Stackelberg Duopoly With Uncertainty And Mean-Standard Deviation Preference

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Resumo

Nesta dissertação, abordamos um jogo de Stackelberg entre uma firma líder e uma firma secundária em que a firma líder tem incerteza sobre a procura. Consideramos a procura como uma variável aleatória discreta que toma um de dois valores - alto ou baixo. A novidade consiste no uso da maximização de um tipo de preferência relativa à média-variância, comum em matemática financeira, em vez da maximização da utilidade esperada como é usual em teoria de jogos. Mais especificamente, usamos uma preferência dealing média-desvio padrão. A melhor resposta para a firma secundária é abordada. Caracterizamos a melhor resposta da firma líder usando maximização da utilidade esperada e mostramos como tal resposta depende das quantidades económicas. Apresentamos exemplos de quantidades económicas em que a vantagem da firma líder desaparece. Sob maximização da preferência média-desvio padrão, caracterizamos completamente a melhor resposta da firma líder como função de um parâmetro de aversão ao risco. Mostramos como para o mesmo nível de aversão ao risco, a melhor resposta da firma líder depende das quantidades económicas. Uma análise ex-ante e ex-post é apresentada para um domínio específico das quantidades económicas e apresentamos condições suficientes para que a empresa secundária tenha um lucro maior. O caso em que a firma líder não arrisca praticar dumping é analisado.

Palavras-chave: Stackelberg, Teoria de Jogos, Risco, Incerteza, Preferência Média-Desvio Padrão

Abstract

We discuss a Stackelberg game between a leading and a follower firm with demand uncertainty for the leader firm. We consider the demand as a discrete random variable that takes one of two values - low or high. The novelty consists in using a type of mean-variance utility maximization, common in mathematical finance, instead of the expected value maximization as is usual in the game theory literature. More specially, we use a mean-standard deviation preference. The best response for the follower firm is determined. We characterize the leading firm's best response for expected utility maximization and show that it depends on the economical quantities. We present examples in which the leading firms's first mover advantage disappears. Under mean-standard deviation, we characterize completely the leading firm's best response as a function of a risk aversion parameter. We show that for the same level of risk aversion, the leading firm's best response depends on the economical quantities. An ex-ante and ex-post static analysis is presented for a specific domain of the economical quantities and we present sufficient conditions for the follower firm to have a higher payoff. The case where the leading firm does not incur in dumping is analyzed.

Keywords: Stackelberg, Game Theory, Risk, Uncertainty, Mean-Standard Deviation Preference

Table of Contents

List of Figures

List of Tables

List of Abbreviations

CHAPTER 1

Introduction

The Stackelberg duopoly model is one of the most used models in industrial organization. It was proposed by [Stackelberg, 1934], and consists of a game where a leader firm moves first (pioneer) and a follower firm moves second. In it's classical setting, each firm chooses the quantity of a certain good to be produced in a given market. It belongs to the category of dynamic games with complete and perfect information. In equilibrium, the leading firm obtains a higher profit than the follower, reflecting the advantage it has from moving first.

The success of pioneering firms has often been attributed to the advantage they have due to being the first on the market. Dell is an example of a successful pioneering firm. It was the first to introduce the direct sale business model into the PC market with great success. Other such example is Amazon. It was the first to introduce an online bookstore and is still able to maintain its number one status on the market. However, many cases of pioneering firms that were not able to sustain a business have ocurred along the years. Examples of this are Pets.com, Webvan, Garden.com and eToys, that were first movers in their respective markets during the dotcom booming era but failed. (see [Liu, 2005] and references within)

Questions about the pioneering firm's success being solely due to first mover advantage arise. Other factors influence the probability of success of the pioneering firm, such as private information, uncertainty and product differentiation.

Studies about the influence of these and other factors have been conducted. For example, [Gal-Or, 1987] studies the first mover disadvantages when there's private information about stochastic demand. The author found a range of parameter values over which the follower is better off compared to the leader. [Liu, 2005] considers a Stackelberg model with demand uncertainty for the first mover only, where the demand follows a uniform distribution. Liu finds that if the realized demand is far from it's expected value, the leadership advantage is overpowered by the follower firm's benefits due to flexibility. [Ferreira et al., 2015] generalizes Liu's model to include product differentiation. A parameter is used that measures to which degree the good produced by each firm is a substitute for the other. They conclude that if the products are differentiated enough, then even with low uncertainty the follower firm can achieve a higher profit.

We consider a Stackelberg game with asymmetric demand uncertainty for the leading firm. The demand is a discrete random variable that can take one of two values - high or low. In a first instance, we use expected utility maximization for the leading firm and find the subgame perfect Nash equilibrium depending on the economical quantities. This equilibrium is found by the method of backwards induction. We then consider a mean-standard deviation preference for the leading firm. This preference reflects the leading firm's risk aversion with a parameter. We find the leading firm's best response as a function of the economical quantities and the risk aversion parameter.

Expected utility theory has been central in the analysis of decision making under risk. In utility theory, the utilities of outcomes are weighted by their probabilities. It is also the usual setting in game theory, used by the agents to find their best response. Thus it is assumed that a rational agent would obey the axioms of the theory. However, several critiques of expected utility theory appeared along the years with perhaps the most well known being a counter-example given by [Allais, 1953]. A variation of Allais' example is found on [Kai-Ineman and Tversky, 1979]. The authors conduct an empirical study where a significant amount of agents were exposed to a set of problems. The answers gathered reflected consistent violations of the axioms of expected utility theory. They present an alternative to expected utility maximization called Prospect Theory. One of the authors, Daniel Kahneman, won the Nobel Memorial Prize in Economics for his work developing that theory.

Such critiques motivate us to search for an alternative that better reflects the degree of risk aversion of an agent. We replace the expected value maximization with a type of mean-variance utility, common in mathematical finance. More specifically we use a mean-standard deviation preference. The agent will choose the strategy that maximizes that preference, which we define below. Let π be a utility function depending on a set of parameters q, $\delta \geq 0$ and $\sigma[\pi]$ the standard deviation.

Definition 1 (Mean-standard deviation preference)**.** *The mean-standard deviation preference for an agent with utility function $\pi(q)$ is*

$$\mathbb{E}[\pi(q,\delta)] = \mathbb{E}[\pi(q)] - \delta\sigma[\pi]$$

where δ is the risk aversion degree.

Remark. *Although we write $\mathbb{E}[\pi(q,\delta)]$, the function that depends on δ is the mean-standard deviation function and not π. Moreover, we will sometimes write $\mathbb{E}[\pi(q)]$ or $\mathbb{E}[\pi(\delta)]$, according to the dependence we want to emphasize. The dependence on the other variable is implicit.*

Mean-variance preference has been used in the context of game theory before. For example, in [Kopányi, 2013] the author considers a game in which two firms simultaneously decide on both the price and the production level of a homogeneous product. Both firms have conjectures about the actions of the other. Our model differs by being a dynamic game in which we use mean-standard deviation preference instead of mean-variance utility.

We also use mean-standard deviation preference instead of variance to avoid the inflation effect that comes with using the variance. This effect is corrected by using the standard deviation. Moreover, there is a more intuitive interpretation with standard deviation, due to it being measured in terms of the units we are dealing with.

1.1 The classical Stackelberg duopoly

We now briefly present the classical Stackelberg duopoly. Our goal is twofold. On one hand, we have a base model that will function as a particular case of the models we consider in this dissertation. On the other hand, with the exposure of the classical version, we introduce important notation that will be used and generalized when considering said models. The presentation in this section follows closely the one found on [Gibbons, 1992].

We start by defining the normal form of a static game.[1] A static game is a game where both players act at the same time. The normal form of a static game specifies:

(i) The strategies available to each player

(ii) The payoff received by the players for each combination of strategies

Let S_i denote the strategy space of player i and let s_i be any element in S_i. Denote by (s_1, s_2) a combination of strategies for player 1 and player 2 respectively. Furthermore, let $\pi_i(s_1, s_2)$ be the payoff function for player i when the combination of strategies (s_1, s_2) is played. We are now ready to define the normal form.

Definition 2. *The normal form representation of a 2 player game specifies the strategy spaces S_1 and S_2 and their payoff functions π_1 and π_2. We denote the game by $G = \{S_1, S_2; \pi_1, \pi_2\}$.*

The most important concept in game theory is perhaps the concept of Nash Equilibrium, which we define next.

Definition 3. *In the 2-player normal form game $G = \{S_1, S_2; \pi_1, \pi_2\}$, the strategies (s_1^*, s_2^*) are a Nash equilibrium if, for player 1, we have*

$$s_1^* = \underset{s_1 \in S_1}{argmax}\ \pi_1(s_1, s_2^*)$$

And for player 2 we have

$$s_2^* = \underset{s_2 \in S_2}{argmax}\ \pi_2(s_1^*, s_2)$$

So a pair of strategies is a Nash equilibrium if each player's strategy is the response that maximizes its utility given the other player's strategy.

In contrast with static games, the classical Stackelberg duopoly is a dynamic game. In a dynamic game, players act sequentially. It belongs to the category of games with complete and

[1] We will give the definitions and notation for games with only 2 players. We do this for simplicity and given that the models we discuss are of that nature.

perfect information. With complete information, each player knows the payoff function of the other. Perfect information means that each player knows exactly the state of the game at their time of play.

The usual way used to find the Nash equilibrium in dynamic games is backwards induction. It goes as follows:

(i) We compute the second mover's best response to an arbitrary strategy s_1 of the first mover, and denote this best response by $s_2^*(s_1)$

(ii) The first mover maximizes it's utility function taking into account the second mover's best response, i.e., it's strategy is

$$s_1^* = \max_{s_1 \in S_1} \pi_1(s_1, s_2^*(s_1))$$

(iii) The second player then plays $s_2^*(s_1^*)$

The Nash equilibrium is (s_1^*, s_2^*). Once we are dealing with dynamic games, this equilibrium is usually called the backwards induction outcome.

The players in a Stackelberg game are firms and their strategy is the quantity of a homogeneous good they will produce. Player i's strategy space is, then, $S_i = [0, \infty)$. The timing of the game is as follows:

1. Firm 1 (the leading firm) chooses a quantity q_1
2. Firm 2 (the follower firm) observes q_1 and chooses it's quantity q_2
3. The payoff to firm i is given by the profit function

$$\pi_i(q_1, q_2) = q_i(P(Q) - c)$$

where $P(Q) = \alpha - Q$ is the market-clearing price when the aggregated quantity on the market is $Q = q_1 + q_2$. The parameter α is the market demand and c is the production cost, equal for both firms. The normal form of this game is given by

$$G = ([0, \infty), [0, \infty); q_1(P(Q) - c), q_2(P(Q) - c))$$

Using backwards induction to find the equilibrium, we start by computing firm 2's best response to an arbitrary q_1:

$$q_2^* = \max_{q_2 \geq 0} \pi_2(q_1, q_2) = \max_{q_2 \geq 0} q_2(\alpha - q_1 - q_2 - c)$$

which, by the first order condition, yields

$$q_2^*(q_1) = \frac{\alpha - q_1 - c}{2}$$

provided that $q_1 < \alpha - c$ which we assume, otherwise firm 1 would necessarily sell at a price lower than the production cost. We now compute the leading firm's best response taking into account $q_2^*(q_1)$. Firm 1 wants to find

$$q_1^* = \max_{q_1 \geq 0} \pi_1(q_1, q_2^*(q_1)) = \max_{q_1 \geq 0} q_1 \frac{\alpha - q_1 - c}{2}$$

which yields

$$q_1^* = \frac{\alpha - c}{2} \text{ and } q_2^*(q_1^*) = \frac{\alpha - c}{4}$$

Given that the price is the same for both firms, the one that produces more has a higher payoff[2]. And so the leading firm has a higher payoff than the follower firm, which reflects the first mover advantage under complete and perfect information.

We now define some concepts that will be used in the following chapters.

Definition 4. *An information set for player i is a collection of sets containing the possible states of the game that i can experience at their time of play.*

In a game with perfect information, each set on the information set of any player contains only sets with one node (i.e., i knows exactly the state of the game). If the information set contains a set with more than one element, then the player is uncertain about the state of the game at their time of play (for example, the follower firm might not know the exact quantity produced by the leading firm).

