Cosmological Solutions in Geometric Modified Gravity

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Resumo

Relatividade Geral é uma das mais profundas conquistas científicas, combinando elegância matemática com um século de validação empírica. Apesar disso, progressos teóricos e observacionais têm exposto limitações na teoria. Em particular, a descoberta da expansão acelerada do universo põe em evidência a dificuldade em explicar energia escura sem recorrer a uma constante cosmológica com ajuste fino. Em resposta a estas lacunas, diversas teorias modificadas da gravidade foram propostas. Dentre elas, a Gravidade Teleparalela e a Gravidade Teleparalela Simétrica, formuladas no contexto da geometria não-Riemanniana, têm ganho destaque nos últimos anos.

Neste trabalho, investigamos extensões dessas teorias, focando na sua capacidade de reproduzir o efeito da constante cosmológica na aceleração do universo e confrontamos estas teorias com restrições locais provenientes de observações do Sistema Solar, procurando soluções em regimes esfericamente simétricos e estáticos.

Os nossos resultados mostram que, embora vários modelos sejam capazes de reproduzir a aceleração cosmológica, muitos deles são incompatíveis com observações no Sistema Solar, especialmente por preverem desvios no parâmetro de Eddington. Isso destaca a importância de combinar testes cosmológicos e locais na avaliação da viabilidade física de teorias modificadas da gravidade.

Palavras-chave: Teleparalelismo, Energia Escura, Reconstrução, Parâmetro de Eddington

Abstract

General Relativity is one the most profound achievements in science, combining mathematical elegance with over a century of empirical validation.

Nevertheless, both observational and theoretical developments have revealed limitations in General Relativity. Particularly, the discovery of the accelerated expansion of the universe highlights its inability to naturally account for dark energy without invoking a fine tuned cosmological constant. In response, a wide range of modified gravity theories have been proposed. Among these, Teleparallel Gravity and Symmetric Teleparallel Gravity, formulated within non-Riemannian geometry, have gained increasing attention in recent years.

In this work, we investigated extensions of these alternative theories, focusing on their ability to reproduce the late-time acceleration of the universe by mimicking the effects of a cosmological constant. We further subjected these models to local gravitational constraints by analyzing their predictions in static, spherically symmetric spacetimes. Our results demonstrate that while several models can successfully replicate cosmological acceleration, many are inconsistent with Solar System observations, notably through deviations in the Eddington parameter. This highlights the importance of combining cosmological and local tests when assessing the physical viability of modified gravity theories.

Keywords: Teleparallelism; Dark Energy; Reconstruction; Eddington Parameter

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Symbols

$g_{\mu\nu}$	metric tensor
$\Gamma^{\alpha}{}_{\mu\nu}$	Affine Connection
$\mathring{\Gamma}^{\alpha}{}_{\mu\nu}$	Levi-Civita Connection
∇_{μ}	Affine Covariant Derivative
$\mathring{\nabla}_{\mu}$	Levi-Civita Covariant Derivative
$R^{\alpha}{}_{\beta\mu\nu}$	Riemann Tensor
$R_{\mu\nu}$	Ricci Tensor
R	Ricci Scalar
$T^{\alpha}{}_{\mu\nu}$	Torsion Tensor
T_{μ}	Torsion Vector
$\mathbb{T}$	Torsion Scalar
$Q_{\alpha\mu\nu}$	Non-Metricity Tensor Tensor
$\mathbb{Q}$	Non-Metricity Scalar
Q_{μ}	First Non-Metricity Vector
$\bar{Q}_{\mu}$	Second Non-Metricity Vector
$T_{\mu\nu}$	Energy Momentum Tensor
$k = 8\pi G$	Gravitation Constant
G	Newton's Gravitational constant
$a(t)$	Scale factor
$H(t)$	Hubble parameter
$\rho(t)$	Density of perfect fluid
$p(t)$	Pressure of a perfect fluid
γ	Eddington parameter

List of Abbreviations

1. Introduction

General Relativity (GR) is one of the most successful theories in physics, providing an elegant and accurate description of gravity, that has surpassed numerous observational tests. However, while GR has deepened our understanding of the universe, it also raises profound challenges, particularly at the interface with cosmology.

The revolutionary insight in General Relativity is that gravity is not a force acting at a distance, as in Newtonian physics, but rather a manifestation of the curvature of spacetime.

Mathematically, GR is formulated using the language of differential geometry. Spacetime is described by a manifold equipped with a metric tensor and a connection. The metric determines distances and causal structure, while the connection defines parallel transport and the motion of particles through geodesics. Conversely, the presence of matter and energy shapes the spacetime geometry, influencing the metric and the connection. Matter and energy tell spacetime how to curve, and curved spacetime tells matter how to move.

This mutual interaction is expressed through the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} \tag{1.1}$$

which relate the geometry of spacetime, represented on the left-hand side by the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R and the metric tensor $g_{\mu\nu}$ with the matter content of the universe, represented by the energy-momentum tensor $T_{\mu\nu}$.

One of the earliest and most striking confirmations of the theory was its explanation of the anomalous precession of Mercury's perihelion, that was a longstanding problem in celestial mechanics.

In 1859, Le Verrier [1] observed that Mercury's orbit precesses at a rate of approximately $5601''$ per century, a value that could not be fully accounted for using Newtonian gravity.

Le Verrier noted that about $5025''$ arises from the use of a non-inertial frame and $532''$ are due to perturbations of Mercury's orbit caused by the gravitational attraction of the other planets of the solar system.

However, an anomalous perihelion shift of

$$\delta\phi = 43,11'' \pm 0,45'' \text{per century} \tag{1.2}$$

remained unexplained. Several hypotheses were proposed, including the existence of an unseen planet between the Sun and Mercury, called Vulcan.

General Relativity, however, naturally predicts a correction to Mercury's orbit [2]. The relativistic correction to the Newtonian perihelion advance is given by

$$\delta\phi = \frac{6\pi G}{c^2}\frac{M_\odot}{a(1-e^2)} \tag{1.3}$$

where $M_\odot$ is the mass of the Sun, a the semi-major axis, and e the orbital eccentricity of Mercury. Plugging in the relevant parameters yields a theoretical prediction of

$$\delta\phi_{GR} = 43,03''\text{per century} \tag{1.4}$$

which matches the observed anomaly with astonishing accuracy. This agreement marked one of the earliest and most dramatic triumphs of Einstein's theory.

General Relativity predicts that a light ray passing near a massive object such as the Sun is deflected due to spacetime curvature. For a light ray passing the Sun at a distance d the deflection angle is given by

$$\delta\theta = \frac{4M_\odot}{d}\frac{1+cos\Phi}{2} \tag{1.5}$$

where Φ is the angle between the Earth-Sun line and the incoming direction of the photon. In the case of a grazing ray, where $d = R_\odot$ and $\Phi = 0$, this yields a deflection angle of

$$\delta\theta = 1''.7505. \tag{1.6}$$

Earlier calculations, such as von Soldner's 1804 corpuscular theory approach [3] or Einstein's 1911 derivation using the equivalence principle, predicted only half this value. This discrepancy is resolved in the full framework of General Relativity, where both the curvature of time and space contribute to the deflection.

The first empirical test of this prediction came in 1919 during a total solar eclipse. Eddington and collaborators measured the apparent shift in the positions of stars near the Sun's limb, reporting deflection angles in the range $1.5'' < \delta\theta < 2.2''$ [4], consistent with General Relativity but mainly confirming that light is deflect by gravity. More recently, the most precise eclipse-based measurement yielded [5]

$$\delta\theta = 1.751'' \pm 0.060'' \tag{1.7}$$

in excellent agreement with the general relativistic prediction. The upcoming solar eclipses of 2026 and 2027 may offer opportunities to further improve this technique.

General Relativity also recovers Newtonian gravity as a limiting case: in the weak and static field, non-relativistic regime, it reduces to Newtonian gravity. Relaxing the static condition, GR also predicts the existence of gravitational waves, linearized perturbations of the gravitational field propagating like ordinary waves on the background geometry.

A major indirect confirmation of gravitational radiation came in 1974 with the discovery of the binary pulsar PSR1913+16 by Hulse and Taylor. The gradual decrease in the orbital period of the system was shown to match the theoretical prediction for energy loss due to gravitational wave emission [6]. This agreement provided strong evidence for the existence of gravitational waves, earning Hulse and Taylor the Nobel Prize in Physics in 1993.

The direct detection of gravitational waves occurred much later. In 2016, the LIGO and Virgo collaborations announced the first observation of gravitational waves from a binary black hole merger [7]. This marked the beginning of gravitational wave astronomy and constituted one of the most significant confirmations of GR in the weak-field regime.

1.1. General Relativity in Cosmology

General Relativity implies that the universe cannot be static. Its field equations, when applied to a homogeneous and isotropic universe, lead to either expansion or contraction. Remarkably, Einstein could have predicted the dynamical nature of the cosmos from this feature of GR. However, at the time, no observational evidence supported such a conclusion. To enforce a static universe, Einstein introduced the cosmological constant Λ into his equations. This modification allowed for a static solution in which the attractive effect of gravity was exactly balanced by a repulsive effect due to Λ. However, such a balance requires extreme fine-tunning and is unstable under perturbations. Thus, even as a theoretical construct, Einstein's static universe was problematic.

The situation changed dramatically in 1929, when Edwin Hubble presented empirical evidence that the universe is expanding [8]. In light of this discovery, Einstein abandoned the cosmological constant, calling it his "biggest blunder." Yet this term would re-emerge in cosmology decades later.

The discovery of the accelerated expansion of the universe—first reported in 1998 through observations of Type Ia supernovae [9]—prompted the reintroduction of the cosmological constant. The data could be well explained by assuming that the acceleration is driven by a mysterious component known as Dark Energy, with the simplest candidate being a positive cosmological constant. This has led to the current cosmological model, known as the ΛCDM model. It fits a wide range of cosmological observations remarkably well but still faces some theoretical and observational challenges.

One of the most significant is the cosmological constant problem. The cosmological constant behaves like a vacuum energy term in the energy-momentum tensor. However, identifying Λ with the vacuum energy density leads to an enormous discrepancy between observations and the predictions from quantum field theory [10]. Its predicted value is vastly larger than observations, requiring an extreme fine-tuning of a bare cosmological constant to cancel it. The small observed value of the cosmological constant remains unexplained, though some argue that it is no more mysterious than the numerous constants in our fundamental theories that cannot be determined from first principles and have to be measured [11].

Closely related is the coincidence problem: why is the energy density of matter, which decreases over time, roughly equal to the energy density of dark energy, which remains constant, precisely today? In an expanding universe, matter dominates at early times and Λ at late times. The fact that we observe these two components to be comparable now seems to require that we are in a very special moment of the history of the universe.

Although ΛCDM remains consistent with cosmological observations from both the cosmic microwave background and late-time universe surveys, recent observations have revealed tensions within the model, particularly discrepancies between early and late-universe measurements of the Hubble constant [12]. Furthermore, to address issues related to the initial conditions of the universe, such as the horizon and flatness problems, an additional phase of exponential expansion, known as inflation, is required beyond the standard ΛCDM framework [13].

These conceptual and empirical difficulties raise a profound question: Is Dark Energy truly a cosmological constant? If so, why is it so small and what determines its value?

The problems with ΛCDM may be tackled by extending or modifying General Relativity itself and numerous models have been proposed that do give rise to a late-time acceleration of the universe [14].

Even before modern observational motivations, modifications to General Relativity have been considered by Weyl, Cartan and Einstein, inspired by the mathematical advances in differential geometry. In this work, we follow a similar path: we investigate modifications to gravity by considering a more general geometric framework than Riemannian geometry. In Chapter 2, we introduce the necessary mathematical formalism, with a focus on non-Riemannian geometry, which forms the basis for the modifications we will study. We then present the Geometric Trinity of Gravity and its extensions in Chapters 3 and 4. In Chapter 5, we explore the cosmological dynamics of these theories using the reconstruction method and identify those capable of explaining the accelerated expansion of the universe. Finally, in Chapter 6 we confront these theories with Solar System constraints by deriving their spherically symmetric static solutions.

2. Mathematical Formulation of GR

A metric-affine geometry or Non-Riemannian geometry is defined by a triple $(\mathcal{M}, g, \Gamma)$, where $\mathcal{M}$ is a 4-dimensional smooth manifold, g is a Lorentzian metric on $\mathcal{M}$ (a smooth, non-degenerate, symmetric tensor field of type $(0,2)$ with signature $(-,+,+,+)$) and Γ a general affine connection.

The metric allows the definition of norm of a vector:

$$||V|| = g_{\mu\nu} V^{\mu} V^{\nu} \tag{2.1}$$

The affine connection defines the covariant derivative of a vector field V^{ν} as:

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\mu\alpha} V^{\alpha} \tag{2.2}$$

Given a curve $x^{\mu}(\lambda)$, we define the direction derivative along the curve as

$$\frac{D}{d\lambda} \equiv \frac{dx^{\mu}}{d\lambda} \nabla_{\mu} \tag{2.3}$$

A vector V^{μ} is said to be parallel transported along the curve $x^{\mu}(\lambda)$ if

$$\frac{D}{d\lambda} V^{\mu} = 0 \tag{2.4}$$

In a metric-affine geometry we can define the geometric quantities:

Riemann Tensor

$$R^{\alpha}{}_{\mu\nu\rho} = \partial_{\nu} \Gamma^{\alpha}{}_{\rho\mu} - \partial_{\rho} \Gamma^{\alpha}{}_{\nu\mu} + \Gamma^{\alpha}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\rho\mu} - \Gamma^{\alpha}{}_{\rho\lambda} \Gamma^{\lambda}{}_{\nu\mu} \tag{2.5}$$

Torsion Tensor

$$T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu} \tag{2.6}$$

Non-Metricity Tensor

$$Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu} \tag{2.7}$$

From the Riemann tensor, we construct the Ricci Tensor $R_{\mu\nu}$ and the Ricci Scalar R as:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{2.8a}$$

$$R = R_{\mu\nu}g^{\mu\nu} \tag{2.8b}$$

Contracting indices of the torsion and non-metricity tensors, we define the Torsion Vector T_μ and the First and Second Non-Metricity Tensor Q_α and $\bar{Q}_\alpha$:

$$T_\mu = T^{\alpha}{}_{\mu\alpha} \tag{2.9a}$$

$$Q_\alpha = g^{\mu\nu}Q_{\alpha\mu\nu} \tag{2.9b}$$

$$\bar{Q}_\alpha = g^{\mu\nu}Q_{\mu\nu\alpha} \tag{2.9c}$$

2.1. Geometric Interpretation of Curvature, Torsion and Non-Metricity

Consider a vector V^μ and parallel transport it around an infinitesimal parallelogram spanned by two vectors A^μ and B^μ: first parallel transport V^μ in the direction of A^μ, then along B^μ and then backward along A^μ and B^μ. The change in the vector V^μ after being parallel transported around this closed loop is given by:

$$\delta V^\mu = R^{\mu}{}_{\nu\alpha\beta}V^\nu A^\alpha B^\beta \tag{2.10}$$

Thus, the curvature tensor measures the change of a vector when it is parallel transported along a close loop, as illustrated in figure 2.1a.

