\beta_0-\beta_1 B_t-\beta_2 B_t^2\,,
\end{aligned}
\tag{3.35}
$$

where $\beta_0=\frac{(\mu-r)^2}{2(\gamma+\theta)}$, $\beta_1=\frac{(\mu-r)(1-\gamma-\theta)\rho\sigma_X}{\gamma+\theta}-\lambda$ and $\beta_2=\frac{\sigma_X^2}{2}(1-\gamma-\theta)\left[1+\frac{(1-\gamma-\theta)\rho^2}{\gamma+\theta}\right]$ and subject to the condition

$$A_T=B_T=0,$$

that results from the boundary condition (3.20).

We want now to prove that (3.26), (3.27) and (3.28) solve the system (3.35).

Proof:

From (3.27) one obtains the left hand-side of the second ODE in (3.35) $\left(\frac{\partial B_t}{\partial t}\right)$:

$$\frac{2\beta_0\left(-\eta e^{-\eta(T-t)}\right)\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right]}{\left(2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right)^2} - \frac{2\beta_0\left(1-e^{-\eta(T-t)}\right)(\beta_1+\eta)\,\eta e^{-\eta(T-t)}}{\left(2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right)^2}. \tag{3.36}$$

Substituting B_t expression in the right hand-side of the second ODE in (3.35), it becomes:

$$\frac{-\beta_0\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right]^2}{\left(2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right)^2} - \frac{2\beta_0\beta_1\left(1-e^{-\eta(T-t)}\right)\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right]}{\left(2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right)^2} - \frac{4\beta_0^2\beta_2\left(1-e^{-\eta(T-t)}\right)^2}{\left(2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right)^2}. \tag{3.37}$$

In order to (3.27) being the solution of second ODE in (3.35) we have to prove that (3.36)

equals (3.37):

$$
\begin{aligned}
& 2\beta_0\left(-\eta e^{-\eta(T-t)}\right)\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right] \\
& -2\beta_0\left(1-e^{-\eta(T-t)}\right)(\beta_1+\eta)\,\eta e^{-\eta(T-t)} \\
& = -\beta_0\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right]^2 \\
& -2\beta_0\beta_1\left(1-e^{-\eta(T-t)}\right)\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right] \\
& -4\beta_0^2\beta_2\left(1-e^{-\eta(T-t)}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\quad & -4\beta_0\eta^2 e^{-\eta(T-t)}+2\beta_0\eta e^{-\eta(T-t)}(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right) \\
& -2\beta_0\eta\left(1-e^{-\eta(T-t)}\right)e^{-\eta(T-t)}(\beta_1+\eta) \\
& = -4\beta_0^2\beta_2\left(1-e^{-\eta(T-t)}\right)^2 \\
& -\left[2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)\right][\beta_0 2\eta \\
& -\beta_0(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)+2\beta_0\beta_1\left(1-e^{-\eta(T-t)}\right)],
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\quad & \left[4\beta_0\eta^2+2\eta(\beta_0\beta_1-\beta_0\eta)-(\beta_1+\eta)\beta_0 2\eta\right]\left(1-e^{-\eta(T-t)}\right) \\
& = \left[-4\beta_0^2\beta_2+(\beta_1+\eta)(\beta_0\beta_1-\beta_0\eta)\right]\left(1-e^{-\eta(T-t)}\right)^2,
\end{aligned}
$$

which, considering that $\eta=\sqrt{\beta_1^2-4\beta_0\beta_2}$ becomes:

$$
\begin{aligned}
\Leftrightarrow 0 &= \left[-4\beta_0^2\beta_2+\beta_0\beta_1^2-\beta_0\beta_1^2+4\beta_0^2\beta_2\right]\left(1-e^{-\eta(T-t)}\right)^2, \\
\Leftrightarrow 0 &= 0,
\end{aligned}
$$

as we wanted to prove.

Having found the solution B_t for the second ODE in (3.35) we are now ready to deduct the solution of the first ODE as:

$$
\begin{aligned}
A_t &= \int_t^T\left(-r-\lambda\overline{X}B_s\right)ds, \\
\Leftrightarrow A_t &= r(T-t)-\frac{2\beta_0\lambda\overline{X}}{(\beta_1+\eta)}(T-t)+\frac{4\beta_0\lambda\overline{X}}{\eta^2-\beta_1^2}ln\left[\frac{(\beta_1+\eta)-(\beta_1-\eta)e^{(T-t)\eta}}{2\eta}\right],
\end{aligned}
$$

which is (3.26). We prove that this expression for A_t solves the first ODE in (3.35).

From (3.26) one gets:

$$\begin{aligned}
\frac{\partial A}{\partial t} &= -r + \frac{2\beta_0\lambda\overline{X}}{(\beta_1+\eta)} + \frac{4\beta_0\lambda\overline{X}}{\left(\eta^2-\beta_1^2\right)}\frac{\eta\,(\beta_1-\eta)\,e^{(T-t)\eta}}{\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]}\,, \\
\Leftrightarrow \frac{\partial A}{\partial t} &= -r + \frac{2\beta_0\lambda\overline{X}\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right] - 4\beta_0\lambda\overline{X}\eta e^{(T-t)\eta}}{(\beta_1+\eta)\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]}\,, \\
\Leftrightarrow \frac{\partial A}{\partial t} &= -r - 2\beta_0\lambda\overline{X}\left[\frac{(\beta_1-\eta)\,e^{(T-t)\eta}-(\beta_1+\eta)+2\eta e^{(T-t)\eta}}{(\beta_1+\eta)\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]}\right].
\end{aligned}$$

Looking at the first ODE in (3.35), in order to (3.26) be its solution it is necessary that:

$$\begin{aligned}
2\beta_0\left[\frac{(\beta_1-\eta)\,e^{(T-t)\eta}-(\beta_1+\eta)+2\eta e^{(T-t)\eta}}{(\beta_1+\eta)\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]}\right] &= B_t\,, \\
\Leftrightarrow 2\beta_0\left[\frac{(\beta_1-\eta)\,e^{(T-t)\eta}-(\beta_1+\eta)+2\eta e^{(T-t)\eta}}{(\beta_1+\eta)\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]}\right] &= \frac{2\beta_0\left(1-e^{-\eta(T-t)}\right)}{2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)}\,, \\
\Leftrightarrow \frac{(\beta_1+\eta)\,e^{(T-t)\eta}-(\beta_1+\eta)}{(\beta_1+\eta)\left[(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}\right]} &= \frac{\left(1-e^{-\eta(T-t)}\right)}{2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)}\,, \\
\Leftrightarrow \frac{e^{(T-t)\eta}-1}{(\beta_1+\eta)-(\beta_1-\eta)\,e^{(T-t)\eta}} &= \frac{1-e^{-\eta(T-t)}}{2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)}\,, \\
\Leftrightarrow 2\eta e^{(T-t)\eta}-2\eta-(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right)e^{(T-t)\eta} & \\
+(\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right) &= (\beta_1+\eta)\left(1-e^{-\eta(T-t)}\right) \\
&\quad -(\beta_1-\eta)\left(e^{\eta(T-t)}-1\right)\,, \\
\Leftrightarrow 2\eta\left(e^{(T-t)\eta}-1\right)-(\beta_1+\eta)\left(e^{\eta(T-t)}-1\right) & \\
+(\beta_1-\eta)\left(e^{\eta(T-t)}-1\right) &= 0\,, \\
\left(\underbrace{2\eta-\beta_1-\eta+\beta_1-\eta}_{=0}\right)\left(e^{\eta(T-t)}-1\right) &= 0\,, \\
0 &= 0\,,
\end{aligned}$$

as we wanted to prove.

$\square$

3.6.5 Logarithmic Utility Investor

Following Maenhout (2004, 2006), we use $\psi_{ln} = \lim_{\gamma \to 1} \frac{\theta}{(1-\gamma)V(W,X,t)}$ with the limit being computed using the analytical expression for the value function given by (3.25) for $\gamma \neq 1$:

$$\begin{aligned}
\psi_{ln} &= \lim_{\gamma \to 1} \frac{\theta}{(1-\gamma)\frac{W^{1-\gamma}}{1-\gamma} e^{[(1-\gamma)(A_t+B_tX)]}}, \\
\Leftrightarrow \psi_{ln} &= \lim_{\gamma \to 1} \frac{\theta}{W^{1-\gamma} e^{[(1-\gamma)(A_t+B_tX)]}}, \\
\Leftrightarrow \psi_{ln} &= \theta. \qquad (3.38)
\end{aligned}$$

The Bellman equation (3.23) subject to $V(W,X,T) = lnW_T$ is solved by $V(W,X,t) = ln(W) + A_t + B_tX$. Going back to (3.24) and considering the value function $V(W,X,t) = ln(W) + A_t + B_tX$, and using (3.38) one obtains the portfolio rule for the robust-log investor:

$$\begin{aligned}
\alpha^*_{ln} &= \frac{-\frac{1}{W}}{\left(-\frac{1}{W^2} - \frac{\theta}{W^2}\right)W}(\mu - r)X + \frac{\frac{B_t\theta}{W}}{\left(-\frac{1}{W^2} - \frac{\theta}{W^2}\right)W}\sigma_X \rho X, \\
\Leftrightarrow \alpha^*_{ln} &= \frac{(\mu - r)}{(1+\theta)}X - \frac{\theta\sigma_X\rho B_t}{(1+\theta)}X. \qquad (3.39)
\end{aligned}$$

Substituting (3.38), (3.39) and the value function $V(W,X,t) = ln(W) + A_t + B_tX$ and its partial derivatives into (3.23), expressions of A_t and B_t are the ones that solve the system (3.35) with

$$\begin{aligned}
\beta_0 &= \frac{(\mu - r)^2}{2(1+\theta)}, \\
\beta_1 &= -\frac{(\mu - r)\theta\sigma_X\rho}{(1+\theta)} - \lambda, \\
\beta_2 &= \frac{\theta\sigma_X^2}{2}\left[\frac{\theta\rho^2}{(1+\theta)} - 1\right],
\end{aligned}$$

subject to $A_T = B_T = 0$. Under this specification, the system is (3.35) for $\gamma = 1$, implying that all results obtained for $\gamma \neq 1$ also apply for $\gamma = 1$.

□

3.6.6 Expression (3.30)

Regarding the sign of elements in (3.28), from parameter assumptions in the model results immediately that:

$$\begin{aligned}\beta_0 &= \frac{(\mu-r)^2}{2(\gamma+\theta)} > 0\,,\\ \beta_2 &= \frac{\sigma_X^2}{2}(1-\gamma-\theta)\left[\frac{(1-\rho^2)(\gamma+\theta)+\rho^2}{\gamma+\theta}\right] < 0\,.\end{aligned}$$

Then:

$$\beta_0 > 0\,,\, \beta_2 < 0 \;\;\Rightarrow\;\; \beta_1^2 - 4\beta_0\beta_2 > 0\,,$$

and therefore $\eta = \sqrt{\beta_1^2 - 4\beta_0\beta_2}$ is always defined in the $\mathbb{R}$ space and is > 0.[9] We can know quickly prove properties (3.30):

- $B_t > 0\,, \forall_{t<T}$

  Considering that for $\eta > 0$ and $t < T$: $0 < \left(1 - e^{-\eta(T-t)}\right) < 1$, from (3.27) the expression for B_t can be rewritten as:

  $$B_t = \frac{2\beta_0}{\frac{2\eta}{\left(1-e^{-\eta(T-t)}\right)} - \beta_1 - \eta}\,. \tag{3.40}$$

  As $\beta_0 > 0$, it implies that $sign\,(B_t) = sign\left(\frac{2\eta}{\left(1-e^{-\eta(T-t)}\right)} - \beta_1 - \eta\right)$. It is immediate to obtain:

  $$\frac{2\eta}{\left(1-e^{-\eta(T-t)}\right)} - \beta_1 - \eta = \frac{\left(1+e^{-\eta(T-t)}\right)\eta - \left(1-e^{-\eta(T-t)}\right)\beta_1}{\left(1-e^{-\eta(T-t)}\right)} > 0\,,$$

  as $\beta_1 < 0$ if $\rho > 0$, $\left(1 \pm e^{-\eta(T-t)}\right) > 0$ and $\eta > 0$. It is therefore proved that $B_t > 0\,, \forall_{t<T}$.

[9] The difference $\beta_1^2 - 4\beta_0\beta_2$ is strictly positive because $\beta_1^2 - 4\beta_0\beta_2 = \frac{\lambda^2(\gamma+\theta)^2 - 2(\gamma+\theta)(\mu-r)(1-\gamma-\theta)\rho\sigma_X\lambda - (\mu-r)^2\sigma_X^2(1-\gamma-\theta)(\gamma+\theta)}{(\gamma+\theta)^2}$, which for $\rho > 0$ and the remaining assumptions on the parameters is always strictly positive.

- $\frac{\partial B_t}{\partial t} < 0\,, \forall_{t<T}$

  Note that from (3.40), the only element of B_t dependent on t is $\left(1 - e^{-\eta(T-t)}\right)$. Its derivative with respect to time is given by:

$$\frac{\partial\left(1 - e^{-\eta(T-t)}\right)}{\partial t} \quad = \quad -\eta e^{-\eta(T-t)} < 0\,.$$

  As $\eta > 0$, the implication is that:

$$\frac{\partial\left(\frac{2\eta}{\left(1-e^{-\eta(T-t)}\right)} - \beta_1 - \eta\right)}{\partial t} > 0\,,$$

  and therefore, from (3.40), $\frac{\partial B_t}{\partial t} < 0$ as we wanted to prove.

□

Chapter 4

The Price of Risk and Ambiguity in an Intertemporal General Equilibrium Model of Asset Prices[1]

[1]An early version of this chapter has been circulating as Faria and Correia-da Silva (2010b). We are grateful to Alejandro Balbás, Cláudia Ribeiro, Frank Riedel, Paolo Porchia and Tony Berrada (discussant) for useful comments and suggestions, and thank seminar participants at the 6th Portuguese Finance Network Conference (Ponta Delgada).

4.1 Introduction

There are two major approaches for the modelling of asset prices and of the implied uncertainty prices: the equilibrium and the arbitrage approach.

The equilibrium approach includes models that start by describing the production sector of the economy, which is typically a set of production processes driven by exogenous state variables whose dynamics are, in turn, described by stochastic processes. The assets are contingent claims to the output of these production processes. With the objective of maximizing an utility function, the representative agent decides how much to consume and how much to invest (either physically in the production processes or financially by acquiring assets). The equilibrium prices of the assets, and the corresponding uncertainty prices, must be such that demand equals supply. An example of this type of setting is the continuous time model of Cox et al. (1985a), which has several applications in the literature (e.g. Cox et al. (1985b), Longstaff and Schwartz (1992) and Gagliardini et al. (2009) on the modeling of interest rate term structure). A pioneering example of the equilibrium approach is the model of Lucas (1978), which has a similar structure to that of Cox et al. (1985a), but has exogenous production, no technological change and is in discrete time. The model of Lucas (1978) has also several applications in the literature, with a recent example being the consumption based general equilibrium model for designing affine asset pricing models by Eraker and Shaliastovich (2008).

The arbitrage approach starts by assuming the dynamics of the state variables, of which the contingent claims depend, and an exogenous specification for the uncertainty prices. Then, by applying Itô's lemma and imposing the condition that there are no arbitrage opportunities in the economy, the prices of contingent claims are obtained. This is the standard approach in the option pricing literature (e.g. Black and Scholes (1973)), with some exceptions (e.g. Amin and Ng (1993)), and has been extensively used in other fields of finance. For example, Vasicek (1977) and Brennan and Schwartz (1979) used it to model the term structure of interest rates.

The equilibrium approach has clear advantages versus the arbitrage approach. As pointed out by Cox et al. (1985b), imposing exogenous uncertainty prices without any underlying economic equilibrium may lead to internal inconsistencies. Conversely, in the equilibrium approach, uncertainty prices are endogenous and therefore part of the equilibrium. Moreover, models under the arbitrage approach say very little about the economic nature of the price of uncertainty.

In this chapter, we consider a continuous time general equilibrium model for contingent claim pricing which is a two state variable version of the model of Cox et al. (1985a).

It is assumed that the two state variables, Y_{1t} and Y_{2t}, are correlated and both impact the expected return of the single production process, Q_t. However, it is also assumed that only shocks in one of the state variables, Y_{2t}, are correlated with the shocks in the return of the production process. We compare the case in which the uncertainty faced by the representative agent is exclusively risk to that in which uncertainty has two dimensions: risk and ambiguity (about Y_{2t}).[2] In both scenarios, we obtain the specification of the equilibrium price of uncertainty associated with Y_{2t} and of the equilibrium interest rate, both used to derive the fundamental partial differential equation satisfied by the price of a contingent claim.

We start by analyzing the case in which uncertainty is exclusively risk. We find out that the equilibrium price of risk associated with the state variable whose shocks are correlated with those of the economy's output rate depends on that correlation and on the diffusion function specification in the stochastic processes that generate the output rate, $\frac{dQ_t}{Q_t}$, and that state variable, Y_{2t}. In this case, the equilibrium interest rate is given by the difference between the economy's expected output rate and the variance of the output rate. A particular case of this setting is the model of Longstaff and Schwartz (1992), for the pricing of interest rate sensitive contingent claims.

Afterwards, it is assumed that the representative agent is ambiguous about the stochastic model that characterizes the dynamics of the investment opportunity set (more precisely, about the stochastic process for the state variable Y_{2t}).[3] The agent considers contaminations, P^h, around the reference belief, P. Aversion towards ambiguity is considered by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative agent chooses the worst possible contamination, i.e., the one associated with the lowest expected utility.[4] The introduction of ambiguity about Y_{2t} impacts the equilibrium asset

[2]The distinction between risk and ambiguity was firstly pointed out by Knight (1921) and latter supported by the empirical experiments of Ellsberg (1961) and others (see Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey). The reason for this distinction is that economic agents may not be able to completely describe the uncertainty that they face by using a single probability distribution. Risk refers to uncertainty that can be represented by a probability distribution, while ambiguity refers to uncertainty that cannot. This distinction has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. That is why a rapidly growing literature on asset pricing under ambiguity aversion is emerging. This literature has been comprehensively surveyed by Epstein and Schneider (2010).

[3]An extensive review on decision theory under ambiguity has been carried out by Etner et al. (2009). Briefly, the two most common approaches being used in the ambiguity literature are: the robust control (RC) approach, associated to an assumption of model uncertainty, which is the one that we follow in the present chapter (as, for e.g., in Maenhout (2004, 2006) and Gagliardini et al. (2009)); the multiple priors (MP) approach, from the seminal work in Gilboa and Schmeidler (1989), where the single probability measure of the standard expected utility model is replaced by a set of probabilities or priors. The relationship between the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001a), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006a).

[4]The approach of Gilboa and Schmeidler (1989) is sometimes criticized because it apparently implies

prices through two channels: the uncertainty price of Y_{2t} and the interest rate.

We find that the price of uncertainty associated with the ambiguous state variable, Y_{2t}, has two components: the price of risk and the price of ambiguity. The latter depends on the level of ambiguity faced by the representative agent, and on the diffusion function in the stochastic process for Y_{2t}. Regarding the effect of ambiguity about Y_{2t} on the interest rate, we find that it depends on the level of ambiguity faced by the representative agent, on the diffusion function in the stochastic process for the output rate, and on the correlation between shocks in Y_{2t} and shocks in the output rate.

The major contribution of this chapter is to develop a two-factor general equilibrium framework for asset pricing under ambiguity when the shocks in two state variables are correlated but only the shocks in the ambiguous state variable are correlated with shocks in the economy's output rate. We believe that this is a simple setting that can be used for many asset pricing problems. As an example, we apply our general results to the specific investment opportunity set of the well known option pricing model of Heston (1993), therefore providing an equilibrium motivation for the price specification of variance risk used in it.

The chapter is organized as follows. In section 4.2, the intertemporal general equilibrium model for contingent claim pricing under ambiguity is developed. In section 4.3, it is applied to a concrete investment opportunity set which contains that of the option pricing model of Heston (1993). In section 4.4, we conclude the chapter with some remarks.

4.2 General Equilibrium Framework

We consider an intertemporal general equilibrium model for contingent claim pricing that is a version of the model of Cox et al. (1985a) with two correlated state variables, a single stochastic constant returns-to-scale production process and logarithmic utility.[5]

extreme ambiguity aversion. However, the implied decision criteria may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. This is why the criteria of Gilboa and Schmeidler (1989) is not so extreme as, for example, the Wald maxmin criteria. As Epstein and Schneider (2010) claimed *"Ultimately, the only way to argue that the model is extreme is to demonstrate extreme behavioral implications of the axioms, something that has not been done"*. More recently, the smooth ambiguity aversion utility theory, has been developed on the back of the seminal work of Klibanoff et al. (2005). It is claimed that this setup distinguishes ambiguity from ambiguity aversion and allows for smooth indifference curves, avoiding the infinite ambiguity aversion implied in Gilboa and Schmeidler (1989) approach. However, there is still a debate in the literature about the axiomatic foundations of this line of models (see Epstein (2010) and Klibanoff et al. (2009a) for a recent exchange on this). Also because of this, the approach of Gilboa and Schmeidler (1989) continues to be the main reference in the literature.

[5]We foresee as interesting extensions of this setting the consideration of heterogeneous agents, where trade would occur in equilibrium, and of a more general setting for preferences (e.g. SDU from Duffie and

There is a single physical good in the economy, that the representative agent can consume or reinvest in the stochastic production process, Q_t. The realized return on the physical investment made through the production process, i.e., the economy's output rate, is driven by two correlated state variables, Y_{1t} and Y_{2t}:

$$\frac{dQ_t}{Q_t} = g_Q\left(Y_{1t}, Y_{2t}\right) dt + \sigma_Q\left(Y_{1t}, Y_{2t}\right) dZ_Q\,, \tag{4.1}$$

where $g_Q\left(Y_{1t}, Y_{2t}\right)$ and $\sigma_Q\left(Y_{1t}, Y_{2t}\right)$ are generic expressions for the drift and diffusion functions of the output rate dynamics, which may depend on both state variables, and Z_Q is a standard Brownian motion.

The dynamics of the state variables, (Y_{1t}, Y_{2t}), is given by:

$$dY_{1t} = g_{Y_1}\left(Y_{1t}, Y_{2t}\right) dt + \sigma_{Y_1}\left(Y_{1t}, Y_{2t}\right) dZ_1\,, \tag{4.2}$$

$$dY_{2t} = g_{Y_2}\left(Y_{1t}, Y_{2t}\right) dt + \sigma_{Y_2}\left(Y_{1t}, Y_{2t}\right) dZ_2\,, \tag{4.3}$$

where the generic functions g_{Y_1}, σ_{Y_1}, g_{Y_2} and σ_{Y_2} have the same meaning, for each of the state variables, as g_Q and σ_Q for the stochastic process (4.1). Z_1 and Z_2 are standard Brownian motions with an instantaneous correlation equal to ρ $(dZ_1 dZ_2 = \rho dt)$.

Both state variables potentially impact the expected output rate, but only one of the state variables, Y_{2t}, has shocks that are correlated with those of economy's output rate. The instantaneous correlation between $\frac{dQ_t}{Q_t}$ and dY_{1t} is zero ($dZ_Q dZ_1 = 0$), while that between $\frac{dQ_t}{Q_t}$ and dY_{2t} is ρ^* ($dZ_Q dZ_2 = \rho^* dt$). The correlation structure of the model is deducted in Appendix 4.5.1.

We analyze two scenarios, which only differ in the nature of the uncertainty faced by the representative agent. In section 4.2.1, there is only risk. In section 4.2.2, there is also ambiguity.

4.2.1 Uncertainty is Risk

Start by assuming that the representative agent is totally sure about the data-generating processes (4.1), (4.2) and (4.3) that, under a probability measure P, characterize the investment opportunity set dynamics. This means that uncertainty is exclusively risk.

The intertemporal budget constraint faced by the agent is given by:

$$dW_t = W_t \frac{dQ_t}{Q_t} - C_t dt\,,$$

Epstein (1992a,b)). We thank Frank Riedel for those suggestions for future work.

where W_t and C_t represent wealth and consumption at time t. Considering (4.1), the dynamic budget constraint can be expressed as:

$$dW_t = g_W\left(W_t, Y_{1t}, Y_{2t}, C_t\right) dt + \sigma_W\left(W_t, Y_{1t}, Y_{2t}\right) dZ_Q\,, \tag{4.4}$$

where $g_W\left(W_t, Y_{1t}, Y_{2t}, C_t\right) = W_t g_Q\left(Y_{1t}, Y_{2t}\right) - C_t$ and $\sigma_W\left(W_t, Y_{1t}, Y_{2t}\right) = W_t \sigma_Q\left(Y_{1t}, Y_{2t}\right)$.

The problem of the representative agent is to find the optimal consumption strategy, $C : [0, +\infty[\rightarrow \mathbb{R}_+$, that maximizes his expected intertemporal utility. Recall that in the setting of Cox et al. (1985a), the optimal consumption strategy is financed by allocating all the wealth in the production process and none in the financial assets (which are in zero net supply). This implies that the only relevant control variable is the consumption flow process. Having a logarithmic instantaneous utility function, the representative agent's problem is therefore represented by:

$$J(W_0, Y_{1,0}, Y_{2,0}) = \underset{C}{sup}\, E\left[\int_0^\infty e^{-\delta s} ln\left(C_s\right) ds\right]\,, \tag{4.5}$$

subject to (4.2), (4.3) and (4.4). The function $J(W_t, Y_{1t}, Y_{2t})\,|_{t=0}$ is the value function of the problem, $E\left[\cdot\right]$ is the expectation operator under the probability measure P, and $\delta > 0$ is the subjective rate of discount of the representative agent.

The corresponding Bellman equation is given by (Appendix 4.5.2):

$$\begin{aligned} 0 \;=\; & \underset{C}{sup}\Bigg\{ ln\left(C_t\right) - \delta J + \frac{\partial J}{\partial W} g_W\left(W_t, Y_{1t}, Y_{2t}, C_t\right) + \frac{\partial J}{\partial Y_1} g_{Y_1}\left(Y_{1t}, Y_{2t}\right) + \frac{\partial J}{\partial Y_2} g_{Y_2}\left(Y_{1t}, Y_{2t}\right) \\ & + \frac{1}{2}\frac{\partial^2 J}{\partial W^2}\sigma_W^2\left(W_t, Y_{1t}, Y_{2t}\right) + \frac{1}{2}\frac{\partial^2 J}{\partial Y_1^2}\sigma_{Y_1}^2\left(Y_{1t}, Y_{2t}\right) + \frac{1}{2}\frac{\partial^2 J}{\partial Y_2^2}\sigma_{Y_2}^2\left(Y_{1t}, Y_{2t}\right) \\ & + \frac{\partial^2 J}{\partial W \partial Y_2}\sigma_W\left(W_t, Y_{1t}, Y_{2t}\right)\sigma_{Y_2}\left(Y_{1t}, Y_{2t}\right)\rho^* \\ & + \frac{\partial^2 J}{\partial Y_1 \partial Y_2}\sigma_{Y_1}\left(Y_{1t}, Y_{2t}\right)\sigma_{Y_2}\left(Y_{1t}, Y_{2t}\right)\rho \Bigg\}\,. \end{aligned} \tag{4.6}$$

As in Merton (1971) and Cox et al. (1985b), the value function that solves (4.6) is partially separable and has the generic form:

$$J(W_t, Y_{1t}, Y_{2t}) = \frac{1}{\delta} ln\left(W_t\right) + G\left(Y_{1t}, Y_{2t}\right)\,. \tag{4.7}$$

It is straightforward to show that the instantaneous optimal consumption rule is given by $C_t^* = \delta W_t$, i.e, the representative agent maximizes his utility by consuming a constant fraction of his instantaneous wealth throughout time (Appendix 4.5.2).

