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Bessel expansions of a class of Dirichlet Series

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UC|UP Joint PhD Program in Mathematics

Faculty of Sciences of the University of Porto and Faculty of Sciences and Technology of the University of Coimbra

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Abstract

Several aspects connecting the topics of analytic number theory and special functions are studied in this thesis. These include:

1. new analogues of the Selberg-Chowla formula for the Epstein zeta function;
2. upper bounds for the gaps between consecutive critical zeros of certain Dirichlet series;
3. new results on the number of critical zeros of combinations of completed Dirichlet series;
4. new summation formulas attached to products of Bessel functions;
5. the use of the Kontorovich-Lebedev transform to understand the structure of some classical summation formulas;
6. the use of Koshliakov zeta function theory to generalize formulas from Ramanujan's notebooks.

Keywords: Dirichlet series, Selberg-Chowla formula, Epstein zeta function, Hardy's theorem, Koshliakov zeta function, Bessel functions, Summation formulas, Index transforms.

Resumo

Nesta tese são abordados diversos tópicos que conectam a teoria analítica de números com a teoria das funções especiais. Alguns destes tópicos são:

1. novos análogos da fórmula de Selberg-Chowla para a função zeta de Epstein;
2. estimativas para a distância entre zeros críticos consecutivos de certas séries de Dirichlet;
3. novos resultados sobre o número de zeros críticos de combinações de séries de Dirichlet completas;
4. novas fórmulas de somação associadas a produtos de funções de Bessel;
5. aplicação da transformada de Kontorovich-Lebedev no estudo de fórmulas de somação clássicas;
6. aplicação da teoria da função zeta de Koshliakov no estudo de generalizações de fórmulas devidas a Ramanujan.

Palavras-chave: Séries de Dirichlet, Fórmula de Selberg-Chowla, Função zeta de Epstein, Teorema de Hardy, Função zeta de Koshliakov, Funções de Bessel, Fórmulas de somação, Transformadas de índice.

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Chapter 0

An Informal Introduction and Motivation

Let $Q(m,n)=am^2+bmn+cn^2$ be a positive definite quadratic form. The classical Epstein zeta function is defined as the Dirichlet series

$$\zeta(s,Q)=\sum_{m,n\neq 0}\frac{1}{Q(m,n)^s},\quad \text{Re}(s)>1, \tag{0.0.1}$$

where the notation given in the subscript, m, $n\neq 0$, here means that only the term $m=n=0$ is omitted from the infinite series in (0.0.1). From a formula due to Chowla and Selberg [201], $\zeta(s,Q)$ admits the following Fourier expansion

$$\begin{aligned}a^s\Gamma(s)\,\zeta(s,Q)&=2\Gamma(s)\zeta(2s)+2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)\\&+\,8k^{\frac{1}{2}-s}\pi^s\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-\frac{1}{2}}\left(2\pi kn\right),\end{aligned} \tag{0.0.2}$$

where $\Delta:=4ac-b^2$ is the discriminant of the quadratic form Q, $k^2:=\Delta/4a^2$ and $\sigma_{\nu}(n)=\sum_{d|n}d^{\nu}$ denotes the divisor function of index ν. Also, $\zeta(s)$ and $K_{\nu}(x)$ represent respectively the Riemann zeta function and the modified Bessel function of the second kind.

Given that $\zeta(s,Q)$ is expressed as an infinite series involving the Bessel function $K_{\nu}(x)$, we refer to formulas of the form (0.0.2) as **Bessel expansions**. The primary objective of this thesis is to explore analogues of (0.0.2) in various contexts, using them to investigate new classes of summation formulas and to analyze the zero distribution of certain Dirichlet series.

To briefly explain our main goals and the content of each chapter, in this informal introduction we will write

$$\phi(s)=\sum_{n=1}^{\infty}\frac{a(n)}{\lambda_n^s},\quad \text{Re}(s)>\sigma_a,\quad 0<\lambda_n\nearrow\infty, \tag{0.0.3}$$

as a Dirichlet series with a finite abscissa of absolute convergence, σ_a, and whose analytic continuation satisfies the functional equation

$$\Gamma(s)\phi(s) = \Gamma(r-s)\phi(r-s), \quad r > 0, \tag{0.0.4}$$

which we will encounter several times later and refer to repeatedly as **Hecke's functional equation**. It is clear from the reflection present in (0.0.4) that the line of symmetry of $\phi(s)$ is the vertical line $\text{Re}(s) = \frac{r}{2}$, which will be referred throughout our work as the **critical line**.

It comes as no surprise that the most important zeros of a Dirichlet series $\phi(s)$ are those located on this line of symmetry. These **critical zeros** are fundamental to many problems in number theory, especially when the Dirichlet series $\phi(s)$ is specified as the Riemann $\zeta-$function or a Dirichlet $L-$function.

Starting only with some general assumptions, such as the functional equation (0.0.4), the primary questions that arose during the course of this investigation were the following.

1. If we define $N_0(T) = \#\left\{t \in [0,T] \,:\, \phi\left(\frac{r}{2}+it\right)=0\right\}$, what is the asymptotic order of $N_0(T)$?

2. Given T sufficiently large, what is the upper bound for $H = H(T)$ so that the interval $[T, T+H]$ contains the ordinate of a critical zero of $\phi(s)$?

3. What kind of information does an identity like (0.0.2) provide about the zero distribution of a general Dirichlet series like $\phi(s)$?

4. What kind of summation formulas are relevant from the point of view of the Fourier expansion (0.0.2)?

5. What is the connection between the Selberg-Chowla formula (0.0.2) and the theory of Index transforms?

To a large extent, the content of this thesis can be found in some of the papers produced during the last three years. Each chapter aligns with the corresponding article, although we have extended some discussions and modified the exposition at some points. The Selberg-Chowla formula (0.0.2) remains as the leitmotiv of all the chapters of this thesis. Every chapter begins with a detailed introduction, providing both context and motivation for the study of its respective topic. Below is a summary of the chapters included in this work.

- **Chapter 1.** Taking a general point of view, in the first chapter of this thesis we essentially prove that several Dirichlet series admit a Fourier expansion involving the modified Bessel function $K_{\nu}(x)$. For example, the Dirichlet series $\zeta(s)\zeta(s-7)$ can be expressed via the formula

$$\begin{aligned}2(2\pi)^{-s}\Gamma(s)\,\zeta(s)\,\zeta(s-7) &= (2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\,\zeta(s-3) + (2\pi)^{4-s}\Gamma(s-4)\,\zeta(s-4)\,\zeta(s-7)\\ &+ 480\sum_{m,n=1}^{\infty}\sigma_3(m)\,\sigma_3(n)\left(\frac{m}{n}\right)^{\frac{s-4}{2}}K_{s-4}\left(4\pi\sqrt{m\,n}\right).\end{aligned} \tag{0.0.5}$$

Some curious applications of expressions like (0.0.5) concern estimates for the gaps between consecutive critical zeros of certain Dirichlet series. To illustrate one of our results, let $\zeta_k(s)$ be the zeta function attached to the sum of $k-$squares. Siegel [202] proved very sharp results about the zero distribution of $\zeta_k(s)$ when $k \geq 4$, including exact estimates for the number of its critical zeros, i.e., zeros located on the line $\mathrm{Re}(s) = \frac{k}{4}$. Siegel admitted that his method did not work in the study of the zeros of $\zeta_3(s)$,

$$\zeta_3(s) := \sum_{m_1,m_2,m_3 \neq 0} \frac{1}{(m_1^2+m_2^2+m_3^2)^s}, \quad \mathrm{Re}(s) > \frac{3}{2}.$$

Combining some of Siegel's arguments with a general Bessel expansion of the form (0.0.5), we are able to establish the first quantitative estimate for a gap between consecutive zeros of $\zeta_3\left(\frac{3}{4}+it\right)$. In fact, as we shall prove later, for any $\epsilon > 0$, there is some $T_0(\epsilon)$ such that, if $T \geq T_0(\epsilon)$, we can always find a critical zero of $\zeta_3(s)$, $s = \frac{3}{4} + i\tau$, with $\tau \in [T,\, T + T^{1/2+\epsilon}]$. This chapter is based on the publication [194].

- **Chapters 2 and 3.** Using a general analogue of the transformation formula for Jacobi's $\theta-$function, in the second and third chapters of this thesis we prove estimates for the number of zeros of shifted combinations of completed Dirichlet series. A very special case of one of the main results given in Chapter 3 is the following. Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty} |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. If $\tilde{N}_0(T)$ denotes the number of critical zeros, $s = \frac{1}{2} + it,\ 0 \leq t \leq T$, of the shifted combination

$$\tilde{F}(s) = \sum_{j=1}^{\infty} c_j \left\{ \pi^{-\frac{1}{2}(s+i\lambda_j)} \Gamma\left(\frac{s+i\lambda_j}{2}\right) \zeta\left(s+i\lambda_j\right) + \pi^{-\frac{1}{2}(s-i\lambda_j)} \Gamma\left(\frac{s-i\lambda_j}{2}\right) \zeta\left(s-i\lambda_j\right) \right\}, \tag{0.0.6}$$

then there exists some $c > 0$ such that

$$\liminf_{T\to\infty} \frac{\tilde{N}_0(T)}{\sqrt{T}/\log(T)} \geq c. \tag{0.0.7}$$

The key takeaway from this result is that, regardless of how the bounded shifts $(\lambda_j)_{j\in\mathbb{N}}$ and the sequence $(c_j)_{j\in\mathbb{N}}$ are chosen, one can always find at least $\gg \sqrt{T}/\log(T)$ zeros of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ within the interval $t \in [0, T]$, for a sufficiently large T. The estimate (0.0.7) can be seen as a quantitative analogue of a result due to Dixit, Robles, Roy and Zaharescu [74]. Besides studying zeros of shifted combinations of the form (0.0.6), in the second chapter we also establish new summation formulas. A particular case of a general identity given there is

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\left(n+x^2+y^2+\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\right)^{11}}$$
$$= \frac{4\pi}{(2y)^{11}} \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{\frac{11}{2}}} J_{11}\left(4\pi\sqrt{n}y\right) K_0\left(4\pi\sqrt{n}\,x\right), \quad x, y > 0,$$

where $\tau(n)$ denotes Ramanujan's $\tau-$function and $J_\nu(x)$ represents the Bessel function of the first kind. These chapters are respectively based on the publication [191] and in the preprint [185].

- **Chapter 4.** Motivated by some of the results deduced in the second chapter, the fourth chapter of our thesis consists in proving several transformation formulas. The driving force behind this chapter is a final remark given by Berndt, Dixit, Kim and Zaharescu in [43], which motivates the study of infinite series of the type

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, I_\mu\left(2\pi\sqrt{ny}\right)\, K_\nu\left(2\pi\sqrt{n}\, x\right), \quad \mathrm{Re}(\nu)>0,\ \ \mathrm{Re}(\mu)>-1,\ \ x>y>0, \tag{0.0.8}$$

where $r_k(n)$ denotes the number of ways in which n can be expressed as the sum of k squares. Also, $I_\mu(x)$ represents the modified Bessel function of the first kind. One particular case of the general results obtained in this chapter is the transformation formula

$$\sum_{n=0}^{\infty} r_2(n)\sqrt{n}\, I_0(2\pi\sqrt{ny})\, K_1\left(2\pi\sqrt{n}\, x\right) = \frac{x}{2\pi^2}\sum_{n=0}^{\infty}\frac{r_2(n)\ \left(n+x^2-y^2\right)}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{3}{2}}},$$

where it is assumed that $x>y>0$. This chapter is based on the preprint [186].

- **Chapter 5.** Inspired by the Fourier-Whittaker expansion in the theory of Maass forms, our fifth chapter is devoted to find new summation formulas for infinite series of the type

$$\sum_{n=1}^{\infty} W_{\mu, i\alpha n}(x), \quad \alpha, x>0,\ \ \mu\in\mathbb{R},$$

where $W_{\mu,\nu}(x)$ denotes the Whittaker function. We have been able to obtain several new results concerning infinite series of this kind, such as the formula

$$\begin{aligned} &W_{-\frac{1}{4},0}(x)\, W_{\frac{1}{4},0}(x) + 2\sum_{n=1}^{\infty} W_{-\frac{1}{4}, i\alpha n}(x)\, W_{\frac{1}{4}, i\alpha n}(x) \\ &= \frac{x}{\sqrt{2}\alpha}\left\{e^{-\frac{x}{2}}K_0\left(\frac{x}{2}\right)+2\sum_{n=1}^{\infty} e^{-\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)}\, K_0\left(\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right\}, \end{aligned}$$

which connects two different classes of special functions. This chapter is based on the preprint [195].

- **Chapter 6.** The Russian Mathematician N. S. Koshliakov, famous for his contributions to the fields of differential equations, special functions and number theory, was sent to a labor camp during the siege of Leningrad. During his time as prisoner, Koshliakov wrote two long memoirs on a general theory of the Riemann zeta function. Motivated by a problem concerning heat conduction, Koshliakov introduced the zeta function [132]

$$\zeta_p(s)=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\lambda_n^s}, \quad \mathrm{Re}(s)>1, \tag{0.0.9}$$

where $p>0$ and $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence defined by the positive roots of the transcendental equation

$$p\sin(\pi y)+y\,\cos(\pi y)=0,\quad p>0. \tag{0.0.10}$$

A simple analysis of the equation (0.0.10) shows that the following limiting cases hold

$$\lim_{p\to\infty}\zeta_p(s)=\zeta(s),\quad \lim_{p\to 0^+}\zeta_p(s)=(2^s-1)\,\zeta(s).$$

Thus, if one studies the theory of the Dirichlet series (0.0.9), one certainly obtains, as a particular case, new results about $\zeta(s)$. The sixth and last chapter of our thesis consists in studying summation formulas in Koshliakov's setting. For example, as a corollary of one of our results, we have been able to deduce the following transformation formula

$$\frac{(p^2-a)\left\{\cos(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})\right\}-e^{-\pi\sqrt{2a}}\left(p^2-\sqrt{2a}\,p+a\right)-\sqrt{2a}\,p\left\{\cos(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right\}}{p^2\left(\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})\right)+\sqrt{2a}\,p\left(\sinh(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right)+a\left(\cosh(\pi\sqrt{2a})+\cos(\pi\sqrt{2a})\right)}$$
$$=\frac{2\sqrt{2a}}{\pi}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\lambda_n^2}{a^2+\lambda_n^4}-1,$$

where, recall, $\lambda_n:=\lambda_n(p)$ is the sequence of positive solutions of (0.0.10). When $p\to\infty$, the previous formula reduces to the elegant identity

$$\sum_{n=1}^{\infty}\frac{n^2}{a^2+n^4}=\frac{\pi}{2\sqrt{2a}}\,\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})},\quad a>0,$$

which is due to Glaisher [94] but also shows up in Ramanujan's second notebook [28]. This chapter is based on the publication [193] and is deeply influenced by a paper of Dixit and Gupta [68], which first introduced us to this topic.

Additional work done during the period of this degree but not covered in this thesis can be found in [183, 184, 187–189].

Chapter 1

On the Epstein zeta function and the zeros of a class of Dirichlet series

1.1 Introduction and preliminary definitions

Let $Q(x,y) = ax^2 + bxy + cy^2$ be a real and positive definite quadratic form. As indicated in the brief introduction above, the classical Epstein zeta function is defined as the Dirichlet series

$$\zeta\left(s,\,Q\right) = \sum_{m,n\neq 0}\frac{1}{Q(m,n)^{s}}, \quad \text{Re}(s)>1, \tag{1.1.1}$$

where the notation given in the subscript, $m,\,n\neq 0$, here means that only the term $m=n=0$ is omitted from the infinite series.

In 1949, S. Chowla and A. Selberg [56] announced the following formula for (1.1.1), valid in the entire complex plane,

$$\begin{aligned} a^{s}\Gamma(s)\,\zeta(s,\,Q) &= 2\Gamma(s)\zeta(2s) + 2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) \\ &\quad + 8k^{\frac{1}{2}-s}\pi^{s}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-\frac{1}{2}}\left(2\pi k\,n\right), \end{aligned} \tag{1.1.2}$$

where $\Delta := 4ac-b^2$ is the discriminant of the quadratic form, $k^2 := \Delta/4a^2$ and $\sigma_{\nu}(n)=\sum_{d|n}d^{\nu}$ is the divisor function of index ν. Also, $\zeta(s)$ denotes the classical Riemann zeta function and $K_{\nu}(z)$ is the modified Bessel function of the second kind.

According to Berndt's revision [25] and the more recent survey [39], after its first announcement, (1.1.2) was proved by Rankin [180], Bateman and Grosswald [15], Chowla and Selberg [201] (in a paper which appeared 18 years after the statement of (1.1.2)) and Motohashi [158], although the latter author had the main goal of proving directly the first Kronecker limit formula. As it should be expected, all these proofs used the theory of Fourier series and the Poisson summation formula. Additionally, Anfertieva [4] provided a proof using the theory of Mellin transforms.

Berndt himself [25] gave a very interesting proof of (1.1.2), which was based on a previous theorem due to him regarding the Bessel expansion of the so-called generalized Dirichlet series [21].

In fact, Berndt's proof of (1.1.2) was considered for a more general Epstein zeta function of the form

$$\zeta(s,\,Q;\,g,\,h)=\sum_{m,n\neq 0}\frac{e^{2\pi i(h_1m+h_2n)}}{Q(m+g_1,\,n+g_2)^s},\quad \text{Re}(s)>1,$$

where g_1, g_2, h_1 and h_2 are certain real parameters. Although only written for the particular case (1.1.2), a simplified version of Berndt's argument was later given by Kuzumaki [136]. It is remarkable that Kuzumaki's proof, as well as Berndt's, depends explicitly on the functional equation for the Hurwitz zeta function, $\zeta(s,\,a)$. Although this functional equation can be derived immediately from the theta transformation formula [92]

$$\frac{1}{2}+\sum_{n=1}^{\infty}e^{-\alpha n^2}\cos(\beta n)=\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\sum_{n=-\infty}^{\infty}e^{-\frac{1}{\alpha}\left(\pi n+\frac{\beta}{2}\right)^2},\tag{1.1.3}$$

it can nevertheless be established in strikingly different ways [1], suggesting a connection between (1.1.2) and other lesser-known summation formulas.

The generalization of (1.1.2) for Epstein zeta functions attached to quadratic forms with $n\geq 3$ variables was given by Terras [222], whose proof employed a multidimensional version of the Poisson summation formula. Suzuki [218] provided analogues of (1.1.2) for other Dirichlet series which apriori have nothing to do with (1.1.1). The set of Dirichlet series studied by Suzuki includes the Riemann $\zeta-$function, Dirichlet $L-$functions attached to primitive characters and $L-$functions associated with holomorphic cusp forms.

The idea behind a Bessel expansion of a Dirichlet series of the form (1.1.1) seems to have been firstly motivated by a formula discovered by Watson [[233], p. 99, eq. (4)] (see (1.2.29) below) and perhaps the first explicit appearance of (1.1.2) in the literature is contained in a paper written by Taylor [219]. Although Taylor himself attributed (1.1.2) to Kober [129], it is in Taylor's paper where the continuation of the Epstein zeta function via (1.1.2) is made explicit for the first time. Taylor's proof employed an elegant Mellin-Barnes representation involving the Gauss hypergeometric function[2], ${}_2F_1(a,b;c;z)$. Although Taylor did not prove the functional equation for $\zeta(s,\,Q)$ directly from (1.1.2), as Selberg and Chowla did, his paper appears to provide the first indication of a connection between (1.1.2) and the functional properties of $\zeta(s,\,Q)$.

The Selberg-Chowla formula has found many applications in Number Theory. By using it, Selberg and Chowla [201] proved the existence of a real zero of $\zeta(s,\,Q)$ in the open interval $(\frac{1}{2},\,1)$, when the quadratic form is $Q(x,\,y)=x^2+cy^2$ and c is large enough. This result was later improved by Bateman and Grosswald [[15], p. 367, Theorem 3], who have shown that, for a general positive definite and real quadratic form Q, $\zeta(s,\,Q)$ has always real zero between $\frac{1}{2}$ and 1, provided $k:=\sqrt{\Delta}/2a>7.0556$. A generalization of this argument to Epstein zeta functions attached to

[1] see, for instance, a very recent paper by A. Dixit and R. Kumar [69], in which the functional equation for $\zeta(s,\,a)$ is proved via its Hermite integral representation.

[2] Although fairly unknown, Taylor's approach has found interesting applications in some aspects of spectral theory. Although unaware of Taylor's earlier representation, this way of using Mellin-Barnes integrals containing the Gauss hypergeometric function is crucial in [[83], p. 120, eq. (4.3)], where it is used to establish an analogue of Hardy-Littlewood's Theorem to $L-$functions attached to Maass cusp forms. We have also used some of Taylor's ideas to prove a new formula of Popov type in [183].

ternary quadratic forms was furnished by Terras [223], who employed prior results due to her concerning a multidimensional version of (1.1.2) [[222], p. 480, eq. (2.3)].

With the aid of (1.1.2), Stark [212] and Fujii [93] gave very detailed descriptions about the distribution of the complex zeros of $\zeta(s,\,Q)$ in bounded regions. Building on the work of many previous authors, H. Ki [126] established a remarkable result regarding the zeros of each finite truncation of the right-hand side of (1.1.2), showing that all but finitely many of them lie on the critical line $\text{Re}(s)=1/2$. The idea of studying the zeros of the Epstein zeta function $\zeta(s,\,Q)$ via equivalent representations to (1.1.2) appears to trace back to Deuring's work [62], which strongly influenced Stark's research on the zeros of $\zeta(s,Q)$.

In another interesting paper [61], Deuring connected the Fourier expansion for the simple Epstein $\zeta-$function $\zeta(s,\,Q_0)$ (where $Q_0(x,y)=x^2+y^2)$[3] with the existence of infinitely many zeros of $\zeta(s)$ on the critical line $\text{Re}(s)=\frac{1}{2}$. Using this relation between two drastically different Dirichlet series, he was able to give a new proof of Hardy's Theorem. Hardy's theorem [106] remains one of the most significant breakthroughs in the theory of the Riemann zeta function. In a brief but remarkable note, Hardy demonstrated that infinitely many zeros of $\zeta(s)$ lie on the critical line $\text{Re}(s)=\frac{1}{2}$.

Although Deuring's proof of this famous result of Hardy requires more motivation than the proofs written earlier by Hardy himself [106], Landau [140] or Fekete [88], it is nevertheless an interesting proof. Like Hardy's original argument, Deuring's proof works by *reductio ad absurdum* and the way it seeks the contradiction is through the subconvexity estimate

$$\zeta\left(\frac{1}{2}+it,\,Q_0\right)=O\left(|t|^{\frac{1}{3}+\epsilon}\right), \tag{1.1.4}$$

due to Titchmarsh [225]. This stands in sharp contrast with the classical proofs of Hardy's Theorem, in which one arrives at a contradiction by using the transformation formula for Jacobi's $\theta-$function [106, 140].[4] Although much more difficult to establish than the vanishing of the $\theta-$function itself, a condition like (1.1.4) is deep, since it connects two fundamental properties of two distinct Dirichlet series, i.e., the zeros of $\zeta(s)$ and the magnitude of $\zeta(s,\,Q_0)$. This correspondence, driven by the Selberg-Chowla formula (1.1.2), introduces an additional degree of freedom in the study of the zeros of $\zeta(s)$. This degree of freedom corresponds precisely to the independent study of the Epstein zeta function $\zeta(s,\,Q_0)$.

Our goal in this chapter is to study analogues of the expansion (1.1.2) in a sufficiently general framework and, by appealing to its symmetries, deduce new identities and novel results about zero distribution of a given Dirichlet series at its critical line. We start by making a distinction between the two main sections of this chapter:

[3]this Epstein zeta function will appear several times in this thesis but with a different symbol to represent it, namely "$\zeta_2(s)$", because $\zeta(s,\,Q_0)$ is none other than the zeta function attached to the sum of two squares.

[4]In the next chapter, we shall study a massive generalization of Hardy's classical method, working for a very large set of zeta functions.

1. **The analytic continuation of a "diagonal" Epstein zeta function**: for $\text{Re}(s)$ sufficiently large, one of the goals of our first chapter is to study an infinite series of the form

$$\sum_{m,n\neq 0}^{\infty}\frac{a_1(m)\,a_2(n)}{(\lambda_m+\lambda_n')^s}, \tag{1.1.5}$$

where $a_1(m)$, $a_2(n)$, λ_m and λ_n' will be specified later. This double series [5] was also studied in Berndt's paper [25] and a formula of Selberg-Chowla type for it has been indicated there. Under additional assumptions, in section 1.2 we give two formulas analogous to (1.1.2) and use them to arrive at a functional equation for this double infinite series. We also extend the scope of these formulas to treat a generalization of (1.1.5) with the summation taken over an arbitrary number of indices (see (1.1.18) below). The methods employed in this section are not unrelated in spirit to the ones used by Suzuki [218], who proved analogues of (1.1.2) for other Dirichlet series which apriori have nothing to do with (1.1.1), including Dirichlet $L-$functions and $L-$functions attached to holomorphic cusp forms. Using the formulas obtained in this section, we write three examples concerning a generalization of a famous identity due to Ramanujan and Guinand [37].

2. **The zeros of a class of Dirichlet series**: In section 1.3, we use the Selberg-Chowla formulas derived to extend a beautiful argument due to Deuring [61] for a large class of Dirichlet series. As mentioned above, Deuring connected the Fourier expansion (1.1.2) for the simple Epstein $\zeta-$function $\zeta(s,\,Q_0)$ (where $Q_0(x,y)=x^2+y^2$) with the existence of infinitely many zeros of $\zeta(s)$ on the critical line $\text{Re}(s)=\frac{1}{2}$. Using this relation between two drastically different Dirichlet series, he was able to deduce Hardy's theorem for $\zeta(s)$. In our Theorem 1.3.1, we extend the analogy pointed out by Deuring and we prove that, in a sense, the existence of infinitely many zeros of the double series (1.1.5) at its critical line implies the existence of infinitely many zeros, also on their critical lines, of the auxiliary series constructing it, namely $\sum a_1(n)/\lambda_n^s$ or $\sum a_2(n)/\lambda_n'^s$. Following the method of the proof of this general theorem, we furnish, as an example, quantitative results concerning a family of zeta functions whose critical zeros have not been considered before. To the best of our knowledge, the results obtained as Example 1.3.3 and Corollary 1.3.5 are novel and they cannot be established by the alternative methods appearing in the literature.

Although written in the formal setting of generalized Dirichlet series, the method used by us to prove the existence of infinitely many zeros of $\phi(s)$ on the line $\text{Re}(s)=\frac{r}{2}$ seems to possess certain advantages.

The first of these is its uniform applicability to a broad class of Dirichlet series, allowing to generalize certain results due to Berndt and Hecke [27, 113] (see Corollaries 1.3.1, 1.3.2 and 1.3.3 below). The method also permits the establishment of a quantitative estimate concerning gaps between the zeros that a Dirichlet series possesses at its critical line (see Corollary 1.3.4 below).

[5]Throughout this chapter and overall thesis we shall use the term "double series" or "multiple series" to denote Dirichlet series attached to more than one arithmetical function. This slightly contrasts with the standard terminology, in which these terms are applied to Dirichlet series with more than one complex variable.

The second advantage is the fact that it provides a certain degree of freedom in applications, since in order to check that Hardy's theorem is valid for a particular Dirichlet series, it suffices to verify (independently) that it holds for at least one (out of the infinitely many!) Epstein zeta functions attached to it. As proved by Suzuki [218], it is possible to obtain analogues of the Selberg-Chowla formula for $\zeta(s)$ (or $L(s,\,\chi)$ and $L(s,\,f)$), so that we can find Dirichlet series to which $\zeta(s)$ (or $L(s,\,\chi)$ and $L(s,\,f)$) plays the same role as the Epstein zeta function (1.1.1) does with $\zeta(s)$. Thus, the results described by Theorem 1.3.1 and Corollary 1.3.4 can be employed, for example, to study the zeros of the Dirichlet series appearing in Suzuki's paper.

This is certainly explored in one of the examples given in our contribution (see Example 1.3.3 below), which gives a uniform result for the gap between the zeros of a family of Dirichlet series which we now introduce. If $\theta(x)$ denotes the classical Jacobi $\theta-$function,

$$\theta(x)=\sum_{n\in\mathbb{Z}}e^{-\pi n^2x},$$

then, for any $\alpha>0$, one can define the arithmetical function $r_\alpha(n)$ by [[218], p. 481, eq. (1.6)]

$$\theta^\alpha(x)-1:=\sum_{n=1}^{\infty}r_\alpha(n)\,e^{-\pi nx}. \tag{1.1.6}$$

These powers of Jacobi's theta function seem to have been introduced in a paper by Lagarias and Rains [137], who studied the properties of their coefficients, $r_\alpha(n)$, with considerable detail. Suzuki [[218], p. 481, eq. (1.6)] studied the Dirichlet series associated with (1.1.6), when $0<\alpha<1$, and he succeeded in proving a formula similar to (1.1.2) for Riemann's zeta function [[218], p. 481, Theorem 1]. For $\mathrm{Re}(s)$ large enough, we can consider the formal Dirichlet series (for a rigorous treatment of the analytic continuation of this series, see Step 1 in Example 1.3.3),

$$\zeta_\alpha(s):=\sum_{n=1}^{\infty}\frac{r_\alpha(n)}{n^s}, \tag{1.1.7}$$

and motivate the study of the distribution of its nontrivial zeros on the line $\mathrm{Re}(s)=\alpha/4$. Of course, when α is an integer, $\zeta_\alpha(s)$ reduces to the zeta function attached to the sum of α squares (note that $\zeta_1(s):=2\zeta(2s)$) and, for this case, there already exist very sharp results due to Siegel [[202], p. 170, Theorem 14]. For $\alpha=k\in\mathbb{N}$ and $k\geq 12$, Siegel proved that all the zeros of the function $\zeta_k(s)$ in the strip $2\leq\mathrm{Re}(s)\leq\frac{k}{2}-2$ are simple and lie on the line $\mathrm{Re}(s)=\frac{k}{4}$, with at most a finite number of exceptions. Furthermore, the interval $0<t<T$ contains exactly $\frac{T}{\pi}\log(2)+O(1)$ zeros of $\zeta_k\left(\frac{k}{4}+it\right)$.

By invoking our general construction of the Epstein zeta functions given in section 1.2, we show that, if $\alpha>4$ is a rational number, then there exists a zero of $\zeta_\alpha\left(\frac{\alpha}{4}+it\right)$ for t in any interval of the form $[T,\,T+T^{\frac{1}{2}+\epsilon}]$, for a sufficiently large T and fixed $\epsilon>0$. In particular, if $N_0(T,\,\zeta_\alpha(s))$ denotes the number of zeros of $\zeta_\alpha\left(\frac{\alpha}{4}+it\right)$ for $0<t<T$, then $N_0(T,\,\zeta_\alpha(s))\gg T^{\frac{1}{2}-\epsilon}$ as $T\to\infty$, for any fixed $\epsilon>0$ and any rational $\alpha>4$. In Corollary 1.3.5, we extend this result for irrational values of α and improve slightly the associated estimate for $N_0(T,\,\zeta_\alpha(s))$. Even though the exponent of the gap 1/2 can be regarded as classical, as far as we know this kind of results cannot be achieved by any of the other methods usually employed to study gaps between nontrivial zeros of certain Dirichlet series (see our Introduction or the third chapter below for a brief description of these).

Although far from being as sharp as Siegel's results in [202] (valid for integral $\alpha \geq 12$), it seems that this method gives a quantitative estimate not covered by the techniques employed in [202]. Siegel writes [[202], p. 363]

" ... *it is possible to discuss the remaining cases for* $3 < k < 12$*, but this requires some numerical computations and we omit it. For the function* $\zeta_3(s)$*, however, our method does not lead to any result.*".

Since the underlying reasoning of example 1.3.3 also works for $\alpha = 3$ (see Remark 1.3.9), the result there obtained gives, as $T \to \infty$, $N_0(T, \zeta_3(s)) \gg T^{\frac{1}{2}-\epsilon}$, for any given $\epsilon > 0$. As far as we know, this seems to be the first quantitative estimate regarding the number of zeros of $\zeta_3(s)$ on the line $\text{Re}(s) = \frac{3}{4}$. Although we omit the full proof, (see Remark 1.3.10 below), we can obtain a slight sharpening of this asymptotic estimate and deduce instead that $N_0(T,\, \zeta_3(s)) \gg T^{1/2}/\log(T),\ T \to \infty$.

Finally, we should remark that some of the formalism exposed in this contribution can be applied in a plethora of contexts. For instance, we have been able to extend our formulas (1.2.1) and (1.2.2) to the case where the arithmetical functions $a_1(m)$ and $a_2(n)$, appearing in (1.1.5), are also functions of s. The study of this general case, together with some consequences to the study of zeros of certain functions, will be one of the topics of Chapters 2 and 4 below (see, for instance, Corollary 2.2.4 of Chapter 2). In an independent investigation [184], we have also seen how introducing a generalized Macdonald function in an expansion like (1.1.2) leads to a theory that can be naturally approached by the methods of this chapter.

We now give the main definitions of the objects studied in this chapter. We employ the setting from Chandrasekharan and Narasimhan's papers [53, 54], as well as from the subsequent extensions given by Berndt [19, 21, 23, 25–27].

Definition 1.1.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers strictly increasing to ∞ and $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$ two sequences of complex numbers not identically zero. Consider the functions $\phi(s)$ and $\psi(s)$ representable as Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s} \tag{1.1.8}$$

with finite abscissas of absolute convergence σ_a and σ_b, respectively. We say that $\phi(s)$ and $\psi(s)$ satisfy the functional equation

$$\Gamma(s)\,\phi(s) = \Gamma(r-s)\,\psi(r-s), \quad r > 0, \tag{1.1.9}$$

if there exists a meromorphic function $\chi(s)$ with the following properties:

1. $\chi(s)=\Gamma(s)\,\phi(s)$ for $\text{Re}(s)>\sigma_a$ and $\chi(s)=\Gamma(r-s)\,\psi(r-s)$ for $\text{Re}(s)<r-\sigma_b$;

2. $\lim_{|\text{Im}(s)|\to\infty}\chi(s)=0$ uniformly in every interval $-\infty<\sigma_1\leq\text{Re}(s)\leq\sigma_2<\infty$.

3. $\phi(s)$ and $\psi(s)$ have analytic continuations to the entire complex plane and are analytic on $\mathbb{C}$ except for possible simple poles located at $s=r$ with residues ρ and $\rho^{\star}$, respectively.

4. The sequences $a(n)$, λ_n and $b(n)$, μ_n can be extended to $n=0$ by the values

$$a(0)=-\phi(0),\quad b(0)=-\psi(0),\quad \lambda_0=\mu_0=0.$$

Remark 1.1.1. By the functional equation (1.1.9) and condition 3. on the previous definition, we have that $\phi(0)=-\rho^{\star}\Gamma(r)$, while $\psi(0)=-\rho\,\Gamma(r)$. In particular, if $\psi(s)$ is an entire Dirichlet series, then $\phi(0)=0$. The purpose of introducing this class is to mimic as much as possible the class of Dirichlet series with a given signature. A similar class is also considered in [[19], p. 221], where interesting generalizations of identities due to Landau are given.

Before proceeding further, let us enumerate some simple conventions that will permeate this chapter.

Notation and Conventions A:

- We shall denote $\sum_{m,n\neq 0}$ as the infinite sum over all integers m, n not simultaneously zero. If we assume that m and n are non-negative, we shall write $\sum_{m,n\neq 0}^{\infty}$ instead.

- We will always use the convention $\lambda_0=\mu_0=0$.

- We will always write $\sigma_a:=\max\{\sigma_{a_1},\,\sigma_{a_2}\}$ and, analogously, $\sigma_b:=\max\{\sigma_{b_1},\,\sigma_{b_2}\}$. Here, σ_{a_i} will always represent the abscissa of absolute convergence of the Dirichlet series $\phi_i(s)$.

We will now define a double Dirichlet series which exhibits a similar behavior as the classical Epstein zeta function (1.1.1). We will often call it by the name "diagonal Epstein zeta function".

Definition 1.1.2. Let $(\lambda_n,\,\lambda_n^{\prime})_{n\in\mathbb{N}}$ be a pair of sequences of positive numbers strictly increasing to ∞ and $(a_1(n),\,a_2(n))_{n\in\mathbb{N}}$ be a pair of sequences of complex numbers not identically zero. Also, let $\phi_1(s)$ and $\phi_2(s)$ be two functions representable as Dirichlet series

$$\phi_1(s)=\sum_{n=1}^{\infty}\frac{a_1(n)}{\lambda_n^s},\quad \phi_2(s)=\sum_{n=1}^{\infty}\frac{a_2(n)}{\lambda_n^{\prime s}} \tag{1.1.10}$$

with finite abscissas of absolute convergence σ_{a_1} and σ_{a_2}, respectively. For the pair $(\phi_1,\,\phi_2)$, we define the generalized diagonal Epstein zeta function as the double Dirichlet series

$$\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda^{\prime}\right)=\sum_{m,n\neq 0}^{\infty}\frac{a_1(m)\,a_2(n)}{\left(\lambda_m+\lambda_n^{\prime}\right)^s},\quad \text{Re}(s)>2\,\max\{\sigma_{a_1},\,\sigma_{a_2}\}:=2\,\sigma_a. \tag{1.1.11}$$

It is easily seen that the double series (1.1.11) is absolutely convergent if $\mathrm{Re}(s) > 2\,\max\{\sigma_{a_1},\,\sigma_{a_2}\} = 2\sigma_a$. Examples of (1.1.11) include the classical Dirichlet series associated with the sum of two squares,

$$\zeta_2(s) = \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4 \sum_{m,n\neq 0}^{\infty} \frac{1}{(m^2+n^2)^s} := \zeta(s,\,Q_0), \qquad \mathrm{Re}(s) > 1, \tag{1.1.12}$$

which can be constructed if we take $\phi_1(s) = \phi_2(s) = 2\zeta(2s)$.

Just as (1.1.12), we may represent a general diagonal Epstein zeta function $\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda'\right)$ via the single Dirichlet series

$$\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda'\right) = \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\,a_2(n)}{\left(\lambda_m+\lambda_n'\right)^s} = \sum_{n=1}^{\infty} \frac{\mathfrak{U}_2(n)}{\Lambda_n^s}, \quad \mathrm{Re}(s) > 2\,\sigma_a, \tag{1.1.13}$$

where $\mathfrak{U}_2(n)$ generalizes the arithmetical function $r_2(n)$,

$$\mathfrak{U}_2(n) = \sum_{j_1,j_2\,:\lambda_{j_1}+\lambda_{j_2}'=\Lambda_n} a_1(j_1)\,a_2(j_2), \tag{1.1.14}$$

and the sequence $(\Lambda_n)_{n\in\mathbb{N}}$ is taken as $\Lambda_n := \left(\lambda_{j_1}+\lambda_{j_2}'\right)_{j_1,j_2\in\mathbb{N}_0}$ and then rearranged in increasing order.

In section 1.2 we study the analytic continuation and explore a functional equation for the series (1.1.13). In particular, we prove that, if $\phi_1(s)$ and $\phi_2(s)$ satisfy definition 1.1.1 with the functional equations

$$\Gamma(s)\,\phi_1(s) = \Gamma(r_1-s)\,\psi_1(r_1-s), \qquad \Gamma(s)\,\phi_2(s) = \Gamma(r_2-s)\,\psi_2(r_2-s), \qquad r_1,\,r_2>0, \tag{1.1.15}$$

with $\psi_1(s)$ and $\psi_2(s)$ representing the Dirichlet series

$$\psi_1(s) = \sum_{n=1}^{\infty} \frac{b_1(n)}{\mu_n^s}, \;\; \mathrm{Re}(s) > \sigma_{b_1}, \quad \psi_2(s) = \sum_{n=1}^{\infty} \frac{b_2(n)}{\mu_n'^s}, \;\; \mathrm{Re}(s) > \sigma_{b_2}, \tag{1.1.16}$$

then it is possible to write two distinct Selberg-Chowla formulas for (1.1.11). The combination of both representations appearing in (1.2.1) and (1.2.2) below will allow to write a functional equation for $\mathcal{Z}_2(s;\,a_1,a_2;\,\lambda,\,\lambda')$. This functional equation will also take the form (1.1.9).

We will end section 1.2 with the study of the analytic continuation of certain analogues of (1.1.11) containing more variables of summation, now defined as follows.

Definition 1.1.3. Let $\{\phi_i(s)\}_{i=1,\ldots,k}$ denote a k-tuple of Dirichlet series given by

$$\phi_i(s) = \sum_{n=1}^{\infty} \frac{a_i(n)}{\lambda_{n,i}^s}, \qquad \mathrm{Re}(s) > \sigma_{a_i}, \tag{1.1.17}$$

where, for each $i \in \{1, ..., k\}$, $(\lambda_{n,i})_{n\in\mathbb{N}}$ is a sequence of positive numbers strictly increasing to ∞. We define the multidimensional Epstein zeta function as the infinite series

$$\mathcal{Z}_k\left(s;\, a_1, ..., a_k;\, \lambda_1,\, ...,\, \lambda_k\right) := \sum_{n_1,...,n_k\neq 0}^{\infty} \frac{a_1(n_1)\cdot ... \cdot a_k(n_k)}{\left(\lambda_{n_1,1} + ... + \lambda_{n_k,k}\right)^s}, \quad \mathrm{Re}(s) > k\sigma_a, \tag{1.1.18}$$

where $\sigma_a := \max_{1\leq i\leq k}\, \sigma_{a_i}$.

As in representation (1.1.13), we can write $\mathcal{Z}_k(s;\,\cdot)$ as the Dirichlet series

$$\mathcal{Z}_k\left(s;\, a_1, ..., a_k;\, \lambda_1,\, ...,\, \lambda_k\right) = \sum_{n=1}^{\infty} \frac{\mathfrak{U}_k(n)}{\Lambda_n^s}, \quad \mathrm{Re}(s) > k\sigma_a, \tag{1.1.19}$$

where, in analogy with $\mathfrak{U}_2(n)$ given in (1.1.14), the arithmetical function $\mathfrak{U}_k(n)$ is described by

$$\mathfrak{U}_k(n) = \sum_{\lambda_{j_1,1}+...+\lambda_{j_k,k}=\Lambda_n} a_1(j_1)\cdot ... \cdot a_k(j_k). \tag{1.1.20}$$

In Corollary 1.2.2 we will establish that, if $\{\phi_i(s)\}_{i=1,...,k}$ are Dirichlet series satisfying Hecke's functional equation

$$\Gamma(s)\,\phi_i(s) = \Gamma(r_i - s)\,\psi_i(r_i - s), \quad r_i > 0, \tag{1.1.21}$$

with $\psi_i(s)$ being represented as

$$\psi_i(s) = \sum_{n=1}^{\infty} \frac{b_i(n)}{\mu_{n,i}^s}, \quad \mathrm{Re}(s) > \sigma_{b_i}, \tag{1.1.22}$$

then $\mathcal{Z}_k\left(s;\, a_1, ..., a_k;\, \lambda_1,\, ...,\, \lambda_k\right)$ will also obey to Hecke's functional equation (1.1.9).

For notation purposes, when the Dirichlet series (1.1.17) building the multidimensional Epstein zeta function (1.1.18) are all the same, say $\phi_1(s) = \phi_2(s) = ... = \phi_k(s) := \phi(s)$, we will always write (1.1.18) as $\mathcal{Z}_k(s;\, a;\, \lambda)$, this is,

$$\mathcal{Z}_k(s;\, a;\, \lambda) = \sum_{n_1,...,n_k\neq 0}^{\infty} \frac{a(n_1)\cdot\, ...\, \cdot\, a(n_k)}{\left(\lambda_{n_1} + ... + \lambda_{n_k}\right)^s}, \quad \mathrm{Re}(s) > k\,\sigma_a. \tag{1.1.23}$$

For example, when $k = 2$,

$$\mathcal{Z}_2(s;\, a;\, \lambda) = \sum_{m,n\neq 0}^{\infty} \frac{a(m)\,a(n)}{\left(\lambda_m + \lambda_n'\right)^s}, \quad \mathrm{Re}(s) > 2\sigma_a. \tag{1.1.24}$$

In several occasions throughout this text, we shall need to estimate the asymptotic order of certain integrals involving a combination of Gamma functions and certain Dirichlet series satisfying Definition 1.1.1. To justify most of the steps, we will often invoke the following version of Stirling's formula

$$\Gamma(\sigma+it)=(2\pi)^{\frac{1}{2}}\,t^{\sigma+it-\frac{1}{2}}\,e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+\frac{1}{12(\sigma+it)}+O\left(\frac{1}{t^{2}}\right)\right), \tag{1.1.25}$$

as $t\to\infty$, uniformly for $-\infty<\sigma_1\leq\sigma\leq\sigma_2<\infty$. A similar formula can be written for t tending to $-\infty$ by using the fact that $\Gamma(\overline{s})=\overline{\Gamma(s)}$.

To estimate the order of $\phi(s)$ at a general vertical line $\text{Re}(s)=\sigma$, we shall need a version of the classical Phragmén-Lindelöf theorem given in [[227], p. 180, 5.65]. Since, for $\sigma>\sigma_b$, $\psi(\sigma+it)=O(1)$ as $|t|\to\infty$, it follows from the functional equation (1.1.9) and Stirling's formula (1.1.25) that, for $\sigma<r-\sigma_b$,

$$\phi(\sigma+it)=O\left(\frac{|\Gamma(r-s)|}{|\Gamma(s)|}\,|\psi(r-s)|\right)=O\left(|t|^{r-2\sigma}\right),\quad |t|\to\infty. \tag{1.1.26}$$

Since $\phi(s)=O(1)$ for $\sigma:=\text{Re}(s)>\sigma_a$, it follows from property 2. in Definition 1.1.1 that, for any $\delta>0$ and $r-\sigma_a-\delta\leq\sigma:=\text{Re}(s)\leq\sigma_a+\delta$,

$$\phi(\sigma+it)=O\left(|t|^{\sigma_a+\delta-\sigma}\right),\quad |t|\to\infty. \tag{1.1.27}$$

Ever since the publication of Hardy's short note [106] and the subsequent quantitative proofs given by Landau, Fekete, Hardy and Littlewood [88, 109, 139], as well as later adaptations to other Dirichlet series by Kober [128] and Hecke [113], the study of the zeros of $\zeta(s)$ and other general Dirichlet series is often dependent on the asymptotic behavior of the associated Jacobi $\theta-$function as a function in the upper half-plane. From now on, if $\phi(s)$ satisfies Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1 and it is representable by the first Dirichlet series in (1.1.8), we shall use $\Theta(z;\,a;\,\lambda)$ to denote the generalized $\theta-$function

$$\Theta(z;\,a;\,\lambda):=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n z},\quad \text{Re}(z)>0. \tag{1.1.28}$$

Following Bochner [45], let us consider, for $\text{Re}(z)>0$, the residual function

$$P(z)=\frac{1}{2\pi i}\int_{C}\chi(s)\,z^{-s}ds, \tag{1.1.29}$$

where C denotes a curve, or curves, encircling the singularities of $\chi(s)$ given in Definition 1.1.1. It was established in [45] that Hecke's functional equation for $\phi(s)$ and $\psi(s)$, (1.1.9), and the "modular relation"

$$\Theta(z;\,a;\,\lambda)=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n z}=z^{-r}\sum_{n=1}^{\infty}b(n)\,e^{-\mu_n/z}+P(z)=z^{-r}\,\Theta\left(\frac{1}{z};\,b;\,\mu\right)+P(z), \tag{1.1.30}$$

are equivalent. It is interesting to note that the observation of this equivalence had its genesis in B. Riemann's revolutionary memoir, where one of the implications was proved for the first time. There, a second proof of the functional equation for $\pi^{-s}\zeta(2s)$ (which satisfies (1.1.9) with $r=\frac{1}{2}$)

was given by invoking the transformation formula for the classical Jacobi $\theta-$function, which itself is a particular case of (1.1.30).

The converse is, of course, obtained upon the use of the Cahen-Mellin integral [[109], p. 120, eq. (1.II)]

$$e^{-z}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\,z^{-s}ds, \tag{1.1.31}$$

valid for $c>0$, $\mathrm{Re}(z)>0$ and z^{-s} having its principal value.

In analogy with (1.1.28), we can introduce the generalized $\theta-$function attached to the Epstein zeta function $\mathcal{Z}_2(s;\,a_1,a_2;\,\lambda,\,\lambda')$, (1.1.11), as the series

$$\Theta_2\left(z;\,a_1,a_2;\,\lambda,\lambda'\right)=\sum_{m,n\neq 0}^{\infty}a_1(m)\,a_2(n)\,e^{-(\lambda_m+\lambda_n')z},\quad \mathrm{Re}(z)>0. \tag{1.1.32}$$

Thus, if we prove that $\mathcal{Z}_2(s;\,a_1,a_2;\,\lambda,\lambda')$ has a functional equation of the form (1.1.9), we may expect a "modular relation" for (1.1.32) in the same lines as Bochner's identity (1.1.30). This will be stated in Remark 1.2.2.

In the sequel, we will employ several times well-known integral representations for the modified Bessel function $K_{\nu}(z)$. For a matter of clarity in our exposition, we will invoke them solely when they are needed. Before presenting our main results, we remark that throughout this chapter we need to adopt some conventions and notations that we now specify.

Notation and Conventions B:

- In order to make our general formulas resemble the original formulation (1.1.2), we will use the symbol $\sigma_z(\nu_j;\,b_1,\,a_2;\,\mu,\,\lambda')$ to denote a generalized version of the divisor function (see Remark 1.2.1). At some points (see, for example, Example 1.2.1), we will simplify the notation and represent this divisor function by $\sigma_z(\nu_j;\,b_1,\,a_2)$.

- Whenever we use the term **critical line** for a given Dirichlet series $\phi(s)$ satisfying Definition 1.1.1, we will be referring to the line of symmetry induced by the functional equation (1.1.9), this is, the vertical line $\mathrm{Re}(s)=\frac{r}{2}$. Moreover, whenever we use the term **critical zeros of** $\phi(s)$ we will be referring to the zeros of $\phi(s)$ that are located on the critical line. Furthermore, the property "$\phi(s)$ has infinitely many zeros at its critical line" will sometimes be informally formulated as "$\phi(s)$ satisfies **Hardy's Theorem**".

- We will write $H_{r_1}(s;\,b_1,\,a_2;\,\mu,\,\lambda')$ and $H_{r_2}(s;\,b_2,\,a_1;\,\mu',\,\lambda)$ to denote the entire complex functions (1.2.11) and (1.2.14) appearing in the Selberg-Chowla representations for the diagonal Epstein zeta function, (1.2.1) and (1.2.2).

- If, in Definition 1.1.2, the Dirichlet series are the same, i.e., $\phi_1(s)=\phi_2(s)=\phi(s)$, we will write $\mathcal{Z}_2(s;\,a;\,\lambda)$ to denote (1.1.11). See (1.1.23) and (1.1.24) above. This notation will be important in Remark 1.2.3 below.

- Throughout this chapter and thesis, we name the product "$\Gamma(s)\,\phi(s)$" as **completed Dirichlet series**.

- We shall use $R_{\phi}(s)$ and $R_{\psi}(s)$ to respectively denote the completed Dirichlet series $\Gamma(s)\,\phi(s)$ and $\Gamma(s)\,\psi(s)$. This notation will be employed mainly during the proof of Theorem 1.3.1. Under some conditions given there, $R_{\phi}(s)$ and $R_{\psi}(s)$ will represent real functions on the critical line $\mathrm{Re}(s)=r/2$.

- Throughout the proof of Theorem 1.3.1, c **will represent the least positive integer such that** $a(c)\neq 0$, where $a(n)$ is the arithmetical function attached to a Dirichlet series, say $\phi(s)$, satisfying Definition 1.1.1. This integer also appears in the statements of Corollaries 1.3.1 and 1.3.2.

1.2 The Selberg-Chowla formula and the analytic continuation of analogues of the Epstein zeta function

Theorem 1.2.1 (A general Selberg-Chowla formula)**.** *Let $\phi_1(s)$ and $\phi_2(s)$ be the pair of Dirichlet series (1.1.10) satisfying definition 1.1.1 and obeying to Hecke's functional equation (1.1.15). Also, let s be a complex number such that Re$(s)>\gamma>\max\left\{2\sigma_a,\,2\sigma_b,\,\frac{r'}{2}\right\}$, where $\sigma_a=\max\{\sigma_{a_1},\,\sigma_{a_2}\}$, $\sigma_b=\max\{\sigma_{b_1},\,\sigma_{b_2}\}$ and $r'=\max\{r_1,\,r_2\}$. Then, for Re$(s)>\gamma$, the following identity for the generalized Epstein zeta function, (1.1.11), holds*

$$\begin{aligned}\Gamma(s)\,\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda'\right)&=-\phi_2(0)\,\Gamma(s)\,\phi_1(s)+\rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,\phi_2\left(s-r_1\right)\\&+2\sum_{m,n=1}^{\infty}b_1(m)\,a_2(n)\,\left(\frac{\mu_m}{\lambda_n'}\right)^{\frac{s-r_1}{2}}K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right),\end{aligned}\tag{1.2.1}$$

where $b_1(m)$ is the arithmetical function associated to the Dirichlet series $\psi_1(s)$ given in (1.1.16). Analogously, whenever Re$(s)>\gamma$, $\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda'\right)$ can also be described by the formula

$$\begin{aligned}\Gamma(s)\,\mathcal{Z}_2\left(s;\,a_1,\,a_2;\,\lambda,\,\lambda'\right)&=-\phi_1(0)\,\Gamma(s)\,\phi_2(s)+\rho_2\Gamma(r_2)\,\Gamma(s-r_2)\,\phi_1\left(s-r_2\right)\\&+2\sum_{m,n=1}^{\infty}b_2(m)\,a_1(n)\,\left(\frac{\mu_m'}{\lambda_n}\right)^{\frac{s-r_2}{2}}K_{r_2-s}\left(2\sqrt{\mu_m'\lambda_n}\right).\end{aligned}\tag{1.2.2}$$

Proof. The idea of the proof follows [[21], p. 311] and its sketched version given in [25]. Throughout our argument we shall use $H_{r_1}\left(s;\,b_1,\,a_2;\,\mu,\,\lambda'\right)$ (resp. $H_{r_2}\left(s;\,b_2,\,a_1;\,\mu',\,\lambda\right)$) to denote the double series involving the modified Bessel function appearing on the right-hand side of (1.2.1) (resp. (1.2.2)). We will also let $\gamma>\max\left\{2\sigma_a,\,2\sigma_b,\,\frac{r'}{2}\right\}$ as in the statement and consider a fixed $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\gamma$. Under this condition, the following integral representation holds [[85], p. 349, 7.3.15]

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(z)\,\Gamma(s-z)\,x^{-z}dz=\frac{\Gamma(s)}{(1+x)^{s}},\quad x>0.\tag{1.2.3}$$

Since our Epstein series (1.1.11) is defined as a double infinite series which converges absolutely for $\mathrm{Re}(s) > \gamma > 2\,\sigma_a$, we may write it as

$$\mathcal{Z}_2\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) = a_1(0)\phi_2(s) \,+\, a_2(0)\phi_1(s) \,+\, \sum_{n=1}^{\infty} a_2(n)\,\ell_n\left(s,\, a_1\right), \tag{1.2.4}$$

where, for a fixed $n\in\mathbb{N}$, $\ell_n(s,\, a_1)$ denotes the generalized Dirichlet series

$$\ell_n\left(s,\, a_1\right) = \sum_{m=1}^{\infty}\frac{a_1(m)}{\left(\lambda_m+\lambda_n'\right)^s},\quad \mathrm{Re}(s) > \sigma_{a_1}.$$

By the Mellin representation (1.2.3), we see that we can write $\ell_n\left(s,\, a_1\right)$ as the contour integral

$$\lambda_n'^s\,\Gamma(s)\,\ell_n\left(s,\, a_1\right) = \sum_{m=1}^{\infty}\frac{a_1(m)\,\Gamma(s)}{\left(1+\frac{\lambda_m}{\lambda_n'}\right)^s} = \sum_{m=1}^{\infty}\frac{a_1(m)}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(z)\,\Gamma(s-z)\left(\frac{\lambda_m}{\lambda_n'}\right)^{-z}dz, \tag{1.2.5}$$

valid whenever $\mathrm{Re}(s) > \sigma_{a_1}$, which is the case as $\mathrm{Re}(s) > \gamma > 2\sigma_a \geq 2\sigma_{a_1}$. From the definition of γ, it is clear that $\sum a_1(m)\,\lambda_m^{-\gamma}$ converges absolutely. Moreover, from a corollary of Stirling's formula,

$$\left|\Gamma(\gamma+it)\,\Gamma(s-\gamma-it)\right| = 2\pi\,|t|^{\mathrm{Re(s)}-1}e^{-\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right),\quad |t|\to\infty, \tag{1.2.6}$$

we are allowed to interchange the orders of integration and summation in (1.2.5). This gives the integral representation

$$\lambda_n'^s\Gamma(s)\,\ell_n(s,\, a_1) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\phi_1(z)\,\Gamma\left(z\right)\Gamma(s-z)\,\lambda_n'^{\,z}\,dz,\quad \gamma > \max\left\{\sigma_a,\, \sigma_b,\, \frac{r_1}{2}\right\}.$$

Let us now move the line of integration to $\mathrm{Re}(z) = r_1-\gamma$ (which is located at the left of $\mathrm{Re}(z)=\gamma$, because $\gamma > \frac{r'}{2} \geq \frac{r_1}{2}$ by hypothesis) and integrate along a positively oriented rectangular contour $\mathcal{R}_1(\gamma)$ containing the vertices $\gamma\pm iT$ and $r_1-\gamma\pm iT$, where $T>0$. By Cauchy's residue theorem, we have the equality

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT}\phi_1(z)\,\Gamma\left(z\right)\Gamma(s-z)\,\lambda_n'^z\,dz &= \frac{1}{2\pi i}\int_{r_1-\gamma-iT}^{r_1-\gamma+iT}\phi_1(z)\,\Gamma\left(z\right)\Gamma(s-z)\,\lambda_n'^z\,dz\\ \mp\frac{1}{2\pi i}\int_{\gamma\pm iT}^{r_1-\gamma\pm iT}\phi_1(z)\,\Gamma\left(z\right)\Gamma(s-z)\,\lambda_n'^z\,dz &+ \sum_{\rho\in\mathcal{R}_1(\gamma)}\mathrm{Res}_{z=\rho}\left(\phi_1(z)\,\Gamma\left(z\right)\Gamma(s-z)\,\lambda_n'^{\,z}\right),\end{aligned} \tag{1.2.7}$$

where the second integral on the right-hand side represents (in abbreviated notation) the integrals over the horizontal segments $[\gamma-iT,\, r_1-\gamma-iT]$ and $[r_1-\gamma+iT,\, \gamma+iT]$. By condition 2 in Definition 1.1.1, it is clear that these integrals tend to zero[6] as $T\to\infty$. Therefore, by letting

[6]Despite this imposition, we should note that this could be obtained from a weaker condition of the form $\phi(s) = O\left(\exp|s|^K\right)$, for some $K>0$, as $|s|\to\infty$, replacing condition 2 in Definition 1.1.1. In this case, the

$T\to\infty$ in (1.2.7), we obtain

$$\lambda_n^{\prime s}\Gamma(s)\,\ell_n(s,\,a_1)=\frac{1}{2\pi i}\int\limits_{r_1-\gamma-i\infty}^{r_1-\gamma+i\infty}\phi_1(z)\,\Gamma\,(z)\,\Gamma(s-z)\,\lambda_n^{\prime\,z}\,dz\\ +\sum_{\rho\in\mathcal{R}_1(\gamma)}\mathrm{Res}_{z=\rho}\left(\phi_1(z)\,\Gamma\,(z)\,\Gamma(s-z)\,\lambda_n^{\prime\,z}\right). \tag{1.2.8}$$

It is now easy to check from (1.1.15) and the definition of γ that all the poles of $\chi_1(z):=\Gamma(z)\,\phi_1(z)$ are on the interior of $\mathcal{R}_1(\gamma)$. Moreover, since $\mathrm{Re}(s)>\gamma$ by hypothesis, we clearly have that $\mathrm{Re}(s-z)>0$ for any $z\in\mathcal{R}_1(\gamma)$. This shows that $\Gamma(s-z)$ has no poles inside $\mathcal{R}_1(\gamma)$ and the only poles taken into account in the sum in (1.2.8) are precisely the ones coming from the function $\Gamma(z)\,\phi_1(z)$. Thus, from the considerations given in definition 1.1.1, the previous equality reduces to

$$\lambda_n^{\prime s}\,\Gamma(s)\,\ell_n(s,\,a_1)=\frac{1}{2\pi i}\int\limits_{r_1-\gamma-i\infty}^{r_1-\gamma+i\infty}\phi_1(z)\,\Gamma\,(z)\,\Gamma(s-z)\,\left(\frac{1}{\lambda_n^{\prime}}\right)^{-z}\,dz\\ +\,\phi_1(0)\,\Gamma(s)\,+\,\rho_1\,\Gamma(r_1)\Gamma(s-r_1)\,\lambda_n^{\prime r_1}.$$

Let us now invoke the functional equation for the Dirichlet series $\phi_1(z)$, (1.1.15), and use it in the contour integral given above. Doing this and employing the change of variables $z\leftrightarrow r_1-z$, we see that the right-hand side of the previous equality is

$$\lambda_n^{\prime r_1}\sum_{m=1}^{\infty}\frac{b_1(m)}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\,(z)\,\Gamma(s-r_1+z)\,\left(\mu_m\lambda_n^{\prime}\right)^{-z}\,dz+\,\phi_1(0)\,\Gamma(s)\,+\,\rho_1\,\Gamma(r_1)\Gamma(s-r_1)\,\lambda_n^{\prime r_1}, \tag{1.2.9}$$

where we have invoked the absolute convergence of the Dirichlet series $\psi_1(z)$ for $\mathrm{Re}(z)=\gamma>\sigma_b\geq\sigma_{b_1}$, together with Stirling's formula (1.2.6).

Recalling the Mellin representation for the Modified Bessel function [[85], p. 349, 7.3 (17)],

$$K_\nu(x)=\left(\frac{x}{2}\right)^{\nu}\frac{1}{4\pi i}\int\limits_{\lambda-i\infty}^{\lambda+i\infty}\Gamma(z-\nu)\,\Gamma(z)\,\left(\frac{x^2}{4}\right)^{-z}\,dz,\quad \lambda>\max\{0,\;\mathrm{Re}(\nu)\}, \tag{1.2.10}$$

we see that, since $\gamma>\max\{0,\,r_1-\mathrm{Re}(s)\}$, the following representation of $\ell_n(s,\,a_1)$ takes place

$$\lambda_n^{\prime s}\,\Gamma(s)\,\ell_n(s,\,a_1)=2\,\lambda_n^{\prime r_1}\sum_{m=1}^{\infty}b_1(m)\,\left(\mu_m\lambda_n^{\prime}\right)^{-\frac{r_1-s}{2}}\,K_{r_1-s}\left(2\,\sqrt{\mu_m\lambda_n^{\prime}}\right)\\ +\,\phi_1(0)\,\Gamma(s)\,+\,\rho_1\,\Gamma(r_1)\Gamma(s-r_1)\,\lambda_n^{\prime r_1}.$$

We can easily check that the double series on the right side of (1.2.4) converges absolutely if $\mathrm{Re}(s)>\sigma_{a_1}+\sigma_{a_2}$ and so it must converge absolutely for $\mathrm{Re}(s)>\gamma$. Let us now denote the residual terms in the previous equality by $R_1(s,\,\lambda_n^{\prime-1})$ (note that, if $\phi_1(s)$ is entire, these terms are both

vanishing of the integrals along the horizontal segments would follow from Stirling's formula (1.2.6), together with the Phragmén-Lindelöf estimate (1.1.27) (cf. [21, 26, 27]).

zero). Since $\phi_1(s)$ is analytic in the region $\mathrm{Re}(s) > \sigma_{a_1}$, it is clear from the expression for $R_1(s,\,x)$ that the series $\sum a_2(n)\,\lambda_n'^{-s}\,R_1(s,\,\lambda_n'^{-1})$ converges absolutely whenever $\mathrm{Re}(s) > \sigma_{a_1}+\sigma_{a_2}$. Hence, recalling that $\mathrm{Re}(s) > \gamma \geq \sigma_{a_1}+\sigma_{a_2}$ and summing $a_2(n)\,\ell_n(s,\,a_1)$ with respect to n as in (1.2.4), we arrive at the representation (1.2.1). To finish, we only need to argue that, under the assumption $\mathrm{Re}(s) > \gamma$, the double series which we have obtained by summing over the summation variable n,

$$H_{r_1}(s;\,b_1,\,a_2;\,\mu,\,\lambda') = \sum_{m,n=1}^{\infty} b_1(m)\,a_2(n)\,\left(\frac{\mu_m}{\lambda_n'}\right)^{\frac{s-r_1}{2}}\,K_{r_1-s}\left(2\sqrt{\mu_m\lambda_n'}\right), \tag{1.2.11}$$

converges absolutely. This is possible to prove due to the asymptotic estimate for the modified Bessel function [[232], pp. 199-202],

$$K_{\nu}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}(1+O(1/x)), \qquad x\to\infty, \tag{1.2.12}$$

valid for any fixed $\nu\in\mathbb{C}$. Thus, for some positive constant C depending only on s,

$$|H_{r_1}(s;\,b_1,\,a_2;\,\mu,\,\lambda')| \leq C\,\sum_{m,n=1}^{\infty}|b_1(m)|\,|a_2(n)|\,\left(\frac{\mu_m}{\lambda_n'}\right)^{\frac{\mathrm{Re}(s)-r_1}{2}}\,(\mu_m\lambda_n')^{-\frac{1}{4}}\,e^{-2\sqrt{\mu_m\lambda_n'}}. \tag{1.2.13}$$

From the absolute convergence of $\phi_2(s)$ and $\psi_1(s)$ for $\mathrm{Re}(s) > \max\{\sigma_a,\,\sigma_b\}$, we immediately see from (1.2.13) that the series defining $H_{r_1}(s;\,\cdot)$, (1.2.11), converges absolutely for $\mathrm{Re}(s) > \gamma$. The bound (1.2.13) also shows that this series, as a function of s, is uniformly convergent on bounded subsets of the complex plane. Since the summands of $H_{r_1}(s;\,b_1,\,a_2;\,\mu;\,\lambda')$ are analytic functions of s, a standard application of Weierstrass' theorem on analytic functions shows that $H_{r_1}(s;\,\cdot)$ defines an entire function of $s\in\mathbb{C}$.

In a completely analogous way, the second formula (1.2.2) may be obtained, by reversing the order of summation in the previous argument (which is a possible operation for $\mathrm{Re}(s) > \gamma$). Since the first summation in this second case is performed with respect to the index n, the comments made above can be rephrased to describe the double infinite series

$$H_{r_2}(s;\,b_2,\,a_1;\,\mu',\,\lambda) = \sum_{m,n=1}^{\infty} b_2(m)\,a_1(n)\,\left(\frac{\mu_m'}{\lambda_n}\right)^{\frac{s-r_2}{2}}\,K_{r_2-s}\left(2\,\sqrt{\mu_m'\lambda_n}\right). \tag{1.2.14}$$

□

Remark 1.2.1 (A generalized divisor function)**.** Let $\{\nu_j\}_{j\in\mathbb{N}}$ represent the "product sequence" $\{\nu_j\}_{j\in\mathbb{N}} = \{\mu_m\cdot\lambda_n'\}_{m,n\in\mathbb{N}}$ and then arrange this product in increasing order. For any $z\in\mathbb{C}$, we define the generalized divisor function as

$$\sigma_z\left(\nu_j;\,b_1,\,a_2;\,\mu,\,\lambda'\right) = \sum_{(m,\,n)\in\mathbb{N}^2\,:\,\mu_m\lambda_n'=\nu_j} b_1(m)\,a_2(n)\,\mu_m^z. \tag{1.2.15}$$

Note that, since $(\mu_m)_{m\in\mathbb{N}}$ and $(\lambda_n')_{n\in\mathbb{N}}$ are strictly increasing and tend to ∞ (see definition 1.1.1), the previous sum is finite. It is clear the analogy with the usual divisor function, $\sigma_z(n)=\sum_{d|n}d^z$.

In the example concerning Ramanujan-Guinand's formula (Example 1.2.1 below), we will simplify this notation and reduce it to $\sigma_z(\nu_j;\, b_1,\, a_2)$, i.e., taking the shortened version

$$\sigma_z(\nu_j;\, b_1,\, a_2) := \sigma_z\left(\nu_j;\, b_1,\, a_2;\, \mu,\, \lambda'\right), \tag{1.2.16}$$

but always bearing in mind the order of the sequences $(\mu_m)_{m\in\mathbb{N}}$ and $(\lambda'_n)_{n\in\mathbb{N}}$ in the definition (1.2.15). With the definition (1.2.15) at our disposal, we may rewrite $H_{r_1}(s;\,\cdot)$, given in (1.2.11), as the infinite series

$$H_{r_1}(s;\, b_1,\, a_2;\, \mu,\, \lambda') = \sum_{j=1}^{\infty} \sigma_{s-r_1}\left(\nu_j;\, b_1,\, a_2;\, \mu,\, \lambda'\right)\, \nu_j^{\frac{r_1-s}{2}}\, K_{r_1-s}\left(2\sqrt{\nu_j}\right), \tag{1.2.17}$$

which strongly resembles the entire function appearing on the right-hand side of the familiar Selberg-Chowla formula (1.1.2). Analogously, if we take the sequence $\{\nu'_j\}_{j\in\mathbb{N}} = \{\mu'_m \lambda_n\}_{m,n\in\mathbb{N}}$, it is also evident that we can describe $H_{r_2}(s;\,\cdot)$, appearing in (1.2.14), by the analogous series

$$H_{r_2}(s;\, b_2,\, a_1;\, \mu',\, \lambda) = \sum_{j=1}^{\infty} \sigma_{s-r_2}\left(\nu'_j;\, b_2,\, a_1;\, \mu',\, \lambda\right)\, {\nu'_j}^{\frac{r_2-s}{2}}\, K_{r_2-s}\left(2\sqrt{\nu'_j}\right).$$

Taking $z=0$ in (1.2.15), one obtains a generalization of the standard divisor function $d(n) = \sum_{d|n} 1$,

$$d\left(\nu_j;\, b_1,\, a_2;\, \mu,\, \lambda'\right) = \sum_{(m,\,n)\in\mathbb{N}^2\,:\,\mu_m\lambda'_n=\nu_j} b_1(m)\, a_2(n). \tag{1.2.18}$$

Having proved two formulas (1.2.1) and (1.2.2) that allow to study the analytic continuation of the Epstein zeta function $\mathcal{Z}_2(s;\, a_1,\, a_2;\, \lambda,\, \lambda')$, the next corollary shows that this new Dirichlet series also satisfies the conditions of Definition 1.1.1, thus possessing a suitable functional equation.

Corollary 1.2.1. *Let $\phi_1(s)$ and $\phi_2(s)$ be two Dirichlet series satisfying the conditions of Definition 1.1.1 and having residues ρ_1 and ρ_2 at $s=r_1$ and $s=r_2$, respectively. Then the Selberg-Chowla formula (1.2.1) provides the analytic continuation of the Epstein $\zeta-$function (1.1.11) as a meromorphic function having at most one simple pole located at $s=r_1+r_2$, with residue given by*

$$Res_{s=r_1+r_2}\, \mathcal{Z}_2\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) = \frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)}\, \rho_1\rho_2. \tag{1.2.19}$$

Moreover, the continuation of $\mathcal{Z}_2(s;\, a_1,\, a_2;\, \lambda,\, \lambda')$ satisfies the functional equation

$$\Gamma\left(s\right) \mathcal{Z}_2\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) = \Gamma(r_1+r_2-s)\, \mathcal{Z}_2\left(r_1+r_2-s;\, b_1,\, b_2;\, \mu,\, \mu'\right). \tag{1.2.20}$$

Proof. We have seen at the end of the previous proof that the double series (1.2.11) and (1.2.14) represent entire functions of s, so that the "meromorphic part" of the Epstein $\zeta-$function $\mathcal{Z}_2\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right)$ comes exclusively from the first two terms in (1.2.1), i.e., from the meromorphic function

$$G_{r_1}\left(s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) = -\phi_2(0)\, \phi_1(s) + \rho_1\Gamma(r_1)\, \frac{\Gamma(s-r_1)}{\Gamma(s)}\, \phi_2\left(s-r_1\right). \tag{1.2.21}$$

Clearly, from a standard verification, $G_{r_1}(s;\,\cdot)$ has removable singularities at $s = r_1 - k$, $k \in \mathbb{N}_0$. Since, by the conditions given in definition 1.1.1, $\phi_1(s)$ and $\Gamma\left(s - r_1\right)$ are analytic in a neighbourhood of $s = r_1 + r_2$, $G_{r_1}(s;\,\cdot)$ has at most one pole located at $s = r_1 + r_2$ coming from the factor $\phi_2(s - r_1)$. The residue is easily seen to be $\frac{\Gamma(r_1)\Gamma(r_2)}{\Gamma(r_1+r_2)}\,\rho_1\rho_2$, proving (1.2.19).

To prove the functional equation (1.2.20), let us use both representations (1.2.1) and (1.2.2) for $\mathcal{Z}_2(s;\,\cdot)$, and replace s by $r_1 + r_2 - s$ on (1.2.1). It is clear from the reflection property of the Modified Bessel function, $K_\nu(z) = K_{-\nu}(z)$, that the following relation holds

$$H_{r_1}\left(r_1 + r_2 - s;\, b_1,\, a_2;\, \mu,\, \lambda'\right) = H_{r_2}\left(s;\, a_2,\, b_1;\, \lambda',\, \mu\right). \tag{1.2.22}$$

Thus, in order to describe $\mathcal{Z}_2(r_1 + r_2 - s;\,\cdot)$, we need to see how the right-hand side of (1.2.21) behaves under the reflection $s \leftrightarrow r_1 + r_2 - s$. Since $\phi_i(0) = -\rho_i^{\star}\Gamma(r_i)$ and $\psi_i(0) = -\rho_i\,\Gamma(r_i)$ (see Remark 1.1.1), we have that

$$\begin{aligned}\Gamma\left(r_1 + r_2 - s\right)\, G_{r_1}\left(r_1 + r_2 - s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) &= -\phi_2(0)\,\Gamma\left(s - r_2\right)\,\psi_1\left(s - r_2\right)\\ +\rho_1\Gamma(r_1)\,\Gamma(s)\,\psi_2(s) &= \rho_2^{\star}\,\Gamma(r_2)\,\Gamma\left(s - r_2\right)\psi_1\left(s - r_2\right) - \psi_1(0)\,\Gamma(s)\,\psi_2(s).\end{aligned} \tag{1.2.23}$$

Now, by equations (1.2.1), (1.2.22) and (1.2.23), we see that

$$\begin{aligned}\Gamma(r_1 + r_2 - s)\,\mathcal{Z}_2\left(r_1 + r_2 - s;\, a_1,\, a_2;\, \lambda,\, \lambda'\right) = -\psi_1(0)\,\Gamma(s)\,\psi_2(s) + \rho_2^{\star}\,\Gamma(r_2)\,\Gamma\left(s - r_2\right)\,\psi_1\left(s - r_2\right)\\ + 2\,H_{r_2}\left(s;\, a_2,\, b_1;\, \lambda',\, \mu\right).\end{aligned} \tag{1.2.24}$$

Using the second representation of the Selberg-Chowla formula (1.2.2), replacing there the arithmetical functions $a_1(m)$ and $a_2(n)$ by $b_1(m)$ and $b_2(n)$, respectively, and, finally, applying the work done in Theorem 1.2.1 for the Epstein $\zeta-$function,

$$\mathcal{Z}_2\left(s;\, b_1,\, b_2;\, \mu,\, \mu'\right) = \sum_{m,n\neq 0}^{\infty} \frac{b_1(m)\,b_2(n)}{\left(\mu_m + \mu_n'\right)^s}, \quad \mathrm{Re}(s) > 2\,\sigma_b, \tag{1.2.25}$$

we find immediately that the right-hand side of (1.2.24) is precisely the same as the analogue of (1.2.2) under the aforementioned substitutions. This establishes the functional equation (1.2.20) and completes the proof. □

Remark 1.2.2. By the well-known equivalence between Hecke's functional equation (1.1.9) and Bochner's modular relation (1.1.30), we can derive from the previous result a transformation formula for the generalized theta function (1.1.32). This transformation can be explicitly stated in the form

$$\begin{aligned}\Theta_2(z;\, a_1,\, a_2;\, \lambda,\, \lambda') = z^{-r_1-r_2}\Theta_2\left(z^{-1};\, b_1,\, b_2;\, \mu,\, \mu'\right) + \Gamma(r_1)\Gamma(r_2)\,\rho_1\rho_2\, z^{-r_1-r_2}\\ - \Gamma(r_1)\Gamma(r_2)\,\rho_1^{\star}\rho_2^{\star},\end{aligned} \tag{1.2.26}$$

valid for any $\mathrm{Re}(z)>0$. Note that, for $i=1,2$, $\rho_i^{\star}$ represents the residue of the Dirichlet series $\psi_i(s)$ at the simple pole $s=r_i$. By Remark 1.1.1 and (1.2.19), we know that

$$\mathcal{Z}_2(0;\,a_1,a_2;\,\lambda,\,\lambda')=-\Gamma(r_1+r_2)\,\mathrm{Res}_{s=r_1+r_2}\mathcal{Z}_2\left(s;\,b_1,b_2;\,\lambda,\lambda'\right)=-\Gamma(r_1)\,\Gamma(r_2)\,\rho_1^{\star}\,\rho_2^{\star}.$$

Therefore, for $\mathrm{Re}(z)>0$, we can also write (1.2.26) as

$$\begin{aligned}\Theta_2(z;\,a_1,\,a_2;\,\lambda,\,\lambda')=z^{-r_1-r_2}\Theta_2\left(z^{-1};\,b_1,\,b_2;\,\mu,\,\mu'\right)+\Gamma(r_1)\Gamma(r_2)\,\rho_1\rho_2\,z^{-r_1-r_2}\\+\mathcal{Z}_2(0;\,a_1,a_2;\,\lambda,\,\lambda').\end{aligned} \tag{1.2.27}$$

Example 1.2.1 (A generalized Ramanujan-Guinand formula and an analogue of Koshliakov's formula)**.** On page 253 of his Lost Notebook [2, 37, 179], Ramanujan recorded the following formula, which we now quote from [[2], p. 94]. Let $s\in\mathbb{C}$ and assume that $\alpha,\beta>0$ are such that $\alpha\beta=\pi^2$. Then the following identity holds

$$\begin{aligned}\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\beta)\\=\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\}.\end{aligned} \tag{1.2.28}$$

This formula was rediscovered by Guinand in 1955 [37, 100], who employed a formula due to G. N. Watson [233] in order to derive it. This formula of Watson is

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+x^2)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{x^{-2s}}{2}+\frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi nx), \tag{1.2.29}$$

which is valid for $x>0$ and $\mathrm{Re}(s)>\frac{1}{2}$. This formula of Watson, (1.2.29), serves as the first "one dimensional" analogue of the Selberg-Chowla formula (1.1.2) applied to diagonal quadratic forms. Ramanujan completes page 253 with two corollaries of (1.2.28). The first of these consists in an identity which is equivalent to the transformation formula for the logarithm of Dedekind's $\eta-$function (see (1.2.48) below). In the second corollary, Ramanujan derives one of the most famous formulas attributed to Koshliakov [134]. Preceding Koshliakov's work by more than a decade, Ramanujan states the beautiful formula

$$\sqrt{\alpha}\left(\frac{\gamma}{4}-\frac{\log(4\beta)}{4}+\sum_{n=1}^{\infty}d(n)\,K_0(2n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma}{4}-\frac{\log(4\alpha)}{4}+\sum_{n=1}^{\infty}d(n)\,K_0(2n\beta)\right), \tag{1.2.30}$$

where α and β are two positive numbers such that $\alpha\beta=\pi^2$.

By comparing the formulas obtained for the study of the analytic continuation of the Epstein zeta function (1.1.11), we now give generalizations of (1.2.28) and (1.2.30) with respect to the divisor functions, (1.2.15) and (1.2.18), introduced in Remark 1.2.1. The method used in this example will be employed several times throughout this text. To do so, for a given $\xi>0$, we write

$\mathcal{Z}_2(s;\, a_1,\, a_2;\, \lambda,\, \xi\lambda')$ as the generalized Epstein zeta function

$$\mathcal{Z}_2(s;\, a_1,\, a_2;\, \lambda,\, \xi\lambda') = \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\, a_2(n)}{(\lambda_m + \lambda_n'\xi)^s}, \quad \mathrm{Re}(s) > 2\,\sigma_a. \tag{1.2.31}$$

By virtue of Hecke's functional equation (1.1.9), if we want to apply formulas (1.2.1) and (1.2.2) to (1.2.31), we just need to replace μ_n' by μ_n'/ξ, $b_2(n)$ by $\xi^{-r_2}b_2(n)$ and ρ_2 by $\xi^{-r_2}\rho_2$ in the formulation of Theorem 1.2.1. Invoking (1.2.1) and appealing to the notation introduced in Remark 1.2.1, namely the shortened version (1.2.16), the following Selberg-Chowla formula holds

$$\begin{aligned}\Gamma(s) \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\, a_2(n)}{(\lambda_m + \lambda_n'\xi)^s} &= -\phi_2(0)\,\Gamma(s)\,\phi_1(s) + \rho_1\Gamma(r_1)\,\Gamma(s-r_1)\,\xi^{r_1-s}\,\phi_2\,(s-r_1)\\ &+2\,\xi^{\frac{r_1-s}{2}} \sum_{j=1}^{\infty} \sigma_{s-r_1}\,(\nu_j;\, b_1,\, a_2)\;\nu_j^{\frac{r_1-s}{2}}\, K_{r_1-s}\left(2\sqrt{\nu_j\,\xi}\right), \quad \xi>0.\end{aligned} \tag{1.2.32}$$

Analogously, by (1.2.2), we can write a second Selberg-Chowla formula for (1.2.31)

$$\begin{aligned}\Gamma(s) \sum_{m,n\neq 0}^{\infty} \frac{a_1(m)\, a_2(n)}{(\lambda_m + \lambda_n'\,\xi)^s} &= -\xi^{-s}\,\phi_1(0)\,\Gamma(s)\,\phi_2(s) + \xi^{-r_2}\rho_2\,\Gamma(r_2)\,\Gamma(s-r_2)\,\phi_1\,(s-r_2)\\ &+2\,\xi^{-\frac{r_2+s}{2}} \sum_{j=1}^{\infty} \sigma_{s-r_2}\left(\nu_j';\, b_2,\, a_1\right)\,\nu_j'^{\frac{r_2-s}{2}}\, K_{r_2-s}\left(2\sqrt{\frac{\nu_j'}{\xi}}\right), \quad \xi>0.\end{aligned} \tag{1.2.33}$$

A comparison between (1.2.32) and (1.2.33) yields a generalization of the Ramanujan-Guinand formula (1.2.28), valid for all $s \in \mathbb{C}$. Stated in a clear way, let $\phi_i(s)$, $i = 1,\, 2$, represent the pair of Dirichlet series given in (1.1.10) and satisfying Hecke's functional equation (1.1.9). Denote also by ρ_i the residue that $\phi_i(s)$ possesses at the simple pole $s = r_i$. Then, for all $s \in \mathbb{C}$ and $x > 0$, the following generalization of the Ramanujan-Guinand formula holds

$$\begin{aligned}&2x^{r_1-s}\sum_{j=1}^{\infty}\sigma_{s-r_1}\,(\nu_j; b_1, a_2)\,\nu_j^{\frac{r_1-s}{2}}\;K_{r_1-s}\,(2x\sqrt{\nu_j}) - \frac{2}{x^{r_2+s}}\sum_{j=1}^{\infty}\sigma_{s-r_2}\left(\nu_j'; b_2, a_1\right)\nu_j'^{\frac{r_2-s}{2}}\,K_{r_2-s}\left(\frac{2}{x}\sqrt{\nu_j'}\right)\\ &\quad=\Gamma(s)\left\{\phi_2(0)\phi_1(s) - \phi_1(0)\phi_2(s)x^{-2s}\right\} + \rho_2\,x^{-2r_2}\,\Gamma(r_2)\,\Gamma(s-r_2)\,\phi_1(s-r_2)\\ &\qquad\qquad - \rho_1 x^{2r_1-2s}\Gamma(r_1)\,\Gamma(s-r_1)\,\phi_2(s-r_1),\end{aligned} \tag{1.2.34}$$

where $\sigma_z(\cdot)$ represents the generalized divisor functions described by (1.2.15), which are respectively attached to the sequences $\{\nu_j\}_{j\in\mathbb{N}} = \{\mu_m \cdot \lambda_n'\}$ and $\{\nu_j'\}_{j\in\mathbb{N}} = \{\mu_m' \cdot \lambda_n\}_{m,n\in\mathbb{N}}$.

Letting $\phi_1(s) = \phi_2(s) = \phi(s)$ (so that $r_1 = r_2 = r$ and $\rho_1 = \rho_2 = \rho$) and replacing s by $r - s$ in (1.2.34), we obtain a more symmetric identity

$$\begin{aligned}&x^r\sum_{j=1}^{\infty}\sigma_{-s}\,(\nu_j;\, b,\, a)\;\nu_j^{\frac{s}{2}}\,K_s\,(2x\,\sqrt{\nu_j}) - x^{-r}\sum_{j=1}^{\infty}\sigma_{-s}\,(\nu_j;\, b,\, a)\;\nu_j^{\frac{s}{2}}\,K_s\left(\frac{2}{x}\,\sqrt{\nu_j}\right)\\ &\quad= \frac{\phi(0)}{2}\,\Gamma(s)\psi(s)\left\{x^{r-s} - x^{s-r}\right\} + \frac{\rho\Gamma(r)}{2}\,\Gamma(-s)\,\phi(-s)\left\{x^{-r-s} - x^{s+r}\right\},\end{aligned} \tag{1.2.35}$$

which closely resembles the classical case[7] (1.2.28). For example, if $\phi_1(s)=\phi_2(s)=\pi^{-s}\zeta(2s)$, an application of (1.2.35) gives (1.2.28) after one takes $x=\frac{\alpha}{\pi}$, $\alpha>0$.

From (1.2.34) it is possible to derive generalizations of the classical Koshliakov formula (1.2.30). Writing the Laurent series for $\phi_i(s)$ around $s=r_i=r$ as

$$\phi_i(s)=\frac{\rho_i}{s-r}+\rho_{0,i}+O\left(s-r\right), \quad i=1,\,2, \tag{1.2.36}$$

assuming the same conditions as in (1.2.34), and taking $r_1=r_2=r$, we have the formula

$$\begin{aligned} &2\sum_{j=1}^{\infty} d\left(\nu_j;\,b_1,\,a_2\right)\,K_0\left(2x\,\sqrt{\nu_j}\right)-2\,x^{-2r}\sum_{j=1}^{\infty} d\left(\nu_j';\,b_2,\,a_1\right)\,K_0\left(\frac{2}{x}\,\sqrt{\nu_j'}\right)\\ &=\Gamma(r)\left\{\phi_2(0)\rho_{0,1}+\phi_2(0)\,\rho_1\,\frac{\Gamma'(r)}{\Gamma(r)}-\rho_1\,\left\{\rho_2^{\star}\Gamma'(r)+\rho_{0,2}^{\star}\Gamma(r)\right\}+2\,\log(x)\,\phi_2(0)\,\rho_1\right\}\\ &+x^{-2r}\Gamma(r)\left\{\left\{\rho_1^{\star}\Gamma'(r)+\rho_{0,1}^{\star}\Gamma(r)\right\}\rho_2-\phi_1(0)\,\rho_{0,2}-\phi_1(0)\,\rho_2\,\frac{\Gamma'(r)}{\Gamma(r)}+2\log(x)\,\phi_1(0)\,\rho_2\right\}, \quad x>0. \end{aligned} \tag{1.2.37}$$

Here, $\rho_i^{\star}$ and $\rho_{0,i}^{\star}$ denote the coefficients of the meromorphic expansion (1.2.36) for $\psi_i(s)$ and $d(\nu_j;\,b_1,a_2)$ is defined by (1.2.18). Of course, (1.2.37) is obtained by letting $s\to r$ in (1.2.34), invoking the uniform and absolute convergence of the series via (1.2.12), and finally using the meromorphic expansions for $\Gamma(s)$ and $\phi_i(s)$ around $s=0$,

$$\Gamma(s)=\frac{1}{s}-\gamma+O(s), \tag{1.2.38}$$

$$\phi_i(s)=\phi_i(0)+\left(\rho_i^{\star}\Gamma'(r_i)+\rho_{0,i}^{\star}\Gamma(r_i)+\gamma\phi_i(0)\right)\,s+O(s^2), \tag{1.2.39}$$

where γ is Euler's constant.

In the examples that follow, we give some new identities that can be derived from our general formulas (1.2.35) and (1.2.37). We refer to the preprint version of the present chapter, [194], for additional results, including extensions of an identity due to Soni [207], which also appears in Ramanujan's Lost Notebook [[37], p. 30, eq. (4.1)].

Example 1.2.2. For $i=1,2$, let $F_i(n)$ denote, as usual, the number of integral ideals of norm n in an imaginary quadratic number field with discriminant D_i, $K_i=\mathbb{Q}\left(\sqrt{-D_i}\right)$. Then it is well-known that the associated Dirichlet series is the classical Dedekind zeta function

$$\zeta_{K_i}(s)=\sum_{n=1}^{\infty}\frac{F_i(n)}{n^s}, \quad \text{Re}(s)>1, \tag{1.2.40}$$

[7] Berndt [[23], p. 343, eq. (9.1)] (see also [[37], pp. 37-40]) established other analogues of (1.2.28). But these formulas of Berndt are only valid for arithmetical functions whose Dirichlet series satisfy Hecke's functional equation, while in general the divisor function $c_z(j):=\sigma_z(\nu_j;\,b_1,\,a_2)$ is not attached to a Dirichlet series obeying to (1.1.9). In fact, this Dirichlet series is nothing but $\varphi_z(s):=\psi_1(s-z)\,\phi_2(s)$ and (1.2.34) is actually equivalent to the functional equation satisfied by $\varphi_z(s)$ (cf. [[23], p. 324]). In Corollary 4.2.7 below, we will see that analogues of Berndt's formula are connected to the theory of the Kontorovich-Lebedev transform.

which satisfies Hecke's functional equation

$$\left(\frac{2\pi}{\sqrt{D_i}}\right)^{-s}\Gamma(s)\,\zeta_{K_i}(s)=\left(\frac{2\pi}{\sqrt{D_i}}\right)^{s-1}\Gamma(1-s)\,\zeta_{K_i}(1-s). \tag{1.2.41}$$

Since $\zeta_{K_i}(s)$ has a continuation to the entire complex plane as an analytic function except at a simple pole located at $s=1$, we see that, for $i=1,\,2$, $\zeta_{K_i}(s)$ satisfies the conditions given by our Definition 1.1.1 with parameter $r_i=1$. Hence, we may take $\phi_i(s):=\zeta_{K_i}(s)$. In order to apply Corollary 1.2.1, we need to employ the following substitutions,

$$a_i(n)=F_i(n),\;\; b_i(n)=\frac{2\pi}{\sqrt{D_i}}F_i(n),\;\; \lambda_{n,i}=n,\;\; \mu_{n,i}=\frac{4\pi^2}{D_i}n,\;\; \rho_i=\frac{2\pi\,h(K_i)R(K_i)}{\sqrt{D_i}\,w(K_i)},$$

where $h(K_i)$, $R(K_i)$ and $w(K_i)$ denote, respectively, the class number of K_i, the regulator of K_i and the number of roots of unity in K_i. Recall that, by our convention (see condition 4 on Definition 1.1.1), $F_i(0):=-\zeta_{K_i}(0)=h(K_i)\,R(K_i)/w(K_i)$, and we can consider the diagonal Epstein zeta function attached to K_1 and K_2 as follows

$$\mathcal{Z}_2\left(s;\,K_1,\,K_2\right):=\sum_{m,n\neq 0}^{\infty}\frac{F_1(m)\,F_2(n)}{(m+n)^s},\quad \mathrm{Re}(s)>2. \tag{1.2.42}$$

Under the aforementioned substitutions, we have from the Selberg-Chowla formula (1.2.1) that (1.2.42) has the expansion

$$\begin{aligned}\Gamma(s)\,\mathcal{Z}_2\left(s;\,K_1,\,K_2\right)&=\frac{h(K_2)\,R(K_2)}{w(K_2)}\Gamma(s)\,\zeta_{K_1}(s)+\frac{2\pi\,h(K_1)R(K_1)}{\sqrt{D_1}\,w(K_1)}\Gamma(s-1)\,\zeta_{K_2}\left(s-1\right)\\&\quad+2\left(\frac{2\pi}{\sqrt{D_1}}\right)^{s}\sum_{m,n=1}^{\infty}F_1(m)\,F_2(n)\left(\frac{m}{n}\right)^{\frac{s-1}{2}}K_{s-1}\left(\frac{4\pi\sqrt{m\,n}}{\sqrt{D_1}}\right),\end{aligned}$$

giving the analytic continuation of $\mathcal{Z}_2(s;\,K_1,\,K_2)$ as a meromorphic function with one simple pole located at $s=2$, whose residue is precisely (compare with (1.2.19)

$$\mathrm{Res}_{s=2}\mathcal{Z}_2(s;\,K_1,\,K_2)=\frac{4\pi^2}{\sqrt{D_1D_2}}\,\frac{h(K_1)h(K_2)\,R(K_1)R(K_2)}{w(K_1)w(K_2)}.$$

By Corollary 1.2.1, we also have the functional equation,

$$\left(\frac{2\pi}{\sqrt{D_1D_2}}\right)^{-s}\Gamma(s)\,\mathcal{Z}_2\left(s;\,K_1,\,K_2\right)=\sqrt{D_1D_2}\left(\frac{2\pi}{\sqrt{D_1D_2}}\right)^{-(2-s)}\Gamma(2-s)\,\mathcal{Z}_2\left(2-s;\,\mathbb{I}_{D_1,D_2},\,K_1,\,K_2\right),$$

where $\mathcal{Z}_2\left(s;\,\mathbb{I}_{D_1,D_2},\,K_1,\,K_2\right)$ denotes the Dirichlet series

$$\mathcal{Z}_2\left(s;\,\mathbb{I}_{D_1,D_2},\,K_1,\,K_2\right):=\sum_{m,n\neq 0}^{\infty}\frac{F_1(m)\,F_2(n)}{(D_2m+D_1n)^s},\quad \mathrm{Re}(s)>2. \tag{1.2.43}$$

If, for the imaginary quadratic fields K_1 and K_2, we define an analogue of the divisor function, $\sigma_{\nu}(n;\,K_1,\,K_2)$, in the following way

$$\sigma_{\nu}(n;\,K_1,\,K_2)=\sum_{d|n}F_1(d)\,F_2\left(\frac{n}{d}\right)\,d^{\nu}, \tag{1.2.44}$$

then, for any $x>0$, the following identity of Ramanujan-Guinand type holds

$$\begin{aligned}
&\left(\frac{2\pi}{\sqrt{D_1}}\right)^{s}x^{1-s}\sum_{n=1}^{\infty}\sigma_{s-1}\left(n;\,K_1,\,K_2\right)\,n^{\frac{1-s}{2}}\,K_{s-1}\left(\frac{4\pi\sqrt{n}}{\sqrt{D_1}}\,x\right)\\
&-\left(\frac{2\pi}{\sqrt{D_2}}\right)^{s}x^{-1-s}\sum_{n=1}^{\infty}\sigma_{s-1}\left(n;\,K_2,\,K_1\right)\,n^{\frac{1-s}{2}}\,K_{s-1}\left(\frac{4\pi\sqrt{n}}{\sqrt{D_2}\,x}\right)\\
&=\frac{h(K_1)\,R(K_1)\,x^{-2s}}{2\,w(K_1)}\left\{\Gamma(s)\,\zeta_{K_2}(s)-\frac{2\pi\,x^{2}}{\sqrt{D_1}}\Gamma(s-1)\,\zeta_{K_2}(s-1)\right\}\\
&+\frac{h(K_2)\,R(K_2)}{2\,w(K_2)}\left\{\frac{2\pi}{\sqrt{D_2}\,x^{2}}\,\Gamma(s-1)\,\zeta_{K_1}(s-1)-\Gamma(s)\,\zeta_{K_1}(s)\right\},
\end{aligned} \tag{1.2.45}$$

which seems to be new (cf. [38]). Taking the quadratic fields as the same, say $K_1=K_2:=K$, and replacing s by $1-s$ in (1.2.45), we can obtain a more appealing formula,

$$\begin{aligned}
&x\sum_{n=1}^{\infty}\sigma_{-s}\left(n;\,K,K\right)\,n^{\frac{s}{2}}\,K_{s}\left(\frac{4\pi\sqrt{n}}{\sqrt{D}}\,x\right)-\frac{1}{x}\sum_{n=1}^{\infty}\sigma_{-s}\left(n;\,K,\,K\right)\,n^{\frac{s}{2}}\,K_{s}\left(\frac{4\pi\sqrt{n}}{\sqrt{D}\,x}\right)\\
&=\frac{h(K)\,R(K)}{2\,w(K)}\left(\frac{2\pi}{\sqrt{D}}\right)^{-s}\Gamma(s)\,\zeta_K(s)\left\{x^{s-1}-x^{1-s}\right\}\\
&+\frac{h(K)\,R(K)}{2\,w(K)}\left(\frac{2\pi}{\sqrt{D}}\right)^{s}\Gamma(-s)\,\zeta_K(-s)\left\{x^{-s-1}-x^{s+1}\right\},\qquad x>0.
\end{aligned} \tag{1.2.46}$$

From the Kronecker limit formula for imaginary quadratic fields, it is also possible to take the limit $s\to 0$ on (1.2.46) in order to derive new analogues of Koshliakov's formula for the divisor function (1.2.44). A special case of this procedure, where we can replace each $\phi_i(s)$ by an Epstein zeta function attached to a binary, real and positive-definite quadratic form, seems to be worthy of note. For $j=1,\,2$, consider the positive-definite quadratic form $Q_j(x,\,y)=a_jx^2+b_jx\,y+c_jy^2$, with $a_j>0$ and $\Delta_j=4a_j\,c_j-b_j^2$. From Kronecker's first limit formula [201, 222], the Laurent expansion of $\zeta\left(s,\,Q_j\right)$, defined by (1.1.1), reads

$$\zeta(s,\,Q_j)=\frac{2\pi}{\sqrt{\Delta_j}}\,\frac{1}{s-1}+\frac{2\pi}{\sqrt{\Delta_j}}\left(2\gamma-\log\left(\frac{\Delta_j}{a_j}\right)-4\log(|\eta(\tau_j)|)\right)+O(s-1), \tag{1.2.47}$$

where γ is Euler's constant, $\tau_j=\frac{b_j+i\sqrt{\Delta_j}}{2a_j}\in\mathbb{H}$ and $\eta(\tau)$ is Dedekind's $\eta-$function,

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right),\quad \mathrm{Im}(\tau)>0. \tag{1.2.48}$$

Using (1.2.47) in Koshliakov's formula (1.2.37), we arrive at the curious identity

$$\frac{1}{\sqrt{\Delta_1}}\sum_{n=1}^{\infty}d\left(n;\,Q_1,Q_2\right)\,K_0\left(\frac{4\pi\sqrt{n}}{\sqrt{\Delta_1}}x\right)-\frac{1}{\sqrt{\Delta_2}\,x^2}\sum_{n=1}^{\infty}d\left(n;\,Q_1,Q_2\right)\,K_0\left(\frac{4\pi\sqrt{n}}{\sqrt{\Delta_2}x}\right)$$
$$=\frac{1}{\sqrt{\Delta_1}}\left\{\log\left[\frac{2\pi}{x}\sqrt{\frac{\Delta_1}{a_1a_2}}\,|\eta(\tau_1)\,\eta(\tau_2)|^2\right]-\gamma\right\}+\frac{1}{\sqrt{\Delta_2}x^2}\left\{\gamma-\log\left[2\pi x\sqrt{\frac{\Delta_2}{a_1a_2}}\,|\eta(\tau_1)\,\eta(\tau_2)|^2\right]\right\}, \tag{1.2.49}$$

valid for all $x>0$. Here, $d(n;\,Q_1,\,Q_2)$ denotes the analogue of the divisor function,

$$d(n;\,Q_1,\,Q_2)=\sum_{\ell|n}r_{Q_1}(\ell)\,r_{Q_2}\left(\frac{n}{\ell}\right),$$

where $r_Q(n)$ denotes the number of representations of n by the quadratic form Q. When $Q_1(x,\,y)=Q_2(x,\,y):=Q(x,\,y):=ax^2+bxy+cy^2$ (and so $\Delta_1=\Delta_2:=\Delta$), a particular case of (1.2.49) is

$$\sum_{n=1}^{\infty}d_Q\left(n\right)K_0\left(\frac{4\pi\sqrt{n}}{\sqrt{\Delta}}x\right)-\frac{1}{x^2}\sum_{n=1}^{\infty}d_Q\left(n\right)K_0\left(\frac{4\pi\sqrt{n}}{\sqrt{\Delta}\,x}\right)$$
$$=\log\left[\frac{2\pi\sqrt{\Delta}}{ax}\,|\eta(\tau)|^4\right]-\gamma+\frac{1}{x^2}\left\{\gamma-\log\left[\frac{2\pi x}{a}\sqrt{\Delta}\,\,|\eta(\tau)|^4\right]\right\},$$

where $d_Q(n)=d(n;\,Q,\,Q):=\sum_{\ell|n}r_Q(\ell)\,r_Q(n/\ell)$. An alternative way of writing the previous identity is via two double infinite series, i.e.,

$$\sum_{m,n=1}^{\infty}r_Q(m)\,r_Q(n)\,K_0\left(\frac{4\pi\sqrt{m\,n}}{\sqrt{\Delta}}x\right)-\frac{1}{x^2}\sum_{m,n=1}^{\infty}r_Q(m)\,r_Q(n)\,K_0\left(\frac{4\pi\sqrt{m\,n}}{\sqrt{\Delta}\,x}\right)$$
$$=\log\left[\frac{2\pi\sqrt{\Delta}}{ax}\,|\eta(\tau)|^4\right]-\gamma+\frac{1}{x^2}\left\{\gamma-\log\left[\frac{2\pi x}{a}\sqrt{\Delta}\,\,|\eta(\tau)|^4\right]\right\},\qquad x>0. \tag{1.2.50}$$

If we take the quadratic form $Q(x,y)=Q_0(x,y):=x^2+y^2$, (1.2.50) reduces to the formula

$$\sum_{m,n=1}^{\infty}r_2(m)\,r_2(n)\,K_0\left(2\pi\sqrt{m\,n}\,x\right)-\frac{1}{x^2}\sum_{m,n=1}^{\infty}r_2(m)\,r_2(n)\,K_0\left(\frac{2\pi\sqrt{m\,n}}{x}\right)$$
$$=\log\left[\frac{4\pi}{x}\,\eta(i)^4\right]-\gamma+\frac{1}{x^2}\left\{\gamma-\log\left[4\pi x\,\eta(i)^4\right]\right\}, \tag{1.2.51}$$

which will be generalized in the fourth chapter. It is also important to note that we can use the following evaluation (coming from Euler's pentagonal theorem),

$$\eta(i)=\frac{\Gamma\left(\frac{1}{4}\right)}{2\pi^{\frac{3}{4}}}, \tag{1.2.52}$$

in order to rewrite (1.2.51) as

$$\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, K_0\left(2\pi\sqrt{m\,n}\,x\right) - \frac{1}{x^2}\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, K_0\left(\frac{2\pi\sqrt{m\,n}}{x}\right)$$
$$= \log\left[\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2 x}\right] - \gamma + \frac{1}{x^2}\left\{\gamma - \log\left[\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2}x\right]\right\}, \qquad x>0. \tag{1.2.53}$$

Example 1.2.3 (Dirichlet series attached to Eisenstein Series)**.** Let $k \geq 3$ be an odd positive integer and consider the divisor function $\sigma_k(n) := \sum_{d|n} d^k$. We know that the associated Dirichlet series is given by the product of zeta functions

$$\zeta(s)\,\zeta(s-k) = \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}, \quad \mathrm{Re}(s) > k+1, \tag{1.2.54}$$

and that $\phi(s) = (2\pi)^{-s}\,\zeta(s)\,\zeta(s-k)$ obeys to Hecke's functional equation [[53], p. 17, eq. (57)]

$$(2\pi)^{-s}\,\Gamma(s)\,\zeta(s)\,\zeta(s-k) = (-1)^{\frac{k+1}{2}}(2\pi)^{-(k+1-s)}\Gamma(k+1-s)\,\zeta(k+1-s)\,\zeta(1-s). \tag{1.2.55}$$

Moreover, since $\zeta(1-k)=0$, we know that $\phi(s)$ satisfies the conditions of definition 1.1.1, and we may take the convention $\sigma_k(0) := -B_{k+1}/2k+2$.

Let $k_1,\, k_2 \geq 3$ be two odd positive integers and consider two Dirichlet series $\phi_i(s)$, $i=1,\,2$, given by $\phi_i(s) = (2\pi)^{-s}\zeta(s)\,\zeta(s-k_i)$. We construct the diagonal Epstein zeta function (1.1.11) as

$$\mathcal{Z}_2(s;\,k_1,\,k_2) = (2\pi)^{-s}\sum_{m,n\neq 0}^{\infty}\frac{\sigma_{k_1}(m)\,\sigma_{k_2}(n)}{(m+n)^s} := (2\pi)^{-s}\,Z_2(s;\,k_1,\,k_2), \quad \mathrm{Re}(s) > 2\,\max\{k_1,\,k_2\}+2. \tag{1.2.56}$$

In order to apply directly Theorem 1.2.1 and its Corollary 1.2.1, we take the substitutions

$$a_i(n) = \sigma_{k_i}(n), \quad b_i(n) = (-1)^{\frac{k_i+1}{2}}\sigma_{k_i}(n), \;\; \lambda_{n,i} = \mu_{n,i} = 2\pi n. \tag{1.2.57}$$

Using Euler's formula for $\zeta(2n)$, it is very simple to find that

$$\rho_i = \frac{(-1)^{\frac{k_i-1}{2}}}{2(k_i+1)!}\,B_{k_i+1}, \quad \phi_i(0) := \zeta(0)\,\zeta(-k_i) = \frac{B_{k_i+1}}{2k_i+2}, \tag{1.2.58}$$

where $(B_n)_{n\in\mathbb{N}}$ denotes the sequence of Bernoulli numbers. Applying these computations to the first Selberg-Chowla formula (1.2.1), we arrive at the representation

$$(2\pi)^{-s}\,\Gamma(s)\,Z_2(s;\,k_1,\,k_2) = -\frac{B_{k_2+1}\,(2\pi)^{-s}}{2k_2+2}\,\Gamma(s)\,\zeta(s)\,\zeta(s-k_1)$$
$$+\frac{(-1)^{\frac{k_1-1}{2}}\,B_{k_1+1}\,(2\pi)^{-s+k_1+1}}{2k_1+2}\,\Gamma(s-k_1-1)\zeta(s-k_1-1)\,\zeta(s-k_1-k_2-1)$$
$$+2\sum_{m,n=1}^{\infty}(-1)^{\frac{k_1+1}{2}}\sigma_{k_1}(m)\,\sigma_{k_2}(n)\left(\frac{m}{n}\right)^{\frac{s-k_1-1}{2}}K_{s-k_1-1}\left(4\pi\sqrt{m\,n}\right), \tag{1.2.59}$$

which gives the analytic continuation of $Z_2(s;\, k_1,\, k_2)$ as a meromorphic function with one simple pole located at $s = k_1 + k_2 + 2$, whose residue is given by

$$\mathrm{Res}_{s=k_1+k_2+2}\, Z_2(s;\, k_1,\, k_2) = \frac{(-1)^{\frac{k_1+k_2}{2}-1}(2\pi)^{k_1+k_2+2}}{4\,(k_1+1)\,(k_2+1)}\,\frac{B_{k_1+1}\,B_{k_2+1}}{(k_1+k_2+1)!}. \tag{1.2.60}$$

The second Selberg-Chowla formula can be analogously written and an application of Corollary 1.2.1 shows that $Z_2(s;\, k_1,\, k_2)$ must satisfy the functional equation

$$\begin{aligned}&(2\pi)^{-s}\,\Gamma(s)\,Z_2(s;\, k_1,\, k_2)\\ &= (-1)^{\frac{k_1+k_2+2}{2}}(2\pi)^{-(k_1+k_2+2-s)}\,\Gamma(k_1+k_2+2-s)\,Z_2(k_1+k_2+2-s;\, k_1,\, k_2).\end{aligned} \tag{1.2.61}$$

We now combine the Bessel expansion (1.2.59) with some celebrated formulas of Ramanujan [178] in order to derive new identities involving the Riemann zeta function. By a very famous paper of Ramanujan [178], we have a formula for the convolution of divisor functions of the form,

$$\sum_{m=0}^{n}\sigma_{k_1}(m)\,\sigma_{k_2}(n-m) = \frac{k_1!\,k_2!}{(k_1+k_2+1)!}\,\frac{\zeta\,(k_1+1)\,\zeta\,(k_2+1)}{\zeta\,(k_1+k_2+2)}\,\sigma_{k_1+k_2+1}(n), \tag{1.2.62}$$

where $k_1 \geq 3$ and $k_2 \geq 3$ are odd integers such that $k_1 + k_2 \in \{6, 8, 12\}$. Using (1.2.62), we can rewrite the Epstein zeta function (1.2.56) in the region $\mathrm{Re}(s) > 2\,\max\{k_1,\, k_2\} + 2$ as

$$\begin{aligned}\mathcal{Z}_2(s;\, k_1,\, k_2) &= (2\pi)^{-s}\left\{\sigma_{k_2}(0)\,\zeta(s)\,\zeta(s-k_1) + \sum_{n=1,m=0}^{\infty}\frac{\sigma_{k_1}(m)\,\sigma_{k_2}(n)}{(m+n)^s}\right\}\\ &= (2\pi)^{-s}\left\{-\frac{B_{k_2+1}}{2k_2+2}\cdot\zeta(s)\,\zeta(s-k_1) + \sum_{n=1}^{\infty}\frac{\sum_{m=0}^{n-1}\sigma_{k_1}(m)\,\sigma_{k_2}(n-m)}{n^s}\right\}\\ &= \frac{k_1!\,k_2!}{(k_1+k_2+1)!}\,\frac{\zeta\,(k_1+1)\,\zeta\,(k_2+1)}{\zeta\,(k_1+k_2+2)}\,(2\pi)^{-s}\zeta(s)\,\zeta(s-k_1-k_2-1).\end{aligned} \tag{1.2.63}$$

By employing Euler's formula for $\zeta(2n)$, $n \in \mathbb{N}$, and combining the representations (1.2.59) and (1.2.63), we are able to obtain the following formula

$$\begin{aligned}&-\frac{(k_1+k_2+2)\;B_{k_1+1}B_{k_2+1}\,(2\pi)^{-s}\Gamma(s)}{2\,(k_1+1)\,(k_2+1)\;B_{k_1+k_2+2}}\,\zeta(s)\,\zeta(s-k_1-k_2-1) = -\frac{B_{k_2+1}\,(2\pi)^{-s}\,\Gamma(s)}{2k_2+2}\,\zeta(s)\,\zeta(s-k_1)\\ &\quad+\frac{(-1)^{\frac{k_1-1}{2}}\,B_{k_1+1}\,(2\pi)^{-s+k_1+1}\,\Gamma(s-k_1-1)}{2k_1+2}\zeta(s-k_1-1)\,\zeta(s-k_1-k_2-1)\\ &\quad+2\sum_{m,n=1}^{\infty}(-1)^{\frac{k_1+1}{2}}\,\sigma_{k_1}(m)\,\sigma_{k_2}(n)\left(\frac{m}{n}\right)^{\frac{s-k_1-1}{2}}\;K_{s-k_1-1}\left(4\pi\sqrt{m\,n}\right),\end{aligned} \tag{1.2.64}$$

which is valid for all $s \in \mathbb{C}$ and odd $k_1, k_2 \geq 3$ such that $k_1 + k_2 \in \{6, 8, 12\}$. This result seems to be new. Take, for example, $k_1 = k_2 = 3$: then (1.2.64) reads

$$(2\pi)^{-s}\Gamma(s)\,\frac{\zeta(s)\,\zeta(s-7)}{120} = \frac{(2\pi)^{-s}}{240}\,\Gamma(s)\,\zeta(s)\,\zeta(s-3) + \frac{(2\pi)^{4-s}\Gamma(s-4)\,\zeta(s-4)\,\zeta(s-7)}{240} + 2\sum_{m,n=1}^{\infty}\sigma_3(m)\,\sigma_3(n)\left(\frac{m}{n}\right)^{\frac{s-4}{2}} K_{s-4}\left(4\pi\sqrt{m\,n}\right), \tag{1.2.65}$$

for any complex number s. Putting $s = 9/2$ in the formula above gives the curious relation

$$\frac{105\,\zeta\left(\frac{7}{2}\right)\zeta\left(\frac{9}{2}\right)}{128\,\pi^4} = \frac{7\,\zeta\left(\frac{3}{2}\right)\zeta\left(\frac{9}{2}\right)}{4\pi} + \zeta\left(\frac{1}{2}\right)\zeta\left(\frac{7}{2}\right) + 1024\,\pi^3\sum_{m,n=1}^{\infty}\frac{\sigma_3(m)\,\sigma_3(n)}{\sqrt{n}}\,e^{-4\pi\sqrt{m\,n}}.$$

There is an extensive bibliography concerning divisor convolutions of the form (1.2.62). See, for instance, [116] for several evaluations. The interested reader may combine our formulas of Selberg-Chowla type with other known convolutions of divisor functions to reach new identities akin to (1.2.64).

Example 1.2.4 (Dirichlet series attached to Cusp Forms)**.** Let $f(\tau)$ be a cusp form with weight $k \geq 12$ for the full modular group and $L(s,\,f)$ be the associated $L-$function,

$$L(s,\,f) = \sum_{n=1}^{\infty}\frac{a_f(n)}{n^s}, \tag{1.2.66}$$

where the arithmetical function $a_f(n)$ represents the Fourier coefficients of $f(\tau)$. The Dirichlet series (1.2.66) is known to be absolutely convergent in the half-plane $\text{Re}(s) > \frac{k+1}{2}$ and can be analytically continued to an entire function [[6], Theorem 6.20]. Furthermore, it obeys to Hecke's functional equation

$$(2\pi)^{-s}\Gamma(s)\,L(s,\,f) = (-1)^{k/2}(2\pi)^{-(k-s)}\Gamma(k-s)\,L(k-s,\,f). \tag{1.2.67}$$

Let us now consider two holomorphic cusp forms $f_1(\tau)$ and $f_2(\tau)$ with weights k_1 and k_2, respectively, and the pairs of Dirichlet series $\phi_i(s) = (2\pi)^{-s}L(s,\,f_i)$ and $\psi_i(s) = (-1)^{k_i/2}(2\pi)^{-s}L(s,\,f_i)$, $i = 1,\,2$, associated with them. Since each $\phi_i(s)$ is entire, it follows from Corollary 1.2.1 that the analogue of Epstein's zeta function,

$$\mathcal{Z}_2(s;\,f_1,\,f_2) = (2\pi)^{-s}\sum_{m,n\neq 0}^{\infty}\frac{a_{f_1}(m)\,a_{f_2}(n)}{(m+n)^s} = (2\pi)^{-s}Z_2(s;\,f_1,\,f_2), \quad \text{Re}(s) > \max\{k_1,\,k_2\} + 1, \tag{1.2.68}$$

is also entire and its analytic continuation can be established through the formulas

$$(2\pi)^{-s}\,\Gamma(s)\,Z_2\left(s;\,f_1,\,f_2\right) = 2\,(-1)^{\frac{k_1}{2}}\sum_{m,n=1}^{\infty}a_{f_1}(m)\,a_{f_2}(n)\left(\frac{m}{n}\right)^{\frac{s-k_1}{2}} K_{k_1-s}\left(4\pi\sqrt{m\,n}\right)$$

and

$$(2\pi)^{-s}\,\Gamma(s)\,Z_2\left(s;\,f_1,\,f_2\right) = 2\,(-1)^{\frac{k_2}{2}}\sum_{m,n=1}^{\infty} a_{f_1}(m)\,a_{f_2}(n)\left(\frac{m}{n}\right)^{\frac{k_2-s}{2}} K_{k_2-s}\left(4\pi\sqrt{m\,n}\right).$$

It also follows from Corollary 1.2.1 that (1.2.68) satisfies the functional equation

$$(2\pi)^{-s}\,\Gamma(s)\,Z_2\left(s;\,f_1,\,f_2\right) = (-1)^{\frac{k_1+k_2}{2}}\,(2\pi)^{-(k_1+k_2-s)}\,\Gamma\left(k_1+k_2-s\right)\,Z_2\left(k_1+k_2-s;\,f_1,\,f_2\right). \tag{1.2.69}$$

Due to the fact that $\phi_1(s)$ and $\phi_2(s)$ are entire, it is now very easy to write analogues of the Ramanujan-Guinand or Koshliakov's formulas. In particular, for the case $k_1 = k_2 = 12$, $\phi_1(s) = \phi_2(s) = \phi(s)$, we may consider the Dirichlet series associated to the Ramanujan $\tau-$function,

$$\phi(s) := (2\pi)^{-s}\,L(s,\,\tau) = (2\pi)^{-s}\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s},\quad \mathrm{Re}(s) > \frac{13}{2},$$

which naturally is the Dirichlet series attached to the cusp form $f(\tau) = \eta(\tau)^{24}$, where $\eta(\tau)$ denotes Dedekind's $\eta-$function (1.2.48). It follows from the previous analysis that the resulting diagonal Epstein zeta function (1.2.68) will satisfy Hecke's functional equation (1.2.69) with $r := k_1+k_2 = 24$. Also, a curious formula of Koshliakov type takes place, namely,

$$\sum_{n=1}^{\infty} d_\tau(n)\,K_0\left(4\pi\,\sqrt{n}\,x\right) = x^{-24}\sum_{n=1}^{\infty} d_\tau(n)\,K_0\left(\frac{4\pi\sqrt{n}}{x}\right),\quad x>0, \tag{1.2.70}$$

where $d_\tau(n)$ denotes the divisor function associated with $\tau(n)$,

$$d_\tau(n) = \sum_{d|n}\tau(d)\,\tau\left(\frac{n}{d}\right).$$

A different proof of (1.2.70) is given in [[22], p. 357, Example 2]. This result is obtained from a generalization of Bochner's modular relation (1.1.30) for Dirichlet series satisfying a functional equation with a factor of the form $\Delta(s) = \Gamma^N(s)$.

In analogy to (1.2.20), one should expect a functional equation connecting (1.1.19) and the multidimensional extrapolation of (1.2.25). In fact, it will be simple to show, via an inductive argument, that the following corollary takes place.

Corollary 1.2.2 (The Analytic Continuation of the Multidimensional diagonal Epstein zeta function)**.** *Assume that $\{\phi_i(s)\}_{i=1,\dots,k}$ are Dirichlet series satisfying Hecke's functional equation (1.1.21) in the sense of definition 1.1.1, each of these having a residue at $s = r_i$ equal to ρ_i. Then the multidimensional Epstein zeta function (1.1.18) has an analytic continuation as a meromorphic function having at most a simple pole located at $s = \sum_{j=1}^{k} r_j$, with residue given by*

$$Res_{s=\sum_{j=1}^{k} r_j}\,\mathcal{Z}_k\left(s;\,a_1,\dots,a_k;\,\lambda_1,\,\dots,\,\lambda_k\right) = \frac{\prod_{j=1}^{k}\Gamma(r_j)\,\rho_j}{\Gamma\left(\sum_{j=1}^{k} r_j\right)}. \tag{1.2.71}$$

Moreover, it satisfies the functional equation

$$\begin{aligned}&\Gamma\left(s\right)\mathcal{Z}_{k}\left(s;\,a_{1},...,\,a_{k};\,\lambda_{1},\,...,\,\lambda_{k}\right)\\ &=\Gamma(r_{1}+...+r_{k}-s)\,\mathcal{Z}_{k}\left(r_{1}+...+r_{k}-s;\,b_{1},...,\,b_{k};\,\mu_{1},...,\mu_{k}\right).\end{aligned}\tag{1.2.72}$$

Proof. Let us first introduce some notation: in analogy with (1.1.19), we start by writing

$$\mathcal{Z}_{k}\left(s;\,b_{1},...,\,b_{k};\,\mu_{1},...,\,\mu_{k}\right)=\sum_{n=1}^{\infty}\frac{\mathfrak{V}_{k}(n)}{\Omega_{n}^{s}},\quad \mathrm{Re}(s)>k\sigma_{b},\tag{1.2.73}$$

where $\mathfrak{V}_{k}(n)$ plays the same role as $\mathfrak{U}_{k}(n)$ in (1.1.20) and $\sigma_{b}=\max_{1\leq i\leq k}\,\sigma_{b_{i}}$. Also, according to the notation introduced in (1.1.19) and (1.2.73), we shall represent $\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)$ and $\mathcal{Z}_{k-1}\left(s;\,b_{1},...,\,b_{k-1};\,\mu_{1},\,...,\,\mu_{k-1}\right)$ as the Dirichlet series

$$\begin{aligned}\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)&=\sum_{n=1}^{\infty}\frac{\mathfrak{U}_{k-1}(n)}{\Lambda_{n,k-1}^{s}},\\ \mathcal{Z}_{k-1}\left(s;\,b_{1},...,\,b_{k-1};\,\mu_{1},\,...,\,\mu_{k-1}\right)&=\sum_{n=1}^{\infty}\frac{\mathfrak{V}_{k-1}(n)}{\Omega_{n,k-1}^{s}},\end{aligned}$$

where the index $k-1$ here denotes the "dimension" of the Epstein zeta function $\mathcal{Z}_{k-1}(s;\,\cdot)$.

The proof of the corollary follows now by induction. For the inductive hypothesis, let us assume that $\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)$ satisfies the functional equation

$$\begin{aligned}&\Gamma\left(s\right)\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)\\ &=\Gamma(r_{1}+...+r_{k-1}-s)\,\mathcal{Z}_{k}\left(r_{1}+...+r_{k-1}-s;\,b_{1},...,\,b_{k-1};\,\mu_{1},...,\mu_{k-1}\right)\end{aligned}\tag{1.2.74}$$

and its continuation to the entire complex plane has at most one simple pole located at $s=\sum_{j=1}^{k-1}r_{j}$, whose residue is

$$\mathrm{Res}_{s=\sum_{j=1}^{k-1}r_{j}}\,\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)=\frac{\prod_{j=1}^{k-1}\Gamma(r_{j})\,\rho_{j}}{\Gamma\left(\sum_{j=1}^{k-1}r_{j}\right)}.$$

When we combine this hypothesis with the fact that $\phi_{k}(s)$ satisfies (1.1.21), we can apply the first Selberg-Chowla formula (1.2.1) and replace there $\phi_{1}(s)$ by $\mathcal{Z}_{k-1}\left(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1}\right)$ and $\phi_{2}(s)$ by $\phi_{k}(s)$. Invoking the representation (1.2.73), we see that $\mathcal{Z}_{k}(s;\,a_{1},...,\,a_{k};\,\lambda_{1},...,\,\lambda_{k})$ can be written as

$$\begin{aligned}&\mathcal{Z}_{k}(s;\,a_{1},...,\,a_{k};\,\lambda_{1},...,\,\lambda_{k})=-\phi_{k}(0)\,\mathcal{Z}_{k-1}(s;\,a_{1},...,\,a_{k-1};\,\lambda_{1},\,...,\,\lambda_{k-1})\\ &+\prod_{j=1}^{k-1}\Gamma(r_{j})\,\rho_{j}\,\frac{\Gamma(s-r_{1}-...-r_{k-1})}{\Gamma(s)}\,\phi_{k}\left(s-r_{1}-...-r_{k-1}\right)\\ &+\frac{2}{\Gamma(s)}\sum_{m,n=1}^{\infty}\mathfrak{V}_{k-1}(m)\,a_{k}(n)\left(\frac{\Omega_{m,k-1}}{\lambda_{n,k}}\right)^{\frac{s-r_{1}-...-r_{k-1}}{2}}K_{r_{1}+...+r_{k-1}-s}\left(2\sqrt{\Omega_{m,k-1}\lambda_{n,k}}\right).\end{aligned}\tag{1.2.75}$$

Using a similar argument to the one given at the end of the proof of Theorem 1.2.1, one sees that the double series appearing in the previous formula represents an entire function of s. Thus, the only possible poles that $\mathcal{Z}_k(s;\,a_1,...,\,a_k;\,\lambda_1,...,\,\lambda_k)$ possesses must come from the sum of the first two terms in (1.2.75). However, from the functional equation for $\phi_k(s)$ and the induction hypothesis on $\mathcal{Z}_{k-1}(s;\,\cdot)$, it is clear that $\mathcal{Z}_k(s;\,\cdot)$ has removable singularities at $s=r_1+...+r_{k-1}-n$, $\forall n\in\mathbb{N}_0$. Thus, the only possible pole of $\mathcal{Z}_k(s;\,\cdot)$ remaining comes from the factor $\phi_k\left(s-r_1-...-r_{k-1}\right)$ on the second term of (1.2.75) and it is located at $s=r_1+...+r_k$. The computation of its residue gives immediately (1.2.71).

Finally, in order to prove the functional equation (1.2.72), it suffices to compare (1.2.75) with the second Selberg-Chowla formula for $\mathcal{Z}_k(s;\,a_1,...,\,a_k;\,\lambda_1,...,\,\lambda_k)$ (analogous to (1.2.2)),

$$\begin{aligned}
&\mathcal{Z}_k(s;\,a_1,...,\,a_k;\,\lambda_1,...,\,\lambda_k)=-\mathcal{Z}_{k-1}(0;a_1,...,a_{k-1};\lambda_1,...,\lambda_{k-1})\phi_k(s)\\
&+\frac{\rho_k\Gamma(r_k)\Gamma(s-r_k)}{\Gamma(s)}\mathcal{Z}_{k-1}\left(s-r_k;a_1,...,a_{k-1};\lambda_1,...,\lambda_{k-1}\right)\\
&+\frac{2}{\Gamma(s)}\sum_{m,n=1}^{\infty}b_k(m)\,\mathfrak{U}_{k-1}(n)\left(\frac{\mu_{m,k}}{\Lambda_{n,k-1}}\right)^{\frac{s-r_k}{2}}K_{r_k-s}\left(2\sqrt{\mu_{m,k}\Lambda_{n,k-1}}\right),
\end{aligned}$$

and use the induction hypothesis (1.2.74). By adapting the argument given in the proof of Corollary 1.2.1 and replacing there the roles of $\phi_1(s)$ and $\phi_2(s)$ by $\phi_k(s)$ and $\mathcal{Z}_{k-1}(s;\,\cdot)$, respectively, we arrive at (1.2.72). □

Remark 1.2.3 (The dyadic Epstein zeta function)**.** When combined with Theorem 1.2.1, the previous result allows the construction of Selberg-Chowla formulas for a family of multidimensional Epstein zeta functions.

Before moving to the next section, in this remark we will introduce a fairly symmetric class of multidimensional Epstein zeta functions, which we shall call "dyadic Epstein zeta functions", and write useful representations for each one of its elements. These formulas will be very important for the inductive argument presented in the proof of Theorem 1.3.1.

Let $k\geq 0$ and consider the case where $\phi_1(s)=\phi_2(s)=...=\phi_{2^k}(s):=\phi(s)$, with $\phi(s)$ satisfying Hecke's functional equation (1.1.9) with parameter r and abscissa of absolute convergence σ_a. One of the $2^{k+1}-1$ ways of constructing the Epstein zeta function $\mathcal{Z}_{2^{k+1}}(s;\,a;\,\lambda)$ (1.1.23) is by taking the symmetric pairing,

$$\mathcal{Z}_{2^k}\left(s;\,a;\,\lambda\right)=\sum_{m=1}^{\infty}\frac{\mathfrak{U}_{2^k}(m)}{\Lambda_m^s},\quad \mathcal{Z}_{2^{k+1}}\left(s;\,a;\,\lambda\right)=\sum_{m,\,n\neq 0}^{\infty}\frac{\mathfrak{U}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)}{\left(\Lambda_m+\Lambda_n\right)^s},\tag{1.2.76}$$

with each series converging absolutely for $\mathrm{Re}(s)>2^k\sigma_a,\ 2^{k+1}\sigma_a$, respectively (note the convention $\mathcal{Z}_{2^0}(s;\,a;\,\lambda):=\phi(s)$).

From Corollary 1.2.2, each $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$ plays in (1.2.76) the same role as $\phi_1(s)$ and $\phi_2(s)$ play in Theorem 1.2.1 and Corollary 1.2.1. By an application of (1.2.1), it is clear that we can write a

Selberg-Chowla formula in the form

$$\Gamma(s)\,\mathcal{Z}_{2^{k+1}}\left(s;\,a;\,\lambda\right)=-\mathcal{Z}_{2^k}\left(0;\,a;\,\lambda\right)\,\Gamma(s)\,\mathcal{Z}_{2^k}\left(s;\,a;\,\lambda\right)+\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(s-2^kr)\,\mathcal{Z}_{2^k}\left(s-2^kr;\,a;\,\lambda\right)$$
$$+\,2\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\,\left(\frac{\Omega_m}{\Lambda_n}\right)^{\frac{s}{2}-2^{k-1}r}K_{2^kr-s}\left(2\sqrt{\Omega_m\Lambda_n}\right), \qquad (1.2.77)$$

with the arithmetical functions $\mathfrak{U}_{2^k}(n)$ and $\mathfrak{V}_{2^k}(n)$ appearing respectively in (1.1.19) and (1.2.73). We know from (1.2.72) that the multidimensional Epstein zeta function $\mathcal{Z}_{2^k}\left(s;\,a;\,\lambda\right)$ satisfies Hecke's functional equation (1.1.9) with parameter 2^kr, and so its critical line must be $\text{Re}(s)=2^{k-1}r$. Using this information, we can give a convex estimate (of the form (1.1.27)) for the order of $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$. Indeed, for any $\delta>0$ and $2^kr-2^k\sigma_a-\delta\leq\sigma\leq 2^k\sigma_a+\delta$, we have that

$$\mathcal{Z}_{2^k}\left(\sigma+it;\,a;\,\lambda\right)=O\left(|t|^{2^k\sigma_a-\sigma+\delta}\right),\quad |t|\to\infty. \qquad (1.2.78)$$

Remark 1.2.4. By the equivalence between Bochner's modular relation (1.1.30) and Hecke's functional equation (1.1.9), one sees that (1.2.72) is equivalent to the transformation formula

$$\Theta_k(z;\,a_1,...,a_k;\,\lambda_1,...,\lambda_k)=z^{-r_1-...-r_k}\,\Theta_k(z^{-1};\,b_1,...,b_k;\,\mu_1,...,\mu_k)$$
$$+\prod_{j=1}^{k}\Gamma(r_j)\,\rho_j\,z^{-r_1-...-r_k}-\prod_{j=1}^{k}\Gamma(r_j)\,\rho_j^{\star},\quad \text{Re}(z)>0, \qquad (1.2.79)$$

where $\Theta_k(z;\,a_1,...,a_k;\,\lambda_1,...,\lambda_k)$ denotes a further extension of the theta functions (1.1.28) and (1.1.32),

$$\Theta_k(z;\,a_1,...,a_k;\,\lambda_1,...,\lambda_k):=\sum_{m_1,...,m_k\neq 0}^{\infty}a_1(m_1)\cdot...\cdot a_k(m_k)\,\exp\left\{-\sum_{i=1}^{k}\lambda_{m_i,i}\,z\right\},\quad \text{Re}(z)>0.$$

1.3 The zeros of a general Class of Dirichlet series

It is immediate to check from the Selberg-Chowla formula (1.2.1) that $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\lambda')$ has trivial zeros located at the negative integers, this is, $s=-n$, $n\in\mathbb{N}$. It is also not difficult to show that there exists some $N\in\mathbb{N}$ such that, for every $n>N$, all trivial zeros of the form $s=-n$ will be simple. Moreover, in a completely analogous way to the results given in [[15], p. 367, Theorem 3] and [223], one can exhibit subclasses of the Epstein zeta function (1.2.31) (given that ξ is a large parameter) which do not obey to a generalized Riemann hypothesis, thus having a real zero on the open interval (r_1,r_1+r_2), for $r_2\leq r_1$.

More can be said about the distribution of the nontrivial zeros of a given Dirichlet series $\phi(s)$ and the Epstein zeta functions attached to it: let $(\phi(s),\,\psi(s))$ be the pair of Dirichlet series (1.1.8) satisfying Definition 1.1.1. Further, let c and d be the least positive integers such that $a(c)\neq 0$ and $b(d)\neq 0$. By using a method of Levinson and Montgomery [144], adapting the main steps given by

Berndt in [[26], p. 246, Theorem 10][8], and invoking Littlewood's lemma [146][9], one can show that, if $\omega=\beta+i\gamma$ denotes a nontrivial zero of $\phi(s)$, then, as T tends to infinity,

$$\sum_{|\gamma|\leq T}\left(\beta-\frac{r}{2}\right)=-\frac{T}{\pi}\,\log\left(\frac{|a(c)|}{|b(d)|}\,\left(\frac{\mu_d}{\lambda_c}\right)^{r/2}\right)+O\left(\log(T)\right). \tag{1.3.1}$$

This asymptotic formula can be helpful in detecting the existence of nontrivial zeros of $\phi(s)$ off the critical line $\mathrm{Re}(s)=\frac{r}{2}$, as well as to study how the distribution of these zeros behaves with respect to this line. If we apply (1.3.1) to $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\lambda')$ and denote by $\omega^{(2)}=\beta^{(2)}+i\gamma^{(2)}$ the nontrivial zeros of this Dirichlet series, we obtain, under the assumption that $\lambda_c<\lambda_c'$ and $\mu_d<\mu_d'$,

$$\sum_{|\gamma^{(2)}|\leq T}\left(\beta^{(2)}-\frac{r_1+r_2}{2}\right)=-\frac{T}{\pi}\,\log\left(\frac{|a_1(c)|}{|b_1(d)|}\,\left|\frac{\rho_2^{\star}}{\rho_2}\right|\,\left(\frac{\mu_d}{\lambda_c}\right)^{\frac{r_1+r_2}{2}}\right)+O\left(\log(T)\right). \tag{1.3.2}$$

Note that this formula provides a very simple method to find Epstein zeta functions with infinitely many zeros off their critical lines, since it is possible to pick Dirichlet series whose properties make the logarithmic factor in (1.3.2) different from zero [10].

Despite the fact that, in general, most particular cases of the Epstein zeta functions studied above have infinitely many zeros off their critical lines, a large class of them will certainly admit infinitely many zeros on the critical line, even under our general setting.

In the next theorem, we show that the presence of infinitely many zeros of $\phi(s)$ at its critical line $\mathrm{Re}(s)=\frac{r}{2}$ is, in a sense, influenced by the presence of infinitely many zeros of $\mathcal{Z}_k(s;\,a;\,\lambda)$ at the critical line $\mathrm{Re}(s)=\frac{k\,r}{2}$, along with the asymptotic order of $\mathcal{Z}_k(s;\,a;\,\lambda)$.

As we shall see now, the Selberg-Chowla formula (1.2.77) resides at the core of the connection between the truth of Hardy's theorem for $\phi(s)$ and for any one of the Epstein zeta functions attached it. However, to make this connection precise, we need to reformulate the Selberg-Chowla formula (1.2.77) in a way which incorporates a contour integral of $\phi(s)$ along the line $\mathrm{Re}(s)=r/2$. The following lemma does precisely this.

Lemma 1.3.1. *Let $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$ be the dyadic Epstein zeta function defined by (1.2.76). Then the following representation of $\mathcal{Z}_{2^{k+1}}(s;\,a;\,\lambda)$ at the critical line, Re$(s)=2^kr$, holds*

$$\begin{aligned}\Gamma(2^kr+2it)\,\mathcal{Z}_{2^{k+1}}\left(2^kr+2it;\,a;\,\lambda\right)&=-2\mathcal{Z}_{2^k}\left(0;\,a;\,\lambda\right)\,\Gamma(2^kr+2it)\,\mathcal{Z}_{2^k}\left(2^kr+2it;\,a;\,\lambda\right)\\ +2\,\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathcal{Z}_{2^k}\left(2it;\,a;\,\lambda\right)&+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(2^{k-1}r+i(y-t)\right)\,\mathcal{Z}_{2^k}\left(2^{k-1}r+i(y-t);\,b;\,\lambda\right)\\ &\times\Gamma\left(2^{k-1}r+i(y+t)\right)\,\mathcal{Z}_{2^k}\left(2^{k-1}r+i(y+t);\,a;\,\lambda\right)dy.\end{aligned} \tag{1.3.3}$$

[8] or by Lekkerkerker in [142].

[9] a detailed statement and proof of this result can be found in [[228], section 9.9, pps. 220, 221].

[10] Although the size of these asymmetric zeros is quite small when compared to the total number of zeros of $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\lambda')$, which is $A\,T\,\log(T)+O(T)$ as $T\to\infty$ (see [[26], p. 246, eq. (2.2)]).

Proof. Start with the Selberg-Chowla formula (1.2.77) and take $s=2^k r+2it$, $t\in\mathbb{R}$ (note that these values of s lie on the critical line of $\mathcal{Z}_{2^{k+1}}(s;\,a;\,\lambda)$). We obtain the following representation

$$\Gamma(2^k r+2it)\,\mathcal{Z}_{2^{k+1}}\left(2^k r+2it;\,a;\,\lambda\right)=-\mathcal{Z}_{2^k}\left(0;\,a;\,\lambda\right)\Gamma(2^k r+2it)\,\mathcal{Z}_{2^k}\left(2^k r+2it;\,a;\,\lambda\right)$$
$$+\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathcal{Z}_{2^k}\left(2it;\,a;\,\lambda\right)+2\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\left(\frac{\Omega_m}{\Lambda_n}\right)^{it}K_{2it}\left(2\sqrt{\Omega_m\Lambda_n}\right),\quad(1.3.4)$$

where the definitions of the arithmetical functions $\mathfrak{U}_{2^k}(n)$ and $\mathfrak{V}_{2^k}(m)$ are given in (1.1.20) and (1.2.73). To proceed further, we need to give an integral representation for the double series appearing on the right-hand side of (1.3.4). Our idea is then to return to the formula (1.2.10) and rewrite it in a more symmetric form

$$K_\nu(x)=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{w-2}\Gamma\left(\frac{w+\nu}{2}\right)\Gamma\left(\frac{w-\nu}{2}\right)x^{-w}\,dw,\quad\mu>|\mathrm{Re}(\nu)|.\qquad(1.3.5)$$

Replacing ν by $2it$ in (1.3.5) and letting $\mu>\max\left\{2^{k+1}\sigma_a,\,2^{k+1}\sigma_b,\,2^{k+1}r\right\}$, we can write $2\,H_{2^k r}(2^k r+2it;\,a;\,\lambda)$ as

$$\sum_{m,n=1}^{\infty}\mathfrak{V}_{2^k}(m)\,\mathfrak{U}_{2^k}(n)\left(\frac{\Omega_m}{\Lambda_n}\right)^{it}\frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{w-2}\Gamma\left(\frac{w+2it}{2}\right)\Gamma\left(\frac{w-2it}{2}\right)\left(2\sqrt{\Omega_m\Lambda_n}\right)^{-w}dw$$
$$=\frac{1}{4\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(\frac{w-2it}{2}\right)\mathcal{Z}_{2^k}\left(\frac{w-2it}{2};\,b;\,\lambda\right)\Gamma\left(\frac{w+2it}{2}\right)\mathcal{Z}_{2^k}\left(\frac{w+2it}{2};\,a;\,\lambda\right)dw,\quad(1.3.6)$$

since this choice of μ allows to interchange the double series and the Mellin integral, a valid step due to absolute convergence (see the proof of Theorem 1.2.1). Let us now shift the line of integration in (1.3.6) to $\mathrm{Re}(w)=2^k r$ which, by the choice of μ, is located on the left of the line $\mathrm{Re}(w)=\mu$. Doing this requires once more to integrate along a positively oriented rectangular contour $\mathcal{R}(\mu)$ with vertices $\mu\pm iT$ and $2^k r\pm iT$, $T>0$. The application of the Phragmén-Lindelöf principle (1.2.78), as well as Stirling's formula (1.1.25) shows that the integrals along the horizontal segments of $\mathcal{R}(\mu)$ vanish when $T\to\infty$. By shifting the line of integration, we note that the integrand has two simple poles on the interior of $\mathcal{R}(\mu)$, which are located at $w=2^{k+1}r\pm 2it$. Denoting their respective residues by $R_{2^{k+1}r\pm 2it}$, we find that

$$R_{2^{k+1}r+2it}=2\,\Gamma^{2^k}(r)\,\rho^{\star 2^k}\Gamma(2^k r+2it)\,\mathcal{Z}_{2^k}\left(2^k r+2it;\,a;\,\lambda\right)$$
$$=-2\,\mathcal{Z}_{2^k}\left(0;\,a;\,\lambda\right)\Gamma(2^k r+2it)\,\mathcal{Z}_{2^k}\left(2^k r+2it;\,a;\,\lambda\right),\qquad(1.3.7)$$

where we have used the functional equation for $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$ (see Corollary 1.2.2), and

$$R_{2^{k+1}r-2it}=2\,\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathcal{Z}_{2^k}\left(2it;\,a;\,\lambda\right).\qquad(1.3.8)$$

Using Cauchy's residue theorem and adding (1.3.7) and (1.3.8) to (1.3.4), we obtain the formula

$$\Gamma(2^k r+2it)\,\mathcal{Z}_{2^{k+1}}\left(2^k r+2it;\,a;\,\lambda\right)=-2\mathcal{Z}_{2^k}\,(0;\,a;\,\lambda)\,\Gamma(2^k r+2it)\,\mathcal{Z}_{2^k}\left(2^k r+2it;\,a;\,\lambda\right)$$
$$+2\,\Gamma^{2^k}(r)\,\rho^{2^k}\,\Gamma(2it)\,\mathcal{Z}_{2^k}\,(2it;\,a;\,\lambda)+\frac{1}{4\pi i}\int\limits_{2^k r-i\infty}^{2^k r+i\infty}\Gamma\left(\frac{w-2it}{2}\right)\,\mathcal{Z}_{2^k}\left(\frac{w-2it}{2};\,b;\,\lambda\right)$$
$$\times\,\Gamma\left(\frac{w+2it}{2}\right)\,\mathcal{Z}_{2^k}\left(\frac{w+2it}{2};\,a;\,\lambda\right)\,dw,$$

which finally yields (1.3.3), once we write $w=2^k r+2iy$ on the contour integral. □

Remark 1.3.1. When $k=0$, the previous lemma gives an integral representation of $\mathcal{Z}_2(s;\,a;\,\lambda)$ in terms of $\phi(s)$, the unique Dirichlet series in its decomposition (due to the convention $Z_{2^0}(s;\,a;\,\lambda):=\phi(s)$). If, however, we have two distinct Dirichlet series $\phi_1(s)$ and $\phi_2(s)$, then, due to the functional equation (1.2.20), the critical line for $\mathcal{Z}_2(s;\,a_1,\,a_2;\,\lambda,\,\lambda')$ will be $\mathrm{Re}(s)=\frac{r_1+r_2}{2}$. In this case, the modification of (1.3.3) for $k=0$ reads

$$R_{\mathcal{Z}_2}\left(\frac{r_1+r_2}{2}+2it;\,a_1,\,a_2;\,\lambda,\,\lambda'\right)=-\phi_2(0)\,R_{\phi_1}\left(\frac{r_1+r_2}{2}+2it\right)-\phi_1(0)\,R_{\phi_2}\left(\frac{r_1+r_2}{2}+2it\right)$$
$$+\rho_1\Gamma(r_1)\,R_{\phi_2}\left(\frac{r_2-r_1}{2}+2it\right)+\rho_2\,\Gamma(r_2)\,R_{\phi_1}\left(\frac{r_1-r_2}{2}+2it\right)\qquad(1.3.9)$$
$$+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}R_{\psi_1}\left(\frac{r_1}{2}+i(y-t)\right)\,R_{\phi_2}\left(\frac{r_2}{2}+i(y+t)\right)\,dy,$$

where, for any Dirichlet series $\varphi(s)$, $R_{\varphi}(s):=\Gamma(s)\,\varphi(s)$ denotes its completed version (see the Notation Remarks B above).

With the extremely useful representation provided by Lemma 1.3.1, we can now prove the following theorem.

Theorem 1.3.1. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1. Suppose also that $\overline{a}(n)=b(n)$ and $\lambda_n=\mu_n$. Moreover, assume that its abscissa of absolute convergence, σ_a, satisfies $\sigma_a<\min\{1,r\}+\frac{1}{2}\max\{1,r\}$.*

Additionally, if one of the following conditions holds:

1. *For some $k\geq 0$, the diagonal Epstein zeta function $\mathcal{Z}_{2^{k+1}}(s;\,a;\,\lambda)$, (1.2.76), has infinitely many zeros on the critical line Re$(s)=2^k r$.*

2. *For some $k\geq 0$, the diagonal Epstein zeta function $\mathcal{Z}_{2^{k+1}}(s;\,a;\,\lambda)$ obeys to a subconvexity estimate of the form*

$$\mathcal{Z}_{2^{k+1}}(2^k r+it;\,a;\,\lambda)=o\left(|t|^{2^k-\frac{1}{2}}\right),\quad |t|\to\infty.\qquad(1.3.10)$$

Then $\phi(s)$ has infinitely many zeros on the critical line Re$(s)=\frac{r}{2}$.

Proof. We start by writing the integral representation (1.3.3) with $k=0$,

$$\Gamma(r+2it)\,\mathcal{Z}_2\,(r+2it;\,a;\,\lambda) = -2\phi(0)\,\Gamma(r+2it)\,\phi\,(r+2it) + 2\,\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\,(2it)$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+i(y-t)\right)\psi\left(\frac{r}{2}+i(y-t)\right)\,\Gamma\left(\frac{r}{2}+i(y+t)\right)\,\phi\left(\frac{r}{2}+i(y+t)\right)\,dy. \qquad (1.3.11)$$

The idea of the forthcoming proof is to estimate both sides of the previous equality when $|t|\to\infty$. We shall give an estimate for the right-hand side of (1.3.11) assuming that $\phi(s)$ does not possess infinitely many zeros on the line $\mathrm{Re}(s)=\frac{r}{2}$, which, after some natural steps, will eventually contradict one of the assumptions of our theorem.

By the hypotheses $b(n)=\overline{a}(n)$ and $\lambda_n=\mu_n$, we first claim that the integrand on the right-hand side of (1.3.11) must be a real function of $y\in\mathbb{R}$. Indeed, under these assumptions, let us first observe that

$$\psi(s)=\sum_{n=1}^{\infty}\frac{b(n)}{\mu_n^s}=\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{\lambda_n^s}=\overline{\sum_{n=1}^{\infty}\frac{a(n)}{\lambda_n^{\overline{s}}}},\qquad \mathrm{Re}(s)>\sigma_a=\sigma_b. \qquad (1.3.12)$$

Therefore, by analytic continuation, we have $\psi(s)=\overline{\phi(\overline{s})}$ for all $s\in\mathbb{C}$. Also, by Hecke's functional equation (1.1.9), we have

$$\Gamma\left(\frac{r}{2}+it\right)\phi\left(\frac{r}{2}+it\right)=\Gamma\left(\frac{r}{2}-it\right)\psi\left(\frac{r}{2}-it\right)=\Gamma\left(\frac{r}{2}-it\right)\overline{\phi\left(\frac{r}{2}+it\right)}$$
$$=\overline{\Gamma\left(\frac{r}{2}+it\right)\phi\left(\frac{r}{2}+it\right)},$$

which implies that the function $\Gamma(s)\,\phi(s)$ is real-valued on the critical line $\mathrm{Re}(s)=\frac{r}{2}$. The same reasoning evidently applies to $\Gamma(s)\,\psi(s)$ and, as a consequence, to the function inside the integral in (1.3.11).