In Euclidean Geometry, the sum of two vectors can be visualized as a parallelogram. The torsion tensor measures the failure of an infinitesimal parallelogram to close. That is, torsion quantifies the non-commutativity of infinitesimal displacements, as represented in figure 2.1b. Consider two curves: $x^\mu = x^\mu(\lambda)$ and $\tilde{x}^\mu = \tilde{x}^\mu(\lambda)$ with tangent vectors $u^\mu = \frac{dx^\mu}{d\lambda}$ and $\tilde{u}^\mu = \frac{d\tilde{x}^\mu}{d\lambda}$

Now, displace u^α infinitesimally along $\tilde{x}^\mu$:

$$u'^\alpha = u^\alpha + (\partial_\mu u^\alpha)d\tilde{x}^\mu \tag{2.11}$$

Since u^α is parallel transported along $\tilde{x}$ it holds that:

$$\frac{d\tilde{x}^\mu}{d\lambda}\partial_\mu u^\alpha + \Gamma^{\alpha}{}_{\nu\mu}\frac{d\tilde{x}^\mu}{d\lambda}u^\nu = 0 \implies (\partial_\mu u^\alpha)d\tilde{x}^\mu = -\Gamma^{\alpha}{}_{\nu\mu}u^\nu\tilde{u}^\mu d\lambda \tag{2.12}$$

Substituting this into equation (2.11):

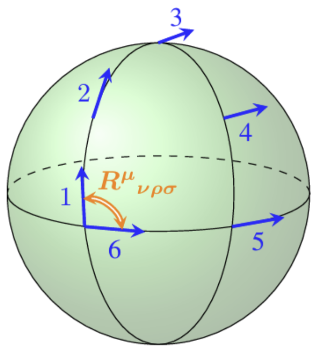
$$u'^{\alpha} = u^{\alpha} - \Gamma^{\alpha}{}_{\nu\mu} u^{\nu} \tilde{u}^{\mu} d\lambda \tag{2.13}$$

Applying the same reasoning to the other vector gives:

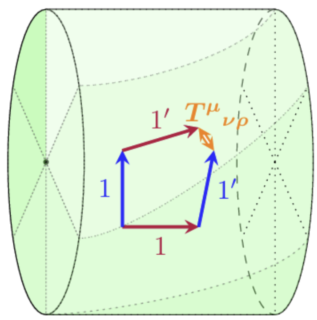
$$\tilde{u}'^{\alpha} = \tilde{u}^{\alpha} - \Gamma^{\alpha}{}_{\nu\mu} \tilde{u}^{\nu} u^{\mu} d\lambda = \tilde{u}^{\alpha} - \Gamma^{\alpha}{}_{\mu\nu} \tilde{u}^{\mu} u^{\nu} d\lambda \tag{2.14}$$

The failure to close the parallelogram is then quantified by:

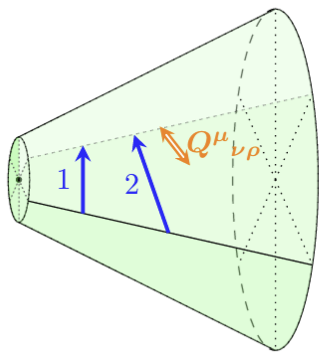
$$(\tilde{u}^{\alpha} + u'^{\alpha}) - (u^{\alpha} + \tilde{u}'^{\alpha}) = \tilde{u}^{\alpha} + u^{\alpha} - \Gamma^{\alpha}{}_{\nu\mu} u^{\nu} \tilde{u}^{\mu} d\lambda - u^{\alpha} - \tilde{u}^{\alpha} + \Gamma^{\alpha}{}_{\mu\nu} \tilde{u}^{\mu} u^{\nu} d\lambda = T^{\alpha}{}_{\mu\nu} \tilde{u}^{\mu} u^{\nu} d\lambda \tag{2.15}$$



(a) Parallel transported vectors changing around a closed loop due to curvature.



(b) Torsion as the non-closure of infinitesimal parallelograms



(c) Lengths of parallel transported vectors changing due to non-metricity.

Figure 2.1: Representation of (a) curvature, (b) torsion and (c) non-metricity for parallel-transported vectors. Images taken from [15]

The non-metricity tensor provides a measure for how the magnitude of a vector changes under parallel transport, as depicted in figure 2.1c. The change in the magnitude of a vector V^{μ} under parallel transport along a curve $x^{\mu} = x^{\mu}(\lambda)$ is:

$$\frac{D}{d\lambda}(V^{\mu} V^{\nu} g_{\mu\nu}) = 2\frac{dx^{\alpha}}{d\lambda}(\nabla_{\alpha} V^{\mu}) V_{\mu} + \frac{dx^{\alpha}}{d\lambda}(\nabla_{\alpha} g_{\mu\nu}) V^{\mu} V^{\nu} \tag{2.16}$$

Since V^μ is parallel transported along the curve, $\frac{dx^\alpha}{d\lambda}(\nabla_\alpha V^\mu) = 0$ by definition (2.4). Thus, we are left with:

$$\frac{D}{d\lambda}(||V^2||) = Q_{\alpha\mu\nu}\frac{dx^\alpha}{d\lambda}V^\mu V^\nu \tag{2.17}$$

This equation shows that non-metricity $Q_{\alpha\mu\nu}$ directly controls the variation of the length of a vector during parallel transport. In Riemannian geometry, where $Q_{\alpha\mu\nu} = 0$ the length of a vector remains unchanged under parallel transport.

2.2. Decomposition of the Connection

GR adopts the Riemannian Geometry framework where it is postulated that $T^\alpha{}_{\mu\nu} = 0$ and $Q_{\alpha\mu\nu} = 0$. These two geometric conditions are satisfied if and only if the connection is the Levi-Civita connection, which is uniquely determined by the metric:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{2.18}$$

Since the Levi-Civita connection is fully determined by the metric, a spacetime in General Relativity is fully described by the pair $(\mathcal{M}, g)$.

In contrast, a metric-affine geometry $(\mathcal{M}, g, \Gamma)$ considers a more general connection Γ that is not necessarily Levi-Civita and can include contributions from both torsion and non-metricity. In this context, the connection can be decomposed into the Levi-Civita part plus extra contributions:

$$\Gamma^\alpha{}_{\mu\nu} = \mathring{\Gamma}^\alpha{}_{\mu\nu} + D^\alpha{}_{\mu\nu} \tag{2.19}$$

where $\mathring{\Gamma}^\alpha{}_{\mu\nu}$ is the Levi-Civita connection and $D^\alpha{}_{\mu\nu}$ is the distorsion tensor, which captures deviations from Riemannian geometry. It is composed of two parts: the contorsion tensor, which encodes torsion:

$$K^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu} + T_{\nu\lambda\mu}) \tag{2.20}$$

and the disformation tensor, which encodes non-metricity:

$$L^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(Q_{\lambda\mu\nu} - Q_{\mu\lambda\nu} - Q_{\nu\lambda\mu}) \tag{2.21}$$

Thus, the total distortion is:

$$D^\alpha{}_{\mu\nu} = K^\alpha{}_{\mu\nu} + L^\alpha{}_{\mu\nu} \tag{2.22}$$

Since the Riemann tensor depends only on the connection, this decomposition induces a corresponding decomposition of the Riemann tensor:

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \mathring{R}^{\alpha}{}_{\beta\mu\nu} + \mathring{\nabla}_{\mu} D^{\alpha}{}_{\nu\beta} - \mathring{\nabla}_{\nu} D^{\alpha}{}_{\mu\beta} + D^{\sigma}{}_{\nu\beta} D^{\alpha}{}_{\mu\sigma} - D^{\sigma}{}_{\mu\beta} D^{\alpha}{}_{\nu\sigma} \tag{2.23}$$

where $\mathring{R}^{\alpha}{}_{\mu\nu\rho}$ and $\mathring{\nabla}_{\mu}$ are the Riemann tensor and covariant derivative associated with the Levi-Civita connection.

We define the following scalar quantities:

Torsion scalar

$$\mathbb{T} = -\frac{1}{4} T^{\alpha\mu\nu} T_{\alpha\mu\nu} - \frac{1}{2} T^{\alpha\mu\nu} T_{\mu\alpha\nu} + T^{\alpha} T_{\alpha} \tag{2.24}$$

Non-metricity scalar

$$\mathbb{Q} = -\frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\alpha\nu} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \bar{Q}^{\alpha} \tag{2.25}$$

Contracting the Riemann tensor decomposition leads to a decomposition of the Ricci scalar:

$$R(\Gamma) = \mathring{R} - \mathbb{T} - \mathbb{Q} + T^{\rho\mu\nu} Q_{\mu\nu\rho} - T^{\mu} Q_{\mu} + T^{\mu} \bar{Q}_{\mu} + \mathring{\nabla}_{\alpha}(Q^{\alpha} - \bar{Q}^{\alpha} + 2T^{\alpha}) \tag{2.26}$$

This result connects the scalar curvature of a general connection to quantities derived from the Levi-Civita connection and the torsion/non-metricity tensors. The special cases with vanishing curvature give useful identities.

In a spacetime with zero curvature and zero non-metricity ($R(\Gamma) = 0$, $Q_{\alpha\mu\nu} = 0$), we obtain:

$$\mathring{R} = \mathbb{T} - 2\mathring{\nabla}_{\alpha} T^{\alpha} \tag{2.27}$$

In a spacetime with zero curvature and zero torsion ($R(\Gamma) = 0$, $T^{\alpha}{}_{\mu\nu} = 0$), we obtain:

$$\mathring{R} = \mathbb{Q} - \mathring{\nabla}_{\alpha}(Q^{\alpha} - \bar{Q}^{\alpha}) \tag{2.28}$$

These identities will later be used to construct alternatives theories of gravity.

We emphasize that a manifold endowed with a metric does not inherently possess curvature, torsion, or non-metricity: these properties are defined only with respect to the affine connection. Nevertheless, we can always compute the Levi-Civita curvature of a manifold with a metric, independently of the connection.

3. Geometric Trinity of Gravity

Alternative theories of gravity can be constructed by adopting different geometric structures. As shown earlier, a general metric-affine spacetime is characterized by three geometric quantities: curvature, torsion, and non-metricity. In General Relativity, both torsion and non-metricity vanish, and gravity is described purely as a manifestation of spacetime curvature. However, one can construct gravitational theories in which any two of these properties are constrained to zero, and the remaining one mediates the gravitational interaction. In fact, different combinations of these variables lead to different geometric frameworks upon which alternative theories of gravity can be constructed. In figure 3.1, we represent the different possibilities of metric-affine geometries classified according to whether or not curvature, torsion and non-metricity vanish.

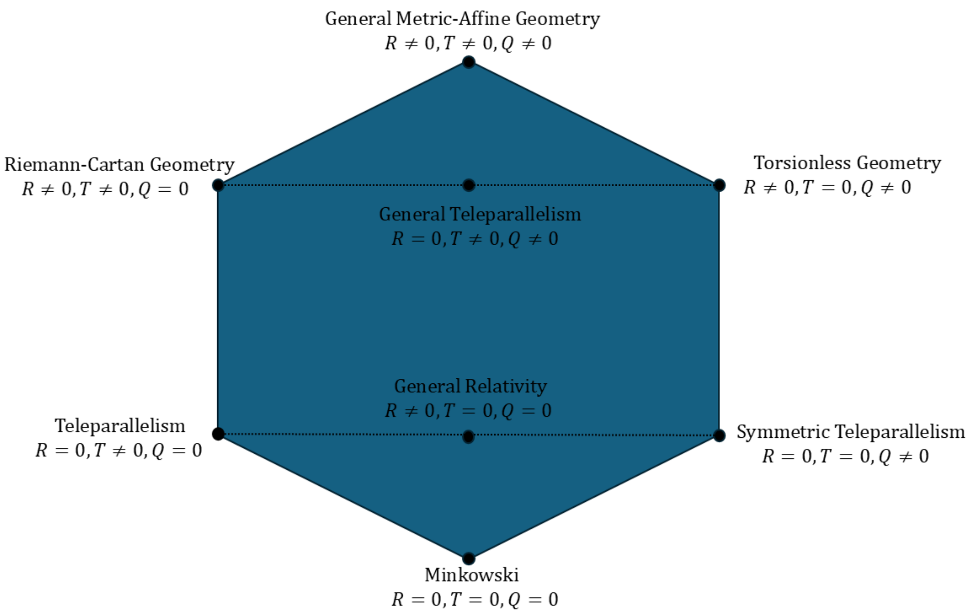


Figure 3.1: The Eightfold Way of Gravity [16]: a diagram of all the possible metric-affine geometries.

In this chapter, we present the geometry trinity of gravity [17] which is the set of theories from figure 3.1 with only one non-vanishing component.

3.1. General Relativity

In General Relativity, it is postulated that both torsion and non-metricity vanish.

$$T^{\alpha}{}_{\mu\nu} = 0 \qquad\qquad Q_{\alpha\mu\nu} = 0 \tag{3.1}$$

As a result, gravity manifests as the curvature of spacetime.

The field equations of General Relativity relate the curvature of spacetime to the energy-momentum content. They can be derived from the Einstein–Hilbert action:

$$S = \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}R + S_M \tag{3.2}$$

where S_M is the matter action and $k = 8\pi G$ in natural units. Varying this action with respect to the metric yields the Einstein field equations:

$$G_{\mu\nu} = kT_{\mu\nu} \tag{3.3}$$

where the Einstein Tensor is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{3.4}$$

Using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, the field equations imply the energy conservation equation

$$\nabla^\mu T_{\mu\nu} = 0 \tag{3.5}$$

The Bianchi identity itself is a consequence of the diffeomorphism invariance of the Einstein–Hilbert action.

More generally, given an action of the form

$$\int dx^4\sqrt{-g}(\mathcal{L}_g + \mathcal{L}_m) \tag{3.6}$$

the field equations can be derived using different variational formalisms. In the metric formalism the connection is the Levi-Civita connection and the metric is the only degree of freedom; in the Palatini formalism the metric and the connection are treated as independent variables and the gravitation Lagrangian $\mathcal{L}_g$ depends on the both the metric and on the connection, but $\mathcal{L}_m$ depends solely the metric; in the Metric-Affine formalism the metric and the connection are independent and both $\mathcal{L}_g$ and $\mathcal{L}_m$ depend on the metric and on the connection.

In the Palatini formalism, the Einstein–Hilbert action still leads to General Relativity. Varying the action with respect to the connection yields that the connection is Levi-Civita up to a projective transformation:

$$\Gamma^\lambda_{\mu\nu} = \mathring{\Gamma}^\lambda_{\mu\nu} - \frac{2}{3}S_\nu\delta^\lambda_\mu = \mathring{\Gamma}^\lambda_{\mu\nu} + \frac{1}{8}\delta^\lambda_\mu Q_\nu \tag{3.7}$$

The connection becomes Levi-Civita if and only if torsion is zero or equivalently if and only if non-metricity is zero. However, this extra term drops out when inserted into the metric field equations. Therefore General Relativity arises naturally if the dynamics are described by the Einstein-Hilbert action, even when using the Palatini Formalism.

3.2. Teleparallelism

In Teleparallel Gravity (TG), spacetime is flat and metric-compatible and gravity is mediated by torsion. Teleparallelism was first introduced by Einstein himself, as an attempt to unify gravity with electromagnetism [18].

Consider a metric-affine geometry $(\mathcal{M}, g, \Gamma)$ where the connection satisfies

$$R^{\alpha}{}_{\mu\nu\rho} = 0 \qquad\qquad Q_{\alpha\mu\nu} = 0 \tag{3.8}$$

The dynamics are specified by the action:

$$S_{TEGR} = \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}\,\mathbb{T} + S_M \tag{3.9}$$

we have shown in equation (2.27),that this Lagrangian differs from the Einstein–Hilbert Lagrangian by a total divergence. Therefore, the metric field equations derived from (3.9) are equivalent to those of General Relativity. This theory is known as Teleparallel Equivalent of General Relativity (TEGR).

Using the Palatini formalism for TEGR, we can incorporate the geometric postulates of the theory directly via Lagrange multipliers:

$$S = \int_{\mathcal{M}} d^4x\left(\frac{1}{2k}\sqrt{-g}\,\mathbb{T} + \lambda_{\alpha}{}^{\mu\nu\rho}R^{\alpha}{}_{\mu\nu\rho} + \chi^{\alpha}{}_{\mu\nu}Q_{\alpha}{}^{\mu\nu}\right) + S_M \tag{3.10}$$

We note that for $f(R)$, the geometric postulates of vanishing torsion and non-metricity are already implicit in the metric formalism.

