Socially optimal land use in a city with central amenities

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Abstract

We provide a full characterisation of the socially optimal allocation of urban land to households and firms in a long narrow city with centrally located amenities. Households support linear costs of commuting to their workplace, and linear costs of travelling to the centre to benefit from amenities. Firms operate under monopolistic competition, having a marginal production cost that increases exponentially with distance to the centre. We compare the socially optimal urban land use with that which emerges as a result of market forces, and propose policy interventions to achieve city-wide welfare gains. Notably, travel subsidies to households are sufficient for the socially optimal allocation of urban land to emerge as the market equilibrium.

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1 Introduction

Urban land use in developed countries appears to be evolving from monocentric city structures, where most jobs are concentrated in the centre of the city, to polycentric city structures, where multiple job subcentres develop [Anas et al., 2009, Goswami and Lall, 2019]. One of the challenges this trend presents for urban planners is the assessment of whether the strength of market forces that are driving the emergence of subcentres is excessive or lacking. Such assessment is an important input for the design of policies that mitigate or reinforce the development of subcentres.

In this paper we characterise the socially optimal land use in a city and compare it with the land use that emerges as the result of market forces. This allows us to identify the existence of a market failure and to design policies that align private incentives with public interest so that the socially optimal city structure emerges.

Our approach is carried out in a stylised environment which builds on the Fujita-Ogawa model [Ogawa and Fujita, 1980, Fujita and Ogawa, 1982, Imai, 1982, Mossay et al., 2020] by incorporating centrally located amenities [Garrido-da-Silva et al., 2022a,b]. Two types of agents, households and firms, locate along a long narrow city with an exogenous centre and support three types of transportation costs. Households commute to their workplace and (with a possibly different frequency) travel to the centre of the city for shopping and other purposes, supporting commuting and shopping costs that are proportional to the distance they travel.[1] Firms support an iceberg cost to transport their output to the city centre which implies that their marginal cost increases exponentially with distance to the centre.[2]

The socially optimal urban land use depends on the relative magnitude of the three transportation costs. An integrated distribution of households and firms whereby each household and the corresponding employer co-locate eliminates commuting costs. By contrast, a totally segregated distribution with households at the centre and firms at the periphery minimises shopping costs, and a distribution with firms at the centre and households at the periphery minimises output transportation costs.

We find that the set of land use patterns that are possibly socially optimal coincides with the set of patterns that possibly emerge in market equilibrium. However, for given parameter values, the socially optimal pattern and the pattern that emerges in equilibrium do not coincide. In a market equilibrium, firms are excessively driven towards the

[1]We use the designation "shopping cost" for the cost of travelling to the centre, but this should not be taken literally. The city centre typically features a high concentration not only of commerce, but also of art, culture and entertainment. Besides shopping, people visit the centre for leisure and social interaction.

[2]The transportation of output to the centre is a proxy for the disadvantages of a peripheral location. It would be equivalent to assume that proximity to the city centre makes firms more innovative so that their marginal production cost increases exponentially with distance to the centre.

periphery because they neglect the negative externality imposed on households (which more than offsets the positive externality imposed on other firms) that results from the increase in marginal production cost from locating farther from the centre. This can be corrected through a subsidy that supports a constant fraction of households' commuting and shopping costs, which diminishes the incentives for households to locate close to the centre. When the magnitude of this subsidy is such that the biases of households and firms cancel out, the socially optimal land use emerges as the market equilibrium.

Relation with the existing literature

In path-breaking contributions, Ogawa and Fujita [1980], Fujita and Ogawa [1982] and Imai [1982] introduced a market equilibrium model of endogenous allocation of urban land to households and firms, thereby dispensing with the traditional assumption of having firms concentrated in a single point at the centre of the city. Subsequent research has mostly focused on the characterisation of urban land use patterns that arise in market equilibrium [Ogawa and Fujita, 1980, Fujita and Ogawa, 1982, Ota and Fujita, 1993, Anas and Kim, 1996, Anas and Xu, 1999, Anas and Rhee, 2006, Lucas and Rossi-Hansberg, 2002, Berliant et al., 2002, Brinkman, 2016, Chatterjee and Eyigungor, 2017, Malykhin and Ushchev, 2018, Mossay et al., 2020, Garrido-da-Silva et al., 2022a,b]. However, there are contributions that focus on the socially optimal land use [Imai, 1982, Rossi-Hansberg, 2004, Arnott et al., 2008], others that characterise it as a benchmark against which they compare market equilibrium land use [Fujita, 1988], and others that do not characterise socially optimal land use but find policy interventions that generate socially optimal land use in a market equilibrium [Zhang and Kockelman, 2016].

The main trade-off in the models of Imai [1982] and Rossi-Hansberg [2004] is between, on the one hand, firm agglomeration economies driving firms to cluster in a single business centre (monocentric city), and, on the other hand, commuting costs driving households and firms to co-locate or at least to form multiple business areas (polycentric city). Arnott et al. [2008] neglect firm agglomeration economies – as we do in the present paper – to focus on the tension between the attraction between households and firms due to commuting costs, and the repulsion of households from firms caused by localised pollution.

In our model, there are three transportation costs which originate two tensions between counteracting forces. The first tension is that both households and firms wish to locate near the centre, due, respectively, to the linear cost of travelling to the centre and the iceberg cost of transporting output to the centre. If the shopping cost dominates, households locate at the centre and firms at the fringe; if the output transportation cost dominates, firms locate at the centre and households at the fringe. In borderline cases, firms locate at the centre and at the fringe, with households in between.[3] The reason

[3]The corresponding land use patters can be described in compact notation by [RBR] (business area

why firms either locate at the centre or at the fringe (and never in between) is that, while households support a cost of travelling to the centre that is linear in distance, the impact of firm location on the cost of delivering one unit of the Dixit-Stiglitz aggregate of differentiated goods is concave. For this reason, it is preferable to have some firms in the centre and some firms in the fringe rather than have all firms in between. The second tension is that commuting costs generate an attraction between matched firm-household pairs that drives housing and industry to blend, thereby interfering with the solution to the previous tension. As a result, an integrated area in which matched firm-household pairs co-locate tends to emerge in the locations that are most distant to business areas.

Land use that results from market forces differs from socially optimal land use due to externalities (the location of an agent potentially affects other agents). In the presence of firm agglomeration economies [Imai, 1982, Rossi-Hansberg, 2004], there is too little agglomeration of industry because a firm takes into account the benefit of agglomeration on own profit but not on the profit of other firms. In the environment considered by Imai [1982], it can be verified (although Imai [1982] did not perform this exercise) that a subsidy that supports a constant fraction of commuting costs implements the optimal land use as an equilibrium. Intuitively, the suboptimal willingness of firms to pay to locate near other firms can be compensated by inducing households to exhibit the same suboptimal willingness to pay to locate near firms, so that the resulting allocation of land is optimal. In the more complex environment considered by Lucas and Rossi-Hansberg [2002] and Rossi-Hansberg [2004], with land-labour substitution in production and location-dependent wages, the first-best policy is a sophisticated location-dependent labour subsidy.

In our environment, which abstracts from firm agglomeration economies, firms exhibit a suboptimal willingness to pay to locate near the centre. We consider standard Dixit-Stiglitz preferences with quasi-linear upper-tier utility and logarithmic sub-utility [Pflüger, 2004, Gaspar et al., 2018], which imply that when the monopolistically competitive firms approach the centre to increase individual profit, there is a negative externality on other firms which exactly offsets the internalised individual gain, and a positive externality on consumers by a factor of $\frac{\sigma}{\sigma-1}$ (where $\sigma > 1$ is the elasticity of substitution between Dixit-Stiglitz varieties, and thus $\frac{\sigma}{\sigma-1}$ is the profit-maximising price-cost ratio). This means that individual firms have suboptimal incentives to approach the centre: the private gain is a fraction $\frac{\sigma-1}{\sigma}$ of the social gain. The optimal allocation of land to households and firms can be attained by inducing households to exhibit the same suboptimal tendency to approach the centre.

We find that subsidising commuting and shopping travel costs of households so that they support only a fraction $\frac{\sigma-1}{\sigma}$ of the total cost is a way of implementing the optimal allocation in a market equilibrium. One of the main take-aways is that taxing

at the city centre), [BRB] (business areas at the fringes of the city), and [BRBRB] (business areas at the city centre and at the fringes of the city).

commuting through congestion tolls, which has advantages that are well known [Anas and Kim, 1996, Anas and Xu, 1999, Pines and Sadka, 1985, Wheaton, 1998, 2004, Zhang and Kockelman, 2016] may also have detrimental effects.[4]

The remainder of the paper is organised as follows: the model is formulated in Section 2; the social welfare function is defined in Subsection 3.1; the optimal urban land use is found in Subsection 3.2; a comparison between the optimal land use and the market equilibrium land use is provided in Subsection 3.3; the first-best policy is found in Subsection 4.1; independence between price efficiency and land use efficiency is established in Subsection 4.2; some concluding remarks are made in Section 5.

2 Formulation of the model

Suppose that a city develops on a linear terrain of land of unit width denoted $\mathcal{B} = \mathbb{R}$. Two types of freely mobile agents, households and firms, compete for land for residential use and production use. Each household resides at a single location with coordinate $x \in \mathcal{B}$ and the density of households at x is quantified by the function $\lambda(x)$, with $\lambda : \mathcal{B} \to [0, 1]$. Each firm produces a differentiated good at a single location with coordinate $y \in \mathcal{B}$ and the density of firms at y is quantified by the function $\mu(y)$, with $\mu : \mathcal{B} \to [0, 1]$. The total mass of households is exogenous and given by $N = \int_{\mathcal{B}} \lambda(x)dx$, whereas the total mass of firms is endogenous and given by $M = \int_{\mathcal{B}} \mu(y)dy$. The supports of the distributions of households and firms are respectively denoted by $\mathcal{X} = \{x \in \mathcal{B} : \lambda(x) > 0\}$ and $\mathcal{Y} = \{y \in \mathcal{B} : \mu(y) > 0\}$.

The city has a predetermined centre, taken as the origin, which households visit regularly and to to which firms' output is transported to. As in Alonso [1964], land is owned by rent-maximising absentee landlords who rent their plots to the highest bidder. The demand of land is inelastic for all agents so that the lot size is fixed at some positive constant, S_{h} for households and S_{f} for firms, such that $S_{\mathrm{h}}\lambda(z)+S_{\mathrm{f}}\mu(z) \leq 1$ for all $z \in \mathcal{B}$. Households earn their income by supplying inelastically labour to firms. Both land and labour markets are assumed to be perfectly competitive at every location.

Let $R(z)$ and $w(z)$ denote respectively the land rent per unit of land and the wage per unit of labour prevailing at $z \in \mathcal{B}$.

Henceforth, $\mathbb{R}_-$ and $\mathbb{R}_+$ denote the sets of non-positive and non-negative real numbers, respectively.

[4] See also Brinkman [2016].

2.1 Households and firms

Households share the same tastes for a composite of differentiated varieties Q and a homogeneous good c_0. The utility function is quasi-linear with logarithmic subutility [Pflüger, 2004]:

$$U(Q, c_0) = \alpha \ln(Q) + c_0, \qquad Q = \left[\int_{\mathcal{Y}} c(y)^{\frac{\sigma-1}{\sigma}} \mu(y) dy\right]^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where $\alpha > 0$ is the expenditure intensity on varieties, $\sigma > 1$ is the constant elasticity of substitution between varieties, and $c(y)$ is the consumption of a variety produced at $y \in \mathcal{Y}$.

The homogeneous good is freely traded everywhere, being chosen as the numeraire. The composite good is exclusively available at the centre[5] at the consumer price index

$$P = \left[\int_{\mathcal{Y}} p_{\mathrm{h}}(y)^{1-\sigma} \mu(y)\right]^{\frac{1}{1-\sigma}}, \tag{2}$$

where $p_{\mathrm{h}}(y)$ is the delivered price of a variety produced at $y \in \mathcal{Y}$. Each household is endowed with one unit of labour, making two independent and costly trips: to work and to purchase the composite good. The *commuting cost* equals a positive constant t_{f} times the distance between the residence and the job site, while the *shopping cost* equals a positive constant t_{s} times the distance between the residence and the centre.

