Towards a Theory of Crisis, Polycrisis and Syndromes



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Resumo

A partir da segunda metade do século vinte, a crescente atividade humana tem impactado o Sistema Terrestre de modo ainda não completamente compreendido. As Fronteiras Planetárias (FP) são indicadores deste impacto, estabelecendo o estado do Sistema Terrestre. Contudo, a conexão entre estes parâmetros e a emergência de Eventos Sociopolíticos (ESP) específicos e a respetiva interação entre eles ainda não foram clarificadas através de um modelo teórico adequado.

Com o intuito de desvendar os processos característicos da influência exercida entre estas FPs e ESPs, propomos um termo de interação matricial que estabecerá o acoplamento entre eles. A caracterização de possíveis cenários evolutivos torna-se, então, num objetivo alcançável. Destes cenários, aqueles associados à existência de crises no Sistema Terrestre são de particular interesse.

Neste trabalho consideram-se alguns casos particulares, nomeadamente Fronteiras Planetárias que obedecem a equação logística contínua e ESPs que crescem de forma exponencial. Em ambas as situações, as equações de evolução são obtidas e, através dos seus limites assimtóticos, os seus efeitos no Sistema Terrestre são interpretados. São também definidos novos objetos, o vetor de policrises e síndromes, que serão de particular interesse na quantificação e comparação do efeito coletivo de múltiplas policrises no Sistema Terrestre.

Por fim, Consideram-se métodos para incluir resiliência no modelo, de modo a melhor descrever o real comportamentos do Sistema Terrestre. Este efeito deverá estar associado a uma intervenção humana que vise minimizar a deterioração do Sistema Terrestre.

Palavras-chave: Fronteiras Planetárias, Sistemas complexos, Eventos Sociopolíticos, Ciência do planeta Terra, Antropocénico, Modelos físicos

Abstract

From the second half of the twentieth century onwards, the steep growth of the human activity has had an impact on the Earth System that is not fully understood. Indicators of this impact can be obtained through the Planetary Boundaries, suitably assessing the state of the Earth System. Nevertheless, the way this translates into the emergence of specific Socio-Political Events in the Earth System and how they back react into the Planetary Boundaries has not been properly framed by a suitable theoretical model.

With the objective of unveiling the inner workings characteristic of these interactions, a matrix interaction model is proposed, establishing a coupling between the Planetary Boundaries and the Socio-Political Events. The evaluation of the possible evolutive scenarios becomes then an attainable objective. Of those, the ones leading to crises in the Earth System are of particular interest.

Some specific situations are addressed in the present work, namely Planetary Boundaries evolving as a continuous logistic function and Socio-Political Events that have a characteristic exponential growth. In both cases, the evolution equations are obtained and the implications for the Earth System are considered through their asymptotic limits. New objects are also defined, the polycrisis vector and syndromes, which prove useful in quantifying and comparing the intertwining of multiple polycrisis on the Earth System.

Additionally, means for the inclusion of resilience features in the model are also discussed to better describe the true behaviour of the Earth System. This includes the response to a planned human intervention aiming to minimize the deterioration of the Earth System.

Keywords: Planetary Boundaries, Complex Systems, Socio-Political Events, Earth Sciences, Anthropocene, Physical models

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List of Abbreviations

AM Anthropocene Misalignment.

ES Earth System.

PB Planetary Boundary.

SOS Safe Operating Space.

SPE Socio-Political Event.

1. Crises in the Earth System and Planetary Boundaries

The Earth System (ES) is defined as the set of Atmosphere, Biosphere, Cryosphere, Hydrosphere, Superior Lithosphere, Humankind and all the established relations amongst them. Its structure is complex, comprised of many subsystems deeply connected amongst themselves. In the last decades, we have witnessed the rise of sociopolitical turmoil and a growing number of extreme climatic phenomena. These come as warning signs of emerging series of crises, which may entangle and mutually aggravate. On these considerations, the present work develops a formal description of the ES, closely following Refs. [1, 2].

Taken that human activity has reached an unprecedented magnitude [3], with impact on the ES at socio-economic [4, 5] and at environmental [6, 7] scales, accurate assessments of the outcome are required. This task can be achieved via the Planetary Boundaries (PBs) formalism, which provides objective indicators of the ES's state [6, 7]. Of course, the impact on an unstable ES resonates on a set of Socio-Political Events (SPEs), globally associated to the deterioration of the human condition.

The emergence of human activities as a geological force, whose effects undoubtedly outpace other natural drivers, has put an end to the climatic stability of the Holocene epoch [8, 9]. As a result, a number of systems are undergoing a change of regime and bordering their critical points [10, 11]. These include Greenland's melting ice sheets [12] along with a noticeable decrease in vegetal resilience [13]. The dense connectivity found between subsystems akin to the ones enumerated foments the general degradation of the ES's resilience mechanisms, fuelling a cascade of critical transitions [14, 15]. Such was the onset of the Anthopocenic epoch [8, 9], where the unprecedented human action on the ES acts as a hyper-connective network amplifying and propagating undesired events amongst the constitutive elements of the ES [5].

Recalling the aforementioned Planetary Boundaries (PBs) and their usefulness as trackers of the ES's deviation from sustainable conditions, one finds in Figure 1.1 a visual depiction of their most recent assessments. There are 9 PBs*, all presented with respect

*The set is comprised of: Climate Change, Change in Biosphere Integrity, Land System Change, Freshwater Change, Modification of Biogeochemical Flows, Introduction of Novel Entities, Ocean Acidification, Atmospheric Aerosol Loading and Stratospheric Ozone Depletion.

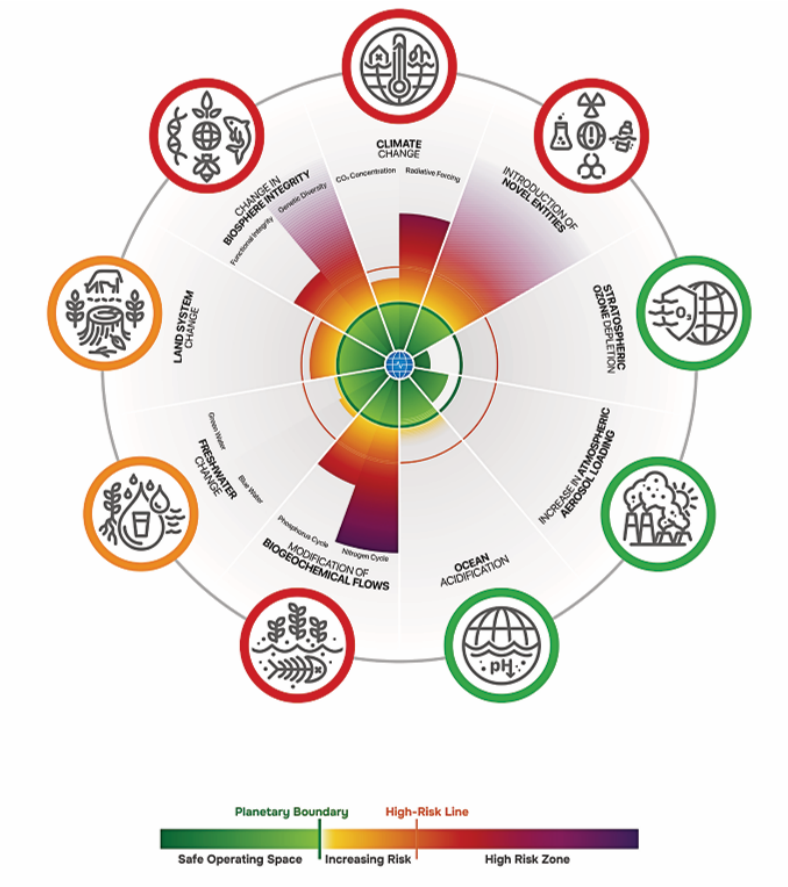


FIGURE 1.1: Representation of the PBs with their associated risk, where the green circle contains the Safe Operating Space (SOS), linked to a stable ES. Their most recent evaluation is marked by the colouring on each of the chart's "slices". It should be noted that the PBs "Novel Entities" and "Biosphere Integrity" are estimated to be beyond the scale. Chart taken from [16].

to their Holocene estimations, depicted by the inner green ring. Any value within this ring ensures the continuity of the ES with Holocene-like conditions and constitutes Safe Operating Space (SOS). If they are evaluated between the green and red rings, then they are at risk of trespassing the critical threshold of regime change [6, 7]. Beyond this outer ring, the PBs are pushing the ES towards a new stable regime.