We define the torsion superpotential

$$S_{\alpha}{}^{\mu\nu} = \frac{1}{4}(T^{\mu}{}_{\alpha}{}^{\nu} + T_{\alpha}{}^{\mu\nu} + T^{\nu\mu}{}_{\alpha}) + \frac{1}{2}\delta^{\mu}_{\alpha}T^{\nu} - \frac{1}{2}\delta^{\nu}_{\alpha}T^{\mu} \tag{3.11}$$

which can be used to write the torsion scalar as:

$$\mathbb{T} = -S_{\alpha}{}^{\mu\nu}T^{\alpha}{}_{\mu\nu} \tag{3.12}$$

The field equations for the metric and the connection are [19]*:

$$2(\nabla_\alpha + T_\alpha)S_{(\mu\nu)}{}^{\alpha} + T_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - 2T_{\alpha\beta\mu}S^{\alpha\beta}{}_\nu - \frac{1}{2}\mathbb{T}g_{\mu\nu} = kT_{\mu\nu} \tag{3.13a}$$

$$(\nabla_\alpha + T_\alpha)(\sqrt{-g}S_{[\mu}{}^{\alpha}{}_{\nu]}) = 0 \tag{3.13b}$$

The metric field equations are just the Einstein Equations in disguise, and the connection field equation does not impose constraints on the connection, but it is trivially satisfied: it corresponds to the Bianchi identities.

3.3. Symmetric Telleparelism

In 1999, Nester and Yo [20] proposed an alternative theory, Symmetric Teleparallel Gravity (STG), constructed by setting curvature and torsion to zero, with gravity mediated by non-metricity.

The action for the Symmetric Teleparallel Equivalent of General Relativity (STEGR) in the Palatini formalism reads:

$$S = \int_{\mathcal{M}} d^4x(\frac{1}{2k}\sqrt{-g}\mathbb{Q} + \lambda_\alpha{}^{\mu\nu\rho}R^\alpha{}_{\mu\nu\rho} + \chi^\alpha{}_{\mu\nu}T_\alpha{}^{\mu\nu}) + S_M \tag{3.14}$$

Define the non-metricity superpotencial:

$$P^\alpha{}_{\mu\nu} = \frac{1}{4}[-Q^\alpha{}_{\mu\nu} + Q_\mu{}^\alpha{}_\nu + Q_\nu{}^\alpha{}_\mu + (Q^\alpha - \bar{Q}^\alpha)g_{\mu\nu} - \frac{1}{2}(\delta^\alpha_\mu Q_\nu + \delta^\alpha_\nu Q_\mu)] \tag{3.15}$$

so that

$$\mathbb{Q} = P_{\alpha\mu\nu}Q^{\alpha\mu\nu} \tag{3.16}$$

The field equations are [19]†:

$$\frac{2}{\sqrt{-g}}\nabla_\alpha[\sqrt{-g}P^\alpha_{\mu\nu}] + P_{(\mu|\alpha\beta}Q_{\nu)}{}^{\mu\nu} - 2P^{\alpha\beta}{}_{(\nu}Q_{\alpha\beta|\mu)} - \frac{1}{2}\mathbb{Q}g_{\mu\nu} = kT_{\mu\nu} \tag{3.17a}$$

$$\nabla_\mu\nabla_\nu(\sqrt{-g}P^{\mu\nu}_\alpha) = 0 \tag{3.17b}$$

3.4. Connection in Teleparallelism

Contrary to General Relativity where the geometric postulates fix the connection to be the Levi connection, which is entirely determined by the metric, in these new theories the

*In [19], the torsion scalar is defined identically to our convention. However, the torsion superpotential is defined as $\mathbb{T} = \frac{1}{2}S_\alpha{}^{\mu\nu}T^\alpha{}_{\mu\nu}$. Thus, the first three terms of equation (3.13a) differ from theirs by a factor of -2.

†In this case, [19] defines the non-metricity superpotencial in the same way, but the the non-metricity scalar is the symmetric of ours.

geometric postulates do not completely fix the connection. In both TG and STG, the curvature is zero. We impose the teleparallel constraint by making the Riemann tensor vanish:

$$R^{\alpha}{}_{\mu\nu\rho} = 2\partial_{[\nu}\Gamma^{\alpha}{}_{\rho]\mu} + 2\Gamma^{\alpha}{}_{[\nu|\lambda}\Gamma^{\lambda}{}_{\rho]\mu} = 0 \tag{3.18}$$

which has the solution

$$\Gamma^{\alpha}_{\mu\nu} = (\Lambda^{-1})^{\alpha}_{\lambda}\partial_{\mu}\Lambda^{\lambda}_{\nu} \tag{3.19}$$

where Λ^{μ}_{ν} is a matrix belonging to $GL(4, \mathbb{R})$. Any connection of this form is flat.

Imposing metric compatibility to this connection leads to:

$$g^{\lambda(\mu}\partial_{\alpha}\Lambda^{\nu)}_{\rho}(\Lambda^{-1})^{\rho}_{\lambda} = \frac{1}{2}\partial_{\alpha}g^{\mu\nu} \tag{3.20}$$

This relation can be used to eliminate the metric in terms of the matrix Λ.

Imposing instead that the flat connection (3.19) is symmetric, we get:

$$\partial_{[\mu}\Lambda^{\alpha}_{\nu]} = 0 \implies \Gamma^{\alpha}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}}\partial_{\mu}\partial_{\nu}\xi^{\lambda} \tag{3.21}$$

For $\xi^{\mu} = M^{\mu}_{\nu}x^{\nu} + \xi^{\mu}_{c}$ where M is a constant matrix, the connection vanishes. This choice corresponds to the coincident gauge. There always exists a coordinate transformation that brings the connection into this trivial form. This can be used to construct a formulation of general relativity based on non-metricity, but without the affine structure, called Coincident General Relativity [21, 22].

A similar gauge can be constructed for TEGR using a different formalism. Instead of working with the metric and connection, one uses the tetrad and the spin connection. In a flat and metric-compatible spacetime, there always exists a local Lorentz transformation that makes the spin connection vanish. This is known as the Weitzenböck gauge.

While it might seem that such gauges simplify calculations, this is not generally the case. The coordinate system in which the connection vanishes may not coincide with the one best adapted to the symmetries of the spacetime. Although the coordinate transformation to the coincident gauge eliminates the affine connection, the same information is transferred into the components of the metric, which becomes more complicated as a result.

In other words, a diffeomorphism that trivializes the connection will typically make the metric more complex. All the information that was encoded in the symmetry-reduced connection is now moved into the metric. Hence, nothing is truly gained by using the coincident gauge.

For this reason, we will not use the coincident gauge or the Weitzenböck gauge in what follows and won't develop the tetrad formalism for TG.

Instead, we adopt a metric-affine approach, where the metric and the connection are treated as independent. The connection is then constrained by the geometric postulates of the theory, and both the metric and the connection are required to be compatible with the symmetries of the spacetime under study. This will be further developed in chapters 5 and 6.

4. Modified Geometric Trinity Gravity

The equivalence of GR, TEGR and STEGR raises profound questions about the fundamental geometric nature of spacetime. While General Relatity describes gravity as a manifestation of spacetime curvature, TEGR and STEGR attribute gravitational effects to torsion and non-metricity, respectively. Yet, it appears that these three geometric frameworks are observationally indistinguishable. This ambiguity is known as the problem of geometric underdetermination [23, 24].

A further consequence of this degeneracy is that these equivalent theories inherit the same limitations. In particular, none of them can provide an explanation for the accelerated expansion of the Universe without resorting to a cosmological constant. If GR requires such a constant to fit cosmological observations, then so do TEGR and STEGR.

This motivates the study of modified gravity theories using the geometric trinity of gravity as a starting point. These modified theories could exhibit distinct phenomenologies and potentially offer alternative explanations for dark energy.

The use of modified versions of these theories, having in mind their discrimination, was explored in [25] by comparing their suitability in describing inflation. In parallel, the foundational role of the Equivalence Principle across these different geometric frameworks has been investigated in [26].

To understand how to modify gravity, it is essential to first recognize what makes GR unique. This uniqueness is formally established by Lovelock's theorem [27], which states that in four dimensions, the most general theory that can be derived from an action constructed solely from the metric, containing at most second-order derivatives of the metric is GR with a cosmological constant.

Consequently, any extension of GR must introduce new ingredients beyond this framework such as extra dimensions, extra fields, higher order derivatives or non-Riemann Geometry. The teleparallel theories introduced earlier already go beyond Riemannian geometry but remain dynamically equivalent to GR. To obtain distinct dynamics, we can combine these geometric approaches with higher-derivative modifications.

4.1. $f(R)$ gravity

A natural extension of GR, first considered by Buchdahl in 1970 [28], is to replace the Ricci scalar R in the Einstein-Hilbert action with an arbitrary function $f(R)$. This modification introduces sufficient generality to capture key features of higher-order gravity while maintaining mathematical simplicity. One of its main motivations is its potential to explain cosmic acceleration without invoking dark energy. Notably, Starobinsky used an R^2 correction to GR to model inflation [29], producing what is still one of the most reliable inflation models. Additionally, $f(R)$ theories have been employed to describe the late-time acceleration of the universe [30], [31], while Nojiri and Odintsov [32] used it to construct an unified description of early and late acceleration.

We define the theory by the action:

$$S = \int_{\mathcal{M}} d^4x \frac{1}{2k}\sqrt{-g}f(R) + S_M \tag{4.1}$$

The $f(R)$ field equations derived in the metric formalism are:

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = 8\pi G T_{\mu\nu} \tag{4.2}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$ and $f_R = \frac{df}{dR}$. The field equations are fourth order non-linear equations for the metric, due to the second order differential operator $g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu$ acting on f_R which already contains second order derivatives of the metric. The theory propagates three degrees of freedom: two degrees of freedom corresponding to a massless graviton and one scalar degree of freedom.

4.2. $f(\mathbb{T})$ gravity

An analogous modification in the teleparallel framework replaces the torsion scalar $\mathbb{T}$ in TEGR with a general function $f(\mathbb{T})$. We define the theory by the action:

$$S = \int_{\mathcal{M}} d^4x \left(\frac{1}{2k}\sqrt{-g}f(\mathbb{T}) + \lambda_\alpha{}^{\mu\nu\rho}R^\alpha{}_{\mu\nu\rho} + \chi^\alpha{}_{\mu\nu}Q_\alpha{}^{\mu\nu}\right) + S_M \tag{4.3}$$

In the Palatini formalism, the $f(\mathbb{T})$ metric and connection field equations are [22]:

$$2(\nabla_\alpha + T_\alpha)[f_\mathbb{T} S_{(\mu\nu)}{}^{\alpha}] + f_\mathbb{T}(T_{\nu\alpha\beta}S_\mu{}^{\alpha\beta} - 2T_{\alpha\beta\mu}S_\mu{}^{\alpha\beta}) - \frac{1}{2}f g_{\mu\nu} = kT_{\mu\nu} \tag{4.4a}$$

$$(\nabla_\mu + T_\mu)(f_\mathbb{T} S_{[\alpha}{}^{\mu}{}_{\beta]}) = 0 \tag{4.4b}$$

Unlike in $f(R)$ gravity, the metric field equations of $f(\mathbb{T})$ gravity are second order.

The metric field equations can be re-written in a form resembling GR:

$$f_{\mathbb{T}} G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f - f_{\mathbb{T}} \mathbb{T}) + 2 f_{\mathbb{T}\mathbb{T}} (\mathbb{T}) S_{(\mu\nu)}{}^{\alpha} \partial_\alpha \mathbb{T} = k T_{\mu\nu} \tag{4.5}$$

From this form, it becomes clear that the case $f(\mathbb{T}) = \mathbb{T}$ is equivalent to General Relativity. In this form, one can readily prove the following result: Given an arbitrary function $f(\mathbb{T})$, if $\mathbb{T}$ is a constant, then the metric field equations reduce to the field equations of General Relativity.

Let us denote the constant value of the torsion scalar by $\mathbb{T}_0$. Then, the field equation (4.5) simplifies to:

$$G_{\mu\nu} - \frac{1}{2} \frac{f(\mathbb{T}_0) - f'(\mathbb{T}_0)\mathbb{T}_0}{f'(\mathbb{T}_0)} g_{\mu\nu} = k \frac{T_{\mu\nu}}{f'(\mathbb{T}_0)} \tag{4.6}$$

This corresponds to the Einstein Field Equations with rescaled energy-momentum tensor and effective cosmological constant given by:

$$\tilde{T}^{\mu\nu} = \frac{T_{\mu\nu}}{f'(\mathbb{T}_0)} \tag{4.7a}$$

$$\Lambda_{eff} = -\frac{1}{2} \frac{f(\mathbb{T}_0) - f'(\mathbb{T}_0)\mathbb{T}_0}{f'(\mathbb{T}_0)} \tag{4.7b}$$

4.3. $f(\mathbb{Q})$ gravity

The corresponding extension of STEGR is defined by the action:

$$S = \int_{\mathcal{M}} d^4x \left(\frac{1}{2k} \sqrt{-g} f(\mathbb{Q}) + \lambda_\alpha{}^{\mu\nu\rho} R^\alpha{}_{\mu\nu\rho} + \chi^\alpha{}_{\mu\nu} T_\alpha{}^{\mu\nu} \right) + S_M \tag{4.8}$$

The $f(\mathbb{Q})$ metric and connection field equations derived in the Palatini formalism are [22]:

$$\frac{2}{\sqrt{-g}} \nabla_\alpha (\sqrt{-g} f_{\mathbb{Q}} P^\alpha{}_{\mu\nu}) + f_{\mathbb{Q}} [P_{(\mu|\alpha\beta} Q_{\nu)}{}^{\mu\nu} - 2 P^{\alpha\beta}{}_{(\nu} Q_{\alpha\beta|\mu)}] - \frac{1}{2} f g_{\mu\nu} = k T_{\mu\nu} \tag{4.9a}$$

$$\nabla_\mu \nabla_\nu (\sqrt{-g} f_{\mathbb{Q}} P^{\mu\nu}{}_\alpha) = 0 \tag{4.9b}$$

The metric field equation can be recast in a form analogous to that of GR:

$$f_{\mathbb{Q}} G_{\mu\nu} - \frac{1}{2} (f - f_{\mathbb{Q}} \mathbb{Q}) + 2 f_{\mathbb{Q}\mathbb{Q}} P^\alpha{}_{\mu\nu} \partial_\alpha \mathbb{Q} = k T_{\mu\nu} \tag{4.10}$$

Again, this form makes it evident that the case $f(\mathbb{Q}) = \mathbb{Q}$ is equivalent to General Relativity.

Just as in the case of $f(\mathbb{T})$, constant $\mathbb{Q}$ solutions in $f(\mathbb{Q})$ gravity are indistinguishable from General Relativity up to a redefinition of the matter coupling and cosmological constant.

4.4. Extended Geometric Trinity of Gravity

The teleparallel theories from the modified trinity can be extended to include $f(\mathbb{T})$, $f(\mathbb{Q})$ and $f(R)$ gravity as special cases. This is achieved by promoting the boundary terms in the equivalence relations (2.27) and (2.28) to independent variables within the action.

4.4.1 Extended Teleparallelism

The equivalence between GR and TEGR was a result of the identity 2.27.

Defining the boundary term:

$$B \equiv 2\mathring{\nabla}_\alpha T^\alpha \tag{4.11}$$

we consider the action:

$$S = \int_{\mathcal{M}} d^4x \left(\frac{1}{2k}\sqrt{-g} f(\mathbb{T}, B) + \lambda_\alpha{}^{\mu\nu\rho} R^\alpha{}_{\mu\nu\rho} + \chi^\alpha{}_{\mu\nu} Q_\alpha{}^{\mu\nu} \right) + S_M \tag{4.12}$$

The corresponding metric field equations, derived in [16], are:

$$G_{\mu\nu} f_{\mathbb{T}} - \frac{1}{2} g_{\mu\nu}(f - f_{\mathbb{T}}\mathbb{T} - f_B B) + 2\partial_\alpha(f_{\mathbb{T}} + f_B) S_{(\mu\nu)}{}^{\alpha} + \nabla_\mu \nabla_\nu f_B - g_{\mu\nu}\Box f_B = kT_{\mu\nu} \tag{4.13}$$

If $f(\mathbb{T}, B) = f(\mathbb{T})$ we get back the field equation for $f(\mathbb{T})$ gravity (4.5). If $f(\mathbb{T}, B) = f(\mathbb{T} - B) = f(R)$, which implies the relation $f_{\mathbb{T}} = f_R = -f_B$, we get back the field equation for $f(R)$ gravity (4.2).

