On the closure of cyclic subgroups of a free group in some profinite topologies

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Abstract

We determine the closure of a cyclic subgroup H of a free group for the pro-$\mathbf{V}$ topology when $\mathbf{V}$ is an extension-closed pseudovariety of finite groups. We show that H is always closed for the pro-nilpotent topology and compute its closure for the pro-$\mathbf{G}_p$ and pro-$\mathbf{V}_p$ topologies, where $\mathbf{G}_p$ is the pseudovariety of finite p-groups and, when p varies through the set of primes, $\mathbf{V}_p \supset \mathbf{G}_p$ denote the pseudovarieties involved in the join decomposition of the pseudovariety of finite supersolvable groups. More generally, given any nonempty set P of primes, we consider the stated problem for the pseudovariety $\mathbf{G}_P$ of all finite groups having order a product of primes in P.

1 Introduction

The concept of pseudovariety is central in the study of finite algebras. A *pseudovariety of finite groups* is a class of finite groups closed under taking subgroups, homomorphic images and finitary direct products. Every pseudovariety $\mathbf{V}$ of finite groups induces a (metrizable) topology on any group G, known as the *pro-$\mathbf{V}$ topology.* The pro-$\mathbf{V}$ topology is the initial topology with respect to all homomorphisms $G \to H \in \mathbf{V}$, where H is endowed with the discrete topology. If $\mathbf{G}$ denotes the pseudovariety of all finite groups, the pro-$\mathbf{G}$ topology is known as the profinite topology. It was introduced by Marshall Hall in [6] and he proved in [5, Theorem 5.1] that every finitely generated subgroup of a free group is closed for the profinite topology.

Over the years, several other pseudovarieties $\mathbf{V}$ were considered [2, 9, 10, 11, 12, 14]. In general, finitely generated subgroups of a free group are not necessarily closed for the pro-$\mathbf{V}$ topology. The goal becomes then deciding whether such a subgroup is closed, or even the membership problem for its closure.

In [14], Ribes and Zalesskiĭ answered these questions positively for the pseudovariety $\mathbf{G}_p$ of all finite p-groups, for an arbitrary prime p. In [9], Margolis, Sapir and Weil dealt successfully with the pseudovariety $\mathbf{N}$ of all finite nilpotent groups (the case of free groups of infinite rank following from Corollary 2.4). The pseudovariety $\mathbf{S}$ of all finite solvable groups has resisted so far all attempts to solve these problems. This paper offers a contribution by proving that every cyclic subgroup of a free group is closed for the pro-**N** topology and consequently for the pro-**S** topology.

We note that cyclic subgroups of free groups were considered by Huang, Pawliuk, Sabok and Wise in [7]. Let $\mathbb{P}$ be the set of all primes. Given a subset P of primes, let $\mathbf{G}_P$ denote the pseudovariety of all finite groups whose order is a product of primes in P. Note that $\mathbf{G}_{\mathbb{P}}$ is the pseudovariety $\mathbf{G}$ of all finite groups and $\mathbf{G}_\emptyset$ is the trivial pseudovariety (consisting of trivial groups). Let $P^\perp$ be the subset of all primes not in P. In [7, Theorem 1.5], it is shown that if F_n is a free group of rank $n \in \mathbb{N}$, $H \leq F_n$ is cyclic, P is a proper subset of primes, and

$$u^p \in H \quad \Rightarrow \quad u \in H$$

holds for all $u \in F_n$ and $p \in P$, then H is $\mathbf{G}_{P^\perp}$-closed (the converse is proved in Corollary 3.5(ii) below). They also relate this property to the Hrushovsky property (the extending property for partial automorphisms) for hypertournaments.

In Section 2, after fixing some notation and giving some preliminary results, we show that when free groups are residually $\mathbf{V}$ (i.e. $\langle 1\rangle$ is $\mathbf{V}$-closed), we can restrict our attention to free groups of finite rank.

In Section 3, we consider the case of extension-closed pseudovarieties $\mathbf{V}$. We determine the closure of a cyclic subgroup H of a free group for the pro-$\mathbf{V}$ topology and show that it is effectively computable under mild decidability conditions (Theorem 3.2). We show also that all cyclic subgroups being closed is equivalent to $\mathbf{V}$ containing all finite abelian groups (Corollary 3.3). We then apply these results to the particular cases of $\mathbf{S}$ and $\mathbf{G}_P$.

In Section 4, we use the computation of the $\mathbf{G}_p$-closure to show that cyclic subgroups are always closed for the pro-**N** topology. This generalizes to any pseudovariety containing **N**, in particular to the pseudovariety **Su** of all finite supersolvable groups. By a theorem of Auinger and Steinberg, **Su** is the join of the pseudovarieties $\mathbf{V}_p$ (finite groups having a normal Sylow p-subgroup with quotient an abelian group of exponent dividing $p-1$), for every prime p. By [2, Theorem 1.1], the pseudovarieties $\mathbf{V}_p$ are precisely the maximal *Hall subpseudovarieties* of **Su** (see [2, Subsection 3.5] for definitions). By [1], $\mathbf{V}_p$ is also the class of all finite groups having a faithful representation by upper triangular matrices over the field $\mathbb{F}_p$ of p elements. We close the paper by proving that the $\mathbf{V}_p$-closure of a cyclic subgroup of a free group is effectively computable (Corollary 4.7).

2 Preliminaries

In this paper, $\mathbb{P}$ denotes the set of all primes. Also if m is a positive integer, then C_m denotes the cyclic group of order m.

