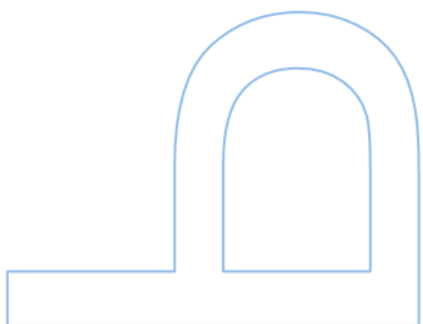




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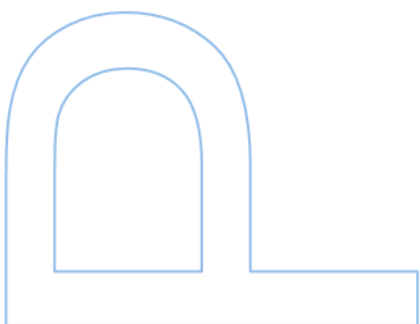
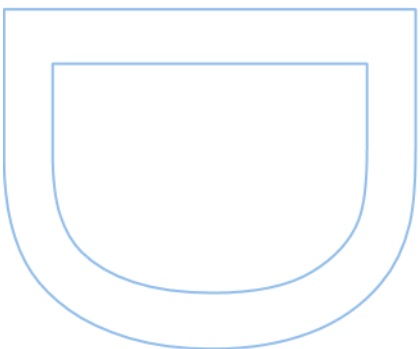
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Universidade do Minho

Attractiveness of cycles in heteroclinic networks

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Doctoral Program in Applied Mathematics
Faculty of Sciences of the University of Porto, University of Aveiro and
University of Minho
2025



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Abstract

This thesis analyzes the dynamics of two types of models in population dynamics and in game theory. The models describe extensions for the Rock-Scissors-Paper (RSP) game that create (i) the Rock-Scissors-Paper-Lizard-Spock (RSPLS) game and (ii) the Jungle Game, both representing distinct and interesting scenarios involving a larger number of either species or game strategies. All the models are discussed in the context of the dynamics and stability of heteroclinic cycles within heteroclinic networks.

We introduce some essential background in the context of heteroclinic dynamics. Accordingly, we explain how the stability of heteroclinic cycles can be obtained from the value of the local stability index along each heteroclinic connection.

We start with the study of the RSPLS game, showing the asymptotic stability of the network for parameter values in a range compatible with both population and game dynamics. We obtain estimates of the relative attractiveness of each one of the cycles by computing their stability indices. For the parameter values ensuring the asymptotic stability of the network we relate the attractiveness properties of each cycle to that of the others. In particular, for three of the cycles we show that if one of them has a weak form of attractiveness, then the other two are completely unstable. We also show the existence of an open region in parameter space where all four cycles are completely unstable and the network is asymptotically stable, giving rise to intricate dynamics that has been observed numerically by other authors.

We proceed with the study of the Jungle Game with four species interacting via a food chain. We prove that, for an open set of parameter values, the heteroclinic network in the Jungle Game with four species is asymptotically stable. The stability of heteroclinic cycles in the network for the Jungle Game with four species determines that only three species coexist in the long-run, interacting under cyclic dominance as a RSP Game. This is in stark contrast with other interactions involving four species, such as cyclic interaction and intraguild predation. We use the Jungle Game with four species to determine the success of a fourth species invading a population of RSP players.

The Jungle Game can be extended to more species. We obtain the stability of the heteroclinic cycles of a Jungle Game with five species. We also prove that, regardless of whether the number of species interacting in the game is four or five, only the cycle involving the bottom, top and second to the top species is stable. We conjecture that the same holds for any number of species. We present a partial proof of this conjecture by showing that this cycle is always stable, while uniqueness remain a conjecture. These three species interact through a Rock-Scissors-Paper game ensuring that cyclic dominance of those three species persists in the long-run. Some of the results are extended to a Jungle Game with an arbitrary number of players.

Resumo

Esta tese analisa a dinâmica de dois tipos de modelos de dinâmica de populações e teoria dos jogos. Os modelos descrevem extensões do jogo Pedra-Papel-Tesoura (RSP) que originam (i) o jogo Pedra-Papel-Tesoura-Lagarto-Spock (RSPLS) e (ii) o Jungle Game. Ambos representam cenários distintos e interessantes envolvendo um largo número de espécies ou estratégias de jogo. Todos os modelos são discutidos em contexto de dinâmica e estabilidade de ciclos heteroclínicos em redes heteroclínicas.

Introduzimos alguns conceitos fundamentais no contexto da dinâmica heteroclínica. Conformemente, explicamos como a estabilidade de ciclos heteroclínicos pode ser determinada pelo valor do índice de estabilidade local ao longo de cada ligação heteroclínica.

Começamos com o estudo do jogo RSPLS, demonstrando a estabilidade assintótica da rede para valores de parâmetros compatíveis com a dinâmica de populações e do jogo. Obtivemos estimativas para a atratividade relativa de cada ciclo através do cálculo dos índices de estabilidade. Para os valores dos parâmetros que garantem a estabilidade assintótica da rede, relacionamos as propriedades de atratividade de cada ciclo em relação aos outros. Em particular, mostramos que, para três dos ciclos, se um deles apresentar uma forma fraca de atratividade, então os outros dois são completamente instáveis. Também demonstramos a existência de uma região aberta no espaço de parâmetros onde todos os quatro ciclos são completamente instáveis, mas a rede é assintoticamente estável, dando origem a dinâmicas complexas que já foram observadas numericamente por outros autores.

Prosseguimos com o estudo do Jungle Game com quatro espécies que interagem por meio de uma cadeia alimentar. Provamos que, para um conjunto aberto de valores dos parâmetros, a rede heteroclínica do Jungle Game com quatro espécies é assintoticamente estável. A estabilidade dos ciclos heteroclínicos determina que, no longo prazo, apenas três espécies coexistem, interagindo sob um regime de dominância cíclica, semelhante ao jogo Pedra-Papel-Tesoura. Isso contrasta fortemente com outros tipos de interação entre quatro espécies, como a interação cíclica e a predação entre espécies. Utilizamos o Jungle Game com quatro espécies para analisar o sucesso da invasão de uma quarta espécie numa população de jogadores num jogo Pedra-Papel-Tesoura.

O Jungle Game pode ser estendido para um número maior de espécies. Obtivemos a estabilidade dos ciclos num Jungle Game com cinco espécies. Além disso, provamos que, independentemente de o número de espécies que interagem no jogo ser quatro ou cinco, apenas o ciclo que envolve a espécie no fundo da cadeia alimentar, a do topo e a segunda a seguir ao topo é estável. Conjecturamos que o mesmo acontece para qualquer número de espécies. Apresentamos uma demonstração parcial desta conjectura, mostrando que esse ciclo é sempre estável, embora a sua unicidade permaneça uma conjectura. Essas três espécies interagem através de um jogo Pedra-Papel-Tesoura, garantindo que a dominância cíclica dessas três espécies persiste a longo prazo. Alguns dos resultados são estendidos a um Jungle Game com um número arbitrário de jogadores.

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Chapter 1

Introduction

1.1 Overview

The study of dynamical systems within the mathematical sciences reveals intricate behaviors emerging from simple deterministic rules. Among these, heteroclinic networks have been an intriguing area of research within dynamical systems for several decades due to their structures and dynamics.

This thesis is concerned with the study of dynamical systems modelling population dynamics and game theory. The models describe the interaction of different species, and consist of systems of differential equations, whose equilibria typically correspond to the survival of only one species. The equilibria in each model are connected by trajectories which indicate that one species wins over the other. The direction of movement is towards the equilibrium representing the winning species whose numbers grow either by feeding on the other or by successfully competing with it. In addition to population dynamics, the models draw on concepts from evolutionary game theory, where species or strategies are viewed as players in a game. Their interactions are governed by payoff matrices that determine reproductive success or survival advantage. The differential equations describe how the proportion of individuals using a certain strategy changes over time based on its performance relative to the population. This framework allows for the analysis of stable strategies and how competitive advantages shift over time depending on the population composition.

If no species wins over all the other, cyclic dominance can occur. This is a phenomenon whereby one species increases in number for a while (i.e., wins) until it becomes prey (or is beaten) by another species. This latter species then increases in number until yet another species becomes stronger. When this process continues in a repeating cycle, the dynamics exhibit cyclic dominance.

The classic example for cyclic dominance in population dynamics is that of the *Uta Stansburiana* lizards whose dynamics is the same as that of a game of Rock-Scissors-Paper (RSP) exhibiting cyclically increasing and decreasing numbers of each variety. See, for instance, the work of Allesina and Levine [2], May and Leonard [30], Sinervo and Lively [44] and Szolnoki *et al.* [46]. The RSP game has been extended to more species, as the examples for four species studied by Durney *et al.* [15], Hua [21], Intoy and Pleimling [22], Roman *et al.* [43], Szabó and Sznaider [45], and Dobramysl *et al.* [14].

In game theory, the equilibria describe the certain use of a single action and cyclic dominance occurs when one action performs better for a while only to be replaced by another cyclically. Examples of the RSP game in this context appear in Hofbauer and

Sigmund [20] and Szolnoki *et al* [46].

Postlethwaite and Rucklidge [42] studied a generalization of the RSP game which includes two new species, Lizard and Spock, with the property that each species wins over half of the remaining species and loses when confronted with the other half, forming the RSPLS game. Cyclic dominance is a mechanism that allows multiple species to coexist by creating a dynamic balance where no single species can outcompete all others.

This research focuses on two types of extensions for the RSP game that create (i) the RSPLS game and (ii) the Jungle Game. Both games represent distinct and interesting scenarios in population dynamics. The RSPLS game is characterized by cyclic dominance with five species, see Figure 1.1, while the Jungle Game is a hierarchical interaction game between an arbitrary number of species, see Figure 1.2.

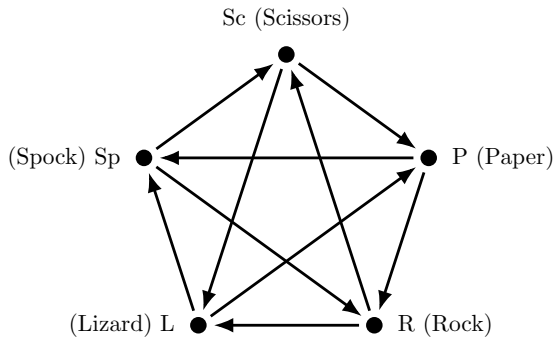


Figure 1.1: The RSPLS game: a directed edge indicates that the starting node beats the end node.

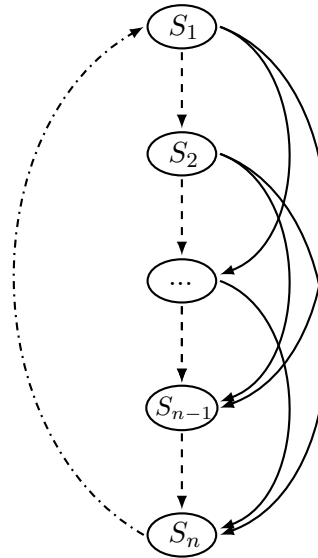


Figure 1.2: The relationships of a Jungle Game with n species. The arrows point from prey to predator and indicate that the coordinate corresponding to the end-point grows as that corresponding to the node at the beginning decreases. We distinguish three strengths of interaction: directly from S_i to S_{i+1} by a dashed line, from S_i to S_j where $j \neq i+1, n$ by a solid line, and by a dash-dotted line the interaction between the lowest species S_n and the top species S_1 .

The literature on population dynamics is vast, and we select only a few studies for clarity based on relevance and our personal preferences. Park and Jang [33] and Kang *et al.* [24, 25] are examples of numerical studies of coexistence of 5 species in a spatial version of RSPLS. There are other studies where the introduction of new species into an existing system can significantly influence both cyclic dominance and species coexistence. An invading species introduces a new interaction pathway. Choices for the invasion and reproduction rates leading to the coexistence of some but not all the original 5 species appear in the work of Vukov *et al.* [49] who extend the work of [24] to contemplate more invasion rates and find that two species become extinct while the remaining three coexist. An analogous outcome is found by Cheng *et al.* [12] by looking at mesoscopic (i.e., intermediate scale) interactions whereas the modelling through PDEs supports an

outcome of only two surviving species in Park *et al.* [32]. In addition to those studies listed above, there are others where the RSP game is extended to more species. Yang and Park [50] used this procedure to obtain a further extension to seven species. They show that in a spatial game, the level of mobility conditions the number of coexisting species. Further examples of spatial competition involving three, four and five species can also be found in the work of Park and co-authors [32], [33], and [34]. All of these studies obtain numerical results.

The Jungle Game extends the RSP by adding species and constructing a hierarchical interaction. It may be interpreted as describing a hierarchical chain such that each species wins over the ones below, except for the first and last species. The last species wins over the first. Figure 1.2 shows the interactions present among species in a generic Jungle Game with an arbitrary number n of species. These can be precisely described in the following way: let S_i , $i = 1, \dots, n$ denote a species so that the lower the index the closer to the top of the hierarchy the species is. Then S_i for $i = 1, \dots, n-1$ preys on S_j where $j = i+1, \dots, n$ while S_n just preys on S_1 . This game appears as a chinese board game, also under the name of *Dou Shou Qi*, with eight species. The original RSP appears in any cycle of three species within the Jungle Game: $[S_1 \rightarrow S_j \rightarrow S_n \rightarrow S_1]$, with $j \neq 1, n$. See Kang *et al.* [23] for a general description of the game and simulations in the case of five species, as well as the work of Buss and Jackson [6] for an account of the Jungle Game in coral reefs. As suggested by [23] the Jungle Game can also describe the growth and destruction of forests, from the low vegetation state to increasing height and finally destruction by fire.

It is important to note that the dynamic model of all the problems addressed in this thesis exhibits a heteroclinic network.

1.2 Stability of cycles in heteroclinic dynamics

A heteroclinic cycle is a union of a finite number of equilibria and the trajectories connecting them in a cyclic way. In turn, a heteroclinic network is a connected union of finitely many heteroclinic cycles. Heteroclinic networks and the stability of their cycles play a crucial role in understanding long-term behavior in population dynamics and game theory. Heteroclinic networks describe trajectories in phase space where the system moves between different saddle equilibria, following heteroclinic cycles. In population models the equilibria correspond to the survival of only one species. The stability of a cycle determines whether a system exhibits persistence, extinction, or cyclic behavior. These networks often appear in models describing competitive interactions ([2, 6, 28]), evolutionary games ([12, 20, 25, 34, 46, 50]), and ecological systems ([19, 47]), where different strategies or species dominate sequentially over time.

Heteroclinic cycles have different levels of complexity. The least complex have been classified by Krupa and Melbourne [26], the so-called simple heteroclinic cycles. A simple heteroclinic cycle is such that the connections between consecutive equilibria are one-dimensional contained in a two-dimensional subspace. This kind of cycle was studied later by many researchers. See for instance, Podvigina and Lohse [36], Castro and Lohse [9] and Voit and Meyer-Ortmanns [48]. Chossat *et al.* and Podvigina and Chossat [13, 40, 41] defined and studied pseudo-simple heteroclinic cycles. Contrasting with simple heteroclinic cycles, a pseudo-simple one has at least one equilibrium with an unstable manifold which is two-dimensional. Recently Garrido-da-Silva and Castro [18] dropped some assumptions and have presented results for another class, the quasi-simple heteroclinic cycles. Quasi-

simple robust heteroclinic cycles are those whose connections are one-dimensional and contained in flow-invariant spaces of equal dimension. These flow-invariant spaces do not necessarily arise from a symmetric setting. All the cycles analyzed in this research are either simple or quasi-simple.

In multi-species ecological models stability is crucial for understanding biodiversity and species' coexistence. We can associate the stability of each of the cycles to the survival of the species that correspond to its equilibria. Around a stable cycle there is a large set of initial conditions whose trajectories follow the cycle. If a cycle is stable, we expect the coexistence of the species associated with the equilibria in that cycle as time evolves. Although heteroclinic cycles in a network cannot attract all initial conditions in any of its neighbourhoods, that is, cannot be asymptotically stable (otherwise we have just a cycle and not a network), they may exhibit weaker notions of stability such as *fragmentary asymptotic stability* (f.a.s.) and *essential asymptotic stability* (e.a.s.) which were introduced by Podvigina [37] and Melbourne [31], respectively. The notion of e.a.s. is strong enough to allow e.a.s. cycles to be visible in simulations. The attraction of an f.a.s., but not e.a.s., cycle is frequently (but not always) too weak to be spotted in simulations or experiments. If the whole network is asymptotically stable it attracts all nearby trajectories. Less stable cycles in an asymptotically stable network may thus become visible. We use previously established and new results concerning stability of networks and cycles (see Podvigina *et al.* [39] and Garrido-da-Silva and Castro [18]) to study the stability of the entire network and of the heteroclinic cycles.

1.3 Contributions and Organization

This thesis focuses on the stability analysis and dynamics of heteroclinic networks, particularly those two mentioned above supported by a system of ordinary differential equations. This research addresses several questions: How can the various stability types be used to predict the outcome of the interaction among species or to play a game? What are the implications for species' coexistence in ecological models where such networks are present? By focusing on these questions, this work contributes to the broader understanding of heteroclinic networks and their applications.

We study two different systems of ordinary differential equations under Lotka-Volterra competition which are used as models in both population dynamics and game theory. Each system exhibits a heteroclinic network describing the RSPLS game and the Jungle Game, respectively. We show the asymptotic stability of the RSPLS network and the Jungle Game with four species network, for parameter values within a range compatible with both population and game dynamics. We provide a thorough analysis of the stability of the cycles within the RSPLS and the Jungle Game with four and five species. In the case of the Jungle Game, we go further by proving that only the cycle involving the bottom, top and second from the top species is stable independently of the number of the species interacting in the game. These three species interact through a Rock-Scissors-Paper game ensuring that cyclic dominance of three species persists in the long-run. We also interpret the Jungle Game with four species as a game describing the introduction of a fourth species, an Alien, into the cyclic dominance of three species modelled by a RSP game. We show that the outcome depends on the nature of the invading species, the Alien. If the Alien is weak, in the sense that it loses against two of the original species, then it becomes extinct and the original species coexist in the long-run. If the Alien is strong, i.e. it loses against only one of the original species, then one of the original species becomes

extinct and is replaced in the RSP by the Alien. We can further determine which species is replaced by the Alien. All our results are analytical.

The thesis proceeds as follows: the next chapter introduces some notation, definitions and results in the literature that are used in the sequel. Chapter 3 is devoted to the study of the RSPLS network. It consists of results published in the joint article with Castro, Labouriau and Garrido-da-Silva in [11]. Chapter 4 is devoted to the study of the Jungle Game with four species. It consists of results published in the joint article with Castro and Labouriau in [8]. Chapter 5 provides new unpublished results on the study of the Jungle Game with five species and some generalization for the case of a Jungle Game with an arbitrary number of species. The final chapter provides some concluding remarks.

Chapter 2

Preliminaries

This chapter establishes the notation and provides an overview of methods and fundamental concepts and other relevant material related to the stability of cycles and the dynamics near heteroclinic structures.

2.1 Population models

There are many ways of modelling the interaction by an ordinary differential equation. Here we use a Lotka-Volterra model. The Lotka-Volterra equations are a class of differential equations widely used to model interactions between species, particularly in the context of predator-prey dynamics and competitive systems.

For a model with n populations S_i , ($i = 1, \dots, n$), in Chapters 4 and 5 we will model it by a system of n differential equations in the variables $x_i > 0$, for $i = 1, \dots, n$ of the form

$$\dot{x}_i = x_i f_i(x), \quad x = (x_1, \dots, x_n) \quad (2.1)$$

where the x_i represent the proportion of the species S_i in the total population and the f_i are affine functions of the x_i . Note that the points where $x_i = 1$ and $x_j = 0$ for all $j \neq i$ are equilibria, that is, only the species S_i survives. In Chapters 4 and 5 we will use the particular form:

$$\dot{x}_i = x_i \left(1 - R + \sum_{j \neq i} \ell_{ij} x_j \right), \quad R = \sum_{j=1}^n x_j \quad (2.2)$$

where the coefficient ℓ_{ij} is positive if the graph contains an arrow from S_i to S_j , negative if the arrow points in the opposite direction, zero if there is no arrow connecting S_i to S_j .

A key structural property of the Lotka-Volterra system is the existence of invariant subspaces corresponding to the extinction of subsets of species. Specifically, the coordinate hyperplanes $x_i = 0$ for each i are invariant under the flow, reflecting the biological constraint that extinct species (i.e., populations reduced to zero) remain extinct. In particular, it follows that the planes where all but two of the coordinates are zero are also invariant. Under suitable assumptions this can be used to show the existence of heteroclinic connections between the equilibria, that persist under small perturbations within this class of systems, as will be explained below.

Models that codify games and population interactions are usually described by a graph where nodes correspond to the axial equilibria and edges represent the heteroclinic connections between them. The edges are arrows that point from each population to the

one it wins over, or preys on. In the models (2.1), the orientation of the heteroclinic connections within the phase space is the opposite of the direction of the arrows in the graph. In the graph, an edge from species i to species j typically signifies that species j outcompetes species i (i.e., $l_{ji} > l_{ij}$). However, the heteroclinic trajectory in the phase space leads from the equilibrium associated with species j towards that of species i , reflecting the fact that species i eventually invades and replaces species j in the competitive process. Thus, the flow on the heteroclinic network is oriented oppositely to the edges in the graph representing direct competition strengths. In what follows, the methods used in the remaining chapters for the analysis of the Lotka-Volterra formulation of the games are presented.

2.2 Definitions

Consider a smooth vector field f from $\mathbb{R}^n$ to itself described by a system of ODEs:

$$\dot{x} = f(x), \quad (2.3)$$

An *equilibrium* $\xi \in \mathbb{R}^n$ of (2.3) satisfies $f(\xi) = 0$. We denote the stable and unstable manifolds of ξ by $W^s(\xi)$ and $W^u(\xi)$, respectively.

Definition 2.2.1. Given two hyperbolic equilibria of (2.3), ξ_i and ξ_j , we call,

$$C_{ij} = W^u(\xi_i) \cap W^s(\xi_j),$$

a *connection*¹ from ξ_i to ξ_j .

We assume that ξ_i and ξ_j are neither the same equilibrium nor symmetry related so that the connection is *heteroclinic*. Note that if $\dim(C_{ij}) > 1$ the connection C_{ij} consists of infinitely many *connecting trajectories* $\kappa_{ij} = [\xi_i \rightarrow \xi_j]$, solutions of (2.3) that converge to ξ_i in backward time and to ξ_j in forward time.

Definition 2.2.2. A *heteroclinic cycle* is a finite union of hyperbolic saddles, $\xi_1, \dots, \xi_m$ such that there exist connections $C_{j,j+1}$ for $j = 1, \dots, m$ with $\xi_{m+1} = \xi_1$.

Generically, a connection between two saddles is not robust but when it is contained in a flow-invariant space where the connection is of saddle-sink type, robustness is the norm. Such flow-invariant spaces appear naturally in symmetric dynamics, in the form of fixed-point spaces, as well as in game theory dynamics, in the form of either coordinate hyperplanes (Lotka-Volterra systems) or hyperfaces of a simplex (replicator dynamics, another formulation of these models, see for example the work of Bomze [5]).

Definition 2.2.3. A connected union of finitely many heteroclinic cycles is a *heteroclinic network*.

In the context of heteroclinic networks and cycles the equilibria are sometimes called *nodes*.

We focus on the stability of heteroclinic cycles that are part of the same heteroclinic network. In such case there is at least one equilibrium that belongs to more than one cycle. It is clear that, in this case, $W^u(\xi_i)$ is not contained in a single heteroclinic cycle

¹Ashwin *et al.* [3] use “full set of connections” instead of “connection”.

and therefore, cycles in a heteroclinic network are never asymptotically stable. Several intermediate notions of stability exist in the literature. The second strongest notion of stability we can find is *essential asymptotic stability* (e.a.s.) introduced by Melbourne [31] and refined by Brannath [4]. An e.a.s. object attracts almost all trajectories that start nearby. A weaker notion of attractiveness, referred by Podvigina [37], is *fragmentary asymptotic stability* (f.a.s.). A f.a.s. object attracts a positive measure set nearby, that may be very small. If a heteroclinic cycle is not, at least, f.a.s., then it is *completely unstable* (c.u.) as in Krupa and Melbourne [26] and attracts almost nothing. Rigorous definitions are given in Section 2.5 below.

A large class of heteroclinic networks is that of *quasi-simple* networks whose stability properties are systematically studied by Garrido-da-Silva and Castro [18]. The stability results in [18] can be used for any heteroclinic cycle along which the return map has a particular form. Let P_j be a flow-invariant sub-space and $\hat{L}_j$ be the vector sub-space spanned by ξ_j in $\mathbb{R}^n$:

Definition 2.2.4. A *quasi-simple* cycle is a robust heteroclinic cycle connecting $m < \infty$ equilibria $\xi_j \in P_j \cap P_{j-1}$ so that for all $j = 1, \dots, m$:

- (i) P_j is a flow-invariant space,
- (ii) $\dim(P_j) = \dim(P_{j+1})$,
- (iii) $\dim(P_j \ominus \hat{L}_j) = 1$, where $P_j \ominus \hat{L}_j$ is the orthogonal complement to $\hat{L}_j$ in P_j .

We then classify the eigenvalues of the Jacobian matrix at ξ_j as into four groups:

- radial eigenvalues $-r_j$, which have eigenvectors in $\hat{L}_j$;
- contracting eigenvalues $-c_{ji}$ which have eigenvectors in $P_{j-1} \ominus \hat{L}_j$;
- expanding eigenvalues e_{jk} which have eigenvectors in $P_j \ominus \hat{L}_j$;
- transverse eigenvalues $t_{j,\ell}$ with $\ell = 1, \dots, s$, otherwise.

The radial and contracting eigenvalues are negative, the expanding eigenvalues are positive but the transverse ones can have either sign.

2.3 Notions for the stability of a network

Definition 2.3.1. An *ac-network* is a heteroclinic network such that all equilibria are located on coordinate axes (not more than one equilibrium per half-axis), with unstable manifold of dimension one or two which are entirely contained in the network.

Podvigina *et al.* [39] provide sufficient conditions for the asymptotic stability of ac-networks. For easy reference, we introduce the necessary notation and definitions and transcribe the relevant results below. Although stated for ac-networks these results apply to a wider set of networks and cycles.

Since a necessary condition for the asymptotic stability of a heteroclinic network is that the unstable manifold of every node be contained in the network, a structure called a Δ -*clique* typically appears, see Definition 2.1 in [39]. Loosely speaking, a Δ -clique Δ_{ijk} , illustrated in Figure 2.1, is a subset of a heteroclinic network, made of 3 nodes and 3 full sets of trajectories, that is not a cycle. Every Δ -clique, Δ_{ijk} , has a beginning point

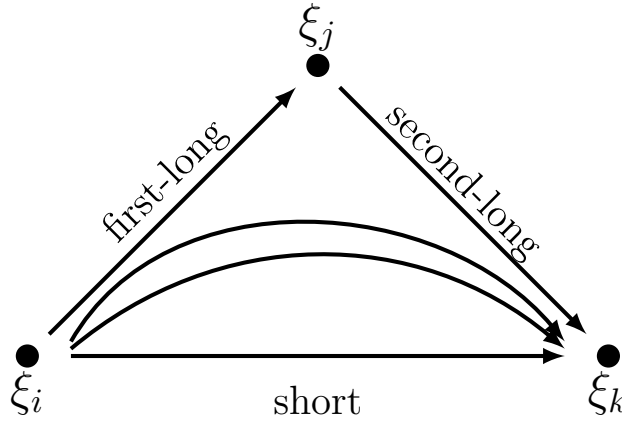


Figure 2.1: A Δ -clique Δ_{ijk} . The short connection is C_{ik} , the first-long connection is C_{ij} and the second-long connection is C_{jk} .

(b-point), a medium point (m-point) and an end point (e-point), denoted respectively by ξ_i , ξ_j , and ξ_k in Figure 2.1. The short connection of a Δ -clique is 2-dimensional and connects the b- and e-points directly. The first-long and second-long connections are 1-dimensional and together they indicate a transition from the b-point to the e-point through the m-point.

The trajectories move away from a b-point in the direction of the first-long connection and approach an e-point along the second-long connection. This corresponds to the way in which trajectories in a Δ -clique are typically drawn. This occurs since we assume that the eigenvalues satisfy the following condition obtained from the linearization of the flow:

$$e_{ik} > e_{ij} \quad \text{and} \quad c_{ki} > c_{kj}. \quad (2.4)$$

We rewrite Lemmas 4.4, 4.6 and 4.8 from [39] according to our context and needs. We use the notation $c = \max c_{ij}$ and $e = \max e_{ij}$. Let ξ_j be an equilibrium in an ac-network Y . The conditions for asymptotic stability of a network are formulated in Corollary 4.18 of [39] in terms of some quantities ρ_j associated to each node in the network that may be computed using the lemmas below.

Lemma 2.3.2 (Lemma 4.4 from [39]). *Suppose that an equilibrium $\xi_j \in X$ is not an m-point for any of the Δ -cliques of X . Then*

$$\rho_j = \min(c/e, 1 - t/e) \quad (2.5)$$

Lemma 2.3.3 (Lemma 4.6 from [39]). *Suppose that an equilibrium $\xi_j \in X$ is an m-point for several (one or more) Δ -cliques of X and it has one expanding eigenvector. Then*

$$\rho_j = \min(c/e, 1) \quad (2.6)$$

Lemma 2.3.4 (Lemma 4.8 from [39]). *Suppose that an equilibrium $\xi_j \in X$ has one contracting eigenvector, two expanding eigenvectors and ξ_j is an m-point for just one Δ -clique. The contracting eigenvector is the f-long vector of the Δ -clique and the expanding eigenvector associated to the eigenvalue e_2 is not a s-long vector of the Δ -clique. Then*

$$\rho_j = \frac{c}{c + e_2} \quad (2.7)$$

Corollary 4.18 from [39] states that if the product of the ρ_j for each cycle in the network is bigger than one then the network is asymptotically stable.