A household residing at $x \in \mathcal{X}$ and working at $x_{\mathrm{w}} \in \mathcal{Y}$ receives a wage of $w(x_{\mathrm{w}})$, incurs a total travel cost of $t_{\mathrm{f}}|x - x_{\mathrm{w}}| + t_{\mathrm{s}}|x|$ and pays land rent $R(x)S_{\mathrm{h}}$ on its S_{h} units of land. The budget constraint can be written as

$$c_0(x, x_{\mathrm{w}}) + PQ + t_{\mathrm{f}}|x - x_{\mathrm{w}}| + t_{\mathrm{s}}|x| + R(x)S_{\mathrm{h}} = w(x_{\mathrm{w}}) + \overline{c}_0 + g \tag{3}$$

where $\overline{c}_0 > 0$ represents the initial endowment of numeraire and $g > 0$ represents a potential flat-rate location subsidy for households. Given prices, wages and land rents, the individual optimisation problem of each household is as follows:

Problem (H). *For each household residing at $x \in \mathcal{X}$ and working at $x_{\mathrm{w}} \in \mathcal{Y}$, choose consumption levels Q, $c(y)$ and $c_0(x, x_{\mathrm{w}})$ so as to maximise utility $U(Q, c_0)$ in* (1) *subject to the budget constraint in* (3).

Firms use the same technology requiring γ units of numeraire per unit of output, L units of labour and S_{f} units of land. Under monopolistic competition, each firm manufactures a single variety of goods which are sold at the centre. There are no barriers to entry and exit from the industry.

[5]It is equivalent to assume that the composite good is available everywhere, but firms have a marginal production cost that is increasing with distance from the centre at the rate τ.

Transportation costs for varieties take *Samuelson's iceberg* form such that if a unit is shipped from location y to the centre, only $\mathrm{e}^{-\tau|y|}$ units actually arrive, where τ is a positive constant.

A firm operating at $y \in \mathcal{Y}$ produces the quantity $c(y)\mathrm{e}^{\tau|y|}$ to fully satisfy consumer's demand $c(y)$ for their variety. By charging the mill price $p(y)$, the delivered price is $p_{\mathrm{h}}(y) = p(y)\,\mathrm{e}^{\tau|y|}$ and the production cost totals $\gamma\, c(y)\mathrm{e}^{\tau|y|}N + w(y)L + R(y)S_{\mathrm{f}}$. The firm's profit is thus expressed as

$$\Pi(y) = [p(y) - \gamma]\, c(y)\mathrm{e}^{\tau|y|}N - [w(y) - s(y;p)]\, L - R(y)S_{\mathrm{f}}, \tag{4}$$

where $s(y;p)$ represents a potential labour subsidy for firms at y to hire each worker. Given wages and land rents, the individual optimisation problem of each firm is as follows:

Problem (F). *For each firm producing at $y \in \mathcal{Y}$, choose mill price $p(y)$ so as to maximise profit $\Pi(y)$ in (4).*

We are interested in designing urban policies that maximise social welfare. Without loss of generality, we normalise $S_{\mathrm{h}} = 1$ and $S_{\mathrm{f}} = 1$.

2.2 Equilibrium conditions

Product market equilibrium is achieved when the behaviours of utility-maximising consumers and profit-maximising producers are such that demand of goods matches supply. Suppose that $\hat{Q}$, $\hat{c}(y)$ and $\hat{c}_0(x, x_{\mathrm{w}})$ are the solutions to Problem (H) and U^* is the equilibrium utility level. Suppose that $\hat{p}(y)$ is the solution to Problem (F) and $\hat{P}$ is the associated price index in (2). All households consume the same amount of goods regardless of where they reside and work. The equilibrium consumption levels are

$$\hat{Q} = \frac{\alpha}{\hat{P}}, \tag{5}$$

$$\hat{c}(y) = \alpha\,\frac{\hat{p}(y)^{-\sigma}\mathrm{e}^{-\sigma\tau|y|}}{\hat{P}^{1-\sigma}} \qquad \text{for all } y \in \mathcal{Y}, \tag{6}$$

$$\hat{c}_0(x, x_{\mathrm{w}}) \equiv \hat{c}_0 = U^* - \alpha\big(\ln\alpha - \ln\hat{P}\big) \qquad \text{for all } x \in \mathcal{X},\ x_{\mathrm{w}} \in \mathcal{Y}, \tag{7}$$

with the equilibrium price of each variety produced at $y \in \mathcal{Y}$ satisfying

$$\frac{\partial s}{\partial p} = \frac{\alpha N}{L}\frac{p(y)^{-1-\sigma}\mathrm{e}^{(1-\sigma)\tau|y|}}{P^{1-\sigma}}\big\{ -p(y) + \sigma\,[p(y) - \gamma]\,\big\}. \tag{8}$$

A commuting trip originating at a location may take any different destination within the city. We introduce a commuting correspondence $Y : \mathcal{B} \to \mathcal{B}$ that associates a potential job site x_{w} with a potential residence x in a one-to-one manner, i.e., $Y(x) = x_{\mathrm{w}}$. Labour market equilibrium requires that households and firms are all located such that $Y(\mathcal{X}) = \mathcal{Y}$ and labour supply equals labour demand under the commuting pattern Y:

$$\int_J \lambda(x)dx = \int_{Y(J)} L\mu(y)dy \qquad \text{for every interval } J \subset \mathcal{B}.$$

From (7), the consumption $\hat{c}_0$ of the numeraire good is maximum at the equilibrium level U^*. Plugging (5) and (7) into (3), it follows that every household residing at x must work at location $Y(x)$ that maximises wage net of commuting cost, $w(y) - t_\mathrm{f}|x-y|$, with respect to job site y. Under this condition, net wage equals the wage work-based income of households working at x, i.e., $w(Y(x)) - t_\mathrm{f}|x - Y(x)| = w(x)$. No worker can benefit from changing their job site as long as $|w'(x)| = t_\mathrm{f}$ if commuting occurs, and $|w'(x)| \leq t_\mathrm{f}$ otherwise. Let $\hat{w}$ be the wage profile prevailing in equilibrium.

It is supposed that the land rent is the only source of landlord income. Land must be allocated to households and firms who bid for any plot via their willingness to pay. By (3), the household bid rent function $\hat{\Psi}_\mathrm{h}(x)$ is

$$\hat{\Psi}_\mathrm{h}(x) = \alpha\big(\ln\alpha - \ln\hat{P}\big) - \alpha + \hat{w}(x) + \overline{c}_0 + g - t_\mathrm{s}|x| - U^*. \tag{9}$$

Competition drives the profit of all firms to zero. Using (4), the firm bid rent function $\hat{\Psi}_\mathrm{f}(y)$ is

$$\hat{\Psi}_\mathrm{f}(y) = \alpha N\left[\hat{p}(y) - \gamma\right]\frac{\hat{p}(y)^{-\sigma}\mathrm{e}^{(1-\sigma)\tau|y|}}{\hat{P}^{1-\sigma}} - \left[\hat{w}(y) - s(y;\hat{p})\right]L. \tag{10}$$

Land market equilibrium establishes that, at each location $z \in \mathcal{B}$, land is occupied by the highest bidder in a way that there is no vacant land in the city. Assuming that the opportunity cost of land is zero, the market rent profile is $\hat{R}(z) = \max\big\{\hat{\Psi}_\mathrm{h}(z), \hat{\Psi}_\mathrm{f}(z), 0\big\}$. When the market rent exceeds the opportunity cost, the land use shares for households and firms offset the available land wherein $\lambda(z) + \mu(z) = 1$.

A *spatial equilibrium* is reached when: taking wages and land rents as given, households optimally choose both their residence and their workplace, and achieve the same utility level U^*; taking wages and land rents as given, firms optimally choose their factory location and have zero profit; land and labour markets clear at all locations. The unknowns are, for each $z \in \mathcal{B}$, the household distribution $\lambda(z)$, the firm distribution $\mu(z)$, the land rent function $\hat{R}(z)$, the wage function $\hat{w}(z)$, the commuting function $Y(z)$, the utility level U^*, the price level $\hat{p}(z)$, plus the policy instruments g and $s(z;p)$. See Appendix B for the free-market case.

2.3 Commuting pattern

Given the commuting correspondence Y above, the total distance of commuting is expressed as

$$\int_\mathcal{B} |z - Y(z)|\,\lambda(z)dz.$$

For an arbitrary location $z \in \mathcal{B}$, let $n(z)$ denote the total excess of labour to the left of z with the functional form

$$n(z) = \int_{-\infty}^{z} \lambda(x)dx - \int_{-\infty}^{z} L\mu(y)dy.$$

The first term gives the total labour supplied by households residing to the left of z. The second term gives the total labour demanded by firms operating to the left of z. When $n(z) \neq 0$, at least $|n(z)|$ commuters have to pass through z so as to meet the labour requirement in the right-hand side of the city. These commuters contribute to the overall commuting distance by the amount of $|n(z)|dz$ at each z. Accordingly,

$$\int_{\mathcal{B}} |z - Y(z)| \, \lambda(z)dz \geq \int_{\mathcal{B}} |n(z)|dz.$$

The term on the right-hand side indicates the level of the minimal distance of commuting. We formulate the following

Lemma 2.1. *Keeping λ and μ fixed, there is a unique (almost everywhere in $\mathcal{B}$) strictly monotone increasing commuting correspondence Y^* that minimises the total commuting distance such that*

$$\int_{\mathcal{B}} |z - Y^*(z)| \, \lambda(z)dz = \int_{\mathcal{B}} |n(z)|dz. \tag{11}$$

Proof. See Appendix C. □

We say that *cross commuting* occurs if one commute path contains another: given a commuting correspondence Y, there exist $z_1, z_2 \in \mathcal{B}$ such that $(z_1 - z_2)[Y(z_1) - Y(z_2)] < 0$. An immediate consequence of Lemma 2.1 is that the socially optimal commuting correspondence (the one that minimises the total commuting distance) does not feature cross commuting. The flow of commuters is unidirectional everywhere: at any $z \in \mathcal{B}$, commuting is exclusively right bound when $n(z) > 0$ and exclusively left bound when $n(z) < 0$. Locations that involve no commuting or a change in commuting direction satisfy $n(z) = 0$.

The Fundamental Theorem of Calculus ensures that n is continuous and the flow of commuters evolves according to the differential equation $n'(z) = \lambda(z) - L\mu(z)$.

3 Optimal land use

Optimality is defined in terms of the maximisation of social welfare. Suppose that all land is owned by a benevolent planner (e.g., the city government) who controls the distributions of households and firms over space.

3.1 Social welfare function

A convenient formulation of optimisation problems for land use theory is the so-called *Herbert-Stevens model* [Herbert and Stevens, 1960]. When all households are assumed identical, the objective of the planner is to maximise welfare among the whole population while the utility level is equal everywhere. The solution is always efficient (or Pareto-optimal), and any optimal allocation is obtained adjusting the prespecified target utility level.

Under the prevailing prices, wages and rents in the city, what is to be consumed is equal to what is to be produced. Welfare is measured by means of the citywide net surplus $\mathscr{S}$ associated with the plan $\{\lambda(x), \mu(y), Y(x), c(y), c_0(x,y);\ x, y \in \mathcal{B}\}$. The net surplus is defined to be the difference between the revenue from the initial endowment of numeraire and the cost of maintaining a common utility $\bar{U}$ for all households:

$$\mathscr{S} = \mathsf{Total\ Revenue} - \mathsf{Total\ Cost}$$

where

$$\mathsf{Total\ Revenue} = \overline{c}_0\, N$$

$$\begin{aligned}\mathsf{Total\ Cost} = {} & \underbrace{t_\mathrm{f} \int_{\mathcal{B}} |z - Y(z)| \lambda(z) dz}_{\text{aggregate commuting costs}} \quad + \quad \underbrace{t_\mathrm{s} \int_{\mathcal{B}} |z| \lambda(z) dz}_{\text{aggregate shopping costs}} \\ & + \underbrace{\int_{\mathcal{B}} c_0(z, Y(z)) \lambda(z) dz}_{\text{household numeraire consumption}} + \underbrace{\gamma \int_{\mathcal{B}} \int_{\mathcal{B}} c(y) \mathrm{e}^{\tau|y|} \mu(y) \lambda(z) dy dz}_{\text{firm production costs}}.\end{aligned}$$

Total cost in the city consists of the expenditures on the numeraire good and the travel costs of the households, on the marginal cost of production of the firms, and on the opportunity cost of land used for urban purposes. The latter is actually zero.