For convenience, from this point on we shall write $\Gamma(s)\,\phi(s)$ as $R_\phi(s)$ and $\Gamma(s)\,\psi(s)$ as $R_\psi(s)$. Supposing that $\phi\,(s)$ has finitely many zeros on the line $\mathrm{Re}(s)=\frac{r}{2}$, there exists some $T_0>0$ such that, if $|y-t|>T_0$ and $|y+t|>T_0$, the integrand in (1.3.11) has a constant sign. Hence, the following equality holds

$$\left|\int_{-t+T_0}^{t-T_0}R_\psi\left(\frac{r}{2}+i\,(y-t)\right)\,R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\,dy\right|$$
$$=\int_{-t+T_0}^{t-T_0}\left|R_\psi\left(\frac{r}{2}+i\,(y-t)\right)\,R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\right|\,dy. \qquad (1.3.13)$$

The idea now is to see that the right-hand side of (1.3.13) will provide a lower bound for the integral on the right-hand side of (1.3.11) which, in its turn, will contradict items 1 and 2 of our statement.

If we take a partition of the integral in (1.3.11) of the form

$$\left\{\int_{t-T_0}^{\infty}+\int_{-\infty}^{-t+T_0}+\int_{-t+T_0}^{t-T_0}\right\} R_{\psi}\left(\frac{r}{2}+i\left(y-t\right)\right)\, R_{\phi}\left(\frac{r}{2}+i\left(y+t\right)\right)\, dy, \tag{1.3.14}$$

we see that, in the third integral, we can apply the contradiction hypothesis (1.3.13). Let us denote each integral in the previous partition by $\mathcal{A}_i(t)$, $i=1,2,3$. We first check that, for any $\delta>0$, the first two integrals have the asymptotic behavior

$$\mathcal{A}_1(t),\, \mathcal{A}_2(t)=O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right), \quad |t|\to\infty, \tag{1.3.15}$$

with the above constant only depending on δ. The estimate (1.3.15) comes directly from Stirling's formula and a simple application of the Phragmén-Lindelöf principle (1.1.27) for $\phi(s)$ and $\psi(s)$. It suffices to verify (1.3.15) only for $\mathcal{A}_1(t)$, as $\mathcal{A}_1(t)=\mathcal{A}_2(t)$. From the fact that $\phi(s)$ and $\psi(s)$ have the same abscissa of absolute convergence, σ_a (because $\overline{a}(n)=b(n)$ and $\lambda_n=\mu_n$ by hypothesis), $R_{\phi}(s)$ and $R_{\psi}(s)$ satisfy the estimate

$$R_{\phi}\left(\frac{r}{2}+it\right),\ R_{\psi}\left(\frac{r}{2}+it\right)=O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}e^{-\frac{\pi}{2}|t|}\right), \quad |t|\to\infty, \tag{1.3.16}$$

which can be used inside the integral for $\mathcal{A}_1(t)$, after noticing that

$$\begin{aligned}\int_{-T_0}^{T_0} R_{\psi}\left(\frac{r}{2}+iu\right)\, R_{\phi}\left(\frac{r}{2}+i\left(u+2t\right)\right)\, du &= O\left(\int_{-T_0}^{T_0}\left|R_{\psi}\left(\frac{r}{2}+iu\right)\right|\, |u+2t|^{\sigma_a-\frac{1}{2}+\delta}e^{-\frac{\pi}{2}|u+2t|}\, du\right)\\ &= O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right),\end{aligned} \tag{1.3.17}$$

while, analogously,

$$\begin{aligned}\int_{T_0}^{\infty} R_{\psi}\left(\frac{r}{2}+iu\right)\, R_{\phi}\left(\frac{r}{2}+i\left(u+2t\right)\right)\, du &= O\left(|t|^{2\sigma_a+2\delta}\, e^{-\pi|t|}\int_{T_0/2t}^{\infty}(u^2+u)^{\sigma_a-\frac{1}{2}+\delta}\, e^{-2\pi ut}\, du\right)\\ &= O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}\right),\end{aligned} \tag{1.3.18}$$

proving (1.3.15). We will now find a lower bound for the third integral, $\mathcal{A}_3(t)$. To do this we need to invoke Stirling's formula once more, because we are under the assumption that T_0 is a large parameter and, in the range of integration of $\mathcal{A}_3(t)$, $|y-t|>T_0$ and $|y+t|>T_0$. Let c be the least positive integer such that $a(c)\neq 0$: invoking (1.1.25) on the right-hand side of (1.3.13), we deduce the inequality

$$\int_{-t+T_0}^{t-T_0} \left| R_\psi\left(\frac{r}{2}+i\,(y-t)\right)\,R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\right|\,dy$$
$$=\lambda_c^{-r}\int_{-t+T_0}^{t-T_0}\left|\lambda_c^{r+2iy}R_\psi\left(\frac{r}{2}+i\,(y-t)\right)\,R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\right|\,dy$$
$$\geq 2\pi\lambda_c^{-r}\,e^{-\pi|t|}\left|\int_{\frac{r}{2}-i(t-T_0)}^{\frac{r}{2}+i(t-T_0)}\lambda_c^{2z}\,\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\,\psi(z-it)\,\phi(z+it)dz\,\left\{1+O\left(|t|^{-1}\right)\right\}\right|. \tag{1.3.19}$$

Applying Cauchy's theorem to the positively oriented rectangular contour with vertices $\frac{r}{2}\pm i\,(t-T_0)$ and $\sigma_a+1\pm i\,(t-T_0)$, we see that the contour integral on the right-hand side of (1.3.19) can be written as

$$\left\{\int_{\frac{r}{2}-i(t-T_0)}^{\sigma_a+1-i(t-T_0)}+\int_{\sigma_a+1-i(t-T_0)}^{\sigma_a+1+i(t-T_0)}+\int_{\sigma_a+1+i(t-T_0)}^{\frac{r}{2}+i(t-T_0)}\right\}\lambda_c^{2z}\,\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\,\psi(z-it)\,\phi(z+it)\,dz. \tag{1.3.20}$$

The first and third integrals (along the horizontal segments) in the above partition can be easily evaluated with the aid of (1.1.27). Note that, in the first of these integrals, the factor $\phi(z+it)$ does not depend on t, since T_0 is some fixed parameter. Hence, we apply (1.1.27) solely to $\psi(z-it)$. By symmetry, the same happens on the third integral of (1.3.20), with the roles of $\phi(z+it)$ and $\psi(z-it)$ being reversed. Denoting each integral on (1.3.20) by $\mathcal{A}_{3j}(t)$, $j=1,2,3$, we find that, for any $\delta>0$,

$$\mathcal{A}_{31}(t),\;\mathcal{A}_{33}(t)=O\left(|t|^{\sigma_a+\frac{r}{2}-1+\delta}\right). \tag{1.3.21}$$

To estimate the integral in the middle, $\mathcal{A}_{32}(t)$, we can use the Dirichlet series representation of the product $\psi(z-it)\,\phi(z+it)$, which involves the generalized divisor function $\sigma_{2it}\,(\nu_j;\,\overline{a},\,a;\,\lambda,\,\lambda)$ (recall (1.2.15) above). We arrive at the equality

$$\int_{\sigma_a+1-i(t-T_0)}^{\sigma_a+1+i(t-T_0)}\lambda_c^{2z}\,\left(t^2+\left(z-\frac{r}{2}\right)^2\right)^{\frac{r-1}{2}}\,\psi(z-it)\,\phi(z+it)\,dz=|a(c)|^2\int_{\sigma_a+1-\frac{r}{2}-i(t-T_0)}^{\sigma_a+1-\frac{r}{2}+i(t-T_0)}\left(t^2+z^2\right)^{\frac{r-1}{2}}\,dz$$
$$+\sum_{n=c+1}^{\infty}\overline{a}(m)\,a(n)\,\left(\frac{\lambda_m}{\lambda_n}\right)^{it}\left(\frac{\lambda_c^2}{\lambda_m\lambda_n}\right)^{\frac{r}{2}}\int_{\sigma_a+1-\frac{r}{2}-i(t-T_0)}^{\sigma_a+1-\frac{r}{2}+i(t-T_0)}\left(t^2+z^2\right)^{\frac{r-1}{2}}\left(\frac{\lambda_c^2}{\lambda_m\lambda_n}\right)^{z}\,dz$$
$$=2i\,|a(c)|^2\,|t|^r+O\left(|t|^{r-1}\right), \tag{1.3.22}$$

where the error term comes from an integration by parts[11]. Since $\sigma_a < \min\{1,r\} + \frac{1}{2}\max\{1,r\}$ by hypothesis, (1.3.19) gives

$$\begin{aligned}
&\int_{-t+T_0}^{t-T_0} \left|R_\psi\left(\frac{r}{2}+i\,(y-t)\right) R_\phi\left(\frac{r}{2}+i\,(y+t)\right)\right|\,dy \\
&\geq 2\pi\lambda_c^{-r}e^{-\pi|t|}\left|2i|a(c)|^2\,|t|^r + O(|t|^{r-1}) + O\left(|t|^{\sigma_a+\frac{r}{2}-1+\delta}\right)\right| \\
&= 4\pi\,|a(c)|^2\lambda_c^{-r}\,|t|^r e^{-\pi|t|} + O\left(|t|^{r-1}\,e^{-\pi|t|}\right) + O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right). \qquad (1.3.23)
\end{aligned}$$

It is well-known that the second O term in (1.3.23) always bounds the first one, as $\sigma_a > \frac{r}{2}+\frac{1}{4}$ [[54], p. 111]. From the contradiction hypothesis (1.3.13), we see that (1.3.23) yields the lower bound

$$|\mathcal{A}_3(t)| \geq 4\pi\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|} + O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right), \qquad |t| > T_0. \qquad (1.3.24)$$

Returning to the Selberg-Chowla formula (1.3.11), the lower bound (1.3.24), the contradiction hypothesis (1.3.13) and the definition of the integrals $\mathcal{A}_i(t)$, we realize that all the information that we have collected leads to the inequality

$$\begin{aligned}
&\Big|\Gamma\,(r+it)\;\mathcal{Z}_2(r+it;\,a;\,\lambda) - 2\phi(0)\,\Gamma(r+2it)\,\phi\,(r+2it) \\
&+2\,\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\,(2it) - \frac{1}{2\pi}\mathcal{A}_1(t) - \frac{1}{2\pi}\mathcal{A}_2(t)\Big| \geq 2\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|} + O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right). \\
&\qquad (1.3.25)
\end{aligned}$$

Finally, in order to contradict one of the hypotheses 1 or 2 for $k=0$, we just need to estimate the remaining terms appearing on the left-hand side of (1.3.25). We have already seen that $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ satisfy (1.3.15), so we just need to deal with the remaining two. The desired estimate easily follows from (1.1.25) and (1.1.26), giving

$$2\,\Gamma(r)\,\rho\,\Gamma(2it)\,\phi\,(2it) - 2\phi(0)\,\Gamma(r+2it)\,\phi\,(r+2it) = O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\,e^{-\pi|t|}\right), \quad |t|\to\infty. \qquad (1.3.26)$$

Using (1.3.25) and the hypothesis that $\sigma_a < r+\frac{1}{2}$, we obtain the lower bound for the Epstein zeta function,

$$\begin{aligned}
|\Gamma\,(r+2it)\;\mathcal{Z}_2\,(r+2it;\,a;\,\lambda)\,| &\geq 2\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|} + O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right) \\
&\quad + O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\,e^{-\pi|t|}\right), \qquad (1.3.27)
\end{aligned}$$

which is valid for all t such that $|t| > T_0$. Using Stirling's formula and enlarging T_0 if needed, (1.3.27) immediately implies that

$$|\mathcal{Z}_2\,(r+2it;\,a;\,\lambda)\,| > B\,|t|^{\frac{1}{2}}, \quad |t| > T_0, \qquad (1.3.28)$$

[11] Although we have used the explicit representation of a Dirichlet series which is dependent on t, $\psi(z-it)\,\phi(z+it)$, the constant obtained from the application of Cauchy's theorem above only depends on $a(c)$ and λ_c. Therefore, the lower bound obtained in (1.3.22) is exactly the same as the one that one gets when the Dirichlet series $\psi(z-it)\,\phi(z+it)$ is replaced by any other, say $\varphi(z)$, not depending on t.

for some $B > 0$. It is now clear that $\mathcal{Z}_2(s;\, a;\, \lambda)$ cannot have infinitely many zeros on the critical line $\mathrm{Re}(s) = r$ since, by (1.3.28), its absolute value is always greater than a positive function whenever $|\mathrm{Im}(s)| > 2T_0$. This also contradicts (1.3.10) for $k = 0$.

To prove the theorem for a general $k \in \mathbb{N}$, we shall use Lemma 1.3.1 and prove by induction that (1.3.28) implies the negation of (1.3.10). The following claim contains the necessary assertion to finish our proof.

Claim 1.3.1. *Suppose that, for some $T_k > 0$, the 2^k−th (dyadic) Epstein zeta function (1.2.76) satisfies the following lower bound on its critical line,*

$$|\mathcal{Z}_{2^k}(2^{k-1}r + it;\, a;\, \lambda)| > B\, |t|^{\frac{2^k-1}{2}}, \quad |t| > T_k, \quad B > 0. \tag{1.3.29}$$

Then $\mathcal{Z}_{2^{k+1}}(s;\, a;\, \lambda)$ satisfies a similar bound of the form

$$|\mathcal{Z}_{2^{k+1}}(2^k r + it;\, a;\, \lambda)| > B'\, |t|^{\frac{2^{k+1}-1}{2}}, \quad |t| > 2T_k, \;\; B' > 0. \tag{1.3.30}$$

Proof of Claim 1.3.1. The idea is to mimic the argument previously outlined. Use (1.3.3) and estimate the residual terms as in (1.3.26). From (1.2.78) and Stirling's formula, we see that these are bounded as $\ll_\delta |t|^{2^k\sigma_a - \frac{1}{2}+\delta}\, e^{-\pi|t|}$. To estimate the integral appearing in (1.3.3), we take once more a partition similar to that of (1.3.14). As before, if we denote each integral on the partition by $\mathcal{A}_i^{(k)}(t)$, $i = 1, 2, 3$, we can mimic the proof of (1.3.15) to obtain $\mathcal{A}_i^{(k)}(t) \ll_\delta |t|^{2^k\,\sigma_a - \frac{1}{2}+\delta}\, e^{-\pi|t|}$, $i = 1, 2$. To deal with the third integral, $\mathcal{A}_3^{(k)}(t)$, we proceed as in (1.3.13): by the inductive hypothesis (1.3.29), we know that $\mathcal{Z}_{2^k}(s;\, a;\, \lambda)$ has constant sign on the line $\mathrm{Re}(s) = 2^{k-1}r$ when $|\mathrm{Im}(s)| > T_k$ and so, employing directly (1.3.29) on the third integral $\mathcal{A}_3^{(k)}(t)$ we obtain,

$$\begin{aligned}\left|\mathcal{A}_3^{(k)}(t)\right| &= \int_{-t+T_k}^{t-T_k} \left|\Gamma\left(2^{k-1}r + i(y-t)\right)\, \mathcal{Z}_{2^k}\left(2^{k-1}r + i(y-t);\, b;\, \lambda\right)\right. \\ &\qquad\qquad \left.\times\, \Gamma\left(2^{k-1}r + i(y+t)\right)\, \mathcal{Z}_{2^k}\left(2^{k-1}r + i(y+t);\, a;\, \lambda\right)\right|\, dy \\ &> B\, e^{-\pi|t|}\left|\int_{-t+T_k}^{t-T_k}\left(t^2-y^2\right)^{2^{k-1}(r+1)-1}\, dy\right| = D\, |t|^{2^k(r+1)-1}\, e^{-\pi|t|} + O\left(|t|^{2^k(r+1)-2}\, e^{-\pi|t|}\right)\end{aligned}$$

for all $|t| > T_k$ and some $D > 0$. Therefore, recalling once more (1.3.3) and the steps leading to (1.3.27), one sees that there exists some positive constant for some positive constant D' such that

$$|\Gamma(2^k r + 2it)\, \mathcal{Z}_{2^{k+1}}(2^k r + 2it;\, a;\, \lambda)| > D'\, |t|^{2^k(r+1)-1}\, e^{-\pi|t|}, \quad |t| > T_k, \tag{1.3.31}$$

giving (1.3.30). □

□

Remark 1.3.2. Let $\mathcal{Z}_k(s;\, a;\, \lambda)$ denote the k^{th}−dimensional Epstein zeta function (1.1.23). Although it was easier to write the previous proof in terms of the dyadic Epstein zeta functions (1.2.76), note that the integral representation (1.3.3) can be given for any pair of Epstein zeta functions $\mathcal{Z}_{m_1}(s;\,a;\,\lambda)$ and $\mathcal{Z}_{m_2}(s;\,a;\,\lambda)$ such that $m_1+m_2=k$. For example, when $m_1=1$ and $m_2=k-1$, a suitable integral representation for $\mathcal{Z}_k(s;\,a;\,\lambda)$ at the critical line $\mathrm{Re}(s)=\frac{k\,r}{2}$ holds, resulting from applying (1.3.9) with $\phi_1(s)=\phi(s)$ and $\phi_2(s)=\mathcal{Z}_{k-1}(s;\,a;\,\lambda)$. This integral representation assumes the following form

$$\begin{aligned}R_{\mathcal{Z}_k}\left(\frac{kr}{2}+2it;\,a;\,\lambda\right)&=-\mathcal{Z}_{k-1}(0;\,a;\,\lambda)\,R_{\phi}\left(\frac{kr}{2}+2it\right)-\phi(0)\,R_{\mathcal{Z}_{k-1}}\left(\frac{kr}{2}+2it\right)\\&+\rho_1\Gamma(r_1)\,R_{\mathcal{Z}_{k-1}}\left(\frac{(k-2)\,r}{2}+2it;\,a;\,\lambda\right)+\rho_2\,\Gamma(r_2)\,R_{\phi}\left(-\frac{(k-2)r}{2}+2it\right)\\&+\frac{1}{2\pi}\int_{-\infty}^{\infty}R_{\psi}\left(\frac{r}{2}+i(y-t)\right)\,R_{\mathcal{Z}_{k-1}}\left(\frac{(k-1)\,r}{2}+i(y+t);\,a;\,\lambda\right)dy.\end{aligned}$$

By using inductively an analogue of the hypothesis (1.3.29), we can replace the statement of Claim 1.3.1 (involving the powers of 2) by a statement involving the index k. Furthermore, the conditions 1 and 2 of our Theorem 1.3.1 can be rewritten by using $\mathcal{Z}_k(s;\,a;\,\lambda)$ instead of $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$. Indeed, if at least one of the multidimensional Epstein zeta functions $\mathcal{Z}_k(s;\,a;\,\lambda)$ possesses infinitely many zeros on its critical line $\mathrm{Re}(s)=\frac{kr}{2}$, then the same will happen to $\phi(s)$. In a similar way, if the order of $\mathcal{Z}_k(kr/2+it;\,a;\,\lambda)$ satisfies $o\left(|t|^{\frac{k-1}{2}}\right)$, $|t|\to\infty$, the same conclusion can be derived. Similar comments can be made about the lower bounds that we shall obtain below in (1.3.55). This simple remark will be helpful in order to explain some steps in Example 1.3.3 below.

Remark 1.3.3. Note that the condition $\sigma_a<1+\frac{r}{2}$ was essential to move the line of integration on the integral $\mathcal{A}_3(t)$ from the critical line $\mathrm{Re}(s)=\frac{r}{2}$ to $\mathrm{Re}(s)=\sigma_a+1$ (recall (1.3.20) and (1.3.21)). This condition was important to prove that the O-term in (1.3.20) (an asymptotic estimate of the horizontal integrals) was actually bounded by $|t|^r$, thus allowing to establish the lower bound (1.3.24). If, however, we disregard such generality and assume that the generalized Dirichlet series $\phi(s)$ is a Titchmarsh series [[175], Chapter 2, p. 39], then a theorem of Ramachandra (see [[11], p. 570, Theorem 3], [[120], p. 23, Theorem 2.4] or Lemma 1.3.2 below) would still allow us to have a lower bound of the form (1.3.28). Examples of "Titchmarsh series" include the class of Hecke Dirichlet series with signature $(\lambda,\,r,\,\gamma)$, and so the condition $\sigma_a<1+\frac{r}{2}$ in the statement of the previous theorem can be removed for this class. In any case, the imposition $\sigma_a<r+\frac{1}{2}$ needs to be kept, as it assures that $|t|^r$ exceeds the estimates of the integrals $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ (1.3.15), as well as the estimates for the residual terms (1.3.26). For an interesting application of Ramachandra's result, see [160]. In this reference, Ramachandra's theorem is combined with the Potter-Titchmarsh method [170] in order to establish Hardy's theorem for certain Dirichlet series in the Selberg class. We will also employ Ramachandra's result in Example 1.3.3 below.

Remark 1.3.4. Condition (1.3.10) on Theorem 1.3.1 was, in a sense, invoked by Deuring [[61], p. 592] to prove the infinitude of zeros of $\zeta(s)$ on $\mathrm{Re}(s)=\frac{1}{2}$. This argument reduces the proof of Hardy's theorem for $\zeta(s)$ to establishing the order of magnitude of a Dirichlet series (in this case, $\zeta_2(s)$) on its critical line. Although this may be extraordinarily difficult to study for general Dirichlet series, note that, in order to apply our method, we do not need explicitly (1.3.10). What is actually required is to verify that at least one lower bound of the form (1.3.30) cannot hold, i.e., to find one dimension k for which the absolute value of the Epstein zeta function, $|\mathcal{Z}_{2^k}(2^{k-1}r+it;\,a;\,\lambda)|$, "oscillates" along a curve of the form $C|t|^{\alpha},\ 0\leq\alpha\leq 2^{k-1}-\frac{1}{2}$. This is why the first condition on the statement of Theorem 1.3.1 may add simplicity to it, since the aforementioned behavior of $\mathcal{Z}_2(s;\,a;\,\lambda)$ can be established independently.[12]

The previous result has several corollaries which are slight generalizations of the results given in [26, 27, 113, 142]. Like Hardy's original proof, these results concern the behavior of the generalized $\theta-$functions attached to $\phi(s)$ and to $\mathcal{Z}_2(s;\,a;\,\lambda)$. The following corollary appears to be new.

Corollary 1.3.1. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1. Suppose also that $\overline{a}(n)=b(n)$ and $\lambda_n=\mu_n$. Moreover, assume that its abscissa of absolute convergence, σ_a, satisfies $\sigma_a<\min\{1,r\}+\frac{1}{2}\,\max\{1,r\}$ and let $\Theta(z;\,a;\,\lambda)$ denote its generalized $\theta-$function (1.1.28). If*

$$\Theta\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\,a;\,\lambda\right)=o\left(\epsilon^{-\frac{r+1}{2}}\right),\quad \epsilon\to 0^{+}, \tag{1.3.32}$$

then $\phi(s)$ has infinitely many zeros on the critical line $Re(s)=\frac{r}{2}$.

Proof. Assume that $\phi(s)$ only has finitely many zeros on the critical line $\text{Re}(s)=\frac{r}{2}$. It follows from the proof of Theorem 1.3.1 that, at the line $\text{Re}(s)=r$, the Epstein zeta function $\mathcal{Z}_2(s;\,a;\,\lambda)$ satisfies the inequality (see eq. (1.3.27) above)

$$|\Gamma\left(r+2it\right)\,\mathcal{Z}_2\left(r+2it;\,a;\,\lambda\right)|\geq 2\,|a(c)|^2\lambda_c^{-r}\,|t|^r\,e^{-\pi|t|}+O\left(|t|^{\sigma_a+\frac{r}{2}-1}\,e^{-\pi|t|}\right)+O\left(|t|^{\sigma_a-\frac{1}{2}+\delta}\,e^{-\pi|t|}\right), \tag{1.3.33}$$

whenever $|t|>T_0$. For $\text{Re}(z)>0$ and $\mu>2\sigma_a$, it follows from (1.1.31) and (1.1.32) that $\Theta_2(z;\,a;\,\lambda)$ admits the representation via the contour integrals

$$\begin{aligned}\Theta_2(z;\,a;\,\lambda)&=\frac{1}{2\pi i}\int\limits_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)\,\mathcal{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds\\&=\frac{1}{2\pi i}\int\limits_{C_1}\Gamma(s)\,\mathcal{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds+\frac{1}{2\pi i}\int\limits_{C_2}\Gamma(s)\,\mathcal{Z}_2(s;\,a;\,\lambda)\,z^{-s}ds,\end{aligned} \tag{1.3.34}$$

where C_1 and C_2 are the paths given by

$$C_1=(r-i\infty,\,r-2iT_0)\cup(r+2iT_0,\,r+i\infty)$$

and

$$C_2=(r-2iT_0,\,\mu-2iT_0)\cup(\mu-2iT_0,\,\mu+2iT_0)\cup(\mu+2iT_0,\,r+2iT_0)\,.$$

[12]For example, a direct application of the o-condition (1.3.10) in the case $\phi(s)=\pi^{-s}\zeta_2(s)$ does not lead to anywhere. This is because the general Epstein zeta function constructed from this Dirichlet series is $\mathcal{Z}_2(s)=\pi^{-s}\,\zeta_4(s)$. From Jacobi's four-square theorem, $\zeta_4(s)=8\,(1-2^{2-2s})\,\zeta(s)\,\zeta(s-1)$. Furthermore, since $\zeta(1+it)=\Omega(1)$, we have that $\mathcal{Z}_2(1+it)=\Omega(|t|^{\frac{1}{2}})$, which is the negation of (1.3.10). However, as stated in Remark 1.3.4, there is no need to invoke the condition (1.3.10) to reach a conclusion here, because $\zeta_4(s)$ has already infinitely many zeros at its critical line $\text{Re}(s)=1$, being of the form $1\pm\frac{\pi i}{\log(2)}k,\ k\in\mathbb{N}$ (see Examples 1.3.1 and 1.3.3 below for more details). This means that $\mathcal{Z}_2(1+it)$ has infinitely many zeros, thus satisfying hypothesis 1 of Theorem 1.3.1.

If we now take $z = \exp\left\{i\left(\frac{\pi}{2}-\epsilon\right)\right\}$ in (1.3.34), we see that

$$\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right) = \frac{1}{2\pi i}\int_{C_1}\Gamma(s)\,\mathcal{Z}_2(s;\, a;\, \lambda)\, e^{-i\left(\frac{\pi}{2}-\epsilon\right)s}ds + O(1). \tag{1.3.35}$$

The functional equation for $\mathcal{Z}_2(s;\, a;\, \lambda)$, (1.2.20), and the assumption that $b(n)=\overline{a}(n)$ ensure that $\Gamma(s)\,\mathcal{Z}_2(s;\, a;\, \lambda)$ is a real-valued function on the line $\text{Re}(s) = r$ (see the beginning of the proof of Theorem 1.3.1). From this observation, together with (1.3.35) and (1.3.33), we see that the inequality,

$$\begin{aligned}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right)\right| &= \frac{1}{2\pi}\int_{|y|>2T_0}|\Gamma\left(r+iy\right)\,\mathcal{Z}_2(r+iy;\, a;\, \lambda)|\; e^{\left(\frac{\pi}{2}-\epsilon\right)y}dy + O(1)\\ &\geq \frac{|a(c)|^2(2\lambda_c)^{-r}}{\pi}\int_{2T_0}^{\infty} y^r\, e^{-\epsilon y}\, dy + O\left(\int_{2T_0}^{\infty} y^{\sigma_a+\frac{\max\{1,r\}}{2}-1}e^{-\epsilon y}dy\right)\\ &= \frac{|a(c)|^2(2\lambda_c)^{-r}\,\Gamma(r+1)}{\pi}\,\epsilon^{-(r+1)} + O\left(\epsilon^{-\sigma_a-\frac{\max\{1,r\}}{2}}\right),\end{aligned} \tag{1.3.36}$$

must be valid for any $\epsilon>0$. However, if we recall the definition of $\Theta_2(z;\, a;\, \lambda)$ (1.1.32),

$$\Theta_2\left(z;\, a;\, \lambda\right) = \sum_{m,n\neq 0}^{\infty} a(m)\, a(n)\, e^{-(\lambda_m+\lambda_n)\, z} = 2a(0)\,\Theta(z;\, a;\, \lambda) + \Theta^2(z;\, a;\, \lambda), \tag{1.3.37}$$

we find that (1.3.32) implies $\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right) = o\left(\epsilon^{-r-1}\right)$ as $\epsilon\to 0^+$. This last assertion contradicts (1.3.36) and finishes our proof. □

Remark 1.3.5. Note that, when restricted to the class of Dirichlet series satisfying Definition 1.1.1, the previous corollary extends a result of Berndt [[27], p. 679, Theorem 1]. Furthermore, under the constraints of the previous corollary, (1.3.32) improves the assumption

$$\Theta\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right) = O(\epsilon^{-\beta}), \quad 0\leq\beta<\frac{r+1}{2}, \tag{1.3.38}$$

which was taken by Hecke in order to establish Hardy's Theorem to certain Dirichlet series (see [[113], p. 74, Satz 1]).

Corollary 1.3.2. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1 with $0<r<1$ and also satisfying the conditions of Corollary 1.3.1. Instead of (1.3.32), assume that $a(n)$ is such that $\arg\left\{a(n)\right\}_{n\in\mathbb{N}} =$ constant. Then $\phi(s)$ has infinitely many zeros on the critical line Re$(s)=\frac{r}{2}$.*

Moreover, if $r=1$ and $a(n)$ satisfies the condition

$$|a(c)| > \sqrt{2\pi\lambda_c}\,|\rho|, \tag{1.3.39}$$

where ρ denotes the residue of ϕ at $s=r=1$ and c represents the least positive integer for which $a(c)\neq 0$. Then $\phi(s)$ has infinitely many zeros on the critical line Re$(s)=\frac{1}{2}$.

Proof. Assume first that $0 < r < 1$ and that $\phi(s)$ does not have infinitely many zeros on the critical line $\text{Re}(s) = \frac{r}{2}$. Then an application of (1.3.36) gives

$$\epsilon^{r+1}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right)\right| \geq C, \tag{1.3.40}$$

for some positive constant C. On the other hand, using Bochner's "modular" relation for $\Theta_2(z;\, a;\, \lambda)$ (1.2.27), as well as the hypothesis $b(n) = \overline{a}(n)$, we may write the transformation formula

$$\Theta_2\left(z;\, a;\, \lambda\right) = z^{-2r}\sum_{m,n=1}^{\infty}\overline{a}(m)\,\overline{a}(n)\,e^{-(\lambda_m+\lambda_n)/z} + \Gamma^2(r)\,\rho^2 z^{-2r} + \mathcal{Z}_2(0,\, a,\, \lambda), \quad \text{Re}(z) > 0. \tag{1.3.41}$$

Using the fact that $\arg_{n\in\mathbb{N}}\{a(n)\}$ is constant, the previous relation gives

$$\begin{aligned}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right)\right| &\leq \sum_{m,n\neq 0}^{\infty}\left|a(m)\,a(n)\,e^{-(\lambda_m+\lambda_n)\sin(\epsilon)}\right| = \left|\sum_{m,n\neq 0}^{\infty} a(m)\,a(n)\,e^{-(\lambda_m+\lambda_n)\sin(\epsilon)}\right| \\ &= \left|\sin^{-2r}(\epsilon)\sum_{m,n=1}^{\infty}\overline{a}(m)\,\overline{a}(n)\,e^{-(\lambda_m+\lambda_n)/\sin(\epsilon)} + \Gamma^2(r)\,\rho^2\sin^{-2r}(\epsilon) + \mathcal{Z}_2(0,\, a,\, \lambda)\right| \\ &= \Gamma^2(r)\,|\rho|^2\sin^{-2r}(\epsilon) + O(1). \end{aligned} \tag{1.3.42}$$

Since, by hypothesis, (1.3.40) holds, then we must have

$$0 < C \leq \epsilon^{r+1}\left|\Theta_2\left(e^{i\left(\frac{\pi}{2}-\epsilon\right)};\, a;\, \lambda\right)\right| = \epsilon^{r+1}\,\Gamma^2(r)\,|\rho|^2\sin^{-2r}(\epsilon) + O(\epsilon^{r+1}) \tag{1.3.43}$$

for any $\epsilon > 0$. However, since $0 < r < 1$, we see that the right-hand side of (1.3.43) tends to zero as $\epsilon \to 0^+$, contradicting (1.3.40). This concludes the proof for the first case.

Assume now that $r = 1$: from (1.3.36) with $r = 1$, we know that the constant C given above can be explicitly written as $C = |a(c)|^2/2\pi\lambda_c$, and so (1.3.43) contradicts (1.3.39). □

Remark 1.3.6. By a theorem of Fekete [87], which in its turn extends a classical theorem of Landau on Dirichlet series [[5], p. 237, Theorem 11.13], the condition that $\arg_{n\in\mathbb{N}}\{a(n)\}$ is constant implies that $\phi(s)$ cannot be entire (cf. [[54], p. 111, remark A.2.]). Moreover, under the conditions of Definition 1.1.1, the previous corollary is only valid for $\sigma_a = r$. In any case, it is not strictly necessary for $\arg_{n\in\mathbb{N}}\{a(n)\}$ to be a constant if we wish to arrive at the conclusion of Corollary 1.3.2. In fact, it suffices to suppose that $|a(n)| \leq \mathfrak{c}(n)\ \forall n \in \mathbb{N}$, with $\mathfrak{c}(n)$ being such that the Dirichlet series $\varphi(s) = \sum \mathfrak{c}(n)\,\lambda_n^{-s}$ satisfies Hecke's functional equation with $r \leq 1$, as well as the conditions of Theorem 1.3.1. For example, Dirichlet L-functions attached to even primitive Dirichlet characters satisfy this condition. On the other hand, Dirichlet $L-$functions attached to odd primitive Dirichlet characters have their $\theta-$function automatically satisfying (1.3.32).

Under our general setting, it seems very difficult to ensure the subconvexity estimates (1.3.10) for the Epstein zeta functions $\mathcal{Z}_{2^k}(s;\, a;\, \lambda)$ on their critical lines, as most of the information in this regard comes exclusively from the Phragmén-Lindelöf principle (1.2.78). However, in our next result we impose an additional condition which will help establishing a generalization of Hardy's theorem by the way of contradicting (1.3.29) for large values of k. We remark that the next corollary was already proved by Berndt [26] under a different setting. Since our proof is drastically different from his and it has the advantage of avoiding the evaluation of certain exponential integrals, typically present in several proofs of Hardy's theorem,[13] we offer our alternative short proof.

Corollary 1.3.3. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1. Suppose also that $\overline{a}(n) = b(n)$ and $\lambda_n = \mu_n$. Moreover, assume that its abscissa of absolute convergence, σ_a, satisfies*

$$\sigma_a < \frac{r+1}{2}. \tag{1.3.44}$$

Then $\phi(s)$ has infinitely many zeros on the critical line $Re(s) = \frac{r}{2}$.

Proof. By the Phragmén-Lindelöf principle applied to the Dirichlet series $\mathcal{Z}_{2^{k+1}}(s;\, a;\, \lambda)$, we have, for any $\delta > 0$,

$$\mathcal{Z}_{2^{k+1}}(2^k r + 2it;\, a;\, \lambda) = O\left(|t|^{2^{k+1}\sigma_a - 2^k r + \delta}\right), \quad |t| \to \infty. \tag{1.3.45}$$

However, if $\phi(s)$ only possesses a finite number of zeros on the line $\text{Re}(s) = \frac{r}{2}$, we can be sure that there is some $T_0 > 0$ such that, for every $k \in \mathbb{N}_0$ (see Claim 1.3.1),

$$|\mathcal{Z}_{2^{k+1}}(2^k r + 2it;\, a;\, \lambda)| > B_k\, |t|^{\frac{2^{k+1}-1}{2}}, \quad |t| > T_k := 2^k\, T_0, \tag{1.3.46}$$

for some $B_k > 0$ depending on k. If we compare (1.3.45) with (1.3.46) and let $|t|$ tend to infinity, we deduce that the inequality $\sigma_a \geq \frac{r+1}{2} - \frac{1}{2^{k+2}} - \delta'$ must hold for every $k \in \mathbb{N}_0$ and any $\delta' > 0$. This, however, contradicts (1.3.44) and our corollary follows. □

As an application of the method described in Theorem 1.3.1, we now give a quantitative result regarding the distribution of the zeros of $\phi(s)$ on the critical line. This means that, for a sufficiently large T, we find $H = H(T)$ such that the interval $[T,\, T + H]$ contains the ordinate of a zero of $\phi(s)$ located at $\text{Re}(s) = \frac{r}{2}$. To the best of our knowledge, this seems to be the first quantitative result concerning the critical zeros of a general Dirichlet series.

Corollary 1.3.4. *Let $\phi(s)$ be a Dirichlet series satisfying Hecke's functional equation (1.1.9) in the sense of Definition 1.1.1. Suppose also that $\overline{a}(n) = b(n)$ and $\lambda_n = \mu_n$. Moreover, assume that its abscissa of absolute convergence, σ_a, satisfies*

$$\sigma_a < \frac{r+1}{2}. \tag{1.3.47}$$

Then, for any fixed $\epsilon > 0$, there exists a positive number $T_0(\epsilon)$ such that, for all $T \geq T_0(\epsilon)$, there is a zero $s = \frac{r}{2} + i\tau$ of $\phi(s)$ with

$$|\tau - T| \leq T^{\sigma_a + \frac{1-r}{2} + \epsilon}. \tag{1.3.48}$$

[13] see, for instance, [55, 139, 160, 170]. In these papers, explicit estimates of exponential integrals are used to deduce analogues of Hardy's result.

Proof. The proof is quite similar to the proof of Theorem 1.3.1, so we shall only indicate the main steps. Instead of taking a fixed parameter T_0 as in the proof of Theorem 1.3.1, we will take a sufficiently large T, which now has the same order as the free parameter t in the argument of Theorem 1.3.1.

If one assumes that $\phi(\frac{r}{2}+iy)$ does not possess a zero whenever $y \in [T,\, T+H]$, then the integrand on (1.3.11) does not possess a zero when $y \in [t-T-H,\, t-T]$. If we let $T+\frac{H}{4} < t \leq T+\frac{H}{2}$, say $t = T+\eta\, H$, $\frac{1}{4} < \eta \leq \frac{1}{2}$, and take a partition of the integral exactly as in (1.3.14), we will now be able to write the integral in (1.3.11) as

$$\left\{\int_{\eta H}^{\infty} + \int_{-\infty}^{-\eta H} + \int_{-\eta H}^{\eta H}\right\} R_{\psi}\left(\frac{r}{2}+i\,(y-t)\right) R_{\phi}\left(\frac{r}{2}+i\,(y+t)\right)\, dy, \tag{1.3.49}$$

where, as we have noted, $\frac{1}{4} < \eta \leq \frac{1}{2}$. Estimates of the first two integrals will be similar to the ones already given in (1.3.15), and both will be of the form $\ll_{\delta} T^{\sigma_a-\frac{1}{2}+\delta}\, e^{-\pi|t|}$, as t and T now have the same order. By the contradiction hypothesis, we are allowed to take the modulus inside the third integral as in (1.3.13) and then, just as we have done in the steps (1.3.19), (1.3.20) and (1.3.22), we can invoke Stirling's formula and Cauchy's theorem to arrive at a lower bound. Since t is of the same order as T, we can write the third integral appearing in the partition (1.3.49) as

$$\begin{aligned} \mathcal{A}_3(t) &\gg T^{r-1}\, e^{-\pi|t|} \int_{\frac{r}{2}-i\eta H}^{\frac{r}{2}+i\eta H} \left|\psi(z-it)\,\phi(z+it)\right|\, dz & (1.3.50)\\ &\gg T^{r-1}\, e^{-\pi|t|} \left|\int_{\sigma_a+1-i\eta H}^{\sigma_a+1+i\eta H} \psi(z-it)\,\phi(z+it)\, dz + O\left(T^{2\sigma_a-r+\delta}\right)\right| & (1.3.51)\\ &\gg T^{r-1}\, e^{-\pi|t|} \left\{H + O\left(T^{2\sigma_a-r+\delta}\right)\right\}, & (1.3.52)\end{aligned}$$

so that $H(T)$ needs to be chosen in order to be greater than the estimates of the integrals along the horizontal segments, which are $\ll_{\delta} T^{2\sigma_a-r+\delta}$. An immediate adaptation of the steps leading to (1.3.27) will give a lower bound for $\mathcal{Z}_2(r+2it;\, a;\, \lambda)$ of the form

$$\left|\mathcal{Z}_2\,(r+2it;\, a;\, \lambda)\,\right| \gg T^{-\frac{1}{2}}H + O\left(T^{2\sigma_a-r-\frac{1}{2}+\delta}\right) + O\left(T^{\sigma_a-r+\delta}\right) > B'\, T^{-\frac{1}{2}}\, H, \tag{1.3.53}$$

for some positive B' and $T+\frac{H}{4} < t \leq T+\frac{H}{2}$. In particular, this shows that

$$\left|\mathcal{Z}_2\,(r+it;\, a;\, \lambda)\,\right| > B'\, T^{-\frac{1}{2}}\, H, \qquad 2T+\frac{H}{2} < t \leq 2T+H. \tag{1.3.54}$$

An inductive reasoning similar to the one given in Claim 1.3.1 will also yield, for every $k \in \mathbb{N}_0$,

$$\left|\mathcal{Z}_{2^{k+1}}\left(2^k r+it;\, a;\, \lambda\right)\right| > C_k\, T^{-\frac{2^{k+1}-1}{2}}\, H^{2^{k+1}-1}, \quad 2^{k+1}T+\frac{2^{k+1}-1}{2^{k+1}}\, H < t \leq 2^{k+1}T+H, \tag{1.3.55}$$

where $C_k > 0$. If our Dirichlet series is such that (1.3.47) holds, then the choice $H(T) = T^{\sigma_a + \frac{1-r}{2} + \epsilon}$ satisfies all the requirements of the above reasoning. In fact, under the hypothesis (1.3.47), the inequality (1.3.55) contradicts, for large values of k, the convex estimate (1.2.78) obtained for $\mathcal{Z}_{2^{k+1}}(s;\, a;\, \lambda)$. This completes the proof. $\square$

Remark 1.3.7. Like in Remark 1.3.2, we note that it is possible to rewrite the previous proof in terms of any multidimensional Epstein zeta function, $\mathcal{Z}_k(s;\, a;\, \lambda)$, and not just for dyadic ones. For example, assuming that $\phi(s)$ has no zeros in the interval $[T,\, T+H]$, one may find that, analogously to (1.3.55),

$$\left|\mathcal{Z}_{k+1}\left(\frac{k+1}{2}r + 2it;\, a;\, \lambda\right)\right| > D_k\, T^{-\frac{k}{2}}\, H^k, \quad (k+1)T + \alpha_k H < t \leq (k+1)T + H, \tag{1.3.56}$$

for some sequence $(\alpha_k)_{k\in\mathbb{N}}$.

Remark 1.3.8. Although (1.3.48) is valid under the general conditions of the previous corollary, it may be seen from our next example and Example 1.3.3 that direct adaptations of the method developed in Theorem 1.3.1 yield better estimates for $H(T)$. This happens when we have enough information regarding the Epstein zeta functions attached to $\phi(s)$, besides only knowing that they obey to (1.2.78). Once we have such information, the condition (1.3.47) used in Corollary 1.3.3 may be discarded, since we can find a lower bound for $\mathcal{A}_3(T)$ in (1.3.50) by invoking Ramachandra's theorem (see Lemma 1.3.2 below).

Example 1.3.1 (Hardy's theorem and the four-square theorem)**.** Based upon Theorem 1.3.1, this example gives a new proof of Hardy's theorem for $\zeta(s)$, which is curious as its conclusion is derived from presumably independent properties of the arithmetical function $r_4(n)$. Let us invoke Theorem 1.3.1 to $\phi(s) = 2\pi^{-s}\zeta(2s)$, which satisfies Hecke's functional equation with $r = \frac{1}{2}$. If $\phi(s)$ does not have infinitely many zeros on the critical line $\mathrm{Re}(s) = \frac{1}{4}$, then, according to the argument given in Theorem 1.3.1, all of its dyadic Epstein zeta functions (1.2.76) will only have finitely many zeros on their critical lines. Consider, for instance, $\mathcal{Z}_4(s)$: it comes immediately from the definition of multidimensional Epstein zeta functions (1.1.18) that, for $\mathrm{Re}(s) > 2$,

$$\mathcal{Z}_4(s) = 16\,\pi^{-s} \sum_{m_1,\ldots,m_4\neq 0}^{\infty} \frac{1}{\left(m_1^2 + m_2^2 + m_3^2 + m_4^2\right)^s} = \pi^{-s} \sum_{n=1}^{\infty} \frac{r_4(n)}{n^s} = \pi^{-s}\zeta_4(s).$$

Hence, according with our assumption, $\zeta_4(s)$ cannot have infinitely many zeros at its critical line $\mathrm{Re}(s) = 1$. On the other hand, by the celebrated Jacobi's four-square theorem, $\zeta_4(s)$ can be described by

$$\zeta_4(s) = 8\left(1 - 2^{2-2s}\right)\zeta(s)\,\zeta(s-1). \tag{1.3.57}$$

Note that the right-hand side of (1.3.57) has infinitely many zeros on the line $\mathrm{Re}(s) = 1$, all of them being of the form $1 \pm \frac{\pi i}{\log(2)}\,k$, $k \in \mathbb{N}$. This means that $\mathcal{Z}_4(s)$ has infinitely many zeros on its critical line $\mathrm{Re}(s) = 1$ and a contradiction is derived, proving Hardy's theorem in its classical form.

This observation can also be used to improve the quantitative result given in Corollary 1.3.4. As it is clear from the proof, the condition $H(T) = T^{\sigma_a + \frac{1-r}{2} + \epsilon}$ is only necessary in order to contrast (1.3.55) with the Phragmén-Lindelöf estimate (1.3.45). However, for the present case where $\phi(s) = 2\,\pi^{-s}\zeta(2s)$, it suffices to assume that $H(T) = T^{\frac{1}{2}+\epsilon}$ until the point where we arrive at the condition (1.3.55) (in fact, it is enough to assume $H(T) = c\,T^{1/2}\,\log(T)$). In this particular case, (1.3.55) says that the inequality

$$|\zeta_4(1+it)| > B'\left(\frac{H}{T^{\frac{1}{2}}}\right)^3, \quad B' > 0, \tag{1.3.58}$$

must be valid for all t satisfying $4T + \frac{3H}{4} < t \leq 4T + H$. This obviously contradicts (1.3.57), which says that there is always a zero of $\zeta_4(1+it)$ in any interval of length greater than $\frac{\pi}{\log(2)}$. This observation is extended in Example 1.3.3 below.

Example 1.3.2. An application of Theorem 1.3.1 also allows to prove lower bounds (at the critical line) for some Epstein zeta functions. In [113] it is constructed an octonary quadratic form whose Epstein zeta function does not possess a single zero at its critical line. Indeed, if $\mathbb{I}_n$ denotes the $n-$dimensional identity matrix and $\mathcal{Q}(x_1, ..., x_8)$ is the octonary quadratic form represented by

$$\mathcal{Q} = \begin{pmatrix} 2\cdot\mathbb{I}_4 & \mathcal{S} \\ -\mathcal{S} & 2\cdot\mathbb{I}_4 \end{pmatrix} \quad \text{with} \quad \mathcal{S} := \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix},$$

then its associated Epstein zeta function, $Z_8(s,\,\mathcal{Q})$, can be written explicitly as

$$Z_8(s,\,\mathcal{Q}) = 240\cdot 2^{-s}\,\zeta(s)\,\zeta(s-3). \tag{1.3.59}$$

It is clear that $Z_8(s,\,\mathcal{Q})$ does not possess any zero at the critical line $\text{Re}(s) = 2$. Thus, if we construct the quadratic form in sixteen variables with a matrix representation of the form

$$\mathcal{P} = \begin{pmatrix} \mathcal{Q} & \mathbf{0}_8 \\ \mathbf{0}_8 & \mathcal{Q} \end{pmatrix},$$

we see by a simple application of item 2 of Theorem 1.3.1 that the associated Epstein zeta function satisfies $|Z_{16}\,(4+it,\,\mathcal{P})\,| > C\,|t|^{\frac{7}{2}}$, $C > 0$, whenever $|t|$ is sufficiently large.[14] This proves that, at most, $Z_{16}(s,\,\mathcal{P})$ has a finite number of zeros in the line $\text{Re}(s) = 4$. Hence, besides violating the Riemann hypothesis quite badly, as the construction (1.3.59) shows, Epstein zeta functions with a high number of variables do not satisfy the Lindelöf hypothesis. Some particular cases covered by this example can also be established as consequences of Ramanujan's formula (1.2.62).

Analogues of Stark's and Fujii's results [93, 212] are similarly subject to generalizations within this context. Consider, for instance, the Dirichlet series $\phi(s) := \zeta(s)\,\zeta(s-3)$: then the analogue of

[14] By the functional equation for the Riemann zeta-function and Stirling's formula, we have that $|Z_8(2+it,\,\mathcal{Q})| \gg |t|^{\frac{3}{2}}$, $|t| \gg 1$. Applying this to the Selberg-Chowla formula (1.3.3) and proceeding just as in the proof of Claim 1.3.1, we can derive the lower bound $|Z_{16}(4+it,\,\mathcal{P})| \gg |t|^{\frac{7}{2}}$, $|t| \gg 1$.

Epstein's zeta function $\mathcal{Z}_2(s;\,3,3)$ attached to $\phi(s)$ is the double Dirichlet series (see (1.2.56))

$$Z_2(s;\,3,3):=\sum_{m,n=1}^{\infty}\frac{\sigma_3(m)\,\sigma_3(n)}{(m+n)^s},\quad \mathrm{Re}(s)>8.$$

Using the convolution formulas of Ramanujan, we have seen in Example 1.2.3 that $Z_2(s;\,3,3)$ admits the simple representation[15]

$$Z_2\,(s;\,3,3):=\sum_{m,n=1}^{\infty}\frac{\sigma_3(m)\,\sigma_3(n)}{(m+n)^s}=\frac{1}{120}\zeta(s)\,\zeta(s-7),\qquad \mathrm{Re}(s)>8,$$

which shows that $Z_2\,(s;\,3,3)$ does not have a single zero at its critical line $\mathrm{Re}(s)=4$. However, if we construct a parametric Epstein zeta function of the form

$$Z_{2,\xi}(s;3,3):=\sum_{m,n\neq 0}^{\infty}\frac{\sigma_3(m)\,\sigma_3(n)}{(m+\xi^2\,n)^s},\quad \xi>0,\quad \mathrm{Re}(s)>8, \tag{1.3.60}$$

then it is possible to show that, for ξ large enough, all but two of the zeros of $Z_2(s;\,\xi)$ in the rectangle $\mathcal{R}_{\xi}:=\{z\in\mathbb{C}\,:\,-1<\mathrm{Re}(z)<9,\,|\mathrm{Im}(z)|\leq 2\xi\}$ are on the critical line $\mathrm{Re}(s)=4$. This observation can be interpreted as an analogue of a theorem of Stark [212] about Epstein zeta functions associated with quadratic forms of large discriminants. The key takeaway of these last lines is that, although (1.3.60) does not have critical zeros when $\xi=1$, there are certainly very large values of ξ for which the first $O(\xi\log(\xi))$ non-real zeros on the critical strip **are all on the critical line**. Further properties of these Dirichlet series attached to convolutions of divisor functions are studied in [189].

To introduce our final results, we need to briefly comment on the proof of Corollary 1.3.4. The condition (1.3.47), which implies that $\phi(s)$ has a fairly narrow critical strip, was necessary in two occasions of the proof. First, to allow that the term involving $H(T)$ was the dominant one on the right-hand side of (1.3.53). Second, it was required in order to deduce a contradiction upon a combination of (1.3.55) and the estimate (1.3.45).

Thus, if one wants to avoid a condition like (1.3.47), one needs to have more information concerning the dyadic Epstein zeta functions $\mathcal{Z}_{2^k}(s;\,a;\,\lambda)$ and the lower bound for the integral (1.3.50). Actually, this lower bound needs to be sharp enough in order to dominate the contributions of the horizontal segments in (1.3.52).

Thanks to a general result of K. Ramachandra, such lower bound for the integral (1.3.50) is possible, yet for its application we will consider a slightly less general class of Dirichlet series. We now give the following weaker version of Ramachandra's theorem appearing in [[160], p. 126, Lemma 3]. For a proof of this useful theorem, we refer to Balasubramanian's paper [[11], Theorem 3, p. 570] (see also [[120], p. 24, Theorem 2.4] for a nice application to the case where the Dirichlet series is $\zeta^k(s)$).

Lemma 1.3.2 (Ramachandra's theorem)**.** *Let* $F(s)=\sum_{n=1}^{\infty}a(n)\,n^{-s}$ *be any Dirichlet series satisfying the following conditions:*

[15] just use (1.2.63) with $k_1=k_2=3$.

1. *The sequence $a(n)$ is not the zero sequence;*

2. *$F(s)$ can be analytically continued to a region of the form $\sigma \geq \sigma_0$, $|t| \geq t_0$ and in this region $F(\sigma+it) = O\left((|t|+10)^A\right)$, for some $A \geq 1$.*

Then, for any $\epsilon > 0$, there exists some $T_0(\epsilon) > 0$ such that, for all $T \geq T_0(\epsilon)$ and $\sigma > \sigma_0$, we have

$$\int_{T-H}^{T+H} |F(\sigma+it)|\, dt \gg H \tag{1.3.61}$$

for all $H \geq \log^{\epsilon}(T)$.

In the next example we will apply the work done in the proofs of Theorem 1.3.1 and Corollary 1.3.4 to the Dirichlet series $\zeta_\alpha(s)$, $\alpha \in \mathbb{Q}_{>4}$, formally introduced in (1.1.7). But to do so it will be required to apply (1.3.61) on the integral appearing in (1.3.50),

$$\int_{-\eta H}^{\eta H} \left|\zeta_\alpha\left(\frac{\alpha}{4}+i(y-t)\right)\zeta_\alpha\left(\frac{\alpha}{4}+i(y+t)\right)\right| dy,$$

where $\frac{1}{4} < \eta \leq \frac{1}{2}$ and $T^{\frac{1}{2}+\epsilon} \leq H \leq T$. Since t was taken in such a way that $T+\frac{H}{4} < t \leq T+\frac{H}{2}$, we can see that the previous integral has the lower bound

$$\int_{T-\frac{H}{4}}^{T+\frac{H}{4}} \left|\zeta_\alpha\left(\frac{\alpha}{4}+i(y-2t)\right)\zeta_\alpha\left(\frac{\alpha}{4}+i\,y\right)\right| dy \gg H, \tag{1.3.62}$$

once we use (1.3.61) with $F(s) := \zeta_\alpha(s-2it)\,\zeta_\alpha(s)$. Note that, just as in (1.3.27), the implied constant does not depend[16] on t.

Example 1.3.3 (Dirichlet series attached to the powers of the $\theta-$function)**.** In this example we prove that, if $\alpha \in \mathbb{Q}_{>4}$ is fixed, then, for any $\epsilon > 0$, there is some $T_0(\epsilon)$ (which also depends on α) such that, if $T \geq T_0(\epsilon)$, there is a zero $\alpha/4+i\tau$ of $\zeta_\alpha(s)$ with $\tau \in [T,\, T+T^{1/2+\epsilon}]$. In particular, this implies the estimate $N_0(T,\, \zeta_\alpha) \gg T^{\frac{1}{2}-\epsilon}$ given at the introduction of this chapter. We should note that the simple qualitative result $\lim_{T\to\infty} N_0(T,\, \zeta_\alpha) = \infty$ could easily be derived from the works of Hecke [[113], p. 74, Satz 1] and Landau [[140], p. 240, section 6], as well as from our Corollary 1.3.1 above. Nevertheless, the higher precision of the statement in this example does not seem to be covered by any of the existing methods.

In the argument that follows, we will assume that α is some dyadic rational, say $\alpha = p\,2^{-j}$ with $p,\, j \in \mathbb{N}$. This assumption is only taken in order to match the reasoning given in Theorems 1.3.1 and Corollary 1.3.4. For a modification to any $\alpha \in \mathbb{Q}_{\alpha>4}$, we refer to the indications given in Remarks 1.3.2 and 1.3.7.

We divide our argument in four steps:

Step 1: Analytic continuation of $\zeta_\alpha(s)$. First, in order to apply the ideas from Corollary 1.3.4, we need to study the analytic continuation of the Dirichlet series attached to $r_\alpha(n)$, $\zeta_\alpha(s)$.

[16] To see this, we draw the reader's attention to the steps leading to (1.3.22) and to the footnote n.7 following it.

To do this, an estimate for $r_{\alpha}(n)$ is needed. The theta function $\vartheta_3(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ is a modular form of weight $\frac{1}{2}$ with a multiplier system with respect to the theta group Γ_θ (see [[137], pp. 15-16]). Therefore, for any $\alpha > 0$, $\vartheta_3^{\alpha}(\tau)$ is a modular form of weight $\alpha/2$ with a multiplier system on the same group.

By definition, $r_{\alpha}(n)$ are the Fourier coefficients of the expansion of $\vartheta_3(\tau)$ at the cusp $i\infty$, this is,

$$\vartheta_3^{\alpha}(ix) := \theta^{\alpha}(x) = 1 + \sum_{n=1}^{\infty} r_{\alpha}(n)\, e^{-\pi n\, x}, \quad x > 0. \tag{1.3.63}$$

The order of growth of $r_{\alpha}(n)$ is now determined by classical estimates due to Petersson and Lehner [137, 141, 218] for the Fourier coefficients of arbitrary modular forms of positive real weight with multiplier systems. These estimates show that $r_{\alpha}(n)$ grows polynomially, more precisely in the form

$$r_{\alpha}(n) \ll_{\alpha} \begin{cases} n^{\alpha/2-1} & \alpha > 4, \\ n^{\alpha/2-1}\log(n) & \alpha = 4, \\ n^{\alpha/4} & 0 < \alpha < 4. \end{cases} \tag{1.3.64}$$

The bounds above displayed determine the existence of a finite abscissa of absolute convergence for $\zeta_{\alpha}(s)$ given in (1.1.7). In particular, due to the estimate

$$r_{\alpha}(n) \ll_{\alpha} n^{\frac{\alpha}{2}-1}, \quad \alpha > 4, \tag{1.3.65}$$

the formal zeta function (1.1.7) can be rigorously studied as a Dirichlet series

$$\zeta_{\alpha}(s) = \sum_{n=1}^{\infty} \frac{r_{\alpha}(n)}{n^s}, \quad \mathrm{Re}(s) > \sigma_{\alpha} := \begin{cases} \frac{\alpha}{2} & \alpha \geq 4 \\ 1 + \frac{\alpha}{4} & 0 < \alpha < 4 \end{cases}.$$

By mimicking Riemann's second proof of the functional equation for $\zeta(s)$ [[228], p. 21] (observe once more that $\zeta_1(s) := 2\zeta(2s)$), it is a very simple matter to prove that $\zeta_{\alpha}(s)$ can be continued as a meromorphic function with a simple pole located at $s = \frac{\alpha}{2}$, whose residue $\mathrm{Res}_{s=\alpha/2}\zeta_{\alpha}(s) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$. Moreover, if we define $\eta_{\alpha}(s)$ as the completed Dirichlet series,

$$\eta_{\alpha}(s) := \pi^{-s}\Gamma(s)\,\zeta_{\alpha}(s), \tag{1.3.66}$$

then we can show (à la Riemann) that $\eta_{\alpha}(s)$ satisfies Hecke's functional equation

$$\eta_{\alpha}(s) := \pi^{-s}\Gamma(s)\,\zeta_{\alpha}(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_{\alpha}\left(\frac{\alpha}{2}-s\right) := \eta_{\alpha}\left(\frac{\alpha}{2}-s\right). \tag{1.3.67}$$

Thus, the Dirichlet series $\phi_{\alpha}(s) := \pi^{-s}\zeta_{\alpha}(s)$ satisfies all the conditions of Definition 1.1.1 and our results can be invoked in the next steps.

Step 2: Identifying the analogues of $\mathcal{Z}_k(s;\, a;\, \lambda)$. Secondly, we note that the multidimensional Epstein zeta function $\mathcal{Z}_k(s;\, \alpha)$ attached to $\phi_{\alpha}(s)$ is $\pi^{-s}\zeta_{\alpha\cdot k}(s)$. Since α is rational by hypothesis, we know that there exists some $k \in \mathbb{N}$ for which $\mathcal{Z}_k(s;\, \alpha)$ is equal to $\pi^{-s}\zeta_{\ell}(s)$, with

$\ell = \alpha \cdot k$ being an integer. Thus, using known results concerning the zeros of $\zeta_\ell(s)$, the zeta function attached to the sum of ℓ squares, we can extract information regarding the zeros of $\zeta_\alpha(s)$.

Step 3: Useful properties of $\zeta_\ell(s)$. Thirdly, we use some formulas due to Siegel to independently study the zeros of $\zeta_\ell(s)$, $\ell \in \mathbb{N}$. Let $f_\ell(\tau) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n\tau} \in S_{\ell/2}(\Gamma_\theta)$ denote the orthogonal projection of $\vartheta_3^\ell(\tau)$ onto the space of cusp forms. By a famous result originally due to Rankin [181], the cuspidal contribution is zero unless $\ell \geq 9$. Assume now that $\ell \equiv 0 \mod 4$: it is well-known [[202], pp. 165, 166] that the Dirichlet series attached to $f_\ell(z)$, say $L(s,\, f_\ell)$, satisfies the convex estimate $L\left(\frac{\ell}{4} + it,\, f_\ell\right) \ll_\delta (1+|t|)^{\frac{1}{2}+\delta}$.

We now consider the Eisenstein series contribution to $\vartheta_3^\ell(\tau)$, whose Dirichlet series is, by definition, $E_\ell(s) := \zeta_\ell(s) - L(s,\, f_\ell)$. It follows from a Formula due to Siegel [[202], p. 169] that $E_\ell(s)$ can be explicitly written as

$$E_\ell(s) = c_\ell \left\{1 - 2^{-s} + (-1)^{\ell/4}\left(2^{\frac{\ell}{2}-2s} - 2^{-s}\right)\right\} \zeta(s)\, \zeta\left(s + 1 - \frac{\ell}{2}\right), \tag{1.3.68}$$

where $c_\ell > 0$. Note that, when $\ell = 4$, (1.3.68) reduces to Jacobi's result (1.3.57). It is clear from this comparison that $E_\ell(s)$ has infinitely many zeros on the line $\mathrm{Re}(s) = \frac{\ell}{4}$, all of them of the form

$$\begin{cases} \omega_n = \frac{\ell}{4} \pm \frac{i\pi n}{\log(2)}, \quad n \in \mathbb{N} & \ell \equiv 4 \mod 8, \\ \omega_n = \frac{\ell}{4} + \frac{i}{\log(2)}\left\{2n\pi \pm \arctan\left(\sqrt{2^{\ell/2}-1}\right)\right\}, \quad n \in \mathbb{Z} & \ell \equiv 0 \mod 8. \end{cases} \tag{1.3.69}$$

Step 4: Finishing the Proof. Fix $\epsilon > 0$ and let $\alpha = p\, 2^{-j}$, $p \in \mathbb{N}$. Assume also that $\zeta_\alpha\left(\frac{\alpha}{4} + it\right)$ does not have a zero between T and $T + H(T)$. The idea is to apply the inductive process in Corollary 1.3.4 a high number of times in order to reach a contradiction. Since $\sigma_a = r = \alpha/2$ (see (1.3.65 and (1.3.67) above), an application of Ramachandra's lower bound (1.3.62) at the step (1.3.50) gives, for $T + \frac{H}{4} < t \leq T + \frac{H}{2}$,

$$|\mathcal{Z}_2(r + 2it;\, \alpha)| := \left|\zeta_{2\alpha}\left(\frac{\alpha}{2} + 2it\right)\right| \gg T^{-\frac{1}{2}} H + O\left(T^{\epsilon/2}\right) > B'\, T^{-\frac{1}{2}} H, \tag{1.3.70}$$

as $T^{\frac{1}{2}+\epsilon} \leq H \leq T$ by hypothesis (and also taking $\delta = \epsilon/2$ in (1.3.53)). Let k be an integer so large that $2^{k+1} > \max\left\{2 + \frac{1}{2\epsilon},\, 2^{j+1}\right\}$: from this choice, we know that $2^{k+1}\alpha = \ell$ for some $\ell \equiv 0 \mod 4$. Following the argument exposed in Theorem 1.3.1 and in Corollary 1.3.4, we know that (1.3.69) must imply

$$\left|\mathcal{Z}_{2^{k+1}}\left(\frac{\ell}{4} + it,\, \alpha\right)\right| := \left|\zeta_\ell\left(\frac{\ell}{4} + it\right)\right| = \left|E_\ell\left(\frac{\ell}{4} + it\right) + L\left(\frac{\ell}{4} + it,\, f_\ell\right)\right| > C_k \left(\frac{H}{T^{\frac{1}{2}}}\right)^{2^{k+1}-1}, \tag{1.3.71}$$

for all t such that $2^{k+1}T + \frac{2^{k+1}-1}{2^{k+1}} H < t \leq 2^{k+1}T + H$ and $C_k > 0$. Since $L\left(\frac{\ell}{4} + it,\, f_\ell\right) \ll_\epsilon T^{\frac{1}{2}+\epsilon}$ (as t has the same order as T) and $H(T) \geq T^{\frac{1}{2}+\epsilon}$ by hypothesis, it follows from our choice of k and (1.3.71) that, for some $C_k' > 0$,

$$\left|E_\ell\left(\frac{\ell}{4} + it\right)\right| > C_k'\, T^{\left(2^{k+1}-1\right)\epsilon} + O\left(T^{\frac{1}{2}+\epsilon}\right), \quad 2^{k+1}T + \frac{2^{k+1}-1}{2^{k+1}} H < t \leq 2^{k+1}T + H. \tag{1.3.72}$$

Finally, since we can choose $T_0(\epsilon(k))$ so large that $H\,2^{-k-1} > 2\pi/\log(2)$ is valid for all $T \geq T_0(k(\epsilon))$, our result follows. This is because (1.3.71) implies that $E_\ell\left(\frac{\ell}{4}+it\right)$ does not vanish on an interval whose length is greater than $2\pi/\log(2)$. This constitutes an immediate contradiction to (1.3.69), which says that the spacing between consecutive zeros of $E_\ell\left(\frac{\ell}{4}+it\right)$ is less than $2\pi/\log(2)$.

Remark 1.3.9. One can extend the previous example to any rational α such that $2<\alpha<4$, although the available estimates for $r_\alpha(n)$ (1.3.64) (cf. [[141], p. 603, eq. (5)]) will not allow us to take $H(T)=T^{\frac{1}{2}+\epsilon}$ as before. However, for $\alpha=3$, the previous steps hold, as it is known that the abscissa of absolute convergence of $\zeta_3(s)$ is equal to 3/2.

As a slight sharpening, we can show a stronger result concerning the gap between the zeros of $\zeta_\alpha(s)$ for intervals of the type $\left[T,\,T+c\,T^{1/2}\log(T)\right]$, for some $c>0$. Although the proof is the same as the one outlined in Example 1.3.3 (modulo estimates), one may be able to extend it to the case where α is irrational as well. We state our final result. To follow the proof, we refer the reader to the published version of this chapter [194].[17]

Corollary 1.3.5. *Let $\alpha>4$ and $\zeta_\alpha(s)$ be the Dirichlet series defined by (1.1.7). There exist T_0 and $c>0$ (depending on α) such that, for any $T\geq T_0$, there is a zero $\rho=\alpha/4+i\tau$ of $\zeta_\alpha(s)$ with $\tau\in[T,\,T+cT^{1/2}\log(T)]$.*

Remark 1.3.10. The proof of the previous corollary is also valid for $\zeta_3(s)$. Sankaranarayanan [197] derived a similar bound for $H(T)$ when the Dirichlet series is the classical Epstein zeta function $\zeta(s,\,Q)$ or the Dedekind zeta function attached to a real quadratic field. However, a modification of Sankaranarayanan's method to the ternary Epstein zeta function $\zeta_3(s)$ does not give anything better than $H(T)=c\,T^{3/4}\log(T)$. It is certainly feasible to combine Sankaranarayanan's argument with the methods of the paper [176] to prove that $N_0(T,\zeta_3(s))\gg T^{1/2}/\log(T)$ as $T\to\infty$. However, this does not provide a better result than the one obtained following the ideas of this chapter. Besides, in Chapter 3 below we will prove a much stronger result than this by using a forgotten method of Fekete [88].

[17] We do not present the proof because in the third chapter we will be able to get a massive generalization of a result analogous to this one. See Remark 3.1.2 below.

Chapter 2

Certain Extensions of results of Siegel, Wilton and Hardy

2.1 Introduction and main results

Let $\zeta(s)$ denote the Riemann zeta function and define[1]

$$\eta(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{2.1.1}$$

A. Dixit, N. Robles, A. Roy and A. Zaharescu [74] proved the following theorem.

Theorem 2.1.1. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$. Also, let $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers that attains its bounds. Then the function*

$$F(s)=\sum_{j=1}^{\infty}c_j\,\eta\left(s+i\lambda_j\right):=\sum_{j=1}^{\infty}c_j\,\pi^{-\frac{s+i\lambda_j}{2}}\Gamma\left(\frac{s+i\lambda_j}{2}\right)\zeta\left(s+i\lambda_j\right)$$

has infinitely many zeros on the critical line $Re(s)=\frac{1}{2}$.

Based on an integral representation of Jacobi's transformation formula due to Dixit [[63], p. 374, eq. (1.13)], A. Dixit, R. Kumar, B. Maji and A. Zaharescu [73] later generalized the aforementioned result and proved a much more general theorem. This new theorem, stated below, involves the combination of Riemann's completed function $\eta(s)$ with Kummer's confluent hypergeometric function, ${}_1F_1(a;c;z)$, which is usually defined by the power series [[164], p. 322, 13.2.2],

$${}_1F_1\left(a;c;z\right):=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.1.2}$$

where $(a)_k$ denotes Pochhammer's symbol,

$$(a)_k := a(a+1)\cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)}. \tag{2.1.3}$$

[1] We warn the reader to not mistake the completed Dirichlet series $\eta(s)$ with Dedekind's $\eta-$function (1.2.48) introduced in Example 1.2.2 of the previous chapter. Note that $\eta(s)$ is defined for any $s\in\mathbb{C}$ while Dedekind's $\eta-$function is only defined for a variable belonging to the upper half-plane $\mathbb{H}$.

Theorem 2.1.2. *Let $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ be as in Theorem 2.1.1. Also, let $\mathscr{R}$ denote the region of the complex plane defined[2] by $\mathscr{R} := \left\{z \in \mathbb{C} \, : \, |Re(z)| < \sqrt{\frac{\pi}{2}}, \, |Im(z)| < \sqrt{\frac{\pi}{2}}\right\}$. Then, for any $z \in \mathscr{R}$, the function*

$$F_z(s) = \sum_{j=1}^{\infty} c_j \eta\left(s + i\lambda_j\right) \left\{ {}_1F_1\left(\frac{1-(s+i\lambda_j)}{2}; \frac{1}{2}; \frac{z^2}{4}\right) + {}_1F_1\left(\frac{1-(\overline{s}-i\lambda_j)}{2}; \frac{1}{2}; \frac{\overline{z}^2}{4}\right)\right\} \tag{2.1.4}$$

has infinitely many zeros on the critical line $Re(s) = \frac{1}{2}$.

The proofs of both theorems used a very elegant generalization of Hardy's method of studying the moments of the real function $\eta\left(\frac{1}{2}+it\right)$ [[106], [228], Chapter X]. One of the crucial steps in Hardy's proof is the transformation formula for Jacobi's theta function, usually defined as

$$\theta(x) := \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} = 2\psi(x) + 1, \tag{2.1.5}$$

with $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. This transformation formula can be explicitly written as

$$x^{1/2}\left(1 + 2\psi(x)\right) = 1 + 2\psi\left(\frac{1}{x}\right). \tag{2.1.6}$$

It was also established by Jacobi that, if we consider a two-variable generalization of $\psi(x)$ in the form,

$$\psi(x,z) := \sum_{n=1}^{\infty} e^{-\pi n^2 x} \cos\left(\sqrt{\pi x}\, n\, z\right), \quad \mathrm{Re}(x) > 0, \ \ z \in \mathbb{C}, \tag{2.1.7}$$

then the transformation formula takes place [[73], p. 312, eq. (2.6)]

$$\sqrt{x}\left(1 + 2\psi(x,z)\right) = e^{-z^2/4}\left(1 + 2\,\psi\left(\frac{1}{x}, iz\right)\right). \tag{2.1.8}$$

Dixit [[63], p. 374, eq. (1.13)] realized that (2.1.8) could be achieved through an integral representation involving the Riemann zeta function, proving the elegant identity

$$2x^{1/4}\psi(x,z) - x^{-1/4}e^{-z^2/4} = 2e^{-z^2/4}x^{-1/4}\,\psi\left(\frac{1}{x},\, iz\right) - x^{1/4}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^{-\frac{1}{4}-\frac{it}{2}}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\zeta\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{4}+\frac{it}{2}; \frac{1}{2}; -\frac{z^2}{4}\right) x^{-\frac{it}{2}}\, dt. \tag{2.1.9}$$

This formula played a fundamental role in proving Theorem 2.1.2 above (see also the very interesting survey [65] for more integral representations of several transformation formulas). The purpose of this contribution is to see how Theorem 2.1.2 can be extended to a class of Dirichlet series satisfying Hecke's functional equation, which we have already presented in the first chapter of this thesis.

Our method is an extension of the one employed in [73], but in order to use it we need to develop a general formulation of the integral identity (2.1.9). Although we proceed by following the

[2]The region indicated in [73] is actually $|\mathrm{Re}(z) - \mathrm{Im}(z)| < \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}}\,\mathrm{Re}(z)\,\mathrm{Im}(z)$, which contains $\mathscr{R}$, but in this chapter we will only focus on rectangular regions as the one indicated in the above statement.

setting of Chandrasekharan, Narasimhan and Berndt [22, 53], for the purpose of introducing the main results of the present chapter, we now provide some examples.

For any $\alpha > 0$, define $r_\alpha(n)$ as the arithmetical function [[218], p. 481, eq. (1.6)]

$$\theta^\alpha(x) - 1 := \sum_{n=1}^{\infty} r_\alpha(n)\, e^{-\pi n x}. \tag{2.1.10}$$

As we have seen in Example 1.3.3 of the previous chapter, for $\mathrm{Re}(s)$ sufficiently large, we can consider formally the Dirichlet series

$$\zeta_\alpha(s) := \sum_{n=1}^{\infty} \frac{r_\alpha(n)}{n^s}, \tag{2.1.11}$$

and motivate its study through the transformation formula for $\theta(x)$, (2.1.6). The zeta function (2.1.11) seems to have been introduced in a paper by Lagarias and Rains [137] who studied, among several other things, how the distribution of its zeros could vary with the index α. Suzuki proved the analytic continuation and the functional equation for $\zeta_\alpha(s)$, with $0 < \alpha < 1$, and derived a Selberg-Chowla formula involving $r_\alpha(n)$. Motivated by Suzuki's approach, and following a general Fourier expansion resembling Selberg-Chowla's formula, in the previous chapter of our thesis we proved a zero density estimate for the zeta functions $\zeta_\alpha(s)$ on their critical lines (see Example 1.3.3 and the statement of Corollary 1.3.5 above).

When $\alpha = k \in \mathbb{N}$, it is effortless to see that $r_\alpha(n)$ reduces to the arithmetical function counting the number of representations of n as a sum of k squares. We have also seen in Example 1.3.3 of the previous chapter that, when $\alpha = 1$, $\zeta_1(s)$ reduces to $2\zeta(2s)$. Like $\zeta_k(s)$ and $\zeta(2s)$, $\zeta_\alpha(s)$ satisfies the functional equation (cf. (1.3.67) above)

$$\eta_\alpha(s) := \pi^{-s}\Gamma(s)\,\zeta_\alpha(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_\alpha\left(\frac{\alpha}{2}-s\right) := \eta_\alpha\left(\frac{\alpha}{2}-s\right), \tag{2.1.12}$$

from which it is possible to conclude that $\eta_\alpha(s)$ is real on the critical line $\mathrm{Re}(s) = \frac{\alpha}{4}$.

One of the first results of the present chapter is the extension of Theorem 2.1.2 above to $\zeta_\alpha(s)$. Its statement is as follows.

Theorem 2.1.3. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-zero real numbers such that $\sum_j |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. Then, for any z satisfying the condition*

$$z \in \mathscr{D}_\alpha := \left\{ z \in \mathbb{C}\,:\, |Re(z)| < \sqrt{\frac{\pi\alpha}{2}},\, |Im(z)| < \sqrt{\frac{\pi\alpha}{2}} \right\}, \tag{2.1.13}$$

the function

$$\begin{aligned} F_{z,\alpha}(s) := &\sum_{j=1}^{\infty} c_j\, \pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_\alpha(s+i\lambda_j) \\ &\times \left\{ {}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\, \frac{\alpha}{2};\, \frac{\overline{z}^2}{4}\right) \right\} \end{aligned} \tag{2.1.14}$$

has infinitely many zeros on the critical line Re$(s) = \frac{\alpha}{4}$.

In particular, for $\alpha = 1$ we can recover Theorem 2.1.2 above. For $\alpha = k \in \mathbb{N}$, our result generalizes a theorem of Siegel [202], discussed in the previous chapter, and Landau [140] (see also [113]), which says that the completed Dirichlet series

$$\eta_k(s) := \pi^{-s}\Gamma(s)\,\zeta_k(s)$$

possesses infinitely many zeros at its critical line[3] $\text{Re}(s) = \frac{k}{4}$.

One of the main ingredients in the proof Theorem 2.1.3 is a generalization of Dixit's integral formula (2.1.9): as it will be seen in Example 2.2.1 of the present contribution, it is possible to prove the transformation formula

$$\begin{aligned}
&\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi\,n\,x}\,z)-\left(\sqrt{\frac{\pi}{x}}\,\frac{z}{2}\right)^{\frac{\alpha}{2}-1}\frac{e^{-z^2/8}}{\Gamma\left(\frac{\alpha}{2}\right)\sqrt{x}}\\
&=\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{\alpha}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\,z\right)-\left(\sqrt{\pi x}\,\frac{z}{2}\right)^{\frac{\alpha}{2}-1}\frac{\sqrt{x}\,e^{z^2/8}}{\Gamma\left(\frac{\alpha}{2}\right)}\\
&=\left(\frac{\sqrt{\pi}z}{2}\right)^{\frac{\alpha}{2}-1}\frac{e^{z^2/8}}{2\pi\,\Gamma\left(\frac{\alpha}{2}\right)}\int_{-\infty}^{\infty}\pi^{-\frac{\alpha}{4}-it}\Gamma\left(\frac{\alpha}{4}+it\right)\zeta_{\alpha}\left(\frac{\alpha}{4}+it\right){}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};-\frac{z^2}{4}\right)x^{-it}\,dt,
\end{aligned} \tag{2.1.15}$$

where $J_\nu(z)$ and $I_\nu(z)$ denote the Bessel functions of the first kind, usually defined by the power series [[164], p. 217, eq. 10.2.2]

$$J_{\nu}(x) := \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu}, \tag{2.1.16}$$

and [[164], p. 249, eq. (10.25.2)]

$$I_{\nu}(x) := \sum_{m=0}^{\infty}\frac{1}{m!\Gamma\left(m+\nu+1\right)}\left(\frac{x}{2}\right)^{2m+\nu}. \tag{2.1.17}$$

As we will see below, formula (2.1.9) can be now recovered from (2.1.15) when we take there $\alpha = 1$.

Since the proof of (2.1.15) will be achieved from a general point of view, identities akin to (2.1.15) are also valid to other Dirichlet series. For example, another version of (2.1.15) holds for Epstein zeta functions attached to binary quadratic forms: recalling the introduction of the previous chapter, if $Q(m,n) = Am^2 + Bmn + Cn^2$ is a binary, integral and positive definite quadratic form, we can consider the Dirichlet series

$$\zeta\left(s,Q\right) = \sum_{m,n\neq 0}\frac{1}{Q(m,n)^s}, \quad \text{Re}(s) > 1, \tag{2.1.18}$$

which is already familiar to us, being known as the Epstein zeta function attached to Q. In 1934, Potter and Titchmarsh [170] extended Hardy's result to $\zeta(s,Q)$. Roughly one year later, Kober

[3] In fact, as we have discussed in the previous chapter, Siegel's result is very sharp and provides detailed information about the zeros of $\zeta_k(s)$ on a fairly large strip of the complex plane.

[128] furnished a considerable simplification of their proof, more similar in spirit to Hardy's own argument.

In this contribution we are also able to extend the result given in Theorem 2.1.2 to a subclass of the zeta functions (2.1.18). Its statement reads as follows.

Theorem 2.1.4. *Let $Q(x,y) = Ax^2+Bxy+Cy^2$ be a binary, integral and positive definite quadratic form and let $\Delta := 4AC - B^2$. Assume also that gcd$(A,\, B,\, C) = 1$, $\sqrt{\Delta} \equiv 2 \mod 4$ and consider the Epstein zeta function attached to Q,*

$$\zeta(s,\, Q) = \sum_{m,n\neq 0} \frac{1}{(Am^2+Bmn+Cn^2)^s}, \quad Re(s) > 1. \tag{2.1.19}$$

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty} |c_j| < \infty$, $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds and z satisfies the condition

$$z \in \mathscr{D}_Q := \left\{ z \in \mathbb{C}\, :\, |Re(z)| < \frac{2\sqrt{\pi}}{\Delta^{3/4}},\ |Im(z)| < \frac{2\sqrt{\pi}}{\Delta^{3/4}} \right\}, \tag{2.1.20}$$

then the function

$$\begin{aligned} F_{z,Q}(s) = &\sum_{j=1}^{\infty} c_j \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(s+i\lambda_j)} \Gamma\left(s+i\lambda_j\right) \zeta(s+i\lambda_j,\, Q) \\ &\times \left\{ {}_1F_1\left(1-s-i\lambda_j;\, 1;\, \frac{z^2}{4}\right) + {}_1F_1\left(1-\overline{s}+i\lambda_j;\, 1;\, \frac{\overline{z}^2}{4}\right)\right\} \end{aligned}$$

has infinitely many zeros on the critical line $Re(s) = \frac{1}{2}$.

Another natural extension of Theorem 2.1.2 has to be concerned with Dirichlet L-functions. At the end of their paper, Dixit, Kumar, Maji and Zaharescu [73] state a character analogue of the identity (2.1.9) and write that *"it would be worthwhile (...) to find a character analogue of Theorem 2 [Theorem 2.1.2 above]"*.

Under certain restrictions on even characters, we have managed to prove a character analogue of Theorem 2.1.2. In fact, motivated by the study of our Theorem 2.1.3, we establish something more general than that. In order to clarify our next statement, note that, in analogy with $\zeta_\alpha(s)$, we can define, just as Suzuki does [[218], p. 483, eq. (1.15)], a new Dirichlet series attached to the powers of the theta function:

$$\theta(x,\chi) := \sum_{n\in\mathbb{Z}} n^{\delta}\chi(n)\, e^{-\frac{\pi n^2 x}{q}}, \quad \delta = \begin{cases} 0 & \text{if } \chi(-1)=1 \\ 1 & \text{if } \chi(-1)=-1 \end{cases}. \tag{2.1.21}$$

However, in order to define $\theta^{\alpha}(x,\chi)$ for any $\alpha > 0$, one would have to assure that χ is real and such that $\theta(x,\chi) > 0$ for any $x > 0$. Whether this condition holds or not, it is always possible to take an arbitrary integer power of $\theta(x,\chi)$, which results in the following series

$$\theta^k(x,\chi) = \sum_{n_1,\dots,n_k\in\mathbb{Z}} n_1^{\delta}\,\chi(n_1)\dots n_k^{\delta}\,\chi(n_k)\, e^{-\frac{\pi(n_1^2+\dots+n_k^2)\,x}{q}}.$$

Thus, in analogy to the Dirichlet series attached to the sum of $k-$squares, we study the zeros of arbitrary combinations of the Dirichlet series[4]

$$L_k\left(s,\chi\right):=\sum_{n_1,...,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\,\chi(n_1)\,...\,n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+...+n_k^2\right)^s},\quad \text{Re}(s)>\frac{k}{2}(1+\delta), \tag{2.1.22}$$

which is clearly attached to the powers of $\theta(x,\chi)$. Note that, if $k=1$, $L_1(s,\chi)=2\,L(2s-\delta,\chi)$. Also, $L_k\left(s,\chi\right)$ can be thought as a "character analogue" of $\zeta_k(s)$. The reader is encouraged to briefly jump to Lemma 2.1.3 below to see the functional equation satisfied by (2.1.22) and compare it with (2.1.12).

Using a class of transformation formulas involving the Dirichlet series $L_k(s,\chi)$, we will study the analytic continuation of (2.1.22) and establish the following theorem.

Theorem 2.1.5. *Let χ be a primitive Dirichlet character modulo q and define $\delta=0$ if χ is even and $\delta=1$ if χ is odd. Furthermore, if χ is even, assume that $q\neq 0 \mod 4$.*

If $L_k\left(s,\chi\right)$ is the Dirichlet series given in (2.1.22), let $\eta_k(s,\chi)$ denote its completed version

$$\eta_k\left(s,\chi\right):=\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k\left(s,\chi\right).$$

Assume that $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_j|c_j|<\infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds. Moreover, assume that z satisfies the condition

$$z\in\mathscr{D}_q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\sqrt{\frac{\pi}{2q}},\,|Im(z)|<\sqrt{\frac{\pi}{2q}}\right\}. \tag{2.1.23}$$

Then the function $F_{z,k,\chi}(s)$ given by

$$\begin{aligned}F_{z,k,\chi}(s):=\sum_{j=1}^{\infty}c_j\,\eta_k\left(s+i\lambda_j,\chi\right)\Bigg\{&{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-s-i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{z^2}{4}\right)\\&+{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-\overline{s}+i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{\overline{z}^2}{4}\right)\Bigg\}\end{aligned}$$

has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}\left(\frac{1}{2}+\delta\right)$.

Finally, we consider an extension of Theorem 2.1.2 to Dirichlet series attached to cusp forms for the full modular group, whose Fourier expansions we have dealt with in Example 1.2.4 of the previous chapter. Wilton [237] used Hardy's method to prove that, if p and q are two integers such that $p^2\equiv 1 \mod q$, then the twisted $L-$function,

$$L_{\tau}\left(s,\frac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{\tau(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\quad \text{Re}(s)>\frac{13}{2}, \tag{2.1.24}$$

where $\tau(n)$ is Ramanujan's $\tau-$function, has infinitely many zeros on the line $\text{Re}(s)=6$. Since $\tau(n)$ defines the Fourier coefficients of a cusp form having weight $k=12$, it is natural that Wilton's

[4]Note that the zero term $(n_1,...,n_k)=(0,...,0)$ is not problematic in the expression of this Dirichlet series because the non-principal characters vanish at zero.

proof extends to $L-$functions attached to different cusp forms. Very recently, Meher, Pujahari and Shankhadhar [156] employed Wilton's argument to prove that $L-$functions attached to half-integral weight cusp forms on $\Gamma_0(4N^2)$ have infinitely many zeros on their critical lines. We have also extended some ideas of de la Valleé Poussin and Lekkerkerker to prove estimates for the number of critical zeros of such $L-$functions of half-integral weight cusp forms (see [187] and the next chapter).

Since any Dirichlet series of the form (2.1.24) satisfies Hecke's functional equation, our general formalism allows to prove the following extension of Wilton's result.

Theorem 2.1.6. *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$ and consider its $L-$function*

$$L_f(s,p/q)=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\quad Re(s)>\frac{k+1}{2},\quad p^2\equiv 1 \mod q. \tag{2.1.25}$$

If $(c_j)_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$, $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers attaining its bounds and z satisfies the condition

$$z\in\mathscr{D}_q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<2\sqrt{\frac{\pi}{q}},\ |Im(z)|<2\sqrt{\frac{\pi}{q}}\right\}, \tag{2.1.26}$$

then the function

$$\begin{aligned}&\sum_{j=1}^{\infty}c_j\left(\frac{2\pi}{q}\right)^{-s-i\lambda_j}\Gamma(s+i\lambda_j)\,L_f\left(s+i\lambda_j,\frac{p}{q}\right)\\&\qquad\times\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{\overline{z}^2}{4}\right)\right\}\end{aligned} \tag{2.1.27}$$

has infinitely many zeros on the critical line $Re(s)=\frac{k}{2}$.

We would like to remark that it is possible to generalize Theorem 2.1.2 to even more Dirichlet series. Although we do not give the details in this thesis, we have obtained similar results as the ones stated above in the following cases:

1. Although $L_k(s,\chi)$ is a nice analogue of the Dirichlet series $\zeta_k(s)$, there are other ways of constructing character analogues of Epstein zeta functions. One such way is due to H. M. Stark [[210, 211]] (see also [29] for more constructions), who introduced the character analogue of (2.1.18) in the form

$$L\left(s,\chi,Q\right):=\sum_{m,n\neq 0}\frac{\chi\left(Q(m,n)\right)}{Q(m,n)^s},\quad \mathrm{Re}(s)>1, \tag{2.1.28}$$

   and studied its analytic properties when χ is a primitive character modulo q. One can prove an analogue of Theorem 2.1.4 for this Dirichlet series, under the additional assumption that $(q,\Delta)=1$ and Δ satisfying the same conditions as in Theorem 2.1.4.

2. Like in the case of (2.1.28), we could get similar results to different analogues of the Epstein zeta function such as

$$\zeta\left(s,Q,\chi\right):=\sum_{m,n\neq0}\frac{\chi(m)\,\chi(n)}{Q(m,n)^{s}},\quad \mathrm{Re}(s)>1,$$

as well as

$$\sum_{m_1,...,m_{2k}\neq0}\frac{\chi\left(m_1^2+...+m_{2k}^2\right)}{\left(m_1^2+...+m_{2k}^2\right)^s},\quad \mathrm{Re}(s)>k.$$

3. There are also extensions of Theorem 2.1.6 to $L-$functions attached to cusp forms whose coefficients are twisted by primitive Dirichlet characters.

4. Instead of considering an Epstein zeta function in Theorem 2.1.4, one may prove a similar result for the Dirichlet series $\zeta(s)\,L(s,\chi)$, where χ is an odd and primitive Dirichlet character whose modulus is a perfect square.

This chapter is organized as follows. In the next subsections we introduce some general definitions in the setting of Dirichlet series. We also give the functional equations of the main L-functions here studied, as well as some background on special functions. Section 2.2 is devoted to prove a general analogue of the theta transformation formula, from which (2.1.9) is derived as a particular case. We also provide examples which will be helpful in establishing our main theorems. Finally, in the remaining sections of the chapter we shall prove each particular theorem stated above.

2.1.1 Definitions

To give a generalization of (2.1.9), we need the following definition, which is a slight generalization of Definition 1.1.1 of the previous chapter.

Definition 2.1.1. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers strictly increasing to ∞ and $(a(n))_{n\in\mathbb{N}}$ and $(b(n))_{n\in\mathbb{N}}$ two sequences of complex numbers not identically zero. Consider the functions $\phi(s)$ and $\psi(s)$ representable as Dirichlet series

$$\phi(s)=\sum_{n=1}^{\infty}\frac{a(n)}{\lambda_n^s}\quad\text{and}\quad\psi(s)=\sum_{n=1}^{\infty}\frac{b(n)}{\mu_n^s},\tag{2.1.29}$$

with finite abscissas of absolute convergence σ_a and σ_b respectively. Let $\Delta(s)$ denote one of the following three gamma factors: $\Gamma(s)$, $\Gamma\left(\frac{s}{2}\right)$ and $\Gamma\left(\frac{s+1}{2}\right)$ and r be an arbitrary positive real number in the first case and 1 in the other two.

We say that $\phi(s)$ and $\psi(s)$ satisfy the functional equation

$$\Delta(s)\,\phi(s)=\Delta(r-s)\,\psi(r-s),\tag{2.1.30}$$

if there exists a meromorphic function $\chi(s)$ with the following properties:

1. $\chi(s)=\Delta(s)\,\phi(s)$ for $\mathrm{Re}(s)>\sigma_a$ and $\chi(s)=\Delta(r-s)\,\psi(r-s)$ for $\mathrm{Re}(s)<r-\sigma_b$;

2. $\lim_{|\text{Im}(s)|\to\infty}\chi(s)=0$ uniformly in every interval $-\infty<\sigma_1\leq \text{Re}(s)\leq\sigma_2<\infty$;

3. The singularities $\chi(s)$ are at most poles and are confined to some compact set.

We say that the pair of functions $(\phi,\,\psi)$ representable as Dirichlet series (2.1.29) satisfy **Hecke's functional equation** if they satisfy the conditions of the previous definition for $\Delta(s)=\Gamma(s)$. This particular case of (2.1.30) reads

$$\Gamma(s)\,\phi(s)=\Gamma(r-s)\,\psi(r-s),\quad r>0. \tag{2.1.31}$$

Similarly, we say that the pair of functions $(\phi,\,\psi)$ representable as (2.1.29) with finite abscissas of absolute convergence σ_a and σ_b satisfy **Bochner's functional equation** if they satisfy the functional equation

$$\Gamma\left(\frac{s+\delta}{2}\right)\phi(s)=\Gamma\left(\frac{1+\delta-s}{2}\right)\,\psi\left(1-s\right), \tag{2.1.32}$$

in the sense of Definition 2.1.1.

Before proceeding further, we recall the "Notation and Convention" remarks A introduced at the first chapter: we shall denote $\sum_{m,n\neq 0}$ as the infinite sum over all integers m, n not simultaneously zero. The same notation is taken for multiple sums. Whenever we use the term **critical line** for a given Dirichlet series $\phi(s)$ satisfying Definition 2.1.1 and (2.1.31), we will be referring to $\text{Re}(s)=\frac{r}{2}$, which is the line of symmetry. To recall some notions and notation, we urge the reader to consult the Notation remarks A and B of the previous chapter.

The Dirichlet series considered in all of our theorems have at most one simple pole as singularity. Therefore, it will be more convenient for the analysis given in this chapter to consider subclasses of the Dirichlet series defined above, with fewer singularities. We now give two definitions introducing these: the first one is the same as Definition 1.1.1 of the previous chapter.

Definition 2.1.2. Let $\phi(s)$ be a Dirichlet series satisfying Definition 2.1.1 with $\Delta(s)=\Gamma(s)$. We say that $\phi(s)$ belongs to the class $\mathcal{A}$ if additionally:

1. $\phi(s)$ and $\psi(s)$ are analytic everywhere in $\mathbb{C}$ except for possible simple poles located at $s=r$, with residues ρ and $\rho^{\star}$ respectively.

Remark 2.1.1. As we have remarked in the previous chapter (see Remark 1.1.1 above), by the functional equation (2.1.31) and the additional condition in the previous definition, $\phi(0)=-\rho^{\star}\Gamma(r)$, while $\psi(0)=-\rho\,\Gamma(r)$. In particular, if $\psi(s)$ is an entire Dirichlet series, then $\phi(0)=0$. It is also clear that $\phi(s)\in\mathcal{A}$ if and only if $\psi(s)\in\mathcal{A}$.

Remark 2.1.2. It is clear from the definition above and the functional equation (2.1.31) that $\phi(-n)=0$ for every $n\in\mathbb{N}$.

Another class of Dirichlet series related to the previous one is given in the following definition, which is a reformulation of the previous definition for Bochner's functional equation (2.1.32).

Definition 2.1.3. Let $\phi(s)$ be a Dirichlet series satisfying Definition 2.1.1 with $\Delta(s)=\Gamma\left(\frac{s+\delta}{2}\right)$, $\delta\in\{0,1\}$, $r=1$ (Bochner class). We say that $\phi(s)$ belongs to the class $\mathcal{B}$ if additionally:

1. For $\delta = 0$, $\phi(s)$ and $\psi(s)$ are analytic everywhere in $\mathbb{C}$ except for possible simple poles located at $s = 1$ with residues ρ and $\rho^{\star}$ respectively.

2. For $\delta = 1$, $\phi(s)$ and $\psi(s)$ can be analytically continued as entire Dirichlet series.

Remark 2.1.3. Note that Definition 2.1.3 mimics in some way the properties that the Dirichlet $L-$functions and the Riemann $\zeta-$function have. Note that $\phi(s) = \pi^{-s/2}\zeta(s)$ is a Dirichlet series belonging to the class $\mathcal{B}$ with $\delta = 0$. By the functional equation for Bochner Dirichlet series (2.1.32), it is effortless to see that $\phi(0) = -\frac{\rho^{\star}}{2}\sqrt{\pi}$, while $\psi(0) = -\frac{\rho}{2}\sqrt{\pi}$. In particular, if $\psi(s)$ is entire then $\phi(0) = 0$.

Remark 2.1.4. If $\phi(s) \in \mathcal{B}$ with $\delta = 0$ and its residue at the simple pole $s = 1$ is ρ, then $\phi'(s) := \phi(2s) \in \mathcal{A}$, having a simple pole located at $s = 1/2$ with residue $\rho/2$. Moreover, $\phi'(s)$ satisfies Hecke's functional equation (2.1.31) with parameter $r = 1/2$. If $\phi(s) \in \mathcal{B}$ with $\delta = 1$, then $\phi'(s) := \phi(2s-1) \in \mathcal{A}$. Since $\phi(s)$ is entire when $\delta = 1$, $\phi'(s)$ will also be entire. Moreover, $\phi'(s)$ satisfies Hecke's functional equation (2.1.31) with parameter $r = 3/2$.

The functional equations (2.1.31) and (2.1.32) can be translated into arithmetical identities involving the sequences $a(n)$ and $b(n)$. From now on, if $\phi(s)$ satisfies Hecke's functional equation in the sense of Definition 2.1.1 and it is representable by the first Dirichlet series in (2.1.29), we shall use $\Phi(x)$ and $\Psi(x)$ to denote the generalized $\theta-$functions (2.1.5)

$$\Phi(x) := \sum_{n=1}^{\infty} a(n)\, e^{-\lambda_n x}, \quad \Psi(x) := \sum_{n=1}^{\infty} b(n)\, e^{-\mu_n x} \quad \mathrm{Re}(x) > 0. \tag{2.1.33}$$

Following Bochner, for $\mathrm{Re}(x) > 0$, let $P(x)$ denote the residual function

$$P(x) = \frac{1}{2\pi i} \int_{C} \Gamma(s)\, \phi(s)\, x^{-s} ds, \tag{2.1.34}$$

where C denotes a curve, or curves, encircling the singularities of $\chi(s)$ given in Definition 2.1.1. It was established in [45] that Hecke's functional equation for $\phi(s)$ and $\psi(s)$, (2.1.31), and the "modular relation"

$$\Phi(x) = x^{-r}\Psi\left(\frac{1}{x}\right) + P(x) \tag{2.1.35}$$

are equivalent. It is interesting to note that the observation of this equivalence has its genesis in Riemann's revolutionary memoir [196], where one of the implications was proved for the first time.

Berndt [[21], [23]] established an expansion involving the modified Bessel function of the second kind which is equivalent to (2.1.35). He proved that, if $x > 0$, the curious formula takes place

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2)^s} = R(s,x) + 2x^{r-s} \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s-r}{2}}\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right), \quad \mathrm{Re}(s) > \sigma_a, \tag{2.1.36}$$

where $R(s,x)$ denotes the sum of residues of the function $\Gamma(w)\, \phi(w)\, \Gamma(s-w)\, x^{2w-2s}$ at the poles of $\Gamma(w)\, \phi(w)$. Since we will generalize (2.1.35) in the next section (see Theorem 2.2.1 and Corollary2.2.2 below), it will be natural to seek a generalization of (2.1.36), which will actually be given in Corollary 2.2.4.

2.1.2 Preliminary results

In several occasions throughout this chapter, we shall need to estimate the asymptotic order of certain integrals involving the Dirichlet series $\phi(s)$ satisfying Definition 2.1.1. To justify most of the steps, we will often invoke the following version of Stirling's formula,

$$\Gamma(\sigma+it)=(2\pi)^{\frac{1}{2}}\,t^{\sigma+it-\frac{1}{2}}\,e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+\frac{1}{12(\sigma+it)}+O\left(\frac{1}{t^{2}}\right)\right), \tag{2.1.37}$$

as $t\to\infty$, uniformly for $-\infty<\sigma_1\leq\sigma\leq\sigma_2<\infty$. A similar formula can be written for $t<0$, i.e., as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s})=\overline{\Gamma(s)}$. Of course, a direct consequence of this exact version is

$$|\Gamma(\sigma+it)|=(2\pi)^{\frac{1}{2}}\,|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right),\quad |t|\to\infty. \tag{2.1.38}$$

To estimate the order of $\phi(s)$ at a generic line $\text{Re}(s)=\sigma$, we shall need a version of the classical Phragmén-Lindelöf theorem that was already explained in the previous chapter, (1.1.27): this version states that, if $\phi(s)$ satisfies Hecke's functional equation (2.1.31) and has abscissa of absolute convergence σ_a, then, as $|t|\to\infty$,

$$\phi(\sigma+it)=O\left(|t|^{\sigma_a+\delta-\sigma}\right),\qquad r-\sigma_a-\delta\leq\sigma\leq\sigma_a+\delta. \tag{2.1.39}$$

The next few lemmas are mainly concerned with the analytic continuation of the Dirichlet series studied in this chapter. We start with $\zeta_\alpha(s)$. Although some basic properties of $\zeta_\alpha(s)$ were already discussed as items in Example 1.3.3 of the previous chapter, here we need more detailed information about this zeta function. The theta function $\vartheta_3(\tau):=\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau}$ is a modular form of weight $\frac{1}{2}$ with a multiplier system with respect to the theta group Γ_θ (see [[137], pp. 15-16]). Therefore, for any $\alpha>0$, $\vartheta_3^\alpha(\tau)$ is a modular form of weight $\alpha/2$ with a multiplier system on the same group.

By definition, $r_\alpha(n)$ are the Fourier coefficients of the expansion of $\vartheta_3(\tau)$ at the cusp $i\infty$, this is,

$$\vartheta_3^\alpha(ix):=\theta^\alpha(x)=1+\sum_{n=1}^{\infty}r_\alpha(n)\,e^{-\pi n\,x},\quad x>0. \tag{2.1.40}$$

It is helpful to see how these coefficients show up: these are computable by using the expansion

$$\begin{aligned}\vartheta_3^\alpha(ix)=\theta^\alpha(x)&=\left(1+2\sum_{n=1}^{\infty}e^{-\pi n^2x}\right)^{\alpha}=\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\left(\sum_{n=1}^{\infty}e^{-\pi n^2x}\right)^{j}\\ &=1+\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\sum_{n_1,\dots,n_j=1}^{\infty}e^{-\pi\left(n_1^2+\dots+n_j^2\right)x}.\end{aligned} \tag{2.1.41}$$

Defining a new summation variable, say $m=n_1^2+...+n_j^2$, we have that the number of $(n_i)_{1\leq i\leq j}$ decomposing n in this way is at most m. Therefore, the sum over j is finite (in fact, a polynomial of degree m in α), so we may define $r_\alpha(m)$ as

$$\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\sum_{n_1,\dots,n_j=1}^{\infty}e^{-\pi\left(n_1^2+\dots+n_j^2\right)x}:=\sum_{m=1}^{\infty}r_\alpha(m)\,e^{-\pi mx}. \tag{2.1.42}$$

The order of growth of $r_\alpha(n)$ as $n \to \infty$ is determined by classical estimates due to Petersson and Lehner [137, 141, 218] for the Fourier coefficients of arbitrary modular forms of positive real weight with multiplier systems. These estimates show that $r_\alpha(n)$ grows polynomially, more precisely in the form

$$r_\alpha(n) \ll_\alpha \begin{cases} n^{\alpha/2-1} & \alpha > 4 \\ n^{\alpha/2-1}\log(n) & \alpha = 4 \\ n^{\alpha/4} & 0 < \alpha < 4 \end{cases}. \tag{2.1.43}$$

See also [[137], p. 18, Theorem 3.3.] for estimates whose constants are independent of α. These bounds determine the existence of a finite abscissa of absolute convergence for $\zeta_\alpha(s)$ given in (2.1.11). Our first lemma concerns the analytic continuation of (2.1.11), which should resemble the analytic continuation of $\zeta_k(s)$. For a proof see [[137], p. 11, Theorem 2.1.].

Lemma 2.1.1. *Let $r_\alpha(n)$ be the sequence defined by (2.1.40). Consider the Dirichlet series:*

$$\zeta_\alpha(s) = \sum_{n=1}^{\infty} \frac{r_\alpha(n)}{n^s}, \quad \mathit{Re}(s) > \sigma_\alpha := \begin{cases} \frac{\alpha}{2} & \alpha \geq 4 \\ 1+\frac{\alpha}{4} & 0 < \alpha < 4 \end{cases}. \tag{2.1.44}$$

Then $\zeta_\alpha(s)$ can be analytically continued to the entire complex plane as a meromorphic function with a simple pole located at $s = \frac{\alpha}{2}$ with residue $\mathit{Res}_{s=\frac{\alpha}{2}}\zeta_\alpha(s) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$. Moreover, it satisfies Hecke's functional equation

$$\pi^{-s}\Gamma(s)\,\zeta_\alpha(s) = \pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_\alpha\left(\frac{\alpha}{2}-s\right). \tag{2.1.45}$$

Our next lemma is due to Paul Epstein [81, 82] and gives the analytic continuation and the functional equation for the Epstein zeta function attached to any positive quadratic form. Before stating it, let $\mathbf{x}$ denote a vector in $\mathbb{R}^n$ and $Q(\mathbf{x})$ a positive definite quadratic form in n variables. We write $Q(\mathbf{x})$ in the matrix form

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\,\mathbf{x}, \tag{2.1.46}$$

where A is a symmetric $n \times n$ matrix of real numbers. Define the discriminant of Q to be

$$D(Q) := \det(A),$$

and the adjoint quadratic form $Q^\dagger(\mathbf{x})$ by

$$Q^\dagger(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A^\dagger\,\mathbf{x}, \tag{2.1.47}$$

with $A^\dagger$ denoting the adjoint matrix of A. The Epstein zeta functions attached to Q are defined by the series

$$\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right) := \sum_{\mathbf{m}\in\mathbb{Z}^n,\,\mathbf{m}+\mathbf{g}\neq\mathbf{0}} \frac{\exp\left(2\pi i\,\mathbf{m}\cdot\mathbf{h}\right)}{Q\left(\mathbf{m}+\mathbf{g}\right)^s}, \quad \mathrm{Re}(s) > \frac{n}{2}, \tag{2.1.48}$$

where the $\mathbf{g}$ and $\mathbf{h}$ are vectors in $\mathbb{R}^n$. The following lemma gives the analytic properties of (2.1.48) (for an even more detailed statement, see [[203], p. 54, Theorem 3]).

Lemma 2.1.2. *The Epstein zeta function $\zeta\left(s,\mathbf{g},\mathbf{h},\,Q\right)$ has an analytic continuation into the entire complex plane as:*

1. *An entire function if $\mathbf{h}\notin\mathbb{Z}^n$;*

2. *A meromorphic function with a simple pole at $s=\frac{n}{2}$ if $\mathbf{h}\in\mathbb{Z}^n$. The residue that $\zeta\left(s,\mathbf{g},\mathbf{h},\,Q\right)$ possesses at $s=\frac{n}{2}$ is given by $\frac{(2\pi)^{\frac{n}{2}}}{\sqrt{D(Q)}\,\Gamma\left(\frac{n}{2}\right)}$.*

Moreover, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ satisfies Hecke's functional equation

$$\left(\frac{2\pi}{D(Q)^{1/n}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)=e^{-2\pi i\mathbf{g}\cdot\mathbf{h}}\left(\frac{2\pi}{D(Q^{\dagger})^{1/n}}\right)^{-\left(\frac{n}{2}-s\right)}\Gamma\left(\frac{n}{2}-s\right)\,\zeta\left(\frac{n}{2}-s,\mathbf{h},-\mathbf{g},Q^{\dagger}\right). \tag{2.1.49}$$

The previous functional equation will be very useful in studying our next result which, although making its first appearance here as a lemma, will be proven in subsection 2.5.2. It concerns the analytic continuation of the series (2.1.22).

Lemma 2.1.3. *Let χ be a primitive Dirichlet character modulo q and set $\delta=0$ if χ is even and $\delta=1$ if χ is odd. If $L_k(s,\chi)$ is the Dirichlet series given by*

$$L_k\left(s,\chi\right):=\sum_{n_1,\ldots,n_k\in\mathbb{Z}}\frac{n_1^{\delta}\,\chi(n_1)\ldots n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+\ldots+n_k^2\right)^s},\quad Re(s)>\frac{k}{2}(1+\delta), \tag{2.1.50}$$

then $L_k(s,\chi)$ can be analytically continued as an entire function and it satisfies the functional equation

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k\left(s,\chi\right)\;=\frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\,\left(\frac{\pi}{q}\right)^{-\left(k(\frac{1}{2}+\delta)-s\right)}\,\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\,L_k\left(k\left(\frac{1}{2}+\delta\right)-s,\,\overline{\chi}\right),$$

where $G(\chi)$ denotes the Gauss sum

$$G(\chi):=\sum_{r=1}^{q-1}\chi(r)\,e^{2\pi i\,r/q}. \tag{2.1.51}$$

We will prove this result on subsection 2.5.2 because we could not track any reference containing it, namely when $\delta=1$. Of course, when $k=1$, (2.1.51) easily reduces to the functional equation for Dirichlet L-functions and for the case $\delta=0$ it actually comes from a more general result of Berndt [29]. For $\delta=1$, however, (2.1.51) seems to be absent in the literature.

To conclude our set of lemmas, let $f(\tau)$ be a cusp form of weight $k\geq 12$ for $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H}:=\{\tau\in\mathbb{C}\,:\,\mathrm{Im}(\tau)>0\}$ with Fourier expansion given by

$$f(\tau)=\sum_{n=1}^{\infty}a_f(n)\,e^{2\pi in\tau}. \tag{2.1.52}$$

Then, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$, $f(\tau)$ satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k}\, f(\tau), \qquad \lim_{\mathrm{Im}(\tau)\to\infty} f(\tau)=0. \tag{2.1.53}$$

The following lemma, due to Wilton [237], gives the analytic continuation of a twisted Dirichlet series attached to $f(\tau)$. We quote Wilton's result in the same form as exposed in Jutila's text [[122], p. 14, Lemma 1.2.].

Lemma 2.1.4. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and let $a_f(n)$ be its Fourier coefficients. Also, assume that p, q are integers such that $(p,q)=1$ and consider the Dirichlet series*

$$L_f(s,p/q) := \sum_{n=1}^{\infty} \frac{a_f(n)\, e^{\frac{2\pi i p}{q}n}}{n^{s}}, \quad Re(s) > \frac{k+1}{2}. \tag{2.1.54}$$

Then $L_f(s,p/q)$ can be analytically continued as an entire function satisfying Hecke's functional equation

$$\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\, L_f\left(s,\frac{p}{q}\right) = (-1)^{k/2}\left(\frac{2\pi}{q}\right)^{-(k-s)}\Gamma(k-s)\, L_f\left(k-s,\, -\overline{p}/q\right), \tag{2.1.55}$$

where $\overline{p}$ is such that $p\,\overline{p} \equiv 1 \mod q$.

Throughout this chapter we will also require the asymptotic formulas for the Bessel functions. It can be seen in [[232], pp. 199-203] that $J_{\nu}(z)$, $I_{\nu}(z)$ and $K_{\nu}(z)$ satisfy the asymptotic expansions

$$\begin{aligned} J_{\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \Bigg\{ &\cos\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\sum_{n=0}^{\infty}\frac{(-1)^n(\nu,2n)}{(2z)^{2n}} \\ &-\sin\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\sum_{n=0}^{\infty}\frac{(-1)^n(\nu,2n+1)}{(2z)^{2n+1}}\Bigg\}, \end{aligned} \tag{2.1.56}$$

$$I_{\nu}(z) \sim \frac{e^{z}}{\sqrt{2\pi z}}\sum_{n=0}^{\infty}\frac{(-1)^n(\nu,n)}{(2z)^n} + \frac{e^{-z\pm(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}}\sum_{n=0}^{\infty}\frac{(\nu,n)}{(2z)^n}, \tag{2.1.57}$$

$$K_{\nu}(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\sum_{n=0}^{\infty}\frac{(\nu,n)}{(2z)^n}. \tag{2.1.58}$$

Formulas (2.1.56) and (2.1.58) are valid in the limit $|z|\to\infty$, $|\arg(z)|<\pi$. The asymptotic expansion (2.1.57) is taken with the plus sign if $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$ and with minus sign if $-\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}$ (see also [[164], pp. 223, 249 and 252, relations 10.7.3, 10.25.3, 10.30.4, 10.30.5]). Here, (ν,n) stands for the Hankel symbol

$$(\nu,n) := \frac{(-1)^n\cos(\pi\nu)}{\pi\, n!}\,\Gamma\left(\frac{1}{2}+\nu+n\right)\,\Gamma\left(\frac{1}{2}-\nu+n\right). \tag{2.1.59}$$

In the sequel, we will employ at some points well-known integral representations for the Bessel functions given above, as well as some expressions involving Kummer's and Gauss' hypergeometric functions. For a matter of clarity in our exposition, we will invoke them solely when they are needed.

2.2 A general theta transformation formula involving Bessel functions

To prove all the theorems given at the introduction, we need to develop a new summation formula which generalizes Jacobi's formula (2.1.8) with respect to the parameter r in the functional equation (2.1.31). In the same way that the "modular relation" (2.1.35) acts as a generalization of (2.1.6), in this section we will try to seek which kind of modular transformation generalizes (2.1.8).

We will restrict ourselves to Dirichlet series belonging to the class $\mathcal{A}$ (see Definition 2.1.2 above) although the study given in this section also works for Dirichlet series with a more general distribution of singularities. We will need to find a way to study a summation formula for

$$\sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\alpha\lambda_n}\, J_{r-1}(\beta\,\sqrt{\lambda_n}), \quad r>0, \tag{2.2.1}$$

where $a(n)$ and λ_n are given in Definition 2.1.1. Here, r is the parameter appearing in Hecke's functional equation (2.1.31).

This study employs an integral representation, known as Hankel's formula (see [[49], p. 8, eq. (15)], [[48], p. 155, eq. (3.10.3.2)] and also [[3], pp. 221-222] for a brief proof),

$$\int_0^{\infty} t^{s-1}e^{-p^2t^2}J_{\nu}(at)\,dt = \frac{\Gamma\left(\frac{s+\nu}{2}\right)a^{\nu}}{2^{\nu+1}p^{s+\nu}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{s+\nu}{2};\,\nu+1;\,-\frac{a^2}{4p^2}\right), \tag{2.2.2}$$

valid for $\mathrm{Re}(s)>-\mathrm{Re}(\nu)$, $\mathrm{Re}(p)>0$, $|\arg(a)|<\pi$. Clearly, for $\nu=r-1$ and $\mathrm{Re}(s)>1-r$, $\mathrm{Re}(\alpha)>0$ and $|\arg(\beta)|<\pi$, the following particular case of the previous formula holds

$$\int_0^{\infty} t^{s-1}e^{-\alpha t}J_{r-1}(\beta\sqrt{t})\,dt = \frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\,{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right). \tag{2.2.3}$$

This suggests the inversion formula, now valid for $\mathrm{Re}(\alpha)>0$, $\beta\in\mathbb{C}$ and $\sigma,\, x>0$,

$$\begin{aligned}\frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma\left(s+\frac{r-1}{2}\right){}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)(\alpha x)^{-s}ds \\ = e^{-\alpha x}J_{r-1}(\beta\sqrt{x}),\end{aligned} \tag{2.2.4}$$

which might be proved by invoking directly the power series expansion of Kummer's function and using the power series for the Bessel function $J_{\nu}(z)$. To represent the series given in (2.2.1), we need an asymptotic formula for the confluent hypergeometric function on the right-hand side of

(2.2.4) which has to be valid when $|s| \to \infty$. Following the reasoning in [[63], p. 379], recall that the Whittaker function $M_{\lambda,\mu}(z)$ has the asymptotic formula [[84], Vol. I, p. 279], [[164], p. 341, eq. (13.21.1)]

$$M_{\lambda,\mu}(z) = \frac{z^{1/4}}{\sqrt{\pi}}\lambda^{-\mu-\frac{1}{4}}\Gamma\left(2\mu+1\right)\cos\left(2\sqrt{\lambda z}-\frac{\pi}{4}-\mu\pi\right)+O\left(|\lambda|^{-\mu-\frac{3}{4}}\right),$$

as $|\lambda| \to \infty$ and z such that $|\arg(\lambda z)| < 2\pi$. Furthermore, by its definition,

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}{}_1F_1\left(\mu-\lambda+\frac{1}{2};\,2\mu+1;\,z\right), \tag{2.2.5}$$

we see that, once we replace λ by $\frac{1}{2}-s$, μ by $\frac{r-1}{2}$ and z by $-\frac{\beta^2}{4\alpha}$,

$$\begin{aligned}{}_1F_1\left(s+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right) &= \frac{e^{-\frac{\beta^2}{8\alpha}}\Gamma(r)}{\sqrt{\pi}}\left(\frac{\beta^2}{4\alpha}\left(s-\frac{1}{2}\right)\right)^{\frac{1}{4}-\frac{r}{2}}\cos\left(\frac{\beta}{\alpha}\sqrt{s-\frac{1}{2}}+\frac{\pi}{4}-\pi\frac{r}{2}\right)\\ &\quad+O\left(|s|^{-\frac{r}{2}-\frac{1}{4}}\right),\end{aligned} \tag{2.2.6}$$

as $|s| \to \infty$ and $\left|\arg\left(\frac{\beta^2}{4\alpha}\left(s-\frac{1}{2}\right)\right)\right| < 2\pi$. For the purposes of this chapter, it will be enough to invoke a simpler bound of the form,

$$\left|{}_1F_1\left(\sigma+it+\frac{r-1}{2};\,r;\,-\frac{\beta^2}{4\alpha}\right)\right| \leq A\,|t|^{\frac{1}{4}-\frac{r}{2}}\,e^{B\sqrt{|t|}}+O\left(|t|^{-\frac{r}{2}-\frac{1}{4}}\right), \quad |t| \to \infty, \tag{2.2.7}$$

where A and B are positive constants depending only on α, β and r and $c \leq \sigma \leq d$.

Since we will use the Mellin inverse representation (2.2.4), the confluent hypergeometric function lying in its kernel will have to satisfy a transformation formula compatible with Hecke's functional equation (2.1.31). The following formula due to Kummer [[3], p. 191, eq. (4.1.11)] will be useful

$${}_1F_1\left(a;\,c;\,x\right) = e^{x}{}_1F_1\left(c-a;\,c;\,-x\right). \tag{2.2.8}$$

2.2.1 General identities for Dirichlet series

We are now ready to prove the following theorem, which has independent interest because it is an identity concerning general Dirichlet series.

Theorem 2.2.1. *Let $\phi(s)$ and $\psi(s)$ be the pair of Dirichlet series satisfying the functional equation (2.1.31). Furthermore, assume that $\phi(s) \in \mathcal{A}$ and let us denote by ρ the residue that $\phi(s)$ possesses at its pole $s = r$.*

Then, for $Re(\alpha) > 0$ and $\beta \in \mathbb{C}$, the following transformation formula holds

$$\begin{aligned}\sum_{n=1}^{\infty} a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\alpha\lambda_n}\,J_{r-1}(\beta\sqrt{\lambda_n}) &= \frac{\phi(0)\beta^{r-1}}{2^{r-1}\Gamma(r)}+\frac{\beta^{r-1}\rho}{2^{r-1}\alpha^{r}}\,e^{-\frac{\beta^2}{4\alpha}}\\ &\quad+\frac{e^{-\frac{\beta^2}{4\alpha}}}{\alpha}\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\mu_n/\alpha}I_{r-1}\left(\frac{\beta\sqrt{\mu_n}}{\alpha}\right).\end{aligned} \tag{2.2.9}$$

Remark 2.2.1. The series on the left-hand side is obviously convergent under the conditions established for the sequences $a(n)$ and λ_n in Definition 2.1.1. The convergence of the series on the right-hand side comes from the Hankel expansion for the modified Bessel function (2.1.57), which gives $|I_\nu(z)| \ll_\nu e^{|\mathrm{Re}(z)|}/\sqrt{2\pi|z|}$ as $|z| \to \infty$.

Proof. Pick $\mu > \max\left\{\sigma_a - \frac{r}{2} + \frac{1}{2}, \sigma_b - \frac{r}{2} + \frac{1}{2}, \frac{r+1}{2}, \frac{3r}{2} - \frac{1}{2}\right\}$. By (2.2.6) and (2.2.4), and arguing by absolute convergence, we can write the series (2.2.1) as

$$\begin{aligned}&\sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\alpha\lambda_n}\, J_{r-1}(\beta\sqrt{\lambda_n})\\ =&\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\, {}_1F_1\left(s+\frac{r-1}{2};\, r;\, -\frac{\beta^2}{4\alpha}\right) ds. \qquad (2.2.10)\end{aligned}$$

We will now integrate along a positively oriented rectangular contour $\mathcal{R}$ containing the vertices $\mu \pm iT$ and $r-\mu \pm iT$ for $T>0$. By the choice of μ, we know that the line $\mathrm{Re}(s) = r-\mu$ is located at the left of the line $\mathrm{Re}(s)=\mu$. By using the residue theorem, we have

$$\begin{aligned}&\int_{\mu-iT}^{\mu+iT} \phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\, {}_1F_1\left(s+\frac{r-1}{2};\, r;\, -\frac{\beta^2}{4\alpha}\right) ds\\ =&\left[\int_{r-\mu+iT}^{\mu+iT} + \int_{r-\mu-iT}^{r-\mu+iT} + \int_{\mu-iT}^{r-\mu-iT}\right] \phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\, {}_1F_1\left(s+\frac{r-1}{2};\, r;\, -\frac{\beta^2}{4\alpha}\right) ds\\ &+2\pi i\sum_{\rho\in\mathcal{R}}\mathrm{Res}_{s=\rho}\left\{\phi\left(s+\frac{r-1}{2}\right)\frac{\Gamma\left(s+\frac{r-1}{2}\right)\beta^{r-1}}{2^{r-1}\alpha^{s+\frac{r-1}{2}}\Gamma(r)}\, {}_1F_1\left(s+\frac{r-1}{2};\, r;\, -\frac{\beta^2}{4\alpha}\right)\right\}. \qquad (2.2.11)\end{aligned}$$

Clearly, by Stirling's formula (2.1.38), the convex estimate (2.1.39) and the asymptotic formula (2.2.6), the integrals over the horizontal segments $[r-\mu \pm iT,\, \mu \pm iT]$ tend to zero as $T \to \infty$. Also, we know that $\phi(s)$ has a simple pole located at $s=r$ and, by Remark 2.1.2, has zeros located at the negative integers. Since ${}_1F_1\left(z;\, r;\, -\frac{\beta^2}{4\alpha}\right)$ is an entire function in the variable z, the only poles that we need to take into account in the sum above are located at the points $\rho = \frac{1-r}{2}$ and $\rho = \frac{r+1}{2}$. Denoting the integrand in (2.2.10) by $\mathcal{I}_{\alpha,\beta}(s)$, we see that the residues at these poles are respectively

$$\mathrm{Res}_{s=\frac{r+1}{2}}\{\mathcal{I}_{\alpha,\beta}(s)\} = \frac{\beta^{r-1}\rho}{2^{r-1}\alpha^r}\, e^{-\frac{\beta^2}{4\alpha}}, \quad \mathrm{Res}_{s=\frac{1-r}{2}}\{\mathcal{I}_{\alpha,\beta}(s)\} = \frac{\phi(0)\beta^{r-1}}{2^{r-1}\Gamma(r)}. \qquad (2.2.12)$$

Letting $T \to \infty$ in (2.2.11), we now need to evaluate the integral

$$\frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}}\frac{1}{2\pi i}\int_{r-\mu-i\infty}^{r-\mu+i\infty}\Gamma\left(s+\frac{r-1}{2}\right)\phi\left(s+\frac{r-1}{2}\right){}_1F_1\left(s+\frac{r-1}{2};r;-\frac{\beta^2}{4\alpha}\right)\alpha^{-s}ds,$$

which we do by appealing to the functional satisfied by $\phi(s)$, (2.1.31), and to Kummer's formula (2.2.8),

$$ {}_1F_1\left(s+\frac{r-1}{2};\, r;\, -\frac{\beta^2}{4\alpha}\right) = e^{-\frac{\beta^2}{4\alpha}}\, {}_1F_1\left(\frac{r+1}{2}-s;\, r;\, \frac{\beta^2}{4\alpha}\right). \tag{2.2.13} $$

Performing such transformations and changing the variable back to the region of absolute convergence of $\psi(s)$ (which is possible due to the fact that we have chosen $\mu > \sigma_b - \frac{r}{2} + \frac{1}{2}$), we get

$$ \begin{aligned} &\frac{1}{\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}} \frac{1}{2\pi i}\int_{r-\mu-i\infty}^{r-\mu+i\infty} \Gamma\left(s+\frac{r-1}{2}\right)\phi\left(s+\frac{r-1}{2}\right) {}_1F_1\left(s+\frac{r-1}{2};r;-\frac{\beta^2}{4\alpha}\right)\alpha^{-s}ds \\ &= \frac{e^{-\frac{\beta^2}{4\alpha}}}{2\pi i\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}} \int_{r-\mu-i\infty}^{r-\mu+i\infty} \Gamma\left(\frac{r+1}{2}-s\right)\psi\left(\frac{r+1}{2}-s\right) {}_1F_1\left(\frac{r+1}{2}-s;r;\frac{\beta^2}{4\alpha}\right)\alpha^{-s}ds \\ &= \frac{e^{-\frac{\beta^2}{4\alpha}}}{2\pi i\Gamma(r)}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}} \alpha^{-r}\int_{\mu-i\infty}^{\mu+i\infty} \Gamma\left(s+\frac{1-r}{2}\right)\psi\left(s+\frac{1-r}{2}\right) {}_1F_1\left(s+\frac{1-r}{2};r;\frac{\beta^2}{4\alpha}\right)\alpha^{s}ds. \end{aligned} \tag{2.2.14} $$

We now use the expansion of $\psi(s)$ as a Dirichlet series (2.1.29) together with the power series expansion for the confluent hypergeometric function. Due to absolute convergence of both series, we can write the right-hand side of (2.2.14) as[5]

$$ \begin{aligned} &e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}} \alpha^{-r}\sum_{n=1}^{\infty} b(n)\mu_n^{\frac{r-1}{2}} \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{\Gamma\left(s+\frac{1-r}{2}\right)}{\Gamma(r)} {}_1F_1\left(s+\frac{1-r}{2};r;\frac{\beta^2}{4\alpha}\right)(\alpha/\mu_n)^s ds \\ &= e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta^2}{4\alpha}\right)^{\frac{r-1}{2}} \alpha^{-r}\sum_{n=1}^{\infty} b(n)\mu_n^{\frac{r-1}{2}} \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\sum_{k=0}^{\infty} \frac{\Gamma\left(s+\frac{1-r}{2}+k\right)}{\Gamma(r+k)k!}\frac{\beta^{2k}}{4^k\alpha^k}(\alpha/\mu_n)^s ds \\ &= e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta}{2}\right)^{r-1} \alpha^{-r}\sum_{n=1}^{\infty} b(n)\sum_{k=0}^{\infty}\frac{1}{\Gamma(r+k)k!}\left(\frac{\beta^2\mu_n}{4\alpha^2}\right)^k \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)(\alpha/\mu_n)^z dz \\ &= e^{-\frac{\beta^2}{4\alpha}}\left(\frac{\beta}{2}\right)^{r-1} \alpha^{-r}\sum_{n=1}^{\infty} b(n)e^{-\mu_n/\alpha}\sum_{k=0}^{\infty}\frac{1}{\Gamma(r+k)k!}\left(\frac{\beta\sqrt{\mu_n}}{2\alpha}\right)^{2k}. \end{aligned} \tag{2.2.15} $$

Recalling the definition of the modified Bessel function of the first kind (2.1.17) (cf. [[221], p. 233, eq. (9.28)]),

$$ \left(\frac{x}{2}\right)^{-\nu} I_\nu(x) = \sum_{m=0}^{\infty}\frac{1}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m}, $$

we see immediately that the latter infinite series in (2.2.15) is represented by the modified Bessel function of the first kind. This proves (2.2.9). □

Remark 2.2.2. Using Voronoï's summation formula, Berndt proved the summation formula (2.2.9) for Dirichlet series with a more general distribution of singularities (cf. [[24], p. 154, eq. (6.12)]). Our identity is also connected with an identity due to A. I. Popov [168], which was rigorously

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[5] Note that, at the third step, the vertical line of integration is unchanged due to Cauchy's theorem and the choice of μ.

proved for the first time in [42]. The advantage of our method of establishing the transformation formula (2.2.9) is that we can relate it with a contour integral involving $\Gamma(s)\phi(s)$ on the critical line $\mathrm{Re}(s) = \frac{r}{2}$ (see Corollary 2.2.2 below).

Since the class $\mathcal{B}$ can be included in the class $\mathcal{A}$, i.e., if $\phi(s) \in \mathcal{B}$ then $\phi(2s-\delta) \in \mathcal{A}$ (see Remark 2.1.4), we can deduce the following formula.

Corollary 2.2.1. *Let $\phi(s)$ be a Dirichlet series representable as (2.1.29) and belonging to the class $\mathcal{B}$ with $\delta = 0$. Then, for any $x > 0$ and $Re(\alpha) > 0$, $\beta \in \mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2}\cos(\beta\lambda_n x) = \phi(0) + \sqrt{\frac{\pi}{\alpha}}\frac{\rho}{2x}\, e^{-\frac{\beta^2}{4\alpha}} + \frac{e^{-\frac{\beta^2}{4\alpha}}}{\sqrt{\alpha}x}\sum_{n=1}^{\infty} b(n)\, e^{-\frac{\mu_n^2}{\alpha x^2}}\cosh\left(\frac{\beta\,\mu_n}{\alpha x}\right). \quad (2.2.16)$$

Moreover, if $\phi(s)$ belongs to the class $\mathcal{B}$ with $\delta = 1$, the analogous formula takes place

$$\sum_{n=1}^{\infty} a(n)\, e^{-\alpha\lambda_n^2 x^2}\sin(\beta\lambda_n x) = \frac{e^{-\frac{\beta^2}{4\alpha}}}{\sqrt{\alpha}x}\sum_{n=1}^{\infty} b(n)\, e^{-\frac{\mu_n^2}{\alpha x^2}}\sinh\left(\frac{\beta\mu_n}{\alpha\, x}\right). \quad (2.2.17)$$

Proof. Let $\phi'(s) = \phi(2s-\delta)$, $\delta \in \{0,1\}$. Then $\phi'(s)$ satisfies Hecke's functional equation

$$\Gamma(s)\,\phi'(s) = \Gamma\left(\frac{1}{2}+\delta-s\right)\,\psi'\left(\frac{1}{2}+\delta-s\right) \quad (2.2.18)$$

and belongs to the class $\mathcal{A}$, with (at most) a simple pole located at $s = \frac{1}{2}$ (if $\delta = 0$) with residue $\rho/2$ (see Remark 2.1.4 above).

In general, if $\phi(s)$ and $\psi(s) \in \mathcal{B}$ are representable as (2.1.29), $\phi'(s)$ and $\psi'(s)$ can be written in the form

$$\phi'(s) = \sum_{n=1}^{\infty}\frac{a(n)\,\lambda_n^{\delta}}{\lambda_n^{2s}}, \quad \psi'(s) = \sum_{n=1}^{\infty}\frac{b(n)\,\mu_n^{\delta}}{\mu_n^{2s}}, \quad \mathrm{Re}(s) > \frac{\sigma_a}{2}.$$

From (2.2.18), we can apply the summation formula (2.2.9) replacing there $a(n)$ by $a(n)\,\lambda_n^{\delta}$, λ_n by λ_n^2, $b(n)$ by $b(n)\,\mu_n^{\delta}$, μ_n by μ_n^2 and finally ρ by $\rho/2$: this gives

$$\sum_{n=1}^{\infty} a(n)\lambda_n^{\frac{1}{2}}e^{-\alpha\lambda_n^2}J_{\delta-\frac{1}{2}}(\beta\lambda_n) = \sqrt{\frac{2}{\pi\beta}}\left\{\phi(0) + \frac{\rho}{2}\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\beta^2}{4\alpha}}\right\}(1-\delta) + \frac{e^{-\frac{\beta^2}{4\alpha}}}{\alpha}\sum_{n=1}^{\infty} b(n)\mu_n^{\frac{1}{2}}e^{-\mu_n^2/\alpha}I_{\delta-\frac{1}{2}}\left(\frac{\beta\mu_n}{\alpha}\right). \quad (2.2.19)$$

Using now the particular values for the Bessel functions [[221], p. 248],

$$J_{\delta-\frac{1}{2}}(x) = \begin{cases}\sqrt{\frac{2}{\pi x}}\cos(x) & \text{if } \delta = 0\\ \sqrt{\frac{2}{\pi x}}\sin(x) & \text{if } \delta = 1\end{cases}, \quad I_{\delta-\frac{1}{2}}(x) = \begin{cases}\sqrt{\frac{2}{\pi x}}\cosh(x) & \text{if } \delta = 0\\ \sqrt{\frac{2}{\pi x}}\sinh(x) & \text{if } \delta = 1\end{cases}, \quad (2.2.20)$$

we obtain immediately the identities (2.2.16) and (2.2.17) after replacing α and β in (2.2.19) respectively by αx^2 and βx, $x > 0$. □

Remark 2.2.3. In particular, since $\phi(s) := \pi^{-s/2}\zeta(s) \in \mathcal{B}$ by Remark 2.1.3, an application of the identity (2.2.16) gives Jacobi's formula (2.1.8). This seems to be the first indication that the general formula (2.2.9) will prove to be the rightful analogue of (2.1.8).

Identities (2.2.16) and (2.2.17) can be employed in order to derive generalizations of the Selberg-Chowla formula for Dirichlet series in the class $\mathcal{B}$ (cf. Remark 2.2.5 below). Based on the "theta-like" features of the series appearing in (2.2.16) and (2.2.17), we now give a definition which introduces a new general analogue of the theta function.

Definition 2.2.1. Let $\text{Re}(x) > 0$, $y \in \mathbb{C}$ and assume that $\phi(s)$ is a Dirichlet series satisfying Hecke's functional equation (2.1.31). We define the generalized $\theta-$function as the following series

$$\Phi(x,y) = 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x}\, J_{r-1}\left(\sqrt{\lambda_n}\, y\right). \tag{2.2.21}$$

Our next corollary, which actually consists in rewriting (2.2.9) in a compact form, gives a transformation formula for $\Phi(x,y)$. Moreover, just like Dixit's formula (2.1.9), we are able to compare this modular relation with an integral involving $\phi(s)$ and the confluent hypergeometric function.

Corollary 2.2.2. *Let $Re(x) > 0$, $y \in \mathbb{C}$ and assume that $\phi(s) \in \mathcal{A}$. Let us consider the generalized $\theta-$functions*

$$\Phi(x,y) := 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x}\, J_{r-1}(\sqrt{\lambda_n}\, y) \tag{2.2.22}$$

and

$$\Psi\left(x,y\right) := 2^{r-1}\Gamma(r)\, y^{1-r} \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{1-r}{2}}\, e^{-\mu_n x}\, J_{r-1}\left(\sqrt{\mu_n}\, y\right). \tag{2.2.23}$$

Then $\Phi(x,y)$ (resp. $\Psi(x,y)$) has the following properties:

1. *It is entire in $y \in \mathbb{C}$ and analytic in $Re(x) > 0$;*

2. *It satisfies the transformation formula*

$$\Phi(x,y) = \phi(0) + \frac{\rho}{x^r}\Gamma(r)\, e^{-\frac{y^2}{4x}} + \frac{e^{-\frac{y^2}{4x}}}{x^r}\,\Psi\left(\frac{1}{x},\, \frac{iy}{x}\right). \tag{2.2.24}$$

Moreover, (2.2.24) can be written in terms of an integral involving $\phi(s)$ at its critical line, this is,

$$\begin{aligned} x^{r/2}\Phi\left(x,y\right) - \frac{\rho\Gamma(r)}{x^{r/2}}\, e^{-\frac{y^2}{4x}} &= e^{-\frac{y^2}{4x}} x^{-r/2}\Psi\left(\frac{1}{x},\, i\,\frac{y}{x}\right) + \phi(0)\, x^{r/2} \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma\left(\frac{r}{2}+it\right)\, \phi\left(\frac{r}{2}+it\right) {}_1F_1\left(\frac{r}{2}+it;\, r;\, -\frac{y^2}{4x}\right) x^{-it}\, dt. \end{aligned} \tag{2.2.25}$$

Proof. We first show that $\Phi(x,y)$ satisfies item 1., i.e., that is entire in $y \in \mathbb{C}$ and analytic in the half-plane $\text{Re}(x) > 0$. We divide this proof in two cases: $r > \frac{1}{2}$ or $0 < r \leq \frac{1}{2}$. For the first case, if $\text{Re}(x) = \sigma \geq \sigma_0 > 0$ and y belongs to a bounded subset of $\mathbb{C}$, contained in $|y| \leq M$ say, then the series defining $\Phi(x,y)$ is absolutely and uniformly convergent, since, for $r>\frac{1}{2}$,

$$\begin{aligned}|\Phi(x,y)| &\leq \Gamma(r)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left|\left(\frac{\sqrt{\lambda_n}\,y}{2}\right)^{1-r}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)\right| \\ &\leq \frac{2\Gamma(r)}{\sqrt{\pi}\Gamma\left(r-\frac{1}{2}\right)}\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\int_0^1\left(1-t^2\right)^{r-\frac{3}{2}}\left|\cos\left(\sqrt{\lambda_n}\,y\,t\right)\right|dt \\ &\leq \sum_{n=1}^{\infty}|a(n)|\,\exp\left(-\lambda_n\sigma_0+\sqrt{\lambda_n}\,M\right)<\infty,\end{aligned}$$

where in the second inequality we have used the well-known Fourier representation (due to Poisson) [[164], p. 224, eq. (10.9.4)]

$$\left(\frac{z}{2}\right)^{-\nu}J_{\nu}(z)=\frac{2}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1\left(1-t^2\right)^{\nu-\frac{1}{2}}\cos(zt)\,dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\ z\in\mathbb{C}, \tag{2.2.26}$$

as well as the fact that the Dirichlet series $\phi(s)$ converges in some half-plane. Note that (2.2.26) can only be invoked for $J_{r-1}(z)$ if $r>\frac{1}{2}$. To address the case where $0<r\leq\frac{1}{2}$, we just bound trivially $\Phi(x,y)$, which gives

$$\begin{aligned}|\Phi(x,y)| &\leq \Gamma(r)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left|\left(\frac{\sqrt{\lambda_n}\,y}{2}\right)^{1-r}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)\right| \\ &\leq \Gamma(r)\sum_{n=1}^{\infty}|a(n)|e^{-\lambda_n\sigma_0}\left(\frac{\sqrt{\lambda_n}\,|y|}{2}\right)^{1-r}I_{r-1}\left(\sqrt{\lambda_n}\,|y|\right)<\infty,\end{aligned}$$

where we have used the well-known bound $|J_{\nu}(z)|\leq I_{\nu}(|z|)$, valid for $\nu\in\mathbb{R}$. The finitude of the last series comes from the asymptotic estimate (2.1.57) for $I_{\nu}(x)$ [[164], p. 252, eq. (10.30(ii))], which gives

$$\left(\frac{\sqrt{\lambda_n}\,|y|}{2}\right)^{1-r}I_{r-1}\left(\sqrt{\lambda_n}\,|y|\right)\ll \exp\left(\sqrt{\lambda_n}\,M\right)\,\lambda_n^{\frac{1}{4}-\frac{r}{2}}|y|^{\frac{1}{2}-r},\quad n\to\infty. \tag{2.2.27}$$

Therefore, for each fixed x in the half-plane $\mathrm{Re}(x)>0$, the function $\Phi(x,\cdot)$ is entire and for each fixed $y\in\mathbb{C}$ the function $\Phi(\cdot,y)$ is holomorphic in the right half-plane $\mathrm{Re}(x)>0$. This proves item 1. above. Now, we prove the integral representation (2.2.25), which immediately gives (2.2.24). We start our proof with the integral on the right-hand side of (2.2.25), transforming it into a contour integral along the critical line $\mathrm{Re}(s) = \frac{r}{2}$,

$$\frac{y^{r-1}x^{-r/2}}{2^r\pi\,\Gamma(r)}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt$$
$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\frac{1}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(s;\,r;\,-\frac{y^2}{4x}\right)x^{-s}\,ds$$
$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds,$$

where we have used Kummer's formula (2.2.13). We now change the line of integration to $\mathrm{Re}(s) = \sigma_a+\delta$, for some positive δ. Doing so we integrate over a (positively oriented) rectangular contour whose vertices are $\frac{r}{2}\pm iT$ and $\sigma_a+\delta\pm iT$, where T is a (sufficiently large) positive real number. The integrals along the horizontal segments vanish as $T\to\infty$ due to the asymptotic formulas (2.1.38) and (2.2.6). Doing this shift and using the information that the simple pole of $\phi(s)$ is located at $s=r$, we find the residue

$$\mathrm{Res}_{s=r}\left\{\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\right\}=\Gamma(r)\rho\,x^{-r}.$$

Since $\phi(s)$ converges absolutely for $\mathrm{Re}(s)>\sigma_a$, we have by absolute convergence that

$$\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\frac{r}{2}-i\infty}^{\frac{r}{2}+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds$$
$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\int_{\sigma_a+\delta-i\infty}^{\sigma_a+\delta+i\infty}\Gamma(s)\,\phi(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)x^{-s}\,ds-\frac{y^{r-1}\rho}{2^{r-1}x^r}\,e^{-\frac{y^2}{4x}}$$
$$=\frac{y^{r-1}}{2^{r-1}\,\Gamma(r)}\frac{e^{-\frac{y^2}{4x}}}{2\pi i}\sum_{n=1}^{\infty}a(n)\int_{\sigma_a+\delta-i\infty}^{\sigma_a+\delta+i\infty}\Gamma(s)\,{}_1F_1\left(r-s;\,r;\,\frac{y^2}{4x}\right)(x\lambda_n)^{-s}\,ds-\frac{y^{r-1}\rho}{2^{r-1}x^r}\,e^{-\frac{y^2}{4x}}$$
$$=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)-\frac{y^{r-1}\rho}{2^{r-1}x^r}\,e^{-\frac{y^2}{4x}},$$

where in the last step we have used the integral representation (2.2.4). This proves the formula

$$\frac{y^{r-1}x^{-r/2}}{2^r\pi\,\Gamma(r)}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt$$
$$=\sum_{n=1}^{\infty}a(n)\,e^{-\lambda_n x}J_{r-1}\left(\sqrt{\lambda_n}\,y\right)-\frac{y^{r-1}\rho}{2^{r-1}x^r}\,e^{-\frac{y^2}{4x}}. \qquad (2.2.28)$$

After multiplying both sides of (2.2.28) by the factor $\Gamma(r)\, 2^{r-1} y^{1-r} x^{r/2}$ and appealing to the definition of the generalized theta function (2.2.22), we derive

$$x^{r/2}\Phi\left(x,y\right)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{y^2}{4x}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt. \quad (2.2.29)$$

Let us now look at the integral on the right-hand side of (2.2.29): by the functional equation for $\phi(s)$, (2.1.31), and Kummer's formula (2.2.8), we can rewrite it as

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}-it;\,r;\,-\frac{y^2}{4x}\right)x^{it}\,dt\\&=\frac{e^{-\frac{y^2}{4x}}}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt. \qquad (2.2.30)\end{aligned}$$

But the last integral in (2.2.30) is actually the same as the one on the right-hand side of (2.2.29) if we replace there x by $1/x$, y by iy/x and $\phi(s)$ by $\psi(s)$: hence, (2.2.29) implies

$$\begin{aligned}&x^{-r/2}\Psi\left(\frac{1}{x},\,i\,\frac{y}{x}\right)-\rho^{\star}\Gamma(r)\,x^{r/2}\,e^{\frac{y^2}{4x}}\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt, \qquad (2.2.31)\end{aligned}$$

where $\rho^{\star}$ is the residue that $\psi(s)$ possesses at its simple pole at $s=r$. From the simple properties of the class $\mathcal{A}$ (see Remark 2.1.1 above), we know that $\rho^{\star}\Gamma(r)=-\phi(0)$, which gives, after using (2.2.31), (2.2.30) and (2.2.29),

$$\begin{aligned}&e^{-\frac{y^2}{4x}}x^{-r/2}\Psi\left(\frac{1}{x},\,i\,\frac{y}{x}\right)+\phi(0)\,x^{r/2}\\&=\frac{e^{-\frac{y^2}{4x}}}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\psi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,\frac{y^2}{4x}\right)x^{it}\,dt\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{y^2}{4x}\right)x^{-it}\,dt\\&=x^{r/2}\Phi\left(x,y\right)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{y^2}{4x}},\end{aligned}$$

completing our proof. □

Since the previous "modular" relation (2.2.24) is an analogue of Bochner's "modular" relation (2.1.35), in the next corollary we note that (2.1.35) can indeed be obtained from (2.2.9).

Corollary 2.2.3. *Assume that $\phi(s) \in \mathcal{A}$. Then, for any Re$(x) > 0$, Bochner's formula (2.1.35) takes place, this is,*

$$\sum_{n=1}^{\infty} a(n)\, e^{-\lambda_n x} = \phi(0) + \rho \Gamma(r)\, x^{-r} + x^{-r} \sum_{n=1}^{\infty} b(n)\, e^{-\mu_n / x}. \tag{2.2.32}$$

Proof. The transformation formula (2.2.24) implies, for $\text{Re}(x) > 0$ and $y \in \mathbb{C}$,

$$\Gamma(r)\, 2^{r-1} y^{1-r} \sum_{n=1}^{\infty} a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n x}\, J_{r-1}(\sqrt{\lambda_n}\, y) = \phi(0) + \frac{\rho}{x^r} \Gamma(r)\, e^{-\frac{y^2}{4x}}$$
$$+ \frac{e^{-\frac{y^2}{4x}}}{x}\, \Gamma(r)\, 2^{r-1} y^{1-r} \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{1-r}{2}}\, e^{-\frac{\mu_n}{x}}\, I_{r-1}\left(\frac{\sqrt{\mu_n} y}{x}\right).$$

We have seen above that the series on both sides converge absolutely and uniformly with respect to y contained in any bounded subset of $\mathbb{C}$. Therefore, we can take $y \to 0^+$ and interchange the orders of limit and summation on both sides. From the limiting relations for the Bessel functions [[164], p. 223, eq. (10.7.3)],

$$\lim_{y \to 0} y^{-\nu} J_\nu(y) = \frac{2^{-\nu}}{\Gamma(\nu+1)}, \quad \lim_{y \to 0} y^{-\nu} I_\nu(y) = \frac{2^{-\nu}}{\Gamma(\nu+1)}, \tag{2.2.33}$$

(2.2.32) is immediately obtained. □

By the equivalence between the identities (2.1.35), (2.1.36) and Hecke's functional equation, the fact that (2.2.24) generalizes (2.1.35) now poses an interesting question: is it possible to find a generalization of (2.1.36) in this context? The following corollary furnishes a positive answer to this question. Before stating it, however, let us briefly explain which kind of generalization we shall get. First, by analogy with the generalized θ-function (2.2.23), one may foresee that the right-hand side of (2.1.36),

$$\sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s-r}{2}}\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right),$$

will be extended to a series of the form

$$\sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s+1}{2}-r}\, J_{r-1}(2\sqrt{\mu_n}\, y)\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right). \tag{2.2.34}$$

In the same manner, the left-hand side of (2.1.36),

$$\sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n + x^2)^s},$$

will be generalized in our next result. In order to understand this generalization, however, we need to introduce a basic special function that was not explicitly defined before in this thesis. Like Kummer's function (2.1.2), the Gauss hypergeometric function, ${}_2F_1(a, b; c; z)$, is usually encountered as the infinite series

$$ {}_2F_1\left(a,b;c;z\right) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!}\, z^k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}\, z^k, \quad |z|<1, \tag{2.2.35} $$

where $(a)_k$ denotes the Pochhammer symbol (2.1.3). The next corollary establishes a correspondence between the series (2.2.34) and another infinite series involving the function ${}_2F_1(a,b;c;z)$.