2.4 Cycles and sub-cycles

It is clear that in a network, where the equilibria lie on different coordinate axes, there exists at least one equilibrium whose unstable manifold has dimension at least 2. Consequently, there must be connections C_{ij} of dimension at least 2. We define a sub-cycle by selecting a specific 1-dimensional connecting trajectory within the 2-dimensional connection, namely the trajectory that is contained in a coordinate hyperplane.

The notion of sub-cycles is essential because it allows us to apply the existing results about the calculation of stability indices. In particular, we use the results from [18] where the authors provide a framework for calculating stability indices under the assumption that the cycles are quasi-simple.

2.5 Notions for the stability of cycles

Let X be a compact set in $\mathbb{R}^n$ invariant under the flow $\Phi_t(x)$ of (2.3). Given a metric d on $\mathbb{R}^n$ and $\epsilon > 0$, an ϵ -neighbourhood of X is:

$$B_\epsilon(X) = \{x \in \mathbb{R}^n : d(x, X) < \epsilon\}.$$

The δ -local basin of attraction of X is:

$$\mathcal{B}_\delta(X) = \left\{ x \in \mathbb{R}^n : d(\Phi_t(x), X) < \delta \text{ for any } t \geq 0 \text{ and } \lim_{t \rightarrow \infty} d(\Phi_t(x), X) = 0 \right\}.$$

Definition 2.5.1. The compact invariant set $X \subset \mathbb{R}^n$ is:

- *essentially asymptotically stable* if the measure of its δ -local basin of attraction, $\mathcal{B}_\delta(X)$, tends to full measure in a ϵ -neighbourhood, $B_\epsilon(X)$, of X as δ and ϵ become small, that is, if $\lim_{\delta \rightarrow 0} \left[\lim_{\epsilon \rightarrow 0} \frac{\ell(B_\epsilon(X) \cap \mathcal{B}_\delta(X))}{\ell(B_\epsilon(X))} \right] = 1$; (thick cusp on the right side of Figure 2.2).
- *fragmentarily asymptotically stable* if the measure of its δ -local basin of attraction is positive, that is, if $\ell(\mathcal{B}_\delta(X)) > 0$ for any $\delta > 0$; (thin cusp on the left side of Figure 2.2).
- *completely unstable* if there exists some $\delta > 0$ such that the δ -local basin of attraction of X is of measure zero, that is, $\ell(\mathcal{B}_\delta(X)) = 0$;

where $\ell(\cdot)$ is the Lebesgue measure on $\mathbb{R}^n$.

2.6 The stability index

The local stability index of a cycle at a point in a connection, defined below, was introduced by Podvigina and Ashwin [35] and measures how much of its neighbourhood returns and is attracted to it after following the cycle to which the connection belongs. Results in [35] and Lohse [29] relate the stability of a cycle with the sign of the stability indices

of every connection in that cycle. If the stability index is positive for every connection then the cycle is e.a.s., if the stability index is negative and greater than $-\infty$ for every connection then the cycle is f.a.s. and if all the stability indices are equal to $-\infty$ then the cycle is c.u. It is clear that an e.a.s. cycle is also f.a.s. but not conversely.

Given $x \in X$, small $\delta > 0$ and $\epsilon > 0$, define the relative size of the δ -local basin of attraction in an ϵ -neighbourhood of x as

$$\Sigma_{\epsilon,\delta}(x) = \frac{\ell(B_\epsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\epsilon(x))}.$$

Definition 2.6.1. The *local stability index*² $\sigma(x)$ of a cycle X at a point $x \in X$ in a connection is the difference $\sigma(x) = \sigma_+(x) - \sigma_-(x)$ where

$$\sigma_+(x) = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\ln(\Sigma_{\epsilon,\delta}(x))}{\ln \epsilon} \quad \sigma_-(x) = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{\ln(1 - \Sigma_{\epsilon,\delta}(x))}{\ln \epsilon}$$

We use the convention that $\sigma_-(x) = \infty$ when $\Sigma_{\epsilon,\delta} = 0$ for some $\epsilon > 0$, $\delta > 0$. Analogously, $\sigma_+(x) = \infty$ if there is an $\epsilon > 0$ such that $\Sigma_{\epsilon,\delta} = 1$. Note that $\sigma_\pm(x) \geq 0$, so we can assume that $\sigma(x) \in [-\infty, +\infty]$; the strongest form of local stability corresponds to $\sigma(x) = +\infty$ while $\sigma(x) = -\infty$ is the weakest.

A positive stability index indicates that X attracts all points in the thick side of a cusp in its neighbourhood. If the stability index is negative, only points in the thin side of the cusp are attracted to X . See Figure 2.2. Note that, as proved in Theorem 2.2 of [35], given a connecting trajectory κ_{ij} , for any point $x \in \kappa_{ij}$, the index $\sigma(x)$ is constant on trajectories, whenever it is defined.

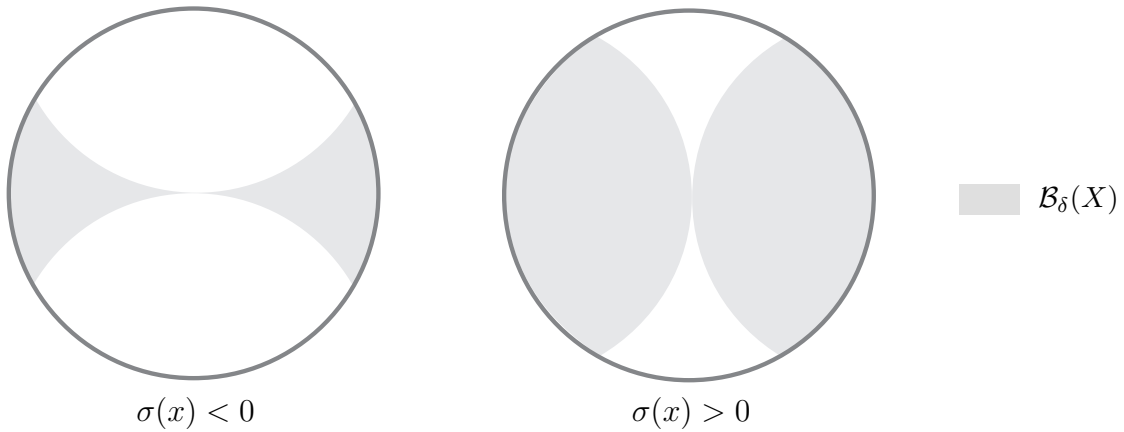


Figure 2.2: A negative stability index (left) indicates that the set of points in $\mathcal{B}_\delta(X)$ are those in the thin (shaded) side of a cusp. A positive stability index (right) corresponds to $\mathcal{B}_\delta(X)$ being in the thick (shaded) side of a cusp.

²We ignore the subscript “loc” used in [35] to distinguish between “stability index” and “local stability index” since we do not use the former.

2.6.1 Calculation of the stability indices

As it is usually done in the literature [18, 37], we study the stability of a cycle by using Poincaré return maps from a cross section. Define cross sections to the flow near an equilibrium ξ_j and denote them by H_j^{in} and H_j^{out} along an incoming and outgoing connection, respectively. If the cycle is part of a network and the equilibrium has more than one incoming/outgoing connection, we distinguish them with a second index. Thus, $H_j^{in,i}$ denotes the cross section, near ξ_j , to the $[\xi_i \rightarrow \xi_j]$ connection whereas $H_j^{out,i}$ denotes the cross section, near ξ_j , to the connection $[\xi_j \rightarrow \xi_i]$. Local maps near ξ_j are such that $H_j^{in} \rightarrow H_j^{out}$ and global maps along a connection $[\xi_i \rightarrow \xi_j]$ are such that $H_i^{out} \rightarrow H_j^{in}$.

We transcribe below the results from [18] which we are going to use in the next chapters, for the calculation of stability indices.

Let n_t be the number of transverse eigenvalues at each equilibrium of the network. The transition $g_j : H_j^{in} \rightarrow H_{j+1}^{in}$ between neighbourhoods of successive equilibria in a cycle is codified by the $N \times N$ *basic transition matrices* M_j (with $N = n_t + 1$), that have the form:

$$M_j = A_j \begin{bmatrix} b_{j,1} & 0 & 0 & \cdots & 0 \\ b_{j,2} & 1 & 0 & \cdots & 0 \\ b_{j,3} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{j,N} & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (2.8)$$

where A_j is a permutation matrix. The dimension of M_j , of course, depends on the dimension of the state-space. The entries of M_j depend on the eigenvalues at an equilibrium ξ_j as follows:

$$b_{j,1} = \frac{c_j}{e_j}, \quad b_{j,l+1} = -\frac{t_{j,l}}{e_j}, \quad l = 1, \dots, s \quad \text{and} \quad j = 1, \dots, m.$$

Given a $N \times N$ matrix M , denote by $\lambda_{\max}$ the biggest eigenvalue in absolute value and by $\mathbf{w}^{\max} = (w_1^{\max}, \dots, w_N^{\max})^T$ the corresponding eigenvector, where the superscript “T” indicates the transpose of a matrix in general.

Lemma 2.6.2. *The necessary and sufficient conditions for stability are (cf [18, Lemma 3.2]):*

- (i) $\lambda_{\max}$ is real,
- (ii) $\lambda_{\max} > 1$,
- (iii) $w_l^{\max} w_q^{\max} > 0$ for all $l, q = 1, \dots, N$.

Consider σ_j to be the stability index along the incoming connection to ξ_j . The following proposition adapts Theorem 3.4 in [18] to our setting.

Proposition 2.6.3 (Theorem 3.4 in [18]). *Let M_j , $j = 1, \dots, m$ be basic transition matrices of a collection of maps associated with a heteroclinic cycle. Suppose that for all $j = 1, \dots, m$, all entries of the matrices are non-negative. Then,*

- (a) *If the transition matrix $M^{(1)} = M_m \dots M_1$ satisfies the condition (ii), then $\sigma_j = +\infty$ for all $j = 1, \dots, m$ and the cycle is asymptotically stable.*

(b) Otherwise, $\sigma_j = -\infty$ for all $j = 1, \dots, m$ and the cycle is c.u..

The calculations required to obtain the local stability index are performed by means of the function F^{index} of [18], which we transcribe bellow.

Definition 2.6.4 (Definition 3.8 in [18]). Let $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ and $\alpha_{\min} = \min_{i=1, \dots, N} \{\alpha_i\}$. The values of the function $F^{\text{index}} : \mathbb{R}^N \rightarrow \mathbb{R}$ are

$$F^{\text{index}}(\alpha) = F^+(\alpha) - F^-(\alpha)$$

where $F^-(\alpha) = F^+(-\alpha)$ and

$$F^+(\alpha) = \begin{cases} +\infty, & \text{if } \alpha_{\min} \geq 0 \\ 0, & \text{if } \sum_{i=1}^N \alpha_i \leq 0 \\ -\frac{1}{\alpha_{\min}} \sum_{i=1}^N \alpha_i, & \text{if } \alpha_{\min} < 0 \text{ and } \sum_{i=1}^N \alpha_i \geq 0. \end{cases}$$

See Appendix of [18] for more details about the construction of F^{index} .

The following proposition adapts Theorem 3.10 in [18] to our setting.

Proposition 2.6.5 (Theorem 3.10 in [18]). Let M_j , $j = 1, \dots, m$ be basic transition matrices of a collection of maps associated with a heteroclinic cycle. Denote by $q = j_1, \dots, j_L$, $L \geq 1$, all the indices for which M_q has at least one negative entry.

(a) If $M^{(j)}$ does not satisfy conditions (i)–(iii) of Lemma 2.6.2 for at least one j , then $\sigma_j = -\infty$ for all $j = 1, \dots, m$ and the cycle is c.u..

(b) If $M^{(j)}$ satisfies conditions (i)–(iii) of Lemma 2.6.2 for all $j = j_p + 1$, $p = 1, \dots, L$, such that $j_p + 1 \notin \{j_1, \dots, j_L\}$, then Σ is f.a.s. Furthermore for each $j = 1, \dots, m$, there exist vectors $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(K)} \in \mathbb{R}^N$ such that

$$\sigma_j = \min_{i=1, \dots, K} \{F^{\text{index}}(\alpha^{(i)})\}.$$

For each $j = 1, \dots, m$, the vectors $\alpha^{(i)}$ that must be considered are the rows of the basic transition matrices M_j and their products:

$$\begin{aligned} M^{(j)} : H_j^{\text{in}} &\rightarrow H_j^{\text{in}}, & M^{(j)} &= M_{j-1} \dots M_1 M_m \dots M_j, \\ M_{(l,j)} : H_j^{\text{in}} &\rightarrow H_{l+1}^{\text{in}}, & M_{(l,j)} &= \begin{cases} M_l \dots M_j, & l > j \\ M_l \dots M_1 M_m \dots M_j, & l < j \\ M_j, & l = j, \end{cases} \end{aligned}$$

where m is the number of equilibria of the cycles in the examples we consider. The number K refers to the number of such rows whenever

$$U^{-\infty}(M^{(j)}) = \left\{ \mathbf{y} \in \mathbb{R}_-^N : \lim_{k \rightarrow +\infty} (M^{(j)})^k \mathbf{y} = -\infty \right\} = \mathbb{R}_-^N,$$

is satisfied, where $\mathbb{R}_-^N = \{\mathbf{y} = (y_1, \dots, y_N) \in \mathbb{R}^N : y_1, \dots, y_N < 0\}$, see [18] for details.

We also reproduce the values of $F^{\text{index}}(\boldsymbol{\alpha})$ for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ from Appendix A.1 of [18], for easy reference:

$$F^{\text{index}}(\boldsymbol{\alpha}) = \begin{cases} +\infty, & \text{if } \min\{\alpha_1, \alpha_2, \alpha_3\} \geq 0 \\ -\infty, & \text{if } \max\{\alpha_1, \alpha_2, \alpha_3\} \leq 0 \\ 0, & \text{if } \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \frac{\alpha_1 + \alpha_2 + \alpha_3}{\max\{\alpha_1, \alpha_2, \alpha_3\}}, & \text{if } \max\{\alpha_1, \alpha_2, \alpha_3\} > 0 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 < 0 \\ -\frac{\alpha_1 + \alpha_2 + \alpha_3}{\min\{\alpha_1, \alpha_2, \alpha_3\}}, & \text{if } \min\{\alpha_1, \alpha_2, \alpha_3\} < 0 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 > 0. \end{cases}$$

2.6.2 A simpler special case

Definition 2.6.6. A simple robust heteroclinic cycle $X \subset \mathbb{R}^4$ is of type B if there is a fixed-point subspace Q with $\dim Q = 3$ such that $X \subset Q$.

The calculation of the stability indices for simple cycles of type B , as defined by Krupa and Melbourne [27], are particular cases of the ones above and are calculated in [35].

Suppose that $\Gamma \subset O(n)$ is a finite Lie group acting linearly on $\mathbb{R}^n$. We transcribe, in the following two lemmas, the results from [35] that we are going to use in chapter 4. The superscript “ $-$ ” indicates that $-I \in \Gamma$ and the subscript “3” indicates the number of equilibria in the cycle. The function f^{index} is used to calculate the local stability indices and is defined as $f^{\text{index}}(\alpha, \beta) = f^+(\alpha, \beta) - f^+(-\alpha, -\beta)$ where

$$f^+(\alpha, \beta) = \begin{cases} +\infty, & \alpha \geq 0, \beta \geq 0 \\ 0, & \alpha \leq 0, \beta \leq 0 \\ -\frac{\beta}{\alpha} - 1, & \alpha < 0, \beta > 0, \frac{\beta}{\alpha} < -1 \\ 0, & \alpha < 0, \beta > 0, \frac{\beta}{\alpha} > -1 \\ -\frac{\alpha}{\beta} - 1, & \alpha > 0, \beta < 0, \frac{\alpha}{\beta} < -1 \\ 0, & \alpha > 0, \beta < 0, \frac{\alpha}{\beta} > -1. \end{cases}$$

For a heteroclinic cycle in $\mathbb{R}^4$ the Jacobian matrix at any equilibrium ξ_j has exactly four eigenvalues, one of each type. For an equilibrium ξ_j in a cycle, let $a_j = \frac{c_{j,j-1}}{e_{j,j+1}}$ and $b_j = -\frac{t_{j,i}}{e_{j,j+1}}$.

Lemma 2.6.7. *For a cycle of type B_3^- with $b_1 < 0, b_2 > 0$ and $b_3 > 0$, the stability indices along connecting trajectories are as follows:*

1. *If $a_1 a_2 a_3 < 1$ or $b_1 a_2 a_3 + b_3 a_2 + b_2 < 0$, then the cycle is c.u. and the stability indices are $-\infty$.*
2. *If $a_1 a_2 a_3 > 1$ and $b_1 a_2 a_3 + b_3 a_2 + b_2 > 0$, then the stability indices are $\sigma_1 = f^{\text{index}}(b_1, 1)$, $\sigma_2 = +\infty$ and $\sigma_3 = f^{\text{index}}(b_3 + b_1 a_3, 1)$.*

Lemma 2.6.8. *For a cycle of type B_3^- with $b_1 < 0, b_2 < 0$ and $b_3 > 0$, the stability indices along connecting trajectories are as follows:*

1. *If $a_1 a_2 a_3 < 1$ or $b_2 a_1 a_3 + b_1 a_3 + b_3 < 0$ or $b_1 a_2 a_3 + a_2 b_3 + b_2 < 0$, then the cycle is c.u. and the stability indices are $-\infty$.*
2. *If $a_1 a_2 a_3 > 1$ and $b_2 a_1 a_3 + b_1 a_3 + b_3 > 0$ and $b_1 a_2 a_3 + a_2 b_3 + b_2 > 0$, then the stability indices are $\sigma_1 = \min(f^{\text{index}}(b_1, 1), f^{\text{index}}(b_1 + b_2 a_1, 1))$, $\sigma_2 = f^{\text{index}}(b_2, 1)$ and $\sigma_3 = f^{\text{index}}(b_3 + b_1 a_3, 1)$.*

All the stability indices above are for a single connecting trajectory and will thus be applied to subcycles as defined in Subsection 2.4.

Chapter 3

Stability of cycles in a game of Rock-Scissors-Paper-Lizard-Spock

This chapter contains results that were published in the article [11], in which I am one of the authors.

The Rock-Scissors-Paper-Lizard-Spock (RSPLS, henceforth) game is an extension of the traditional Rock-Scissors-Paper (RSP) game and has become ubiquitous¹ in the dynamical systems literature, associated especially to population dynamics. Additionally to Rock beating Scissors, Scissors beating Paper and Paper beating Rock, two more actions, Lizard and Spock, are added to construct the following relations

Rock	wins over and	Scissors Lizard
Scissors	win over and	Paper Lizard
Paper	wins over and	Rock Spock
Lizard	wins over and	Paper Spock
Spock	wins over and	Rock Scissors

In the context of game theory, these are considered actions chosen by a player, while in that of population dynamics these represent types or species in a population. In this way, each type/action wins over two other types/actions while it loses when confronted with the remaining two types/actions. These interactions can be described by the graph in Figure 3.1, where each node corresponds to a type or choice of an action and a directed edge indicates that the starting node beats the end node. The equilibria correspond to the nodes and the connecting trajectories correspond to edges in the graph. Note that the graph remains the same under cyclic permutation $(Rock, Paper, Scissors, Spock, Lizard) \rightarrow (Paper, Scissors, Spock, Lizard, Rock)$, a symmetry of the graph.

We contribute to a theoretical understanding of the dynamics generated by the RSPLS game by studying the stability properties of the four types of cycles present in this network. Namely, those shown in Figure 3.2 and those obtained from them by the symmetry, as we proceed to describe:

¹It also appears in less scientific environments such as the television show “The Big Bang Theory”.

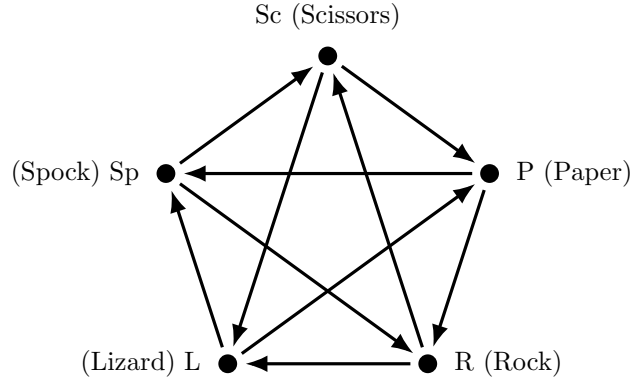


Figure 3.1: The RSPLS game: a directed edge indicates that the starting node beats the end node. In a heteroclinic network the connections have the opposite orientation.

- the Rock-to-Paper cycle corresponding to the cyclic dominance of Rock over Lizard, Lizard over Spock, Spock over Scissors, Scissors over Paper, and finally Paper over Rock;
- the Rock-to-Spock or Star cycle corresponding to the other cyclic dominance among the actions, namely, Rock over Scissors, Scissors over Lizard, Lizard over Paper, Paper over Spock, and at last Spock over Rock;
- the RSP cycle corresponding in the above literature to the coexistence of only three of the five species;
- the Four-node cycle corresponding to the coexistence of four of the five species.

The Rock-to-Paper sub-cycle consists of the trajectories (1-dimensional) in the 2-dimensional connections (solid lines in Figure 3.2, referred to as of type A in [42]) that are contained in coordinate planes. In the Star or Rock-to-Spock cycle all connections are 1-dimensional (dashed lines in Figure 3.2, referred to as of type B in [42]). The RSP sub-cycle has two trajectories which are part of two connections of dimension 2 and one connection of dimension 1. It corresponds to a sequence AAB in [42]. The Four-node sub-cycle comprises one trajectory belonging to one connection of dimension 2 and three connections of dimension 1. It corresponds to a sequence $Q = ABBB$ in [42]. Due to symmetry, the RSP and Four-node cycles appear each in five equivalent configurations as follows:

Rock $\rightarrow$ **Scissors** $\rightarrow$ **Paper** $\rightarrow$ Rock
 Paper $\rightarrow$ Rock $\rightarrow$ Lizard $\rightarrow$ Paper
 Scissors $\rightarrow$ Paper $\rightarrow$ Spock $\rightarrow$ Scissors
 Spock $\rightarrow$ Scissors $\rightarrow$ Lizard $\rightarrow$ Spock
 Lizard $\rightarrow$ Spock $\rightarrow$ Rock $\rightarrow$ Lizard

and

Rock $\rightarrow$ **Scissors** $\rightarrow$ **Paper** $\rightarrow$ **Lizard** $\rightarrow$ Rock
 Paper $\rightarrow$ Spock $\rightarrow$ Scissors $\rightarrow$ Lizard $\rightarrow$ Paper
 Scissors $\rightarrow$ Lizard $\rightarrow$ Spock $\rightarrow$ Rock $\rightarrow$ Scissors
 Spock $\rightarrow$ Rock $\rightarrow$ Lizard $\rightarrow$ Paper $\rightarrow$ Spock
 Lizard $\rightarrow$ Paper $\rightarrow$ Rock $\rightarrow$ Scissors $\rightarrow$ Lizard.

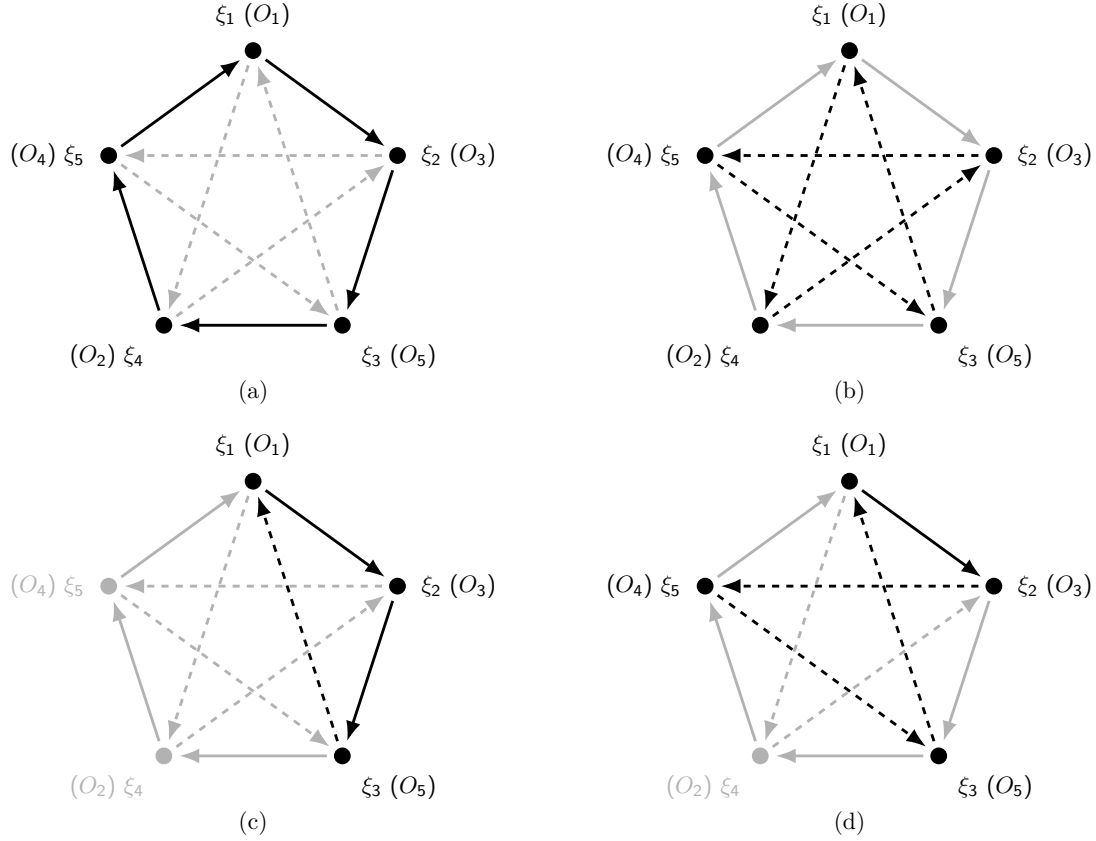


Figure 3.2: The types of cycles present in the RSPLS network:

- (a) The Rock-to-Paper sub-cycle;
- (b) The Star or Rock-to-Spock cycle;
- (c) The RSP sub-cycle;
- (d) The Four-node sub-cycle.

Solid lines represent 2-dimensional connections, dashed lines are 1-dimensional. The sub-cycles are obtained by selecting one particular (1-dimensional) connecting trajectory from the 2-dimensional connection, as discussed in Section 2.4.

Regarding the content of the following sections please note that not all cycles in the RSLPS network are quasi-simple. In fact, the only quasi-simple cycle is the Star cycle. The Rock-to-Paper cycle does not satisfy (iii) of Definition 2.2.4 and the remaining cycles do not satisfy (ii) of the same Definition. In Section 3.3, we focus on the quasi-simple (sub-)cycles whose connections are contained in the flow-invariant coordinate planes.

3.1 The RSPLS network

Using the notation of [1], a dynamical system describing a Lotka-Volterra system is one where the ODE in (2.1) takes the form² (see Equation (1) in [1]):

$$\dot{x}_i = x_i \left(\tau_i - \sum_{j=1}^n \rho_{ij} x_j \right) \quad \text{for } i = 1, \dots, n. \quad (3.1)$$

²In [1] the notation is σ_i instead of τ_i . We make this change to avoid confusion with the stability indices.

All the parameters τ_i and ρ_{ij} are positive and $\rho_{ii} = 1$. To ensure biological meaning, the state space is $\mathbb{R}_+^n$, the subspace of $\mathbb{R}^n$ where all coordinates are non-negative.

For RSPLS, it is $n = 5$. In [42] the dynamics of the game of RSPLS is described by looking at a particular case of (3.1), namely, $\tau_j = 1$ for all j and

$$\rho_{j,j+1} = 1 + c_A, \rho_{j,j+2} = 1 - e_B, \rho_{j,j+3} = 1 + c_B, \rho_{j,j+4} = 1 - e_A, \pmod{5}. \quad (3.2)$$

The dynamics of (3.1) supports a heteroclinic network with connections of dimension 1 and 2 between saddles. All the saddles are located on the coordinate axes and have 2-dimensional unstable manifolds. We use O_j to denote equilibria when referring to the more general dynamics of (3.1) and ξ_j otherwise. Each equilibrium O_j is located at a point where only the j^{th} coordinate is non-zero and equal to τ_j . In the context of the RSPLS game, it is natural to set $\tau_j = 1$ since this equilibrium represents the availability of only type j .

The Jacobian matrix of the system (3.1) evaluated at each equilibrium O_k is upper triangular with eigenvalues given by (see [1], Section 2)

$$-\tau_k \quad \text{and} \quad \tau_j - \rho_{jk}\tau_k, \quad j \neq k.$$

The first eigenvalue is radial and negative. In order to guarantee existence of the heteroclinic network the following assumptions are made (see Equations (3) and (4) in [1])

$$\min_{i=1,2} \{\tau_{k+i} - \rho_{k+i,k}\tau_k\} > 0 \quad (3.3)$$

and

$$\tau_j - \rho_{jk}\tau_k < 0, \quad \text{for } j \neq k, k+1, k+2, \quad (3.4)$$

where all indices are $\pmod{n}$. We note that in [42] the first assumption holds for $i = 1, 3$ so that an equilibrium ξ_k has connections to ξ_{k+1} and ξ_{k+3} . The heteroclinic networks are equivalent under the following correspondence: $O_1 \equiv \xi_1$, $O_2 \equiv \xi_4$, $O_3 \equiv \xi_2$, $O_4 \equiv \xi_5$ and $O_5 \equiv \xi_3$.