The connection field equation is:

$$(\nabla_\alpha + T_\alpha)[S_{[\mu}{}^{\alpha}{}_{\nu]} \sqrt{-g}(f_{\mathbb{T}} + f_B)] = 0 \tag{4.14}$$

which reduces to the equation (4.4b) in the case $f(\mathbb{T}, B) = f(\mathbb{T})$ and becomes an identity in the case $f(\mathbb{T}, B) = f(\mathbb{T} - B) = f(R)$, consistent with the absence of a dynamical connection in metric $f(R)$.

4.4.2 Extended Symmetric Teleparallelism

Analogously for STG, we use the identity 2.28 to define a boundary term variable

$$C \equiv \mathring{\nabla}_\alpha (Q^\alpha - \bar{Q}^\alpha) \tag{4.15}$$

and construct the action:

$$S = \int_{\mathcal{M}} d^4x \left(\frac{1}{2k}\sqrt{-g} f(\mathbb{Q}, C) + \lambda_\alpha{}^{\mu\nu\rho} R^\alpha{}_{\mu\nu\rho} + \chi^\alpha{}_{\mu\nu} T_\alpha{}^{\mu\nu} \right) + S_M \tag{4.16}$$

The metric field equations, derived in [33], are:

$$G_{\mu\nu} f_{\mathbb{Q}} - \frac{1}{2} g_{\mu\nu}(f - f_{\mathbb{Q}}\mathbb{Q} - f_C C) + 2\partial_\alpha(f_{\mathbb{Q}} + f_C) P^\alpha{}_{\mu\nu} + \nabla_\mu \nabla_\nu f_C - g_{\mu\nu} \Box f_C = k T_{\mu\nu} \tag{4.17}$$

Again if $f(\mathbb{Q}, B) = f(\mathbb{Q})$ we get back the field equation for $f(\mathbb{Q})$ gravity (4.10) and if $f(\mathbb{Q}, C) = f(\mathbb{Q} - C) = f(R)$ we get back the field equation for $f(R)$ gravity (4.2).

The connection field equation is:

$$\nabla_\mu \nabla_\nu [P^{\mu\nu}{}_\alpha \sqrt{-g}(f_{\mathbb{Q}} + f_C)] = 0 \tag{4.18}$$

which reduces to connection field equation of $f(\mathbb{Q})$ in the appropriate limit.

5. Cosmology

The discovery of the Universe's accelerating expansion [9] sparked decades of intense theoretical and observational efforts aimed at uncovering the nature and origin of dark energy. The most observationally successful model is ΛCDM, constructed upon GR and using a cosmological constant Λ as the source of dark energy. As discussed in the Introduction, this model faces several challenges which motivate the search for alternative theories of gravity.

In this chapter, we construct gravitational theories capable of reproducing the effects of a cosmological constant, particularly the accelerated expansion of the Universe.

We begin by reviewing the standard cosmological model ΛCDM, within the framework of General Relativity. We then derive the corresponding Friedmann equations in the context of teleparallel theories of gravity, emphasizing the particular care required when determining the affine connection in such frameworks. Finally, we introduce and apply the reconstruction method, a technique used to construct gravitational Lagrangians that yield a desired cosmological evolution.

5.1. Cosmology in General Relativity

The cosmological principle, which posits that the Universe is homogeneous and isotropic on large scales, leads to the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right) \tag{5.1}$$

where κ denotes the spatial curvature. The matter content of the Universe is modeled as a perfect fluid with energy-momentum tensor

$$T_\mu^\nu = diag(-\rho, p, p, p) \tag{5.2}$$

where ρ is the energy density and p the pressure of the fluid.

Substituting the FLRW metric and energy-momentum tensor into the Einstein field equations with a cosmological constant, we obtain the Friedmann equations:

$$3H^2 + \frac{3\kappa}{a^2} = k\rho + \Lambda \tag{5.3a}$$

$$2\dot{H} + 3H^2 + \frac{\kappa}{a^2} = -kp + \Lambda \tag{5.3b}$$

These equations imply the continuity equation, expressing the conservation of energy:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{5.4}$$

Since we have two independent equations and three variables $\{a; p, \rho\}$, an additional equation of state is needed. For instance, dust-like matter is characterized by $p = 0$ while a component with $p = -\rho$ would imply (by the continuity equation) a constant energy density and therefore would contribute to the Friedmann Equations in the same way as cosmological constant.

In ΛCDM, the recent universe is spatially flat and dominated by dust-like matter and dark energy. The first Friedmann equation for such composition reads:

$$3H^2 = k\rho + \Lambda \tag{5.5}$$

where the matter density evolves as $\rho = \frac{\rho_0}{a^3}$, following from the conservation equation for pressureless matter.

This yields a differential equation for the scale factor $a(t)$, which admits the exact solution:

$$a(t) = \left(\frac{8\pi G\rho_0}{\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}t\right) \tag{5.6}$$

This solution describes a Universe transitioning from matter domination to accelerated expansion driven by Λ and is in excellent agreement with current cosmological observations.

In the subsequent sections, we aim to reproduce this solution within the alternative geometric frameworks of TG and STG. Our goal is to recover (5.5) but without introducing a cosmological constant explicitly in the action.

5.2. Symmetry Reduction in Teleparallelism

In contrast to GR or metric $f(R)$, where the geometric postulates of vanishing torsion and metric compatibility completely determine the affine connection in terms of the metric, TG and STG do not fully constrain the connection. While it is always possible to choose coordinates in which the connection vanishes identically—namely, the Weitzenböck gauge in TG and the coincident gauge in STG—these coordinate systems may not be naturally adapted to the symmetries of the spacetime under consideration. For instance, in cosmological settings, the standard FLRW metric expressed in spherical coordinates cannot be consistently combined with the coincident gauge. Implementing the coincident gauge for that metric requires transforming the FLRW metric to Cartesian coordinates. Furthermore, adopting such gauges can obscure alternative connection that are also compatible with the geometric postulates. For these reasons, there has been a recent shift toward abandoning the coincident gauge in favor of a fully covariant formulation of $f(\mathbb{T})$ [34] and $f(\mathbb{Q})$ gravity [35], where the metric and connection are treated as a priori independent variables. In this approach, the connection is subsequently constrained by the geometric postulates and by the imposition of spacetime symmetries.

In a cosmological context, this amounts to requiring that both the metric and the connection be invariant under the symmetries of spatial homogeneity and isotropy. Mathematically, this symmetry requirement is enforced by demanding that the Lie derivatives of the metric and the connection with respect to the Killing vectors generating rotations and translations vanish:

$$\mathcal{L}_\xi g^{\mu\nu} = 0 \tag{5.7a}$$

$$\mathcal{L}_\xi \Gamma^{\alpha}{}_{\mu\nu} = 0 \tag{5.7b}$$

The imposition of these symmetry constraints on the metric yields the FLRW metric:

$$ds^2 = g_{tt}dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right) \tag{5.8}$$

The component g_{tt} can always be set to $g_{tt} = -1$ by an appropriate coordinate transformation. For the remainder of this work, we will adopt this convention and fix $g_{tt} = -1$. Furthermore, since observational data strongly supports a spatially flat universe, we will assume the spatial curvature $\kappa = 0$ from this point onwards.

For the affine connection, the symmetry constraints (5.7) are supplemented by the geometric postulates of the theory. For both $f(\mathbb{T})$ and $f(\mathbb{Q})$ gravity, we impose vanishing curvature:

$$R^{\alpha}{}_{\mu\nu\rho} = 0 \tag{5.9}$$

In addition, we impose vanishing torsion $T^{\alpha}{}_{\mu\nu} = 0$ for $f(\mathbb{Q})$ and for $f(\mathbb{T})$ vanishing non-metricity $Q_{\alpha\mu\nu} = 0$.

Since the torsion tensor is antisymmetric in its lower indices, the condition $T^{\alpha}{}_{\mu\nu} = 0$ imposes 24 independent constraints, reducing the number of independent connection components from 64 to 40. In contrast, the non-metricity tensor is symmetric in its last two indices, so $Q_{\alpha\mu\nu} = 0$ imposes 40 independent constraints. Thus, we expect that the

STG connection is more strongly constrained than in the torsional case.

This procedure has been carried out systematically in [36]. After imposing both the geometric and symmetry conditions, the following results emerge: In $f(\mathbb{T})$ gravity, there exists a unique spatially flat connection compatible with the FLRW metric, denoted Γ^0, which is presented in equation (5.10). If spatial curvature is permitted, four distinct classes of connections arise, all of which converge to Γ^0 in limit $\kappa \to 0$. In all cases, the connection is completely determined by the metric, indicating that no additional degrees of freedom are introduced by the connection in this theory.

Connection Γ^0

$$
\begin{aligned}
&\Gamma^t{}_{\mu\nu} = \mathbf{0}^{(4\times 4)} \qquad && \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix} \\
&\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & H & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad && \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & H \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix}
\end{aligned}
\tag{5.10}
$$

In $f(\mathbb{Q})$ gravity, three spatially flat connections are compatible with the symmetry and geometric constraints. All three include a free function $\gamma(t)$, which appears as a potential degree of freedom. Whether this function represents a propagating degree of freedom depends on the behavior of the connection field equations. It has been shown [36] that for connection Γ^1, the connection field equation becomes an identity, and $\gamma(t)$ is left undetermined and can be set to zero. However, for the connections Γ^2 and Γ^3, $\gamma(t)$ remains dynamical and may correspond to an additional propagating degree of freedom.

Connection Γ^1

$$\Gamma^t{}_{\mu\nu} = \begin{pmatrix} \gamma(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{5.11}$$

Connection Γ^2

$$\Gamma^t{}_{\mu\nu} = \begin{pmatrix} \gamma + \frac{\dot\gamma}{\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & \gamma & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & \gamma & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \gamma & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \gamma & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{5.12}$$

Connection Γ^3

$$\Gamma^t{}_{\mu\nu} = \begin{pmatrix} -\frac{\dot\gamma}{\gamma} & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma r^2 & 0 \\ 0 & 0 & 0 & \gamma r^2\sin^2\theta \end{pmatrix} \qquad \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix} \tag{5.13}$$

5.2.1 Friedmann Equations for Teleparallelism

In $f(\mathbb{T})$ gravity, the imposition of homogeneity and isotropy for a spatially flat universe uniquely determines the affine connection to be that given in (5.10). For this connection, the torsion scalar is found to be:

$$\mathbb{T} = -6H^2 \tag{5.14}$$

and the boundary term is given by

$$B = -6(\dot{H} + 3H^2) \tag{5.15}$$

The components tt and rr of the $f(\mathbb{T})$ metric field equation (4.4a) give the modified Friedmann equations:

$$\frac{f}{2} - \mathbb{T} f_{\mathbb{T}} = k\rho \tag{5.16a}$$

$$-\frac{f}{2} + f_{\mathbb{T}}(\mathbb{T} - 2\dot{H}) - 2H\dot{\mathbb{T}} f_{\mathbb{T}\mathbb{T}} = kp \tag{5.16b}$$

It is often convenient to rewrite the action as $f(\mathbb{T}) = \mathbb{T} + F(\mathbb{T})$ where F encodes deviations from GR. The first Friedmann equation then becomes:

$$3H^2 = k\rho + F_{\mathbb{T}}\mathbb{T} - \frac{F}{2} \tag{5.17}$$

In comparison to the standard Friedmann equation in GR (5.3a), the function $F(\mathbb{T})$ acts as a source to the matter sector. This motivates the definition of an effective total energy density:

$$k\rho_{eff} \equiv k\rho + F_{\mathbb{T}}\mathbb{T} - \frac{F}{2} \tag{5.18}$$

so that the Friedmann equation assumes its standard form:

$$3H^2 = k\rho_{eff} \tag{5.19}$$

We may also define a dark energy density contribution as

$$\rho_{DE} = F_{\mathbb{T}}\mathbb{T} - \frac{F}{2} \tag{5.20}$$

such that $\rho_{eff} = \rho + \rho_{DE}$.

In the case of $f(\mathbb{T},B)$ the modified Friedmann equations become:

$$\frac{f}{2}-f_{\mathbb{T}}\mathbb{T}-\frac{1}{2}f_B B-3H\dot{f}_B=k\rho \tag{5.21a}$$

$$-\frac{f}{2}+f_{\mathbb{T}}\mathbb{T}+\frac{1}{2}f_B B-2\dot{H}f_{\mathbb{T}}-2H\dot{f}_{\mathbb{T}}+\ddot{f}_B=kp \tag{5.21b}$$

Assuming again the decomposition $f(\mathbb{T},B)=\mathbb{T}+F(\mathbb{T},B)$, we obtain the effective energy density:

$$3H^2=k\rho_{eff}\equiv k\rho+TF_{\mathbb{T}}-\frac{F}{2}+\frac{1}{2}F_B B+3H\dot{F}_B \tag{5.22}$$

In this framework, the dark energy contribution reads:

$$\rho_{DE}\equiv TF_{\mathbb{T}}-\frac{F}{2}+\frac{1}{2}F_B B+3H\dot{F}_B \tag{5.23}$$

Similarly, for $f(\mathbb{Q},C)$ gravity, with the choice of connection Γ^1, one finds:

$$Q=-6H^2 \tag{5.24a}$$

$$C=-6(3H^2+\dot{H}) \tag{5.24b}$$

and the Friedmann equations become:

$$\frac{f}{2}-f_{\mathbb{Q}}\mathbb{Q}-\frac{1}{2}f_C C-3H\dot{f}_C=k\rho \tag{5.25a}$$

$$-\frac{f}{2}+f_{\mathbb{Q}}\mathbb{Q}+\frac{1}{2}f_C C-2\dot{H}f_{\mathbb{Q}}-2H\dot{f}_{\mathbb{Q}}+\ddot{f}_C=kp \tag{5.25b}$$

We see that for connection Γ^0 in TG and Γ^1 in STG, the structure of the Friedmann equations in $f(\mathbb{Q},C)$ coincides with $f(\mathbb{T},B)$. This is a case of correspondence between connection Γ^0 and Γ^1. Two affine connection are said to be correspondent if the non-metricity scalar in $f(\mathbb{Q})$ has the same value as the torsion scalar for $f(\mathbb{T})$ and the equation of motion take the same form regardless of f [37].

However, for the alternative connections Γ^2 and Γ^3 additional dynamical degrees of freedom provide a richer cosmology to STG. For the spatially flat FLRW metric, the $f(\mathbb{T})$ solutions are just a subset of $f(\mathbb{Q})$.

5.3. Reconstruction

A natural approach to studying the cosmology of modified gravity theories is to postulate a specific functional form for f, derive the corresponding Friedmann equations, and solve for the Hubble parameter given a particular matter content in the universe. Under this method, cosmology on teleparallel theories has been studied in [36, 38–40].

An alternative approach is the reconstruction method. This procedure begins with a predetermined Hubble function $H(t)$, representing a particular phase of cosmological evolution, and works backward to deduce the form of the gravitational Lagrangian that admits such a solution. That is, the field equations are inverted to determine which class of f-theories give rise to the desired cosmological dynamics.

This method was pioneered by Nojiri and Odintsov [41], who showed that for any given scale factor $a(t)$ and matter composition of the universe there exists a corresponding $f(R)$ theory that admits $a(t)$ as a solution. Reconstruction of ΛCDM in the context of $f(R)$ was investigated in [42]. The reconstruction method has since been extended to a variety of modified theories, including scalar-tensor theories [43], Gauss-Bonnet gravity [44] and more recently, to TG and STG: [45–51].

In this section, we first illustrate the reconstruction method for $f(R)$ gravity and subsequently adapt it to the $f(\mathbb{T})$ and $f(\mathbb{Q})$ gravity. In the cases where the connection does not introduce new degrees of freedom, we shall see that the reconstruction leads to results similar to those of General Relativity. This motivates the exploration of more general theories that include dynamical connections or theories with boundary terms.

5.3.1 Reconstruction of ΛCDM in $f(R)$ gravity

The general reconstruction procedure is as follows: given a desired $a(t)$, one computes the relevant scalar quantity (R, $\mathbb{T}$ or $\mathbb{Q}$), as a function of the $a(t)$, and inverts this relation to express the scale factor as a function of the scalar. Substituting this into the modified Friedmann equation transforms it into a differential equation for the function f. Solving this equation yields the class of gravitational Lagrangians that reproduce the chosen cosmological behavior.

