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Numerical Semigroups: a conjecture of Wilf and related topics

Neeraj Kumar

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UC|UP Joint PhD Program in Mathematics
Faculty of Sciences of the University of Porto and Faculty of Sciences and Technology of the University of Coimbra
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PhD

Numerical Semigroups: a conjecture of Wilf and related topics

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Thesis carried out as part of the UC|UP Joint PhD Program in Mathematics

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The Supervisor,

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Abstract

This work consists of the study of the enumeration problem of numerical semigroups and Wilf's conjecture. We introduce a new way of counting numerical semigroups, namely by their maximum primitive, and discuss its various properties. Given a positive integer n, let A_n denote the number of numerical semigroups with maximum primitive n and let N_n denote the number of numerical semigroups with Frobenius number n. We show in particular the monotonic behavior of the sequences (A_{2n}) and (A_{2n+1}). We also discuss various bounds for A_n and its asymptotic growth. We establish that the sequences (A_n) and (N_n) are Möbius transforms of each other, and that they are asymptotically equal. We also study counting by Frobenius number and improve the existing lower bound for N_n. We address a problem posed by Eliahou and Delgado concerning the asymptotic proportion of numerical semigroups S which have at least $\mathrm{m}(S)/3$ primitives under counting by the Frobenius number.

We discuss several special cases of Wilf's conjecture. We provide a simpler proof of Wilf's conjecture for numerical semigroups of depth 2. We also discuss an alternative solution to the Frobenius problem for the numerical semigroups generated by arithmetic progressions, and give a direct verification of Wilf's conjecture for this class. We also establish that almost all numerical semigroups with large enough maximum primitive satisfy Wilf's conjecture.

Resumo

Este trabalho consiste no estudo do problema de enumeração de semigrupos númericos e da conjectura de Wilf. Introduzimos uma nova forma de contar semigrupos númericos, nomeadamente pelos seus primitivos máximos, e discutimos as suas várias propriedades. Dado um inteiro positivo n, sejam A_n o número de semigrupos com primitivo máximo n, e N_n o número de semigrupos com número de Frobenius n. Mostramos, em particular, a monotonia das sucessões (A_{2n}) e (A_{2n+1}). Discutimos vários limitantes de A_n e o crescimento assimptótico de A_n. Estabelecemos que as sucessões (A_n) e (N_n) são iguais assimptoticamente, e que uma é a transformada de Möbius da outra. Estudamos também a contagem por número de Frobenius e melhoramos o limitante inferior de N_n. Discutimos também um problema mencionado por Eliahou e Delgado sobre a proporção assimptótica de semigrupos númericos S que têm pelo menos $m(S)/3$ primitivos, quando contados por número de Frobenius.

Estudamos vários casos específicos da conjectura de Wilf. Damos uma prova mais simples da conjectura de Wilf para semigrupos números de profundidade 2. Discutimos uma solução alternativa do problema de Frobenius para semigrupos números gerados por progressões aritméticas, e damos uma verificação imediata da conjectura de Wilf para esta classe. Estabelecemos finalmente que quase todos os semigrupos números com primitivo máximo suficientemente grande satisfazem a conjectura de Wilf.

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Conventions

We assume the following conventions throughout this work.

We denote the set of integers by $\mathbb{Z}$, the set of nonnegative integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. Unless specified otherwise, the term *number* refers to an element of $\mathbb{N}$.

We denote the group $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$.

Given a set $A \subset \mathbb{N}$, $\langle A \rangle$ denotes the set of all the linear combinations of elements of A with coefficients in $\mathbb{N}$. Moreover, for $k \in \mathbb{N}$ and $a = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$, $\langle a \rangle$ denotes $\langle \{a_1, a_2, \ldots, a_k\} \rangle$.

We denote a sequence by the infinite tuple $(a_1, a_2, \ldots)$ or (a_i) with index i varying over $\mathbb{N}$.

Unless specified otherwise, the term *semigroup* will be used interchangeably with *numerical semigroup*[1].

Given a set X its power set will be denoted by 2^X.

Given $f \in \mathbb{N}$ we denote $\left\lfloor \frac{f-1}{2} \right\rfloor$ by $\bar{f}$.

Given sets $A, B \subset \mathbb{Z}$ we denote the set $\{a+b : a \in A, b \in B\}$ by $A+B$. In particular, $A+A$ is denoted by $2A$. Iteratively, we define nA as $A + (n-1)A$ for all $n > 1$.

Given $n \in \mathbb{Q}$ and $B \subseteq \mathbb{Z}$ we define $n \cdot B := \{nb : b \in B\}$. Note that the set $n \cdot B$ is not always the same as nB.

Given $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$ we define $n + A := \{n\} + A$, and $A + n := n + A$. For instance, we denote the set of positive integers by $\mathbb{N} + 1$.

We write $[n]$ to denote the set $\{1, 2, \ldots, n\}$.

Let $A \subset \mathbb{N}$ be a finite set. We denote its cardinality by $|A|$.

Unless specified otherwise, the intervals we consider will be the integer intervals in $\mathbb{N}$ i.e. given $a, b \in \mathbb{N}$, the interval $[a, b]$ denotes the set $\{a, a+1, \ldots, b-1, b\}$. Analogously we have $[a, b)$, $(a, b]$ and (a, b). Moreover we denote the infinite sets $\{a, a+1, \ldots\}$ and $\{a+1, a+2, \ldots\}$ by $[a, \infty)$ and (a, ∞) respectively, and refer to them as infinite intervals.

[1] see Definition 1.1

If a set B is the union of a finite set $\{b_1, b_2, \ldots, b_k\}$ and an infinite interval $[b, \infty)$, for some $b > b_i$ for all $i \leq k$, then we write $B = \{b_1, b_2, \ldots, b_k, b, \rightarrow\}$.

We usually denote the subsets of $\mathbb{N}$ by uppercase letters A, B, etc., and the elements of these sets by the lowercase letters a, b, etc. In particular we reserve the letter S to denote numerical semigroups.

A set $A \subset \mathbb{Z}$ such that $A = \{a + kd : k \in [0, n]\}$ for some $a, d \in \mathbb{Z}$ and $n \in \mathbb{N} + 1$, with $\gcd(a, d) = 1$, will be called an *arithmetic sequence* or *arithmetic progression*.

Given a positive real number a and function $f : \mathbb{N} \to \mathbb{N}$, we will write $f(a)$ to denote $f(\lceil a \rceil)$ unless specified otherwise. For instance we may take f to be the factorial function, which can additionally be used to define expressions like $\binom{a}{\sqrt{2a}}$, $\binom{a}{a/3}$, etc.

For functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f(x) = o(g(x))$ or $f = o(g)$ when $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$ and $g(x) \neq 0$ for all $x > N$, for some $N \in \mathbb{N}$.

For functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f(x) = O\big(g(x)\big)$ or $f = O\big(g\big)$ if there exists a positive real number M, and a nonnegative integer x_0 such that

$$|f(x)| \leq M\, |g(x)| \quad \text{for all } x \geq x_0\ .$$

Let F_n denote the n-th *Fibonacci number* defined using the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

with $F_0 = 0$ and $F_1 = 1$.

Chapter 1

Introduction

Definition 1.1. *A numerical semigroup S is a subset of $\mathbb{N}$ which satisfies the following axioms:*

1. $0 \in S$.
2. *S is additively closed, i.e.* $2S \subseteq S$.
3. $\mathbb{N} \setminus S$ *is finite.*

We note that the first and second axiom can be summarized by the axiom $S = 2S$. Furthermore, given the first two axioms, we can equivalently replace the third by either of the following axioms:

$3'$. $\gcd(S) = 1$.

$3''$. For some $x \in \mathbb{N}$, we have $\{x, x+1\} \subset S$.

The notion of a numerical semigroup has its origin in a famous open problem posed by Frobenius in the 1880s, known as the *coin problem* or *Frobenius coin problem* or *Frobenius problem*.

Problem 1.2. *(Frobenius Problem) What is the largest amount that cannot be obtained using coins of a given set of denominations?*

In other words, given a set $A = \{a_1, \cdots, a_n\} \subseteq \mathbb{N}$ the problem asks for a formula for the largest number $\mathrm{F}(A)$ which cannot be expressed as a linear combination of elements of A over $\mathbb{N}$.

Remark 1.3. Note that the number $\mathrm{F}(A)$ only exists when $\gcd(A) = 1$, since if $\gcd(A) = d$ for some $d \neq 1$ then no element of the unbounded set $d\mathbb{N} + 1$ can be expressed as a linear combination of elements of A over $\mathbb{N}$.[1]

Let $\mathbb{S}$ denote the set of all numerical semigroups. Throughout this section, let S be a numerical semigroup and we further assume $S \neq \mathbb{N}$ for non triviality. Similar to any algebraic structure, we have the notion of a *set of generators* for numerical semigroups as well. Given a set $A \subset \mathbb{N}$, let $\langle A \rangle$ denote the set of all $\mathbb{N}$-linear combinations of elements of A i.e.

$$\langle A \rangle := \left\{ \sum_{i \leq n} x_i a_i : n \in \mathbb{N}, a_i \in A, x_i \in \mathbb{N}, \text{ for all } i \leq n \right\}.$$

[1] This argument explains why Axiom 3 implies Axiom $3'$. The reverse implication from Axiom $3'$ to Axiom 3 is obtained by Bezout's identity for several variables.

Lemma 1.4. *Let $A \subset \mathbb{N}$ and $\gcd(A) = 1$. Then $\langle A \rangle$ is a numerical semigroup.*

Proof. $\langle A \rangle$ is closed under addition by definition, contains 0 (since $0a \in \langle A \rangle$ for $a \in A$) and $\gcd(\langle A \rangle) = \gcd(A) = 1$. □

Let S be equal to $\langle A \rangle$ for some $A \subset \mathbb{N}$. The set A in this case is called a *set of generators* of S or a *generating set* of S. Every numerical semigroup has a finite generating set.

Lemma 1.5. *Every numerical semigroup S is generated by a finite set of generators.*

Proof. We first note that for any positive integer c the set $[c, \infty)$ is generated by the set $[c, 2c-1]$. We know from Axiom 3 that S is cofinite, so there exists some $c(S)$ such that $[c(S), \infty) \subseteq S$. Thus S is generated by the finite set $S \cap [0, 2c(S)-1]$. □

Moreover if we partially order all the generating sets of S under inclusion then it is well known that there exists a unique set minimal under this ordering which is known as *the set of minimal generators of S* or *the set of primitives* of S which we denote by $\mathrm{P}(S)$.

Remark 1.6. The set of primitives $\mathrm{P}(S)$ is equal to $S^* \setminus 2S^*$, where S^* denotes $S \setminus \{0\}$. In particular, $\mathrm{P}(S)$ is finite since it is contained in the finite generating set given by Lemma 1.5.

The elements of $\mathrm{P}(S)$ are called the *minimal generators* of S or the *primitives* of S. In particular, the smallest primitive $\min(\mathrm{P}(S))$ is referred to as the *multiplicity* of S and denoted by $\mathrm{m}(S)$, and the largest primitive is referred to as the *maximum primitive* of S and denoted by $\nu(S)$. The cardinality of $\mathrm{P}(S)$ is called the *embedding dimension* of S and it is denoted $p(S)$ or $e(S)$.

Lemma 1.7. $p(S) \leq \mathrm{m}(S)$.

Proof. There can be at most one element in $\mathrm{P}(S)$ per residue class modulo $\mathrm{m}(S)$ since if we have $x = y \mod \mathrm{m}(S)$ for some $x > y$, then for some n we have $x = n\,\mathrm{m}(S) + y$ and hence $x \notin S^* \setminus 2S^*$. □

The elements of $\mathbb{N} \setminus S$ are known as the *gaps* of S, and the number of gaps of S is called the *genus* of S and denoted by $\mathrm{g}(S)$. The largest, and arguably the most important, gap of S is known as the *Frobenius number* of S and is denoted by $\mathrm{F}(S)$. We note that $\mathbb{N}$ has no gaps, and by convention we set $\mathrm{F}(\mathbb{N})$ to be -1. The number $\mathrm{F}(S)+1$ is called the conductor of S and denoted by $\mathrm{c}(S)$. The elements of S in the interval $[0, \mathrm{c}(S))$ are known as the *left elements* and the set of left elements of S is denoted by $\mathrm{L}(S)$. The set $\mathrm{L}(S) \cup \{\mathrm{F}(S)\}$ is called the *extended set of left elements* of S and is denoted by $\mathrm{L}'(S)$.

Remark 1.8. $S = (\mathrm{L}'(S) \setminus \max(\mathrm{L}'(S))) \cup [\max(\mathrm{L}'(S)) + 1, \infty)$. Thus the numerical semigroup S is determined by the set $\mathrm{L}'(S)$.

The integer $\lceil \mathrm{c}(S)/\mathrm{m}(S) \rceil$ is called the *depth* of S and denoted by $\mathrm{q}(S)$. The *primitive depth* of S is defined as $\lceil \nu(S)/\mathrm{m}(S) \rceil$ and denoted by $\pi(S)$. We next define the notion of a quotient of numerical semigroups.

Definition 1.9. *Given a numerical semigroup S and a positive integer n, we define the* quotient *of S by n as*

$$\frac{S}{n} := \{x \in \mathbb{N} : nx \in S\}.$$

We note that $\frac{S}{n}$ is a numerical semigroup (see [34, Proposition 5.1]). Moreover for $n \neq 1$, the semigroup $\frac{S}{n}$ is not the same as $\frac{1}{n} \cdot S = \left\{\frac{s}{n} : s \in S\right\}$.

Lemma 1.10. *Let S be a numerical semigroup with Frobenius number f and suppose that d is a positive divisor of* $\gcd(\mathrm{L}'(S))$. *Then $\frac{S}{d} = \frac{1}{d} \cdot \mathrm{L}(S) \cup \left(\frac{f}{d}, \infty\right)$. In particular the Frobenius number of $\frac{S}{d}$ is equal to $\frac{f}{d}$.*

Proof. Let $x \in \frac{S}{d}$, so that $dx \in S$. If $dx \in \mathrm{L}(S)$ then $x \in \frac{1}{d} \cdot \mathrm{L}(S)$. Otherwise $dx \in (f, \infty)$ in which case $x \in \left(\frac{f}{d}, \infty\right)$. Thus $x \in \frac{1}{d} \cdot \mathrm{L}(S) \cup \left(\frac{f}{d}, \infty\right)$, and therefore $\frac{S}{d} \subseteq \frac{1}{d} \cdot \mathrm{L}(S) \cup \left(\frac{f}{d}, \infty\right)$.
Conversely, let $x \in \frac{1}{d} \cdot \mathrm{L}(S) \cup \left(\frac{f}{d}, \infty\right)$. Then $dx \in S$ and thus by definition $x \in \frac{S}{d}$. Therefore $\frac{S}{d} = \frac{1}{d} \cdot \mathrm{L}(S) \cup \left(\frac{f}{d}, \infty\right)$. We finally note that f/d is not an element of $\frac{S}{d}$ since f is not an element of S, and thus the Frobenius number of $\frac{S}{d}$ is equal to $\frac{f}{d}$. □

An important open problem related to numerical semigroups is a conjectured inequality involving $p(S)$, $|\mathrm{L}(S)|$ and $\mathrm{c}(S)$, which was first asked as a question by Wilf in [38].

Conjecture 1.11. *(Wilf) For every numerical semigroup S, the following inequality holds:*

$$p(S) \cdot |\mathrm{L}(S)| \geq \mathrm{c}(S).$$

This conjecture has been proven for several special cases, some of which are discussed in Chapter 3. When the numerical semigroup S under consideration is clear from the context, we will often write m, p, F, etc., instead of $\mathrm{m}(S)$, $p(S)$, $\mathrm{F}(S)$, etc. We refer the reader to the book by Rosales and García-Sánchez [34] for a detailed background on the subject.

We next briefly describe the notion of *counting* numerical semigroups which constitutes a major part of this thesis. This is further discussed in a detailed manner in Chapter 2. Let f be a positive integer. We define

$$\mathscr{N}_f := \{S \in \mathbb{S} : \mathrm{F}(S) = f\}$$

and denote its cardinality by N_f. The study of properties of $\mathscr{N}_f$ as a function of f is referred to as the study of numerical semigroups under *counting by Frobenius number* or *counting numerical semigroups by Frobenius number*. Note that the problem of estimating N_f as a function of f is included in this as a special case. Wilf [38] posed the problem of determining the asymptotic growth of N_f which was later studied by Backelin (see [2, Proposition 1]). Moreover we have a notion of probability associated to counting by Frobenius number given by the uniform probability distribution on $\mathscr{N}_f$. Given $Q \subseteq \mathbb{S}$ we define the probability that a randomly chosen numerical semigroup in $\mathscr{N}_f$ belongs to Q as

$$\mathbb{P}_{\mathscr{N}_f}(Q) = \frac{|Q \cap \mathscr{N}_f|}{|\mathscr{N}_f|}.$$

Remark 1.12. Note that counting by Frobenius number can also be referred to as *counting by maximum gap*.

Analogously, we introduce a new way of counting numerical semigroups based on the maximum primitive of a numerical semigroup. Given a positive integer n, let

$$\mathscr{A}_n = \{S \in \mathbb{S} : \nu(S) = n\}$$

and let A_n denote the cardinality of $\mathscr{A}_n$. The study of properties of $\mathscr{A}_n$ as a function of n will be referred to as the study of numerical semigroups under *counting by maximum primitive* or *counting numerical semigroups by maximum primitive*. We have a notion of probability associated to counting by maximum primitive through the uniform probability distribution in $\mathscr{A}_n$. Given $n \neq 2$ and $Q \subseteq \mathbb{S}$, we define the probability that a randomly chosen numerical semigroup in $\mathscr{A}_n$ belongs to Q as

$$\mathbb{P}_{\mathscr{A}_n}(Q) = \frac{|Q \cap \mathscr{A}_n|}{|\mathscr{A}_n|}.$$

Remark 1.13. In the above definition we take $n \neq 2$ as $\mathscr{A}_2 = \emptyset$, see Lemma 2.31(ii).

In Chapter 2, we discuss the notion of counting numerical semigroups in further detail. In Section 2.1.1 we describe the most common ways of counting- namely, counting by Frobenius number, counting by genus, and box counting. Among these we particularly focus on counting by Frobenius number. We provide the following improvement to Backelin's lower bound for N_n (from [2]).

Proposition 1.14. *For $n > 12$, we have*

$$N_n \in \left(2.06 \times 2^{\lfloor \frac{n-1}{2} \rfloor}, 4 \times 2^{\lfloor \frac{n-1}{2} \rfloor}\right).$$

Moreover, if n is a positive even number, then $N_n < N_{n+1}$.

In Section 2.2 we discuss counting by maximum primitive in detail. We prove some basic properties of A_n, and compare it with N_n. For small values of n these are given in Table 2.1. For the general case, we prove the following result.

Proposition 1.15. *For every $n > 1$, $A_n < N_n$. Moreover, for a prime p, $A_p = N_p - 1$.*

Therefore in particular N_n is an upper bound for A_n and this bound is tight when n is a prime. Let F_n denote the n-th *Fibonacci number* (see Conventions). We obtain the following lower bound for A_n.

Proposition 1.16. *Let $n \geq 3$ be an integer. Then*

$$A_n \geq 2^{\lfloor \frac{n-1}{2} \rfloor} - F_{\lfloor \frac{n-1}{2} \rfloor + 1}.$$

Moreover we show that A_n and N_n, viewed as functions of n, are asymptotically equivalent.

Theorem 1.17. *Let $A, N : \mathbb{N} \to \mathbb{N}$ be given by $A(n) = A_n$ and $N(n) = N_n$ when $n > 0$ and set $A(0) = N(0) = 0$. Then we have*

$$A(n) = N(n) - 2^{\left(\frac{1}{4} + o(1)\right)n}.$$

As a consequence of this and a result of Backelin (see [2, Proposition 1]) we obtain the following corollary.

Corollary 1.18. *The following assertion holds:*

$$\lim_{n\to\infty}\frac{A_n}{N_n}=1.$$

The subsequences (A_{2n}) *and* (A_{2n+1}) *of* (A_n) *grow asymptotically as* $c_0 2^{n/2}$ *and* $c_1 2^{n/2}$ *for some positive constants* $c_0, c_1 \in \mathbb{R}$.

Analogous to a result of Singhal [36, Theorem 4], we prove the monotonicity of subsequences (A_{2n}) and (A_{2n+1}) by the following result.

Theorem 1.19. *For* $n > 2$, *we have* $A_n < N_n \leq A_{n+2}$.

