Algorithms for the free inverse monoid

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This section is written in Portuguese as to be better understood by its recipients.

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Resumo

Esta dissertação aborda as propriedades algorítmicas e de decidibilidade dos monóides inversos livres, com foco particular no caso monogénico, $FIM_{\{a\}}$, que denotamos por F.

Começamos por fornecer o enquadramento necessário nas áreas relacionadas com o nosso trabalho, nomeadamente em teoria de semigrupos e monóides, teoria de autómatos e de linguagens formais, e problemas de decidibilidade. Apresentamos os resultados fundamentais de decidibilidade para monóides inversos livres e o papel que as árvores de Munn desempenham nos mesmos, bem como os desenvolvimentos mais recentes por Silva.

Mostramos que os conjuntos racionais de F podem ser escritos como uma união finita de conjuntos onde o operador estrela é aplicado apenas em *singletons*, e no máximo em três deles.

Provamos que é decidível se dois conjuntos racionais de F dados se intersetam de forma não trivial. Além disso, concluímos que a interseção de conjuntos racionais em F pode não ser racional, mas que é livre de contexto, e listamos algumas classes de conjuntos racionais para os quais a interseção com outros conjuntos racionais ainda é racional.

Por fim, generalizamos o problema da pertença a conjuntos racionais ao produto direto de monóides finite $\mathscr{J}$-above, dos quais os monóides inversos livres são um caso particular.

Palavras-chave: monóides inversos livres, conjuntos racionais, problemas de decidibilidade.

Abstract

This dissertation covers algorithmic and decidability properties of free inverse monoids, with a focus on the monogenic case, $FIM_{\{a\}}$ which we denote by F.

We begin by providing the necessary background in the areas surrounding our work, namely semigroup and monoid theory, automata and formal language theory, and decidability problems. We present the foundational decidability results for free inverse monoids and the role Munn trees play in them, as well as the most recent developments by Silva.

We prove that rational sets of F can be written as a finite union of sets where the star operator is applied only to singletons, and at most three of them.

We show that it is decidable whether or not two given rational sets of F intersect non-trivially. Furthermore, we prove the intersection of rational sets in F may not be rational, but that it is context free, and list some classes of rational sets for which the intersection with other rational sets continues being rational.

Lastly, we generalize the membership problem for rational sets to the direct product of finite $\mathscr{J}$-above monoids, of which free inverse monoids are a special case.

Keywords: free inverse monoids, rational sets, decidability problems.

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Introduction

The study of inverse semigroups corresponds to one of the central branches of semigroup theory, arising, for example, whenever injective partial transformations of any set are in question. Among these, free inverse monoids, denoted FIM_A for an alphabet A, play an important role due to their universal property, which could allow known properties of these monoids to extend to many others.

The independent seminal work of Scheiblich and Munn in the 1970s provided the first solutions to the word problem in FIM_A, and revealed combinatorial structure inherent to these monoids. In particular, Munn's approach builds an automaton for each element in the free inverse monoid, which became known as its Munn tree. The study of these automata provides both algebraic and language related properties, which are crucial to the work developed here.

The most recent advancements by Silva deepened our understanding of rational sets in free inverse monoids. In the general case, one of the possible generalizations of the word problem, the membership problem to rational sets, was solved. When restricted to the monogenic free inverse monoid, usually denoted $FIM_{\{a\}}$ or simply F, the problem of inclusion and equality of rational sets was proven decidable. Furthermore, a computable description of any rational set of F using only regular expressions of star-height 1 was found.

Similarly, in this dissertation, we focus our study mostly on rational sets of the free inverse monoid over F. This simpler case turned out thus to be more fertile for new results. In particular, we refine the previous description of rational sets of F and investigate the properties of their intersection. Some attention was given to the direct product of free inverse monoids.

Chapter 1 gives the necessary preliminaries on related areas, such as an introduction to the tools of semigroup and monoid theory required, the basic notions of formal language theory and a discussion on the parameters of the algorithms used.

In Chapter 2, the construction of the free inverse monoid on an alphabet A and the previously known results are presented in further detail. We include a definition of the Munn tree of an element and showcase its use in deducing algebraic properties of FIM_A. Lastly, we provide the most important results on the rational sets of F, by Silva.

Next, we prove that a rational set in F can be written as the union of sets where the star operator is applied to at most three singletons. Chapter 3 is dedicated to building up to and proving this theorem.

We explore the intersection of rational sets in Chapter 4, concluding that it is decidable whether or not two such sets intersect non-trivially. Moreover, we show that this intersection may not be rational, but that it is context-free, and that a grammar which recognizes it is constructible.

Finally, in Chapter 5, we broaden our study to direct products of free inverse monoids by generalizing Silva's result on finite $\mathscr{J}$-above monoids.

1. Preliminaries

Throughout this dissertation, we assume that 0 is not a natural number. For every $n \in \mathbb{N}$, we denote by $[n]$ the set $\{1, \dots, n\}$.

1.1. Semigroup and monoid theory

During our study, due to our familiarity with them, the main reference were the lecture notes [1]. We add [5] for an overview on semigroup theory, including some more advanced results.

A set S along with a binary operation $\cdot$ is said to be a *semigroup* when $\cdot$ is associative, that is,

$$\forall s, t, u \in S, (s \cdot t) \cdot u = s \cdot (t \cdot u).$$

Moreover, if $\cdot$ admits an identity element, that is, an element $e \in S$ such that for all $s \in S, s \cdot e = e \cdot s = s$, then $(S, \cdot)$ is said to be a *monoid*. Frequently, the symbol for the operation will be omitted whenever the operation in question is not ambiguous. For any element u in a monoid, when we write u^0 we refer to the identity of the respective monoid.

Any semigroup S can be turned into a monoid, usually denoted S^1, by adding one element, usually denoted 1, supposed distinct from all previous elements of S and defining the new operation $*$ as follows:

$$s * t = \begin{cases} s \cdot t & \text{if } s, t \in S \\ s & \text{if } t = 1 \\ t & \text{if } s = 1 \end{cases}.$$

In S^1, 1 acts as the identity element. If S was already a monoid to begin with, we set S^1 to be S.

A *right ideal* of a semigroup S is a nonempty set $I \subseteq S$ that absorbs multiplication to the right, that is: if $i \in I$ and $s \in S$, $is \in I$. For a set $X \subseteq S$, the *right ideal generated by* X is the smallest (in the sense of being contained in any other) right ideal that contains X. It always exists, since the intersection of a family of right ideals is still a right ideal, and it is XS^1. *Left* and *two-sided ideals* (*generated by a set*) are defined analogously. Two-sided ideals may also be referred to simply as *ideals*.

Given S a semigroup, we say $T \subseteq S$ is a *subsemigroup* of S if it is a semigroup for the restricted multiplication. Since associativity is inherited, it is sufficient to check that for all $s, t \in T$, $st \in T$. If S is a monoid, for a *submonoid* we additionally demand that the identity of S is in T (and not that T is simply a monoid, for a different identity element).

Next, we present some properties we may want from an element s of a semigroup. It is said to be *idempotent* if $s^2 = s$. The set of all idempotents in a semigroup S is denoted by $E(S)$. It is said to be a *zero* if, for all $t \in S$, $st = ts = s$. It is said to be *regular* if there is $t \in S$ such that $sts = s$. If, moreover, $tst = t$, t is said to be an *inverse* of s.

A *regular* semigroup is one where all its elements are regular. If, moreover, all its idempotents commute, it is said to be *inverse*. Equivalently, and perhaps more intuitively, an inverse semigroup may be defined as a semigroup where any element has one and only one inverse. For each $s \in S$, its unique inverse is denoted by s^{-1}. It is easy to check that, for any s, t elements in an inverse semigroup, we have that $(st)^{-1} = t^{-1}s^{-1}$. A monoid is said to be *regular* (respectively *inverse*) if it is regular (respectively inverse) as a semigroup. In an inverse semigroup, the product of two idempotents is idempotent: given $e, f \in E(S)$, since idempotents commute, $(ef)^2 = efef = eeff = ef$.

Some important examples of inverse semigroups are, for any set X, the set of injective partial transformations on X under composition and many of its subsemigroups, such as that of injective functions. In fact, both these examples constitute monoids.

Given two semigroups $(S, \cdot)$ and $(T, *)$, and a map $\phi : S \to T$, we say ϕ is a *semigroup homomorphism* if, for all $s_1, s_2 \in S$, we have that $(s_1 \cdot s_2)\varphi = (s_1\varphi) * (s_2\varphi)$. If $(S, \cdot)$ and $(T, *)$ are monoids, with identities e_S and e_T, respectively, we say φ is a *monoid homomorphism* if, besides being a semigroup homomorphism, $e_S\varphi = e_T$. Lastly, we note that if S, T are inverse semigroups, then, for all $s \in S$,

$$s\varphi = (ss^{-1}s)\varphi = (s\varphi)(s^{-1}\varphi)(s\varphi),$$
$$s^{-1}\varphi = (s^{-1}ss^{-1})\varphi = (s^{-1}\varphi)(s\varphi)(s^{-1}\varphi),$$

hence, $(s\varphi)^{-1} = s^{-1}\varphi$.

If $(S, \cdot) = (T, *)$, then φ is called an *endomorphism*, and, if φ is also bijective, it is said to be an *automorphism*. If the map $\varphi : S \to T$ instead has the property that, for any $s, t \in S$, $(st)\varphi = (t\varphi)(s\varphi)$, then φ is said to be an *antihomomorphism*. The map $^{-1} : S \to S$ in an inverse semigroup is an example of a self-inverse anti-automorphism. An *isomorphism* is a bijective homomorphism.

A *right congruence* in a semigroup S is an equivalence relation τ with the property that, for any $s, t, u \in S$, if $s \;\tau\; t$, then $su \;\tau\; tu$. A *left congruence* is defined dually, and a (*two-sided*) *congruence* is an equivalence relation which is simultaneously a right and a left congruence. Given a congruence τ on a semigroup S, and writing S/τ to mean the set of equivalence classes of S in relation to τ, then we can define a semigroup structure on S/τ, called a *quotient*, by $[s]_\tau[t]_\tau = [st]_\tau$ for all $s, t \in S$. Setting $\theta_\tau : S \to S/\tau$ as the canonical projection $s \mapsto [s]_\tau$, then it follows from the definition that θ_τ is a semigroup homomorphism. If S is a monoid, then so is S/τ, with identity $[e_S]_\tau$, and θ_τ is a monoid homomorphism.

Given a binary relation ρ, we define the *congruence generated by* ρ as the least congruence that contains ρ. It always exists, since the intersection of a family of congruences is still a congruence.

We provide a particular example of a congruence and quotient: let I be an ideal of a semigroup S, and define τ_I as the union of the identity relation with $I \times I$. Then τ_I is a congruence on S and S/τ_I, more often denoted S/I, is the *Rees quotient* of S by I. In S/I the class of I is a zero, and $I = I\theta_I^{-1}$.

Some particularly useful tools in the study of semigroups are *Green's relations*. Green's partial orders are defined by

- $s \leq_{\mathscr{R}} t \;\Leftrightarrow\; sS^1 \subseteq tS^1 \;\Leftrightarrow\; su = t$, for some $u \in S^1$;
- $s \leq_{\mathscr{L}} t \;\Leftrightarrow\; S^1 s \subseteq S^1 t \;\Leftrightarrow\; us = t$, for some $u \in S^1$;
- $s \leq_{\mathscr{H}} t \Leftrightarrow\; s \leq_{\mathscr{L}} t \wedge s \leq_{\mathscr{R}} t \;\Leftrightarrow\; S^1 s \subseteq S^1 t \wedge sS^1 \subseteq tS^1 \;\Leftrightarrow\; us = t \wedge sv = t$, for some $u, v \in S^1$;
- $s \leq_{\mathscr{J}} t \;\Leftrightarrow\; S^1 s S^1 \subseteq S^1 t S^1 \;\Leftrightarrow\; usv = t$, for some $u, v \in S^1$.

These partial orders define the following equivalence relations:

- $\mathscr{R} = \leq_{\mathscr{R}} \cap \geq_{\mathscr{R}}$;
- $\mathscr{L} = \leq_{\mathscr{L}} \cap \geq_{\mathscr{L}}$;
- $\mathscr{H} = \leq_{\mathscr{H}} \cap \geq_{\mathscr{H}}$;

- $\mathscr{J} = \leq_{\mathscr{J}} \cap \geq_{\mathscr{J}}$;
- $\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$.

where the composition of two binary relations, σ, τ, is defined by

$$s\ (\sigma \circ \tau)\ t \Leftrightarrow \exists u : s\ \sigma\ u \wedge u\ \tau\ t.$$

Green's relations encode which elements can be obtained by which others through multiplication. They are not used in the study of algebraic objects with a stronger structure, such as groups, as in this context, all of these relations become the universal relation: if g and h are elements of a group, then $g = h(h^{-1}g) = (gh^{-1})h$, and so $g\ \mathscr{H}\ h$. However, they are intimately related with the study of ideals. The right ideal generated by $s \in S$ is sS^1; the corresponding left ideal is S^1s and the two-sided ideal generated by s is S^1sS^1. Thus, two elements $s, t \in S$ satisfy $s \leq_{\mathscr{R}} t$ if and only if the right ideal of s is contained in t, and so on for the relations $\mathscr{L}$ and $\mathscr{J}$.

They are interrelated as follows:

$$\begin{tikzcd}
 & \mathscr{R} \arrow[dr, hook] & & \\
\mathscr{H} \arrow[ur, hook] \arrow[dr, hook] & & \mathscr{D} \arrow[r, hook] & \mathscr{J} \\
 & \mathscr{L} \arrow[ur, hook] & &
\end{tikzcd}$$

where $\hookrightarrow$ should be interpreted as the inclusion map, and is thus equivalent in this case to $\subseteq$.

The study of these relations in many classes of semigroups has proven productive, specially in light of Green's Lemma, which sheds light on a semigroup's $\mathscr{D}$-class structure [5, Lemma 2.2.1].

An important property, somewhat related to our work, is that in a regular semigroup each $\mathscr{R}$-class and each $\mathscr{L}$-class contain at least one idempotent, and if it is inverse, then this idempotent is unique.

1.2. Formal languages and automata

For more information on the topics covered in this section, we recommend [4].

Given a set A, usually called an *alphabet*, whose elements are called *letters* or *symbols*, we define the *free monoid over* A, denoted A^*, as the monoid with:

- elements: any finite sequence/string of letters, usually called *words*, including the empty sequence, which is the identity of this monoid and we denote by 1;
- operation: concatenation of words.

The word "free" refers to one of its properties: given any monoid M, and any function $f : A \to M$, then there is one and only one monoid homomorphism φ such that, defining $\iota : A \to A^*$ as the inclusion map, the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "f"'] & A^* \arrow[dl, dashed, "\varphi"] \\ M & \end{tikzcd}.$$

This property can be seen as a generalization of how a basis operates in a vector space, and, in some sense, this means that A^* is the most general monoid that can be generated by $|A|$ elements. An object with this property can be found in other classes of algebraic structures such as the free group, free algebra, and, as we will see with greater detail, free inverse monoid.

Often, any subset of A^* will be denominated a *language*. Our work will very often mention a particular class of languages of A^*: *rational languages*, also denoted by $\mathrm{Rat}(A^*)$. It can be defined as the least class of languages which

- contains the empty set and $\{a\}$ for each $a \in A$;
- is closed under finite products: if $L, K \in \mathrm{Rat}(A^*)$, then $LK \in \mathrm{Rat}(A^*)$;
- is closed under finite unions: if $L, K \in \mathrm{Rat}(A^*)$, then $L \cup K \in \mathrm{Rat}(A^*)$;
- is closed under application of the star operator.

The *star operator*, $*$, also called *Kleene's star operator*, maps a language L to the language

$$L^* = \bigcup_{n \in \mathbb{N}_0} L^n.$$

A description of a rational language by successive applications of the mentioned operations from elements of A is called a *regular expression*. For instance, if $A = \{a, b\}$, some possible regular expressions for $L = \{aa^{2n}b^m : n, m \in \mathbb{N}_0\} \cup \{aba\}$ are

$$L = a(a^2)^*b^* \cup aba = a((a^2)^*b^* \cup ba).$$

The set X^*, for any $X \subseteq A^*$, coincides with the submonoid generated by X and $(X^*)^* = X^*$.

Along with the notion of rational sets and their regular expressions, a second way to construct important subsets of A^* is through *automata*. An (A-)automaton is a tuple $\mathcal{A} = (Q, I, T, E)$ where:

- Q is a *finite* set, whose elements are called *states* or *vertices*;
- $I, T \subseteq Q$, and their elements are called, respectively, the *initial* and *terminal* vertices;
- E is a finite subset of $Q \times A \times Q$, its elements are called *edges*, their second component is called the edge *label*. and an edge (q_1, a, q_2) will usually be denoted by $q_1 \xrightarrow{a} q_2$.

An initial vertex will be represented by an incoming arrow $\to \cdot$ or $\circ$, a terminal vertex by an outgoing arrow $\cdot \to$ or $\bullet$ and an edge (q_1, a, q_2) by $q_1 \xrightarrow{a} q_2$.

In the literature, this type of automaton is referred to as a *finite non-deterministic* automaton, but infinite automata do not feature the work we have developed, and deterministic and non-deterministic automata are interchangeable for most purposes [4, Theorem 2.5.2].

Each automaton $\mathcal{A}$ defines a set, called *its language*, or the set it *recognizes*:

$$L(\mathcal{A}) = \{u \in A^* : \text{there is some path } i \xrightarrow{u} t, \text{with } i \in I, t \in T\}.$$

Automata function as an abstraction of very simple machines, where the alphabet represent the accepted inputs, and a word recognized by the automaton would correspond to a sequence of inputs that produces some desired outcome.

We say an automaton is *trim* if, for each of its vertices there is a path from an initial point to it, and from it to a terminal point. Given an automaton with nonempty language, we can create a trim automaton that recognizes the same set by removing each vertex that can not be reached from an initial point or from which a terminal point can not be reached.

This does not alter the recognized set as any walk that crossed them would not go from an initial vertex to a terminal one, and thus did not contribute to the automaton's language.

One of the foundational results of the study of automata and languages is Kleene's Theorem [**uoig**], which states that a language is rational if and only if it is recognized by a finite automaton, and provides an algorithm for building such an automaton from a given regular expression, and vice-versa.

Rational languages enjoy some properties, from which we highlight that if $L \in \mathrm{Rat}(A^*)$ then $A^* \setminus L$ is also rational and computable, and hence the intersection of rational languages is still rational and computable.