If A is a set, we set F_A to be the free group on A. Also if H is a finitely generated subgroup of F_A, we write $H \leq_{f.g.} F_A$. We denote by F_n the free group of rank $n \in \mathbb{N}$. Recall that an element w of F_A is primitive if there is an automorphism of F_A sending w to an element of A.

We call a pseudovariety of finite groups *nontrivial* if it contains some nontrivial group. Besides the pseudovarieties defined in the introduction, we consider also, for each $m \geq 1$, the pseudovariety $\mathbf{Ab}(m)$ of all finite abelian groups of exponent dividing m. This pseudovariety is an *equational* pseudovariety because it is defined by the identities $[x,y] = 1$ and $x^m = 1$. This amounts to saying that a finite group G is in $\mathbf{Ab}(m)$ if and only if $[g,h] = g^m = 1$ for all $g, h \in G$.

For all $m, n \geq 1$, write

$$F_n^{\mathbf{Ab}(m)} = \langle \{[u,v] \mid u, v \in F_n\} \cup \{u^m \mid u \in F_n\}\rangle \trianglelefteq F_n.$$

Given a basis A of F_n, it is easy to check that

$$F_n/F_n^{\mathbf{Ab}(m)} \cong C_m^n \tag{2.1}$$

is the free object of $\mathbf{Ab}(m)$ on the set A: we have $F_n/F_n^{\mathbf{Ab}(m)} \in \mathbf{Ab}(m)$ and every map $\varphi : A \to G \in \mathbf{Ab}(m)$ extends to a (unique) homomorphism $\Phi : F_n/F_n^{\mathbf{Ab}(m)} \to G$ (viewing A as a subset of $F_n/F_n^{\mathbf{Ab}(m)}$).

Following [2], we may use the semidirect product of pseudovarieties to get the decomposition $\mathbf{V}_p = \mathbf{G}_p * \mathbf{Ab}(p-1)$. This alternative description of $\mathbf{V}_p$ follows from the Kalužnin-Krasner theorem [13].

A finite group is *supersolvable* if each of its chief factors is cyclic of prime order. By [2, Corollary 2.7], we have

$$\mathbf{Su} = \bigvee_{p\in\mathbb{P}} \mathbf{V}_p.$$

We say that a pseudovariety $\mathbf{V}$ of finite groups is *extension-closed* if

$$N, G/N \in \mathbf{V} \quad \Rightarrow \quad G \in \mathbf{V}$$

holds for every (finite) group G and every $N \trianglelefteq G$.

It is well known that $\mathbf{S}$ is extension-closed, and it is obvious that $\mathbf{G}_P$ is extension-closed for every $P \subseteq \mathbb{P}$.

The smallest non-nilpotent group is the symmetric group S_3 (which is not simple), hence $\mathbf{N}$ is not extension-closed. On the other hand, the alternating group A_4 is the smallest non-supersolvable group, hence $\mathbf{Su}$ is not extension-closed either.

By considering the cyclic group C_{m^2}, we see that $\mathbf{Ab}(m)$ is not extension-closed for any $m \geq 2$, and by considering $C_{(p-1)^2}$ we see that $\mathbf{V}_p$ is not extension-closed either for any prime $p > 2$.

Given a pseudovariety $\mathbf{V}$ of finite groups, where we consider finite groups endowed with the discrete topology, the pro-$\mathbf{V}$ topology on a group G is defined as the coarsest topology which makes all homomorphisms from G to groups in $\mathbf{V}$ continuous. Equivalently, G is a topological group where the normal subgroups K of G such that $G/K \in \mathbf{V}$ form a basis of neighbourhoods of the identity.

For a topological property $\mathcal{P}$ and a subset S of a group G, we say that S is $\mathbf{V}$-$\mathcal{P}$ if S has property $\mathcal{P}$ in the pro-$\mathbf{V}$ topology on G.

Given a subgroup H of a group G, the core $\mathrm{Core}_G(H)$ of H in G is the largest normal subgroup of G contained in H and is equal to $\bigcap_{g\in G} g^{-1}Hg$.

Theorem 2.1. [9, Proposition 1.2] *Given a group G and $H \leq G$, the following conditions are equivalent:*

(i) H is $\mathbf{V}$-open;

(ii) H is $\mathbf{V}$-clopen;

(iii) $G/\mathrm{Core}_G(H) \in \mathbf{V}$.

It is easy to prove (see [11, Proposition 2.6]) the following:

$$\text{If } [G : H] < \infty, \text{ then } H \text{ is } \mathbf{V}\text{-open if and only if } H \text{ is } \mathbf{V}\text{-closed.} \tag{2.2}$$

We note that a subgroup H of G is $\mathbf{V}$-closed if and only if, for every $g \in G \setminus H$, there exists some $\mathbf{V}$-clopen $K \leq G$ such that $H \leq K$ and $g \notin K$. On the other hand, a subgroup H of G is $\mathbf{V}$-dense if and only if $HN = G$ for every normal subgroup N of G such that $G/N \in \mathbf{V}$.

Given a group G and $S \subseteq G$, we denote by $\mathrm{Cl}^G_{\mathbf{V}}(S)$ the $\mathbf{V}$-closure of S in G. We can omit the superscript if G is clear from the context.

Note that if $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of finite groups, then

$$\mathbf{W} \subseteq \mathbf{V} \quad \Rightarrow \quad \mathrm{Cl}^G_{\mathbf{V}}(H) \leq \mathrm{Cl}^G_{\mathbf{W}}(H) \tag{2.3}$$

holds for every subgroup H of a group G by [9, Corollary 3.2].