We also show an equivalence between counting by maximum primitive and counting by Frobenius number. More precisely, we prove using the Möbius inversion formula that the sequences (A_n) and (N_n) are Möbius transforms of one another.

Theorem 1.20. *Let* $n > 0$. *Then*

$$N_n=\sum_{d|n}A_{\frac{n}{d}}.$$

Theorem 1.21. *Let* $n > 2$. *Then*

$$A_n=\sum_{d|n}\mu\left(\frac{n}{d}\right)\cdot N_d$$

where μ *is the Mobius function.*

In Chapter 3 we discuss Wilf's conjecture in detail. We first mention some known cases in which Wilf's conjecture has been proven. We then provide a simplified proof of a result of Kaplan [25, Proposition 26] which establishes that numerical semigroups with depth less than or equal to 2 satisfy Wilf's conjecture. We then show the following result.

Theorem 1.22. *Let* S *be a numerical semigroup such that* $|S\cap(\mathrm{m}(S),2\,\mathrm{m}(S))|\geq\sqrt{3\,\mathrm{m}(S)}$. *Then* S *satisfies Wilf's conjecture.*

We finally prove the asymptotic version of Wilf's conjecture under counting by maximum primitive. Let

$$\mathscr{W}:=\{S\in\mathbb{S}:S\text{ satisfies Wilf's conjecture}\}.$$

We show that the probability that a randomly chosen numerical semigroup in $\mathscr{A}_n$ satisfies Wilf's conjecture tends to 1 as n tends to infinity.

Theorem 1.23. *The following assertion holds:*

$$\lim_{n\to\infty}\mathbb{P}_{\mathscr{A}_n}(\mathscr{W})=\lim_{n\to\infty}\frac{|\{S\in\mathscr{A}_n:|\mathrm{P}(S)|\cdot|\mathrm{L}(S)|>\mathrm{F}(S)\}|}{|\mathscr{A}_n|}=1.$$

In Chapter 4 we discuss some experimental results and observed patterns related to the topics from Chapters 2 and 3. In particular, we pose some conjectures related to counting numerical semigroups by Frobenius number.

This work contains the content of the published article *Monotonicity of some sequences related to the counting by maximum primitive* [28], and the joint work with Manuel Delgado and Claude Marion titled *On counting numerical semigroups by maximum primitive and Wilf's conjecture* [13].

Chapter 2

Counting Numerical Semigroups

Counting numerical semigroups primarily refers to the enumeration of numerical semigroups under a specified constraint. It has, with time, also been used to discuss the study of numerical semigroups which satisfy that constraint. We see this with an example. Consider the numerical semigroups with a given positive integer f as the Frobenius number. The counting under this constraint refers to the study of problems such as: What is the number of numerical semigroups with a given Frobenius number f? What proportion of numerical semigroups S with Frobenius number f satisfy the relation $p(S) \geq \mathrm{m}(S)/3$? In particular, we are also interested in how the answers to these questions vary with change in f. Thus, to discuss the problems of such kind, we formally define the notion of counting in the following manner.

Let $(\mathscr{S}_i)$ be a sequence of sets $\mathscr{S}_i \subseteq \mathbb{S}$ such that $\bigcup \mathscr{S}_i = \mathbb{S}$. We refer to the study of properties of this sequence as *counting numerical semigroups* by this sequence. Usually such sequences are obtained from the fibers of some natural functions from $\mathbb{S}$ to $\mathbb{N}$.

Let $\mu : \mathbb{S} \to \mathbb{N}$ be a function. Corresponding to this function, we have the sequence

$$\left(\mu^{-1}(n)\right)_{n\in\mathbb{N}} = \left(\mu^{-1}(0), \mu^{-1}(1), \mu^{-1}(2)\ldots\right).$$

We will refer to the study of properties of this sequence as *counting numerical semigroups by* μ or *counting by* μ. For example, we may take μ to be the genus map $\mathrm{g} : \mathbb{S} \to \mathbb{N}$.

We consider two broad classes of properties. Firstly, we consider the cardinality of the set $\mu^{-1}(n)$ i.e. the *cardinality sequence* $(|\mu^{-1}(n)|)$. In particular, we consider the following problems: What are some good estimates for $|\mu^{-1}(n)|$? How does this number grow asymptotically as n tends to infinity? Is this sequence monotonous?

Secondly, we consider the statistical and probabilistic properties of $\mu^{-1}(n)$. In particular, this includes the following problems: What proportion of numerical semigroups in $\mu^{-1}(n)$ satisfy Wilf's conjecture as n tends to infinity? How are the properties like m, F, etc., distributed among the semigroups in $\mu^{-1}(n)$ as n tends to infinity? In particular, the properties of $\mu^{-1}(n)$ as n tends to infinity will be referred to as the *asymptotic properties* of numerical semigroups under counting by μ.

Remark 2.1. We note that for certain functions, such as the multiplicity map $\mathrm{m} : \mathbb{S} \to \mathbb{N}$, the corresponding sequence of counting contains infinite sets. For instance $\mathrm{m}^{-1}(n)$ contains $\langle\{n, p_i\}\rangle$ for any $n > 1$ and p_i the i-th prime greater than n. Therefore the sequence of cardinality gives us

$$\left(|\,\mathrm{m}^{-1}(n)|\right)_{n\in\mathbb{N}} = (0, 1, \infty, \infty, \ldots).$$

We will call such ways of counting as *trivial*. We will only consider the ways of counting which are not trivial.

We assume from now onward that $|\mu^{-1}(n)|$ is finite for all n. We assign a probability measure using the uniform distribution on the finite set $\mu^{-1}(n)$: for $S \in \mu^{-1}(n)$, whenever $\mu^{-1}(n)$ is nonempty, we define

$$\mathbb{P}_\mu(\{S\}) = \frac{1}{|\mu^{-1}(n)|}$$

and given $Q \subseteq \mathbb{S}$ we define

$$\mathbb{P}_\mu(Q) = \frac{|Q \cap \mu^{-1}(n)|}{|\mu^{-1}(n)|}.$$

In what follows we refer to a condition (inequality or equality) defined on the elements of $\mathbb{S}$ as a *property* ψ, and thus a property can equivalently be identified with the subset of $\mathbb{S}$ which satisfies it.

Definition 2.2. *A* generic *or* typical *property ψ of numerical semigroups under counting by μ refers to a property for which the proportion of semigroups in $\mu^{-1}(n)$ that satisfy ψ tend to 1 as n tends to infinity–*

$$\lim_{n\to\infty} \mathbb{P}_\mu(\{S \in \mu^{-1}(n) : S \textit{ satisfies } \psi\}) = 1.$$

This equivalently defines a *generic class* or *generic subset* in $\mathbb{S}$ under counting by μ. Alternatively, we also say that a *typical* or *generic* element of $\mu^{-1}(n)$ has the property ψ as n tends to infinity, or that a *randomly chosen* element x in $\mu^{-1}(n)$ has the property ψ almost surely as n tends to infinity. Such generic properties play an important role in the study of various other asymptotic properties.

2.1 Some known ways of counting numerical semigroups

We discuss the established ways of counting in this section. First we consider the way of counting defined by the Frobenius number map in some detail. We next mention the counting defined by the genus map. Finally, we briefly mention a different way of counting known as *box counting*, which does not arise from a natural function on $\mathbb{S}$.

2.1.1 Counting by Frobenius number

We first note that by our convention $\mathrm{F}(\mathbb{N})$ is -1, and therefore the image of Frobenius number map F is not contained in $\mathbb{N}$. Thus we consider the restriction of F to the set $\mathbb{S} \setminus \{\mathbb{N}\}$. Moreover we abuse notation and denote this restriction of the function F by the same symbol F. Thus we consider the map $\mathrm{F} : \mathbb{S} \setminus \{\mathbb{N}\} \to \mathbb{N}$ and refer to the associated way of counting as *counting numerical semigroups by Frobenius number* or *counting by Frobenius number*.

Let $\mathscr{N}_f$ denote the set of all numerical semigroups with Frobenius number f, and let N_f denote its cardinality. Thus we get the sequence $(\mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \ldots)$ which is as follows:

$$(\mathscr{N}_n) = (\emptyset, \{\langle 2,3\rangle\}, \{\langle 3,4,5\rangle\}, \{\langle 5,6,7,8\rangle, \langle 3,5,7,8\rangle\}, \ldots).$$

The corresponding sequence of cardinality is as follows:

$$(N_f) = (0,1,1,2,2,5,4,11,10,21,22,51,40,106,103,\ldots).$$

We observe that the sequence (N_f) is not monotonous. Wilf [38] posed the problem of determining how N_f grow asymptotically as a function of f. Backelin answered this in [2] through the following result.

Theorem 2.3. *[2, Theorem 1] The following assertion holds:*

$$0 < \liminf_{f\to\infty} 2^{-\frac{f}{2}} N_f < \limsup_{f\to\infty} 2^{-\frac{f}{2}} N_f < \infty.$$

The proof of Theorem 2.3 follows from the following two results of Backelin which are stated at the beginning of his article. The first result gives bounds for N_f while the second one establishes existence of certain limits.

Proposition 2.4. *For any $f \in \mathbb{N}$, we have $N_f \in \left(2^{\lfloor \frac{f-1}{2} \rfloor}, 4\times 2^{\lfloor \frac{f-1}{2} \rfloor}\right)$.*

Proposition 2.5. *[2, Proposition 1] The following limits exist:*

$$\lim_{f \text{ odd}} 2^{-\frac{f}{2}} N_f, \quad \lim_{f \text{ even}} 2^{-\frac{f}{2}} N_f.$$

In other words, the sequences (N_{2f}) and (N_{2f+1}), grow asymptotically as $c_0 2^{\frac{f}{2}}$ and $c_1 2^{\frac{f}{2}}$, where c_0 and c_1 are some positive real numbers.

As a consequence, this implies that $N_f < N_{f+2}$ for all positive integer $f > d$, for some large enough d. Singhal showed that, in fact, this inequality holds for every nonnegative integer f.

Proposition 2.6. *[36, Theorem 4] For every positive integer f, $N_f < N_{f+2}$.*

Thus although the sequence (N_f) is not monotonous, its subsequences (N_{2f}) and (N_{2f+1}) are. We observe that the above results do not provide much information about the ratio of the consecutive terms N_{f+1}/N_f in (N_f), except that it belongs to the interval $[0.25, 8]$ in real numbers. We address this partially by providing in Proposition 1.14 an improvement to the bounds in Proposition 2.4. We make conjectures on the case when f is a positive odd number (see Conjecture 4.23), and on the asymptotic limit of the ratio N_{f+1}/N_f (see Conjecture 4.22).

Backelin also proves an interesting result regarding the distribution of multiplicities of numerical semigroups in $\mathscr{N}_f$, namely that almost all semigroups in $\mathscr{N}_f$ have their multiplicities close to $\frac{f}{2}$.

Proposition 2.7. *[2, Proposition 2] For any real number $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for every $f \in \mathbb{N}$*

$$\left|\left\{S \in \mathscr{N}_f : \left|\mathrm{m}(S) - \frac{f}{2}\right| > N_\varepsilon\right\}\right| < \varepsilon \cdot 2^{f/2}.$$

Note that for a given ε and the corresponding N_ε in the above theorem we have that for most numerical semigroups S, $\mathrm{m}(S) \in (\frac{f}{2} - N_\varepsilon, \frac{f}{2} + N_\varepsilon)$. In particular, if f is large enough compared to N_ε, this implies that $\mathrm{m}(S) \in (\frac{f}{3}, f)$. Therefore Proposition 2.7 implies that the probability that a randomly chosen numerical semigroup in $\mathscr{N}_f$ has depth either 2 or 3 tends to 1 as f tends to infinity. Thus this distribution condition is a generic property under counting by Frobenius number.

We now summarize some basic results related to counting by Frobenius number. Recall that $\mathrm{L}'(S) = \mathrm{L}(S) \cup \{\mathrm{F}(S)\}$.

Definition 2.8. *Let f and d be positive integers. We define*

$$\mathscr{N}_f(d) = \{S \in \mathscr{N}_f : \gcd(\mathrm{L}'(S)) = d\}$$

and let $N_f(d)$ denote the cardinality of $\mathscr{N}_f(d)$.

Proposition 2.9. *Let $f \geq 3$ be an integer. The following assertions hold.*

(i) $\mathscr{N}_f = \bigcup_{d \mid f} \mathscr{N}_f(d)$ *is a disjoint union.*

(ii) $N_f = \sum_{d \mid f} N_f(d)$.

Proof. Part (i) follows from the fact that every numerical semigroup has a unique Frobenius number. Since the cardinality of a disjoint union of sets is the sum of the cardinalities of the sets involved, part (ii) is a consequence of part (i). □

Definition 2.10. *Let f be a positive integer and d be a positive divisor of f. We define the map*

$$\delta_d : \mathscr{N}_f(d) \to \mathscr{N}_{\frac{f}{d}}(1)$$

by

$$\delta_d(S) = \left\{\frac{x}{d} : x \in \mathrm{L}(S)\right\} \cup \left(\frac{f}{d}, \infty\right).$$

Let

$$\Delta : \mathscr{N}_f \to \bigcup_{d \mid f} \mathscr{N}_{\frac{f}{d}}(1)$$

be the map defined by $\Delta(S) = \delta_d(S)$ where $d = \gcd(\mathrm{L}'(S))$. Note that this map restricted to $\mathscr{N}_f(1)$ gives the identity map.

Lemma 2.11. *Let f be a positive integer and d be a positive divisor of f. Then δ_d and Δ are bijective maps.*

Proof. We observe that by Lemma 1.10, the map $\delta_d : \mathscr{N}_f(d) \to \mathscr{N}_{\frac{f}{d}}(1)$ is given by

$$S \mapsto \frac{S}{d}.$$

We first verify that δ_d is injective. Let $R,S \in \mathscr{N}_f(d)$ such that $\delta_d(R) = \delta_d(S) = T$, for some $T \in \mathscr{N}_{\frac{f}{d}}(1)$. Thus $T = \frac{S}{d} = \frac{R}{d}$. Since $\mathrm{F}(T) = \mathrm{F}\left(\frac{S}{d}\right) = \mathrm{F}\left(\frac{R}{d}\right) = \frac{f}{d}$, we have

$$\mathrm{L}(T) = \left\{\frac{x}{d} : x \in \mathrm{L}(S)\right\} = \left\{\frac{x}{d} : x \in \mathrm{L}(R)\right\}.$$

This implies that $\mathrm{L}(S) = \mathrm{L}(R) = d \cdot \mathrm{L}(T)$. Moreover as $\mathrm{F}(S) = \mathrm{F}(R)$, we obtain $S = R$ and so δ_d is injective.

We now verify the surjectivity of δ_d. Let $T \in \mathscr{N}_{\frac{f}{d}}(1)$ and $T' = d \cdot \mathrm{L}(T) \cup (f, \infty)$. We observe that $T' \in \mathscr{N}_f(d)$, and $\delta_d(T') = T$. Thus δ_d is surjective.

Therefore δ_d is bijective. Note that the inverse of δ_d is given by the map

$$R \mapsto d \cdot \mathrm{L}(R) \cup (d \cdot \mathrm{F}(R), \infty).$$

By Proposition 2.9, $\mathscr{N}_f$ is partitioned into the sets $\mathscr{N}_f(d)$'s which in turn are bijectively mapped to the corresponding sets $\mathscr{N}_{\frac{f}{d}}(1)$'s. Thus Δ is a bijection. □

Definition 2.12. *We denote the set of numerical semigroups with Frobenius number f and depth d by $\mathscr{D}_f(d)$, and its cardinality by $D_f(d)$.*

Definition 2.13. *We denote the set of numerical semigroups with multiplicity m and Frobenius number f by $\mathscr{N}(m,f)$, and its cardinality by $N(m,f)$.*

Given $m \in \mathbb{Q} \setminus \mathbb{N}$, we set $N(m,f)$ to be 0.

Backelin [2, p. 208] showed the following result in his proof of Proposition 2.5 (referred to as Proposition 1 in his article), where he used an argument involving a graph structure on certain subsets of a numerical semigroup with depth 2 or 3.

Proposition 2.14. *For $m \in (\frac{f}{3}, f)$, $N(m+1, f+2) = 2N(m,f)$.*

Corollary 2.15.

$$\sum_{\frac{f}{3} < m < f} [N(m+1,f+2) - N(m,f)] = \sum_{\frac{f}{3} < m < f} N(m,f) = D_f(2) + D_f(3).$$

We next mention a result of Singhal [36] which is a part of the proof of Proposition 2.6 in [36].

Proposition 2.16. *For $f > 37$,*

$$\sum_{m > f/3} [N(m+1,f+2) - N(m,f)] \geq 2.06 \times 2^{\left\lfloor \frac{f-1}{2} \right\rfloor} \tag{2.1}$$

and

$$\sum_{m < f/3} N(m,f) < 1.9 \times 2^{\left\lfloor \frac{f-1}{2} \right\rfloor}. \tag{2.2}$$

Using Corollary 2.15 and the inequality (2.1), we obtain the proof of Proposition 1.14.

Proof of Proposition 1.14 : Let $f \in \mathbb{N}$ with $f > 37$. We have by Corollary 2.15

$$N_f \geq D_f(2) + D_f(3) = \sum_{f>m>f/3} N(m,f) = \sum_{f>m>f/3} [N(m+1,f+2) - N(m,f)]$$

and thus by applying Proposition 2.16 (2.1), we get

$$N_f \geq (2.06) \times 2^{\lfloor \frac{f-1}{2} \rfloor}$$

and the upper bound is given by Proposition 2.4.

For $f \in (12,38) \cap \mathbb{N}$, one can verify the result from Table 2.1. Therefore for f an even number greater than 12, we have

$$N_{f+1} > 2 \times 2^{\lfloor \frac{f}{2} \rfloor} = 2^{\frac{f}{2}+1} = 4 \times 2^{\frac{f}{2}-1} = 4 \times 2^{\lfloor \frac{f-1}{2} \rfloor} \geq N_f.$$

We verify by direct calculations that $N_{f+1} > N_f$ for every even positive integer f less than 12. Hence this inequality holds for every positive even number f. □

Considering the difference $N_{f+2} - N_f$ by applying Proposition 2.16, we get the following.

Proposition 2.17. *Given $f > 37$, we have*

$$N_{f+2} - N_f \geq \sum_{d=4}^{f+2} D_{f+2}(d) + (0.16) \times 2^{\lfloor \frac{f-1}{2} \rfloor}. \tag{2.3}$$

Proof. For any $f \in \mathbb{N}$, we can write N_{f+2} as

$$N_{f+2} = \sum_{m \leq f+2} N(m+1,f+2) = \sum_{m \leq f/3} N(m+1,f+2) + \sum_{f/3 < m \leq f+2} N(m+1,f+2).$$

Using the inequalities (2.1) and (2.2), we get

$$\begin{aligned}
N_{f+2} - N_f &\geq \sum_{m \leq f/3} [N(m+1,f+2) - N(m,f)] + \sum_{m > f/3} [N(m+1,f+2) - N(m,f)]. \\
&\geq \sum_{m \leq f/3} [N(m+1,f+2) - N(m,f)] + (2.06) \times 2^{\lfloor \frac{f-1}{2} \rfloor}. \\
&\geq \sum_{m \leq f/3} [N(m+1,f+2)] + (2.06) \times 2^{\lfloor \frac{f-1}{2} \rfloor} - \sum_{m \leq f/3} [N(m,f)]. \\
&\geq \sum_{m \leq f/3} [N(m+1,f+2)] + (2.06) \times 2^{\lfloor \frac{f-1}{2} \rfloor} - (1.9) \times 2^{\lfloor \frac{f-1}{2} \rfloor}. \\
&\geq \sum_{m \leq f/3} [N(m+1,f+2)] + (0.16) \times 2^{\lfloor \frac{f-1}{2} \rfloor}.
\end{aligned}$$

Re-indexing this, we get

$$\begin{aligned} N_{f+2}-N_f &\geq \sum_{m<\frac{f}{3}+1} N(m,f+2)+(0.16)\times 2^{\left\lfloor\frac{f-1}{2}\right\rfloor}. \\ &\geq \sum_{d=4}^{\infty} D_{f+2}(d)+(0.16)\times 2^{\left\lfloor\frac{f-1}{2}\right\rfloor}. \end{aligned}$$

□

Taking the first term in the summation on right side of the latter inequality, we get the following corollary.

Corollary 2.18. *Given $f > 37$,*

$$N_{f+2}-N_f \geq D_{f+2}(4).$$

We use the following consequence of Proposition 2.4 for calculations.

