# A VERIFICATION OF WILF'S CONJECTURE UP TO GENUS 100

M. DELGADO, S. ELIAHOU AND J. FROMENTIN

Abstract. For a numerical semigroup $S \subseteq \mathbb{N}$, let $m, e, c, g$ denote its multiplicity, embedding dimension, conductor and genus, respectively. Wilf's conjecture (1978) states that $e(c-g) \geq c$. As of 2023, Wilf's conjecture has been verified by computer up to genus $g \leq 66$. In this paper, we extend the verification of Wilf's conjecture up to genus $g \leq 100$. This is achieved by combining three main ingredients: (1) a theorem in 2020 settling Wilf's conjecture in the case $e \geq m/3$, (2) an efficient trimming of the tree $\mathcal{T}$ of numerical groups identifying and cutting out irrelevant subtrees, and (3) the implementation of a fast parallelized algorithm to construct the tree $\mathcal{T}$ up to a given genus. We further push the verification of Wilf's conjecture up to genus 120 in the particular case where $m$ divides $c$. Finally, we unlock three previously unknown values of the number $n_g$ of numerical semigroups of genus $g$, namely for $g = 73, 74, 75$.

## 1. Introduction

A *numerical semigroup* is a cofinite submonoid $S$ of $\mathbb{N}$, i.e. a subset containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a set of the form $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ where $a_1, \dots, a_n$ are positive integers with $\gcd(a_1, \dots, a_n) = 1$, called *generators* of $S$. The least such $n$ is usually denoted $e$ and called the *embedding dimension* of $S$, see below.

Let $S$ be a numerical semigroup and $S^* = S \setminus \{0\}$. A *primitive element* of $S$ is an element $a \in S^* \setminus (S^* + S^*)$, i.e. a nonzero element of $S$ which is not the sum of two nonzero elements of $S$. Let $P = P(S)$ denote the set of primitive elements of $S$. It is easy to see that $P$ is finite and is the unique minimal generating set of $S$. The *embedding dimension* of $S$ is $e = e(S) = |P|$, the *multiplicity* of $S$ is $m = m(S) = \min S^*$, the *Frobenius number* of $S$ is $F = F(S) = \max(\mathbb{Z} \setminus S)$ and the *conductor* of $S$ is $c = c(S) = F + 1$, satisfying $c + \mathbb{N} \subseteq S$ and minimal with respect to that property. The *genus* of $S$ is $g = g(S) = |\mathbb{N} \setminus S|$, its number of gaps. We partition $S$ as

$$S = L \sqcup R,$$

where $L = L(S) = \{a \in S \mid a < F(S)\}$ and $R = R(S) = \{a \in S \mid a > F(S)\}$, the *left part* and *right part* of $S$, respectively.

Wilf's conjecture is the inequality

$$e|L| \geq c.$$

*Date*: June 2, 2026.

2010 *Mathematics Subject Classification.* 20M14.

*Key words and phrases.* Numerical semigroup; genus.

The first author was partially supported by CMUP, a member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

The authors gratefully acknowledge support from ULCO for visits of the first author to Calais in May 2019 and May 2023.

While this conjecture remains open since 1978, it has been verified in several cases. For convenience, we list some of them in a single statement together with their respective references.

**Theorem 1.1.** *Wilf's conjecture holds for all numerical semigroups satisfying one of the following conditions:*

*(1)* $e \leq 3$ [17]
*(2)* $c \leq 3m$ [12]
*(3)* $e \geq m/3$ [13]
*(4)* $|L| \leq 12$ [16]
*(5)* $m \leq 19$ [20]

See also [10] for an extensive recent survey of partial results on Wilf's conjecture, and [2, 5, 8, 11, 15, 18, 19, 21, 24, 25, 26, 27] for some other relevant papers.

The first significant verification by computer of Wilf's conjecture up to a given genus was accomplished in 2008 by Bras-Amorós [2]. There, Wilf's conjecture was announced to hold for all numerical semigroups of genus $g \leq 50$. This result was extended to genus $g \leq 60$ in 2016 by Fromentin-Hivert [18], to genus $g \leq 65$ in 2021 by Bras-Amorós and Rodríguez [3], and finally to genus $g \leq 66$ in 2023 by Bras-Amorós [4].

In this paper, we extend the verification of Wilf's conjecture up to genus $g \leq 100$. See Theorem 3.5 below. This is achieved by combining three main ingredients:

(1) The main result of [13] stating that Wilf's conjecture holds in case $e \geq m/3$.
(2) The method from [9] to exploit the above result by efficiently trimming the tree $\mathcal{T}$ of numerical semigroups (see below) and thereby drastically reduce the number of numerical semigroups up to a given large genus to test for Wilf's conjecture.
(3) A fast parallelisable algorithm to enumerate all numerical semigroups up to a given large genus [18].

For general reference books on numerical semigroups, see [22, 23].