We will consider a refinement for the Nash equilibrium in the context of dynamic games. For that, we must give the definition of subgame, which pertains dynamic games.

Definition 5. *A subgame begins at any singleton information set, and includes all the other game stages that go from that set onwards.*

In other words, a subgame is the game that starts at a player's time of play (other than the first) in which he knows exactly the state of the game. We now define the refinement.

Definition 6. *A Nash equilibrium is subgame perfect if the player's strategies constitute a Nash equilibrium in every subgame.*

This refinement eliminates Nash equilibrium in which one of the players makes non-credible threats, and is a subset of the set of Nash equilibria. To illustrate, let $\gamma \geq 0$. Suppose the follower firm's strategy is: if the leading firm produces $q_1 > \gamma$, then the follower will produce a quantity $q_{2,H}$ that makes the selling price dip below production cost; if the leading firm produces $q_1 \leq \gamma$, then the follower firm plays its optimal quantity. Here, the follower firm makes a non-credible threat in case the leading firm plays $q_1 > \gamma$. If we consider the subgame that starts after the leading firm plays $q_1 > \gamma$, then playing $q_{2,H}$ would not be the follower firm's best response, and thus its strategy is not subgame perfect.

After analyzing the classical Stackelberg game and providing the necessary definitions and notation, we present our model.

[2]Provided that the price is positive, which does happen here.

CHAPTER 2

Stackelberg duopoly with uncertainty

We consider a Stackelberg Duopoly with asymmetric uncertainty about the demand level for the leader firm. The leader does not know the exact demand at their time of play - to them, the demand is a random variable following a certain distribution. In our model, the demand is a discrete random variable α that takes one of two values: $\mathbb{P}(\alpha = \beta_L - c) = \mu_L$ and $\mathbb{P}(\alpha = \beta_H - c) = \mu_H$. The parameters β_L and β_H are the possible levels of demand: low or high, respectively. We consider $0 < \beta_L < \beta_H$. For simplicity, we set $\alpha_L = \beta_L - c$ and $\alpha_H = \beta_H - c$

In this chapter, we will use the principle of expected utility maximization for the leader firm. Backwards induction will be used to find the subgame perfect nash equilibrium. This model belongs to the category of dynamic games with complete but imperfect information. The information is imperfect in an asymmetric way, given that the only information set containing more than one state belongs to the leading firm.

2.1 Notation and the normal form

A normal form representation of this game can be given as follows. The set of players is $P = \{F_1, F_2\}$, where F_1 is the leader firm and F_2 is the follower firm. This notation should bring no ambiguity: think of the subscript as the time of play of each firm. The set of actions is the same for each player: $S_1 = S_2 = [0, \alpha_H]$. We denote an action taken by player i as $q_i \in S_i$.

Each player has an information set at their time of play. Player 1's information set, I_1, contains one element (representing the asymmetry in information) and Player 2's information set, I_2 contains 2 elements, as follows:

$$I_1 = \{\{\alpha_L, \alpha_H\}\} \qquad I_2 = \{\{(q_1, \alpha_L)\}, \{(q_1, \alpha_H)\}\}$$

A strategy for the leader firm consists of a choice of quantity from its action set, while a strategy for the follower firm consists of a choice of quantity for each of the possible sets it can receive. Thus, a strategy for F_1 is a single element $q_1 \in S_1$ while a strategy for F_2 is a pair of choices $(q_{2,L}, q_{2,H}) \in S_2 \,\mathrm{x}\, S_2$. The follower firm chooses $q_{2,L}$ in case the information set it receives is (q_1, α_L) and chooses $q_{2,H}$ if the information set it receives is (q_1, α_H).

The only thing left to specify are the payoff functions. Both of these are given by the profit of each firm and are equal to the amount produced multiplied by the price of the good. Before writing these functions explicitly, we need to specify how the price is determined. Let $Q_\alpha = q_1 + q_2(q_1,\alpha)$ denote the aggregated quantity produced at the end of the game. Note the subscript α: the aggregated quantity depends on the realization of the random variable α because the choice of quantity by F_2 depends on that realization. The price is then given as in the classical Stackelberg Game, by the linear inverse demand function $P_\alpha = \alpha - Q_\alpha = \alpha - q_1 - q_2(q_1,\alpha)$.

The utility functions for the leader and the follower firm respectively are given by:

$$\pi_1(q_1,q_2(q_1,\alpha)) = q_1 P_\alpha \quad \text{and} \quad \pi_2(q_1,q_2(q_1,\alpha)) = q_2 P_\alpha$$

We should note that the model we study in this chapter is a particular case of the one to be studied in the next chapter. However, we find it appropriate to treat it separately, as it is the usual setting in game theory, i.e., expected utility maximization. In the next section, we characterize both firm's best response using backwards induction.

2.2 Backwards induction and expected utility maximization

The first step towards finding equilibria using backwards induction is to find the best response from the last player to each of their sets of information. In our model, this means finding the best response of the follower firm once it realizes the level of demand and the quantity chosen by the leader firm. Denote this best response by $q_2^*(q_1,\alpha)$.

Lemma 2.2.1. *The best response for the follower firm is*

$$q_2^*(q_1,\alpha) = max\left\{\frac{\alpha - q_1}{2}, 0\right\}$$

Proof. We are going to solve this generally, thinking about $\{\alpha, q_1\}$ as any of F_2's possible sets of information. F_2 wants to maximize it's utility function, i.e., wants to find

$$q_2^*(q_1,\alpha) = \underset{q_2\in S_2}{\arg\max}\ \pi_2 = \underset{q_2\in S_2}{\arg\max}\ q_2(\alpha - q_1 - q_2)$$

As a function of q_2, this is a quadratic polynomial with negative second derivative. To find its maximizer we differentiate the expression with respect to q_2, equate it to zero and solve for q_2. This yields

$$q_2^*(q_1,\alpha) = \frac{\alpha - q_1}{2}$$

If $\alpha < q_1$, the best response for F_2 would be to produce a negative quantity. In the context of our model, this does not make sense. So the best response for F_2 is set to be

$$q_2^*(q_1,\alpha) = \max\left\{\frac{\alpha - q_1}{2}, 0\right\}$$

$\square$

Once analyzed, this expression for the best response makes sense: F_2 will only enter the market if F_1 produces less than the realized demand.

Given that F_1 can solve F_2's maximization problem as well, it should anticipate that F_2 will play $q_2^*(q_1, \alpha)$ and choose q_1 to maximize its expected utility accordingly. F_1 wants to find

$$q_1^* := \underset{q_1 \in S_1}{\arg\max}\, \mathbb{E}[\pi_1(q_1, q_2^*(q_1, \alpha))]$$

If $q_1 < \alpha_L$, F_2 always enters the market. But if $\alpha_L \leq q_1 < \alpha_H$, F_2 only enters the market if the realized demand is α_H. Taking into account F_2's best response, the price is then given by

$$P_\alpha = \begin{cases} \frac{\alpha - q_1}{2} & q_1 < \alpha \\ \alpha - q_1 & \alpha \leq q_1 \leq \alpha_H \end{cases}$$

Both P_α and π_1[1] are themselves random variables because they depend on the value of the demand. $\mathbb{E}[\pi_1(q_1)]$ is a piecewise function with 2 pieces. Each piece is the restriction of a different quadratic polynomial of $q_1 \in \mathbb{R}$, defined as follows:[2]

$$\pi_{1,L}(q_1) = \mu_L q_1 (\frac{\alpha_L - q_1}{2}) + \mu_H q_1 (\frac{\alpha_H - q_1}{2})$$

$$\pi_{1,H}(q_1) = \mu_L q_1 (\alpha_L - q_1) + \mu_H q_1 (\frac{\alpha_H - q_1}{2})$$

The expression for $\mathbb{E}[\pi_1]$ is

$$\mathbb{E}[\pi_1(q_1)] = \begin{cases} \pi_{1,L}(q_1) & 0 \leq q_1 < \alpha_L \\ \pi_{1,H}(q_1) & \alpha_L \leq q_1 \leq \alpha_H \end{cases}$$

We observe $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ are both differentiable with negative second derivative. This implies that $\mathbb{E}[\pi_1(q_1)]$ is piecewise differentiable. Moreover, $\pi_{1,L}(\alpha_L) = \pi_{1,H}(\alpha_L)$ and so $\mathbb{E}[\pi_1(q_1)]$ is continuous. Next, we characterize the leading firm's best response given the follower firm's best response.

2.3 Leading firm's best response

In this section, we start by finding the quantity that maximizes each of $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$. We then investigate the position of these maximizers relative to each other and determine the maximizer of each of $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ as a function of $q_1 \in [0, \alpha_L)$ and $q_1 \in [\alpha_L, \alpha_H]$ respectively. We will often refer to these intervals for q_1 as the domain in which $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ are valid, respectively. Lastly, we characterize the maximizer of $\mathbb{E}[\pi_1]$ (i.e., the leading firm's best response) under three hypothesis that cover all possible cases.

[1] To avoid an overcomplex notation, we will sometimes write, for example, π_1 instead of $\pi_1(q_1, q_2^*(q_1, \alpha))$ or q_2^* instead of $q_2^*(q_1, \alpha)$. At times, we write the dependence on one or two of the variables only with the goal of emphasizing the dependence on them.

[2] Recall that μ_L and μ_H are the probabilities of low and high demand respectively, and $\mu_L + \mu_H = 1$.

We will use $q_{1,L}^M$ and $q_{1,H}^M$ to denote the maximizers of $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ for $q_1 \in \mathbb{R}$.

Theorem 2.3.1. *The maximizers of $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ are, respectively:*

(i) $q_{1,L}^M = \frac{\mathbb{E}[\alpha]}{2}$

(ii) $q_{1,H}^M = \frac{\mathbb{E}[\alpha]}{2}\frac{1-\frac{\mu_H\alpha_H}{2\mathbb{E}[\alpha]}}{1-\frac{\mu_H}{2}}$

Proof. We proceed analogously for both maximizers: differentiate each function with respect to q_1, equate to zero and solve for q_1.

(i)

$$\begin{aligned}\frac{\mathrm{d}\pi_{1,L}}{\mathrm{d}q_1} &= \mu_L(\frac{\alpha_L - q_1}{2} - \frac{q_1}{2}) + \mu_H(\frac{\alpha_H - q_1}{2} - \frac{q_1}{2})\\ &= \mu_L(\frac{\alpha_L - 2q_1}{2}) + \mu_H(\frac{\alpha_H - 2q_1}{2})\\ &= \frac{1}{2}(\mu_L\alpha_l + \mu_H\alpha_H - 2q_1(\mu_L + \mu_H))\\ &= \frac{1}{2}(\mathbb{E}[\alpha] - 2q_1)\end{aligned} \tag{2.1}$$

(ii)

$$\begin{aligned}\frac{\mathrm{d}\pi_{1,H}}{\mathrm{d}q_1} &= \mu_L(\alpha_L - 2q_1) + \mu_H(\frac{\alpha_H - 2q_1}{2})\\ &= (\mu_L\alpha_L + \mu_H\alpha_H) - \frac{\mu_H\alpha_H}{2} - 2\mu_L q_1 - \mu_H q_1\\ &= \mathbb{E}[\alpha] - \frac{\mu_H\alpha_H}{2} - q_1(1+\mu_L)\end{aligned} \tag{2.2}$$

Now we solve for q_1:

$$\begin{aligned}0 &= \mathbb{E}[\alpha] - \frac{\mu_H\alpha_H}{2} - q_1(1+\mu_L)\\ q_1 &= \frac{\mathbb{E}[\alpha] - \frac{\mu_H\alpha_H}{2}}{1+\mu_L}\\ &= \frac{\mathbb{E}[\alpha]}{2}\frac{1-\frac{\mu_H\alpha_H}{2\mathbb{E}[\alpha]}}{1-\frac{\mu_H}{2}}\end{aligned} \tag{2.3}$$

□

We express it as a multiple of $\mathbb{E}[\alpha]/2$ for an easier comparison to $q_{1,L}^M$. Regarding this comparison, their position is always the same relative to each other, as the next lemma states.