Corollary 2.2.4. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 2.1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Furthermore, let x, y be two positive real numbers and s be any complex number such that $Re(s)>\sigma_a$.*

Then the following identity holds

$$ \begin{aligned} &\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\, y^{r-1}}{\Gamma(r)} \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n+x^2+y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, r;\, \frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right) \\ &= \frac{\rho\, y^{r-1}\, x^{r-s}\, \Gamma(s-r)}{2} + \frac{x^{s-r} y^{r-1} \phi(0)\, \Gamma(s)}{2\Gamma(r)\, (x^2+y^2)^s} + \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s+1}{2}-r} J_{r-1}(2\sqrt{\mu_n}\, y)\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right), \end{aligned} \tag{2.2.36} $$

where ${}_2F_1(a,b;c;z)$ denotes Gauss' hypergeometric function, (2.2.35).

Moreover, if x and y are two positive numbers such that $x>y$ and $Re(s)>\sigma_a$, then the analogous identity holds

$$ \begin{aligned} &\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\, y^{r-1}}{\Gamma(r)} \sum_{n=1}^{\infty} \frac{a(n)}{(\lambda_n+x^2-y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, r;\, -\frac{4y^2\lambda_n}{(\lambda_n+x^2-y^2)^2}\right) \\ &= \frac{\rho\, y^{r-1} x^{r-s}\, \Gamma(s-r)}{2} + \frac{x^{s-r} y^{r-1} \phi(0)\, \Gamma(s)}{2\Gamma(r)\, (x^2-y^2)^s} + \sum_{n=1}^{\infty} b(n)\, \mu_n^{\frac{s+1}{2}-r} I_{r-1}(2\sqrt{\mu_n}\, y)\, K_{s-r}\left(2\sqrt{\mu_n}\, x\right). \end{aligned} \tag{2.2.37} $$

Proof. Since $I_\nu(x)=i^{-\nu}J_\nu(ix)$ for $x>0$, it is simple to see that (2.2.37) can be derived from (2.2.36) once we assure the convergence of both sides of (2.2.37). Indeed, the series on the right-hand side of (2.2.37) converges absolutely for any $s\in\mathbb{C}$ because of the fact that $x>y>0$ and the asymptotic formulas (2.1.57) and (2.1.58) [[232], pp. 199-203],

$$ K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1+O\left(\frac{1}{x}\right)\right), \quad I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1+O\left(\frac{1}{x}\right)\right), \quad x\to\infty. \tag{2.2.38} $$

To justify the convergence of the series on the left-hand side of (2.2.37), note that, as $n\to\infty$,

$$ \frac{1}{(\lambda_n+x^2-y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, r;\, -\frac{4y^2\lambda_n}{(\lambda_n+x^2-y^2)^2}\right) \sim \frac{1}{(\lambda_n+x^2-y^2)^s}, $$

since the hypergeometric function factor tends to 1 as $\lambda_n\to\infty$, which is evident from the power series (2.2.35). Thus, if $\mathrm{Re}(s)>\sigma_a$, the series on the left of (2.2.37) converges absolutely. Similar comments can be made about the series on both sides of (2.2.36) without the additional hypothesis that $x>y$.

From the previous observation, we just need to prove the identity (2.2.36). To that end, we look first at the infinite series on its right-hand side and invoke the well-known representation for the modified Bessel function of the second kind [[95], p. 368, eq. (3.471.9)],

$$\int_0^{\infty} x^{s-1}e^{-\beta x}\,e^{-\frac{\gamma}{x}}\,dx=2\left(\frac{\gamma}{\beta}\right)^{s/2}\,K_s\left(2\sqrt{\beta\,\gamma}\right),\quad \text{Re}(\beta),\,\text{Re}(\gamma)>0. \tag{2.2.39}$$

Using (2.2.39) on the infinite series appearing on the right side of (2.2.36), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{s+1-2r}{2}}\,J_{r-1}(2\sqrt{\mu_n}\,y)\,K_{s-r}\left(2\sqrt{\mu_n}\,x\right)\\&=\frac{x^{s-r}}{2}\int_0^{\infty} t^{s-r-1}e^{-x^2t}\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\frac{\mu_n}{t}}\,J_{r-1}\left(2\sqrt{\mu_n}\,y\right)\,dt,\end{aligned}$$

where interchanging the orders of summation and integration is possible due to absolute convergence. Now we invoke the transformation formula (2.2.9) with the roles of $\phi(s)$ and $\psi(s)$ being reversed and we get

$$\begin{aligned}&\sum_{n=1}^{\infty} b(n)\,\mu_n^{\frac{s+1-2r}{2}}\,J_{r-1}(2\sqrt{\mu_n}\,y)\,K_{s-r}\left(2\sqrt{\mu_n}\,x\right)\\&=\frac{x^{s-r}}{2}\int_0^{\infty} t^{s-r-1}e^{-x^2t}\left\{\frac{\psi(0)y^{r-1}}{\Gamma(r)}+y^{r-1}\rho^{\star}\,t^r\,e^{-y^2t}\right\}\,dt\\&+\frac{x^{s-r}}{2}\int_0^{\infty} t^{s-r-1}e^{-x^2t}\left\{t\,e^{-y^2t}\sum_{n=1}^{\infty} a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n t}\,I_{r-1}\left(2y\,t\,\sqrt{\lambda_n}\right)\right\}\,dt.\end{aligned} \tag{2.2.40}$$

The computation of the first integral is simple and it is equal to[6]

$$\begin{aligned}\frac{x^{s-r}}{2}\int_0^{\infty} t^{s-r-1}e^{-x^2t}&\left\{\frac{\psi(0)y^{r-1}}{\Gamma(r)}+y^{r-1}\rho^{\star}\,t^r\,e^{-y^2t}\right\}\,dt\\&=\frac{\psi(0)\,y^{r-1}\,x^{r-s}\,\Gamma(s-r)}{2\Gamma(r)}+\frac{x^{s-r}y^{r-1}\rho^{\star}\,\Gamma(s)}{2\,(x^2+y^2)^s}\\&=-\frac{\rho\,y^{r-1}\,x^{r-s}\,\Gamma(s-r)}{2}-\frac{x^{s-r}y^{r-1}\phi(0)\,\Gamma(s)}{2\Gamma(r)\,\left(x^2+y^2\right)^s},\end{aligned} \tag{2.2.41}$$

where we have used the simple properties of the class $\mathcal{A}$, namely that $\psi(0)=-\rho\,\Gamma(r)$ and $\rho^{\star}=-\phi(0)/\Gamma(r)$ (see Remark 2.1.1). To evaluate the second integral, we use absolute convergence

[6] In order to check the convergence of this integral, note that, if $\rho\neq 0$, then it comes from Definition 2.1.2 that $\phi(s)$ possesses a simple pole located at $s=r$. This means that $\sigma_a\geq r$ and so, since $\text{Re}(s)>\sigma_a$ by hypothesis of Corollary 2.2.4, then $\text{Re}(s)>r$. On the other hand, if $\phi(s)$ is entire then it comes from Remark 2.1.1 that $\psi(0)=0$, which shows that the term involving the integral $\int_0^{\infty} t^{s-r-1}e^{-x^2t}dt$ is zero.

(assured by item 1. of Corollary 2.2.2 above) and the integral given in [[48], p. 191, eq. (3.13.2.1)],

$$\int_0^\infty x^{z-1}\, e^{-Ax} I_\nu(Bx)\, dx = A^{-z-\nu}\left(\frac{B}{2}\right)^\nu \frac{\Gamma(z+\nu)}{\Gamma(\nu+1)}\, {}_2F_1\left(\frac{z+\nu}{2},\, \frac{z+\nu+1}{2};\, \nu+1;\, \frac{B^2}{A^2}\right), \tag{2.2.42}$$

valid for $\mathrm{Re}(A) > |\mathrm{Re}(B)|$ and $\mathrm{Re}(z) > -\mathrm{Re}(\nu)$. Invoking (2.2.42) on the last integral in (2.2.40), we obtain

$$\begin{aligned}
&\int_0^\infty t^{s-r} e^{-x^2 t}\, e^{-y^2 t} \sum_{n=1}^\infty a(n)\, \lambda_n^{\frac{1-r}{2}}\, e^{-\lambda_n t}\, I_{r-1}\left(2y\, t\, \sqrt{\lambda_n}\right)\, dt \\
&= \sum_{n=1}^\infty a(n)\, \lambda_n^{\frac{1-r}{2}} \int_0^\infty t^{s-r}\, e^{-\left(x^2+y^2+\lambda_n\right)t}\, I_{r-1}\left(2yt\,\sqrt{\lambda_n}\right)\, dt \\
&= \frac{\Gamma(s)\, y^{r-1}}{\Gamma(r)} \sum_{n=1}^\infty \frac{a(n)}{(\lambda_n+x^2+y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, r;\, \frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right),
\end{aligned} \tag{2.2.43}$$

which is legitimate through (2.2.42) because $\mathrm{Re}(s) > \sigma_a \geq \frac{r}{2} > 0$ by hypothesis and $x^2+y^2+\lambda_n > y^2+\lambda_n > 2y\sqrt{\lambda_n}$. Combining (2.2.43) with (2.2.40) and (2.2.41) proves our formula (2.2.36). $\square$

Remark 2.2.4. A formula similar to (2.2.37) involving the product $I_\nu(y)\, K_\nu(z)$ was found by Berndt, Dixit, Kim and Zaharescu in [43] and later generalized to a class of Dirichlet series by three of the authors and Gupta in [[41], Theorem 11.1]. Restricted to the case where the Dirichlet series is attached to the sum of $k-$squares, $\zeta_k(s)$, their identity reads [[43], p. 315, Theorem 1.6.]

$$\begin{aligned}
&\sum_{n=1}^\infty r_k(n)\, I_\nu\left(\pi\sqrt{n}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right)\, K_\nu\left(\pi\sqrt{n}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right) \\
&= -\frac{1}{2\nu}\left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right)^\nu + \frac{\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}2^{k-1}\Gamma(\nu+1)} \sum_{n=0}^\infty \frac{r_k(n)}{\sqrt{n+\alpha}\,\sqrt{n+\beta}}\left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^\nu \\
&\times\left(\frac{1}{\sqrt{n+\alpha}}+\frac{1}{\sqrt{n+\beta}}\right)^{k-2} {}_2F_1\left(1-\frac{k}{2}+\nu,\, 1-\frac{k}{2};\, \nu+1;\, \left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^2\right),
\end{aligned} \tag{2.2.44}$$

where $\mathrm{Re}(\sqrt{\alpha}) \geq \mathrm{Re}(\sqrt{\beta}) > 0$ and $\mathrm{Re}(\nu) > 0$. Compare formula (2.2.44) with (2.2.75) below. Our identity (2.2.37) and its consequences are somewhat akin to (2.2.44) but they seem to be independent because in (2.2.44) it is required that the indices of the Bessel functions are the same. In our formula (2.2.37) we have a free complex parameter s, although the index of the modified Bessel function $I_\nu(x)$ is fixed by the functional equation, being equal to $r-1$. In the case of (2.2.44) the indices of the Bessel functions are kept equal but they are independent of the functional equation for $\zeta_k(s)$, this is, they do not depend on k. In the fourth chapter of this thesis, we will make this connection more precise, as we shall prove a general summation formula from which (2.2.44) and (2.2.75) arise as corollaries (see Theorem 4.4.1 below).

The formula obtained by Berndt, Dixit, Kim and Zaharescu (2.2.44) can be used to extend known formulas due to Dixon and Ferrar [75, 77, 78]. For example, by using (2.2.44), they were able to establish the beautiful identity due to Popov (see [[43], p. 329, Corollary 4.6.])

$$\frac{\beta^{\nu/2}\Gamma\left(\nu+\frac{k}{2}\right)}{2\pi^{\nu+\frac{k}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+\beta)^{\nu+\frac{k}{2}}}=\sum_{n=0}^{\infty}r_k(n)\,n^{\nu/2}K_{\nu}\left(2\pi\sqrt{n\beta}\right), \tag{2.2.45}$$

which is valid for any positive integer $k>1$ and $\mathrm{Re}(\sqrt{\beta})$, $\mathrm{Re}(\nu)>0$. On the other hand, (2.2.45) can be established immediately from Berndt's general Bessel expansion (2.1.36) because $\zeta_k(s)$ satisfies Hecke's functional equation.

In a general context, we now prove that (2.2.36) implies (2.1.36) in the same way that the similar formula (2.2.44) gives (2.2.45) (cf. [38]).

Corollary 2.2.5. *Let Re(s) $>\sigma_a$ and $\phi(s)$ be a Dirichlet series belonging to the class $\mathcal{A}$. Then, for any $x>0$, the following Bessel expansion takes place*

$$\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2)^s}=\frac{\rho\,\Gamma(r)\,x^{2r-2s}\,\Gamma(s-r)}{\Gamma(s)}+x^{-2s}\phi(0)+\frac{2x^{r-s}}{\Gamma(s)}\sum_{m=1}^{\infty}b(m)\,\mu_m^{\frac{s-r}{2}}\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right). \tag{2.2.46}$$

Proof. Multiply both sides of (2.2.36) by $\Gamma(r)\,y^{1-r}$: we obtain

$$\Gamma(r)\,y^{1-r}\sum_{m=1}^{\infty}b(m)\,\mu_m^{\frac{s+1-2r}{2}}\,J_{r-1}(2\sqrt{\mu_m}\,y)\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right)+\frac{\rho\,\Gamma(r)\,x^{r-s}\,\Gamma(s-r)}{2}+\frac{x^{s-r}\phi(0)\,\Gamma(s)}{2\,(x^2+y^2)^s}$$
$$=\frac{x^{s-r}}{2}\,\Gamma(s)\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right).$$

We have argued at the beginning of the proof of Corollary 2.2.4 that both series above converge absolutely and uniformly with respect to y for $0\leq y<M$. Thus, letting $y\to 0^{+}$, using the first limiting relation in (2.2.33) and recalling that ${}_2F_1(a,b;\,c;\,0)=1$, we obtain the identity

$$\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2)^s}=\frac{\rho\,\Gamma(r)\,x^{2r-2s}\,\Gamma(s-r)}{\Gamma(s)}+x^{-2s}\phi(0)+\frac{2x^{r-s}}{\Gamma(s)}\sum_{m=1}^{\infty}b(m)\,\mu_m^{\frac{s-r}{2}}\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right),$$

which is Berndt's formula (2.1.36) restricted to the class $\mathcal{A}$. □

Since (2.2.46) can be thought as a generalization of Watson's formula [[233], p. 299, eq. (4)], we now see that Corollary 2.2.4 also gives another analogue of Watson's formula for the class $\mathcal{B}$.

Corollary 2.2.6. *Let $Q(x,y)=Ax^2+Bxy+Cy^2$ be a positive definite and real quadratic form, i.e., $\Delta:=4AC-B^2>0$ and $A>0$. Also, define $k:=\sqrt{\Delta}/2A$.*

If $\phi(s)$ is a Dirichlet series belonging to the class $\mathcal{B}$ and s is a complex number such that Re(s) $>\frac{\sigma_a}{2}$, then the following formulas take place

$$-2\phi(0)\,C^{-s}+\sum_{n=1}^{\infty}\frac{a(n)}{(A\lambda_n^2+B\,\lambda_n+C)^s}+\sum_{n=1}^{\infty}\frac{a(n)}{(A\lambda_n^2-B\,\lambda_n+C)^s}$$
$$=\rho\sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,A^{-s}k^{1-2s}+\frac{4k^{\frac{1}{2}-s}A^{-s}}{\Gamma(s)}\sum_{n=1}^{\infty}b(n)\mu_n^{s-\frac{1}{2}}\,\cos\left(\frac{B\mu_n}{A}\right)K_{s-\frac{1}{2}}\left(2k\,\mu_n\right),\quad \delta=0, \tag{2.2.47}$$

$$\sum_{n=1}^{\infty}\frac{a(n)}{(A\lambda_n^2+B\,\lambda_n+C)^s}-\sum_{n=1}^{\infty}\frac{a(n)}{(A\lambda_n^2-B\,\lambda_n+C)^s}$$
$$=-\frac{4k^{\frac{1}{2}-s}A^{-s}}{\Gamma(s)}\sum_{n=1}^{\infty}b(n)\,\mu_n^{s-\frac{1}{2}}\sin\left(\frac{B\mu_n}{A}\right)\,K_{s-\frac{1}{2}}\left(2k\,\mu_n\right),\ \ \delta=1. \tag{2.2.48}$$

Proof. Starting with (2.2.47), we use the fact that if $\phi(s)\in\mathcal{B}$ with $\delta=0$, then $\phi(2s)\in\mathcal{A}$ with $r=\frac{1}{2}$. Thus, to obtain (2.2.47), we start by using (2.2.36) with $\phi(s)$ being replaced by $\phi(2s)$ and $r=\frac{1}{2}$. Appealing to the well-known particular case of Gauss' hypergeometric function [[172], Vol. III, p. 461, eq. (7.3.1.106)],

$${}_2F_1\left(a,a+\frac{1}{2};\,\frac{1}{2};\,z\right)=\frac{1}{2}\left[(1+\sqrt{z})^{-2a}+(1-\sqrt{z})^{-2a}\right], \tag{2.2.49}$$

the left-hand side of (2.2.36) can be simplified to

$$\frac{x^{s-\frac{1}{2}}}{2}\,\Gamma(s)\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n^2+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,\frac{1}{2};\,\frac{4y^2\lambda_n^2}{(\lambda_n^2+x^2+y^2)^2}\right)$$
$$=\frac{x^{s-\frac{1}{2}}}{4}\,\Gamma(s)\sum_{n=1}^{\infty}a(n)\,\left\{\left(\lambda_n^2+2y\,\lambda_n+x^2+y^2\right)^{-s}+\left(\lambda_n^2-2y\,\lambda_n+x^2+y^2\right)^{-s}\right\},\ \ \text{Re}(s)>\frac{\sigma_a}{2}. \tag{2.2.50}$$

On the other hand, for any $s\in\mathbb{C}$, we can write the right-hand side of (2.2.36) as follows

$$\sum_{m=1}^{\infty}b(m)\,\mu_m^{s-\frac{1}{2}}\,\cos\left(2\mu_m y\right)\,K_{s-\frac{1}{2}}\left(2\mu_m x\right)+\frac{\sqrt{\pi}\,\rho\,x^{\frac{1}{2}-s}\,\Gamma(s-\frac{1}{2})}{4}+\frac{x^{s-\frac{1}{2}}\,\phi(0)\,\Gamma(s)}{2\,(x^2+y^2)^s}, \tag{2.2.51}$$

where we have used the particular case $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\cos(x)$, (2.2.20), and the fact that $\phi(2s)$ has a residue equal to $\rho/2$ at the simple pole $s=1/2$ (see Remark 2.1.4). Comparing (2.2.50) with (2.2.51) and assuming that $\text{Re}(s)>\frac{\sigma_a}{2}$ gives the formula

$$-\frac{2\phi(0)}{(x^2+y^2)^s}+\sum_{n=1}^{\infty}a(n)\,\left\{\left(\lambda_n^2+2y\,\lambda_n+x^2+y^2\right)^{-s}+\left(\lambda_n^2-2y\,\lambda_n+x^2+y^2\right)^{-s}\right\}$$
$$=\frac{\sqrt{\pi}\,\rho\,x^{1-2s}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}+\frac{4x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}b(m)\,\mu_m^{s-\frac{1}{2}}\cos\left(2\mu_m y\right)\,K_{s-\frac{1}{2}}\left(2\,\mu_m\,x\right).$$

Multiplying both sides of the previous equality by A^{-s} and defining $y=\frac{B}{2A}$ and $x=\frac{\sqrt{\Delta}}{2A}:=k$, we get immediately (2.2.47).

The proof of (2.2.48) is analogous: in this case, note that $\phi(2s-1)\in\mathcal{A}$ with $r=\frac{3}{2}$ (see Remark 2.1.4), so we just need to use (2.2.36) with $r=\frac{3}{2}$. Appealing to the formula [[172], Vol. III, p. 461, eq. (7.3.1.107)],

$${}_2F_1\left(a,a+\frac{1}{2};\,\frac{3}{2};\,z\right)=\frac{1}{2(2a-1)\sqrt{z}}\left\{(1-\sqrt{z})^{1-2a}-(1+\sqrt{z})^{1-2a}\right\},$$

we see after some straightforward simplifications that the left-hand side of (2.2.36) is reduced to

$$\frac{x^{s-\frac{3}{2}}}{4\sqrt{\pi y}}\,\Gamma(s-1)\sum_{n=1}^{\infty}a(n)\left\{\frac{1}{\left(\lambda_n^2-2y\lambda_n+x^2+y^2\right)^{s-1}}-\frac{1}{\left(\lambda_n^2+2y\lambda_n+x^2+y^2\right)^{s-1}}\right\}, \tag{2.2.52}$$

for $\text{Re}(s)>\frac{\sigma_a}{2}+1$. In what concerns the right-hand side of (2.2.36), note that when $\delta=1$ in the class $\mathcal{B}$, we have $\rho=\phi(0)=0$, so we just need to simplify the series involving the Bessel functions. For any $s\in\mathbb{C}$, this series can be written as

$$\sum_{n=1}^{\infty}b(n)\,\mu_n^{s-1}\,J_{\frac{1}{2}}(2\mu_n\,y)\,K_{s-\frac{3}{2}}(2\mu_n\,x)=\frac{1}{\sqrt{\pi y}}\sum_{n=1}^{\infty}b(n)\,\mu_n^{s-\frac{3}{2}}\,\sin(2\mu_n\,y)\,K_{s-\frac{3}{2}}(2\mu_n\,x), \tag{2.2.53}$$

where we have used the particular case $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\,\sin(x)$, (2.2.20). Comparing (2.2.52) and (2.2.53), replacing in both expressions s by $s+1$ and defining $y=\frac{B}{2A}$ and $x=\frac{\sqrt{\Delta}}{2A}:=k$, we obtain (2.2.48). $\square$

Remark 2.2.5. The previous corollary is somewhat remindful of the classical Selberg-Chowla formula for the Epstein zeta function [15, 201], studied with considerable detail in the previous chapter. Being a generalization of Watson's formula, it is expected that (2.2.36) can be used to derive new analogues of Guinand and Koshliakov's formulas [34, 35, 37, 100, 134, 163, 207], as well as new extensions of the classical Epstein zeta function. This study will be given in Chapter 4 below. Particularly, Theorem 4.5.1 will be one of many general analogues of the Ramanujan-Guinand formula that one may get from the formula (2.2.36).

Like (2.2.44) (cf. [[43], p. 331]), our formulas (2.2.36) and (2.2.37) can be extended to a larger region of the complex plane containing $\text{Re}(s)>\sigma_a$. In the following corollary, we prove the continuation of (2.2.36) to the region $\text{Re}(s)>\sigma_a-1$.

Corollary 2.2.7. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 2.1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Furthermore, let x, y be two positive real numbers and s be any complex number such that $Re(s)>\sigma_a-1$. Then the following formula holds*

$$\begin{aligned}\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\left\{\frac{a(n)}{\left(\lambda_n+x^2+y^2\right)^{s}}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{\left(\lambda_n+x^2+y^2\right)^2}\right)-\frac{a(n)}{\lambda_n^{s}}\right\}\\ +\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\phi(s)=\frac{\rho\,y^{r-1}\,x^{r-s}\,\Gamma(s-r)}{2}+\frac{x^{s-r}y^{r-1}\phi(0)\,\Gamma(s)}{2\Gamma(r)\,\left(x^2+y^2\right)^{s}}\\ +\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{s+1}{2}-r}\,J_{r-1}(2\sqrt{\mu_n}\,y)\,K_{s-r}\left(2\sqrt{\mu_n}\,x\right),\end{aligned} \tag{2.2.54}$$

which constitutes the analytic continuation of (2.2.36) to the region $Re(s)>\sigma_a-1$.

Proof. Note that, as $n\to\infty$, the terms on the infinite series at the left-hand side of (2.2.36) are of the form

$$\left|\frac{1}{\left(\lambda_n+x^2+y^2\right)^{s}}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{\left(\lambda_n+x^2+y^2\right)^2}\right)\right|\sim\frac{1}{\lambda_n^{\text{Re}(s)}}+O\left(\frac{1}{\lambda_n^{\text{Re}(s)+1}}\right).$$

Therefore, for $\mathrm{Re}(s) > \sigma_a$, we can add and subtract the first term of this asymptotic expansion and have

$$\begin{aligned}
&\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\frac{a(n)}{(\lambda_n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,r;\,\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right)\\
&=\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\sum_{n=1}^{\infty}\left\{\frac{a(n)}{(\lambda_n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};r;\,\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right)-\frac{a(n)}{\lambda_n^s}\right\}\\
&\quad+\frac{x^{s-r}}{2}\,\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\phi(s).
\end{aligned}$$

The previous expression is the analytic continuation of the first series to the region $\mathrm{Re}(s) > \sigma_a - 1$. Thus, since the right-hand side of (2.2.54) already admits a continuation to all $s \in \mathbb{C}$, we conclude that (2.2.54) holds when $\mathrm{Re}(s) > \sigma_a - 1$. □

As a particular case of our previous result, one can extend Berndt's formula (2.2.46) to the half-plane $\mathrm{Re}(s) > \sigma_a - 1$ in the following form.

Corollary 2.2.8. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 2.1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Furthermore, let x, y be two positive real numbers and s be any complex number such that $Re(s) > \sigma_a - 1$. Then the following identity holds*

$$\begin{aligned}
\sum_{n=1}^{\infty} a(n)\left\{\frac{1}{(\lambda_n+x^2)^s}-\frac{1}{\lambda_n^s}\right\} &= \frac{\rho\,\Gamma(r)\,x^{2r-2s}\,\Gamma(s-r)}{\Gamma(s)} - \phi(s) + x^{-2s}\phi(0)\\
&\quad+\frac{2x^{r-s}}{\Gamma(s)}\sum_{m=1}^{\infty} b(m)\,\mu_m^{\frac{s-r}{2}}\,K_{s-r}\left(2\sqrt{\mu_m}\,x\right). \qquad (2.2.55)
\end{aligned}$$

Before concluding this section, we now prove summation formulas that can be derived from the continuation given in Corollary 2.2.7. The first summation formula holds when one takes $s = r$ in Corollary 2.2.7, while the second results from setting $s = 2r - 1$.

Corollary 2.2.9. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 2.1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Assume that $\sigma_a < r + 1$. Moreover, suppose that $\phi(s)$ has the following expansion around its (possible) simple pole $s = r$,*

$$\phi(s) = \frac{\rho}{s-r} + \rho_0 + O(s-r). \qquad (2.2.56)$$

If x, y are two positive real numbers, then the following identity holds

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\frac{2^{r-1}a(n)}{\sqrt{(\lambda_n+x^2+y^2)^2-4y^2\lambda_n}\left(\lambda_n+x^2+y^2+\sqrt{(\lambda_n+x^2+y^2)^2-4y^2\lambda_n}\right)^{r-1}}-\frac{a(n)}{\lambda_n^r}\right\}\\
&=\frac{\phi(0)}{(x^2+y^2)^r}-\rho\left(\frac{\Gamma'(r)}{\Gamma(r)}+\gamma+2\log(x)\right)-\rho_0+\frac{2}{y^{r-1}}\sum_{n=1}^{\infty} b(n)\mu_n^{\frac{1-r}{2}}\,J_{r-1}(2\sqrt{\mu_n}\,y)\,K_0\left(2\sqrt{\mu_n}\,x\right).
\end{aligned}$$

(2.2.57)

Proof. Since $\sigma_a < r+1$ by hypothesis, (2.2.54) is valid for $s=r$. Taking the limit $s\to r$ in (2.2.54), we have

$$-\frac{y^{r-1}\phi(0)}{2\,(x^2+y^2)^r}+\frac{y^{r-1}}{2}\sum_{n=1}^{\infty}\left\{\frac{a(n)}{(\lambda_n+x^2+y^2)^r}\,{}_2F_1\left(\frac{r}{2},\frac{r+1}{2};r;\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right)-\frac{a(n)}{\lambda_n^r}\right\}$$
$$=\lim_{s\to r}\left\{\frac{\rho y^{r-1}x^{r-s}\Gamma(s-r)}{2}-\frac{x^{s-r}}{2}\frac{\Gamma(s)y^{r-1}}{\Gamma(r)}\phi(s)\right\}+\sum_{n=1}^{\infty}b(n)\mu_n^{\frac{1-r}{2}}J_{r-1}(2\sqrt{\mu_n}y)K_0\left(2\sqrt{\mu_n}x\right).$$

Invoking the Laurent expansion for $\phi(s)$ around the possible simple pole $s=r$, (2.2.56), together with

$$\Gamma(s-r)=\frac{1}{s-r}-\gamma+O(s-r)$$

and

$$\frac{1}{\Gamma(s)}=\frac{1}{\Gamma(r)}-\frac{\Gamma'(r)}{\Gamma^2(r)}(s-r)+O\left((s-r)^2\right),$$

we find that

$$\lim_{s\to r}\left\{\frac{\rho\, y^{r-1}\,x^{r-s}\,\Gamma(s-r)}{2}-\frac{x^{s-r}}{2}\frac{\Gamma(s)\,y^{r-1}}{\Gamma(r)}\phi(s)\right\}=-\frac{y^{r-1}}{2}\left\{\rho\frac{\Gamma'(r)}{\Gamma(r)}+\rho\gamma+2\rho\log(x)+\rho_0\right\}. \tag{2.2.58}$$

In the meantime, let us use the following particular case of the hypergeometric function [[172], Vol. III, p. 461, eq. (7.3.104)],

$${}_2F_1\left(a,a+\frac{1}{2};2a;z\right)=\frac{2^{2a-1}\left(1+\sqrt{1-z}\right)^{1-2a}}{\sqrt{1-z}},$$

from which we obtain that

$${}_2F_1\left(\frac{r}{2},\frac{r+1}{2};r;\frac{4y^2\lambda_n}{(\lambda_n+x^2+y^2)^2}\right)$$
$$=\frac{2^{r-1}\left(\lambda_n+x^2+y^2\right)^r}{\sqrt{(\lambda_n+x^2+y^2)^2-4y^2\lambda_n}\left(\lambda_n+x^2+y^2+\sqrt{(\lambda_n+x^2+y^2)^2-4y^2\lambda_n}\right)^{r-1}}. \tag{2.2.59}$$

Combining (2.2.59) with (2.2.58) and (2.2.54) gives (2.2.57), completing our proof. □

Corollary 2.2.10. *Let $\phi(s)$ and $\psi(s)$ be two Dirichlet series satisfying Definition 2.1.2 and let σ_a be the abscissa of absolute convergence of $\phi(s)$. Assume that $\sigma_a<2r$ and that $\phi(s)$ admits the expansion (2.2.56) around $s=r$.*

1. *If $r=1$ and x,y are two positive real numbers, then the following identity holds*

$$-\frac{\phi(0)}{x^2+y^2}+\sum_{n=1}^{\infty}\left\{\frac{a(n)}{\sqrt{\lambda_n^2+2\lambda_n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}-\frac{a(n)}{\lambda_n}\right\}$$
$$=-2\rho\log(x)-\rho_0+2\sum_{n=1}^{\infty}b(n)\,J_0(2\sqrt{\mu_n}\,y)\,K_0\left(2\sqrt{\mu_n}\,x\right). \tag{2.2.60}$$

2. If $r \neq 1$ and x, y are two positive real numbers, one has that

$$-\frac{\phi(0)}{(x^2+y^2)^{2r-1}}+\sum_{n=1}^{\infty}\left\{\frac{a(n)}{\left(\lambda_n^2+2\lambda_n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2\right)^{r-\frac{1}{2}}}-\frac{a(n)}{\lambda_n^{2r-1}}\right\}+\phi(2r-1)$$
$$=\frac{\rho x^{2-2r}\Gamma(r)\Gamma(r-1)}{\Gamma(2r-1)}+\frac{2\Gamma(r)}{\Gamma(2r-1)(xy)^{r-1}}\sum_{n=1}^{\infty}b(n)J_{r-1}(2\sqrt{\mu_n}y)K_{r-1}\left(2\sqrt{\mu_n}x\right). \tag{2.2.61}$$

Proof. We assume that $r \neq 1$ because for $r=1$ the resulting formula is already given in the Corollary 2.2.9. Taking $s = 2r-1$ in formula (2.2.54) (which is a valid step, since $\sigma_a < 2r$ by hypothesis) and appealing to the elementary relation

$$_2F_1\left(a, b; a; z\right) = (1-z)^{-b},$$

one deduces immediately (2.2.61). $\square$

Remark 2.2.6. In most of the examples below, we will have that $2r-1 > \sigma_a$. If this condition holds and $r \neq 1$, then we can write (2.2.61) in the form

$$-\frac{\phi(0)}{(x^2+y^2)^{2r-1}}+\sum_{n=1}^{\infty}\frac{a(n)}{\left(\lambda_n^2+2\lambda_n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2\right)^{r-\frac{1}{2}}}$$
$$=\frac{\rho\, x^{2-2r}\,\Gamma(r)\,\Gamma(r-1)}{\Gamma(2r-1)}+\frac{2\Gamma(r)}{\Gamma(2r-1)(xy)^{r-1}}\sum_{n=1}^{\infty}b(n)\,J_{r-1}(2\sqrt{\mu_n}y)K_{r-1}\left(2\sqrt{\mu_n}\,x\right). \tag{2.2.62}$$

2.2.2 Some useful examples

In their paper [[73], p. 312] Dixit, Kumar, Maji and Zaharescu use a variant of Jacobi's $\psi-$function of the form

$$\psi(x,z) := \sum_{n=1}^{\infty}e^{-\pi n^2 x}\cos(\sqrt{\pi\, x}\, nz), \quad \mathrm{Re}(x)>0, \quad z\in\mathbb{C}. \tag{2.2.63}$$

The parameter z in this consideration is exactly the same parameter that appears in the statement of their theorem (see eq. (2.1.4) above). So, in order to extend their result, we need to convert the generalized theta function (2.2.22) into the more symmetric version (2.2.63) depending on a new parameter z. This is done in the following definition.

Definition 2.2.2. Let $\phi(s)$ be any Dirichlet series belonging to the class $\mathcal{A}$ in the sense of Definition (2.1.2). Then, for $\mathrm{Re}(x)>0$ and $z\in\mathbb{C}$, we define the generalized Jacobi's $\psi-$function attached to $\phi(s)$ as

$$\psi_\phi(x,z) := 2^{r-1}\,\Gamma(r)\,\left(\sqrt{x}\,z\right)^{1-r}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}\left(\sqrt{\lambda_n x}\,z\right)=\Phi\left(x,\sqrt{x}\,z\right), \tag{2.2.64}$$

where the definition of $\Phi(x,y)$ is given in (2.2.22). Analogously, one may also define the associated $\psi-$function as follows

$$\tilde{\psi}_{\phi}(x,z):=2^{r-1}\Gamma(r)\,\left(\sqrt{x}\,z\right)^{1-r}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\mu_n x}\,J_{r-1}\left(\sqrt{\mu_n x}\,z\right)=\Psi\left(x,\sqrt{x}\,z\right), \qquad (2.2.65)$$

where $\Psi(x,y)$ is defined by (2.2.23).

With this new parameter $z\in\mathbb{C}$, we note that the transformation formula (2.2.24) can be rewritten as

$$\begin{aligned}\psi_{\phi}(x,z)&:=2^{r-1}\,\Gamma(r)\,x^{\frac{1-r}{2}}\,z^{1-r}\sum_{n=1}^{\infty}a(n)\,\lambda_n^{\frac{1-r}{2}}\,e^{-\lambda_n x}\,J_{r-1}\left(\sqrt{\lambda_n x}\,z\right)\\ &=\phi(0)+\frac{\rho}{x^r}\Gamma(r)\,e^{-\frac{z^2}{4}}+2^{r-1}\Gamma(r)\,z^{1-r}\,x^{-\frac{r+1}{2}}\,e^{-\frac{z^2}{4}}\sum_{n=1}^{\infty}b(n)\,\mu_n^{\frac{1-r}{2}}\,e^{-\frac{\mu_n}{x}}\,I_{r-1}\left(\sqrt{\frac{\mu_n}{x}}\,z\right)\\ &:=\phi(0)+\frac{\rho\Gamma(r)}{x^r}\,e^{-\frac{z^2}{4}}+\frac{e^{-\frac{z^2}{4}}}{x^r}\,\tilde{\psi}_{\phi}\left(\frac{1}{x},iz\right). \qquad (2.2.66)\end{aligned}$$

Also, the integral formulation (2.2.25) takes the new form

$$\begin{aligned}x^{r/2}\psi_{\phi}(x,z)-\frac{\rho\Gamma(r)}{x^{r/2}}\,e^{-\frac{z^2}{4}}&=e^{-\frac{z^2}{4}}\,x^{-r/2}\,\tilde{\psi}_{\phi}\left(\frac{1}{x},iz\right)+\phi(0)\,x^{r/2}\\ =\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma\left(\frac{r}{2}+it\right)\,\phi\left(\frac{r}{2}+it\right){}_1F_1\left(\frac{r}{2}+it;\,r;\,-\frac{z^2}{4}\right)\,x^{-it}\,dt, \qquad (2.2.67)\end{aligned}$$

which is valid for any $z\in\mathbb{C}$ and $\mathrm{Re}(x)>0$.

Example 2.2.1. Let $\alpha>0$ and $r_{\alpha}(n)$ be defined as the coefficients of the expansion (2.1.40). Recall the Dirichlet series attached to it,

$$\zeta_{\alpha}(s)=\sum_{n=1}^{\infty}\frac{r_{\alpha}(n)}{n^s},\quad \mathrm{Re}(s)>\sigma_{\alpha}:=\begin{cases}\frac{\alpha}{2} & \text{if } \alpha\geq 4\\ 1+\frac{\alpha}{4} & \text{if } 0<\alpha<4\end{cases}. \qquad (2.2.68)$$

We know by Lemma 2.1.1 that $\phi(s)=\pi^{-s}\zeta_{\alpha}(s)$ satisfies Hecke's functional equation (2.1.31) with parameter $r=\frac{\alpha}{2}$ and belongs to the class $\mathcal{A}$. For this particular example, the analogue of Jacobi's $\psi-$function (2.2.64) is

$$\psi_{\alpha}(x,z):=2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}r_{\alpha}(n)n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi nx}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi nx}\,z),\quad \mathrm{Re}(x)>0, \qquad (2.2.69)$$

for any $z\in\mathbb{C}$. Note that we can recover the classical Jacobi's function (2.2.63) from (2.2.69). Indeed, using the particular cases

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\,\cos(x),\quad r_1(n)=\begin{cases}2 & \text{if } n \text{ is a square}\\ 0 & \text{otherwise}\end{cases},$$

we obtain from (2.2.69) and (2.2.20)

$$\psi_1(x,z) := \sqrt{2\pi z}\,(\pi x)^{1/4} \sum_{n=1}^{\infty} \sqrt{n}\, e^{-\pi n^2 x}\, J_{-\frac{1}{2}}(\sqrt{\pi\, x}\, n\, z) = 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}\, \cos\left(\sqrt{\pi x}\, n\, z\right)$$
$$:= 2\,\psi(x,z).$$

From the transformation formula (2.2.66), we see that the following identity holds

$$\psi_\alpha(x,z) = -1 + \frac{e^{-z^2/4}}{x^{\alpha/2}} + \frac{e^{-z^2/4}}{x^{\alpha/2}}\,\psi_\alpha\left(\frac{1}{x},\, iz\right), \tag{2.2.70}$$

from which we can recover (2.1.8) when $\alpha = 1$. Furthermore, we can write (2.2.70) in the suitable integral form (2.2.67). Note that the critical line for $\phi(s) := \pi^{-s}\zeta_\alpha(s)$ is the line $\mathrm{Re}(s) = \frac{\alpha}{4}$, so that (2.2.67) yields the representation

$$x^{\alpha/4}\psi_\alpha(x,z) - x^{-\alpha/4}\, e^{-\frac{z^2}{4}} = e^{-\frac{z^2}{4}}\, x^{-\alpha/4}\,\psi_\alpha\left(\frac{1}{x},\, i\,z\right) - x^{\alpha/4}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_\alpha\left(\frac{\alpha}{4} + it\right) {}_1F_1\left(\frac{\alpha}{4} + it;\, \frac{\alpha}{2};\, -\frac{z^2}{4}\right)\, x^{-it}\, dt, \tag{2.2.71}$$

where

$$\eta_\alpha(s) = \pi^{-s}\,\Gamma(s)\,\zeta_\alpha(s). \tag{2.2.72}$$

From Corollary 2.2.4 we can derive the curious formula

$$\frac{1}{(x^2+y^2)^s} + \sum_{n=1}^{\infty} \frac{r_\alpha(n)}{(n+x^2+y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, \frac{\alpha}{2};\, \frac{4n\,y^2}{(n+x^2+y^2)^2}\right) = \frac{\pi^{\frac{\alpha}{2}}\,\Gamma(s-\frac{\alpha}{2})\,x^{\alpha-2s}}{\Gamma(s)}$$
$$+ \frac{2\Gamma\left(\frac{\alpha}{2}\right)\pi^{s+1-\frac{\alpha}{2}}\, x^{\frac{\alpha}{2}-s} y^{1-\frac{\alpha}{2}}}{\Gamma(s)} \sum_{n=1}^{\infty} r_\alpha(n) n^{\frac{s+1-\alpha}{2}}\, J_{\frac{\alpha}{2}-1}(2\pi\sqrt{n}y)\, K_{s-\frac{\alpha}{2}}\left(2\pi\sqrt{n}\,x\right), \tag{2.2.73}$$

valid for any pair of positive numbers x and y and s such that $\mathrm{Re}(s) > \sigma_\alpha$, with σ_α being given by (2.2.68). When $\alpha = 1$, (2.2.73) yields an extension of Watson's formula due to Kober [[129], p. 614]: since $r_1(n) = 2$ only when n is a perfect square, we can use the steps leading to the proof of Corollary 2.2.6 to obtain the formula

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+2yn+x^2+y^2)^s} = \frac{\sqrt{\pi}x^{1-2s}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \frac{4\pi^s}{x^{s-\frac{1}{2}}\,\Gamma(s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\, \cos(2\pi n y)\, K_{s-\frac{1}{2}}\left(2\pi n x\right), \tag{2.2.74}$$

which is valid for any $\mathrm{Re}(s) > \frac{1}{2}$ and $x, y > 0$. By letting $y = 0$ in (2.2.74), one can deduce Watson's formula [[233], p. 299, eq. (4)]. On the other hand, when $\alpha = k \in \mathbb{N}_{\geq 2}$, (2.2.73) gives the identity

$$\frac{1}{(x^2+y^2)^s} + \sum_{n=1}^{\infty} \frac{r_k(n)}{(n+x^2+y^2)^s}\, {}_2F_1\left(\frac{s}{2},\, \frac{s+1}{2};\, \frac{k}{2};\, \frac{4n\,y^2}{(n+x^2+y^2)^2}\right) = \frac{\pi^{\frac{k}{2}}\,\Gamma(s-\frac{k}{2})\,x^{k-2s}}{\Gamma(s)}$$
$$+ \frac{2\,\Gamma\left(\frac{k}{2}\right)\,\pi^{s+1-\frac{k}{2}}\, x^{\frac{k}{2}-s} y^{1-\frac{k}{2}}}{\Gamma(s)} \sum_{n=1}^{\infty} r_k(n)\, n^{\frac{s+1-k}{2}}\, J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_{s-\frac{k}{2}}\left(2\pi\sqrt{n}\,x\right), \tag{2.2.75}$$

which is valid whenever $\mathrm{Re}(s) > \frac{k}{2}$ and $x, y > 0$. Compare this with formula (2.2.44) of Berndt et al. [43]. Using the Laurent expansion of $\zeta_k(s)$ around $s = \frac{k}{2}$ due to Epstein [[81], p. 644], it is possible to provide examples of the identities given in Corollaries 2.2.9 and 2.2.10. For instance, according to Remark 2.2.6, if $k \geq 3$ we can write (2.2.61) in the form

$$\frac{1}{(x^2+y^2)^{k-1}} + \sum_{n=1}^{\infty} \frac{r_k(n)}{\left(n^2+2n\,(x^2-y^2)+(x^2+y^2)^2\right)^{\frac{k-1}{2}}}$$
$$= \frac{\Gamma\left(\frac{k}{2}-1\right)\pi^{\frac{k}{2}}}{(k-2)!x^{k-2}} + \frac{2\,\Gamma\left(\frac{k}{2}\right)\pi^{\frac{k}{2}}}{(k-2)!(xy)^{\frac{k}{2}-1}} \sum_{n=1}^{\infty} r_k(n)\, J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_{\frac{k}{2}-1}\left(2\pi\sqrt{n}\,x\right), \tag{2.2.76}$$

which is also a particular case of (2.2.44) when $\nu = \frac{k}{2} - 1$, $k \geq 3$. The analogue of the summation formula (2.2.76) for $k = 2$ will be given in the next example. Moreover, the analogue of (2.2.57) in the case of $\zeta_k(s)$ can be proved by using the Laurent expansion for $\zeta_k(s)$ provided in Lemma 4.4.5 of the fourth chapter.

Note that, when $k = 1$, it is actually easy (and without invoking the higher analogues of Kronecker's limit formula given in Terras' paper [[222], p. 485]) to get a particular case of (2.2.57). Indeed, recalling that $\zeta_1(s) := 2\zeta(2s)$ and using (2.2.57) with

$$\rho = \frac{1}{\sqrt{\pi}}, \quad \rho_0 = \frac{1}{\sqrt{\pi}}\left(2\gamma - \log(\pi)\right),$$

we obtain the following identity

$$\frac{1}{\sqrt{x^2+y^2}} + 2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{n^2+x^2+y^2+\sqrt{n^4+2n^2\,(x^2-y^2)+(x^2+y^2)^2}}}{\sqrt{n^4+2n^2\,(x^2-y^2)+(x^2+y^2)^2}} - \frac{1}{n}\right\}$$
$$= 2\log\left(\frac{2}{x}\right) - 2\gamma + 4\sum_{n=1}^{\infty}\cos\left(2\pi ny\right) K_0\left(2\pi nx\right), \tag{2.2.77}$$

which seems to be new. In particular, when we take $y \to 0^+$ in (2.2.77), we are able to deduce another formula due to Watson [[233], p. 301, eq. (6)]

$$\frac{1}{x} + 2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{n^2+x^2}} - \frac{1}{n}\right\} = 2\log\left(\frac{2}{x}\right) - 2\gamma + 4\sum_{n=1}^{\infty} K_0(2\pi nx), \tag{2.2.78}$$

which can be also obtained from (2.2.55) when $\phi(s) := \pi^{-s}\zeta(2s)$.

Example 2.2.2. Let Q be a real, binary and positive definite quadratic form and let $\Delta := 4AC - B^2$ be its discriminant. The Epstein zeta function attached to Q, (2.1.18), satisfies Hecke's functional equation

$$\eta_Q(s) := \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta(s,Q) = \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(1-s)}\Gamma(1-s)\,\zeta(1-s,Q) := \eta_Q(1-s).$$

Thus, we can apply the previous formalism to the pair of Dirichlet series

$$\phi(s)=\psi(s)=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_Q(n)}{n^s},\quad \text{Re}(s)>1, \tag{2.2.79}$$

where $r_Q(n)$ denotes the representation number of n by Q. Since $\zeta(s,Q)$ has a simple pole at $s=1$ with residue $2\pi/\sqrt{\Delta}$, $\phi(s)$ has a simple pole at $s=1$ with residue $\rho=1$. Moreover, $\phi(0)=-1$. From (2.2.64), we can write the analogue of Jacobi's $\psi-$function (2.2.64) as follows

$$\psi_Q(x,z)=\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi n x}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi n x}{\Delta^{1/2}}}\,z\right),\quad \text{Re}(x)>0,\ \ z\in\mathbb{C}. \tag{2.2.80}$$

Using (2.2.24), the transformation formula for (2.2.80) reads

$$\begin{aligned}\sqrt{x}\,\psi_Q(x,z)-\frac{e^{-\frac{z^2}{4}}}{\sqrt{x}}&=\frac{e^{-\frac{z^2}{4}}}{\sqrt{x}}\,\psi_Q\left(\frac{1}{x},\,iz\right)-\sqrt{x}\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_Q\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{2}+it;\,1;\,-\frac{z^2}{4}\right)x^{-it}\,dt,\end{aligned} \tag{2.2.81}$$

where

$$\eta_Q(s):=\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta(s,Q). \tag{2.2.82}$$

Further identities can be obtained. From a straightforward application of Corollary 2.2.4 we can derive the formula

$$\begin{aligned}&\frac{1}{(x^2+y^2)^s}+\sum_{n=1}^{\infty}\frac{r_Q(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,1;\,\frac{4ny^2}{(n+x^2+y^2)^2}\right)=\frac{2\pi}{\sqrt{\Delta}}\,\frac{x^{2-2s}}{s-1}\\ &+\frac{2x^{1-s}}{\Gamma(s)}\left(\frac{2\pi}{\sqrt{\Delta}}\right)^{s}\sum_{n=1}^{\infty}r_Q(n)\,n^{\frac{s-1}{2}}\,J_0\left(4\pi\sqrt{\frac{n}{\Delta}}\,y\right)K_{s-1}\left(4\pi\sqrt{\frac{n}{\Delta}}\,x\right),\quad \text{Re}(s)>1,\end{aligned}$$

which seems to be novel. We can also prove the continuation of the previous formula to the point $s=1$. Since $\zeta(s,Q)$ satisfies Hecke's functional equation (2.1.31) with $r=1$, the results given as Corollary 2.2.9 and Corollary 2.2.10 yield the same formula. Appealing to Kronecker's limit formula [201, 222], we know that

$$\zeta(s,Q)=\frac{2\pi}{\sqrt{\Delta}}\,\frac{1}{s-1}+\frac{2\pi}{\sqrt{\Delta}}\left(2\gamma-\log\left(\frac{\Delta}{A}\right)-4\log\left(|\eta(\tau)|\right)\right)+O(s-1), \tag{2.2.83}$$

where γ is the Euler-Mascheroni constant, $\tau:=\frac{B+i\sqrt{\Delta}}{2A}\in\mathbb{H}$ and $\eta(\tau)$ denotes the Dedekind $\eta-$function,

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right),\quad \text{Im}(\tau)>0. \tag{2.2.84}$$

Hence, using the summation formula (2.2.60) for $\phi(s)$ in (2.2.79) and appealing to the substitutions

$$\rho = 1, \quad \rho_0 = 2\gamma - \log\left(\frac{2\pi\sqrt{\Delta}}{A}\right) - 4\log\left(|\eta(\tau)|\right),$$

we obtain

$$\begin{aligned}
&\frac{\sqrt{\Delta}}{2\pi(x^2+y^2)} + \frac{\sqrt{\Delta}}{2\pi}\sum_{n=1}^{\infty} r_Q(n)\left\{\frac{1}{\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}} - \frac{1}{n}\right\} \\
&= \log\left(\frac{\Delta}{A}\right) + 4\log\left(|\eta(\tau)|\right) - 2\gamma - 2\log(x) + 2\sum_{n=1}^{\infty} r_Q(n)\, J_0\left(4\pi\sqrt{\frac{n}{\Delta}}\,y\right) K_0\left(4\pi\sqrt{\frac{n}{\Delta}}\,x\right). \quad (2.2.85)
\end{aligned}$$

Replacing y by iy in (2.2.85) and assuming that $x > y > 0$, we can also get the formula

$$\begin{aligned}
&\frac{\sqrt{\Delta}}{2\pi(x^2-y^2)} + \frac{\sqrt{\Delta}}{2\pi}\sum_{n=1}^{\infty} r_Q(n)\left\{\frac{1}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}} - \frac{1}{n}\right\} \\
&= \log\left(\frac{\Delta}{A}\right) + 4\log\left(|\eta(\tau)|\right) - 2\gamma - 2\log(x) + 2\sum_{n=1}^{\infty} r_Q(n) I_0\left(4\pi\sqrt{\frac{n}{\Delta}}y\right) K_0\left(4\pi\sqrt{\frac{n}{\Delta}}x\right). \quad (2.2.86)
\end{aligned}$$

Consider $Q(m,n) = m^2 + n^2$ and take the change of variables $x := \frac{\sqrt{\alpha}+\sqrt{\beta}}{2}$, $y := \frac{\sqrt{\alpha}-\sqrt{\beta}}{2}$, $\alpha, \beta > 0$, in (2.2.85) and (2.2.86). We find the equivalent expression

$$\begin{aligned}
&\frac{1}{\pi(\alpha+\beta)} + \frac{1}{2\pi}\sum_{n=1}^{\infty} r_2(n)\left\{\frac{1}{\sqrt{n^2+2n\sqrt{\alpha\beta}+\frac{1}{4}\left(\alpha+\beta\right)^2}} - \frac{1}{n}\right\} \\
&= 2\log(2) + 2\log\left(\eta(i)\right) - \gamma - \log\left(\sqrt{\alpha}+\sqrt{\beta}\right) \\
&+ \sum_{n=1}^{\infty} r_2(n)\, J_0\left(\pi\sqrt{n}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right) K_0\left(\pi\sqrt{n}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right),
\end{aligned}$$

and, under the additional assumption that $\alpha > \beta$,

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{\alpha\beta}} + \frac{1}{2\pi}\sum_{n=1}^{\infty} r_2(n)\left\{\frac{1}{\sqrt{n^2+n\left(\alpha+\beta\right)+\alpha\beta}} - \frac{1}{n}\right\} \\
&= 2\log(2) + 2\log\left(\eta(i)\right) - \gamma - \log\left(\sqrt{\alpha}+\sqrt{\beta}\right) \\
&+ \sum_{n=1}^{\infty} r_2(n)\, I_0\left(\pi\sqrt{n}\left(\sqrt{\alpha}-\sqrt{\beta}\right)\right) K_0\left(\pi\sqrt{n}\left(\sqrt{\alpha}+\sqrt{\beta}\right)\right).
\end{aligned}$$

Our previous formula actually gives a particular case of the continuation of (2.2.44) (cf. [[43], p. 333, eq. (5.9)]). Note that (2.2.85) is the extension of the identity (2.2.76) to $k = 2$ when $Q(m,n) = m^2 + n^2$.

Example 2.2.3. Let χ be a primitive Dirichlet character modulo q. If χ is even, we define $\delta = 0$ and if χ is odd, we take $\delta = 1$. In this example we consider the Dirichlet series

$$L_k(s,\chi) := \sum_{n_1,\dots,n_k\in\mathbb{Z}} \frac{n_1^{\delta}\,\chi(n_1)\dots n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+\dots+n_k^2\right)^s}, \quad \mathrm{Re}(s) > \frac{k}{2}(1+\delta).$$

First, let us note that $L_k(s,\chi)$ can be rewritten as

$$L_k(s,\chi) := \sum_{n_1,\dots,n_k\in\mathbb{Z}} \frac{n_1^{\delta}\,\chi(n_1)\dots n_k^{\delta}\,\chi(n_k)}{\left(n_1^2+\dots+n_k^2\right)^s} = \sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{n^s}, \quad \mathrm{Re}(s) > \frac{k}{2}(1+\delta),$$

where $r_{k,\chi}(n)$ is the character analogue of $r_k(n)$,

$$r_{k,\chi}(n) := \sum_{n_1^2+\dots+n_k^2=n} n_1^{\delta}\chi(n_1)\,\cdot\,\dots\,\cdot\, n_k^{\delta}\,\chi(n_k), \tag{2.2.87}$$

with the sum running over any decomposition of n as a sum of k squares.

It will be shown in Subsection 2.5.2 of this chapter that $L_k(s,\chi)$ can be analytically continued to an entire function satisfying Hecke's functional equation

$$\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_k(s,\chi) = \frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\left(\frac{\pi}{q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)L_k\left(k\left(\frac{1}{2}+\delta\right)-s;\overline{\chi}\right). \tag{2.2.88}$$

Therefore, we can apply the previous formulas to the pair of Dirichlet series

$$\phi(s) := \left(\frac{\pi}{q}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{n^s}, \quad \psi(s) := \frac{(-i)^{\delta k}G^k(\chi)}{q^{k/2}}\left(\frac{\pi}{q}\right)^{-s}\sum_{n=1}^{\infty}\frac{r_{k,\overline{\chi}}(n)}{n^s}. \tag{2.2.89}$$

From the definition (2.2.64), we can consider the analogue of Jacobi's $\psi-$function attached to $L_k(s,\chi)$,

$$\psi_{\chi,k}(x,z) := 2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}}x\,z\right)^{1-\frac{k}{2}-k\delta}\sum_{n=1}^{\infty} r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}x}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}nx}\,z\right). \tag{2.2.90}$$

When $k=1$, (2.2.90) reduces to the character analogue of Jacobi's $\psi-$function given at the end of the paper [[73], p. 321] and in [[71], Theorem 1.3.]. Indeed, since $r_{1,\chi}(n) = 2\sqrt{n}^{\delta}\chi(\sqrt{n})$ only if n is a perfect square, we see that (2.2.90) gives

$$\begin{aligned}\psi_{\chi,1}(x,z) &:= 2^{\frac{1}{2}+\delta}\,\Gamma\left(\frac{1}{2}+\delta\right)\left(\sqrt{\frac{\pi}{q}}x\,z\right)^{\frac{1}{2}-\delta}\sum_{n=1}^{\infty}\chi(n)\,\sqrt{n}\,e^{-\frac{\pi n^2}{q}x}\,J_{\delta-\frac{1}{2}}\left(\sqrt{\frac{\pi}{q}}x\,nz\right)\\ &= \begin{cases} 2\,\sum_{n=1}^{\infty}\chi(n)\,e^{-\frac{\pi n^2}{q}x}\,\cos\left(\sqrt{\frac{\pi}{q}}x\,nz\right) & \delta=0\\ \frac{2}{z}\sqrt{\frac{q}{\pi x}}\,\sum_{n=1}^{\infty}\chi(n)\,e^{-\frac{\pi n^2}{q}x}\,\sin\left(\sqrt{\frac{\pi}{q}}x\,nz\right) & \delta=1\end{cases}.\end{aligned}$$

Since $\phi(0)=\rho=0$ for $\phi(s)$ given by (2.2.89), we can deduce from our general formulas that

$$\sqrt{x}\sum_{n=1}^{\infty}r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}x}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}nx}\,z\right)$$
$$=\frac{(-i)^{\delta k}G^{k}(\chi)}{q^{k/2}}\,\frac{e^{-\frac{z^2}{4}}}{\sqrt{x}}\sum_{n=1}^{\infty}r_{k,\overline{\chi}}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{qx}}\,I_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi n}{qx}}\,z\right),$$

which, by virtue of relation (2.2.67), can be rewritten as

$$x^{\frac{k}{4}+\frac{k\delta}{2}}\,\psi_{\chi,k}\left(x,z\right)=\frac{(-i)^{\delta k}G^{k}(\chi)\,e^{-\frac{z^2}{4}}}{q^{k/2}}\,x^{-\frac{k}{4}-\frac{k}{2}\delta}\,\psi_{\overline{\chi},k}\left(\frac{1}{x},iz\right)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{k}\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right){}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}+it;\,\frac{k}{2}+k\delta;\,-\frac{z^2}{4}\right)x^{-it}\,dt, \tag{2.2.91}$$

where

$$\eta_{k}(s,\chi):=\left(\frac{\pi}{q}\right)^{-s}\Gamma(s)\,L_{k}\left(s,\chi\right). \tag{2.2.92}$$

From Corollary 2.2.4, we may derive a character analogue of (2.2.73), namely

$$\frac{q^{k/2}i^{\delta k}}{G^{k}(\chi)}\sum_{n=1}^{\infty}\frac{r_{k,\chi}(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,\frac{k}{2}+k\delta;\,\frac{4y^2n}{(n+x^2+y^2)^2}\right)$$
$$=\frac{2\Gamma\left(\frac{k}{2}+k\delta\right)}{\Gamma(s)y^{\frac{k}{2}+k\delta-1}x^{s-\frac{k}{2}-k\delta}}\left(\frac{\pi}{q}\right)^{1+s-\frac{k}{2}-k\delta}$$
$$\times\sum_{n=1}^{\infty}r_{k,\overline{\chi}}(n)n^{\frac{s+1-k}{2}-k\delta}J_{\frac{k}{2}+k\delta-1}\left(\frac{2\pi\sqrt{n}}{q}y\right)K_{s-\frac{k}{2}-k\delta}\left(\frac{2\pi\sqrt{n}}{q}x\right), \tag{2.2.93}$$

which is valid for $\mathrm{Re}(s)>\frac{k}{2}(1+\delta)$ and $x,y>0$. If $k=1$ and $\delta=0$, we can also appeal to the particular case (2.2.49) to obtain a character analogue of Watson's formula, valid for $\mathrm{Re}(s)>\frac{1}{2}$,

$$\sum_{n\in\mathbb{Z}}\frac{\chi(n)}{(n^2+2yn+x^2+y^2)^s}=\frac{4\,x^{\frac{1}{2}-s}\,G(\chi)}{\Gamma(s)\sqrt{\pi}}\left(\frac{\pi}{q}\right)^{\frac{1}{2}+s}\sum_{n=1}^{\infty}\overline{\chi}(n)\,n^{s-\frac{1}{2}}\cos\left(\frac{2\pi n}{q}y\right)K_{s-\frac{1}{2}}\left(\frac{2\pi n}{q}x\right).$$

By letting $y=0$ above, we rederive a formula due to Berndt, Dixit and Sohn [[34], p. 56, eq. (2.9.)]. On the other hand, when $k=1$ and $\delta=1$, we see that $L_1\left(s,\chi\right)=2\,L(2s-1,\chi)$, with χ being an odd Dirichlet character. In this case, (2.2.93) implies the identity

$$\sum_{n\in\mathbb{Z}}\frac{\chi(n)}{(n^2+2yn+x^2+y^2)^s}=\frac{4i\,x^{\frac{1}{2}-s}\,G(\chi)}{\Gamma(s)\sqrt{\pi}}\left(\frac{\pi}{q}\right)^{s+\frac{1}{2}}\sum_{n=1}^{\infty}\overline{\chi}(n)\,n^{s-\frac{1}{2}}\sin\left(\frac{2\pi n}{q}y\right)K_{s-\frac{1}{2}}\left(\frac{2\pi n}{q}x\right),$$

valid for $\mathrm{Re}(s)>\frac{1}{2}$ and $x,y>0$. This is another character analogue of Watson's formula. If we differentiate the latter expression with respect to y (a procedure valid in the region $\mathrm{Re}(s)>1$) and then let $y\to0^{+}$, we can derive a formula due to Berndt given for the first time in [[34], p. 56, eq. (2.10)]. Using Corollaries 2.2.9 and 2.2.10, we may also obtain the continuation of the formula (2.2.93) to a larger domain of complex numbers.

Example 2.2.4. Let $f(\tau)$ be a holomorphic cusp form with weight $k\geq 12$ for the full modular group and $L(s,f)$ the associated $L-$function,

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s},\quad \text{Re}(s)>\frac{k+1}{2}. \tag{2.2.94}$$

From Lemma 2.1.4 we know that $L(s,f)$ can be analytically continued as an entire function with functional equation

$$(2\pi)^{-s}\Gamma(s)\,L(s,f)=(-1)^{k/2}\,(2\pi)^{-(k-s)}\,\Gamma(k-s)\,L(k-s,f). \tag{2.2.95}$$

From the previous results we can conceive the analogue of Jacobi's $\psi-$function, (2.2.64), in the following form

$$\psi_f(x,z):=(k-1)!\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi nx}\,J_{k-1}\left(\sqrt{2\pi nx}\,z\right), \tag{2.2.96}$$

for $\text{Re}(x)>0$ and any $z\in\mathbb{C}$. Moreover, we can establish the formula

$$\begin{aligned} x^{k/2}\psi_f(x,z)&=(-1)^{k/2}\,e^{-\frac{z^2}{4}}x^{-k/2}\,\psi_f\left(\frac{1}{x},\,iz\right)\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_f\left(\frac{k}{2}+it\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)\,x^{-it}dt, \end{aligned} \tag{2.2.97}$$

where

$$\eta_f(s):=(2\pi)^{-s}\Gamma(s)\,L(s,f). \tag{2.2.98}$$

Also, for $\text{Re}(s)>\frac{k+1}{2}$ and $x,y>0$, the following particular case of Corollary 2.2.4 holds

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{a_f(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,k;\,\frac{4n\,y^2}{(n+x^2+y^2)^2}\right)\\ &=\frac{2\,(2\pi)^{s+1-k}\,(k-1)!\,(-1)^{k/2}}{\Gamma(s)\,y^{k-1}x^{s-k}}\sum_{n=1}^{\infty}a_f(n)\,n^{\frac{s+1}{2}-k}\,J_{k-1}(4\pi\sqrt{n}\,y)\,K_{s-k}\left(4\pi\sqrt{n}\,x\right). \end{aligned} \tag{2.2.99}$$

For example, when $L(s,f)$ is the Dirichlet series associated with Ramanujan's $\tau-$function,

$$L(s,\tau)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s},\quad \text{Re}(s)>\frac{13}{2},$$

we obtain from (2.2.99)

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\tau(n)}{(n+x^2+y^2)^s}\,{}_2F_1\left(\frac{s}{2},\,\frac{s+1}{2};\,12;\,\frac{4ny^2}{(n+x^2+y^2)^2}\right)\\ &=\frac{2\times 11!\times(2\pi)^{s-11}}{\Gamma(s)\,y^{11}x^{s-12}}\sum_{n=1}^{\infty}\tau(n)\,n^{\frac{s-23}{2}}\,J_{11}(4\pi\sqrt{n}\,y)\,K_{s-12}\left(4\pi\sqrt{n}\,x\right), \end{aligned}$$

whenever $x, y > 0$ and $\text{Re}(s) > \frac{13}{2}$. This formula appears to be new (see [[41], eq. (13.1)] for a companion formula). Finally, applying Corollary 2.2.9 gives the interesting formula

$$\sum_{n=1}^{\infty} \frac{a_f(n)}{\sqrt{n^2 + 2n\,(x^2 - y^2) + (x^2 + y^2)^2}\,\left(n + x^2 + y^2 + \sqrt{n^2 + 2n\,(x^2 - y^2) + (x^2 + y^2)^2}\right)^{k-1}}$$
$$= \frac{(-1)^{k/2} 4\pi}{(2y)^{k-1}} \sum_{n=1}^{\infty} a_f(n)\, n^{\frac{1-k}{2}}\, J_{k-1}\left(4\pi\sqrt{n}y\right)\, K_0\left(4\pi\sqrt{n}\,x\right),$$

which also seems to be a new identity. Moreover, an application of Corollary 2.2.10 yields

$$\sum_{n=1}^{\infty} \frac{a_f(n)}{\left(n^2 + 2n\,(x^2 - y^2) + (x^2 + y^2)^2\right)^{k-\frac{1}{2}}}$$
$$= \frac{2(-1)^{k/2} (2\pi)^k (k-1)!}{(xy)^{k-1}\,(2k-2)!} \sum_{n=1}^{\infty} a_f(n)\, J_{k-1}\left(4\pi\sqrt{n}\,y\right)\, K_{k-1}\left(4\pi\sqrt{n}\,x\right).$$

Example 2.2.5. Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $a_f(n)$ its Fourier coefficients. Also, assume that p, q are integers such that $(p, q) = 1$ and consider the Dirichlet series

$$L_f(s, p/q) := \sum_{n=1}^{\infty} \frac{a_f(n)\, e^{\frac{2\pi i p}{q} n}}{n^s}, \quad \text{Re}(s) > \frac{k+1}{2}.$$

We know from Lemma 2.1.4 that $L_f(s, p/q)$ is an entire function and satisfies the functional equation

$$\left(\frac{2\pi}{q}\right)^{-s} \Gamma(s)\, L_f\left(s, \frac{p}{q}\right) = (-1)^{k/2} \left(\frac{2\pi}{q}\right)^{-(k-s)} \Gamma(k-s)\, L_f\left(k-s,\, -\overline{p}/q\right),$$

where $\overline{p}$ is such that $p\,\overline{p} \equiv 1 \mod q$. From this, we can create the analogue of Jacobi's $\psi-$function, (2.2.64), as follows

$$\psi_{f,p/q}(x, z) := (k-1)! \left(\sqrt{\frac{\pi x}{2q}}\, z\right)^{1-k} \sum_{n=1}^{\infty} a_f(n)\, e^{\frac{2\pi i p}{q} n}\, n^{\frac{1-k}{2}}\, e^{-\frac{2\pi n}{q} x}\, J_{k-1}\left(\sqrt{\frac{2\pi n}{q} x}\, z\right), \qquad (2.2.100)$$

defined for $\text{Re}(x) > 0$ and $z \in \mathbb{C}$. Clearly, the transformation formula for $\psi_{f,p/q}(x, z)$ (2.2.66) is explicitly given by

$$\sum_{n=1}^{\infty} a_f(n)\, e^{\frac{2\pi i p}{q} n}\, n^{\frac{1-k}{2}}\, e^{-\frac{2\pi n}{q} x}\, J_{k-1}\left(\sqrt{\frac{2\pi n}{q} x}\, z\right)$$
$$= \frac{(-1)^{k/2}\, e^{-z^2/4}}{x} \sum_{n=1}^{\infty} a_f(n)\, e^{-\frac{2\pi i \overline{p}}{q} n}\, n^{\frac{1-k}{2}}\, e^{-\frac{2\pi n}{q x}}\, I_{k-1}\left(\sqrt{\frac{2\pi n}{q x}}\, z\right), \qquad (2.2.101)$$

and it admits the integral representation

$$\begin{aligned}x^{k/2}\psi_{f,p/q}(x,z) &= \frac{e^{-z^2/4}(-1)^{k/2}}{x^{k/2}}\,\psi_{f,-\frac{\overline{p}}{q}}\left(\frac{1}{x},iz\right)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_f\left(\frac{k}{2}+it,\frac{p}{q}\right){}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^2}{4}\right)x^{-it}dt,\end{aligned} \tag{2.2.102}$$

where

$$\eta_f\left(s,\frac{p}{q}\right):=\left(\frac{2\pi}{q}\right)^{-s}\Gamma(s)\,L_f\left(s,\frac{p}{q}\right). \tag{2.2.103}$$

The integral formulas (2.2.71), (2.2.81), (2.2.91) and (2.2.102) will be instrumental in the arguments that we shall present in the next sections of this chapter.

2.3 Zeros of combinations attached to $\zeta_\alpha(s)$

2.3.1 Dirichlet series attached to powers of Jacobi's theta functions

At the core of Hardy's proof of his theorem is the fact that Jacobi's $\theta-$function satisfies

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^n}{d\omega^n}\,\theta\left(e^{2i\omega}\right)=0,\qquad \forall n\in\mathbb{N}_0. \tag{2.3.1}$$

If we replace the traditional role of $\theta(x)$ in (2.3.1) by the analogue $\psi_\alpha(x,z)$, we will see below that the study of the limit (2.3.1) can be made through the evaluation of

$$\lim_{\delta\to 0^+}\psi_\alpha(i+\delta,z).$$

However, to inspect $\psi_\alpha(i+\delta,z)$ is the same as to study the series

$$2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}(-1)^n\,r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\delta}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi n(i+\delta)}\,z) \tag{2.3.2}$$

as $\delta\to 0^+$. For $\alpha=1$, this study was done in [[73], p. 314] and it represents an easier case because $r_1(n)=2$ if n is a perfect square and it is zero otherwise. Indeed, the expression (2.3.2) reduces to

$$\psi_1(i+\delta,z)=2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2\delta}\cos\left(\sqrt{\pi(i+\delta)}\,nz\right),$$

which can be evaluated simply as (once we separate the previous series for n even and n odd)

$$\psi_1(i+\delta,z)=2\,\psi_1\left(4\delta,\sqrt{\frac{i+\delta}{\delta}}\,z\right)-\psi_1\left(\delta,\sqrt{\frac{i+\delta}{\delta}}\,z\right).$$

Thus, the study of the limit $\delta\to 0^+$ is complete once we apply the transformation formula (2.2.70) for $\psi_1(x,z)$!

By looking at (2.3.2), one can see that it may be difficult to employ the previous trick (i.e., to separate the sum for even and odd n) for an arbitrary $\alpha > 0$, and so we need to have more information about the coefficients $(-1)^n r_\alpha(n)$ and the Dirichlet series attached to them. This study will be developed through a sequence of important lemmas.

These lemmas concern the powers of certain variants of Jacobi's theta function. Like in the case of $\vartheta_3(\tau)$, (2.1.41), we can consider arbitrary powers of the remaining thetanulls,

$$\vartheta_2(\tau) = 2\sum_{n=0}^{\infty} e^{\pi i\tau\left(n+\frac{1}{2}\right)^2}, \quad \vartheta_4(\tau) = 1+2\sum_{n=1}^{\infty}(-1)^n e^{\pi i\tau n^2}, \quad \mathrm{Im}(\tau)>0, \tag{2.3.3}$$

and study their expansions at $i\infty$ by taking

$$\theta_2(x) := \vartheta_2(ix) = 2\sum_{n=0}^{\infty} e^{-\pi\left(n+\frac{1}{2}\right)^2 x}, \quad \theta_4(x) := \vartheta_4(ix) = 1+2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}.$$

It follows from Jacobi's formula (2.1.8) that $\theta_2(x)$ and $\theta_4(x)$ are connected via the transformation formula

$$\theta_2\left(\frac{1}{x}\right) = \sqrt{x}\,\theta_4\left(x\right). \tag{2.3.4}$$

Arithmetical functions akin to $r_\alpha(n)$ arise from the study of the powers of $\theta_2(x)$ and $\theta_4(x)$. For example, modifying slightly the computations in (2.1.41) and (2.1.42), we can define the Fourier coefficients of $\theta_2^\alpha(x)$ as follows

$$\begin{aligned}
\vartheta_2^\alpha(ix) := \theta_2^\alpha(x) &= \left(2\sum_{n=0}^{\infty} e^{-\pi x\left(n+\frac{1}{2}\right)^2}\right)^\alpha = \left(2\,e^{-\frac{\pi x}{4}}+2\sum_{n=1}^{\infty} e^{-\pi x(n+\frac{1}{2})^2}\right)^\alpha \\
&= 2^\alpha e^{-\frac{\pi\alpha x}{4}}\left(1+\sum_{n=1}^{\infty} e^{-\pi x\left(n^2+n\right)}\right)^\alpha = 2^\alpha\, e^{-\frac{\pi\alpha x}{4}}\sum_{j=0}^{\infty}\binom{\alpha}{j}\left(\sum_{n=1}^{\infty} e^{-\pi x\left(n^2+n\right)}\right)^j \\
&:= \sum_{m=0}^{\infty}\tilde{r}_\alpha(m)\, e^{-\pi\left(m+\frac{\alpha}{4}\right)x},
\end{aligned} \tag{2.3.5}$$

where we have taken the new variable of summation as $m := n_1(n_1+1)+...+n_j(n_j+1)$. Analogously to $r_\alpha(n)$, it is possible to show that the coefficients $\tilde{r}_\alpha(n)$ grow polynomially[7] with n. Note also that $\tilde{r}_\alpha(0) := 2^\alpha$ by construction. Hence, it is meaningful to introduce a Dirichlet series attached to the coefficients $\tilde{r}_\alpha(n)$, this is, for some finite $\tilde{\sigma}_\alpha$, we define

$$\tilde{\zeta}_\alpha(s) := \sum_{n=0}^{\infty}\frac{\tilde{r}_\alpha(n)}{(n+\frac{\alpha}{4})^s}, \quad \mathrm{Re}(s) > \tilde{\sigma}_\alpha. \tag{2.3.6}$$

[7]We do not need a bound of the form (2.1.43) to proceed with our reasoning. It suffices to check that the argument by Lagarias and Rains [[137], p. 18, Theorem 3.3] works for $\vartheta_2(\tau)$. See Lemma 2.3.1 below.

We can also introduce a Dirichlet series attached to any positive power of $\theta_4(x)$. Proceed once more as in (2.1.41) and write an arbitrary power of $\theta_4(x)$ in the form

$$\begin{aligned}\vartheta_4^{\alpha}(ix) := \theta_4^{\alpha}(x) &= \left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}\right)^{\alpha}\\ &= 1+\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\left(\sum_{n=1}^{\infty}(-1)^n e^{-\pi n^2 x}\right)^j\\ &= 1+\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\sum_{n_1,\dots,n_j=1}^{\infty}(-1)^{n_1+n_2+\dots+n_j}\,e^{-\pi(n_1^2+\dots+n_j^2)\,x}\\ &= 1+\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\sum_{n_1,\dots,n_j=1}^{\infty}(-1)^{n_1^2+n_2^2+\dots+n_j^2}\,e^{-\pi(n_1^2+\dots+n_j^2)\,x}.\end{aligned}$$

If we now define a new variable of summation $m=n_1^2+\dots+n_j^2$, then the finite sum over j is exactly the same as we have obtained in defining (2.1.42) but with an extra factor containing $(-1)^m$. Following (2.1.42), we see that

$$\begin{aligned}\theta_4^{\alpha}(x) &= 1+\sum_{j=1}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)2^j\sum_{n_1,\dots,n_j=1}^{\infty}(-1)^{n_1^2+n_2^2+\dots+n_j^2}\,e^{-\pi(n_1^2+\dots+n_j^2)\,x}\\ &= 1+\sum_{m=1}^{\infty}(-1)^m\,r_\alpha(m)\,e^{-\pi m x}.\end{aligned} \tag{2.3.7}$$

The computations above show that $(-1)^n r_\alpha(n)$ are the coefficients of the Fourier expansion of $\vartheta_4^{\alpha}(\tau)$ at the cusp $i\infty$. The Dirichlet series attached to these coefficients is then given by

$$\zeta_\alpha^{\star}(s) := \sum_{n=1}^{\infty}\frac{(-1)^n\,r_\alpha(n)}{n^s},\quad \mathrm{Re}(s)>\sigma_\alpha := \begin{cases}\frac{\alpha}{2} & \text{if } \alpha\geq 4\\ 1+\frac{\alpha}{4} & \text{if } 0<\alpha<4\end{cases}. \tag{2.3.8}$$

In the following subsection, we connect the Dirichlet series (2.3.6) and (2.3.8) through a functional equation and prove a new summation formula for their coefficients.

2.3.2 Summation formulas

To establish the connection between the Dirichlet series $\zeta_\alpha^{\star}(s)$ and $\tilde{\zeta}_\alpha(s)$, we need to assure that the latter is actually a Dirichlet series, i.e., that $\tilde{\sigma}_\alpha$ in (2.3.6) is a finite number. The next lemma gives an estimate for $\tilde{r}_\alpha(n)$ similar in spirit to the bounds found in [137].

Lemma 2.3.1. *Let $\alpha>0$ and $\tilde{r}_\alpha(n)$ be defined by (2.3.5). Then, for any $n\in\mathbb{N}$, $\tilde{r}_\alpha(n)$ satisfies the inequality*

$$|\tilde{r}_\alpha(n)| < 540\,(2n)^{\alpha/2}. \tag{2.3.9}$$

Therefore, the Dirichlet series defined by (2.3.6) converges absolutely in the half-plane Re$(s)>\tilde{\sigma}_\alpha := \frac{\alpha}{2}+1$.

Proof. Our proof is nothing but a simple adaptation of [[137], p. 19, Thm 3.3, eq. (3.12)]. Write $\varphi_2(q)=2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}$ for $|q|<1$: invoking Cauchy's integral formula, the following bound takes place

$$\begin{aligned}|\tilde{r}_{\alpha}(n)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\varphi_2^{\alpha}\left(Re^{i\phi}\right)\,R^{-n}e^{-in\phi}\,d\phi\right|\leq R^{-n}\max_{\phi\in[-\pi,\pi]}\left|\varphi_2^{\alpha}\left(Re^{i\phi}\right)\right|\\ &= R^{-n}\left(\max_{\phi\in[-\pi,\pi]}\left|\varphi_2\left(Re^{i\phi}\right)\right|\right)^{\alpha},\quad 0<R<1,\end{aligned}$$

since $\alpha>0$ by hypothesis. Clearly, for any $\phi\in[-\pi,\pi]$,

$$\left|\varphi_2\left(Re^{i\phi}\right)\right|\leq\varphi_2\left(R\right),$$

which immediately gives

$$|\tilde{r}_{\alpha}(n)|\leq R^{-n}\,\varphi_2^{\alpha}\left(R\right), \tag{2.3.10}$$

for any choice of $R\in(0,1)$. For each n, take $R:=R(n)=e^{-\pi A/n}$ for a fixed $A>1$. Then (2.3.10) yields

$$\begin{aligned}|\tilde{r}_{\alpha}(n)| &\leq e^{\pi A}\,\varphi_2^{\alpha}\left(e^{-\pi A/n}\right)=e^{\pi A}\,\theta_2^{\alpha}\left(\frac{A}{n}\right)=e^{\pi A}\left(\frac{n}{A}\right)^{\alpha/2}\theta_4^{\alpha}\left(\frac{n}{A}\right)\\ &= \frac{e^{\pi A}}{A^{\alpha/2}}\left\{1+2\sum_{k=1}^{\infty}(-1)^k e^{-\pi k^2\frac{n}{A}}\right\}^{\alpha}n^{\alpha/2}\leq\frac{e^{\pi A}\,n^{\alpha/2}}{A^{\alpha/2}}\left\{1+\frac{2}{e^{\pi n/A}-1}\right\}^{\alpha}\\ &\leq\frac{e^{\pi A}}{A^{\alpha/2}}\left\{1+\frac{2}{e^{\pi/A}-1}\right\}^{\alpha}n^{\alpha/2}.\end{aligned} \tag{2.3.11}$$

where we have used (2.3.4) together with elementary bounds on the series defining $\theta_4(x)$. The desired estimate (2.3.9) now follows once we take $A=2$ in (2.3.11). $\square$

Our next lemma gives a functional equation for the zeta functions $\zeta_{\alpha}^{\star}(s)$ and $\tilde{\zeta}_{\alpha}(s)$.

Lemma 2.3.2. *Let $r_{\alpha}(n)$ and $\tilde{r}_{\alpha}(n)$ be given by (2.1.40) and (2.3.5), respectively. Then the Dirichlet series defined by (2.3.8) can be analytically continued as an entire function satisfying Hecke's functional equation*

$$\pi^{-s}\Gamma(s)\,\zeta_{\alpha}^{\star}(s)=\pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\tilde{\zeta}_{\alpha}\left(\frac{\alpha}{2}-s\right), \tag{2.3.12}$$

where $\tilde{\zeta}_{\alpha}(s)$ is given by (2.3.6).

Proof. We have deduced above that $(-1)^n r_{\alpha}(n)$ are the coefficients of the Fourier expansion of $\vartheta_4^{\alpha}(\tau)$ at the cusp $i\infty$. It is clear from this that the Mellin transform holds

$$\pi^{-s}\Gamma\left(s\right)\,\zeta_{\alpha}^{\star}(s)=\int_0^{\infty}x^{s-1}\left\{\theta_4^{\alpha}(x)-1\right\}\,dx,\quad\mathrm{Re}(s)>\sigma_{\alpha},$$

where σ_α is explicitly given by (2.3.8). By following Riemann's paper [196, 228], we study the functional equation for $\zeta_\alpha^\star(s)$: indeed, for $\mathrm{Re}(s) > \sigma_\alpha$,

$$\pi^{-s}\Gamma(s)\,\zeta_\alpha^\star(s) = \int_0^\infty x^{s-1}\left\{\theta_4^\alpha(x)-1\right\}\,dx = \int_0^1 x^{s-1}\left\{\theta_4^\alpha(x)-1\right\}\,dx + \int_1^\infty x^{s-1}\left\{\theta_4^\alpha(x)-1\right\}\,dx. \tag{2.3.13}$$

Using the particular case of Jacobi's formula (2.1.8) given in (2.3.4),

$$\theta_4^\alpha(1/x) - 1 = x^{\alpha/2}\,\theta_2^\alpha(x) - 1,$$

we see that the first integral on the right of (2.3.13) can be written as

$$\int_0^1 x^{s-1}\left\{\theta_4^\alpha(x)-1\right\}\,dx = \int_1^\infty x^{-s-1}\left\{\theta_4^\alpha\left(1/x\right)-1\right\}\,dx = \int_1^\infty x^{\frac{\alpha}{2}-s-1}\theta_2^\alpha(x)\,dx - \frac{1}{s}. \tag{2.3.14}$$

Combining (2.3.13) and (2.3.14) leads to the representation

$$\pi^{-s}\Gamma\left(s\right)\,\zeta_\alpha^\star(s) = \int_1^\infty \left[x^{s-1}\left\{\theta_4^\alpha(x)-1\right\} + x^{\frac{\alpha}{2}-s-1}\,\theta_2^\alpha(x)\right]\,dx - \frac{1}{s}. \tag{2.3.15}$$

Since the integral on the right-hand side of (2.3.15) converges absolutely for any $s\in\mathbb{C}$, it represents an entire function of $s\in\mathbb{C}$. Hence, the previous representation gives the analytic continuation of $\zeta_\alpha^\star(s)$ as an entire function. Furthermore, since $\Gamma(s)$ has a simple pole at $s=0$, it is clear to see that $\zeta_\alpha^\star(0)=-1$.

We can do the same kind of computations with $\tilde{\zeta}_\alpha(s)$. We find that, for $\mathrm{Re}(s) > \tilde{\sigma}_\alpha := \frac{\alpha}{2}+1$ (see (2.3.6) above)

$$\begin{aligned}
\pi^{-s}\Gamma(s)\,\tilde{\zeta}_\alpha(s) &= \int_0^\infty x^{s-1}\theta_2^\alpha(x)\,dx = \int_0^1 x^{s-1}\theta_2^\alpha(x)\,dx + \int_1^\infty x^{s-1}\theta_2^\alpha(x)\,dx \\
&= \int_1^\infty x^{-s-1}\theta_2^\alpha(1/x)\,dx + \int_1^\infty x^{s-1}\,\theta_2^\alpha(x)\,dx \\
&= \int_1^\infty x^{-s-1}\left(x^{\alpha/2}\left(\theta_4^\alpha(x)-1\right)+x^{\alpha/2}\right)\,dx + \int_1^\infty x^{s-1}\theta_2^\alpha(x)\,dx \\
&= \int_1^\infty \left(x^{\frac{\alpha}{2}-s-1}\left(\theta_4^\alpha(x)-1\right)+x^{s-1}\theta_2^\alpha(x)\right)\,dx + \int_1^\infty x^{-s-1+\frac{\alpha}{2}}\,dx \\
&= \int_1^\infty \left(x^{\frac{\alpha}{2}-s-1}\left(\theta_4^\alpha(x)-1\right)+x^{s-1}\theta_2^\alpha(x)\right)\,dx + \frac{1}{s-\frac{\alpha}{2}}.
\end{aligned} \tag{2.3.16}$$

From the above expression we can see that $\tilde{\zeta}_\alpha(s)$ can be continued to the complex plane as a meromorphic function having a simple pole at $s=\frac{\alpha}{2}$ with residue $\frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2})}$. Furthermore, the right-hand sides of (2.3.15) and (2.3.16) can be turned into one another under the reflection $s \leftrightarrow \frac{\alpha}{2} - s$. This implies the functional equation (2.3.12) and completes the proof. $\square$

Since the functional equation (2.3.12) holds, we might expect that the summation formula (2.2.9) must be valid for the coefficients $(-1)^n r_\alpha(n)$ and $\tilde{r}_\alpha(n)$. The next lemma gives precisely this.

Lemma 2.3.3. *Let $r_\alpha(n)$ and $\tilde{r}_\alpha(n)$ be given by (2.1.40) and (2.3.5), respectively. Then, for $Re(x) > 0$ and any $y \in \mathbb{C}$, the following identity holds*

$$\sum_{n=1}^{\infty}(-1)^n\, r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(\sqrt{\pi\, n}\, y) = -\frac{y^{\frac{\alpha}{2}-1}\pi^{\frac{\alpha}{4}-\frac{1}{2}}}{2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)}$$
$$+\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,y}{x}\right). \tag{2.3.17}$$

Proof. By the previous result, we have that the pair of Dirichlet series

$$\phi(s) = \pi^{-s}\sum_{n=1}^{\infty}\frac{(-1)^n r_\alpha(n)}{n^s}, \quad \psi(s) = \pi^{-s}\sum_{n=0}^{\infty}\frac{\tilde{r}_\alpha(n)}{\left(n+\frac{\alpha}{4}\right)^s},$$

satisfies Hecke's functional equation (2.1.31). Thus, by an application of our summation formulas (2.2.9) or (2.2.24) to this pair, we just need to take the simple substitutions $r = \frac{\alpha}{2}$, $\rho = 0$ (because $\phi(s)$ is entire by the previous lemma) and $\phi(0) = -1$, thus obtaining (2.3.17). $\square$

2.3.3 The behavior of $\psi_\alpha(x, z)$

Using the previous Lemma 2.3.3, we are now able to show that $\psi_\alpha(x, z)$ has the same kind of behavior as the classical Jacobi's theta function when x approaches the imaginary axis. We now establish one of the most important results of the present chapter.

Lemma 2.3.4. *Let $\alpha > 0$ and $r_\alpha(n)$ be defined as the coefficients of the Fourier expansion of $\theta^\alpha(x)$, (2.1.40). For $Re(x) > 0$ and $z \in \mathbb{C}$, let $\psi_\alpha(x, z)$ denote the analogue of Jacobi's $\psi-$function (2.2.69),*

$$\psi_\alpha(x, z) = 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi x}\, z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty} r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(\sqrt{\pi n x}\, z). \tag{2.3.18}$$

Then, for any z satisfying the condition

$$z \in \mathscr{D}_\alpha := \left\{z \in \mathbb{C} \,:\, |Re(z)| < \sqrt{\frac{\pi\alpha}{2}},\, |Im(z)| < \sqrt{\frac{\pi\alpha}{2}}\right\}, \tag{2.3.19}$$

and every $m \in \mathbb{N}_0$, one has that

$$\frac{d^m}{d\omega^m}\left(1+\psi_\alpha\left(e^{2i\omega}, z\right)\right) \to 0, \quad as \quad \omega \to \frac{\pi}{4}^{-}. \tag{2.3.20}$$

Proof. Note that $e^{2i\omega}\to i$ as $\omega\to\frac{\pi}{4}^-$ along a circular path where $\mathrm{Re}(e^{2i\omega}),\mathrm{Im}(e^{2i\omega})>0$. Thus, we can write $e^{2i\omega}$ as $i+\delta$, where $\delta\to 0$ along any path in the region $|\arg(\delta)|<\frac{\pi}{2}$. With this change of variable and Faà di Bruno's formula, we can write

$$\frac{d^m}{d\omega^m}\psi_\alpha\left(e^{2i\omega},z\right)=(2i)^m\sum_{b_1,\ldots,b_m}\frac{m!}{b_1!\cdot\ldots\cdot b_m!}\prod_{j=1}^{m}\left(\frac{i+\delta}{j!}\right)^{b_j}\frac{d^{b_1+\ldots+b_m}}{d\delta^{b_1+\ldots+b_m}}\psi_\alpha\left(i+\delta,z\right), \qquad (2.3.21)$$

where the sum is over all nonnegative integers $b_1,\ldots,b_m$ such that $b_1+2b_2+\ldots+mb_m=m$. Therefore, to prove (2.3.20), we shall evaluate

$$\begin{aligned}&\psi_\alpha(i+\delta,\,z)\\&=2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n(i+\delta)}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n(i+\delta)}\,z\right)\\&=2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}(-1)^n\,r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\delta}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n(i+\delta)}\,z\right).\end{aligned}$$

Using the previous formula (2.3.17) with $x=\delta$ and $y=\sqrt{i+\delta}\,z$, we obtain

$$\begin{aligned}&1+\psi_\alpha(i+\delta,z)\\&=1+2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}(-1)^n\,r_\alpha(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\delta}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n(i+\delta)}\,z\right)\\&=2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\\&\times\sum_{n=0}^{\infty}\tilde{r}_\alpha(n)\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}}e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})(i+\delta)}\,z}{\delta}\right). \qquad (2.3.22)\end{aligned}$$

Note that (2.3.20) is established once we prove that any derivative (with respect to δ) of the last expression on the right-hand side of (2.3.22) tends to zero as $\delta\to 0$ along any path in the region $|\arg(\delta)|<\frac{\pi}{2}$. We will prove first that (2.3.20) holds for $m=0$ and the remaining cases will follow. In order to check this, we use the fact that $\psi_\alpha(x,z)$ is analytic as a function of both x in $\mathrm{Re}(x)>0$ and $z\in\mathbb{C}$,[8] so it suffices to bound each term of the series as $\delta\to 0^+$.

Like in Corollary 2.2.2, we divide the proof in two cases: $\alpha>1$ or $0<\alpha\leq 1$. For the first case, we use the integral representation for the modified Bessel function [[164], p. 252, eq. (10.32.2)], connected to the Poisson integral (2.2.26),

$$\left(\frac{z}{2}\right)^{-\nu}I_\nu(z)=\frac{1}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-t^2\right)^{\nu-\frac{1}{2}}e^{zt}dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\ \ z\in\mathbb{C}, \qquad (2.3.23)$$

[8]The analyticity of $\psi_\alpha(x,z)$ is, of course, a direct consequence of the more general result given in Corollary 2.2.2.

from which one can immediately obtain the bound (with real $\nu > -\frac{1}{2}$ and $z \in \mathbb{C}$)

$$\left|\left(\frac{z}{2}\right)^{-\nu} I_{\nu}(z)\right| \leq \frac{e^{|\text{Re}(z)|}}{\Gamma(\nu+1)}. \tag{2.3.24}$$

Using (2.3.24), Lemma 2.3.1 and the fact that $\tilde{r}_{\alpha}(0) = 2^{\alpha}$, we can bound $|1+\psi_{\alpha}(i+\delta,z)|$ simply as follows

$$\begin{aligned}
&|1+\psi_{\alpha}(i+\delta,z)| \\
&\leq \delta^{-\frac{\alpha}{2}}\, e^{-\frac{\text{Re}\left((i+\delta)\,z^2\right)}{4\delta}} \sum_{n=0}^{\infty} |\tilde{r}_{\alpha}(n)|\, e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)} \exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\text{Re}\left(\sqrt{i+\delta}\, z\right)\right|\right) \\
&< 2^{\alpha}\delta^{-\frac{\alpha}{2}}\, e^{-\frac{\text{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\, e^{-\frac{\pi\alpha}{4\delta}} \exp\left(\frac{\sqrt{\pi\alpha}}{2\delta}\left|\text{Re}\left(\sqrt{i+\delta}\, z\right)\right|\right) \\
&+ 540\,\delta^{-\frac{\alpha}{2}}\, e^{-\frac{\text{Re}\left((i+\delta)\,z^2\right)}{4\delta}} \sum_{n=1}^{\infty} (2n)^{\alpha/2}\, e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)} \exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\text{Re}\left(\sqrt{i+\delta}\, z\right)\right|\right).
\end{aligned}$$

Thus, to prove (2.3.20) it is sufficient to show that, for every fixed $n \in \mathbb{N}_0$ and $z \in \mathscr{D}_{\alpha}$ (see condition (2.3.19)),

$$\lim_{\delta\to 0^+} \delta^{-\frac{\alpha}{2}} \exp\left(-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right) + \frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\text{Re}\left(\sqrt{i+\delta}\, z\right)\right| - \frac{\text{Re}\left((i+\delta)\ z^2\right)}{4\delta}\right) = 0. \tag{2.3.25}$$

Using the fact that $\text{Re}\left(\sqrt{i+\delta}\, z\right) \to \text{Re}(\sqrt{i}\, z) = \frac{1}{\sqrt{2}}\left(\text{Re}(z) - \text{Im}(z)\right)$ as $\delta \to 0^+$ and $\text{Re}\left((i+\delta)\, z^2\right) \to -2\,\text{Re}(z)\,\text{Im}(z)$ also in this limit, we get

$$\begin{aligned}
&\lim_{\delta\to 0^+} \delta^{-\frac{\alpha}{2}}\, e^{-\frac{\text{Re}\left((i+\delta)\,z^2\right)}{4\delta}}\, e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)} \exp\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\text{Re}\left(\sqrt{i+\delta}\, z\right)\right|\right) \\
&= \lim_{\delta\to 0^+} \delta^{-\frac{\alpha}{2}} \exp\left[-\frac{\pi}{\delta}\left(\left(n+\frac{\alpha}{4}\right) - \frac{|\text{Re}(z)-\text{Im}(z)|}{\sqrt{2\pi}}\sqrt{n+\frac{\alpha}{4}} - \frac{\text{Re}(z)\,\text{Im}(z)}{2\pi}\right)\right].
\end{aligned} \tag{2.3.26}$$

Note that the term in the exponential is a quadratic polynomial of variable $X = \sqrt{n+\frac{\alpha}{4}}$, with the polynomial being explicitly given by

$$P(X) = X^2 - \frac{|\text{Re}(z)-\text{Im}(z)|}{\sqrt{2\pi}}\, X - \frac{\text{Re}(z)\,\text{Im}(z)}{2\pi}. \tag{2.3.27}$$

We know that, for a given $X_0 \in \mathbb{R}_{>0}$, $P(X_0) > 0$ if $X_0 > \sup\{y \in \mathbb{R} : P(y) = 0\}$. It is an easy task to compute the zeros of the polynomial $P(X)$, so the previous condition on X_0 reduces to

$$P(X_0) > 0 \ \text{ if } \ \frac{|\text{Re}(z)-\text{Im}(z)|}{2\sqrt{2\pi}} + \frac{|\text{Re}(z)+\text{Im}(z)|}{2\sqrt{2\pi}} < X_0. \tag{2.3.28}$$

Since the polynomial in the exponential is defined for $X=\sqrt{n+\frac{\alpha}{4}}$, $n\in\mathbb{N}_0$, a sufficient condition to assure that $P\left(\sqrt{n+\frac{\alpha}{4}}\right)>0$ for every $n\in\mathbb{N}_0$ is that z must satisfy

$$|\mathrm{Re}(z)-\mathrm{Im}(z)|+|\mathrm{Re}(z)+\mathrm{Im}(z)|<\sqrt{2\pi\alpha}, \tag{2.3.29}$$

which actually is the case if $z\in\mathscr{D}_\alpha=\left\{z\in\mathbb{C}\,:\,|\mathrm{Re}(z)|<\sqrt{\frac{\pi\alpha}{2}},\ |\mathrm{Im}(z)|<\sqrt{\frac{\pi\alpha}{2}}\right\}$. Therefore, under this hypothesis one sees that the limit in (2.3.26) goes to zero as $\delta\to0^{+}$. Hence,

$$1+\psi_\alpha\left(e^{2i\omega},z\right)\to0,\qquad \text{as}\ \ \omega\to\frac{\pi}{4}^{-}\ \ \text{and}\ \ z\in\mathscr{D}_\alpha.$$

Finally, since $\exp\left(-A/\delta\right)$, $A>0$, tends to zero faster than any power δ^{N} as $\delta\to0^{+}$, any derivative (with respect to δ) of $\psi_\alpha(i+\delta,\,z)$ will also go to zero in this limit, proving (2.3.20) for every $m\geq1$. This concludes the proof of the lemma for the case where $\alpha>1$. Assume now that $0<\alpha\leq1$: then the same argument must follow but we cannot invoke the integral representation (2.3.23) for $I_{\frac{\alpha}{2}-1}(z)$. Just like in (2.3.25), we need to evaluate the limit

$$\lim_{\delta\to0^{+}}\left|2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\frac{e^{-\frac{(i+\delta)z^{2}}{4\delta}}}{\delta}\,e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})(i+\delta)}\,z}{\delta}\right)\right| \tag{2.3.30}$$

for every fixed $n\in\mathbb{N}_0$. This can be done by appealing to the well-known Hankel expansion for the modified Bessel function (see [[164], p. 255, eq. (10.40.5)] and (2.1.57) above),

$$\begin{aligned}I_\nu(z)\sim&\ \frac{e^{z}}{\sqrt{2\pi z}}\,\frac{\cos(\pi\nu)}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\nu+n\right)\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^{n}n!}\\&+\frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}}\,\frac{\cos(\pi\nu)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma\left(\frac{1}{2}+\nu+n\right)\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^{n}n!},\end{aligned} \tag{2.3.31}$$

which is valid for $-\frac{\pi}{2}<\arg(z)<\frac{3\pi}{2}$, $|z|\to\infty$. Alternatively, in the range $-\frac{3}{2}\pi<\arg(z)<\frac{\pi}{2}$, $|z|\to\infty$, we have the expansion

$$\begin{aligned}I_\nu(z)\sim&\ \frac{e^{z}}{\sqrt{2\pi z}}\,\frac{\cos(\pi\nu)}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\nu+n\right)\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^{n}\,n!}\\&+\frac{e^{-z-(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}}\,\frac{\cos(\pi\nu)}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n}\Gamma\left(\frac{1}{2}+\nu+n\right)\Gamma\left(\frac{1}{2}-\nu+n\right)}{(2z)^{n}n!}.\end{aligned} \tag{2.3.32}$$

In any case, from (2.3.31) and (2.3.32) and real ν, one can obtain the simple bound

$$|I_\nu(z)|\leq2\left(\frac{e^{\mathrm{Re}(z)}}{\sqrt{2\pi|z|}}+\frac{e^{-\mathrm{Re}(z)}}{\sqrt{2\pi|z|}}\right)\leq\sqrt{\frac{8}{\pi|z|}}\,\exp\left(|\mathrm{Re}(z)|\right), \tag{2.3.33}$$

which becomes valid for any $|z|>M$, with M being a sufficiently large number. We now apply (2.3.33) to (2.3.30) and we obtain, for a sufficiently small δ, some fixed $n\in\mathbb{N}_0$ and $z\in\mathscr{D}_{\alpha}$,

$$\left|2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,\left(\sqrt{\pi(i+\delta)}\,z\right)^{1-\frac{\alpha}{2}}\,\frac{e^{-\frac{(i+\delta)z^{2}}{4\delta}}}{\delta}\,e^{-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})\,(i+\delta)}\,z}{\delta}\right)\right|$$
$$<\frac{\pi^{-\frac{\alpha+1}{4}}2^{\frac{\alpha}{2}+1}\Gamma\left(\frac{\alpha}{2}\right)|z|^{\frac{1-\alpha}{2}}}{(n+\frac{\alpha}{4})^{\frac{1}{4}}\sqrt{\delta}}\exp\left(-\frac{\pi}{\delta}\left(n+\frac{\alpha}{4}\right)+\frac{\sqrt{\pi(n+\frac{\alpha}{4})}}{\delta}\left|\mathrm{Re}\left(\sqrt{i+\delta}z\right)\right|-\frac{\mathrm{Re}((i+\delta)z^{2})}{4\delta}\right).$$

Note that there is no problem if z is arbitrarily close to the origin because we are supposing that $0<\alpha\leq 1$. Thus, to prove that (2.3.30) tends to zero, we just follow the same logic as before, because the term in the exponential is exactly the same as in the previous case (2.3.26) and it involves the quadratic polynomial (2.3.27). Thus, the condition $z\in\mathscr{D}_{\alpha}$ needs to be kept in this range of α and this completes the proof of (2.3.20). □

2.3.4 Some additional lemmas

With Lemma 2.3.4 completely established, we are ready to prove Theorem 2.1.3. Before doing this we provide two additional lemmas, one of which being an easy consequence of Lemma 2.3.4.

Lemma 2.3.5. *Let $h:\mathbb{C}\longmapsto\mathbb{C}$ be an analytic function. Then, for any $z\in\mathscr{D}_{\alpha}$, (2.3.19), and arbitrary $m\in\mathbb{N}_0$, we have*

$$\lim_{\omega\to\frac{\pi}{4}^{-}}\frac{d^{m}}{d\omega^{m}}\left\{h(\omega)\,\left(e^{-z^{2}/8}+e^{z^{2}/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\right\}=-2\,h^{(m)}\left(\frac{\pi}{4}\right)\,\sinh\left(\frac{z^{2}}{8}\right). \tag{2.3.34}$$

Proof. Applying the Leibniz formula for the derivative, we obtain

$$\begin{aligned}&\frac{d^{m}}{d\omega^{m}}\left\{h(\omega)\,\left(e^{-z^{2}/8}+e^{z^{2}/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\right\}\\&=\sum_{k=0}^{m}\left(\begin{array}{c}m\\k\end{array}\right)\,h^{(m-k)}(\omega)\,\frac{d^{k}}{d\omega^{k}}\left(e^{-z^{2}/8}+e^{z^{2}/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right).\end{aligned} \tag{2.3.35}$$

Letting $\omega\to\frac{\pi}{4}^{-}$, we see from Lemma 2.3.4 that any derivative of order $k\geq 1$ of $\psi_{\alpha}\left(e^{2i\omega},z\right)$ tends to zero as $\omega\to\frac{\pi}{4}^{-}$. Thus, the only surviving term is when $k=0$, which immediately gives (2.3.34). □

Since our proof will follow very closely the argument in [73], we will also require Kronecker's lemma [111].

Lemma 2.3.6. *If $x\notin\mathbb{Q}$, then the sequence of fractional parts $(\{n\,x\})_{n\in\mathbb{N}}$ is dense in the interval $(0,1)$.*

2.3.5 Proof of Theorem 2.1.3

The proof that follows uses the previous lemmas and integral representations, but the main ideas are due to Hardy and to a variation of his argument given by Dixit et al. in the papers [73, 74].

We start by using the integral representation given in Example 2.2.1, (2.2.71), replacing there x by $e^{2i\omega}$, $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right){}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}e^{-\frac{z^2}{4}}.$$

Using Kummer's formula (2.2.8), we can get the more symmetric formulation

$$\frac{e^{-\frac{z^2}{8}}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right){}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,e^{\frac{z^2}{8}}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\alpha}{2}}e^{-\frac{z^2}{8}}. \quad (2.3.36)$$

If we add and subtract the term $e^{-z^2/8}\,e^{i\omega\alpha/2}$ on the right-hand side of the previous equality, we can rewrite (2.3.36) in the form

$$\begin{aligned}&\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right){}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\&=-2\,e^{-z^2/8}\cos\left(\frac{\omega\alpha}{2}\right)+e^{\frac{i\omega\alpha}{2}}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right).\end{aligned} \quad (2.3.37)$$

After having an identity like (2.3.37), the structure of our proof follows [73] almost line by line. Since we want to study the zeros of an infinite combination of $\eta_\alpha(s+i\lambda_j)\,{}_1F_1(s+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4})$, we change the variable to get

$$\begin{aligned}&\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+i(t+\lambda_j)\right){}_1F_1\left(\frac{\alpha}{4}-i(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\&=\frac{e^{-z^2/8}e^{-2\omega\lambda_j}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right){}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\&=e^{-2\omega\lambda_j}\left\{-2\,e^{-z^2/8}\cos\left(\frac{\omega\alpha}{2}\right)+e^{\frac{i\omega\alpha}{2}}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}\\&=-\,e^{-z^2/8}\,e^{-2\omega\lambda_j+i\,\frac{\omega\alpha}{2}}-\,e^{-z^2/8}\,e^{-2\omega\lambda_j-i\,\frac{\omega\alpha}{2}}+e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right).\end{aligned} \quad (2.3.38)$$

Proceeding now as in Hardy's own proof [106], let us take the p^{th} derivative of both sides of (2.3.38) with respect to ω: since the integral on the left of (2.3.38) converges absolutely and uniformly with respect to $\omega\in\left(-\frac{\pi}{4},\,\frac{\pi}{4}\right)$, one can apply the Leibniz rule and deduce the equality

$$\begin{aligned}&\frac{2^p\,e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}t^p\,\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right){}_1F_1\left(\frac{\alpha}{4}-i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\&=-\,e^{-z^2/8}\left(-2\lambda_j+i\frac{\alpha}{2}\right)^p e^{-2\omega\lambda_j+i\,\frac{\omega\alpha}{2}}-\,e^{-z^2/8}\left(-2\lambda_j-i\frac{\alpha}{2}\right)^p e^{-2\omega\lambda_j-i\,\frac{\omega\alpha}{2}}\\&\quad+\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}.\end{aligned} \quad (2.3.39)$$

We can now consider the change of variables,

$$\frac{i\alpha}{2}-2\lambda_j:=r_j\,e^{i\theta_j},\ \ 0<\theta_j<\pi, \tag{2.3.40}$$

and, using these new coordinates, we are able to write (2.3.39) in a more suitable form

$$\begin{aligned}&\int_{-\infty}^{\infty}t^p\,\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right){}_1F_1\left(\frac{\alpha}{4}-i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\,e^{2\omega t}\,dt\\&=-4\pi\,\left(\frac{r_j}{2}\right)^p\,e^{-2\omega\lambda_j}\,\cos\left(p\,\theta_j+\frac{\omega\alpha}{2}\right)\\&\quad+\frac{2\pi}{2^p}\,e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}.\end{aligned} \tag{2.3.41}$$

In order to apply Hardy's method, we need to have a real integrand on the left-hand side of (2.3.41) (to count its sign changes). This is done if we sum another integral

$$\int_{-\infty}^{\infty}t^p\,\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right){}_1F_1\left(\frac{\alpha}{4}+i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{\overline{z}^2}{4}\right)\,e^{2\omega t}\,dt,$$

whose integrand is the complex conjugate of the integrand of (2.3.41). Taking the real part in both sides of (2.3.41), multiplying by $(c_j)_{j\in\mathbb{N}}\in\ell^1$ and summing over j, we can formally derive the relation

$$\begin{aligned}&\int_{-\infty}^{\infty}t^p\,F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,e^{2\omega t}\,dt\\&=-\frac{8\pi}{2^p}\,\sum_{j=1}^{\infty}c_j\,r_j^pe^{-2\omega\lambda_j}\,\cos\left(p\,\theta_j+\frac{\omega\alpha}{2}\right)\\&\quad+\frac{4\pi}{2^p}\,\mathrm{Re}\left(e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}\right),\end{aligned} \tag{2.3.42}$$

where $F_{z,\alpha}(s)$ is the function given by (2.1.14),

$$F_{z,\alpha}(s)=\sum_{j=1}^{\infty}c_j\eta_\alpha\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\frac{\alpha}{2};\frac{z^2}{4}\right)+{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\frac{\alpha}{2};\frac{\overline{z}^2}{4}\right)\right\},$$

whose zeros we will now study.

But before this, we justify (2.3.42) by following a similar reasoning to that given in [[73], p. 316]. By Stirling's formula and convex estimates for $\zeta_\alpha(s)$ (obtained from the general Phragmén-Lindelöf principle (2.1.39), we know that

$$\left|\eta_\alpha\left(\frac{\alpha}{4}+it\right)\right|\ll_\alpha\ |t|^{A(\alpha)}\,e^{-\frac{\pi}{2}|t|},\quad |t|\to\infty.$$

This, together with the asymptotic estimate for Kummer's function (2.2.6),

$$\left|{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right| \ll_{\alpha,z} |t|^{\frac{1}{4}-\frac{\alpha}{4}}\exp\left(|z|\sqrt{|t|}\right), \quad |t|\to\infty,$$

yields

$$\sum_{j=1}^{\infty}\left|c_j\,\eta_\alpha\left(\frac{\alpha}{4}+i\,(t+\lambda_j)\right)\mathrm{Re}\left({}_1F_1\left(\frac{\alpha}{4}-i\,(t+\lambda_j);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right)\right|$$
$$\ll_{\alpha,z} C_\lambda\,|t|^{B(\alpha)}\,e^{-\frac{\pi}{2}|t|+|z|\sqrt{|t|}}\sum_{j=1}^{\infty}|c_j|<\infty, \tag{2.3.43}$$

where we have used the fact that $(\lambda_j)_{j\in\mathbb{N}}$ is a bounded sequence and $(c_j)_{j\in\mathbb{N}}\in\ell^1$. The term C_λ only stands for a positive constant which depends on the bounds of the sequence $(\lambda_j)_{j\in\mathbb{N}}$. This observation now allows the interchange of the orders of integration and summation in the process leading to the left-hand side of (2.3.42). The very same reasoning can be used to show the procedure

$$\sum_{j=1}^{\infty}c_j\,\frac{d^p}{d\omega^p}\left\{e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}$$
$$=\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}.$$

The strategy now is to let $\omega\to\frac{\pi}{4}^-$ on both sides of (2.3.42). The previous lemmas are indispensable to study the behavior of the right-hand side under this limit. Since $(c_j)_{j\in\mathbb{N}}\in\ell^1$, it is clear that the function

$$h_\alpha(\omega):=\sum_{j=1}^{\infty}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}$$

is analytic for any $\omega\in\mathbb{C}$. Hence, according to Lemma 2.3.5 and appealing to the polar coordinates (2.3.40),

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_je^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}$$
$$=\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^p}{d\omega^p}\left\{h_\alpha(\omega)\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)\right)\right\}$$
$$=-2\sinh\left(\frac{z^2}{8}\right)\sum_{j=1}^{\infty}c_j\left(\frac{i\alpha}{2}-2\lambda_j\right)^p e^{\frac{i\pi\alpha}{8}-\frac{\pi}{2}\lambda_j}$$
$$=-2\sinh\left(\frac{z^2}{8}\right)\sum_{j=1}^{\infty}c_j\,r_j^p\,e^{-\frac{\pi}{2}\lambda_j}\,e^{i\left(\frac{\pi\alpha}{8}+p\,\theta_j\right)}. \tag{2.3.44}$$

Returning to the identity (2.3.42) and combining it with (2.3.44) gives

$$\begin{aligned}
&\lim_{\omega\to\frac{\pi}{4}^{-}}\int_{-\infty}^{\infty} t^{p}F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)e^{2\omega t}\,dt\\
&=-\frac{8\pi}{2^{p}}\sum_{j=1}^{\infty}c_{j}r_{j}^{p}\,e^{-\frac{\pi}{2}\lambda_{j}}\left[\cos\left(p\,\theta_{j}+\frac{\pi}{8}\alpha\right)+\text{Re}\left(e^{z^{2}/8}\sinh\left(\frac{z^{2}}{8}\right)\,e^{i\left(\frac{\pi\alpha}{8}+p\,\theta_{j}\right)}\right)\right]\\
&=-\frac{8\pi}{2^{p}}\sum_{j=1}^{\infty}c_{j}r_{j}^{p}e^{-\frac{\pi}{2}\lambda_{j}}\left[\cos\left(p\,\theta_{j}+\frac{\pi}{8}\alpha\right)\left(1+\text{Re}\left(e^{z^{2}/8}\sinh\left(\frac{z^{2}}{8}\right)\right)\right)\right.\\
&\quad\left.-\sin\left(p\theta_{j}+\frac{\pi}{8}\alpha\right)\,\text{Im}\left(e^{z^{2}/8}\sinh\left(\frac{z^{2}}{8}\right)\right)\right].
\end{aligned}\tag{2.3.45}$$

If we continue to follow [73] and set

$$u_{z}:=1+\text{Re}\left(e^{z^{2}/8}\sinh\left(\frac{z^{2}}{8}\right)\right),\quad v_{z}:=\text{Im}\left(e^{z^{2}/8}\sinh\left(\frac{z^{2}}{8}\right)\right),\tag{2.3.46}$$

we are able to write the right-hand side of (2.3.45) in the appropriate form

$$-\frac{8\pi}{2^{p}}w_{z}\sum_{j=1}^{\infty}c_{j}r_{j}^{p}e^{-\frac{\pi}{2}\lambda_{j}}\cos\left(p\theta_{j}+\frac{\pi\alpha}{8}+\beta_{z}\right),\ \text{for}\ \ w_{z}=\sqrt{u_{z}^{2}+v_{z}^{2}},\ \ \beta_{z}:=\arctan\left(\frac{v_{z}}{u_{z}}\right).\tag{2.3.47}$$

Since all the parameters in the previous expression are fixed either by the Dirichlet series or by the sequence taken in the statement, $(\theta_j)_{j\in\mathbb{N}}$, the only free parameter is the integer p, which is the number of times we have differentiated both sides of (2.3.38). Therefore, the previous expression can be thought of as a sequence of real numbers, say $(s_p)_{p\in\mathbb{N}}$. Under only some minor modifications of the argument given in [[73], pp. 317-321], we now show that the sequence $(s_p)_{p\in\mathbb{N}}$ defined by (2.3.47) changes its sign for infinitely many values of p. Since $(\lambda_j)_{j\in\mathbb{N}}$ attains its bounds and is made of distinct elements, we know that there is some positive integer M such that

$$|\lambda_M|=\max_{j\in\mathbb{N}}\left\{|\lambda_j|\right\},\quad \lambda_j\neq\lambda_M\ \ \text{for}\ j\neq M.\tag{2.3.48}$$

We have now two possibilities: either there is an element of the sequence $(\lambda_j)_{j\in\mathbb{N}}$, say λ_N, such that $\lambda_N=-\lambda_M$ or does not. If such an element exists, our condition (2.3.48) implies that

$$r_j<r_M\ \ \text{for}\ j\neq M,N\ \text{and}\ r_M=r_N,\tag{2.3.49}$$

where r_M and r_N are the radial coordinates of λ_M and λ_N (see (2.3.40) above). On the other hand, if λ_N does not exist, then (2.3.48) implies that

$$r_j<r_M\ \ \text{for}\ j\neq M.\tag{2.3.50}$$

Let us suppose that the latter case is valid, i.e., that the element λ_N does not exist. The case where λ_N exists will be addressed at the end of the proof. Observe that, under this assumption, we may write the series in (2.3.47) as

$$-\frac{8\pi}{2^p}\,w_z\,c_M r_M^p\,e^{-\frac{\pi}{2}\lambda_M}\,\cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)\{1+E(X,z)+H(X,z)\}\,, \tag{2.3.51}$$

where

$$E(X,z):=\sum_{j\neq M,\,j\leq X}\frac{c_j}{c_M}\,e^{-\frac{\pi}{2}(\lambda_j-\lambda_M)}\left(\frac{r_j}{r_M}\right)^p\frac{\cos\left(p\theta_j+\frac{\pi\alpha}{8}+\beta_z\right)}{\cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)}$$

and

$$H(X,z):=\sum_{j\neq M,\,j>X}\frac{c_j}{c_M}\,e^{-\frac{\pi}{2}(\lambda_j-\lambda_M)}\left(\frac{r_j}{r_M}\right)^p\frac{\cos\left(p\theta_j+\frac{\pi\alpha}{8}+\beta_z\right)}{\cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)},$$

with X being some large parameter that will be chosen at a later stage of the proof.

We will now see that, as a function of p, the sign of $\cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)$ will prevail in (2.3.51), making the sign of the integral on the left of (2.3.45) to change infinitely often. We divide the proof of this fact in two items: in the first item, we show it for $\frac{\theta_M}{2\pi}\notin\mathbb{Q}$ and in the second we argue that the case $\frac{\theta_M}{2\pi}\in\mathbb{Q}$ ultimately reduces to the former.

1. Assume that $\frac{\theta_M}{2\pi}\notin\mathbb{Q}$: by Kronecker's lemma (see Lemma 2.3.6), $\left(\left\{n\,\frac{\theta_M}{2\pi}\right\}\right)_{n\in\mathbb{N}}$ is a dense subset of $(0,1)$. Hence,

$$\left(\cos\left(n\,\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)\right)_{n\in\mathbb{N}}=\left(\cos\left(2\pi\left\{\frac{n\,\theta_M}{2\pi}\right\}+\frac{\pi\alpha}{8}+\beta_z\right)\right)_{n\in\mathbb{N}}$$

is a dense subset of $[-1,1]$ due to continuity and surjectivity of the function $f(X)=\cos(X+a)$. Therefore, we can find two sequences of integers $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ such that, as $n\to\infty$,

$$\cos\left(2\pi\left\{\frac{q_n\,\theta_M}{2\pi}\right\}+\frac{\pi\alpha}{8}+\beta_z\right)\to\frac{1}{2},\quad \cos\left(2\pi\left\{\frac{r_n\,\theta_M}{2\pi}\right\}+\frac{\pi\alpha}{8}+\beta_z\right)\to-\frac{1}{2}. \tag{2.3.52}$$

Thus, for a sufficiently large N_0 and $p\in(q_n)_{n\geq N_0}\cup(r_n)_{n\geq N_0}$, we have the inequality

$$\left|\cos\left(p\theta_M+\frac{\pi\alpha}{8}+\beta_z\right)\right|\geq\frac{1}{3}.$$

Since $(\lambda_k)_{k\in\mathbb{N}}$ is bounded and attains its bounds, under this choice of N_0 we have the bound for $H(X,z)$,

$$|H(X,z)|\leq\frac{3}{|c_M|}\,e^{\frac{\pi}{2}\max_{k,\ell}|\lambda_k-\lambda_\ell|}\sum_{j\neq M,\,j>X}|c_j|<\frac{1}{3},$$

once we take $X\geq X_0$ large enough. With the same choice of X, we bound $E(X,z)$ as follows

$$|E(X,z)|\leq\frac{3\mu_X^p}{|c_M|}\,e^{\frac{\pi}{2}\max_{k,\ell}|\lambda_k-\lambda_\ell|}\sum_{j\neq M,\,j\leq X}|c_j|,$$

where

$$\mu_X=\max_{j\leq X}\left\{\frac{r_j}{r_M}\right\}.$$

By (2.3.50), we know that $\mu_X < 1$ and so, for N_0 sufficiently large and $p \geq \max\{q_{N_0}, r_{N_0}\}$,

$$|E(X,z)| < \frac{1}{3}.$$

Therefore, under our choices of N_0 and X, $1 + E(X,z) + H(X,z) > \frac{1}{3}$, and so, for $p \in (q_n)_{n\geq N_0} \cup (r_n)_{n\geq N_0}$, the limit

$$\lim_{\omega\to\frac{\pi}{4}^-} \int_{-\infty}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\, dt$$

must have the same sign (as a sequence depending on $p \in (q_n)_{n\in\mathbb{N}} \cup (r_n)_{n\in\mathbb{N}}$) as the sequence of real numbers

$$\left(s_p'\right)_{p\in(q_n)_{n\geq N_0}\cup(r_n)_{n\geq N_0}} := -\frac{8\pi}{2^p}\, w_z\, c_M r_M^p\, e^{-\frac{\pi}{2}\lambda_M}\, \cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right),$$

whose sign changes infinitely often by the construction (2.3.52).

2. We now deal with the case where $\frac{\theta_M}{2\pi} \in \mathbb{Q}$ making the same remark as in [[73], pp. 320-321]. Note that, for a fixed small $\delta > 0$, Theorem 2.1.3 is equivalent to the statement that $F_{z,\alpha}(s+i\delta)$ has infinitely many zeros at the line $\mathrm{Re}(s) = \frac{\alpha}{4}$. However, throughout the course of the previous argument we just need to replace the elements of the sequence $(\lambda_j)_{j\in\mathbb{N}}$ by $\lambda_j^\star := \lambda_j + \delta$. But now in this new analysis we need to rewrite (2.3.40) as $\frac{i\alpha}{2} - 2\lambda_j^\star = r_j^\star\, e^{i\theta_j^\star}$. Since we have the freedom of choosing δ, we may take it in such a way that $\frac{\theta_M^\star}{2\pi} \notin \mathbb{Q}$, due to the density of the irrational numbers in the real line. Therefore, if we prove Theorem 2.1.3 under the hypothesis $\frac{\theta_M}{2\pi} \notin \mathbb{Q}$, we can easily see that the same strategy will work for $\frac{\theta_M}{2\pi} \in \mathbb{Q}$.

We are ready to finish the proof of Theorem 2.1.3. By contradiction, let us assume that $F_{z,\alpha}(s)$ has only a finite number of zeros located on the line $\mathrm{Re}(s) = \frac{\alpha}{4}$. Under this hypothesis, there exists some $T_0 > 0$ such that one of the following situations takes place:

1. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $|t| > T_0$;
2. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) < 0$ for $|t| > T_0$;
3. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $t > T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) < 0$ for $t < -T_0$;
4. $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) < 0$ for $t > T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $t < -T_0$.

But the second contradiction hypothesis easily reduces to the first if we carry out the proof with $(c_j)_{j\in\mathbb{N}}$ replaced by $(-c_j)_{j\in\mathbb{N}}$. Analogously, the fourth hypothesis reduces to the third. Thus, for the sake of completing the proof, we just need to deal with cases 1 and 3 above.

By hypotheses 1. and 3., we know that $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $t > T_0$: this implies the inequality

$$\lim_{\omega\to\frac{\pi}{4}^-} \int_{T_0}^{T} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\, dt \leq \lim_{\omega\to\frac{\pi}{4}^-} \int_{T_0}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\, dt := L_{z,p}(T_0), \tag{2.3.53}$$

where $L_{z,p}(T_0)$ exists and is finite, according to (2.3.45). Therefore, for any $T \geq T_0$, the following inequality holds

$$\int_{T_0}^{T} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t}\, dt \leq L_{z,p}\left(T_0\right). \tag{2.3.54}$$

But since $T \geq T_0$ is arbitrary, (2.3.54) implies that

$$\int_{T_0}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t}\, dt \leq L_{z,p}\left(T_0\right). \tag{2.3.55}$$

Using (2.3.55) and one of the contradiction hypotheses 1 or 3, we now prove that

$$\int_{-\infty}^{\infty} t^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t}\, dt < \infty. \tag{2.3.56}$$

In fact, since the hypotheses 1 and 3 imply that $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) = \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|$ for $t > T_0$ and $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) = \pm\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|$ for[9] $t < -T_0$,

$$\begin{aligned}
\int_{-\infty}^{\infty} t^p F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} dt &= \int_{0}^{\infty} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&= \int_{0}^{T_0} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&+ \int_{T_0}^{\infty} \left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t} \pm (-1)^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right| e^{-\frac{\pi}{2}t}\right\} t^p dt.
\end{aligned} \tag{2.3.57}$$

If $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) > 0$ for $t < -T_0$ (condition 1 above), we take p as an even integer and, if $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) < 0$ for $t < -T_0$ (condition 3 above), we assume p to be odd. In any case, the second integral in (2.3.57) reduces to

$$\begin{aligned}
&\int_{T_0}^{\infty} \left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t} + \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right| e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&= \int_{-\infty}^{\infty} |t|^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t} dt - \int_{-T_0}^{T_0} |t|^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t} dt,
\end{aligned} \tag{2.3.58}$$

[9] here, the + sign is under hypothesis 1. while the − sign is under hypothesis 3.

and so, combining (2.3.57) and (2.3.58),

$$\begin{aligned}
&\int_{-\infty}^{\infty} |t|^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t}dt \\
&= \int_{-\infty}^{\infty} t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t}dt + \int_{-T_0}^{T_0} |t|^p \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t}dt \\
&- \int_{0}^{T_0} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt < \infty, \qquad (2.3.59)
\end{aligned}$$

establishing (2.3.56). We are now free to use the dominated convergence theorem: by (2.3.45), (2.3.51) and (2.3.56),

$$\begin{aligned}
&\int_{0}^{\infty} \left\{t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, t^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p\, dt \\
&= -\frac{8\pi}{2^p}\, w_z\, c_M r_M^p\, e^{\frac{\pi}{2}\lambda_M} \cos\left(p\theta_M + \frac{\pi\alpha}{8} + \beta_z\right) \{1 + E(X,z) + H(X,z)\}\,. \qquad (2.3.60)
\end{aligned}$$

But we have already seen that there are infinitely many integers $p \in (q_n)_{n\geq N_0} \cup (r_n)_{n\geq N_0}$ for which the right-hand side of (2.3.60) is negative, this is, there are infinitely many p such that

$$\begin{aligned}
&\int_{T_0}^{\infty} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&< -\int_{0}^{T_0} \left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt \\
&< T_0^p \int_{0}^{T_0} \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t}\right| dt \leq K(T_0)\, T_0^p, \qquad (2.3.61)
\end{aligned}$$

where K only depends on T_0 but not on p. By our hypotheses (1 or 3) over $F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ (and for the choice of p depending whether 1 or 3 takes place), we know that there exists some $\epsilon := \epsilon(T_0) > 0$ such that

$$\begin{aligned}
&F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{\frac{\pi}{2}t} + (-1)^p\, F_{z,\alpha}\left(\frac{\alpha}{4}-it\right) e^{-\frac{\pi}{2}t} \\
&= \left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| e^{\frac{\pi}{2}t} + \left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right| e^{-\frac{\pi}{2}t} \geq \epsilon(T_0), \qquad \forall t \in [2T_0,\, 2T_0+1]\,.
\end{aligned}$$

Under the existence of $\epsilon(T_0)$, we have the lower bound

$$\int\limits_{T_0}^{\infty}\left\{F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)e^{\frac{\pi}{2}t}+(-1)^p\,F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)e^{-\frac{\pi}{2}t}\right\}t^p dt$$
$$>\int\limits_{2T_0}^{2T_0+1}\left\{\left|F_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right|e^{\frac{\pi}{2}t}+\left|F_{z,\alpha}\left(\frac{\alpha}{4}-it\right)\right|e^{-\frac{\pi}{2}t}\right\}t^p dt\geq\epsilon\left(T_0\right)\left(2T_0\right)^p. \tag{2.3.62}$$

However, if we compare (2.3.61) with (2.3.62), we conclude that the inequality

$$2^p\leq\frac{K(T_0)}{\epsilon(T_0)}$$

must hold for infinitely many values of $p\in(p_n)_{n\geq N_0}\cup(q_n)_{n\geq N_0}$. This is absurd because we can take p as large as desired.

It only remains to prove Theorem 2.1.3 in the case where there is some $N\in\mathbb{N}$ such that $\lambda_N=-\lambda_M$. We reduce this case to the assumption treated above, this is, when there is no integer N such that $\lambda_N=-\lambda_M$. Without loss of generality, let us assume that λ_M constructed by (2.3.48) is positive and, for a fixed positive real number δ (smaller than λ_M), consider the function $F_{z,\alpha}(s+i\delta)$. As we have seen in the case where $\frac{\theta_M}{2\pi}\in\mathbb{Q}$, proving that $F_{z,\alpha}(s)$ has infinitely many zeros at the critical line $\mathrm{Re}(s)=\frac{\alpha}{4}$ is equivalent to proving the same statement for $F_{z,\alpha}(s+i\delta)$. Since the function $F_{z,\alpha}(s+i\delta)$ satisfies the conditions of our Theorem 2.1.3 with the same $z\in\mathscr{D}_\alpha$ and the same set of coefficients $(c_j)_{j\in\mathbb{N}}$, the only modification is that we need to replace the sequence $(\lambda_j)_{j\in\mathbb{N}}$ by $\lambda_j^\star:=\lambda_j+\delta$. It is clear that $\max_{j\in\mathbb{N}}\left\{|\lambda_j^\star|\right\}=\lambda_M^\star:=\lambda_M+\delta$ but $|\lambda_N^\star|=|-\lambda_M+\delta|=\lambda_M-\delta<\lambda_M^\star$. Therefore, for the new sequence $(\lambda_j^\star)_{j\in\mathbb{N}}$ we have that $r_j^\star<r_M^\star$, (2.3.50), for every $j\neq M$. □

2.4 Zeros of combinations attached to $\zeta(s,Q)$

In this section we apply all the previous work to prove a version of Theorem 2.1.3 where $\zeta_\alpha(s)$ is replaced by the Epstein zeta function attached to a binary, integral and positive definite quadratic form. To adapt the work done in the proof of Theorem 2.1.3, as well as to apply the summation formula (2.2.9), we need some lemmas concerning twists of the Epstein zeta function by an additive character $e^{2\pi inp/q}$, $p/q\in\mathbb{Q}$.

The first lemma given in the next subsection is essentially given in [[51], Theorems 2 and 3] and was used by Jutila to derive exponential sums attached to Epstein zeta functions in [123]. However, to match the notation introduced in our thesis and to be easier to apply the next result to our summation formula, we will give a brief proof of it.

2.4.1 Exponential sums attached to quadratic forms

Lemma 2.4.1. *Let $(p,\,q)=1$ and $Q(m,\,n)=Am^2+Bmn+Cn^2$ be a binary, positive and integral quadratic form with discriminant $\Delta:=4AC-B^2$. Also, consider the periodic Epstein zeta function*

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right):=\sum_{m,n\neq 0}\frac{\exp\left(-\frac{2\pi ip}{q}\left(Am^2+Bmn+Cn^2\right)\right)}{\left(Am^2+Bmn+Cn^2\right)^s},\quad Re(s)>1. \tag{2.4.1}$$

Then $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ can be analytically continued as a meromorphic function with (at most) one simple pole located at $s=1$, whose residue is

$$Res_{s=1}\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\frac{2\pi}{q^2\sqrt{\Delta}}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}, \tag{2.4.2}$$

under the convention that, if the previous value is zero, $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ is an entire function. Moreover, $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ satisfies the functional equation

$$\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-(1-s)}\Gamma\left(1-s\right)\,\tilde{\zeta}\left(1-s,\,Q,\,\frac{p}{q}\right), \tag{2.4.3}$$

where $\tilde{\zeta}(s;\,Q;\,p/q)$ represents the Dirichlet series

$$\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right):=\sum_{m,n\neq 0}\frac{b_Q(m,n,p/q)}{Q(m,n)^s},\quad Re(s)>1, \tag{2.4.4}$$

with

$$b_Q(m,n,p/q):=\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}\,e^{\frac{2\pi i}{q}(mk_1+nk_2)}. \tag{2.4.5}$$

Proof. For $\mathrm{Re}(s)>1$, we start by writing (2.4.1) as

$$\begin{aligned}\zeta\left(s,Q,\frac{p}{q}\right):&=\sum_{m,n\neq 0}\frac{e^{-2\pi i\frac{p}{q}Q(m,n)}}{Q(m,n)^s}=\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\frac{p}{q}Q(k_1,k_2)}\sum_{\ell_1,\ell_2\neq 0}\frac{1}{Q\left(\ell_1q+k_1,\ell_2q+k_2\right)^s}\\&=q^{-2s}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\frac{p}{q}Q(k_1,k_2)}\sum_{\ell_1,\ell_2\neq 0}\frac{1}{Q\left(\ell_1+\frac{k_1}{q},\ell_2+\frac{k_2}{q}\right)^s}.\end{aligned} \tag{2.4.6}$$

Note that the infinite series with respect to ℓ_1,ℓ_2 is well defined because the denominator displayed there cannot be zero. For $\mathrm{Re}(s)>1$, it represents a particular example of the Epstein zeta function (2.1.48),

$$\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right):=\sum_{\mathbf{m}\in\mathbb{Z}^n,\,\mathbf{m}+\mathbf{g}\neq\mathbf{0}}\frac{\exp\left(2\pi i\,\mathbf{m}\cdot\mathbf{h}\right)}{Q\left(\mathbf{m}+\mathbf{g}\right)^s},\quad\mathrm{Re}(s)>\frac{n}{2}, \tag{2.4.7}$$

when Q is binary and $\mathbf{h}=(0,0)$. We know from Lemma 2.1.2 that, for $\mathbf{h}\notin\mathbb{Z}^n$, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ can be analytically continued as an entire function. If, however, $\mathbf{h}\in\mathbb{Z}^n$, $\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)$ has an analytic continuation into the complex plane as a meromorphic function possessing a simple pole located at

$s=n/2$ with residue

$$\mathrm{Res}_{s=\frac{n}{2}}\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)=\frac{(2\pi)^{n/2}}{\sqrt{D(Q)}\,\Gamma\left(\frac{n}{2}\right)}. \tag{2.4.8}$$

For $n=2$, we know that $D(Q)$ reduces to the discriminant $\Delta:=4AC-B^2$ of the binary quadratic form. Hence, for $\mathrm{Re}(s)>1$ and taking $\mathbf{k}:=(k_1,k_2)$, we can reduce (2.4.6) to

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right)=q^{-2s}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(s,\,\frac{\mathbf{k}}{q},\,\mathbf{0},\,Q\right), \tag{2.4.9}$$

which proves that $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ has an analytic continuation as a meromorphic function with a simple pole located at $s=1$ with residue

$$\mathrm{Res}_{s=1}\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\frac{2\pi}{q^2\sqrt{\Delta}}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}.$$

This proves the first part of the lemma. To prove the second part, we use the functional equation for the Epstein zeta function (2.1.49)

$$\begin{aligned}&\left(\frac{2\pi}{D(Q)^{1/n}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,\mathbf{g},\,\mathbf{h},\,Q\right)\\&=\exp\left(-2\pi i\,\mathbf{g}\cdot\mathbf{h}\right)\left(\frac{2\pi}{D(Q^{\dagger})^{1/n}}\right)^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2}-s\right)\,\zeta\left(\frac{n}{2}-s,\mathbf{h},\,-\mathbf{g},\,Q^{\dagger}\right),\end{aligned} \tag{2.4.10}$$

where $Q^{\dagger}$ is the adjoint quadratic form (2.1.47). Since $\mathbf{h}=(0,0)$ and Q is binary, $\zeta\left(s,\mathbf{0},-\frac{\mathbf{k}}{q},\,Q^{\dagger}\right)=\zeta\left(s,\mathbf{0},\frac{\mathbf{k}}{q},\,Q\right)$, so combining (2.4.6) and (2.4.10) with $n=2$ and $D(Q)=\Delta$, we obtain

$$\begin{aligned}&\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\Gamma(s)\,\zeta\left(s,\,Q,\,\frac{p}{q}\right)\\&=\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-(1-s)}\Gamma\left(1-s\right)\,\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(1-s,\,\mathbf{0},\,\frac{\mathbf{k}}{q},\,Q\right).\end{aligned} \tag{2.4.11}$$

Now, for $\mathrm{Re}(s)>1$, it is simple to see that

$$\begin{aligned}&\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\,\zeta\left(s,\,\frac{\mathbf{k}}{q},\,\mathbf{0},\,Q\right)=\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\,\frac{p}{q}Q(k_1,\,k_2)}\sum_{m,n\neq0}\frac{e^{\frac{2\pi i}{q}(m\,k_1+n\,k_2)}}{Q\left(m,n\right)^s}\\&=\sum_{m,n\neq0}\frac{1}{Q(m,n)^s}\,\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}\exp\left(-2\pi i\,\frac{p}{q}Q\left(k_1,\,k_2\right)+\frac{2\pi i}{q}\left(m\,k_1+n\,k_2\right)\right)\\&=\sum_{m,n\neq0}\frac{b_Q(m,n,p/q)}{Q(m,n)^s},\end{aligned}$$

which proves (2.4.3). $\square$

Remark 2.4.1. Note that both $\zeta\left(s,\,Q,\,\frac{p}{q}\right)$ and $\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right)$ can be written as Dirichlet series over one variable of summation. It is not hard to see that, if $r_Q(n)$ denotes the number of representations of n by Q, then

$$\zeta\left(s,\,Q,\,\frac{p}{q}\right)=\sum_{n=1}^{\infty}\frac{r_Q(n)\,\exp\left(-\frac{2\pi i\,p}{q}n\right)}{n^s},\quad \mathrm{Re}(s)>1 \tag{2.4.12}$$

and

$$\tilde{\zeta}\left(s,\,Q,\,\frac{p}{q}\right):=\sum_{n=1}^{\infty}\frac{\tilde{b}_Q(n,p/q)}{n^s},\quad \mathrm{Re}(s)>1, \tag{2.4.13}$$

where

$$\tilde{b}_Q(n,p/q):=\frac{1}{q}\sum_{Q(\alpha,\beta)=n}\,\sum_{k_1,k_2=0}^{q-1}\exp\left(-\frac{2\pi ip}{q}Q(k_1,k_2)+\frac{2\pi i}{q}(\alpha k_1+\beta k_2)\right). \tag{2.4.14}$$

In the expression above we are summing over pairs of integers (α,β) such that $Q(\alpha,\beta)=n$. Note that $|\tilde{b}_Q(n,p/q)|\leq r_Q(n)$, so the absolute convergence of the Dirichlet series on the right-hand side of (2.4.13) is assured for $\mathrm{Re}(s)>1$.

From the previous lemma, it is natural to apply the summation formula given in Theorem 2.2.1 not only to $\zeta(s,Q)$ (as we have done in Example 2.2.2) but also to $\zeta(s,Q,p/q)$. The next result gives one particular case of our summation formula which will be crucial in the proof of Theorem 2.1.4.

Lemma 2.4.2. *Let $Q(m,n)=Am^2+Bmn+Cn^2$ be an integral and positive definite quadratic form and $\Delta:=4AC-B^2$ be its discriminant. Then for Re$(x)>0$, $y\in\mathbb{C}$ and $(p,q)=1$, the following summation formula takes place*

$$\begin{aligned}1+&\sum_{n=1}^{\infty}r_Q(n)\exp\left(-\frac{2\pi ip}{q}n\right)e^{-\frac{2\pi nx}{q\sqrt{\Delta}}}J_0\left(\frac{\sqrt{2\pi ny}}{\sqrt{q}\Delta^{1/4}}\right)\\ &=\frac{e^{-\frac{y^2}{4x}}}{qx}\sum_{k_1,k_2=0}^{q-1}e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}+\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=1}^{\infty}\tilde{b}_Q(n,p/q)e^{-\frac{2\pi n}{q\sqrt{\Delta}x}}I_0\left(\frac{\sqrt{2\pi ny}}{\sqrt{q}\Delta^{1/4}x}\right),\end{aligned} \tag{2.4.15}$$

where $\tilde{b}_Q(n,p/q)$ is explicitly given by (2.4.14).

Proof. Since the functional equation (2.4.3) holds and the analytic continuation of $\zeta(s,\,Q,\,p/q)$ satisfies all the properties of the class $\mathcal{A}$, we can invoke (2.2.9) for the pair of Dirichlet series

$$\phi(s)=\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\exp\left(-\frac{2\pi i\,p}{q}n\right)\,r_Q(n)}{n^s},\quad \psi(s)=\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\tilde{b}_Q(n,p/q)}{n^s},$$

both absolutely convergent in the region $\mathrm{Re}(s)>1$.

Under this template, from (2.4.2) we know that the residue of $\phi(s)$ is $\rho=q^{-1}\sum_{k_1,k_2=0}^{q-1}$ $e^{-\frac{2\pi ip}{q}Q(k_1,k_2)}$. To compute $\phi(0)$, it suffices to know $\zeta(0,Q,p/q)$. To this end, we use (2.4.11):

multiplying both sides of this equation by s and letting $s\to 0$, we obtain

$$\zeta\left(0,Q,\frac{p}{q}\right)=\lim_{s\to 0}s\left(\frac{2\pi q}{\sqrt{\Delta}}\right)^{s-1}\Gamma\left(1-s\right)\frac{1}{q}\sum_{k_1,k_2=0}^{q-1}e^{-2\pi i\frac{p}{q}Q(k_1,k_2)}\zeta\left(1-s,\mathbf{0},\frac{\mathbf{k}}{q},Q\right)$$

$$=\lim_{s\to 0}s\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{s-1}\Gamma\left(1-s\right)\frac{1}{q}\left\{\zeta\left(1-s,\mathbf{0},\mathbf{0},Q\right)+\sum_{k_1,k_2\neq 0}^{q-1}e^{-2\pi i\frac{p}{q}Q(k_1,k_2)}\zeta\left(1-s,\mathbf{0},\frac{\mathbf{k}}{q},Q\right)\right\}.$$

If $k_1\neq 0$ or $k_2\neq 0$, then $\mathbf{k}/q\notin\mathbb{Z}^2$, and so each Dirichlet series on the second sum $\zeta(1-s,\mathbf{0},\mathbf{k}/q,Q)$ is an entire function by Lemma 2.1.2. Therefore, the only singularity comes from $\zeta(1-s,\mathbf{0},\mathbf{0},Q)$ at $s=0$, giving

$$\zeta\left(0,\,Q,\,\frac{p}{q}\right)=-\left(\frac{2\pi}{q\sqrt{\Delta}}\right)^{-1}\frac{1}{q}\lim_{s\to 1}\left(s-1\right)\zeta\left(s,\,\mathbf{0},\,\mathbf{0},\,Q\right)=-1,\qquad(2.4.16)$$

where we have used (2.4.8). An application of the summation formula (2.2.9) yields (2.4.15) immediately. □

2.4.2 The behavior of $\psi_Q(x,z)$

Using the previous lemma, we can now study the behavior of the analogue of Jacobi's $\psi-$function attached to $\zeta(s,Q)$. Our next result essentially says the same as Lemma 2.3.4 in the present situation.