The RSPLS network is represented by the graphs depicted in Figure 3.3. This is equivalent to Figure 1 in [1] and [42], and appears in Figure 13 of [39]. Each node of the graph corresponds to an equilibrium of (3.1) where only one type is present. On the right-hand side, the nodes $\xi_1, \dots, \xi_5$ are ordered so that they correspond to the sequence Rock, Scissors, Paper, Lizard, Spock. On the left, they have the ordering used in [1].

On the right-hand side of Figure 3.3, the sequence of connections $C_{j,j+1} \pmod{5}$ $j = 1, \dots, 5$, together with the ordered equilibria constitute a heteroclinic cycle with 2-dimensional connections. On the left, this same heteroclinic cycle is made of the sequence of connections $C_{j,j+2} \pmod{5}$ $j = 1, \dots, 5$, and the sequence of nodes O_j, O_{j+2} . We refer to this as the *Rock-to-Paper cycle*.

Another heteroclinic cycle consists of all the nodes (in suitable order) and the sequence of 1-dimensional connections $C_{j,j+3} \pmod{5}$ $j = 1, \dots, 5$ on the right-hand side; $C_{j,j+1} \pmod{5}$ $j = 1, \dots, 5$ on the left. We call this Rock-to-Spock or the *Star cycle* due to its shape in the graph of Figure 3.3 (right).

We point out that the Rock-Scissors-Paper game appears as the heteroclinic cycle with three consecutive nodes and the connections $C_{j,j+1}, C_{j+1,j+2}, C_{j+2,j}$ on the right-hand side; this cycle has nodes O_j, O_{j+2}, O_{j+4} and the connections between each two on the left. This is the *RSP cycle* in what follows.

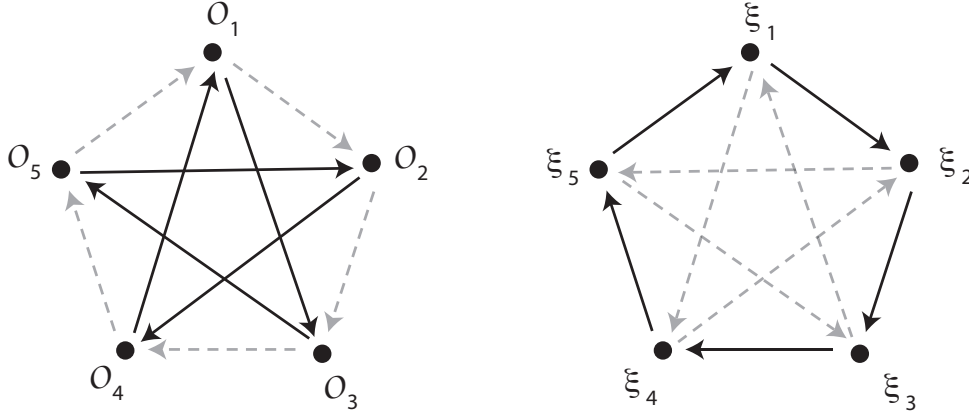


Figure 3.3: The RSPLS network: on the left with the labelling of [1] and on the right with that of [42]. On the left, the 2-dimensional connections are those shown as a star in the innermost part of the graph (solid lines); the connections on the outermost part, sequentially connecting $O_1, \dots, O_5$ are all 1-dimensional (dashed lines). On the right, it is the connections on the outside of the graph (solid), connecting in sequence $\xi_1, \dots, \xi_5$, that are 2-dimensional.

Finally, heteroclinic cycles with four nodes exist. They are described by sequences of nodes $O_j, O_{j+2}, O_{j+3}, O_{j+4}$ on the right-hand side; and by sequences of nodes $\xi_j, \xi_{j+1}, \xi_{j+4}, \xi_{j+2}$ on the left.

We refer to the four cycles described above as the *elementary heteroclinic cycles*. Many heteroclinic cycles are available as combinations of these four types if we allow for repetition of one or more nodes. For instance, we may have the sequence $\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_4 \rightarrow \xi_5 \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_1$, or the sequence $\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_4 \rightarrow \xi_5 \rightarrow \xi_3 \rightarrow \xi_1$, among many other.

The connections among ξ_j, ξ_{j+1} and ξ_{j+3} on the right-hand side of Figure 3.3, namely, $C_{j,j+1}$, $C_{j+3,j+1}$, and $C_{j,j+3}$ form what Ashwin *et al.* [3] call a Δ -clique, as shown in Figure 3.4. In [39, Definition 2.1], the term Δ -clique is reserved for such pieces of graph so that all trajectories starting near ξ_j end at ξ_{j+1} . The connection $C_{j,j+1}$ is called the short-connection while $C_{j+3,j+1}$, and $C_{j,j+3}$ are the second-long and the first-long connections, respectively. The short connection is 2-dimensional.

On the left-hand side of Figure 3.3, the Δ -cliques appear associated to the connections $C_{j,j+1}$, $C_{j+1,j+2}$, and $C_{j,j+2}$, this last being the short-connection, see also Figure 3.4.

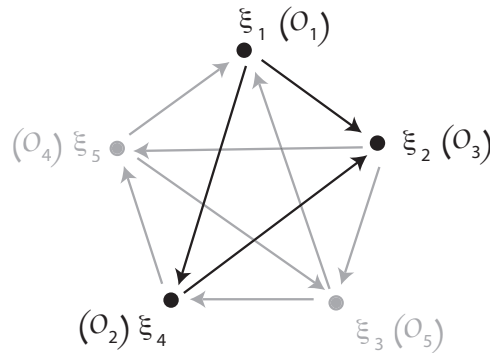


Figure 3.4: One of the Δ -cliques of the RSPLS.

3.2 Stability of the network

In this section we keep to the notation of [1] and show that, for most parameter values in [42] the RSPLS network is asymptotically stable. We start by finding a set that attracts all trajectories that do not start at the origin.

Lemma 3.2.1. *If $\tau_j > 0$, $j = 1, \dots, n$, then (3.1) admits a flow-invariant globally attracting $(n - 1)$ -sphere.*

Proof. We transform the ODE (3.1) by changing coordinates as $x_i = X_i^2$. We obtain

$$\dot{X}_i = \frac{X_i}{2} \left(\tau_i - \sum_{j=1}^n \rho_{ij} X_j^2 \right) \quad \text{for } i = 1, \dots, n. \quad (3.5)$$

The equilibria in the network remain on the coordinate axes but the non-zero coordinate is now represented by $\sqrt{\tau_i}$. At each equilibrium O_j , the Jacobian matrix is diagonal. The radial eigenvalues are preserved and the remaining eigenvalues appear divided by 2. They thus satisfy the assumptions in (3.3) and (3.4).

The nonlinear part of (3.1) is contracting and homogeneous of degree 3. Therefore the Invariant Sphere Theorem of Field [16] holds, ensuring the existence of an attracting invariant $(n - 1)$ -sphere. $\square$

It follows from Lemma 3.2.1 that the radial eigenvalue is negative, since the invariant sphere is attracting. Then, at each equilibrium the radial eigenvalue does not have to be taken into account for the stability of the RSPLS network.

Theorem 2.3 in [1] provides sufficient conditions for the asymptotic stability of the RSPLS network. Other than (3.3) and (3.4), for each k , these are that

$$\frac{\tau_{k+1}}{\rho_{k+1,k}} \leq \frac{\tau_{k+2}}{\rho_{k+2,k}} \quad (3.6)$$

and

$$\max_{i=1,2} \{ \tau_{k+i} - \rho_{k+i,k} \tau_k \} < \min_{j \neq k, k+1, k+2} \{ |\tau_j - \rho_{jk} \tau_k|, \tau_k \}. \quad (3.7)$$

It is a straightforward consequence of Lemma 3.2.1 that condition (3.7) can be simplified to

$$\max_{i=1,2} \{ \tau_{k+i} - \rho_{k+i,k} \tau_k \} < \min_{j \neq k, k+1, k+2} \{ |\tau_j - \rho_{jk} \tau_k| \}, \quad (3.8)$$

since the radial eigenvalue ceases to play a role.

Conditions for the asymptotic stability of the RSPLS network are the focus of the next result, with the aim of covering the cases treated in [42] where $e_A = 1$. From the previous correspondence (3.2), we see that $\rho_{j,j+4} = 1 - e_A = 0$ when $e_A = 1$ and thus, does not satisfy the restriction imposed in [1] that $\rho_{jk} > 0$. Furthermore, the hypotheses in Theorem 2.3 of [1] have to be adapted so that the outgoing connections at each node ξ_j are to ξ_{j+1} and ξ_{j+3} . Thus conditions (3.3), (3.4), (3.6) and (3.8) become, respectively,

$$\min_{i=1,3} \{ \tau_{k+i} - \rho_{k+i,k} \tau_k \} > 0, \quad (3.9)$$

$$\tau_j - \rho_{jk} \tau_k < 0, \quad \text{for } j \neq k, k+1, k+3, \quad (3.10)$$

$$\frac{\tau_{k+3}}{\rho_{k+3,k}} \leq \frac{\tau_{k+1}}{\rho_{k+1,k}} \quad (3.11)$$

and

$$\max_{i=1,3} \{ \tau_{k+i} - \rho_{k+i,k} \tau_k \} < \min_{j \neq k, k+1, k+3} \{ |\tau_j - \rho_{jk} \tau_k| \}. \quad (3.12)$$

Proposition 3.2.2. *If $0 < e_B < e_A < \min\{c_A, c_B\}$ and $e_A \leq 1$, then the RSPLS network in [42] is asymptotically stable.*

Proof. We follow the ideas of the proof of Theorem 2.3 of [1] and relax their parameter space to obtain the same result when $e_A = 1$. The dynamical system describing the RSPLS game in [42] is

$$\dot{x}_i = x_i \left[1 - \left(x_i + (1 + c_A)x_{i+1} + (1 - e_B)x_{i+2} + (1 + c_B)x_{i+3} + (1 - e_A)x_{i+4} \right) \right]. \quad (3.13)$$

As previously stated, $\tau_i = 1$ and the remaining coefficients are given in (3.2). Conditions (3.9) and (3.10) are trivially satisfied. Condition (3.12) leads to $\max\{e_A, e_B\} < \min\{c_A, c_B\}$, implied by our hypothesis. Condition (3.11) reads as $\frac{1}{1 - e_B} \leq \frac{1}{1 - e_A}$. If $e_A \neq 1 \neq e_B$, this is implied by $0 < e_B < e_A$. It follows from direct application of the sequence of Lemmas 3.2–3.7 in [1] that the hypotheses of their Theorem 2.3 hold and the network is asymptotically stable.

When $e_A = 1$, condition (3.11) cannot be verified. We prove that, in this case, the unstable manifold of each equilibrium is contained in the heteroclinic network. We consider the Δ -clique defined by the equilibria ξ_1 , ξ_2 and ξ_4 and show that the 2-dimensional unstable manifold of ξ_1 is contained in the Δ -clique, which in fact is a Δ -clique in the more restrictive sense of [39]. We show that there are no equilibria in the portion of state space defined by $x_i > 0$, $i = 1, 2, 4$ and $x_3 = x_5 = 0$. Such equilibria, if they exist, are in the intersection of the following three planes

$$\begin{aligned} P_1 &= \{1 - x_1 - (1 + c_A)x_2 - (1 + c_B)x_4 = 0\} \\ P_2 &= \{1 - (1 - e_A)x_1 - x_2 - (1 - e_B)x_4 = 0\} \\ P_4 &= \{1 - (1 - e_B)x_1 - (1 + c_B)x_2 - x_4 = 0\} \end{aligned}$$

To see that the planes P_1 , P_2 and P_4 do not intersect in the interior of the Δ -clique we show that P_2 dominates the other two planes. We say, as in [1], that the plane P_2 dominates P_1 if, when representing each plane by the graph of a function $x_2 = z_2(x_1, x_4)$ and $x_1 = z_1(x_2, x_4)$, the graph representing P_2 is always above that representing P_1 . Analogously, for the statement that P_2 dominates P_4 . The intersections of the planes with the coordinate axes are as follows, when $e_A = 1$:

plane \ axis	x_1	x_2	x_4
P_1	$x_1 = 1$	$x_2 = 1/(1 + c_A)$	$x_4 = 1/(1 + c_B)$
P_2	$\emptyset$	$x_2 = 1$	$x_4 = 1/(1 - e_B)$
P_4	$x_1 = 1/(1 - e_B)$	$x_2 = 1/(1 + c_B)$	$x_4 = 1$

Since $1/(1 + c_A), 1/(1 + c_B) < 1$ and $1/(1 - e_B) > 1$, it is easy to see that the intersection of P_2 with the axes x_2 and x_4 is larger than those of either P_1 or P_4 . Hence, P_2 is always above the other two planes for $x_i > 0$, $i = 1, 2, 4$ and $x_3 = x_5 = 0$. The proof that the Δ -clique exists follows analogously to the case $e_A \neq 1$. Note that we are working in an attracting invariant topological sphere in three-dimensional space so that trajectories do not go to infinity. Since the invariant sphere is compact and 2-dimensional, Poincaré-Bendixson requires an equilibrium for the existence of a periodic orbit. Since there are no equilibria, no period orbits exist.

The proof for the remaining Δ -cliques in the network is done by permutation of the indices. $\square$

The next result establishes the asymptotic stability of the RSPLS network for most values in Figure 7 of [42] that correspond to the existence of ‘sausages’³. These correspond to fragmentary asymptotic stability regions for various sequences other than the Rock-to-Spock, the Star and the RSP cycles referred to above. Establishing the asymptotic stability of the whole network supports the visibility of the sausages of [42]. Note that for some portion of the region depicted in [42], namely $c_A \in (0.8, 1]$, the sufficient conditions for asymptotic stability of the network given in [1] do not apply.

Corollary 3.2.3. *The RSPLS network in [42] is asymptotically stable if $e_A = 1$, $e_B = 0.8$, $c_A \in (1.0, 1.8)$ and $c_B \in (1, 4.5)$.*

3.3 Stability of the four elementary cycles

In this section we present the stability indices for the sub-cycles of 1-dimensional heteroclinic connections of the four cycles: Rock-to-Paper, Star, RSP and Four-node. From now on we remain with the formulation of [42] given in (3.13). Recall the relation between our cycles and those of [42]: our Rock-to-Paper cycle is of type A, our Star cycle is of type B, our RSP cycle is of type AAB, and our Four-node cycle is of type Q = ABBB.

3.3.1 Previous results

As is shown in [18] the stability indices can be calculated for the general class of quasi-simple cycles. It is easily seen that the four cycles of interest are either quasi-simple or have quasi-simple sub-cycles when restricted to the flow-invariant coordinate planes. This restriction ensures that Definition 2.2.4 is satisfied since, for these sub-cycles, all invariant P_j ’s are coordinate planes and $\dim(P_j \ominus \hat{L}_j) = 1$. Actually, the sub-cycles so obtained admit at every equilibrium one radial, one contracting, one expanding and two transverse eigenvalues. We refer the reader to [18] for more detail on the classification of the eigenvalues. All the connections in the Star cycle are one-dimensional, in this case the sub-cycle coincides with the cycle. We label the sub-cycles as $\Sigma_{\text{R-to-P}}$, Σ_{Star} , Σ_{RSP} and $\Sigma_{\text{4-node}}$, respectively.

For every $j = 1, \dots, 5$, the eigenvalues of ξ_j are -1 , e_A , $-c_B$, e_B and $-c_A$, with eigenvectors in the x_j , x_{j+1} , x_{j+2} , x_{j+3} and x_{j+4} directions (mod 5), respectively. This naturally adds symmetry to the problem under the action of the group $\mathbb{Z}_5(\varphi)$ with $\varphi(x_1, x_2, x_3, x_4, x_5) = (x_5, x_1, x_2, x_3, x_4)$ as in [42]. Let H_j^{in} stand for the cross-section to the flow at an incoming connection to ξ_j . Since the radial direction can be omitted all cross-sections are 3-dimensional – we take cross-sections within the invariant 4-sphere. The dynamics near each sub-cycle is approximated by basic transition matrices⁴ $M_j : H_j^{\text{in}} \rightarrow H_{j+1}^{\text{in}}$ whose entries are rational functions of the eigenvalues at ξ_j , where we change the indexing, so now ξ_{j+1} is the equilibrium with a connection $\xi_j \rightarrow \xi_{j+1}$ in the

³The term ‘sausage’ has been used by the authors of [42] to describe small intertwined regions in parameter space with different dynamics and depicted in their Figures 2, 7 and 8.

⁴A basic transition matrix provides a convenient description of the dynamics from one incoming cross-section to the next. Its entries are 0’s and 1’s, except for one column which consists of quotients between the modulus of the contracting and expanding eigenvalues (for one entry) and between the symmetric of transverse eigenvalues and the expanding eigenvalue (for the remaining rows). The definition of a transition matrix goes back to the work of Field and Swift [17]. A detailed construction of basic transition matrices in the context of cycles of type Z (a subset of quasi-simple cycles) can be found in [37]. Here we use the work of [18].

sub-cycle under study. The basic transition matrices coincide with those presented in [42, Subsection 4.1]. The results from [18] hold in the present case and the stability of the sub-cycles is governed by properties of the basic transition matrices and their product as explained in Subsection 2.6.1.

In virtue of one repelling transverse direction at every ξ_j , all basic transition matrices M_j have one negative entry.

Considering Proposition 2.6.5, for each $j = 1, \dots, m$, the vectors $\alpha^{(i)}$ that must be considered for the calculations that follow are the rows of the transition matrices $M_{(j,j)} = M_j$, $M_{(j+1,j)} = M_{j+1}M_j$, $M_{(j+2,j)} = M_{j+2}M_{j+1}M_j$, $\dots$, $M_{(j-1,j)} = M^{(j)}$. The number K , as defined in Proposition 2.6.5, refers to the number of such rows whenever

$$U^{-\infty}(M^{(j)}) = \left\{ \mathbf{y} \in \mathbb{R}_-^3 : \lim_{k \rightarrow +\infty} (M^{(j)})^k \mathbf{y} = -\infty \right\} = \mathbb{R}_-^3, \quad (3.14)$$

is satisfied, where $\mathbb{R}_-^3 = \{\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1, y_2, y_3 < 0\}$, see [18] for details.

3.3.2 Stability of the elementary cycles

In this subsection we provide the stability results for each of the four elementary cycles in the RSPLS network.

The Rock-to-Paper sub-cycle: The Rock-to-Paper sub-cycle comprises five equilibria and five 1-dimensional heteroclinic connections in the order, see Figure 3.2(a):

$$\Sigma_{\text{R-to-P}} = [\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_4 \rightarrow \xi_5 \rightarrow \xi_1].$$

The behaviour of trajectories between any two consecutive equilibria is captured up to a permutation by the basic transition matrix $M_2 : H_2^{\text{in}} \rightarrow H_3^{\text{in}}$ with

$$M_2 = \begin{bmatrix} \frac{c_B}{e_A} & 0 & 1 \\ \frac{c_A}{e_A} & 0 & 0 \\ -\frac{e_B}{e_A} & 1 & 0 \end{bmatrix}.$$

Starting near each equilibrium, the powers $(M_2)^l$, $l = 1, \dots, 5$ provide an approximation of a trajectory that visits once a neighbourhood of each equilibrium of $\Sigma_{\text{R-to-P}}$. The stability indices may thus be computed from the rows of M_2 .

We have the following:

Proposition 3.3.1. *The local stability indices for the Rock-to-Paper sub-cycle $\Sigma_{\text{R-to-P}}$ are all equal and:*

- (a) *if either $c_A + c_B < e_A + e_B$ or $c_A e_A < c_B e_B$ or $c_A c_B^3 < e_A e_B^3$, then $\sigma_{\text{R-to-P}} = -\infty$.*
- (b) *if $c_A + c_B > e_A + e_B$ and $c_A e_A > c_B e_B$ and $c_A c_B^3 > e_A e_B^3$, then*

$$-\infty < \sigma_{\text{R-to-P}} \leq F^{\text{index}}\left(-\frac{e_B}{e_A}, 1, 0\right).$$

Proof. According to Proposition 2.6.5, the stability of $\Sigma_{\text{R-to-P}}$ depends on whether or not M_2 satisfies conditions (i)–(iii). Eigenvalues of M_2 are the roots of the characteristic polynomial

$$p(\lambda) = -\lambda^3 + a_2\lambda^2 + a_3\lambda + a_1,$$

where

$$a_1 = \frac{c_A}{e_A}, \quad a_2 = \frac{c_B}{e_A}, \quad a_3 = -\frac{e_B}{e_A}.$$

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ be the eigenvalues of M_2 such that $\lambda_1 = \lambda_{\max}$ and $\mathbf{w}^{\max}$ is the eigenvector associated with λ_1 . Vieta's formulas applied to cubic polynomials give

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{c_B}{e_A}, \quad \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{e_B}{e_A}, \quad \lambda_1\lambda_2\lambda_3 = \frac{c_A}{e_A}. \quad (3.15)$$

Using Lemma 10 in [38] we find that conditions (i)–(iii) for M_2 are individually fulfilled if and only if

$$\frac{c_A}{e_A} + \frac{c_B}{e_A} - \frac{e_B}{e_A} > 1 \Leftrightarrow c_A + c_B > e_A + e_B, \quad (3.16)$$

$$-\frac{c_B}{e_A} \frac{e_B}{e_A} + \frac{c_A}{e_A} > 0 \Leftrightarrow c_A e_A > c_B e_B, \quad (3.17)$$

$$\frac{c_A}{e_A} \frac{c_B^3}{e_A^3} - \frac{e_B^3}{e_A^3} > 0 \Leftrightarrow c_A c_B^3 > e_A e_B^3. \quad (3.18)$$

When one of the relations (3.16) to (3.18) does not hold, statement (a) is immediate from Proposition 2.6.5(a).

Suppose now that (3.16)–(3.18) hold true. Then, $\lambda_1 = \lambda_{\max} > 1$ and the components of $\mathbf{w}^{\max}$ have all the same sign. The identities in (3.15) enable one to disclose that λ_1 is the only eigenvalue with positive real part. It follows that $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 > 0$. We check that $U^{-\infty}(M_2) = \mathbb{R}_-^3$ in (3.14). This is equivalent to showing that any $\mathbf{y} \in \mathbb{R}_-^3$ written in the eigenbasis of M_2 must have a negative coefficient for the largest eigenvector. The coefficient writes as $(\mathbf{v}^{\max})^T \mathbf{y}$, where $\mathbf{v}^{\max}$ is a vector multiple of

$$\left(\lambda_1, \frac{1}{\lambda_1}, 1 \right)^T.$$

Because $\lambda_1 > 0$ we get $(\mathbf{v}^{\max})^T \mathbf{y} < 0$ for any $\mathbf{y} \in \mathbb{R}_-^3$. Proposition 2.6.5(b) applies and we need to take into account the rows with at least one negative entry of M_2 , $(M_2)^2$, $(M_2)^3$, $(M_2)^4$ and $(M_2)^5$ in Appendix A.1, so that

$$\sigma_{\text{R-to-P}} \leq F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right).$$

□

The Star cycle: The Star cycle comprises five equilibria and five 1-dimensional heteroclinic connections in the order, see Figure 3.2(b):

$$\Sigma_{\text{Star}} = [\xi_1 \rightarrow \xi_4 \rightarrow \xi_2 \rightarrow \xi_5 \rightarrow \xi_3 \rightarrow \xi_1].$$

The transition between any two consecutive equilibria is described up to a permutation by the basic transition matrix $M_4 : H_4^{\text{in}} \rightarrow H_2^{\text{in}}$ with

$$M_4 = \begin{bmatrix} 0 & \frac{c_A}{e_B} & 1 \\ 1 & -\frac{e_A}{e_B} & 0 \\ 0 & \frac{c_B}{e_B} & 0 \end{bmatrix}.$$

Again we are reduced to establishing the stability properties of Σ_{Star} by taking the rows of M_4 as follows.

Proposition 3.3.2. *The local stability indices for the Star cycle Σ_{Star} are all equal and:*

(a) *if either $c_A + c_B < e_A + e_B$ or $c_B e_B < c_A e_A$ or $c_A^3 e_B < c_B e_A^3$, then $\sigma_{\text{Star}} = -\infty$.*

(b) *if $c_A + c_B > e_A + e_B$ and $c_B e_B > c_A e_A$ and $c_A^3 e_B > c_B e_A^3$, then*

$$-\infty < \sigma_{\text{Star}} \leq F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right).$$

Proof. This follows by the same method of the proof of Proposition 3.3.1, making use of the transition matrix M_4 whose characteristic polynomial is $p(\lambda) = -\lambda^3 - \frac{e_A}{e_B} \lambda^2 + \frac{c_A}{e_B} \lambda + \frac{c_B}{e_B}$. For (b) the negative entries of M_4 , $(M_4)^2$, $(M_4)^3$, $(M_4)^4$ and $(M_4)^5$ in Appendix A.2 must be considered. $\square$

The RSP sub-cycle: The RSP sub-cycle comprises three equilibria and three 1-dimensional connections in the order, see Figure 3.2(c):

$$\Sigma_{\text{RSP}} = [\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_1].$$

We write down the three basic transition matrices $M_j : H_j^{\text{in}} \rightarrow H_{j+1}^{\text{in}}$, $j = 1, 2, 3 \pmod{3}$, with

$$M_1 = \begin{bmatrix} \frac{c_B}{e_A} & 0 & 0 \\ \frac{c_A}{e_A} & 0 & 1 \\ -\frac{e_B}{e_A} & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \frac{c_B}{e_A} & 0 & 1 \\ \frac{c_A}{e_A} & 0 & 0 \\ -\frac{e_B}{e_A} & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & \frac{c_A}{e_B} & 0 \\ 1 & -\frac{e_A}{e_B} & 0 \\ 0 & \frac{c_B}{e_B} & 1 \end{bmatrix}.$$

The products $M_{(j+1,j)} = M_{j+1} M_j : H_j^{\text{in}} \rightarrow H_{j+2}^{\text{in}}$ and $M^{(j)} = M_{j+2} M_{j+1} M_j : H_j^{\text{in}} \rightarrow H_j^{\text{in}}$, $j = 1, 2, 3 \pmod{3}$ can be found in Appendix A.3. The following quantities are useful:

$$\begin{aligned} \delta_T &= \frac{c_A^2 c_B}{e_A^2 e_B} & \gamma_T &= \frac{c_A^3}{e_A^2 e_B} + \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \\ \alpha_T &= \frac{c_B^2}{e_A^2} - \frac{c_A c_B}{e_B e_A} - \frac{e_B}{e_A} & \theta_T &= -\frac{c_A^2}{e_A^2} + \frac{c_B}{e_A} - \frac{c_A}{e_B} \\ \beta_T &= \frac{c_B^2 c_A}{e_A^2 e_B} - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} & \mu_T &= \frac{c_B^2 c_A}{e_A^2 e_B} - \frac{c_A}{e_A} - \frac{e_A}{e_B} \\ & & \nu_T &= -\frac{c_B c_A}{e_A^2} + \frac{c_A^2}{e_A e_B} + \frac{c_B}{e_B}. \end{aligned}$$

In the next result the conditions imposed in (a) and (b) are complementary, in view of Lemma B.1.1 in Appendix B.1. We denote by σ_{ij} the stability index along the trajectory connecting ξ_i to ξ_j .

