On a pseudovariety of finite supersolvable groups

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Abstract

We introduce the pseudovariety of finite groups $\mathbf{U} = \bigvee_{p \in \mathbb{P}} \mathbf{Ab}(p) * \mathbf{Ab}(p-1)$, where $\mathbb{P}$ is the set of all primes. We show that $\mathbf{U}$ consists of all finite supersolvable groups with elementary abelian derived subgroup and abelian Sylow subgroups, and so $\mathbf{U}$ has decidable membership problem. We prove that it is decidable whether or not a finitely generated subgroup of a free group is closed or dense for the pro-$\mathbf{U}$ topology. We consider also the pseudovariety of finite groups $\mathbf{Ab}(p) * \mathbf{Ab}(d)$ (where p is a prime and d divides $p-1$). We study the pro-$(\mathbf{Ab}(p) * \mathbf{Ab}(d))$ topology on a free group and construct the unique generator of minimum size of the pseudovariety $\mathbf{Ab}(p) * \mathbf{Ab}(d)$. Finally, we prove that the variety of groups generated by $\mathbf{U}$ is the variety of all metabelian groups, obtaining also results on the varieties generated by a Baumslag-Solitar group of the form $BS(1, q)$ for q prime.

1 Introduction

The classification of finite groups is usually interpreted as the classification of finite simple groups, but universal algebra provides an alternative approach through the concept of a pseudovariety (a class of finite algebras closed under taking subalgebras, homomorphic images and finitary direct products). This is common practice with more general classes of algebras such as semigroups or monoids, where the role played by simple groups has no equivalent.

One of the interesting features of pseudovarieties of finite groups is that they induce a (pseudometrizable) topology on any group G. Given such a pseudovariety $\mathbf{V}$, the pro-$\mathbf{V}$ topology on G is the initial topology with respect to all homomorphisms $G \to H \in \mathbf{V}$, where H is endowed with the discrete topology. If $\mathbf{G}$ denotes the pseudovariety of all finite groups, the pro-$\mathbf{G}$ topology is known as the profinite topology.

The profinite topology was introduced by Marshall Hall in [11] and he proved in [10, Theorem 5.1] that every finitely generated subgroup of a free group is closed for the profinite topology.

Over the years, other pseudovarieties $\mathbf{V}$ were considered, and the following decidability questions became objects of study for an arbitrary finitely generated subgroup H of a free group F (assuming that F is a free group over A and we are given a set of generators of H expressed as reduced words over $A \cup A^{-1}$):

- Can we decide whether H is closed in the pro-$\mathbf{V}$ topology?
- Can we decide whether H is dense in the pro-$\mathbf{V}$ topology?
- Does the pro-$\mathbf{V}$ closure $\mathrm{Cl}_{\mathbf{V}}(H)$ of H have decidable membership problem?
- Can we decide whether $\mathrm{Cl}_{\mathbf{V}}(H)$ is finitely generated and can we compute a basis in the affirmative case?

In [28], Ribes and Zalesskiĭ answered all these questions positively for the pseudovariety $\mathbf{G}_p$ of all finite p-groups, for an arbitrary prime p. In [20], Margolis, Sapir and Weil dealt successfully with the pseudovariety $\mathbf{N}$ of all finite nilpotent groups (the case where F is of infinite rank follows from [21, Corollary 2.4]). The cases of the pseudovariety $\mathbf{Ab}$ of all finite abelian groups and the pseudovariety $\mathbf{M}$ of all finite metabelian groups are settled in [22]. It is understood that the case of extension-closed pseudovarieties is in general more favourable. However, the most famous open problems concern the extension-closed pseudovariety $\mathbf{S}$ of finite solvable groups.

Among the most important pseudovarieties lying between $\mathbf{N}$ and $\mathbf{S}$, we find the pseudovariety $\mathbf{Su}$ of finite supersolvable groups. A finite group is *supersolvable* if its chief factors are all cyclic. Equivalently, a finite group is supersolvable if *all its maximal subgroups are* of prime index. In [3], Auinger and Steinberg studied $\mathbf{Su}$ and its Hall subpseudovarieties, and this motivated us to study the pro-$\mathbf{Su}$ topology in [23]. In particular, we related the property of being dense for the pro-$\mathbf{Su}$ topology with groups of the form $C_p \rtimes C_d$, where p is a prime and d divides $p-1$. In [21] we prove that cyclic subgroups of a free group of finite rank are $\mathbf{N}$-closed and consequently $\mathbf{Su}$-closed and $\mathbf{S}$-closed. On the other hand, Auinger and Steinberg proved in [3, Corollary 2.7] the decomposition

$$\mathbf{Su} = \bigvee_{p \in \mathbb{P}} \mathbf{G}_p * \mathbf{Ab}(p-1),$$

where $\mathbb{P}$ denotes the set of all primes, $\mathbf{Ab}(n)$ denotes the class of finite groups whose exponent divides n and $*$ is the semidirect product of pseudovarieties.

It seemed thus a natural idea to study the pseudovariety

$$\mathbf{U} = \bigvee_{p \in \mathbb{P}} \mathbf{Ab}(p) * \mathbf{Ab}(p-1).$$

We note that the aforementioned semidirect products $C_p \rtimes C_d$, which play an important role in the pro-$\mathbf{Su}$ topology, belong to the pseudovariety $\mathbf{Ab}(p) * \mathbf{Ab}(p-1)$.

We give now a brief description of the structure of the paper. After introducing basic concepts and fixing notation in Section 2, we recall in Section 3 some known facts on the pseudovarieties $\mathbf{Ab}(p) * \mathbf{Ab}(d)$ (where $p \in \mathbb{P}$ and d divides $p-1$) and compute the closure of $H \leqslant_{f.g.} F_n$ for the pro-$(\mathbf{Ab}(p) * \mathbf{Ab}(d))$ topology. We compute also the unique generator of minimum size (up to isomorphism) for each pseudovariety $\mathbf{Ab}(p) * \mathbf{Ab}(d)$.

In Section 4, we start the study of $\mathbf{U}$. The main result states that a finite group G belongs to $\mathbf{U}$ if and only if it is supersolvable, its derived subgroup is elementary abelian and has abelian Sylow subgroups. As a corollary, $\mathbf{U}$ has decidable membership problem (i.e., there is an algorithm which decides whether or not a given arbitrary finite group belongs to $\mathbf{U}$).

In Section 5, we prove that the variety of groups generated by $\mathbf{U}$ is the variety of all metabelian groups, obtaining also results on the varieties generated by a Baumslag-Solitar group of the form $BS(1,q)$ for q prime. The stronger version of some results depends on a generalized form of the Riemann hypothesis. An intriguing conjecture is also suggested, involving the variety generated by families of groups of the form $C_p \rtimes C_{p-1}$ (p prime).

We note that $\mathbf{U}$ is not extension-closed: it is easy to check that $A_4 \notin \mathbf{U}$, but $[A_4, A_4] \cong C_2^2 \in \mathbf{U}$ and $A_4/[A_4, A_4] \cong C_3 \in \mathbf{U}$.

In Section 6, we show that $\mathbf{U} \subset \mathbf{Su} \cap \mathbf{M}$ and study the pro-$\mathbf{U}$ topology on F_n. In particular, we provide a noneffective description of the pro-$\mathbf{U}$ closure of $H \leqslant_{f.g.} F_n$ and show that is decidable whether or not H is closed (or dense) for the pro-$\mathbf{U}$ topology on F_n. If $H \leqslant_{f.g.} F$ and F is a free group of infinite rank, we show that H is neither closed nor dense.

Finally, we produce a finitely generated subgroup of the free object of the variety generated by $\mathbf{U}$ which is not $\mathbf{U}$-closed, unlike the abelian and metabelian cases.

2 Preliminaries

Given a group G and elements $g, h \in G$, we use the notation $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h, and denote the derived subgroup of G by $[G, G]$. The second derived subgroup $[[G, G], [G, G]]$ is denoted by $G^{(2)}$. Given subgroups H and K of G, we let $[H, K] = \langle [h, k] : h \in H, k \in K\rangle$. If H and K are normal in G then so is $[H, K]$. Given $H \trianglelefteq G$ and $k \geqslant 1$, we denote by $\mathrm{pow}_k(H)$ the (normal) subgroup of G generated by all powers of the form u^k with $u \in H$. On the other hand, given a group H, H^k denotes the direct product of k copies of H. Given a subset X of a group G, we denote by $\langle\langle X\rangle\rangle_G$ the normal closure of X in G, i.e. the smallest normal subgroup of G containing X. We denote by $o(g)$ the order of an element g of a group.

Given a positive integer m, we let C_m or $\mathbb{Z}/m\mathbb{Z}$ denote the cyclic group of order m. When the context is clear, we sometimes view $\mathbb{Z}/m\mathbb{Z}$ as a ring (namely the ring of integers modulo m), and denote by $(\mathbb{Z}/m\mathbb{Z})^*$ the group of units of $\mathbb{Z}/m\mathbb{Z}$. When p is a prime, we also write $\mathbb{F}_p$ for $\mathbb{Z}/p\mathbb{Z}$.

Let p be a prime and let $d > 1$ be a divisor of $p-1$. It is well known that $(\mathbb{Z}/p\mathbb{Z})^*$ is a cyclic group. Hence $(\mathbb{Z}/p\mathbb{Z})^*$ has a unique subgroup of order d, which is itself cyclic. Given a positive integer k, we let $[k] = \{1, \ldots, k\} \subset \mathbb{N}$. We shall use the notation

$$Q_{p,d} = \{q \in [p-1] \mid q^d \equiv 1 \pmod{p}\} \text{ and } Q'_{p,d} = \{q \in [p-1] \mid q \text{ has order } d \text{ in } \mathbb{F}_p^*\}.$$

Given a ring R and elements $k_1, \ldots, k_n \in R$, we let $\mathrm{diag}(k_1, \ldots, k_n)$ be the matrix having entries $k_1, \ldots, k_n$ in the main diagonal. Given a direct product $R = R_1 \times \ldots \times R_k$ of rings with unity and $i \in [k]$, we use the notation e_i for the element of R having 1 in the ith component and 0 everywhere else. Let 1^k be shorthand for a sequence $1, 1, \ldots, 1$ of k 1's.

If A is a set, we set F_A to be the free group on A. The length of an element $u \in F_A$ is the length of the reduced form of u and is denoted by $|u|$. Also if H is a finitely generated subgroup of F_A, we write $H \leqslant_{f.g.} F_A$. We denote by F_n the free group of rank $n \in \mathbb{N}$ and denote by A or A_n a fixed basis of F_n.

In this paper a (finite) *automaton* is a structure of the form $\mathcal{A} = (A, Q, q_0, E)$, where the *alphabet* A and the *vertex set* Q are (finite) sets, $q_0 \in Q$ is the *basepoint* and $E \subseteq Q \times A \times Q$ is the *edge set*.

Given a subgroup H of a free group F_A, we define the *Schreier automaton* $S(H) = (A \cup A^{-1}, Q, H, E)$ by

$$Q = \{Hu \mid u \in F_A\}, \quad E = \{(Hu, a, Hua) \mid u \in F_A, a \in A \cup A^{-1}\}.$$

If H is finitely generated, the subautomaton $\mathcal{A}(H)$ induced by the set of vertices of the form Hu, where u is a prefix of some $h \in H$, is a finite automaton known as the *Stallings automaton* of H. In [32], Stallings introduced an algorithm to compute $\mathcal{A}(H)$ from any finite generating set of H through the *folding* operation), and provided several important applications of the concept. For the basic properties of Stallings automata, the reader is referred to [4, Section 3].

A *pseudovariety of finite groups* is a class of finite groups closed under taking subgroups, homomorphic images and finitary direct products. Since no other pseudovarieties occur in the paper, we will just write *pseudovariety*. For the general theory of pseudovarieties, the reader is referred to [26].

Classical examples include the pseudovariety $\mathbf{Ab}$ of all finite abelian groups and the Burnside pseudovarieties $\mathbf{B}(n)$. For every $n \geqslant 1$, $\mathbf{B}(n)$ denotes the class of all finite groups G such that $g^n = 1$ for every $g \in G$. Since the intersection of pseudovarieties is still a pseudovariety,

$$\mathbf{Ab}(n) = \mathbf{Ab} \cap \mathbf{B}(n)$$

is also a pseudovariety for every $n \geqslant 1$. In other words $\mathbf{Ab}(n)$ is the pseudovariety of all finite abelian groups of exponent dividing n.

If $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties, the pseudovariety $\mathbf{V}$-by-$\mathbf{W}$ contains all finite groups G admitting a normal subgroup $N \in \mathbf{V}$ such that $G/N \in \mathbf{W}$. By [8, Proposition 10.1], we have $\mathbf{V}$-by-$\mathbf{W} = \mathbf{V} * \mathbf{W}$, i.e. the pseudovariety of finite groups containing all subgroups of semidirect products $V \rtimes W$ where $V \in \mathbf{V}$ and $W \in \mathbf{W}$.