Substituting the optimal consumption rule into (4.4), one obtains the equilibrium dynamics of wealth. Then, with $H(Y_{1t}, Y_{2t}, \tau)$ denoting the price of a contingent claim with maturity τ, dependent on both state variables but not on wealth, and making use of Theorem 3 in Cox et al. (1985a), the fundamental partial differential equation that has to be satisfied by $H(Y_{1t}, Y_{2t}, \tau)$ is obtained (Appendix 4.5.3):

$$\begin{aligned}\frac{\partial H}{\partial \tau} &= \frac{1}{2}\sigma_{Y_1}^2(Y_{1t}, Y_{2t})\frac{\partial^2 H}{\partial Y_1^2} + \frac{1}{2}\sigma_{Y_2}^2(Y_{1t}, Y_{2t})\frac{\partial^2 H}{\partial Y_2^2} \\ &+\rho\,\sigma_{Y_1}(Y_{1t}, Y_{2t})\,\sigma_{Y_2}(Y_{1t}, Y_{2t})\frac{\partial^2 H}{\partial Y_1 \partial Y_2} + g_{Y_1}(Y_{1t}, Y_{2t})\frac{\partial H}{\partial Y_1} \\ &+\left[g_{Y_2}(Y_{1t}, Y_{2t}) - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\right) Cov(W_t, Y_{2t})\right]\frac{\partial H}{\partial Y_2} \\ &-r_t H + \Lambda(W_t, Y_{1t}, Y_{2t}, t)\,, \end{aligned} \tag{4.8}$$

where $Cov(W_t, Y_{2t})$ is the instantaneous covariance between W_t and Y_{2t}, and $\Lambda(W_t, Y_{1t}, Y_{2t}, t)$ represents the instantaneous payoff of the contingent claim, which depends on its specific contractual conditions.

The equilibrium instantaneous interest rate r_t is obtained by applying Theorem 1 of Cox et al. (1985a), which yields (Appendix 4.5.3):

$$r_t = g_Q(Y_{1t}, Y_{2t}) - \sigma_Q^2(Y_{1t}, Y_{2t})\,. \tag{4.9}$$

The equilibrium interest rate when uncertainty is exclusively risk, is, therefore, given by the difference between the expected output rate and the variance of the output rate. Depending on the specifications of g_Q and σ_Q, conditions on its parameters have to be imposed in order to guarantee that the equilibrium interest rate is non-negative.[6]

The term $\left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\right) Cov(W_t, Y_{2t})$ in (4.8) represents the equilibrium price of risk associated to Y_{2t}, the only state variable whose shocks are correlated with those of the economy's output rate. Considering the value function (4.7), the fact that $dZ_Q dZ_2 = \rho^* dt$ and $\sigma_W(W_t, Y_{1t}, Y_{2t}) = W_t \sigma_Q(Y_{1t}, Y_{2t})$ from (4.4), the general expression for the equilibrium price of risk associated with Y_{2t} is given by (Appendix 4.5.3):

$$\sigma_{Y_2}(Y_{1t}, Y_{2t})\,\sigma_Q(Y_{1t}, Y_{2t})\,\rho^*. \tag{4.10}$$

[6] As explained by Longstaff and Schwartz (1992), the lower bound of zero for the interest rate is consistent with the basic properties of the economy under study, because as the single good produced in this economy can be consumed or invested in the production process, it can be seen as storable.

From (4.10), one concludes that, in our setting, the equilibrium price of risk associated with Y_{2t} is positive (negative) when the correlation between its shocks and those of the output rate is positive (negative).[7]

A particular case of this setting is presented in Longstaff and Schwartz (1992), where an intertemporal general equilibrium setting for valuing interest rate sensitive contingent claims is developed starting from a two state variable version of the model of Cox et al. (1985a). There, it is also assumed that both state variables impact the expected output rate in the economy, that only one of the state variables has shocks correlated with those of the economy's output rate, and that the representative agent has a logarithmic utility function. They assume, however, that the state variables are uncorrelated. The model of Longstaff and Schwartz (1992) is, therefore, a particular case of our setting (see Appendix 4.5.4).

4.2.2 Uncertainty is Risk and Ambiguity

In this section, we introduce ambiguity about the investment opportunity set through a "constraint preferences" robust control approach, following the extension of the model of Cox et al. (1985b) made by Gagliardini et al. (2009). Apart from ambiguity aversion, the preferences of the representative agent are the same as in the previous section.

It is assumed that the representative agent is ambiguous about the stochastic process (4.3), that evolves under the probability measure P, describing the dynamics of Y_{2t}. The agent considers contaminations (alternative models), P^h, around that reference belief, P. The contaminations are assumed to be absolutely continuous with respect to P, and, therefore, are equivalently described by contaminating drift processes, h. A Brownian motion Z_2^h under each of the alternative models represented by P^h is given by $Z_2^h(t) = Z_2(t) + \int_0^t h(s)\,ds$.[8]

Existence of ambiguity is analytically represented by perturbations of the drift, with respect to the reference belief, in the dynamics of the ambiguous state variable, Y_{2t}. Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative agent chooses from all the contaminations, P^h, the one that corresponds to the worst case scenario, i.e., the one associated with lower expected utility.

[7]Diffusion functions σ_Q and σ_{Y_2} are assumed to be positive, as it is usual in the literature, due to the analogy with the statistical concept of standard deviation, although probabilistically it is not required so. We thank Paolo Porchia for this insight.

[8]Gagliardini et al. (2009) explain that, for tractability reasons, the analysis is restricted to the class of Markov-Girsanov kernels. The absolute continuity assumption between P and P^h guarantees the equivalence property between the probability measures and, consequently, that Cameron-Martin-Girsanov theorem can be applied. Moreover, from this theorem and considering the diffusion family of models under consideration, all that a probability measure change implies is the change of the drift function of the stochastic processes.

An upper bound is imposed on the contaminating drift process, h:

$$h'h \leqslant 2\eta, \tag{4.11}$$

where $\eta \geqslant 0$ is a parameter that can be interpreted as the level of ambiguity. The class of admissible Markovian drift contaminations satisfying this bound is denoted by $\mathscr{H}$.

As Gagliardini et al. (2009) highlight, the bound (4.11) should be such that alternative models are statistically close to the "reference belief" model: otherwise the agent would easily distinguish among them and, consequently, would not face ambiguity. That is, η should be small. Moreover, the bound (4.11) constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference belief, P, and any admissible contaminated belief, P^h. Trojani and Vanini (2004) explain that the set $\{h : h'h \in [0, 2\eta], \forall t \geqslant 0\}$ defines a rectangular set of priors because any process h (and therefore any probability measure P^h) in this set corresponds to a selection of transition densities from t to $t+dt$, $t \geqslant 0$, such that $h'h \in [0, 2\eta]$. The fact that the specification of the ambiguity aversion is based on a rectangular set of priors, guarantees a dynamic consistent preference ordering supported by the Multiple Priors Recursive Utility. More generally, in Hansen and Sargent (2006) there is a comprehensive discussion of the dynamic consistency issue under the robust control approach.[9]

Under an admissible contamination, P^h, the investment opportunity is described by equations (4.1), (4.2), and the contaminated process for variable Y_{2t} given by:

$$dY_{2t} = g_{Y_2}(Y_{1t}, Y_{2t})\, dt + \sigma_{Y_2}(Y_{1t}, Y_{2t})\,(dZ_2 + hdt)\,. \tag{4.12}$$

In Appendix 4.5.1, we show the conditions so that (4.12) holds. Moreover, it is also proved that the relationship between dZ_Q, dZ_1 and dZ_2 is unchanged by the contamination, versus the previous section where uncertainty is exclusively risk.

Applying result [6] from Gagliardini et al. (2009) and considering that, in equilibrium, in the model of Cox et al. (1985a), wealth that is not consumed is totally invested in the production process, the intertemporal budget constraint faced by the agent is given by:

$$dW_t = g_{W_{Ab}}(W_t, Y_{1t}, Y_{2t}, C_t)\, dt + \sigma_{W_{Ab}}(W_t, Y_{1t}, Y_{2t})\, dZ_Q, \tag{4.13}$$

where $g_{W_{Ab}}(W_t, Y_{1t}, Y_{2t}, C_t) = [W_t g_Q(Y_{1t}, Y_{2t}) - C_t]$ and $\sigma_{W_{Ab}}(W_t, Y_{1t}, Y_{2t}) = W_t \sigma_Q(Y_{1t}, Y_{2t})$.[10]

[9] See Epstein and Schneider (2003) for the definition of the rectangularity property. Additionally, in Trojani and Vanini (2004), pp.289, there is a detailed explanation supporting the rectangularity property of the present set of priors built under the constraint (4.11), and how this rectangular set of priors can be defined under Chen and Epstein's (2002) *k-ignorance* metric.

[10] Notation Ab is introduced to indicate that uncertainty contains an ambiguity component.

We continue to assume that the ambiguity-averse representative agent has a logarithmic utility function. The difference is that now there isn't a single probability measure, P, but, instead, a set of probability measures, P^h.

The ambiguity averse representative agent solves the following Maxmin expected utility program:

$$J^{Ab}(W_0, Y_{1,0}, Y_{2,0}) = \sup_{C} \inf_{h \in \mathscr{H}} E^h \left[\int_0^\infty e^{-\delta s} ln(C_s)\, ds \right], \tag{4.14}$$

subject to (4.2), (4.12) and (4.13), where the operator E^h denotes expectations under the measure P^h, and $J^{Ab}(W_t, Y_{1t}, Y_{2t})\,|_{t=0}$ denotes the value function of the problem.

Applying Proposition 1 in Gagliardini et al. (2009), the value function of the ambiguity-averse agent is given by :

$$J^{Ab}(W_0, Y_{1,0}, Y_{2,0}) = -\frac{1}{\delta} + \frac{ln(\delta W_0)}{\delta} + \frac{1}{\delta} V(Y_{1,0}, Y_{2,0}), \tag{4.15}$$

where

$$V(Y_{1,0}, Y_{2,0}) = \inf_{h \in \mathscr{H}} E^h \left[\int_0^\infty e^{-\delta s} \left(g_Q - \frac{1}{2}\sigma\sigma^\top + \sigma h_s \right) ds \right], \tag{4.16}$$

subject to (4.2) and (4.12), where σ is the standard deviation matrix associated with the stochastic process of the output rate. The corresponding Bellman equation solved by the value function $V(Y_{1,0}, Y_{2,0})$ is given by:

$$\begin{aligned} 0 &= V_Y^\top g_Y + \frac{1}{2} trace\left[\Xi^\top V_{YY} \Xi\right] - \sqrt{2\eta}\sqrt{\left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right)} \\ &\quad + g_Q - \frac{1}{2}\sigma\sigma^\top - \delta V, \end{aligned} \tag{4.17}$$

where: (i) V_Y and V_{YY} are the gradient and Hessian matrices of the value function $V(Y_{1,0}, Y_{2,0})$ with respect to the state variables; (ii) Ξ is the standard deviation matrix associated with the stochastic processes of the state variables; and (iii) g_Y is the vector of drift functions of the state variables.

Considering that $dZ_Q = \rho^* dZ_2 + \sqrt{1-\rho^{*2}}\, dZ_3$ (Appendix 4.5.1), Proposition 1 in Gagliardini et al. (2009) can also be directly used to obtain the equilibrium contamination drift,

h^*, that solves the model selection problem (Appendix 4.5.5):

$$h^* = \begin{cases} -\sqrt{2\eta} & if\ V_{Y_2} > -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \\ \\ \sqrt{2\eta} & if\ V_{Y_2} < -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \end{cases} \tag{4.18}$$

where V_{Y_2} is the gradient of the value function $V(Y_{1t},Y_{2t})$ in (4.16), with respect to the ambiguous state variable, Y_{2t}.

From the Cameron-Martin-Girsanov theorem coupled with the fact that in our setting only diffusion models are considered, results that the change from one probability measure to an equivalent probability measure only leads to a change of drift in the stochastic processes of the state variables. Considering the reference belief P and an equivalent "uncertainty-neutralized" probability measure, then the change of drift associated with each of the state variables, represented by the matrix $\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}^\top$, is the equilibrium price of uncertainty associated to each of the state variables.

In our setting, only the state variable Y_{2t} has an associated market price of uncertainty, as it is the only state variable whose shocks are correlated with those of economy's output rate and about which the representative agent is ambiguous. We can therefore advance that the existence of ambiguity about the stochastic process of Y_{2t} implies an additional drift adjustment, that adds to the equilibrium price of risk associated to Y_{2t} given by the term $\left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\right) Cov(W_t, Y_{2t})$ in (4.8). Note that if the agent was also ambiguous about the state variable Y_{1t}, not correlated with the economy's output rate, that state variable would also have a corresponding ambiguity price (although not having a risk price).

In the present setting, following Gagliardini et al. (2009), the equilibrium price of uncertainty associated to Y_{2t} is given by (Appendix 4.5.5):

$$\phi_2 = \begin{cases} \sigma_{Y_2}(Y_{1t},Y_{2t})\,\sigma_Q(Y_{1t},Y_{2t})\,\rho^* + \sigma_{Y_2}(Y_{1t},Y_{2t})\sqrt{2\eta} & if\ V_{Y_2} > -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \\ \\ \sigma_{Y_2}(Y_{1t},Y_{2t})\,\sigma_Q(Y_{1t},Y_{2t})\,\rho^* - \sigma_{Y_2}(Y_{1t},Y_{2t})\sqrt{2\eta} & if\ V_{Y_2} < -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \end{cases} \tag{4.19}$$

where, in each of the two scenarios, the first parcel $(\sigma_{Y_2}\sigma_Q\rho^*)$ is the equilibrium price of risk, unchanged versus (4.10) in section 4.2.1, and the second parcel $\left(\pm\sigma_{Y_2}\sqrt{2\eta}\right)$ is the equilibrium price of ambiguity.

From (4.19), we conclude that the consideration of ambiguity implies an extra pricing component in equilibrium, related with the ambiguity dimension of the uncertainty in

the model. If there exists ambiguity ($\eta > 0$), and Y_{2t} is non-deterministic ($\sigma_{Y_2} \neq 0$), then from (4.19) it results that the equilibrium market price of ambiguity is non zero and can be positive or negative. Consequently, the equilibrium uncertainty price associated with Y_{2t} can either increase or decrease when the agent is ambiguous about its stochastic process, depending on the relation between: (i) the impact on the indirect utility (value function) from changes in the ambiguous state variable (V_{Y_2}); (ii) the correlation between shocks in that variable and in the economy's output rate (ρ^*); and (iii) the diffusion functions of the stochastic processes for the ambiguous state variable (σ_{Y_2}) and economy's output rate (σ_Q).

Note also that the uncertainty price becomes preference-dependent when ambiguity aversion is considered (as it includes the η parameter). This does not happen when uncertainty is exclusively risk.

In order to derive the partial differential equation (PDE) satisfied by the contingent claim price under the present setting, it is still necessary to obtain the equilibrium instantaneous interest rate, which we denominate as r_t^{Ab}. The generic expression for r_t^{Ab} is given by (Appendix 4.5.5):

$$r_t^{Ab} = \begin{cases} g_Q(Y_{1t}, Y_{2t}) - \sigma_Q^2(Y_{1t}, Y_{2t}) - \sigma_Q(Y_{1t}, Y_{2t})\rho^*\sqrt{2\eta} & if\ V_{Y_2} > -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \\ \\ g_Q(Y_{1t}, Y_{2t}) - \sigma_Q^2(Y_{1t}, Y_{2t}) + \sigma_Q(Y_{1t}, Y_{2t})\rho^*\sqrt{2\eta} & if\ V_{Y_2} < -\frac{\sigma_Q(Y_{1t},Y_{2t})\rho^*}{\sigma_{Y_2}(Y_{1t},Y_{2t})}, \end{cases} \tag{4.20}$$

where the first two parcels, in each of the scenarios, give the equilibrium instantaneous interest rate when uncertainty is exclusively risk, r_t, unchanged versus (4.9) in section 4.2.1, and the third parcel is the new component resulting from the existence of ambiguity.

Looking carefully at this new component, we conclude that ambiguity about the stochastic process of Y_{2t} does not impact the equilibrium interest rate only when the economy's output rate is deterministic ($\sigma_Q = 0$) or shocks in the ambiguous state variable dynamics are uncorrelated with those of economy's output rate ($\rho^* = 0$). When this does not happen, the equilibrium instantaneous interest rate under ambiguity, r_t^{Ab}, can be higher or lower than when uncertainty is exclusively risk, r_t, depending on the sign of ρ^* and on the relation between ρ^* and: (i) the gradient of the value function with respect to the ambiguous state variable (V_{Y_2}); and (ii) the diffusion functions of the stochastic processes for the ambiguous state variable (σ_{Y_2}) and the economy's output rate (σ_Q).[11] Again, as stated

[11] As highlighted in Epstein and Schneider (2010), agent's willingness to save is a positive function of his level of uncertainty and the more the agent tries to save the lower tends to be the equilibrium interest rate. Considering ambiguity as an extra source of uncertainty (alongside risk), we may therefore conclude that the most intuitive scenario is that when ambiguity is considered, the equilibrium interest rate decreases

in section 4.2.1, depending on the concrete specifications of functions g_Q and σ_Q, adequate conditions about its parameters need to be set in order to guarantee a non-negative equilibrium interest rate.

Given the equilibrium price of uncertainty associated with Y_{2t} (4.19) and the equilibrium instantaneous interest rate (4.20), the fundamental partial differential equation (PDE) satisfied by the price of a contingent claim with maturity τ, $H^{Ab}(Y_{1t}, Y_{2t}, \tau)$, assumed to depend on both state variables but not on wealth, as in section 4.2.1, is given by (Appendix 4.5.5):

$$\begin{aligned}
\frac{\partial H^{Ab}}{\partial \tau} &= \frac{1}{2}\sigma_{Y_1}^2(Y_{1t}, Y_{2t})\frac{\partial^2 H^{Ab}}{\partial Y_1^2} + \frac{1}{2}\sigma_{Y_2}^2(Y_{1t}, Y_{2t})\frac{\partial^2 H^{Ab}}{\partial Y_2^2} \\
&\quad + \rho\sigma_{Y_1}(Y_{1t}, Y_{2t})\sigma_{Y_2}(Y_{1t}, Y_{2t})\frac{\partial^2 H^{Ab}}{\partial Y_1 \partial Y_2} + g_{Y_1}(Y_{1t}, Y_{2t})\frac{\partial H^{Ab}}{\partial Y_1} \\
&\quad + \left[g_{Y_2}(Y_{1t}, Y_{2t}) - \left(\sigma_{Y_2}(Y_{1t}, Y_{2t})\sigma_Q(Y_{1t}, Y_{2t})\rho^* \pm \sigma_{Y_2}(Y_{1t}, Y_{2t})\sqrt{2\eta}\right)\right]\frac{\partial H^{Ab}}{\partial Y_2} \\
&\quad - r_t^{Ab} H^{Ab} + \Lambda^{Ab}(W_t, Y_{1t}, Y_{2t}, t)\,. \qquad (4.21)
\end{aligned}$$

Note that the expression within brackets associated with $\frac{\partial H^{Ab}}{\partial Y_2}$ and r_t^{Ab} in (4.21) are consistent with (4.19) and (4.20), respectively (Appendix 4.5.5).

Comparing the PDE's (4.8) and (4.21), it is clear that ambiguity aversion changes the fundamental pricing equation through two preference-dependent inputs: the equilibrium instantaneous interest rate and the equilibrium price of uncertainty associated with Y_{2t}.

4.3 An Example of Investment Opportunity Set

In the previous two sections, we developed, for a general investment opportunity set with endogenous production driven by two correlated state variables, an intertemporal equilibrium setting for contingent claim pricing following Cox et al. (1985a). In section 4.2.1, uncertainty was entirely risk while in section 4.2.2 uncertainty included an ambiguity dimension. A key issue of our investment set is that state variables are correlated but only one of them is correlated with shocks in the economy's output rate.

In this section, as an example, we apply results of the previous two sections considering an investment opportunity set where Y_{1t} is an economic variable and Y_{2t} represents the variance of changes in Y_{1t}, with both variables being correlated. It is assumed that only Y_{2t}

(everything else constant). This is also the result under the general equilibrium model with ambiguity in Trojani and Vanini (2004).

has shocks correlated with those of economy's output rate process and the representative agent is ambiguous about the stochastic process of Y_{2t}.

The return of economy's production process (output rate) is assumed to be given by:

$$\frac{dQ_t}{Q_t} = g_Q(Y_{1t}, Y_{2t})dt + l\sqrt{Y_{2t}}dZ_Q, \tag{4.22}$$

where the drift function, g_Q, is, for now, still unspecified and $l > 0$.

It is assumed that Y_{1t} follows the geometric Brownian motion:

$$dY_{1t} = \mu Y_{1t}dt + Y_{1t}\sqrt{Y_{2t}}dZ_1, \tag{4.23}$$

where μ is Y_{1t}'s expected change, Y_{2t} is the instantaneous variance of Y_{1t}'s change and it is assumed that $dZ_Q dZ_1 = 0$.

Regarding the ambiguous state variable Y_{2t}, the "reference belief" dynamics for the representative agent is assumed to be given by the mean reverting square-root process (as used, for e.g., in Cox et al. (1985b)):

$$dY_{2t} = \kappa(\theta - Y_{2t})dt + \varepsilon\sqrt{Y_{2t}}dZ_2, \tag{4.24}$$

where θ is the expected value of Y_{2t}, $\kappa > 0$ is the mean reverting parameter and $\varepsilon > 0$. It is assumed that $dZ_1 dZ_2 = \rho dt$ and $dZ_Q dZ_2 = \rho^* dt$.

From the previous two sections, it is straightforward to obtain the specification of ϕ_2, the equilibrium market price of uncertainty associated with the variable Y_{2t}, as well as the equilibrium instantaneous interest rate, r_t^{Ab}. By recognizing that under this concrete setting:

$$\begin{aligned} \sigma_{Y_2}(Y_{1t}, Y_{2t}) &= \varepsilon\sqrt{Y_{2t}}, \\ \sigma_Q(Y_{1t}, Y_{2t}) &= l\sqrt{Y_{2t}}, \end{aligned}$$

then, from (4.19), one obtains:

$$\begin{aligned} \phi_2 &= \varepsilon\sqrt{Y_{2t}}l\sqrt{Y_{2t}}\rho^* \pm \varepsilon\sqrt{Y_{2t}}\sqrt{2\eta}, \\ \Leftrightarrow \phi_2 &= \underbrace{\varepsilon l\rho^* Y_{2t}}_{risk\,price}\underbrace{\pm\varepsilon\sqrt{2\eta Y_{2t}}}_{ambiguity\,price}, \end{aligned}$$

which, by defining $\lambda_1 = \varepsilon l\rho^*$ and $\lambda_2 = \pm\varepsilon\sqrt{2\eta}$, is simplified to:

$$\phi_2 = \lambda_1 Y_{2t} + \lambda_2\sqrt{Y_{2t}}. \tag{4.25}$$

The equilibrium market price of uncertainty associated with the variance of changes in Y_{1t} has two components: variance risk price, which is linear on the instantaneous level of variance, Y_{2t}, and variance ambiguity price, which is nonlinear on Y_{2t}. Variance risk price depends on: (i) the parameter l of the diffusion function in the stochastic process describing the economy's output rate; (ii) on the parameter ε, representing the standard deviation of Y_{2t}; and (iii) on the correlation ρ^* between shocks in dY_{2t} and in the output rate. The variance risk price is positive (negative) when $\rho^* > 0$ ($\rho^* < 0$). The variance ambiguity price depends on ε and on the degree of ambiguity η faced by the representative agent, and can also be either positive or negative.

Note that the specification for the state variables dynamics (4.23) and (4.24) is the one that is used in Heston's (1993) stochastic volatility option price model. In Heston (1993), Y_{1t} represents the option's underlying asset spot price and, consequently, Y_{2t} is the variance of the underlying asset return, with both being correlated. The specification for the market price of variance risk used in Heston (1993), where uncertainty is exclusively risk, is a scalar multiplied by the instantaneous level of variance. This is consistent with our findings under the developed equilibrium approach: it corresponds to the $\lambda_1 Y_{2t}$ component in (4.25). We have therefore provided an equilibrium motivation for the price specification of variance risk in Heston's (1993) model. We also conclude that a potential extension of Heston's (1993) model by incorporating ambiguity aversion about the stochastic variance process of the underlying asset return could use (4.25) as the specification for the variance uncertainty price, with the sign of the ambiguity component depending on the concrete calibration to be used.

Moreover, from (4.20), it is straightforward to obtain the expression for the equilibrium interest rate:

$$r_t^{Ab} = g_Q - l^2 Y_{2t} \pm l\rho^* \sqrt{2\eta}\sqrt{Y_{2t}}\,, \tag{4.26}$$

where the new component emerging from the ambiguity consideration is $\pm l\rho^* \sqrt{2\eta}\sqrt{Y_{2t}}$.

In order to study the sign of the ambiguity components in both (4.25) and (4.26), we must specify the output rate drift function g_Q and, subsequently, solve the corresponding Bellman equation (4.17). This is done in the next subsection.