Lemma 2.4.3. *Let $Q(m,n)=Am^2+Bmn+Cn^2$ be a binary, integral and positive definite quadratic form and let $\Delta:=4AC-B^2$. Assume also that $gcd(A,\,B,\,C)=1$ and $\sqrt{\Delta}\equiv 2 \mod 4$. Consider the generalized Jacobi's $\psi-$function attached to Q, (2.2.80),*

$$\psi_Q(x,z)=\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi nx}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi nx}{\Delta^{1/2}}}\,z\right),\quad Re(x)>0,\ \ z\in\mathbb{C}.\qquad(2.4.17)$$

Then, for any z satisfying the condition

$$z\in\mathscr{D}_Q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}},\ |Im(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}}\right\},\qquad(2.4.18)$$

every $m\in\mathbb{N}_0$ and every analytic function $h:\mathbb{C}\longmapsto\mathbb{C}$, the asymptotic formula holds

$$\lim_{\omega\to\frac{\pi}{4}^{-}}\frac{d^m}{d\omega^m}\left\{h(\omega)\,\left(e^{-z^2/8}+e^{z^2/8}\,\psi_Q\left(e^{2i\omega},z\right)\right)\right\}=-2\,h^{(m)}\left(\frac{\pi}{4}\right)\,\sinh\left(\frac{z^2}{8}\right).\qquad(2.4.19)$$

Proof. We proceed as in the proof of Lemma 2.3.4. By Lemma 2.3.5 and Corollary 2.2.2, it suffices to compute $\psi_Q(i+\delta,\,z)$ as $\delta\to 0^{+}$. Note that

$$1+\psi_Q(i+\delta,z):=1+\sum_{n=1}^{\infty}r_Q(n)\,e^{-\frac{2\pi ni}{\sqrt{\Delta}}}\,e^{-\frac{2\pi n\delta}{\sqrt{\Delta}}}\,J_0\left(\sqrt{\frac{2\pi n(i+\delta)}{\Delta^{1/2}}}\,z\right),$$

and so, applying the previous summation formula (2.4.15) with $p = 1$, $q = \sqrt{\Delta}$, $x = \sqrt{\Delta}\,\delta$ and $y = \sqrt{i+\delta}\,z\,\Delta^{1/4}$, we obtain

$$\begin{aligned}1 + \psi_{Q}(i+\delta, z) &:= 1 + \sum_{n=1}^{\infty} r_{Q}(n)\, e^{-\frac{2\pi n i}{\sqrt{\Delta}}}\, e^{-\frac{2\pi n \delta}{\sqrt{\Delta}}}\, J_{0}\left(\sqrt{\frac{2\pi n(i+\delta)}{\Delta^{1/2}}}\, z\right) \\ = \frac{e^{-\frac{(i+\delta)z^{2}}{4\delta}}}{\Delta\delta} \sum_{k_1,k_2=0}^{\sqrt{\Delta}-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} &+ \frac{e^{-\frac{(i+\delta)z^{2}}{4\delta}}}{\sqrt{\Delta}\delta} \sum_{n=1}^{\infty} \tilde{b}_{Q}\left(n, \frac{1}{\sqrt{\Delta}}\right) e^{-\frac{2\pi n}{\Delta^{3/2}\delta}} I_{0}\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\, \frac{z}{\delta}\right).\end{aligned} \tag{2.4.20}$$

To have the conclusion (2.4.19), we need to show that the right-hand side of (2.4.20) tends to zero exponentially fast as $\delta \to 0^{+}$. To do this, one needs to get rid of the residual term involving δ^{-1}. We now proceed to show that the residual term (2.4.2) is zero: by hypothesis, we know that $\sqrt{\Delta}$ is an integer and $\sqrt{\Delta}^{2} + B^{2} = 4AC$. Since the sum of two odd numbers is not divisible by 4, this automatically shows that B and $\sqrt{\Delta}$ must be even. Since $(A,\, B,\, C) = 1$ and B is even, at least A or C is odd and without any loss of generality we may suppose that A is odd. We now evaluate the Gauss sum (2.4.2), restricting ourselves to the sum with respect to k_1. We will see that this sum vanishes if $\sqrt{\Delta} \equiv 2 \mod 4$: indeed

$$\begin{aligned}\sum_{k_1,k_2=0}^{\sqrt{\Delta}-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} &= \sum_{k_2=0}^{\sqrt{\Delta}} \left\{ \sum_{k_1=0}^{\sqrt{\Delta}/2-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} + \sum_{k_1=\sqrt{\Delta}/2}^{\sqrt{\Delta}-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} \right\} \\ &= \sum_{k_2=0}^{\sqrt{\Delta}} \left\{ \sum_{k_1=0}^{\sqrt{\Delta}/2-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} + \sum_{k_1=0}^{\sqrt{\Delta}/2-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q\left(k_1+\frac{\sqrt{\Delta}}{2},k_2\right)} \right\}.\end{aligned}$$

However, one easily checks that

$$Q\left(k_1 + \frac{\sqrt{\Delta}}{2},\, k_2\right) = Q\left(k_1, k_2\right) + \sqrt{\Delta}\left(\frac{A}{4}\sqrt{\Delta} + A\,k_1 + \frac{B}{2}k_2\right), \tag{2.4.21}$$

and so,

$$\begin{aligned}\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q\left(k_1+\frac{\sqrt{\Delta}}{2}, k_2\right)\right) &= \exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2) - 2\pi i\left(\frac{A}{4}\sqrt{\Delta} + Ak_1 + \frac{B}{2}k_2\right)\right) \\ = \exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)\right)\cdot \exp\left(-\frac{\pi i}{2}A\sqrt{\Delta}\right) &= -\exp\left(-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)\right),\end{aligned}$$

because A is odd and $\sqrt{\Delta} \equiv 2 \mod 4$. Thus, we have that

$$\sum_{k_1,k_2=0}^{\sqrt{\Delta}-1} e^{-\frac{2\pi i}{\sqrt{\Delta}}Q(k_1,k_2)} = 0,$$

and so the summation formula (2.4.20) is reduced to

$$1+\tilde{\psi}_Q(i+\delta,z)=\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\sqrt{\Delta}\,\delta}\sum_{n=1}^{\infty}\tilde{b}_Q\left(n,\frac{1}{\sqrt{\Delta}}\right)e^{-\frac{2\pi n}{\Delta^{3/2}\delta}}I_0\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\frac{z}{\delta}\right). \tag{2.4.22}$$

Concluding the proof of this lemma is now a matter of following the proof of Lemma 2.3.4: we need to show that

$$\lim_{\delta\to0^+}\left|\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}e^{-\frac{2\pi n}{\Delta^{3/2}\delta}}I_0\left(\frac{\sqrt{2\pi n(i+\delta)}}{\Delta^{3/4}}\frac{z}{\delta}\right)\right|=0$$

for any fixed $n\in\mathbb{N}$. To do this, we just need to invoke the simple bound (2.3.24) and to recall that $\operatorname{Re}\left(\sqrt{i+\delta}\,z\right)\to\operatorname{Re}(\sqrt{i}\,z)=\frac{1}{\sqrt{2}}\left(\operatorname{Re}(z)-\operatorname{Im}(z)\right)$ and $\operatorname{Re}\left((i+\delta)\,z^2\right)\to-2\operatorname{Re}(z)\operatorname{Im}(z)$ as $\delta\to0^+$, so that the term in the previous equation is bounded by

$$\lim_{\delta\to0^+}\frac{1}{\delta}\exp\left(-\frac{2\pi}{\Delta^{3/2}\delta}\left(n-\frac{\Delta^{3/4}}{2\sqrt{\pi}}\left|\operatorname{Re}(z)-\operatorname{Im}(z)\right|\sqrt{n}-\frac{\operatorname{Re}(z)\operatorname{Im}(z)\,\Delta^{3/2}}{4\pi}\right)\right).$$

As before, the expression in the exponential is a quadratic polynomial with variable $X=\sqrt{n}$ of the form

$$P(X)=X^2-\frac{\Delta^{3/4}}{2\sqrt{\pi}}\left|\operatorname{Re}(z)-\operatorname{Im}(z)\right|X-\frac{\operatorname{Re}(z)\operatorname{Im}(z)\,\Delta^{3/2}}{4\pi}.$$

After computing the zero of $P(X)$ with highest value, we know that a sufficient condition for the positivity of $P(X)$ at a point $X=X_0$ is that

$$\frac{\Delta^{3/4}}{4\sqrt{\pi}}\left[\left|\operatorname{Re}(z)-\operatorname{Im}(z)\right|+\left|\operatorname{Re}(z)+\operatorname{Im}(z)\right|\right]<X_0. \tag{2.4.23}$$

Thus, we want (2.4.23) to hold for every $X_0=\sqrt{n}$, $n\in\mathbb{N}$: this is satisfied if

$$\left|\operatorname{Re}(z)-\operatorname{Im}(z)\right|+\left|\operatorname{Re}(z)+\operatorname{Im}(z)\right|<\frac{4\sqrt{\pi}}{\Delta^{3/4}}, \tag{2.4.24}$$

which is the case whenever $z\in\mathscr{D}_Q=\left\{z\in\mathbb{C}\,:\,|\operatorname{Re}(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}},\ |\operatorname{Im}(z)|<\frac{2\sqrt{\pi}}{\Delta^{3/4}}\right\}$. Now it is just a matter of following the conclusion of Lemmas 2.3.4 and 2.3.5 to deduce (2.4.19). □

Remark 2.4.2. Note that the condition (2.4.24) is indicated to assure that (2.4.23) holds for every $X_0=\sqrt{n}$, $n\in\mathbb{N}$. But this condition can be relaxed and replaced by

$$\left|\operatorname{Re}(z)-\operatorname{Im}(z)\right|+\left|\operatorname{Re}(z)+\operatorname{Im}(z)\right|<\frac{4\sqrt{\pi m_Q}}{\Delta^{3/4}}, \tag{2.4.25}$$

where m_Q is the least integer for which $\tilde{b}_Q\left(m_Q,\frac{1}{\sqrt{\Delta}}\right)\neq0$. For example, when $Q(m,n)=Q_0(m,n):=m^2+n^2$, $\Delta=4$, the condition (2.4.18) does not reduce to (2.1.13) for $\alpha=2$. This is because

$$\tilde{b}_{Q_0}\left(1,\frac{1}{2}\right):=\frac{1}{2}\sum_{\alpha^2+\beta^2=1}\sum_{k_1,k_2=0}^{1}\exp\left(-\pi i\left(k_1^2+k_2^2\right)+\pi i\left(\alpha k_1+\beta k_2\right)\right)=0,$$

while $\tilde{b}_{Q_0}\left(2,\frac{1}{2}\right) \neq 0$. Thus, $m_{Q_0} = 2$, and then (2.4.25) now reduces to (2.1.13). It is a nice exercise of notation and arrangement of variables to check that (2.4.22) reduces to (2.3.17) when $Q(m,n) = m^2 + n^2$.

2.4.3 Proof of Theorem 2.1.4

The proof of our Theorem 2.1.4 is almost the same (modulo different computations) as the proof of Theorem 2.1.3. Therefore, we will brief and just indicate the main steps. Start with the integral representation (2.2.81) given in Example 2.2.2 and take there $x = e^{2i\omega}$, $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$: this gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_Q\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{2}+it;\,1;\,-\frac{z^2}{4}\right)e^{2\omega t}\,dt = e^{i\omega}\,\psi_Q\left(e^{2i\omega},z\right) - e^{-i\omega}\,e^{-\frac{z^2}{4}}.$$

By using Kummer's formula (2.2.8) and adding and subtracting the term $e^{-z^2/8}\,e^{i\omega}$, the previous expression reduces to

$$\begin{aligned}&\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_Q\left(\frac{1}{2}+it\right){}_1F_1\left(\frac{1}{2}-it;\,1;\,-\frac{z^2}{4}\right)e^{2\omega t}\,dt\\ &= -2\,e^{-z^2/8}\,\cos\left(\omega\right) + e^{i\omega}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_Q\left(e^{2i\omega},z\right)\right),\end{aligned} \tag{2.4.26}$$

which is analogous to (2.3.37) for $\alpha = 2$. Therefore, we can use the same computations as in the previous section: changing the variable t to $t+\lambda_j$, differentiating p times on both sides of (2.4.26) and appealing to the notation (2.3.40), we deduce

$$\begin{aligned}&\int_{-\infty}^{\infty}t^p\,\eta_Q\left(\frac{1}{4}+i\left(t+\lambda_j\right)\right){}_1F_1\left(\frac{1}{2}-i\left(t+\lambda_j\right);\,1;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\ &= -4\pi\left(\frac{r_j}{2}\right)^p e^{-2\omega\lambda_j}\,\cos\left(p\,\theta_j+\omega\right)\\ &\quad+\frac{2\pi}{2^p}\,e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{e^{i\omega-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_Q\left(e^{2i\omega},z\right)\right)\right\}.\end{aligned}$$

Continuing to argue in the same manner, we can fully justify the equality (cf. (2.3.42) above)

$$\begin{aligned}&\int_{-\infty}^{\infty}t^p\,F_{z,Q}\left(\frac{1}{2}+it\right)e^{2\omega t}\,dt\\ &= -\frac{8\pi}{2^p}\sum_{j=1}^{\infty}c_j\,r_j^p e^{-2\omega\lambda_j}\,\cos\left(p\,\theta_j+\omega\right)\\ &\quad+\frac{4\pi}{2^p}\,\mathrm{Re}\left(e^{z^2/8}\,\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_j e^{i\omega-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_Q\left(e^{2i\omega},z\right)\right)\right\}\right),\end{aligned} \tag{2.4.27}$$

where

$$F_{z,Q}(s) = \sum_{j=1}^{\infty} c_j \left(\frac{2\pi}{\sqrt{\Delta}}\right)^{-(s+i\lambda_j)} \Gamma\left(s+i\lambda_j\right) \zeta(s+i\lambda_j,\, Q)$$
$$\times \left\{ {}_1F_1\left(1-s-i\lambda_j;\, 1;\, \frac{z^2}{4}\right) + {}_1F_1\left(1-\overline{s}+i\lambda_j;\, 1;\, \frac{\overline{z}^2}{4}\right)\right\}.$$

Letting $\omega \to \frac{\pi}{4}^{-}$ on both sides of (2.4.27), we see from Lemma 2.4.3 above (namely, relation (2.4.19)) that the right-hand side of (2.4.27) can be simplified to

$$-\frac{8\pi}{2^p} \sum_{j=1}^{\infty} c_j r_j^p e^{-\frac{\pi}{2}\lambda_j} \left[\cos\left(p\,\theta_j + \frac{\pi}{4}\right)\left(1 + \mathrm{Re}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\right)\right.$$
$$\left. - \sin\left(p\theta_j + \frac{\pi}{4}\right) \mathrm{Im}\left(e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\right],$$

which is exactly the same as the right-hand side of (2.3.45) with $\alpha = 2$. From this point onwards, the proof follows exactly the same steps as the proof of Theorem 2.1.3. □

2.5 Zeros of combinations attached to $L_k(s,\chi)$

Similarly to the previous sections, in order to study the zeros of $L_k(s,\chi)$ we need a lemma which gives the analytic continuation of a periodic Dirichlet series containing information about the behavior of $\psi_{k,\chi}(x,z)$, (2.2.90). Before proving such a result, we first need an auxiliary proposition, which will be the first statement on this section.

Although the proof of our next result is quite standard and it is a known result for $\delta=0$ (see Lemma 2.1.2 above), we prove it bearing in mind the case where $\delta=1$, because this might be instructive to some readers. Although straightforward, we could not track any reference containing explicitly its statement.

2.5.1 Exponential sums attached to Dirichlet characters

Lemma 2.5.1. *Let $\delta\in\{0,1\}$ and $0<a_i<1$, for $i\in\{1,...,k\}$. Consider the Dirichlet series*

$$\varphi_{k,\delta}(s;a_1,...,a_k):=\sum_{n_1,...,n_k\in\mathbb{Z}}\frac{\left(n_1+a_1\right)^{\delta}\cdot...\cdot\left(n_k+a_k\right)^{\delta}}{\left(\left(n_1+a_1\right)^2+...+\left(n_k+a_k\right)^2\right)^{s}},\quad Re(s)>\frac{k}{2}\left(1+\delta\right). \tag{2.5.1}$$

Then $\varphi_{k,\delta}(s;\,a_1,...,a_k)$ has the following properties:

1. *If $\delta=0$, it can be continued as a meromorphic function with a simple pole located at $s=\frac{k}{2}$ with residue $\text{Res}_{s=\frac{k}{2}}\ \varphi_{k,0}(s;\,a_1,...,a_k)=\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}$;*
2. *If $\delta=1$, it can be continued as an entire function.*

Moreover, it satisfies the functional equation

$$\begin{aligned}&\pi^{-s}\Gamma(s)\,\varphi_{k,\delta}(s;\,a_1,...,a_k)\\ &=(-i)^{\delta k}\,\pi^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\,\tilde{\varphi}_{k,\delta}\left(k\left(\frac{1}{2}+\delta\right)-s;\,a_1,...,a_k\right),\end{aligned} \tag{2.5.2}$$

where, for $Re(s)>\frac{k}{2}(1+\delta)$, $\tilde{\varphi}_{k,\delta}(s;\,a_1,...,a_k)$ can be written as the Dirichlet series

$$\tilde{\varphi}_{k,\delta}\left(s;\,a_1,...,a_k\right)=\sum_{n_1,...,n_k\neq0}\frac{n_1^{\delta}\cdot...\cdot n_k^{\delta}\,e^{2\pi i(n_1,...,n_k)\cdot(a_1,...,a_k)}}{\left(n_1^2+...+n_k^2\right)^{s}}. \tag{2.5.3}$$

Furthermore, $\tilde{\varphi}_{k,\delta}\left(s;\,a_1,...,a_k\right)$ can be analytically continued as an entire function no matter the value of δ.

Proof. Note that the condition $0<a_i<1$ implies that the multiple series (2.5.1) is well-defined for any $(n_1,...,n_k)\in\mathbb{Z}^k$. Note also that when $\delta=0$ this result is already known because the Dirichlet series (2.5.1) is actually an Epstein zeta function (2.1.48), this is

$$\varphi_{k,0}(s;a_1,...,a_k)=\zeta\left(s,\mathbf{a},\,\mathbf{0},\,\mathbf{I}_k\right),$$

where $\mathbf{a} = (a_1, ..., a_k)$ and $\mathbf{I}_k$ denotes the diagonal quadratic form $n_1^2 + ... + n_k^2$. The pole of $\varphi_{k,0}(s; a_1, ..., a_k)$ comes from the pole of the Epstein zeta function at $s = \frac{k}{2}$. Furthermore,

$$\tilde{\varphi}_{k,0}\,(s;\, a_1, ..., a_k) = \zeta\,(s,\, \mathbf{0},\, \mathbf{a},\, \mathbf{I}_k)\,.$$

Since $(a_1, ..., a_k) \notin \mathbb{Z}^k$, by the analytic continuation of Epstein's zeta function we see that $\tilde{\varphi}_{k,0}\,(s;\, a_1, ..., a_k)$ must be entire. The functional equation (2.5.2) is an immediate corollary of (2.1.49).

We now focus on the case where $\delta = 1$. We start by writing (2.5.1) as a Mellin transform

$$\pi^{-s}\Gamma(s)\,\varphi_{k,1}(s; a_1, ..., a_k) = \int_0^{\infty} x^{s-1} \prod_{j=1}^{k} \sum_{n_j\in\mathbb{Z}} (n_j + a_j)\; e^{-\pi(n_j+a_j)^2 x}\, dx, \quad \text{Re}(s) > k, \tag{2.5.4}$$

so that the proof of (2.5.2) relies on applying Poisson's summation formula to each factor. This is standard and we obtain for each $j \in \{1, ..., k\}$,

$$\sum_{n_j\in\mathbb{Z}} (n_j + a_j)\, e^{-\pi(n_j+a_j)^2 x} = -\frac{i}{x^{3/2}} \sum_{n_j\in\mathbb{Z}} n_j\, \exp\left(-\frac{\pi}{x} n_j^2 + 2\pi i\, n_j\, a_j\right). \tag{2.5.5}$$

Hence, breaking the integral on (2.5.4) into $(0,1)$ and $(1,\infty)$ and applying (2.5.5) on the first integral, we get the representation

$$\begin{aligned} &\pi^{-s}\Gamma(s)\,\varphi_{k,1}(s; a_1, ..., a_k) \\ &= \int_1^{\infty} x^{s-1} \sum_{n_1,...,n_k\in\mathbb{Z}} (n_1 + a_1)\cdot ... \cdot (n_k + a_k) \exp\Big(-\pi\left((n_1 + a_1)^2 + ... + (n_k + a_k)^2\right)x\Big)\, dx \\ &+ \int_1^{\infty} x^{\frac{3}{2}k-s-1} (-i)^k \sum_{n_1,...,n_k\in\mathbb{Z}} n_1\, e^{2\pi i\, n_1\, a_1} \cdot ... \cdot n_k\, e^{2\pi i\, n_k\, a_k} \cdot \exp\left(-\pi x\left(n_1^2 + ... + n_k^2\right)\right)\, dx, \end{aligned} \tag{2.5.6}$$

which is invariant once we replace s by $\frac{3k}{2} - s$ and $\varphi_{k,1}(s; a_1, ..., a_k)$ by $\tilde{\varphi}_{k,1}\,(s;\, a_1, ..., a_k)$. Moreover, the uniform and absolute convergence of the integrals given in (2.5.6) assure that $\varphi_{k,1}(s; a_1, ..., a_k)$ and $\tilde{\varphi}_{k,1}\,(s;\, a_1, ..., a_k)$ are entire functions of s. $\square$

The next lemma has a role similar to Lemmas 2.3.2 and 2.4.1 of the previous sections.

Lemma 2.5.2. *Let χ be a Dirichlet character modulo q and p be an integer such that $(p,q) = 1$. If χ is even, define $\delta = 0$. Otherwise, if χ is odd, set $\delta = 1$. Consider the Dirichlet series*

$$L_k(s, \chi, p/q) := \sum_{n_1,...,n_k\neq 0} \frac{n_1^{\delta}\,\chi(n_1)\, ...\, n_k^{\delta}\,\chi(n_k)}{\left(n_1^2 + ... + n_k^2\right)^s}\, e^{-\frac{i\pi p}{q}\left(n_1^2+...+n_k^2\right)}, \quad Re(s) > \frac{k}{2}(1+\delta). \tag{2.5.7}$$

Then $L_k\,(s, \chi, p/q)$ has the following properties:

1. *If χ is even, it can be analytically continued as a meromorphic function with at most one simple pole located at $s=\frac{k}{2}$, whose residue is*

$$Res_{s=\frac{k}{2}}L_k(s,\chi,p/q)=\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)(2q)^k}\left(\sum_{r=1}^{2q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}\right)^k. \tag{2.5.8}$$

In particular, if p,q are odd integers, $L_k(s,\chi,p/q)$ is entire.

2. *If χ is odd, $L_k\left(s,\chi,p/q\right)$ can be analytically continued as an entire function.*

Moreover, it satisfies the functional equation

$$\begin{aligned}&\left(\frac{\pi}{2q}\right)^{-s}\Gamma(s)\,L_k(s,\chi,p/q)\\&=(-i)^{\delta k}\left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\tilde{L}_k\left(k\left(\frac{1}{2}+\delta\right)-s,\chi,p/q\right),\end{aligned} \tag{2.5.9}$$

where $\tilde{L}_k\left(s,\chi,p/q\right)$ is representable by the series

$$\tilde{L}_k\left(s,\chi,p/q\right)=\sum_{n_1,...,n_k\neq0}\frac{b_\chi(n_1,...,n_k;\,p/q)\,\cdot\,n_1^\delta\cdot...\cdot n_k^\delta}{\left(n_1^2+...+n_k^2\right)^s},\quad Re(s)>\frac{k}{2}(1+\delta), \tag{2.5.10}$$

with

$$\begin{aligned}b_\chi(n_1,...,n_k;\,p/q):=(2q)^{-\frac{k}{2}}&\sum_{r_1,...,r_k=0}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\\&\times\exp\left(-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)+2\pi i\,(n_1,...,n_k)\cdot\left(\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\right).\end{aligned} \tag{2.5.11}$$

Furthermore, $\tilde{L}_k\left(s,\chi,p/q\right)$ can be continued as an entire function no matter the parity of χ.

Proof. Since the arithmetical function $\chi(n)\,e^{-\frac{i\pi p}{q}n^2}$ has period $2q$, we may decompose the series (2.5.7) into residue classes modulo $2q$. Assuming that $\text{Re}(s)>\frac{k}{2}(1+\delta)$, such decomposition gives

$$\begin{aligned}L_k(s,\chi,p/q)&:=\sum_{n_1,...,n_k\neq0}\frac{n_1^\delta\,\chi(n_1)\,...\,n_k^\delta\,\chi(n_k)}{\left(n_1^2+...+n_k^2\right)^s}\,e^{-\frac{i\pi p}{q}\left(n_1^2+...+n_k^2\right)}\\&=(2q)^{k\delta-2s}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\\&\times\sum_{m_1,...,m_k\in\mathbb{Z}}\frac{\left(m_1+\frac{r_1}{2q}\right)^\delta\cdot...\cdot\left(m_k+\frac{r_k}{2q}\right)^\delta}{\left(\left(m_1+\frac{r_1}{2q}\right)^2+...+\left(m_k+\frac{r_k}{2q}\right)^2\right)^s}\\&:=(2q)^{k\delta-2s}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\varphi_{k,\delta}\left(s;\frac{r_1}{2q},...,\frac{r_k}{2q}\right),\end{aligned} \tag{2.5.12}$$

where $\varphi_{k,\delta}(s;\,a_1,...,a_k)$ is defined by (2.5.1). We now apply the previous lemma: if $\delta=1$, i.e., if χ is an odd Dirichlet character, we have that $L_k(s,\chi,p/q)$ is entire because $\varphi_{k,1}(s;\,a_1,...,a_k)$ is entire. On the other hand, if χ is even, then $L_k(s,\chi,p/q)$ must have a simple pole at $s=\frac{k}{2}$ with residue given explicitly given by (2.5.8), due to item 1. of Lemma 2.5.1. In particular, if χ is even and p,q are odd integers,

$$\begin{aligned}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)} &= \left(\sum_{r=1}^{2q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}\right)^k\\ &= \left(\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}+\sum_{r=q+1}^{2q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}\right)^k\\ &= \left(\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}-\sum_{r=1}^{q-1}\chi(r)\,e^{-\frac{\pi ip}{q}r^2}\right)^k=0,\end{aligned}$$

which shows that, under these conditions, $L_k(s,\chi,p/q)$ must be entire.

The functional equation (2.5.9) now follows from (2.5.3): invoking (2.5.12) and (2.5.3), we see that

$$\begin{aligned}\left(\frac{\pi}{2q}\right)^{-s}\Gamma(s)\,L_k(s,\chi,p/q) &= \frac{(-i)^{\delta k}}{(2q)^{k/2}}\left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\\ &\times\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(k\left(\frac{1}{2}+\delta\right)-s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right).\end{aligned}$$

However, appealing to the definition of $\tilde{\varphi}_{k,\delta}(s;\,a_1,...,a_k)$ as a Dirichlet series, (2.5.3), we have, for $\mathrm{Re}(s)>\frac{k}{2}(1+\delta)$,

$$\begin{aligned}&\frac{1}{(2q)^{k/2}}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\\ &=\sum_{n_1,...,n_k\neq0}\frac{n_1^\delta\cdot...\cdot n_k^\delta}{\left(n_1^2+...+n_k^2\right)^s}\,(2q)^{-\frac{k}{2}}\sum_{r_1,...,r_k=0}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\\ &\quad\times\exp\left(-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)+2\pi i\,(n_1,...,n_k)\cdot\left(\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\right)\\ &:=\sum_{n_1,...,n_k\neq0}\frac{b_\chi(n_1,...,n_k;\,p/q)\,n_1^\delta\cdot...\cdot n_k^\delta}{\left(n_1^2+...+n_k^2\right)^s}=\tilde{L}_k\left(s,\chi,p/q\right).\end{aligned}$$

To conclude, we just need to show that $\tilde{L}_k(s,\chi,p/q)$ is an entire function: this comes from the previous expression of $\tilde{L}_k(s,\chi,p/q)$ as the sum

$$\frac{1}{(2q)^{k/2}}\sum_{r_1,...,r_k=1}^{2q-1}\chi(r_1)\cdot...\cdot\chi(r_k)\,e^{-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right)}\,\tilde{\varphi}_{k,\delta}\left(s;\,\frac{r_1}{2q},...,\frac{r_k}{2q}\right).$$

Since $0 < r_j/(2q) < 1$, we know that $\tilde{\varphi}_{k,\delta}\left(s;\, \frac{r_1}{2q}, ..., \frac{r_k}{2q}\right)$ is always entire (no matter if δ is 0 or 1) and so $\tilde{L}_k(s,\chi,p/q)$ must be entire as well. □

Remark 2.5.1. Note that we can write the multiple series defining (2.5.7) and (2.5.10) as Dirichlet series involving a single variable of summation. In fact, we may express (2.5.7) as

$$L_k(s,\chi,p/q) = \sum_{n=1}^{\infty} \frac{r_{k,\chi}(n)\, e^{-\frac{i\pi p}{q}n}}{n^s}, \quad \mathrm{Re}(s) > \frac{k}{2}\,(1+\delta)\,, \tag{2.5.13}$$

where $r_{k,\chi}(n)$ is explicitly given by (2.2.87). We can also put (2.5.10) in the form

$$\tilde{L}_k\,(s,\chi,p/q) = \sum_{n=1}^{\infty} \frac{\tilde{b}_{k,\chi}(n,\, p/q)}{n^s}, \quad \mathrm{Re}(s) > \frac{k}{2}(1+\delta), \tag{2.5.14}$$

with $\tilde{b}_{k,\chi}(n,p/q)$ being defined as

$$\begin{aligned}\tilde{b}_{k,\chi}(n,p/q) := (2q)^{-\frac{k}{2}} & \sum_{n_1^2+...+n_k^2=n} n_1^{\delta}\cdot ...\cdot n_k^{\delta} \sum_{r_1,...,r_k=0}^{2q-1} \chi(r_1)\cdot ...\cdot \chi(r_k) \\ & \times \exp\left(-\frac{i\pi p}{q}\left(r_1^2+...+r_k^2\right) + 2\pi i\,(n_1,...,n_k)\cdot\left(\frac{r_1}{2q},...,\frac{r_k}{2q}\right)\right).\end{aligned} \tag{2.5.15}$$

Remark 2.5.2. Note that we did not use the primitivity of the character χ in the proof of Lemma 2.5.2. In the next subsection we will see that, under this additional hypothesis and by setting $p=0$ in (2.5.9), we can recover the functional equation (2.1.50) given at the beginning of the chapter.

2.5.2 Proof of Lemma 2.1.3

Since, by hypothesis, χ is a primitive character, we know that the Gauss sum splits in the form

$$G(n,\overline{\chi}) := \sum_{j=1}^{q-1} \overline{\chi}(j)\, e^{2\pi i n j/q} = \chi(n)\, G(\overline{\chi}), \tag{2.5.16}$$

where $G(\chi) := G(1,\chi)$. By the representations (2.5.7) and (2.1.50), we clearly have that $L_k\,(s,\chi) = L_k(s,\chi,0)$. Since (2.5.8) is zero when $p=0$, we conclude that $L_k\,(s,\chi)$ is entire. To prove the functional equation (2.1.51), we look at (2.5.9) and see that

$$\left(\frac{\pi}{2q}\right)^{-s}\Gamma(s)\, L_k(s,\chi) = (-i)^{\delta k}\left(\frac{\pi}{2q}\right)^{-\left(k\left(\frac{1}{2}+\delta\right)-s\right)}\Gamma\left(k\left(\frac{1}{2}+\delta\right)-s\right)\,\tilde{L}_k\left(k\left(\frac{1}{2}+\delta\right)-s,\chi,0\right). \tag{2.5.17}$$

Thus, it remains to simplify the expression for $\tilde{L}_k(s,\chi,p/q)$, (2.5.10), when $p=0$. By (2.5.11), we know that

$$\tilde{L}_k\left(s,\chi,0\right)=(2q)^{-\frac{k}{2}}\sum_{n_1,\dots,n_k\neq 0}\sum_{r_1,\dots,r_k=0}^{2q-1}\chi(r_1)\cdot\ldots\cdot\chi(r_k)$$
$$\times\exp\left(2\pi i\,\left(n_1,\dots,n_k\right)\cdot\left(\frac{r_1}{2q},\dots,\frac{r_k}{2q}\right)\right)\,\frac{n_1^{\delta}\cdot\ldots\cdot n_k^{\delta}}{\left(n_1^2+\ldots+n_k^2\right)^s},\tag{2.5.18}$$

for $\mathrm{Re}(s)>\frac{k}{2}(1+\delta)$. Now, for every pair (n_j,r_j), the resulting Gauss sum is

$$\begin{aligned}&\sum_{r_j=0}^{2q-1}\chi(r_j)\,\exp\left(2\pi i\,n_j\,\frac{r_j}{2q}\right)\\&=\sum_{r_j=1}^{q-1}\chi(r_j)\,\exp\left(2\pi i\,n_j\,\frac{r_j}{2q}\right)+\sum_{j=1}^{q-1}\chi(r_j)\,\exp\left(2\pi i\,n_j\,\frac{r_j+q}{2q}\right)\\&=\left(1+(-1)^{n_j}\right)\sum_{r_j=1}^{q-1}\chi(r_j)\exp\left(2\pi i\,n_j\,\frac{r_j}{2q}\right)\\&=\begin{cases}2\,\sum_{r_j=1}^{q-1}\chi(r_j)\,\exp\left(2\pi i\,m_j\,\frac{r_j}{q}\right)=2\,G\left(m_j,\chi\right) & n_j=2m_j\\ 0 & n_j\ \text{ odd}\end{cases}.\end{aligned}\tag{2.5.19}$$

Hence, the sum in (2.5.18) over $(n_1,\dots,n_k)$ does not vanish only if each n_j is an even integer: writing $n_j=2m_j$, we derive from (2.5.18) and (2.5.19),

$$\begin{aligned}\tilde{L}_k\left(s,\chi,0\right)&=(2q)^{-\frac{k}{2}}2^{-2s+k(\delta+1)}\sum_{m_1,\dots,m_k\neq 0}\frac{G\left(m_1,\chi\right)m_1^{\delta}\cdot\ldots\cdot G\left(m_k,\chi\right)m_k^{\delta}}{\left(m_1^2+\ldots+m_k^2\right)^s}\\&=\frac{G^k(\chi)}{q^{k/2}}\,2^{-2s+k\delta+\frac{k}{2}}\sum_{m_1,\dots,m_k\neq 0}\frac{\overline{\chi}(m_1)m_1^{\delta}\cdot\ldots\cdot\overline{\chi}(m_k)m_k^{\delta}}{\left(m_1^2+\ldots+m_k^2\right)^s}\\&=\frac{G^k(\chi)}{q^{k/2}}\,2^{-2s+k\delta+\frac{k}{2}}\,L_k\left(s,\,\overline{\chi}\right),\end{aligned}\tag{2.5.20}$$

where in the second step we have used the fact that χ is primitive, i.e., (2.5.16). Joining (2.5.20) and (2.5.17) gives (2.1.51). $\square$

Remark 2.5.3. Another way of proving the above lemma is by using Corollary 1.2.2, which establishes the functional equation of a general class of multidimensional Epstein zeta functions. See also the published version of the previous chapter, [194], for some complementary remarks.

2.5.3 The behavior of $\psi_{\chi,k}(x,z)$

We finally prove that the analogue of Jacobi's $\psi-$function given at (2.2.90) has the same vanishing properties as the ones given in previous sections. Once again, this study starts with a summation formula involving $r_{k,\chi}(n)$.

Lemma 2.5.3. *Let χ be a primitive Dirichlet character modulo q and let $r_{k,\chi}(n)$ and $\tilde{b}_{k,\chi}(n, p/q)$ be the arithmetical functions given in (2.2.87) and (2.5.15). Furthermore, suppose that $q \neq 0$* mod 4 *when χ is even.*

Then, for $Re(x) > 0$ and any $y \in \mathbb{C}$, the following transformation formula takes place

$$\sum_{n=1}^{\infty} r_{k,\chi}(n)\, e^{-\frac{i\pi}{q}n}\, n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\, e^{-\frac{\pi}{2q}nx}\, J_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n}{2q}}\, y\right)$$
$$= \frac{e^{-\frac{y^2}{4x}}}{x} \sum_{n=1}^{\infty} (-i)^{\delta k} \tilde{b}_{k,\chi}(n,\, 1/q)\, n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\, e^{-\frac{\pi n}{2qx}}\, I_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n}{2q}}\, \frac{y}{x}\right). \tag{2.5.21}$$

Proof. Assume that $\mathrm{Re}(s) > \frac{k}{2}(1+\delta)$ and apply the summation formula (2.2.9) to the pair of Dirichlet series

$$\phi(s) = \left(\frac{\pi}{2q}\right)^{-s} L_k\left(s, \chi, 1/q\right) := \left(\frac{\pi}{2q}\right)^{-s} \sum_{n=1}^{\infty} \frac{r_{k,\chi}(n)\, e^{-\frac{i\pi}{q}n}}{n^s} \tag{2.5.22}$$

and

$$\psi(s) = \left(\frac{\pi}{2q}\right)^{-s} (-i)^{\delta k} \tilde{L}_k\left(s, \chi, 1/q\right) := \left(\frac{\pi}{2q}\right)^{-s} \sum_{n=1}^{\infty} \frac{(-i)^{\delta k} \tilde{b}_{k,\chi}(n,\, 1/q)}{n^s}, \tag{2.5.23}$$

which, by Lemma 2.5.2, satisfy Hecke's functional equation and belong to the class $\mathcal{A}$. This requires to take the simple substitutions: $\lambda_n = \frac{\pi n}{2q}$, $\mu_n = \frac{\pi n}{2q}$, $a(n) = r_{k,\chi}(n)\, e^{-\frac{i\pi p}{q}n}$, $b(n) = (-i)^{\delta k} \tilde{b}_{k,\chi}(n, 1/q)$ and $r = k\left(\frac{1}{2}+\delta\right)$ in (2.2.9).

We now claim that $\phi(s)$, (2.5.22), is an entire function. First, if χ is odd, this assertion follows from our Lemma 2.5.2. On the other hand, if χ is even, then q must be odd due to our imposition that $q \neq 0 \mod 4$ and the fact that there are no primitive characters (mod q) such that $q \equiv 2$ mod 4. Hence, by item 1. of Lemma 2.5.2, $\phi(s)$ must be an entire function for even primitive χ and odd q.

Thus, $\rho = 0$ regardless of χ being even or odd. In order to apply (2.2.9), we need to find also $\phi(0)$. Since $\phi(s) \in \mathcal{A}$ and $\psi(s)$ given by (2.5.22) is an entire function (cf. Lemma 2.5.2), we know by Remark 2.1.1 that $\phi(0)$ must be zero. Finally, replacing $\rho = \phi(0) = 0$ in (2.2.9), we are able to derive (2.5.21). □

We now prove the character analogue of Lemmas 2.3.4 and 2.4.3 given at the previous sections.

Lemma 2.5.4. *Let χ be a primitive Dirichlet character modulo q and $r_{k,\chi}(n)$ be defined by (2.2.87). For $Re(x) > 0$ and $z \in \mathbb{C}$, consider the analogue of Jacobi's $\psi-$function, (2.2.90),*

$$\psi_{\chi,k}(x,z) := 2^{\frac{k}{2}+k\delta-1}\, \Gamma\left(\frac{k}{2}+k\delta\right) \left(\sqrt{\frac{\pi}{q}}x\, z\right)^{1-\frac{k}{2}-k\delta}$$
$$\times \sum_{n=1}^{\infty} r_{k,\chi}(n)\, n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\, e^{-\frac{\pi n}{q}x}\, J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}nx}\, z\right).$$

Furthermore, let $h : \mathbb{C} \longmapsto \mathbb{C}$ be an analytic function and assume that z satisfies the condition

$$z \in \mathscr{D}_{\chi} := \left\{z \in \mathbb{C} \,:\, |Re(z)| < \sqrt{\frac{\pi}{2q}},\ |Im(z)| < \sqrt{\frac{\pi}{2q}}\right\}. \tag{2.5.24}$$

If we assume one of the conditions:

1. χ *is even with* $q \neq 0 \mod 4$ *or*

2. χ *is odd with arbitrary* q,

then the following limit holds

$$\lim_{\omega\to\frac{\pi}{4}^-} \frac{d^m}{d\omega^m}\left\{h(\omega)\,\psi_{\chi,k}\left(e^{2i\omega},\,z\right)\right\}=0, \qquad \forall m\in\mathbb{N}_0. \tag{2.5.25}$$

Proof. Following the proofs of Lemmas 2.3.4 and 2.3.5 (in particular, see (2.3.21) and (2.3.35)) and changing the $\delta-$notation there to ϵ, we just need to consider the behavior of $\psi_{\chi,k}(i+\epsilon,z)$ as $\epsilon\to 0^+$. From (2.2.90), the expression for $\psi_{\chi,k}(i+\epsilon,z)$ is

$$\begin{aligned}
&2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\\
&\quad\times\sum_{n=1}^{\infty} r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}(i+\epsilon)}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n(i+\epsilon)}\,z\right)\\
&=2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\\
&\quad\times\sum_{n=1}^{\infty} r_{k,\chi}(n)\,n^{\frac{1}{2}-\frac{k}{4}-\frac{k\delta}{2}}\,e^{-\frac{\pi n}{q}i}\,e^{-\frac{\pi n}{q}\epsilon}\,J_{\frac{k}{2}+k\delta-1}\left(\sqrt{\frac{\pi}{q}n(i+\epsilon)}\,z\right).
\end{aligned} \tag{2.5.26}$$

We may apply the summation formula (2.5.21) to the right-hand side of (2.5.26) with x in (2.5.21) being replaced by 2ϵ and y by $\sqrt{2(i+\epsilon)}\,z$. This gives the curious transformation

$$\begin{aligned}
\psi_{\chi,k}(i+\epsilon,z)&=2^{\frac{k}{2}+k\delta-1}\,\Gamma\left(\frac{k}{2}+k\delta\right)\left(\sqrt{\frac{\pi}{q}(i+\epsilon)}\,z\right)^{1-\frac{k}{2}-k\delta}\frac{e^{-\frac{(i+\epsilon)z^2}{4\epsilon}}}{2\epsilon}\\
&\times\sum_{n=1}^{\infty}(-i)^{\delta k}\tilde{b}_{k,\chi}\left(n,\,1/q\right)\,n^{\frac{1}{2}-\frac{k}{2}\left(\frac{1}{2}+\delta\right)}\,e^{-\frac{\pi n}{4q\epsilon}}\,I_{k\left(\frac{1}{2}+\delta\right)-1}\left(\sqrt{\frac{\pi n(i+\epsilon)}{q}}\,\frac{z}{2\epsilon}\right).
\end{aligned} \tag{2.5.27}$$

If we bound $I_{\nu}(z)$ in the same way as before (see (2.3.24)), a sufficient condition ensuring that (2.5.27) tends to zero as $\epsilon\to 0^+$ is

$$\lim_{\epsilon\to 0^+}\epsilon^{-\frac{k}{2}-k\delta}\exp\left(-\frac{\pi n}{4q\epsilon}+\frac{1}{2\epsilon}\sqrt{\frac{\pi n}{q}}\,\frac{|\mathrm{Re}(z)-\mathrm{Im}(z)|}{\sqrt{2}}+\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{2\epsilon}\right)=0.$$

By examining the quadratic polynomial (with variable $\sqrt{n}$),

$$P(X)=X^2-X\sqrt{\frac{2q}{\pi}}\,|\mathrm{Re}(z)-\mathrm{Im}(z)|-\frac{2q\,\mathrm{Re}(z)\,\mathrm{Im}(z)}{\pi},$$

one can see that $P(\sqrt{n})>0$ for every $n\in\mathbb{N}$ if

$$|\mathrm{Re}(z)-\mathrm{Im}(z)|+|\mathrm{Re}(z)+\mathrm{Im}(z)|<\sqrt{\frac{2\pi}{q}}, \tag{2.5.28}$$

which is satisfied whenever z satisfies the condition (2.5.24). □

Remark 2.5.4. In contrast with the statement of Theorem 2.1.3, the domain in (2.5.24) does not depend on k. Like in Remark 2.4.2, enlarging this domain requires to get more information about the arithmetical function $\tilde{b}_{k,\chi}\left(n,\,1/q\right)$. As in (2.4.25), (2.5.28) can be replaced by

$$|\mathrm{Re}(z)-\mathrm{Im}(z)|+|\mathrm{Re}(z)+\mathrm{Im}(z)|<\sqrt{\frac{2\pi m_{\chi,k}}{q}},$$

where $m_{\chi,k}$ is the least integer for which $\tilde{b}_{k,\chi}\left(n,\,1/q\right)\neq 0$.

2.5.4 Proof of Theorem 2.1.5

Since the Dirichlet series $L_k(s,\chi)$ is entire, the argument in the proof of Theorem 2.1.3 becomes easier, because we do not need to count the sign changes of the residual terms like in the expression (2.3.45). We shall see how the proof in this case goes: start with the integral representation (2.2.91) obtained in Example 2.2.3 and replace there x by $e^{2i\omega}$, $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}+it;\,\frac{k}{2}+k\delta;\,-\frac{z^2}{4}\right)e^{2\omega t}\,dt=e^{i\omega k\left(\frac{1}{2}+\delta\right)}\,\psi_{\chi,k}\left(e^{2i\omega},z\right).$$

From Kummer's formula (2.2.13), we can rewrite the previous equality in the symmetric form

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right){}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-it;\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\=e^{i\omega k\left(\frac{1}{2}+\delta\right)}\,e^{z^2/4}\,\psi_{\chi,k}\left(e^{2i\omega},z\right),\end{aligned}\tag{2.5.29}$$

which is somewhat remindful of the previous steps given in (2.3.36) and (2.4.26). In the present case, however, we are in a much easier situation because there are no residual terms on the right-hand side of (2.5.29). After changing the variable $t\to t+\lambda_j$ and differentiating (2.5.29) p times with respect to ω, we are able to get the expression

$$\begin{aligned}\int_{-\infty}^{\infty}t^p\,\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+i\,(t+\lambda_j),\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-i(t+\lambda_j);\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\=\frac{2\pi}{2^p}\,e^{z^2/4}\,\frac{d^p}{d\omega^p}\left\{e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j}\,\psi_{\chi,k}\left(e^{2i\omega},z\right)\right\}.\end{aligned}\tag{2.5.30}$$

Now, let us make an important remark: in order to apply Hardy's method, we need the integrand on the left of (2.5.30) to be a real function of t. In the previous two cases, it was enough to take the real part of the hypergeometric factor. In this case, however, we note that the function

$$\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$$

is not real-valued. This can be easily seen from the asymmetry in the functional equation (2.1.51),

$$\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)=\left(\frac{(-i)^{\delta}G(\chi)}{\sqrt{q}}\right)^k\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}-it,\overline{\chi}\right). \tag{2.5.31}$$

However, since $|G(\chi)/\sqrt{q}|=1$, we can write $(-i)^{\delta}\,G(\chi)/\sqrt{q}:=e^{i\gamma}$ for some real γ and then set

$$\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right):=e^{-i\gamma k/2}\,\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right). \tag{2.5.32}$$

It is now clear from (2.5.31) that $\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$ is real-valued and its zeros are exactly the same as the ones of $\eta_k\left(\frac{k}{4}+\frac{k\delta}{2}+it,\chi\right)$. Thus, we may multiply both sides of (2.5.30) by $e^{-i\gamma k/2}$ and have

$$\begin{aligned}\int_{-\infty}^{\infty}t^p\,\tilde{\eta}_k\left(\frac{k}{4}+\frac{k\delta}{2}+i\,(t+\lambda_j),\chi\right)\,{}_1F_1\left(\frac{k}{4}+\frac{k\delta}{2}-i(t+\lambda_j);\,\frac{k}{2}+k\delta;\,\frac{z^2}{4}\right)e^{2\omega t}\,dt\\=\frac{2\pi e^{-i\gamma k/2}}{2^p}\,e^{z^2/4}\,\frac{d^p}{d\omega^p}\left\{e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j}\,\psi_{\chi,k}\left(e^{2i\omega},z\right)\right\}.\end{aligned}$$

In the previous equality, take the real part of the hypergeometric factor, multiply by $(c_j)_{j\in\mathbb{N}}\in\ell^1$ and sum over j: with the same kind of justifications as in the proof of Theorem 2.1.3, we can deduce the expression

$$\begin{aligned}&\int_{-\infty}^{\infty}t^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right)e^{2\omega t}\,dt\\&=\frac{4\pi}{2^p}\,\mathrm{Re}\left[e^{-i\gamma k/2+z^2/4}\,\frac{d^p}{d\omega^p}\left\{\sum_{j=1}^{\infty}c_j\,e^{i\omega k\left(\frac{1}{2}+\delta\right)-2\omega\lambda_j}\,\psi_{\chi,k}\left(e^{2i\omega},z\right)\right\}\right],\end{aligned} \tag{2.5.33}$$

where

$$\tilde{F}_{z,k,\chi}(s):=e^{-i\gamma k/2}\,F_{z,k,\chi}(s),$$

with $F_{z,k,\chi}(s)$ being

$$\begin{aligned}\sum_{j=1}^{\infty}c_j\,\eta_k\left(s+i\lambda_j,\chi\right)&\left\{{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-s-i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{z^2}{4}\right)\right.\\&\left.+\,{}_1F_1\left(k\left(\frac{1}{2}+\delta\right)-\overline{s}+i\lambda_j;\,k\left(\frac{1}{2}+\delta\right);\,\frac{\overline{z}^2}{4}\right)\right\}.\end{aligned}$$

If we show that the function $\tilde{F}_{z,k,\chi}(s)$ has infinitely many zeros on the line $\mathrm{Re}(s)=\frac{k}{2}\left(\frac{1}{2}+\delta\right)$, we are done. This can be proved similarly as in Theorem 2.1.3: taking the limit $\omega\to\frac{\pi}{4}^-$ on both sides of (2.5.33) and recalling (2.5.25), we deduce that

$$\lim_{\omega\to\frac{\pi}{4}^-}\int_{-\infty}^{\infty}t^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right)e^{2\omega t}\,dt=0. \tag{2.5.34}$$

Assuming that the integrand in (2.5.34) has only a finite number of zeros, we can use our previous reasoning (cf. (2.3.57), (2.3.58) and (2.3.59) above) to show that

$$\int_{-\infty}^{\infty} t^p \left|\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+i\left(t+\lambda_j\right)\right)\right| e^{\frac{\pi}{2}t}\,dt<\infty,$$

and so, by (2.5.34) and the dominated convergence theorem,

$$\int_{-\infty}^{\infty} t^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}\,dt=0.$$

Mimicking (2.3.61) and (2.3.62), we find that

$$\begin{aligned}
&\int_{T_0}^{\infty}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt\\
&=-\int_{0}^{T_0}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt\\
&\leq K(T_0)\,T_0^p \qquad (2.5.35)
\end{aligned}$$

where T_0 is a (sufficiently large) parameter such that $\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right)\neq 0$ when $t>T_0$. As in (2.3.62), the contradiction hypothesis on the zeros of $F_{z,k,\chi}(s)$ yields the lower bound

$$\int_{T_0}^{\infty}\left\{\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}+it\right) e^{\frac{\pi}{2}t}+(-1)^p\,\tilde{F}_{z,k,\chi}\left(\frac{k}{4}+\frac{k\delta}{2}-it\right) e^{-\frac{\pi}{2}t}\right\} t^p dt\geq \epsilon(T_0)\,(2T_0)^p. \qquad (2.5.36)$$

Comparing (2.5.35) and (2.5.36), we are led to the conclusion that the inequality

$$2^p\leq \frac{K(T_0)}{\epsilon(T_0)}$$

must hold for every $p\in\mathbb{N}$, which is absurd. □

Remark 2.5.5. As it is clear from the proof, we need the condition that the sequence $\left(\lambda_j\right)_{j\in\mathbb{N}}$ is bounded, but we have not used the fact that it attains its bounds. Thus, we may relax this condition in the statement of Theorem 2.1.5. Similar comments can be made for Theorem 2.1.6.

2.6 Zeros of combinations attached to $L_f(s,p/q)$

In this final section we extend Wilton's result by proving Theorem 2.1.6. We will be very brief because, unlike in the previous sections, we do not need to develop "exponential summation formulas" for the Dirichlet series under study. The basic Dirichlet series in this section is

$$L_f(s,p/q):=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\quad \text{Re}(s)>\frac{k+1}{2},\quad (p,q)=1, \tag{2.6.1}$$

where $a_f(n)$ are the Fourier coefficients of the cusp form $f(\tau)$ and are assumed to be real. From the functional equation for $L_f(s,p/q)$ (see (2.1.55) and (2.2.103) above)

$$\eta_f\left(s,\frac{p}{q}\right)=(-1)^{k/2}\,\eta_f\left(k-s,\,-\frac{\overline{p}}{q}\right),$$

one can check that $\eta_f(k/2+it,p/q)$ is only real (or purely imaginary) when $p^2\equiv 1 \mod q$.

Henceforth, we shall assume that $p^2\equiv 1 \mod q$. Since $L_f(s,p/q)$ is entire, we expect that the proof of Theorem 2.1.6 will have exactly the same nature as the proof of our Theorem 2.1.5 and so we shall omit it. The only purpose of this section is to show that the analogue of Jacobi's $\psi-$function for $L_f(s,p/q)$, (2.2.100), vanishes exponentially fast when $x\to i$ through the semicircle $|x|=1$, $\text{Re}(x)>0$. This is the last result of this chapter. After showing it, Theorem 2.1.6 is automatically proved once we follow the lines of the previous sections.

Lemma 2.6.1. *Let $f(\tau)$ be a cusp form of weight k for the full modular group. Consider the Dirichlet series*

$$L_f(s,p/q)=\sum_{n=1}^{\infty}\frac{a_f(n)\,e^{\frac{2\pi ip}{q}n}}{n^s},\quad (p,q)=1,\;\; p^2\equiv 1 \mod q,$$

as well as its analogue of Jacobi's $\psi-$function,

$$\psi_{f,p/q}(x,z):=(k-1)!\,\left(\sqrt{\frac{\pi x}{2q}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)\,e^{\frac{2\pi ip}{q}n}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{q}x}\,J_{k-1}\left(\sqrt{\frac{2\pi n}{q}}x\,z\right),$$

defined for Re$(x)>0$ and $z\in\mathbb{C}$. Suppose further that z belongs to the region

$$z\in\mathscr{D}_q:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<2\sqrt{\frac{\pi}{q}},\;|Im(z)|<2\sqrt{\frac{\pi}{q}}\right\}. \tag{2.6.2}$$

Then, for any z satisfying (2.6.2), every $m\in\mathbb{N}_0$ and any analytic function $h:\mathbb{C}\longmapsto\mathbb{C}$, the following relation takes place

$$\lim_{\omega\to\frac{\pi}{4}^-}\frac{d^m}{d\omega^m}\left\{h(\omega)\,\psi_{f,p/q}\left(e^{2i\omega},z\right)\right\}=0. \tag{2.6.3}$$

Proof. The proof is instantaneous because we already have the transformation (2.2.101) in the right template. Indeed,

$$\psi_{f,p/q}(i+\delta,z)=(k-1)!\,2^{k-1}(\sqrt{i+\delta}\,z)^{1-k}\left(\frac{2\pi}{q}\right)^{\frac{1-k}{2}}\\ \times\sum_{n=1}^{\infty}a_f(n)\,e^{\frac{2\pi i(p-1)}{q}n}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{q}\delta}\,J_{k-1}\left(\sqrt{\frac{2\pi n}{q}}\,(i+\delta)\,z\right).$$

Applying (2.2.101) and replacing there x by δ, z by $\frac{\sqrt{i+\delta}}{\sqrt{\delta}}z$ and p by $p-1$, we get

$$\psi_{f,p/q}(i+\delta,z)=(k-1)!\,2^{k-1}(\sqrt{i+\delta}\,z)^{1-k}\left(\frac{2\pi}{q}\right)^{\frac{1-k}{2}}\frac{e^{-\frac{(i+\delta)z^2}{4\delta}}}{\delta}\\ \times(-1)^{k/2}\sum_{n=1}^{\infty}a_f(n)\,e^{-\frac{2\pi iRn}{q}}\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{q\delta}}\,I_{k-1}\left(\sqrt{\frac{2\pi n(i+\delta)}{q}}\,\frac{z}{\delta}\right),$$

where R is such that $R\,(p-1)\equiv 1 \mod q$. From the bound (2.3.24) and the work done in Lemmas 2.3.4 and 2.3.5, the proof of (2.6.3) is complete once we prove that

$$\lim_{\delta\to0^+}\delta^{-1}\exp\left(-\frac{2\pi n}{q\delta}+\sqrt{\frac{2\pi n}{q}}\,\frac{|\mathrm{Re}(z)-\mathrm{Im}(z)|}{\sqrt{2}\delta}+\frac{\mathrm{Re}(z)\,\mathrm{Im}(z)}{2\delta}\right)=0.$$

But a sufficient condition for this to hold is that z satisfies (2.6.2). $\square$

Chapter 3

Estimates for the number of zeros of shifted combinations of completed Dirichlet series

3.1 Introduction and main results

Since Riemann established the connection between the analytic structure of an infinite series and the distribution of the prime numbers, the investigation of the zeros of $\zeta(s)$ has emerged as the main motivation driving some of the major breakthroughs in Number Theory.

As we had the opportunity to mention in previous chapters, one of the most astonishing results towards addressing Riemann's conjecture came from Hardy [106]. Hardy's result establishes that infinitely many nontrivial zeros of $\zeta(s)$ lie on the critical line, this is, on $\text{Re}(s) = \frac{1}{2}$. Shortly after Hardy's proof, Landau [140] extended this theorem to include Dirichlet $L-$functions, as well as diagonal Epstein zeta functions, usually defined as the Dirichlet series

$$\zeta_k(s) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}, \quad \text{Re}(s) > \frac{k}{2},$$

where, recall, $r_k(n)$ denotes the number of ways in which n can be written as the sum of k squares.

Landau also proved an estimate for the number of critical zeros of $\zeta(s)$ up to a certain height. In order to state precisely his result, let $N_0(T)$ denote the number of nontrivial zeros on the critical line whose imaginary parts are between 0 and T. Landau was able to show that $N_0(T) > A \log\log(T)$, $A > 0$, as $T \to \infty$. This estimate marked the beginning of a series of improvements in the lower bounds for $N_0(T)$. Hardy and Littlewood indicate in their paper [[109], p. 125] that, in their first attempt to improve on Landau's result, they proved that, for a sufficiently large T, there should exist a zero of $\zeta(s)$ of the form $s = \frac{1}{2} + i\tau$ with $\tau \in [T, T + T^{\frac{1}{2}+\epsilon}]$. However, they had abandoned this first result because, throughout their investigations, they had been able to establish the better estimate

$$N_0(T) \gg T^{\frac{3}{4}-\epsilon}. \tag{3.1.1}$$

As it is expected from Riemann's conjecture, this lower bound still has a large margin for improvements. First, it was surpassed by the same authors, i.e., by Hardy and Littlewood themselves, who have shown that (3.1.1) can be improved to the more interesting result [110]

$$N_0(T) \gg T. \tag{3.1.2}$$

A generalization of Hardy and Littlewood's method, suited to include Dirichlet $L-$functions, was quickly realized by Suetuna [217], who showed that the number of critical zeros of any Dirichlet $L-$function, $L(s,\chi)$ (with χ being primitive), also satisfies (3.1.2).

Nearly a decade after the publication of Hardy and Littlewood's result, significant progress was made by C. L. Siegel, who introduced a novel method for deriving the estimate (3.1.2). Using the now famous Riemann-Siegel formula, C. L. Siegel [204] proved in 1932 that it is possible to provide an explicit numerical value for the constant appearing in Hardy-Littlewood's result (3.1.2), leading to the more precise estimate

$$N_0(T) > \frac{3}{8\pi} e^{-\frac{3}{2}} T + \mathrm{o}(T). \tag{3.1.3}$$

Although (3.1.3) did not surpass the asymptotic nature of (3.1.2), Siegel's approach would prove to be highly influential in subsequent research on $N_0(T)$. Siegel made further progress on these questions and employed a similar method to find an explicit constant for the number of critical zeros of Dirichlet $L-$functions [202], thus extending Suetuna's early findings [217]. Using the theory of modular forms, Siegel also proved in [202] a precise estimate for the number of zeros of $\zeta_k(s)$, $k \geq 4$, that lie on the critical line $\mathrm{Re}(s) = \frac{k}{4}$. The reader is encouraged to revise the introduction of Chapter 1 to check the precise content of Siegel's result.

Around the same time as this second paper of Siegel appeared, Selberg [200] remarkably improved (3.1.2) and (3.1.3) to $N_0(T) \gg T\,\log(T)$. The main point in Selberg's approach is the replacement of $\zeta(s)$ by $\zeta(s)\,\mathcal{M}(s)$ in the method of Hardy and Littlewood. Here, the function $\mathcal{M}(s)$ is called a mollifier and the fundamental idea behind Selberg's proof is that some inequalities (such as the Cauchy-Schwarz inequality) become sharper when used with $\zeta(s)\,\mathcal{M}(s)$ instead of $\zeta(s)$.[1] If $N(T)$ denotes the number of zeros of the Riemann zeta function $\zeta(s)$ with imaginary part between 0 and T, we know from the Riemann Von-Mangoldt formula that

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O\left(\log(T)\right),$$

and so, according to Selberg's result,

$$\kappa := \liminf_{T\to\infty} \frac{N_0(T)}{N(T)} > 0. \tag{3.1.4}$$

Hence, we may interpret Selberg's theorem by saying that a positive proportion of the nontrivial zeros of $\zeta(s)$ lie on the critical line. However, the lower bound for the proportion κ, implicit in Selberg's argument, is very small.

[1] In fact, the idea behind the mollifier method can be found in a paper by Bohr and Landau [47], which precedes Hardy's note [106]. They used this method to prove the first zero density estimate for $\zeta(s)$.

As we have mentioned above, Selberg's main idea consisted in refining the proof by Hardy and Littlewood. Levinson [143] combined Siegel's alternative method with the mollifier approach to prove the stronger bound $\kappa \geq 0.3474$. Conrey [57] raised the value of κ to at least 0.4088 by using deep techniques involving Kloosterman sums. Currently the record for the best lower bound for κ is held by Pratt, Robles, Zaharescu and Zeindler [171], who showed that $\kappa \geq 5/12$. For other intermediate developments concerning the proportion of κ, we refer to the papers [12, 73], as well as to Chapter 10 of Ivić's treatise [119]. The reader is also encouraged to read Rezvyakova's wonderful paper [182] that investigates analogues of Selberg's result (3.1.4) for linear combinations of Dirichlet series in the Selberg class, offering also a very nice historical overview of these results.

It may not come as a surprise knowing that the aforementioned results admit interesting generalizations to various settings. For instance, as mentioned in the previous chapter, Wilton [237] was the first mathematician to show that $L-$functions attached to holomorphic cusp forms have infinitely many zeros on their critical lines. The underlying philosophy of Wilton's proof mirrors Hardy's original argument [106]. An analogue of the estimate (3.1.2) for these $L-$functions was provided by Lekkerkerker in his doctoral thesis [142]. This method was later adapted by Epstein, Hafner and Sarnak [83] to establish the same estimate, i.e. (3.1.2), for $L-$functions associated with Maass cusp forms. The first analogue of Selberg's theorem for these automorphic $L-$functions was proved by Hafner [102, 103]. Hafner was, therefore, the first mathematician to ever show that the positive proportion result (3.1.4) was valid for some $L-$functions in the Selberg class with degree 2 (cf. [[182], p. 603]).

There are other proofs and arguments concerning estimates for $N_0(T)$ that, while significant, did not gain widespread recognition. For example, using Ingham's result on the fourth moment of $\zeta\left(\frac{1}{2}+it\right)$ [118], Atkinson was able to demonstrate that $N_0(T) \gg T/\log(T)$ [7]. Although this estimate is less sharp than the one found by Hardy and Littlewood (3.1.2), the method employed by Atkinson is somewhat easier to follow. More recently, Baluyot [12] combined Atkinson's approach with the mollifier method to give yet another proof of Selberg's result (3.1.4).

Another approach to Hardy's theorem is due to Deuring [61], who developed a beautiful connection between the critical zeros of $\zeta(s)$ and the growth of $|\zeta_2(s)|$ on the critical line $\mathrm{Re}(s)=\frac{1}{2}$. In the first chapter of this thesis, we explored an inductive generalization of Deuring's idea, demonstrating that a variant of it works not only for $\zeta(s)$ but also for several Dirichlet series. Later in this chapter, we will delve into two lesser-known methods for proving Hardy's theorem, respectively due to Fekete and to de la Vallée Poussin, which will play a crucial role in the proof of the main results of this work.

In the previous chapter we have studied zeros of shifted combinations involving several Dirichlet series. One of the main results obtained there, namely Theorem 2.1.3, concerned zeros of shifted combinations involving the $\Gamma-$function and $\zeta_\alpha(s)$ (see (2.1.10) above). For clarity in our exposition, we shall restate it here.

Theorem 3.1.1 (see Theorem 2.1.3 above)**.** *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. Then,* *for any z satisfying the condition*

$$z \in \mathscr{D}_{\alpha} := \left\{ z \in \mathbb{C} \, : \, |Re(z)| < \sqrt{\frac{\pi \alpha}{2}}, \, |Im(z)| < \sqrt{\frac{\pi \alpha}{2}} \right\}, \tag{3.1.5}$$

the function

$$\begin{aligned} F_{z,\alpha}(s) := \sum_{j=1}^{\infty} c_j \, \pi^{-(s+i\lambda_j)} \Gamma\left(s+i\lambda_j\right) \zeta_{\alpha}(s+i\lambda_j) \\ \times \left\{ {}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j; \, \frac{\alpha}{2}; \, \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j; \, \frac{\alpha}{2}; \, \frac{\overline{z}^2}{4}\right) \right\} \end{aligned} \tag{3.1.6}$$

has infinitely many zeros on the critical line $Re(s) = \frac{\alpha}{4}$.

One of the main ingredients in the proof of this theorem is a generalization of Dixit's integral formula (2.1.9), obtained in the previous chapter. Recalling (2.2.71), this identity takes the form

$$\begin{aligned} x^{\alpha/4}\psi_{\alpha}(x,z) - x^{-\alpha/4}\, e^{-\frac{z^2}{4}} &= e^{-\frac{z^2}{4}} x^{-\alpha/4}\, \psi_{\alpha}\left(\frac{1}{x}, \, iz\right) - x^{\alpha/4} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta_{\alpha}\left(\frac{\alpha}{4}+it\right) {}_1F_1\left(\frac{\alpha}{4}+it; \, \frac{\alpha}{2}; \, -\frac{z^2}{4}\right) x^{-it}\, dt, \end{aligned} \tag{3.1.7}$$

where $\eta_{\alpha}(s)$ is the completed Dirichlet series (2.2.72) and $\psi_{\alpha}(x,z)$ is the generalization of Jacobi's $\psi-$function (see (2.2.69) above),

$$\psi_{\alpha}(x,z) = 2^{\frac{\alpha}{2}-1}\, \Gamma\left(\frac{\alpha}{2}\right) \left(\sqrt{\pi x}\, z\right)^{1-\frac{\alpha}{2}} \sum_{n=1}^{\infty} r_{\alpha}(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-\pi n x}\, J_{\frac{\alpha}{2}-1}(\sqrt{\pi\, n\, x}\, z), \quad \mathrm{Re}(x) > 0, \;\; z \in \mathbb{C}. \tag{3.1.8}$$

As we proved in Corollary 2.2.2 of the previous chapter, the transformation formula (3.1.7) can be also taken in a general setting, with $\zeta_{\alpha}(s)$ being replaced by any Dirichlet series satisfying Definition 2.1.2. For example, let $f(\tau)$ be a holomorphic cusp form with weight $k \geq 12$ for the full modular group whose Fourier expansion is given by

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)\, e^{2\pi i n \tau}, \quad \mathrm{Im}(\tau) > 0. \tag{3.1.9}$$

If we construct the Dirichlet series associated to $f(\tau)$,

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \mathrm{Re}(s) > \frac{k+1}{2}, \tag{3.1.10}$$

we know that $L(s,f)$ can be analytically continued to an entire function obeying Hecke's functional equation (see (1.2.67) and (2.2.95) above)

$$\eta_f(s) := (2\pi)^{-s}\Gamma(s)\, L(s,f) = (-1)^{k/2}\eta_f\left(k-s\right). \tag{3.1.11}$$

Analogously to the construction $\psi_{\alpha}(x,z)$ (3.1.8), we can take a generalization of the cusp form $f(\tau)$ in the form[2]

$$\psi_{f}(x,z):=(k-1)!\,\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_{f}(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n\,x}\,z\right),\ \mathrm{Re}(x)>0,\ z\in\mathbb{C},\tag{3.1.12}$$

and show that it obeys to the transformation formula (see (2.2.97) above)

$$\begin{aligned}
x^{\frac{k}{2}}\psi_{f}(x,z) &= (-1)^{k/2}\,e^{-\frac{z^{2}}{4}}x^{-k/2}\,\psi_{f}\left(\frac{1}{x},\,iz\right)\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}(2\pi)^{-\frac{k}{2}-it}\,\Gamma\left(\frac{k}{2}+it\right)L\left(\frac{k}{2}+it,f\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^{2}}{4}\right)\,x^{-it}\,dt\\
&:=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_{f}\left(\frac{k}{2}+it\right)\,{}_1F_1\left(\frac{k}{2}+it;\,k;\,-\frac{z^{2}}{4}\right)\,x^{-it}dt,
\end{aligned}\tag{3.1.13}$$

where $\eta_{f}(s)$ is the completed Dirichlet series (3.1.11). Using (3.1.13), we have been able to give a sketch of the proof of Theorem 2.1.6 in the final section of the previous chapter. We now state this result of ours.

Theorem 3.1.2 (see Theorem 2.1.6 above)**.** *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_{f}(n)$. Consider the Dirichlet series,*

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_{f}(n)}{n^{s}},\quad Re(s)>\frac{k+1}{2}.\tag{3.1.14}$$

If $(c_{j})_{j\in\mathbb{N}}$ is a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty}|c_{j}|<\infty$, $(\lambda_{j})_{j\in\mathbb{N}}$ is a bounded sequence of distinct real numbers[3] and z satisfies the condition

$$z\in\mathscr{D}:=\left\{z\in\mathbb{C}\,:\,|Re(z)|<2\sqrt{\pi},\ |Im(z)|<2\sqrt{\pi}\right\},\tag{3.1.15}$$

then the function

$$\begin{aligned}
G_{z,f}\left(s\right) &:=\sum_{j=1}^{\infty}c_{j}\,\left(2\pi\right)^{-s-i\lambda_{j}}\,\Gamma(s+i\lambda_{j})\,L\left(s+i\lambda_{j},f\right)\\
&\times\left\{{}_1F_1\left(k-s-i\lambda_{j};\,k;\,\frac{z^{2}}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_{j};\,k;\,\frac{\overline{z}^{2}}{4}\right)\right\}
\end{aligned}\tag{3.1.16}$$

has infinitely many zeros at the critical line $Re(s)=\frac{k}{2}$.

Since the previous chapter consisted in applications of Hardy's well-known method to study critical zeros of the cumbersome functions $F_{z,\alpha}(s)$ and $G_{f,z}(s)$, respectively defined by (3.1.6) and (3.1.16), one may pose the following natural question.

[2]Note that, when $z=0$, $\psi_{f}(x,0)=f(ix)$.

[3]At the previous chapter, the statement of Theorem 2.1.6 used the condition that the sequence $(\lambda_{j})_{j\in\mathbb{N}}$ attains its bounds. However, in view of Remark 2.5.5, this hypothesis is not necessary when we work with combinations involving entire Dirichlet series.

Question 3.1.1. *We know that the number of critical zeros of the functions (3.1.6) and (3.1.16) is infinite. However, for a given $T \gg 1$, can we estimate the number of critical zeros of $F_{z,\alpha}(s)$ and $G_{f,z}(s)$ whose imaginary parts lie between 0 and T?*

In this chapter, our primary objective is to answer to the previous question, this is, to establish quantitative analogues of Theorems 3.1.1 and 3.1.2 for a particular subclass of shifted combinations. This approach aims to deepen the understanding of the critical zeros in this context and to provide new insights that may be applied to broader classes of functions (see Remarks 3.1.3 and 3.1.5 below). Specifically, we focus on determining lower bounds for the number of critical zeros of the functions $F_{z,\alpha}(s)$ and $G_{z,f}(s)$ under additional constraints on the sequence $(\lambda_j)_{j\in\mathbb{N}}$. To achieve this, we shall work within a set of assumptions, which we now introduce.

1. **Assumption 1:** If the shift $s \to s + i\lambda_j$ is introduced in one of the combinations (3.1.6) or (3.1.16), then the symmetric shift $s \to s - i\lambda_j$ needs to be introduced as well. Moreover, this symmetric shift must be labeled with the same weight as λ_j, i.e., with c_j. Concerning Theorem 3.1.1, for example, this means that if the term

$$c_j\,\eta_\alpha\left(s+i\lambda_j\right)\mathrm{Re}\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}$$

belongs to the combination (3.1.6), then the term

$$c_j\,\eta_\alpha\left(s-i\lambda_j\right)\mathrm{Re}\left\{{}_1F_1\left(\frac{\alpha}{2}-s+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}$$

also belongs to it. This assumption essentially says that we are enlarging the sequences $\left(\lambda_j\right)_{j\in\mathbb{N}}$ in (3.1.6) and (3.1.16) in such a way that each element has a symmetric pair contained in it. Therefore, we can write it as a sequence over $\mathbb{Z}\setminus\{0\}$ in the form $\left(\lambda_j\right)_{j\in\mathbb{Z}\setminus\{0\}} = \left(\lambda_j\right)_{j\in\mathbb{N}} \cup \left(-\lambda_j\right)_{j\in\mathbb{N}}$, once we take the convention that $\lambda_{-j} := -\lambda_j$. The same can be done to $(c_j)_{j\in\mathbb{N}}$ once we take the convention $c_{-j} := c_j$.

2. **Assumption 2:** In each one of the cases (3.1.6) and (3.1.16), the parameter z in the hypergeometric functions will be a real number belonging to an interval contained in the regions (3.1.5) and (3.1.15).

Under these assumptions, we can rewrite the shifted combinations (3.1.6) and (3.1.16) in the symmetric forms

$$\tilde{F}_{z,\alpha}(s) := \sum_{j\neq 0} c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_\alpha\left(s+i\lambda_j\right)$$
$$\times\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\} \qquad (3.1.17)$$

and

$$\tilde{G}_{z,f}(s) := \sum_{j\neq 0} c_j \, (2\pi)^{-s-i\lambda_j}\, \Gamma(s+i\lambda_j)\, L\left(s+i\lambda_j, f\right)$$
$$\times \left\{ {}_1F_1\left(k-s-i\lambda_j;\, k;\, \frac{z^2}{4}\right) + {}_1F_1\left(k-\overline{s}+i\lambda_j;\, k;\, \frac{z^2}{4}\right)\right\}. \tag{3.1.18}$$

In both cases, the sum is taken over $\mathbb{Z}\setminus\{0\}$, which make sense because, according to Assumption 1, we are taking the conventions that $c_{-j} := c_j$ and $\lambda_{-j} := -\lambda_j$.

Note also that $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ is a real-valued and even function of $t\in\mathbb{R}$. These properties come immediately from the functional equation (2.1.12) and from the Assumptions 1 and 2. In the same lines, one can easily check that, when $a_f(n)\in\mathbb{R}$, the function $i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ is real-valued as well. Additionally, $i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ is an even function if $k\equiv 0 \mod 4$ and it is an odd function if $k\equiv 2 \mod 4$.

Although the introduction of the Assumptions 1 and 2 may restrict some of the general aspects of Theorems 3.1.1 and 3.1.2, they prove to be very helpful in the derivation of some estimates for the number of zeros of the functions (3.1.17) and (3.1.18). We begin by stating a result about the number of zeros of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$.

Theorem 3.1.3. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_{j=1}^{\infty}|c_j|<\infty$, and let $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers that attains its bounds. Let us impose that $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ can be extended to the set $\mathbb{Z}\setminus\{0\}$ with the properties $c_{-j}=c_j$ and $\lambda_{-j}=-\lambda_j$. Additionally, assume that the real parameter z satisfies the condition*

$$z\in\left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}},\, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]. \tag{3.1.19}$$

Let $N_{\alpha,z}(T)$ denote the number of zeros of the function

$$\tilde{F}_{z,\alpha}(s) := \sum_{j\neq 0} c_j\, \pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\, \zeta_\alpha\left(s+i\lambda_j\right)$$
$$\times \left\{ {}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\, \frac{\alpha}{2};\, \frac{z^2}{4}\right)\right\} \tag{3.1.20}$$

that lie on the critical line $s=\frac{\alpha}{4}+it$, with $0\leq t\leq T$.

Then, there exists a constant $c>0$ such that

$$\liminf_{T\to\infty} \frac{N_{\alpha,z}(T)}{\sqrt{T}/\log(T)} \geq c. \tag{3.1.21}$$

The quantitative estimate given in (3.1.21) is fairly general, and improving on the power of T using the same method appears to be quite challenging. In the final section of this chapter, we shall propose some conjectures regarding potential improvements of this exponent. The most important takeaway from this result is that, regardless of how the bounded shifts $(\lambda_j)_{j\in\mathbb{N}}$, the sequence $(c_j)_{j\in\mathbb{N}}$, or the bounded parameter z in the hypergeometric factor are chosen, one can always find at least $\gg \sqrt{T}/\log(T)$ zeros of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ within the interval $t\in[0,T]$.

If we set $z=0$ in Theorem 3.1.3, the following corollary naturally follows.

Corollary 3.1.1. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j|<\infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. If $N_{0,\alpha}(T)$ denotes the number of zeros $s=\frac{\alpha}{4}+it,\ 0\leq t\leq T$, of the shifted combination*

$$\tilde{F}_{\alpha}(s)=\sum_{j=1}^{\infty}c_j\left\{\pi^{-s-i\lambda_j}\Gamma\left(s+i\lambda_j\right)\zeta_{\alpha}\left(s+i\lambda_j\right)+\pi^{-s+i\lambda_j}\Gamma\left(s-i\lambda_j\right)\zeta_{\alpha}\left(s-i\lambda_j\right)\right\}, \tag{3.1.22}$$

then there exists some constant $c>0$ such that

$$\liminf_{T\to\infty}\frac{N_{0,\alpha}(T)}{\sqrt{T}/\log(T)}\geq c. \tag{3.1.23}$$

When the shifted combination in (3.1.22) is trivial (i.e., when $c_j=0$ for every $j\geq 2$ and $\lambda_1=0$), our estimate (3.1.23) extends, for all $\alpha>0$, Corollary 1.3.5 obtained at the first chapter (see Remark 3.1.2 below). Furthermore, since $\zeta_1(s):=2\zeta(2s)$, we can specialize the previous corollary to the following result.

Corollary 3.1.2. *Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j|<\infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers attaining its bounds. If $\tilde{N}_0(T)$ denotes the number of zeros $s=\frac{1}{2}+it,\ 0\leq t\leq T$, of the shifted combination*

$$\tilde{F}(s):=\sum_{j\neq 0}c_j\left\{\pi^{-\frac{1}{2}(s+i\lambda_j)}\Gamma\left(\frac{s+i\lambda_j}{2}\right)\zeta\left(s+i\lambda_j\right)+\pi^{-\frac{1}{2}(s-i\lambda_j)}\Gamma\left(\frac{s-i\lambda_j}{2}\right)\zeta\left(s-i\lambda_j\right)\right\}, \tag{3.1.24}$$

then there exists some constant $c>0$ such that

$$\liminf_{T\to\infty}\frac{\tilde{N}_0(T)}{\sqrt{T}/\log(T)}\geq c. \tag{3.1.25}$$

The generality of this result, with respect to the shifted combination and the explicit estimate for the number of critical zeros, seems to be unnoticed in the literature. We should remark that Selberg's outstanding result about the positive proportion of combinations of $L-$functions with degree one [199] does not seem to cover the case (3.1.24), because we are considering shifted combinations of $\eta(s)$ and not combinations of distinct $L-$functions.

In our proof of Theorem 3.1.3, we draw upon a method originally developed by Fekete [88], a technique that, to the best of our knowledge, has rarely been used for explicit zero counting estimates, aside from Fekete's own work. This approach is one of the many variants of Hardy's result that has largely been forgotten over the last century. Even in Titchmarsh's text [[228], p. 259], only a portion of Fekete's main argument is explained in detail. This portion (which only corresponds to the first half of Fekete's note) only establishes the infinitude of critical zeros of $\zeta(s)$ without offering insight into how to extract an estimate from the underlying idea. A very interesting paper by Berlowitz uses the same method [[18], p. 206, Lemma 2] but, again, only the first half of Fekete's paper is important in his argument.

The reasons why Fekete's paper is not so well-known is because it was written in a rapidly evolving period in the study of the Riemann zeta function. Fekete himself acknowledged that his result did not add any new information about the zero distribution of $\zeta(s)$, as by the time of his publication, Hardy and Littlewood had already established the estimate $N_0(T) \gg T$.[4]

In any case, we found out that Fekete's main ideas are particularly well-suited for studying zeros of shifted combinations of the form (3.1.20). Fekete's argument is essentially inspired by a lemma of Fejér [86] (see Lemma 3.3.1 below), which addresses the number of zeros of the sequence of moments of a given continuous function. In order to apply Fejér's lemma, however, it will be crucial in our proof below to use the fact that $\zeta_\alpha(s)$ has a simple pole located at $s = \frac{\alpha}{2}$.

This limits the scope of Fekete's method, because it cannot be applied to extend Theorem 3.1.2. Given that $L(s, f)$ is an entire function of s, deriving a quantitative analogue of Theorem 3.1.2 will require a different approach. In any case, we have succeeded in proving the following result.

Theorem 3.1.4. *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$ and let $L(s, f)$ be the $L-$function attached to it. Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of non-zero real numbers such that $\sum_{j=1}^{\infty} |c_j| < \infty$, and let $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers. Let us impose that $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ can be extended to the set $\mathbb{Z}\setminus\{0\}$ with the properties $c_{-j} = c_j$ and $\lambda_{-j} = -\lambda_j$. Additionally, assume that the real parameter z satisfies the condition*

$$z \in \left[-\frac{\sqrt{\pi}}{3}, \frac{\sqrt{\pi}}{3}\right]. \tag{3.1.26}$$

Let $N_{f,z}(T)$ denote the number of zeros of the function of the function

$$\tilde{G}_{z,f}(s) := \sum_{j\neq 0} c_j \, (2\pi)^{-s-i\lambda_j}\, \Gamma(s+i\lambda_j)\, L\left(s+i\lambda_j, f\right) \\ \times \left\{ {}_1F_1\left(k-s-i\lambda_j;\, k;\, \frac{z^2}{4}\right) + {}_1F_1\left(k-\overline{s}+i\lambda_j;\, k;\, \frac{z^2}{4}\right)\right\} \tag{3.1.27}$$

that lie on the critical line $s = \frac{k}{2} + it$, with $0 \leq t \leq T$.

Then there exists a constant $d > 0$ such that

$$\limsup_{T\to\infty} \frac{N_{f,z}(T)}{\sqrt{T}} \geq d. \tag{3.1.28}$$

Analogously to Corollary 3.1.1, setting $z = 0$ in the previous result immediately implies the following result.

[4] It is worth noting, however, that Hardy and Littlewood alluded to a similar idea to Fekete's on page 125 of their paper [109]. They have never published the proof supporting this idea, because (3.1.2) quickly superseded it. Also, the proof of (3.1.2) seemed to be more elementary and easier to expose, avoiding to handle with the properties of the $\theta-$function and with the Cahen-Mellin integral (1.1.31). Quoting Hardy and Littlewood *"Our proof of this result [i.e., (3.1.2)] is now free from any reference either to the Cahen-Mellin integral or to the theory of elliptic functions."*.

Corollary 3.1.3. *Let $f(\tau)$ be a cusp form of weight k for the full modular group with real Fourier coefficients $a_f(n)$ and let $L(s,f)$ be the $L-$function attached to it. Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers such that $\sum_j |c_j| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$ be a bounded sequence of distinct real numbers. If $N_{0,f}(T)$ denotes the number of zeros $s = \frac{k}{2} + it,\ 0 \leq t \leq T,$ of the shifted combination*

$$\tilde{G}_f(s) = \sum_{j=1}^{\infty} c_j \left\{(2\pi)^{-s-i\lambda_j}\Gamma\left(s+i\lambda_j\right) L\left(s+i\lambda_j, f\right) + (2\pi)^{-s+i\lambda_j}\,\Gamma\left(s-i\lambda_j\right) L\left(s-i\lambda_j, f\right)\right\}, \tag{3.1.29}$$

then there exists a constant $d > 0$ such that

$$\limsup_{T\to\infty} \frac{N_{0,f}(T)}{\sqrt{T}} \geq d. \tag{3.1.30}$$

Our proof of Theorem 3.1.4 draws inspiration from a beautiful argument due to de la Vallée Poussin [58], supplemented by some ideas from our paper [187]. The Belgian mathematician Charles-Jean de la Valleé Poussin is mostly famous for his proof of the Prime Number Theorem. However, his contributions to the study of the zeros of $\zeta(s)$ on the line $\mathrm{Re}(s) = \frac{1}{2}$ are not so well-known. Shortly after the appearance of the papers by Hardy and Landau [106, 140] on this topic, de la Vallée Poussin published two short notes. In one of them, he proved that the number of zeros of $\zeta(s)$ of the form $s = \frac{1}{2} + it$, $0 < t < T$, satisfies the estimate $N_0(T) = \Omega\left(T^{1/2}\right)$, as $T \to \infty$. However, as he remarks [[58], p. 421] *"Cette méthode ne s'applique pas à toutes les fonctions qui interviennent dans l'étude de la progression arithmétique".* [5] In order to circumvent this difficulty, de la Vallée Poussin develops in a later note an argument providing estimates for the difference $N_0(T+H) - N_0(T),\ \ T^{\frac{3}{4}+\epsilon} \leq H(T) \leq T$. His second note was the first improvement of the results of Landau about the zeros of Dirichlet $L-$functions.

Although the arguments in these short notes were elegant, the contributions of de la Vallée Poussin, like those of Fekete, quickly faded into obscurity [50]. The sharper estimates provided by Hardy and Littlewood [109, 110] around the same time[6], and later given by Selberg and Levinson [143, 200] overshadowed his work. Even Titchmarsh's renowned text on the zeta function, celebrated for its meticulous historical background, overlooks de la Vallée Poussin's contributions in this regard, notwithstanding the fact that it exposes (just for the sake of historical interest) five different proofs of Hardy's theorem [[228], Chapter X]. [7]

With the proof of Theorem 3.1.4, we hope to bring de la Vallée's Poussin elegant arguments to a modern mathematical audience, as we believe that the method employed in proving the result (3.1.28) holds significant theoretical interest in its own right.

This chapter is organized as follows. We begin by presenting the essential technical lemmas required for establishing Theorem 3.1.3. A particular emphasis shall be reserved for Lemma 3.2.2,

[5] by other words, his method does not work for certain Dirichlet $L-$functions.

[6] The first improvements on the number of critical zeros of $\zeta\left(s\right)$ by Hardy and Littlewood were in fact independent of those of de la Vallée Poussin. An additional note to the paper [[109], p. 196] was written by Hardy and Littlewood just to acknowledge de la Vallée Poussin's contributions.

[7] It should be mentioned, however, that de la Vallée Poussin's papers [58, 59] appear as references in Titchmarsh's book but they are not mentioned in the main text.

which will be derived from a generalization of the theta transformation formula discussed in the previous chapter, namely, in Lemma 2.3.3. Section 3.3 is devoted to a proof of Theorem 3.1.3. An analogous structure is followed in sections 3.4 and 3.5. Finally, this chapter concludes with some conjectures about potential extensions of our Theorems 3.1.3 and 3.1.4.

Before advancing further, we introduce a couple of remarks which describe some interesting particular cases of our Theorems 3.1.3 and 3.1.4, as well as other results that can be proved using the same methods.

Remark 3.1.1. The intervals (3.1.19) and (3.1.26) are respectively contained in the regions (3.1.5) and (3.1.15) of Theorems 3.1.1 and 3.1.2. As we shall see below, it is possible to enlarge the length of the interval (3.1.19) by a more careful choice of the parameter λ in the proof of Lemma 3.2.2 and a more precise inequality for $|I_\nu(x)|$. See Remark 3.2.1 below.

Remark 3.1.2. Using an inductive method involving generalized Epstein zeta functions, we have proved in [194] and stated at the first chapter that, when $\alpha > 4$, there always exist a sufficiently large $T_0(\alpha)$ and $c := c(\alpha) > 0$ such that, for any $T \geq T_0(\alpha)$, there is a zero $\rho = \frac{\alpha}{4} + i\gamma$ of $\zeta_\alpha(s)$ with $\gamma \in [T, T + cT^{1/2}\log(T)]$ (see Corollary 1.3.5 above). In particular, if $N_0(T,\, \zeta_\alpha(s))$ denotes the number of zeros of $\zeta_\alpha\left(\frac{\alpha}{4} + it\right)$ with $0 < t < T$, then

$$\liminf_{T\to\infty} \frac{N_0(T,\, \zeta_\alpha(s))}{\sqrt{T}/\log(T)} \geq c' > 0, \quad \alpha > 4. \tag{3.1.31}$$

Our Theorem 3.1.3 above generalizes the estimate (3.1.31) to shifted combinations of $\zeta_\alpha(s)$. Moreover, when $z = 0$ and $c_1 = 1$, $\lambda_1 = 0$, $c_2 = c_3 = ... = 0$, it actually extends (3.1.31) to the range $0 < \alpha \leq 4$. As remarked in the first chapter above, when α is any integer greater than 3, it is known that $N_0(T,\, \zeta_\alpha(s)) \asymp T$, which is a sharp estimate due to C. L. Siegel [202]. Thus, when $\alpha \in \mathbb{N}_{\geq 4}$, (3.1.31) does not say anything new, but it seems to be a novel result when $\alpha \notin \mathbb{N}$.

Remark 3.1.3. Analogues of our Theorem 3.1.3 can be proved when $\zeta_\alpha(s)$ is replaced by any of the meromorphic Dirichlet series mentioned in the previous chapter. For example, it is still valid if we replace $\zeta_\alpha(s)$ by an Epstein zeta function $\zeta(s, Q)$ attached to a binary and integral quadratic form $Q(m, n) = Am^2 + Bmn + Cn^2$ with the additional property that $\sqrt{4AC - B^2} \equiv 2 \mod 4$. Of course, the admissible region for z in this case would have to depend on the discriminant of Q (see Theorem 2.1.4 above).

Remark 3.1.4. A lower bound for d appearing in (3.1.28) can be explicitly computed. A simple numerical computation (see the proof of Lemma 3.4.2 below) shows that the value $d = \frac{1}{36\pi}$ is admissible.

Remark 3.1.5. Since the method of proof of Theorem 3.1.4 works well for any entire Dirichlet series, there is an analogue of this result for some Dirichlet $L-$functions. For the details, the reader is encouraged to check the proof of Theorem 2.1.5 on the previous chapter. Furthermore, we can use this method to prove (3.1.28) for a general combination involving shifts of $\zeta_\alpha(s)$. Since $s\left(s - \frac{\alpha}{2}\right)\eta_\alpha(s)$ is an entire function of s, then our proof of Theorem 3.1.4 can be applied to study the number of zeros of the function

$$\sum_{j\neq 0} c_j\,(s+i\lambda_j)\left(s+i\lambda_j-\frac{\alpha}{2}\right)\eta_\alpha\,(s+i\lambda_j)$$
$$\times\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\}, \tag{3.1.32}$$

whenever z satisfies (3.1.19). If $N_{\alpha,z}^{\star}(T)$ denotes the number of zeros of the above combination (3.1.32) that can be written in the form $s=\frac{\alpha}{4}+it$, $0\leq t\leq T$, then the following estimate takes place

$$\limsup_{T\to\infty}\frac{N_{\alpha,z}^{\star}(T)}{\sqrt{T}}\geq\tilde{c}, \tag{3.1.33}$$

for some $\tilde{c}>0$. In particular, when we reduce the combination (3.1.32) to the case where $\lambda_1=0$ and $c_j=0$ for any $|j|\geq 2$, we see that, besides (3.1.31), we also have $N_0(T,\,\zeta_\alpha(s))=\Omega\left(T^{\frac{1}{2}}\right)$ for any $\alpha>0$.

3.2 Lemmas for the proof of Theorem 3.1.3

As we have done in the previous two chapters, we start by recalling Stirling's formula for the Gamma function,

$$\Gamma(\sigma+it)=(2\pi)^{\frac{1}{2}}\,t^{\sigma+it-\frac{1}{2}}\,e^{-\frac{\pi t}{2}-it+\frac{i\pi}{2}(\sigma-\frac{1}{2})}\left(1+\frac{1}{12(\sigma+it)}+O\left(\frac{1}{t^2}\right)\right),\quad t\to\infty,$$

valid whenever $-\infty<\sigma_1\leq\sigma\leq\sigma_2<\infty$. A similar formula can be written for $t<0$ as t tends to $-\infty$ by using the fact that $\Gamma(\overline{s})=\overline{\Gamma(s)}$. Of course, a direct consequence of this exact version is

$$|\Gamma(\sigma+it)|=(2\pi)^{\frac{1}{2}}\,|t|^{\sigma-\frac{1}{2}}\,e^{-\frac{\pi}{2}|t|}\left(1+O\left(\frac{1}{|t|}\right)\right),\quad |t|\to\infty. \tag{3.2.1}$$

As seen in earlier stages of this thesis, we can rigorously define the Dirichlet series attached to $\zeta_\alpha(s)$ in the following way (see (2.1.44) above)

$$\zeta_\alpha(s):=\sum_{n=1}^{\infty}\frac{r_\alpha(n)}{n^s},\quad \mathrm{Re}(s)>\sigma_\alpha:=\begin{cases}\frac{\alpha}{2} & \alpha\geq 4\\ 1+\frac{\alpha}{4} & 0<\alpha<4\end{cases}. \tag{3.2.2}$$

Besides, if one mimics Riemann's seminal work, one is able to easily show that $\zeta_\alpha(s)$ can be analytically continued to the entire complex plane as a meromorphic function with a simple pole located at $s=\frac{\alpha}{2}$, whose residue is $\mathrm{Res}_{s=\alpha/2}\zeta_\alpha(s)=\frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}$. Furthermore, $\zeta_\alpha(s)$ satisfies Hecke's functional equation

$$\eta_\alpha(s):=\pi^{-s}\Gamma(s)\,\zeta_\alpha(s)=\pi^{-\left(\frac{\alpha}{2}-s\right)}\Gamma\left(\frac{\alpha}{2}-s\right)\,\zeta_\alpha\left(\frac{\alpha}{2}-s\right):=\eta_\alpha\left(\frac{\alpha}{2}-s\right).$$

According to the Phragmén-Lindelöf principle (2.1.39), $\zeta_{\alpha}(s)$ also adheres to the convex estimate

$$\zeta_{\alpha}\left(\sigma+it\right)\ll_{\alpha}|t|^{\sigma_{\alpha}-\sigma+\delta},\quad \frac{\alpha}{2}-\sigma_{\alpha}-\delta<\sigma<\sigma_{\alpha}+\delta, \tag{3.2.3}$$

for any $\delta>0$, where σ_{α} is defined by (3.2.2). For a more detailed discussion, we recommend the reader to review Example 1.3.3 of the first chapter or Section 2.1.2 above.

In order for us to estimate the function $\tilde{F}_{z,\alpha}(s)$, (3.1.17), an asymptotic formula for the confluent hypergeometric function valid when $|s|\to\infty$ will be greatly needed. To state it, we have to recall the introductory lines of Section 2.2 of the previous chapter. It is known that the Whittaker function of the first kind, $M_{\lambda,\mu}(z)$, has the asymptotic behavior [[164], p. 341, eq. (13.21.1)]

$$M_{\lambda,\mu}(z)=\frac{z^{1/4}}{\sqrt{\pi}}\lambda^{-\mu-\frac{1}{4}}\Gamma\left(2\mu+1\right)\cos\left(2\sqrt{\lambda z}-\frac{\pi}{4}-\mu\pi\right)+O\left(|\lambda|^{-\mu-\frac{3}{4}}\right), \tag{3.2.4}$$

as $|\lambda|\to\infty$ and z being such that $|\arg(\lambda z)|<2\pi$. Furthermore, since $M_{\lambda,\mu}(z)$ is connected to Kummer's hypergeometric function via the relation

$$M_{\lambda,\mu}(z)=z^{\mu+\frac{1}{2}}e^{-z/2}\,{}_1F_1\left(\mu-\lambda+\frac{1}{2};\,2\mu+1;\,z\right), \tag{3.2.5}$$

we see that, for any fixed $z\in\mathbb{R}$, the substitutions in (3.2.4) and (3.2.5) yield the useful bound

$$\left|{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right|=\left(\frac{|z|}{2}\right)^{-\frac{\alpha}{2}}e^{\frac{z^2}{8}}\left\{\Gamma\left(\frac{\alpha}{2}\right)\sqrt{\frac{|z|}{2\pi}}\,|t|^{\frac{1-\alpha}{4}}\exp\left(\sqrt{\frac{|t|}{2}}\,|z|\right)+O\left(|t|^{-\frac{\alpha}{4}-\frac{1}{4}}\right)\right\}, \tag{3.2.6}$$

as $|t|\to\infty$. This estimate shall be more than enough to justify most of the steps in this chapter, as it was in the previous one. On the other hand, merging Stirling's formula (3.2.1) with (3.2.3) shows the bound

$$\left|\eta_{\alpha}\left(\frac{\alpha}{4}+it\right)\right|\ll_{\alpha}|t|^{A(\alpha)}\,e^{-\frac{\pi}{2}|t|},\quad |t|\to\infty, \tag{3.2.7}$$

where $A(\alpha)=\sigma_{\alpha}-\frac{1}{2}$, with σ_{α} being defined by (3.2.2). When combined with (3.2.6), the convex estimate (3.2.7) yields

$$\begin{aligned}\left|\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| &\leq\sum_{j\neq0}\left|c_j\,\eta_{\alpha}\left(\frac{\alpha}{4}+i\left(t+\lambda_j\right)\right)\mathrm{Re}\left({}_1F_1\left(\frac{\alpha}{4}-i\left(t+\lambda_j\right);\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right)\right|\\ &\ll_{\alpha,z}C_{\lambda}\sum_{j=1}^{\infty}|c_j|\,|t|^{B(\alpha)}\,e^{-\frac{\pi}{2}|t|+|z|\sqrt{|t|}},\end{aligned} \tag{3.2.8}$$

where we have used the fact that $\left(\lambda_j\right)_{j\in\mathbb{N}}$ is a bounded sequence and $\left(c_j\right)_{j\in\mathbb{N}}\in\ell^1$. The term C_{λ} on the right-hand side of (3.2.8) only stands for a positive constant depending on the bounds of the sequence $\left(\lambda_j\right)_{j\in\mathbb{N}}$.

Next, let us note that an explicit way of writing the first equality in (3.1.7) is

$$1+2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\,(\sqrt{\pi x}\,z)^{1-\frac{\alpha}{2}}\,\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n\,x}\,J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n\,x}\,z\right)$$
$$=\frac{e^{-\frac{z^2}{4}}}{x^{\alpha/2}}\left\{1+2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\frac{\pi}{x}}\,z\right)^{1-\frac{\alpha}{2}}\sum_{n=1}^{\infty}r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{\alpha}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\,z\right)\right\},\qquad(3.2.9)$$

which is valid whenever $\mathrm{Re}(x)>0$ and $z\in\mathbb{C}$. Since the summation formula (3.2.9) involves a generalized $\psi-$function containing the Bessel functions of the first kind, it will be useful in our next argument to find a bound for the modified Bessel function $I_{\nu}(z)$, $\nu>-1$. In the previous chapter, we used the famous Hankel expansion for $I_{\nu}(z)$, $|z|\to\infty$, or an integral representation of Poisson type (see (2.3.23) and (2.3.31) above). However, for our alternative argument we need a bound for $|I_{\nu}(z)|$ that is somewhat uniform in ν and z.

The following simple argument will be more than enough to find such a bound. First, let us note that, if $\nu\geq 0$ and $k\in\mathbb{N}_0$,

$$\Gamma(k+\nu+1)=\Gamma(\nu+1)(\nu+1)...(\nu+k)\geq k!\,\Gamma(\nu+1).$$

Using the power series expansion for the Modified Bessel function $I_{\nu}(z)$, (2.1.17), the previous elementary inequality shows that

$$|I_{\nu}(z)|\leq\left(\frac{|z|}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{2^{2k}k!\,\Gamma(k+\nu+1)}\leq\left(\frac{|z|}{2}\right)^{\nu}\frac{1}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{2^{2k}(k!)^2}$$
$$\leq\left(\frac{|z|}{2}\right)^{\nu}\frac{1}{\Gamma(\nu+1)}\left\{\sum_{k=0}^{\infty}\frac{|z|^{k}}{2^{k}k!}\right\}^{2}=\left(\frac{|z|}{2}\right)^{\nu}\frac{e^{|z|}}{\Gamma(\nu+1)},\qquad\nu\geq 0.\qquad(3.2.10)$$

If, on the other hand, we want to extend (3.2.10) to the interval $-1<\nu<0$, we begin to note that, for $k\geq 1$,

$$\Gamma(k+\nu+1)=\Gamma(\nu+2)\,(\nu+2)...(\nu+k)\geq(k-1)!\,\Gamma(\nu+2).$$

Hence, if $-1<\nu<0$,

$$|I_{\nu}(z)|\leq\left(\frac{|z|}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{2^{2k}k!\,\Gamma(k+\nu+1)}\leq\frac{(|z|/2)^{\nu}}{\Gamma(\nu+1)}+\frac{(|z|/2)^{\nu+2}}{\Gamma(\nu+2)}\sum_{k=0}^{\infty}\frac{|z|^{2k}}{2^{2k}(k!)^2}$$
$$<\left(\frac{|z|}{2}\right)^{\nu}\frac{e^{|z|}}{\Gamma(\nu+2)}\left(1+\frac{|z|^2}{4}\right).\qquad(3.2.11)$$

As stated at the introduction, the transformation formula for $\psi_{\alpha}(x,z)$, (3.1.8), given by the summation formula (3.2.9) will play a major role in this chapter. We will also need some auxiliary formulas already given in this thesis. Let us recall that, analogously to $r_{\alpha}(n)$, one may also consider the positive powers of the theta function $\vartheta_2(\tau)$,

$$\vartheta_2(\tau)=2\sum_{n=0}^{\infty}e^{\pi i\tau\left(n+\frac{1}{2}\right)^2},\quad\mathrm{Im}(\tau)>0,\qquad(3.2.12)$$

by taking its Fourier expansion at the cusp $\tau = i\infty$. As such, we can define a new arithmetical function, $\tilde{r}_{\alpha}(m)$, as the corresponding Fourier coefficients, i.e., (see Section 2.3.1 above)

$$\begin{aligned}\vartheta_2^{\alpha}(ix) := \theta_2^{\alpha}(x) &= \left(2\sum_{n=0}^{\infty} e^{-\pi x\left(n+\frac{1}{2}\right)^2}\right)^{\alpha} = \left(2\,e^{-\frac{\pi x}{4}} + 2\sum_{n=1}^{\infty} e^{-\pi x(n+\frac{1}{2})^2}\right)^{\alpha}\\ &= 2^{\alpha} e^{-\frac{\pi\alpha x}{4}}\left(1+\sum_{n=1}^{\infty} e^{-\pi x\left(n^2+n\right)}\right)^{\alpha} = 2^{\alpha}\,e^{-\frac{\pi\alpha x}{4}}\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha\\ j\end{array}\right)\left(\sum_{n=1}^{\infty} e^{-\pi x\left(n^2+n\right)}\right)^{j}\\ &:= \sum_{m=0}^{\infty}\tilde{r}_{\alpha}(m)\,e^{-\pi\left(m+\frac{\alpha}{4}\right)x}, \qquad \mathrm{Re}(x)>0.\end{aligned} \tag{3.2.13}$$

Analogously to $r_{\alpha}(n)$, the coefficients $\tilde{r}_{\alpha}(n)$ grow polynomially with n, according to Lemma 2.3.1. Moreover, by construction, $\tilde{r}_{\alpha}(0) := 2^{\alpha}$. Using Jacobi's transformation formula for $\vartheta_2(\tau)$, one can derive a summation formula linking the arithmetical functions $\tilde{r}_{\alpha}(n)$ and $r_{\alpha}(n)$. The next lemma, which is a restatement of Lemma 2.3.3, establishes this in a precise form.

Lemma 3.2.1 (see Lemma 2.3.3)**.** *Let $r_{\alpha}(n)$ be the coefficients of the series expansion of $\theta^{\alpha}(x)-1$, (2.1.42), and $\tilde{r}_{\alpha}(n)$ be defined by (3.2.13). Then, for Re$(x)>0$ and $y\in\mathbb{C}$, the following identity holds*

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\,r_{\alpha}(n)\,n^{\frac{1}{2}-\frac{\alpha}{4}}\,e^{-\pi n x}\,J_{\frac{\alpha}{2}-1}(\sqrt{\pi n}\,y) &= -\frac{y^{\frac{\alpha}{2}-1}\pi^{\frac{\alpha}{4}-\frac{1}{2}}}{2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)}\\ +\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=0}^{\infty}\tilde{r}_{\alpha}(n)\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}} &e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,y}{x}\right).\end{aligned} \tag{3.2.14}$$

The previous transformation formula was necessary in the second chapter to prove that, for z satisfying (3.1.5) and every $m\in\mathbb{N}_0$,

$$\frac{d^m}{d\omega^m}\left(1+\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\to 0, \quad \text{as} \quad \omega\to\frac{\pi}{4}^{-}. \tag{3.2.15}$$

This key lemma will now be pivotal in refining (3.2.15). In our next result, we shall provide a uniform estimate for the derivatives of the function $\psi_{\alpha}\left(ie^{-2iu},z\right)$, $0<u<\frac{\pi}{4}$.

Lemma 3.2.2. *Let $\psi_{\alpha}(x,z)$ be the generalized Jacobi's $\psi-$function defined by (3.1.8) and assume that $z\in\mathbb{R}$ satisfies the condition*

$$-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\leq z\leq\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}. \tag{3.2.16}$$

Then there are two positive constants A and C (depending only on α) such that, for any $0<u<\frac{\pi}{4}$,

$$\left|\frac{d^k}{du^k}\left(1+\psi_{\alpha}\left(ie^{-2iu},z\right)\right)\right| < C\,\frac{2^{7k}k!}{u^{\frac{\alpha}{2}+k}}e^{-\frac{A}{u}}, \quad \alpha>2 \tag{3.2.17}$$

and

$$\left|\frac{d^k}{du^k}\left(1+\psi_{\alpha}\left(ie^{-2iu},z\right)\right)\right| < C\,\frac{2^{7k}k!}{u^{\frac{\alpha}{2}+k+2}}e^{-\frac{A}{u}}, \quad 0\leq\alpha\leq 2. \tag{3.2.18}$$

Proof. Let us fix $0 < u_0 < \frac{\pi}{4}$. We shall prove (3.2.17) only, as the bound (3.2.18) can be similarly deduced. At the end of the proof, we just outline briefly the main differences in the process of obtaining (3.2.18). The derivative of the function $\psi_\alpha\left(ie^{-2iu}, z\right)$ at the point u_0 can be evaluated by integrating along a positively oriented circle with center u_0 and radius λu_0, $0 < \lambda < 1$, which we shall denote by $C_{\lambda u_0}(u_0)$. Throughout the forthcoming proof, this parameter λ will be arbitrary up to the point where the condition (3.2.16) is finally used in the argument. Let $D_{\lambda u_0}(u_0) := \text{int}\left(C_{\lambda u_0}(u_0)\right) = \{w \in \mathbb{C} : |w - u_0| < \lambda u_0\}$. For any $w \in D_{\lambda u_0}(u_0)$, it is simple to check that $\text{Re}\left(ie^{-2iw}\right) > 0$, and so, for any fixed z, $\psi_\alpha\left(ie^{-2iw}, z\right)$ defines an analytic function inside the circle $C_{\lambda u_0}(u_0)$, according to Lemma 2.2.2. Therefore, Cauchy's integral formula is valid and it gives the representation

$$\left[\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right]_{u=u_0} = \frac{k!}{2\pi i}\int_{C_{\lambda u_0}(u_0)} \frac{1+\psi_\alpha\left(ie^{-2iw}, z\right)}{(w-u_0)^{k+1}}\, dw. \tag{3.2.19}$$

The core of the argument that follows consists in bounding the contour integral lying at the right-hand side of (3.2.19). The transformation formula (3.2.14) will be instrumental in most of the technical steps below. Indeed, since $ie^{-2iw} = i + 2e^{-iw}\sin(w)$ and, by definition (3.1.8),

$$1+\psi_\alpha(ie^{-2iw}, z) := 1 + 2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi}\, e^{i\frac{\pi}{4}} e^{-iw}\, z\right)^{1-\frac{\alpha}{2}}$$
$$\times \sum_{n=1}^{\infty}(-1)^n r_\alpha(n)\, n^{\frac{1}{2}-\frac{\alpha}{4}}\, e^{-2\pi n e^{-iw}\sin(w)}\, J_{\frac{\alpha}{2}-1}\left(\sqrt{\pi n}\, e^{i\left(\frac{\pi}{4}-w\right)}\, z\right),$$

an application of (3.2.14) with $x = 2e^{-iw}\sin(w)$ and $y = e^{i\left(\frac{\pi}{4}-w\right)}\, z$ gives

$$1+\psi_\alpha(ie^{-2iw}, z) = 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi}\, e^{i\left(\frac{\pi}{4}-w\right)}z\right)^{1-\frac{\alpha}{2}}$$
$$\times \frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)} \sum_{n=0}^{\infty} \tilde{r}_\alpha(n) e^{-\frac{\pi i}{2}\left(n+\frac{\alpha}{4}\right)}\left(n+\frac{\alpha}{4}\right)^{\frac{1}{2}-\frac{\alpha}{4}} e^{-\frac{\pi}{2\tan(w)}\left(n+\frac{\alpha}{4}\right)} I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi\left(n+\frac{\alpha}{4}\right)}\, e^{\frac{i\pi}{4}}\, z}{2\sin(w)}\right). \tag{3.2.20}$$

We shall bound $\left|1+\psi_\alpha\left(ie^{-2iw}, z\right)\right|$ by estimating the second expression on the right-hand side of (3.2.20). Estimating this expression requires that we track each factor independently. Clearly, for any $w \in C_{\lambda u_0}(u_0)$,

$$\left|2^{\frac{\alpha}{2}-1}\,\Gamma\left(\frac{\alpha}{2}\right)\left(\sqrt{\pi}\, e^{i\frac{\pi}{4}} e^{-iw} z\right)^{1-\frac{\alpha}{2}}\right| \leq 2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{1}{2}-\frac{\alpha}{4}} e^{\frac{\pi}{4}\left|1-\frac{\alpha}{2}\right|}|z|^{1-\frac{\alpha}{2}},$$

where we have used the condition $\lambda u_0 < \frac{\pi}{4}$. Next,

$$\left|\frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\right| = \frac{e^{-\frac{z^2}{8}-\mathrm{Im}(w)}\exp\left(\frac{z^2}{8}\mathrm{Im}\left(\frac{1}{\tan(w)}\right)\right)}{2\sqrt{\sin^2(\mathrm{Re}(w))+\sinh^2(\mathrm{Im}(w))}}$$
$$= \frac{e^{-\frac{z^2}{8}-\mathrm{Im}(w)}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right)}{2\sqrt{\sin^2(\mathrm{Re}(w))+\sinh^2(\mathrm{Im}(w))}}$$
$$\leq \frac{e^{-\frac{z^2}{8}+\lambda u_0}}{2\sin(\mathrm{Re}(w))}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right),$$

where in the last step we just have used the fact that, for any $w \in C_{\lambda u_0}(u_0)$, $|\mathrm{Im}(w)| \leq \lambda u_0$. Moreover, we have employed the elementary relation

$$\frac{1}{\tan(w)} = \frac{\sin(2\mathrm{Re}(w)) - i\,\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}. \tag{3.2.21}$$

Using now Jordan's inequality $\sin(x) > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, and the fact that $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$, the previous set of inequalities implies

$$\left|\frac{e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\right| < \frac{\pi e^{-\frac{z^2}{8}+\lambda u_0}}{4\mathrm{Re}(w)}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right)$$
$$\leq \frac{\pi e^{-\frac{z^2}{8}+\lambda u_0}}{4(1-\lambda)u_0}\exp\left(-\frac{z^2}{8}\cdot\frac{\sinh(2\mathrm{Im}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\right), \quad w \in C_{\lambda u_0}(u_0). \tag{3.2.22}$$

We now address each term of the series on the right-hand side of (3.2.20). Since, at this first stage, we are aiming at a proof of (3.2.17), we may assume here that $\alpha > 2$, i.e., that $\frac{\alpha}{2} - 1 > 0$. Using the first uniform inequality for $I_\nu(z)$, (3.2.10), we have that the term involving the Bessel function in (3.2.20) admits the bound

$$\left|I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,e^{\frac{i\pi}{4}}z}{2\sin(w)}\right)\right| \leq \frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)|\sin(w)|^{\frac{\alpha}{2}-1}}\exp\left(\frac{\sqrt{\pi\left(n+\frac{\alpha}{4}\right)}\,|z|}{2\,|\sin(w)|}\right)$$
$$\leq \frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)\sin^{\frac{\alpha}{2}-1}(\mathrm{Re}(w))}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right)$$
$$\leq \frac{\pi^{\frac{3\alpha}{4}-\frac{3}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}-\frac{1}{2}}|z|^{\frac{\alpha}{2}-1}}{2^{\frac{3\alpha}{2}-3}\Gamma\left(\frac{\alpha}{2}\right)(1-\lambda)^{\frac{\alpha}{2}-1}u_0^{\frac{\alpha}{2}-1}}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right). \tag{3.2.23}$$

In the second inequality above we have used the fact that $|\sin(w)| \geq |\sin(\mathrm{Re}(w))|$. Also, the third inequality is a combination of Jordan's result, $\sin(x) > \frac{2x}{\pi}$, $0 < x < \frac{\pi}{2}$, and the fact that $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$. Returning to (3.2.20) and recalling once more (3.2.21), we deduce from (3.2.22) and (3.2.23) that

$$\begin{aligned}&\left|1+\psi_{\alpha}(ie^{-2iw},z)\right|\\&<\frac{d_{\alpha}\,\pi^{\frac{\alpha}{2}}e^{-\frac{z^2}{8}+\lambda u_0}}{2^{\alpha}\,(1-\lambda)^{\frac{\alpha}{2}}\,u_0^{\frac{\alpha}{2}}}\sum_{n=0}^{\infty}|\tilde{r}_{\alpha}(n)|\,\exp\left\{-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\,P_w\left(\sqrt{n+\frac{\alpha}{4}}\right)\right\},\end{aligned} \tag{3.2.24}$$

where d_{α} is some constant depending on α and $P_w\,(X)$ is the real-valued polynomial

$$P_w(X):=X^2-\sqrt{\frac{2}{\pi}}\frac{|z|}{\sin(2\mathrm{Re}(w))}\sqrt{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\,X+\frac{z^2}{4\pi}\frac{\sinh(2\mathrm{Im}(w))}{\sin(2\mathrm{Re}(w))}. \tag{3.2.25}$$

The inequality (3.2.24) is valid for any $w\in C_{\lambda u_0}(u_0)$. Taking out the first term of the series on (3.2.24) and using the fact that $\tilde{r}_{\alpha}(0):=2^{\alpha}$, we arrive at the inequality

$$\begin{aligned}\left|1+\psi_{\alpha}(ie^{-2iw},z)\right|<\frac{d_{\alpha}\,\pi^{\frac{\alpha}{2}}e^{-\frac{z^2}{8}+\frac{u_0}{2}}}{(1-\lambda)^{\frac{\alpha}{2}}\,u_0^{\frac{\alpha}{2}}}\exp\left[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right]\\ \times\sum_{n=0}^{\infty}\left|\frac{\tilde{r}_{\alpha}(n)}{\tilde{r}_{\alpha}(0)}\right|\exp\left[-Q_w(n)\right],\end{aligned} \tag{3.2.26}$$

where $P_w(X)$ is the polynomial defined by (3.2.25) and the exponent $Q_w(n)$ is explicitly

$$Q_w(n):=\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\left[P_w\left(\sqrt{n+\frac{\alpha}{4}}\right)-P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right].$$

We will show below that there is a choice of λ such that, for any z satisfying (3.2.16), $Q_w(n)>0$, $\forall n\in\mathbb{N}$. After establishing this, the use of Lemma 2.3.1 ensures that the infinite series showing up on the right-hand side of (3.2.26) must be convergent. The remaining part of our proof will consist in finding a uniform bound for this series, whose constant only depends on λ and α. Hence, our main goal now is to make a good choice of λ allowing us to find an upper bound for the expression

$$\begin{aligned}\exp\left[-Q_w(n)\right]:=\exp\Bigg[-\frac{\pi}{2}\frac{\sin\left(2\mathrm{Re}(w)\right)}{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}\\ \times\left\{n-\sqrt{\frac{2}{\pi}}\,|z|\frac{\sqrt{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}}{\sin(2\mathrm{Re}(w))}\left(\sqrt{n+\frac{\alpha}{4}}-\frac{\sqrt{\alpha}}{2}\right)\right\}\Bigg].\end{aligned}$$

Since $(1-\lambda)u_0\leq\mathrm{Re}(w)\leq(1+\lambda)u_0$, we know from Jordan's inequality that

$$\begin{aligned}&\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))\geq1-\cos\left(2(1-\lambda)u_0\right)\\&=\int_0^{2(1-\lambda)u_0}\sin(t)\,dt>\frac{2}{\pi}\int_0^{2(1-\lambda)u_0}t\,dt=\frac{4(1-\lambda)^2}{\pi}u_0^2,\end{aligned}$$

implying

$$\frac{1}{\sqrt{\cosh\left(2\mathrm{Im}(w)\right)-\cos\left(2\mathrm{Re}(w)\right)}}<\frac{\sqrt{\pi}}{2(1-\lambda)u_0}. \tag{3.2.27}$$

On the other hand, we can also find an upper bound for the expression $\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))$ as follows

$$\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w)) = \int_0^{2\mathrm{Im}(w)} \sinh(t)\,dt + \int_0^{2\mathrm{Re}(w)} \sin(t)\,dt$$

$$\leq \int_0^{2\mathrm{Im}(w)} t\,e^{\frac{t^2}{6}}\,dt + \int_0^{2\mathrm{Re}(w)} t\,dt \leq 2\mathrm{Im}(w)^2\,e^{\frac{2}{3}\mathrm{Im}(w)^2} + 2\mathrm{Re}(w)^2 < 4e^{\frac{2\lambda^2}{3}u_0^2}\,(1+\lambda)^2\,u_0^2, \tag{3.2.28}$$

where we have employed the assumptions $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0$ and $-\lambda u_0 \leq \mathrm{Im}(w) \leq \lambda u_0$. At the second step of (3.2.28), we have also used the known inequality

$$\sinh(x) \leq x\,e^{\frac{x^2}{6}}, \quad x > 0. \tag{3.2.29}$$

Since $(1-\lambda)u_0 \leq \mathrm{Re}(w) \leq (1+\lambda)u_0 < \frac{\pi}{2}$, yet another variant of Jordan's result gives

$$\sin\left(2\mathrm{Re}(w)\right) > \begin{cases} \frac{4}{\pi}\mathrm{Re}(w), & 0 < \mathrm{Re}(w) < \frac{\pi}{4} \\ 2 - \frac{4}{\pi}\mathrm{Re}(w), & \frac{\pi}{4} \leq \mathrm{Re}(w) < \frac{\pi}{2} \end{cases} \geq \frac{4}{\pi}\,(1-\lambda)\,u_0, \tag{3.2.30}$$

and so, combining (3.2.28) with (3.2.30) and recalling again the hypothesis $0 < u_0 < \frac{\pi}{4}$, we are able to find that

$$\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))} > \frac{(1-\lambda)}{\pi e^{\frac{2\lambda^2}{3}u_0^2}\,(1+\lambda)^2\,u_0} > \frac{(1-\lambda)}{\pi e^{\frac{\pi^2\lambda^2}{24}}\,(1+\lambda)^2\,u_0}. \tag{3.2.31}$$

Therefore, the infinite series on the right-hand side of (3.2.26) satisfies

$$\sum_{n=0}^{\infty} \left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-Q_w(n)\right]$$
$$< \sum_{n=0}^{\infty} \left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\,(1+\lambda)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right].$$

Using the condition (3.2.16), one sees that the exponential term within the infinite series is uniformly bounded by

$$\exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\,(1+\lambda)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right]$$
$$\leq \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}\,(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}\sqrt{\alpha}}{24(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right]$$
$$\leq \exp\left[-\frac{n}{u_0}\left\{\frac{1-\lambda}{2e^{\frac{\pi^2\lambda^2}{24}}\,(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}}{24(1-\lambda)}\right\}\right],$$

where in the last inequality we have used the mean value theorem for the real function $f(x) = \sqrt{x + \frac{\alpha}{4}}$. If we select the value $\lambda = 0.01$ in the previous expression, it is easy to check that

$$\frac{1-\lambda}{2e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}}{24(1-\lambda)} > \frac{1}{4},$$

and so, recalling again the hypothesis $0 < u_0 < \frac{\pi}{4}$, the following inequality must take place

$$\begin{aligned}\sum_{n=0}^{\infty} \left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-\frac{1}{u_0}\left\{\frac{n(1-\lambda)}{2e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi\,|z|}{2\sqrt{2}(1-\lambda)}\left(\sqrt{n+\frac{\alpha}{4}} - \frac{\sqrt{\alpha}}{2}\right)\right\}\right] \\ \leq \sum_{n=0}^{\infty} \left|\frac{\tilde{r}_\alpha(n)}{\tilde{r}_\alpha(0)}\right| \exp\left[-\frac{n}{4u_0}\right] < \frac{A_\alpha}{2^\alpha}\sum_{n=0}^{\infty} n^{\frac{\alpha}{2}} \exp\left[-\frac{n}{\pi}\right] \leq M_\alpha. \qquad (3.2.32)\end{aligned}$$

The penultimate step is, of course, justified by the inequality $|\tilde{r}_\alpha(n)| < A_\alpha\, n^{\frac{\alpha}{2}}$, which we proved already in Lemma 2.3.1. Combining (3.2.32) with (3.2.26), we see that the following inequality is true

$$\left|1 + \psi_\alpha(ie^{-2iw}, z)\right| < \frac{d_\alpha M_\alpha e^{\frac{\pi}{8}}\,\pi^{\frac{\alpha}{2}}\,e^{-\frac{z^2}{8}}}{(2u_0)^{\frac{\alpha}{2}}} \exp\left[-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right], \qquad (3.2.33)$$

which is almost (3.2.17) with $k = 0$. To conclude the proof in this direction, we just need to bound uniformly the term

$$\exp\left[-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right], \quad w \in C_{\lambda u_0}(u_0)$$

under the hypothesis on z (3.2.16) and the choice $\lambda = 0.01$. This can be done by appealing to (3.2.29), (3.2.27) and (3.2.31), which provide the set of inequalities

$$\begin{aligned}&\exp\left[-\frac{\pi}{2}\frac{\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w)) - \cos(2\mathrm{Re}(w))}\,P_w\left(\frac{\sqrt{\alpha}}{2}\right)\right] \\ &< \exp\left[-\frac{\alpha}{8}\,\frac{(1-\lambda)}{e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2\,u_0} + \sqrt{\frac{\alpha}{2}}\,\frac{\pi\,|z|}{4(1-\lambda)u_0} + \frac{\pi z^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{16(1-\lambda)^2 u_0}\right] \\ &\leq \exp\left[-\frac{\alpha}{u_0}\left\{\frac{1-\lambda}{8e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}}{48(1-\lambda)} - \frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{1152(1-\lambda)^2}\right\}\right]. \qquad (3.2.34)\end{aligned}$$

However, since we already took the choice $\lambda = 0.01$, a numerical confirmation gives

$$\frac{1-\lambda}{8e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2} - \frac{\pi^{\frac{3}{2}}}{48(1-\lambda)} - \frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{1152(1-\lambda)^2} > 0.004,$$

so that, if we return (3.2.33), we finally reach

$$\left|1+\psi_\alpha(ie^{-2iw},z)\right| < \frac{d_\alpha M_\alpha e^{\frac{\pi}{8}}\,\pi^{\frac{\alpha}{2}}e^{-\frac{z^2}{8}}}{(2u_0)^{\frac{\alpha}{2}}}\exp\left[-\frac{0.04\,\alpha}{u_0}\right] = \frac{C}{u_0^{\alpha/2}}\,e^{-\frac{A}{u_0}} \tag{3.2.35}$$

where $C := C(\alpha)$ and $A := 0.04\,\alpha$ are two constants only depending on α. This proves (3.2.17) when $k=0$. For the remaining cases $k \geq 1$, we invoke Cauchy's integral formula (3.2.19) and use (3.2.35), which yields

$$\begin{aligned}\left|\left[\frac{d^k}{du^k}\psi_\alpha\left(ie^{-2iu},z\right)\right]_{u=u_0}\right| &\leq \frac{k!}{2\pi\left(\lambda u_0\right)^{k+1}}\int_{C_{\lambda u_0}(u_0)} \left|1+\psi_\alpha\left(ie^{-2iw},z\right)\right||dw| \\ &< C\,\frac{k!}{u_0^{\frac{\alpha}{2}+k}\lambda^k}e^{-\frac{A}{u_0}}.\end{aligned} \tag{3.2.36}$$

Since we chose $\lambda = 0.01$, we know that $\frac{1}{\lambda} < 2^7$. Using this elementary inequality in (3.2.36), we derive (3.2.17) in its final form.

To get the second formula (3.2.18), we use the bound (3.2.11) for $I_\nu(z)$, $-1<\nu<0$, and use the same inequalities as in (3.2.23) to obtain

$$\begin{aligned}&\left|I_{\frac{\alpha}{2}-1}\left(\frac{\sqrt{\pi(n+\frac{\alpha}{4})}\,e^{\frac{i\pi}{4}}z}{2\sin(w)}\right)\right| \\ &\leq \mathcal{A}\,\frac{\pi^{\frac{\alpha}{4}-\frac{1}{2}}\left(n+\frac{\alpha}{4}\right)^{\frac{\alpha}{4}+\frac{1}{2}}|z|^{\frac{\alpha}{2}+1}}{2^{\frac{3\alpha}{2}-3}\Gamma\left(\frac{\alpha}{2}+1\right)(1-\lambda)^{\frac{\alpha}{2}+1}u_0^{\frac{\alpha}{2}+1}}\exp\left(\frac{\sqrt{\frac{\pi}{2}\left(n+\frac{\alpha}{4}\right)}\,|z|}{\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}}\right),\end{aligned} \tag{3.2.37}$$

for some large constant $\mathcal{A}$. The only formal difference between (3.2.37) and (3.2.23) lies in the power of $1/u_0$. Thus, the same argument carries through with an extra factor of $1/u_0^2$. This gives (3.2.18) and finishes the proof. □

Remark 3.2.1. It is evident from the preceding argument that the condition on the parameter z given in (3.1.19) can be further refined by making a more careful selection of the parameter λ. Additionally, when $\nu > -\frac{1}{2}$, the uniform bounds (3.2.10) and (3.2.11) can be replaced by the much better expression (see (2.3.24) above),

$$|I_\nu(z)| \leq \left(\frac{|z|}{2}\right)^\nu \frac{e^{|\mathrm{Re}(z)|}}{\Gamma(\nu+1)}, \quad \nu > -\frac{1}{2}. \tag{3.2.38}$$

By using (3.2.38) in place of (3.2.10) in the above proof, one may enlarge the interval $\left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}},\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ in the condition (3.2.16) to $\left[-\frac{1}{4}\sqrt{\frac{\pi\alpha}{2}},\frac{1}{4}\sqrt{\frac{\pi\alpha}{2}}\right]$, at least when $\alpha \geq 1$. Thus, in some instances, technical refinements yield stronger results than those presented in our Theorems 3.1.3 and 3.1.4.

We now present another lemma that will be essential for proving our Theorem 3.1.3.

Lemma 3.2.3. *Let $h : \mathbb{C} \longmapsto \mathbb{C}$ be an analytic function. If $z \in \left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}, \frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ then, for every $p \in \mathbb{N}_0$, the following relation holds*

$$\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu},z\right)\right)\right\}=-2\sinh\left(\frac{z^2}{8}\right)h^{(2p)}(u)+\mathcal{A}_{\alpha,p}(u,z), \tag{3.2.39}$$

where, for any $0<u<\frac{\pi}{4}$, $\mathcal{A}_{\alpha,p}(u,z)$ satisfies the inequalities

$$|\mathcal{A}_{\alpha,p}(u,z)|<D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\|h\|_{L^\infty(C_1(0))}\,,\quad \alpha>2, \tag{3.2.40}$$

$$|\mathcal{A}_{\alpha,p}(u,z)|<D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2+2p}}\|h\|_{L^\infty(C_1(0))}\,,\quad 0\leq\alpha\leq 2, \tag{3.2.41}$$

with A and D depending only on α. Moreover, $C_1(0):=\partial D(0,1)$ denotes the unit circle in the complex plane and $\|h\|_{L^\infty(C_1(0))}:=\max_{w\in C_1(0)}|h(w)|$.

Proof. First observe that

$$\begin{aligned}&\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(e^{-z^2/8}+e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu},z\right)\right)\right\}\\&=-2\sinh\left(\frac{z^2}{8}\right)h^{(2p)}(u)+e^{\frac{z^2}{8}}\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right\}\\&:=-2\sinh\left(\frac{z^2}{8}\right)h^{(2p)}(u)+\mathcal{A}_{\alpha,p}(u,z).\end{aligned} \tag{3.2.42}$$

By the well-known Cauchy estimates for the coefficients of analytic functions and (3.2.17), we obtain

$$\begin{aligned}&\left|e^{\frac{z^2}{8}}\frac{d^{2p}}{du^{2p}}\left\{h(u)\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right\}\right|\\&\leq e^{\frac{\pi\alpha}{576}}\sum_{k=0}^{2p}\binom{2p}{k}\left|h^{(2p-k)}(u)\right|\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right|\\&\leq e^{\frac{\pi\alpha}{576}}\|h\|_{L^\infty(D(0,1))}\sum_{k=0}^{2p}\binom{2p}{k}(2p-k)!\left|\frac{d^k}{du^k}\left(1+\psi_\alpha\left(ie^{-2iu},z\right)\right)\right|\\&<Ce^{\frac{\pi\alpha}{576}}\|h\|_{L^\infty(D(0,1))}\frac{(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}}}\sum_{k=1}^{2p}\frac{2^{7k}}{u^k}\\&<C'\|h\|_{L^\infty(C_1(0))}\frac{(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\frac{(128)^{2p+1}-u^{2p+1}}{128-u}\\&<C''\|h\|_{L^\infty(C_1(0))}\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\end{aligned}$$

with the Maximum Modulus principle having been invoked in the fourth inequality. This completes the proof of (3.2.40). Establishing the inequality (3.2.41) is totally analogous, so we skip the details. ∎

To conclude this section, let us recall the useful notation introduced in the previous chapter. It will be convenient to express the terms of the sequence $(\lambda_j)_{j\in\mathbb{N}}$ in polar coordinates as follows:

$$\frac{i\alpha}{2}-2\lambda_j := r_j\, e^{i\theta_j},\ \ 0<\theta_j<\pi. \tag{3.2.43}$$

In accordance with Assumption 1, we extend this definition to include negative indices. Given that $\lambda_{-j}=-\lambda_j$, it follows that $r_{-j}=r_j$ and $\theta_{-j}=\pi-\theta_j$. With these polar coordinates, we now present a straightforward lemma that provides an integral representation for the moments of $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$, (3.1.17). Its proof closely mirrors the steps outlined in (2.3.38), (2.3.41) and (2.3.42), so we will omit some of the details here.

Lemma 3.2.4. *Let $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$, $p\in\mathbb{N}_0$ and $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ be the function defined by (3.1.20). Then the following integral representation holds*

$$\begin{aligned}&\int_0^{\infty} t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\,\cosh\left(2\omega t\right)\,dt\\ &=-\frac{8\pi}{2^{2p}}\sum_{j=1}^{\infty}c_j r_j^{2p}\left[\cos\left(2p\theta_j\right)\,\cos\left(\frac{\omega\alpha}{2}\right)\cosh\left(2\omega\lambda_j\right)+\sin\left(2p\theta_j\right)\,\sin\left(\frac{\omega\alpha}{2}\right)\sinh\left(2\omega\lambda_j\right)\right]\\ &+\frac{2\pi\,e^{z^2/8}}{2^{2p}}\,Re\left(\frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq 0}c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j}\left(e^{-z^2/8}+e^{z^2/8}\,\psi_{\alpha}\left(e^{2i\omega},z\right)\right)\right\}\right),\end{aligned} \tag{3.2.44}$$

where $\psi_\alpha(x,z)$ is the generalized Jacobi's $\psi-$function (3.1.8) and (r_j,θ_j) are the polar coordinates attached to $(\lambda_j)_{j\in\mathbb{N}}$, defined by (3.2.43).

Proof. We start by using the integral representation given at the introduction to this chapter, (3.1.7), and by replacing there x by $e^{2i\omega}$, with $-\frac{\pi}{4}<\omega<\frac{\pi}{4}$. This easily gives the expression

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}+it;\,\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\,e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\,\alpha}{2}}e^{-z^2/4}. \tag{3.2.45}$$

Recalling Kummer's transformation formula,

$${}_1F_1\left(a;\,c;\,x\right)=e^{x}\,{}_1F_1\left(c-a;\,c;\,-x\right), \tag{3.2.46}$$

an equivalent version of the integral representation (3.2.45) is

$$\frac{e^{-z^2/8}}{2\pi}\int_{-\infty}^{\infty}\eta_\alpha\left(\frac{\alpha}{4}+it\right)\,{}_1F_1\left(\frac{\alpha}{4}-it;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\,e^{2\omega t}\,dt=e^{\frac{i\omega\alpha}{2}}\,e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega},z\right)-e^{-\frac{i\omega\,\alpha}{2}}\,e^{-z^2/8}. \tag{3.2.47}$$

In the previous chapter (see (2.3.42) above), we made use of (3.2.47) and the estimates given in (3.2.8), to find

$$\int_{-\infty}^{\infty} t^{2p}\, F_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\, dt = -\frac{8\pi}{2^{2p}} \sum_{j=1}^{\infty} c_j\, r_j^{2p} e^{-2\omega\lambda_j} \cos\left(2p\,\theta_j + \frac{\omega\alpha}{2}\right)$$
$$+\frac{4\pi}{2^{2p}} \operatorname{Re}\left(e^{z^2/8} \frac{d^{2p}}{d\omega^{2p}} \left\{ \sum_{j=1}^{\infty} c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j} \left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega}, z\right)\right)\right\}\right),$$

where $0<\omega<\frac{\pi}{4}$, $z\in\mathbb{C}$, $\psi_\alpha(x,z)$ is the generalized theta function (3.1.8) and $F_{z,\alpha}(s)$ is the function defined by (cf. (3.1.6) above)

$$F_{z,\alpha}(s) := \sum_{j=1}^{\infty} c_j\, \eta_\alpha\left(s+i\lambda_j\right) \left\{ {}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\ \frac{\alpha}{2};\ \frac{z^2}{4}\right) + {}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\ \frac{\alpha}{2};\ \frac{\overline{z}^2}{4}\right)\right\}.$$

According to our Assumption 1, the symmetric shifted combination $\tilde{F}_{z,\alpha}(s)$ has the same expression as $F_{z,\alpha}(s)$ with an additional extension to the negative integers (see (3.1.17) above). Thus, using the additional hypothesis that $z\in\mathbb{R}$ (Assumption 2), we immediately find the integral

$$\int_{-\infty}^{\infty} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) e^{2\omega t}\, dt = -\frac{8\pi}{2^{2p}} \sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\omega\lambda_j} \cos\left(2p\,\theta_j + \frac{\omega\alpha}{2}\right)$$
$$+\frac{4\pi}{2^{2p}} e^{z^2/8} \operatorname{Re}\left(\frac{d^{2p}}{d\omega^{2p}} \left\{ \sum_{j\neq 0} c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j} \left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega}, z\right)\right)\right\}\right). \quad (3.2.48)$$

But it is clear that the first infinite series admits the expression

$$\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\omega\lambda_j} \cos\left(2p\,\theta_j + \frac{\omega\alpha}{2}\right)$$
$$= \sum_{j=1}^{\infty} c_j r_j^{2p} \left[e^{-2\omega\lambda_j} \cos\left(2p\,\theta_j + \frac{\omega\alpha}{2}\right) + e^{2\omega\lambda_j} \cos\left(2p\,\theta_j - \frac{\omega\alpha}{2}\right)\right]$$
$$= 2\sum_{j=1}^{\infty} c_j r_j^{2p} \left[\cos\left(2p\theta_j\right) \cos\left(\frac{\omega\alpha}{2}\right) \cosh\left(2\omega\lambda_j\right) + \sin\left(2p\theta_j\right) \sin\left(\frac{\omega\alpha}{2}\right) \sinh\left(2\omega\lambda_j\right)\right], \quad (3.2.49)$$

because $\lambda_{-j} = -\lambda_j$, $r_{-j} = r_j$ and $\theta_{-j} = \pi - \theta_j$ by hypothesis. Since $\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)$ is a real and even function of t, using the previous expression allows to rewrite (3.2.48) in the form

$$\int_{0}^{\infty} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(2\omega t\right)\, dt$$
$$= -\frac{8\pi}{2^{2p}} \sum_{j=1}^{\infty} c_j r_j^{2p} \left[\cos\left(2p\theta_j\right) \cos\left(\frac{\omega\alpha}{2}\right) \cosh\left(2\omega\lambda_j\right) + \sin\left(2p\theta_j\right) \sin\left(\frac{\omega\alpha}{2}\right) \sinh\left(2\omega\lambda_j\right)\right]$$
$$+\frac{2\pi\, e^{z^2/8}}{2^{2p}} \operatorname{Re}\left(\frac{d^{2p}}{d\omega^{2p}} \left\{ \sum_{j\neq 0} c_j e^{\frac{i\omega\alpha}{2}-2\omega\lambda_j} \left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(e^{2i\omega}, z\right)\right)\right\}\right),$$

completing the proof. □

To finish this section, we shall recall some notions about sequences of real numbers that are uniformly distributed modulo one. A sequence of real numbers $(\gamma_n)_{n\in\mathbb{N}}$ is said to be uniformly distributed modulo 1 if, for any pair of real numbers $0 \le a < b \le 1$, we have

$$\lim_{N\to\infty} \frac{\#\{n \le N:\ a \le \{\gamma_n\} < b\}}{N} = b - a, \tag{3.2.50}$$

where $\{x\}$ denotes the fractional part of x. A necessary and sufficient condition for a sequence to be uniformly distributed modulo 1 is Weyl's criterion [235]: it says that a sequence $(\gamma_n)_{n\in\mathbb{N}}$ satisfies (3.2.50) if and only if

$$\sum_{n=1}^{N} e^{2\pi i m\gamma_n} = \mathrm{o}(N), \tag{3.2.51}$$

for every $m \in \mathbb{Z}\setminus\{0\}$. A consequence of Weyl's criterion is that, if γ is an irrational number, then the sequence $(n\gamma)_{n\in\mathbb{N}}$ is uniformly distributed modulo one [[214], p. 107, Theorem 2.1]. In particular, the sequence $\{(n\gamma)_{n\in\mathbb{N}}\}$ is dense in $[0,1)$, which is essentially the content of Kronecker's lemma.

3.3 Proof of Theorem 3.1.3

3.3.1 Outline of the Proof

As explained in the previous sections, at the core of the method of this chapter is the following lemma, which is an immediate consequence of a theorem of Fejér [86, 88, 228].

Lemma 3.3.1. *Let $(p_n)_{n\in\mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$ with $p_0 := 0$. Then the number of variations of sign of a real continuous function $f(x)$ in the interval $[0,a]$ is **not less** than the number of variations of sign of the following sequence*

$$\mathcal{F}_{p_n}(a) := \begin{cases} f(0) & n = 0 \\ \int_0^a f(t)\, t^{p_n - 1} dt & n \ge 1. \end{cases}$$

Our proof of Theorem 3.1.3 is essentially divided in four parts: in the first of these, we employ Lemmas 3.2.3 and 3.2.4 to rewrite (3.2.44) in a form which is useful to apply the zero counting method suggested by the previous lemma. By the conditions of our Theorem 3.1.3, the sequence $(\lambda_j)_{j\in\mathbb{N}}$ attains its bounds and is made of distinct elements. Hence, we know that there exists some $M \in \mathbb{N}$ such that

$$|\lambda_M| = \max_{j\in\mathbb{N}}\{|\lambda_j|\}, \quad \lambda_j \ne \lambda_M \ \text{ for } \ j \ne M. \tag{3.3.1}$$

Recalling the definition of the polar coordinates (3.2.43),

$$\frac{i\alpha}{2} - 2\lambda_j := r_j\, e^{i\theta_j},\ \ 0 < \theta_j < \pi,$$

we see that (3.3.1) implies $r_j < r_M$ for $j \neq M$. During the proof of our result, we will assume without any loss of generality that $\lambda_M < 0$.[8] According to the representation (3.2.43), this assumption implies that $\theta_M \in (0, \frac{\pi}{2})$.

Using the existence of λ_M given in (3.3.1), the first section of our proof is devoted to show that, for $0 < u < \frac{\pi}{4}$,

$$\int_0^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right) \cosh\left(\left(\frac{\pi}{2} - 2u\right) t\right) dt = -\frac{8\pi c_M r_M^{2p}}{2^{2p}} \left(1 + e^{z^2/8} \sinh\left(\frac{z^2}{8}\right)\right) \mathcal{G}_p\left(\theta_M\right)$$
$$\times \left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p\left(\theta_M\right)} + \tilde{E}\left(X, z\right) + \tilde{H}\left(X, z\right)\right\} + \frac{2\pi\, e^{z^2/8}}{2^{2p}} \operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u, z)\right], \tag{3.3.2}$$

where $\mathcal{A}_{\alpha,p}(u, z)$ and the terms inside the braces will be specified later. Moreover, θ_M is the angular coordinate appearing in the expression $\frac{i\alpha}{2} - 2\lambda_M := r_M e^{i\theta_M}$, according to the convention (3.2.43). Finally, the term $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$ denotes the sequence defined by

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} := \cos(2p\theta_M)\, \cos\left(\frac{\pi\alpha}{8}\right) \cosh\left(\frac{\pi\lambda_M}{2}\right) + \sin(2p\theta_M)\, \sin\left(\frac{\pi\alpha}{8}\right) \sinh\left(\frac{\pi\lambda_M}{2}\right). \tag{3.3.3}$$

In the second part of our proof, we construct a sequence of integers $(q_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ have always different signs. In the third part of our proof, we show that the terms

$$\mathcal{A}_{\alpha,p}(u, z),\ \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p\left(\theta_M\right)},\ \tilde{E}\left(X, z\right),\ \tilde{H}\left(X, z\right)$$

are very small once we pick u small and choose p as a very large integer. Finally, in the fourth part of our proof we combine our collected data with Lemma 3.3.1 to reach the desired conclusion.

We should remark that, throughout this proof, we will only work with the assumption that $\alpha > 2$, so the relevant bounds in this study will be (3.2.17) and (3.2.40). In the range $0 < \alpha \leq 2$ the proof is entirely analogous, but the set of inequalities will be slightly different in some technical passages. We omit the proof for this case, but we certainly encourage the reader to check that the argument presented below follows *mutatis mutandis*.

3.3.2 A suitable integral representation

Let $0 < u < \frac{\pi}{4}$ and take $\omega = \frac{\pi}{4} - u$ on the integral representation (3.2.44). If $\mathcal{I}_{2p}(u)$ denotes the integral

$$\mathcal{I}_{2p}(u) := \int_0^\infty t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4} + it\right) \cosh\left(\left(\frac{\pi}{2} - 2u\right) t\right) dt, \quad 0 < u < \frac{\pi}{4},$$

[8] There is no loss of generality because we can interchange λ_j with $-\lambda_j$ in the function $\tilde{F}_{z,\alpha}\left(s\right)$.

then, according to (3.2.44), $\mathcal{I}_{2p}(u)$ can be explicitly given by

$$\mathcal{I}_{2p}(u) = -\frac{8\pi}{2^{2p}} \sum_{j=1}^{\infty} c_j r_j^{2p} \left[\cos(2p\theta_j)\, \cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right) \cosh\left(2\lambda_j\left(\frac{\pi}{4}-u\right)\right) \right.$$
$$\left. + \sin(2p\theta_j)\, \sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right) \sinh\left(2\lambda_j\left(\frac{\pi}{4}-u\right)\right)\right]$$
$$+\frac{2\pi\, e^{z^2/8}}{2^{2p}} \mathrm{Re}\left(\frac{d^{2p}}{du^{2p}}\left\{e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)} \left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right)\right\}\right). \quad (3.3.4)$$

We shall rewrite (3.3.4) by appealing to Lemma 3.2.3. Consider the following function

$$h_\alpha(u) = e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)}. \quad (3.3.5)$$

Since $\sum |c_j| < \infty$, it is clear that $h_\alpha(u)$ is an analytic function of u. Therefore, by (3.2.42) and the notation (3.2.43),

$$\frac{d^{2p}}{du^{2p}}\left\{\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right) e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\lambda_j\left(\frac{\pi}{4}-u\right)}\right\}$$
$$= -2\sinh\left(\frac{z^2}{8}\right) h_\alpha^{(2p)}(u) + \mathcal{A}_{\alpha,p}(u,z)$$
$$= -2\sinh\left(\frac{z^2}{8}\right) e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j r_j^{2p} e^{2ip\theta_j}\, e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} + \mathcal{A}_{\alpha,p}(u,z). \quad (3.3.6)$$

We will employ Lemma 3.2.3 with the precise aim to reach an appropriate bound for $\mathcal{A}_{\alpha,p}(u,z)$. Since $(\lambda_j)_{j\in\mathbb{N}}$ is bounded by hypothesis, there exists some μ such that $|\lambda_j| < \mu$ for every $j \in \mathbb{N}$: hence

$$\|h_\alpha\|_{L^\infty(C_0(1))} \leq e^{\frac{\alpha}{2}} \sum_{j\neq 0} |c_j| e^{\left(2-\frac{\pi}{2}\right)\lambda_j} < e^{(4-\pi+\alpha)\mu} \sum_{j\neq 0} |c_j| \leq \mathcal{M}, \quad (3.3.7)$$

where $\mathcal{M}$ only depends on numerical the value of the convergent series $\sum_{j\neq 0} |c_j| := 2\sum_{j=1}^{\infty} |c_j|$. Recalling (3.2.40), we arrive at the following bound

$$|\mathcal{A}_{\alpha,p}(u,z)| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}\, \|h\|_{L^\infty(C_1(0))} \leq \mathscr{D}\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}, \quad (3.3.8)$$

for some $\mathscr{D}$ depending only on the sequence $(c_j)_{j\in\mathbb{N}}$ and α. This suggests that we may represent the second term of (3.3.4) in the following form

$$e^{z^2/8}\mathrm{Re}\left(\frac{d^{2p}}{du^{2p}}\left\{\left(e^{-z^2/8} + e^{z^2/8}\,\psi_\alpha\left(ie^{-2iu}, z\right)\right) e^{\frac{i\alpha}{2}\left(\frac{\pi}{4}-u\right)} \sum_{j\neq 0} c_j e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\right\}\right)$$
$$= e^{z^2/8}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right] - 2e^{\frac{z^2}{8}} \sinh\left(\frac{z^2}{8}\right) \sum_{j\neq 0} c_j r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right). \quad (3.3.9)$$

We can rewrite the infinite series on the previous expression with the index j ranging over $\mathbb{N}$ instead of $\mathbb{Z}\setminus\{0\}$. This requires to repeat the somewhat tedious algebraic manipulations leading to (3.2.49), from which one obtains

$$\begin{aligned} 2\sum_{j=1}^{\infty} c_j r_j^{2p} \Big\{ &\cos(2p\theta_j)\,\cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\cosh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right) \\ &+\sin(2p\theta_j)\,\sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\sinh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right)\Big\} \\ &=\sum_{j\neq 0} c_j r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\cos\left(2p\theta_j+\frac{\pi\alpha}{8}-\frac{\alpha}{2}u\right). \end{aligned} \tag{3.3.10}$$

Combining (3.3.10) with (3.3.4) yields the important formula for the integral $\mathcal{I}_{2p}(u)$,

$$\begin{aligned} \mathcal{I}_{2p}(u) = \;& \frac{2\pi\, e^{z^2/8}}{2^{2p}}\operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right] - \frac{8\pi}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right) \\ &\times\sum_{j=1}^{\infty} c_j\, r_j^{2p}\Big\{\cos(2p\theta_j)\,\cos\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\cosh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right) \\ &\qquad+\sin(2p\theta_j)\,\sin\left(\frac{\alpha}{2}\left(\frac{\pi}{4}-u\right)\right)\sinh\left(2\left(\frac{\pi}{4}-u\right)\lambda_j\right)\Big\}. \end{aligned} \tag{3.3.11}$$

Of course, by (3.3.9) and (3.3.4), an equivalent way of writing (3.3.11) also works the other way around. A particularly compact version is

$$\begin{aligned} \mathcal{I}_{2p}(u) = &-\frac{4\pi}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\cos\left(2p\,\theta_j+\frac{\pi\alpha}{8}-\frac{\alpha}{2}u\right) \\ &+\frac{2\pi\, e^{z^2/8}}{2^{2p}}\operatorname{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \end{aligned} \tag{3.3.12}$$

whose symmetric shape will be very helpful below.

We will now analyze the infinite series on the first term on the right side of (3.3.11) and approximate it by its value for $u=0$. For this analysis it will be easier to manipulate the more compact alternative expression (3.3.12). By the mean value theorem and the existence of an element λ_M satisfying (3.3.1), we know that

$$\left|\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j}\left\{\cos\left(2p\,\theta_j+\frac{\pi\alpha}{8}-\frac{\alpha}{2}u\right)-\cos\left(2p\,\theta_j+\frac{\pi\alpha}{8}\right)\right\}\right| \le C_\alpha\, r_M^{2p}\, u, \tag{3.3.13}$$

where, by the definition (3.2.43), $\frac{i\alpha}{2} - 2\lambda_M = r_M e^{i\theta_M}$. Thus, we can write the infinite series in (3.3.12) in the approximate form

$$\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right) = \sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8}\right) + \mathcal{B}_{\alpha,p}(u)$$
$$= 2\sum_{j=1}^{\infty} c_j\, r_j^{2p} \left\{\cos\left(2p\theta_j\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin\left(2p\theta_j\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_j\right)\right\} + \mathcal{B}_{\alpha,p}(u), \tag{3.3.14}$$

where, according to (3.3.13), $\mathcal{B}_{\alpha,p}(u)$ should satisfy the estimate

$$|\mathcal{B}_{\alpha,p}(u)| \leq C_\alpha\, r_M^{2p}\, u. \tag{3.3.15}$$

Let $p \in \mathbb{N}$ be such that

$$\mathcal{G}_p(\theta_M) := \cos\left(2p\theta_M\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin\left(2p\theta_M\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_M\right) \neq 0. \tag{3.3.16}$$

Then, for every p satisfying (3.3.16), we can write (3.3.14) as

$$\sum_{j\neq 0} c_j\, r_j^{2p} e^{-2\left(\frac{\pi}{4}-u\right)\lambda_j} \cos\left(2p\,\theta_j + \frac{\pi\alpha}{8} - \frac{\alpha}{2}u\right) = 2\sum_{j=1}^{\infty} c_j r_j^{2p}\, \mathcal{G}_p\left(\theta_j\right) + \mathcal{B}_{\alpha,p}(u)$$
$$= 2c_M r_M^{2p}\, \mathcal{G}_p\left(\theta_M\right)\left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p\left(\theta_M\right)} + \tilde{E}\left(X,z\right) + \tilde{H}\left(X,z\right)\right\}, \tag{3.3.17}$$

where

$$\tilde{E}(X,z) := \sum_{j\neq M,\, j\leq X} \frac{c_j}{c_M}\left(\frac{r_j}{r_M}\right)^{2p} \frac{\cos\left(2p\theta_j\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin\left(2p\theta_j\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_j\right)}{\cos\left(2p\theta_M\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin\left(2p\theta_M\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_M\right)} \tag{3.3.18}$$

and

$$\tilde{H}\left(X,z\right) := \sum_{j\neq M,\, j> X} \frac{c_j}{c_M}\left(\frac{r_j}{r_M}\right)^{2p} \frac{\cos\left(2p\theta_j\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_j\right) + \sin\left(2p\theta_j\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_j\right)}{\cos\left(2p\theta_M\right)\, \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_M\right) + \sin\left(2p\theta_M\right)\, \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_M\right)}, \tag{3.3.19}$$

for some large parameter X that will be chosen later. Returning to (3.3.12), we can write the following expression for $\mathcal{I}_{2p}(u)$,

$$\mathcal{I}_{2p}(u) = -\frac{8\pi c_M r_M^{2p}}{2^{2p}}\left(1 + e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\mathcal{G}_p\left(\theta_M\right)\left\{1 + \frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\, \mathcal{G}_p\left(\theta_M\right)} + \tilde{E}\left(X,z\right) + \tilde{H}\left(X,z\right)\right\}$$
$$+ \frac{2\pi\, e^{z^2/8}}{2^{2p}}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \tag{3.3.20}$$

which is no other than the one claimed at the introduction of our proof, (3.3.2).