A formal description the PBs' evolution would provide guidelines for the planning of adaptation and mitigation actions, fomenting the ES's resilience [17–19]. Unfortunately, six PBs are well beyond their SOS, deviating the ES from a sustainable trajectory. We define this discrepancy on the PBs as Anthropocene Misalignments (AMs).

The human impact has been already identified motivated as the current driving force behind the ES's demise. As such, one can see these regime changes as thermodynamic phase transitions, which are accurately described by the Landau-Ginsburg model [20, 21].

Considering the ES's free energy shift with respect to Holocene values, then [20]:

$$F(\psi, H) - F_0 = a\psi^2 + b\psi^4 - cH\psi, \tag{1.1}$$

where $\psi = (T - \langle T \rangle_H)/\langle T \rangle_H$ is the respective relative temperature deviation and $\{a, b, c\}$ phenomenological parameters of the model. These are admitted constant as the timescales of other relevant driving mechanisms did not significantly changed in the timescale typical of the human activity. In Eq. (1.1), the human action factor, H, is expressed at lowest order as the sum over all PB. However, considering $h_i(t)$, with i ranging from 1 to 9, as the PB's change with respect to the Holocene and that these can mutually affect each other, one must also consider non-trivial interactions amongst them. Hence, the human action factor must be of the form [19, 22]:

$$H(t) = \sum_{i=1}^{9} h_i(t) + \sum_{i,j=1}^{9} g_{ij} h_i(t) h_j(t) + \sum_{i,j,k=1}^{9} \alpha_{ijk} h_i(t) h_j(t) h_k(t) + ... \tag{1.2}$$

with g_{ij} and α_{ijk} as a second and third order interaction terms, respectively. Indeed, non-linear terms are relevant, as they can reach proportions of, at least, one order of magnitude below the dominant term [22].

The importance of the type of time dependence found in Eq. (1.2) is manifest. For instance, assuming a linear growth of human activities

$$H(t) = H_0 t \tag{1.3}$$

leads to the ES evolving towards an Hothouse state scenario [23], regardless of initial conditions [21]. Explicitly, one obtains phase-space orbits around an ever increasing associated relative temperature [21]

$$\psi(t) = \psi_0 \cos(\omega t) + \beta t. \tag{1.4}$$

Of course, this assumption is conservative. Considering that characteristic dynamic connectivity of the ES is expected to increase in the foreseeable future, one can expect higher orders terms being relevant.

Alternatively, different conditions may be imposed on the model. If one considers that the PBs evolve as a discrete logistic function

$$h_{i(n+1)} = r h_{i(n)} \left(1 - \alpha^{-1} h_{i(n)}\right), \tag{1.5}$$

where r and α are the model's growth constant and the carrying capacity parameter, respectively. Each discrete iteration can be considered as a time step [24]. This assumption implies that the ES shows a wider variety of trajectories, including chaotic behaviour as well as instabilities and bifurcations [24].

The loss of long time predictability arising from the behaviour described by Eq. (1.5), turns impossible considering long term mitigation strategies, accelerating the current evolution of the PBs and the ES away from the Holocene [16, 25].

Mitigation actions have the ultimate objective of enhancing the resilience of the ES [19]. Throughout this work, however, resilience is considered as the resistance the ES shows in evolving towards states that compromise Holocene-like conditions, such as the Hothouse Earth.

Remarkably, the ES's free energy, Eq. (1.1), admits resilience features with the addition of auxiliary terms and imposing constraints on the model's parameters, such that [19]:

$$F(\psi, H) - F_0 = a\psi^2 - b|\psi^3| + c\psi^4 - \gamma H(t)\psi, \tag{1.6}$$

requiring that constants b, c and γ are positive, while a can be negative. To obtain the desired relevant features, extrema and global stability conditions must be imposed on the free energy. Thus, the cubic temperature term's inclusion allows for metastable states by creating new locally stable minima, as depicted in Figure 1.2.

Complementarily, a more elaborate dependency of the human action, $H(t)$, on the PBs must also be considered. Considering only a single PB, one must consider the extension [19]:

$$H(t) = h_i(t) + g_i h_i^2(t) + \alpha_i h_i^3(t), \tag{1.7}$$

ensuring g_i is negative and α_i is positive. As a consequence, the ES is now able to withstand growth in the PBs while retaining minimal temperature deviations, until a threshold is surpassed. From that point onwards, lies a global free energy minimum, which onsets a significant increase in the ES's temperature, quintessence of the Hothouse Earth.

The above extension of the original model based on the Landau-Ginsburg theory allows for predictions on the ES's evolution. Such conclusions entail an indirect involvement of human activity via the PBs's growth. More detail can be extracted from these interactions if PBs affect specific SPEs, and vice-versa. In this work we aim to establish the couplings between PBs and SPEs by a matrix interaction model. This might allow for the emergence

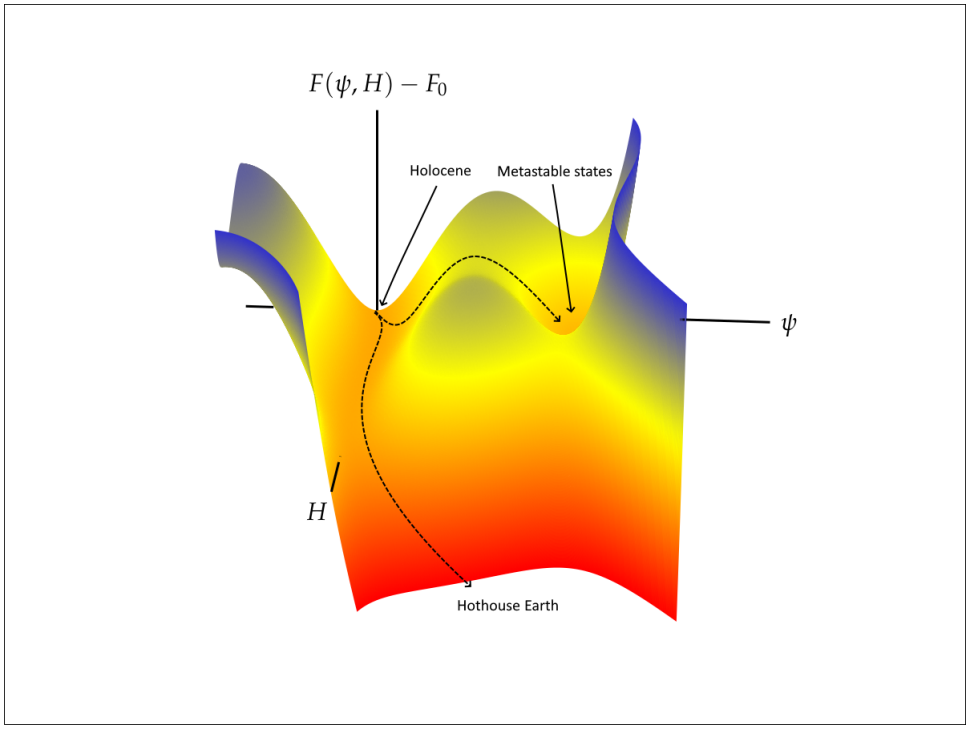


FIGURE 1.2: The ES's free energy as a function of relative temperature deviation from the Holocene, ψ and human activity, represented by growing PBs, H, given by Eq. (1.1).

of crises in the ES, as well as their entanglement and subsequent genesis of a polycrisis.