Proposition 3.3.3. *The local stability indices for the RSP sub-cycle Σ_{RSP} are:*

(a) *if either $\delta_T < 1$ or $\alpha_T < 0$ or $\beta_T < 0$ or $\gamma_T < 0$ or $\theta_T < 0$ or $\mu_T < 0$, or $\nu_T < 0$, then $\sigma_{31} = \sigma_{12} = \sigma_{23} = -\infty$.*

(b) *if $\delta_T > 1$ and $\theta_T > 0$ and $\nu_T > 0$, then*

$$\sigma_{31} = \begin{cases} \min \left\{ 1 - \frac{e_B}{e_A}, 1 - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \right\} < 0 & \text{if } \frac{e_B}{e_A} > \max \left\{ 1, \frac{e_A + c_A}{c_B} \right\} \\ 1 - \frac{e_B}{e_A} < 0 & \text{if } 1 < \frac{e_B}{e_A} < \frac{e_A + c_A}{c_B} \\ 1 - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} < 0 & \text{if } \frac{e_A + c_A}{c_B} \leq \frac{e_B}{e_A} < 1 \\ \min \left\{ -1 + \frac{e_A}{e_B}, -1 + \frac{e_A^2}{e_B c_B - c_A e_A} \right\} > 0 & \text{if } \frac{e_B}{e_A} < \min \left\{ 1, \frac{e_A + c_A}{c_B} \right\} \end{cases}$$

$$\sigma_{12} = \begin{cases} 1 - \frac{e_B}{e_A} < 0 & \text{if } 1 < \frac{e_B}{e_A} \leq \frac{c_B c_A}{e_B e_A} \text{ or } \max \left\{ 1, \frac{c_B c_A}{e_B e_A} \right\} < \frac{e_B}{e_A} < 1 + \frac{c_B c_A}{e_B e_A} \\ & \text{or } \frac{e_B}{e_A} \geq 1 + \frac{c_B c_A}{e_B e_A} \\ -1 + \frac{e_A}{e_B} > 0 & \text{if } \frac{e_B}{e_A} < \min \left\{ 1, \frac{c_B c_A}{e_B e_A} \right\} \text{ or } \frac{c_B c_A}{e_B e_A} < \frac{e_B}{e_A} < 1 \end{cases}$$

$$\sigma_{23} = \begin{cases} 1 - \frac{c_A}{e_A} - \frac{e_A}{e_B} < 0 & \text{if } \frac{e_B}{e_A} < 1 \text{ or } 1 < \frac{e_B}{e_A} < \frac{e_A}{e_A - c_A} \\ 1 - \frac{e_A}{e_B} < 0 & \text{if } \frac{e_B}{e_A} > \frac{e_A}{e_A - c_A}. \end{cases}$$

Proof. This follows by the same method of the proof of Proposition 3.3.1, see Appendix B.1 where Lemma B.1.1 explains why some of the quantities do not appear in statement (b). $\square$

The Four-node sub-cycle: The Four-node sub-cycle comprises four equilibria and four 1-dimensional heteroclinic connections in the order, see Figure 3.2(d):

$$\Sigma_{4\text{-node}} = [\xi_1 \rightarrow \xi_2 \rightarrow \xi_5 \rightarrow \xi_3 \rightarrow \xi_1].$$

The four basic transition matrices between consecutive equilibria are⁵

$$\widehat{M}_1 : \widehat{H}_1^{\text{in}} \rightarrow \widehat{H}_2^{\text{in}}, \quad \widehat{M}_2 : \widehat{H}_2^{\text{in}} \rightarrow \widehat{H}_5^{\text{in}}, \quad \widehat{M}_5 : \widehat{H}_5^{\text{in}} \rightarrow \widehat{H}_3^{\text{in}}, \quad \widehat{M}_3 : \widehat{H}_3^{\text{in}} \rightarrow \widehat{H}_1^{\text{in}}$$

⁵Based on the type of heteroclinic connection, we have the following correspondence: $\widehat{M}_1 = M_1$, $\widehat{M}_2 = M_3$, $\widehat{M}_3 = \widehat{M}_5 = M_4$.

where

$$\widehat{M}_1 = \begin{bmatrix} \frac{c_B}{e_A} & 0 & 0 \\ \frac{c_A}{e_A} & 0 & 1 \\ -\frac{e_B}{e_A} & 1 & 0 \end{bmatrix}, \quad \widehat{M}_2 = \begin{bmatrix} 0 & \frac{c_A}{e_B} & 0 \\ 1 & -\frac{e_A}{e_B} & 0 \\ 0 & \frac{c_B}{e_B} & 1 \end{bmatrix}, \quad \widehat{M}_3 = \widehat{M}_5 = \begin{bmatrix} 0 & \frac{c_A}{e_B} & 1 \\ 1 & -\frac{e_A}{e_B} & 0 \\ 0 & \frac{c_B}{e_B} & 0 \end{bmatrix}.$$

The products of the basic transition matrices with respect to the Four-node sub-cycle near ξ_1 are $\widehat{M}_{(2,1)} = \widehat{M}_2 \widehat{M}_1 : \widehat{H}_1^{\text{in}} \rightarrow \widehat{H}_5^{\text{in}}$, $\widehat{M}_{(5,1)} = \widehat{M}_5 \widehat{M}_2 \widehat{M}_1 : \widehat{H}_1^{\text{in}} \rightarrow \widehat{H}_3^{\text{in}}$ and $\widehat{M}^{(1)} = \widehat{M}_3 \widehat{M}_5 \widehat{M}_2 \widehat{M}_1 : \widehat{H}_1^{\text{in}} \rightarrow \widehat{H}_1^{\text{in}}$. In the same manner we obtain the products near ξ_j , $j = 2, 3, 5$. All transition matrix products can be found in Appendix A.4.

The next result makes use of notation introduced in Appendix B.2.

Proposition 3.3.4. *The local stability indices for the Four-node sub-cycle $\Sigma_{4\text{-node}}$ are:*

- (a) *if either $\alpha_{11} + \alpha_{33} < \min \left\{ 2, 1 + \frac{c_B^3 c_A}{e_B^3 e_A} \right\}$ or $\theta_T > 0$ or $\nu_T > 0$ or $w_2^{\max,1} < 0$ or $w_3^{\max,2} < 0$ or $w_3^{\max,5} < 0$ or $w_1^{\max,3} < 0$, then $\sigma_{31} = \sigma_{12} = \sigma_{25} = \sigma_{53} = -\infty$.*
- (b) *if $\alpha_{11} + \alpha_{33} > \min \left\{ 2, 1 + \frac{c_B^3 c_A}{e_B^3 e_A} \right\}$ and $\theta_T < 0$ and $\nu_T < 0$ and $w_3^{\max,2} > 0$, then*

$$-\infty < \sigma_{31} \leq F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right) \\ -\infty < \sigma_{12}, \sigma_{25}, \sigma_{53} \leq F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right).$$

Proof. This follows by the same method of the proof of Proposition 3.3.1, see Appendix B.2. $\square$

It is clear that the Four-node cycle cannot be e.a.s. In fact,

$$F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right) \cdot F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right) = - \left(1 - \frac{e_A}{e_B} \right)^2 < 0$$

so that the stability indices cannot all be positive.

Regardless of the stability exhibited by the heteroclinic network as a whole, as expected, not all cycles can be simultaneously stable.

Lemma 3.3.5. *At most one of the 5-node cycles in the RSPLS network is f.a.s. At most either the 3-node sub-cycle or the 4-node sub-cycle is f.a.s. Furthermore, if the sub-cycle $\Sigma_{R\text{-to-P}}$ is f.a.s. then the sub-cycle $\Sigma_{4\text{-node}}$ is c.u.*

Proof. The stability is obtained by using Lemma 2.5 in [18] to relate a finite stability index to f.a.s. It is clear from Propositions 3.3.1 and 3.3.2 that the sufficient condition $c_A e_A < c_B e_B$ for the sub-cycle $\Sigma_{R\text{-to-P}}$ to be c.u. is satisfied when the cycle Σ_{Star} has a finite stability index. Analogously, the sufficient condition $c_B e_B < c_A e_A$ for the cycle Σ_{Star} to be c.u. is satisfied when the sub-cycle $\Sigma_{R\text{-to-P}}$ has a finite stability index.

The conditions on the sign of θ_T and ν_T for the cycles Σ_{RSP} and $\Sigma_{4\text{-node}}$ are exclusive. Hence, at most one of these two cycles is f.a.s.

The condition $c_B e_B < c_A e_A$ that is satisfied if $\Sigma_{R\text{-to-P}}$ is f.a.s. guarantees that $\nu_T > 0$ and therefore $\Sigma_{4\text{-node}}$ is c.u. $\square$

3.3.3 Stability of cycles in an asymptotically stable network

From now on, we assume that the sufficient condition for the asymptotic stability (a.s.) of the whole heteroclinic network holds. From Proposition 3.2.2 we restrict the parameter space to

$$0 < e_B < e_A < \min\{c_A, c_B\} \quad \text{and} \quad e_A \leq 1. \quad (3.19)$$

Lemma 3.3.6. *The following relations hold:*

(a) if (3.19) is satisfied, then

$$(a1) \quad \delta_T > 1, \beta_T > 0 \text{ and } \gamma_T > 0;$$

$$(a2) \quad c_A + c_B > e_A + e_B;$$

$$(a3) \quad c_A c_B^3 > e_A e_B^3.$$

(b) if either $\alpha_T > 0$ or $\theta_T > 0$ or $\beta_T < 0$ or $\nu_T < 0$, then $c_A e_A < c_B e_B$.

(c) if $\theta_T > 0$ and $\nu_T > 0$, then $c_A^3 e_B < c_B e_A^3$. Otherwise, if $\theta_T < 0$ and $\nu_T < 0$, then $c_A^3 e_B > c_B e_A^3$.

Proof. In (a), given (3.19), it is immediate that $\delta_T > 1$, $c_A + c_B > e_A + e_B$ and $c_A c_B^3 > e_A e_B^3$. Write

$$\beta_T = \frac{c_B(c_B c_A - e_B^2) + c_A e_A e_B}{e_A^2 e_B} \quad \text{and} \quad \gamma_T = \frac{c_A^3 + e_A(c_B c_A - e_B^2)}{e_A^2 e_B}.$$

Now $c_B c_A - e_B^2 > 0$ from (3.19) and the signs of β_T and γ_T are respectively given by

$$c_B(c_B c_A - e_B^2) + c_A e_A e_B > 0 \quad \text{and} \quad c_A^3 + e_A(c_B c_A - e_B^2) > 0.$$

We establish (b) by expressing

$$\alpha_T = \frac{-c_B(c_A e_A - c_B e_B) - e_B^2 e_A}{e_A^2 e_B}, \quad \theta_T = \frac{-c_A^2 e_B - e_A(c_A e_A - c_B e_B)}{e_A^2 e_B}, \quad (3.20)$$

$$\beta_T = \frac{c_B^2 c_A + e_B(c_A e_A - c_B e_B)}{e_A^2 e_B}, \quad \nu_T = \frac{c_A(c_A e_A - c_B e_B) + e_A^2 c_B}{e_A^2 e_B}. \quad (3.21)$$

In (c), observe that

$$\theta_T > 0 \Leftrightarrow c_A(-c_A^2 e_B + c_B e_A e_B - e_A^2 c_A) > 0 \quad \Leftrightarrow c_A c_B e_A e_B - e_A^2 c_A^2 > c_A^3 e_B,$$

$$\nu_T > 0 \Leftrightarrow e_A(-c_A c_B e_B + c_A^2 e_A + e_A^2 c_B) > 0 \quad \Leftrightarrow c_B e_A^3 > c_A c_B e_A e_B - e_A^2 c_A^2.$$

Accordingly, $c_A^3 e_B < c_A c_B e_A e_B - e_A^2 c_A^2 < c_B e_A^3$. The second statement is immediate by reversing the direction of the above inequalities. $\square$

The hypotheses in Propositions 3.3.1–3.3.3 can be simplified in view of the previous lemma. We obtain the following more specific results concerning the stability regions of the three sub-cycles, illustrated in Figures 3.5 and 3.6.

Proposition 3.3.7. *Let (3.19) be satisfied. Then:*

(a) the sub-cycle $\Sigma_{R\text{-to-}P}$ is

- (a1) c.u. if $c_A e_A < c_B e_B$;
- (a2) f.a.s. if $c_A e_A > c_B e_B$;
- (a3) e.a.s. if $c_A e_A - c_B e_B > e_A e_B$.

(b) the cycle Σ_{Star} is

- (b1) c.u. if either $c_B e_B < c_A e_A$ or $c_A^3 e_B < c_B e_A^3$;
- (b2) f.a.s. if $c_B e_B > c_A e_A$ and $c_A^3 e_B > c_B e_A^3$.

(c) the sub-cycle Σ_{RSP} is

- (c1) c.u. if either $\alpha_T < 0$, or $\theta_T < 0$, or $\mu_T < 0$, or $\nu_T < 0$;
- (c2) f.a.s. if $\theta_T > 0$ and $\nu_T > 0$.

Proof. This proof is deferred to Appendix B.3.

We use Theorem 3.1 in Lohse [29] showing that if all stability indices are positive then the cycle is e.a.s. whereas f.a.s. is obtained from Lemma 2.5 in [18]. It follows from its definition in [35] that a stability index equal to $-\infty$ implies the complete instability of the cycle.

For the sub-cycle $\Sigma_{R\text{-to-}P}$ the proof consists in checking which entries in the transition matrices may be negative under the constraint (3.19) and using this information to obtain the stability indices. For the other two sub-cycles this can be done directly from Propositions 3.3.2 and 3.3.3. $\square$

Remark 3.3.8. The conditions for the f.a.s. of the sub-cycle Σ_{RSP} can be written as a function of the eigenvalues as

$$\frac{c_A^2 e_B + e_A^2 c_A}{e_A} < c_B e_B < \frac{c_A^2 e_A + e_A^2 c_B}{c_A}.$$

Hypothesis (3.19) does not provide a complete description of the stability of the Four-node sub-cycle in parameter space. This can, of course, be obtained if values are assigned to all eigenvalues. The next result lists the most general results.

Lemma 3.3.9. *Let (3.19) be satisfied.*

(a) The sub-cycle $\Sigma_{4\text{-node}}$ is c.u. if at least one of the following holds

- (a1) $c_A > c_B$;
- (a2) $c_A^3 e_B < c_B e_A^3$.

(b) If the sub-cycle $\Sigma_{4\text{-node}}$ is f.a.s. then so is the cycle Σ_{Star} .

Proof. Let $c_A > c_B$ and write ν_T , from Proposition 3.3.4, as in (3.21). We have $\nu_T > 0$ since $c_A e_A - c_B e_B > 0$.

Consider now two necessary conditions for $\Sigma_{4\text{-node}}$ not to be c.u: $\theta_T < 0$ and $\nu_T < 0$. Given (c) in Lemma 3.3.6, we find that $c_A^3 e_B < c_B e_A^3$ makes the conditions incompatible.

Assuming that the Four-node cycle is f.a.s. we must have $\theta_T < 0$ and $\nu_T < 0$. Again from (c) in Lemma 3.3.6, this implies $c_A^3 e_B > c_B e_A^3$. From (3.21), it is easy to see that $c_B e_B > c_A e_A$ follows from $\nu_T < 0$. $\square$

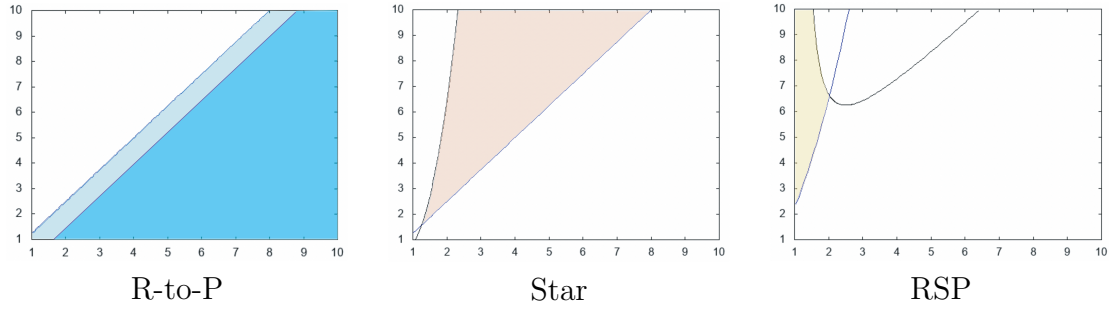


Figure 3.5: Regions of stability in the (c_A, c_B) plane for the three elementary cycles, with $e_A = 1$, $e_B = 0.8$, in the region where the network is asymptotically stable. In the coloured region the (sub-)cycles R-to-P, Star and RSP are f.a.s. and the sub-cycle R-to-P is e.a.s. in the darker region. The Four-node sub-cycle is c.u. for at least those parameter values for which the Star cycle is c.u.

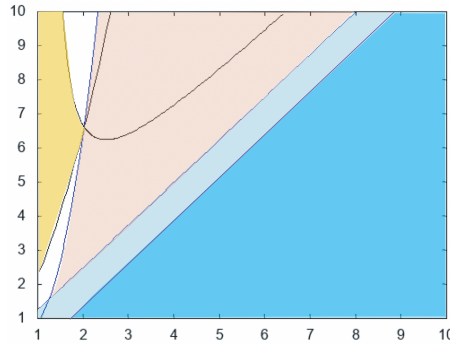


Figure 3.6: Regions of stability in the (c_A, c_B) plane for the three elementary cycles, with $e_A = 1$, $e_B = 0.8$, in the region where the network is asymptotically stable. The sub-cycle RSP is f.a.s. at the yellow region on the left, Star is f.a.s. on the pink region at the centre and R-to-P is f.a.s. on the light blue region and e.a.s. on the darker region at the right. The Four-node sub-cycle is c.u. for at least those parameter values for which the Star cycle is c.u. In the two white regions all the elementary cycles are c.u. The sausages found in [42] lie in the lower component of the white region.

A straightforward consequence of Lemma 3.3.9 is that the region of stability (f.a.s.) of the Four-node sub-cycle is contained in that of the Star cycle.

Our final result establishes some stability combinations for the cycles in the RSPLS network. The fact that all four (sub-)cycles may be c.u. indicates that other sequences may be visible in simulations.

Proposition 3.3.10. *Consider the three elementary (sub-)cycles with an odd number of nodes. If (3.19) holds and one of these three elementary (sub-)cycles satisfies the conditions above to be f.a.s. then the other two elementary (sub-)cycles are c.u. Moreover, if $\nu_T < 0$ and $c_A^3 e_B < c_B e_A^3$, then all four (sub-)cycles are c.u.*

Proof. According to Proposition 3.3.7, Σ_{Star} and Σ_{RSP} are at most f.a.s. If $\Sigma_{\text{R-to-P}}$ is not c.u. then it is automatically f.a.s.

Suppose that $\Sigma_{\text{R-to-P}}$ is f.a.s. From (a1) in Proposition 3.3.7 we get $c_A e_A > c_B e_B$. Lemma 3.3.5 leads to Σ_{Star} being c.u. The contrapositive of (b) in Lemma 3.3.6 determines that $c_A e_A > c_B e_B$ yields $\alpha_T, \theta_T < 0$ and $\nu_T > 0$. Proposition 3.3.3 states that the stability indices for Σ_{RSP} are all equal to $-\infty$ and this sub-cycle is also c.u.

Suppose that Σ_{Star} is f.a.s. Recall that Lemma 3.3.5 already establishes that $\Sigma_{\text{R-to-P}}$ is c.u. From the contrapositive of (c) in Lemma 3.3.6, if $c_A^3 e_B > c_B e_A^3$ then either $\theta_T < 0$ or $\nu_T < 0$, and thus Σ_{RSP} is also c.u.

Suppose that Σ_{RSP} is f.a.s. By virtue of (b) in Lemma 3.3.6, when $\theta_T > 0$ we have $c_A e_A < c_B e_B$. That $\Sigma_{\text{R-to-P}}$ is c.u. is a consequence of (a1) in Proposition 3.3.7. Given (c) in Lemma 3.3.6, it follows that $c_A^3 e_B < c_B e_A^3$, and hence Σ_{Star} is c.u.

To prove the second statement we use (b) in Lemma 3.3.6 whence $\nu_T < 0$ implies $c_A e_A < c_B e_B$, ensuring that Σ_{RSP} and $\Sigma_{\text{R-to-P}}$ are both c.u. The remaining condition asserts that Σ_{Star} and $\Sigma_{4\text{-node}}$ are also c.u. $\square$

We finish this section by considering the parameter range depicted in Figure 7 of [42] and add the information provided by our analysis. This allows us to distinguish f.a.s. from e.a.s. when a cycle is f.a.s. but not e.a.s.

Since the Star cycle is not e.a.s. for the chosen values of e_A and e_B , and the stability indices are the same along all its connections, our results coincide in determining the stability region of this cycle. However, for the remaining two sub-cycles, we can add that there are smaller regions inside those identified in Figure 7 of [42] where stronger attraction properties occur.

Propositions 3.3.1, 3.3.2 and 3.3.3 also provide proof that the (sub-)cycles are f.a.s, but not e.a.s, in the regions depicted for their stability in [42]. This is achieved by replacing $e_A = 1$ and $e_B = 0.8$ in the expressions for the values of the function F^{index} in Propositions 3.3.1 and 3.3.2 and by replacing these values directly into the stability indices calculated in Proposition 3.3.3.

When the RSPLS network is a.s., the Rock-to-Paper sub-cycle is e.a.s. under condition (a3) in Proposition 3.3.7 and, if we allow c_A or c_B to be smaller than e_A , then a lower bound appears to guarantee that the last of the values of F^{index} in the proof of Proposition 3.3.1 is positive. Although the RSP sub-cycle is not e.a.s., it may be f.a.s. with either just one or two connections with a positive stability index ($\sigma_{23} < 0$ always). The values obtained by replacing $e_A = 1$ and $e_B = 0.8$ in Proposition 3.3.3 show that

$\sigma_{12} > 0$ and

$$\sigma_{31} = \begin{cases} 1 - 0.8c_B + c_A (< 0) & \text{if } \frac{1 + c_A}{c_B} \leq 0.8 \\ \min \left\{ 0.25, -1 + \frac{1}{0.8c_B - c_A} (> 0) \right\} & \text{if } 0.8 < \frac{1 + c_A}{c_B}. \end{cases}$$

Hence, the existence of at least one positive index may support the visibility of the RSP sub-cycle in simulations. In the region of stability but closer to its lower bound there are two connections along which the stability index is positive. This promotes the attraction properties of the cycle.

Note that the parameter region depicted in our figures is much wider than that analysed in [42]. In fact, our results are analytic and therefore extend to values, not previously considered, of all four eigenvalues: e_A , e_B , c_A and c_B .

Chapter 4

Stability of cycles and survival in a Jungle Game with four species.

This chapter contains results that were published in the article [8], in which I am one of the authors.

In the present chapter we look at a game that extends RSP by adding a fourth species and constructing an interaction to obtain what is called a Jungle Game. The Jungle Game consists of a hierarchical chain where each species wins over all the ones below, except for the first and last species. The last species wins over the first. This game appears as a chinese board game, also under the name of Dou Shou Qi, with eight species.

The Jungle Game with four species describes the hierarchical interaction among four species, S_1 , S_2 , S_3 , and S_4 . We assume S_1 is at the top and S_4 at the bottom of the hierarchy so that S_1 wins over S_2 and S_3 , S_2 wins over S_3 and S_4 , S_3 wins over S_4 , and S_4 wins over S_1 . The relations among the species and the graph of the corresponding heteroclinic network appear in Figure 4.1.

There are three cycles in the heteroclinic network in the dynamics of the Jungle Game with four species: $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_3 \rightarrow \xi_2 \rightarrow \xi_1]$, $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_2 \rightarrow \xi_1]$ and $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_3 \rightarrow \xi_1]$ and two Δ -cliques (Δ_{321} and Δ_{432}).

We can associate the stability of each of these cycles to the survival of the species that correspond to its equilibria. Around a stable cycle there is a large set of initial conditions whose trajectories follow the cycle. For the cycles with three species this means they coexist in cyclic dominance, that is, playing a RSP game. The survival of a subset of the species also appears as an outcome of the extension of RSP to RSPLS; we do not pursue this similarity further.

4.1 Asymptotic stability of the network

We prove that, for an open set of parameter values, the heteroclinic network in the Jungle Game with four species is asymptotically stable. This guarantees that all trajectories that begin in a neighbourhood of the network remain close to, and eventually converge to, the network. When combined with the stability of each cycle, this provides a complete description of the long-run dynamics. In a population model this property is essential in order to ensure that the future outcome around the network is only that of the prescribed relationships of the species present.

Modelling the interactions as Lotka-Volterra (2.2), we obtain the following system of ODEs:

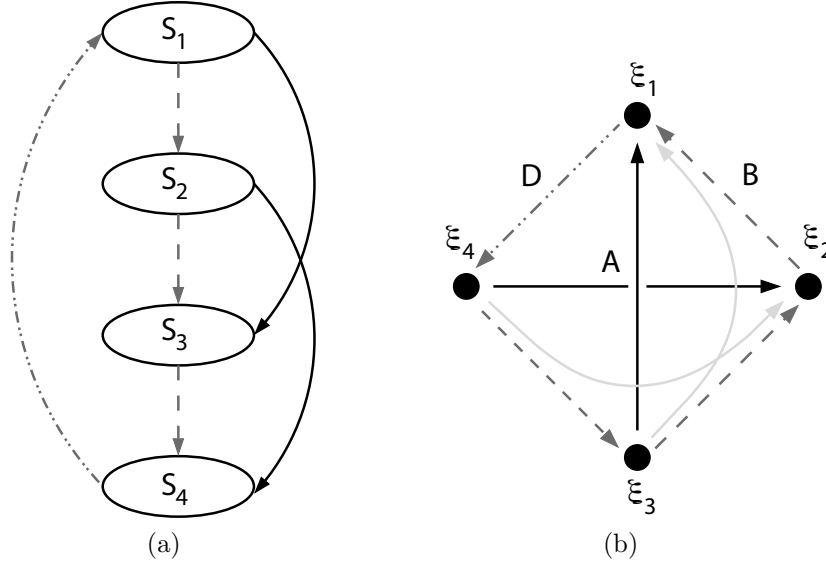


Figure 4.1: (a) The relationships of four species in the Jungle Game; arrows point from winner to loser. (b) The four species Jungle Game network: 2-dimensional connections are indicated by the letter A and solid lines, 1-dimensional connections in a Δ -clique by B and dashed lines, and other 1-dimensional connections by D and dash-dotted lines. The grey arrows signal the existence of Δ -cliques. There are two Δ -cliques in the heteroclinic network of the Jungle Game with four species: Δ_{321} and Δ_{432} .

$$\begin{cases} \dot{x}_1 = x_1 (1 - R + e_{21}x_2 + e_{31}x_3 - c_{41}x_4) \\ \dot{x}_2 = x_2 (1 - R - c_{12}x_1 + e_{32}x_3 + e_{42}x_4) \\ \dot{x}_3 = x_3 (1 - R - c_{13}x_1 - c_{23}x_2 + e_{43}x_4) \\ \dot{x}_4 = x_4 (1 - R + e_{14}x_1 - c_{24}x_2 - c_{34}x_3) \end{cases} \quad (4.1)$$

where $R = x_1 + x_2 + x_3 + x_4$ and $x_i \in \mathbb{R}_+$ denotes the proportion of individuals of the species S_i . The equilibrium corresponding to the survival of only species S_j is denoted by ξ_j . We label the coefficients in such a way that the indices ij denote an eigenvalue of the Jacobian matrix at ξ_i in the direction of x_j .

Theorem 4.1.1. *If $e_{ij} < 1$ for all i, j then there exists an attracting invariant sphere for the dynamics of (4.1). Furthermore, system (4.1) supports the heteroclinic network of Figure 4.1 (b).*

Proof. We use Theorem 5.1 in Field [16] to prove the existence of an attracting invariant sphere. In order to verify the hypotheses we change variables to obtain a vector field whose nonlinear part is homogeneous of degree 3. It is easy to verify that defining $X_i^2 = x_i$ with $X = (X_1, \dots, X_4)$, and denoting by $Q(X)$ their non-linear part, the ODEs in the new variables X_i satisfy the hypotheses of this theorem provided $e_{ij} < 1$ for all i, j . In fact,

using the standard inner product, $\langle \cdot, \cdot \rangle$, we obtain

$$\begin{aligned} \langle Q(X), X \rangle &= \frac{1}{2}X_1^2(-X_1^2 - (1 - e_{21})X_2^2 - (1 - e_{31})X_3^2 - (1 + c_{41})X_4^2) + \\ &+ \frac{1}{2}X_2^2(-(1 + c_{12})X_1^2 - X_2^2 - (1 - e_{32})X_3^2 - (1 - e_{42})X_4^2) + \\ &+ \frac{1}{2}X_3^2(-(1 + c_{13})X_1^2 - (1 + c_{23})X_2^2 - X_3^2 - (1 - e_{43})X_4^2) + \\ &+ \frac{1}{2}X_4^2(-(1 - e_{14})X_1^2 - (1 + c_{24})X_2^2 - (1 + c_{34})X_3^2 - X_4^2) \end{aligned}$$

It is clear that $\langle Q(X), X \rangle < 0$ if $e_{ij} < 1$ for all i, j .

That the vector field supports a heteroclinic network, follows from the facts that there are no equilibria outside of the coordinate axes in $\mathbb{R}_+^4$ and that the subspaces $\{x_i = 0\}$ are flow-invariant. This can be checked directly from (4.1). $\square$

We assume that the interaction between any two of the species has the same strength when they occur along edges of similar type. This allows us to group the eigenvalues of the Jacobian matrix at each equilibrium according to the type of connection, as suggested by Figure 4.1(b): species S_i wins over species S_{i+1} along a connection of type B, species S_i wins over species S_{i+2} along a connection of type A, and S_4 wins over S_1 along a connection of type D. We define

$$\begin{array}{lll} e_A = e_{31} = e_{42} & e_B = e_{43} = e_{32} = e_{21} & e_D = e_{14} \\ c_A = c_{13} = c_{24} & c_B = c_{34} = c_{23} = c_{12} & c_D = c_{41}. \end{array}$$

We use the results described in Subsection 2.3 and, without loss of generality, assume further that

$$\min c > \max e, \quad e_A > e_B, \quad \text{and} \quad c_A > c_B. \quad (4.2)$$

The first condition is the weakest usual necessary condition for stability. The second and third conditions determine the tangency of trajectories in a Δ -clique.

Note that the authors in [39] are considering as transverse the eigenvalues that are neither radial, contracting nor expanding for any cycle through ξ_j . These eigenvalues are called global transverse in Castro and Garrido-da-silva [10]. The Jungle Game network does not have global transverse eigenvalues.

Theorem 4.1.2. *Assume that (4.2) holds. The heteroclinic network supported by (4.1) is asymptotically stable if*

$$c_B^2 c_D > (c_B + e_A) e_B e_D. \quad (4.3)$$

Proof. The quantity ρ_j is obtained from Lemmas 2.3.2, 2.3.3 and 2.3.4 for all equilibria in each one of the cycles and depends on whether the node ξ_j is, or is not, an m-point in one or more Δ -cliques of the network. Since the Jungle Game network does not have global transverse eigenvalues, for this network in Lemma 2.3.2 we have $\rho_j = c/e$.

From Corollary 4.18 of [39], the Jungle Game network of four species is asymptotically stable if, for any cycle Σ of the network, $\rho(\Sigma) = \prod \rho_j > 1$, that is, if the following conditions are satisfied:

$$\rho(\Sigma_{142}) = \rho_1 \rho_4 \rho_2 = \frac{c_B}{e_D} \cdot \frac{c_D}{e_A} > 1 \quad (4.4)$$

$$\rho(\Sigma_{143}) = \bar{\rho}_1 \bar{\rho}_4 \bar{\rho}_3 = \frac{c_A}{e_D} \cdot \frac{c_D}{e_B} \cdot \frac{c_B}{c_B + e_A} > 1 \quad (4.5)$$

$$\rho(\Sigma_{1432}) = \tilde{\rho}_1 \tilde{\rho}_4 \tilde{\rho}_3 \tilde{\rho}_2 = \frac{c_B}{e_D} \cdot \frac{c_D}{e_B} \cdot \frac{c_B}{c_B + e_A} > 1 \quad (4.6)$$

Equations (4.4), (4.5) and (4.6) are the $\rho(X)$ when X is the cycle Σ_{142} , Σ_{143} and Σ_{1432} respectively. Once condition (4.2) holds, then condition (4.4) is automatically satisfied and moreover, if condition (4.6) holds, then condition (4.5) holds also. Condition (4.6) can be written as (4.3). $\square$

4.2 Stability of the cycles

Using the results from [35]¹, we can calculate the stability indices for points in all the connections in the network of the Jungle Game with four species which are contained in the coordinate planes. Therefore, when referring to the cycles in the Jungle Game network we study trajectories that occur in coordinate planes. This is a natural choice since the interactions are between two species at a time: when S_4 is eating S_1 , the species S_2 and S_3 are not directly involved.