Lastly, we define a second class of *context-free languages*. They can be seen as the languages that can be recognized by a stronger type of automaton: a pushdown automaton. With this extension, besides the finite states and edges, the automaton also carries a stack, whose top element can determine which state transitions are permitted and when the process is allowed to stop, but we present a different, equivalent definition. A *context-free grammar* is a tuple $G = (V, A, R, \varepsilon)$ where:

- V, A and R are finite sets, and V and A should be disjoint;
- the elements of V are called *variables*, *non-terminal characters* or *non-terminal symbols*;
- $\varepsilon \in V$ is called the *starting variable*;
- the elements of A are called *terminals* or *terminal symbols*;
- $R \subseteq V \times (V \cup A)^*$, its elements are called *productions* and are often represented by $\alpha \to u$, for $\alpha \in V$ and $u \in (V \cup A)^*$.

When we say a production $\alpha \to u$ is applied to a word $v \in (V \cup A)^*$, we mean that v admits α as a factor, $v = v_0 \alpha v_1$, and that factor was replaced by u, obtaining the new word $v_0 u v_1$. The language recognized by G is the set of all elements in A^* that can be obtained by repeatedly applying productions, starting from the starting variable, until only terminal symbols are left. We say a language is context-free if it is recognized by some context-free grammar.

All rational languages are context-free, as, given an automaton $\mathcal{A} = (Q, I, T, E)$, it is easy to construct a context-free grammar $G = (Q \cup \{\varepsilon\}, A, R, \varepsilon)$ that mimics it, by considering the productions:

$$\begin{aligned} \varepsilon &\to i, && \text{for each } i \in I, \\ q &\to xq', && \text{whenever } q \xrightarrow{x} q', x \in A, \\ t &\to 1, && \text{for each } t \in T. \end{aligned}$$

We may expand the definition of rational sets to a general monoid. If M is a monoid, we define a *rational set of* M as any homomorphic image of a rational set from A^*, for some alphabet A. Since the composition of homomorphisms is a homomorphism, it is easy to see that if M' is a second monoid, $L \in \mathrm{Rat}(M)$ and $\varphi : M \to M'$ is a homomorphism, then $L\varphi \in \mathrm{Rat}(M')$. Analogously, a context-free set of M is defined as one which can be seen as the image of a context-free language of some A^* by a homomorphism.

A related notion is that of a *recognizable subset* of a monoid M. A set $L \subseteq M$ can be equivalently defined by:

- L is recognizable when there is some homomorphism $\varphi : M \to F$, with F a finite monoid, and some $X \subseteq F$ such that $L = X\varphi^{-1}$;
- define a congruence $\sim_L$ in M by, given $u, v \in M$:

  $$u \sim_L v \iff \forall s, t \in M, aub \in L \text{ if and only if } avb \in L;$$

  then L is recognizable when $\sim_L$ has finitely many equivalence classes in M.

If M is finitely generated, and A is a finite set of generators, then all recognizable sets are rational and setting $\pi : A^* \to M$ as the unique homomorphism that maps A to A, L is recognizable if and only if $L\pi^{-1} \in \mathrm{Rat}(A^*)$. The class of all recognizable languages in M is denoted by $\mathrm{Rec}(M)$.

From the first definition, it becomes clear that the preimage of a recognizable set by a monoid homomorphism is still recognizable. In a finitely generated monoid, all recognizable sets are rational and, furthermore, the intersection of a recognizable set with a rational one is always rational and computable.

1.3. Algorithms and decidability problems

An *algorithm* for a class of problems can be generally understood as a finite set of (unambiguous) instructions that can produce the answer for any questions from that class after some time. A more rigorous definition exists, with recourse to Turing machines, but such detail is not necessary for our purpose.

As an example, to solve the problem “is there an odd perfect number?”, the naïve approach of taking each odd number in increasing order, computing its proper divisors and calculating their sum before moving on to the next does not constitute an algorithm: it is not guaranteed that the process would ever stop, as if the answer is negative, there is no halting condition. However, if it was known that such a number exists and the problem was instead “what is the smallest odd perfect number?” the same process would be an algorithm, as it is now guaranteed to stop eventually, even though we can not, a priori, determine when.

Problems where the possible answers are “yes” and “no” are called decidability problems. A problem is undecidable if there is no algorithm that answers it.

Given some alphabet A, a homomorphism $\theta : A^* \to M$ (usually A is mapped to a set of generators, whereby θ is surjective) and $\mathcal{K}$ a class of languages, some classical decidability problems for a monoid M include:

- the word problem: given two words $u, v \in A^*$, is $u\theta = v\theta$ or not?
- the membership problem for $\mathcal{K}$: given $u \in A^*$ and $L \in \mathcal{K}$, does $u\theta$ belong to $L\theta$ or not?
- the equality problem for $\mathcal{K}$: given $K, L \in \mathcal{K}$, is $K\theta = L\theta$ or not?
- the intersection problems for $\mathcal{K}$: given $K, L \in \mathcal{K}$, is $K\theta \cap L\theta$ empty or not, and is it in $\mathcal{K}\theta$?

It is not viable to study these problems for all subsets ($\mathcal{K} = \mathcal{P}(A^*)$), as there is no finite description for a general set, and so no algorithm would be capable of even reading the input. For having a concise description, either as a regular expression or an automaton (which can be obtained algorithmically from each other, by Kleene’s Theorem), rational sets are a particularly common object for these studies, and the focus of ours.

Some known results of this type are:

- all of the above mentioned problems are decidable for the free monoid over A (that is, $\theta = \mathrm{id}$) and the class of rational languages;

- for a context-free grammar, it is decidable whether it produces an empty, finite or infinite language [3, Theorem 6.6], and, given a word in A^*, it is decidable whether it belongs to its language or not; however, it is not decidable whether the intersection of the languages of two context-free grammars is empty, nor whether it is context-free [3, Theorems 8.10 & 8.13];
- while, in general, any of these problems may be undecidable even when M is a group, there are many classes of groups, such as free groups, for which these problems have been tackled in particular and are decidable.

When creating an algorithm, we assume that whenever we receive as input an element of a monoid M, with a set of generators A, it is provided as a product in A^*, and when the input is a rational set, it provided as a regular expression in A^* or an A-automaton. The same standards apply when we want to construct an element or rational set of M.

2. The Free Inverse Monoid

The study of inverse semigroups in particular was pioneered by Vagner [9] (see also [2]). Later, in the 1970s, Scheiblich [7] and Munn [6] presented the first two independent solutions to the word problem in FIM_A, by creating tools which also answer other questions about these monoids. We present a construction similar to Munn's and some of the results that can be obtained from it. The most recent advancements regarding rational sets of free inverse monoids are due to Silva [8], from which we present the ones we shall make use of.

Let A be a (nonempty) alphabet. For each letter $a \in A$, create a new letter, denoted a^{-1}, that we assume was not already in A. We write $\bar{A}$ to refer to the set $A \cup A^{-1}$, where $A^{-1} = \{a^{-1} : a \in A\}$. We define $^{-1} : \bar{A}^* \to \bar{A}^*$ by:

$$1^{-1} = 1, \quad \text{for } a \in \bar{A},\ (a^{-1})^{-1} = a, \quad \text{for } u = a_1 \cdots a_n,\ a_i \in \bar{A},\ u^{-1} = a_n^{-1} \cdots a_1^{-1}.$$

We denote by ν the *Vagner congruence*, which is the congruence on $\bar{A}^*$ generated by

$$\{(uu^{-1}u, u) : u \in \bar{A}^*\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) : u, v \in \bar{A}^*\}.$$

The *free inverse monoid on* A is defined as the quotient $\bar{A}^*/\nu$, and is denoted by FIM_A. The first set of relations ensure that FIM_A is regular and the second that idempotents commute: given $e, f \in E(FIM_A)$, then it is easy to check that $e^{-1} = e, f^{-1} = f$ and so

$$ef = ee^{-1}ff^{-1} = ff^{-1}ee^{-1} = fe,$$

thus FIM_A is in fact an inverse monoid. We denote by θ the homomorphism that sends each $u \in \bar{A}^*$ to $[u]_\nu$ in FIM_A.

It is called free for having the *universal* property, which consists in: for any inverse monoid M and any mapping $f : A \to M$, there is one and only one monoid homomorphism φ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, "f"'] & FIM_A \arrow[dl, dashed, "\varphi"] \\
M &
\end{tikzcd}$$

where ι maps each $a \in A$ to $a\theta$. This property determines this monoid uniquely, up to isomorphism [5, Section 5.10].

2.1. By Munn

Next, to each $u = a_1 \cdots a_n \in \bar{A}^*$, for $a_i \in \bar{A}$, we associate an automaton, introduced by Munn, that shall be useful in solving decidability problems on FIM_A.

We start by creating the *linear automaton* of u:

$$\begin{tikzcd} {} \arrow[r] & q_0 \arrow[r, bend left, "a_1"] & q_1 \arrow[l, bend left, "a_1^{-1}"] \arrow[r, bend left, "a_2"] & \cdots \arrow[l, bend left, "a_2^{-1}"] \arrow[r, bend left, "a_n"] & q_n \arrow[l, bend left, "a_n^{-1}"] \arrow[r] & {} \end{tikzcd}$$

Next, whenever edges of the form

$$\begin{tikzcd} p_1 \arrow[r, bend left, "a"] & p_2 \arrow[l, bend left, "a^{-1}"] \arrow[r, bend right, "a^{-1}"'] & p_3 \arrow[l, bend right, "a"'] \end{tikzcd}$$

occur, for some $a \in \bar{A}$, then we identify the vertices p_1 and p_3, and ignore the now duplicate edges. After applying this operation for as long as it is applicable, the resultant automaton is known as the *Munn tree*, $\mathrm{MT}(u)$, of u. The underlying graph (obtained by ignoring which vertices are initial and terminal) is denoted by $\mathrm{mt}(u)$.

We shall represent each pair of edges $\begin{tikzcd} p_1 \arrow[r, bend left, "a"] & p_2 \arrow[l, bend left, "a^{-1}"] \end{tikzcd}$, $a \in A$ by $p_1 \xrightarrow{a} p_2$, while not forgetting to allow it to be traversed in the opposite direction with the label a^{-1}. If each of these pairs of edges is instead considered a single undirected edge, the underlying graph is a tree, hence the name.

Example 1. Let $A = \{a, b\}$ and $u = ab^2b^{-1}a^{-1}a^2a^{-1}$. The linear automaton of u is

$$\circ \xrightarrow{a} \cdot \xrightarrow{b} \cdot \xrightarrow{b} \cdot \xleftarrow{b} \cdot \xleftarrow{a} \cdot \xrightarrow{a} \cdot \xrightarrow{a} \cdot \xleftarrow{a} \bullet$$

and the corresponding automaton $\mathrm{MT}(u)$ is

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[r, "a"] & \bullet \arrow[u, "b"] \arrow[r, "a"] & \cdot \\ \circ \arrow[r, "a"] & \cdot \arrow[u, "b"] & \end{tikzcd}$$

The process of building a Munn tree can also be pictured edge by edge. Given $u = a_1 \cdots a_n \in \bar{A}^*$:

1. start from a single vertex, it will be the initial vertex of the automaton;

2. draw an edge from this vertex, outgoing and labelled a_1 if $a_1 \in A$ and incoming and labelled a_1^{-1} if $a_1 \in A^{-1}$;

3. starting from the vertex on the opposite end of the drawn edge, repeat the process from the previous step with the next letter, until the entire word has been read;

4. if, at any step during the previous, there was already an edge drawn at the vertex in consideration like that one that we are about to draw, that edge is traversed, but no new edge is added;

5. the ending point is the terminal vertex for the automaton.

This construction leads to a solution for the word problem of FIM_A, for any alphabet A:

Theorem 2.1.1. For all $u, v \in \bar{A}^*$, $u\theta = v\theta$ if and only if $\mathrm{MT}(u) \cong \mathrm{MT}(v)$.

In fact, Munn trees over an alphabet A are in bijection with elements of FIM_A: given a Munn tree, since it is connected, we can construct a spanning walk γ for it, with some starting and ending points s, e; again by connectedness, we can find a walk from the initial point of the Munn tree to s, γ_1, and from e to the terminal point, γ_2; then the label of $\gamma_1\gamma\gamma_2$ is in $\bar{A}^*$ which, after applying θ, results in an element of FIM_A whose Munn tree is the desired one.

This theorem guarantees that it makes sense to define a Munn tree of an element $u \in FIM_A$ as there is a unique one among words in $\bar{A}^*$ that are mapped to u. For an element $u \in FIM_A$, we define its *norm*, $\|u\|$ as the number of positive edges in $\mathrm{MT}(u)$.

This approach makes it clear that multiplication can only add new edges, and never make existing ones disappear. This implies that for all $u, v \in FIM_A$, $\|u\|, \|v\| \leq \|uv\| \leq \|u\| + \|v\|$. Furthermore, multiplication on the right alters the terminal vertex but not the initial one, and vice-versa. We define some notions from graph theory to establish the following results.

In this context, we define a *directed labelled graph* over the alphabet A, Γ, as a tuple $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a finite set, whose elements are said to be vertices, and $E(\Gamma)$ is a finite set, whose elements are said to be edges. To each edge e is associated an element $(p_1, a, p_2) \in V(\Gamma) \times A \times V(\Gamma)$, where we would say e connects p_1 to p_2 with label a. Note that, in general, there could be more than one edge connecting the same points with the same label.

Given two such graphs Γ, Γ', by a *(directed labelled) graph homomorphism* $\Gamma \to \Gamma'$ we mean a tuple (f, g) such that:

- $f : V(\Gamma) \to V(\Gamma')$ and $g : E(\Gamma) \to E(\Gamma')$;
- if $e \in E(\Gamma)$ is an edge connecting points from $V(\Gamma)$ p_1 to p_2, then eg is an edge connecting $p_1 g$ to $p_2 g$ with the same label.

Such an homomorphism is said to be an *embedding* if both f and g are injective, and an *isomorphism* if they are bijective. A *subgraph* of a graph Γ is a graph Γ' for which $V(\Gamma') \subseteq V(\Gamma), E(\Gamma') \subseteq E(\Gamma)$ and the identity constitutes an embedding.

Lemma 2.1.2. For $u, v \in FIM_A$, we have that:

- $v \leq_{\mathscr{R}} u$ if and only if there is an embedding $\mathrm{MT}(u) \to \mathrm{MT}(v)$ where the initial vertex of $\mathrm{MT}(u)$ is mapped to the initial vertex of $\mathrm{MT}(v)$;
- $v \leq_{\mathscr{L}} u$ if and only if there is an embedding $\mathrm{MT}(u) \to \mathrm{MT}(v)$ where the terminal vertex of $\mathrm{MT}(u)$ is mapped to the terminal vertex of $\mathrm{MT}(v)$;
- $v \leq_{\mathscr{J}} u$ if and only if there is an embedding $\mathrm{MT}(u) \to \mathrm{MT}(v)$.

In particular, all of Green's orders in FIM_A are decidable.

Note that this means that for any $u \in FIM_A$ and $v \in \bar{A}^*$, $v \in L(\mathrm{MT}(u))$ if and only if $u\theta \leq_{\mathscr{H}} v\theta$.

Extending these results to the Green's equivalence relations, we conclude the following.

Theorem 2.1.3. For $u, v \in FIM_A$, we have that:

- $u \mathrel{\mathscr{R}} v$ if and only if $\mathrm{mt}(u) \cong \mathrm{mt}(v)$ with an isomorphism that maps the initial vertices to each other;
- $u \mathrel{\mathscr{L}} v$ if and only if $\mathrm{mt}(u) \cong \mathrm{mt}(v)$ with an isomorphism that maps the terminal vertices to each other;
- $u \mathrel{\mathscr{H}} v$ if and only if $u = v$;
- $u \mathrel{\mathscr{J}} v$ if and only if $\mathrm{mt}(u) \cong \mathrm{mt}(v)$;
- $\mathscr{D} = \mathscr{J}$.

In particular, all of Green's equivalence relations in FIM_A are decidable.

Lastly, we mention that, for $u \in FIM_A$, we have that $\mathrm{mt}(u) = \mathrm{mt}(u^{-1})$ and $\mathrm{MT}(u^{-1})$ can be obtained from $\mathrm{MT}(u)$ by swapping the initial and terminal vertices. This way, since an idempotent is its own inverse, it becomes easy to check that an element is idempotent if and only if the initial and terminal vertices of its Munn tree coincide.

Thus, the single idempotent in an $\mathscr{R}$-class is the element with the mt and initial vertex that determine the class, and terminal vertex equal to the initial one. The unique idempotent in an $\mathscr{L}$-class is the one with mt and terminal vertex matching all other elements in the class, and initial point in the terminal vertex.

2.2. By Silva

Next we turn our gaze to results due to Silva [8] which go deeper in the matter of decidability results for free inverse monoids.

Consider a monoid M and $u \in M$, and define $u\mu = \{v \in M : u \leq_{\mathscr{J}} v\}$. We say that M is *finite $\mathscr{J}$-above* if, for every $u \in M$, $u\mu$ is finite.

Theorem 2.2.1. Let M be a finitely generated, finite $\mathscr{J}$-above monoid such that:

1. M has decidable word problem;*
2. the $\mathscr{J}$-order on M is decidable.

Then the membership problem for rational subsets of M is decidable.

Corollary 2.2.2. The membership problem for rational subsets of a free inverse monoid is decidable.

Proof. Let A be some alphabet and $L \in \mathrm{Rat}(FIM_A)$. Only finitely many letters occur in L, since a regular expression must be finite. Thus, w may assume that A is finite, as elements whose words include other letters will never belong to L.

By Theorem 2.1.1, the word problem for FIM_A is decidable. By Lemma 2.1.2, the $\mathscr{J}$-order in FIM_A is decidable and since, for each $u \in FIM_A$, $\mathrm{mt}(u)$ is a finite graph and so it has finitely many subgraphs, $u\mu$ is finite.

Therefore, Theorem 2.2.1 gives us the desired result. $\square$

*It has been pointed out that this condition is not necessary. Any finitely generated, finite $\mathscr{J}$-above monoid has finite and constructible Schützenberger graphs, which provide a solution to the word problem. We maintain it out of consistency with the original statement.

This result shall be generalized later.