Lemma 2.19. *Let $n \geq 2$ be an integer. Let $d \geq 2$ be a divisor of n. Then*

$$N_{\frac{n}{d}} \leq 2\sqrt{2}\times 2^{\frac{n}{2d}} \leq 2\sqrt{2}\times 2^{\frac{n}{4}}.$$

In particular, for $d \geq 3$,

$$N_{\frac{n}{d}} \leq 2\sqrt{2}\times 2^{\frac{n}{6}}.$$

Proof. By Proposition 2.4, we have

$$N_{\frac{n}{d}} \leq 4\times 2^{\left\lfloor\frac{(n/d)-1}{2}\right\rfloor} \leq 4\times 2^{\frac{(n/d)-1}{2}} = 2\sqrt{2}\times 2^{\frac{n}{2d}}.$$

In particular, when $d \geq 2$ we get the result since $2^{\frac{n}{2d}} \leq 2^{\frac{n}{4}}$ and for $d \geq 3$ we get $2^{\frac{n}{2d}} \leq 2^{\frac{n}{6}}$. □

We next give a lower bound for the number of depth 4 semigroups in $\mathscr{N}_f$ when $f \geq 13$, which will be required later.

Lemma 2.20. *Given a positive integer f, the following assertions hold.*

(i) *We have $\mathscr{D}_f(4) \neq \emptyset$ if and only if $f \in \{10,11\}$ or $f \geq 13$.*

(ii) *Let $f \geq 10$, $f \neq 12$, $m = \left\lfloor\frac{f-1}{3}\right\rfloor$ and*

$$\mathscr{T}_f = \left\{ Y \cup \{0,m,2m,3m\} \cup (f,\infty) : Y \subset \left(\frac{2f}{3}+1, f\right) \setminus \{3m\} \right\}.$$

Then $\mathscr{T}_f$ is a nonempty subset of $\mathscr{D}_f(4)$ whose members have multiplicity m and Frobenius number f. Moreover we have $|\mathscr{T}_f| \geq 2^{\frac{f}{3}-3}$.

(iii) *If $f \geq 10$ and $f \neq 12$, then $|\mathscr{D}_f(4)| \geq 2^{\frac{f}{3}-3}$.*

Proof. We first consider part (i). Let $S \in \mathscr{D}_f(4)$. The only numerical semigroup with multiplicity 1 is $\mathbb{N}$ which does not have depth 4 and so $m(S) \geq 2$. Since by definition $\operatorname{depth}(S) = \left\lceil\frac{f(S)+1}{m(S)}\right\rceil$ and

here depth$(S)=4$ we obtain $f(S)\geq 7$. Now $m(S)\in\left(\frac{f(S)}{4},\frac{f(S)}{3}\right)$ and so $\left(\frac{f(S)}{4},\frac{f(S)}{3}\right)\cap\mathbb{N}\neq\emptyset$ yielding $f(S)\geq 10$.
Consider $S=\langle 3,11,13\rangle$ and $T=\langle 3,13,14\rangle$. Then S and T are numerical semigroups of depth 4 and respective Frobenius numbers 10 and 11. Note that there is no numerical semigroup S of depth 4 and Frobenius number $f(S)$ equal to 12 as otherwise $m(S)$ would be equal to 4 contradicting the fact that 12 is the Frobenius number.
Suppose that $f\geq 13$. Let $1\leq\varepsilon\leq 4$ be an integer such that $f+1\equiv\varepsilon \mod 4$. Set $m=\frac{f+1-\varepsilon}{4}+1$ and $S=\langle m,f+1,f+2,\ldots,f+m\rangle$. Then S is a numerical semigroup of multiplicity m, Frobenius number f and depth 4, in particular $S\in\mathscr{D}_f(4)$. This establishes part (i).

We now consider part (ii). It is clear that $\mathscr{T}_f\neq 0$. Let $T\in\mathscr{T}_f$. We claim that T is a numerical semigroup with multiplicity m and Frobenius number f. By definition, T is cofinite in $\mathbb{N}$, $0\in T$, m is the smallest positive integer in T and f is the greatest positive integer not in T. It therefore remains to check that T is closed under addition. Let $x,y\in T$ and say $x\leq y$. If $x=0$ then $x+y\in T$. Suppose that $x>0$. If $y\in\left(\frac{2f}{3}+1,f\right)$ then $x+y\geq m+y>\frac{f}{3}-1+\frac{2f}{3}+1=f$, and so $x+y\in T$. If $y\notin\left(\frac{2f}{3}+1,f\right)$ then $y\in\{m,2m\}$ and so $x\in\{m,2m\}$ and $x+y\in T$. If $y>f$ then it is trivial that $x+y$ belongs to T. Hence T is a numerical semigroup as claimed. It is now clear that $T\in\mathscr{D}_f(4)$, hence $\mathscr{T}_f\subset\mathscr{D}_f(4)$. By definition the cardinality of $\mathscr{T}_f$ is the number of subsets of the set $\left(\frac{2f}{3}+1,f\right)\setminus\{3m\}$. The number of such subsets is $\left\lceil\frac{f}{3}-3\right\rceil$. Therefore

$$|\mathscr{T}_f|=2^{\left\lceil\frac{f}{3}-3\right\rceil}\geq 2^{\frac{f}{3}-3}.$$

We finally consider part (iii). Since $\mathscr{T}_f\subset\mathscr{D}_f(4)$, the result follows from part (ii). □

We refer the reader to [4] for more on enumeration of numerical semigroups by Frobenius number. More recently, counting by Frobenius number regained some interest in works like [29, 36] where various asymptotic problems are discussed.

2.1.2 Counting by genus

We now consider the genus map $\mathrm{g}:\mathbb{S}\to\mathbb{N}$ given by $\mathrm{g}(S)=|\mathbb{N}\setminus S|$ and the counting associated to it is known as *counting numerical semigroups by genus*, or *counting by genus*. This problem originated within some conjectures of Bras-Amorós given in [5] in 2008. Let g be a nonnegative integer. Let

$$\mathscr{G}_g=\{S\in\mathbb{S}:\mathrm{g}(S)=g\}$$

and let n_g denote the cardinality of $\mathscr{G}_g$. Analogously as in the case of counting by Frobenius number we define the probability that a randomly chosen numerical semigroup in $\mathscr{G}_g$ belongs to a subset Q of $\mathbb{S}$ as

$$\mathbb{P}_{\mathscr{G}_g}(Q)=\frac{|Q\cap\mathscr{G}_g|}{|\mathscr{G}_g|}.$$

The set $\mathbb{S}$ has a graph structure of an infinite tree (a cycle-free connected graph)–the vertex set $\mathbb{S}$ has an edge between every two vertices S and R with $g(S)<g(R)$ if and only if $S=R\cup\{\mathrm{F}(R)\}$.

Algorithmically, for every $S \in \mathbb{S}$ we remove a right primitive to get a numerical semigroup with genus $g(S)+1$ and join it to S with an edge. We do this for all right primitives of S. We construct the graph starting with the vertex $\mathbb{N}$ and generate the tree inductively.

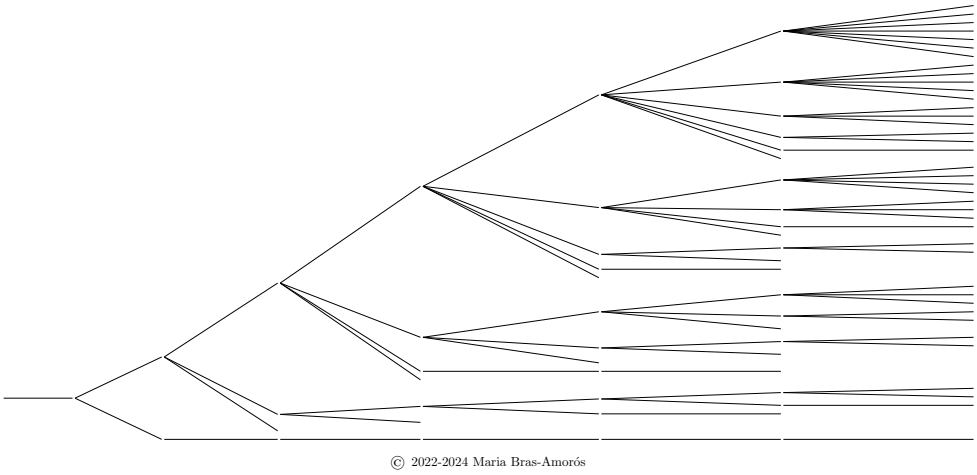


Fig. 2.1 The classical semigroup tree

We observe that the tree drawn in Figure 2.1[1] has numerical semigroups with genus g at that the g-th level from the left. Thus this provides a visual representation of the sequence (n_g), and it is known as the *classical semigroup tree*. Bras-Amorós [5] observed that this tree grows in a Fibonacci-like manner and posed the following conjectures.

Conjecture 2.21. *The sequence (n_g) has the following properties:*

(i) Super-Fibonacci *growth inequality:*

$$n_g \geq n_{g-1} + n_{g-2}, \text{ for } g \geq 2.$$

(ii) Fibonacci-like asymptotic relation:

$$\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1.$$

(iii) The asymptotic ratio of the consecutive terms of the sequence is the golden ratio $\phi = \left(\frac{1+\sqrt{5}}{2}\right)$, i.e.

$$\lim_{g\to\infty} \frac{n_g}{n_{g-1}} = \phi.$$

Moreover Bras-Amorós [6] showed the following bounds for n_g:

$$2F_g \leq n_g \leq 1 + 3 \times 2^{g-3}$$

[1] This figure is generated using the Drawsgtree tool by Maria Brás-Amorós [8].

where F_g is the g-th Fibonacci number. The lower bound was improved by Eliahou et al.(see [17, Corollary 5.7]) to

$$(7.84)F_g \leq n_g \quad \text{for } g \geq 59.$$

Conjecture 2.21(i) remains open. The Fibonacci-like asymptotic growth of (n_g) from Conjecture 2.21 parts (ii) and (iii) was proved by Zhai [39], and it was recently reproved by Zhu [41] using other methods.

We next look at some asymptotic properties under counting by genus. The following result was conjectured by Zhao [40] and later proved by Zhai as a key ingredient in his proof of the asymptotic Fibonacci-like growth of (n_g).

Proposition 2.22. *[39] The probability that a randomly chosen numerical semigroup S in $\mathscr{G}_g$ satisfies the condition* $\mathrm{F}(S) < 3\,\mathrm{m}(S)$ *tends to* 1 *as g tends to infinity.*

Kaplan and Singhal established another asymptotic result under counting by genus.

Proposition 2.23. *[27, Corollary 7] The probability that a randomly chosen numerical semigroup S in $\mathscr{G}_g$ satisfies the condition* $\frac{\mathrm{m}(S)}{2} \leq |\mathrm{P}(S)|$ *tends to* 1 *as g tends to infinity.*

We refer the reader to [19] and [7] for the computational aspects of the problem of counting numerical semigroups up to a given genus. For a survey on this topic see [26].

2.1.3 Box Counting

We next describe a way of counting which does not originate from any of the natural properties of numerical semigroups. It was first studied by Sinai and Bourgain in [3]. A similar way of counting was previously described by Arnold in [1], which only differs from it in terms of using the ℓ_1-norm instead of the ℓ_∞-norm.

Let $\mathbb{N}_+$ denote $\mathbb{N}+1$, and let $n,t \in \mathbb{N}_+$. We define the set of n-tuples

$$G(n,t) := \{a = (a_1,\ldots,a_n) \in \mathbb{N}_+^n : \ \|a\|_\infty := \max_i(\{a_i : i \leq n\}) \leq t, \ \gcd(a) = 1\}$$

and assign the uniform probability distribution to it. We consider each tuple in this set as a generating set, and denote the set of numerical semigroups generated by them as

$$\mathscr{B}(n,t) := \{\langle A\rangle : A \in G(n,t)\}$$

and include all possible values of n together to define

$$\mathscr{B}(t) := \bigcup_{n\leq t} \mathscr{B}(n,t).$$

We note that $|\mathscr{B}(n,t)|$ is less than t^n, and thus $|\mathscr{B}(t)|$ is less than $\sum_{n=1}^{t} t^n$. However this bound is far from optimal, which can be seen in the following way. We observe a simple relation between the box counting and counting by maximum primitive[2].

[2]defined in Chapter 1, and discussed in a detailed manner in the next section

Proposition 2.24. *Given $t \in \mathbb{N}$ we have*

$$\mathscr{B}(t) = \bigcup_{i \leq t} \mathscr{A}_i.$$

Proof. Let $S \in \mathscr{B}(t)$. Then the maximum primitive of S is less than or equal to t, and thus $S \in \mathscr{A}_i$ for some $i \leq t$. Conversely, if $i \leq t$ then any numerical semigroup $S \in \mathscr{A}_i$ is generated by a subset of $[0,i]$ which is also a subset of $[0,t]$, and thus $S \in \mathscr{B}(t)$. □

Proposition 2.25. *We have*

$$\sum_{i=1}^{t} \left(2^{\bar{i}} - F_{\bar{i}+1}\right) < |\mathscr{B}(t)| < 16 \times 2^{\bar{t}}.$$

Proof. These bounds are obtained using the bounds on A_n. The lower bound is directly a substitution from Proposition 1.16 into Proposition 2.24. The upper bound is obtained from using Proposition 1.15 and Proposition 2.4 in Proposition 2.24 which, after some simplifications, gives

$$|\mathscr{B}(t)| < \sum_{i=1}^{t} N_i < 2 \sum_{i=1}^{\bar{t}} \left(4 \times 2^{\bar{i}}\right) < 16 \times 2^{\bar{t}}.$$ □

A notion related to box counting is that of a *random* numerical semigroup, inspired by the Erdös-Renyi model of random graphs, which has been studied in [30].

2.2 Counting by the maximum primitive

In this section we introduce a new way of counting numerical semigroups based on the maximum primitive map $v : \mathbb{S} \to \mathbb{N}$. Given a positive integer n, let

$$\mathscr{A}_n = \{S \in \mathbb{S} : \max \mathrm{P}(S) = n\}$$

and let A_n denote the cardinality of $\mathscr{A}_n$. The study of $\mathscr{A}_n$ and, in particular, determining A_n for a given n will be referred to as *counting by maximum primitive*.

As before, we have the notion of probability associated to counting by maximum primitive given by the uniform distribution on $\mathscr{A}_n$. Given a positive integer $n \neq 2$ and a set $Q \subseteq \mathbb{S}$, we define the probability that a randomly chosen numerical semigroup in $\mathscr{A}_n$ belongs to Q as

$$\mathbb{P}_{\mathscr{A}_n}(Q) = \frac{|Q \cap \mathscr{A}_n|}{|\mathscr{A}_n|}.$$

In the above definition we take $n \neq 2$ as $\mathscr{A}_2 = \emptyset$, see Lemma 2.31(ii).

We next define a map from $\mathscr{A}_n$ to $\mathscr{N}_n$ and discuss some of the properties it satisfies.

Definition 2.26. *Given a positive integer n, we define the map $\Phi_n : \mathscr{A}_n \to \mathscr{N}_n$ by*

$$\Phi_n(S) = (S \setminus \{n\}) \cup (n, \infty).$$

When there is no risk of confusion we omit the subscript and write simply Φ *for* Φ_n.

Example 2.27. This example aims at illustrating that commonly studied combinatorial invariants (such as the genus, embedding dimension, or Frobenius number) of S are seemingly unrelated to the corresponding invariants in $\Phi(S)$.

We give in Figures 2.2 and 2.3 pictorial views of a couple of semigroups and their images under the map Φ.

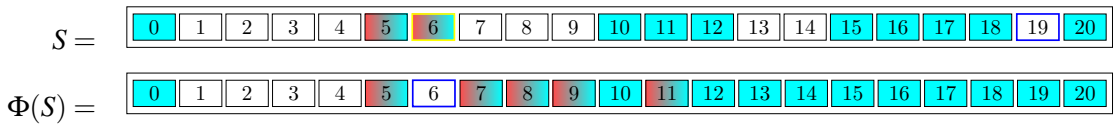


Fig. 2.2 Pictorial views of $S = \langle 5,6\rangle$ and $\Phi(S) = \langle 5,7,8,9,11\rangle$.

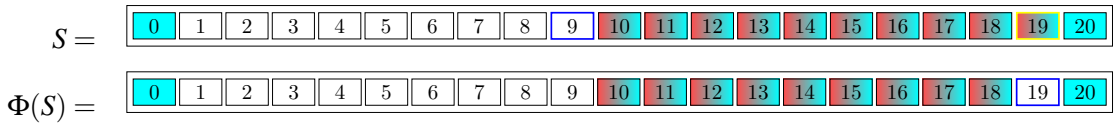


Fig. 2.3 Pictorial views of $S = \langle 10,\dots,19\rangle$ and $\Phi(S) = \langle 10,\dots,18\rangle$.

Let n be a positive integer. We denote by O_n the numerical semigroup that consists of 0 and all integers greater than or equal to n. The numerical semigroups in $\{O_n \mid n \in \mathbb{N}\}$ are often referred to as *ordinary* semigroups. Note that $O_n = \langle n,\dots,2n-1\rangle$.

Example 2.28. We have $\Phi_1(\mathbb{N}) = O_2$. As $\mathscr{A}_1 = \{\mathbb{N}\}$ and $\mathscr{N}_1 = \{O_2\} = \{\langle 2,3\rangle\}$, we have that Φ_1 is a bijection. By Lemma 2.31(ii) $\mathscr{A}_2 = \emptyset$, thus Φ_2 is injective vacuously.

Example 2.28 treats the pathological cases. From now on, unless otherwise stated, we assume that $n \geq 3$.

Lemma 2.29. *Let* $n \geq 3$ *be an integer and let* $S \in \mathscr{A}_n$. *Then the primitive depth of* S *coincides with the depth of* $\Phi(S)$. *In other words,*

$$\pi(S) = \mathrm{q}(\Phi(S)).$$

Proof. Recall that for a numerical semigroup $S \neq \mathbb{N}$, $\mathrm{q}(S) = \left\lceil \frac{\mathrm{F}(S)}{\mathrm{m}(S)} \right\rceil$. It follows from the definitions that $n = \max \mathrm{P}(S) = \mathrm{F}(\Phi(S))$ and $\mathrm{m}(S) = \min \mathrm{P}(S) = \mathrm{m}(\Phi(S))$. Therefore

$$\pi(S) = \left\lceil \frac{\max \mathrm{P}(S)}{\min \mathrm{P}(S)} \right\rceil = \left\lceil \frac{\mathrm{F}(\Phi(S))}{\mathrm{m}(\Phi(S))} \right\rceil = \mathrm{q}(\Phi(S)). \qquad \square$$

We now prove some properties of the map Φ.

Proposition 2.30. *Let* $n \geq 3$ *be an integer. The following assertions hold.*

(i) $\Phi(\mathscr{A}_n) = \{S \in \mathscr{N}_n : \gcd(\mathrm{L}'(S)) = 1\}$.

(ii) $O_{n+1} \in \mathscr{N}_n \setminus \Phi(\mathscr{A}_n)$. *In particular,* Φ *is not surjective.*

(iii) *The map* Φ *is injective.*

Proof. We first consider part (i). We first show that $\Phi(\mathscr{A}_n) \subseteq \{S \in \mathscr{N}_n : \gcd(\mathrm{L}'(S)) = 1\}$. Let $S \in \Phi(\mathscr{A}_n)$. Then there exists $T \in \mathscr{A}_n$ such that $S = \Phi(T)$. Therefore

$$\mathrm{P}(T) \subseteq T \cap [0,n] = (\Phi(T) \cup \{n\}) \cap [0,n] = \mathrm{L}'(S).$$

Since $\gcd(\mathrm{P}(T)) = 1$, we obtain $\gcd \mathrm{L}'(S) = 1$ and hence the claimed inclusion is established.

We next show that $\{S \in \mathscr{N}_n : \gcd(\mathrm{L}'(S)) = 1\} \subseteq \Phi(\mathscr{A}_n)$. Let $R \in \mathscr{N}_n$ with $\gcd(\mathrm{L}'(R)) = 1$ and let

$$S = \langle \mathrm{L}'(R) \rangle = \langle (R \cap [0,n]) \cup \{n\} \rangle \in \mathscr{A}_n.$$

Then $\Phi(S) = R$ and so $R \in \Phi(\mathscr{A}_n)$, establishing the claimed inclusion.