The next three propositions show that only the cycle among species S_1 , S_2 , and S_4 exhibits any form of stability, indeed the strongest type of stability admissible for a heteroclinic cycle in a network. Together with Theorem 4.1.2, we know that for almost any initial condition close to any part of the network its trajectory converges to the cycle Σ_{142} .

The stability of the heteroclinic cycles, or lack thereof, provides information about the survival of the species S_i , $i = 1, \dots, 4$. Only the species corresponding to the nodes of a stable heteroclinic cycle are expected to coexist. Because the heteroclinic cycle with four nodes, $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_3 \rightarrow \xi_2 \rightarrow \xi_1]$, is always unstable we know that the coexistence of all four species is not expected in the long-run. The same applies to the heteroclinic cycle with three nodes, $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_3 \rightarrow \xi_1]$, meaning that these three species are not expected to coexist in the long-run. The essential asymptotic stability of the heteroclinic cycle with three nodes, $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_2 \rightarrow \xi_1]$, shows that the species S_1, S_2, S_4 coexist in the long-run while the species S_3 becomes extinct. Furthermore, the connections in this heteroclinic cycle indicate that the surviving species interact as a RSP game.

Proposition 4.2.1. *Assume that (4.2) and (4.3) hold. Then the cycle Σ_{142} is essentially asymptotically stable and the local stability indices are:*

$$\sigma_{14} = 1 - \frac{e_B}{e_A} > 0, \quad \sigma_{42} = +\infty \quad \text{and} \quad \sigma_{21} = +\infty$$

Proof. For this cycle, only for the Jacobian matrix at ξ_4 there is a positive transverse eigenvalue. Re-ordering the indices so that the quantity b_1 in Lemma 2.6.7 corresponds

¹Note that the change of variables $x_i = X_i^2$ for each species, transforms the equations such that they become symmetric under the sign change $X_i \mapsto -X_i$. After this change, each equation takes the form $\dot{X}_i = \frac{1}{2} X_i f_i(X^2)$. The non-radial eigenvalues are transformed by a factor of $\frac{1}{2}$ which does not affect the stability analysis.

to ξ_4 , we obtain

$$a_4 a_2 a_1 = \frac{c_A c_B c_D}{e_A e_B e_D} > 1$$

and

$$\begin{aligned} b_4 a_2 a_1 + b_1 a_2 + b_2 &= -\frac{e_B c_A c_B}{e_A e_B e_D} + \frac{c_A^2}{e_B e_D} + \frac{c_B}{e_B} \\ &= \frac{c_B e_A e_D + c_A (c_A e_A - c_B e_B)}{e_A e_B e_D} > 0. \end{aligned}$$

The first inequality follows from the first in (4.2). The last inequality follows from the last two in (4.2). Then,

$$\begin{aligned} \sigma_{14} &= f^{\text{index}}(b_4, 1) = f^{\text{index}}\left(-\frac{e_B}{e_A}, 1\right) = \frac{e_A}{e_B} - 1 > 0 \\ \sigma_{42} &= +\infty \\ \sigma_{21} &= f^{\text{index}}(b_1 + b_4 a_1, 1) = f^{\text{index}}\left(\frac{c_A e_A - c_B e_B}{e_A e_D}, 1\right) = +\infty \end{aligned}$$

□

We note that the connection C_{42} is 2-dimensional. Proposition 4.2.1 calculates the stability index for the 1-dimensional connecting trajectory $\kappa_{42} \subset C_{42}$ contained in the coordinate plane spanned by the axes containing ξ_2 and ξ_4 .

Proposition 4.2.2. *Assume that (4.2) and (4.3) hold. Then the cycle Σ_{143} is completely unstable.*

Proof. This cycle satisfies the hypotheses of Lemma 2.6.8, where two transverse eigenvalues are positive: at ξ_3 and at ξ_4 . Denoting the quantities of Subsection 2.5 for this cycle by $\bar{a}_j$ and $\bar{b}_j$ and again re-ordering the indices we obtain, as in Proposition 4.2.1,

$$\bar{a}_4 \bar{a}_3 \bar{a}_1 = \frac{c_A c_B c_D}{e_A e_B e_D} > 1$$

and

$$\begin{aligned} \bar{b}_3 \bar{a}_4 \bar{a}_1 + \bar{b}_4 \bar{a}_1 + \bar{b}_1 &= -\frac{e_B c_A c_D}{e_A e_B e_D} - \frac{e_A c_A}{e_B e_D} + \frac{c_B}{e_D} < 0 \Leftrightarrow \\ &= c_B < c_A \left(\frac{c_D}{e_A} + \frac{e_A}{e_B} \right). \end{aligned}$$

The last inequality holds from (4.2) since $c_B < c_A$ and the term in brackets is greater than 2. Then all $\sigma_j = -\infty$ and the cycle Σ_{143} is not an attractor. □

Proposition 4.2.3. *Assume that (4.2) and (4.3) hold. Then the cycle Σ_{1432} is completely unstable.*

Proof. In cycle Σ_{142} the stability index for the connecting trajectory in Σ_{142} from equilibrium ξ_2 to ξ_1 is $\sigma_{21} = +\infty$, according to Proposition 4.2.1. Hence the stability index for the same connecting trajectory in any other cycle of the network is equal to $-\infty$. It follows from Corollary 4.1 in [35] that all stability indices are equal to $-\infty$ and the cycle is completely unstable. □

4.3 Numerical results

In order to illustrate the results which we have analytically proved in Subsection 4.2, we present some images obtained by simulating the dynamics of (4.1). We note that the same behaviour is observed for different parameter values as long as they satisfy the sufficient condition for the asymptotic stability of the network. We do not reproduce the figures here since they are identical to Figures 4.2 and 4.3.

Figure 4.2 represents the time course of a trajectory of (4.1) for some parameter values which fulfill the conditions for asymptotic stability of the network in Theorem 4.1.2. After some time the trajectory approaches Σ_{142} , the e.a.s. cycle. As time increases the trajectory stays longer near each equilibrium. The survival of the three species S_1, S_2, S_4 and their cyclic dominance can be seen by the successive periods of x_i close to 1 for $i \neq 3$. The fact that x_3 becomes equal to zero indicates the extinction of species S_3 . Note that this happens even though the initial condition in this example is close to the equilibrium $x_3 = 1, x_1 = x_2 = x_4 = 0$.

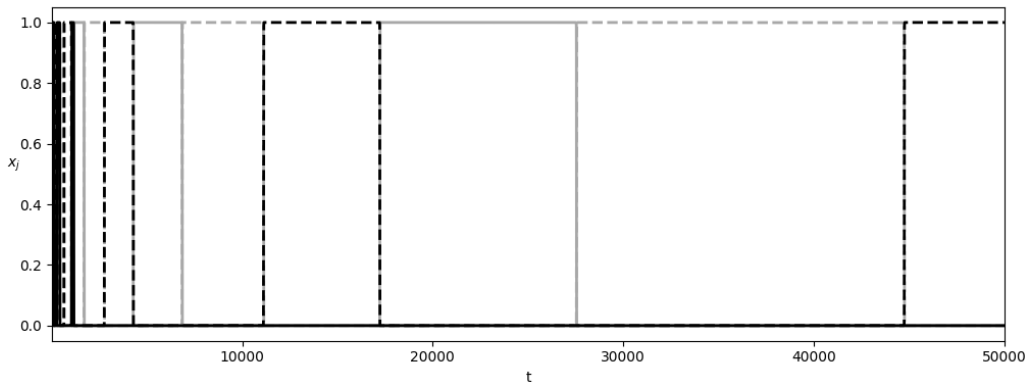


Figure 4.2: A typical time course for (4.1). The lines in dashed grey, solid grey, solid black and dashed black are the coordinates x_1, x_2, x_3 and x_4 respectively. The parameter values for the simulation are: $c_A = 1.2, c_B = 1, c_D = 1.1, e_A = 0.7, e_B = 0.65, e_D = 0.72$. The initial condition is $x_1 = x_2 = x_4 = 0.1$ and $x_3 = 1$.

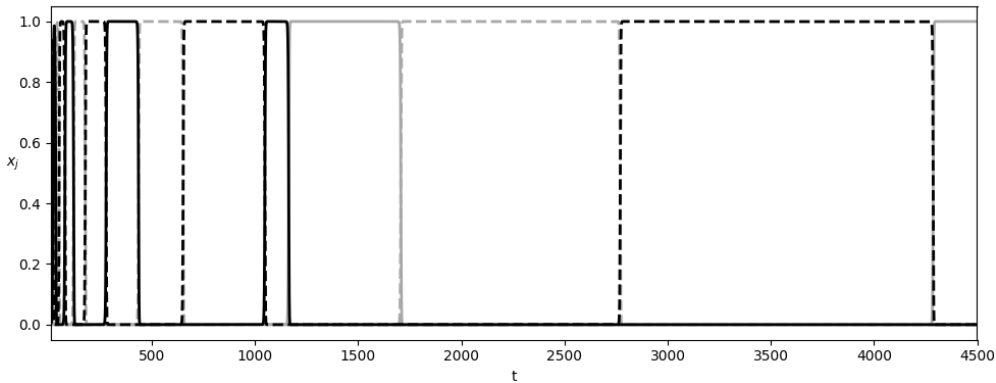


Figure 4.3: Detail of the trajectory of Figure 4.2 over a shorter time interval, showing a transient visit to ξ_3 (solid black). Conventions as in Figure 4.2.

The same trajectory as in Figure 4.2 is shown in a shorter time scale in Figure 4.3. Before the trajectory reaches the e.a.s. cycle, some other sequences of equilibria are visited. The trajectory visits the cycle Σ_{143} during some small period of time and once the

trajectory leaves that cycle and visits the equilibrium ξ_2 , then, for the remaining time it cyclically follows the sequence $[\xi_1 \rightarrow \xi_4 \rightarrow \xi_2 \rightarrow \xi_1]$, i.e., the e.a.s. cycle.

4.4 An alien in a game of three species

In what remains of this section we interpret the Jungle Game with four species to obtain information about the outcome of having a fourth species trying to invade three species that play a RSP game. Let S_1 , S_2 and S_3 be the original species playing RSP and assume that S_{i-1} wins when confronted with S_i , $i \bmod 3$. Let A denote the invading species, an *alien*, which can be weak or strong depending on whether it wins when confronted with either one or two of the original species, respectively. We denote a weak alien by A_w and a strong alien by A_s . The graphs of the interaction are represented in Figure 4.4. Without loss of generality, we choose S_3 as the only species that wins against A_s ; we choose S_3 as the only species that loses against A_w .

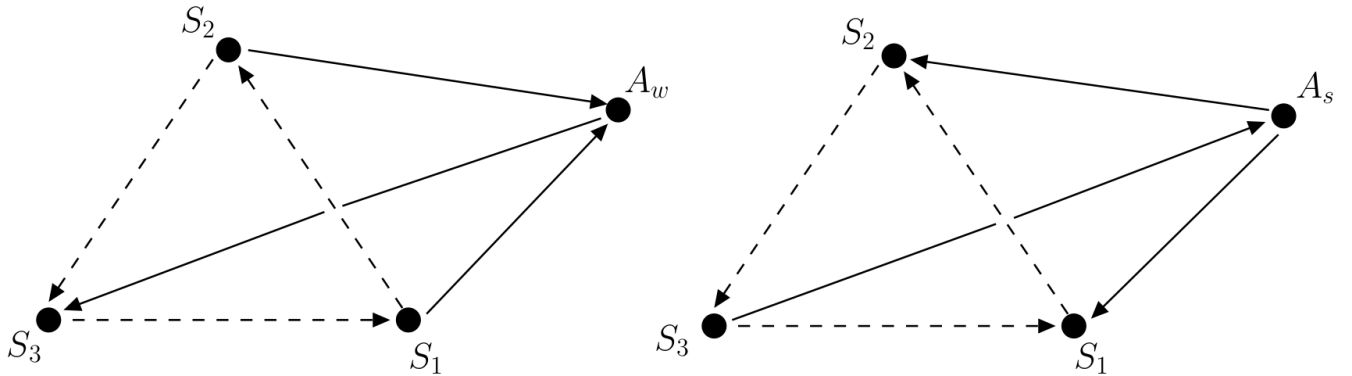
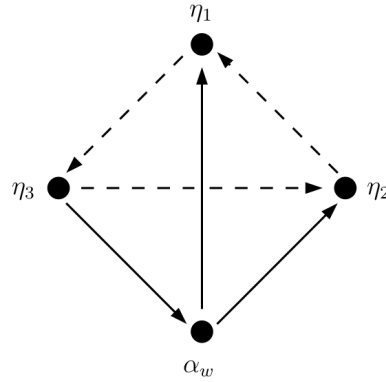
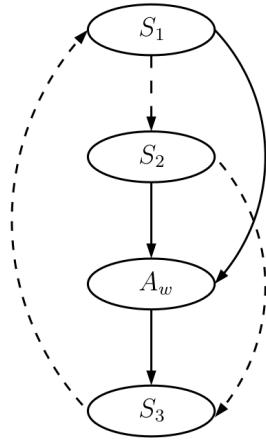


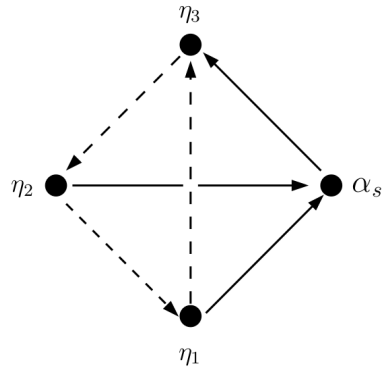
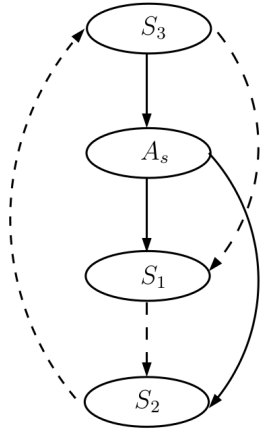
Figure 4.4: Relationships of species in an invasion of a RSP game by an alien: the predator-prey relationships among the original population of RSP players and a weak alien, A_w (left), and a strong alien, A_s (right). The dashed lines indicate the interactions among the original population of RSP players and the solid lines correspond to interactions with the alien.

It is simple to see that these correspond to the heteroclinic network of the Jungle Game with four species by comparing the relationships in Figure 4.1(a) with those in Figure 4.5. The results in Subsection 4.2 establish that in the interaction with a weak alien the only e.a.s. cycle is that connecting the equilibria η_1 , η_2 , and η_3 ; when the fourth species is a strong alien the only e.a.s. cycle is that connecting the equilibria η_2 , η_3 , and α_s . Thus the weak alien A_w is suppressed when interacting with a population of RSP players, whereas the strong alien A_s replaces one of the original species in playing a RSP game with the other two. It is noteworthy, although not surprising, that the original species which is replaced by A_s is the species that loses when confronted with the species that beats the alien.

From Figure 4.5 it is easy to see that after the introduction of a weak alien the original three species survive. However, introducing a strong alien leads to the survival of the strong alien together with S_2 and S_3 . This was anticipated by Case *et al.* [7] in their maxim “the prey of the prey of the weakest is the least likely to survive” albeit for a different model. From Figure 4.5 (b) we see that S_2 is the weakest species and its prey is S_3 . The prey of S_3 is S_1 which is the original species that becomes extinct by being



(a) The weak alien only wins against one species.



(b) The strong alien wins against two species.

Figure 4.5: On the left, the predator-prey relationships among the original population of RSP players and a weak alien in panel (a) and a strong alien in panel (b). The interactions are the same even though the species change place corresponding to different survival outcomes. On the right, the network of an invasion of a RSP game with nodes η_j by a weak alien, α_w , in panel (a) and a strong alien, α_s , in panel (b). Dashed lines indicate the interactions among the original population of RSP players and the solid lines correspond to interactions with the alien, as in Figure 4.4.

replaced by the strong alien. Of S_1 and S_2 that are the two weakest, in the sense of losing to two others, the one who is also a prey of prey of one of the weakest does not survive.

Chapter 5

The Jungle Game of five species and some generalizations

In the present chapter we present the Jungle Game as in the previous chapter but adding more species. The Jungle Game with n species describes the hierarchical interaction among n species, $S_1, S_2, \dots$, and S_n . We assume again S_1 is at the top and S_n at the bottom of the hierarchy, so that the main features about the relations among the species remain as in the Chapter 4.

We start with the analysis of the five species Jungle Game. The interactions in a five-species Jungle Game are depicted in Figure 5.1 (a): S_1 preys on S_2, S_3 and S_4 , S_2 preys on S_3, S_4 and S_5 , S_3 preys on S_4 and S_5 , S_4 preys on S_5 , while S_5 preys on S_1 . The five species Jungle Game network is depicted in 5.1 (b): The arrows point from prey to predator and indicate that the coordinate corresponding to the end-point grows as that corresponding to the node at the beginning decreases. The 2-dimensional connections are indicated by red lines, the 1-dimensional connections in a Δ -clique by blue lines, and other 1-dimensional connections by black lines. The green lines represent the 3-dimensional connections that emerge upon the introduction of the fifth species. Note that as we increase the number of species, we inevitably increase the complexity in terms of network dynamics. Specifically because the greater the number of species, the greater the number of connections that emerge, each with increasingly larger dimensions.

As usual the stability of each cycle depends on the eigenvalues of the Jacobian matrix at each equilibrium. We group the eigenvalues that correspond to some Δ -clique in four categories: with the subscript “A” corresponding to a 2-dimensional connection, with the subscript “B” corresponding to a 1-dimensional connection and with the subscript “F” corresponding to a 3-dimensional connection. We label the eigenvalues that are not part of any Δ -clique with the subscript “D”. Let $-c_{ij} < 0$ and $e_{ik} > 0$ be the contracting and expanding eigenvalues at the equilibrium ξ_i in the direction of ξ_j and ξ_k respectively. Then:

$$\begin{aligned} e_A &= e_{53} = e_{31} = e_{42}, & c_A &= c_{35} = c_{13} = c_{24}, \\ e_B &= e_{54} = e_{43} = e_{32} = e_{21}, & c_B &= c_{45} = c_{34} = c_{23} = c_{12}, \\ e_F &= e_{52} = e_{41}, & c_F &= c_{25} = c_{14}, \\ e_D &= e_{15}, & c_D &= c_{51}. \end{aligned} \tag{5.1}$$

Analogously to the Δ -clique’s tangency condition in 2.4 we assume that:

$$e_F \geq e_A > e_B \quad \text{and} \quad c_F \geq c_A > c_B \tag{5.2}$$

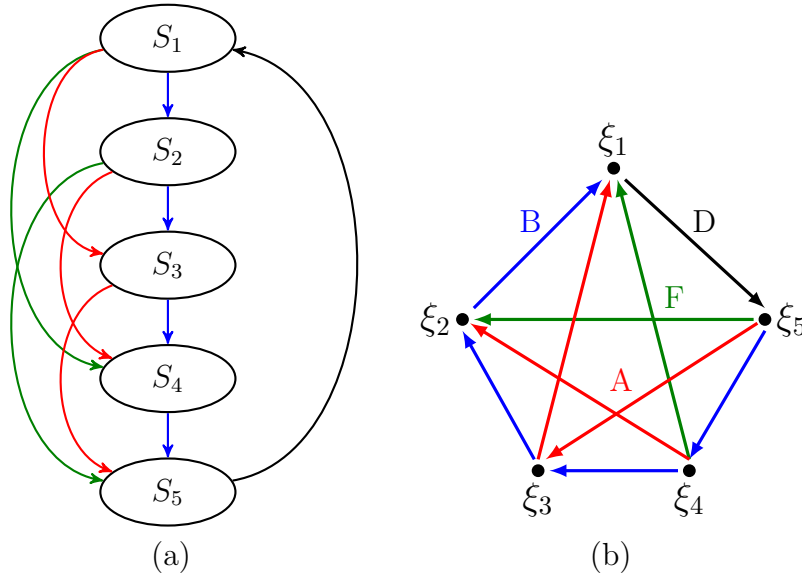


Figure 5.1: (a) The interactions among species and (b) the heteroclinic network in the 5-species Jungle Game.

We also make the common assumption that $\min c_J > \max e_J$, $J \in \{A, B, C, D\}$.

Modelling the interactions as Lotka-Volterra equations, we obtain the following system of ODEs

$$\begin{cases} \dot{x}_1 = x_1 (1 - R + e_B x_2 + e_A x_3 + e_F x_4 - c_D x_5) \\ \dot{x}_2 = x_2 (1 - R - c_B x_1 + e_B x_3 + e_A x_4 + e_F x_5) \\ \dot{x}_3 = x_3 (1 - R - c_A x_1 - c_B x_2 + e_B x_4 + e_A x_5) \\ \dot{x}_4 = x_4 (1 - R - c_F x_1 - c_A x_2 - c_B x_3 + e_B x_5) \\ \dot{x}_5 = x_5 (1 - R + e_D x_1 - c_F x_2 - c_A x_3 - c_B x_4) \end{cases} \quad (5.3)$$

where $R = x_1 + x_2 + x_3 + x_4 + x_5$ and $x_i \in \mathbb{R}_+$ denotes individuals of the species S_i . The equilibrium corresponding to the survival of only species S_j is denoted by ξ_j . We label the coefficients according to the choices in (5.1).

Theorem 5.0.1. *If $e_{ij} < 1$ for all i, j then there exists an attracting invariant sphere for the dynamics of (5.3). Furthermore, system (5.3) supports the heteroclinic network of Figure 5.1 (b).*

Proof. The proof is analogous to the proof of Theorem 4.1.1. □

5.1 Stability of the three-node sub-cycles

In this section we determine the stability of the subcycles¹ with three nodes consisting of equilibria and one-dimensional connections in the invariant planes. Before stating and proving our results we present the basic transition matrices for each cycle which are obtained from (2.8). Recall that for cycles with three nodes the global map is the identity, see [18].

¹Recall that as mentioned in Section 2.4 when we refer to a sub-cycle we are resctring the trajectories to the flow-invariant coordinate planes.

All basic transition matrices, and consequently their products, have the following form:

$$M_j = \begin{bmatrix} c_1 & 0 & 0 \\ c_2 & & \\ c_3 & I & \end{bmatrix},$$

where C_j , $C^{(j)}$, $C_{(i,j)} = (c_1, c_2, c_3)^T$ are the entries of the first column of the matrix M_j , $M^{(j)}$, $M_{(i,j)}$ respectively and I is the 2x2 identity matrix.

Transition matrices for sub-cycle $\Sigma_{152} = [S_1 \rightarrow S_5 \rightarrow S_2 \rightarrow S_1]$ The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{\text{in}} \rightarrow H_5^{\text{in}}, \quad M_5 : H_5^{\text{in}} \rightarrow H_2^{\text{in}}, \quad M_2 : H_2^{\text{in}} \rightarrow H_1^{\text{in}}$$

where $C_1 = \left(\frac{c_B}{e_D}, \frac{c_A}{e_D}, \frac{c_F}{e_D}\right)^T$, $C_5 = \left(\frac{c_D}{e_F}, -\frac{e_A}{e_F}, -\frac{e_B}{e_F}\right)^T$ and $C_2 = \left(\frac{c_F}{e_B}, \frac{c_B}{e_B}, \frac{c_A}{e_B}\right)^T$.

The products of the basic transition matrices with respect to this sub-cycle starting at H_1^{in} are $M_{(5,1)} = M_5 M_1 : H_1^{\text{in}} \rightarrow H_2^{\text{in}}$, $M^{(1)} = M_2 M_5 M_1 : H_1^{\text{in}} \rightarrow H_1^{\text{in}}$. In the same manner we obtain the products starting at H_j^{in} for $j = 5, 2$. The products of basic transition matrices have as first column:

$$\begin{aligned} C^1 &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, \frac{c_A e_B e_F - c_B e_A e_B + c_B^2 c_D}{e_B e_D e_F}, \frac{c_F e_B e_F - c_B e_B^2 + c_A c_B c_D}{e_B e_D e_F} \right)^T, \\ C^{(5)} &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, \frac{c_A c_D c_F - (e_A e_B - c_B c_D) e_D}{e_B e_D e_F}, \frac{c_D c_F^2 - (e_B^2 - c_A c_D) e_D}{e_B e_D e_F} \right)^T, \\ C^{(2)} &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, \frac{(c_B e_D + c_A c_F) e_F - c_B c_F e_A}{e_B e_D e_F}, \frac{(c_A e_D + c_F^2) e_F - c_B c_F e_B}{e_B e_D e_F} \right)^T, \\ C_{(5,1)} &= \left(\frac{c_B c_D}{e_D e_F}, \frac{c_A e_F - c_B e_A}{e_D e_F}, \frac{c_F e_F - c_B e_B}{e_D e_F} \right)^T, \\ C_{(2,5)} &= \left(\frac{c_D c_F}{e_B e_F}, \frac{c_B c_D - e_A e_B}{e_B e_F}, \frac{c_A c_D - e_B^2}{e_B e_F} \right)^T, \\ C_{(1,2)} &= \left(\frac{c_B c_F}{e_B e_D}, \frac{c_B e_D + c_A c_F}{e_B e_D}, \frac{c_A e_D + c_F^2}{e_B e_D} \right)^T. \end{aligned}$$

Transition matrices for sub-cycle $\Sigma_{153} = [S_1 \rightarrow S_5 \rightarrow S_3 \rightarrow S_1]$ The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{\text{in}} \rightarrow H_5^{\text{in}}, \quad M_5 : H_5^{\text{in}} \rightarrow H_3^{\text{in}}, \quad M_3 : H_3^{\text{in}} \rightarrow H_1^{\text{in}}$$

where $C_1 = \left(\frac{c_A}{e_D}, \frac{c_B}{e_D}, \frac{c_F}{e_D}\right)^T$, $C_5 = \left(\frac{c_D}{e_A}, -\frac{e_F}{e_A}, -\frac{e_B}{e_A}\right)^T$ and $C_3 = \left(\frac{c_A}{e_A} - \frac{e_B}{e_A}, \frac{c_B}{e_A}\right)^T$.

The products of the basic transition matrices with respect to this sub-cycle starting at H_1^{in} are $M_{(5,1)} = M_5 M_1 : H_1^{\text{in}} \rightarrow H_3^{\text{in}}$, $M^{(1)} = M_3 M_5 M_1 : H_1^{\text{in}} \rightarrow H_1^{\text{in}}$. In the same manner we obtain the products starting at H_j^{in} for $j = 5, 3$. The products of basic transition

matrices have as first column:

$$\begin{aligned}
C^{(1)} &= \left(\frac{c_A^2 c_D}{e_A^2 e_D}, -\frac{c_A e_A e_F + c_A c_D e_B - c_B e_A^2}{e_A^2 e_D}, -\frac{c_A e_A e_B - c_F e_A^2 - c_A c_B c_D}{e_A^2 e_D} \right)^T \\
C^{(5)} &= \left(\frac{c_A^2 c_D}{e_A^2 e_D}, \frac{c_A c_B c_D - e_A e_D e_F - c_D e_B e_D}{e_A^2 e_D}, \frac{c_A c_D c_F - (e_A e_B - c_B c_D) e_D}{e_A^2 e_D} \right)^T \\
C^{(3)} &= \left(\frac{c_A^2 c_D}{e_A^2 e_D}, -\frac{c_A^2 e_F + e_A e_B e_D - c_A c_B e_A}{e_A^2 e_D}, \frac{c_B e_A e_D - c_A^2 e_B + c_A c_F e_A}{e_A^2 e_D} \right)^T \\
C_{(5,1)} &= \left(\frac{c_A c_D}{e_A e_D}, \frac{c_B e_A - c_A e_F}{e_A e_D}, \frac{c_F e_A - c_A e_B}{e_A e_D} \right)^T \\
C_{(3,5)} &= \left(\frac{c_A c_D}{e_A^2}, -\frac{e_A e_F + c_D e_B}{e_A^2}, \frac{c_B c_D - e_A e_B}{e_A^2} \right)^T \\
C_{(1,3)} &= \left(\frac{c_A^2}{e_A e_D}, \frac{c_A c_B - e_B e_D}{e_A e_D}, \frac{c_B e_D + c_A c_F}{e_A e_D} \right)^T
\end{aligned}$$

Transition matrices for sub-cycle $\Sigma_{154} = [S_1 \rightarrow S_5 \rightarrow S_4 \rightarrow S_1]$ The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{\text{in}} \rightarrow H_5^{\text{in}}, \quad M_5 : H_5^{\text{in}} \rightarrow H_4^{\text{in}}, \quad M_4 : H_4^{\text{in}} \rightarrow H_1^{\text{in}}$$

where $C_1 = \left(\frac{c_F}{e_D}, \frac{c_B}{e_D}, \frac{c_A}{e_D} \right)^T$, $C_5 = \left(\frac{c_D}{e_B}, -\frac{e_F}{e_B}, -\frac{e_A}{e_B} \right)^T$ and $C_4 = \left(\frac{c_B}{e_F}, -\frac{e_A}{e_F}, -\frac{e_B}{e_F} \right)^T$

The products of the basic transition matrices with respect to this sub-cycle starting at H_1^{in} are $M_{(5,1)} = M_5 M_1 : H_1^{\text{in}} \rightarrow H_4^{\text{in}}$, $M^{(1)} = M_4 M_5 M_1 : H_1^{\text{in}} \rightarrow H_1^{\text{in}}$. In the same manner we obtain the products starting at H_j^{in} for $j = 5, 4$. The products of basic transition matrices have as first column:

$$\begin{aligned}
C^{(1)} &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, -\frac{c_F e_F^2 - c_B e_B e_F + c_D c_F e_A}{e_B e_D e_F}, \frac{(c_A e_B - c_F e_A) e_F - c_D c_F e_B}{e_B e_D e_F} \right)^T \\
C^{(5)} &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, -\frac{e_D e_F^2 + c_D e_A e_D - c_B^2 c_D}{e_B e_D e_F}, \frac{c_A c_B c_D - e_A e_D e_F - c_D e_B e_D}{e_B e_D e_F} \right)^T \\
C^{(4)} &= \left(\frac{c_B c_D c_F}{e_B e_D e_F}, -\frac{c_B c_F e_F + e_A e_B e_D - c_B^2 e_B}{e_B e_D e_F}, -\frac{e_B^2 e_D - c_A c_B e_B + c_B c_F e_A}{e_B e_D e_F} \right)^T \\
C_{(5,1)} &= \left(\frac{c_D c_F}{e_B e_D}, \frac{c_B e_B - c_F e_F}{e_B e_D}, \frac{c_A e_B - c_F e_A}{e_B e_D} \right)^T \\
C_{(4,5)} &= \left(\frac{c_B c_D}{e_B e_F}, -\frac{e_F^2 + c_D e_A}{e_B e_F}, -\frac{e_A e_F + c_D e_B}{e_B e_F} \right)^T \\
C_{(1,4)} &= \left(\frac{c_B c_F}{e_D e_F}, -\frac{e_A e_D - c_B^2}{e_D e_F}, -\frac{e_B e_D - c_A c_B}{e_D e_F} \right)^T
\end{aligned}$$

Proposition 5.1.1. *The subcycle Σ_{152} is e.a.s. with the following stability indices: $\sigma_{15} > 0$, $\sigma_{52} = +\infty$ and $\sigma_{21} = +\infty$.*

Proof. Checking conditions (i)–(iii) in Lemma 2.6.2 for each $M^{(j)}$, $j = 1, 5, 2$, we verify that the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_B c_D c_F}{e_B e_D e_F} \quad \text{and} \quad \bar{\lambda} = 1 \quad (\text{multiplicity } 2).$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds. From this it also follows that the largest eigenvalue is $\lambda = \frac{c_B c_D c_F}{e_B e_D e_F}$.

