On linear response for tent maps

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Abstract

This dissertation explores the phenomenon of linear response in the context of dynamical systems, with a particular focus on tent maps. We begin by considering the transfer operator and some of its spectral properties, with the main goal of investigating how physical measures change under small deterministic perturbations of the system. We examine a family of tent maps with varying slopes and show that these systems lack linear response when the critical point is perturbed. However, by modifying the map to include a cusp singularity, we recover linear response.

Keywords: physical measure, transfer operator, tent map, spectral gap, quasi-compactness, linear response.

Resumo

Esta dissertação explora o fenómeno de resposta linear no contexto de sistemas dinâmicos, com particular ênfase nas aplicações do tipo tenda. Começamos por considerar o operador de transferência e algumas das suas propriedades espetrais, tendo como principal objetivo investigar como medidas físicas variam sob pequenas perturbações determinísticas do sistema. Examinamos uma família de aplicações do tipo tenda com diferentes declives e mostramos que estes sistemas não apresentam resposta linear quando o ponto crítico é perturbado. No entanto, ao modificar a aplicação de modo a incluir uma singularidade do tipo cúspide, recuperamos a resposta linear.

Palavras-chave: medida física, operador de transferência, tenda, *spectral gap*, quasi-compacidade, resposta linear.

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Chapter 1

Introduction

In the study of dynamical systems, we seek to understand the behavior of orbits over time. In an ideal world, for a given dynamical system $f : M \to M$, where M is a certain adequate space, one would be able to describe to the set $\{f^n(x) : n \in \mathbb{N}\}$ for every $x \in M$. However, such a goal is almost always impossible to achieve, hence one often seeks to describe the orbits of large sets of initial conditions, in a certain sense. For instance, one may study topologically large sets of initial conditions, e.g. dense sets; if we are dealing with a measure space, then another choice is to study sets which are large in the sense of measure.

Let $(M, \mathcal{B}, \mu)$ be a probability measure space and μ be invariant by the dynamics, that is $\mu(f^{-1}(A)) = \mu(A)$, for any measurable set A. Many classical results shed light on how the dynamical system evolves over time. One of the many examples of such results is Birkhoff's ergodic theorem, which tells us that, for any integrable observable $\varphi : M \to \mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

exists at μ-almost every point $x \in M$. If the measure μ is also ergodic, i.e. there is no measurable set such that $f^{-1}(A) = A$ and $0 < \mu(A) < 1$, then it also tells us that the limit above equals $\int \varphi \, d\mu$. Oftentimes, this is quite satisfactory: we conclude that, given an integrable observable φ, the time averages converge to the spatial average, for μ-almost every point. However, it is sometimes the case that sets A with $\mu(A) = 1$ are quite small, maybe even a single point, and so this result is actually not giving us that much information. Therefore, it is interesting to consider a certain class of measures, called physical measures, which avoid this problem.

To better formalize these notions, consider a probability measure space $(M, \mathcal{B}, \mu)$, where M is a compact differentiable Riemannian manifold with Lebesgue measure (volume), $f : M \to M$ and μ is f-invariant. We say that an invariant measure μ is physical if the set

$$\left\{ x \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int_M \varphi \, d\mu \right\}$$

has positive Lebesgue measure. An easy way to conclude a certain probability measure is physical is to check that it is invariant, ergodic, and absolutely continuous with respect to Lebesgue measure. In this case, since M is compact, Birkhoff's ergodic theorem tells us that such a measure must be physical.

In this work, we study this type of measures, in particular how they vary (in an appropriate topology) under suitable perturbations of the map f. More precisely, let $f_0 = f$ and $(f_\varepsilon)_{\varepsilon \geq 0}$ be a family of maps such that $f_\varepsilon \to f_0$ (in an appropriate topology) and each f_ε admits a unique physical measure μ_ε. A natural question one could ask is what properties the function $\varepsilon \mapsto \mu_\varepsilon$ possesses.

We are specifically interested in the differentiability, in a certain sense, of this function. If the measures μ_ε are absolutely continuous with respect to Lebesgue measure, then we can consider their densities h_ε and how they vary in L^1.

Definition 1.0.1. If the function $\varepsilon \mapsto h_\varepsilon$ is differentiable in L^1 at $\varepsilon = 0$, then we say that the family $(f_\varepsilon)_{\varepsilon \geq 0}$ has strong linear response.

This notion of strong linear response is, however, too ambitious for some dynamical systems, so one may look for linear response in a weaker sense.

Definition 1.0.2. If the function

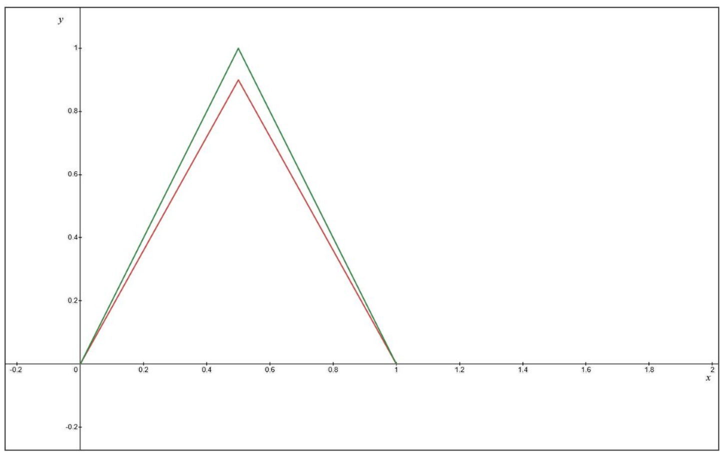
$$\varepsilon \mapsto \int \varphi \, d\mu_\varepsilon$$

is differentiable at $\varepsilon = 0$, for φ in a suitable class of functions, then we say that $(f_\varepsilon)_{\varepsilon \geq 0}$ has weak linear response.

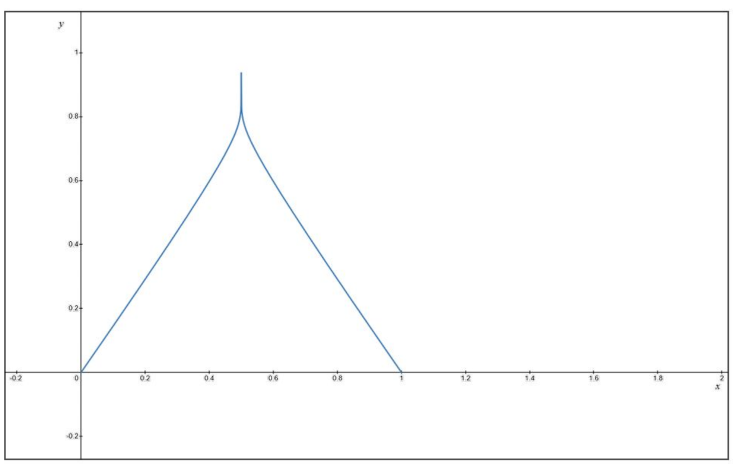
In this work, our main goal is to present two results. The first one, in Chapter 4, establishes the lack of linear response for perturbations of tent maps $f_t : [0, 1] \to [0, 1]$ with slope $\pm t$, $1 < t \leq 2$, that change the image of the critical point $1/2$.

The second result, treated in Chapter 5, follows the approach of [2] and shows that there is a way of recovering linear response, even in the strong sense, by introducing a cusp-like singularity at the turning point.

In Chapter 3, we introduce the transfer operator and some of its properties, as well as a few results regarding the spectral properties of this operator, which will be especially useful in Chapter 5.



Tent maps with slope $t = 2$ and $t = 1.8$



Map with a cusp singularity

Chapter 2

Functional spaces

2.1 Functions of bounded variation

We refer to the beginning of Chapter 3 of [15] and Section 4 of [13] for the results presented throughout this section.

Let $I = [a, b]$ be a bounded interval and let $\mathcal{P}$ be a partition of I, $a = x_0 < \cdots < x_n = b$.

Definition 2.1.1. Let $f : I \to \mathbb{R}$. We define the variation of f on I as

$$V_I f = \sup_{\mathcal{P}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all finite partitions of I. We may sometimes use Vf instead of $V_I f$.

Definition 2.1.2. We say that a function $f : I \to \mathbb{R}$ is of bounded variation if $V_I f < +\infty$.

We list some elementary properties of functions of bounded variation.

Proposition 2.1.3. *Let $f, g, h : I \to \mathbb{R}$ be functions of bounded variation, with h of class C^1. Then:*

1. $V_I(f + g) \leq V_I(f) + V_I(g)$;

2. $V_I(f \cdot g) \leq V_I f \sup |g| + \sup |f| V_I(g)$;

3. $V_I(f \cdot h) \leq V_I f \sup |h| + \sup |h'| \|f\|_1$;

4. $V_I|f| \leq V_I f$;

5. $V_I(f \circ H) = V_{H(I)}(f)$, *if* $H : I \to H(I)$ *is a homeomorphism.*

Theorem 2.1.4. *A function* $f : I \to \mathbb{R}$ *is of bounded variation if and only if* f *can be expressed as the difference of two increasing functions.*

Remark 2.1.5. It follows that the set of points of discontinuity of a function of bounded variation is countable, since monotonic functions can only have a countable number of discontinuities.

Definition 2.1.6. The space

$$BV(I) = \{f \in L^1(I) : V_I f < +\infty\}$$

is the space of functions of bounded variation.

We can make it into a Banach space by defining

$$\|f\|_{BV} = \|f\|_1 + V_I f_1$$

Theorem 2.1.7 (Helly). *Let* $f_n : I \to \mathbb{R}, n \geq 1$, *be a sequence of functions. Suppose that there exists a constant* $K > 0$ *such that, for all* $n \geq 1$,

1. $|f_n(x)| \leq K$, *for all* $x \in I$,

2. $V_I f_n \leq K$.

Then there exists a subsequence $(f_{n_k})_k$ *and a function* $f : I \to \mathbb{R}$ *satisfying* $|f(x)| \leq K$, *for all* $x \in I$, *and* $V_I f \leq K$ *such that* f_{n_k} *converges to* f *a.e. and in* $L^1(I)$.

Theorem 2.1.8. $BV(I)$ *is compactly embedded in* $L^1(I)$.

2.2 Sobolev spaces

We present the strictly necessary results for what will be done in the following chapters. Their proofs can be found in Section 8.2 of [6] and Section 6.1 of [1].

Let $I = (a, b)$ be an open interval (possibly unbounded) and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 2.2.1. The Sobolev space $W^{1,p}(I)$ is defined as

$$W^{1,p}(I) = \left\{ f \in L^p(I) \mid \exists\, g \in L^p(I) : \int_I f\varphi'\, dm = -\int_I g\varphi\, dm, \forall\, \varphi \in C_c^1(I) \right\},$$

where $C_c^1(I) = \{\varphi : I \to \mathbb{R} \mid \varphi$ is once continuously differentiable and has compact support$\}$ and m is the Lebesgue measure on I.

Remark 2.2.2. For $f \in W^{1,p}(I)$, we denote g by f'. Moreover, if $f \in C^1(I) \cap L^p(I)$, where $C^1(I) = \{\varphi : I \to \mathbb{R} \mid \varphi'$ exists and is continuous$\}$, and its usual derivative f' is such that $f' \in L^p(I)$, then the usual derivative of f and its derivative in the $W^{1,p}$ sense coincide.

We shall equip this space with the norm

$$\|f\|_{W^{1,p}} = \|f\|_p + \|f'\|_p.$$

Proposition 2.2.3. *The Sobolev space $W^{1,p}(I)$ is a Banach space, for $1 \leq p \leq \infty$.*

Theorem 2.2.4. *If I is bounded, then the injection $W^{1,1}(I) \to L^p(I)$ is compact, for all $1 \leq p < \infty$.*

Remark 2.2.5. An essential consequence of Theorem 2.2.4 is that any uniformly bounded sequence in $W^{1,1}(I)$ has a subsequence that converges in $L^p(I), 1 \leq p < \infty$.

Proposition 2.2.6 (differentiation of a product)**.** *Let $f, g \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Then $fg \in W^{1,p}(I)$,*

$$(fg)' = f'g + fg'$$

and

$$\int_y^x f'g = f(x)g(x) - f(y)g(y) - \int_y^x fg', \forall\, x, y \in \bar{I}$$

Proposition 2.2.7 (differentiation of a composition)**.** *Suppose I is a bounded interval. Let $g \in C^1(\mathbb{R})$ and $f \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Then $g \circ f \in W^{1,p}(I)$ and $(g \circ f)' = (g' \circ f)f'$.*

Remark 2.2.8. Since we will work with bounded intervals, we present this version of the result. However:

(i) if I is unbounded and $p = \infty$, this result is still valid;

(ii) if I is unbounded and $p \neq \infty$, then this result is not necessarily true; if we additionally suppose that $g(0) = 0$, then the conclusion still holds.

Up until this point, we are only considering the first derivatives of functions, but it will be useful to consider higher order derivatives.

Definition 2.2.9 (the Sobolev spaces $W^{m,p}$)**.** Given an integer $m \geq 2$ and a real number $1 \leq p \leq \infty$, we define

$$W^{m,p}(I) = \left\{ f \in W^{m-1,p}(I) : f' \in W^{m-1,p}(I) \right\}$$

Notice that $f \in W^{m,p}(I)$ if and only if there exist m functions $g_1, \ldots, g_m \in L^p(I)$ such that

$$\int_I f\varphi^{(j)} = (-1)^j \int_I g_j \varphi, \ \forall\, \varphi \in C_c^\infty(I), \forall\, j = 1, 2, \ldots, m,$$

where $\varphi^{(j)}$ denotes the jth derivative of φ and $C_c^\infty(I)$ denotes the space of infinitely differentiable functions from I to $\mathbb{R}$ with compact support. We can then consider higher order derivatives of f: $f' = g_1, f'' = g_2, \ldots,$ up to order m. We also denote them by $f^{(j)}$.
We equip this space with the norm

$$\|f\|_{W^{m,p}} = \sum_{j=0}^{m} \|f^{(j)}\|_p$$

One can extend the results shown for $W^{1,p}$ to analogous results for $W^{m,p}$. Since we will only work with $W^{1,1}$ and $W^{2,1}$, we will only mention the following specific result:

Proposition 2.2.10. *If I is bounded, then the injection $W^{2,1}(I) \to W^{1,1}(I)$ is compact.*

Chapter 3

The transfer operator

Throughout this section we will work on an interval $I = [a, b]$ with its Borel σ-algebra $\mathcal{B}$. Let m be the normalized Lebesgue measure on I and $T : I \to I$ be a non-singular map with respect to m, that is, given a measurable set A, if $m(A) = 0$ then $m(T^{-1}(A)) = 0$.

3.1 Definition and basic properties

Definition 3.1.1. The transfer operator $L_T : L^1(I) \to L^1(I)$ is defined as follows: for each $f \in L^1(I)$, let $L_T f$ be the unique (a.e.) function in $L^1(I)$ that satisfies

$$\int_A L_T f \, dm = \int_{T^{-1}(A)} f \, dm, \quad \forall A \in \mathcal{B}.$$

Remark 3.1.2. Notice that this operator is well defined by the Radon-Nikodym Theorem (since the measure $\mu(A) = \int_{T^{-1}(A)} f \, dm$ is absolutely continuous with respect to m).

We will now list many useful properties of the transfer operator.