In the subsequent sections, our focus will be to demonstrate that the sign of $\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}}$ oscillates infinitely often and its absolute value emerges as the dominant term in both (3.3.17) and

(3.3.20). This conclusion will follow from an analysis showing that the terms $\mathcal{A}_{\alpha,p}(u,z)$, $\mathcal{B}_{\alpha,p}(u)$, $\tilde{E}(X,z)$ and $\tilde{H}(X,z)$ become negligible for infinitely many values of p and for sufficiently small values of u. Given that $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$ changes its sign infinitely often for an appropriately chosen sequence of integers $(q_n)_{n\in\mathbb{N}}$, our proof will be concluded after a careful use of Lemma 3.3.1.

3.3.3 Studying the sign changes of $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$

In this section we will show that it is always possible to construct a sequence $(q_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ have different signs. To this end, we divide our construction in two cases: first, by assuming that $\frac{\theta_M}{\pi} \notin \mathbb{Q}$ and, secondly, by assuming that $\frac{\theta_M}{\pi} \in \mathbb{Q}$.

1st case: $\theta_M/\pi \notin \mathbb{Q}$ Since we took the hypothesis $\lambda_M < 0$ (so that $\theta_M \in (0, \frac{\pi}{2})$), we can write the sequence (3.3.3) as follows

$$(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}} = \cos(2p\theta_M)\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi|\lambda_M|}{2}\right) - \sin(2p\theta_M)\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi|\lambda_M|}{2}\right).$$

The study of the sign changes of the previous sequence is actually equivalent to the study of the sign of the function of real variable

$$G(\phi) := \cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi|\lambda_M|}{2}\right)\cos(\phi) - \sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi|\lambda_M|}{2}\right)\sin(\phi).$$

In order to examine $G(\phi)$, let us assume that $8k < \alpha < 8k+4$, for some $k \in \mathbb{N}_0$. The remaining cases $8k+4 < \alpha < 8k+8$ and $\alpha \equiv 0 \mod 4$ are analogous and will be sketched at the end of the argument. Under the assumption $8k < \alpha < 8k+4$, we know that $\cos\left(\frac{\pi\alpha}{8}\right)$ and $\sin\left(\frac{\pi\alpha}{8}\right)$ must have the same sign and, without loss of generality, we may assume that both are positive.

Let us consider the sets

$$\mathcal{A}^+ := \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \pi,\ \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right)\right) \tag{3.3.21}$$

and

$$\mathcal{A}^- := \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right),\ \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \pi\right). \tag{3.3.22}$$

Clearly, if $\phi \in \mathcal{A}^+$ then $G(\phi) > 0$, while $G(\phi) < 0$ whenever $\phi \in \mathcal{A}^-$. The sets $\mathcal{A}^+$ and $\mathcal{A}^-$ contain subintervals that will be pivotal in our argument. These subintervals are defined by

$$\mathcal{I}^+ := \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} - \theta_M,\ \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} + \theta_M\right),$$

$$\mathcal{I}^- := \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} - \theta_M,\ \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M\right).$$

It is very simple to check that $\mathcal{I}^+$ (resp. $\mathcal{I}^-$) is an arc lying on the center of $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) and having length equal to $2\theta_M$. Thus, we can write the partition of $\mathcal{A}^+$ as follows (see Fig. 3.1 below)

$$\mathcal{A}^+ = \mathcal{A}_1^+ \cup \mathcal{I}^+ \cup \mathcal{A}_2^+, \tag{3.3.23}$$

where

$$\mathcal{A}_1^+ = \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \pi, \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} - \theta_M\right),$$

while

$$\mathcal{A}_2^+ = \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) - \frac{\pi}{2} + \theta_M, \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right)\right).$$

Analogously, we can find the partition of $\mathcal{A}^-$,

$$\mathcal{A}^- = \mathcal{A}_1^- \cup \mathcal{I}^- \cup \mathcal{A}_2^-,$$

where

$$\mathcal{A}_1^- = \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right), \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} - \theta_M\right)$$

and

$$\mathcal{A}_2^- = \left(\tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M, \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \pi\right).$$

Since $\theta_M/\pi \notin \mathbb{Q}$, by Kronecker's lemma (see Lemma 2.3.6 above) we know that the set of points $\{n\theta_M/\pi\}_{n\in\mathbb{N}}$ is dense in $(0,1)$. Thus, there exist two infinite sequences $(q_n^+)_{n\in\mathbb{N}}$ and $(q_n^-)_{n\in\mathbb{N}}$ such that

$$\left(q_n^+\right)_{n\in\mathbb{N}} := \left\{q \in \mathbb{N} \,:\, 2q\theta_M \in \mathcal{I}^+\right\}, \quad (q_n^-)_{n\in\mathbb{N}} := \{q \in \mathbb{N} \,:\, 2q\theta_M \in \mathcal{I}^-\}. \tag{3.3.24}$$

The main claim of this section states that $(q_n^+)_{n\in\mathbb{N}}$ and $(q_n^-)_{n\in\mathbb{N}}$ possess a very useful interlacing property.

Claim 3.3.1. *For every $n \in \mathbb{N}$, we have that*

$$q_n^+ < q_n^- < q_{n+1}^+ < q_{n+1}^- < q_{n+2}^+ < \dots .$$

Proof. First, let us observe that, if $a \in (q_n^+)_{n\in\mathbb{N}}$ then $a+1 \notin (q_n^+)_{n\in\mathbb{N}}$, because $2(a+1)\theta_M = 2a\theta_M + 2\theta_M$ and the length of $\mathcal{I}^+$ is, by construction, equal to $2\theta_M$. Therefore, either $2(a+1)\theta_M \in \mathcal{I}^-$ (i.e., $a+1 \in (q_n^-)_{n\in\mathbb{N}}$) or $2(a+1)\theta_M \in \mathcal{A}_2^+ \cup \mathcal{A}_1^-$. This actually happens because $2\theta_M < \pi$ and, since $2a\theta_M \in \mathcal{I}^+$ by hypothesis,

$$2a\theta_M + 2\theta_M < \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) + \frac{\pi}{2} + \theta_M,$$

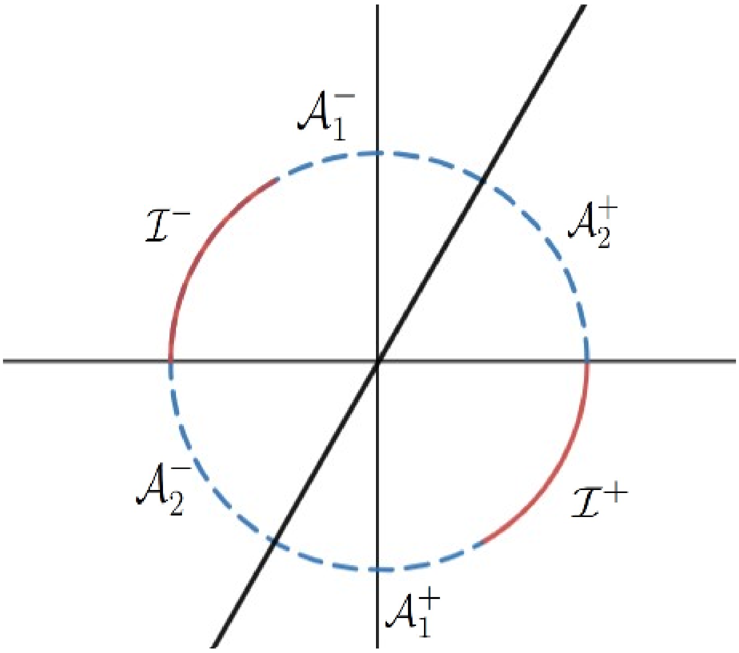

Figure 3.1 The partition of the unit circle into $\mathcal{A}^+$ and $\mathcal{A}^-$ and the representations of the subsets $\mathcal{A}_1^{\pm}$, $\mathcal{A}_2^{\pm}$ and $\mathcal{I}^{\pm}$. The line dividing the unit circle is defined by the equation $Y = \tan^{-1}\left(\cot\left(\frac{\pi\alpha}{8}\right)\coth\left(\frac{\pi|\lambda_M|}{2}\right)\right) X$.

which ensures that $2(a+1)\theta_M \notin \mathcal{A}_2^-$. Since $a \in (q_n^+)_{n\in\mathbb{N}}$, there exists some $m_0 \in \mathbb{N}$ for which $a = q_{m_0}^+$: thus, if $a+1 \in (q_n^-)_{n\in\mathbb{N}}$, then we have that $q_{m_0}^+ < q_{m_0}^-$. On the other hand, if $2(a+1)\theta_M \in \mathcal{A}_2^+ \cup \mathcal{A}_1^-$ then there is some $j \geq 2$ such that $2(a+j)\theta_M \in \mathcal{I}^-$ because the length of $\mathcal{I}^-$ is equal to $2\theta_M$. This simple argument demonstrates that the translation $a+j$, $j \in \mathbb{N}_0$, cannot go from $q_{m_0}^+$ to $q_{m_0+1}^+$ without passing first by an element of the sequence $(q_n^-)_{n\in\mathbb{N}}$. This completes the proof of the claim. □

The interlacing property of q_n^+ and q_n^- allows us to construct a new sequence

$$(q_n)_{n\in\mathbb{N}} := \left(q_n^+\right)_{n\in\mathbb{N}} \cup \left(q_n^-\right)_{n\in\mathbb{N}}$$

in such a way that $q_{2n-1} := q_n^+$ and $q_{2n} := q_n^-$. Since q_m and q_{m+1} are such that $2q_m\theta_M$ and $2q_{m+1}\theta_M$ belong to opposite sides of Fig. 3.1, we necessarily conclude that $\mathcal{G}_{q_m}(\theta_M)$ and $\mathcal{G}_{q_{m+1}}(\theta_M)$ must have opposite signs. The work proposed for this section is, thus, finished under the assumption that $\theta_M/\pi \notin \mathbb{Q}$.

Before moving on to the case where $\theta_M/\pi \in \mathbb{Q}$, we remark that an analogous idea works well when $8k+4 < \alpha < 8k+8$ and $\alpha \equiv 0 \mod 4$. In the first scenario, $\cos\left(\frac{\pi\alpha}{8}\right)$ and $\sin\left(\frac{\pi\alpha}{8}\right)$ have opposite signs. However, a similar construction as the one given above works and the only modification consists in replacing $\mathcal{A}_1^+, \mathcal{I}^+$ and $\mathcal{A}_2^+$ by $\mathcal{A}_1^-, \mathcal{I}^-$ and $\mathcal{A}_2^-$, respectively.

On the other hand, if $\alpha = 8k$ or $\alpha = 8k+4$ for some $k \in \mathbb{N}_0$, then

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} = (-1)^k \cosh\left(\frac{\pi|\lambda_M|}{2}\right)\cos\left(2p\theta_M\right), \text{ if } \alpha = 8k$$

and

$$\left(\mathcal{G}_p(\theta_M)\right)_{p\in\mathbb{N}} = (-1)^{k-1}\sinh\left(\frac{\pi|\lambda_M|}{2}\right)\sin\left(2p\theta_M\right), \ \text{ if } \alpha = 8k+4.$$

In any of these cases a similar construction of the sets $\mathcal{A}_1^{\pm}$, $\mathcal{A}_2^{\pm}$ and $\mathcal{I}^{\pm}$ can be done.

2^nd case: $\theta_M/\pi \in \mathbb{Q}$**.** The conclusions given above are analogous in the case where $\theta_M/\pi \in \mathbb{Q}$. However, instead of arguing via Kronecker's lemma, the sequences (3.3.24) can now be explicitly constructed. Since $|\mathcal{I}^+| = |\mathcal{I}^-| = 2\theta_M$, we know that there exist q_1 and $q_2 > q_1$ such that $2q_1\theta_M \in \mathcal{I}^+$ and $2q_2\theta_M \in \mathcal{I}^-$. Since, by hypothesis, $\theta_M = \frac{a}{b}\pi$, $(a,b)=1$, then $2q_1\theta_M \equiv 2\left(q_1+nb\right)\theta_M \mod 2\pi$ and $2q_2\theta_M \equiv 2\left(q_2+nb\right)\theta_M \mod 2\pi$. Hence, if we construct the sequence $(q_n)_{n\in\mathbb{N}}$ in such a way that[9]

$$q_{2n-1} := q_1 + nb, \quad q_{2n} := q_2 + nb, \tag{3.3.25}$$

then $\mathcal{G}_{q_n}(\theta_M)\cdot\mathcal{G}_{q_{n+1}}(\theta_M) < 0$ or, by other words, $\mathcal{G}_{q_n}(\theta_M)$ and $\mathcal{G}_{q_{n+1}}(\theta_M)$ have opposite signs.

[9]Note that the interlacing property established by Claim 3.3.1 is obviously valid in this case.

3.3.4 The dominance of $(\mathcal{G}_p(\theta_M))_{p\in\mathbb{N}}$.

Having constructed a sequence $(q_n)_{n\in\mathbb{N}}$ for which $(\mathcal{G}_{q_n}(\theta_M))_{n\in\mathbb{N}}$ always alternates its sign, we are now ready to proceed with the proof of Theorem 3.1.3. Our next claim establishes that the sequence $\mathcal{G}_{q_n}(\theta_M)$ dominates, in some sense, the second term (involving $\mathcal{A}_{\alpha,p}(u,z)$) appearing in (3.3.20).

Claim 3.3.2. *There exists a sufficiently large η_0 such that, for any z satisfying (3.2.16), $\mathcal{A}_{\alpha,p}(\cdot,z)$ obeys to the inequality*

$$e^{z^2/8}\left|Re\left[\mathcal{A}_{\alpha,p}\left(\frac{1}{\eta_0\, p\log(p)},z\right)\right]\right| < |c_M|\, r_M^{2p}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)|\mathcal{G}_p\left(\theta_M\right)|, \tag{3.3.26}$$

for any $p\in(q_n)_{n\in\mathbb{N}}$.

Proof. By (3.3.8),

$$|\mathcal{A}_{\alpha,p}(u,z)| \leq \mathscr{D}\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{\frac{\alpha}{2}+2p}}, \quad 0<u<\frac{\pi}{4},$$

so that the inequality (3.3.26) holds for any $p\in(q_n)_{n\in\mathbb{N}}$ if

$$\frac{\mathscr{D}2^{14p}(2p)!\, e^{z^2/8}}{|c_M|r_M^{2p}\, u^{\frac{\alpha}{2}+2p}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)|\mathcal{G}_p\left(\theta_M\right)|} < e^{A/u} \tag{3.3.27}$$

holds for any $p\in(q_n)_{n\in\mathbb{N}}$. By our construction of the sets $\mathcal{I}^+$ and $\mathcal{I}^-$, if $p\in(q_n)_{n\in\mathbb{N}}$ then

$$|\mathcal{G}_p\left(\theta_M\right)| = \left|\cos\left(2p\theta_M\right)\,\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_M\right)+\sin\left(2p\theta_M\right)\,\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_M\right)\right| \geq \epsilon_0, \tag{3.3.28}$$

for some $\epsilon_0>0$ (only depending on θ_M). Therefore, since $e^{z^2/8}<e^{\frac{\pi\alpha}{576}}$ (by condition (3.1.19)), (3.3.27) is plainly satisfied if, for any $p\in(q_n)_{n\in\mathbb{N}}$, the following inequality holds

$$\frac{\mathscr{D}\exp\left(\frac{\pi\alpha}{576}\right)}{\epsilon_0|c_M|}\,\frac{2^{14p}(2p)!}{r_M^{2p}u^{\frac{\alpha}{2}+2p}} < e^{A/u}. \tag{3.3.29}$$

Setting $u=\frac{1}{\eta_0\, p\log(p)}$ and taking the logarithm on both sides of (3.3.29), one may check that (3.3.29) holds if and only if

$$\begin{aligned}&\frac{1}{Ap\log(p)}\log\left(\frac{\mathscr{D}\exp\left(\frac{\pi\alpha}{576}\right)}{\epsilon_0|c_M|}\right)+\frac{2\,\log\left(128/r_M\right)}{A\log(p)}+\frac{\log(2p)!}{Ap\log(p)}\\&\quad+\frac{1}{A\log(p)}\left(\log(\eta)+\log(p)+\log(\log(p))\right)\left(2+\frac{\alpha}{2p}\right)<\eta_0.\end{aligned} \tag{3.3.30}$$

From Stirling's formula, we know that $\log(2p)! = O\left(p\log(p)\right)$, and so (3.3.30) must be valid for every $p\in(q_n)_{n\in\mathbb{N}}$ once we take η_0 very large. This completes the proof of (3.3.26). □

We just proved that the second term of (3.3.20) is dominated by $\mathcal{G}_p\left(\theta_M\right)$, as long as p belongs to an appropriate sequence of integers. Just like the proof of Theorem 2.1.3 in the previous chapter, we will require a bound for the residual functions $\tilde{E}(X,z)$ and $\tilde{H}(X,z)$, which will be offered by our next claim.

Claim 3.3.3. *There exists a sufficiently large* N_0 *such that, for any* $p \in (q_n)_{n \geq N_0}$, $\eta > 0$ *and* z *satisfying (3.1.19), the following inequalities hold*

$$|\tilde{E}(X,z)| < \frac{1}{6}, \; |\tilde{H}\,(X,z)\,| < \frac{1}{6} \tag{3.3.31}$$

and

$$\frac{\left|\mathcal{B}_{\alpha,p}\left(\frac{1}{\eta p \log(p)}\right)\right|}{2\,|c_M| r_M^{2p}\,|\mathcal{G}_p\,(\theta_M)|} < \frac{1}{6}, \tag{3.3.32}$$

with $\mathcal{B}_{\alpha,p}(u)$ *being defined by (3.3.14).*

Proof. Since $p \in (q_n)_{n \geq N_0}$, then (3.3.28) holds for some $\epsilon_0 \geq 0$. Thus, since $(c_j)_{j\in\mathbb{N}} \in \ell^1$, we can choose a sufficiently large $X \geq X_0$ such that

$$\begin{aligned}|\tilde{H}(X,z)| \leq \frac{1}{\epsilon_0} \sum_{j\neq M,\, j>X} \frac{|c_j|}{|c_M|}\left(\frac{r_j}{r_M}\right)^{2p} &\left|\cos\left(2p\theta_j\right)\,\cos\left(\frac{\pi\alpha}{8}\right)\cosh\left(\frac{\pi}{2}\lambda_j\right)\right. \\ &\left. + \sin\left(2p\theta_j\right)\,\sin\left(\frac{\pi\alpha}{8}\right)\sinh\left(\frac{\pi}{2}\lambda_j\right)\right| \leq \frac{2e^{\frac{\pi}{2}\lambda_M}}{\epsilon_0|c_M|}\sum_{j\neq M, j>X}|c_j| < \frac{1}{6}.\end{aligned}$$

On the other hand, we may bound $\tilde{E}(X,z)$ in the following form

$$|\tilde{E}(X,z)| \leq \frac{2e^{\frac{\pi}{2}\lambda_M}}{\epsilon_0|c_M|}\mu_X^{2p}\sum_{j\neq M,\, j\leq X}|c_j|,$$

where

$$\mu_X = \max_{j\leq X}\left\{\frac{r_j}{r_M}\right\}.$$

By property (3.3.1), we know that $\mu_X < 1$ and so, for a sufficiently large N_0 and for $p \in (q_n)_{n\geq N_0}$,

$$|\tilde{E}(X,z)| < \frac{1}{6},$$

which establishes (3.3.31). Finally, in order to obtain (3.3.32), we just need to use (3.3.15) with $u = \frac{1}{\eta p\, \log(p)}$,

$$\frac{\left|\mathcal{B}_{\alpha,p}\left(\frac{1}{\eta p \log(p)}\right)\right|}{2\,|c_M| r_M^{2p}\,|\mathcal{G}_p\,(\theta_M)|} \leq \frac{C_\alpha}{2\epsilon_0|c_M|\eta p\,\log(p)} < \frac{1}{6},$$

which holds whenever $p \geq q_{N_0}$ and N_0 is picked as a sufficiently large number. This completes the proof of the claim.

$\square$

3.3.5 Conclusion of the argument

In these final steps towards the proof of Theorem 3.1.3, we are ready to collect all the information of the previous sections. We return to the integral representation (3.3.20),

$$\int_0^\infty t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt = -\frac{8\pi c_M r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\mathcal{G}_p\left(\theta_M\right)$$
$$\times\left\{1+\frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\,\mathcal{G}_p\left(\theta_M\right)}+\tilde{E}\left(X,z\right)+\tilde{H}\left(X,z\right)\right\}+\frac{2\pi\, e^{z^2/8}}{2^{2p}}\,\mathrm{Re}\left[\mathcal{A}_{\alpha,p}(u,z)\right], \qquad (3.3.33)$$

and we take there $p\in(q_n)_{N_0\le n\le N}$, where $N\ge 2N_0+2$ and N_0 is a sufficiently large number for which (3.3.31) and (3.3.32) hold. Replacing u by $\frac{1}{\eta q_N\log(q_N)}$ in (3.3.33), our choices of N_0 and X in Claim 3.3.3 allow us to get the inequality

$$1+\frac{\mathcal{B}_{\alpha,p}(u)}{2c_M r_M^{2p}\,\mathcal{G}_p\left(\theta_M\right)}+\tilde{E}\left(X,z\right)+\tilde{H}\left(X,z\right)>\frac{1}{2}. \qquad (3.3.34)$$

Moreover, according to Claim 3.3.2,

$$\frac{2\pi\, e^{z^2/8}}{2^{2p}}\left|\mathrm{Re}\left[\mathcal{A}_{\alpha,p}\left(\frac{1}{\eta\, p\log(p)},z\right)\right]\right|<\frac{1}{4}\cdot\frac{8\pi|c_M|\,r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\left|\mathcal{G}_p\left(\theta_M\right)\right|, \qquad (3.3.35)$$

for any $\eta\ge\eta_0$ and η_0 sufficiently large (such that (3.3.30) holds). Combining (3.3.34) with (3.3.35), we see that, for any $p\in(q_n)_{N_0\le n\le N}$ and η_0 chosen as in the proof of Claim 3.3.2, the sequence of moments

$$\left(M_p\right)_{p\in(q_n)_{N_0\le n\le N}} := \int_0^\infty t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\cosh\left(\left(\frac{\pi}{2}-\frac{2}{\eta_0 q_N\log(q_N)}\right)t\right)\,dt \qquad (3.3.36)$$

must have the same sign as the numerical sequence defined by

$$\left(s_p\right)_{p\in(q_n)_{N_0\le n\le N}} := -\frac{8\pi c_M r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\mathcal{G}_p(\theta_M). \qquad (3.3.37)$$

Moreover, from (3.3.34) and (3.3.35), the modulus of M_p, (3.3.36), must satisfy the inequality

$$\left|\int_0^\infty t^{2p}\,\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\cosh\left(\left(\frac{\pi}{2}-\frac{2}{\eta_0 q_N\log(q_N)}\right)t\right)\,dt\right|$$
$$>\frac{1}{4}\cdot\frac{8\pi|c_M|r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\left|\mathcal{G}_p(\theta_M)\right|$$
$$=\frac{2\pi|c_M|r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)\left|\mathcal{G}_p(\theta_M)\right|. \qquad (3.3.38)$$

We are almost ready to apply Lemma 3.3.1. In order to meet its conditions, we need to transform the integral on the right-hand side of (3.3.36) into an integral over a finite interval and still manage to know something about its sign as p changes. The following lemma is the final necessary step to achieve this.

Claim 3.3.4. *There exists $\eta_1 > \eta_0$ such that, for any $\eta \geq \eta_1$ and z satisfying (3.1.19), the following inequality holds*

$$\left|\int_{\eta^2 q_N^2 \log^2(q_N)}^{\infty} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2}-\frac{2}{\eta_0 q_N \log(q_N)}\right)t\right) dt\right| < \frac{2\pi|c_M| r_M^{2p}}{2^{2p}}\left(1+e^{z^2/8}\sinh\left(\frac{z^2}{8}\right)\right)|\mathcal{G}_p(\theta_M)|, \tag{3.3.39}$$

whenever $p \in (q_n)_{N_0\leq n\leq N}$.

Proof. Using (3.2.8), it is simple to check that, for a sufficiently large η_1 and $\eta \geq \eta_1$,

$$\begin{aligned}
&\int_{\eta^2 q_N^2 \log^2(q_N)}^{\infty} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2}-\frac{2}{\eta_0 q_N \log(q_N)}\right)t\right) dt \\
&\leq e^{-\frac{\eta^2}{\eta_0}q_N\log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^{\infty} t^{2q_N} \left|\tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right)\right| \exp\left(\left(\frac{\pi}{2}-\frac{1}{\eta_0 q_N \log(q_N)}\right)t\right) dt \\
&\ll_{\alpha,z,\Lambda} e^{-\frac{\eta^2}{\eta_0}q_N\log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^{\infty} t^{2q_N+B(\alpha)} \exp\left(-\frac{t}{\eta_0 q_N \log(q_N)}+|z|\sqrt{t}\right) dt \\
&\ll_{\alpha,z,\Lambda} e^{-\frac{\eta^2}{\eta_0}q_N\log(q_N)} \int_{\eta^2 q_N^2 \log^2(q_N)}^{\infty} t^{2q_N+B(\alpha)} \exp\left(-\frac{t}{2\eta_0 q_N \log(q_N)}\right) dt \\
&\ll_{\alpha,z,\Lambda} (2\eta_0 q_N\log(q_N))^{2q_N+B(\alpha)+1}\, e^{-\frac{\eta^2}{\eta_0}q_N\log(q_N)}\, (2q_N+[B(\alpha)]+1)!\,.
\end{aligned}$$

In the above inequalities, $\ll_{\alpha,z,\Lambda}$ stands for a constant depending on α, z and the pair of sequences $(c_j,\lambda_j)_{j\in\mathbb{Z}\setminus\{0\}}$. By Stirling's formula, we have that the latter expression is bounded by

$$\begin{aligned}
\exp\Big\{&-\frac{\eta^2}{\eta_0}q_N\log(q_N)+(2q_N+B(\alpha)+1)\log(2\eta_0 q_N\log(q_N)) \\
&+(2q_N+[B(\alpha)]+1)\log(2q_N+[B(\alpha)]+1)+O(q_N)\Big\}.
\end{aligned}$$

It is now clear that, if η_0 is sufficiently large and we take $\eta \geq \eta_1 > \eta_0$, the dominant term inside the braces will be $-\frac{\eta^2}{\eta_0}q_N\log(q_N)$. This now proves (3.3.39). $\square$

By the previous claim and (3.3.38), we conclude that, for each $p \in (q_n)_{N_0\leq n\leq N}$, there exists some sufficiently large $\eta_1 > \eta_0$ such that the sequence of moments

$$\left(\tilde{M}_p\right)_{p\in(q_n)_{N_0\leq n\leq N}} := \int_0^{\eta_1^2 q_N^2 \log^2(q_N)} t^{2p}\, \tilde{F}_{z,\alpha}\left(\frac{\alpha}{4}+it\right) \cosh\left(\left(\frac{\pi}{2}-\frac{2}{\eta_0 q_N \log(q_N)}\right)t\right) dt$$

must have the same sign as the numerical sequence $(s_p)_{p\in(q_n)_{N_0\leq n\leq N}}$, defined by (3.3.37). According to Claim 3.3.1, s_ℓ and $s_{\ell+1}$ have always distinct signs, and so the sequence $(s_p)_{p\in(q_n)_{N_0\leq n\leq N}}$ has exactly $N-N_0-1$ sign changes. By Lemma 3.3.1,

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) \geq N - N_0 - 1 > \frac{N}{2}, \tag{3.3.40}$$

since $N \geq 2N_0+2$ by hypothesis. To conclude, we just need to connect q_N with N. Once more, we divide this final argument in two cases[10].

1. If $\theta_M/\pi \in \mathbb{Q}$, say $\theta_M = \pi\frac{a}{b}$, then the explicit construction (3.3.25) shows that, for $N \geq 3$, $N > \frac{q_N - q_2}{b} \geq \frac{q_N}{2b}$. Thus, (3.3.40) implies

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) > \frac{q_N}{4b},$$

which is equivalent to (3.1.21).

2. If $\theta_M/\pi \notin \mathbb{Q}$ then, by Weyl's criterion (3.2.51), the sequence $\left(\left\{n\,\frac{2\theta_M}{\pi}\right\}\right)_{n\in\mathbb{N}}$ is uniformly distributed on the interval $[0,1]$. This means that

$$\lim_{X\to\infty}\frac{\#\left\{1\leq m\leq X\,:\,2m\theta_M\in\left(\mathcal{I}^+\cup\mathcal{I}^-\right)\right\}}{X} = \frac{|\mathcal{I}^+\cup\mathcal{I}^-|}{2\pi} = \frac{2\theta_M}{\pi}. \tag{3.3.41}$$

Thus, for a sufficiently large N_0 and $N \geq 2N_0+2$, if we take $X = q_N$ in (3.3.41), we find that (3.3.40) yields

$$N_{\alpha,z}\left(\eta_1^2\, q_N^2 \log^2(q_N)\right) > \frac{\theta_M}{\pi} q_N,$$

which is equivalent to (3.1.21) and completes the proof of the Theorem. $\square$

3.4 Lemmas for the proof of Theorem 3.1.4

This section is devoted to establish (without too many details) the necessary lemmas for the proof of Theorem 3.1.4. We initiate it by noting that, in analogy to (3.2.8), we may use (3.2.6) to prove the estimate

$$\begin{aligned}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right| &\leq \sum_{j\neq 0}\left|c_j\,\eta_f\left(\frac{k}{2}+i\,(t+\lambda_j)\right)\mathrm{Re}\left({}_1F_1\left(\frac{k}{2}-i\,(t+\lambda_j);\,k;\,\frac{z^2}{4}\right)\right)\right| \\ &\ll_{k,z} C_\lambda\sum_{j=1}^{\infty}|c_j|\,|t|^{B(k)}e^{-\frac{\pi}{2}|t|+|z|\sqrt{|t|}},\end{aligned} \tag{3.4.1}$$

where $B(k)$ is some positive constant depending on the weight of the cusp form, k. The next lemma is an analogue of Lemma 3.2.1 above. We note that we already proved this result in the previous chapter (see (2.2.97) above), so we forego its proof.

Lemma 3.4.1. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $a_f(n)$ be its Fourier coefficients. For $Re(x)>0$ and any $y\in\mathbb{C}$, the following transformation*

[10] We urge the reader to recall that we are assuming that $0<\theta_M<\frac{\pi}{2}$ in this proof.

formula takes place

$$\sum_{n=1}^{\infty}a_f(n)\,n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n}\,y\right)=\frac{(-1)^{k/2}\,e^{-\frac{y^2}{4x}}}{x}\,\sum_{n=1}^{\infty}a_f(n)\,n^{\frac{1-k}{2}}\,e^{-\frac{2\pi n}{x}}\,I_{k-1}\left(\sqrt{2\pi n}\,\frac{y}{x}\right). \tag{3.4.2}$$

Equivalently,

$$\psi_f(x,z)=(-1)^{k/2}\,e^{-\frac{z^2}{4}}\,x^{-k}\,\psi_f\left(\frac{1}{x},\,iz\right),\quad Re(x)>0,\,z\in\mathbb{C}, \tag{3.4.3}$$

where $\psi_f(x,z)$ is the analogue of Jacobi's $\psi-$function (3.1.12),

$$\psi_f(x,z):=(k-1)!\,\left(\sqrt{\frac{\pi x}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\,x}\,J_{k-1}\left(\sqrt{2\pi n\,x}\,z\right),\;Re(x)>0,\;\;z\in\mathbb{C}. \tag{3.4.4}$$

The next lemma is an analogue of Lemma 3.2.2. Since the proof is very similar, we shall only indicate its main steps.

Lemma 3.4.2. *Let $f(\tau)$ be a holomorphic cusp form of weight k for the full modular group and $\psi_f(x,z)$ be the generalized Jacobi's $\psi-$function attached to it, i.e., (3.4.4). Furthermore, assume that $z\in\mathbb{R}$ satisfies the condition*

$$-\frac{\sqrt{\pi}}{3}\leq z\leq\frac{\sqrt{\pi}}{3}. \tag{3.4.5}$$

Then there are two positive constants A and C (depending only on k) such that, for any $0<u<\frac{\pi}{4}$ and every $p\in\mathbb{N}_0$,

$$\left|\frac{d^p}{du^p}\psi_f\left(ie^{-2iu},z\right)\right|<C\,\frac{2^{7p}p!}{u^{k+p}}e^{-\frac{A}{u}}. \tag{3.4.6}$$

Proof. The proof is exactly the same as in Lemma 3.2.2, so we only give a brief sketch. Starting with Cauchy's formula,

$$\left[\frac{d^p}{du^p}\psi_f\left(ie^{-2iu},z\right)\right]_{u=u_0}=\frac{p!}{2\pi i}\int\limits_{C_{\lambda u_0}(u_0)}\frac{\psi_f\left(ie^{-2iw},z\right)}{(w-u_0)^{p+1}}\,dw, \tag{3.4.7}$$

we need to find a suitable bound for $\psi_f\left(ie^{-2iw},z\right)$ when $w\in C_{\lambda u_0}(u_0)$. This is done through Lemma 3.4.1: replacing there x by $2e^{-iw}\sin(w)$ and y by $e^{i\left(\frac{\pi}{4}-w\right)}\,z$, we obtain the transformation formula

$$\psi_f\left(ie^{-2iw},z\right)=(k-1)!\left(\sqrt{\frac{\pi ie^{-2iw}}{2}}\,z\right)^{1-k}\sum_{n=1}^{\infty}a_f(n)n^{\frac{1-k}{2}}\,e^{-2\pi n\cdot 2e^{-iw}\sin(w)}\,J_{k-1}\left(\sqrt{2\pi nie^{-2iw}}\,z\right)$$

$$=(k-1)!\,\left(\sqrt{\frac{\pi}{2}}e^{i\left(\frac{\pi}{4}-w\right)}z\right)^{1-k}\frac{(-1)^{k/2}\,e^{-\frac{ie^{-iw}z^2}{8\sin(w)}}}{2e^{-iw}\sin(w)}\,\sum_{n=1}^{\infty}(-1)^n a_f(n)\,n^{\frac{1-k}{2}}\,e^{-\frac{\pi n}{\tan(w)}}\,I_{k-1}\left(\sqrt{\frac{\pi n}{2}}\frac{e^{i\frac{\pi}{4}}z}{\sin(w)}\right).$$

From this point on, we can easily adapt the steps given in the proof of the bounds (3.2.24) and (3.2.26). Extracting the first term of the series in the above expression,[11] we obtain

$$\begin{aligned}\left|\psi_f\left(ie^{-2iw},z\right)\right| &\leq d_k\,\frac{\pi^k e^{-\frac{z^2}{8}+\frac{u_0}{2}}}{(2u_0)^k}\exp\left\{-\frac{\pi\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}P_w^{\star}(1)\right\}\\ &\times\sum_{n=1}^{\infty}\left|\frac{a_f(n)}{a_f(1)}\right|\exp\left[-\frac{\pi\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\left(P_w^{\star}(\sqrt{n})-P_w^{\star}(1)\right)\right],\end{aligned}\tag{3.4.8}$$

where $P_w^{\star}(X)$ is the real-valued polynomial

$$P_w^{\star}(X):=X^2-\frac{|z|}{\sqrt{\pi}\sin(2\mathrm{Re}(w))}\sqrt{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}\,X+\frac{z^2}{8\pi}\frac{\sinh(2\mathrm{Im}(w))}{\sin(2\mathrm{Re}(w))}.$$

Note that our expression (3.4.8) is completely analogous to (3.2.26). Repeating the steps in (3.2.32), we find that the series on the right-hand side of (3.4.8) is uniformly bounded for any $w\in C_{0.01u_0}(u_0)$ and z satisfying (3.4.5). The resulting constant, M_k, will only dependent on k. Analogously to (3.2.33) and (3.2.34),

$$\begin{aligned}\left|\psi_f\left(ie^{-2iw},z\right)\right| &\leq \frac{d_kM_ke^{\frac{\pi}{8}}\,\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left\{-\frac{\pi\sin(2\mathrm{Re}(w))}{\cosh(2\mathrm{Im}(w))-\cos(2\mathrm{Re}(w))}P_w^{\star}(1)\right\}\\ &< \frac{d_kM_ke^{\frac{\pi}{8}}\,\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{1}{u_0}\left\{\frac{1-\lambda}{e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2}-\frac{\pi|z|}{2(1-\lambda)}-\frac{\pi z^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{16(1-\lambda)^2}\right\}\right]\\ &< \frac{d_kM_ke^{\frac{\pi}{8}}\,\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{1}{u_0}\left\{\frac{1-\lambda}{e^{\frac{\pi^2\lambda^2}{24}}(1+\lambda)^2}-\frac{\pi^{\frac{3}{2}}}{6(1-\lambda)}-\frac{\pi^2\lambda e^{\frac{\pi^2\lambda^2}{24}}}{144(1-\lambda)^2}\right\}\right].\end{aligned}$$

Thus, if we pick $\lambda=0.01$, the term on the braces is greater than 0.03: this implies that

$$\left|\psi_f\left(ie^{-2iw},z\right)\right| < \frac{d_kM_ke^{\frac{\pi}{8}}\,\pi^k e^{-\frac{z^2}{8}}}{(2u_0)^k}\exp\left[-\frac{0.03}{u_0}\right]=\frac{C}{u_0^k}\,e^{-\frac{A}{u_0}},$$

where C depends on k and A is an absolute constant, say $A=0.03$. Using Cauchy's formula (3.4.7) and mimicking the steps in (3.2.36), we find that (3.4.6) must hold for every $p\in\mathbb{N}_0$. $\square$

We finish this section with two lemmas that are completely analogous to Lemmas 3.2.3 and 3.2.4 of Section 3.2. Since their proofs are exactly the same, we shall omit them.

Lemma 3.4.3. *Let $g:\mathbb{C}\longmapsto\mathbb{C}$ be an analytic function. If $z\in\left[-\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}},\frac{1}{6}\sqrt{\frac{\pi\alpha}{2}}\right]$ then, for arbitrary $p\in\mathbb{N}_0$ and $0<u<\frac{\pi}{4}$, the following inequality holds*

$$\left|\frac{d^{2p}}{du^{2p}}\left\{g(u)\,\psi_f\left(ie^{-2iu},z\right)\right\}\right| < D\frac{2^{14p}(2p)!e^{-\frac{A}{u}}}{u^{k+2p}}\,\|g\|_{L^{\infty}(C_1(0))},\tag{3.4.9}$$

[11] without any loss of generality, we suppose that $a_f(1)\neq 0$. If c is the smallest integer such that $a_f(c)\neq 0$, then it would be possible to enlarge the interval (3.1.26) to $\left[-\frac{\sqrt{\pi c}}{3},\frac{\sqrt{\pi c}}{3}\right]$.

where A is some absolute constant and D only depends on k. Moreover, $C_1(0) := \partial D(0,1)$ denotes the unit circle in the complex plane and $\|g\|_{L^\infty(C_1(0))} := \max_{w\in C_1(0)} |g(w)|$.

Lemma 3.4.4. *Let $-\frac{\pi}{4} < \omega < \frac{\pi}{4}$, $p \in \mathbb{N}$ and $\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ be the function defined by (3.1.20). Furthermore, let $\psi_f(x,z)$ denote the "cusp form analogue" of Jacobi's $\psi-$function (3.4.4). Then the following integral representations hold:*[12]

1. *If $k \equiv 0 \mod 4$, one has that*

$$\int_0^\infty i^{-\frac{k}{2}}\, t^{2p} \tilde{G}_{z,f}\left(\frac{k}{2}+it\right) \cosh(2\omega t)\, dt$$
$$= \frac{2\pi\, e^{\frac{z^2}{4}}}{2^{2p}}\, Re\left(i^{-\frac{k}{2}}\, \frac{d^{2p}}{d\omega^{2p}}\left\{\sum_{j\neq 0} c_j\, e^{i\omega k - 2\omega\lambda_j}\, \psi_f\left(e^{2i\omega}, z\right)\right\}\right). \tag{3.4.10}$$

2. *If $k \equiv 2 \mod 4$, then*

$$\int_0^\infty i^{-\frac{k}{2}}\, t^{2p+1} \tilde{G}_{z,f}\left(\frac{k}{2}+it\right) \cosh(2\omega t)\, dt$$
$$= \frac{2\pi\, e^{\frac{z^2}{4}}}{2^{2p}}\, Re\left(i^{-\frac{k}{2}}\, \frac{d^{2p+1}}{d\omega^{2p+1}}\left\{\sum_{j\neq 0} c_j\, e^{i\omega k - 2\omega\lambda_j}\, \psi_f\left(e^{2i\omega}, z\right)\right\}\right). \tag{3.4.11}$$

3.5 Proof of Theorem 3.1.4

In this section we follow closely our variant of de la Vallée Poussin's method [187]. The details of the proof are mostly given for the case where $k \equiv 0 \mod 4$ and a similar argument, valid when $k \equiv 2 \mod 4$, will be devised at the end of this section (see Remark 3.5.1 below). Let $(\rho_n)_{n\in\mathbb{N}}$ denote the sequence of zeros of odd order of $\tilde{G}_{z,f}(s)$ such that $\text{Re}(\rho_n) = \frac{k}{2}$. Then we can write $\rho_n := \frac{k}{2} + i\tau_n$, with $\tau_n > 0$ being an increasing sequence[13]. If we show that there exists some $h > 0$ such that, for infinitely many values of n, $\tau_n < h\, n^2$, then our Theorem 3.1.4 is proved. This is the case because, if we choose the sequence $T_n := hn^2$, we are able to find that $N_{f,z}(T_n) \geq N_{f,z}(\tau_n) = n = \sqrt{\frac{T_n}{h}}$, thus establishing

$$\limsup_{T\to\infty} \frac{N_{f,z}(T)}{\sqrt{T}} > \frac{1}{\sqrt{h}}, \quad \text{or equivalently } N_{f,z}(T) = \Omega\left(T^{\frac{1}{2}}\right). \tag{3.5.1}$$

For the sake of contradiction, let us then assume that there is some N_0 such that, for every $n \geq N_0$ and any $h > 0$, $\tau_n \geq h\, n^2$. We will prove the existence of some (large enough) h for which

[12] Note that the integral is written in terms of $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ because it defines a real function of $t \in \mathbb{R}$.
[13] If $\tilde{G}_{z,f}\left(\frac{k}{2}\right) = 0$, we exclude this real zero from the sequence.

this assumption leads to a contradiction. Indeed, if we construct the real and entire function[14]

$$\varphi_{f,z}(y)=\prod_{\ell=1}^{\infty}\left(1-\frac{y^2}{\tau_\ell^2}\right)=\sum_{\ell=0}^{\infty}(-1)^\ell a_{2\ell}\,y^{2\ell},\tag{3.5.2}$$

we see that $a_0=1$ and, for $\ell\geq 1$,

$$a_{2\ell}=\sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_\ell>r_{\ell-1}}\frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_\ell}^2}=\sum_{1\leq r_1<r_2<\ldots<r_\ell}\frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_\ell}^2},\tag{3.5.3}$$

where we are summing over the ℓ-tuple $(r_1,...,r_\ell)\in\mathbb{N}^\ell$ such that $r_1<r_2<...<r_\ell$. Note that, in the k^{th} nested series in (3.5.3), the index r_k always satisfies $r_k\geq k$, due to the condition $r_k>r_{k-1}>...>r_1\geq 1$. From this point on, we just need to find a suitable bound for $a_{2\ell}$. Considering the nested sum above, we have two possibilities: if, on one hand, $1\leq k\leq\ell$ and $r_k\geq N_0$, we know by the contradiction hypothesis that $\tau_{r_k}^{-2}\leq\frac{r_k^{-4}}{h^2}$. On the other hand, if $1\leq r_k\leq N_0-1$, then $\tau_{r_k}^{-2}\leq\frac{r_k^{-4}}{h^{\star 2}}$, where $h^\star:=\min_{1\leq n\leq N_0-1}\left\{\frac{\tau_n}{n^2}\right\}$. Let us take a generic series in (3.5.3): by the above reasoning, if we first suppose that $r_{k-1}\geq N_0-1$, then the contradiction hypothesis gives

$$\sum_{r_k>r_{k-1}}\frac{1}{\tau_{r_k}^2}=\sum_{r_k>r_{k-1}\geq N_0-1}\frac{1}{\tau_{r_k}^2}\leq\frac{1}{h^2}\sum_{r_k>r_{k-1}}\frac{1}{r_k^4}.\tag{3.5.4}$$

On the other hand, if we suppose that $1\leq r_{k-1}<N_0-1$, is is not difficult to reach the following inequality

$$\begin{aligned}\sum_{r_k>r_{k-1}}\frac{1}{\tau_{r_k}^2}&=\sum_{r_k=r_{k-1}+1}^{N_0-1}\frac{1}{\tau_{r_k}^2}+\sum_{r_k=N_0}^{\infty}\frac{1}{\tau_{r_k}^2}\leq\frac{1}{h^{\star 2}}\sum_{r_k=r_{k-1}+1}^{N_0-1}\frac{1}{r_k^4}+\frac{1}{h^2}\sum_{r_k=N_0}^{\infty}\frac{1}{r_k^4}\\&=\frac{1}{h^2}\sum_{r_k>r_{k-1}}\frac{1}{r_k^4}+\frac{1}{h^2}\left(\frac{h^2}{h^{\star 2}}-1\right)\sum_{r_k=r_{k-1}+1}^{N_0-1}\frac{1}{r_k^4}\\&\leq\frac{1}{h^2}\sum_{r_k>r_{k-1}}\frac{1}{r_k^4}+\frac{1}{h^2}\left|\frac{h^2}{h^{\star 2}}-1\right|\sum_{j=1}^{N_0-1}\frac{1}{j^4}\leq\frac{\mathscr{A}}{h^2}\sum_{r_k>r_{k-1}}\frac{1}{r_k^4},\end{aligned}\tag{3.5.5}$$

for some positive constant $\mathscr{A}$ depending on h and N_0, but not on k. We can explicitly choose this constant by taking the observation that there exists some $A>0$ such that

$$\left|\frac{h^2}{h^{\star 2}}-1\right|\sum_{j=1}^{N_0-1}\frac{1}{j^4}\leq A\sum_{j\geq N_0}\frac{1}{j^4}\leq A\sum_{r_k>r_{k-1}}\frac{1}{r_k^4},$$

where the last inequality is valid since $1\leq r_{k-1}<N_0-1$. Picking $\mathscr{A}:=1+A\geq 1$ gives (3.5.5). Since $r_\ell\geq\ell$ for every $\ell\in\mathbb{N}$, a sufficient condition ensuring $r_{k-1}\geq N_0-1$ is that $k\geq N_0$. Hence, we have at most N_0-1 infinite series in the nested sum (3.5.3) where we need to apply (3.5.5),

[14]This infinite product can be written because, due to Theorem 2.1.6, $\tilde{G}_{z,f}(s)$ has infinitely many zeros on the critical line $\mathrm{Re}(s)=\frac{k}{2}$.

allowing us to bound $a_{2\ell}$ in the following simple way

$$
\begin{aligned}
a_{2\ell} &= \sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_\ell>r_{\ell-1}} \frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_j}^2} = \sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_{N_0}>r_{N_0-1}\geq N_0-1}\cdots\sum_{r_\ell>r_{\ell-1}} \frac{1}{\tau_{r_1}^2\cdot\ldots\cdot\tau_{r_\ell}^2} \\
&\leq \frac{\mathscr{A}^{N_0-1}}{h^{2\ell}}\sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_\ell>r_{\ell-1}} \frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4} := \frac{\mathscr{B}}{h^{2\ell}}\sum_{1\leq r_1<r_2<\ldots<r_\ell}\frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4},
\end{aligned} \tag{3.5.6}
$$

where $\mathscr{B} := \mathscr{A}^{N_0-1}$ is a positive constant that only depends on h and N_0. It is now evident that determining a bound for $a_{2\ell}$ reduces to analyzing the new coefficient

$$
b_{2\ell} := \sum_{r_1\geq 1}\sum_{r_2>r_1}\cdots\sum_{r_\ell>r_{\ell-1}} \frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4} = \sum_{1\leq r_1<\ldots<r_\ell}\frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4}. \tag{3.5.7}
$$

Whoever is familiar with one of Euler's proofs of the formula for $\zeta(2n)$ recognizes this sequence of numbers as coming from the Weierstrass factorization theorem applied to the sine function, which takes the form

$$
\frac{\sinh(\pi\sqrt{y})\sin(\pi\sqrt{y})}{\pi^2 y} = \prod_{\ell=1}^{\infty}\left(1+\frac{y}{\ell^2}\right)\prod_{\ell=1}^{\infty}\left(1-\frac{y}{\ell^2}\right) = \prod_{\ell=1}^{\infty}\left(1-\frac{y^2}{\ell^4}\right) := 1+\sum_{\ell=1}^{\infty}(-1)^{\ell} b_{2\ell}\, y^{2\ell}. \tag{3.5.8}
$$

Thus, we can get precise information about the sequence $b_{2\ell}$ (and, consequently, about $a_{2\ell}$) by interpreting it as the sequence of Taylor coefficients of the function on the left-hand side of (3.5.8).[15] Indeed, it is quite simple[16] to see that, for any $y \in \mathbb{R}$, this Taylor expansion is

$$
\begin{aligned}
\sinh(\pi\sqrt{y})\,\sin(\pi\sqrt{y}) &= -\frac{i}{2}\left[\cos\left((1-i)\pi\sqrt{y}\right) - \cos\left((1+i)\pi\sqrt{y}\right)\right] \\
&= -\frac{i}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}}{(2k)!}y^k\left\{(1-i)^{2k}-(1+i)^{2k}\right\} \\
&= -\sum_{k=0}^{\infty}\frac{(-1)^k\pi^{2k}}{(2k)!}(2y)^k\sin\left(\frac{\pi}{2}k\right) \\
&= \sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}2^{2\ell+1}\pi^{4\ell+2}}{(4\ell+2)!}y^{2\ell+1}.
\end{aligned} \tag{3.5.9}
$$

By comparing (3.5.8) with (3.5.9) and revisiting the inequality (3.5.6), we arrive at the conclusion that

$$
b_{2\ell} := \sum_{1\leq r_1<\ldots<r_\ell}\frac{1}{r_1^4\cdot\ldots\cdot r_\ell^4} = \frac{2^{2\ell+1}\pi^{4\ell}}{(4\ell+2)!} \implies a_{2\ell}\leq \mathscr{B}\,\frac{2^{2\ell+1}\pi^{4\ell}}{h^{2\ell}(4\ell+2)!}. \tag{3.5.10}
$$

To motivate the conclusion of our proof, we observe that establishing the validity of (3.5.10) will not only contradict (3.4.10) but also the bounds presented in Lemma 3.4.2. Recall that, under the assumption $k \equiv 0 \mod 4$, the function $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$ is real-valued and even. Moreover, in accordance with the construction (3.5.2), $\varphi_{f,z}(t)$ possesses the same odd zeros as $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)$.

[15] It is somewhat poetic that the idea that led Euler to have the first grasp on the nature of $\zeta(2n)$ can be also used to understand the zero distribution of $\zeta(s)$ and the combinations of $L-$functions studied in this chapter.

[16] Although these computations are standard, we did not find any reference containing this Taylor expansion. In order to be self-contained, we have decided to briefly present it.

This means that the product function $i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\cdot\varphi_{f,z}(t)$ must be real, even and having constant sign for any $t\in\mathbb{R}$. Hence, if we consider the continuous function $Q_f\,:\,\left(0,\,\frac{\pi}{4}\right)\longmapsto\mathbb{R}$ defined by the integral

$$Q_f(u) := \int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\varphi_{f,z}(t)\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt,\quad 0<u<\frac{\pi}{4}, \tag{3.5.11}$$

we know that $|Q_f(u)|$ has to be a positive decreasing function. Our proof will now show that this cannot be true if the bound (3.5.10) is valid. If we use the power series (3.5.2) for $\varphi_{f,z}(t)$, we see that $Q_f(u)$ can be written as an infinite series of the form

$$\begin{aligned}Q_f(u) &= \int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\varphi_{f,z}(t)\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &= \sum_{j=0}^{\infty}(-1)^{\ell}a_{2\ell}\int_0^\infty i^{-\frac{k}{2}}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\,t^{2\ell}\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt,\end{aligned} \tag{3.5.12}$$

where the interchange of the orders of summation and integration in (3.5.12) can be justified by Fubini's theorem and the bounds (3.5.10). Indeed, it is not difficult to check that

$$\begin{aligned}&\int_0^\infty\sum_{\ell=0}^{\infty}|a_{2\ell}|t^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &\leq \mathscr{B}\int_0^\infty\sum_{j=0}^{\infty}\frac{2^{2\ell+1}\pi^{4\ell}}{h^{2\ell}(4\ell+2)!}t^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &\leq 2\mathscr{B}\int_0^\infty\sum_{j=0}^{\infty}\frac{1}{(4\ell)!}\left(\frac{2\pi^2 t}{h}\right)^{2\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &\leq 2\mathscr{B}\int_0^\infty\sum_{j=0}^{\infty}\frac{1}{\ell!}\left(\pi\sqrt{\frac{2t}{h}}\right)^{\ell}\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &< 2\mathscr{B}\int_0^\infty\left|\tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\right|\exp\left(\pi\sqrt{\frac{2t}{h}}+\left(\frac{\pi}{2}-2u\right)t\right)\,dt\\ &\ll_{k,z,\Lambda}\int_0^\infty|t|^{B(k)}\,\exp\left(-2ut+\left(\pi\sqrt{\frac{2}{h}}+|z|\right)\sqrt{t}\right)\,dt<\infty,\end{aligned}$$

where the last step is completely justified by the estimate (3.4.1). Having assured that we can perform the operation (3.5.12), it now follows from Lemma 3.4.4 that $Q_f(u)$ admits the following expression

$$Q_f(u) = \sum_{j=0}^{\infty}(-1)^{\ell} a_{2\ell} \int_0^{\infty} i^{-k/2}\tilde{G}_{z,f}\left(\frac{k}{2}+it\right) t^{2\ell} \cosh\left(\left(\frac{\pi}{2}-2u\right)t\right) dt$$

$$= \sum_{\ell=0}^{\infty}(-1)^{\ell} a_{2\ell}\, \frac{2\pi}{2^{2\ell}}\, e^{\frac{z^2}{4}} \operatorname{Re}\left(i^{-\frac{k}{2}} \frac{d^{2\ell}}{du^{2\ell}}\left\{\sum_{j\neq 0} c_j\, e^{i\left(\frac{\pi}{4}-u\right)k-\left(\frac{\pi}{2}-2u\right)\lambda_j}\, \psi_f\left(ie^{-2iu}, z\right)\right\}\right). \tag{3.5.13}$$

Using Lemma 3.4.3 together with (3.5.10), we can bound the previous series uniformly with respect to u. Doing so requires to recall the path leading to (3.3.7): since the analytic function defined by

$$g(u) = e^{i\left(\frac{\pi}{4}-u\right)k} \sum_{j\neq 0} c_j\, e^{-\left(\frac{\pi}{2}-2u\right)\lambda_j}$$

satisfies $\|g(u)\|_{L^{\infty}(C_0(1))} \leq \mathcal{M}$, with $\mathcal{M}$ only depending on the numerical value $\sum_{j\neq 0}|c_j|$, we can use (3.4.9) to see that

$$\left|\frac{d^{2\ell}}{du^{2\ell}}\left\{g(u)\,\psi_f\left(ie^{-2iu}, z\right)\right\}\right| < \mathcal{D}\frac{2^{14\ell}(2\ell)!e^{-\frac{A}{u}}}{u^{k+2\ell}}, \tag{3.5.14}$$

for some constant $\mathcal{D}$ depending on the weight of the cusp form, k, and on the sequences $(c_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$. Returning to (3.5.13) and invoking (3.5.10), we can reach the following inequality satisfied by $|Q_f(u)|$,

$$\begin{aligned}|Q_f(u)| &< 2\pi\mathcal{D}e^{\frac{z^2}{4}}\frac{e^{-A/u}}{u^k}\sum_{\ell=0}^{\infty}|a_{2\ell}|\frac{2^{12\ell}(2\ell)!}{u^{2\ell}} \leq 2\pi\mathscr{C}e^{\frac{z^2}{4}}\frac{e^{-A/u}}{u^k}\sum_{\ell=0}^{\infty}\frac{2^{14\ell+1}\pi^{4\ell}(2\ell)!}{h^{2\ell}(4\ell+2)!}\cdot\frac{1}{u^{2\ell}}\\ &= \mathscr{C}\pi^{\frac{3}{2}}e^{\frac{z^2}{4}}\frac{e^{-A/u}}{u^k}\sum_{\ell=0}^{\infty}\frac{2^{10\ell}\pi^{4\ell}}{\Gamma\left(2\ell+\frac{3}{2}\right)(2\ell+1)(hu)^{2\ell}} < \mathscr{C}\pi^{\frac{3}{2}}e^{\frac{z^2}{4}}\frac{e^{-A/u}}{u^k}\cosh\left(\frac{32\pi^2}{hu}\right)\\ &< \mathscr{C}\frac{\pi^{\frac{3}{2}}e^{\pi/36}}{u^k}\exp\left(-\left(A-\frac{32\pi^2}{h}\right)\frac{1}{u}\right),\end{aligned} \tag{3.5.15}$$

where we have used the hypothesis that $|z| < \frac{1}{3}\sqrt{\pi}$, (3.1.26). Based on the inequality in (3.5.15), if we select the constant h in such a way that $h > \frac{32\pi^2}{A}$, then

$$\lim_{u\to 0^+}|Q_f(u)| < \mathscr{C}\pi^{\frac{3}{2}}e^{\pi/36}\lim_{u\to 0^+}u^{-k}\exp\left(-\left(A-\frac{32\pi^2}{h}\right)\frac{1}{u}\right) = 0,$$

contradicting the fact that $|Q_f(u)|$ is positive decreasing. Consequently, we have found $h > \frac{32\pi^2}{A}$ such that $\tau_n < h\,n^2$ for infinitely many values of n. This shows (3.5.1) and so we complete the proof of our Theorem. Finally, we note that we can replace the value of d in (3.1.28) by an explicit constant. Taking $A = 0.03$ (roughly calculated in the proof of Lemma 3.4.2) and setting $h = \frac{36\pi^2}{0.03}$, we find from (3.5.1) that the constant $d = \frac{1}{36\pi}$ works in the estimate (3.1.28). □

Remark 3.5.1. To address the case where $k \equiv 2 \mod 4$, the core argument remains largely unchanged, requiring only a minor adjustment. Rather than working with the function $Q_f(u)$, as defined in (3.5.11), we must instead examine the following alternative integral

$$P_f(u) := \int_0^\infty i^{-\frac{k}{2}}\, t\, \tilde{G}_{z,f}\left(\frac{k}{2}+it\right)\varphi_{f,z}(t)\,\cosh\left(\left(\frac{\pi}{2}-2u\right)t\right)\,dt,\quad 0<u<\frac{\pi}{4},$$

where $\varphi_{f,z}(t)$ is the function constructed in (3.5.2). Invoking the second integral representation (3.4.11) and using the bounds for $a_{2\ell}$ found in (3.5.10), one can prove that $|P_f(u)| \to 0$ as $u \to 0^+$, thereby confirming the desired result.

3.6 Concluding Remarks

In this chapter we have proved estimates for the number of critical zeros of the arbitrary shifted combinations

$$\tilde{F}_{z,\alpha}(s) := \sum_{j\neq 0} c_j\,\eta_\alpha\left(s+i\lambda_j\right)\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\frac{\alpha}{2};\,\frac{z^2}{4}\right)\right\},\quad (3.6.1)$$

$$\tilde{G}_{z,f}(s) := \sum_{j\neq 0} c_j\,\eta_f\left(s+i\lambda_j\right)\left\{{}_1F_1\left(k-s-i\lambda_j;\,k;\,\frac{z^2}{4}\right)+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,k;\,\frac{z^2}{4}\right)\right\}.\quad (3.6.2)$$

The proofs of both estimates were based on the summation formulas (3.2.14) and (3.4.2), established in the previous chapter. A natural question arises regarding how these results might be extended in new directions. For instance, one could explore whether it is possible to relax the class of hypergeometric functions involved in the combinations (3.6.1) and (3.6.2). Note that the second parameter of the hypergeometric functions above depends on the Dirichlet series that precedes them: in the first case, it is equal to $\alpha/2$ and, in the second, equals k. This invites further investigation on the critical zeros of the following functions

$$\begin{aligned}\tilde{F}_{z,\alpha}(s;\nu) := \sum_{j\neq 0} c_j\,\eta_\alpha\left(s+i\lambda_j\right)&\left\{{}_1F_1\left(\frac{\alpha}{2}-s-i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right.\\ &\left.+\,{}_1F_1\left(\frac{\alpha}{2}-\overline{s}+i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right\},\end{aligned}\quad (3.6.3)$$

$$\begin{aligned}\tilde{G}_{z,f}(s;\nu) := \sum_{j\neq 0} c_j\,\eta_f\left(s+i\lambda_j\right)&\left\{{}_1F_1\left(k-s-i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right.\\ &\left.+\,{}_1F_1\left(k-\overline{s}+i\lambda_j;\,\nu+1;\,\frac{z^2}{4}\right)\right\},\end{aligned}\quad (3.6.4)$$

where ν is now a real parameter independent of α in the first case and of k in the second case.

We note that analogues of Theorems 3.1.3 and 3.1.4 can indeed be derived for the shifted combinations given in (3.6.3) and (3.6.4), provided that $\nu \geq \frac{\alpha}{2}-1$ in the first case and $\nu \geq k-1$ in the second. However, instead of relying on the summation formulas (3.2.14) and (3.4.2), more general transformations will be required. One such transformation[17], which will be discussed in the next chapter, states that, when $\mathrm{Re}(\nu) > -1$ and $\mathrm{Re}(x) > 0$, $y \in \mathbb{C}$,

$$\sum_{n=1}^{\infty}(-1)^n r_\alpha(n)\, n^{-\nu/2}\, e^{-\pi n x}\, J_\nu(\sqrt{\pi n} y) = -\frac{y^\nu \pi^{\nu/2}}{2^\nu \Gamma(\nu+1)}$$
$$+\frac{\pi^{\nu/2} y^\nu e^{-\frac{y^2}{4x}}}{2^\nu \Gamma(\nu+1) x^{\alpha/2}} \sum_{n=1}^{\infty} \tilde{r}_\alpha(n) e^{-\frac{\pi}{x}\left(n+\frac{\alpha}{4}\right)}\, \Phi_3\left(1-\frac{\alpha}{2}+\nu; \nu+1; \frac{y^2}{4x}, \frac{\pi y^2}{4x^2}\left(n+\frac{\alpha}{4}\right)\right), \qquad (3.6.5)$$

where $\Phi_3(b;c;w,z)$ is the usual Humbert function,

$$\Phi_3(b;c;\, w,z) = \sum_{k,m=0}^{\infty} \frac{(b)_k}{(c)_{k+m}} \frac{w^k z^m}{k!\, m!}, \qquad (3.6.6)$$

whose series converges absolutely for any $w, z \in \mathbb{C}$. By using (3.6.5) instead of (3.2.14), we can establish a lower bound for the number of critical zeros of the function $\tilde{F}_{z,\alpha}(s;\, \nu)$ defined by (3.6.3). Analogously, Theorem 3.1.4 can be extended to shifted combinations of the form (3.6.4) by using the transformation formula

$$\sum_{n=1}^{\infty} a_f(n)\, n^{-\frac{\nu}{2}}\, e^{-2\pi n x}\, J_\nu\left(\sqrt{2\pi n}\, y\right)$$
$$= \frac{(-1)^{k/2}\pi^{\nu/2} y^\nu x^{-k}}{2^{\nu/2}\Gamma(\nu+1)} e^{-\frac{y^2}{4x}} \sum_{n=1}^{\infty} a_f(n) e^{-\frac{2\pi n}{x}}\, \Phi_3\left(1-k+\nu; \nu+1; \frac{y^2}{4x}, \frac{\pi y^2 n}{2x^2}\right). \qquad (3.6.7)$$

Note that (3.6.5) and (3.6.7) reduce to (3.2.14) and (3.4.2) when $\nu = \frac{\alpha}{2} - 1$ and $\nu = k-1$, respectively. This is the case because the Humbert function satisfies the reduction formula

$$\Phi_3\left(0; c; w, z\right) = \Gamma(c)\, z^{-\frac{c-1}{2}} I_{c-1}\left(2\sqrt{z}\right),$$

which is an immediate consequence of the power series (3.6.6). A plethora of new results can be derived when the hypergeometric function ${}_1F_1\left(a;c;z\right)$, contained in the shifted combinations (3.6.1) and (3.6.2), is replaced by the generalized hypergeometric function ${}_2F_2\left(a,b;c,d;z\right)$ or by Gauss' function ${}_2F_1(a,b;c;z)$.

Another direction that this work can take concerns even more general shifted combinations of completed Dirichlet series. To continue, recall again the notation $\eta(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$. In the paper [73], the authors have posed the following interesting problem: if one considers a combination

[17] Although the identity (3.6.5) may not be of inherent interest, it serves a crucial role in deriving new summation formulas for expressions of the form $\sum_{n=1}^{\infty} r_k(n) n^{\frac{\nu-\mu}{2}} I_\mu\left(Y\sqrt{n}\right) K_\nu\left(X\sqrt{n}\right)$, where $X > Y$ and $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$ (cf. Theorem 4.4.1 below). This development represents a generalization of a formula established by Berndt, Dixit, Kim, and Zaharescu [43], which will be presented in the following chapter.

of shifted products of the form

$$H_z(s) := \sum_{m,n=1}^{\infty} c_{m,n}\, \eta\left(s+i\lambda_m\right)\, \eta\left(s+i\lambda_n\right) \\ \times \operatorname{Re}\left({}_1F_1\left(\frac{1-s-i\lambda_m}{2};\, \frac{1}{2};\, \frac{z^2}{4}\right)\right) \operatorname{Re}\left({}_1F_1\left(\frac{1-s-i\lambda_n}{2};\, \frac{1}{2};\, \frac{z^2}{4}\right)\right), \quad (3.6.8)$$

under what circumstances will $H_z(s)$ have infinitely many zeros on the critical line? Even when restricted to $z = 0$, it seems very complicated to formulate a correct conjecture to attack. Indeed, consider the case where $c_{m,n} := d_m\, \delta_{m,n}$, $d_m \geq 0$, and set $z = 0$ in (3.6.8): we are then dealing with the study of the zeros of the function

$$\mathcal{H}(s) = \sum_{m=1}^{\infty} d_m\, \eta^2\left(s+i\lambda_m\right), \quad d_m \geq 0. \quad (3.6.9)$$

It is simple to give an example where $\mathcal{H}(s)$ does not have a single zero located at the critical line $\operatorname{Re}(s) = \frac{1}{2}$. For some sufficiently large M, consider the finite sequence $\lambda_m = m$, $1 \leq m \leq M$, and assume that $d_j = 0$ when $m \geq M+1$. If the function $\mathcal{H}(s) := \sum_{1\leq m\leq M} d_m\, \eta^2\left(s+im\right)$ had a zero on the critical line, say $s_0 = \frac{1}{2} + i\tau$, then, due to the non-negativity of d_m, one would deduce that

$$\eta\left(\frac{1}{2} + i(\tau+m)\right) = 0, \quad \text{for } 1 \leq m \leq M. \quad (3.6.10)$$

But (3.6.10) is impossible to meet when M is sufficiently large, because every arithmetical sequence contains infinitely many elements which are not zeros of $\zeta(z)$.[18] Therefore, it seems difficult to secure a correct result regarding combinations of shifted products. However, one may conjecture that a combination of the form (3.6.9) has infinitely many critical zeros for "infinitely many" pairs of bounded shifts.

Conjecture 3.6.1. *Let $(c_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence of non-zero real numbers such that $\sum_{j,k=1}^{\infty} |c_{j,k}| < \infty$ and $(\lambda_j)_{j\in\mathbb{N}}$, $(\lambda_k')_{k\in\mathbb{N}}$ be two bounded sequences of distinct real numbers that attain their bounds. Then there are infinitely many $\tau \neq 0$ such that the function*

$$G(s;\, \tau) := \sum_{j,k=1}^{\infty} c_{j,k}\, \eta_{\alpha}\left(s+i\lambda_j\right)\, \eta_{\alpha}\left(s+i\tau+i\lambda_k'\right)$$

possesses infinitely many zeros on the critical line $Re(s) = \frac{\alpha}{4}$.

Lastly, we should address a situation where a zero counting problem, similar to those examined in this chapter, yields a more precise estimate than the ones provided by our results. Specifically,

[18] This is a beautiful observation (with a simple proof) made by Putnam [173, 174]. Note that, if we pick $\lambda_m := \frac{2\pi m}{\log(2)}$ in (3.6.9), then our reasoning above would also contradict a very interesting result due to Steuding and Wegert [216], which states that

$$\frac{1}{M}\sum_{0\leq m<M} \zeta\left(\frac{1}{2} + i\tau + i\,\frac{2\pi m}{\log(2)}\right) = \frac{1}{1-2^{-\frac{1}{2}-i\tau}} + O\left(\frac{\log(M)}{\sqrt{M}}\right), \quad M \to \infty.$$

under the conditions outlined in Theorem 3.1.3, one can explore the problem of counting the zeros of the function

$$\mathcal{Z}(t) := \sum_{j=1}^{\infty} c_j \, Z(t+\lambda_j), \tag{3.6.11}$$

where $Z(t)$ represents Hardy's $Z-$function [120],

$$Z(t) := \zeta\left(\frac{1}{2}+it\right)\left(\chi\left(\frac{1}{2}+it\right)\right)^{-1/2}, \quad \chi(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s). \tag{3.6.12}$$

Building on Ingham's work [118], and following an approach very similar to Atkinson's [7], Hall [[104], p. 103, Theorem 5] deduced a curious integral formula involving the product of $\mathcal{Z}(t)$ with a shifted version of itself. He proved the asymptotic formula

$$\int_0^T \mathcal{Z}(t)\, \mathcal{Z}\left(t+\frac{\beta}{\log(T)}\right) dt = \sum_{j=1}^{\infty} c_j^2 \cdot \frac{\sin(\beta/2)}{\beta/2} T\log(T) + O(T), \tag{3.6.13}$$

where $0<\beta<1$. A formula akin to (3.6.13) was derived for the first time by Atkinson [7] [19]and it was used by him to give a very short proof that the Riemann zeta function has $\gg T/\log(T)$ critical zeros[20]. Assuming that $\sum c_j^2 < \infty$ (which is actually implied by the condition $\sum |c_j| < \infty$ present in our Theorems 3.1.3 and 3.1.4), Hall proved that, if $\mathcal{N}_0(T)$ denotes the number of critical zeros, $s=\frac{1}{2}+it,\ 0<t<T$, of $\mathcal{Z}(t)$, then $\mathcal{N}_0(T)$ obeys to the estimate

$$\mathcal{N}_0(T) \gg \frac{T}{\log(T)}. \tag{3.6.14}$$

While the function described in (3.6.11) differs significantly from the shifted combinations in (3.1.20), it is reasonable to conjecture that lower bounds similar to (3.6.14) might also apply to the shifted combinations discussed in this chapter. We now state a conjecture which, if proven, would significantly improve the result given in Corollary 3.1.1.

Conjecture 3.6.2. *Assume the conditions of Corollary 3.1.1 and, again, let $N_{0,\alpha}(T)$ denote the number of critical zeros, $s=\frac{\alpha}{4}+it$, $0\leq t\leq T$, of the following function*

$$\tilde{F}_\alpha(s) = \sum_{j=1}^{\infty} c_j \left\{\pi^{-s-i\lambda_j}\Gamma(s+i\lambda_j)\,\zeta_\alpha(s+i\lambda_j) + \pi^{-s+i\lambda_j}\,\Gamma(s-i\lambda_j)\,\zeta_\alpha(s-i\lambda_j)\right\}.$$

Then there exists some constant $c>0$ such that

$$\liminf_{T\to\infty} \frac{N_{0,\alpha}(T)}{T} \geq c. \tag{3.6.15}$$

As it can be seen from Jacobi's four-square Theorem,

$$\zeta_4(s) = 8(1-2^{2-2s})\,\zeta(s)\,\zeta(s-1),$$

[19]The reader may check [[[120], p. 121] to find a careful analysis of the main differences between the results of Hall and Atkinson.

[20]see the above introduction or Baluyot's paper [12] for a concise description of Atkinson's results.

the estimate (3.6.15) cannot be in general improved to $N_{0,\alpha}(T) \gg T\,\log(T)$. By the result of Siegel already mentioned at the introduction to this chapter, whenever $\alpha \in \mathbb{N}_{\geq 4}$, $\zeta_\alpha(s)$ has $\asymp T$ zeros of the form $s = \frac{\alpha}{4} + it$, $0 \leq t \leq T$. This result underscores why (3.6.15) might be the best estimate that is possible to prove. In any case, we conjecture that for $\zeta(s)$, which corresponds to the case where $\alpha = 1$, the following estimate holds

$$\liminf_{T\to\infty} \frac{N_{0,1}(T)}{T\log(T)} \geq c > 0, \tag{3.6.16}$$

which can be seen as another generalization of Selberg's famous result (3.1.4).

Following the structure of this chapter, one may also formulate the following conjecture for $L-$functions associated with holomorphic cusp forms. If true, this result would constitute a massive generalization of Hafner's theorem [102], presented in a different setting than the one considered by Rezvyakova [182].

Conjecture 3.6.3. *Assume the conditions of Corollary 3.1.3 and, again, let $N_{0,f}(T)$ denote the number of critical zeros, $s = \frac{k}{2} + it,\ \ 0 \leq t \leq T$, of the following function*

$$\tilde{G}_f(s) = \sum_{j=1}^{\infty} c_j \left\{(2\pi)^{-s-i\lambda_j}\Gamma\left(s+i\lambda_j\right)\,L\left(s+i\lambda_j, f\right) + (2\pi)^{-s+i\lambda_j}\,\Gamma\left(s-i\lambda_j\right)\,L\left(s-i\lambda_j, f\right)\right\}.$$

Then there exists some constant $d > 0$ such that

$$\liminf_{T\to\infty} \frac{N_{0,f}(T)}{T\log(T)} \geq d.$$

Chapter 4

A class of Summation Formulas attached to products of Bessel functions

4.1 Introduction

The insights contained in Ramanujan's lost notebook [179] still inspire a lot of new mathematical ideas. Recently, the discovery of some overlooked works from the Russian school, namely from the mathematician N. S. Koshliakov [132] has produced a similar effect on mathematical research (cf. [40, 43, 68] and our next chapter). Belonging to the same tradition as Koshliakov, the mathematician and linguist Alexander Ivanovich Popov (1899-1973) made his contribution to Mathematics by providing new interesting summation formulas.

In a fairly unknown paper [168], Popov states the following beautiful result. If $\text{Re}(x) > 0$ and $z \in \mathbb{C}$, then

$$\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{z^2/8}+\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi n x}z)$$
$$=\frac{z^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}x^{-\frac{k}{4}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}\,e^{-z^2/8}+\frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}z\right), \tag{4.1.1}$$

where $r_k(n)$ denotes the number of representations of the positive integer n as a sum of k squares and, as usual, $J_\nu(z)$ and $I_\nu(z)$ respectively denote the Bessel and modified Bessel functions of the first kind. A couple of reasons why this identity is fascinating are already provided by Berndt, Dixit, Kim and Zaharescu in [[42], pp. 3795-3796]. For the purposes of our discussion, we shall enumerate them.[1]

[1] In the introduction to this chapter we will follow closely the brilliant expositions that can be found in [39, 42, 43].

1. If we construct the Dirichlet series attached to $r_k(n)$,

$$\zeta_k(s)=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^s},\quad \text{Re}(s)>\frac{k}{2},\tag{4.1.2}$$

then $\zeta_k(s)$ can be continued to the complex plane as a meromorphic function possessing only a simple pole at $s=\frac{k}{2}$ with residue $\pi^{k/2}/\Gamma(k/2)$. Moreover, as we have seen in countless ways before in this thesis, it satisfies the functional equation

$$\eta_k(s):=\pi^{-s}\Gamma\left(s\right)\zeta_k(s)=\pi^{s-\frac{k}{2}}\Gamma\left(\frac{k}{2}-s\right)\zeta_k\left(\frac{k}{2}-s\right):=\eta_k\left(\frac{k}{2}-s\right).\tag{4.1.3}$$

Note that, when $k=1$, $r_1(n)=2$ if and only if n is a perfect square and zero otherwise. Therefore, (4.1.2) reduces to

$$\zeta_1(s):=\sum_{n=1}^{\infty}\frac{r_1(n)}{n^s}=2\sum_{n=1}^{\infty}\frac{1}{n^{2s}}=2\zeta(2s),\quad \text{Re}(s)>\frac{1}{2}.\tag{4.1.4}$$

Furthermore, (4.1.3) with $k=1$ gives the functional equation for Riemann's $\zeta-$function

$$\pi^{-s}\Gamma\left(s\right)\zeta(2s)=\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1}{2}-s\right)\zeta\left(1-2s\right).\tag{4.1.5}$$

The first point highlighted in [42] is that the powers of n in the denominators of both sides of (4.1.1) are remindful of the functional equation (4.1.3).

2. Riemann's second proof of the functional equation for $\zeta(s)$, (4.1.5), employs the transformation formula for Jacobi's $\theta-$function,

$$\theta(x):=\sum_{n\in\mathbb{Z}}e^{-\pi n^2x}=\frac{1}{\sqrt{x}}\sum_{n\in\mathbb{Z}}e^{-\frac{\pi n^2}{x}}:=\frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right),\quad \text{Re}(x)>0.\tag{4.1.6}$$

The theta transformation formula associated to the Dirichlet series $\zeta_k(s)$ can be obtained by taking the k^{th} power on both sides of (4.1.6). This results in

$$\sum_{n=0}^{\infty}r_k(n)\,e^{-\pi nx}=x^{-\frac{k}{2}}\sum_{n=0}^{\infty}r_k(n)\,e^{-\frac{\pi n}{x}},\quad \text{Re}(x)>0,\tag{4.1.7}$$

where we take the convention $r_k(0):=1$.[2] Of course, the exponential factors on both sides of (4.1.1) remind us of the theta transformation formula (4.1.7). In fact, (4.1.7) is a particular case of (4.1.1) when we let $z\to 0$, due to the limiting relations for the Bessel functions [[164], p. 223, eq. (10.7.3)]

$$\lim_{y\to 0}y^{-\nu}J_{\nu}(y)=\lim_{y\to 0}y^{-\nu}I_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)}.$$

[2]Which makes perfect sense because there is only one way in which 0 can be expressed as a sum of any number of squares. It also agrees with the convention given in item 4 of Definition 1.1.1 in the first chapter.

3. Chandrasekharan and Narasimhan [[53], p. 19, eq. (65)] proved yet another equivalent identity to (4.1.3) and (4.1.7). If $x>0$ and $q>\frac{k-1}{2}$, then

$$\frac{1}{\Gamma(q+1)}\sum_{0\leq n\leq x}{}^{'} r_k(n)\,(x-n)^q=\frac{\pi^{\frac{k}{2}}x^{\frac{k}{2}+q}}{\Gamma\left(q+1+\frac{k}{2}\right)}+\pi^{-q}\sum_{n=1}^{\infty}r_k(n)\left(\frac{x}{n}\right)^{\frac{k}{4}+\frac{q}{2}}J_{\frac{k}{2}+q}\left(2\pi\sqrt{nx}\right), \tag{4.1.8}$$

where the Bessel series on the right-hand side converges absolutely. The prime on the summation sign indicates that, if $q=0$ and x is an integer, then the last contribution in this Riesz sum is just $\frac{1}{2}r_k(x)$. The appearance of the Bessel functions in (4.1.1) reminds us of (4.1.8).

Berndt, Dixit, Kim and Zaharescu [42] emphasized the importance of Popov's result by explaining its connection with the formulas stated in the previous three items. We would like to add another item to the list, which is perhaps more directly related to the second item above. When $k=1$, (4.1.1) gives the identity[3]

$$x^{\frac{1}{4}}\,e^{z^2/8}\left\{1+2\sum_{n=1}^{\infty}e^{-\pi n^2x}\cos\left(\sqrt{\pi x}\,nz\right)\right\}=x^{-\frac{1}{4}}\,e^{-z^2/8}\left\{1+2\sum_{n=1}^{\infty}e^{-\frac{\pi n^2}{x}}\cosh\left(\sqrt{\frac{\pi}{x}}\,nz\right)\right\}, \tag{4.1.9}$$

which is mainly due to the particular cases of the Bessel functions [[221], p. 248] (cf. (2.2.20) above),

$$J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos(x),\quad I_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cosh(x).$$

Popov provided a sketch of a proof of his beautiful formula (4.1.1). However, as remarked in [[42], p. 3796], his proof contained some flaws and the argument was purely formal. To give some context into Popov's argument, we start by briefly commenting on the work of Voronoï.

In his general study of summation formulas, Voronoï [24, 215, 230] stated for the first time the following identity

$$\sum_{0<n\leq x}{}^{'} r_2(n)=\pi x-1+\sqrt{x}\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{n}}J_1\left(2\pi\sqrt{nx}\right), \tag{4.1.10}$$

although disclaiming to have a fully rigorous proof. The first known proof of (4.1.10) seems to have been given by Hardy [105] during his study on the average order of the arithmetical function $r_2(n)$. Another proof of (4.1.10) was later provided by Hardy and Landau [107, 108]. For more details about Voronoï's summation formula, we refer to the papers [24, 39, 157], where several proofs (4.1.10) are mentioned.[4] Popov claimed that, for any $k\geq 2$, the generalization of (4.1.10)

[3] This implication was proved in a general form in Corollary 2.2.1 above.

[4] We would like to mention, as a footnote, our favorite argument to obtain (4.1.10), due to Ivić [121]. Ivić's proof uses the injectivity of the Laplace transform and the transformation formula for the theta function

$$\sum_{n=0}^{\infty}r_2(n)\,e^{-\pi nx}=\frac{1}{x}\sum_{n=0}^{\infty}r_2(n)\,e^{-\frac{\pi n}{x}},\quad \mathrm{Re}(x)>0,$$

resulting in a very short argument.

takes place

$$\sum_{0<n\leq x}{}^{\prime} r_k(n) = \frac{(\pi x)^{k/2}}{\Gamma\left(\frac{k}{2}+1\right)} - 1 + x^{\frac{k}{4}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}}} J_{\frac{k}{2}}\left(2\pi\sqrt{nx}\right), \tag{4.1.11}$$

and he employed this very strong version of (4.1.8) to derive an analogue of Voronoï's summation formula

$$\begin{aligned}\lim_{x\to 0} \frac{h(x)}{x^{\frac{k}{4}-\frac{1}{2}}} &+ \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} h(n) \\ = \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \int_0^{\infty} x^{\frac{k}{4}-\frac{1}{2}} h(x)\, dx &+ \pi \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} \int_0^{\infty} h(x)\, J_{\frac{k}{2}-1}\left(2\pi\sqrt{nx}\right) dx,\end{aligned} \tag{4.1.12}$$

where h is some "suitable function".[5] First, as noted in [42, 43], Popov does not impose any conditions on $h(x)$ for the validity of the formula (4.1.12). Besides, in most versions of Voronoï's summation formula, such as (4.1.12), the conditions imposed on $h(x)$ are often too restrictive to directly derive (4.1.1) from an application of (4.1.12).

Popov asserted (4.1.11) because he continued to work under the assumption that (4.1.8) is valid for any $q > -1$. This stands in sharp contrast to the condition imposed by Chandrasekharan and Narasimhan [53, Theorem III, p. 14], which requires that $q > \frac{k-3}{2}$ and appears to be optimal. Using a variant of Voronoï's summation formula due to Koshliakov, which applies to analytic functions and whose proof does not rely on (4.1.8), Berndt, Dixit, Kim and Zaharescu [42] provided the first rigorous proof of (4.1.1).

Aiming to study the zeros of shifted combinations of Dirichlet series, we have established, independently of Popov, an extension of (4.1.1) to a class of Hecke Dirichlet series.[6] One of the main ideas permeating our second chapter is that (4.1.1) can be achieved through an integral representation of the form[7]

$$\begin{aligned}&\sqrt{x}\, e^{z^2/8} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\pi n x}\, J_{\frac{k}{2}-1}(\sqrt{\pi n x}\, z) - \frac{\pi^{\frac{k}{4}-\frac{1}{2}} z^{\frac{k}{2}-1} e^{-z^2/8}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right) x^{\frac{k}{4}}} \\ &= \frac{e^{-z^2/8}}{\sqrt{x}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\frac{\pi n}{x}}\, I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\, z\right) - \frac{\pi^{\frac{k}{4}-\frac{1}{2}} z^{\frac{k}{2}-1} x^{\frac{k}{4}}\, e^{z^2/8}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \\ &= \left(\frac{\sqrt{\pi} z}{2}\right)^{\frac{k}{2}-1} \frac{e^{z^2/8}}{2\pi\Gamma\left(\frac{k}{2}\right)} \int_{-\infty}^{\infty} \pi^{-\frac{k}{4}-it}\Gamma\left(\frac{k}{4}+it\right) \zeta_k\left(\frac{k}{4}+it\right) {}_1F_1\left(\frac{k}{4}+it;\, \frac{k}{2};\, -\frac{z^2}{4}\right) x^{-it}\, dt.\end{aligned} \tag{4.1.13}$$

[5] When $k \geq 3$, the series on the right-hand side of (4.1.11) does not converge but can be studied as a "Riesz summable" series. For details, we refer to the works of Dixon, Ferrar and Oppenheim [78, 165]. The Riesz sum (4.1.11) is given in Oppenheim's paper [[165], p. 297, Theorem 2] and Oppenheim proves that the series on the right-hand side of (4.1.11) is Riesz summable $\left(R, n, \frac{k-3}{2}+\epsilon\right)$, for any $\epsilon > 0$. However, it is not clear how Popov could use (4.1.11) to prove (4.1.12) rigorously.

[6] Such extension was stated for the first time by Berndt in [24]. Berndt used a generalized version of Voronoï's summation formula (4.1.12). Note that, throughout this thesis, we have stubbornly avoided this approach of proving general summation formulas and we have always prefered to start with a Mellin-Barnes integral and, afterwards, study its symmetries.

[7] When $k = 1$, (4.1.13) gives a formula due to Dixit [[63], p. 374, eq. (1.13)].

The main advantage of the formula (4.1.13) is that Popov's result (4.1.1) can be now expressed as a direct consequence of an integral representation. For instance, (4.1.13) allowed us to show that a shifted combination of the form

$$F_{z,\alpha}(s)=\sum_{j=1}^{\infty}c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_k(s+i\lambda_j)$$
$$\times\left\{{}_1F_1\left(\frac{k}{2}-s-i\lambda_j;\,\frac{k}{2};\,\frac{z^2}{4}\right)+{}_1F_1\left(\frac{k}{2}-\overline{s}+i\lambda_j;\,\frac{k}{2};\,\frac{\overline{z}^2}{4}\right)\right\}$$

contains infinitely many zeros at its critical line. Moreover, the proof via the integral representation (4.1.13) is far simpler than any proof that requires the use of Voronoï's summation formula (4.1.12).

Besides helping in the study of the zeros of shifted combinations of the Dirichlet series $\zeta_k(s)$, this formula of Popov proved to be very useful to derive new identities. In particular, for $\text{Re}(\nu)>0$ and $x,y>0$, we have been able to use (4.1.1) to deduce the following formula (see Corollary 2.2.4 and (2.2.74) above)

$$\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)=-\frac{\Gamma(\nu)\pi^{\frac{k}{2}-\nu-1}y^{\frac{k}{2}-1}}{2x^{\nu}\,\Gamma\left(\frac{k}{2}\right)}$$
$$+\frac{y^{\frac{k}{2}-1}x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\,\Gamma\left(\frac{k}{2}\right)\,\pi^{\nu+1}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+x^2+y^2)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4ny^2}{(n+x^2+y^2)^2}\right),\quad(4.1.14)$$

which, as we aim to clarify below, serves as a key motivation for this entire chapter.

As we have seen in the second chapter of this thesis, it is possible to rewrite (4.1.14) by replacing the $J-$Bessel function by a modified $I-$Bessel function. In fact, if $x>y>0$, an equivalent version of (4.1.14) reads

$$\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)=-\frac{\Gamma(\nu)\pi^{\frac{k}{2}-\nu-1}y^{\frac{k}{2}-1}}{2x^{\nu}\,\Gamma\left(\frac{k}{2}\right)}$$
$$+\frac{y^{\frac{k}{2}-1}x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\,\Gamma\left(\frac{k}{2}\right)\,\pi^{\nu+1}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+x^2-y^2)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};-\frac{4ny^2}{(n+x^2-y^2)^2}\right).\quad(4.1.15)$$

These formulas involving products of Bessel functions yielded interesting particular cases. For example, in (2.2.75), we have obtained the interesting summation formula

$$\frac{1}{\sqrt{x^2+y^2}}+2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{n^2+x^2+y^2+\sqrt{n^4+2n^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}}{\sqrt{n^4+2n^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}-\frac{1}{n}\right\}$$
$$=2\log\left(\frac{2}{x}\right)-2\gamma+4\sum_{n=1}^{\infty}\cos\left(2\pi ny\right)K_0\left(2\pi nx\right).$$

Other particular cases immediately arise from (4.1.14) and (4.1.15). If we let $y \to 0^{+}$ in both (4.1.14) and (4.1.15), we are able to obtain

$$\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,x^{\nu}}{2\pi^{\nu+\frac{k}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+x^2)^{\nu+\frac{k}{2}}}=\frac{\Gamma(\nu)}{2\pi^{\nu}x^{\nu}}+\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu}{2}}K_{\nu}\left(2\pi\sqrt{n}\,x\right), \tag{4.1.16}$$

being a valid identity for any $x>0$ and $\text{Re}(\nu)>0$.[8] Curiously, just like (4.1.1), we owe (4.1.16) to Popov [169]. Note that, due to the limiting relation for the Macdonald function [[164], p. 252, eq. 10.30.2]

$$\lim_{x\to 0}x^{\nu}\,K_{\nu}(x)=2^{\nu-1}\Gamma(\nu),\quad \text{Re}(\nu)>0, \tag{4.1.17}$$

one may find an interpretation for the term "$n=0$" in the Bessel series (4.1.20) and then rewrite (4.1.20) in a more compact form,[9]

$$\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,x^{\nu}}{2\pi^{\nu+\frac{k}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+x^2)^{\nu+\frac{k}{2}}}=\sum_{n=0}^{\infty}r_k(n)\,n^{\frac{\nu}{2}}K_{\nu}\left(2\pi\sqrt{n}\,x\right). \tag{4.1.18}$$

There are several particular cases of (4.1.16) that we found worth highlighting repeatedly throughout this thesis. The most elementary particular case is when we take $k=1$. Since $\zeta_1(s):=2\zeta(2s)$, one sees that (4.1.16) reduces to the formula

$$\frac{\Gamma\left(\nu+\frac{1}{2}\right)\,x^{\nu}}{\pi^{\nu+\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{1}{(n^2+x^2)^{\nu+\frac{1}{2}}}=\frac{\Gamma(\nu)}{2\pi^{\nu}x^{\nu}}+2\sum_{n=1}^{\infty}n^{\nu}K_{\nu}\left(2\pi nx\right),\quad x,\ \text{Re}(\nu)>0, \tag{4.1.19}$$

which was established by Watson during his formal study of self-reciprocal functions with respect to Hankel's transform [233], four years before the appearance of (4.1.18) in Popov's work.

Yet another interesting particular case of (4.1.16) emerges when we set $k=2$, resulting in the beautiful formula

$$\frac{\Gamma\left(\nu+1\right)\,x^{\nu}}{2\pi^{\nu+1}}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+x^2)^{\nu+1}}=\frac{\Gamma(\nu)}{2\pi^{\nu}x^{\nu}}+\sum_{n=1}^{\infty}r_2(n)\,n^{\frac{\nu}{2}}K_{\nu}\left(2\pi\sqrt{n}\,x\right), \tag{4.1.20}$$

which is valid under the restrictions $\text{Re}(\nu)>0$ and $x>0$. Analogously to (4.1.18), this particular formula can be written in the more compact form

$$\frac{\Gamma\left(\nu+1\right)\,x^{\nu}}{2\pi^{\nu+1}}\sum_{n=0}^{\infty}\frac{r_2(n)}{(n+x^2)^{\nu+1}}=\sum_{n=0}^{\infty}r_2(n)\,n^{\frac{\nu}{2}}K_{\nu}\left(2\pi\sqrt{n}\,x\right). \tag{4.1.21}$$

Using a variant of Voronoï's summation formula (4.1.12), Dixon and Ferrar [77] proved (4.1.20) for the first time. It is reasonable to anticipate that several interesting formulas may emerge as direct corollaries of (4.1.20). As remarked in [43], when we set $\nu=\frac{1}{2}$ and use the elementary

[8] In fact, it valid for a much more general case, where x is complex and satisfies $\text{Re}(x)>0$.

[9] Throughout this chapter, we shall use the different versions (4.1.20) (4.1.21) in the contexts in which one or the other is more convenient. For example, we have recorded the compact version (4.1.18) before in this thesis as formula (2.2.45) of the second chapter.

reduction formula [[164], p. 254, 10.39.2]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}, \tag{4.1.22}$$

one is able to deduce a result of Hardy [105]

$$\sum_{n=1}^{\infty} r_2(n)\, e^{-\sqrt{n}x} = \frac{2\pi}{x^2} - 1 + 2\pi x \sum_{n=1}^{\infty} \frac{r_2(n)}{(x^2+4\pi^2 n)^{\frac{3}{2}}}, \qquad x>0. \tag{4.1.23}$$

Hardy [36, 105] readily employed (4.1.23) to derive a lower bound for the error term in the circle problem, which he proved to be[10]

$$P(x) := \sum_{0\leq n\leq x}{}^{'} r_2(n) - \pi x = \Omega_{\pm}\left(x^{\frac{1}{4}}\right). \tag{4.1.24}$$

At the end of his paper, Hardy concluded that his formula (4.1.23) could be presented as a particular case of a beautiful formula of Ramanujan, which says that, for $a, b > 0$,[11]

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{b(n+a)}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{a(n+b)}}. \tag{4.1.25}$$

According to [[36], p. 108] and [[32], p. 126], this identity is not given elsewhere in any of Ramanujan's published or unpublished work. The beautiful symmetry of the parameters a and b in (4.1.25) is indeed striking. Hardy's sketched proof of (4.1.25) used the transformation formula for Jacobi's theta function. The full details of Hardy's suggested proof were completed by Chan and Toh [52] and these authors remark *"We surmise that our approach is very similar to that of Ramanujan."*. As we hope to show later (see Remark 4.2.2), the proof suggested by Hardy, which takes the point of view of the theta transformation formula, may not be the initial approach taken by Ramanujan to deduce (4.1.25).

Besides proving (4.1.20), Dixon and Ferrar also gave a proof of (4.1.25) following a totally different principle than the one sketched by Hardy. In fact, using again their variant of Voronoï's summation formula and invoking some integral evaluations of Sonin type, they obtained the general formula

$$\sum_{n=0}^{\infty} r_2(n) \left(\frac{b}{n+a}\right)^{\frac{\nu}{2}} K_{\nu}\left(2\pi\sqrt{(n+a)\,b}\right) = \sum_{n=0}^{\infty} r_2(n) \left(\frac{a}{n+b}\right)^{\frac{1-\nu}{2}} K_{1-\nu}\left(2\pi\sqrt{(n+b)\,a}\right), \tag{4.1.26}$$

which is valid for any $\nu\in\mathbb{C}$ and $a \geq 0$, $\mathrm{Re}\left(\sqrt{b}\right) > 0$.[12]. This interesting result generalizes both Ramanujan's formula (4.1.25) and the particular case of Popov's formula (4.1.20). Ramanujan's formula easily results from (4.1.26) after we set $\nu=\frac{1}{2}$ and then use (4.1.22). Obtaining (4.1.20)

[10] i.e., there exist two sequences of real numbers $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ tending to infinity and two positive constants A and B such that, for every $n\in\mathbb{N}$, $P(x_n) > Ax_n^{1/4}$ and $P(y_n) < -B\, y_n^{1/4}$.

[11] in fact, it is valid for the much wider region $\mathrm{Re}(a), \mathrm{Re}(b) > 0$, but for the sake of exposition of our next results we will keep all these parameters as real and positive.

[12] It is possible to extend this to $\mathrm{Re}(a)\geq 0$ and $\mathrm{Re}\left(\sqrt{b}\right) > 0$ as later shown in [23, 37, 41].

needs some further restrictions on (4.1.26): for example, if we replace ν by $-\nu$ in (4.1.26) and then suppose that $\text{Re}(\nu) > 0$, (4.1.26) implies

$$\sum_{n=0}^{\infty} r_2(n) \left(\frac{b}{n+a}\right)^{-\frac{\nu}{2}} K_{\nu}\left(2\pi\sqrt{(n+a)\,b}\right) = \sum_{n=0}^{\infty} r_2(n) \left(\frac{a}{n+b}\right)^{\frac{\nu+1}{2}} K_{\nu+1}\left(2\pi\sqrt{(n+b)\,a}\right), \tag{4.1.27}$$

and so, if we let $a \to 0^{+}$ in (4.1.27) and invoke the limiting relation (4.1.17), we find that Popov's formula (4.1.21) must hold, this is

$$\sum_{n=0}^{\infty} r_2(n)\, n^{\frac{\nu}{2}}\, K_{\nu}\left(2\pi\sqrt{n\,b}\right) = \frac{\Gamma(\nu+1) b^{\frac{\nu}{2}}}{2\pi^{\nu+1}} \sum_{n=0}^{\infty} \frac{r_2(n)}{(n+b)^{\nu+1}}.$$

The collaboration between Dixon and Ferrar continued to generate new ideas in the theory of summation formulas. In the sequel to the paper where (4.1.27) is proved, these authors studied, for example, the summability in Cesàro means of a series of the form

$$\sum_{n=0}^{\infty} r_2(n)\, J_0(2\pi\sqrt{n\lambda}),$$

where $\lambda \in \mathbb{N}$ is such that $r_2(\lambda) = 0$. They also extended earlier proofs, primarily due to Wilton [238, 239], of Poisson's and Voronoï's formulas to a broader class of functions.

As stated in [[24], p. 140], Ferrar seems to have been the first mathematician to clearly draw a connection between the functional aspects of a summation formula and the behavior of the Dirichlet series underlying it. Around the same time as the formula (4.1.27) was published, Ferrar embarked on a study of summation formulas that possess symmetric properties [89]. To give an example, let us recall Koshliakov's formula (1.2.30), introduced in Example 1.2.1 of the Chapter 1,

$$\sqrt{x}\left\{\gamma - \log\left(\frac{4\pi}{x}\right) + 4\sum_{n=1}^{\infty} d(n)\, K_0\left(2\pi n x\right)\right\} = \frac{1}{\sqrt{x}}\left\{\gamma - \log\left(4\pi x\right) + 4\sum_{n=1}^{\infty} d(n)\, K_0\left(\frac{2\pi n}{x}\right)\right\}, \tag{4.1.28}$$

where $d(n) := \sum_{d|n} 1$ is the classical divisor function and $x > 0$. Just like (4.1.26) or the theta transformation formula (4.1.7), this particular formula is quite symmetrical. A formal way to view it is as just one example of a "modular" relation of the form

$$\mathcal{F}(x) = \mathcal{F}\left(\frac{1}{x}\right), \quad x > 0, \tag{4.1.29}$$

where $\mathcal{F}$ is a function of arithmetical interest.[13]

[13]In the case of (4.1.28), $\mathcal{F}(x)$ represents an infinite series containing $d(n)$ and some residual terms.

Besides Koshliakov's formula (4.1.28), Ferrar provided another interesting transformation of the form (4.1.29). He showed that, if α and β are two positive real numbers such that $\alpha\beta = 1$, then

$$\begin{aligned}&2\sqrt{\alpha}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\alpha^2}{2}}K_0\left(\frac{\pi n^2\alpha^2}{2}\right)-\frac{1}{\alpha n}\right\}+\frac{1}{\sqrt{\alpha}}\left(\gamma-\log(16\pi)-2\log\left(\alpha\right)\right)\\&=2\sqrt{\beta}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\beta^2}{2}}K_0\left(\frac{\pi n^2\beta^2}{2}\right)-\frac{1}{\beta n}\right\}+\frac{1}{\sqrt{\beta}}\left(\gamma-\log(16\pi)-2\log\left(\beta\right)\right).\end{aligned}\tag{4.1.30}$$

Around the same time of the publication of [89], Popov continued his study of summation formulas attached to the arithmetical function $r_2(n)$ [169]. He stated that, if $\mathrm{Re}(\nu) > 0$ and $\alpha \geq \beta > 0$, then the following beautiful formula holds

$$\begin{aligned}&\frac{2\pi}{(\alpha-\beta)^{\nu}}\sum_{n=1}^{\infty}r_2(n)\,I_{\nu}\left(\pi\left(\sqrt{n\alpha}-\sqrt{n\beta}\right)\right)\,K_{\nu}\left(\pi\left(\sqrt{n\alpha}+\sqrt{n\beta}\right)\right)\\&=\frac{\nu-\pi\sqrt{\alpha\beta}}{\nu\sqrt{\alpha\beta}\left(\sqrt{\alpha}+\sqrt{\beta}\right)^{2\nu}}+\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{n+\alpha}\,\sqrt{n+\beta}\left(\sqrt{n+\alpha}+\sqrt{n+\beta}\right)^{2\nu}}.\end{aligned}\tag{4.1.31}$$

Taking the convention that $r_2(0) := 0$, the previous formula can be rewritten in the following compact form

$$\begin{aligned}&\frac{2\pi}{(\alpha-\beta)^{\nu}}\sum_{n=0}^{\infty}r_2(n)\,I_{\nu}\left(\pi\left(\sqrt{n\alpha}-\sqrt{n\beta}\right)\right)\,K_{\nu}\left(\pi\left(\sqrt{n\alpha}+\sqrt{n\beta}\right)\right)\\&=\sum_{n=0}^{\infty}\frac{r_2(n)}{\sqrt{n+\alpha}\,\sqrt{n+\beta}\left(\sqrt{n+\alpha}+\sqrt{n+\beta}\right)^{2\nu}}.\end{aligned}\tag{4.1.32}$$

Popov's result (4.1.31) is already evocative of our identity (4.1.15), because it contains an infinite series involving a product of two modified Bessel functions. Note that, just as Dixon and Ferrar's formula (4.1.27), Popov's result (4.1.32) is also a generalization of (4.1.21)! This can be easily seen once we take $\alpha - \beta \to 0^{+}$ and then use the limiting relation for the Macdonald function, (4.1.17), as well as

$$\lim_{y\to 0}y^{-\nu}I_{\nu}(y)=\frac{2^{-\nu}}{\Gamma(\nu+1)}.\tag{4.1.33}$$

Popov neither gave a proof of (4.1.32) nor did he indicate how to prove it. The authors of [43] speculate about the possible method that could have led Popov to the formula (4.1.32), linking Popov's reasoning to reach (4.1.32) with his flawed, sketched proof of (4.1.1). The main issue in Popov's hypothetical argument seems to be once more his unfounded claim concerning the Riesz sum (4.1.11).

Up to this point, we have discussed two very general formulas, namely (4.1.27) and (4.1.32), that imply (4.1.21). A question that one may ask is whether there are generalizations of (4.1.27) and (4.1.32) when the arithmetical function $r_2(n)$ is replaced by $r_k(n)$, for a general $k \in \mathbb{N}$.

From the point of view of Voronoï's summation formula (4.1.12), the extension of these formulas may seem delicate, because one can no longer rely on a formula of the form (4.1.8) for $q = 0$ and

$k \geq 3$. In any case, there are always clever ways to attain this goal and surpass the technical obstacles posed by this challenge. Based on the general work of Berndt [23], Berndt, Lee and Sohn [[37], p. 39, eq. (5.5)] established the formula

$$\sum_{n=0}^{\infty} r_k(n) \left(\frac{b}{n+a}\right)^{\frac{\nu}{2}} K_{\nu}\left(2\pi\sqrt{(n+a)\,b}\right) = \sum_{n=0}^{\infty} r_k(n) \left(\frac{a}{n+b}\right)^{\frac{k}{4}-\frac{\nu}{2}} K_{\frac{k}{2}-\nu}\left(2\pi\sqrt{(n+b)\,a}\right), \tag{4.1.34}$$

valid for any $\nu \in \mathbb{C}$, $\text{Re}(a)$, $\text{Re}(b) > 0$ and $k \geq 2$. Recently, some new interesting analogues of (4.1.34) have been derived by Berndt, Dixit, Gupta and Zaharescu [41] where, for instance, the role of $r_k(n)$ is replaced by a wide variety of arithmetical functions.

On the other hand, Popov's formula (4.1.32) was generalized by Berndt, Dixit, Kim and Zaharescu [43] to incorporate the arithmetical function $r_k(n)$. We have encountered this generalization before in this thesis, namely in Remark 2.2.4 of the second chapter. The main formula obtained by these authors is [[43], Theorem 1.6]

$$\begin{aligned}\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(\pi\left(\sqrt{n\alpha}-\sqrt{n\beta}\right)\right) K_{\nu}\left(\pi\left(\sqrt{n\alpha}+\sqrt{n\beta}\right)\right) &= -\frac{1}{2\nu}\left(\frac{\sqrt{\alpha}-\sqrt{\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right)^{\nu} \\ + \frac{\Gamma\left(\nu+\frac{k}{2}\right)}{2^{k-1}\pi^{\frac{k}{2}}\,\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\sqrt{n+\alpha}\,\sqrt{n+\beta}} \left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^{\nu} & \\ \times \left(\frac{1}{\sqrt{n+\alpha}}+\frac{1}{\sqrt{n+\beta}}\right)^{k-2} {}_2F_1\left(\nu+1-\frac{k}{2}, 1-\frac{k}{2}; \nu+1; \left(\frac{\sqrt{n+\alpha}-\sqrt{n+\beta}}{\sqrt{n+\alpha}+\sqrt{n+\beta}}\right)^2\right), & \end{aligned} \tag{4.1.35}$$

which is valid for every integer $k \geq 2$, $\text{Re}(\nu) > 0$ and $\text{Re}\left(\sqrt{\alpha}\right) \geq \text{Re}\left(\sqrt{\beta}\right) > 0$.

Like our formulas from the second chapter, namely (4.1.14) and (4.1.15), the identity (4.1.35) seems to be very complicated because it involves Gauss' hypergeometric function in one of its main infinite series. However, as we had the opportunity to explore in the second chapter, these kind of formulas possess many interesting particular cases. We now describe the main steps guiding the proof of (4.1.35).

1. The authors of [43] begin by invoking a variant of Voronoï's summation formula developed by Guinand [98], which they use to rigorously derive the summation formula that Popov left purely formal, (4.1.12). They show that, under certain conditions over a given real function and its Hankel transform[14] [[43], p. 314, Theorem 1.5], the following formula of Voronoï type

[14]These conditions are essentially concerned with the decay at infinity of the function $f(x)$ and of its Hankel transform, $g(x)$, (4.1.37). Similar conditions for (4.1.36) to hold have been studied by Yakubovich in [246] and the class of functions there introduced, the so-called Müntz class, seems appropriate to apply in the study given in [43]. See also [145] and the fourth chapter of [190] for a variant of (4.1.36) in the case $k = 2$.

holds

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}f(n)-\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\int_0^{\infty}x^{\frac{k}{4}-\frac{1}{2}}f(x)\,dx$$
$$=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}g(n)-\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\int_0^{\infty}x^{\frac{k}{4}-\frac{1}{2}}g(x)\,dx, \tag{4.1.36}$$

where $g(x)$ is defined as the Hankel transform

$$g(x):=\pi\int_0^{\infty}f(t)\,J_{\frac{k}{2}-1}\left(2\pi\sqrt{xt}\right)\,dt. \tag{4.1.37}$$

2. After proving the summation formula (4.1.36), they apply it to the function

$$f_{\alpha,\beta}(x):=\frac{x^{\frac{k}{4}-\frac{1}{2}}}{\sqrt{x+\alpha}\,\sqrt{x+\beta}}\left(\frac{\sqrt{x+\alpha}-\sqrt{x+\beta}}{\sqrt{x+\alpha}+\sqrt{x+\beta}}\right)^{\nu}\left(\frac{1}{\sqrt{x+\alpha}}+\frac{1}{\sqrt{x+\beta}}\right)^{k-2}$$
$$\times\ {}_2F_1\left(\nu+1-\frac{k}{2},1-\frac{k}{2};\ \nu+1;\ \left(\frac{\sqrt{x+\alpha}-\sqrt{x+\beta}}{\sqrt{x+\alpha}+\sqrt{x+\beta}}\right)^2\right), \tag{4.1.38}$$

under the supposition that α and β are two real numbers such that $\alpha>\beta>0$. In order to find the corresponding Hankel transform, $g(x)$, defined by (4.1.37), they employ an integral formula due to Koshliakov [[133], p. 146, eq. (1)],

$$\int_0^{\infty}J_{\mu}(\rho u)\left(\frac{\sqrt{u^2+z^2}-\sqrt{u^2+w^2}}{\sqrt{u^2+z^2}+\sqrt{u^2+w^2}}\right)^{\nu}\frac{u^{\mu+1}}{\sqrt{u^2+z^2}\,\sqrt{u^2+w^2}}\left(\frac{1}{\sqrt{u^2+z^2}}+\frac{1}{\sqrt{u^2+w^2}}\right)^{2\mu}$$
$$\times\ {}_2F_1\left(\nu-\mu,-\mu;\ \nu+1;\ \left(\frac{\sqrt{u^2+z^2}-\sqrt{u^2+w^2}}{\sqrt{u^2+z^2}+\sqrt{u^2+w^2}}\right)^2\right)\,du$$
$$=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}\,(2\rho)^{\mu}\,I_{\nu}\left(\frac{\rho(z-w)}{2}\right)\,K_{\nu}\left(\frac{\rho(z+w)}{2}\right), \tag{4.1.39}$$

which is valid for $\mathrm{Re}(\mu)>-1$, $\mathrm{Re}\left(\mu+2\nu+\frac{3}{2}\right)>0$ and $\mathrm{Re}(z)>\mathrm{Re}(w)>0$.

3. Combining (4.1.36) and (4.1.39), they reach (4.1.35) almost immediately. The last steps in the argument of [43] are made of several transformation formulas for the hypergeometric ${}_2F_1(a,b;c;z)$. Although the application of (4.1.36) to the function $f_{\alpha,\beta}(x)$, (4.1.38), is only valid under the assumption $\alpha>\beta>0$, the authors of [43] extend the resulting summation formula, (4.1.35), to the region $\mathrm{Re}(\sqrt{\alpha})\geq\mathrm{Re}(\sqrt{\beta})>0$ by applying the principle of analytic continuation.

Although the formula (4.1.35) possesses several interesting particular cases, the restriction that the indices of the Bessel functions, $I_{\nu}(X)$ and $K_{\nu}(x)$, must be the same may be seen as somewhat restrictive in the exploration of new summation formulas.

On this point, Berndt, Dixit, Kim, and Zaharescu conclude their paper with the following remark:

"The summands in (1.31)[15] *contain a product of Bessel functions $I_{\nu}(X)$ and $K_{\nu}(x)$. Dixon and Ferrar obtained integral representations for the product $I_{\mu}(X)\,K_{\nu}(x)$, where the orders μ and ν are not necessarily equal. Therefore, perhaps exists a more general transformation than our formula."*

Since we have obtained a summation formula totally analogous to (4.1.35) but involving, instead, the product $I_{\frac{k}{2}-1}(X)\,K_{\nu}(x)$ (see (4.1.15) above), we may ask if the formulas (4.1.15) and (4.1.35) are in some way connected to one another and if they can be written as particular cases of a greater summation formula.

Our hope to this chapter is, then, to give a general transformation formula that includes both formulas (4.1.35) and (4.1.15) as particular cases. Our main goal is to see what kind of transformation will emerge when considering the infinite series

$$S_k(x,y;\,\mu,\nu):=\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu-\mu}{2}}\,I_{\mu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right), \tag{4.1.40}$$

where $k\in\mathbb{N}$, $x>y>0$ and μ and ν are complex numbers satisfying certain conditions (given later in this chapter).

Looking at the overall shape of $S_k(x,y;\,\mu,\nu)$, we can easily see that it reduces to the left-hand side of (4.4.39) when $\mu=\nu$. Moreover, when $\mu=\frac{k}{2}-1$, it becomes precisely the infinite series studied in the second chapter, (4.1.15). Although the study of the general series (4.1.40) is only given in the third technical section composing this chapter, we consider the first two technical sections preceding it as very important, because we establish there new proofs of the already known formulas (4.1.25) and (4.1.35) using the formalism of index transforms. Besides, we believe that our proof of Ramanujan's formula (4.1.25) is of some interest because, as we speculate in Remark 4.2.2, a similar reasoning might have led Ramanujan to its derivation.[16]

This chapter is organized as follows:

1. In the second section, we give a totally new proof of Berndt, Lee and Sohn's formula (4.1.34) by essentially exploiting the symmetries of the integral

$$\int_{-\infty}^{\infty}\eta_k\left(\frac{k}{4}+it\right)\,K_{it}(x)\,\xi^{-it}dt, \tag{4.1.41}$$

where, as noted before, (4.1.3), $\eta_k(s)$ denotes the completed Dirichlet series

$$\eta_k(s):=\pi^{-s}\Gamma(s)\,\zeta_k(s).$$

[15] formula (4.1.35) above.

[16] Even if our proof is not the same as Ramanujan's, our reasoning might have appealed to Ramanujan because we employ the same method as he used in several of his papers.

In particular, the study of (4.1.41) in the case where $k=2$ leads to a new proof of Ramanujan's beautiful result (4.1.25). Given Ramanujan's tendency to express summation formulas through symmetric integral representations, it is not unreasonable to wonder if our approach is close to Ramanujan's original method for deriving (4.1.25).

2. In the third section of this chapter we give a new proof of the formula due to Berndt, Dixit, Kim and Zaharescu (4.1.35) that avoids any use of Voronoï's summation formula (4.1.36). Instead, our approach takes the formula (4.1.16) as a starting point. We believe that our proof is much simpler than the proof presented in [43], because we no longer need to rely on Guinand's result [98], as the authors of [43] do. We also write a remark on how another proof can be achieved by invoking the theta transformation formula,

$$\sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n x} = x^{-\frac{k}{2}} \sum_{n=0}^{\infty} r_k(n)\, e^{-\frac{\pi n}{x}}, \quad \mathrm{Re}(x)>0.$$

3. In the fourth section we prove a generalized summation formula for the infinite series

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, I_{\mu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right), \quad \mathrm{Re}(\mu)>-1, \ \ \mathrm{Re}(\nu)>0. \tag{4.1.42}$$

As we shall see, the future transformation formula will require that we handle with the so-called Appell hypergeometric series [[84], p. 222],

$$F_4\left(\alpha;\, \beta;\, \gamma, \gamma';\, w, z\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!\, n!\, (\gamma)_m(\gamma')_n}\, w^m z^n, \quad \sqrt{|w|}+\sqrt{|z|}<1. \tag{4.1.43}$$

As one should expect, (4.1.43) is the special function that will naturally expand the role of the hypergeometric function ${}_2F_1(a,b;c;z)$ in formulas (4.1.35) and (4.1.15). After this, we take certain combinations of the parameters μ and ν in (4.1.42) and look for meaningful reduction formulas that the function $F_4\left(\alpha;\, \beta;\, \gamma, \gamma';\, w, z\right)$ might have. The following infinite series (making sense for $\mathrm{Re}(\nu)>0$ and $x>y>0$)

$$\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right), \tag{4.1.44}$$

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu}{2}-\frac{k}{4}+\frac{1}{2}}\, I_{\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right) \tag{4.1.45}$$

and

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, I_{\nu+\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right) \tag{4.1.46}$$

will be studied with great detail below. Although the first two series, (4.1.44) and (4.1.45), are, respectively, partially studied in [43] and in the second chapter above, in this chapter we take a more detailed approach and try to furnish several new particular cases.

4. In the fifth section of this chapter, we use the symmetries lying in the study of the infinite series (4.1.45) to prove new formulas of Guinand and Koshliakov type, already introduced in the first chapter (recall (1.2.28) and (1.2.30) above).

5. The sixth and final section of this chapter is devoted to present a new analogue of Ferrar's formula,[17]

$$2\sqrt{\alpha}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\alpha^2}{2}}K_0\left(\frac{\pi n^2\alpha^2}{2}\right)-\frac{1}{\alpha n}\right\}+\frac{1}{\sqrt{\alpha}}\left(\gamma-\log(16\pi)-2\log\left(\alpha\right)\right)$$
$$=2\sqrt{\beta}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\beta^2}{2}}K_0\left(\frac{\pi n^2\beta^2}{2}\right)-\frac{1}{\beta n}\right\}+\frac{1}{\sqrt{\beta}}\left(\gamma-\log(16\pi)-2\log\left(\beta\right)\right),$$

in which the summation index, "n^2", is replaced by its "multidimensional analogue", this is, "$n_1^2+...+n_k^2$".

4.2 A new proof of Ramanujan's formula using the Kontorovich-Lebedev transform

Although one may suppose that Ramanujan obtained (4.1.25) by using the reflection formula for Jacobi's theta function, as Chan and Toh remark [52], it is the purpose of this section to give a new proof of Ramanujan's formula (4.1.25) that, in our view (see Remark 4.2.2 below), contains some ideas that were familiar and firstly used by Ramanujan himself.

Before doing this, we will need some preliminary results. We start with an informal definition that, although not explicitly needed to prove (4.1.25), will be of great use in the exposition. For a real function $f(\tau)$ belonging to a "suitable class", one may informally introduce the Kontorovich-Lebedev transform as

$$(\mathrm{KL}(f))(x):=\int_0^{\infty}K_{i\tau}(x)\,f(\tau)\,d\tau,\quad x>0, \tag{4.2.1}$$

which depends on how the special function $K_{\nu}(x)$ behaves with respect to its order. For a complete study of this transform restricted to functions belonging to the weighted space $L_{\nu,p}\left(\mathbb{R}_+\right)$, see [[243], Chapter 2].

We shall see in this section that Ramanujan's formula (4.1.25) and, more generally, (4.1.34), can be proved using an integral transform resembling (4.2.1). This means that, if we take the index integral transform (4.2.1) as a starting point, the resulting integral symmetries will prove to be instrumental in getting new formulas.

First, let us see some asymptotic formulas for the kernel $K_{i\tau}(x)$ to our study. By [[243], p. 20, Theorem 1.7], this kernel has the asymptotic behavior

$$K_{i\tau}(x)=\sqrt{\frac{2\pi}{\tau}}\,e^{-\frac{\pi\tau}{2}}\sin\left(\tau\,\log\left(\frac{2\tau}{x}\right)-\tau+\frac{\pi}{4}+\frac{x^2}{4\tau}\right)\left\{1+O\left(\frac{1}{\tau}\right)\right\},\quad\tau\to\infty, \tag{4.2.2}$$

[17] recall that $\alpha,\beta>0$ are such that $\alpha\beta=1$.

with $x \in (0, X]$, $X > 0$. Although the exponential decay of (4.2.2) is very precise, the dependence with respect to x is not very useful in the applications that follow, because, as $x \to \infty$ or $x \to 0^+$ we cannot extract nice decay properties from (4.2.2).

However, there are powerful uniform bounds of the kernel $K_{i\tau}(x)$ which behave quite nicely with respect to the variable x. The most famous and useful of these is essentially due to Lebedev and it takes the form

$$|K_{i\tau}(x)| \leq A \, \frac{x^{-\frac{1}{4}}}{\sqrt{\sinh(\pi\tau)}}, \quad x, \tau > 0, \tag{4.2.3}$$

where $A > 0$ is some absolute constant. This inequality was very useful in [[243], p. 219] to justify an inversion formula of an integral transform involving the Meijer G-function.[18]

In this section, however, we will only need a uniform bound for $K_{i\tau}(x)$ that, although uniform, does not need the steep decay $e^{-\pi\tau/2}$ as $\tau \to \infty$, just like (4.2.2) and (4.2.3) do. In most of the occasions here, we only need to invoke the following inequality, valid for $x, \tau > 0$, [[243], p. 15, eq. (1.100)]

$$|K_{i\tau}(x)| \leq e^{-\delta\tau} \, K_0\left(x \, \cos(\delta)\right), \quad 0 \leq \delta < \frac{\pi}{2}. \tag{4.2.4}$$

Although the exponential decay with respect to τ is not as steep as in (4.2.2) and (4.2.3) (because $\delta < \frac{\pi}{2}$), the behavior with respect to x is far better because, for any fixed $\delta \in [0, \frac{\pi}{2})$,

$$K_0\left(x \cos(\delta)\right) = \left(\frac{\pi}{2x \cos(\delta)}\right)^{1/2} e^{-x\cos(\delta)} \left[1 + O\left(\frac{1}{x\cos(\delta)}\right)\right], \quad x \to \infty,$$

$$K_0\left(x \cos(\delta)\right) = -\log\left(x \cos(\delta)\right) + O(1), \quad x \to 0^+.$$

It is not necessarily needed that the index of $K_\nu(x)$ is a purely imaginary number to reach an uniform bound of the form (4.2.4). As it is proved in [[248], p. 280, eq. (1.8)], under the same conditions as in (4.2.4) the following generalization is valid

$$|K_{\sigma+i\tau}(x)| \leq e^{-\delta|\tau|} \, K_\sigma\left(x \, \cos(\delta)\right), \quad \sigma \in \mathbb{R}, \quad 0 \leq \delta < \frac{\pi}{2}, \tag{4.2.5}$$

which will be instrumental in the sequel. It is because of inequalities like (4.2.4) and (4.2.5) that the integral (4.2.1) exists as an absolutely convergent integral for f belonging to a very large class of functions.

Like any integral transform, it is very important to provide conditions allowing an inversion formula for $\mathrm{KL}(f)$, (4.2.1). Although there are several results about the structure of this transform in countless function spaces (cf. [247]), in the next lemma we present the classical inversion formula for (4.2.1) when f belongs to a "nice" space of functions. The proof of the next result can be found in [[249], p. 340, Theorem 1].

Lemma 4.2.1. *Suppose that $f(\tau)$ is a function whose Fourier cosine transform,*

$$(F_c f)(x) := \sqrt{\frac{2}{\pi}} \int_0^\infty f(\tau) \, \cos(x\tau) \, d\tau, \tag{4.2.6}$$

[18] See also [251] for more inequalities of the form (4.2.3) with explicit constants.

belongs to the space $L_1(\mathbb{R}_+)$.[19] *Then, for all* $\tau > 0$, *the following representation of* $f(\tau)$ *is valid*

$$f(\tau) = \frac{2\tau}{\pi^2} \sinh(\pi\tau) \lim_{\epsilon \to 0^+} \int_0^\infty x^{\epsilon-1} K_{i\tau}(x) \, (KL(f))(x) \, dx. \tag{4.2.7}$$

Our next set of results concern the evaluation of some integrals of the form (4.2.1) when we take the function $f(\tau)$ as dependent on the Dirichlet series $\zeta_k(s)$. Our first theorem is nothing but a useful integral evaluation.

Theorem 4.2.1. *For any* $\nu \in \mathbb{C}$, $\mu > \frac{k}{2}$ *and* $x, \xi > 0$, *let* $\mathcal{I}_k(x,\xi;\nu)$ *denote the following Mellin-Barnes integral*

$$\mathcal{I}_k(x,\xi;\nu) = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} \eta_k(s) \, K_{s+\nu}(x) \, \xi^{-s} ds, \tag{4.2.8}$$

where $\eta_k(s)$ *is the usual completed Dirichlet series,*

$$\eta_k(s) := \pi^{-s}\Gamma(s) \, \zeta_k(s). \tag{4.2.9}$$

Then $\mathcal{I}_k(x,\xi;\nu)$ *can be explicitly expressed as the infinite series*

$$\mathcal{I}_k(x,\xi;\nu) = x^{-\frac{\nu}{2}} \sum_{n=1}^\infty r_k(n) \, (x + 2\pi\xi \, n)^{\nu/2} \, K_\nu\left(\sqrt{x^2 + 2\pi\xi x \, n}\right). \tag{4.2.10}$$

Proof. First, let us check that the definition (4.2.8) makes sense, i.e., that the integral $\mathcal{I}_k(x,\xi;\nu)$ does not depend on which line of integration, $\text{Re}(s) = \mu > \frac{k}{2}$, we start with. This comes as an immediate consequence of the fact that $\zeta_k(s)$ is analytic in the half-plane of absolute convergence $\text{Re}(s) > \frac{k}{2}$ and $K_{s+\nu}(x)$ defines an entire function of s, for any complex s. Second, let us check that the integral defining $\mathcal{I}_k(x,\xi;\nu)$ converges absolutely for any $\nu \in \mathbb{C}$. Since $\mu > \frac{k}{2}$, we know that $|\zeta_k(\mu + it)| \leq \zeta_k(\mu)$, so that, writing $\nu = \sigma + i\tau$,

$$\int_{\mu-i\infty}^{\mu+i\infty} \left|\eta_k(s) \, K_{s+\nu}(x) \, \xi^{-s}\right| \, |ds| \leq \zeta_k(\mu) \, (\pi\xi)^{-\mu} \int_{-\infty}^\infty \left|\Gamma(\mu + it) \, K_{\mu+\sigma+i(t+\tau)}(x)\right| \, dt.$$

Clearly, the last integral exists. This can be justified by using the uniform bound (4.2.5) for the Macdonald function and Stirling's formula for the Gamma function (2.1.37). They show that, for a sufficiently large T_0 and $|t| \geq T_0 > 2|\tau|$,

$$\begin{aligned} \left|\Gamma(\mu + it) \, K_{\mu+\sigma+i(t+\tau)}(x)\right| &< 2\sqrt{2\pi}|t|^{\mu-\frac{1}{2}} e^{-\frac{\pi}{2}|t|} e^{-\delta|t+\tau|} \, K_{\mu+\sigma}(x\cos(\delta)) \\ &< 2\sqrt{2\pi}|t|^{\mu-\frac{1}{2}} e^{-(\pi+\delta)\frac{|t|}{2}} \, K_{\mu+\sigma}(x\cos(\delta)), \end{aligned} \tag{4.2.11}$$

where $0 \leq \delta < \frac{\pi}{2}$. The bound (4.2.11) shows the absolute convergence of the integral defining $\mathcal{I}_k(x,\xi;\nu)$ for any $x, \xi > 0$ and $\nu \in \mathbb{C}$. Since $\mu > \frac{k}{2}$ by hypothesis, we know that $\zeta_k(s)$ admits

[19] It is worthwhile noting that this space of functions forms a Banach space, $L^\star(\mathbb{R}_+)$, whose norm is $\|f\|_{L^\star} := \|F_c f\|_{L_1(\mathbb{R}_+)}$.

an expansion as an absolutely convergent Dirichlet series on the line $\mathrm{Re}(s) = \mu$. By absolute convergence and the estimates given in (4.2.11)

$$\begin{aligned}\mathcal{I}_k(x,\xi;\nu) &:= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \eta_k(s)\, K_{s+\nu}(x)\, \xi^{-s} ds \\ &= \sum_{n=1}^{\infty} \frac{r_k(n)}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \Gamma(s)\, K_{s+\nu}(x)\, (\pi\xi n)^{-s}\, ds \\ &= \sum_{n=1}^{\infty} r_k(n)\int_0^{\infty} g(u)\, e^{-\frac{\pi\xi n}{u}}\, \frac{du}{u},\end{aligned} \tag{4.2.12}$$

with $g(u)$ being

$$g(u) := \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} K_{s+\nu}(x)\, u^{-s} ds. \tag{4.2.13}$$

Note that the last step in (4.2.12) results from an application of the Parseval formula for Mellin transforms [[166], p. 83, eq. (3.3.10)]. To complete the proof, we just need to evaluate $g(u)$ and the integral on the last member of (4.2.12). First, let us note that the integral (4.2.13) is absolutely convergent due to the uniform estimate (4.2.5). Also, we know that [[48], p. 20, relation 2.2.1.8]

$$\int_0^{\infty} u^{s-1}\, e^{-\alpha u} e^{-\frac{\beta}{u}}\, du = 2\left(\frac{\beta}{\alpha}\right)^{\frac{s}{2}} K_s\left(2\sqrt{\alpha\beta}\right), \quad \mathrm{Re}(\alpha),\, \mathrm{Re}(\beta) > 0, \tag{4.2.14}$$

which implies

$$\int_0^{\infty} u^{s+\nu-1} e^{-\frac{x}{2}\left(u+\frac{1}{u}\right)}\, du = 2\, K_{s+\nu}(x), \quad \mathrm{Re}(x) > 0, \tag{4.2.15}$$

and then, according to Mellin's inversion formula [[166], p. 80], this gives the evaluation

$$g(u) := \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} K_{s+\nu}(x)\, u^{-s} ds = \frac{1}{2} u^{\nu} e^{-\frac{x}{2}\left(u+\frac{1}{u}\right)}. \tag{4.2.16}$$

Returning to (4.2.12), and using this explicit evaluation of $g(u)$ in the last integral, we find that

$$\begin{aligned}\mathcal{I}_k(x,\xi;\nu) &= \sum_{n=1}^{\infty} r_k(n)\int_0^{\infty} g(u)\, e^{-\frac{\pi\xi n}{u}}\, \frac{du}{u} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} r_k(n)\int_0^{\infty} u^{\nu-1} e^{-\frac{x}{2}u} e^{-\frac{1}{u}\left(\frac{x}{2}+\pi\xi n\right)}\, du \\ &= x^{-\frac{\nu}{2}}\sum_{n=1}^{\infty} r_k(n)\, (x+2\pi\xi\, n)^{\nu/2}\, K_{\nu}\left(\sqrt{x^2+2\pi\xi x\, n}\right),\end{aligned}$$

where the last step results from (4.2.14). This completes the proof of our result.

□

In the next corollary, we take a very special case of the previous result, moving the contour integral to a region where part of the integrand is a real function. By the functional equation, (4.1.3), it is well-known that $\eta_k(s)$ is a real-valued function at the critical line $\text{Re}(s) = \frac{k}{4}$. Moreover, by the property $K_\nu(x) = K_{-\nu}(x)$, we also know that, when $x \in \mathbb{R}$, the kernel $K_{it}(x)$ is a real-valued function of $t \in \mathbb{R}$. Consequently, it is not unreasonable to consider the following interesting integral

$$\mathcal{F}_k(x,\xi) := \int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{it}(x)\, \xi^{-it} dt, \tag{4.2.17}$$

which can be interpreted as a Fourier transform of the real function

$$\tilde{\Xi}(t) := \eta_k\left(\frac{k}{4}+it\right) K_{it}(x).$$

The integral defining $\mathcal{F}_k(x,\xi)$, (4.2.17), can be also seen as the Kontorovich-Lebedev transform of the function

$$\Xi^{\star}(t) := \eta_k\left(\frac{k}{4}+it\right) \xi^{-it}.$$

In the next result, we evaluate explicitly the integral (4.2.17).

Corollary 4.2.1. *For $k \in \mathbb{N}$, let $\mathcal{F}_k(x,\xi)$ denote the Kontorovich-Lebedev like integral defined by (4.2.17). Then $\mathcal{F}_k(x,\xi)$ can be expressed as the infinite Bessel series*

$$\mathcal{F}_k(x,\xi) = 2\pi \left(\sqrt{x}\xi\right)^{\frac{k}{4}} \sum_{n=1}^{\infty} \frac{r_k(n)}{(x+2\pi\xi n)^{\frac{k}{8}}} K_{\frac{k}{4}}\left(\sqrt{x^2+2\pi\xi x\, n}\right) - 2\pi\, K_{\frac{k}{4}}(x)\, \xi^{-\frac{k}{4}}. \tag{4.2.18}$$

Proof. We start by writing the integral (4.2.17) in terms of a contour integral, this is

$$\int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{it}(x)\, \xi^{-it} dt = \frac{\xi^{k/4}}{i} \int_{\frac{k}{4}-i\infty}^{\frac{k}{4}+i\infty} \eta_k(s)\, K_{s-\frac{k}{4}}(x)\, \xi^{-s} ds. \tag{4.2.19}$$

Next, we use Cauchy's residue theorem by shifting the line of integration from $\text{Re}(s) = \frac{k}{4}$ to $\text{Re}(s) = \frac{k}{2}+\epsilon$, $\epsilon > 0$. In the process, we take an integration along a closed, positively oriented rectangular contour $\mathscr{R}(T)$ with vertices $\frac{k}{4} \pm iT$ and $\frac{k}{2}+\epsilon \pm iT$. By the Phragmén-Lindelöf principle, obtained in (1.1.27), we know that, for any $\delta > 0$, $\zeta_k(\sigma+it) \ll_\delta |t|^{A(\sigma)+\delta}$, where

$$A(\sigma) = \begin{cases} 0 & \sigma > \frac{k}{2}+\delta \\ \frac{k}{2}-\sigma & -\delta \leq \sigma \leq \frac{k}{2}+\delta \\ \frac{k}{2}-2\sigma & \sigma < -\delta \end{cases}. \tag{4.2.20}$$

Writing $\nu = \sigma + i\tau$, we have that, when $\text{Im}(s) = T \geq T_0 > 2|\tau|$ and $\frac{k}{4} \leq \text{Re}(s) \leq \frac{k}{2} + \delta$ (recall (4.2.11) above)

$$\left|\eta_k(s)\, K_{s+\nu}(x)\, \xi^{-s}\right| \ll_\epsilon (\pi\xi)^{-\text{Re}(s)} K_{\text{Re}(s)+\sigma}\left(x\cos(\delta)\right) T^{\frac{k-1}{2}+\epsilon}\, e^{-(\pi+\delta)\frac{T}{2}},$$

where $0 \leq \delta < \frac{\pi}{2}$. Thus, when T is sufficiently large

$$\left|\int_{\frac{k}{4}\pm iT}^{\frac{k}{2}+\epsilon\pm iT} \eta_k(s)\, K_{s-\frac{k}{4}}(x)\, \xi^{-s} ds\right| \ll_\epsilon (\pi\xi)^{-\text{Re}(s)} K_{\text{Re}(s)+\sigma}\left(x\cos(\delta)\right) T^{\text{Re}(s)-\frac{1}{2}}\, e^{-(\pi+\delta)\frac{T}{2}},$$

which clearly tends to zero as $T \to \infty$. Since the integrand in (4.2.19) is analytic everywhere except at the points $s = 0$ and $s = \frac{k}{2}$, the only pole inside the contour $\mathscr{R}(T)$ is located precisely at the point $s = \frac{k}{2}$. Its residue is simply

$$\text{Res}_{s=\frac{k}{2}} \left\{\eta_k(s)\, K_{s-\frac{k}{4}}(x)\, \xi^{-s}\right\} = K_{\frac{k}{4}}(x)\, \xi^{-\frac{k}{2}},$$

and so we obtain the formula

$$\int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4} + it\right) K_{it}(x)\, \xi^{-it} dt = \frac{\xi^{k/4}}{i} \int_{\frac{k}{4}-i\infty}^{\frac{k}{4}+i\infty} \eta_k(s)\, K_{s-\frac{k}{4}}(x)\, \xi^{-s} ds$$

$$= \frac{\xi^{k/4}}{i} \int_{\mu-i\infty}^{\mu+i\infty} \eta_k(s)\, K_{s-\frac{k}{4}}(x)\, \xi^{-s} ds - 2\pi\, K_{\frac{k}{4}}(x)\, \xi^{-\frac{k}{4}}$$

$$= 2\pi \left(\sqrt{x}\,\xi\right)^{\frac{k}{4}} \sum_{n=1}^{\infty} r_k(n)\, (x + 2\pi\xi n)^{-\frac{k}{8}}\, K_{\frac{k}{4}}\left(\sqrt{x^2 + 2\pi\xi x\, n}\right) - 2\pi\, K_{\frac{k}{4}}(x)\, \xi^{-\frac{k}{4}},$$

where the last step consisted in using (4.2.10) with $\nu = -\frac{k}{4}$. □

In the next corollary we use the inversion formula for the Kontorovich-Lebedev transform (4.2.7) to achieve a new integral representation for the completed Dirichlet series $\eta_k(s)$.

Corollary 4.2.2. *For all $\tau > 0$, the following integral representation holds*

$$\eta_k\left(\frac{k}{4} + i\tau\right) = \frac{2\tau}{\pi} \sinh(\pi\tau) \lim_{\epsilon\to 0^+} \int_0^{\infty} x^{\epsilon-1} K_{i\tau}(x) \left\{\sum_{n=1}^{\infty} \frac{r_k(n)}{\left(1 + \frac{2\pi n}{x}\right)^{\frac{k}{8}}} K_{\frac{k}{4}}\left(\sqrt{x^2 + 2\pi x n}\right) - K_{\frac{k}{4}}(x)\right\} dx. \tag{4.2.21}$$

Proof. We know from the previous corollary that, if $f(\tau) := \eta_k\left(\frac{k}{4} + i\tau\right)$,

$$(KLf)(x) := \int_0^{\infty} K_{i\tau}(x)\, f(\tau)\, d\tau = \int_0^{\infty} K_{i\tau}(x)\, \eta_k\left(\frac{k}{4} + i\tau\right) d\tau = \frac{1}{2}\mathcal{F}_k\left(x,\, 1\right)$$

$$= \pi x^{\frac{k}{8}} \sum_{n=1}^{\infty} \frac{r_k(n)}{(x + 2\pi n)^{\frac{k}{8}}} K_{\frac{k}{4}}\left(\sqrt{x^2 + 2\pi x n}\right) - \pi\, K_{\frac{k}{4}}(x).$$

Our formula (4.2.21) now follows from Lemma 4.2.1. In order to check that the inversion formula (4.2.7) is valid in this case, we just need to check that the Fourier cosine transform of $\eta_k\left(\frac{k}{4}+i\tau\right)$, (4.2.6), belongs to the space $L_1(\mathbb{R}_+)$.

Although this can be done in an indirect way, i.e., by showing that the real function $f(\tau) := \eta_k\left(\frac{k}{4}+i\tau\right)$ belongs to a certain class of "nice functions" (see Remark 5.2.1 of the next chapter), we will prove this by a direct calculation. The completion of our argument is just a matter of following the straightforward computations[20]

$$\begin{aligned}
\int_0^\infty \eta_k\left(\frac{k}{4}+i\tau\right)\cos(x\tau)\,d\tau &= \frac{1}{2}\int_{-\infty}^\infty \eta_k\left(\frac{k}{4}+i\tau\right)e^{-ix\tau}\,d\tau \\
&= \frac{e^{xk/4}}{2i}\int_{\frac{k}{4}-i\infty}^{\frac{k}{4}+i\infty}\eta_k(s)\,e^{-xs}\,ds = -\pi e^{-xk/4}+\frac{e^{xk/4}}{2i}\int_{\frac{k}{2}+\epsilon-i\infty}^{\frac{k}{2}+\epsilon+i\infty}\eta_k(s)\,e^{-xs}\,ds \\
&= -\pi e^{-xk/4}+\pi e^{xk/4}\sum_{n=1}^\infty\frac{r_k(n)}{2\pi i}\int_{\frac{k}{2}+\epsilon-i\infty}^{\frac{k}{2}+\epsilon+i\infty}\Gamma(s)\,\left(\pi e^x n\right)^{-s}\,ds \\
&= -\pi e^{-xk/4}+\pi e^{xk/4}\sum_{n=1}^\infty r_k(n)\,e^{-\pi e^x n},
\end{aligned}$$

where the last step comes from the use of the Cahen-Mellin integral (1.1.31). Thus,

$$(F_c f)(x) := \sqrt{\frac{2}{\pi}}\int_0^\infty \eta_k\left(\frac{k}{4}+i\tau\right)\cos(x\tau)\,d\tau = -\sqrt{2\pi}\,e^{-\frac{xk}{4}}+\sqrt{2\pi}\,e^{\frac{xk}{4}}\sum_{n=1}^\infty r_k(n)\,e^{-\pi e^x n}, \quad (4.2.22)$$

which clearly belongs to the space $L_1(\mathbb{R}_+)$. This completes the proof. $\square$

Taking $k=1$ in the previous corollary and using the fact that $\eta_1(s) := 2\pi^{-s}\Gamma(s)\zeta(2s)$, we are able to get the following integral representation for the Riemann $\zeta-$function on the critical line, $\text{Re}(s)=\frac{1}{2}$.

Corollary 4.2.3. *For any $\tau>0$, the following representation of the Riemann zeta function is valid*

$$\zeta\left(\frac{1}{2}+i\tau\right) = \frac{\tau\,\sinh\left(\frac{\pi\tau}{2}\right)}{\pi^{\frac{5}{4}+i\frac{\tau}{2}}\,\Gamma\left(\frac{1}{4}+i\frac{\tau}{2}\right)}\lim_{\epsilon\to0^+}\int_0^\infty x^{\epsilon-1}K_{i\frac{\tau}{2}}(x)\left\{2x^{\frac{k}{8}}\sum_{n=1}^\infty\frac{K_{\frac{1}{4}}\left(\sqrt{x^2+2\pi xn^2}\right)}{\left(x+2\pi n^2\right)^{\frac{1}{8}}}-K_{\frac{1}{4}}(x)\right\}dx.$$

An idea coming from Ramanujan's work consists in obtaining transformation formulas by looking at the symmetries of their integral manifestations (see Remark 4.2.2 below). In the next corollary, we use (4.2.18) together with the parity of the integrand in (4.2.17) to produce a transformation formula.

[20] where we are just performing Cauchy's residue theorem and using the parity of the function $\eta_k\left(\frac{k}{4}+it\right)$, which in its turn is justified by the functional equation (4.1.3).

Corollary 4.2.4. *For any $x, \xi > 0$, the following identity holds*

$$\begin{aligned}
&(\sqrt{x}\,\xi)^{\frac{k}{4}} \sum_{n=1}^{\infty} \frac{r_k(n)}{(x+2\pi\xi n)^{\frac{k}{8}}} K_{\frac{k}{4}}\left(\sqrt{x^2+2\pi\xi x\, n}\right) - K_{\frac{k}{4}}(x)\,\xi^{-\frac{k}{4}} \\
&= \left(\frac{\sqrt{x}}{\xi}\right)^{\frac{k}{4}} \sum_{n=1}^{\infty} \frac{r_k(n)}{\left(x+\frac{2\pi n}{\xi}\right)^{\frac{k}{8}}} K_{\frac{k}{4}}\left(\sqrt{x^2+\frac{2\pi x}{\xi}\,n}\right) - K_{\frac{k}{4}}(x)\,\xi^{\frac{k}{4}} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{it}(x)\,\xi^{-it}dt. \qquad (4.2.23)
\end{aligned}$$

Proof. We have already proved that the first and third terms in (4.2.23) are equal. In order to get the equality between the first and second terms in (4.2.23), we just need to look at the term containing the integral. Using the functional equation for $\eta_k(s)$, (4.1.3), and the fact that $K_\nu(z) = K_{-\nu}(z)$, we can express the integral $\mathcal{F}_k(x,\xi)$, (4.2.17), in the following way

$$\begin{aligned}
\mathcal{F}_k(x,\xi) &:= \int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{it}(x)\,\xi^{-it}dt = \int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}-it\right) K_{it}(x)\,\xi^{-it}dt \\
&= \int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{-it}(x)\,\xi^{it}dt = \int_{-\infty}^{\infty} \eta_k\left(\frac{k}{4}+it\right) K_{it}(x)\,\left(\frac{1}{\xi}\right)^{-it} dt \\
&= \mathcal{F}_k\left(x,\frac{1}{\xi}\right),
\end{aligned}$$

which completes the proof of Corollary 4.2.4. $\square$

In the next corollary, we specialize the previous result to the case where $k=1$, deriving an interesting integral transform containing the Riemann zeta function. Since the result is a trivial particular case of (4.2.23), we shall omit its proof.

Corollary 4.2.5. *For any $x, \xi > 0$, the following identity holds*

$$\begin{aligned}
2x^{\frac{1}{8}}\sqrt{\xi}\sum_{n=1}^{\infty}\frac{K_{\frac{1}{4}}\left(\sqrt{x^2+2\pi x\,\xi^2 n^2}\right)}{(x+2\pi\xi^2n^2)^{\frac{1}{8}}} - \frac{K_{\frac{1}{4}}(x)}{\sqrt{\xi}} &= \frac{2x^{\frac{1}{8}}}{\sqrt{\xi}}\sum_{n=1}^{\infty}\frac{K_{\frac{1}{4}}\left(\sqrt{x^2+\frac{2\pi x}{\xi^2}n^2}\right)}{\left(x+\frac{2\pi n^2}{\xi^2}\right)^{\frac{1}{8}}} - K_{\frac{1}{4}}(x)\sqrt{\xi} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta\left(\frac{1}{2}+it\right)K_{\frac{it}{2}}(x)\,\xi^{-it}\,dt,
\end{aligned}$$

where $\eta(s)$ denotes the completed version of Riemann's $\zeta-$function

$$\eta(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Taking $k=2$ in (4.2.23) and appealing again to the reduction formula for the Macdonald function (4.1.22), we are able to find a new way to prove Ramanujan's formula stated at the introduction, (4.1.25). We state Ramanujan's formula in our next corollary.

Corollary 4.2.6. *For any $x,\xi>0$, the following identity holds*

$$\sqrt{\frac{\pi\xi}{2}}\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{x+2\pi\xi n}}\,e^{-\sqrt{x^2+2\pi\xi x\,n}}-\sqrt{\frac{\pi}{2x\xi}}e^{-x}=\sqrt{\frac{\pi}{2\xi}}\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{x+\frac{2\pi n}{\xi}}}\,e^{-\sqrt{x^2+\frac{2\pi x}{\xi}n}}-\sqrt{\frac{\pi\xi}{2x}}e^{-x}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_2\left(\frac{1}{2}+it\right)\,K_{it}(x)\,\xi^{-it}dt. \tag{4.2.24}$$

Remark 4.2.1. The previous corollary not only establishes Ramanujan's formula, but also shows that its equivalent to an integral transformation. In order to see that (4.2.24) implies Ramanujan's formula (4.1.25), we just need to take the first equality in (4.2.24) and then change the variables to $a:=x/(2\pi\xi)$ and $b:=x\xi/(2\pi)$.

Remark 4.2.2. Ramanujan notoriously deduced several summation formulas by transforming them into integrals. For example, in [177], Ramanujan studies the following integral

$$\frac{1}{4\pi^{3/2}}\int_{0}^{\infty}\Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\,\xi\left(\frac{1+it}{2}\right)\,\cos\left(\frac{t}{2}\log(\alpha)\right)\,dt, \tag{4.2.25}$$

where $\alpha>0$ and $\xi(s)$ denotes Riemann's $\xi-$function, which is defined by

$$\xi(s):=\frac{1}{2}s(s-1)\,\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{4.2.26}$$

Using what is now known as Parseval's formula for the Fourier transform, Ramanujan connected (4.2.25) with another integral, namely, with the expression (cf. [[67], p. 360])

$$\alpha^{-\frac{1}{2}}-4\pi\alpha^{-\frac{3}{2}}\int_{0}^{\infty}\frac{x\,e^{-\frac{\pi x^2}{\alpha^2}}}{e^{2\pi x}-1}dx. \tag{4.2.27}$$

The relationship between (4.2.25) and (4.2.27) is unsurprising, as it arises naturally from Mellin or Fourier convolutions. In fact, it is not much more surprising than our evaluation of the integral $\mathcal{I}_k\left(x,\xi;\nu\right)$, (4.2.8), given in (4.2.10). However, what we can achieve by using the symmetries of the integrand in (4.2.25) to obtain a transformation formula for (4.2.27) reveals something entirely different. Note that (4.2.25) stays the same if we replace α by $\beta:=1/\alpha$, and so (4.2.27) should also be invariant under this transformation of parameters. Thus, if α and β are two positive real numbers such that $\alpha\beta=1$, then the interesting transformation must take place

$$\alpha^{-\frac{1}{2}}-4\pi\alpha^{-\frac{3}{2}}\int_{0}^{\infty}\frac{x\,e^{-\frac{\pi x^2}{\alpha^2}}}{e^{2\pi x}-1}dx=\beta^{-\frac{1}{2}}-4\pi\beta^{-\frac{3}{2}}\int_{0}^{\infty}\frac{x\,e^{-\frac{\pi x^2}{\beta^2}}}{e^{2\pi x}-1}dx.$$

This method of Ramanujan is invaluable. There are other instances in his work where Ramanujan reached new identities via this approach. The reader is encouraged to check [65, 67] for more examples. In light of this, it is not unreasonable to conjecture that Ramanujan knew how to manipulate the integral $\mathcal{I}_k(x,\xi;\nu)$, (4.2.8), in order to get (4.2.24).