## 2. The tree $\mathcal{T}$

The set of all numerical semigroups can be organised in a rooted tree $\mathcal{T}$, with root $\mathbb{N}$ of genus 0, such that all numerical semigroups of genus $g$ lie at distance $g$ from the root. Before recalling its construction, we introduce some terminology.

### 2.1. **Left and right primitive elements.**

Let $S$ be a numerical semigroup. Recall the above partition $S = L \sqcup R$ relative to the Frobenius number $F(S)$. Accordingly, we partition the set $P$ of primitive elements of $S$ as

$$P = (P \cap L) \sqcup (P \cap R).$$

We call *left primitive* the elements of $P \cap L$ and *right primitive* those of $P \cap R$. We denote

$$e = |P|,$$
$$e_l = |P \cap L|,$$
$$e_r = |P \cap R|.$$

Thus

$$e = e_l + e_r.$$

2.2. **The children of $S$.** We now briefly recall the construction of the tree $\mathcal{T}$. Let $S$ be a numerical semigroup, and let $a \in P \cap R$ be a right primitive element of $S$, if any. Then the set $S' = S \setminus \{a\}$ is still a numerical semigroup, as easily seen.

A *child* of $S$ in $\mathcal{T}$ is any numerical semigroup of the above form

$$S' = S \setminus \{a\}$$

where $a \in P \cap R$. Clearly, the number of children of $S$ in $\mathcal{T}$ is equal to $e_r$.

The Frobenius number, genus and multiplicity of a child $S' = S \setminus \{a\}$ of $S$ with $a \in P \cap R$ are easy to determine. Indeed, one has

$$\begin{cases} F(S') = a, \\ g(S') = g(S) + 1 \end{cases}$$

as easily seen. As for the multiplicity of $S'$, one has

$$m(S') = m(S)$$

whenever $c(S) > m(S)$, see Proposition 3.2. The only numerical semigroups $S$ for which $c(S) = m(S)$ are the so-called *ordinary* or *superficial* numerical semigroups $O_m$, defined for any $m \geq 1$ by

$$O_m = \langle m, m+1, \ldots, 2m-1 \rangle = \{0\} \sqcup (m + \mathbb{N}).$$

Then $O_m$ is of multiplicity $m$ and conductor $c = m$. It has exactly $m$ children, namely $O_m \setminus \{m+i\}$ for $0 \leq i \leq m-1$. They still have multiplicity $m$ for $1 \leq i \leq m-1$. The only case where the multiplicity differs is at $i = 0$, for which $O_m \setminus \{m\} = O_{m+1}$ is of multiplicity $m+1$.

Note finally that any numerical semigroup of multiplicity $m \geq 1$ is a descendant of $O_m$ in the tree $\mathcal{T}$.

2.3. **Numerical semigroups of given genus.** For $g \in \mathbb{N}$, let $n_g$ denote the number of numerical semigroups of genus $g$. It is well known that $n_g$ is finite. The first few values of $n_g$ are

$$(n_0, n_1, n_2, n_3, n_4, n_5, n_6) = (1, 1, 2, 4, 7, 12, 23).$$

In her famous paper [2], Maria Bras-Amorós conjectured that $n_g$ behaves like the $g$th Fibonacci number $F_g$, with a growth rate tending to the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and satisfying

$$n_g \geq n_{g-1} + n_{g-2} \tag{1}$$

for all $g \geq 2$. The conjectured growth rate of $n_g$ was subsequently confirmed by A. Zhai [28], implying $n_g \geq n_{g-1}$ for $g$ large enough. Yet the conjectured inequality (1) remains widely open to this day, as is the case for the much weaker inequality $n_g \geq n_{g-1}$ for all $g \geq 1$.

Until now, the exact value of $n_g$ had been computed up to $g \leq 72$. In this paper, with massive computations using a distributed version of the fast algorithms in [18], we unlock three new values, namely

$$\begin{aligned} n_{73} &= 6\,832\,823\,876\,813\,577, \\ n_{74} &= 11\,067\,092\,660\,179\,522, \\ n_{75} &= 17\,924\,213\,336\,425\,401. \end{aligned}$$

See Section 5 for more details.

## 3. Trimming $\mathcal{T}$

The main ideas proposed in [9] to trim the tree $\mathcal{T}$ so as to drastically reduce the verification of Wilf's conjecture up to a given genus $G$ are the following ones.

**Notation 3.1.** *Let $G \geq 1$. We denote by $\mathcal{T}_G$ the subtree of $\mathcal{T}$ consisting of all numerical semigroups $S$ of genus $g(S) \leq G$.*

**Proposition 3.2.** *Let $S \neq O_m$ be a numerical semigroup of multiplicity $m$. Let $S'$ be a child of $S$. Then*

*(1) $m(S') = m(S)$*
*(2) $e_l(S') \geq e_l(S)$*
*(3) $e(S) - 1 \leq e(S') \leq e(S)$*

*Proof.* We have $S' = S \setminus \{a\}$ for some right primitive element $a \in P \cap R$. Denote $m, c, P, L$ the multiplicity, conductor, primitive elements and left part of $S$, respectively, and $m', c', P', L'$ the corresponding objects for $S'$.