The resource constraints for the maximisation are:

the target utility constraint	$U(Q, c_0) = \bar{U}$
the commuting flow constraint	$n'(z) = \lambda(z) - L\mu(z)$
the land constraint	$\lambda(z) + \mu(z) \leq 1$
the population constraints	$\int_{\mathcal{B}} \lambda(z) dz = N, \qquad \int_{\mathcal{B}} \mu(z) dz = M$
the full employment constraint	$N = ML$

and the non-negativity constraints on all choice variables.

Since $\overline{c}_0\, N$ is a constant, the maximisation of the net surplus is equivalent to the minimisation of the total cost. From Lemma 2.1 the commuting correspondence Y^*

minimises aggregate commuting costs. We use (11) and solve $U(Q, c_0) = \bar{U}$ for $c_0 = \bar{U} - \alpha \ln(Q)$, which is independent of $Y = Y^*$, so the objective function becomes

$$\mathscr{S} = \bar{c}_0 N - \int_{\mathcal{B}} \left\{ t_{\mathrm{f}} |n(z)| + \left[t_{\mathrm{s}} |z| + \bar{U} \right] \lambda(z) \right\} dz + \int_{\mathcal{B}} \left[\alpha \ln(Q) - \gamma \int_{\mathcal{B}} c(y) \mathrm{e}^{\tau |y|} \mu(y) dy \right] \lambda(z) \, dz. \tag{12}$$

By the fact that money transfers are welfare-neutral, there is no loss of generality in assuming $g = 0$ and $s(\cdot) = 0$ in the remainder of this section.

3.2 Social optimum

We study the first-best problem in which the optimal price level and land use pattern are chosen without any market constraints.

The third term of the right-hand side in (12) can be maximised with respect to Q and $c(y)$ independently of all other variables. In other words,

$$\max_{Q, c(y)} \left\{ \alpha \ln(Q) - \gamma \int_{\mathcal{B}} c(y) \mathrm{e}^{\tau |y|} \mu(y) dy \right\} \quad \text{subject to} \quad Q = \left[\int_{\mathcal{B}} c(y)^{\frac{\sigma - 1}{\sigma}} \mu(y) dy \right]^{\frac{\sigma}{\sigma - 1}}.$$

The method of Lagrange multipliers says to look for solutions of the Lagrangian

$$\mathcal{F}(Q, c(y), \nu) = \alpha \ln(Q) - \gamma \int_{\mathcal{B}} c(y) \mathrm{e}^{\tau |y|} \mu(y) dy - \nu \left\{ Q - \left[\int_{\mathcal{B}} c(y)^{\frac{\sigma - 1}{\sigma}} \mu(y) dy \right]^{\frac{\sigma}{\sigma - 1}} \right\},$$

where $\nu \geq 0$ stands for the Lagrange multiplier. The first order conditions yield

$$\frac{\partial \mathcal{F}}{\partial Q} = \frac{\alpha}{Q} - \nu = 0$$

$$\frac{\partial \mathcal{F}}{\partial c} = -\gamma \mathrm{e}^{\tau |y|} \mu(y) + \nu \left[\int_{\mathcal{B}} c(y)^{\frac{\sigma - 1}{\sigma}} \mu(y) dy \right]^{\frac{1}{\sigma - 1}} c(y)^{-\frac{1}{\sigma}} \mu(y) = 0.$$

After replacing ν by $\frac{\alpha}{Q}$ into $\frac{\partial \mathcal{F}}{\partial c} = 0$ and using the constraint, we obtain

$$\frac{\alpha}{Q} \left[\int_{\mathcal{B}} c(y)^{\frac{\sigma - 1}{\sigma}} \mu(y) dy \right]^{\frac{1}{\sigma - 1}} c(y)^{-\frac{1}{\sigma}} = \gamma \mathrm{e}^{\tau |y|} \Leftrightarrow \alpha \frac{c(y)^{-\frac{1}{\sigma}}}{Q^{\frac{\sigma - 1}{\sigma}}} = \gamma \mathrm{e}^{\tau |y|}. \tag{13}$$

For arbitrary $y_1, y_2 \in \mathcal{Y}$ the following equality holds

$$\frac{\alpha}{\gamma \mathrm{e}^{\tau |y_1|}} \frac{c(y_1)^{-\frac{1}{\sigma}}}{Q^{\frac{\sigma - 1}{\sigma}}} = \frac{\alpha}{\gamma \mathrm{e}^{\tau |y_2|}} \frac{c(y_2)^{-\frac{1}{\sigma}}}{Q^{\frac{\sigma - 1}{\sigma}}} \Leftrightarrow c(y_1) = c(y_2) \mathrm{e}^{\sigma \tau (|y_2| - |y_1|)}.$$

Again from the constraint,

$$Q = \left[\int_{\mathcal{B}} c(y_2) \mathrm{e}^{(\sigma - 1) \tau (|y_2| - |y_1|)} \mu(y_1) dy_1 \right]^{\frac{\sigma}{\sigma - 1}} = c(y_2) \mathrm{e}^{\sigma \tau |y_2|} T^{-\sigma} \tag{14}$$

with

$$T \equiv \left[\int_{\mathcal{B}} \mathrm{e}^{(1-\sigma)\tau|y|}\mu(y)dy\right]^{\frac{1}{1-\sigma}}.$$

We combine (13) with (14) to get

$$c^*(y) = \frac{\alpha}{\gamma}\frac{\mathrm{e}^{-\sigma\tau|y|}}{T^{1-\sigma}} \qquad \text{and} \qquad Q^* = \frac{\alpha}{\gamma T}. \tag{15}$$

As expected, *the optimal variety consumption level* $c^*(y)$ *(and hence the optimal composite consumption level* Q^**) in* (15) *corresponds to the equilibrium demand distribution under marginal cost pricing*[6], i.e., $p^*(y) = \gamma$.

Next, we substitute (15) into (12) and, since $\int_{\mathcal{B}} \lambda(z)dz = N$, the social optimum problem can be stated as:

Problem (O). *Choose density distributions* $\{\lambda(x), \mu(y);\ x, y \in \mathcal{B}\}$ *so as to maximise*

$$\mathscr{S} = \int_{\mathcal{B}} \left\{ \left[\bar{c}_0 - t_{\mathrm{s}}\,|z| - \bar{U} + \alpha\left(\ln\tfrac{\alpha}{\gamma} - \ln T\right) - \alpha\right]\lambda(z) - t_{\mathrm{f}}|n(z)|\right\} dz \tag{16}$$

subject to

$$n'(z) = \lambda(z) - L\mu(z) \tag{17}$$

$$\lambda(z) + \mu(z) \leq 1 \tag{18}$$

$$\int_{\mathcal{B}} \lambda(z)dz = N, \qquad \int_{\mathcal{B}} \mu(z)dz = M \tag{19}$$

$$N = ML \tag{20}$$

$$\lambda(z) \geq 0, \qquad \mu(z) \geq 0, \quad \textit{for all } z \in \mathcal{B}. \tag{21}$$

A solution set $\{\lambda(x), \mu(y);\ x, y \in \mathcal{B}\}$ of Problem (O) is called an *optimal urban configuration*

We solve Problem (O) by means of optimal control theory (see Kanemoto [1980, Appendix IV]). The state variable is the stock of labour $n(z)$; the control variables are the household density and firm density, $\lambda(z)$ and $\mu(z)$; and the control parameters are the household population and firm population, N and M. The Hamiltonian function $\mathcal{H} \equiv \mathcal{H}(z, n, \lambda, \mu, w_{\mathrm{s}}, \theta, \phi)$ is given by

$$\begin{aligned}\mathcal{H} = \Big[\bar{c}_0 - t_{\mathrm{s}}\,|z| - \bar{U} + \alpha\Big(\ln\tfrac{\alpha}{\gamma} - \ln T\Big) - \alpha\Big]\lambda(z) - t_{\mathrm{f}}|n(z)| \\ + w_{\mathrm{s}}(z)\left[\lambda(z) - L\mu(z)\right] + \theta\lambda(z) - \phi\mu(z),\end{aligned} \tag{22}$$

[6] Set $\hat{p}(y) = \gamma$. From the equilibrium demand of each y-variety in (6), we get $c(y) = \dfrac{\alpha}{\gamma}\dfrac{\mathrm{e}^{-\sigma\tau|y|}}{T^{1-\sigma}}$.

where $w_{\rm s}(z)$ is the co-state variable associated with (17), and θ and ϕ are the multipliers associated with (19). Applying the Maximum Principle, the Hamiltonian must be maximised under the constraints (18) and (20). We build the Lagrangian $\mathcal{L} \equiv \mathcal{L}(z, n, \lambda, \mu, w_{\rm s}, \theta, \phi, R_{\rm s}, \delta)$ where

$$\mathcal{L} = \mathcal{H} + R_{\rm s}(z)\,[1 - \lambda(z) - \mu(z)] + \delta(N - ML),$$

and $R_{\rm s}(z)$ and δ are respectively the Lagrange multipliers associated with (18) and (20).

Incorporating the non-negativity restrictions in (21), we derive the first order conditions

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{c}_0 - t_{\rm s}|z| - \bar{U} + \alpha\left(\ln\tfrac{\alpha}{\gamma} - \ln T\right) - \alpha + w_{\rm s}(z) + \theta - R_{\rm s}(z) \le 0, \tag{23}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \mu} &= \frac{d}{d\mu}\Big(-\alpha\ln T\Big)\int_{\mathcal{B}} \lambda(z)dz - w_{\rm s}(z)L - \phi - R_{\rm s}(z)\\ &= -\frac{\alpha N}{1-\sigma}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w_{\rm s}(z)L - \phi - R_{\rm s}(z) \le 0.\end{aligned} \tag{24}$$

In either case, the equality holds when $\lambda(z) > 0$ and $\mu(z) > 0$.