Proposition 3.1.3 (Linearity)**.** *L_T is a linear operator.*

Proof. Let $A \subseteq I$ be measurable and let $a, b \in \mathbb{R}$. Then, for $f, g \in L^1(I)$, we have

$$\begin{aligned} \int_A L_T(af + bg) \, dm &= \int_{T^{-1}(A)} (af + bg) \, dm = a \int_{T^{-1}(A)} f \, dm + b \int_{T^{-1}(A)} g \, dm \\ &= a \int_A L_T f \, dm + b \int_A L_T g \, dm \\ &= \int_A (aL_T f + bL_T g \, dm). \end{aligned}$$

Since this holds for any measurable $A \subseteq I$, then $L_T(af + bg) = aL_T f + bL_T g$ a.e. ☐

Proposition 3.1.4 (Positivity). *If $f \in L^1(I)$, with $f \geq 0$, then $L_T f \geq 0$.*

Proof. Let $A \subseteq I$ be measurable. Then

$$\int_A L_T f \, dm = \int_{T^{-1}(A)} f \, dm \geq 0.$$

Therefore, $L_T f \geq 0$. ☐

Proposition 3.1.5 (Preservation of integrals). *The transfer operator satisfies*

$$\int_I L_T f \, dm = \int_I f \, dm$$

Proof. We have

$$\int_I L_T f \, dm = \int_{T^{-1}(I)} f \, dm = \int_I f \, dm.$$

☐

Proposition 3.1.6 (Contraction). *The transfer operator satisfies $\|L_T f\|_1 \leq \|f\|_1$, for any $f \in L^1(I)$.*

Proof. Let $f \in L^1(I)$ and write $f = f^+ - f^-$, where $f+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. By linearity, we have

$$L_T f = L_T(f^+ - f^-) = L_T f^+ - L_T f^-.$$

Since $|f| = f^+ + f^-$, we have

$$|L_T f| \leq |L_T f^+| + |L_T f^-| = L_T f^+ + L_T f^- = L_T |f|,$$

where the positivity of the transfer operator was used in the first equality. We conclude that

$$\|L_T f\|_1 = \int_I |L_T f| \, dm \leq \int_I L_T |f| \, dm = \int_I |f| \, dm = \|f\|_1,$$

where we used the preservation of integrals in the second equality. ☐

Proposition 3.1.7 (Composition). *If $T_1 : I \to I$ and $T_2 : I \to I$ are two nonsingular maps, then $L_{T_1 \circ T_2} = L_{T_1} \circ L_{T_2}$. Therefore, $L_{T^n} f = L_T^n f$.*

Proof. Let $f \in L^1(I)$ and consider the measure μ defined by

$$\mu(A) = \int_{(T_1 \circ T_2)^{-1}(A)} f\, dm, \text{ for each } A \subseteq I \text{ measurable.}$$

Since T_1 and T_2 are non-singular, $\mu \ll m$. By the Radon-Nikodym theorem, there exists a function $L_{T_1 \circ T_2} f$ such that

$$\mu(A) = \int_A L_{T_1 \circ T_2} f\, dm.$$

We have

$$\int_A L_{T_1 \circ T_2} f\, dm = \int_{T_2^{-1}(T_1^{-1}(A))} f\, dm = \int_{T_1^{-1}(A)} L_{T_2} f\, dm = \int_A L_{T_1}(L_{T_2} f)\, dm.$$

Therefore, $L_{T_1 \circ T_2} = L_{T_1} \circ L_{T_2}$. By induction, one concludes that $L_{T^n} = L_T^n$. □

Proposition 3.1.8 (Adjoint)**.** *For any $f \in L^1(I)$ and any $g \in L^\infty(I)$, we have*

$$\int_I (L_T f) \cdot g\, dm = \int_I f \cdot (g \circ T)\, dm.$$

Proof. It suffices to check the equality for characteristic functions. Let $A \subseteq I$ be measurable and consider $g = \chi_A$. We have

$$\int_I (L_T f) \cdot g\, dm = \int_A L_T f\, dm = \int_{T^{-1}(A)} f\, dm$$

and

$$\int_I f \cdot (g \circ T)\, dm = \int_I f \cdot \chi_{T^{-1}(A)}\, dm = \int_{T^{-1}(A)} f\, dm.$$

Thus we have the equality for characteristic functions. By linearity, this holds for all simple functions, and since any $g \in L^\infty$ is the limit of a sequence of simple functions, the dominated convergence theorem gives us the desired conclusion. □

The following property relates fixed points of the transfer operator with absolutely continuous T-invariant probability measures (we will refer to these as acip's).

Proposition 3.1.9. *Let $h \in L^1(I)$, with $h \geq 0$ and $\|h\|_1 = 1$. Then $L_T h = h$ if and only if the measure $\mu = h \cdot m$ defined by $\mu(A) = \int_A h\, dm$ is T-invariant.*

Proof. Let $A \subseteq I$ be a measurable set. If $L_T h = h$, then

$$\int_A L_T h\, dm = \int_A h\, dm = \mu(A).$$

By definition of the transfer operator, we also have

$$\int_A L_T h\, dm = \int_{T^{-1}(A)} h\, dm = \mu(T^{-1}(A)).$$

We conclude that $\mu(T^{-1}(A)) = \mu(A)$.

If $\mu(T^{-1}(A)) = \mu(A)$, then

$$\int_A L_T h\, dm = \int_{T^{-1}(A)} h\, dm = \int_A h\, dm.$$

Since A is arbitrary, $L_T h = h$. □

Remark 3.1.10. This proposition gives us a way of looking for acip's: we need to find non-negative eigenfunctions associated to the eigenvalue 1.

Proposition 3.1.11. *The transfer operator cannot have eigenvalues with modulus larger than* 1.

Proof. Suppose there exists an eigenvalue $z \in \mathbb{C}$, $|z| > 1$, with eigenfunction g, $\|g\|_1 \neq 0$. Then,

$$\|L_T g\|_1 = |z| \|g\|_1$$

and, by the contraction property of transfer operators,

$$\|L_T g\|_1 \leq \|g\|_1.$$

Thus

$$|z| \|g\|_1 \leq \|g\|_1$$

which implies $|z| \leq 1$ (since $\|g\|_1 \neq 0$) and contradicts the assumption $|z| > 1$. □

The transfer operator has a nice representation for the following class of functions:

Definition 3.1.12. The function $T : I \to I$ is called piecewise monotonic if there exists a partition of I, say $a = t_0 < t_1 < \cdots < t_m = b$ such that:

(a) $T_{|(t_{i-1},t_i)}$ is a C^1 function, for all $i = 1, \ldots, m$ which can be extended to a C^1 function on $[t_{i-1}, t_i]$, for $i = 1, \ldots, m$;

(b) $|T'(x)| > 0$ on (t_{i-1}, t_i), for $i = 1, \ldots, m$.

Theorem 3.1.13. *If $T : I \to I$ is piecewise monotonic, then for any $f \in L^1(I)$,*

$$L_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

Proof. Let $A \subseteq I$ be a measurable set. By definition of the transfer operator,

$$\int_A L_T f \, dm = \int_{T^{-1}(A)} f \, dm.$$

Changing the value of T on a finite set, we may assume that T is right-continuous. Notice that T is invertible on each $I_i = [t_{i-1}, t_i), i = 1, \ldots, m-1$ and on $I_m = [t_{m-1}, t_m]$. Letting $T_i = T|_{(a_{i-1},a_i)}$ and $B_i = T(I_i)$, we have

$$T^{-1}(A) = \bigcup_{i=1}^{m} T_i^{-1}(B_i \cap A),$$

where the union is disjoint. Hence,

$$\begin{aligned}
\int_A L_T f \, dm &= \sum_{i=1}^{m} \int_{T_i^{-1}(B_i \cap A)} f \, dm = \sum_{i=1}^{m} \int_{B_i \cap A} f(T_i^{-1}(x)) \cdot |(T_i^{-1})'(x)| \, dm \\
&= \sum_{i=1}^{m} \int_A f(T_i^{-1}(x)) \cdot |(T_i^{-1})'(x)| \chi_{B_i}(x) \, dm = \int_A \sum_{i=1}^{m} \frac{f(T_i^{-1}(x))}{|T'(x)|} \chi_{T(I_i)}(x) \, dm.
\end{aligned}$$

Since A is arbitrary, we conclude that

$$L_T f(x) = \sum_{i=1}^{m} \frac{f(T_i^{-1}(x))}{|T'(T^{-1}(x))|} \chi_{T(I_i)}(x),$$

which can be rewritten as

$$L_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

□

Example 3.1.14. Consider the doubling map $T : [0,1] \to [0,1]$ defined by $T(x) = 2x \bmod 1$. Then

$$L_T f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|} = \frac{1}{2} \left(f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) \right).$$

3.2 Spectral gap and quasi-compactness

We now introduce a concept which will prove itself essential in guaranteeing the existence and uniqueness of acip's. We follow Sections 2.1 and 2.2 of [14].

Definition 3.2.1 (Spectral gap)**.** Let B be a Banach space and let $L : B \to B$ be a linear operator. We say that L has spectral gap if we can write $L = \lambda P + N$, where:

(a) P is a projection ($P^2 = P$) and $\dim(\mathrm{Im}(P)) = 1$;

(b) N is a bounded operator with $r(N) < |\lambda|$, where $r(N)$ denotes the spectral radius of N;

(c) $PN = NP = 0$.

Remark 3.2.2. Notice that (c), together with (a), imply that $L^n = \lambda^n P + N^n$. Moreover, for any $x \in B$, we have $\|L^n x - \lambda^n P x\| = \|N^n x\|$ and so, by (b), we conclude that $\lambda^{-n} L^n x \to Px$, as $n \to \infty$.

Remark 3.2.3. The reason for the name "spectral gap" may not be obvious. However, if an operator L has spectral gap, then its spectrum can be decomposed as

$$\sigma(L) = \{\lambda\} \cup \Sigma,$$

where Σ is a subset of the ball of radius r centered at 0, for some $r < |\lambda|$. This fact will not be explicitely used, and so we do not prove it here.

Checking wheather an operator has spectral gap on a given Banach space may not be so easy. With this in mind, we define an additional notion which is easier to deal with.

Definition 3.2.4 (Essential spectral radius and Quasi-compact operator)**.** Let B be a Banach space. We define the essential spectral radius of $L : B \to B$, $r_e(L)$, to be the infimum of $r(L)$ and the real numbers $r' \geq 0$ for which B can be decomposed as $B = F \oplus H$, where:

1. F, H are closed and L-invariant subspaces ($L(F) \subseteq F, L(H) \subseteq H$);

2. $\dim(F) < \infty$ and each eigenvalue of $L_{|F}$ has modulus greater than r';

3. $r(L_{|H}) < r'$.

If $r_e(L) < r(L)$, we say that L is quasi-compact.

Remark 3.2.5. Notice that this is weaker notion than spectral gap, since if $L : B \to B$ has spectral gap, then we have the decomposition $B = \mathrm{Im}(P) \oplus \mathrm{Im}(I - P)$ and quasi-compactness readily follows.

We now relate this quasi-compactness definition with the spectral gap property.

Theorem 3.2.6. *Suppose $L : B \to B$ is a quasi-compact operator on a Banach space B. Suppose that L has a unique eigenvalue on $\{z \in \mathbb{C} : |z| = r(L)\}$ with algebraic multiplicity 1. Then L has spectral gap.*

Proof. We write $B = F \oplus H$ as in the definition of quasi-compactness. Let λ be the unique eigenvalue on $\{z \in \mathbb{C} : |z| = r(L)\}$ and let λ_i be the (possibly) remaining eigenvalues of $L_{|F}$. Let v be an eigenvector associated with λ, i.e., $Lv = \lambda v$. Notice that we can decompose F as $F = \mathrm{span}\{v\} \oplus F'$, where F' is L-invariant and $r(L|_{F'}) < |\lambda|$, since all other eigenvalues λ_i are such that $|\lambda_i| < |\lambda|$. Now define $H' = F' \oplus H$. We have $B = \mathrm{span}\{v\} \oplus H'$. Let $P : B \to \mathrm{span}\{v\}$ be the projection onto $\mathrm{span}\{v\}$ and $N = L \circ \pi_{H'} : B \to H'$, where $\pi_{H'} : B \to H'$ is the projection onto H'. First we check that $L = \lambda P + N$. Indeed, given $x \in B$, we can write it (uniquely) as $x = av + h'$, with $h' \in H'$ and for some scalar a, thus

$$Lx = L(av + h') = \lambda av + Lh' = \lambda Px + (L \circ \pi_{H'})x.$$

(a) P is a projection (by definition) and $\dim(\mathrm{Im}(P)) = 1$;

(b) N is bounded $r(N) = r(L|_{H'}) < |\lambda|$;

(c) For any $x \in B$, $P(Nx) = P(L(\pi_{H'}x) = 0$, since H' is L-invariant; analogously, $N(Px) = 0$, for any $x \in B$.

We conclude that L has spectral gap. □

Finally, we give a general result that will allow us to show that certain operators are quasi-compact.

Proposition 3.2.7. *Let $(B, \|\cdot\|_s)$ be a Banach space, $\|\cdot\|_w$ be a continuous semi-norm on B. Suppose that any sequence (x_n) with $\|x_n\|_s \leq 1$ contains a Cauchy subsequence for $\|\cdot\|_w$. Let L be a bounded linear operator on $(B, \|\cdot\|_s)$ and assume there exist $\lambda \geq 0$ and $C > 0$ such that, for any $f \in B$,*

$$\|Lf\|_s \leq \lambda \|f\|_s + C\|f\|_w$$

Then $r_e(L) \leq \lambda$. In particular, if $\lambda < r(L)$, then L is quasi-compact.

To prove Proposition 3.2.7, we introduce another way of computing the essential spectral radius of an operator.

For $A \subseteq B$, let $\Gamma(A) = \{\delta > 0 : A \text{ can be covered by finitely many balls of radius } \delta\}$, $\gamma(A) = \inf \Gamma(A)$ and $\Gamma(L) = \{k \geq 0 : \forall\, A \subseteq B, \gamma(L(A)) \leq k\gamma(A)\}$, $\gamma(L) = \inf \Gamma(L)$. Let B_1 be the unit ball in $(B, \|\cdot\|_s)$. Then $\gamma(L) = \gamma(L(B_1)) \leq \|L\|_{s\to s}$ and, moreover, the sequence $((\gamma(L^n)^{\frac{1}{n}})_n$ is submultiplicative. Letting $r_\gamma(L) = \lim_n(\gamma(L^n))^{\frac{1}{n}}$, a result by Nussbaum [11] tells us that $r_\gamma(L) = r_e(L)$.

We can now proceed with the proof.

Proof of Proposition 3.2.7. We begin by proving the result assuming a stronger hypothesis. Suppose that, for all $n \geq 1$, there exist $\lambda'_n \geq 0$ and $C'_n > 0$, with $\liminf_n(\lambda'_n)^{\frac{1}{n}} = \lambda' < r(L)$, such that, for any $f \in B$,

$$\|L^n f\|_s \leq \lambda'_n \|f\|_s + C'_n \|f\|_w.$$

For $\delta > 0$, $f \in B$, let

$$B_s(f, \delta) = \{g \in B : \|g - f\|_s < \delta\}, \quad B_w(f, \delta) = \{g \in B : \|g - f\|_w < \delta\}.$$

Let $M = \|L\|_{s\to s}$ and $k \geq 1$. We will estimate $\gamma(L^{k+1})$. By hypothesis, the set $L(B_1)$ is relatively compact in $(B, \|\cdot\|_w)$, so for any $\varepsilon > 0$, there exist $(f_i)_{1\leq i\leq n}$ such that, if $g_i = Lf_i$,

$$L(B_1) \subseteq \bigcup_{i=1}^{n} \left(B_w(g_i, \varepsilon) \cap B_s(0, \|L\|_{s\to s})\right).$$

If $g \in B_w(g_i, \varepsilon) \cap B_s(0, \|L\|_{s\to s})$, then

$$\|L^k g_i - L^k g\|_s \leq \lambda'_k \|g_i - g\|_s + C'_k \|g_i - g\|_w \leq 2\lambda'_k \|L\|_{s\to s} + C'_k \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $\gamma(L^{k+1}) \leq 2\|L\|_{s\to s}\lambda'_k$ and so $r_\gamma(L) \leq \liminf_k(\lambda'_k)^{\frac{1}{k}} = \lambda'$.

To prove the general case, suppose now that there exist $\lambda \geq 0, C > 0$ such that, for any $f \in B$,

$$\|Lf\|_s \leq \lambda\|f\|_s + C\|f\|_w.$$

We will define a new seminorm $\|\cdot\|'_w$ on B which will allow us to iterate this inequality and thus reduce this to the particular case we just proved before. Define

$$\|f\|'_w = \sum_{n=0}^{+\infty} \frac{\|L^n f\|_w}{2^n M^n}.$$

This new seminorm $\|\cdot\|'_w$ satisfies the same compactness assumptions as $\|\cdot\|_w$. Furthermore, L is bounded for this semi-norm, since

$$\begin{aligned}\|Lf\|'_w = \sum_{n=0}^{+\infty} \frac{\|L^{n+1} f\|_w}{2^n M^n} &= 2M \sum_{n=1}^{\infty} \frac{\|L^n f\|_w}{2^n M^n} \\ &= 2M(\|f\|'_w - \|f\|_w) \leq 2M\|f\|'_w.\end{aligned}$$

Therefore we can iterate this inequality and we get that, for each $n \geq 1$, there exist constants $C_n > 0$ such that

$$\|L^n f\| \leq \lambda^n\|f\|_s + C_n\|f\|'_w.$$

We are now in the particular case from before, and the result follows. □

3.3 Spectral projections

Throughout this section, let $L : B \to B$ be a bounded linear operator on a complex Banach space B. Let $\lambda_0 \in \sigma(L)$ be an isolated point of $\sigma(L)$ and let Γ_{λ_0} be a simple closed curve contour around λ_0 such that the region enclosed by Γ_{λ_0} intersects $\sigma(L)$ only at the point λ_0 (this is possible since λ_0 is an isolated point of $\sigma(L)$).

Consider

$$P^{\lambda_0} = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_0}} (\lambda I - L)^{-1}\, d\lambda = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_0}} R_\lambda(L)\, d\lambda.$$

Recall that the resolvent $R_\lambda(L)$ is analytic in $\rho(L)$ and so P^{λ_0} does not depend on the choice of contour Γ_{λ_0}. We can thus define the following:

Definition 3.3.1 (Spectral projection)**.** The spectral projection for L and λ_0 is defined as

$$P^{\lambda_0} = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_0}} R_\lambda(L)\,d\lambda.$$

Indeed, the name "projection" is accurate, by the following proposition.