Since $a \geq c > m$, it follows that $\min(S')^* = \min S^*$, i.e. $m' = m$. Moreover, since $c' = a + 1 > c$, it follows that $L \subset L'$, and any left primitive element $a \in P \cap L$ of $S$ remains left primitive in $S'$. Thus $P \cap L \subseteq P' \cap L'$, implying $e_l(S') \geq e_l(S)$. Finally, as easily seen, one has either $P' = P \setminus \{a\}$ or $P' = P \setminus \{a\} \sqcup \{a + m\}$. The former occurs when $a + m = s_1 + s_2$ for some pair $\{s_1, s_2\} \subset S^*$ distinct from $\{a, m\}$, whereas the latter occurs when $a + m$ has no other representation as an element of $S^* + S^*$. Therefore $e(S) - 1 \leq e(S') \leq e(S)$. □

**Corollary 3.3.** *Let $S \neq O_m$ be a numerical semigroup of multiplicity $m$ such that $e_l \geq m/3$. Then all descendants $T$ of $S$ in $\mathcal{T}$ satisfy Wilf's conjecture.*

*Proof.* Let $T$ be a descendant of $S$. Then, by a repeated application of Proposition 3.2 (2) for each generation of children going down from $S$ to $T$, we have

$$e(T) \geq e_l(T) \geq e_l(S) \geq m/3 = m(T)/3.$$

It then follows from Theorem 1.1 (3) that $T$ satisfies Wilf's conjecture. □

**Corollary 3.4.** *Let $G \in \mathbb{N}^*$. Let $S$ be a numerical semigroup of genus $g \leq G$ such that $e \geq m/3 + (G - g)$. Then all descendants $T$ of $S$ of genus $g(T) \leq G$ satisfy Wilf's conjecture.*

*Proof.* Let $h = g(T)$. Then $g \leq h \leq G$ by hypothesis, and $T$ is an $(h - g)$th descendant of $S$. Now at each step from $S$ down to $T$ in $\mathcal{T}$, the number of primitive elements diminishes by at most 1 as stated in Proposition 3.2 (3). Hence

$$e(T) \geq e - (h - g) \geq e + g - G \geq m/3 + (G - g) + g - G = m(T)/3.$$

It then follows from Theorem 1.1 (3) that $T$ satisfies Wilf's conjecture. □

Consequently, when exploring $\mathcal{T}_G$ to probe Wilf's conjecture up to a given maximal genus $G$, the subtree rooted at any numerical semigroup $S$ satisfying Corollary 3.3 or 3.4 can be completely cut off from $\mathcal{T}_G$. What remains after this systematic trimming of $\mathcal{T}_G$ is a subtree $\mathcal{T}_G(3)$ all of whose numerical semigroups $S$ satisfy

(1) $e_l(S) < m(S)/3$,
(2) $e(S) < m(S)/3 + (G - g(S))$.

Summarizing, to probe Wilf's conjecture up to genus $G$, we only need to test those numerical semigroups $S$ in $\mathcal{T}_G(3)$. This is a significant reduction, as the subtree $\mathcal{T}_G(3)$ turns out to be much smaller than $\mathcal{T}_G$. For instance, for $G = 100$, we found that $\mathcal{T}_{100}(3)$ counts approximately $4.5 \times 10^{15}$ nodes, as compared to the full tree $\mathcal{T}_{100}$ counting roughly $n_{75} \times \phi^{25} \approx 3.2 \times 10^{19}$ nodes. (See Section 5.) This level of reduction allowed us to reach one of the main computational results of this paper.

**Theorem 3.5.** *Wilf's conjecture holds for all numerical semigroups of genus $g \leq 100$.* □

*Proof.* By computer, we constructed the much smaller subtree $\mathcal{T}_G(3) \subset \mathcal{T}_G$ for $G = 100$, and checked that all of its nodes satisfy Wilf's conjecture. The claimed statement follows from Corollary 3.3 and 3.4. □

3.1. **Further trimming.** There are ways to further trim $\mathcal{T}_G(3)$ and thus further reduce the number of numerical semigroups of genus $g \leq G$ to test for Wilf's conjecture. They may be used whenever the added computational cost remains reasonable.

Here is a useful instance, exploiting the result that numerical semigroups with embedding dimension $e \leq 3$ satisfy Wilf's conjecture [17], as recalled in Theorem 1.1 (1).