A subgroup K of a group G is called a *retract* if there exists a (surjective) homomorphism $\varphi : G \to K$ fixing the elements of K. If $\mathbf{V}$ is a pseudovariety of finite groups, the group G is said to be *residually* $\mathbf{V}$ if $\langle 1 \rangle$ is a $\mathbf{V}$-closed subgroup of G. The following result is well known, but we include a proof for the sake of completeness:

Lemma 2.2. *Every free group is residually* $\mathbf{V}$ *for every nontrivial extension-closed pseudovariety* $\mathbf{V}$ *of finite groups.*

Proof. Since $\mathbf{V}$ is nontrivial, then $C_p \in \mathbf{V}$ for some prime p. Since every finite p-group admits a composition series where the factor groups are cyclic of order p, it follows from $\mathbf{V}$ being extension-closed that $\mathbf{G}_p \subseteq \mathbf{V}$.

By a theorem of Iwasawa [8], every free group is residually $\mathbf{G}_p$, and therefore residually $\mathbf{V}$ in view of (2.3). □

Now we can prove the following:

Proposition 2.3. *Let* $\mathbf{V}$ *be a pseudovariety of finite groups. Let* G *be a residually* $\mathbf{V}$ *group and let* $K \leq G$ *be a retract of* G*. Let* $H \leq K$*. Then* $\mathrm{Cl}^G_{\mathbf{V}}(H) = \mathrm{Cl}^K_{\mathbf{V}}(H)$*.*

Proof. Write $C_1 = \mathrm{Cl}^G_{\mathbf{V}}(H)$ and $C_2 = \mathrm{Cl}^K_{\mathbf{V}}(H)$. By [9, Proposition 1.6], the pro-$\mathbf{V}$ topology of K coincides with the subspace topology of K with respect to the pro-$\mathbf{V}$ topology of G. Hence $C_2 = C_1 \cap K$. Since G is residually $\mathbf{V}$, the retract K is $\mathbf{V}$-closed in G by [9, Corollary 1.8]. Hence $C_1 = C_1 \cap K = C_2$. □

Corollary 2.4. *Let* H *be a finitely generated subgroup of a free group* F_A *of arbitrary rank. Let* $B \subseteq A$ *be a finite set such that* $H \leq F_B$*. Let* $\mathbf{V}$ *be a pseudovariety of finite groups for which* F_A *is residually* $\mathbf{V}$*. Then:*

(i) $\mathrm{Cl}^{F_A}_{\mathbf{V}}(H) = \mathrm{Cl}^{F_B}_{\mathbf{V}}(H)$*;*

(ii) H *is* $\mathbf{V}$*-closed in* F_A *if and only if* H *is* $\mathbf{V}$*-closed in* F_B*;*

(iii) if A *is infinite then* H *is not* $\mathbf{V}$*-dense in* F_A*.*

This applies in particular to $\mathbf{N}$*,* $\mathbf{Su}$*,* $\mathbf{V}_p$ *(for every prime* p*) and every nontrivial extension-closed pseudovariety (including* $\mathbf{S}$ *and* $\mathbf{G}_P$ *(for every nonempty* $P \subseteq \mathbb{P}$*)).*

Proof. Part (i) follows from Proposition 2.3, since F_B is a retract of F_A. Now parts (ii) and (iii) follow from part (i).

By Lemma 2.2, this applies to every nontrivial extension-closed pseudovariety. Since the remaining pseudovarieties $\mathbf{N}$, $\mathbf{Su}$, $\mathbf{V}_p$ contain $\mathbf{G}_p$ for some prime p, it follows from (2.3) that they inherit the same property. □

As a consequence, the results to be proved in Sections 3 and 4 reduce to the setting of free groups of finite rank.

3 Extension-closed pseudovarieties

We discuss in this section the extension-closed case. The following result combines some of the special properties shared by extension-closed pseudovarieties:

Theorem 3.1. [9, Proposition 2.17], [14, Corollary 3.3 and Proposition 3.4] *Let $\mathbf{V}$ be a nontrivial extension-closed pseudovariety of finite groups and let F be a free group. Let $H \leq_{f.g.} F$. Then:*

(i) H is $\mathbf{V}$-closed if and only if H is a free factor of a $\mathbf{V}$-clopen subgroup of F;

(ii) the rank of $\mathrm{Cl}_{\mathbf{V}}(H)$ is at most the rank of H;

(iii) the pro-$\mathbf{V}$ topology on $\mathrm{Cl}_{\mathbf{V}}(H)$ is the subspace topology with respect to the pro-$\mathbf{V}$ topology on F.

We define the *root* and the *exponent* (in F) of a word $w \in F \setminus \{1\}$ as follows. Clearly, w can be written as a power u^r for some $u \in F$ and $r \geq 1$, and there exist finitely many such decompositions. The maximum such r is the exponent of w and is denoted by $\exp(w)$. Then the root of w is defined through the equality $w = \mathrm{root}(w)^{\exp(w)}$ (well defined since the implication

$$x^n = y^n \qquad \Rightarrow \qquad x = y$$

holds in free groups).