A *variety of groups* is a class of groups closed under taking subgroups, homomorphic images and arbitrary direct products. By Birkhoff's Theorem, a class of groups constitutes a variety if and only if it is the class of all groups satisfying a certain set of group identities. If X denotes a countable alphabet, a *group identity* is a formal equality of the type $u = v$, where $u, v \in F_X$. A group G satisfies this identity if $\varphi(u) = \varphi(v)$ for every homomorphism $\varphi : F_X \to G$. We then write $G \models u = v$. Given a set of identities $\Sigma \subseteq F_X \times F_X$, we denote by $[\Sigma]$ the variety of all groups satisfying each one of the identities in Σ.

We consider two important examples of varieties in this paper. One is the variety of all abelian groups, denoted by $\mathcal{AB}$ and defined by the single identity $xy = yx$. The other is the variety of all metabelian groups, denoted by $\mathcal{M}$. A group G is *metabelian* if $[G, G]$ is abelian, hence $\mathcal{M}$ is defined by the single identity $[x_1, x_2][x_3, x_4] = [x_3, x_4][x_1, x_2]$.

Given a class C of groups, we denote by $\mathcal{V}(C)$ the variety of groups *generated* by C. This is the smallest variety containing C and consists of all homomorphic images of subgroups of direct products of groups in C.

Let $\mathcal{V} = [\Sigma]$. Given an alphabet A, let

$$F_A^{\mathcal{V}} = \langle\langle \varphi(uv^{-1}) \mid (u, v) \in \Sigma,\, \varphi : F_X \to F_A \text{ homomorphism}\rangle\rangle_{F_A} \trianglelefteq F_A.$$

It follows easily from the definitions and the universal property of F_A that $F_A(\mathcal{V}) = F_A/F_A^{\mathcal{V}}$ is the *free object* of $\mathcal{V}$ on A. For details on varieties of groups, the reader is referred to [24].

Now $\mathcal{V}^f = \mathcal{V} \cap \mathbf{G}$ is clearly a pseudovariety, known as the *finite trace of* $\mathcal{V}$. In view of Birkhoff's Theorem, a pseudovariety $\mathbf{V}$ is of the form $\mathbf{V} = \mathcal{V}^f$ if and only if it is *equational*, that is, there exists a set Σ of group identities such that $\mathbf{V} = [\Sigma] \cap \mathbf{G}$. Note that arbitrary pseudovarieties require in general pseudoidentities (in view of Reiterman's Theorem). The reader is referred to [26, Section 7.2] for further details.

Let $\mathbf{V}$ denote a pseudovariety and let Σ be a set of identities. We write $\mathbf{V} \models \Sigma$ if $G \models \sigma$ for all $G \in \mathbf{V}$ and $\sigma \in \Sigma$. Write $[[\Sigma]] = [\Sigma]^f$.

Given a finite group G, we denote by $\langle G \rangle$ the pseudovariety generated by G (consisting of all homomorphic images of subgroups of direct powers of the form G^n). Such a pseudovariety is said to be *finitely generated*. Note that the smallest pseudovariety containing the finite groups $G_1, \ldots, G_n$ is the pseudovariety generated by $G_1 \times \ldots \times G_n$ (this explains the terminology).

Every finitely generated pseudovariety $\mathbf{V}$ admits *free objects*. That is, given any finite set A, there exists some $F_A(\mathbf{V}) \in \mathbf{V}$ and some mapping $\iota : A \to F_A(\mathbf{V})$ such that, for every mapping $\varphi : A \to G \in \mathbf{V}$, there exists some (unique) homomorphism $\Phi : F_A(\mathbf{V}) \to G$ such that $\Phi \circ \iota = \varphi$. Moreover, $F_A(\mathbf{V})$ is unique up to isomorphism. On the other hand, it follows from [25] that if $\mathbf{V}$ is finitely generated then $\mathbf{V} = [[\Sigma]]$ for some set Σ of identities.

Given a pseudovariety $\mathbf{V}$, where we consider finite groups endowed with the discrete topology, the pro-$\mathbf{V}$ topology on a group G is defined as the coarsest topology which makes all morphisms from G into elements of $\mathbf{V}$ continuous. Equivalently, G is a topological group where the normal subgroups K of G such that $G/K \in \mathbf{V}$ form a basis of neighbourhoods of the identity. It is well known (see e.g. [28, Lemmas 2.1 and 2.2]) that a subgroup H of G is pro-$\mathbf{V}$ open if and only if it is pro-$\mathbf{V}$ clopen, and if and only if $G/\mathrm{Core}_G(H)$ belongs to $\mathbf{V}$ (the core $\mathrm{Core}_G(H)$ of H in G is the largest normal subgroup of G contained in H and is equal to $\bigcap_{g\in G} g^{-1}Hg$). Note that H is pro-$\mathbf{V}$ closed if and only if, for every $g \in G \setminus H$, there exists some pro-$\mathbf{V}$ clopen $K \leqslant G$ such that $H \leqslant K$ and $g \notin K$. Moreover, a subgroup H of G is pro-$\mathbf{V}$ dense if and only if $HN = G$ for every normal subgroup N of G such that $G/N \in \mathbf{V}$.

For a topological property $\mathcal{P}$ and a subset S of G, we say that S is $\mathbf{V}$-$\mathcal{P}$ if S has property $\mathcal{P}$ in the pro-$\mathbf{V}$ topology on G. Given $S \subseteq G$, we also denote by $\mathrm{Cl}^G_{\mathbf{V}}(S)$ the $\mathbf{V}$-closure of S in G and we omit the superscript when no confusion is possible. In view of [11, Theorem 3.3], for every $H \leqslant G$, $\mathrm{Cl}^G_{\mathbf{V}}(H)$ is the intersection of all the $\mathbf{V}$-open subgroups of G containing H (and is therefore a subgroup itself).

Suppose that $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of finite groups. Then:

$$\text{If } \mathbf{W} \subseteq \mathbf{V}, \text{ then } \mathrm{Cl}^G_{\mathbf{V}}(S) \subseteq \mathrm{Cl}^G_{\mathbf{W}}(S) \text{ for every } S \subseteq G. \tag{1}$$

This follows from the pro-$\mathbf{W}$ topology being coarser than the pro-$\mathbf{V}$ topology on G.

In this paper we mainly consider the case $G = F_n$.

We consider now the case of locally finite pseudovarieties. We say that a pseudovariety $\mathbf{V}$ of groups is *locally finite* if, for every $n \geqslant 1$, there exist only finitely many n-generated groups in $\mathbf{V}$, up to isomorphism. This is equivalent to saying that $\mathbf{V}$ contains a finite free object $F_n(\mathbf{V})$ on any finite set with n elements. Then we may write $F_n(\mathbf{V}) = F_n/L_n(\mathbf{V})$ for some finite index $L_n(\mathbf{V}) \trianglelefteq F_n$. It is well known that every finitely generated pseudovariety of groups is locally finite.

The following two results are well known to experts, but we include short proofs for lack of an adequate reference:

Proposition 2.1 *Let $\mathbf{V}$ be a locally finite pseudovariety of groups and let $H \leqslant F_n$. Then:*

(i) $\mathrm{Cl}_{\mathbf{V}}(H) = HL_n(\mathbf{V})$*;*

(ii) $\mathrm{Cl}_{\mathbf{V}}(H)$ *has finite index in* F_n*;*

(iii) $\mathrm{Cl}_{\mathbf{V}}(H)$ *is finitely generated.*

Proof. (i) Since $F_n/L_n(\mathbf{V}) \in \mathbf{V}$, then $L_n(\mathbf{V})$ is $\mathbf{V}$-clopen. Hence $\mathrm{Cl}_{\mathbf{V}}(1) \leqslant L_n(\mathbf{V})$.

Now let $M \leqslant F_n$ be $\mathbf{V}$-open. Then $F_n/\mathrm{Core}_{F_n}(M) \in \mathbf{V}$, so $L_n(\mathbf{V}) \leqslant \mathrm{Core}_{F_n}(M)$ since $F_n/L_n(\mathbf{V})$ is a free object of $\mathbf{V}$. Hence $L_n(\mathbf{V}) \subseteq M$ and so $L_n(\mathbf{V}) \subseteq \mathrm{Cl}_{\mathbf{V}}(1)$. Thus $\mathrm{Cl}_{\mathbf{V}}(1) = L_n(\mathbf{V})$.

It follows that $H, L_n(\mathbf{V}) \leqslant \mathrm{Cl}_{\mathbf{V}}(H)$ and so $HL_n(\mathbf{V}) \leqslant \mathrm{Cl}_{\mathbf{V}}(H)$. Since $L_n(\mathbf{V})$ is $\mathbf{V}$-clopen, so is $HL_n(\mathbf{V})$, therefore $\mathrm{Cl}_{\mathbf{V}}(H) = HL_n(\mathbf{V})$.

(ii) This follows from the fact that $F_n/L_n(\mathbf{V}) \in \mathbf{V}$ and $L_n(\mathbf{V}) \leq HL_n(\mathbf{V}) = \mathrm{Cl}_{\mathbf{V}}(H)$.

(iii) This follows from part (ii), since F_n is finitely generated. $\square$

Corollary 2.2 *Let* $\mathbf{V}$ *be a locally finite pseudovariety of groups and let* $H \leqslant_{f.g.} F_n$*. If* $F_n(\mathbf{V})$ *is computable, then* $\mathrm{Cl}_{\mathbf{V}}(H)$ *is computable from a finite generating subset of* H*.*

Proof. Since $F_n(\mathbf{V}) = F_n/L_n(\mathbf{V})$ is computable, then $L_n(\mathbf{V})$ is itself computable: we can construct the Cayley graph of $F_n(V)$ with respect to A_n, which is also the Stallings automaton $\mathcal{A}(L_n(\mathbf{V}))$ with respect to the basis A_n of F_n. We can also construct $\mathcal{A}(H)$ since we are given a finite generating set of H. By Proposition 2.1(i), we have

$$\mathrm{Cl}_{\mathbf{V}}(H) = HL_n(\mathbf{V}) = H \vee L_n(\mathbf{V}).$$

Now it follows from Stallings' algorithm [4, Section 3] that $\mathcal{A}(H \vee L_n(\mathbf{V}))$ is obtained by identifying the basepoints of $\mathcal{A}(L_n(\mathbf{V}))$ and $\mathcal{A}(H)$ and performing a complete folding. Therefore $\mathcal{A}(\mathrm{Cl}_{\mathbf{V}}(H))$ is effectively constructible, which is equivalent to saying that $\mathrm{Cl}_{\mathbf{V}}(H)$ is computable. $\square$

Throughout the paper, unless otherwise stated, we fix a prime $p > 2$ and a divisor $d > 1$ of $p - 1$.

3 The pseudovarieties $\mathbf{Ab}(p) * \mathbf{Ab}(d)$

Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p - 1$.

By a theorem of Šmel'kin [31] (see also [24, Theorem 24.64]), the pseudovariety $\mathbf{Ab}(p)*\mathbf{Ab}(d)$ is finitely generated (hence locally finite) and admits therefore free objects. For every $n \geqslant 1$, let $F_n(p, d)$ denote the free object of $\mathbf{Ab}(p) * \mathbf{Ab}(d)$ on a set of size n. By a theorem of Higman [13] (see also [24, Item 24.65]), we have

$$\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle F_2(p, d) \rangle.$$

How big is $F_2(p, d)$? For all $m, n \geqslant 1$, we have $\mathbf{Ab}(m) = \langle C_m \rangle$ and it follows easily that

$$F_n(\mathbf{Ab}(m)) \cong C_m^n.$$

Now it follows from [24, Corollary 21.13] that $F_n(p, d)$ admits a semidirect product decomposition

$$F_n(p, d) = C_p^{(n-1)d^n+1} \rtimes C_d^n.$$

In particular, $|F_2(p, d)| = p(d^2 + 1)d^2$.