For each $j = 1, 5, 2$, denote by $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, w_3^{\max, j})$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$ that are given by:

$$\begin{aligned} \mathbf{w}^{\max, 1} &= \left(1, \frac{c_B e_A e_B - c_B^2 c_D - c_A e_B e_F}{e_B e_D e_F - c_B c_D c_F}, \frac{c_B e_B^2 - c_A c_B c_D - c_F e_B e_F}{e_B e_D e_F - c_B c_D c_F} \right)^T \\ \mathbf{w}^{\max, 5} &= \left(1, \frac{(e_A e_B - c_B c_D) e_D - c_A c_D c_F}{e_B e_D e_F - c_B c_D c_F}, \frac{(e_B^2 - c_A c_D) e_D - c_D c_F^2}{e_B e_D e_F - c_B c_D c_F} \right)^T \\ \mathbf{w}^{\max, 2} &= \left(1, \frac{c_B c_F e_A - (c_B e_D + c_A c_F) e_F}{e_B e_D e_F - c_B c_D c_F}, \frac{c_B c_F e_B - (c_A e_D + c_F^2) e_F}{e_B e_D e_F - c_B c_D c_F} \right)^T. \end{aligned}$$

By condition (ii) we know that $e_B e_D e_F - c_B c_D c_F < 0$. Then, condition (iii) is satisfied if the numerators $w_{i, \text{Num}}^{\max, j}$ of $w_2^{\max, j}$ and $w_3^{\max, j}$ are negative.

$$\begin{aligned} w_{2, \text{Num}}^{\max, 1} &= c_B e_A e_B - c_B^2 c_D - c_A e_B e_F \\ &= c_B (e_A e_B - c_B c_D) - c_A e_B e_F < 0, \text{ since } \min c > \max e. \\ w_{3, \text{Num}}^{\max, 1} &= c_B e_B^2 - c_A c_B c_D - c_F e_B e_F \\ &= e_B (c_B e_B - c_F e_F) - c_A c_B c_D < 0, \text{ since } c_B e_B < c_F e_F \text{ by (5.2).} \\ w_{2, \text{Num}}^{\max, 5} &= (e_A e_B - c_B c_D) e_D - c_A c_D c_F < 0, \text{ since } \min c > \max e. \\ w_{3, \text{Num}}^{\max, 5} &= (e_B^2 - c_A c_D) e_D - c_D c_F^2 < 0, \text{ since } \min c > \max e. \\ w_{2, \text{Num}}^{\max, 2} &= c_B c_F e_A - c_B e_D e_F - c_A c_F e_F \\ &= c_F (c_B e_A - c_A e_F) - c_B e_D e_F < 0, \text{ since } c_B e_A < c_A e_F \text{ by (5.2).} \\ w_{3, \text{Num}}^{\max, 2} &= c_B c_F e_B - e_F (c_A e_D + c_F^2) \\ &= c_F (c_B e_B - c_F e_F) - c_A e_D e_F < 0, \text{ since } c_B e_B < c_F e_F \text{ by (5.2).} \end{aligned}$$

Then, condition (iii) is also satisfied.

We compute the stability indices by applying the results in Appendix A.1 of [18]. Recall that the function F^{index} takes values $< +\infty$ only if an entry in a row of the respective matrix is negative. We thus compute the value of F^{index} for the rows of the transition matrices M_j , $M_{(l, j)}$, $M^{(j)}$ with at least one negative entry:

$$\sigma_{15} = \min \left\{ F^{\text{index}} \left(-\frac{e_A}{e_F}, 1, 0 \right), F^{\text{index}} \left(-\frac{e_B}{e_F}, 0, 1 \right) \right\} > 0, \text{ by (5.2).}$$

$$\sigma_{52} = +\infty$$

$$\sigma_{21} = +\infty.$$

□

Proposition 5.1.2. *The subcycle Σ_{153} is c.u.*

Proof. Checking conditions (i)–(iii) in Lemma 2.6.2 for each $M^{(j)}$, $j = 1, 5, 3$, we verify that the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_A^2 c_D}{e_A^2 e_D} \quad \text{and} \quad \bar{\lambda} = 1 \quad (\text{multiplicity } 2).$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds. From this it also follows that the largest eigenvalue is $\frac{c_A^2 c_D}{e_A^2 e_D}$.

For each $j = 1, 5, 3$, denote by $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, w_3^{\max, j})$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$ that are given by:

$$\mathbf{w}^{\max, 1} = \left(1, \frac{c_A e_A e_F + c_A c_D e_B - c_B e_A^2}{e_A^2 e_D - c_A^2 c_D}, \frac{c_A e_A e_B - c_F e_A^2 - c_A c_B c_D}{e_A^2 e_D - c_A^2 c_D} \right)^T$$

By condition (ii) we know that $e_A^2 e_D - c_A^2 c_D < 0$. Then, condition (iii) is satisfied if the numerators $w_{i, \text{Num}}^{\max, j}$ of $w_2^{\max, j}$ and $w_3^{\max, j}$ are negative. However,

$$\begin{aligned} w_{2, \text{Num}}^{\max, 1} &= c_A e_A e_F + c_A c_D e_B - c_B e_A^2 \\ &= e_A (c_A e_F - c_B e_A) + c_A c_D e_B > 0 \text{ by (5.2)}. \end{aligned}$$

Then, condition (iii) is not satisfied. By Lemma 3 in [37], the sub-cycle is completely unstable with all $\sigma_{ij} = -\infty$. □

Proposition 5.1.3. *The subcycle Σ_{154} is c.u.*

Proof. The proof follows that of Proposition 5.1.2 verifying that for $j = 1, 5, 4$ the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_B c_D c_F}{e_B e_D e_F} \quad \text{and} \quad \bar{\lambda} = 1 \quad (\text{multiplicity } 2).$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds. From this it also follows that the largest eigenvalue is $\frac{c_B c_D c_F}{e_B e_D e_F}$.

We can write $\mathbf{w}^{\max, 1}$ as follows:

$$\mathbf{w}^{\max, 1} = \left(1, \frac{c_F e_F^2 - c_B e_B e_F + c_D c_F e_A}{e_B e_D e_F - c_B c_D c_F}, \frac{c_D c_F e_B - (c_A e_B - c_F e_A) e_F}{e_B e_D e_F - c_B c_D c_F} \right)^T.$$

By condition (ii) we know that $e_B e_D e_F - c_B c_D c_F < 0$. However,

$$\begin{aligned} w_{2, \text{Num}}^{\max, 1} &= c_F e_F^2 - c_B e_B e_F + c_D c_F e_A \\ &= e_F (c_F e_F - c_B e_B) + c_D c_F e_A > 0 \text{ by (5.2)}. \end{aligned}$$

Then, condition (iii) is not satisfied. By Lemma (3) in [37], the sub-cycle Σ_{154} is completely unstable with all $\sigma_{ij} = -\infty$. □

5.2 Stability of the four-node cycles

We prove that all the sub-cycles with four nodes and 1-dimensional connections in invariant planes are completely unstable.

Proposition 5.2.1. *The sub-cycles Σ_{1542} , Σ_{1532} and Σ_{1543} are c.u.*

Proof. The proof is divided into two parts, one concerning the cycles Σ_{1542} and Σ_{1532} and another concerning Σ_{1543} .

Part 1: The cycles Σ_{1542} and Σ_{1532} both contain the connection $[\xi_2 \rightarrow \xi_1]$, which from Proposition 5.1.1 has stability index $+\infty$ for the 3-node cycle Σ_{152} . Therefore, both 4-node cycles have stability index for this connection equal to $-\infty$. Thus, by Lemma 3.5 of [18] all connections of the cycles have stability index equal to $-\infty$ and the cycles are completely unstable.

Part 2: To prove that Σ_{1543} is c.u. we observe that a sufficient condition for the stability index along a connection to be equal to $-\infty$ is, for the corresponding basic transition matrix, to have a row with non positive entries.

In fact, from Proposition 2.6.5 we know that $\sigma_j = \min_{i=1,\dots,K} \{F^{\text{index}}(\alpha^{(i)})\}$. We know also from the function F^{index} that if $\max\{\alpha_1, \alpha_2, \alpha_3\} \leq 0$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ are the rows of the relevant transition matrices, then $F^{\text{index}}(\alpha) = -\infty$.

Consider the transition matrix $M_{(5,3)}$,

$$M_{(5,3)} = M_5 M_1 M_3 = \begin{bmatrix} -\frac{(c_B e_D + c_A c_F) e_F + e_B^2 e_D - c_A c_B e_B}{e_A e_B e_D} & 1 & -\frac{c_F e_F - c_B e_B}{e_B e_D} \\ -\frac{c_B e_A e_D - c_A^2 e_B + c_A c_F e_A}{e_A e_B e_D} & 0 & \frac{c_A e_B - c_F e_A}{e_B e_D} \\ \frac{c_B c_D e_D + c_A c_D c_F}{e_A e_B e_D} & 0 & \frac{c_D c_F}{e_B e_D} \end{bmatrix}.$$

The inequalities (5.2) suffice to show that the non-zero entries in the second row of $M_{(5,3)}$ are negative:

$$\begin{aligned} -\frac{c_B e_A e_D - c_A^2 e_B + c_A c_F e_A}{e_A e_B e_D} < 0 &\Leftrightarrow c_B e_A e_D - c_A^2 e_B + c_A c_F e_A > 0 \\ &\Leftrightarrow c_B e_A e_D + c_A (c_F e_A - c_A e_B) > 0; \end{aligned}$$

and

$$\frac{c_A e_B - c_F e_A}{e_B e_D} < 0 \Leftrightarrow c_F e_A - c_A e_B > 0.$$

Hence, the stability index σ_{43} equals $-\infty$. So, the four-node cycle Σ_{1543} is completely unstable with all $\sigma_{ij} = -\infty$. $\square$

5.3 The five-node sub-cycle

Proposition 5.3.1. *The sub-cycle Σ_{15432} is c.u.*

Proof. The cycle Σ_{15432} contains the connection κ_{21} , which from Proposition 5.1.1 has stability index $+\infty$ for the three-node cycle Σ_{152} . Therefore, the five-node cycle has stability index for this connection equal to $-\infty$ and is completely unstable. $\square$

5.4 Numerical results

In order to illustrate the results which we have analytically proved in previous subsections, we model the interactions between the species using Lotka-Volterra equations. We present in Figure 5.2 the image obtained by simulating the dynamics of the system of ODEs in (5.3):

We note that the same behaviour is observed for different parameter values as long as they satisfy conditions in (5.2) but we do not reproduce them here since they are similar to Figure 5.2.

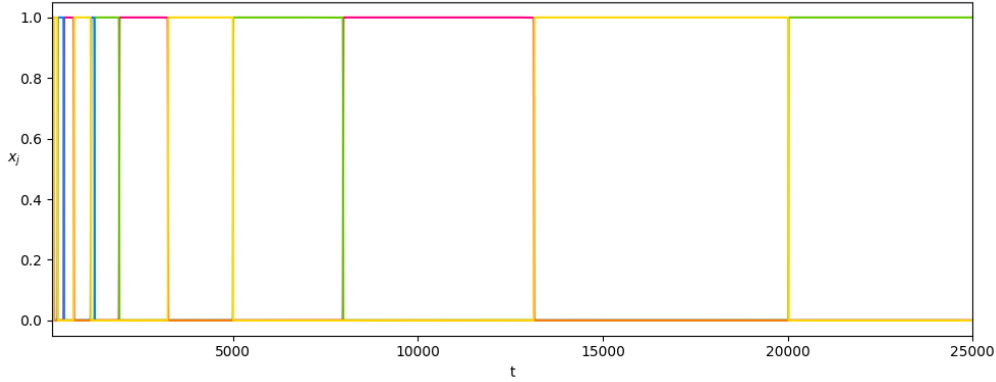


Figure 5.2: A typical time course of the dynamics of (5.3) illustrating the attractivity of the cycle Σ_{152} . The lines in pink, green, blue, orange and yellow are the coordinates x_1, x_2, x_3, x_4 and x_5 respectively. The parameter values for the simulation are: $c_A = 1.2$, $c_B = 1$, $c_D = 1.1$, $c_F = 1.25$, $e_A = 0.7$, $e_B = 0.65$, $e_D = 0.72$ and $e_F = 0.75$. The initial condition is $x_3 = x_4 = 1$ and $x_1 = x_2 = x_5 = 0.01$.

Figure 5.2 represents the time course of a trajectory of (5.3) for some parameter values which fulfill the conditions in (5.2). After some time the trajectory approaches Σ_{152} , the e.a.s. cycle. As time increases the trajectory stays longer near each equilibrium. The survival of the three species S_1, S_2, S_5 and their cyclic dominance can be seen by the successive periods of x_i close to 1 for $i \neq \{3, 4\}$. The fact that x_3 and x_4 become equal to zero indicates the extinction of species S_3 and S_4 . Note that this happens even though the initial condition in this example is close to the point where $x_3 = x_4 = 1$ and $x_1 = x_2 = x_5 = 0$.

5.5 Some generalizations

The interactions in a n -species Jungle Game are the followings: let S_i , $i = 1, \dots, n$ denote a species so that the lower the index the closer to the top of the hierarchy the species is. The species at the top of the hierarchy preys on all others except on S_n , while S_n just preys on S_1 . For $i = 2, \dots, n - 1$, S_i preys on all from S_{i+1} to S_n , see Figure 5.3.

As we increase the number of species in a Jungle Game, the calculations become increasingly complicated to handle. To avoid this, in a Lotka-Volterra model (2.2), we label the eigenvalues of the Jacobian matrix at each equilibrium in only two groups: with the subscript “B” corresponding to a one-dimensional connection and with the subscript “A” for those corresponding to a two or higher dimensional connection. Let $-c_{ji}$ and e_{jk}

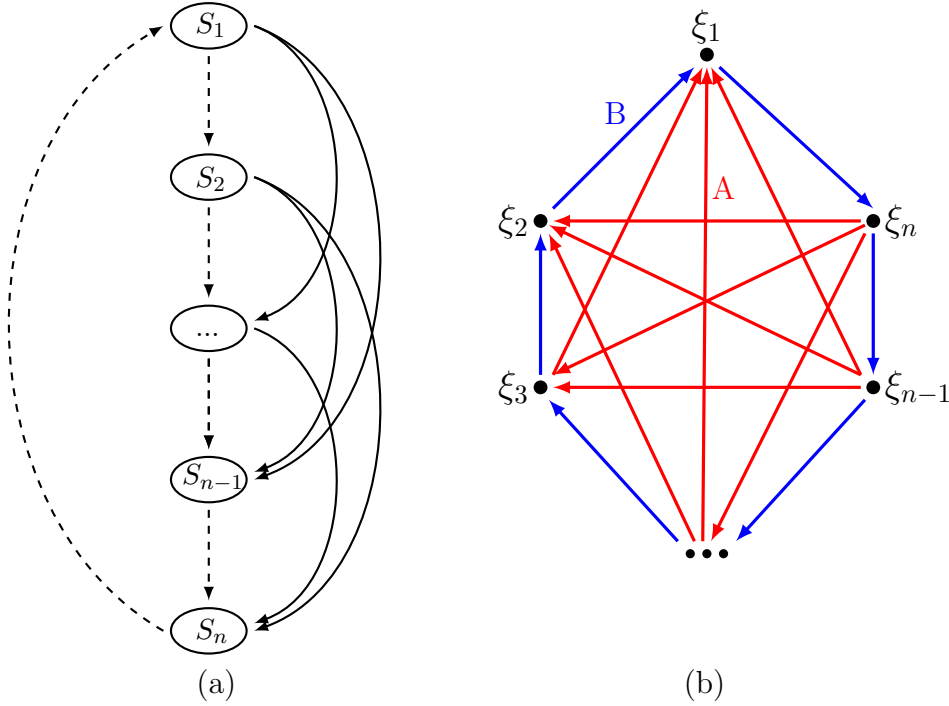


Figure 5.3: (a) The relationships of a Jungle Game with n species. The arrows point from prey to predator and indicate that the coordinate corresponding to the end-point grows as that corresponding to the node at the beginning decreases. (b) The heteroclinic network of the n -species Jungle Game. 1-dimensional connections are indicated by letter B while all other connections with dimension greater than 1 are indicated by letter A.

be the contracting and expanding eigenvalues from the equilibrium ξ_j in the direction of ξ_i and ξ_k , respectively. Then:

$$\begin{aligned}
 e_A &= e_{i1}, \text{ for all } i = 3, \dots, n-1 \\
 &= e_{ij}, \text{ for all } i = j+2, \dots, n; \text{ and for all } j = 2, \dots, n-2 \\
 c_A &= c_{1i}, \text{ for all } i = 3, \dots, n-1 \\
 &= c_{ij}, \text{ for all } i = 2, \dots, n-2; \text{ and for all } j = i+2, \dots, n \\
 e_B &= e_{n(n-1)} = e_{(n-1)(n-2)} = \dots = e_{(j+1)j} = \dots = e_{21} = e_{1n} \\
 c_B &= c_{(n-1)n} = c_{(n-2)(n-1)} = \dots = c_{j(j+1)} = \dots = c_{12} = c_{n1}
 \end{aligned}$$

For $n = 5$, this becomes:

$$\begin{aligned}
 e_A &= e_{53} = e_{31} = e_{42} = e_{52} = e_{41} \\
 c_A &= c_{35} = c_{13} = c_{24} = c_{25} = c_{14} \\
 e_B &= e_{54} = e_{43} = e_{32} = e_{21} = e_{15} \\
 c_B &= c_{45} = c_{34} = c_{23} = c_{12} = c_{51}
 \end{aligned}$$

that may be compared with (5.1) above.

From the Δ -clique's tangency conditions in Equation (2.4) we assume that:

$$e_A > e_B \quad \text{and} \quad c_A > c_B. \quad (5.4)$$

We also assume the condition

$$\min c > \max e. \quad (5.5)$$

5.5.1 Stability of the three-node cycles

In this section we determine the stability of any sub-cycle with just three nodes consisting of equilibria and one-dimensional connections in the invariant planes. Before stating and proving our results we present the basic transition matrices for each cycle. Recall that for cycles with three nodes the global map is the identity.

Lemma 5.5.1. *Let M be a $n \times n$ matrix of the form $M = \begin{bmatrix} q_0 & 0 \\ Q & I \end{bmatrix}$, where $Q = (q_1, \dots, q_{n-1})^T$ and I is the $(n-1) \times (n-1)$ identity matrix. Then the product of two matrices of this form also has the same form. The eigenvalues of M are $\lambda = q_0$ with multiplicity 1 and $\lambda = 1$ with multiplicity $(n-1)$ and an eigenvector w of M associated to the eigenvalue $\lambda = q_0$ is $w = \left(1, \frac{Q}{q_0}\right)$*

Thus, the layout of a basic transition matrix M_j is the following:

$$\left[\begin{array}{c|c} \frac{c_{j,j-1}}{e_{j,j+1}} & 0 \\ \hline Q_j & I \end{array} \right],$$

where $Q_j = \left(\frac{t_{j,s}}{e_{j,j+1}}, \dots, \frac{t_{j,N}}{e_{j,j+1}} \right)^T$ consists of quotients between each transverse eigenvalue at ξ_j and the expanding eigenvalue at ξ_j and I is the identity matrix.

Transition matrices for $\Sigma_{1n2} = [S_1 \rightarrow S_n \rightarrow S_2 \rightarrow S_1]$:

The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{in} \rightarrow H_n^{in}, \quad M_n : H_n^{in} \rightarrow H_2^{in}, \quad M_2 : H_2^{in} \rightarrow H_1^{in}$$

where

$$\begin{aligned} Q_1 &= \left(\frac{c_A}{e_B}, \frac{c_A}{e_B}, \dots, \frac{c_A}{e_B} \right)^T, \\ Q_n &= \left(-1, \dots, -1, -\frac{e_B}{e_A} \right)^T, \\ Q_2 &= \left(\frac{c_B}{e_B}, \frac{c_A}{e_B}, \dots, \frac{c_A}{e_B} \right)^T. \end{aligned}$$

Transition matrices for $\Sigma_{1nj} = [S_1 \rightarrow S_n \rightarrow S_j \rightarrow S_1]$ for $j \neq \{2, n-1\}$

The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{in} \rightarrow H_n^{in}, \quad M_n : H_n^{in} \rightarrow H_j^{in}, \quad M_j : H_j^{in} \rightarrow H_1^{in}$$

where

$$\begin{aligned} Q_1 &= \left(\frac{c_B}{e_B}, \frac{c_A}{e_B}, \dots, \frac{c_A}{e_B} \right)^T, \\ Q_n &= \left(-1, \dots, -1, -\frac{e_B}{e_A} \right)^T, \\ Q_j &= \left(-1, \dots, -\frac{e_B}{e_A}, \frac{c_B}{e_A}, \frac{c_A}{e_A}, \dots \right)^T. \end{aligned}$$

The matrix Q_j has $n - 3$ entries that are the transverse eigenvalues to the cycle at ξ_j . As we take an increasing j , the number of positive transverse eigenvalues increases and the number of negative eigenvalues decreases. Consequently, as j increases, the number of entries with the value -1 increases in Q_j and the number of entries $\frac{c_A}{e_A}$ decreases.

Transition matrices for $\Sigma_{1n(n-1)} = [S_1 \rightarrow S_n \rightarrow S_{n-1} \rightarrow S_1]$ The basic transition matrices between consecutive equilibria are:

$$M_1 : H_1^{in} \rightarrow H_n^{in}, \quad M_n : H_n^{in} \rightarrow H_{n-1}^{in}, \quad M_{n-1} : H_{n-1}^{in} \rightarrow H_1^{in}$$

where

$$\begin{aligned} Q_1 &= \left(\frac{c_B}{e_B}, \frac{c_A}{e_B}, \dots, \frac{c_A}{e_B} \right)^T, \\ Q_n &= \left(-\frac{e_A}{e_B}, \dots, -\frac{e_A}{e_B} \right)^T, \\ Q_{n-1} &= \left(-1, \dots, -1, -\frac{e_B}{e_A} \right)^T. \end{aligned}$$

Proposition 5.5.2. *The subcycle Σ_{1n2} is f.a.s. with the following stability indices: $\sigma_{1n} = 0$, $\sigma_{n2} = +\infty$ and $\sigma_{21} = +\infty$.*

Proof. Checking conditions (i)–(iii) in Lemma 2.6.2 for each $M^{(j)}$, $j = 1, n, 2$, we verify that the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_A c_B^2}{e_A e_B^2} \quad \text{and} \quad \bar{\lambda} = 1 \quad (\text{multiplicity } n - 1).$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds.

For each $j = 1, n, 2$ denote by $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, \dots, w_{n-2}^{\max, j})$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$ that, by lemma 5.5.1, are given by:

$$\begin{aligned} \mathbf{w}^{\max, 1} &= \left(1, \frac{(c_B - c_A)e_A e_B - c_B^3}{e_A e_B^2 - c_A c_B^2}, \frac{(c_B - c_A)e_A e_B - c_A c_B^2}{e_A e_B^2 - c_A c_B^2}, \dots, \frac{(c_B - c_A)e_A e_B - c_A c_B^2}{e_A e_B^2 - c_A c_B^2}, \right. \\ &\quad \left. \frac{(c_B e_B - c_A e_A)e_B - c_A c_B^2}{e_A e_B^2 - c_A c_B^2} \right)^T \\ \mathbf{w}^{\max, n} &= \left(1, \frac{(e_A e_B - c_B^2)e_B - c_A^2 c_B}{e_A e_B^2 - c_A c_B^2}, \frac{(e_A e_B - c_A c_B)e_B - c_A^2 c_B}{e_A e_B^2 - c_A c_B^2}, \dots, \right. \\ &\quad \left. \frac{(e_A e_B - c_A c_B)e_B - c_A^2 c_B}{e_A e_B^2 - c_A c_B^2}, \frac{(e_B^2 - c_A c_B)e_B - c_A^2 c_B}{e_A e_B^2 - c_A c_B^2} \right)^T \\ \mathbf{w}^{\max, 2} &= \left(1, -\frac{c_B e_A e_B + c_A e_A (c_A - c_B)}{e_A e_B^2 - c_A c_B^2}, -\frac{c_A e_A e_B + c_A e_A (c_A - c_B)}{e_A e_B^2 - c_A c_B^2}, \dots, \right. \\ &\quad \left. -\frac{c_A e_A e_B + c_A e_A (c_A - c_B)}{e_A e_B^2 - c_A c_B^2}, -\frac{c_A (c_A e_A - c_B e_B) + c_A e_A e_B}{e_A e_B^2 - c_A c_B^2} \right)^T. \end{aligned}$$

By condition (ii) we know that $e_A e_B^2 - c_A c_B^2 < 0$. Then, condition (iii) is satisfied if the numerators of $w_2^{\max,j}, \dots, w_{n-2}^{\max,j}$ are negative.

$$\begin{aligned}
w_{2,\text{Num}}^{\max,1} &= (c_B - c_A)e_A e_B - c_B^3 < 0, \text{ since } c_A > c_B. \\
w_{3,\text{Num}}^{\max,1} &= w_{3,\text{Num}}^{\max,1} = w_{4,\text{Num}}^{\max,1} = \dots = w_{n-3,\text{Num}}^{\max,1} \\
&= (c_B - c_A)e_A e_B - c_A c_B^2 < 0, \text{ since } c_A > c_B \text{ by (5.4).} \\
w_{n-2,\text{Num}}^{\max,1} &= (c_B e_B - c_A e_A)e_B - c_A c_B^2 < 0, \text{ since } c_A > c_B \text{ and } e_A > e_B \text{ by (5.4).} \\
w_{2,\text{Num}}^{\max,n} &= (e_A e_B - c_B^2)e_B - c_A^2 c_B < 0, \text{ since } \min c > \max e, \text{ by (5.5).} \\
w_{3,\text{Num}}^{\max,n} &= w_{3,\text{Num}}^{\max,n} = w_{4,\text{Num}}^{\max,n} = \dots = w_{n-3,\text{Num}}^{\max,n} \\
&= (e_A e_B - c_A c_B)e_B - c_A^2 c_B < 0, \text{ since } \min c > \max e, \text{ by (5.5).} \\
w_{n-2,\text{Num}}^{\max,n} &= (e_B^2 - c_A c_B)e_B - c_A^2 c_B < 0, \text{ since } \min c > \max e, \text{ by (5.5).} \\
w_{2,\text{Num}}^{\max,2} &= c_B e_A e_B + c_A e_A (c_A - c_B) > 0, \text{ since } c_A > c_B \text{ by (5.4).} \\
w_{3,\text{Num}}^{\max,2} &= w_{3,\text{Num}}^{\max,2} = w_{4,\text{Num}}^{\max,2} = \dots = w_{n-3,\text{Num}}^{\max,2} \\
&= c_A e_A e_B + c_A e_A (c_A - c_B) > 0, \text{ since } c_A > c_B \text{ by (5.5).} \\
w_{n-2,\text{Num}}^{\max,2} &= (c_A e_A - c_B e_B)c_A + c_A e_A e_B > 0, \text{ since } c_A > c_B \text{ and } e_A > e_B \text{ by (5.4).}
\end{aligned}$$

Then, condition (iii) is also satisfied.