In many cases, the reconstructed equations are not analytically solvable or result in very complex Lagrangians, restricting their usefulness beyond the background cosmological scale.

We exemplify this method by reconstructing ΛCDM in $f(R)$ gravity for a universe filled only with pressureless matter, following [42].

Using the $f(R)$ field equations for the FLRW metric and perfect fluid energy momentum-tensor we get the modified Friedmann Equations:

$$\frac{f}{2} - 3f_R(\dot{H} + H^2) + 3H\dot{R}f_{RR} = k\rho \tag{5.26a}$$

$$\frac{f}{2} - f_R(\dot{H} + 3H^2) + \ddot{R}f_{RR} + (\dot{R})^2 f_{RRR} + 2H\dot{R}f_{RR} = -kp \tag{5.26b}$$

The ΛCDM equation is:

$$3H^2 = k\rho_M a^{-3} + \Lambda \tag{5.27}$$

Differentiating with respect to time gives:

$$6H\dot{H} = (-3)k\rho_M a^{-4}\dot{a} \implies \dot{H} = -\frac{k}{2}\rho_M a^{-3} \tag{5.28}$$

The Ricci scalar for the FLRW metric is:

$$R = 6(\dot{H} + 2H^2) \tag{5.29}$$

Substituting the previous relations for H and $\dot{H}$ we get:

$$R(a) = k\rho_M a^{-3} + 4\Lambda \tag{5.30}$$

which implies, when inverted:

$$a(R) = \left[\frac{k\rho_M}{R - 4\Lambda}\right]^{\frac{1}{3}} \tag{5.31}$$

We also find:

$$H^2 = \frac{R}{3} - \Lambda \tag{5.32a}$$

$$\dot{H} = -\frac{R}{2} + 2\Lambda \tag{5.32b}$$

The first Friedmann Equation in $f(R)$ gravity is:

$$-3f_R(\dot{H} + H^2) + \frac{1}{2}f + 3H\dot{R}f_{RR} = k\rho \tag{5.33}$$

Using the above expressions to rewrite all quantities (H,$\dot{H}$, and ρ) in terms of R, we obtain the reconstruction equation:

$$-3(R - 3\Lambda)(R - 4\Lambda)f_{RR} + \left(\frac{R}{2} - 3\Lambda\right)f_R + \frac{1}{2}f = R - 4\Lambda \tag{5.34}$$

The general solution to this equation is:

$$f = c_1 F(\alpha_+, \alpha_-; \frac{-1}{2}; x) + c_2 x^{\frac{3}{2}} F(\beta_+, \beta_-; \frac{5}{2}; x) + R - 2\Lambda \tag{5.35}$$

where $x = -3 + \frac{R}{\Lambda}$, $\alpha_\pm = \frac{-7\pm\sqrt{73}}{12}$, $\beta_\pm = \frac{-11\pm\sqrt{73}}{12}$ and F is the Gauss Hypergeometric Function, gives as series expansion as:

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \tag{5.36}$$

where $(a)_n$ is the Pochhammer's symbol defined by:

$$(a)_0 = 1; \quad (a)_{n>0} = a(a+1)...(a+n-1) \tag{5.37}$$

The physical condition $R > 4\Lambda$ coming form the positivity of the scale factor implies $x > 1$ but convergence of the hypergeometric series restricts $|x| < 1$. These conditions are incompatible unless the integration constants vanish: $c_1 = c_2 = 0$. Thus, the only consistent solution is:

$$f(R) = R - 2\Lambda \tag{5.38}$$

which is General Relativity with a cosmological constant. No non-trivial $f(R)$ reproduces the exact ΛCDM background for a universe filled with dust-like matter.

We note that, although only the first Friedmann equation was used explicitly during the reconstruction procedure, the second Friedmann equation was implicitly used via the conservation equation.

5.3.2 Reconstruction in $f(\mathbb{T})$ gravity

We want to reproduce the effect of the cosmological constant in $f(\mathbb{T})$ gravity. Using the ΛCDM differential equation (5.5) and the value of the scale factor $\mathbb{T} = -6H^2$, we can write:

$$k\rho = 3H^2 - \Lambda = -\frac{\mathbb{T}}{2} - \Lambda \tag{5.39}$$

and the first Friedmann equation becomes:

$$\frac{f}{2} - \mathbb{T} f_{\mathbb{T}} = -\frac{\mathbb{T}}{2} - \Lambda \tag{5.40}$$

Introducing the function F previous defined as $f(\mathbb{T}) = \mathbb{T} + F(\mathbb{T})$, the Friedmann equation can be written as:

$$\mathbb{T} F_{\mathbb{T}} - \frac{F}{2} = \Lambda \tag{5.41}$$

This equation could have been immediately obtained from $\rho_{DE} = \Lambda$, using the definition of ρ_{DE} in (5.20). The general solution is:

$$F(\mathbb{T}) = -2\Lambda + \alpha\sqrt{-\mathbb{T}} \tag{5.42}$$

We see that the cosmological constant is still necessary and the additional term $\alpha\sqrt{-\mathbb{T}}$ corresponds to the solution of the homogeneous part of the Friedmann equation and therefore does not influence the background.

5.3.3 Reconstruction in $f(\mathbb{T}, B)$ gravity

Applying the same strategy of equating $\rho_{DE} = \Lambda$, the reconstruction equation for $f(\mathbb{T}, B)$ is:

$$TF_{\mathbb{T}} - \frac{F}{2} + \frac{1}{2}F_B B + 3H\dot{F}_B = \Lambda \tag{5.43}$$

Reconstruction cannot be straightforwardly applied to a general $F(\mathbb{T}, B)$. The reason is that the method, as introduced earlier, relies on expressing all relevant quantities as functions of a single scalar argument of the Lagrangian. This is feasible when there is only one scalar. However, in $f(\mathbb{T}, B)$, the torsion scalar and the boundary term are, in principle, independent variables. To proceed, one must restrict the functional form of $f(\mathbb{T}, B)$ so that the reconstruction equation can be treated consistently without compromising the independence of $\mathbb{T}$ and B. We will use as an ansatz:

$$F(\mathbb{T}, B) = g(\mathbb{T}) + Bh(\mathbb{T}) \tag{5.44}$$

from which we obtain, when substituting in the reconstruction equation (5.43):

$$\mathbb{T}g_{\mathbb{T}} - \frac{g}{2} + 3\mathbb{T}^2 h_{\mathbb{T}} = \Lambda \tag{5.45}$$

For any function g there is always a function h solving (5.45) that reconstructs ΛCDM. For instance, choosing the simplest case $g(\mathbb{T}) = 0$, the solution is:

$$h(\mathbb{T}) = -\frac{\Lambda}{3\mathbb{T}} + c_1 \tag{5.46}$$

The arbitrary constants c_1 is irrelevant as it will corresponds to a term $c_1 B$ in the Lagrangian, which is just a boundary term. Without it the resulting Lagrangian reads:

$$f(\mathbb{T}, B) = \mathbb{T} - \frac{\Lambda B}{3\mathbb{T}} \tag{5.47}$$

This model reproduces the effect of a cosmological constant through the coupling between B and $\mathbb{T}$ in the term $\frac{\Lambda B}{3\mathbb{T}}$.

The result generalizes for $g(\mathbb{T}) = \alpha\mathbb{T}$, yielding:

$$f(\mathbb{T}) = \alpha\mathbb{T} - \frac{B}{3}\left(\frac{\Lambda}{\mathbb{T}} + \frac{\alpha - 1}{2}\log\left(-\frac{\mathbb{T}}{\Lambda}\right)\right) \tag{5.48}$$

The chosen ansatz proved to be highly flexible, allowing for an infinite number of solutions while maintaining their relative simplicity compared to the more complex functions such as hypergeometric functions that frequently emerge from reconstruction methods. Moreover, if the function g is free of integration constants, the resulting Lagrangian will never contain a cosmological constant term.

5.3.4 Reconstruction in $f(\mathbb{Q})$ and $f(\mathbb{Q}, C)$ gravity

The above conclusions apply mutatis mutandis to STG, when using the connection Γ^1: in $f(\mathbb{Q})$ gravity one again requires an explicit cosmological constant with the correction $\sqrt{-\mathbb{Q}}$ having no effect on the background; in $f(\mathbb{Q}, C)$ the role of the cosmological constant can be mimicked by a term $\frac{C}{\mathbb{Q}}$.

From this perspective, TG and STG yield the same cosmological behavior. Consequently, the geometric identity of spacetime (torsion vs. non-metricity) remains observationally indistinguishable. This degeneracy arises due to the specific connections chosen. However, STG allows for a broader class of connections, including those that introduce additional degrees of freedom.

We present the Friedmann equations, connection field equations, and the non-metricity scalar $\mathbb{Q}$ for the dynamical connections Γ^2 and Γ^3.

For connection Γ^2, the non-metricity scalar is:

$$\mathbb{Q} = \frac{3}{2}(6h\gamma + 2\dot{\gamma} - 4H^2) \tag{5.49}$$

The Friedmann Equations are:

$$\frac{3}{4}f_{\mathbb{Q}}(-6H\gamma - 2\dot{\gamma} + 8H^2) + \frac{1}{2}(3\gamma f_{\mathbb{Q}\mathbb{Q}}\dot{\mathbb{Q}} + f) = k\rho \tag{5.50a}$$

$$\frac{f_{\mathbb{Q}}}{4}(18H\gamma + 6\dot{\gamma} - 8\dot{H} - 24H^2) - \frac{1}{2}(f_{\mathbb{Q}\mathbb{Q}}\dot{\mathbb{Q}}(4H - 3\gamma) + f) = kp \tag{5.50b}$$

The only non-identical connection equation is:

$$-\frac{3\gamma}{4}(f_{\mathbb{QQ}}\dot{\mathbb{Q}}6H + 2f_{\mathbb{QQQ}}\dot{\mathbb{Q}}^2 + 2f_{\mathbb{QQ}}\ddot{\mathbb{Q}}) = 0 \tag{5.51}$$

For connection Γ^3, the non-metricity scalar and the metric and connection field equations are:

$$\mathbb{Q} = \frac{3}{2}(\gamma 6H + 2\dot{\gamma} - 4H^2) \tag{5.52a}$$
$$\frac{3}{4}f_{\mathbb{Q}}(6H\gamma + 2\dot{\gamma} - 8H^2) + \frac{1}{2}(3\gamma f_{\mathbb{QQ}}\dot{Q} - f) = -k\rho \tag{5.52b}$$
$$\frac{f_{\mathbb{Q}}}{4}(18H\gamma + 6\gamma - 8\dot{H} - 24H^2) - \frac{1}{2}(f_{\mathbb{QQ}}\dot{Q}(4H - \gamma) + f) = kp \tag{5.52c}$$
$$\frac{3\gamma}{4}(f_{\mathbb{QQ}}\dot{Q}10H) + 2f_{\mathbb{QQQ}}\dot{Q}^2 + 2f_{\mathbb{QQ}}\ddot{Q}) + 3\dot{\gamma}f_{\mathbb{QQ}}\dot{Q} = 0 \tag{5.52d}$$

These equations were first derived in [52]. Since then, analytical and numerical solutions have been explored in [53, 54], the stability of the solutions was investigated in [55] and observation constraints studied in [56] and [57].

We now prove a general result applicable to both dynamical connections, illustrating their capacity to reconstruct cosmological evolution.

Given a function $f(\mathbb{Q})$, if there is a constant $\mathbb{Q}_0$ such that:

$$f'(\mathbb{Q}_0) = 1 \quad \text{and} \quad \mathbb{Q}_0 - f(\mathbb{Q}_0) > 0 \tag{5.53}$$

then in a universe filled with pressureless matter, the Friedmann equation becomes:

$$3H^2 = k\rho + \Lambda_{eff} \tag{5.54}$$

with $\Lambda_{eff} = \frac{\mathbb{Q}_0 - f(\mathbb{Q}_0)}{2}$.

We demonstrate this for connection Γ^2. The proof proceeds identically for Γ^3.

First, if we choose $\mathbb{Q} = \mathbb{Q}_0$ then the connection field equation (5.51) is trivially satisfied, regardless of f and γ.

From (5.49), we express γ in terms of $\mathbb{Q}_0$ and H:

$$\frac{2}{3}\mathbb{Q}_0 + 4H^2 = 6H\gamma + 2\dot{\gamma} \tag{5.55}$$

Substituting into the Friedmann equation (5.50a),we get:

$$\frac{3}{4}f'(\mathbb{Q}_0)(-\frac{2}{3}\mathbb{Q}_0+4H^2)+\frac{1}{2}f=k\rho \tag{5.56}$$

This simplifies to:

$$\frac{f}{2}-\frac{f'(\mathbb{Q}_0)\mathbb{Q}_0}{2}+3H^2f'(\mathbb{Q}_0)=k\rho \tag{5.57}$$

Using $f'(\mathbb{Q}_0)=1$, the equation becomes:

$$3H^2=k\rho+\frac{\mathbb{Q}_0-f(\mathbb{Q}_0)}{2} \tag{5.58}$$

Thus, the function $f(\mathbb{Q})$ reproduces ΛCDM evolution with an effective cosmological constant $\Lambda_{eff}=\frac{\mathbb{Q}_0-f(\mathbb{Q}_0)}{2}$. This matches the effective cosmological constant derived previously in (4.7b) for constant scalar solutions. Hence, this reconstruction result can be viewed as a direct consequence of that more general property. However, this reconstruction is only possible when the chosen connection admits a constant $\mathbb{Q}$ without constraining the Hubble parameter. In particular, such reconstruction cannot be achieved using connection Γ^1. In that case, constant $\mathbb{Q}$ implies constant H, which corresponds to de Sitter space. Therefore, the full ΛCDM behavior cannot be recovered. This conclusion is made explicit by the Friedmann equation for $f(\mathbb{Q})$ gravity with constant $\mathbb{Q}$ under connection Γ^1, which reads:

$$\frac{f(\mathbb{Q}_0)}{2}-\mathbb{Q}_0f'(\mathbb{Q}_0)=k\rho \tag{5.59}$$

Since the left-hand side is constant, the energy density ρ must also be constant. However, energy conservation in an expanding universe implies that ρ must decay over time unless it is zero. Therefore, for connection Γ^1, constant $\mathbb{Q}$ is incompatible with ΛCDM.

On the other hand, for connection Γ^2 and Γ^3, the presence of an extra degree of freedom allows one to maintain $\mathbb{Q}$ constant, without constraining H.

In summary, multiple modified theories can exactly reproduce the background dynamics of ΛCDM expansion: distinguishing these theories from General Relativity requires studying perturbations such as the growth of structure, gravitational wave propagation, and anisotropies in the CMB. Theories which coincide at the background level may yield observationally distinct signatures at the perturbative level.

Moreover, these theories must also pass local tests, a topic which we address in the next chapter.

6. Solar System Tests

GR has been remarkably successful in describing gravitational phenomena within the Solar System. A cornerstone of this success is the Schwarzschild solution, which is the unique spherically symmetric and static vacuum solution to the Einstein field equations. This uniqueness is formally established by the Birkhoff theorem [58], which states that any spherically symmetric vacuum solution in GR must be the Schwarzschild solution, even if not static a priori. Therefore, any viable modification to GR must reproduce the Schwarzschild solution (or a spacetime observationally indistinguishable from it) in regimes where it has been tested with high precision.

The Schwarzschild metric is:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{1 - \frac{2GM}{r}} dr^2 + r^2 d\Omega^2 \tag{6.1}$$

and its linearized form given by:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2 \tag{6.2}$$

We can parametrize deviations from GR by introducing the Eddington parameter γ into the linearized metric, leading to the more general form:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \gamma\frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2 \tag{6.3}$$

For the metric (6.3), the bending of light is given by a generalization of equation (1.5):

$$\delta\theta = \frac{1+\gamma}{2}\frac{4GM}{d}\frac{1+cos\Phi}{2} \tag{6.4}$$

Precise measurements of light deflection thus offer stringent constraints on γ and, by extension, on deviations from GR. General Relativity predicts $\gamma = 1$, a value strongly supported by current experimental data. For instance, Very Long Baseline Interferometry yields $\gamma - 1 = (-1.6 \pm 1.5) \times 10^{-4}$ [59], while time-delay measurements from the Cassini spacecraft report $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ [60].

