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Dynamics of vector fields with univalued solutions

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Faculty of Sciences of the University of Porto and Faculty of Sciences and Technology of the University of Coimbra
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Abstract

Whether or not a vector field X has univalued solutions, in other words, is semicomplete, depends to a large extent on its behavior near its pole divisor $(X)_\infty$. In turn, $(X)_\infty$ is invariant under the underlying foliation $\mathcal{F}$ of X, and the leaves of $\mathcal{F}$ contained in $(X)_\infty$ are naturally equipped with an affine structure induced by X, as proved in [23].

The pair constituted by the foliation restricted to $(X)_\infty$ along with its foliated affine structure, encodes much information on the behavior of X in a neighborhood of $(X)_\infty$ and, in this way, on the problem of determining whether a given X is semicomplete. Assuming that X is tangent to a codimension-1 foliation $\mathcal{D}$ leaving $(X)_\infty$ invariant, we are able to provide an interpretation of the monodromy homomorphism of the affine structure on the leaves of $\mathcal{F}$ by means of the holonomy of the leaf of $\mathcal{D}$ contained in $(X)_\infty$.

This situation occurs in particular when X has a suitable first integral. When the linear part of the holonomy group in question is unimodular, which always happens if a first integral exists, we are able to build on this affine structure to conclude that X must be semicomplete, provided that it is semicomplete in a neighborhood of a certain "singular set" contained in $(X)_\infty$. It is to be noted that the "singular set" is a proper Zariski-closed subset of $(X)_\infty$ and, hence, a Zariski-closed subset of the ambient manifold having codimension at least 2. This allows one to "localize" the problem of deciding the global semicomplete character of X in a relatively small region.

In a similar order of ideas, we also prove that the foliated affine structure on the restriction of $\mathcal{F}$ to $(X)_\infty$ is uniformizable. Finally, we also address the problem of determining how wild the dynamics of such a semicomplete vector field can be. In this direction, we establish a criterion for the leaves of the foliation to have a parabolic nature, and consequently, the corresponding dynamics exhibit some "tame" characteristics.

Key words: Holomorphic foliations, dynamics, meromorphic semicomplete vector fields, affine structures, holonomy.

Resumo

Se um campo vetorial X tem ou não soluções univaluadas, ou seja, se é semicompleto, depende em grande medida do seu comportamento na vizinhança do seu divisor de polos $(X)_\infty$. Por sua vez, $(X)_\infty$ é invariante sob a folheação subjacente $\mathcal{F}$ de X, e as folhas de $\mathcal{F}$ contidas em $(X)_\infty$ estão naturalmente equipadas com uma estrutura afim induzida por X, conforme demonstrado em [23].

O par constituído pela folheação restrita a $(X)_\infty$ junto com a sua estrutura afim folheada codifica muita informação sobre o comportamento de X numa vizinhança de $(X)_\infty$ e, portanto, sobre o problema de determinar se um dado X é semicompleto. Supondo que X é tangente a uma folheação de codimensão-1, $\mathcal{D}$, que deixa $(X)_\infty$ invariante, apresentamos uma interpretação do homomorfismo de monodromia da estrutura afim nas folhas de $\mathcal{F}$ por meio da holonomia da folha de $\mathcal{D}$ contida em $(X)_\infty$.

Esta situação ocorre, em particular, quando X tem uma integral primeira adequada. Quando a parte linear do grupo de holonomia em questão é unimodular –o que sempre acontece se existir uma integral primeira– conseguimos basear-nos nesta estrutura afim para concluir que X deve ser semicompleto, desde que seja semicompleto numa vizinhança de um certo “conjunto singular” contido em $(X)_\infty$. Nota-se que o “conjunto singular” é um subconjunto de Zariski fechado próprio de $(X)_\infty$ e, consequentemente, um subconjunto de Zariski fechado da variedade ambiente com codimensão pelo menos 2. Isto permite “localizar” o problema de decidir o caráter semicompleto global de X numa região relativamente pequena.

Numa linha de ideias semelhante, também provamos que a estrutura afim folheada na restrição de $\mathcal{F}$ a $(X)_\infty$ é uniformizável. Finalmente, também abordamos o problema de determinar quão complexa pode ser a dinâmica de um campo vetorial semicompleto desse tipo. Nessa direção, estabelecemos um critério para assegurar que as folhas da foliação apresentam uma natureza parabólica e, como consequência, indica que a sua dinâmica exibe algumas características “domáveis”.

Palavras-chave: Folheações holomorfas, sistemas dinâmicos, campos vetoriais meromorfos semicompactos, estruturas afins, holonomia.

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Chapter 1

Introduction

This work deals with 1-dimensional holomorphic (singular) foliations induced by a particular class of vector fields on n-dimensional complex manifolds, namely, the class of vector fields presenting only *univalued solutions*. This kind of vector fields frequently appears in Mathematical Physics, and are closely related to Painlevé Property. This work focuses on two main topics:

(T1) Identifying vector fields having only univalued solutions.

(T2) Describing the global dynamics of the foliation induced by such class of vector fields.

In this introduction we present the motivation as well as the main results of the present work. Naturally, in order to make the main results intelligible, some background along with some intuitive explanations have to be presented as well. Precise definitions will be made explicit in the following chapters.

For convenience, we will begin by introducing topic (T2) and then proceed to the topic (T1). In the sequel, M will stand for a compact complex manifold and most of the time, the dimension of M will be assumed to be at least 3.

1.1 Nature of leaves and dynamics of a foliation

Global dynamics of a foliation can be understood from several perspectives. One approach involves studying the *nature of leaves*, a notion that, in turn, is related with topological, geometrical and dynamical features, as we will now proceed to explain.

1.1.1 Uniformization and nature of leaves

The problem of uniformization of Riemann surfaces goes back to Poincaré and it is still a subject of current research. In particular, a standard way to define the *nature of a Riemann surface* S, is by means of its universal covering $\tilde{S}$. In fact, the Uniformization Theorem ([39], [26]) provides a classification of Riemann surfaces into three categories:

- $\tilde{S} = \mathbb{C}$ and then S is said to be a *parabolic* Riemann surface

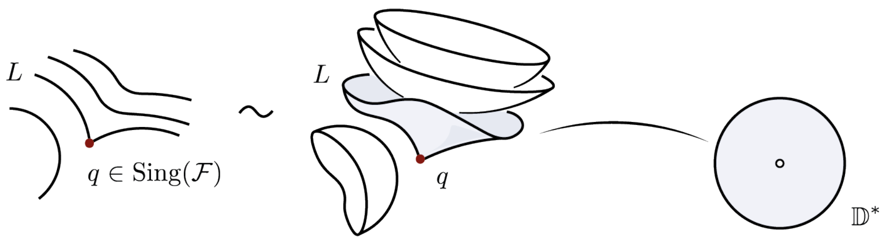
- $\tilde{S} = \mathbb{D}$ and then S is said to be a *hyperbolic* Riemann surface

- $\tilde{S} = \mathbb{CP}^1$ and then S is said to be a *rational* Riemann surface.

Leaves of a 1-dimensional holomorphic foliation inherit a Riemann surface structure and, hence, the notion of nature of leaves (parabolic, hyperbolic, rational) is well defined. Which class of leaves are more likely to appear is a natural question as well as how they co-exist in a foliation. With respect to the first question, typical (or generic) foliations present non-degenerate singularities (see [31]). A. Lins-Neto shows in [27], [28] that foliations in $\mathbb{CP}^n$ with $n \geq 2$ possessing non-degenerate singularities have only hyperbolic leaves, in other words parabolic foliations (i.e. foliations having only parabolic leaves) are “rare”. As for the second question, a few results in that direction show that these mixed cases are quite limited; the well-known Reeb Stability Theorem, ensures that if one leaf L of a foliation is of rational type, then all leaves in a saturated neighborhood of L are rational. On the other hand, regarding non-rational leaves, M. Brunella proved that parabolic and hyperbolic leaves do not coexist much either (cf. [3]). More precisely he showed that when M is a projective manifold, then either all leaves of $\mathcal{F}$ are parabolic, or the set of parabolic leaves is contained in a complete pluripolar set of M. Without getting into details, a pluripolar set is a “small set”, in the sense that it is contained in the set of poles of a certain plurisubharmonic function (see, for example, [10]). In other words, either $\mathcal{F}$ has only parabolic leaves, or has very few.

This classification should actually incorporate the notion of *entire leaf*, i.e., a leaf whose closure coincides with the image of an entire curve $f : \mathbb{C} \to M$. In fact, some subsets of hyperbolic leaves exhibit geometric and dynamical properties quite similar to those of parabolic leaves (as we will explain in the sequel). This resemblance has a concrete justification concerning a subtle aspect on the “choice” of the universal covering of separatrices, i.e., invariant closed analytic curves passing through the singular set $\mathrm{Sing}(\mathcal{F})$.

Indeed, by definition of leaf, the latter does not contain singular points of a foliation. However, we may have leaves “passing through” singularities of the foliation that can be “completed” by adding such point. Locally, these leaves are isomorphic to the punctured disc $\mathbb{D}^*$, but not its “completion”.



By considering the singular point as a point of the leaf, under the above assumptions, it may happen that, while L is hyperbolic, we have that $\widetilde{L'}$ (i.e. the universal covering of $\tilde{L}'$, where L' stands for the completion of L) is parabolic. A leaf can be said “truly” hyperbolic if both L and L' are hyperbolic. In his work, Brunella redefined the notion of leaves and his results cited above should be read with this definition in mind.

Parabolic, or more precisely entire leaves, and the “true” hyperbolic leaves have distinct properties that justify to consider them as separate classes of objects. In fact, their nature can be related with a broad spectrum of fields (such as geometry, dynamical systems, geometric groups theory and ergodic theory, among others). What follows is a brief explanation of the rich interplay between these areas, aiming to motivate the interest of identifying the nature of leaves and how the latter helps understanding the behavior of the foliation.

1.1.2 Geometry and dynamics of foliations

Below, we remind some dynamical notions that are related to the dynamics of group actions and, up to adapting them to pseudogroups, can be considered in the context of foliation (see [6], [15], [24], [46], [47]).

A dynamical point of view: Combinatorial growth

The notion of holonomy bridges the gap between a rather “static” perspective of foliations and a more dynamical point of view. Essentially, it consists in the following. Fix a leaf L and two transverse sections Σ_1, Σ_2 of L at $p_1, p_2 \in L$. Fix also a path γ on L, joining p_1 to p_2, that can be lifted to the leaves passing through the points in Σ_1. Then, the holonomy with respect to γ maps each point in Σ_1 to a point in Σ_2 by following the lifted path. This only depends on the homotopy class of γ. The behavior of the foliation along that leaf is captured by the holonomy map.

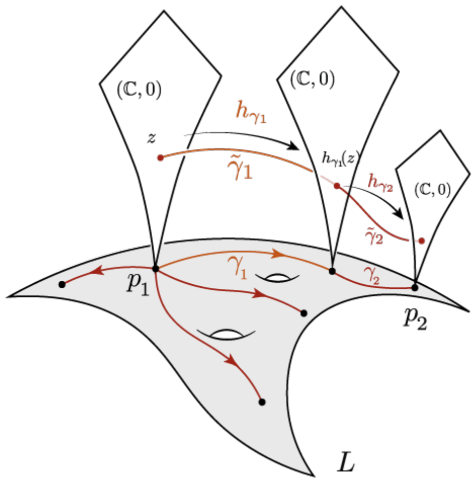


Figure 1.1 Holonomy of a leaf for $n = 2$. The path $\tilde{\gamma}_i$ is the lift of γ_i and induces local diffeomorphism h_{γ_i} on the transverse space.

By letting Σ_1 and Σ_2 to coincide (and so do p_1 and p_2) the holonomy representation is defined as a map

$$\rho : \pi_1(L) \to \mathrm{Diff}(\mathbb{C}^{n-1}, 0)$$

and its image

$$\mathrm{Hol}(L) = \rho(\pi_1(L)) \subseteq \mathrm{Diff}(\mathbb{C}^{n-1}, 0)$$

defines a pseudogroup. Note that for foliations admitting a global transverse section (e.g. Riccati foliations), then $\mathrm{Hol}(L)$ is a group. However, this situation is uncommon and one must, in general, settle for working with pseudogroups.

The complexity of the dynamics of a foliation can then be measured by means of the complexity of its holonomy pseudogroup. For simplification, we will explain this in the context of groups. Given a group H having finite generating set S, the growth function $G_H(n)$ counts the number of distinct group elements that can be represented by words of length at most $n \in \mathbb{N}$ in S. The growth function can be classified as polynomial, exponential, or of intermediate type, depending on the asymptotic behavior of $G_H(n)$. As a matter of example, the free group F_2 generated by two elements, has exponential growth. In contrast, an abelian group grows at polynomial rate. In a wider context, a group that does not contain a free group of two generators is called amenable. This is an important concept in geometric group theory, as it provides a framework for studying dynamical properties of groups. Not going into details, examples of amenable groups include solvable groups and groups of subexponential growth, which exhibit a certain form of "assymptotic tameness". The key idea here is that, even though the dynamics generated by an abelian group (or more generally by an amenable group) can be complicated, their complexity cannot reach to the level as one arising from a free group action.

Recall that all notions that one has for groups can be adapted for pseudogroups and, consequetly, to foliated spaces through the corresponding holonomy pseudogroup, see for instance [6]. However, dealing with pseudogroups is not an easy task. Fortunately, there is another notion of leaf growth that establishes a connection between the foliation and its geometric properties and that is equivalent to this one.

A geometric point of view: Geometric growth

A standard way to identify the nature of leaves is by examining the volume growth of "balls" centered at a fixed point. To be more precise, if M is equipped with an auxiliary Riemannian metric g, its restriction to L induces a metric g_L on the immersed leaf itself. The volume growth function $G_z : \mathbb{N} \to \mathbb{R}^+$ is defined by

$$G_z(r) := \mathrm{Vol}(B_r(z))\,,$$

where $B_r(z)$ denotes the ball centered at $z \in L$ and with radius $r \in \mathbb{N}$. The volume may be computed using the volume form ω induced by g on L, i.e.

$$\mathrm{Vol}(B_r(z)) = \int_{B_r(z)} \omega.$$

The asymptotic behaviour of G_z when r tends to infinity, allows to define the growth type $gr(L)$. When M is compact, the growth type is well defined in the sense that it does not depend neither on the metric g, nor on the point $z \in L$. For example, the Poincaré disc (with its hyperbolic metric) has exponential volume growth type, and the flat space $\mathbb{C}$ has polynomial volume growth type. Note that this can be shown directly by computing the volume formula.

In turn, when L is a parabolic or entire leaf, $gr_V(L)$ is of polynomial type. However, if L is a "true hyperbolic leaf", then $gr_V(L)$ is of exponential type.

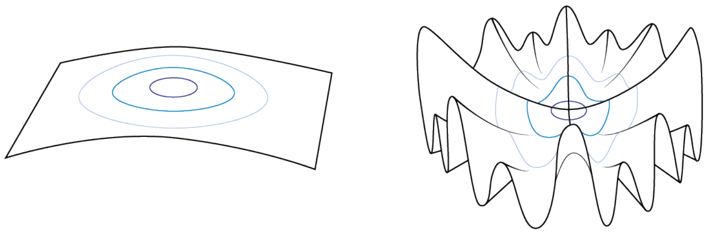


Figure 1.2 Volume growth of a parabolic leaf (left) and a "true" hyperbolic leaf (right)

One of the advantages of this definition of type of volume growth is that it is much easier to handle than the one provided by the holonomy pseudogroup. It also allows to define geometric conditions on the foliation, such as the property of "closedness at ∞" (c.f. [46]), which in turn gives information on the asymptotic behaviour of leaves.

Now, we can wonder to say something about the growth type of the holonomy pseudogroup for parabolic and hyperbolic leaves. It turns out that if M is a compact manifold and L is a leaf of a holomorphic foliation $\mathcal{F}$ on M, then the two notions of growth type coincide, i.e.

$$\mathrm{gr}_V(L) = \mathrm{gr}(\mathrm{Hol(L)}).$$

For a proof of this equality see for example [38] or [6] .

It then follows that if L is a "true" hyperbolic leaf, the dynamics of its holonomy pseudogroup will be rich. In contrast, if L is a parabolic leaf, its dynamics tend to be milder. Despite of this, we emphasize that this tameness or complexity is relative. From the perspective of dynamical systems, there are still examples of parabolic foliations that present interesting properties and are worth studying.

As a matter of fact, having parabolic leaves –or more generally leaves with sub-exponential growth type– ensures the existence of an holonomy invariant transverse measure [38]. The existence of such a measure can be thought as the foundation for developing "standard" ergodic theory; it allows to define notions such as ergodicity, entropy, mixing (among others), which describe the global, or coarse behaviour of the orbits. This measure no longer exists in the hyperbolic case. Nevertheless, all is not lost: in [12], it was shown that one can define a harmonic measure, which takes the role of the invariant measure and enables the development of ergodic theory in hyperbolic foliations. Both types of measures find their analogues (or duals) in complex geometry with the notion of positive (resp. harmonic) closed currents directed by $\mathcal{F}$, see [46], [1] among others. These topics go beyond what is covered in this thesis, however, the continuation of this work addresses them and is part of an ongoing project. For an overview of the subject, see for instance the survey [35].

As a summary of all this discussion, identifying if a foliation has parabolic or true hyperbolic leaves, determines the growth type of the leaves (polynomial or exponential).

Then by exploiting all the interconnections discussed earlier, one can establish a wide amount of properties satisfied by $\mathcal{F}$.

1.2 On semicomplete vector fields

Recall that, given a vector field X, the existence and uniqueness theorem ensures the existence of *local* solutions for the differential equation associated with X, for every initial condition that we fix. In other words, for every $p \in M$ there is an open neighborhood $U \subseteq \mathbb{C}$ of the origin along with a holomorphic map $\Phi_p : U \to M$ such that $\Phi(0) = p$ and $\Phi_p'(t) = X(\Phi(t))$ for every $y \in U$. While in the real case solutions can always be extended in order to obtain a maximal domain of definition on $\mathbb{R}$ (which is an open interval on $\mathbb{R}$), in the complex case, however, it is not always possible to obtain a maximal domain of definition contained in $\mathbb{C}$ due to the monodromy effect. More precisely, a maximal domain of definition can always be constructed by analytic continuation. This maximal domain of definition is a Riemann surface that need not be in $\mathbb{C}$.

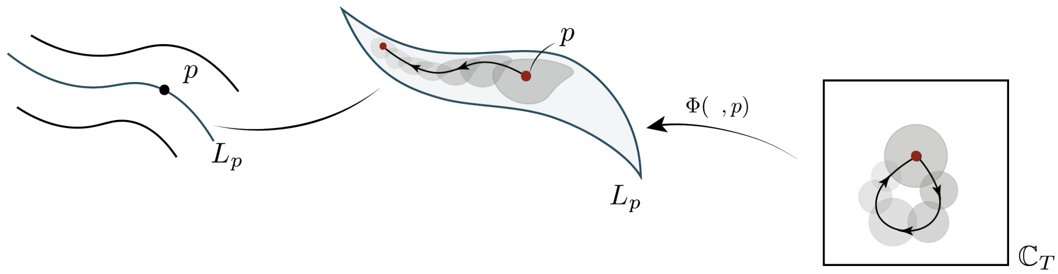


Figure 1.3 Multivaluation of the flow

The vector fields satisfying the property that every solution of the differential equation associated to it has a maximal domain of definition in $\mathbb{C}$ are called *semicomplete*. They correspond to the vector fields admitting only univalued solutions. The notion of semicomplete vector fields was introduced by J. Rebelo in [40] and has been largely studied in the recent years.

The notion of being semicomplete extends naturally to the case of meromorphic vector fields (see for instance [23]) and they will be part of the object of study of this thesis. To be precise, a meromorphic vector field X is said to be semicomplete of X is semicomplete away from its pole divisor.

Suppose we are given a semicomplete vector field on an open set U. It is well-known that the restriction of X to every open subset V of U is semicomplete as well. In this sense, the notion of semicomplete vector fields can be taken at the level of points. In other words, we will say that X is semicomplete at p if X is semicomplete on some open neighborhood of p. Since complete vector fields (i.e. those whose solutions have $\mathbb{C}$ as domain of definition) are, in particular, semicomplete, the latter can be viewed as the local version of the first ones. Furthermore, if a vector field X is not semicomplete at a singular point p, then it cannot be realized as a complete vector field on a complex manifold.

We should highlight the fact that the classes of vector field that are semicomplete and that satisfy the Painlevé property (cf. [36], [9]) have non-negligible overlap, although the two classes do not coincide. They often exhibit some sort of complete integrability, which in turn can be understood as having relatively tame dynamics. Furthermore, both appear in many examples concerning Mathematical Physics, for instance in Chazy or Garnier's works, [7], [21], [13].

Before proceeding, let us just recall that the notion of semicompletness does not apply to foliations. In fact, a (germ of) foliation at a singularity may have more than one representative vector field and not all of them need to be semicomplete.

Extensive work on semicomplete vector fields has been done in recent years (at local and/or global nature). In [40], J. Rebelo proved that germs of semicomplete vector fields on 1-dimensional and 2-dimensional manifolds, cannot be too "degenerate". To be more precise, he has shown that vector fields, on these low dimensional manifold, that are semicomplete at an isolated singular point p necessarily have a non-vanishing second jet at p, i.e., $\mathscr{J}_p^2 X \not\equiv 0$. He also proved that semicomplete vector fields on $(\mathbb{C}, 0)$ are given, up to analytic conjugation and a multiplicative constant, by $\partial/\partial z$, $z\,\partial/\partial z$ and $z^2\,\partial/\partial z$. In turn, no strictly meromorphic vector fields on $(\mathbb{C}, 0)$ is semicomplete.

As for vector fields on $(\mathbb{C}^2, 0)$, a complete characterizartion of holomorphic semicomplete vector fields was obtained by Ghys and Rebelo ([14]), while a local and global description of meromorphic semicomplete vector fields was provided by A. Guillot and J. Rebelo in the paper [23]. Their analysis is based on studying the behavior of the vector field "at infinity", i.e., in a neighborhood of the divisor of poles, which will be a central idea in this thesis.

In dimension 3 (and higher), however, the study of 1-dimensional holomorphic foliations becomes significantly more complicated; in terms of the ambient manifold ("geometry"), it is well-known that going from dimension 2 to dimension 3 always entails a significantly increased difficulty (compare Enriques-Kodaira classification with advances in "Mori theory"). Similarly, in terms of the vector fields/foliations themselves, they are basically determined by the action of the holonomy pseudogroup on the transverse space ("dynamics"). So, compared to the 2-dimensional setting, we move from (complex) 1-dimensional dynamical systems to 2-dimensional dynamics systems where the theory is still in a much less developed stage. Other difficulties can also be enumerated.

(1) One of the main challenges comes from the fact that these singularities in dimension at least 3 encode some global dynamics on the divisors naturally associated to them. As a matter of example, consider the one-point blow-up of a "generic" homogeneous vector field on $\mathbb{C}^3$. Understanding the initial singularity requires understanding the global foliation induced on the projective space $\mathbb{CP}^2$ identified with the exceptional divisor by the homogeneous vector field in question. In general, this foliation possesses a very complicated dynamics leaving no algebraic "object" invariant (cf. [32]). In dimension 2, however, the exceptional divisor consists of the union of a unique leaf with finitely many singular points and, consequently, the dynamics obtained on the divisor is rather trivial.

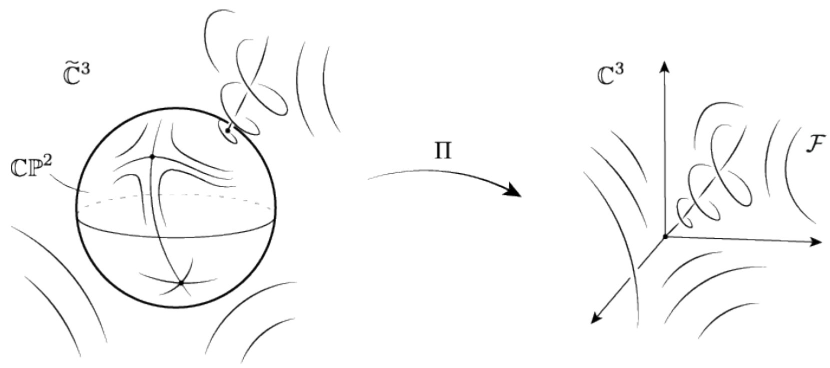


Figure 1.4 Blow up in $(\mathbb{C}^3, 0)$ and the induced foliation on the exceptional divisor

(2) Also, a desingularization procedure as effective as Siedenberg's Theorem fails to exist in dimension at least 3, as follows from a well-known example of Sancho-Sanz (cf. introduction in [37]). Nevertheless, we note that in dimension 3, there are some useful theorems on the resolution of foliations (as can be seen in [34], [42]) and of vector fields ([42]).

(3) Furthermore, Camacho-Sad Theorem ([5]) asserts the existence of separatrices (i.e. invariant analytic curves) though a singularity for foliations on (smooth) complex surfaces. A well-known example by Gómez-Mont and Luengo [17]) shows that the existence of separatrices no longer holds for 1-dimensional foliations, even in 3-manifolds.

Seidenberg Theorem as well as Camacho-Sad Theorem played an important role on the classification of semicomplete vector fields in dimension 2. Albeit those difficulties, advances on the study of semicomplete vector fields on 3-manifolds have been made by Guillot, Rebelo and Reis both at local and global setting.

At local setting, although the question of whether the second jet of a semicomplete vector field at an isolated singularity cannot be identically zero has not been settled, it has been established under certain conditions (cf. [44]) which, at least, fits the goal of trying to give bounds for the automorphism group of a complex manifold.

In this work, we are interested in global aspects of vector fields. At global level, we may highlight Guillot's work on the dynamics of quadratic semicomplete vector fields on $\mathbb{C}^3$, with emphasis to his work on Halphen vector fields (see [20]). On the latter, Guillot exhibited the first examples of semicomplete vector fields presenting a wild dynamics (dynamics comparable with the triangular groups). Furthermore, he proved that these vector fields can be realized as a complete vector fields on some (non-Kähler) complex manifold. These examples show that a lot is yet to be understood in dimension 3 and hence open to further exploration. For an overview of some results, we refer to [45]. More specifically, regarding local analysis, [22] provides a comprehensive review of 2−dimensional semicomplete singularities, while [43] focuses on singularities of 3-dimensional holomorphic vector fields.

Let us now discuss our main goal concerning item (T1), mentioned at the beginning of this introduction. As already mentioned, if a vector field X is semicomplete on a complex manifold M, then M must be semicomplete at every singular point of X. We wonder if a kind of converse is valid, namely:

Question: If X is a vector field on a compact complex manifold M that is semicomplete in an open subset of M (e.g. on the union of neighborhoods of the singular set), is it true that X is semicomplete on the entire M?

So far, there is no proof or counterexample for this question. In this work we establish conditions that yield a positive answer. For better understanding, let us restrict ourselves to polynomial vector fields on $\mathbb{C}^n$. As it is well known, polynomial vector fields on $\mathbb{C}^n$ admit a meromorphic extension to $\mathbb{CP}^n \simeq \mathbb{C}^n \cup \Delta_\infty$, where Δ_∞ stands for the hyperplane at infinity. The extension of X to $\mathbb{CP}^n$ induces foliation $\mathcal{F}$ on there which may or may not leave the hyperplane at infinity invariant. We prove the following.