A classical theorem states that two nonempty words $x, y \in F$ commute if and only if x, y are powers of a same word if and only if $x^m = y^k$ for some nonzero $m, k \in \mathbb{Z}$. It follows that, for all $w \in F \setminus \{1\}$ and $s \in \mathbb{Z} \setminus \{0\}$,

$$w = v^s \text{ implies } v = (\mathrm{root}(w))^k \text{ for some } k \in \mathbb{Z} \setminus \{0\}. \tag{3.1}$$

From the same theorem we may deduce that, for every $s \in \mathbb{Z} \setminus \{0\}$,

$$\mathrm{root}(w^s) = \begin{cases} \mathrm{root}(w) & \text{if } s > 0 \\ \mathrm{root}(w)^{-1} & \text{if } s < 0 \end{cases} \text{ and } \exp(w^s) = |s|\exp(w). \tag{3.2}$$

Indeed, since

$$(\mathrm{root}(w))^{s\exp(w)} = w^s = (\mathrm{root}(w^s))^{\exp(w^s)},$$

it follows that $\mathrm{root}(w)$ and $\mathrm{root}(w^s)$ are powers of a same word and so by maximality of exponents $\mathrm{root}(w^s) = \mathrm{root}(w)^{\pm 1}$: Moreover, $\mathrm{root}(w^s) = \mathrm{root}(w)$ if and only if $s > 0$. Hence

$$(\mathrm{root}(w))^{s\exp(w)} = \begin{cases} (\mathrm{root}(w))^{\exp(w^s)} & \text{if } s > 0 \\ (\mathrm{root}(w)^{-1})^{\exp(w^s)} & \text{if } s < 0 \end{cases}$$

and so (3.2) holds. In [4] Berlai and Ferov generalize the concepts of root and exponent to right-angled Artin groups.

Theorem 3.2. *Let $\mathbf{V}$ be a nontrivial extension-closed pseudovariety of finite groups and let F be a free group. Let $w \in F \setminus \{1\}$, $u = \mathrm{root}(w)$ and $e = \exp(w)$. Then:*

(i) $\mathrm{Cl}_{\mathbf{V}}(\langle 1 \rangle) = \langle 1 \rangle$;

(ii) $\mathrm{Cl}_{\mathbf{V}}(\langle w \rangle) \leq \langle u \rangle$;

(iii) $\langle w \rangle$ is $\mathbf{V}$-closed if and only if $C_e \in \mathbf{V}$;

(iv) $\mathrm{Cl}_{\mathbf{V}}(\langle w\rangle) = \langle u^m\rangle$ *for*

$$m = \max\{k \geq 1 \mid k \textit{ divides } e \textit{ and } C_k \in \mathbf{V}\}.$$

(v) $\mathrm{Cl}_{\mathbf{V}}(\langle z\rangle)$ *is effectively computable for every* $z \in F$ *if and only if we can decide, for each* $n \geq 2$*, whether or not* $C_n \in \mathbf{V}$.

Proof. Part (i) follows from Lemma 2.2.

We now consider part (ii). Write $C = \mathrm{Cl}_{\mathbf{V}}(\langle w\rangle)$. By Theorem 3.1(ii), we have $C = \langle v\rangle$ for some $v \in F$. Hence $w = v^r$ for some $r \in \mathbb{Z}$ and so $v \in \langle u\rangle$ by (3.1). Thus $C \leq \langle u\rangle$.

We now consider part (iii). Since $\mathrm{root}(u) = \mathrm{root}(u^e) = u$ by (3.2), it follows from part (ii) that $\langle u\rangle$ is $\mathbf{V}$-closed. By Theorem 3.1(iii), $\langle w\rangle$ is $\mathbf{V}$-closed in F if and only if it is $\mathbf{V}$-closed in $\langle u\rangle$. Since $\langle w\rangle$ is a finite index (normal) subgroup of $\langle u\rangle$, it follows from Theorem 2.1 and (2.2) that $\langle w\rangle$ is $\mathbf{V}$-closed in F if and only if $\langle u\rangle/\langle w\rangle \in \mathbf{V}$. Since $\langle u\rangle/\langle w\rangle = \langle u\rangle/\langle u^e\rangle \cong C_e$, we get the desired equivalence.

We now consider part (iv). By parts (ii) and (iii), $\mathrm{Cl}_{\mathbf{V}}(\langle w\rangle) = \langle u^m\rangle$ for some $m \geq 1$ such that $C_m \in \mathbf{V}$. Since $w = u^e \in \langle u^m\rangle$, then m divides e. Suppose that $k \geq 1$ divides e and $C_k \in \mathbf{V}$. By part (iii), $\langle u^k\rangle$ is $\mathbf{V}$-closed, and we have $w = u^e \in \langle u^k\rangle$. Hence $\langle w\rangle \leq \langle u^k\rangle$ and by minimality of the closure we get $\langle u^m\rangle \leq \langle u^k\rangle$. Thus k divides m and so $k \leq m$ as required.

We finally consider part (v). If we can decide, for each $n \geq 2$, whether or not $C_n \in \mathbf{V}$ then, by parts (i) and (iv), $\mathrm{Cl}_{\mathbf{V}}(\langle z\rangle)$ is effectively computable for every $z \in F$.
Suppose that $a \in F$ is a primitive word and $n \geq 2$. It follows from part (iii) that $\langle a^n\rangle$ is $\mathbf{V}$-closed if and only if $C_n \in \mathbf{V}$. Thus we have the equivalence: $C_n \in \mathbf{V}$ if and only if $\mathrm{Cl}_{\mathbf{V}}(\langle a^n\rangle) = \langle a^n\rangle$, and this establishes the claimed equivalence. $\square$

Corollary 3.3. *Let* $\mathbf{V}$ *be an extension-closed pseudovariety of finite groups. Then the following conditions are equivalent:*

(i) every cyclic subgroup of a free group is $\mathbf{V}$*-closed;*

(ii) $\mathbf{Ab} \subseteq \mathbf{V}$.