2.2.1 The monogenic free inverse monoid

Now, we focus our study on the case $A = \{a\}$. For the rest of this dissertation, we denote by F the monogenic free inverse monoid $FIM_{\{a\}}$ and by θ the homomorphism $\bar{A}^* \to F$ such that $u\theta = [u]_\nu$. The Munn tree mt for any element in this free inverse monoid must be of the form

$$\cdot \xrightarrow{a} \cdot \xrightarrow{a} \cdot \xrightarrow{a} \cdots \xrightarrow{a} \cdot \xrightarrow{a} \cdot$$

where we would then indicate an initial and terminal vertices to specify each element.

Let $\pi : F \to \mathbb{Z}$ be the unique monoid homomorphism where $a\pi = 1$. It is easy to check that π corresponds to the distance in MT between the initial and terminal points, with positive sign if, in the above graph, the terminal is to the right of the initial, negative if it is to the left, and null if they coincide (which corresponds to the element being idempotent). We define two other maps from F to $\mathbb{Z}$:

- λ corresponding to the distance between the starting point and the leftmost vertex in $\mathrm{MT}(u)$ for each $u \in F$, with negative or null sign;
- and ρ corresponding to the distance between the starting point and the rightmost vertex in $\mathrm{MT}(u)$ for each $u \in F$, with positive or null sign.

These may equivalently be defined recursively by:

$$\begin{aligned} &1\lambda = 0,\, a\lambda = 0,\, a^{-1}\lambda = -1, \qquad &(uv)\lambda = \min\{u\lambda, u\pi + v\lambda\} \\ &1\rho = 0,\, a\rho = 1,\, a^{-1}\rho = 0, \qquad &(uv)\rho = \max\{u\rho, u\pi + v\rho\} \end{aligned}$$

for all $u, v \in F$. These three functions shall be referred to as the *coordinates* of an element in F. For any $u, v \in F$, if $u\lambda = v\lambda$, $u\pi = v\pi$ and $u\rho = v\rho$, then $\mathrm{MT}(u) = \mathrm{MT}(v)$ and so $u = v$, by Theorem 2.1.1.

We define $F^+ = \{u \in F : u\pi \geq 0\}$ and $F^- = \{u \in F : u\pi \leq 0\}$. Next, we list some basic properties of these coordinates.

Proposition 2.2.3. Let $u, v \in F$, and $u = x_1 \cdots x_n$ for $x_1, \ldots, x_n \in F$. Then:

1. $u\lambda \leq u\pi \leq u\rho$;
2. $u^{-1}\lambda = u\lambda - u\pi$, $u^{-1}\pi = -u\pi$, $u^{-1}\rho = u\rho - u\pi$;

3. $u\lambda = \min\{x_1\pi + \cdots x_{i-1}\pi + x_i\lambda : i \in [k]\}$;

4. $u\rho = \max\{x_1\pi + \cdots x_{i-1}\pi + x_i\rho : i \in [k]\}$;

5. $(uv)\lambda \leq u\lambda$ and $(uv)\rho \geq u\rho$;

6. u can be written as $a^{u\lambda}a^{-u\lambda}a^{u\rho}a^{-u\rho}a^{u\pi}$;

7. $\|u\| = u\rho - u\lambda$ and is greater than or equal to the distance between any two points in $\mathrm{mt}(u)$, including $u\rho - u\pi, u\rho, u\pi - u\lambda, -u\lambda$ and $|u\pi|$;

8. if $u \in F^-$ and for $k \in \mathbb{N}$, then $u^k\lambda = (k-1)(u\pi) + u\lambda, u^k\pi - u^k\lambda = u\pi - u\lambda$ and $u^k\rho = u\rho$;

9. if $u \in F^+$ and for $k \in \mathbb{N}$, then $u^k\rho = (k-1)(u\pi) + u\rho, u^k\rho - u^k\pi = u\rho - u\pi$ and $u^k\lambda = u\lambda$;

10. for $k \in \mathbb{Z}$, $u^k\lambda \geq \min\{-\|u\|, u^k\pi - \|u\|\}$ and $u^k\rho \leq \max\{\|u\|, u^k\pi + \|u\|\}$.

Proof. No reference was found for these results explicitly, so we provide our own proofs.

(1) Seeing $\mathrm{mt}(u)$ as a numbered line, with origin at the initial vertex, $u\pi$ corresponds to the coordinate of the terminal vertex. Thus, it must be greater than or equal to the coordinate of the leftmost vertex ($u\lambda$) and lesser than or equal to the coordinate of the rightmost vertex ($u\rho$).

(2) Let $u' \in F$ be the element with coordinates $u'\lambda = u\lambda - u\pi$, $u'\pi = -u\pi$, $u'\rho = u\rho - u\pi$. We have that $(uu'u)\pi = u\pi - u\pi + u\pi = u\pi$. Furthermore, $(uu'u)\lambda$ (respectively ρ) equals $u\lambda$ (resp. ρ) since all elements in the $\min$ (resp. $\max$) simplify to that value. Thus, $uu'u = u$. Analogously, we can prove that $u'uu' = u'$ and so $u' = u^{-1}$.

(3) and (4) can be obtained by repeatedly applying the recursive definition of λ and ρ. (5) follows immediately from it as well.

(6) We have that

$$(a^{u\lambda}a^{-u\lambda}a^{u\rho}a^{-u\rho}a^{u\pi})\pi = u\lambda - u\lambda + u\rho - u\rho + u\pi = u\pi.$$

Next, consider that $(a^{u\lambda}a^{-u\lambda})\pi = (a^{u\lambda}a^{-u\lambda})\rho = 0$ and $(a^{u\rho}a^{-u\rho})\pi = (a^{u\rho}a^{-u\rho})\lambda = 0$. The result follows from (3) and (4).

(7) This result can be obtained by a simple analysis of the Munn tree of an arbitrary element. Since it is a linear graph, the total number of positive edges is the distance between

the two extremes, which can be obtained by adding the distances from each to the initial vertex: $|u\lambda| = -u\lambda$ and $u\rho$. The other mentioned quantities correspond to distances between some combination of the leftmost, rightmost, initial and terminal vertices, and thus cannot surpass it.

(8) and (9) are given without issue by (3) and (4).

(10) For $k = 0$, the result is trivial. For $k > 0$, it follows from either (8) or (9), taken with (7). For $k < 0$, we may proceed likewise, considering u^{-1} instead. $\square$

We denote by α the (unique) inverse monoid homomorphism from F to F defined by $a \mapsto a^{-1}$. Note that $(a^{-1})\alpha = (a\alpha)^{-1} = a$, and so the action of α permutes a and a^{-1}, thus α is a self-inverse automorphism. In terms of Munn trees, it corresponds to inverting the direction of all edges:

$$
\begin{array}{lll}
\text{if } u\pi \geq 0, & \mathrm{MT}(u): & \xrightarrow{\quad a^i \quad} \circ \xrightarrow{a^j} \bullet \xrightarrow{\quad a^k \quad} \\[2em]
 & \mathrm{MT}(u\alpha): & \xleftarrow{\quad a^i \quad} \circ \xleftarrow{a^j} \bullet \xleftarrow{\quad a^k \quad} \\[2em]
\text{if } u\pi \leq 0, & \mathrm{MT}(u): & \xrightarrow{\quad a^i \quad} \bullet \xrightarrow{a^j} \circ \xrightarrow{\quad a^k \quad} \\[2em]
 & \mathrm{MT}(u\alpha): & \xleftarrow{\quad a^i \quad} \bullet \xleftarrow{a^j} \circ \xleftarrow{\quad a^k \quad}
\end{array}
$$

for some $i, j, k \in \mathbb{N}_0$.

Given any $u \in F$, this automorphism produces the following effects on its coordinates:

$$u\alpha\lambda = -u\rho, \quad u\alpha\pi = -u\pi, \quad u\alpha\rho = -u\lambda.$$

Furthermore, we define β as the anti-automorphism given by $u\beta = (u\alpha)^{-1}$, for all $u \in F$. This map also has order 2, and for $u \in F$, $u\beta\pi = u\pi$. If $u\pi \geq 0$,

$$u\beta\lambda = -(u\rho - u\pi), \qquad u\beta\rho - u\beta\pi = -u\lambda,$$

while if $u\pi \leq 0$,

$$u\beta\lambda - u\beta\pi = -u\rho, \qquad u\beta\rho = -(u\lambda - u\pi).$$

These properties can be easily checked resorting to the previous Munn trees, recalling that inverting an element keeps the same mt and swaps the initial and terminal vertices:

$$\text{if } u\pi \geq 0, \qquad \mathrm{MT}(u): \quad \xrightarrow{\quad a^i \quad} \circ \xrightarrow{a^j} \bullet \xrightarrow{\qquad a^k \qquad}$$

$$\mathrm{MT}(u\beta): \quad \xleftarrow{\quad a^i \quad} \bullet \xleftarrow{a^j} \circ \xleftarrow{\qquad a^k \qquad}$$

$$\text{if } u\pi \leq 0, \qquad \mathrm{MT}(u): \quad \xrightarrow{\quad a^i \quad} \bullet \xrightarrow{a^j} \circ \xrightarrow{\qquad a^k \qquad}$$

$$\mathrm{MT}(u\beta): \quad \xleftarrow{\quad a^i \quad} \circ \xleftarrow{a^j} \bullet \xleftarrow{\qquad a^k \qquad}$$

for some $i, j, k \in \mathbb{N}_0$.

We remark that if L is a rational set of F, then L^{-1}, $L\alpha$ and $L\beta$ are also rational. This can be seen by applying $^{-1}$, α or β to a description of L as a regular expression, or by obtaining automata that recognize them. If $\mathcal{A}$ is an automaton that recognizes L, then we may obtain such automata by:

- for $L\alpha$, replacing the labels of the edges in $\mathcal{A}$ with label a with the label a^{-1}, and vice-versa;
- for $L\beta$, inverting the direction of each edge in $\mathcal{A}$ and exchanging the initial and terminal vertices;
- for L^{-1}, applying both those procedures, since $L\alpha\beta = (L\alpha\alpha)^{-1} = L^{-1}$.

For every $n \geq 1$, consider the sets

$$G_{n,1} = \{u \in F^+ : u\lambda \leq -n\},$$
$$G_{n,2} = \{u \in F^+ : u\pi \geq n\},$$
$$G_{n,3} = \{u \in F^+ : u\rho - u\pi \geq n\}.$$

These are an important piece in defining maps $\xi_{n,i} : G_{n,i} \to F, i \in [3]$, which have the following effects:

- $\xi_{n,1}$ sends each $u \in G_{n,1}$ to the element of F with the same π and ρ as u, and λ greater than $u\lambda$ by n;
- $\xi_{n,2}$ sends each $u \in G_{n,2}$ to the element of F with the same λ and $(\rho - \pi)$ as u, and π lesser than $u\pi$ by n;

- $\xi_{n,3}$ sends each $u \in G_{n,3}$ to the element of F with the same λ and π as u, and ρ lesser than $u\rho$ by n.

The definition makes it clear that $\xi_{n,i}, i \in [3]$, commute: each of these maps alters exclusively one of λ, π, or $\rho - \pi$, while leaving the other two unchanged, thus, if we would be able to apply these functions in some order to an element, we are able to do it in any order and results, depending only on how many subtractions are applied to each coordinate, do not vary.

Lemma 2.2.4. Let $L \in \mathrm{Rat}(F)$. Then there exist some computable $n' \geq n \geq 1$ such that

$$u \in L \Leftrightarrow u\xi_{n,i} \in L \tag{2.2.1}$$

holds for all $i \in [3]$ and $u \in G_{n',i}$.

Corollary 2.2.5. Let $L \in \mathrm{Rat}(F)$. For all $k > 1$ and $n'' \geq n' + kn$, we can replace the constants $n, n' \geq 1$ of Lemma 2.2.4 by kn, n''.

The previous lemma inspired the following definition of our creation, to summarize the property stated in the next corollary.

Definition 2.2.6. For a set $L \subseteq F$ and $n', n \in \mathbb{N}$, we say the property $\star(L, n, n')$ holds when for $K \in \{L, L^{-1}\}$, and for any $i \in [3]$ and $u \in G_{n',i}$, it holds that

$$u \in K \Leftrightarrow u\xi_{n,i} \in K.$$

Corollary 2.2.7. For any $L \in \mathrm{Rat}(F)$, there are computable constants $n, n' \geq 1$ such that $\star(L, n, n')$ holds.

Proof. Let $L \in \mathrm{Rat}(F)$. Since, as seen before, $L^{-1} \in \mathrm{Rat}(F)$, Lemma 2.2.4 provides constants n, n' and $\bar{n}, \bar{n}'$ for L and L^{-1} respectively. By Corollary 2.2.5, the equivalence from the lemma holds for both sets and constants $n_0 = \mathrm{lcm}\{n, \bar{n}\}, n_0' = \max\{n', \bar{n}'\} + \mathrm{lcm}\{n, \bar{n}\}$. Therefore, $\star(L, n_0, n_0')$ holds. $\square$

Lemma 2.2.4 allowed for a solution of the problems of comparison and equality of rational sets for F, by reducing the elements that must be checked to a finite set:

Theorem 2.2.8. Let $K, L \in \mathrm{Rat}(F)$. Then it is decidable whether or not $K \subseteq L$ and $K = L$.

Next, we present the theorem which establishes the best previously know description of arbitrary rational sets in F.

Theorem 2.2.9. Let $L \subseteq F$. The following conditions are equivalent:

1. $L \in \mathrm{Rat}(F)$;
2. L is a finite union of subsets of the form

$$u_0 X_1^* u_1 X_2^* u_2 \cdots X_k^* u_k \tag{2.2.2}$$

for $k \in \mathbb{N}$, $u_0, \dots, u_k \in F$ and finite $X_1, \dots, X_k \subset F$.

Moreover, considering we shall mention recognizable sets of F later, we present the following result by Silva:

Theorem 2.2.10. Given $L \in \mathrm{Rat}(F)$, it is decidable whether or not $L \in \mathrm{Rec}(F)$.

Lastly, we set some notations that shall be used throughout the rest of this dissertation.

For each $n \in \mathbb{N}_0$, let

$$W_n = \{u \in F^+ : -n < u\lambda,\ \ u\pi < n,\ \ u\rho - u\pi < n\} = F \setminus (G_{n,1} \cup G_{n,2} \cup G_{n,3}),$$

and for each $n' \geq n \geq 1$, denote by $\zeta_{n,n'}$ the function $F^+ \to W_{n'}$ that, for each $u \in F^+$, corresponds to, in any order:

- applying $\xi_{n,1}$ repeatedly until the product's λ is greater than $-n'$, and so in $F^+ \setminus G_{n',1}$;
- applying $\xi_{n,2}$ repeatedly until the product's π is lesser than n', and so in $F^+ \setminus G_{n',2}$;
- applying $\xi_{n,3}$ repeatedly until the product's $(\rho - \pi)$ is lesser than n', and so in $F^+ \setminus G_{n',3}$.

3. Description of rational sets

The goal of this chapter is to build up to Theorem 3.2.4, which serves as an optimization of Theorem 2.2.9, providing a more precise description of rational sets. Unless specifically stated, all the results from now on are original.

We begin by presenting an auxiliary lemma and a first proposition which slightly limits the sets we must consider later.

Lemma 3.0.1. Suppose an element $u \in F$ may be written as $v_1 \cdots v_k$ for some $k \in \mathbb{N}$, and $v_i \in F$ for $i \in [k]$. Then, u may be written as $v_{i_1} \cdots v_{i_l}$, for some $l \leq k$, such that $i_j < i_{j+1}, \forall j \in \{1, \dots, l-1\}$, and there are at most two idempotents in $\{v_{i_j} : j \in [l]\}$.

Proof. Let $j_1, j_2 \in [k]$ be such that $u\lambda = v_1\pi + \cdots + v_{j_1-1}\pi + v_{j_1}\lambda$ and $u\rho = v_1\pi + \cdots + v_{j_2-1}\pi + v_{j_2}\rho$.

Suppose now that, for some $j \in [k] \setminus \{j_1, j_2\}$, $v_j \in E(F)$, and set $u' = v_1 \cdots v_{j-1}v_{j+1} \cdots v_k$. Since $v_j\pi = 0$, we have that:

$$u'\pi = v_1\pi + \cdots + v_{j-1}\pi + v_{j+1}\pi + \cdots v_k\pi = v_1\pi + \cdots + v_k\pi = u\pi.$$

By choice of j_1, $v_1\pi + \cdots + v_{j_1-1}\pi + v_{j_1}\lambda \leq v_1\pi + \cdots + v_{i-1}\pi + v_i\lambda$, for all $i \in [k]$, and neither side of the inequality is affected by adding or subtracting $v_j\pi$ (as it is null), so $u'\lambda = v_1\pi + \cdots + v_{j_1-1}\pi + v_{j_1}\lambda = u\lambda$, and, analogously, $u'\rho = u\rho$. Therefore, $u' = u$.

By repeating the process for each idempotent v_j, we conclude that all idempotents from $\{v_i : i \in [k] \setminus \{j_1, j_2\}\}$ may be removed from the expression for u without altering the resulting product, and thus, v_{j_1} and v_{j_2} are the only possible idempotents in the final factorization. □

Proposition 3.0.2. Let $L \subseteq F$. The following conditions are equivalent:

1. $L \in \mathrm{Rat}(F)$;

2. L is a finite union of subsets of the form

$$u_0X_1^*u_1X_2^*u_2 \dots X_k^*u_k$$

for $n \in \mathbb{N}$, $u_0, \dots, u_n \in F$ and finite $X_1, \dots, X_n \subseteq F \setminus E(F)$.

Proof. Fix $X \in \{X_i : i \in [k]\}$ and let $X' = X \setminus E(F)$ and $X^E = (X \cap E(F)) \cup \{1\}$. We claim that

$$X^* = \bigcup_{e,f \in X^E} X'^* e X'^* f X'^*.$$

The inclusion $\supseteq$ holds trivially. Let u be an arbitrary element of X^*, and $x_1 \cdots x_s$ be a factorization of u using only elements from $X \cup \{1\}$. By Lemma 3.0.1, we may suppose that, at most, two of the factors $x_i, i \in [s]$ are idempotent, and by adding 1 as a factor, we may indeed suppose that exactly two of the factors $x_i, i \in [s]$ are idempotent. We denote them by x_{j_1} and x_{j_2}, with $j_1 < j_2$.

Then, $u = (x_1 \cdots x_{j_1-1}) x_{j_1} (x_{j_1+1} \cdots x_{j_2-1}) x_{j_2} (x_{j_2+1} \cdots x_s) \in X'^* x_{j_1} X'^* x_{j_2} X'^*$.