Theorem II.B. *Let X be a rational vector field on $\mathbb{CP}^n$, with $n \geq 3$, and assume that Δ_∞ is invariant by $\mathcal{F}$. In addition, assume that $f : \mathbb{CP}^n \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *f is constant at generic points of Δ_∞.*
- *X is semicomplete on a neighborhood of $I(f) \cap \Delta_\infty$.*

Then X is semicomplete on $\mathbb{CP}^n$.

Recall that given a meromorphic function $f : \mathbb{CP}^n \to \mathbb{C}$, $I(f)$ stands for the indeterminacy set of f. The existence of such a meromorphic first integral is not a necessary condition. In fact, we show that it arises as a particular case of a more general framework involving the existence of a suitable codimension-one foliation (see Theorem 1.3).

1.2.1 Nature of leaves for semicomplete vector fields

When dealing with semicomplete vector fields, the nature of leaves can be defined in terms of the nature of the maximal domain of definition of the flow. In fact, if X is semicomplete, then for each $p \in M$, the flow $\Phi_p : \Omega_p \to M$ can be thought as a pair (Φ_p, Ω_p) where $\Omega_p \subseteq \mathbb{C}$ denotes its maximal domain of definition for the solution through p. One can then wonder what the the “shape” of the subsets Ω_p is.

Let L_p stand for the leaf through p. It can be proved that $\Phi_p : \Omega_p \to L_p$ is actually a proper map –hence a covering– of L_p (cf. [40]). Furthermore, Ω_p is isomorphic to the so-called *monodromy covering* of L_p, which will be denoted by $\hat{L}_p$ (cf. definition in Chapter 3). Note, in addition, that $\Omega_p \simeq \hat{L}_p$ is itself covered by the universal covering $\tilde{L}_p$ of L_p. As a matter of fact, the description of the nature of leaves in terms of the monodromy covering provides some sort of refinement of the definition of nature of leaves, and is particularly well-suited for the study of semicomplete vector fields. The following table summarizes the relation between the universal/monodromy coverings and entire/non-entire leaves.

	$\hat{L}_p \simeq \Omega_p$	$\tilde{L}_p$
"True" hyperbolic	$\mathbb{D}$	Hyperbolic leaves $\mathbb{D}$
	$\mathbb{C} \setminus \{\text{non discrete set}\}$	
Entire leaves	$\mathbb{C} \setminus \underset{\# \geq 2}{\{\text{discrete set}\}}$	
	$\mathbb{C} \setminus \{1 \text{ point}\}$	Parabolic leaves $\mathbb{C}$
	$\mathbb{C}$	

Taking advantage of this terminology, note that in particular if X is a complete vector field, then it induces a parabolic foliation.

In the paper [23], it was shown that all meromorphic semicomplete vector fields in complex manifolds of dimension 2 possess entire leaves; in other words, their dynamic is not particularly wild. On the other hand, the Halphen vector field is an example of a semicomplete vector field in dimension 3 with true hyperbolic (non-entire) leaves, and hence exhibiting much more complex dynamics.

Theorem III.B below, concerns the issue (T2) of identifying the nature of the maximal domain of definition. More concretely, we provided criteria to determine when a foliation induced by a meromorphic semicomplete vector field on $\mathbb{CP}^n$, $n \geq 3$, has entire leaves, thereby, not so complicated dynamics. Specifically, by recovering the setting of Theorem II.B, we have

Theorem III.B. *Let X be a rational vector field on $\mathbb{CP}^n$, $n \geq 3$. Assume that X is semicomplete and $f : \mathbb{CP}^n \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *f is constant at generic points of Δ_∞.*
- *Ends of leaves of local branches are contained on a discrete set of points.*

Then, all the leaves of $\mathcal{F}$ are entire leaves.

1.3 Statements of the main theorems and outline of the thesis

I will now present the outline of the thesis and provide an overview of the main ideas behind the problem, hoping this will serve as a guideline for the reader.

Chapter 2

Chapter 2 is devoted to offering the very basics in foliation theory that will be needed for the thesis. Notions such as one dimensional holomorphic foliation and codimension-1 foliation will be recalled along with a few related concepts.

In a second part, we define the notions of translation and affine structure on a Riemann surface and we prove that the flow of a holomorphic vector field X induces an affine structure on every leaf of the foliation. This result was previously established in [23], or [11]. We present an alternative proof of it that will be better suited for the work.

Chapter 3

This chapter focuses on studying the behaviour of leaves, and their relation with the affine structure previously defined, when restricted to a regular part of $\mathcal{F}$ near an irreducible component E of the divisor of poles. The goal is to establish conditions that allow to "control" the dynamics induced by the affine constants on such a regular part.

We begin by introducing the notion of developing map of an affine structure. In the previous section it was shown that, when comparing two consecutive charts, the local vector fields restricted to consecutive plaques of a leaf L differs from one chart to the other by a multiplicative constant. The developing map relative to a leaf L, provides a tool to "extend" a chart, and it allows to transition from local to global data by keeping these constants as a cumulative product. In other words, the developing map of leaves contained in the divisor of zeros or poles will somehow encode all the dynamics of its affine structure. However, due to possible monodromy phenomena, one needs to define the developing map on an appropriate cover of L; this is when the monodromy cover of L comes into play. Roughly speaking, the monodromy cover is the smallest cover in which the developing map is well-defined, although its key role will become clear in Chapter 4.

We will introduce all these notions and then adapt them for *foliated affine structures* (cf. 2) in order to "compare" developing maps of nearby leaves. However, this first requires clarifying how the comparison should be made. In fact, a priori, the monodromy covers of individual leaves cannot necessarily be assembled in a coherent way within a single unified space. And the existence of such a space where all the monodromy covers coexist and possess some sort of regularity, is a non-trivial problem (see, for instance, Ilyashenko and Brunella's works, [25], [2] concerning simultaneous uniformization). For our purposes, it will suffice to consider a notion of uniform convergence following "leafwise paths" for sufficiently close leaves. This leads to a certain form of transverse regularity for "leafwise" developing maps when restricted to simple connected regions of the leaves.

In a second section of this chapter, we fix a *finite* foliated cover in a neighborhood of a compact part "away‘" from the singular set intersecting E, and we investigate precisely how these affine structures affect the size and shape of the image of the developing map when restricted to such a regular compact part of the foliation.

Once this has be done, we introduce a codimension-1 foliation $\mathcal{D}$ tangent to the leaves of $\mathcal{F}$ and such that E (or to be precise $E_0 = E \setminus \mathrm{Sing}(\mathcal{D}) \cap E$) is a leaf for $\mathcal{D}$. This codimension-1 foliation provides a way to interpret the affine constants of the developing maps as the derivatives of the holonomy representation of $\mathcal{D}$. This connection allows us to relate the growth of the images of the developing maps –which are, by definition, subsets of $\mathbb{C}$– to the holonomy of $\mathcal{D}$.

More precisely, the hypothesis $\mathrm{Hol}'(E_0) \subseteq S^1$ on the holonomy imposes constraints on how much the local images of the developing maps may shrink. We study separately this extent of contraction of the image of the developing map for two situations: for leaves contained in the divisor of zeros or poles (Proposition 3.2.8), and for leaves that are not contained in the divisor of zeros an poles (Proposition 3.2.10). The proof of the later relies on the path leafwise uniform convergence of the developing maps demonstrated before. For the former case –namely, leaves contained in the divisor of zeros or poles– we show that by asking the "taming" hypothesis on the holonomy, the images of balls on the leaves having a fixed radius (read on charts) under the action of the developing map, are uniformly bounded by below.

Chapter 4

At the beginning of Chapter 4, we will offer a brief review of the formal definition of semicomplete vector fields, along with some essential properties that will be useful in the sequel. The remainder of the chapter is devoted to proving the main theorems of the thesis. We will now provide a brief review of them.

For what follows, X is a meromorphic vector field on a compact complex manifold M of dimension $n \geq 3$. The divisor of poles of X is denoted by $(X)_\infty$, and for sake of simplicity, one can assume that it consists of a single smooth, connected component, E, which is invariant by $\mathcal{F}$. It is possible to adapt the statements to the general case. The foliation induced by X will be denoted by $\mathcal{F}$. To ground the previous setting, one can think on the particular case of a polynomial vector field on $\mathbb{C}^n$, regarded as a meromorphic (rational) vector field on $\mathbb{CP}^n$. Then its pole divisor coincides with the hyperplane at infinity Δ_∞.

For Theorems I and II, we need to assume the existence of a codimension one foliation $\mathcal{D}$, satisfying the following:

(D1) At regular points, $\mathcal{F}$ is tangent to the leaves of $\mathcal{D}$

(D2) E coincides with the closure of a leaf of $\mathcal{D}$

Also, $\mathrm{Sing}(\mathcal{F})$ and $\mathrm{Sing}(\mathcal{D})$ stand for the singularity set of $\mathcal{F}$ and $\mathcal{D}$ respectively.

Under the above considerations, let $E_0 \subseteq E$ be the leaf referred on item (D2). Associated with E_0, one can define the holonomy representation $\rho_{\mathcal{D}} : \Pi(E_0) \to \mathrm{Diff}(\mathbb{C}, 0)$ which arises from the codimension one foliation $\mathcal{D}$ (not to be confused with the holonomy of the leaves of the foliation $\mathcal{F}$). By differentiating elements in $\mathrm{Diff}(\mathbb{C}, 0)$ at the origin, we obtain the *derivative representation* $\rho'_{\mathcal{D}} : \pi_1(E_0) \to \mathbb{C}^*$. We denote the image of the derivative representation as $\mathrm{Hol}'(E_0) = \rho'_{\mathcal{D}}(\pi_1(E_0))$.

Regarding (T1): Proving (global) semicompletness Theorems I and II, together with the simplified versions Theorem II.A and II.B, concern the first line of work (T1) presented at the beginning of the introduction. The key idea that serves as the foundation of both theorems is the following: If $\mathsf{Dev}_L : \hat{L} \to \mathbb{C}$ is injective, thus invertible, its inverse allows to define a global (univalued) flow on L. Indeed the image of the developing map denoted by $\Omega_L = \mathsf{Dev}_L(\hat{L})$, will take the role of the maximal domain of definition of $\Phi|_L$.

$$\begin{tikzcd}
\hat{L} \arrow[d, "\rho"'] \arrow[dr, shift left, "\mathrm{Dev}_L"] & \\
L & \Omega_L \subseteq \mathbb{C} \arrow[l, dashed, "\Phi_L"] \arrow[ul, dashed, shift left]
\end{tikzcd}$$

Figure 1.5 If Dev_L is invertible, $\rho \circ \mathsf{Dev}_L^{-1}$ defines the flow

This motivates the notion of *uniformizable affine structure.* Although its initial definition, introduced by A. Guillot on [19], was formulated differently, it is equivalent to the injectivity of the (monodromy) developing map relative to the affine structure in question. Here the role of the monodromy covering becomes evident, and the notion of semicompletness finally emerges on stage in a natural way.

One of the advantages of this property is that it applies for every leaf of the foliation, independently on whether the leaf in question is or is not contained in the divisor of zeros and poles. In fact:

- For leaves in the divisor, it provides information on the structure of the leaf. Namely, $L \simeq U/\Gamma$ where U is an open set of $\mathbb{C}$ and Γ is a subgroup of the affine group leaving U invariant and acting properly discontinuously on U (see the original definition in [19]).

- For leaves outside the divisor, the notion is equivalent to say that the flow is univalued.

More concretely: a vector field is semicomplete if and only if the induced affine structure of every leaf is uniformizable (cf. [23], or Proposition 4.1.6).

The proof of both theorems will be based on a careful study of the boundary of the image of the developing maps. Roughly speaking, while the hypotheses on the singularities determine what might occur in the image of a leaf near a singularity, the results proved in Chapter 3 controls what may happen in the image of the regular part.

Theorem I concerns leaves that are contained in the divisor of zeros or poles of X. Its proof relies on studying closely the relation between the affine constants and the holonomy of the codimension-1 foliation.

Theorem I. *Let M, X, $\mathcal{F}$, E, E_0 and $\mathcal{D}$ be as before. Assume also that*

- $Hol'(E_0) \subseteq S^1$

- *X is semicomplete on a neighborhood of* $(\mathrm{Sing}(\mathcal{F}) \cap E) \cup (\mathrm{Sing}(\mathcal{D}) \cap E)$.

Then the affine structure of every leaf $L \subseteq E$ is uniformizable.

We should note that this theorem also applies in the case whete E is an irreducible component of the divisor zeros of X.

In turn, Theorem II concerns leaves that are not contained in $(X)_0 \cup (X)_\infty$, but that accumulate on E. Its proof is based on similar ideas as Theorem I (actually, the later can be interpreted as an infinitesimal version of Theorem II) but involves additional challenges. We will show that the hypothesis on the holonomy of $\mathcal{D}$, slows the approach of leaves to the divisor, which in turn influences the shape of the developing map's image. Regarding the singular set, the theorem requires an additional assumption in order to rule out particularly chaotic scenarios. Specifically, we will assume that X has a *friendly singular set* (see the definition in Section 4.3).

Theorem II. *Let M, X, $\mathcal{F}$, E, E_0 and $\mathcal{D}$ be as before. Assume, in addition, that the following holds:*

- *$Hol'(E_0) \subseteq S^1$,*
- *X is semicomplete on some neighborhood of $(\mathrm{Sing}(\mathcal{F}) \cap E) \cup (\mathrm{Sing}(\mathcal{D}) \cap E)$ and this singular set is friendly.*

Then X is semicomplete on M.

Although the hypotheses of the previous theorem may seem slightly sophisticated, they become significantly simpler when applied to vector fields possessing a meromorphic first integral. The last results we will present, fit within this context, namely, for the remainder of the introduction, we will assume that X is equipped with a meromorphic first integral $f : M \to \mathbb{C}$. In fact, the level sets of f define a natural codimension-1 foliation, which additionally satisfies the hypothesis related to the holonomy $\mathrm{Hol}'(E_0) \subseteq S^1$. In the present context, E_0 is defined to be $E \setminus I(f)$ where $I(f)$ denotes the indeterminacy set of f. Besides, the singular set turns out to be friendly. Therefore, as a consequence of the previous result, in Section 4.4, we proved:

Theorem II.A. *Let M, X, $\mathcal{F}$, E and E_0 be as before. Assume that $f : M \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *There is a constant $c_0 \in \mathbb{CP}^1$, such that $E_0 = f^{-1}(c_0)$.*
- *X is semicomplete on a neighborhood of the set $I(f) \cap E$*

Then X is semicomplete on M.

Note that Theorem II.B is an immediate consequence of the previous theorem in the context of meromorphic vector fields on the projective space $\mathbb{CP}^n$. Since a large amount of differential equations appearing in physics come along with some constant of motion (i.e., a first integral), it is promising to find a wide variety of examples where the theorem applies. One of our directions of future work is to study some concrete examples.

Regarding nature of leaves (T2) Finally, within the scope of a vector field possessing a first integral, we will prove one final result regarding the nature of leaves of the foliation. Its proof is presented in Section 4.4, and it is based on Theorem II.A, along with a careful study of the ends of leaves and an application of the Remmert-Stein theorem.

Before stating the result, a comment concerning the distinction between global and local leaves needs to be made. Fix a connected neighborhood U of the singular set and consider the foliation induced by the restriction of $\mathcal{F}$ to U. A leaf of $\mathcal{F}|_U$ is called a local branch of leaf. To exemplify this distinction, let us consider the following example: Let U be a neighborhood of the origin, and assume that X restricted to U looks like the radial vector field $X|_U = x\,\partial/\partial x + y\,\partial/\partial y$. While $\mathcal{F}|_U$ possesses an infinite number of distinct separatrices, it could happen that globally, many of the local leaves of $\mathcal{F}|_U$, are contained in a same leaf of L (see Figure 1.6).

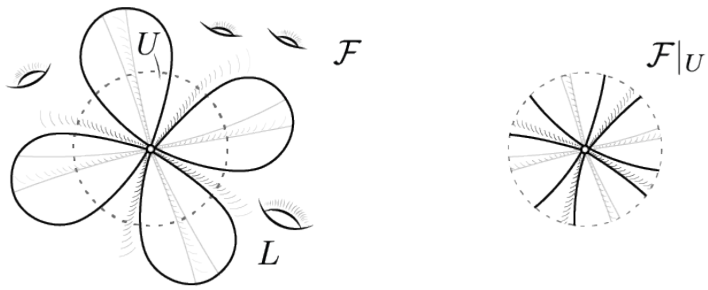


Figure 1.6 Distinction between local and global leaves.

An additional hypothesis concerning the ends of local branches will be required.

Theorem III. *Let M, X, $\mathcal{F}$, E and E_0 be as before. Assume that $f : M \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *There is a constant $c_0 \in \mathbb{CP}^1$, such that $E_0 = f^{-1}(c_0)$.*
- *X is semicomplete on a neighborhood of the set $(I(f) \cap E)$.*

In addition, assume that ends of leaves of local branches are contained in a discrete set. Then all the leaves of $\mathcal{F}$ are entire leaves.

Finally, as a corollary, we obtain the following theorem together with simplified version stated previously in the introduction concerning meromorphic vector fields on $\mathbb{CP}^n$.

Theorem III.A. *Let M, X, $\mathcal{F}$, E and E_0 be as before with X semicomplete. Assume that $f : M \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *There is a constant $c_0 \in \mathbb{CP}^1$, such that $E_0 = f^{-1}(c_0)$.*
- *Ends of leaves of local branches are contained in a discrete set.*

Then all the leaves of $\mathcal{F}$ are entire leaves.

Finally, to close the thesis, we present a concrete example of a quadratic vector field on $\mathbb{C}^3$, in connection with to the so-called Lins-Neto foliations, that illustrates the last theorem. In [18] A. Guillot demonstrated that this vector field is semicomplete, and that all the solutions are defined on the complex plane except for a discrete set of points. We provide an alternative proof of this facts by applying our theorems.

Chapter 2

Preliminaries

Let M be a complex manifold. Throughout this work, the phrase meromorphic vector field X always mean a meromorphic vector field that is not trivial in the sense that neither X vanishes identically, nor it is constant equal to $+\infty$. Nonetheless referring to X as a meromorphic vector field does not rules out the possibility of X being, indeed, holomorphic. Let us first recall the definition of a (global) vector field and of a (regular) foliation.

2.1 1-dimensional holomorphic foliations and vector fields

Definition 2.1.1. A regular holomorphic foliation of dimension k on M consists of a distinguished complex atlas $\{(U_i, \psi_i)\}$, $\psi_i : U_i \to V_i \subseteq \mathbb{C}^n$ such that whenever $U_i \cap U_j \neq \emptyset$, the corresponding change of coordinate map takes on the form

$$\psi_i \circ \psi_j^{-1}(u, v) = (f_{ij}(u, v), g_{ij}(v))$$

where $(u, v) \in \mathbb{C}^k \times \mathbb{C}^{n-k}$, where n stands for the dimension of the manifold M. The foliation may also be called a codimension $n - k$ foliation.

The standard notions such as leaves, plaques and holonomy of leaves associated with the general theory of foliations can naturally be applied in this holomorphic context (see [4]). In particular, each leaf is an immersed complex submanifold of M (though not necessarily properly embedded).

In this work we will be particularly interested in foliations of dimension 1, though foliations of codimension 1 will also play an important role in this discussion. In terms of notation foliations of dimension 1 will typically be denoted along the text by $\mathcal{F}$ whereas $\mathcal{D}$ will stand for codimension 1 foliations.

Now let us make precise the definition of 1-dimensional holomorphic singular foliations by means of holomorphic vector fields.

Definition 2.1.2. A 1-dimensional singular holomorphic foliation on M consists of the following data:

1. An atlas $\{(U_i, \psi_i)\}$ compatible with the complex structure on M, where $\psi_i : U_i \to V_i \subseteq \mathbb{C}^n$.

2. Holomorphic vector fields X_i, defined on each V_i, with singular set of codimension at least 2, i.e., $\mathrm{codim}(\mathrm{Sing}(X_i)) \geq 2$, and such that whenever $U_i \cap U_j \neq \emptyset$ there exist holomorphic functions $h_{ij} : \psi_i(U_i \cap U_j) \to \mathbb{C}^*$ satisfying

$$d(\psi_j \circ \psi_i^{-1})|_{\psi_i(p)} X_i(\psi_i(p)) = h_{ij}(\psi_i(p)) X_j(\psi_j(p))$$

for every $p \in U_i \cap U_j$. Analogously as in the definition of the regular case, we will write $(\psi_j \circ \psi_i^{-1})_* X_i = h_{ij} X_j$.

The singular set of a (1-dimensional) foliation $\mathcal{F}$ is a subset of M of codimension at least 2. In fact, being $\mathcal{F}$ defined as in Definition 2.1.2, its singular set corresponds to

$$\mathrm{Sing}(\mathcal{F}) = \bigcup_i \psi_i^{-1}\left(\{p \in V_i : X_i(p) = 0\}\right)$$

and each X_i has, by definition, a singular set of codimension at least 2.

Now let us recall the definition of (global) holomorphic vector field on M.

Definition 2.1.3. A holomorphic (resp. meromorphic) vector field X on M consists on a collection of holomorphic (resp. meromorphic) vector fields X_i defined on each V_i such that the change of coordinates map satisfies the following equation:

$$d(\psi_j \circ \psi_i^{-1}) X_i(\psi_i(U_i \cap U_j)) = X_j(\psi_j(U_i \cap U_j))$$

or equivalently:

$$(\psi_j \circ \psi_i^{-1})_* X_i = X_j.$$

The standard Flow Box Theorem for holomorphic vector fields ensures the existence of local diffeomorphims in neighborhooods of non-singular points, that transforms the vector field into the constant one.

This shows that the collection of vector fields $\{X_i\}$ actually define a regular holomorphic foliation on the open manifold $M \setminus \mathrm{Sing}(\mathcal{F})$. *In particular, leaves of a singular foliation are defined to be the leaves of the regular foliation in* $M \setminus \mathrm{Sing}(\mathcal{F})$. Holonomy groups and similar notions are analogously defined.

Consider now a meromorphic vector field X on M with singular set of codimension one. Whereas X may vanish (or have poles) over codimension-1 subvarieties –namely, the zero and pole divisor of X– it still defines a 1-dimensional holomorphic foliation on all of M. Let us make this statement accurate. In local coordinates $(x_1, \ldots x_n)$, X takes on the form

$$X = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n},$$

where $f_1, \ldots f_n$ are meromorphic functions. Now, if the zero-set of X has a codimension-1 component, then there follows from Hilbert's Nulstellensatz that $f_1, f_2, \ldots f_n$ have a non-invertible common factor, denoted by g. In other words,

$$X = gY$$

where Y is a meromorphic vector field with the same poles as X and whose zero-set has codimension at least 2. The analogous argument applying to poles of X, we can, indeed, write

$$X = \frac{g}{h} Y$$

for holomorphic functions g, h and where Y is a holomorphic vector fields whose zero-set has codimension at least two. The integral curves of X and Y naturally coincide away from the singular set of X and, in this sense, the foliation induced by X coincides with the foliation associated with Y. The vector field Y is said to be a representative of $\mathcal{F}$. Note, however, that this notion of vector field representing a foliation is well defined only up to multiplying by an invertible function.

2.2 Codimension-1 holomorphic foliations

As to singular codimension-1 foliations, integrable 1-forms can be used to provide a definition analogous to Definition 2.1.2.

Definition 2.2.1. A (singular) codimension-1 foliation $\mathcal{D}$ on M, consists of:

1. an atlas $\{(U_i, \psi_i)\}$ compatible with the complex structure on M, where $\psi_i : U_i \to V_i \subseteq \mathbb{C}^n$;

2. a holomorphic 1-form ω_i on each V_i, having singular set of codimension at least 2, that is integrable in the sense that
$$d\omega_i \wedge \omega_i = 0\,.$$

3. whenever $U_i \cap U_j \neq \emptyset$, there exist holomorphic functions $g_{ij} : U_i \cap U_j \to \mathbb{C}^*$, such that
$$\omega_i(\psi_i(p)) = g_{ij}(p)\omega_j(\psi_j(p))$$
for each $p \in U_i \cap U_j$. This is called the compatibility condition.

The singular set of a codimension-1 foliation as in Definition 2.2.1 is given by

$$\mathrm{Sing}(\mathcal{D}) = \bigcup_i \psi_i^{-1} \left(\{p \in V_i : (\omega_i)_p \equiv 0\} \right).$$

Again, Frobenius theorem implies that the collection of integrable 1-forms $\{\omega_i\}$ define a regular codimension-1 holomorphic foliation on $M \setminus \mathrm{Sing}(\mathcal{D})$. It is therefore straightforward to introduce the corresponding notions of leaves, holonomy groups and so on.

Codimension-1 foliation induced by a meromorphic map

Let $f : M \to \mathbb{C}$ be a meromorphic map. Then f locally takes the form $f = g/h$ where g and h are holomorphic functions, and where h is not identically zero. Furthermore, this quotient can be taken in such a way g and h have no common irreducible factors.

Definition 2.2.2. The indeterminacy set (or indeterminacy locus) of f is defined as the common zeros of g and h. Namely:

$$I(f) := \{p \in M : g(p) = 0 = h(p)\}$$

The map f induces a codimension-1 foliation, $\mathcal{D}_f$ whose leaves coincide with the level sets of f. Indeed, it is immediate to show that the distribution defined by the kernel of df, is tangent to the level sets of f.

Note also that the 1-form df is clearly integrable in the sense of Definition 2.2.1 and since $df = (gdh - hdg)/h^2$, one can show that

$$df = \frac{g_1}{h_1}\omega$$

where $I(g_1/h_1) = I(f)$ and ω is a holomorphic 1-form whose singular set has codimension at least 2. Consequently, it satisfies Definition 2.2.1.

2.3 Translation and affine structures on Riemann surfaces and a foliated variant

Let us begin by introducing the notion of affine and translation structures on a Riemann surface.

Definition 2.3.1. Let S be a Riemann surface. An affine (resp. translation) structure on S consists of an atlas such that all the changes of coordinates are restrictions of affine (resp. translation) maps of $\mathbb{C}$.

Clearly a translation structure is a particular case of an affine one.

Definition 2.3.2. A singular affine (resp. translation) structure on a Riemann surface S is an atlas defining an affine (resp. translation) structure on $S \setminus \Upsilon$ where Υ is a discrete subset of S.

If a Riemann surface S is equipped with an affine structure, we will sometimes refer to the atlas in question as the *affine atlas* of S.