Proof. (i) $\Rightarrow$ (ii). In view of the Classification Theorem of finite abelian groups, it suffices to show that $C_n \in \mathbf{V}$ for every $n \geq 2$. Let u be a primitive word of F. Then $\mathrm{root}(u) = u$ and $\exp(u^n) = n$ by (3.2). Since $\langle u^n\rangle$ is $\mathbf{V}$-closed, it follows from Theorem 3.2(iii) that $C_n \in \mathbf{V}$ and so $\mathbf{Ab} \subseteq \mathbf{V}$.

(ii) $\Rightarrow$ (i). This follows from Theorem 3.2(i) and (iii). $\square$

Since it is well known that the pseudovariety $\mathbf{S}$ of all finite solvable groups is extension-closed, Corollary 3.3 and Theorem 3.1(i) yield:

Corollary 3.4. *Let* H *be a cyclic subgroup of a free group* F*. Then:*

(i) H *is* $\mathbf{S}$*-closed;*

(ii) H *is a free factor of an* $\mathbf{S}$*-clopen subgroup of* F.

Given $P \subseteq \mathbb{P}$ nonempty and $k \geq 1$, we let $\nu_P(k) = 1$ if k is coprime to every prime in P, otherwise we set $\nu_P(k)$ to be the largest divisor of k which is a product of primes in P. Since $\mathbf{G}_P$ is extension-closed, we can now deduce the following result:

Corollary 3.5. *Let* $P \subseteq \mathbb{P}$ *be nonempty and let* F *be a free group. Let* $w \in F \setminus \{1\}$, $u = \mathrm{root}(w)$ *and* $e = \exp(w)$*. Then:*

(i) $\mathrm{Cl}_{\mathbf{G}_P}(\langle 1\rangle) = \langle 1\rangle$;

(ii) $\langle w\rangle$ *is* $\mathbf{G}_P$*-closed if and only if* e *is a product of primes in* P *if and only if*

$$v^p \in \langle w\rangle \Rightarrow v \in \langle w\rangle \tag{3.3}$$

holds for all $v \in F$ *and* $p \in P^\perp$;

(iii) $\mathrm{Cl}_{\mathbf{G}_P}(\langle w\rangle) = \langle u^{\nu_P(e)}\rangle$.

(iv) membership in P *is decidable if and only if* $\mathrm{Cl}_{\mathbf{G}_P}(\langle z\rangle)$ *is effectively computable for every* $z \in F$.

Proof. Part (i) follows from Lemma 2.2.

We now consider part (ii). By Theorem 3.2(iii), $\langle w\rangle$ is $\mathbf{G}_P$-closed if and only if $C_e \in \mathbf{G}_P$, that is, e is a product of primes in P.

Suppose that some $p \in P^\perp$ divides e. Then (3.3) fails for p and $v = u^{\frac{e}{p}}$. The converse implication follows from [7, Theorem 1.5], but we include a proof for the sake of completeness.

Assume that e is a product of primes in P. Let $v \in F$ and $p \in P^\perp$ be such that $v^p \in \langle w\rangle$. By (3.2), we get $\mathrm{root}(v) = \mathrm{root}(v^p) = \mathrm{root}(w)^{\pm 1} = u^{\pm 1}$.

Hence $v = u^k$ for some $k \in \mathbb{Z}$ and so

$$v^e = (u^k)^e = (u^e)^k = w^k \in \langle w\rangle.$$

Now $\gcd(p, e) = 1$ yields $1 = pr + es$ for some $r, s \in \mathbb{Z}$ and so $v = v^{pr+es} = (v^p)^r(v^e)^s \in \langle w\rangle$. Therefore (3.3) holds as required.

Part (iii) follows from Theorem 3.2(iv), noting that $\nu_P(e)$ is the maximum divisor k of e such that $C_k \in \mathbf{G}_P$.

We finally consider part (iv). If membership in P is decidable, then $\nu_p(e)$ is computable for every $e \geq 1$, hence it follows from parts (i) and (iii) that $\mathrm{Cl}_{\mathbf{G}_P}(\langle z\rangle)$ is effectively computable for every $z \in F$.

Conversely, let $a \in F$ be primitive and let $p \in \mathbb{P}$. Since

$$p \in P \quad \Longleftrightarrow \quad \nu_P(p) = p \quad \Longleftrightarrow \quad \mathrm{Cl}_{\mathbf{G}_P}(\langle a^p\rangle) = \langle a^p\rangle,$$

then computing $\mathrm{Cl}_{\mathbf{G}_P}(\langle z\rangle)$ (for every $z \in F$) allows us to decide whether or not $p \in P$. □

By considering a single prime, we get:

Corollary 3.6. *Let* $p \in \mathbb{P}$ *and let* F *be a free group. Let* $w \in F \setminus \{1\}$, $u = \mathrm{root}(w)$ *and* $e = \exp(w)$. *Then:*

(i) $\mathrm{Cl}_{\mathbf{G}_p}(\langle 1\rangle) = \langle 1\rangle$;

(ii) $\langle w\rangle$ *is* $\mathbf{G}_p$*-closed if and only if* $e = p^s$ *for some* $s \geq 0$ *if and only if* $v^q \in \langle w\rangle \Rightarrow v \in \langle w\rangle$ *holds for all* $v \in F$ *and* $q \in \{p\}^\perp$;

(iii) $\mathrm{Cl}_{\mathbf{G}_p}(\langle w\rangle) = \langle u^{\nu_p(e)}\rangle$.