Since $\mathbf{Ab}(m) = \big[\big[[x, y], x^m\big]\big]$ for every $m \geqslant 1$, we have

$$F_n(\mathbf{Ab}(m)) \cong F_n/\langle\langle [F_n, F_n] \cup \mathrm{pow}_m(F_n) \rangle\rangle_{F_n}.$$

Write $K_{n,m} = [F_n, F_n]\mathrm{pow}_m(F_n)$. It is routine to check that $K_{n,m} \trianglelefteq F_n$, hence $\langle\langle [F_n, F_n] \cup \mathrm{pow}_m(F_n) \rangle\rangle_{F_n} = K_{n,m}$ and so

$$F_n(\mathbf{Ab}(m)) \cong F_n/K_{n,m}.$$

Write

$$L_{n,p,d} = [K_{n,d}, K_{n,d}]\mathrm{pow}_p(K_{n,d}).$$

It follows from [24, Item 21.12] that

$$\mathbf{Ab}(p) * \mathbf{Ab}(d) = \Big[\Big[w = 1 \ (w \in \bigcup_{n \geqslant 1} L_{n,p,d})\Big]\Big]. \tag{2}$$

Now we can derive easily the following alternative characterization:

Proposition 3.1 *Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p-1$. If $n \geqslant 1$ then $F_n(p,d) \cong F_n/L_{n,p,d}$.*

Proof. In view of (2), it suffices to show that $F_n/L_{n,p,d} \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$, which amounts to proving that $F_n/L_{n,p,d} \models w = 1$ for all $w \in L_{m,p,d}$ and $m \geqslant 1$. It is therefore enough to show that $\varphi(w) \in L_{n,p,d}$ for every homomorphism $\varphi : F_m \to F_n$ and every $w \in L_{m,p,d}$. This can be further reduced to proving that $\varphi(u) \in K_{n,d}$ for every homomorphism $\varphi : F_m \to F_n$ and every $u \in K_{m,d}$, which is straightforward. $\square$

Thus $L_{n,p,d} = L_n(\mathbf{Ab}(p) * \mathbf{Ab}(d))$ and it follows from Proposition 2.1(i) that

$$\mathrm{Cl}_{\mathbf{Ab}(p)*\mathbf{Ab}(d)}(H) = HL_{n,p,d} \text{ for every } H \leqslant F_n. \tag{3}$$

Moreover, it follows from Corollary 2.2 that if $H \leqslant_{f.g.} F_n$, then $\mathrm{Cl}_{\mathbf{Ab}(p)*\mathbf{Ab}(d)}(H)$ is computable from a finite generating subset of H.

The remainder of this section is devoted to finding a generator of minimum size for $\mathbf{Ab}(p) * \mathbf{Ab}(d)$.

Since p and d are coprime, it follows from Schur-Zassenhaus theorem [30, Theorem 7.41] that the pseudovariety $\mathbf{Ab}(p)$-by-$\mathbf{Ab}(d) = \mathbf{Ab}(p) * \mathbf{Ab}(d)$ consists of all semidirect products of the form $N \rtimes H$ with $N \in \mathbf{Ab}(p)$ and $H \in \mathbf{Ab}(d)$.

The (outer) semidirect product of two groups N and H is defined by a group action

$$\begin{array}{rcl} \varphi : H & \to & \mathrm{Aut}(N) \\ h & \mapsto & \varphi_h \end{array}$$

which induces a group operation on $N \times H$ by

$$(u,h)(u',h') = (u \cdot \varphi_h(u'), hh').$$

This group is denoted by $N \rtimes_\varphi H$. Note that:

- $N \cong N \times \{1\} \trianglelefteq N \rtimes_\varphi H$;
- $H \cong \{1\} \times H \leqslant N \rtimes_\varphi H$;
- $(1,h)(u,1)(1,h)^{-1} = (\varphi_h(u),h)(1,h^{-1}) = (\varphi_h(u),1)$ for all $u \in N$ and $h \in H$.

It is well known how to build a presentation for $N \rtimes_\varphi H$ from presentations for N and H. Assume that $\langle X \mid R \rangle$ is a presentation for N and $\langle Y \mid S \rangle$ is a presentation for H, with $X \cap Y = \emptyset$. For all $x \in X$ and $y \in Y$, fix a word $\overline{\varphi}_y(x) \in (X \cup X^{-1})^*$ (the free monoid on $X \cup X^{-1}$) representing $\varphi_y(x) \in N$. Then

$$\langle X \cup Y \mid R \cup S, \, yxy^{-1} = \overline{\varphi}_y(x) \ (x \in X, \, y \in Y) \rangle. \tag{4}$$

is a presentation of $N \rtimes_\varphi H$.

We claim that we can restrict the type of semidirect product required to produce all the groups in $\mathbf{Ab}(p) * \mathbf{Ab}(d)$. Indeed, assume that $N \in \mathbf{Ab}(p)$. Then N is a direct product of n

copies of $\mathbb{Z}/p\mathbb{Z}$ for some $n \geqslant 1$ and so $\mathrm{Aut}(N)$ is isomorphic to the group of linear automorphisms of a vector space of dimension n over the field $\mathbb{F}_p$, which in turn is isomorphic to $\mathrm{GL}_n(\mathbb{F}_p)$.

Since we are writing functions on the left, we must be careful on defining this isomorphism. Let $\{e_1, \ldots, e_n\}$ denote the canonical basis of $\mathbb{F}_p^n$. Given $\theta \in \mathrm{Aut}(N)$, let $M_\theta \in \mathrm{GL}_n(\mathbb{F}_p)$ be the matrix where the ith row vector is given by the coordinates of $\theta(e_i)$ in the canonical basis. Then

$$\mathrm{Aut}(N) \to \mathrm{GL}_n(\mathbb{F}_p) : \theta \mapsto M_\theta$$

is really an anti-isomorphism. To get a true isomorphism, we must define instead

$$\begin{array}{rcl} \Lambda : \mathrm{Aut}(N) & \to & \mathrm{GL}_n(\mathbb{F}_p) \\ \theta & \mapsto & M_{\theta^{-1}} \end{array}$$

We say that $\varphi : H \to \mathrm{Aut}(N)$ is a *diagonalizable* action if there exists some $P \in \mathrm{GL}_n(\mathbb{F}_p)$ such that $P^{-1}M_{\varphi_h}P$ is a diagonal matrix for every $h \in H$.

The following is a special case from [2, Theorem 4.1]:

Lemma 3.2 *Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p-1$. Let $N \in \mathbf{Ab}(p)$ and $H \in \mathbf{Ab}(d)$. Every action $\varphi : H \to Aut(N)$ is diagonalizable.*

Lemma 3.3 *Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p-1$. The following conditions are equivalent for a group G:*

(i) $G \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$;

(ii) G admits a presentation of the form

$$\begin{aligned} \langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid x_i^p = 1,\, y_j^{d_j} = 1,\, x_i x_{i'} = x_{i'} x_i,\, y_j y_{j'} = y_{j'} y_j, \\ y_j x_i y_j^{-1} = x_i^{q_{ij}} \ (i, i' \in [n],\, j, j' \in [m])\rangle \end{aligned} \tag{5}$$

for some $n, m \geqslant 0$, divisors $d_j > 1$ of d and $q_{ij} \in Q_{p,d_j}$ $(i \in [n],\, j \in [m])$.

Proof. (i) $\Rightarrow$ (ii). Assume that $G = N \rtimes_\varphi H$ with $N \in \mathbf{Ab}(p)$ and $H \in \mathbf{Ab}(d)$. By Lemma 3.2, the action $\varphi : H \to \mathrm{Aut}(N)$ is diagonalizable. We may assume that both N and H are nontrivial. Then $N \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some $n \geqslant 1$ and $H \cong (\mathbb{Z}/d_1\mathbb{Z}) \times \ldots \times (\mathbb{Z}/d_m\mathbb{Z})$ for some divisors $d_1, \ldots, d_m > 1$ of d. Let $a_i \in N$ correspond to $e_i \in (\mathbb{Z}/p\mathbb{Z})^n$ for each $i \in [n]$, and let $y_j \in H$ correspond to $e_j \in (\mathbb{Z}/d_1\mathbb{Z}) \times \ldots \times (\mathbb{Z}/d_m\mathbb{Z})$ for each $j \in [m]$. Then $N = \langle a_1, \ldots, a_n\rangle$ and $H = \langle y_1, \ldots, y_m\rangle$.

Since φ is diagonalizable, there exists some $P \in \mathrm{GL}_n(\mathbb{F}_p)$ such that $P^{-1}M_{\varphi_h}P$ is a diagonal matrix for every $h \in H$. Note that M_{φ_h} is invertible since $M_{\varphi_h}M_{\varphi_{h^{-1}}}$ is the identity matrix. Then there exist $q_{ij} \in [p-1]$ $(i \in [n],\, j \in [m])$ such that

$$P^{-1}M_{\varphi_{y_j}}P = \mathrm{diag}(q_{1j}, \ldots, q_{nj})$$

for each $j \in [m]$.

For each $i \in [n]$, write the Kronecker vector $e_i \in \mathbb{F}_p^n$ as $(\delta_{1i}, \ldots, \delta_{ni})$ and let $x_i \in N$ correspond to the ith column of the matrix $P = (p_{ii'})$. Then

$$\begin{aligned} M_{\varphi_{y_j}}(p_{1i}, p_{2i}, \ldots, p_{ni})^T &= PP^{-1}M_{\varphi_{y_j}}Pe_i^T = P\mathrm{diag}(q_{1j}, \ldots, q_{nj})e_i^T \\ &= P(q_{1j}\delta_{1i}, \ldots, q_{nj}\delta_{ni})^T = (p_{1i}q_{ij}, p_{2i}q_{ij}, \ldots, p_{ni}q_{ij})^T \end{aligned}$$

and so $\varphi_{y_j}(x_i) = x_i^{q_{ij}}$ for all $i \in [n]$ and $j \in [m]$ (in multiplicative notation). Hence

$$(1, y_j)(x_i, 1)(1, y_j)^{-1} = (\varphi_{y_j}(x_i), y_j)(1, y_j^{-1}) = (x_i^{q_{ij}}, 1).$$

We may abuse notation by identifying x_i with $(x_i, 1)$ and y_j with $(1, y_j)$. It follows that

$$G = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$$

and G satisfies all the relations in (5).

Given $i \in [n]$ and $j \in [m]$, it is easy to prove by induction on $k \geqslant 0$ that

$$y_j^k x_i y_j^{-k} = x_i^{q_{ij}^k} \tag{6}$$

holds for the presentation (5). It follows that

$$x_i = y_j^{d_j} x_i y_j^{-d_j} = x_i^{q_{ij}^{d_j}},$$

hence $q_{ij}^{d_j} \equiv 1 \pmod p$ and so $q_{ij} \in Q_{p,d_j}$.

Let G' be the group defined by (5). Since G is a quotient of G', it suffices to show that $|G'| \leqslant |G|$. Clearly, $|G| = p^n d_1 \ldots d_m$. Now, using the relations in (5), we see that every element of G' can be written in the form

$$x_1^{r_1} \ldots x_n^{r_n} y_1^{s_1} \ldots y_m^{s_m}$$

with $r_i \in \{0, \ldots, p-1\}$ for $i \in [n]$ and $s_j \in \{0, \ldots, d_j - 1\}$ for $j \in [m]$. Therefore $|G'| \leqslant p^n d_1 \ldots d_m$ and so (5) is a presentation of G as desired.

(ii) $\Rightarrow$ (i). This is straightforward. $\square$

Given $q \in Q'_{p,d}$, we denote by $G_{p,d}^{(q)}$ the group generated by the presentation

$$\langle x, y \mid x^p = 1,\, y^d = 1,\, yxy^{-1} = x^q \rangle. \tag{7}$$

Lemma 3.4 *Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p - 1$. If $q, r \in Q'_{p,d}$, then $G_{p,d}^{(q)} \cong G_{p,d}^{(r)}$.*

Proof. Since $\mathbb{F}_p^*$ has a unique (cyclic) subgroup of order d, there exist $m, k \geqslant 1$ such that $r^m = q$ and $q^k = r$ modulo p. Consider $\{x, y\}$ as a basis for F_2. Let φ, ψ be the endomorphisms of F_2 defined by

$$\varphi(x) = \psi(x) = x, \quad \varphi(y) = y^m, \quad \psi(y) = y^k.$$

It is routine to check that φ and ψ induce mutually inverse isomorphisms $\overline{\varphi} : G_{p,d}^{(q)} \to G_{p,d}^{(r)}$ and $\overline{\psi} : G_{p,d}^{(r)} \to G_{p,d}^{(q)}$. For instance,

$$\overline{\varphi}(yxy^{-1}) = y^m x y^{-m} = x^{r^m} = x^q = \overline{\varphi}(x^q),$$

while $r^{mk} \equiv r \pmod p$ yields $mk \equiv 1 \pmod d$ and consequently

$$\overline{\psi}(\overline{\varphi}(y)) = y^{mk} = y.$$

Therefore $G_{p,d}^{(q)} \cong G_{p,d}^{(r)}$. $\square$

Therefore we can omit the exponent in $G_{p,d}^{(q)}$.

Proposition 3.5 *Let $p > 2$ be a prime and let $d > 1$ be a divisor of $p-1$. Then*

$$\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle G_{p,d} \rangle.$$

Moreover, $G_{p,d}$ is up to isomorphism the unique generator of minimum size of $\mathbf{Ab}(p) * \mathbf{Ab}(d)$.

Proof. By Lemma 3.3, we have $G_{p,d} \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$, hence $\langle G_{p,d} \rangle \subseteq \mathbf{Ab}(p) * \mathbf{Ab}(d)$.