Suppose that the 9 PBs constitute the components of a vector $\vec{h}(\vec{r}, t)$ and the SPEs form another vector $\vec{E}(\vec{r}, t)$. To avoid unnecessary complications, the space dependence of both vectors shall be dropped and $\vec{E}$ is assumed, for simplicity, to lie in a 9-dimensional vector space. Thus, one describes the interaction amongst both vectors' components by the means of a matrix R_{ij}, where the indexes assume values ranging $i, j = 1, 2, ..., 9$, resulting in a generalised term of the form $E_i R_{ij} h_j$.

The remaining parts of this thesis are organised as follows. Chapter 2 contains the setting of the Lagrangian/Hamiltonian formalism. This allows us to obtain the equations of motion of the vector $\vec{E}$ once a logistic profile is imposed on the PBs. Solutions are then obtained. It is also shown that the "Malthusian" growth of the PBs, leads to an exponential evolution of the SPEs. Crises are identified with SPEs with positive time derivative. In Chapter 3 we discuss the polycrisis and the concept of "Syndromes". This construction precedes the discussion on how resilience can be incorporated as a physical property. Chapter 4 contains our conclusions.

2. Coupling between Planetary Boundaries and Socio-Political Events

We now present our model. The ES's crises and how they are associated to the degradation of the human condition can be qualitatively modelled by a matrix interaction akin to condensed matter models. In particular, those that describe the coupling between a lattice spin and an external magnetic field. Thus, SPEs and PBs are coupled and will also influence each other. These entities are described as vectors belonging to 9-dimensional vector spaces. Indeed, to define what is a SPE one faces a certain degree of arbitrariness. Still, there must be an association to some class of events that can affect/be affected by the PBs' growth out of the SOS. The following set can then be considered as representative components of the SPEs' vector space: "Economic Growth", "Harmful Technological Developments", "Bellicose Actions", "Social Inequalities", "Poverty and Famine", "Pandemics", Ecological Disasters", "Unemployment" and "Disruptive Political Developments". This set instantiates anthropogenic activity, thereby fuelling the PBs' divergence from their original Holocene conditions.

As previously mentioned, a PB is influenced by the remaining set of PBs' evolution [16, 22]. Similarly, SPEs also show this dynamical property. The complex nature of human society and it's hyperconnectivity ensure high correlation amid this class of events. However, the focus of this work will not be directed towards such influences. It is intended to construct and elaborate on a theory that establishes the coupling of SPEs with PBs. As a consequence, the human impact on the ES, or AMs, is being assessed through the PBs.

The general socioeconomic trends from second the half of the last century onwards evidence a high impact on the ES [3]. This deterioration could be regarded as Anthropocene Traps, that lock the ES in trajectories associated with negative impacts on human well-being, as elaborated in Ref. [25]. That being said, Anthropocene traps resemble and inspired the concept of SPEs, which are tweaked to better express the impact on the PBs in what was defined as AMs. Aiming to couple vectors $\vec{h}(t)$ and $\vec{E}(t)$, matrix models are a natural choice. As mentioned, condensed matter theory contains pertinent descriptions of complex systems that interact in similar fashion. Such is the case of ferromagnetic models and quantum hall effect or topological insulators [26–28].

Specifically, one considers two vectors, $\vec{h}(t)$ and $\vec{E}(t)$, the first gathering the 9 PBs' deviations and the second the 9 chosen SPEs. These shall then interact by the means of a matrix element R_{ij}. It is intended that such term establishes the degree of alignment between the vectors, therefore each entry is bounded as $-1 \leq R_{ij} \leq 1$. The matrix term can also exhibit dynamical properties on its own. Spatial and time dependence are also feasible. In fact, the ES's connectivity is expected to increase in the foreseeable future [5], leading to extreme misalignments, with $\left|R_{ij}\right| \to 1$. Still, in this work all entries of the matrix will be assumed as constant. Both vectors, $\vec{h}(t)$ and $\vec{E}(t)$, already bear enough complexity to thoroughly describe the anthropogenic impact on the ES.

With the ultimate objective of obtaining evolution equations for both the PBs and the SPEs vectors, one needs to set up the relevant Lagrangian function:

$$L(\vec{h}, \vec{E}, \dot{\vec{h}}, \dot{\vec{E}}; t) = \frac{1}{2}\mu \sum_{i=1}^{9} \dot{h_i}^2 + \frac{1}{2}\nu \sum_{i=1}^{9} \dot{E_i}^2 - \sum_{i,j=1}^{9} E_i R_{ij} h_j, \tag{2.1}$$

where μ and ν are phenomenological constants and time derivatives are denoted as dots. Computing the associated conjugate momenta from the Lagrangian function Eq. (2.1):

$$p_{hi} = \frac{dL}{d\dot{h}_i} = \mu \dot{h}_i,$$
$$p_{Ei} = \frac{dL}{d\dot{E}_i} = \nu \dot{E}_i,$$

whose respective Hamiltonian function can now be obtained:

$$H(\vec{h}, \vec{E}, \vec{p}_h, \vec{p}_E; t) = \sum_{i=1}^{9} \frac{p_{hi}^2}{2\mu} + \sum_{i=1}^{9} \frac{p_{Ei}^2}{2\nu} + \sum_{i,j=1}^{9} R_{ij} E_i h_j. \tag{2.2}$$

From equation (2.2) come the dynamical equations of the model's variables.

$$\dot{p}_{hi} = -\sum_{j=1}^{9} E_j R_{ji}, \tag{2.3}$$

$$\dot{h}_i = \frac{p_{hi}}{\mu}, \tag{2.4}$$

$$\dot{p}_{Ei} = -\sum_{j=1}^{9} R_{ij} h_i, \tag{2.5}$$

$$\dot{E}_i = \frac{p_{Ei}}{\nu}. \tag{2.6}$$

From this set of equations, it is evident that the SPE's evolution is constrained by the PBs and vice-versa. The main features of the model are sketched in Figure 2.1.

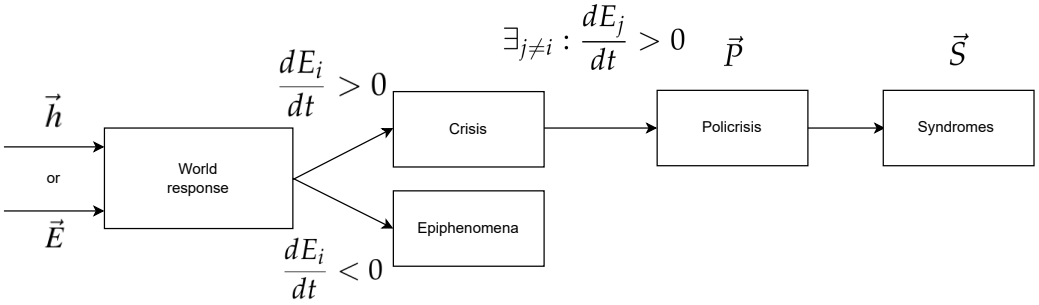


FIGURE 2.1: Pictorial scheme of the model's structure.

The model's equations of motion dictate a symmetry in the interaction. An important feature of the Lagrangian function, Eq. (2.1), is that the "external force", that being the SPEs, is also affected by the interaction with the PBs. A resultant complementarity ensues, in the sense that both vectors can be used as input to obtain the other's evolution equation. This property will be explored in the next subsection, where the two approaches are considered.

Given a certain evolution of the PBs, vector $\vec{h}(t)$, the existence of a crisis is dependent on the time derivative of the components of vector $\vec{E}(t)$. As Figure 2.1 indicates, if a SPE increases in time, $dE_i(t)/dt > 0$, there is an emergent crisis in the system. Otherwise, if $dE_i(t)/dt < 0$ one gets an epiphenomenon, where that class of events will, in due time, dissipate. When more than one component of the SPE vector has positive time derivatives, a policrisis arises in the system. As such, it is in the interest of this work to assess the degree of influence these crises exert amongst themselves. In order to do so, the concept of a kth-order Syndrome is constructed from the vector product of k SPEs vectors, whose components behave as crisis. This construction shall be formalised later on in chapter 3.