4.3.1 A Particular Solution

Assuming $g_Q(Y_{1t}, Y_{2t}) = lnY_{1t} + \alpha Y_{2t}$, where α is a scalar parameter, the output rate process (4.22) is given by:

$$\frac{dQ_t}{Q_t} = (lnY_{1t} + \alpha Y_{2t})dt + l\sqrt{Y_{2t}}dZ_Q\,. \tag{4.27}$$

Considering this concrete specification for the output rate process and the processes (4.23) and (4.24) for the state variables, we start solving the corresponding Bellman equation (4.17) given by (Appendix 4.5.6.1):

$$\begin{aligned}\delta V &= \mu V_{Y_1} Y_{1t} + \kappa(\theta - Y_{2t}) V_{Y_2} + \frac{1}{2} Y_{1t}^2 Y_{2t} V_{Y_1 Y_1} + \frac{1}{2}\varepsilon^2 Y_{2t} V_{Y_2 Y_2} + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t} \\ &\quad - \sqrt{2\eta\left(l^2 Y_{2t} + Y_{1t}^2 Y_{2t} V_{Y_1}^2 + \varepsilon^2 Y_{2t} V_{Y_2}^2 + 2\varepsilon l \rho^* Y_{2t} V_{Y_2}\right)}\,. \end{aligned} \tag{4.28}$$

We obtain a solution that is exact when there is no ambiguity and approximate in the presence of ambiguity (Appendix 4.5.6.2):

$$\begin{aligned} V(Y_{1t}, Y_{2t}) &= alnY_{1t} + bY_{2t} + c\,, \\ &\quad with \\ a &= \frac{1}{\delta}\,, \\ b &= \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{(\kappa + \delta)}\,, \\ c &= \frac{\mu}{\delta^2} + \frac{\kappa\theta}{\delta} b\,. \end{aligned} \tag{4.29}$$

The value function (4.29) is an approximate solution of (4.28) under the domain $0 < \eta < \Psi$ (with Ψ being an arbitrarily small positive number).[12] The major reason we ended with an approximate solution is that it is difficult to obtain an exact solution for (4.28) under ambiguity ($\eta > 0$). We suspect that, if that solution exists, it is not separable in the state variables, making it difficult to study its gradient regarding Y_{2t}, and probably could be solved only through numerical procedures. Moreover, there is a reason to believe that the accuracy of the approximation is reasonable: the domain $0 < \eta < \Psi$ must be very tight, for the reasoning previously invoked that alternative models must be statistically close to the "reference belief" model, so that the representative agent has difficulty to distinguish

[12] It is immediate to conclude that the value function (4.15) with the specification (4.29) for $V(Y_{1t}, Y_{2t})$ is consistent with the obtained value function when uncertainty is exclusively risk ($\eta = 0$) in (4.7), by making $G(Y_{1t}, Y_{2t}) = \frac{1}{\delta}(alnY_{1t} + bY_{2t} + c + ln\delta - 1)$.

them and therefore faces ambiguity.

This asymptotic methodology to find a solution of the problem is intuitively close to the perturbation theory used in Trojani and Vanini (2004) to solve intertemporal general equilibrium models under ambiguity. The rationale is provide by the authors (pp.291) "*the basic idea of asymptotic methods is to formulate a general problem, find a particular relevant case that has a known solution, and use this as a starting point for computing the solution to nearby problems*". As in our case, in Trojani and Vanini (2004), the asymptotic solutions of the problems under ambiguity "... *hold for neighborhoods of a model with log utility of consumption and no ambiguity aversion*".

From (4.29) it is immediate that $V_{Y_2} = b$, and the expression for the equilibrium price of uncertainty associated with Y_2 (4.25) can be clarified (Appendix 4.5.6.3):

$$\phi_2 \;=\; \lambda_1 Y_{2t} + \lambda_2 \sqrt{Y_{2t}}\,,\; with\;\; \lambda_2 > 0\;(<0)\;\;\; if\;\;\; \alpha > \omega\;(<\omega)\,, \tag{4.30}$$

with the threshold value ω being given by:

$$\omega \;=\; \frac{l^2\delta\varepsilon + \varepsilon - 2\delta\,(\kappa+\delta)\,l\rho^*}{2\delta\varepsilon}\,. \tag{4.31}$$

The equilibrium uncertainty price associated with Y_{2t} can therefore increase or decrease when ambiguity about its process is considered. That depends on the relative magnitude of the parameter α, which measures the sensitivity of the expected output rate of the economy relatively to changes on Y_{2t}, versus a benchmark value that synthesizes some information of the investment opportunity set (parameters l, ε, κ and ρ^*) and the subjective rate of discount of the representative agent, δ. If $\alpha > \omega$, ambiguity about the stochastic process of Y_{2t} increases its equilibrium uncertainty price, and the contrary when $\alpha < \omega$.

Regarding the expression for the equilibrium instantaneous interest rate, r_t^{Ab}, under this concrete setting, from (4.20) it is given by (Appendix 4.5.6.3):

$$r_t^{Ab} \;=\; \begin{cases} (lnY_{1t} + \alpha Y_{2t}) - l^2 Y_{2t} - l\rho^*\sqrt{2\eta}\sqrt{Y_{2t}} & if\;\; \alpha > \omega \\ \\ (lnY_{1t} + \alpha Y_{2t}) - l^2 Y_{2t} + l\rho^*\sqrt{2\eta}\sqrt{Y_{2t}} & if\;\; \alpha < \omega \end{cases}\,. \tag{4.32}$$

From (4.32), one concludes that if there exists a negative correlation between shocks in the ambiguous state variable and the economy's output rate ($\rho^* < 0$), the impact on the equilibrium interest rate from ambiguity has the same direction (increase or decrease) as on the equilibrium uncertainty price. The contrary happens when $\rho^* > 0$.

Overall, the effects on ϕ_2 and r_t^{Ab} from the consideration of ambiguity about the stochastic process of Y_{2t} are summarized in Table 4.1:

Table 4.1: Impact on ϕ_2 and r_t^{Ab} from ambiguity about the stochastic process of Y_{2t}

	$\alpha > \omega$		$\alpha < \omega$	
	$\rho^* < 0$	$\rho^* > 0$	$\rho^* < 0$	$\rho^* > 0$
ϕ_2	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$
r_t^{Ab}	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$

Note: The sign $\uparrow$ ($\downarrow$) indicates that ambiguity about Y_{2t} stochastic process increases (decreases) the considered channel: ϕ_2 and r_t^{Ab}.

4.4 Concluding Remarks

We developed a general intertemporal equilibrium setting for asset pricing using a two state variable version of Cox et al. (1985a) model. All the physical investment is delivered by a single stochastic production process whose realized return (economy's output rate) is driven by the state variables Y_{1t} and Y_{2t}. It is assumed that both state variables impact the economy's expected output rate but only one of the two state variables (Y_{2t}) has shocks correlated with those of output rate. A key assumption in our setting is that the state variables are correlated, which we believe to be quite useful for modeling economic problems, particularly regarding asset pricing.

Assuming that the representative agent has a logarithmic utility function, two scenarios regarding the uncertainty he faces about the investment opportunity set are analyzed. First, it is assumed that uncertainty is exclusively risk, i.e., the agent is totally sure about the probability measure P under which the investment opportunity evolves. We find out that the equilibrium market price of risk associated with Y_{2t}, the only state variable whose changes should be hedged as it is the only with shocks correlated with those of economy's output rate, depends on the level and sign of that correlation and on the diffusion function of the stochastic processes of economy's output rate and of Y_{2t}. The equilibrium interest rate is found to be given by the difference between the expected output rate of the economy and the variance of the output rate. We show that Longstaff and Schwartz's (1992) general equilibrium model for the interest rate term structure can be obtained as a particular case of our setting.

In a second step, under the same general setting, ambiguity aversion is introduced. The source of ambiguity is the stochastic process of the state variable Y_{2t}. The representative agent considers contaminations around his reference belief and aversion towards ambiguity is introduced by assuming that, under the spirit of Gilboa and Schmeidler (1989), the representative agent chooses from all the contaminations the one associated with lower

expected utility. Ambiguity aversion changes the fundamental pricing equation satisfied by the contingent claim price through two inputs, that become preference-dependent: the equilibrium instantaneous interest rate and the equilibrium price of uncertainty associated with Y_{2t}.

We conclude that when ambiguity is considered, the equilibrium price of uncertainty and equilibrium interest rate can increase or decrease, depending on the sign of the correlation between shocks in the ambiguous state variable and in the output rate and on the relation between (i) that correlation, (ii) the diffusion functions of the stochastic processes for the ambiguous state variable and for the output rate and (iii) the gradient of the value function with respect to the ambiguous state variable.

As an example, we apply the obtained general results to a specific investment opportunity set, where Y_{2t} is the instantaneous variance of the change of Y_{1t}, assuming that both are correlated. This is the investment opportunity set of the well known option pricing model of Heston (1993), by letting Y_{1t} to be the option's underlying asset price. The obtained equilibrium market price of variance risk is linear on its instantaneous level, which in fact is the specification used by Heston (1993): we therefore provide an equilibrium motivation for the price specification of variance risk used in Heston (1993) model. Conversely, the obtained equilibrium market price of ambiguity about stochastic variance is nonlinear on its instantaneous level. Through this example, we have therefore obtained a specification of the equilibrium variance uncertainty price that can be used in an extension of Heston (1993) model accommodating ambiguity aversion about the stochastic variance process of the option's underlying asset return. This extension is made in Faria and Correia-da Silva (2010a).

Appendix

4.5 Mathematical Proofs

4.5.1 Correlation and Contamination Structures

Suppose that the Brownian motions Z_Q, Z_1, Z_2 and Z_3 (an instrumental Brownian motion, assumed to be independent of Z_2, useful in section 4.2.2) are a linear combination of four independent Brownian motions, X_1, X_2, X_3 and X_4:

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_Q \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}. \tag{4.33}$$

In the following, we deduct conditions that matrix A has to fulfill in order to induce the desired correlation structure between Z_1, Z_2, Z_3 and Z_Q as well as the contamination structure in section 4.2.2.

Since $(dX_i)^2 = dt$ (Kamien and Schwartz (1991) ch. 22) and $dX_i dX_j = 0$ for $j \neq i$ (independent Brownian Motions), the instantaneous correlation between dZ_1, dZ_2, dZ_3 and dZ_Q is given by:

$$\begin{aligned} dZ_1 dZ_2 &= (a_{11}dX_1 + a_{12}dX_2 + a_{13}dX_3 + a_{14}dX_4)(a_{21}dX_1 + a_{22}dX_2 + a_{23}dX_3 + a_{24}dX_4) \\ &= (a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} + a_{14}a_{24})\,dt\,, \end{aligned}$$

$$\begin{aligned} dZ_1 dZ_3 &= (a_{11}dX_1 + a_{12}dX_2 + a_{13}dX_3 + a_{14}dX_4)(a_{31}dX_1 + a_{32}dX_2 + a_{33}dX_3 + a_{34}dX_4) \\ &= (a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} + a_{14}a_{34})\,dt\,, \end{aligned}$$

$$\begin{aligned} dZ_1 dZ_Q &= (a_{11}dX_1 + a_{12}dX_2 + a_{13}dX_3 + a_{14}dX_4)(a_{41}dX_1 + a_{42}dX_2 + a_{43}dX_3 + a_{44}dX_4) \\ &= (a_{11}a_{41} + a_{12}a_{42} + a_{13}a_{43} + a_{14}a_{44})\,dt\,, \end{aligned}$$

$$
\begin{aligned}
dZ_2dZ_3 &= (a_{21}dX_1 + a_{22}dX_2 + a_{23}dX_3 + a_{24}dX_4)(a_{31}dX_1 + a_{32}dX_2 + a_{33}dX_3 + a_{34}dX_4) \\
&= (a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} + a_{24}a_{34})\,dt\,,
\end{aligned}
$$

$$
\begin{aligned}
dZ_2dZ_Q &= (a_{21}dX_1 + a_{22}dX_2 + a_{23}dX_3 + a_{24}dX_4)(a_{41}dX_1 + a_{42}dX_2 + a_{43}dX_3 + a_{44}dX_4) \\
&= (a_{21}a_{41} + a_{22}a_{42} + a_{23}a_{43} + a_{24}a_{44})\,dt\,,
\end{aligned}
$$

$$
\begin{aligned}
dZ_3dZ_Q &= (a_{31}dX_1 + a_{32}dX_2 + a_{33}dX_3 + a_{34}dX_4)(a_{41}dX_1 + a_{42}dX_2 + a_{43}dX_3 + a_{44}dX_4) \\
&= (a_{31}a_{41} + a_{32}a_{42} + a_{33}a_{43} + a_{34}a_{44})\,dt\,,
\end{aligned}
$$

The general conditions for having $dZ_1dZ_Q = 0$, $dZ_1dZ_2 = \rho dt$, $dZ_2dZ_3 = 0$ and $dZ_2dZ_Q = \rho^* dt$ therefore imply:

$$
\begin{aligned}
\sum_{i=1}^{4} a_{1i}a_{2i} &= \rho\,, \\
\sum_{i=1}^{4} a_{1i}a_{3i} &= \frac{-\rho\rho^*}{\sqrt{1-\rho^{*2}}}\,, \\
\sum_{i=1}^{4} a_{1i}a_{4i} &= 0\,, \\
\sum_{i=1}^{4} a_{2i}a_{3i} &= 0\,, \\
\sum_{i=1}^{4} a_{2i}a_{4i} &= \rho^*\,, \\
\sum_{i=1}^{4} a_{3i}a_{4i} &= \sqrt{1-\rho^{*2}}\,.
\end{aligned}
\tag{4.34}
$$

The condition that instantaneous correlation between Z_1 and Z_3 is given by $\frac{-\rho\rho^*}{\sqrt{1-\rho^{*2}}}dt$ results from the fact that, otherwise, Z_1 and Z_Q would have a non-zero correlation. Consequently, it can be written $dZ_Q = \rho^* dZ_2 + \sqrt{1-\rho^{*2}}dZ_3$.

In order to guarantee the standard variance of Brownian motions Z_1, Z_2, Z_3 and Z_Q another set of conditions on A is imposed by the correlations structure of our setting:

$$
\sum_{i=1}^{4} (a_{1i})^2 = \sum_{i=1}^{4} (a_{2i})^2 = \sum_{i=1}^{4} (a_{3i})^2 = \sum_{i=1}^{4} (a_{4i})^2 = 1. \tag{4.35}
$$

The introduction of contaminations in the system (4.33) is represented by:

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_Q \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} X_1 + h_1 \\ X_2 + h_2 \\ X_3 + h_3 \\ X_4 + h_4 \end{bmatrix},$$

where h_j, $j = 1,2,3,4$ represent instrumental contaminations drift processes. Considering that in our setting, contamination is exclusively about Y_{2t} process, results:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} 0 \\ h \\ 0 \\ 0 \end{bmatrix},$$

where h is the contaminating drift process in (4.12). The relationship between h_j, $j = 1,2,3,4$ and h is given by:

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ h \\ 0 \\ 0 \end{bmatrix}, \tag{4.36}$$

which implies that matrix A is invertible, i.e.:

$$|\, A \,| \neq 0, \tag{4.37}$$

where $|\, A \,|$ denotes the determinant of matrix A.

Regarding the correlation structure in (4.34), nothing changes with the introduction of contamination over Y_{2t} process (as expected from Cameron-Martin-Girsanov theorem),

as:

$$
\begin{aligned}
dZ_1\left(dZ_2+hdt\right) &= dZ_1dZ_2+dZ_1hdt \\
\Leftrightarrow dZ_1\left(dZ_2+hdt\right) &= dZ_1dZ_2 \\
\Leftrightarrow dZ_1\left(dZ_2+hdt\right) &= \rho dt\,,
\end{aligned}
$$

$$
\begin{aligned}
dZ_3\left(dZ_2+hdt\right) &= dZ_3dZ_2+dZ_3hdt \\
\Leftrightarrow dZ_3\left(dZ_2+hdt\right) &= dZ_3dZ_2 \\
\Leftrightarrow dZ_3\left(dZ_2+hdt\right) &= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
dZ_Q\left(dZ_2+hdt\right) &= dZ_QdZ_2+dZ_Qhdt \\
\Leftrightarrow dZ_Q\left(dZ_2+hdt\right) &= dZ_QdZ_2 \\
\Leftrightarrow dZ_Q\left(dZ_2+hdt\right) &= \rho^* dt\,,
\end{aligned}
$$

as $dZ_1dt = dZ_3dt = dZ_Qdt = 0$ (Kamien and Schwartz (1991) ch. 22).

Overall, the required conditions on matrix A in order to have the structure of correlations and contaminations of our setting are given by (4.34), (4.35), (4.36) and (4.37). It is immediate to conclude that there are different combinations for the elements of matrix A that satisfy these conditions.

□

4.5.2 Bellman Equation (4.6) and Optimal Consumption Rule

We start with the general expression of the problem, except that the utility program in (4.5) is set to the instantaneous time t. With the corresponding value function denoted by $\overline{V}(t,W_t,Y_{1t},Y_{2t})$, the general expression for this non-autonomous problem is given by:

$$
\begin{aligned}
\overline{V}\left(t,W_t,Y_{1t},Y_{2t}\right) &= \sup_{C}\Big\{ ln\left(C_t\right)dt + \\
&\qquad \overline{V}\left(t+dt,W_t+dW_t,Y_{1t}+dY_{1t},Y_{2t}+dY_{2t}\right)\Big\}. \qquad (4.38)
\end{aligned}
$$

For a more parsimonious presentation, we drop out the arguments of the functions g_i, σ_i $(i = Y_1, Y_2, W, Q)$ and of $\overline{V}(t, W_t, Y_{1t}, Y_{2t})$ and, from Itô Theorem, one obtains:

$$\begin{aligned}
\overline{V}(t+dt, W_t+dW_t, Y_{1t}+dY_{1t}, Y_{2t}+dY_{2t}) &= \overline{V} + \frac{\partial \overline{V}}{\partial t}dt + \frac{\partial \overline{V}}{\partial W}[g_W dt + \sigma_W dZ_Q] \\
&+ \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial W^2}\sigma_W^2 dt + \frac{\partial \overline{V}}{\partial Y_1}[g_{Y_1}dt + \sigma_{Y_1}dZ_1] \\
&+ \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_1^2}\sigma_{Y_1}^2 dt + \frac{\partial \overline{V}}{\partial Y_2}[g_{Y_2}dt + \sigma_{Y_2}dZ_2] \\
&+ \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_2^2}\sigma_{Y_2}^2 dt + \frac{\partial^2 \overline{V}}{\partial W \partial Y_1}\sigma_W \sigma_{Y_1} dZ_Q dZ_1 \\
&+ \frac{\partial^2 \overline{V}}{\partial W \partial Y_2}\sigma_W \sigma_{Y_2} dZ_Q dZ_2 \\
&+ \frac{\partial^2 \overline{V}}{\partial Y_1 \partial Y_2}\sigma_{Y_1}\sigma_{Y_2} dZ_1 dZ_2 ,
\end{aligned}$$

which, from Appendix 4.5.1, equals:

$$\begin{aligned}
\overline{V}(t+dt, W_t+dW_t, Y_{1t}+dY_{1t}, Y_{2t}+dY_{2t}) &= \overline{V} + \frac{\partial \overline{V}}{\partial t}dt + \frac{\partial \overline{V}}{\partial W}g_W dt + \frac{\partial \overline{V}}{\partial W}\sigma_W dZ_Q \\
&+ \frac{\partial \overline{V}}{\partial Y_1}g_{Y_1}dt + \frac{\partial \overline{V}}{\partial Y_1}\sigma_{Y_1}dZ_1 + \frac{\partial \overline{V}}{\partial Y_2}g_{Y_2}dt \\
&+ \frac{\partial \overline{V}}{\partial Y_2}\sigma_{Y_2}dZ_2 + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial W^2}\sigma_W^2 dt + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_1^2}\sigma_{Y_1}^2 dt \\
&+ \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_2^2}\sigma_{Y_2}^2 dt + \frac{\partial^2 \overline{V}}{\partial W \partial Y_2}\sigma_W \sigma_{Y_2}\rho^* dt \\
&+ \frac{\partial^2 \overline{V}}{\partial Y_1 \partial Y_2}\sigma_{Y_1}\sigma_{Y_2}\rho dt .
\end{aligned}$$

Substituting this result into (4.38), it becomes:

$$\begin{aligned}
\overline{V} &= \sup_C \Bigg\{ ln(C_t)dt + \overline{V} + \frac{\partial \overline{V}}{\partial t}dt + \frac{\partial \overline{V}}{\partial W}g_W dt + \frac{\partial \overline{V}}{\partial W}\sigma_W dZ_Q + \frac{\partial \overline{V}}{\partial Y_1}g_{Y_1}dt + \frac{\partial \overline{V}}{\partial Y_1}\sigma_{Y_1}dZ_1 \\
&+ \frac{\partial \overline{V}}{\partial Y_2}g_{Y_2}dt + \frac{\partial \overline{V}}{\partial Y_2}\sigma_{Y_2}dZ_2 + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial W^2}\sigma_W^2 dt + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_1^2}\sigma_{Y_1}^2 dt + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial Y_2^2}\sigma_{Y_2}^2 dt \\
&+ \frac{\partial^2 \overline{V}}{\partial W \partial Y_2}\sigma_W \sigma_{Y_2}\rho^* dt + \frac{\partial^2 \overline{V}}{\partial Y_1 \partial Y_2}\sigma_{Y_1}\sigma_{Y_2}\rho dt \Bigg\} .
\end{aligned}$$

Taking expectations on both sides of this expression (recall that $E(dZ_Q) = E(dZ_1) = E(dZ_2) = 0$), and dividing by dt, we obtain:[13]

$$\begin{aligned} 0 &= \sup_C \left\{ ln(C_t) + \frac{\partial \overline{V}}{\partial t} + \frac{\partial \overline{V}}{\partial W} g_W + \frac{\partial \overline{V}}{\partial Y_1} g_{Y_1} + \frac{\partial \overline{V}}{\partial Y_2} g_{Y_2} + \frac{1}{2} \frac{\partial^2 \overline{V}}{\partial W^2} \sigma_W^2 + \frac{1}{2} \frac{\partial^2 \overline{V}}{\partial Y_1^2} \sigma_{Y_1}^2 \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 \overline{V}}{\partial Y_2^2} \sigma_{Y_2}^2 + \frac{\partial^2 \overline{V}}{\partial W \partial Y_2} \sigma_W \sigma_{Y_2} \rho^* + \frac{\partial^2 \overline{V}}{\partial Y_1 \partial Y_2} \sigma_{Y_1} \sigma_{Y_2} \rho \right\}. \end{aligned} \tag{4.39}$$

Following Kamien and Schwartz (1991) ch.22, in order to obtain the Bellman equation for the corresponding autonomous problem, we use the relationship:

$$\overline{V}(W_t, Y_{1t}, Y_{2t}, t) = e^{-\delta t} J(W_t, Y_{1t}, Y_{2t}), \tag{4.40}$$

where $J(W_t, Y_{1t}, Y_{2t})$ and $\delta > 0$ are the corresponding autonomous value function and the subjective rate of discount. From (4.40), at $t = 0$ (as in (4.5)):

$$\begin{aligned} \frac{\partial \overline{V}}{\partial t} &= -\delta e^{-\delta t} J = -\delta J, \\ \frac{\partial \overline{V}}{\partial i} &= e^{-\delta t} \frac{\partial J}{\partial i} = \frac{\partial J}{\partial i}, (i = Y_1, Y_2, W), \\ \frac{\partial^2 \overline{V}}{\partial j \partial i} &= e^{-\delta t} \frac{\partial^2 J}{\partial j \partial i} = \frac{\partial^2 J}{\partial j \partial i}, (i, j = Y_1, Y_2, W). \end{aligned}$$

Substituting in (4.39), we obtain:

$$\begin{aligned} 0 &= \sup_C \left\{ ln(C_t) - \delta J + \frac{\partial J}{\partial W} g_W + \frac{\partial J}{\partial Y_1} g_{Y_1} + \frac{\partial J}{\partial Y_2} g_{Y_2} + \frac{1}{2} \frac{\partial^2 J}{\partial W^2} \sigma_W^2 + \frac{1}{2} \frac{\partial^2 J}{\partial Y_1^2} \sigma_{Y_1}^2 \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 J}{\partial Y_2^2} \sigma_{Y_2}^2 + \frac{\partial^2 J}{\partial W \partial Y_2} \sigma_W \sigma_{Y_2} \rho^* + \frac{\partial^2 J}{\partial Y_1 \partial Y_2} \sigma_{Y_1} \sigma_{Y_2} \rho \right\}, \end{aligned}$$

which is the autonomous Bellman equation (4.6).

Regarding the optimal consumption policy, observe that only the first and third terms in the right hand side of (4.6) depend on the instantaneous consumption, C_t. Considering the expression for the value function (4.7) and for $g_W = (W_t g_Q - C_t)$ previously obtained, the

[13] This differential generator $\frac{E[d\overline{V}]}{dt}$ is commonly referred in the literature as the differential expectations operator or Dynkin generator.

first order condition for the optimal consumption rule is:

$$\begin{aligned} 0 &= \frac{1}{C_t} - \frac{\partial J}{\partial W} \\ \Leftrightarrow 0 &= \frac{1}{C_t} - \frac{1}{\delta W_t} \\ \Leftrightarrow C_t &= \delta W_t. \end{aligned}$$

It is immediate that the second order condition is also verified and, consequently, $C_t^* = \delta W_t$, as we wanted to prove.

□

4.5.3 Equations (4.8), (4.9) and Expression (4.10)

As in the Appendix 4.5.2 we drop out the arguments of the functions g_i, σ_i $(i = Y_1, Y_2, W, Q)$ as well as in $J(W_t, Y_{1t}, Y_{2t})$ and $\Lambda(W_t, Y_{1t}, Y_{2t}, t)$. We start by deducting the equilibrium wealth dynamics. Substituting the optimal consumption rule into (4.4), the equilibrium wealth dynamics is obtained:

$$dW_t = W_t(g_Q - \delta)\,dt + W_t \sigma_Q dZ_Q.$$

In order to obtain the partial differential equation (PDE) satisfied by the price of the contingent claim, $H(Y_{1t}, Y_{2t}, \tau)$, we start from Theorem 3 in Cox et al. (1985a):

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2} Var(W_t) \frac{\partial^2 H}{\partial W^2} + Cov(Y_{1t}, W_t) \frac{\partial^2 H}{\partial W \partial Y_1} + Cov(Y_{2t}, W_t) \frac{\partial^2 H}{\partial W \partial Y_2} \\ &+ \frac{1}{2} Var(Y_{1t}) \frac{\partial^2 H}{\partial Y_1^2} + \frac{1}{2} Var(Y_{2t}) \frac{\partial^2 H}{\partial Y_2^2} + \frac{1}{2} Cov(Y_{2t}, Y_{1t}) \frac{\partial^2 H}{\partial Y_2 \partial Y_1} \\ &+ \frac{1}{2} Cov(Y_{1t}, Y_{2t}) \frac{\partial^2 H}{\partial Y_1 \partial Y_2} + (r_t W_t - \delta W_t) \frac{\partial H}{\partial W} \\ &+ \left[g_{Y_1} - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov(W_t, Y_{1t}) - \left(\frac{-\frac{\partial^2 J}{\partial W \partial Y_2}}{\frac{\partial J}{\partial W}} \right) Cov(Y_{1t}, Y_{2t}) \right] \frac{\partial H}{\partial Y_1} \\ &+ \left[g_{Y_2} - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov(W_t, Y_{2t}) - \left(\frac{-\frac{\partial^2 J}{\partial W \partial Y_1}}{\frac{\partial J}{\partial W}} \right) Cov(Y_{2t}, Y_{1t}) \right] \frac{\partial H}{\partial Y_2} - r_t H + \Lambda, \end{aligned}$$

where $Var(\cdot)$ and $Cov(\cdot)$ represent the instantaneous variance and covariance, respectively.