Conversely, let $G \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$. By Lemma 3.3, we may assume that G is defined by a presentation of the form (5). For every $i \in [n]$, let G_i be the group defined by the presentation

$$\langle x_i, y_1, \ldots, y_m \mid x_i^p = 1,\, y_j^{d_j} = 1,\, y_j y_{j'} = y_{j'} y_j,\, y_j x_i y_j^{-1} = x_i^{q_{ij}} \; (j, j' \in [m]) \rangle.$$

We define a homomorphism $\theta : G \to G_1 \times \ldots \times G_n$ by

$$\theta(x_i) = (1^{i-1}, x_i, 1^{n-i}) \quad \text{and} \quad \theta(y_j) = (y_j, \ldots, y_j) \quad (i \in [n],\ j \in [m]).$$

It is straightforward to check that θ is a well-defined homomorphism. For instance,

$$\begin{aligned}
\theta(y_j x_i y_j^{-1}) &= (y_j, \ldots, y_j)(1^{i-1}, x_i, 1^{n-i})(y_j^{-1}, \ldots, y_j^{-1}) = (1^{i-1}, y_j x_i y_j^{-1}, 1^{n-i}) \\
&= (1^{i-1}, x_i^{q_{ij}}, 1^{n-i}) = \theta(x_i^{q_{ij}}).
\end{aligned}$$

Let $g \in \mathrm{Ker}(\theta)$. Then we may write

$$g = x_1^{r_1} \ldots x_n^{r_n} y_1^{s_1} \ldots y_m^{s_m}$$

for some $r_i \in \{0, \ldots, p-1\}$ and $s_j \in \{0, \ldots, d_j - 1\}$ ($i \in [n]$, $j \in [m]$). Hence

$$(1^n) = \theta(g) = \theta(x_1^{r_1} \ldots x_n^{r_n} y_1^{s_1} \ldots y_m^{s_m}) = (x_1^{r_1} y_1^{s_1} \ldots y_m^{s_m}, \ldots, x_n^{r_n} y_1^{s_1} \ldots y_m^{s_m})$$

and so

$$r_1 = \ldots = r_n = s_1 = \ldots = s_m = 0.$$

Thus θ is injective and so $G \in \langle G_1, \ldots, G_n \rangle$. Therefore we may assume that G is defined by the presentation

$$\langle x, y_1, \ldots, y_m \mid x^p = 1,\, y_j^{d_j} = 1,\, y_j y_{j'} = y_{j'} y_j,\, y_j x y_j^{-1} = x^{q_j} \; (j, j' \in [m]) \rangle,$$

where $q_j \in Q_{p,d_j}$ for $j \in [m]$.

In fact, we claim that we may assume that $d_j = d$ for every $j \in [m]$. Let G' be defined by the presentation

$$\langle x, y_1, \ldots, y_m \mid x^p = 1,\, y_j^d = 1,\, y_j y_{j'} = y_{j'} y_j,\, y_j x y_j^{-1} = x^{q_j} \; (j, j' \in [m]) \rangle. \tag{8}$$

Note that $Q_{p,d_j} \subseteq Q_{p,d}$ for $j \in [m]$. Then G is the image of G' by the canonical homomorphism and so $G \in \langle G' \rangle$.

Therefore we assume that $d_j = d$ for every $j \in [m]$, that is, G is defined by the presentation (8). Now let $r \in Q'_{p,d}$ and let G_r be the group defined by the presentation obtained from (8) by replacing the relation $y_1 x y_1^{-1} = x^{q_1}$ by the relation $y_1 x y_1^{-1} = x^r$. We show that

$$\text{there exists an injective homomorphism } \varphi : G \to G_r \times \mathbb{Z}/d\mathbb{Z}. \tag{9}$$

If q_1 has order d' in $\mathbb{F}_p^*$, then the subgroup generated by r has a subgroup of order d', which must necessarily contain q_1 because a cyclic group has a unique subgroup of a given order. Thus $q_1 \equiv r^k \,(\mathrm{mod}\, p)$ for some $k \geqslant 0$. We define a homomorphism $\varphi : G \to G_r \times \mathbb{Z}/d\mathbb{Z}$ by $\varphi(x) = (x, 0)$, $\varphi(y_1) = (y_1^k, 1)$ and $\varphi(y_j) = (y_j, 0)$ for $j = 2, \ldots, m$. We have

$$\varphi(y_1 x y_1^{-1}) = (y_1^k x y_1^{-k}, 0) = (x^{r^k}, 0) = (x^{q_1}, 0) = \varphi(x^{q_1})$$

and it follows easily that the homomorphism is well defined.

Let $g \in \mathrm{Ker}(\varphi)$. Then $g = x^i y_1^{t_1} \ldots y_m^{t_m}$ for some $i \in \{0, \ldots, p-1\}$ and $t_1, \ldots, t_m \in \{0, \ldots, d-1\}$. Then

$$(1, 0) = \varphi(g) = \varphi(x^i y_1^{t_1} \ldots y_m^{t_m}) = (x^i y_1^{kt_1} y_2^{t_2} \ldots y_m^{t_m}, t_1),$$

hence $i = t_1 = \ldots = t_m = 0$ and φ is injective. Therefore (9) holds.

Now we assume that G is defined by the presentation (8) and we prove that $G \in \langle G_{p,d} \rangle$ by induction on m. We prove first the case $m = 1$. Let G be the group defined by the presentation

$$\langle x, y \mid x^p = 1,\, y^d = 1,\, yxy^{-1} = x^q \rangle.$$

This group is not necessarily isomorphic to $G_{p,d}$ because $q \in Q_{p,d}$ needs not be in $Q'_{p,d}$. Let $r \in Q'_{p,d}$. By (9), G is isomorphic to a subgroup of $G_{p,d}^{(r)} \times \mathbb{Z}/d\mathbb{Z} = G_{p,d} \times \mathbb{Z}/d\mathbb{Z}$. Since $\mathbb{Z}/d\mathbb{Z} \in \langle G_{p,d} \rangle$, it follows that $G \in \langle G_{p,d} \rangle$ and the case $m = 1$ is proved.

Assume now that $m > 1$ and the claim holds for $m-1$. In view of (9), and exchanging y_1 and y_2 if needed, we may assume that $q_2 \in Q'_{p,d}$. Since $q_1 \in Q_{p,d}$ and cyclic groups have a unique subgroup of a given order, it follows that there exists some $k \geqslant 0$ such that $q_1 q_2^k \equiv 1 \,(\mathrm{mod}\, p)$. Hence

$$y_2^k y_1 x y_1^{-1} y_2^{-k} = y_2^k x^{q_1} y_2^{-k} = x^{q_1 q_2^k} = x.$$

Now we replace y_1 by $y_1' = y_2^k y_1$ in the generating set of G, adjusting the presentation accordingly. The unique relation that changes is $y_1 x y_1^{-1} = x^{q_1}$, which is now $y_1' x (y_1')^{-1} = x$. Thus G admits the alternative presentation

$$\langle x, y_1, y_2, \ldots, y_m \mid x^p = 1,\, y_j^d = 1,\, y_j y_{j'} = y_{j'} y_j,\, y_1 x y_1^{-1} = x,\, y_r x y_r^{-1} = x^{q_j}\ (j, j', r \in [m],\, r > 1) \rangle.$$

Denoting by G' the group defined by the presentation

$$\langle x, y_2, \ldots, y_m \mid x^p = 1,\, y_j^d = 1,\, y_j y_{j'} = y_{j'} y_j,\, y_j x y_j^{-1} = x^{q_j}\ (j, j' \in \{2, \ldots, m\}) \rangle,$$

it is now evident that $G \cong G' \times \mathbb{Z}/d\mathbb{Z}$. Since $G' \in \langle G_{p,d} \rangle$ by the induction hypothesis, we get $G \in \langle G_{p,d} \rangle$ and so $\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle G_{p,d} \rangle$.

Suppose now that $\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle H \rangle$ with $|H|$ minimum. The order of any element of any $G \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$ must divide $|H|$: this follows from this property being preserved through (finitary) direct powers, subgroups and homomorphic images. Since $C_p, C_d \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$, then p and d divide both $|H|$. Since $|G_{p,d}| = pd$, then $|H| = pd$ by minimality.

Moreover, since $d < p$, it follows from the third Sylow theorem that H has a normal subgroup of order p, generated, say, by a. Then $|H/\langle a \rangle| = d$, and, since $H \in \mathbf{Ab}(p) * \mathbf{Ab}(d)$, $H/\langle a \rangle$ is abelian. Suppose that $H/\langle a \rangle$ is not cyclic. By the structure theorem of abelian groups, we have $H/\langle a \rangle \cong C_{d_1} \times \ldots \times C_{d_m}$ for some $d_1, \ldots, d_m > 1$ not all coprime. Hence $H/\langle a \rangle$ has exponent $d' < d$. For every $h \in H$, we have then $(h\langle a \rangle)^{d'} = \langle a \rangle$, yielding $h^{d'} \in \langle a \rangle$ and $h^{pd'} = 1$. Thus $H \models x^{pd'} = 1$ and so does $\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle H \rangle$. In particular, $G_{p,d} \models x^{pd'} = 1$. Using the presentation (7) of $G_{p,d}$, and letting b be the generator of order d therein, we obtain $b^{pd'} = 1$. Since $o(b) = d$ in $G_{p,d}$, then $d \mid pd'$ and therefore $d \mid d'$, contradicting $d' < d$. Thus $H/\langle a\rangle \cong C_d$.

Since p and d are coprime, it follows from the Schur-Zassenhaus theorem that $H \cong C_p \rtimes C_d$. Let $b \in H$ correspond to a generator of C_d. We have $a^p = b^d = 1$. Since $\langle a\rangle \lhd H$, we have $bab^{-1} = a^j$ for some $j \in [p-1]$. Applying Lemma 3.3 to the case of H (where $m = n = 1$), we get $j \in Q_{p,d}$, i.e. $j^d \equiv 1 \pmod{p}$. It is now enough to prove that $o(j) = d$ in $\mathbb{F}_p^*$.

Let $d' = o(j)$ in $\mathbb{F}_p^*$. Since $j^d \equiv 1 \pmod{p}$, we have $d' \mid d$. Suppose that $d' < d$. We show that $H \models x^{d'}y^{d'} = y^{d'}x^{d'}$.

Indeed, $\langle a\rangle \lhd H$ implies that $h^{d'} \in \langle a, b^{d'}\rangle$ for every $h \in H$. Hence it suffices to show that a and $b^{d'}$ commute in H. By (6), we have $b^{d'}ab^{-d'} = a^{j^{d'}} = a$, hence $H \models x^{d'}y^{d'} = y^{d'}x^{d'}$ and so does $\mathbf{Ab}(p) * \mathbf{Ab}(d) = \langle H\rangle$. In particular, $G_{p,d} \models x^{d'}y^{d'} = y^{d'}x^{d'}$. Using the presentation (7) of $G_{p,d}$, we obtain $a^{d'}b^{d'} = b^{d'}a^{d'} = a^{d'q^{d'}}b^{d'}$. Hence $p \mid d'(q^{d'} - 1)$. Since $d' < p$, it follows that $q^{d'} \equiv 1 \pmod{p}$, contradicting $q \in Q'_{p,d}$. Therefore $o(j) = d$ in $\mathbb{F}_p^*$ as required. $\square$

Note that $|G_{p,d}| = pd$ constitutes a clear improvement with respect to $|F_2(p,d)| = p(d^2+1)d^2$.

4 The pseudovariety $\mathbf{U} = \bigvee_{p\in\mathbb{P}} \mathbf{Ab}(p) * \mathbf{Ab}(p-1)$

This section is devoted to the pseudovariety $\mathbf{U}$, which is also the join of all the pseudovarieties $\mathbf{Ab}(p) * \mathbf{Ab}(d)$ studied in the preceding section.

Let $\mathbf{E}$ denote the pseudovariety generated by all groups of prime order (equivalently, by all finite elementary abelian groups).

Recall that $\mathbf{Su}$ denotes the pseudovariety of all finite supersolvable groups.

For every prime p, write $\mathbf{U}_p = \mathbf{Ab}(p) * \mathbf{Ab}(p-1)$. Hence $\mathbf{U} = \bigvee_{p\in\mathbb{P}} \mathbf{U}_p$.

Given a pseudovariety $\mathbf{V}$ of finite groups, let

- $\mathrm{Der}(\mathbf{V})$ denote the class of all finite groups G such that $[G,G] \in \mathbf{V}$;
- $\mathrm{Syl}(\mathbf{V})$ denote the class of all finite groups G such that $H \in \mathbf{V}$ for every Sylow subgroup H of G.