Lemma 2.3.2. $q_{1,H}^M \leq q_{1,L}^M$

Proof.

$$q_{1,H}^M \leq q_{1,L}^M \iff \frac{1-\frac{\mu_H}{2}\frac{\alpha_H}{\mathbb{E}[\alpha]}}{1-\frac{\mu_H}{2}} \leq 1$$

Noting that $\alpha_H/\mathbb{E}[\alpha] > 1$, the inequality above holds as long as μ_H, μ_L are different from zero, which we are assuming. $\square$

We can only consider $q_{1,L}^M$ as a candidate to be the maximizer of $\mathbb{E}[\pi_1]$ if $q_{1,L}^M$ belongs to the domain for which $\mathbb{E}[\pi_1(q_1)] = \pi_{1,L}(q_1)$. Analogously for $q_{1,H}^M(q_1)$. The next lemma presents conditions for each of the maximizers to be valid candidates to the maximizer of $\mathbb{E}[\pi_1]$.

Lemma 2.3.3. *The conditions for $q_{1,L}^M$ and $q_{1,H}^M$ to be valid candidates to the maximizer of $\mathbb{E}[\pi_1]$ are, respectively:*

(i) $\frac{\mathbb{E}[\alpha]}{2} \leq \alpha_L$

(ii) $\alpha_L \leq \frac{\mu_H \alpha_H}{2}$

Proof. We prove one point at a time.

(i) This statement follows directly from the expression of $q_{1,L}^M$.

(ii) We would like $\alpha_L \leq q_{1,H}^M$, and

$$\begin{aligned} \alpha_L &\leq q_{1,H}^M \\ \alpha_L &\leq \frac{\mathbb{E}[\alpha]}{2} \frac{1 - \frac{\mu_H \alpha_H}{2\mathbb{E}[\alpha]}}{1 - \frac{\mu_H}{2}} \\ \alpha_L &\leq \frac{\mu_L \alpha_L + \mu_H \alpha_H - \frac{\mu_H \alpha_H}{2}}{2 - \mu_H} \\ \alpha_L &\leq \frac{\mu_L \alpha_L + \frac{\mu_H \alpha_H}{2}}{1 + \mu_L} \\ \alpha_L(1 + \mu_L) &\leq \mu_L \alpha_L + \frac{\mu_H \alpha_H}{2} \\ \alpha_L &\leq \frac{\mu_H \alpha_H}{2} \end{aligned} \tag{2.4}$$

$\square$

We state another result that will be important to determine the leading firm's best response.

Lemma 2.3.4. *Regarding the conditions for the maximizers to be valid candidates, we have*

$$\frac{\mu_H \alpha_H}{2} \leq \frac{\mathbb{E}[\alpha]}{2}$$

Proof. We just substitute $\mathbb{E}[\alpha]$ for its expression:

$$\begin{aligned} \frac{\mu_H \alpha_H}{2} &\leq \frac{\mathbb{E}[\alpha]}{2} \\ \mu_H \alpha_H &\leq \mu_L \alpha_L + \mu_H \alpha_H \\ 0 &\leq \mu_L \alpha_L \end{aligned} \tag{2.5}$$

$\square$

We are now ready to compute the quantity that maximizes $\mathbb{E}[\pi_1]$, which we will denote by q_1^*. Before that, we have the following lemma.

Lemma 2.3.5. *The maximizers of $\pi_{1,L}$ for $q_1 \in [0, \alpha_L)$ and $\pi_{1,H}$ for $q_1 \in [\alpha_L, \alpha_H]$ are, respectively:*

(i) $q_{1,L}^* := min\{q_{1,L}^M, \alpha_L\}$

(ii) $q_{1,H}^* := max\{q_{1,H}^M, \alpha_L\}$

Proof. Both statements follow from the fact that $\pi_{1,L}$ and $\pi_{1,H}$ are quadratic polynomials with negative second derivative and given the domain for which they define $\mathbb{E}[\pi_1]$. □

We present three hypothesis that cover all the possible cases and compute q_1^* for each one.

Hypothesis 1: $\mathbb{E}[\alpha] \leq 2\alpha_L$ **(Large low intersect demand)**

Recall that this implies $q_{1,L}^M \leq \alpha_L$ and, according to Lemma 2.3.4, that $\mu_H \alpha_H \leq \mathbb{E}[\alpha] \leq 2\alpha_L$.

Theorem 2.3.6. *Under the conditions of Hypothesis 1,*

$$q_1^* = q_{1,L}^M = \frac{\mathbb{E}[\alpha]}{2}$$

Proof. Hypothesis 1 implies that $q_{1,H}^M \leq \alpha_L$, i.e., $q_{1,H}^* = \alpha_L$. $\mathbb{E}[\pi_1]$ being continuous at $q_1 = \alpha_L$ together with the fact that both $\pi_{1,L}(q_1)$ and $\pi_{1,H}(q_1)$ are quadratic polynomials with negative second derivative implies that $q_{1,L}^* = q_{1,L}^M$ and that $q_1^* = q_{1,L}^* = q_{1,L}^M = \mathbb{E}[\alpha]/2$. □

Under Hypothesis 1, F_2 competes for both low and high demand. Its best response is given in the following lemma.

Lemma 2.3.7. *Under hypothesis 1, the follower firm's best response is:*

(i) $q_{2,L}^* = \frac{2\alpha_L - \mathbb{E}[\alpha]}{4}$

(ii) $q_{2,H}^* = \frac{2\alpha_H - \mathbb{E}[\alpha]}{4}$

Proof. Follows directly from substituting the expression of q_1^* under hypothesis 1 into the expression for $q_{2,L}^*$ and $q_{2,H}^*$. □

The subgame perfect Nash equilibrium is

$$(q_1^*, (q_{2,L}^*, q_{2,H}^*)) = \left(\frac{\mathbb{E}[\alpha]}{2}, \left(\frac{2\alpha_L - \mathbb{E}[\alpha]}{4}, \frac{2\alpha_H - \mathbb{E}[\alpha]}{4}\right)\right)$$

Hypothesis 2: $\mu_H \alpha_H \leq 2\alpha_L \leq \mathbb{E}[\alpha]$ **(Intermediate low intersect demand)**

When hypothesis 2 holds, get a frontier case:

Theorem 2.3.8. *Under hypothesis 2, the leading firm's best response is* $q_1^* = \alpha_L$.

Proof. From hypothesis 2, we get immediately that $q_{1,L}^* = \alpha_L$ and $q_{1,H}^* = \alpha_L$. This implies that $q_1^* = \alpha_L$. □

Under hypothesis 2, F_2 stays out of the market if the realized demand is low but competes if the realized demand is high. Its best response is stated in the following lemma.

Lemma 2.3.9. *The follower firm's best response under hypothesis 2 is:*

(i) $q_{2,L}^* = 0$

(ii) $q_{2,H}^* = \frac{\alpha_H - \alpha_L}{2}$

Proof. If $\alpha = \alpha_L$ then
$$q_{2,L}^* = \frac{\alpha_L - \alpha_L}{2} = 0$$
The case where $\alpha = \alpha_H$ follows similarly. □

This is a specially interesting case, given that the optimal quantity for F_1 is the quantity that keeps F_2 out of the market in case the demand is low (the monopoly quantity). In these conditions, the subgame perfect Nash equilibria is

$$(q_1^*, (q_{2,L}^*, q_{2,H}^*)) = (\alpha_L, (0, \frac{\alpha_H - \alpha_L}{2}))$$

Hypothesis 3: $2\alpha_L \leq \mu_H \alpha_H \leq \mathbb{E}[\alpha]$ **(Small low intersect demand)**

The optimal quantity for F_1 if hypothesis 3 holds is given by $q_{1,H}^M$. We state and prove this in the following theorem.

Theorem 2.3.10. *Under hypothesis 3, the leading firm's best response*

$$q_1^* = q_{1,H}^M = \frac{\mathbb{E}[\alpha] - \frac{\mu_H \alpha_H}{2}}{2 - \mu_H}$$

Proof. The thought process is entirely analogous to the previous cases. Hypothesis 3 implies that $\alpha_L \leq q_{1,L}^M$, so $q_{1,L}^* = \alpha_L$. It also implies that $\alpha_L \leq q_{1,H}^M$, which means that $q_{1,H}^* = q_{1,H}^M$, and so $q_1^* = q_{1,H}^* = q_{1,H}^M$. □

The follower firm is out of the market if the observed demand is low and competes if the observed demand is high. Its best response is given in the following lemma.

Lemma 2.3.11. *If hypothesis 3 holds, then the best response for the follower firm is:*

(i) $q_{2,L}^* = 0$

(ii) $q_{2,H}^* = \frac{\alpha_H}{2} - \frac{\mathbb{E}[\alpha] - \frac{\mu_H \alpha_H}{2}}{2(2-\mu_H)}$

Proof. The proof is entirely analogous to the one in Lemma 2.3.9. □

The subgame perfect Nash equilibrium is

$$(q_1^*, (q_{2,L}^*, q_{2,H}^*)) = \left(\frac{\mathbb{E}[\alpha] - \frac{\mu_H \alpha_H}{2}}{2-\mu_H}, \left(0, \frac{\alpha_H}{2} - \frac{\mathbb{E}[\alpha] - \frac{\mu_H \alpha_H}{2}}{2(2-\mu_H)}\right)\right)$$

In the next section, we conduct an ex-post analysis.

2.4 Ex-post analysis

We conduct an ex-post analysis for each of the hypothesis. We show that the follower firm cannot earn more than the leading firm under low demand in hypothesis 1. Under high demand, we provide an example illustrating $\pi_2 > \pi_1$ under each hypothesis. Our goal is to emphasize that when uncertainty is present and the demand is high, the first mover advantage vanishes under certain economical quantities.

Hypothesis 1

Here, $q_1^* < \alpha_L$ and F_2 competes in both low and high demand. However, with low demand the follower firm cannot have a profit higher than the leading firm.

Lemma 2.4.1. *If hypothesis 1 holds and the realized demand is low, F_2 never has a higher payoff than F_1.*

Proof. Given that the price is the same for both firms, the one that produces more has the higher profit:

$$q_{2,L}^* > q_1^*$$
$$\frac{2\alpha_L - \mathbb{E}[\alpha]}{4} > \frac{\mathbb{E}[\alpha]}{2}$$
$$\alpha_L > \frac{3}{2}\mathbb{E}[\alpha]$$

Which is impossible, given that $\alpha_L < \mathbb{E}[\alpha] < (3/2)\mathbb{E}[\alpha]$. □

The following example shows that when the demand is high, the follower firm can earn more than the leading firm under hypothesis 1.

Example 2.4.1. *We consider the following economical quantities:* $\mu_L = 2/3, \mu_H = 1/3, \alpha_L = 3/2, \alpha_H = 4$. *First we check that with these, we are under hypothesis 1, i.e., that* $\mathbb{E}[\alpha]/2 < \alpha_L$*:*

$$\mathbb{E}[\alpha] \le 2\alpha_L \iff 1 + \frac{4}{3} \le 3 \iff \frac{7}{3} \le 3$$

Now we check that the quantity produced by the follower firm is higher:

$$\begin{aligned} q_1^* &< q_2^* \\ \frac{\mathbb{E}[\alpha]}{2} &< \frac{2\alpha_H - \mathbb{E}[\alpha]}{4} \\ \frac{7}{6} &< \frac{8 - \frac{7}{3}}{4} \\ \frac{7}{6} &< \frac{17}{12} \end{aligned}$$

Given that the price is the same for both firms, F_2 *has a higher payoff.*

Hypothesis 2

Here, the leading firm produces the monopoly quantity α_L

If the realized demand is α_L, then F_2 does not compete. F_1 sells at cost price and has a payoff of 0. For $\alpha = \alpha_H$, we provide a set of economical quantities such that hypothesis 2 holds and the follower firm has a higher payoff.

Example 2.4.2. *We consider* $\mu_L = \mu_H = 1/2, \alpha_L = 6, \alpha_H = 20$. *We check that with these parameters, we are under hypothesis 2. We verify one inequality of the hypothesis at a time:*

(i)

$$\mu_H \alpha_H \le 2\alpha_L \iff 10 \le 12$$

(ii)

$$2\alpha_L \le \mathbb{E}[\alpha] \iff 12 \le 13$$

We notice that

$$q_1^* < q_2^* \iff \alpha_L < \frac{\alpha_H - \alpha_L}{2} \iff 6 < 7$$

Once the price is the same for both firms, F_2 *has a higher profit.*

Hypothesis 3

Under hypothesis 3, the leading firm produces a quantity higher than α_L. If $\alpha = \alpha_L$, this implies that it will sell at a price lower than what it costs to produce and incurs in dumping.