We calculate the stability indices by applying the results in Appendix A.1 of [18]. Recall that the function F^{index} takes values $< +\infty$ only if an entry in a row of the respective matrix is negative. We thus calculate the value of F^{index} for the rows of the transition matrices $M_j, M_{(l,j)}, M^{(j)}$ with at least one negative entry:

$$\sigma_{1n} = F^{\text{index}}(-1, 1, 0) = 0$$

$$\sigma_{n2} = +\infty$$

$$\sigma_{21} = +\infty$$

□

Proposition 5.5.3. *The subcycle $\Sigma_{1nj}, j \neq 2$ is c.u.*

Proof. Checking conditions (i)–(iii) in Lemma 2.6.2 for each $M^{(j)}, j = 1, n, j$, we verify that the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_A^2 c_B}{e_A^2 e_B} \quad \text{and} \quad \bar{\lambda} = 1 \text{ (multiplicity } n-1 \text{)}.$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds.

For each $j = 1, n, j$ denote by $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, \dots, w_{n-2}^{\max, j})$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$ that, by lemma 5.5.1, are given by:

$$\mathbf{w}^{\max, 1} = \left(1, \frac{(c_B - c_A)e_A^2 - c_A c_B e_A}{e_A^2 e_B - c_A^2 c_B}, \frac{c_A c_B e_B}{e_A^2 e_B - c_A^2 c_B}, -\frac{c_A c_B^2}{e_A^2 e_B - c_A^2 c_B}, \dots, -\frac{c_A c_B^2}{e_A^2 e_B - c_A^2 c_B}, \right. \\ \left. \frac{(e_B - e_A)c_A e_A - c_A^2 c_B}{e_A^2 e_B - c_A^2 c_B} \right)^T$$

By condition (ii) we know that $e_A^2 e_B - c_A^2 c_B < 0$. Then, condition (iii) is not satisfied since the numerator of $w_3^{\max, 1}$ is positive. $\square$

Proposition 5.5.4. *The subcycle $\Sigma_{1n(n-1)}$ is c.u.*

Proof. Checking conditions (i)–(iii) in Lemma 2.6.2 for each $M^{(j)}$, $j = 1, n, n-1$, we verify that the transition matrices $M^{(j)}$ all have the same eigenvalues:

$$\lambda = \frac{c_A c_B^2}{e_A e_B^2} \quad \text{and} \quad \bar{\lambda} = 1 \quad (\text{multiplicity } n-1).$$

Conditions (i) and (ii) are satisfied since the eigenvalues are all real and our assumption $\min c > \max e$ holds.

For each $j = 1, n, n-1$ denote by $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, \dots, w_{n-2}^{\max, j})$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$. By lemma 5.5.1:

$$\mathbf{w}^{\max, 1} = \left(1, \frac{(c_A e_A - c_B e_B)e_A + c_A c_B e_A}{e_A e_B^2 - c_A c_B^2}, \frac{(e_A - e_B)c_A e_A + c_A c_B e_A}{e_A e_B^2 - c_A c_B^2}, \dots, \right. \\ \left. -\frac{(e_A - e_B)c_A e_A + c_A c_B e_A}{e_A e_B^2 - c_A c_B^2}, \frac{(e_A - e_B)c_A e_A + c_A c_B e_B}{e_A e_B^2 - c_A c_B^2} \right)^T$$

By condition (ii) we know that $e_A e_B^2 - c_A c_B^2 < 0$. Then, condition (iii) is not satisfied since the numerator of $w_2^{\max, 1}$ is positive. $\square$

5.5.2 Stability of other sub-cycles

Proposition 5.5.5. *All sub-cycles that contain the connection κ_{21} , with the exception of sub-cycle Σ_{1n2} , are c.u.*

Proof. The cycle $\Sigma_{1n2} = [S_1 \rightarrow S_n \rightarrow S_{n-1} \rightarrow \dots \rightarrow S_2 \rightarrow S_1]$ contains the connection κ_{21} , which from Proposition 5.5.2 has stability index $+\infty$ for the three-node cycle Σ_{1n2} . So, all other sub-cycles that contain the connection κ_{21} have stability index for this connection equal to $-\infty$ and, therefore, are completely unstable. $\square$

We conjecture that, regardless the number of species interacting in a Jungle Game, only the cycle involving the bottom, top and second to the top species is stable. Above is presented a partial proof of this conjecture by showing that this cycle is always stable while others are completely unstable.

Chapter 6

Concluding remarks

In this thesis, we studied the dynamics and stability of heteroclinic networks arising from models of population dynamics and game theory. By analyzing two extensions of the classic Rock-Scissors-Paper (RSP) game, namely the Rock-Scissors-Paper-Lizard-Spock (RSPLS) game and the Jungle Game, we showed how the interactions among multiple species can generate interesting and occasionally unexpected dynamical patterns. All the results presented are analytical and rely on the technique of using Poincaré return maps from a cross section.

We presented a new and thorough analysis of the stability for the heteroclinic network describing the RSPLS game. A detailed study of the stability of all sub-cycles was conducted, with a full description of the transition matrices it involves. Our analytical results support the findings of Postlethwaite and Rucklidge [42], as well as numerical simulations by other authors. Importantly, we established stability results for a parameter range significantly broader than that considered in [42]. These findings led us to conjecture that interesting and complex dynamics may be found in the white region at the top of Figure 3.5.

In the case of the Jungle Game, we studied the stability of heteroclinic cycles in models with four, five and higher number of species under Lotka-Volterra competition. We proved that, for the game with four and five species, only the cycle involving the bottom, top, and second-to-top species is stable. For any higher number of species, we proved that the mentioned cycle is e.a.s. and we conjecture that it is only one. Consequently, any initial configuration evolves through competition toward a state where only three species persist in the long term — precisely those involved in the e.a.s. cycle connecting S_1 , S_2 and S_n . These three species interact via a Rock-Scissors-Paper dynamic, ensuring that cyclic dominance among them persists indefinitely.

Furthermore, the Jungle Game with four species was re-interpreted as modeling the invasion of a population governed by RSP dynamics. This reinterpretation allowed us to explore a novel type of ecological question — not just coexistence or extinction, but replacement of one species by another in the presence of an invader. Our results show that, depending on the invader's strength, it either goes extinct or replaces one of the original RSP players. Crucially, we proved that cyclic dominance among four species is not a possible outcome in this setting.

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Appendix A

Transition matrices

A.1 The Rock-to-Paper sub-cycle

The products of basic transition matrices with respect to the Rock-to-Paper sub-cycle near ξ_2 are:

$$M_{(j+1,2)} = (M_2)^j : H_2^{\text{in}} \rightarrow H_{j+2}^{\text{in}}, \quad j = 1, \dots, 5 \pmod{5},$$

where

$$M_{(2,2)} = M_2 = \begin{bmatrix} \frac{c_B}{e_A} & 0 & 1 \\ \frac{c_A}{e_A} & 0 & 0 \\ -\frac{e_B}{e_A} & 1 & 0 \end{bmatrix}, \quad M_{(3,2)} = (M_2)^2 = \begin{bmatrix} \frac{c_B^2}{e_A^2} - \frac{e_B}{e_A} & 1 & \frac{c_B}{e_A} \\ \frac{c_A c_B}{e_A^2} & 0 & \frac{c_A}{e_A} \\ -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} & 0 & -\frac{e_B}{e_A} \end{bmatrix},$$

$$M_{(4,2)} = (M_2)^3 = \begin{bmatrix} \frac{c_B^3}{e_A^3} - \frac{2 e_B c_B}{e_A^2} + \frac{c_A}{e_A} & \frac{c_B}{e_A} & \frac{c_B^2}{e_A^2} - \frac{e_B}{e_A} \\ \frac{c_B^2 c_A}{e_A^3} - \frac{c_A e_B}{e_A^2} & \frac{c_A}{e_A} & \frac{c_A c_B}{e_A^2} \\ -\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} & -\frac{e_B}{e_A} & -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \end{bmatrix},$$

$$M_{(5,2)} = (M_2)^4 = \begin{bmatrix} \frac{c_B^4}{e_A^4} - \frac{3 c_B^2 e_B}{e_A^3} + \frac{2 c_A c_B + e_B^2}{e_A^2} & \frac{c_B^2}{e_A^2} - \frac{e_B}{e_A} & \frac{c_B^3}{e_A^3} - \frac{2 e_B c_B}{e_A^2} + \frac{c_A}{e_A} \\ \frac{c_B^3 c_A}{e_A^4} - \frac{2 c_A c_B e_B}{e_A^3} + \frac{c_A^2}{e_A^2} & \frac{c_A c_B}{e_A^2} & \frac{c_B^2 c_A}{e_A^3} - \frac{c_A e_B}{e_A^2} \\ -\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2} & -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} & -\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} \end{bmatrix},$$

and

$$M_{(1,2)} = (M_2)^5 = \begin{bmatrix} \frac{c_B^5}{e_A^5} - \frac{4c_B^3e_B}{e_A^4} + \frac{3c_B^2c_A + 3e_B^2c_B}{e_A^3} - \frac{2c_Ae_B}{e_A^2} \\ \frac{c_B^4c_A}{e_A^5} - \frac{3c_B^2c_Ae_B}{e_A^4} + \frac{2c_A^2c_B + e_B^2c_A}{e_A^3} \\ -\frac{c_B^4e_B}{e_A^5} + \frac{c_B^3c_A + 3c_B^2e_B^2}{e_A^4} - \frac{4c_Ac_Be_B + e_B^3}{e_A^3} + \frac{c_A^2}{e_A^2} \\ \frac{c_B^3}{e_A^3} - \frac{2e_Bc_B}{e_A^2} + \frac{c_A}{e_A} & \frac{c_B^4}{e_A^4} - \frac{3c_B^2e_B}{e_A^3} + \frac{2c_Ac_B + e_B^2}{e_A^2} \\ \frac{c_B^2c_A}{e_A^3} - \frac{c_Ae_B}{e_A^2} & \frac{c_B^3c_A}{e_A^4} - \frac{2c_Ac_Be_B}{e_A^3} + \frac{c_A^2}{e_A^2} \\ -\frac{c_B^2c_A}{e_A^3} + \frac{c_Ac_B + e_B^2}{e_A^2} & -\frac{c_B^3e_B}{e_A^4} + \frac{c_B^2c_A + 2e_B^2c_B}{e_A^3} - \frac{2c_Ae_B}{e_A^2} \end{bmatrix}.$$

A.2 The Star cycle

The products of basic transition matrices with respect to the Star cycle near ξ_4 are:

$$M_{(3j+1,4)} = (M_4)^j : H_4^{\text{in}} \rightarrow H_{3j+4}^{\text{in}},$$

where

$$M_{(4,4)} = M_4 = \begin{bmatrix} 0 & \frac{c_A}{e_B} & 1 \\ 1 & -\frac{e_A}{e_B} & 0 \\ 0 & \frac{c_B}{e_B} & 0 \end{bmatrix}, \quad M_{(2,4)} = (M_4)^2 = \begin{bmatrix} \frac{c_A}{e_B} & -\frac{c_Ae_A}{e_B^2} + \frac{c_B}{e_B} & 0 \\ -\frac{e_A}{e_B} & \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} & 1 \\ \frac{c_B}{e_B} & -\frac{c_Be_A}{e_B^2} & 0 \end{bmatrix},$$

$$M_{(5,4)} = (M_4)^3 = \begin{bmatrix} -\frac{c_Ae_A}{e_B^2} + \frac{c_B}{e_B} & \frac{e_A^2c_A}{e_B^3} + \frac{c_A^2}{e_B^2} - \frac{c_Be_A}{e_B^2} & \frac{c_A}{e_B} \\ \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} & -\frac{e_A^3}{e_B^3} - \frac{2c_Ae_A}{e_B^2} + \frac{c_B}{e_B} & -\frac{e_A}{e_B} \\ -\frac{c_Be_A}{e_B^2} & \frac{e_A^2c_B}{e_B^3} + \frac{c_Bc_A}{e_B^2} & \frac{c_B}{e_B} \end{bmatrix},$$

$$M_{(3,4)} = (M_4)^4 = \begin{bmatrix} \frac{e_A^2c_A}{e_B^3} + \frac{c_A^2 - c_Be_A}{e_B^2} & -\frac{e_A^3c_A}{e_B^4} + \frac{e_A^2c_B - 2c_A^2e_A}{e_B^3} + \frac{2c_Bc_A}{e_B^2} & -\frac{c_Ae_A}{e_B^2} + \frac{c_B}{e_B} \\ -\frac{e_A^3}{e_B^3} - \frac{2c_Ae_A}{e_B^2} + \frac{c_B}{e_B} & \frac{e_A^4}{e_B^4} + \frac{3e_A^2c_A}{e_B^3} + \frac{c_A^2 - 2c_Be_A}{e_B^2} & \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} \\ \frac{e_A^2c_B}{e_B^3} + \frac{c_Bc_A}{e_B^2} & -\frac{e_A^3c_B}{e_B^4} - \frac{2c_Be_Ac_A}{e_B^3} + \frac{c_B^2}{e_B^2} & -\frac{c_Be_A}{e_B^2} \end{bmatrix},$$

and

$$M_{(1,4)} = (M_4)^5 = \begin{bmatrix} -\frac{e_A^3 c_A}{e_B^4} + \frac{e_A^2 c_B - 2 c_A^2 e_A}{e_B^3} + \frac{2 c_B c_A}{e_B^2} \\ \frac{e_A^4}{e_B^4} + \frac{3 e_A^2 c_A}{e_B^3} + \frac{c_A^2 - 2 c_B e_A}{e_B^2} \\ -\frac{e_A^3 c_B}{e_B^4} - \frac{2 c_B e_A c_A}{e_B^3} + \frac{c_B^2}{e_B^2} \\ \frac{e_A^4 c_A}{e_B^5} + \frac{3 c_A^2 e_A^2 - e_A^3 c_B}{e_B^4} + \frac{c_A^3 - 4 c_B e_A c_A}{e_B^3} + \frac{c_B^2}{e_B^2} & \frac{e_A^2 c_A}{e_B^3} + \frac{c_A^2 - c_B e_A}{e_B^2} \\ -\frac{e_A^5}{e_B^5} - \frac{4 e_A^3 c_A}{e_B^4} + \frac{3 e_A^2 c_B - 3 c_A^2 e_A}{e_B^3} + \frac{2 c_B c_A}{e_B^2} & -\frac{e_A^3}{e_B^3} - \frac{2 c_A e_A}{e_B^2} + \frac{c_B}{e_B} \\ \frac{e_A^4 c_B}{e_B^5} + \frac{3 e_A^2 c_A c_B}{e_B^4} + \frac{c_A^2 c_B - 2 c_B^2 e_A}{e_B^3} & \frac{e_A^2 c_B}{e_B^3} + \frac{c_B c_A}{e_B^2} \end{bmatrix}.$$

A.3 The RSP sub-cycle

The products of basic transition matrices with respect to the RSP sub-cycle are:

$$\begin{aligned} M_{(j,j+1)} &= M_{j+1} M_j & : H_j^{\text{in}} &\rightarrow H_{j+2}^{\text{in}}, \\ M^{(j)} &= M_{j+2} M_{j+1} M_j & : H_j^{\text{in}} &\rightarrow H_j^{\text{in}}, \quad j = 1, 2, 3 \pmod{3} \end{aligned}$$

where

$$M_{(2,1)} = M_2 M_1 = \begin{bmatrix} \frac{c_B^2}{e_A^2} - \frac{e_B}{e_A} & 1 & 0 \\ \frac{c_A c_B}{e_A^2} & 0 & 0 \\ -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} & 0 & 1 \end{bmatrix}, \quad M^{(1)} = M_3 M_2 M_1 = \begin{bmatrix} \delta_T & 0 & 0 \\ \alpha_T & 1 & 0 \\ \beta_T & 0 & 1 \end{bmatrix} \quad (\text{A.1})$$

$$M_{(3,2)} = M_3 M_2 = \begin{bmatrix} \frac{c_A^2}{e_B e_A} & 0 & 0 \\ \frac{c_B}{e_A} - \frac{c_A}{e_B} & 0 & 1 \\ \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} & 1 & 0 \end{bmatrix}, \quad M^{(2)} = M_1 M_3 M_2 = \begin{bmatrix} \delta_T & 0 & 0 \\ \gamma_T & 1 & 0 \\ \theta_T & 0 & 1 \end{bmatrix} \quad (\text{A.2})$$

$$M_{(1,3)} = M_1 M_3 = \begin{bmatrix} 0 & \frac{c_B c_A}{e_B e_A} & 0 \\ 0 & \frac{c_A^2}{e_B e_A} + \frac{c_B}{e_B} & 1 \\ 1 & -\frac{c_A}{e_A} - \frac{e_A}{e_B} & 0 \end{bmatrix}, \quad M^{(3)} = M_2 M_1 M_3 = \begin{bmatrix} 1 & \mu_T & 0 \\ 0 & \delta_T & 0 \\ 0 & \nu_T & 1 \end{bmatrix} \quad (\text{A.3})$$

and

$$\begin{aligned}
\delta_T &= \frac{c_A^2 c_B}{e_A^2 e_B} & \gamma_T &= \frac{c_A^3}{e_A^2 e_B} + \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \\
\alpha_T &= \frac{c_B^2}{e_A^2} - \frac{c_A c_B}{e_B e_A} - \frac{e_B}{e_A} & \theta_T &= -\frac{c_A^2}{e_A^2} + \frac{c_B}{e_A} - \frac{c_A}{e_B} \\
\beta_T &= \frac{c_B^2 c_A}{e_A^2 e_B} - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} & \mu_T &= \frac{c_B^2 c_A}{e_A^2 e_B} - \frac{c_A}{e_A} - \frac{e_A}{e_B} \\
& & \nu_T &= -\frac{c_B c_A}{e_A^2} + \frac{c_A^2}{e_A e_B} + \frac{c_B}{e_B}.
\end{aligned}$$

A.4 The Four-node sub-cycle

The product of the basic transition matrices with respect to the Four-node sub-cycle are:

- near ξ_1

$$\begin{aligned}
\hat{M}_{(2,1)} &= \hat{M}_2 \hat{M}_1 = \begin{bmatrix} \frac{c_A^2}{e_B e_A} & 0 & \frac{c_A}{e_B} \\ \frac{c_B}{e_A} - \frac{c_A}{e_B} & 0 & -\frac{e_A}{e_B} \\ \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} & 1 & \frac{c_B}{e_B} \end{bmatrix}, \\
\hat{M}_{(5,1)} &= \hat{M}_5 \hat{M}_2 \hat{M}_1 = \begin{bmatrix} -\frac{c_A^2}{e_B^2} + \frac{2 c_B c_A}{e_B e_A} - \frac{e_B}{e_A} & 1 & -\frac{c_A e_A}{e_B^2} + \frac{c_B}{e_B} \\ \frac{c_A e_A}{e_B^2} + \frac{c_A^2}{e_B e_A} - \frac{c_B}{e_B} & 0 & \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} \\ \frac{c_B^2}{e_B e_A} - \frac{c_B c_A}{e_B^2} & 0 & -\frac{c_B e_A}{e_B^2} \end{bmatrix}, \\
\hat{M}^{(1)} &= \hat{M}_3 \hat{M}_5 \hat{M}_2 \hat{M}_1 = \begin{bmatrix} \frac{c_A^2 e_A}{e_B^3} + \frac{c_A^3}{e_B^2 e_A} - \frac{2 c_B c_A}{e_B^2} + \frac{c_B^2}{e_B e_A} & 0 & \frac{e_A^2 c_A}{e_B^3} + \frac{c_A^2 - c_B e_A}{e_B^2} \\ -\frac{e_A^2 c_A}{e_B^3} - \frac{2 c_A^2 - c_B e_A}{e_B^2} + \frac{2 c_B c_A}{e_B e_A} - \frac{e_B}{e_A} & 1 & -\frac{e_A^3}{e_B^3} - \frac{2 c_A e_A}{e_B^2} + \frac{c_B}{e_B} \\ \frac{c_A e_A c_B}{e_B^3} + \frac{c_A^2 c_B}{e_B^2 e_A} - \frac{c_B^2}{e_B^2} & 0 & \frac{e_A^2 c_B}{e_B^3} + \frac{c_B c_A}{e_B^2} \end{bmatrix};
\end{aligned}$$

- near ξ_2

$$\hat{M}_{(5,2)} = \hat{M}_5 \hat{M}_2 = \begin{bmatrix} \frac{c_A}{e_B} & -\frac{c_A e_A}{e_B^2} + \frac{c_B}{e_B} & 1 \\ -\frac{e_A}{e_B} & \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} & 0 \\ \frac{c_B}{e_B} & \frac{c_B e_A}{e_B^2} & 0 \end{bmatrix},$$

$$\widehat{M}_{(3,2)} = \widehat{M}_3 \widehat{M}_5 \widehat{M}_2 = \begin{bmatrix} -\frac{c_A e_A}{e_B^2} + \frac{c_A}{e_B} & \frac{e_A^2 c_A}{e_B^3} + \frac{c_A^2 - c_B e_A}{e_B^2} & 0 \\ \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} & -\frac{e_A^3}{e_B^3} - \frac{2 c_A e_A}{e_B^2} + \frac{c_B}{e_B} & 1 \\ -\frac{c_B e_A}{e_B^2} & \frac{c_B e_A^2}{e_B^3} + \frac{c_B c_A}{e_B^2} & 0 \end{bmatrix},$$

$$\widehat{M}^{(2)} = \widehat{M}_1 \widehat{M}_3 \widehat{M}_5 \widehat{M}_2 = \begin{bmatrix} -\frac{c_B c_A}{e_B^2} + \frac{c_B^2}{e_B e_A} & \frac{e_A c_B c_A}{e_B^3} + \frac{c_A^2 c_B}{e_B^2 e_A} - \frac{c_B^2}{e_B^2} & 0 \\ -\frac{c_A^2 + c_B e_A}{e_B^2} + \frac{c_B c_A}{e_B e_A} & \frac{c_A^2 e_A + e_A^2 c_B}{e_B^3} + \frac{c_A^3}{e_B^2 e_A} & 0 \\ \frac{e_A^2}{e_B^2} + \frac{2 c_A}{e_B} - \frac{c_B}{e_A} & -\frac{e_A^3}{e_B^3} - \frac{3 c_A e_A}{e_B^2} - \frac{c_A^2}{e_B e_A} + \frac{2 c_B}{e_B} & 1 \end{bmatrix};$$

- near ξ_5

$$\widehat{M}_{(3,5)} = \widehat{M}_3 \widehat{M}_5 = \begin{bmatrix} \frac{c_A}{e_B} & -\frac{c_A e_A}{e_B^2} + \frac{c_B}{e_B} & 0 \\ -\frac{e_A}{e_B} & \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B} & 1 \\ \frac{c_B}{e_B} & -\frac{c_B e_A}{e_B^2} & 0 \end{bmatrix},$$

$$\widehat{M}_{(1,5)} = \widehat{M}_1 \widehat{M}_3 \widehat{M}_5 = \begin{bmatrix} \frac{c_B c_A}{e_B e_A} & -\frac{c_B c_A}{e_B^2} + \frac{c_B^2}{e_B e_A} & 0 \\ \frac{c_A^2}{e_B e_A} + \frac{c_B}{e_B} & -\frac{c_A^2 + c_B e_A}{e_B^2} + \frac{c_B c_A}{e_B e_A} & 0 \\ -\frac{c_A}{e_A} - \frac{e_A}{e_B} & \frac{e_A^2}{e_B^2} + \frac{2 c_A}{e_B} - \frac{c_B}{e_A} & 1 \end{bmatrix},$$

$$\widehat{M}^{(5)} = \widehat{M}_2 \widehat{M}_1 \widehat{M}_3 \widehat{M}_5 = \begin{bmatrix} \frac{c_A^3}{e_B^2 e_A} + \frac{c_B c_A}{e_B^2} & -\frac{c_A^3 + e_A c_B c_A}{e_B^3} + \frac{c_A^2 c_B}{e_B^2 e_A} & 0 \\ -\frac{c_A^2 + c_B e_A}{e_B^2} + \frac{c_B c_A}{e_B e_A} & \frac{c_A^2 e_A + e_A^2 c_B}{e_B^3} - \frac{2 c_B c_A}{e_B^2} + \frac{c_B^2}{e_B e_A} & 0 \\ \frac{c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B} & -\frac{c_A^2 c_B + c_B^2 e_A}{e_B^3} + \frac{c_B^2 c_A}{e_B^2 e_A} + \frac{e_A^2}{e_B^2} + \frac{2 c_A}{e_B} - \frac{c_B}{e_A} & 1 \end{bmatrix};$$

- near ξ_3

$$\widehat{M}_{(1,3)} = \widehat{M}_1 \widehat{M}_3 = \begin{bmatrix} 0 & \frac{c_B c_A}{e_B e_A} & \frac{c_B}{e_A} \\ 0 & \frac{c_A^2}{e_B e_A} + \frac{c_B}{e_B} & \frac{c_A}{e_A} \\ 1 & -\frac{c_A}{e_A} - \frac{e_A}{e_B} & -\frac{e_B}{e_A} \end{bmatrix},$$

$$\widehat{M}_{(2,3)} = \widehat{M}_2 \widehat{M}_1 \widehat{M}_3 = \begin{bmatrix} 0 & \frac{c_A^3}{e_B^2 e_A} + \frac{c_B c_A}{e_B^2} & \frac{c_A^2}{e_B e_A} \\ 0 & -\frac{c_A^2 + c_B e_A}{e_B^2} + \frac{c_B c_A}{e_B e_A} & \frac{c_B}{e_A} - \frac{c_A}{e_B} \\ 1 & \frac{c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B} & \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \end{bmatrix},$$

$$\widehat{M}^{(3)} = \widehat{M}_5 \widehat{M}_2 \widehat{M}_1 \widehat{M}_3 = \begin{bmatrix} 1 & -\frac{c_A^3 + e_A c_B c_A}{e_B^3} + \frac{2 c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B} & -\frac{c_A^2}{e_B^2} + \frac{2 c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \\ 0 & \frac{c_A^2 e_A + e_A^2 c_B}{e_B^3} + \frac{c_A^3}{e_B^2 e_A} & \frac{c_A e_A}{e_B^2} + \frac{c_A^2}{e_B e_A} - \frac{c_B}{e_B} \\ 0 & -\frac{c_A^2 c_B + c_B^2 e_A}{e_B^3} + \frac{c_B^2 c_A}{e_B^2 e_A} & -\frac{c_B c_A}{e_B^2} + \frac{c_B^2}{e_B e_A} \end{bmatrix}.$$

Appendix B

Some proofs

B.1 Proposition 3.3.3

Given the entries of the transition matrices $M^{(j)}$, $j = 1, 2, 3$, in (A.1)–(A.3) we derive the relations below

$$\begin{aligned} \frac{c_B}{e_B}(\delta_T - 1) + \nu_T &= \frac{c_A}{e_B}\beta_T, & \frac{c_B}{e_A}\theta_T + \frac{e_B}{e_A}(\delta_T - 1) &= \alpha_T, \\ \frac{c_A}{e_A}(\delta_T - 1) + \beta_T &= \frac{c_B}{e_A}\gamma_T, & \frac{c_A}{e_A}\nu_T + \frac{e_B}{e_A}(\delta_T - 1) &= \gamma_T, \\ \frac{c_B}{e_A}(\delta_T - 1) + \theta_T &= \frac{c_A}{e_A}\mu_T, & \frac{c_A}{e_B}\alpha_T + \frac{e_A}{e_B}(\delta_T - 1) &= \mu_T, \end{aligned}$$

which enable one to formulate the following:

Lemma B.1.1. *Suppose that $\delta_T > 1$.*

- (a) *If $\theta_T > 0$ and $\nu_T > 0$, then $\alpha_T > 0$, $\beta_T > 0$, $\gamma_T > 0$ and $\mu_T > 0$.*
- (b) *If $\gamma_T < 0$ and $\mu_T < 0$, then $\alpha_T < 0$, $\beta_T < 0$, $\theta_T < 0$ and $\nu_T < 0$.*

Proof of Proposition 3.3.3: We start with checking conditions (i)–(iii) for each $M^{(j)}$, $j = 1, 2, 3$, in (A.1)–(A.3). Due to similarity, the transition matrices $M^{(j)}$ have all the same eigenvalues. By the fact that $M^{(1)}$ is a lower triangular matrix, the eigenvalues are the entries in the main diagonal: $\lambda_1 = \delta_T$ and $\lambda_2 = \lambda_3 = 1$. Condition (i) is naturally satisfied by taking $\lambda_{\max} = \delta_T$. For each $j = 1, 2, 3$, denote by $\mathbf{w}^{\max, j}$ the eigenvector of $M^{(j)}$ associated with the eigenvalue $\lambda_{\max}$. An easy computation shows that $\mathbf{w}^{\max, 1} = (\delta_T - 1, \alpha_T, \beta_T)^T$, $\mathbf{w}^{\max, 2} = (\delta_T - 1, \gamma_T, \theta_T)^T$ and $\mathbf{w}^{\max, 3} = (\mu_T, \delta_T - 1, \nu_T)^T$.

Condition (ii) is violated when $\delta_T < 1$ while condition (iii) is violated for some j when $\delta_T < 1$ or $\alpha_T < 0$ or $\beta_T < 0$ or $\gamma_T < 0$ or $\theta_T < 0$ or $\mu_T < 0$ or $\nu_T < 0$. Proposition 2.6.5(a) then establishes statement in (a).

On the other hand, that conditions (ii)–(iii) hold true when $\delta_T > 1$, $\theta_T > 0$ and $\nu_T > 0$ follows from Lemma B.1.1. Under these inequalities all $M^{(j)}$ meet (3.14) as a result of any $\mathbf{y} \in \mathbb{R}_-^3$ written in the eigenbasis of $M^{(j)}$ having a negative coefficient for the largest eigenvector. Indeed, such a coefficient is of the form $(\mathbf{v}^{\max, j})^T \mathbf{y} < 0$ given

that $\mathbf{v}^{\max,1} = \mathbf{v}^{\max,2} = \left(\frac{1}{\delta_T-1}, 0, 0\right)^T$ and $\mathbf{v}^{\max,3} = \left(0, \frac{1}{\delta_T-1}, 0\right)^T$ admit all non-negative components.