Lemma 4.1 *Let $\mathbf{V}$ be a pseudovariety of finite groups. Then $\mathrm{Der}(\mathbf{V})$ and $\mathrm{Syl}(\mathbf{V})$ are pseudovarieties of finite groups.*

Proof. We must show that $\mathrm{Der}(\mathbf{V})$ and $\mathrm{Syl}(\mathbf{V})$ are closed under taking subgroups, homomorphic images and direct products.

Let $H \leqslant G \in \mathrm{Der}(\mathbf{V})$. Then $[H,H] \leqslant [G,G] \in \mathbf{V}$ yields $H \in \mathrm{Der}(\mathbf{V})$.

Assume now that $\varphi : G \to H$ is a surjective homomorphism and $G \in \mathrm{Der}(\mathbf{V})$. Then $[G,G] \in \mathbf{V}$ yields

$$[H,H] = [\varphi(G),\varphi(G)] = \varphi([G,G]) \in \mathbf{V}$$

and so $H \in \mathrm{Der}(\mathbf{V})$.

Finally, assume that $G_1, G_2 \in \mathrm{Der}(\mathbf{V})$. Then $[G_1,G_1], [G_2,G_2] \in \mathbf{V}$ yields

$$[G_1 \times G_2, G_1 \times G_2] \leqslant [G_1,G_1] \times [G_2,G_2] \in \mathbf{V}$$

and consequently $G_1 \times G_2 \in \mathrm{Der}(\mathbf{V})$.

Therefore $\mathrm{Der}(\mathbf{V})$ is a pseudovariety of finite groups.

Now let $H \leqslant G \in \mathrm{Syl}(\mathbf{V})$ and let K be a Sylow subgroup of H. By the first Sylow theorem, K is a subgroup of some Sylow subgroup K' of G. Since $K' \in \mathbf{V}$, we get $K \in \mathbf{V}$, thus $H \in \mathrm{Syl}(\mathbf{V})$.

Assume now that $\varphi : G \to H$ is a surjective homomorphism and $G \in \mathrm{Syl}(\mathbf{V})$. Let K be a Sylow subgroup of H. By [29, 1.6.18], $K = \varphi(K')$ for some Sylow subgroup K' of G. Now $K' \in \mathbf{V}$ yields $K \in \mathbf{V}$ and so $H \in \mathrm{Syl}(\mathbf{V})$.

Finally, assume that $G_1, G_2 \in \mathrm{Syl}(\mathbf{V})$ and let K be a Sylow p-subgroup of $G_1 \times G_2$. Let $\pi_i : G_1 \times G_2 \to G_i$ be the canonical projection for $i = 1, 2$. Then $\pi_1(K) \times \pi_2(K)$ is a p-subgroup of $G_1 \times G_2$ containing K, hence $K = \pi_1(K) \times \pi_2(K)$ by maximality of K. Again by the first Sylow theorem, $\pi_i(K)$ is a subgroup of some Sylow subgroup K_i of G_i for $i = 1, 2$. It follows that $K_1, K_2 \in \mathbf{V}$, hence $\pi_1(K), \pi_2(K) \in \mathbf{V}$ and so $K = \pi_1(K) \times \pi_2(K) \in \mathbf{V}$. Thus $G_1 \times G_2 \in \mathrm{Syl}(\mathbf{V})$.

Therefore $\mathrm{Syl}(\mathbf{V})$ is also a pseudovariety of finite groups. $\square$

Theorem 4.2 *The following conditions are equivalent for any finite group G:*

(i) $G \in \mathbf{U}$*;*

(ii) $G \in \mathbf{Su} \cap \mathrm{Der}(\mathbf{E}) \cap \mathrm{Syl}(\mathbf{Ab})$*;*

(iii) *G embeds into $(C_{p_1} \rtimes C_{p_1-1}) \times \ldots \times (C_{p_s} \rtimes C_{p_s-1})$ for some $p_1, \ldots, p_s \in \mathbb{P}$, not necessarily distinct;*

(iv) *G embeds into $(C_{p_1} \rtimes C_{p_1-1}) \times \ldots \times (C_{p_s} \rtimes C_{p_s-1}) \times C_{|G|}^m$ for some prime divisors $p_1, \ldots, p_s$ of $|G|$, not necessarily distinct, and some $m \in \mathbb{N}$.*

Proof. For every prime p, let G_p denote the group $C_p \rtimes C_{p-1}$ presented by

$$\langle x, y \mid x^p = 1,\, y^{p-1} = 1,\, yxy^{-1} = x^{q_p} \rangle,$$

for some fixed $q_p \in Q'_{p,p-1}$. By Proposition 3.5, we have $\mathbf{U}_p = \langle G_p \rangle$, hence

$$\mathbf{U} = \langle G_p \mid p \in \mathbb{P} \rangle. \tag{10}$$

Write

$$\mathbf{U}' = \mathbf{Su} \cap \mathrm{Der}(\mathbf{E}) \cap \mathrm{Syl}(\mathbf{Ab})$$

and let $\mathbf{U}''$ be the class of all finite groups satisfying condition (iii).

(i) $\Rightarrow$ (ii). We know that $\mathbf{Su}$, $\mathbf{E}$ and $\mathbf{Ab}$ are pseudovarieties, and so are $\mathrm{Der}(\mathbf{E})$ and $\mathrm{Syl}(\mathbf{Ab})$ in view of Lemma 4.1. Hence $\mathbf{U}'$ is a pseudovariety. In view of (10), it suffices to show that $G_p \in \mathbf{U}'$ for every $p \in \mathbb{P}$.

Let $p \in \mathbb{P}$. We have $[G_p, G_p] \leqslant \langle x \rangle \cong C_p$, hence $G_p \in \mathrm{Der}(\mathbf{E})$. On the other hand, $G_p \in \mathbf{G}_p * \mathbf{Ab}(p-1)$ yields $G_p \in \mathbf{Su}$ by [3, Corollary 2.7].

Finally, let $H \leqslant G_p$ be a Sylow q-subgroup for some $q \in \mathbb{P}$. If $q = p$, then $H = \langle x \rangle \in \mathbf{Ab}$, hence we may assume that $q \neq p = o(x)$. Then

$$[H, H] \leqslant [G_p, G_p] \cap H \leqslant \langle x \rangle \cap H = \{1\},$$

hence $H \in \mathbf{Ab}$ and so $G_p \in \mathrm{Syl}(\mathbf{Ab})$. Therefore $G_p \in \mathbf{U}'$ as required.

(ii) $\Rightarrow$ (iii). By Dirichlet's theorem, every finite cyclic group embeds into C_{p-1} for some $p \in \mathbb{P}$. Now it follows from the fundamental theorem of finite abelian groups that $\mathbf{Ab} \subseteq \mathbf{U}''$.

Suppose that $\mathbf{U}' \not\subseteq \mathbf{U}''$ and choose $G \in \mathbf{U}' \setminus \mathbf{U}''$ with $|G|$ minimal. We show that

$$\text{if } 1 \neq N, N' \trianglelefteq G, \text{ then } N \cap N' \neq 1. \tag{11}$$

Indeed, suppose that $N \cap N' = 1$. Then the mapping

$$\begin{array}{rcl} \varphi : G & \to & G/N \times G/N' \\ g & \to & (Ng, N'g) \end{array}$$

is injective. Since $G \in \mathbf{U}'$, we have $G/N, G/N' \in \mathbf{U}'$. Since $|G/N|, |G/N'| < |G|$, it follows from the minimality of G that $G/N, G/N' \in \mathbf{U}''$. But then G would be in $\mathbf{U}''$, since φ is an embedding into a direct product of groups in $\mathbf{U}''$. We have reached a contradiction, therefore (11) holds.

For each $p \in \mathbb{P}$, let $O_p(G)$ denote the intersection of all the Sylow p-subgroups of G. This is a normal subgroup of G, which is in fact nilpotent (being a p-group). The *Fitting subgroup* $F(G)$ is the (unique) largest normal nilpotent subgroup of G. Hence $O_p(G) \leqslant F(G)$ for every $p \in \mathbb{P}$.

By the first Sylow theorem, $O_p(G)$ is a subgroup of some Sylow subgroup S_p of $F(G)$, which is itself a subgroup of some Sylow subgroup S'_p of G. In view of the second Sylow theorem, we get

$$O_p(G) = \bigcap_{g \in G} gO_p(G)g^{-1} \leqslant \bigcap_{g \in G} gS_pg^{-1} \leqslant \bigcap_{g \in G} gS'_pg^{-1} = O_p(G),$$

hence $\bigcap_{g \in G} gS_pg^{-1} = O_p(G)$. On the other hand, since $F(G) \trianglelefteq G$, we get $gS_pg^{-1} \leqslant F(G)$ for every $g \in G$, and since a nilpotent group has a unique Sylow p-subgroup, $gS_pg^{-1} = S_p$ for every $g \in G$. Thus $O_p(G)$ is the (unique) Sylow p-subgroup of $F(G)$. Since a nilpotent group is always the direct product of its Sylow subgroups, we get

$$F(G) = \prod_{p \in \mathbb{P}} O_p(G).$$

Since $[G, G]$ is abelian, and therefore nilpotent, we have $[G, G] \leqslant F(G)$. Since $\mathbf{Ab} \subseteq \mathbf{U}''$, then G is not abelian and so $F(G) \neq 1$. By (11), there is at most one prime p such that $O_p(G) \neq 1$, thus $F(G) = O_p(G) = S_p$ for this only prime p, which we now fix.

Since $G \in \mathrm{Syl}(\mathbf{Ab})$ then $F(G) = O_p(G) = S_p$ is abelian. Since G is a finite solvable group, $C_G(F(G)) \leqslant F(G)$ (see [9, Theorem 6.1.3]). We therefore obtain $C_G(O_p(G)) = O_p(G)$ and so $O_p(G)$ is the unique Sylow p-subgroup of G. Note also that $G \neq O_p(G)$ as otherwise G is abelian, contradicting the choice of G.

Let $Y = \Omega_1(O_p(G))$ (i.e. the group generated by all elements of $O_p(G)$ of order p). Recall that $\Omega_1(O_p(G))$ is a characteristic subgroup of $O_p(G)$, and since $O_p(G) \trianglelefteq G$, $\Omega_1(O_p(G))$ is a normal subgroup of G. Since $O_p(G)$ is abelian, then Y is elementary abelian. Moreover, G acts on Y by conjugation. It follows that Y is a G-module over the finite field $\mathbb{F}_p$, and the G-submodules of Y are exactly the G-normal subgroups of Y. Since $C_G(Y) \geqslant O_p(G)$, then $G/C_G(Y)$ is a quotient of $G/O_p(G)$, hence $G/C_G(Y)$ has order coprime to p (since $O_p(G)$ is a Sylow p-subgroup of G). Now it follows from Maschke's theorem that Y can be written in the form $Y = Y_1 \times \ldots \times Y_e$, where each Y_i is an irreducible G-submodule of Y (that is, a minimal G-normal subgroup of Y).

Thus each Y_i is a chief factor of G. Since G is supersolvable, each Y_i has prime order, hence Y_i is cyclic of order p. If $e \geqslant 2$, then it would follow that Y_1 and Y_2 are two normal subgroups of G intersecting trivially, contradicting (11). Thus Y is cyclic of order p. Since $O_p(G)$ is abelian, it follows that $O_p(G)$ is itself cyclic.

Since $[G, G] \leqslant F(G) = O_p(G)$, the quotient group $A = G/O_p(G)$ is abelian. Since $|A|$ is coprime to p, it follows from Schur-Zassenhaus theorem that we can write $G = O_p(G) \rtimes A$. Thus there exists some $A' \leqslant G$, isomorphic to A, such that $G = O_p(G)A'$, and the action of A' on $O_p(G)$ is by conjugation.

Denoting by $C_{A'}(O_p(G)) = \{a \in A' \mid aga^{-1} = g\,\forall g \in O_p(G)\}$ the centralizer of this action, and since $C_G(O_p(G)) = O_p(G)$, we have

$$C_{A'}(O_p(G)) = A' \cap C_G(O_p(G)) = A' \cap O_p(G) = 1,$$

hence the action of A on $O_p(G)$ is faithful and so A embeds in the automorphism group of $O_p(G)$.

Since A' acts by automorphisms (in fact, by conjugation) on the abelian group $O_p(G)$ and $|A'|$ is coprime to $|O_p(G)|$, it follows from a theorem of Fitting [15, Theorem 4.34] that we also have $O_p(G) = [O_p(G), A'] \times C_{O_p(G)}(A')$.