If the realized demand is α_H, then F_2 competes. We provide an example in which the follower firm has a higher payoff.

Example 2.4.3. *We consider $\mu_L = \mu_H = 1/2, \alpha_L = 1/2, \alpha_H = 4$. To verify that we are under hypothesis 3, it is enough to check that $\mu_H \alpha_H \geq 2\alpha_L$:*

$$\mu_H \alpha_H \geq 2\alpha_L \iff 2 \geq 1$$

Comparing the quantities produced, we have:

$$\begin{aligned} q_1^* &< q_2^* \\ \frac{2\mathbb{E}[\alpha] - \mu_H \alpha_H}{2(1+\mu_L)} &< \frac{\alpha_H}{2} - \frac{2\mathbb{E}[\alpha] - \mu_H \alpha_H}{4(1+\mu_L)} \\ 6\mathbb{E}[\alpha] - 3\mu_H \alpha_H &< \alpha_H(2+2\mu_L) \\ 6\mathbb{E}[\alpha] &< \alpha_H(4+\mu_H) \\ \frac{6\mathbb{E}[\alpha]}{(4+\mu_H)} &< \alpha_H \end{aligned}$$

Substituting with the economical quantities given in the example, we get $3 < 4$. So we have $\pi_1 < \pi_2$.

2.5 Dumping aversion

We consider now what happens if the leading firm decides it will not produce more than α_L under any of the hypothesis to avoid dumping. The situation under hypothesis 1 and 2 remains unchanged because F_1 produced at most α_L under these. We consider what happens **under hypothesis 3** both ex-ante and ex-post in the no-dumping case.

Ex-ante analysis with dumping aversion

We compute $\mathbb{E}[\pi_1]$ and $\mathbb{E}[\pi_2]$, respectively.

$$\mathbb{E}[\pi_1] = \mu_H \alpha_L (\frac{\alpha_H - \alpha_L}{2})$$

$$\mathbb{E}[\pi_2] = \mu_H (\frac{\alpha_H - \alpha_L}{2})^2$$

Next, we derive conditions for the expected profit of the follower firm to be higher than the expected profit from the leading firm.

$$\mathbb{E}[\pi_1] < \mathbb{E}[\pi_2] \iff \mu_H \alpha_L (\frac{\alpha_H - \alpha_L}{2}) < \mu_H (\frac{\alpha_H - \alpha_L}{2})^2$$

$$\alpha_L < \frac{\alpha_H - \alpha_L}{2} \iff 3\alpha_L < \alpha_H$$

We have the following lemma.

Lemma 2.5.1. *Under hypothesis 3 and dumping aversion, if $\mu_H < 2/3$ then F_2 has a higher expected payoff than F_1.*

Proof. The condition for F_2's higher profit is equivalent to

$$\frac{\alpha_L}{\alpha_H} < \frac{1}{3}$$

On the other hand, the condition for H_3 is equivalent to

$$\frac{\alpha_L}{\alpha_H} \leq \frac{\mu_H}{2}$$

So if we we choose μ_H such that

$$\frac{\mu_H}{2} < \frac{1}{3}$$

then the condition for F_2's higher profit is guaranteed under hypothesis 3. But for that we only need to consider $\mu_H < 2/3$. □

Ex-post analysis with dumping aversion

In case the realized demand is α_L, F_2 stays out of the market and F_1 sells at cost price with a payoff of 0. If the realized demand is α_H, then the follower firm competes and the payoffs are:

$$\pi_1 = \alpha_L(\frac{\alpha_H - \alpha_L}{2})$$

$$\pi_2 = (\frac{\alpha_H - \alpha_L}{2})(\frac{\alpha_H - \alpha_L}{2})$$

Comparing the quantities produced:

$$\alpha_L < \frac{\alpha_H - \alpha_L}{2} \iff 3\alpha_L < \alpha_H$$

which is the same condition as in the ex-ante analysis in the dump aversion case, above. We have the following lemma.

Lemma 2.5.2. *Under hypothesis 3 and dumping aversion, if the realized demand is $\alpha = \alpha_H$ and $\mu_H < 2/3$ then F_2 has a higher expected payoff than F_1.*

Proof. The proof is analogous to the proof of Lemma 2.5.1. □

Under hypothesis 3 and dumping aversion, if $\mu_H < 2/3$ then F_1 knows that F_2 has a higher expected payoff. In ex-post, the leading firm could either earn 0 or have a lower payoff when compared to the follower firm. So there is no incentive for a firm to enter the market as a leading firm.

CHAPTER 3

Generalized Stackelberg duopoly with uncertainty

In this chapter, we study the leading firm's best response under mean-standard deviation preference. It is structured as follows. First, we compute the standard deviation for each of the pieces of $\mathbb{E}[\pi_1]$, with the goal of writing an expression for $\mathbb{E}[\pi_1(\delta)]$ explicitly. Secondly, we compute the maximizer of each piece of $\mathbb{E}[\pi_1(\delta)]$ as a function of $q_1 \in \mathbb{R}$. A series of lemmas and theorems is presented in which we fully characterize the leading firm's best response depending on different cases for the economical quantities. We conduct an ex-ante and ex-post static analysis of a specific case of the economical quantities and consider, for the same case, the leading firm's best response under dumping aversion, comparing them.

3.1 Mean-standard deviation preference

To emphasize the dependence of $\mathbb{E}[\pi_1] - \delta\sigma[\pi_1]$ on δ, we will denote

$$\mathbb{E}[\pi_1(\delta)] = \mathbb{E}[\pi_1] - \delta\sigma[\pi_1]$$

We note that $\mathbb{E}[\pi_1(0)] = \mathbb{E}[\pi_1]$, and so the previous chapter corresponds to a particular case of the one we analyze now.

Since $\mathbb{E}[\pi_1]$ is defined piecewise depending on the value of q_1, then so will $\mathbb{E}[\pi_1(\delta)]$. In particular, the standard deviation of π_1 is also going to be defined piecewise. To make the notation clear and in an effort to generalize the notation used in the previous chapter, we will use the following notation. Let $\sigma_L[\pi_1]$ denote the standard deviation of π_1 for $q_1 \in [0, \alpha_L)$ and $\sigma_H[\pi_1]$ denote the standard deviation of π_1 for $q_1 \in [\alpha_L, \alpha_H)$. Analogous for the variance, where we will write $\mathbb{V}_L[\pi_1]$ and $\mathbb{V}_H[\pi_1]$. $\mathbb{E}[\pi_1(\delta)]$ is, then, given By

$$\mathbb{E}[\pi_1(\delta)] = \begin{cases} \pi_{1,L}(\delta) := \pi_{1,L} - \delta\sigma_L[\pi_1] & 0 \le q_1 < \alpha_L \\ \pi_{1,H}(\delta) := \pi_{1,H} - \delta\sigma_H[\pi_1] & \alpha_L \le q_1 \le \alpha_H \end{cases}$$

Our goal is to find the maximizer of $\pi_{1,L}(\delta)$ and $\pi_{1,H}(\delta)$ as functions of $q_1 \in \mathbb{R}$, similarly to what we did in the previous chapter. The analysis is divided in two cases. In each case we start by computing the variance of each piece, and then compute their maximizer as a function of $q_1 \in \mathbb{R}$.

Previously, $q^M_{1,L}$ and $q^M_{1,H}$ were depending on the economical quantities. Now, the maximizers will also depend on δ. Let $q^M_{1,L}(\delta)$ and $q^M_{1,H}(\delta)$ denote the maximizers of $\pi_{1,L}(\delta)$ and $\pi_{1,H}(\delta)$ for $q_1 \in \mathbb{R}$, respectively. The next subsection starts with the analysis of the first case.

3.1.1 Case $0 \leq q_1 < \alpha_L$

Here we have

$$\mathbb{E}[\pi_1] = \pi_{1,L} = \mu_L q_1 (\frac{\alpha_L - q_1}{2}) + \mu_H q_1 (\frac{\alpha_H - q_1}{2})$$

In the next theorem, we state the expression of $\sigma_L[\pi_1]$ explicitly.

Lemma 3.1.1. *If $0 \leq q_1 < \alpha_L$, the standard deviation of π_1 is*

$$\sigma_L[\pi_1] = \sqrt{\mu_L \mu_H} q_1 (\frac{\alpha_H - \alpha_L}{2})$$

Proof. We compute the variance of π_1 in this case directly. We will denote

$$A = \frac{\alpha_L - q_1}{2} \quad \text{and} \quad B = \frac{\alpha_H - q_1}{2}$$

to make the computation simpler, as follows:

$$\begin{aligned}
\mathbb{V}_L[\pi_1] &= \mathbb{E}[\pi_1]^2 - \mathbb{E}[\pi_1^2] \\
&= \mu_L q_1^2 A^2 + \mu_H q_1^2 B^2 - (\mu_L q_1 (A) + \mu_H q_1 (B))^2 \\
&= \mu_L q_1^2 A^2 + \mu_H q_1^2 B^2 - \mu_L^2 q_1^2 A^2 - \mu_H^2 q_1^2 B^2 - 2\mu_L \mu_H q_1^2 AB \\
&= \mu_L q_1^2 A^2 (1 - \mu_L) + \mu_H q_1^2 B^2 (1 - \mu_H) - 2\mu_L \mu_H q_1^2 AB \\
&= \mu_L \mu_H q_1^2 A^2 + \mu_L \mu_H q_1^2 B^2 - 2\mu_L \mu_H q_1^2 AB \\
&= \mu_L \mu_H q_1^2 (B - A)^2 \\
&= \mu_L \mu_H q_1^2 (\frac{\alpha_H - \alpha_L}{2})^2
\end{aligned}$$

As such, the standard deviation is

$$\sigma_L[\pi_1] = \sqrt{\mathbb{V}_L[\pi_1]} = \sqrt{\mu_L \mu_H q_1^2 (\frac{\alpha_H - \alpha_L}{2})^2} = \sqrt{\mu_L \mu_H} q_1 (\frac{\alpha_H - \alpha_L}{2})$$

□

Here, the standard deviation increases as the quantity produced increases. Substituting into the expression for the first piece of $\mathbb{E}[\pi_1(\delta)]$, we get

$$\pi_{1,L}(\delta) = \pi_{1,L} - \delta \sqrt{\mu_L \mu_H} q_1 (\frac{\alpha_H - \alpha_L}{2})$$

The next theorem characterizes $q^M_{1,L}(\delta)$.

Theorem 3.1.2. *The maximizer of $\pi_{1,L}(\delta)$ for $q_1 \in \mathbb{R}$ is*

$$q^M_{1,L}(\delta) = \frac{\mathbb{E}[\alpha]}{2} - \delta\sqrt{\mu_L\mu_H}(\frac{\alpha_H - \alpha_L}{2})$$

Proof. We take the derivative of $\pi^M_{1,L}(\delta)$ with respect to q_1 and solve for q_1:

$$\begin{aligned}
\frac{\partial(\pi_{1,L} - \delta\sqrt{\mu_L\mu_H}q_1(\frac{\alpha_H-\alpha_L}{2}))}{\partial q_1} &= \frac{\partial \pi_{1,L}}{\partial q_1} - \frac{\partial(\delta\sqrt{\mu_L\mu_H}q_1(\frac{\alpha_H-\alpha_L}{2}))}{\partial q_1} \\
&= \frac{\partial \pi_{1,L}}{\partial q_1} - \delta\sqrt{\mu_L\mu_H}(\frac{\alpha_H - \alpha_L}{2}) \\
&= \frac{1}{2}(\mathbb{E}[\alpha] - 2q_1) - \delta\sqrt{\mu_L\mu_H}(\frac{\alpha_H - \alpha_L}{2})
\end{aligned}$$

Equating this to zero and solving for q_1, we get that

$$q^M_{1,L}(\delta) = \frac{\mathbb{E}[\alpha]}{2} - \delta\sqrt{\mu_L\mu_H}(\frac{\alpha_H - \alpha_L}{2})$$

□

We observe that $q^M_{1,L}(\delta) < q^M_{1,L}(0) = q^M_{1,L}$ for any $\delta > 0$. So the more risk adverse the leading firm is, the less the maximizer of $\pi_{1,L}(\delta)$ is. We characterize $q^M_{1,H}(\delta)$ in the next section.