In (2.2.2), by replacing each X_i by the finite union $\bigcup_{e,f \in X_i^E} X_i'^* e X_i'^* f X_i'^*$, and (conducting the necessary distributions), we obtain a rewriting of L as a union of sets of the desired form. □

Next, we start treating each factor of the form X_i^* in equation (2.2.2) separately, depending on whether X_i is contained exclusively in F^+ or F^-, named the positive and negative cases, respectively, or if otherwise it has elements with positive and negative π, named the mixed case.

3.1. The positive and negative cases

Proposition 3.1.1. Let X be a finite subset of F, contained in F^+ or F^-, and with no idempotents. Then, there is a computable N such that, for any multiple of N, n, X^* may be constructively written as a finite union of sets of the form

$$u_0 x^* u_1$$

with $u_0, u_1, x \in F$ and $|x\pi| = n$.

Proof. We cover the case where $X \subseteq F^+$. The opposite one would follow dually.

Let $N' = \operatorname{lcm}\{x\pi : x \in X\}$ and choose N to be its least multiple which is larger than $\max\{\|x\| : x \in X\}$. Fix n as an arbitrary multiple of N.

For each $x \in X$, denote by n_x the natural number such that $n_x(x\pi) = x^{n_x}\pi = n$. Let S be the set of all elements of X^* where each element x is used at most n_x times.

Let $d = \gcd\{|x\pi| : x \in X\}$. Any $u \in X^*$ is such that $d|u\pi$. By the Euclidean algorithm, we may compute an integer c_x for each $x \in X$ such that $d = \sum_{x\in X} c_x(x\pi)$. Besides, by adding or subtracting some multiple of $n_x|x\pi|$, we conclude that there is some $c_x \in \{0, \ldots, n_x - 1\}$ for each $x \in X$ such that

$$d \equiv \sum_{x\in X} c_x(x\pi) \mod n.$$

Similarly, given any $m \in \mathbb{Z}$, there is a coefficient $c_{m,x} \in \{0, \ldots, n_x - 1\}$ for each $x \in X$ such that

$$md \equiv \sum_{x\in X} c_{m,x}(x\pi) \mod n.$$

Lastly, notice that any element of X^* which uses each $x \in X$ exactly $c_{m,x}$ times is in S. Therefore, any value of $\pi \pmod n$ which can be arithmetically obtained by elements of $X^*\pi$ can be attained with an element in $S\pi$.

Denote by x'_L and x'_R elements of X such that

$$\begin{aligned} x'_L\lambda &= \max\{x\lambda : x \in X\}, \\ x'_R\rho &= \min\{x\rho - x\pi : x \in X\}, \end{aligned}$$

and set $x_i = (x'_i)^{n_{x'_i}}$, for $i \in \{L, R\}$. Note that $x_i\pi = n_{x'_i}(x'_i\pi) = n$ and that, from Proposition 2.2.3, $x_L\lambda = x'_L\lambda$ and $x_R\rho - x_R\pi = x'_R\rho - x'_R\pi$. Lastly, we mention that $x_L, x_R \in S$.

Let $l = \max\{\|u\| : u \in S\}$, z be the least natural number such that $zn \geq 2l$ and $M = 2zn + 3l$. We claim that

$$X^* = \{u \in X^* : \|u\| \leq M\} \cup SXx_L^z Sx_R^* XS. \tag{3.1.1}$$

Clearly, the inclusion $\supseteq$ holds. Take u, an arbitrary element of X^*, which can be written as $u = x_1 \cdots x_k$, for some $k \in \mathbb{N}_0$ and $x_i \in X$ for $i \in [k]$. If $\|u\| \leq M$, is an element of the right hand side of the equation trivially. Suppose, otherwise, that $\|u\| > M$.

Take $i, j \in [k]$ such that

$$\begin{aligned} u\lambda &= x_1\pi + \cdots + x_{i-1}\pi + x_i\lambda \\ u\rho &= x_1\pi + \cdots + x_{j-1}\pi + x_j\rho. \end{aligned}$$

Note that

$$0 \geq u\lambda = (x_1 \cdots x_{i-1})\pi + x_i\lambda \Leftrightarrow (x_1 \cdots x_{i-1})\pi \leq -x_i\lambda \leq \|x_i\| < N \leq n$$

and so, for each $b \in [i-1]$, x_b occurs less than n_{x_b} times in $(x_1, \ldots, x_{i-1})$, thus $s_1 = x_1 \cdots x_{i-1} \in S$. Analogously, $s_3 = x_{j+1} \cdots x_k \in S$.

This allows us to conclude that $i < j$ as otherwise,

$$\begin{aligned} \|u\| = \|x_1 \cdots x_j \cdots x_i \cdots x_k\| \leq \\ \leq \|x_1 \cdots x_{i-1}\| + \|x_i \cdots x_k\| \leq \\ \leq \|x_1 \cdots x_{i-1}\| + \|x_{j+1} \cdots x_k\| = \|s_1\| + \|s_3\| \leq 2l < M, \end{aligned}$$

which would contradict our assumption on u.

Furthermore, we have that $(x_1 \cdots x_j)\lambda = u\lambda$ since

$$\begin{aligned} (x_1 \cdots x_j)\lambda \leq (x_1 \cdots x_{i-1})\pi + x_i\lambda = u\lambda \leq \\ \leq \min\{(x_1 \cdots x_{k-1})\pi + x_k\lambda : k \in [j]\} = (x_1 \cdots x_j)\lambda. \end{aligned}$$

Take $s_2 \in S$ such that $s_2\pi \equiv (x_{i+1} \cdots x_{j-1})\pi \mod n \Leftrightarrow (s_1 x_i)\pi + s_2\pi \equiv (x_1 \cdots x_{j-1})\pi \mod n$.

We notice that

$$\begin{aligned} M <&\|u\| = u\rho - u\lambda = \\ =&(x_1 \cdots x_{j-1})\pi + x_j\rho - (x_1 \cdots x_{i-1})\pi - x_i\lambda = \\ =&x_j\rho + (x_{i+1} \cdots x_{j-1})\pi + (x_i\pi - x_i\lambda) \leq \\ \leq&(x_{i+1} \cdots x_{j-1})\pi + \|x_i\| + \|x_j\| \leq \\ \leq&(x_{i+1} \cdots x_{j-1})\pi + 2l \\ &\implies 2zn + l < (x_{i+1} \cdots x_{j-1})\pi, \end{aligned}$$

while at the same time,

$$(x_L^z s_2)\pi = zn + s_2\pi \leq zn + \|s_2\| \leq zn + l.$$

Joining these two facts, we conclude that

$$(x_L^z s_2)\pi + zn < (x_{i+1}\cdots x_{j-1})\pi \Leftrightarrow$$
$$\Leftrightarrow (s_1 x_i x_L^z s_2)\pi + zn < (x_1\cdots x_{j-1})\pi$$

and so, since $(s_1 x_i x_L^z s_2)\pi \equiv (s_1 x_i)\pi + s_2\pi \equiv (x_1\cdots x_{j-1})\pi \mod n$, there is some $m \in \mathbb{N}$ such that $(x_1\cdots x_{j-1})\pi = (s_1 x_i x_L^z s_2)\pi + (z+m)n = (s_1 x_i x_L^z s_2 x_R^{z+m})\pi$, from which $(s_1 x_i x_L^z s_2 x_R^{z+m} x_j)\pi = (x_1\cdots x_j)\pi$.

We now determine the values of λ and ρ of $s_1 x_i x_L^z s_2 x_R^{z+m} x_j$. Notice that

$$(s_1 x_i x_L^z s_2 x_R^{z+m} x_j)\lambda = \min\left\{\begin{array}{c} (s_1x_i)\lambda, \\ (s_1x_i)\pi + x_L^z\lambda, \\ (s_1x_ix_L^z)\pi + (s_2x_R^{z+m})\lambda, \\ (s_1x_ix_L^zs_2x_R^{z+m})\pi + x_j\lambda \end{array}\right\}, \qquad \begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

and so we proceed as follows:

$$\begin{aligned}
&(1) \quad (s_1x_i)\lambda = s_1\pi + x_i\lambda = u\lambda,\\
&(2) \quad (s_1x_i)\pi + x_L\lambda \geq (s_1x_i)\pi + x_i\lambda > s_1\pi + x_i\lambda,\\
&(3) \quad (s_2x_R^{z+m})\lambda = \min\{s_2\lambda, s_2\pi + x_R^{z+m}\lambda\} = \min\{s_2\lambda, s_2\pi + x_R\lambda\} \geq\\
&\qquad \geq \min\{s_2\lambda, x_R\lambda\} \geq \min\{-\|s_2\|, -\|x_R\|\} \geq -l,\\
&\qquad (s_1x_i)\pi + (x_L^z)\pi + (s_2x_R^{z+m})\lambda \geq zn - l > 0 \geq u\lambda,\\
&(4) \quad (s_1x_ix_L^zs_2x_R^{z+m})\pi + x_j\lambda = (x_1\cdots x_{j-1})\pi + x_j\lambda \leq u\lambda,
\end{aligned}$$

which allows us to conclude that $(s_1 x_i x_L^z s_2 x_R^{z+m} x_j)\lambda = u\lambda = (x_1\cdots x_j)\lambda$.

On the other hand, considering that

$$(s_1 x_i x_L^z s_2 x_R^{z+m} x_j)\rho = \max\left\{\begin{array}{c} (s_1x_i)\rho, \\ (s_1x_i)\pi + (x_L^zs_2)\rho, \\ (s_1x_ix_L^zs_2)\pi + x_R^{z+m}\rho, \\ (s_1x_ix_L^zs_2x_R^{z+m})\pi + x_j\rho \end{array}\right\}, \qquad \begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

and that

$$
\begin{aligned}
&(1) \quad (s_1x_i)\rho = (x_1\cdots x_i)\rho \leq u\rho,\\
&(2) \quad x_i\lambda \leq s_1\pi + x_i\lambda = u\lambda \Rightarrow -u\lambda + x_i\lambda \leq 0,\\
&\qquad (x_L^z s_2)\rho = \max\{x_L^z\rho, x_L^z\pi + s_2\rho\} = \max\{x_L^{z-1}\pi + x_L\rho, zn + s_2\rho\} \leq\\
&\qquad \leq \max\{(z-1)n + l, zn + l\} = zn + l,\\
&\qquad (s_1x_i)\pi + (x_L^z s_2)\rho \leq \|s_1\| + \|x_i\| + zn + l \leq 2l + zn + l <\\
&\qquad < M - l < \|u\| - l = u\rho - u\lambda - l \leq u\rho - u\lambda + x_i\lambda \leq u\rho,
\end{aligned}
$$

$$
\begin{aligned}
&(3) \quad x_R\rho - x_R\pi = x'_R\rho - x'_R\pi \leq x_j\rho - x_j\pi \Rightarrow x_R\rho < x_R\rho + x_j\pi \leq x_R\pi + x_j\rho \Rightarrow\\
&\Rightarrow (s_1x_ix_L^z s_2)\pi + x_R^{z+m}\rho = (s_1x_ix_L^z s_2 x_R^{z+m-1})\pi + x_R\rho < (s_1x_ix_L^z s_2 x_R^{z+m})\pi + x_j\rho\\
&(4) \quad (s_1x_ix_L^z s_2 x_R^{z+m})\pi + x_j\rho = (x_1\cdots x_{j-1})\pi + x_j\rho = u\rho,
\end{aligned}
$$

from which follows that $(s_1x_ix_L^z s_2 x_R^{z+m}x_j)\rho = u\rho = (x_1\cdots x_j)\rho$.

In conclusion, $s_1x_ix_L^z s_2 x_R^{z+m}x_j = x_1\cdots x_j$ and so we can rewrite $s_1x_ix_L^z s_2 x_R^{z+m}x_j s_3 = (x_1\cdots x_j)(x_{j+1}\cdots x_k) = u$. Therefore, u is an element of the right hand side of equation (3.1.1), and that equation holds as desired.

Since $\{u \in X^* : \|u\| \leq M\}$ is finite, it can be written as the union of the finite set $\{\{u_0x^*u_1\} : u_0 \in \{u \in X^* : \|u\| \leq M\}\}$ for $x = u_1 = 1$, and since X and S are finite, the rest of the right hand side of the equation may also be expressed as a finite union of rational sets, where the star operator is applied just once, to x_R, which therefore have the desired form. □

To simplify calculations in the proof of the next proposition, we introduce the following lemma.

Lemma 3.1.2.

1. Let $w, w' \in F$ be such that $(ww')\lambda = w\lambda$ and $(ww')\rho = w\pi + w'\rho$. Given $y \in F^+$,
   - if $y\lambda \geq w\lambda - w\pi$, then $(wy^nw')\lambda = w\lambda$ for every $n \in \mathbb{N}_0$;
   - if $y\rho - y\pi \leq w'\rho$, then $(wy^nw')\rho = y^n\pi + (ww')\rho$ for every $n \in \mathbb{N}_0$.
2. On the other hand, if $(ww')\lambda = w\pi + w'\lambda$, $(ww')\rho = w\rho$ and $y \in F^-$,
   - if $y\rho \leq w\rho - w\pi$, then $(wy^nw')\rho = w\rho$ for every $n \in \mathbb{N}_0$;

- if $y\lambda - y\pi \geq w'\lambda$, then $(wy^n w')\lambda = y^n\pi + (ww')\lambda$ for every $n \in \mathbb{N}_0$.

Proof. For $n = 0$, all results follow trivially. Choose an arbitrary $n \in \mathbb{N}$.

(1) Let w, w' be elements of F satisfying the desired conditions for the first set of properties. Take $y \in F^+$ and suppose that $y\lambda \geq w\lambda - w\pi$.

Recall that $(wy^n w')\lambda = \min\{w\lambda, w\pi + y^n\lambda, w\pi + y^n\pi + w'\lambda\}$. Firstly, we note that, since $y \in F^+$, $w\pi + y^n\pi + w'\lambda \geq w\pi + w'\lambda \geq (ww')\lambda = w\lambda$. Next, recalling Proposition 2.2.3, we have that $y^n\lambda = y\lambda$, and so, by hypothesis, $w\pi + y^n\lambda \geq w\lambda$. Therefore, $(wy^n w')\lambda = w\lambda$.

Suppose now that $y\rho - y\pi \leq w'\rho$. We know that $(wy^n w')\rho = \max\{w\rho, w\pi + y^n\rho, w\pi + y^n\pi + w'\rho\}$. Since $w\rho \leq (ww')\rho = w\pi + w'\rho$ and $y\pi \geq 0$, then $w\rho \leq w\pi + y^n\pi + w'\rho$. Lastly, by the condition imposed on y, we know that $y\rho \leq y\pi + w'\rho$, hence $w\pi + y^n\rho = w\pi + y^{n-1}\pi + y\rho \leq w\pi + y^n\pi + w'\rho$. In conclusion, $(wy^n w')\rho = w\pi + y^n\pi + w'\rho = y^n\pi + (ww')\rho$.

Using the properties of the automorphism α, we can obtain (2) dually. □

Now, we establish the next proposition which allows us to reduce the number of elements with π of the same sign where the star operator is applied in a row in a regular expression, down to a single one.

Proposition 3.1.3. The set x^*uy^*, for any $u \in F$ and $x, y \in F^+$ or $x, y \in F^-$, can constructively be written as a finite union of sets of the form $u_0v^*u_1$, with $u_0, u_1 \in F$ and $v \in \{1, x, y\}$.

Proof. Suppose $x, y \in F^+$. If x or y are idempotents, the result follows trivially. Suppose, otherwise, that $x\pi, y\pi > 0$.

Since $x\pi, y\pi > 0$, there exists some $z \in \mathbb{N}$ such that $(x^z uy^z)\lambda = (x^z y^z)\lambda = x\lambda$, $(x^z uy^z)\rho = x^z\pi + u\pi + y^z\rho$ and $(x^z y^z)\rho = x^z\pi + y^z\rho$.

Let $n \in \mathbb{N}_0$. We have $(x^n u)\pi = (ux^n)\pi$. By Lemma 3.1.2, $(x^z x^n uy^z)\lambda = (x^z ux^n y^z)\lambda = x\lambda$ and $(x^z x^n uy^z)\rho = (x^z ux^n y^z)\rho = x^z\pi + x^n\pi + u\pi + y^z\rho$.

Let $N = \operatorname{lcm}\{x\pi, y\pi\}$, and $N_1 = {}^N/_{x\pi}, N_2 = {}^N/_{y\pi}$. With the same z as before, we claim that $x^z x^{N_1} y^z = x^z y^{N_2} y^z$. Clearly, $(x^z x^{N_1} y^z)\pi = zp_1 + N + zp_2 = (x^z y^{N_2} y^z)\pi$, but from the choice of z, we also obtain that

$$(x^z x^{N_1} y^z)\lambda = x\lambda = (x^z y^{N_2} y^z)\lambda \qquad \text{and}$$
$$(x^z x^{N_1} y^z)\rho = zp_1 + N + y^z\rho = (x^z y^{N_2} y^z)\rho.$$

With this information, we conclude that:

$$n \geq z,\, m \geq z + N_2 \implies x^n y^m = x^{n+N_1} y^{m-N_2},$$
$$n \geq z + N_1,\, m \geq z \implies x^n y^m = x^{n-N_1} y^{m+N_2}.$$

This way, any element of x^*y^* with exponent in x of at least z can be written with an exponent in y of less than $z + N_2$, and any element with exponent in y of at least z can be written with an exponent in x of less than $z + N_1$. Thus,

$$x^*y^* = \bigcup_{0 \leq m < z+N_2} x^* y^m \cup \bigcup_{0 \leq n < z+N_1} x^n y^*.$$

If $x, y \in F^-$, then $(x^*uy^*)^{-1} = (y^{-1})^* u^{-1} (x^{-1})^*$ is a set of the form being studied but $x^{-1}, y^{-1} \in F^+$. Thus, $(x^*uy^*)^{-1}$ can be written as $\cup_{i \in I} u_i v_i^* w_i$, for some finite set I, and $u_i, w_i \in F$ and $v_i \in \{1, x^{-1}, y^{-1}\}$ for each $i \in I$. Therefore,

$$x^*uy^* = ((x^*uy^*)^{-1})^{-1} = \left(\bigcup_{i \in I} u_i v_i^* w_i\right)^{-1} = \bigcup_{i \in I} (u_i v_i^* w_i)^{-1} = \bigcup_{i \in I} w_i^{-1} (v_i^{-1})^* u_i^{-1},$$

which is a finite union of the desired form. □

Propositions 3.1.1 and 3.1.3, taken together, imply that any finite set X (free of idempotents) with in the positive and negative cases can be written as a finite union of sets of the form $u_0 v^* u_1, u_0, u_1, v \in F$. This is an important stepping stone, and the end goal of this section.