Since $O_p(G)$ is a cyclic p-group, it does not admit nontrivial direct product decompositions. Suppose that $C_{O_p(G)}(A') \neq 1$. Then $[O_p(G), A'] = 1$ and so $C_{O_p(G)}(A') = O_p(G)$. This gives $C_{A'}(O_p(G)) = A'$ and so $A' = 1$ and $G = O_p(G)$, a contradiction. Hence $C_{O_p(G)}(A') = 1$ and so

$$O_p(G) = [O_p(G), A'] \times C_{O_p(G)}(A') = [O_p(G), A'] \leqslant [G, G] \leqslant F(G) = O_p(G),$$

yielding $O_p(G) = [G, G]$. Since $O_p(G)$ is a cyclic p-group and $[G, G] \in \mathbf{E}$, it follows that $O_p(G) \cong C_p$. Since A embeds in the automorphism group of $O_p(G)$, then A embeds into C_{p-1}.

But then $G = O_p(G) \rtimes A \in \mathbf{U}''$, a contradiction. Therefore $\mathbf{U}' \subseteq \mathbf{U}''$ and (iii) holds.

(iii) $\Rightarrow$ (iv). Assume there exists an embedding $\varphi : G \to G_{p_1} \times \ldots \times G_{p_s}$ for some $p_1, \ldots, p_s \in \mathbb{P}$. For $i \in [s]$, let $\pi_i : G_{p_1} \times \ldots \times G_{p_s} \to G_{p_i}$ be the canonical projection. Then

$$\varphi(G) \leqslant \pi_1(\varphi(G)) \times \ldots \times \pi_s(\varphi(G)), \tag{12}$$

and we may assume that $\pi_i(\varphi(G)) \neq 1$ for every $i \in [s]$.

Suppose that p_i does not divide $|G|$ for some i. We claim that $\pi_i(\varphi(G)) \in \mathbf{Ab}$. Indeed, if $\pi_i(\varphi(G)) \notin \mathbf{Ab}$, then $\pi_i(\varphi(G))$ contains a nontrivial commutator c of G_{p_i}, which has necessarily order p_i. Let $g \in G$ be such that $\pi_i(\varphi(g)) = c$. Then

$$c^{|G|} = (\pi_i(\varphi(g)))^{|G|} = \pi_i(\varphi(g^{|G|})) = 1$$

and so p_i divides $|G|$, a contradiction. Thus $\pi_i(\varphi(G)) \in \mathbf{Ab}$. Since $|\pi_i(\varphi(G))|$ divides $|G|$, it follows from the fundamental theorem of finite abelian groups that $\pi_i(\varphi(G))$ embeds into $C_{|G|}^{m_i}$ for some $m_i \in \mathbb{N}$. In view of (12), we get the desired embedding.

(iv) $\Rightarrow$ (i). By Dirichlet's theorem, $C_{|G|}$ embeds into C_{p-1} for some $p \in \mathbb{P}$ and we are done. $\square$

We immediately get:

Corollary 4.3 *The pseudovariety* $\mathbf{U}$ *has decidable membership problem.*

We recall that $\mathbf{M}$ denotes the pseudovariety of all finite metabelian groups.

Proposition 4.4 *We have* $\mathbf{Ab} \subset \mathbf{U} \subset \mathbf{Su} \cap \mathbf{M}$.

Proof. Since $\mathbf{M} = \mathrm{Der}(\mathbf{Ab}) \supseteq \mathrm{Der}(\mathbf{E})$, Theorem 4.2 yields $\mathbf{U} \subseteq \mathbf{Su} \cap \mathbf{M}$.

Let Q_8 denote the quaternion group, which is well known to be both supersolvable and metabelian. Since Q_8 is not abelian and is its unique Sylow subgroup, it follows from Theorem 4.2 that $Q_8 \notin \mathbf{U}$. Therefore $\mathbf{U} \subset \mathbf{Su} \cap \mathbf{M}$.

The inclusion $\mathbf{Ab} \subset \mathbf{U}$ follows from Theorem 4.2 and is obviously strict. $\square$

5 The variety generated by U

The main result of this section proves that $\mathbf{U}$ generates the variety $\mathcal{M}$ of all metabelian groups. It is known that $\mathcal{M}$ is generated by $F_2(\mathcal{M})$, the free metabelian group of rank 2 (see the comments before [24, Subsection 16.41]).

On the other hand, it follows from (10) that

$$\mathcal{V}(\mathbf{U}) = \mathcal{V}(\{G_p \mid p \in \mathbb{P}\}). \tag{13}$$

We present now a concrete description of $F_2(\mathcal{M})$, which generalizes in the obvious way to any finite rank. We fix an alphabet $A = \{a, b\}$. The Cayley graph $\mathrm{Cay}_A(\mathbb{Z} \times \mathbb{Z})$ is an infinite grid with vertex set $\mathbb{Z} \times \mathbb{Z}$ and edges of the form

$$(m, n) \xrightarrow{a} (m + 1, n), \quad (m, n) \xrightarrow{b} (m, n + 1) \quad (m, n \in \mathbb{Z}).$$

We denote by E the set of all such edges. Each edge can be read in both directions: in the direct sense, with label a or b; in the opposite sense, with label a^{-1} or b^{-1} (so we may consider edges labelled by a^{-1} or b^{-1} whenever convenient). We consider the vertex $(0, 0)$ as the basepoint of $\mathrm{Cay}_A(\mathbb{Z} \times \mathbb{Z})$. It is immediate that each word $u \in (A \cup A^{-1})^*$ labels a unique path π_u off the basepoint in $\mathrm{Cay}_A(\mathbb{Z}\times\mathbb{Z})$. For every $e \in E$, we denote by $|u|_e$ the number of times π_u traverses e in the positive direction minus the number of times π_u traverses e in the opposite direction. Note that if $\overline{u}$ denotes the reduced form of u, then $|u|_e = |\overline{u}|_e$ and so $|g|_e$ is well defined for every $g \in F_2$. It follows from a general result of Almeida [1, Theorem 2.2] (see also [17]) that $F_2(\mathcal{M}) = F_2/N$ for

$$N = \{g \in F_2 \bigm| |g|_e = 0 \text{ for every } e \in E\}.$$

Theorem 5.1 *We have $\mathcal{V}(\mathbf{U}) = \mathcal{M}$.*

Proof. It follows from Proposition 4.4 that $\mathbf{U} \subset \mathbf{M} \subset \mathcal{M}$, hence $\mathcal{V}(\mathbf{U}) \subseteq \mathcal{M}$.

To prove the reverse inclusion, since $\mathcal{M} = \mathcal{V}(F_2(\mathcal{M}))$, it suffices to show that $F_2(\mathcal{M}) = F_2/N \in \mathcal{V}(\mathbf{U})$. Note that the proof below of the latter assertion generalizes to free metabelian groups of any finite rank. It suffices to show that

$$\begin{array}{r}\text{for every } u \in F_2 \setminus N, \text{ there exists some } H_u \in \mathbf{U} \text{ and some homomorphism}\\ \varphi_u : F_2/N \to H_u \text{ such that } \varphi_u(uN) \neq 1.\end{array} \tag{14}$$

Indeed, if (14) holds, then

$$\varphi : F_2/N \to \prod_{u \in F_2 \setminus N} H_u$$

defined by $\varphi(wN) = (\varphi_u(wN))$ is an injective homomorphism. Since this direct product is in $\mathcal{V}(\mathbf{U})$, the theorem follows.

Let $u \in F_2 \setminus N$. Then $|u|_e \neq 0$ for some $e \in E$. Fix some prime $p > |u|$ and let π'_u be the unique path labelled by u off the basepoint in $\mathrm{Cay}_A(\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$, whose set of positive edges is denoted by E'.

For every $e' \in E'$, we denote by $|u|_{e'}$ the number of times π'_u traverses e' in the positive direction minus the number of times π'_u traverses e' in the opposite direction. Note that our choice of p ensures that $|u|_{e'_0} \neq 0$ for some $e'_0 \in E'$.

Now we recall a construction from [3, Subsection 3.3] and apply it to the particular case $G = \mathbb{Z}/(p-1)\mathbb{Z} \times Z/(p-1)\mathbb{Z}$. Let $\theta : F_2 \to G$ be the canonical homomorphism defined by

$\theta(a) = (1,0)$ and $\theta(b) = (0,1)$. Let M denote the $\mathbb{F}_p$-vector space with basis E'. Let e_a and e_b denote the edges $(0,0) \xrightarrow{a} (1,0)$ and $(0,0) \xrightarrow{b} (0,1)$ in $\mathrm{Cay}_A(\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$, respectively. Let $e_{a^{-1}}$ and $e_{b^{-1}}$ denote the edges $(0,0) \xrightarrow{a^{-1}} (-1,0)$ and $(0,0) \xrightarrow{b^{-1}} (0,-1)$, respectively. The group G acts on E' (and consequently on M) by left translation. Then $G^{\mathbf{Ab}(p)}$ is defined as

$$G^{\mathbf{Ab}(p)} = \langle (e_a, \theta(a)), (e_b, \theta(b)) \rangle \leq M \rtimes G.$$

By [3, Lemma 3.2], $G^{\mathbf{Ab}(p)}$ is an extension by G of an elementary abelian p-group, hence $G^{\mathbf{Ab}(p)} \in \mathbf{Ab}(p) * \mathbf{Ab}(p-1) \subseteq \mathbf{U}$. Note that $(e_c, \theta(c))^{-1} = (e_{c^{-1}}, \theta(c^{-1}))$ for $c \in \{a, b\}$.

By the universal property, we can define a (surjective) homomorphism $\theta' : F_2 \to G^{\mathbf{Ab}(p)}$ by $\theta'(c) = (e_c, \theta(c))$ for $c \in \{a, b\}$. We claim that

$$\mathrm{Ker}(\theta') = \{g \in \mathrm{Ker}(\theta) \bigm| |g|_{e'} \in p\mathbb{Z} \text{ for every } e' \in E'\}. \tag{15}$$

Let $g = c_1 \dots c_m \in F_2$ with $c_1, \dots, c_m \in A \cup A^{-1}$. Then

$$\theta'(g) = (e_{c_1}, \theta(c_1)) \dots (e_{c_m}, \theta(c_m)) = (e_{c_1} + \theta(c_1)e_{c_2} + \dots + \theta(c_1 \dots c_{m-1})e_{c_m}, \theta(g)),$$

hence $g \in \mathrm{Ker}(\theta')$ if and only if $g \in \mathrm{Ker}(\theta)$ and the coefficient of each $e' \in E'$ in $e_{c_1} + \theta(c_1)e_{c_2} + \dots + \theta(c_1 \dots c_{m-1})e_{c_m}$ is zero in $\mathbb{F}_p$. If e' is the edge $h \xrightarrow{c} hc$, its coefficient is precisely

$$\begin{array}{rl} & |\{j \in \{1, \dots, m\} \mid \theta(c_1 \dots c_{j-1}) = h,\, c_j = c\}| \\ - & |\{j \in \{1, \dots, m\} \mid \theta(c_1 \dots c_{j-1}) = hc,\, c_j = c^{-1}\}| = |g|_{e'}, \end{array}$$

viewed as an element of $\mathbb{F}_p$. Therefore (15) holds.

It is easy to see that $N \leqslant N'$: every $g \in N$ must label a closed path in $\mathrm{Cay}_A(\mathbb{Z} \times \mathbb{Z})$, which is a covering of $\mathrm{Cay}_A(\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z})$ (so $|g|_{e'} = 0$ for every $e' \in E'$). Hence it follows from (15) that we have a mapping $\varphi_u : F_2/N \to G^{\mathbf{Ab}(p)} \in \mathbf{U}$ such that $\varphi_u(uN) \neq 1$ (since $|u|_{e'_0} \neq 0$ and $|u| < p$). Therefore (14) holds as required. $\square$

Recall that, if $\mathcal{V}$ is a variety of groups, we denote by $\mathcal{V}^f$ the pseudovariety of all finite groups in $\mathcal{V}$. Such pseudovarieties have some positive features (see [22]), but this is not the case of $\mathbf{U}$:

Corollary 5.2 *There exists no variety $\mathcal{V}$ of groups satisfying $\mathbf{U} = \mathcal{V}^f$.*

Proof. Indeed, suppose that $\mathbf{U} = \mathcal{W}^f$ for some variety $\mathcal{W}$ of groups. Then $\mathbf{U} \subset \mathcal{W}$ and Theorem 5.1 yields $\mathcal{M} = \mathcal{V}(\mathbf{U}) \subseteq \mathcal{W}$. Thus $\mathbf{M} = \mathcal{M}^f \subseteq \mathcal{W}^f = \mathbf{U}$, contradicting Proposition 4.4. $\square$

Given a prime $p \in \mathbb{P}$, we say that $q \in \mathbb{Z}$ is a *primitive root modulo p* if q corresponds to a generator of $\mathbb{F}_p^*$. This is equivalent to saying that the remainder of the division of q by p is in $Q'_{p,p-1}$. Given $q \in \mathbb{Z}$, we write

$$\mathrm{PR}(q) = \{p \in \mathbb{P} \mid q \text{ is a primitive root modulo } p\}.$$

Given $q \in \mathbb{P}$, the *Baumslag-Solitar group $BS(1,q)$* is defined by the presentation

$$\langle a, b \mid bab^{-1} = a^q \rangle.$$

Baumslag-Solitar groups were introduced in [5] and have deserved full attention in geometric group theory ever since, constituting often the source of interesting counter-examples.