3.1.2 Case $\alpha_L \leq q_1 \leq \alpha_H$

We proceed in the same manner as in the previous case. The next theorem gives the expression for $\sigma_H(\pi_1)$.

Lemma 3.1.3. *If $\alpha_L \leq q_1 \leq \alpha_H$, the standard deviation of π_1 is given by*

$$\sigma_H(\pi_1) = \sqrt{\mu_L\mu_H}q_1(\frac{\alpha_H - 2\alpha_L + q_1}{2})$$

Proof. We will denote

$$A = \alpha_L - q_1 \quad \text{and} \quad B = \frac{\alpha_H - q_1}{2}$$

to make the computations simpler. We proceed similarly to the case $\sigma_L(\pi_1)$. In fact, the calculations are the same up to the point where we compute $B - A$. We have:

$$\begin{aligned}
\mathbb{V}_L[\pi_1] &= \mathbb{E}[\pi_1]^2 - \mathbb{E}[\pi_1^2] \\
&= \mu_L q_1^2 A^2 + \mu_H q_1^2 B^2 - (\mu_L q_1(A) + \mu_H q_1(B))^2 \\
&= \mu_L q_1^2 A^2 + \mu_H q_1^2 B^2 - \mu_L^2 q_1^2 A^2 - \mu_H^2 q_1^2 B^2 - 2\mu_L\mu_H q_1^2 AB \\
&= \mu_L q_1^2 A^2(1 - \mu_L) + \mu_H q_1^2 B^2(1 - \mu_H) - 2\mu_L\mu_H q_1^2 AB \\
&= \mu_L\mu_H q_1^2 A^2 + \mu_L\mu_H q_1^2 B^2 - 2\mu_L\mu_H q_1^2 AB \\
&= \mu_L\mu_H q_1^2 (B - A)^2 \\
&= \mu_L\mu_H q_1^2 (\frac{\alpha_H - 2\alpha_L + q_1}{2})^2
\end{aligned}$$

As such, the standard deviation is

$$\sigma_H[\pi_1] = \sqrt{\mu_L\mu_H q_1^2(\frac{\alpha_H - 2\alpha_L + q_1}{2})^2} = \sqrt{\mu_L\mu_H} q_1(\frac{\alpha_H - 2\alpha_L + q_1}{2})$$

□

Regarding $\sigma_H[\pi_1]$, if $\alpha_H - 2\alpha_L + q_1 < 0$, the expression would be negative. This would not make sense, as the standard deviation is always either positive or 0. However, $\sigma_H[\pi_1]$ is only defined for $\alpha_L \leq q_1 \leq \alpha_H$, and as such

$$\alpha_H - 2\alpha_L + q_1 \geq \alpha_H - 2\alpha_L + \alpha_L \geq \alpha_H - \alpha_L > 0$$

therefore this poses no problem for our analysis.

The next theorem characterizes $q_{1,H}^M(\delta)$.

Theorem 3.1.4. *The maximizer of $\pi_{1,H}(\delta)$ for $q_1 \in \mathbb{R}$ is*

$$q_{1,H}^M(\delta) = \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L)}{2(1 + \mu_L + \delta\sqrt{\mu_L\mu_H})}$$

Proof. As before, we differentiate $\pi_{1,H}(\delta)$ with respect to q_1 and solve for q_1:

$$\begin{aligned}\frac{\partial(\pi_{1,H} - \delta\sqrt{\mu_L\mu_H} q_1(\frac{\alpha_H - 2\alpha_L + q_1}{2})))}{\partial q_1} &= \frac{\partial \pi_{1,H}}{\partial q_1} - \frac{\partial(\delta\sqrt{\mu_L\mu_H} q_1(\frac{\alpha_H - 2\alpha_L + q_1}{2})))}{\partial q_1} \\ &= \mathbb{E}[\alpha] - \frac{\mu_H\alpha_H}{2} - q_1(1 + \mu_L) - \delta\sqrt{\mu_L\mu_H}(\frac{\alpha_H - 2\alpha_L}{2} + q_1)\end{aligned}$$

We now solve for q_1:

$$\begin{aligned}0 &= 2\mathbb{E}[\alpha] - \mu_H\alpha_H - 2q_1(1 + \mu_L) - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L + 2q_1) \\ 0 &= \mathbb{E}[\alpha] + \mu_L\alpha_L - 2q_1(1 + \mu_L) - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L + 2q_1) \\ q_1 2(1 + \mu_L + \delta\sqrt{\mu_L\mu_H}) &= \mathbb{E}[\alpha] + \mu_L\alpha_L - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L) \\ q_{1,H}^M(\delta) &= \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L)}{2(1 + \mu_L + \delta\sqrt{\mu_L\mu_H})}\end{aligned}$$

□

The next section studies the function $\pi_{1,L}(\delta)$ in more detail.

3.1.3 Some properties of $\pi_{1,L}(\delta)$

In this section, we will study the quantities q_1 for which $\pi_{1,L}(\delta, q_1)$ is zero for each δ. This section will allow us to make a brief remark after Theorem 3.2.3.

Lemma 3.1.5. *For a given δ, the quantities $q_1 \in \mathbb{R}$ for which $\pi_{1,L}(\delta, q_1)$ is zero are*

(i) $q_{0,1} = 0$

(ii) $q_{0,2} = \mathbb{E}[\alpha] - \delta\sqrt{\alpha_L \alpha_H}(\alpha_H - \alpha_L)$

Furthermore, if

$$\delta > \frac{\mathbb{E}[\alpha]}{\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_H)}$$

then $q_{0,2}$ is negative.

Proof. We may write $\pi_{1,L}(\delta, q_1) = 0$ as

$$q_1(\mu_L(\frac{\alpha_L - q_1}{2}) + \mu_H(\frac{\alpha_H - q_1}{2}) - \sqrt{\mu_L \mu_H}(\frac{\alpha_H - \alpha_L}{2})) = 0$$

from which we get that either $q_1 = 0$ (i.e., $q_{0,1}$) or

$$\begin{aligned}
&\mu_L(\frac{\alpha_L - q_1}{2}) + \mu_H(\frac{\alpha_H - q_1}{2}) - \sqrt{\mu_L \mu_H}(\frac{\alpha_H - \alpha_L}{2}) = 0 \\
&-\mu_L q_1 - \mu_H q_1 + \mu_L \alpha_L + \mu_H \alpha_H = \delta\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_L) \\
&q_1(-1) + \mathbb{E}[\alpha] = \delta\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_L) \\
&q_1 = \mathbb{E}[\alpha] - \delta\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_L)
\end{aligned}$$

which is $q_{0,2}$. The second part of the lemma comes from solving $q_{0,2} < 0$ for δ:

$$\begin{aligned}
0 &> \mathbb{E}[\alpha] - \delta\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_L) \\
\delta &> \frac{\mathbb{E}[\alpha]}{\sqrt{\mu_L \mu_H}(\alpha_H - \alpha_L)}
\end{aligned}$$

□

In the next section, we characterize the leading firm's best response.

3.2 Leading firm's best response

In this section, we are going to characterize the best response for the leading firm. We will separate the results depending on the economical quantities present in the model. To avoid clutter in the computations, we will write $X = \sqrt{\mu_L \mu_H}$.

The position of $q^M_{1,L}(\delta)$ and $q^M_{1,H}(\delta)$ relative to α_L is of special importance when determining $q^*_1(\delta)$. Given that these maximizers are a function of δ, we would like to find for which values of δ this relative position changes. With that in mind, we give the following definition.

Definition 7. *δ_0, δ_L and δ_H are thresholds for δ defined by:*

(i) δ_0 is the minimum $\delta \geq 0$ such that $q^M_{1,L}(\delta) \leq 0$

(ii) δ_L is the minimum $\delta \geq 0$ such that $q^M_{1,L}(\delta) \leq \alpha_L$

(iii) δ_H *is the minimum* $\delta \geq 0$ *such that* $q^M_{1,H}(\delta) \leq \alpha_L$

Remark: If $q^M_{1,L}(0) \leq \alpha_L$, then $\delta_L = 0$. We have an analogous result for the other cases.

Once these thresholds are defined, we can express the maximizers of $\pi_{1,L}$ and $\pi_{1,H}$ in the domain in which they are valid. We denote these maximizers by $q^*_{1,L}(\delta)$ and $q^*_{1,H}(\delta)$, respectively.

Lemma 3.2.1. *The possible cases for the maximizer of* $\pi_{1,L}$ *and* $\pi_{1,H}$ *are, respectively:*

(i)

$$q^*_{1,L}(\delta) = \begin{cases} \alpha_L & \delta \leq \delta_L \\ q^M_{1,L}(\delta) & \delta_L < \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

(ii)

$$q^*_{1,H}(\delta) = \begin{cases} q^M_{1,H}(\delta) & \delta < \delta_H \\ \alpha_L & \delta_H \leq \delta \end{cases}$$

Proof. Follows from Definition 7 directly. □

We compute these thresholds and state the results in the following lemma.

Lemma 3.2.2. *The values for* δ_0, δ_L *and* δ_H *are:*

(i)

$$\delta_0 = \frac{\mathbb{E}[\alpha]}{X(\alpha_H - \alpha_L)}$$

(ii)

$$\delta_L = max\left\{\frac{\mathbb{E}[\alpha] - 2\alpha_L}{X(\alpha_H - \alpha_L)}, 0\right\}$$

(iii)

$$\delta_H = max\left\{\frac{\mathbb{E}[\alpha] - \alpha_L(2 + \mu_L)}{X\alpha_H}, 0\right\}$$

Proof. (i) We solve $q^M_{1,L}(\delta) \leq 0$ for δ:

$$\begin{aligned} q^M_{1,L}(\delta) &\leq 0 \\ \frac{\mathbb{E}[\alpha]}{2} - \delta X \frac{\alpha_H - \alpha_L}{2} &\leq 0 \\ \frac{\mathbb{E}[\alpha]}{X(\alpha_H - \alpha_L)} &\leq \delta \end{aligned}$$

The result follows from Definition 7 and this inequality.

(ii) Solving $q^M_{1,L}(\delta) \leq \alpha_L$ for δ, we get:

$$\begin{aligned} q^M_{1,L}(\delta) &\leq \alpha_L \\ \frac{\mathbb{E}[\alpha]}{2} - \delta X \frac{\alpha_H - \alpha_L}{2} &\leq \alpha_L \\ \frac{\mathbb{E}[\alpha] - 2\alpha_L}{X(\alpha_H - \alpha_L)} &\leq \delta \end{aligned}$$

This expression can be negative, and that happens if $q^M_{1,L}(0) < \alpha_L$. Then by Definition 7,

$$\delta_L = \max\left\{\frac{\mathbb{E}[\alpha] - 2\alpha_L}{X(\alpha_H - \alpha_L)}, 0\right\}$$

(iii) Analogously, we solve $q^M_{1,H}(\delta) \leq \alpha_L$ for δ :

$$\begin{aligned} q^M_{1,H}(\delta) &\leq \alpha_L \\ \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta X(\alpha_H - 2\alpha_L)}{2(1 + \mu_L + \delta X)} &\leq \alpha_L \\ \mathbb{E}[\alpha] + \mu_L\alpha_L - \delta X\alpha_H + 2\delta X\alpha_L &\leq 2\alpha_L + 2\mu_L\alpha_L + 2\delta X\alpha_L \\ \frac{\mathbb{E}[\alpha] - \alpha_L(2 + \mu_L)}{X\alpha_H} &\leq \delta \end{aligned}$$

This expression is negative if the numerator is:

$$\mathbb{E}[\alpha] - \alpha_L(2 + \mu_L) < 0 \iff \frac{\mathbb{E}[\alpha]}{2} < \alpha_L(1 + \frac{\mu_L}{2})$$

and for that to occur it is sufficient that $q^M_{1,L}(0) < \alpha_L$. Following Definition 7, we then have

$$\delta_H = \max\left\{\frac{\mathbb{E}[\alpha] - \alpha_L(2 + \mu_L)}{X\alpha_H}, 0\right\}$$

□

From this lemma we note that the fist piece of $q^*_{1,L}(\delta)$ and $q^*_{1,H}(\delta)$ in Lemma 3.2.1 can disappear if $\delta_L = 0$ and $\delta_H = 0$, respectively.