As the price of the contingent claim, $H(Y_{1t}, Y_{2t}, \tau)$, does not depend on wealth, we have

$\frac{\partial H}{\partial W} = \frac{\partial^2 H}{\partial W^2} = \frac{\partial^2 H}{\partial W \partial Y_1} = \frac{\partial^2 H}{\partial W \partial Y_2} = 0$. Additionally, from (4.7), we have $\frac{\partial J}{\partial W} = \frac{1}{\delta W_t}$, implying that $\frac{\partial^2 J}{\partial W \partial Y_1} = \frac{\partial^2 J}{\partial W \partial Y_2} = 0$. Therefore, the PDE becomes:

$$\begin{aligned}
\frac{\partial H}{\partial \tau} &= \frac{1}{2} Var\left(Y_{1t}\right) \frac{\partial^2 H}{\partial Y_1^2} + \frac{1}{2} Var\left(Y_{2t}\right) \frac{\partial^2 H}{\partial Y_2^2} + Cov\left(Y_{1t}, Y_{2t}\right) \frac{\partial^2 H}{\partial Y_1 \partial Y_2} \\
&\quad + \left[g_{Y_1} - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov\left(W_t, Y_{1t}\right) \right] \frac{\partial H}{\partial Y_1} \\
&\quad + \left[g_{Y_2} - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov\left(W_t, Y_{2t}\right) \right] \frac{\partial H}{\partial Y_2} - r_t H + \Lambda.
\end{aligned}$$

The instantaneous correlation between any two variables, k_1 and k_2, is given by $\rho_{k_1 k_2} = \frac{Cov(k_1, k_2)}{\sigma_{k_1} \sigma_{k_2}}$, where $Cov(k_1, k_2)$ is the instantaneous covariance between the two variables and $\sigma_{k_1} \sigma_{k_2}$ represents the product of their standard deviations. In our setting, as $dZ_Q dZ_1 = 0$, we have $Cov\left(W_t, Y_{1t}\right) = 0$, because the Brownian Motion in (4.4) is Z_Q. Using this result and assumptions in Appendix 4.5.1, obtained PDE can be further simplified to:

$$\begin{aligned}
\frac{\partial H}{\partial \tau} &= \frac{1}{2} \sigma_{Y_1}^2 \frac{\partial^2 H}{\partial Y_1^2} + \frac{1}{2} \sigma_{Y_2}^2 \frac{\partial^2 H}{\partial Y_2^2} + \rho \sigma_{Y_1} \sigma_{Y_2} \frac{\partial^2 H}{\partial Y_1 \partial Y_2} + g_{Y_1} \frac{\partial H}{\partial Y_1} \\
&\quad + \left[g_{Y_2} - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov\left(W_t, Y_{2t}\right) \right] \frac{\partial H}{\partial Y_2} - r_t H + \Lambda,
\end{aligned}$$

which is (4.8).

□

4.5.3.1 Equilibrium Instantaneous Interest Rate r_t

From Theorem 1 in Cox et al. (1985a), one obtains:

$$\begin{aligned}
r_t &= g_Q - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) \left(\frac{Var\left(W_t\right)}{W_t} \right) - \left(\frac{-\frac{\partial^2 J}{\partial W \partial Y_1}}{\frac{\partial J}{\partial W}} \right) \left(\frac{Cov\left(W_t, Y_{1t}\right)}{W_t} \right) \\
&\quad - \left(\frac{-\frac{\partial^2 J}{\partial W \partial Y_2}}{\frac{\partial J}{\partial W}} \right) \left(\frac{Cov\left(W_t, Y_{2t}\right)}{W_t} \right).
\end{aligned}$$

From (4.7), we know that $\frac{\partial^2 J}{\partial W \partial Y_1} = \frac{\partial^2 J}{\partial W \partial Y_2} = 0$ and that:

$$\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} = \frac{\frac{1}{\delta W_t^2}}{\frac{1}{\delta W_t}} = \frac{1}{W_t}, \tag{4.41}$$

thus, the expression for the equilibrium interest rate becomes:

$$r_t = g_Q - \frac{1}{W_t}\left(\frac{Var(W_t)}{W_t}\right).$$

Going back to (4.4), $Var(W_t) = W_t^2 \sigma_Q^2$, which, substituting above, gives:

$$r_t = g_Q - \sigma_Q^2,$$

which is (4.9).

□

4.5.3.2 Equilibrium Price of Risk associated with Y_{2t}

The utility dependent term associated to $\frac{\partial H}{\partial Y_2}$ in (4.8) represents the equilibrium market price of risk associated with Y_{2t}. In order to obtain expression (4.10), we start by pointing out that $dZ_Q dZ_2 = \rho^* dt \Rightarrow Cov(W_t, Y_{2t}) = \rho^* \sigma_W \sigma_{Y_2}$. Considering (4.41) and the fact that $\sigma_W = W_t \sigma_Q$, from (4.4), one gets:

$$\left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\right) Cov(W_t, Y_{2t}) = \frac{1}{W_t}\rho^* \sigma_W \sigma_{Y_2} = \sigma_{Y_2}\sigma_Q \rho^*,$$

which is (4.10).

□

4.5.4 Longstaff and Schwartz (1992) Model: a Particular Case

The model of Longstaff and Schwartz (1992) can be obtained as a particular case of our setting. Start by assuming:

$$g_Q = (zY_{1t} + uY_{2t}) \ \ and \ \ \sigma_Q = \nu\sqrt{Y_{2t}},$$
$$g_{Y_1} = (a - bY_{1t}) \ \ and \ \ \sigma_{Y_1} = c\sqrt{Y_{1t}},$$
$$g_{Y_2} = (d - eY_{2t}) \ \ and \ \ \sigma_{Y_2} = f\sqrt{Y_{2t}},$$

where z, u, ν, a, b, c, d, e and f are positive parameters.

And make the following change of variables , $y_{1t} = \frac{Y_{1t}}{c^2}$ and $y_{2t} = \frac{Y_{2t}}{f^2}$, which implies:

$$g_{y_1} = (\gamma - \vartheta y_{1t}) \ \ and \ \ \sigma_{y_1} = \sqrt{y_{1t}}$$
$$g_{y_2} = (\zeta - \xi y_{2t}) \ \ and \ \ \sigma_{y_2} = \sqrt{y_{2t}},$$

where $\gamma = \frac{a}{c^2}$, $\vartheta = b$, $\zeta = \frac{d}{f^2}$ and $\xi = e$.

In the model of Longstaff and Schwartz (1992), the relationship between the shocks in $\frac{dQ_t}{Q_t}$, dY_{1t} and dY_{2t} are the same as in Appendix 4.5.1, except for the fact that state variables are uncorrelated, i.e., $\rho = 0$. Thus, the specification of (4.34) is given by:

$$\begin{aligned}
\sum_{i=1}^{3} a_{1i}a_{2i} &= 0, \\
\sum_{i=1}^{3} a_{1i}a_{3i} &= 0, \\
\sum_{i=1}^{3} a_{1i}a_{4i} &= 0, \\
\sum_{i=1}^{3} a_{2i}a_{3i} &= 0, \\
\sum_{i=1}^{3} a_{2i}a_{4i} &= \rho^*, \\
\sum_{i=1}^{3} a_{3i}a_{4i} &= \sqrt{1-\rho^{*2}}.
\end{aligned}$$

Conditions (4.35) continue to apply. As there is no ambiguity:

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The utility specification is the same as in our setting but the expectation operator in (4.5) is set at instantaneous time t and not at the initial moment $(t = 0)$. This implies that the

value function J in (Longstaff and Schwartz (1992)) is not autonomous being given by:

$$J(W_t, Y_{1t}, Y_{2t}, t) = \frac{e^{-\delta t}}{\delta} ln(W_t) + G(Y_{1t}, Y_{2t}, t). \tag{4.42}$$

However, using the correspondent Bellman equation[14], optimal consumption rule continues to be given by $C_t^* = \delta W_t$. The resulting equilibrium dynamics of wealth are defined by:

$$\begin{aligned} & g_W = \left(zc^2 y_{1t} + uf^2 y_{2t} - \delta\right) W_t \ and \ \sigma_W = f\nu W_t \sqrt{y_{2t}}, \\ \Leftrightarrow \ & g_W = (zY_{1t} + uY_{2t} - \delta) W_t \ and \ \sigma_W = \nu W_t \sqrt{Y_{2t}}. \end{aligned}$$

Additionally, Longstaff and Schwartz (1992) assume that $\Lambda(W_t, Y_{1t}, Y_{2t}, t) = 0$.

Substituting those specifications into (4.8) the fundamental PDE satisfied by the contingent claim price $H(y_{1t}, y_{2t}, \tau)$ is given by:

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{1}{2}(\sqrt{y_{1t}})^2 \frac{\partial^2 H}{\partial y_1^2} + \frac{1}{2}(\sqrt{y_{2t}})^2 \frac{\partial^2 H}{\partial y_2^2} + (\gamma - \vartheta y_{1t}) \frac{\partial H}{\partial y_1} + \\ &\quad + \left[(\zeta - \xi y_{2t}) - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov(W_t, y_{2t}) \right] \frac{\partial H}{\partial y_2} - r_t H, \\ \Leftrightarrow \frac{\partial H}{\partial \tau} &= \frac{y_{1t}}{2} \frac{\partial^2 H}{\partial y_1^2} + \frac{y_{2t}}{2} \frac{\partial^2 H}{\partial y_2^2} + (\gamma - \vartheta y_{1t}) \frac{\partial H}{\partial y_1} + \\ &\quad + \left[(\zeta - \xi y_{2t}) - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} \right) Cov(W_t, y_{2t}) \right] \frac{\partial H}{\partial y_2} - r_t H, \end{aligned}$$

which is the PDE obtained by Longstaff and Schwartz (1992) (their equation (8)).

From (4.42) results that

$$\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} = \frac{\frac{e^{-\delta t}}{\delta W_t^2}}{\frac{e^{-\delta t}}{\delta W_t}} = \frac{1}{W_t},$$

as in our setting (4.41). We can therefore apply (4.10) to obtain the market price of risk associated with variable y_{2t}:

$$\begin{aligned} \frac{1}{W_t} \rho^* \sigma_W \sigma_{y_2} &= \frac{\rho^*}{W_t} f\nu W_t \sqrt{y_{2t}} \sqrt{y_{2t}}, \\ &= \rho^* f\nu y_{2t}, \\ &= \lambda y_{2t}, \ with \ \lambda = \rho^* f\nu. \end{aligned}$$

[14]Differs from (4.6) by considering (i) the discounted utility to instantaneous time t, so the first term in the Bellman argument is $e^{-\delta t} ln(C_t)$, and (ii) the partial derivative of J to time $\left(\frac{\partial J}{\partial t}\right)$replaces expression $-\delta J$, in consistency with comments in Appendix 4.5.2.

which is the expression reached by Longstaff and Schwartz (1992) (their equation (9)).[15]

Regarding the equilibrium interest rate, from Theorem 1 in Cox et al. (1985a) and considering (i) the result $\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}} = \frac{1}{W_t}$, (ii) from value function (4.42) that $\frac{\partial^2 J}{\partial W \partial Y_1} = \frac{\partial^2 J}{\partial W \partial Y_2} = 0$ and (iii) expressions above obtained $g_Q = \left(zc^2 y_{1t} + uf^2 y_{2t}\right)$ and $\sigma_W = f\nu W_t \sqrt{y_{2t}}$, then expression for r_t is given by:

$$
\begin{aligned}
r_t &= g_Q - \left(\frac{-\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\right)\left(\frac{Var(W_t)}{W_t}\right), \\
\Leftrightarrow r_t &= zc^2 y_{1t} + uf^2 y_{2t} - \frac{1}{W_t}\left[\frac{\left(f\nu W_t \sqrt{y_{2t}}\right)^2}{W_t}\right], \\
\Leftrightarrow r_t &= zc^2 y_{1t} + uf^2 y_{2t} - f^2 \nu^2 y_{2t}, \\
\Leftrightarrow r_t &= zc^2 y_{1t} + f^2\left(u - \nu^2\right) y_{2t},
\end{aligned}
$$

which is the expression reached by Longstaff and Schwartz (1992) (their equation (10)).

□

4.5.5 Expressions (4.18), (4.19), (4.20) and Equation (4.21)

As in previous Appendices, for a more parsimonious presentation, we drop out the arguments of the functions g_i, σ_i $(i = Y_1, Y_2, W, Q)$ and for the instantaneous pay-off function $\Lambda^{Ab}(W_t, Y_{1t}, Y_{2t}, t)$.

4.5.5.1 Optimal Contamination Drift (4.18)

From $dZ_Q = \rho^* dZ_2 + \sqrt{1-\rho^{*2}}\, dZ_3$, we obtain the standard deviation matrix associated with the economy's output rate process:

$$
\sigma = \begin{bmatrix} \sigma_Q \sqrt{1-\rho^{*2}} & 0 & \sigma_Q \rho^* \end{bmatrix}. \tag{4.43}
$$

Additionally, the standard deviation matrix associated with the stochastic processes of state variables is given by:

$$
\Xi = \begin{bmatrix} 0 & \sigma_{Y_1} & 0 \\ 0 & 0 & \sigma_{Y_2} \end{bmatrix}. \tag{4.44}
$$

[15] The equilibrium market price of risk associated with the "original" state variable Y_{2t} is also linear on its instantaneous level with the same coefficient λ, as $\frac{\rho^*}{W_t}\sigma_W \sigma_{Y_2} = \frac{\rho^*}{W_t}\nu W_t \sqrt{Y_{2t}} f \sqrt{Y_{2t}} = \rho^* \nu f Y_{2t} = \lambda Y_{2t}$, with $\lambda = \rho^* f \nu$.

In order to obtain the equilibrium contamination drift, we make use of the result in Proposition 1 in Gagliardini et al. (2009):

$$h^* = -\sqrt{2\eta}\frac{\Xi^\top V_Y + \sigma^\top}{\sqrt{\left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right)}}, \tag{4.45}$$

where V_Y is the vector representing the gradient of the value function $V(Y_1, Y_2)$ in (4.16) with respect to the state variables Y_1 and Y_2. Starting with the numerator in (4.45):

$$\begin{aligned}
\Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} 0 & 0 \\ \sigma_{Y_1} & 0 \\ 0 & \sigma_{Y_2} \end{bmatrix} \begin{bmatrix} V_{Y_1} \\ \\ V_{Y_2} \end{bmatrix} + \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* \end{bmatrix} \\
\Leftrightarrow \Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ \sigma_{Y_1}V_{Y_1} \\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* \end{bmatrix}.
\end{aligned}$$

With this result and considering that, by assumption $h^* = \begin{bmatrix} 0 & 0 & h \end{bmatrix}^\top$, then:

$$h = -\sqrt{2\eta}\frac{\sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^*}{\sqrt{\left(\sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^*\right)^2}} = \begin{cases} -\sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* > 0 \\ \\ \sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* < 0 \end{cases}$$

as we wanted to prove.

□

4.5.5.2 Equilibrium Price of Uncertainty (4.19)

From Corollary 1 in Gagliardini et al. (2009) the equilibrium market premium of risk and ambiguity (MP) associated with the state variables Y_{1t} and Y_{2t} and the production process Q_t is given by:

$$MP = \sigma^\top + \sqrt{2\eta}\frac{\Xi^\top V_Y + \sigma^\top}{\sqrt{\left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right)}},$$

where matrices σ and Ξ are those given in (4.43) and (4.44), respectively, and V_Y maintains the same meaning of the previous subsection. Considering (4.45), it is immediate to

conclude that MP is given by:

$$
\begin{aligned}
MP &= \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* \end{bmatrix} \pm \sqrt{2\eta}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
\Leftrightarrow MP &= \begin{cases} \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* + \sqrt{2\eta} \end{bmatrix} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* > 0 \\ \\ \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* - \sqrt{2\eta} \end{bmatrix} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* < 0\,. \end{cases}
\end{aligned} \tag{4.46}
$$

Following expression [18] in Gagliardini et al. (2009), the equilibrium market price of uncertainty associated with each of the state variables Y_1 and Y_2 is given by:

$$
\begin{aligned}
\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} &= \Xi MP, \\
\Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} &= \begin{bmatrix} 0 & \sigma_{Y_1} & 0 \\ 0 & 0 & \sigma_{Y_2} \end{bmatrix} \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* \pm \sqrt{2\eta} \end{bmatrix}, \\
\Leftrightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ \\ \sigma_{Y_2}\sigma_Q\rho^* \pm \sigma_{Y_2}\sqrt{2\eta} \end{bmatrix}.
\end{aligned}
$$

As expected, the equilibrium price of uncertainty associated with the uncorrelated (with output rate process) and unambiguous state variable Y_1 is zero ($\phi_1 = 0$). Regarding Y_2, the equilibrium uncertainty price is therefore given by:

$$
\phi_2 = \begin{cases} \sigma_{Y_2}\sigma_Q\rho^* + \sigma_{Y_2}\sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* > 0 \\ \\ \sigma_{Y_2}\sigma_Q\rho^* - \sigma_{Y_2}\sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* < 0 \end{cases},
$$

which is (4.19).

□

4.5.5.3 Equilibrium Interest Rate (4.20)

From Corollary 1 in Gagliardini et al. (2009) the equilibrium instantaneous interest rate r_t^{Ab} is given by:

$$r_t^{Ab} = g_Q - \sigma MP,$$

where g_Q is the drift function in (4.1) and matrices σ and MP are those given in (4.43) and (4.46), respectively. Consequently, r_t^{Ab} is given by:

$$r_t^{Ab} = g_Q - \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} & 0 & \sigma_Q\rho^* \end{bmatrix} \begin{bmatrix} \sigma_Q\sqrt{1-\rho^{*2}} \\ 0 \\ \sigma_Q\rho^* \pm \sqrt{2\eta} \end{bmatrix},$$

$$\Leftrightarrow r_t^{Ab} = \begin{cases} g_Q - \sigma_Q^2\left(1-\rho^{*2}\right) - \sigma_Q^2\rho^{*2} - \sigma_Q\rho^*\sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* > 0 \\ \\ g_Q - \sigma_Q^2\left(1-\rho^{*2}\right) - \sigma_Q^2\rho^{*2} + \sigma_Q\rho^*\sqrt{2\eta} & if\ \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* < 0, \end{cases}$$

which is (4.20).

□

4.5.5.4 Equation (4.21)

From Proposition 2 in Gagliardini et al. (2009), the PDE satisfied by the price $H^{Ab}(Y_{1t}, Y_{2t}, \tau)$ of the contingent claim is given by:

$$\begin{aligned} \frac{\partial H^{Ab}}{\partial \tau} &= \frac{1}{2}\left[Var(Y_{1t})\frac{\partial^2 H^{Ab}}{\partial Y_1^2} + 2Cov(Y_{1t}, Y_{2t})\frac{\partial^2 H^{Ab}}{\partial Y_1 \partial Y_2} + Var(Y_{2t})\frac{\partial^2 H^{Ab}}{\partial Y_2^2}\right] + \\ &\quad + [g_{Y_1} - \phi_{Y_1}]\frac{\partial H^{Ab}}{\partial Y_1} + [g_{Y_2} - \phi_{Y_2}]\frac{\partial H^{Ab}}{\partial Y_2} - r_t^{Ab}H^{Ab} + \Lambda^{Ab}, \end{aligned}$$

where r_t^{Ab} is given by (4.20) and ϕ_{Y_1} and ϕ_{Y_2} are the equilibrium prices of uncertainty associated to each of the state variables, which, from the previous subsection 4.5.5.2 we know to be $\phi_{Y_1} = 0$ and ϕ_{Y_2} given by (4.19). Obtained PDE can therefore be written as:

$$
\begin{aligned}
\frac{\partial H^{Ab}}{\partial \tau} &= \frac{1}{2}\sigma_{Y_1}^2 \frac{\partial^2 H^{Ab}}{\partial Y_1^2} + \rho\,\sigma_{Y_1}\sigma_{Y_2}\frac{\partial^2 H^{Ab}}{\partial Y_1 \partial Y_2} + \frac{1}{2}\sigma_{Y_2}^2 \frac{\partial^2 H^{Ab}}{\partial Y_2^2} + \\
&+ g_{Y_1}\frac{\partial H^{Ab}}{\partial Y_1} + \left[g_{Y_2} - \left(\sigma_{Y_2}\sigma_Q\rho^* + \sigma_{Y_2}\sqrt{2\eta}\right)\right]\frac{\partial H^{Ab}}{\partial Y_2} \\
&- r_t^{Ab} H^{Ab} + \Lambda^{Ab} \;\; if \;\; \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* > 0\,,
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\partial H^{Ab}}{\partial \tau} &= \frac{1}{2}\sigma_{Y_1}^2 \frac{\partial^2 H^{Ab}}{\partial Y_1^2} + \rho\,\sigma_{Y_1}\sigma_{Y_2}\frac{\partial^2 H^{Ab}}{\partial Y_1 \partial Y_2} + \frac{1}{2}\sigma_{Y_2}^2 \frac{\partial^2 H^{Ab}}{\partial Y_2^2} + \\
&+ g_{Y_1}\frac{\partial H^{Ab}}{\partial Y_1} + \left[g_{Y_2} - \left(\sigma_{Y_2}\sigma_Q\rho^* - \sigma_{Y_2}\sqrt{2\eta}\right)\right]\frac{\partial H^{Ab}}{\partial Y_2} \\
&- r_t^{Ab} H^{Ab} + \Lambda^{Ab} \;\; if \;\; \sigma_{Y_2}V_{Y_2} + \sigma_Q\rho^* < 0\,,
\end{aligned}
$$

where functions g_i and σ_i $(i = Q, Y_1, Y_2)$ are those in equations (4.1), (4.2) and (4.12). Obtained PDE is (4.21) as we wanted to prove.

□

4.5.6 Concrete Investment Opportunity Set

4.5.6.1 Bellman Equation (4.28)

Under the investment opportunity set (4.23), (4.24) and (4.27):

$$
\begin{aligned}
g_Y &= \begin{bmatrix} \mu Y_{1t} \\ \kappa(\theta - Y_{2t}) \end{bmatrix}; \\
\Xi &= \begin{bmatrix} 0 & Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & 0 & \varepsilon\sqrt{Y_{2t}} \end{bmatrix}; \\
\sigma^\top &= \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\rho^{*2}} \\ 0 \\ l\sqrt{Y_{2t}}\rho^* \end{bmatrix}.
\end{aligned}
$$

From which, it is immediate to obtain:

$$V_Y^\top g_Y = \begin{bmatrix} V_{Y_1} & V_{Y_2} \end{bmatrix} \begin{bmatrix} \mu Y_{1t} \\ \kappa(\theta - Y_{2t}) \end{bmatrix} = \mu V_{Y_1} Y_{1t} + \kappa(\theta - Y_{2t}) V_{Y_2};$$

$$\begin{aligned}
\Xi^\top V_{YY} \Xi &= \begin{bmatrix} 0 & 0 \\ Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & \varepsilon\sqrt{Y_{2t}} \end{bmatrix} \begin{bmatrix} V_{Y_1Y_1} & V_{Y_1Y_2} \\ V_{Y_2Y_1} & V_{Y_2Y_2} \end{bmatrix} \begin{bmatrix} 0 & Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & 0 & \varepsilon\sqrt{Y_{2t}} \end{bmatrix}, \\
\Leftrightarrow \Xi^\top V_{YY} \Xi &= \begin{bmatrix} 0 & 0 \\ Y_{1t}\sqrt{Y_{2t}} V_{Y_1Y_1} & Y_{1t}\sqrt{Y_{2t}} V_{Y_1Y_2} \\ \varepsilon\sqrt{Y_{2t}} V_{Y_2Y_1} & \varepsilon\sqrt{Y_{2t}} V_{Y_2Y_2} \end{bmatrix} \begin{bmatrix} 0 & Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & 0 & \varepsilon\sqrt{Y_{2t}} \end{bmatrix}, \\
\Leftrightarrow \Xi^\top V_{YY} \Xi &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & Y_{1t}^2 Y_{2t} V_{Y_1Y_1} & \varepsilon Y_{1t} Y_{2t} V_{Y_1Y_2} \\ 0 & \varepsilon Y_{1t} Y_{2t} V_{Y_2Y_1} & \varepsilon^2 Y_{2t} V_{Y_2Y_2} \end{bmatrix}, \\
\Rightarrow trace\left[\Xi^\top V_{YY} \Xi\right] &= Y_{1t}^2 Y_{2t} V_{Y_1Y_1} + \varepsilon^2 Y_{2t} V_{Y_2Y_2};
\end{aligned}$$

$$\begin{aligned}
\Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} 0 & 0 \\ Y_{1t}\sqrt{Y_{2t}} & 0 \\ 0 & \varepsilon\sqrt{Y_{2t}} \end{bmatrix} \begin{bmatrix} V_{Y_1} \\ V_{Y_2} \end{bmatrix} \\
&\quad + \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\rho^{*2}} \\ 0 \\ l\sqrt{Y_{2t}}\rho^* \end{bmatrix}, \\
\Leftrightarrow \Xi^\top V_Y + \sigma^\top &= \begin{bmatrix} l\sqrt{Y_{2t}}\sqrt{1-\rho^{*2}} \\ V_{Y_1} Y_{1t}\sqrt{Y_{2t}} \\ \varepsilon\sqrt{Y_{2t}} V_{Y_2} + l\sqrt{Y_{2t}}\rho^* \end{bmatrix}, \\
\Rightarrow \left(\Xi^\top V_Y + \sigma^\top\right)^\top \left(\Xi^\top V_Y + \sigma^\top\right) &= l^2 Y_{2t} + Y_{1t}^2 Y_{2t} V_{Y_1}^2 + \varepsilon^2 Y_{2t} V_{Y_2}^2 + 2\varepsilon l \rho^* Y_{2t} V_{Y_2};
\end{aligned}$$

Going back to (4.17), and substituting obtained expressions:

$$
\begin{aligned}
0 &= V_Y^\top g_Y + \frac{1}{2} trace\left[\Xi^\top V_{YY}\Xi\right] - \sqrt{2\eta}\sqrt{\left(\Xi^\top V_Y + \sigma^\top\right)^\top\left(\Xi^\top V_Y + \sigma^\top\right)} \\
&\quad + g_Q - \frac{1}{2}\sigma\sigma^\top - \delta V\,, \\
\Leftrightarrow \delta V &= \mu V_{Y_1} Y_{1t} + \kappa\left(\theta - Y_{2t}\right) V_{Y_2} + \frac{1}{2}\left(Y_{1t}^2 Y_{2t} V_{Y_1 Y_1} + \varepsilon^2 Y_{2t} V_{Y_2 Y_2}\right) + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t} \\
&\quad - \sqrt{2\eta\left(l^2 Y_{2t} + Y_{1t}^2 Y_{2t} V_{Y_1}^2 + \varepsilon^2 Y_{2t} V_{Y_2}^2 + 2\varepsilon l \rho^* Y_{2t} V_{Y_2}\right)}\,,
\end{aligned}
$$

which is (4.28).