In the language of [4], Corollary 3.6(ii) is equivalent to saying that given a prime p, a cyclic subgroup of a free group is pro-p closed if and only if it is a p-isolated subgroup. More generally in [4] it is shown that given a prime p, a p-isolated cyclic subgroup of a right-angled Artin group is pro-p closed.

The pseudovariety $\mathbf{O}$ of all (finite) groups of odd order is equal to $\mathbf{G}_{2^\perp}$, thus we also have:

Corollary 3.7. *Let F be a free group. Let $w \in F \setminus \{1\}$, $u = \mathrm{root}(w)$ and $e = \exp(w)$. Then:*

(i) $\mathrm{Cl}_{\mathbf{O}}(\langle 1 \rangle) = \langle 1 \rangle$*;*

(ii) $\langle w \rangle$ *is* $\mathbf{O}$*-closed if and only if e is odd if and only if $v^2 \in \langle w \rangle \Rightarrow v \in \langle w \rangle$ holds for every $v \in F$;*

(iii) $\mathrm{Cl}_{\mathbf{O}}(\langle w \rangle) = \langle u^{\nu_{\{2\}^\perp}(e)} \rangle = \langle u^{\frac{e}{\nu_2(e)}} \rangle$.

4 Non extension-closed pseudovarieties of solvable groups

We start by considering one of the most important cases: the pseudovariety of finite nilpotent groups.

Theorem 4.1. *Every cyclic subgroup of a free group is* $\mathbf{N}$*-closed.*

Proof. Let F be a free group and let $w \in F$. By [9, Corollary 4.2(2)], we have

$$\mathrm{Cl}_{\mathbf{N}}(\langle w \rangle) = \bigcap_{p \in \mathbb{P}} \mathrm{Cl}_{\mathbf{G}_p}(\langle w \rangle).$$

If $w = 1$, then Corollary 3.6(i) implies that $\langle 1 \rangle$ is $\mathbf{N}$-closed, hence we may assume that $w \neq 1$. Let $u = \mathrm{root}(w)$ and $e = \exp(e)$. Write $e = p_1^{s_1} \dots p_m^{s_m}$ with $p_1, \dots, p_m$ distinct primes and $s_1, \dots, s_m \geq 1$.

Let $v \in \mathrm{Cl}_{\mathbf{N}}(\langle w \rangle)$. Then $v \in \mathrm{Cl}_{\mathbf{G}_{p_i}}(\langle w \rangle)$ for $i = 1, \dots, s$. It follows from Corollary 3.6(iii) that

$$v \in \langle u^{\nu_{p_i}(e)} \rangle = \langle u^{p_i^{s_i}} \rangle$$

for $i = 1, \dots, s$. Since the primes p_i are distinct, we get

$$v \in \langle u^{p_1^{s_1} \dots p_m^{s_m}} \rangle = \langle u^e \rangle = \langle w \rangle.$$

Therefore $\langle w \rangle$ is $\mathbf{N}$-closed. □

Corollary 4.2. *Let $\mathbf{V}$ be a pseudovariety of finite groups containing $\mathbf{N}$. Let H be a cyclic subgroup of a free group F. Then:*

(i) H is $\mathbf{V}$*-closed;*

(ii) H is a free factor of a $\mathbf{V}$*-clopen subgroup of F.*

Proof. Part (i) follows from Theorem 4.1 and (2.3).

Part (ii) follows from part (i) and [14, Corollary 3.3(i)]. □

In particular, this applies to the pseudovariety $\mathbf{Su}$ and provides an alternative proof to Corollary 3.4. Since

$$\mathbf{Su} = \bigvee_{p \in \mathbb{P}} \mathbf{V}_p,$$

we concentrate now our efforts on the pseudovariety $\mathbf{V}_p$, for a given prime p which we assume fixed throughout the rest of the paper. In view of Corollary 2.4, we may restrict our attention to free groups of finite rank. Write $K_n = F_n^{\mathbf{Ab}(p-1)} \trianglelefteq F_n$.

We can derive from the proof of [2, Proposition 4.4] the following result:

Proposition 4.3. *Let $H \leq_{f.g.} F_n$ and let $p \in \mathbb{P}$. The following conditions are equivalent:*

(i) H is $\mathbf{V}_p$-closed in F_n;

(ii) $H \cap K_n$ is $\mathbf{G}_p$-closed in K_n.

Proof. Write $F = F_n$ and $K = K_n$. Since $[F : K] < \infty$, we have $[H : H \cap K] < \infty$ as well. Taking $m = [H : H \cap K] - 1$, we may write

$$H = \bigcup_{i=0}^{m} (H \cap K)t_i \tag{4.1}$$

for some $t_0, \ldots, t_m \in F$ with $t_0 = 1$.

Since the closure of a finite union of sets is the finite union of the closures and translations are homeomorphisms in our setting, we get

$$\mathrm{Cl}^F_{\mathbf{V}_p}(H) = \bigcup_{i=0}^{m} \mathrm{Cl}^F_{\mathbf{V}_p}((H \cap K)t_i) = \bigcup_{i=0}^{m} (\mathrm{Cl}^F_{\mathbf{V}_p}(H \cap K))t_i$$

as in the proof of [2, Proposition 4.4], where it is also proved that

$$\mathrm{Cl}^F_{\mathbf{V}_p}(H \cap K) = \mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K).$$

It follows that

$$\mathrm{Cl}^F_{\mathbf{V}_p}(H) = \bigcup_{i=0}^{m} (\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K))t_i. \tag{4.2}$$

Now we prove the stated equivalence.