3.2. The mixed case

To tackle the case where $X\pi$ contains positive and negative values, we start by determining which elements we can find in such a set, with this first proposition.

Proposition 3.2.1. Let X be a finite subset of F, which contains elements from F^+ and F^-, but no idempotents. Set $d = \gcd\{|x\pi| : x \in X\}$.

Then $X^*\pi = d\mathbb{Z}$, and elements $u, v \in X^*$ such that $u\pi = d$ and $v\pi = -d$ are constructible.

Proof. We prove the second part of the result by induction on the size of X.

If $|X| = 2$, denote by x and y the elements of X so that $x\pi > 0$ and $y\pi < 0$. By applying the Euclidean algorithm, we obtain $b, c \in \mathbb{Z}$ such that $d = b(x\pi) + c(y\pi)$. Choose $n, m \in \mathbb{N}$ such that $n(x\pi) + m(y\pi) = 0$. By choosing $k \in \mathbb{N}_0$ large enough, we may write $d = (b + kn)(x\pi) + (c + km)(y\pi)$ and $b + kn, c + km \geq 0$.

This way, writing $u = x^{b+kn}y^{c+km}$ and $v = y^m u^{n(x\pi)/d-1}$, we have that $u\pi = d$ and $v\pi = -d$.

Suppose now that $|X| > 2$. Take $x \in X$ such that $X' = X \setminus \{x\}$ still includes elements from F^+ and from F^-. From our induction hypothesis, for $d' = \gcd\{|y\pi| : y \in X'\}$, we can construct elements $u', v' \in X'^*$ such that $u'\pi = d = -v'\pi$.

Note that $d = \gcd\{|y\pi| : y \in X\} = \gcd\{|x\pi|, d'\}$. By considering Y as the set containing x and either u' or v' (whichever has a sign of π opposite that of $x\pi$), we can obtain through the process detailed for $|X| = 2$ elements $u, v \in Y^* \subseteq X^*$ such that $u\pi = d = -v\pi$.

Since $u^l, v^l \in X^*$ for any $l \in \mathbb{N}_0$, then $d\mathbb{Z} \subseteq X^*\pi$. For the oposite inclusion, consider any $w = x_1 \cdots x_s \in X^*$, for some $s \in \mathbb{N}_0$ and $x_1, \ldots, x_s \in X$. Since $d|x_i\pi$ for all $i \in [s]$, then $d| \sum_{i=1}^{s} x_i\pi = w\pi$ and thus $w\pi \in d\mathbb{Z}$. $\square$

Next, we decompose the sets of the form X^* which fall in the mixed case similarly to what we did in the previous section.

Proposition 3.2.2. Let X be a finite subset of F, which contains elements from F^+ and F^-. Then, X^* can constructively be written as a finite union of sets of the form

$$u_0(y^{k_1})^*u_1(y^{k_2})^*u_2(y^{k_3})^*u_3 \tag{3.2.1}$$

for some $u_0, \ldots, u_3, y \in F$ and $k_1, k_2, k_3 \in \{-1, 0, 1\}$.

Proof. Later in this proof, we shall make use of the following lemma to simplify calculations:

Lemma 3.2.3. Let $u = x_1 \cdots x_s$, for some $x_1, \ldots, x_s \in F$. Given integers $0 = k_0 < k_1 < \cdots < k_n = s, n \in \mathbb{N}$, define $w_i = x_{k_{i-1}+1} \cdots x_{k_i}$, for $i \in [n]$. If $u\lambda = (x_1 \cdots x_{i-1})\pi + x_i\lambda$ and $u\rho = (x_1 \cdots x_{j-1})\pi + x_j\rho$, and $k_{b-1} + 1 \leq i \leq k_b, k_{c-1} + 1 \leq j \leq k_c$, then

$$u\lambda = (w_1 \cdots w_{b-1})\pi + w_b\lambda \quad \text{and} \quad u\rho = (w_1 \cdots w_{c-1})\pi + w_c\rho.$$

Proof. For simplicity of notation, let $w = w_1 \cdots w_{b-1}$, $w' = w_{b+1} \cdots w_n$ and $v = w_1 \cdots w_{c-1}$, $v' = w_{c+1} \cdots w_n$. Then, keeping in mind Proposition 2.2.3,

$$w\pi + w_b\lambda \leq w\pi + (x_{k_{b-1}+1} \cdots x_{i-1})\pi + x_i\lambda = u\lambda \leq (ww_b)\lambda \leq w\pi + w_b\lambda$$

$$v\pi + w_c\rho \geq v\pi + (x_{k_{c-1}+1} \cdots x_{j-1})\pi + x_j\rho = u\rho \geq (vw_c)\rho \geq v\pi + w_c\rho.$$

□

Let $d = \gcd\{|x\pi| : x \in X^*\}$. By Proposition 3.2.1, we can find $y_0, y_1 \in X^*$ such that $y_0\pi = -d$ and $y_1\pi = d$. Let $z \in \mathbb{N}$ be the least natural with the property that $-zd \leq \min\{y_0\lambda + d, y_1\lambda\}$ and $zd \geq \max\{y_0\rho, y_1\rho - d\}$.

Define $e = y_0^z y_1^z$ and $f = y_1^z y_0^z$. Since $e\pi = f\pi = 0$, $e, f \in E(F)$. Recalling Proposition 2.2.3, note that:

$$e\lambda \leq y_0^z\lambda \leq y_0^z\pi = -zd$$

$$e\rho = \max\{y_0^z\rho,\ y_0^z\pi + y_1^z\rho\} = \max\{y_0\rho,\ -d + y_1\rho\}$$

$$f\lambda = \min\{y_1^z\lambda,\ y_1^z\pi + y_0^z\lambda\} = \min\{y_1\lambda,\ d + y_0\lambda\}$$

$$f\rho \geq y_1^z\rho \geq y_1^z\pi = zd.$$

Set $y = ey_1f$ and $y' = fy_0e$. Clearly, $y, y' \in X^*$ and $y\pi = d = -y'\pi$. We calculate the other coordinates of y and y' as follows:

$$y\lambda = \min\{e\lambda,\ y_1\lambda,\ y_1\pi + f\lambda\} = e\lambda$$

$$y\rho = \max\{e\rho,\ y_1\rho,\ y_1\pi + f\rho\} = d + f\rho$$

$$y'\lambda = \min\{f\lambda,\ y_0\lambda,\ y_0\pi + e\lambda\} = -d + e\lambda$$

$$y'\rho = \max\{f\rho,\ y_0\rho,\ y_0\pi + e\rho\} = f\rho.$$

Thus, $y' = y^{-1}$.

We shall build a finite collection of sets of the form (3.2.1), each contained in X^*, such that its union covers X^*. Firstly, we remark that it is enough to consider sufficiently large elements (in the sense of the norm) from $(X^*)^+$ since:

- we may add elements of lesser norm one by one, as there are finitely many of them;

- repeating the process for $(X^{-1})^*$ would cover $((X^{-1})^*)^+ = ((X^*)^-)^{-1}$. Then, we could take the inverse of the regular expressions obtained for each set of the form (3.2.1), which keep that form, to find a cover for $(X^*)^-$.

Let $N = \max\{\|u\| : u \in X \cup \{y\}\}$.

Case 1: covering elements $u \in (X^*)^+$ such that $u\lambda = l_0 > -3N$ and $u\rho - u\pi = r_0 < 3N$, and $u\pi \geq 3N$ (for $u\pi < 3N$, there are finitely many such elements, and thus they may be disregarded). For each pair of l_0 and r_0, consider v the element of this class with the least value of π, which can be found through trial and error, since membership in L is decidable, and Lemma 2.2.4 guarantees that there is some constant n' where, if this class is not empty, $v\pi < n'$. Let $x_1, \dots, x_s$ be elements of X such that $x_1 \cdots x_s = v$. This factorization can eventually be obtained by testing all products of an incrementally increasing length. Fix indexes $i, j \in [s]$ such that $v\lambda = (x_1 \cdots x_{i-1})\pi + x_i\lambda$ and $v\rho = (x_1 \cdots x_{j-1})\pi + x_j\rho$. Lastly, consider that

$$\begin{aligned}(x_1 \cdots x_i)\pi &\leq (x_1 \cdots x_{i-1})\pi + x_i\lambda + \|x_i\| \leq \\ &\leq v\lambda + N \leq \\ &\leq N\end{aligned}$$

and that

$$\begin{aligned}(x_1 \cdots x_j)\pi &\geq (x_1 \cdots x_{j-1})\pi + x_j\rho - \|x_j\| \geq \\ &\geq v\rho - N \geq \\ &\geq v\pi - N \geq \\ &\geq 2N.\end{aligned}$$

If $i < j$, since each element x_k for $k \in [s]$ is in X, it is such that $|x_k\pi| \leq \|x_k\| \leq N$, thus there is some $k \in \{i, i+1, \dots, j-1\}$ such that $(x_1 \cdots x_k)\pi \in \{N, N+1, \dots, 2N\}$. Setting $w = x_1 \cdots x_k$ and $w' = x_{k+1} \cdots x_s$, we claim that

$$\{u \in (X^*)^+ : u\lambda = l_0, u\rho - u\pi = r_0, u\pi \geq 3N\} \subseteq wy^*w'.$$

Let u be an element of the left-hand side of the equation. Then $u\lambda = v\lambda$, $u\rho - u\pi = v\rho - v\pi$ and $u\pi \geq v\pi$. Hence, $u\pi = v\pi + nd$, for some $n \in \mathbb{N}_0$, and thus $u\rho = v\rho + nd$. Clearly, $(wy^nw')\pi = v\pi + nd$. Considering that x_i occurs in w and x_j in w', we have that $v\lambda = w\lambda$ and $v\rho = w\pi + w'\rho$. Note also that

- $w\lambda - w\pi \leq -w\pi \leq -N \leq -\|y\| \leq y\lambda$ and
- $w'\rho = v\rho - w\pi \geq 3N - w\pi \geq 3N - 2N = N \geq \|y\| \geq y\rho - y\pi.$

We are, therefore, in condition to apply Lemma 3.1.2 to conclude that

$$(wy^n w')\lambda = w\lambda = u\lambda, \quad \text{and} \quad (wy^n w')\rho = v\rho + nd = u\rho,$$

which ensures that $u = wy^n w' \in wy^* w'$.

On the other hand, if $j < i$, by the same argument employed in the previous case, since $u\pi \geq 3N$, we can find indexes $k_1 < j$, $j < k_2 < i$ and $i < k_3$ such that $(x_1 \cdots x_{k_b})\pi \in \{N, N+1, \ldots, 2N\}$ for each $b \in \{1,2,3\}$. Let $p_b = (x_1 \cdots x_{k_b})\pi$, for each $b \in \{1,2,3\}$. Setting $k_0 = 0$ and $k_4 = s$, define $w_b = x_{k_{b-1}+1} \cdots x_{k_b}$, for each $b \in [4]$. Then, we shall prove that we may rewrite:

$$v = w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d} w_2 w_3 w_4.$$

First of all, this expression is well defined: for each $b \in [3]$, since p_b is the value of π for some element in X^*, then $d | p_b$ and so p_b/d is an integer.

Note that $w_1\pi = p_1$, $w_2\pi = p_2 - p_1$, $w_3\pi = p_3 - p_2$ and $w_4\pi = v\pi - p_3$. By Lemma 3.2.3, we have that $w_1\pi + w_2\rho = v\rho$ e $(w_1w_2)\pi + w_3\lambda = v\lambda$. We may also calculate π for each of the relevant prefixes of the rewriting:

$$\begin{aligned}
w_1\pi &= p_1\\
(w_1 y^{(p_2 - p_1)/d})\pi &= p_1 + (p_2 - p_1) = p_2\\
(w_1 y^{(p_2 - p_1)/d} w_3)\pi &= p_2 + (p_3 - p_2) = p_3\\
(w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d})\pi &= p_3 + (p_1 - p_3) = p_1\\
(w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d} w_2)\pi &= p_1 + (p_2 - p_1) = p_2\\
(w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d} w_2 w_3)\pi &= p_2 + (p_3 - p_2) = p_3\\
(w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d} w_2 w_3 w_4)\pi &= p_3 + (v\pi - p_3) = v\pi.
\end{aligned}$$

This allows us to make the following remarks:

- Whenever a factor of the form $y^n, n \in \mathbb{Z}$, occurs, setting w' as the product of all preceding factors, both $w'\pi$ and $(w'y^n)\pi$ are between N and $2N$. By Proposition 2.2.3, this guarantees that $w'\pi + y^n\lambda$ is no lesser than $0 \geq u\lambda$ and $w'\pi + y^n\rho$ is no greater than $3N \leq u\pi \leq u\rho$, since $y^n\lambda$ and $y^n\rho$ are produced at one of the ends of the power, and are at most N-distant from 0 and $y^n\pi$.

- Whenever a factor of the form $w_b, b \in [4]$, occurs, the π of the preceding product is the same as in the original writing of v. Hence, no factor of this form can create λ under $v\lambda$ nor ρ above $v\rho$, as in the original writing of v, the sum of that same π and $w_b\lambda$ would be one of the values from which the minimum was selected to be $v\lambda$, and analogously for ρ. In particular,

$$(w_1 y^{p_2 - p_1/d})\pi + w_3\lambda = (w_1 w_2)\pi + w_3\lambda = v\lambda, \qquad \text{and}$$
$$(w_1 y^{(p_2 - p_1)/d} w_3 y^{(p_1 - p_3)/d})\pi + w_2\rho = w_1\pi + w_2\rho = v\rho.$$

Thus, this product meets, but does not surpass, $v\lambda$ and $v\rho$. Therefore, the expression we presented corresponds to v, and w_3 occurs (for the first time) before w_2. We may choose indexes i' and j' relative to the new writing, such that $x_{i'}$ is a factor in the first occurrence of w_3 and $x_{j'}$ is in w_2, and that $v\lambda$ and $v\rho$ are obtained in i' and j', respectively. This way, $i' < j'$ and so we may execute the process detailed for $i < j$.

Lastly, it is impossible to have $i = j$, since if that were that case, we would have

$$\|v\| = v\rho - v\lambda = (x_1 \cdots x_{i-1})\pi + x_i\rho - ((x_1 \cdots x_{i-1})\pi + x_i\lambda) = \|x_i\| \leq N < v\pi.$$

Case 2: covering elements $u \in (X^*)^+$ such that $u\lambda = l_0 > -3N$, $u\rho - u\pi \geq 3N$, and $u\rho - u\pi \equiv r \mod d, r \in \{0, \ldots, d-1\}$. For each pair of l_0 and r, we set v as an element with the least value of π attainable in this class. Similarly to the previous case, we can use Lemma 2.2.4 to limit how large an element we must test before finding v or concluding that the class is empty, and we can find a factorization $v = x_1 \cdots x_s$, with $x_i \in X, i \in [s]$. Note that there are finitely many elements with the fixed λ and ρ under $v\rho$, which may be ignored. Define $i, j \in [s]$ as indexes where $v\lambda$ and $v\rho$ are obtained, respectively.

Although $v\pi$ may not be large, $v\rho$ is, and so $(x_1 \cdots x_{j-1})\pi$ is under the same conditions as in the previous case. Thus, we may assume that $i < j$, and find $k_1 \in \{i, i+1, \ldots, j-1\}$ such that $(x_1 \cdots x_{k_1})\pi \in \{N, N+1, \ldots, 2N\}$. Furthermore, take $k_2 > j$ to be the greatest index such that $N \leq (x_1 \cdots x_{k_2})\pi$.

Setting $w_1 = x_1 \cdots x_{k_1}$, $w_2 = x_{k_1+1} \cdots x_{k_2}$ and $w_3 = x_{k_2+1} \cdots x_s$, we claim that

$$\{u \in (X^*)^+ : u\lambda = l_0,\, u\rho \geq v\rho,\, u\rho - u\pi \equiv r \mod d\} \subseteq w_1 y^* w_2 (y^* \cup (y^{-1})^*) w_3.$$

Firstly, we must have $v\pi < N$ since, otherwise, vy^{-1} would be in the same class, with a strictly lower π, as $v\pi + y^{-1}\lambda \geq N - \|y\| \geq 0 \geq v\lambda$ and $v\pi + y^{-1}\rho \leq v\pi + N < v\rho$. From $v\pi = (w_1w_2)\pi + w_3\pi \geq N + w_3\pi$, this implies that $w_3\pi < 0$. From the choice of k_2 this implies that $(w_1w_2)\pi < 2N$, and thus $0 \leq v\pi = (w_1w_2)\pi + w_3\pi < 2N + w_3\pi$. Therefore, $-2N < w_3\pi < 0$.

Let u be any element in this class. Since $u\pi \equiv 0 \equiv v\pi \mod d$ (by Proposition 3.2.1), there is some $n_1 \in \mathbb{N}_0$ such that $u\pi = v\pi + n_1d$, because, by choice of v, $v\pi \leq u\pi$. This also implies that $u\rho \equiv r \equiv v\rho \mod d$, hence, since we are disregarding the elements where $u\rho < v\rho$, there is $n_2 \in \mathbb{N}_0$ where $u\rho = v\rho + n_2d$. Note that $(w_1y^{n_2}w_2y^{n_1-n_2}w_3)\pi = w_1\pi + n_2d + w_2\pi + (n_1 - n_2)d + w_3\pi = v\pi + n_1d = u\pi$, and that

$$w_1\lambda - w_1\pi \leq -w_1\pi \leq -N \leq -\|y\| \leq y\lambda$$
$$w_2\rho = v\rho - w_1\pi \geq 3N - 2N = N \geq \|y\| \geq y\rho - y\pi.$$

Thus, we can apply Lemma 3.1.2 to conclude that

$$(w_1y^{n_2}w_2)\lambda = v\lambda = u\lambda, \quad \text{and} \quad (w_1y^{n_2}w_2)\rho = v\rho + n_2d = u\rho.$$

Next, we note that, by Proposition 2.2.3, $(w_1y^{n_2}w_2)\pi + y^{n_1-n_2}\lambda$ is greater than or equal to either:

$$(w_1y^{n_2}w_2)\pi - \|y\| = (w_1w_2)\pi + n_2d - \|y\| \geq N + n_2d - N \geq 0 \qquad \text{or}$$
$$(w_1y^{n_2}w_2)\pi + y^{n_1-n_2}\pi - \|y\| = u\pi - w_3\pi - \|y\| \geq v\pi - w_3\pi - \|y\| =$$
$$= (x_1 \cdots x_{k_2})\pi - \|y\| \geq N - N = 0.$$

Furthermore, $(w_1y^{n_2}w_2y^{n_1-n_2})\pi + w_3\lambda = (w_1w_2)\pi + w_3\lambda + n_1d \geq v\lambda + n_1d \geq v\lambda = u\lambda$. Thus we conclude that the factor $y^{n_1-n_2}w_3$ does not alter the product's λ.