**Proposition 3.6.** *Let $S$ be a numerical semigroup such that $c \geq 4g/3$. Then $S$ satisfies Wilf's conjecture.*

*Proof.* As $g = c - |L|$, the hypothesis yields $c \geq 4(c - |L|)/3$, i.e. $4|L| \geq c$. If $e \geq 4$ then $e|L| \geq 4|L| \geq c$, so $S$ satisfies Wilf's conjecture. And if $e \leq 3$, the same conclusion holds by Theorem 1.1 (1). □

**Corollary 3.7.** *Let $G \in \mathbb{N}^*$. Let $S$ be a numerical semigroup of genus $g \leq G$ such that $|L| \geq G/3$. Then all descendants $T$ of $S$ of genus $g(T) \leq G$ satisfy Wilf's conjecture.*

*Proof.* Let $T$ be a descendant of $S$ of genus $g(T) \leq G$. We have

$$\begin{aligned} c(T) &= |L(T)| + g(T) \\ &\geq |L| + g(T) \\ &\geq G/3 + g(T) \\ &\geq g(T)/3 + g(T) \\ &= 4g(T)/3. \end{aligned}$$

Proposition 3.6 then implies that $T$ satisfies Wilf's conjecture. □

## 4. The case $c \in m\mathbb{N}$

Given a numerical semigroup $S$, consider the Euclidean division of its conductor $c$ by its multiplicity $m$ with nonpositive remainder:

$$c = qm - \rho, \quad 0 \leq \rho \leq m - 1. \tag{2}$$

As argued in [14], the particular case $c = qm$, i.e. with $\rho = 0$, might well be the heart of Wilf's conjecture. Indeed, the proofs of Wilf's conjecture in either case $c \leq 3m$ [12] or $e \geq m/3$ [13] can be significantly shortened in this particular case. Moreover, the first five instances of the rare occurrence $W_0(S) < 0$ all belong to this case [15].

**Definition 4.1.** *A numerical semigroup $S$ is* special *if its multiplicity $m$ divides its conductor $c$.*

For instance, the ordinary numerical semigroup $O_m = \{0\} \cup (m + \mathbb{N})$ is special since it satisfies $c = m$. The above discussion leads one to think that, if Wilf's conjecture is false, then some counterexamples might well be special. Thus, the special case should be given priority in related research work.

In this context, the validity of Wilf's conjecture has been extended to the case $e \geq m/4$ for special numerical semigroups. More generally, using the notation in (2), the result reads as follows [14].

**Theorem 4.2.** *Let $S$ be a numerical semigroup satisfying $e \geq m/4$. Then $e|L| \geq c - m + \rho$. Morever, if $S$ is special, then $e|L| \geq c$, i.e. $S$ satisfies Wilf's conjecture.*

Let us see how we used here this theorem to push the verification of Wilf's conjecture up to genus $G = 120$ in the special case $c \in m\mathbb{N}$.

To start with, the method in Section 3 to trim $\mathcal{T}$ or its bounded version $\mathcal{T}_G$ to get the greatly reduced relevant subtree $\mathcal{T}_G(3)$ in case $e < m/3$ works as well to retain only those numerical semigroups such that $e < m/d$ for some fixed integer $d \geq 3$. This yields a subtree $\mathcal{T}_G(d)$ of $\mathcal{T}_G$ all of whose numerical semigroups $S$ satisfy

(1) $e_l(S) < m(S)/d$,
(2) $e(S) < m(S)/d + (G - g(S))$.

By Theorem 4.2, the case of relevance to us here is $d = 4$. The subtree $\mathcal{T}_G(4)$ may be further trimmed by exploiting the added hypothesis $c = qm$, as explained below. For convenience, we introduce the following definition.

4.1. **Special trimming.** We start with a condition on a non-special numerical semigroup $S$ ensuring that no descendant of $S$ is special.

**Proposition 4.3.** *Let $S$ be a non-special numerical semigroup. Assume further that $S$ has no right primitive element $a \in P \cap R$ such that $a \equiv -1 \bmod m$. Then no descendant $T$ of $S$ is special.*

*Proof.* Let $S' = S \setminus \{a\}$ with $a \in P \cap R$ be a child of $S$. As usual, we denote by $m, F, c, P$ the multiplicity, Frobenius number, conductor and primitive elements of $S$, and by $m', F', c', P'$ the corresponding objects for $S'$. First of all, $S \neq O_m$ since $O_m$ is special. Hence $m' = m$ by Proposition 3.2. We have $F' = a$ and so $c' = a + 1$. Since $a \not\equiv -1 \bmod m$ by hypothesis, and $m' = m$, we have $c' \not\equiv 0 \bmod m'$. Therefore $S'$ is not special. Moreover, no right primitive element of $S'$ can be congruent to $-1 \bmod m'$ since $P' = P \setminus \{a\}$ or $P \setminus \{a\} \sqcup \{a + m\}$ as seen in the proof of Proposition 3.2. Hence $S'$ satisfies the same hypotheses as $S$ and we are done by induction on the distance between $S$ and any of its descendant $T$. □

We denote by $\mathcal{T}'_G(d)$ the subtree obtained from $\mathcal{T}_G(d)$ by the above special trimming.

4.2. **Outcome.** Having constructed the subtree $\mathcal{T}'_G(d)$ for $d = 4$ and maximal genus $G = 120$, we ended up with the following computational result.