Many modified gravity theories, particularly those proposed to explain the observed acceleration of the Universe without invoking dark energy, struggle to recover the correct Solar System phenomenology. A notable example is the model proposed by Carroll et al.

[31], in which the gravitational action is given by:

$$f(R) = R - \frac{\mu^4}{R} \tag{6.5}$$

This theory successfully predicted an accelerated cosmological expansion. However, it was later shown by Erickcek et al. [61] that it fails to reproduce Solar System observations.

Originally, Carroll et al. argued that the vacuum field equations in their theory admitted the SdS solution. Given that the effective cosmological constant would be many orders of magnitude too small to affect gravitational dynamics at Solar System scales, they concluded that the resulting spacetime would be observationally equivalent to the Schwarzschild solution. However, Erickcek et al. demonstrated that this conclusion was incorrect.

Although the SdS metric is indeed a spherically symmetric vacuum solution in this theory, it is not the unique solution. To determine the physically relevant solution, one must match the vacuum exterior to a solution for the interior of a matter distribution, such as a star. Erickcek et al. showed that when this matching is performed, the theory predicts $\gamma = \frac{1}{2}$.

In this section, we perform a similar analysis for the class of modified gravity theories introduced in the previous chapter, by determining the vacuum spherically symmetric and static solutions. We begin by presenting the most general spherically symmetric and static affine connection, consistent with the geometric constraints of the theory and discuss the criteria that guide the selection of such connections. Finally, we solve the field equations and investigate whether the SdS solution arises as a result, discussing whether the method developed by Erickcek et al. is relevant or whether the field equations fully determine the exterior spacetime from the outset.

6.1. Symmetry Reduction of the Connection

Using the same approach described in Section 5.2, we now present the most general static and spherically symmetric metric and connection compatible with the geometric postulates of teleparallel gravity theories, as classified in [52].

The most general static and spherically symmetric metric, allowing for potential off-diagonal terms, is given by:

$$ds^2 = g_{tt}dt^2 + g_{tr}dtdr + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\theta\theta}\sin^2\theta d\phi^2 \tag{6.6}$$

The imposition on the connection to be static and spherically symmetric, under the geometric postulates of STG (flat and torsionless connection), leads to two inequivalent solution families for the connection, as detailed in [52].

Solution 1

The first class of solutions admits the following non-vanishing components for the affine connection:

$$\Gamma^{t}{}_{\mu\nu} = \begin{pmatrix} c & \Gamma^{\phi}{}_{r\phi} & 0 & 0 \\ \Gamma^{\phi}{}_{r\phi} & \Gamma^{t}{}_{rr} & 0 & 0 \\ 0 & 0 & -\frac{1}{c} & 0 \\ 0 & 0 & 0 & -\frac{\sin^2\theta}{c} \end{pmatrix} \qquad \Gamma^{r}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Gamma^{r}{}_{rr} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\Gamma^{\theta}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & \Gamma^{\phi}{}_{r\phi} & 0 \\ c & \Gamma^{\phi}{}_{r\phi} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \quad \Gamma^{\phi}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & \Gamma^{\phi}{}_{r\phi} \\ 0 & 0 & 0 & \cot\theta \\ c & \Gamma^{\phi}{}_{r\phi} & \cot\theta & 0 \end{pmatrix} \tag{6.7}$$

The undetermined components Γ^{t}_{rr}, Γ^{r}_{rr}, and $\Gamma^{\phi}_{r\phi}$ are constrained by the following relation:

$$\partial_r \Gamma^{\phi}{}_{r\phi} = c\Gamma^{t}{}_{rr} + \Gamma^{\phi}{}_{r\phi}(\Gamma^{r}{}_{rr} - \Gamma^{\phi}{}_{\rho\phi}) \tag{6.8}$$

Solution 2

$$\Gamma^{t}{}_{\mu\nu} = \begin{pmatrix} k - c - c(2c-k)\Gamma^{t}{}_{\theta\theta} & \frac{(2c-k)\Gamma^{t}{}_{\theta\theta}(1+c\Gamma^{t}{}_{\theta\theta})}{\Gamma^{r}{}_{\theta\theta}} & 0 & 0 \\ \frac{(2c-k)\Gamma^{t}{}_{\theta\theta}(1+c\Gamma^{t}{}_{\theta\theta})}{\Gamma^{r}{}_{\theta\theta}} & \Gamma^{t}{}_{rr} & 0 & 0 \\ 0 & 0 & \Gamma^{t}{}_{\theta\theta} & 0 \\ 0 & 0 & 0 & \Gamma^{t}{}_{\theta\theta}\sin^2\theta \end{pmatrix}$$
$$\Gamma^{r}{}_{\mu\nu} = \begin{pmatrix} -c(2c-k)\Gamma^{t}{}_{\theta\theta} & c(2c-k)\Gamma^{t}{}_{\theta\theta} + c & 0 & 0 \\ c(2c-k)\Gamma^{t}{}_{\theta\theta} + c & \Gamma^{r}{}_{rr} & 0 & 0 \\ 0 & 0 & \Gamma^{r}{}_{\theta\theta} & 0 \\ 0 & 0 & 0 & \Gamma^{r}{}_{\theta\theta}\sin^2\theta \end{pmatrix} \tag{6.9}$$
$$\Gamma^{\theta}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & c & 0 \\ 0 & 0 & \frac{-c\Gamma^{t}{}_{\theta\theta}-1}{\Gamma^{r}{}_{\theta\theta}} & 0 \\ c & \frac{-c\Gamma^{t}{}_{\theta\theta}-1}{\Gamma^{r}{}_{\theta\theta}} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \Gamma^{\phi}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & \frac{-c\Gamma^{t}{}_{\theta\theta}-1}{\Gamma^{r}{}_{\theta\theta}} \\ 0 & 0 & 0 & \cot\theta \\ c & \frac{-c\Gamma^{t}{}_{\theta\theta}-1}{\Gamma^{r}{}_{\theta\theta}} & \cot\theta & 0 \end{pmatrix}$$

These components satisfy the differential constraints:

$$\partial_r \Gamma^t{}_{\theta\theta} = -\frac{\Gamma^t{}_{\theta\theta}}{\Gamma^r{}_{\theta\theta}}[1 + \Gamma^t{}_{\theta\theta}(3c - k + (2c - k)] - \Gamma^t{}_{rr}\Gamma^r{}_{\theta\theta} \tag{6.10a}$$

$$\partial_r \Gamma^r{}_{\theta\theta} = -1 - c\Gamma^t{}_{\theta\theta}(2 + (2c - k)\Gamma^t{}_{\theta\theta}) - \Gamma^r{}_{rr}\Gamma^r{}_{\theta\theta} \tag{6.10b}$$

The general metric (6.6) can be brought into the diagonal form

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 \tag{6.11}$$

via an appropriate coordinate transformation. While such a diffeomorphism generally modifies the affine connection, it preserves the structure of the two solution classes: each solution class is mapped onto itself under the transformation. This means that one can consistently study $f(\mathbb{Q})$ field equations using the diagonal metric (6.11) together with either Solution 1 or Solution 2 for the connection.

In section 3.4, we have shown that there is always a coordinate system where the connection vanishes. These coordinate system were worked out in [62] for both Solution 1 and 2. They also showed that the metric tensor written in that coordinate systems has a much more complicated form, as it now encodes all the information previously contained in the connection. They conclude that nothing is gained by using the coincident gauge.

These connections can be further constrained using the metric field equations. The off-diagonal rt component of the metric field equation (4.10) as the form:

$$f_{\mathbb{Q}\mathbb{Q}}\partial_r \mathbb{Q} P^r{}_{rt} = 0 \tag{6.12}$$

Since $\mathbb{Q}$ being constant leads to General Relativity, the only way to get new solutions is if the equation is an identity, choosing a connection for which $P^r{}_{rt} = 0$.

It has been proven in [52] that this is impossible for Solution 1 which therefore yields only GR solutions. However, Solution 2 allows for this condition to be satisfied when additional constraints are imposed. Two distinct branches of viable connections emerge:

Branch 1:

$$\Gamma^t{}_{\theta\theta} = \frac{k}{2c(2c-k)} \quad \Gamma^t{}_{rr} = \frac{k(8c^2 + 2ck - k^2)}{8c^2(2c-k)^2(\Gamma^r{}_{\theta\theta})^2} \quad c \neq 0; k \neq 2c \tag{6.13}$$

Branch 2:

$$\Gamma^t{}_{rr} = -\frac{\Gamma^t{}_{\theta\theta}}{(\Gamma^r{}_{\theta\theta})^2} \quad c = k = 0 \tag{6.14}$$

A particular solution within Branch 2, which is fully determined by the metric and is consistent with the connection field equations, is given by setting:

$$\Gamma^{r}{}_{\theta\theta} = \pm \frac{r}{\sqrt{g_{rr}}} \tag{6.15}$$

These connections (6.15) were used to study static spherically symmetric solutions beyond GR in the theory $f(\mathbb{Q}) = \mathbb{Q} + \alpha \mathbb{Q}^n - 2\Lambda$ in [63]. Static, spherically symmetric solutions in the coincident gauge were analyzed in [64] and [65], where it was found that the exterior solutions reduce to those of GR.

In $f(\mathbb{T})$ gravity, the most general flat, metric-compatible spherically symmetric and static connection is:

$$\begin{aligned}
\Gamma^{t}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial_r g_{tt}}{2g_{tt}} & \Gamma^{t}_{rr} & 0 & 0 \\ 0 & 0 & \Gamma^{t}_{\theta\theta} & \frac{\Gamma^{r}_{\theta\phi}}{\Gamma^{r}_{\theta\theta}} \sin\theta \Gamma^{t}_{\theta\theta} \\ 0 & 0 & -\frac{\Gamma^{r}_{\theta\phi}}{\Gamma^{r}_{\theta\theta}} \sin\theta \Gamma^{t}_{\theta\theta} & \sin^2\theta \Gamma^{t}_{\theta\theta} \end{pmatrix} \\
\Gamma^{r}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{g_{tt}}{g_{rr}} \Gamma^{t}_{rr} & \frac{\partial_r g_{rr}}{2g_{rr}} & 0 & 0 \\ 0 & 0 & \Gamma^{r}_{\theta\theta} & \Gamma^{r}_{\theta\phi} \\ 0 & 0 & -\Gamma^{r}_{\theta\phi} & \sin^2\theta \Gamma^{r}_{\theta\theta} \end{pmatrix} \\
\Gamma^{\theta}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & \Gamma^{\theta}_{r\phi} \\ -\frac{g_{tt}}{r^2} \Gamma^{t}_{\theta\theta} & -\frac{g_{rr}}{r^2} \Gamma^{r}_{\theta\theta} & 0 & 0 \\ -\frac{g_{tt}}{r^2} \Gamma^{t}_{\phi\theta} & \frac{g_{rr}}{r^2} \Gamma^{r}_{\theta\phi} & 0 & -\cos\theta \sin\theta \end{pmatrix} \\
\Gamma^{\phi}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sin^2\theta} \Gamma^{\theta}_{r\phi} & \frac{1}{r} \\ -\frac{g_{tt}}{\sin^2\theta r^2 \Gamma^{r}_{\theta\theta}} \Gamma^{r}_{\theta\phi} \Gamma^{t}_{\theta\theta} & -\frac{g_{rr}}{\sin^2\theta r^2} \Gamma^{r}_{\theta\phi} & 0 & \cot\theta \\ -\frac{g_{tt}}{r^2} \Gamma^{t}_{\theta\theta} & -\frac{g_{rr}}{r^2} \Gamma^{r}_{\theta\theta} & \cot\theta & 0 \end{pmatrix}
\end{aligned} \tag{6.16}$$

The components that were not specified are given by:

$$\Gamma^t_{rr} = \pm \frac{r({\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2)\partial_r g_{rr} - 2g_{rr}({\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2 - r\Gamma^r_{\theta\theta}\partial_r\Gamma^r_{\theta\theta} - r\frac{1}{\sin^2\theta}\Gamma^r_{\theta\phi}\partial_r\Gamma^r_{\theta\phi})}{2r\sqrt{g_{tt}}\sqrt{{\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2}\sqrt{r^2 - g_{rr}({\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2)}} \tag{6.17a}$$

$$\Gamma^t_{\theta\theta} = \pm \frac{\Gamma^r_{\theta\theta}\sqrt{r^2 - g_{rr}({\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2)}}{\sqrt{g_{tt}}\sqrt{{\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2}} \tag{6.17b}$$

$$\Gamma^\theta_{r\phi} = \frac{\Gamma^r_{\theta\theta}\partial_r\Gamma^r_{\theta\phi} - \Gamma^r_{\theta\phi}\partial_r\Gamma^r_{\theta\theta}}{{\Gamma^r_{\theta\theta}}^2 + \frac{1}{\sin^2\theta}{\Gamma^r_{\theta\phi}}^2} \tag{6.17c}$$

The only free components of the connection are $\Gamma^r{}_{\theta\theta}$ and $\Gamma^r{}_{\theta\phi}$.

Again using the non-diagonal rt component of the metric field equation (4.5), it has proven in [52] that the only way to have solution beyond GR is by setting

$$\Gamma^r{}_{\theta\theta} = \pm\frac{r}{\sqrt{g_{rr}}} \quad \Gamma^r{}_{\theta\phi} = 0 \tag{6.18}$$

So the only non-trivial connections $\Gamma^\pm$ in $f(\mathbb{T})$ are:

$$\Gamma^t{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial_r g_{tt}}{2g_{tt}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\partial_r g_{rr}}{2g_{rr}} & 0 & 0 \\ 0 & 0 & \pm\frac{r}{g_{rr}} & 0 \\ 0 & 0 & 0 & \pm\frac{r}{g_{rr}}\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ c & \mp\frac{g_{rr}}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \mp\frac{g_{rr}}{r} & \cot\theta & 0 \end{pmatrix} \tag{6.19}$$

These connections are analogous to the connections (6.15) found in $f(\mathbb{Q})$. In fact, the resulting field equations are identical in both frameworks (again this is a case of correspondence, as we found in cosmology). Using this connection, approximate solutions beyond General Relativity were derived, and their implications for classical solar system tests (such as light bending, Shapiro time delay, perihelion precession, and gravitational redshift) were thoroughly investigated in [66–69].

it

6.2. Physical Viability of the Connection

When these connections were first determined in [52], one of the main motivations was to find new black hole solutions that depart from GR. However, here we would like to keep the standard SdS solution in order to keep our theories consistent with solar system observation. Using the above analysis, it seems that there is an infinite number of connections that could do so. However, we will argue that the choice of connection cannot be just made in order to recover the SdS but it must obey certain physical criteria.

Bahamonde et al. [62] point out a problem with the connections from STG that also extends to standard TG. It is reasonable to expect that in the Minkowski limit all effects of gravity should vanish. In particular, both the torsion scalar and torsion tensor are expected to be zero in this limit.

Computing the torsion scalar for the connections $\Gamma^{\pm}$ (6.19) we get:

$$\mathbb{T}^{\pm} = \left(1 \pm \sqrt{g_{rr}}\right) \frac{2}{r^2 g_{rr}} \left(\frac{r\partial_r g_{tt}}{g_{tt}} + 1 \pm \sqrt{g_{rr}} \right) \tag{6.20}$$

In the case of the Minkowski metric:

$$\mathbb{T}^{+} = \frac{8}{r^2} \qquad \mathbb{T}^{-} = 0 \tag{6.21}$$

We see that for connection Γ^{+}, the value of $\mathbb{T}$ for the Minkowski metric does not vanish. Only Γ^{-} satisfies the natural expectation that the torsion will vanish in the Minkowski limit. It is then plausible that only this second connection is physically meaningful.

Nevertheless, Bahamonde et al. also caution that the vanishing of torsion in the Minkowski limit may not be strictly necessary. Since matter follows Levi-Civita geodesics, particles are sensitive only to the metric, not to the affine structure. Therefore, as long as the metric is flat, any non-trivial geometry in the background may not be detectable.