We now consider part (ii). It is clear that $O_{n+1} \in \mathscr{N}_n$. Let $T \in \mathscr{A}_n$. Since $\gcd \mathrm{P}(T) = 1$, there exists at least one primitive $t \in \mathrm{P}(T)$ such that $0 < t < n$. From the definition of Φ it follows that $t \in \Phi(T)$. As $t \notin O_{n+1}$ it follows that $O_{n+1} \notin \Phi(\mathscr{A}_n)$, concluding the proof of the assertion.

We finally consider part (iii). Let $S, R \in \mathscr{A}_n$ be such that $\Phi(S) = \Phi(R)$. Then

$$(S \setminus \{n\}) \cup (n, \infty) = (R \setminus \{n\}) \cup (n, \infty),$$

and since $n \in S \cap R$, we have $S \cap [0,n] = R \cap [0,n]$. In particular, $\mathrm{P}(R) \subseteq S$ and $\mathrm{P}(S) \subseteq R$, and therefore $S = R$. □

We can now prove Proposition 1.15.

Proof of Proposition 1.15. We have $A_2 = 0$ and $N_2 = 1$, thus $A_2 < N_2$. Given $n > 2$, we consider the map $\Phi : \mathscr{A}_n \to \mathscr{N}_n$ from Definition 2.26. By Proposition 2.30, Φ is injective but not surjective. Therefore $A_n < N_n$.

Suppose finally that p is prime. Let $S \in \mathscr{N}_p \setminus \Phi(\mathscr{A}_p)$. By Proposition 2.30, $\gcd(\mathrm{L}'(S)) \neq 1$ so $\mathrm{L}'(S) = \{0, p\}$. In particular, S has no primitive less than p and thus $S = O_{p+1}$. Hence $\mathscr{N}_p = \Phi(\mathscr{A}_p) \cup \{O_{p+1}\}$. By Proposition 2.30, Φ is injective and so $N_p = A_p + 1$. □

Lemma 2.31. *Let* n *be a positive integer. The following assertions hold.*

(i) *If* $n \neq 2$, *then the set* $\mathscr{A}_n$ *is finite and non-empty.*

(ii) $\mathscr{A}_2 = \emptyset$.

Proof. First we consider part (i). The statement holds for $n = 1$, since $\mathscr{A}_1 = \{\mathbb{N}\}$.
For $n > 2$, $\langle n-1, n \rangle \in \mathscr{A}_n$, thus $\mathscr{A}_n$ is non-empty. Furthermore, the number of numerical semigroups with maximum primitive n does not exceed 2^n, as all of them are generated by subsets of $[1,n]$. Thus $\mathscr{A}_n$ is finite, which concludes the proof part of (i).

Next we consider part (ii). First note that $\mathbb{N} \notin \mathscr{A}_2$, since $\mathbb{N} = \langle 1 \rangle$. Suppose that there exists $S \in \mathscr{A}_2$. As $1 \notin S$ (otherwise we would have $S = \mathbb{N}$), S is an additive submonoid of $\mathbb{N}$ generated by 2. Hence S consists of even integers and therefore is not a numerical semigroup, a contradiction. □

Lemma 2.32. *The set* $\{\mathscr{A}_n : n \in \mathbb{N}\}$ *is a partition of* $\mathbb{S}$.

Proof. The result follows from the fact that every numerical semigroup has a unique maximum primitive. □

For any positive integer f, the sets $\mathscr{A}_f$ and $\mathscr{N}_f$ are disjoint. Indeed, the integer f belongs to every semigroup in $\mathscr{A}_f$ but f does not belong to any semigroup of $\mathscr{N}_f$. Note that any semigroup in $\mathscr{A}_n$ belongs to $\mathscr{N}_f$, for some $f \neq n$. These numbers n and f can be very different, as the following example illustrates.

Example 2.33. Let $S = \langle 6,7,8\rangle$ and $T = \langle 10,11,\ldots,19\rangle$. Then $S \in \mathscr{A}_8 \cap \mathscr{N}_{17}$ and $T \in \mathscr{A}_{19} \cap \mathscr{N}_9$. Figure 2.4 is a pictorial representation of the semigroups S and T where some of the elements are emphasized. Our illustration is made using pictorial views of the semigroups obtained from the TikZ code produced using the `GAP` [20] package `intpic` [11].

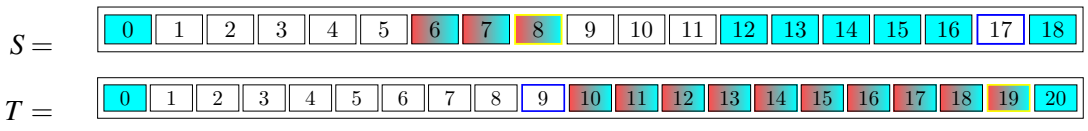


Fig. 2.4 Pictorial representation of $S = \langle 6,7,8\rangle$ and $T = \langle 10,11,\ldots,19\rangle$.

In Table 2.1 below we collect some experimental data consisting of the first 60 elements of the sequences (A_n) and (N_n). All the elements of the sequence (A_n) in the table were obtained by using `GAP` code that is part of the `numericalsgps` [12] `GAP` package (from Version 1.4.0), and is distributed by `GAP` [20] (from Version 4.14.0). Regarding the sequence (N_n), the first 39 elements have been known for some time [21, 33]. We used Theorem 1.20 to complete the table.

Despite being significantly different, as Example 2.33 suggests, the sets $\mathscr{A}_f$ and $\mathscr{N}_f$ have similar cardinalities. This fact can be verified using Table 2.1 for small values of f. Asymptotically, the similarity of the cardinalities of the sequences under consideration is given by Theorem 1.17.

From the data presented in Table 2.1 one notes that for any prime $p < 60$, $N_p = A_p + 1$. This is in fact true for any prime p by Proposition 1.15. Furthermore, we see in Theorem 1.21 that for $n \in \mathbb{N}$, the difference between A_n and N_n has a clear dependency on the divisors of n.

Lemma 2.34. *Let* f *be a positive integer. The following assertions hold.*

(i) $\mathscr{N}_f(d) \neq \emptyset$ *if and only if* $d \mid f$.

(ii) $N_f(1) = A_f$.

(iii) *If* d_1 *and* d_2 *are distinct positive divisors of* f*, then* $\mathscr{N}_f(d_1)$ *and* $\mathscr{N}_f(d_2)$ *are disjoint.*

Proof. We first consider part (i). The numerical semigroup $d \cdot \mathbb{N} \cup (f,\infty)$ belongs to $\mathscr{N}_f(d)$ if d divides f. If d does not divide f, then $\gcd(\mathrm{L}(S), f)$ is not equal to d for any S in $\mathscr{N}_f$, and thus $\mathscr{N}_f(d) = \emptyset$. This proves part (i).
We next consider part (ii). From Proposition 2.30 we have $N_f(1) = |\Phi(\mathscr{A}_f)|$, and thus the injectivity

n	A_n	N_n	n	A_n	N_n	n	A_n	N_n
1	1	1	21	1816	1828	41	2425710	2425711
2	0	1	22	1862	1913	42	2285281	2287203
3	1	2	23	4095	4096	43	4889433	4889434
4	1	2	24	3530	3578	44	4783757	4785671
5	4	5	25	8268	8273	45	9574948	9575167
6	2	4	26	8069	8175	46	9674748	9678844
7	10	11	27	16111	16132	47	19919901	19919902
8	8	10	28	16163	16267	48	18893119	18896892
9	19	21	29	34902	34903	49	40010840	40010851
10	17	22	30	31603	31822	50	39437596	39445886
11	50	51	31	70853	70854	51	78793811	78794277
12	35	40	32	68476	68681	52	78922130	78930306
13	105	106	33	137339	137391	53	162306074	162306075
14	92	103	34	140196	140661	54	155991666	156008182
15	194	200	35	292066	292081	55	325800242	325800297
16	195	205	36	269817	270258	56	320507004	320523279
17	464	465	37	591442	591443	57	643198150	643199112
18	382	405	38	581492	582453	58	644611930	644646833
19	960	961	39	1155905	1156012	59	1317118755	1317118756
20	877	900	40	1160411	1161319	60	1269732856	1269765591

Table 2.1 Counting by maximum primitive and by Frobenius number

of Φ gives the result.

We finally consider part (iii). Suppose for contradiction that $S \in \mathscr{N}_f(d_1) \cap \mathscr{N}_f(d_2)$. Then $d_1 = \gcd(\mathrm{L}'(S)) = d_2$, a contradiction. □

Next we prove Theorem 1.17.

Proof of Theorem 1.17. Recall that given a positive integer n, $\mathscr{N}_n(1)$ denotes $\{S \in \mathscr{N}_n : \gcd(\mathrm{L}'(S)) = 1\}$. By Proposition 2.30(i), if $n \geq 3$ then

$$\Phi(\mathscr{A}_n) = \mathscr{N}_n(1).$$

Since for any positive divisor d of n, $\mathscr{N}_{\frac{n}{d}}(1) \subseteq \mathscr{N}_{\frac{n}{d}}$, by Lemma 2.11 we obtain

$$|\mathscr{N}_n \setminus \mathscr{N}_n(1)| \leq \left| \bigcup_{d \neq 1, d|n} \mathscr{N}_{\frac{n}{d}} \right|.$$

Since the sets $\mathscr{N}_{\frac{n}{d}}$'s are disjoint, we have

$$\left| \bigcup_{d \neq 1, d|n} \mathscr{N}_{\frac{n}{d}} \right| = \sum_{d \neq 1, d|n} N_{\frac{n}{d}}.$$

Lemma 2.19 now yields

$$|\mathscr{N}_n \setminus \mathscr{N}_n(1)| \leq \sum_{d\neq 1, d|n} N_{\frac{n}{d}} \leq \sum_{d\neq 1, d|n} 2\sqrt{2}\times 2^{\frac{n}{2d}} \leq \sum_{d\neq 1, d|n} 2\sqrt{2}\times 2^{\frac{n}{4}}.$$

By Lemma A.3, we get

$$|\mathscr{N}_n \setminus \mathscr{N}_n(1)| = n^{o(1)}\cdot 2\sqrt{2}\cdot 2^{\frac{n}{4}}.$$

The result now follows from the fact that $n^{o(1)}\cdot 2\sqrt{2}\cdot 2^{\frac{n}{4}} = 2^{o(1)\log_2 n}\cdot 2^{\frac{n}{4}} = 2^{\left(\frac{n}{4}+o(\log n)\right)} = 2^{\left(\frac{1}{4}+o(1)\right)n}$. □

We can now prove Corollary 1.18.

Proof of Corollary 1.18. By Theorem 1.17

$$\frac{A_n}{N_n} = 1 - \frac{2^{\left(\frac{1}{4}+o(1)\right)n}}{N_n}$$

which, by Theorem 2.3, tends to 1 as n tends to infinity. By Proposition 2.5, we have that the limit

$$\lim_{n \text{ even}} 2^{-\frac{n}{2}} N_n = \lim_{n\to\infty} \frac{N_{2n}}{2^{\frac{2n}{2}}}$$

exists and is a positive real number. Thus the limit

$$\lim_{n\to\infty} \frac{A_{2n}}{2^{\frac{2n}{2}}} = \lim_{n \text{ even}} 2^{-\frac{n}{2}} A_n$$

exists and is a positive real number. Thus the sequence (A_{2n}) grows asymptotically as $c_o 2^{\frac{n}{2}}$ for some positive real number c_o. A similar argument yields the corresponding result for the sequence (A_{2n+1}). □

2.2.1 Bounds on A_n

By Proposition 1.15, for an integer $n \geq 3$, we have the upper bound $A_n < N_n$. We obtain a lower bound of A_n by an application of Bertrand's postulate

Lemma 2.35. *For $n \geq 8$, $A_n \geq \frac{3}{4}\times 2^{\left\lfloor \frac{n-1}{2}\right\rfloor}$.*

Proof. By Bertrand's postulate, for any integer $x > 3$ the interval $(x, 2x-2)$ contains a prime. For $n > 8$, therefore, there exists at least one prime $p \in \left(\left\lceil \frac{n}{2}\right\rceil, 2\times\left\lceil \frac{n}{2}\right\rceil - 2\right) \subset \left(\frac{n}{2}, n-1\right)$. Let us fix such a prime p. Any subset of $X \subseteq (\frac{n}{2}, n)$ which contains p or $n-1$ gives us the set $X\cup\{n\}$ which has greatest common divisor equal to 1. Thus we get the following subset of $\mathscr{A}_n$

$$\left\{\langle X\cup\{y,n\}\rangle : X \subseteq \left(\frac{n}{2}, n\right)\setminus\{p, n-1\}, y\in\{p, n-1\}\right\}.$$

The cardinality of the above subset is $2^{\lfloor \frac{n-1}{2} \rfloor - 1}$. Moreover we have the following subset of $\mathscr{A}_n$, which is clearly disjoint from the one constructed above:

$$\left\{ \langle X \cup \{p, n-1, n\} \rangle : X \subseteq \left(\frac{n}{2}, n-1 \right) \right\}.$$

The cardinality of this subset is $2^{\lfloor \frac{n-1}{2} \rfloor - 2}$. Thus in total we have constructed $\frac{3}{4} \times 2^{\lfloor \frac{n-1}{2} \rfloor}$ numerical semigroups in $\mathscr{A}_n$, which gives us the result. □

Let $n > 2$ be an integer. Recall that for $a, m \in \mathbb{N}$ with $m > 1$ we define

$$T_a(m) = \{X \subseteq [a+1, a+m] : \text{ for some } x \in X, \{x, x+1\} \subseteq X \cup \{a+m+1\}\}.$$

We consider the special case $a = m = \lfloor \frac{n-1}{2} \rfloor$ below and denote $T_a(m)$ by T in this case.

Lemma 2.36. *Let $n > 2$ be an integer and*

$$T = \left\{ X \subseteq \left(\frac{n}{2}, n \right) : \textit{ for some } x \in X, \{x, x+1\} \subseteq X \cup \{n\} \right\}.$$

Suppose $X \in T$. Then $\langle X \cup \{n\} \rangle \in \mathscr{A}_n$ and the primitive depth $\pi(\langle X \cup \{n\} \rangle)$ is 2. In particular

$$|T| \leq |\{S \in \mathscr{A}_n : \pi(S) = 2\}| \leq A_n.$$

Proof. Let $X \in T$, and let S_X denote $\langle X \cup \{n\} \rangle$. Note that $\gcd(X \cup \{n\})$ is 1, since $X \cup \{n\}$ contains at least one pair of consecutive numbers. Thus S_X is a numerical semigroup.

Suppose $s \in X \cup \{n\}$. If $s = a + b$ for some positive integers a and b, then either $a \leq \frac{n}{2}$ or $b \leq \frac{n}{2}$. Thus $\{a, b\} \not\subseteq S_X$ and hence $s \in \mathrm{P}(S_X)$. Thus $\mathrm{P}(S_X) = X \cup \{n\}$ and $S_X \in \mathscr{A}_n$. Also, since $\mathrm{P}(S_X) \subseteq \left(\frac{n}{2}, n \right]$ we obtain $\pi(S_X) = 2$.

Thus every set $X \in T$ determines a unique numerical semigroup $S_X \in \mathscr{A}_n$ with $\pi(S_X) = 2$. Therefore

$$|T| \leq |\{S \in \mathscr{A}_n : \pi(S) = 2|\} \leq A_n.$$ □

We can now prove Proposition 1.16 using Lemma A.1 which gives an estimate for the cardinality of T. Recall that F_n denotes the n-th Fibonacci number.

Proof of Proposition 1.16. Lemma A.1 and Lemma 2.36 yield $A_n \geq |T| \geq 2^{\lfloor \frac{n-1}{2} \rfloor} - F_{\lfloor \frac{n-1}{2} \rfloor + 1}$. □

It is straightforward to check that the lower bound for A_n provided in Lemma 2.36 is better than the one given in Lemma 2.35 for $n > 10$.

2.2.2 A correspondence between A_n and N_n

In this section we establish a number-theoretic relation between the quantities A_n and N_n.

Recall that

$$N_n = \sum_{d|n} N_n(d).$$

By Proposition 2.30, $A_n = |\mathscr{N}_n(1)|$. Therefore

$$N_n - A_n = \sum_{d \neq 1, d|n} N_n(d). \tag{2.4}$$

Note that $|\mathscr{N}_n(n)| = 1 = A_1$.

Lemma 2.37. *Let $n > 2$ be an integer and let d be a positive divisor of n. Then*

$$N_n(d) = A_{\frac{n}{d}}.$$

Proof. We have the following sequence of bijections

$$\mathscr{A}_{\frac{n}{d}} \longleftrightarrow \mathscr{N}_{\frac{n}{d}}(1) \longleftrightarrow \mathscr{N}_n(d)$$

given by $\Phi_{\frac{n}{d}}$ (via Proposition 2.30) and δ_d (via Lemma 2.11), respectively. □

We can now prove Theorem 1.20.

Proof of Theorem 1.20 . The result follows from (2.4) and Lemma 2.37. □

Corollary 2.38. *Let f be a positive integer and let q be a positive divisor of f. Then*

$$N_q \leq N_f.$$

Proof. By Theorem 1.20

$$N_f = \sum_{d|f} A_{\frac{f}{d}} = \sum_{d|f} A_d = \sum_{d|q} A_d + \sum_{h|f, h\nmid q} A_h.$$

Thus

$$N_f = \sum_{d|q} A_{\frac{q}{d}} + \sum_{h|f, h\nmid q} A_h = N_q + \sum_{d|f, d\nmid q} A_d.$$

Since all terms in the above sum are nonnegative, $N_q \leq N_f$. □

We now show that each term A_n in the sequence (A_n) is determined by the sequence (N_n). We begin with the following special case.

Lemma 2.39. *Let p be a prime and s be a positive integer. Then*

$$A_{p^s} = N_{p^s} - N_{p^{s-1}}.$$

Proof. We prove the statement by induction on s. The case when $s = 1$ follows from Proposition 1.15. We assume that the statement is true for all $i < s$ for some fixed $s > 1$. Applying Lemma 2.37 we get

$$N_{p^s} = \sum_{d|p^s} N_{p^s}(d) = \sum_{d|p^s} A_{p^s/d}.$$

Since the divisors of p^s are of the form p^i where $0 \leq i \leq s$, we have

$$N_{p^s} = \sum_{i=0}^{s} A_{p^{s-i}} = \sum_{i=0}^{s} A_{p^i} = A_{p^s} + \sum_{i=1}^{s-1} A_{p^i} + A_1.$$

Using the induction hypothesis for the terms A_{p^i} in the sum, and the fact that $N_1 = A_1 = 1$ we obtain

$$N_{p^s} = A_{p^s} + \sum_{i=1}^{s-1} \left(N_{p^i} - N_{p^{i-1}}\right) + N_1 = A_{p^s} + \left(N_{p^{s-1}} - N_1\right) + N_1 = A_{p^s} + N_{p^{s-1}}. \qquad \square$$

We use the well known Möbius inversion formula from number theory to obtain the result for the general case. We give the definition below for the sake of completeness.

Definition 2.40. *The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined as follows. Given a nonnegative integer n, $\mu(n)$ is the sum of all n-th roots of unity. More explicitly*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ has a factor which is a square} \\ (-1)^k & \text{if } n \text{ is square-free with } k \text{ distinct prime factors.} \end{cases}$$

We next state the Möbius inversion formula.

Theorem 2.41. *[24, Theorem 266] Let f and $F : \mathbb{N} \to \mathbb{Z}$ satisfy*

$$F(n) = \sum_{d|n} f(d) \quad \text{for } n \geq 1.$$

Then

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \cdot F(d) \quad \text{for } n \geq 1.$$

The functions f and F are said to be Möbius transforms of one another.

We can now prove Theorem 1.21.

Proof of Theorem 1.21. Let n be a positive integer. The result follows from Theorem 1.20 and Theorem 2.41 by taking A_n for $f(n)$ and N_n for $F(n)$. $\square$

2.2.3 Monotonicity and related problems

We show the following relation between terms of the sequence (A_n).

Proposition 2.42. *Let $n > 2$ be an integer and let d be a divisor of n, with $d \neq 1$. The following assertions hold.*

(i) *There exist injective maps $\psi_d : \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_n$ and $\psi : \bigcup_{d \neq 1, d|n} \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_n$.*

(ii) *There exist injective maps $\Psi_d : \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_{n+1}$ and $\Psi : \bigcup_{d \neq 1, d|n} \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_{n+1}$.*

(iii)

$$\sum_{d \neq 1, d|n} A_{\frac{n}{d}} \leq A_n \quad \text{and} \quad \sum_{d \neq 1, d|n} A_{\frac{n}{d}} \leq A_{n+1}.$$

In particular $A_{\frac{n}{d}} \leq A_n$ and $A_{\frac{n}{d}} \leq A_{n+1}$.