**Theorem 4.4.** *Wilf's conjecture holds for all special numerical semigroups $S$ of genus $g \leq 120$.*

*Proof.* For each numerical semigroup $S$ in the subtree $\mathcal{T}'_{120}(4)$, we computed $W(S) = e|L| - c$ and found that $W(S) \geq 0$, i.e. that $S$ satisfies Wilf's conjecture. All numerical semigroups of genus $g \leq 120$ outside $\mathcal{T}'_{120}(4)$ satisfy $e \geq m/4$ or are not special. Combining the computational results on $\mathcal{T}'_{120}(4)$ and Theorem 4.2, we conclude that all special numerical semigroups of genus $g \leq 120$ satisfy Wilf's conjecture. □

## 5. Computational results

Our experiments were carried out on the computational platform CALCULCO [1] using an adapted and distributed version of algorithms given in [18]. Source codes are available on GitHub [6]. These experiments were run for two weeks during summer 2023, when the platform CALCULCO, shared by the whole university, had a much better availability. The distributed computation ran in parallel on up to 1500 cores, with a special-purpose finely tuned dynamic balancing of the tasks of the respective cores. Early experiments were conducted using the GAP package *NumericalSgps* [7].

5.1. **Exploration of $\mathcal{T}_{100}(3)$.** The following table gives the number $t_g$ of numerical semigroups of genus $g$ in the trimmed subtree $\mathcal{T}_{100}(3)$.

| $g$ | $t_g$ | $g$ | $t_g$ | $g$ | $t_g$ |
|---|---|---|---|---|---|
| 0 | 1 | 34 | 183 029 | 68 | 78 371 434 661 |
| 1 | 1 | 35 | 268 072 | 69 | 114 677 728 452 |
| 2 | 1 | 36 | 392 646 | 70 | 167 759 612 028 |
| 3 | 1 | 37 | 575 237 | 71 | 245 327 971 537 |
| 4 | 2 | 38 | 842 632 | 72 | 358 502 883 157 |
| 5 | 3 | 39 | 1 234 294 | 73 | 523 268 737 918 |
| 6 | 4 | 40 | 1 808 003 | 74 | 762 512 542 535 |
| 7 | 6 | 41 | 2 648 088 | 75 | 1 108 797 952 894 |
| 8 | 9 | 42 | 3 878 863 | 76 | 1 608 029 199 893 |
| 9 | 13 | 43 | 5 681 044 | 77 | 2 323 793 898 612 |
| 10 | 19 | 44 | 8 320 312 | 78 | 3 343 540 732 459 |
| 11 | 28 | 45 | 12 184 995 | 79 | 4 786 270 172 173 |
| 12 | 41 | 46 | 17 844 810 | 80 | 6 811 932 935 500 |
| 13 | 60 | 47 | 26 134 470 | 81 | 9 633 271 340 874 |
| 14 | 88 | 48 | 38 275 824 | 82 | 13 524 365 031 892 |
| 15 | 129 | 49 | 56 052 677 | 83 | 18 835 200 708 312 |
| 16 | 189 | 50 | 82 079 784 | 84 | 26 006 592 640 071 |
| 17 | 277 | 51 | 120 191 188 | 85 | 35 586 447 144 420 |
| 18 | 406 | 52 | 176 010 965 | 86 | 48 235 329 094 317 |
| 19 | 595 | 53 | 257 743 713 | 87 | 64 707 333 203 651 |
| 20 | 872 | 54 | 377 377 331 | 88 | 85 854 587 472 809 |
| 21 | 1 278 | 55 | 552 530 112 | 89 | 112 592 214 454 082 |
| 22 | 1 870 | 56 | 809 003 680 | 90 | 145 836 255 324 616 |
| 23 | 2 741 | 57 | 1 184 568 132 | 91 | 186 464 879 487 116 |
| 24 | 4 019 | 58 | 1 734 367 942 | 92 | 234 882 687 403 501 |
| 25 | 5 888 | 59 | 2 539 101 162 | 93 | 290 865 320 646 279 |
| 26 | 8 622 | 60 | 3 717 160 466 | 94 | 353 167 513 519 152 |
| 27 | 12 634 | 61 | 5 441 979 825 | 95 | 419 043 410 131 476 |
| 28 | 18 513 | 62 | 7 967 290 270 | 96 | 483 141 727 918 288 |
| 29 | 27 128 | 63 | 11 663 422 314 | 97 | 534 768 932 735 380 |
| 30 | 39 749 | 64 | 17 072 801 062 | 98 | 557 018 016 635 015 |
| 31 | 58 192 | 65 | 24 990 316 134 | 99 | 522 447 041 258 147 |
| 32 | 85 285 | 66 | 36 581 421 194 | 100 | 389 883 092 218 470 |
| 33 | 124 928 | 67 | 53 548 048 989 | | |

The number of nodes in $\mathcal{T}_{100}(3)$ is equal to $4\,554\,895\,996\,302\,538 \approx 4.5\times10^{15}$. We checked that all of these numerical semigroups satisfy Wilf's conjecture, whence Theorem 3.5.