It is noteworthy that the formal definition of crisis in social sciences is coherent with this model's counterpart, a SPE depicting positive growth. In Ref. [5], a crisis is seen as a sudden event (or series of events) that significantly harms the well-being of a considerable share of the human population. Hence, the classes of events enumerated before as components of the SPE vector represent phenomena associated to the degradation of the human condition. Again, the elevated degree of connectivity on the ES, extensively enhanced by anthropogenic factors, entangles these crises, which are mutually aggravated.

In the next section, solutions of this model are considered for different inputs. At first, it is

considered that PBs evolve as logistic functions and an equation for vector $\vec{E}$ is calculated. Secondly, the reverse process is examined, where a full blown crises, with SPEs exhibiting exponential evolution, is shown to lead to a similar behaviour of the PBs.

2.1. Hypothesis of logistical constraints on Planetary Boundaries

From Eq. (2.2), it is seen that a solution for the behaviour of the SPE vector's may be determined, granted an already known progression of the PBs. One must consider the question of a representative evolution of the PBs. Previously, it was argued that the PBs, gathered in the singular term $H(t)$, could evolve linearly, as long as the considered time span remains short [21]. Alternatively, the impending finiteness of Earth's resources and the impact human activities have on the ES are more accurately described by a continuous logistic function. By similar reasoning, Ref. [24] adopts the discrete logistic map, Eq. (1.5). However, chaotic patterns arise and the of discreteness of time steps handicaps the predictability of such model. This alone motivates choosing its continuous counterpart, capturing both the initial exponential evolution, associated to impact of the Great Acceleration [3], and the later logistic stabilization imposed by the ES. Therefore, each PB's growth is determined by [30]:

$$\dot{h}_i(t) = \lambda_i h_i(t)(1 - K_i h_i(t)), \tag{2.7}$$

with λ_i as the growth factor of the PB and the inverse of K_i defining the upper bound, assuming $K_i > 0$ hereafter as to ensure the upper bound to be positive valued. Solving this differential equation gets us to the general solution [30]:

$$h_i(t) = \frac{e^{\lambda_i t}}{C_i + K_i e^{\lambda_i t}}, \tag{2.8}$$

where C_i is an integration constant, also assumed positive.

Introducing Eq. (2.5) in the time derivative of Eq. (2.6) results in:

$$\ddot{E}_i(t) = -\sum_{j=1}^{9} \frac{R_{ij}}{\nu} \frac{e^{\lambda_j t}}{C_j + K_j e^{\lambda_j t}}. \tag{2.9}$$

Evidently, each component $E_i(t)$ receives differently weighted contributions from the PBs, which may even be at different stages of the logistic curve. There is, then, margin for such a set of solutions. Proceeding with the integration over $[0, t]$ of Eq. (2.9) yields:

$$\dot{E}_i(t) - \dot{E}_{i0} = -\sum_{j=1}^{9} \frac{R_{ij}}{K_j \lambda_j \nu} \left[\log \left| C_j e^{-\lambda_j t} + K_j \right| + \lambda_j t - \log \left| C_j + K_j \right| \right]. \tag{2.10}$$

Performing the last integration step over the same interval leads to,

$$\begin{aligned} E_i(t) - \dot{\tilde{E}}_{i0}t - E_{i0} &= -\sum_{j=1}^{9} \frac{R_{ij}}{K_j\lambda_j\nu} \left[\int_0^t \log\left| C_j e^{-\lambda_j t'} + K_j \right| dt' + t^2 \frac{\lambda_j}{2} \right] \\ &= -\sum_{j=1}^{9} \frac{R_{ij}}{K_j\lambda_j\nu} \left[- \int_{-\frac{C_j}{K_j}}^{-\frac{C_j e^{-\lambda_j t}}{K_j}} \frac{\log\left|-z'+1\right|}{\lambda_j z'} dz' + t\log|K_j| + t^2\frac{\lambda_j}{2} \right], \end{aligned} \tag{2.11}$$

with $\dot{\tilde{E}}_{i0}$ as a readjusted constant $\dot{E}_{i0}$ and a redefined integration variable, $z' = -C_j e^{-\lambda_j t'}/K_j$, unveiling the presence of the dilogarithm function, defined as follows:

$$Li_2(z) \equiv -\int_0^z \frac{\log(1-z')}{z'} dz', \tag{2.12}$$

for $z \in \mathbb{C}\backslash_{[1,+\infty]}$. Tweaking the integration limits in Eq. (2.11) so to set the path's starting point in 0, culminates in the final exact form of the integral:

$$E_i(t) - \dot{\tilde{E}}_{i0}t - E_{i0} = -\sum_{j=1}^{9} \frac{R_{ij}}{K_j\lambda_j^2\nu} \left[Li_2\left(-\frac{C_j e^{-\lambda_j t}}{K_j}\right) - Li_2\left(-\frac{C_j}{K_j}\right) + t\lambda_j \log|K_j| + t^2\frac{\lambda_j^2}{2} \right], \tag{2.13}$$

arbitrarily defined by 2 constants $\{E_{i0}, \dot{E}_{i0}\}$. The general solution for the SPEs permits its evaluation at any given time, determined by the interaction matrix. Such predictability may be exploited by the means of a more fundamental expression of Eq. (2.13). Noticeably, the error of any approximation involving the dilogarithmic function will be tied to the region of interest, with respect to the integration path. This region is bounded by z, whose magnitude is dictated by the ratio between constants $r \equiv C_j/K_j$. It is possible to create a sound approximation by adequately choosing a Padé approximant, more versatile and robust than the usual methods [31, 32]. The discussion of these issues is presented in Appendix A. There, it is argued that a reliable approximation for Eq. (2.12) is given by:

$$\begin{aligned} Li_2(z) &\simeq -\int_0^z \frac{11z'^2 + 60z' - 60}{3z'^3 - 36z'^2 + 90z' - 60} dz' \\ &= -\left[\frac{25}{6\sqrt{15}} \log\left| \frac{z - \sqrt{15} - 5}{z + \sqrt{15} - 5} \right| + \frac{25}{18}\log\left|z^2 - 10z + 10\right| + \frac{8}{9}\log|z-2| \right]_0^{-\frac{C_j}{K_j}e^{-\lambda_j t}}, \end{aligned} \tag{2.14}$$

where the integral's argument was replaced by Eq. (A.5). Suitable substitution of the integration limits and incorporating the remaining dilogarithm of (2.13) results in:

$$\begin{aligned} Li_2\left(-\frac{C_j e^{-\lambda_j t}}{K_j}\right) - Li_2\left(-\frac{C_j}{K_j}\right) \simeq -\frac{25}{6}\Bigg[\frac{1}{3}\log\left|\frac{C_j^2 e^{-2\lambda_j t} + 10C_j K_j e^{-\lambda_j t} + 10K_j^2}{C_j^2 + 10C_j K_j + 10K_j^2}\right| \\ + \frac{1}{\sqrt{15}}\log\left|\frac{C_j e^{-\lambda_j t} + 5K_j + K_j\sqrt{15}}{C_j e^{-\lambda_j t} + 5K_j - K_j\sqrt{15}} \cdot \frac{C_j + 5K_j - K_j\sqrt{15}}{C_j + 5K_j + K_j\sqrt{15}}\right|\Bigg] \\ - \frac{8}{9}\log\left|\frac{C_j e^{-\lambda_j t} + 2K_j}{2K_j + C_j}\right|. \end{aligned} \tag{2.15}$$