Assume first that $H \cap K$ is $\mathbf{G}_p$-closed in K. In other words $\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K) = H \cap K$, and so it follows from (4.1) and (4.2) that

$$\mathrm{Cl}^F_{\mathbf{V}_p}(H) = \bigcup_{i=0}^{m} (\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K))t_i = \bigcup_{i=0}^{m} (H \cap K)t_i = H.$$

Thus H is $\mathbf{V}_p$-closed in F.

Conversely, assume that H is $\mathbf{V}_p$-closed in F, that is, $\mathrm{Cl}^F_{\mathbf{V}_p}(H) = H$. Then in view of (4.2) we get

$$H = \bigcup_{i=0}^{m} (\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K))t_i.$$

Since $t_0 = 1$, this implies $\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K) \subseteq H$ and consequently $\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K) \subseteq H \cap K$. Thus $\mathrm{Cl}^K_{\mathbf{G}_p}(H \cap K) = H \cap K$ and $H \cap K$ is $\mathbf{G}_p$-closed in K as required. $\square$

In view of Corollary 2.4, $\langle 1 \rangle$ is $\mathbf{V}_p$-closed in F_n. For a nontrivial cyclic subgroup of F_n, more work is needed in order to determine its $\mathbf{V}_p$-closure.

We fix now a basis $A = \{a_1, \ldots, a_n\}$ of F_n and consider the n homomorphisms $f_{a_i} : F_n \to \mathbb{Z}$ where $1 \leq i \leq n$ defined by

$$f_{a_i}(a_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq j \leq n$.

Given $w \in F_n$, write

$$h_w = \frac{p-1}{\gcd(p-1, f_{a_1}(w), \ldots, f_{a_n}(w))}.$$

Lemma 4.4. *Let $w \in F_n$. Then $h_w = \min\{r \geq 1 \mid w^r \in K_n\}$.*

Proof. Since $K_n = F_n^{\mathbf{Ab}(p-1)}$, it follows from (2.1) that

$$\begin{aligned} w^r \in K_n \quad &\Leftrightarrow \quad \forall i \in \{1, \ldots, n\} : \ (p-1) \mid f_{a_i}(w^r) \\ &\Leftrightarrow \quad (p-1) \mid \gcd(f_{a_1}(w^r), \ldots, f_{a_n}(w^r)) \\ &\Leftrightarrow \quad (p-1) \mid r \gcd(f_{a_1}(w), \ldots, f_{a_n}(w)) \\ &\Leftrightarrow \quad \tfrac{p-1}{\gcd(p-1, f_{a_1}(w), \ldots, f_{a_n}(w))} \mid r \\ &\Leftrightarrow \quad h_w \mid r \end{aligned}$$

and the claim follows. □

Given $w \in K_n \setminus \{1\}$, we now characterize the root and the exponent in K_n of w in terms of the root and the exponent in F_n of w.

Lemma 4.5. *Let $w \in K_n \setminus \{1\}$. Let u (respectively u_K) denote the root of w in F_n (respectively K_n). Let e (respectively e_K) denote the exponent of w in F_n (respectively K_n). Then:*

(i) $u_K = u^{h_u}$;

(ii) $e_K = \frac{e}{h_u}$;

(iii) $\nu_p(e_K) = \nu_p(e)$.

Proof. Since $u_K^{e_K} = w = u^e$, it follows from (3.1) that $u_K = u^q$ for some positive integer q. Moreover, $e_K = \frac{e}{q}$. Now $u_K \in K_n$ and we must have in fact

$$q = \min\{r \geq 1 \mid u^r \in K_n \text{ and } r \mid e\}. \tag{4.3}$$

But $u^e = w \in K_n$, hence it follows from the proof of Lemma 4.4 that $h_u \mid e$. Now Lemma 4.4 and (4.3) together yield $q = h_u$. Thus parts (i) and (ii) hold.

Finally, to prove part (iii) we remark that part (ii) and the fact that h_u divides $p-1$ yield $\nu_p(e_K) = \nu_p(\frac{e}{h_u}) = \nu_p(e)$. □

We now prove the following criterion for a cyclic subgroup of F_n to be $\mathbf{V}_p$-closed.

Proposition 4.6. *Let $w \in F_n \setminus \{1\}$. Let $u = \mathrm{root}(w)$ and $e = \exp(w)$. Then $h_w \mid h_u$ and the following conditions are equivalent:*

(i) $\langle w \rangle$ *is $\mathbf{V}_p$-closed in F_n;*

(ii) $e = \frac{h_u}{h_w} p^s$ *for some $s \geq 0$.*

Proof. Write $F = F_n$ and $K = K_n$. We have

$$h_w \mid h_u \quad \Leftrightarrow \quad \gcd(p-1, f_{a_1}(u), \ldots, f_{a_n}(u)) \mid \gcd(p-1, f_{a_1}(w), \ldots, f_{a_n}(w)).$$

Since $f_{a_i}(w) = f_{a_i}(u^e) = e f_{a_i}(u)$ for $i = 1, \ldots, n$, it follows that $h_w \mid h_u$.