On a Riemann surface, the notions of translation structure and nowhere zero holomorphic vector field are equivalent. In fact, if S is a Riemann surface equipped with an atlas $\{(U_i, \varphi_i)\}$ determining a translation structure on S, then the collection $\{\varphi_i^*\left(\frac{\partial}{\partial z}\right)\}_{i\in I}$ defines a holomorphic nowhere zero vector field on S. Conversely, if X is a nowhere vanishing vector field on S it is easy to check that the atlas given by the local inverses of the flow $\{(U_i, \Phi_i^{-1})\}_{i\in I}$ define a translation structure on S. For reference, we state the following result:

Lemma 2.3.3. *Let S be a Riemann surface equipped with a non-trivial meromorphic vector field X (possibly with zeros and poles). Then X induces a singular translation structure on S.* □

As a generalization of the previous fact, we have the following lemma:

Lemma 2.3.4. *Let S be a Riemann surface equipped with a collection of (locally defined) meromorphic vector fields $\{X_i\}_{i\in I}$, where each X_i is defined on an open set U_i. Assume that $\{U_i\}_{i\in I}$ is a cover of S and that on each intersection $U_i \cap U_j$, the vector fields X_i and X_j differ by a non-zero multiplicative constant, i.e., $X_i = \lambda_{ij} X_j$. Then the collection of $\{X_i\}$ defines a singular affine structure on S.*

Proof. The proof consists of noticing that X_i and $Y_j = \lambda_{ij} X_j$ induce a translation structure on $U_i \cup U_j$ and that if Φ_j is the flow of X_j, then φ given as $\varphi(t) = \Phi_j(\lambda_{ij} t)$ is the flow of $\lambda_{ij} X_j$. □

Let S be a Riemann surface equipped with a collection of (local) meromorphic non-trivial vector fields $\{X_i\}_{i\in I}$, each X_i defined on a open set $U_i \subseteq S$. Assume that $S = \bigcup_{i\in I} U_i$ and that the collection of $\{X_i\}$ endows S with a (singular) affine structure. Then, the collection of all multiples $\{\lambda X_i\}_{\lambda\in\mathbb{C}^*}$ of the vector fields $\{X_i\}$ still defines the same affine structure on S. Furthermore, the family $\{\lambda X_i\}$ yields a maximal atlas for the affine structure in question, in the sense that any chart compatible with the initial affine structure, locally coincides with the inverse of the flow of a vector field in the family $\{\lambda X_i\}$.

In particular, if S is equipped with a singular translation structure, i.e., with a non trivial meromorphic vector field X, then the family $\{\lambda X\}$ −consisting of constant mutiples of restrictions of X to open sets− defines a maximal singular affine atlas for S.

Definition 2.3.5. The (singular) affine structure defined above will be referred to as *the underlying affine structure* relative to the original translation structure.

Next, let us consider a foliated analogue of what precedes. We consider a 1−dimensional singular holomorphic foliation and remind the reader that the corresponding leaves inherit of a natural Riemann surface structure. We would like to consider then families of Riemann surfaces equipped with affine structures. This idea is materialized by the notion of *foliated affine structure* (cf. the definition given in [11]), which also makes accurate the intuitive idea of affine structures *varying holomorphically with the leaves.*

Definition 2.3.6. Let M be a complex manifold of dimension n and let $\mathcal{F}$ be a 1−dimensional foliation on M. A foliated affine structure on $\mathcal{F}$ consists of an open cover $\{U_i\}_{i\in I}$ of $M \backslash \mathrm{Sing}(\mathcal{F})$ and a collection of vector fields $\{Y_i\}$, each Y_i defined on U_i, such that for every $i \in I$:

1. Y_i is tangent to $\mathcal{F}$
2. for every leaf L, the inverse of the flow of Y_i restricted any plaque $P \subseteq U_i$ of L, belongs to the affine atlas of the leaf L.

2.3.1 Foliated affine structures and vector fields

Consider now a complex manifold M of dimension n equipped with a meromorphic vector field X and denote by $\mathcal{F}$ the (singular) holomorphic foliation induced by X on M.

Let $(X)_0$ denote the divisor of zeros of X and let $(X)_\infty$ be the divisor of poles. If $(X)_0$ (resp. $(X)_\infty$) is not empty and E is an irreducible component of $(X)_0$ (resp. $(X)_\infty$), then a generic point in E is regular for the foliation $\mathcal{F}$. Moreover E may or may not be invariant by $\mathcal{F}$.

Let L be a regular leaf for $\mathcal{F}$. Then one of the following situations occurs:

(a) L is a leaf in which the restriction $X|_L$ of X to L, yields a (non-trivial) meromorphic vector field (on the Riemann surface L);

(b) L is contained in an irreducible and $\mathcal{F}-$invariant component of the divisor of zeros and poles of X, i.e., $L \subseteq (X)_0 \cup (X)_\infty$.

According to Lemma 2.3.3, in case (a), the vector field endows L with a singular translation structure. On the other hand, if L is contained in a component of $(X)_0 \cup (X)_\infty$ that is invariant by $\mathcal{F}$, then Lemma 2.3.3 no longer applies. However, in the latter case, X still defines a canonical *singular affine structure* on L. Indeed, the underlying affine structure arising on the leaves where the translation structure is defined can be extended to all leaves in $\mathcal{F}$ as shown in [11], [23]. This result can accurately be stated as follows.

Proposition 2.3.7 ([11], [23]). *Let X be a (non-trivial) meromorphic vector field on M, and denote by $\mathcal{F}$ its associated foliation. Then $\mathcal{F}$ possesses a foliated affine structure induced by X.*

In the sequel, we provide a proof of Proposition 2.3.7 that, albeit slightly more computational than the original argument in [23], is particularly adapted to the way its existence will be exploited in this paper.

Let $E \subseteq M$ be an irreducible component of the divisor of zeros and poles of X. Assume E to be invariant by $\mathcal{F}$. It suffices to introduce a suitable affine structure on *generic leaves* of $\mathcal{F}$ contained in $(X)_0 \cup (X)_\infty$. In other words, we can assume that the leaves are not contained in the intersection of two different components of $(X)_0 \cup (X)_\infty$. Indeed, once we will have proved that the affine structure extends to the "generic" leaves in question, there will follow that $\mathcal{F}$ is endowed with a (holomorphic) foliated affine structure defined away from an analytic set of codimension at least 2: the foliated affine structure must then extends holomorphically to all of $\mathcal{F}$.

Remark 2.3.8. Similarly to the foliated tangent bundle of a foliation $\mathcal{F}$, foliated higher order jet bundles can be defined as well. Naturally, as it happens with all the geometric structures, a foliated affine structure is nothing more than a section of a suitable foliated jet bundle over M. In particular, the existence of a holomorphic extension to an analytic subset of codimension larger than 2 follows from standard results in complex analysis.

Let L be a leaf of $\mathcal{F}$ contained in E and $p \in L$. Consider local coordinates $(x_1, \dots x_n)$ around p such that

(a) E is identified with $\{x_n = 0\}$ and p with $(0, \dots 0)$.

(b) The vector field X is locally given by

$$X = x_n^k h(x_1, \dots x_n) \frac{\partial}{\partial x_1}$$

for some $k \in \mathbb{Z}$ and some holomorphic function h with $h(x_1, \ldots x_{n-1}, 0)$ non-identically zero.

Note that k does not depend on the choice of the coordinates $(x_1, \ldots x_n)$ around p satisfying (a) and (b). In fact, the integer k is said to be the order of X over E. Furthermore, we define the vector field

$$X^R = h(x_1, \ldots x_n)\partial/\partial x_1$$

as the *renormalized vector field* of X in the present coordinates. Unlike the integer k, the vector field X^R does depend on the choice of coordinates satisfying (a) and (b). The exact extent of this dependence will be worked out in what follows.

Note that X and X^R define the same foliation $\mathcal{F}$ on their domains of definition. Naturally, X^R has the advantage that its local flow on L can be considered. Also, for leaves of $\mathcal{F}$ lying away from E, we note that the corresponding restrictions of X and of X^R differ by a multiplicative constant. In fact, locally, such a leaf L_c is identified with the plaque $L_c = \{(x_1, c_2, \ldots c_n)\}$ for constants $c_2, \ldots c_n$, with $c_n \neq 0$, and clearly the vector fields $X|_{L_c}$ and $X^R|_{L_c}$ verify

$$X|_{L_c} = c_n^k X^R|_{L_c} .$$

In particular, for leaves not contained in $(X)_0 \cup (X)_\infty$, the translation structures induced by X and by X^R have the same underlying affine structure. Therefore, to prove Proposition 2.3.7, it suffices to check that the charts provided by the local flows of the vector fields X^R, fit together into an affine structure for the leaves contained in E.

Since, in general, X^R is not a globally defined vector field, to derive the existence of a suitable affine structure on L, we need to work out the dependence of X^R on the choice of coordinates. For this, consider a pair of (different) local coordinates $(x_1, \ldots x_n)$ and $(u_1, \ldots u_n)$ around p such that:

- $E \simeq \{x_n = 0\}$ and $E \simeq \{u_n = 0\}$.
- In the above coordinates, the vector field X is respectively given by

$$X_x = x_n^k h(x_1, \ldots x_n)\frac{\partial}{\partial x_1} \quad \text{and} \quad X_u = {u_n}^k g(u_1, \ldots u_n)\frac{\partial}{\partial u_1}.$$

  In particular the corresponding renormalized vector fields take on the forms

$$X_x^R = h(x_1, \ldots x_n)\frac{\partial}{\partial x_1} \quad \text{and} \quad X_u^R = g(u_1, \ldots u_n)\frac{\partial}{\partial u_1}.$$

Consider now a leaf $L \subseteq E$ which, in addition, is "generic" in the previously mentioned sense. Recalling that $E \simeq \{x_n = 0\} \simeq \{u_n = 0\}$, a plaque of L in these local coordinates is respectively parametrized by $x_1 \to (x_1, c_2, \ldots c_{n-1}, 0)$ and $u_1 \to (u_1, c_2', \ldots c_{n-1}', 0)$ for suitable constants $c_2, \ldots, c_{n-1}$ and $c_2', \ldots, c_{n-1}'$. To prove Proposition 2.3.7, we need to prove that the underlying affine structures on leaves not contained in E extends to leaves contained in E. For this, it suffices to show that the local charts on L, given by the inverses of the local flows of X_x^R and of X_u^R restricted to L, belong to a same affine atlas. In view of the discussion in Section 2.3, it is enough to prove the following proposition.

Proposition 2.3.9. *The local vector fields on L given by X_x^R and by X_u^R differ by a multiplicative constant.*

Proof. Let $\Psi = (\Psi_1, \dots \Psi_n)$ be the change of coordinates from $(x_1, \dots, x_n)$ to $(u_1, \dots u_n)$, the statement amounts to checking that

$$\Psi_* X_x^R|_L = C_{xu} \cdot X_u^R|_L$$

for some $C_{xu} \in \mathbb{C}^*$. Without loss of generality, assume that $p = (0, \dots, 0)$ in both coordinates. It then follows that $L = \{u_2 = \cdots = u_n = 0\}$ and $L = \{x_2 = \cdots = x_n = 0\}$. Recall that Ψ satisfy the following properties:

1. Ψ preserves leaves, i.e., it sends "horizontal" lines to "horizontal" lines:

   $$\Psi(x_1, a_2, \dots a_n) = (\Psi_1(x_1, a_2, \dots a_n), \tilde{a}_2, \dots \tilde{a}_n) = (u_1, \tilde{a}_2, \dots \tilde{a}_n)$$

   for constants a_i and $\tilde{a}_i$, $i = 2, \dots n$. In other words, Ψ_i does not depend on the first variable x_1.

2. Ψ leaves L invariant, in the sense that

   $$\Psi(x_1, 0 \dots 0) = (\Psi_1(x_1, 0 \dots 0), 0 \dots, 0).$$

3. Ψ leaves E invariant, in the sense that $\Psi(x_1, \dots x_{n-1}, 0) = (\Psi_1, \dots \Psi_{n-1}, 0)$ where Ψ_i stands for $\Psi_i(x_1, \dots x_{n-1}, 0)$. Therefore Ψ_n takes on the form $\Psi_n(x_1, \dots x_n) = x_n \Psi_n^{x_n}(x_2, \dots x_n)$ for some holomorphic function $\Psi_n^{x_n}$.

Finally since X is a global vector field, the local vector fields X_x, X_u must satisfy the relation $\Psi_* X_x = X_u$. This yields:

$$d\Psi|_{\vec{x}} X_x(\vec{x}) = X_u(\Psi(\vec{x}))$$

where $\vec{x} = (x_1, \dots x_n)$. It follows that

$$\frac{\partial \Psi_1}{\partial x_1}(\vec{x}) \cdot x_n^k h(\vec{x}) = (\Psi_n(\vec{x}))^k g(\Psi(\vec{x}))$$

and hence

$$\frac{\partial \Psi_1}{\partial x_1}(\vec{x}) \cdot x_n^k h(\vec{x}) = x_n^k (\Psi_n^{x_n}(x_2, \dots x_n))^k g(\Psi(\vec{x})).$$

After cancelling the common factor, we obtain:

$$\frac{\partial \Psi_1}{\partial x_1}(\vec{x}) \cdot h(\vec{x}) = (\Psi_n^{x_n}(x_2, \dots x_n))^k g(\Psi(\vec{x})). \tag{2.1}$$

In particular, restricted to the corresponding plaque L, we have:

$$\frac{\partial \Psi_1}{\partial x_1}(x_1, 0, \ldots, 0) \cdot h(x_1, 0, \ldots, 0) = (\Psi_n^{x_n}(0, \ldots 0))^k \, g(\Psi_1(x_1, 0, \ldots 0), 0, \ldots 0).$$

Denote by $C_{xu} = (\Psi_n^{x_n}(0, \ldots 0))^k$, then this is just the pushforward of the vector field $X_x^R|_L$ by Ψ_1, i.e.,

$$\Psi_{1*} X_x^R|_L = C_{xu} \cdot X_u^R|_L. \tag{2.2}$$

In other words the vector fields $X_x^R|_L$ and $X_u^R|_L$ differ by a multiplicative constant. This proves Proposition 2.3.9 and the proof of Proposition 2.3.7 follows immediately as well. □

Remark 2.3.10. Summarizing what precedes, we single out the following crucial observations, namely:

(a) Every leaf is equipped with an affine structure induced by X.

(b) For every leaf L such that the restriction of X to L yields a non-trivial vector field, i.e., L not contained in $(X)_0 \cup (X)_\infty$, the affine structure in question coincides with the underlying affine structure arising from the translation structure defined by X.

(c) The affine structure defined on each leaf varies holomorphically from leaf to leaf, i.e., the foliation $\mathcal{F}$ is equipped with *a foliated affine structure.*

Chapter 3

Control of the dynamics in the regular part of $\mathcal{F}$

Consider now a compact complex manifold M of dimension n, along with a meromorphic vector field X on M. As mentioned in the previous section, components of the divisor of zeros and poles of X may or may not be invariant by $\mathcal{F}$.

Let E be an irreducible component of $(X)_0 \cup (X)_\infty$ invariant by $\mathcal{F}$. Note that E can have non-empty intersection with other components of the divisor of zeros and poles of X. In this regard, let us define the "extended singular" set relative to E and denoted by $\mathcal{S}_0$ by letting

$$\mathcal{S}_0 = \mathrm{Sing}(\mathcal{F}) \cup \bigcup_j (D_j \cap E)$$

where D_j runs over all irreducible components of $(X)_0 \cup (X)_\infty$ different from E.

Let $\mathscr{V} = \{V_i, \psi_i\}_{i \in I}$ be a foliated atlas of $E \setminus \mathcal{S}_0$. And for each $i \in I$, let $(x_1^i, \dots x_n^i)$ be the coordinates on $\psi_i(V_i) \subseteq \mathbb{C}^n$. As in the previous section, we may assume that

1. On each of the charts, we have $E = \{x_n^i = 0\}$.

2. The local vector field X_i on $\psi_i(V_i)$ takes on the form

$$X_i = (x_n^i)^k h_i(x_1^i, \dots, x_n^i) \frac{\partial}{\partial x_1^i}$$

   where $k \in \mathbb{Z}$, and h_i is holomorphic and nowhere vanishing. We denote the corresponding renormalized vector field X_i^R by

$$X_i^R = h_i(x_1^i, \dots, x_n^i) \frac{\partial}{\partial x_1^i},$$

   and let Φ_i the corresponding (local) flow of X_i^R.

We can now introduce the main tool of this section, namely the developing map of the affine structure induced by X on the leaf L of $\mathcal{F}$. Let us begin with a brief review of this construction for general Riemann surfaces equipped with affine structures (see for example [16] for a detailed presentation of the subject).

3.1 Developing maps for foliated affine structures

3.1.1 Developing map of a Riemann surface equipped with an affine structure

Let S be a Riemann surface equipped with an affine structure, and denote by $\mathcal{A}_S$ an affine atlas of S. Fix a base point $p_0 \in S$ and some chart $(W_0, \varphi_0) \in \mathcal{A}_S$, such that $p_0 \in W_0$. Viewed as a function valued in $\mathbb{C}$, the coordinate $\varphi_0 : W_0 \to \mathbb{C}$ can be extended over paths $\gamma : [0,1] \to S$ with $\gamma(0) = p_0$ as follows. Let $\gamma : [0,1] \to S$ be a given path with initial point p_0, we cover γ by finitely many charts in $\mathcal{A}_S$, namely $(W_0, \varphi_0), \dots, (W_k, \varphi_k)$ such that for every $i \in 1, \dots, k$

- the intersection $W_{i-1} \cap W_i$ is connected,
- $W_i \cap W_j = \emptyset$ if $|i - j| \geq 2$.

The changes of coordinates for consecutive charts, $\varphi_i \circ \varphi_j^{-1}$ (resp. $\varphi_j \circ \varphi_i^{-1}$) with $j = i-1$, are affine maps defined in suitable subsets of $\mathbb{C}$. In particular, they have a natural extension to all of $\mathbb{C}$ as an affine map. We denote the respective affine map by $\varphi_{j,i} : \mathbb{C} \to \mathbb{C}$ (resp. $\varphi_{i,j}$).

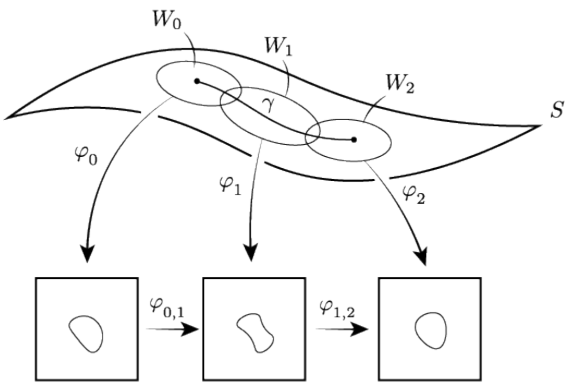


Figure 3.1 Affine change of coordinates.

The first step of the construction consists of modifying in an appropriate way the chart φ_1 so that the new coordinate coincides with φ_0 in the intersection $W_0 \cap W_1$. This is done as follows. Let $\tilde{\varphi}_1 : W_1 \to \mathbb{C}$ be defined as $\tilde{\varphi}_1 = \varphi_{1,0} \circ \varphi_1$. Now, since $\tilde{\varphi}_1$ differs from φ_1 by an affine map, $(W_1, \tilde{\varphi}_1)$ also belongs to the affine atlas $\mathcal{A}_S$. Let $\mathsf{D}_1 : W_0 \cup W_1 \to \mathbb{C}$, be the extension of φ_0 to $W_0 \cup W_1$ defined by:

$$\mathsf{D}_1(x) = \begin{cases} \varphi_0(x) & \text{if} \quad x \in W_0 \\ \tilde{\varphi}_1(x) & \text{if} \quad x \in W_1. \end{cases}$$

Proceeding inductively, this construction gives rise to a map $\mathsf{D}_k : W_0 \cup \cdots \cup W_k \to \mathbb{C}$, being the extension of φ_0 along the path γ.

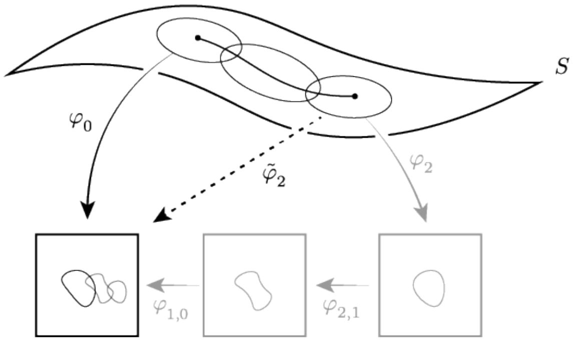


Figure 3.2 The map $\tilde{\varphi}_2$ takes the form $\tilde{\varphi}_2(x) = C_{01}^{-1} C_{12}^{-1} \cdot \varphi_2(x) + const.$ for $x \in W_2$.

The above construction has a few basic features that are worth pointing out, namely:

(i). $\mathsf{D}_k(\gamma(1))$ remains unchanged under deformation of γ with fixed endpoints. Therefore $\mathsf{D}_k(\gamma(1))$ only depends on the homotopy class of γ (with fixed endpoints).

(ii). Up to weakening our condition to allow the intersection $W_0 \cap W_k$ to be non empty, we can try to extend φ_0 over loops based at p_0. On $W_0 \cap W_k$ one has two affine charts, namely (φ_0, W_0) and $(\tilde{\varphi}_k, W_k)$ where $\tilde{\varphi}_k = \mathsf{D}_k|_{W_k}$ so that φ_0 and $\tilde{\varphi}_k$ differ by an affine map, which will be denoted by μ_γ. In view of the preceding, μ_γ depends only on the homotopy class of γ so that we obtain a map $\mu : \pi_1(S, p_0) \longrightarrow \mathrm{Aff}(\mathbb{C})$ assigning to each loop γ based on p_0, the affine map $\mu(\gamma) = \mu_\gamma$.

(iii). It turns out that, $\mu : \pi_1(S, p_0) \longrightarrow \mathrm{Aff}(\mathbb{C})$ is a homomorphism called the *monodromy homomorphism* associated with the affine structure. Its conjugacy class does not depend on the choice of the base point.

(iv). $\mu(\pi_1(S, p_0)) \subseteq \mathrm{Aff}(\mathbb{C})$ is called the *monodromy group* of the affine structure and it encodes the obstruction to be able to extend the map φ_0 globally over S.

A standard way to handle this (potential) monodromy phenomenon consists of defining the so-called *developing map* on the universal covering of S, which will be denoted by $\mathsf{Dev} : \tilde{S} \to \mathbb{C}$. This developing map together with the monodromy homomorphism satisfy the following *equivariance relation*

$$\mathsf{Dev}([\gamma]p) = \mu_\gamma(\mathsf{Dev}(p)) \tag{3.1}$$

where $p \in \tilde{S}$, and where $[\gamma]$ stands for the corresponding deck transformation associated with γ. It is not difficult to show that the previous map can be defined not only on the universal covering space of S but also in every covering $\hat{S}$ of S satisfying $\pi_1(\hat{S}) \leq \ker(\mu)$. In particular, let $\hat{S}$ be the covering space of S such that $\pi_1(\hat{S}) = \ker(\mu)$. We will refer to $\hat{S}$ as the *monodromy covering space* of S and for sake of simplicity, we will maintain the notation $\mathsf{Dev} : \hat{S} \to \mathbb{C}$ for the developing map defined on $\hat{S}$. Then $\mathsf{Dev} : \hat{S} \to \mathbb{C}$ together with μ still satisfy Relation (3.1). As will soon be clear, the monodromy developing map is better suited for our purposes.

Remark 3.1.1. In closing this paragraph, it is convenient to highlight two consequences of the preceding construction, namely:

(a). The developing map is canonically defined up to composition with an affine map of $\mathbb{C}$.

(b). For every simply connected open set $\mathcal{W} \subseteq S$ containing $p_0 \in \mathcal{W}$, the developing map restricted to $\mathcal{W}$, $\mathsf{Dev}|_{\mathcal{W}} : \mathcal{W} \to \mathbb{C}$, is well defined.

3.1.2 Developing maps of foliated affine structures and transverse regularity

Recalling that, by Proposition 2.3.7, the leaves of $\mathcal{F}$ are equipped with a foliated affine structure induced by X, for every leaf L of $\mathcal{F}$, the corresponding developing map can be considered. Namely $\mathsf{Dev}_L : \hat{L} \to \mathbb{C}$, or simply Dev when there is no risk of confusion. Our purpose will be to understand the developing map especially in the case of leaves contained in the divisor of zeros and poles of X.

We place ourselves in the context of Proposition 2.3.7 and the beginning of this chapter, fix a leaf L equipped with the singular affine structure induced by X. Recall that a foliated covering $\{(V_i, \psi_i)\}_{i\in I}$ for $\mathcal{F}$ was fixed on a neighborhood of $E \setminus \mathcal{S}_0$. With respect to this covering, the renormalized vector fields X_i^R for $i \in I$ were defined. Furthermore, the $\mathbb{C}$-valued maps obtained as inverses of their local flows restricted to plaques of L, provide charts for the affine structure on L.

The next step basically consists of comparing developing maps arising from nearby leaves. This is, however, a bit tricky in that the coverings where developing maps are defined may vary "discontinuously" with the leaf, pending on whether or not the leaves carry a non trivial monodromy. For our purposes, this issue can be avoided since it suffices to compare the extension of a given coordinate along paths. Alternatively, we may consider convergence of developing maps restricted to simply connected domains in the leaf L. Indeed, nearby leaves restricted to simple connected domains are essentially "the same", and therefore the notion of uniform convergence makes sense. Let us make this idea accurate.

Assume that $p_0 \in E \setminus \mathcal{S}_0$, and denote by L_0 the leaf passing through p_0, clearly $L_0 \subseteq E$. Consider a simply connected and relatively compact open set $\mathcal{W}$ of L_0 such that $p_0 \in \mathcal{W}$. In view of Remark 3.1.1, the developing map relative to the leaf L_0 restricted to $\mathcal{W}$, namely $\mathsf{Dev}_0|_{\mathcal{W}} : \mathcal{W} \to \mathbb{C}$, is well defined.

Let $\widehat{\mathcal{W}}$ be a $\mathcal{C}^\infty$-tubular neighborhood of $\mathcal{W}$ and let $\Pi : \widehat{\mathcal{W}} \to \mathcal{W}$ be the bundle projection. Since $\mathcal{W}$ is contractible (being homeomorphic to $\mathbb{R}^2$), there follows that the normal bundle of $\mathcal{W}$ is trivial and hence $\widehat{\mathcal{W}}$ can be identified to a product of the form $\mathcal{W} \times \mathbb{D}^{n-1}$. Since $\mathcal{W}$ is simply connected (and relatively compact), Reeb Stability Theorem (see [5]) implies that $\widehat{\mathcal{W}}$ can be chosen foliated by $\mathcal{F}$. More precisely the following holds:

(1) The leaves L_ε of the restriction of $\mathcal{F}$ to $\widehat{\mathcal{W}}$ are transverse to the fibers of Π.

(2) Restricted to one of those leaves, $\Pi : L_\varepsilon \to \mathcal{W}$ is a global diffeomorphism. In particular $\Pi|_{\mathcal{W}}$ is the identity map.

Condition (2) can alternatively be formulated by saying that the leaves L_ε are graphs of smooth functions $g : \mathcal{W} \to \mathbb{D}^{n-1}$ with the previous identification of $\widehat{\mathcal{W}} \simeq \mathcal{W} \times \mathbb{D}^{n-1}$.