Proposition 3.3.2. *Let P^{λ_0} be the spectral projection for L and λ_0. Then P^{λ_0} is a projection, i.e. $(P^{\lambda_0})^2 = P^{\lambda_0}$.*

Proof. We omit λ_0 in this proof. Suppose that $\tilde{\Gamma}$ is another contour in the same conditions as Γ, such that Γ is contained in the interior of the region bounded by $\tilde{\Gamma}$. Then

$$\begin{aligned}
P^2 &= \frac{1}{(2\pi i)^2}\int_\Gamma R_\mu(L)\,d\mu \int_{\tilde{\Gamma}} R_\lambda(L)\,d\lambda \\
&= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\tilde{\Gamma}} R_\mu(L)R_\lambda(L)\,d\lambda\,d\mu \\
&= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\tilde{\Gamma}} \frac{1}{\lambda-\mu}(R_\mu(L)-R_\lambda(L))\,d\lambda\,d\mu \\
&= \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\tilde{\Gamma}} \frac{R_\mu(L)}{\lambda-\mu}\,d\lambda\,d\mu - \frac{1}{(2\pi i)^2}\int_\Gamma\int_{\tilde{\Gamma}} \frac{R_\lambda(L)}{\lambda-\mu}\,d\lambda\,d\mu \\
&= \frac{1}{(2\pi i)^2}\int_\Gamma \left(R_\mu(L)\int_{\tilde{\Gamma}} \frac{1}{\lambda-\mu}\,d\lambda\right)d\mu - \frac{1}{(2\pi i)^2}\int_{\tilde{\Gamma}} R_\lambda(L)\left(\int_\Gamma \frac{1}{\lambda-\mu}\,d\mu\right)d\lambda \\
&= \frac{1}{2\pi i}\int_\Gamma R_\mu(L)\,d\mu = P.
\end{aligned}$$

□

Proposition 3.3.3. *If λ_0 is an isolated eigenvalue with finite algebraic multiplicity, then* $\dim(\mathrm{Im}(P^{\lambda_0}))$ *is equal to the algebraic multiplicity of λ_0.*

Proof. We will begin working under slightly more general conditions by using the following auxiliary result, whose proof can be found in Section 49 of [8].

Theorem 3.3.4. *Let B be a complex Banach space and let σ_1, σ_2 be two closed disjoint subsets of $\sigma(L)$ such that $\sigma(L) = \sigma_1 \cup \sigma_2$. Let Γ be a simple closed curve such that σ_1 is in its interior and Γ crosses no other part of $\sigma(L)$. Then, the operator*

$$P^{\sigma_1} = \frac{1}{2\pi i}\int_\Gamma R_\lambda(L)\,d\lambda$$

generates a decomposition

$$B = \mathrm{Im}(P^{\sigma_1}) \oplus \ker(P^{\sigma_1}),$$

where both subspaces are invariant under L (and even under every $f(L)$, where f is analytic). Moreover, $\sigma(L|_{\mathrm{Im}(P^{\sigma_1})}) = \sigma_1$ and $\sigma(L|_{\ker(P^{\sigma_1})}) = \sigma_2$.

Suppose that, in the conditions of this auxilliary theorem, there exists a circle $B_\delta(\alpha)$ such that $\partial B_\delta(\alpha) \subseteq \rho(L)$ and its interior contains σ_1 (and no other points of $\sigma(L)$) - notice that, in the case where σ_1 is an isolated eigenvalue, this is clearly satisfied. Then, we have, for each $n \geq 1$,

$$(\alpha I - L)^n P^{\sigma_1} = \frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n R_\lambda(L)\, d\lambda$$

and so, given $f \in B$, we have

$$(\alpha I - L)^n P^{\sigma_1} f = \frac{1}{2\pi i} \int_\Gamma (\alpha - \lambda)^n R_\lambda(L) f\, d\lambda.$$

Given $f \in \mathrm{Im}(P^{\sigma_1})$, we have $P^{\sigma_1} f = f$ and so

$$\|(\alpha I - L)^n f\| \leq \frac{1}{2\pi} 2\pi\delta \cdot \delta^n \max_{\lambda \in \Gamma} \|R_\lambda(L)\| \|f\|.$$

Since Γ lies inside $\rho(L)$, this also holds for some $\delta' < \delta$, and therefore

$$\limsup \|(\alpha I - L)^n f\|^{\frac{1}{n}} < \delta.$$

Conversely, if this inequality is valid, then

$$\sum_{n=0}^{\infty} \left(\frac{\alpha I - L}{\alpha - \lambda} \right)^n f$$

converges, for all $\lambda \in \Gamma$. If $g(\lambda)$ is its sum, then it satisfies

$$\left(I - \frac{\alpha I - L}{\alpha - \lambda} \right) g(\lambda) = f,$$

from where one obtains

$$g(\lambda) = (\lambda - \alpha) R_\lambda(L) f.$$

Therefore, for all $\lambda \in \Gamma$,

$$R_\lambda(L)f = -\sum_{n=0}^{\infty} \frac{(\alpha I - L)^n f}{(\alpha - \lambda)^{n+1}}.$$

Integrating term by term, we obtain

$$P^{\sigma_1} f = \frac{1}{2\pi i} \int_\Gamma R_\lambda(L) f \, d\lambda = -\frac{1}{2\pi i} \int_\Gamma \frac{1}{\alpha - \lambda} f \, d\lambda = f.$$

This shows that if the set σ_1 is enclosed in an adequate circle of radius δ as above, then

$$\mathrm{Im}(P^{\sigma_1}) = \left\{ f \in B : \limsup \|(\alpha I - L)^n f\|^{\frac{1}{n}} < \delta \right\}.$$

Applying this to the particular case of $\sigma_1 = \{\lambda_0\}$, where λ_0 is an isolated point of the spectrum, then one obtains

$$\mathrm{Im}(P^{\lambda_0}) = \left\{ f \in B : \lim \|(\lambda_0 I - L)^n f\|^{\frac{1}{n}} = 0 \right\}.$$

Now, suppose that λ_0 is an isolated eigenvalue of finite algebraic multiplicity, i.e. there exists $p \geq 1$ such that $\ker(L - \lambda_0 I)^n = \ker(L - \lambda_0 I)^p$, for any $n \geq p$. It is immediate that any generalised eigenvector must be in $\mathrm{Im}(P^{\lambda_0})$. We now show the reverse inclusion. Define $T = L - \lambda_0 I$. We have the decomposition

$$B = \ker(T^p) \oplus \mathrm{Im}(T^p).$$

It suffices to show that $\mathrm{Im}(P^{\lambda_0}) \subseteq \ker(T^m)$, for some m. Indeed, if $f \in \mathrm{Im}(P^{\lambda_0})$, then write $f = g + h$, with $g \in \ker(T^p), h \in \mathrm{Im}(T^p)$. We have $T^n f = T^n h$, for $n \geq p$. It follows that $h \in \mathrm{Im}(P^{\lambda_0})$, so $h \in \mathrm{Im}(T^p) \cap \mathrm{Im}(P^{\lambda_0})$. All that remains is to show that this intersection is $\{0\}$, because then we will have $f = g \in \ker(T^p)$. Let $D = \mathrm{Im}(T^p) \cap \mathrm{Im}(P^{\lambda_0})$. We prove $D = \{0\}$ by showing that the restriction L_D of L to D has empty spectrum. We will make use of the following facts (see, respectively, Sections 38 and 43 of [8]):

1. T maps $\mathrm{Im}(T^p)$ bijectively onto itself;

2. If A is an endomorphism of a vector space E, $f_1, \ldots, f_n$ are pairwise relatively prime polynomials and $f = f_1 f_2 \ldots f_n$, then

$$\ker(f(A)) = \ker(f_1(A)) \oplus \cdots \oplus \ker(f_n(A)).$$

By fact 1, given $u \in D$, there exists only one $v \in \mathrm{Im}(T^p)$ such that $u = Tv$. It follows that $v \in D$, i.e. T is bijective on D, therefore $\lambda_0 \in \rho(L_D)$. If $\lambda \neq \lambda_0$, then (by the auxiliary theorem) $\lambda \in \rho(L|_{\mathrm{Im}(P^{\lambda_0})})$. Therefore, for every $u \in D$, there exists only one $v \in \mathrm{Im}(P^{\lambda_0})$ such that $u = (L - \lambda I)v$. Writing $v = g + h$, with $g \in \ker(T^p), h \in \mathrm{Im}(T^p)$, we obtain

$$u - (L - \lambda I)(g + h) = 0$$

and so $u-(L-\lambda I)h = (L-\lambda I)g$. Notice that $u-(L-\lambda I)h \in \mathrm{Im}(T^p)$ and $(L-\lambda I)g \in \ker(T^p)$, so we conclude that $(L - \lambda I)g = 0$, i.e.

$$g \in \ker(L - \lambda_0 I)^p \cap \ker(L - \lambda I).$$

By fact 2, it follows that $g = 0$, hence $v = h \in \mathrm{Im}(T^p)$, which implies $v \in D$ and thus $L_D - \lambda I$ is bijective, i.e., $\lambda \in \rho(L_D)$. We conclude that $\rho(L_D) = \mathbb{C}$ and so $\sigma(L_D) = \emptyset$, which finishes the proof.

□

We have previously seen that it will be useful to show that 1 is the only eigenvalue on the unit circle (and that it is isolated) for certain transfer operators. To this effect, we prove the following quite technical result:

Theorem 3.3.5. *Let $(B, \|\cdot\|_s)$ be a Banach space and let $\|\cdot\|_w \leq \|\cdot\|_s$ be a semi-norm on B. For any bounded linear operator $L : B \to B$, let*

$$\|L\|_{s\to w} = \sup_{\|f\|_s \leq 1} \|Lf\|_w.$$

Let $(L_\varepsilon)_{\varepsilon\geq 0}$ be a family of bounded linear operators on B that satisfy:

(a) there exist $C_1, M > 0$ such that, for all $\varepsilon \geq 0$,

$$\|L_\varepsilon^n\|_w \leq C_1 M^n, \,\forall\, n \in \mathbb{N};$$

(b) there exist $C_2, C_3 > 0$ and $\alpha \in (0, 1)$, with $\alpha < M$ such that, for all $\varepsilon \geq 0$

$$\|L_\varepsilon^n f\|_s \leq C_2\alpha^n \|f\|_s + C_3 M^n \|f\|_w, \,\forall\, n \in \mathbb{N}, \,\forall\, f \in B;$$

(c) if $z \in \sigma(L_\varepsilon)$, with $|z| > \alpha$, then z is not in the residual spectrum of L_ε;

(d) *there exists a monotone upper-semicontinuous function* $\tau : [0, +\infty) \to [0, +\infty)$ *such that* $\tau_\varepsilon > 0$ *if* $\varepsilon > 0$ *and*

$$\|L_0 - L_\varepsilon\|_{s\to w} \leq \tau_\varepsilon \to 0, \text{ as } \varepsilon \to 0.$$

For $\delta > 0$ *and* $r \in (\alpha, M)$, *let*

$$\eta = \frac{\log \frac{r}{\alpha}}{\log \frac{M}{\alpha}}, \quad V_{\delta,r} = \{z \in \mathbb{C} : |z| \leq r\} \cup \{z \in \mathbb{C} : d(z, \sigma(L_0)) \leq \delta\},$$

where $d(z, \sigma(L_0))$ *denotes the distance from the point* z *to the set* $\sigma(L_0)$. *Then there exist constants* $\varepsilon_0, a, b, c, d > 0$ *such that, for* $0 \leq \varepsilon \leq \varepsilon_0$ *and* $z \in \mathbb{C} \setminus V_{\delta,r}$, *we have*

$$\|(zI - L_\varepsilon)^{-1} f\|_s \leq a\|f\|_s + b\|f\|_w, \forall f \in B$$

and

$$\left\|(zI - L_\varepsilon)^{-1} - (zI - L_0)^{-1}\right\|_{s\to w} \leq \tau_\varepsilon^\eta \left(c\left\|(zI - P_0)^{-1}\right\|_s + d\left\|(zI - P_0)^{-1}\right\|_s^2\right).$$

Proof. Notice that (b) implies that, for all $\varepsilon \geq 0$, for all $n \in \mathbb{N}$,

$$\|L_\varepsilon^n\|_{s\to s} \leq C_4 M^n, \quad \text{where } C_4 = C_2 + C_3.$$

Fix the notation $Q_\varepsilon = (zI - L_\varepsilon)$ (Q_ε depends on z, but we omit this in the notation). Since $z \notin V_{\delta,r}$, then we always have $|z| \geq r > \alpha$ and Q_0^{-1} exists and is a bounded linear operator on B. We first prove an auxiliary lemma.

Lemma 3.3.6. *Let* $r \in (\alpha, M)$. *Then, for all* $f \in B$ *and for all* $\varepsilon \geq 0$,

$$\|f\|_s \leq C_5\|Q_\varepsilon f\|_s + C_6\|f\|_w,$$

where

$$C_5 = \frac{2C_4}{M - r}\left(\frac{M}{r}\right)^{n_1}, \quad C_6 = 2C_3\left(\frac{M}{r}\right)^{n_1}, \quad n_1 = \left\lceil \frac{\log 2C_2}{\log \frac{r}{\alpha}} \right\rceil.$$

Proof of lemma. For all $f \in B$, we have

$$\begin{aligned} |z|^n\|f\|_s = \|z^n f\|_s &\leq \|(z^n - L_\varepsilon^n)f\|_s + \|L_\varepsilon^n f\|_s \\ &\leq \sum_{j=0}^{n-1} |z^j| \|L_\varepsilon^{n-1-j}\|_s \|(z - L_\varepsilon)f\|_s + C_2\alpha^n\|f\|_s + C_3 M^n\|f\|_w, \end{aligned}$$

where hypothesis (b) was used to obtain the last inequality. Using the fact that $\|L_\varepsilon^n\|_{s\to s} \;\le\; C_4 M^n$, we obtain

$$\|f\|_s \le \frac{C_4}{M-r}\left(\frac{M}{r}\right)^n \|Q_\varepsilon f\|_s + C_2\left(\frac{\alpha}{r}\right)^n \|f\|_s + C_3\left(\frac{M}{r}\right)^n \|f\|_w.$$

Now if we let $n = n_1$, then $C_2\left(\frac{\alpha}{r}\right)^{n_1} \le 1/2$, and so

$$\|f\|_s \le \frac{2C_4}{M-r}\left(\frac{M}{r}\right)^{n_1} \|Q_\varepsilon f\|_s + 2C_3\left(\frac{M}{r}\right)^{n_1} \|f\|_w,$$

which proves the lemma. □

Now we proceed with the proof of the theorem. Let $h \in B$ and $g = Q_0 h$. Since

$$Q_0^{-1} = (zI - L_0)^{-1} = z^{-n} Q_0^{-1} L_0^n + \sum_{j=0}^{n-1} z^{-j-1} L_0^j,$$

we have

$$\begin{aligned}
\|h\|_w = \|Q_0^{-1} g\|_w &\le |z|^{-n}\|Q_0^{-1} L_0^n g\|_s + \sum_{j=0}^{n-1} |z|^{-j-1}\|L_0^j g\|_w \\
&\le |z|^{-n}\|Q_0^{-1}\|_{s\to s}\|L_0^n g\|_s + \sum_{j=0}^{n-1} |z|^{-j-1} C_1 M^j \|g\|_w \\
&\le \|Q_0^{-1}\|_{s\to s} C_2\left(\frac{\alpha}{r}\right)^n \|g\|_s + \|Q_0^{-1}\|_{s\to s} C_3\left(\frac{M}{r}\right)^n \|g\|_w + r^{-1} C_1 \sum_{j=0}^{n-1}\left(\frac{M}{r}\right)^j \|g\|_w \\
&\le \|Q_0^{-1}\|_{s\to s} C_2\left(\frac{\alpha}{r}\right)^n \|g\|_s + \|Q_0^{-1}\|_{s\to s} C_3\left(\frac{M}{r}\right)^n \|g\|_w + r^{-1} C_1 \frac{(M/r)^n}{M/r-1}\|g\|_w \\
&= \|Q_0^{-1}\|_{s\to s} C_2\left(\frac{\alpha}{r}\right)^n \|g\|_s + \|Q_0^{-1}\|_{s\to s} C_3\left(\frac{M}{r}\right)^n \|g\|_w + \frac{C_1}{M-r}\left(\frac{M}{r}\right)^n \|g\|_w.
\end{aligned}$$

Moreover, since $g = (Q_0 - Q_\varepsilon)Q_0^{-1} g + Q_\varepsilon Q_0^{-1} g = (L_\varepsilon - L_0)h + Q_\varepsilon h$, then

$$\|g\|_w \le \|(L_\varepsilon - L_0)h\|_w + \|Q_\varepsilon h\|_w \le \tau_\varepsilon \|h\|_s + \|Q_\varepsilon h\|_w.$$

Combining this inequality with the previous one, we obtain

$$\|h\|_w \le \|Q_0^{-1}\|_{s\to s} C_2\left(\frac{\alpha}{r}\right)^n \|g\|_s + \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^n (\tau_\varepsilon\|h\|_s + \|Q_\varepsilon h\|_w).$$