We now state and prove a theorem about the position of the thresholds relative to each other that will be important when characterizing $q^*_1(\delta)$.

Theorem 3.2.3. *The position of the thresholds relative to each other is*

$$\delta_H \underset{(i)}{<} \delta_L \underset{(ii)}{<} \delta_0$$

Proof. We prove one inequality at a time.

(i) The proof given is by contradiction. Suppose $\exists \delta : \delta_L \leq \delta \leq \delta_H$. By the definition of δ_L, δ_H, for such δ we would have $q^M_{1,L}(\delta) \underset{(a)}{\leq} \alpha_L \underset{(b)}{\leq} q^M_{1,H}(\delta)$. We develop one inequality at a time:

(a)

$$\frac{\mathbb{E}[\alpha]}{2} - \delta X\left(\frac{\alpha_H - \alpha_L}{2}\right) \leq \alpha_L$$
$$\mathbb{E}[\alpha] - \delta\alpha_H X \leq \alpha_L(2 - \delta X) \tag{3.1}$$

(b)

$$\alpha_L \leq \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta X(\alpha_H - 2\alpha_L)}{2(1 + \mu_L + \delta X)}$$
$$\alpha_L(2(1 + \mu_L + \delta X) - \mu_L - 2\delta X) \leq \mathbb{E}[\alpha] - \delta\alpha_H X$$
$$\alpha_L(2 + \mu_L) \leq \mathbb{E}[\alpha] - \delta\alpha_H X \tag{3.2}$$

But Eq. (3.1) and Eq. (3.2) imply

$$\alpha_L(2 + \mu_L) \leq \mathbb{E}[\alpha] - \delta\alpha_H X \leq \alpha_L(2 - \delta X)$$

which is not possible given that $2 - \delta X < 2 + \mu_L$.

(ii) We can re-write δ_L as

$$\delta_L = \frac{\mathbb{E}[\alpha] - 2\alpha_L}{X(\alpha_H - \alpha_L)} = \delta_0 - \frac{2\alpha_L}{X(\alpha_H - \alpha_L)}$$

and the inequality follows.

□

Inequality (i) in Theorem 3.2.3 implies in particular that $q^M_{1,L}(\delta) \leq \alpha_L \leq q^M_{1,H}(\delta)$ cannot happen. For example, if we find δ such that $\alpha_L \leq q^M_{1,H}(\delta)$, then by Theorem 3.2.3 together with the continuity of $\mathbb{E}[\pi_1(\delta)]$ at α_L we immediately get that $q^*_1(\delta) = q^M_{1,H}(\delta)$.

Theorem 3.2.3 also implies that $\delta_H < \delta_0$. This means that for δ such that $q^M_{1,L}(\delta) \leq 0$, we have $q^M_{1,H}(\delta) \leq \alpha_L$ which implies $q^*_{1,H}(\delta) = \alpha_L$. Again by continuity of $\mathbb{E}[\pi_1(\delta)]$ at α_L, for such δ this means that $q^*_1(\delta) = 0$, and so the leading firm chooses not to compete, i.e., the game does not happen. Furthermore, by Lemma 3.1.5, if F_1 decided to enter the game, not only would it not be its optimal response but it would also have a negative payoff no matter the quantity produced. Thus, we will set the maximum value for δ in our analysis to be δ_0, and call it the "game ending threshold" for the risk aversion.

We are now ready to characterize the leading firm's best response in each case.

3.2.1 Case $\alpha_H < 2\alpha_L$

Theorem 3.2.4. *If $\alpha_H < 2\alpha_L$, then:*

$$q^*_1(\delta) = \begin{cases} q^M_{1,L}(\delta) & 0 \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

Proof. We prove $q^M_{1,H}(\delta) < \alpha_L$ for all δ (which means that $q^*_{1,H}(\delta) = \alpha_L$) and $q^M_{1,L}(0) < \alpha_L$. The cases stated in the theorem follow from these and the consideration in the begining of the chapter.

Recall that from the last chapter we get $q^M_{1,H}(0) \leq q^M_{1,L}(0)$. We have

$$q^M_{1,H}(0) = \frac{\mathbb{E}[\alpha] + \mu_L \alpha_L}{2(1+\mu_L)} = \frac{\mu_H \alpha_H + 2\mu_L \alpha_L}{2(1+\mu_L)} < \frac{(1-\mu_L)2\alpha_L + 2\mu_L \alpha_L}{2(1+\mu_L)} < \frac{\alpha_L}{1+\mu_L} < \alpha_L$$

On the other hand,

$$\lim_{\delta\to\infty} q^M_{1,H}(\delta) = \frac{2\alpha_L - \alpha_H}{2} < \alpha_L$$

A condition stronger than the two above holds, however. In fact, we have

$$\begin{aligned}
q^M_{1,H}(\delta) &< \alpha_L \\
\mathbb{E}[\alpha] + \mu_L \alpha_L - \delta X(\alpha_H - 2\alpha_L) &< \alpha_L(2 + 2\mu_L + 2\delta X) \\
\mathbb{E}[\alpha] + \mu_L \alpha_L - \delta X \alpha_H + 2\delta X \alpha_L &< 2\alpha_L + 2\mu_L \alpha_L + 2\delta X \alpha_L \\
\mu_L \alpha_L + \mu_H \alpha_H - \delta X \alpha_H &< 2\alpha_L + \mu_L \alpha_L \\
\alpha_H(\mu_H - \delta X) &< 2\alpha_L
\end{aligned}$$

which is true given that $\alpha_H < 2\alpha_L$ and $\mu_H - \delta X < 1$. This means that $q^M_{1,H}(\delta)$ is not a valid choice for any δ and thus the maximizer for $\pi_{1,H}(\delta)$ is α_L. This, together with the consideration in the beginning of the chapter, gives:

$$q^*_1(\delta) = \begin{cases} \alpha_L & 0 \leq \delta < \delta_L \\ q^M_{1,L} & \delta_L \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

Under the hypothesis we are working with, i.e., $\alpha_H < 2\alpha_L$, we have

$$q^M_{1,L}(0) = \frac{\mathbb{E}[\alpha]}{2} = \frac{\mu_L \alpha_L + \mu_H \alpha_H}{2} < \frac{\mu_L \alpha_L + (1-\mu_L)2\alpha_L}{2} < \frac{2\alpha_L - \mu_L \alpha_L}{2} < \alpha_L$$

And so $\delta_L = 0$ which means that $q^*_1(\delta) = q^M_{1,L}(\delta)$ for $\delta \in (0, \delta_0)$, and the first case disappears. □

We could have proven this theorem in another way: the result follows from Theorem 3.2.3 after noting that $q^M_{1,L}(\delta) < \alpha_L$ for $\delta \in [0, \delta_L)$. However, we find the characterization of the behavior of $q^M_{1,H}(\delta)$ to be of notice in its own right.

3.2.2 Case $\alpha_H = 2\alpha_L$

In this case,

$$\begin{aligned} q_{1,H}^M(\delta) &= \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta\sqrt{\mu_L\mu_H}(\alpha_H - 2\alpha_L)}{2(1+\mu_L+\delta\sqrt{\mu_L\mu_H})} \\ &= \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L}{2(1+\mu_L+\delta X)} \end{aligned}$$

So $q_{1,H}^M(\delta)$ is decreasing as δ increases. The following lemma states that $q_{1,H}^M(\delta)$ is not a valid choice under this hypothesis.

Lemma 3.2.5. *If $\alpha_H = 2\alpha_L$ then $q_{1,H}^M(\delta) < \alpha_L$ for all $\delta \geq 0$.*

Proof. We develop the expression for $q_{1,H}^M(0)$.

$$\begin{aligned} q_{1,H}^M(0) &= \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L}{2(1+\mu_L)} \\ &= \frac{2\mu_L\alpha_L + \mu_H\alpha_H}{2(1+\mu_L)} \\ &= \frac{2\mu_L\alpha_L + 2\mu_H\alpha_L}{2(1+\mu_L)} \\ &= \frac{2\alpha_L}{2(1+\mu_L)} \\ &< \alpha_L \end{aligned}$$

The result now follows after noting that $q_{1,H}^M(\delta)$ is decreasing as δ increases. □

The next theorem gives F_1's best response in this case.

Theorem 3.2.6. *If $\alpha_H = 2\alpha_L$, then*

$$q_1^*(\delta) = \begin{cases} q_{1,L}^M(\delta) & 0 \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

Proof. We prove that $q_{1,L}^M(0) < \alpha_L$:

$$\begin{aligned} q_{1,L}^M(0) &= \frac{\mathbb{E}[\alpha]}{2} \\ &= \frac{\mu_L\alpha_L + \mu_H\alpha_H}{2} \\ &= \frac{\mu_L\alpha_L + \mu_H 2\alpha_L}{2} \\ &= \frac{\alpha_L(1+\mu_H)}{2} \\ &< \alpha_L \end{aligned}$$

given that $(1+\mu_H)/2 < 1$. The result then follows. □

So if $\alpha_H = 2\alpha_L$, the leading firm either produces $q^M_{1,L}(\delta)$ or has enough risk aversion that the game does not happen.

3.2.3 Case $\alpha_H > 2\alpha_L$

If $\alpha_H > 2\alpha_L$, there $\exists\gamma > 1$ such that $\alpha_H = 2\gamma\alpha_L$. Equivalently, $\exists\beta > 0$ such that $\gamma = (1+\beta/(1-\mu_L))$, and thus $\alpha_H = 2(1+\beta/(1-\mu_L))\alpha_L$.

In this case, $q^M_{1,H}(\delta)$ is decreasing in δ. Asymptotically, we have

$$\lim_{\delta\to\infty} q^M_{1,H}(\delta) = \frac{-X(\alpha_H - 2\alpha_L)}{2X} = \frac{2\alpha_L - \alpha_H}{2} < 0$$

On the other extreme, we have

$$\begin{aligned} q^M_{1,H}(0) &= \frac{\mathbb{E}[\alpha] + \mu_L\alpha_L}{2(1+\mu_L)} = \frac{(1-\mu_L)\alpha_H + 2\mu_L\alpha_L}{2(1+\mu_L)} \\ &= \frac{(1-\mu_L)(2+\frac{2\beta}{1-\mu_L})\alpha_L + 2\mu_L\alpha_L}{2(1+\mu_L)} \\ &= \frac{1+\beta}{1+\mu_L}\alpha_L \end{aligned}$$

If $\beta > \mu_L$ then $q^M_{1,H}(0) > \alpha_L$, and if $\beta \le \mu_L$, then $q^M_{1,H}(0) \le \alpha_L$. We divide the analysis of the case $\alpha_H > 2\alpha_L$ into these two subcases.

Theorem 3.2.7. *If $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$ then:*

$$q^*_1(\delta) = \begin{cases} q^M_{1,H}(\delta) & 0 \le \delta < \delta_H \\ \alpha_L & \delta_H \le \delta < \delta_L \\ q^M_{1,L}(\delta) & \delta_L \le \delta < \delta_0 \\ 0 & \delta_0 \le \delta \end{cases}$$

Proof. If $0 \le \delta < \delta_H$, then $q^M_{1,H}(\delta) > \alpha_L$ and we conclude $q^*_1(\delta) = q^M_{1,H}(\delta)$ by applying Theorem 3.2.3. If $\delta_H \le \delta$, then $q^*_{1,H}(\delta) = \alpha_L$. Again by Theorem 3.2.3, for each $\delta \in [0,\delta_L)$, we have $q^M_{1,L}(\delta) > \alpha_L$, which implies $q^*_{1,L}(\delta) = \alpha_L$. Thus, for $\delta \in [\delta_H,\delta_L)$, $q^*_1(\delta) = \alpha_L$. If $\delta \in [\delta_L,\delta_0)$, then $q^*_{1,H}(\delta) = \alpha_L$ as before (and will be as δ increases), and $q^*_{1,M}(\delta) \le \alpha_L$, which implies $q^*_1(\delta) = q^M_{1,L}(\delta)$. □

Now if $\beta \le \mu_L$ then $q^M_{1,H}(0) \le \alpha_L$, and it will be less than α_L as δ increases, i.e., $q^*_{1,H}(\delta) = \alpha_L$ for any $\delta \ge 0$. We now check if $q^*_1(\delta)$ is α_L, $q^M_{1,L}(\delta)$, or 0. To evaluate that, we want to know if $q^M_{1,L}(\delta)$ is higher or lower than α_L. The next lemma deals with this question.