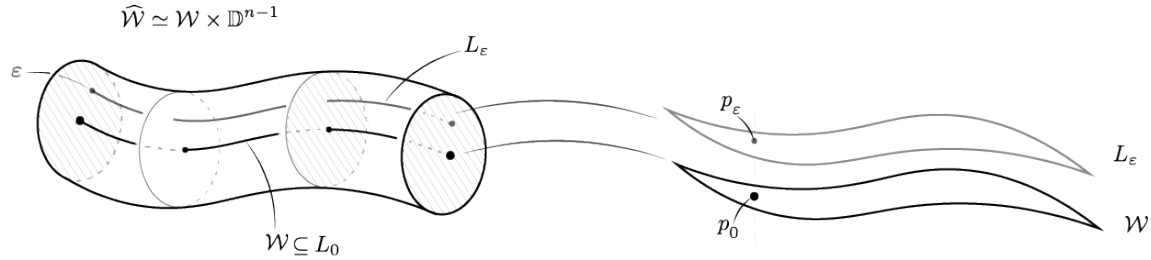


Figure 3.3 Foliated tubular neighborhood.

Remark 3.1.2. It is possible to prove that in the present case, the tubular neighborhood $\widehat{\mathcal{W}}$ can be chosen holomorphic, but this will not be needed in what follows.

Recalling that $p_0 \in \mathcal{W}$ was fixed let $\Sigma = \Pi^{-1}(p_0)$ and set $p_\varepsilon \in \Sigma \cap L_\varepsilon$. In what follows, $\boldsymbol{\varepsilon}$ will be the $(n-1)$-dimensional parameter associated with the position of the leaf L_ε over the polydisc $\mathbb{D}^{n-1}$ by means of the identification $\widehat{\mathcal{W}} \simeq \mathcal{W} \times \mathbb{D}^{n-1}$. More precisely $\boldsymbol{\varepsilon} = (\varepsilon_2, \varepsilon_3, \dots \varepsilon_n)$. To simplify notation, subscripts will use non bold typefont. Note that for p_ε sufficiently close to p_0, we may say that L_ε tends to $\mathcal{W} = L_0 \cap \widehat{\mathcal{W}}$ as p_ε tends to p_0, or alternatively as $\boldsymbol{\varepsilon}$ tends to the origin 0. Denote by $f_\varepsilon := \Pi|_{L_\varepsilon}^{-1} : \mathcal{W} \to L_\varepsilon$ the diffeomorphism given by (2), in other words, $f_\varepsilon \in \mathcal{C}^\infty(\mathcal{W}, L_\varepsilon)$ is a section of the bundle. Note that f_ε converges uniformly on $\mathcal{W}$ to the identity map as $\boldsymbol{\varepsilon}$ tends to 0.

The construction of the developing map of L_ε with base point p_ε is made by choosing the "same" charts as with Dev_0, i.e., using the charts provided by the flow of the renormalized vector field on each of the open sets V_i. More precisely, for each open set V_i intersecting $\mathcal{W}$, the charts appearing in the construction of the developing maps of L_0 and L_ε are given by:

$$\varphi_i^0 = (\Phi_i|_{\mathcal{W} \cap V_i})^{-1} : \mathcal{W} \to \mathbb{C} \text{ and } \varphi_i^{\varepsilon} = (\Phi_i|_{\mathcal{W}_\varepsilon \cap V_i})^{-1} : L_\varepsilon \to \mathbb{C}$$

In other words, φ_i^0 and φ_i^{ε} are nothing but the inverses of the parametrizations of the corresponding plaques by the flow of X_i^R with initial conditions p_0 and p_ε respectively.

Remark 3.1.3. We point out that the composition $\varphi_i^{\varepsilon} \circ f_\varepsilon : \mathcal{W} \cap V_i \to \mathbb{C}$, and φ_i^0 have the same domain of definition. Therefore, from the continuity of the inverse of the flow and the fact that $\lim_{\varepsilon \to 0} f_\varepsilon = id$, it follows that $\varphi_i^{\varepsilon} \circ f_\varepsilon$ converges uniformly to φ_i^0 in $\mathcal{W} \cap V_i$, as $\boldsymbol{\varepsilon}$ tends to 0 (or equivalently as L_ε approaches L_0 on $\widehat{\mathcal{W}}$).

Denote by $\mathsf{Dev}_\varepsilon : L_\varepsilon \to \mathbb{C}$ the developing map of L_ε constructed as above. Now we can define the "normalized" developing map of the leaf L_ε restricted to $\mathcal{W}$ as

$$\mathsf{Dev}^{(\varepsilon)} = \mathsf{Dev}_\varepsilon \circ f_\varepsilon : \mathcal{W} \to \mathbb{C}.$$

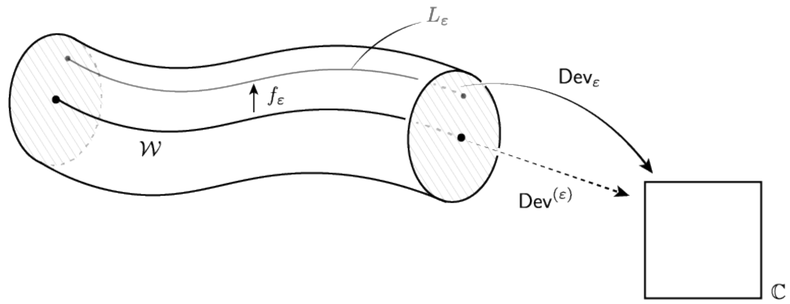


Figure 3.4 Normalized developing map on $\mathcal{W}$

Since for every small enough ε, all the maps $\mathsf{Dev}^{(\varepsilon)}$ have the same domain of definition, the previous discussion allows us to formalize the notion of uniform convergence of developing maps. In fact, we may now state the following lemma:

Lemma 3.1.4. *For any open simply connected relatively compact set $\mathcal{W}$ containing p_0 as before, the normalized developing maps $\mathsf{Dev}^{(\varepsilon)} : \mathcal{W} \to \mathbb{C}$ converge uniformly to Dev_0 as ε tends to* 0.

Proof. Since $\mathcal{W}$ is a relatively compact subset, we may assume that $\mathcal{W}$, and $\widehat{W}$, are contained in a finite union of open sets of the atlas $\mathscr{V}$, namely $V_0 \cup \cdots \cup V_j$. From the definition of developing map, for points in $x \in L_\varepsilon \cap V_j$, the following identity holds

$$\mathsf{Dev}_\varepsilon(x) = (C^\varepsilon_{\mathcal{W}})^{-1} \cdot \varphi^\varepsilon_j(x) + \Delta^\varepsilon_{\mathcal{W}}$$

where $C^\varepsilon_{\mathcal{W}} = C^\varepsilon_{0,1} \cdot \ldots \cdot C^\varepsilon_{j-1,j}$ and each $C^\varepsilon_{i,i+1}$ stands for the affine constants arising from the change of coordinates on $V_i \cap V_{i+1} \cap L_\varepsilon$ as proved in Proposition 2.3.7, and $\Delta^\varepsilon_{\mathcal{W}}$ is a constant. Thus, for any $x \in \mathcal{W} \cap V_j$ we have that

$$\mathsf{Dev}^{(\varepsilon)}(x) = (C^\varepsilon_{\mathcal{W}})^{-1} \cdot \varphi^\varepsilon_j \circ f_\varepsilon(x) + \Delta^\varepsilon_{\mathcal{W}}.$$

The statement of this lemma is a consequence from the following facts:

- From Remark 3.1.3, $\varphi^\varepsilon_j \circ f_\varepsilon$ converges uniformly to φ^0_j on $\mathcal{W}$.
- Since the affine structure varies holomorphically from leaf to leaf, we have that $C^\varepsilon_{i,i+1}$ tends to $C^0_{i,i+1}$ as ε tends to 0, and therefore the product $(C^\varepsilon_{\mathcal{W}})^{-1}$ converges to $\left(C^0_{\mathcal{W}}\right)^{-1}$. The convergence of the translation constants $\Delta^\varepsilon_{\mathcal{W}}$ follows from the same fact.

$\square$

Now we would like to interpret the constants arising from the affine structure of L_0 – which in turn affect the computation of Dev_0 – in terms of data encoded by the behavior of leaves L_ε near L_0 (in the sense that their distance to p_0 is small), but lying away from E.

Recall that the proof of Proposition 2.3.9 hinged from the fact that $X = \{X_i\}$ defines a global vector field on the leaves not contained in E. By cancelling a common factor in the equation stemming from the definition of the vector field X, we were able to extend the underlying affine structure to leaves inside the divisor of zeros (or poles). These common factor, however, conceal some fundamental information needed to understand the nature of the developing maps associated with the affine structure contained in the divisor in question. Let us make it accurate.

Assume that $\widehat{\mathcal{W}} \subseteq V_0 \cup V_1 \cup \cdots \cup V_j$ with $V_i \cap V_{i+1} \neq \emptyset$. In the sequel, we resume the notation and the setting of Proposition 2.3.7 (with slight changes) and of the beginning of the present section. To abridge notation, the change of coordinates Ψ_{01} from V_0 to V_1 will be denoted by $\Psi_{01} = \Psi = (\Psi_1, \ldots \Psi_n)$. Recall that since E is identified with $\{x_n^0 = 0\}$, Ψ_n takes on the form

$$\Psi_n = x_n^0 \tilde{\Psi}_n(x_2^0, \ldots x_n^0).$$

Assume without loss of generality that L_0 is identified with $\{(x_1^i, 0, \ldots, 0)\}$ for every $i = 0, \ldots, j$. If L_ε is a leaf in $\widehat{\mathcal{W}}$ not contained in E, then L_ε can be parametrized in V_0 as $L_\varepsilon \simeq \{(x_1^0, \varepsilon_2^0, \ldots \varepsilon_{n-1}^0, \nu_0)\}$ for some constants $\varepsilon_2^0, \ldots \varepsilon_{n-1}^0 \in \mathbb{C}$ and $\nu_0 \in \mathbb{C}^*$. We emphasize the fact that these constants may change from one coordinate to another. In particular, the value of ν_0 in V_1 coordinates is determined by $\Psi_n|_{L_\varepsilon}$. Denote by

$$\nu_1 = \Psi_n|_{L_\varepsilon} = \nu_0 \tilde{\Psi}_n(\varepsilon_2^0, \ldots, \nu_0). \tag{3.2}$$

Now recall from the proof of Proposition 2.3.9, that the constant defining the affine structure of L_ε on $V_0 \cup V_1$ is given by $C_{01}^\varepsilon = \left(\tilde{\Psi}_n(\varepsilon_2^0, \ldots, \nu_0)\right)^k$ which, in turn, can be written as

$$C_{01}^\varepsilon = \left(\tilde{\Psi}_n(\varepsilon_2^0, \ldots, \nu_0)\right)^k = \left(\frac{\nu_1}{\nu_0}\right)^k. \tag{3.3}$$

Recall also that the constant relative to the leaf L_0 is obtained by taking the limit of C_{01}^ε as $\boldsymbol{\varepsilon}$ tends to 0. Therefore, owing to Equations (3.2) and (3.3), we conclude that

$$\left(C_{01}^0\right)^{1/k} = \lim_{\varepsilon \to 0} \frac{\nu_1}{\nu_0} = \lim_{\varepsilon \to 0} \frac{\Psi_n(\varepsilon_2^0, \ldots, \nu_0)}{\nu_0} = \frac{\partial}{\partial x_n^0}\Big|_{\nu_0 = 0} \Psi_n(0, \ldots, 0, \nu_0).$$

Repeating the previous argument in the next change of coordinates intersecting L_ε (namely, when L_ε quits V_1), it follows that the affine constant relative to the affine change of coordinates from V_1 to V_2 when restricted to L_ε is given by

$$C_{12}^\varepsilon = \left(\frac{\nu_2}{\nu_1}\right)^k$$

and, by taking the limit, we obtain an analogue formula for C_{12}^0. The procedure can then be inductively continued.

Therefore, the developing map of L_ε takes on the form

$$\mathsf{Dev}_\varepsilon(x) = \prod_{i=0}^{j} \left(\frac{\nu_{i+1}}{\nu_i}\right)^{-k} \varphi_j^\varepsilon(x) + cte \tag{3.4}$$

for any $x \in L_\varepsilon \cap V_j$. In the previous equation ν_i is the n-th coordinate of the leaf L_ε in coordinates V_i.

Note that Formula (3.4) only holds for leaves away from the divisor of zeros and poles. However, denoting by $\Psi_n^{(i,i+1)}$ the last coordinate of the change of coordinates from V_i to V_{i+1}, what precedes implies that

$$\left(C^0_{i,i+1}\right)^{1/k} = \lim_{\varepsilon\to 0} \frac{\nu_{i+1}}{\nu_i} = \lim_{\varepsilon\to 0} \frac{\Psi_n^{(i,i+1)}(\varepsilon_2^i, \ldots, \nu_i)}{\nu_i} = \frac{\partial}{\partial x_n^i}\Big|_{\nu_i=0} \Psi_n^{(i,i+1)}(0, \ldots, 0, \nu_i).$$

We can now summarize the content of the discussion above by stating the following:

Proposition 3.1.5. *Let $\mathcal{W}$ be a simply connected and relatively compact open set of $L_0 \subseteq E$ such that $p_0 \in \mathcal{W}$. Assume that $\mathcal{W} \subseteq V_0 \cup \cdots \cup V_j$. Up to an additive constant, the developing map of the leaf L_0 restricted to $\mathcal{W}$ takes on the following form:*

$$\mathsf{Dev}_0\,|_{\mathcal{W}}(x) = \left(C^0_{\mathcal{W}}\right)^{-1} \cdot \varphi_j^0(x),$$

for any $x \in V_j \cap \mathcal{W}$, where

(a) $C^0_{\mathcal{W}} = \prod_{i=0}^{j-1} C^0_{i,i+1}$, *and*

(b) $\left(C^0_{i,i+1}\right)^{1/k} = \frac{\partial}{\partial x_n^i}\Big|_{\nu_i=0} \Psi_n^{(i,i+1)}(0, \ldots, 0, \nu_i)$.

3.2 Estimate of the image of developing maps and control of the dynamics

Let V_s be a small neighborhood of the extended singular set $\mathcal{S}_0$ and define K_E as the closure of the set $E \setminus V_s$. It is then clear that K_E is compact. Let $\{V_0, \psi_0\}, \ldots, \{V_N, \psi_N\}$ be a finite subcover of the foliated atlas $\mathscr{V}$ such that $K_E \subseteq \bigcup_{i=0}^N V_i$. For each $i \in \{0, \ldots N\}$, let $(x_1^i, \ldots x_n^i)$ be the coordinates on $\psi_i(V_i) \subseteq \mathbb{C}^n$. This subcover satisfies conditions (1) and (2) as in the beginning of the present chapter (see page 27). Note that up to taking a refinement of the cover, we may assume that X_i^R and ψ_i are defined on $\overline{V_i}$ as well.

To avoid difficulties with points lying in the boundary of $\bigcup_{i=0}^N V_i$, a minor technical adjustment will be required. Up to adjusting the sets V_i, we can assume the existence of a second cover $\{(U_i, \psi_i)\}_{i=0}^N$ of K_E satisfying the following conditions:

- For every $i = 0, \ldots N$, the set U_i is properly contained in V_i.
- The coordinate $\psi_i : U_i \to \mathbb{C}^n$ is the restriction of the coordinate $\psi_i : V_i \to \mathbb{C}^n$.

Let $K_1 = \bigcup_{i=0}^N \overline{U}_i$ and $K_2 = \bigcup_{i=0}^N \overline{V}_i$.

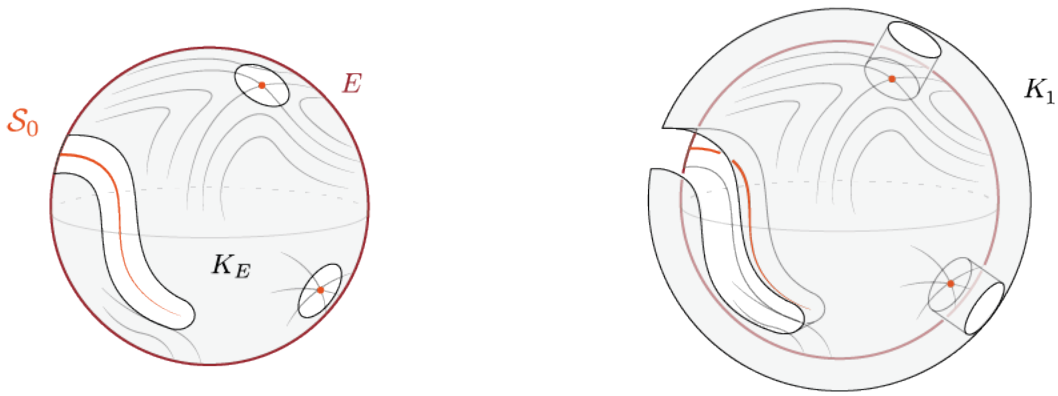


Figure 3.5

It then follows that K_1 is properly contained in K_2. Since K_1 is compact, the following lemma is standard:

Lemma 3.2.1. *There exists a uniform $\beta > 0$, called the Lebesgue number of the cover $\{V_i\}_{i=0}^N$, satisfying that for every $x \in K_1$, there is an open set V_j containing the (closed) ball of radius β and center x, i.e., $\overline{\mathcal{B}}_\beta(x) \subseteq V_j$.*

Remark 3.2.2. In particular, the following property is satisfied: for every $x \in U_i$, the ball $\overline{\mathcal{B}}_\beta(x)$ is contained in V_i. This will be used in the sequel.

On the other hand, note that the vector field X_i^R is bounded on the compact set $\overline{V}_i$. Let $||X_i^R|| = \sup\{||X_i^R(p)||_{\mathbb{C}^n} : p \in \psi_i(\overline{V}_i)\}$ and let $M_K = \max\{||X_i^R|| : i = 0, \dots N\}$. Since i varies on a finite set, then $M_K < \infty$. We will say that the collection of local renormalized vector fields $\{X_i^R\}_{i=1}^N$ is *uniformly bounded* by M_K on K_2.

Lemma 3.2.3. *There exists some uniform $\delta > 0$ such that the following holds: for every $x \in K_1$ and $t_0 \in \mathbb{C}$, there is a local vector field X_i^R (for some $i \in \{0, \dots N\}$) defined around x such that the solution Φ_i of X_i^R satisfying $\Phi_i(t_0, x) = x$ is defined on a disc of radius δ around t_0.*

Proof. Let V_i be the neighborhood of x given by Lemma 3.2.1, in particular $\mathcal{B}_\beta(x) \subseteq V_i$. On the other hand $||X_i^R|| \leq M_K$, thus the statement follows for $\delta = \beta/M_K$ due to the Existence and Uniqueness Theorem, together with the Mean Value Theorem applied to $\Phi_i(. , x)$. $\square$

In other words, for every point $x \in U_j$ (thereby $\overline{\mathcal{B}}_\beta(x) \subseteq V_j$), the time needed to reach the boundary of $\mathcal{B}_\beta(x)$ following the flow of the leaf in question is *at least* $\delta = \beta/M_K$.

Recalling that the charts defining the affine structures on leaves are the inverses of the local flows, the previous lemma can be expressed in terms of the affine atlas $\mathcal{A}_L$ (cf. Section 3.1.1). In fact, let P_j be a plaque of L contained in U_i, and denote by $\varphi_i^j = \left(\Phi_i|_{P_j}\right)^{-1}$ the chart on P_j arising from the inverse of the local flow of $X_i^R|_{P_j}$, up to translations. (Recall that in general the intersection $L \cap V_i$ is not connected, this is why a distinction of indices needs to be made). Now Lemma 3.2.3 can be reformulated as follows

Lemma 3.2.4. *There is some uniform $\delta > 0$ such that the following holds: for every $x \in K_1$ there is a open set V_i of the finite cover $\{V_k\}_{k=0}^N$ containing $x \in P_j \cap L$, such that the subset $\varphi_i^j(P_j) \subseteq \mathbb{C}$ contains a disc of radius $\delta > 0$ around $\varphi_i^j(x)$.*

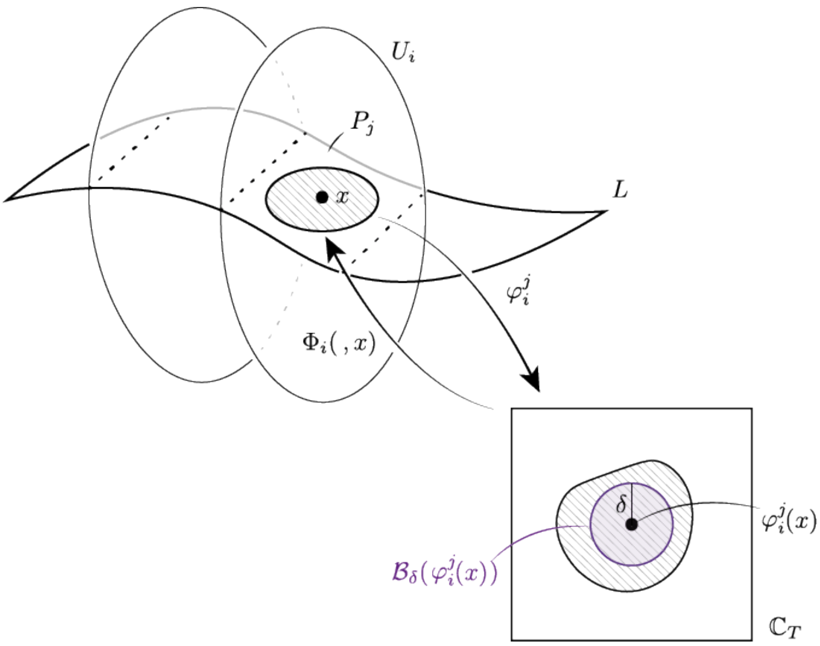


Figure 3.6 For every $x \in K_1$, $\mathcal{B}_\delta(\varphi_i^j(x)) \subseteq \varphi_i^j(P_j)$

Remark 3.2.5. We proved that, regardless of the chosen leaf, the images of the coordinate charts cannot be arbitrarily reduced. In other words they are "δ-thick".

When trying to "glue" two images of charts, the effect of the affine structure becomes apparent. Indeed, since the changes of coordinates are affine maps, the affine constants may shrink the image of Dev.

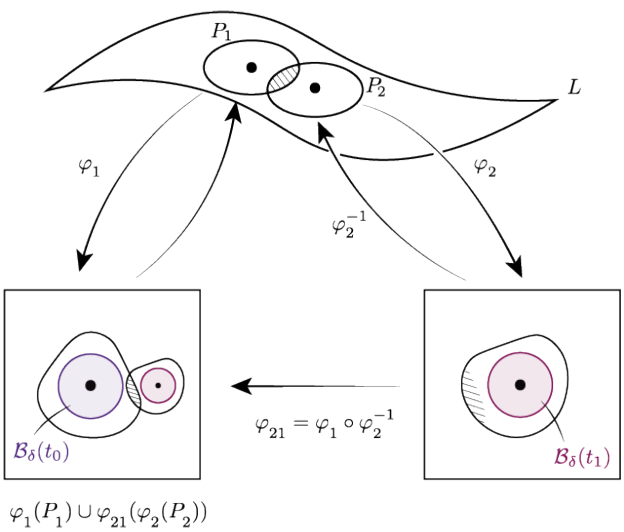


Figure 3.7 Effect of the constant C_{12}^{-1} on the discs of radius δ on a leaf L.

The purpose of the next section is to provide conditions that allow to control this potential reduction of size.

3.2.1 Controling the dynamics: foliation of codimension 1

In what follows we keep the notation of the previous section. From now on, in addition to the already established setting, we will also assume the existence of a second foliation. In fact, let $\mathcal{D}$ be a codimension 1 (singular) foliation such that

(D1) $\mathcal{F}$ is tangent to the leaves of $\mathcal{D}$,

(D2) E coincides with the closure of a (single) leaf $E_0 \subseteq E$ of $\mathcal{D}$.

Recall that at the beginning of Chapter 3 we defined $\mathcal{S}_0$ as the "extended" singular set of $\mathcal{F}$ restricted to E. To this extended singular set, we will add the singularities of the foliation $\mathcal{D}$. Namely we set

$$\mathcal{S}_0' := \mathcal{S}_0 \cup \mathrm{Sing}(\mathcal{D}).$$

Recall also that we had chosen an adapted foliated atlas suited for our work, now the existence of $\mathcal{D}$ allows one to have more control on the choice of this atlas. In fact, we can adjust the charts such that the following holds:

Lemma 3.2.6. *There is an adapted atlas of M such that around any point on $E \setminus \mathcal{S}_0'$ there are local coordinates $(x_1, \dots x_n)$ satisfying the following:*

- *Leaves of $\mathcal{D}$ are identified with hyperplanes $\{x_n = cst.\}$. In particular, E locally coincides with $\{x_n = 0\}$.*
- *The leaves of $\mathcal{F}$ are generated by $\frac{\partial}{\partial x_1}$. In other words, the vector field X is locally given by*

$$X = x_n^k h(x_1, \dots x_n) \frac{\partial}{\partial x_1}.$$

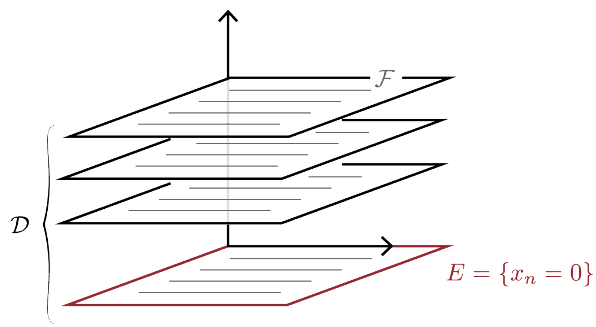


Figure 3.8 Foliated charts of the addapted atlas.

Proof. Basically, the proof consists of choosing foliated coordinates for $\mathcal{D}$ so that the first condition is satisfied. Now because $\mathcal{F}$ is tangent to $\mathcal{D}$, it becomes clear that we can find foliated coordinates for $\mathcal{F}$ satisfying also the second condition. $\square$

Let $\Psi = (\Psi_1, \dots \Psi_n)$ be the change of coordinates relative to two consecutive charts of this adapted atlas. An important fact that follows from the previous lemma, is that, besides properties 1. 2. and 3. of the proof of Proposition 2.3.9, Ψ also satisfies the following condition:

4. Ψ preserves leaves of $\mathcal{D}$. In other words, the last coordinate Ψ_n only depends on the "last variable" x_n.

Also, as in Section 3.2, we can consider a compact part K_E contained in $E \setminus \mathcal{S}'_0$ and extract a finite cover of it, namely $\{(V_i, \psi_i)\}_{i=0}^N$ and corresponding $\{(U_i, \psi_i)\}_{i=0}^N$. Note that each of these charts satisfies the conditions stated in Lemma 3.2.6. Again, as in the preceding section, each set U_i in the atlas in question is endowed with $(x_1^i, \dots x_n^i)$ coordinates.