However, when considering the cosmological limit, the adequacy of these connections must be re-examined. The de Sitter metric, when expressed in FLRW coordinates, is manifestly homogeneous and isotropic. Nonetheless, a coordinate transformation reveals that it is also spherically symmetric and static. In FLRW coordinates, the de Sitter metric takes the form:

$$ds^2 = -dT^2 + \exp(2HT)[dR^2 + R^2 d\Omega^2] \tag{6.22}$$

while in static Schwarzschild-like coordinates it reads:

$$ds^2 = -(1 - H^2r^2)dt^2 + \frac{1}{1 - H^2r^2}dr^2 + r^2 d\Omega^2 \tag{6.23}$$

These two forms are related via the coordinate transformation:

$$r = \exp(HT)R \tag{6.24a}$$

$$t = T - \frac{1}{2H}\log(1 - H^2r^2) \tag{6.24b}$$

Since de Sitter metric possesses both symmetries (homogeneity/isotropy and spherical/static), it is natural to expect that the homogeneous and isotropic connection used in cosmology should, under this coordinate transformation, match one of the spherically symmetric and static connections. In particular, the torsion scalar should remain invariant across these representations of the same space. From the cosmological analysis, we found that $\mathbb{T} = -6H^2$, which correctly vanishes in the Minkowski limit. So the connection Γ^+ which did not have the correct Minkowski limit also does not have the correct de Sitter limit. One concrete manifestation of this inconsistency is that the term $\sqrt{-\mathbb{T}}$ that comes from cosmological reconstruction is not even defined for connection Γ^+ as the torsion scalar background value is positive. While Γ^- avoids this issue in the Minkowski limit, it too becomes problematic in the de Sitter regime.

If we compute the torsion scalar for Γ^-, in the de Sitter metric as a series in H^2, we find:

$$\mathbb{T} = \frac{5}{2}H^4r^2 + \mathcal{O}(H^6r^4) \tag{6.25}$$

which is incompatible with $\mathbb{T} = -6H^2$.

We will formalize this concept in the following definition: a spherically symmetric and static connection is said to be de Sitter compatible if, when the metric is the de Sitter metric, the connection can be obtained from a homogeneous and isotropic connection via the coordinate transformation 6.24. Because the de Sitter space, exhibits both symmetries, it is reasonable to expect such a connection to exist.

The cosmological connection Γ^0 for de Sitter spacetime, expressed in FLRW coordinates, is given by:

$$\Gamma^{t}{}_{\mu\nu} = \mathbf{0}^{(4\times4)} \qquad \Gamma^{r}{}_{\mu\nu} = \begin{pmatrix} 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R & 0 \\ 0 & 0 & 0 & -R\sin^2\theta \end{pmatrix}$$
$$\Gamma^{\theta}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & H & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & \frac{1}{R} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^{\phi}{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & H \\ 0 & 0 & 0 & \frac{1}{R} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{R} & \cot\theta & 0 \end{pmatrix} \tag{6.26}$$

where H is constant. We now apply the coordinate transformation (6.24) to this connection. Connections transform non-homogeneously following the transformation rule:

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = \partial_\beta x^\alpha \partial_\mu x^\rho \partial_\nu x^\sigma \Gamma^{\beta}_{\rho\sigma} + \partial_\lambda x^\alpha \partial_\mu \partial_\nu x^\lambda \tag{6.27}$$

Applying this to the connection (6.26), we obtain the transformed connection in Schwarzschild coordinates:

$$\begin{aligned}
\Gamma^{t}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{-H^2 r}{1-H^2r^2} & \frac{-H}{(1-H^2r^2)^2} & 0 & 0 \\ 0 & 0 & \frac{-Hr^2}{1-H^2r^2} & 0 \\ 0 & 0 & 0 & \frac{-Hr^2}{1-H^2r^2}\sin^2\theta \end{pmatrix} \\
\Gamma^{r}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -H & \frac{H^2 r}{1-H^2r^2} & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix} \\
\Gamma^{\theta}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ -H & \frac{1}{r}\frac{1}{1-H^2r^2} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \\
\Gamma^{\phi}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ & \frac{1}{r}\frac{1}{1-H^2r^2} & 0 & \cot\theta \\ -H & \frac{1}{r}\frac{1}{1-H^2r^2} & \cot\theta & 0 \end{pmatrix}
\end{aligned} \tag{6.28}$$

In particular, we have $\Gamma^{r}_{\theta\theta} = -r$ and $\Gamma^{r}_{\theta\phi} = 0$. These are the two degrees of freedom that define the most general static and spherically symmetric connection in TG (6.16). If we substitute them in connection (6.16), we obtain a de Sitter compatible connection:

$$
\begin{aligned}
\Gamma^{t}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\partial_r g_{tt}}{2g_{tt}} & -\frac{\partial_r g_{rr}}{2\sqrt{-g_{tt}}\sqrt{g_{rr}-1}} & 0 & 0 \\ 0 & 0 & -r\frac{\sqrt{g_{rr}-1}}{\sqrt{-g_{tt}}} & 0 \\ 0 & 0 & 0 & -r\sin^2\frac{\sqrt{g_{rr}-1}}{\sqrt{-g_{tt}}} \end{pmatrix} \\
\Gamma^{r}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{\partial_r g_{rr}}{2g_{rr}}\frac{\sqrt{-g_{tt}}}{\sqrt{g_{rr}-1}} & \frac{\partial_r g_{rr}}{2g_{rr}} & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & & -r\sin^2\theta \end{pmatrix} \\
\Gamma^{\theta}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ -\frac{\sqrt{-g_{tt}}\sqrt{g_{rr}-1}}{r} & \frac{g_{rr}}{r} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \\
\Gamma^{\phi}{}_{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ -\frac{\sqrt{-g_{tt}}\sqrt{g_{rr}-1}}{r} & \frac{g_{rr}}{r} & \cot\theta & 0 \end{pmatrix}
\end{aligned}
\tag{6.29}
$$

For the de Sitter metric, this connection (6.29) reproduces the transformed connection (6.28).

The torsion scalar for this connection (6.29) reads:

$$\mathbb{T} = -\frac{2}{r^2 g_{rr} g_{tt}}[r(1-g_{rr})\partial_r g_{tt} + g_{tt}(1-g_{rr}-r\partial_r g_{rr})] \tag{6.30}$$

which evaluates to $\mathbb{T} = -6H^2$ in de Sitter spacetime, confirming consistency.

As discussed earlier, the connections $\Gamma^{\pm}$ are the only ones in $f(\mathbb{T})$ that allow deviations from GR. The requirement of a de Sitter compatible connection imposes that $\mathbb{T}$ must be constant, inevitably leading to the SdS solution.

Another way to see that this de Sitter compatible connection does not lead to anything different from GR is by solving the linearized field equations for a perturbed metric:

$$ds^2 = [-1 + H^2r^2 + A(r)]dt^2 + [1 + H^2r^2 + B(r)]dt^2dr^2 + r^2d\Omega^2 \tag{6.31}$$

For this connection, the torsion scalar to linear order in the perturbations is

$$\mathbb{T} = -6H^2 - \frac{2B'}{r} - \frac{2B}{r^2} \tag{6.32}$$

Since this scalar has to be constant, $B'r + B = 0$ implying $B \sim \frac{1}{r}$. Substituting this scalar in the vacuum field equations (4.5), the tt will be an identity and the rr component, to first order in the perturbations, becomes $A'r + B = 0$ which implies $A(r) = B(r)$ and thus $\gamma = 1$.

However, this restriction does not necessarily apply to $f(\mathbb{T}, B)$ gravity. While the considerations made for de Sitter compatibility are theory-independent, the exclusivity of $\Gamma^{\pm}$ in enabling deviations from GR is a particular feature of $f(\mathbb{T})$.

In $F(\mathbb{T}, B)$ gravity, the geometric and symmetry postulates are unchanged, leading to the same class of admissible connections as in $f(\mathbb{T})$. The field equations for $f(\mathbb{T}, B)$ gravity are:

$$G_{\mu\nu} f_T - \frac{1}{2} g_{\mu\nu}(f - f_T T - f_B B) + 2\partial_\alpha (f_T + f_B) S_{(\mu\nu)}{}^{\alpha} + \nabla_\mu \nabla_\nu f_B - g_{\mu\nu} \Box f_B = k T_{\mu\nu} \tag{4.13}$$

Evaluating the rt component of the field equations using the de Sitter compatible connection yields:

$$\partial_r (f_{\mathbb{T}} + f_B) \frac{\sqrt{g_{rr} - 1}}{\sqrt{-g_{tt}} r} = 0 \tag{6.33}$$

For this choice of connection, this is not an identity, and therefore imposes the constraint:

$$\partial_r (f_{\mathbb{T}} + f_B) = 0 \tag{6.34}$$

But now, contrary to what happens in $f(\mathbb{T})$ gravity, where this equation implied $\partial_r \mathbb{T} = 0$ and this for its turn implied that we cannot have solutions beyond GR, this equation can be satisfied without the scalars $\mathbb{T}$ and B necessarily being constant. Instead, it simply requires a relation between them, allowing for more general solutions beyond SdS.

We now consider the specific model:

$$f(\mathbb{T}, B) = \mathbb{T} - H^2 \frac{B}{\mathbb{T}} \tag{6.35}$$

where H is constant.

For this theory, equation (6.34) becomes:

$$1 + H^2 \frac{B}{\mathbb{T}^2} - \frac{H^2}{\mathbb{T}} = \beta. \tag{6.36}$$

where β is a constant.

We now consider the most general static and spherically symmetric perturbation around de Sitter spacetime:

$$ds^2 = [-1 + H^2r^2 + A(r)]dt^2 + [1 + H^2r^2 + B(r)]dt^2dr^2 + r^2d\Omega^2 \tag{6.31}$$

The unperturbed value of the torsion scalar is $\mathbb{T} = -6H^2$. We define a new function:

$$c(r) \equiv \frac{1}{6} + \frac{H^2}{\mathbb{T}} \tag{6.37}$$

This function $c(r) << 1$ measures the departure of the torsion scalar from the unperturbed value. We expect that $c(r) \to 0$ as we depart from the source of the perturbation and that it can be treated as a perturbation of the same order as $A(r)$ and $B(r)$.

Inverting the definition (6.37), we can write the perturbed torsion to linear order in c.

$$\mathbb{T} = -6H^2 - 36H^2c \tag{6.38}$$

Inserting this relation in condition (6.36) and solving for B we get:

$$B = -6H^2[1 - 6(\beta - 1) + 6c(1 - 12(\beta - 1))] \tag{6.39}$$

To recover the de Sitter value $B = -18H^2$ when we turn off the perturbation, we must have $\beta = \frac{2}{3}$. Then the perturbed scalars can all be written in terms of the function $c(r)$:

$$\mathbb{T} = -6H^2(1 + 6c) \tag{6.40a}$$

$$B = -18H^2(1 + 10c) \tag{6.40b}$$

$$R = 12H^2(1 + 12c) \tag{6.40c}$$

The trace of the field equation (4.13) is:

$$-Rf_T - 2(f - f_T T - f_B B) - 3\Box f_B = kT \tag{6.41}$$

For our model $f(\mathbb{T}, B) = \mathbb{T} - H^2 \frac{B}{\mathbb{T}}$, the trace equation reads:

$$-R\left(1+H^2\frac{B}{\mathbb{T}^2}\right)+2H^2\frac{B}{\mathbb{T}}+3\Box\frac{H^2}{\mathbb{T}}=kT \tag{6.42}$$

Inserting the perturbed scalars from (6.40) in equation (6.42) and linearizing in $c(r)$, we find:

$$\nabla^2 c(r)=\frac{kT}{3} \tag{6.43}$$

where ∇^2 is the flat-space Laplacian operator.

We assume that the de Sitter space is perturbed by the presence of pressureless matter, case in which $T=-\rho$. Using that $k=8\pi G$, the equation becomes:

$$\nabla^2 c(r)=-\frac{8\pi G\rho}{3} \tag{6.44}$$

Defining the enclosed mass $m(r)=\int_0^r 4\pi r'^2\rho(r')dr'$, we integrate to find:

$$\frac{dc}{dr}=-\frac{2}{3}G\frac{m(r)}{r^2} \tag{6.45}$$

Outside the matter distribution $r>R_\odot$, the enclosed mass is a constant $m(r)=M$. Therefore for $r>R_\odot$ we obtain:

$$\frac{dc}{dr}=-\frac{2}{3}G\frac{M}{r^2}\implies c(r)=\frac{2}{3}\frac{GM}{r} \tag{6.46}$$

This is a vacuum solution but depends on the interior matter distribution and matches the interior solution at the boundary, reinforcing that matching conditions determine the correct solution. If instead we had assumed $\rho=0$, then the equation $\nabla^2 c=0$ would admit any solution of the form $c(r)=\frac{c_1}{r}$, in particular the case $c=0$, corresponding to the SdS solution.

Inserting the perturbed metric (6.31) and scalar quantities (6.40) into the the tt and rr components of the field equations (4.13), we obtain the linearized equations:

$$\frac{B+B'r}{2r^2}-\nabla^2 c=k\rho \tag{6.47a}$$

$$4c'r=B+A'r \tag{6.47b}$$

Using (6.44) in (6.47a) we get:

$$B+B'r=\frac{4k\rho r^2}{3} \tag{6.48}$$

This leads to:

$$\partial_r(rB) = \frac{4k\rho r^2}{3} \implies B(r) = \frac{8GM}{3r} \tag{6.49}$$

Using this result together with (6.46) in (6.53b) we get:

$$-\frac{8GM}{3r} = \frac{8GM}{3r} + A'r \implies A(r) = \frac{16}{3}\frac{GM}{r} \tag{6.50}$$

The solution is:

$$ds^2 = \left(-1 + H^2r^2 + \frac{16GM}{3r}\right)dt^2 + \left(1 + H^2r^2 + \frac{8GM}{3r}\right)dr^2 \tag{6.51}$$

We observe that this solution does not reproduce the Newtonian limit $A(r) = \frac{2GM}{r}$, although this discrepancy can be circumvented by redefining the gravitational constant k. However, the main issue lies in the fact that it predicts $\gamma = \frac{1}{2}$.

If we had naively set $\rho = 0$, the trace equation would lead to:

$$\nabla^2 c = 0 \implies c(r) = \frac{c_1}{r} \tag{6.52}$$

c_1 is an arbitrary constant that can be determined with the boundary conditions, but by making $\rho = 0$ we lost that information. Now the field equations read:

$$B + B'r = 0 \tag{6.53a}$$

$$-4\frac{c_1}{r} = B + A'r \tag{6.53b}$$

The first equation implies that $B(r) = \frac{c_2}{r}$ and plugging this in the second we get $A = \frac{4c_1+c_2}{r}$. The Eddington parameter for this solution is

$$\gamma = \frac{1}{4\frac{c_1}{c_2} + 1} \tag{6.54}$$

In particular, for $\rho = 0$, the field equations admit $c_1 = 0$ as a solution, which corresponds to $\gamma = 1$.

Birkhoff's theorem is lost in this theory and there may be several spherically symmetric vacuum solutions. However, the Solar System spacetime is determined uniquely by matching the exterior vacuum solution to the interior solution. When this is done correctly, the theory predicts:

$$\gamma = \frac{1}{2} \tag{6.55}$$

in conflict with Solar System tests, which require γ to be extremely close to unity as predicted by GR.

We can generalize the previous results to the theory defined by the Lagrangian:

$$f(\mathbb{T}) = \alpha\mathbb{T} - \frac{B}{3}\left(\frac{\Lambda}{\mathbb{T}} + \frac{\alpha-1}{2}\log\left(-\frac{\mathbb{T}}{\Lambda}\right)\right) \tag{5.48}$$

which also reconstructs the ΛCDM background.

Applying the method developed above to this Lagrangian yields the perturbed static spherically symmetric solution:

$$ds^2 = \left(-1 + H^2r^2 + \frac{16GM}{3\alpha r}\right)dt^2 + \left(1 + H^2r^2 + \frac{8GM}{3\alpha r}\right) \tag{6.56}$$

This solution again leads to an Eddington parameter $\gamma = \frac{1}{2}$ independently of α, in violation of Solar System tests.