Since Ramanujan's formula (4.1.25) is implied by (4.1.26), due to Dixon and Ferrar, it is natural to ask if we can use our Theorem 4.2.1 to get a new proof (4.1.26). The answer to this question is positive and, more than this, our formula (4.2.10) can be also used to give a new proof of (4.1.34).

Corollary 4.2.7. *For every $k \in \mathbb{N}$, any complex ν and $x, \xi > 0$, the following formula is valid*

$$\begin{aligned} & K_\nu(x) + x^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, (x + 2\pi\xi\, n)^{\frac{\nu}{2}}\, K_\nu\left(\sqrt{x^2 + 2\pi\xi x\, n}\right) \\ & = \xi^{-\frac{k}{2}}\, K_{\nu+\frac{k}{2}}(x) + x^{\frac{k}{4}+\frac{\nu}{2}}\, \xi^{-\frac{k}{2}} \sum_{n=1}^{\infty} r_k(n) \left(x + \frac{2\pi n}{\xi}\right)^{-\frac{k}{4}-\frac{\nu}{2}} K_{\frac{k}{2}+\nu}\left(\sqrt{x^2 + \frac{2\pi x n}{\xi}}\right). \end{aligned} \tag{4.2.28}$$

Proof. In Theorem 4.2.1, we have seen that, for any $\epsilon > 0$, the following representation takes place

$$x^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, (x + 2\pi\xi n)^{\nu/2}\, K_\nu\left(\sqrt{x^2 + 2\pi\xi x\, n}\right) = \frac{1}{2\pi i} \int_{\frac{k}{2}+\epsilon-i\infty}^{\frac{k}{2}+\epsilon+i\infty} \eta_k(s)\, K_{s+\nu}(x)\, \xi^{-s} ds. \tag{4.2.29}$$

Taking the integral on the right of (4.2.29) as a starting point and moving the line of integration from $\text{Re}(s) = \frac{k}{2} + \epsilon$ to $\text{Re}(s) = -\epsilon$, Cauchy's residue theorem yields

$$\begin{aligned} & x^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, (x + 2\pi\xi n)^{\nu/2}\, K_\nu\left(\sqrt{x^2 + 2\pi\xi x\, n}\right) \\ & = K_{\nu+\frac{k}{2}}(x)\, \xi^{-\frac{k}{2}} - K_\nu(x) + \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \eta_k(s)\, K_{s+\nu}(x)\, \xi^{-s} ds, \end{aligned} \tag{4.2.30}$$

where the justification for its application is totally analogous to the one offered in the proof of Corollary 4.2.1. The isolated terms appearing on the right-hand side of (4.2.30) come from the contribution of the residues that the function $\eta_k(s)$ possesses at the points $s = \frac{k}{2}$ and $s = 0$. Looking at the last integral and invoking the functional equation for $\zeta_k(s)$, (4.1.3), there, we are then able to find that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \eta_k(s)\, K_{s+\nu}(x) \xi^{-s} ds = \frac{\xi^{-\frac{k}{2}}}{2\pi i} \int_{\frac{k}{2}+\epsilon-i\infty}^{\frac{k}{2}+\epsilon+i\infty} \eta_k(s)\, K_{\frac{k}{2}-s+\nu}(x)\, \xi^{s} ds \\ & = \frac{\xi^{-\frac{k}{2}}}{2\pi i} \int_{\frac{k}{2}+\epsilon-i\infty}^{\frac{k}{2}+\epsilon+i\infty} \eta_k(s)\, K_{s-\frac{k}{2}-\nu}(x)\, \xi^{s} ds = x^{\frac{k}{4}+\frac{\nu}{2}} \xi^{-\frac{k}{2}} \sum_{n=1}^{\infty} r_k(n) \left(x + \frac{2\pi n}{\xi}\right)^{-\frac{k}{4}-\frac{\nu}{2}} K_{\frac{k}{2}+\nu}\left(\sqrt{x^2 + \frac{2\pi x n}{\xi}}\right), \end{aligned}$$

where the last step is just an application of (4.2.10) with ν replaced by $-\frac{k}{2} - \nu$. This establishes the desired formula (4.2.28). $\square$

Remark 4.2.3. Consider the change of variables $a := x/(2\pi\xi)$ and $b := x\xi/(2\pi)$: then our previous corollary is equivalent to the elegant identity

$$K_{\nu}\left(2\pi\sqrt{ab}\right)+a^{-\frac{\nu}{2}}\sum_{n=1}^{\infty}r_{k}(n)\,\,(n+a)^{\frac{\nu}{2}}\,\,K_{\nu}\left(2\pi\sqrt{(n+a)\,b}\right)$$
$$=\left(\frac{b}{a}\right)^{-\frac{k}{4}}\,K_{\nu+\frac{k}{2}}\left(2\pi\sqrt{ab}\right)+a^{\frac{k}{4}}\,b^{\frac{\nu}{2}}\sum_{n=1}^{\infty}r_{k}(n)\,\,(n+b)^{-\frac{k}{4}-\frac{\nu}{2}}\,\,K_{\frac{k}{2}+\nu}\left(2\pi\sqrt{(n+b)\,a}\right).$$

Moreover, recalling the convention that $r_k(0) := 1$, we can rewrite the previous identity in the form

$$a^{-\frac{\nu}{2}}\sum_{n=0}^{\infty}r_{k}(n)\,\,(n+a)^{\frac{\nu}{2}}\,\,K_{\nu}\left(2\pi\sqrt{(n+a)\,b}\right)$$
$$=a^{\frac{k}{4}}\,b^{\frac{\nu}{2}}\sum_{n=0}^{\infty}r_{k}(n)\,\,(n+b)^{-\frac{k}{4}-\frac{\nu}{2}}\,\,K_{\frac{k}{2}+\nu}\left(2\pi\sqrt{(n+b)\,a}\right), \qquad (4.2.31)$$

which is equivalent to the formula of Berndt, Lee and Sohn [[37], p. 39, eq. (5.5)], provided at the introduction, (4.1.34).[21]

[21] It suffices to multiply both sides of (4.2.31) by $(a/b)^{\nu/2}$ and then replacing ν by $-\nu$.

4.3 A new proof of a formula involving the product $I_\nu(X)\,K_\nu(x)$

This section is devoted to give a new proof of formula (4.1.35) due to Berndt, Dixit, Kim and Zaharescu. Our proof is totally inspired by the method given in the previous section. Since the formula (4.1.35) contains Gauss' hypergeometric function ${}_2F_1(a, b;\, c;\, z)$ at its core, it may be helpful to remind some basic features of this special function, just as we have done in the second chapter.

Recalling the brief definition (2.2.35), for $|z| < 1$, the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by the Gauss series,

$$
{}_2F_1\left(a, b; c; z\right) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell \ell!}\, z^\ell = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{\ell=0}^{\infty} \frac{\Gamma(a+\ell)\Gamma(b+\ell)}{\Gamma(c+\ell)\ell!}\, z^\ell, \tag{4.3.1}
$$

and defined elsewhere by analytic continuation. For instance, this continuation can be given via Slater's theorem [243], which is based on the Mellin-Barnes integral representation (cf. relation 3.31.1.1, page 461 of [48]),

$$
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(a, b; c; -z\right) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\,\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}\, z^{-s} ds, \tag{4.3.2}
$$

where $0 < \gamma < \max\{\mathrm{Re}(a),\, \mathrm{Re}(b)\}$, $c \neq 0, -1, -2, ...$ and $|\arg(z)| < \pi$.[22]

There is, however, another method of continuing the series (4.3.1) via Euler's integral representation: if $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|\arg(1-z)| < \pi$, then ${}_2F_1(a, b; c; z)$ satisfies the formula

$$
{}_2F_1\left(a, b; c; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{4.3.3}
$$

where we have chosen the principal branch $(1-tz)^{-a} = e^{-a\log(1-tz)}$, with $\log(1-tz)$ being real for $z \in [0, 1]$. There are several applications of Euler's formula. Using it, one is able to derive the two most famous transformation formulas for the hypergeometric function,

$$
{}_2F_1\left(a, b; c; z\right) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{4.3.4}
$$

$$
{}_2F_1\left(a, b; c; z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(c-a, c-b; c; z\right), \tag{4.3.5}
$$

known respectively as Pfaff's and Euler's formulas[23]. It is also known that, for any $c \neq 0, -1, -2, ...$ and fixed z, the function ${}_2F_1\left(a, b; c; z\right)$ is an entire function of a, b and c [[164], p. 384, 15.2(ii)].

In this section we will invoke several transformation formulas for the function ${}_2F_1(a, b; c; z)$, so we think that it will be quite helpful to make some changes in the parameters of the formula (4.1.35) and introduce further assumptions.

[22] For a very clear historical exposition about this integral representation, the reader is encouraged to check [[3], pp. 85-87].

[23] Alternatively, under the terminology of [243], they are also called Boltz formula and self-transformation formula.

Notation and assumptions: Although (4.1.35) is valid for $\text{Re}(\sqrt{\alpha}) \geq \text{Re}(\sqrt{\beta}) > 0$, we will assume that α and β are positive real numbers, ignoring the continuation to complex values. Furthermore, we shall adopt the new variables:

$$x := \frac{\sqrt{\alpha}+\sqrt{\beta}}{2}, \quad y := \frac{\sqrt{\alpha}-\sqrt{\beta}}{2}. \tag{4.3.6}$$

According to this new setting, one may see that, for $x > y > 0$ and $\text{Re}(\nu) > 0$, the transformation formula (4.1.35) is actually equivalent to

$$\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right) = -\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu} + \frac{\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}2^{k-1}\Gamma(\nu+1)}$$
$$\times \sum_{n=0}^{\infty} r_k(n) \frac{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{k-1}{2}}} \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}$$
$$\times {}_2F_1\left(1-\frac{k}{2}+\nu,\, 1-\frac{k}{2};\, \nu+1;\, \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^2\right). \tag{4.3.7}$$

Thus, our goal in this section is to establish (4.1.35) in the form (4.3.7) and assuming the (less general) condition that $x > y > 0$.[24] Our main point of departure will be the following integral representation [[48], p. 229, relation 3.14.15.5],

$$I_{\nu}\left(b\sqrt{u^2+a^2}-ab\right)\, K_{\nu}\left(b\sqrt{u^2+a^2}+ab\right)$$
$$= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{2\sqrt{\pi}}\left(\frac{a}{b}\right)^{s/2} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} K_{\frac{s}{2}}(2ab)\, u^{-s}\, ds, \tag{4.3.8}$$

where a, u, $\text{Re}(b) > 0$ and $-2\text{Re}(\nu) < \mu < 1$. Taking $x > y > 0$, $n \in \mathbb{N}$, and using (4.3.8) with $a = \frac{x-y}{2}$, $b = 2\pi\sqrt{n}$ and, finally, $u = \sqrt{xy}$, we find the equivalent Mellin-Barnes integral

$$I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right)$$
$$= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{2\sqrt{\pi}}\left(\frac{x-y}{4\pi xy}\right)^{\frac{s}{2}} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\, n^{-\frac{s}{4}}\, K_{\frac{s}{2}}\left(2\pi\,(x-y)\sqrt{n}\right)\, ds, \tag{4.3.9}$$

which makes sense when $-2\text{Re}(\nu) < \mu < 1$. Clearly, this appears to be a good starting point for our proof of (4.3.7). In the next result, we establish an equivalent formula to (4.3.7).

[24] It is not problematic to set $y \geq 0$, but since the case $y = 0$ is better viewed as a limiting form of the asymptotic behavior of the Bessel function $I_{\nu}(\cdot)$, (4.1.33), we leave the convention in this way.

Theorem 4.3.1. *Let $k \in \mathbb{N}$ and assume that $Re(\nu) > 0$ and $x > y > 0$. Then the following summation formula holds*

$$\sum_{n=1}^{\infty} r_k(n)\, I_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}x\right) = -\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu} + \frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}$$
$$\times \sum_{n=0}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\frac{k}{2}+\nu}}\, {}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\, \frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\, \nu+1;\, \left(\frac{2xy}{n+x^2+y^2}\right)^2\right). \qquad (4.3.10)$$

Proof. Throughout this proof, we shall assume that $\max\left\{-2\mathrm{Re}(\nu), -\frac{k}{2}-\mathrm{Re}(\nu)\right\} < \mu < 0$. The idea is to start with the infinite series at the left-hand side of (4.3.10) and use the integral representation (4.3.9). Since $|K_{s/2}(x)| \le K_{\mu/2}(x)$ when $\mathrm{Re}(s) = \mu$, we can invoke Fubini's theorem to interchange the orders of integration and summation during this process. Indeed,

$$\sum_{n=1}^{\infty} r_k(n) \int_{\mu-i\infty}^{\mu+i\infty}\left|\left(\frac{x-y}{4\pi xy}\right)^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\, n^{-\frac{s}{4}}\, K_{\frac{s}{2}}\left(2\pi\left(x-y\right)\sqrt{n}\right)\right|\,|ds|$$
$$\le \left(\frac{x-y}{4\pi xy}\right)^{\frac{\mu}{2}}\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\mu}{4}} K_{\frac{\mu}{2}}\left(2\pi(x-y)\sqrt{n}\right)\int_{\mu-i\infty}^{\mu+i\infty}\left|\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\right|\,|ds| < \infty, \qquad (4.3.11)$$

because, when $|t| \to \infty$, the ratio of Gamma functions has the asymptotic behavior

$$\left|\frac{\Gamma\left(\frac{\mu+it+2\nu}{2}\right)\Gamma\left(\frac{1-\mu-it}{2}\right)}{\Gamma\left(\frac{2-\mu-it+2\nu}{2}\right)}\right| = |t|^{\frac{\mu}{2}-1}\, e^{-\frac{\pi|t|}{4}}\left(1+O\left(\frac{1}{|t|}\right)\right), \quad |t| \to \infty.$$

Moreover, the infinite series in (4.3.11) converges because

$$K_{\frac{\mu}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}\left(1+O(z^{-1})\right), \quad z \to \infty.$$

Hence, the following way to represent the series on the left-hand side of (4.3.10) must be true

$$\sum_{n=1}^{\infty} r_k(n)\, I_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}x\right)$$
$$= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{1}{2\sqrt{\pi}}\left(\frac{x-y}{4\pi xy}\right)^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\sum_{n=1}^{\infty} r_k(n) n^{-\frac{s}{4}}\, K_{\frac{s}{2}}\left(2\pi\left(x-y\right)\sqrt{n}\right)\, ds. \qquad (4.3.12)$$

Since we started this argument with the assumption that $\mu < 0$, we can invoke the Bessel expansion (4.1.16), due to Popov, in order to evaluate the infinite series inside the integral in (4.3.12). Applying (4.1.16) with x being replaced by $x-y$ and taking $\nu = -\frac{s}{2}$, we find the transformation formula

$$\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{s}{4}}\, K_{s/2}\left(2\pi(x-y)\sqrt{n}\right) = -\frac{\pi^{\frac{s}{2}}\Gamma\left(-\frac{s}{2}\right)(x-y)^{\frac{s}{2}}}{2} + \frac{\Gamma\left(\frac{k-s}{2}\right)}{2\pi^{\frac{k-s}{2}}\,(x-y)^{\frac{s}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k-s}{2}}},$$

which, when written inside the integral (4.3.12), yields

$$\sum_{n=1}^{\infty} r_k(n)\, I_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}x\right)$$
$$= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{4\pi^{\frac{k+1}{2}}} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k-s}{2}}} \left(\frac{1}{2\sqrt{xy}}\right)^s ds$$
$$- \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \left(\frac{x-y}{2\sqrt{xy}}\right)^s ds. \tag{4.3.13}$$

We start by evaluating the first integral on the right-hand side of (4.3.13), which is far more technical than the second. First, let us check that it is possible to change the orders of integration and summation in this first integral. This verification is easy to handle since

$$\int_{\mu-i\infty}^{\mu+i\infty} \sum_{n=0}^{\infty} r_k(n) \left| \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \frac{1}{\left(n+(x-y)^2\right)^{\frac{k-s}{2}}} \left(\frac{1}{2\sqrt{xy}}\right)^s \right| |ds|$$
$$= \left(\frac{1}{2\sqrt{xy}}\right)^{\mu} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k-\mu}{2}}} \int_{\mu-i\infty}^{\mu+i\infty} \left| \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \right| |ds| < \infty, \tag{4.3.14}$$

where the absolute convergence of the integral is due to Stirling's formula and the convergence of the series can be justified by the hypothesis $\mu < 0$ and the well-known asymptotic estimate $r_k(n) = O\left(n^{\frac{k}{2}-1}\right)$. Therefore, the following equality must hold

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{4\pi^{\frac{k+1}{2}}} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k-s}{2}}} \left(\frac{1}{2\sqrt{xy}}\right)^s ds$$
$$= \frac{1}{4\pi^{\frac{k+1}{2}}} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k}{2}}} \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)} \left(\frac{2\sqrt{xy}}{\sqrt{n+(x-y)^2}}\right)^{-s} ds. \tag{4.3.15}$$

In the remainder of our argument, we will evaluate the integral showing up on the right-hand side of (4.3.15). This only requires to recall (4.3.2) with $z \in \mathbb{R}_+$, i.e., to write the Mellin-Barnes integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1\left(a,b;c;-z\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\,\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}\, z^{-s} ds, \tag{4.3.16}$$

for $0 < \gamma < \max\{\mathrm{Re}(a),\, \mathrm{Re}(b)\}$, $c \neq 0, -1, -2, ...$ and $z > 0$. Although the right-hand side of (4.3.15) is still not presented in the right shape to directly apply (4.3.16), we can perform a simple change of variables to obtain

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\left(\frac{2\sqrt{xy}}{\sqrt{n+(x-y)^2}}\right)^{-s}ds$$
$$=\left(\frac{4xy}{n+(x-y)^2}\right)^{\nu}\frac{2}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s)\,\Gamma\left(\frac{1}{2}+\nu-s\right)\Gamma\left(\frac{k}{2}+\nu-s\right)}{\Gamma(1+2\nu-s)}\left(\frac{4xy}{n+(x-y)^2}\right)^{-s}ds,$$

where $\sigma:=\frac{\mu}{2}+\text{Re}(\nu)$. Since $\max\left\{-2\text{Re}(\nu),-\frac{k}{2}-\text{Re}(\nu)\right\}<\mu<0$, we have that $\max\left\{0,-\frac{k}{4}+\frac{\text{Re}(\nu)}{2}\right\}<\sigma<\text{Re}(\nu)$, and so we can use (4.3.16) to get the evaluation

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\left(\frac{2\sqrt{xy}}{\sqrt{n+(x-y)^2}}\right)^{-s}ds$$
$$=\left(\frac{4xy}{n+(x-y)^2}\right)^{\nu}\frac{2\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{k}{2}+\nu\right)}{\Gamma(1+2\nu)}\,{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right)$$
$$=\frac{2\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+\nu\right)}{\Gamma(\nu+1)}\left(\frac{xy}{n+(x-y)^2}\right)^{\nu}{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right),\qquad(4.3.17)$$

where in the last step we have used Legendre's duplication formula for $\Gamma(s)$ [[221], p. 46],

$$\Gamma(2s)=\frac{2^{2s-1}\Gamma(s)\,\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}}.\qquad(4.3.18)$$

Analogously, the evaluation of the second and easier integral in (4.3.13) gives

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma\left(\frac{s+2\nu}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{2-s+2\nu}{2}\right)}\left(\frac{x-y}{2\sqrt{xy}}\right)^{s}ds$$
$$=\frac{2\sqrt{\pi}}{\nu}\left(\frac{xy}{(x-y)^2}\right)^{\nu}{}_2F_1\left(\nu,\frac{1}{2}+\nu;1+2\nu;-\frac{4xy}{(x-y)^2}\right)=\frac{2\sqrt{\pi}}{\nu}\left(\frac{y}{x}\right)^{\nu},\qquad(4.3.19)$$

where the last step consisted in using the reduction formula [[172], Vol. III, p. 461, eq. (7.3.1.105)],

$${}_2F_1\left(a,a+\frac{1}{2};2a+1;z\right)=\left(\frac{2}{1+\sqrt{1-z}}\right)^{2a}.\qquad(4.3.20)$$

Returning to (4.3.13) and invoking the evaluations (4.3.17) and (4.3.19), we are able to deduce the formula

$$\sum_{n=1}^{\infty}r_k(n)\,I_\nu\left(2\pi\sqrt{ny}\right)\,K_\nu\left(2\pi\sqrt{nx}\right)$$
$$=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\,\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+(x-y)^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right),$$
$$(4.3.21)$$

which is very close to what we want to achieve, (4.3.10). To conclude the proof of our theorem, we just need to apply the transformation formula for the hypergeometric function [[95], p. 1009, eq. (9.134.1)],

$$ {}_2F_1\left(a,b;2b;z\right)=\left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};b+\frac{1}{2};\left(\frac{z}{2-z}\right)^2\right), $$

we can see that

$$ \begin{aligned} &\frac{1}{\left(n+(x-y)^2\right)^{\frac{k}{2}+\nu}}{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right)\\ &=\frac{1}{\left(n+x^2+y^2\right)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\frac{k}{4}+\frac{\nu+1}{2};\,\nu+1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right). \end{aligned} \tag{4.3.22} $$

Replacing the right-hand side of (4.3.22) in (4.3.21) proves our Theorem 4.3.1. □

The formula just proved, (4.3.10), closely resembles (4.3.7), although it is not exactly the same identity. This is because we still need to transform the hypergeometric factor in (4.3.10) in order to achieve the symmetric form of (4.3.7). This will be addressed in the next corollary.

Corollary 4.3.1. *Let Re(ν) > 0 and $x>y>0$. Then formula (4.3.7) is valid, this is*

$$ \begin{aligned} &\sum_{n=1}^{\infty}r_k(n)\,I_\nu\left(2\pi\sqrt{n}y\right)\,K_\nu\left(2\pi\sqrt{n}x\right)=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}2^{k-1}\Gamma(\nu+1)}\\ &\times\sum_{n=0}^{\infty}r_k(n)\,\frac{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{k-1}{2}}}\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}\\ &\times{}_2F_1\left(1-\frac{k}{2}+\nu,\,1-\frac{k}{2};\,\nu+1;\,\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^2\right). \end{aligned} \tag{4.3.23} $$

Proof. Part of the steps shown below are already given in [43]. To simplify our notation, let $u(:=u(n))$ be the variable defined by

$$ u:=\frac{2xy}{n+x^2+y^2}, \tag{4.3.24} $$

where $n\in\mathbb{N}_0$ runs over the summation variable. Clearly, since $x>y>0$, we have that $0<u<1$. We have seen in (4.3.22) that the following transformation is valid

$$ \begin{aligned} &\frac{1}{\left(n+(x-y)^2\right)^{\frac{k}{2}+\nu}}{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right)\\ &=\frac{1}{\left(n+x^2+y^2\right)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\frac{k}{4}+\frac{\nu+1}{2};\,\nu+1;\,u^2\right). \end{aligned} $$

To reach (4.3.23), we shall invoke yet another transformation [[95], p. 1009, eq. (9.134.2)]

$$ {}_2F_1\left(a,b;a-b+1;z\right)=(1+z)^{-a}\,{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\,\frac{4z}{(1+z)^2}\right),\quad |z|<1, \tag{4.3.25} $$

and apply it taking the parameters

$$a = \frac{k}{2} + \nu,\ b = \frac{k}{2},\ z = \frac{1-\sqrt{1-u^2}}{1+\sqrt{1-u^2}}. \tag{4.3.26}$$

This yields

$$\begin{aligned}
&\frac{1}{(n+(x-y)^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{1}{2}+\nu, \frac{k}{2}+\nu; 1+2\nu; -\frac{4xy}{n+(x-y)^2}\right)\\
&= \frac{1}{(n+x^2+y^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2}, \frac{k}{4}+\frac{\nu+1}{2};\ \nu+1;\ u^2\right)\\
&= \frac{1}{(n+x^2+y^2)^{\frac{k}{2}+\nu}}\,\frac{2^{\frac{k}{2}+\nu}}{\left(1+\sqrt{1-u^2}\right)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{2}+\nu, \frac{k}{2};\ \nu+1;\ \frac{1-\sqrt{1-u^2}}{1+\sqrt{1-u^2}}\right)\\
&= \frac{1}{(n+x^2+y^2)^{\frac{k}{2}+\nu}}\,\frac{2^{\nu+1-\frac{k}{2}}\,(1-u^2)^{\frac{1-k}{2}}}{\left(1+\sqrt{1-u^2}\right)^{\nu+1-\frac{k}{2}}}\,{}_2F_1\left(\nu+1-\frac{k}{2}, 1-\frac{k}{2}; \nu+1; \frac{1-\sqrt{1-u^2}}{1+\sqrt{1-u^2}}\right),
\end{aligned} \tag{4.3.27}$$

where in the last step we have used Euler's transformation formula (4.3.5). We can simplify the last expression in (4.3.27) even further. Simple algebraic procedures show that, according to the definition of u, (4.3.24),

$$\sqrt{1-u^2} = \frac{\sqrt{(n+x^2+y^2)^2-4x^2y^2}}{n+x^2+y^2} = \frac{\sqrt{(n+(x+y)^2)\,(n+(x-y)^2)}}{n+x^2+y^2},$$

and so

$$\begin{aligned}
\frac{1-\sqrt{1-u^2}}{1+\sqrt{1-u^2}} &= \frac{n+x^2+y^2-\sqrt{(n+(x+y)^2)\,(n+(x-y)^2)}}{n+x^2+y^2+\sqrt{(n+(x+y)^2)\,(n+(x-y)^2)}}\\
&= \frac{n+x^2+y^2-2\sqrt{\left(\frac{1}{2}n+\frac{1}{2}(x+y)^2\right)\left(\frac{1}{2}n+\frac{1}{2}(x-y)^2\right)}}{n+x^2+y^2+2\sqrt{\left(\frac{1}{2}n+\frac{1}{2}(x+y)^2\right)\left(\frac{1}{2}n+\frac{1}{2}(x-y)^2\right)}}\\
&= \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^2.
\end{aligned}$$

These simplifications show that the last hypergeometric factor in (4.3.27) is given by

$$\begin{aligned}
&{}_2F_1\left(\nu+1-\frac{k}{2}, 1-\frac{k}{2}; \nu+1; \frac{1-\sqrt{1-u^2}}{1+\sqrt{1-u^2}}\right)\\
&= {}_2F_1\left(\nu+1-\frac{k}{2}, 1-\frac{k}{2}; \nu+1; \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^2\right),
\end{aligned} \tag{4.3.28}$$

which agrees with the corresponding term in (4.3.7). In order to manipulate the fractions in (4.3.27) that are dependent on u, we just recall the elementary algebraic expressions

$$\frac{(4xy)^{\nu}}{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{2\nu}}=\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu},$$

$$\sqrt{1-u^2}\left(n+x^2+y^2\right)=\sqrt{(n+(x+y)^2)\,(n+(x-y)^2)}=\sqrt{n^2+2n\,(x^2+y^2)+(x^2-y^2)^2},$$

so that

$$\begin{aligned}
&\frac{1}{(n+x^2+y^2)^{\frac{k}{2}+\nu}}\,\frac{2^{\nu+1-\frac{k}{2}}\left(1-u^2\right)^{\frac{1-k}{2}}}{\left(1+\sqrt{1-u^2}\right)^{\nu+1-\frac{k}{2}}}\\
&=\frac{2^{\nu+1-\frac{k}{2}}\left(1-u^2\right)^{\frac{1-k}{2}}\left(n+x^2+y^2\right)^{1-k}}{\left(n+x^2+y^2+\sqrt{(n+(x+y)^2)\,(n+(x-y)^2)}\right)^{\nu+1-\frac{k}{2}}}\\
&=\frac{2^{\nu+1-\frac{k}{2}}\left(n^2+2n\,(x^2+y^2)+\left(x^2-y^2\right)^2\right)^{\frac{1-k}{2}}}{\left(n+x^2+y^2+2\sqrt{\left(\frac{1}{2}n+\frac{1}{2}(x+y)^2\right)\left(\frac{1}{2}n+\frac{1}{2}(x-y)^2\right)}\right)^{\nu+1-\frac{k}{2}}}\\
&=\frac{2^{2\nu+2-k}\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\,(x^2+y^2)+(x^2-y^2)^2\right)^{\frac{k-1}{2}}\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{2\nu}}\\
&=\frac{2^{2-k}(xy)^{-\nu}\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\,(x^2+y^2)+(x^2-y^2)^2\right)^{\frac{k-1}{2}}}\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}.
\end{aligned}\tag{4.3.29}$$

Combining (4.3.29) with (4.3.28) and (4.3.27) and, at last, returning to (4.3.21), we find that

$$\begin{aligned}
&\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\,\Gamma\left(\nu+1\right)}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+(x-y)^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{1}{2}+\nu,\frac{k}{2}+\nu;1+2\nu;-\frac{4xy}{n+(x-y)^2}\right)=\frac{\Gamma\left(\frac{k}{2}+\nu\right)}{2^{k-1}\pi^{\frac{k}{2}}\,\Gamma\left(\nu+1\right)}\\
&\times\sum_{n=0}^{\infty}r_k(n)\,\frac{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\,(x^2+y^2)+(x^2-y^2)^2\right)^{\frac{k-1}{2}}}\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}\\
&\times\,{}_2F_1\left(\nu+1-\frac{k}{2},1-\frac{k}{2};\nu+1;\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{2}\right),
\end{aligned}$$

which completes the proof of Corollary 4.3.1. $\square$

An entirely analogous proof to the one presented above can be written while studying the infinite series

$$\sum_{n=1}^{\infty}r_k(n)\,J_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right),$$

where the Bessel $J-$function takes the role of the Bessel $I-$function in (4.3.10). This study analogously starts with the representation

$$J_\nu\left(b\sqrt{\sqrt{u^2+a^2}-a}\right)K_\nu\left(b\sqrt{\sqrt{u^2+a^2}+a}\right)$$
$$=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{\frac{3s}{2}-1}\left(\frac{a}{b^2}\right)^{\frac{s}{2}}\frac{\Gamma\left(\frac{s+\nu}{2}\right)}{\Gamma\left(\frac{2-s+\nu}{2}\right)}K_s\left(\sqrt{2a}\,b\right)u^{-s}ds,$$

which is valid for $-\mathrm{Re}(\nu)<\mu<0$ and $a,\,b,\,u>0$. If we use the previous integral with $a=\frac{x^2-y^2}{2}$, $b=2\pi\sqrt{n}$ and $u=xy$, we obtain the Mellin-Barnes representation

$$J_\nu\left(2\pi\sqrt{n}y\right)\,K_\nu\left(2\pi\sqrt{n}x\right)=\frac{1}{4\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\left(\frac{1}{y^2}-\frac{1}{x^2}\right)^{\frac{s}{2}}\frac{\pi^{-s}\Gamma\left(\frac{s+\nu}{2}\right)}{\Gamma\left(\frac{2-s+\nu}{2}\right)}n^{-\frac{s}{2}}K_s\left(2\pi\sqrt{(x^2-y^2)\,n}\right)ds, \tag{4.3.30}$$

holding when $x>y>0$ and $-\mathrm{Re}(\nu)<\mu<0$. Using the same procedure as in the proof of Theorem 4.3.1, i.e., inserting the arithmetical function $r_k(n)$ in (4.3.30), taking the summation over $n\in\mathbb{N}$ and, in the end, appealing to (4.1.16), we are able to prove the following result.

Theorem 4.3.2. *Let Re$(\nu)>0$ and $x>y>0$. Then the following summation formula holds*

$$\sum_{n=1}^{\infty}r_k(n)\,J_\nu\left(2\pi\sqrt{n}y\right)\,K_\nu\left(2\pi\sqrt{n}x\right)=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^\nu}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}$$
$$\times\sum_{n=0}^{\infty}\frac{r_k(n)}{\left(n+x^2-y^2\right)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\,\frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\,\nu+1;\,-\left(\frac{2xy}{n+x^2-y^2}\right)^2\right). \tag{4.3.31}$$

Remark 4.3.1. We have started this section with an integral representation for the product of Bessel functions that involves the index of the Macdonald function (see (4.3.8) above). Here we remark that yet another proof of Theorem 4.3.1 is available. Instead of using (4.1.16), this alternative proof only invokes the equivalent theta transformation formula (4.1.7). We may start with the product representation[25]

$$I_\nu\left(2\pi\sqrt{n}y\right)\,K_\nu\left(2\pi\sqrt{n}x\right)=\frac{1}{2}\int_0^\infty\frac{1}{u}\exp\left\{-\frac{1}{4}\left(x^2+y^2\right)u-\frac{4\pi^2n}{u}\right\}I_\nu\left(\frac{xy}{2}u\right)du, \tag{4.3.32}$$

which can be found in [[172], Vol. II, p. 307, relation 2.15.6.4]. Multiplying both sides of (4.3.32) by $r_k(n)$ and taking the summation over n, one can easily argue by absolute convergence and Fubini's

[25] the integral representation (4.3.32) is used by Koshliakov [133] to arrive at his integral formula (4.1.39). Since the approach in [43] uses (4.1.39) to derive Theorem 4.3.1, the proof presented in this remark is deeply connected with the proof of (4.3.10) that uses Voronoï's summation formula.

theorem that

$$\begin{aligned}\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right) &= \frac{1}{2}\int_0^{\infty}\frac{1}{u}\sum_{n=1}^{\infty} r_k(n)\, e^{-\frac{4\pi^2}{u}n}\, e^{-\frac{x^2+y^2}{4}u}\, I_{\nu}\left(\frac{xy}{2}u\right)\, du \\ &= \frac{1}{2}\int_0^{\infty}\frac{1}{u}\, e^{-\frac{x^2+y^2}{4}u}\left\{\left(\frac{u}{4\pi}\right)^{\frac{k}{2}}\sum_{n=0}^{\infty} r_k(n)\, e^{-\frac{u}{4}n}-1\right\} I_{\nu}\left(\frac{xy}{2}u\right)\, du \\ &= \frac{1}{2}\,(4\pi)^{-\frac{k}{2}}\sum_{n=0}^{\infty} r_k(n)\int_0^{\infty} u^{\frac{k}{2}-1}e^{-\frac{n+x^2+y^2}{4}u} I_{\nu}\left(\frac{xy}{2}u\right)\, du \\ &\quad -\frac{1}{2}\int_0^{\infty}\frac{1}{u}\, e^{-\frac{x^2+y^2}{4}u}\, I_{\nu}\left(\frac{xy}{2}u\right)\, du,\end{aligned}$$

where the second equality results from the theta transformation formula (4.1.7). The remaining integrals can now be found by using relation 3.13.2.1 on page 191 of [48], which presents the Mellin tranform

$$\begin{aligned}\int_0^{\infty} u^{\frac{k}{2}-1}e^{-\frac{n+x^2+y^2}{4}u} I_{\nu}\left(\frac{xy}{2}u\right)\, du &= \left(\frac{xy}{n+x^2+y^2}\right)^{\nu}\left(\frac{2}{\sqrt{n+x^2+y^2}}\right)^{k}\frac{\Gamma\left(\frac{k}{2}+\nu\right)}{\Gamma\left(\nu+1\right)} \\ &\quad\times\, {}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\, \frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\, \nu+1;\, \left(\frac{2xy}{n+x^2+y^2}\right)^2\right),\end{aligned}$$

yielding the transformation formula

$$\begin{aligned}\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right) &= -\frac{1}{2\nu}\left(\frac{x^2+y^2}{xy}\right)^{-\nu}\, {}_2F_1\left(\frac{\nu}{2},\, \frac{\nu+1}{2};\, \nu+1;\, \left(\frac{2xy}{x^2+y^2}\right)^2\right) \\ +\frac{\Gamma\left(\frac{k}{2}+\nu\right)\,(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}\sum_{n=0}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\frac{k}{2}+\nu}}\, &{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\, \frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\, \nu+1;\, \left(\frac{2xy}{n+x^2+y^2}\right)^2\right).\end{aligned}$$

Finally, if we use (4.3.20), we can see that the first term on the right-hand side of the previous equality is

$$\begin{aligned}&-\frac{1}{2\nu}\left(\frac{x^2+y^2}{xy}\right)^{-\nu}\, {}_2F_1\left(\frac{\nu}{2},\, \frac{\nu+1}{2};\, \nu+1;\, \left(\frac{2xy}{x^2+y^2}\right)^2\right) \\ &= -\frac{1}{2\nu}\,(xy)^{\nu}\left(\frac{2}{x^2+y^2+\sqrt{(x^2+y^2)^2-4x^2y^2}}\right)^{\nu} = -\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu},\end{aligned}$$

resulting in a new proof of Theorem 4.3.1. □

4.4 A general summation formula attached to products of Bessel functions

4.4.1 Proof of a general formula involving the product $I_{\mu}(X)\, K_{\nu}(x)$

As we had the opportunity to mention at the introduction to this chapter, our goal is to study transformation formulas for the infinite series

$$S_k(x,y;\,\mu,\nu) := \sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, I_{\mu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right), \quad x>y>0. \tag{4.4.1}$$

Since both formulas (4.4.39) and (4.1.15) contain the Gauss hypergeometric function, we can expect that, in the less general possible scenario, a transformation formula for (4.4.1) would also contain it. However, as we hope to show below, (4.4.1) will transform into a series involving an even more general hypergeometric function, called Appell series, which we will briefly introduce now, following closely the concise introduction to this topic given by Schlosser [198].

Appell series are the natural extensions of the hypergeometric series (4.3.1) to two variables. There are four nontrivial such functions, which are defined by

$$F_1\left(\alpha;\,\beta,\beta';\,\gamma;\,w,z\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m\,(\beta')_n}{m!\,n!\,(\gamma)_{m+n}}\, w^m z^n, \quad |w|,|z|<1; \tag{4.4.2}$$

$$F_2\left(\alpha;\,\beta,\beta';\gamma,\gamma';\,w,z\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_m(\gamma')_n}\, w^m z^n, \quad |w|+|z|<1; \tag{4.4.3}$$

$$F_3\left(\alpha,\alpha';\beta,\beta';\gamma;w,z\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{m!\,n!\,(\gamma)_{m+n}}\, w^m z^n, \quad |w|,\,|z|<1; \tag{4.4.4}$$

$$F_4\left(\alpha;\,\beta;\,\gamma,\gamma';\,w,z\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!\,n!\,(\gamma)_m(\gamma')_n}\, w^m z^n, \quad \sqrt{|w|}+\sqrt{|z|}<1. \tag{4.4.5}$$

For a nice discussion about the convergence of such series, the reader is encouraged to consult [[205], pp. 211-213].

Since the Appell function F_4 generalizes the classical hypergeometric function ${}_2F_1(a,b;c;z)$, it is also helpful to our study to give an introduction to generalizations of the Kummer function, ${}_1F_1(a;c;z)$, known as Humbert functions. The study of Humbert functions started with the work of P. Humbert [117] in 1922. In the same way that Kummer's hypergeometric function ${}_1F_1(a;c;z)$ can be defined as a limiting case of ${}_2F_1(a,b;c;z)$, Humbert defined the functions in his paper as limiting forms of the series (4.4.2) - (4.4.5). The functions of Humbert type that will be important in this chapter are respectively defined by the double series

$$\Psi_2\left(a;\,b,c;\,w,z\right) := \sum_{k,m=0}^{\infty} \frac{(a)_{k+m}}{(b)_k(c)_m}\, \frac{w^k z^m}{k!\,m!} \tag{4.4.6}$$

and

$$\Phi_3\left(b;c;\,w,z\right)=\sum_{k,m=0}^{\infty}\frac{(b)_k}{(c)_{k+m}}\,\frac{w^k z^m}{k!\,m!}, \tag{4.4.7}$$

both converging absolutely for any $w,z\in\mathbb{C}$. Although these functions initially appear to be independent of each other, their structure conceals the transformation formulas satisfied by Appell's hypergeometric functions. As far as we know, Manako [150] seems to have been the first mathematician to have noticed the following interesting connection.

Lemma 4.4.1. *For $b,c\neq 0,-1,-2,...,$ $\Psi_2\left(a;\,b,c;\,w,z\right)$ and $\Phi_3\left(b;c;\,w,z\right)$ are connected by*

$$\Psi_2\left(b;\,b,c;\,w,z\right)=e^{w+z}\,\Phi_3\left(c-b;\,c;\,-z,\,wz\right). \tag{4.4.8}$$

Recall from Corollary 2.2.4 that our proof of (4.1.15) employed Popov's formula (4.1.1). But since we want to reach a transformation formula for (4.4.1), we may need to start our study with the more general infinite series

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\nu}{2}}}\,e^{-\pi n x}\,J_{\nu}(\sqrt{\pi n x}\,z), \tag{4.4.9}$$

where $\text{Re}(\nu)>-1$, $\text{Re}(x)>0$ and $z\in\mathbb{C}$. Note that, when $\nu=\frac{k}{2}-1$, (4.4.9) reduces to the first infinite series on Popov's formula (4.1.1). The next lemma establishes a transformation formula for a general series of the form (4.4.9).

Lemma 4.4.2. *For any $Re(\nu)>-1$ and $Re(x)>0$, $z\in\mathbb{C}$, the following summation formula holds*

$$\begin{aligned}\pi^{-\frac{\nu}{2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\nu}{2}}}\,e^{-\pi n x}\,J_{\nu}(\sqrt{\pi n x}\,z)&=-\frac{x^{\frac{\nu}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}+\frac{x^{\frac{\nu-k}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{k}{2};\,\nu+1;\,-\frac{z^2}{4}\right)\\&+\frac{x^{\frac{\nu-k}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}e^{-\frac{z^2}{4}}\sum_{n=1}^{\infty}r_k(n)e^{-\frac{\pi n}{x}}\,\Phi_3\left(1-\frac{k}{2}+\nu;\nu+1;\frac{z^2}{4},\frac{\pi z^2 n}{4x}\right),\end{aligned} \tag{4.4.10}$$

where $\Phi_3(b;c;w,z)$ is the usual Humbert function,

$$\Phi_3(b;c;\,w,z)=\sum_{k,m=0}^{\infty}\frac{(b)_k}{(c)_{k+m}}\,\frac{w^k z^m}{k!\,m!},\quad z,w\in\mathbb{C}. \tag{4.4.11}$$

Under the same conditions, the analogous summation formula takes place

$$\begin{aligned}\pi^{-\frac{\nu}{2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\nu}{2}}}\,e^{-\pi n x}\,I_{\nu}(\sqrt{\pi n x}\,z)&=-\frac{x^{\frac{\nu}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}+\frac{x^{\frac{\nu-k}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}\,{}_1F_1\left(\frac{k}{2};\,\nu+1;\,\frac{z^2}{4}\right)\\&+\frac{x^{\frac{\nu-k}{2}}z^{\nu}}{2^{\nu}\Gamma(\nu+1)}e^{\frac{z^2}{4}}\sum_{n=1}^{\infty}r_k(n)e^{-\frac{\pi n}{x}}\,\Phi_3\left(1-\frac{k}{2}+\nu;\nu+1;-\frac{z^2}{4},-\frac{\pi z^2 n}{4x}\right),\end{aligned} \tag{4.4.12}$$

Note that we have already stated this result in Section 3.6 of the previous chapter. We shall omit its proof because it is fairly uninteresting and it contains no more ideas than the proof of Theorem 2.2.1 itself. The only technical aspect that seems to not be fully covered by the argument in the second chapter is the convergence of the series involving Humbert's function in (4.4.11) and (4.4.12). The next lemma is more than enough to justify this particular issue.

Lemma 4.4.3. *For any complex z, $w \in \mathbb{C}$ and $Re(c) > 0$, the following inequality holds*

$$|\Phi_3(b; c;\, w, z)| \leq 1 + |z|\, e^{2\sqrt{|z|}} + e^{2\sqrt{|z|}+2|w|+\frac{|bw|}{|c|}}. \tag{4.4.13}$$

Proof. We know that, for $\mathrm{Re}(\mu) \geq 0$ and $m \in \mathbb{N}$, the elementary inequality is valid

$$|\Gamma(\mu + m + 1)| = |\Gamma(\mu + 1)(1 + \mu) \cdot \ldots \cdot (m + \mu)| \geq m!\, |\Gamma(\mu + 1)|. \tag{4.4.14}$$

Therefore, for $\mathrm{Re}(c) > 0$, the Humbert function obeys to the set of inequalities

$$\begin{aligned}
|\Phi_3(b; c;\, w, z)| &\leq \left|\sum_{m=0}^{\infty} \frac{z^m}{m!(c)_m}\right| + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_{k+m}}\right| \frac{|w|^k|z|^m}{k!\, m!} \\
&= \left|\Gamma(c) \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(c+m)}\right| + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \left|\frac{\Gamma(c+k)}{\Gamma(c+k+m)}\right| \frac{|w|^k|z|^m}{k!\, m!} \\
&\leq 1 + \sum_{m=1}^{\infty} \frac{|z|^m}{m!\,(m-1)!} + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \frac{|w|^k|z|^m}{k!\,(m!)^2} \\
&\leq 1 + |z| \sum_{m=0}^{\infty} \frac{|z|^m}{m!^2} + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \frac{|w|^k|z|^m}{k!\,(m!)^2} \\
&\leq 1 + |z| \left(\sum_{m=0}^{\infty} \frac{|z|^{\frac{m}{2}}}{m!}\right)^2 + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \frac{|w|^k|z|^m}{k!\,(m!)^2} \\
&= 1 + |z|\, e^{2\sqrt{|z|}} + \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \frac{|w|^k|z|^m}{k!\,(m!)^2},
\end{aligned} \tag{4.4.15}$$

where in the third step we have invoked (4.4.14) with $\mu = c + k - 1$. To finally bound the last double infinite series in (4.4.15), we note the simple relation

$$\left|\frac{(b)_k}{(c)_k}\right| = \prod_{j=0}^{k-1} \left|\frac{b+j}{c+j}\right| = \prod_{j=0}^{k-1} \left|1 + \frac{b-c}{c+j}\right| \leq \left(2 + \left|\frac{b}{c}\right|\right)^k,$$

which establishes

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \left|\frac{(b)_k}{(c)_k}\right| \frac{|w|^k|z|^m}{k!\,(m!)^2} \leq \sum_{m=0}^{\infty} \frac{|z|^m}{m!^2} \cdot \sum_{k=1}^{\infty} \frac{|w|^k}{k!} \left(2 + \left|\frac{b}{c}\right|\right)^k \\
&\leq \left(\sum_{m=0}^{\infty} \frac{|z|^{\frac{m}{2}}}{m!}\right)^2 \left(e^{2|w|+\frac{|bw|}{|c|}} - 1\right) \leq e^{2\sqrt{|z|}+2|w|+\frac{|bw|}{|c|}}.
\end{aligned}$$

□

In the next corollary we establish an identity equivalent to (4.4.10), in which the role of Φ_3 is replaced by Ψ_2. Its proof is a simple consequence of Manako's formula (4.4.8), and so we shall omit it as well.

Corollary 4.4.1. *For any positive integer $k \geq 1$ and $Re(x) > 0$, $z \in \mathbb{C}$, the following transformation formula holds*

$$\pi^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{\nu}{2}}} e^{-\pi x x} J_\nu\left(\sqrt{\pi x\, n}\, z\right) = -\frac{x^{\nu/2} z^\nu}{2^\nu \Gamma(\nu+1)} + \frac{x^{\frac{\nu-k}{2}} z^\nu}{2^\nu \Gamma(\nu+1)}\, {}_1F_1\left(\frac{k}{2};\, \nu+1;\, -\frac{z^2}{4}\right)$$
$$+ \frac{x^{\frac{\nu-k}{2}} z^\nu}{2^\nu \Gamma(\nu+1)} \sum_{n=1}^{\infty} r_k(n)\, \Psi_2\left(\frac{k}{2};\, \frac{k}{2}, \nu+1; -\frac{\pi n}{x},\, -\frac{z^2}{4}\right), \quad (4.4.16)$$

where $\Psi_2(a; b, c; w, z)$ denotes the Humbert function,

$$\Psi_2\left(a;\, b, c;\, w, z\right) := \sum_{k,m=0}^{\infty} \frac{(a)_{k+m}}{(b)_k (c)_m} \frac{w^k z^m}{k!\, m!}. \quad (4.4.17)$$

Remark 4.4.1. Combining the inequality for Humbert's function Φ_3, (4.4.13), with Manako's formula (4.4.13), we have that the Humbert function $\Psi_2\left(b; b, c; w, z\right)$ obeys to the following inequality

$$\left|\Psi_2\left(b;\, b, c;\, w, z\right)\right| \leq e^{\mathrm{Re}(w+z)} \left(1 + |wz|\, e^{2\sqrt{|wz|}} + e^{2\sqrt{|wz|} + 2|z| + \frac{|c-b|}{|c|}|z|}\right),$$

which shows that, as $n \to \infty$

$$\left|\Psi_2\left(\frac{k}{2};\, \frac{k}{2}, \nu+1; -\frac{\pi n}{x},\, -\frac{z^2}{4}\right)\right| \leq e^{-\pi n \mathrm{Re}\left(\frac{1}{x}\right) - \frac{1}{4}\mathrm{Re}(z^2)} \left\{1 + \left(\frac{\pi |z|^2 n}{4x} + e^{\frac{|z|^2}{2}\left(1 + \frac{|2\nu+2-k|}{4|\nu+1|}\right)}\right) e^{|z|\sqrt{\frac{\pi n}{x}}}\right\}. \quad (4.4.18)$$

The previous estimate assures the absolute convergence of the infinite series lying at the right-hand side of (4.4.16).

In the next corollary we show that (4.4.10) implies Popov's formula (4.1.1), indicating that we are in the right direction to get a generalization of the formulas (4.1.15) and (4.1.35).

Corollary 4.4.2. *For $Re(x) > 0$ and $z \in \mathbb{C}$, the following transformation formula holds*

$$\frac{z^{\frac{k}{2}-1} \pi^{\frac{k}{4}-\frac{1}{2}} x^{\frac{k}{4}}}{2^{\frac{k}{2}-1} \Gamma\left(\frac{k}{2}\right)} e^{z^2/8} + \sqrt{x}\, e^{z^2/8} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\pi n x} J_{\frac{k}{2}-1}(\sqrt{\pi n x} z)$$
$$= \frac{z^{\frac{k}{2}-1} \pi^{\frac{k}{4}-\frac{1}{2}} x^{-\frac{k}{4}}}{2^{\frac{k}{2}-1} \Gamma\left(\frac{k}{2}\right)} e^{-z^2/8} + \frac{e^{-z^2/8}}{\sqrt{x}} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\frac{\pi n}{x}} I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}} z\right). \quad (4.4.19)$$

Proof. If we take $\nu = \frac{k}{2} - 1$ in (4.4.10) and appeal to the particular values

$${}_1F_1\left(c; c;\, z\right) = e^z,$$

as well as

$$\Phi_3(0; c;\, w, z) = \sum_{k,m=0}^{\infty} \frac{(0)_k}{(c)_{k+m}} \frac{w^k z^m}{k!\, m!} = \Gamma(c) \sum_{m=0}^{\infty} \frac{z^m}{m! \Gamma(c+m)}$$
$$= \Gamma(c) z^{-\frac{c-1}{2}} I_{c-1}\left(2\sqrt{z}\right),$$

we find (4.4.19) immediately.

□

We are now ready to prove the main theorem of this section. Before proceeding to its detailed proof, we still need a Mellin transform that can be found in [[48], p. 532, relation 3.35.2.10].

Lemma 4.4.4. *For $Re(p)$, $Re(z)$, $Re(w)$, $Re(s)>0$, the following Mellin transform holds*

$$\int_0^{\infty} x^{s-1}e^{-px}\Psi_2\left(a;c,c';wx,zx\right)\,dx=\frac{\Gamma(s)}{p^s}\,F_4\left(s;a;c,c';\,\frac{w}{p},\frac{z}{p}\right). \tag{4.4.20}$$

Theorem 4.4.1. *Let $Re(\mu)>-1$, $Re(\nu)>0$ and $x>y>0$. Then the following summation formula holds*

$$\begin{aligned}&\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu-\mu}{2}}\,J_{\mu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)\\&=-\frac{\Gamma(\nu)\,\pi^{\frac{\mu-\nu}{2}}x^{-\nu}y^{\mu}}{2\Gamma(\mu+1)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,\pi^{\frac{\mu-\nu-k}{2}}x^{-\nu-k}y^{\mu}}{2\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\mu+1;\,-\frac{y^2}{x^2}\right)\\&+\frac{x^{\nu}y^{\mu}\pi^{\frac{\mu-\nu-k}{2}}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\nu+\frac{k}{2}}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\mu+1,\,\nu+1,\,\frac{y^2}{n},\,-\frac{x^2}{n}\right),\end{aligned} \tag{4.4.21}$$

where F_4 is Appell's double hypergeometric function (4.4.5), defined by the power series

$$F_4\left(\alpha;\beta;\gamma,\gamma';w,z\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_{m+n}}{(\gamma)_m\,(\gamma')_n}\,\frac{w^m z^n}{m!\,n!},\quad |w|^{\frac{1}{2}}+|z|^{\frac{1}{2}}<1. \tag{4.4.22}$$

Under the same conditions, we have that

$$\begin{aligned}&\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu-\mu}{2}}\,I_{\mu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)\\&=-\frac{\Gamma(\nu)\,\pi^{\frac{\mu-\nu}{2}}x^{-\nu}y^{\mu}}{2\Gamma(\mu+1)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,\pi^{\frac{\mu-\nu-k}{2}}x^{-\nu-k}y^{\mu}}{2\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\mu+1;\,\frac{y^2}{x^2}\right)\\&+\frac{x^{\nu}y^{\mu}\pi^{\frac{\mu-\nu-k}{2}}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\nu+\frac{k}{2}}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\mu+1,\,\nu+1,\,-\frac{y^2}{n},\,-\frac{x^2}{n}\right).\end{aligned} \tag{4.4.23}$$

Proof. First, let us note that, for fixed $x>y>0$, there exists a sufficiently large N such that, for any $n>N$, $y^2/n<1$ and $x^2/n<1$. Thus, using the power series for the Appell function F_4, (4.4.5), one easily sees that

$$F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\mu+1,\,\nu+1,\,\frac{y^2}{n},\,-\frac{x^2}{n}\right)\sim 1,\quad \text{as } n\to\infty,$$

and so, since $r_k(n) = O\left(n^{\frac{k}{2}-1}\right)$, we get the bound

$$\left|\frac{r_k(n)}{n^{\nu+\frac{k}{2}}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\mu+1,\,\nu+1,\,\frac{y^2}{n},\,-\frac{x^2}{n}\right)\right| \ll_k n^{-\text{Re}(\nu)-1},$$

which assures the absolute convergence of the infinite series on the right-hand side of (4.4.21) when $\text{Re}(\nu)>0$. We now address the explicit proof of the formula (4.4.21) and start by evaluating the following infinite series

$$\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, J_\mu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}\,x\right), \tag{4.4.24}$$

and by appealing to the integral representation

$$\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}e^{-\frac{\pi n}{u}}\,du = 2x^{-\nu}n^{\frac{\nu}{2}}\,K_\nu\left(2\pi\sqrt{n}x\right),$$

which we have already used in (4.2.14). An argument by absolute convergence is enough to justify the following relation

$$\begin{aligned}
&\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, J_\mu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}\,x\right)\\
=&\frac{x^\nu\pi^{\frac{\nu-\mu}{2}}}{2}\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\mu}{2}}\, J_\mu\left(2\pi\sqrt{n}y\right)\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}e^{-\frac{\pi n}{u}}\,du\\
=&\frac{x^\nu\pi^{\frac{\nu}{2}}}{2}\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}\,\pi^{-\frac{\mu}{2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\mu}{2}}}\,J_\mu\left(2\pi\sqrt{n}y\right)e^{-\frac{\pi n}{u}}\,du,
\end{aligned}\tag{4.4.25}$$

which is now presented in the right template to apply the summation formula (4.4.16). Since $\text{Re}(\mu)>-1$ by hypothesis, we immediately find

$$\begin{aligned}
&\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, J_\mu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}\,x\right)\\
=&\frac{x^\nu\pi^{\frac{\nu}{2}}}{2}\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}\,\pi^{-\frac{\mu}{2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{\mu}{2}}}\,J_\mu\left(2\pi\sqrt{n}y\right)e^{-\frac{\pi n}{u}}\,du\\
=&\frac{x^\nu\pi^{\frac{\nu}{2}}}{2}\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}\left\{-\frac{(\sqrt{\pi}y)^\mu}{\Gamma(\mu+1)}+\frac{u^{\frac{k}{2}}(\sqrt{\pi}y)^\mu}{\Gamma(\mu+1)}\,{}_1F_1\left(\frac{k}{2};\,\mu+1;\,-\pi u y^2\right)\right\}du\\
+&\frac{x^\nu\pi^{\frac{\nu}{2}}}{2}\int_0^\infty u^{\nu-1}\,e^{-\pi x^2 u}\,\frac{u^{\frac{k}{2}}\,(\sqrt{\pi}y)^\mu}{\Gamma(\mu+1)}\sum_{n=1}^{\infty} r_k(n)\,\Psi_2\left(\frac{k}{2};\,\frac{k}{2},\mu+1;-\pi n u,\,-\pi u y^2\right)du.
\end{aligned}\tag{4.4.26}$$

The first integral in (4.4.26) can be explicitly evaluated as follows

$$\begin{aligned}
&\frac{x^{\nu}\pi^{\frac{\nu}{2}}}{2}\int_0^{\infty}u^{\nu-1}\,e^{-\pi x^2 u}\left\{-\frac{(\sqrt{\pi}y)^{\mu}}{\Gamma(\mu+1)}+\frac{u^{\frac{k}{2}}(\sqrt{\pi}y)^{\mu}}{\Gamma(\mu+1)}\,{}_1F_1\left(\frac{k}{2};\,\mu+1;\,-\pi u y^2\right)\right\}du\\
&=-\frac{\pi^{\frac{\mu+\nu}{2}}x^{\nu}y^{\mu}}{2\Gamma(\mu+1)}\int_0^{\infty}u^{\nu-1}e^{-\pi x^2 u}du+\frac{\pi^{\frac{\mu+\nu}{2}}x^{\nu}y^{\mu}}{2\Gamma(\mu+1)}\int_0^{\infty}u^{\nu+\frac{k}{2}-1}e^{-\pi x^2 u}\,{}_1F_1\left(\frac{k}{2};\,\mu+1;\,-\pi u y^2\right)du\\
&=-\frac{\Gamma(\nu)\,\pi^{\frac{\mu-\nu}{2}}x^{-\nu}y^{\mu}}{2\Gamma(\mu+1)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,\pi^{\frac{\mu-\nu-k}{2}}x^{-\nu-k}y^{\mu}}{2\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\mu+1;\,-\frac{y^2}{x^2}\right),
\end{aligned}\tag{4.4.27}$$

where the last step is a direct consequence of the Mellin transform [[48], p. 409, relation 3.28.2.1]

$$\int_0^{\infty}x^{s-1}e^{-\sigma x}\,{}_1F_1\left(a;c;\omega x\right)\,dx=\frac{\Gamma(s)}{\sigma^s}\,{}_2F_1\left(a,\,s;\,c;\,\frac{\omega}{\sigma}\right),\tag{4.4.28}$$

provided $|\omega/\sigma|<1$, which is the case because $x>y>0$ by hypothesis. Let us now evaluate the latter term in (4.4.26): by absolute convergence and (4.4.20),

$$\begin{aligned}
&\frac{x^{\nu}\pi^{\frac{\nu}{2}}}{2}\int_0^{\infty}u^{\nu-1}\,e^{-\pi x^2 u}\,\frac{u^{\frac{k}{2}}\,(\sqrt{\pi}y)^{\mu}}{\Gamma(\mu+1)}\sum_{n=1}^{\infty}r_k(n)\,\Psi_2\left(\frac{k}{2};\,\frac{k}{2},\mu+1;-\pi n u,\,-\pi u y^2\right)du\\
&=\frac{x^{\nu}y^{\mu}\pi^{\frac{\mu+\nu}{2}}}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty}r_k(n)\int_0^{\infty}u^{\nu+\frac{k}{2}-1}\,e^{-\pi x^2 u}\,\Psi_2\left(\frac{k}{2};\,\frac{k}{2},\mu+1;-\pi n u,\,-\pi y^2 u\right)du\\
&=\frac{x^{-\nu-k}y^{\mu}\pi^{\frac{\mu-\nu-k}{2}}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty}r_k(n)\,F_4\left(\nu+\frac{k}{2},\,\frac{k}{2};\,\frac{k}{2},\,\mu+1;\,-\frac{n}{x^2},-\frac{y^2}{x^2}\right).
\end{aligned}\tag{4.4.29}$$

To justify this interchange of operations a bit better, let us note that, by appealing to the inequality (4.4.18), it is possible to find a sufficiently large N_0 such that

$$\begin{aligned}
&\sum_{n=N_0}^{\infty}r_k(n)\int_0^{\infty}u^{\mathrm{Re}(\nu)+\frac{k}{2}-1}\,e^{-\pi x^2 u}\left|\Psi_2\left(\frac{k}{2};\,\frac{k}{2},\mu+1;-\pi n u,\,-\pi y^2 u\right)\right|du\\
&\leq\sum_{n=N_0}^{\infty}r_k(n)\int_0^{\infty}u^{\mathrm{Re}(\nu)+\frac{k}{2}-1}\,e^{-\pi x^2 u}\,e^{-\pi n u-\pi y^2 u}\left\{1+\left(\pi^2y^2u^2n+e^{2\pi y^2 u\left(1+\frac{|2\mu+2-k|}{4|\mu+1|}\right)}\right)\,e^{2\pi y\sqrt{n u}}\right\}du\\
&=\pi^{-\mathrm{Re}(\nu)-\frac{k}{2}}\sum_{n=N_0}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\mathrm{Re}(\nu)+\frac{k}{2}}}+\pi^{-\mathrm{Re}(\nu)-\frac{k}{2}}y^2\sum_{n=N_0}^{\infty}\frac{r_k(n)\,n}{\left(\left(\sqrt{n}-y\right)^2+x^2\right)^{\mathrm{Re}(\nu)+\frac{k}{2}+2}}\\
&+\pi^{-\mathrm{Re}(\nu)-\frac{k}{2}}\sum_{n=N_0}^{\infty}\frac{r_k(n)}{\left(\left(\sqrt{n}-y\right)^2+x^2-2y^2-\frac{y^2|2\mu+2-k|}{2|\mu+1|}\right)^{\mathrm{Re}(\nu)+\frac{k}{2}}}<\infty,
\end{aligned}\tag{4.4.30}$$

where we have picked N_0 so large that

$$\left(\sqrt{n}-y\right)^2+x^2-2y^2-\frac{y^2}{2}\,\frac{|2\mu+2-k|}{|\mu+1|}>0,\quad \forall n\geq N_0.$$

Combining (4.4.29), (4.4.27) and (4.4.26), we find the transformation formula

$$\begin{aligned}
&\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu-\mu}{2}}\,J_{\mu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)\\
&=-\frac{\Gamma(\nu)\,\pi^{\frac{\mu-\nu}{2}}x^{-\nu}y^{\mu}}{2\Gamma(\mu+1)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,\pi^{\frac{\mu-\nu-k}{2}}x^{-\nu-k}y^{\mu}}{2\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\mu+1;\,-\frac{y^2}{x^2}\right)\\
&+\frac{x^{-\nu-k}y^{\mu}\pi^{\frac{\mu-\nu-k}{2}}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty}r_k(n)\,F_4\left(\nu+\frac{k}{2};\,\frac{k}{2};\,\frac{k}{2},\,\mu+1;\,-\frac{n}{x^2},-\frac{y^2}{x^2}\right),
\end{aligned}\tag{4.4.31}$$

which is very similar to (4.4.21), but not exactly (4.4.21). The essential difference between both formulas (4.4.31) and (4.4.21) is the shape of the summands lying on their right-hand sides. However, we can easily check that these are related by a functional relation for Appell's function F_4. Since we have the formula [[95], p. 1020, 9.183.3]

$$\begin{aligned}
F_4\left(\alpha,\beta;\,\gamma,\gamma';\,w,z\right)=&\frac{\Gamma\left(\gamma'\right)\Gamma\left(\beta-\alpha\right)}{\Gamma\left(\gamma'-\alpha\right)\Gamma\left(\beta\right)}\left(-z\right)^{-\alpha}\,F_4\left(\alpha;\,\alpha+1-\gamma';\,\gamma,\alpha+1-\beta;\,\frac{w}{z},\frac{1}{z}\right)\\
&+\frac{\Gamma\left(\gamma'\right)\Gamma\left(\alpha-\beta\right)}{\Gamma\left(\gamma'-\beta\right)\Gamma\left(\alpha\right)}\left(-z\right)^{\beta}\,F_4\left(\beta+1-\gamma';\,\beta;\,\gamma,\beta+1-\alpha;\,\frac{w}{z},\,\frac{1}{z}\right),
\end{aligned}$$

we see that, when $\gamma'=\beta$,

$$F_4\left(\alpha,\beta;\,\gamma,\beta;\,w,z\right)=\left(-z\right)^{-\alpha}\,F_4\left(\alpha;\,\alpha+1-\beta;\,\gamma,\alpha+1-\beta;\,\frac{w}{z},\frac{1}{z}\right),$$

and so the double hypergeometric factor in (4.4.31) becomes

$$\begin{aligned}
F_4\left(\nu+\frac{k}{2},\,\frac{k}{2};\,\frac{k}{2},\,\mu+1;\,-\frac{n}{x^2},-\frac{y^2}{x^2}\right)&=F_4\left(\nu+\frac{k}{2};\,\frac{k}{2};\,\mu+1,\,\frac{k}{2};-\frac{y^2}{x^2},-\frac{n}{x^2}\right)\\
&=\frac{x^{2\nu+k}}{n^{\nu+\frac{k}{2}}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\mu+1,\,\nu+1,\,\frac{y^2}{n},\,-\frac{x^2}{n}\right),
\end{aligned}$$

and (4.4.31) turns into (4.4.21). The proof of (4.4.23) is totally analogous. □

Remark 4.4.2. The intermediate step (4.4.31) will be very useful to rederive a formula obtained in the second chapter, namely (4.4.101). When we replace the Bessel $J-$function on the left-hand side of (4.4.31) by the Bessel $I-$function, (4.4.31) is equivalent to

$$\pi^{\frac{\nu-\mu}{2}}\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, I_{\mu}\left(2\pi\sqrt{ny}\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$=-\frac{\Gamma(\nu)\,\pi^{\frac{\mu-\nu}{2}}x^{-\nu}y^{\mu}}{2\Gamma(\mu+1)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,\pi^{\frac{\mu-\nu-k}{2}}x^{-\nu-k}y^{\mu}}{2\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\mu+1;\,\frac{y^2}{x^2}\right)$$
$$+\frac{x^{-\nu-k}y^{\mu}\pi^{\frac{\mu-\nu-k}{2}}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\Gamma(\mu+1)}\sum_{n=1}^{\infty} r_k(n)\,F_4\left(\nu+\frac{k}{2};\,\frac{k}{2};\,\frac{k}{2},\,\mu+1;\,-\frac{n}{x^2},\,\frac{y^2}{x^2}\right),\tag{4.4.32}$$

which will be very helpful in the proof of Theorem 4.4.3 below.

Remark 4.4.3. Another proof of our Theorem 4.4.1 can be written if we use the same method as in [43]. Once proving that a function of the form

$$h(t):=t^{\frac{\nu-\mu}{2}}\,J_{\mu}(2\pi\sqrt{t}y)\,K_{\nu}(2\pi\sqrt{t}x),\quad x>y>0,\ \ \mathrm{Re}(\nu)>0,\ \mathrm{Re}(\mu)>-1$$

satisfies the necessary conditions to apply Voronoï's formula, we may appeal to a formula due to Bailey [[8], p. 38] (see also [[48], p. 232, eq. (3.14.17)])

$$\int_0^{\infty} t^{z-1}\,J_{\lambda}(at)\,I_{\mu}(bt)\,K_{\nu}(ct)\,dt=\frac{2^{z-2}a^{\lambda}b^{\mu}}{c^{z+\lambda+\mu}}\,\frac{\Gamma\left(\frac{z+\lambda+\mu-\nu}{2}\right)\Gamma\left(\frac{z+\lambda+\mu+\nu}{2}\right)}{\Gamma\left(\mu+1\right)\Gamma\left(\lambda+1\right)}$$
$$\times F_4\left(\frac{z+\lambda+\mu-\nu}{2};\,\frac{z+\lambda+\mu+\nu}{2};\,\lambda+1,\mu+1;\,-\frac{a^2}{c^2},\frac{b^2}{c^2}\right),\tag{4.4.33}$$

which is valid for $c>|b|$, $\mathrm{Re}(z+\lambda+\mu)>|\mathrm{Re}(\nu)|$. Using (4.4.33) in Voronoï's summation formula (4.1.36) leads to a formal proof of (4.4.23).

4.4.2 Reduction I: The case $\mu=\nu$

Here we will state the first particular case that can be obtained from Theorem 4.4.1. Let $\mu=\nu$ in (4.4.23): we have that

$$\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{ny}\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)x^{-\nu-k}y^{\nu}}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\,{}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\nu+1;\,\frac{y^2}{x^2}\right)$$
$$+\frac{x^{\nu}y^{\nu}\Gamma\left(\nu+\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\nu+\frac{k}{2}}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\nu+1,\,\nu+1,\,-\frac{y^2}{n},\,-\frac{x^2}{n}\right).\tag{4.4.34}$$

It is expected that we get (4.3.10) from (4.4.34). In order to reach exactly (4.3.10), we need to perform some transformations. First, by [[95], p. 1009, eq. 9.134.2], we know that the hypergeometric function obeys to the transformation formula

$$ {}_2F_1\left(\alpha+1-\gamma,\,\alpha;\,\gamma;\,z\right)=(1+z)^{-\alpha}\ {}_2F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\,\gamma;\,\frac{4z}{(1+z)^2}\right). \tag{4.4.35}$$

Using (4.4.35) with $\alpha=\nu+\frac{k}{2}$ and $\gamma=\nu+1$, the second term on the right-hand side of (4.4.34) can be written as

$$\begin{aligned}&\frac{\Gamma\left(\nu+\frac{k}{2}\right)\,x^{-\nu-k}y^{\nu}}{2\pi^{\frac{k}{2}}\,\Gamma(\nu+1)}\ {}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\nu+1;\,\frac{y^2}{x^2}\right)\\ &=\frac{\Gamma\left(\nu+\frac{k}{2}\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\,\Gamma\left(\nu+1\right)\left(x^2+y^2\right)^{\nu+\frac{k}{2}}}\ {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{x^2+y^2}\right)^2\right).\end{aligned} \tag{4.4.36}$$

Moreover, the Appell function F_4, (4.4.5), also obeys to the reduction formula [[172], Vol. III, p. 453, relation 7.2.4.79] (see also [[95], p. 1019, 9.182.7] and [[84], Vol. I, p. 238, eq. 5.10.(7)])

$$F_4\left(\alpha,\beta;\beta,\beta;w,z\right)=\left(1-w-z\right)^{-\alpha}\ {}_2F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2};\,\beta;\,\frac{4wz}{\left(1-w-z\right)^2}\right), \tag{4.4.37}$$

so that, when $\mu=\nu$, the term involving Appell's function inside the summation formula (4.4.23) has the representation

$$\begin{aligned}&F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\nu+1,\,\nu+1,\,-\frac{y^2}{n},\,-\frac{x^2}{n}\right)\\ &=\left(\frac{n}{n+x^2+y^2}\right)^{\nu+\frac{k}{2}}\ {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right).\end{aligned} \tag{4.4.38}$$

Combining (4.4.34) with (4.4.36) and (4.4.38) gives the formula

$$\begin{aligned}&\sum_{n=1}^{\infty}r_k(n)\,I_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}\\ &+\frac{\Gamma\left(\nu+\frac{k}{2}\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\,\Gamma\left(\nu+1\right)\left(x^2+y^2\right)^{\nu+\frac{k}{2}}}\ {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{x^2+y^2}\right)^2\right)\\ &+\frac{\Gamma\left(\nu+\frac{k}{2}\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\sum_{n=1}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\nu+\frac{k}{2}}}\ {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right),\end{aligned}$$

which is precisely (4.3.10). Using the limiting relations for the Bessel functions (4.1.17) and (4.1.33), as well as the convention $r_k(0):=1$, we can rewrite the previous formula in the compact and convenient form (compare this with (4.1.32) above)

$$\sum_{n=0}^{\infty} r_k(n)\, I_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}\,x\right)$$
$$= \frac{\Gamma\left(\nu+\frac{k}{2}\right)(xy)^\nu}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{r_k(n)}{(n+x^2+y^2)^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4}, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \nu+1;\, \left(\frac{2xy}{n+x^2+y^2}\right)^2\right). \tag{4.4.39}$$

Analogously, one may get the summation formula, equivalent to (4.3.31),

$$\sum_{n=0}^{\infty} r_k(n)\, J_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}\,x\right)$$
$$= \frac{\Gamma\left(\nu+\frac{k}{2}\right)(xy)^\nu}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{r_k(n)}{(n+x^2-y^2)^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4}, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \nu+1;\, -\left(\frac{2xy}{n+x^2-y^2}\right)^2\right). \tag{4.4.40}$$

This proves that Theorems 4.3.1 and 4.3.2 are immediate corollaries from Theorem 4.4.1.

4.4.3 Particular cases of Reduction I

Several particular cases of the general summation formulas (4.4.39) and (4.4.40) can now be stated. Here, we shall restate some of the particular cases already present in the literature and that were observed by Berndt, Dixit, Kim and Zaharescu. We start by stating the first particular case obtained in [[43], p. 328, Corollary 4.4].

Corollary 4.4.3. *For any positive integer $k \geq 2$ and $x > y > 0$, the following identities hold*

$$\sum_{n=0}^{\infty} \frac{r_k(n)}{\sqrt{n}} \sinh\left(2\pi\sqrt{n}y\right)\, e^{-2\pi\sqrt{n}x} = \frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} r_k(n) \left(\frac{1}{(n+(x-y)^2)^{\frac{k-1}{2}}} - \frac{1}{(n+(x+y)^2)^{\frac{k-1}{2}}}\right),$$

$$\sum_{n=0}^{\infty} \frac{r_k(n)}{\sqrt{n}} \sin\left(2\pi\sqrt{n}y\right)\, e^{-2\pi\sqrt{n}x} = \frac{\Gamma\left(\frac{k-1}{2}\right)}{2i\pi^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} r_k(n) \left(\frac{1}{(n+(x-iy)^2)^{\frac{k-1}{2}}} - \frac{1}{(n+(x+iy)^2)^{\frac{k-1}{2}}}\right).$$

The previous result is just a consequence of Theorem 4.3.1 with $\nu = \frac{1}{2}$, which is then combined with the reduction formulas

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}, \quad I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, \sinh(x), \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, \sin(x), \tag{4.4.41}$$

and, finally, [[172], Vol. III, p. 461, 7.3.1.107]

$${}_2F_1\left(a, a+\frac{1}{2}; \frac{3}{2}; z\right) = \frac{1}{2(2a-1)\sqrt{z}} \left\{\left(1-\sqrt{z}\right)^{1-2a} - \left(1+\sqrt{z}\right)^{1-2a}\right\}. \tag{4.4.42}$$

In the next corollary we specialize the parameter ν as equal to $\frac{k}{2}-1$. In order to apply the summation formulas (4.4.39) and (4.4.40) in a direct form, the number of squares in $r_k(n)$, k, needs to be taken as $k \geq 3$.

Corollary 4.4.4. *For any positive integer $k \geq 3$ and $x > y > 0$, the following summation formulas hold*

$$\sum_{n=0}^{\infty} r_k(n)\, I_{\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\, K_{\frac{k}{2}-1}\left(2\pi\sqrt{n}x\right) = \frac{(k-2)!\,(xy)^{\frac{k}{2}-1}}{2\pi^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n^2+2n\,\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{k-1}{2}}}, \tag{4.4.43}$$

$$\sum_{n=0}^{\infty} r_k(n)\, J_{\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\, K_{\frac{k}{2}-1}\left(2\pi\sqrt{n}x\right) = \frac{(k-2)!\,(xy)^{\frac{k}{2}-1}}{2\pi^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \sum_{n=0}^{\infty} \frac{r_k(n)}{\left(n^2+2n\,\left(x^2-y^2\right)+\left(x^2+y^2\right)^2\right)^{\frac{k-1}{2}}}. \tag{4.4.44}$$

Proof. Just take $\nu = \frac{k}{2}-1$ in the summation formulas (4.4.39) and (4.4.40) and invoke the elementary relation for Gauss' hypergeometric function

$$ {}_2F_1\left(a, b; a; z\right) = \left(1-z\right)^{-b}. \tag{4.4.45}$$

□

Remark 4.4.4. The previous corollary was also found in the second chapter as a consequence of the identity (4.1.14). The cases $k = 1, 2$ are omitted from the statement of Corollary 4.4.4 because the formulas (4.4.39) and (4.4.40) proved above are only valid when $\mathrm{Re}(\nu) > 0$. In Theorem 4.4.2 below, we will prove an extension of (4.4.39) and (4.4.40) to the region $\mathrm{Re}(\nu) > -1$.

Taking $k = 2$ in (4.4.39) and invoking the reduction formula for Gauss' hypergeometric function [[172], Vol. III, p. 461, 7.3.1.104],

$$ {}_2F_1\left(a, a+\frac{1}{2}; 2a; z\right) = \frac{1}{\sqrt{1-z}}\left(\frac{2}{1+\sqrt{1-z}}\right)^{2a-1}, \tag{4.4.46}$$

we are able to deduce the result of Popov (4.1.32) that we stated at the introduction to this chapter.[26]

[26]Taking into account the substitutions $y := \frac{\sqrt{\alpha}-\sqrt{\beta}}{2}$ and $x := \frac{\sqrt{\alpha}+\sqrt{\beta}}{2}$ mentioned at the previous section.

Corollary 4.4.5. *For Re$(\nu)>0$ and $x>y>0$, the following identity holds*

$$\frac{2\pi}{(2xy)^{\nu}}\sum_{n=0}^{\infty}r_2(n)\,I_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right)$$
$$=\sum_{n=0}^{\infty}\frac{r_2(n)}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\,\left(\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}+n+x^2+y^2\right)^{\nu}}. \tag{4.4.47}$$

Moreover, under the same conditions, the analogous formula holds

$$\frac{2\pi}{(2xy)^{\nu}}\sum_{n=0}^{\infty}r_2(n)\,J_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right)$$
$$=\sum_{n=0}^{\infty}\frac{r_2(n)}{\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\,\left(\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}+n+x^2-y^2\right)^{\nu}}. \tag{4.4.48}$$

As remarked at the introduction, the proof of Popov's formula (4.4.47) was furnished for the first time by Berndt, Dixit, Kim and Zaharescu in [43]. The reason for the interest of the results (4.4.47) and (4.4.48) is that, no matter the value of ν (as long as it belongs to the half-plane $\mathrm{Re}(\nu)>0$), when we set $k=2$, the hypergeometric factor in (4.4.39) collapses into an elementary function. One crucial ingredient in the derivation of (4.4.47) and (4.4.48) is the relation (4.4.46). Since this transformation formula for Gauss' hypergeometric function has a companion of the form [[172], Vol. III, p. 461, 7.3.1.103],

$${}_2F_1\left(a,a+\frac{1}{2};2a-1;z\right)=\frac{2^{2a-2}}{2a-1}\cdot\frac{(2a-3)\left(\sqrt{1-z}-1\right)+2(a-1)z}{z(1-z)^{\frac{3}{2}}\left(\sqrt{1-z}+1\right)^{2a-3}}, \tag{4.4.49}$$

we may use (4.4.49) to get a general identity like (4.4.47). In fact, we will now prove that the use of (4.4.49) agrees quite well with the particular case $k=4$, which allows us to give yet another analogue of Popov's formula (4.4.47). We now state our next result, which does not seem to be noticed in previous references.

Corollary 4.4.6. *Let ν be a complex number such that Re$(\nu)>0$ and $x>y>0$. Then the following transformation formulas hold*

$$\sum_{n=0}^{\infty}r_4(n)\,I_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right)$$
$$=\frac{(xy)^{\nu-2}2^{\nu-3}}{\pi^2}\sum_{n=0}^{\infty}r_4(n)\,\frac{(\nu-1)(n+x^2+y^2)\left(\sqrt{(n+x^2+y^2)^2-4x^2y^2}-(n+x^2+y^2)\right)+4\nu x^2y^2}{\left((n+x^2+y^2)^2-4x^2y^2\right)^{\frac{3}{2}}\left(\sqrt{(n+x^2+y^2)^2-4x^2y^2}+n+x^2+y^2\right)^{\nu-1}}, \tag{4.4.50}$$

$$\sum_{n=0}^{\infty} r_4(n)\, J_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}x\right)$$
$$= \frac{(xy)^{\nu-2}2^{\nu-3}}{\pi^2}\sum_{n=0}^{\infty} r_4(n)\,\frac{4\nu x^2y^2-(\nu-1)(n+x^2-y^2)\left(\sqrt{(n+x^2-y^2)^2+4x^2y^2}-(n+x^2-y^2)\right)}{\left((n+x^2-y^2)^2+4x^2y^2\right)^{\frac{3}{2}}\left(\sqrt{(n+x^2-y^2)^2+4x^2y^2}+n+x^2-y^2\right)^{\nu-1}}. \tag{4.4.51}$$

Proof. We will only prove (4.4.50) because the analogous formula (4.4.51) has a similar proof. Let us use (4.4.49) with $a=1+\frac{\nu}{2}$ and $z=\left(\frac{2xy}{n+x^2+y^2}\right)^2$. Elementary simplifications show that

$$_2F_1\left(\frac{\nu}{2}+1,\,\frac{\nu}{2}+\frac{3}{2};\,\nu+1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right)=\frac{2^\nu\left(n+x^2+y^2\right)^{\nu+2}}{4x^2y^2(\nu+1)}$$
$$\times\,\frac{(\nu-1)(n+x^2+y^2)\left(\sqrt{n^2+2n(x^2+y^2)+(x^2-y^2)^2}-(n+x^2+y^2)\right)+4\nu x^2y^2}{(n^2+2n(x^2+y^2)+(x^2-y^2)^2)^{\frac{3}{2}}\left(\sqrt{n^2+2n(x^2+y^2)+(x^2-y^2)^2}+n+x^2+y^2\right)^{\nu-1}},$$

and so an application of (4.4.43) gives the result

$$\sum_{n=0}^{\infty} r_4(n)\, I_\nu\left(2\pi\sqrt{n}y\right)\, K_\nu\left(2\pi\sqrt{n}x\right)$$
$$= \frac{(xy)^{\nu-2}2^{\nu-3}}{\pi^2}\sum_{n=0}^{\infty} r_4(n)\,\frac{(\nu-1)(n+x^2+y^2)\left(\sqrt{(n+x^2+y^2)^2-4x^2y^2}-(n+x^2+y^2)\right)+4\nu x^2y^2}{\left((n+x^2+y^2)^2-4x^2y^2\right)^{\frac{3}{2}}\left(\sqrt{(n+x^2+y^2)^2-4x^2y^2}+n+x^2+y^2\right)^{\nu-1}}.$$

□

Remark 4.4.5. In particular, when we set $\nu=1$ in (4.4.50) and (4.4.51), we are then led to the interesting summation formulas

$$\sum_{n=0}^{\infty} r_4(n)\, I_1\left(2\pi\sqrt{n}y\right)\, K_1\left(2\pi\sqrt{n}x\right)=\frac{xy}{\pi^2}\sum_{n=0}^{\infty}\frac{r_4(n)}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{3}{2}}}, \tag{4.4.52}$$

$$\sum_{n=0}^{\infty} r_4(n)\, J_1\left(2\pi\sqrt{n}y\right)\, K_1\left(2\pi\sqrt{n}x\right)=\frac{xy}{\pi^2}\sum_{n=0}^{\infty}\frac{r_4(n)}{\left(n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2\right)^{\frac{3}{2}}}, \tag{4.4.53}$$

which can also be seen as particular cases of the formulas given in Corollary 4.4.4 above.

Analytic continuation of Reduction I: General case

As previously mentioned, our formulas (4.4.43) and (4.4.44) cannot be extended to the values $k=1,2$ because one crucial hypothesis of Theorem 4.3.1 is that $\text{Re}(\nu)>0$. In the next theorem we give two formulas that serve as the analytic continuations of (4.4.39) and (4.4.40) to the region $\text{Re}(\nu)>-1$. It should be noted that the next result is already proved by Berndt, Dixit, Kim and Zaharescu in [[43], p. 331] but we need to state it in order to explain our next results.

Theorem 4.4.2. *For any $Re(\nu)>-1$ and $k\in\mathbb{N}$, the following continuation of the formula (4.4.40) holds*

$$\sum_{n=1}^{\infty}r_k(n)\,I_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right)=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}\zeta_k\left(\frac{k}{2}+\nu\right)+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}$$
$$\times\sum_{n=1}^{\infty}r_k(n)\left\{\frac{1}{(n+x^2+y^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\,\frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right)-\frac{1}{n^{\frac{k}{2}+\nu}}\right\}$$
$$+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)(x^2+y^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\,\frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\,\nu+1;\,\left(\frac{2xy}{x^2+y^2}\right)^2\right).\tag{4.4.54}$$

Moreover, the analogous formula takes place

$$\sum_{n=1}^{\infty}r_k(n)\,J_{\nu}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}x\right)=-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}\zeta_k\left(\frac{k}{2}+\nu\right)+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}$$
$$\times\sum_{n=1}^{\infty}r_k(n)\left\{\frac{1}{(n+x^2-y^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\,\frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\,\nu+1;\,-\left(\frac{2xy}{n+x^2-y^2}\right)^2\right)-\frac{1}{n^{\frac{k}{2}+\nu}}\right\}$$
$$+\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)(x^2-y^2)^{\frac{k}{2}+\nu}}\,{}_2F_1\left(\frac{k}{4}+\frac{\nu}{2},\,\frac{k}{4}+\frac{\nu}{2}+\frac{1}{2};\,\nu+1;\,-\left(\frac{2xy}{x^2-y^2}\right)^2\right).\tag{4.4.55}$$

There are several corollaries of (4.4.54) and (4.4.55). We start with the following result, which can be also seen as a particular case of a previous corollary already established by us before.[27]

Corollary 4.4.7. *Let $k\in\mathbb{N}$ and $x>0$. Then, for $Re(\nu)>-1$ the continuation of Popov's formula, (4.1.16), takes place*

$$\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu}{2}}\,K_{\nu}\left(2\pi\sqrt{n}x\right)=-\frac{\Gamma(\nu)}{2\pi^{\nu}x^{\nu}}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)}{2\pi^{\frac{k}{2}+\nu}x^{k+\nu}}$$
$$+\frac{\Gamma\left(\frac{k}{2}+\nu\right)x^{\nu}}{2\pi^{\frac{k}{2}+\nu}}\sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{(n+x^2)^{\frac{k}{2}+\nu}}-\frac{r_k(n)}{n^{\frac{k}{2}+\nu}}\right\}+\frac{\Gamma\left(\frac{k}{2}+\nu\right)x^{\nu}}{2\pi^{\frac{k}{2}+\nu}}\zeta_k\left(\frac{k}{2}+\nu\right).\tag{4.4.56}$$

[27]See, for instance, Corollary 2.2.8 on the second chapter and take there $\phi(s):=\pi^{-s}\,\zeta_k(s)$.

Proof. Clearly, the series (4.4.39) and (4.4.40) converge absolutely and uniformly with respect to y for $y \in [0, M)$. This is due to the asymptotic behaviors of $K_\nu(z)$ and $I_\nu(z)$, $J_\nu(z)$ when $z \to \infty$ (cf. (2.1.56), (2.1.57) and (2.1.58)). Taking this into consideration, let us now multiply both sides of (4.4.54) by $y^{-\nu}$ and then let $y \to 0^+$. Using the limiting relation

$$\lim_{y\to 0} y^{-\nu} I_\nu(y) = \frac{2^{-\nu}}{\Gamma(\nu+1)}, \tag{4.4.57}$$

we can easily deduce the formula

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu}{2}}\, K_\nu\left(2\pi\sqrt{n}x\right) = -\frac{\Gamma(\nu)}{2\pi^\nu x^\nu} + \frac{\Gamma\left(\frac{k}{2}+\nu\right)}{2\pi^{\frac{k}{2}+\nu} x^{k+\nu}}$$
$$+\frac{\Gamma\left(\frac{k}{2}+\nu\right)x^\nu}{2\pi^{\frac{k}{2}+\nu}} \sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{(n+x^2)^{\frac{k}{2}+\nu}} - \frac{r_k(n)}{n^{\frac{k}{2}+\nu}}\right\} + \frac{\Gamma\left(\frac{k}{2}+\nu\right)x^\nu}{2\pi^{\frac{k}{2}+\nu}}\,\zeta_k\left(\frac{k}{2}+\nu\right).$$

completing the proof. □

Remark 4.4.6. Note that, when $k = 1$, (4.4.56) reduces to

$$2\sum_{n=1}^{\infty} n^\nu K_\nu\left(2\pi n x\right) = -\frac{\Gamma(\nu)}{2\pi^\nu x^\nu} + \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{2\pi^{\nu+\frac{1}{2}} x^{\nu+1}}$$
$$+\frac{\Gamma\left(\nu+\frac{1}{2}\right)x^\nu}{\pi^{\nu+\frac{1}{2}}} \sum_{n=1}^{\infty}\left\{\frac{1}{(n^2+x^2)^{\nu+\frac{1}{2}}} - \frac{1}{n^{2\nu+1}}\right\} + \frac{\Gamma\left(\nu+\frac{1}{2}\right)x^\nu}{\pi^{\nu+\frac{1}{2}}}\,\zeta\left(2\nu+1\right),$$

which is the continuation of Watson's formula to the region $\mathrm{Re}(\nu) > -1$ [[233], p. 300, eq. (5)]. After an elementary change of variables, one is able to get the representation for $\zeta(s)$,

$$\zeta\left(s\right) = \sum_{n=1}^{\infty}\left\{\frac{1}{n^s} - \frac{1}{(n^2+x^2)^{\frac{s}{2}}}\right\} - \frac{x^{-s}}{2} + \frac{\sqrt{\pi}x^{1-s}}{2\Gamma\left(\frac{s}{2}\right)}\Gamma\left(\frac{s-1}{2}\right) + \frac{2\pi^{\frac{s}{2}}x^{\frac{1-s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\sum_{n=1}^{\infty} n^{\frac{s-1}{2}} K_{\frac{s-1}{2}}\left(2\pi n x\right), \tag{4.4.58}$$

which is valid for any s belonging to the region $\mathrm{Re}(s) > -1$. In the final chapter of this thesis, namely in Corollary 6.3.4 below, we will obtain (4.4.58) as a consequence of an interesting formula involving a theory due to Koshliakov.

Analytic continuation of Reduction I: particular cases

One of the most interesting particular values for ν that is covered by the "continuation formula" (4.4.54) but not by (4.4.39), is the case where $\nu = 0$. In order to let ν approach 0 in (4.4.54), we need to have more information concerning how $\zeta_k(s)$ behaves near its simple pole, $s = \frac{k}{2}$. The reason for this is as follows. Let $\nu \to 0$ in (4.4.54): the left-hand side of (4.4.54) clearly makes sense under this new value for ν, as well as the infinite series contining Gauss' hypergeometric function, which was constructed to converge in the region $\mathrm{Re}(\nu) > -1$. We are able to get the general formula

$$\sum_{n=1}^{\infty}r_{k}(n)\,I_{0}\left(2\pi\sqrt{ny}\right)\,K_{0}\left(2\pi\sqrt{nx}\right)=\frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\left(x^{2}+y^{2}\right)^{\frac{k}{2}}}\,{}_{2}F_{1}\left(\frac{k}{4},\,\frac{k}{4}+\frac{1}{2};\,1;\,\left(\frac{2xy}{x^{2}+y^{2}}\right)^{2}\right)$$
$$+\lim_{\nu\to0}\left\{\frac{\Gamma\left(\frac{k}{2}+\nu\right)\,(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}\,\zeta_{k}\left(\frac{k}{2}+\nu\right)-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}\right\}$$
$$+\frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\sum_{n=1}^{\infty}r_{k}(n)\left\{\frac{1}{\left(n+x^{2}+y^{2}\right)^{\frac{k}{2}}}\,{}_{2}F_{1}\left(\frac{k}{4},\,\frac{k}{4}+\frac{1}{2};\,1;\,\left(\frac{2xy}{n+x^{2}+y^{2}}\right)^{2}\right)-\frac{1}{n^{\frac{k}{2}}}\right\}.\qquad(4.4.59)$$

The only unknown quantity in (4.4.59) is the limit

$$\lim_{\nu\to0}\left\{\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma\left(\nu+1\right)}\,\zeta_{k}\left(\frac{k}{2}+\nu\right)-\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}\right\},\qquad(4.4.60)$$

which can only be understood once we know how $\zeta_k(s)$ can be expanded near the point $s=\frac{k}{2}$. Berndt, Dixit, Kim and Zaharescu calculate (4.4.60) only for the cases where $k=2,4,6,8$. The reason for these limits imposed on k is the following. The arithmetical function $r_{2k}(n)$ (i.e., the number of ways expressing n as an even number of squares) can be explicitly expressed in the nice form, [28]

$$r_{2k}(n)=\rho_{2k}(n)+a_{f_{2k}}(n),$$

where $\rho_{2k}(n)$ is a combination of "divisor functions" and $a_{f_{2k}}(n)$ is the n^{th} coefficient of a cusp form. Rankin [181] seems to have been the first mathematician to prove that $a_{f_{2k}}(n)=0$ for any $n\in\mathbb{N}$ iff $2\leq k\leq 4$. In the language of the theory of modular forms, Rankin's result says that the cusp form composing the form

$$\theta^{2k}(\tau):=\sum_{n=1}^{\infty}r_{2k}(n)\,e^{-\pi n\tau},\qquad \mathrm{Im}(\tau)>0,$$

vanishes identically only when $k=2$, 3 and 4. This has yet another translation into the realm of Dirichlet series: when $k=2$, 3 and 4, we can express $\zeta_{2k}(s)$ as

$$\zeta_{4}(s):=\sum_{n=1}^{\infty}\frac{r_{4}(n)}{n^{s}}=8\left(1-2^{2-2s}\right)\zeta(s)\,\zeta(s-1),\quad \mathrm{Re}(s)>2,\qquad(4.4.61)$$

$$\zeta_{6}(s):=\sum_{n=1}^{\infty}\frac{r_{6}(n)}{n^{s}}=16\zeta(s-2)\,L(s,\chi_{4})-4\zeta(s)\,L(s-2,\chi_{4}),\quad \mathrm{Re}(s)>3,\qquad(4.4.62)$$

$$\zeta_{8}(s):=\sum_{n=1}^{\infty}\frac{r_{8}(n)}{n^{s}}=16\left(1-2^{1-s}+2^{4-2s}\right)\zeta(s)\,\zeta(s-3),\quad \mathrm{Re}(s)>4,\qquad(4.4.63)$$

[28] as a matter of fact, the representation is valid for $r_k(n)$, with k not necessarily even. For the purposes of the discussion in this section, however, we write $r_{2k}(n)$ instead of $r_k(n)$.

where $L(s,\chi_4)$ denotes the Dirichlet beta-function, defined by the series

$$L(s,\chi_4) := \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \text{Re}(s) > 0.$$

We also have a formula of this type for $\zeta_2(s)$, which is an easy consequence of Legendre's two-square theorem,

$$\zeta_2(s) := \sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4\zeta(s)\, L(s,\chi_4), \quad \text{Re}(s) > 1. \tag{4.4.64}$$

Recall that the above formula for $\zeta_4(s)$ was one of the main motivations behind the first chapter of this thesis. In Example 1.3.1, for instance, it was explicitly invoked to sketch a new proof of Hardy's theorem. The unknown limit (4.4.60) can now be evaluated in terms of particular values of $\zeta(s)$ and $L(s,\chi_4)$.

However, as proved by Rankin, there is only a finite amount of nice representations like (4.4.61)-(4.4.64) and, if we want to avoid the contributions that the $L-$functions associated to cusp forms might have in the calculation of (4.4.60), we need a different strategy to study the quantity (4.4.60).

In this section we propose to use a generalization of Kronecker's limit formula due to Epstein [[81], p. 644], which shall be presented in the next lemma. Although a proof of the forthcoming expansion is already provided in Terras' paper [[222], p. 485], our notation and method is slightly different and we rely on the generalization of the Selberg-Chowla formula obtained in our first chapter (see Theorem 1.2.1 above).

Lemma 4.4.5. *Let $k \in \mathbb{N}$ and $\zeta_k(s)$ be the Dirichlet series attached to the sum of k squares. Moreover, for a vector $\mathbf{x} \in \mathbb{R}^n$, let $|\mathbf{x}|$ denote its Euclidean norm. Then $\zeta_k(s)$ admits the following Laurent expansion around its simple pole $s = \frac{k}{2}$,*

$$\zeta_k(s) = \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \frac{1}{s-\frac{k}{2}} + \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} \left(\gamma - 2\log(2) - \psi\left(\frac{k}{2}\right) - 2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}} \left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}\right)$$
$$+\zeta_{k-1}\left(\frac{k}{2}\right) + O\left(s-\frac{k}{2}\right), \tag{4.4.65}$$

where $\psi(z)$ denotes Euler's digamma function[29]*,*

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

Proof. The key to the proof is to write $\zeta_k(s)$ as a generalized Epstein zeta function (1.1.11), the natural object introduced in the first chapter. Let us note that, for $k \geq 2$ and $\text{Re}(s) > \frac{k}{2}$,

$$\zeta_k(s) = \sum_{n_1,\dots,n_k \neq 0} \frac{1}{\left(n_1^2 + \dots + n_{k-1}^2 + n_k^2\right)^s} = \sum_{m,n\neq 0}^{\infty} \frac{r_{k-1}(m)\, r_1(n)}{(m+n)^s} \tag{4.4.66}$$

[29]We warn the reader to not make any confusion with Jacobi's $\psi-$function, $\psi(x)$, given in the introduction of the second Chapter in (2.1.5).

Applying our generalization of the Selberg-Chowla formula (1.2.1) with $\phi_1(s)=\pi^{-s}\zeta_{k-1}(s)$ and $\phi_2(s)=\pi^{-s}\zeta_1(s):=2\pi^{-s}\zeta(2s)$ (see Theorem 1.2.1 of the first chapter of this thesis), we get the following formula

$$\begin{aligned}\zeta_k(s)=\zeta_{k-1}(s)&+\frac{2\pi^{\frac{k-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{k-1}{2}\right)\zeta\left(2s-k+1\right)\\&+\frac{4\pi^s}{\Gamma(s)}\sum_{m,n=1}^{\infty}r_{k-1}(m)\left(\frac{m}{n^2}\right)^{\frac{s}{2}-\frac{k-1}{4}}K_{s-\frac{k-1}{2}}\left(2\pi\sqrt{m}\,n\right).\end{aligned}\tag{4.4.67}$$

Let us study the right-hand side of this representation around the point $s=\frac{k}{2}$. Recalling the definition of the digamma function, $\psi(z):=\Gamma'(z)/\Gamma(z)$, and using the well-known Laurent expansions,

$$\frac{1}{\Gamma(s)}=\frac{1}{\Gamma\left(\frac{k}{2}\right)}\left(1-\psi\left(\frac{k}{2}\right)\left(s-\frac{k}{2}\right)+O\left(\left(s-\frac{k}{2}\right)^2\right)\right),$$

$$\Gamma\left(s-\frac{k-1}{2}\right)=\sqrt{\pi}\left(1-\left(2\log(2)+\gamma\right)\left(s-\frac{k}{2}\right)+O\left(\left(s-\frac{k}{2}\right)^2\right)\right),$$

$$\zeta\left(2s-k+1\right)=\frac{1}{2s-k}+\gamma+O\left(s-\frac{k}{2}\right),$$

we have

$$\begin{aligned}&\frac{2\pi^{\frac{k-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{k-1}{2}\right)\zeta\left(2s-k+1\right)\\&=\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\frac{1}{s-\frac{k}{2}}+\frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(\gamma-2\log(2)-\psi\left(\frac{k}{2}\right)\right)+O\left(s-\frac{k}{2}\right).\end{aligned}\tag{4.4.68}$$

By (4.1.22), the series involving the Modified Bessel function in (4.4.67) has the following value at $s=\frac{k}{2}$,

$$\begin{aligned}\frac{4\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m,n=1}^{\infty}r_{k-1}(m)\left(\frac{m}{n^2}\right)^{\frac{1}{4}}K_{\frac{1}{2}}\left(2\pi\sqrt{m}\,n\right)&=\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m,n=1}^{\infty}\frac{r_{k-1}(m)\,e^{-2\pi\sqrt{m}\,n}}{n}\\=\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m=1}^{\infty}r_{k-1}(m)\sum_{n=1}^{\infty}\frac{e^{-2\pi\sqrt{m}n}}{n}&=-\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m=1}^{\infty}r_{k-1}(m)\log\left(1-e^{-2\pi\sqrt{m}}\right).\end{aligned}$$

Using the definition of $r_{k-1}(m)$ and writing the last series into a multiple sum, we arrive at the expression

$$\begin{aligned}-\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m=1}^{\infty}r_{k-1}(m)\log\left(1-e^{-2\pi\sqrt{m}}\right)&=-\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\sum_{m_1,\ldots,m_{k-1}\neq0}\log\left(1-e^{-2\pi\sqrt{m_1^2+\ldots+m_{k-1}^2}}\right)\\=-\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\log\left\{\prod_{m_1,\ldots,m_{k-1}\neq0}\left(1-e^{-2\pi\sqrt{m_1^2+\ldots+m_{k-1}^2}}\right)\right\}&=-\frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\},\end{aligned}\tag{4.4.69}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $|\mathbf{x}|$ denotes the usual Euclidean norm. Combining (4.4.69) with (4.4.68) and returning to the Selberg-Chowla formula (4.4.67), we see that $\zeta_k(s)$ admits the Laurent expansion,

$$\zeta_k(s) = \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\frac{1}{s-\frac{k}{2}} + \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(\gamma - 2\log(2) - \psi\left(\frac{k}{2}\right) - 2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}\right)$$
$$+\zeta_{k-1}\left(\frac{k}{2}\right) + O\left(s-\frac{k}{2}\right),$$

which is exactly (4.4.65). □

Remark 4.4.7. The infinite product

$$\Phi_k(y) := \prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|y}\right), \quad y>0, \tag{4.4.70}$$

can be considered as an "analogue" of the square of Dedekind's $\eta-$function (see Example 1.2.2 of the first chapter),

$$\eta(\tau) := e^{\frac{\pi i\tau}{12}}\prod_{m=1}^{\infty}\left(1-e^{2\pi i m\tau}\right), \quad \mathrm{Im}(\tau)>0, \tag{4.4.71}$$

when restricted to the imaginary axis $\tau = iy$, $y>0$. To see why, let us note that, when $k=2$, $\Phi_k(y)$ reduces to

$$\Phi_2(y) := \prod_{m\in\mathbb{Z}\setminus\{0\}}\left(1-e^{-2\pi|m|y}\right) = \left\{\prod_{m=1}^{\infty}\left(1-e^{-2\pi m y}\right)\right\}^2 := e^{\frac{\pi y}{6}}\,\eta\left(iy\right)^2. \tag{4.4.72}$$

Therefore, as expected, (4.4.65) must imply a particular case of Kronecker's limit formula, whose classical form arises when $k=2$. Indeed, using the fact that $\psi(1)=-\gamma$, we have from (4.4.65) and (4.4.72) that

$$\zeta_2(s) = \frac{\pi}{s-1} + \zeta_1\left(1\right) + \pi\left(2\gamma - 2\log(2) - 4\log\left(\eta(i)\right) - \frac{\pi}{3}\right) + O\left(s-1\right).$$

But it is also well-known that $\zeta_1(1) := 2\zeta(2) = \frac{\pi^2}{3}$ and so, after elementary simplifications, we get the Laurent expansion

$$\zeta_2(s) = \frac{\pi}{s-1} + \pi\left(2\gamma - 2\log(2) - 4\log\left(\eta(i)\right)\right) + O(s-1), \tag{4.4.73}$$

which is none other than Kronecker's famous result, (1.2.47).

Remark 4.4.8. When $k=1$, we know that neither $\zeta_{k-1}(s)$ nor $\Phi_k(1) := \prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)$ make much sense (besides, our proof above uses a formula, (4.4.66), that only makes sense for $k\geq 2$). Therefore, in order to (4.4.65) be valid for any $k\in\mathbb{N}$, we take the convention

$$\zeta_0(s) := 0, \quad \Phi_1(y) := 1. \tag{4.4.74}$$

Recalling that $\zeta_1(s) = 2\zeta(2s)$ and $\psi\left(\frac{1}{2}\right) = -2\log(2) - \gamma$, a particular case of (4.4.65) under the convention (4.4.74) is

$$2\zeta(2s) = \frac{1}{s-\frac{1}{2}} + 2\gamma + O\left(s-\frac{1}{2}\right),$$

which is the well-known Laurent expansion for Riemann's zeta function.

The previous lemma can now be employed to compute the limit (4.4.60). Using the previous Laurent expansion in (4.4.59), we are now able to state the following corollary.

Corollary 4.4.8. *For any positive integer $k \geq 1$ and $x > y > 0$, the following summation formula holds*

$$\sum_{n=1}^{\infty} r_k(n)\, I_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right) = \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\left(x^2+y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; \left(\frac{2xy}{x^2+y^2}\right)^2\right)$$
$$+ \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\zeta_{k-1}\left(\frac{k}{2}\right) + \gamma + \log\left(\frac{x}{2}\right) - \log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}$$
$$+ \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\sum_{n=1}^{\infty} r_k(n)\left\{\frac{1}{\left(n+x^2+y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; \left(\frac{2xy}{n+x^2+y^2}\right)^2\right) - \frac{1}{n^{\frac{k}{2}}}\right\}. \quad (4.4.75)$$

Analogously, the following identity is valid

$$\sum_{n=1}^{\infty} r_k(n)\, J_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right) = \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\left(x^2-y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; -\left(\frac{2xy}{x^2-y^2}\right)^2\right)$$
$$+ \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\zeta_{k-1}\left(\frac{k}{2}\right) + \gamma + \log\left(\frac{x}{2}\right) - \log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}$$
$$+ \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\sum_{n=1}^{\infty} r_k(n)\left\{\frac{1}{\left(n+x^2-y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; -\left(\frac{2xy}{n+x^2-y^2}\right)^2\right) - \frac{1}{n^{\frac{k}{2}}}\right\}. \quad (4.4.76)$$

Proof. We only prove (4.4.75), since the proof of (4.4.76) is analogous. Letting $\nu \to 0$ in (4.4.54), we are able to deduce that

$$\sum_{n=1}^{\infty} r_k(n)\, I_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right) = \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}\left(x^2+y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; \left(\frac{2xy}{x^2+y^2}\right)^2\right)$$
$$+ \lim_{\nu\to 0}\left\{\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\zeta_k\left(\frac{k}{2}+\nu\right) - \frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}\right\}$$
$$+ \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\sum_{n=1}^{\infty} r_k(n)\left\{\frac{1}{\left(n+x^2+y^2\right)^{\frac{k}{2}}}\, {}_2F_1\left(\frac{k}{4}, \frac{k}{4}+\frac{1}{2}; 1; \left(\frac{2xy}{n+x^2+y^2}\right)^2\right) - \frac{1}{n^{\frac{k}{2}}}\right\}. \quad (4.4.77)$$

Thus, if we evaluate the limit on the right-hand side of (4.4.77), we are done. To simplify the notation, we denote the constant term of the Laurent expansion (4.4.65) by $\mathscr{C}_k$, this is,

$$\mathscr{C}_k := \zeta_{k-1}\left(\frac{k}{2}\right) + \frac{\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(\gamma - 2\log(2) - \psi\left(\frac{k}{2}\right) - 2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}\right). \quad (4.4.78)$$

Elementary calculations give the following Laurent expansion around $\nu = 0$,

$$\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\zeta_k\left(\frac{k}{2}+\nu\right) = \frac{1}{2\nu} + \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\mathscr{C}_k + \frac{1}{2}\left\{\log(xy)+\gamma+\psi\left(\frac{k}{2}\right)\right\} + O(\nu),$$

allowing the calculation

$$\lim_{\nu\to 0}\left\{\frac{\Gamma\left(\frac{k}{2}+\nu\right)(xy)^{\nu}}{2\pi^{\frac{k}{2}}\Gamma(\nu+1)}\zeta_k\left(\frac{k}{2}+\nu\right) - \frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}\right\} = \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\mathscr{C}_k + \log(x) + \frac{1}{2}\left\{\gamma+\psi\left(\frac{k}{2}\right)\right\}$$
$$:= \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\zeta_{k-1}\left(\frac{k}{2}\right) - \log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\} + \gamma + \log\left(\frac{x}{2}\right). \quad (4.4.79)$$

A combination of (4.4.79) and (4.4.77) produces the desired formula (4.4.75), completing the proof of our corollary.

□

A natural consequence of the previous result follows once we let $y \to 0^+$ in (4.4.75) and (4.4.76).

Corollary 4.4.9. *For any positive integer $k \geq 1$ and $x > 0$, the following formula holds*

$$\sum_{n=1}^{\infty} r_k(n)\, K_0\left(2\pi\sqrt{n}x\right) = \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}x^k} + \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\zeta_{k-1}\left(\frac{k}{2}\right) + \gamma + \log\left(\frac{x}{2}\right)$$
$$-\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\} + \frac{\Gamma\left(\frac{k}{2}\right)}{2\pi^{\frac{k}{2}}}\sum_{n=1}^{\infty} r_k(n)\left\{\frac{1}{(n+x^2)^{\frac{k}{2}}} - \frac{1}{n^{\frac{k}{2}}}\right\}. \quad (4.4.80)$$

Remark 4.4.9. Using the convention that $\zeta_0(s) := 0$ and $\Phi_1(1) := 1$ (see Remark 4.4.8 above), as well as the fact that $r_1(n) := 2$ iff n is a perfect square and zero otherwise, the summation formula (4.4.80) implies a classical result of Watson [233], which states that, for any $x > 0$,

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{n^2+x^2}} - \frac{1}{n}\right\} + \frac{1}{2x} + \gamma + \log\left(\frac{x}{2}\right) = 2\sum_{n=1}^{\infty} K_0(2\pi n\, x). \quad (4.4.81)$$

We have generalized this formula before (see (2.2.77) above) and in Chapter 6 we will again encounter (4.4.81) as a particular case resulting from Koshliakov's theory (see Corollary 6.3.3 below).

Corollary 4.4.8 above possesses several interesting particular cases. We shall only state them for summation formulas involving the product $I_0(2\pi\sqrt{n}y)\, K_0(2\pi\sqrt{n}x)$. The analogous summation formulas involving $J_0(2\pi\sqrt{ny})\,K_0(2\pi\sqrt{nx})$ are left to the reader. As one can easily see, they are quite straightforward when k is even, but when k is odd one has to introduce the notion of elliptic functions [221].

When $k=1$, (4.4.75) reduces to a formula obtained by Berndt, Dixit, Gupta and Zaharescu in [[41], Corollary 15.1]. This formula can be stated as follows: if we let $K(z)$ denote the complete elliptic integral of the first kind,

$$K(z) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-z^2\sin^2(\theta)}}, \qquad 0\leq |z|<1, \tag{4.4.82}$$

then, for $x>y>0$, (4.4.75) yields the result

$$\sum_{n=1}^{\infty} I_0\left(2\pi n y\right)\,K_0\left(2\pi n x\right) = \frac{1}{2\pi x}\,K\left(\frac{y^2}{x^2}\right)+\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{x}{2}\right)$$
$$+\sum_{n=1}^{\infty}\left\{\frac{2}{\pi\left(\sqrt{n^2+(x+y)^2}+\sqrt{n^2+(x-y)^2}\right)}\,K\left(\left(\frac{\sqrt{n^2+(x+y)^2}-\sqrt{n^2+(x-y)^2}}{\sqrt{n^2+(x+y)^2}+\sqrt{n^2+(x-y)^2}}\right)^2\right)-\frac{1}{2n}\right\}. \tag{4.4.83}$$

We also state a particular case of the previous corollary when $k=2,4$. Although the next result is already given in [[43], p. 333], we note that the formulas presented in [43] depend on the quantities $L'(0,\chi_4)$ and $\zeta'(2)$. We present an alternative expression involving the logarithm of Dedekind's η-function (4.4.71) and its generalization (4.4.70) in the case $k=4$. We forego the proof of the next corollary, because it is a consequence of (4.4.75) combined with the well-known relations

$${}_2F_1\left(\frac{1}{2},1;1;z\right)=\frac{1}{\sqrt{1-z}},\quad {}_2F_1\left(1,\frac{3}{2};1;z\right)=\frac{1}{(1-z)^{\frac{3}{2}}},$$

which are nothing but particular cases of (4.4.45). For a more detailed description of the proof of our next formula (4.4.84), we refer to Example 2.2.2 of the second chapter. We should also note that we can rewrite it using the evaluation of $\eta(i)$ given in (1.2.52).

Corollary 4.4.10. *For $x>y>0$, the following formula holds*

$$\sum_{n=1}^{\infty} r_2(n)\,I_0\left(2\pi\sqrt{ny}\right)K_0\left(2\pi\sqrt{nx}\right)=\frac{1}{2\pi\left(x^2-y^2\right)}+\gamma+\log\left(\frac{x}{2}\right)-2\log\left(\eta(i)\right)$$
$$+\frac{1}{2\pi}\sum_{n=1}^{\infty} r_2(n)\left\{\frac{1}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}-\frac{1}{n}\right\}, \tag{4.4.84}$$

where $\eta(\tau)$, $Im(\tau)>0$, denotes Dedekind's $\eta-$function, defined by (4.4.71). Moreover, we have that

$$\sum_{n=1}^{\infty} r_4(n)\, I_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right)=\frac{x^2+y^2}{2\pi^2\left(x^2-y^2\right)^3}+\frac{\zeta_3(2)}{2\pi^2}+\gamma+\log\left(\frac{x}{2}\right)$$
$$-\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^3\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}+\frac{1}{2\pi^2}\sum_{n=1}^{\infty} r_4(n)\left\{\frac{(n+x^2+y^2)}{(n^2+2n(x^2+y^2)+(x^2-y^2)^2)^{\frac{3}{2}}}-\frac{1}{n^2}\right\}. \tag{4.4.85}$$

One curious aspect about the formulas (4.4.84) and (4.4.85) above is that, just like Popov's formula (4.4.47), no matter how complicated the summation regarding the product of Bessel functions might seem, the right-hand sides of both (4.4.84) and (4.4.85) contain an infinite series involving elementary functions. When we consider k odd in the transformation (4.4.75), the reduction formulas available for the hypergeometric function,

$$_2F_1\left(\frac{k}{4},\,\frac{k}{4}+\frac{1}{2};\,1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right),$$

no longer yield elementary functions, as we have seen in the case where $k=1$, (4.4.83). In the next corollary we establish a new summation formula involving the arithmetical function $r_3(n)$, which is akin to (4.4.83).

Corollary 4.4.11. *Let $E(z)$ denote the complete elliptic integral of the second kind, defined by the integral*

$$E(z)=\int_0^{\frac{\pi}{2}}\sqrt{1-z^2\sin^2(\varphi)}\,d\varphi,\qquad 0\leq|z|<1. \tag{4.4.86}$$

Then the following summation formula holds

$$\sum_{n=1}^{\infty} r_3(n)\, I_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right)=\frac{E\left(\sqrt{\frac{4xy}{(x+y)^2}}\right)}{2\pi^2(x+y)(x-y)^2}+\frac{1}{4\pi}\,\zeta_2\left(\frac{3}{2}\right)+\gamma+\log\left(\frac{x}{2}\right)$$
$$-\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}+\frac{1}{4\pi}\sum_{n=1}^{\infty} r_3(n)\left\{\frac{2E\left(\sqrt{\frac{4xy}{(n+x+y)^2}}\right)}{\pi\sqrt{n+(x+y)^2}\,\left(n+(x-y)^2\right)}-\frac{1}{n^{\frac{3}{2}}}\right\}. \tag{4.4.87}$$

Proof. Taking $k=3$ in (4.4.75), we find that, for $x>y>0$,

$$\sum_{n=1}^{\infty} r_3(n)\, I_0\left(2\pi\sqrt{n}y\right)\, K_0\left(2\pi\sqrt{n}x\right)=\frac{1}{4\pi\left(x^2+y^2\right)^{\frac{3}{2}}}\,{}_2F_1\left(\frac{3}{4},\,\frac{5}{4};\,1;\,\left(\frac{2xy}{x^2+y^2}\right)^2\right)$$
$$+\frac{1}{4\pi}\,\zeta_2\left(\frac{3}{2}\right)+\gamma+\log\left(\frac{x}{2}\right)-\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^2\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}$$
$$+\frac{1}{4\pi}\sum_{n=1}^{\infty} r_3(n)\left\{\frac{1}{(n+x^2+y^2)^{\frac{3}{2}}}\,{}_2F_1\left(\frac{3}{4},\,\frac{5}{4};\,1;\,\left(\frac{2xy}{n+x^2+y^2}\right)^2\right)-\frac{1}{n^{\frac{3}{2}}}\right\}.$$

Using the reduction formula for the hypergeometric function [[172], Vol. III, p. 479, eq. (7.3.2.204)],

$$ {}_2F_1\left(\frac{3}{4},\frac{5}{4};1;z\right)=\frac{2E\left(\sqrt{\frac{2\sqrt{z}}{1+\sqrt{z}}}\right)}{\pi\sqrt{1+\sqrt{z}}\,(1-\sqrt{z})}, \tag{4.4.88}$$

we find that, after some straightforward simplifications,

$$\frac{1}{(n+x^2+y^2)^{\frac{3}{2}}}\,{}_2F_1\left(\frac{3}{4},\frac{5}{4};1;\left(\frac{2xy}{n+x^2+y^2}\right)^2\right)=\frac{2}{\pi}\,\frac{E\left(\sqrt{\frac{4xy}{(n+x+y)^2}}\right)}{\sqrt{n+(x+y)^2}\,\left(n+(x-y)^2\right)}.$$

□

We can now use Theorem 4.4.2 in order to set $\nu=-\frac{1}{2}$ in (4.4.54). Like the case where $\nu=0$, there are some quantities that need to be understood in a different perspective. For example, if we take $\nu=-\frac{1}{2}$ in (4.4.54), the second term on the right-hand side becomes

$$\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k+1}{2}}}\zeta_k\left(\frac{k-1}{2}\right), \tag{4.4.89}$$

which cannot be expressed otherwise, unless we have an expression that defines $\zeta_k(s)$ on the half-plane $\mathrm{Re}(s)<\frac{k}{2}$.[30] One way to rewrite this unknown term (4.4.89) is by assuming that k is within a certain range of values and then use one of the formulas (4.4.61) - (4.4.64). For example, when $k=2$, one can invoke (4.4.64) to write this term as

$$\frac{\Gamma\left(\frac{1}{2}\right)}{2\pi^{\frac{3}{2}}}\zeta_2\left(\frac{1}{2}\right)=\frac{2}{\pi}\zeta\left(\frac{1}{2}\right)L\left(\frac{1}{2},\chi_4\right),$$

which now offers a simpler way to look at it, because each factor in the above formula can be expressed as conditionally convergent series

$$\zeta\left(\frac{1}{2}\right)=\frac{1}{\sqrt{2}-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}},\qquad L\left(\frac{1}{2},\chi_4\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{2n+1}}.$$

Since the approach of using analogues of the formulas (4.4.61) - (4.4.64) only works well when $k=2,4,6,8$, in the next lemma we follow the approach of the proof of Lemma 4.4.5 in order to establish a formula for the value $\zeta_k\left(\frac{k-1}{2}\right)$ that uses the Selberg-Chowla formula.

Lemma 4.4.6. *For any positive integer $k\geq 2$, the following formula holds*

$$\zeta_k\left(\frac{k-1}{2}\right)=\zeta_{k-2}\left(\frac{k-1}{2}\right)+\frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\left(2\gamma-2\log(4\pi)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-2}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}\right)$$
$$+\frac{4\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\sum_{m,n=1}^{\infty}r_{k-1}(m)\,K_0\left(2\pi\sqrt{m}\,n\right). \tag{4.4.90}$$

[30] Note that the expansion as Dirichlet series, $\zeta_k(s)=\sum_{n=1}^{\infty}\frac{r_k(n)}{n^s}$, is only valid in the half-plane $\mathrm{Re}(s)>\frac{k}{2}$.

Remark 4.4.10. Taking again the convention that $\zeta_0(s) := 0$ (see Remark 4.4.8 above) and that the infinite product

$$\prod_{\mathbf{m}\in\mathbb{Z}^{k-2}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)$$

is equal to 1 when $k=2$, the previous relation gives the formula

$$\zeta_2\left(\frac{1}{2}\right)=2\gamma-2\log(4\pi)+4\sum_{m,n=1}^{\infty}r_1(m)\,K_0(2\pi\sqrt{m}\,n).$$

Since $r_1(m)=2$ iff m is a perfect square and equals zero otherwise, we find that

$$\begin{aligned}\zeta_2\left(\frac{1}{2}\right)&=2\gamma-2\log(4\pi)+4\sum_{m,n=1}^{\infty}r_1(m)\,K_0(2\pi\sqrt{m}\,n)\\&=2\gamma-2\log(4\pi)+8\sum_{m,n=1}^{\infty}K_0(2\pi m\,n)\\&=2\gamma-2\log(4\pi)+8\sum_{n=1}^{\infty}d(n)\,K_0(2\pi n),\end{aligned}\tag{4.4.91}$$

where $d(n)=\sum_{d|n}1$ is the usual divisor function. The formula just proved is a particular case of a more general identity due to Selberg and Chowla [[201], p. 94, eq. (29)]. Using an analogue of the identity (4.4.91), Bateman and Grosswald [[15], p. 367, Theorem 3] proved that, when the discriminant of a certain real quadratic form is taken large enough, the associated Epstein zeta function, $\zeta\left(s,Q\right)$, always possesses a real zero belonging to the interval $\left(\frac{1}{2},1\right)$.

Proof of Lemma 4.4.6. If we start with the Selberg-Chowla formula (4.4.67), we know that

$$\begin{aligned}\zeta_k\left(\frac{k-1}{2}\right)=&\lim_{s\to\frac{k-1}{2}}\left\{\zeta_{k-1}(s)+\frac{2\pi^{\frac{k-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{k-1}{2}\right)\zeta\left(2s-k+1\right)\right\}\\&+\frac{4\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\sum_{m,n=1}^{\infty}r_{k-1}(m)\,K_0\left(2\pi\sqrt{m}\,n\right),\end{aligned}\tag{4.4.92}$$

and so the conclusion of the proof follows once we compute the limit above, i.e.,

$$\lim_{s\to\frac{k-1}{2}}\left\{\zeta_{k-1}(s)+\frac{2\pi^{\frac{k-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{k-1}{2}\right)\zeta\left(2s-k+1\right)\right\}.\tag{4.4.93}$$

We shall evaluate (4.4.93) by using the Laurent expansion for both terms contained in it. To get the expansion of the first term, this is, $\zeta_{k-1}(s)$, we start by using the Kronecker limit formula (see

(4.4.65) above),

$$\zeta_{k-1}(s)=\frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\frac{1}{s-\frac{k-1}{2}}+\zeta_{k-2}\left(\frac{k-1}{2}\right)$$
$$+\frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\left(\gamma-2\log(2)-\psi\left(\frac{k-1}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-2}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}\right)$$
$$+O\left(s-\frac{k-1}{2}\right). \tag{4.4.94}$$

On the other hand, we have seen in (4.4.68) that the second term contained in (4.4.93) satisfies the expansion

$$\frac{2\pi^{\frac{k-1}{2}}}{\Gamma(s)}\Gamma\left(s-\frac{k-1}{2}\right)\zeta\left(2s-k+1\right)=-\frac{\pi^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}\left\{\frac{1}{s-\frac{k-1}{2}}-\psi\left(\frac{k-1}{2}\right)+2\log(2\pi)-\gamma\right\}$$
$$+O\left(s-\frac{k-1}{2}\right). \tag{4.4.95}$$

Combining (4.4.94) and (4.4.95) immediately gives (4.4.90). □

We can now set $\nu=-1/2$ in (4.4.54) and (4.4.55) and write the term (4.4.89) by using the previous lemma. In the next result we obtain an interesting formula valid for every $k\geq 2$.

Corollary 4.4.12. *For any positive integer $k\geq 2$ and $x>y>0$, the following summation formula is valid*

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{\sqrt{n}}\cosh\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}=2\pi x+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}\left(x^2+y^2\right)^{\frac{k-1}{2}}}\left\{\frac{1}{(x+y)^{k-1}}+\frac{1}{(x-y)^{k-1}}\right\}$$
$$+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\,\pi^{\frac{k-1}{2}}}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{1}{\left(n+(x+y)^2\right)^{\frac{k-1}{2}}}+\frac{1}{\left(n+(x-y)^2\right)^{\frac{k-1}{2}}}-\frac{2}{n^{\frac{k-1}{2}}}\right\}$$
$$+\pi^{-\frac{k-1}{2}}\Gamma\left(\frac{k-1}{2}\right)\zeta_{k-2}\left(\frac{k-1}{2}\right)+2\gamma-2\log(4\pi)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-2}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}$$
$$+4\sum_{m,n=1}^{\infty}r_{k-1}(m)\,K_0\left(2\pi\sqrt{m}\,n\right). \tag{4.4.96}$$

Moreover, one has the analogous identity

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{\sqrt{n}}\cos\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}=2\pi x+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}\left(x^2-y^2\right)^{\frac{k-1}{2}}}\left\{\frac{1}{(x+iy)^{k-1}}+\frac{1}{(x-iy)^{k-1}}\right\}$$
$$+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\,\pi^{\frac{k-1}{2}}}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{1}{(n+(x+iy)^2)^{\frac{k-1}{2}}}+\frac{1}{(n+(x-iy)^2)^{\frac{k-1}{2}}}-\frac{2}{n^{\frac{k-1}{2}}}\right\}$$
$$+\pi^{-\frac{k-1}{2}}\Gamma\left(\frac{k-1}{2}\right)\zeta_{k-2}\left(\frac{k-1}{2}\right)+2\gamma-2\log(4\pi)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-2}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}$$
$$+4\sum_{m,n=1}^{\infty}r_{k-1}(m)\,K_0\left(2\pi\sqrt{m}\,n\right). \tag{4.4.97}$$

Proof. As usual, it is enough for us to prove the first identity (4.4.96), as the second one can be analogously obtained. In order to reach (4.4.96), we take $\nu=-\frac{1}{2}$ in (4.4.54) and use the reduction formulas (cf. (2.2.20) and [[172], Vol. III, p. 461, 7.3.1.106])

$$K_{-\frac{1}{2}}(x)=K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}\,e^{-x},\quad I_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cosh(x),\quad J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos(x), \tag{4.4.98}$$

$${}_2F_1\left(a,a+\frac{1}{2};\frac{1}{2};z\right)=\frac{1}{2}\left\{\frac{1}{(1-\sqrt{z})^{2a}}+\frac{1}{(1+\sqrt{z})^{2a}}\right\}, \tag{4.4.99}$$

whose combination results in the identity

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{\sqrt{n}}\cosh\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}=2\pi x+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}\left(x^2+y^2\right)^{\frac{k-1}{2}}}\left\{\frac{1}{(x+y)^{k-1}}+\frac{1}{(x-y)^{k-1}}\right\}$$
$$+\frac{\Gamma\left(\frac{k-1}{2}\right)}{2\,\pi^{\frac{k-1}{2}}}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{1}{(n+(x+y)^2)^{\frac{k-1}{2}}}+\frac{1}{(n+(x-y)^2)^{\frac{k-1}{2}}}-\frac{2}{n^{\frac{k-1}{2}}}\right\}$$
$$+\pi^{-\frac{k-1}{2}}\Gamma\left(\frac{k-1}{2}\right)\zeta_k\left(\frac{k-1}{2}\right).$$

Using the result of the previous lemma, we may write $\zeta_k\left(\frac{k-1}{2}\right)$ in a more explicit form, which is (4.4.90). This completes the proof of the formula (4.4.96). □

We end this section with a particular case of the previous corollary holding for $k=2$. Although this result is already given in [[43], p. 335, Theorem 5.6], we note that the formula presented in [43] involves terms such as the central values $\zeta\left(\frac{1}{2}\right)$ and $L\left(\frac{1}{2},\chi_4\right)$. The advantage of our next corollary is that we present an alternative expression involving an infinite series containing $d(n)$ and the Macdonald function $K_0(x)$.

Corollary 4.4.13. *Let $x > y > 0$. Then the following summation formula holds*

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} \cosh\left(2\pi\sqrt{n}y\right) e^{-2\pi\sqrt{n}x} = \frac{1}{2}\sum_{n=1}^{\infty} r_2(n) \left\{ \frac{1}{\sqrt{n+(x+y)^2}} + \frac{1}{\sqrt{n+(x-y)^2}} - \frac{2}{\sqrt{n}} \right\}$$
$$+ 2\pi x + \frac{x}{\sqrt{x^2+y^2}\,(x^2-y^2)} + 2\gamma - 2\log(4\pi) + 8\sum_{n=1}^{\infty} d(n)\, K_0(2\pi n).$$

4.4.4 Reduction II: The case $\mu = \frac{k}{2} - 1$

We will now see that (4.1.14) and (4.1.15) can be obtained as particular cases of Theorem 4.4.1 when $\mu = \frac{k}{2} - 1$. For our convenience, we will state the resulting summation formulas in their most compact form.

Theorem 4.4.3. *For any positive integer $k \geq 1$, $x > y > 0$ and $Re(\nu) > 0$, the following identity holds*

$$\sum_{n=0}^{\infty} r_k(n)\, n^{\frac{\nu+1}{2}-\frac{k}{4}}\, I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$= \frac{y^{\frac{k}{2}-1} x^{\nu}\, \Gamma\left(\nu+\frac{k}{2}\right)}{2\,\Gamma\left(\frac{k}{2}\right)\, \pi^{\nu+1}} \sum_{n=0}^{\infty} \frac{r_k(n)}{(n+x^2-y^2)^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4}, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2}; \frac{k}{2}; -\frac{4ny^2}{(n+x^2-y^2)^2}\right). \tag{4.4.100}$$

Analogously, under the same conditions, we have the formula[31]

$$\sum_{n=0}^{\infty} r_k(n)\, n^{\frac{\nu+1}{2}-\frac{k}{4}}\, J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$= \frac{y^{\frac{k}{2}-1} x^{\nu}\, \Gamma\left(\nu+\frac{k}{2}\right)}{2\,\Gamma\left(\frac{k}{2}\right)\, \pi^{\nu+1}} \sum_{n=0}^{\infty} \frac{r_k(n)}{(n+x^2+y^2)^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4}, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2}; \frac{k}{2}; \frac{4ny^2}{(n+x^2+y^2)^2}\right). \tag{4.4.101}$$

Proof. As in the previous results, it is enough to show the summation formula (4.4.100), because the proof of (4.4.101) is totally analogous. We set $\mu = \frac{k}{2} - 1$ in (4.4.32), which results in

$$\pi^{\frac{\nu+1}{2}-\frac{k}{4}} \sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu+1}{2}-\frac{k}{4}}\, I_{\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$= -\frac{\Gamma(\nu)\, y^{\frac{k}{2}-1}}{2\pi^{\nu+1-\frac{k}{2}} x^{\nu}\Gamma\left(\frac{k}{2}\right)} + \frac{\Gamma\left(\nu+\frac{k}{2}\right) y^{\frac{k}{2}-1}}{2\pi^{\nu+1} x^{\nu+k}\Gamma\left(\frac{k}{2}\right)}\, {}_2F_1\left(\frac{k}{2}, \nu+\frac{k}{2}; \frac{k}{2}; \frac{y^2}{x^2}\right)$$
$$+ \frac{y^{\frac{k}{2}-1}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2x^{\nu+k}\pi^{\nu+1}\Gamma\left(\frac{k}{2}\right)} \sum_{n=1}^{\infty} r_k(n)\, F_4\left(\nu+\frac{k}{2}; \frac{k}{2}; \frac{k}{2}, \frac{k}{2}; -\frac{n}{x^2}, \frac{y^2}{x^2}\right). \tag{4.4.102}$$

[31] in fact, we only need to assume that $x, y > 0$ for (4.4.101) to hold. See the conditions of Corollary 2.2.4 above. We shall state all the particular cases of Theorem 4.4.3 with the condition $x > y > 0$, although the summation formulas associated to the Bessel $J-$function hold without this supposition.

To simplify (4.4.102), we start by noting that

$$ {}_2F_1\left(\frac{k}{2},\,\nu+\frac{k}{2};\,\frac{k}{2};\,\frac{y^2}{x^2}\right)=\frac{x^{2\nu+k}}{\left(x^2-y^2\right)^{\nu+\frac{k}{2}}}, \tag{4.4.103} $$

which is nothing but the relation already recorded in (4.4.45). Also, we can apply the reduction formula (4.4.37) inside the infinite series on the right of (4.4.102). Using (4.4.37) with $\alpha=\nu+\frac{k}{2}$ and $\beta=\frac{k}{2}$, we are able to get the transformation

$$ F_4\left(\nu+\frac{k}{2};\,\frac{k}{2};\,\frac{k}{2},\,\frac{k}{2};\,-\frac{n}{x^2},\,\frac{y^2}{x^2}\right)=\frac{x^{2\nu+k}}{\left(n+x^2-y^2\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};-\frac{4ny^2}{\left(n+x^2-y^2\right)^2}\right), \tag{4.4.104} $$

which already agrees with the terms of the infinite series on the right-hand side of (4.4.100) when $n\geq1$. The combination of (4.4.102), (4.4.103) and (4.4.104) yields

$$ \begin{aligned} \sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)&=-\frac{\Gamma(\nu)\pi^{\frac{k}{2}-\nu-1}\,y^{\frac{k}{2}-1}}{2x^{\nu}\Gamma\left(\frac{k}{2}\right)}+\frac{\Gamma\left(\nu+\frac{k}{2}\right)x^{\nu}\,y^{\frac{k}{2}-1}}{2\pi^{\nu+1}\Gamma\left(\frac{k}{2}\right)\left(x^2-y^2\right)^{\nu+\frac{k}{2}}}\\ &+\frac{\Gamma\left(\nu+\frac{k}{2}\right)x^{\nu}y^{\frac{k}{2}-1}}{2\pi^{\nu+1}\Gamma\left(\frac{k}{2}\right)}\sum_{n=1}^{\infty}\frac{r_k(n)}{\left(n+x^2-y^2\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};-\frac{4ny^2}{\left(n+x^2-y^2\right)^2}\right), \end{aligned} $$

which can now be rewritten as (4.4.100) due to the limiting relations (4.1.17) and (4.1.33). This completes the proof. □

Remark 4.4.11. A much more laborious proof of this formula could be written if we set $\mu=\frac{k}{2}-1$ in (4.4.23) and use the following formula [[172], Vol. III, p. 453, relation 7.2.4.89]

$$ F_4\left(\alpha,\beta;\,1+\alpha-\beta,\,\beta;-\frac{w}{(1-w)(1-z)},\,-\frac{z}{(1-w)(1-z)}\right) \tag{4.4.105} $$

$$ =(1-z)^{\alpha}\,{}_2F_1\left(\alpha,\beta;\,1+\alpha-\beta;\,-\frac{w(1-z)}{1-w}\right). \tag{4.4.106} $$

4.4.5 Particular cases of Reduction II

If we take $k=1$ in (4.4.100) and use the fact that $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos(x)$, $I_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cosh(x)$ (cf. (4.4.98) above), as well as the reduction formula for ${}_2F_1\left(a,a+\frac{1}{2};\frac{1}{2};z\right)$, (4.4.99), we can derive the following corollary.

Corollary 4.4.14. *For any complex number ν such that $Re(\nu)>0$ and $x>y>0$, the following identity holds*

$$ \sum_{n=1}^{\infty}n^{\nu}\cos(2\pi ny)\,K_{\nu}\left(2\pi nx\right)=-\frac{x^{-\nu}\Gamma\left(\nu\right)}{4\pi^{\nu}}+\frac{x^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{4\pi^{\nu+\frac{1}{2}}}\sum_{n\in\mathbb{Z}}\frac{1}{\left(n^2+2yn+x^2+y^2\right)^{\nu+\frac{1}{2}}}. \tag{4.4.107} $$

Analogously,

$$\sum_{n=1}^{\infty} n^{\nu} \cosh(2\pi n y)\, K_{\nu}\left(2\pi n x\right) = -\frac{x^{-\nu}\Gamma\left(\nu\right)}{4\pi^{\nu}} + \frac{x^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}{4\pi^{\nu+\frac{1}{2}}} \sum_{n\in\mathbb{Z}} \frac{1}{\left(n^2+2iyn+x^2-y^2\right)^{\nu+\frac{1}{2}}}. \tag{4.4.108}$$

We remark that the previous result is actually due to Kober [129] and one can frequently step into the evaluation of the infinite series

$$\sum_{n=1}^{\infty} n^{\nu} \cos(2\pi n y)\, K_{\nu}\left(2\pi n x\right)$$

while studying the Selberg-Chowla formula for the binary Epstein zeta function $\zeta\left(s,Q\right)$. Setting $y:=0$ in (4.4.107) and (4.4.108) yields Watson's formula (4.1.19). Another interesting corollary can be deduced once we take $\nu=1$ in Theorem 4.4.3.

Corollary 4.4.15. *For any $x>y>0$, the following formula holds*

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{4-k}{4}}\, J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_1\left(2\pi\sqrt{n}\,x\right) = -\frac{\pi^{\frac{k}{2}-2}y^{\frac{k}{2}-1}}{2x\,\Gamma\left(\frac{k}{2}\right)} + \frac{k\,y^{\frac{k}{2}-1}x}{4\pi^2\left(x^2+y^2\right)^{\frac{k}{2}+1}} + \frac{2^{\frac{k}{2}-4}y^{\frac{k}{2}-3}\,x}{\pi^2}$$
$$\times \sum_{n=1}^{\infty}\frac{r_k(n)}{n}\, \frac{\left(\frac{k}{2}-2\right)\left(n+x^2+y^2\right)\left(\sqrt{\left(n+x^2+y^2\right)^2-4ny^2}-\left(n+x^2+y^2\right)\right)+4ny^2\left(\frac{k}{2}-1\right)}{\left(\left(n+x^2+y^2\right)^2-4ny^2\right)^{\frac{3}{2}}\left(\sqrt{\left(n+x^2+y^2\right)^2-4ny^2}+n+x^2+y^2\right)^{\frac{k}{2}-2}}. \tag{4.4.109}$$

Analogously,

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{4-k}{4}}\, I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_1\left(2\pi\sqrt{n}\,x\right) = -\frac{\pi^{\frac{k}{2}-2}y^{\frac{k}{2}-1}}{2x\,\Gamma\left(\frac{k}{2}\right)} + \frac{k\,y^{\frac{k}{2}-1}x}{4\pi^2\left(x^2-y^2\right)^{\frac{k}{2}+1}} + \frac{2^{\frac{k}{2}-4}y^{\frac{k}{2}-3}\,x}{\pi^2}$$
$$\times \sum_{n=1}^{\infty}\frac{r_k(n)}{n}\, \frac{4ny^2\left(\frac{k}{2}-1\right)-\left(\frac{k}{2}-2\right)\left(n+x^2-y^2\right)\left(\sqrt{\left(n+x^2-y^2\right)^2+4ny^2}-\left(n+x^2-y^2\right)\right)}{\left(\left(n+x^2-y^2\right)^2+4ny^2\right)^{\frac{3}{2}}\left(\sqrt{\left(n+x^2-y^2\right)^2+4ny^2}+n+x^2-y^2\right)^{\frac{k}{2}-2}}. \tag{4.4.110}$$

Proof. Take $\nu=1$ in (4.4.101). We find the formula

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{4-k}{4}}\, J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_1\left(2\pi\sqrt{n}\,x\right) = -\frac{\pi^{\frac{k}{2}-2}y^{\frac{k}{2}-1}}{2x\,\Gamma\left(\frac{k}{2}\right)} + \frac{k\,y^{\frac{k}{2}-1}x}{4\pi^2\left(x^2+y^2\right)^{\frac{k}{2}+1}}$$
$$+\frac{k\,y^{\frac{k}{2}-1}x}{4\pi^2}\sum_{n=1}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\frac{k}{2}+1}}\,{}_2F_1\left(\frac{1}{2}+\frac{k}{4},\,1+\frac{k}{4};\,\frac{k}{2};\,\frac{4ny^2}{\left(n+x^2+y^2\right)^2}\right). \tag{4.4.111}$$

The hypergeometric factor on the series at the right-hand side of (4.4.111) can be simplified by appealing to the reduction formula (4.4.49). After some straightforward manipulations, we may be able to obtain the identity

$$\frac{1}{(n+x^2+y^2)^{\frac{k}{2}+1}}\,{}_2F_1\left(\frac{1}{2}+\frac{k}{4},\,1+\frac{k}{4};\,\frac{k}{2};\,\frac{4ny^2}{(n+x^2+y^2)^2}\right)$$
$$=\frac{2^{\frac{k}{2}}}{k}\cdot\frac{\left(\frac{k}{2}-2\right)(n+x^2+y^2)\left(\sqrt{(n+x^2+y^2)^2-4ny^2}-(n+x^2+y^2)\right)+4ny^2\left(\frac{k}{2}-1\right)}{4ny^2\left((n+x^2+y^2)^2-4ny^2\right)^{\frac{3}{2}}\left(\sqrt{(n+x^2+y^2)^2-4ny^2}+n+x^2+y^2\right)^{\frac{k}{2}-2}},$$

which proves (4.4.109). The second summation formula (4.4.110) can be analogously derived. □

If we take $k=4$, the previous corollary implies the identities (4.4.53) and (4.4.52). Another very interesting formula can be deduced when we impose $k=2$ in the previous corollary.

Corollary 4.4.16. *For any $x>y>0$, the following identities hold*

$$\sum_{n=0}^{\infty}r_2(n)\sqrt{n}\,J_0(2\pi\sqrt{n}y)\,K_1\left(2\pi\sqrt{n}\,x\right)=\frac{x}{2\pi^2}\sum_{n=0}^{\infty}\frac{r_2(n)\,\left(n+x^2+y^2\right)}{\left(n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2\right)^{\frac{3}{2}}},$$

$$\sum_{n=0}^{\infty}r_2(n)\sqrt{n}\,I_0(2\pi\sqrt{n}y)\,K_1\left(2\pi\sqrt{n}\,x\right)=\frac{x}{2\pi^2}\sum_{n=0}^{\infty}\frac{r_2(n)\,\left(n+x^2-y^2\right)}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{3}{2}}}.$$

Remark 4.4.12. Like in Theorem 4.4.2, it is possible to prove the continuation of the previous formula (4.4.100) and (4.4.101) to the region $\mathrm{Re}(\nu)>-1$. But this is already done in Corollary 2.2.7. For example, to get a summation formula for the infinite series

$$\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_0\left(2\pi\sqrt{n}\,x\right),$$

we may use Corollary 2.2.9 of the second chapter. All the additional work only consists in replacing the residual terms ρ and ρ_0 given at the right-hand side of (2.2.57) by those found in Lemma 4.4.5 above.

4.4.6 Reduction III: The case $\mu = \nu + \frac{k}{2} - 1$

In this part of the chapter we will prove a new summation formula that can be derived when we take $\mu = \nu + \frac{k}{2} - 1$ in our Theorem 4.4.1.

Theorem 4.4.4. *For any positive integer k, $x > y > 0$ and $Re(\nu) > 0$, the following transformation formula holds*

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} I_{\nu+\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right) K_{\nu}\left(2\pi\sqrt{n}\,x\right) = -\frac{\pi^{\frac{k}{2}-1}\,\Gamma(\nu)\,x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\Gamma\left(\nu+\frac{k}{2}\right)} + \frac{x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\pi\left(x^2-y^2\right)^{\frac{k}{2}}}$$
$$+\frac{2^{\nu+\frac{k}{2}-2}\,x^{\nu}y^{\nu+\frac{k}{2}-1}}{\pi}\sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{\left(n+x^2-y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\right)^{\frac{k}{2}-1}}\right.$$
$$\left.\times\frac{1}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\left(n+x^2+y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\right)^{\nu}}\right\}. \tag{4.4.112}$$

Analogously, one has the companion formula

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} J_{\nu+\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right) K_{\nu}\left(2\pi\sqrt{n}\,x\right) = -\frac{\pi^{\frac{k}{2}-1}\,\Gamma(\nu)\,x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\Gamma\left(\nu+\frac{k}{2}\right)} + \frac{x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\pi\left(x^2+y^2\right)^{\frac{k}{2}}}$$
$$+\frac{2^{\nu+\frac{k}{2}-2}\,x^{\nu}y^{\nu+\frac{k}{2}-1}}{\pi}\sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{\left(n+x^2+y^2+\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\right)^{\frac{k}{2}-1}}\right.$$
$$\left.\times\frac{1}{\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\left(n+x^2-y^2+\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\right)^{\nu}}\right\}. \tag{4.4.113}$$

Proof. We take $\mu = \nu + \frac{k}{2} - 1$ in (4.4.23). Since $\text{Re}(\nu) > 0$ by hypothesis, we know that $\text{Re}(\mu) > \frac{k}{2} - 1 > -1$ and so we can fully apply (4.4.21). Using the reduction formula (4.4.45) on the second term at the right-hand side of (4.4.23), we have that

$$\pi^{\frac{1}{2}-\frac{k}{4}}\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} I_{\nu+\frac{k}{2}-1}\left(2\pi\sqrt{n}y\right) K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$
$$= -\frac{\Gamma(\nu)\,\pi^{\frac{k}{4}-\frac{1}{2}}x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\Gamma\left(\nu+\frac{k}{2}\right)} + \frac{\pi^{-\frac{k}{4}-\frac{1}{2}}x^{-\nu}y^{\nu+\frac{k}{2}-1}}{2\left(x^2-y^2\right)^{\frac{k}{2}}}$$
$$+\frac{x^{\nu}y^{\nu+\frac{k}{2}-1}\,\pi^{-\frac{k}{4}-\frac{1}{2}}}{2}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\nu+\frac{k}{2}}}F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\nu+\frac{k}{2},\,\nu+1;\,-\frac{y^2}{n},\,-\frac{x^2}{n}\right).$$

In order to simplify the term on the infinite series involving Appell's hypergeometric function, we appeal to [[172], Vol. III, p. 453, relation 7.2.4.87] (cf. [[95], p. 1019, 9.182.6]), which states that

$$F_4\left(\alpha;\,\beta;\,\alpha,\beta;-\frac{w}{(1-w)(1-z)},-\frac{z}{(1-w)(1-z)}\right)=\frac{(1-w)^{\beta}\,(1-z)^{\alpha}}{1-wz},\tag{4.4.114}$$

where we assume w,z to be real numbers such that $0<z<1$ and $0<w<1$. Taking $\alpha=\nu+\frac{k}{2}$, $\beta=\nu+1$ and, finally[32],

$$w=\frac{n+x^2+y^2-\sqrt{\left(n+x^2+y^2\right)^2-4x^2y^2}}{2x^2},\tag{4.4.115}$$

$$z=\frac{n+x^2+y^2-\sqrt{\left(n+x^2+y^2\right)^2-4x^2y^2}}{2y^2},\tag{4.4.116}$$

we find after several simplifications that the main term on the infinite series at the right of (4.4.23) can be written as

$$\begin{aligned}
&n^{-\nu-\frac{k}{2}}\,F_4\left(\nu+\frac{k}{2};\,\nu+1;\,\nu+\frac{k}{2},\,\nu+1;\,-\frac{y^2}{n},\,-\frac{x^2}{n}\right)\\
&=\frac{2^{\nu+\frac{k}{2}-1}}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\,\left(n+x^2+y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\right)^{\nu}}\\
&\times\frac{1}{\left(n+x^2-y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\right)^{\frac{k}{2}-1}},
\end{aligned}$$

completing the proof. □

Clearly, taking $k=2$ in (4.4.112) and (4.4.113) yields Popov's formulas (4.4.47) and (4.4.48), proved in Corollary 4.4.5 of the previous section. In the next corollary we get an interesting summation formula that results from setting $\nu=\frac{1}{2}$ in (4.4.112).