Now fix again $p_0 \in U_0 \cap E$. Let $L_0 \subseteq E$ be the leaf of $\mathcal{F}$ passing through p_0 and let q be another point in L_0. Then, owing to Proposition 3.1.5, the following equality holds

$$\mathsf{Dev}_0 |_{\mathcal{W}}(q) = \left(C^0_{\mathcal{W}}\right)^{-1} \cdot \varphi_j^0(q).$$

For our purposes, we will be only interested in the constant $(C^0_{\mathcal{W}})^{-1}$ and, more precisely in the "map" that assigns to every simply connected set $\mathcal{W}$ the constant $(C^0_{\mathcal{W}})^{-1}$. This can alternatively be formulated as a map assigning to every path γ with starting point p_0, the constant $(C^0_{\mathcal{W}})^{-1}$. In the sequel, we will mainly denote this constant by C^0_γ, as opposed to $(C^0_{\mathcal{W}})^{-1}$. In other words, we set

$$C^0_\gamma := \left(C^0_{\mathcal{W}}\right)^{-1}.$$

On the other hand, since $E_0 = E \setminus \mathcal{S}'_0$ is a leaf of $\mathcal{D}$, one can also consider the "parallel transport" along leaves of $\mathcal{D}$ associated with paths $\gamma \subseteq E \setminus \mathcal{S}'_0$. Taking advantage of the collection $(x_1^i, \dots, x_n^i)$ of local coordinates previously fixed in connection with the atlas $\{(U_i, \psi_i)\}_{i=0}^N$, to every path $\gamma \subseteq E \setminus \mathcal{S}'_0$ we can associate a local diffeomorphism

$$h_\gamma : (\mathbb{C}, 0) \to (\mathbb{C}, 0).$$

The connection between the constants C^0_γ and the map h_γ will be made clear in the next proposition. For its statement, L_0 denotes a leaf of $\mathcal{F}$ contained in E,

Lemma 3.2.7. *Let $\gamma \subseteq L_0 \cap E_0$ be a path with initial point $p_0 \in L_0$, and $\tilde{\gamma}$ its lift into a leaf L_ε of $\mathcal{F}$ which is not contained in E. Then the constants $C^\varepsilon_{\tilde{\gamma}}$ (resp. C^0_γ) and the local diffeomorphism h_γ satisfy the following relations:*

(a) $C^\varepsilon_{\tilde{\gamma}} = (h_\gamma(\nu_0))^{-k}$,

(b) $C^0_\gamma = (h'_\gamma(0))^{-k}$,

where ν_0 stands for the $n-$th coordinate within the chart U_0 containing p_0.

Proof. The path $\gamma \subseteq L_0$ is covered by a finite number of open sets belonging to the adapted atlas $\{(U_i, \psi_i)\}_{i=0}^N$, namely $U_0, \dots U_j$. Then γ can be split as a concatenation of small paths $\gamma = \gamma_0 * \cdots * \gamma_{j-1}$ so that there is a partition of the interval $[0, 1]$, denoted by $\{t_0 = 0, t_1, \dots t_j = 1\}$, satisfying the following:

- For every $i \in \{0, \dots j-1\}$, $\gamma_i : [t_i, t_{i+1}] \to E_0$ coincides with γ restricted to $[t_i, t_{i+1}]$ and $\gamma_i([t_i, t_{i+1}]) \subseteq U_i$.
- $\gamma_0(t_0) = p_0$, $\gamma_j(t_j) = q$ and for every $i \in \{1, \dots j-1\}$, the point $\gamma_i(t_i)$ lies in the intersection $U_{i-1} \cap U_i$.

Using, as always, the notation of Section 2.3, the change of coordinates from consecutive foliated charts, U_i to U_{i+1}, is denoted by $\Psi_{i,i+1} = (\Psi_1^{(i,i+1)}, \dots, \Psi_n^{(i,i+1)})$ and it satisfies the properties (1) to (4) stated before. Since for this adapted foliated atlas, the leaves of the foliation $\mathcal{D}$ are hyperplanes (given by $\{x_n^i = cst.\}$ on each U_i), it then follows that the parallel transport associated $\gamma \in E \setminus \mathcal{S}_0'$ is defined as being the variation of the last coordinate map $\Psi_n^{(i,i+1)}$, which, as mentioned in item (4), only depends on x_n^i. In other words, the transition map from the transverse sections $\Sigma_{\gamma_i(t_i)}$ to $\Sigma_{\gamma_{i+1}(t_{i+1})}$ read in coordinates, is nothing but $\Psi_n^{(i,i+1)}$. Consequently, the parallel transport of leaves of $\mathcal{D}$ sufficiently close to E, is given by the composition of consecutive changes of the last coordinates

$$h_\gamma = \Psi_n^{(j-1,j)} \circ \dots \circ \Psi_n^{(0,1)}. \tag{3.5}$$

Part (a) of the statement follows by considering the lift $\tilde{\gamma}$ of γ to L_ε, and from Equation 3.5 together with the very definition of the constants $C_{\tilde{\gamma}}^{\varepsilon}$ (cf. Proposition 2.3.9 and Section 3.1.1).

As for part (b), since the divisor E is identified with $\{x_n^i = 0\}$, it follows that $\Psi_n^{(i,i+1)}(0) = 0$ for every $i \in \{0, \dots j-1\}$. Therefore, by a direct application of the chain rule, the derivative of this composition of maps coincides with the product of derivatives evaluated at 0,

$$h'_\gamma(0) = \prod_{i=0}^{j-1} \left(\Psi_n^{(i,i+1)}\right)'(0). \tag{3.6}$$

On the other hand, after getting rid of the dependence on the first $n-1$ variables for the last coordinate $\Psi_n^{(i,i+1)}$, we may write the formula of the constant C_γ^0 obtained in Proposition 3.1.5 but adapted to this new foliated atlas as follows:

$$\left(C_{i,i+1}^0\right)^{1/k} = \frac{\partial}{\partial x_n^i}\Big|_{\nu_i=0} \Psi_n^{(i,i+1)}(\nu_i) = \left(\Psi_n^{(i,i+1)}\right)'(0).$$

Hence, the constant C_γ^0 takes on the simpler form

$$C_\gamma^0 = \left(C_{\mathcal{W}}^0\right)^{-1} = \left(\prod_{i=0}^{j-1} C_{i,i+1}^0\right)^{-1} = \left(\prod_{i=0}^{j-1} \left(\Psi_n^{(i,i+1)}\right)'(0)\right)^{-k}.$$

It is now clear that $C_\gamma^0 = \left(h'_\gamma(0)\right)^{-k}$ and the proof of the lemma is complete.

□

Now let us consider the following situation: assume that $L_0 \cap U_0$ is a non connected set, in other words, the leaf L_0 intersects several times the open set U_0. Let $q \in L_0 \cap U_0$ be a point that does not belong to the same connected component as p_0, and let $\gamma \subseteq K_E \cap L_0$ be a path connecting p_0 and q. Owing to Lemma 3.2.7, we have the equality $C^0_\gamma = (h'_\gamma(0))^{-k}$. On the other hand, one can extend the path γ so as to obtain a loop by adding to γ a small segment. Indeed, let us define the concatenation

$$\hat{\gamma} = \gamma * \gamma_{q,p_0}$$

where γ_{q,p_0} is a small enough path connecting q to p_0 and such that $\gamma_{q,p_0} \subseteq U_0 \cap E$. Note that γ_{q,p_0} does not lie anymore within L_0. However, since $q, p_0 \in U_0$ and γ_{q,p_0} is contained in $U_0 \cap E$, then h_γ coincides with $h_{\hat{\gamma}}$ (see Figure 3.9). Therefore

$$h'_{\hat{\gamma}}(0) = h'_\gamma(0) = \left(C^0_\gamma\right)^{-1/k}.$$

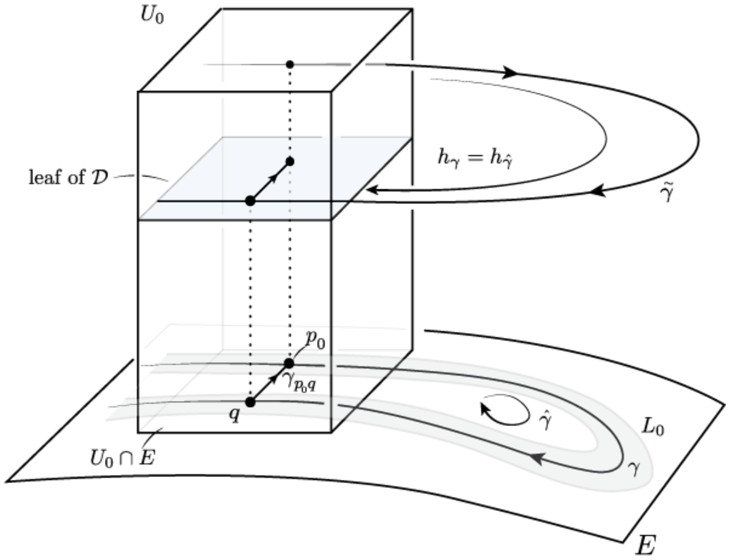


Figure 3.9 The leaf L_0 returns to $U_0 \cap E$ and $h_\gamma = h_{\hat{\gamma}}$.

The above discussion sheds light on the fact that the behavior of the affine constants arising from the developing map of L will only rely on the holonomy group of the leaf E_0 of the codimension 1 foliation $\mathcal{D}$. Let us denote by $\mathrm{Hol}(E_0, p_0) \subseteq \mathrm{Diff}(\mathbb{C}, 0)$ (or simply by $\mathrm{Hol}(E_0)$), the holonomy group of the leaf E_0 based at p_0, and let $\mathrm{Hol}'(E_0, p_0) \subseteq (\mathbb{C}, 0)$ be the subgroup of $\mathbb{C}$ defined by:

$$\mathrm{Hol}'(E_0, p_0) := \{h(\gamma)'(0) : \gamma \in \pi_1(E_0, p_0)\}.$$

To be able to control the behavior of the constants arising from the affine structure of the leaves, in the sequel we will impose that the derivative of every holonomy map has norm 1 at p_0, i.e.,

(H1) *Condition Holonomy of $\mathcal{D}$:* Assume that $\mathrm{Hol}'(E_0, p_0) \subseteq S^1$.

Although this assumption is certainly not a generic one, it is of particular interest to us since, as shown in Chapter 4, vector fields possessing a first integral satisfy conditions

(D1), (D2) settled at the beginning of Section 3.2.1, and (H1) as well. Also note that a large amount of potential interesting examples arise in the context of physics, where such systems often exhibit constants of motion (e.g. Hamiltonian systems).

We can now state the final propositions of this section, which summarize the construction carried out so far. First, recall that we had defined two covers $\{U_i\}_{i=0}^N$ and $\{V_i\}_{i=0}^N$ of the compact set K_E so that each U_i is properly contained in V_i and satisfying the requirements described at the beginning of Section 3.2.

To abridge notation in the discussion below, we will drop the sub-index 0 of L_0. Thus, provided that there is no risk of misunderstanding, L will denote the leaf of $\mathcal{F}$ contained in E, and Dev will denote the (monodromy) developing map of the leaf L in question. Given a point $x \in L \cap K_E$, let U_j be the open set for which $x \in U_j$, we recall that P_x denotes the plaque that contains x within V_j. Next, we state:

Proposition 3.2.8. *With the same notation as above, assume that $Hol'(E_0, p_0) \subseteq S^1$. Then, there exists a uniform $\tilde{\delta} > 0$ such that the following holds: for every $x \in L \cap K_E$ the image under Dev of the plaque P_x, contains a disc of radius $\tilde{\delta}$ around $\mathsf{Dev}(x)$. In other words:*

$$\mathcal{B}_{\tilde{\delta}}(\mathsf{Dev}(x)) \subseteq \mathsf{Dev}(P_x).$$

Proof. Let x be a point in $L \cap K_E$ and assume that $x \in U_j$, then by Remark 3.2.2, we have $\overline{\mathcal{B}}_\beta(x) \subseteq V_j$. Let $\gamma \subseteq L$ be an injective path contained in L joining p_0 and x. For every $y \in \partial\mathcal{B}_\beta(x) \subseteq P_x$, one has

$$|\,\mathsf{Dev}(x) - \mathsf{Dev}(y)| \quad = \quad |C_\gamma| \cdot |\varphi_j^{P_x}(y) - \varphi_j^{P_x}(x)|$$

where $\varphi_j^{P_x}$ is the chart defined in V_j restricted to the plaque P_x and C_γ denotes the constant arising from the definition of the developing map following the path γ.

By Lemma 3.2.4 $|\varphi_j^{P_x}(y) - \varphi_j^{P_x}(x)|$ is bounded from below by δ. Moreover according to Lemma 3.2.7, the affine constant arising from the developing map C_γ is expressed in terms of the holonomy of E along γ, namely we have $C_\gamma = \left(h'_\gamma(0)\right)^{-k}$. This said, to prove the proposition it then suffices to show that the derivative of this holonomy is also uniformly bounded.

Being K_E compact, any two points can be connected by a path of minimal length in K_E. Let $\sigma \subseteq K_E$ such a minimizing path with extreme points $\sigma(0) = x$ and $\sigma(1) = p_0$ and consider the concatenation of paths $\gamma = \gamma * \sigma * \sigma^{-1}$. It then follows that the holonomy of E along γ can be split as $h_\gamma = h_{\sigma^{-1}} \circ h_{\gamma*\sigma}$ and hence

$$h'_\gamma(0) = h'_{\sigma^{-1}}(0) \cdot h'_{\gamma*\sigma}(0).$$

Since $\gamma * \sigma \in \pi_1(E_0, p_0)$ and $\mathrm{Hol}'(E_0, p_0) \subseteq S^1$, it follows that $|h'_\gamma(0)| = |h'_{\sigma^{-1}}(0)|$. The proposition is reduced to proving that $|h'_{\sigma^{-1}}(0)|$ can be bounded by a constant that does not depend on the chosen minimal path $\sigma \subseteq K_E$.

Note that since σ is of minimal length, it can intersect each open set U_i of the finite cover $\{U_0 \ldots, U_N\}$ at most once. This fact together with Formula (3.5), implies that the map $h_{\sigma^{-1}}$

is obtained as a composition of, at most, $N-1$ changes of coordinates $\Psi_n^{(i,j)}$. Define the constants

$$\mu = \min\{|\Psi_n^{(i,j)'}(0)| : i,j \in \{0,\dots,N\} \text{ such that } U_i \cap U_j \neq \emptyset\}.$$

and

$$\mu_1 = \min\{\mu^l : 1 \leq l \leq N-1\}.$$

In turn, denote by $\tilde{\mu}$ and μ_2 the respective maxima. It follows that $\mu_1 \leq |h'_{\sigma^{-1}}(0)| \leq \mu_2$. Note that μ_1 and μ_2 only depend on the choice of the finite cover $\{U_0 \dots, U_N\}$. Consequently:

- if E is an irreducible component of the divisor of poles $(X)_\infty$, then $-k > 0$ and hence $|h'_\gamma(0)|^{-k} \geq \mu_1^{-k}$.
- if E is an irreducible component of the divisor of zeros $(X)_0$, then $-k < 0$ and therefore $|h'_\gamma(0)|^{-k} \geq \mu_2^{-k}$.

The constants $\tilde{\delta} := \mu_1^{-k} \cdot \delta$ (resp. $\tilde{\delta} := \mu_2^{-k} \cdot \delta$) then satisfy the assertion in the statement of the proposition.

□

We present, as a corollary, an alternative way of stating Proposition 3.2.8. For this, recall that $\hat{L}$ stands for the monodromy cover of the leaf L. In this cover, the (monodromy) developing map $\mathsf{Dev} : \hat{L} \to \mathbb{C}$ with base point p_0 is well-defined (cf. Section 3.1.1).

Corollary 3.2.9. *With the same notation as above, assume that $Hol'(E_0, p_0) \subseteq S^1$. Then, there exists an uniform $\tilde{\delta} > 0$ such that for every point $x \in L \cap K_E$ one has*

$$\mathcal{B}_{\tilde{\delta}}(\mathsf{Dev}(x)) \subseteq \mathsf{Dev}(\hat{L}).$$

To close this section, we provide an analogous proposition corresponding to leaves lying outside the divisor of zeros and poles of X.

Proposition 3.2.10. *Let L_ε be a leaf that is not contained in $(X)_\infty \cup (X)_0$. Assume that $L_\varepsilon \subseteq \mathcal{V}_E$ where $\mathcal{V}_E$ stands for a neighborhood of E. Then for every $x \in K_1 \cap L_\varepsilon$,*

$$\mathcal{B}_{r_x}(\mathsf{Dev}_\varepsilon(x)) \subseteq \mathsf{Dev}_\varepsilon(\hat{L})$$

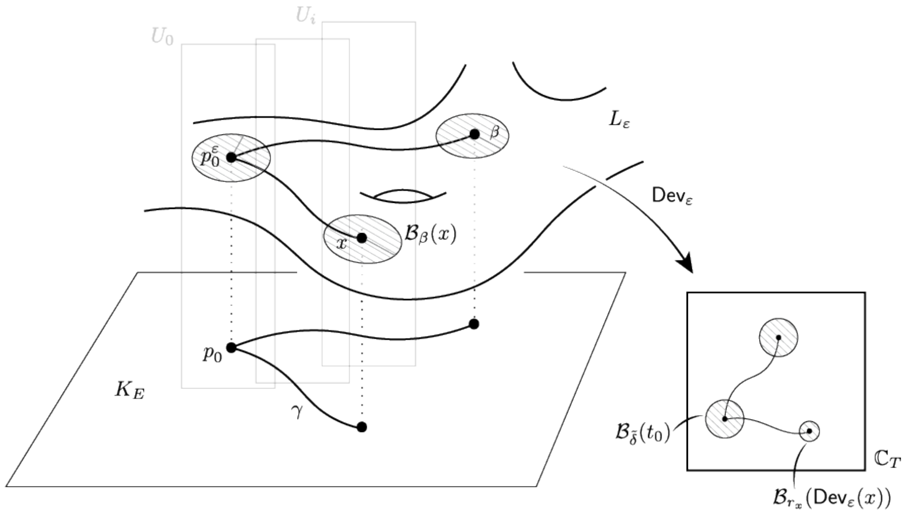
where $r_x = |C^\varepsilon_{\tilde{\gamma}}| \cdot \delta$ and $\tilde{\gamma}$ is a path from p_0^ε to x.

Proof. We rely on the notation of Proposition 3.2.8. By Remark 3.2.2, let V_j be such that $\mathcal{B}_\beta(x) \subseteq V_j$, then for any $y \in \partial\mathcal{B}_\beta(x)$, we have

$$|\,\mathsf{Dev}_\varepsilon(x) - \mathsf{Dev}_\varepsilon(y)| \;=\; |C^\varepsilon_{\tilde{\gamma}}| \cdot |\varphi_j^{P_x}(y) - \varphi_j^{P_x}(x)| \;\geq\; |C^\varepsilon_{\tilde{\gamma}}| \cdot \delta \;=\; r_x$$

This completes the proof.

□



U_0
U_i
L_ε
p_0^ε
β
Dev_ε
x
$\mathcal{B}_\beta(x)$
p_0
K_E
γ
$\mathcal{B}_{\tilde{\delta}}(t_0)$
$\mathbb{C}_T$
$\mathcal{B}_{r_x}(\mathsf{Dev}_\varepsilon(x))$

Figure 3.10

Chapter 4

Main Results

4.1 Preliminaries: Semicomplete vector fields

We keep the notation and the global setting as before. Namely, $\mathcal{F}$ is a one dimensional foliation induced by a meromorphic vector field X, and E an irreducible component of the zero (pole) divisor of X invariant by $\mathcal{F}$. Besides, $\mathcal{D}$ stands for a codimension-1 foliation satisfying conditions $(D1)$ and $(D2)$ as in Section 3.2.1. Finally, as mentioned at the end of the previous section, we will assume that condition $(H1)$, related to the holonomy of E_0, is satisfied.

In the sequel, we shall be concerned with semicomplete vector fields. These vector fields were introduced by J. Rebelo in [40] and they are defined as follows:

Definition 4.1.1. Let X be a holomorphic vector field defined on an open set U of a complex manifold M. Then X is said to be semicomplete on U, if for every $p \in U$ there exists a connected domain $\Omega_p \subseteq \mathbb{C}$ with $0 \in \Omega_p$ and a map $\Phi_p : \Omega_p \to U$ satisfying the following conditions:

1. $\Phi_p(0) = p$ and $\Phi_p'(t) = X(\Phi_p(t))$ for every $t \in \Omega_p$.
2. For every sequence $\{t_n\} \subseteq \Omega_p$ that converges to some $\hat{t} \in \partial\Omega_p$, the sequence $\{\Phi_p(t_n)\}$ escapes from every compact subset of U.

The definition is equivalent to saying that a vector field is semicomplete in U if for every $p \in U$ the integral curve Φ_p satisfying $\Phi(0) = p$ has a maximal domain of definition Ω_p *contained in* $\mathbb{C}$. This notion is extended to meromorphic vector fields simply by considering the complement of the pole divisor (see [23]).

Remark 4.1.2. Definition 4.1.1 can be formulated in a few equivalent ways. In fact X is semicomplete if and only if the developing map of every leaf L on which the vector field is well defined, is injective. See [40] and [41].

The above notion has a few properties that should be pointed out. In fact, note that if X is a complete vector field on M, then it is in particular semicomplete on every open subset

$U \subseteq M$. More generally speaking, if X is semicomplete on U and $V \subseteq U$ is also an open subset, then X is semicomplete on V. The later allows us to define de notion of germ of semicomplete vector field of X.

Definition 4.1.3. Let X be a vector field defined an open set U of M and let $p \in U$ be a singular point of X. Then p is said to be a semicomplete singularity of X if there is an open neighborhood of p, $V \subseteq U$ such that X is semicomplete on V.

Although the definition of a meromorphic vector field does not directly take into account its pole (or zero) divisor, being semicomplete infers constraints on the geometry associated with the underlying foliation on this divisor. In fact, it is easy to check that every irreducible component of $(X)_\infty$ is invariant under the underlying foliation $\mathcal{F}$ provided that X is semicomplete (see [23]). In particular, every leaf L of $\mathcal{F}$ contained in $(X)_\infty$ is endowed with an affine structure induced by X as in Section 2.3.1. The analogous statement holds for every component of $(X)_0$ that happens to be invariant by $\mathcal{F}$. To exploit the connection between the affine structures and the semicomplete nature of X, the following definition is needed.

Definition 4.1.4. A singular affine structure on a Riemann surface S is said to be uniformizable if it gives rise to a monodromy developing map $\mathsf{Dev} : \hat{S} \to \mathbb{C}$ that is injective.

Remark 4.1.5. For leaves not contained in $(X)_0 \cup (X)_\infty$, the singular translation structure yields a "preferred" developing map. It is defined as the integration along paths $\gamma : [0,1] \to L$ of the *time-form*, that is, the unique one 1-form denoted by $dT|_L$ satisfying $dT|_L(X) \equiv 1$ (see [40]);

$$\int_\gamma dT|_L.$$

Note that since two developing maps arising from a same affine structure on a Riemann surface coincide up to composition with an affine map (cf. Remark 3.1.1), the injectivity of $\int_\gamma dT$ implies the injectivity of Dev and conversely.

In the sequel, by a leaf $L \subseteq E$ of $\mathcal{F}$ it is meant an actual leaf of $\mathcal{F}$ viewed as a foliation in the ambient n-dimensional space, as opposed to a leaf of the foliation induced by $\mathcal{F}$ on E. Indeed recall that $\mathrm{Sing}(\mathcal{F}) \cap E$ may have components of dimension $n-2$. Since the singular set of a foliation has codimension at least 2, if we look at the restriction of $\mathcal{F}$ to E "from within E", i.e., for the foliation induced on E by $\mathcal{F}$, then this foliation extends to generic points of $\mathrm{Sing}(\mathcal{F}) \cap E$.

Let E denote an irreducible component of the divisor of zeros and poles of a vector field X (not necessarily semicomplete). The following proposition can be found in [23], see also [19]. We present an alternative version of it, and for sake of completeness we include a sketch of the proof.

Proposition 4.1.6 (cf. [23], [19]). *Assume that E is invariant by the foliation $\mathcal{F}$ associated with X. If X is semicomplete in a neighborhood of E, then the affine structure induced by X on every leaf $L \subseteq E$ of $\mathcal{F}$, is uniformizable.*

Proof. In fact, with the notation of section 3.1.2, let L_0 be a leaf contained in E, and let $\mathcal{W} \subseteq L \setminus \Lambda$ be a simply connected set containing p_0. By Lemma 3.1.4, the normalized developing maps of close enough leaves $\mathsf{Dev}^{(\varepsilon)}$, converge uniformly on $\mathcal{W}$ to Dev_0. Since X is semicomplete on a neighborhood of E, then $\mathsf{Dev}^{(\varepsilon)}$ is injective on $\mathcal{W}$. The statement then follows from the facts that the uniform limit of a sequence of injective maps is injective provided that it is not constant.

□

Before concluding this preliminary section, we recall a well-known result regarding complete vector fields on compact manifolds. See, for instance, [30] for a proof.

Theorem 4.1.7. *Let X be a holomorphic vector field on a compact manifold M. Then X is complete.*

4.2 Theorem I

Theorem 4.2.1 (Theorem I)**.** *Let X be a meromorphic vector field on a compact manifold M and denote by $\mathcal{F}$ the foliation associated with X. Consider an irreducible component E of $(X)_0 \cup (X)_\infty$ that is invariant under $\mathcal{F}$. Let $\mathcal{D}$ be a codimension-1 foliation satisfying conditions $(D1)$ and $(D2)$ (see Section 3.2.1), and assume also that*

- *$Hol'(E_0, p_0) \subseteq S^1$*
- *X is semicomplete on a neighborhood of $\mathcal{S}_0' \cap E$.*

Then the affine structure of every leaf $L \subseteq E$ is uniformizable.

Note that, in principle, it is difficult to construct examples of foliations with uniformizable affine structures on its leaves due to the need of controlling the global behavior of the mentioned leaves. Along the same lines, it is difficult to ensure that a vector field is semicomplete on a neighborhood of an invariant hypersurface. In this direction, the most of the interest of Theorem 4.2.1 lies in the fact that it ensures that foliated affine structures are uniformizable on an invariant hypersurface out of local information localized at the (extended) singular set $\mathcal{S}_0' \cap E$. We emphasize the fact that this set is "small" in the sense that it has codimension larger than 2.

Let us begin our approach to Theorem 4.2.1. Let $\mathcal{V}_E$ be a neighborhood of E and let $\mathcal{V}_s^1 \cup \dots \mathcal{V}_s^l$ be a fixed neighborhood of the "extended singular set" $\mathcal{S}_0'$ (cf. Section 3.2.1) in such a way each $\mathcal{V}_s^i$ is an open set of M contained in $\mathcal{V}_E$ and such that X is semicomplete on $\mathcal{V}_s^i$. Let $K_E = E \setminus \left(\bigcup_{i=0}^{l} \mathcal{V}_s^i \right)$ and, as before, $\{U_i\}_{i=0}^{N}$ will stand for the adapted atlas covering K_E (cf. Lemma 3.2.6).