Proof of Proposition 2.42 : Let d be a divisor of n greater than 1, and let $S \in \mathscr{A}_{\frac{n}{d}}$. Let $\mathrm{P}(S) = \{a_1, \ldots, a_k, n/d\}$ where $a_i < n/d$ for all i.

We first consider part (i). Let $\psi_d : \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_n$ be given by

$$\psi_d(S) = \langle d \cdot a_1, \ldots, d \cdot a_k, n-1, n \rangle.$$

We note that $\mathrm{P}(\psi_d(S)) = \{d \cdot a_1, \ldots, d \cdot a_k, n-1, n\}$, and n is the maximum primitive of $\psi_d(S)$. In particular $\psi_d : \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_n$ is well defined and injective.
Let $\psi : \bigcup_{d|n, d\neq 1} \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_n$ be given by

$$\psi(S) = \psi_d(S) \quad \text{for } S \in \mathscr{A}_{\frac{n}{d}}.$$

It remains to show that ψ is injective. Suppose that $\psi(S) = \psi(T)$ for some $S \in \mathscr{A}_{\frac{n}{d}}$ and $T \in \mathscr{A}_{\frac{n}{q}}$ where d, q are divisors of n with $d, q \neq 1$. Then $\psi_d(S) = \psi_q(T)$ and so

$$\gcd(\mathrm{P}(\psi_d(S)) \setminus \{n-1, n\}) = \gcd(\mathrm{P}(\psi_d(T)) \setminus \{n-1, n\}).$$

Hence $d = q$. Since ψ_d is injective we obtain $S = T$ and so ψ is injective.

We now consider part (ii). We can argue as in part (i) where $\Psi_d : \mathscr{A}_{\frac{n}{q}} \to \mathscr{A}_{n+1}$ is given by

$$\psi_d(S) = \langle d \cdot a_1, \ldots, q \cdot a_k, n, n+1 \rangle$$

and $\Psi : \bigcup_{d|n, d\neq 1} \mathscr{A}_{\frac{n}{d}} \to \mathscr{A}_{n+1}$ is given by

$$\Psi(S) = \Psi_d(S) \quad \text{for } S \in \mathscr{A}_{\frac{n}{d}}.$$

We finally consider part (iii). This follows from parts (i) and (ii) by considering the cardinality of the sets and the injectivity of the maps. □

Next we have a lemma that we will use to prove Theorem 1.19. Let n be a positive integer. We denote the set $\{d : d|n+2 \text{ and } d > 2\}$ by X. We also adopt the convention that $N_r = 0$ and $A_r = 0$ if $r \in \mathbb{Q} \setminus \mathbb{Z}$.

Lemma 2.43. *Let $n \geq 2$ be an integer. Then*

$$N_{n+2} - A_{n+2} \leq 4 \times 2^{\frac{n}{4}} + 8\sqrt{n} \cdot 2^{\frac{n}{6}}.$$

Proof. By Theorem 1.20

$$N_{n+2} - A_{n+2} = \sum_{d \in X \cup \{2\}} A_{\frac{n+2}{d}}.$$

Proposition 1.15 applied to the right hand side of the above yields

$$\sum_{d \in X \cup \{2\}} A_{\frac{n+2}{d}} \leq N_{\frac{n+2}{2}} + \sum_{d \in X} N_{\frac{n+2}{d}}.$$

Let $n \geq 2$. By Lemmas 2.19 and A.3, and Proposition 2.4 in turn give

$$\sum_{d \in X \cup \{2\}} A_{\frac{n+2}{d}} \leq N_{\frac{n+2}{2}} + |X| \left(4 \times 2^{\frac{n}{6}}\right) \leq N_{\frac{n+2}{2}} + 2\sqrt{n} \left(4 \times 2^{\frac{n}{6}}\right) < 4 \times 2^{\frac{n}{4}} + 8\sqrt{n} \cdot 2^{\frac{n}{6}}.$$

The result follows. □

We finally prove Theorem 1.19.

Proof of Theorem 1.19 : For $n \leq 58$, one can verify the result using Table 2.1. For $n = 59$, we have $n+2$ as prime, and thus we have by Proposition 1.15, $A_{61} = N_{61} - 1$. Moreover, using Corollary 2.18 and Lemma 2.20(iii), we have $N_{61} - N_{59} > 2^{\frac{61}{3}-3} > 2^{17}$, and thus $A_{59} < N_{59} < A_{61}$. Therefore we assume from now onward that $n \geq 60$. By Lemma 2.43

$$N_{n+2} - A_{n+2} \leq 4 \times 2^{\frac{n}{4}} + 8\sqrt{n} \cdot 2^{\frac{n}{6}}.$$

One can check using calculus that for $n \geq 60$ we have

$$4 \times 2^{\frac{n}{4}} + 8\sqrt{n} \cdot 2^{\frac{n}{6}} \leq 2^{\frac{n+2}{3}-3}.$$

Thus Lemma 2.20 yields

$$N_{n+2} - A_{n+2} \leq 2^{\frac{n+2}{3}-3} \leq |\mathscr{D}_{n+2}(4)|,$$

and combining this with Corollary 2.18, we get

$$N_{n+2} - A_{n+2} \leq |\mathscr{D}_{n+2}(4)| \leq N_{n+2} - N_n.$$

Hence $A_{n+2} \geq N_n$. Finally, we use Proposition 1.15 to get

$$A_n < N_n \leq A_{n+2}. \quad \square$$

2.2.4 Distribution of Frobenius number under fixed multiplicity

Let m be a positive integer and let I be the (integer) interval $[m, 2m)$. In this section we discuss the distribution of Frobenius number in the set

$$\Omega_m = \{\langle A \rangle : A \subseteq [m, 2m), \gcd(A) = 1\}$$

as m tends to infinity.

Remark 2.44. Since the number of sets $A \subseteq [m, 2m)$ with $\gcd(A) > 1$ is negligible compared to the size of the set $2^{[m,2m)}$ in the sense of Corollary A.2, we can ignore this constraint for asymptotic considerations.

Let $N(m)$ be a monotonically increasing function in the variable m bounded from above pointwise by $m/2$ such that $m\left(\frac{3}{4}\right)^{N(m)} \to 0$. Note that, in particular, we may take $N(m)$ to be εm for some real number $\varepsilon < 1/2$. Let

$$X_m := \{A \subseteq [m, 2m) : [2m + N(m), 3m + N(m)] \subseteq 2A\}.$$

Theorem 2.45. *The following assertion holds.*

$$\lim_{m\to\infty} \frac{|X_m|}{2^m} = 1.$$

Proof. Consider any element $x \in [2m+2N(m), 3m+2N(m)]$, and thus $\frac{x}{2} \in [m+N(m), \frac{3m}{2}+N(m))$. We consider a set $B \subseteq [m,2m)$ such that $x \notin 2B$. Clearly, this is equivalent to not having any rational number r such that $\{\frac{x}{2}-r, \frac{x}{2}+r\} \subseteq B$. We refer to this by saying that B is *anti-symmetric* about $\frac{x}{2}$. Thus, in other words, if $\frac{x}{2}-r$ is assumed to be an integer then either $\frac{x}{2}+r \notin B$ or $\frac{x}{2}-r \notin B$ or both $\frac{x}{2}-r$ and $\frac{x}{2}+r$ are not in B. Thus exactly three subsets of $\{\frac{x}{2}-r, \frac{x}{2}+r\}$ can be a contained as a subset of an anti-symmetric set, and this is true independently for each r for which $\frac{x}{2}-r$ is an integer.

We therefore conclude that the number of antisymmetric subsets of $[m,2m)$ about a point $k \in [m,2m)$ is $3^{\alpha_k} \times 2^{m-2\alpha_k}$ where $\alpha_k = \min\{k-m,\ 2m-k\}$. Thus the proportion of such subsets in the power set $2^{[m,2m)}$ as m tends to 0 as

$$\lim_{m\to\infty} \left(\frac{3^{\alpha_k}}{2^{2\alpha_k}}\right) = \lim_{m\to\infty} \left(\frac{3}{4}\right)^{\alpha_k}.$$

If $k = \frac{x}{2}$ then $\alpha_k > N(m)$ and thus

$$\lim_{m\to\infty} \left(\frac{3}{4}\right)^{\alpha_k} = 0.$$

Next we consider the subsets $C \subseteq [m,2m)$ such that for some $x \in [2m+N(m), 3m+N(m))$ we have $x \notin 2C$. Thus C is antisymmetric about at least one rational number $k \in [m,2m)$. The proportion of such subsets in the power set $2^{[m,2m)}$ is at most

$$\sum_{i\le m} \max_k \left\{\left(\frac{3}{4}\right)^{\alpha_k}\right\} \le m \times \max_k \left\{\left(\frac{3}{4}\right)^{\alpha_k}\right\} \le m \cdot \left(\frac{3}{4}\right)^{N(m)}$$

which tends to 0 as m tends to infinity. This completes the proof. □

Corollary 2.46. *Given any positive real number $\varepsilon < 1/2$, as m tends to infinity, almost every semigroup in Ω_m has its Frobenius number less than $2m+\varepsilon m$. In particular, almost all numerical semigroups in Ω_m satisfy* $\mathrm{F} < 3\,\mathrm{m}$.

Chapter 3

On Wilf's conjecture

Let S be a numerical semigroup. Wilf asked the following question in [38] regarding the fraction of gaps in the interval $[0, \mathrm{c}(S)]$: Is the fraction $\frac{\mathrm{g}(S)}{\mathrm{c}(S)}$ always less than or equal to $1 - \frac{1}{p(S)}$? Nowadays the conjecture that this question has an affirmative answer is referred to as *Wilf's conjecture*. In equivalent terms, this can be expressed in the following way.

Conjecture 3.1. *The following inequality holds for every numerical semigroup S:*

$$p(S) \cdot |\mathrm{L}(S)| \geq \mathrm{c}(S).$$

This simple-to-state inequality has been the subject of a lot of investigations for several decades, and although it remains unresolved, it has been shown to be true in numerous special cases (see [10] for a survey). We next define the related notion of Wilf number of a numerical semigroup. Let $\ell(S)$ denote $|L(S)|$.

Definition 3.2. *We have a map $\mathrm{W} : \mathbb{S} \to \mathbb{Z}$ associated with Wilf's conjecture given by*

$$\mathrm{W}(S) := p(S)\ell(S) - c(S)$$

and the number $\mathrm{W}(S)$ is known as the Wilf number *of S.*

Thus Wilf's conjecture can equivalently be posed as the claim that the inequality $W(S) \geq 0$ holds for every numerical semigroup S. Moreover, we say that a numerical semigroup S is a *Wilf semigroup* (or simply *Wilf*) if it satisfies Wilf's conjecture.

Wilf considered the special case $W(S) = 0$ and asked the following problem:

Question 3.3.

$$\text{Is } W^{-1}(0) = \{O_n : n \in \mathbb{N}\} \cup \{\langle a, b\rangle : a, b \in \mathbb{N} + 1, \gcd(a, b) = 1\}?$$

The answer to this is *no*, since if we consider the case of numerical semigroups

$$m\mathbb{N} \cup O_{km} = m\mathbb{N} \cup (\mathbb{N} + km) = \{0, m, 2m, \ldots, km, \to\}$$

for $m,k \in \mathbb{N}+1$, then we observe that $W(m\mathbb{N} \cup O_{km}) = 0$. These semigroups together with the ones mentioned in Question 3.3 are the only known semigroups S that satisfy $W(S) = 0$, and it is an open question whether these are the only ones that satisfy it.

Remark 3.4. We observe that when c is a prime and $p\ell = c$, then we have $p = c$ and $\ell = 1$, and thus

$$W^{-1}(0) \cap \mathscr{N}_{c-1} = \{O_c\}.$$

We thus note that not only do the experiments suggest that numerical semigroups satisfy this conjecture, but only a very particular class of semigroups have been observed to satisfy the equality $W(S) = 0$. In other words, experimental observations seem to suggest that for most numerical semigroup the inequality in Wilf's conjecture is *easily* attained, i.e. the Wilf number is large. This will be quantified later by considering some *typical* cases in a more precise sense. We further discuss the problem of how *close* some specified class of numerical semigroups is to being Wilf using the notion of *extent of Wilfness* in Chapter 4.

Another useful notion related to the Wilf number is that of $W_0(S)$ introduced by Eliahou in [15]. The number $W_0(S)$, which we define next in an equivalent way, has been referred to as *Eliahou number* of S in their work by various authors, and we will adopt this nomenclature and denote it by $E(S)$.

Definition 3.5. *Let p_r denote the number of right primitives of a numerical semigroup S i.e. $p_r = |P(S) \cap (\mathrm{F}(S), \infty)|$. Let q be the depth and ℓ be the number of left elements of S. We define the Eliahou number $E(S)$ as*

$$E(S) := W(S) - p_r(\ell - q).$$

We note that $E(S) \leq W(S)$ since $\ell - q$ and p_r are always greater than or equal to 0. Thus if $E(S) \geq 0$ then S satisfies Wilf's conjecture. Moreover, we observe that if p_r is 0 then $E(S)$ is equal to $W(S)$.

We now give a brief survey of the known results that will be important for our work and which, in the precise sense of counting by Frobenius number and genus, are the largest known classes of Wilf semigroups. We finally collect some results related to Wilf's conjecture.

Theorem 3.6. *[25, Proposition 26] Every numerical semigroup S with $\mathrm{F}(S) < 2\,\mathrm{m}(S)$ satisfies Wilf's conjecture.*

A short proof for this was later provided by Eliahou using the notion of Eliahou number in [15, Proposition 3.13], and the result was further improved through the following theorem.

Theorem 3.7. *[15] Numerical semigroups with $2m < \mathrm{F}(S) < 3\,\mathrm{m}(S)$ satisfy Wilf's conjecture.*

Sammartano showed the following result regarding Wilf's conjecture for numerical semigroups with a large enough embedding dimension (with respect to the multiplicity).

Theorem 3.8. *[35] Numerical semigroups which satisfy $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{2}$ also satisfy Wilf's conjecture.*

Theorem 3.9. *[16] Numerical semigroups which satisfy $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{3}$ also satisfy Wilf's conjecture.*

Therefore, as observed by Kaplan and Singhal, this result together with Proposition 2.23 implies the following corollary.

Corollary 3.10. *The probability that a randomly chosen numerical semigroup in $\mathscr{G}_g$ satisfies Wilf's conjecture tends to* 1 *as g tends to infinity.*

Note that the above corollary can also be obtained by using Proposition 2.22 and Theorems 3.6 and 3.7. Similarly, Proposition 2.7 together with Theorems 3.6 and 3.7 yields the following corollary.

Corollary 3.11. *The probability that a randomly chosen numerical semigroup in $\mathscr{N}_f$ satisfies Wilf's conjecture tends to* 1 *as f tends to infinity.*

We next discuss the analogous asymptotic problem to Corollaries 3.10 and 3.11 under counting numerical semigroups by maximum primitive – Does almost every semigroup in $\mathscr{A}_n$ satisfy Wilf's conjecture as n tends to infinity? We answer this question positively. To prove this, we observe an elementary criterion on the number of primitives in the interval $(\mathrm{m}(S), 2\,\mathrm{m}(S))$ of a numerical semigroup S for it to satisfy Wilf's conjecture.

We begin by stating the well known case of the numerical semigroups S which satisfy $|\mathrm{P}(S)| = \mathrm{m}(S)$. Such numerical semigroups are referred to as having *maximal embedding dimension* due to Lemma 1.7. The following result can be found in [14, Corollary 2.4] but we observe that a simpler direct calculation suffices.

Lemma 3.12. *A numerical semigroup with maximal embedding dimension satisfies Wilf's conjecture.*

Proof. Let S be a numerical semigroup and let m be its multiplicity. Suppose now that S has maximal embedding dimension so that $|\mathrm{P}(S)| = m$. Recall that $\mathrm{q}(S)$ is the depth of S. We have $|\mathrm{L}(S)| \geq \mathrm{q}(S)$ since

$$\{0, m, 2m, \ldots, (q-1)m\} \subseteq \mathrm{L}(S).$$

Therefore

$$|\mathrm{P}(S)| \cdot |\mathrm{L}(S)| = m \cdot |\mathrm{L}(S)| \geq \mathrm{q}(S)m > \mathrm{F}(S). \qquad \square$$

We now give a simplified proof of the result of Kaplan [25, Proposition 26] regarding Wilf's conjecture for numerical semigroups with depth less than or equal to 2.

Proposition 3.13. *Every numerical semigroup S with* $\mathrm{F}(S) < 2\,\mathrm{m}(S)$ *satisfies Wilf's conjecture.*

Proof. Let f, m be integers such that $f < 2m$. Let S be a numerical semigroup with Frobenius number $\mathrm{F}(S) = f$ and multiplicity $\mathrm{m}(S) = m$. We note that if $\mathrm{F}(S) \leq 0$ then $S = \mathbb{N}$ which satisfies Wilf's conjecture since $|\mathrm{P}(\mathbb{N})| \cdot |\mathrm{L}(\mathbb{N})| = 0$ and $\mathrm{F}(\mathbb{N}) = -1$. Therefore we assume that f and m are positive. Let r denote the number of primitives of S that are greater than f. Let ℓ denote $|\mathrm{L}(S)|$. Note that the number of primitives that are smaller than f is precisely $|\mathrm{L}(S) \setminus \{0\}| = \ell - 1$. We divide the proof into two cases depending on whether $r \geq \frac{f}{2}$ or $r < \frac{f}{2}$.

Suppose first that $r \geq \frac{f}{2}$. We consider three possibilities, namely $\ell \geq 3$, $\ell = 2$ and $\ell = 1$. We first assume that $\ell \geq 3$. Then S satisfies Wilf's conjecture since $|\mathrm{P}(S)| \cdot |\mathrm{L}(S)| \geq \frac{f}{2} \times 3 > f$. We next assume that $\ell = 2$. Then $\mathrm{L}(S) = \{0, m\}$ and therefore $r = m - 1$. Thus $S = \langle \{m\} \cup \{f+1, f+2, \ldots, f+m\} \setminus \{2m\} \rangle$ has maximal embedding dimension. Thus S satisfies Wilf's conjecture by Lemma 3.12. We finally assume that $\ell = 1$. Then $\mathrm{L}(S) = \{0\}$, and thus $S = \langle m, m+1, \cdots, 2m-1 \rangle$ has maximal embedding dimension and it satisfies Wilf's conjecture by Lemma 3.12.

We now suppose that $r < \frac{f}{2}$. Consider the interval $I = (f, f+m+1)$ which contains all the r primitives greater than f since the elements greater than $f+m+1$ are not primitives in S. Assume that $x \in I$ is not a primitive. Then $x = a+b$ for some primitives $a, b \in \mathrm{L}(S)$. Since there are $\ell - 1$ primitives in $\mathrm{L}(S)$, the number of elements in I which are not primitives is less than $\binom{\ell}{2}$. Thus

$$\frac{f}{2} - r < m - r < \binom{\ell}{2}.$$

In particular

$$f < \ell(\ell-1) + 2r.$$

Assume for contradiction that S does not satisfy Wilf's conjecture. Then

$$|\mathrm{P}(S)| \cdot |\mathrm{L}(S)| = (r + \ell - 1) \cdot \ell \leq f.$$

Combining the above two inequalities, we get

$$\ell(r+\ell-1) < \ell(\ell-1) + 2r$$

and so

$$r + \ell - 1 < \ell - 1 + \frac{2r}{\ell}.$$

Therefore $\ell = 1$ and $S = \langle m, m+1, \cdots, 2m-1 \rangle$. Since S has maximal embedding dimension it satisfies Wilf's conjecture by Lemma 3.12, a contradiction. □

Analogous to Proposition 2.7, we show the following result regarding the distribution of multiplicities in $\mathscr{A}_n$ which will be required later.