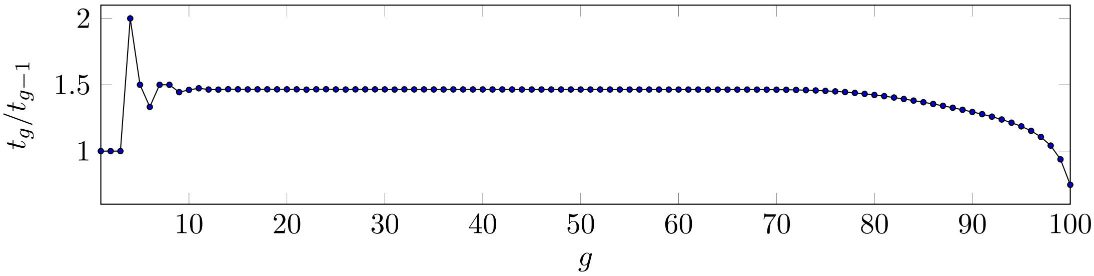
5.2. **Exploration of $\mathcal{T}'_{120}(4)$.** The following table gives the number $t'_g$ of numerical semigroups of genus $g$ in the subtree $\mathcal{T}'_{120}(4)$.

| $g$ | $t'_g$ | $g$ | $t'_g$ | $g$ | $t'_g$ |
|---|---|---|---|---|---|
| 0 | 1 | 41 | 200 122 | 82 | 98 068 856 236 |
| 1 | 1 | 42 | 278 371 | 83 | 136 816 899 688 |
| 2 | 1 | 43 | 369 269 | 84 | 188 124 954 369 |
| 3 | 1 | 44 | 510 693 | 85 | 246 090 486 177 |
| 4 | 1 | 45 | 699 711 | 86 | 336 614 411 642 |
| 5 | 2 | 46 | 975 178 | 87 | 446 053 184 686 |
| 6 | 3 | 47 | 1 342 072 | 88 | 606 309 920 447 |
| 7 | 4 | 48 | 1 876 236 | 89 | 801 094 605 082 |
| 8 | 5 | 49 | 2 608 650 | 90 | 1 075 828 933 040 |
| 9 | 7 | 50 | 3 458 914 | 91 | 1 412 830 185 991 |
| 10 | 10 | 51 | 4 794 003 | 92 | 1 734 561 883 001 |
| 11 | 14 | 52 | 6 551 846 | 93 | 2 241 022 409 143 |
| 12 | 19 | 53 | 9 147 280 | 94 | 2 760 341 832 836 |
| 13 | 26 | 54 | 12 582 317 | 95 | 3 516 881 984 622 |
| 14 | 36 | 55 | 17 614 571 | 96 | 4 299 648 368 842 |
| 15 | 49 | 56 | 24 483 065 | 97 | 5 388 292 247 874 |
| 16 | 67 | 57 | 32 457 488 | 98 | 6 544 140 003 541 |
| 17 | 93 | 58 | 45 063 305 | 99 | 7 117 478 807 043 |
| 18 | 128 | 59 | 61 488 392 | 100 | 8 451 858 568 006 |
| 19 | 177 | 60 | 85 953 600 | 101 | 9 261 220 874 872 |
| 20 | 245 | 61 | 118 226 772 | 102 | 10 843 978 899 677 |
| 21 | 340 | 62 | 165 696 926 | 103 | 11 904 592 718 137 |
| 22 | 455 | 63 | 230 209 288 | 104 | 13 736 474 753 272 |
| 23 | 624 | 64 | 305 247 236 | 105 | 15 172 362 267 910 |
| 24 | 863 | 65 | 424 326 522 | 106 | 13 564 852 076 075 |
| 25 | 1 194 | 66 | 578 281 772 | 107 | 14 519 416 932 134 |
| 26 | 1 647 | 67 | 809 156 311 | 108 | 13 448 628 571 779 |
| 27 | 2 286 | 68 | 1 112 860 410 | 109 | 14 268 658 755 506 |
| 28 | 3 180 | 69 | 1 561 100 560 | 110 | 13 799 851 102 125 |
| 29 | 4 234 | 70 | 2 168 306 879 | 111 | 14 508 263 526 352 |
| 30 | 5 823 | 71 | 2 876 214 827 | 112 | 14 534 198 939 692 |
| 31 | 8 035 | 72 | 4 002 364 427 | 113 | 11 099 577 260 819 |
| 32 | 11 135 | 73 | 5 449 537 905 | 114 | 10 699 792 288 594 |
| 33 | 15 341 | 74 | 7 630 005 823 | 115 | 9 284 396 042 115 |
| 34 | 21 369 | 75 | 10 492 890 135 | 116 | 8 422 462 308 726 |
| 35 | 29 722 | 76 | 14 726 118 585 | 117 | 6 954 667 530 694 |
| 36 | 39 491 | 77 | 20 444 255 810 | 118 | 5 013 091 736 917 |
| 37 | 54 511 | 78 | 27 121 425 859 | 119 | 2 599 964 149 312 |
| 38 | 74 910 | 79 | 37 736 682 161 | 120 | 289 298 823 487 |
| 39 | 104 183 | 80 | 51 267 633 069 | | |
| 40 | 143 431 | 81 | 71 625 262 707 | | |