We can prove the following:

Theorem 5.3 *The following assertions hold:*

(i) The equality

$$\mathcal{V}(BS(1,q)) = \mathcal{V}(\{G_p \mid p \in \mathrm{PR}(q)\}). \tag{16}$$

fails for at most two primes $q \in \mathbb{P}$.

(ii) The equality (16) holds for $q \in \mathbb{P}$ *if the Riemann hypothesis holds for the Dedekind zeta function of each field* $\mathbb{Q}(\sqrt[p]{q})$ *for every* $p \in \mathbb{P}$.

Proof. We show that (16) holds whenever $\mathrm{PR}(q)$ is infinite. Part (i) then follows from [12, Corollary 2]. On the other hand, part (ii) follows then from a theorem of Hooley [14], who showed that $\mathrm{PR}(q)$ is infinite if the Riemann hypothesis holds for the Dedekind zeta function of each field $\mathbb{Q}(\sqrt[p]{q})$ for every $p \in \mathbb{P}$.

Assume therefore that $\mathrm{PR}(q)$ is infinite. Let $p \in \mathrm{PR}(q)$. Then

$$\langle a, b \mid a^p = b^{p-1} = 1,\, bab^{-1} = a^q \rangle$$

is a presentation for G_p and so we have a canonical homomorphism from $BS(1,q)$ onto G_p. Hence $G_p \in \mathcal{V}(BS(1,q))$ for every $p \in \mathrm{PR}(q)$ and so $\mathcal{V}(\{G_p \mid p \in \mathrm{PR}(q)\}) \subseteq \mathcal{V}(BS(1,q))$.

For the converse, let $(p_1, p_2, \ldots)$ be an enumeration of the elements of $\mathrm{PR}(q)$. For each $n \geqslant 1$, let $\varphi_n : BS(1,q) \to G_{p_n}$ be the canonical homomorphism. Then

$$\begin{array}{rcl} \varphi : BS(1,q) & \to & \prod_{n \geqslant 1} G_{p_n} \\ g & \mapsto & (\varphi_n(g)) \end{array}$$

is a homomorphism. We claim that φ is injective. Let $g \in \mathrm{Ker}(\varphi)$. Since $BS(1,q)$ is an HNN extension, it follows from Britton's normal form [18, Theorem IV.2.1] that $g = a^i b^j a^k$ for some $i, j, k \in \mathbb{Z}$. Since $g \in \mathrm{Ker}(\varphi)$ if and only if $g^{-1} \in \mathrm{Ker}(\varphi)$, we may assume that $j \geqslant 0$. Since $\mathrm{PR}(q)$ is infinite, there exists some $n \geqslant 1$ such that $p_n > \max(|i + kq^j|, j+1)$. Then

$$1 = \varphi_n(g) = \varphi_n(a^i b^j a^k) = a^i b^j a^k = a^{i+kq^j} b^j.$$

Since $o(a) = p_n$ and $o(b) = p_n - 1$ in G_{p_n}, it follows from our choice of p_n that $i + kq^j = j = 0$. Hence $k = -i$ and so $g = a^i b^j a^k = a^i a^{-i} = 1$. Thus φ is injective and so $\mathcal{V}(BS(1,q)) \subseteq \mathcal{V}(\{G_p \mid p \in \mathrm{PR}(q)\})$. Therefore $\mathcal{V}(BS(1,q)) = \mathcal{V}(\{G_p \mid p \in \mathrm{PR}(q)\})$ as required. $\square$

Given $P \subseteq \mathbb{P}$, let

$$\mathcal{M}_P = \mathcal{V}(\{G_p \mid p \in P\}).$$

It would be tempting to think that $\mathcal{M}_P \neq \mathcal{M}_Q$ for different subsets $P, Q \subseteq \mathbb{P}$, but that is far from true. Indeed, it follows from the theorems of Cohen [6] that there exist only countably many varieties of metabelian groups. Hence there exists an uncountable subset $\mathcal{P}$ of $2^{\mathbb{P}}$ such that $\mathcal{M}_P = \mathcal{M}_Q$ for all $P, Q \in \mathcal{P}$. Since $\mathcal{P}$ is uncountable, there must exist some $P, Q \in \mathcal{P}$ with $P \setminus Q$ infinite. Let $R = \mathbb{P} \setminus (P \cup Q)$. Then

$$\mathcal{M} = \mathcal{M}_{\mathbb{P}} = \mathcal{M}_{P \cup Q \cup R} = \mathcal{M}_{Q \cup R}.$$

Since $\mathbb{P} \setminus (Q \cup R) = P \setminus Q$, it follows that there exists some $S \subset \mathbb{P}$ such that $\mathcal{M} = \mathcal{M}_S$ and $\mathbb{P} \setminus S$ is infinite.

Thus we can discard infinitely many primes and still generate $\mathcal{M}$. For that purpose, we need on the other hand infinitely many primes since $\mathcal{M}$ is not finitely generated. This suggests the following conjecture:

Conjecture 5.4 *If $P \subseteq \mathbb{P}$ is infinite, then $\mathcal{M}_P = \mathcal{M}$.*

If the conjecture holds, it follows from Theorem 5.3(i) that the equality $\mathcal{V}(BS(1,q)) = \mathcal{M}$ fails for at most two primes q.

6 The pro-U topology

We discuss in this section the pro-**U** topology, starting with a general result concerning the **U**-closure of a finitely generated subgroup of a free group of finite rank. However, it does not provide decidability for being finitely generated, neither it is assuredly computable in that case.

Proposition 6.1 *Let $H \leqslant_{f.g.} F_n$. Then*

$$\mathrm{Cl}_{\mathbf{U}}(H) = \bigcap_{p \in \mathbb{P}} \mathrm{Cl}_{\mathbf{U}_p}(H) = \bigcap_{p \in \mathbb{P}} HL_{n,p,p-1} \geqslant H\mathrm{Cl}_{\mathbf{U}}(1) = H \bigcap_{p \in \mathbb{P}} L_{n,p,p-1} \geqslant HF_n^{(2)} = \mathrm{Cl}_{\mathbf{M}}(H).$$

Proof. Since $\mathbf{U}_p \subseteq \mathbf{U}$, it follows from (1) and the fact that the closure of a subgroup is always a subgroup that $\mathrm{Cl}_{\mathbf{U}}(H) \leqslant \mathrm{Cl}_{\mathbf{U}_p}(H)$ for every $p \in \mathbb{P}$, therefore

$$\mathrm{Cl}_{\mathbf{U}}(H) \leqslant \bigcap_{p \in \mathbb{P}} \mathrm{Cl}_{\mathbf{U}_p}(H).$$

We prove next the opposite inclusion. Write $F = F_n$ and $C = \bigcap_{p \in \mathbb{P}} \mathrm{Cl}_{\mathbf{U}_p}(H)$. It is well known (see e.g. [20, Proposition 1.3]) that we have

$$\mathrm{Cl}_{\mathbf{U}}(H) = \bigcap \{N \trianglelefteq F \mid F/N \in \mathbf{U} \text{ and } H \leqslant N\}.$$

Hence it suffices to show that $C \leqslant N$ for every normal subgroup N of F containing H with $F/N \in \mathbf{U}$.

Now, it follows from Theorem 4.2 that F/N embeds into $G_{p_1} \times \ldots \times G_{p_s}$ for some $p_1, \ldots, p_s \in \mathbb{P}$. For $i \in [s]$, let $\pi_i : G_{p_1} \times \ldots \times G_{p_s} \to G_{p_i}$ be the canonical projection, and let $\varphi : F \to F/N$ be the canonical homomorphism. Let $N_i = \mathrm{Ker}(\pi_i \circ \varphi)$. We claim that

$$\bigcap_{i=1}^{s} N_i \leqslant N. \tag{17}$$

Let $u \in \bigcap_{i=1}^{s} N_i$. Then $\pi_i(\varphi(u)) = 1$ for every $i \in [s]$, hence $\varphi(u) = 1$ and so $u \in N$. Thus (17) holds.

For every $i \in [s]$, we have

$$F/N_i = F/\mathrm{Ker}(\pi_i \circ \varphi) \cong \mathrm{Im}(\pi_i \circ \varphi) \leqslant G_{p_i} \in \mathbf{U}_{p_i},$$

hence $F/N_i \in \mathbf{U}_{p_i}$. On the other hand, given $i \in [s]$, $H \leqslant N$ yields $\pi_i(\varphi(H)) = \pi_i(1) = 1$, thus $H \leqslant N_i$. It follows that $\mathrm{Cl}_{\mathbf{U}_{p_i}}(H) \leqslant N_i$ for every $i \in [s]$, hence (17) yields

$$C = \bigcap_{p \in \mathbb{P}} \mathrm{Cl}_{\mathbf{U}_p}(H) \leqslant \bigcap_{i=1}^{s} \mathrm{Cl}_{\mathbf{U}_{p_i}}(H) \leqslant \bigcap_{i=1}^{s} N_i \leqslant N$$

and so the equality $\mathrm{Cl}_{\mathbf{U}}(H) = \bigcap_{p\in\mathbb{P}} \mathrm{Cl}_{\mathbf{U}_p}(H)$ holds.

The second and third equalities follow from (3). The fourth equality follows from [22, Theorem 4.7(i)].

Now $H\mathrm{Cl}_{\mathbf{U}}(1) \leqslant \mathrm{Cl}_{\mathbf{U}}(H)$ follows from $H \leqslant \mathrm{Cl}_{\mathbf{U}}(H)$ and $\mathrm{Cl}_{\mathbf{U}}(1) \leqslant \mathrm{Cl}_{\mathbf{U}}(H)$.

Finally, $\mathbf{U} \subseteq \mathbf{M}$ yields $\mathrm{Cl}_{\mathbf{M}}(1) \leqslant \mathrm{Cl}_{\mathbf{U}}(1)$ and consequently $F^{(2)} \leqslant \mathrm{Cl}_{\mathbf{U}}(1)$ by [22, Theorem 4.7(i)]. Thus $HF^{(2)} \leqslant H\mathrm{Cl}_{\mathbf{U}}(1)$ and the proof is complete. $\square$

We can prove however, for an arbitrary pseudovariety and any finite index subgroup of an arbitrary group, that the closure is computable. We first need a lemma.

Lemma 6.2 [22, Proposition 2.6] *Let $\mathbf{V}$ be a pseudovariety of finite groups. Let G be a group and let $H \leqslant G$ have finite index. Then the following conditions are equivalent:*

(i) *H is $\mathbf{V}$-closed;*

(ii) *H is $\mathbf{V}$-clopen;*

(iii) $G/\mathrm{Core}_G(H) \in \mathbf{V}$.

Proposition 6.3 *Let $\mathbf{V}$ be a pseudovariety of finite groups with decidable membership problem. Let G be a group and let $H \leqslant G$ have finite index. Then:*

(i) *it is decidable whether or not H is $\mathbf{V}$-closed;*

(ii) $\mathrm{Cl}_{\mathbf{V}}(H)$ *is computable from H.*

Proof. (i) Since $[G : H] < \infty$, $\mathrm{Core}_G(H)$ is a finite intersection of conjugates of H in G, and so is computable. In particular $G/\mathrm{Core}_G(H)$ is computable. Since $\mathbf{V}$ has decidable membership problem, the result follows from Lemma 6.2.

(ii) Since $[G : H] < \infty$, there exist only finitely many subgroups of G containing H. Now we can decide for each one of them whether it is $\mathbf{V}$-closed or not. The intersection of the $\mathbf{V}$-closed ones must be $\mathrm{Cl}_{\mathbf{V}}(H)$. $\square$

In the next theorem, we describe the $\mathbf{V}$-closed finitely generated subgroups of F_n when $\mathbf{V}$ is a pseudovariety of finite groups which does not generate the variety $\mathcal{G}$ of all groups. In view of Birkhoff's Theorem, this is equivalent to saying that $\mathbf{V}$ satisfies some group identity. It is well known that $\mathcal{V}(\mathbf{G}_p) = \mathcal{G}$ for every prime p, and the same happens with any pseudovariety containing $\mathbf{G}_p$ for some prime p (for example $\mathbf{N}$, $\mathbf{Su}$, $\mathbf{S}$ and the pseudovariety $\mathbf{O}$ of all finite groups of odd order).

Theorem 6.4 *Let $H \leqslant_{f.g.} F_n$ and let $\mathbf{V}$ be a pseudovariety of finite groups such that $\mathcal{V}(\mathbf{V}) \neq \mathcal{G}$. Then the following conditions are equivalent:*

(i) *H is $\mathbf{V}$-closed;*

(ii) *H is $\mathbf{V}$-clopen;*

(iii) $[F_n : H] < \infty$ *and* $F_n/\mathrm{Core}_{F_n}(H) \in \mathbf{V}$.