Corollary 4.4.17. *For any $x>y>0$, the following formula holds*

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k-1}{4}}}\,I_{\frac{k-1}{2}}\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}=-\frac{\pi^{\frac{k-1}{2}}y^{\frac{k-1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}+\frac{y^{\frac{k-1}{2}}}{\pi\left(x^2-y^2\right)^{\frac{k}{2}}}\\
&+\frac{2^{\frac{k-1}{2}}y^{\frac{k-1}{2}}\,x}{\pi}\sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{\left(n+x^2-y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}\right)^{\frac{k}{2}-1}}\right.\\
&\left.\times\frac{1}{\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}\,\frac{1}{\sqrt{n+x^2+y^2+\sqrt{n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}}\right\}.
\end{aligned}$$

[32]It is easy to check that $0<w<1$ and $0<z<1$, for w and z defined by (4.4.115) and (4.4.116).

Analogously, one has the identity

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k-1}{4}}}\,J_{\frac{k-1}{2}}\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}=-\frac{\pi^{\frac{k-1}{2}}y^{\frac{k-1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}+\frac{y^{\frac{k-1}{2}}}{\pi\left(x^2+y^2\right)^{\frac{k}{2}}}+$$
$$+\frac{2^{\frac{k-1}{2}}y^{\frac{k-1}{2}}\,x}{\pi}\sum_{n=1}^{\infty}\left\{\frac{r_k(n)}{\left(n+x^2+y^2+\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}\right)^{\frac{k}{2}-1}}\right.$$
$$\left.\times\frac{1}{\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}\,\frac{1}{\sqrt{n+x^2-y^2+\sqrt{n^2+2n\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}}\right\}.$$

When we set $k=1$ in the previous result, we get a new formula that involves the exponential function and the Bessel functions of index 0.

Corollary 4.4.18. *For $x>y>0$, the following summation formula holds*

$$1+2\sum_{n=1}^{\infty}I_0\left(2\pi ny\right)\,e^{-2\pi nx}=\frac{2x}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sqrt{n^2+x^2-y^2+\sqrt{n^4+2n^2\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}}{\sqrt{n^4+2n^2\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}\right.$$
$$\left.\times\frac{1}{\sqrt{n^2+x^2+y^2+\sqrt{n^4+2n^2\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}}\right\}+\frac{1}{\pi\sqrt{x^2-y^2}}.$$

Analogously,

$$1+2\sum_{n=1}^{\infty}J_0\left(2\pi ny\right)\,e^{-2\pi nx}=\frac{2x}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\sqrt{n^2+x^2+y^2+\sqrt{n^4+2n^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}}{\sqrt{n^4+2n^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}\right.$$
$$\left.\times\frac{1}{\sqrt{n^2+x^2-y^2+\sqrt{n^4+2n^2\left(x^2-y^2\right)+\left(x^2+y^2\right)^2}}}\right\}+\frac{1}{\pi\sqrt{x^2+y^2}}.$$

Remark 4.4.13. It is also possible to prove the analytic continuation of formulas (4.4.112) and (4.4.113) to the region $\mathrm{Re}(\nu)>-1$, allowing us to prove new summation formulas involving series of the form

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k-1}{4}}}\,I_{\frac{k-3}{2}}\left(2\pi\sqrt{n}y\right)\,e^{-2\pi\sqrt{n}x}.$$

We leave the details for the interested reader.

Remark 4.4.14. The reader may be surprised that we took the reduction formulas (4.4.37), (4.4.106) and (4.4.114) for Appell's function F_4 but not the most important one, due to Bailey

[9, 10][33]

$$\begin{aligned}
&F_4\left(\alpha;\,\beta;\gamma,\alpha+\beta-\gamma+1;w(1-z),z(1-w)\right)\\
&={}_2F_1\left(\alpha,\beta;\gamma;w\right)\,{}_2F_1\left(\alpha,\beta;\alpha+\beta-\gamma+1;z\right),
\end{aligned}\tag{4.4.117}$$

valid inside simply-connected regions surrounding $(w,z)=(0,0)$ for which

$$|w(1-z)|^{\frac{1}{2}}+|w(1-z)|^{\frac{1}{2}}<1.$$

The reason for this omission is that we did not find the transformation formulas for (4.4.1) resulting from (4.4.117) very interesting, because in all of its particular cases we would always have an incomplete beta function and not elementary function as in the previous cases.[34]

4.5 New analogues of Guinand's and Koshliakov's formulas

This section is devoted to establish new analogues of the Ramanujan-Guinand and Koshliakov's formulas. We have addressed these results before, namely in Example 1.2.1 of the first chapter. In order to motivate the results of this section, let us recall what the Ramanujan-Guinand formula says, by following the way it is stated in Ramanujan's Lost Notebook [179].

Entry 3.3.1. *Let $\sigma_k(n)=\sum_{d|n}d^k$ and let $K_\nu(z)$ be the modified Bessel function of the second kind. If α and β are two positive real numbers such that $\alpha\beta=\pi^2$ and if ν is any complex number, then*

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}K_{\frac{\nu}{2}}(2n\alpha)-\sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\nu/2}K_{\frac{\nu}{2}}(2n\beta)\\
&=\frac{1}{4}\Gamma\left(-\frac{\nu}{2}\right)\zeta(-\nu)\left\{\beta^{(1+\nu)/2}-\alpha^{(1+\nu)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{\nu}{2}\right)\zeta(\nu)\left\{\beta^{(1-\nu)/2}-\alpha^{(1-\nu)/2}\right\}.
\end{aligned}\tag{4.5.1}$$

As mentioned in the first chapter of this thesis, (4.5.1) was rediscovered by Guinand in 1955 [100], who employed Watson's formula [233] in order to derive it (see (1.2.29) above). Using an idea similar to the one employed by Guinand, we propose to prove the following generalization of (4.5.1).

[33] in fact, while writing the obituary notice of Ernest William Barnes for the London Mathematical Society, Bailey found that, in 1907, Barnes had derived (4.4.117) in a long manuscript that was never published. For more interesting historical accounts like these, the reader is encouraged to consult [256].

[34] The transformation formulas that one can get from (4.4.117) are those with $\mu=\nu+\frac{k}{2}$.

Theorem 4.5.1. *For every $k \in \mathbb{N}$, if $x, y > 0$ and ν is any complex number, then the following formula holds*

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} (mn)^{\frac{1}{2}-\frac{k}{4}} J_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right) K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right)$$
$$-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2+y^2} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} (mn)^{\frac{1}{2}-\frac{k}{4}} J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right)$$
$$= x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\} + x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}+\nu}}-1\right\}, \tag{4.5.2}$$

where, as usual, $\eta_k(s)$ denotes the completed Dirichlet series (4.2.9),

$$\eta_k(s) = \pi^{-s}\Gamma(s)\,\zeta_k(s).$$

Moreover, if ν is any complex number and $x > y > 0$, then the analogous formula is valid

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} (mn)^{\frac{1}{2}-\frac{k}{4}} I_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right) K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right)$$
$$-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2-y^2} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} (mn)^{\frac{1}{2}-\frac{k}{4}} I_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2-y^2}\right) K_{\nu}\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2-y^2}\right)$$
$$= x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2-y^2)^{\frac{k}{2}-\nu}}-1\right\} + x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2-y^2)^{\frac{k}{2}+\nu}}-1\right\}. \tag{4.5.3}$$

At first glance, (4.5.2) and (4.5.3) do not seem to be related to (4.5.1), despite the fact that both formulas share the modified Bessel function, $K_{\nu}(z)$, as an element in their infinite series. In the next few lines, we shall argue why (4.5.2) and (4.5.3) are, in fact, generalizations of (4.5.1). Recalling that $\zeta_1(s) = 2\zeta(2s)$ (see (4.1.4) above) and that $r_1(n) = 2$ iff n is a perfect square and 0 otherwise, we see that a particular case of (4.5.2) is

$$8\pi\sqrt{y} \sum_{m,n=1}^{\infty} \left(\frac{m}{n}\right)^{\nu} \sqrt{m\,n}\, J_{-\frac{1}{2}}\left(2\pi m\,n\,y\right) K_{\nu}\left(2\pi m\,n\,x\right)$$
$$-\frac{8\pi\sqrt{y}}{x^2+y^2} \sum_{m,n=1}^{\infty} \left(\frac{m}{n}\right)^{\nu} \sqrt{m\,n}\, J_{-\frac{1}{2}}\left(\frac{2\pi m\,n\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{2\pi m\,n\,x}{x^2+y^2}\right)$$
$$= 2(\pi x)^{-\nu}\,\Gamma(\nu)\zeta\left(2\nu\right)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}-\nu}}-1\right\} + 2(\pi x)^{\nu}\,\Gamma(-\nu)\zeta\left(-2\nu\right)\left\{\frac{1}{(x^2+y^2)^{\frac{1}{2}+\nu}}-1\right\}. \tag{4.5.4}$$

However, this expression can be simplified further. Indeed, recalling the reduction formula for the Bessel function,

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\cos(x),$$

and making the substitution of ν by $\nu/2$ in (4.5.4), we can now rewrite (4.5.4) in the following manner

$$\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\nu}{2}}\cos\left(2\pi m\,n\,y\right)\,K_{\frac{\nu}{2}}\left(2\pi mn\,x\right)-\frac{1}{\sqrt{x^2+y^2}}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{\nu}{2}}\cos\left(\frac{2\pi m\,n\,y}{x^2+y^2}\right)\,K_{\frac{\nu}{2}}\left(\frac{2\pi mn\,x}{x^2+y^2}\right)$$
$$=\frac{1}{4}(\pi x)^{-\frac{\nu}{2}}\,\Gamma\left(\frac{\nu}{2}\right)\zeta\left(\nu\right)\left\{\frac{1}{\left(x^2+y^2\right)^{\frac{1-\nu}{2}}}-1\right\}+\frac{1}{4}(\pi x)^{\frac{\nu}{2}}\,\Gamma\left(-\frac{\nu}{2}\right)\zeta\left(-\nu\right)\left\{\frac{1}{\left(x^2+y^2\right)^{\frac{1+\nu}{2}}}-1\right\}. \tag{4.5.5}$$

The main formal difference between (4.5.1) and the newly derived formula (4.5.5) is that (4.5.1) is expressed via two single infinite series, whilst the series in (4.5.5) contain two variables of summation, m and n. A way to circumvent this is by recalling the definition of the divisor function,

$$\sigma_z(n)=\sum_{d|n}d^z, \tag{4.5.6}$$

which helps us to see that (4.5.5) can be rewritten in the following manner

$$\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\frac{\nu}{2}}\cos\left(2\pi ny\right)\,K_{\frac{\nu}{2}}\left(2\pi n\,x\right)-\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\sigma_{-\nu}(n)\,n^{\frac{\nu}{2}}\cos\left(\frac{2\pi n\,y}{x^2+y^2}\right)\,K_{\frac{\nu}{2}}\left(\frac{2\pi n\,x}{x^2+y^2}\right)$$
$$=\frac{1}{4}(\pi x)^{-\frac{\nu}{2}}\,\Gamma\left(\frac{\nu}{2}\right)\zeta\left(\nu\right)\left\{\frac{1}{\left(x^2+y^2\right)^{\frac{1-\nu}{2}}}-1\right\}+\frac{1}{4}(\pi x)^{\frac{\nu}{2}}\Gamma\left(-\frac{\nu}{2}\right)\zeta(-\nu)\left\{\frac{1}{\left(x^2+y^2\right)^{\frac{1+\nu}{2}}}-1\right\}. \tag{4.5.7}$$

Now, let us observe that equation (4.5.7) closely resembles (4.5.1), while being even more general. By substituting x by α/π and setting $y=0$ in (4.5.7), it is possible to retrieve the Ramanujan-Guinand formula in the same form as in (4.5.1). We are now ready to prove our Theorem 4.5.1.

Proof of Theorem 4.5.1. We only establish (4.5.2), because (4.5.3) can be analogously deduced. First, let us note that (4.4.101), which is valid for $x,y>0$, can be rearranged in the following way

$$\sum_{n=1}^{\infty}\frac{r_k(n)}{\left(n+x^2+y^2\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4ny^2}{\left(n+x^2+y^2\right)^2}\right)=-\frac{1}{\left(x^2+y^2\right)^{\nu+\frac{k}{2}}}$$
$$+\frac{\Gamma(\nu)\pi^{\frac{k}{2}}}{x^{2\nu}\Gamma\left(\nu+\frac{k}{2}\right)}+\frac{2\Gamma\left(\frac{k}{2}\right)\pi^{\nu+1}}{y^{\frac{k}{2}-1}x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,J_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right), \tag{4.5.8}$$

for any $\text{Re}(\nu)>0$. For $x,y>0$, we now consider the following double infinite series

$$\sum_{m,n=1}^{\infty}\frac{r_k(m)\,r_k(n)}{\left(m+n\left(x^2+y^2\right)\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2}\right),\quad \text{Re}(\nu)>\frac{k}{2}. \tag{4.5.9}$$

Let us note first that this double series is absolutely convergent in the half-plane $\text{Re}(\nu) > \frac{k}{2}$. Since $m + n\,\left(x^2+y^2\right) > m + ny^2 > 2y\sqrt{m\,n}$, we have that

$$0 < \frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2} \leq \eta < 1.$$

Therefore, for any $M, N \in \mathbb{N}$, the following inequality must hold

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\left|\frac{r_k(m)\,r_k(n)}{\left(m+n\,(x^2+y^2)\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2}\right)\right|$$
$$< \frac{\Gamma\left(\frac{k}{2}+\ell\right)\left(2\sqrt{x^2+y^2}\right)^{-\text{Re}(\nu)-\frac{k}{2}}}{\left|\Gamma\left(\frac{\nu}{2}+\frac{k}{4}\right)\Gamma\left(\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2}\right)\right|}\sum_{\ell=0}^{\infty}\frac{\Gamma\left(\frac{\text{Re}(\nu)}{2}+\frac{k}{4}+\ell\right)\Gamma\left(\frac{\text{Re}(\nu)}{2}+\frac{k}{4}+\frac{1}{2}+\ell\right)}{\Gamma\left(\frac{k}{2}+\ell\right)\,\ell!}\,\eta^{\ell}\left(\sum_{m=1}^{\infty}\frac{r_k(m)}{m^{\frac{\text{Re}(\nu)}{2}+\frac{k}{4}}}\right)^2.$$

Since, by hypothesis, $\text{Re}(\nu) > \frac{k}{2}$, the last term is finite. In the double series (4.5.9), let us sum firstly with respect to m, i.e., let us consider the iterated series

$$\sum_{n=1}^{\infty}r_k(n)\sum_{m=1}^{\infty}\frac{r_k(m)}{\left(m+n\,(x^2+y^2)\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2}\right). \quad (4.5.10)$$

Applying (4.5.8) with x being replaced by $x\sqrt{n}$ and y replaced by $y\sqrt{n}$, we have that the infinite series with respect to m in (4.5.10) is explicitly given by

$$\sum_{m=1}^{\infty}\frac{r_k(m)}{\left(m+n\,(x^2+y^2)\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2}\right)$$
$$= -\frac{n^{-\nu-\frac{k}{2}}}{\left(x^2+y^2\right)^{\nu+\frac{k}{2}}}+\frac{\Gamma(\nu)\pi^{\frac{k}{2}}}{x^{2\nu}\Gamma\left(\nu+\frac{k}{2}\right)}\,n^{-\nu}$$
$$+\frac{2\Gamma\left(\frac{k}{2}\right)\pi^{\nu+1}}{y^{\frac{k}{2}-1}x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}\sum_{m=1}^{\infty}r_k(m)\,\left(m\,n\right)^{\frac{1}{2}-\frac{k}{4}}\left(\frac{m}{n}\right)^{\frac{\nu}{2}}J_{\frac{k}{2}-1}(2\pi\sqrt{m\,n}y)\,K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right). \quad (4.5.11)$$

In order to define the double series (4.5.9), we have made the imposition $\text{Re}(\nu) > \frac{k}{2}$. Therefore, it is possible to multiply both sides of (4.5.11) by $r_k(n)$ and then take the summation over n. This results in the formula

$$\sum_{m,n=1}^{\infty}\frac{r_k(m)\,r_k(n)}{\left(m+n\,(x^2+y^2)\right)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\,\frac{k}{2};\,\frac{4mn\,y^2}{\left(m+n\,\left(x^2+y^2\right)\right)^2}\right)$$
$$= -\frac{1}{\left(x^2+y^2\right)^{\nu+\frac{k}{2}}}\,\zeta_k\left(\frac{k}{2}+\nu\right)+\frac{\Gamma(\nu)\pi^{\frac{k}{2}}}{x^{2\nu}\Gamma\left(\nu+\frac{k}{2}\right)}\,\zeta_k\left(\nu\right)$$
$$+\frac{2\Gamma\left(\frac{k}{2}\right)\pi^{\nu+1}}{y^{\frac{k}{2}-1}x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,\left(m\,n\right)^{\frac{1}{2}-\frac{k}{4}}\left(\frac{m}{n}\right)^{\frac{\nu}{2}}J_{\frac{k}{2}-1}(2\pi\sqrt{m\,n}y)\,K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right)$$
$$(4.5.12)$$

Since the double series (4.5.9) converges absolutely in the region $\mathrm{Re}(\nu) > \frac{k}{2}$, we may switch the orders of summation and now consider a second iterated series

$$\sum_{m=1}^{\infty} r_k(m) \sum_{n=1}^{\infty} \frac{r_k(n)}{(m+n\,(x^2+y^2))^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \frac{k}{2};\, \frac{4mn\,y^2}{(m+n\,(x^2+y^2))^2}\right), \quad (4.5.13)$$

which results from firstly summing with respect to the variable n. In order to put the inner sum in (4.5.13) in the same shape as (4.5.10), we take the following change of variables

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{(m+n\,(x^2+y^2))^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \frac{k}{2};\, \frac{4mn\,y^2}{(m+n\,(x^2+y^2))^2}\right)$$
$$= \frac{1}{(x^2+y^2)^{\nu+\frac{k}{2}}} \sum_{n=1}^{\infty} \frac{r_k(n)}{(n+m\,(X^2+Y^2))^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \frac{k}{2};\, \frac{4mn\,Y^2}{(n+m\,(X^2+Y^2))^2}\right),$$

where $X := x/(x^2+y^2)$ and $Y := y/(x^2+y^2)$. Note that X^2+Y^2 is just $1/(x^2+y^2)$, but putting (4.5.13) in this template now shows that we can apply (4.5.11) with x being replaced by X, y being replaced by Y and with the roles of m and n being reversed. This now results in a second transformation that is similar to (4.5.11), i.e.,

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{(m+n\,(x^2+y^2))^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \frac{k}{2};\, \frac{4mn\,y^2}{(m+n\,(x^2+y^2))^2}\right)$$
$$= -m^{-\nu-\frac{k}{2}} + \left(x^2+y^2\right)^{\nu-\frac{k}{2}} \frac{x^{-2\nu}\,\Gamma(\nu)\pi^{\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)}\, m^{-\nu}$$
$$+ \frac{2\Gamma\left(\frac{k}{2}\right)\pi^{\nu+1}x^{-\nu}y^{1-\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)(x^2+y^2)} \sum_{n=1}^{\infty} r_k(n)\,(m\,n)^{\frac{1}{2}-\frac{k}{4}} \left(\frac{n}{m}\right)^{\frac{\nu}{2}} J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right). \quad (4.5.14)$$

Using again our initial hypothesis that $\mathrm{Re}(\nu) > \frac{k}{2}$, we can multiply (4.5.14) by $r_k(m)$ and then sum over m. This results in the alternative formula for the double series (4.5.9),

$$\sum_{m,n=1}^{\infty} \frac{r_k(m)\,r_k(n)}{(m+n\,(x^2+y^2))^{\nu+\frac{k}{2}}}\, {}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\, \frac{\nu}{2}+\frac{k}{4}+\frac{1}{2};\, \frac{k}{2};\, \frac{4mn\,y^2}{(m+n\,(x^2+y^2))^2}\right)$$
$$= -\zeta_k\left(\nu+\frac{k}{2}\right) + \left(x^2+y^2\right)^{\nu-\frac{k}{2}} \frac{x^{-2\nu}\,\Gamma(\nu)\pi^{\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)}\, \zeta_k(\nu)$$
$$+ \frac{2\Gamma\left(\frac{k}{2}\right)\pi^{\nu+1}x^{-\nu}y^{1-\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)(x^2+y^2)} \sum_{m,n=1}^{\infty} r_k(m)\,r_k(n)\,(m\,n)^{\frac{1}{2}-\frac{k}{4}} \left(\frac{m}{n}\right)^{\frac{\nu}{2}} J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right). \quad (4.5.15)$$

Comparing (4.5.12) with (4.5.15) proves (4.5.2) under the assumption that $\mathrm{Re}(\nu) > \frac{k}{2}$. But since both sides of (4.5.2) represent analytic functions of ν, the same identity must hold for any $\nu \in \mathbb{C}$ by the principle of analytic continuation. To show that the left-hand side of (4.5.2) is an entire function of $\nu \in \mathbb{C}$, one just needs to check that its infinite series are uniformly convergent on bounded subsets of the complex plane. But this is completely analogous to what we have done in the first chapter (recall (1.2.13) above) and so our proof finishes with this observation. □

We have seen above that, when we impose $k = 1$ in (4.5.2), we are able to get the generalization of the Ramanujan-Guinand formula (4.5.5). The next corollary provides yet another generalization of (4.5.1).

Corollary 4.5.1. *For any complex ν and $x > 0$, the following formula holds*

$$x^{\frac{k}{2}} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} K_{\nu}\left(2\pi\sqrt{m\,n}\,x\right) - x^{-\frac{k}{2}} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} K_{\nu}\left(\frac{2\pi\sqrt{m\,n}}{x}\right)$$
$$= \frac{1}{2}\eta_k(\nu)\left\{x^{\nu-\frac{k}{2}} - x^{-\nu+\frac{k}{2}}\right\} + \frac{1}{2}\eta_k(-\nu)\left\{x^{-\nu-\frac{k}{2}} - x^{\nu+\frac{k}{2}}\right\}. \tag{4.5.16}$$

Remark 4.5.1. It is worthwhile to remark that (4.5.16) is also a particular case of the general formula (1.2.34) obtained in the first chapter. To deduce (4.5.16) from (1.2.34) we just need to take there $\phi_1(s) = \phi_2(s) = \pi^{-s}\,\zeta_k(s)$.

In the remainder of this section, we establish a generalization of the most famous formula of Koshliakov [134], which establishes that

$$\sqrt{\alpha}\left(\frac{\gamma}{4} - \frac{\log(4\beta)}{4} + \sum_{n=1}^{\infty} d(n)\, K_0(2n\alpha)\right) = \sqrt{\beta}\left(\frac{\gamma}{4} - \frac{\log(4\alpha)}{4} + \sum_{n=1}^{\infty} d(n)\, K_0(2n\beta)\right), \tag{4.5.17}$$

where α and β are two positive real numbers satisfying $\alpha\beta = \pi^2$. This is given in the next corollary.

Corollary 4.5.2. *Let $k \in \mathbb{N}$ and recall the definition (4.4.70),*

$$\Phi_k(u) := \prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}} \left(1 - e^{-2\pi|\mathbf{m}|u}\right), \quad u > 0. \tag{4.5.18}$$

For $x, y > 0$, the following analogue of Koshliakov's formula (4.5.17) holds

$$2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n)\,(mn)^{\frac{1}{2}-\frac{k}{4}}\, J_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\, K_0\left(2\pi\sqrt{m\,n}\,x\right)$$
$$- \frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2+y^2} \sum_{m,n=1}^{\infty} r_k(m)\, r_k(n)\,(mn)^{\frac{1}{2}-\frac{k}{4}}\, J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right)$$
$$= \frac{1}{(x^2+y^2)^{\frac{k}{2}}}\left\{2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right) + 2\gamma - 4\log\left(\Phi_k(1)\right) - 2\log\left(\frac{4\pi}{x}\left(x^2+y^2\right)\right)\right\}$$
$$- 2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right) - 2\gamma + 4\log\left(\Phi_k(1)\right) - 2\log\left(\frac{x}{4\pi}\right). \tag{4.5.19}$$

Moreover, if $x>y>0$, then the analogous identity takes place

$$\begin{aligned}
&2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,(mn)^{\frac{1}{2}-\frac{k}{4}}\;I_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\;K_0\left(2\pi\sqrt{m\,n}\,x\right)\\
&-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2-y^2}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,(mn)^{\frac{1}{2}-\frac{k}{4}}\;I_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2-y^2}\right)\;K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2-y^2}\right)\\
&=\frac{1}{(x^2-y^2)^{\frac{k}{2}}}\left\{2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right)+2\gamma-4\log\left(\Phi_k(1)\right)-2\log\left(\frac{4\pi}{x}\left(x^2-y^2\right)\right)\right\}\\
&-2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right)-2\gamma+4\log\left(\Phi_k(1)\right)-2\log\left(\frac{x}{4\pi}\right).
\end{aligned}\tag{4.5.20}$$

Proof. We shall only prove (4.5.19), as the proof of (4.5.20) is similar. If we let $\nu\to 0$ in Ramanujan-Guinand's formula (4.5.2), we find, due to the uniform and absolute convergence of the Bessel series composing it,

$$\begin{aligned}
&2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,(m\,n)^{\frac{1}{2}-\frac{k}{4}}\;J_{\frac{k}{2}-1}\left(2\pi\sqrt{m\,n}\,y\right)\;K_0\left(2\pi\sqrt{m\,n}\,x\right)\\
&-\frac{2\Gamma\left(\frac{k}{2}\right)(\pi y)^{1-\frac{k}{2}}}{x^2+y^2}\sum_{m,n=1}^{\infty}r_k(m)\,r_k(n)\,(mn)^{\frac{1}{2}-\frac{k}{4}}\;J_{\frac{k}{2}-1}\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right)\;K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right)\\
&=\lim_{\nu\to 0}x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}+\lim_{\nu\to 0}x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}+\nu}}-1\right\}.
\end{aligned}$$

The proof of (4.5.19) will be complete once we evaluate the limits appearing in the previous equality. Note that, by the functional equation (4.1.3),

$$x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}=x^{-\nu}\eta_k\left(\frac{k}{2}-\nu\right)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\},$$

and so, using (4.4.65), we can easily calculate the Laurent expansion

$$\begin{aligned}
&x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}=\frac{1}{\nu}\left\{1-\left(x^2+y^2\right)^{-\frac{k}{2}}\right\}\\
&+\frac{1}{(x^2+y^2)^{\frac{k}{2}}}\left\{\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\mathscr{C}_k+\psi\left(\frac{k}{2}\right)-\log\left(\frac{\pi}{x}\left(x^2+y^2\right)\right)\right\}\\
&-\pi^{-\frac{k}{2}}\,\Gamma\left(\frac{k}{2}\right)\mathscr{C}_k-\psi\left(\frac{k}{2}\right)+\log\left(\frac{\pi}{x}\right)+O\left(\nu\right),
\end{aligned}$$

where $\mathscr{C}_k$ is the constant term in the generalized Kronecker limit formula (4.4.65), and it is explicitly described by (4.4.78). This implies that

$$\begin{aligned}
&\lim_{\nu\to 0} x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}+\lim_{\nu\to 0} x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}+\nu}}-1\right\}\\
&=\frac{2}{(x^2+y^2)^{\frac{k}{2}}}\left\{\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\mathscr{C}_k+\psi\left(\frac{k}{2}\right)-\log\left(\frac{\pi}{x}\left(x^2+y^2\right)\right)\right\}\\
&-2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\mathscr{C}_k-2\psi\left(\frac{k}{2}\right)+2\log\left(\frac{\pi}{x}\right).
\end{aligned}$$

Appealing to the definition of the constant term $\mathscr{C}_k$, (4.4.78), we may find the expression

$$\begin{aligned}
&\lim_{\nu\to 0} x^{-\nu}\eta_k(\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}-\nu}}-1\right\}+\lim_{\nu\to 0} x^{\nu}\eta_k(-\nu)\left\{\frac{1}{(x^2+y^2)^{\frac{k}{2}+\nu}}-1\right\}\\
&=\frac{1}{(x^2+y^2)^{\frac{k}{2}}}\left\{2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right)+2\gamma-4\log\left(\Phi_k(1)\right)-2\log\left(\frac{4\pi}{x}\left(x^2+y^2\right)\right)\right\}\\
&-2\pi^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)\zeta_{k-1}\left(\frac{k}{2}\right)-2\gamma+4\log\left(\Phi_k(1)\right)-2\log\left(\frac{x}{4\pi}\right),
\end{aligned}$$

which completes the proof of (4.5.19). □

Taking $k=1$ in Corollary 4.5.2, we are now able to deduce the following generalization of Koshliakov's formula (4.5.17), which seems to be new.

Corollary 4.5.3. *For any $x,y>0$, the following generalization of Koshliakov's formula (4.5.17) holds*

$$\begin{aligned}
&\sum_{n=1}^{\infty} d(n)\,\cos\left(2\pi ny\right)\,K_0\left(2\pi n\,x\right)-\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty} d(n)\,\cos\left(\frac{2\pi n\,y}{x^2+y^2}\right)\,K_0\left(\frac{2\pi n\,x}{x^2+y^2}\right)\\
&=\frac{1}{4\sqrt{x^2+y^2}}\left\{\gamma-\log\left(\frac{4\pi}{x}\left(x^2+y^2\right)\right)\right\}-\frac{1}{4}\left(\gamma-\log\left(\frac{4\pi}{x}\right)\right).
\end{aligned}$$

Moreover, for $x>y>0$, the analogous formula takes place

$$\begin{aligned}
&\sum_{n=1}^{\infty} d(n)\,\cosh\left(2\pi ny\right)\,K_0\left(2\pi n\,x\right)-\frac{1}{\sqrt{x^2-y^2}}\sum_{n=1}^{\infty} d(n)\,\cosh\left(\frac{2\pi n\,y}{x^2-y^2}\right)\,K_0\left(\frac{2\pi n\,x}{x^2-y^2}\right)\\
&=\frac{1}{4\sqrt{x^2-y^2}}\left\{\gamma-\log\left(\frac{4\pi}{x}\left(x^2-y^2\right)\right)\right\}-\frac{1}{4}\left(\gamma-\log\left(\frac{4\pi}{x}\right)\right).
\end{aligned}$$

Using the relation (4.4.72), we can express our general analogue of Koshliakov's formula in terms of Dedekind's $\eta-$function.

Corollary 4.5.4. *For $x > 0$ and $y \geq 0$, the following identity holds*

$$
\begin{aligned}
&\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, J_0\left(2\pi\sqrt{m\,n}\,y\right)\, K_0\left(2\pi\sqrt{m\,n}\,x\right) \\
&-\frac{1}{x^2+y^2}\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, J_0\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right) \\
&=\frac{\gamma-\log\left(\frac{4\pi}{x}\left(x^2+y^2\right)\eta(i)^4\right)}{x^2+y^2}-\gamma+\log\left(\frac{4\pi}{x}\eta(i)^4\right),
\end{aligned}
\tag{4.5.21}
$$

where $\eta(\tau)$, $Im(\tau) > 0$, represents the Dedekind $\eta-$function, (4.4.71). Moreover, if $x > y \geq 0$,

$$
\begin{aligned}
&\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, I_0\left(2\pi\sqrt{m\,n}\,y\right)\, K_0\left(2\pi\sqrt{m\,n}\,x\right) \\
&-\frac{1}{x^2-y^2}\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, I_0\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2-y^2}\right) K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2-y^2}\right) \\
&=\frac{\gamma-\log\left(\frac{4\pi}{x}\left(x^2-y^2\right)\eta(i)^4\right)}{x^2-y^2}-\gamma+\log\left(\frac{4\pi}{x}\eta(i)^4\right).
\end{aligned}
\tag{4.5.22}
$$

Remark 4.5.2. Taking $y = 0$ in (4.5.21) and (4.5.22), we are able to recover the formula (1.2.51) obtained in the first chapter of this thesis. Furthermore, appealing to the evaluation of Dedekind's $\eta-$function (1.2.52), we can rewrite (4.5.21) in the following form

$$
\begin{aligned}
&\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, J_0\left(2\pi\sqrt{m\,n}\,y\right)\, K_0\left(2\pi\sqrt{m\,n}\,x\right) \\
&-\frac{1}{x^2+y^2}\sum_{m,n=1}^{\infty} r_2(m)\, r_2(n)\, J_0\left(\frac{2\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_0\left(\frac{2\pi\sqrt{m\,n}\,x}{x^2+y^2}\right) \\
&=\frac{\gamma-\log\left(\frac{x^2+y^2}{4\pi^2 x}\Gamma^4\left(\frac{1}{4}\right)\right)}{x^2+y^2}-\gamma+\log\left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2 x}\right),
\end{aligned}
$$

an identity that is valid for $x > 0$, $y \geq 0$. Taking $y = 0$ in the previous equality yields (1.2.53).

Remark 4.5.3. It is possible to find certain analogues of the transformation formulas (4.5.2) and (4.5.3) involving more elementary functions. For example, in [183] we found a new and meaningful analogue of Popov's formula (4.1.1) taking the form

$$
\begin{aligned}
&\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)}+\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\pi n x} J_{\frac{k}{4}-\frac{1}{2}}\left(\pi n y\right) \\
&=\frac{(\pi y)^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)\left(x^2+y^2\right)^{\frac{k}{4}}}+\frac{1}{\sqrt{x^2+y^2}}\sum_{n=1}^{\infty}\frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}}\, e^{-\frac{\pi n x}{x^2+y^2}} J_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2+y^2}\right),
\end{aligned}
$$

where it is assumed that $x > y > 0$. Note that the modular structure of this formula is very similar to the one that is present in Theorem 4.5.1.

Remark 4.5.4. The results of this chapter are, of course, valid for the class of Dirichlet series studied in Chapters 1 and 2. Just to give an example, let $f_1(\tau)$ and $f_2(\tau)$ be two holomorphic cusp forms of the same weight, $k \geq 12$, for the full modular group. Also, let $a_{f_1}(n)$ and $a_{f_2}(n)$ denote their respective Fourier coefficients. It is not difficult to generalize the proofs given in this section to get the following analogue of (4.5.2),

$$\sum_{m,n=1}^{\infty} a_{f_1}(m)\, a_{f_2}(n) \left(\frac{m}{n}\right)^{\frac{\nu}{2}} (m\,n)^{\frac{1-k}{2}}\, J_{k-1}\left(4\pi\sqrt{m\,n}\,y\right)\, K_{\nu}\left(4\pi\sqrt{m\,n}\,x\right)$$
$$= \frac{1}{x^2+y^2} \sum_{m,n=1}^{\infty} a_{f_1}(m)\, a_{f_2}(n) \left(\frac{n}{m}\right)^{\frac{\nu}{2}} (m\,n)^{\frac{1-k}{2}}\, J_{k-1}\left(\frac{4\pi\sqrt{m\,n}\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{4\pi\sqrt{m\,n}\,x}{x^2+y^2}\right), \quad (4.5.23)$$

which is valid for any complex ν and $x, y > 0$. Just like we have done in the first chapter, if $f_{\alpha}(\tau)$ and $f_{\beta}(\tau)$ are two holomorphic cusp forms whose Fourier coefficients are, respectively, $a_{f_{\alpha}}(n)$ and $a_{f_{\beta}}(n)$, we may come up with a generalized divisor function of the form

$$\sigma_z(n; f_{\alpha}, f_{\beta}) := \sum_{d|n} a_{f_{\alpha}}(d)\, a_{f_{\beta}}\left(\frac{n}{d}\right) d^z, \quad (4.5.24)$$

which is clearly a particular case of the notion introduced in Remark 1.2.1 of the first chapter. Appealing to the definition (4.5.24), we may now rewrite (4.5.23) in the following compact form

$$\sum_{n=1}^{\infty} \sigma_{-\nu}(n;\, f_2, f_1)\, n^{\frac{\nu+1-k}{2}}\, J_{k-1}\left(4\pi\sqrt{n}\,y\right)\, K_{\nu}\left(4\pi\sqrt{n}\,x\right)$$
$$= \frac{1}{x^2+y^2} \sum_{n=1}^{\infty} \sigma_{-\nu}(n;\, f_1, f_2)\, n^{\frac{\nu+1-k}{2}}\, J_{k-1}\left(\frac{4\pi\sqrt{n}\,y}{x^2+y^2}\right) K_{\nu}\left(\frac{4\pi\sqrt{n}\,x}{x^2+y^2}\right), \quad (4.5.25)$$

which truly resembles (4.5.1)! Moreover, letting $\nu \to 0$ in (4.5.25), one is able to find the following analogue of Koshliakov's formula

$$\sum_{n=1}^{\infty} d(n;\, f_2, f_1)\, n^{\frac{1-k}{2}}\, J_{k-1}\left(4\pi\sqrt{n}\,y\right)\, K_0\left(4\pi\sqrt{n}\,x\right)$$
$$= \frac{1}{x^2+y^2} \sum_{n=1}^{\infty} d(n;\, f_1, f_2)\, n^{\frac{1-k}{2}}\, J_{k-1}\left(\frac{4\pi\sqrt{n}\,y}{x^2+y^2}\right) K_0\left(\frac{4\pi\sqrt{n}\,x}{x^2+y^2}\right), \quad (4.5.26)$$

where

$$d(n;\, f_1, f_2) := \sigma_0(n;\, f_1, f_2) = \sum_{d|n} a_{f_1}(d)\; a_{f_2}\left(\frac{n}{d}\right).$$

Taking $k = 12$ and $a_{f_1}(n) = a_{f_2}(n) = \tau(n)$, where $\tau(n)$ denotes Ramanujan's $\tau-$function, it is simple to check that (4.5.26) implies formula (1.2.70) provided in Example 1.2.4 of the first chapter.

4.6 A new analogue of Ferrar's formula

As stated at the introduction to this chapter, in a paper concerning the study of modular relations of the form $\mathcal{F}(\alpha)=\mathcal{F}(\beta)$, with $\alpha\beta=1$, Ferrar [[89], p. 103] offered the following interesting example.

Theorem 4.6.1. *If $\alpha,\beta>0$ are such that $\alpha\beta=1$, then the following transformation formula takes place*

$$2\sqrt{\alpha}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\alpha^2}{2}}K_0\left(\frac{\pi n^2\alpha^2}{2}\right)-\frac{1}{\alpha n}\right\}+\frac{1}{\sqrt{\alpha}}\left(\gamma-\log(16\pi)-2\log\left(\alpha\right)\right)$$
$$=2\sqrt{\beta}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\beta^2}{2}}K_0\left(\frac{\pi n^2\beta^2}{2}\right)-\frac{1}{\beta n}\right\}+\frac{1}{\sqrt{\beta}}\left(\gamma-\log(16\pi)-2\log\left(\beta\right)\right). \tag{4.6.1}$$

Since the summation index "n^2" is suggestive of the one-dimensional case of $\zeta_1(s)$, one may ask what is the generalization of Ferrar's formula when the summation variable, "n^2", is replaced by a sum of $k-$squares, this is, "$n_1^2+...+n_k^2$".

Before presenting this generalization, we need to introduce a new special function that will help us finding it. The Whittaker function $W_{\mu,\nu}(x)$ is the solution of Whittaker's differential equation [[236], p. 337],

$$\frac{d^2u}{dx^2}+\left(\frac{\mu}{x}+\frac{1}{4x^2}-\frac{\nu^2}{4x^2}-\frac{1}{4}\right)u=0,$$

which is determined uniquely by the property[35]

$$W_{\mu,\nu}(x)\sim x^{\mu}e^{-\frac{x}{2}},\quad x\to\infty.$$

The theory of Whittaker's functions is usually derived from the study of confluent hypergeometric functions (see the next chapter for more details on this). One can see from this theory that Whittaker's function has the asymptotic behavior

$$W_{\mu,\nu}(x)=O\left(x^{\mathrm{Re}(\nu)+\frac{1}{2}}\right)+O\left(x^{\frac{1}{2}-\mathrm{Re}(\nu)}\right),\quad \nu\neq 0\quad x\to 0^{+},$$

$$W_{\mu,0}(x)=O\left(x^{\frac{1}{2}}\,\log(x)\right),\quad x\to 0^{+}$$

and

$$W_{\mu,\nu}(x)=O\left(x^{\mu}\,e^{-x/2}\right),\quad x\to\infty. \tag{4.6.2}$$

When we set its first index, μ, as zero, the function $W_{\mu,\nu}(x)$ reduces to the modified Bessel function of the second kind, $K_{\nu}(x)$, and, in a different arrangement of the indices, it can be reduced to an elementary function [[164], p. 338, 13.18.2, 13.18.9]. This means that we have the reduction formulas

$$W_{0,\nu}\left(2x\right)=\sqrt{\frac{2x}{\pi}}\,K_{\nu}(x), \tag{4.6.3}$$

[35] here, we suppose that $\mu\in\mathbb{R}$.

$$W_{\nu+\frac{1}{2},\nu}(x)=x^{\nu+\frac{1}{2}}e^{-\frac{x}{2}}. \tag{4.6.4}$$

The first of these formulas, (4.6.3), will be of special importance in this section, as we hope to show in Corollary 4.6.1 below. The next result establishes a generalization of Ferrar's formula (4.6.1).

Theorem 4.6.2. *Let $x>0$ and $k\in\mathbb{N}$. Then the following generalization of Ferrar's formula (4.6.1) holds*

$$\begin{aligned}
&x^{\frac{k}{4}-\frac{1}{2}}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi nx}{2}}}{\sqrt{n}}W_{\frac{1-k}{2},0}(\pi nx)-\frac{(\pi x)^{\frac{1-k}{2}}}{n^{\frac{k}{2}}}\right\}+\pi^{\frac{1-k}{2}}x^{-\frac{k}{4}}\zeta_{k-1}\left(\frac{k}{2}\right)\\
&+\frac{\sqrt{\pi}x^{-\frac{k}{4}}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log(4\pi x)\right)\\
&=x^{\frac{1}{2}-\frac{k}{4}}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi n}{2x}}}{\sqrt{n}}W_{\frac{1-k}{2},0}\left(\frac{\pi n}{x}\right)-\left(\frac{\pi}{x}\right)^{\frac{1-k}{2}}\frac{1}{n^{\frac{k}{2}}}\right\}+\pi^{\frac{1-k}{2}}x^{\frac{k}{4}}\zeta_{k-1}\left(\frac{k}{2}\right)\\
&+\frac{\sqrt{\pi}\,x^{\frac{k}{4}}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log\left(\frac{4\pi}{x}\right)\right),
\end{aligned} \tag{4.6.5}$$

where γ is Euler-Mascheroni's constant, $\psi(z)$ is Euler's digamma function and $W_{\mu,\nu}(z)$ denotes Whittaker's function.

Proof. We start the argument by considering the infinite series

$$\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi nx}{2}}}{\sqrt{n}}W_{\frac{1-k}{2},0}(\pi nx)-\frac{(\pi x)^{\frac{1-k}{2}}}{n^{\frac{k}{2}}}\right\}. \tag{4.6.6}$$

Since $r_k(n)=O\left(n^{\frac{k}{2}-1}\right)$ and $W_{-\rho,\sigma}(x)$, $\rho,\sigma\in\mathbb{R}$, has the asymptotic behavior (cf. (4.6.2) above)

$$W_{-\rho,\nu}(x)=x^{-\rho}e^{-\frac{x}{2}}\left\{1+O\left(\frac{1}{x}\right)\right\},\quad x\to\infty,$$

we conclude the absolute convergence of (4.6.6) after a simple application of the mean value theorem. In order to study this first series, we shall consider the useful integral representation [[48], p. 458, relation (3.30.2.5)]

$$e^{x/2}W_{-\rho,\sigma}(x)=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\Gamma\left(s-\sigma+\frac{1}{2}\right)\Gamma\left(s+\sigma+\frac{1}{2}\right)\Gamma\left(-s+\rho\right)}{\Gamma\left(\frac{1}{2}+\rho-\sigma\right)\Gamma\left(\frac{1}{2}+\rho+\sigma\right)}x^{-s}ds, \tag{4.6.7}$$

where $|\mathrm{Re}(\sigma)|-\frac{1}{2}<\mu<\mathrm{Re}(\rho)$. If we shift the line of integration from this range to $\mathrm{Re}(\rho)<\mu'<1+\mathrm{Re}(\rho)$, we find that the integrand in (4.6.7) has a simple pole located at $s=\rho$. Thus, by the residue theorem,[36] we have

$$e^{x/2}W_{-\rho,\sigma}(x)-x^{-\rho}=\frac{1}{2\pi i}\int\limits_{\mu'-i\infty}^{\mu'+i\infty}\frac{\Gamma\left(s-\sigma+\frac{1}{2}\right)\Gamma\left(s+\sigma+\frac{1}{2}\right)\Gamma\left(-s+\rho\right)}{\Gamma\left(\frac{1}{2}+\rho-\sigma\right)\Gamma\left(\frac{1}{2}+\rho+\sigma\right)}\,x^{-s}ds. \tag{4.6.8}$$

Taking $\sigma=0$ and $\rho=\frac{k-1}{2}$, we have that (4.6.8) is valid in the region $\frac{k-1}{2}<\mu'<\frac{k+1}{2}$ and so, by absolute convergence, the following integral representation holds

$$\begin{aligned}&\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi nx}{2}}}{\sqrt{n}}W_{\frac{1-k}{2},0}\left(\pi nx\right)-\frac{(\pi x)^{\frac{1-k}{2}}}{n^{\frac{k}{2}}}\right\}\\&=\frac{1}{2\pi i}\int\limits_{\mu'-i\infty}^{\mu'+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)(\pi x)^{-s}ds.\end{aligned} \tag{4.6.9}$$

We shift the line of integration in (4.6.9) from $\text{Re}(s)=\mu'$ to $\text{Re}(s)=\frac{k}{2}-\mu'$, taking an integration along a positively oriented rectangular contour, $\mathscr{R}_{\mu'}(T)$, with vertices $\mu'\pm iT$ and $\frac{k}{2}-\mu'\pm iT$. Since $\frac{k-1}{2}<\mu'<\frac{k+1}{2}$ by hypothesis, we know that $\Gamma^2\left(s+\frac{1}{2}\right)$ contains no poles inside the rectangular contour $\mathscr{R}_{\mu'}(T)$. Moreover, since $\mu'<\frac{k+1}{2}$, the only pole that $\Gamma\left(\frac{k-1}{2}-s\right)$ contains inside $\mathscr{R}_{\mu'}(T)$ is located at $s=\frac{k-1}{2}$ and this is the exact same pole that $\zeta_k\left(s+\frac{1}{2}\right)$ possesses. The computation of the residue at the point $s=\frac{k-1}{2}$ comes from the Laurent expansion

$$\Gamma\left(\frac{k-1}{2}-s\right)=\frac{2}{k-1-2s}-\gamma+O\left(s-\frac{k-1}{2}\right),$$

together with the generalized Kronecker limit formula obtained in Lemma 4.4.5 above. Straightforward simplifications yield

$$\begin{aligned}&\text{Res}_{s=\frac{k-1}{2}}\left\{\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)(\pi x)^{-s}\right\}\\&=-\frac{\sqrt{\pi}x^{\frac{1-k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log(4\pi x)\right)\\&-(\pi x)^{-\frac{k-1}{2}}\zeta_{k-1}\left(\frac{k}{2}\right).\end{aligned} \tag{4.6.10}$$

Moreover, by Stirling's formula and the Phragmén-Lindelöf principle (4.2.20), we know that the integrals along the horizontal segments of $\mathscr{R}_{\mu'}(T)$ satisfy the bound

$$\int\limits_{\mu'}^{\frac{k}{2}-\mu'}\left|T^{A(\sigma)+\frac{k}{2}-1}e^{-\frac{3\pi}{2}T}(\pi x)^{-\sigma}\right|d\sigma\ll e^{-\frac{3\pi}{2}T}T^{\frac{k}{2}-1},$$

[36] the residue theorem can be trivially applied due to the asymptotic behavior of the integrand with respect to $\text{Im}(s)$, as $|\text{Im}(s)|\to\infty$.

becoming negligible as $T \to \infty$. Hence, by Cauchy's residue theorem and (4.6.10), we have the equality

$$\frac{1}{2\pi i}\int_{\mu'-i\infty}^{\mu'+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)(\pi x)^{-s}ds=-(\pi x)^{-\frac{k-1}{2}}\zeta_{k-1}\left(\frac{k}{2}\right)$$
$$-\frac{\sqrt{\pi}x^{\frac{1-k}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log(4\pi x)\right)$$
$$+\frac{1}{2\pi i}\int_{\frac{k}{2}-\mu'-i\infty}^{\frac{k}{2}-\mu'+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)(\pi x)^{-s}ds. \tag{4.6.11}$$

Our proof will be complete once we evaluate the last integral in (4.6.11). In order to do this, we appeal to the functional equation for $\zeta_k(s)$, (4.1.3), and make the change of variables $s \leftrightarrow \frac{k}{2}-s$, resulting in

$$\frac{1}{2\pi i}\int_{\frac{k}{2}-\mu'-i\infty}^{\frac{k}{2}-\mu'+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)(\pi x)^{-s}ds$$
$$=\frac{\pi}{2\pi i}\int_{\mu'-i\infty}^{\mu'+i\infty}\frac{\Gamma\left(\frac{k+1}{2}-s\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma^2\left(\frac{k}{2}\right)}\pi^{-s}\,\Gamma\left(s-\frac{1}{2}\right)\zeta_k\left(s-\frac{1}{2}\right)x^{s-\frac{k}{2}}\,ds$$
$$=\frac{x^{1-\frac{k}{2}}}{2\pi i}\int_{\mu'-1-i\infty}^{\mu'-1+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)\left(\frac{\pi}{x}\right)^{-s}ds, \tag{4.6.12}$$

where $\frac{k-3}{2}<\mu'-1<\frac{k-1}{2}$. It would be highly convenient to use (4.6.9) in order to evaluate the last integral in (4.6.12). But to be able to do this requires that we shift the line of integration from $\mathrm{Re}(s)=\mu'-1$ back to $\mathrm{Re}(s)=\mu'$. To make this procedure, we need to apply the residue theorem again. Using the calculations made in (4.6.10), we easily find that

$$\mathrm{Res}_{s=\frac{k-1}{2}}\left\{\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)\left(\frac{\pi}{x}\right)^{-s}\right\}$$
$$=-\frac{\sqrt{\pi}x^{\frac{k-1}{2}}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log\left(\frac{4\pi}{x}\right)\right)$$
$$-\left(\frac{\pi}{x}\right)^{-\frac{k-1}{2}}\zeta_{k-1}\left(\frac{k}{2}\right),$$

and so the last member of (4.6.12) can be written as

$$\frac{x^{1-\frac{k}{2}}}{2\pi i}\int_{\mu'-1-i\infty}^{\mu'-1+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)\left(\frac{\pi}{x}\right)^{-s}ds$$

$$=\frac{x^{1-\frac{k}{2}}}{2\pi i}\int_{\mu'-i\infty}^{\mu'+i\infty}\frac{\Gamma^2\left(s+\frac{1}{2}\right)\Gamma\left(\frac{k-1}{2}-s\right)}{\Gamma^2\left(\frac{k}{2}\right)}\zeta_k\left(s+\frac{1}{2}\right)\left(\frac{\pi}{x}\right)^{-s}ds+\sqrt{x}\pi^{\frac{1-k}{2}}\zeta_{k-1}\left(\frac{k}{2}\right)$$

$$+\frac{\sqrt{\pi x}}{\Gamma\left(\frac{k}{2}\right)}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log\left(\frac{4\pi}{x}\right)\right). \tag{4.6.13}$$

The summation formula (4.6.5) now results from a combination of formulas (4.6.9)-(4.6.13). □

Remark 4.6.1. Note that, if we take $x=\alpha^2$ in (4.6.5) and then define $\beta=1/\alpha$, the previous identity may be rewritten in the form

$$\alpha^{\frac{k}{2}-1}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi n\alpha^2}{2}}}{\sqrt{n}}W_{\frac{1-k}{2},0}\left(\pi n\alpha^2\right)-\frac{\pi^{\frac{1-k}{2}}}{n^{\frac{k}{2}}\alpha^{k-1}}\right\}+\pi^{\frac{1-k}{2}}\alpha^{-\frac{k}{2}}\zeta_{k-1}\left(\frac{k}{2}\right)$$

$$+\frac{\sqrt{\pi}}{\Gamma\left(\frac{k}{2}\right)\alpha^{\frac{k}{2}}}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log\left(4\pi\alpha^2\right)\right)$$

$$=\beta^{\frac{k}{2}-1}\sum_{n=1}^{\infty}r_k(n)\left\{\frac{e^{\frac{\pi n\beta^2}{2}}}{\sqrt{n}}W_{\frac{1-k}{2},0}\left(\pi n\beta^2\right)-\frac{\pi^{\frac{1-k}{2}}}{n^{\frac{k}{2}}\beta^{k-1}}\right\}+\pi^{\frac{1-k}{2}}\beta^{-\frac{k}{2}}\zeta_{k-1}\left(\frac{k}{2}\right)$$

$$+\frac{\sqrt{\pi}}{\Gamma\left(\frac{k}{2}\right)\beta^{\frac{k}{2}}}\left(2\gamma+\psi\left(\frac{k}{2}\right)-2\log\left\{\prod_{\mathbf{m}\in\mathbb{Z}^{k-1}\setminus\{\mathbf{0}\}}\left(1-e^{-2\pi|\mathbf{m}|}\right)\right\}-\log\left(4\pi\beta^2\right)\right), \tag{4.6.14}$$

which actually resembles (4.6.1)!

Corollary 4.6.1. *Let $\alpha,\beta>0$ be such that $\alpha\beta=1$. Then Ferrar's formula (4.6.1) holds, this is,*

$$2\sqrt{\alpha}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\alpha^2}{2}}K_0\left(\frac{\pi n^2\alpha^2}{2}\right)-\frac{1}{\alpha n}\right\}+\frac{1}{\sqrt{\alpha}}\left(\gamma-\log(16\pi)-2\log\left(\alpha\right)\right)$$

$$=2\sqrt{\beta}\sum_{n=1}^{\infty}\left\{e^{\frac{\pi n^2\beta^2}{2}}K_0\left(\frac{\pi n^2\beta^2}{2}\right)-\frac{1}{\beta n}\right\}+\frac{1}{\sqrt{\beta}}\left(\gamma-\log(16\pi)-2\log\left(\beta\right)\right). \tag{4.6.15}$$

Proof. We set $k=1$ in (4.6.14). Once again, we recall the conventions that $\Phi_1(y):=1$ and $\zeta_0(s)=0$ (see Remark 4.4.8 above). Remembering also that $r_1(n):=2$ iff n is a perfect square and zero otherwise and, finally, appealing to the special value for Euler's digamma function,

$\psi\left(\frac{1}{2}\right) = -2\log(2) - \gamma$, we can deduce the transformation formula

$$\frac{2}{\sqrt{\alpha}}\sum_{n=1}^{\infty}\left\{\frac{e^{\frac{\pi n^2\alpha^2}{2}}}{n}W_{0,0}\left(\pi n^2\alpha^2\right) - \frac{1}{n}\right\} + \frac{1}{\sqrt{\alpha}}\left(\gamma - \log(16\pi) - 2\log(\alpha)\right)$$
$$= \frac{2}{\sqrt{\beta}}\sum_{n=1}^{\infty}\left\{\frac{e^{\frac{\pi n^2\beta^2}{2}}}{n}W_{0,0}\left(\pi n^2\beta^2\right) - \frac{1}{n}\right\} + \frac{1}{\sqrt{\beta}}\left(\gamma - \log(16\pi) - 2\log(\beta)\right). \quad (4.6.16)$$

Using the reduction formula (4.6.3),

$$W_{0,0}(x) = \sqrt{\frac{x}{\pi}}\,K_0\left(\frac{x}{2}\right),$$

we find that (4.6.16) yields Ferrar's identity (4.6.15). □

Corollary 4.6.2. *Let $\alpha, \beta > 0$ be such that $\alpha\beta = 1$. Then the following summation formula is valid*

$$\alpha\sum_{n=1}^{\infty} r_2(n)\left\{e^{\pi n\alpha^2}\,Ei\left(-\pi n\alpha^2\right) + \frac{1}{\pi\alpha^2 n}\right\} - \frac{1}{\alpha}\left(\gamma - 4\log\left(\eta(i)\right) - \log\left(4\pi\alpha^2\right)\right)$$
$$= \beta\sum_{n=1}^{\infty} r_2(n)\left\{e^{\pi n\beta^2}\,Ei\left(-\pi n\beta^2\right) + \frac{1}{\pi\beta^2 n}\right\} - \frac{1}{\beta}\left(\gamma - 4\log\left(\eta(i)\right) - \log\left(4\pi\beta^2\right)\right), \quad (4.6.17)$$

where, for $Im(\tau) > 0$, $\eta(\tau)$ denotes Dedekind's $\eta-$function (4.4.71) and $Ei(-x)$ represents the exponential integral function

$$Ei(-x) := -E_1(x) = -\int_x^{\infty}\frac{e^{-u}}{u}\,du. \quad (4.6.18)$$

Proof. We set $k = 2$ in (4.6.14) and use the fact that $\Phi_2(y)$ is connected to $\eta(iy)$ by (4.4.72). This procedure yields the formula

$$\sum_{n=1}^{\infty} r_2(n)\left\{\frac{e^{\frac{\pi n\alpha^2}{2}}}{\sqrt{n}}W_{-\frac{1}{2},0}\left(\pi n\alpha^2\right) - \frac{1}{\sqrt{\pi}\,\alpha n}\right\} + \frac{\sqrt{\pi}}{\alpha}\left(\gamma - 4\log\left(\eta(i)\right) - \log\left(4\pi\alpha^2\right)\right)$$
$$= \sum_{n=1}^{\infty} r_2(n)\left\{\frac{e^{\frac{\pi n\beta^2}{2}}}{\sqrt{n}}W_{-\frac{1}{2},0}\left(\pi n\beta^2\right) - \frac{1}{\sqrt{\pi}\,\beta n}\right\} + \frac{\sqrt{\pi}}{\beta}\left(\gamma - 4\log\left(\eta(i)\right) - \log\left(4\pi\beta^2\right)\right), \quad (4.6.19)$$

Next, comparing the Mellin-Barnes integral (4.6.7) with [[48], p. 102, relation 3.3.2.1],

$$e^{x}\mathrm{Ei}(-x) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma^2(s)\,\Gamma(1-s)\,x^{-s}ds, \quad 0 < \sigma < 1,$$

we find out that, for any $x > 0$,

$$W_{-\frac{1}{2},0}(x) = -\sqrt{x}e^{\frac{x}{2}}\,\mathrm{Ei}(-x), \quad (4.6.20)$$

which completes the proof of (4.6.17). □

Remark 4.6.2. Using the evaluation (1.2.52) for $\eta(i)$, we can rewrite the two-dimensional analogue of Ferrar's formula (4.6.17) in the following elegant manner

$$\alpha\sum_{n=1}^{\infty} r_2(n)\left\{e^{\pi n\alpha^2}\,\mathrm{Ei}\left(-\pi n\alpha^2\right)+\frac{1}{\pi\alpha^2 n}\right\}-\frac{1}{\alpha}\left(\gamma-2\log\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi}\alpha\right)\right)$$
$$=\beta\sum_{n=1}^{\infty} r_2(n)\left\{e^{\pi n\beta^2}\mathrm{Ei}\left(-\pi n\beta^2\right)+\frac{1}{\pi\beta^2 n}\right\}-\frac{1}{\beta}\left(\gamma-2\log\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi}\beta\right)\right), \quad (4.6.21)$$

where $\alpha,\beta>0$ are such that $\alpha\beta=1$. Using the definition of the arithmetical function $r_2(n)$, we can rewrite both infinite series of (4.6.17) as double series, resulting in a transformation formula of the form

$$\alpha\sum_{m,n\neq 0}\left\{e^{\pi\alpha^2\left(m^2+n^2\right)}\,\mathrm{Ei}\left(-\pi\alpha^2\left(m^2+n^2\right)\right)+\frac{1}{\pi\alpha^2\left(m^2+n^2\right)}\right\}-\frac{1}{\alpha}\left(\gamma-2\log\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi}\alpha\right)\right)$$
$$=\beta\sum_{m,n\neq 0}\left\{e^{\pi\beta^2\left(m^2+n^2\right)}\mathrm{Ei}\left(-\pi\beta^2\left(m^2+n^2\right)\right)+\frac{1}{\pi\beta^2\left(m^2+n^2\right)}\right\}-\frac{1}{\beta}\left(\gamma-2\log\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi}\beta\right)\right).$$

Other interesting examples can be deduced. For example, using the reduction formula for the Whittaker function

$$W_{-1,0}(x)=\frac{2\sqrt{x}(x+1)}{\sqrt{\pi}}K_0\left(\frac{x}{2}\right)-\frac{2x^{\frac{3}{2}}}{\sqrt{\pi}}K_1\left(\frac{x}{2}\right),$$

one can write down yet another particular case of Ferrar's formula (4.6.5) for the arithmetical function $r_3(n)$.

Chapter 5

A class of summation formulas attached to the indices of special functions

5.1 Introduction

In the previous chapters, we examined various methods for generalizing the well-known Bessel expansion of the Epstein zeta function. As mentioned in several sources (cf. [37, 39]), the Bessel expansion commonly referred to as the "Selberg-Chowla formula" is entirely equivalent to the Fourier expansion of the non-holomorphic Eisenstein series. Let us briefly explain what this means: take $\tau \in \mathbb{H} := \{z \in \mathbb{C} \, : \, \mathrm{Im}(z) > 0\}$ and consider the double infinite series

$$E(s,\tau) := \sum_{m,n\neq 0} \frac{y^{s}}{|m+n\tau|^{2s}}, \quad \mathrm{Re}(s) > 1, \quad \tau := x+iy, \tag{5.1.1}$$

commonly presented as the non-holomorphic Eisenstein series. Since each positive definite and real quadratic form $Q(m,n) := am^{2}+bmn+cn^{2}$ can be attached to an element of $\mathbb{H}$, say τ_{Q}, by writing this element as $\tau_{Q} := \frac{b+i\sqrt{\Delta}}{2a}$, we may translate the Selberg-Chowla formula (1.1.2) into

$$\begin{aligned} E(s,\,\tau) &= 2y^{s}\,\zeta(2s) + 2\sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}\,\zeta\left(2s-1\right) \\ &\quad + \frac{8\pi^{s}\sqrt{y}}{\Gamma(s)}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\,\cos\left(2\pi nx\right)\,K_{s-\frac{1}{2}}(2\pi ny), \end{aligned} \tag{5.1.2}$$

where $x=\mathrm{Re}(\tau)$ and $y=\mathrm{Im}(\tau)$. Just like in the classical case of the Epstein zeta function, the Fourier expansion (5.1.2) establishes the analytic continuation of the infinite series (5.1.1), and in particular says that $E(s,\,\tau)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\,E(s,\tau) = \pi^{s-1}\Gamma(1-s)\,E(1-s,\tau). \tag{5.1.3}$$

Formula (5.1.2) also serves as one of the first steps into the theory of Maass forms [148, 149]. We refrain ourselves to give the details about this rich topic within analytic number theory and refer the reader to the recent book [159] about these objects. One crucial aspect highlighting the importance of the expansion (5.1.2) is that it allows to show[1]

$$\Delta E(s,\tau) := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)E(s,\tau) = s\,(s-1)\,E\,(s,\tau)\,, \tag{5.1.4}$$

for all $s\in\mathbb{C}$. There exist several applications of the differential property (5.1.4). One of these is the derivation of the Maass-Selberg relation (cf. [[228], pp. 39-41] for details). Using this relation for the nonholomorphic Eisenstein series, $E(s,\tau)$, Selberg found an alternative proof that $\zeta(s)$ does not vanish on the line $\text{Re}(s)=1$.[2] Another very interesting application, due to Hejhal [114], is that when we fix $y\geq 1$ and consider the function[3]

$$F(s,y) := \pi^{-s}\Gamma(s)\,\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\,\zeta(2-2s)\,y^{1-s},$$

then all of its zeros are located on the critical line $\text{Re}(s)=\frac{1}{2}$ with exception of only two real zeros.[4] For more details about results of this type, we refer the reader to the paper by Lagarias and Suzuki [138] and to the very interesting approaches due to Haseo Ki [126, 127].[5]

The nonholomorphic Eisenstein series and its Fourier expansion (5.1.2) admit generalizations in different settings. We have already seen two of them in this thesis. In the first place, we extended the usual divisor function $\sigma_\nu(n)=\sum_{d|n}d^\nu$ to a more general arithmetical function, whose introduction was given in Remark 1.2.1 of the first chapter. Secondly, an analogue of (5.1.2) was introduced in the previous chapter, where the cosine function on the right-hand side of (5.1.2) was replaced by a Bessel or a modified Bessel function of the first kind (see (4.1.40) above).

Another natural approach to generalize (5.1.2), concerned with a reformulation of Hecke's correspondence theorem in terms of automorphic forms, is due to Maass [148, 149]. Maass [[149], Chapter IV] considered the following extension of (5.1.1),[6]

$$G(a,b;\,\tau,\,\overline{\tau}) = \sum_{m,n\neq 0}\frac{1}{(m+n\tau)^a(m+n\overline{\tau})^b} = \sum_{m,n\neq 0}\frac{(m+n\tau)^{b-a}}{|m+n\tau|^{2b}},\quad \text{Re}(a+b)>2, \tag{5.1.5}$$

[1] Of course, (5.1.4) could be established in a simple way by only looking at the action of the hyperbolic Laplacian, $\Delta F(x,y) := -y^2\,\nabla^2 F(x,y)$, on the elements of the infinite series (5.1.1).

[2] At least according to the notes written by Heath-Brown in Titchmarsh's text. We have not been able to track a paper written by Selberg containing the proof described in [[228], p. 69].

[3] This function is usually known as the "constant term" of the Selberg-Chowla formula. The study of the zeros of symmetric versions of the completed Riemann $\zeta-$function was started by Paul Rowland Taylor [220], in a paper published after his death during active service as a Flight-Lieutenant in Egypt, during the second world war.

[4] In fact, Lagarias and Suzuki strengthened Hejhal's result by showing that, when $1\leq y\leq 4\pi e^{-\gamma}$, all the zeros of $F(s,y)$ are on the critical line. The two real zeros that show up when $y>4\pi e^{-\gamma}$ behave very much like Landau-Siegel zeros. See [[138], p. 103] and the first chapter of this thesis for further interesting remarks.

[5] Ki [127] presented a completely novel approach to the study of the zeros of the function $F(s,y)$, based on the Hermite-Biehler theorem. Besides, Ki's method allows to show that all the nonreal zeros of $F(s,y)$ are simple when $y\geq 1$.

[6] To be concise with the presentation in this thesis, we took the liberty of slightly changing Maass' notation.

in which it is assumed that $\mathrm{Im}(\tau) > 0$, $a - b \in 2\mathbb{Z}$ and that the sum runs over all pairs of integers that are not simultaneously zero. It is natural to define the function $G(a, b;\, \tau,\, \overline{\tau})$ by the second infinite series in (5.1.5), in which one takes the principal branch of the logarithm. When $a = b = s$, we see that, for $\mathrm{Re}(s) > 1$, the Maass Eisenstein series $G(a, b;\, \tau, \overline{\tau})$ reduces to

$$G(s, s;\, \tau,\, \overline{\tau}) = \sum_{m,n\neq 0} \frac{1}{|m + n\tau|^{2s}} := y^{-s}\, E\left(s, \tau\right), \tag{5.1.6}$$

so $G(a, b;\, \tau,\, \overline{\tau})$ is a generalization of $E(s, \tau)$.

Like $E(s, \tau)$, the "weighted" Eisenstein series (5.1.5) admits a Fourier expansion. Maass [[149], p. 210] proved that this expansion takes the form

$$\begin{aligned} G(a,\, b;\, \tau,\, \overline{\tau}) &= 2\zeta(a+b) + 4\pi\,(-1)^{\frac{a-b}{2}}\, 2^{1-a-b}\, \frac{\Gamma\left(a+b-1\right)}{\Gamma\left(a\right)\Gamma(b)}\, \zeta(a+b-1)\, y^{1-a-b} \\ &\quad + 2\,(-1)^{\frac{a-b}{2}} \sum_{n\neq 0} \frac{\sigma_{a+b-1}(|n|)\, e^{2\pi i n x}}{\Gamma\left(\frac{a+b}{2} + \mathrm{sgn}(n)\, \frac{a-b}{2}\right)}\, (2\pi|n|\, y)^{-\frac{a+b}{2}}\, W_{\mathrm{sgn}(n)\,\frac{a-b}{2},\, \frac{a+b-1}{2}}\left(4\pi|n|\, y\right), \end{aligned} \tag{5.1.7}$$

where $W_{\mu,\nu}(z)$ denotes the Whittaker function, which we have already encountered in the previous chapter. Combining (5.1.6) with the reduction formula for Whittaker's function,

$$W_{0,\nu}(x) = \sqrt{\frac{x}{\pi}}\, K_{\nu}\left(\frac{x}{2}\right), \tag{5.1.8}$$

one sees that (5.1.7) is, in fact, a generalization of (5.1.2). Just like $E(s, \tau)$, $G(a, b;\, \tau, \overline{\tau})$ is an automorphic form, satisfies a functional equation with respect to the variables a and b (thus generalizing (5.1.3)) and it is an eigenfunction for a generalized Laplacian operator, such as (5.1.4). These properties are essentially listed in [[149], pp. 213-214].

The presence of Whittaker functions in the expansion (5.1.7) offers a compelling motivation to investigate analogues of Guinand's and Koshliakov's formulas, (4.5.1) and (4.5.17), which may result from substituting the Modified Bessel functions, ubiquitous in our previous chapters, by Whittaker functions. While such a study is certainly feasible and worth pursuing, the Fourier-Whittaker series (5.1.7) also suggests an opportunity to explore a fundamentally different class of summation formulas.

A purely informal description of some of our previous chapters is that we have primarily focused on summation formulas of the form

$$\sum_{n=0}^{\infty} a(n)\, g(ny)\, K_{\nu}(nx), \tag{5.1.9}$$

where g denotes some function (often a "special function") and $a(n)$ is some arithmetical function that we deemed relevant in some way or another. In the previous chapter, we have also seen how integrals with respect to the index of the Macdonald function, $K_{\nu}(x)$, were useful to study summation formulas of the form (5.1.9). Let us recall that, if we start with a real-valued function, say $f(t)$, belonging to a "suitable" class, then the Kontorovich-Lebedev transform is explicitly defined by the integral (cf. (4.2.1))

$$(\mathrm{KL}(f))(x) := \int_{-\infty}^{\infty} K_{it}(x)\, f(t)\, dt, \qquad x > 0. \tag{5.1.10}$$

Examples of this transform were computed before in the case where $f_{\xi}(t) = \eta_k\left(\frac{k}{4} + it\right)\xi^{-it}$. For instance, referring back to Corollary 4.2.6, the evaluation of the index transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \eta_2\left(\frac{1}{2} + it\right) K_{it}(x)\, \xi^{-it} dt, \qquad x,\ \xi > 0, \tag{5.1.11}$$

served as motivation for presenting a new proof of Ramanujan's beautiful formula (cf. (4.2.24))

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{x + 2\pi\xi n}}\, e^{-\sqrt{x^2+2\pi\xi x n}} = \frac{1}{\xi}\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{x + \frac{2\pi n}{\xi}}}\, e^{-\sqrt{x^2+\frac{2\pi x}{\xi}n}}, \qquad x,\ \xi > 0.$$

One may wonder now which kind of summation formulas can result from considering discrete analogues of the transform (5.1.10), i.e., what can be said about formulas that may complement (5.1.9) if we move the variable of summation, say n, from the argument inside the Bessel function to the index, i.e., taking the conversion

$$\sum_{n=0}^{\infty} a(n)\, g(ny)\, K_{\nu}(nx) \Longrightarrow \sum_{n=0}^{\infty} a(n)\, g(ny)\, K_{in\alpha}(x). \tag{5.1.12}$$

As far as we know, Yakubovich [241] took the first step in the study of summation formulas of the form (5.1.12), proving that the identity

$$K_0(x) + 2\sum_{n=1}^{\infty} K_{in\alpha}(x) = \frac{\pi}{\alpha}e^{-x} + \frac{2\pi}{\alpha}\sum_{n=1}^{\infty} e^{-x\cosh\left(\frac{2\pi n}{\alpha}\right)} \tag{5.1.13}$$

holds for any positive real numbers x and α.

We can take the class of summation formulas of the form (5.1.12) and (5.1.13) even further, following the analogy between the Fourier expansions (5.1.2) and (5.1.7) or, by other words, by trying to convert the Macdonald function in (5.1.12) into a Whittaker function.

It is no surprise that this study has already been considered in the setting of integral tranforms. In 1964, Wimp [240] discovered an inversion formula for an integral transformation involving the Whittaker function $W_{\mu,i\tau}(x)$, producing the pair of integral transforms

$$F(\tau) = \int_{0}^{\infty} W_{\mu,i\tau}(x)\, f(x)\, \frac{dx}{x^2}, \quad \mu \in \mathbb{R},\ \tau > 0, \tag{5.1.14}$$

$$f(x) = \frac{1}{\pi^2} \int_0^\infty \tau \sinh(2\pi\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x)\, F(\tau)\, d\tau, \quad \mu \in \mathbb{R},\ x > 0. \tag{5.1.15}$$

Due to the reduction formula (5.1.8), when $\mu = 0$, the integral defining $F(\tau)$ reduces to the direct Kontorovich-Lebedev transform [243], this is,

$$\frac{1}{\sqrt{\pi}} \int_0^\infty K_{i\tau}\left(\frac{x}{2}\right) f(x)\, \frac{dx}{x^{3/2}}. \tag{5.1.16}$$

There are several variants of Wimp's result. Srivastava, Vasil'ev and Yakubovich [209], for example, considered an interesting class of index transforms with respect to Whittaker's function. Later, Yakubovich studied inversion formulas for integral transforms associated with products of Whittaker functions, as well as discrete transforms [242, 245].

In the same way that (5.1.14) generalizes (5.1.16) and that (5.1.7) has the potential to furnish new summation formulas of the form

$$\sum_{n=1}^{\infty} a(n)\, g(ny)\, W_{\mu,\nu}(nx),$$

what can we say about summation formulas involving the index of Whittaker's function? In other words, in what conditions can we establish new results for the infinite series

$$\sum_{n=1}^{\infty} a(n)\, W_{\mu,i\alpha n}(x), \quad \alpha, x > 0,\ \ \mu \in \mathbb{R}\ ?$$

Our main goal in this chapter is, then, to analyze and prove a new transformation formula for an infinite series of the form

$$\sum_{n=1}^{\infty} W_{\mu,i\alpha n}(x), \quad \alpha, x > 0,\ \ \mu \in \mathbb{R}. \tag{5.1.17}$$

Despite the formal similarity with the Kapteyn and Neumann series (cf. [13, 14, 152–154, 232]), the nature and structure of the summation formulas attached to the imaginary indices of the functions $W_{\mu,i\tau}(x)$ and $K_{i\tau}(x)$ drastically differ from these classical settings.

We shall evaluate the series (5.1.17) by using a weak variant of the Poisson summation formula. This approach has the advantage to give new integral representations involving the index of a class of special functions. In fact, the kind of integrals derived in this chapter cannot be found in any of the standard references containing explicit index transforms, such as [95] and the third volume of [172]. This class is also not examined in papers focused on the evaluation of such integrals, such as [16, 17].

This chapter is organized in the following simple manner: in the next section, we briefly present the basic special functions that make their first serious appearance in this part of the thesis, and in the third (i.e., final) section, we compute the Fourier transform of $f(\tau) := W_{\mu,i\tau}(x)$, for fixed parameters $\mu \in \mathbb{R}$ and $x > 0$. Using this transform and appealing to a very simple version of Poisson's summation formula using residue theory [213, 214], we find an interesting transformation formula for the series (5.1.17), connecting it with the parabolic cylinder function. Several new results are deduced from our main transformation formula. For example, the identity (valid for $x, \alpha > 0$ and $\mu \in \mathbb{R}$),

$$W_{-\frac{1}{4},0}(x)\, W_{\frac{1}{4},0}(x) + 2\sum_{n=1}^{\infty} W_{-\frac{1}{4},i\alpha n}(x)\, W_{\frac{1}{4},i\alpha n}(x)$$
$$= \frac{x}{\sqrt{2\alpha}}\left\{e^{-\frac{x}{2}} K_0\left(\frac{x}{2}\right) + 2\sum_{n=1}^{\infty} e^{-\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)} K_0\left(\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right\}$$

will be obtained as Corollary 5.3.3 below. Additionally, in Corollary 5.3.4 we will be able to compute the Fourier transform

$$\int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1}{2} - \mu + i\tau\right)\right|^2 W_{\mu,i\tau}(x)\, e^{-i\tau\xi}\, d\tau$$
$$= \pi\, 2^{\mu+\frac{1}{2}} \sqrt{x}\, \Gamma(1-2\mu)\, e^{\frac{x \sinh^2(\xi/2)}{2}}\, D_{2\mu-1}\left(\sqrt{2x}\cosh\left(\frac{\xi}{2}\right)\right),$$

which we could not find elsewhere in the literature.

5.2 Preliminaries

As we had the opportunity to mention before, the notion of Whittaker's function arises in the study of the Kummer confluent hypergeometric function, ${}_1F_1(a;c;z)$, which is usually defined by the power series [[164], p. 322, 13.2.2],

$$ {}_1F_1\left(a;c;z\right) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}, \tag{5.2.1}$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$ denotes the Pochhammer symbol. A special combination of Kummer's functions leads to one of the most important confluent hypergeometric functions, known as Tricomi's function $\Psi(a,c;z)$. This special function is typically defined by

$$\Psi(a,c;z) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1(a;c;z) + z^{1-c}\, \frac{\Gamma(c-1)}{\Gamma(a)}\, {}_1F_1(a-c+1;2-c;z). \tag{5.2.2}$$

As we had the opportunity to mention in Section 4.6, the Whittaker function $W_{\mu,\nu}(x)$ is the solution of Whittaker's differential equation,

$$\frac{d^2u}{dx^2} + \left(\frac{\mu}{x} + \frac{1}{4x^2} - \frac{\nu^2}{4x^2} - \frac{1}{4}\right)u = 0, \tag{5.2.3}$$

which is determined uniquely by the property (here, we suppose that $\mu \in \mathbb{R}$)

$$W_{\mu,\nu}(x) \sim x^{\mu} e^{-\frac{x}{2}}, \quad x \to \infty. \tag{5.2.4}$$

The Whittaker function can be also defined in terms of the Tricomi function, $\Psi(a,c;z)$, in the following manner

$$W_{\mu,\nu}(z)=e^{-z/2}z^{c/2}\Psi\left(a,c;z\right), \tag{5.2.5}$$

where $a=\frac{1}{2}+\nu-\mu$ and $c=2\nu+1$. Invoking the asymptotic expansions for the Kummer and Tricomi functions, we can check that both definitions of the function $W_{\mu,\nu}(x)$ are equivalent, because the right-hand side of (5.2.5) satisfies the differential equation (5.2.3), but also

$$W_{\mu,\nu}(x)=O\left(x^{\mathrm{Re}(\nu)+\frac{1}{2}}\right)+O\left(x^{\frac{1}{2}-\mathrm{Re}(\nu)}\right),\quad \nu\neq 0\quad x\to 0^{+},$$

$$W_{\mu,0}(x)=O\left(x^{\frac{1}{2}}\,\log(x)\right),\quad x\to 0^{+}$$

and (supposing $\mu\in\mathbb{R}$)

$$W_{\mu,\nu}(x)=O\left(x^{\mu}\,e^{-x/2}\right),\quad x\to\infty.$$

There are several integral representations of the Whittaker function that will be very useful in the sequel. For example, recall that in the previous chapter we employed an interesting integral representation, namely (4.6.7), to deduce a new analogue of Ferrar's formula (4.6.5). There are other very important representations of the function $W_{\mu,\nu}(z)$ that can be regarded as classical. For instance [[164], p. 337, 13.16.5],

$$W_{\mu,\nu}(z)=\frac{z^{\nu+\frac{1}{2}}2^{-2\nu}}{\Gamma\left(\frac{1}{2}+\nu-\mu\right)}\int_{1}^{\infty}e^{-\frac{tz}{2}}(t-1)^{\nu-\mu-\frac{1}{2}}(t+1)^{\nu+\mu-\frac{1}{2}}\,dt,\ \mathrm{Re}(\nu-\mu)+\frac{1}{2}>0,\ |\arg(z)|<\frac{\pi}{2}. \tag{5.2.6}$$

As we have seen in Section 4.6, the Whittaker function possesses two important reduction formulas, namely

$$W_{0,\nu}\left(2x\right)=\sqrt{\frac{2x}{\pi}}\,K_{\nu}(x), \tag{5.2.7}$$

$$W_{\nu+\frac{1}{2},\nu}(x)=x^{\nu+\frac{1}{2}}e^{-\frac{x}{2}}. \tag{5.2.8}$$

The first reduction formula (5.2.7) was crucial to obtain Ferrar's formula (4.6.1) as a particular case of our Theorem 4.6.2. Let us now briefly recall some properties of $K_{\nu}(x)$. Although we have employed several of these properties in the previous chapters, it is quite helpful to remind the reader of some of them, because they will play an even more important role here. Generally, it can be seen from (5.2.3) that the modified Bessel function of the second kind (or Macdonald function), $K_{\nu}(x)$, satisfies the differential equation

$$x^{2}\frac{d^{2}u}{dx^{2}}+x\,\frac{du}{dx}-\left(x^{2}+\nu^{2}\right)u=0,$$

and has the asymptotic behavior [[164], relations 10.25.3, 10.30.2, 10.30.3]

$$K_{\nu}(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x}\left\{1+O\left(\frac{1}{x}\right)\right\},\quad x\to\infty, \tag{5.2.9}$$

$$K_{\nu}(x) = 2^{\nu-1}\Gamma(\nu)x^{-|\mathrm{Re}(\nu)|} + \mathrm{o}\left(x^{-|\mathrm{Re}(\nu)|}\right), \quad x \to 0^{+},\ \nu \neq 0, \tag{5.2.10}$$

$$K_0(x) = -\log(x) + O(1), \quad x \to 0^{+}. \tag{5.2.11}$$

The study of summation formulas attached to the function $K_{\nu}(x)$ depends on many of its integral representations. For example, the only feature distinguishing different proofs of Watson's formula (4.1.19) [[233], p. 299, eq. (4)] is the integral representation used for the Macdonald function $K_{\nu}(x)$.[7] One consequence of (5.2.6) and the reduction formula (5.2.7) is the integral representation [[164], p. 252, 10.32.8],

$$K_{\nu}(z) = \frac{\sqrt{\pi}z^{\nu}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{1}^{\infty} e^{-zu}\left(u^2-1\right)^{\nu-\frac{1}{2}}du, \quad \mathrm{Re}(\nu) > -\frac{1}{2},\ |\arg(z)| < \frac{\pi}{2},$$

which, incidentally, will be very useful in the next chapter on Koshliakov's theory. In 1871, Schläfli proved that this representation for the Modified Bessel function is equivalent to the integral [[164], p. 252, 10.32.9]

$$K_{\nu}(z) = \int_{0}^{\infty} e^{-z\cosh(u)}\cosh(\nu u)\,du, \quad \nu \in \mathbb{C}, \quad |\arg(z)| < \frac{\pi}{2}. \tag{5.2.12}$$

In particular, if we take $\nu = i\tau$, $\tau \in \mathbb{R}$,(5.2.12) gives

$$K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh(u)}e^{i\tau u}du, \quad x > 0, \tag{5.2.13}$$

which will be very important in the sequel. In fact, seeking the right generalization of (5.2.13) in the Whittaker setting is fundamental to obtain a transformation formula for the infinite series (5.1.17).

From the representation (5.2.13), Yakubovich obtained a uniform estimate of the kernel $K_{i\tau}(x)$ with respect to its index $\tau \in \mathbb{R}$ and argument $x > 0$, namely [[243], p. 15, eq. (1.100)]

$$|K_{i\tau}(x)| \leq e^{-\delta|\tau|}\,K_0\left(x\cos(\delta)\right), \qquad 0 \leq \delta < \frac{\pi}{2}. \tag{5.2.14}$$

This estimate admits a generalization, which was also found by Yakubovich [248],

$$|K_{\sigma+i\tau}(x)| \leq e^{-\delta|\tau|}\,K_{\sigma}\left(x\cos(\delta)\right), \quad \sigma \in \mathbb{R}, \quad 0 \leq \delta < \frac{\pi}{2}, \tag{5.2.15}$$

and whose proof follows exactly the same ideas as (5.2.14). Formula (5.2.15) proved to be very useful in the previous chapter, as it was more than enough to justify some technical steps in the proof of Theorem 4.2.1 above. The estimate (5.2.14) will be invoked several times throughout this chapter and we shall always make explicit reference to it in the main text.

[7] Watson confesses in his paper (see pages 299 and 300) that the referee suggested a much easier proof of his formula based on the Poisson summation formula and the Basset representation for $K_{\nu}(x)$.

The Whittaker function has several other reductions that, for historical reasons (and just like the Macdonald function $K_\nu(z)$), possess independent interest. One of such reductions is the well-known parabolic cylinder function, $D_\mu(z)$. In the classical approach of [236], it can be also defined as the solution of the differential equation

$$\frac{d^2u}{dx^2}+\left(\mu+\frac{1}{2}-\frac{x^2}{4}\right)u=0,$$

which can be then rewritten in an equation of the form (5.2.3), describing $D_\mu(z)$ in terms of the Whittaker function,

$$D_\mu(z)=2^{\frac{\mu}{2}+\frac{1}{4}}z^{-\frac{1}{2}}W_{\frac{\mu}{2}+\frac{1}{4},\frac{1}{4}}\left(\frac{z^2}{2}\right). \tag{5.2.16}$$

It is known that $D_\mu(z)$ is an entire function of its parameter μ. This function makes its appearance in our treatment of the infinite series (5.1.17). Due to its connection with the function $W_{\mu,\nu}(z)$, there are several useful representations of the parabolic cylinder function that can be derived from integral representations available for the Whittaker function. One such representation, which will be very important in this chapter, is [[48], p. 20, 2.2.1.6] (see also relation 2.3.15.3 on page 343 of vol. I of [172]),

$$\int_0^\infty u^{\mu-1}e^{-xu^2-yu}du=\frac{\Gamma(\mu)}{(2x)^{\mu/2}}\,e^{\frac{y^2}{8x}}\,D_{-\mu}\left(\frac{y}{\sqrt{2x}}\right),\quad \mathrm{Re}(\mu),\ \mathrm{Re}(x)>0,\ y\in\mathbb{C}. \tag{5.2.17}$$

From the reduction formulas for the Whittaker function, one can easily check [[164], p. 308, 12.7.1]

$$D_0(z)=e^{-\frac{z^2}{4}} \tag{5.2.18}$$

and that [[164], p. 308, 12.7.5] (cf. [[236], p. 341] and [[164], p. 338, 13.18.7])

$$D_{-1}(z)=\sqrt{\frac{\pi}{2}}e^{\frac{z^2}{4}}\,\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right), \tag{5.2.19}$$

where erfc$(\cdot)$ denotes the complementary error function, usually defined by [[164], p. 160, 7.2(i)]

$$\mathrm{erfc}\,(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt. \tag{5.2.20}$$

It can be inferred from (5.2.4) that, when x is taken large, $D_\mu(x)$ will satisfy the asymptotic formula (cf. [[164], p. 309, 12.9 (i)], [[84], Vol. II, p. 122, eq. 8.4(1)]),

$$D_\mu(x)\sim x^\mu e^{-\frac{x^2}{4}},\quad x\to\infty. \tag{5.2.21}$$

In particular, from the reduction formula (5.2.19), we can also write the following asymptotic formula for the complementary error function [[164], p. 164, 7.12.1]

$$\operatorname{erfc}(x) \sim \frac{1}{\sqrt{\pi}} \frac{e^{-x^2}}{x}, \quad x \to \infty. \tag{5.2.22}$$

For a comprehensive approach to the theory of the confluent hypergeometric function, which also includes a detailed study of the particular examples $W_{\mu,\nu}(z)$ and $D_{\mu}(z)$, the reader is encouraged to check Buchholz's book [49].

In order to study the class of summation formulas that we aim to consider in this chapter, we will need to employ the classical Poisson summation formula (cf. [96]). Although this particular result from Fourier analysis could be presented and motivated to the reader in a self-contained (but rather technical) manner,[8] we shall present it in a more pragmatic way, without even having the need to state it.[9] To apply Poisson's summation formula we shall need to introduce the classical Fourier transform,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(\tau)\, e^{-i\tau\xi} d\tau, \tag{5.2.23}$$

which makes sense when $f(\tau)$ satisfies some proper conditions of regularity and decaying.

There are several analogues of the Fourier inversion theorem, and each one is labeled with the regularity and decay properties that one assumes for f in the first place. For the details given in this chapter, we will only require a version of Fourier's inversion formula that works for a subclass of complex functions having moderate decrease. This class is introduced in the reference [[213], pp. 112-114] and it is a natural class to consider if one wants to approach Fourier analysis via the tools of complex function theory.

Definition 5.2.1. For each $a > 0$, let $\mathfrak{F}_a$ denote the class of all functions f that satisfy the following two conditions:

1. The function f is analytic in the horizontal strip

$$S_a := \{z \in \mathbb{C} \, : \, |\mathrm{Im}(z)| < a\}. \tag{5.2.24}$$

2. There exists a positive constant A such that

$$|f(\eta + i\zeta)| \leq \frac{A}{1+\eta^2} \quad \text{for all } \eta \in \mathbb{R} \text{ and } |\zeta| < a. \tag{5.2.25}$$

Under the assumptions of this class, the next lemma establishes the Fourier inversion formula, which will be fundamental in our proof of Theorem 5.3.1.

[8] i.e., by being a particular case of the several summation formulas (formally known as Voronoï's summation formulas (4.1.36)) that arise in the Hecke class.

[9] The primary goal of this chapter is to study new summation formulas involving new aspects of special functions rather than developing a more general theory of Dirichlet series associated to this kind of formulas.

Lemma 5.2.1 ([[213], p. 115, Theorem 2.2])**.** *If there exists some $a > 0$ such that $f \in \mathfrak{F}_a$, then the Fourier inversion holds, this is, for all $\tau \in \mathbb{R}$, we have the representation of $f(\tau)$ by the integral*

$$f(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\tau\xi}\, d\xi. \tag{5.2.26}$$

Remark 5.2.1. If f belongs to the class $\mathfrak{F}_a$, for some $a > 0$, then one can guarantee that $\hat{f}(\xi)$ will have exponential decay. In fact, for any $\xi \in \mathbb{R}$, $|\hat{f}(\xi)| \leq B\, e^{-b|\xi|}$ for any $0 < b < a$ and some $B > 0$ [[213], p. 114, Theorem 2.1].

Having presented all the main special functions and tools of classical analysis comprising our summation formulas, we are now ready to move to the next section and prove the main results of this chapter.

5.3 A class of Summation formulas attached to the index of Whittaker's function

As remarked at the introduction, the primary goal in this chapter is to find a transformation formula for an infinite series of the form

$$\sum_{n=1}^{\infty} W_{\mu, i\alpha n}(x), \quad \alpha, x > 0, \;\; \mu \in \mathbb{R}. \tag{5.3.1}$$

Such transformation will necessarily possess at its core the classical Poisson summation formula [96, 213, 214, 226]. In order to apply this classical result from Analysis and Number Theory, we need to compute the Fourier transform

$$F_{\mu,\xi}(x) := \int_{-\infty}^{\infty} W_{\mu,i\tau}(x)\, e^{-i\tau\xi}\, d\tau. \tag{5.3.2}$$

First, let us note that the infinite series (5.3.1) converges for any $x \in (0, X]$, $X > 0$, by virtue of the asymptotic expansion (cf. [[243], p. 25, eq. (1.172)] and [254])

$$W_{\mu,i\tau}(x) = \sqrt{2x}\, e^{-\frac{\pi\tau}{2}} \tau^{\mu-\frac{1}{2}} \cos\left(\tau \log\left(\frac{x}{4\tau}\right) - \frac{\pi}{2}\left(\mu - \frac{1}{2}\right) + \tau\right) \left\{1 + O\left(\frac{1}{\tau}\right)\right\}, \quad \tau \to \infty. \tag{5.3.3}$$

In an analogous way, if x is any fixed positive real number, then the integral (5.3.2) exists as an absolutely convergent integral.

In order to compute the transform (5.3.2), we appeal to the following representation for $W_{\mu,i\tau}(x)$ [[48], p. 458, eq. (3.30.2.4)],

$$e^{-x/2} W_{\mu,i\tau}(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\Gamma\left(\frac{2s-2i\tau+1}{2}\right) \Gamma\left(\frac{2s+2i\tau+1}{2}\right)}{\Gamma\left(s - \mu + 1\right)}\, x^{-s} ds, \tag{5.3.4}$$

valid for $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$.[10] Now, our Fourier transform reduces to the evaluation of the integral

$$F_{\mu,\xi}(x) = e^{x/2} \int_{-\infty}^{\infty} \frac{e^{-i\tau\xi}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma\left(\frac{2s-2i\tau+1}{2}\right)\Gamma\left(\frac{2s+2i\tau+1}{2}\right)}{\Gamma\left(s-\mu+1\right)} x^{-s} ds\, d\tau, \tag{5.3.5}$$

and, in order to compute it, we will interchange the orders of integration by appealing to Fubini's theorem. In order to check that this procedure is valid, it suffices to show that

$$\int_0^\infty \int_0^\infty \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt\, d\tau < \infty, \tag{5.3.6}$$

since the resulting iterated absolute integral is even with respect to the variables t and τ. In the next argument, we shall invoke Stirling's formula,

$$|\Gamma(\sigma+it)| \asymp_\sigma \begin{cases} |t|^{\sigma-\frac{1}{2}} e^{-\frac{\pi|t|}{2}} & |t| \geq 1 \\ 1 & |t| \leq 1 \end{cases}. \tag{5.3.7}$$

Assume first $0 \leq \tau \leq 1$: then we split the inner integral (with respect to t) in the following form

$$\begin{aligned} &\int_0^\infty \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt \\ &= \left\{ \int_0^{\tau+1} + \int_{\tau+1}^\infty \right\} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt. \end{aligned}$$

Since $0 \leq \tau \leq 1$ by hypothesis, the first integral will be obviously bounded by continuity of the integrand in the range $0 \leq t \leq \tau+1$. Next, in the second range we have that $t-\tau \geq 1$ and $t \geq 1$, and so we may appeal to Stirling's formula (5.3.7) to obtain

$$\int_{\tau+1}^\infty \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt \ll_\sigma \int_{\tau+1}^\infty t^{\sigma+\mu-\frac{1}{2}} e^{-\frac{\pi t}{2}}\, dt,$$

which is obviously a finite quantity. Therefore,

$$\begin{aligned} &\int_0^1 \int_0^\infty \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt d\tau \\ &= \int_0^1 \int_0^{\tau+1} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt d\tau \\ &+ \int_0^1 \int_{\tau+1}^\infty t^{\sigma+\mu-\frac{1}{2}} e^{-\frac{\pi t}{2}}\, dt\, d\tau < \infty. \end{aligned}$$

[10] In fact, this representation is even valid for $\sigma > -\frac{1}{2}$. For simplicity, however, we work under the assumption $\sigma > 0$.

Next, assume that $\tau \geq 1$: under this assumption, we can split the inner integral in (5.3.6) in the form

$$\left\{ \int_0^{\tau-1} + \int_{\tau-1}^{\tau+1} + \int_{\tau+1}^{\infty} \right\} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt.$$

For $0 \leq t \leq \tau-1$ or $t \geq \tau+1$, we know that $|t-\tau| \geq 1$, so that

$$\int_0^{\tau-1} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt \leq \int_0^{\tau-1} \frac{\left(\tau^2-t^2\right)^{\sigma} e^{-\pi\tau}}{\left|\Gamma\left(\sigma-\mu+1+it\right)\right|} dt.$$

If $1 \leq \tau \leq 2$, then the integral is trivially estimated as $\ll \tau^{2\sigma+1}e^{-\pi\tau}$. On the other hand, if $\tau > 2$, we have that

$$\begin{aligned} \int_0^{\tau-1} \frac{\left(\tau^2-t^2\right)^{\sigma} e^{-\pi\tau}}{\left|\Gamma\left(\sigma-\mu+1+it\right)\right|} dt &\leq \tau^{2\sigma+1}e^{-\pi\tau} + \int_1^{\tau-1} \frac{\left(\tau^2-t^2\right)^{\sigma} e^{-\pi\tau}}{t^{\sigma-\mu+\frac{1}{2}}} e^{\frac{\pi}{2}t}\, dt \\ &\leq \tau^{2\sigma+1}e^{-\pi\tau} + \tau^{\eta}e^{-\frac{\pi}{2}\tau}, \end{aligned} \tag{5.3.8}$$

where $\eta := \max\left\{2\sigma+1, \sigma+\mu+\frac{1}{2}\right\}$. This proves the following bound

$$\begin{aligned} &\int_1^{\infty}\int_0^{\tau-1} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt\, d\tau \\ &\leq \int_1^{2} \tau^{2\sigma+1}e^{-\pi\tau}\, d\tau + \int_2^{\infty} \left(\tau^{2\sigma+1}e^{-\pi\tau} + \tau^{\eta}e^{-\frac{\pi}{2}\tau}\right) d\tau < \infty. \end{aligned}$$

Next, in the region $\tau-1 \leq t \leq \tau+1$ we have that $|t-\tau| \leq 1$, so that Stirling's formula (5.3.7) gives

$$\int_{\tau-1}^{\tau+1} \left| \frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)} \right| dt \leq \int_{\tau-1}^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}(t+\tau)}}{|\Gamma(\sigma-\mu+1+it)|}\, dt.$$

Once more, if $1 \leq \tau \leq 2$, then the integral can be estimated as

$$\begin{aligned} \int_{\tau-1}^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}(t+\tau)}}{|\Gamma(\sigma-\mu+1+it)|}\, dt &\leq \int_{\tau-1}^{1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}(t+\tau)}}{|\Gamma(\sigma-\mu+1+it)|}\, dt + \int_{1}^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}(t+\tau)}}{|\Gamma(\sigma-\mu+1+it)|}\, dt \\ &\ll \tau^{\sigma}e^{-\pi\tau} + \int_1^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}\tau}}{t^{\sigma-\mu+\frac{1}{2}}}\, dt \leq \tau^{\sigma}e^{-\pi\tau} + \tau^{\eta-\sigma}e^{-\frac{\pi}{2}\tau}, \end{aligned}$$

where, as before, $\eta := \max\left\{2\sigma+1, \sigma+\mu+\frac{1}{2}\right\}$. On the other hand, if $\tau \geq 2$, we obtain

$$\int_{\tau-1}^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}(t+\tau)}}{|\Gamma(\sigma-\mu+1+it)|}\, dt \leq \int_{\tau-1}^{\tau+1} \frac{(t+\tau)^{\sigma}e^{-\frac{\pi}{2}\tau}}{t^{\sigma-\mu+\frac{1}{2}}}\, dt \ll \tau^{\mu+\frac{1}{2}}e^{-\frac{\pi}{2}\tau},$$

which results in

$$\int_1^{\infty}\int_{\tau-1}^{\tau+1}\left|\frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)}\right|dt\,d\tau$$
$$\leq\int_1^2\left(\tau^{\sigma}e^{-\pi\tau}+\tau^{\eta-\sigma}e^{-\frac{\pi}{2}\tau}\right)d\tau+\int_2^{\infty}\tau^{\mu+\frac{1}{2}}e^{-\frac{\pi}{2}\tau}d\tau<\infty.$$

Finally, for $t\geq\tau+1$, we can apply the first estimate in (5.3.7) in all of the Gamma factors and get the inequality

$$\int_{\tau+1}^{\infty}\left|\frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)}\right|dt\leq\int_{\tau+1}^{\infty}\frac{\left(t^2-\tau^2\right)^{\sigma}e^{-\frac{\pi}{2}t}}{t^{\sigma-\mu+\frac{1}{2}}}\,dt\leq\int_{\tau+1}^{\infty}t^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}t}dt$$
$$=e^{-\frac{\pi}{2}\tau}\int_1^{\infty}(u+\tau)^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}u}du\ll\tau^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}\tau}+e^{-\frac{\pi}{2}\tau}\int_{\tau}^{\infty}u^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}u}du$$
$$\ll\tau^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}\tau}+e^{-\frac{\pi}{2}\tau},$$

which proves that

$$\int_1^{\infty}\int_{\tau+1}^{\infty}\left|\frac{\Gamma\left(\frac{1}{2}+\sigma+i(t-\tau)\right)\Gamma\left(\frac{1}{2}+\sigma+i(t+\tau)\right)}{\Gamma\left(\sigma-\mu+1+it\right)}\right|dt\,d\tau\ll\int_1^{\infty}\left\{\tau^{\sigma+\mu-\frac{1}{2}}e^{-\frac{\pi}{2}\tau}+e^{-\frac{\pi}{2}\tau}\right\}d\tau<\infty.$$

Thus, by Fubini's theorem, it is possible to interchange the orders of integration and obtain the integral

$$F_{\mu,\xi}(x):=\int_{-\infty}^{\infty}W_{\mu,i\tau}(x)\,e^{-i\tau\xi}\,d\tau=e^{x/2}\int_{-\infty}^{\infty}\frac{e^{-i\tau\xi}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma\left(\frac{2s-2i\tau+1}{2}\right)\Gamma\left(\frac{2s+2i\tau+1}{2}\right)}{\Gamma\left(s-\mu+1\right)}\,x^{-s}ds\,d\tau$$
$$=\frac{e^{x/2}}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^{-s}}{\Gamma(s+1-\mu)}\int_{-\infty}^{\infty}\Gamma\left(\frac{1}{2}+s-i\tau\right)\Gamma\left(\frac{1}{2}+s+i\tau\right)\,e^{-i\tau\xi}\,d\tau\,ds. \tag{5.3.9}$$

The integal with respect to the product of $\Gamma-$functions is a well-known Fourier transform, and it is usually presented as an application of Parseval's formula [[226], p. 105]. Indeed, we know from [[243], p. 1.104] (cf. [[172], Vol. II, 2.2.4.5]) that the Fourier transform holds

$$\int_{-\infty}^{\infty}\Gamma(z+it)\,\Gamma(z-it)\,e^{-i\xi t}\,dt=\frac{\sqrt{\pi}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)}{\cosh^{2z}\left(\frac{\xi}{2}\right)},\qquad \mathrm{Re}(z)>0, \tag{5.3.10}$$

and so, making elementary substitutions in (5.3.9), we now obtain the expression for $F_{\mu,\xi}(x)$ in the form

$$F_{\mu,\xi}(x)=\frac{\sqrt{\pi}e^{\frac{x}{2}}}{2\pi i\,\cosh\left(\frac{\xi}{2}\right)}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(s+1\right)}{\Gamma(s+1-\mu)}\left(x\cosh^2\left(\frac{\xi}{2}\right)\right)^{-s}ds. \tag{5.3.11}$$

In the meantime, recalling entry 8.4.44.1 in [[172], Vol. III] or [[48], p. 458, eq. (3.30.2.4)], we know that the representation (5.3.4) can be generalized via the Mellin-Barnes integral

$$e^{-x/2}W_{\rho,\nu}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(s-\nu+\frac{1}{2}\right)\Gamma\left(s+\nu+\frac{1}{2}\right)}{\Gamma\left(s-\rho+1\right)}\,x^{-s}ds,\quad x>0,\ c>|\mathrm{Re}(\nu)|-\frac{1}{2}. \tag{5.3.12}$$

Thus, replacing ρ by $\frac{1}{4}+\mu$ and ν by $-\frac{1}{4}$ in (5.3.12), we see that, for any $c>-\frac{1}{4}$ and $x>0$,

$$\begin{aligned} e^{-x/2}W_{\frac{1}{4}+\mu,-\frac{1}{4}}(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(s+\frac{3}{4}\right)\Gamma\left(s+\frac{1}{4}\right)}{\Gamma\left(s-\mu+\frac{3}{4}\right)}\,x^{-s}ds \\ &= \frac{x^{-\frac{1}{4}}}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{\Gamma\left(s+1\right)\Gamma\left(s+\frac{1}{2}\right)}{\Gamma\left(s+1-\mu\right)}\,x^{-s}\,ds, \end{aligned} \tag{5.3.13}$$

where $d:=c-\frac{1}{4}>-\frac{1}{2}$. Since (5.3.13) holds for any $d>-\frac{1}{2}$, we can replace x by $x\cosh^2(\xi/2)$ there and use this modification of (5.3.13) in (5.3.11) to produce the formula

$$\begin{aligned} F_{\mu,\xi}(x) &= \frac{\sqrt{\pi}x^{\frac{1}{4}}e^{\frac{x}{4}}}{\cosh^{\frac{1}{2}}\left(\frac{\xi}{2}\right)}\,e^{-\frac{x}{4}\cosh(\xi)}\,W_{\frac{1}{4}+\mu,-\frac{1}{4}}\left(x\cosh^2\left(\frac{\xi}{2}\right)\right) \\ &= 2^{-\mu}\sqrt{\pi x}\,e^{\frac{x}{4}(1-\cosh(\xi))}\,D_{2\mu}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right), \end{aligned}$$

where we have invoked the definition of the parabolic cylinder function, (5.2.16). Since, according to (5.2.21),

$$D_{2\mu}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right)\sim(2x)^{\mu}\cosh^{2\mu}(\xi/2)\,e^{-\frac{x}{2}\cosh^2(\xi/2)},\quad \xi\to\infty,$$

we find that[11]

$$|F_{\mu,\xi}(x)|\sim\sqrt{\pi}\,x^{\mu+\frac{1}{2}}\cosh^{2\mu}\left(\frac{\xi}{2}\right)\,e^{-\frac{x}{2}\cosh(\xi)},\quad \xi\to\infty.$$

This asymptotic behavior of the Fourier transform $F_{\mu,\xi}(x)$ now suggests that we are in a position to find an inversion formula. This formula shall be stated in our next lemma.

Lemma 5.3.1. *For any $\mu\in\mathbb{R}$, $x>0$ and $\tau,\xi\in\mathbb{R}$, we have the pair of index Fourier transforms*

$$\int_{-\infty}^{\infty}W_{\mu,i\tau}(x)\,e^{-i\tau\xi}\,d\tau=2^{-\mu}\sqrt{\pi x}\,e^{\frac{x}{4}(1-\cosh(\xi))}\,D_{2\mu}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right), \tag{5.3.14}$$

$$W_{\mu,i\tau}(x)=\sqrt{\frac{x}{\pi}}\,\frac{e^{x/4}}{2^{\mu+1}}\int_{-\infty}^{\infty}e^{-\frac{x}{4}\cosh(\xi)}\,D_{2\mu}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right)\,e^{i\tau\xi}\,d\xi. \tag{5.3.15}$$

Proof. In accordance with Lemma 5.2.1, we just need to show that, for $z\in\mathbb{C}$ and fixed $\mu\in\mathbb{R}$, $x>0$, the function $f(z):=W_{\mu,iz}(x)$ belongs to the class $\mathfrak{F}_a$, for some $a>0$. Since, for a fixed x, $W_{\mu,\nu}(x)$ is an entire function of the first and second parameters [[164], p. 335, 13.14 (ii)], we see

[11] A weaker asymptotic estimate for $F_{\mu,\xi}(x)$ can be immediately inferred by Remark 5.2.1.

that condition 1 in Definition 5.2.1 is automatically satisfied, because, for any $a>0$, $W_{\mu,iz}(x)$ will always be analytic in S_a.

In order to establish (5.2.25) we need to find, for a fixed $x \in (0,X]$, a suitable estimate for $W_{\mu,\sigma+i\tau}(x)$, when τ is large and σ is a fixed real number. Taking the definition of the Whittaker function as the combination of confluent hypergeometric functions (cf. (5.2.2) and (5.2.4) above)

$$W_{\mu,\sigma+i\tau}(x) = e^{-x/2}x^{\frac{1}{2}+\sigma+i\tau}\left\{\frac{\Gamma(-2\sigma-2i\tau)}{\Gamma\left(\frac{1}{2}-\sigma-i\tau-\mu\right)}\,{}_1F_1\left(\frac{1}{2}+\sigma+i\tau-\mu;\,2\sigma+2i\tau+1;x\right)\right.$$
$$\left.+x^{-2\sigma-2i\tau}\,\frac{\Gamma(2\sigma+2i\tau)}{\Gamma\left(\frac{1}{2}+\sigma+i\tau-\mu\right)}\,{}_1F_1\left(\frac{1}{2}-\sigma-i\tau-\mu;\,1-2\sigma-2i\tau;\,x\right)\right\}, \quad (5.3.16)$$

we see that, if we write the power series for ${}_1F_1(a;c;z)$, (5.2.1), we are then able to obtain the asymptotic expansion

$${}_1F_1\left(\frac{1}{2}+\sigma+i\tau-\mu;\,2\sigma+2i\tau+1;x\right) = \lim_{M\to\infty}\sum_{k=0}^{M}\frac{\left(\frac{1}{2}+\sigma+i\tau-\mu\right)_k}{(1+2\sigma+2i\tau)_k}\,\frac{x^k}{k!}$$
$$= \lim_{M\to\infty}\sum_{k=0}^{M}\frac{(i\tau)^k}{(2i\tau)^k}\,\frac{x^k}{k!}\left[1+O\left(\frac{1}{\tau}\right)\right] = e^{\frac{x}{2}}\left[1+O\left(\frac{1}{\tau}\right)\right], \quad \tau\to\infty,$$

with an analogous result result holding for the second confluent hypergeometric function in (5.3.16).[12] Using Stirling's formula for the ratios containing the $\Gamma-$functions in (5.3.16), we are able to deduce the bound

$$|W_{\mu,\sigma+i\tau}(x)| \leq \frac{\tau^{\mu-\frac{1}{2}}}{\sqrt{2}}x^{\frac{1}{2}+\sigma}e^{-\frac{\pi\tau}{2}}\left\{2^{-2\sigma}\tau^{-\sigma}+2^{2\sigma}\,x^{-2\sigma}\tau^{\sigma}\right\}\left[1+O\left(\frac{1}{\tau}\right)\right], \quad (5.3.17)$$

which holds for any $\tau\geq T\gg 1$, $\sigma\in\mathbb{R}$ and $x\in(0,X]$. Note that, if $\tau\to-\infty$, the modification of (5.3.17) reads

$$|W_{\mu,\sigma+i\tau}(x)| \leq \frac{|\tau|^{\mu-\frac{1}{2}}}{\sqrt{2}}x^{\frac{1}{2}+\sigma}e^{-\frac{\pi|\tau|}{2}}\left\{2^{-2\sigma}|\tau|^{-\sigma}+2^{2\sigma}\,x^{-2\sigma}|\tau|^{\sigma}\right\}\left[1+O\left(\frac{1}{|\tau|}\right)\right]. \quad (5.3.18)$$

If we write $z=\eta+i\zeta$, say, then, with the proper substitutions, (5.3.17) and (5.3.18) imply the inequality

$$|f\,(\eta+i\zeta)\,| := |W_{\mu,i\eta-\zeta}(x)| \leq \frac{|\eta|^{\mu-\frac{1}{2}}}{\sqrt{2}}x^{\frac{1}{2}-\zeta}e^{-\frac{\pi|\eta|}{2}}\left\{2^{2\zeta}|\eta|^{\zeta}+2^{-2\zeta}\,x^{2\zeta}|\,\eta|^{-\zeta}\right\}\left[1+O\left(\frac{1}{|\eta|}\right)\right], \quad (5.3.19)$$

which establishes an even stronger decay than (5.2.25). $\square$

[12]The $O-$term in the sum $\sum_{k=0}^{M}$ is uniform with respect to $0\leq k\leq M$ [[243], p. 20, eq. (1.145)].

Remark 5.3.1. By the reduction formulas (5.2.7) and (5.2.18), the previous result offers the following pair of Fourier transforms

$$\int_{-\infty}^{\infty} K_{i\tau}\left(x\right)\, e^{-i\tau\xi}\, d\tau = \pi e^{-x\cosh(\xi)}, \quad K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh(\xi)}\, e^{i\tau\xi}\, d\xi, \tag{5.3.20}$$

which we have already presented at the introduction to this chapter (see (5.2.13) above).

After having the Fourier transform of $W_{\mu,i\tau}(x)$, it is now a matter of using a classical version of the Poisson summation formula [96] to get an explicit transformation of the infinite series (5.3.1). However, in the next theorem we provide a purely complex analytic proof of the desired transformation.

Theorem 5.3.1. *Let $x, \alpha > 0$ and $\mu \in \mathbb{R}$. Then the following summation formula holds*

$$W_{\mu,0}(x)+2\sum_{n=1}^{\infty} W_{\mu,i\alpha n}(x) = \frac{2^{-\mu}\sqrt{\pi x}}{\alpha}\left(D_{2\mu}\left(\sqrt{2x}\right)+2e^{\frac{x}{4}}\sum_{n=1}^{\infty} e^{-\frac{x}{4}\cosh\left(\frac{2\pi n}{\alpha}\right)} D_{2\mu}\left(\sqrt{2x}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right). \tag{5.3.21}$$

Proof. For a fixed $x > 0$ and $\mu \in \mathbb{R}$, let $g(z) := W_{\mu,i\alpha z}(x)/\left(e^{2\pi i z}-1\right)$. Since $W_{\mu,\nu}(x)$ is an entire function of the first and second parameters [[164], p. 335, 13.14 (ii)], we know that $g(z)$ is analytic everywhere except at the integers, where it has simple poles with residues $W_{\mu,i\alpha n}(x)/2\pi i$, $n \in \mathbb{Z}$. Thus, if $b > 0$ and $\gamma_{N,b}$ denotes the rectangular contour with vertices $N+\frac{1}{2}\pm ib$ and $-N-\frac{1}{2}\pm ib$, we have by Cauchy's residue theorem

$$\sum_{|n|\leq N} W_{\mu,i\alpha n}(x) = \int_{\gamma_{N,b}} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi i z}-1} dz. \tag{5.3.22}$$

Note that the contribution of the vertical segments of $\gamma_{N,b}$ tends to zero as $N \to \infty$. Indeed, by (5.3.17) and (5.3.18),

$$\left|\int_{N+\frac{1}{2}-ib}^{N+\frac{1}{2}+ib} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi i z}-1} dz\right| \leq \int_{-b}^{b} \frac{\left|W_{\mu,-\alpha u+i\alpha\left(N+\frac{1}{2}\right)}(x)\right|}{1+e^{-2\pi u}}\, du$$
$$\ll b\, N^{\mu-\frac{1}{2}} e^{-\frac{\pi N}{2}} x^{\frac{1}{2}+\alpha b}\left\{2^{2\alpha b}N^{\alpha b}+2^{2\alpha b}\, x^{2\alpha b}N^{\alpha b}\right\}\left[1+O\left(\frac{1}{N}\right)\right],$$

which becomes negligible as $N \to \infty$. Recalling (5.3.3), we see that the sum on the left-hand side of (5.3.22) converges to $\sum_{n\in\mathbb{Z}} W_{\mu,i\alpha n}(x)$. Therefore, we have the identity

$$\sum_{n\in\mathbb{Z}} W_{\mu,i\alpha n}(x) = \int_{\Gamma_1^{-}(b)} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi i z}-1} dz - \int_{\Gamma_1^{+}(b)} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi i z}-1} dz, \tag{5.3.23}$$

where $\Gamma_1^-(b)$ and $\Gamma_1^+(b)$ are the horizontal lines described by the parametric equation $\gamma_1^{\pm}(t) := t \pm ib$, $t \in \mathbb{R}$. Note that, for $z \in \Gamma_1^-(b)$, $|e^{2\pi iz}| > 1$, and so we can express the first integral in (5.3.23) as

$$\int_{\Gamma_1^-(b)} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi iz}-1} dz = \sum_{n=1}^{\infty} \int_{-ib-\infty}^{-ib+\infty} W_{\mu,i\alpha z}(x)\, e^{-2\pi inz} dz = \frac{1}{\alpha} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} W_{\mu,i\tau}(x)\, e^{-2\pi in\frac{\tau}{\alpha}}\, d\tau$$
$$= \frac{2^{-\mu}}{\alpha} \sqrt{\pi x}\, e^{\frac{x}{4}} \sum_{n=1}^{\infty} e^{-\frac{x}{4}\cosh\left(\frac{2\pi n}{\alpha}\right)} D_{2\mu}\left(\sqrt{2x}\, \cosh\left(\frac{\pi n}{\alpha}\right)\right), \tag{5.3.24}$$

where, in the last step, we have used the evaluation of the Fourier transform (5.3.14). The first equality is justified by absolute convergence, because $\sum_{n=1}^{\infty} e^{-2\pi nb} < \infty$ and $W_{\mu,b\alpha+i\alpha t}(x)$ has the asymptotic behaviors (5.3.17) and (5.3.18), as t tends to ∞ or $-\infty$ respectively. Finally, the second equality in (5.3.24) is justified by Cauchy's residue theorem, which can be invoked because $h(z) := W_{\mu,i\alpha z}(x)$ is an analytic function of z. Analogously, one can prove that

$$\int_{\Gamma_1^+(b)} \frac{W_{\mu,i\alpha z}(x)}{e^{2\pi iz}-1} dz = -\sum_{n=0}^{\infty} \int_{ib-\infty}^{ib+\infty} W_{\mu,i\alpha z}(x)\, e^{2\pi inz} dz$$
$$= -\frac{2^{-\mu}}{\alpha} \sqrt{\pi x}\, e^{\frac{x}{4}} \sum_{n=0}^{\infty} e^{-\frac{x}{4}\cosh\left(\frac{2\pi n}{\alpha}\right)} D_{2\mu}\left(\sqrt{2x}\, \cosh\left(\frac{\pi n}{\alpha}\right)\right). \tag{5.3.25}$$

Combining (5.3.24) and (5.3.25) gives immediately our formula (5.3.21). □

Remark 5.3.2. Since our proof of (5.3.21) "only" used the classical Poisson summation formula, it is natural to consider character analogues of (5.3.21) (cf. [20, 34, 190]). In fact, it is not hard to show that, if χ is a nonprincipal, primitive and even Dirichlet character modulo q, then the following summation formula holds

$$\sum_{n=1}^{\infty} \chi(n)\, W_{\mu,i\alpha n}(x) = \frac{2^{-\mu}\sqrt{\pi x}\, G(\chi) e^{\frac{x}{4}}}{q\alpha} \sum_{n=1}^{\infty} \overline{\chi}(n)\, e^{-\frac{x}{4}\cosh\left(\frac{2\pi n}{q\alpha}\right)} D_{2\mu}\left(\sqrt{2x}\, \cosh\left(\frac{\pi n}{q\alpha}\right)\right),$$

where $G(\chi)$ denotes the Gauss sum,

$$G(\chi) := \sum_{r=1}^{q-1} \chi(r)\, e^{2\pi ir/q}.$$

Analogously, if χ is a nonprincipal, primitive and odd Dirichlet character modulo q, then the analogous summation formula takes place

$$\sum_{n=1}^{\infty} \chi(n)\, n\, W_{\mu,i\alpha n}(x)$$
$$= -\frac{i2^{-\mu-\frac{1}{2}}\sqrt{\pi}\, x\, G(\chi)\, e^{\frac{x}{4}}}{q\alpha} \sum_{n=1}^{\infty} \overline{\chi}(n)\, e^{-\frac{x}{4}\cosh\left(\frac{2\pi n}{q\alpha}\right)} \sinh\left(\frac{\pi n}{q\alpha}\right) D_{2\mu+1}\left(\sqrt{2x}\, \cosh\left(\frac{\pi n}{q\alpha}\right)\right).$$

Although these formulas involving Dirichlet characters are valid and well motivated, we do not know how to extend the scope of Theorem 5.3.1 to the class of Dirichlet series studied in Chapters

1 and 2. For example, we do not know which kind of transformation formulas one can get while studying the infinite series

$$\sum_{n=1}^{\infty} r_k(n)\, W_{\mu,i\sqrt{n}}(x), \quad \sum_{n=1}^{\infty} a_f(n)\, W_{\mu,i\sqrt{n}}(x), \quad \mu \in \mathbb{R}, \quad x > 0,$$

where $r_k(n)$ denotes the number of ways in which we can write n as a sum of k squares and $a_f(n)$ denotes the n^{th} Fourier coefficient of a holomorphic cusp form, $f(\tau)$, for the full modular group. It is definitely worthwhile to explore this, and we encourage the reader to engage in studying these formulas.

Remark 5.3.3. We have studied in the previous chapter how the Kontorovich-Lebedev transform (5.1.10) acts on the Riemann's $\eta-$function, $\eta(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s)$. Recalling Corollary 4.2.5, we proved that, for any $x, \xi > 0$, the following integral transform holds

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \eta\left(\frac{1}{2}+it\right) K_{\frac{it}{2}}(x)\,\xi^{-it}\,dt = 2x^{\frac{1}{8}}\sqrt{\xi}\sum_{n=1}^{\infty}\frac{K_{\frac{1}{4}}\left(\sqrt{x^2+2\pi x\,\xi^2 n^2}\right)}{(x+2\pi\xi^2 n^2)^{\frac{1}{8}}} - \frac{K_{\frac{1}{4}}(x)}{\sqrt{\xi}}. \tag{5.3.26}$$

In principle, the same ideas that led to (5.3.26) could be used to compute the integral

$$\mathcal{I}(x,\mu;\,\xi) := \frac{1}{2\sqrt{2\pi x}}\int_{-\infty}^{\infty} \eta\left(\frac{1}{2}+it\right) W_{\mu,\frac{it}{2}}(2x)\;\xi^{-it}\,dt, \quad x,\xi > 0,\ \mu \in \mathbb{R}, \tag{5.3.27}$$

which reduces to the left-hand side (5.3.26) when $\mu = 0$. Although we have the Fourier transform of the function $g(t) := W_{\mu,\frac{it}{2}}(x)\,\xi^{-it}$ at our disposal (resulting from (5.3.14)), we have not been able to find a generalization of (5.3.26), i.e., to write the computation of the integral (5.3.27) in terms of an interesting infinite series of special functions. We also leave this problem to the interested reader.

Recalling the particular cases (5.2.7) and (5.2.18), our formula (5.3.21) can be seen as a generalization of an analogous identity involving the Macdonald function. The following corollary was given for the first time in [241], and it is due to Yakubovich.

Corollary 5.3.1. *For any $x, \alpha > 0$, the following identity holds*

$$K_0(x) + 2\sum_{n=1}^{\infty} K_{in\alpha}(x) = \frac{\pi}{\alpha}e^{-x} + \frac{2\pi}{\alpha}\sum_{n=1}^{\infty} e^{-x\cosh\left(\frac{2\pi n}{\alpha}\right)}. \tag{5.3.28}$$

Some immediate consequences of (5.3.28) are already provided in [241]. The most important of these is the transformation formula [[241], p. 299, formula (24)]

$$\sum_{n\in\mathbb{Z}}\Gamma\left(\frac{s}{2}+i\alpha n\right)\Gamma\left(\frac{s}{2}-i\alpha n\right) = \frac{\sqrt{\pi}}{\alpha}\,\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\sum_{n\in\mathbb{Z}}\frac{1}{\cosh^s\left(\frac{\pi n}{\alpha}\right)}, \quad \text{Re}(s) > 0,\ \alpha > 0. \tag{5.3.29}$$

In the remainder of this chapter we derive several new corollaries of the formulas (5.3.21) and (5.3.28). We start with a summation formula over the imaginary indices of two distinct products of Macdonald functions.

Corollary 5.3.2. *Let $x, y, \alpha > 0$. Then the following formula holds*

$$K_0(x)\,K_0(y)+2\sum_{n=1}^{\infty} K_{i\alpha n}\,(x)\,K_{i\alpha n}(y) = \frac{\pi}{\alpha}K_0\,(x+y)+\frac{2\pi}{\alpha}\sum_{n=1}^{\infty} K_0\left(\sqrt{x^2+y^2+2xy\cosh\left(\frac{2\pi n}{\alpha}\right)}\right). \tag{5.3.30}$$

Moreover, if $\sigma \in \mathbb{R}$ and $x, \alpha > 0$, the analogous summation formula takes place

$$K_\sigma^2(x) + 2\sum_{n=1}^{\infty} K_{\sigma+i\alpha n}(x)\,K_{\sigma-i\alpha n}(x) = \frac{\pi}{\alpha}K_{2\sigma}(2x) + \frac{2\pi}{\alpha}\sum_{n=1}^{\infty} K_{2\sigma}\left(2x\cosh\left(\frac{\pi n}{\alpha}\right)\right). \tag{5.3.31}$$

Proof. The first formula (5.3.30) can be proved by appealing to relation 2.16.9.1 on page 354 of [[172], Vol. II], which states that

$$K_{i\tau}(x)\,K_{i\tau}(y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u + \frac{xy}{u}\right)\right) K_{i\tau}(u)\,\frac{du}{u},$$

for $x, y > 0$. Using this product formula and Corollary 5.3.1, we find the equalities

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} K_{i\alpha n}(x)\,K_{i\alpha n}(y) &= \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u + \frac{xy}{u}\right)\right) \sum_{n\in\mathbb{Z}} K_{i\alpha n}(u)\,\frac{du}{u} \\
&= \frac{\pi}{2\alpha}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u + \frac{xy}{u}\right)\right) \sum_{n\in\mathbb{Z}} e^{-u\cosh\left(\frac{2\pi n}{\alpha}\right)}\,\frac{du}{u} \\
&= \frac{\pi}{2\alpha}\sum_{n\in\mathbb{Z}}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2+2xy\cosh\left(\frac{2\pi n}{\alpha}\right)}{xy}u + \frac{xy}{u}\right)\right)\frac{du}{u} \\
&= \frac{\pi}{\alpha}\sum_{n\in\mathbb{Z}} K_0\left(\sqrt{x^2+y^2+2xy\cosh\left(\frac{2\pi n}{\alpha}\right)}\right),
\end{aligned} \tag{5.3.32}$$

where in the last step we employed the integral representation [[164], p. 253, 10.32.10],

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty \exp\left(-u - \frac{x^2}{4u}\right)\frac{du}{u^{\nu+1}}, \quad x > 0, \;\; \nu \in \mathbb{C}. \tag{5.3.33}$$

Note that (5.3.32) is equivalent to (5.3.30). The interchange of the orders of integration and summation in the steps leading to (5.3.32) can be justified by absolute convergence and Fubini's theorem. Indeed, since $|K_{i\alpha n}(u)| \leq K_0\left(\frac{u}{2}\right) e^{-\frac{\pi}{3}\alpha|n|}$ (see (5.2.14) above), we can justify the first equality in (5.3.32) by

$$\sum_{n\in\mathbb{Z}}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u+\frac{xy}{u}\right)\right)|K_{i\alpha n}(u)|\,du$$
$$\leq \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{3}\alpha|n|}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u+\frac{xy}{u}\right)\right)K_0\left(\frac{u}{2}\right)\,du<\infty.$$

Analogously, the third equality in (5.3.32) is made possible by the following steps

$$\sum_{n=1}^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u+\frac{xy}{u}\right)\right)\,e^{-u\cosh\left(\frac{2\pi n}{\alpha}\right)}\,\frac{du}{u}$$
$$=\sum_{n=1}^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u+\frac{xy}{u}\right)\right)\,e^{-u-2u\sinh^2\left(\frac{\pi n}{\alpha}\right)}\,\frac{du}{u}$$
$$\leq\sum_{n=1}^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2}{xy}u+\frac{xy}{u}\right)\right)\,\frac{e^{-u}}{1+2u\sinh^2\left(\frac{\pi n}{\alpha}\right)}\,\frac{du}{u}$$
$$\leq\frac{1}{2\sqrt{2}}\sum_{n=1}^\infty\frac{1}{\sinh\left(\frac{\pi n}{\alpha}\right)}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{x^2+y^2+2xy}{xy}u+\frac{xy}{u}\right)\right)\,\frac{du}{u^{\frac{3}{2}}}<\infty. \tag{5.3.34}$$

To prove the second formula (5.3.31), we start by using the following integral [[172], p. 349, relation 2.16.5.4],

$$\int_0^\infty u^{2\sigma-1}K_{2i\tau}\left(xu+\frac{x}{u}\right)du=K_{\sigma+i\tau}(x)\,K_{\sigma-i\tau}(x),$$

which is valid for any $x>0$ and $\sigma\in\mathbb{R}$. Using the appropriate substitutions, we can write the left-hand side of (5.3.31) as

$$\sum_{n\in\mathbb{Z}}K_{\sigma+i\alpha n}(x)\,K_{\sigma-i\alpha n}(x)=\sum_{n\in\mathbb{Z}}\int_0^\infty u^{2\sigma-1}K_{2i\alpha n}\left(xu+\frac{x}{u}\right)du. \tag{5.3.35}$$

Now, since $|K_{i\tau}(x)|\leq e^{-\delta|\tau|}K_0(x\cos(\delta))$ for any $\delta\in[0,\frac{\pi}{2})$ (see (5.2.14) above), we know that

$$\sum_{n\in\mathbb{Z}}\int_0^\infty u^{2\sigma-1}\left|K_{2i\alpha n}\left(xu+\frac{x}{u}\right)\right|du\leq\sum_{n\in\mathbb{Z}}e^{-2\alpha|n|\delta}\int_0^\infty u^{2\sigma-1}K_0\left(x\cos(\delta)\left(u+\frac{1}{u}\right)\right)\,du$$
$$=\frac{1+e^{-2\alpha\delta}}{1-e^{-2\alpha\delta}}\,K_\sigma^2\left(x\cos(\delta)\right),$$

which proves that the interchange of the orders of summation and integration is permitted in (5.3.35). Therefore, we have that

$$\int_0^\infty u^{2\sigma-1}\sum_{n\in\mathbb{Z}} K_{2i\alpha n}\left(xu+\frac{x}{u}\right)du = \frac{\pi}{2\alpha}\int_0^\infty u^{2\sigma-1}\sum_{n\in\mathbb{Z}} e^{-\left(xu+\frac{x}{u}\right)\cosh\left(\frac{\pi n}{\alpha}\right)}\,du$$
$$= \frac{\pi}{2\alpha}\sum_{n\in\mathbb{Z}}\int_0^\infty u^{2\sigma-1}e^{-x\cosh\left(\frac{\pi n}{\alpha}\right)\left(u+\frac{1}{u}\right)}du = \frac{\pi}{\alpha}\sum_{n\in\mathbb{Z}} K_{2\sigma}\left(2x\cosh\left(\frac{\pi n}{\alpha}\right)\right), \tag{5.3.36}$$

which completes our proof, as (5.3.36) is equivalent to (5.3.31). The last equality is clearly a direct consequence of (5.3.33) and the relation $K_\nu(z)=K_{-\nu}(z)$. The second equality in (5.3.36) is again justified via absolute convergence because

$$\sum_{n=1}^\infty\int_0^\infty u^{2\sigma-1}e^{-x\left(u+\frac{1}{u}\right)\cosh\left(\frac{\pi n}{\alpha}\right)}du = \sum_{n=1}^\infty\int_0^\infty u^{2\sigma-1}e^{-x\left(u+\frac{1}{u}\right)\left(1+2\sinh^2\left(\frac{\pi n}{2\alpha}\right)\right)}du$$
$$\leq \sum_{n=1}^\infty\int_0^\infty \frac{u^{2\sigma-1}\,e^{-xu-\frac{x}{u}}}{1+2x\left(u+\frac{1}{u}\right)\sinh^2\left(\frac{\pi n}{2\alpha}\right)}du \leq \frac{1}{2\sqrt{2x}}\sum_{n=1}^\infty\frac{1}{\sinh\left(\frac{\pi n}{2\alpha}\right)}\int_0^\infty\frac{u^{2\sigma-\frac{1}{2}}e^{-xu-\frac{x}{u}}}{\sqrt{1+u^2}}du<\infty.$$

□

We now prove a new transformation formula concerning an infinite series involving the product of two Whittaker functions.

Corollary 5.3.3. *For $x,\alpha>0$, the following formula holds*

$$W_{-\frac{1}{4},0}(x)\,W_{\frac{1}{4},0}(x)+2\sum_{n=1}^\infty W_{-\frac{1}{4},i\alpha n}(x)\,W_{\frac{1}{4},i\alpha n}(x)$$
$$=\frac{x}{\sqrt{2}\alpha}\left\{e^{-\frac{x}{2}}K_0\left(\frac{x}{2}\right)+2\sum_{n=1}^\infty e^{-\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)}K_0\left(\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right\}. \tag{5.3.37}$$

Moreover, for all $\tau,\xi\in\mathbb{R}$, we have the pair of index Fourier transforms

$$\int_{-\infty}^\infty W_{-\frac{1}{4},i\tau}(x)\,W_{\frac{1}{4},i\tau}(x)\,e^{-i\tau\xi}\,d\tau=\frac{x}{\sqrt{2}}e^{-\frac{x}{2}\cosh\left(\frac{\xi}{2}\right)}K_0\left(\frac{x}{2}\cosh\left(\frac{\xi}{2}\right)\right), \tag{5.3.38}$$

$$W_{-\frac{1}{4},i\tau}(x)\,W_{\frac{1}{4},i\tau}(x)=\frac{x}{2\sqrt{2}\pi}\int_{-\infty}^\infty e^{-\frac{x}{2}\cosh\left(\frac{\xi}{2}\right)}K_0\left(\frac{x}{2}\cosh\left(\frac{\xi}{2}\right)\right)e^{i\tau\xi}\,d\xi. \tag{5.3.39}$$

Proof. We start our argument by appealing to the integral representation [[48], p. 460, relation 3.30.6.7],

$$W_{-\frac{1}{4},i\alpha n}(x)\,W_{\frac{1}{4},i\alpha n}(x)=\frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(s+1)\,\Gamma\left(\frac{s+1}{2}+i\alpha n\right)\Gamma\left(\frac{s+1}{2}-i\alpha n\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}x^{-s}ds, \tag{5.3.40}$$

where $\sigma > -1$. We take the summation of the previous expression over $n \in \mathbb{Z}$. Using exactly the same justification as in (5.3.9), it is not difficult to show that[13]

$$\sum_{n\in\mathbb{Z}} W_{-\frac{1}{4},i\alpha n}(x)\, W_{\frac{1}{4},i\alpha n}(x)$$
$$= \frac{1}{4\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)} \sum_{n\in\mathbb{Z}} \Gamma\left(\frac{s+1}{2}+i\alpha n\right)\Gamma\left(\frac{s+1}{2}-i\alpha n\right) x^{-s}ds. \tag{5.3.41}$$

Since $\mathrm{Re}\,(s+1) = \sigma+1 > 0$ by hypothesis, we can use (5.3.29) in order to establish the equality

$$\sum_{n\in\mathbb{Z}} W_{-\frac{1}{4},i\alpha n}(x)\, W_{\frac{1}{4},i\alpha n}(x) = \frac{1}{4\sqrt{\pi} i\alpha}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s+1)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)} \sum_{n\in\mathbb{Z}} \frac{1}{\cosh^{s+1}\left(\frac{\pi n}{\alpha}\right)}\, x^{-s}ds$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\pi i\alpha}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma^2(s+1)}{\Gamma\left(s+\frac{3}{2}\right)} \sum_{n\in\mathbb{Z}} \frac{1}{\cosh^{s+1}\left(\frac{\pi n}{\alpha}\right)}\, x^{-s}ds, \tag{5.3.42}$$

where, at the last step, we have simply used Legendre's duplication formula (4.3.18). It is clear that

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \sum_{n\in\mathbb{Z}} \left|\frac{\Gamma(s+1)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}\frac{1}{\cosh^{s+1}\left(\frac{\pi n}{\alpha}\right)}\, x^{-s}\right| |ds|$$
$$\leq \sum_{n\in\mathbb{Z}} \frac{x^{-\sigma}}{\cosh^{\sigma+1}\left(\frac{\pi n}{\alpha}\right)} \int_{\sigma-i\infty}^{\sigma+i\infty} \left|\frac{\Gamma(s+1)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{s}{2}+\frac{5}{4}\right)}\right| |ds| < \infty, \tag{5.3.43}$$

since the absolute convergence of the infinite series in (5.3.43) is justified by the fact that $\sigma > -1$. Moreover, when $|t| \gg 1$, we know from Stirling's formula that

$$\left|\frac{\Gamma(\sigma+1+it)\,\Gamma\left(\frac{\sigma+1+it}{2}\right)\Gamma\left(\frac{\sigma}{2}+1+\frac{it}{2}\right)}{\Gamma\left(\frac{\sigma}{2}+\frac{3}{4}+\frac{it}{2}\right)\Gamma\left(\frac{\sigma}{2}+\frac{5}{4}+\frac{it}{2}\right)}\right| \ll |t|^{\sigma}e^{-\frac{\pi|t|}{2}},$$

which fully justifies the absolute convergence claimed in (5.3.43). By Fubini's theorem, we are now able to obtain the summation formula

$$\sum_{n\in\mathbb{Z}} W_{-\frac{1}{4},i\alpha n}(x)\, W_{\frac{1}{4},i\alpha n}(x) = \sqrt{\frac{\pi}{2}} \sum_{n\in\mathbb{Z}} \frac{x}{2\pi i\alpha}\int_{\sigma+1-i\infty}^{\sigma+1+i\infty} \frac{\Gamma^2(s)}{\Gamma(s+\frac{1}{2})} \left(x\cosh\left(\frac{\pi n}{\alpha}\right)\right)^{-s} ds$$
$$= \frac{x}{\alpha\sqrt{2}} \sum_{n\in\mathbb{Z}} e^{-\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)} K_0\left(\frac{x}{2}\cosh\left(\frac{\pi n}{\alpha}\right)\right),$$

where the last step consisted in an application of the Mellin-Barnes integral [[48], page 209, relation 3.14.3.1],

$$e^{-x}K_{\nu}(x) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{\sqrt{\pi}}{2^s}\,\frac{\Gamma(s-\nu)\Gamma(s+\nu)}{\Gamma\left(s+\frac{1}{2}\right)}\, x^{-s}ds, \quad \mu > |\mathrm{Re}(\nu)|,$$

[13]In fact, (5.3.41) is nothing but a discrete analogue of (5.3.9).

with $\nu = 0$. Finally, (5.3.38) and (5.3.39) are immediate corollaries of the Fourier inversion formula (5.2.26). $\square$

This chapter is reaching its end. We conclude it by presenting a new summation formula involving products of Gamma and Whittaker functions. The kernel presented in our next summation formula dates back to Wimp's work [240], and the reader can find some inversion formulas concerning it in [209, 242].

Corollary 5.3.4. *Let $x, \alpha > 0$ and assume that $\mu < \frac{1}{2}$. Then the following summation formula takes place*

$$\Gamma^2\left(\frac{1}{2}-\mu\right) W_{\mu,0}(x) + 2\sum_{n=1}^{\infty}\left|\Gamma\left(\frac{1}{2}-\mu+i\alpha n\right)\right|^2 W_{\mu,i\alpha n}(x)$$

$$= \frac{\pi 2^{\mu+\frac{1}{2}}\sqrt{x}}{\alpha}\,\Gamma(1-2\mu)\left\{D_{2\mu-1}\left(\sqrt{2x}\right) + 2\sum_{n=1}^{\infty} e^{\frac{x\sinh^2(\pi n/\alpha)}{2}}\, D_{2\mu-1}\left(\sqrt{2x}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right\}, \quad (5.3.44)$$

where $D_{2\mu-1}(\cdot)$ denotes the parabolic cylinder function, (5.2.16). Moreover, we have the pair of index Fourier transforms

$$\int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x)\, e^{-i\tau\xi}\, d\tau = \pi\, 2^{\mu+\frac{1}{2}}\sqrt{x}\,\Gamma(1-2\mu)\, e^{\frac{x\sinh^2(\xi/2)}{2}}\, D_{2\mu-1}\left(\sqrt{2x}\cosh\left(\frac{\xi}{2}\right)\right), \quad (5.3.45)$$

$$\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x) = 2^{\mu-\frac{1}{2}}\sqrt{x}\,\Gamma(1-2\mu)\int_{-\infty}^{\infty} e^{\frac{x\sinh^2(\xi/2)}{2}}\, D_{2\mu-1}\left(\sqrt{2x}\cosh\left(\frac{\xi}{2}\right)\right) e^{i\tau\xi}\, d\xi. \quad (5.3.46)$$

Proof. In contrast with the previous corollary, we only compute the Fourier transform (5.3.45) and leave the proof of the summation formula (5.3.44) to the reader. We start with an integral representation that can be found in [[172], Vol. II, page 352, relation 2.16.8.4]

$$\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x) = 2(4x)^{\mu}e^{-x/2}\int_0^{\infty} t^{-2\mu}e^{-\frac{t^2}{4x}}\, K_{2i\tau}(t)\, dt, \quad (5.3.47)$$

which is valid for $x > 0$, $\tau \in \mathbb{R}$ and $\mu < \frac{1}{2}$ (see also [[48], p. 210, eq. 3.14.3.10]). Thus, the Fourier transform of the left-hand side of (5.3.47) is

$$G_{\mu,\xi}(x) := \int_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x)\, e^{-i\tau\xi}\, d\tau$$

$$= 2(4x)^{\mu}e^{-x/2}\int_{-\infty}^{\infty}\int_0^{\infty} t^{-2\mu}e^{-\frac{t^2}{4x}}\, K_{2i\tau}(t)\, dt\, e^{-i\tau\xi}\, d\tau. \quad (5.3.48)$$

Since $\mu < \frac{1}{2}$ and $|K_{2i\tau}(t)| \le e^{-2\delta|\tau|}\, K_0(t\cos(\delta))$ for any $0 \le \delta < \frac{\pi}{2}$, it is clear from the asymptotic behaviour $K_0(x) = -\log(x) + O(1)$, $x \to 0^+$ (cf. (5.2.11)), that the double integral appearing in

(5.3.48) is absolutely convergent. In fact, fixing some $\delta \in [0, \frac{\pi}{2})$, we have

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|t^{-2\mu}e^{-\frac{t^2}{4x}}\,K_{2i\tau}(t)\right|\,dt\,d\tau \leq \int_{-\infty}^{\infty}\int_{0}^{\infty}t^{-2\mu}e^{-\frac{t^2}{4x}}\,e^{-2\delta|\tau|}\,K_0(t\,\cos(\delta))\,dt\,d\tau$$
$$= \frac{1}{\delta}\int_{0}^{\infty}t^{-2\mu}e^{-\frac{t^2}{4x}}K_0\left(t\cos(\delta)\right)\,dt \leq \frac{1}{\delta}\int_{0}^{\infty}t^{-2\mu}K_0\left(t\cos(\delta)\right)\,dt$$
$$= \delta^{-1}2^{-2\mu-1}\cos^{2\mu-1}(\delta)\,\Gamma^2\left(\frac{1}{2}-\mu\right), \tag{5.3.49}$$

which proves that we can reverse the orders of integration in (5.3.48). Hence, we are able to reach the following Fourier transform

$$G_{\mu,\xi}(x) = 2(4x)^{\mu}e^{-x/2}\int_{0}^{\infty}t^{-2\mu}e^{-\frac{t^2}{4x}}\int_{-\infty}^{\infty}K_{2i\tau}(t)\,e^{-i\tau\xi}\,d\tau\,dt$$
$$= \pi(4x)^{\mu}e^{-x/2}\int_{0}^{\infty}t^{-2\mu}e^{-\frac{t^2}{4x}}\,e^{-t\cosh\left(\frac{\xi}{2}\right)}\,dt$$
$$= \pi 2^{\mu+\frac{1}{2}}\sqrt{x}\,\Gamma(1-2\mu)\,e^{\frac{x\sinh^2(\xi/2)}{2}}\,D_{2\mu-1}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right),$$

where we have invoked (5.3.20) to get the integral with respect to τ and, in the last step, we have used (5.2.17).[14]

By the asymptotic behavior of the parabolic cylinder function (5.2.21), we also know that

$$\left|e^{\frac{x\sinh^2(\xi/2)}{2}}\,D_{2\mu-1}\left(\sqrt{2x}\,\cosh\left(\frac{\xi}{2}\right)\right)\right| \sim (2x)^{\mu-\frac{1}{2}}e^{-\frac{x}{2}}\cosh^{2\mu-1}\left(\frac{\xi}{2}\right),\quad \xi\to\infty. \tag{5.3.50}$$

Thus, for a fixed $x \in (0, X]$, the functions $g(\tau) := \left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x)$ and $\hat{g}(\xi) := G_{\mu,\xi}(x)$ both belong to the class of functions with moderate decrease, $\mathscr{F}$, in the sense of [[214], p. 131]. Hence, the Fourier inversion formula is valid [[214], p. 144] and we have the pair of transforms (5.3.45) and (5.3.46).[15] □

Remark 5.3.4. In the previous two corollaries we have respectively proved a formula involving the product of two Whittaker functions and a formula covering a combination of Gamma and Whittaker functions. The main restriction in (5.3.37) is the fact that $\mu = \frac{1}{4}, -\frac{1}{4}$. One may wonder if there exist summation formulas (akin to the ones studied here) for the infinite series

$$\sum_{n\in\mathbb{Z}}W_{\mu,in\alpha}(x)\,W_{\mu,in\alpha}(y),\quad \sum_{n\in\mathbb{Z}}\left|\Gamma\left(\frac{1}{2}-\mu+in\alpha\right)\right|^2 W_{\mu,in\alpha}(x)\,W_{\mu,in\alpha}(y),\quad x,y>0, \tag{5.3.51}$$

[14] Note that (5.2.17) is valid in this case because $\mu < \frac{1}{2}$ by hypothesis.

[15] It is straightforward to check that, for fixed x and $\mu < \frac{1}{2}$, the function $g(\tau) := \left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 W_{\mu,i\tau}(x)$ satisfies the conditions of Definition 5.2.1 and so we may also use Lemma 5.2.1 to arrive at the pair (5.3.45) and (5.3.46).

where one may take $\mu < \frac{1}{2}$ in the latter series. One possible way to study these hypothetical summation formulas is by invoking the product representation [208][16],

$$W_{\mu,i\tau}(x)\, W_{\mu,i\tau}(y)$$
$$= 2^{-\mu-1}\sqrt{\frac{xy}{\pi}}e^{\frac{x+y}{4}}\int_0^{\infty} W_{\mu,i\tau}(u)\, u^{-\frac{3}{2}} \exp\left(\frac{u}{2}-\frac{(x+y)^2u}{8xy}-\frac{xy}{8u}\right) D_{2\mu}\left(\sqrt{\frac{xy}{2u}}+\frac{x+y}{\sqrt{2xy}}\sqrt{u}\right) du. \tag{5.3.52}$$

We have tried to use the summation formula (5.3.21) inside the integral in (5.3.52) in order to study the first series in (5.3.51). However, our modest attempts proved to be unsuccessful and we vividly encourage the reader to find new summation formulas for the infinite series given in (5.3.51).

Our final corollary gives a variant of the summation formula (5.3.28) when we combine its general term $K_{i\alpha n}(x)$ with a rapidly converging factor. We omit its proof, because it only consists in taking $\mu = 0$ in (5.3.44) and then use the reduction formula for Whittaker's function (5.2.19).