In the spirit of presenting compact expressions, the summation of all the constant terms present above shall henceforth be hidden through a redefinition of E_{i0}. Thus, one is able to present a solution of the SPE's dynamics, expressed exclusively through fundamental functions:

$$\begin{aligned} E_i(t) - \dot{\tilde{E}}_{i0} t - \tilde{E}_{i0} \simeq -\sum_{j=1}^{9} \frac{R_{ij}}{K_j \lambda_j^2 \nu}\Bigg\{ t\lambda_j \log|K_j| + t^2\frac{\lambda_j^2}{2} - \frac{8}{9}\log\left|C_j e^{-\lambda_j t} + 2K_j\right| \\ - \frac{25}{6}\Bigg[\frac{1}{3}\log\left|C_j^2 e^{-2\lambda_j t} + 10C_j K_j e^{-\lambda_j t} + 10K_j^2\right| \\ + \frac{1}{\sqrt{15}}\log\left|\frac{C_j e^{-\lambda_j t} + 5K_j + K_j\sqrt{15}}{C_j e^{-\lambda_j t} + 5K_j - K_j\sqrt{15}}\right|\Bigg]\Bigg\}, \end{aligned} \tag{2.16}$$

recalling that both integration constants were suitably redefined as:

$$\begin{aligned} \dot{\tilde{E}}_{i0} &\equiv \dot{E}_{i0} + \sum_{j=1}^{9} \frac{R_{ij}}{\nu K_j \lambda_j}\log\left|C_j + K_j\right|, \\ \tilde{E}_{i0} &\equiv E_{i0} + \sum_{j=1}^{9} \frac{R_{ij}}{\nu K_j \lambda_j^2}\Bigg\{\frac{25}{6\sqrt{15}}\log\left|\frac{C_j + 5K_j - K_j\sqrt{15}}{C_j + 5K_j + K_j\sqrt{15}}\right| \\ &\quad - \frac{25}{15}\log\left|C_j^2 + 10C_j K_j + 10K_j^2\right| - \frac{8}{9}\log\left|C_j + 2K_j\right|\Bigg\}. \end{aligned} \tag{2.17}$$

We consider now some relevant limits of the SPE's evolution. For $t \ll \lambda_j^{-1}$, the characteristic time scale of jth PB, for all $j = 1, ..., 9$, the logarithmic contributions are the slowest

and the quadratic one is negligible. Hence, $E_i(t)$ is controlled by the linear term:

$$E_i(t) \sim \dot{\tilde{E}}_{i0}t + \tilde{E}_{i0} - \sum_{j=1}^{9} \frac{R_{ij}}{\nu K_j \lambda_j^2} \Bigg\{ (\log|K_j|)\lambda_j t + \frac{\lambda_j^2}{2}t^2 - \Bigg(\frac{25}{6} \Bigg[\frac{1}{\sqrt{15}} \log \left| \frac{C_j(1-\lambda_j t) + K_j\sqrt{15} + 5K_j}{C_j(1-\lambda_j t) + K_j\sqrt{15} - 5K_j} \right| + \frac{1}{3} \log \left| -(2C_j^2 + 10C_jK_j)\lambda_j t + C_j^2 + 10C_jK_j + 10K_j^2 \right| \Bigg] + \frac{8}{9} \log \left| C_j(1-\lambda_j t) + 2K_j \right| \Bigg) \Bigg\}. \tag{2.18}$$

On the other hand, as $t \gg \lambda_j^{-1}$ for all the PBs, the dominant terms are of the quadratic order, also present in the previous expansion, while the logarithms' contributions stall and become constants:

$$E_i(t) \sim \dot{\tilde{E}}_{i0}t + \tilde{E}_{i0} - \sum_{j=1}^{9} \frac{R_{ij}}{\nu K_j \lambda_j^2} \Bigg\{ (\log|K_j|)\lambda_j t + \frac{\lambda_j^2}{2}t^2 - \Bigg(\frac{25}{6} \Bigg[\frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{15}+5}{\sqrt{15}-5} \right| + \frac{1}{3} \log \left| 10K_j^2 \right| \Bigg] + \frac{8}{9} \log \left(2K_j \right) \Bigg) \Bigg\}, \tag{2.19}$$

which simplifies to

$$E_i(t) \sim \dot{\tilde{E}}_{i0}t + \tilde{E}_{i0} - \sum_{j=1}^{9} \frac{R_{ij}}{\nu K_j \lambda_j^2} \left\{ \lambda_j t \log|K_j| + \frac{\lambda_j^2}{2}t^2 \right\}. \tag{2.20}$$

Ultimately, the hypothesis of PBs following a logistic function equates to a linear evolution of the SPEs for small time, as observed in Eq. (2.18). When considering large times, however, Eq. (2.20) depicts a quadratic progression of the SPEs.

Alternatively, taking the limit of $K_j \to 0$ in Eq. (2.7) results in exponential growth of the PBs. In their turn, SPEs' components would depict the same steep evolution, due to Eq. (2.9):

$$\ddot{E}_i(t) = -\sum_{j=1}^{9} \frac{R_{ij}}{\nu} \frac{e^{\lambda_j t}}{C_j} \tag{2.21}$$

$$E_i(t) = \dot{E}_{i0}t + E_{i0} - \sum_{j=1}^{9} \frac{R_{ij}}{\nu C_j \lambda_j^2} e^{\lambda_j t}. \tag{2.22}$$

This illustrates the tight relationship between the evolution of the PBs and the ES's crises.

It is worth remarking that both Eq. (2.20) and Eq. (2.21) require, for a long lasting crisis whose criterion was already established as $dE_i(t)/dt > 0$, that R_{ij} coefficients must have negative entries. In rigour, if SPE $E_i(t)$ depicts an evolving crisis, R_{ij} contains at least one negative valued entry with j spanning from 1 to 9. Notice the parameters $\{C_j, K_j\}$ are positively defined and that ν can be arbitrarily absorbed into the interaction matrix. As such, this matrix fully determines whether a SPE evolves as a crisis or rather into an epiphenomenon (see Figure 2.1).

We can consider the opposite situation, where the SPE vector is taken as input and its impact on the PBs examined. The model's symmetry guarantees that such approach is feasible. Suppose the worst case scenario, where the crises evolve exponentially:

$$E_i(t) = A_i e^{B_i t}, \tag{2.23}$$

taking A_i as an initial value and B_i controlling the growth rate. The coupling between these crises and PBs is achieved through Eqs. (2.4) and (2.3), resulting in the following equation of motion:

$$\ddot{h}_i(t) = -\sum_{j=1}^{9} \frac{R_{ji}}{\mu} E_j(t), \tag{2.24}$$

from which, a double integration yields

$$h_i(t) - \dot{h}_{i0} t - h_{i0} = -\sum_{j=1}^{9} \frac{R_{ji}}{\mu} \frac{A_j}{B_j^2} e^{B_j t}, \tag{2.25}$$

where the set $\{\dot{h}_{i0}, h_{i0}\}$ corresponds to the integration constants. As expected, the PBs will degrade while swiftly diverging exponentially. Of course, the impact of such evolution on the ES, even with a small growth parameter B_j, is severe [6, 7]. Its appearance as the squared denominator of Eq. (2.25) partially counteracts the decelerated exponential.

Here it is stressed how an imposed upper bound on the PBs curbs SPEs' dynamics, which backtracks the "malthusian" PBs, perpetuating the crises. In the next chapter, we describe SPEs becoming crises as a consequence of the highly intra-connected ES and introduce the concepts of Syndromes and of a vector of polycrisis.

3. From crisis and polycrisis into syndromes and resilience

At this stage, with the inner workings of the coupled model addressed in the previous chapter, we turn to the issue of defining a hierarchy between sets of ES's crises. As it stands, the concept of polycrisis remains fortuitous and the entanglement amongst them not entirely objective. Further characterisation would prove its usefulness providing guidance for mitigation and adaptation policies. To accomplish that, the concept of Syndromes shall be promptly introduced.