Within the set of admissible locations, θ, ϕ and δ are non-negative constants and can be interpreted as the shadow prices for households, firms and workers in utility terms. The transversality conditions for N and M lead to

$$\left.\begin{aligned}\delta + (-\theta) = 0\\ -\delta L + \phi = 0\end{aligned}\right\} \;\Rightarrow\; \theta L = \phi. \tag{25}$$

The multiplier $R_{\rm s}(z)$ is required to be piecewise continuous and represents the shadow rent of land at each location z such that

$$R_{\rm s}(z) \ge 0 \quad \text{and} \quad R_{\rm s}(z)\,[1 - \lambda(z) - \mu(z)] = 0. \tag{26}$$

The co-state variable $w_{\rm s}(z)$ is required to be continuous and have piecewise continuous derivative. Its behaviour realises the shadow wage at each location z under an extra condition provided by

$$w_s'(z) = -\frac{\partial \mathcal{H}}{\partial n} = \begin{cases} -t_{\rm f} & \text{if } n(z) < 0\\ \in [-t_{\rm f}, t_{\rm f}] & \text{if } n(z) = 0\\ t_{\rm f} & \text{if } n(z) > 0.\end{cases} \tag{27}$$

To specify the optimality conditions, it is convenient to recast shadow prices in terms of output. Accordingly, define the wage and land rent functions

$$w(z) = w_{\rm s}(z) + \theta \qquad \text{and} \qquad R(z) = R_{\rm s}(z). \tag{28}$$

On account of expressions in (23)–(24), the household and firm bid rent functions proposed by the planner can be written as

$$\Psi_{\mathrm{h}}^{\mathrm{opt}}(z) = \alpha\left(\ln\tfrac{\alpha}{\gamma} - \ln T\right) - \alpha + w(z) + \bar{c}_0 - t_{\mathrm{s}}|z| - \bar{U}, \tag{29}$$

$$\Psi_{\mathrm{f}}^{\mathrm{opt}}(z) = \frac{\alpha N}{\sigma - 1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w(z)L. \tag{30}$$

The optimality conditions are now summarised as follows:

(i) optimal allocation of land: at each $z \in \mathcal{B}$,

$$\begin{aligned} &R(z) = \max\left\{\Psi_h^{opt}(z), \Psi_f^{opt}(z), 0\right\} && && (31)\\ &R(z) = \Psi_h^{opt}(z) && \text{if } z \in \mathcal{X} = \left\{x \in \mathcal{B} : \lambda(x) > 0\right\} \\ &R(z) = \Psi_f^{opt}(z) && \text{if } z \in \mathcal{Y} = \left\{y \in \mathcal{B} : \mu(y) > 0\right\} \end{aligned}$$

and

$$\begin{aligned} \lambda(z) + \mu(z) &\leq 1 && \text{if } R(z) = 0 \\ \lambda(z) + \mu(z) &= 1 && \text{if } R(z) > 0 \end{aligned}$$

and

$$\lambda(z) = \begin{cases} 1 & \text{if } \Psi_h^{opt}(z) - \Psi_f^{opt}(z) > 0 \\ \in [0,1] & \text{if } \Psi_h^{opt}(z) - \Psi_f^{opt}(z) = 0 \\ 0 & \text{if } \Psi_h^{opt}(z) - \Psi_f^{opt}(z) < 0 \end{cases} \tag{32}$$

(ii) commuting pattern: at each $z \in \mathcal{B}$,

$$n(z) = \int_{-\infty}^{z} \lambda(x)dx - \int_{-\infty}^{z} L\mu(y)dy$$

(iii) optimal allocation of labour: at each $z \in \mathcal{B}$,

$$w'(z) = \begin{cases} -t_f & \text{if } n(z) < 0 \\ \in [-t_f, t_f] & \text{if } n(z) = 0 \\ t_f & \text{if } n(z) > 0 \end{cases} \tag{33}$$

and

$$\int_{\mathcal{B}} \lambda(z)dz = N, \qquad \int_{\mathcal{B}} \mu(z)dz = M = \frac{N}{L}.$$

Part (i) is a direct consequence of (26) together with (23) and (24). The planner allows for a perfectly competitive land market to determine the allocation of land to households and firms depending on whose bid is higher. Part (ii) comes from Lemma 2.1 applied to the commuting flow constraint. An additional number of commuters passing the infinitesimal interval dz on their way to work exactly offsets the excess of labour generated in dz with the minimum aggregate commuting cost. Part (iii) is the result of (27) and the population constraints. Under the equal-utility condition, all alternative worplace choices must be equally attractive. This requires a wage gradient that varies with the marginal cost of commuting.

The Maximum Principle gives conditions that are necessary for an admissible pair $\{\lambda(\cdot), \mu(\cdot)\}$ to solve Problem (O). We have just shown that

Proposition 3.1. *Let $\{\lambda^*(x), \mu^*(y);\ x, y \in \mathcal{B}\}$ be an admissible urban configuration for Problem (O). If $\{\lambda^*(x), \mu^*(y);\ x, y \in \mathcal{B}\}$ is optimal, then there exists a price system $\{w^*(z), R^*(z);\ z \in \mathcal{B}\}$ satisfying (i)–(iii).*

A sufficient condition for a maximum is the concavity of the Hamiltonian evaluated at the candidate solution. With $\lambda = 1 - \mu$ that the Maximum Principle produces, the function $n \mapsto \max_\mu \mathcal{H}(z, n, 1-\mu, \mu, w_{\mathrm{s}}, \theta, \phi)$ is linear, and hence concave for each z. The Arrow sufficiency theorem establishes that (n, λ, μ) is a local maximiser of $\mathcal{H}$. At the same time, the function $\mu \mapsto \mathcal{H}(z, n, 1-\mu, \mu, w_{\mathrm{s}}, \theta, \phi)$ is strictly concave in μ for each z. Since the control region $[0, 1]$ is a convex and compact set, $\mathcal{H}$ does have a unique maximum. Immediately, we state that

Proposition 3.2. *Let $\{\lambda^*(x), \mu^*(y);\ x, y \in \mathcal{B}\}$ be an admissible urban configuration for Problem (O). If there exists a price system $\{w^*(z), R^*(z);\ z \in \mathcal{B}\}$ satisfying (i)–(iii), then $\{\lambda^*(x), \mu^*(y);\ x, y \in \mathcal{B}\}$ is optimal and uniquely solves Problem (O).*

In the end, the first-best optimal solution consists in the combination of the optimal variety consumption $c^*(y)$ in (15) and the optimal land use pattern $\{\lambda^*(\cdot), \mu^*(\cdot)\}$ from Proposition 3.2.

3.2.1 Properties of optimal land use patterns

The planner understands that it is wasteful to have vacant land within the city. Because the populations of households and firms are fixed, the support of the city $\mathcal{B}^*$ is restricted to the closure of $\mathcal{X} \cup \mathcal{Y}$, which is a bounded and convex subset of $\mathbb{R}$. In consequence, the hinterland $\mathcal{B} \setminus \mathcal{B}^*$ is vacant.

Our analysis is limited to the case of *symmetric urban configurations* with respect to the centre. Without loss of generality, let $\mathcal{B}^* = [-b, b]$ where $\pm b$ are the city fringes. The absence of vacant land together with full employment lead to

$$2b = N + M \;\Leftrightarrow\; b = \frac{1+L}{2L}\, N.$$

We see that the sign of the bid rent differential $\Psi_{\mathrm{f}} - \Psi_{\mathrm{h}}$ controls the allocation of economic activity over $\mathcal{B}^*$. The urban sprawl is partitioned into a finite number of sections of the form $[b_j, b_{j+1}]$ for consecutive locations $b_j < b_{j+1}$ with $\Psi_{\mathrm{f}}(b_j) = \Psi_{\mathrm{h}}(b_j)$ and $\Psi_{\mathrm{f}}(b_{j+1}) = \Psi_{\mathrm{h}}(b_{j+1})$. It suffices to evaluate $\Psi_{\mathrm{f}} - \Psi_{\mathrm{h}}$ at each of those sections.

Recall that journeys to work do not exhibit a cross commuting pattern. Given the traffic flow of labour described by (17) across the city, we say that $[b_j, b_{j+1}]$ is a *residential section* if $n'(z) > 0$ on $[b_j, b_{j+1}]$. We say that $[b_j, b_{j+1}]$ is a *business section* if $n'(z) < 0$ on $[b_j, b_{j+1}]$. We say that $[b_j, b_{j+1}]$ is an *integrated section* if $n(z) = n'(z) = 0$ on $[b_j, b_{j+1}]$.

By making use of properties of bid rent functions stated in Appendix A, for every $z \in [b_j, b_{j+1}]$, we can generically compute

$$\begin{aligned}
&\Psi_{\mathrm{f}}^{\mathrm{opt}}(z) - \Psi_{\mathrm{h}}^{\mathrm{opt}}(z) = \\
&\quad = \int_{b_j}^{z} \left[(\Psi_{\mathrm{f}}^{\mathrm{opt}})'(u) - (\Psi_{\mathrm{h}}^{\mathrm{opt}})'(u) \right] du \\
&\quad = \int_{b_j}^{z} \left\{ -\mathrm{sgn}(u) \frac{\alpha \tau N}{T^{1-\sigma}} \mathrm{e}^{(1-\sigma)\tau |u|} - \mathrm{sgn}[n(u)](1+L)t_{\mathrm{f}} + \mathrm{sgn}(u) t_{\mathrm{s}} \right\} du \\
&\quad = \frac{\alpha N}{\sigma - 1} \frac{\mathrm{e}^{(1-\sigma)\tau |z|} - \mathrm{e}^{(1-\sigma)\tau |b_j|}}{T^{1-\sigma}} - (1+L)\, t_{\mathrm{f}} \left\{ \mathrm{sgn}[n(z)]\, z - \mathrm{sgn}[n(b_j)]\, b_j \right\} \qquad (34) \\
&\qquad\qquad + t_{\mathrm{s}} \big\{ \mathrm{sgn}(z)\, z - \mathrm{sgn}(b_j)\, b_j \big\}.
\end{aligned}$$

From here, we check at once that:

Lemma 3.3. *The optimal urban configuration is independent of the utility level $\bar{U}$.*

3.3 Optimum versus equilibrium

We compare the optimal solution with the one arising from the competitive market mechanism. A brief exposition of the free-market equilibrium solution can be found in Appendix B. The set of optimality conditions *(i)–(iii)* above and the set of equilibrium conditions *(i)–(iii)* in Definition B.1 have the same formulation. The only difference concerns the functional form of the bid rents, and so the bid rent differential.

Due to the symmetry of the city, we can focus on the right-hand side $[0, b]$. Coordinate locations are positive and the function $n(z)$ has the same sign in each section $[b_j, b_{j+1}]$, so that at the optimum,

$$\begin{aligned}
&\Psi_{\mathrm{f}}^{\mathrm{opt}}(z) - \Psi_{\mathrm{h}}^{\mathrm{opt}}(z) = \\
&\quad = \frac{\alpha N}{\sigma - 1} \frac{\mathrm{e}^{(1-\sigma)\tau z} - \mathrm{e}^{(1-\sigma)\tau b_j}}{T^{1-\sigma}} + \big\{ -(1+L)\,\mathrm{sgn}[n(z)]\, t_{\mathrm{f}} + t_{\mathrm{s}} \big\} (z - b_j), \qquad (35)
\end{aligned}$$

and in equilibrium,

$$\begin{aligned}
&\Psi_{\mathrm{f}}^{\mathrm{eq}}(z) - \Psi_{\mathrm{h}}^{\mathrm{eq}}(z) = \\
&\quad = \frac{\alpha N}{\sigma} \frac{\mathrm{e}^{(1-\sigma)\tau z} - \mathrm{e}^{(1-\sigma)\tau b_j}}{T^{1-\sigma}} + \big\{ -(1+L)\,\mathrm{sgn}[n(z)]\, t_{\mathrm{f}} + t_{\mathrm{s}} \big\} (z - b_j). \qquad (36)
\end{aligned}$$

We further prove the invariance with respect to changes in parameters applicable to both solutions.

Proposition 3.4. *Cities have the same optimal urban spatial structure if and only if they have the same ratios* $\frac{t_f}{\alpha}$ *and* $\frac{t_s}{\alpha}$.

Proof. See Appendix C. $\square$

An immediate consequence is that the value of the parameter α can be always normalised to one. We oppose (35) and (36) and attain the relation

$$\Psi_{\mathrm{f}}^{\mathrm{opt}}(z; t_{\mathrm{f}}, t_{\mathrm{s}}) - \Psi_{\mathrm{h}}^{\mathrm{opt}}(z; t_{\mathrm{f}}, t_{\mathrm{s}}) = \frac{\sigma}{\sigma - 1}\left[\Psi_{\mathrm{f}}^{\mathrm{eq}}(z; \tfrac{\sigma-1}{\sigma} t_{\mathrm{f}}, \tfrac{\sigma-1}{\sigma} t_{\mathrm{s}}) - \Psi_{\mathrm{h}}^{\mathrm{eq}}(z; \tfrac{\sigma-1}{\sigma} t_{\mathrm{f}}, \tfrac{\sigma-1}{\sigma} t_{\mathrm{s}})\right]$$

leading to the following

Theorem 3.5. *The optimality conditions in a city under parameter set* $\{\alpha, \sigma, \tau, N, L, t_f, t_s\}$ *are equivalent to the equilibrium conditions in a city under parameter set* $\{\alpha, \sigma, \tau, N, L, \rho\, t_f, \rho\, t_s\}$*, where* $\rho = \frac{\sigma-1}{\sigma}$.