Let $L \subseteq E$ be a leaf of $\mathcal{F}$ where the affine structure induced by X is well defined, $p_0 \in L \cap K_E$ and consider the monodromy cover $\hat{L}$ and its monodromy developing map $\mathsf{Dev} : \hat{L} \to \mathbb{C}$ with base point p_0. The monodromy cover $\hat{L}$ comes along with the usual covering projection $\rho : \hat{L} \to L$. Denote the image of $\hat{L}$ through Dev by

$$\Omega := \mathsf{Dev}(\hat{L}).$$

Let $L_{K_E} := L \cap K_E$ and let $\hat{L}_{K_E}$ be its lifting to the monodromy cover $\hat{L}$. The image of $\hat{L}$ under Dev can be split as

$$\Omega = \Omega_r \cup \Omega_s, \tag{4.1}$$

where

- $\Omega_r = \mathsf{Dev}(\hat{L}_{K_E})$ is the image of the "regular" part of the leaf, in the sense that the projection of $\hat{L}_{K_E}$ in L lies away from the singular set $\mathcal{S}_0$,
- $\Omega_s = \Omega \setminus \Omega_r$ denotes the image of the part of the leaf that lies in a neighborhood of the singular set.

Note that Ω is an open set of $\mathbb{C}$. We are interested in the existence (or more precisely on the non-existence) of points of $\partial\Omega$.

Let us give a more precise description of $\partial\Omega$. Points of $\partial\Omega$, in principle, may lie in

(a) $\partial\Omega_s$

(b) or $\partial\Omega_r$.

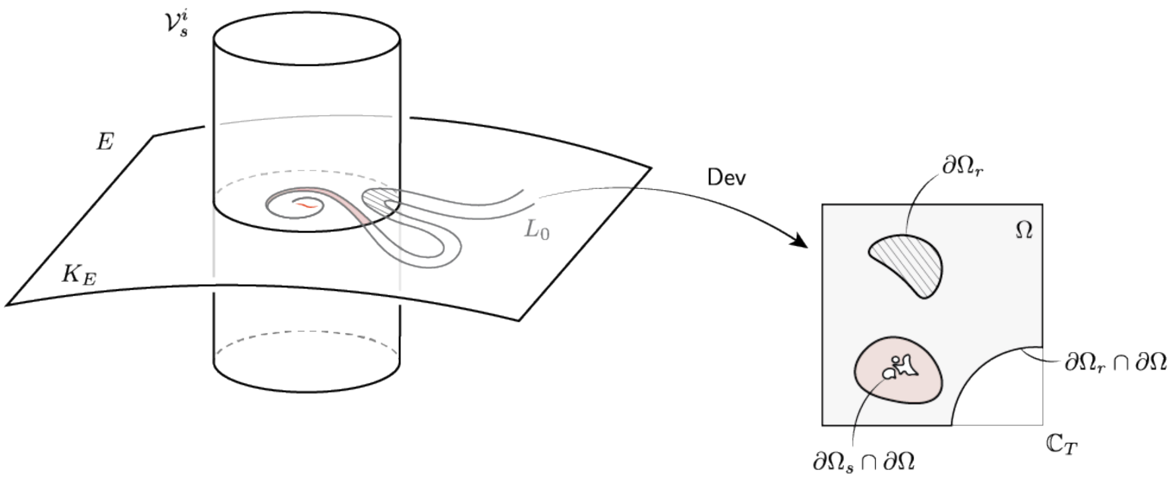


Figure 4.1 Representation of points in $\partial\Omega$ for cases (a) and (b).

Note that the above items are not mutually exclusive situations. Nonetheless, in the present setting, the lemma below shows that item (b) can be ruled out.

Lemma 4.2.2. *Under the assumption of Theorem 4.2.1, points in $\partial\Omega$ of type (b) do not exist.*

Proof. Suppose, aiming a contradiction, that t_∞ is a point in $\partial\Omega$ of type (b), i.e., $t_\infty \in \partial\Omega \cap \partial\Omega_r$, and consider the neighborhood $\mathcal{B}_{\tilde{\delta}/2}(t_\infty)$, where $\tilde{\delta}$ is the uniform constant of Corollary 3.2.9. Let $t \in \mathcal{B}_{\tilde{\delta}/2}(t_\infty) \cap \Omega_r$ and $x \in L \cap K_E$ such that $\mathsf{Dev}(x) = t$. Then, by Proposition 3.2.9, we have

$$\mathcal{B}_{\tilde{\delta}}(\mathsf{Dev}(x)) \subseteq \Omega.$$

This contradicts the fact that t_∞ lies in $\partial\Omega$. $\square$

Remark 4.2.3. A by-product of the combination of Lemma 4.2.2 and Corollary 3.2.9, is that for any point $t \in \partial\Omega_r$, a disc of size $\tilde{\delta}$ centered at t is contained in Ω, i.e., $\mathcal{B}_{\tilde{\delta}}(t) \subseteq \Omega$.

This implies that points in $\partial\Omega$ are not only contained in $\overline{\Omega}_s$, but that they stay away (actually a $\tilde{\delta}$-apart) from the common border $\partial\Omega_r \cap \partial\Omega_s$. Now we are ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. In order to show that $\mathsf{Dev} : \hat{L} \to \mathbb{C}$ is injective, we will construct a well-defined inverse map Φ for Dev from Ω to $\hat{L}$, where $\Omega \subseteq \mathbb{C}$ stands for the image of Dev. The construction of $\Phi : \Omega \to \hat{L}$ will be based on the analytic continuation of an initial germ. First, recall that Dev is locally invertible on $\hat{L}$, and that by construction, its local inverses are the (local) flows of the renormalized vector fields adjusted by suitable multiplicative constants (Section 2.3.1).

Let (Φ_0, D_0) be the local map around $0 \in \mathbb{C}$, where Φ_0 denotes the local inverse of Dev in a neighborhood of p_0, and D_0 is a sufficiently small disc centered at $0 = \mathsf{Dev}(p_0)$. Since Ω is the image of $\hat{L}$ under Dev, it follows that Φ_0 admits an analytic continuation along paths on the whole Ω. The proof then amounts to showing that the latter is univalued. In other words, we want to prove that the analytic continuation of Φ_0 along every loop in Ω based at 0 locally coincides with Φ_0.

For this, let $\gamma \in \pi_1(\Omega, 0)$.

- If γ is homotopic in Ω to the constant path, then by the Monodromy Theorem, the analytic continuation of Φ_0 along γ is not multivalued.
- If γ is *not* homotopic to the constant path, by Lemma 4.2.2 it follows that γ needs to wind around a subset of $\partial\Omega \cap \partial\Omega_s$.

We place ourselves in the second scenario. Note that Ω_s may have an infinite number of disjoint connected components depending on how many times the leaf L enters and exits a singular neighborhood. We can split $\Omega_s = \bigcup_{i \in I} \Omega_{s_i}$, where Ω_{s_i} stands for a connected component of Ω_s, the cardinality of I can be finite or infinite. Then $\partial\Omega_{s_j} \cap \partial\Omega_{s_i} = \emptyset$. Denote the bounded component surrounded by γ as $\mathrm{int}(\gamma)$ and define the set

$$B = \{t \in \partial\Omega | t \in \mathrm{int}(\gamma)\}.$$

We will show that the number of components Ω_{s_i} surrounded by γ whose closure contains points of $\partial\Omega$, is finite.

Claim 4.2.4. *There is some $N \in \mathbb{N}$ such that $B \cap \overline{\Omega}_{s_i} = \emptyset$ for $i > N$.*

Proof of the claim. Assume the contrary. Let t_i be a sequence of points $t_i \in \overline{\Omega}_{s_i} \cap \partial\Omega \cap \mathrm{int}(\gamma)$. Being $\overline{\mathrm{int}(\gamma)}$ a compact set, there must exist a point $t \in \overline{\mathrm{int}(\gamma)}$ to which t_n (or a subsequence of it) converges. Note that since $t_n \in \partial\Omega$, then $t \in \partial\Omega$.

Since each point of the sequence $\{t_n\}$ is contained in a different singular component and these components have disjoint closures, it follows that t should also be accumulated by points in Ω_r. This implies that $t \in \partial\Omega_r$ contradicting Lemma 4.2.2.

□

In other words, γ can only wind around a finite number of "holed" components of Ω_s. Assume first that γ winds around a unique holed connected component of Ω_s, say Ω_s^1. Therefore, by Lemma 4.2.2 and Remark 4.2.3, γ can be homotopically deformed into a path $\tilde{\gamma} = \sigma * \eta * \sigma^{-1}$ where:

- σ is an open path with $\sigma(0) = t_0$ and $t_1 = \sigma(1) \in \Omega_s^1$.
- η is a loop contained in Ω_s^1 based at $t_1 = \sigma(1)$ homotopic to $\partial\Omega_s^1$.

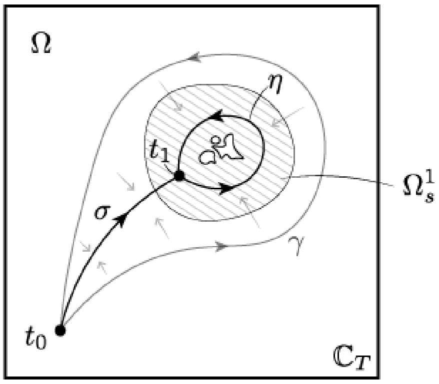


Figure 4.2 The path $\tilde{\gamma} = \sigma * \eta * \sigma^{-1}$.

Denote by (Φ_1, D_1) the local map obtained by analytic continuation of (Φ_0, D_0) along σ, it coincides with the local inverse of Dev in a small disk around t_1, $D_1 \subseteq \Omega_s^1$. Note that the analytic continuation along the trivial path $\sigma * \sigma^{-1}$ leads to the same germ of (Φ_0, D_0). Therefore, to show that the analytic continuation along $\sigma * \eta * \sigma^{-1}$ is not multivalued, it is enough to prove that the analytic continuation of Φ_1 along η defines the same germ as (Φ_1, D_1) at t_1. This is, however, a consequence of the assumption that X is semicomplete on $\mathcal{V}_s^1$. Indeed, as according to Proposition 4.1.6, the developing map of L restricted to $\mathcal{V}_s^1$ is injective, hence invertible. Its inverse $\Phi : \Omega_s^1 \to \hat{L}$ needs to coincide with Φ_1 on D_1 and thus yields the analytic continuation of Φ_1 along η.

If γ winds around a finite number of holed connected components of Ω_s, the same argument applies by deforming homotopically γ into a finite number of paths, each of them having the form described above, and will wind around an unique component of Ω_s.

This shows that the analytic continuation of Φ_0 along *any* loop $\gamma \in \pi_1(\Omega, 0)$ leads to a unique univalued map $\hat{\Phi} : \Omega \to \hat{L}$ satisfying $\hat{\Phi}(0) = p_0$, by construction $\hat{\Phi}$ is the inverse of Dev. This ends the proof.

□

4.3 Theorem II

Before stating the theorem, we need to define the type of singular set we will be concerned with. This will require an analysis of the possible situations that might occur in a neighborhood of the singular set.

4.3.1 Description of the behavior of a leaf nearby a singularity.

Consider the extended singular set $\mathcal{S}_0'$ (see 3.2.1), and let Υ be a connected component of $\mathcal{S}_0'$. To simplify notation, in what follows, $\mathcal{V}_s$ stands for the corresponding component of the singular neighborhood that contains Υ (cf. the notation on p. 47). Let L be a leaf of $\mathcal{F}$ that intersects the neighborhood $\mathcal{V}_E$ of E and denote

$$L_{\text{sing}} = L \cap \mathcal{V}_s \quad \text{and} \quad L_{\text{reg}} = L \cap K_1.$$

The above sets stand for the singular and the regular parts of the leaf L, respectively. Besides, note that L_{sing} may consist of a disjoint (maybe infinite) union of connected components of the leaf (see Figure 4.3). We will denote the latter as follows

$$L \cap \mathcal{V}_s = \bigsqcup_{i \in I} L_i,$$

with I a set that is at most countable.

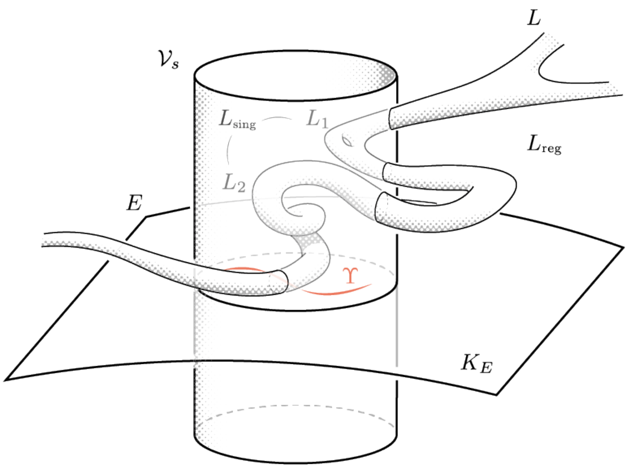


Figure 4.3

Now assume that L approaches E in the sense that $d(E, L) = 0$ where d denotes the distance induced by an auxiliary Riemannian metric on M. Then the following (non-exclusive) situations may take place:

Case 1. L approaches E through the regular part, i.e. $d(L_{\text{reg}}, E) = 0$. If so, L must intersect an infinite number of times at least one of the sets of the open cover $\{U_i\}_{i=0}^N$.

Case 2. L gets close to E through the singular part, i.e., $d(L_{\text{sing}}, E) = 0$. In this case, two scenarios may arise:

i. There is a path $\gamma \subseteq L_{\text{sing}}$, $\gamma : [0, 1) \to L$ converging to Υ, i.e., such that $d(\gamma(t), \Upsilon) \to 0$ whenever $t \to 1$ (see Figure 4.4 left side).

ii. There is a sequence of connected components $\{L_i\}$ of L_{sing} and positive constants c_i such that $d(L_i, \Upsilon) \to 0$ and $d(L_i, \Upsilon) > c_i > 0$ (see Figure 4.4 right side).

The Case 2.ii requires a special treatment. In fact, note that such a situation implies the existence of paths $\gamma : [0, 1) \to L$, with $\gamma(0) = \tilde{p}_0$ satisfying the following:

(i) $d(\gamma, \Upsilon) = 0$.

(ii) γ intersects $\mathcal{V}_s$ in an infinite number of (disjoint) subpaths such that each of them is contained in *a different* connected component of $\mathcal{V}_s \cap L$. Namely

$$\mathcal{V}_s \cap \gamma = \bigsqcup_{i=0}^{\infty} \gamma_s^i$$

with $\gamma_s^i \subseteq L_i$, and $L_i \cap L_j = \emptyset$.

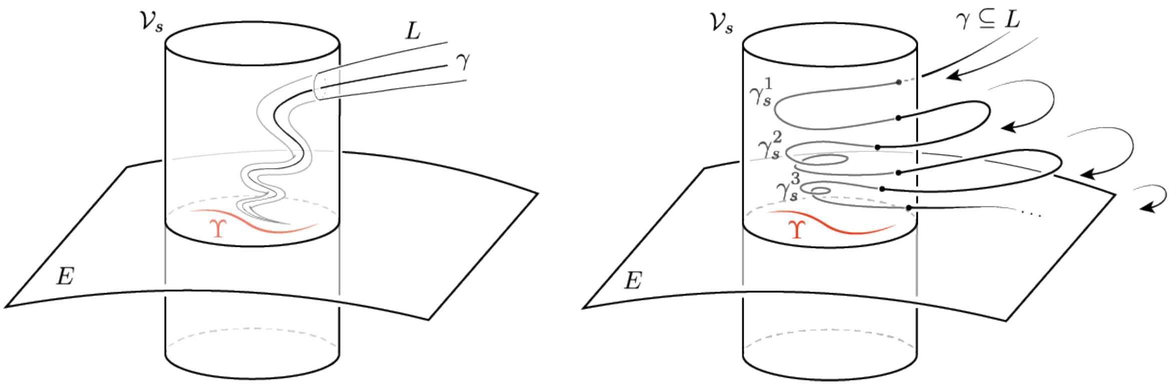


Figure 4.4 Cases 2.i (left) and 2.ii (right)

For our purposes, in addition to properties (i) and (ii), we are also interested in paths that satisfy the following condition:

(iii) The subpaths γ_s^i cannot be deformed homotopically inside the leaf into a subpath contained in the boundary $\partial\mathcal{V}_s \cap L$.

We will denote the set of paths satisfying the previous items as Γ_L

$$\Gamma_L = \{\gamma : [0, 1) \to L : \gamma \text{ satisfies } (i), (ii) \text{ and } (iii)\}.$$

For what comes next, we need to be careful regarding on whether the path under consideration is within the leaf, or it is the projection of the path over E. In that context, in the definition below –and as needed during the proof of the forthcoming theorem– we will use the following notation:

- $\tilde{\gamma}$ stands for a path in L, such that $\tilde{\gamma} \in \Gamma_L$ and $\tilde{\gamma}_s^i$ denotes the restriction of $\tilde{\gamma}$ to a subpath contained on some $L_i \subseteq \mathcal{V}_s$.

- $p^\varepsilon \in \tilde{\gamma} \cap \partial\mathcal{V}_s$ denotes an entry point of $\tilde{\gamma}$ to the singular set $\mathcal{V}_s$ and $q^\varepsilon \in \tilde{\gamma} \cap \partial\mathcal{V}_s$ the exit point after following $\tilde{\gamma}_s^i$.

- γ and γ_s^i represent the projected paths onto E, and p and q denote the projections of p^ε and q^ε over E. Let z (resp. w) be the last coordinate of p_0^ε (resp. p^ε), in other words z (resp. w) lies in a transversal section $\Sigma_{p_0} \simeq (\mathbb{C}, 0)$ (resp. Σ_p).

Note that the superscript ε is used to distinguish points in the leaf L –which is not contained in E– from projections that lie in E.

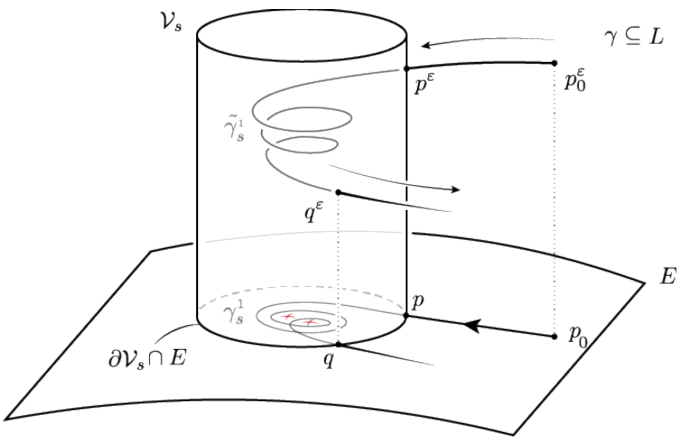


Figure 4.5

The subpath $\gamma_s^i \subseteq \mathcal{V}_s$ of γ can be artificially closed by means of a minimal path from p to q staying in $\partial\mathcal{V}_s$, leading to the notion of length of γ_s^i. Alternatively, $\gamma_s^i \subseteq \mathcal{V}_s$ can be split as $\gamma_s^i \simeq \alpha_{i_1} * \ldots \alpha_{i_m} * \sigma_{i_m}$, where each α_{i_j} represents a generating element of the fundamental group $\Pi(E_0, p)$ based at $p \in \partial\mathcal{V}_s \cap L_i$, and where σ_{i_m} stands for an open path of minimal length from p to q, contained in $\partial\mathcal{V}_s \cap E$. Note that there may be repetitions of generating loops. For sake of simplicity, denote as α_p^m the combination of m non trivial loops $\alpha_{i_1} * \cdots * \alpha_{i_m}$. If γ_s^i takes the previous decomposition, we will say that, both α_q^m and γ_s^i have length m, and we will write $\ell(\gamma_s^i) = \ell(\alpha_q^m) = m$.

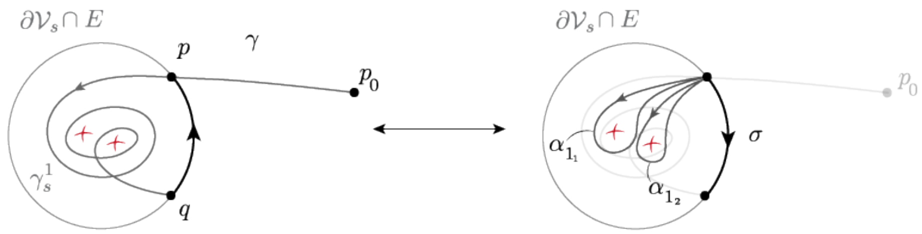


Figure 4.6 $\gamma_s^1 \simeq \alpha_{1_1}^{(2)} * \alpha_{1_2}^{(3)} * \sigma$, and $\ell(\gamma_s^1) = 5$

Finally recall that for an arbitrary path η, h_η denotes the holonomy, or "parallel transport", of E following the path η.

Definition 4.3.1. Denote by $\mathcal{V}_s$ a sufficiently small singular neighborhood of Υ. We will say that the component Υ is a *friendly singular set* if for every leaf L such that $\Gamma_L \neq \emptyset$, and for every $\tilde{\gamma} \in \Gamma_L$ at least one of the following conditions is satisfied:

($\mathtt{S}_1$) There exists a uniform $\nu > 0$ such that

$$\left| \int_{\tilde{\gamma}_s^i} dT_L \right| > \nu$$

for every $i \in \mathbb{N}$.

($\mathtt{S}_2$) For every $i \in \mathbb{N}$, the corresponding subpath $\tilde{\gamma}_s^i \subseteq L_i$ and subsequent loop α_p^m defined as before, satisfies the following inequality

$$|h_{\sigma * \alpha_p^m * \sigma^{-1}}(z)| \geq \sum_{j=1}^{m} \psi^j(|z|),$$

where σ is a path of minimal length from p_0 to p contained in K_1 and ψ^j denotes the $j-$th iteration of the map $\psi(z) = z - z^2$.

Note that condition ($\mathtt{S}_2$) implies that each time a path $\gamma \in \Gamma_L$ enters $\mathcal{V}_s$, the exit point cannot be arbitrarily close to the pole divisor. The choice of bounding from below by powers of ψ is not arbitrary; it will be essential in the proof of Lemma 4.3.4.

4.3.2 Statement and proof of Theorem II

In the sequel, the divisor of poles will be assumed to have only irreducible connected components. This means that $(X)_\infty$ consists of a disjoint union of irreducible components. The generalization of this theorem to non irreducible divisor of poles follows the same lines of the proof that will be done, but Definition 4.3.1 needs to be adapted.

Theorem 4.3.2 (Theorem II). *Let M be a compact $n-$dimensional complex manifold, let X be a meromorphic vector field on M and let $\mathcal{F}$ be the 1-dimensional foliation induced by X. Assume, in addition, the existence of a codimension-1 foliation $\mathcal{D}$ such that, for any arbitrary invariant irreducible component E of $(X)_\infty$, the following holds:*

- *$\mathcal{D}$ satisfies the conditions $(D1)$ and $(D2)$,*
- *$Hol'(E_0, p_0) \subseteq S^1$.*
- *X is semicomplete in some neighborhood of $\mathcal{S}_0' \cap E$ and this singular set is friendly.*

Then X is semicomplete on M.

The proof of this theorem requires a fine analysis of the behavior of leaves in a neighborhood of an irreducible component of the divisor of poles $E \subseteq (X)_\infty$.

Let L_ε be a leaf that is not contained in E but that stays in a neighborhood $\mathcal{V}_E$ of E and such that $d(E, L_\varepsilon) = 0$. Again denote by $\Omega \subseteq \mathbb{C}$ the image of the developing map of L_ε, $\Omega = \mathsf{Dev}_\varepsilon(\hat{L}_\varepsilon)$, then the description given by Equation (4.1) holds. The idea of the proof is essentially the same: first to show that $\partial\Omega$ has no points of type (b), and secondly to conclude that the developing map Dev_ε is injective by constructing its inverse. The crucial difference is that the identity obtained in Lemma 3.2.7, is no longer applicable in the present case, and hence, the proof of the statement in Lemma 4.2.2 requires some non trivial adjustments. Once that has been done, the proof will follow the same lines of the proof of 4.2.1 on page 49.

Let us begin by showing a simple result concerning the dynamics of the map $g : [0, 1] \to [0, 1]$, given by $g(x) = x - x^2$.

Lemma 4.3.3. *Let g be the real valued function defined as $g(x) = x - x^2$, then, for any $x \in (0, 1)$, the series*

$$\sum_{i=0}^{\infty} g^i(x)$$

diverges.

Proof. First, note that for $0 < x < 1$, the function g and its iterations are positive and remain in the interval $[0, 1]$. We will show that the sequence of partial sums is not a Cauchy sequence. In fact, fix some large integer $k \in \mathbb{N}$ and let $\mu = g^k(x)$. Note that the iterations $g^i(x)$ tend to 0 as $i \to \infty$, hence we may assume without loss of generality that $\mu < 1/2$. Moreover, let $N_k > k$ be the smallest positive integer such that $g^{N_k}(x) < \mu/2$. It then follows that $g^i(x) \geq \mu/2$ for $i < N_k$, and thus

$$\sum_{i=k}^{N_k} g^i(x) \geq (N_k - k) \cdot \frac{\mu}{2}.$$

We want to estimate the difference $N_k - k$. In that direction, let us consider the conjugation of g by the function $\phi : x \to 1/x$. The conjugated map, denoted by f, takes on the form

$$f(y) = \phi \circ g \circ \phi^{-1}(y) = \frac{1}{\phi(y)(1 - \phi(y))}.$$

Which in turn yields to

$$f(y) = y + 1 + \frac{1}{y} + \frac{1}{y^2} + \dots$$

for $y > 1$, where $y = 1/x$. Proceeding by induction, it is not difficult to show that

$$y + n \leq f^n(y) \leq y + 2n$$

for any $n \in \mathbb{N}$ and $y > 2$. It then follows that, starting from $y = 1/\mu$, the number of iterations of f needed to surpass $2/\mu$ must be greater than $1/2\mu$.

Returning to the $x-$variable, we conclude that the iterations required to reach $g^{N_k}(\mu)$ starting from $\mu = g^k(x)$, i.e., $N_k - k$, are inferiorly bounded by $1/2\mu$. Therefore

$$\sum_{i=k}^{N_k} g^i(x) \geq \frac{1}{2\mu} \cdot \frac{\mu}{2} = \frac{1}{4},$$

and this ends the proof. $\square$

The second lemma that we will need relates the previous result with the holonomy of the leaf E . Since K_E is compact, the fundamental group $\pi_1(K_E, p_0)$ is finitely generated and so is $\mathrm{Hol}(K_E, p_0)$. Let $\mathcal{G} = \{h_1, \dots h_l\}$ be a generating set of $\mathrm{Hol}(K_E, p_0)$. Having fixed a path $\gamma : [0, 1) \to K_E$ that returns an infinite number of times to p_0, the holonomy of a nearby leaf of $\mathcal{D}$ following γ, can be written in terms of consecutive compositions of elements in $\mathcal{G}$. In that context, the notation $\mathbf{h^n}$, stands for a combination of n iterations of elements in $\mathcal{G}$ (or a *word* of length n). More precisely, if $z \in (\mathbb{C}, 0)$, then

$$\mathbf{h^n}(z) = h_{i_n} \circ \cdots \circ h_{i_2} \circ h_{i_1}(z)$$

where $h_{i_j} \in \mathcal{G}$. Therefore, a $n+1$-iteration takes the form

$$\mathbf{h^{n+1}}(z) = h_{i_{n+1}} \circ \mathbf{h^n}(z)$$

for some $h_{i_{n+1}} \in \mathcal{G}$.