Prior discussion concluded that PBs, through matrix R_{ij}, impact the SPEs' growth. Thus, the evolution of the latter is quickly assessed via their equations of motion (from Eq. (2.3) to Eq. (2.6)). However, the dynamics of the coupling were conveniently set aside. While it made sense as a means to obtain the broad behaviour of both vector $\vec{E}$ and vector $\vec{h}$, considering only a single set of SPEs, it is a subset of all possible behaviours. Given that the connectivity of the ES changes with time, as previously discussed, it is a natural conclusion that SPEs shall couple differently to the PBs. Hence, SPEs of the same type differ on their evolution depending on the time they arise, that is, to which set of SPEs they belong to. To distinguish between sets, one adopts an upper index l, so that $\vec{E}^{(l)}(t)$ encompasses all the components of the SPE vector at a given time. Therefore, our definition of crisis, for a given set of SPEs, is that one of its components shall grow in time $dE_i^{(l)}(t)/dt > 0$. Thus, a polycrisis arises as two or more of these components show positive time derivatives. Remains to be defined how the contributions from these sets can be gathered and simultaneously analysed.

It should be noted that, as regarded in Eq. (2.25), the worst degradation of the PBs occurs as crises acquire uncontrolled exponential growth. Indeed, this is a signature of AMs imparting ill consequences on the ES. The connection with social sciences is still attainable noticing that these misalignments resemble the Anthropocene traps, as defined in Ref. [25]. Concerning the trajectories of the ES, strong AMs are associated to an evolution towards a less resilient and degraded ES. Such unsustainable trajectories are also characteristic of activated by Anthropocene traps. In spite of all this, the exact relation between the former and the latter is out of the scope of this work and shall be developed elsewhere.

The vector nature of each set of SPEs allows to consider the cross product amongst multiple sets, which results in a generalised volume, achieved through Cartan's product for p-forms [33]. Then, it is natural to explore such quantity, hereafter to be referred to as kth-order Syndrome, given k sets of SPEs vectors depicting polycrisis behaviour. Such is the case when more than one SPE component evolve with positive time derivative.

As an auxiliary intermediate step between SPEs and Syndromes, the Polycrisis vector $\vec{P}_l^{(1)}(t)$ is introduced, containing $2 \leq l \leq 9$ components. These are gathered from the subspace of $\vec{E}^{(1)}(t)$ whose elements are classified as crises. In practice, from

$$\vec{E}^{(1)}(t) = \left(E_1^{(1)}, \cdots, E_i^{(1)}, \cdots, E_j^{(1)}, \cdots, E_9^{(1)}\right), \tag{3.1}$$

where i and j indicate the crises in this specific set. It follows the respective polycrisis vector:

$$\vec{P}_2^{(1)}(t) = \left(E_i^{(1)}, E_j^{(1)}\right). \tag{3.2}$$

Now, it is possible to assemble a set of k Polycrisis vectors into a kth-order Syndrome $S_l^{(k)}(t)$. Formally, its structure is as shown below

$$\vec{S}_l^{(k)}(t) = f(t) \underbrace{\vec{P}_l^{(1)}(t) \times \vec{P}_l^{(2)}(t) \times ... \times \vec{P}_l^{(k)}(t)}_{k \text{ times}}, \tag{3.3}$$

with $f(t)$ as an arbitrary time dependent function. One should also remark that SPEs of different sets do not emerge simultaneously. In order to avoid unnecessary complications, $f(t)$ shall be assumed constant and implicit in subsequent computations. Also, one sees that the orthogonality of the SPEs vector space requires that $l \leq 9$ and $k \leq l$.

Let us now consider the simplest case for a Syndrome. Suppose $E_\alpha^{(1)}(t)$ and $E_\beta^{(1)}(t)$ are two escalating components of a single SPE vector, $\vec{E}^{(1)}(t)$, depicting a polycrisis. The first order Syndrome lies in the vector subspace of vector $\vec{P}(t)$, that is:

$$\begin{aligned} \vec{S}_1^{(1)}(t) =& \vec{P}_2^{(1)}(t) \\ =& \left(E_\alpha^{(1)}(t), E_\beta^{(1)}(t)\right). \end{aligned} \tag{3.4}$$

Evidently, Eq. (3.4) does not feature intertwining amongst polycrises, as is only allowed in higher order Syndromes. This novel property of Eq. (3.3) may be instantiated with a second order Syndrome:

$$\vec{S}_2^{(2)}(t) = \vec{P}_2^{(1)}(t) \times \vec{P}_2^{(2)}(t), \tag{3.5}$$

where $\vec{P}_2^{(1)}(t)$ and $\vec{P}_2^{(2)}(t)$ contain each at least two growing in time SPEs. Assuming they

are respectively $\left\{E_{\alpha}^{(1)}(t), E_{\beta}^{(1)}(t)\right\}$ and $\left\{E_{\gamma}^{(2)}(t), E_{\delta}^{(2)}(t)\right\}$, Eq. (3.5) unfolds into a single component:

$$S_2^{(2)}(t) = \left(E_{\alpha}^{(1)} E_{\delta}^{(2)} - E_{\beta}^{(1)} E_{\gamma}^{(2)}\right), \tag{3.6}$$

where the greek indexes may range from 1 to 9.

In similar manner, higher order Syndromes are constructed from successive cross products amongst multiple Polycrisis vectors. Whenever these vectors present mismatched dimensions, one extends the subspace of the vectors that have less components with null valued entries. As a result, all Polycrisis vectors must be dimensionally equal to each other, casting them into an extended vector space if required.

As it stands, the concept of Syndromes manages to complement this description of the ES. All desired interactions amongst SPEs are easily encapsulated, while providing a useful metric by which the intertwining of multiple Polycrisis is objectively assessed, with their respective ”volumes” acting as the discriminating factor. This implies that a fair comparison between sets of Polycrisis vectors is only achievable when components of Syndromes of the same kth-order are considered. A Syndrome's absolute growth indicates strengthening sets of Polycrisis, with worsening misalignments between the associated SPEs and PBs. From then on, the ES faces deterioration.

At last, this work addresses how the resilience of the ES may be described in the context of this model. That is, how the ES resists the gross impact of anthropogenic activity, be it through already existing natural mechanisms or through artificially crafted ones. The thermodynamic contextualization developed in Ref. [19] proposes that energy dissipation terms properly feature this natural resistance. Moreover, metastable states are introduced in this model by the means of mitigation strategies, restoration of ecosystems, amongst others. Physically, they come as local minima of the free energy function Eq. (1.6), also depicted in Figure 1.2.

Turning our attention to the pivotal assumption of this model, Eq. (2.7), one sees that the response of the ES through the PBs may be influenced. Indeed, that is achieved when the parameters of the logistic function are considered to be dynamic, what is to say:

$$\begin{aligned} \lambda_j &\to \lambda_j(t) \\ K_j &\to K_j(t) \end{aligned} \tag{3.7}$$

where the exact dependence in time of these parameters must be determined empirically. The evolution of SPEs is still dictated by Eq. (2.9), although the complexity of the

description may increase with such considerations. In spite of this, the crises of the ES are averted if the growth of the logistic function is hindered in accordance with the newly acquired functional time dependence.

Additionally, resilience also comes at the expenses of a less connected ES. Relaxing the entries of the interaction matrix, by making $\left|R_{ij}\right| \rightarrow 0$, reduces the number of ways an uncontrolled PB may fuel emerging crises. As a result, an attenuation ensues of SPEs evolving as crises into epiphenomena. These are then stabilized by the ES's internal mechanisms.

All in all, the gathering of crises into their respective sets of polycrisis vectors allows to evidence the classes of SPEs harming the ES. Additionally, by successive cross products multiple syndromes are obtained, which broadly represent the the intertwining of polycrises belonging to different sets. On its turn, resilience properties can be described by the means of functional parameters and a vanishing interaction matrix.