Since $g = Q_0 h = (zI - L_0)h$, then $\|g\|_s = \|(zI - L_0)h\|_s \le (C_4 M + |z|)\|h\|_s$, applying this to the previous inequality, we obtain

$$\begin{aligned}
\|h\|_w \le &\left(\|Q_0^{-1}\|_{s\to s} C_2 (C_4 M + |z|)\left(\frac{\alpha}{r}\right)^n + \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\tau_\varepsilon\left(\frac{M}{r}\right)^n\right)\|h\|_s \\
&+ \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^n \|Q_\varepsilon h\|_w.
\end{aligned}$$

Now we wish to obtain the first conclusion of the theorem. We separate the argument in two cases. If $|z| > 2M$, then we can write

$$(zI - L_\varepsilon)^{-1} = \frac{1}{z}(I - z^{-1}L_\varepsilon)^{-1} = \frac{1}{z}\sum_{j=0}^{+\infty} z^{-j} L_\varepsilon^j.$$

Since $|z| > 2M$, and thus $\frac{1}{|z|} < \frac{1}{2M}$, we have

$$\|(zI - L_\varepsilon)^{-1} f\|_s = \frac{1}{|z|}\sum_{j=0}^{+\infty} |z|^{-j}\|L_\varepsilon^j f\|_s \leq \frac{1}{2M}\sum_{j=0}^{+\infty}\frac{1}{2^j M^j} C_4 M^j \|f\|_s = \frac{C_4}{M}\|f\|_s.$$

Now we consider the case $|z| \leq 2M$. Let

$$\begin{aligned}
H &= H(\delta, r) = \sup_{z\in\mathbb{C}\setminus V_{\delta,r}} \|(zI - L_0)^{-1}\|_{s\to s},\\
n_2 &= n_2(\delta, r) = \left\lceil \frac{\log(4C_6 H C_2(C_4+2)M)}{\log\frac{r}{\alpha}} \right\rceil,\\
\varepsilon_1 &= \varepsilon_1(\delta, r) = \sup\left\{\varepsilon > 0 : \tau_\varepsilon\left(HC_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^{n_2} \leq \frac{1}{4C_6}\right\}.
\end{aligned}$$

Recall the auxiliary lemma we proved at the start of this proof. It gives us that

$$\|h\|_s \leq C_5\|Q_\varepsilon h\|_s + C_6\|h\|_w.$$

Combining this with the inequality from before for $\|h\|_w$, we obtain, with $n = n_2$ and for $0 \leq \varepsilon \leq \varepsilon_1$,

$$\begin{aligned}
\|h\|_w &\leq \left(\|Q_0^{-1}\|_{s\to s} C_2(C_4 M + |z|)\left(\frac{\alpha}{r}\right)^n + \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\tau_\varepsilon\left(\frac{M}{r}\right)^n\right)\|h\|_s\\
&\quad + \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^n \|Q_\varepsilon h\|_w\\
&\leq \left(HC_2(C_4+2)M\left(\frac{\alpha}{r}\right)^{n_2} + \left(HC_3 + \frac{C_1}{M-r}\right)\tau_\varepsilon\left(\frac{M}{r}\right)^{n_2}\right)\|h\|_s\\
&\quad + \left(HC_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^{n_2}\|Q_\varepsilon h\|_w\\
&\leq \frac{1}{4C_6}\|h\|_s + \frac{1}{4C_6}\|h\|_s + \frac{1}{4\tau_{\varepsilon_1} C_6}\|Q_\varepsilon h\|_w\\
&\leq \frac{1}{2C_6}\left(C_5\|Q_\varepsilon h\|_s + C_6\|h\|_w\right) + \frac{1}{4\tau_{\varepsilon_1} C_6}\|Q_\varepsilon h\|_w\\
&= \frac{1}{2}\|h\|_w + \frac{C_5}{2C_6}\|Q_\varepsilon h\|_s + \frac{1}{4\tau_{\varepsilon_1} C_6}\|Q_\varepsilon h\|_w.
\end{aligned}$$

Therefore,

$$\|h\|_w \le \frac{C_5}{C_6}\|Q_\varepsilon h\|_s + \frac{1}{2\tau_{\varepsilon_1} C_6}\|Q_\varepsilon h\|_w.$$

Applying this to the inequality $\|h\|_s \le C_5\|Q_\varepsilon h\|_s + C_6\|h\|_w$ yields

$$\begin{aligned}\|h\|_s &\le C_5\|Q_\varepsilon h\|_s + C_6\left(\frac{C_5}{C_6}\|Q_\varepsilon h\|_s + \frac{1}{2\tau_\varepsilon C_6}\|Q_\varepsilon h\|_w\right)\\ &= 2C_5\|Q_\varepsilon h\|_s + \frac{1}{2\tau_{\varepsilon_1}}\|Q_\varepsilon h\|_w \le \left(2C_5 + \frac{1}{2\tau_{\varepsilon_1}}\right)\|Q_\varepsilon\|_s,\end{aligned}$$

for all $0 \le \varepsilon \le \varepsilon_1$. It follows that Q_ε^{-1} exists as a bounded operator on B, by (c). If $f \in B$ is arbitrary, letting $h = Q_\varepsilon^{-1} f$, recalling that $Q_\varepsilon = zI - L_\varepsilon$, we conclude that

$$\|(zI - L_\varepsilon)^{-1} f\|_s \le 2C_5\|f\|_s + \frac{1}{2\tau_{\varepsilon_1}}\|f\|_w.$$

Thus, the first conclusion of the theorem holds, with $a = \max\{2C_5, C_4 M^{-1}\}$ and $b = \frac{1}{2\tau_{\varepsilon_1}}$.

It remains to show that there exist constants c, d such that

$$\|(zI - L_\varepsilon)^{-1} - (zI - L_0)^{-1}\|_{s\to w} \le \tau_\varepsilon^n\left(c\|(zI - L_0)^{-1}\|_{s\to s} + d\|(zI - L_0)^{-1}\|_{s\to s}^2\right).$$

Once again, for $f \in B$, let $h = Q_\varepsilon^{-1} f$. Recall the inequality

$$\begin{aligned}\|h\|_w \le &\left(\|Q_0^{-1}\|_{s\to s} C_2(C_4 M + |z|)\left(\frac{\alpha}{r}\right)^n + \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\tau_\varepsilon\left(\frac{M}{r}\right)^n\right)\|h\|_s\\ &+ \left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\left(\frac{M}{r}\right)^n\|Q_\varepsilon h\|_w.\end{aligned}$$

Letting $n = \left\lceil \frac{\log \tau_\varepsilon}{\log(\alpha/M)}\right\rceil = \left\lceil \eta\frac{\log \tau_\varepsilon}{\log(\alpha/r)}\right\rceil = \left\lceil (\eta - 1)\frac{\log \tau_\varepsilon}{log(M/r)}\right\rceil$, we have

$$\begin{aligned}\|h\|_w \le &\,\tau_\varepsilon^\eta\left(\|Q_0^{-1}\|_{s\to s}(C_2(C_4 M + |z|) + C_3) + \frac{C_1}{M-r}\right)\|h\|_s\\ &+ \tau_\varepsilon^{\eta-1}\left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\|f\|_w.\end{aligned}$$

If $|z| \ge 2M$, then (using a von Neumann expansion as before) $|z|\|Q_0^{-1}\|_{s\to s} \le 2C_4$ and thus

$$\begin{aligned}\|h\|_w \le &\,\tau_\varepsilon^\eta\left(\|Q_0^{-1}\|_{s\to s}(C_2(C_4 + 2)M + C_3) + \left(2C_2 C_4 + \frac{C_1}{M-r}\right)\right)\|h\|_s\\ &+ \tau_\varepsilon^{\eta-1}\left(\|Q_0^{-1}\|_{s\to s} C_3 + \frac{C_1}{M-r}\right)\|f\|_w.\end{aligned}$$

To simplify notation, let $K = C_2(C_4+2)M + C_3$ and $K' = 2C_2C_4 + \frac{C_1}{M-r}$. We once again apply the auxiliary lemma to obtain

$$\|h\|_w \le \tau_\varepsilon^\eta \left(\|Q_0^{-1}\|_{s\to s}K + K'\right) C_5\|f\|_s + \tau_\varepsilon^\eta \left(\|Q_0^{-1}\|_{s\to s}K + K'\right) C_6\|h\|_w$$
$$+ \tau_\varepsilon^{\eta-1}\left(\|Q_0^{-1}\|_{s\to s}C_3 + \frac{C_1}{M-r}\right)\|f\|_w.$$

Recall the previously defined

$$H = H(\delta, r) = \sup_{z\in\mathbb{C}\setminus V_{\delta,r}} \|(zI - L_0)^{-1}\|_{s\to s}$$

and let

$$\varepsilon_0 = \varepsilon_0(\delta, r) = \sup\left\{0 < \varepsilon \le \varepsilon_1 : \tau_\varepsilon^\eta(HK + K')C_6 \le \frac{1}{2}\right\}.$$

We have, for $0 \le \varepsilon \le \varepsilon_0$,

$$\|Q_\varepsilon^{-1} f\|_w = \|h\|_w \le 2\tau_\varepsilon^\eta\left(\left(\|Q_0^{-1}\|_{s\to s}K + K'\right) C_5\|f\|_s + \tau_\varepsilon^{-1}\left(\|Q_0^{-1}\|_{s\to s}C_3 + \frac{C_1}{M-r}\right)\|f\|_w\right).$$

Since $Q_\varepsilon^{-1} - Q_0^{-1} = Q_\varepsilon^{-1}(L_\varepsilon - L_0)Q_0^{-1}$, we will apply this inequality to $(L_\varepsilon - L_0)Q_0^{-1}f$, which gives us

$$\begin{aligned}
\|(Q_\varepsilon^{-1} - Q_0^{-1})f\|_w &= \|Q_\varepsilon^{-1}(L_\varepsilon - L_0)Q_0^{-1}f\|_w \\
&\le 2\tau_\varepsilon^\eta\Bigg(\left(\|Q_0^{-1}\|_{s\to s}K + K'\right) C_5\|(L_\varepsilon - L_0)Q_0^{-1}f\|_s \\
&+ \tau_\varepsilon^{-1}\left(\|Q_0^{-1}\|_{s\to s}C_3 + \frac{C_1}{M-r}\right)\|(L_\varepsilon - L_0)Q_0^{-1}f\|_w\Bigg) \\
&\le 2\|Q_0^{-1}\|_{s\to s}^2\tau_\varepsilon^\eta(2KC_4C_5M + C_3)\|f\|_s \\
&+ 2\|Q_0^{-1}\|_{s\to s}\tau_\varepsilon^\eta\left(2K'C_4C_5M + \frac{C_1}{M-r}\right)\|f\|_s.
\end{aligned}$$

Letting

$$c = 2\left(2K'C_4C_5M + \frac{C_1}{M-r}\right), \quad d = 2(2KC_4C_5M + C_3),$$

then we have the desired conclusion, for all $0 \le \varepsilon \le \varepsilon_0$. □

It directly follows from the first conclusion of this theorem that

$$\sup_{0\le\varepsilon\le\varepsilon_0,\, z\in\mathbb{C}\setminus V_{\delta,r}} \left\|(zI - L_\varepsilon)^{-1}\right\|_{s\to s} < +\infty,\ \forall\,\delta > 0,\ \forall\, r \in]\alpha, M[.$$

Letting $\sigma_\alpha(L_\varepsilon) = \sigma(L_\varepsilon) \cup \{z \in \mathbb{C} : |z| \le \alpha\}$, then (as $\varepsilon \to 0$) all accumulation points of spectral values in $\sigma_\alpha(L_\varepsilon)$ are contained in $\sigma_\alpha(L_0)$. Therefore, if $\lambda \in \sigma(L_0)$ is an isolated eigenvalue of L_0 with $|\lambda| > \alpha$, then there exists some $\delta > 0$ such that $B_\delta(\lambda) \cap \sigma(L_0) = \{\lambda\}$. This allows us to consider (for $\delta > 0$ sufficiently small) the spectral projections

$$P_\varepsilon^\lambda = \frac{1}{2\pi i}\int_{\partial B_\delta(\lambda)} (zI - L_\varepsilon)^{-1} dz.$$

Notice that they do not depend on $\delta > 0$, as long as $B_\delta(\lambda) \cap \sigma_\alpha(L_0) = \{\lambda\}$ and ε is sufficiently small.

Corollary 3.3.7. *In the conditions of Theorem 3.3.5, let λ be an isolated eigenvalue of L_0 with $|\lambda| > r$. If $\delta > 0$ is such that $B_\delta(\lambda) \cap \sigma_\alpha(L_0) = \{\lambda\}$, then:*

1. *there exists a constant $K_1 > 0$ such that $\left\|P_\varepsilon^\lambda - P_0^\lambda\right\|_{s\to w} \le K_1 \tau_\varepsilon^\eta$, $\forall\, 0 \le \varepsilon \le \varepsilon_0$;*
2. *there exist constants $K_2 > 0$ and $\delta_0 > 0$ such that $\|P_\varepsilon^\lambda f\|_s \le K_2 \|P_\varepsilon^\lambda f\|_w$, for all $f \in B, 0 < \delta \le \delta_0, 0 \le \varepsilon \le \varepsilon_0$;*
3. *if $0 < \delta \le \delta_0$, then rank(P_ε^λ) = rank(P_0^λ), for sufficiently small ε.*

Remark 3.3.8. Notice that if the isolated eigenvalue λ has finite algebraic multiplicity, then (for sufficiently small ε and δ) $\sigma_\alpha(L_\varepsilon) \cap B_\delta(\lambda)$ consists of eigenvalues $\lambda_j^{(\varepsilon)}$ such that, for all j, $\lim_{\varepsilon\to 0} \lambda_j^{(\varepsilon)} = \lambda$. Moreover, the total multiplicity of $\lambda_j^{(\varepsilon)}$ equals the multiplicity of λ.

Proof. 1. We have, letting $H = H(\delta, r)$ as in the previous proof,

$$\|P_\varepsilon^\lambda - P_0^\lambda\|_{s\to w} \le \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} \|(zI - L_\varepsilon)^{-1} - (zI - L_\varepsilon)^{-1}\|_{s\to w}\, dz \le \delta \tau_\varepsilon^\eta (cH + dH^2),$$

and so we have the desired conclusion, with $K_1 = \delta(cH + dH)$. It is worth noting that this implies that $\|P_\varepsilon^\lambda - P_0^\lambda\|_{s\to w} \to 0$, as $\varepsilon \to 0$.

2. By the first conclusion of Theorem 3.3.5, for any $f \in B$,

$$\begin{aligned}
\|P_\varepsilon^\lambda f\|_s &\le \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} \|(zI - L_\varepsilon)^{-1} f\|_s\, dz \\
&\le \frac{1}{2\pi}\int_{\partial B_\delta(\lambda)} (a\|f\|_s + b\|f\|_w)\, dz \\
&\le \delta a \|f\|_s + \delta b \|f\|_w.
\end{aligned}$$

Fixing r, we may choose $\delta > 0$ such that $\delta a(r) \leq \frac{1}{2}$. For sufficiently small ε, we have

$$\|P_\varepsilon^\lambda f\|_s \leq \frac{1}{2}\|f\|_s + K''\|f\|_w,$$

where $K'' > 0$ is some constant. Now we apply this inequality to $P_\varepsilon^\lambda f$, which yields (since P_ε^λ is a projection), for a suitable constant $K_2 > 0$,

$$\|P_\varepsilon^\lambda(P_\varepsilon^\lambda f)\|_s = \|P_\varepsilon^\lambda f\|_s \leq \frac{1}{2}\|P_\varepsilon^\lambda f\|_s + K_2\|P_\varepsilon^\lambda f\|_w,$$

which implies $\|P_\varepsilon^\lambda f\|_s \leq K_2\|P_\varepsilon^\lambda f\|_w$.

3. We will make use of the following result, the proof of which can be found in Section II.1 of [12].

Theorem 3.3.9. *Let X_{n+1} be any $(n+1)$-dimensional subspace of a normed space X and let $S(X_{n+1})$ denote the unit ball of X_{n+1}. Then*

$$d_k(S(X_{n+1};X) = 1, \quad k = 0, 1, \dots, n,$$

where, for a subset $A \subseteq X$, $d_k(A;X)$ is defined as

$$d_k(A;X) = \inf_{X_k}\sup_{x\in A}\inf_{y\in X_k}\|x-y\|,$$

where the infimum is taken over all k-dimensional subspaces X_k of X.

Let $n \geq 1$ and V_n be an artbitrary n-dimensional subspace of $P_\varepsilon^\lambda(B)$. By 1, we can choose ε and δ small enough such that $\|P_\varepsilon^\lambda - P_0^\lambda\|_{s\to w} < \frac{1}{2K_2}$. For $f \in V_n$, we have

$$\|f - P_0^\lambda\|_w = \|P_\varepsilon^\lambda f - P_0^\lambda f\|_w \leq \frac{1}{2K_2}\|f\|_s \leq \frac{1}{2}\|f\|_w,$$

where we used 2 to obtain the last inequality. By the Theorem we just stated above, we obtain $n \leq rank(P_0^\lambda)$. Since n is arbitrary, then $rank(P_\varepsilon^\lambda) \leq rank(P_0^\lambda)$. Switching the roles of P_ε^λ and P_0^λ, we have the reverse inequality.