On the other hand, by Proposition 2.2.3, $(w_1y^{n_2}w_2)\pi + y^{n_1-n_2}\rho$ is lesser than or equal to either

$$(w_1y^{n_2}w_2)\pi + \|y\| = (w_1w_2)\pi + n_2d + ||y|| \leq 3N + n_2d \leq v\rho + n_2d = u\rho \quad \text{or}$$
$$(w_1y^{n_2}w_2)\pi + y^{n_1-n_2}\pi + \|y\| \leq u\pi - w_3\pi + N \leq u\pi + 3N \leq u\rho.$$

Therefore, $y^{n_1-n_2}$ does not alter the product's ρ and so

$$(w_1y^{n_2}w_2y^{n_1-n_2}w_3)\lambda = u\lambda, \quad \text{and} \quad (w_1y^{n_2}w_2y^{n_1-n_2}w_3)\rho = u\rho.$$

In conclusion, $u = w_1 y^{n_2} w_2 y^{n_1-n_2} w_3 \in w_1 y^* w_2 (y^* \cup (y^{-1})^*) w_3$.

Case 3: covering elements $u \in (X^*)^+$ such that $u\rho - u\pi = r_0 < 3N$, $u\lambda \leq -3N$, and $u\lambda \equiv l \mod d, l \in \{0, \ldots, d-1\}$. Let S denote this class. Then, $S\beta$ is a set of the form in case 2. Thus, it can be written as a finite union $\cup_{i \in I} S'_i$ of sets of the desired form, with $y\beta$ being used instead of y. Therefore, $S = S\beta\beta = \cup_{i \in I}(S'_i\beta)$ is of the desired form.

Case 4: we have left to consider the elements $u \in (X^*)^+$ such that $u\lambda \leq -3N$, $u\lambda \equiv l \mod d$, $u\rho - u\pi \geq 3N$, and $u\rho - u\pi \equiv r \mod d, l, r \in \{0, \ldots, d-1\}$. We can find an element v which has the minimal π in this class, as Lemma 2.2.4 limits how large an element we must test (both λ and $\rho - \pi$ are limited). In fact, it is guaranteed that $v\pi = 0$: otherwise, since $v\pi \in d\mathbb{N}$, vy^{-1} would be another element in the same class with lesser π. We disregard elements with larger λ and smaller ρ than v, since they are finite in number.

As before, we know we can find an expression for v as a product of elements in X, $x_1 \cdots x_s$, where $i < j$, for some $i, j \in [s]$ such that $v\lambda = (x_1 \cdots x_{i-1})\pi + x_i\lambda$ and $v\rho = (x_1 \cdots x_{j-1})\pi + x_j\rho$. Let $k \in [s]$ be an index such that $i < k < j$ and $(x_1 \cdots x_k)\pi \in \{N, \ldots, 2N\}$. Lastly, set $w = x_1 \cdots x_k, w' = x_{k+1} \cdots x_s$. We claim that

$$\begin{aligned} & \{u \in (X^*)^+ : u\lambda \leq v\lambda,\ u\rho - u\pi \geq v\rho - v\pi,\ u\lambda \equiv l,\ u\rho - u\pi \equiv r \mod d\} = \\ = \quad & (y^{-1})^* w y^* w' (y^* \cup (y^{-1})^*). \end{aligned}$$

Note that, in accordance with Lemma 3.2.3, $v\lambda = w\lambda$ and $v\rho = w\pi + w'\rho$.

Let u be an element in the left-hand side of the equation. Then for some $n_1, n_2, n_3 \in \mathbb{N}_0$ it holds that

$$u\lambda = v\lambda - n_1 d, \quad u\pi = n_2 d, \quad u\rho = v\rho + n_3 d.$$

We shall prove that $u = y^{-n_1} w y^{n_1+n_3} w' y^{n_2-n_3}$. Clearly, $(y^{-n_1} w y^{n_1+n_3} w' y^{n_2-n_3})\pi = w\pi + w'\pi + n_2 d = u\pi$.

Now, we focus on $y^{-n_1} w y^{n_1}$. If $n_1 > 0$, in terms of λ,

$$\begin{aligned} y^{-n_1}\lambda = -(n_1 - 1)d + y^{-1}\lambda > -n_1 d - N \geq -n_1 d + v\lambda = u\lambda, \\ y^{-n_1}\pi + w\lambda = -n_1 d + v\lambda = u\lambda, \\ (y^{-n_1} w)\pi + y^{n_1}\lambda = -n_1 d + w\pi + y\lambda \geq -n_1 d + N - N = -n_1 d > u\lambda, \end{aligned}$$

and in terms of ρ,

$$y^{-n_1}\rho = y^{-1}\rho \leq N < u\rho,$$
$$y^{-n_1}\pi + w\rho \leq -n_1 d + v\rho < v\rho \leq u\rho,$$
$$(y^{-n_1}w)\pi + y^{n_1}\rho = -n_1 d + w\pi + (n_1 - 1)d + y\rho < w\pi + y\rho \leq 3N \leq u\rho.$$

From this, we conclude that $y^{-n_1}wy^{n_1}$ meets but does not go under $u\lambda$, nor does it surpass $u\rho$. Further, $(y^{-n_1}wy^{n_1})\pi = w\pi$ For $n_1 = 0$ we would directly obtain the same result.

Next, we consider the factor $y^{n_3}w'$, when $n_3 > 0$. On its λ:

$$y^{n_3}\lambda = y\lambda \geq -N,$$
$$y^{n_3}\pi + w'\lambda > w'\lambda,$$

and on its ρ:

$$y^{n_3}\rho = (n_3 - 1)d + y\rho < n_3 d + w'\rho = y^{n_3}\pi + w'\rho,$$

and so $(y^{n_3}w')\lambda \geq \min\{-N, w'\lambda\}$ and $(y^{n_3}w')\rho = n_3 d + w'\rho$. The same results are obtained if $n_3 = 0$, trivially. Thus,

$$(y^{-n_1}wy^{n_1})\pi + (y^{n_3}w')\lambda \geq w\pi + \min\{-N, w'\lambda\} \geq v\lambda \geq u\lambda,$$
$$(y^{-n_1}wy^{n_1})\pi + (y^{n_3}w')\rho = w\pi + n_3 d + w'\rho = v\rho + n_3 d = u\rho.$$

We have left to consider the impacts of $y^{n_2-n_3}$ on the products' coordinates. We have that

$$(y^{-n_1}wy^{n_1+n_3}w')\pi = v\pi + n_3 d \geq v\pi = 0,$$
$$(y^{-n_1}wy^{n_1+n_3}w')\pi = v\pi + n_3 d \leq v\rho - 3N + n_3 d = u\rho - 3N,$$
$$(y^{-n_1}wy^{n_1+n_3}w'y^{n_2-n_3})\pi = u\pi \in [0, u\rho - 3N].$$

Thus, by Proposition 2.2.3, $(y^{-n_1}wy^{n_1+n_3}w')\pi + y^{n_2-n_3}\lambda$ is no lesser than $0 - \|y\| \geq -N > u\lambda$, while, on the other hand, $(y^{-n_1}wy^{n_1+n_3}w')\pi + y^{n_2-n_3}\rho$ is no greater than $u\rho - 3N + \|y\| \leq u\rho - 2N < u\rho$.

In conclusion, the product $y^{-n_1}wy^{n_1+n_3}w'y^{n_2-n_3}$ has the same coordinates as u and so $u = y^{-n_1}wy^{n_1+n_3}w'y^{n_2-n_3} \in (y^{-1})^*wy^*w'(y^* \cup (y^{-1})^*)$.

This way, we have covered all elements in $(X^*)^+$ with four finite classes of rational sets of the desired form, each of which is contained in X^*, and explained what analogous process could be preformed to cover $(X^*)^-$. □

To give some insight into the process that has been developed, consider the case of $X = \{a^{-2}, a^2a^{-1}, a^2\}$. We shall name each of its elements as follows:

$$s = a^{-2} \qquad t = a^2a^{-1} \qquad u = a^2$$

Our goal is to cover $(X^*)^+$ with a finite collection of sets contained in X^*, each of which is of the form $u_0x^*u_1y^*u_2z^*u_3$, with $u_0, u_1, u_2, u_3, x, y, z \in F$. The elements t and st or ts are useful to alter the π of a product by ± 1.

Case 1: covering elements with $\lambda \geq -1,\ \rho - \pi \leq 1$.

$\lambda = 0, \rho - \pi = 0 : t\,t^*u$	covers all elements with $\pi \geq 3$
$\lambda = 1, \rho - \pi = 0 : tst\,t^*u$	covers all elements with $\pi \geq 2$
$\lambda = 0, \rho - \pi = 1 : t\,t^*$	covers all elements with $\pi \geq 1$
$\lambda = 1, \rho - \pi = 1 : tst\,t^*$	covers all elements with $\pi \geq 0$

This missed a finite number of elements we can consider individually, namely, in the case $\lambda = 0, \rho - \pi = 0$: $1, u$.

Case 2: covering elements with $\lambda \geq -1$ and any $\rho - \pi \geq 2$.

$$\lambda = 0 : t^*\,us(t \cup ts)^*$$
$$\lambda = 1 : tst\,t^*\,us\,(t \cup ts)^*$$

Case 3: covering elements with $\rho - \pi \leq 1$ and any $\lambda \leq -2$.

$$\rho - \pi = 0 : (st)^*\,s\,t^*\,u$$
$$\rho - \pi = 1 : (st)^*\,st\,t^*\,t$$

Case 4: covering elements where both λ and $\rho - \pi$ may be large.

$$(ts)^*\,su\,t^*\,us(t \cup ts)^*$$

For a general set X, this procedure could branch out further (we consider each possible λ and $\rho - \pi$ modulo $\gcd\{|x\pi| : x \in X\}$, as an example), but the global structure would be the same.

We are now in position to compile the previous results into the main theorem of the chapter.

Theorem 3.2.4. Let $L \in \mathrm{Rat}(F)$. Then L can constructively be written as a finite union of sets of the form

$$u_0x^*u_1y^*u_2z^*u_3$$

with $u_0, u_1, u_2, u_3, x, y, z \in F$.

Proof. Taking the results from Propositions 3.0.2, 3.1.1, 3.2.2 and 3.1.3, we conclude that L can constructively be written as a finite union of sets of the form

$$u_0x_1^*u_1\cdots x_k^*u_k$$

with $k \in \mathbb{N}_0$ and $u_0, \dots, u_k \in F, x_1, \dots, x_k \in F \setminus E(F)$ such that the signs of $x_i\pi$ are alternating. We have left to prove that we may always choose $k \leq 3$. By induction, it is enough to consider sets of the form

$$x_1^*u_0y_1^*u_1x_2^*u_2y_2^*$$

with $\mathrm{sign}(x_1\pi) = \mathrm{sign}(x_2\pi) \neq \mathrm{sign}(y_1\pi) = \mathrm{sign}(y_2\pi)$.

We start by assuming $x_i\pi > 0 > y_i\pi$ for $i \in [2]$.

Consider that an arbitrary element u of this set is of the form $u = x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2}u_3y_2^{m_2}$, for some $n_1, n_2, m_1, m_2 \in \mathbb{N}_0$.

We may suppose $n_1, n_2, m_1, m_2 > B$ for any $B \in \mathbb{N}_0$, as otherwise, this element is covered by one or more of the following sets:

$$\begin{array}{cc} x_1^bu_0y_1^*u_1x_2^*u_2y_2^* & x_1^*u_0y_1^bu_1x_2^*u_2y_2^* \\ x_1^*u_0y_1^*u_1x_2^bu_2y_2^* & x_1^*u_0y_1^*u_1x_2^*u_2y_2^b \end{array}$$

with $b \leq B$, which can be turned into the desired form by applying Proposition 3.1.3. Thus, we may consider only elements that have:

- a large enough m_1 so that $u\lambda$ is not obtained at u_1;
- a large enough n_2 so that $u\rho$ is not obtained at u_2;
- a large enough m_2 so that $u\lambda$ is not obtained at u_3.

Therefore, we have that

$$u\lambda = \min\{x_1\lambda,\ (x_1^{n_1}u_1y_1^{m_1-1})\pi + (y_1u_2x_2)\lambda,\ (x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2}u_3y_2^{m_2-1})\pi + y_2\lambda\},$$
$$u\rho = \max\{x_1^{n_1-1}\pi + (x_1u_1y_1)\rho,\ (x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2-1})\pi + (x_2u_3y_2)\rho\}.$$

Take N as the least multiple of $\operatorname{lcm}\{x_1\pi, x_2\pi, -y_1\pi, -y_2\pi\}$ that is greater than the norms of u_0, u_1, u_2, u_3 and x_1, x_2, y_1, y_2. Lastly, we define the following notation:

$$\begin{aligned} N_1 &= \frac{N}{x_1\pi}, & X_1 &= \{1, x_1, \ldots, x_1^{2N_1-1}\},\\ M_1 &= -\frac{N}{y_1\pi}, & Y_1 &= \{1, y_1, \ldots, y_1^{2M_1-1}\},\\ N_2 &= \frac{N}{x_2\pi}, & X_2 &= \{1, x_2, \ldots, x_2^{2N_2-1}\},\\ M_2 &= -\frac{N}{y_2\pi}, & Y_2 &= \{1, y_2, \ldots, y_2^{2M_2-1}\}. \end{aligned}$$

We separate our proof into cases, depending on which factor determine $u\rho$.

Case 1: suppose $u\rho = x_1^{n_1-1}\pi + (x_1u_1y_1)\rho$. We shall prove that u is an element of $x_1^*u_1y_1^*u_2X_2u_3y_2^*$ or $x_1^*u_1y_1^*u_2x_2^*u_3Y_2$. If $n_2 < 2N_2$ or $m_2 < 2M_2$, this is immediately evident. Suppose $n_2 \geq 2N_2$ and $m_2 \geq 2M_2$, we claim that

$$u = x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2}u_3y_2^{m_2} = x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2-N_2}u_3y_2^{m_2-M_2}.$$

This change removes the factors $x_2^{N_2}$ and $y_2^{M_2}$, which have π of respectively N and $-N$, so the product's π is unaltered.

Since $n_2 \geq 2N_2$, then $n_2 - N_2 \geq N_2$. Hence, $x_2^{n_2-N_2}\pi + u_3\lambda \geq N - \|u_3\| > 0$. Thus,

$$\begin{aligned} (x_2^{n_2-N_2}u_3y_2^{m_2-M_2})\lambda &= \min\{x_2\lambda, (x_2^{n_2-N_2}u_3y_2^{m_2-M_2-1})\pi + y_2\lambda\} =\\ &= \min\{x_2\lambda, (x_2^{n_2}u_3y_2^{m_2-1})\pi + y_2\lambda\} =\\ &= (x_2^{n_2}u_3y_2^{m_2})\lambda, \end{aligned}$$

and so λ is also not affected by the change.

Lastly, we have that $(x_2^{n_2-N_2}u_3y_2^{m_2-M_2})\rho < (x_2^{n_2}u_3y_2^{m_2})\rho$, since

$$x_2^{n_2-N_2}\rho = x_2^{n_2-N_2-1}\pi + x_2\rho < x_2^{n_2-1}\pi + x_2\rho = x_2^{n_2}\rho,$$
$$x_2^{n_2-N_2}\pi + u_3\rho < x_2^{n_2}\pi + u_3\rho,$$
$$(x_2^{n_2-N_2}u_3)\pi + y_2^{m_2-M_2}\rho = (x_2^{n_2-N_2}u_3)\pi + y_2\rho < (x_2^{n_2}u_3)\pi + y_2\rho = (x_2^{n_2}u_3)\pi + y_2^{m_2}\rho,$$

meaning that, since $u\rho$ was not (exclusively) obtained at the altered factor, it is still unchanged.

By repeatedly reducing the exponents of x_2 and y_2 by N_2 and M_2 respectively, we eventually end up at an expression for u where $n_2 < 2N_2$ or $m_2 < 2M_2$, and so u is in $x_1^*u_1y_1^*u_2X_2u_3y_2^*$ or $x_1^*u_1y_1^*u_2x_2^*u_3Y_2$, which are each a finite union of sets of the desired form (or one application of Proposition 3.1.3 away from it).

Case 2: suppose $u\rho = (x_1^{n_1}u_1y_1^{m_1}u_2x_2^{m_2-1})\pi + (x_2u_3y_2)\rho$. We shall prove that u is an element of $X_1u_1y_1^*u_2x_2^*u_3y_2^*$ or $x_1^*u_1Y_1u_2x_2^*u_3y_2^*$. If $n_1 < 2N_1$ or $m_1 < 2M_1$, this is immediately evident. Suppose $n_1 \geq 2N_1$ and $m_1 \geq 2M_1$, we claim that

$$u = x_1^{n_1}u_1y_1^{m_1}u_2x_2^{n_2}u_3y_2^{m_2} = x_1^{n_1-N_1}u_1y_1^{m_1-M_1}u_2x_2^{n_2}u_3y_2^{m_2}.$$

This change removes the factors $x_1^{N_1}$ and $y_1^{M_1}$, which have π of respectively N and $-N$, so the product's π is unaltered. This also ensures that the ρ and λ attained at the factors posterior to the change are not altered.

As before, $n_1 - N_1 \geq N_1$. Hence, $x_1^{n_1-N_1}\pi + u_1\lambda \geq N - \|u_1\| > 0$. Thus,

$$\begin{aligned}(x_1^{n_1-N_1}u_1y_1^{m_1-M_1})\lambda &= \min\{x_1\lambda, (x_1^{n_1-N_1}u_1y_1^{m_1-M_1-1})\pi + y_1\lambda\} = \\ &= \min\{x_1\lambda, (x_1^{n_1}u_1y_1^{m_1-1})\pi + y_1\lambda\} = \\ &= (x_1^{n_1}u_1y_1^{m_1})\lambda,\end{aligned}$$

and so λ is also not affected by the change.