Lemma 4.3.4. *Let $z \in (\mathbb{C}, 0)$ be small enough. Assume that $Hol'(E_0, p_0) \subseteq S^1$. Then for any combination of iterations of elements in $\mathcal{G}$, the series*

$$\sum_{i=0}^{\infty} |\mathbf{h^i}(z)|$$

diverges.

Proof. Let $h : (\mathbb{C}, 0) \to \mathbb{C}$ be a map such that $h(0) = 0$. Then its series expansion

$$h(z) = h'(0)z + a_2 z^2 + a_3 z^3 \dots$$

converges for $|z| < R$, where R denotes the radius of convergence of the power series. If $|h'(0)| = 1$, it is not difficult to show that

$$|h(z)| > |z| - C|z|^2$$

where C is defined as the constant $\sum_{i=2}^{\infty} |a_i| R^{i-2}$.

Repeating the argument for each $h_i \in \mathcal{G}$ and taking C as the minimum of the constants arising on each inequality it follows that

$$|h_i(z)| \geq |z| - C|z|^2 > 0$$

for every $i \in \{0, \dots l\}$ and $z \in (\mathbb{C}, 0)$ small enough.

Furthermore, up to an uniform renormalization of $\mathcal{G}$ (more precisely, a suitable conjugation by the map $z \to \lambda z$), C can be assumed to be such that $C < 1$. Therefore, for any $i \in \{1, \dots l\}$ and $0 < |z| < 1$,

$$|h_i(z)| \geq |z| - C|z|^2 > |z| - |z|^2 > 0.$$

Let $\psi(z) = z - z^2$. From the previous inequality along with the fact that ψ restricted to the segment $[0, 1/2)$ is an increasing function, it follows that

$$|\mathbf{h^2}(z)| = |h_i(h_j(z))| \geq \psi(|h_j(z)|) \geq \psi(\psi(|z|)).$$

Proceeding inductively, we obtain

$$|\mathbf{h^n}(z)| \geq \psi^n(|z|). \tag{4.2}$$

Finally, Lemma 4.3.3 ensures that the series $\sum \psi^n(|z|)$ diverges, so that the same applies to $\sum |\mathbf{h^n}(z)|$.

□

The following lemma is an equivalent version of Corollary 3.2.9, and we will use it several times in the sequel.

Lemma 4.3.5. *Assume that $E = (X)_\infty$, and let $S \subseteq M$ be a subset such that $d(S, E) > c > 0$ for some constant c. There is a uniform $\delta' > 0$ such that for any leaf L of $\mathcal{F}$ whose intersection with S is non empty, the following property is satisfied: given any $x \in S \cap L$, one has*

$$\mathcal{B}_{\delta'}(\mathsf{Dev}(x)) \subseteq \mathsf{Dev}(\hat{L})$$

Remark 4.3.6. Once more, we emphasize that δ' only depends on the chosen $c > 0$, it does not depend on the leaf L.

Proof. Denote by $L_S = L \cap S$. Since $d(L_S, E) > c > 0$, there is a neighborhood of E, namely $\mathcal{W}_E$, such that L_S is contained in the compact set $M \setminus \mathcal{W}_E$. In other words, L_S stays away of E. Since E is the unique component of the divisor of poles, X restricted to $M \setminus \mathcal{W}_E$ is a holomorphic vector field uniformly bounded, therefore, the theorem of existence and uniqueness ensures the existence of a $\delta' > 0$ such that for every point $x \in M \setminus \mathcal{W}_E$ (and thus in L_S) the local flow is well defined in a ball of size δ'. Since L is not contained in E, the flow of X induces a translation structure on L, and it then follows that the δ'−ball can be carried out along L_S without having its size reduced. In other words

$$\mathcal{B}_{\delta'}(\mathsf{Dev}(x)) \subseteq \mathsf{Dev}(\hat{L})$$

□

Lemma 4.3.7 (Main Lemma). *Ω has no boundary points of type (b), i.e., $\partial\Omega \cap \partial\Omega_r = \emptyset$.*

Proof. Aiming at a contradiction, assume that $t_\infty \in \mathbb{C}$ is a point lying in $\partial\Omega \cap \partial\Omega_r$. The argument can can be split into two cases:

Case 1: t_∞ is attainable by a path $\eta : [0,1) \to \Omega$ such that $\eta(s) \to t_\infty$ as s tends to 1 and $\eta \subseteq \Omega_r$, i.e., η stays in the "regular part".

Case 2: The previous case doesn't hold. In other words there is no path remaining within the regular part that converges to t_∞. Since $t_\infty \in \partial\Omega_r$, it can be shown that there must exist a path $\eta : [0,1) \to \Omega$ intersecting an infinite number of distinct components of Ω_s and such that $\eta(s) \to t_\infty$ as s tends to 1 (see the description of the connected components of Ω_s in the proof of Theorem 4.2.1).

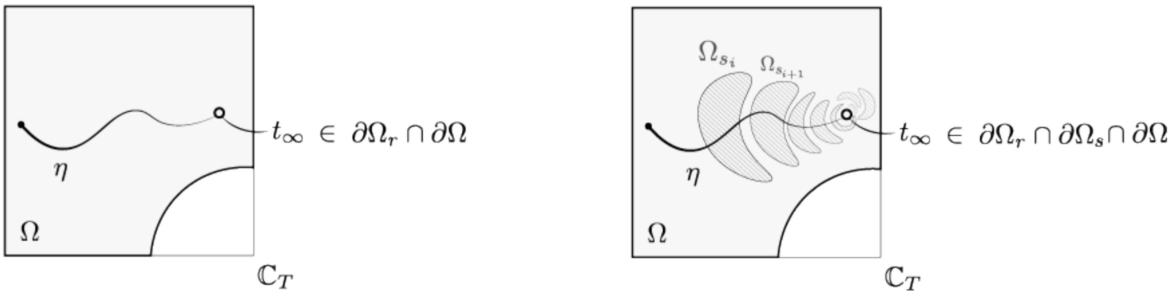


Figure 4.7 Cases 1 (left) and 2 (right)

The final goal is to find a suitable sequence of points in η converging to t_∞, in such a way, the sum of radius of the the balls provided by Proposition 3.2.10 bounds by below $|t_\infty|$ and diverges, leading to a contradiction. Bearing this purpose in mind, without lose of generality, η can be assumed to be a segment from $t_0 = 0$ to t_∞, satisfying either that $[0, t_\infty) \subseteq \Omega_r$ (Case 1), or such that the cardinality of the indices $i \in I$ such that $[0, t_\infty) \cap \Omega_i \neq \emptyset$ is not finite (Case 2).

Now let $\tilde{\gamma}$ be a path in L such that $\mathsf{Dev}_\varepsilon(\tilde{\gamma}) = [0, t_\infty)$.

Claim 4.3.8. $d(\tilde{\gamma}, E) = 0$

Proof of the claim. In fact, if $d(\tilde{\gamma}, E) > c > 0$ Lemma 4.3.5 applied to $S = \tilde{\gamma}$ leads to the existence of a δ'−neighborhood centered all along $\eta = [0, \infty)$ and thus contradicts the fact that $t_\infty \in \partial\Omega$. □

Let $\gamma \subseteq E$ be the projection of $\tilde{\gamma}$ over E. Alternatively, we can see $\tilde{\gamma}$ as the lift of γ on the leaf L. Now we will discuss both cases separately.

Case 1 Since $[0, t_\infty) \subseteq \Omega_r$, then $\tilde{\gamma}$ remains in the regular neighborhood $\bigcup_{i=0}^{N} U_i$, and therefore, the projected path γ will be contained in $K_E \subseteq E$.

As $t \in [0, t_\infty)$ moves forward, the path $\gamma|_{[0,t)}$ wanders into the compact set K_E. Since the latter is covered by $\{U_i\}_{i=0}^{N}$, it follows that whenever γ passes through more than N open sets, it must have intersected at least one open set of the cover twice (see Figure 4.8).

Consider the first time γ returns to an open set of the cover it has already passed through. Namely, assume γ returned to the set U_1, the latter will henceforth be referred as a *return set*. Then γ intersects U_1 in two pieces of paths. Let p be a point in the first piece of path and $q = \gamma(t) \in U_1$, for some $t \in [0, t_\infty)$, be a returning point within the second piece of path.

Remark 4.3.9 (Notation). In the sequel, the notation σ_{xy} will stand for a path σ from x to y. In the particular case of γ, if x and y are two points lying on its image, the expression γ_{xy} will denote the restriction of γ to that subpath.

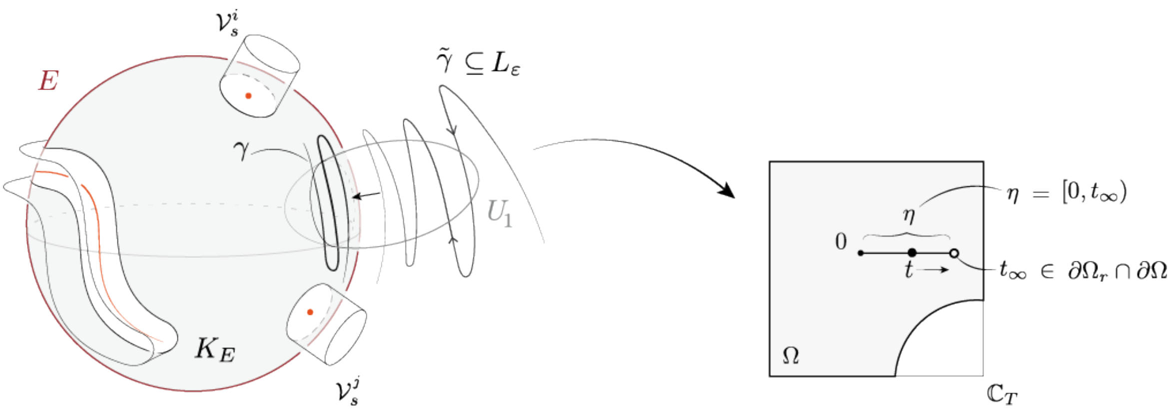


Figure 4.8

Following the above notation, let $\gamma_{p_0q} = \gamma|_{[0,t)}$. Consider the subpath γ_{pq} and a segment σ_{qp} contained in U_1. Therefore,

$$\alpha_p = \gamma_{pq} * \sigma_{qp}$$

is a loop based at p and then it can be written in terms of the generators of $\pi_1(K_E, p)$. Denote its length $\ell(\alpha_p)$ by m.

Note that since the number of holes, or singular neighborhoods, surrounded by α_p, must necessarily be strictly smaller than N it follows than $m < N$. As a consequence, we can pick $m-1$ points $p_1, \ldots p_{m-1}$ in γ_{pq} such that

(a) No two points belong to a same open set of the cover.

(b) Each pair of points is separated by a distance of at least β (where recall that β was defined as the Lebesgue number of the cover $\{U_i\}_{i=0}^{N}$).

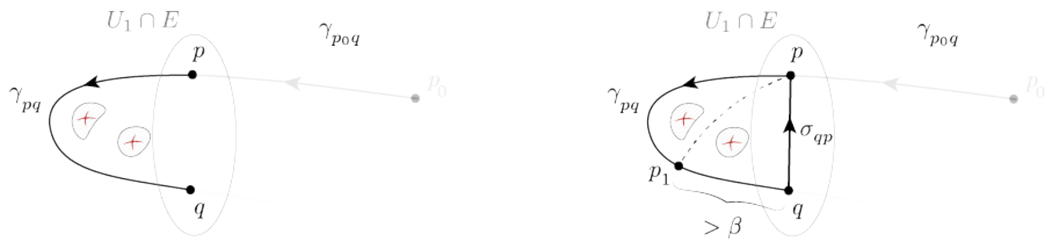


Figure 4.9

Let $\tilde{p}_1, \dots, \tilde{p}_{m-1}$ be the corresponding points lying in the lifted path $\tilde{\gamma}$, and denote their images under Dev_ε by $t_i = \mathsf{Dev}_\varepsilon(\tilde{p}_i) \in [0, t_\infty)$. For each t_i, Proposition 3.2.10 ensures the existence of a ball around t_i staying in Ω, with radius $r_i = |C^\varepsilon_{\tilde{\gamma}_i}| \cdot \delta/2$, where $\tilde{\gamma}_i$ denotes the path going from $\tilde{p}_0$ to $\tilde{p}_i$. In other words, we have that for each i, $\mathcal{B}_{r_i}(t_i) \subseteq \Omega$. By virtue of items (a) and (b), it follows that $t_{i+1} \notin B_{r_i}(t_i)$, i.e.,

$$|t_\infty - t_i| - |t_\infty - t_{i+1}| > r_i.$$

Consequently

$$|t_\infty| > \sum_{i=0}^{m} |t_\infty - t_i| - |t_\infty - t_{i+1}| > \sum_{i=0}^{m} r_i.$$

Replacing the definition of the radii, we obtain

$$|t_\infty| > \left(|C^\varepsilon_{\tilde{\gamma}_1}| \cdot + \cdots + |C^\varepsilon_{\tilde{\gamma}_m}|\right) \cdot \delta/2.$$

Let us denote by z the last coordinate of the leaf L_ε read in charts U_0, i.e., the coordinate belonging to a transverse section $\Sigma_{p_0} \simeq (\mathbb{C}, 0)$. Finally, using the identity in Lemma 3.2.7 that relates the affine constants with the holonomy, the previous inequality is reformulated as follows

$$|t_\infty| \geq \sum_{i=1}^{m} |h_{\gamma_i}(z)|^{-k} \cdot \delta/2.$$

For sake of simplicity, we may assume that the order k of the irreducible component E of the pole divisor equals to -1. Note that this requires adjusting the generating set $\mathcal{G}$.

Now let us turn our attention to the projected paths γ_i lying on E. Note that $\gamma_i(0) = p_0$, $\gamma_i(1) = p_i$ and $\gamma_i \subseteq \gamma_{i+1}$. Along similar lines as before, each γ_i can be homotopically decomposed in terms of the generators of $\pi_1(K_E, p_0)$, followed by a path of minimal length:

$$\gamma_i \simeq \underbrace{\alpha_1 * \cdots * \alpha_i}_{\text{loop based at } p_0} * \sigma_i.$$

where σ_i is a minimal path from p_0 to p_i contained in $\bigcup_{i=0}^{N} U_i$.

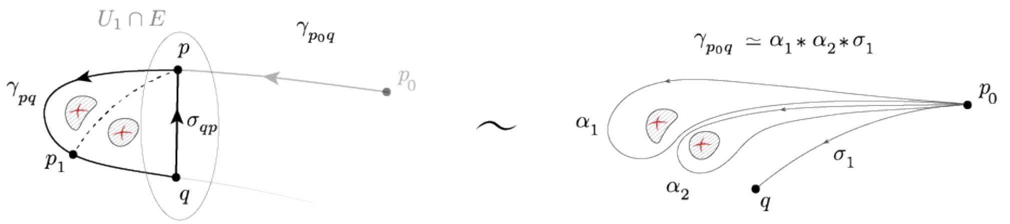


Figure 4.10

Therefore,

$$h_{\gamma_i} = h_{\sigma_i} \circ h_{\alpha_1 * \cdots * \alpha_i} = h_{\sigma_i} \circ \mathbf{h^i},$$

where we recall that $\mathbf{h^i}$ denotes a recursive composition of i elements of $\mathcal{G}$.

We remind that K_1 was defined in Section 3.2 as $\bigcup_{i=0}^{N} \overline{U}_i$. Since σ_i is of minimal length, it intersects at most once each of the open sets of the cover $\{U_i\}_{i=0}^{N}$. Besides, the fundamental group $\pi_1(K_E, p_0)$ and the cover $\{U_i\}_{i=0}^{N}$ are finite sets, therefore the set

$$\mathcal{G}_1 = \{h_\sigma : (\mathbb{C}, 0) \to (\mathbb{C}, 0) : \sigma(0) = p_0,\ \sigma \subseteq K_1 \text{ and } \sigma \text{ is of minimal lenght}\}$$

is also finite. It follows that there is a uniform constant $A \in (0, 1)$ such that

$$|h_\sigma(z)| > A|z|$$

for every $h_\sigma \in \mathcal{G}_1$.

All the previous discussion, boils down into the following inequalities:

$$\begin{aligned} |t_\infty| &> \sum_{i=1}^{m} |h_{\sigma_i} \circ \mathbf{h^i}(z)| \cdot \delta/2 \\ &> A \sum_{i=1}^{m} |\mathbf{h^i}(z)| \cdot \delta/2. \end{aligned}$$

As we continue moving $t \in [0, t_\infty)$ forward, we apply the same argument to the subpath of γ that begins with σ_m. Since it is a minimal path, it cannot intersect an element of the open cover more than once. Within at most $N - 1$ steps (changes of coordinates), it must revisit a set it has already passed through. Repeating the process and stopping each time at a return set, we obtain a sum

$$|t_\infty| \quad > \quad A \sum_{i=1}^{m_1 + \ldots m_n} |\mathbf{h^i}(z)| \cdot \delta/2 \quad \geq \quad A \sum_{i=1}^{n} |\mathbf{h^i}(z)| \cdot \delta/2.$$

By Lemma 4.3.4, the series on the right diverges. This leads to a contradiction and ends the proof of case 1.

Case 2. Let $[0, t_\infty) \subseteq \Omega$ be a path intersecting an infinite number of singular components of Ω_s and $\tilde{\gamma} \subseteq L_\varepsilon$ be a path such that $\mathsf{Dev}_\varepsilon(\tilde{\gamma}) = [0, \infty)$. Then $\tilde{\gamma}$ belongs to the set Γ_{L_ε}. We refer to the definition and keep the notation of Section 4.3.1 together with the notation used in the previous case. By hypothesis, two cases may happen.

($\mathtt{S}_1$). Assume $\tilde{\gamma}$ satisfies ($\mathtt{S}_1$). Then there is an uniform $\nu > 0$ such that

$$\left| \int_{\tilde{\gamma}_s^i} dT_{L_\varepsilon} \right| > \nu$$

for every subpath of $\tilde{\gamma}$, $\tilde{\gamma}_s^i \subseteq L_i \subseteq \mathcal{V}_s$ as described in Subsection 4.3.1.

As pointed out in Remarks 4.1.5 and 3.1.1, the previous integral differs from Dev_ε only by a multiplicative constant. It then follows that for every $i \in I$

$$|\,\mathsf{Dev}_\varepsilon(p_i) - \mathsf{Dev}_\varepsilon(q_i)| > \tilde{\nu},$$

where $p_i^\varepsilon = \tilde{\gamma}_s^i(0)$ (entry point to $\mathcal{V}_s$) and $q_i^\varepsilon = \tilde{\gamma}_s^i(1)$ (exit point).

Now we choose a partition of $[0, t_\infty)$ such that for each i, t_{2i-1} (resp. t_{2i}) stands for the entrance point (resp. exit point) in the boundary of Ω_{s_i}. We obtain that

$$|t_\infty| > \sum_{i=1}^{n} |t_i - t_{i+1}| = \sum_{i=1}^{n} |\,\mathsf{Dev}_\varepsilon(\tilde{p}_i) - \mathsf{Dev}_\varepsilon(\tilde{q}_i)| > n \cdot \tilde{\nu}$$

which clearly diverges as n tends to infinity.

($\mathtt{S}_2$). Now assume that $\tilde{\gamma}$ satisfies ($\mathtt{S}_2$). Let $t_p \in [0, t_\infty)$ be the first point where $\tilde{\gamma}$ enters $\mathcal{V}_s$ and t_q the subsequent exit point, i.e., $t_p, t_q \in \partial\Omega_r \cap \partial\Omega_{s_1}$. Let us consider separately the segments $[0, t_p]$ and $[t_p, t_q]$. Write $\gamma_{p_0 q} = \gamma_r * \gamma_s$, where γ_r (resp. γ_s) stands for the piece of projected path in M contained in K_1 (resp. in $\mathcal{V}_s$). Again, p and q stand for the projections onto $\gamma \subseteq E$ of the entry and exit points of $\tilde{\gamma}$ into $\mathcal{V}_s$, respectively. Since $[0, t_p] \subseteq \Omega_r$, the same construction as in Case 1 applies and it follows that

$$|t_p - 0| > \sum_{i=0}^{m_1} |h_{\sigma_i} \circ \mathbf{h^i}(z)| \cdot \delta/2 > A \cdot \sum_{i=0}^{m_1} \psi^i(|z|) \cdot \delta/2$$

where $m_1 = \ell(\gamma_r)$. Keeping the notation used in Case 1, we have that the last summand of the previous sum is defined as $h_{\sigma_{m_1}} \circ \mathbf{h^{m_1}}(z) = h_{\alpha_{p_0} * \sigma_{m_1}}(z)$ where α_{p_0} stands for a combination, with no repetitions, of m_1 elements of generating elements in $\pi_1(K_E, p_0)$.

As for the subsegment $[t_p, t_q]$, let r_q be half of the radius of the ball centered at t_q provided by Proposition 3.2.10. Then

$$|t_p - t_q| > r_q = |h_{\gamma_{p_0 q}}(z)| \cdot \delta/2.$$

Note that since $\gamma_r \simeq \alpha_{p_0} * \sigma_{m_1}$ and $\gamma_s \simeq \alpha_p * \sigma_{m_2}$, where $m_2 = \ell(\gamma_s)$, we have that

$$\gamma_{p_0 q} \quad \simeq \quad \alpha_{p_0} * \sigma_{m_1} \quad * \quad \alpha_q * \sigma_{m_2}.$$

Let us conveniently write γ_{p_0q} as follows:

$$\gamma_{p_0q} \simeq \alpha_{p_0} * \underbrace{\sigma_{m_1} * \alpha_q * \sigma_{m_1}^{-1}}_{\beta_{p_0}} * \sigma_{m_1} * \sigma_{m_2},$$

where $\beta_{p_0} = \sigma_{m_1} * \alpha_q * \sigma_{m_1}^{-1}$ is now a loop based at p_0. The holonomy along γ_{p_0q} can then be expressed as

$$h_{\gamma_{p_0q}}(z) = h_{\sigma_{m_1}*\sigma_{m_2}} \circ h_{\beta_{p_0}}(w)$$

where $w = h_{\alpha_{p_0}}(z)$. Without lose of generality we may assume that $\sigma_{m_1} * \sigma_{m_2}$ is of minimal length, and hence $h_{\sigma_1*\sigma_2} \in \mathcal{G}_1$. Then, hypothesis ($\mathtt{S}_2$) applied to $h_{\beta_{p_0}}(w)$ yields

$$|h_{\gamma_{p_0q}}(w)| \geq A \cdot \sum_{i=1}^{m_2} \psi^i(|w|).$$

According to identity (4.2) from Lemma 4.3.4, $|w| = |h_{\alpha_{p_0}}(z)| > \psi^{m_1}(|z|)$, it follows that

$$|h_{\gamma_{p_0q}}(w)| \geq A \cdot \sum_{i=1}^{m_2} \psi^{m_1+i}(|z|) = A \cdot \sum_{i=m_1+1}^{m_2} \psi^i(|z|).$$

Finally bringing all together, we obtain the following estimations:

$$\begin{aligned} |t_\infty| &> \quad |t_p - 0| \quad + \quad |t_q - t_p| \\ &> A \cdot \sum_{i=0}^{m_1} \psi^i(|z|) \cdot \delta/2 \;+\; A \cdot \sum_{i=m_1+1}^{m_2} \psi^i(|z|) \cdot \delta/2. \end{aligned}$$

Which leads to

$$|t_\infty| > A \cdot \sum_{i=0}^{m_1+m_2} \psi^i(|z|) \cdot \delta/2.$$

Repeating the argument with the remaining piece of path $\sigma_1 * \sigma_2$ and moving forward $t \in [0, t_\infty)$, we obtain the divergence of the sum.

□

We are finally able to establish the proof of Theorem 4.3.2. We keep the notation of all the previous discussion.

Proof of Theorem 4.3.2. Let L_ε be a leaf of $\mathcal{F}$ not contained in $(X)_0 \cup (X)_\infty$. If L_ε does not accumulate on any irreducible component of the divisor of poles, then there is a positive constant such that $d(L_\varepsilon, (X)_\infty) > c > 0$. Then, L_ε is contained in a compact component of M where X –and thus X restricted to L_ε– is uniformly bounded. Owing to Theorem 4.1.7, the restriction of X to L_ε is complete, in particular, L_ε is a parabolic leaf.

Assume that L_ε accumulates on an unique irreducible component E of the divisor of poles $E \subseteq (X)_\infty$, i.e., $d(L_\varepsilon, E) = 0$. Let $\mathcal{V}_E$ be a small neighborhood of E.

(1) If L_ε stays in $\mathcal{V}_E$, then the theorem is a straightforward consequence of Lemma 4.3.7, followed by the same proof of Theorem 4.2.1.

(2) Finally, if L_ε approaches E without staying in $\mathcal{V}_E$, a similar argument can be deduced. Indeed, Lemma 4.3.5 leads to the existence of a uniform $\delta' > 0$ and thus a time-ball with uniform size in Ω, corresponding to the image under Dev_ε of points lying in the intersection of L_ε with $M \setminus \mathcal{V}_E$. It follows that if t_∞ was a point in $\partial\Omega \cap \partial\Omega_r$, it could not be approached by a sequence of points in Ω such that their preimages under Dev_ε belong to $L_\varepsilon \cap (M \setminus \mathcal{V}_E)$. Thus, the proof is reduced to case (1).

If L_ε accumulates into several components of $(X)_\infty$, an analogous argument can be carried out.

□

4.4 Vector fields possessing a first integral: Theorem II.A and Theorem III

Assume that X comes along with a meromorphic first integral $f : M \to \mathbb{C}$, that is constant at generic points of $E = (X)_\infty$, i.e., away from $I(f)$, the indeterminacy set of f. Let us denote

$$E_0 = E \setminus I(f)$$

and let $c_0 \in \mathbb{CP}^1$ be the value of f when restricted to E_0.

The first thing to observe is that the map $f : M \to \mathbb{C}$, defines a codimension-1 foliation $\mathcal{D}$ whose leaves are given by the level sets of f;

$$S_c = \{x \in M : f(x) = c\}$$

for $c \in \mathbb{CP}^1$. In particular, since $f|_{E_0} \equiv c_0$, then E_0 is itself a leaf of $\mathcal{D}$. Assume also that $f^{-1}(c_0) = E_0$, i.e., that E_0 is the unique leaf of $\mathcal{D}$ mapped through f to the value c_0.

Note that since f is a first integral for X, then every leaf L of $\mathcal{F}$ is contained within some S_c for a certain $c \in \mathbb{CP}^1$. Since f is holomorphic away from $I(f)$ and $f^{-1}(c_0) = E_0$, and according to the standard theory of foliations having a first integral (see, for instance, [33]) we highlight the following two facts:

(a) The leaves of $\mathcal{D}$ (and consequently the leaves of $\mathcal{F}$) cannot accumulate over $E \setminus I(f)$.

(b) Owing to the previous fact, the holonomy $\mathrm{Hol}(E_0, p_0)$ is finite, and consequently the linear coefficient of the holonomy map cannot be hyperbolic, i.e., $\mathrm{Hol}'(E_0, p_0) \subseteq \mathbb{S}^1$.