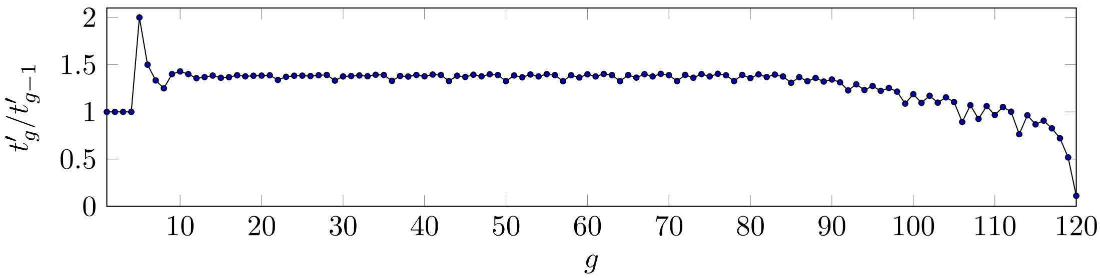
The number of nodes in $\mathcal{T}'_{120}(4)$ is equal to 261 588 966 883 192 $\approx 2.6 \times 10^{14}$. We checked that all of these numerical semigroups satisfy Wilf's conjecture, whence Theorem 4.4.

5.3. **Growth rates.** Recall that $t_g$ and $t'_g$ denote the number of numerical semigroups of genus $g$ in $\mathcal{T}_{100}(3)$ and $\mathcal{T}'_{120}(4)$, respectively. Recall also that the growth rate of $n_g$ tends to $\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$ [28].

As illustrated by the following figure, the growth rate of $t_g$ seems to stabilize around 1.46 for $g \in [10, 70]$.




The growth rate of $t'_g$ is a little more chaotic but seems to have the same behavior, as illustrated by the following figure.




5.4. **New values of $n_g$.** We took advantage of our distributed version of the algorithms in [18] to compute the values of $n_g$ for all $g \leq 75$. See Table 1. The total number of numerical semigroups of genus $g \leq 75$ is equal to $46\,844\,766\,834\,597\,649 \approx 4.6 \times 10^{16}$.

| $g$ | $n_g$ | $g$ | $n_g$ | $g$ | $n_g$ |
|---|---|---|---|---|---|
| 0 | 1 | 26 | 770 832 | 52 | 266 815 155 103 |
| 1 | 1 | 27 | 1 270 267 | 53 | 433 317 458 741 |
| 2 | 2 | 28 | 2 091 030 | 54 | 703 569 992 121 |
| 3 | 4 | 29 | 3 437 839 | 55 | 1 142 140 736 859 |
| 4 | 7 | 30 | 5 646 773 | 56 | 1 853 737 832 107 |
| 5 | 12 | 31 | 9 266 788 | 57 | 3 008 140 981 820 |
| 6 | 23 | 32 | 15 195 070 | 58 | 4 880 606 790 010 |
| 7 | 39 | 33 | 24 896 206 | 59 | 7 917 344 087 695 |
| 8 | 67 | 34 | 40 761 087 | 60 | 12 841 603 251 351 |
| 9 | 118 | 35 | 66 687 201 | 61 | 20 825 558 002 053 |
| 10 | 204 | 36 | 109 032 500 | 62 | 33 768 763 536 686 |
| 11 | 343 | 37 | 178 158 289 | 63 | 54 749 244 915 730 |
| 12 | 592 | 38 | 290 939 807 | 64 | 88 754 191 073 328 |
| 13 | 1 001 | 39 | 474 851 445 | 65 | 143 863 484 925 550 |
| 14 | 1 693 | 40 | 774 614 284 | 66 | 233 166 577 125 714 |
| 15 | 2 857 | 41 | 1 262 992 840 | 67 | 377 866 907 506 273 |
| 16 | 4 806 | 42 | 2 058 356 522 | 68 | 612 309 308 257 800 |
| 17 | 8 045 | 43 | 3 353 191 846 | 69 | 992 121 118 414 851 |
| 18 | 13 467 | 44 | 5 460 401 576 | 70 | 1 607 394 814 170 158 |
| 19 | 22 464 | 45 | 8 888 486 816 | 71 | 2 604 033 182 682 582 |
| 20 | 37 396 | 46 | 14 463 633 648 | 72 | 4 218 309 716 540 814 |
| 21 | 62 194 | 47 | 23 527 845 502 | 73 | **6 832 823 876 813 577** |
| 22 | 103 246 | 48 | 38 260 496 374 | 74 | **11 067 092 660 179 522** |
| 23 | 170 963 | 49 | 62 200 036 752 | 75 | **17 924 213 336 425 401** |
| 24 | 282 828 | 50 | 101 090 300 128 | | |
| 25 | 467 224 | 51 | 164 253 200 784 | | |

TABLE 1. Number $n_g$ of numerical semigroups of genus $g \leq 75$. Previously unknown values are in bold font.