Lastly, we have that $(x_1^{n_1-N_1}u_1y_1^{m_1-M_1})\rho < (x_1^{n_1}u_1y_1^{m_1})\rho$, since

$$x_1^{n_1-N_1}\rho = x_1^{n_1-N_1-1}\pi + x_1\rho < x_1^{n_1-1}\pi + x_1\rho = x_1^{n_1}\rho,$$

$$x_1^{n_1-N_1}\pi + u_1\rho < x_1^{n_1}\pi + u_1\rho,$$

$$(x_1^{n_1-N_1}u_1)\pi + y_1^{m_1-M_1}\rho = (x_1^{n_1-N_1}u_1)\pi + y_1\rho < (x_1^{n_1}u_1)\pi + y_1\rho = (x_1^{n_1}u_1)\pi + y_1^{m_1}\rho,$$

meaning that, since $u\rho$ was not (exclusively) obtained at the altered factor, it is still unchanged.

By repeatedly reducing the exponents of x_1 and y_1 by N_1 and M_1 respectively, we eventually end up at an expression for u where $n_1 < 2N_1$ or $m_1 < 2M_1$, and so u is in $X_1u_1y_1^*u_2x_2^*u_3y_2^*$ or $x_1^*u_1Y_1u_2x_2^*u_3y_2^*$, which are each a finite union of sets of the desired form (or one application of Proposition 3.1.3 away from it).

Assuming $x_i\pi < 0 < y_i\pi$, for $i \in [2]$, we consider its inverse $(x_1^*u_1y_1^*u_2x_2^*u_3y_2^*)^{-1} = (y_2^{-1})^*u_3^{-1}(x_2^{-1})^*u_2^{-1}(y_1^{-1})^*u_1^{-1}(x_1^{-1})^*$, which is a set of the form in study and falls into the previous case. Thus, it can be decomposed into a union of sets of the desired form. Then, applying $^{-1}$ to a regular expression for each, we cover the original set, $x_1^*u_1y_1^*u_2x_2^*u_3y_2^*$, with sets of the desired form. □

Lastly, we note that, in some sense, the description of rational sets of F provided by Theorem 3.2.4 is the most precise possible.

Remark. In general, a rational set of F cannot be written as a finite union of sets of the form ux^*vy^*w with $u, v, w, x, y \in F$. As an example, consider $L \in \mathrm{Rat}(F)$, which contains cofinitely many idempotents. There exist such rational sets: F itself is an example and, later, Proposition 4.0.6 will produce an infinite family.

Suppose that L admits such a decomposition. Then, we may write

$$L = \bigcup_{i \in I} u_i x_i^* v_i y_i^* w_i$$

for some finite set I. Define $L_i = u_i x_i^* v_i y_i^* w_i$ for each $i \in I$. Fix some $i \in I$. To simplify notation, we shall omit i from the indexes of elements.

Case 1: if $x, y \in F^+$ or $x, y \in F^-$, by Proposition 3.1.3, L_i can be written as a finite union of sets of the form $u'z^*v'$, for $u', v', z \in F$. For each such set, if $z \in E(F)$ it is finite, and otherwise, since $u'z^nv'$ is an idempotent if and only if $(u'v')\pi + n(z\pi) = 0$, which has at most one solution for $z\pi \neq 0$, it contains at most one idempotent. Hence, L_i contains a finite number of idempotents.

Case 2: if $x\pi > 0 > y\pi$, since ux^nvy^mw is idempotent if and only if $(uvw)\pi + n(x\pi) + m(y\pi) = 0$, for each $n \in \mathbb{N}_0$, there is at most one $m_n \in \mathbb{N}_0$ such that $ux^nvy^{m_n}w \in E(F)$. For sufficiently large n, as $x\pi > 0$, λ of the product is never obtained at v, and so (if m_n exists)

$$(ux^nvy^{m_n}w)\lambda = \min\{(ux)\lambda, y\lambda - y\pi - w\pi, w\lambda - w\pi\}.$$

Importantly, this value does not depend on n, and so L_i includes infinitely many idempotents, but only finitely many values of λ.

Case 3: if $x\pi < 0 < y\pi$, by the same process, we would obtain the dual conclusion: L_i includes infinitely many idempotents, but only finitely many values of ρ.

Thus, for some value of ρ, r_0, large enough such that only L_i's from case 2 can reach it, $L \cap \{e \in E(F) : e\rho = r_0\} = \cup_{i \in I}(L_i \cap \{e \in E(F) : e\rho = r_0\})$ must be finite. Thus, $\{e \in E(F) : e\rho = r_0\} \setminus L \subseteq E(F) \setminus L$ is infinite, which contradicts the condition imposed on L.

4. Intersection of rational sets

In this chapter we tackle questions about the intersection of rational sets in F.

To start with, we show that the question of whether or not two rational sets meet is decidable.

Proposition 4.0.1. Let $L, K \in \mathrm{Rat}(F)$. It is decidable whether or not $L \cap K = \varnothing$.

Proof. Let n_L, n'_L and n_K, n'_k be the constants given by Lemma 2.2.4 for L and K respectively. By Corollary 2.2.5, we may choose $n = \mathrm{lcm}(n_L, n_K)$ and $n' = \max(n'_L, n'_K) + n$ so that 2.2.1 holds for L and K simultaneously. We begin by showing that

$$L^+ \cap K^+ = \varnothing \Leftrightarrow (L^+ \cap W_{n'}) \cap (K^+ \cap W_{n'}) = \varnothing.$$

The direct implication follows trivially.

The converse we show by its contrapositive. Suppose that there is $u \in L^+ \cap K^+$. By choice of n and n',

$$u \in L^+ \Rightarrow u\zeta_{n,n'} \in L \cap W_{n'}$$
$$u \in K^+ \Rightarrow u\zeta_{n,n'} \in K \cap W_{n'}$$

Thus, $u\zeta_{n,n'} \in (L^+ \cap W_{n'}) \cap (K^+ \cap W_{n'})$, so it is not empty.

By applying the same process to $(L^-)^{-1} \cap (K^-)^{-1}$, we conclude that

$$L^- \cap K^- = \varnothing \Leftrightarrow (L^- \cap W_{n'}^{-1}) \cap (K^- \cap W_{n'}^{-1}) = \varnothing.$$

Thus,

$$\begin{aligned} L \cap K = \varnothing \Leftrightarrow &(L^+ \cap W_{n'}) \cap (K^+ \cap W_{n'}) = \varnothing \\ &\wedge (L^- \cap W_{n'}^{-1}) \cap (K^- \cap W_{n'}^{-1}) = \varnothing. \end{aligned}$$

Since $W_{n'}$ is finite, and membership in L and K is decidable, the truth value of the right side of the equivalence can be determined. ◻

Next, we show that the intersection between two rational sets may not be rational. For that, we begin by proving the following proposition which is mostly helpful in showing that certain sets are not rational.

Proposition 4.0.2. Let $L \in \mathrm{Rat}(FIM_A)$ for some alphabet A. Let $\theta_A : \bar{A}^* \to FIM_A$ and $\pi_A : FIM_A \to FG_A$ denote the natural projections. The following conditions are equivalent:

1. L is finite;
2. $L\pi_A$ is finite;
3. in any regular expression for L, whenever the star operator is applied to a set X, $X \subseteq E(FIM_A)$;
4. there is a regular expression for L where the star operator is only applied to sets of idempotents;
5. for any trim $\bar{A}$-automaton $\mathcal{A}$ such that $L(\mathcal{A})\theta_A = L$, the label of any loop in $\mathcal{A}$ maps to an idempotent in FIM_A;
6. there is a trim $\bar{A}$-automaton $\mathcal{A}$ such that $L(\mathcal{A})\theta_A = L$ and the label of a loop in $\mathcal{A}$ maps to an idempotent in FIM_A.

Proof. During this proof we shall omit the mention of the alphabet in the maps θ_A and π_A, as it is fixed, and abuse notation by applying π directly to elements of $\bar{A}^*$, to mean $\theta\pi$.

The implications (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) are trivial.

We prove (2) $\Rightarrow$ (4) by its contrapositive. Suppose there is $\bar{L} \in \mathrm{Rat}(\bar{A}^*)$, $X \subseteq \bar{A}^*$ and $x \in X$, such that:

- $\bar{L}\theta = L$;
- X^* occurs in some regular expression describing $\bar{L}$;
- $x\theta$ is not an idempotent.

Under these conditions, there are $y, z \in \bar{A}^*$ such that $yx^*z \subseteq \bar{L}$; that is, for any $n \in \mathbb{N}_0$, $yx^nz \in \bar{L}$. In addition, $(yx^nz)\pi = (yx^mz)\pi \Leftrightarrow x^n\pi = x^m\pi \Leftrightarrow 1_{FG_A} = (x^{m-n})\pi \Leftrightarrow m - n = 0$, so yx^nz is unique for each $n \in \mathbb{N}_0$ and so $L\pi = \bar{L}\pi$ is infinite.

On (4) $\Rightarrow$ (1), it is enough to prove that if $X \subseteq E(FIM_A)$ is a finite set, then X^* is still finite.

Suppose the elements of X are numbered as $x_1, \dots, x_n$, $n = |X|$. For any $x_{i_1} \cdots x_{i_k} \in X^*$, since idempotents commute, we can assume $i_1 < \cdots < i_k$. This way, each element in X^* can be characterized (not necessarily uniquely) by $\{i_1, \dots, i_k\} \in \mathcal{P}([n])$. Therefore, $|X^*| \leq 2^{|X|}$ which means X^* is finite.

We consider the contrapositive of (6) $\Rightarrow$ (2): suppose $L\pi$ infinite and let $\mathcal{A} = (Q, I, T, E)$ be a trim $\bar{A}$-automaton such that $L(\mathcal{A})\theta = L$.

Denote by $\|g\|$ the number of symbols in the reduced word for $g \in FG_A$. Given $u \in FIM_A$, $\|u\pi\|$ corresponds to the length of the shortest path between $\circ$ and $\bullet$ in the Munn tree for u, so u is idempotent if and only if $\|u\pi\| = 0$. Moreover, $\|u\pi\| \leq \|u\|$.

Let $m = |Q|$. Since $L\pi$ is infinite, there must be $u \in L$ with $\|u\pi\| > m$. In $\mathcal{A}$, there is some sequence of edges

$$\rightarrow q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} q_n \rightarrow$$

such that $q_0 \in I$, $q_n \in T$, each $a_i \in \bar{A}, i \in [n]$ and $(a_1 \cdots a_n)\theta = u$. Since $m < \|u\pi\| \leq \|u\| \leq n$, there must be indexes $i < j$ where $q_i = q_j$.

If $a_{i+1} \cdots a_j$ is not an idempotent in FIM_A, we have found a loop in the desired conditions.

Otherwise, consider $u' = a_1 \cdots a_i a_{j+1} \cdots a_n \in L$. Since $(a_{i+1} \cdots a_j)\pi = 1_{FG_A}$, $u\pi = u'\pi$, so we may repeat the process considering u' instead of u, until a desired loop is found.

Lastly, regarding the contrapositive of (2) $\Rightarrow$ (5), suppose there is some trim $\bar{A}$-automaton $\mathcal{A} = (Q, I, T, E)$ whose language is mapped to F and it contains a loop at some vertex p with label $u \in \bar{A}^*$, such that $u\theta$ is not idempotent. Since $\mathcal{A}$ is trim, there is a walk from some initial vertex to p, with label x, and from p to some terminal vertex, with label y.

For each $n \in \mathbb{N}_0$, $xu^n y \in L(\mathcal{A})$ and so $(xu^n y)\theta \in L$. Since $u\theta$ is not idempotent, each $(xu^n y)\pi$ is distinct. Therefore, $L\pi$ is infinite.

$\square$

Remark. Given $L, K \in \mathrm{Rat}(F)$, $L \cap K$ may not be rational.

Example 2. Consider $L = aa^{-1}(a^{-1})^* a^*$ and $K = (a^{-1})^* a^* aa^{-1}$.

An element $u \in L$ may be written as $aa^{-1}a^{-n}a^m$, for some $n, m \in \mathbb{N}_0$. In this case, we conclude that:

- $u\lambda = ((aa^{-1})(a^{-n}a^m))\lambda = \min\{0, 0+(a^{-n}a^m)\lambda\} = \min\{0, \min\{-n, -n+0\}\} = -n$;
- $u\pi = 1 - 1 - n + m = -n + m$;
- $u\rho = ((aa^{-1})(a^{-n}a^m))\rho = \max\{1, 0 + (a^{-n}a^m)\rho\} = \max\{1, \max\{0, -n + m\}\} = \max\{1, u\pi\}$.

Since $n, m \in \mathbb{N}_0$ are arbitrary, we conclude that an element $u \in L$ may be described as: $u\lambda$ and $u\pi$ are arbitrary (as long as the general rule of $u\lambda \leq u\pi$ is kept), and $u\rho = \max\{1, u\pi\}$; that is, $L = \{u \in F : u\rho = \max\{1, u\pi\}\}$.

Analogously, we may describe an element $u \in K$ as: $u\lambda \leq u\pi$ are arbitrary, and $u\rho = \max\{0, u\pi + 1\}$; that is, $K = \{u \in F : u\rho = \max\{0, u\pi + 1\}\}$.

Suppose now that $u \in L \cap K$. Then $\max\{1, u\pi\} = u\rho = \max\{0, u\pi + 1\}$. Consider the following cases:

- if $u\pi \geq 1$, $u\pi = u\rho = u\pi + 1$, so no element in $L \cap K$ may have positive π;
- if $u\pi < 0$, then $1 = u\rho = 0$, which again is absurd.

So $L \cap K$ coincides with $\{u \in F : u\pi = 0 \wedge u\rho = 1\}$. Thus, $L \cap K$ is infinite, while $(L \cap K)\pi = \{0\}$ is finite. Therefore, by Proposition 4.0.2, $L \cap K$ cannot be rational.

On the other hand, we are able to identify a wider class of languages containing every intersection of rational sets.

Theorem 4.0.3. Let $L \subseteq F$ be a set such that there are $n', n \in \mathbb{N}$ for which $\star(L, n, n')$ holds. Then L is context-free.

Proof. Define the following sets (if membership in L is decidable, all are computable since $W_{n'}$ is finite):

$$\begin{aligned} O_0 =& L \cap W_{n'} \\ O_1 =& O_0 \cap G_{n'-n,1} \\ O_2 =& O_0 \cap G_{n'-n,2} \\ O_3 =& O_0 \cap G_{n'-n,3}. \end{aligned}$$

Notice that $v\xi_{n,i} \in G_{n'-n,i}$ if and only if $v \in G_{n',i}$, for each $i \in [3]$, so, from $\star(L,n,n')$, an element $u \in F$ is in $O_i, i \in [3]$, if and only if it can be obtained by $v\xi_{n,i}$ for some $v \in L$, and it is a fixed point by $\zeta_{n,n'}$.

Consider also the sets

$$\begin{aligned}
O_{1,2} =& O_1 \cap O_2 \\
O_{1,3} =& O_1 \cap O_3 \\
O_{2,3} =& O_2 \cap O_3 \\
O_{1,2,3} =& O_1 \cap O_2 \cap O_3.
\end{aligned}$$

Lastly, we define the following sets:

$$\begin{array}{rclll}
\mathcal{O}_i = \{ & u\xi_{n,i}^{-k_i} & : u \in O_i,\ k_i \in \mathbb{N}\}, & & \text{for } i \in [3] \\
\mathcal{O}_{i,j} = \{ & u\xi_{n,i}^{-k_i}\xi_{n,j}^{-k_j} & : u \in O_{i,j},\ k_i, k_j \in \mathbb{N}\}, & & \text{for } i,j \in [3], i<j \\
\mathcal{O}_{1,2,3} = \{ & u\xi_{n,1}^{-k_1}\xi_{n,2}^{-k_2}\xi_{n,3}^{-k_3} & : u \in O_{1,2,3},\ k_1,k_2,k_3 \in \mathbb{N}\}. & &
\end{array}$$

These sets can be written in the following ways:

$$\begin{aligned}
\mathcal{O}_1 \quad &= \quad \{v \in F : v\lambda = u\lambda - k_1 n, v\pi = u\pi, v\rho = u\rho, \text{ for some } u \in O_1,\ k_1 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-k_1 n}a^{k_1 n}a^{-u\lambda}a^{u\rho}a^{u\pi - u\rho} : u \in O_1,\ k_1 \in \mathbb{N}\} \\
\mathcal{O}_2 \quad &= \quad \{v \in F : v\lambda = u\lambda, v\pi = u\pi + k_2 n, v\rho = u\rho + k_2 n, \text{ for some } u \in O_2,\ k_2 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-u\lambda}a^{u\rho}a^{k_2 n}a^{u\pi - u\rho} : u \in O_2,\ k_2 \in \mathbb{N}\} \\
\mathcal{O}_3 \quad &= \quad \{v \in F : v\lambda = u\lambda, v\pi = u\pi, v\rho = u\rho + k_3 n, \text{ for some } u \in O_3,\ k_3 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-u\lambda}a^{u\rho}a^{k_3 n}a^{-k_3 n}a^{u\pi - u\rho} : u \in O_3,\ k_3 \in \mathbb{N}\} \\
\mathcal{O}_{1,2} \quad &= \quad \{v \in F : v\lambda = u\lambda - k_1 n, v\pi = u\pi + k_2 n, v\rho = u\rho + k_2 n, \\
&\qquad\qquad \text{for some } u \in O_{1,2},\ k_1, k_2 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-k_1 n}a^{k_1 n}a^{-u\lambda}a^{u\rho}a^{k_2 n}a^{u\pi - u\rho} : u \in O_{1,2},\ k_1, k_2 \in \mathbb{N}\} \\
\mathcal{O}_{1,3} \quad &= \quad \{v \in F : v\lambda = u\lambda - k_1 n, v\pi = u\pi, v\rho = u\rho + k_3 n, \\
&\qquad\qquad \text{for some } u \in O_{1,3},\ k_1, k_3 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-k_1 n}a^{k_1 n}a^{-u\lambda}a^{u\rho}a^{k_3 n}a^{-k_3 n}a^{u\pi - u\rho} : u \in O_{1,3},\ k_1, k_2 \in \mathbb{N}\} \\
\mathcal{O}_{2,3} \quad &= \quad \{v \in F : v\lambda = u\lambda, v\pi = u\pi + k_2 n, v\rho = u\rho + (k_2 + k_3), \\
&\qquad\qquad \text{for some } u \in O_{2,3},\ k_2, k_3 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-u\lambda}a^{u\rho}a^{(k_2+k_3)n}a^{-k_3 n}a^{u\pi - u\rho} : u \in O_{2,3},\ k_2, k_3 \in \mathbb{N}\} \\
\mathcal{O}_{1,2,3} \quad &= \quad \{v \in F : v\lambda = u\lambda - k_1 n, v\pi = u\pi + k_2 n, v\rho = u\rho + (k_2 + k_3)n, \\
&\qquad\qquad \text{for some } u \in O_{1,2,3},\ k_1, k_2, k_3 \in \mathbb{N}\} \\
&= \quad \{a^{u\lambda}a^{-k_1 n}a^{k_1 n}a^{-u\lambda}a^{u\rho}a^{(k_2+k_3)n}a^{-k_3 n}a^{u\pi - u\rho} : u \in O_{1,2,3},\ k_1, k_2, k_3 \in \mathbb{N}\}.
\end{aligned}$$

We shall prove that $L^+ = O_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_{1,2} \cup \mathcal{O}_{1,3} \cup \mathcal{O}_{2,3} \cup \mathcal{O}_{1,2,3}$.