This method assumes that the scalars are constant to first order and that the function $c(r)$ appears in second order. If we calculate the torsion scalar $\mathbb{T}$, the boundary scalar B and the Ricci scalar R for the metric 6.31, to first order in the perturbations:

$$\mathbb{T} = -6H^2 - \frac{2B'}{r} - \frac{2B}{r^2} \tag{6.57a}$$

$$B = -18H^2 - \frac{4B'}{r} - \frac{4B}{r^2} - A'' - \frac{2}{r}A' \tag{6.57b}$$

$$R = 12H^2 + \frac{2B'}{r} + \frac{2B}{r^2} + A'' + \frac{2}{r}A' \tag{6.57c}$$

This is consistent with the expressions (6.40) because if $A(r) \sim \frac{1}{r} \sim B(r)$ then the effect of the perturbations exactly cancel in first order.

Why is the matching onto the interior solution not relevant in the $f(\mathbb{T})$ case? This can be understood by examining the role of the function $c(r)$. In $f(\mathbb{T}, B)$ gravity, this function enters the field equations through the D'Alembertian term $\Box f_B$. In contrast, in $f(\mathbb{T})$ gravity the D'Alembertian term is absent. As a result, the function $c(r)$ does not enter the field equations and can be consistently set to zero. Besides, in $f(\mathbb{T})$ gravity, de Sitter compatibility of the connection implied a constant torsion scalar and therefore there is no choice but to make $c = 0$.

To emphasize the importance of using a de Sitter compatible connection, we will now solve the linearized equations in $f(\mathbb{T}, B)$ for the Γ^+ connection which is not de Sitter compatible.

For this connection, the perturbed scalars are:

$$\mathbb{T} = \frac{8}{r^2}(1 - \frac{B}{2} - \frac{rA'}{2}) \tag{6.58a}$$

$$B = \frac{2}{r^2}(4 - \frac{A''}{2}r^2 - B'r - 3B - 3rA') \tag{6.58b}$$

and the field equations are:

$$\frac{f}{2} - f_{\mathbb{T}}\left(\frac{\mathbb{T}}{2} + \frac{1}{r^2 g_{rr}} - \frac{1}{r^2} - \frac{\partial_r g_{rr}}{r g_{rr} g_{rr}}\right) - \frac{f_B B}{2} - \frac{2}{r g_{rr}}\partial_r(f_{\mathbb{T}} + f_B)(1 + \sqrt{g_{rr}}) + \frac{\partial_r^2 f_B}{g_{rr}} + \frac{\partial_r f_B}{g_{rr}}\left(\frac{2}{r} - \frac{\partial_r g_{rr}}{2 g_{rr}}\right) = k\rho \tag{6.59a}$$

$$-\frac{f}{2} + f_{\mathbb{T}}\left(\frac{\mathbb{T}}{2} + \frac{1}{r}\frac{\partial_r g_{tt}}{g_{tt} g_{rr}} + \frac{1}{g_{rr} r^2} - \frac{1}{r^2}\right) + \frac{f_B B}{2} - \frac{\partial_r f_B}{g_{rr}}\left(\frac{\partial_r g_{tt}}{2 g_{tt}} + \frac{2}{r}\right) = kp \tag{6.59b}$$

When we substitute the ansatz metric (6.31) and the linearized scalars in the field equations, and assuming dust-like matter:

$$B + A'r + 2H^2r^2 = 0 \tag{6.60a}$$

$$B'r + B + \frac{7}{4}H^2r^2 = k\rho \tag{6.60b}$$

The exterior solution is $B(r) = \frac{2GM}{r} - \frac{7H^2r^2}{12}$ and $A(r) = \frac{2GM}{r} - \frac{17}{24}H^2r^2$ and the metric is:

$$ds^2 = [-1 + \frac{2GM}{r} + \frac{7}{24}H^2r^2]dt^2 + [1 + \frac{2GM}{r} + \frac{5}{12}H^2r^2] + r^2 d\Omega^2 \tag{6.61}$$

While this metric recovers the correct behavior in the $r \to 0$ limit, corresponding to the Schwarzschild solution, it fails to asymptotically approach the de Sitter metric as $r \to \infty$. This discrepancy reflects the fact that the connection Γ^+ is not de Sitter compatible.

For $f(\mathbb{Q})$ gravity, the cosmological connection in flat FLRW spacetime was found to be:

$$\Gamma^t{}_{\mu\nu} = \mathbf{0}^{(4\times 4)} \qquad \Gamma^r{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -R & 0 \\ 0 & 0 & 0 & -R\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{R} & 0 \\ 0 & \frac{1}{R} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \qquad \Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{R} & \cot\theta & 0 \end{pmatrix} \tag{5.11}$$

Applying the coordinate transformation that relates the FLRW and static spherically symmetric representations of de Sitter spacetime, we obtain:

$$\Gamma^t{}_{\mu\nu} = \begin{pmatrix} \frac{H^3r^2}{1-H^2r^2} & \frac{-H^2r}{(1-H^2r^2)^2} & 0 & 0 \\ \frac{-H^2r}{(1-H^2r^2)^2} & \frac{-H+2H^3r^2}{(1-H^2r^2)^3} & 0 & 0 \\ 0 & 0 & \frac{-Hr^2}{1-H^2r^2} & 0 \\ 0 & 0 & 0 & \frac{-Hr^2}{1-H^2r^2}\sin^2\theta \end{pmatrix}$$
$$\Gamma^r{}_{\mu\nu} = \begin{pmatrix} H^2r & \frac{-H}{1-H^2r^2} & 0 & 0 \\ \frac{-H}{1-H^2r^2} & \frac{H^2r(2-H^2r^2)}{(1-H^2r^2)^2} & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}$$
$$\Gamma^\theta{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & -H & 0 \\ 0 & 0 & \frac{1}{r}\frac{1}{1-H^2r^2} & 0 \\ -H & \frac{1}{r}\frac{1}{1-H^2r^2} & 0 & 0 \\ 0 & 0 & 0 & -\cos\theta\sin\theta \end{pmatrix} \tag{6.62}$$
$$\Gamma^\phi{}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -H \\ 0 & 0 & 0 & \frac{1}{r}\frac{1}{1-H^2r^2} \\ 0 & 0 & 0 & \cot\theta \\ -H & \frac{1}{r}\frac{1}{1-H^2r^2} & \cot\theta & 0 \end{pmatrix}$$

This rules out the connection from solution 1 (6.7), as in that case $\Gamma^t{}_{\theta\theta}$ is constant.

If we plug $c = k = -H$, $\Gamma^r{}_{\theta\theta} = -r$ and $\Gamma^t{}_{\theta\theta} = \frac{-Hr^2}{1-H^2r^2}$ in Solution 2, the connection is equal to 6.62, confirming that Solution 2 is de Sitter compatible.

Even though Solution 1 is not de Sitter compatible, it cannot generate any solutions that deviate from General Relativity for any choice of the theory. Choosing this connection results in the SdS metric independently of the form of $f(\mathbb{Q})$, and thus leaves the free parameters of the theory completely unconstrained. In contrast, Solution 2 allows for deviations from GR, but only under specific conditions. For instance, Option 1 is again ruled out, as it would require $\Gamma^t{}_{\theta\theta}$ to be constant. Option 2 implies a relation $\Gamma^t{}_{rr} = -\frac{\Gamma^t{}_{\theta\theta}}{(\Gamma^r{}_{\theta\theta})^2}$ which is not satisfied by the de Sitter compatible connection. Therefore, no deviation from the SdS solution arises here either.

We conclude that for $f(\mathbb{Q})$, de Sitter compatibility of the connection leads to the SdS solution, independently of the theory. In particular, for $f(\mathbb{Q}) = \mathbb{Q} + \alpha\sqrt{-\mathbb{Q}} - 2\Lambda$, this solution does not depend on the parameter α. The parameter is not constrained either by cosmology or Solar System tests.

In the case of $f(\mathbb{Q}, C)$ gravity, the method previously applied to $f(\mathbb{T}, B)$ gravity is still applicable. The derivation only references the connection in the non-diagonal components of the field equations. These either vanish identically for some connection choices (which were never de Sitter compatible) or enforce a constraint $\partial_r(f_\mathbb{Q} + f_C) = 0$. The connection does not influence the remaining derivation in any other way. It will just depend on the Lagrangian and the metric ansatz. Therefore, the results we found in $f(\mathbb{T}, B)$ gravity apply to $f(\mathbb{Q}, C)$ as well.

Despite $f(\mathbb{Q})$ having more connections available, it behaves very similarly to $f(\mathbb{T})$. When choosing a connection that is de Sitter compatible, every $f(\mathbb{Q})$ and $f(\mathbb{T})$ theory leads to SdS metric. In particular, $f(\mathbb{Q}) = \mathbb{Q} + \alpha\sqrt{-\mathbb{Q}} - 2\Lambda$ is indistinguishable from General Relativity with a Cosmological Constant for any value of α and analogously for $f(\mathbb{T})$. Consequently, Occam's razor would suggest that these extensions are redundant unless additional evidence is found in cosmological perturbations.

Theories that use boundary terms to reconstruct a cosmological constant predict an Eddington parameter $\gamma = \frac{1}{2}$ when a de Sitter compatible connection is used.

7. Conclusion

In this work, we have investigated whether modified gravity theories within the Geometric Trinity of Gravity can reproduce a ΛCDM-like cosmological expansion while remaining compatible with Solar System tests, where GR is highly successful. In particular, we examined the implications of such theories for light bending through the computation of the Eddington parameter γ, which is tightly constrained by observations.

We focused on modified theories constructed from non-Riemannian geometries, which generalize the geometric foundations of GR by including torsion and non-metricity. Specifically, we considered extensions of the three dynamically equivalent formulations — General Relativity , TEGR and STEGR — by promoting their Lagrangians to functions of the respective scalars: $f(R), f(\mathbb{T}), f(\mathbb{Q})$ and the two extensions $f(\mathbb{T}, B)$ and $f(\mathbb{Q}, C)$.

In these non-Riemannian formulations, the connection is not determined by the metric and must be fixed independently, by imposing symmetry requirements. This freedom enables a richer landscape of connections and potentially introduces new degrees of freedom.

For cosmological studies, we considered homogeneous, isotropic, and spatially flat spacetimes and explored their compatible connections, determined in [36]. In TG, there is a unique such connection (Γ^0), whereas in the STG there are three connections (Γ^1, Γ^2, Γ^3), with the latter two supporting dynamical degrees of freedom.

To study the background cosmology, we applied the reconstruction method, which constructs the gravitational Lagrangian based on a given cosmological behavior stemming from a specific choices of $H(t)$.

We have shown that:

- For $f(R); f(\mathbb{T}); f(\mathbb{Q})$ the reconstruction of the ΛCDM model requires an explicit cosmological constant. However, $f(\mathbb{T})$ and $f(\mathbb{Q})$ allow for additional terms ($\sqrt{-\mathbb{T}}$ and $\sqrt{-\mathbb{Q}}$), which do not affect the background evolution, as already proven in [42, 46]
- For the extension $f(\mathbb{T}, B)$ we introduced the ansatz $f(\mathbb{T}, B) = f_1(\mathbb{T}) + Bf_2(\mathbb{T})$ and and showed that infinitely many pairs (f_1, f_2) can reconstruct ΛCDM. For instance, the model $f(\mathbb{T}, B) = \mathbb{T} - \alpha\frac{B}{T}$ yields and effective cosmological constant $\Lambda_{eff} = 3\alpha$.

- For connection Γ_1, $f(\mathbb{Q}, C)$ gives the same results as $f(\mathbb{T}, B)$. In fact, $f(\mathbb{T}, B)$ is a subset of $f(\mathbb{Q}, C)$.

- For the dynamical connections Γ^2 and Γ^3, we showed that ΛCDM reconstruction is possible even for constant $\mathbb{Q}$ thanks to the presence of extra degrees of freedom. In contrast, for Γ^1, constant $\mathbb{Q}$ implies constant H, and thus only de Sitter expansion is allowed.

We then analyzed the Solar System behavior of these reconstructed theories, aiming to compute spherically symmetric and static solutions and evaluate the Eddington parameter γ. Following the methodology of [61], we emphasized the importance of matching the exterior vacuum solution to the interior solution of the Sun, and of perturbing around a de Sitter background.

However, we identified a key issue: many connections presented in the literature are not consistent with a de Sitter background. To address this, we introduced the concept of a de Sitter compatible connection, defined as a static, spherically symmetric connection that can be obtained from a homogeneous and isotropic connection through the appropriate coordinate transformation of de Sitter spacetime. We demonstrated that such connections exist in both TG and STG.

With this notion in place, we proved that:

- in $f(\mathbb{T})$ (or $f(\mathbb{Q})$), a de Sitter compatible connection implies constant $\mathbb{T}$ (or $\mathbb{Q}$). The field equations then reduce to those of GR with a cosmological constant. The resulting solution is the SdS metric, and the theory predicts $\gamma = 1$, independently of the function f.

- in $f(\mathbb{T}, B)$ (or $f(\mathbb{Q}, C)$), even with de Sitter compatible connections $\mathbb{T}$ (or $\mathbb{Q}$) need not be constant. Applying the matching procedure to the reconstructed theory $f(\mathbb{T}, B) = \mathbb{T} - \alpha \frac{B}{T}$ (and its STG analog), we found that it yield $\gamma = \frac{1}{2}$, which is inconsistent with observations.

These results are summarized in Table 7.1.

Regarding the distinction among the three types of geometry and the identification of the most promising geometric mediator of gravity, our results reaffirm the ambiguity between

Theory	Reconstruction	Has Cosmological Constant?	γ
$f(R)$	$R-2\Lambda$	Yes	1
$f(\mathbb{T})$	$\mathbb{T}-\alpha\sqrt{-\mathbb{T}}-2\Lambda$	Yes	1
$f(\mathbb{Q})$	$\mathbb{Q}-\alpha\sqrt{-\mathbb{Q}}-2\Lambda$	Yes	1
$f(\mathbb{T},B)$	$\mathbb{T}-\frac{\Lambda B}{3T}$	No	$\frac{1}{2}$
$f(\mathbb{Q},C)$	$\mathbb{Q}-\frac{\Lambda Q}{3C}$	No	$\frac{1}{2}$

Figure 7.1: Summary of reconstruction and observational viability of different modified gravity models considered in this work.

them. In particular, when restricted to single-variable theories all three require the introduction of an explicit cosmological constant to reproduce ΛCDM evolution and when extended to include boundary terms they present the same solutions.

The fundamental difference between these theories seems to be the flexibility displayed by the connection. From this perspective, STG seems to be the most promising of the three as it admits a wider variety of connections, that could enrich its phenomenology, as we have seen in cosmology. However, most of those connections may be ruled out by physical criteria, as showed for the spherically symmetric connections.

7.1. Future Prospects

In the study of background cosmology within modified gravity theories, three prominent methodologies have emerged in the literature: reconstruction, the Noether symmetry approach, and dynamical systems analysis. In this work, our focus has been on the reconstruction method. This approach can be further developed by exploring alternative ansatz for the boundary-term theories or by considering more general extensions of the geometric trinity, including non-minimal couplings to matter, of the form:

$$S=\int d^4x\sqrt{-g}\left(f_1(R)+f_2(R)\mathcal{L}_M\right) \tag{7.1}$$

and analogously for theories based on torsion or non-metricity [70–72].

In our analysis, we proposed an ansatz that allowed for an infinite family of Lagrangians capable of reproducing ΛCDM dynamics at the background level. We then examined one representative model in detail and found it to be incompatible with Solar System constraints due to deviations in the Eddington parameter. A natural extension of this work would be to systematically analyze the remaining models within the same ansatz to identify whether any satisfy both cosmological and local constraints.

Regarding the other approaches, although the Noether symmetry method and dynamical systems techniques have already been applied to $f(T,B)$ [73–78], their application to $f(Q,C)$, the more recent of the two, remains relatively unexplored.

Finally, the homogeneous square-root terms that arise in $f(T)$ and $f(Q)$ reconstructions remain unconstrained by Solar System tests, as they do not affect the background nor the weak-field limit. Cosmological perturbation theory thus offers the most promising avenue for detecting or constraining such terms, and future work should aim to explore their imprints on large-scale structure and gravitational wave propagation.

Ultimately, the quest to understand the true geometry of gravity continues and with it, the promise of new physics at the intersection of geometry, cosmology, and observation.

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