Proposition 3.14. *For any real number $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for every $n \in \mathbb{N}$*

$$\left| \left\{ S \in \mathscr{A}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| > N_\varepsilon \right\} \right| < \varepsilon 2^{n/2}.$$

Proof. Let $\varepsilon > 0$. By Proposition 2.7, there exists $N_\varepsilon \in \mathbb{N}$ such that for every $n \in \mathbb{N}$

$$\left| \left\{ S \in \mathscr{N}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| > N_\varepsilon \right\} \right| < \varepsilon 2^{n/2}.$$

For $S \in \mathscr{A}_n$, $\mathrm{m}(S) = \mathrm{m}(\Phi(S))$ and $n = \mathrm{F}(\Phi(S))$ (by definition of the map Φ). Moreover, the injectivity of Φ yields

$$\begin{aligned} \left| \left\{ S \in \mathscr{A}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| > N_\varepsilon \right\} \right| &\leq \left| \left\{ \Phi(S) \in \mathscr{N}_n : \left| \mathrm{m}(\Phi(S)) - \frac{\mathrm{F}(\Phi(S))}{2} \right| > N_\varepsilon \right\} \right| \\ &= \left| \left\{ \Phi(S) \in \mathscr{N}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| > N_\varepsilon \right\} \right| \\ &< \varepsilon 2^{n/2}. \end{aligned}$$

□

We note that for every numerical semigroup S, we have $S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S)) \subseteq \mathrm{P}(S)$ and therefore

$$S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S)) = \mathrm{P}(S) \cap (\mathrm{m}(S), 2\,\mathrm{m}(S)). \tag{3.1}$$

Lemma 3.15. *The following assertion holds:*

$$\lim_{n\to\infty} \frac{\left|\left\{S \in \mathscr{A}_n : |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)}\right\}\right|}{A_n} = 0.$$

In other words, the probability that a randomly chosen numerical semigroup S from $\mathscr{A}_n$ satisfies the condition $|S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| \geq \sqrt{3\,\mathrm{m}(S)}$ tends to 1 as n tends to infinity.

Proof. Given $\varepsilon > 0$ let $N_\varepsilon \in \mathbb{N}$ be as in Proposition 3.14. In case this N_ε is less than 5, we instead replace it by setting N_ε to be 5 for reasons that will be evident later. Let n be an integer larger than $50N_\varepsilon^2$. Throughout for $x \in \mathbb{R}$ we abuse notation and write $\binom{n}{x}$ for $\binom{n}{\lceil x \rceil}$. Let $m \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2} + N_\varepsilon\right]$. We split the calculations into two cases depending on whether m is larger than $\frac{n}{2}$ or not.

We first suppose that $m \in \left(\frac{n}{2}, \frac{n}{2} + N_\varepsilon\right]$. Let α be the number of semigroups $S \in \mathscr{A}_n$ with multiplicity $\mathrm{m}(S) = m$ and $|S \cap (m, 2m)| < \sqrt{3m}$. Note that α is equal to the number of subsets X of (m, n) with size less than $\sqrt{3m} - 1$ such that the greatest common divisor of $X \cup \{m, n\}$ is equal to 1. In particular, α is bounded above by the number of subsets of (m, n) with size less than $\sqrt{3m}$. Thus an upper bound for α is given by

$$\sum_{m \in \left(\frac{n}{2}, \frac{n}{2} + N_\varepsilon\right]} \sum_{k < \sqrt{3m}} \binom{n-m}{k}.$$

Let n' be a positive integer. We note that for integers $x, y \in (0, n')$ with $x < y$ such that $x + y < n'$, we have $\binom{n'}{x} < \binom{n'}{y}$. This is true, in particular, if we additionally have $x < y < \frac{n'}{2}$. Let

$$x = k < \sqrt{3m}, \quad y = \sqrt{2n} \quad \text{and } n' = n - m.$$

The inequality $x < y$ is satisfied since $k < \sqrt{3m} < \sqrt{2n}$ if we choose n larger than $6N_\varepsilon$. The inequality $y < \frac{n'}{2}$ is satisfied by choosing n larger than $50N_\varepsilon^2$, which is larger than $6N_\varepsilon$. Therefore taking $n > 50N_\varepsilon^2$ we get $\binom{n-m}{k} < \binom{n-m}{\sqrt{2n}}$ for all $k < \sqrt{3m}$. Thus

$$\begin{aligned} \sum_{m \in \left(\frac{n}{2}, \frac{n}{2} + N_\varepsilon\right]} \sum_{k < \sqrt{3m}} \binom{n-m}{k} &\leq \sum_{m \in \left(\frac{n}{2}, \frac{n}{2} + N_\varepsilon\right]} \sqrt{2n} \cdot \binom{n-m}{\sqrt{2n}} \\ &\leq N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}}. \end{aligned}$$

One can verify the following limit (see Lemma B.1)

$$\lim_{n\to\infty} \frac{N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}}}{2^{n/2}} = 0.$$

Thus there exists $M_0 \in \mathbb{N}$ such that for every $n > M_0$ we have

$$\frac{N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}}}{2^{n/2}} < \frac{\varepsilon}{8}.$$

By Lemma 2.35, $A_n \geq \frac{3}{4} \times 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ for $n > 8$. In particular $A_n \geq \frac{1}{4} \times 2^{n/2}$. This inequality holds provided that M_0 is sufficiently large. Without loss of generality we therefore have

$$N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}} < \frac{\varepsilon}{8} \cdot 2^{n/2} \leq \frac{\varepsilon}{2} \cdot A_n.$$

Therefore there exists $M_0 \in \mathbb{N}$ such that for all $n > M_0$

$$\left| \left\{ S \in \mathscr{A}_n : \mathrm{m}(S) \in \left(\frac{n}{2}, \frac{n}{2} + N_\varepsilon \right], |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)} \right\} \right| < \frac{\varepsilon}{2} A_n.$$

We suppose next that $m \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2} \right)$. Let β be the number of semigroups $S \in \mathscr{A}_n$ with multiplicity $\mathrm{m}(S) = m$ and $|S \cap (m, 2m)| < \sqrt{3m}$. Note that by (3.1) every element in $(m, 2m)$ is primitive. Therefore such numerical semigroups can be constructed by first choosing fewer than $\sqrt{3m}$ elements in $(m, 2m)$ and then choosing more primitives from $(2m, n)$ such that the greatest common divisor of the set of primitives is 1. Therefore, arguing as before, under the assumption that $n > 2N_\varepsilon^2$ and $N_\varepsilon > 5$, the number β is bounded above by

$$\begin{aligned} \sum_{m \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2} \right)} \sum_{k < \sqrt{3m}} \binom{m}{k} \cdot 2^{n-2m} &\leq \sum_{m \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2} \right)} \sqrt{2n} \cdot \binom{m}{\sqrt{2n}} \cdot 2^{n-2m} \\ &\leq N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}} \cdot 2^{2N_\varepsilon}. \end{aligned}$$

Therefore in case the N_ε provided by Proposition 3.14 is less than 5, we replace it by 5. We then choose n larger than $2N_\varepsilon^2$. Also one can verify (from the calculations in Lemma B.1) that

$$\lim_{n \to \infty} \frac{N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}} \cdot 2^{2N_\varepsilon}}{2^{n/2}} = 0.$$

Thus arguing as before we obtain for some sufficiently large integer M_1 and $n > M_1$

$$N_\varepsilon \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}} \cdot 2^{2N_\varepsilon} < \frac{\varepsilon}{8} \cdot 2^{n/2} \leq \frac{\varepsilon}{2} \cdot A_n.$$

Thus there exists $M_1 \in \mathbb{N}$ such that for all $M > M_1$

$$\left| \left\{ S \in \mathscr{A}_n : \mathrm{m}(S) \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2} \right), |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)} \right\} \right| < \frac{\varepsilon}{2} A_n.$$

Now taking $M' = \max\{M_0, M_1\}$, we obtain the following result:

Given $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and $M' \in \mathbb{N}$ such that for all $n > M'$

$$\left| \left\{ S \in \mathscr{A}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| \leq N_\varepsilon, |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)} \right\} \right| < \varepsilon A_n.$$

By Proposition 3.14

$$\left| \left\{ S \in \mathscr{A}_n : \left| \mathrm{m}(S) - \frac{n}{2} \right| > N_\varepsilon, |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)} \right\} \right| < \varepsilon A_n.$$

Therefore given $\varepsilon > 0$ there exists $M' \in \mathbb{N}$ such that for all $n > M'$

$$\left|\left\{S \in \mathscr{A}_n : |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)}\right\}\right| < 2\varepsilon A_n.$$

In other words, we conclude that

$$\lim_{n\to\infty} \frac{\left|\{S \in \mathscr{A}_n : |S \cap (\mathrm{m}(S), 2\,\mathrm{m}(S))| < \sqrt{3\,\mathrm{m}(S)}\}\right|}{A_n} = 0. \qquad \square$$

Corollary 3.16. *The probability that a randomly chosen numerical semigroup $S \in \mathscr{A}_n$ satisfies the condition $|\mathrm{P}(S)| \geq \sqrt{3\,\mathrm{m}(S)}$ tends to 1 as n tends to infinity.*

We now prove Theorem 1.22.

Proof of Theorem 1.22. Let $n, d, m \in \mathbb{N}$ and $S \in \mathscr{A}_n$ be such that the depth $\mathrm{q}(S) = d$ and the multiplicity $\mathrm{m}(S) = m$. Suppose that $|S \cap (m, 2m)| \geq \sqrt{3m}$. By (3.1) we then have $|\mathrm{P}(S) \cap (m, 2m)| \geq \sqrt{3m}$. We also note that $dm \geq \mathrm{F}(S) + 1$. We have the following possibilities.

If $d \leq 2$ then S satisfies Wilf's conjecture by Proposition 3.13. Suppose that $d \geq 3$. Then for every $x \in \mathrm{P}(S) \cap (m, 2m)$, we have

$$\{x, x+m, \ldots, x+(d-3)m\} \subset \mathrm{L}(S).$$

Therefore

$$|\mathrm{L}(S)| \geq (d-2) \cdot |\mathrm{P}(S) \cap (m, 2m)|.$$

Since $|\mathrm{P}(S) \cap (m, 2m)| \geq \sqrt{3m}$, and $d \geq 3$ we get

$$|\mathrm{P}(S)| \cdot |\mathrm{L}(S)| \geq |\mathrm{P}(S)| \cdot (d-2) \cdot |\mathrm{P}(S) \cap (m, 2m)| \geq (d-2) \cdot 3m \geq dm \geq \mathrm{F}(S) + 1.$$

Thus S satisfies Wilf's conjecture, and we are done. $\square$

We finally prove Theorem 1.23.

Proof of Theorem 1.23. This follows from Lemma 3.15 and Theorem 1.22. $\square$

We observe that Theorem 1.22 provides an infinite class of numerical semigroups which were previously not known to satisfy Wilf's conjecture. We justify this by comparing this class with the two known large classes of numerical semigroups which satisfy Wilf's conjecture and which have non-trivial intersections with the one in Theorem 1.22 – the numerical semigroups which satisfy $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{3}$ (see [16]), and those which satisfy $\mathrm{F}(S) < 3\,\mathrm{m}(S)$ (see [25] and [15]). The following construction provides a subset of this infinite class.

Let $m > 30$ be an integer and A be the interval $\left[m, m + \left\lfloor \frac{m-2}{3} \right\rfloor\right]$. We note that since $m > (3\sqrt{105} + 31)/2 \simeq 30.87$ we have $|A| \geq \frac{m-2}{3} > \sqrt{3m}$, and thus A contains subsets that have at least $\sqrt{3m}$ elements. Consider a set $B \subseteq A$ such that $m \in B$, $|B| \in \left[\sqrt{3m}, m/3\right)$ and $\gcd(B) = 1$. Let S_1 and S_2 be the numerical semigroups $\langle A \rangle$ and $\langle B \rangle$ respectively, for which we note that $\mathrm{m}(S_1) = \mathrm{m}(S_2) = m$.

Proposition 3.17. *The numerical semigroup S_2 satisfies the following conditions:*

(i) $|\mathrm{P}(S_2)| < \frac{m}{3}$.

(ii) $|S_2 \cap (m, 2m)| \geq \sqrt{3m}$.

(iii) $\mathrm{F}(S_2) > 3m$.

Proof. We first consider (i) and (ii), which are satisfied since $\sqrt{3m} < |B| < m/3$ and $B = \mathrm{P}(S_2)$. We then consider (iii). We observe that by [22, Corollary 5] we have

$$\mathrm{F}(S_1) = \left\lceil \frac{m-1}{\lfloor (m-2)/3 \rfloor} \right\rceil m - 1 = 4m - 1 > 3m.$$

Since $S_2 \subseteq S_1$, we have $\mathrm{F}(S_2) \geq \mathrm{F}(S_1)$ and therefore $\mathrm{F}(S_2) > 3m$.

□

Thus S_2 satisfies the condition $|S_2 \cap (m, 2m)| \geq \sqrt{3m}$ of Theorem 1.22 whereas $\mathrm{F}(S_2) > 3m$ and $|\mathrm{P}(S_2)| < \frac{m}{3}$. Finally, we observe that by varying m and for different choices of B we obtain infinitely many such numerical semigroups S_2.

Chapter 4

Discussion

In this chapter we explore various details of the topics mentioned in the previous chapters. In the first section, we discuss a direct verification of Wilf's conjecture for numerical semigroups generated by arithmetic sequences, which differs from the proof in [35] where this was first proved. Taking motivation from this calculation, we quantify how *easily* a numerical semigroup satisfies the inequality of Wilf's conjecture in the second section. In the third section, we discuss the overlap between two known large classes of numerical semigroups that satisfy Wilf's conjecture. This proves a conjecture of Delgado and Eliahou on the asymptotic proportion of numerical semigroups with embedding dimension at least a third of their multiplicity (see [16]), which was originally posed under counting by genus, under counting by Frobenius number as well. In the fourth section, we discuss relations between some well-known problems in additive combinatorics and counting by maximum primitive and the Frobenius problem. In particular, we mention some experimental results and discuss their implications for future directions.

4.1 Wilf's Conjecture for semigroups generated by APs

In this section we discuss Wilf's conjecture for numerical semigroups generated by arithmetic progressions. We provide an alternate verification of Wilf's conjecture in this case, a result which was first proved by Sammartano in [35] where he used an inequality involving the *type*[1] of the semigroup to prove the result. In particular we give an alternate derivation for the Frobenius number of an arithmetic sequence, which was first proven in [32].

We recall the notion of Apery elements of a numerical semigroup modulo its multiplicity.

Definition 4.1. *The Apery set corresponding of a numerical semigroup S corresponding to its multiplicity m is*

$$\{s \in S : s - m \notin S\}.$$

In other words it is the collection of all the elements of S which are minimal in the residue classes modulo m.

Proposition 4.2. *Numerical semigroups generated by arithmetic progressions are Wilf.*

[1] the *type* is the number of elements x in $\mathbb{N} \setminus S$ for which $x + S \setminus \{0\} \subseteq S$.

Proof. Let $a,d,n \in \mathbb{N}$ with $\gcd(a,d) = 1$, and $n < a$. Also let $A = \{a, a+d, \ldots, a+nd\}$ and $S = \langle A \rangle$. Let

$$k := \left\lfloor \frac{a-2}{n} \right\rfloor \quad \text{and} \quad y := (d-1)(a-1).$$

We consider corresponding to A in $\mathbb{Z}_a$ the following set

$$A' := \{i \mod a : i \in A\} \subseteq \mathbb{Z}_a.$$

We identify this set with A by denoting $x \mod a$ in $\mathbb{Z}_a$ by x. We thus note that d (which denotes $d \mod a$ in $\mathbb{Z}_a$) is a generator of $\mathbb{Z}_a$. Thus the multiples of d cover the entire group $\mathbb{Z}_a$, i.e. the set $\{0,d,2d,\ldots,(a-1)d\}$ modulo a is equal to $\mathbb{Z}_a$. We observe that $(a-1)d$ corresponds to the residue class modulo a which is generated the last, i.e. the Apery element in the class $(a-1)d$ is larger than all other Apery elements. In particular this means that $(a-1)d$ corresponds to the residue class of the Frobenius number, i.e. $\mathrm{F}(S)$ is $(a-1)d$ modulo a, or $-d$ modulo a.

We next estimate the exact value $\mathrm{F}(S)$. We observe that

$$sA' = \{0,d,2d,\ldots,snd\}$$

for all $s \leq n$, and the elements in sA' are all distinct modulo a if $sA' \neq \mathbb{Z}_a$. Thus we have a sequence of containments $A' \subset 2A' \subset \ldots \subseteq \mathbb{Z}_a$ which, due to finiteness, contains only finitely many distinct terms. Let k be the largest positive integer such that $(k+1)A' = \mathbb{Z}_a$ i.e. k is the largest positive multiple of n such that $|kA| = kn+1 \leq a-1$. Thus

$$k = \left\lfloor \frac{a-2}{n} \right\rfloor.$$

Considering A as a subset of $\mathbb{Z}$ we observe that kA consists of $ka, ka+d, \ldots$, and the largest Apery element is $ka+(a-1)d+a$ where the last term a is added since the set A is preceded by an a-sized interval $[0,a-1]$ which needs to be added prior to considerations involving the sumsets sA for different values of s. Thus the conductor is

$$\mathrm{c}(S) = \mathrm{F}(S)+1 = ((k+1)a+(a-1)d-a)+1 = (k+1)a+(a-1)(d-1).$$

From [31, Theorem 2.3] we have the formula[2]

$$\mathrm{g}(S) = \frac{(k+1)(2a-2-nk)+y}{2}.$$

and thus

$$\ell(S) = \mathrm{c}(S) - \frac{(k+1)(2a-2-nk)+y}{2}$$

which is equal to

$$(k+1)a+(a-1)(d-1) - \left(\frac{(k+1)(2a-2-nk)+(a-1)(d-1)}{2} \right)$$

[2]Note that in [31] there is a small mistake – the formula states $\mathrm{g}(S) = \frac{(k+1)(2a-2-k)+y}{2}$, which is incorrect. There should be $2a-2-nk$ in the numerator of that expression in place of $2a-2-k$.

which simplifies to

$$\frac{1}{2}(a-1)(d-1)+\frac{(k+1)(2+nk)}{2}.$$

Thus the Wilf's conjecture is satisfied since

$$\begin{aligned} p\ell &= (n+1)\left(\frac{1}{2}(a-1)(d-1)+\left(\frac{(k+1)(nk+2)}{2}\right)\right) \\ &= \frac{(n+1)(a-1)(d-1)}{2}+\frac{(k+1)(n+1)(nk+2)}{2} \\ &= \frac{(n-1)(a-1)(d-1)}{2}+\frac{(k+1)(n+1)(nk+2)}{2}+\mathrm{c}(S)-(k+1)a \\ &= \frac{(n-1)(a-1)(d-1)}{2}+(k+1)\left(\frac{(n+1)(nk+2)}{2}-a\right)+\mathrm{c}(S). \end{aligned}$$

Moreover we have $nk \geq a-n-1$ by definition, and therefore

$$\frac{(n+1)(nk+2)}{2}-a \geq \frac{(n+1)(a-n+1)}{2}-a = \frac{a(n-1)-(n+1)(n-1)}{2} \geq 0$$

where the last inequality holds since the embedding dimension $n+1$ is not greater than the multiplicity a. Thus $p\ell \geq c$. □

4.2 The extent of Wilfness

We recall that the Wilf number is a measure of how far a given numerical semigroup is from satisfying the equality in Wilf's conjecture. However, this number is less useful when we deal with large values of p, ℓ, and c, since we are often interested in bounds for these quantities and the accuracy of those bounds often reduces in the absolute term compared to these quantities but may improve relatively. In particular, several asymptotic bounds improve relative to the exact value when these terms become large. For example, consider the numerical semigroup $T = \langle a, b\rangle$. The Frobenius number is given by Sylvester's formula $\mathrm{F}(T) = ab-a-b$. Consider the upper bound on Frobenius number of a numerical semigroup $S = \langle a_1, a_2, \ldots, a_n\rangle$ with $\gcd(a_1, a_n) = 1$ given by $F(S) < a_1 a_n$ (which follows from the Sylvester's formula). Thus, as the maximum primitive b tends to infinity, we observe that the absolute error (the difference of the actual value of the Frobenius number from the bound) is $ab-(ab-a-b)$ which is $a+b$ which tends to infinity. The relative value on the other hand (the ratio of the Frobenius number with the bound) is $\frac{ab-a-b}{ab}$ which is close to 1 as b tends to infinity.

Therefore we define the following rational number to measure how far a given numerical semigroup is from satisfying the equality in Wilf's conjecture in a *relative* sense.

Definition 4.3. *The* extent of Wilf property *or* extent of Wilfness *of a numerical semigroup $S(\neq \mathbb{N})$ is the positive rational number $\omega(S) := p\ell/\mathrm{c}$.*

Thus Wilf's conjecture is equivalent to the extent of Wilfness of every numerical semigroup being greater than or equal to 1. The key idea regarding this concept is that if the extent of Wilfness of a numerical semigroup S is *large enough* (in some sense) then making small changes (addition or

deletion of a *few* primitives) still keeps the extent of Wilfness to be more than 1. We next compute the extent of Wilfness of a few well known classes of numerical semigroups.