Let $v \in L^+$. If $v \in W_{n'}$, then $v \in O_0$. Suppose now that $v \notin W_{n'}$ and set $u = v\zeta_{n,n'}$. We have that $u = v\xi_{n,1}^{k_1}\xi_{n,2}^{k_2}\xi_{n,3}^{k_3}$, for some $k_1, k_2, k_3 \in \mathbb{N}_0$, not all zero (otherwise, $v \in W_{n'}$). Therefore:

- $v\lambda = u\lambda - k_1 n$;
- $v\pi = u\pi + k_2 n$;
- $v\rho = u\rho + k_3 n$.

All that is left to prove is that, for $i \in [3]$, if $k_i > 0$, then $u \in O_i$. Firstly, when each $\xi_{n,i}, i \in [3]$ was applied, the previous element was still in $G_{n',i}$, and so Lemma 2.2.4 implies that the resulting element is still in L. This induction justifies that $u \in L$.

Since u is the result of applying $\zeta_{n,n'}$ to some element in F^+, $u\lambda > -n'$ and $u\pi, u\rho - u\pi < n'$.

Suppose $k_1 > 0$. Since $v\xi_{n,1}^{k_1-1}$ was reduced further, $v\xi_{n,1}^{k_1-1}\lambda \leq -n'$ and so $v\xi_{n,1}^{k_1}\lambda \leq -n' + n$. As $\xi_{n,2}$ and $\xi_{n,3}$ do not affect λ, $u\lambda = v\xi_{n,1}^{k_1}\lambda \leq -n' + n$. In conclusion, $-n' < u\lambda \leq -n' + n$, so $u \in W_n' \cap G_{n'-n,1}$ and thus $u \in O_1$.

Considering $\xi_{n,1}, \xi_{n,2}$ and $\xi_{n,3}$ commute, we may proceed analogously as follows:

Denote λ by γ_1, π by γ_2 and $\rho - \pi$ by γ_3, let $i = 2, 3$ and let $j \in \{2, 3\}$ such that $i \neq j$. Suppose $k_i > 0$. Since $v\xi_{n,i}^{k_i-1}$ was reduced further, $v\xi_{n,i}^{k_i-1}\gamma_i \geq n'$ and so $v\xi_{n,i}^{k_i}\gamma_i \geq n' - n$. In conclusion, as $\xi_{n,1}$ and $\xi_{n,j}$ do not affect γ_i, $n' - n \leq u\gamma_i < n'$, so $u \in W_n' \cap G_{n'-n,i}$ and thus $u \in O_i$.

So, independently of which $k_i, i \in [3]$, are non-zero, we have ensured that u belongs to the correct $O_i, i \in [3]$, to allow for v to be in some $\mathcal{O}_i, \mathcal{O}_{i,j}$ or $\mathcal{O}_{1,2,3}, i, j \in [3], i < j$.

Now, let $v \in O_0$. By definition, $v \in L$. Let $v \in \mathcal{O}_s$ for $s \in \mathcal{P}([3]) \setminus \{\varnothing\}$), and let $u \in O_s$ be the element such that $v = u\xi_{n,1}^{-k_1}\xi_{n,2}^{-k_2}\xi_{n,3}^{-k_3}$, for $k_1, k_2, k_3 \in \mathbb{N}_0$, with $k_i \neq 0$ if and only if $i \in s$.

For each $i \in s$, $u \in O_i$, so $u\xi_{n,i}^{-1} \in G_{n',i}$ and thus, according to Lemma 2.2.4, $u\xi_{n,i}^{-1} \in L$. Repeating this process k_i times, we obtain that $u\xi_{n,i}^{-k_i} \in L$. Again, since $\xi_{n,j}^{-1}$ for $j \neq i$ does not alter γ_i and commutes with $\xi_{n,i}^{-1}$, we obtain that $v = u\xi_{n,1}^{-k_1}\xi_{n,2}^{-k_2}\xi_{n,3}^{-k_3} \in L$.

Therefore, we obtain the claimed set equality. The set O_0 is finite. As seen in the last description for each $\mathcal{O}_s, s \in \mathcal{P}([3]) \setminus \{\varnothing\}$, they are context-free, as they are recognized by the grammars $G_s = (V_s, \{1, a, a^{-1}\}, R_s, \varepsilon)$, with $V_s = \{\varepsilon, \alpha.\beta, \gamma\}$ and R_s:

$$
\begin{aligned}
&\varepsilon \to a^{u\lambda}\alpha a^{-u\lambda}a^{u\rho}\beta\gamma a^{u\pi-u\rho} && \text{for each } u \in O_s\\
&\alpha \to a^{-n}\alpha a^{n} && \text{if } 1 \in s\\
&\beta \to a^{n}\beta && \text{if } 2 \in s\\
&\gamma \to a^{n}\gamma a^{-n} && \text{if } 3 \in s\\
&\alpha \to 1 &&\\
&\beta \to 1 &&\\
&\gamma \to 1. &&
\end{aligned}
$$

So L^+ is a union of context-free sets, and thus is context-free.

Repeating the previous steps of the proof taking in consideration $L^{-1} \in \mathrm{Rat}(F)$, we obtain that $(L^{-1})^+ = L^-$ is context-free, and thus $L = L^+ \cup L^-$ is context-free. □

Corollary 4.0.4*.* Given $L_1, \ldots, L_k \in \mathrm{Rat}(F)$, $\cap_{i=1}^{k} L_i$ is context-free.

Proof. We may compute n, n' such that $\star$ holds for all L_i, $i \in [k]$, namely: for each $i \in [k]$, let n_i, n_i' be the constants given by Lemma 2.2.4, and set $n = \mathrm{lcm}\{n_i : i \in [k]\}$ and $n' = \max\{n_i' : i \in [k]\} + n$.

By Corollary 2.2.5, we have that $\star(L_i, n, n')$ for all $i \in [k]$. Thus $\star(\cap_{i=1,\ldots,k} L_i, n, n')$ also holds. Therefore, by Theorem 4.0.3, $\cap_{i=1,\ldots,k} L_i$ is context-free. □

Lastly, we provide two families of rational sets for which the intersection with another rational set is still rational.

Proposition 4.0.5*.* Let M be a finitely generated monoid, and suppose I is a cofinite ideal of M. Then, for any $L \in \mathrm{Rat}(M)$, $L \cap I$ is rational and constructible.

Proof. In fact, $I \in \mathrm{Rec}(M)$: since I is cofinite, the Rees quotient M/I is a finite monoid, and $I = (I\theta_I)\theta_I^{-1}$. Thus, $L \cap I$ is rational and constructible. □

Corollary 4.0.6*.* Given $L \in \mathrm{Rat}(FIM_A)$, for some alphabet A, and $X \subseteq FIM_A$ finite, then $L \setminus X$ is rational and constructible.

Proof. Only finitely many elements of $\bar{A}$ occur in any regular expression for L. Let $\bar{A}'$ be a finite alphabet that can write L. We may assume that $X \subseteq FIM_{A'}$, as other elements in X would not belong to L.

Set $N = \max\{\|x\| : x \in X\}$. Let $S_N = \{u \in FIM_{A'} : \|u\| \leq N\}$. Then $I = FIM_{A'} \setminus S_N$ is a cofinite ideal, since the norm in $FIM_{A'}$ can not decrease under multiplication, and for finite A', the set $\{u \in FIM_{A'} : \|u\| = l\}$ is finite for all $l \in \mathbb{N}_0$. Thus, by Proposition 4.0.5, $L \cap I$ is rational and constructible.

Since S_N is finite and equality is decidable in $FIM_{A'}$ we can compute the rational set $L \cap (S_N \setminus X)$. Therefore,

$$L \setminus X = (L \cap (S_N \setminus X)) \cup (L \cap I)$$

which is a union of two computable rational sets, and so one itself. □

Proposition 4.0.7. Take $L \in \mathrm{Rat}(F)$ which can be written as a finite union of sets of the form

$$ux^*v \tag{4.0.1}$$

for some $u, v, x \in F$. Then, for any $K \in \mathrm{Rat}(F)$, $L \cap K$ is rational.

Proof. Let $K \in \mathrm{Rat}(F)$. If $L \in \mathrm{Rat}(F)$ is of the form

$$\bigcup_{i \in I} u_i x_i^* v_i$$

for some finite set of indices I, and, for each $i \in I, u_i, v_i, x_i \in F$, then

$$L \cap K = \bigcup_{i \in I} (u_i x_i^* v_i \cap K),$$

and so, since a finite union of rational sets is rational, it is enough to show that the result holds for $L = ux^*v,\ u, v, x \in F$.

If $x \in E(F)$, then $L = ux^*v = \{uv, uxv\}$, and so $L \cap K$ is either empty or finite, and thus rational.

Otherwise, suppose $x\pi > 0$ and set $n_0 = x\pi$. For each $k \in \mathbb{N}_0$, denote by x_k the element $ux^k v \in L$. For each $k \in \mathbb{N}_0$, we have the following equalities:

$$\begin{aligned}
x_k\pi =& u\pi + v\pi + kn_0 \\
x_k\lambda =& \min\{u\lambda, u\pi + x^k\lambda, (ux^k)\pi + v\lambda\} \\
=& \min\{u\lambda, u\pi + x\lambda, u\pi + kn_0 + v\lambda\} \\
x_k\rho =& \max\{u\rho, u\pi + x^k\rho, (ux^k)\pi + v\rho\} \\
=& \max\{u\rho, u\pi + (k-1)n_0 + x\rho, u\pi + kn_0 + v\rho\}.
\end{aligned}$$

Since $n_0 > 0$, there is $k_0 \in \mathbb{N}$ such that for $k > k_0$,

$$\begin{aligned}
x_k\lambda =& \min\{u\lambda, u\pi + x\lambda\} \\
x_k\pi \geq& 0 \\
x_k\rho =& \max\{u\pi + (k-1)n_0 + x\rho, u\pi + kn_0 + v\rho\} \\
x_k\rho - x_k\pi =& - n_0 + v\pi + \max\{x\rho, n_0 + v\rho\}.
\end{aligned}$$

Let $L_0 = \{x_k : k \leq k_0\}$ and $L' = \{x_k : k > k_0\}$. Clearly, $L = L_0 \cup L'$, and, since L' can be obtained by removing a finite set from L, by Proposition 4.0.6, L' is rational.

From Lemma 2.2.4, we can obtain some constants $n_{L'}, n'_{L'}$ such that $\star(L', n_{L'}, n'_{L'})$ and by Corollary 2.2.5, we may assume that $n_0 | n_{L'}$ and $n'_{L'} > -\min\{u\lambda, u\pi + x\lambda\} + n_{L'}$, $-n_0 + v\pi + \max\{x\rho, n_0 + v\rho\} + n_{L'}$.

Taking now new constants n', n such that $n' > n'_{L'} + n$ and $n_{L'} | n$ and $\star(L' \cap K, n, n')$, then, recalling the notation from the proof of Theorem 4.0.3, applied to $L \cap K$, we have that $O_1 = O_3 = \varnothing$, since L' does not contain any elements with λ of at most $-n' + n$, nor elements with $\rho - \pi$ of at least $n' - n$. Therefore,

$$\begin{aligned}
L' \cap K =& O_0 \cup \mathcal{O}_2 = \\
=& O_0 \cup \{a^{w\lambda} a^{-w\lambda} a^{w\rho} a^{-w\rho} a^{w\pi} a^{in} : w \in O_2, i \in \mathbb{N}\} = \\
=& O_0 \cup \bigcup_{w \in O_2} a^{w\lambda} a^{-w\lambda} a^{w\rho} a^{-w\rho} a^{w\pi + n} (a^n)^*
\end{aligned}$$

so, since O_2 is finite, $L' \cap K$ is a finite union of rational sets, and thus rational.

On the other hand, $L_0 \cap K$ is finite (and thus rational). So

$$L \cap K = (L_0 \cap K) \cup (L' \cap K)$$

is rational. □

Even with the results obtained in this chapter regarding intersections of rational sets in F, some open questions remain, such as:

- Is it decidable whether the intersection of two given rational sets is still rational?
- Are recognizable sets, sets of the form ux^*v and their finite unions (of which we proved cofinite sets are a particular case) the only rational sets for which the intersection with any other rational set is rational?

5. Beyond F

Lastly, we provide a simple generalization of a previous result to direct products of appropriate monoids.

Corollary 5.0.1. Let M_1 and M_2 be finitely generated, finite $\mathscr{J}$-above monoids, such that, for each $i = 1, 2$:

- M_i has decidable word problem;*
- the $\mathscr{J}$-order on M_i is decidable.

Then the membership problem for rational subsets of $M = M_1 \times M_2$ is decidable.

Proof. Let G_1 and G_2 denote a finite generating set and e_1 and e_2 the identity of M_1 and M_2, respectively. Then the set

$$\{(g, e_2) : g \in G_1\} \cup \{(e_1, g) : g \in G_2\}$$

is a finite generating set for M.

For the rest of this proof, let u and v be arbitrary elements in M, and let $u_1, v_1 \in M_1$ and $u_2, v_2 \in M_2$ be such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

The equality $u = v$ holds if and only if $u_1 = v_1$ and $u_2 = v_2$. Since the word problem is decidable in M_1 and M_2, the word problem in M is decidable.

Consider now that $u \leq_{\mathscr{J}} v$ if and only if there are $s, t \in M$ such that $u = svt$, that is, exist $s_1, t_1 \in M_1, s_2, t_2 \in M_2$ such that $u_1 = s_1 v_1 t_1$ and $u_2 = s_2 v_2 t_2$, which coincides with $u_1 \leq_{\mathscr{J}} v_1$ and $u_2 \leq_{\mathscr{J}} v_2$. Since $\leq_{\mathscr{J}}$ is decidable in M_1 and M_2, we can decide whether $u \leq_{\mathscr{J}} v$.

Furthermore, $u\mu = (u_1\mu) \times (u_2\mu)$, therefore it is finite.

Thus, M fulfills all the necessary conditions to apply Theorem 2.2.1, resulting in the decidability of membership to its rational sets. □

By repeatedly applying the previous corollary we obtain the same result for any finite direct product of such monoids. In particular, we may apply it to free inverse monoids as follows.

*As in Theorem 2.2.1, it is unnecessary to state this condition. We mantain it for consistency.

Corollary 5.0.2. For any $n \in \mathbb{N}$ and alphabets $A_1, \dots, A_n$, $M = FIM_{A_1} \times \cdots \times FIM_{A_n}$ has decidable membership for rational subsets.

Proof. For each $L \in \mathrm{Rat}(M)$, only finitely many generators occur in a regular expression for L, and so we may assume each alphabet is finite. Thus, we may assume as well that M is finitely generated.

By Theorem 2.1.1, the word problem is decidable in each component. By Lemma 2.1.2, the $\mathscr{J}$-order is decidable in each component.

By the same argument as in the previous proof, for each $u = (u_1, \dots, u_n) \in M$, $u\mu \subseteq u_1\mu \times \cdots \times u_n\mu$, and so M is finite $\mathscr{J}$-above.

Therefore, by applying the previous corollary $n-1$ times, we conclude that M has decidable membership for rational sets. $\square$

Clearly, outside of F, there is still much that is unknown about rational sets, either in the direction of free inverse monoids over alphabets of higher cardinality or direct products of free inverse monoids, even when restricted to just copies of F. Naturally with a bias towards the properties known about rational sets of F, we highlight that the inclusion (and equality) problem is still open, and that no results can be mentioned at the moment regarding their intersection.

Bibliography

[1] Jorge Almeida. *Lecture notes in Semigroups and Automata*. Faculdade de Ciências da Universidade do Porto, 2024.

[2] Christopher D. Hollings and Mark V. Lawson. “Generalised Groups”. In: *Wagner’s Theory of Generalised Heaps*. Springer International Publishing, 2017, pp. 43–47. ISBN: 9783319636214. DOI: `10.1007/978-3-319-63621-4_7`.

[3] John E Hopcroft and Jeffrey D Ullman. *An introduction to automata theory, languages, and computation*. en. Addison-Wesley series in computer science. Upper Saddle River, NJ: Pearson, Jan. 1979.

[4] John M Howie. *Automata and Languages*. en. Oxford, England: Clarendon Press, Oct. 1991. ISBN: 9780198534426.

[5] John M Howie. *Fundamentals of semigroup theory*. en. London Mathematical Society Monographs. Oxford, England: Clarendon Press, Dec. 1995.

[6] W. D. Munn. “Free inverse semigroups”. In: *Semigroup Forum* 5 (Dec. 1972), pp. 262–269. ISSN: 1432-2137. DOI: `10.1007/bf02572897`.

[7] H. E. Scheiblich. “Free inverse semigroups”. In: *Semigroup Forum* 4 (Dec. 1972), pp. 351–359. ISSN: 1432-2137. DOI: `10.1007/bf02570809`.

[8] Pedro V. Silva. “On the rational subsets of the monogenic free inverse monoid”. In: *Journal of Algebra* 618 (Dec. 2022), pp. 214–240. ISSN: 0021-8693. DOI: `10.1016/j.jalgebra.2022.12.006`.

[9] V. V. Vagner. “Generalized groups”. In: *Dokl. Akad. Nauk. SSSR* 84.6 (1952), pp. 1119–1122.