Corollary 5.3.5. *Let $x, \alpha > 0$ and erfc$(\cdot)$ denote the complementary error function (5.2.20). Then the following transformation formula holds*

$$K_0(x)+2\sum_{n=1}^{\infty}\frac{K_{i\alpha n}(x)}{\cosh(\pi\alpha n)} = \frac{\pi}{\alpha}e^{x}\,\mathit{erfc}\left(\sqrt{2x}\right)+\frac{2\pi}{\alpha}\sum_{n=1}^{\infty}e^{x\cosh\left(\frac{2\pi n}{\alpha}\right)}\,\mathit{erfc}\left(\sqrt{2x}\cosh\left(\frac{\pi n}{\alpha}\right)\right). \tag{5.3.53}$$

Remark 5.3.5. It is possible to employ Theorem 5.3.1 to get even more index summation formulas. For details, the reader is encouraged to check the preprint version of this chapter [195]. For example, it is shown there that, for any $x, \alpha > 0$, the following summation formula holds

$$\mathrm{E}_1(x)+\frac{2}{\alpha}\sqrt{\frac{x}{\pi}}\,e^{-\frac{x}{2}}\sum_{n=1}^{\infty}\frac{\mathrm{Im}\left\{K_{\frac{1}{2}+i\alpha n}\left(\frac{x}{2}\right)\right\}}{n} = \frac{\pi}{\alpha}\left(\mathrm{erfc}\left(\sqrt{x}\right)+2\sum_{n=1}^{\infty}\mathrm{erfc}\left(\sqrt{x}\cosh\left(\frac{\pi n}{\alpha}\right)\right)\right), \tag{5.3.54}$$

where erfc$(\cdot)$ denotes the complementary error function (5.2.20) and $\mathrm{E}_1(\cdot)$ is the exponential integral function [[164], p. 150, 6.2.1], usually defined by (cf. (4.6.18) and (4.6.20) above)

$$\mathrm{E}_1(x) := \int_x^{\infty}\frac{e^{-t}}{t}dt, \quad x > 0.$$

This summation formula can be seen as an analogue of (5.3.53).

[16] We should remark that (5.3.52) can be derived as a consequence of (5.3.15), together with the injectivity of the Fourier transform and the convolution theorem. This yields an interesting proof of it, which is much shorter than the proof presented in [208].

Chapter 6

Generalizations of some formulas from Ramanujan's Lost Notebook

This final chapter is dedicated to the memory of my Friend José Carlos Petronilho

6.1 Introduction

On entry 8 of Chapter 15 of his second notebook [28, 31], Ramanujan stated the beautiful formula for $\zeta(1/2)$.

Entry 8. *If $x > 0$, then the following identity holds*

$$\sum_{n=1}^{\infty}\frac{1}{e^{n^2x}-1}=\frac{1}{4}+\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta\left(\frac{1}{2}\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{\cos(2\pi\sqrt{\pi n/x})-\sin(2\pi\sqrt{\pi n/x})-e^{-2\pi\sqrt{\pi n/x}}}{\cosh(2\pi\sqrt{\pi n/x})-\cos(2\pi\sqrt{\pi n/x})}\right). \tag{6.1.1}$$

On Entry 8.3.1, page 332 of his Lost Notebook, Ramanujan restates (6.1.1) in two different ways, which might be an indication that this representation for $\zeta(1/2)$ held significant importance to him.

The first proof of (6.1.1) was given by Berndt and Evans [31], who employed the Poisson summation formula and the transformation formula for Jacobi's $\theta-$function. After this, several generalizations of (6.1.1) have been obtained, including character analogues [80] as well as evaluations of the Riemann zeta function at certain rational arguments aside 1/2 [125]. The reader may find in [[2], pp. 191-193] an exhaustive account of the work surrounding (6.1.1).

Naturally, (6.1.1) is accompanied by numerous other remarkable formulas found in Ramanujan's Lost Notebook. As discussed earlier in Chapters 1 and 4 of this thesis, Ramanujan presents the following formula on page 253 of [179]:

Entry 3.3.1 (p. 253)**.** *Let $\sigma_k(n) = \sum_{d|n} d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are two positive real numbers such that $\alpha\beta = \pi^2$ and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\beta)$$
$$= \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2} - \alpha^{(1+s)/2}\right\} + \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\right\}. \qquad (6.1.2)$$

The appearance of the Bessel and the divisor functions on the infinite series on the left side of (6.1.2) is somewhat remindful of the Fourier expansion of the nonholomorphic Eisenstein series or of the Epstein zeta function attached to a binary quadratic form. We have used this analogy in two distinct instances before, namely in Example 1.2.1 and in Theorem 4.5.1. As previously mentioned in our text, the first proof of (6.1.2), due to Guinand [100], used a formula of Watson [233], which states that[1]

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+x^2)^s} = \frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right) - \frac{x^{-2s}}{2} + \frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi n x), \qquad (6.1.3)$$

for any $x > 0$ and $\mathrm{Re}(s) > \frac{1}{2}$. Due to the fact that Guinand's proof seems to be the first appearing in the literature, (6.1.2) is usually called "Ramanujan-Guinand formula". As remarked by Berndt, Lee and Sohn [[37], p. 23], Guinand's proof of (6.1.2) via (6.1.3) is *"completely independent of any considerations of nonanalytic Eisenstein series"*.

However, proving (6.1.3) is in fact the first step in almost all the proofs of the aforementioned Fourier expansion of Epstein zeta functions. We know from the previous sections that, from the historical point of view, this expansion was firstly considered by A. Selberg and S. Chowla in 1949, who started with the Epstein zeta function [56]

$$\zeta(s,\,Q) = \sum_{m,n\neq 0}\frac{1}{(am^2+bmn+cn^2)^s}, \quad \mathrm{Re}(s) > 1, \qquad (6.1.4)$$

(where $m,\,n \neq 0$ here means that only the term $m = n = 0$ is omitted from the sum) and announced the following formula, valid in the entire complex plane,

$$a^s\Gamma(s)\,\zeta(s,\,Q) = 2\Gamma(s)\zeta(2s) + 2k^{1-2s}\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)$$
$$+ 8k^{\frac{1}{2}-s}\pi^s\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\,\cos\left(n\pi b/a\right)K_{s-\frac{1}{2}}\left(2\pi k n\right). \qquad (6.1.5)$$

Here, $Q(x,y) = ax^2 + bxy + cy^2$ denotes a real and positive definite quadratic form, $\Delta := 4ac - b^2$ its discriminant, $k^2 := \Delta/4a^2$ and $\sigma_\nu(n) = \sum_{d|n} d^\nu$. As we have mentioned in the Introduction of the first chapter, the first explicit appearance of (6.1.5) in the literature is given in a paper of Taylor [[219], p. 182], a student of Titchmarsh, which was written nine years before the announcement made by Selberg and Chowla. In fact, Taylor attributed (6.1.5) to Kober, although Taylor's proof

[1] We had the opportunity to encounter this formula in the first, second and fourth chapters of this thesis. For clarity in this exposition, however, we have decided to write it in this introduction.

of this formula is drastically different. Kober was interested in extending Watson's formula (6.1.3) and one of the results obtained by him was the identity [[129], p. 614, eq. (2b)]

$$\sum_{n\in\mathbb{Z}}\frac{1}{\left((n+\beta)^2+x^2\right)^s}=\frac{\sqrt{\pi}x^{1-2s}}{\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)+\frac{4\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-\frac{1}{2}}\,\cos(2\pi n\beta)\,K_{s-\frac{1}{2}}(2\pi nx), \tag{6.1.6}$$

valid for any $x>0$, $\beta\in\mathbb{R}$ and $\text{Re}(s)>\frac{1}{2}$.

It is now clear that the appearance of the divisor function in (6.1.5) is in some form connected to the identity of Ramanujan (6.1.2) and if one analyzes the underlying structure of the proofs of (6.1.5), they tend to rely in one way or another on Watson's or Kober's formulas (6.1.3) and (6.1.6). By using (6.1.5), Selberg and Chowla [201] established several new results and reproved others by a novel approach. For example, they were able to derive a new proof of the functional equation for $\zeta(s,\,Q)$. They also obtained from (6.1.5) a very simple proof of Kronecker's limit formula,

$$\zeta(s,\,Q)=\frac{2\pi}{\sqrt{\Delta}}\,\frac{1}{s-1}+\frac{2\pi}{\sqrt{\Delta}}\,\left(2\gamma-\log\left(\frac{\Delta}{a}\right)-4\log\left(|\eta(\tau)|\right)\right)+O(s-1), \tag{6.1.7}$$

where γ is the Euler-Mascheroni constant, $\tau:=\frac{b+i\sqrt{\Delta}}{2a}\in\mathbb{H}$ and $\eta(\tau)$ denotes the Dedekind $\eta-$function,

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{m=1}^{\infty}\left(1-e^{2\pi im\tau}\right),\quad \text{Im}(\tau)>0. \tag{6.1.8}$$

We encountered (6.1.7) before in this thesis, and we have employed it to get a "quadratic form analogue" of Koshliakov's formula in Example 1.2.2 of the first chapter. In particular, when the quadratic form is diagonal, this is $Q(m,n)=m^2+cn^2$, $c>0$, the following case of (6.1.7) holds

$$\zeta(s,\,Q):=\zeta_2(s,c)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\,\left(2\gamma-\log\left(4c\right)-4\log\left(\eta\left(i\sqrt{c}\right)\right)\right)+O(s-1). \tag{6.1.9}$$

The function defined by (6.1.8) also played an important role in Ramanujan's lost notebook, right after the statement of (6.1.2). Ramanujan completes page 253 with two corollaries, the first of them being equivalent to a transformation formula for $\eta(\tau)$.

Entry 3.3.2 (p. 253)**.** *Let α and β be two positive real numbers such that $\alpha\beta=\pi^2$. Then*

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\alpha}-\sum_{n=1}^{\infty}\sigma_{-1}(n)\,e^{-2n\beta}=\frac{\beta-\alpha}{12}+\frac{1}{4}\log\left(\frac{\alpha}{\beta}\right). \tag{6.1.10}$$

Finally, Ramanujan uses (6.1.2) to derive the most famous formula attributed to Koshliakov [134]. Preceding Koshliakov's contributions by more than a decade, Ramanujan states:

Entry 3.3.3 (p. 253)**.** *Let α and β be two positive real numbers such that $\alpha\beta=\pi^2$. Then the following identity holds*

$$\sqrt{\alpha}\left(\frac{\gamma}{4}-\frac{\log(4\beta)}{4}+\sum_{n=1}^{\infty}d(n)\,K_0(2n\alpha)\right)=\sqrt{\beta}\left(\frac{\gamma}{4}-\frac{\log(4\alpha)}{4}+\sum_{n=1}^{\infty}d(n)\,K_0(2n\beta)\right). \tag{6.1.11}$$

The aim of this chapter is to extend these classical results of Ramanujan, namely entries (6.1.1), (6.1.2), (6.1.10) and (6.1.11). In exploring the extensions of (6.1.2) and (6.1.10), we were naturally led to examine generalizations of Watson's formula (6.1.3) and expand upon Kronecker's limit formula (6.1.7) (see Sections 6.4 and 6.5 below).

We now outline the nature of the generalizations proposed in this work, which are connected to a manuscript by N. S. Koshliakov that has recently garnered renewed interest. In a very long paper [132], written during his stay in a sharashka,[2] Koshliakov introduced the zeta function [132, p. 6] [3]

$$\zeta_p(s)=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\lambda_n^s},\quad \mathrm{Re}(s)>1, \tag{6.1.12}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence defined by the positive roots of the transcendental equation

$$p\sin(\pi y)+y\cos(\pi y)=0,\quad p>0. \tag{6.1.13}$$

From the fact that $n-\frac{1}{2}<\lambda_n<n$, the series (6.1.12) converges absolutely in the half-plane $\mathrm{Re}(s)>1$. In particular, $\zeta_p(s)$ generalizes the Riemann zeta function, since the structure of the transcendental equation (6.1.13) yields the limiting cases

$$\lim_{p\to\infty}\zeta_p(s)=\zeta(s),\quad \lim_{p\to 0^+}\zeta_p(s)=(2^s-1)\,\zeta(s).$$

Koshliakov connected the study of the zeta function $\zeta_p(s)$ with a second generalized zeta function, $\eta_p(s)$, being defined by the series of Dirichlet-type[4]

$$\eta_p(s):=\sum_{k=1}^{\infty}\frac{(s,2\pi pk)_k}{k^s},\quad \mathrm{Re}(s)>1, \tag{6.1.14}$$

where[5]

$$(s,\nu k)_k:=\frac{1}{\Gamma(s)}\int\limits_0^{\infty}x^{s-1}e^{-x}\left(\frac{k\nu-x}{k\nu+x}\right)^k\,dx. \tag{6.1.15}$$

On page 20 of Koshliakov's memoir [132], it is possible to look at an argument concerning the uniform and absolute convergence of the series (6.1.14) in the region $\mathrm{Re}(s)>1$. This property was also rederived on page 26, formula (57), where it is established that

$$\lim_{k\to\infty}(s,\lambda k)_k=\frac{1}{\left(1+\frac{2}{\lambda}\right)^s}, \tag{6.1.16}$$

[2] which were secret research laboratories working within the Soviet Gulag labor camp system. For details about Koshliakov's imprisonment and the group of scientists from Leningrad that were accused and sentenced during the blockade of the city, we recommend the very interesting article [147].

[3] we advise the reader not to mistake Koshliakov's zeta function, $\zeta_p(s)$, for the zeta function linked to $\theta^{\alpha}(x)$, $\zeta_{\alpha}(s)$, which was explored in previous chapters (cf. (1.1.7)).

[4] we once again caution the reader not to confuse Koshliakov's second zeta function, $\eta_p(s)$, with the completed Dirichlet series attached to $\zeta_{\alpha}(s)$, also studied in previous chapters (cf. (1.3.66)).

[5] Note that $\eta_p(s)$ **is not a Dirichlet series** because the "arithmetical function" $(s,2\pi pk)_k$ depends on k and s.

for any $\lambda > 0$. Besides studying the analytic continuations of (6.1.12) and (6.1.14), Koshliakov proved that $\zeta_p(s)$ and $\eta_p(s)$ are connected through the functional equation

$$\zeta_p(1-s) = \frac{2\cos\left(\frac{\pi s}{2}\right)\Gamma(s)}{(2\pi)^s}\eta_p(s). \tag{6.1.17}$$

After developing an analytic structure for the functions $\zeta_p(s)$ and $\eta_p(s)$, Koshliakov enters the realm of summation formulas, proving new analogues of Poisson's summation formula, introducing new generalizations of the digamma function and Bernoulli numbers[6] and even extending an integral formula of Ramanujan.[7]

Koshliakov also introduced two analogues of the Hurwitz zeta function [[132], p. 107, eq. (16)]. For $0 < a < 1$ and $\mathrm{Re}(s) > 1$, one may consider the infinite series

$$\eta_p(s,a) = \sum_{k=0}^{\infty}\frac{(s, 2\pi p(k+a))_k}{(k+a)^s}, \tag{6.1.18}$$

which, in the limit $p \to \infty$, reduces to the classical Hurwitz zeta function,

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \qquad \mathrm{Re}(s) > 1, \quad 0 < a \leq 1.$$

Koshliakov [[132], pp. 108 and 109] established that $\eta_p(s,a)$ can be analytically continued to the entire complex plane as a meromorphic function with a simple pole located at $s = 1$, whose residue is $\mathrm{Res}_{s=1}\{\eta_p(s,a)\} = \frac{1}{1+\frac{1}{\pi p}}$. Moreover, like the standard Hurwitz zeta function, it has a continuation to the plane $\mathrm{Re}(s) < 0$ via the formula [[132], p. 109, eq. (23)] [8]

$$\begin{aligned}\eta_p(s,a) = 2(2\pi)^{s-1}\Gamma(1-s)\Bigg\{ &\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\cos(2\pi\lambda_n a)}{\lambda_n^{1-s}} \\ &+\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\sin(2\pi\lambda_n a)}{\lambda_n^{1-s}}\Bigg\},\end{aligned} \tag{6.1.19}$$

which, in the limit $p \to \infty$, reduces to the familiar result [[228], p. 37, eq. (2.17.3)]

$$\zeta(s,a) = 2(2\pi)^{s-1}\Gamma(1-s)\left\{\sin\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi n a)}{n^{1-s}} + \cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi n a)}{n^{1-s}}\right\}, \tag{6.1.20}$$

also valid for $0 < a < 1$ [9]and $\mathrm{Re}(s) < 0$.

The transformation formula (6.1.19) coexists with another extension of Hurwitz's relation (6.1.20) proved by Koshliakov [[132], p. 64, eq. (51)]. It is because of identities like (6.1.19) that we feel that it is not an overstatement to declare Koshliakov's manuscript as a true masterpiece!

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[6]The construction of the generalized Bernoulli numbers seems to be motivated by a previous work of his, [130].

[7]See [132, p. 150, eq. (19)], where a generalization of the main formula from Ramanujan's paper [177] is derived.

[8]since $0 < a < 1$, this formula is also valid for $\mathrm{Re}(s) < 1$ in the limiting cases $p \to 0^+$ and $p \to \infty$.

[9]Actually, (6.1.20) is also valid when $a = 1$, in which case it is reduced to the functional equation for $\zeta(s)$. But since $\eta_p(s,a)$ does not reduce to $\eta_p(s)$ in the same way as $\zeta(s,a)$ reduces to $\zeta(s)$, there is no loss of generality in omitting this case here.

Unfortunately, it remained in relative obscurity since its publication in 1949, but thanks to the recent and outstanding work of A. Dixit and R. Gupta [68], Koshliakov's key results have been revisited and extended in numerous directions.

In addition to bringing Koshliakov's ideas to the attention of a modern mathematical community, Dixit and Gupta also built upon his framework to derive new formulas. Among their many contributions, they extended a famous formula of Ramanujan for $\zeta(2n+1)$ [[68], p. 13, Theorem 4.1][10]. From this extension, they discovered new identities by making a suitable choice of the parameter p in the transcendental equation (6.1.13) defining $\zeta_p(s)$ and $\eta_p(s)$. Following this remarkable work, Dixit and Gupta, together with Berndt and Zaharescu, further explored these ideas in [40], where new analogues of the Abel-Plana formula were examined.

It is in the line of reasoning of the contributions given in [68] and [40] that our main results are here presented. The work presented in this chapter is organized as follows: in Section 6.2 we establish a generalization (in Koshliakov's setting) of Ramanujan's formula for $\zeta(1/2)$ (6.1.1). We also prove an analogue of this formula involving the generalized Hurwitz zeta function, $\eta_p(1/2, a)$, (6.1.18). In Sections 6.3 and 6.4 we generalize Watson's formula (6.1.3), as well as the Kronecker limit formula for 'diagonal quadratic forms', (6.1.9). At last, we devote the fifth section of our chapter to the generalization of Entries 3.3.1, 3.3.2 and 3.3.3 stated above and establish, at the very end, the transcendence of two infinite series motivated by this work.

Before going to the main sections of this chapter, we overview some additional facts about the zeta function (6.1.14), which will be useful in the sequel. Since $\eta_p(s)$ is not a Dirichlet series in the classical sense, we need some particular properties that can be extracted from the Mellin representation (6.1.15). In fact, Koshliakov [[132], p. 25, eq. (48)] could write (6.1.15) as a combination of fractional integrals of the form

$$\Gamma(s)\,(s,\lambda)_k = \Gamma(s)(-1)^k + \Gamma(s)\,k^s e^{\lambda} \sum_{\ell=1}^{k} \binom{k}{\ell} \left(\frac{2\lambda}{k}\right)^{\ell} (-1)^{k-\ell} \int_k^{\infty} t^{-s} e^{-\frac{\lambda}{k}t}\, \frac{(t-k)^{\ell-1}}{(\ell-1)!}\, dt, \tag{6.1.21}$$

which, when used in the definition (6.1.14), gives the expression for $\eta_p(s)$

$$\begin{aligned}\eta_p(s) := \sum_{k=1}^{\infty} \frac{(s, 2\pi p k)_k}{k^s} &= \left(2^{1-s}-1\right)\zeta(s) \\ &+ \sum_{k=1}^{\infty} e^{2\pi p k} \sum_{\ell=1}^{k} \binom{k}{\ell} (4\pi p)^{\ell} (-1)^{k-\ell} \int_k^{\infty} t^{-s} e^{-2\pi p t}\, \frac{(t-k)^{\ell-1}}{(\ell-1)!}\, dt, \end{aligned}\tag{6.1.22}$$

being valid when $\mathrm{Re}(s) > 1$. Representation (6.1.22) will prove to be very useful in the third section below, where a generalization of Watson's formula (6.1.3) will be given. Throughout this work, we shall also employ the notation introduced by Koshliakov [[132], p. 6]: if $p, p' \in \mathbb{R}^+$, we define the functions $\sigma(t)$ and $\sigma'(t)$ as[11]

$$\sigma(t) := \frac{p+t}{p-t}, \quad \sigma'(t) := \frac{p'+t}{p'-t}. \tag{6.1.23}$$

[10] See also [33, 66] for two excellent surveys on Ramanujan's formula for $\zeta(2n+1)$.

[11] the reader is urged to not make any confusion with the divisor function $\sigma(n) := \sum_{d|n} d$.

For additional standard facts about the zeta functions (6.1.12) and (6.1.14), we shall refer to Koshliakov's own manuscript [132] and to Dixit and Gupta's paper [68].

6.2 Generalization of Entry 8.3.1, p. 332 of Ramanujan's Lost Notebook

We are now ready to establish a general version of Entry 8.3.1.

Theorem 6.2.1. *Let $p, p' \in \mathbb{R}_+$ and $(\lambda_n)_{n\in\mathbb{N}}$, $(\lambda'_n)_{n\in\mathbb{N}}$ be the sequences of numbers satisfying respectively the transcendental equations*

$$p\sin(\pi y) + y\cos(\pi y) = 0, \quad p'\sin(\pi y) + y\cos(\pi y) = 0. \tag{6.2.1}$$

Moreover, let $\sigma(t)$, $\sigma'(t)$ be defined by (6.1.23). Then, for any $x > 0$, the following identity holds

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right)e^{\lambda_n^2 x}-1} &= \frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{\pi^2}{6x}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}\\ &+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{1}{\sqrt{\lambda'_n}}\,G_p\left(\frac{2\pi\lambda'_n}{x}\right),\end{aligned} \tag{6.2.2}$$

where, for $a > 0$, $G_p(a)$ is explicitly given by

$$G_p(a) = \frac{G_{1,p}(a)}{G_{2,p}(a)}, \tag{6.2.3}$$

with $G_{1,p}(a)$ and $G_{2,p}(a)$ being the following functions

$$\begin{aligned}G_{1,p}(a) &= \left(p^2-a\right)\left(\cos(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})\right)\\ &\quad - e^{-\pi\sqrt{2a}}\left(p^2-\sqrt{2a}p+a\right)-\sqrt{2a}p\left(\cos(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right),\end{aligned} \tag{6.2.4}$$

and

$$\begin{aligned}G_{2,p}(a) &= p^2\left(\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})\right)\\ &\quad + \sqrt{2a}p\left(\sinh(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right)+a\left(\cosh(\pi\sqrt{2a})+\cos(\pi\sqrt{2a})\right).\end{aligned} \tag{6.2.5}$$

Proof. By [[132], p. 32, Chapter 1], one has the representation

$$\zeta_{p'}(1-s) = 2\cos\left(\frac{\pi s}{2}\right)\int_0^{\infty}\frac{x^{s-1}}{\sigma'(x)\,e^{2\pi x}-1}\,dx, \qquad \mathrm{Re}(s) > 1.$$

Therefore, from Mellin's inversion formula and the absolute convergence of the series (6.1.12) in the half-plane $\mathrm{Re}(s) > \mu > 1$, we can write the infinite series on the left-hand side of (6.2.2) as the contour integral

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right) e^{\lambda_n^2 x}-1} = \frac{1}{2\pi i} \int\limits_{\mu-i\infty}^{\mu+i\infty} \frac{\zeta_{p'}(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)} \left(\frac{x}{2\pi}\right)^{-s} ds. \tag{6.2.6}$$

Taking the previous Mellin-Barnes integral as a starting point, we shift its line of integration from $\mathrm{Re}(s)=\mu$ to $\mathrm{Re}(s)=\frac{1}{2}-\mu$. To do this, we just need to integrate along a positively oriented rectangular contour $\mathscr{R}_\mu(T)$ with vertices $\mu \pm iT$ and $\frac{1}{2}-\mu\pm iT$: by an application of Cauchy's residue theorem, we have that

$$\left\{\int\limits_{\mu-iT}^{\mu+iT} + \int\limits_{\mu+iT}^{\frac{1}{2}-\mu+iT} + \int\limits_{\frac{1}{2}-\mu+iT}^{\frac{1}{2}-\mu-iT} + \int\limits_{\frac{1}{2}-\mu-iT}^{\mu-iT}\right\} \frac{\zeta_{p'}(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)} \left(\frac{x}{2\pi}\right)^{-s} ds = 2\pi i\, R_{p,p'}(x), \tag{6.2.7}$$

where $R_{p,p'}(x)$ denotes the sum of the residues of the integrand inside $\mathscr{R}_\mu(T)$. Since $\zeta_p(-2k)=0$ for every $k\in\mathbb{N}$ [[132], p. 22, eq. (37)], we know that the points $s=2k+1$, $k\in\mathbb{Z}\setminus\{0\}$ are removable singularities of the complex function under integration. Thus, the above integrand has only three simple poles, which are located at the points $s=0,\ \frac{1}{2},\ 1$. By a simple computation of their residues, one may see that $R_{p,p'}(x)$ can be explicitly written as

$$R_{p,p'}(x) = \frac{1}{4\left(1+\frac{1}{\pi p}\right)} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right) + \frac{\pi^2}{6x} \frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}, \tag{6.2.8}$$

because, just like Riemann's $\zeta-$function, $\zeta_p(s)$ possesses the particular values [[132], p. 22, eqs. (34), (39)][12]

$$\zeta_p(0) = -\frac{1}{2}\frac{1}{1+\frac{1}{\pi p}}, \qquad \zeta_p(2) = \frac{\pi^2}{6}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2}.$$

We now estimate in an elementary way the integrals along the horizontal segments, $\left[\frac{1}{2}-\mu\pm iT,\ \mu\pm iT\right]$, when $T\to\infty$: indeed,

$$\int\limits_{\frac{1}{2}-\mu\pm iT}^{\mu\pm iT} \left|\frac{\zeta_p(1-s)\,\zeta_p(2s)}{2\cos\left(\frac{\pi s}{2}\right)}\left(\frac{x}{2\pi}\right)^{-s}\right| |ds| \leq \frac{T^A e^{-\frac{\pi}{2}T}}{\log\left(\frac{x}{2\pi}\right)}\left[\left(\frac{x}{2\pi}\right)^{\mu-\frac{1}{2}} - \left(\frac{x}{2\pi}\right)^{-\mu}\right], \tag{6.2.9}$$

which clearly tends to zero as $T\to\infty$. Here, $A:=A(\mu)$ is a positive number depending on μ: we can write it explicitly by using the convex estimates for $\zeta_p(s)$ found by Koshliakov [[132], p. 24, eq. (44)],

$$\zeta_p(\sigma+it) = \begin{cases} O\left(|t|^{1-\sigma}\log|t|\right) & \frac{1}{2}\leq\sigma\leq 1, \\ O\left(|t|^{\frac{1}{2}}\log|t|\right) & 0<\sigma\leq\frac{1}{2}, \quad |t|\to\infty. \\ O\left(|t|^{\frac{1}{2}-\sigma}\log|t|\right) & \sigma\leq 0, \end{cases} \tag{6.2.10}$$

[12] which, of course, reduce to the well-known particular values $\zeta(0)=-\frac{1}{2}$ and $\zeta(2)=\frac{\pi^2}{6}$, as long as $p\to\infty$.

Combining (6.2.6) with (6.2.7) and using the previous bounds (6.2.9) for the integrals along the horizontal segments, we are able to obtain the intermediate formula

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right)e^{\lambda_n^2 x}-1}=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)\,\zeta_{p'}(s+\frac{1}{2})}{\sqrt{2}\left[\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)\right]}\left(\frac{x}{2\pi}\right)^{s-\frac{1}{2}}ds$$
$$+\frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)+\frac{\pi^2}{6x}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}, \tag{6.2.11}$$

where the last step is just a consequence of the change of variables $s\leftrightarrow\frac{1}{2}-s$ and the explicit expression (6.2.8). Since we have chosen $\mu>1$ at the beginning of this argument, we can use the representation of $\zeta_{p'}\left(s+\frac{1}{2}\right)$ as a Dirichlet series, (6.1.12), in order to write the integral on the right side of (6.2.11) as

$$\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\sqrt{\frac{\pi}{\lambda_n' x}}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)}{\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)}\left(\frac{x}{2\pi\lambda_n'}\right)^{s}ds. \tag{6.2.12}$$

Finally, we evaluate the remaining integral by using the functional equation for $\zeta_p(s)$, (6.1.17), and making some elementary reductions, which yield

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{\zeta_p(1-2s)}{\cos\left(\frac{\pi s}{2}\right)+\sin\left(\frac{\pi s}{2}\right)}\left(\frac{x}{2\pi\lambda_n'}\right)^{s}ds$$
$$=\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\Gamma(s)\,(s,\,2\pi pm)_m\left(\frac{x}{8\pi^3 m^2\lambda_n'}\right)^{s/2}ds. \tag{6.2.13}$$

The last equality is totally justified by the absolute convergence of the zeta function $\eta_p(s)$ in the half-plane $\mathrm{Re}(s)>2\mu$ (see (6.1.16) above). Combining (6.1.21) with Stirling's formula for $\Gamma(s)$, when we write $s=\sigma+it$, $a\le\sigma\le b$, and take $z\in\mathbb{C}$, we find the inequality

$$\left|\Gamma(s)\,(s,\lambda)_k\,z^{-s}\right|\le e^{-\sigma\log|z|}\,C(\sigma,k,\lambda)\,|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|+\arg(z)\,t},\quad |t|\gg 1,$$

where $C(\sigma,k,\lambda)$ is a positive constant only depending on the fixed parameters $\sigma,\ k$ and λ. Since $\Gamma(s)\,(s,\lambda)_k\,z^{-s}$ defines an analytic function in the region $\mathrm{Re}(s)>0$, we have from (6.1.15), Mellin's inversion formula and an argument by analytic continuation in the half-plane $\mathrm{Re}(z)>0$, that the following integral representation must be valid

$$f_k(z,\lambda):=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\,(s,\lambda)_k\,z^{-s}ds=\left(\frac{\lambda-z}{\lambda+z}\right)^{k}e^{-z},\quad \sigma>0,\ \mathrm{Re}(z)>0. \tag{6.2.14}$$

Applying the previous equality (6.2.12) and using (6.2.13) along with (6.2.14), we immediately obtain the formula

$$\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\Gamma(s)\,(s,\,2\pi pm)_{m}\left(\frac{x}{8\pi^{3}m^{2}\lambda_{n}^{\prime}}\right)^{s/2}$$
$$=\sqrt{2}\,\mathrm{Im}\left\{e^{i\frac{\pi}{4}}\sum_{m=1}^{\infty}\exp\left(-\sqrt{\frac{8\pi^{3}m^{2}\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}\right)\left(\frac{p-\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}}{p+\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}}\right)^{m}\right\}$$
$$=\sqrt{2}\,\mathrm{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}\right)e^{2\pi\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}}-1}\right\}, \tag{6.2.15}$$

where in the last step we have used the geometric series (recall once more (6.1.23))

$$\frac{1}{\sigma(z)\,e^{2\pi z}-1}=\sum_{m=1}^{\infty}\left(\frac{p-z}{p+z}\right)^{m}e^{-2\pi mz},\quad \mathrm{Re}(z)>0. \tag{6.2.16}$$

Returning to (6.2.11) and to (6.2.12), and collecting (6.2.15) in the way, we are able to derive the identity

$$\sum_{n=1}^{\infty}\frac{p^{2}+\lambda_{n}^{2}}{p\left(p+\frac{1}{\pi}\right)+\lambda_{n}^{2}}\frac{1}{\sigma^{\prime}\left(\frac{\lambda_{n}^{2}x}{2\pi}\right)e^{\lambda_{n}^{2}x}-1}=\frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p^{\prime}}\left(\frac{1}{2}\right)+\frac{\pi^{2}}{6x}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p^{\prime}}\right)\left(1+\frac{1}{\pi p}\right)^{2}}$$
$$+\sqrt{\frac{2\pi}{x}}\sum_{n=1}^{\infty}\frac{p^{\prime 2}+\lambda_{n}^{\prime 2}}{p^{\prime}\left(p^{\prime}+\frac{1}{\pi}\right)+\lambda_{n}^{\prime 2}}\frac{1}{\sqrt{\lambda_{n}^{\prime}}}\,\mathrm{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}\right)e^{2\pi\sqrt{\frac{2\pi\lambda_{n}^{\prime}}{x}}\,e^{i\frac{\pi}{4}}}-1}\right\}. \tag{6.2.17}$$

Our proof is nearly finished, with the only step remaining being to simplify the imaginary part of the term in the braces at (6.2.17). This comes after straightforward manipulations: it is very simple to show that the denominator of the fraction inside the braces in (6.2.17) can be simplified to (here, we put $a:=\sqrt{2\pi\lambda_{n}^{\prime}/x}$)

$$\frac{e^{\pi\sqrt{2a}}}{p^{2}-\sqrt{2a}\,p+a}\left\{\cos\left(\pi\sqrt{2a}\right)\left(p^{2}-a\right)-\sqrt{2a}p\sin\left(\pi\sqrt{2a}\right)\right.$$
$$\left.+\,i\left[\sqrt{2a}p\cos\left(\pi\sqrt{2a}\right)+\left(p^{2}-a\right)\sin\left(\pi\sqrt{2a}\right)\right]\right\}-1, \tag{6.2.18}$$

and, from this, we can write the expression

$$\left|\sigma\left(\sqrt{a}\,e^{i\frac{\pi}{4}}\right)e^{2\pi\sqrt{a}\,e^{i\frac{\pi}{4}}}-1\right|^{2}=\frac{e^{2\pi\sqrt{2a}}\left(p^{2}+\sqrt{2a}\,p+a\right)}{p^{2}-\sqrt{2a}\,p+a}$$
$$-\frac{2\,e^{\pi\sqrt{2a}}}{p^{2}-\sqrt{2a}\,p+a}\left\{\cos(\pi\sqrt{2a})\left(p^{2}-a\right)-\sqrt{2a}\,p\,\sin(\pi\sqrt{2a})\right\}+1. \tag{6.2.19}$$

Taking the conjugate of (6.2.18) and using (6.2.19), we have after routine algebraic simplifications that

$$\operatorname{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{a}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{a}\,e^{i\frac{\pi}{4}}}-1}\right\}=\frac{G_p(a)}{2\sqrt{2}}, \tag{6.2.20}$$

where $G_p(a)$ is the functions defined by (6.2.3), (6.2.4) and (6.2.5). This now completes the proof of our extension of Ramanujan's formula for $\zeta(1/2)$. □

Remark 6.2.1. Of course, a more direct way to present the function $G_p(a)$, (6.2.3), is by writing it as the fraction

$$\frac{(p^2-a)\left\{\cos(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})\right\}-e^{-\pi\sqrt{2a}}\left(p^2-\sqrt{2a}\,p+a\right)-\sqrt{2a}\,p\left\{\cos(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right\}}{p^2\left(\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})\right)+\sqrt{2a}\,p\left(\sinh(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right)+a\left(\cosh(\pi\sqrt{2a})+\cos(\pi\sqrt{2a})\right)},$$

which, in the classical limiting case $p\to\infty$ (and setting $a:=2\pi n/x$) should yield the expression

$$\frac{\cos(2\pi\sqrt{\pi n/x})-\sin(2\pi\sqrt{\pi n/x})-e^{-2\pi\sqrt{\pi n/x}}}{\cosh(2\pi\sqrt{\pi n/x})-\cos(2\pi\sqrt{\pi n/x})},$$

which appears in the classical Ramanujan formula (6.1.1). The next corollaries explore several reduction formulas of the above general fraction, depending on the values for $a:=2\pi\lambda_n'/x$ and p.[13]

We will now derive particular cases of our generalization of Entry 8.3.1, (6.2.2), recovering some results previously known in the literature while also uncovering new insights. Since the proofs of all the corollaries below are nothing but a specialization of (6.2.2) as p or p' tend to 0^+ or ∞, we will omit them. We start with the following result, which can be obtained by letting $p'\to\infty$ in our previous Theorem. It offers an infinite number of representations for $\zeta(1/2)$, with one of them being, of course, Ramanujan's formula (6.1.1).

Corollary 6.2.1. *For any $p\in\mathbb{R}^+$ and $x>0$, we have the representation for $\zeta(1/2)$,*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{e^{\lambda_n^2 x}-1}=\frac{1}{4\left(1+\frac{1}{\pi p}\right)}$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\zeta\left(\frac{1}{2}\right)+\frac{\pi^2}{6x}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{G_p\left(\frac{2\pi n}{x}\right)}{\sqrt{n}},$$

[13]Note that, when we write $a:=2\pi\lambda_n'/x$, we are already saying that a depends on p', because λ_n' is defined implicitly by (6.2.1).

where $G_p(a)$, $a>0$, is defined by (6.2.3), (6.2.4) and (6.2.5). In particular, Ramanujan's formula (6.1.1) holds, together with the identity

$$\sum_{n=1}^{\infty}\frac{1}{e^{(2n-1)^2x}-1}=\frac{1}{4}\sqrt{\frac{\pi}{x}}\,\zeta\left(\frac{1}{2}\right)+\frac{\pi^2}{8x}$$
$$-\frac{1}{4}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\,\frac{\cos\left(\pi\sqrt{\frac{\pi n}{x}}\right)-\sin\left(\pi\sqrt{\frac{\pi n}{x}}\right)+e^{-\pi\sqrt{\frac{\pi n}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi n}{x}}\right)+\cos\left(\pi\sqrt{\frac{\pi n}{x}}\right)},\qquad x>0.$$

If, instead, we take $p\to\infty$ and fix p' in (6.2.2), we are then able to get to the following result.

Corollary 6.2.2. *For any $p'\in\mathbb{R}^+$ and $x>0$, the following representation of $\zeta_{p'}(1/2)$ holds*

$$\sum_{n=1}^{\infty}\frac{1}{\sigma'\left(\frac{n^2x}{2\pi}\right)e^{n^2x}-1}=\frac{1}{4}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)+\frac{\pi^2}{6x}\,\frac{1}{1+\frac{1}{\pi p'}}$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{\left(p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2\right)\sqrt{\lambda_n'}}\,\frac{\cos\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\sin\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-e^{-2\pi\sqrt{\frac{\pi\lambda_n'}{x}}}}{\cosh\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\cos\left(2\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)}.$$

In particular, we have a companion of Ramanujan's formula

$$-\sum_{n=1}^{\infty}\frac{1}{e^{n^2x}+1}=\frac{1}{4}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\left(\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right)$$
$$+\sqrt{\frac{\pi}{2x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}\left\{\frac{\cos\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-\sin\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-e^{-\pi\sqrt{\frac{\pi(4n-2)}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)-\cos\left(\pi\sqrt{\frac{\pi(4n-2)}{x}}\right)}\right\}.$$

If we let $p'\to 0^+$ and fix $p\in\mathbb{R}^+$ in our Theorem 6.2.1, we are still able to find alternative representations for $\zeta\left(\frac{1}{2}\right)$. These are given in the following corollary.

Corollary 6.2.3. *For any $p\in\mathbb{R}^+$ and $x>0$, we have the representation for $\zeta(1/2)$,*

$$-\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{e^{\lambda_n^2x}+1}=\frac{1}{4\left(1+\frac{1}{\pi p}\right)}$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\left(\sqrt{2}-1\right)\zeta\left(\frac{1}{2}\right)+\sqrt{\frac{\pi}{2x}}\sum_{n=1}^{\infty}\frac{G_p\left(\frac{\pi(2n-1)}{x}\right)}{\sqrt{2n-1}},$$

where $G_p(a)$, $a>0$, is given by (6.2.3), (6.2.4) and (6.2.5). In particular, the following relation, analogous to Ramanujan's, takes place

$$\sum_{n=1}^{\infty}\frac{1}{e^{(2n-1)^2x}+1}=\frac{1}{4}\sqrt{\frac{\pi}{x}}\left(1-\sqrt{2}\right)\zeta\left(\frac{1}{2}\right)$$
$$+\frac{1}{4}\sqrt{\frac{2\pi}{x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n-1}}\frac{\cos\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)-\sin\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)+e^{-\pi\sqrt{\frac{\pi(2n-1)}{2x}}}}{\cosh\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)+\cos\left(\pi\sqrt{\frac{\pi(2n-1)}{2x}}\right)}.$$

Finally, we state the corollary that we obtain once we take $p\to 0^+$ in our Theorem 6.2.1.

Corollary 6.2.4. *For any $p'\in\mathbb{R}^+$ and $x>0$, the following identity holds*

$$\sum_{n=1}^{\infty}\frac{1}{\sigma'\left(\frac{(2n-1)^2x}{2\pi}\right)e^{(2n-1)^2x}-1}=\frac{\pi^2}{8x}\frac{1}{1+\frac{1}{\pi p'}}+\frac{1}{4}\sqrt{\frac{\pi}{x}}\zeta_{p'}\left(\frac{1}{2}\right)$$
$$+\frac{1}{4}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\frac{1}{\sqrt{\lambda_n'}}\left\{\frac{\sin\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-\cos\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)-e^{-\pi\sqrt{\frac{\pi\lambda_n'}{x}}}}{\cosh\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)+\cos\left(\pi\sqrt{\frac{\pi\lambda_n'}{x}}\right)}\right\}.$$

Our proof of Theorem 6.2.1 relied on the classical theory of Mellin transforms. This method of proving summation formulas was somewhat popularized by Ferrar [90], more than ten years after Ramanujan's death. Therefore, it seems very unlikely that Ramanujan's original proof used the same kind of ideas. In fact, we have some evidence about the method used by Ramanujan to get (6.1.1). He indicated that the equality [[31], p. 124, eq. (8.2)]

$$\sum_{n=1}^{\infty}\frac{n^2}{a^2+n^4}=\frac{\pi}{2\sqrt{2a}}\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})},\quad a>0, \tag{6.2.21}$$

played a fundamental role in his argument. The interesting identity (6.2.21) is prior to Ramanujan, and it seems to appear for the first time in a paper of Glaisher [94]. Ramanujan's suggestion for a proof of (6.1.1) consisted of a combination of (6.2.21) and

$$\sum_{n=1}^{\infty}\frac{1}{e^{n^2x}-1}=\frac{1}{4}+\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\zeta\left(\frac{1}{2}\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left\{4\sqrt{\frac{n}{\pi x}}\sum_{m=1}^{\infty}\frac{m^2}{m^4+\left(\frac{2\pi n}{x}\right)^2}-1\right\}. \tag{6.2.22}$$

Berndt and Evans [31] prove (6.2.22) by using the Poisson summation formula and a classical integral representation for $\zeta(s)$ in the strip $0<\mathrm{Re}(s)<1$. In the next corollary, we reverse Ramanujan's reasoning and prove the "auxiliary identity", (6.2.22), as a consequence of our Theorem 6.2.1.

Corollary 6.2.5. *Assume the same conditions as in Theorem 6.2.1. Then, for any $x>0$, the following identity holds*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\sigma'\left(\frac{\lambda_n^2 x}{2\pi}\right)e^{\lambda_n^2 x}-1}$$
$$=\frac{1}{4\left(1+\frac{1}{\pi p}\right)}+\frac{\pi^2}{6x}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\zeta_{p'}\left(\frac{1}{2}\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{\left(p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2\right)\sqrt{\lambda_n'}}\left\{4\sqrt{\frac{\lambda_n'}{\pi x}}\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{\lambda_m^2}{\lambda_m^4+\left(\frac{2\pi\lambda_n'}{x}\right)^2}-1\right\}.$$

Proof. According to Theorem 6.2.1, it is enough to show that, for any $a>0$,

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\lambda_n^2}{a^2+\lambda_n^4}=\frac{\pi}{2\sqrt{2a}}+\frac{\pi}{2\sqrt{2a}}G_p(a), \tag{6.2.23}$$

with $G_p(a)$ being the function described by (6.2.3). A more explicit form of writing (6.2.23) is

$$\frac{(p^2-a)\left\{\cos(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})\right\}-e^{-\pi\sqrt{2a}}\left(p^2-\sqrt{2a}\,p+a\right)-\sqrt{2a}\,p\left\{\cos(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right\}}{p^2\left(\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})\right)+\sqrt{2a}\,p\left(\sinh(\pi\sqrt{2a})+\sin(\pi\sqrt{2a})\right)+a\left(\cosh(\pi\sqrt{2a})+\cos(\pi\sqrt{2a})\right)}$$
$$=\frac{2\sqrt{2a}}{\pi}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\lambda_n^2}{a^2+\lambda_n^4}-1. \tag{6.2.24}$$

The proof of the formula (6.2.23) will employ the theory of residues, rather than the functional equation for $\zeta_p(s)$. The method to get this summation formula is completely analogous to the one employed in the proof of Theorem 5.3.1 on the previous chapter.[14] For brevity, we write

$$f(z):=\frac{z^2}{a^2+z^4}$$

and define $g(z)$ as the complex function

$$g(z):=\frac{f(z)}{\sigma(iz)\,e^{2\pi iz}-1}.$$

We extend the Koshliakov sequence $(\lambda_n)_{n\in\mathbb{N}}$ to negative indices by taking[15] $\lambda_{-n}:=-\lambda_n$ and $\lambda_0:=0$. It is straightforward to check that $g(z)$ is an analytic function everywhere except at the points[16] $(\lambda_n)_{n\in\mathbb{Z}}$ and at $\pm e^{\frac{i\pi}{4}}\sqrt{a}$, $\pm e^{\frac{3\pi i}{4}}\sqrt{a}$, the latter being the simple poles of $f(z)$. Using the

[14]We could also prove Corollary 6.2.5 by using Koshliakov's analogue of the Poisson summation formula [[132], p. 112], but we follow here the "minimal" conventions of the previous chapter.

[15]Note that y satisfies the transcendental equation (6.1.13) if and only if $-y$ also satisfies it. Therefore, this convention is well motivated.

[16]It is straightforward to see that $\sigma(iz)\,e^{2\pi iz}=1$ only when $z=0,\pm\lambda_1,\pm\lambda_2,\ldots$.

basic formula for computing the residue of a complex function, one has that

$$\operatorname{Res}_{z=\lambda_n} g(z) = \frac{1}{2\pi i} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} f(\lambda_n).$$

Therefore, by Cauchy's residue theorem,

$$\sum_{|n|\leq N} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} f(\lambda_n) = \int_{\gamma_N} \frac{f(z)}{\sigma(iz)\, e^{2\pi iz}-1}\, dz, \tag{6.2.25}$$

where γ_N is the positively oriented rectangular contour with vertices $-N-\frac{1}{2}\pm i\xi$ and $N+\frac{1}{2}\pm i\xi$, for $N\in\mathbb{N}$ and $0<\xi<\sqrt{a}$. Letting N tend to infinity and using the parity of $f(z)=\frac{z^2}{a^2+z^4}$, together with the convention $\lambda_{-n}=-\lambda_n$, we see that (6.2.25) implies

$$2\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} f(\lambda_n) = \int_{\Gamma_1^-(\xi)} \frac{f(z)}{\sigma(iz)\, e^{2\pi iz}-1}\, dz - \int_{\Gamma_1^+(\xi)} \frac{f(z)}{\sigma(iz)\, e^{2\pi iz}-1}\, dz, \tag{6.2.26}$$

where $\Gamma_1^{\pm}(t):=t\pm i\xi,\ \ t\in\mathbb{R}$. Note that, on $\Gamma_1^-(\xi)$, $|e^{2\pi iz}|>1$ and $|\sigma(iz)|>1$, so that, for any $z\in\Gamma_1^-(\xi)$ (see (6.2.16) above),

$$\frac{1}{\sigma(iz)\, e^{2\pi iz}-1} = \sum_{k=1}^{\infty}\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz}.$$

Meanwhile, if $z\in\Gamma_1^+(\xi)$, then $|e^{2\pi iz}|<1$ and $|\sigma(iz)|<1$, which gives the alternative expansion

$$\frac{1}{\sigma(iz)\, e^{2\pi iz}-1} = -\sum_{k=0}^{\infty}\left(\frac{p+iz}{p-iz}\right)^k e^{2\pi ik\, z} = -\sum_{k\leq 0}\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz}.$$

Combining both representations with (6.2.26), we find, for any $0<\xi<\sqrt{a}$,

$$\begin{aligned}
&\int_{\Gamma_1^-(\xi)} \frac{f(z)}{\sigma(iz)\, e^{2\pi iz}-1}\, dz - \int_{\Gamma_1^+(\xi)} \frac{f(z)}{\sigma(iz)\, e^{2\pi iz}-1}\, dz \\
&= \int_{\Gamma_1^-(\xi)} \sum_{k=1}^{\infty}\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz} f(z)\, dz + \int_{\Gamma_1^+(\xi)} \sum_{k\leq 0}\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz} f\,(z)\ dz \\
&= \sum_{k=1}^{\infty} \int_{\Gamma_1^-(\xi)} \left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz} f(z)\, dz + \sum_{k\leq 0} \int_{\Gamma_1^+(\xi)} \left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz} f(z)\, dz,
\end{aligned} \tag{6.2.27}$$

where the interchange of the series with the integrals can be justified via Fubini's Theorem, because

$$\sum_{k=1}^{\infty}\int_{\Gamma_1^-(\xi)}\left|\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz}f(z)\right||dz|\leq\sum_{k=1}^{\infty}e^{-2\pi k\xi}\int_{-\infty}^{\infty}\left(\frac{(p-\xi)^2+t^2}{(p+\xi)^2+t^2}\right)^{k/2}|f\,(t+i\xi)|\,dt$$
$$\leq\frac{1}{e^{2\pi\xi}-1}\int_{-\infty}^{\infty}\frac{dt}{t^2+\xi^2}.$$

A similar justification holds for the second series in (6.2.27). Therefore, by (6.2.26), (6.2.27) and Cauchy's theorem applied to the functions[17]

$$h_k(z)=\left(\frac{p-iz}{p+iz}\right)^k e^{-2\pi ikz}f(z),$$

we are able to deduce the equality

$$2\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\lambda_n^2}{a^2+\lambda_n^4}$$
$$=\frac{\pi}{\sqrt{2a}}+\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{a^2+x^4}\left\{\left(\frac{p+ix}{p-ix}\right)^k e^{2\pi ikx}+\left(\frac{p-ix}{p+ix}\right)^k e^{-2\pi ikx}\right\}dx. \tag{6.2.28}$$

We now evaluate the Fourier integrals at the right-hand side of (6.2.28) using the residue theorem. Indeed, let $C_{R^{\pm}}$ be the half-circle defined by $C_{R^{\pm}}(\theta):=R\,e^{\pm i\theta}$, $\theta\in(0,\pi)$, and consider the closed path $\gamma_{R^{\pm}}=[-R,R]\cup C_{R^{\pm}}$, $R>\sqrt{a}$. When its argument lies inside the contour γ_{R^+}, the function

$$f_1(z)=\frac{z^2}{a^2+z^4}\left(\frac{p+iz}{p-iz}\right)^k e^{2\pi ikz},\quad \text{fixed } k\in\mathbb{N},$$

has two simple poles located at $z_1=e^{\frac{i\pi}{4}}\sqrt{a}$ and $z_2=e^{\frac{3\pi i}{4}}\sqrt{a}$, because $p\geq 0$ by hypothesis. Moreover, we can bound trivially the integral over the boundary C_{R^+} as

$$\left|\int_{C_{R^+}}f_1(z)\,dz\right|\leq\int_0^{\pi}\left|\frac{z^2}{a^2+z^4}\right|e^{-2\pi kR\sin(\theta)}\,R\,d\theta\leq\frac{\pi R^3}{R^4-a^2}, \tag{6.2.29}$$

which clearly tends to zero as $R\to\infty$. On the other hand, it is straightforward to compute the residues of $f_1(z)$, which are explicitly

$$\operatorname{Res}_{z=\sqrt{a}e^{\frac{i\pi}{4}}}f_1(z)=\frac{e^{-i\frac{\pi}{4}}}{4\sqrt{a}}\left(\frac{p-\sqrt{a}e^{-\frac{\pi i}{4}}}{p+\sqrt{a}e^{-\frac{\pi i}{4}}}\right)^k\exp\left(-2\pi k\sqrt{a}\,e^{-\frac{\pi i}{4}}\right)$$

and

$$\operatorname{Res}_{z=\sqrt{a}e^{\frac{3\pi i}{4}}}f_1(z)=-\frac{e^{\frac{i\pi}{4}}}{4\sqrt{a}}\left(\frac{p-\sqrt{a}e^{\pi i/4}}{p+\sqrt{a}e^{\pi i/4}}\right)^k\exp\left(-2\pi k\sqrt{a}e^{\pi i/4}\right).$$

[17] note that, if $k\geq 1$, then $h_k(z)$ is analytic in the strip $-\xi\leq\operatorname{Im}(z)\leq 0$. If $k\leq 0$, it is also straightforward to see that $h_k(z)$ is analytic in the strip $0\leq\operatorname{Im}(z)\leq\xi$.

Once more, applying Cauchy's residue theorem to the contour γ_{R^+}, invoking (6.2.29) and letting $R \to \infty$, we get the evaluation of the Fourier transform

$$\frac{2\sqrt{a}}{\pi i}\int_{-\infty}^{\infty}\frac{x^2}{a^2+x^4}\left(\frac{p+ix}{p-ix}\right)^k e^{2\pi ikx}dx = e^{-i\frac{\pi}{4}}\left(\frac{p-\sqrt{a}e^{-\frac{\pi i}{4}}}{p+\sqrt{a}e^{-\frac{\pi i}{4}}}\right)^k \exp\left(-2\pi k\sqrt{a}e^{-\frac{\pi i}{4}}\right)$$
$$- e^{\frac{i\pi}{4}}\left(\frac{p-\sqrt{a}e^{\pi i/4}}{p+\sqrt{a}e^{\pi i/4}}\right)^k \exp\left(-2\pi k\sqrt{a}e^{\frac{\pi i}{4}}\right).$$

Therefore, summing over k and using (6.2.16) and (6.2.20), we find that

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{a^2+x^4}\left(\frac{p+ix}{p-ix}\right)^k e^{2\pi ikx}dx\\
&= -\frac{\pi i}{2\sqrt{a}}\left\{\frac{e^{\frac{i\pi}{4}}}{\sigma\left(\sqrt{a}e^{\frac{i\pi}{4}}\right)e^{2\pi\sqrt{a}e^{i\pi/4}}-1} - \frac{e^{-\frac{i\pi}{4}}}{\sigma\left(\sqrt{a}e^{-\frac{i\pi}{4}}\right)e^{2\pi\sqrt{a}e^{-i\pi/4}}-1}\right\}\\
&= \frac{\pi}{\sqrt{a}}\operatorname{Im}\left\{\frac{e^{\frac{i\pi}{4}}}{\sigma\left(\sqrt{a}e^{\frac{i\pi}{4}}\right)e^{2\pi\sqrt{a}e^{i\pi/4}}-1}\right\} = \frac{\pi G_p(a)}{2\sqrt{2a}}.\end{aligned} \tag{6.2.30}$$

Analogously, by choosing the contour γ_{R^-}, one can also show the equality

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{a^2+x^4}\left(\frac{p-ix}{p+ix}\right)^k e^{-2\pi ikx}dx = \frac{\pi G_p(a)}{2\sqrt{2a}}. \tag{6.2.31}$$

At last, combining (6.2.31) with (6.2.30) and (6.2.28) yields (6.2.23). □

Remark 6.2.2. When $p \to \infty$, (6.2.23) gives, of course, (6.2.21). On the other hand, when we take $p \to 0^+$ we get

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2}{a^2+(2n-1)^4} = \frac{\pi}{4\sqrt{2a}}\,\frac{\sin(\pi\sqrt{\frac{a}{2}})+\sinh(\pi\sqrt{\frac{a}{2}})}{\cosh(\pi\sqrt{\frac{a}{2}})+\cos(\pi\sqrt{\frac{a}{2}})}, \quad a>0.$$

If we use the relations (cf. [31])

$$\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{1}{e^{\lambda_m^2 x}-1} = \sum_{m,n=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,e^{-\lambda_m^2 nx}$$

and

$$\int_0^{\infty}\sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,e^{-\lambda_m^2 x}\cos(ax)\,dx = \sum_{m=1}^{\infty}\frac{p^2+\lambda_m^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_m^2}\,\frac{\lambda_m^2}{a^2+\lambda_m^4},$$

we can furnish an alternative proof of Corollary 6.2.1 by using (6.2.23) and the classical Poisson summation formula (because the second variable of summation in (6.2.23) runs over the integers). The resulting argument would now be the same as the one given by Berndt and Evans [31]. A proof of Corollary 6.2.3 could also be achieved in this way.

By the functional equation (6.1.17), it is immediate to see that $\zeta_p\left(\frac{1}{2}\right)=\eta_p\left(\frac{1}{2}\right)$, and so (6.2.2) can also be regarded as an analogue of Ramanujan's formula for $\eta_{p'}\left(\frac{1}{2}\right)$. Hence, one may ask if there is an identity of the form (6.2.2) involving the generalized Hurwitz zeta function (6.1.18). The following theorem provides a positive answer to this.

Theorem 6.2.2. *Assume the same conditions as in Theorem 6.2.1. Then, for any* $0<a<1$ *and* $x>0$*, the following identity holds*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{e^{-a\lambda_n^2 x}}{1-\sigma'\left(-\frac{\lambda_n^2 x}{2\pi}\right)e^{-\lambda_n^2 x}}$$
$$=\frac{1}{2\left(1+\frac{1}{\pi p}\right)}\left\{\frac{a}{1+\frac{1}{\pi p'}}-\frac{1}{2}\right\}+\frac{\pi^2}{6x}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2\left(1+\frac{1}{\pi p'}\right)}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\eta_{p'}\left(\frac{1}{2},a\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\frac{\cos\left(2\pi\lambda_n' a\right)}{\sqrt{\lambda_n'}}G_p\left(\frac{2\pi\lambda_n'}{x}\right)$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\frac{\sin\left(2\pi\lambda_n' a\right)}{\sqrt{\lambda_n'}}\,H_p\left(\frac{2\pi\lambda_n'}{x}\right), \tag{6.2.32}$$

where, for $y>0$, $G_p(y)$ *is the function given by (6.2.3), (6.2.4) and (6.2.5). Moreover,* $H_p(y)$ *is the function defined as*

$$H_p(y)=\frac{H_{1,p}(y)}{H_{2,p}(y)}, \tag{6.2.33}$$

where $H_{1,p}(y)$ *and* $H_{2,p}(y)$ *are explicitly*

$$H_{1,p}(y)=\left(p^2-y\right)\left(\cos(\pi\sqrt{2y})+\sin(\pi\sqrt{2y})\right)$$
$$-e^{-\pi\sqrt{2y}}\left(p^2-\sqrt{2y}\,p+y\right)-\sqrt{2y}\,p\left(\sin\left(\pi\sqrt{2y}\right)-\cos\left(\pi\sqrt{2y}\right)\right),$$

and

$$H_{2,p}(y)=p^2\left(\cosh(\pi\sqrt{2y})-\cos(\pi\sqrt{2y})\right)$$
$$+\sqrt{2y}\,p\left(\sinh(\pi\sqrt{2y})+\sin(\pi\sqrt{2y})\right)+y\left(\cosh(\pi\sqrt{2y})+\cos(\pi\sqrt{2y})\right).$$

Proof. Since the proof that follows contains the same ideas as the proof of our Theorem 6.2.1, we will be brief. We start with the representation,

$$\eta_p(s,a)=\frac{(2\pi)^s}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-2\pi a t}}{1-\sigma(-t)e^{-2\pi t}}dt,\quad 0<a<1,\quad \mathrm{Re}(s)>1,$$

which can be immediately obtained from the definition of $\eta_p(s,a)$. Therefore, from Mellin's inversion formula and the absolute convergence of the series for $\zeta_p(s)$, together with Cauchy's residue theorem and the estimates given in (6.2.10) [18], one finds that, whenever $\mu > 1$,

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{e^{-a\lambda_n^2 x}}{1-\sigma'\left(-\frac{\lambda_n^2 x}{2\pi}\right)\,e^{-\lambda_n^2 x}}=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(s)\,\eta_{p'}\left(s,a\right)\zeta_p(2s)\,x^{-s}ds$$
$$=\frac{1}{2\pi i}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(s)\,\eta_{p'}\left(s,a\right)\zeta_p(2s)\,x^{-s}ds+\frac{1}{2\left(1+\frac{1}{\pi p}\right)}\left\{\frac{a}{1+\frac{1}{\pi p'}}-\frac{1}{2}\right\}$$
$$+\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\eta_{p'}\left(\frac{1}{2},a\right)+\frac{\pi^2}{6x}\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2\left(1+\frac{1}{\pi p'}\right)}. \tag{6.2.34}$$

The computation of the residues of the integrand essentially followed (6.2.8) and the fact that $\text{Res}_{s=1}\left\{\eta_{p'}(s,a)\right\}=\frac{1}{1+\frac{1}{\pi p'}}$ and $\eta_{p'}(0,a)=\frac{1}{2}-\frac{a}{1+\frac{1}{\pi p'}}$ [[132], p. 108, eqs. (17), (18)]. In the meantime, we can use the functional equation for $\zeta_p(2s)$, (6.1.17), on the integral along the line $\text{Re}(s)=\frac{1}{2}-\mu$, as well as the transformation formula (6.1.19) for $\eta_p(s,a)$ (which is valid because $\text{Re}(s)=\frac{1}{2}-\mu<0$). After a standard change of variables and using the absolute convergence of both infinite series on the right-hand side of (6.1.19) when $\text{Re}(s)<0$, we get the expression

$$\sqrt{\frac{x}{\pi}}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(s)\,\eta_{p'}(s,a)\zeta_p(2s)\,x^{-s}ds$$
$$=\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\cos\left(2\pi\lambda_n' a\right)}{\sqrt{\lambda_n'}}\int_{2\mu-i\infty}^{2\mu+i\infty}\Gamma(s)\,\eta_p(s)\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\left(\frac{x}{8\pi^3\lambda_n'}\right)^{\frac{s}{2}}ds$$
$$+\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\sin\left(2\pi\lambda_n' a\right)}{\sqrt{\lambda_n'}}\int_{2\mu-i\infty}^{2\mu+i\infty}\Gamma(s)\,\eta_p(s)\left\{\cos\left(\frac{\pi s}{4}\right)+\sin\left(\frac{\pi s}{4}\right)\right\}\left(\frac{x}{8\pi^3\lambda_n'}\right)^{\frac{s}{2}}ds. \tag{6.2.35}$$

Once more, since $\eta_p(s)$ is defined as an absolutely convergent series in the half-plane $\text{Re}(s)>2\mu>2$, we can invoke the same steps as the ones that led us to (6.2.13). We note that the first infinite series on the right-hand side of (6.2.35) was already evaluated in the proof of Theorem 6.2.1. In fact, following (6.2.20),

$$\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\Gamma(s)\eta_p(s)\,\left\{\cos\left(\frac{\pi s}{4}\right)-\sin\left(\frac{\pi s}{4}\right)\right\}\left(\frac{x}{8\pi^3\lambda_n'}\right)^{\frac{s}{2}}\,ds$$
$$=\sqrt{2}\,\text{Im}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}\right)e^{2\pi\sqrt{\frac{2\pi\lambda_n'}{x}}e^{i\frac{\pi}{4}}}-1}\right\}=\frac{1}{2}G_p\left(\frac{2\pi\lambda_n'}{x}\right),$$

[18] which are also valid for $\eta_p(s,a)$, due to the Phragmén-Lindelöf principle and the formula (6.1.19).

where $G_p(y)$ is given by (6.2.3). Analogously, the use of (6.2.18) and (6.2.19) gives the evaluation of the second integral in the form

$$\frac{1}{2\pi i}\int_{2\mu-i\infty}^{2\mu+i\infty}\Gamma(s)\eta_p(s)\left\{\cos\left(\frac{\pi s}{4}\right)+\sin\left(\frac{\pi s}{4}\right)\right\}\left(\frac{x}{8\pi^3\lambda_n'}\right)^{\frac{s}{2}}ds$$

$$=\sqrt{2}\,\mathrm{Re}\left\{\frac{e^{i\frac{\pi}{4}}}{\sigma\left(\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}\right)\,e^{2\pi\sqrt{\frac{2\pi\lambda_n'}{x}}\,e^{i\frac{\pi}{4}}}-1}\right\}=\frac{1}{2}H_p\left(\frac{2\pi\lambda_n'}{x}\right),$$

where, for $y>0$, $H_p(y)$ is defined by (6.2.33). Returning to (6.2.35) and (6.2.34), we are able to finally deduce (6.2.32). □

Remark 6.2.3. Another way of writing the function $H_p(y)$, (6.2.33), is via the explicit fraction

$$\frac{(p^2-y)\,\{\cos(\pi\sqrt{2y})+\sin(\pi\sqrt{2y})\}-e^{-\pi\sqrt{2y}}\,(p^2-\sqrt{2y}\,p+y)-\sqrt{2y}\,p\,\{\sin(\pi\sqrt{2y})-\cos(\pi\sqrt{2y})\}}{p^2\,(\cosh(\pi\sqrt{2y})-\cos(\pi\sqrt{2y}))+\sqrt{2y}\,p\,(\sinh(\pi\sqrt{2y})+\sin(\pi\sqrt{2y}))+y\,(\cosh(\pi\sqrt{2y})+\cos(\pi\sqrt{2y}))},$$

which, in analogy to what we wrote in Remark 6.2.1, should yield interesting particular cases depending on the limiting cases for p and $a:=2\pi\lambda_n'/x$ (which depends on p').

In the case when both p and p' tend to infinity, our previous result implies an interesting summation formula given in [[72], p. 17, Theorem 2.13]. We now state the four particular cases that can be obtained from (6.2.32). When we take $p'\to 0^+$ or $p'\to\infty$, it is well known that (6.1.19) holds for $s=\frac{1}{2}$ (see Footnote 8). Therefore, in these limiting cases we can replace $\eta_{p'}\left(\frac{1}{2},a\right)$ by the conditionally convergent infinite series at the right-hand side of (6.1.19). The following result is the one that we get by letting p, $p'\to\infty$ in Theorem 6.2.2.

Corollary 6.2.6 ([[72], p. 17, Theorem 2.13])**.** *For any* $0<a<1$ *and* $x>0$*, the following identity holds*

$$\sum_{n=1}^{\infty}\frac{e^{-an^2x}}{1-e^{-n^2x}}=-\frac{1}{4}+\frac{a}{2}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{\sqrt{n}}\,\frac{\sinh\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)-\sin\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}{\cosh\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)-\cos\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}$$
$$+\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\sum_{n=1}^{\infty}\frac{\sin(2\pi na)}{\sqrt{n}}\,\frac{\sinh\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)+\sin\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}{\cosh\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)-\cos\left(2\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}.$$

In particular, when $a=\frac{1}{2}$,

$$\sum_{n=1}^{\infty}\frac{1}{\sinh(n^2x)}=\frac{\pi^2}{6x}+\sqrt{\frac{\pi}{2x}}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\,\frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n}{x}}\right)-\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n}{x}}\right)-\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n}{x}}\right)}.$$

The next corollary is deduced when we take $p\to\infty$ and $p'\to 0^+$. In particular, when $a=\frac{1}{2}$, it gives a seemingly new identity involving an infinite series containing $\mathrm{sech}(n^2x)$ (cf. [[72], p. 17, Remark 7] for an interesting observation concerning identities of this type).

Corollary 6.2.7. *For any $0 < a < 1$ and $x > 0$, the following identity holds*

$$\sum_{n=1}^{\infty} \frac{e^{-an^2x}}{1+e^{-n^2x}} = \sqrt{\frac{\pi}{2x}} \sum_{n=1}^{\infty} \frac{\cos(\pi(2n-1)a)}{\sqrt{2n-1}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)-\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)-\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)}$$
$$-\frac{1}{4}+\sqrt{\frac{\pi}{2x}} \sum_{n=1}^{\infty} \frac{\sin(\pi(2n-1)a)}{\sqrt{2n-1}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)+\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)-\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{4n-2}{x}}\right)}.$$

In particular, when $a = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \frac{1}{\cosh(n^2x)} = -\frac{1}{2}+\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n-1}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{x}}\right)+\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{x}}\right)-\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{x}}\right)}.$$

The next result is obtained when we take p, $p' \to 0^+$ in Theorem 6.2.2.

Corollary 6.2.8. *For any $0 < a < 1$ and $x > 0$, the following identity holds*

$$\sum_{n=1}^{\infty} \frac{e^{-a(2n-1)^2x}}{1+e^{-(2n-1)^2x}}$$
$$= \frac{1}{4}\sqrt{\frac{2\pi}{x}} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi a)}{\sqrt{2n-1}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)+\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)+\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)}$$
$$+\frac{1}{4}\sqrt{\frac{2\pi}{x}} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi a)}{\sqrt{2n-1}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)-\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)+\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{2n-1}{2x}}\right)}.$$

In particular, when $a = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \frac{1}{\cosh\left((2n-1)^2x\right)} = \frac{1}{2}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{2n-1}} \frac{\sinh\left(\frac{\pi^{\frac{3}{2}}}{2}\sqrt{\frac{2n-1}{x}}\right)-\sin\left(\frac{\pi^{\frac{3}{2}}}{2}\sqrt{\frac{2n-1}{x}}\right)}{\cosh\left(\frac{\pi^{\frac{3}{2}}}{2}\sqrt{\frac{2n-1}{x}}\right)+\cos\left(\frac{\pi^{\frac{3}{2}}}{2}\sqrt{\frac{2n-1}{x}}\right)}.$$

Finally, if we take $p \to 0^+$ and $p' \to \infty$ in (6.2.32), we achieve the last corollary of this section.

Corollary 6.2.9. *For any $0 < a < 1$ and $x > 0$, the following identity holds*

$$\sum_{n=1}^{\infty} \frac{e^{-a(2n-1)^2x}}{1-e^{-(2n-1)^2x}} = \frac{\pi^2}{8x}+\frac{1}{4}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\cos(2\pi na)}{\sqrt{n}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)+\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)+\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}$$
$$+\frac{1}{4}\sqrt{\frac{\pi}{x}} \sum_{n=1}^{\infty} \frac{\sin(2\pi na)}{\sqrt{n}} \frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)-\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)+\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{x}}\right)}.$$

In particular, when $a=\frac{1}{2}$,

$$\sum_{n=1}^{\infty}\frac{1}{\sinh\left((2n-1)^2x\right)}=\frac{\pi^2}{8x}+\frac{1}{2}\sqrt{\frac{\pi}{2x}}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}\,\frac{\sinh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{2x}}\right)+\sin\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{2x}}\right)}{\cosh\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{2x}}\right)+\cos\left(\pi^{\frac{3}{2}}\sqrt{\frac{n}{2x}}\right)}.$$

6.3 A Generalization of Watson's formula

In order to extend Watson's result (6.1.3) to Koshliakov's setting, we first need to generalize the series on the left-hand side of (6.1.3). For $x>0$ and $\mathrm{Re}(s)>\frac{1}{2}$, we will examine the following infinite series

$$\varphi_p(s,x):=\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s},\quad \mathrm{Re}(s)>\frac{1}{2}. \tag{6.3.1}$$

Note that, when $p\to\infty$, (6.3.1) reduces to the left-hand side of (6.1.3), so our hope is that $\varphi_p(s,x)$ satisfies a formula akin to (6.1.3). Such formula constitutes the content of our next result.

Theorem 6.3.1. *Let $x>0$ and Re$(s)>\frac{1}{2}$. Then the following generalization of Watson's formula (6.1.3) takes place*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}$$
$$+\frac{2\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}(-1)^m m^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi m x)+\frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}e^{2\pi p m}m^{s-\frac{1}{2}}$$
$$\times\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\left(4\pi m p\right)^{\ell}\int_1^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi m p t}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi x m t\right)\,dt. \tag{6.3.2}$$

In particular, for $\frac{1}{2}<$ Re$(s)<1$, one has the integral representation

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}$$
$$+2^{2-2s}\,x^{1-2s}\sin(\pi s)\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}dy, \tag{6.3.3}$$

with the right-hand side of the previous expression being the analytic continuation of the series (6.3.1) to the half-plane Re$(s)<\frac{1}{2}$.

Proof. Let $\mu>\frac{1}{2}$ and consider a fixed $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\mu$. Under this condition, the following integral representation holds [[85], p. 348, eq. 7.3.15]

$$\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,x^{-2z}dz=\frac{\Gamma(s)}{(1+x^2)^s},\quad x>0. \tag{6.3.4}$$

Using this Mellin-Barnes integral in the series defining $\varphi_p(s,x)$, (6.3.1), we have that

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{x^{-2s}}{\Gamma(s)}\,\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz,$$

where the interchange of the infinite series with the contour integral (6.3.4) is justified by the absolute convergence of $\zeta_p(2z)$ in the half-plane $\mathrm{Re}(z)>\mu>\frac{1}{2}$, together with Stirling's formula, which implies that $\left|\Gamma\left(\mu+it\right)\Gamma\left(s-\mu-it\right)\right|=O\left(|t|^{\mathrm{Re}(s)-1}e^{-\pi|t|}\right)$, $|t|\to\infty$.

As in the proof of Theorem 6.2.1, let us now move the line of integration from $\mathrm{Re}(z)=\mu$ to $\mathrm{Re}(z)=\frac{1}{2}-\mu$ and integrate along a positively oriented rectangular contour, $\mathscr{R}_\mu(T)$, containing the vertices $\mu\pm iT$ and $\frac{1}{2}-\mu\pm iT$, $T>0$. Due to the location of the trivial zeros of $\zeta_p(2z)$ at the negative integers, as well as our initial choice $\mathrm{Re}(s)>\mu>\frac{1}{2}$, the only singularities of the integrand inside $\mathscr{R}_\mu(T)$ are located at $z=\frac{1}{2}$ and at $z=0$. Like in (6.2.9), we can estimate trivially the integrals along the horizontal segments $\left[\frac{1}{2}-\mu\pm iT,\,\mu\pm iT\right]$, as $T\to\infty$, and show that they vanish in this limit. As such, an application of Cauchy's residue theorem gives immediately

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz&=\frac{1}{2\pi i}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz\\&\quad-\frac{1}{2}\,\frac{\Gamma(s)}{1+\frac{1}{\pi p}}+\frac{\sqrt{\pi}\,x}{2}\Gamma\left(s-\frac{1}{2}\right).\end{aligned}\tag{6.3.5}$$

Invoking the functional equation for $\zeta_p(z)$, (6.1.17), we can simplify the integral on the right-hand side of (6.3.5) as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}-\mu-i\infty}^{\frac{1}{2}-\mu+i\infty}\Gamma(z)\,\Gamma(s-z)\,\zeta_p(2z)\,x^{2z}dz\\&=\sqrt{\pi}\,x\sum_{m=1}^{\infty}\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,\left(2z,2\pi pm\right)_m\,\left(\pi xm\right)^{-2z}\,dz,\end{aligned}$$

because the representation of $\eta_p(2z)$, (6.1.14), converges absolutely in the half-plane $\mathrm{Re}(z)>\frac{1}{2}$. For each fixed $m\in\mathbb{N}$, our task now is to evaluate

$$I_{m,p}(s,x):=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\left(2z,2\pi pm\right)_m\,\Gamma\left(z\right)\,\Gamma\left(s+z-\frac{1}{2}\right)\,\left(\pi\,x\,m\right)^{-2z}\,dz.$$

This task can be simplified if we use the representation (6.1.21), which shows that

$$I_{m,p}(s,x)=I_{m,p}^{(1)}(s,x)+I_{m,p}^{(2)}(s,x),$$

with

$$I_{m,p}^{(1)}(s,x)=\frac{(-1)^m}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma(z)\,\Gamma\left(s+z-\frac{1}{2}\right)(\pi x m)^{-2z}\,dz$$
$$=2\,(-1)^m\,(\pi x\,m)^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi x\,m) \tag{6.3.6}$$

and[19]

$$I_{m,p}^{(2)}(s,x)=2\,(\pi x)^{s-\frac{1}{2}}e^{2\pi p m}m^{s-\frac{1}{2}}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi p m)^{\ell} \tag{6.3.7}$$
$$\times\int_{1}^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi m p t}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}(2\pi x m t)\,dt. \tag{6.3.8}$$

We should note that, in both formulas (6.3.6) and (6.3.8), we have used the well-known integral representation for the Macdonald function [[85], p. 349, eq. 7.3.17]

$$K_{\nu}(x)=\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}2^{s-2}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)x^{-s}ds,\quad x>0,\ \ \mu>\max\{0,\mathrm{Re}(\nu)\},$$

which has been very useful throughout this thesis.[20] Combining all the expressions given above yields the formula for $\varphi_p(s,x)$,

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left(\lambda_n^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\,\frac{x^{-2s}}{1+\frac{1}{\pi p}}$$
$$+\frac{2\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}(-1)^m m^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi m x)+\frac{2\,\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{m=1}^{\infty}e^{2\pi p m}m^{s-\frac{1}{2}}$$
$$\times\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi m p)^{\ell}\int_{1}^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi m p t}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}(2\pi x m t)\;dt. \tag{6.3.9}$$

Although (6.3.9) already constitutes a generalization of the classical case (6.1.3), we will now write the latter series in a more elegant form, which will be well-defined in the half-plane $\mathrm{Re}(s)\leq\frac{1}{2}$. We claim that the right-hand side of (6.3.9) constitutes the analytic continuation of the series $\varphi_p(s,x)$, (6.3.1), to the entire complex plane. We prove this by showing that the last series on the right-hand side of (6.3.9) defines an entire function of $s\in\mathbb{C}$. Indeed, it is enough to check this for s belonging to the half-plane $\mathrm{Re}(s)\leq\frac{1}{2}$, because the analyticity in the region $\mathrm{Re}(s)>\frac{1}{2}$ is already ensured by the absolute convergence of the left-hand side of (6.3.9).

From now on, we analyze the last infinite series on the right of (6.3.9) under the hypothesis that $\mathrm{Re}(s)<1$, which obviously includes the case $\mathrm{Re}(s)\leq\frac{1}{2}$. To that end, recall the well-known

[19]the interchange of the operations is justified by absolute convergence.
[20]It was instrumental to establish the integral formulation of the Selberg-Chowla formula in Lemma 1.3.1.

integral formula [[164], p. 252, relation 10.32.8]

$$K_{\nu}(z)=\frac{\sqrt{\pi}z^{\nu}}{2^{\nu}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{1}^{\infty}e^{-zt}\left(t^{2}-1\right)^{\nu-\frac{1}{2}}dt,\quad \mathrm{Re}(\nu)>-\frac{1}{2},\ \ |\arg(z)|<\frac{\pi}{2}. \tag{6.3.10}$$

From the fact that $K_{-\nu}(z)=K_{\nu}(z)$, (6.3.10) can be rewritten in the equivalent form[21]

$$\left(\frac{z}{2}\right)^{\nu}K_{\nu}(z)=\frac{2^{-2\nu}\sqrt{\pi}\,e^{-z}}{\Gamma\left(\frac{1}{2}-\nu\right)}\int_{0}^{\infty}e^{-2zt}t^{-\nu-\frac{1}{2}}(t+1)^{-\nu-\frac{1}{2}}dt,\quad \mathrm{Re}(\nu)<\frac{1}{2},\ \ |\arg(z)|<\frac{\pi}{2}. \tag{6.3.11}$$

Assuming that s lies in the region $\mathrm{Re}(s)\leq\frac{1}{2}$, we may invoke (6.3.11) to see that the fractional integral appearing on the right-hand side of (6.3.9) can be simplified in the following manner

$$\begin{aligned}J_{p,\ell}(m,x)&:=\int_{1}^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi mpt}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi xmt\right)\,dt\\&=\frac{2^{1-2s}\sqrt{\pi}\,(\pi xm)^{\frac{1}{2}-s}}{\Gamma\left(1-s\right)}\int_{0}^{\infty}\frac{\exp\left(-2\pi m\left(p+\left(2y+1\right)x\right)\right)}{\left(2\pi m\left(p+\left(2y+1\right)x\right)\right)^{\ell}}\,\frac{dy}{y^{s}\left(y+1\right)^{s}},\end{aligned}$$

where the interchange of the orders of integration comes from the absolute convergence of both integrals.[22] Note now that the integral in the last expression converges absolutely, which allows us to take the finite sum over $\ell\in\{1,...,m\}$ in the last term of (6.3.9). This results in

$$\begin{aligned}&\sum_{m=1}^{\infty}e^{2\pi pm}m^{s-\frac{1}{2}}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\left(4\pi mp\right)^{\ell}\,J_{p,\ell}(m,x)\\&=\frac{2^{1-2s}\,\pi^{1-s}x^{\frac{1}{2}-s}}{\Gamma\left(1-s\right)}\sum_{m=1}^{\infty}e^{2\pi pm}\int_{0}^{\infty}\left\{\left(\frac{p-(2y+1)x}{p+(2y+1)x}\right)^{m}+(-1)^{m-1}\right\}\frac{e^{-2\pi m(p+(2y+1)x)}}{y^{s}\left(y+1\right)^{s}}\,dy\\&=\frac{2^{1-2s}\,\pi^{1-s}x^{\frac{1}{2}-s}}{\Gamma\left(1-s\right)}\left\{\int_{0}^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\left(2y+1\right)x\right)e^{2\pi(2y+1)x}-1}dy+\sum_{m=1}^{\infty}(-1)^{m-1}\int_{0}^{\infty}\frac{e^{-2\pi m(2y+1)x}}{y^{s}(y+1)^{s}}\,dy\right\}\\&=\frac{2^{1-2s}\,\pi^{1-s}x^{\frac{1}{2}-s}}{\Gamma\left(1-s\right)}\int_{0}^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\left(2y+1\right)x\right)e^{2\pi(2y+1)x}-1}dy-\sum_{m=1}^{\infty}(-1)^{m}m^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi xm\right),\end{aligned} \tag{6.3.12}$$

where the last step results from employing (6.3.11) again, as well as from the use of the geometric series (6.2.16). In fact, to justify the interchange of the series and the integral in this procedure,

[21] Recall that we have encountered this integral representation before in a previous section about a generalized Ferrar's formula (4.6.1).

[22] note that we are working under the hypothesis $\mathrm{Re}(s)\leq\frac{1}{2}<1$.

one needs to check that, if $\mathrm{Re}(s)<1$,

$$\begin{aligned}&\sum_{m=1}^{\infty}\int_0^{\infty}\left|\frac{p-(2y+1)x}{p+(2y+1)x}\right|^m\frac{e^{-2\pi m(2y+1)x}}{y^{\mathrm{Re}(s)}(y+1)^{\mathrm{Re}(s)}}\,dy\leq\sum_{m=1}^{\infty}e^{-2\pi mx}\int_0^{\infty}\frac{e^{-4\pi mxy}}{y^{\mathrm{Re}(s)}(y+1)^{\mathrm{Re}(s)}}dy\\&=\frac{2^{2\mathrm{Re}(s)-1}\Gamma\left(1-\mathrm{Re}(s)\right)}{\sqrt{\pi}}\sum_{m=1}^{\infty}\left(\pi mx\right)^{\mathrm{Re}(s)-\frac{1}{2}}K_{\mathrm{Re}(s)-\frac{1}{2}}\left(2\pi mx\right)\\&\leq C_s\,x^{\mathrm{Re}(s)-1}\sum_{m=1}^{\infty}m^{\mathrm{Re}(s)-1}e^{-2\pi mx},\end{aligned}$$

where C_s is some positive constant depending on s, which is fixed. In the last steps, we utilized the integral representation (6.3.11) and the asymptotic estimate for the modified Bessel function,

$$K_{\nu}(x)=\left(\frac{\pi}{2x}\right)^{\frac{1}{2}}e^{-x}\left\{1+O\left(\frac{1}{x}\right)\right\},\qquad x\to\infty,\tag{6.3.13}$$

which has been part of all the chapters of this thesis. Clearly, the integral on the right-hand side of (6.3.12) converges absolutely and uniformly for any s in the half-plane $\mathrm{Re}(s)<1$, as the elementary bound shows

$$\left|\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}\,dy\right|\leq\int_0^{\infty}\frac{y^{-\sigma}(y+1)^{-\sigma}}{e^{2\pi(2y+1)x}-1}dy.$$

This proves that the last infinite series on the right of (6.3.9) defines an entire function of $s\in\mathbb{C}$ [[236], p. 92], and so (6.3.2) constitutes the analytic continuation of the series (6.3.1) to the half-plane $\mathrm{Re}(s)<\frac{1}{2}$. Finally, joining (6.3.12) and (6.3.9) leads to (6.3.3) as an equivalent continuation of (6.3.1) to $\mathrm{Re}(s)<\frac{1}{2}$. $\square$

Remark 6.3.1. It is also possible to write the last infinite series on the right-hand side of (6.3.2) as a series involving integrals of the Whittaker function, $W_{\mu,\nu}(z)$, described in previous chapters. We could also replace the integral in (6.3.3) by an integral with respect to a Hankel contour. This would allow a representation of $\varphi_p(s,x)$ valid for any $s\in\mathbb{C}$.

One may expect that, once we let $p\to\infty$ in (6.3.2) or in (6.3.3), Watson's formula (6.1.3) may be recovered. Moreover, taking $p\to0^+$ may yield a new analogue of it. We state this analogue in the next corollary.

Corollary 6.3.1. *Let s be any complex number such that $Re(s)>\frac{1}{2}$ and $x>0$. Then Watson's formula (6.1.3) holds true. Moreover, one has the companion identity*

$$\sum_{n=1}^{\infty}\frac{1}{\left((2n-1)^2+x^2\right)^s}=\frac{\sqrt{\pi}\,x^{1-2s}}{4\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)+\frac{2^{\frac{1}{2}-s}\,x^{\frac{1}{2}-s}\,\pi^s}{\Gamma(s)}\sum_{m=1}^{\infty}(-1)^m m^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(\pi xm).\tag{6.3.14}$$

Proof. Since the proofs of (6.3.14) and (6.1.3) are analogous, we will only see that (6.3.3) reduces to (6.1.3) as $p\to\infty$. For this, it suffices to analyze, under the present limiting case, the continuation of the integral in (6.3.3) to the region $\mathrm{Re}(s)>1$. Assume that $\mathrm{Re}(s)<1$: by an application of the dominated convergence theorem, we can take $p\to\infty$ on the right-hand side of (6.3.3). Since the only term of (6.3.3) that is modified by this limit is the integral, we see that it results in the expression

$$
\begin{aligned}
&2^{1-2s}\,x^{1-2s}\sin(\pi s)\int_0^\infty \frac{y^{-s}(y+1)^{-s}}{e^{2\pi(2y+1)x}-1}dy\\
&=2^{1-2s}\,x^{1-2s}\sin(\pi s)\sum_{n=1}^\infty e^{-2\pi nx}\int_0^\infty y^{-s}(y+1)^{-s}e^{-4\pi nx\,y}dy\\
&=\frac{\pi^s x^{\frac{1}{2}-s}}{\Gamma(s)}\sum_{n=1}^\infty n^{s-\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi nx),\qquad \mathrm{Re}(s)<\frac{1}{2},
\end{aligned}
\tag{6.3.15}
$$

where the second step is justified by invoking the geometric series, as well as by absolute convergence (recall (6.3.13)), while the last equality results from (6.3.11). Since the series on the right-hand side of (6.3.15) converges absolutely and uniformly for any $s\in\mathbb{C}$ and its summands are entire functions of s, we see that the expression (6.3.15) is valid for any $s\in\mathbb{C}\setminus\{\frac{1}{2}-k,\,k\in\mathbb{N}_0\}$. Therefore, (6.3.15) and the series (6.3.1) must coincide whenever $\mathrm{Re}(s)>\frac{1}{2}$. □

Remark 6.3.2. Note that (6.3.14) can also be derived from Kober's formula (6.1.6) with $\beta=\frac{1}{2}$ or, in a general form, by the results given in [194].

We now obtain a result in which the restriction $\frac{1}{2}<\mathrm{Re}(s)<1$ in our Theorem 6.3.1 is relaxed. Our proof is completely analogous to Watson's [[233], p. 300].

Corollary 6.3.2. *Let $N\in\mathbb{N}$. For any $x>0$ and any complex number s satisfying the condition $-N+\frac{1}{2}<Re(s)<1$, we have the identity*

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\left(\lambda_n^2+x^2\right)^s}-\sum_{m=0}^{N-1}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}\\
&+\sum_{m=0}^{N-1}\binom{-s}{m}x^{2m}\zeta_p\left(2s+2m\right)=\frac{\sqrt{\pi}\,x^{1-2s}}{2\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)-\frac{1}{2}\frac{x^{-2s}}{1+\frac{1}{\pi p}}\\
&+2^{2-2s}\,x^{1-2s}\sin(\pi s)\int_0^\infty\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)\,x\right)e^{2\pi(2y+1)x}-1}dy.
\end{aligned}
\tag{6.3.16}
$$

Proof. Note that, for $\mathrm{Re}(s)>\frac{1}{2}$ and every $N\in\mathbb{N}$, we can write the left-hand side of (6.3.3) in the following form

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\left\{\frac{1}{\left(\lambda_n^2+x^2\right)^s}-\sum_{m=0}^{N-1}\binom{-s}{m}\frac{x^{2m}}{\lambda_n^{2s+2m}}\right\}\\
&+\sum_{m=0}^{N-1}\binom{-s}{m}x^{2m}\zeta_p\left(2s+2m\right).
\end{aligned}
\tag{6.3.17}
$$

Let M be a positive integer so large that, for $n > M$, $x/\lambda_n < 1$. Using the generalized binomial theorem and picking $n > M$, we have that

$$\frac{1}{(\lambda_n^2 + x^2)^s} = \sum_{m=0}^{\infty} \binom{-s}{m} \frac{x^{2m}}{\lambda_n^{2s+2m}}.$$

Thus, (6.3.17) can be actually expressed as

$$\sum_{n=1}^{M} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \left\{ \frac{1}{(\lambda_n^2 + x^2)^s} - \sum_{m=0}^{N-1} \binom{-s}{m} \frac{x^{2m}}{\lambda_n^{2s+2m}} \right\}$$
$$+ \sum_{n=M+1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \left\{ \sum_{m=N}^{\infty} \binom{-s}{m} \frac{x^{2m}}{\lambda_n^{2s+2m}} \right\} + \sum_{m=0}^{N-1} \binom{-s}{m} x^{2m} \zeta_p \left(2s + 2m\right).$$

Note that the general term of the infinite series $\sum_{n\geq M+1}$ is $O\left(\lambda_n^{-2\sigma-2N}\right)$ for large n. This means that this series converges absolutely and uniformly with respect to s in the region $\mathrm{Re}(s) > -N + \frac{1}{2}$.[23] Since its terms are analytic functions of s, an application of Weierstrass' M-test shows that (6.3.17) represents an analytic function in the region $\mathrm{Re}(s) > -N + \frac{1}{2}$. By the above argument, we know that the right-hand side of (6.3.16) also defines an analytic function in the half-plane $\mathrm{Re}(s) < 1$. Thus, (6.3.16) must hold by analytic continuation. $\square$

From the previous result, we can obtain the following interesting identities.

Corollary 6.3.3. *Let $p \in \mathbb{R}_+$ and $x > 0$. Then, for any s in the strip $-1 < Re(s) < 2$, the following representation holds*

$$\zeta_p(s) = \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \left\{ \frac{1}{\lambda_n^s} - \frac{1}{(\lambda_n^2 + x^2)^{\frac{s}{2}}} \right\} + \frac{\sqrt{\pi}\,\Gamma\left(\frac{s-1}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} x^{1-s} - \frac{1}{2} \frac{x^{-s}}{1 + \frac{1}{\pi p}}$$
$$+ 2^{2-s} x^{1-s} \sin\left(\frac{\pi s}{2}\right) \int_0^{\infty} \frac{y^{-\frac{s}{2}} (y+1)^{-\frac{s}{2}}}{\sigma\left((2y+1)\,x\right) e^{2\pi(2y+1)x} - 1} dy. \tag{6.3.18}$$

In particular, we have the formula

$$\sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \left\{ \frac{1}{\sqrt{\lambda_n^2 + x^2}} - \frac{1}{\lambda_n} \right\} + C_p^{(1)} + \log\left(\frac{x}{2}\right) + \frac{1}{2x} \frac{1}{1 + \frac{1}{\pi p}}$$
$$= 2 \int_0^{\infty} \frac{1}{\sqrt{y^2 + y}} \frac{dy}{\sigma\left((2y+1)\,x\right) e^{2\pi(2y+1)x} - 1}, \tag{6.3.19}$$

where $C_p^{(1)}$ denotes Koshliakov's generalization of Euler's constant γ [[132], p. 46, eq. (46)]

$$C_p^{(1)} := \lim_{n\to\infty} \left\{ \sum_{j=1}^{n-1} \frac{p^2 + \lambda_j^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_j^2} \frac{1}{\lambda_j} - \log\left(\lambda_n\right) \right\}. \tag{6.3.20}$$

[23] because $\lambda_n > n - \frac{1}{2}$ for any $p \in \mathbb{R}_+$.

Proof. To prove (6.3.18) it is enough to take $N = 1$ in (6.3.16). In order to prove the second formula, (6.3.19), we take the limit $s \to \frac{1}{2}$ on both sides of (6.3.16): this yields

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \left\{ \frac{1}{\sqrt{\lambda_n^2+x^2}} - \frac{1}{\lambda_n} \right\} + \lim_{s\to\frac{1}{2}} \left[\zeta_p(2s) - \frac{\sqrt{\pi}\, x^{1-2s}}{2\Gamma(s)} \Gamma\left(s-\frac{1}{2}\right) \right]$$
$$= -\frac{1}{2x} \frac{1}{1+\frac{1}{\pi p}} + 2\int_0^{\infty} \frac{1}{\sqrt{y^2+y}} \frac{dy}{\sigma\left((2y+1)\,x\right) e^{2\pi(2y+1)x}-1}.$$

To finish the computation of the above limit, we use the well-known Laurent expansions,

$$\frac{1}{\Gamma(s)} = \frac{1}{\sqrt{\pi}} + \frac{\gamma+2\log(2)}{\sqrt{\pi}} \left(s-\frac{1}{2}\right) + O\left(\left(s-\frac{1}{2}\right)^2\right), \tag{6.3.21}$$

$$x^{-2s} = x^{-1}\left(1-2\log(x)\left(s-\frac{1}{2}\right) + O\left(\left(s-\frac{1}{2}\right)^2\right)\right),$$

$$\Gamma\left(s-\frac{1}{2}\right) = \frac{1}{s-\frac{1}{2}} - \gamma + O\left(s-\frac{1}{2}\right) \tag{6.3.22}$$

and finally [[132], p. 48, eq. (51)],

$$\zeta_p(2s) = \frac{1}{2s-1} + C_p^{(1)} + O\left(s-\frac{1}{2}\right). \tag{6.3.23}$$

Deducing our corollary from (6.3.21)-(6.3.23) is now immediate. □

By letting $p \to \infty$ and $p \to 0^+$ in the previous result, we are able to obtain a formula also due to Watson [[233], p. 301, eq. (6)]. Furthermore, we can derive an analogue of it, which seems to be new.

Corollary 6.3.4. *For any* $Re(s) > -1$ *and* $x > 0$*, the following representations take place*

$$\zeta(s) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n^s} - \frac{1}{(n^2+x^2)^{\frac{s}{2}}} \right\} + \frac{\sqrt{\pi} x^{1-s}}{2\Gamma\left(\frac{s}{2}\right)} \Gamma\left(\frac{s-1}{2}\right) - \frac{x^{-s}}{2} + \frac{2\pi^{\frac{s}{2}} x^{\frac{1-s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \sum_{n=1}^{\infty} n^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(2\pi n x),$$

and

$$(1-2^{-s})\,\zeta(s) = \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n-1)^s} - \frac{1}{\left((2n-1)^2+x^2\right)^{\frac{s}{2}}} \right\}$$
$$+ \frac{\sqrt{\pi}\Gamma\left(\frac{s-1}{2}\right) x^{1-s}}{4\Gamma\left(\frac{s}{2}\right)} + \frac{2^{\frac{1-s}{2}} \pi^{\frac{s}{2}} x^{\frac{1-s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \sum_{n=1}^{\infty} (-1)^n n^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(\pi n x).$$

In particular, the following formulas are valid

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{\sqrt{n^2+x^2}} - \frac{1}{n} \right\} + \frac{1}{2x} + \gamma + \log\left(\frac{x}{2}\right) = 2\sum_{n=1}^{\infty} K_0(2\pi n x),$$

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{(2n-1)^2+x^2}}-\frac{1}{2n-1}\right\}+\frac{\gamma}{2}+\frac{\log(x)}{2}=\sum_{n=1}^{\infty}(-1)^n\,K_0(\pi nx).$$

Remark 6.3.3. We have dealt with formulas of this type in Chatpers 2 and 4 of this thesis. Recall, for instance, (2.2.77) which represents a twisted analogue of Watson's formula (2.2.78).

As a particular case of our generalization of Watson's result, we can evaluate the series (6.3.1) when s is an integer. In the next corollary we shall do this for $s=1,\,2.$[24]

Corollary 6.3.5. *Let $p\in\mathbb{R}_+$ and $x>0$. Then the following identities are valid*

$$\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{x^2+\lambda_n^2}=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\,\frac{1}{2x^2}+\frac{\pi}{x}\,\frac{1}{\sigma(x)\,e^{2\pi x}-1},\tag{6.3.24}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{(\lambda_n^2+x^2)^2}&=\frac{\pi}{4x^3}-\frac{1}{2x^4}\,\frac{1}{1+\frac{1}{\pi p}}\\&+\frac{\pi^2}{x^2\,(\sigma(x)\,e^{2\pi x}-1)}\left[\frac{1}{2\pi x}+\left(1+\frac{p}{\pi\,(p^2-x^2)}\right)\frac{\sigma(x)\,e^{2\pi x}}{\sigma(x)\,e^{2\pi x}-1}\right].\end{aligned}\tag{6.3.25}$$

Proof. We only give a proof of the first formula (6.3.24), as the proof leading to (6.3.25) is analogous. Taking $s=1$ in formula (6.3.2) and using the special case for the Macdonald function,

$$K_{\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}\,e^{-x},\quad x>0,\tag{6.3.26}$$

we deduce

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\lambda_n^2+x^2}&=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\,\frac{1}{2x^2}+\frac{\pi}{x}\,\sum_{m=1}^{\infty}(-1)^m\,e^{-2\pi mx}\\&+\frac{\pi}{x}\sum_{m=1}^{\infty}e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi mp)^{\ell}\int_1^{\infty}e^{-2\pi mt(x+p)}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,dt\\&=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\,\frac{1}{2x^2}+\frac{\pi}{x}\sum_{m=1}^{\infty}e^{-2\pi mx}\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi mp)^{\ell}\,(2\pi m(x+p))^{-\ell}\\&=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\,\frac{1}{2x^2}+\frac{\pi}{x}\sum_{m=1}^{\infty}\left(\frac{p-x}{p+x}\right)^m e^{-2\pi mx}\\&=\frac{\pi}{2x}-\frac{1}{1+\frac{1}{\pi p}}\,\frac{1}{2x^2}+\frac{\pi}{x}\,\frac{1}{\sigma(x)\,e^{2\pi x}-1},\end{aligned}\tag{6.3.27}$$

completing the proof. $\square$

[24]We remark that there are other ways of proving both formulas of our next corollary, for instance involving the theory of residues. In fact, Koshliakov gave a direct proof of (6.3.24) on page 34 of his paper [132].

Remark 6.3.4. By invoking (6.3.2) and appealing to the definition of Bessel polynomials

$$K_{n-\frac{1}{2}}(x)=\sqrt{\frac{\pi}{2x}}\,e^{-x}\sum_{k=0}^{n-1}\frac{(n-1+k)!}{k!\,(n-1-k)!\,(2x)^{k}},\quad n\in\mathbb{N},$$

it is possible to compute the values of (6.3.1) for every $s\in\mathbb{N}$. Moreover, we may connect the resulting expressions with identities involving $\zeta_p(2n-1)$ which are reminiscent of Terras' formulas (cf. [124, 224]).

Remark 6.3.5. We remark that it is also possible to obtain an analogue of Kober's formula (6.1.6) in Koshliakov's setting. In fact, the modified version of (6.3.1),

$$\sum_{n\in\mathbb{Z}}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{1}{\left((\lambda_n+\beta)^2+x^2\right)^s},\quad \text{Re}(s)>\frac{1}{2},\quad x>0,\quad \beta\in\mathbb{R},$$

admits a Fourier expansion very similar to the one given in (6.3.3) when $\frac{1}{2}<\text{Re}(s)<1$. However, we need to replace the integral on the right-hand side of (6.3.3) by a generalization of the form

$$\frac{1}{2}\int_0^\infty\left\{\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)x+i\beta\right)\exp\left(2\pi(2y+1)x+2\pi i\beta\right)-1}\right.$$
$$\left.+\frac{y^{-s}(y+1)^{-s}}{\sigma\left((2y+1)x-i\beta\right)\exp\left(2\pi(2y+1)x-2\pi i\beta\right)-1}\right\}dy.$$

6.4 A generalization of the Epstein zeta function

In this section we will consider a generalization of the Epstein zeta function (6.1.4) when the summation indices are replaced by two Koshliakov sequences $(\lambda_m,\lambda_n)_{m,n\in\mathbb{Z}}$. Once more, let us note that if y is a root of the equation (6.1.13) then $-y$ also is. Therefore, we can extend Koshliakov's sequence $(\lambda_n)_{n\in\mathbb{N}}$ to the nonpositive integers by simply setting $\lambda_{-n}:=-\lambda_n$ and $\lambda_0:=0$.

Although it is possible to proceed with any positive quadratic form $Q(x,y)=ax^2+bxy+cy^2$ (see Remark 6.3.5 above), it is enough for our purposes to consider an analogue of (6.1.4) for diagonal forms, this is,

$$\zeta_{p,p'}(s,c):=\sum_{m,n\neq 0}\frac{\left(p^2+\lambda_m^2\right)\left(p'^2+\lambda_n'^2\right)}{\left(p(p+\frac{1}{\pi})+\lambda_m^2\right)\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\,\frac{1}{(\lambda_m^2+c\lambda_n'^2)^s},\quad \text{Re}(s)>1,\tag{6.4.1}$$

where $c>0$ and $m,n\neq 0$ here means that only the term $m=n=0$ is omitted from the sum above. It is simple to see that the Dirichlet series defining $\zeta_{p,p'}(s,c)$ converges absolutely in the half-plane $\text{Re}(s)>1$. In the next result we give a formula which provides the continuation of (6.4.1) to $\text{Re}(s)<1$, thus being analogous to the Selberg-Chowla formula (6.1.5).

Theorem 6.4.1. *For any $c>0$ and $p,p'\in\mathbb{R}_+$, let $\zeta_{p,p'}(s)$ denote the analogue of Epstein's zeta function given by (6.4.1). Then $\zeta_{p,p'}(s,c)$ can be continued to the half-plane Re$(s)<1$ by the* *following integral formula*

$$\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,c)=\frac{2c^{s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+2\,c^{\frac{1-s}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)$$
$$+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{1-s}{2}}}{\Gamma(1-s)}\sum_{n=1}^{\infty}\frac{(p'^2+\lambda_n'^2)\lambda_n'^{1-2s}}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'\,(2y+1)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy. \tag{6.4.2}$$

Furthermore, (6.4.2) provides the continuation of $\zeta_{p,p'}(s,c)$ to the entire complex plane as an analytic function everywhere except at the point $s=1$, where it possesses a simple pole whose residue is

$$Res_{s=1}\zeta_{p,p'}(s,c)=\frac{\pi}{\sqrt{c}}.$$

Proof. The proof comes immediately from our analogue of Watson's formula (6.3.3). We start by choosing $\mathrm{Re}(s)>\mu>1$ and by writing the generalized Epstein zeta function in the following form

$$\zeta_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+\frac{2c^{-s}}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)$$
$$+4\sum_{m,n=1}^{\infty}\frac{(p^2+\lambda_m^2)\,(p'^2+\lambda_n'^2)}{\left(p(p+\frac{1}{\pi})+\lambda_m^2\right)\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\,\frac{1}{(\lambda_m^2+c\lambda_n'^2)^s}. \tag{6.4.3}$$

If, on the double series in (6.4.3), we fix the variable of summation n and sum over m, we can use (6.3.2) with x replaced by $\sqrt{c}\lambda_n'$ and then get

$$\zeta_{p,p'}(s,c)=\frac{2}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta_{p'}(2s-1)+H_{p,p'}(s,c), \tag{6.4.4}$$

where $\mathrm{Re}(s)>\mu>1$ and the function $H_{p,p'}(s,c)$ is described by

$$\frac{\Gamma(s)\,H_{p,p'}(s,c)}{8\pi^s c^{\frac{1}{4}-\frac{s}{2}}}=\sum_{m,n=1}^{\infty}\frac{(p'^2+\lambda_n'^2)\,(-1)^m}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{m}{\lambda_n'}\right)^{s-\frac{1}{2}}K_{s-\frac{1}{2}}\left(2\pi m\,\lambda_n'\sqrt{c}\right)$$
$$+\sum_{m,n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\left(\frac{m}{\lambda_n'}\right)^{s-\frac{1}{2}}e^{2\pi pm}\sum_{\ell=1}^{m}\binom{m}{\ell}(-1)^{m-\ell}\,(4\pi mp)^{\ell}$$
$$\times\int_1^{\infty}t^{s-\frac{1}{2}}\,e^{-2\pi mpt}\,\frac{(t-1)^{\ell-1}}{(\ell-1)!}\,K_{s-\frac{1}{2}}\left(2\pi m\sqrt{c}\lambda_n't\right)\,dt. \tag{6.4.5}$$

We now claim that the right-hand side of (6.4.4) constitutes the analytic continuation of $\zeta_{p,p'}(s,c)$ to the entire complex plane and, when $\mathrm{Re}(s)<1$, its expression reduces to (6.4.2). Since the continuations of $\zeta_p(s)$ and $\zeta_{p'}(s)$ are already provided in Koshliakov's paper [132], we only need to focus on the continuation of $H_{p,p'}(s,c)$. This is now analogous to what we have done in the previous section, and it is possible to show that it defines an entire function of $s\in\mathbb{C}$.[25]

[25]Compare this with the way we have studied the series on the right-hand side of (6.3.9).

Take now the expression defining $H_{p,p'}(s,c)$, (6.4.5), and assume that $\mathrm{Re}(s)<1$. We see from the computations leading to (6.3.12) that

$$H_{p,p'}(s,c) = 2^{4-2s}c^{\frac{1}{2}-s}\sin(\pi s)\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\lambda_n'^{1-2s}$$
$$\times\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'\,(2y+1)\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy,$$

which gives (6.4.2). By a simple analysis of the right-hand side of (6.4.4), it is easily seen that $\zeta_{p,p'}(s,c)$ has removable singularities located at the points $s=\frac{1}{2}-k$, $k\in\mathbb{N}_0$. Since $\zeta_p(2s)$ and $\Gamma\left(s-\frac{1}{2}\right)$ are analytic in a neighborhood of the point $s=1$, we can also conclude that $\zeta_{p,p'}(s,c)$ must have a simple pole located at $s=1$, which comes from the factor $\zeta_{p'}(2s-1)$ on the right-hand side of (6.4.4). Its residue is easily seen to be $\pi/\sqrt{c}$, which completes our proof. □

Remark 6.4.1. It is possible to deform the path of integration on the right-hand side of (6.4.2) in order to make it valid for any $s\in\mathbb{C}$. See Remark 6.3.1 above.

Remark 6.4.2. It is clear from the proof of Corollary 6.3.1 (see the steps leading to (6.3.15)) that (6.4.2) and (6.4.5) are generalizations of the Selberg-Chowla formula (6.1.5) when the quadratic form is diagonal, i.e., $Q(m,n)=m^2+cn^2$, $c>0$.

Remark 6.4.3. Although it is possible to write a functional equation for (6.4.1) using the general Selberg-Chowla formula (6.4.2), we shall omit it here because there is an asymmetry between the zeta functions appearing on both sides of it. This asymmetry is due to the drastic differences between Koshliakov's zeta functions $\zeta_p(s)$ and $\eta_p(s)$, which are themselves connected by a functional equation, (6.1.17). In the preprint version of this chapter [192], we have considered yet another analogue of Epstein's zeta function that makes sense within Koshliakov's framework. Its functional equation is much easier to state and to prove (see [[192], Corollary 4.9]).

Having proved that $\zeta_{p,p'}(s,c)$ possesses a simple pole located at $s=1$, we now study how $\zeta_{p,p'}(s,c)$ behaves around this singularity, generalizing a classical formula due to Kronecker [203] (see (6.1.9) above).

Corollary 6.4.1. *Let $p,p'\in\mathbb{R}_+$ and let $\zeta_{p,p'}(s,c)$ be the generalized Epstein $\zeta-$function (6.4.1). Moreover, let $\sigma(t)$ be defined by (6.1.23). Then $\zeta_{p,p'}(s,c)$ admits the meromorphic expansion around $s=1$,*

$$\zeta_{p,p'}(s,\,c) = \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi^2}{3}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2}$$
$$+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log(4c)+4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\lambda_n'^{-1}}{\sigma\left(\sqrt{c}\lambda_n'\right)e^{2\pi\sqrt{c}\lambda_n'}-1}\right)$$
$$+O(s-1), \tag{6.4.6}$$

where $C_p^{(1)}$ is Koshliakov's analogue of the Euler-Mascheroni constant (6.3.20), i.e.,

$$C_p^{(1)} := \lim_{n\to\infty}\left\{\sum_{j=1}^{n-1}\frac{p^2+\lambda_j^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_j^2}\,\frac{1}{\lambda_j}-\log\left(\lambda_n\right)\right\}.$$

Proof. The proof consists in evaluating the right-hand side of (6.4.4) when $s\to 1$. Let us recall the following expansions around the point $s=1$,

$$\frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}=\pi-2\pi\log(2)\,(s-1)+O\left(s-1\right)^2, \tag{6.4.7}$$

and [[132], p. 48, eq. (51)]

$$\zeta_{p'}(2s-1)=\frac{1}{2(s-1)}+C_{p'}^{(1)}+O\left(s-1\right). \tag{6.4.8}$$

Combining (6.4.7) and (6.4.8), we see that the second term in (6.4.4) can be written as

$$\frac{2\sqrt{\pi}\,c^{\frac{1}{2}-s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\zeta_{p'}(2s-1)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log\left(4c\right)\right)+O\left(s-1\right). \tag{6.4.9}$$

Using (6.4.4), (6.4.9) and invoking the generalized Basel identity [[132], p. 22, eq. (39)],

$$\zeta_p(2)=\frac{\pi^2}{6}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{(1+\frac{1}{\pi p})^2},$$

we get the meromorphic expansion for the Epstein zeta function $\zeta_{p,p'}(s,c)$,

$$\zeta_{p,p'}(s,\,c)=\frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log\left(4c\right)\right)+\frac{\pi^2}{3}\,\frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p'}\right)\left(1+\frac{1}{\pi p}\right)^2} \tag{6.4.10}$$

$$+H_{p,p'}(1,c)+O\left(s-1\right). \tag{6.4.11}$$

Appealing once more to the relation (6.3.26) and doing no more than the calculations made in (6.3.27), it is straightforward to derive that

$$H_{p,p'}(1,c)=\frac{4\pi}{\sqrt{c}}\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\lambda_n'^{-1}}{\sigma\left(\sqrt{c}\lambda_n'\right)e^{2\pi\sqrt{c}\lambda_n'}-1}$$

which, when combined with (6.4.11), gives immediately (6.4.6). □

Remark 6.4.4. The infinite series appearing in (6.4.6) can be regarded as a generalization of the logarithm of Dedekind's $\eta-$function (6.1.8). Taking $p,\,p'\to\infty$ in (6.4.6), the particular case (6.1.9) can be obtained. Analogously to Section 6.2, we can obtain several new analogues of Kronecker's limit formula by fixing one of the parameters and varying the other. Of course, when p and p' both tend to zero or to infinity (not necessarily the same limit), the resulting formula (6.4.6) may be also achieved through Kronecker's second limit formula [[203], p. 22].

Our first corollary of (6.4.6) is obtained when we take $p' \to 0^+$, while p is kept fixed.

Corollary 6.4.2. *For any $p \in \mathbb{R}_+$ and $c > 0$, consider the Epstein zeta function defined by*

$$\zeta_{p,0}(s,c) := \sum_{m,n\neq 0} \frac{p^2+\lambda_m^2}{p(p+\frac{1}{\pi})+\lambda_m^2} \frac{1}{\left(\lambda_m^2 + c\left(n-\frac{1}{2}\right)^2\right)^s}, \quad Re(s) > 1.$$

Then $\zeta_{p,0}(s,c)$ admits the meromorphic expansion around $s=1$

$$\begin{aligned}\zeta_{p,0}(s,c) &= \frac{\pi}{\sqrt{c}} \frac{1}{s-1} \\ &+ \frac{\pi}{\sqrt{c}}\left(2\gamma - \log\left(\frac{c}{4}\right) + 8\sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{1}{\sigma\left(\sqrt{c}(n-\frac{1}{2})\right) e^{\pi\sqrt{c}(2n-1)} - 1}\right) + O(s-1).\end{aligned}$$

In particular, the standard expansion holds

$$\begin{aligned}\zeta_{0,0}(s,c) = \frac{\pi}{\sqrt{c}} \frac{1}{s-1} + \frac{\pi}{\sqrt{c}}\left(2\gamma - \log\left(\frac{c}{4}\right) - 8\sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{1}{e^{\pi\sqrt{c}(2n-1)} + 1}\right) \\ + O(s-1).\end{aligned}$$

Now, letting $p' \to \infty$ and fixing $p \in \mathbb{R}_+$, we can deduce the following result from (6.4.6).

Corollary 6.4.3. *For any $p \in \mathbb{R}_+$ and $c > 0$, consider the Epstein zeta function defined by*

$$\zeta_{p,\infty}(s,c) = \sum_{m,n\neq 0} \frac{p^2+\lambda_m^2}{p(p+\frac{1}{\pi})+\lambda_m^2} \frac{1}{(\lambda_m^2 + cn^2)^s}, \quad Re(s) > 1.$$

Then $\zeta_{p,\infty}(s,c)$ has the following meromorphic expansion around $s=1$,

$$\begin{aligned}\zeta_{p,\infty}(s,\, c) &= \frac{\pi}{\sqrt{c}} \frac{1}{s-1} + \frac{\pi^2}{3} \frac{1+\frac{3}{\pi p}(1+\frac{1}{\pi p})}{\left(1+\frac{1}{\pi p}\right)^2} \\ &+ \frac{\pi}{\sqrt{c}}\left(2\gamma - \log(4c) + 4\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sigma(\sqrt{c}n)\, e^{2\pi\sqrt{c}n} - 1}\right) + O(s-1).\end{aligned}$$

In particular, the usual Kronecker limit formula (6.1.9) takes place. Moreover,

$$\zeta_{0,\infty}(s,c) = \frac{\pi}{\sqrt{c}} \frac{1}{s-1} + \pi^2 + \frac{\pi}{\sqrt{c}}\left(2\gamma - \log(4c) - 4\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{e^{2\pi\sqrt{c}n} + 1}\right) + O(s-1).$$

If we now take $p \to \infty$ and fix $p' \in \mathbb{R}_+$, we obtain the following formula.

Corollary 6.4.4. *For any $p' \in \mathbb{R}_+$ and $c > 0$, consider the Epstein zeta function defined by*

$$\zeta_{\infty,p'}(s,c) := \sum_{m,n\neq 0} \frac{p'^2+\lambda_n'^2}{p'(p'+\frac{1}{\pi})+\lambda_n'^2} \frac{1}{(m^2 + c\lambda_n'^2)^s}, \quad Re(s) > 1.$$

Then $\zeta_{\infty,p'}(s,c)$ has a Laurent expansion around $s=1$ given by

$$\begin{aligned}\zeta_{\infty,p'}(s,\,c) &= \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi^2}{3}\,\frac{1}{1+\frac{1}{\pi p'}}\\ &+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log\left(4c\right)+4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\lambda_n'^{-1}}{e^{2\pi\sqrt{c}\lambda_n'}-1}\right)\\ &+O\left(s-1\right).\end{aligned}$$

In particular, we have the expansion

$$\begin{aligned}\zeta_{\infty,0}(s,\,c) &= \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi}{\sqrt{c}}\left(2\gamma-\log\left(\frac{c}{4}\right)+8\sum_{n=1}^{\infty}\frac{1}{2n-1}\,\frac{1}{e^{\pi\sqrt{c}(2n-1)}-1}\right)\\ &+O\left(s-1\right).\end{aligned}$$

At last, our final corollary is the result of taking the limit $p\to 0^{+}$ in (6.4.6).

Corollary 6.4.5. *For any $p'\in\mathbb{R}_+$ and $c>0$, consider the Epstein zeta function defined by*

$$\zeta_{0,p'}(s,c):=\sum_{m,n\neq 0}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{1}{\left(\left(m-\frac{1}{2}\right)^2+c\,\lambda_n'^2\right)^s},\quad Re(s)>1.$$

Then $\zeta_{0,p'}(s,c)$ admits the meromorphic expansion around $s=1$,

$$\begin{aligned}\zeta_{0,p'}(s,\,c) &= \frac{\pi}{\sqrt{c}}\,\frac{1}{s-1}+\frac{\pi^2}{1+\frac{1}{\pi p'}}\\ &+\frac{\pi}{\sqrt{c}}\left(2C_{p'}^{(1)}-\log\left(4c\right)-4\sum_{n=1}^{\infty}\frac{p'^2+\lambda_n'^2}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\,\frac{\lambda_n'^{-1}}{e^{2\pi\sqrt{c}\lambda_n'}+1}\right)\\ &+O(s-1).\end{aligned}$$

6.5 Generalizations of Entries 3.3.1-3.3.3, pp. 253-254 of Ramanujan's Lost Notebook

Building on the work of the previous two sections, we now establish a generalization of the Ramanujan-Guinand formula (6.1.2), stated at the introduction to this chapter as Entry 3.3.1.

Theorem 6.5.1. *If α and β are two positive real numbers such that $\alpha\beta=\pi^2$ and if s is a complex number such that $Re(s)<1$, then the following identity holds*

$$\begin{aligned}&\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\,\mathcal{K}_{\frac{s}{2},p}\left(2\lambda_n\alpha\right)\\&-\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\beta^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\,\mathcal{K}_{\frac{s}{2},p}\left(2\lambda_n\beta\right)\\&=\frac{\Gamma\left(-\frac{s}{2}\right)\eta_p(-s)}{4\left(1+\frac{1}{\pi p}\right)}\left\{\beta^{\frac{1+s}{2}}-\alpha^{\frac{1+s}{2}}\right\}+\frac{\Gamma\left(\frac{s}{2}\right)\zeta_p(s)}{4}\left\{\beta^{\frac{1-s}{2}}-\alpha^{\frac{1-s}{2}}\right\},\end{aligned}\tag{6.5.1}$$

where $\mathcal{K}_{\nu,p}(x)$ denotes the integral

$$\mathcal{K}_{\nu,p}(x)=\int_0^{\infty}\frac{y^{-\nu-\frac{1}{2}}(y+1)^{-\nu-\frac{1}{2}}}{\sigma\left(\frac{x}{2\pi}(2y+1)\right)e^{(2y+1)x}-1}dy,\quad x>0,\ Re(\nu)<\frac{1}{2},\tag{6.5.2}$$

and $\sigma(t)$ is defined by (6.1.23).

Proof. In Theorem 6.4.1 we have seen a representation (namely (6.4.2)) which extends the Epstein zeta function, $\zeta_{p,p'}(s,c)$, to the half-plane $\mathrm{Re}(s)<1$. It was deduced by summing the infinite series in (6.4.3) with respect to the variable of summation m and then by applying our generalization of Watson's formula (6.3.2). But in the region of absolute convergence of $\zeta_{p,p'}(s,c)$, where it is representable as the series (6.4.1), it is possible to reverse the orders of summation and start to sum the double series with respect to the index n. Indeed, for $\mathrm{Re}(s)>1$,

$$\begin{aligned}&\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,c)\\&=\left(\frac{\pi}{\sqrt{1/c}}\right)^{-s}\Gamma(s)\sum_{m,n\neq0}\frac{\left(p^2+\lambda_m^2\right)\left(p'^2+\lambda_n'^2\right)}{\left(p(p+\frac{1}{\pi})+\lambda_m^2\right)\left(p'(p'+\frac{1}{\pi})+\lambda_n'^2\right)}\frac{1}{\left(\lambda_n'^2+\lambda_m^2c^{-1}\right)^s}\\&=\left(\frac{\pi}{\sqrt{1/c}}\right)^{-s}\Gamma(s)\,\zeta_{p',p}\left(s,\frac{1}{c}\right).\end{aligned}\tag{6.5.3}$$

Therefore, an alternative representation of $\zeta_{p,p'}(s,c)$ can be obtained once we apply Theorem 6.4.1 to $\zeta_{p',p}\left(s,\frac{1}{c}\right)$.[26] In fact, using (6.4.2) with c replaced by $1/c$ and with p replaced by p', we find that an equivalent continuation of $\zeta_{p,p'}\left(s,c\right)$ to the half-plane $\mathrm{Re}(s)<1$ is

$$\begin{aligned}\left(\frac{\pi}{\sqrt{c}}\right)^{-s}\Gamma(s)\,\zeta_{p,p'}(s,\,c)&=\frac{2c^{-s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)+2\,c^{\frac{s-1}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_p(2s-1)\\&+\frac{2^{4-2s}\,\pi^{1-s}\,c^{\frac{s-1}{2}}}{\Gamma(1-s)}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{1-2s}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma'\left(\frac{\lambda_n}{\sqrt{c}}\,(2y+1)\right)e^{\frac{2\pi\lambda_n}{\sqrt{c}}(2y+1)}-1}\,dy.\end{aligned}\tag{6.5.4}$$

[26] This trick of playing with Watson's formula in the region of absolute convergence of a symmetric double series has been applied before in this thesis, namely in Example 1.2.1 and in Theorem 4.5.1.

Since (6.4.2) and (6.5.4) both represent the Epstein zeta function $\zeta_{p,p'}(s,c)$ in the region $\mathrm{Re}(s)<1$, by uniqueness of the analytic continuation, we must have the equality

$$\begin{aligned}
&\frac{2c^{s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p'}}\,\zeta_p(2s)+2\,c^{\frac{1-s}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_{p'}(2s-1)\\
&+\frac{2^{4-2s}\,(\pi\sqrt{c})^{1-s}}{\Gamma(1-s)}\sum_{n=1}^{\infty}\frac{(p'^2+\lambda_n'^2)\lambda_n'^{1-2s}}{p'\left(p'+\frac{1}{\pi}\right)+\lambda_n'^2}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma\left(\lambda_n'\,(2y+1)\,\sqrt{c}\right)e^{2\pi\lambda_n'(2y+1)\sqrt{c}}-1}dy\\
&=\frac{2c^{-s/2}\,\pi^{-s}\Gamma(s)}{1+\frac{1}{\pi p}}\,\zeta_{p'}(2s)+2\,c^{\frac{s-1}{2}}\,\pi^{-\left(s-\frac{1}{2}\right)}\Gamma\left(s-\frac{1}{2}\right)\zeta_p(2s-1)\\
&+\frac{2^{4-2s}}{\Gamma(1-s)}\left(\frac{\pi}{\sqrt{c}}\right)^{1-s}\sum_{n=1}^{\infty}\frac{(p^2+\lambda_n^2)\lambda_n^{1-2s}}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\int_0^{\infty}\frac{y^{-s}(y+1)^{-s}}{\sigma'\left(\frac{\lambda_n}{\sqrt{c}}\,(2y+1)\right)e^{\frac{2\pi\lambda_n'}{\sqrt{c}}(2y+1)}-1}dy,
\end{aligned}\tag{6.5.5}$$

whenever $\mathrm{Re}(s)<1$. Take the substitution $c=\alpha^2/\pi^2$, replace s by $\frac{1+s}{2}$ and let $p=p'$ in (6.5.5): recalling the condition $\alpha\beta=\pi^2$, we may reduce (6.5.5) to

$$\begin{aligned}
&\frac{\pi^{-(s+1)}\,\Gamma\left(\frac{s+1}{2}\right)}{4\left(1+\frac{1}{\pi p}\right)}\,\zeta_p(s+1)\left\{\alpha^{\frac{1+s}{2}}-\beta^{\frac{1+s}{2}}\right\}+\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\zeta_p(s)\left\{\alpha^{\frac{1-s}{2}}-\beta^{\frac{1-s}{2}}\right\}\\
&=\frac{2^{-s}}{\Gamma\left(\frac{1-s}{2}\right)}\beta^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{\sigma\left(\frac{\lambda_n\beta}{\pi}\,(2y+1)\right)e^{2\beta\lambda_n(2y+1)}-1}\,dy\\
&-\frac{2^{-s}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\lambda_n^{-s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{\sigma\left(\frac{\lambda_n\alpha}{\pi}\,(2y+1)\right)e^{2\alpha\lambda_n(2y+1)}-1}dy.
\end{aligned}\tag{6.5.6}$$

Now, (6.5.1) is finally obtained once we use the functional equation for $\zeta_p(s+1)$ on the left-hand side of (6.5.6) and the definition of $\mathcal{K}_{\nu,p}(x)$, (6.5.2). □

Remark 6.5.1. By taking a proper deformation of the contour defining the integral (6.5.2), it is possible to have a formula similar to (6.5.1) and valid for any $s\in\mathbb{C}$. See Remark 6.3.1 above.

With a slight change in the sequence of Ramanujan's statements in [179], we now establish a generalized version of Entry 3.3.3, also known as Koshliakov's formula (6.1.11).

Corollary 6.5.1. *Let α and β denote two positive real numbers such that $\alpha\beta=\pi^2$. Define*

$$\gamma_p:=C_p^{(2)}+\frac{C_p^{(1)}}{1+\frac{1}{\pi p}}-\gamma-\frac{2e^{2\pi p}\,Q_{2\pi p}(0)}{1+\frac{1}{\pi p}}-\log(2\pi)\left(1-\frac{1}{1+\frac{1}{\pi p}}\right),$$

where $C_p^{(1)}$ is given by (6.3.20) and $C_p^{(2)}$ denotes Koshliakov's second analogue of the Euler-Mascheroni constant,

$$C_p^{(2)}:=\lim_{n\to\infty}\left\{\sum_{k=1}^{n-1}\frac{(1,2\pi pk)_k}{k}-\frac{\log(n)}{1+\frac{1}{\pi p}}\right\}.\tag{6.5.7}$$

Moreover, $Q_{\mu}(s)$ represents (in Koshliakov's notation) the incomplete gamma function $\Gamma(s,\mu)$,

$$Q_{\mu}(s)=\int_{\mu}^{\infty} t^{s-1}e^{-t}dt.$$

Then the following generalization of Entry 3.3.3, (6.1.11), holds

$$\begin{aligned}&\sqrt{\alpha}\left(\frac{1}{4}\gamma_p-\frac{\log(4\beta)}{4\left(1+\frac{1}{\pi p}\right)}+\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\mathcal{K}_{0,p}\left(2\alpha\lambda_n\right)\right)\\ =&\sqrt{\beta}\left(\frac{1}{4}\gamma_p-\frac{\log(4\alpha)}{4\left(1+\frac{1}{\pi p}\right)}+\sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\mathcal{K}_{0,p}\left(2\beta\lambda_n\right)\right),\end{aligned}\tag{6.5.8}$$

where $\mathcal{K}_{\nu,p}(x)$ denotes the function defined by (6.5.2) and $\sigma(t)$ is (6.1.23).

Proof. The proof comes after letting $s\to 0$ in the generalized Ramanujan-Guinand formula (6.5.1). Invoking the Laurent expansion for $\Gamma(s)$ around $s=0$, (6.3.22), along with the formulas [[132], p. 22, pp. 48 and 49, eqs. (52), (54)]

$$\zeta_p(s)=-\frac{1}{2}\frac{1}{1+\frac{1}{\pi p}}+\left(\frac{1}{2}C_p^{(2)}-\frac{\gamma}{2}-\frac{e^{2\pi p}}{1+\frac{1}{\pi p}}Q_{2\pi p}(0)-\frac{\log(2\pi)}{2}\right)s+O\left(s^2\right)$$

and

$$\eta_p(s)=-\frac{1}{2}+\left(\frac{1}{2}C_p^{(1)}-\frac{\gamma}{2}-\frac{\log(2\pi)}{2}\right)s+O\left(s^2\right),$$

we can immediately derive (6.5.8). □

By letting $p\to 0^+$ or $p\to\infty$, we are able to deduce formulas akin to the classical ones, because in these cases the integral (6.5.2) can be written as a series of Bessel functions.

Corollary 6.5.2. *If α and β are two positive real numbers such that $\alpha\beta=\pi^2$ and if s is any complex number, then the following formulas hold*

$$\begin{aligned}&\sqrt{\alpha}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2mn\alpha\right)-\sqrt{\beta}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}\left(2mn\beta\right)\\ =&\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\left\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\right\}+\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\left\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\right\},\end{aligned}\tag{6.5.9}$$

$$\begin{aligned}&\sqrt{\alpha}\sum_{m,n=1}^{\infty}\frac{(-1)^m m^{s/2}}{(2n-1)^{s/2}}K_{\frac{s}{2}}\left((2n-1)m\alpha\right)-\sqrt{\beta}\sum_{m,n=1}^{\infty}\frac{(-1)^m m^{s/2}}{(2n-1)^{s/2}}K_{\frac{s}{2}}\left((2n-1)m\beta\right)\\ &=\frac{\Gamma\left(\frac{s}{2}\right)\left(2^{s/2}-2^{-s/2}\right)\zeta(s)}{4}\left\{\beta^{\frac{1-s}{2}}-\alpha^{\frac{1-s}{2}}\right\}.\end{aligned}\tag{6.5.10}$$

Moreover, we have the following formulas of Koshliakov-type

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\beta)+\sum_{m,n=1}^{\infty}K_0(2mn\alpha)\right)=\sqrt{\beta}\left(\frac{1}{4}\gamma-\frac{1}{4}\log(4\alpha)+\sum_{m,n=1}^{\infty}K_0(2mn\beta)\right), \quad (6.5.11)$$

$$\sqrt{\alpha}\left(\sum_{m,n=1}^{\infty}(-1)^m K_0\left((2n-1)m\alpha\right)-\frac{\log(2)}{4}\right)=\sqrt{\beta}\left(\sum_{m,n=1}^{\infty}(-1)^m K_0\left((2n-1)m\beta\right)-\frac{\log(2)}{4}\right). \quad (6.5.12)$$

Proof. We only prove the classical Ramanujan-Guinand formula (6.5.9) from our generalized version of Entry 3.3.1. The proofs of the remaining formulas (6.5.10), (6.5.11) and (6.5.12) follow the same principle. The only part that it is not immediate is the recovery of the Modified Bessel function appearing in (6.5.9) from the left-hand side of (6.5.1).

Indeed, assuming that $\text{Re}(s)<1$ and justifying once more all the intermediate steps by absolute convergence, we find that the first term on the left of (6.5.1) is

$$\begin{aligned}
&\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty}\frac{y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}}{e^{2\alpha n(2y+1)}-1}dy\\
&=\frac{2^{-s}\sqrt{\pi}}{\Gamma\left(\frac{1-s}{2}\right)}\alpha^{\frac{1-s}{2}}\sum_{m,n=1}^{\infty}\frac{1}{n^s}\int_0^{\infty}e^{-2\alpha mn(2y+1)}y^{-\frac{s+1}{2}}(y+1)^{-\frac{s+1}{2}}dy\\
&=\sqrt{\alpha}\sum_{m,n=1}^{\infty}\left(\frac{m}{n}\right)^{\frac{s}{2}}K_{\frac{s}{2}}(2mn\alpha)=\sqrt{\alpha}\sum_{n=1}^{\infty}\sum_{d|n}d^{-s}n^{\frac{s}{2}}K_{\frac{s}{2}}(2n\alpha)\\
&=\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)\,n^{s/2}K_{\frac{s}{2}}(2n\alpha),
\end{aligned} \quad (6.5.13)$$

where the second equality is justified by the integral representation (6.3.11). Note now that the expression on the extreme right-hand side of (6.5.13) is an infinite series that converges uniformly and absolutely for any $s\in\mathbb{C}$. Since its summands are analytic functions of s, this series defines an analytic function of s as well. By the principle of analytic continuation, we now see that (6.5.9) must hold for any $s\in\mathbb{C}$. □

Finally, we derive a generalized transformation formula for the logarithm of the Dedekind $\eta-$function, (6.1.8). It is noteworthy that a special case of the forthcoming formula (arising from the condition $p=p'$) was previously established by Dixit and Gupta [[68], p. 15, Theorem 4.5]. However, our approach significantly differs from theirs, as it relies on the generalized Kronecker limit formula (6.4.6) introduced in the preceding section.

Corollary 6.5.3. *Let $p, p' \in \mathbb{R}_+$ and $\sigma(t)$, $\sigma'(t)$ be defined by (6.1.23). If $\alpha, \beta > 0$ are two positive real numbers such that $\alpha\beta = \pi^2$, then the following generalization of Entry 3.3.2 holds*

$$\sum_{n=1}^{\infty} \frac{p'^2 + \lambda_n'^2}{p'\left(p' + \frac{1}{\pi}\right) + \lambda_n'^2} \frac{\lambda_n'^{-1}}{\sigma\left(\frac{\lambda_n'\alpha}{\pi}\right) e^{2\alpha\lambda_n'} - 1} - \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{\lambda_n^{-1}}{\sigma'\left(\frac{\lambda_n\beta}{\pi}\right) e^{2\beta\lambda_n} - 1}$$
$$= \frac{1}{12\left(1 + \frac{1}{\pi p}\right)\left(1 + \frac{1}{\pi p'}\right)} \left\{\beta \frac{1 + \frac{3}{\pi p'}(1 + \frac{1}{\pi p'})}{1 + \frac{1}{\pi p'}} - \alpha \frac{1 + \frac{3}{\pi p}(1 + \frac{1}{\pi p})}{1 + \frac{1}{\pi p}}\right\}$$
$$+ \frac{C_p^{(1)} - C_{p'}^{(1)}}{2} + \frac{1}{4} \log\left(\frac{\alpha}{\beta}\right). \tag{6.5.14}$$

Proof. We have seen in the proof of Theorem 6.5.1 (see (6.5.3) above) that, for any $s \in \mathbb{C}$, the following relation holds

$$\zeta_{p,p'}(s,\, c) = c^{-s}\, \zeta_{p',p}\left(s,\, \frac{1}{c}\right). \tag{6.5.15}$$

Hence, applying Kronecker's limit formula to the function $\zeta_{p',p}\left(s,\, \frac{1}{c}\right)$ must lead to the same Laurent expansion as (6.4.6). In order to apply Kronecker's limit formula to the right-hand side of (6.5.15), we need to use (6.4.6) with c replaced by c^{-1} and p by p': from these simple substitutions, it is not hard to deduce that $c^{-s}\zeta_{p',p}\left(s,\, \frac{1}{c}\right)$ must have the meromorphic expansion

$$c^{-s}\zeta_{p,p'}\left(s,\, \frac{1}{c}\right) = \frac{\pi}{\sqrt{c}} \frac{1}{s-1} - \frac{\pi \log(c)}{\sqrt{c}} + \frac{\pi^2}{3c} \frac{1 + \frac{3}{\pi p'}(1 + \frac{1}{\pi p'})}{\left(1 + \frac{1}{\pi p}\right)\left(1 + \frac{1}{\pi p'}\right)^2}$$
$$+ \pi\sqrt{c}\left(2C_p^{(1)} - \log\left(\frac{4}{c}\right) + 4\sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{\lambda_n^{-1}}{\sigma'\left(\sqrt{\frac{1}{c}}\lambda_n\right) e^{2\pi\sqrt{\frac{1}{c}}\lambda_n} - 1}\right)$$
$$+ O\left(s - 1\right). \tag{6.5.16}$$

Since (6.5.15) is valid for all $s \in \mathbb{C}$ by analytic continuation, the constant terms of the Laurent expansions (6.5.16) and (6.4.6) must coincide. By equating these constant terms and substituting $c = \alpha^2/\pi^2$, we immediately arrive at (6.5.14). □

Our next corollary presents a particular case of the previous result when we take $p \to 0^+$ and $p' \to \infty$. It is not explicitly given in [68], so we shall state it here.

Corollary 6.5.4. *Let $\alpha, \beta > 0$ be such that $\alpha\beta = \pi^2$. Then the following identity takes place*

$$\sum_{n=1}^{\infty} \frac{1}{n\left(e^{2\alpha n} + 1\right)} + 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \frac{1}{\left(e^{\beta(2n-1)} - 1\right)} = \frac{\alpha}{4} - \log(2) - \frac{1}{4}\log\left(\frac{\alpha}{\beta}\right).$$

Using the same method as in the previous Corollary 6.5.3, we state the final general formula of this chapter. This result is also an extension of a formula that can be found in one of Ramanujan's notebooks [[179], p. 318], although originally discovered by Schlömlich.[27] We will omit its proof, because it consists in using (6.5.14) with $p = p'$, dividing by $\beta - \alpha$ and then letting $\alpha \to \beta$.

[27]See [[28], p. 256] for a historical account of this particular formula of Schlömlich, i.e., (6.5.18).

Corollary 6.5.5. *Let $p \in \mathbb{R}_+$ and let $\sigma(t)$ be defined by (6.1.23). Then the following identity holds*

$$\sum_{n=1}^{\infty} \frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2} \frac{e^{2\pi\lambda_n}}{\left(\sigma(\lambda_n)e^{2\pi\lambda_n}-1\right)^2} \left\{\pi\sigma(\lambda_n)+\frac{p}{(p-\lambda_n)^2}\right\}=\frac{\pi}{24}\frac{1+\frac{3}{\pi p}\left(1+\frac{1}{\pi p}\right)}{\left(1+\frac{1}{\pi p}\right)^3}-\frac{1}{8}. \tag{6.5.17}$$

Taking $p \to \infty$ in (6.5.17) results in the following relation

$$\sum_{n=1}^{\infty} \frac{e^{2\pi n}}{\left(e^{2\pi n}-1\right)^2} = \frac{1}{24} - \frac{1}{8\pi},$$

which is equivalent to Schlömlich's formula,[28]

$$\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1} = \frac{1}{24} - \frac{1}{8\pi}. \tag{6.5.18}$$

On the other hand, by letting $p \to 0^+$ in (6.5.17), we are led to a companion of (6.5.18), which seems to be new.

Corollary 6.5.6. *The following identity holds*

$$\sum_{n=1}^{\infty} \frac{e^{\pi(2n-1)}}{\left(e^{\pi(2n-1)}+1\right)^2} = \frac{1}{8\pi}.$$

Note that Corollary 6.5.5 has a particularly interesting consequence. It implies that, for any algebraic p, the infinite series on the left-hand side of (6.5.17) is always a transcendental number. The proof of this assertion is, of course, an immediate consequence of the transcendence of π. The study of the transcendental nature of some important infinite series has a fascinating history. In his doctoral thesis, Weatherby [234] analyzed the arithmetic nature of several infinite series. While some of his results rely on Schanuel's conjecture, most of them are unconditional (see, for instance, [[234], p. 98, Corollary 5.38]). The classes of infinite series treated in the fifth chapter of [234], as well as the proofs of their transcendence, are primarily inspired by an interesting paper of Pilehrood and Pilehrood [115]. The work presented in Weatherby's thesis was later generalized in [161] and, more recently, in [155].

Following Weatherby's approach, we now demonstrate that two infinite series examined in this chapter are indeed transcendental numbers. We begin by introducing two classical lemmas that play a key role in proving our final theorem. The first is a powerful result due to Nesterenko [162].

Lemma 6.5.1 (Nesterenko's Theorem [162])**.** *Let $N \in \mathbb{N}$. Then the numbers π and $e^{\pi\sqrt{N}}$ are algebraically independent.*

[28] we should remark that Dixit and Gupta [[68], Corollary 5.4] also obtain a general formula that reduces to (6.5.18). But their formula seems to differ from (6.5.17), as it is written as a particular case of their generalization of Ramanujan's formula for $\zeta(2n+1)$.

Lemma 6.5.2 ([[234], p. 73, Lemma 5.3]). *If $z_1, ..., z_\ell$ are algebraically independent over a field K, then, for any pairwise distinct nonzero rational numbers $\frac{\alpha_1}{\beta_1}, ..., \frac{\alpha_\ell}{\beta_\ell}$, the elements $z_1^{\frac{\alpha_1}{\beta_1}}, ..., z_\ell^{\frac{\alpha_\ell}{\beta_\ell}}$ are also algebraically independent over the field K.*

Remark 6.5.2. A combination of both lemmas shows that, for $\alpha, \beta \in \mathbb{N}$ such that $(\alpha, \beta) = 1$, π and $e^{\pi\sqrt{\frac{\alpha}{\beta}}}$ are algebraically independent.

Theorem 6.5.2. *Let p be a positive algebraic number and $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of numbers satisfying the transcendental equation*

$$p\sin(\pi y) + y\cos(\pi y) = 0.$$

Furthermore, suppose that $\alpha, \beta \in \mathbb{N}$ are such that $(\alpha, \beta) = 1$. Then the following infinite series

$$S_p(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{1}{\alpha + \beta\,\lambda_n^2} \tag{6.5.19}$$

and

$$T_p(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{\lambda_n^2}{\alpha^4 + 4\beta^4\lambda_n^4} \tag{6.5.20}$$

are transcendental numbers.

Proof. From (6.3.24), we see that the infinite series $S_p(\alpha, \beta)$, (6.5.19), satisfies the identity

$$S_p(\alpha, \beta) = \frac{\pi}{2\sqrt{\alpha\beta}} - \frac{1}{1 + \frac{1}{\pi p}} \frac{1}{2\alpha} + \frac{\pi}{\sqrt{\alpha\beta}} \frac{1}{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right) e^{2\pi\sqrt{\frac{\alpha}{\beta}}} - 1}. \tag{6.5.21}$$

Clearly, if $p = \sqrt{\frac{\alpha}{\beta}}$, then it follows from the transcendence of π that $S_p(\alpha, \beta)$ is a transcendental number. Now, to deal with the case where $p \neq \sqrt{\frac{\alpha}{\beta}}$, let us assume that $S_p(\alpha, \beta)$ is algebraic. Multiplying both sides of (6.5.21) by $\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right) e^{2\pi\sqrt{\alpha/\beta}} - 1$ and rearranging, we have the equality

$$\begin{aligned} &-\frac{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right)}{2\sqrt{\alpha\beta}}\pi\, e^{2\pi\sqrt{\frac{\alpha}{\beta}}} + \left\{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right) S_p(\alpha, \beta) + \frac{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right)}{2\alpha\left(1 + \frac{1}{\pi p}\right)}\right\} e^{2\pi\sqrt{\frac{\alpha}{\beta}}} \\ &-\frac{\pi}{2\sqrt{\alpha\beta}} - \frac{1}{2\alpha\left(1 + \frac{1}{\pi p}\right)} - S_p(\alpha, \beta) = 0. \end{aligned}$$

Since $\alpha/\beta \in \mathbb{Q}$ and p is algebraic, then $\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right) = \frac{\sqrt{\beta}p + \sqrt{\alpha}}{\sqrt{\beta}p - \sqrt{\alpha}}$ is algebraic as well. Therefore, it follows from the contradiction hypothesis that $\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right) S_p(\alpha, \beta)$ is also algebraic, and so the nonzero polynomial in two variables defined by

$$P(X,Y) := -\frac{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right)}{2\sqrt{\alpha\beta}}XY + \left\{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right)S_p(\alpha,\beta) + \frac{\sigma\left(\sqrt{\frac{\alpha}{\beta}}\right)}{2\alpha\left(1+\frac{1}{\pi p}\right)}\right\}X$$
$$- \frac{Y}{2\sqrt{\alpha\beta}} - \frac{1}{2\alpha\left(1+\frac{1}{\pi p}\right)} - S_p(\alpha,\beta)$$

has algebraic coefficients and satisfies $P\left(e^{2\pi\sqrt{\frac{\alpha}{\beta}}},\pi\right) = 0$. But this contradicts Nesterenko's theorem, which says that $e^{2\pi\sqrt{\frac{\alpha}{\beta}}}$ and π are algebraically independent (see Remark 6.5.2). Therefore, $S_p(\alpha,\beta)$ must be transcendental.

Proving the transcendence of the infinite series $T_p(\alpha,\beta)$, (6.5.20), is analogous. First, note that the series $T_p(\alpha,\beta)$ is never zero because all of its terms are positive. Using the summation formula (6.2.23) and recalling (6.2.3), we have that

$$T_p(\alpha,\beta) := \sum_{n=1}^{\infty}\frac{p^2+\lambda_n^2}{p\left(p+\frac{1}{\pi}\right)+\lambda_n^2}\,\frac{\lambda_n^2}{\alpha^4+4\beta^4\lambda_n^4} = \frac{\pi}{8\alpha\beta^3}\left\{1+G_p\left(\frac{\alpha^2}{2\beta^2}\right)\right\} = \frac{\pi}{8\alpha\beta^3}\,\frac{A_p(\alpha,\beta)}{B_p(\alpha,\beta)}, \tag{6.5.22}$$

where $A_p(\alpha,\beta)$ and $B_p(\alpha,\beta)$ are explicitly given by

$$A_p(\alpha,\beta) = 2\beta^2p^2\left(\sinh\left(\pi\frac{\alpha}{\beta}\right) - \sin\left(\pi\frac{\alpha}{\beta}\right)\right)$$
$$+ 2\alpha\beta\,p\left(\cosh\left(\pi\frac{\alpha}{\beta}\right) - \cos\left(\pi\frac{\alpha}{\beta}\right)\right) + \alpha^2\left(\sinh\left(\pi\frac{\alpha}{\beta}\right) + \sin\left(\pi\frac{\alpha}{\beta}\right)\right)$$

and

$$B_p(\alpha,\beta) = 2\beta^2p^2\left(\cosh\left(\pi\frac{\alpha}{\beta}\right) - \cos\left(\pi\frac{\alpha}{\beta}\right)\right)$$
$$+ 2\alpha\beta\,p\left(\sinh\left(\pi\frac{\alpha}{\beta}\right) + \sin\left(\pi\frac{\alpha}{\beta}\right)\right) + \alpha^2\left(\cosh\left(\pi\frac{\alpha}{\beta}\right) + \cos\left(\pi\frac{\alpha}{\beta}\right)\right).$$

Since $\cos\left(\pi\frac{\alpha}{\beta}\right)$ and $\sin\left(\pi\frac{\alpha}{\beta}\right)$ are always algebraic numbers for $\alpha,\beta\in\mathbb{N}$ and p is algebraic by hypothesis, it is clear that (6.5.22) can be written as

$$\frac{\pi}{T_p(\alpha,\beta)} = \frac{Ae^{2\pi\frac{\alpha}{\beta}} + Be^{\pi\frac{\alpha}{\beta}} + C}{De^{2\pi\frac{\alpha}{\beta}} + Ee^{\pi\frac{\alpha}{\beta}} + F}$$
$$\Longleftrightarrow \frac{D}{T_p(\alpha,\beta)}\pi e^{2\pi\frac{\alpha}{\beta}} + \frac{E}{T_p(\alpha,\beta)}\pi e^{\pi\frac{\alpha}{\beta}} + \frac{F}{T_p(\alpha,\beta)}\pi - Ae^{2\pi\frac{\alpha}{\beta}} - Be^{\pi\frac{\alpha}{\beta}} - C = 0,$$

where A,B,C,D,E,F are nonzero algebraic numbers. If we assume that $T_p(\alpha,\beta)$ is also algebraic, then the nonzero polynomial

$$Q(X,Y) := \frac{D}{T_p(\alpha,\beta)}X^2Y + \frac{E}{T_p(\alpha,\beta)}XY - AX^2 - BX + \frac{F}{T_p(\alpha,\beta)}Y - C$$

has algebraic coefficients and satisfies $Q\left(e^{\pi\frac{\alpha}{\beta}}, \pi\right) = 0$, which once more contradicts Lemma 6.5.1. $\square$

Remark 6.5.3. Using the summation formulas deduced in this chapter, we can prove analogues of the results given in [115, 155, 161, 234]. In particular, by differentiation, it is also possible to prove the transcendence of the infinite series

$$S_{p,N}(\alpha, \beta) := \sum_{n=1}^{\infty} \frac{p^2 + \lambda_n^2}{p\left(p + \frac{1}{\pi}\right) + \lambda_n^2} \frac{1}{(\alpha + \beta\,\lambda_n^2)^N}, \quad N \in \mathbb{N},$$

whenever p is algebraic. We leave the details and further extensions to the interested reader.

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