□

4.5.6.2 Value Function (4.29)

Start noting that the value function $V\left(Y_{1t}, Y_{2t}\right)$ is a function of two state variables Y_{1t}, Y_{2t}, for a given ambiguity parameter η, and therefore can be denoted as $V\left(Y_{1t}, Y_{2t}, \eta\right)$:

$$
V:\ \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}\,.
$$

When there is no ambiguity ($\eta = 0$), (4.28) becomes:

$$
\begin{aligned}
\delta V &= \mu V_{Y_1} Y_{1t} + \kappa\left(\theta - Y_{2t}\right) V_{Y_2} + \frac{1}{2}\left(Y_{1t}^2 Y_{2t} V_{Y_1 Y_1} + \varepsilon^2 Y_{2t} V_{Y_2 Y_2}\right) \\
&\quad + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2} l^2 Y_{2t}\,,
\end{aligned} \tag{4.47}
$$

which is solved by the value function:

$$
V\left(Y_{1t}, Y_{2t}, \eta = 0\right) = alnY_{1t} + bY_{2t} + c\,, \tag{4.48}
$$

where:

$$
\begin{aligned}
a &= \frac{1}{\delta}\,, \\
b &= \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{\left(\kappa + \delta\right)}\,, \\
c &= \frac{\mu}{\delta^2} + \frac{\kappa\theta}{\delta} b\,.
\end{aligned} \tag{4.49}
$$

Proof:

From (4.48) results:

$$
\begin{aligned}
V_{Y_1} &= \frac{a}{Y_{1t}}\,,\; V_{Y_1Y_1} = -\frac{a}{Y_{1t}^2}\,,\\
V_{Y_2} &= b\,,\; V_{Y_2Y_2} = 0\,.
\end{aligned}
$$

Considering those results, (4.48) and (4.49) and plugging them into (4.47) one gets:

$$
\begin{aligned}
\delta\left(\frac{1}{\delta}lnY_{1t} + bY_{2t} + c\right) &= \mu\frac{a}{Y_{1t}}Y_{1t} + \kappa\left(\theta - Y_{2t}\right)b + \frac{1}{2}\left(-Y_{1t}^2Y_{2t}\frac{a}{Y_{1t}^2}\right) + lnY_{1t}\\
&\quad + \alpha Y_{2t} - \frac{1}{2}l^2Y_{2t}\,,\\
\Leftrightarrow lnY_{1t} + \delta bY_{2t} + \delta c &= \mu a + \kappa\left(\theta - Y_{2t}\right)b - \frac{1}{2}Y_{2t}a + lnY_{1t} + \alpha Y_{2t} - \frac{1}{2}l^2Y_{2t}\,,\\
\Leftrightarrow \delta c - \frac{\mu}{\delta} - b\kappa\theta &= \left[\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta} - b\left(\kappa + \delta\right)\right]Y_{2t}\,,\\
\Leftrightarrow \frac{\mu}{\delta} + \kappa\theta b - \frac{\mu}{\delta} - b\kappa\theta &= \left[\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta} - \frac{\left(\alpha - \frac{l^2}{2} - \frac{1}{2\delta}\right)}{\left(\kappa + \delta\right)}\left(\kappa + \delta\right)\right]Y_{2t}\,,\\
\Leftrightarrow 0 &= 0,
\end{aligned}
$$

as we wanted to prove.

We then make the assumptions, that under the domain $0 < \eta < \Psi$ (with Ψ being a small positive number):

- $\frac{\partial V}{\partial Y_2} = V_{Y_2}$ exists,
- $\frac{\partial V_{Y_2}}{\partial \eta}$ exists.

This means that under this domain for η, $V\left(Y_{1t}, Y_{2t}, \eta > 0\right) \approx V\left(Y_{1t}, Y_{2t}, \eta = 0\right)$, which implies:

$$V_{Y_2}\mid_{\eta>0} \approx V_{Y_2}\mid_{\eta=0} = b\,,$$

where b is given in (4.49).

□

4.5.6.3 Equilibrium Uncertainty Price (4.30) and Interest Rate (4.32)

From (4.19) and (4.20), a key issue is the sign of $V_{Y_2}\sigma_{Y_2} + \sigma_Q\rho^*$. Under the concrete investment opportunity:

$$V_{Y_2}\sigma_{Y_2} + \sigma_Q\rho^* = \left[\frac{\varepsilon}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho^*\right]\sqrt{Y_{2t}},$$

implying that the sign of $V_{Y_2}\sigma_{Y_2} + \sigma_Q\rho^*$ is given by the sign of $\left[\frac{\varepsilon}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho^*\right]$. The threshold value ω in (4.30) and (4.32) is therefore obtained:

$$\begin{aligned} 0 &= \frac{\varepsilon}{(\kappa+\delta)}\left(\alpha - \frac{1}{2}l^2 - \frac{1}{2\delta}\right) + l\rho^*, \\ \Leftrightarrow \alpha &= \underbrace{\frac{l^2\delta\varepsilon + \varepsilon - 2\delta(\kappa+\delta)l\rho^*}{2\delta\varepsilon}}_{\omega}. \end{aligned}$$

When $\alpha > \omega\,(\alpha < \omega)$ then $V_{Y_2}\sigma_{Y_2} + \sigma_Q\rho^* > 0\,(< 0)$, making it immediate to obtain (4.30) from (4.19) and (4.32) from (4.20).

□

Chapter 5

Closed-Form Solution for Options with Ambiguity about Stochastic Volatility[1]

[1]An early version of this chapter has been circulating as Faria and Correia-da Silva (2010a). We are grateful to Alejandro Balbás, Cláudia Ribeiro (discussant) and Frank Riedel for useful comments and suggestions, and thank seminar participants at the 6th Doctoral Workshop at U.Porto.

5.1 Introduction

We derive a closed-form solution for the price of a European call option on an asset whose return's variance is stochastic, in the presence of ambiguity about the stochastic process of the variance.

Since the seminal works of Black and Scholes (1973) and Merton (1973b), the option pricing literature had an impressive development: models with stochastic variance of the underlying asset return (e.g. Hull and White (1987), Johnson and Shanno (1987), Wiggins (1987), Scott (1987), Stein and Stein (1991), Heston (1993)), with jumps in the underlying asset price process with and without stochastic interest rate (e.g. Merton (1976), Bates (1996), Bakshi et al. (1997)), models with jumps in both the price and the variance processes (e.g. Duffie et al. (2000) and Barndorff-Nielsen and Shephard (2001)), jumps with finite activity (e.g. Bates (2000) and Pan (2002)) and infinite activity (e.g. Madan et al. (1998) and Carr and Wu (2003)), models using ARCH processes (e.g. Bollerslev and Ole Mikkelsen (1996) and Heston and Nandi (2000)), among others. In Carr and Wu (2004) a general theoretical framework for "closed-form" option prices is provided, as well as a comprehensive survey on the option pricing literature.

The core of this literature focuses on the relation between the underlying asset's price dynamics and the option price. This is understandable: that relationship is the central issue on option pricing, as an option represents a claim on the underlying asset. In this chapter we explore a different angle of analysis: the nature of the uncertainty involved in the option pricing problem.

Due to the option's forward looking nature, uncertainty is intrinsically linked with its pricing. There are two standard approaches in the option pricing literature: no-arbitrage valuation and risk-neutral valuation. The former is centered in obtaining a "riskless" portfolio excluding arbitrage possibilities between the option and the underlying asset, while the latter is centered in a change of probability measure, from the physical "real world" measure to an equivalent martingale "risk-neutral" measure under which the option price is obtained.[2] Under both approaches, which are equivalent in their results, probability measures are assumed to be perfectly known by the investor. That is, as in the mainstream theory of choice under uncertainty in economics, the standard option pricing

[2]When markets are incomplete, in the sense that a perfect hedge strategy is not available (as it is not possible to perfectly replicate the option using the available assets), there is an infinite number of equivalent martingale measures (EMM) under which the option price may be obtained (implying that the option price is not unique under incomplete markets). This is what happens when stochastic variance is considered (if there are not assets whose payoffs are contingent on the observed variance), where each of that EMM reflects a different variance uncertainty price. Superhedging, Mean-Variance hedging and Shortfall hedging are examples of hedging strategies under incomplete markets that have been studied in the literature (e.g. Cvitanic et al. (1999), Follmer and Sondermann (1986) and Follmer and Leukert (2000), respectively).

literature has been treating uncertainty as something that can be entirely described by a probability distribution.

The conceptual distinction between risk and ambiguity was firstly pointed out by Knight (1921) and later supported by the empirical experiments of Ellsberg (1961) and others (see, for e.g., Camerer and Weber (1992) and Epstein and Schneider (2010) for a survey). The reason for this distinction is that economic agents may not be able to completely describe the uncertainty that they face by using a single probability distribution. Risk refers to uncertainty that can be represented by a probability distribution, while ambiguity refers to uncertainty that cannot. This distinction has relevant implications for the behavior of economic agents, and, therefore, for economic theory in general. That is why a rapidly growing literature on asset pricing under ambiguity aversion is emerging. This literature has been comprehensively surveyed by Epstein and Schneider (2010).

Regarding the option literature, ambiguity is starting to be taken into account, even if there are still very few works and almost all of them are related to the exercise strategies of American options. In Miao and Wang (2009), a real options problem is studied, adopting a recursive multiple-priors utility model,[3] with the source of ambiguity being the state process which influences the continuation and termination payoffs associated with the option. It is concluded that ambiguity lowers the option value. In Kast et al. (2010) and Jaimungal (2010), a similar problem regarding real options is studied by using Choquet Brownian motions and robust control approach, respectively.

In Riedel (2009), a general theory of optimal stopping under time-consistent ambiguity in discrete time is developed, using the recursive multiple-priors setting of Epstein and Schneider (2003), and one of the applications is to American options. The source of ambiguity is the payoff of exercising the option. This work is extended in Riedel (2010) for a continuous-time environment. Chudjakow and Vorbrink (2009) use Riedel's (2009) framework to study the exercise strategies for several American exotic options under ambiguity. In Liu et al. (2005), ambiguity aversion about rare events in the economy's endowment is considered, generating a correspondent equilibrium premium for rare-events (which added to the standard risk premiums for diffusive and jump risks, gives the equity premium in the model). The authors conclude that this ambiguity can help explain the well documented "smirk" pattern of option-implied volatility observed for options across

[3] An extensive review on decision theory under ambiguity has been carried out by Etner et al. (2009). Briefly, the two most common approaches being used in ambiguity literature are: the robust control (RC) approach, associated to an assumption of model uncertainty, and the multiple priors (MP) approach, whereby the single probability measure of the standard expected utility model is replaced by a set of probabilities or priors. The relationship between the robust control and multiple priors approaches has been widely discussed in the literature, for e.g., in Hansen and Sargent (2001a), Hansen et al. (2002), Epstein and Schneider (2003), and Maccheroni et al. (2006a).

moneyness levels (e.g. Rubinstein (1994)).[4]

In this chapter, we derive a closed-form solution for the price of a European call option when: (i) the underlying asset return's variance is stochastic and is correlated with the spot asset return and (ii) there exists ambiguity (aversion) about the variance stochastic process.

The major motivation to develop a stochastic variance option pricing model is to get a more realistic assumption regarding the variance behavior: there is empirical evidence supporting the stochastic nature of risky assets return's variance (e.g. Eraker et al. (2003) and Eraker (2004)). Moreover, assuming stochastic variance allows to obtain more realistic return distributions, namely with higher kurtosis than that of normal distribution (as assumed in Black and Scholes (1973)), non-zero skewness (negative skewness in case of a negative correlation between shocks in the return and in its variance, and positive skewness for positive correlation) and implied volatility surfaces closer to those verified in reality. It is also recognized in the literature (e.g., Bakshi et al. (1997)) that the most significant improvement over the Black and Scholes (1973) model came from the introduction of stochastic variance. Once this is done, introduction of stochastic interest rates or jumps bring marginal improvements. The trade-off for obtaining a more realistic model is that stochastic variance option pricing models are more difficult to calibrate and, in most of the cases, are only approximately solved through time consuming numerical methods.

Regarding the source of ambiguity of our setting, the stochastic process of the variance, it has been advocated in the literature (Cao et al. (2005); Garlappi et al. (2007); Ui (2009)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we assume ambiguity about the stochastic process for the variance of the risky asset's return because: (1) there is no "a priori" reason to assume that investors are not ambiguous about it; (2) we are able to find a closed-form solution for a European

[4]Liu et al. (2005) extend the economy of Lucas (1978) by considering rare events, which are modeled through a jump component in the economy's endowment process. Those rare events in the economy's endowment jump-diffusion process are the exclusive source of ambiguity for the representative agent. Ambiguity aversion is formally treated under the robust control approach. In Liu et al. (2005), the price of the underlying asset of the European option follows a jump-diffusion process with no stochastic variance, and the European option pricing is the modification of the Black and Scholes (1973) formula established in Merton (1976).

call option price; (3) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world, with the option implied variance frequently differing both in level and dynamics from the statistical measure (e.g. Drechsler and Yaron (2008) and Todorov (2010)); and (4) the relevance of variance as an option pricing input.

We start from the stochastic variance option price model of Heston (1993). This is a well established model in the option pricing literature, offering a good trade-off between analytical and computational tractability and empirically realistic assumptions and results. Concretely, Heston's (1993) option price model has a closed-form pricing formula,[5] without imposing any restriction regarding the correlation between the variance and the underlying asset return. Moreover, Heston's (1993) setting accounts for relevant stylized facts in financial data (apart from stochastic variance) as non-normal distribution of the assets returns, leverage effect (negative correlation between return and variance in some asset classes) and volatility clustering. Additionally, Black-Scholes option implied volatility surfaces generated by Heston's model are closer to those empirically observed (see, for e.g., Mikhailov and Nogel (2003)).

The option pricing formula obtained in this chapter differs from that in Heston (1993) exclusively because a new specification for the variance uncertainty price is considered. Our variance uncertainty price specification takes into account the ambiguity about the variance stochastic process, being decomposed in two components: variance risk price and variance ambiguity price. It is shown that the model of Heston (1993) can be obtained as a particular case of our option pricing model, when ambiguity does not exist. The specification of the variance uncertainty price used in this chapter is theoretically motivated by the general equilibrium model in Faria and Correia-da Silva (2010b), where ambiguity is formally introduced through a "constraint preferences" robust control methodology. However, it is important to highlight that the option pricing formula obtained in the present chapter is reached through an arbitrage approach and does not depend on the remaining assumptions in the model of Faria and Correia-da Silva (2010b).

We therefore obtain a closed-form solution for the price of a European call option when the underlying asset return's variance is stochastic, correlated with the asset spot return, and when there exists ambiguity about the stochastic process of the variance. This is the main result of the chapter. Then, we use the calibration in Heston (1993) and simulate our

[5]Rigorously speaking, Heston (1993) option pricing model delivers a semi closed-form solution, as it includes two integrals that cannot be evaluated exactly. They can however be approximated by using some numerical integration methods, e.g, Gauss Lagendre or Gauss Lobatto integration. Notwithstanding, both in Heston (1993) and in the literature in general, the Heston (1993) option pricing formula is said to be a closed-form solution. Consistently, the option pricing formula obtained in this chapter will also be designated as a closed-form solution in this wider sense.

option pricing model.

It is found that, if the variance ambiguity price is positive, ambiguity aversion about the variance stochastic process leads to a decrease on option prices. Considering that a European option price (either Call or Put) is a positive function of variance, this means that under this scenario our option pricing model helps explaining the variance premium puzzle. This puzzle refers to the fact that option-implied variance for a certain maturity tends to be higher than the conditional expectation of realized variance for that period of time (see, for e.g., Drechsler and Yaron (2008) and Todorov (2010)). When the variance ambiguity price is negative, obtained option prices are higher when ambiguity is considered versus when stochastic variance is considered without ambiguity.

Additionally, analyzing the impact from a non-zero correlation between shocks in the spot asset return and its variance and from a non-zero standard deviation of variance, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process implies, across all moneyness levels, a relative increase of option prices generated by our model versus those obtained under the Black and Scholes (1973) model with comparable variance. The opposite happens when the variance ambiguity price is negative.

In our view, this chapter brings three major contributions: (1) it is the first time that ambiguity aversion within an option pricing problem with stochastic variance is considered, and where the latter's process is the source of ambiguity; (2) it is obtained a closed-form solution for an extension to the Heston (1993) model with a non-linear specification for the variance uncertainty price; (3) in a specific scenario, it helps explaining the variance premium puzzle.

The chapter is organized as follows. In section 5.2, the European call option closed-form solution is deducted. In section 5.3, simulation outputs are shown. In section 5.4, concluding remarks are presented.

5.2 European Call Option Closed-Form Solution

It is assumed that the spot price, S_t, of the European call option's underlying asset follows the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_S, \tag{5.1}$$

where μ is the asset's expected return, v_t represents the instantaneous variance of the underlying asset return and W_S is a standard Brownian motion. For simplicity it is assumed,

as in Heston (1993), that the underlying asset does not pay any income during the life of the option (e.g. stock with no dividends). However, it is well known from the literature that it is analytically straightforward to accommodate that feature (for e.g., Taylor (2005) ch.14).

Regarding the instantaneous variance v_t, start by assuming that $\sqrt{v_t}$ follows an Ornstein-Uhlenbeck process given by,

$$d\sqrt{v_t} = -\frac{\kappa}{2}\sqrt{v_t}dt + \frac{\sigma}{2}dW_v, \tag{5.2}$$

with $k > 0$ and $\sigma > 0$ (with economic meaning given below), and W_v being a Brownian motion with an instantaneous correlation ρ with W_S, i.e., $dW_S dW_v = \rho dt$. From Itô's Lemma, the instantaneous variance, v_t, follows the process (Appendix 5.5.1):

$$dv_t = \left(\frac{\sigma^2}{4} - \kappa v_t\right)dt + \sigma\sqrt{v_t}dW_v, \tag{5.3}$$

which, assuming $\theta = \frac{\sigma^2}{4\kappa}$ (as in Heston (1993)) that guarantees non-negativity of v_t,[6] can be rewritten as the mean reverting square-root process:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_v. \tag{5.4}$$

From (5.4) it results that k represents the variance mean reversion parameter, θ is the expected value of variance and σ represents the standard deviation of variance. For simplicity, as in Heston's (1993) base case, a constant interest rate, r, is assumed.

In order to obtain the contingent claim pricing formula through standard arbitrage arguments (e.g. Black and Scholes (1973) and Merton (1973b)), we need the specification of the variance uncertainty price. In the model of Heston (1993), where uncertainty is exclusively risk, the variance uncertainty price is linear on the instantaneous variance v_t. However, by introducing ambiguity about the variance stochastic process (5.4), the variance uncertainty price specification will differ from that of Heston (1993).

The variance uncertainty price specification to be used in this chapter is theoretically motivated by the general equilibrium model with ambiguity developed in Faria and Correia-da Silva (2010b).[7] As ambiguity is the key issue of this chapter, we describe briefly how am-

[6]Under the condition $0 \leqslant 2\kappa\theta \leqslant \sigma^2$, which contains the scenario $\theta = \frac{\sigma^2}{4\kappa}$, it is guaranteed that the instantaneous variance v_t is non-negative. In Cox et al. (1985b) it is assumed the condition $2\kappa\theta \geqslant \sigma^2$, which guaranties that v_t is strictly positive.

[7]This is an intertemporal general equilibrium model that is a version of that of Cox et al. (1985a) with two correlated state variables, a single production process and logarithmic utility. Both state variables impact the expected return of the single production process in the economy, Q_t, being assumed that only

biguity aversion about the stochastic process (5.4) is considered in Faria and Correia-da Silva (2010b).

It is assumed that the representative investor is ambiguous about the stochastic process (5.4), that evolves under a probability measure P, describing v_t dynamics. As a consequence, the ambiguous investor considers contaminations, P^h, around his reference belief, P. Those contaminations, representing alternative models for v_t dynamics, are assumed to be absolutely continuous with respect to P, and therefore, P^h are equivalently described by contaminating drift processes h_v. The Brownian motion under each of the measures P^h is given by $W_v^h = W_v(t) + \int_0^t h_v(s)\,ds$.[8] Aversion towards ambiguity is introduced by assuming that, in the spirit of Gilboa and Schmeidler (1989), the representative investor chooses the worst possible contamination, i.e., the one associated with lower expected utility.[9]

As a result, under the investment opportunity set given by (5.1) and (5.4), from Faria and Correia-da Silva (2010b), the variance uncertainty price specification to be used contains a new non-linear component on v_t, and is given by (see Faria and Correia-da Silva (2010b) for full details):

$$\lambda_v(v_t) \quad = \quad \underbrace{\lambda_1 v_t}_{risk\,price} \; + \; \underbrace{\lambda_2 \sqrt{v_t}}_{ambiguity\,price} \quad , \tag{5.5}$$

shocks in one of them are correlated with those in the production process return (output rate). Ambiguity about the stochastic process of that state variable is introduced, following the extension of the model of Cox et al. (1985b) made by Gagliardini et al. (2009), implying that only changes in that state variable should be hedged and its uncertainty (risk and ambiguity) priced. In Faria and Correia-da Silva (2010b), as a concrete example, it is deducted the equilibrium uncertainty price specification for a setting where unambiguous and ambiguous state variables follow the stochastic processes (5.1) and (5.4), respectively, from which results the variance uncertainty price specification to be used in this chapter.

[8]As explained in Gagliardini et al. (2009) the analysis is restricted, for tractability, to the class of Markov-Girsanov kernels $h_v(Y)$, where Y is the vector of state variables. Moreover, it is assumed an upper bound for the contaminating drift process h_v: $h_v' h_v \leqslant 2\eta$, where $\eta \geqslant 0$ is a parameter representing the level of ambiguity. This bound constrains both the instantaneous time variation and the continuation value of the relative entropy between the reference model, P, and any admissible contaminated model, P^h. This guarantees the rectangularity property of the set of priors, i.e, a dynamic consistent preferences ordering (see Epstein and Schneider (2003) for the definition of this property and Trojani and Vanini (2004) pp. 289 for a detailed explanation supporting the rectangularity property of the set of priors under the setting in Faria and Correia-da Silva (2010b)).

[9]The Gilboa and Schmeidler (1989) approach is sometimes criticized because it apparently implies extreme ambiguity aversion. However, the implied decision criteria may not be so extreme as it seems. The reasoning for this is that the set of priors is not an independent object including all logically possible priors, being instead part of the representation of the concrete problem under analysis. This is why Gilboa and Schmeidler (1989) criteria is not so extreme as, for example, the Wald maxmin criteria. Klibanoff et al. (2005) developed a setting which claims a smoother implied level of aversion towards ambiguity. However, there is still a debate in the literature about the axiomatic foundations of their model (see Epstein (2010) and Klibanoff et al. (2009a) for a recent exchange on this). Also because of this, the approach of Gilboa and Schmeidler (1989) (choosing the worst possible outcome) continues to be the most used in the literature.

where λ_1 and $\lambda_2 \in \mathbb{R}$.

Two comments on (5.5). First, it is immediate to conclude that if there is no ambiguity ($\lambda_2 = 0$), then variance uncertainty price continues to be linear on the instantaneous variance, v_t, as in Heston's (1993) model. Moreover, this implies that the drift function of the variance process under the uncertainty neutral measure has the same (linear) structure of the "physical" measure. When ambiguity is considered, two scenarios are possible: if $\lambda_2 > 0$, implying that the variance ambiguity price is positive, variance uncertainty price becomes a concave function of the instantaneous level of variance, v_t; if $\lambda_2 < 0$, i.e. the variance ambiguity price is negative, then the variance uncertainty price becomes a convex function of v_t. In either cases, there is a change of structure of the variance drift function from the physical measure (linear) to the uncertainty neutral measure (non-linear). The second comment on the specification (5.5) is that it guarantees that when variance approaches zero ($v_t \to 0$), its uncertainty price also converges to zero, hence excluding arbitrage opportunities (see, for e.g. Cheridito et al. (2007), for a discussion on state variables uncertainty price specifications and arbitrages opportunities).

Additionally, although the specification (5.5) is theoretically motivated by the model of Faria and Correia-da Silva (2010b), it should be stressed that contingent claim pricing results are obtained through a standard arbitrage approach, not depending on the remaining assumptions of that model.

We have now all the required inputs to deduct the closed-form solution for a European call option price when: (i) the underlying asset dynamics is given by (5.1) and (5.4); (ii) the representative investor is ambiguous about the stochastic process (5.4) of the underlying asset's return variance; and (iii) the investor is ambiguity averse.

Considering the variance uncertainty price given by (5.5), applying Itô's lemma and making use of the standard arbitrage rational, it results that the price of a contingent claim $U(S_t, v_t, t)$ has to satisfy the following partial differential equation (PDE) (Appendix 5.5.2):

$$\begin{aligned} &\frac{1}{2} v_t S_t^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma v_t S_t \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 U}{\partial v^2} + r S_t \frac{\partial U}{\partial S} \\ &\quad + \left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})\right] \frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} \quad = \quad 0. \end{aligned} \tag{5.6}$$

We follow the methodology of Heston (1993), based on characteristic functions, in order to obtain the closed-form solution for a European call option with ambiguous stochastic variance.

A European call option with strike price K and maturing at time T satisfies the PDE (5.6) subject to the boundary conditions:

$$\begin{aligned} U(S_T, v_T, T) &= Max(0, S_T - K), \\ U(0, v_t, t) &= 0, \\ \frac{\partial U}{\partial S}(\infty, v_t, t) &= 1, \\ rS_t \frac{\partial U}{\partial S}(S_t, 0, t) + \kappa\theta \frac{\partial U}{\partial v}(S_t, 0, t) - rU(S_t, 0, t) + \frac{\partial U}{\partial t}(S_t, 0, t) &= 0, \\ U(S_t, \infty, t) &= S_t. \end{aligned} \tag{5.7}$$

In the spirit of the Black and Scholes (1973) formula, the guess for the European call option price formula is

$$Call(S_t, v_t, t) = S_t P_1 - Ke^{-r(T-t)} P_2, \tag{5.8}$$

where $S_t P_1$ and $Ke^{-r(T-t)} P_2$ are the present value of the spot asset price upon option exercise and of the strike-price payment, respectively. Both of these terms have to satisfy (5.6). Substituting the guess formula (5.8) into (5.6), it results that both P_j $(j = 1, 2)$ must satisfy the partial differential equations given by (Appendix 5.5.3):

$$\begin{aligned} \frac{1}{2} v_t \frac{\partial^2 P_j}{\partial x^2} + \sigma\rho v_t \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_j}{\partial v^2} + \left(u_j v_t + r\right) \frac{\partial P_j}{\partial x} \\ + \left[a - b_j v_t - c\sqrt{v_t}\right] \frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} &= 0, \end{aligned} \tag{5.9}$$

where $x_t = ln(S_t)$, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \lambda_1 - \rho\sigma$, $b_2 = \kappa + \lambda_1$ and $c = \lambda_2$.

The obtained PDEs (5.9) are subject to the terminal condition $P_j(x, v, T; ln(K)) = 1_{\{x_T \geqslant ln(K)\}}$ $j = 1, 2$, so that the option price satisfies the boundary conditions in (5.7).

A key issue in the option price formula (5.8) is understanding the nature of probabilities P_j $(j = 1, 2)$, and obtain their analytical expressions. For that, we start from the implied "uncertainty-neutral" processes of x_t and v_t:

$$dx_t = \left(u_j v_t + r\right) dt + \sqrt{v_t} dW_S, \tag{5.10}$$

$$dv_t = \left(a - b_j v_t - c\sqrt{v_t}\right) dt + \sigma\sqrt{v_t} dW_v, \tag{5.11}$$

where the parameters u_j, r, a, b_j, c and σ are defined as before.