Therefore, the present setting satisfies the hypotheses (D_1) and (D_2) of Theorem 4.3.2. In this context, the extended singular set is defined by

$$\mathcal{S}_0' := E \cap I(f).$$

Besides, we recall that K_E is a compact set contained in $E \setminus \mathcal{S}_0'$ and again K_1 is a suitable neighborhood in M of K_E. The following lemma is a direct consequence of Lemma 4.3.5 applied to $S = S_c \cap K_1$ and will be used in the proof of Theorem III.

Lemma 4.4.1. *Let S_c be a leaf of the codimension-1 foliation $\mathcal{D}$. There is a uniform $\delta' > 0$ such that for any leaf of $\mathcal{F}$, namely $L \subseteq S_c$, and any $x \in K_1 \cap L$, one has*

$$\mathcal{B}_{\delta'}(\mathsf{Dev}(x)) \subseteq \mathsf{Dev}(\hat{L})$$

□

The presence of the holomorphic first integral restricts the ways in which a leaf can come close to E. In fact, the previous remarks imply that the only way L can approach E is through paths converging to $I(f) \cap E$ (note that, a priori, $I(f)$ could intersect points from both $\mathrm{Sing}(\mathcal{F}) \cap E$ and $E \setminus \mathrm{Sing}(\mathcal{F})$). In terms of the description in Subsection 4.3.1, Case 1 is immediately ruled out. On the other hand, Case 2.*ii* can only happen in the following situation:

Lemma 4.4.2. *With the previous settings, Case 2.ii in Subsection 4.3.1 can only occur in infinite time.*

Proof. Assume we have a situation as in Case 2.*ii* and let γ be a path in Γ_L such that $\mathsf{Dev}(\gamma) = \eta$ converges to some point $\hat{t} \in \partial\Omega$. Then γ intersects the regular part K_1 an infinite number of times. In other words $\gamma \cap K_1$ consists of an infinite union of disjoint subpaths γ_r^i. Let $x_i \in \gamma_r^i$. By Lemma 4.4.1, $\mathcal{B}_{\delta'}(\mathsf{Dev}(x_i)) \subseteq \Omega$, γ cannot reach the singular set in finite time. In other words, $\mathsf{Dev}(x_i)$ (and hence η) cannot converge to a boundary point of Ω.

□

It immediately follows that the singular set is friendly. Consequently, the present setting satisfies all the hypothesis of Theorem 4.3.2 provided that X is assumed to be semicomplete on a neighborhood of the set $I(f) \cap E$. All the previous discussion proves the following theorem:

Theorem 4.4.3 (Theorem II.A)**.** *Let M, X, $\mathcal{F}$, E and E_0 be as before. Assume that $f : M \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *There is a constant $c_0 \in \mathbb{CP}^1$ such that $f^{-1}(c_0) = E_0$.*
- *X is semicomplete on a neighborhood of the set $I(f) \cap E$.*

Then X is semicomplete on M.

□

Before presenting Theorem 4.4.6, we introduce some notions that will be needed for its proof.

Definition 4.4.4. Let $\mathcal{V}_s$ be a neighborhood of $\mathrm{Sing}(\mathcal{F})$ and consider the restriction of the foliation $\mathcal{F}$ to $\mathcal{V}_s$, denoted by $\mathcal{F}_{\mathcal{V}_s}$. Let L be a leaf of $\mathcal{F}$ such that $L \cap \mathcal{V}_s \neq \emptyset$. $L_{\mathcal{V}_s}$ is said to be a local branch of L if $L_{\mathcal{V}_s}$ is a leaf of $\mathcal{F}_{\mathcal{V}_s}$ satisfying $L_{\mathcal{V}_s} \subseteq L$.

Definition 4.4.5. Let L be a leaf of $\mathcal{F}$, and let y be a point in the closure of the leaf L, $y \in \overline{L}$. The point y is said to be an *end of the leaf* L, if there is a sequence of points $\{x_n\}$ contained in L, converging to y in M and such that $\{x_n\}$ escapes from every compact subset of L.

Let us denote by $\mathscr{E}_L$, the set of ends of the leaf L. Now note that, when X is semicomplete, the ends of the leaf L are in correspondence with the ends of the maximal domain of definition $\Omega \subseteq \mathbb{C}$. The latter consists of boundary points $\partial\Omega$ together with adding "the point at infinity". In other words, if $\Phi_p : \Omega \to L$ is the semiglobal flow provided by the definition of semicomplete vector field satisfying $\Phi_p(0) = p$, then for every $y \in \mathscr{E}_L$, there is a sequence $\{t_n\} \subseteq \Omega$ escaping from every compact set, and such that $\Phi_p(t_n) \to y$. The converse also holds by definition of semicompletness. We emphasize again the two possibilities: either $y \in \mathscr{E}_L$ is reached in finite time, or it is reached in infinite time (see Figure 4.11).

Note that, in both cases, the ends of leaves of L can be of different types: either $y \in \mathrm{Sing}(\mathcal{F})$, or y is a regular point. The latter occurs when L either accumulates on itself (if $y \in L$) or on another leaf (if $y \in L'$). Figure 4.11 illustrates all the possible combinations of the previous situations.

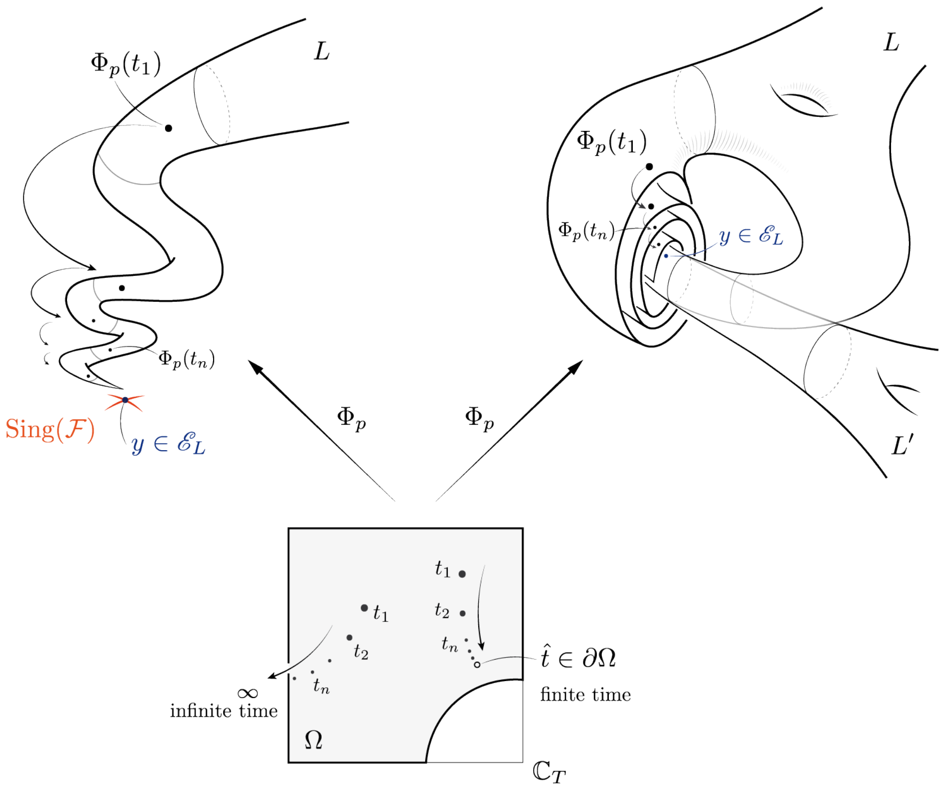


Figure 4.11 $y \in \mathscr{E}_L$ belonging to $\mathrm{Sing}(\mathcal{F})$ (left), or to the regular part of $\mathcal{F}$ (right). Both cases, a priori, can happen in finite or infinite time.

As in the previous section, fix a neighborhood $\mathcal{V}_s$ of $\mathcal{S}'_0$. In order to simplify the notation, assume that $\mathcal{V}_s$ consists of a unique connected component. In the theorem below, when referring to local branches we will always mean leaves of the foliation restricted to the fixed neighborhood $\mathcal{V}_s$.

Theorem 4.4.6 (Theorem III)**.** *Let M, X, $\mathcal{F}$, E and E_0 be as before. Assume that $f : M \to \mathbb{C}$ is a meromorphic first integral for X such that*

- *There is a constant $c_0 \in \mathbb{CP}^1$ such that $f^{-1}(c_0) = E_0$.*
- *X is semicomplete on a neighborhood of the set $E \cap I(f)$.*

In addition, assume that ends of leaves of local branches with respect to $\mathcal{V}_s$ are contained in a discrete set. Then every leaf of $\mathcal{F}$ is in the image of an entire curve.

Remark 4.4.7. In the proof of Theorem 4.4.6, we will only consider leaves that are outside the divisor of poles. However, owing to Theorem 4.2.1, leaves inside the pole divisor possess a uniformizable affine structure, and the proof can easily be adapted to show that, indeed, those leaves are tangent to entire curves as well. Alternatively, M. Brunella proved in [3] that if not all the leaves are parabolic (entire with our notation), then they must be contained in a pluripolar set. Having proved that every leaf outside the divisor of poles is an entire leaf, it immediately follows that leaves within E cannot be hyperbolic.

Lemma 4.4.8. *With the hypothesis of Theorem 4.4.6, $\mathscr{E}_{L_{\mathcal{V}_s}}$ is contained in a discrete set of* $\mathrm{Sing}(\mathcal{F})$*.*

Proof. Let $L_{\mathcal{V}_s}$ be a local branch of leaf approaching E. Then, it can only get close to E by accumulating points within the indeterminacy set of f, in other words $\mathscr{E}_{L_{\mathcal{V}_s}} \cap E \subseteq I(f) \cap E$. Now, let $y \in \mathscr{E}_{L_{\mathcal{V}_s}} \cap E$ and assume that $y \notin \mathrm{Sing}(\mathcal{F})$. It follows that there is a foliated neighborhood U_y around y such that $U_y \cap L_{\mathcal{V}_s}$ consists on a infinite union of disconnected components clustering over the plaque $P_y \subseteq E$ that contains y. Therefore $P_y \subseteq \mathscr{E}_{L_{\mathcal{V}_s}}$ and hence $\dim(\mathscr{E}_{L_{\mathcal{V}_s}}) > 0$, contradicting the theorem's hypothesis. □

We recall that a subset $A \subseteq M$ is said to be analytic in M, if for each point of A there exists a neighborhood of it, such that A is locally written as the common zeros of a certain finite family of holomorphic functions. We refer to the book [8] for a discussion on the definition of analytic set as well as for a statement of Remmert-Stein theorem, which will be crucial in what follows.

Lemma 4.4.9. *Under the assumptions of Theorem 4.4.6, every local branch $L_{\mathcal{V}_s}$ is an analytic set in M.*

Proof. Let $L_{\mathcal{V}_s}$ be a local branch of $\mathcal{F}$ with respect to $\mathcal{V}_s$. If $L_{\mathcal{V}_s}$ were not analytic, there should be a point $x \in L_{\mathcal{V}_s}$ and an open set U around x, such that $U \cap L_{\mathcal{V}_s}$ is defined by the common zeros of an *infinite* family of holomorphic functions. Since x is regular for the foliation $\mathcal{F}$, then $L_{\mathcal{V}_s}$ accumulates on itself in U, and the same argument as in Lemma 4.4.8 applies, contradicting the fact that $\dim(\mathscr{E}_{L_{\mathcal{V}_s}}) = 0$.

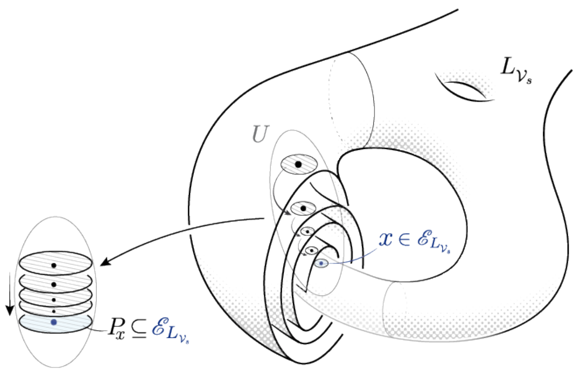


Figure 4.12

□

Remark 4.4.10. As a by-product of these lemmas, we have that $L_{\mathcal{V}_s}$ cannot accumulate on itself. It then follows that $L_{\mathcal{V}_s}$ is an analytic subset of $M \setminus \mathscr{E}_{L_{\mathcal{V}_s}}$.

Lemma 4.4.11. *Under the previous assumptions, the closure of the branch $L_{\mathcal{V}_s}$, denoted by $\overline{L}_{\mathcal{V}_s}$, is an analytic set in M.*

Proof. The statement follows from the Lemma 4.4.9 combined with Remmert-Stein's theorem. In fact, $L_{\mathcal{V}_s}$ is an analytic set having, by hypothesis, discrete ends of leaf (or possibly empty ends of leaf).

- If $\mathscr{E}_{L_{\mathcal{V}_s}} = \emptyset$ then $L_{\mathcal{V}_s}$ is closed in $\mathcal{V}_s$ and Lemma 4.4.9 gives the desired result.
- If $\mathscr{E}_{L_{\mathcal{V}_s}} \neq \emptyset$, since the local branch $L_{\mathcal{V}_s}$ only accumulates on a discrete set of points in $\mathcal{V}_s$, then $\mathscr{E}_{L_{\mathcal{V}_s}}$ is an analytic subset of dimension 0. Finally, owing to Remark 4.4.10 and since $\dim(L_{\mathcal{V}_s}) = 1$, Remmert-Stein theorem applies and leads to the desired conclusion.

□

Recall that a *separatrix* $\mathscr{S}$ for the foliation $\mathcal{F}$, is a germ of analytic curve passing through a singular point y of $\mathcal{F}$, i.e., $y \in \mathscr{S}$, and such that $\mathscr{S} \setminus y$ is a local leaf of $\mathcal{F}$. In the context of the proof of the previous lemma, if $\mathscr{E}_{L_{\mathcal{V}_s}} \neq \emptyset$, then, after possibly reducing $\mathcal{V}_s$, it follows that $L_{\mathcal{V}_s}$ is a separatrix of $\mathcal{F}_{\mathcal{V}_s}$ (and of $\mathcal{F}$).

We can now proceed to the proof of the theorem.

Proof of Theorem III. Let L be a leaf that is not contained in the divisor of zeros or poles of X. If L doesn't approach E, Theorem 4.1.7 implies that when restricted to L, X is complete. Hence L is parabolic (thus entire). Assume that L approaches E. By Theorem 4.4.3 X is semicomplete, hence, up to fixing a point $p \in L$, we can consider the semiglobal flow of X restricted to L. In other words let $\Phi_p : \Omega \subseteq \mathbb{C} \to L$ such that, if $\{t_n\} \subseteq \Omega$ converges to a point in the boundary $\hat{t} \in \partial\Omega$, then $\Phi_p(t_n)$ escapes from every compact set of L.

In what follows, we keep the notation from Section 4.2 and Section 4.3. Since L can only accumulate at points belonging to $I(f) \cap E$, then $\mathcal{V}_s \cap L \neq \emptyset$. Consider the regular and singular parts of L with respect to the fixed singular neighborhood $\mathcal{V}_s$, i.e., the sets $L_{sing} = L \cap \mathcal{V}_s$ and $L_{reg} = L \cap K_1$. We want to show that $\partial\Omega$ is a discrete set of points of $\mathbb{C}$ and that Φ_p can be extended to all of $\mathbb{C}$. In that direction, assume that $\partial\Omega \neq 0$ and let $\hat{t} \in \partial\Omega$. Lemma 4.4.1 allows to carry out the same proof of Lemma 4.2.2 in the present setting. It follows that $\hat{t} \in \partial\Omega \cap \partial\Omega_s$.

The presence of $\hat{t}$ indicates the presence of an end of leaf of L, namely $y \in \mathscr{E}_L$, with $y \in \mathcal{V}_s$. In other words y is contained in the closure of L_{sing}, denoted by $\overline{L}_{\text{sing}}$. Moreover, Lemma 4.4.8 ensures that $y \in \operatorname{Sing}(\mathcal{F})$.

Let $\{t_n\} \subseteq \Omega$ be a sequence of points converging to $\hat{t}$, such that $\Phi_p(t_n) \to y$. Then, Lemma 12 implies that, eventually, $\Phi_p(t_n)$ remains within a local branch of $\mathcal{F}$ that accumulates on y. In other words:

Claim 4.4.12. *For a large enough n, $\{\Phi_p(t_n)\} \subseteq L_{\mathcal{V}_s}$ where $L_{\mathcal{V}_s}$ stands for some local branch of $\mathcal{F}$ with respect to $\mathcal{V}_s$.*

□

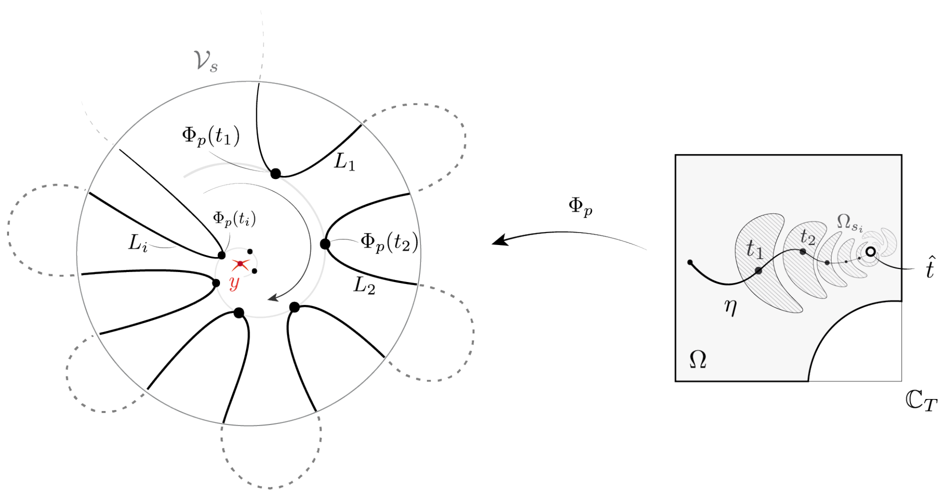


Figure 4.13 The situation represented in this schematic figure cannot occur as t_n approaches the boundary point $\hat{t}$.

Having fixed this local branch that accumulates on y, Lemma 4.4.9 and Lemma 4.4.11 ensure that $\overline{L}_{\mathcal{V}_s}$ is analytic, that is to say, $L_{\mathcal{V}_s}$ induces a separatrix for $\mathcal{F}|_{\mathcal{V}_s}$ at y.

Let $P : \mathbb{D} \to L$ be the Puiseux parametrization of the separatrix $\mathscr{S} = \overline{L}_{\mathcal{V}_s}$. Then $P(0) = y$, P is injective on $\mathbb{D}$ and it is a diffeomorphism of $\mathbb{D}^*$ onto its image. Consider the pullback of the restriction of X to $\mathscr{S}$ given by

$$Z = P_*(X|_{\mathscr{S}})$$

Since $X|_{\mathscr{S}}$ is semicomplete, so is Z. It follows (c.f. [23]) that Z needs to be holomorphic at the origin. Besides, according to [40], the only possibilities for 1-dimensional semicomplete vector fields must satisfy $\mathrm{ord}|_{u=0}Z = \{0, 1, 2\}$. Then, up to a suitable change of coordinates, Z takes on the form:

$$Z(u) = h(u)\frac{\partial}{\partial u}$$

where h is one of the following functions: $h(u) = ct$, $h(u) = u$, or $h(u) = u^2$.

The last two cases are ruled out since their flows reach 0 at infinite time. Thus, the only possibility for h is to be the constant function. In such a case, Φ_p can be holomorphically extended to $\hat{t}$ by setting $\Phi_p(\hat{t}) = y$. This completes the proof of Theorem 4.4.6.

□

To end this work, we provide an example as an application of Theorem 4.4.6.

Example 4.4.13 (Lins-Neto-Guillot)**.** *Let*

$$X = \left(-3x^2 + y^2 + 2xz - 2xy + 2yz\right)\frac{\partial}{\partial x} + \left(3x^2 - y^2 - 6xy + 4yz\right)\frac{\partial}{\partial y} + \\ + \left(-2z^2 + 6xz + 2yz\right)\frac{\partial}{\partial z}$$

and

$$Y = 2y(-x + z)\frac{\partial}{\partial x} + \left(3x^2 - y^2\right)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z}.$$

Consider a linear combination of X and Y given by

$$Z_\alpha = X + \alpha Y$$

where α is any complex number.

Inspired by the work of A. Lins-Neto ([29]), the dynamics of the family of vector fields Z_α were studied by A. Guillot in his paper [18]. He proved that X and Y are semicomplete vector fields, and since X and Y commute (as can be shown by a simple calculation), it follows that Z_α is semicomplete as well. He also showed that the maximal domain of definition of the solutions of Z_α, is all of $\mathbb{C}$ minus a discrete set of points. In other words, the leaves of the induced foliation are entire leaves. Below, we provide an alternative proof of the previous facts, only by assuming that the singularities of Z_α are semicomplete.

First, note that the functions $z(x - y)$ and $z\left(3x^2 + y^2 - 2yz\right)$ are two independent first integrals for X. Similarly $z(-2x + z)$ and $z\left(3x^2 - 6xz + y^2 + 2z^2\right)$ are first integrals for Y. Hence X and Y are completely integrable. On the other hand,

$$f = z^2\left(9x^4 + 6x^2y^2 + y^4 - 4x^3z - 12xy^2z + 4y^2z^2\right)$$

is a common first integral for both X and Y, and hence for the linear combination Z_α as well.

All the previous vector fields can be extended to $\mathbb{CP}^3$ by performing the usual changes of coordinates. Namely, in coordinates $(x, y, z) \to (1/x, y/x, z/x) = (u_1, u_2, u_3)$, the vector field Z_α can be written as $\widetilde{Z}_\alpha = (1/u_1) \cdot Z_\alpha^R$ where

$$Z_\alpha^R = u_1F(u_2, u_3)\frac{\partial}{\partial u_1} + G(u_2, u_3) + \frac{\partial}{\partial u_2} + u_3H(u_2, u_3)\frac{\partial}{\partial u_3}.$$

and F, G and H are holomorphic functions. Note that $\{u_1 = 0\} \simeq \Delta_\infty$ is indeed invariant by $\tilde{Z}_\alpha$. We will focus on studying the vector field in these coordinates, an analogous analysis can be carried out in the remaining charts.

Note that the first integral f in coordinates (u_1, u_2, u_3) takes the form $\tilde{f} = (u_3^2/u_1^6) \cdot f_1$ where f_1 is the irreducible polynomial defined by:

$$f_1 = 9 + u_2^4 - 4u_3 + 2u_2^2(3 - 6u_3 + 2u_3^2)$$

It follows that (in this chart) the indeterminacy set of $\tilde{f}$ consists of two complex lines which intersect at the point $(0, -i\sqrt{3}, 0)$. Namely;

$$I(\tilde{f}) = \underbrace{\{(0, u_2, 0)\}}_{I_1} \cup \underbrace{(\{f_1 = 0\} \cap \{u_1 = 0\})}_{I_2}$$

Note also that $\Delta_\infty \setminus I(\tilde{f})$ is the unique leaf of the codimension-1 foliation induced by $\tilde{f}$ mapped to ∞.

Three singular points lie in the first component I_1 of the indeterminacy set $\{(0, u_2, 0)\}$, namely, they are given by $u_2 = -i\sqrt{3}$, $u_2 = i\sqrt{3}$, and $u_2 = 1 + \alpha$. And four singular points lie in the second component I_2 of the indeterminacy set: $(0, -i\sqrt{3}, 0)$, $(0, -1, 2)$, $(0, 1, 2)$ and

$$u_2 = -\frac{2(1+\alpha)}{1 + (1+\alpha)^2}, \quad u_3 = \frac{1}{4}\left(5 + (1+\alpha)^2 + \frac{4}{1 + (1+\alpha)^2}\right).$$

The linear part of the holonomy of these singular points can be explicitly computed by comparing the eigenvalues corresponding to each singularity, and it can be shown that none of them is hyperbolic (this also holds for the missing singular point that appears in the other charts). In other words $Hol'(\Delta_\infty, q_i) \subseteq S^1$, where $q_i \in \mathrm{Sing}(\mathcal{F}) \cap \Delta_\infty$.

It remains to show that there is no hyperbolic holonomy arising in the regular part. Note that whereas I_1 is basically isomorphic to $\mathbb{CP}^1$, I_2 is a torus, and hence there might be holonomy arising from the interaction of its meridian and parallel lines.

Since all the solutions are tangent to the level sets of $\tilde{f}$, we can restrict the analysis to an arbitrary level set, S_c, of $\tilde{f}$. Note that in dimension 2, the existence of a codimension-1 foliation is trivial. Therefore, in order to apply our results, we need to show that when restricted to any S_c, the holonomy is not hyperbolic, i.e., $Hol'(I(f), q) \subseteq S^1$ for $q \in I(f)$. As already mentioned, it is enough to verify the claim for the case of I_2. Recall that, by construction, the leaves of $\tilde{Y}$ are also tangent to level sets of $\tilde{f}$. To simplify notation, in the sequel, the vector fields $\tilde{Z}_\alpha$ and $\tilde{Y}$ restricted to S_c will be denoted in the same way.

Let q be a regular point for $\mathcal{F}$, with $q \in I_2$, and Σ_q be a transverse section to the flow of $\tilde{Z}_\alpha$ contained in S_c. In rectified coordinates (x, y) around q, $\tilde{Z}_\alpha = y^{-1} g(x, y) \partial/\partial x$, where $y = 0$ is identified with Δ_∞. Since $\tilde{Z}_\alpha$ and $\tilde{Y}$ commute, then by a simple computation, it can be proved that $\tilde{Y}$ takes the form

$$\tilde{Y} = y^{-1}\left(a(x, y)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}\right)$$

where a and b are holomorphic functions. Note that b can be written as

$$b(y) = y^k(1 + o(z)) = y^k(1 + \dots)$$

where k could be any integer (possibly 0). Since b only depends on the variable y, it is invariant by the holonomy group of the curve I_2. In other words, if $W = b(y)\partial/\partial y$, then for any $\gamma \in I_2$,

$$(h_\gamma)_* W = W. \tag{4.3}$$

Now assume that the holonomy map h_γ is hyperbolic, i.e., that $|h'_\gamma(0)| \neq 1$. Hence, by Koening's theorem it can be linearized and without loss of generality we can assume that $h_\gamma(y) = \lambda y$ with $|\lambda| \neq 1$. Therefore, Equation 4.3, takes the form

$$\lambda^{k-1} y^k(1 + \dots) = y^k(1 + \dots)$$

If $k > 1$, then $|\lambda| = 1$, contradicting the assumption that h_γ is hyperbolic. If $k = 1$, then it can be shown that $\tilde{Y}$ is actually a complete vector field, and the commutativity of $\tilde{Y}$ and $\tilde{Z}_\alpha$ yields that $\tilde{Z}_\alpha$ is complete as well.

Now, Theorem 4.3.2 and Theorem 4.4.6 apply proving that $\tilde{Z}_\alpha$ is indeed globally semicomplete, and also that its leaves are entire leaves.

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