We first observe, from the previous discussion regarding the case when $W(S) = 0$, that the extent of Wilfness for the semigroups O_n, $m\mathbb{N} \cup (\mathbb{N} + km)$ and $\langle a, b\rangle$ is exactly the minimum possible value 1, for $n, a, b, k \in \mathbb{N} + 1$ where $\gcd(a, b) = 1$.

We next consider the extent of Wilfness of numerical semigroups generated by arithmetic sequences. We note that the articles [32] and [31] contain the closed form formulas for F and g (and hence for ℓ as well) respectively. One can use them to directly verify the extent of Wilfness of the semigroups generated by arithmetic progressions.

Let $a, d, n \in \mathbb{N}$ with $\gcd(a, d) = 1$ and $n < a - 1$. Also, let

$$x := \left\lfloor \frac{a-2}{n} \right\rfloor \quad \text{and} \quad y := (d-1)(a-1).$$

Proposition 4.4. *Let $A = \{a, a+d, \ldots, a+nd\}$ and $S = \langle A\rangle$. Then the extent of Wilfness $\omega(S)$ is equal to*

$$\frac{n+1}{2}\left(\frac{y + (x+1)(nx+2)}{y + (x+1)a}\right).$$

Proof. From the main result in [32] we obtain

$$\mathrm{F}(S) + 1 = \mathrm{c}(S) = (x+1)a + y.$$

and from [31, Theorem 2.3][3] we have

$$\mathrm{g}(S) = \frac{(x+1)(2a - 2 - nx) + y}{2}.$$

Thus $\ell(S) = \mathrm{F}(S) + 1 - \mathrm{g}(S)$ is equal to

$$\frac{y + (x+1)(nx+2)}{2}.$$

Thus the extent of Wilfness of S is equal to

$$\omega(S) = \frac{(n+1)(y + (x+1)(nx+2))}{2(y + (x+1)a)}.$$ □

We next consider some special cases of the numerical semigroups generated by arithmetic progessions:

1. $A = [a, a+n]$ is an interval, with $n \approx a$ i.e. $n = a - k$ for some small $k \in \left(2, \frac{a}{2}\right)$.
   Thus $d = 1$ and thus in particular $y = 0$. Moreover

$$x = \left\lfloor \frac{a-2}{a-k} \right\rfloor = 1.$$

[3] with the previously mentioned correction

Thus

$$\mathrm{c}(S) = (x+1)a = 2a$$

and

$$\ell(S) = \frac{1}{2}(2(n+2)) = n+2.$$

Therefore the extent of Wilfness becomes

$$\omega(S) = \left(\frac{(n+1)(n+2)}{2a}\right) = \frac{n+1}{2}\left(\frac{n+2}{a}\right) > \frac{n+1}{2}$$

which is much larger than 1 for large n.

2. $A = \{a, a+d\}$.
   This corresponds to the case of the numerical semigroups generated by two generators. Thus in this case we have $x = a-2$. From previous discussions the extent of Wilfness is the minimum possible value 1, and we verify it as follows:

$$\omega(S) = \frac{(1+1)(y+(a-2+1)(a-2+2))}{2(y+(a-2+1)a)} = 1.$$

3. A is such that $n \approx a-k$ for some small $k \ll a$, and $d \ll a$.
   Then $x = 1$, and hence the extent of Wilfness is approximately

$$\omega(S) \approx \frac{n+1}{2}\left(\frac{y+2a}{y+2a}\right) = \frac{n+1}{2}.$$

   Thus this is much larger than 1 for large a.

4. A is such that $a \ll d$ and thus $a \ll nd$.
   In this case, the extent of Wilfness is approximately

$$\omega(S) \approx \frac{n+1}{2}\left(\frac{a(d-1)+(\frac{a}{n})^2}{a(d-1)+(\frac{a^2}{n})}\right) \approx \frac{n+1}{2}$$

   which is much larger than 1.

We have the following result regarding a typical numerical semigroup with depth 2.

Proposition 4.5. *The numerical semigroups with large multiplicity and depth equal to* 2*, typically, have large extent of Wilfness.*

Proof. Let S be a randomly chosen depth 2 numerical semigroup with multiplicity m. This is equivalent to randomly choosing a subset of $[m, 2m)$ under uniform probability distribution. Thus typically it contains at least $m/3$ elements as its left elements (see Appendix B.2) which are also primitives. Thus

$$\omega(S) = \frac{p\ell}{c} \geq \frac{m^2/9}{c} \geq \frac{m^2/9}{2m} = \frac{m}{18}.$$

Thus for large m typically the extent of Wilfness is large. □

4.3 On a question of Delgado and Eliahou

Eliahou showed in [16] that numerical semigroups that satisfy $p(S) \geq \frac{m(S)}{3}$ are Wilf. Delgado observed through experiments that the proportion of such semigroups in $\mathscr{G}_g$ is high (more than 99.999% of semigroups with genus $g \leq 45$ satisfy this criterion). This led them to ask whether the proportion of numerical semigroups S in $\mathscr{G}_g$ that satisfy $p(S) \geq \frac{m(S)}{3}$ tends to 1 as g tends to infinity. Singhal and Kaplan in [27] showed that this result holds. In fact they showed a stronger version – it is sufficient to consider the condition $p(S) \geq \frac{m(S)}{2}$ instead of $p(S) \geq \frac{m(S)}{3}$. We show corresponding result under counting by Frobenius number – that a generic element in $\mathscr{N}_n$ satisfies the condition $p(S) \geq \frac{m(S)}{3}$ as n tends to infinity.

Let n be a positive integer. We recall that a subset of $\mathbb{S}$ is called a generic class under counting by μ if the proportion of size of $\mathbb{S} \cap \mu^{-1}(n)$ in $\mu^{-1}(n)$ tends to 1 as n tends to infinity. We next consider two generic classes which are known to satisfy Wilf's conjecture. The first is the set of numerical semigroups which satisfy the criterion $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{3}$ (see [16]) which is generic under counting by genus by [27, Corollary 7]. The second is the set of those numerical semigroups which satisfy $\mathrm{F}(S) < 3\,\mathrm{m}(S)$ (see [15]) which is generic under counting by genus (see [39]) and under counting by Frobenius number (see Proposition 2.7). We show that the intersection of these two classes is generic in $\mathscr{N}_n$ as n tends to infinity, thus proving an analogous result to Proposition 2.22 proved in [39].

We consider the generic class provided by Proposition 2.7 – given a positive real number ε the numerical semigroups $S \in \mathscr{N}_n$ with $\mathrm{m}(S) \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2} + N_\varepsilon\right)$, where N_ε is the constant provided by that proposition. We investigate the proportion of numerical semigroups in this generic class which satisfy the condition $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{3}$.

Definition 4.6. *Let n be a positive integer. Let $\varepsilon < 1/2$ be a positive real number and N_ε be the integer obtained from Proposition 2.7 corresponding to the given ε. Let $\Lambda_{n,\varepsilon} \subseteq \mathscr{N}_n$ be the set of numerical semigroups S which satisfy the following:*

(i) $\mathrm{m}(S) \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2} + N_\varepsilon\right)$.

(ii) $|\mathrm{P}(S)| \geq \frac{\mathrm{m}(S)}{3}$.

Proposition 4.7. *The probability that a randomly chosen numerical semigroup in $\mathscr{N}_n$ lies in $\Lambda_{n,\varepsilon}$ tends to* 1 *as n tends to infinity.*

Proof. In $\mathscr{N}_n$, a generic element is given by a numerical semigroup S with $\mathrm{m}(S) \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2} + N_\varepsilon\right)$, for any given ε. Let us first consider the semigroups for which $\mathrm{m}(S) = m \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2}\right)$ and $\mathrm{P}(S) < \frac{m}{3}$. In particular, we note that all such numerical semigroups satisfy $|(m,2m) \cap S| < \frac{m}{3}$. Therefore we consider the proportion of numerical semigroups which satisfy $|(m,2m) \cap S| < \frac{m}{3}$ in $\mathscr{N}_n$ as an upper bound for the proportion of those in $\mathscr{N}_n$ which satisfy $\mathrm{P}(S) < \frac{m}{3}$. We then consider the limit of this proportion as n tends to infinity. We have

$$\left|\left\{S \in \mathscr{N}_n : \mathrm{m}(S) = m \in \left[\frac{n}{2} - N_\varepsilon, \frac{n}{2}\right), |S \cap (m,2m)| < \frac{m}{3}\right\}\right| < \sum_{k \leq \frac{m}{3}} \binom{m}{k} \cdot 2^{2N_\varepsilon}.$$

Moreover Lemma B.2 together with Proposition 2.5 implies that

$$\lim_{n\to\infty} \frac{\sum_{k\leq \frac{m}{3}} \binom{m}{k} \cdot 2^{2N_\varepsilon}}{N_n} = 0.$$

Thus the proportion of numerical semigroups S in $\mathscr{N}_n$ with $\mathrm{m}(S) = m \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2}\right)$ and $|(m,2m) \cap S| < \frac{m}{3}$ goes to 0 as n tends to infinity.

Next we consider the numerical semigroups in $S \in \mathscr{N}_n$ with $\mathrm{m}(S) \in \left(\frac{n}{2} + N_\varepsilon, n\right)$ and $|(m,2m) \cap S| < \frac{m}{3}$. We argue as before. In this case we have $2m-n-2$ elements in $(m,2m) \cap S$ since $n+1$, $n+2$, …, $2m-1$ are all contained in $(n,\infty) \subseteq S$. Therefore to get at most $m/3$ elements in $(m,2m) \cap S$ we need to select at most $\frac{m}{3} - 2m + n + 2$ elements in (m,n). Thus the number of numerical semigroups which satisfy this is bounded above by

$$\sum_{k\leq \frac{m}{3}} \binom{m}{k}.$$

Thus the limit of the proportion of such numerical semigroups in $\mathscr{N}_n$, as n tends to infinity, using the calculation in Lemma B.2 is

$$\lim_{n\to\infty} \frac{\sum_{k\leq \frac{m}{3}} \binom{m}{k}}{N_n} \leq \lim_{m\to\infty} \frac{\sum_{k\leq \frac{m}{3}} \binom{m}{k}}{c \cdot 2^m} = 0$$

for some real number c. Thus the proportion of such numerical semigroups in $\mathscr{N}_n$ goes to 0 as n tends to infinity.

Thus combining both the cases, we obtain that the proportion of numerical semigroups $S \in \mathscr{N}_n$ with $\mathrm{m}(S) \in \left(\frac{n}{2} - N_\varepsilon, \frac{n}{2} + N_\varepsilon\right)$ and $|(m,2m) \cap S| < \frac{m}{3}$ tends to 0 as n tends to infinity. In particular this implies that the proportion of numerical semigroups which satisfy $|\mathrm{P}(S)| \geq \frac{m}{3}$ goes to 1 as n tends to infinity. □

Corollary 4.8. *Almost every numerical semigroup S in $\mathscr{N}_n$ with multiplicity m satisfies $|\mathrm{P}(S) \cap (m,2m)| \geq \frac{m}{3}$ as n tends to infinity. In particular, almost every numerical semigroup S in $\mathscr{N}_n$ satisfies $p(S) \geq \frac{m}{3}$ as n tends to infinity.*

4.4 Some connections with Additive Combinatorics

In this section we discuss the relation between the Frobenius problem, in particular for those numerical semigroups with primitive depth 2, and the study of sumsets in additive combinatorics.

We start with a definition. Consider $k,n \in \mathbb{N}$ and let A be a subset of $\mathbb{Z}$ or $\mathbb{Z}_n$.

Definition 4.9. *A restricted k-fold sumset of A is*

$$k^*A = \{a_1 + a_2 + \cdots + a_k \in kA : a_i \neq a_j \text{ for all } i \neq j \text{ where } i,j \in [k]\}.$$

Let S be a numerical semigroup with multiplicity $\mathrm{m}(S) = n$ and $\pi(S) = 2$. Then $\mathrm{P}(S) \subseteq [n,2n)$ and thus the set $A = \mathrm{P}(S) - n$ is a subset of $[0,n)$ and thus it can be seen as a subset of $\mathbb{Z}_n$. Conversely,

for any $A \subset \mathbb{Z}_n$ with $\gcd A = 1$, we can consider A as a subset of $[0,n) \subseteq \mathbb{Z}$, which can be associated with the numerical semigroup $S_A = \langle A+n \rangle$. Note that the $\gcd(A+n)$ might not be the same as $\gcd A$ so we assume the case when they are both equal to 1. This is almost always true asymptotically in the sense of Corollary A.2. Assuming that $\gcd(S_A)$ is 1, we observe the following.

Lemma 4.10. *We have* $\mathrm{q}(S_A) = \left\lceil \frac{\mathrm{F}(S_A)+1}{\mathrm{m}(S_A)} \right\rceil \leq 2 \cdot \min\{k : kA = \mathbb{Z}_n\} \leq 2 \cdot \min\{k : k^*A = \mathbb{Z}_n\}$.

Thus estimates on the size of sumset and restricted sumset of A provide upper bounds for the Frobenius number of S_A in terms of multiples of its multiplicity. We consider the following estimates to see this explicitly.

Proposition 4.11. *[37] If n is odd and $A \subseteq \mathbb{Z}_n$ with $|A| \geq 0.4045n$ then $4^*A = \mathbb{Z}_n$.*

Let S be the numerical semigroup $\langle A \rangle$. Then in particular the above proposition implies that $\mathrm{q}(\langle A \rangle) \leq 8$, since $\max(A)$ is bounded by $2\,\mathrm{m}(S)$, and thus $\mathrm{F}(S) < 8\,\mathrm{m}(S)$. From [15], we know that $\mathrm{F} < 3\,\mathrm{m}$ is a sufficient condition for Wilf's conjecture to hold. Thus an improvement to the above proposition – the conditions on A under which we have $2^*A = \mathbb{Z}_n$, and an improvement to the aforementioned criterion for Wilf's conjecture to hold to $\mathrm{F} < 4\,\mathrm{m}$ will imply that such sets A would generate numerical semigroups of depth 4 which satisfy Wilf's conjecture.

In general, for S with $\pi(S)$ equal to 2 and the $\mathrm{m}(S)$ equal to m, the identification of $\mathrm{P}(S)$ with a subset of $\mathbb{Z}_m$ can be used to study such numerical semigroups.

We next consider a basic notion from additive combinatorics.

Definition 4.12. *A sum-free set $A \subseteq \mathbb{N}$ is a set such that if $a,b \in A$ then $a+b \notin A$.*

Remark 4.13. We observe that for a numerical semigroup S the primitive set $\mathrm{P}(S)$ is a sum-free set since the sum of two primitives cannot be another primitive.

Cameron and Erdös conjectured the following regarding the number of sum-free subsets of $[n]$ in [9], which was later proved by Green.

Theorem 4.14 (Cameron-Erdös-Green)**.** *[23] The number of sum-free subsets contained in $[n]$ is $O(2^{n/2})$ where the constant involved in the big-O notation depends on the parity of n.*

We note that since every set of primitives uniquely determines a numerical semigroup, therefore to count the number of numerical semigroups in $\mathscr{A}_n$ we can, equivalently, count the number of primitive sets contained in $[n]$. Therefore we have the following corollary which gives an asymptotic upper bound for A_n.

Corollary 4.15. *We have $A_n = O(2^{n/2})$, where the constant involved in the big-O notation depends on the parity of n.*

We note that this result can be concluded from Corollary 1.18 as well.

4.5 Some experimental results

Let S be a numerical semigroup with multiplicity m and primitive set P. We denote $|P|$ by p, $|\mathrm{L}|$ by ℓ, and $\mathrm{c}(S)$ by c. We have observed previously that for some classes of numerical semigroups, the inequality in Wilf's conjecture is satisfied by a large margin. This suggests that given such numerical semigroups, a *small* change to them will still preserve the inequality. This leads us to the following definition.

Definition 4.16. *Given a numerical semigroup S and a positive integer t we define the* deformation *of S at t to be the numerical semigroup $S_t := S \cup [t, \infty)$.*

We note that this notion was also used by Eliahou in [15, Notation 3.9]. In fact it was observed that all the numerical semigroups which have Eliahou number negative and genus less than or equal to 60 are actually deformations of numerical semigroup generated by three elements.

We observe that $S_0 = \mathbb{N}$, and for $t \geq \mathrm{c}(S)$ we have $S_t = S_{\mathrm{c}(S)}$. Thus corresponding to S we may consider the finite sequence of numerical semigroups

$$(S_i) = (S_1, S_2, \ldots, S_{\mathrm{c}(S)}).$$

We refer to the numerical semigroups $S_1, S_2, \ldots, S_{\mathrm{c}(S)}$ as the deformations of S. Note that this sequence may contain repetitions – for instance if $t, t+1$ are elements of S then the numerical semigroup S_t is the same as S_{t+1}. We observe how various properties of numerical semigroup S_i change when i varies over $[1, \mathrm{c}(S)]$.

Remark 4.17. We observe the following:

- The embedding dimension p is a non increasing function on the sequence (S_i) for $i \geq m$. For $i < m$, $S_i = O_i$ which has embedding dimension equal to i, thus p increases in $[1, m]$.
- The number of left elements ℓ is a non decreasing function on the sequence.
- The conductor c is a non decreasing function on the sequence.

Moreover, we consider the function $w : [1, \mathrm{c}(S)] \to \mathbb{N}$ given by the Wilf number $w(i) := W(S_i)$. We show this for a couple of cases in Figures 4.1 and 4.3. We also plot the extent of Wilfness in Figure 4.2 for a comparison.

Remark 4.18. The experimental evidence suggests the following patterns on average.

- Initially for small i, the value of $w(i)$ is small and, on average, the function is increasing.
- For i close to the $\mathrm{c}(S)$, on average the value of $w(i)$ decreases with i, with locally the smallest value lying in $[w(\mathrm{c}(S) - \mathrm{m}), w(\mathrm{c}(S))]$.
- For intermediate values of i, the Wilf number is typically large.

The above observations suggest that the Wilf number is typically small in the two extremes (when t in S_t is close to m or close to $\mathrm{c}(S)$), and large for most numerical semigroups in between. Moreover, one of the extremes (when the depth is small) is already known to satisfy Wilf's conjecture

by Propositions 3.6 and 3.7, so we turn to the other extreme, i.e. the case when i is close to $\mathrm{c}(S)$. In particular we observe that the Wilf number is locally the smallest for numerical semigroups which have few right primitives. Thus we pose the following problem.

Question 4.19. Given a numerical semigroup S with $\gcd(\mathrm{L}(S))$ equal to 1, let S' be the numerical semigroup generated by $\mathrm{L}(S)$. Does $W(S') \geq 0$ imply that $W(S'_t) \geq 0$ for all $t \in [m, \mathrm{c}(S')]$?

Note that when $\gcd(\mathrm{L}(S))$ is not equal to 1 then S is known to satisfy Wilf's conjecture (essentially due to having large embedding dimension) as proved in Proposition 7.6 in [15]. Note that an affirmative answer to the above implies that it is sufficient to prove Wilf's conjecture for the numerical semigroups which have no right primitives i.e. when $\mathrm{P}(S) \cap [\mathrm{c}(S), \infty]$ is the empty set. These semigroups are precisely the leaves of the classical semigroup tree described in Figure 2.1.

We prove a particular case from the above discussion, showing that all the numerical semigroups for which the Wilf number is plotted in Figure 4.1 satisfy Wilf's conjecture.

Theorem 4.20. *Let m, b, t be positive integers such that $m < b$ and $\gcd(m,b) = 1$. Then the numerical semigroup $\langle m, b \rangle_t$ is Wilf.*

Proof. Let S denote the numerical semigroup $\langle m, b \rangle_t$, with $\mathrm{F}(S)$ denoted by f. For $t \leq b$ we have the maximum embedding dimension case, so we assume that $t > b$. Let k denote the integer $\left\lfloor \frac{f}{b} \right\rfloor$, and thus we note that $f \in (kb, kb+b)$.