□

Chapter 4

Tent maps

The main goal of this chapter is to exhibit an explicit family of tent maps which lacks linear response. In the first section, we follow Sections 3.1 and 3.2 of [15] and we show that a certain class of maps admits an invariant density (in particular, tent maps). We also obtain a decomposition of the spectrum of the transfer operator acting on the space of functions of bounded variation, which will prove itself useful in the last chapter. In the second section of this chapter, we follow [3] to conclude the lack of linear response for the family of tent maps.

4.1 Topological mixing, existence and uniqueness of acip

We will work with $I = [0, 1]$, although of course the theory could be developed considering an arbitrary interval. Given a partition $\mathcal{P}$, $0 = a_0 < a_1 < \cdots < a_k = 1$, and a map $T : I \to I$, we define the partition $\mathcal{P}^{(n)}$ by $\mathcal{P}^{(n)}(x) = \mathcal{P}^{(n)}(y) \iff \mathcal{P}(T^j(x)) = \mathcal{P}(T^j(y))$, for all $0 \leq j < n$ (where $\mathcal{P}(x)$ denotes the interval of the partition $\mathcal{P}$ that contains x). As in the proof of Theorem 3.1.13, we will use the notation $I_i = [a_{i-1}, a_i), 1 \leq i < k$, $I_k = [a_{k-1}, a_k]$ and assume that T is right-continuous. Moreover, for $J \in \mathcal{P}^{(n)}$, denote $g_J^{(n)} = \frac{1}{|(T^n|_J)'|}$.

Definition 4.1.1 (Piecewise expanding map)**.** A map $T : I \to I$ is said to be piecewise expanding if:

(a) there exists a partition $\mathcal{P}$, $0 = a_0 < a_1 < \cdots < a_k = 1$ such that $T_i = T|_{(a_{i-1}, a_i)}$ is of class C^1, with $|T_i'| > 0$, $i = 1, \ldots, k$;

(b) the function $g_i(x) = \frac{1}{|T_i'(x)|}$ is of bounded variation;

(c) There exist constants $C_1 > 0$ and $K_1 < 1$ such that $\sup g_J^{(n)} \le C_1 K_1^n$, for all $J \in \mathcal{P}^{(n)}$ and all $n \ge 1$.

Example 4.1.2. Let $0 = a_0 < a_1 < \cdots < a_k = 1$ and $T : [0,1] \to [0,1]$ be a map such that each T_i admits a C^2 extension to $[a_{i-1}, a_i]$ satisfying $|T'(x)| \ge K > 1$, for any $x \in [a_{i-1}, a_i]$. Then T is a piecewise expanding map.

Our aim will be to arrive at some conditions that guarantee the existence and uniqueness of an absolutely continuous invariant probability measure.

Lemma 4.1.3. *There exist constants $C_2 > 0$ and $K_2 < 1$ such that, for all $J \in \mathcal{P}^{(n)}$ and all $n \ge 1$,*

$$V g_J^{(n)} \le C_2 K_2^n$$

Proof. Let $C_1 > 0$ be large enough that $V_{(a_{i-1}, a_i)} g_i \le C_1$, for all $1 \le i \le k$. Given $J \in \mathcal{P}^{(n)}$ and $0 \le j < n$, let $\alpha_j \in \mathcal{P}^{(j)}, \beta_j \in \mathcal{P}$ and $\gamma_j \in \mathcal{P}^{(n-j-1)}$ be defined by $J \subseteq \alpha_j$, $T^j(J) \subseteq \beta_j$ and $T^{j+1}(J) \subseteq \gamma_j$. Then

$$\begin{aligned} V g_J^{(n)} &\le \sum_{j=0}^{n-1} \sup g_{\gamma_j}^{(n-j-1)} \cdot V g_{\beta_j} \cdot \sup g_{\alpha_j}^{(j)} \\ &\le \sum_{j=0}^{n-1} C_1 K_1^{n-j-1} \cdot C_1 \cdot C_1 K_1^j = \frac{C_1^3 n K_1^n}{K_1}. \end{aligned}$$

Let $K_2 \in (K_1, 1)$ and $C_2 = \sup\{(C_1^3/K_1) n (K_1/K_2)^n : n \ge 1\}$. Then

$$V g_J^{(n)} \le C_2 K_2^n, \quad \forall\, J \in \mathcal{P}^{(n)}, \forall\, n \ge 1.$$

□

Lemma 4.1.4. *There exist constants $C_0 > 0$ and $K_0 < 1$ such that*

$$V_I(L_T^n f) \le C_0 K_0^n V_I f + C_0 \|f\|_1,$$

for all $n \ge 1$ and for every function f of bounded variation.

Proof. Since T is piecewise monotonic, we have, for each $n \geq 1$,

$$L_T^n f = \sum_{J \in \mathcal{P}^{(n)}} g_J^{(n)} f \circ (T^n|_J)^{-1} \chi_{T^n(J)}.$$

By the properties of functions of bounded variation and noting that $V\chi_{f^n(J)} = 2$, we have

$$V(L_T^n f) \leq \sum_{J \in \mathcal{P}^{(n)}} \left(\left(V g_J^{(n)} + 2 \sup g_J^{(n)} \right) \cdot \sup |f|_J| + \sup g_J^{(n)} \cdot V_J f \right).$$

By the mean value theorem for integrals, there exists $y \in J$ such that

$$|f(y)| = \frac{1}{m(J)} \int_J |f| \, dm.$$

Therefore, we have

$$\sup |f|_J| \leq V_J|f| + |f(y)| = V_J|f| + \frac{1}{m(J)} \int_J |f| \, dm \leq V(f|_J) + \frac{1}{m(J)} \|f\|_1,$$

and so

$$\begin{aligned} V(L_T^n f) &\leq \sum_{J \in \mathcal{P}^{(n)}} \left((C_2 K_2^n + 2C_1 K_1^n) \cdot \left(V(f|_J) + \frac{1}{m(J)} \|f\|_1 \right) + C_1 K_1^n V(f|_J) \right) \\ &\leq 4C_2 K_2^n \sum_{J \in \mathcal{P}^{(n)}} V(f|_J) + 3C_2 K_2^2 \sum_{J \in \mathcal{P}^{(n)}} \frac{1}{m(J)} \|f\|_1 \\ &\leq C_3 K_3^n V f + M(n) \|f\|_1, \end{aligned}$$

where $C_3 = 4C_2, K_3 = K_2$ and $M(n) = 3C_2 K_2^n (\#\mathcal{P}^{(n)}) \sup \left\{ \frac{1}{m(J)} : J \in \mathcal{P}^{(n)} \right\}$. All that remains is to remove the dependence on n of $M(n)$. To this effect, let $N \geq 1$ such that $C_3 K_3^N \leq 1/2$ and let $\tilde{M} = \max\{M(n) : 1 \leq n \leq N\}$. For $n \geq 1$, we can write $n = qN + r$, with $q \geq 0, 1 \leq r \leq N$, and thus

$$\begin{aligned} V(L_T^n f) &\leq \tilde{M} \int |L_T^{n-N} f| \, dm + C_3 K_3^N V(L_T^{n-N} f) \\ &\leq \tilde{M} \|f\|_1 + \frac{1}{2} V(L^{n-N} f) \\ &\leq \left(1 + \cdots + \frac{1}{2^{q-1}} \right) \tilde{M} \|f\|_1 + \frac{1}{2^q} V(L_T^r f) \\ &\leq \left(1 + \cdots + \frac{1}{2^q} \right) \tilde{M} \|f\|_1 + \frac{1}{2^q} C_3 K_3^r V f. \end{aligned}$$

Choosing $C_0 \geq \max\{2\tilde{M}, C_3\}, 1 < K_0 \leq \max\{2^{-1/N}, K_3\}$ gives us the desired result. ∎

We have all the necessary tools to deduce the desired existence of an invariant density (of bounded variation) for piecewise expanding maps.

Theorem 4.1.5. *If $T : I \to I$ is a piecewise expanding map, then T admits an absolutely continuous invariant measure, whose density is of bounded variation. Additionally, if μ is an absolutely continuous invariant probability measure, then $\mu = fm$, where f is a function of bounded variation.*

Proof. Consider the sequence

$$f_n = \frac{1}{n}\sum_{j=0}^{n-1} L_T^j \chi_I.$$

We wish to apply Helly's theorem to this sequence. We have

$$Vf_n \le \frac{1}{n}\sum_{j=0}^{n-1} V(L_T^j \chi_I) \le C_0.$$

Since $\|f_n\|_1 = 1$, for all $n \ge 1$, we also have

$$\sup |f_n| \le V|f_n| + \|f_n\|_1 \le C_0 + 1.$$

We can thus apply Helly's theorem to this sequence to extract a subsequence $(f_{n_k})_k$ that converges in $L^1(I)$ to some function f^* of bounded variation. Notice that

$$L_T f_{n_k} = f_{n_k} + \frac{1}{n_k}(L_T^{n_k}\chi_I - \chi_I).$$

Recall that the transfer operator is continuous in $L^1(I)$ and that $\|L_T^n \chi_I\|_1 = 1$, for all $n \ge 1$. Therefore, taking the limit $k \to +\infty$ in the equality above, we obtain

$$L_T f^* = f^*,$$

where f^* is such that $f^* \ge 0$ and $\|f^*\|_1 = 1$. Therefore, the measure $\mu = f^* m$ is an acip.

Now suppose that $\mu = h \cdot m$ is an acip. Then $\|h\|_1 = 1$ and $L_T h = h$. Since functions of bounded variation are dense in $L^1(I)$, let $(h_k)_k$ be a sequence of functions of bounded variation converging to h in $L^1(I)$. We may suppose that $\|h_k\|_1 \le 2$, for all $k \ge 1$. We have, by lemma 4.1.4,

$$V\left(\frac{1}{n}\sum_{j=0}^{n-1} L_T^j h_k\right) \le \frac{1}{n}\sum_{j=0}^{n-1} C_0\left(K_0^j V h_k + \|h_k\|_1\right).$$

Letting $n \to +\infty$ above, the right-hand side converges to $C_0\|h_k\|_1 \leq 2C_0$. Moreover,

$$\left\| \frac{1}{n} \sum_{j=0}^{n-1} L_T^j h_k \right\|_1 \leq \|h_k\|_1 \leq 2.$$

Therefore, for each fixed k, we can apply Helly's theorem to the sequence $\left(\frac{1}{n}\sum_{j=0}^{n-1} L_T^j h_k\right)_n$ to conclude that, for each k, there exists a function $\tilde{h}_k$ and a subsequence $\left(\frac{1}{n_l}\sum_{j=0}^{n_l-1} L_T^j h_k\right)_l$ that converges to $\tilde{h}_k$ in $L^1(I)$, where $V\tilde{h}_k \leq 2C_0$ and $\|\tilde{h}_k\|_1 \leq 2$, for all $k \geq 1$. We are free to apply Helly's theorem again, this time to the sequence $(\tilde{h}_k)_k$, to obtain a subsequence $(\tilde{h}_{k_m})$ that converges in $L^1(I)$ to some function f^* with $Vf^* \leq 2C_0$. We also have

$$\|\tilde{h}_k - h\|_1 = \lim_{l \to +\infty} \left\| \frac{1}{n_l} \sum_{j=0}^{n_l-1} L_T^j (h_k - h) \right\|_1 \leq \|h_k - h\|_1.$$

Recall that (h_k) converges to h in $L^1(I)$, so $(\tilde{h}_k)$ also converges to h in $L^1(I)$, which implies $f^* = h$ a.e., and so $\mu = h \cdot m = f^* \cdot m$. □

Corollary 4.1.6. *The map T has only finitely many ergodic absolutely continuous invariant probability measures.*

Proof. Suppose μ is an ergodic acip. By the previous result, there exists $h \in BV$ such that $\mu = h \cdot m$ and there also exists an interval J such that $h > 0$ on J: indeed, since h has only countably many discontinuities and $m(\{x \in I : h(x) > 0\} > 0$, we conclude that the set of points x such that h is continuous on x and $h(x) > 0$ has positive Lebesgue measure. Therefore, such an interval J exists. Notice that h is thus also positive on $T^n(J)$, for all $n \geq 0$: since $h = L_T^n h$ and

$$L_T^n h(x) = \sum_{y \in T^{-n}(x)} \frac{h(y)}{|(T^n)'(y)|},$$

then if $x \in J$ and applying this to $T^n(x)$, we have

$$L_T^n h(T^n(x)) = \sum_{y \in T^{-n}(T^n(x))} \frac{h(y)}{|(T^n)'(y)|} \geq \frac{h(x)}{|(T^n)'(x)|} > 0,$$

and so $h(T^n(x)) = L_T h(T^n(x)) > 0$. This implies that $m \ll \mu$ on $T^n(J)$, hence m and μ are equivalent measures on $T^n(J)$, for all $n \geq 0$. By Birkhoff's Ergodic Theorem, we know that, for every continuous function φ, $\frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(x))$ converges to $\int \varphi\, d\mu$, μ-a.e. and thus this convergence also holds Lebesgue-a.e.

Recall the partition $\mathcal{P}$, $a_0 < a_1 < \ldots a_k$ (from the definition of a piecewise expanding map). There must exist an $1 \leq i \leq k-1$ and an N such that $a_i \in T^N(J)$, since if that were not the case, $m(T^n(J))$ would be unbounded (since the map is expanding on each monotonicity interval). Since intervals $T^N(J)$ obtained this way, for different ergodic measures, are necessarily disjoint. Therefore, there are at most $k-1$ ergodic acips.

□

Uniqueness of this acip is not guaranteed with the current assumptions. With an additional hypothesis on the map T, we can achieve the uniqueness of such an ergodic acip.

Definition 4.1.7 (topological mixing)**.** A map $T : I \to I$ is said to be topologically mixing if there exists an interval $I^* \subseteq I$ such that:

(a) $T(I^*) = I^*$;

(b) every orbit $T^n(x), 0 < x < 1$, eventually enters I^*;

(c) for every interval $J \subseteq I^*$, there exists $n \geq 1$ such that $T^n(J) = I^*$.

Example 4.1.8. Consider $f_t : I \to I$ be a tent map defined by $f_t(x) = tx$, if $0 \leq x \leq 1/2$, $f_t(x) = t(1-x)$, if $1/2 \leq x \leq 1$, with $t > \sqrt{2}$. Let $I^* = [f_t^2(1/2), f_t(1/2)]$. It is easy to check that $f_t(I^*) = I^*$ and that any orbit $f_t^n(x)$, with $0 < x < 1$ eventually enters I^*.

We check condition (c). Let $J \subseteq I^*$ be an interval and let $x_t \in (1/2, 1)$ be the fixed point of f_t. Then $f_t^n(J)$ must eventually contain x_t. Suppose this is not the case and let $J_0 = J$ and define the intervals J_n by induction:

- if $f_t^{n-1}(J_{n-1})$ does not contain $1/2$, define $J_n = J_{n-1}$. In this case, we have $m(f_t^n(J_n)) = tm(f^{n-1}(J_{n-1}))$;
- if $f_t^{n-1}(J_{n-1})$ contains $1/2$, then define $J_n \subseteq J_{n-1}$ such that $f_t^{n-1}(J_n)$ is equal to the largest of the two intervals $f_t^{n-1}(J_{n-1}) \cap [f_t^2(1/2), 1/2]$ or $f^{n-1}(J_{n-1}) \cap [1/2, f(1/2)]$. In this case, we have $m(f_t^n(J_n)) \geq \frac{t}{2} m(f^{n-1}(J_{n-1})$.

Furthermore, $f_t^n(J_n)$ does not contain $1/2$, since we are supposing that $x_t \notin f^{n+1}(J_n)$. Thus, $J_{n+1} = J_n$, which implies

$$m(f_t^{n+1}(J_{n+1})) \geq tm(f_t^n(J_n)) \geq \frac{t^2}{2} m(f^{n-1}(J_{n-1})).$$

Since $t > \sqrt{2}$, the sequence $(m(f_t^n(J_n)))_n$ would be unbounded, which cannot happen. Therefore, there exists some n_1 such that $x_t \in f_t^{n_1}(J_{n_1})$ and hence $x_t \in f_t^n(J)$, for all $n \geq n_1$. It also follows that $[1/2, x_t] \subseteq f_t^{n_2}(J)$, for some $n_2 \geq n_1$, and that $[f_t^3(1/2), f_t(1/2)] \subseteq f_t^{n_2+3}(J)$. We have $f_t^3(1/2) < x_t$, for all $t > \sqrt{2}$. If $f_t^3(1/2) \leq 1/2$, then $[f_t^2(1/2), f(c)] \subseteq f_t^{n_2+4}(J)$ and we have (c). If $f_t^3(1/2) > 1/2$, then there must exist some odd number $p > 3$ such that $f_t^p(1/2) < 1/2$ and $[f_t^k(1/2), f(1/2)] \subseteq f_t^{n_2+k}(J)$. In this case, $[f_t^2(c), f(c)] \subseteq f_t^{n_2+k+1}(J)$ and we have (c).