Lemma 3.2.8. *Under the hypothesis $\alpha_H > 2\alpha_L$ and $\beta \le \mu_L$, have that $q^M_{1,L}(0) \ge \alpha_L$ if $\mu_L/2 \le \beta \le \mu_L$ and $q^M_{1,L}(0) < \alpha_L$ if $\beta < \mu_L/2$.*

Proof. Making the substitution $\alpha_H = 2(1+\beta/(1-\mu_L))\alpha_L$ into $q^M_{1,L}(0)$ we have:

$$q_{1,L}^M(0) = \frac{\mathbb{E}[\alpha]}{2} = \frac{\mu_L\alpha_L + \mu_H\alpha_H}{2} = \frac{\mu_L\alpha_L + (1-\mu_L)2(1+\frac{\beta}{1-\mu_L})\alpha_L}{2} =$$
$$= \frac{\mu_L\alpha_L + (2-2\mu_L+2\beta)\alpha_L}{2} = \frac{(2-\mu_L+2\beta)\alpha_L}{2}$$

And this is higher than α_L if:

$$(2-\mu_L+2\beta)\alpha_L \geq 2\alpha_L \iff -\mu_L+2\beta \geq 0 \iff \beta \geq \frac{\mu_L}{2}$$

And so $q_{1,L}^M(\delta) \geq \alpha_L$ if $\beta \geq \frac{\mu_L}{2}$, and lower than α_L otherwise. □

The following two theorems present the leading firm's best response for $\beta < \mu_L/2$ and $\mu_L/2 \leq \beta \leq \mu_L$ respectively.

Theorem 3.2.9. *If $\beta < \mu_L/2$, then:*

$$q_1^*(\delta) = \begin{cases} q_{1,L}^M(\delta) & 0 \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

Proof. If $\beta < \mu_L/2$ then $q_{1,H}^*(\delta) = \alpha_L$. From Lemma 3.2.8, for $\beta < \mu_L/2$ we have $q_{1,L}^M(0) < \alpha_L$ and this implies $q_1^*(\delta) = q_{1,L}^M(\delta)$ for $\delta \in [0, \delta_0)$. □

Theorem 3.2.10. *If $\mu_L/2 \leq \beta \leq \mu_L$, then:*

$$q_1^*(\delta) = \begin{cases} \alpha_L & 0 \leq \delta < \delta_L \\ q_{1,L}^M(\delta) & \delta_L \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases}$$

Proof. By Lemma 3.2.8, $\beta \geq \mu_L/2$ implies $q_{1,L}^M(0) \geq \alpha_L$. So for $\delta \in [0, \delta_L]$ we have $q_{1,L}^M(\delta) \geq \alpha_L$. Given that $q_{1,H}^*(\delta) = \alpha_L$ under if $\beta < \mu_L$, we have $q_1^*(\delta) = \alpha_L$ for $\delta \in [0, \delta_L)$. Recall that here $q_{1,H}^*(\delta)$ is a decreasing function of δ, so in fact $q_{1,H}^*(\delta) = \alpha_L$ for any $\delta \in [0, \infty)$. As a consequence, if $\delta_L \leq \delta < \delta_0$, then $q_1^*(\delta) = q_{1,L}^M(\delta)$. □

The leading firm's best response $q_1^*(\delta)$ is then fully characterized. Note that in each situation, $q_1^*(\delta)$ decreases as the risk aversion parameter δ increases. We now give possible economical interpretations for the results.

There's only one case of the economical quantities under which F_1's best response is to play $q_{1,H}^M(\delta)$ and risk losing in case the realized demand is α_L. This happens when $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$, which is the case where α_H is the highest when compared to α_L. The attractiveness that comes with the value for high-demand being higher compensates for the risk of the loss that happens if $\alpha = \alpha_L$.

If $\alpha_H < 2\alpha_L$, the value for high demand is not attractive enough for F_1 to ignore the risk of losing in case $\alpha = \alpha_L$. The same happens when $\alpha_H > 2\alpha_L$ and $\beta < \frac{\mu_L}{2}$ for the same reason. If

$\mu_L/2 \leq \beta < \mu_L$, there are values of the risk aversion parameter δ for which F_1's best response is to play the monopoly quantity α_L and risk not having a profit in case $\alpha = \alpha_L$ - the higher value for β implies a higher value for α_H when compared to α_L, which is just enough to cause this choice.

3.3 Static analysis - Case $\alpha_L > 2\alpha_L$, $\beta > \mu_L$

We conduct a static analysis, comparing the ex-post profits of both firms in the case $\alpha_H > 2\alpha_L$, $\beta > \mu_L$. Recall that the best response for the follower firm, once it realizes the demand and the quantity produced by the leading firm, is $q_2^* = (\alpha - q_1(\delta))/2$. Recall also that when F_2 competes, the price is the same for both firms which means we only need to compare the quantities produced.

This section is divided in two cases, each divided into the intervals for δ defined in Theorem 3.2.7. We characterize F_2's best response and compare the profits in each case.

3.3.1 Case $\alpha = \alpha_L$

We state the results in the following lemma.

Lemma 3.3.1. *If $\alpha_H > 2\alpha_L$, $\beta > \mu_L$ and the realized demand is $\alpha = \alpha_L$, the follower firm has a higher payoff if*

$$\delta \in \left(\delta_0 - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)}, \delta_0\right)$$

Proof. For $0 \leq \delta < \delta_L$ we have $q_1^*(\delta) \geq \alpha_L$, so the follower firm stays out of the market. If $0 \leq \delta < \delta_H$, F_1 sells at a price lower than its production cost, and thus incurs in dumping. If $\delta_H \leq \delta < \delta_L$, then F_1 produces the monopoly quantity α_L and sells the product at production cost, obtaining a profit of $\pi_1(\delta) = 0$. If $\delta_0 \leq \delta$, the game does not happen, so it is only left to check $\delta_L \leq \delta < \delta_0$. In this case, $q_1^*(\delta) = q_{1,L}^M(\delta)$ and the follower firm competes. The quantity it produces is:

$$q_2^*(\delta) = \frac{\alpha_L - q_{1,L}^M(\delta)}{2} = \frac{2\alpha_L - \mathbb{E}[\alpha] + \delta X(\alpha_H - \alpha_L)}{4}$$

We compare this quantity with $q_1^*(\delta)$ and solve for δ. Our goal is to investigate if there is $\delta \in [\delta_L, \delta_0)$ such that F_2 has a higher profit.

$$\begin{aligned}
\pi_1 &< \pi_2 \\
\frac{\mathbb{E}[\alpha]}{2} - \delta X\left(\frac{\alpha_H - \alpha_L}{2}\right) &< \frac{2\alpha_L - \mathbb{E}[\alpha] + \delta X(\alpha_H - \alpha_L)}{4} \\
2\mathbb{E}[\alpha] - 2\delta X(\alpha_H - \alpha_L) &< 2\alpha_L - \mathbb{E}[\alpha] + \delta X(\alpha_H - \alpha_L) \\
3\mathbb{E}[\alpha] - 2\alpha_L &< 3\delta X(\alpha_H - \alpha_L) \\
\frac{3\mathbb{E}[\alpha] - 2\alpha_L}{3X(\alpha_H - \alpha_L)} &< \delta
\end{aligned}$$

But this last expression is equal to

$$\delta > \frac{\mathbb{E}[\alpha]}{X(\alpha_H - \alpha_L)} - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)} = \delta_0 - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)} \tag{3.3}$$

which is strictly less than δ_0. It is also higher than δ_L:

$$\delta_L < \delta_0 - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)}$$
$$\delta_0 - \frac{2\alpha_L}{X(\alpha_H - \alpha_L)} < \delta_0 - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)}$$
$$\frac{2\alpha_L}{3X(\alpha_H - \alpha_L)} < \frac{2\alpha_L}{X(\alpha_H - \alpha_L)}$$

So for $\delta \in \left(\delta_0 - \frac{2\alpha_L}{3X(\alpha_H-\alpha_L)}, \delta_0\right)$, the follower firm has a higher profit. □

This theorem implies that even if the realized demand is low, the follower firm can still achieve a higher profit if the leading firm has enough aversion to risk.

3.3.2 Case $\alpha = \alpha_H$

In this case, F_2 competes in every $\delta < \delta_0$. We have the following lemma.

Lemma 3.3.2. *If $\alpha_H > 2\alpha_L$, $\beta > \mu_L$ and the realized demand is $\alpha = \alpha_H$, we have the following cases:*

(i) For $\delta \in [0, \delta_H)$: if $\mu_L \geq 1/3$ there are values for δ such that F_2 has a higher payoff

(ii) For $\delta_H \leq \delta < \delta_L$: if $\mu_L \geq 1/3$ then F_2 has a higher payoff

(iii) For $\delta_L \leq \delta < \delta_0$: F_2 has a higher payoff if

$$\delta \in \left(max\left\{ \delta_L, \delta_0 - \frac{2\alpha_H}{3X(\alpha_H - \alpha_L)} \right\}, \delta_0 \right)$$

Proof. (i) Here, $q_1^*(\delta) = q_{1,H}^M(\delta)$, and $q_2^*(\delta)$ is:

$$q_2^*(\delta) = \frac{\alpha_H}{2} - \frac{1}{2} q_{1,H}^M(\delta)$$

We compare the quantities produced:

$$q_1^*(\delta) < q_2^*(\delta) \iff q_{1,H}^M(\delta) < \frac{\alpha_H}{2} - \frac{1}{2} q_{1,H}^M(\delta) \iff 3q_{1,H}^M(\delta) < \alpha_H$$

$$3\left[\frac{\mathbb{E}[\alpha] + \mu_L\alpha_L - \delta X(\alpha_H - 2\alpha_L)}{2(1+\mu_L+\delta X)}\right] < \alpha_H$$

$$3(\mathbb{E}[\alpha] + \mu_L\alpha_L) - \delta X(3\alpha_H - 6\alpha_L) < 2\alpha_H(1+\mu_L) + \delta X 2\alpha_H$$

$$3(\mathbb{E}[\alpha]+\mu_L\alpha_L)-2\alpha_H(1+\mu_L) < \delta X(5\alpha_H-6\alpha_L)$$
$$\frac{3(\mathbb{E}[\alpha]+\mu_L\alpha_L)-2\alpha_H(1+\mu_L)}{X(5\alpha_H-6\alpha_L)} < \delta$$

We would like to know if

$$[0,\delta_H)\cap\left(\frac{3(\mathbb{E}[\alpha]+\mu_L\alpha_L)-2\alpha_H(1+\mu_L)}{X(5\alpha_H-6\alpha_L)},\delta_0\right)\neq\varnothing$$

and for that we check if

$$\frac{3(\mathbb{E}[\alpha]+\mu_L\alpha_L)-2\alpha_H(1+\mu_L)}{X(5\alpha_H-6\alpha_L)} < \delta_H$$

$$\frac{3(\mathbb{E}[\alpha]+\mu_L\alpha_L)-2\alpha_H(1+\mu_L)}{X(5\alpha_H-6\alpha_L)} < \frac{\mathbb{E}[\alpha]-\alpha_L(2+\mu_L)}{X\alpha_H} \tag{3.4}$$