Optimal urban land use coincides with market equilibrium land use in a city where commuting and shopping travel costs are scaled down by a factor of $\rho < 1$. This result meets previous ones such as in Imai [1982] and Fujita [1985]. The way the three sorts of urban areas (residential, business, integrated) can be optimally distributed is stated as follows. For the proof we refer the reader to Proposition 4.1 of Garrido-da-Silva et al. [2022a], which makes use of convexity of bid rents in the free-market case.

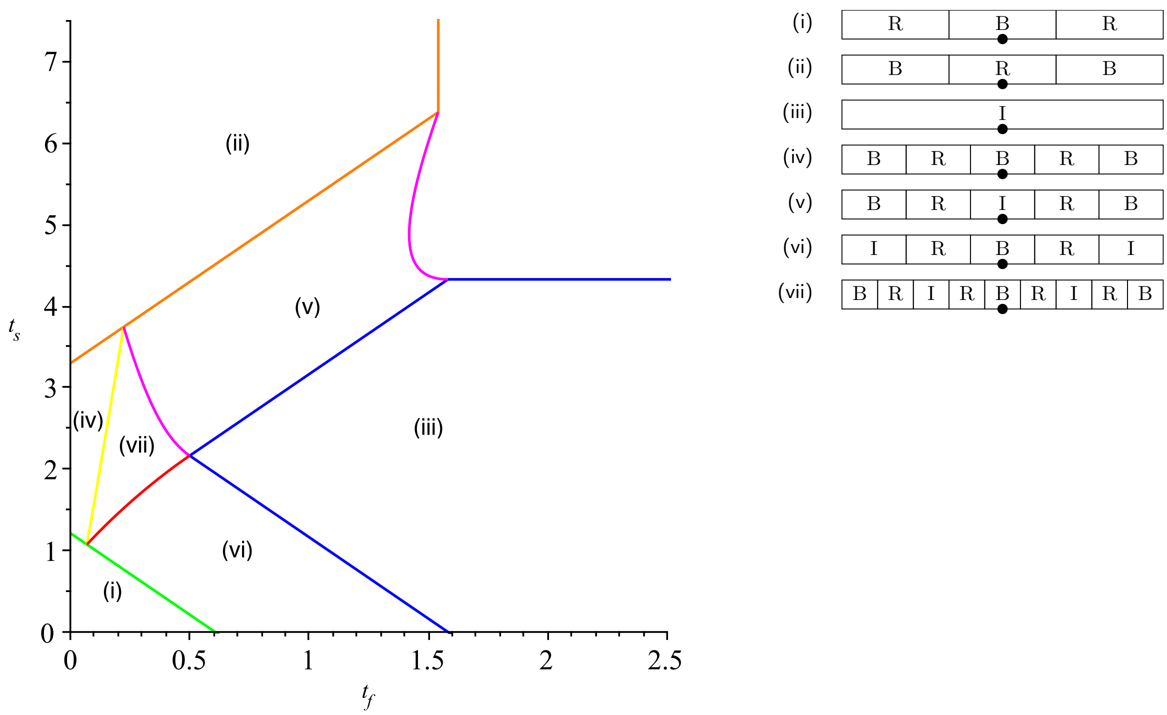

Figure 1: Optimal urban configurations. The black dot represents the centre of the city ($\alpha = 1,\ \sigma = 1.5,\ \tau = 2,\ N = 1,\ L = 1$).

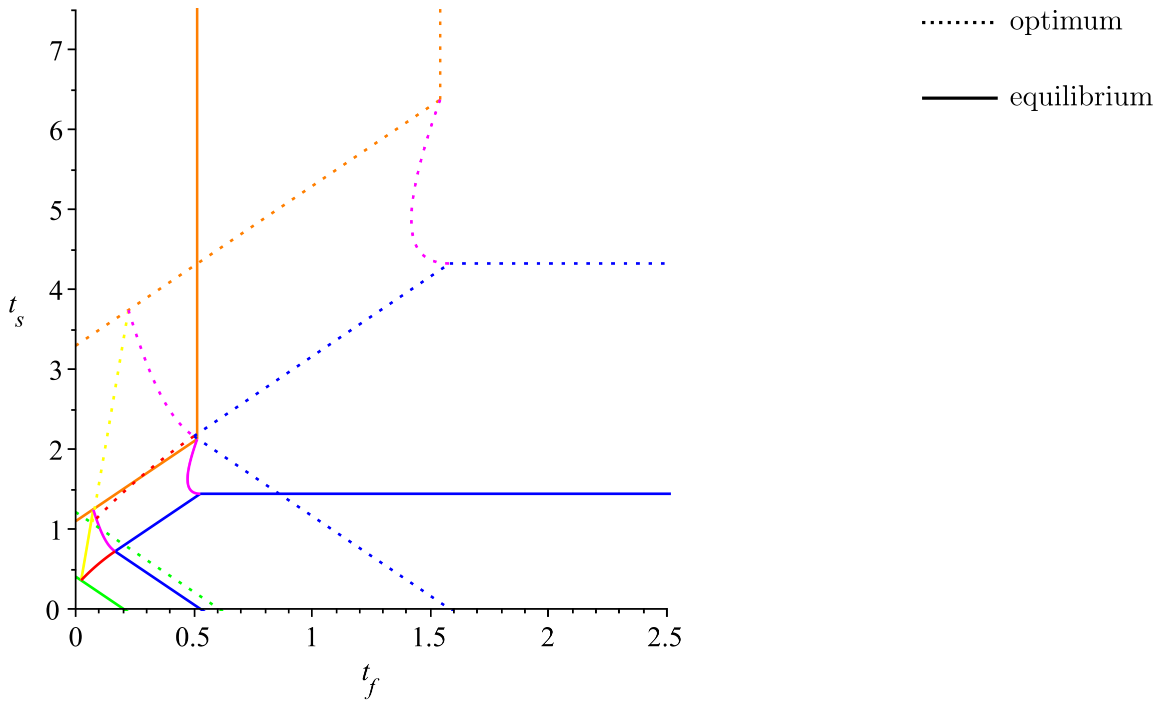

Figure 2: Relationship between optimal and equilibrium urban configurations ($\alpha = 1$, $\sigma = 1.5$, $\tau = 2$, $N = 1$, $L = 1$).

Proposition 3.6. *In commuting and shopping travel cost parameters, the optimal land use structure is unique and exhibits one of the seven patterns, see Figures 1: (i) business monocentric RBR, (ii) residential monocentric BRB, (iii) completely integrated I, (iv) residential duocentric pattern BRBRB, (v) partially integrated pattern BRIRB, (vi) partially integrated pattern IRBRI, or (vii) partially integrated pattern BRIRBRIRB.*

Figure 2 overlaps both the optimal and free-market solutions in the two-parameter region $(t_\mathrm{f}, t_\mathrm{s})$. In view of Proposition 3.5 they are qualitatively the same. Relying on the fact that $\rho < 1$, economic activities tend to be more concentrated in the optimal solution. On the one hand, if RBR is an equilibrium configuration, then it is also an optimum. On the other hand, there are parameter values for which BRBRB, BRIRBRIRB and IRBRI are equilibrium configurations but RBR is the optimum. Configuration BRB, the only one with a central residential area, is more likely to appear under market equilibrium than social optimum, i.e., it is present for smaller values of the travel cost parameters. Configurations I and BRIRB, the only ones with an integrated area at the centre, are more likely to appear under equilibrium than social optimum, i.e., they occur for smaller values of the travel cost parameters. The city is viable for a wider range of parameters under social intervention. Indeed, wages become negative for high values of t_f or t_s and no equilibrium configuration arises. The same reason explains why there is a parameter region in which no optimum exists. The social planner is more likely to choose a configuration with firms located at the centre than the competitive market mechanism.

4 Policy options

Under the absence of congestion and agglomeration externalities, the three unit costs (iceberg transport cost, shopping cost and commuting cost) are the only forces balancing agglomeration and dispersion. Firms wish to be near the centre due to the iceberg transport cost (τ). Households wish to be near the centre due to the shopping cost (t_{s}). Firms and households wish to be near one another due to the commuting cost (t_{f}). A locational choice of one agent (household or firm) affects the interaction benefits with other agents and its own with the centre, resulting in externalities. Individual decisions may naturally be wrong from the social welfare point of view, which causes the distinction between the free-market equilibrium and the social optimum. Market inefficiency further raises the problem of finding policy instruments that implement the social optimum as an equilibrium.

Theorem 3.5 makes it obvious that subsidising a fraction $\frac{1}{\sigma}\,(=1-\rho)$ of households' commuting and shopping costs implements the social optimum as an equilibrium. In this section, we also explore alternative policy instruments: location subsidies to households and firms.

Suppose that the planner has the possibility of providing two kinds of subsidies: a flat-rate location subsidy for households, g, and a labour subsidy for firms at z to support each worker's salary, $s(z;p)$. Household and firm behaviour are described in Subsection 2.1. For convenience, we treat the price level of varieties $p(z)$ as a policy parameter. Set $p(z)=\hat{p}$. The associated price index in (2) thus writes as $\hat{P}=\hat{p}\,T$.

Policy interventions are designed by setting function values for $\{\hat{p},g,s(z;p)\}$, which vary across three spatial equilibria. Accordingly, the social optimum can be attained by three types of instruments described in the following subsections.

As seen in Subsection 2.2, conditions for spatial equilibrium establish that households pursue goods' consumption by maximising utility; firms pursue price by maximising profits; land is used by its highest bidder; and rents and wages clear the land and labour markets. In view of the optimal solution being an equilibrium, we need only look to the individuals Problems (H) and (F). All other conditions result from Problem (O). So the question is: given what $\hat{p}$, g and $s(z;p)$ will make the equilibrium utility U^* equal the target utility $\bar{U}$ and clear product market.

Maximising utility subject to a budget constraint is the dual of minimising expenditure subject to a utility constraint. We see immediately that g is determined so that the cheapest way households at z enjoy the utility level $U^*=\bar{U}$ amounts to the net income $\hat{w}(z)+\bar{c}_0+g-t_{\mathrm{s}}|z|$.

4.1 First-best policy

In a socially optimal city, locational externalities are fully corrected by first-best instruments $\{\hat{p}, g, s(z;p)\}$ as follows. Given $\lambda^*(z)$, $\mu^*(z)$, $w^*(z)$ and $R^*(z)$ in Subsection 3.2, we calculate bid rents and wage that, in an equilibrium context, would induce households and firms to choose the same quantity of consumption and to value land as the planner.

Recall that the optimal variety consumption coincides with the equilibrium variety consumption under the marginal cost price. In achieving the double objectives of the optimal variety consumption and optimal land use pattern, the equilibrium price must be regulated by $\hat{p} = \gamma$. When firms' profits are maximised, from (8), we should obtain

$$\frac{\partial s}{\partial p}(z;\hat{p}) = -\frac{\alpha N}{L}\frac{\hat{p}^{-\sigma}\mathrm{e}^{(1-\sigma)\tau|z|}}{\hat{P}^{1-\sigma}} \ (\neq 0)$$

whose primitive integral is given by

$$s(z;\hat{p}) = \frac{\alpha N}{(\sigma-1)L}\frac{\hat{p}^{1-\sigma}\mathrm{e}^{(1-\sigma)\tau|z|}}{\hat{P}^{1-\sigma}} + k(z), \tag{37}$$

for some function $k(z)$.

Proposition 4.1. *First-best instruments to implement the optimal allocation as an equilibrium satisfy:*

$$\hat{p} = \gamma \quad \textit{and} \quad g = 0 \quad \textit{and} \quad s(z;\hat{p}) = \frac{\alpha N}{(\sigma-1)L}\frac{\hat{p}^{1-\sigma}\mathrm{e}^{(1-\sigma)\tau|z|}}{\hat{P}^{1-\sigma}}. \tag{38}$$

Furthermore, the net surplus provided by the first-best optimum equals the total differential rent $\int_{\mathcal{B}} R^*(z)\left[\lambda^*(z) + \mu^*(z)\right] dz$ *minus the total subsidy* $\int_{\mathcal{B}} s(z;\hat{p})L\mu^*(z)dz$ *from the city.*

Proof. See Appendix C. □

Combining (37) and (38), it follows that $k(z) = 0$. Firm subsidy scheme alone can be introduced either as a labour subsidy to hire workers or as a product subsidy to support the sales of goods. In the latter case, the first term in (37) indicates that generically firms at z must receive $\frac{p(z)}{\sigma-1}$ per unit sales of its good for L workers hired. Profit in (4) thus changes as

$$\Pi(z) = \alpha N\left[p(z) + \frac{p(z)}{\sigma-1} - \gamma\right]\frac{p(z)^{-\sigma}\mathrm{e}^{(1-\sigma)\tau|z|}}{P^{1-\sigma}} - w(z)L - R(z),$$

leading to the equilibrium price $p(z) = \gamma$.