## References


[1] CALCULCO platform. `https://www-calculco.univ-littoral.fr`, 2023.

[2] Maria Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, 76(2):379–384, 2008.

[3] Maria Bras-Amorós and César Marín Rodríguez. New Eliahou semigroups and verification of the Wilf conjecture for genus up to 65. 12898:17–27, [2021] ©2021.

[4] Maria Bras-Amorós. On the seeds and the great-grandchildren of a numerical semigroup, 2023. To appear in Mathematics of Computation.

[5] Winfried Bruns, Pedro García-Sánchez, Christopher O'Neill, and Dane Wilburne. Wilf's conjecture in fixed multiplicity. *Internat. J. Algebra Comput.*, 30(4):861–882, 2020.

[6] M. Delgado, S. Eliahou, and J. Fromentin. Github - trim_tree_semigroups. `https://github.com/jfromentin/trim_tree_semigroups`, 2023.

[7] M. Delgado, P. A. Garcia-Sanchez, and J. Morais. NumericalSgps, a package for numerical semigroups, Version 1.3.1, July 2022. Refereed GAP package.

[8] Manuel Delgado. On a question of Eliahou and a conjecture of Wilf. *Math. Z.*, 288(1-2):595–627, 2018.

[9] Manuel Delgado. Trimming the numerical semigroups tree to probe Wilf's conjecture to higher genus, 2019. 40 pages.

[10] Manuel Delgado. Conjecture of Wilf: A survey. In V. Barucci, S. Chapman, M. D'Anna, and R. Fröberg, editors, *Numerical Semigroups*, volume 40 of *Springer INdAM Ser.*, pages 39–62. Springer, Cham, 2020.

[11] David E. Dobbs and Gretchen L. Matthews. On a question of Wilf concerning numerical semigroups. In *Focus on commutative rings research*, pages 193–202. Nova Sci. Publ., New York, 2006.

[12] Shalom Eliahou. Wilf's conjecture and Macaulay's theorem. *J. Eur. Math. Soc. (JEMS)*, 20(9):2105–2129, 2018.

[13] Shalom Eliahou. A graph-theoretic approach to Wilf 's conjecture. *Electron. J. Combin.*, 27 (2):Article No P2.15, 31 pages, 2020.

[14] Shalom Eliahou. Divsets, numerical semigroups and Wilf's conjecture. `https://hal.science/hal-04234167`, October 2023.

[15] Shalom Eliahou and Jean Fromentin. Near-misses in Wilf's conjecture. *Semigroup Forum*, 98(2):285–298, 2019.

[16] Shalom Eliahou and Daniel Marín-Aragón. On numerical semigroups with at most 12 left elements. *Comm. Algebra*, 49(6):2402–2422, 2021.

[17] R. Fröberg, C. Gottlieb, and R. Häggkvist. On numerical semigroups. *Semigroup Forum*, 35(1):63–83, 1987.

[18] Jean Fromentin and Florent Hivert. Exploring the tree of numerical semigroups. *Math. Comp.*, 85(301):2553–2568, 2016.

[19] Nathan Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf. *J. Pure Appl. Algebra*, 216(5):1016–1032, 2012.

[20] Jonathan Kliem and Christian Stump. A new face iterator for polyhedra and for more general finite locally branched lattices. *Discrete Comput. Geom.*, 67(4):1147–1173, 2022.

[21] Alessio Moscariello and Alessio Sammartano. On a conjecture by Wilf about the Frobenius number. *Math. Z.*, 280(1-2):47–53, 2015.

[22] J. L. Ramírez-Alfonsín. *The Diophantine Frobenius problem*, volume 30 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2005.

[23] J. C. Rosales and P. A. García Sánchez. *Numerical semigroups*, volume 20 of *Developments in Mathematics*. Springer, New York, 2009.

[24] Alessio Sammartano. Numerical semigroups with large embedding dimension satisfy Wilf's conjecture. *Semigroup Forum*, 85(3):439–447, 2012.

[25] Ernst S. Selmer. On the linear Diophantine problem of Frobenius. *J. Reine Angew. Math.*, 293(294):1–17, 1977.

[26] J. J. Sylvester. On Subinvariants, i.e. Semi-Invariants to Binary Quantics of an Unlimited Order. *Amer. J. Math.*, 5(1-4):79–136, 1882.

[27] Herbert S. Wilf. A circle-of-lights algorithm for the "money-changing problem". *Amer. Math. Monthly*, 85(7):562–565, 1978.

[28] Alex Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, 86(3):634–662, 2013.

**Authors addresses**

• Manuel Delgado,
CMUP–Centro de Matemática da Universidade do Porto, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre s/n, 4169– 007 Porto, Portugal
mdelgado@fc.up.pt

• Shalom Eliahou[a,b],
[a]Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
[b]CNRS, FR2037, France
eliahou@univ-littoral.fr

• Jean Fromentin[a,b],
[a]Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
[b]CNRS, FR2037, France
fromentin@univ-littoral.fr