We prove that there are values for the economical quantities such that the expression above is true. Note that the denominator on the LHS of Eq. (3.4) is bigger than the denominator on the RHS:

$$X(5\alpha_H-6\alpha_L) > X\delta_H \iff 4\alpha_H-6\alpha_L>0 \tag{3.5}$$

We are under the hypothesis that $\alpha_H > 2\alpha_L$, and by substituting we get

$$8\alpha_L-6\alpha_L>0 \iff 2\alpha_L>0$$

which is enough to prove Eq. (3.5). Now we derive conditions for the LHS numerator of Eq. (3.4) to be smaller than the numerator on the RHS.

$$3\mathbb{E}[\alpha]+3\mu_L\alpha_L-2\alpha_H(1+\mu_L) < \mathbb{E}[\alpha]-\alpha_L(2+\mu_L)$$
$$2\mu_L\alpha_L+2\mu_L\alpha_H+3\mu_L\alpha_L+\alpha_L(2+\mu_L) < 2\alpha_H(1+\mu_L)$$
$$\alpha_L(2\mu_L+3\mu_L+2+\mu_L) < 2\alpha_H(1+\mu_L-\mu_H)$$
$$\alpha_L(6\mu_L+2) < 2\alpha_H 2\mu_L$$
$$\alpha_L\left(\frac{6\mu_L+2}{4\mu_L}\right) < \alpha_H$$

We are working under the hypothesis that $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$. The above inequality is guaranteed by these conditions if

$$\begin{aligned}
\left(\frac{6\mu_L+2}{4\mu_L}\right) &\leq 2\left(1+\frac{\mu_L}{1-\mu_L}\right)\\
\frac{3\mu_L+1}{4\mu_L} &\leq 1+\frac{\mu_L}{1-\mu_L}\\
\frac{1-\mu_L}{4\mu_L} &\leq \frac{\mu_L}{1-\mu_L}\\
(1-\mu_L)^2 &\leq (2\mu_L)^2\\
1-\mu_L &\leq 2\mu_L\\
\frac{1}{3} &\leq \mu_L
\end{aligned}$$

So under the conditions $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$, having $\mu_L \geq \frac{1}{3}$ guarantees that there are values for δ such that F_2 earns more.

(ii) If $\delta_H \leq \delta < \delta_L$, then $q_1^*(\delta) = \alpha_L$ and $q_2^*(\delta) = \frac{\alpha_H-\alpha_L}{2}$.

$$q_1^*(\delta) < q_2^*(\delta) \iff \alpha_L < \frac{\alpha_H-\alpha_L}{2} \iff 3\alpha_L < \alpha_H$$

Since $\beta > \mu_L$, if $3 \leq 2(1+\mu_L/(1-\mu_L))$ then $3\alpha_L < \alpha_L$ is guaranteed:

$$3 \leq 2(1+\frac{\mu_L}{1-\mu_L}) \iff \frac{1}{2} \leq \frac{\mu_L}{1-\mu_L} \iff \frac{1}{3} \leq \mu_L$$

So if $\mu_L \geq 1/3$, F_2 produces more and thus has a higher payoff.

(iii) If $\delta_L \leq \delta < \delta_0$, then $q_1^*(\delta) = q_{1,L}^M(\delta)$, and $q_2^*(\delta)$ is:

$$q_2^*(\delta) = \frac{\alpha_H - q_{1,L}^M(\delta)}{2} = \frac{2\alpha_H - \mathbb{E}[\alpha] + \delta X(\alpha_H-\alpha_L)}{4}$$

We now check if there is $\delta \in [\delta_L, \delta_0)$ such that $q_1^*(\delta) < q_2^*(\delta)$.

$$\begin{gathered}
\frac{\mathbb{E}[\alpha]}{2} - \delta X(\frac{\alpha_H-\alpha_L}{2}) < \frac{2\alpha_H - \mathbb{E}[\alpha] + \delta X(\alpha_H-\alpha_L)}{4}\\
2\mathbb{E}[\alpha] - 2\delta X(\alpha_H-\alpha_L) < 2\alpha_H - \mathbb{E}[\alpha] + \delta X(\alpha_H-\alpha_L)\\
3\mathbb{E}[\alpha] - 2\alpha_H < 3\delta X(\alpha_H-\alpha_L)\\
\frac{3\mathbb{E}[\alpha]-2\alpha_H}{3X(\alpha_H-\alpha_L)} < \delta
\end{gathered}$$

And this expression is equivalent to

$$\delta > \delta_0 - \frac{2\alpha_H}{3X(\alpha_H-\alpha_L)}$$

which is strictly smaller than δ_0. Furthermore, this lower bound for δ is smaller than δ_L if:

$$\delta_0 - \frac{2\alpha_H}{3X(\alpha_H-\alpha_L)} < \delta_0 - \frac{2\alpha_L}{X(\alpha_H-\alpha_L)}$$
$$-\frac{2\alpha_H}{3X(\alpha_H-\alpha_L)} < -\frac{2\alpha_L}{X(\alpha_H-\alpha_L)}$$
$$3\alpha_L < \alpha_H$$

From the proof of (ii) and (i), we know this is guaranteed to happen if $\mu_L \geq 1/3$. So we need to set the interval for δ as the maximum of δ_L and the lower bound obtained, i.e.,

$$\delta \in \left(\max\left\{\delta_L, \delta_0 - \frac{2\alpha_H}{3X(\alpha_H-\alpha_L)}\right\}, \delta_0\right)$$

□

From this lemma we get that if the value for high demand is significantly higher than low demand (i.e., $\alpha_H > 2\alpha_L$) and the probability of low demand is high enough ($\mu_L \geq 1/3$), then the only situation in which F_2 does not have a bigger payoff is when δ is close to zero.

3.4 Dumping aversion

We study what happens in case the leading firm refuses to produce more than α_L. The only situation in which the leading firm's best response is to play a quantity higher than α_L is when $\alpha_H > 2\alpha_L$, $\beta > \mu_L$. Observing Theorem 3.2.7, we get that F_1's best response under dumping aversion is

$$q_1^*(\delta) = \begin{cases} \alpha_L & 0 \leq \delta < \delta_L \\ q_{1,L}^M(\delta) & \delta_L \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases} \tag{3.6}$$

We conduct an ex-post analysis under dumping aversion and compare the results with the ones obtained on Section 3.3.

3.4.1 Case $\alpha = \alpha_L$

Recall that in this case the follower firm's best response is $q_2^* = \max((\alpha_L - q_1)/2, 0)$. We have the following lemma.

Lemma 3.4.1. *Under dumping aversion, $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$, F_2's best response if $\alpha = \alpha_L$ is:*

$$q_2^*(\delta) = \begin{cases} 0 & 0 \leq \delta < \delta_L \\ \frac{\alpha_L}{2} - \frac{\mathbb{E}[\alpha] - \delta X(\alpha_H - \alpha_L)}{4} & \delta_L \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases} \tag{3.7}$$

Furthermore, the follower firm has a higher payoff if

$$\delta \in \left(\delta_0 - \frac{2\alpha_L}{3X(\alpha_H - \alpha_L)}, \delta_0 \right) \tag{3.8}$$

Proof. Equation (3.7) results from computing $q_2^* = \max((\alpha_L - q_1^*)/2, 0)$ using Eq. (3.6).

The interval in (3.8) results from comparing the quantities produced. If $\delta < \delta_L$ or $\delta_0 \leq \delta$, both firms earn a payoff of 0. If $\delta \in [\delta_L, \delta_0)$, we use the proof of Lemma 3.3.1 to obtain the stated interval for δ. □

3.4.2 Case $\alpha = \alpha_H$

Here we have $q_2^* = \max((\alpha_H - q_1)/2, 0)$. We have the following lemma:

Lemma 3.4.2. *Under dumping aversion, $\alpha_H > 2\alpha_L$ and $\beta > \mu_L$, F_2's best response if $\alpha = \alpha_H$ is:*

$$q_2^*(\delta) = \begin{cases} \frac{\alpha_H - \alpha_L}{2} & 0 \leq \delta < \delta_L \\ \frac{\alpha_H}{2} - \frac{\mathbb{E}[\alpha] - \delta X(\alpha_H - \alpha_L)}{4} & \delta_L \leq \delta < \delta_0 \\ 0 & \delta_0 \leq \delta \end{cases} \tag{3.9}$$

Furthermore:

(i) For $\delta \in [0, \delta_L)$: if $\mu_L \geq 1/3$, then F_2 has a higher profit than F_1; if $\mu_L < 1/3$, then F_2 earns a higher payoff if $\beta > (1 - \mu_L)/2$

(ii) For $\delta \in [\delta_L, \delta_0)$: F_2 earns a higher payoff if

$$\delta \in \left(max\left\{ \delta_L, \delta_0 - \frac{2\alpha_H}{3X(\alpha_H - \alpha_L)} \right\}, \delta_0 \right)$$

Proof. Equation (3.9) results from computing $q_2^* = \max((\alpha_H - q_1^*)/2, 0)$ using Eq. (3.6). Now we compare the quantities produced.

(i) We compare the profits if $0 \leq \delta < \delta_L$:

$$\begin{aligned} \pi_1 &< \pi_2 \\ \alpha_L &< \frac{\alpha_H - \alpha_L}{2} \\ 3\alpha_L &< \alpha_H \end{aligned}$$

Under the hypothesis of the lemma, this is guaranteed to happen if

$$\begin{aligned} 3\alpha_L &< 2\left(1 + \frac{\beta}{1 - \mu_L}\right)\alpha_L \\ \frac{1 - \mu_L}{2} &< \beta \end{aligned}$$

We know that $\beta > \mu_L$, so F_2 is guaranteed to earn a higher payoff if $(1-\mu_L)/2 \leq \mu_L$:

$$\frac{1-\mu_L}{2} \leq \mu_L$$
$$1-\mu_L \leq 2\mu_L$$
$$\frac{1}{3} \leq \mu_L$$

which means that $\mu_L \geq \frac{1}{3}$ guarantees it.

If $(1-\mu_L)/2 > \mu_L$, then we need to add the extra condition $\beta > (1-\mu_L)/2$ to assure F_2 has a higher payoff.

(ii) This statement follows directly from Lemma 3.3.2.

□

The main difference versus the case where F_1 produces a quantity higher than α_L is that the conditions for F_2 to earn a higher payoff than F_1 are less restrictive, i.e., F_2 can earn a higher payoff for lower values of δ.

CHAPTER 4

Conclusions and future work

4.1 Conclusions

In this dissertation, we studied two versions of the Stackelberg duopoly with asymmetric uncertainty about the demand, where the uncertainty is only present for the leader firm. In both versions, the demand is a discrete random variable that takes one of two values - high or low.

In Stackelberg with uncertainty, we used expected utility maximization for the leading firm and characterized its best response under three hypothesis that cover all the possible cases for the economical quantities. We derived the subgame perfect Nash equilibria for each of the hypothesis. We concluded that there are values for the economical quantities for which the follower firm has a higher payoff if the realized demand is high.

We analized the case where the leading firm does not incur in dumping. With no-dumping we saw that under the hypothesis where the low demand is the smallest when compared to two thresholds, if the probability of high demand is low enough then there is no incentive for a firm to enter the market as the leading firm.

On Stackelberg with uncertainty and mean-standard deviation preference, we fully characterized the leading firm's best response using a mean-standard deviation preference maximization. This preference is given in terms of the economical quantities but also as a function of a risk aversion parameter. We separate the analysis into cases and in each case we state the leading firm's best response for every valid value of the risk aversion parameter. For one of these cases, we conducted an ex-post analysis both when the leading firm incurs in dumping and when it is dumping averse. We concluded that if the leading firm is enough risk averse, then the follower firm can earn a higher profit. We also concluded that under no-dumping, the follower firm earns a higher profit than the leading firm for lower values of risk aversion.

4.2 Future work

We can extend our analysis by considering non-linear price functions. We considered a simple distribution for the demand random variable. Our analysis can be further extended by considering different distributions for the demand random variable (see [Liu, 2005]). We can incorporate different costs for each firm in both versions presented, as well as uncertainty for each firm about the costs of the other. Product differentiation ([Ferreira et al., 2015]) can be added to both versions of the model, as well as the component of technology innovation to reduce costs and shorten the time of production.

Comparing the results obtained under mean-standard deviation preference with the results that would have been obtained in a Cournot setting would be interesting. Different preference functions can be considered, with the goal of further studying the impact of risk aversion on the attitude of the leading firm.

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