In the calculation of σ_j we evaluate the function F^{index} for the rows of the transition matrices M_j , $M_{j+1}M_j$ and $M^{(j)} = M_{j+2}M_{j+1}M_j$ ($j \bmod 3$) so that

$$\sigma_{31} = \min \left\{ F^{\text{index}} \left(\frac{c_B}{e_A}, 0, 0 \right), F^{\text{index}} \left(\frac{c_A}{e_A}, 0, 1 \right), F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right), \right. \\ \left. F^{\text{index}} \left(\frac{c_B^2}{e_A^2} - \frac{e_B}{e_A}, 1, 0 \right), F^{\text{index}} \left(\frac{c_A c_B}{e_A^2}, 0, 0 \right), F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 1, 0 \right), \right. \\ \left. F^{\text{index}} (\delta_T, 0, 0), F^{\text{index}} (\alpha_T, 1, 0), F^{\text{index}} (\beta_T, 0, 1) \right\},$$

$$\sigma_{12} = \min \left\{ F^{\text{index}} \left(\frac{c_B}{e_A}, 0, 1 \right), F^{\text{index}} \left(\frac{c_A}{e_A}, 0, 0 \right), F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right), \right. \\ \left. F^{\text{index}} \left(\frac{c_A^2}{e_B e_A}, 0, 0 \right), F^{\text{index}} \left(\frac{c_B}{e_A} - \frac{c_A}{e_B}, 0, 1 \right), F^{\text{index}} \left(\frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, 0 \right), \right. \\ \left. F^{\text{index}} (\delta_T, 0, 0), F^{\text{index}} (\gamma_T, 1, 0), F^{\text{index}} (\theta_T, 0, 1) \right\},$$

and

$$\sigma_{23} = \min \left\{ F^{\text{index}} \left(0, \frac{c_A}{e_B}, 0 \right), F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right), F^{\text{index}} \left(0, \frac{c_B}{e_B}, 1 \right), \right. \\ \left. F^{\text{index}} \left(0, \frac{c_B c_A}{e_B e_A}, 0 \right), F^{\text{index}} \left(0, \frac{c_A^2}{e_B e_A} + \frac{c_B}{e_B}, 0 \right), F^{\text{index}} \left(1, -\frac{c_A}{e_A} - \frac{e_A}{e_B}, 0 \right), \right. \\ \left. F^{\text{index}} (1, \mu_T, 0), F^{\text{index}} (0, \delta_T, 0), F^{\text{index}} (0, \nu_T, 1) \right\}.$$

It is immediate that

$$\begin{aligned} F^{\text{index}} \left(\frac{c_B}{e_A}, 0, 0 \right) &= F^{\text{index}} \left(\frac{c_A}{e_A}, 0, 1 \right) = +\infty \\ F^{\text{index}} \left(\frac{c_B^2}{e_A^2} - \frac{e_B}{e_A}, 1, 0 \right) &= +\infty \\ F^{\text{index}} (\delta_T, 0, 0) &= F^{\text{index}} (\alpha_T, 1, 0) = F^{\text{index}} (\beta_T, 0, 1) = +\infty, \\ F^{\text{index}} \left(\frac{c_B}{e_A}, 0, 1 \right) &= F^{\text{index}} \left(\frac{c_A}{e_A}, 0, 0 \right) = +\infty \\ F^{\text{index}} \left(\frac{c_A^2}{e_B e_A}, 0, 0 \right) &= +\infty \\ F^{\text{index}} (\delta_T, 0, 0) &= F^{\text{index}} (\gamma_T, 1, 0) = F^{\text{index}} (\theta_T, 0, 1) = +\infty, \end{aligned}$$

$$\begin{aligned}
F^{\text{index}}\left(0, \frac{c_A}{e_B}, 0\right) &= F^{\text{index}}\left(0, \frac{c_B}{e_B}, 1\right) = +\infty \\
F^{\text{index}}\left(0, \frac{c_B c_A}{e_B e_A}, 0\right) &= F^{\text{index}}\left(0, \frac{c_A^2}{e_B e_A} + \frac{c_B}{e_B}, 0\right) = +\infty \\
F^{\text{index}}(1, \mu_T, 0) &= F^{\text{index}}(0, \delta_T, 0) = F^{\text{index}}(0, \nu_T, 1) = +\infty.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\alpha_T > 0 &\Leftrightarrow \frac{c_B^2}{e_A^2} - \frac{e_B}{e_A} > \frac{c_A c_B}{e_B e_A} > 0 \\
\theta_T > 0 &\Leftrightarrow \frac{c_B}{e_A} - \frac{c_A}{e_B} > \frac{c_A^2}{e_A^2} > 0 \Leftrightarrow -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} = -\frac{e_B}{e_A} \left(\frac{c_B}{e_A} - \frac{c_A}{e_B} \right) < 0
\end{aligned}$$

and

$$F^{\text{index}}\left(\frac{c_B^2}{e_A^2} - \frac{e_B}{e_A}, 1, 0\right) = F^{\text{index}}\left(\frac{c_B}{e_A} - \frac{c_A}{e_B}, 0, 1\right) = +\infty.$$

It follows that

$$\begin{aligned}
\sigma_{31} &= \min \left\{ F^{\text{index}}\left(-\frac{e_B}{e_A}, 1, 0\right), F^{\text{index}}\left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 1, 0\right) \right\} \\
\sigma_{12} &= \min \left\{ F^{\text{index}}\left(-\frac{e_B}{e_A}, 1, 0\right), F^{\text{index}}\left(\frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, 0\right) \right\} \\
\sigma_{23} &= \min \left\{ F^{\text{index}}\left(1, -\frac{e_A}{e_B}, 0\right), F^{\text{index}}\left(1, -\frac{c_A}{e_A} - \frac{e_A}{e_B}, 0\right) \right\}.
\end{aligned}$$

We get

$$\begin{aligned}
F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right) &= \begin{cases} 1 - \frac{e_B}{e_A} (< 0) & \text{if } \frac{e_B}{e_A} > 1 \\ -1 + \frac{e_A}{e_B} (> 0) & \text{if } \frac{e_B}{e_A} < 1 \end{cases} \\
F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 1, 0 \right) &= \begin{cases} 1 - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} (< 0) & \text{if } -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \leq -1 \\ -1 + \frac{e_A^2}{e_B c_B - c_A e_A} (> 0) & \text{if } -1 < -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} < 0, \end{cases} \\
F^{\text{index}} \left(\frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, 0 \right) &= \begin{cases} +\infty & \text{if } \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \geq 0 \\ 1 + \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} (< 0) & \text{if } \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \leq -1 \\ -1 + \frac{e_A e_B}{e_B^2 - c_A c_B} (> 0) & \text{if } -1 < \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} < 0, \end{cases} \\
F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right) &= \begin{cases} 1 - \frac{e_A}{e_B} (< 0) & \text{if } \frac{e_B}{e_A} < 1 \\ -1 + \frac{e_B}{e_A} (> 0) & \text{if } \frac{e_B}{e_A} > 1 \end{cases} \\
F^{\text{index}} \left(1, -\frac{c_A}{e_A} - \frac{e_A}{e_B}, 0 \right) &= \begin{cases} 1 - \frac{c_A}{e_A} - \frac{e_A}{e_B} (< 0) & \text{if } \frac{c_A}{e_A} + \frac{e_A}{e_B} > 1 \\ -1 + \frac{e_A e_B}{e_A^2 + c_A e_B} (> 0) & \text{if } \frac{c_A}{e_A} + \frac{e_A}{e_B} < 1. \end{cases}
\end{aligned}$$

By combining all suitable branches for each σ_{ij} , the proof is completed.

B.2 Proposition 3.3.4

We determine the eigenvalues and eigenvectors of $\hat{M}^{(j)}$, $j = 1, 2, 3, 5$, in Appendix A.4. To simplify consider the following notation:

$$\begin{aligned}
\hat{M}^{(1)} &= \begin{bmatrix} \alpha_{11} & 0 & \alpha_{13} \\ \alpha_{21} & 1 & \alpha_{23} \\ \alpha_{31} & 0 & \alpha_{33} \end{bmatrix}, & \hat{M}^{(2)} &= \begin{bmatrix} \beta_{11} & \beta_{12} & 0 \\ \beta_{21} & \beta_{22} & 0 \\ \beta_{31} & \beta_{32} & 1 \end{bmatrix}, \\
\hat{M}^{(5)} &= \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ \gamma_{31} & \gamma_{32} & 1 \end{bmatrix}, & \hat{M}^{(3)} &= \begin{bmatrix} 1 & \delta_{12} & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{32} & \delta_{33} \end{bmatrix}.
\end{aligned}$$

We observe that

$$\alpha_{13} = \frac{c_B}{e_A} \alpha_{31} = \frac{e_A}{c_B} \beta_{12} = \frac{e_A}{e_B} \delta_{23} = -\frac{e_A^2}{e_B^2} \theta_T \quad (\text{B.1})$$

$$\gamma_{12} = \frac{c_A}{e_B} \gamma_{21} = \frac{e_B}{c_A} \beta_{21} = \frac{c_A}{c_B} \delta_{32} = -\frac{c_A e_A}{e_B^2} \nu_T. \quad (\text{B.2})$$

As $\widehat{\mathbf{M}}^{(j)}$ are similar, all have the same eigenvalues. For $\widehat{\mathbf{M}}^{(1)}$ as defined above the eigenvalues are

$$\begin{aligned} \lambda_1 &= \frac{\alpha_{11} + \alpha_{33} + \sqrt{(\alpha_{11} + \alpha_{33})^2 - 4 \frac{c_B^3 c_A}{e_B^3 e_A}}}{2} = \frac{\alpha_{11} + \alpha_{33} + \sqrt{(\alpha_{11} - \alpha_{33})^2 + 4 \alpha_{13} \alpha_{31}}}{2}, \\ \lambda_2 &= \frac{\alpha_{11} + \alpha_{33} - \sqrt{(\alpha_{11} + \alpha_{33})^2 - 4 \frac{c_B^3 c_A}{e_B^3 e_A}}}{2} = \frac{\alpha_{11} + \alpha_{33} - \sqrt{(\alpha_{11} - \alpha_{33})^2 + 4 \alpha_{13} \alpha_{31}}}{2}, \\ \lambda_3 &= 1. \end{aligned}$$

The candidate for $\lambda_{\max}$ satisfying conditions (i)–(iii) is λ_1 . From (B.1) we find that

$$\alpha_{13} \alpha_{31} = \frac{c_B}{e_A} \alpha_{13}^2 = \frac{e_A^4}{e_B^4} \theta_T^2 \geq 0, \quad (\text{B.3})$$

which ensures that λ_1 and λ_2 are real. Condition (i) is immediately true. Moreover,

$$\alpha_{11} + \alpha_{33} = \frac{c_A^3 e_B + (c_A e_A - c_B e_B)^2 + c_A c_B e_A e_B + c_B e_A^3}{e_B^3 e_A} > 0$$

and, in consequence, $\lambda_1 > |\lambda_2|$.

Let $\lambda_{\max} = \lambda_1$ and $\mathbf{w}^{\max, j} = (w_1^{\max, j}, w_2^{\max, j}, w_3^{\max, j})^T$ the corresponding eigenvector of each $\widehat{\mathbf{M}}^{(j)}$. We get

$$\begin{aligned} \mathbf{w}^{\max, 1} &= \left(\alpha_{13} (\lambda_{\max} - 1), \alpha_{13} \alpha_{21} + \alpha_{23} (\lambda_{\max} - \alpha_{11}), (\lambda_{\max} - \alpha_{11}) (\lambda_{\max} - 1) \right)^T \\ \mathbf{w}^{\max, 2} &= \left(\beta_{12} (\lambda_{\max} - 1), (\lambda_{\max} - \beta_{11}) (\lambda_{\max} - 1), \beta_{12} \beta_{31} + \beta_{32} (\lambda_{\max} - \beta_{11}) \right)^T \\ \mathbf{w}^{\max, 5} &= \left(\gamma_{12} (\lambda_{\max} - 1), (\lambda_{\max} - \gamma_{11}) (\lambda_{\max} - 1), \gamma_{12} \gamma_{31} + \gamma_{32} (\lambda_{\max} - \gamma_{11}) \right)^T \\ \mathbf{w}^{\max, 3} &= \left(\delta_{12} \delta_{23} + \delta_{13} (\lambda_{\max} - \delta_{22}), \delta_{23} (\lambda_{\max} - 1), (\lambda_{\max} - \delta_{22}) (\lambda_{\max} - 1) \right)^T. \end{aligned}$$

Condition (ii) is satisfied if and only if

$$\alpha_{11} + \alpha_{33} > \min \left\{ 2, 1 + \frac{c_B^3 c_A}{e_B^3 e_A} \right\}.$$

Condition (iii) requires the evaluation of the signs of the components of $\mathbf{w}^{\max, j}$. Note that

$$\begin{aligned} (\alpha_{11} - \alpha_{33})^2 + 4 \alpha_{13} \alpha_{31} &= (\beta_{11} - \beta_{22})^2 + 4 \beta_{12} \beta_{21} \\ &= (\gamma_{11} - \gamma_{22})^2 + 4 \gamma_{12} \gamma_{21} \\ &= (\delta_{22} - \delta_{33})^2 + 4 \delta_{23} \delta_{32}. \end{aligned}$$

It is easily seen that $\lambda_{\max} - \alpha_{11} > 0$ and $\lambda_{\max} - \gamma_{11} > 0$ because of (B.3) and $\gamma_{12}\gamma_{21} = \frac{c_A}{e_B}\gamma_{21}^2 = \frac{c_A^2 e_A^2}{e_B^4}\nu_T^2 \geq 0$, respectively. Assuming $\lambda_{\max} > 1$ to hold for every j , we have $w_3^{\max,1} > 0$ and $w_2^{\max,5} > 0$. By (B.1) all components of $\mathbf{w}^{\max,j}$ have the same sign when

$$\begin{aligned} [j = 1] \quad & \alpha_{13} > 0 \quad \text{and} \quad w_2^{\max,1} = \alpha_{13}\alpha_{21} + \alpha_{23}(\lambda_{\max} - \alpha_{11}) > 0, \\ [j = 2] \quad & \lambda_{\max} - \beta_{11} > 0 \quad \text{and} \quad w_3^{\max,2} = \beta_{12}\beta_{31} + \beta_{32}(\lambda_{\max} - \beta_{11}) > 0, \\ [j = 5] \quad & \gamma_{12} > 0 \quad \text{and} \quad w_3^{\max,5} = \gamma_{12}\gamma_{31} + \gamma_{32}(\lambda_{\max} - \gamma_{11}) > 0, \\ [j = 3] \quad & w_1^{\max,3} = \delta_{12}\delta_{23} + \delta_{13}(\lambda_{\max} - \delta_{22}) > 0 \quad \text{and} \quad \lambda_{\max} - \delta_{22} > 0. \end{aligned}$$

Given (B.2) it means that $\alpha_{13} > 0$ and $\gamma_{12} > 0$ imply $\beta_{12}\beta_{21} = \frac{c_A c_B}{e_A e_B}\alpha_{13}\gamma_{12} > 0$ and $\delta_{23}\delta_{32} = \frac{c_A e_B}{c_B e_A}\alpha_{13}\gamma_{12} > 0$. Hence, $\lambda_{\max} - \beta_{11} > 0$ and $\lambda_{\max} - \delta_{22} > 0$. What is left is to check

$$w_2^{\max,1} > 0, \quad w_3^{\max,2} > 0, \quad w_3^{\max,5} > 0 \quad \text{and} \quad w_1^{\max,3} > 0. \quad (\text{B.4})$$

Using similarity, we can establish

$$\begin{aligned} \mathbf{w}^{\max,2} &= \hat{M}_1 \mathbf{w}^{\max,1} \\ \mathbf{w}^{\max,5} &= \frac{e_B}{c_A} \frac{\gamma_{12}}{\lambda_{\max} - \beta_{11}} \hat{M}_2 \mathbf{w}^{\max,2} \\ \mathbf{w}^{\max,3} &= \frac{e_B}{c_B} \frac{\lambda_{\max} - \delta_{22}}{\lambda_{\max} - \gamma_{11}} \hat{M}_5 \mathbf{w}^{\max,5}. \end{aligned}$$

It follows that

$$\begin{aligned} w_3^{\max,2} &= -\frac{e_A}{e_B} \alpha_{13} (\lambda_{\max} - 1) + w_2^{\max,1} \\ w_3^{\max,5} &= \frac{e_B}{c_A} \frac{\gamma_{12}}{\lambda_{\max} - \beta_{11}} \left[\frac{c_B}{e_B} (\lambda_{\max} - \beta_{11}) (\lambda_{\max} - 1) + w_3^{\max,2} \right] \\ w_1^{\max,3} &= \frac{e_B}{c_B} \frac{\lambda_{\max} - \delta_{22}}{\lambda_{\max} - \gamma_{11}} \left[\frac{c_A}{e_B} (\lambda_{\max} - \gamma_{11}) (\lambda_{\max} - 1) + w_3^{\max,5} \right]. \end{aligned}$$

Accordingly, if $w_3^{\max,2} > 0$, then $w_2^{\max,1} > 0$, $w_3^{\max,5} > 0$ and $w_1^{\max,3} > 0$. We conclude that condition (iii) is fulfilled for

$$\begin{aligned} \alpha_{13} > 0 &\Leftrightarrow \theta_T < 0 \\ \gamma_{12} > 0 &\Leftrightarrow \nu_T < 0 \\ w_3^{\max,2} > 0 &\Leftrightarrow c_A^3 c_B + 2 c_B^2 c_A e_A - c_B^3 e_A e_B \\ &\quad + (-e_A^4 - 3 e_A^2 c_A e_B - c_A^2 e_B^2 + 2 e_B^2 c_B e_A) \lambda_{\max} > 0. \end{aligned}$$

The proof of (a) is immediate. Under the hypotheses of (b), the equality (3.14) holds for all j . In fact, any $\mathbf{y} \in \mathbb{R}_-^3$ written in the eigenbasis of $\hat{M}^{(j)}$ has a negative coefficient

of the form $(\mathbf{v}^{\max,j})^T \mathbf{y}$, where

$$\begin{aligned}\mathbf{v}^{\max,1} &= \left(\frac{\alpha_{11} - \lambda_2}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)\alpha_{13}}, 0, \frac{1}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)} \right)^T \\ \mathbf{v}^{\max,2} &= \left(\frac{\beta_{11} - \lambda_2}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)\beta_{12}}, \frac{1}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)}, 0 \right)^T \\ \mathbf{v}^{\max,5} &= \left(\frac{\gamma_{11} - \lambda_2}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)\gamma_{12}}, \frac{1}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)}, 0 \right)^T \\ \mathbf{v}^{\max,3} &= \left(0, \frac{\delta_{22} - \lambda_2}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)\delta_{13}}, \frac{1}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)} \right)^T.\end{aligned}$$

A trivial verification shows that $\lambda_1 - \lambda_2 > 0$, $\alpha_{11} - \lambda_2 > 0$, $\beta_{11} - \lambda_2 > 0$, $\gamma_{11} - \lambda_2 > 0$ and $\delta_{22} - \lambda_2 > 0$. Therefore, all $\mathbf{v}^{\max,j}$ admit non-negative components.

We calculate the stability indices by plugging the rows of the transition matrices $\hat{M}_j$, $\hat{M}_{(l,j)}^{(j)}$ with at least one negative entry into F^{index} . We thus get

$$\begin{aligned}\sigma_{31} &= \min \left\{ F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right), \right. \\ &\quad F^{\text{index}} \left(\frac{c_B}{e_A} - \frac{c_A}{e_B}, 0, -\frac{e_A}{e_B} \right), F^{\text{index}} \left(\frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, \frac{c_B}{e_B} \right), \\ &\quad F^{\text{index}} \left(-\frac{c_A^2}{e_B^2} + \frac{2 c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, -\frac{c_A e_A}{e_B^2} + \frac{c_B}{e_B} \right), \\ &\quad \left. F^{\text{index}} \left(-\frac{e_A^2 c_A}{e_B^3} - \frac{2 c_A^2 - c_B e_A}{e_B^2} + \frac{2 c_B c_A}{e_B e_A} - \frac{e_B}{e_A}, 1, -\frac{e_A^3}{e_B^3} - \frac{2 c_A e_A}{e_B^2} + \frac{c_B}{e_B} \right) \right\} \\ &\leq F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right);\end{aligned}$$

$$\begin{aligned}\sigma_{12} &= \min \left\{ F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right), \right. \\ &\quad F^{\text{index}} \left(-\frac{e_A}{e_B}, \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B}, 0 \right), \\ &\quad F^{\text{index}} \left(\frac{e_A^2}{e_B^2} + \frac{c_A}{e_B}, -\frac{e_A^3}{e_B^3} - \frac{2 c_A e_A}{e_B^2} + \frac{c_B}{e_B}, 1 \right), F^{\text{index}} \left(-\frac{c_B e_A}{e_B^2}, \frac{c_B e_A^2}{e_B^3} + \frac{c_B c_A}{e_B^2}, 0 \right), \\ &\quad \left. F^{\text{index}} \left(\frac{e_A^2}{e_B^2} + \frac{2 c_A}{e_B} - \frac{c_B}{e_A}, -\frac{e_A^3}{e_B^3} - \frac{3 c_A e_A}{e_B^2} - \frac{c_A^2}{e_B e_A} + \frac{2 c_B}{e_B}, 1 \right) \right\} \\ &\leq F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right);\end{aligned}$$

$$\begin{aligned}
\sigma_{25} = \min & \left\{ F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right), \right. \\
& F^{\text{index}} \left(-\frac{e_A}{e_B}, \frac{e_A^2}{e_B^2} + \frac{c_A}{e_B}, 1 \right), F^{\text{index}} \left(\frac{c_B}{e_B}, -\frac{c_B e_A}{e_B^2}, 0 \right), \\
& F^{\text{index}} \left(-\frac{c_A}{e_A} - \frac{e_A}{e_B}, \frac{e_A^2}{e_B^2} + \frac{2c_A}{e_B} - \frac{c_B}{e_A}, 1 \right), \\
& \left. F^{\text{index}} \left(\frac{c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B}, -\frac{c_A^2 c_B + c_B^2 e_A}{e_B^3} + \frac{c_B^2 c_A}{e_B^2 e_A} + \frac{e_A^2}{e_B^2} + \frac{2c_A}{e_B} - \frac{c_B}{e_A}, 1 \right) \right\} \\
& \leq F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right);
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{53} = \min & \left\{ F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right), \right. \\
& F^{\text{index}} \left(1, -\frac{c_A}{e_A} - \frac{e_A}{e_B}, -\frac{e_B}{e_A} \right), \\
& F^{\text{index}} \left(1, \frac{c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B}, \frac{c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \right), \\
& \left. F^{\text{index}} \left(1, -\frac{c_A^3 + e_A c_B c_A}{e_B^3} + \frac{2c_A^2 c_B}{e_B^2 e_A} + \frac{c_B^2}{e_B^2} - \frac{c_A}{e_A} - \frac{e_A}{e_B}, -\frac{c_A^2}{e_B^2} + \frac{2c_B c_A}{e_B e_A} - \frac{e_B}{e_A} \right) \right\} \\
& \leq F^{\text{index}} \left(1, -\frac{e_B}{e_A}, 0 \right).
\end{aligned}$$

B.3 Proposition 3.3.7

We use Theorem 3.1 in Lohse [29] showing that if all stability indices are positive then the cycle is e.a.s. whereas f.a.s. is obtained from Lemma 2.5 in [18]. It follows from its definition in [35] that a stability index equal to $-\infty$ implies the complete instability of the cycle.

The relations (a2) and (a3) in Lemma 5.6 determine the first and third conditions in Proposition 5.2. The remaining condition is $c_A e_A - c_B e_B < 0$. If it is satisfied, then $\sigma_{R\text{-to-P}} = -\infty$, otherwise $\sigma_{R\text{-to-P}} > -\infty$. In the latter case, we further check the sign of $\sigma_{R\text{-to-P}}$. Taking (20) negative entries can only occur in the last row of each $(M_2)^j$,

$j = 1, \dots, 5$. It follows that

$\sigma_{\text{R-to-P}} =$

$$\min \left\{ F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right), \right. \\ F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 0, -\frac{e_B}{e_A} \right), \\ F^{\text{index}} \left(-\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, -\frac{e_B}{e_A}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \right), \\ F^{\text{index}} \left(-\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, -\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} \right), \\ F^{\text{index}} \left(-\frac{c_B^4 e_B}{e_A^5} + \frac{c_B^3 c_A + 3 c_B^2 e_B^2}{e_A^4} - \frac{4 c_A c_B e_B + e_B^3}{e_A^3} + \frac{c_A^2}{e_A^2}, -\frac{c_B^2 c_A}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, \right. \\ \left. -\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2} \right) \left. \right\}.$$

According to the function F^{index} , consider the rows in $(M_2)^j$ with at least with negative entries and define the sums of the row elements as follows:

$$\begin{aligned} \mathfrak{s}_1 &= -\frac{e_B}{e_A} + 1 \\ \mathfrak{s}_2 &= -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} - \frac{e_B}{e_A} \\ \mathfrak{s}_3 &= -\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} - \frac{e_B}{e_A} - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \\ \mathfrak{s}_4 &= -\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2} - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} - \frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} \\ \mathfrak{s}_5 &= -\frac{c_B^4 e_B}{e_A^5} + \frac{c_B^3 c_A + 3 c_B^2 e_B^2}{e_A^4} - \frac{4 c_A c_B e_B + e_B^3}{e_A^3} + \frac{c_A^2}{e_A^2} - \frac{c_B^2 c_A}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} \\ &\quad - \frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2}. \end{aligned}$$

A trivial verification shows that

$$\mathfrak{s}_2 \leq \mathfrak{s}_3 < \frac{c_A e_A - c_B e_B}{e_A^2} < \mathfrak{s}_5 \leq \mathfrak{s}_4.$$

By virtue of $e_B < e_A$ and $c_A e_A - c_B e_B > 0$, we get immediately $\mathfrak{s}_1 > 0$ and $\mathfrak{s}_4 \geq \mathfrak{s}_5 > 0$. Hence,

$$\begin{aligned} F^{\text{index}} \left(-\frac{e_B}{e_A}, 1, 0 \right) &> 0 \\ F^{\text{index}} \left(-\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, -\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2} \right) &> 0 \\ F^{\text{index}} \left(-\frac{c_B^4 e_B}{e_A^5} + \frac{c_B^3 c_A + 3 c_B^2 e_B^2}{e_A^4} - \frac{4 c_A c_B e_B + e_B^3}{e_A^3} + \frac{c_A^2}{e_A^2}, -\frac{c_B^2 c_A}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, \right. \\ &\quad \left. -\frac{c_B^3 e_B}{e_A^4} + \frac{c_B^2 c_A + 2 e_B^2 c_B}{e_A^3} - \frac{2 c_A e_B}{e_A^2} \right) > 0. \end{aligned}$$

We are now reduced to three possibilities:

(I) $0 < \mathfrak{s}_2 \leq \mathfrak{s}_3$, which is equivalent to $c_A e_A - c_B e_B > e_A e_B$. We obtain

$$\begin{aligned} F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 0, -\frac{e_B}{e_A} \right) &> 0 \\ F^{\text{index}} \left(-\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, -\frac{e_B}{e_A}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \right) &> 0 \end{aligned}$$

and consequently $\sigma_{\text{R-to-P}} > 0$.

(II) $\mathfrak{s}_2 \leq 0 \leq \mathfrak{s}_3$. We obtain

$$\begin{aligned} F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 0, -\frac{e_B}{e_A} \right) &< 0 \\ F^{\text{index}} \left(-\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, -\frac{e_B}{e_A}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \right) &> 0 \end{aligned}$$

and consequently $\sigma_{\text{R-to-P}} < 0$.

(III) $\mathfrak{s}_2 \leq \mathfrak{s}_3 < 0$. We obtain

$$\begin{aligned} F^{\text{index}} \left(-\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A}, 0, -\frac{e_B}{e_A} \right) &< 0 \\ F^{\text{index}} \left(-\frac{c_B^2 e_B}{e_A^3} + \frac{c_A c_B + e_B^2}{e_A^2}, -\frac{e_B}{e_A}, -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \right) &< 0 \end{aligned}$$

and consequently $\sigma_{\text{R-to-P}} < 0$.

In Proposition 3.3.2 only the first condition in (a) and (b) is determined by (3.19). When the stability index σ_{Star} is finite, it is negative since $e_B < e_A$ and hence $F^{\text{index}} \left(1, -\frac{e_A}{e_B}, 0 \right) = 1 - \frac{e_A}{e_B} < 0$. The Star cycle is at most f.a.s.

From Proposition 3.3.3, the stability indices for Σ_{RSP} reduce to

$$\begin{aligned} \sigma_{31} &= \begin{cases} 1 - \frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} < 0 & \text{if } -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} \leq -1 \\ \min \left\{ -1 + \frac{e_A}{e_B}, -1 + \frac{e_A^2}{e_B c_B - c_A e_A} \right\} > 0 & \text{if } -1 < -\frac{e_B c_B}{e_A^2} + \frac{c_A}{e_A} < 0 \end{cases} \\ \sigma_{12} &= -1 + \frac{e_A}{e_B} > 0 \\ \sigma_{23} &= 1 - \frac{c_A}{e_A} - \frac{e_A}{e_B} < 0. \end{aligned}$$

It is easy to see that at least $\sigma_{23} < 0$ always preventing this sub-cycle from being e.a.s.