Theorem 4.1.9. *If $T : I \to I$ is piecewise expanding and topologically mixing, then there exists a unique absolutely continuous invariant probability measure μ, which is ergodic and whose support is I^*.*

Proof. Suppose that h_1 and h_2 are such that $\int_I h_1 = \int_I h_2 = 1, h_1 \geq 0, h_2 \geq 0$ and $L_T h_1 = h_1, L_T h_2 = h_2$. We wish to show that $h_1 = h_2$ a.e. Consider the sets

$$X_1 = \{x \in I : h_1(x) \geq h_2(x)\}, \quad X_2 = \{x \in I : h_1(x) < h_2(x)\}$$

and define the measures $\mu_1 = (h_1 - h_2)\chi_{X_1} \cdot m$, $\mu_2 = (h_2 - h_1)\chi_{X_2} \cdot m$. We have

$$\begin{aligned} \int_{X_1} (h_1 - h_2)\, dm &= \int_{X_1} L_T(h_1 - h_2)\, dm = \int_{T^{-1}(X_1)} (h_1 - h_2)\, dm \\ &\leq \int_{T^{-1}(X_1) \cap X_1} (h_1 - h_2)\, dm \leq \int_{X_1} (h_1 - h_2)\, dm. \end{aligned}$$

Hence $m\,(T^{-1}(X_1) \cap X_1) = m(X_1) = m\,(T^{-1}(X_1))$. Analogously, one arrives at a similar conclusion for X_2. This implies that the measures μ_1 and μ_2 are T-invariant. By Theorem 4.1.5, we know that $(h_1 - h_2)\chi_{X_1}$ and $(h_2 - h_1)\chi_{X_2}$ are functions of bounded variation.

Assume that $(h_1 - h_2)\chi_{X_1} \neq 0$ a.e. Then there exists an interval $J \in I^*$ such that $(h_1 - h_2)\chi_{X_1}(x) > 0$, for all $x \in J$. By topological mixing, $J = I^*$, which means that the support of μ_1 must contain I^*. Notice that the supports of μ_1 and μ_2 are disjoint, so this would imply that $(h_2 - h_1)\chi_{X_2}$ is identically null, so $h_1 \geq h_2$ a.e. and since $\int_I h_1\, dm =$

$\int_I h_2\, dm$, we conclude that $h_1 = h_2$ a.e. The case $(h_2 - h_1)\chi_{X_2} \neq 0$ is entirely analogous and one reaches the same conclusion $h_1 = h_2$.

It remains to check that μ is ergodic. Let $A \subseteq I^*$ be a measurable set with $\mu(A) > 0$ and $T(A) = A$. Given B measurable, we define a new measure

$$\nu(B) = \frac{\mu(B \cap A)}{\mu(A)}.$$

Notice that then ν is an absolutely continuous probability measure and it is T-invariant, as we have

$$\nu(T^{-1}(B)) = \frac{\mu(T^{-1}(B) \cap A)}{\mu(A)} = \frac{\mu(T^{-1}(B \cap A))}{\mu(A)} = \nu(B).$$

By uniqueness, $\mu = \nu$ and so $\mu(A) = \nu(A) = 1$. $\square$

We conclude this section with a fact about the spectrum of the transfer operator when acting on the space $BV([0,1])$. We have the following result.

Theorem 4.1.10. *The spectrum of $L_T : BV([0,1]) \to BV([0,1])$ may be written as $\sigma(L_T) = \{1\} \cup \Sigma_0$, where Σ_0 is contained in a disc of radius $\lambda < 1$ and 1 is an eigenvalue with algebraic multiplicity 1. Furthermore, we have the spectral splitting $BV = \mathbb{R}h \oplus BV_0$, where h is the invariant density and $BV_0 = \{f \in BV : \int f\, dm = 0\}$.*

In order to prove this theorem, we will state the following fact, the proof of which can be found in Section 3.2 of [15]:

Proposition 4.1.11. *If $T : I \to I$ is piecewise expanding and topologically mixing, then the acip μ is mixing.*

We also have the following lemma.

Lemma 4.1.12. *The spectrum of $L_T : BV \to BV$ is of the form $\sigma(L_T) = \{\lambda_0 = 1, \lambda_1, \ldots, \lambda_m\} \cup \Sigma_0$, where Σ_0 is in the conditions of the theorem and where each λ_i is an eigenvalue on the unit circle with finite algebraic multiplicity.*

Proof. By lemma 4.1.4, there exist $C_0 > 0, K_0 < 1$ such that, given $f \in BV$, we have

$$\|L_T f\|_{BV} = V(L_T f) + \|f\|_1 \leq C_0 K_0 V f + (C_0 + 1)\|f\|_1.$$

Since $BV(I)$ is compactly embedded in $L^1(I)$, we are free to apply Proposition 3.2.7 to conclude that $L_T : BV \to BV$ is quasi-compact, with essential spectral radius $r_e(L_T) \leq K_0 < 1$. Since the transfer operator cannot have eigenvalues with modulus larger than 1, the result follows by definition of a quasi-compact operator. $\square$

Proof of Theorem 4.1.10. Suppose that $\lambda_i \in \mathbb{C}$ is an eigenvalue with $|\lambda_i| = 1$ and there exists some $\varphi_i \in BV$ such that $L_\varphi = \lambda_i \varphi_i$. Taking h to be the invariant density, recall that h is supported on I^*, so φ/h is defined a.e. on I^* and $\varphi_i/h \in L^1(I, \mu)$, since

$$\int \left| \frac{\varphi_i}{h} \right| d\mu = \int |\varphi_i| \, dm < +\infty.$$

Since μ is mixing, for every $\varphi \in L^\infty(I, \mu)$, by duality, we have

$$\int \varphi \cdot (L_T^n \varphi_i) \, dm = \int (\varphi \circ T^n) \varphi_i \, dm = \int (\varphi \circ T^n) \frac{\varphi_i}{h} \, d\mu.$$

Taking the limit $n \to +\infty$, since μ is mixing, we obtain

$$\int (\varphi \circ T^n) \frac{\varphi}{h} \, d\mu \to \int \varphi \, d\mu \int \frac{\varphi_i}{h} \, d\mu = \int \varphi \left(h \int \varphi_i \, dm \right) dm.$$

Therefore, $L_T^n \varphi_i = \lambda_i^n \varphi_i$ converges (pointwise) to $h \int \varphi_i \, dm$, which can only happen if $\lambda_i = 1$ and $\varphi_i = h \int \varphi_i \, dm$, so 1 is the only eigenvalue on the unit circle.

Regarding the algebraic multiplicity of 1, if it were not equal to 1, then there would exist some $g \in BV$ such that $L_T^n g = h + ng$, for every $n \geq 1$. But we know that $\|L_T^n\|_{BV}$ is uniformly bounded, so such a g cannot exist.

It remains to check the last statement, but it is easy to see that both $\mathbb{R}h$ and X_0 are invariant under L_T, and so we must have

$$\sigma(L_T) = \{1\} \cup \sigma(L_T|_{X_0})$$

and thus $\sigma(L_{X_0}) = \Sigma_0$.

$\square$

4.2 Lack of linear response for tent maps

We now present a classical example of lack of linear response. Let $1 < t \leq 2$ and consider $f_t : [0,1] \to [0,1]$ be defined by

$$f_t(x) = \begin{cases} tx & x \in \left[0, \frac{1}{2}\right] \\ t - tx & x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Let $c_n(t) = f_t^n(\frac{1}{2})$. A few remarks are worth mentioning. Notice that the interval $[c_2(t), c_1(t)]$ is invariant by f_t and, for any $x \in (0,1)$, there exists some n such that $f_t^n(x) \in [c_2(t), c_1(t)]$, so any invariant density must be supported on $[c_2(t), c_1(t)]$. Furthermore, by Corollary 4.1.6, there is only one ergodic acip.

We can say more about the invariant density of f_t, call it h_t. Let $\{x_n\}$ be a finite or countable subset of $[0,1]$. Then the series $S(x) = \sum H_n(x)$, where $H_n(x) = 0$, if $x < x_n$, $H_n(x_n) = u_n$, $H_n(x) = u_n + v_n$, if $x > x_n$, converges (assuming $\sum u_n$ and $\sum v_n$ converge absolutely) and S is known as a saltus function.

A general fact about functions of bounded variation is that they can be written as $h_t = h_t^s + h_t^c$, where h_t^s is a saltus function and h_t^c is continuous, with both h_t^s and h_t^c being functions of bounded variation (see section 7 of [13]). Let $\varphi = \frac{1}{c_1(t) - c_2(t)} \chi_{[c_2(t), c_1(t)]}$. By what we saw in the previous chapter, we know that $L_{f_t}^n(\varphi)$ converges, in the BV norm, to h_t. Moreover, $|f_t'|$ is constant, and so for each n, $L_{f_t}^n(\varphi)$ is also a saltus function, with discontinuities at the points c_j, for $1 \leq j \leq n$. It follows that h_t will also be a saltus function, with discontinuities at the points $(c_j)_{j=1}^{\infty}$. Notice that if $1/2$ is preperiodic, then the set of discontinuities is finite and h_t is piecewise constant, with discontinuities at the finite set of points c_j.

Theorem 4.2.1. *There exists a C^1 function φ, with $\varphi(0) = \varphi(1) = 0$, a sequence $(t_k)_k$, with $1 < t_k < 2$ and $\lim_{k \to +\infty} t_k = 2$, such that $c_{k+2}(t_k)$ is a fixed point of f_{t_k}. Moreover, there exists a constant $C > 0$ such that*

$$\int \varphi h_{t_k} \, dx - \int \varphi h_2 \, dx \geq Ck(2 - t_k), \quad \forall\, k,$$

where h_{t_k} is the unique invariant density of f_{t_k}.

Proof. We know that $f_t\left(\frac{t}{1+t}\right) = \frac{t}{1+t} > \frac{1}{2}$. Denote the fixed point $\frac{t}{1+t}$ by x_t, its preimage in $\left[0, \frac{1}{2}\right]$ by $y_t = \frac{1}{1+t}$ and the preimage of y_t in $\left[0, \frac{1}{2}\right]$ by $z_t = \frac{y_t}{t}$.

Notice that $c_1(t) = t/2 > 1/2$ and thus $c_2(t) = \frac{(2-t)t}{2}$. If $t = 2$, then $c_1(2) = 1, c_2(2) = c_3(2) = 0$ and h_2 is constant equal to 1 on $[0, 1]$.

If $1 < t < 2$, we wish to find t_k such that $c_{k+2}(t_k) = x_{t_k}$. We can achieve this by finding t_k such that $c_{k+1}(t_k) = y_{t_k}$. This will be the case if

$$t_k = 2 - \frac{2}{t_k^{k+1}(1 + t_k)}, \; k \geq 1.$$

We know that the invariant density h_{t_k} is supported on $[f_{t_k}^2(1/2), f_{t_k}(1/2)] = [c_2(t_k), c_1(t_k)]$. By what was observed immediately before this theorem, h_{t_k} is constant on (c_{j+1}, c_{j+2}), for $1 \leq j \leq k$, and on (c_{k+2}, c_1).

Let $v_j > 0$ be the value of h_{t_k} on (c_{j+1}, c_{j+2}) and $v_{k+1} > 0$ be the value of h_{t_k} on (c_{k+2}, c_1). Then we must have

$$\begin{aligned}
&v_{k+1} = t_k v_1, \\
&v_j + v_{k+1} = t_k v_{j+1}, \; 1 \leq j \leq k-1, \\
&2v_k = t_k v_{k+1}.
\end{aligned}$$

We can then obtain that the sequence $(v_j)_j$ is strictly increasing. In fact, we have

$$v_{k+1} = 2\frac{v_k}{t_k} > v_k$$

and now we proceed by descending induction. Taking $k - 1 \geq j \geq 1$, then we wish to show that $v_{j+1} > v_j$. We have

$$v_{j+1} = \frac{v_j + v_{k+1}}{t_k} > \frac{v_j + v_{k+1}}{2} > \frac{v_j + v_{j+1}}{2},$$

where we used the induction hypothesis in the last inequality, and so $v_{j+1} > v_j$. We conclude that $(v_j)_j$ is strictly increasing.

Now take a C^1 function φ supported in $(2/3, 3/4)$. For all k sufficiently large, φ is also supported in (c_{k+2}, c_1). We may suppose that

$$\int \varphi(x) h_2(x)\, dx = \int \varphi(x)\, dx = 1.$$

To obtain the desired result, it is enough to show that there exists a constant $D > 0$ such that

$$\int \varphi(x) h_{t_k}(x)\, dx \geq 1 + \frac{Dk}{t_k^k},$$

since t_k satisfies

$$2 - t_k = \frac{2}{t_k^{k+1}(1+t_k)}.$$

Notice that

$$\int \varphi(x) h_{t_k}(x)\, dx = v_{k+1} \int_{c_{k+2}}^{c_1} \varphi(x)\, dx = v_{k+1},$$

so we need to bound v_{k+1} from below. We have $\int h_{t_k}\, dx = 1$ and, since h_{t_k} is piecewise constant, we get the equality

$$\int h_{t_k}(x)\, dx = \sum_{j=1}^{k} (c_{j+2} - c_{j+1}) v_j + v_{k+1}(c_{k+2} - c_1),$$

which can be rewritten as

$$v_{k+1}(c_1 - c_2) - \sum_{j=1}^{k} (c_{j+2} - c_{j+1})(v_{k+1} - v_j).$$

Firstly, notice that $c_{j+2} - c_{j+1} = t_k^{j-1}(c_3 - c_2)$ and $c_3 - c_2 \geq A t_k^{-k}$ (where A does not depend on k). Furthermore, we have

$$(v_{k+1} - v_j) \geq (v_{j+1} - v_j) = \frac{v_{k+1} t_k^{k-j}(2 - t_k)}{2} \geq \frac{1}{t_k^{j+1}(1+t_k)}, \quad \text{for } 1 \leq j \leq k-1,$$

The first inequality is immediate from the fact that the sequence $(v_j)_j$ is increasing and the equality is obtained by using

$$v_i + v_{k+1} = t_k v_{i+1}$$

with $i = j$ and $i = j+1$, repeating this $k - j - 1$ times and finally applying $2v_k = t_k v_{k+1}$. By the inequalities for $c_{j+2} - c_{j+1}$ and $v_{k+1} - v_j$ we just deduced, there exists a constant E such that

$$\sum_{j=1}^{k} (c_{j+2} - c_{j+1})(v_{k+1} - v_j) \geq E k t_k^{-k-1}$$

and therefore

$$1 \leq v_{k+1}(c_1 - c_2) - Ekt_k^{-k-1}$$

which implies

$$v_{k+1} \geq \frac{1 + Et_k^{-k-1}}{c_1 - c_2} \geq 1 + Et_k^{-k-1}.$$

This gives us the desired inequality and proves the result. □

Letting $\varepsilon = 2 - t, \varepsilon_k = 2 - t_k,\ g_\varepsilon = f_{2-t}$ and if H_ε is the unique invariant density of g_ε, this shows that we cannot write $H_\varepsilon = H_0 + \varepsilon q + o(\varepsilon)$, where $q \in L^1$ and o is in the L^1 topology, for $0 < \varepsilon < \varepsilon_0$, for some ε_0. In fact, if this were the case for some small ε_0, then we would have, for φ as in the theorem and $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned}\frac{\int |\varphi H_\varepsilon \, dm - \int \varphi H_0| \, dm}{\varepsilon} &= \frac{\int |\varphi \cdot (H_\varepsilon - H_0)| \, dm}{\varepsilon} \\ = \frac{\int |\varphi \cdot (\varepsilon q + o(\varepsilon))| \, dm}{\varepsilon} &= \int |\varphi q| \, dm + \int \left| \varphi \cdot \frac{o(\varepsilon)}{\varepsilon} \right| \, dm.\end{aligned}$$

The first integral is bounded and the second integral goes to 0, as $\varepsilon \to 0$, so this quantity would be bounded, for sufficiently small ε, which is a contradiction, given the sequence exhibited in the theorem.

Remark 4.2.2. In fact, this is enough to show lack of linear response even in the weaker sense for the observable φ, though proving this requires a bit more work. We refer to [3] for the details.

Chapter 5

Tent-like maps with a cusp singularity

We have seen that the family of tent maps with bounded derivative obtained by changing the turning point does not have linear response. We wish to alter this family of functions in order to achieve linear response. The main change is the introduction of a cusp singularity at the turning point, which will provide a certain regularization of the action of the associated transfer operator. We follow the approach taken in [2].