4.2 Second-best policy

We have seen that socially optimal price is marginal cost pricing: $p^{\mathrm{opt}}(y) = \gamma$. However, in the absence of public intervention, firms would set the profit-maximising price: $p^{\mathrm{eq}}(y) = \frac{\gamma}{\rho}$, where $\rho = \frac{\sigma-1}{\sigma}$, see Appendix B. Note that both $p^{\mathrm{opt}}(y)$ and $p^{\mathrm{eq}}(y)$ are independent of location y.

While first-best interventions are the best choice for a planner wishing to pursue welfare improvements, they may be associated with major construction and operations costs. Consider a second-best situation in which the planner can induce firms to set their price as $\hat{p} = \xi\gamma$ with $1 \leq \xi \leq \frac{\sigma}{\sigma-1}$. In that case, consumed quantities are:

$$Q = \frac{\alpha}{\xi\gamma T} \qquad \text{and} \qquad c(y) = \frac{\alpha}{\xi\gamma}\frac{\mathrm{e}^{-\sigma\tau|y|}}{T^{1-\sigma}}.$$

Plugging quantities into the total surplus $\mathscr{S}$ in (12), the planner's purpose described in Problem (O) reads as follows

$$\max_{\{\lambda,\mu\}} \mathscr{S} = \int_{\mathcal{B}} \left\{ \left[\overline{c}_0 - t_{\mathrm{s}}\,|z| - \bar{U} + \alpha\left(\ln\tfrac{\alpha}{\xi\gamma} - \ln T\right) - \frac{\alpha}{\xi}\right]\lambda(z) - t_{\mathrm{f}}|n(z)| \right\} dz$$

subject to (17)–(21).

Application of the Maximum Principle derives that, for $\{\lambda^{**}(x), \mu^{**}(y);\ x, y \in \mathcal{B}\}$ to be a second-best optimal solution, it is necessary and sufficient that there exist multipliers $w^{**}(z)$ and $R^{**}(z)$ under which *(i)–(iii)* are satisfied. Accordingly, $R^{**}(z) = \max\left\{\Psi_{\mathrm{h}}^{\mathrm{opt},2}(z)\Psi_{\mathrm{f}}^{\mathrm{opt},2}(z), 0\right\}$ with the new household and firm bid rent functions:

$$\Psi_{\mathrm{h}}^{\mathrm{opt},2}(z) = \alpha\left(\ln\tfrac{\alpha}{\xi\gamma} - \ln T\right) - \frac{\alpha}{\xi} + w^{**}(z) + \overline{c}_0 - t_{\mathrm{s}}|z| - \bar{U},$$

$$\Psi_{\mathrm{f}}^{\mathrm{opt},2}(z) = \frac{\alpha N}{\sigma-1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w^{**}(z)L.$$

Comparing the latter with the homologous (29) and (30), we observe that the only difference lies on the bid rent function of households. Specifically, the markup ξ is incorporated into the constants $\alpha\ln\frac{\alpha}{\xi\gamma}$ and $-\frac{\alpha}{\xi}$, which makes analytical properties of bid rents in Proposition A.1 unchanged. The differential bid rent $\Psi_{\mathrm{f}}^{\mathrm{opt},2} - \Psi_{\mathrm{h}}^{\mathrm{opt},2}$ has the same form as in (34). This immediately proves the following

Proposition 4.2. *The optimal urban configuration is independent of the markup ξ with $1 \leq \xi \leq \frac{\sigma}{\sigma-1} = \frac{1}{\rho}$.*

We further calculate bid rents and wage that, in an equilibrium context, would lead households and firms to value land as the planner. For an arbitrary price $\hat{p}$, taking (9)–(10), the bid rents households and firms are willing to pay under $\hat{P} = \hat{p}\,T$ and $U^* = \bar{U}$

are, respectively,

$$\hat{\Psi}_{\mathrm{h}}(z) = \alpha\left(\ln\tfrac{\alpha}{\hat{p}} - \ln T\right) - \alpha + \hat{w}(z) + \overline{c}_0 + g - t_{\mathrm{s}}|z| - \bar{U}, \tag{39}$$

$$\hat{\Psi}_{\mathrm{f}}(z) = \alpha N\,\frac{\hat{p}-\gamma}{\hat{p}}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - \left[\hat{w}(z) - s(z;\hat{p})\right]L. \tag{40}$$

At the optimum, $\Psi_{\mathrm{h}}^{\mathrm{opt},2}(z) = \hat{\Psi}_{\mathrm{h}}(z)$ and $\Psi_{\mathrm{f}}^{\mathrm{opt},2} = \hat{\Psi}_{\mathrm{f}}(z)$ should be satisfied. Then,

$$-\frac{\alpha\gamma}{\hat{p}} + w^{**}(z) = -\alpha + \hat{w}(z) + g$$

$$\frac{\alpha N}{\sigma-1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w^{**}(z)L = \alpha N\,\frac{\hat{p}-\gamma}{\hat{p}}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - \left[\hat{w}(z) - s(z;\hat{p})\right]L,$$

which implies

$$\hat{w}(z) = w^{**}(z) + \alpha\left(1 - \frac{\gamma}{\hat{p}}\right) - g$$

$$-\alpha\left(1-\frac{\gamma}{\hat{p}}\right) + g = \frac{\alpha N}{L}\left(\frac{1}{\sigma-1} - \frac{\hat{p}-\gamma}{\hat{p}}\right)\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - s(z;\hat{p}).$$

The previous relations hold for each location z through a combination of two instruments

$$g = \alpha\left(1-\frac{\gamma}{\hat{p}}\right) \qquad \text{and} \qquad s(z;\hat{p}) = \frac{\alpha N}{L}\left(\frac{1}{\sigma-1} - \frac{\hat{p}-\gamma}{\hat{p}}\right)\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}}. \tag{41}$$

Proposition 4.3. *With $s(z;\hat{p})$ given by* (41)*, profit-maximising condition* (8) *is satisfied if either $\hat{p} = \gamma$ or $\hat{p} = \frac{\gamma}{\rho}$.*

Proof. See Appendix C. □

We have seen that firms will charge either the marginal cost pricing or the monopolistic price. When $\hat{p} = \gamma$, the social optimum can be achieved via first-best instruments described in Proposition 4.1. Otherwise, when $\hat{p} = \frac{\gamma}{\rho}$, it means that one optimality condition cannot be satisfied and the planner should impose second-best policies as follows. The second-best optimal solution subject to this price is obtained from the previous analysis substituting ξ for $\frac{1}{\rho}$.

Because $\hat{p} = \frac{\gamma}{\rho}$ is the equilibrium price in the free-market case, the potential subsidy for firms is naturally independent of the mill price and the first-order condition (8) gives $\frac{\partial s}{\partial p}(z;\hat{p}) = 0$, as in the proof of Proposition 4.3. It is immediate that here a product subsidy is not viable. Second-best instruments in practice have two forms of imposition: labour subsidy and travel subsidy.

Proposition 4.4. *Second-best pricing instruments to implement the optimal allocation as an equilibrium satisfy:*

$$\hat{p} = \frac{\gamma}{\rho} \qquad \textit{and} \qquad g + s(z;\hat{p}) = \alpha(1-\rho) + \frac{\alpha N}{\sigma(\sigma-1)L}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}}.$$

Furthermore, the net surplus provided by the second-best optimum equals the total differential rent $\int_{\mathcal{B}} R^{**}(z)\left[\lambda^{**}(z)+\mu^{**}(z)\right]dz$ *minus the total subsidy* $gN+\int_{\mathcal{B}} s(z;\hat{p})L\mu^{**}(z)dz$ *from the city.*

Proof. See Appendix C. □

Theorem 3.5 suggests that subsidies can be affected to travel costs rather than labour costs. For this purpose, consider that the location subsidy for households depends on z but does not depend on the consumption plan. We denote it $g(z)$.

As before suppose that firms charge the monopolistic price $\hat{p}=\frac{\gamma}{\rho}$. Accordingly, if both the unit cost of commuting to work and the unit cost of travelling to shop can be reduced up to $\rho\, t_{\mathrm{f}}$ and $\rho\, t_{\mathrm{s}}$, then an optimal land use will be achieved in equilibrium.

Proposition 4.5. *Second-best travel-based instruments to implement the optimal allocation as an equilibrium satisfy:*

$$\hat{p}=\frac{\gamma}{\rho} \quad \text{and} \quad g(z)+s(z;\hat{p})=(1-\rho)t_f|z-Y^*(z)|+(1-\rho)t_s|z|.$$

Furthermore, the net surplus provided by the second-best optimum equals the total differential rent $\int_{\mathcal{B}} R^{**}(z)\left[\lambda^{**}(z)+\mu^{**}(z)\right]dz$ *minus the total subsidy* $\int_{\mathcal{B}}\big[g(z)\lambda^{**}(z)+s(z;\hat{p})L\mu^{**}(z)\big]dz$ *from the city.*

Proof. See Appendix C. □

We summarise policy interventions in Table 1.

Policy Interventions	Equations
Free-Market Case	$\hat{p}=\dfrac{\gamma}{\rho};\ g=0;\ s(z;p)=0$
First-Best Case	$\hat{p}=\gamma;\ \ g=0;\ s(z;p)=\dfrac{\alpha N}{(\sigma-1)L}\dfrac{\mathrm{e}^{(1-\sigma)\tau\|z\|}}{T^{1-\sigma}}$
Second-Best Labour Subsidy	$\hat{p}=\dfrac{\gamma}{\rho};\ g+s(z,p)=\alpha(1-\rho)+\dfrac{\alpha N}{\sigma(\sigma-1)L}\dfrac{\mathrm{e}^{(1-\sigma)\tau\|z\|}}{T^{1-\sigma}}$
Second-Best Travel Subsidy	$\hat{p}=\dfrac{\gamma}{\rho};\ g(z)+s(z,p)=(1-\rho)\big[t_{\mathrm{f}}\|z-Y^*(z)\|+t_{\mathrm{s}}\|z\|\big]$

Table 1: Policy instrument values $\{\hat{p},g,s(z;p)\}$ for spatial equilibria under three interventions.

5 Conclusion

We discuss the optimality of the urban land market in a location model of residential and business sectors under increasing returns and product differentiation. We characterise the optimal allocation of urban land to households and firms in a model that differs from existing studies by combining city centre amenities with endogenous land use decisions in a linear city. Households commute to work on-site and travel to shop at the city centre. Firms need to transport the goods they produce to the centre. The geographic proximity among the centre, firms and households depends on the magnitude of three types of transportation costs, namely for shipping by firms, and commuting and shopping by households.

We establish existence and uniqueness of the social optimum and compare it with the market equilibrium discussed by Garrido-da-Silva et al. [2022b]. We show that the optimal and equilibrium solutions exhibit identical land use patterns, but generally they do not coincide. In fact, in the optimal city residential areas tend to be more peripheral, supporting previous results by Imai [1982] and Fujita [1985]. Our findings determine that, keeping the remaining parameters fixed, an urban configuration is optimal with commuting and shopping costs $(t_{\mathrm{f}}, t_{\mathrm{s}})$ if and only if the same urban configuration emerges in equilibrium with commuting and shopping costs $(\frac{\sigma-1}{\sigma}\, t_{\mathrm{f}}, \frac{\sigma-1}{\sigma}\, t_{\mathrm{s}})$.