It is shown in Appendix 5.5.4 that, under the "uncertainty-neutral" processes (5.10)-(5.11), probabilities P_j $(j = 1, 2)$ are conditional probabilities of the option expiring in-

the-money, under different probability measures:

$$P_j\left(x,v,T;ln(K)\right)=Pr\left[x_T\geqslant ln\left(K\right)\mid x_t=x,\,v_t=v\right],\,j=1,2. \tag{5.12}$$

From the guess formula (5.8), it is also clear that P_1 represents the delta of the European call option. However, as highlighted by Heston (1993), probabilities P_j $(j=1,2)$ haven't a straightforward closed-form but using the conditional characteristic functions of x for each $j=1,2$, denominated as $f_j\left(x,v,T;\phi\right)$, $j=1,2$, it is possible to obtain that closed-form. The characteristic functions $f_j\left(x,v,T;\phi\right)$, $j=1,2$, continue to satisfy PDEs (5.9), subject to the terminal condition $f_j\left(x,v,T;\phi\right)=e^{i\phi x}$ (Appendix 5.5.4).

The characteristic function solution is:

$$f_j\left(x,v,t;\phi\right) \;=\; e^{C(\tau;\phi)+D(\tau;\phi)v_t+E(\tau;\phi)\sqrt{v_t}+i\phi x}\,,\,j=1,2\,, \tag{5.13}$$

where $\tau=T-t$ and

$$\begin{aligned} C\left(\tau;\phi\right) &= r\phi i\tau+\frac{a}{\sigma^2}\left\{\left(b_j-\rho\sigma\phi i+d\right)\tau-2ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\} \\ &\quad +\omega\left(\tau;\phi\right)-\Omega\,, \end{aligned} \tag{5.14}$$

$$D(\tau;\phi) \;=\; \frac{b_j-\rho\sigma\phi i+d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right], \tag{5.15}$$

$$E(\tau;\phi) \;=\; \frac{\left(b_j-\rho\sigma\phi i+d\right)c}{\left(ge^{d\tau}-1\right)\sigma^2 d}\left[4e^{\frac{d\tau}{2}}-2e^{d\tau}-2\right], \tag{5.16}$$

and

$$
\begin{aligned}
g &= \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}, \\
d &= \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}, \\
\omega(\tau;\phi) &= \frac{-\left(b_j - \rho\sigma\phi i + d\right)c^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left[1 - \left(4e^{\frac{d\tau}{2}} - 6\right)g - \left(4e^{\frac{d\tau}{2}} - 1\right)g^2\right]}{g^2\left(ge^{d\tau} - 1\right)}\right. \\
&\quad - \left(b_j - \rho\sigma\phi i - d\right)d\tau \\
&\quad - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i\right]ArcTanh\left[e^{\frac{d\tau}{2}}\sqrt{g}\right]}{g\sqrt{g}} \\
&\quad \left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i\right]ln\left(ge^{d\tau} - 1\right)}{g^2}\right], \\
\Omega &= \frac{-\left(b_j - \rho\sigma\phi i + d\right)c^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left[1 + 2g - 3g^2\right]}{g^2(g-1)}\right. \\
&\quad - \frac{4\left[b(1-g) + d(1+g) + (g-1)\rho\sigma\phi i\right]ArcTanh\left[\sqrt{g}\right]}{g\sqrt{g}} \\
&\quad \left. + \frac{(1+g)\left[b(g-1) - d(g+1) - (g-1)\rho\sigma\phi i\right]ln(g-1)}{g^2}\right],
\end{aligned}
$$

where *ArcTanh* represents the hyperbolic arc tangent of the complex number as argument.

It is straightforward to conclude that Heston (1993) model can be obtained as a particular case of our setting when $c = \lambda_2 = 0$, i.e., when there is no ambiguity and therefore uncertainty is exclusively risk (Appendix 5.5.5). This is consistent with the fact that, when $c = \lambda_2 = 0$, PDEs (5.6) and (5.9) are the ones in Heston (1993).

With the known characteristic functions $f_j(x,v,t;\phi)$, $j = 1,2$, the expressions of probabilities P_j $(j = 1,2)$, are obtained through the inverse Fourier transformation:

$$
P_j(x,v,T;ln(K)) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi ln[K]}f_j(x,v,T;\phi)}{i\phi}\right]d\phi, \; j = 1,2. \quad (5.17)
$$

The closed-form solution for the European call option, under our setting, is therefore given by equations (5.8), (5.13) and (5.17).

5.3 Simulation Results

In this section, we analyze the effect on option prices of the ambiguity about the stochastic process of the underlying asset return's variance.[10]

Under the uncertainty "neutralized" pricing probabilities, the underlying asset's return variance follows the square-root process:

$$dv_t = (\kappa\theta - (\kappa + \lambda_1)\, v_t - \lambda_2 \sqrt{v_t})\, dt + \sigma \sqrt{v_t} dW_v\,. \tag{5.18}$$

Model simulations are made under this process, and not the physical one (5.4). As known from the option pricing literature (e.g. Black and Scholes (1973) and Heston (1993)), no-arbitrage implies the irrelevance of the expected return of the underlying asset for option pricing, and therefore the process (5.18) exclusively determines option prices.

In the following simulation, the default calibration is the one used in Heston (1993). Two reasons motivate this procedure. First, in this chapter we are not focused on the model estimation and consequently we need to use some calibration from the literature.[11] Second, as our option price model is an extension of Heston (1993), the calibration used by the latter is the natural candidate as it enhances comparative analysis. The default parameters values for the model implementation are presented in Table 5.1:

Table 5.1: Default Parameters Values

Parameter	**Value**
Mean Reversion - κ	2
Long-run variance - θ	0.01
Current variance - v_t	0.01
Std.Deviation of variance - σ	0.1
Option Maturity (years) - T	0.5
Interest rate - r	0
Strike Price - K	100
Correlation - ρ	0

The specification (5.5) for the variance uncertainty price, implies that two further parameters have to be calibrated: λ_1 and λ_2. We maintain Heston (1993) calibration of $\lambda_1 = 0$, which imply a zero variance risk price. As a curiosity, under the equilibrium specification for λ_1 in the model of Faria and Correia-da Silva (2010b), $\lambda_1 = 0$ when: (i) the variance

[10] Implementation using Dynare version 3.065 and MatLab version 7.0.0.19920 R14. The computation of the integrals in (5.17) is done through numerical integration, using an adaptive Gauss Lobatto rule.

[11] There is an extensive literature on Heston's model calibration (e.g. Mikhailov and Nogel (2003) and Zhang and Shu (2003)).

is deterministic; (ii) the output rate of the economy is deterministic; or (iii) there is no correlation between shocks in the variance and in the output rate of the economy. In each of this scenarios, it is immediate to conclude that the motivation to hedge against adverse shocks (when they exist) in variance disappears, which consistently implies a zero price for the variance risk.

With $\lambda_1 = 0$, we run simulations for different values of λ_2, which enables the isolation of the ambiguity effect on option pricing. The issue is how to calibrate λ_2. Alongside the well documented difficulties in estimating parameters in stochastic volatility models, the calibration of λ_2 has an extra difficulty: it economically embeds the ambiguity level of the representative investor, and estimating ambiguity is in itself a challenging task (see, for e.g., Maenhout (2006)).

In such context, we adopt a simple approach in order to get a qualitative perception of the impact of ambiguity about stochastic variance on option pricing: we simulate the option pricing model with arbitrary values for λ_2. In order to sort out those values, we consider the comment in Gagliardini et al. (2009) regarding the entropy-bound (η) for model contaminations, which is an indirect measure of the ambiguity level: it *"should not imply a too wide discrepancy between the reference model P and alternatives P^h"*. By other words, it should be a small number. Having said this, we simulate our option pricing model with three arbitrary values for λ_2: $-0.02, 0, 0.02$.[12] Heston's scenario is given by $\lambda_2 = 0$ (no ambiguity).

At a first stage, the simulation analysis is focused on the difference between the European call option prices obtained from our model and from the Black-Scholes (B-S) model with comparable volatility.[13] The volatility parameter used in the B-S model is obtained in the following way. The process (5.18) is simulated n times (in our case $n = 100$), resulting from each simulation an average variance value. Then it is calculated the average of those n average variance values. The square root of that average value is the volatility parameter input for the B-S option price computation.

The B-S model assumes constant variance and a normal distribution of the underlying asset return. However, as it is well documented in the literature, financial returns in general are skewed and show greater kurtosis that the normal distribution allows. The following analysis, as in Heston (1993), is therefore centered in two parameters of the investment

[12]The η value implied by all calibrations in Gagliardini et al. (2009) is lower than 0.0136. Using the indicative specification $\lambda_2 = \pm\sigma\sqrt{2\eta}$ from Faria and Correia-da Silva (2010b), $\lambda_2 = \pm 0.02$ and $\sigma = 0.1$ (Table 5.1) imply $\eta = 0.02$. With $\sigma = 0.2$, which we will use under some simulations, the implied η value is 0.005.

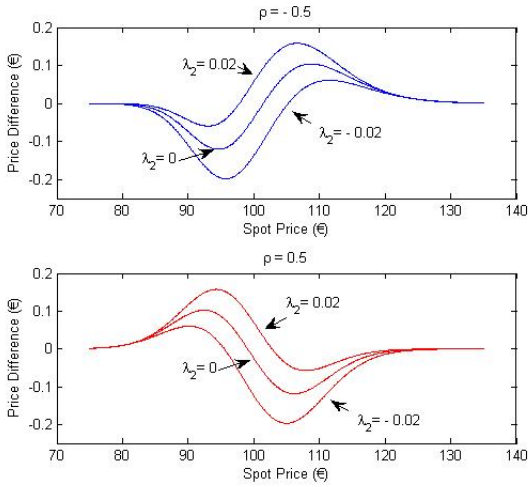
[13]In the literature, the term "volatility" sometimes is used as "variance" and other times as "standard deviation". In this simulation, following Heston (1993) terminology, volatility is the square root of variance, i.e., means "standard deviation".

opportunity set that directly impact the skewness and the kurtosis of the underlying asset return distribution: ρ and σ, respectively.

We start by analyzing the difference between the option prices generated by our model and the B-S model, when a non-zero correlation ρ between shocks in variance and return is allowed. A positive value of ρ implies a positive skewness of the spot return distribution: if there is a higher variance when spot asset price rises, then a fatter right tail of the spot return distribution is generated. The opposite happens when $\rho < 0$.

For a complete spectrum of option moneyness (spot prices S from 75 to 135), Figure 5.1 discloses the option price differences when $\rho = -0.5$ and $\rho = 0.5$ and, for each of the ρ values, considering the three values for $\lambda_2\,(-0.02, 0, 0.02)$.

Figure 5.1: Option Price Differences: ρ and λ_2 sensitivity.



Note: Price Difference is the difference between option prices from our stochastic variance model and from the Black-Scholes model with equal volatility to option maturity; Except for ρ and λ_2, parameter values are those in Table 5.1.

The first conclusion from Figure 5.1 is that for both values of ρ, the introduction of ambiguity aversion about the stochastic variance process does not change the shape of the curve presented in Heston (1993) representing the differences between the option prices from a stochastic variance model and from the B-S model with comparable volatility. For a positive correlation $(\rho = 0.5)$ between shocks in the spot asset return and in its variance, the prices of out-of-the-money (OTM) options are higher than those obtained in the B-S model, and lower for in-the-money (ITM) options. The contrary happens when the correlation is negative $(\rho = -0.5)$.

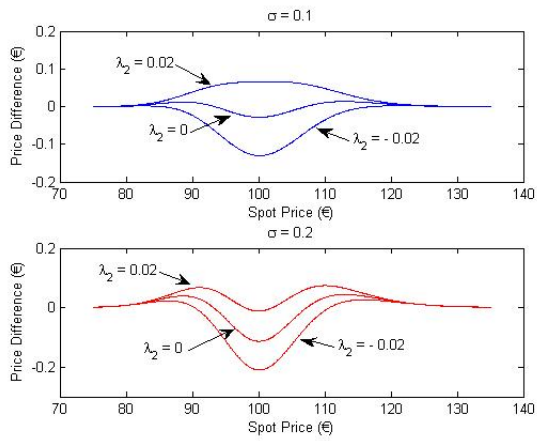
However, Figure 5.1 also shows that, for both values of ρ and from far OTM to far ITM moneyness spectrum, when ambiguity is considered the graphical representation of option prices differences shifts upwards when $\lambda_2 = 0.02$ and downwards when $\lambda_2 = -0.02$.

The conclusion is that ambiguity about the variance stochastic process implies a relative increase of option prices generated by our model, that accommodates for that ambiguity, versus B-S option prices with comparable volatility when the variance ambiguity price is positive (and the opposite when it is negative).

We now analyze the impact from changes in σ, the standard deviation of variance, on the difference between option prices obtained through our model and the B-S model with comparable volatility. When $\sigma = 0$, the variance is deterministic and spot returns have a normal distribution. When $\sigma > 0$, as it is assumed in our setting, the kurtosis of the spot return distribution increases.

We use the default parameters in Table 5.1 and one alternative value for σ $(= 0.2)$, for a complete spectrum of option moneyness (spot prices S from 75 to 135). For each of the σ values, the same three values for λ_2 $(-0.02, 0, 0.02)$ are considered. Simulation results are disclosed in Figure 5.2:

Figure 5.2: Option Price Differences: σ and λ_2 sensitivity.



Note: Price Difference is the difference between option prices from our stochastic variance model and from the Black-Scholes model with equal volatility to option maturity; Except for σ and λ_2, parameter values are those in Table 5.1.

It can be seen that when there is no ambiguity $(\lambda_2 = 0)$, as in Heston (1993), the price of far OTM and far ITM options increases versus B-S prices with comparable volatility, and the opposite happens for near-the-money options. Figure 5.2 also shows that when ambiguity is considered, for both σ values under analysis, there is a relative increase of option prices from far OTM to far ITM moneyness spectrum when the variance ambiguity price is positive ($\lambda_2 = 0.02$) and the opposite when it is negative.

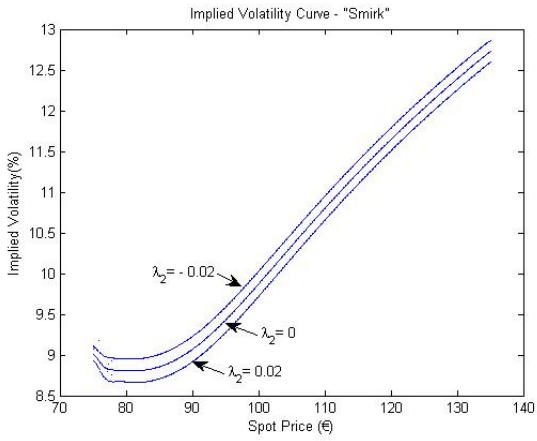
Note that in both Figure 5.1 and Figure 5.2 there is convergence between option prices from our stochastic variance model and the B-S model when the underlying asset spot price converges to zero or increases infinitely. This is expectable as, when the underlying

asset price decreases, call price tends to zero, and when the underlying asset price increases, the call price tends to the difference between the underlying asset spot price and the option strike price (Rouah and Vainberg (2007), ch.10).

Until now, the analysis has been focused on the difference between option prices given by our model and by the B-S model with comparable volatility. However, as the same absolute price difference can have different meanings, it is convenient to analyze the implied volatility curve. This is done by making use of the one-to-one relationship between volatility and call option prices: option prices generated by our model are introduced in the B-S formula to obtain the implied volatility.

If $\rho < 0$, the underlying asset return distribution is negatively skewed which, in consistency with the option price differences disclosed in Figure 5.1, corresponds to an upwards "smirk" shape of the implied volatility curve from option prices. This happens when ambiguity is null. When ambiguity is considered, this upward "smirk" of the implied volatility curve remains unchanged but the curve moves slightly downwards (for all moneyness spectrum) when the variance ambiguity price is positive, and the contrary when it is negative, as it is illustrated in Figure 5.3.[14]

Figure 5.3: Implied Volatility Curve ($\rho = -0.5$)



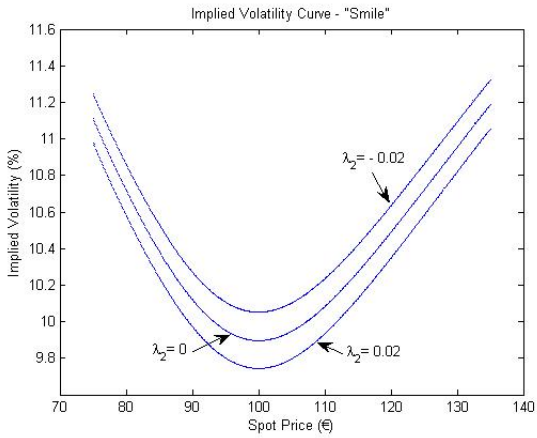
Note: Implied volatility is obtained by considering the option price generated by our model into the B-S formula and then calculating the implied volatility input; Except for ρ and λ_2, parameter values are those in Table 5.1. The B-S volatility input with ($\lambda_2 = -0.02; \sigma = 0.1$) is 10.5%, with ($\lambda_2 = 0; \sigma = 0.1$) is 10% and with ($\lambda_2 = 0.02; \sigma = 0.1$) is 9.5%.

When $\sigma > 0$, kurtosis of the spot return distribution increases which, in consistency with the option price differences disclosed in Figure 5.2, corresponds to a "smile" shape of the implied volatility curve from option prices. This happens when ambiguity is null. When

[14]When $\rho > 0$, the implied volatility curve has a downward "smirk" shape. Although not disclosed here, simulations run for $\rho = 0.5$ imply exactly the same conclusions as those obtained with $\rho = -0.5$ regarding the introduction of ambiguity.

ambiguity is considered that "smile" shape is kept but the implied volatility curve moves slightly downwards (for all moneyness spectrum) when the variance ambiguity price is positive, and the contrary when it is negative. This is illustrated in Figure 5.4.

Figure 5.4: Implied Volatility Curve ($\sigma = 0.1$)



Note: Implied volatility is obtained by considering the option price generated by our model into the B-S formula and then calculating the implied volatility input; Except for λ_2, parameter values are those in Table 5.1. The B-S volatility input with $(\lambda_2 = -0.02; \sigma = 0.1)$ is 10.5%, with $(\lambda_2 = 0; \sigma = 0.1)$ is 10% and with $(\lambda_2 = 0.02; \sigma = 0.1)$ is 9.5%.

From Figures 5.3 and 5.4 it is immediate to conclude that, with the variance uncertainty price specification (5.5), when ambiguity aversion about the variance stochastic process is considered option prices decreases versus a scenario with stochastic variance without ambiguity when the price for that ambiguity is positive and the contrary when it is negative. Moreover, considering the vertical scale of Figures 5.3 and 5.4 and the small shifts between implied volatility curves, we conclude that the magnitude of that price changes is small.

There is a "puzzle" in the literature concerning the fact that option-implied variance for a certain maturity tends to be higher than the conditional expectation of realized variance for that period of time (see, for e.g., Drechsler and Yaron (2008) and Todorov (2010)). This spread is commonly designated as the variance premium. This is an important puzzle to be solved, namely for the design of variance trading strategies (e.g. Bondarenko (2004) and Egloff et al. (2007)). The present chapter gives a contribution in that direction. Concretely, under our option pricing formula when the variance ambiguity price is positive ($\lambda_2 > 0$), the option implied volatility decreases when ambiguity is taken into account. This means that the variance premium can effectively be smaller: the puzzle is partially explained by the consideration of ambiguity about the variance stochastic process.

Additionally, assume that the absolute value of the variance ambiguity price $\mid \lambda_2 \mid$ is a

positive function of the level of ambiguity about the stochastic variance process.[15] When that ambiguity strongly increases, as it is expected to occur during periods of high "turbulence" in the financial markets, our option pricing formula implies (when $\lambda_2 > 0$) that the variance premium eventually disappears or even becomes negative. As shown in the next subsection, this seems to be consistent with the empirical evidence of the immediate months following the Lehman Brothers collapse in September 2008.

5.3.1 Variance Premium Data Analysis

Formally, the most common definition in the literature of the variance premium at date t (vp_t) is the spread of expectations under the uncertainty-neutral and the physical measures ($\mathbb{Q}$ and $\mathbb{P}$, respectively) of the variance of returns for the period $t+1$ (for simplicity we adopt a discrete time approach in this analysis):

$$vp_t = E_t^{\mathbb{Q}} v_{t+1} - E_t^{\mathbb{P}} v_{t+1}.$$

We follow the approach of Drechsler and Yaron (2008) for the estimation of vp and use data for both the VIX Index[16] and the S&P500 Spot Index from January 1st 1990 (90m1) until November 26th 2008 (08m11) from Bloomberg.

- $E_t^{\mathbb{Q}} v_{t+1}$: uses the VIX Index at date t, squaring its value (in order to be expressed in variance terms) and then dividing it by 12 (in order to get a monthly figure). The VIX value for a particular month is the value of the last observation for that month.
- $E_t^{\mathbb{P}} v_{t+1}$: uses the forecast at date t of the sum of the S&P500 Spot Index squared daily log-returns for the next month ($\hat{RV}_{t+1}$), obtained from the estimated regression of conditional variance given by $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$.[17] In Table 5.3 (Appendix 5.5.6) the estimation results for this regression are presented.

The (conditional) variance premium vp is therefore calculated on a monthly basis as:

$$vp_t = E_t^{\mathbb{Q}} v_{t+1} - E_t^{\mathbb{P}} v_{t+1} = \frac{VIX_t^2}{12} - \hat{RV}_{t+1},$$

[15] Which is an intuitive assumption and is theoretically supported by Faria and Correia-da Silva (2010b).

[16] The VIX Index from CBOE is probably the most used volatility index, both in the literature and in the industry. It measures the one-month implied volatility in the S&P 500 Index option prices. For full details on the VIX Index construction methodology please see http://www.cboe.com/micro/vix/.

[17] In Drechsler and Yaron (2008), this is one of the three regressions of realized variance on lagged predictors. The other two regressions are based on squares of the log returns over 5-minute intervals during a month of the S&P500 Futures and S&P500 Index data. The reason why we only use S&P500 Index daily data, is availability. The sample period in Drechsler and Yaron (2008) is monthly and covers 90m1 to 07m3.

with the key descriptive statistics being those in the following table:

Table 5.2: Descriptive Statistics for the Variance Premium (vp)

vp	$90m1-07m3$ - DY	$90m1-07m3$	$90m1-08m8$	$90m1-08m11$
Mean	12.67	12.71	12.21	8.93
Median	7.97	8.03	7.56	5.12
Minimum	-4.02	-4.11	-4.00	-162.69
Std. Deviation	14.38	14.40	14.08	22.47
Skewness	2.45	2.43	2.56	-2.03
Kurtosis	12.62	12.48	13.78	22.2

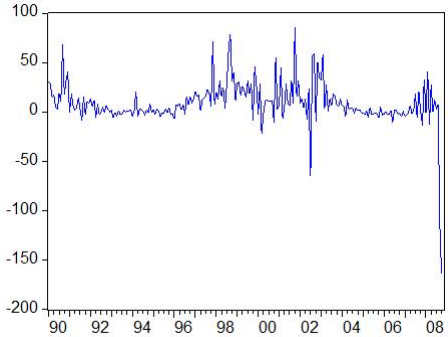
Note: Descriptive statistics in the column entitled $90m1-07m3$ - DY are those obtained in Drechsler and Yaron (2008) for the sample period $90m1-07m3$. Descriptive statistics in the remaining three columns are obtained for different sampling periods, as indicated in their respective titles.

The first sample period (90m1- 07m03) is used for comparison with the result of Drechsler and Yaron (2008), just to cross-check our vp estimations. Regarding the other two sampling periods, they differ by three months (September 08, October 08 and November 08), which are relevant months as they represent the immediate post-Lehman Brothers collapse time period. Uncertainty about the global financial system reached a very high level during that period, which Blanchard (2009) suggestively named as "Knight time". We therefore believe those are relevant months to test the hypothesis that when ambiguity strongly increases, the variance premium vp eventually disappears or even becomes negative.

Comparing the second and third columns in Table 5.2, it is clear that our vp computation is very close to that in Drechsler and Yaron (2008) (DY) for the same period $90m1-07m3$. Additionally, Table 5.2 allows to conclude that vp is relevant in absolute terms: average value of 9 percentage points between 90m1 to 08m11. More importantly for the analysis under way, vp statistics remain relatively unchanged when estimated using $90m1-07m3$ or $90m1-08m8$ data. However, they change significantly when the months from September to November 2008 are included.[18] In fact, as shown in Figure 5.5, between September and November 2008 the variance premium becomes strongly negative.

[18]This is formally confirmed with the rejection of the null hypothesis of no structural change in the estimated regression of conditional variance $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$, using the Chow breakpoint test at September 2008 (F-Statistic and Log likelihood ratio of 216.4193 and 310.5268, respectively).

Figure 5.5: Variance Premium $(90m1 - 08m11)$



Note: Vertical axis is measured in percentage points. Horizontal axis respects to the time period: 1990m1 until 2008m11

This is also suggested by results in Table 5.2, by comparing the vp minimum value and the sign of its skewness under the last sampling period $90m1 - 08m11$ versus the others. Overall, this evidence supports the claim from our option pricing model that, if $\lambda_2 > 0$ and ambiguity increases, the variance premium can become negative.[19]

5.4 Concluding Remarks

We extended the option pricing model of Heston (1993) by considering ambiguity about the variance stochastic process. A new variance uncertainty price specification is used for the deduction of the option pricing formula. It contains two components: the variance risk price, which is linear on the instantaneous variance, v_t, and the variance ambiguity price, which is a non-linear function of v_t. This specification is theoretically motivated by the general equilibrium model in Faria and Correia-da Silva (2010b). The model of Heston (1993) is obtained as a particular case when uncertainty is exclusively risk.

The main result of the chapter is a closed-form solution for the price of a European call option when the underlying asset return's variance is stochastic, correlated with the asset spot return, and there exists ambiguity (aversion) about the variance stochastic process.

Analyzing the impact of correlation between shocks in the spot asset return and its variance and of the variance of variance, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process implies, across all moneyness levels, a relative increase of option prices generated by our model versus those obtained under the Black and Scholes (1973) model with comparable variance. The opposite happens when the variance ambiguity price is negative.

[19]The sign of λ_2, and therefore the sign of the variance ambiguity price, needs to be estimated. This is left for future work.

Moreover, we conclude that when the variance ambiguity price is positive, ambiguity about the variance stochastic process induces a decrease on option prices versus a scenario with stochastic variance and no ambiguity. The contrary happens when the variance ambiguity price is negative.

The implied volatility curve from our model has a "smirk" shape, when the correlation between shocks in price and variance is non-zero, and a "smile" shape, when the variance of variance is non-zero. When the variance ambiguity price is positive, there is a downwards shift of the implied volatility curve when ambiguity is considered, for all option's moneyness spectrum. This means that, under that scenario for the variance ambiguity price, the developed option pricing model is a contribution for the explanation of the variance premium puzzle, particularly relevant for the design of variance trading strategies.