Let $H = \langle w \rangle$. By Proposition 4.3, H is $\mathbf{V}_p$-closed in F if and only if $H \cap K$ is $\mathbf{G}_p$-closed in K. Since $H \cap K$ is a subgroup of a cyclic group, it is itself cyclic, and it follows from Lemma 4.4 that

$$H \cap K = \langle w^{h_w} \rangle. \tag{4.4}$$

Now we have to find out when $H \cap K$ is $\mathbf{G}_p$-closed in K. By Corollary 3.6(ii), this is equivalent to having

$$\exp_K(w^{h_w}) = p^s$$

for some $s \geq 0$. By Lemma 4.5(ii), this is equivalent to having

$$\exp(w^{h_w}) = h_u p^s$$

for some $s \geq 0$. Since $\exp(w^{h_w}) = h_w \exp(w)$ by (3.2), the result follows. $\square$

Finally we determine the $\mathbf{V}_p$-closure of a nontrivial cyclic subgroup of F_n.

Corollary 4.7. *Let $w \in F_n \setminus \{1\}$. Let $u = \mathrm{root}(w)$ and $e = \exp(w)$. Let $d_p = \gcd(e, h_u)\nu_p(e)$. Then $\mathrm{Cl}_{\mathbf{V}_p}(\langle w \rangle) = \langle u^{d_p} \rangle$.*

Proof. Write $F = F_n$ and $K = K_n$. Write also $H = \langle w \rangle$, $p^s = \nu_p(e)$ and $h = h_w$. By (4.4), we have $H \cap K = \langle w^h \rangle$. Since

$$\langle w \rangle = \bigcup_{i=0}^{h-1} \langle w^h \rangle w^i,$$

it follows from (4.2) that

$$\mathrm{Cl}^F_{\mathbf{V}_p}(\langle w \rangle) = \bigcup_{i=0}^{h-1} (\mathrm{Cl}^K_{\mathbf{G}_p}(\langle w^h \rangle)) w^i. \tag{4.5}$$

Now Corollary 3.6(iii) yields

$$\mathrm{Cl}^K_{\mathbf{G}_p}(\langle w^h \rangle) = \langle \mathrm{root}_K(w^h)^{\nu_p(\exp_K(w^h))} \rangle.$$

By (3.2) and Lemma 4.5(i), we get

$$\mathrm{root}_K(w^h) = \mathrm{root}_K(w) = u^{h_u}.$$

By Lemma 4.5(iii) and (3.2), we get

$$\nu_p(\exp_K(w^h)) = \nu_p(\exp(w^h)) = \nu_p(he),$$

hence

$$\mathrm{Cl}^K_{\mathbf{G}_p}(\langle w^h \rangle) = \langle u^{h_u \nu_p(he)} \rangle.$$

Since $h \mid p-1$, we get

$$\mathrm{Cl}^K_{\mathbf{G}_p}(\langle w^h \rangle) = \langle u^{h_u \nu_p(e)} \rangle = \langle u^{h_u p^s} \rangle$$

and it follows from (4.5) that

$$\mathrm{Cl}^F_{\mathbf{V}_p}(\langle w \rangle) = \bigcup_{i=0}^{h-1} \langle u^{h_u p^s} \rangle w^i.$$

Thus we only need to show that

$$\bigcup_{i=0}^{h-1} \langle u^{h_u p^s} \rangle w^i = \langle u^{d_p} \rangle. \tag{4.6}$$

First, we show that

$$\langle w^h \rangle \leq \langle u^{h_u p^s} \rangle. \tag{4.7}$$

Indeed, we have $w^h = u^{eh}$, so it suffices to show that $h_u p^s$ divides eh. Clearly,

$$\nu_p(h_u p^s) = p^s = \nu_p(e) = \nu_p(eh).$$

Hence we only need to show that h_u divides eh. Since $u^{eh} = w^h \in K$ and $u^{h_u} \in K$ by Lemma 4.4, then also $u^{\gcd(eh,h_u)} \in K$. Now it follows from Lemma 4.4 that $h_u = \gcd(eh, h_u)$, hence h_u divides eh. Therefore (4.7) holds.

Since d_p divides $h_u p^s$, we get $u^{h_u p^s} \in \langle u^{d_p} \rangle$ and consequently $\langle u^{h_u p^s} \rangle \leq \langle u^{d_p} \rangle$. On the other hand, d_p also divides e and so $w = u^e \in \langle u^{d_p} \rangle$. Therefore

$$\bigcup_{i=0}^{h-1} \langle u^{h_u p^s} \rangle w^i \leq \langle u^{d_p} \rangle.$$

Conversely, let $v \in \langle u^{d_p} \rangle$. Then $v = u^{kd_p}$ for some $k \in \mathbb{Z}$. Let $x, y \in \mathbb{Z}$ be such that $\gcd(e, h_u) = xe + yh_u$. There exist some $z \in \mathbb{Z}$ and $i \in \{0, \dots, h-1\}$ such that $kxp^s = zh + i$. In view of (4.7), we get

$$v = u^{kd_p} = u^{kxep^s + kyh_u p^s} = (u^{h_u p^s})^{ky} w^{kxp^s} = (u^{h_u p^s})^{ky} (w^h)^z w^i \in \langle u^{h_u p^s} \rangle w^i$$

and so

$$\langle u^{d_p} \rangle \leq \bigcup_{i=0}^{h-1} \langle u^{h_u p^s} \rangle w^i.$$

This establishes (4.6). $\square$

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