We consider the group $\mathbb{Z}_m$. For any integer x, we abuse notation and let x mod m also be denoted by x. We first observe that b is a generator in $\mathbb{Z}_m$. Thus for any x, y with $x < y < m$ we have xb and yb as two different elements in $\mathbb{Z}_m$, since if $xb - yb = zm$ then $x - y$ is divisible by m, and hence $x = y$. Therefore in S we have the elements $0, b, 2b, \ldots,$ and kb as the smallest elements in their respective residue classes modulo m. Thus the number of left elements in S is

$$\ell(S) = \sum_{i=0}^{k} \left\lceil \frac{f - ib}{m} \right\rceil$$

Thus

$$\ell(S) \geq \sum_{i=0}^{k} \left(\frac{f - ib}{m} \right) + \sum_{i=1}^{k+1} \left(\frac{i}{m} \right)$$

which simplifies to

$$\begin{aligned} \ell(S) &\geq \frac{f}{m} \sum_{i=0}^{k} 1 - \frac{b}{m} \sum_{i=0}^{k} i + \frac{1}{m} \sum_{i=1}^{k+1} i \\ &= \frac{f}{m}(k+1) - \frac{b}{m} \cdot \frac{k(k+1)}{2} + \frac{1}{m} \cdot \frac{(k+2)(k+1)}{2} \\ &= \left(\frac{k+1}{2m} \right) (2f - bk + k + 2) \end{aligned}$$

We next observe that there are two primitives less than f, namely m and b, since we assume that $t > b$. Moreover there are at least $m - (k+2)$ primitives in the interval $(f, f+m)$, which correspond to the elements in residue classes $(k+2)b, \ldots, (m-1)b$. Thus the embedding dimension is at least

$m-k$. Thus the Wilf number $W(S)$ satisfies

$$W(S) = p\ell - \mathrm{c} \geq (m-k)\left(\frac{k+1}{2m}\right)(2f - bk + k + 2) - (f+1).$$

Since $f > bk$ we have

$$W(S) \geq \left(\frac{(m-k)(k+1)}{2m}\right)(f+k+2) - (f+1)$$

which is in fact greater than 0 if

$$\frac{(m-k)(k+1)}{2m} \geq 1 \quad \text{or } m-k \geq 2 + \frac{2}{k-1}.$$

If k is 1, then the embedding dimension is at least $m-1$, and thus this satisfies Wilf's conjecture by Theorem 3.9. If k is greater than 1 and the embedding dimension $m-k$ is at least 4, then $W(S)$ is nonnegative. If $m-k$ is less than or equal to 3 then the result holds via the remark following Theorem 20 in [18], which completes the proof. □

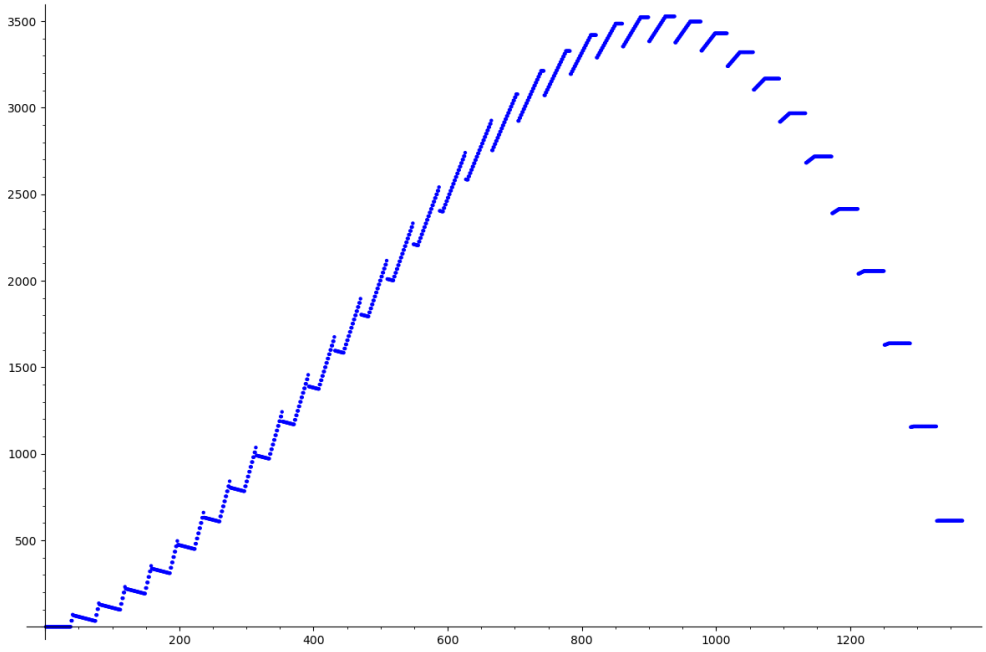


Fig. 4.1 Variation of Wilf number along the deformations of $S = \langle 37, 39 \rangle$.

4.6 Conjectures

We end with some conjectures based on some observed patterns during experiments. We first have an elementary consequence of Proposition 2.5 which gives the limit of the ratio N_{f+2}/N_f as f tends to infinity.

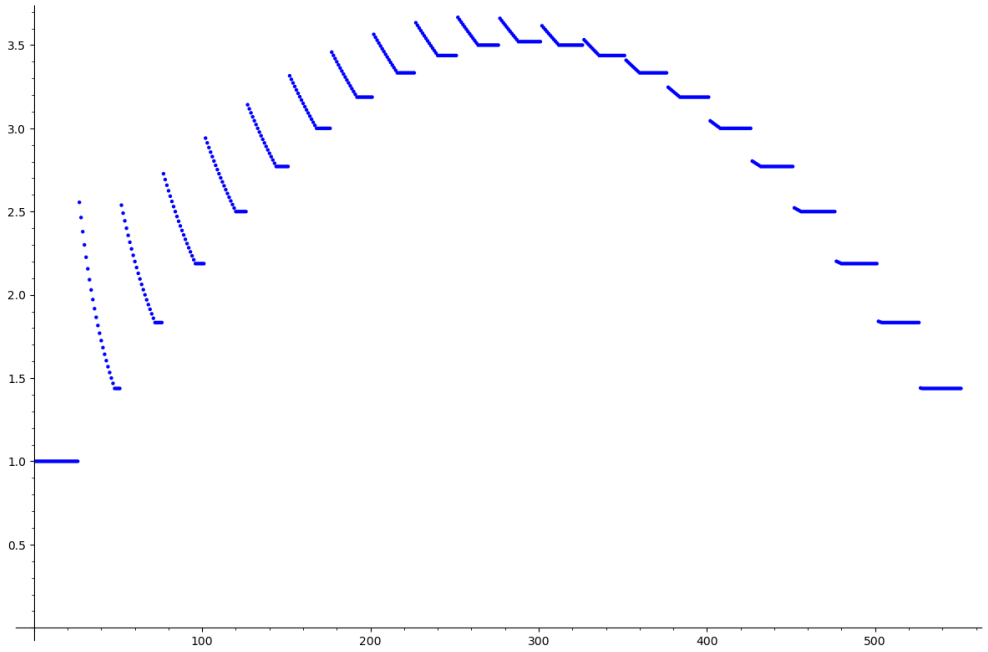


Fig. 4.2 Variations in extent of Wilfness along the deformations of $S = \langle 24, 25 \rangle$.

Corollary 4.21. *The following assertion holds:*

$$\lim_{f \text{ odd}} \left(\frac{N_{f+2}}{N_f} \right) = \lim_{f \text{ even}} \left(\frac{N_{f+2}}{N_f} \right) = 2$$

Proof. By Proposition 2.5 we get

$$\lim_{f \text{ odd}} \left(\frac{N_{f+2}}{N_f} \right) = \lim_{f \text{ odd}} \left(\frac{N_{f+2}/2^{\frac{f}{2}}}{N_f/2^{\frac{f}{2}}} \right) = 2$$

and the same argument gives the result for the case when f is even. □

However not much is known about the ratio $\left(\frac{N_{f+1}}{N_f} \right)$ and its limit. Looking at the Table 2.1 we conjecture the following regarding ratio of the consecutive terms for different parities.

Conjecture 4.22. *The ratio of consecutive terms in (N_f), for different parities, converges to a limit which is as follows:*

$$\lim_{f \text{ even}} \frac{N_{f+1}}{N_f} = 2 \quad \textit{and} \quad \lim_{f \text{ odd}} \frac{N_{f+1}}{N_f} = 1.$$

Also, we have seen in Proposition 1.14 that for f even, $N_f < N_{f+1}$. Thus we observe that the deviation from monotonous behaviour in the sequence (N_f) happens when, for f an odd number, we go from N_f to N_{f+1}. We conjecture that this is true only for finitely many odd values of f.

Conjecture 4.23. *The sequence (N_f) is eventually monotonous i.e. there exists an integer $N \in \mathbb{N}$ such that for all $f > N$, we have $N_f < N_{f+1}$.*

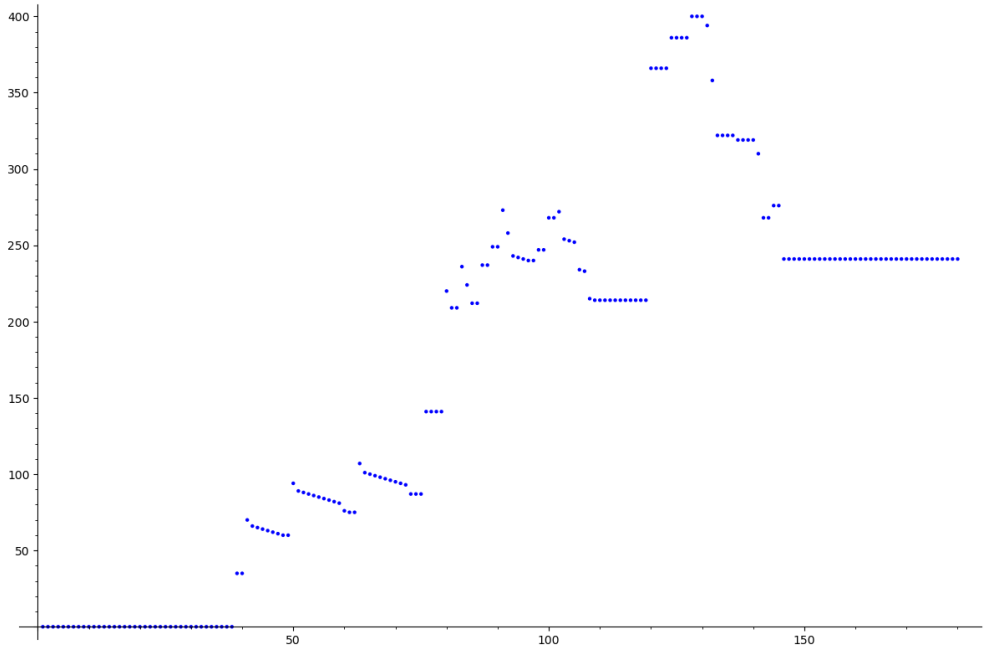


Fig. 4.3 Variation of Wilf number along the deformations of $\langle 37, 39, 48, 61, 73, 77, 81, 85, 89 \rangle$.

Appendix A

Some elementary results

We give a result regarding the number of sets that contain at least one pair of consecutive integers. Given $a,n \in \mathbb{N}$ with $n \geq 1$, we define

$$T_a(n) := \{X \subseteq [a+1, a+n] : \text{ for some } x \in X, \{x, x+1\} \subseteq X \cup \{a+n+1\}\}.$$

and consider its complement

$$R_a(n) = \{X \subseteq [a+1, a+n] : \{x, x+1\} \not\subseteq X \cup \{a+n+1\} \text{ for all } x \in X\}$$

Recall that F_n denotes the n-th *Fibonacci number*.

Lemma A.1. *Given $a,n \in \mathbb{N}$ with $n \geq 1$*

$$|T_a(n)| = 2^n - F_{n+1}.$$

Proof. Note that $|T_a(n)|$ is independent of a. For simplicity, we take $a = 0$ and denote $T_0(n)$ by $T(n)$ and $R_0(n)$ by $R(n)$. The cases where $n \in \{1,2\}$ are easily checked. We therefore assume that $n \geq 3$. We obtain the result by considering the set $R(n)$ which is the complement of $T(n)$ in the power set of $[1,n]$. We denote $|R(n)|$ by r_n. We have the map

$$\gamma : R(n) \to R(n-1) \cup R(n-2)$$

given by

$$\gamma(X) = \begin{cases} X \cap [1, n-2] & \text{if } n-1 \in X \\ X & \text{if } n-1 \notin X. \end{cases}$$

One can easily verify that γ is a bijection. Moreover the inverse of γ, which we denote by γ^{-1}, is given by

$$\gamma^{-1}(Y) = \begin{cases} Y \cup \{n-1\} & \text{if } Y \in R(n-2) \\ Y & \text{if } Y \in R(n-1). \end{cases}$$

Thus, considering the cardinality of the sets involved in this bijection, we get

$$r_n = r_{n-1} + r_{n-2}.$$

Since $r_1 = 1$ and $r_2 = 2$, we can solve this Fibonacci recurrence relation to get

$$r_n = F_{n+1} = \frac{(\varphi_+)^{n+1} - (\varphi_-)^{n+1}}{\sqrt{5}}$$

where $\varphi_+ = \frac{1+\sqrt{5}}{2}$ and $\varphi_- = \frac{1-\sqrt{5}}{2}$. Therefore

$$|T(n)| = 2^n - r_n = 2^n - F_{n+1}.$$ □

Corollary A.2. *Almost every subset of* $[n]$ *has at least one pair of consecutive elements in it. In particular, almost every subset of* $[n]$ *has greatest common divisor* 1*, as n tends to infinity.*

We recall below a well known result regarding the asymptotic size of the set of divisors of a positive number n, and a trivial exact bound for every positive integer n. This will be needed later for calculations.

Lemma A.3. *[24] Let n be a positive integer and let* $\mathrm{d}(n)$ *be the number of positive divisors of n. Then*

(i) $\mathrm{d}(n) = n^{o(1)}$.

(ii) $\mathrm{d}(n) < 2\sqrt{n}$.

Appendix B

Calculations

In this chapter we perform some calculations and obtain their corollaries. We begin with the following inequality:

$$\binom{n}{k} < \left(\frac{n \cdot e}{k}\right)^k \quad \text{if } 0 < k \leq n. \tag{B.1}$$

This is obtained from the power series expansion of the exponential function e^x, which gives $e^k > \frac{k^k}{k!}$ and hence

$$n^k \cdot \frac{e^k}{k^k} > n^k \cdot \frac{1}{k!} > \frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \binom{n}{k}.$$

Lemma B.1. *Let N be a constant. Then the following assertion holds:*

$$\lim_{n\to\infty} \frac{N \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}}}{2^{n/2}} = 0.$$

Proof. Using the above inequality we have

$$\frac{N \cdot \sqrt{2n} \cdot \binom{n/2}{\sqrt{2n}}}{2^{n/2}} < \frac{N \cdot \sqrt{2n} \left(\frac{ne}{2\sqrt{2n}}\right)^{\sqrt{2n}}}{2^{n/2}}.$$

This further simplifies, for some constant b, to

$$\frac{b\sqrt{n} \cdot (\sqrt{n})^{\sqrt{2n}} \cdot (\frac{e}{2\sqrt{2}})^{\sqrt{2n}}}{2^{n/2}} \leq b \cdot \frac{\sqrt{n}^{\sqrt{2n}+1} \cdot (\frac{e}{2\sqrt{2}})^{\sqrt{2n}}}{2^{n/2}}$$

and since $\frac{e}{2\sqrt{2}} (= 0.961...) < 1$ we have

$$b \cdot \frac{\sqrt{n}^{\sqrt{2n}+1} \cdot (\frac{e}{2\sqrt{2}})^{\sqrt{2n}}}{2^{n/2}} \leq b \cdot \frac{\sqrt{n}^{\sqrt{2n}+1}}{2^{n/2}}$$

Hence we have

$$\lim_{n\to\infty} \frac{N \cdot \sqrt{2n} \cdot (\frac{ne}{2\sqrt{2n}})^{\sqrt{2n}}}{2^{n/2}} \leq b \cdot \lim_{n\to\infty} \frac{(\sqrt{n})^{\sqrt{2n}+1}}{\sqrt{2}^n}.$$

Let us define

$$X(n) := \frac{\sqrt{n}^{\sqrt{2n}+1}}{\sqrt{2}^{n}} \leq \left(\frac{\sqrt{n}^{\sqrt{3}}}{\sqrt{2}^{\sqrt{n}}}\right)^{\sqrt{n}}.$$

Note that $X(n)$ is eventually strictly decreasing since $\left(\frac{\sqrt{n}^{\sqrt{3}}}{\sqrt{2}^{\sqrt{n}}}\right)$ becomes less than 1 for all $n > N$ for some large enough $N \in \mathbb{N}$. Thus for some positive real number $\varepsilon < 1$, we have

$$\left(\frac{\sqrt{n}^{\sqrt{3}}}{\sqrt{2}^{\sqrt{n}}}\right) < 1-\varepsilon \quad \text{for all } n > N$$

and therefore $X(n)$ is bounded above by $(1-\varepsilon)^{\sqrt{n}}$, and hence we have

$$\lim_{n\to\infty} X(n) = 0$$

and thus

$$\lim_{n\to\infty} \frac{N\cdot\sqrt{2n}\cdot\binom{n/2}{\sqrt{2n}}}{2^{n/2}} = 0. \qquad \square$$

Lemma B.2. *The following assertion holds:*

$$\lim_{m\to\infty} \frac{\sum_{k\leq\frac{m}{3}}\binom{m}{k}}{2^m} = 0.$$

Remark B.3. Note that by applying the inequality B.1 we have

$$\binom{m}{m/3} \leq \left(\frac{m\cdot e}{\frac{m}{3}}\right)^{\frac{m}{3}} = (3e)^{\frac{m}{3}} \approx (2.0128)^m$$

which, being greater than 2^m, is insufficient to show the required limit. However, since 2.0128 is close to 2, we only need to use a slightly better bound to serve our purpose. Stirling's approximation fulfills this role.

Proof. Note that

$$\sum_{k\leq\frac{m}{3}}\binom{m}{k} \leq \frac{m}{3}\binom{m}{m/3}$$

since

$$\binom{m}{i} \leq \binom{m}{j} \quad \text{for all } i < j < \frac{m}{2}.$$

By Stirling's approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right)$$

therefore for large values of m and some constant C we get

$$\binom{m}{m/3} < \frac{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m}{\left(\sqrt{2\pi\frac{m}{3}}\left(\frac{m}{3e}\right)^{m/3}\right)\left(\sqrt{2\pi\frac{2m}{3}}\left(\frac{2m}{3e}\right)^{2m/3}\right)}\left(1+\frac{1}{12m}+\frac{C}{m^2}\right).$$

This simplifies to

$$\binom{m}{m/3} < \frac{3^m \times 3}{\left(4^{1/3}\right)^m \times \left(2\sqrt{\pi m}\right)} \left(1 + \frac{1}{12m} + \frac{C}{m^2}\right) < \left(\frac{3^m}{(1.587)^m} \times \frac{3}{2\sqrt{\pi m}}\right) \times 2 < (1.8899)^m$$

Thus, since the limit is non negative, we have

$$\lim_{m\to\infty} \left(\frac{\frac{m}{3}\binom{m}{m/3}}{2^m} \right) = \lim_{m\to\infty} \left(\frac{m}{3} \times \left(\frac{1.8899}{2} \right)^m \right) = 0. \qquad \square$$

Corollary B.4. *A randomly chosen subset of $[m]$ has at least $m/3$ elements almost surely under the uniform probability distribution on $[m]$, as m tends to infinity.*

Proof. The calculation in Lemma B.2 shows that the proportion of subsets of $[m]$ with size at most $m/3$ tends to zero, as m tends to infinity. Considering the complementary condition gives the result. $\square$

Corollary B.5. *A randomly chosen semigroup in $\mathscr{N}_f$ with depth equal to 2 has at least $f/6$ left elements almost surely, as f tends to infinity.*

Proof. Recall that we denote $\left\lfloor \frac{f-1}{2} \right\rfloor$ by $\bar{f}$. A set $X \subseteq (\bar{f}, f)$ determines the numerical semigroup $\{0\} \cup X \cup (f, \infty) \in \mathscr{N}_f$ which has depth 2, and conversely every numerical semigroup $S \in \mathscr{N}_f$ with depth 2 determines the set $(\bar{f}, f) \cap S$. Moreover $X \cup \{0\}$ is precisely the set of left elements of such a numerical semigroup.

Thus, after a shift by $\bar{f}$ to the left, the choice of a subset of $[\bar{f}]$ with cardinality k is equivalent to choosing a numerical semigroup of depth 2 with $k+1$ left elements. By Corollary B.4, a generic subset of $[\bar{f}]$ has at least $\bar{f}/3$ elements and thus a typical depth 2 semigroup has at least $\frac{\bar{f}}{3} + 1$ left elements. Thus a generic depth 2 semigroup in $\mathscr{N}_f$ has at least $f/6$ elements. $\square$

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