Let $\delta > 0$ and, for $\varepsilon \in [0, \delta)$, let $T_\varepsilon : I \to I$ (where $I = [0, 1]$) be a family of nonsingular maps with respect to the Lebesgue measure m. Let $C \geq 0, c \in (0, 1), \beta \in \left(-1, -\frac{3}{4}\right)$. Suppose that:

(H1) $T_\varepsilon|_{[0,c)}$ and $T_\varepsilon|_{(c,1]}$ are injective;

(H2) $T_\varepsilon(0) = 0$, $T_\varepsilon(1) = 0$, $\lim_{x\to c^\pm} T_\varepsilon = a_\varepsilon \in [0, 1]$;

(H3) $T_{\varepsilon|[0,1]\setminus\{c\}} \in C^3$;

(H4) $\sup_{\varepsilon\in[0,\delta)} \inf_{x\in[0,1]\setminus\{c\}} |T'_\varepsilon(x)| \geq \theta > 1$;

(H5) T_0 is topologically mixing;

(H6) $\forall\, \varepsilon \in [0, \delta)$, $\lim_{x\to c^\pm} T'_\varepsilon(x) = \pm\infty$ and $\lim_{x\to c^\pm} \frac{|T'_\varepsilon(x)|}{|x-c|^\beta} = C_{\varepsilon,1}$;

(H7) $\forall\, \varepsilon \in [0, \delta)$, $\lim_{x\to c^\pm} \frac{T''_\varepsilon(x)}{|x-c|^{\beta-1}} = C_{\varepsilon,2}$ and $\lim_{x\to c^\pm} \frac{|T'''_\varepsilon(x)|}{|x-c|^{\beta-2}} = C_{\varepsilon,3}$;

(H8)

$$\sup_{\varepsilon\in[0,\delta),i\in\{0,1,2\},x\in[0,1]} \left|\frac{T_\varepsilon^{(i+1)}(x)}{(x-c)^{\beta-i}}\right| < \infty \text{ and } \inf_{\varepsilon\in[0,\delta[,x\in[0,1]} \left|\frac{T'_\varepsilon(x)}{(x-c)^{\beta}}\right| > 0;$$

(H9) For $f \in W^{1,1}$,

$$\sup_{\|f\|_{W^{1,1}}\leq 1} \|(L_{T_0} - L_{T_\varepsilon})f\|_{L^2} \to 0, \text{ as } \varepsilon \to 0;$$

(H10) For $f \in W^{2,1}$, there exists $\frac{\partial L_{T_\varepsilon} f}{\partial \varepsilon}|_{\varepsilon=0} \in W^{1,1}$ such that

$$\left\| L_{T_\varepsilon} f - L_{T_0} f - \varepsilon \left(\frac{\partial L_{T_\varepsilon} f}{\partial \varepsilon}|_{\varepsilon=0}\right)\right\|_{W^{1,1}} = o(\varepsilon).$$

We will first show that, for sufficiently small ε, the map T_ε admits a unique invariant density in $h_\varepsilon \in W^{2,1}(I)$ and then we will show that the function $\varepsilon \mapsto h_\varepsilon$ is differentiable at $\varepsilon = 0$ in $L^1(I)$.

5.1 Existence and uniqueness of acip

To prove the existence and uniqueness of such a density h_ε, we will make of use of the spectral gap results developed in the previous chapter. We begin by showing that the transfer operators L_{T_ε} are quasi-compact.

Lemma 5.1.1. *There exist* $0 < \lambda < 1$, $M \geq 0$ *such that for any* $\varepsilon \in [0, \delta)$, $f \in W^{1,1}(I)$ *we have*

$$\|L_{T_\varepsilon} f\|_{W^{1,1}} \leq \lambda \|f\|_{W^{1,1}} + M\|f\|_2$$

Proof. By Theorem 3.1.13, given $f \in W^{1,1}$, $L_{T_\varepsilon} f$ can be written as

$$L_{T_\varepsilon} f(x) = \begin{cases} \sum_{y\in T^{-1}(x)} \frac{f(y)}{T'_\varepsilon(y)} & x \in [0, a_\varepsilon] \\ 0 & x \in (a_\varepsilon, 1] \end{cases}$$

Differentiating, we obtain

$$(L_{T_\varepsilon} f)'(x) = \begin{cases} \sum_{y\in T^{-1}(x)} \frac{f'(y)}{|T'_\varepsilon(y)|T'_\varepsilon(y)} - \frac{T''_\varepsilon(y)}{(T'_\varepsilon(y))^3} f(y) & x \in [0, a_\varepsilon] \\ 0 & x \in]a_\varepsilon, 1]. \end{cases}$$

Notice that $L_{T_\varepsilon} f$ is continuous, since $\lim_{x\to a_\varepsilon} L_{T_\varepsilon} f = 0$. Furthermore, it is differentiable a.e.; when $x \to a_\varepsilon$, we have $y \to c$ and (by hypothesis (H7))

$$\lim_{y\to c} \frac{T_\varepsilon''(y)}{(T_\varepsilon')^3(y)} = \lim_{y\to c} \frac{|y-c|^{\beta-1}}{|y-c|^{3\beta}} = \lim_{y\to c} |y-c|^{-2\beta-1} = 0.$$

Hence $L_{T_\varepsilon} f \in W^{1,1}$. Notice that $(L_{T_\varepsilon} f)' = L_{T_\varepsilon}\left(\frac{f'}{T_\varepsilon'}\right) - L_{T_\varepsilon}\left(\frac{T_\varepsilon''}{|T_\varepsilon'|T_\varepsilon'} f\right)$ and so

$$\begin{aligned}
\|(L_{T_\varepsilon} f)'\|_1 &\leq \left\| L_{T_\varepsilon}\left(\frac{f'}{T_\varepsilon'}\right)\right\|_1 + \left\| L_{T_\varepsilon}\left(\frac{T_\varepsilon''}{(T_\varepsilon')^2} f\right)\right\|_1 \\
&\leq \left\|\frac{f'}{T_\varepsilon'}\right\|_1 + \left\|\frac{T_\varepsilon''}{(T_\varepsilon')^2} f\right\|_1 \text{ (by proposition 3.1.6)} \\
&\leq \left\|\frac{1}{T_\varepsilon'}\right\|_\infty \|f'\|_1 + \left\|\frac{T_\varepsilon''}{(T_\varepsilon')^2} f\right\|_1 \\
&\leq \lambda_\varepsilon \|f'\|_1 + \left\|\frac{T_\varepsilon''}{(T_\varepsilon')^2}\right\|_2 \|f\|_2 \text{ (by Hölder's inequality),}
\end{aligned}$$

where $\lambda_\varepsilon = \left\|\frac{1}{T_\varepsilon'}\right\|_\infty < 1$ and $\left\|\frac{T_\varepsilon''}{(T_\varepsilon')^2}\right\|_2 < \infty$. By hypothesis (H4) and (H8), $\left\|\frac{1}{T_\varepsilon'}\right\|_\infty$ and $\left\|\frac{T_\varepsilon''}{(T_\varepsilon')^2}\right\|_2$ can be bounded uniformly for $\varepsilon \in [0,\delta)$, say by $\lambda < 1$ and C, respectively.

This inequality is enough to conclude what we need. In fact,

$$\begin{aligned}
\|L_{T_\varepsilon} f\|_{W^{1,1}} &= \|L_{T_\varepsilon} f\|_1 + \|(L_{T_\varepsilon} f)'\|_1 \\
&\leq \|f\|_1 + \lambda\|f'\|_1 + C\|f\|_2 \\
&= \lambda\|f\|_1 + \lambda\|f'\|_1 + (1-\lambda)\|f\|_1 + C\|f\|_2 \\
&\leq \lambda\|f\|_{W^{1,1}} + (1-\lambda)\|f\|_2 + C\|f\|_2.
\end{aligned}$$

Therefore there exists a constant $M > 0$ such that

$$\|L_{T_\varepsilon} f\|_{W^{1,1}} \leq \lambda\|f\|_{W^{1,1}} + M\|f\|_2.$$

□

This establishes the first Lasota Yorke inequality with $\|\cdot\|_2$ as the weak norm. We now prove another Lasota Yorke inequality, but this time with $\|\cdot\|_{W^{2,1}}$ as the strong norm and $\|\cdot\|_{W^{1,1}}$ as the weak norm.

Lemma 5.1.2. *There exists $M \geq 0$ such that, for each $\varepsilon \in [0,\delta)$,*

$$\|L_{T_\varepsilon} f\|_{W^{2,1}} \leq \lambda^2\|f\|_{W^{2,1}} + M\|f\|_{W^{1,1}},$$

where λ is the same as in lemma 5.1.1.

Proof. For $f \in W^{2,1}$, we differentiate $(L_{T_\varepsilon} f)' = L_{T_\varepsilon}\left(\frac{f'}{T'_\varepsilon}\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)$ to obtain

$$
\begin{aligned}
(L_{T_\varepsilon} f)'' &= \left(L_{T_\varepsilon}\left(\frac{f'}{T'_\varepsilon}\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)\right)' \\
&= \left(L_{T_\varepsilon}\left(\frac{f'}{T'_\varepsilon}\right)\right)' - \left(L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)\right)' \\
&= L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{f'}{T'_\varepsilon}\right)'\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{f'}{T'_\varepsilon}\right) \\
&\quad - \left[L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)'\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)\right].
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{f'}{T'_\varepsilon}\right)'\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{f'}{T'_\varepsilon}\right) &= L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{-T''_\varepsilon}{(T'_\varepsilon)^2} f' + \frac{1}{T'_\varepsilon} f''\right)\right) - L_{T_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{1}{T'_\varepsilon} f'\right) \\
&= L_{T_\varepsilon}\left(\left(\frac{1}{T'_\varepsilon}\right)^2 f''\right) - L_{T_\varepsilon}\left(\left(\frac{T''_\varepsilon}{(T'_\varepsilon)^2} + \frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\right)\frac{1}{T'_\varepsilon} f'\right).
\end{aligned}
$$

Also

$$
\begin{aligned}
L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon} f\right)'\right) &= L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left[\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\right)' f + \left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\right) f'\right]\right) \\
&= L_{T_\varepsilon}\left(\frac{1}{T'_\varepsilon}\left(\frac{T'''_\varepsilon |T'_\varepsilon| T'_\varepsilon - 2(|T'_\varepsilon| T''_\varepsilon) T''_\varepsilon}{(T'_\varepsilon)^4}\right) f + \frac{1}{T'_\varepsilon}\left(\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\right) f'\right).
\end{aligned}
$$

Using the fact that $\beta \in \left(-1, -\frac{3}{4}\right)$ and the various hypothesis we obtain:

1. $\left(\frac{1}{T'_\varepsilon}\right)^2 \in L^\infty$, by (H4);

Using the notation $f \sim g$ to mean $\lim_{y \to c} \frac{|f(y)|}{|g(y)|} = K$, with $0 < K < +\infty$, (2)-(6) follow from hypothesis (H6) and (H7):

2. $g_{1,a} = \frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{1}{T'_\varepsilon} \in L^\infty$, since

$$\frac{T''_\varepsilon}{|T'_\varepsilon| T'_\varepsilon}\frac{1}{T'_\varepsilon} \sim |y-c|^{-2\beta-1};$$

3. $g_{1,b} = \frac{T''_\varepsilon}{(T'_\varepsilon)^2}\frac{1}{T'_\varepsilon} \in L^\infty$, since

$$\frac{T''_\varepsilon}{(T'_\varepsilon)^2}\frac{1}{T'_\varepsilon} \sim |y-c|^{-2\beta-1};$$

4. $g_2 = \frac{1}{T_\varepsilon'}\left(\frac{T_\varepsilon'''|T_\varepsilon'T_\varepsilon'}{(T_\varepsilon')^4}\right) \in L^2$, since

$$\frac{1}{T_\varepsilon'}\left(\frac{T_\varepsilon'''|T_\varepsilon'|T_\varepsilon'}{(T_\varepsilon')^4}\right) \sim |y-c|^{-2\beta-2}$$

5. $g_3 = \frac{1}{T_\varepsilon'}\left(\frac{2(|T_\varepsilon'|T_\varepsilon'')T_\varepsilon''}{(T_\varepsilon')^4}\right) \in L^2$, since

$$\frac{1}{T_\varepsilon'}\left(\frac{2(|T_\varepsilon'|T_\varepsilon'')T_\varepsilon''}{(T_\varepsilon')^4}\right) \sim |y-c|^{-2-2\beta}$$

6. $g_4 = \frac{T_\varepsilon''}{|T_\varepsilon'|T_\varepsilon'}\frac{T_\varepsilon''}{|T_\varepsilon'|T_\varepsilon'} \in L^2$, since

$$\frac{T_\varepsilon''}{|T_\varepsilon'|T_\varepsilon'}\frac{T_\varepsilon''}{|T_\varepsilon'|T_\varepsilon'} \sim |y-c|^{-2\beta-2}.$$

By (H8), all the norms $\|g_{1,a}\|_\infty, \|g_{1,b}\|_\infty, \|g_2\|_2, \|g_3\|_2$ and $\|g_4\|_2$ are uniformly bounded for $\varepsilon \in [0,\delta)$. This shows that $L_{T_\varepsilon} f \in W^{2,1}$, if $f \in W^{2,1}$. We can write, using the notation and calculations above,

$$\begin{aligned}
(L_{T_\varepsilon} f)'' &= L_{T_\varepsilon}\left(\left(\frac{1}{T_\varepsilon}\right)^2 f''\right) - L_{T_\varepsilon}([g_{1,a}+g_{1,b}]f') - L_{T_\varepsilon}([g_2-g_3]f) - L_{T_\varepsilon}(g_{1,a}f') + L_{T_\varepsilon}(g_4 f)\\
&= L_{T_\varepsilon}\left(\left(\frac{1}{T_\varepsilon'}\right)^2 f''\right) - L_{T_\varepsilon}([2g_{1,a}+g_{1,b}]f') - L_{T_\varepsilon}([g_2-g_3-g_4]f).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|(L_{T_\varepsilon} f)''\|_1 &\le \left\| L_{T_\varepsilon}\left(\left(\frac{1}{T_\varepsilon'}\right)^2\right)\right\|_1 + \|L_{T_\varepsilon}([2g_{1,a}+g1,b]f')\|_1 + \|L_{T_\varepsilon}([g_2-g_3-g_4]f)\|_1\\
&\le \lambda^2\|f''\|_1 + \|2g_{1,a}+g_{1,b}\|_\infty\|f'\|_1 + \|g_2-g_3-g_4\|_2\|f\|_2.
\end{aligned}$$

Notice that the injection $W^{1,1}(I) \to L^2(I)$ is compact (Theorem 2.2.4) and so, in particular, it is continuous, thus there exists a constant K such that $\|f\|_2 \le K\|f\|_{W^{1,1}}$. We can then conclude the following:

$$\begin{aligned}
\|L_{T_\varepsilon} f\|_{W^{2,1}} &= \|L_{T_\varepsilon} f\|_1 + \|(L_{T_\varepsilon} f)'\|_1 + \|(L_{T_\varepsilon} f)''\|_1\\
&\le \lambda\|f\|_{W^{1,1}} + M\|f\|_2 + \lambda^2\|f''\|_1 + C_1\|f'\|_1 + C_2\|f\|_2\\
&= \lambda\|f\|_1 + \lambda\|f'\|_1 + \lambda^2\|f''\|_1 + C_1\|f'\|_1 + C_3\|f\|_2\\
&= \lambda^2\|f\|_{W^{2,1}} + (\lambda-\lambda^2)\|f\|_1 + (\lambda-\lambda^2)\|f'\|_1 + C_1\|f'\|_1 + C_3\|f\|_2\\
&= \lambda^2\|f\|_{W^{2,1}} + M\|f\|_{W^{1,1}},
\end{aligned}$$

for some constants $C_1, C_2, C_3, M \ge 0$ independent of ε. $\square$

Proposition 5.1.3. *Considering the transfer operator* $L_{T_\varepsilon} : W^{1,1} \to W^{1,1}$*, there exists* $\lambda < 1$ *such that* $r_e(L_{T_\varepsilon}) \leq \lambda$.

Proof. Given the tools we have previously developed, we aim to use Proposition 3.2.7. Using the notation of the proposition, we take $(B, \|\cdot\|_s)$ to be $(W^{1,1}(I), \|\cdot\|_{W^{1,1}})$ and $\|\cdot\|_w$ to be $\|\cdot\|_2$.

Moreover, since the injection $W^{1,1}(I) \to L^2(I)$ is compact (Theorem 2.2.4), then any sequence satisfying $\|x_n\|_{W^{1,1}} \leq 1$ contains a Cauchy subsequence for $\|\cdot\|_2$. It is also clear that L_{T_ε} is a bounded linear operator on $(W^{1,1}, \|\cdot\|_{W^{1,1}})$ and by lemma 5.1.1 we get the desired Lasota-Yorke inequality:

$$\|L_{T_\varepsilon} f\|_{W^{1,1}} \leq \lambda \|f\|_{W^{1,1}} + M\|f\|_2.$$

Since $\lambda < 1$, we conclude that the essential spectral radius satisfies $r_e(L_{T_\varepsilon}) \leq \lambda < 1$. □

This proposition establishes that there exists only a finite number of eigenvalues (with finite algebraic multiplicity) $\lambda_i \in \sigma(L_{T_\varepsilon})$ such that $|\lambda_i| > \lambda$.