Aggregate welfare distortions raise the problem of designing policies that implement the social optimum as an equilibrium. A first-best intervention is to induce firms to set price equal to marginal cost, for example by subsidising a fraction of marginal cost. If this is not possible, then second-best interventions that induce the optimal land use should be considered. There are several alternatives. One is to subsidise a fraction $\frac{1}{\sigma}$ of households' commuting and shopping costs. Another is to offer location-specific subsidies to firms.

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A Properties of functions

For abbreviation, we refer to $\mathsf{R} = \{z \in \mathcal{B} : \ \lambda(z) = 1 \text{ and } \mu(z) = 0\}$ as the *residential area*; to $\mathsf{B} = \{z \in \mathcal{B} : \ \lambda(z) = 0 \text{ and } \mu(z) = 1\}$ as the *business area*; and to $\mathsf{I} = \{z \in \mathcal{B} : \ \lambda(z) > 0 \text{ and } \mu(z) > 0\}$ as the *integrated area*.

Set $\mathcal{B}^* = [-b, b]$. The distribution of urban land is subject to the possibility of commuting triggered by the function $n(z)$ at each $z \in \mathcal{B}^*$. Recall that journeys to work do not exhibit a cross commuting pattern. Given the flow of labour described by (17) across the city, we have either $n'(z) \neq 0$ if and only if commuting takes places at z, or $n'(z) = 0$ otherwise. In the former, commuters are departing from z when $n'(z) > 0$, and commuters are arriving in z when $n'(z) < 0$. In the latter, the number of employed residents equals the number of jobs at z and the aggregate condition for the labour market clearance without cross commuting further implies $n(z) = 0$. Let define the set of locations where commuting changes directions as

$$\mathcal{D}_1 = \left\{ z \in \mathcal{B}^* : \ \lim_{\epsilon \to 0} n(z - \epsilon) \cdot n(z + \epsilon) < 0 \right\},$$

the set of locations where commuting ceases to occur as

$$\mathcal{D}_2 = \left\{ z \in \mathcal{B}^* : \ \lim_{\epsilon \to 0} n(z - \epsilon) \cdot n(z + \epsilon) = 0 \text{ and } \lim_{\epsilon \to 0} n'(z \pm \epsilon) \neq 0 \right\},$$

and the set of locations that host infinitely close households and firms as

$$\mathcal{D}_3 = \left\{ z \in \mathcal{B}^* : \ \lim_{\epsilon \to 0} n'(z - \epsilon) \cdot n'(z + \epsilon) < 0 \right\}.$$

The socially optimal city must thus look like Figure 3 the red line represents a location in $\mathcal{D}_1$, the green lines represent locations in $\mathcal{D}_2$ and the blue lines represent locations in $\mathcal{D}_3$.

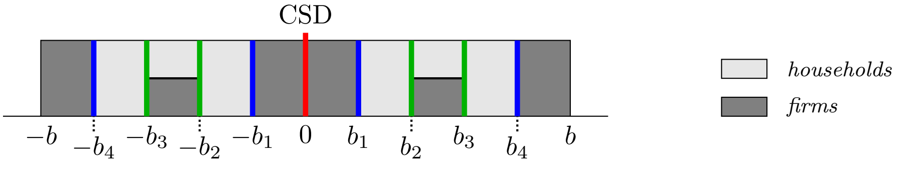

Figure 3: Sample land use pattern.

We say that a property holds *almost everywhere* (abbreviated as *a.e.*) if the set of elements for which the property does not hold is a set of measure zero with respect to the Lebesgue measure. The closure of a set $S \subset \mathbb{R}$ is denoted by $\mathrm{cl}(S)$. The function $\mathrm{sgn}(u)$ returns the sign of u as -1, 0, 1 provided $u < 0$, $u = 0$, $u > 0$, respectively.

Proposition A.1. *Consider the functions on $\mathcal{B}^* = [-b, b]$ given by w in (28), Ψ_h^{opt} in (29), Ψ_f^{opt} in (30), and R in (31). Then,*

(a) w, Ψ_h^{opt}, Ψ_f^{opt} and R are continuous.

(b) w, Ψ_h^{opt}, Ψ_f^{opt} and R are differentiable except for the null set $\mathcal{D}_1 \cup \mathcal{D}_2$ such that

i. for almost every $z \in \mathrm{cl}(\mathsf{R} \cup \mathsf{B})$

$$w'(z) = \mathrm{sgn}[n(z)]t_f \tag{42}$$

$$(\Psi_h^{opt})'(z) = \mathrm{sgn}[n(z)]t_f - \mathrm{sgn}(z)t_s \tag{43}$$

$$(\Psi_f^{opt})'(z) = -\mathrm{sgn}(z)\frac{\alpha\tau N}{T^{1-\sigma}}\mathrm{e}^{(1-\sigma)\tau|z|} - \mathrm{sgn}[n(z)]Lt_f \tag{44}$$

ii. for almost every $z \in \mathrm{cl}(\mathsf{I})$

$$w'(z) = -\mathrm{sgn}(z)\frac{1}{1+L}\left[\frac{\alpha\tau N}{T^{1-\sigma}}\mathrm{e}^{(1-\sigma)\tau|z|} - t_s\right] \tag{45}$$

$$R'(z) = -\mathrm{sgn}(z)\frac{1}{1+L}\left[\frac{\alpha\tau N}{T^{1-\sigma}}\mathrm{e}^{(1-\sigma)\tau|z|} + Lt_s\right]. \tag{46}$$

Proof. (a) The continuity of the shadow wage $w_\mathrm{s}(z)$ at every $z \in \mathcal{B}^*$ ensures the continuity of the wage function $w(z)$ defined in (28). Moreover, $-t_\mathrm{s}|z|$ and $\mathrm{e}^{(1-\sigma)\tau|z|}$ are naturally continuous at every $z \in \mathcal{B}^*$. The bid rents Ψ_h in (29) and Ψ_f in (30) are written as the sum of continuous functions on $\mathcal{B}^*$, and so they are continuous on $\mathcal{B}^*$. Since the maximum of continuous functions is continuous, R is also continuous on $\mathcal{B}^*$.

(b) Suppose that $z \in \mathrm{cl}(\mathsf{R} \cup \mathsf{B})$ with $n(z) \neq 0$. It means that commuting takes place at z. By differentiating expressions (29) and (30) with respect to z, we have

$$(\Psi_\mathrm{h}^\mathrm{opt})'(z) = -t_\mathrm{s}\,\mathrm{sgn}(z) + w'(z),$$

$$(\Psi_\mathrm{f}^\mathrm{opt})'(z) = -\frac{\alpha N}{T^{1-\sigma}}\,\tau\,\mathrm{sgn}(z)\,\mathrm{e}^{(1-\sigma)\tau|z|} - w'(z)L.$$

By virtue of (33) we simplify $w'(z) = \mathrm{sgn}[n(z)]\,t_\mathrm{f}$ whenever $n(z) \neq 0$. Therefore,

$$(\Psi_\mathrm{h}^\mathrm{opt})'(z) = -\mathrm{sgn}(z)\,t_\mathrm{s} + \mathrm{sgn}[n(z)]\,t_\mathrm{f},$$

$$(\Psi_\mathrm{f}^\mathrm{opt})'(z) = -\mathrm{sgn}(z)\,\frac{\alpha\tau N}{T^{1-\sigma}}\,\mathrm{e}^{(1-\sigma)\tau|z|} - \mathrm{sgn}[n(z)]\,t_\mathrm{f}L.$$

Suppose now that $z \in \mathrm{cl}(\mathsf{I})$ with $n(z) = n'(z) = 0$. It means that households and firms coexist and communting does not occur at z. Bid rents then equate at z. From $\Psi_{\mathrm{h}}(z) = \Psi_{\mathrm{f}}(z)$, we get

$$w(z) = \frac{1}{1+L}\left[\frac{\alpha N}{\sigma-1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - \alpha\left(\ln\tfrac{\alpha}{\gamma\mathrm{e}} - \ln T\right) - \bar{c}_0 + t_{\mathrm{s}}|z| + U^\star\right].$$

Substituting back into the land rent function $R(z) = \Psi_{\mathrm{h}}(z) = \Psi_{\mathrm{f}}(z)$ yields

$$R(z) = \frac{L}{1+L}\left[\frac{\alpha N}{(\sigma-1)L}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} + \alpha\left(\ln\tfrac{\alpha}{\gamma\mathrm{e}} - \ln T\right) + \bar{c}_0 - t_{\mathrm{s}}|z| - U^\star\right].$$

We differentiate the latter with respect to z and obtain

$$\begin{aligned}
R'(z) &= \frac{L}{1+L}\left[-\frac{\alpha N}{L}\,\tau\,\mathrm{sgn}(z)\,\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - t_{\mathrm{s}}\,\mathrm{sgn}(z)\right]\\
&= -\mathrm{sgn}(z)\,\frac{1}{1+L}\left[\frac{\alpha\tau N}{T^{1-\sigma}}\,\mathrm{e}^{(1-\sigma)\tau|z|} + t_{\mathrm{s}}L\right].
\end{aligned}$$

By definition, the function $\mathrm{sgn}(u)$ is not differentiable at $u = 0$. In view of the first derivatives above it is clear that $\Psi_{\mathrm{h}}^{\mathrm{opt}}$ and $\Psi_{\mathrm{f}}^{\mathrm{opt}}$ (and hence R) fail to be differentiable at locations such that either $z = 0$ or $n(z) = 0$, equivalently $z \in \mathcal{D}_1 \cup \mathcal{D}_2$. As the union of two sets of measure zero $\mathcal{D}_1 \cup \mathcal{D}_2$ has measure zero, and so we say that $\Psi_{\mathrm{h}}^{\mathrm{opt}}$, $\Psi_{\mathrm{f}}^{\mathrm{opt}}$ and R are differentiable almost everywhere on $\mathcal{B}^*$. □

B Free-market equilibrium

We briefly discuss the free-market equilibrium without policy intervention, i.e., $g = 0$ and $s(y; p) = 0$.

From the first-order condition of profit maximisation with respect to $p(y)$ the equilibrium price for each firm at $y \in \mathcal{Y}$ is given by

$$p^{\mathrm{eq}}(y) = \frac{\gamma\sigma}{\sigma-1}.$$

Set $\rho = \frac{\sigma-1}{\sigma}$. For every $z \in \mathcal{B}$, the bid rent functions of households and firms, $\Psi_{\mathrm{h}}^{\mathrm{eq}}$ and $\Psi_{\mathrm{f}}^{\mathrm{eq}}$ respectively, take the form

$$\begin{aligned}
\Psi_{\mathrm{h}}^{\mathrm{eq}}(z) &= \alpha\left(\ln\tfrac{\alpha\rho}{\gamma} - \ln T\right) - \alpha + w(z) + \bar{c}_0 - t_{\mathrm{s}}|z| - U^*\\
\Psi_{\mathrm{f}}^{\mathrm{eq}}(z) &= \frac{\alpha N}{\sigma}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w(z)L.
\end{aligned}$$

We combine equilibrium conditions for product, labour and land markets producing a competitive spatial equilibrium for the city as follows:

Definition B.1. *We say that* $\{\lambda(\cdot), \mu(\cdot), Y(\cdot), w(\cdot), R(\cdot), U^*\}$, *with* $\lambda : \mathcal{B} \to [0,1]$, $\int_{\mathcal{B}} \lambda(z)dz = N$, $\mu : \mathcal{B} \to [0,1]$, $Y : \mathcal{B} \to \mathcal{B}$, $w : \mathcal{B} \to \mathbb{R}_+$, $R : \mathcal{B} \to \mathbb{R}_+$ *and* $U^* \in \mathbb{R}$, *represents a* competitive spatial equilibrium *if and only if the following conditions hold:*