4. Final remarks

We now proceed to gather the most relevant ideas previously discussed. As a starting point, the PBs formalism was considered, which accurately provides information about the state of the ES. One argued that, when coupled with an additional set of SPEs, a mathematical description of the human impact on the ES can be achieved. Moreover, the formalism was extended by introducing the concepts of polycrisis and syndromes, allowing for the comparison of more intricate phenomena.

Detailed solutions are obtained by assuming that PBs and SPEs both are elements of a 9-dimensional vector space and by subsequent application of the Lagrangian/Hamiltonian formalism. To provide the framework of the dynamics of the ES, a matrix interaction term is considered along with the simplest kinetic terms in order to capture the extensive connectivity found between the elements of the ES. The relevant equations of motion are dependent on the characteristic evolution law of one of either vector $\vec{E}$ or vector $\vec{h}$. By imposing a continuous logistic function for the PBs, meaningful results ensue. These can be looked upon in their asymptotic approximations, in particular to observe their short term and long term dominant behaviours. The associated symmetry of the model is also exploited when analysing the runaway effects on the PBs of exponentially growing SPEs.

Parallels with existing literature are established by identifying the concept of crisis as a component of the SPE vector that has positive time evolution, that is $dE_i(t)/dt > 0$. When taking a set of l such components it is possible to aggregate them into a polycrisis vector, $\vec{P}_l(t)$. From k sets of polycrisis, one can build up the pivotal kth-order syndrome by applying successive vector products on the k vectors $\vec{P}(t)$. Overall, the importance of syndromes lies on the possibility of objectively comparing them, provided they are of the same order, and evaluate their relative prominence. In this manner one enables the quantification of particularly harmful AMs and their detrimental impact on the ES.

Lastly, in an effort to introduce means of resilience into the model previous assumptions were relaxed. Indeed, by imposing a functional time dependence on the parameters of the model and by pointing out a preferred long term tendency of the interaction matrix entries, dampening properties emerge. Such should be the consequence of mitigatory and adaptative actions like, for instance, restoring ecosystems, envisaged to dampen the

unfolding evolution of the PBs. All these efforts pertain the possibility of keeping the ES in meta-stable states, close as possible to the climatically stable Holocene conditions.

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A. Details of Padé approximants

Consider the following function, defined in the real interval $\mathcal{A} = [-C/K, 0]$:

$$f(z) = -\frac{\log(1-z)}{z}, z \in \mathcal{A}. \tag{A.1}$$

Immediately one notices that a primitive function of $f(z)$ cannot be expressed in terms of elementary functions. Despite this, a similarly behaved function can be considered by a suitable approximation of the argument of the integral in Eq. (2.12) given by (A.1). The first step is to take the MacLaurin series of the function:

$$f(z) = \sum_{q=0}^{\infty} \frac{z^q}{q+1}, |z| < 1, \tag{A.2}$$

which has the desired behaviour when the ratio $r \equiv C/K$ is less than 1. The reparametrization of t into z is bounded between 0 and r. However, the convergence radius of this approximation is $r = 1$, which effectively binds the constants to satisfy $C < K$.

For $z > 1$ the discrepancy between Eq. (A.1) and Eq. (A.2) is substantial, as Figure A.1 and Figure A.2 depict:

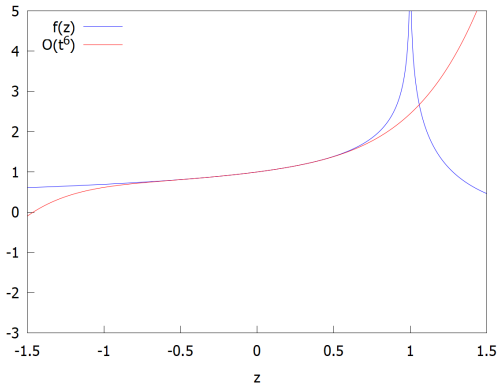


FIGURE A.1: Plot of function (A.1) and its 5^{th} order MacLaurin series (A.2).

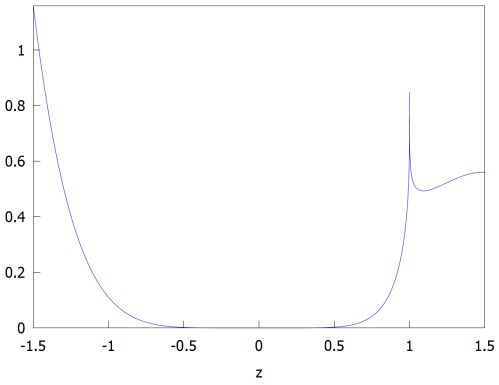


FIGURE A.2: Relative error of approximation given by Eq. (A.2) up to $\mathcal{O}(z^6)$. The divergence becomes significative for $|z| \gtrsim 1$.

The restriction imposed by r is due to the singularity of $f(z)$ in the complex plane for $z = 1$. The fact that it lies outside our region of interest $z \in \left[Ce^{-\lambda t}/K, 0\right]$, with $t \in \mathbb{R}^+$, and still affects the validity of the approximation calls for a different approach. The disruption caused by singularities can be suitably suppressed by the means of Padé approximants

[31]. This formalism ensues from the following consideration:

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} c_n z^n &= \frac{a_0 + a_1 z + a_2 z^2 + ... + a_L z^L}{b_0 + b_1 z + b_2 z^2 + ... + b_M z^M} + \mathcal{O}(z^{L+M+1}) \\ &= \frac{P^{[L/M]}(z)}{Q^{[L/M]}(z)} + \mathcal{O}(z^{L+M+1}) \\ &\equiv [L/M]\,(z) + \mathcal{O}(z^{L+M+1}), \end{aligned} \tag{A.3}$$

ensuring that $b_0 \neq 0$ and noting that L and M are respectively the degrees of the numerator and denominator polynomials, $P^{[L/M]}(z)$ and $Q^{[L/M]}(z)$. However, the multiple ways of summing L and M into the same value allow for plenty of approximants to be chosen from with a given order $L + M$. Extra care needs to be taken, as is the case that just opting for an higher order does not guarantee a more reliable approximations. Particularly, the rational function $[L/M](z)$ must be such that the broad profile of the approximated function is asymptotically recovered. For $z \approx 0$ the fit accurately represents the original function, as it is the point were the MacLaurin series is based upon. Padé approximants in eq. (A.3) are constructed with the MacLaurin series' coefficients, ensuring a reliable behaviour near the origin. Nonetheless, criteria to evaluate the global stability and convergence is needed. Hence, by extending function $f(z)$ into the complex plane, Order Stars will become the way to obtain the result [31, 32]. By performing the ratio between function and approximant

$$s(z) = \frac{[L/M](z)}{f(z)}, z \in \mathbb{C} \tag{A.4}$$

not only the presence of undesired divergences becomes evident but also the stability of $[L/M](z)$ in any given region of interest. Evidently, the approximation is as reliable as $s(z)$ is closer to 1.

Retracing our steps back to the task at hand, Order Stars referring to Eq. (A.1) were plotted as visual auxiliaries in picking an adequate pair $\{L, M\}$ (Figures A.3, A.4, A.5, A.6, A.7, A.8, A.9 and A.10). As mentioned, our interest lies in the interval $\mathcal{A}$ and so any comment with respect to the approximation refers to said sub-region of $\mathbb{C}$.