Proposition 5.1.4. *Consider* $L_{T_\varepsilon} : W^{1,1} \to W^{1,1}$*. Then* $r(L_{T_\varepsilon}) = 1$ *and there exists some* $\varepsilon_0 > 0$ *such that, for all* $0 \leq \varepsilon \leq \varepsilon_0$*,* 1 *is the only eigenvalue on the unit circle and it has algebraic multiplicity* 1.

Proof. First, notice that $\chi_I \circ T_\varepsilon = \chi_I$, so $1 \in \sigma(L_{T_\varepsilon})$. As we have just noticed, the only possible elements $z \in \sigma(L_{T_\varepsilon})$ with $|z| > 1$ are isolated eigenvalues and we know that the transfer operator cannot have eigenvalues with modulus larger than 1. This shows that $r(L_{T_\varepsilon}) = 1$.

We must now show that there are no other eigenvalues on the unit circle and that 1 has algebraic multiplicity 1. As we have previously seen, this is the case for L_{T_0}, since we assumed in (H5) that T_0 is topologically mixing: indeed, notice that the change from the space BV to $W^{1,1}$ is not a problem, since if there was a function $g \in W^{1,1}$ and some $1 \neq z \in \mathbb{C}, |z| = 1$, such that $L_T g = zg$, then certainly z would be an eigenvalue of the operator $L_T : BV \to BV$, but we know there are no such eigenvalues. Now we use Corollary 3.3.7, more precisely, remark 3.3.8. Since 1 has algebraic multiplicity 1 for L_{T_0}, then (for $0 \leq \varepsilon \leq \varepsilon_0$ and δ' small enough) $\sigma(L_{T_\varepsilon}) \cap B_{\delta'}(1)$ consists of eigenvalues $z_j^{(\varepsilon)}$ such that $z_j^{(\varepsilon)} \to 1$ as $\varepsilon \to 0$ and the total multiplicity of the $z_j^{(\varepsilon)}$ equals the multiplicity of 1. But we already know that 1 is an eigenvalue of L_{T_ε} and that the algebraic multiplicity of 1 when seen as an eigenvalue of L_{T_0} is 1. This means that 1 is the only eigenvalue on the unit circle of L_{T_ε} and that it has multiplicity 1 (when seen as an eigenvalue of L_{T_ε}). $\square$

We have all the necessary tools to show the existence and uniqueness of an invariant probability density $h_\varepsilon \in W^{1,1}$ for T_ε (for ε sufficiently small).

Theorem 5.1.5. *There exists $\varepsilon_0 > 0$ such that, for all $0 \leq \varepsilon \leq \varepsilon_0$, T_ε admits a unique invariant probability density $h_\varepsilon \in W^{1,1}$.*

Proof. Let ε_0 be the one given in Proposition 5.1.4. We wish to show that $L_{T_\varepsilon} : W^{1,1} \to W^{1,1}$ has spectral gap and from there deduce the desired result. The previous two propositions allow us to conclude that $L_{T_\varepsilon} : W^{1,1} \to W^{1,1}$ is quasi-compact, since $r_e(L_{T_\varepsilon}) \leq \lambda < 1 = r(L_{T_\varepsilon})$. Together with the fact that, for $0 \leq \varepsilon \leq \varepsilon_0$, 1 is the only eigenvalue on the unit circle and it has multiplicity 1, we obtain (by Theorem 3.2.6) that $L_{T_\varepsilon} : W^{1,1} \to W^{1,1}, 0 \leq \varepsilon \leq \varepsilon_0$, has spectral gap and so we can write $L_{T_\varepsilon} = P_\varepsilon + N_\varepsilon$.

Now fix $0 \leq \varepsilon \leq \varepsilon_0$ and consider the sequence $\left(L^n_{T_\varepsilon}(\chi_I)\right)_n$. By remark 3.2.2, this sequence converges in $W^{1,1}$ to $P_\varepsilon(\chi_I) \in W^{1,1}$. Notice that this sequence also converges in $L^1(I)$ and so, since $L^n_{T_\varepsilon}(\chi_I) \geq 0$ and $\int L^n_{T_\varepsilon}(\chi_I)\, dm = 1$, for all $n \geq 1$, we get that $P_\varepsilon(\chi_I) \geq 0$; letting $h_\varepsilon = P_\varepsilon(\chi_I)$, we have $h_\varepsilon \geq 0$, $\int h_\varepsilon\, dm = 1$ and $L_{T_\varepsilon} h_\varepsilon = h_\varepsilon$. Therefore h_ε is an invariant probability density for T_ε, with $0 \leq \varepsilon \leq \varepsilon_0$. Uniqueness follows from the fact that the eigenspace associated to 1 is one dimensional. $\square$

Corollary 5.1.6. *There exists a unique invariant probability density $h_\varepsilon \in W^{2,1}$ for T_ε, with $0 \leq \varepsilon \leq \varepsilon_0$.*

Proof. By lemma 5.1.2, we can apply lemma 3.2.7 with $(B, \|\cdot\|_s) = (W^{2,1}, \|\cdot\|_{W^{2,1}})$ and $\|\cdot\|_w = \|\cdot\|_{W^{1,1}}$. Using an entirely analogous procedure, we conclude that there exists a unique acip in $W^{2,1}$, for sufficiently small ε. Since $W^{2,1} \subseteq W^{1,1}$, then (by the previous theorem) this acip must be equal to h_ε. $\square$

5.2 Linear response formula

Now we show that the function $\varepsilon \mapsto h_\varepsilon$ is differentiable in L^1 at $\varepsilon = 0$.

Lemma 5.2.1. *For all $\varepsilon \in [0, \delta)$, we have*

$$\|L_{T_\varepsilon}\|_{L^2 \to L^2} \leq 2.$$

Proof. Recall that

$$L_{T_\varepsilon} f(x) = \begin{cases} \sum_{y \in T^{-1}(x)} \frac{f(y)}{T'_\varepsilon(y)} & x \in [0, a_\varepsilon] \\ 0 & x \in (a_\varepsilon, 1] \end{cases}$$

and so we have

$$L_{T_\varepsilon} f(x) = \left(\frac{f}{|T'_\varepsilon|}\right) \circ (T_\varepsilon|_{[0,c)})^{-1}(x) \cdot \chi_{T_\varepsilon([0,c))}(x) + \left(\frac{f}{|T'_\varepsilon|}\right) \circ (T_\varepsilon|_{(c,1]})^{-1}(x) \cdot \chi_{T_\varepsilon((c,1])}(x).$$

By the triangle inequality,

$$\|L_{T_\varepsilon} f\|_2 \leq \left\| \left(\frac{f}{|T'_\varepsilon|}\right) \circ (T_\varepsilon|_{[0,c)})^{-1} \cdot \chi_{T_\varepsilon([0,c))} \right\|_2 + \left\| \left(\frac{f}{|T'_\varepsilon|}\right) \circ (T_\varepsilon|_{(c,1]})^{-1} \cdot \chi_{T_\varepsilon((c,1])} \right\|_2 .$$

We can now bound these norms. We have

$$\begin{aligned}
&\int_I \left(\left(\frac{f}{|T'_\varepsilon|}\right) \circ (T_\varepsilon|_{[0,c)})^{-1}(x) \cdot \chi_{T_\varepsilon([0,c))}(x) \right)^2 dx \\
&= \int_{[0,a_\varepsilon]} \frac{1}{\left|T'_\varepsilon((T_\varepsilon|_{[0,c)})^{-1}(x))\right|} \frac{1}{\left|T'_\varepsilon((T_\varepsilon|_{[0,c)})^{-1}(x))\right|} \cdot \left(f\left((T_\varepsilon|_{[0,c)})^{-1}(x)\right)\right)^2 dx \\
&\leq \left\|\frac{1}{T'}\right\|_\infty \int_{[0,a_\varepsilon]} \frac{1}{\left|T'_\varepsilon((T_\varepsilon|_{[0,c)})^{-1}(x))\right|} \cdot \left(f\left((T_\varepsilon|_{[0,c)})^{-1}(x)\right)\right)^2 dx \\
&= \left\|\frac{1}{T'}\right\|_\infty \left\|L_{T_\varepsilon}(f^2 \cdot \chi_{[0,c)})\right\|_1 \\
&\leq \left\|\frac{1}{T'}\right\|_\infty \|f^2 \cdot \chi_{[0,c)}\|_1 \\
&\leq \left\|\frac{1}{T'}\right\|_\infty \|f\|_2^2.
\end{aligned}$$

Using a similar argument, one can also show that

$$\int_I \left(\left(\frac{f}{|T'_\varepsilon|}\right) \circ T^{-1}_{\varepsilon|]c,1]}(x) \cdot \chi_{T_\varepsilon(]c,1])}(x) \right)^2 dx \leq \left\|\frac{1}{T'}\right\|_\infty \|f\|_2^2.$$

Since $\left\|\frac{1}{T'}\right\|_\infty \leq 1$, we get the desired result. $\square$

Lemma 5.2.2. *Let* $W_0^{1,1} = \left\{ f \in W^{1,1} : \int_0^1 f\,dm = 0 \right\}$*. There exists a constant* $C \geq 0$ *such that, for all* $\varepsilon \in [0, \varepsilon_0]$ *(where* ε_0 *is the same as in Corollary 5.1.6), we have*

$$\left\| (I - L_{T_\varepsilon})^{-1} \right\|_{W_0^{1,1} \to W^{1,1}} \leq C$$

Proof. We wish to apply Theorem 3.3.5, with $(B, \|\cdot\|_s) = (W_0^{1,1}, \|\cdot\|_{W^{1,1}}\|)$ and $\|\cdot\|_w = \|\cdot\|_2$. Using the notation of said theorem, we will get our desired conclusion if $z = 1$. We check that conditions (a) through (d) are verified:

(a) is verified by lemma 5.2.1;

(b) is verified by the Lasota-Yorke inequality of lemma 5.1.1 and lemma 5.2.1;

(c) 1 is not in the spectrum of $L_{T_\varepsilon} : W_0^{1,1} \to W_0^{1,1}$;

(d) is verified by assumption (H9).

Therefore, we conclude that there exists a constant $C > 0$ such that, for any $0 \leq \varepsilon \leq \varepsilon_0$,

$$\left\| (I - L_{T_\varepsilon})^{-1} \right\|_{W_0^{1,1} \to L^2} \leq C.$$

□

Theorem 5.2.3. *The function* $\varepsilon \mapsto h_\varepsilon$ *is differentiable at* $\varepsilon = 0$ *in* L^1*. In particular,*

$$h_\varepsilon = h_0 + \varepsilon (I - L_{T_0})^{-1}(q) + o(\varepsilon),$$

where

$$q(x) = \begin{cases} L_{T_0}[A_0 h_0' + B_0 h_0](x) & x \in [0, a_0) \\ 0 & x \in [a_0, 1] \end{cases}$$

with

$$A_\varepsilon = -\frac{\partial_\varepsilon T_\varepsilon}{T_\varepsilon'}, \quad B_\varepsilon = A_\varepsilon' = \frac{\partial_\varepsilon T_\varepsilon \cdot T_\varepsilon''}{T_\varepsilon'^2} - \frac{\partial_\varepsilon T_\varepsilon'}{T_\varepsilon'},$$

and o *is in the* L^1 *topology.*

Proof. We first introduce some notation. Let

$$H_\varepsilon = L_{T_\varepsilon} - L_{T_0},$$
$$G_\varepsilon = (I - L_{T_\varepsilon})^{-1}.$$

Notice that G_ε is defined only on $W_0^{1,1}$. We have

$$h_\varepsilon = G_\varepsilon H_\varepsilon h_0 + h_0.$$

Since $h_0 \in W^{2,1}$, then (by (H10)) there exists $\partial_\varepsilon L_{T_\varepsilon} h_0 \in W^{1,1}$ such that

$$\left\| L_{T_\varepsilon} h_0 - L_{T_0} h_0 - \varepsilon(\partial_\varepsilon L_{T_\varepsilon} h_{0|\varepsilon=0}) \right\|_{W^{1,1}} = o(\varepsilon)$$

and so we can write

$$H_\varepsilon h_0 = \varepsilon q + o(\varepsilon),$$

for some $q \in W_0^{1,1}$ and where o is in the $W^{1,1}$ topology.

In order to obtain a formula for q, we again introduce some notation. Let $g_{0,\varepsilon} = (T_\varepsilon|_{[0,c)})^{-1}$ and $g_{1,\varepsilon} = (T_\varepsilon|_{(c,1]})^{-1}$. Then, for $j = 0, 1$, we have

$$\begin{aligned}\partial_\varepsilon(h_0 \circ g_{j,\varepsilon} g'_{j,\varepsilon}) &= \partial_\varepsilon(h_0 \circ g_{j,\varepsilon}) g'_{j,\varepsilon} + h_0 \circ g_{j,\varepsilon} \cdot \partial_\varepsilon g'_{j,\varepsilon} \\ &= h'_0 \circ g_{j,\varepsilon} \cdot \partial_\varepsilon g_{j,\varepsilon} g'_{j,\varepsilon} + h_0 \circ g_{j,\varepsilon} \partial_\varepsilon g'_{j,\varepsilon}.\end{aligned}$$

Since $T_\varepsilon \circ g_{j,\varepsilon}(x) = x$, differentiating with respect to ε gives us

$$T'_\varepsilon \circ g_{j,\varepsilon} \cdot \partial_\varepsilon g_{j,\varepsilon} + \partial_\varepsilon T_\varepsilon \circ g_{j,\varepsilon} = 0.$$

Recall that

$$A_\varepsilon = -\frac{\partial_\varepsilon T_\varepsilon}{T'_\varepsilon}$$

and so we get

$$\partial_\varepsilon g_{j,\varepsilon} = A_\varepsilon \circ g_{j,\varepsilon}.$$

Since

$$A'_\varepsilon = B_\varepsilon$$

we also obtain

$$\partial_\varepsilon g'_{j,\varepsilon} = A'_\varepsilon \circ g_{j,\varepsilon} \cdot g'_{j,\varepsilon} = B_\varepsilon \circ g_{j,\varepsilon} \cdot g'_{j,\varepsilon}.$$

This gives us the desired formula for q. Finally, recalling lemma 5.2.2, we know that $G_\varepsilon : W_0^{1,1} \to W^{1,1}$ is uniformly bounded, hence we have

$$G_\varepsilon H_\varepsilon h_0 = \varepsilon G_\varepsilon q + o(\varepsilon),$$

where once again o is in the $W^{1,1}$ topology.

Applying the second conclusion of Theorem 3.3.5, we get that, in particular,

$$\|(I - L_{T_\varepsilon})^{-1}(q) - (I - L_{T_0})^{-1}(q)\|_1 = \|G_\varepsilon(q) - G_0(q)\|_1 \to 0, \text{ as } \varepsilon \to 0.$$

Therefore

$$h_\varepsilon = h_0 + \varepsilon G_0(q) + o(\varepsilon) = h_0 + \varepsilon(I - L_{T_0})^{-1}(q) + o(\varepsilon)$$

where o is in the L^1 topology, which is exactly what we wanted to show. $\square$

Bibliography

[1] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Elsevier, 2003.

[2] Wael Bahsoun and Stefano Galatolo. Linear response due to singularities. *Nonlinearity*, 37(7):075010, 2024.

[3] Viviane Baladi. On the susceptibility function of piecewise expanding interval maps. *Communications in mathematical physics*, 275(3):839–859, 2007.

[4] Jean-Baptiste Bardet, Sébastien Gouëzel, and Gerhard Keller. Limit theorems for coupled interval maps. *Stochastics and Dynamics*, 7(01):17–36, 2007.

[5] Abraham Boyarsky and Paweł Góra. Laws of chaos invariant measures and dynamical systems in one dimension.

[6] Haim Brezis and Haim Brézis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.

[7] Hubert Hennion. Sur un théorème spectral et son application aux noyaux lipchitziens. *Proceedings of the American Mathematical Society*, 118(2):627–634, 1993.

[8] H.G. Heuser. *Functional Analysis*. Wiley, 1981.

[9] Peter D Hislop and Israel Michael Sigal. *Introduction to spectral theory: With applications to Schrödinger operators*, volume 113. Springer Science & Business Media, 2012.

[10] Gerhard Keller and Carlangelo Liverani. Stability of the spectrum for transfer operators. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 28(1):141–152, 1999.

[11] Roger D Nussbaum. The radius of the essential spectrum. 1970.

[12] Allan Pinkus. *N-widths in Approximation Theory*, volume 7. Springer Science & Business Media, 2012.

[13] Frigyes Riesz and Béla Sz Nagy. *Functional analysis.* Courier Corporation, 2012.

[14] Omri Sarig. Introduction to the transfer operator method. *Second Brazilian School on Dynamical Systems, Lecture Notes*, 2012.

[15] Marcelo Viana. *Stochastic dynamics of deterministic systems*, volume 21. IMPA Rio de Janeiro, 1997.