As future research topics on the back of this chapter, we highlight the pricing and hedging of variance and volatility derivatives (including variance and volatility swaps) considering ambiguity about the stochastic variance process,[20] and the estimation of (i) the option's implied variance ambiguity price (adjusting the obtained pricing formula for the case in which the underlying asset pays some income during the life of the option) and (ii) the option's implied ambiguity level, studying its dynamics and empirical strength as market leading indicator (at least for the market of that option's underlying asset). We believe this latter information could be relevant not only for investors but also for policy makers.

[20]We thank Alejandro Balbás for this suggestion.

Appendix

5.5 Mathematical Proofs

5.5.1 Equation (5.3)

From (5.2) and recognizing that $v_t = \left(\sqrt{v_t}\right)^2$, application of Itô's Lemma gives:

$$\begin{aligned}
dv_t &= \left(\frac{\partial\left(\sqrt{v_t}\right)^2}{\partial t} + \frac{\partial\left(\sqrt{v_t}\right)^2}{\partial\sqrt{v_t}}\left(-\frac{\kappa}{2}\sqrt{v_t}\right) + \frac{1}{2}\frac{\partial^2\left(\sqrt{v_t}\right)^2}{\partial^2\sqrt{v_t}}\frac{\sigma^2}{4}\right)dt + \frac{\partial\left(\sqrt{v_t}\right)^2}{\partial\sqrt{v_t}}\frac{\sigma}{2}dW_v \\
\Leftrightarrow dv_t &= \left(2\sqrt{v_t}\left(-\frac{\kappa}{2}\sqrt{v_t}\right) + \frac{\sigma^2}{4}\right)dt + 2\sqrt{v_t}\frac{\sigma}{2}dW_v\,, \\
\Leftrightarrow dv_t &= \left(-\kappa v_t + \frac{\sigma^2}{4}\right)dt + \sqrt{v_t}\sigma dW_v\,,
\end{aligned}$$

which is (5.3).

□

5.5.2 Equation (5.6)

The uncertainty neutral dynamics for S_t and v_t are given by:

$$dS_t = rS_t dt + \sqrt{v_t}S_t dW_S, \tag{5.19}$$

$$dv_t = \left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})\right]dt + \sigma\sqrt{v_t}dW_v, \tag{5.20}$$

where the drift in (5.19) is the risk free rate instead of the expected return μ as given in (5.1), in consistency of no-arbitrage arguments, and (5.20) has a drift adjustment given by the variance uncertainty price specification (5.5).

Following Itô's theorem (see, for e.g., Kamien and Schwartz (1991) ch. 22), $U(S_t, v_t, t)$ dynamics is given by:

$$\begin{aligned}
dU &= \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}(dS)^2 \\
&\quad + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}(dv)^2 + \frac{\partial^2 U}{\partial S\partial v}dvdS,
\end{aligned} \tag{5.21}$$

where:

$$\begin{aligned}
(dS)^2 &= r^2 S_t^2 (dt)^2 + 2rS_t^2 \sqrt{v_t} dt dW_S + v_t S_t^2 (dW_S)^2 \,, \\
\Leftrightarrow (dS)^2 &= 0 + 0 + v_t S_t^2 dt \,; \\
\\
(dv)^2 &= [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})]^2 (dt)^2 \\
&\quad + 2[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] \sigma \sqrt{v_t} dt dW_v + \sigma^2 v_t (dW_v)^2 \,, \\
\Leftrightarrow (dv)^2 &= 0 + 0 + \sigma^2 v_t dt \,; \\
\\
dvdS &= rS_t [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] (dt)^2 + rS_t \sigma \sqrt{v_t} dt dW_v \,, \\
&\quad + \sqrt{v_t} S_t [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] dW_S dt + \sigma v_t S_t dW_S dW_v \,, \\
\Leftrightarrow dvdS &= 0 + 0 + 0 + \rho \sigma v_t S_t dt.
\end{aligned}$$

Equation (5.21) can therefore be written as:

$$dU = \frac{\partial U}{\partial S} dS + \frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial t} dt + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} v_t S_t^2 dt + \frac{1}{2} \frac{\partial^2 U}{\partial v^2} \sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v} \rho \sigma v_t S_t dt. \quad (5.22)$$

The equity value of a hedged position is given by (see, for e.g., Black and Scholes (1973)):

$$S_t - \frac{U}{\frac{\partial U}{\partial S}} \,,$$

where $\frac{1}{\frac{\partial U}{\partial S}}$ is the number of options to buy . The change in the equity value of the hedged position (which is not a perfectly hedged position due to market incompleteness - see footnote 2) is therefore given by:

$$dS - dU \frac{1}{\frac{\partial U}{\partial S}} \,,$$

which from (5.22) equals:

$$-\left(\frac{\partial U}{\partial v} dv + \frac{\partial U}{\partial t} dt + \frac{1}{2} \frac{\partial^2 U}{\partial S^2} v_t S_t^2 dt + \frac{1}{2} \frac{\partial^2 U}{\partial v^2} \sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v} \rho \sigma v_t S_t dt \right) \frac{1}{\frac{\partial U}{\partial S}} \,.$$

Considering the no-arbitrage rational that the change in the equity value of the hedged

position must be equal to the time effect (rdt) on the value of that position, one obtains:

$$\begin{aligned}
\left(S_t - \frac{U}{\frac{\partial U}{\partial S}}\right) rdt &= -\left(\frac{\partial U}{\partial v}dv + \frac{\partial U}{\partial t}dt + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}v_t S_t^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt \right. \\
&\quad \left. + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t dt\right)\frac{1}{\frac{\partial U}{\partial S}} \\
\Leftrightarrow \left(S_t \frac{\partial U}{\partial S} - U\right)\frac{1}{\frac{\partial U}{\partial S}} rdt &= -\left(\frac{\partial U}{\partial v}\left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})\right]dt \right. \\
&\quad + \frac{\partial U}{\partial v}\sigma\sqrt{v_t}dW_v + \frac{\partial U}{\partial t}dt \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}v_t S_t^2 dt + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t dt\right)\frac{1}{\frac{\partial U}{\partial S}}.
\end{aligned}$$

Taking expectations on both sides of this equation ($E(dW_v) = 0$) and eliminating on both sides the common term $dt\frac{1}{\frac{\partial U}{\partial S}}$:

$$\begin{aligned}
\Leftrightarrow \left(S_t \frac{\partial U}{\partial S} - U\right) r &= -\left(\frac{\partial U}{\partial v}\left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})\right] + \frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}v_t S_t^2 \right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 U}{\partial v^2}\sigma^2 v_t + \frac{\partial^2 U}{\partial S \partial v}\rho\sigma v_t S_t\right) \\
\Leftrightarrow 0 &= \frac{1}{2}v_t S_t^2 \frac{\partial^2 U}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 U}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 U}{\partial v^2} + rS_t\frac{\partial U}{\partial S} \\
&\quad + \left[\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})\right]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t},
\end{aligned}$$

which is the partial differential equation (5.6).

□

5.5.3 Equation (5.9)

We prove that substituting (5.8) into (5.6) implies that probabilities P_j $(j = 1, 2)$, satisfy the PDEs (5.9). We analyze individually each of the terms in (5.8).

Starting with the first one and making $U_1 = S_t P_1$ one gets:

$$
\begin{aligned}
\frac{\partial U_1}{\partial S} &= P_1 + S_t \frac{\partial P_1}{\partial S}, \\
\frac{\partial^2 U_1}{\partial S^2} &= \frac{\partial P_1}{\partial S} + \frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2} = 2\frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2}, \\
\frac{\partial^2 U_1}{\partial S \partial v} &= \frac{\partial P_1}{\partial v} + S_t \frac{\partial^2 P_1}{\partial S \partial v}, \\
\frac{\partial U_1}{\partial v} &= S_t \frac{\partial P_1}{\partial v}, \\
\frac{\partial^2 U_1}{\partial v^2} &= S_t \frac{\partial^2 P_1}{\partial v^2}, \\
\frac{\partial U_1}{\partial t} &= S_t \frac{\partial P_1}{\partial t}.
\end{aligned}
$$

As $U_1 = S_t P_1$ must satisfy the PDE (5.6), then:

$$
\begin{aligned}
0 &= \frac{1}{2} v_t S_t^2 \frac{\partial^2 U_1}{\partial S^2} + \rho \sigma v_t S_t \frac{\partial^2 U_1}{\partial S \partial v} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 U_1}{\partial v^2} + r S_t \frac{\partial U_1}{\partial S} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] \frac{\partial U_1}{\partial v} - rU + \frac{\partial U_1}{\partial t}, \\
\Leftrightarrow 0 &= \frac{1}{2} v_t S_t^2 \left(2\frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2} \right) + \rho \sigma v_t S_t \left(\frac{\partial P_1}{\partial v} + S_t \frac{\partial^2 P_1}{\partial S \partial v} \right) + \frac{1}{2} \sigma^2 v_t S_t \frac{\partial^2 P_1}{\partial v^2} \\
&\quad + r S_t \left(P_1 + S_t \frac{\partial P_1}{\partial S} \right) + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] S_t \frac{\partial P_1}{\partial v} - r S_t P_1 + S_t \frac{\partial P_1}{\partial t}, \\
\Leftrightarrow 0 &= \frac{1}{2} v_t S_t \left(2\frac{\partial P_1}{\partial S} + S_t \frac{\partial^2 P_1}{\partial S^2} \right) + \rho \sigma v_t \left(\frac{\partial P_1}{\partial v} + S_t \frac{\partial^2 P_1}{\partial S \partial v} \right) + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} + r S_t \frac{\partial P_1}{\partial S} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] \frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t}, \\
\Leftrightarrow 0 &= v_t S_t \frac{\partial P_1}{\partial S} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 P_1}{\partial S^2} + \rho \sigma v_t \frac{\partial P_1}{\partial v} + \rho \sigma v_t S_t \frac{\partial^2 P_1}{\partial S \partial v} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} + r S_t \frac{\partial P_1}{\partial S} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] \frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t}, \\
\Leftrightarrow 0 &= (v_t + r) S_t \frac{\partial P_1}{\partial S} + \frac{1}{2} v_t S_t^2 \frac{\partial^2 P_1}{\partial S^2} + \rho \sigma v_t S_t \frac{\partial^2 P_1}{\partial S \partial v} + \frac{1}{2} \sigma^2 v_t \frac{\partial^2 P_1}{\partial v^2} \\
&\quad + [\rho \sigma v_t + \kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2 \sqrt{v_t})] \frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t}. \qquad (5.23)
\end{aligned}
$$

Making $x_t = ln\,(S_t)$, one obtains:

$$\begin{aligned}
\frac{\partial P_1}{\partial S} &= \frac{\partial P_1}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial P_1}{\partial x}\frac{1}{S_t}, \\
\frac{\partial^2 P_1}{\partial S \partial v} &= \frac{1}{S_t}\frac{\partial^2 P_1}{\partial x \partial v}, \\
\frac{\partial^2 P_1}{\partial S^2} &= -\frac{1}{S_t^2}\frac{\partial P_1}{\partial x} + \frac{1}{S_t}\frac{\partial^2 P_1}{\partial x^2}\frac{\partial x}{\partial S} = -\frac{1}{S_t^2}\frac{\partial P_1}{\partial x} + \frac{1}{S_t^2}\frac{\partial^2 P_1}{\partial x^2}.
\end{aligned} \tag{5.24}$$

With those results, equation (5.23) becomes:

$$\begin{aligned}
(v_t + r)\frac{\partial P_1}{\partial x} + \frac{1}{2}v_t\frac{\partial^2 P_1}{\partial x^2} - \frac{1}{2}v_t\frac{\partial P_1}{\partial x} + \rho\sigma v_t\frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 P_1}{\partial v^2} \\
+ [\rho\sigma v_t + \kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})]\frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} &= 0 \\
\Leftrightarrow \left(\frac{1}{2}v_t + r\right)\frac{\partial P_1}{\partial x} + \frac{1}{2}v_t\frac{\partial^2 P_1}{\partial x^2} + \rho\sigma v_t\frac{\partial^2 P_1}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t\frac{\partial^2 P_1}{\partial v^2} \\
+ [\rho\sigma v_t + \kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})]\frac{\partial P_1}{\partial v} + \frac{\partial P_1}{\partial t} &= 0,
\end{aligned}$$

which is equation (5.9) when $j = 1$.

Regarding the second term in (5.8), the procedure is the same. Start by making $U_2 = Ke^{-r(T-t)}P_2$ one gets:

$$\begin{aligned}
\frac{\partial U_2}{\partial S} &= Ke^{-r(T-t)}\frac{\partial P_2}{\partial S}, \\
\frac{\partial^2 U_2}{\partial S^2} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial S^2}, \\
\frac{\partial^2 U_2}{\partial S \partial v} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial S \partial v}, \\
\frac{\partial U_2}{\partial v} &= Ke^{-r(T-t)}\frac{\partial P_2}{\partial v}, \\
\frac{\partial^2 U_2}{\partial v^2} &= Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial v^2}, \\
\frac{\partial U_2}{\partial t} &= Kre^{-r(T-t)}P_2 + Ke^{-r(T-t)}\frac{\partial P_2}{\partial t}.
\end{aligned}$$

As $U_2 = Ke^{-r(T-t)}P_2$ must satisfy the PDE (5.6), then:

$$\begin{aligned}
0 &= \frac{1}{2}v_t S_t^2 \frac{\partial^2 U_2}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 U_2}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 U_2}{\partial v^2} + rS_t \frac{\partial U_2}{\partial S} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})] \frac{\partial U_2}{\partial v} - rU_2 + \frac{\partial U_2}{\partial t}, \\
\Leftrightarrow 0 &= \frac{1}{2}v_t S_t^2 Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial S^2} + \rho\sigma v_t S_t Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t Ke^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2} \\
&\quad + rS_t Ke^{-r(T-t)} \frac{\partial P_2}{\partial S} + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})] Ke^{-r(T-t)} \frac{\partial P_2}{\partial v} \\
&\quad - rKe^{-r(T-t)}P_2 + Kre^{-r(T-t)}P_2 + Ke^{-r(T-t)} \frac{\partial P_2}{\partial t}, \\
\Leftrightarrow 0 &= \frac{1}{2}v_t S_t^2 \frac{\partial^2 P_2}{\partial S^2} + \rho\sigma v_t S_t \frac{\partial^2 P_2}{\partial S \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} + rS_t \frac{\partial P_2}{\partial S} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t}. \qquad (5.25)
\end{aligned}$$

With $x_t = ln(S_t)$ and using results (5.24), equation (5.25) becomes:

$$\begin{aligned}
0 &= \frac{1}{2}v_t \left(\frac{\partial^2 P_2}{\partial x^2} - \frac{\partial P_2}{\partial x} \right) + \rho\sigma v_t \frac{\partial^2 P_2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} + r\frac{\partial P_2}{\partial x} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t}, \\
\Leftrightarrow 0 &= \frac{1}{2}v_t \frac{\partial^2 P_2}{\partial x^2} + \left(-\frac{1}{2}v_t + r \right) \frac{\partial P_2}{\partial x} + \rho\sigma v_t \frac{\partial^2 P_2}{\partial x \partial v} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 P_2}{\partial v^2} \\
&\quad + [\kappa(\theta - v_t) - (\lambda_1 v_t + \lambda_2\sqrt{v_t})] \frac{\partial P_2}{\partial v} + \frac{\partial P_2}{\partial t},
\end{aligned}$$

which is equation (5.9) when $j = 2$.

□

5.5.4 Characteristic Functions

We follow closely the explanation in Heston (1993), in order to prove that (i) probabilities P_j $(j = 1,2)$, are two conditional probabilities, under different measures, that the option expires in-the-money; (ii) obtain the characteristic function expression in (5.13).

Assume that x_t and v_t follow the "uncertainty-neutral" processes (5.10)-(5.11). Consider a function $f(x,v,t)$ that is a conditional expectation of some other function $g(\cdot)$ of x_t and

v_t at a posterior date T:

$$f(x,v,t) = E\left[g(x_T, v_T) \mid x_t = x,\, v_t = v\right]. \tag{5.26}$$

It is immediate to conclude that equation (5.26) imposes the terminal condition:

$$f(x,v,T) = g(x,v). \tag{5.27}$$

Following Itô's theorem (see, for e.g., Kamien and Schwartz (1991) ch.22), $f(x,v,t)$ dynamics is given by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}(dv)^2 + \frac{\partial^2 f}{\partial x \partial v}dxdv. \tag{5.28}$$

Recalling that $dW_S dW_v = \rho dt\,, dW_m dt = 0\,, (dt)^2 = 0\,, (dW_m)^2 = dt$ and $m = S,\, v$, one obtains:

$$\begin{aligned}
(dx)^2 &= \left(u_j v_t + r\right)^2 (dt)^2 + 2\left(u_j v_t + r\right)\sqrt{v_t}dtdW_S + v_t (dW_S)^2 \\
&= 0 + 0 + v_t dt\,,
\end{aligned}$$

$$\begin{aligned}
(dv)^2 &= \left(a - b_j v_t - c\sqrt{v_t}\right)^2 (dt)^2 + 2\left(a - b_j v_t - c\sqrt{v_t}\right)\sigma\sqrt{v_t}dtdW_v + \sigma^2 v_t (dW_v)^2 \\
&= 0 + 0 + \sigma^2 v_t dt\,,
\end{aligned}$$

$$\begin{aligned}
dxdv &= \left(u_j v_t + r\right)\left(a - b_j v_t - c\sqrt{v_t}\right)(dt)^2 + \sqrt{v_t}\left(a - b_j v_t - c\sqrt{v_t}\right)dW_S dt \\
&\quad + \left(u_j v_t + r\right)\sigma\sqrt{v_t}dtdW_v + v_t \sigma dW_S dW_v \\
&= 0 + 0 + 0 + v_t \sigma\rho dt.
\end{aligned}$$

Introducing these results in equation (5.28) and considering the "uncertainty-neutral" dynamics (5.10)-(5.11) the former becomes:

$$\begin{aligned}
df &= \frac{\partial f}{\partial x}\left[\left(u_j v_t + r\right)dt + \sqrt{v_t}dW_S\right] + \frac{\partial f}{\partial v}\left[\left(a - b_j v_t - c\sqrt{v_t}\right)dt + \sigma\sqrt{v_t}dW_v\right] \\
&\quad + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}v_t dt + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma^2 v_t dt + \frac{\partial^2 f}{\partial x \partial v}v_t \sigma\rho dt\,, \\
\Leftrightarrow df &= \left(\frac{\partial f}{\partial x}\left(u_j v_t + r\right) + \frac{\partial f}{\partial v}\left(a - b_j v_t - c\sqrt{v_t}\right) + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}v_t\right. \\
&\quad \left. + \frac{1}{2}\frac{\partial^2 f}{\partial v^2}\sigma^2 v_t + \frac{\partial^2 f}{\partial x \partial v}v_t \sigma\rho\right)dt + \frac{\partial f}{\partial x}\sqrt{v_t}dW_S + \frac{\partial f}{\partial v}\sigma\sqrt{v_t}dW_v.
\end{aligned} \tag{5.29}$$

Considering that by iterated expectations $f(x,v,t)$ must be a martingale, then $E(df)=0$. Taking expectations on both sides of equation (5.29) one obtains:

$$0=\frac{1}{2}v_t\frac{\partial^2 f}{\partial x^2}+v_t\sigma\rho\frac{\partial^2 f}{\partial x\partial v}+\frac{1}{2}\sigma^2 v_t\frac{\partial^2 f}{\partial v^2}+\left(u_j v_t+r\right)\frac{\partial f}{\partial x}+\left(a-b_j v_t-c\sqrt{v_t}\right)\frac{\partial f}{\partial v}+\frac{\partial f}{\partial t}. \tag{5.30}$$

We are now ready to obtain the desired proofs.

5.5.4.1 Probabilities P_j are conditional probabilities that the option expires in-the-money

Comparing equations (5.9) and (5.30) it is immediate to conclude that, when x_t and v_t follow the uncertainty neutral processes (5.10)-(5.11), probabilities P_j $(j=1,2)$, and function f satisfy an equal PDE.

Now recalling the terminal condition imposed by (5.27), it is immediate to conclude that if

$$f(x,v,T)=g(x,v)=1_{\{x\geqslant ln(K)\}},$$

then the solution of (5.30) is the conditional probability at time t that $x_T\geqslant ln(K)$. This is the same to say that probabilities P_j are conditional probabilities that the option expires in-the-money:

$$P_j(x,v,T;ln(K))=Pr\left[x_T\geqslant ln(K)\mid x_t=x,\,v_t=v\right],$$

as we wanted to prove. Distinct dynamics under each of the uncertainty neutral measures corresponding to P_1 and P_2 result from differences between u_j and b_j $(j=1,2)$ specifications in (5.9), proved above (Appendix 5.5.3).

□

5.5.4.2 Characteristic Function (5.13)

If $g(x,v)=e^{i\phi x}$ the solution of equation (5.30) is the characteristic function of $x=ln(S)$ (Heston (1993)). We guess the following functional form for the characteristic function:

$$f(x,v,t;\phi) \quad = \quad e^{C(T-t;\phi)+D(T-t;\phi)v_t+E(T-t;\phi)\sqrt{v_t}+i\phi x}. \tag{5.31}$$

This guess form is close to that of Heston (1993), with the difference being a new term $E(T-t;\phi)\sqrt{v_t}$ within the exponential argument. This new term results from the introduction of ambiguity aversion about the stochastic variance process.

The purpose is to obtain the expressions of $C(T-t;\phi), D(T-t;\phi)$ and $E(T-t;\phi)$. From (5.31):

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= i\phi e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial x^2} &= -\phi^2 e^{[\cdot]}, \\
\frac{\partial f}{\partial v} &= \left(D+\frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial v^2} &= \left[\left(D+\frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)^2-\frac{1}{4}\frac{1}{v_t\sqrt{v_t}}E\right]e^{[\cdot]}, \\
\frac{\partial^2 f}{\partial x \partial v} &= i\phi\left(D+\frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)e^{[\cdot]}, \\
\frac{\partial f}{\partial t} &= \left(\frac{\partial C}{\partial t}+\frac{\partial D}{\partial t}v_t+\frac{\partial E}{\partial t}\sqrt{v_t}\right)e^{[\cdot]},
\end{aligned}
\tag{5.32}
$$

where $e^{[\cdot]} = e^{C(T-t)+D(T-t)v_t+E(T-t)\sqrt{v_t}+i\phi x}$. Introducing those results into equation (5.30)

one obtains:

$$
\begin{aligned}
0 &= -\frac{1}{2}\phi^2 v_t e^{[\cdot]} + i\phi v_t \sigma\rho \left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right) e^{[\cdot]} \\
&\quad + \frac{1}{2}\sigma^2 v_t \left[\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)^2 - \frac{1}{4}\frac{1}{v_t\sqrt{v_t}}E\right] e^{[\cdot]} + i\phi e^{[\cdot]}\left(u_j v_t + r\right) \\
&\quad + \left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right) e^{[\cdot]}\left(a - b_j v_t - c\sqrt{v_t}\right) + \left(\frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v_t + \frac{\partial E}{\partial t}\sqrt{v_t}\right) e^{[\cdot]}, \\
\Leftrightarrow 0 &= -\frac{1}{2}\phi^2 v_t + i\phi D\sigma\rho v_t + \frac{1}{2}Ei\phi\sigma\rho\sqrt{v_t} + \frac{1}{2}\sigma^2\left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)^2 v_t - \frac{1}{8}\sigma^2 E\frac{1}{\sqrt{v_t}} \\
&\quad + i\phi u_j v_t + i\phi r + \left(D + \frac{1}{2}\frac{1}{\sqrt{v_t}}E\right)\left(a - b_j v_t - c\sqrt{v_t}\right) + \frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v_t + \frac{\partial E}{\partial t}\sqrt{v_t}, \\
\Leftrightarrow 0 &= -\frac{1}{2}\phi^2 v_t + i\phi D\sigma\rho v_t + \frac{1}{2}Ei\phi\sigma\rho\sqrt{v_t} + \frac{1}{2}\sigma^2 D^2 v_t + \frac{1}{2}\sigma^2 DE\sqrt{v_t} + \frac{1}{8}\sigma^2 E^2 \\
&\quad - \frac{1}{8}\sigma^2 E\frac{1}{\sqrt{v_t}} + i\phi u_j v_t + i\phi r + Da + \frac{1}{2}Ea\frac{1}{\sqrt{v_t}} - Db_j v_t - \frac{1}{2}Eb_j\sqrt{v_t} - cD\sqrt{v_t} \\
&\quad - \frac{1}{2}cE + \frac{\partial C}{\partial t} + \frac{\partial D}{\partial t}v_t + \frac{\partial E}{\partial t}\sqrt{v_t}, \\
\Leftrightarrow 0 &= \left(i\phi r + Da + \frac{1}{8}\sigma^2 E^2 - \frac{1}{2}cE + \frac{\partial C}{\partial t}\right) \\
&\quad + \left(-\frac{1}{2}\phi^2 + i\phi D\sigma\rho + \frac{1}{2}\sigma^2 D^2 + i\phi u_j - Db_j + \frac{\partial D}{\partial t}\right) v_t \\
&\quad + \left(\frac{1}{2}Ei\phi\sigma\rho + \frac{1}{2}\sigma^2 DE - \frac{1}{2}Eb_j - cD + \frac{\partial E}{\partial t}\right)\sqrt{v_t} \\
&\quad + \left(\frac{1}{2}Ea - \frac{1}{8}\sigma^2 E\right)\frac{1}{\sqrt{v_t}}. \qquad (5.33)
\end{aligned}
$$

Equalizing the first coefficient and each of the others associated to v_t, $\sqrt{v_t}$, $\frac{1}{\sqrt{v_t}}$ to zero, equation (5.33) is equivalent to the system of the following four equations:

$$
\begin{aligned}
0 &= i\phi r + aD + \frac{1}{8}\sigma^2 E^2 - \frac{1}{2}cE + \frac{\partial C}{\partial t} && (5.34) \\
0 &= -\frac{1}{2}\phi^2 + i\phi u_j + i\phi\sigma\rho D - b_j D + \frac{1}{2}\sigma^2 D^2 + \frac{\partial D}{\partial t} && (5.35) \\
0 &= \frac{1}{2}i\phi\sigma\rho E - \frac{1}{2}b_j E + \frac{1}{2}\sigma^2 DE - cD + \frac{\partial E}{\partial t} && (5.36) \\
0 &= \frac{1}{2}\left(a - \frac{1}{4}\sigma^2\right)E. && (5.37)
\end{aligned}
$$

It is immediate from the parameter specifications $a = \kappa\theta$ (equation (5.9)) and $\theta = \frac{\sigma^2}{4\kappa}$

(equation (5.4)), that equation (5.37) is always verified,[21] and can therefore be ignored.