From Figure A.3 and Figure A.4 one concludes that $[L/L]$ class approximants are not well suited. Both respective Order Stars grow rapidly in the $\mathbb{C}^-$ semi-plane, an obvious signal that the asymptotic limit of equation (A.1), $f(z \to -\infty) \sim z^{-1}$, is not replicated. Clearly, the ideal choice would be $[L/L+1]$ class quotients. Depicting the lowest order approximant of such class, Figure A.5, and Figure A.6 support this expectation despite the loss of reliability as $r \gtrsim 50$. In fact, by simply raising the denominator's degree to

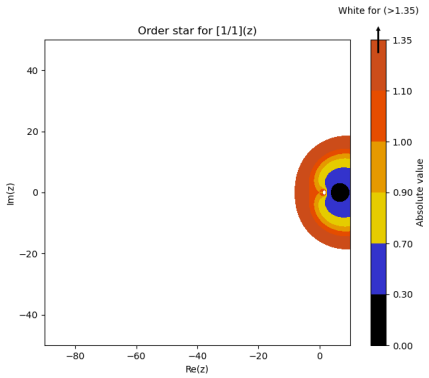


FIGURE A.3: Order Star, given by Eq. (A.4), referring to Padé approximant [1/1](z). The colourbar indicates the ratio between Eq. (A.1) and the approximant, where, in the white region, $s(z)$ starts to grow significantly, eventually reaching infinity. To obtain the best approximation, $s(z)$ should the close to 1 in the region $z \in]-\infty, 0[$.

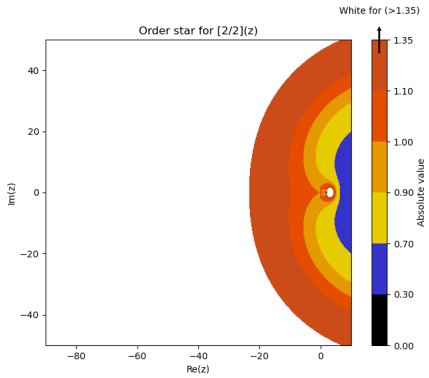


FIGURE A.4: Padé approximant [2/2](z).

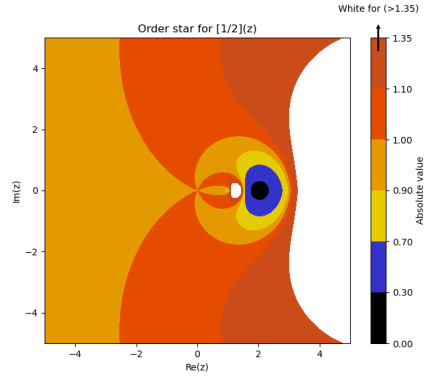


FIGURE A.5: Padé approximant [1/2](z) zoomed in.

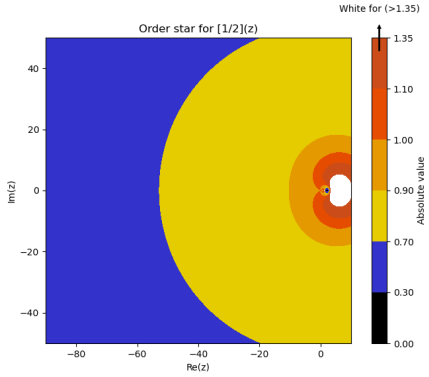


FIGURE A.6: Padé approximant [1/2](z).

$L = M + 2$ the fit no longer matches $f(z)$, as evinced by Figure A.9, where a singularity globally disrupts the fit. As a side note, the Order Star plotted in Figure A.10 depicts a similar problem occurring when considering the series in Eq. (A.2), re-obtaining the divergence found in Figure A.1.

Having shown that the best fit is obtained when $L = M - 1$, the order of the polynomials can be decided by the required accuracy one aims to achieve. For instance, Figure A.7 and Figure A.8 show that picking $L = 2$ and $M = 3$ results in a better fit than the lowest order approximant of the same class. This particular fit is constructed with the following polynomials:

$$[2/3](z) = \frac{11z^2 + 60z - 60}{3z^3 - 36z^2 + 90z - 60}. \tag{A.5}$$

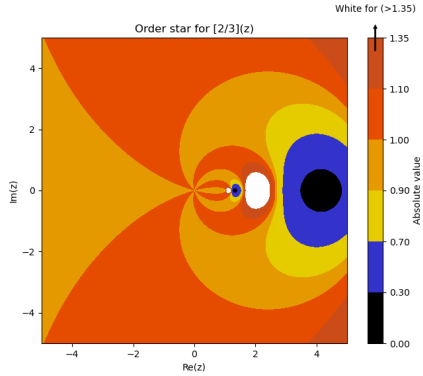


FIGURE A.7: Padé approximant [2/3](z) zoomed in.

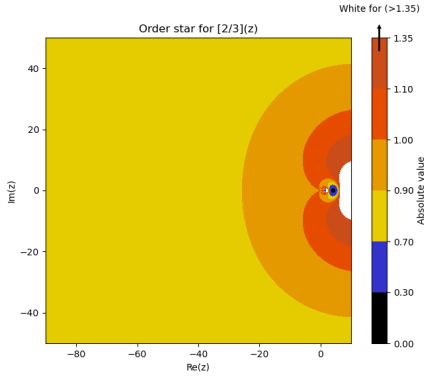


FIGURE A.8: Padé approximant [2/3](z).

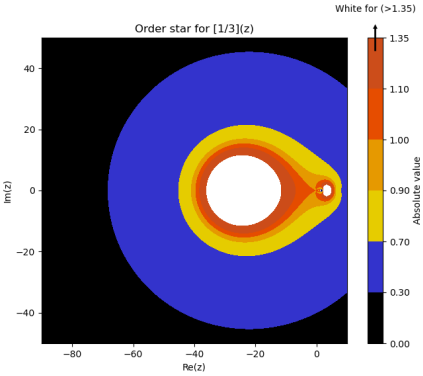


FIGURE A.9: Padé approximant [1/3](z).

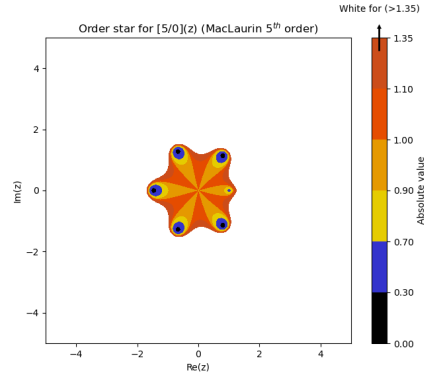


FIGURE A.10: Padé approximant [5/0](z) zoomed in.

As a matter of fact, the maximum error of these approximants is tied to the ratio $r = C/K$ and Eq. (A.5) guarantees a relative error below 3% assuming $r \lesssim 10$, when integrated. Recalling that $z = -Ce^{-\lambda t}/K$, the asymptotic limit $t \to \infty$ is the same as $z \to 0$. Hence, as t increases, a decrease in the relative error is expected, noticing that Eq. (A.5) is constructed by the means of parameters relative to the region $z \approx 0$ (see Eq. (A.2)). Specifically, the relative error's percentage dependence on r is as follows:

$$\Delta_{\%}(r) = \left| 100 - \frac{11r^2 - 60r + 60}{3r^3 - 36r^2 + 90r - 60} \frac{100r}{\log|1-r|} \right|, \tag{A.6}$$

conveniently plotted in Figure A.11. The relative error's growth with $-r$ eventually stalls and ultimately evolves as:

$$\Delta_{\%}(r \to -\infty) = \left| 100 - \frac{1100}{3\log|1-r|} \right|. \tag{A.7}$$

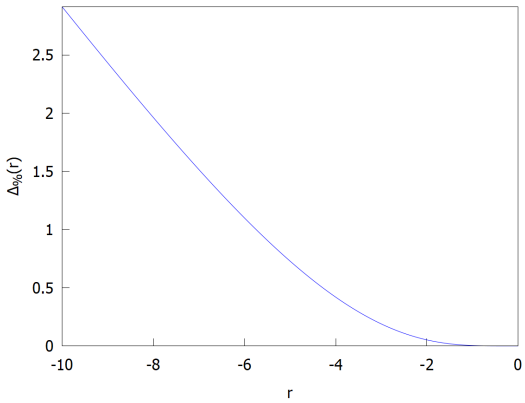


FIGURE A.11: Relative error of Padé approximant $[2/3]$.