In particular, the above conditions are equivalent when $\mathbf{V}$ is the pseudovariety $\mathbf{U}$.

Proof. We start by noting that, by Theorem 5.1, $\mathcal{V}(\mathbf{U}) = \mathcal{M}$ and so $\mathbf{U}$ does not generate the variety of all groups.

(i) $\Rightarrow$ (iii). Let $\varphi : F_n \to F_n'$ be the canonical morphism of F_n onto the free object of $\mathcal{V}(\mathbf{V})$. Then $\{1\} < \mathrm{Ker}(\varphi) \trianglelefteq F_n$. Any morphism from F_n onto any $G \in \mathbf{V}$ factors through φ, so $\mathrm{Ker}(\varphi)$

is contained in all the V-open subgroups of F_n, and therefore $\mathrm{Ker}(\varphi) \leq \mathrm{Cl}_{\mathbf{V}}(1) \leq \mathrm{Cl}_{\mathbf{V}}(H) = H$. By a well-known result [16, Theorem 1], since H is finitely generated and contains a nontrivial normal subgroup of F_n, it must have finite index. Finally, the fact that $F/\mathrm{Core}_F(H) \in \mathbf{V}$ follows now from Lemma 6.2.

(iii) $\Rightarrow$ (ii). This follows from the characterization of $\mathbf{V}$-open subgroups in Section 2.

(ii) $\Rightarrow$ (i). This is trivial. $\square$

Together with [27, Lemma 3.12.(a)], this yields the following result, but we include a short proof for completion:

Corollary 6.5 *Given $H \leqslant_{f.g.} F_n$, it is decidable whether or not H is $\mathbf{U}$-closed.*

Proof. Assume that H is given through a finite generating set. As remarked before, it is decidable whether or not H has finite index in $F = F_n$. Assuming now that $[F : H] < \infty$, we can now compute the finitely many conjugates of H [4, Subsection 2.4] and compute their intersection [4, Theorem 2.13] to get $\mathrm{Core}_F(H)$. And we can decide whether or not $F/\mathrm{Core}_F(H) \in \mathbf{U}$ by Corollary 4.3. Therefore condition (iii) of Theorem 6.4 is decidable and the claim follows. $\square$

The next result provides sufficient conditions for a finitely generated subgroup of F_n to have a non finitely generated $\mathbf{U}$-closure.

Fix a basis $\{a_1, \ldots, a_n\}$ of F_n. For $j \in [n]$, unless otherwise stated, we denote by $\pi_j : F_n \to \mathbb{Z}$ the surjective homomorphism defined by

$$\pi_j(a_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.6 *Let $H \leqslant_{f.g.} F_n$ be such that $\pi_j(H) = 0$ for some $j \in [n]$. Then:*

(i) $[F_n : \mathrm{Cl}_{\mathbf{U}}(H)] = \infty$

(ii) $\mathrm{Cl}_{\mathbf{U}}(H)$ *is not finitely generated.*

Proof. (i) Write $F = F_n$ and $C = \mathrm{Core}_F(\mathrm{Cl}_{\mathbf{U}}(H)) \trianglelefteq F$. Suppose that $[F : \mathrm{Cl}_{\mathbf{U}}(H)]$ is finite. Then $[F : C]$ is also finite. Write $r = [F : C]$. Then $a_j^r C = (a_j C)^r = C$ and so $a_j^r \in C$. Choose a prime $p > r$. Then $a_j^r \in C \leqslant \mathrm{Cl}_{\mathbf{U}}(H) \leqslant HL_{n,p,p-1}$ yields

$$\begin{aligned} r &\in \pi_j(HL_{n,p,p-1}) = \pi_j(H) + \pi_j(L_{n,p,p-1}) = \pi_j([K_{n,p-1}, K_{n,p-1}]K_{n,p-1}^p) \\ &= \pi_j([K_{n,p-1}, K_{n,p-1}]) + \pi_j(K_{n,p-1}^p) = \pi_j([F,F]F^{p-1})^p \\ &= p(\pi_j([F,F]) + \pi_j(F^{p-1})) = p(p-1)\mathbb{Z}, \end{aligned}$$

contradicting $p > r$. Thus $[F : C] = \infty$.

(ii) This follows from part (i) and Theorem 6.4. $\square$

Next, we consider the related concept of $\mathbf{U}$-denseness:

Theorem 6.7 *Let $H \leqslant_{f.g.} F_n$. Then the following conditions are equivalent:*

(i) H is $\mathbf{U}$-dense;

(ii) H is $\mathbf{Su}$-dense.

Proof. (i) $\Rightarrow$ (ii). By a theorem in [23], H is $\mathbf{Su}$-dense if and only if, for every prime p, we have $HN = F_n$ for every $N \trianglelefteq F_n$ such that $F_n/N \in \mathbf{U}_p$. Since $\mathbf{U}_p \subseteq \mathbf{U}$, the latter property follows from H being $\mathbf{U}$-dense.

(ii) $\Rightarrow$ (i). Since $\mathbf{U} \subseteq \mathbf{Su}$ by Theorem 4.2, the implication follows from (1). $\square$

Corollary 6.8 *Given $H \leqslant_{f.g.} F_n$, it is decidable whether or not H is* $\mathbf{U}$*-dense.*

Proof. It was proved in [23] that it is decidable whether or not H is $\mathbf{Su}$-dense, therefore the result follows from Theorem 6.7. $\square$

The next example shows that at least one of the inclusions in Proposition 6.1 can be strict.

Example 6.9 *For every $n \geq 2$, there exists some $H \trianglelefteq F_n$ of finite index such that* $\mathrm{Cl}_{\mathbf{M}}(H) < \mathrm{Cl}_{\mathbf{U}}(H)$.

Indeed, by Proposition 4.4 there exists some 2-generated $G \in \mathbf{M} \setminus \mathbf{U}$. Let $\varphi : F_n \to G$ be a surjective homomorphism and let $H = \mathrm{Ker}(\varphi)$. By Lemma 6.2, H is $\mathbf{M}$-closed but not $\mathbf{U}$-closed. Therefore $\mathrm{Cl}_{\mathbf{M}}(H) = H < \mathrm{Cl}_{\mathbf{U}}(H)$. $\square$

The case of free groups of infinite rank is easily settled in the following result:

Proposition 6.10 *Let F be a free group of infinite rank and let $H \leqslant_{f.g.} F$. Then H is neither* $\mathbf{U}$*-closed nor* $\mathbf{U}$*-dense.*

Proof. Let A be a basis of F. For every subset B of A, there exists a homomorphism $\varphi : F \to F_B$ such that $\varphi|_{F_B}$ is the identity, i.e. F_B is a *retract* of F_A. By [20, Proposition 1.6], the pro-$\mathbf{U}$ topology on F_B is the subspace topology with respect to the pro-$\mathbf{U}$ topology on F. It follows from general topology that

$$\mathrm{Cl}_{\mathbf{U}}^{F_B}(X) = \mathrm{Cl}_{\mathbf{U}}^{F}(X) \cap F_B \text{ for all } B \subseteq A \text{ and } X \subseteq F_B. \tag{18}$$

Since H is finitely generated and A is infinite, there exists some finite $B \subset A$ such that $H \leqslant F_B$. Let $c \in A \setminus B$ and write $C = B \cup \{c\}$.

Suppose that H is $\mathbf{U}$-closed. Then (18) yields $\mathrm{Cl}_{\mathbf{U}}^{F_C}(H) = H$ and so $[F_C : H] < \infty$ by Theorem 6.4. Since $H \leqslant F_B < F_C$, we get $[F_C : F_B] < \infty$, a contradiction. Therefore H is not $\mathbf{U}$-closed.

We define next a homomorphism $\theta : F \to C_2$ as follows. For every $a \in A$, let

$$\theta(a) = \begin{cases} 1 & \text{if } a = c \\ 0 & \text{otherwise.} \end{cases}$$

Write $K = \mathrm{Ker}(\theta)$. Then $K \triangleleft F$ and $F/K \cong C_2 \in \mathbf{U}$. Since $HK = K < F$, then H is not $\mathbf{U}$-dense. $\square$

A more challenging problem consists of studying the pro-$\mathbf{U}$ topology for finitely generated subgroups of the free object $F_A(\mathcal{V}(\mathbf{U}))$ on some set A. The following condition (which $\mathbf{U}$ does not satisfy in view of Corollary 5.2) certainly simplifies the matters at issue:

Proposition 6.11 *Let $\mathbf{V}$ be a pseudovariety of finite groups such that $(\mathcal{V}(\mathbf{V}))^f = \mathbf{V}$. Then the pro-$\mathbf{V}$ and the profinite topologies on $F_A(\mathcal{V}(\mathbf{V}))$ coincide for every set A.*

Proof. We know that, for the pro-$\mathbf{V}$ (respectively profinite) topology, $G = F_A(\mathcal{V}(\mathbf{V}))$ is a topological group where the normal subgroups K of G such that $G/K \in \mathbf{V}$ (respectively $G/K \in \mathbf{G}$) form a basis of neighbourhoods of the identity. Hence it suffices to show that, for every $K \trianglelefteq G$, we have $G/K \in \mathbf{V}$ if and only if $G/K \in \mathbf{G}$.

The direct implication holds trivially. Conversely, assume that $G/K \in \mathbf{G}$. Since $G \in \mathcal{V}(\mathbf{V})$, we have $G/K \in \mathcal{V}(\mathbf{V})$ as well. Now G/K finite yields $G/K \in (\mathcal{V}(\mathbf{V}))^f = \mathbf{V}$ and we are done. $\square$

Corollary 6.12 *Let A be an arbitrary set. Then:*

(i) every finitely generated subgroup of $F_A(\mathcal{V}(\mathbf{Ab}))$ is $\mathbf{Ab}$-closed;

(ii) every finitely generated subgroup of $F_A(\mathcal{V}(\mathbf{M}))$ is $\mathbf{M}$-closed.

Proof. (i) Since free abelian groups are direct sums of copies of $\mathbb{Z}$, we have

$$\mathcal{AB} = \mathcal{V}(\mathbb{Z}) = \mathcal{V}(\mathbf{Ab}).$$

Hence $(\mathcal{V}(\mathbf{Ab}))^f = \mathbf{Ab}$. In 1958, Mal'cev proved that free abelian groups of finite rank are LERF (i.e. all their finitely generated subgroups are $\mathbf{G}$-closed) [19]. The arbitrary rank case follows easily from the finite case [22, Lemma 4.3]. Now the claim follows from Proposition 6.11.

(ii) By Proposition 4.4 and Theorem 5.1 we respectively have $\mathbf{U} \subseteq \mathbf{M}$ and $\mathcal{V}(\mathbf{U}) = \mathcal{M}$. Hence $\mathbf{U} \subseteq \mathbf{M} \subset \mathcal{M}$ and so

$$\mathcal{M} = \mathcal{V}(\mathbf{U}) \subseteq \mathcal{V}(\mathbf{M}) \subseteq \mathcal{M}.$$

Thus $\mathcal{M} = \mathcal{V}(\mathbf{M})$ and so $(\mathcal{V}(\mathbf{M}))^f = \mathbf{M}$.

In 2000, Coulbois proved that free metabelian groups of finite rank are LERF [7]. The arbitrary rank case follows easily from the finite case [22, Lemma 4.6]. Now the claim follows from Proposition 6.11. $\square$

Although $\mathbf{Ab} \subset \mathbf{U} \subset \mathbf{M}$ by Proposition 4.4, the next example shows that Corollary 6.12 cannot be generalized to $\mathbf{U}$. It is a particular instance of a well-known phenomenon.

Example 6.13 *There exists a finitely generated subgroup of $F_2(\mathcal{V}(\mathbf{U}))$ which is not $\mathbf{U}$-closed.*

Indeed, we know that $\mathcal{V}(\mathbf{U}) = \mathcal{M}$ by Theorem 5.1. Since the quaternion group Q_8 is metabelian and 2-generated, there exists some surjective homomorphism $\varphi : F_2(\mathcal{M}) \to Q_8$. Let $N = \mathrm{Ker}(\varphi)$. Since N is a finite index subgroup of a finitely generated group, it is itself finitely generated.

Suppose that N is $\mathbf{U}$-closed in $G = F_2(\mathcal{M})$. By Lemma 6.2, we get $G/\mathrm{Core}_G(N) \in \mathbf{U}$. Since $N \lhd G$, then

$$Q_8 \cong G/N = G/\mathrm{Core}_G(N) \in \mathbf{U},$$

a contradiction (recall the proof of Proposition 4.4). Therefore N is not $\mathbf{U}$-closed. This establishes the claim.

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