Consequently the relevant system to be solved in order to obtain the analytical expression of the characteristic functions consists in the three differential equations (5.34)-(5.35)-(5.36) subject to $C(0;\phi)=0, D(0;\phi)=0$ and $E(0;\phi)=0$.

In order to solve this system, we start noting that the non-linear differential equation (5.35) is unchanged versus Heston (1993) (pp.341).[22] As this equation only depends on D, Heston (1993) solution for this equation continues to apply. We therefore have:

$$D(\tau;\phi) = \frac{b_j-\rho\sigma\phi i+d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right], \tag{5.38}$$

where $\tau = T-t$ and:

$$\begin{aligned} g &= \frac{b_j-\rho\sigma\phi i+d}{b_j-\rho\sigma\phi i-d}, \\ d &= \sqrt{\left(\rho\sigma\phi i-b_j\right)^2-\sigma^2\left(2u_j\phi i-\phi^2\right)}. \end{aligned}$$

In order to obtain the expression for $E(\tau;\phi)$, we start substituting $D(\tau;\phi)$ expression into equation (5.36):

$$\begin{aligned} 0 &= \frac{1}{2}i\phi\sigma\rho E-\frac{1}{2}b_jE+\frac{1}{2}\sigma^2\frac{\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]E \\ &\quad -c\frac{\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]+\frac{\partial E}{\partial t}, \\ \Leftrightarrow \frac{\partial E}{\partial \tau} &= \left[\frac{1}{2}\left(i\phi\sigma\rho-b_j\right)+\frac{1}{2}\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{d\tau}\right)}{\left(1-ge^{d\tau}\right)}\right]E \\ &\quad -c\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{d\tau}\right)}{\sigma^2\left(1-ge^{d\tau}\right)}, \end{aligned} \tag{5.39}$$

as $\tau = T-t \Rightarrow \frac{\partial E}{\partial \tau} = -\frac{\partial E}{\partial t}$.

Equation (5.39) is a standard linear non-homogeneous first order differential equation, which solution is known to be of the type (Gothen (1997)):

$$E(\tau;\phi) = E_H(\tau;\phi)+E_P(\tau;\phi), \tag{5.40}$$

where $E_H(\tau;\phi)$ and $E_P(\tau;\phi)$ are the solutions of the homogeneous and non-homogeneous

[21] As $a-\frac{1}{4}\sigma^2 = \kappa\theta-\kappa\theta = 0$.

[22] Note that there is a typo in Heston (1993) regarding (5.35). See, for e.g., Mikhailov and Nogel (2003) for the correct expression of the equation, which is equal to (5.35).

parts of equation (5.39), respectively. Consider two new functions $p(\tau;\phi)$ and $q(\tau;\phi)$ defined as:

$$\begin{aligned} p(\tau;\phi) &= -\left[\frac{1}{2}\left(i\phi\sigma\rho - b_j\right) + \frac{1}{2}\frac{\left(b_j - \rho\sigma\phi i + d\right)\left(1-e^{d\tau}\right)}{\left(1-ge^{d\tau}\right)}\right], \\ q(\tau;\phi) &= c\frac{\left(b_j - \rho\sigma\phi i + d\right)\left(1-e^{d\tau}\right)}{\sigma^2\left(1-ge^{d\tau}\right)}. \end{aligned}$$

Additionally, let $P(\tau;\phi)$ be a primitive of $p(\tau;\phi)$. Expressions for $E_H(\tau;\phi)$ and $E_P(\tau;\phi)$ are given by:

$$\begin{aligned} E_H(\tau;\phi) &= \vartheta e^{(-P(\tau;\phi))}, \\ E_P(\tau;\phi) &= -e^{(-P(\tau;\phi))}\left[\int_0^\tau e^{(P(s;\phi))}q(s;\phi)\,ds\right], \end{aligned}$$

with ϑ being an unknown constant.

In order to obtain the expression for $E(\tau;\phi)$ we still have to find out the constant ϑ and the analytical expression for $P(\tau;\phi)$. Starting with the former, we use the restriction $E(0;\phi)=0$:

$$\begin{aligned} E(0;\phi) &= -e^{(-P(0;\phi))}\left[\int_0^0 e^{(P(s;\phi))}q(s;\phi)\,ds\right] + \vartheta e^{(-P(0;\phi))} = 0 \\ &= 0 + \vartheta = 0 \\ \Rightarrow \vartheta &= 0. \end{aligned}$$

The expression of $E(\tau;\phi)$ in (5.40) is therefore given by:

$$E(\tau;\phi) = -e^{(-P(\tau;\phi))}\left[\int_0^\tau e^{(P(s;\phi))}q(s;\phi)\,ds\right]. \tag{5.41}$$

It remains to obtain the analytical expression of $P(\tau;\phi)$. For that, we go back to Heston

(1993) where:

$$\begin{aligned}
\frac{\partial C_H(\tau;\phi)}{\partial \tau} &= r\phi i + aD(\tau;\phi), \\
\Rightarrow C_H(\tau;\phi) &= r\phi i\tau + \frac{a}{\sigma^2}\int_0^\tau \left(b_j - \rho\sigma\phi i + d\right)\left[\frac{1-e^{ds}}{1-ge^{ds}}\right] ds.
\end{aligned}$$

Moreover, Heston (1993) discloses a simplified expression of $C_H(\tau;\phi)$:[23]

$$C_H(\tau;\phi) = r\phi i\tau + \frac{a}{\sigma^2}\left\{\left(b_j - \rho\sigma\phi i + d\right)\tau - 2ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}.$$

This implies:

$$\left(b_j - \rho\sigma\phi i + d\right)\tau - 2ln\left[\frac{1-ge^{d\tau}}{1-g}\right] = \int_0^\tau \left(b_j - \rho\sigma\phi i + d\right)\left[\frac{1-e^{ds}}{1-ge^{ds}}\right] ds. \quad (5.42)$$

Considering that $P(\tau;\phi)$ is a primitive of $p(\tau;\phi)$ given above, by using (5.42) one obtains:

$$\begin{aligned}
P(\tau;\phi) &= \int_0^\tau -\left[\frac{1}{2}\left(i\phi\sigma\rho - b_j\right) + \frac{1}{2}\frac{\left(b_j - \rho\sigma\phi i + d\right)\left(1-e^{ds}\right)}{\left(1-ge^{ds}\right)}\right] ds, \\
\Leftrightarrow P(\tau;\phi) &= -\frac{1}{2}\left(i\phi\sigma\rho - b_j\right)\tau - \frac{1}{2}\int_0^\tau \frac{\left(b_j - \rho\sigma\phi i + d\right)\left(1-e^{ds}\right)}{\left(1-ge^{ds}\right)} ds, \\
\Leftrightarrow P(\tau;\phi) &= -\frac{1}{2}\left(i\phi\sigma\rho - b_j\right)\tau - \frac{1}{2}\left\{\left(b_j - \rho\sigma\phi i + d\right)\tau - 2ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}, \\
\Leftrightarrow P(\tau;\phi) &= -\frac{1}{2}d\tau + ln\left[\frac{1-ge^{d\tau}}{1-g}\right].
\end{aligned}$$

This implies that:

$$\begin{aligned}
e^{P(\tau;\phi)} &= e^{-\frac{d\tau}{2}}\left[\frac{1-ge^{d\tau}}{1-g}\right], \\
\Leftrightarrow e^{(-P(\tau;\phi))} &= e^{\frac{d\tau}{2}}\left[\frac{1-g}{1-ge^{d\tau}}\right].
\end{aligned}$$

[23]Note that $C_H(\tau;\phi)$ refers to $C(\tau;\phi)$ in Heston (1993) model. We make this slight adjustment in notation in order to distinguish from "our" $C(\tau;\phi)$.

Going back to (5.41) and substituting the obtained results we get:

$$
\begin{aligned}
E(\tau;\phi) &= e^{\frac{d\tau}{2}}\left(\frac{g-1}{1-ge^{d\tau}}\right)\left[\int_0^{\tau} e^{-\frac{ds}{2}}\left[\frac{1-ge^{ds}}{1-g}\right]c\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(1-e^{ds}\right)}{\sigma^2\left(1-ge^{ds}\right)}ds\right],\\
\Leftrightarrow E(\tau;\phi) &= e^{\frac{d\tau}{2}}\frac{(g-1)\,c\left(b_j-\rho\sigma\phi i+d\right)}{\left(1-ge^{d\tau}\right)(1-g)\,\sigma^2}\left[\int_0^{\tau} e^{-\frac{ds}{2}}\left(1-e^{ds}\right)ds\right],\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{c\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\left[\int_0^{\tau}\left(e^{-\frac{ds}{2}}-e^{\frac{ds}{2}}\right)ds\right],\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{c\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\left[\frac{-2e^{-\frac{ds}{2}}}{d}-\frac{2e^{\frac{ds}{2}}}{d}\right]_0^{\tau},\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{c\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\left[-\frac{2e^{-\frac{d\tau}{2}}}{d}-\frac{2e^{\frac{d\tau}{2}}}{d}+\frac{4}{d}\right],\\
\Leftrightarrow E(\tau;\phi) &= -e^{\frac{d\tau}{2}}\frac{c\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2\left(1-ge^{d\tau}\right)}\left[\frac{4-2e^{-\frac{d\tau}{2}}-2e^{\frac{d\tau}{2}}}{d}\right],\\
\Leftrightarrow E(\tau;\phi) &= \frac{\left(b_j-\rho\sigma\phi i+d\right)c}{\left(ge^{d\tau}-1\right)\sigma^2 d}\left[4e^{\frac{d\tau}{2}}-2e^{d\tau}-2\right]. \qquad (5.43)
\end{aligned}
$$

Expression (5.43) satisfies the condition $E(0;\phi)=0$.

At last, going back to the equation (5.34), substituting the obtained expressions for $D(\tau;\phi)$

and $E(\tau;\phi)$, and considering that $\frac{\partial C}{\partial \tau} = -\frac{\partial C}{\partial t}$, $C(\tau;\phi)$ is given by:

$$
\begin{aligned}
C(\tau;\phi) &= \int_0^\tau \left(i\phi r + aD(\tau;\phi) + \frac{1}{8}\sigma^2 (E(\tau;\phi))^2 - \frac{1}{2}cE(\tau;\phi) \right) ds, \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau \left(i\phi r + a\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \left[\frac{1-e^{ds}}{1-ge^{ds}} \right] \right. \\
&\quad + \frac{1}{8}\sigma^2 \left[\frac{(b_j - \rho\sigma\phi i + d)\, c}{(ge^{ds}-1)\,\sigma^2 d} \left[4e^{\frac{ds}{2}} - 2e^{ds} - 2 \right] \right]^2 \\
&\quad \left. - \frac{1}{2}c \left[\frac{(b_j - \rho\sigma\phi i + d)\, c}{(ge^{ds}-1)\,\sigma^2 d} \left[4e^{\frac{ds}{2}} - 2e^{ds} - 2 \right] \right] \right) ds, \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau \left(i\phi r + a\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \left[\frac{1-e^{ds}}{1-ge^{ds}} \right] \right. \\
&\quad + \frac{(b_j - \rho\sigma\phi i + d)^2 c^2}{2(ge^{ds}-1)^2 \sigma^2} \left[\frac{2e^{\frac{ds}{2}} - e^{ds} - 1}{d} \right]^2 \\
&\quad \left. - \frac{(b_j - \rho\sigma\phi i + d)\, c^2}{(ge^{ds}-1)\,\sigma^2} \left[\frac{2e^{\frac{ds}{2}} - e^{ds} - 1}{d} \right] \right) ds, \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau \left(i\phi r + a\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \left[\frac{1-e^{ds}}{1-ge^{ds}} \right] \right. \\
&\quad + \frac{(b_j - \rho\sigma\phi i + d)\, c^2 \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{(ge^{ds}-1)\,\sigma^2 d} \\
&\quad \left. \left[\frac{(b_j - \rho\sigma\phi i + d) \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{2(ge^{ds}-1)\, d} - 1 \right] \right) ds, \\
\Leftrightarrow C(\tau;\phi) &= \int_0^\tau \left(i\phi r + a\frac{(b_j - \rho\sigma\phi i + d)}{\sigma^2} \left[\frac{1-e^{ds}}{1-ge^{ds}} \right] \right) ds \\
&\quad + \int_0^\tau \left(\frac{(b_j - \rho\sigma\phi i + d)\, c^2 \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{(ge^{ds}-1)\,\sigma^2 d} \right. \\
&\quad \left. \left[\frac{(b_j - \rho\sigma\phi i + d) \left(2e^{\frac{ds}{2}} - e^{ds} - 1 \right)}{2(ge^{ds}-1)\, d} - 1 \right] \right) ds.
\end{aligned}
$$

It is already known (see above) that:

$$\int_0^\tau \left(i\phi r + a\frac{\left(b_j - \rho\sigma\phi i + d\right)}{\sigma^2}\left[\frac{1-e^{ds}}{1-ge^{ds}}\right]\right) ds \quad =$$

$$r\phi i\tau + \frac{a}{\sigma^2}\left\{\left(b_j - \rho\sigma\phi i + d\right)\tau - 2ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}.$$

So the remaining parcel to be simplified in order to obtain $C(\tau;\phi)$ analytical expression is the integral $\int_0^\tau \left(\frac{\left(b_j-\rho\sigma\phi i+d\right)c^2\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{2\left(ge^{ds}-1\right)d}-1\right]\right) ds$. Let us focus on it:

$$\int_0^\tau \left(\frac{\left(b_j - \rho\sigma\phi i + d\right)c^2\left(2e^{\frac{ds}{2}} - e^{ds} - 1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left(2e^{\frac{ds}{2}} - e^{ds} - 1\right)}{2\left(ge^{ds}-1\right)d} - 1\right]\right) ds$$

$$= \frac{-\left(b_j - \rho\sigma\phi i + d\right)c^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left[1-\left(4e^{\frac{ds}{2}}-6\right)g-\left(4e^{\frac{ds}{2}}-1\right)g^2\right]}{g^2\left(ge^{ds}-1\right)}\right.$$

$$-\left(b_j - \rho\sigma\phi i - d\right)ds - \frac{4\left[b(1-g)+d(1+g)+(g-1)\rho\sigma\phi i\right]ArcTanh\left[e^{\frac{ds}{2}}\sqrt{g}\right]}{g\sqrt{g}}$$

$$\left.+\frac{(1+g)\left[b(g-1)-d(g+1)-(g-1)\rho\sigma\phi i\right]ln\left(ge^{ds}-1\right)}{g^2}\right]_0^\tau$$

where the expression $ArcTanh\left[e^{\frac{ds}{2}}\sqrt{g}\right]$ represents the hyperbolic arc tangent (or inverse hyperbolic tangent) of the complex number $\left(e^{\frac{ds}{2}}\sqrt{g}\right)$. Denominating:

$$\frac{-\left(b_j - \rho\sigma\phi i + d\right)c^2}{2d^3\sigma^2}\left[\frac{\left(b_j - \rho\sigma\phi i + d\right)\left[1-\left(4e^{\frac{d\tau}{2}}-6\right)g-\left(4e^{\frac{d\tau}{2}}-1\right)g^2\right]}{g^2\left(ge^{d\tau}-1\right)}\right.$$

$$-\left(b_j - \rho\sigma\phi i - d\right)d\tau - \frac{4\left[b(1-g)+d(1+g)+(g-1)\rho\sigma\phi i\right]ArcTanh\left[e^{\frac{d\tau}{2}}\sqrt{g}\right]}{g\sqrt{g}}$$

$$\left.+\frac{(1+g)\left[b(g-1)-d(g+1)-(g-1)\rho\sigma\phi i\right]ln\left(ge^{d\tau}-1\right)}{g^2}\right]$$

$$= \omega(\tau;\phi),$$

and

$$\frac{-\left(b_j-\rho\sigma\phi i+d\right)c^2}{2d^3\sigma^2}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left[1+2g-3g^2\right]}{g^2\left(g-1\right)}\right.$$
$$-\frac{4\left[b\left(1-g\right)+d\left(1+g\right)+\left(g-1\right)\rho\sigma\phi i\right]ArcTanh\left[\sqrt{g}\right]}{g\sqrt{g}}$$
$$\left.+\frac{\left(1+g\right)\left[b\left(g-1\right)-d\left(g+1\right)-\left(g-1\right)\rho\sigma\phi i\right]ln\left(g-1\right)}{g^2}\right] = \Omega,$$

one gets $[\omega(\tau;\phi)-\Omega]$ given by:

$$\int_0^\tau\left(\frac{\left(b_j-\rho\sigma\phi i+d\right)c^2\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{2\left(ge^{ds}-1\right)d}-1\right]\right)ds.$$

We can now write the analytical expression for $C(\tau;\phi)$:

$$\begin{aligned}
C(\tau;\phi) &= \int_0^\tau\left(i\phi r+a\frac{\left(b_j-\rho\sigma\phi i+d\right)}{\sigma^2}\left[\frac{1-e^{ds}}{1-ge^{ds}}\right]\right)ds \\
&\quad+\int_0^\tau\left(\frac{\left(b_j-\rho\sigma\phi i+d\right)c^2\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{\left(ge^{ds}-1\right)\sigma^2 d}\right. \\
&\quad\left.\left[\frac{\left(b_j-\rho\sigma\phi i+d\right)\left(2e^{\frac{ds}{2}}-e^{ds}-1\right)}{2\left(ge^{ds}-1\right)d}-1\right]\right)ds, \\
\Leftrightarrow C(\tau;\phi) &= r\phi i\tau+\frac{a}{\sigma^2}\left\{\left(b_j-\rho\sigma\phi i+d\right)\tau-2ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}+\omega(\tau;\phi)-\Omega.
\end{aligned}$$

Note that the condition $C(0;\phi)=0$ is satisfied, as $\omega(0;\phi)=\Omega$.

□

5.5.5 Heston (1993) Model: a Particular Case

If $c=\lambda_2=0$, i.e. there is no ambiguity, then from (5.43), $E(\tau;\phi)=0$ for any τ. With $E(\tau;\phi)=0$, the differential equation (5.34) becomes:

$$i\phi r+aD+\frac{\partial C}{\partial t} = 0,$$

the same obtained by Heston (1993) and from which his expression of $C_H(\tau;\phi)$ is derived.

$D(\tau;\phi)$ analytical expression in our model is the same as in Heston (1993), as previously stated (see comment before equation (5.38)).

So when $c = \lambda_2 = 0$, $C(\tau;\phi)$and $D(\tau;\phi)$ expressions are the same as in Heston (1993), and $E(\tau;\phi)$ becomes zero.

It remains to examine the expression for $f_j(x,v,t;\phi)$ when $c = 0$. It is immediate to conclude from (5.13) that, in this scenario, it becomes $f_j(x,v,t;\phi) = e^{[C(\tau;\phi)+D(\tau;\phi)v_t+i\phi x]}$, which is the same as in Heston (1993).

It is therefore proved that Heston (1993) model can be obtained as a particular case of our setting when there is no ambiguity ($\lambda_2 = 0$).

□

5.5.6 Conditional Expectations of Realized Variance: Estimations

Table 5.3: Conditional Realized Variance

$Sample\,Period$	b_0	b_1	b_2	R^2
$90m1-07m3$	20.59	0.82	-0.34	0.40
$(t-stat)$	(4.81)	(13.26)	(-3.43)	
$90m1-08m8$	21.81	0.83	-0.39	0.39
$(t-stat)$	(5.20)	(14.53)	(-4.11)	
$90m1-08m11$	27.37	0.49	0.50	0.55
$(t-stat)$	(4.10)	(5.50)	(5.30)	

Note: This table presents our estimations results of the regression for the conditional realized variance: $\hat{RV}_{t+1} = b_0 + b_1 * RV_t + b_2 * MA(1) + e_{t+1}$. Results are disclosed for the three sampling periods considered. OLS method is used.

Chapter 6

Future Research

In this chapter, I summarize some research topics, related with this thesis, that I foresee as interesting to be explored in the future.

Starting with the portfolio choice problem, one research topic for future work was already pointed out in chapter 2: to study the impact on consumption and portfolio rules from ambiguity about the unconditional expected precision of the risky asset return (parameter θ) through the recursive multiple priors utility specification of Chen and Epstein (2002). It is a natural extension of the preferences setting of Chacko and Viceira (2005), as Chen and Epstein (2002) start from the SDU function of Duffie and Epstein (1992b) by replacing the single prior by a set of priors. The major advantage of following Chen and Epstein's (2002) $k-ambiguity$ setting, is that it guarantees that ambiguity is introduced in a dynamically consistent way.

Moreover, compared with the "multiplier preferences" robust control approach adopted in chapter 3, the multiple priors approach allows ambiguity to have "first-order" effects on portfolio choice (as pointed out by Epstein and Schneider (2010)), that is, it enables the expansion of the range of qualitative behavior that can be rationalized versus the standard expected utility theory. This contrasts with the "multiplier preferences" robust control approach, which is observationally equivalent to the expected utility theory: it enables reinterpretations of some results in the expected utility theory, that can be quantitatively more appealing,[1] but does not enlarge the spectrum of qualitative behavior that can be rationalized. In this sense, the "multiplier preferences" robust control approach only brings "second-order" effects on portfolio choice.

Alternatively, the "constraint preferences" robust control approach could also be adopted, which as shown by Trojani and Vanini (2004), also imply preferences that can be represented by the recursive multiple priors utility specification. This means that the "first-order" effects and the dynamic consistency property above mentioned would be kept (see Hansen and Sargent (2006) for the issue of dynamic consistency under robust control approaches). However, in this case (as in chapter 3), the representative investor would be ambiguous about the model describing the dynamics of precision, and not about a particular unobservable parameter of that process. The general SDU preferences setting with Kreps and Porteus (1978) specification could be kept and, using the perturbation theory methodology applied in Trojani and Vanini (2004), ambiguity about the stochastic precision process could be introduced in the model of Chacko and Viceira (2005).

A second research topic related with the portfolio choice problem is the calibration of the model developed in chapter 3, which would permit the quantification of the effects, under

[1] For example, as ambiguity aversion translates into a higher effective risk aversion, it is a contribution for the explanation of the equity premium puzzle.

that model, from ambiguity about the stochastic variance process on optimal portfolio decisions. Moreover, it would be interesting to compare the obtained empirical results with those obtained in chapter 2. The model calibration requires the new ambiguity preference parameter (θ) to be estimated through a well-founded way and, for that, the detection-error probabilities methodology provided by Anderson et al. (2003) could be applied (as, for e.g., in Barillas et al. (2009)). Moreover, as highlighted by Anderson et al. (2003) and Maenhout (2004, 2006), the new ambiguity preference parameter is an "environment-specific" parameter, in the sense that it cannot be transferred from one decision problem to another.

The intuition behind this detection-error probabilities methodology is that, based on some time series of finite length, the new ambiguity preference parameter is considered reasonable if the worst case model associated with it is sufficient close to the benchmark model and therefore generates a model selection problem. The new parameter is therefore chosen such that the detection-error probability is sufficiently high. Moreover, Maenhout (2006) gave an important contribution for the use of the detection-error probability technique, by developing a methodology for the analytical calculation of the detection-error probabilities. A semi-closed form for the detection-error probability is obtained and, therefore, I foresee that methodology as an interesting way of going ahead with this research topic.[2]

Still regarding the portfolio choice problem, I foresee two further research topics to explore. First, market frictions referred in chapter 2 could be formally introduced within a portfolio problem with ambiguity about the stochastic variance process by assuming, for e.g., information costs. This should be enough to generate an optimal portfolio rule not implying an instantaneous update of the portfolio *vis-a-vis* the instantaneous precision of the risky asset. Under this context, the expectation of future precision, and not the instantaneous precision, would become the driver of the risky asset demand, corresponding to the "expectation-driven" scenario identified in chapter 2. The "inattentiveness" theory literature (e.g. Reis (2006) and Abel et al. (2007)) provides the tools to address this potential research topic. Second, it could be interesting to consider simultaneously ambiguity about the return of the risky asset and its variance, and compare the quantitative and qualitative impact of both sources of ambiguity on the optimal portfolio rule.

Regarding the general equilibrium model for asset pricing under ambiguity developed in chapter 4, I foresee as interesting extensions of that setting the consideration of heterogeneous agents, where trade would occur in equilibrium,[3] and the consideration of a

[2]This methodology consists in two major two steps. First, the conditional characteristic functions of the Radon-Nikodym derivatives between the alternative models are found, by solving the Kolmogorov backward equations from the Feynman-Kac Theorem. Then, through the Fourier inversion of these characteristic functions, a semi-closed form of the detection-error probabilities is obtained.

[3]In Epstein and Miao (2003) and Trojani and Vanini (2004) equilibrium for continuous time heteroge-

more general setting for preferences, as the SDU from Duffie and Epstein (1992a,b)). As explained in chapter 2, this preference setting exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution by not constraining to be reciprocals of one another (as in standard additive intertemporal utility function). Additionally, if the SDU function is such that represents Kreps and Porteus (1978) utility, the investor's preference for the timing of the resolution of risk could also be studied.

Related with option pricing under ambiguity about the stochastic variance process, I highlight three research topics. First, regarding the option pricing formula obtained in chapter 5, its calibration with data from different underlying assets, namely equity indexes, interest rates, bonds, currencies, and analyze how obtained conclusions change (adjusting the obtained pricing formula for the case in which the underlying asset pays some income during the life of the option). It is of particular relevance the sign of both the variance risk price and of the variance ambiguity price. Moreover, instead of adopting the standard no-arbitrage approach for option pricing, an equilibrium approach could be explored implying that the second ambiguity channel identified in chapter 4, the equilibrium interest rate, would also have to be considered (instead of a constant interest rate).

Second, the estimation of the ambiguity level implied in option's prices, studying its dynamics and empirical strength as a market leading indicator (at least for the market of that option's underlying asset). This information could be relevant not only for investors but also for policy makers. It could be achieved through an option pricing closed-form formula with ambiguity considerations, as the one presented in chapter 5, or through a non-parametric approach as it is usual in the option implied estimation literature (so that results are not anchored in a specific pricing model). In this latter scenario, as the parameter is question is a preference parameter, and not a distribution moment (variance, skewness, kurtosis,...) as is more common, a good methodological guidance could be given by works focused in estimating risk aversion from option prices, as for example Ait-Sahalia and Lo (2000), Rosenberg and Engle (2002) and Bliss and Panigirtzoglou (2004). The corollary could be the construction of an option implied ambiguity index, in the same line as the existing option implied variance indexes (from which the most well known is the VIX Index from CBOE).

At last, I foresee the pricing and hedging of variance and volatility derivatives (including variance and volatility swaps) considering ambiguity about the stochastic variance process, as a very interesting extension of the analysis made in chapter 5. For this purpose,

neous economies under ambiguity is studied. In Epstein and Miao (2003), logarithmic utility agents differ in their subjective degrees of ambiguity aversion. In Trojani and Vanini (2004), power utility agents differ in their degrees of risk and ambiguity aversion.

the extended Heston (1993) model developed in chapter 5, that incorporates that ambiguity, could be used. As a reference, under a context where uncertainty is exclusively risk, Broadie and Jain (2008) study the pricing and hedging of variance and volatility derivatives within the Heston (1993) model framework.

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