(i) land market equilibrium: at each $z \in \mathcal{B}$,

$$\begin{aligned} &R(z) = \max\left\{\Psi_h^{eq}(z), \Psi_f^{eq}(z), 0\right\} \\ &R(z) = \Psi_h^{eq}(z) && \text{if } z \in \mathcal{X} = \left\{x \in \mathcal{B} : \lambda(x) > 0\right\} \\ &R(z) = \Psi_f^{eq}(z) && \text{if } z \in \mathcal{Y} = \left\{y \in \mathcal{B} : \mu(y) > 0\right\} \end{aligned}$$

and

$$\lambda(z) + \mu(z) = 1 \qquad \text{if } R(z) > 0$$

and

$$\lambda(z) = \begin{cases} 1 & \text{if } \Psi_f(z) < \Psi_h(z) \\ \in [0,1] & \text{if } \Psi_f(z) = \Psi_h(z) \\ 0 & \text{if } \Psi_f(z) > \Psi_h(z) \end{cases}$$

(ii) commuting equilibrium: at each $z \in \mathcal{B}$,

$$\begin{aligned} w(Y(z)) - t_f|z - Y(z)| &= \max_{y \in \mathcal{B}, w(y) \in \mathbb{R}_+} \left\{w(y) - t_f|z - y|\right\} \\ U(z, Y(z)) &= U^* \end{aligned}$$

(iii) labour market equilibrium: for every interval $J \subset \mathcal{B}$,

$$\int_J \lambda(z)dz = \int_{Y(J)} L\mu(z)dz.$$

C Proofs

Proof of Lemma 2.1. Write $n(z) = f_s(z;p) - f_d(z)$ where $f_s(z;p) = \int_{-\infty}^{z} \lambda(x)dx$ and $f_d(z) = \int_{-\infty}^{z} L\mu(y)dy$. By definition, both f_s and f_d are continuous and non-decreasing function from $\mathcal{B}$ onto $[0, N]$, with $N = LM$, f_s being strictly increasing on $\mathcal{X}$ and f_d being strictly increasing on $\mathcal{Y}$. If $z \in \mathcal{X}$, then there exists a unique $z_{\mathrm{w}} \in \mathcal{Y}$ such that $f_s(z;p) = f_d(z_{\mathrm{w}})$. We define the one-to-one correspondence Y^* such that $Y^*(z) = z_{\mathrm{w}}$. It is easily seen that Y^* is strictly increasing from $\mathcal{X}$ to $\mathcal{Y}$. In addition, take any $z_1, z_2 \in \mathcal{X}$ with $z_1 \leq z_2$ and $Y^*(z_i) = z_{\mathrm{w},i}$, for $i = 1, 2$. Set $J = \mathcal{X} \cap (z_1, z_2)$, and so $Y^*(J) = \mathcal{Y} \cap (z_{\mathrm{w},1}, z_{\mathrm{w},2})$. We get

$$\begin{aligned} \int_J \lambda(x)dx = \int_{z_1}^{z_2} \lambda(x)dx = f_s(z_2) - f_s(z_1) \\ = f_d(z_{\mathrm{w},2}) - f_d(z_{\mathrm{w},1}) = \int_{z_{\mathrm{w},1}}^{z_{\mathrm{w},2}} L\mu(y)dy = \int_{Y^*(J)} L\mu(y)dy. \end{aligned}$$

Accordingly, Y^* describes a commuting function compatible with the city. What is left is to show that $\int_{\mathcal{B}} |z - Y^*(z)|\lambda(z)dz = \int_{\mathcal{B}} |n(z)|dz$. We use the notation of Appendix A. Due to the monotonicity of Y^* we can write

$$\int_{\mathcal{B}} |z - Y^*(z)|\lambda(z)dz = \int_{\mathbf{R}\cap\{z>Y^*(z)\}} \big[z - Y^*(z)\big]dz + \int_{\mathbf{R}\cap\{z<Y^*(z)\}} \big[Y^*(z) - z\big]dz.$$

On the other hand,

$$\begin{aligned}
\int_{\mathbf{R}\cap\{z>Y^*(z)\}} \big[z - Y^*(z)\big]dz &= \int_{\mathbf{R}\cap\{z>Y^*(z)\}} \left[\int_{Y^*(z)}^{z} \lambda(x)dx + \int_{Y^*(z)}^{z} L\mu(y)dy\right] dz \\
&= \int_{\mathbf{B}\cap\{(Y^*)^{-1}(z)>z\}} \int_{z}^{(Y^*)^{-1}(z)} \lambda(x)dxdz + \int_{\mathbf{R}\cap\{z>Y^*(z)\}} \int_{Y^*(z)}^{z} L\mu(y)dydz \\
&= \int_{\mathbf{B}\cap\{(Y^*)^{-1}(z)>z\}} \big[f_s((Y^*)^{-1}(z)) - f_s(z;p)\big]dz + \int_{\mathbf{R}\cap\{z>Y^*(z)\}} \big[f_d(z) - f_d(Y^*(z))\big]dz \\
&= \int_{\mathbf{B}\cap\{(Y^*)^{-1}(z)>z\}} \big[f_d(z) - f_s(z;p)\big]dz + \int_{\mathbf{R}\cap\{z>Y^*(z)\}} \big[f_d(z) - f_s(z;p)\big]dz \\
&= \int_{\mathcal{B}\cap\{z>Y^*(z)\}} \big[f_d(z) - f_s(z;p)\big]dz \\
&= \int_{\mathcal{B}\cap\{z>Y^*(z)\}} |n(z)|dz.
\end{aligned}$$

Similarly,

$$\int_{\mathbf{R}\cap\{z<Y^*(z)\}} \big[Y^*(z) - z\big]dz = \int_{\mathcal{B}\cap\{z<Y^*(z)\}} |n(z)|dz,$$

and the proof is complete.

Proof of Proposition 3.4. The sign of the bid rent differential $\Psi_{\mathrm{f}} - \Psi_{\mathrm{h}}$ over the city fully establishes its spatial structure. Hence, the latter remains unchanged as long as the sign of $\Psi_{\mathrm{f}} - \Psi_{\mathrm{h}}$ is preserved. We replace in both (35) and (36) the parameters α, t_{f}, t_{s} by multiples $\tilde{\alpha} = k \cdot \alpha$, $\tilde{t}_{\mathrm{f}} = k \cdot t_{\mathrm{f}}$, $\tilde{t}_s = k \cdot t_{\mathrm{s}}$, with $k \in \mathbb{R}_+$. The updated differential is written as $\tilde{\Psi}_{\mathrm{f}}(z) - \tilde{\Psi}_{\mathrm{h}}(z) = k \cdot [\Psi_{\mathrm{f}}(z) - \Psi_{\mathrm{h}}(z)]$ for $z \in \mathcal{B}^*$. Accordingly, $\tilde{\Psi}_{\mathrm{f}} - \tilde{\Psi}_{\mathrm{f}}$ admits the same sign of $\Psi_{\mathrm{f}} - \Psi_{\mathrm{h}}$ everywhere. All in all, the urban configuration is shown to be invariant whenever the parameter ratios $\frac{t_{\mathrm{f}}}{\alpha} = \frac{\tilde{t}_{\mathrm{f}}}{\tilde{\alpha}}$ and $\frac{t_{\mathrm{s}}}{\alpha} = \frac{\tilde{t}_s}{\tilde{\alpha}}$ (and so $\frac{t_{\mathrm{f}}}{t_{\mathrm{s}}} = \frac{\tilde{t}_{\mathrm{f}}}{\tilde{t}_s}$) are constants, *ceteris paribus*.

Proof of Proposition 4.1. We replace $\hat{p}$ by γ, $\hat{P}$ by γT and U^* by $\bar{U}$ into (9)–(10). For each location z, the bid rents from households and firms rise to their maximum values with

$$\hat{\Psi}_{\mathrm{h}}(z) = \alpha\left(\ln\tfrac{\alpha}{\gamma} - \ln T\right) - \alpha + \hat{w}(z) + \overline{c}_0 + g - t_{\mathrm{s}}|z| - \bar{U}$$

$$\hat{\Psi}_{\mathrm{f}}(z) = -\left[\hat{w}(z) - s(z;\hat{p})\right] L.$$

At the optimum, $\Psi_{\mathrm{h}}^{\mathrm{opt}}(z) = \hat{\Psi}_{\mathrm{h}}(z)$ and $\Psi_{\mathrm{f}}^{\mathrm{opt}} = \hat{\Psi}_{\mathrm{f}}(z)$ should hold. By virtue of (29)–(30), this yields

$$\begin{gathered} w^*(z) = \hat{w}(z) + g \\ \frac{\alpha N}{\sigma - 1} \frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w^*(z)L = -\left[\hat{w}(z) - s(z;\hat{p})\right]L. \end{gathered}$$

Hence,

$$\hat{w}(z) = w^*(z) - g \qquad \text{and} \qquad g = \frac{\alpha N}{(\sigma-1)L} \frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - s(z;\hat{p}).$$

In order to fulfill the previous relations for each location z, we have

$$g = 0 \qquad \text{and} \qquad s(z;\hat{p}) = \frac{\alpha N}{(\sigma-1)L} \frac{\hat{p}^{1-\sigma}\,\mathrm{e}^{(1-\sigma)\tau|z|}}{(\hat{p}\,T)^{1-\sigma}}.$$

Using (11) and the fact that $\int_{\mathcal{B}} w(Y(z))\lambda(z)dz = \int_{\mathcal{B}} w(z)\mu(z)dz$, the maximised net surplus related to the first-best optimum in (16) can be written as

$$\begin{aligned} \mathscr{S} = &\int_{\mathcal{B}} \Bigg\{ \left[\alpha\left(\ln\tfrac{\alpha}{\gamma} - \ln T\right) - \alpha + w^*(Y^*(z)) - t_{\mathrm{f}}|z - Y^*(z)| + \bar{c}_0 - t_{\mathrm{s}}|z| - \bar{U}\right]\lambda^*(z) \\ &\qquad + \left[\frac{\alpha N}{\sigma-1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w^*(z)L\right]\mu^*(z)\Bigg\}dz - \int_{\mathcal{B}} \frac{\alpha N}{(\sigma-1)L}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} L\mu^*(z)dz \\ = &\underbrace{\int_{\mathcal{B}} R^*(z)\left[\lambda^*(z) + \mu^*(z)\right]dz}_{\text{TDR}} - \underbrace{\int_{\mathcal{B}} s(z;\hat{p})L\mu^*(z)dz}_{\text{TS}}, \end{aligned}$$

where TDR and TS stand for the total differential rent and the total subsidy from the city, respectively.

Proof of Proposition 4.3 We evaluate profit-maximising condition (8) for $p(z) = \hat{p}$ (and hence $\hat{P} = \hat{p}\,T$):

$$\frac{\partial s}{\partial p}(s;\hat{p}) = \frac{\alpha N}{L} \frac{\mathrm{e}^{(1-\sigma)\tau|y|}}{\hat{p}^2\,T^{1-\sigma}}\left[-\hat{p} + \sigma\left(\hat{p} - \gamma\right)\right].$$

It follows that either $\frac{\partial s}{\partial p}(s;\hat{p}) = 0$ or $\frac{\partial s}{\partial p}(s;\hat{p}) \neq 0$. In the former case, we get

$$-\hat{p} + \sigma\left(\hat{p} - \gamma\right) = 0 \;\Leftrightarrow\; \hat{p} = \frac{\sigma}{\sigma-1}\gamma = \frac{\rho}{\gamma}.$$

In the latter case, we differentiate $s(z;\hat{p})$ in (41) with respect to $\hat{p}$:

$$\frac{\partial s}{\partial p}(z;\hat{p}) = -\frac{\alpha N}{L}\frac{\gamma}{\hat{p}^2}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}}.$$

Then, the two previous expressions for $\frac{\partial s}{\partial p}(z;\hat{p})$ must be equal and so

$$-\hat{p} + \sigma\left(\hat{p} - \gamma\right) = -\gamma \;\Leftrightarrow\; \hat{p} = \gamma.$$

Proof of Proposition 4.4 We replace $\hat{p}$ by $\frac{\gamma}{\rho}$ into (41) and subsidy levels follow.

The maximised net surplus related to the second-best optimum can then be expressed as

$$\begin{aligned}
\mathscr{S} = &\int_{\mathcal{B}} \Bigg\{ \left[\alpha\left(\ln\tfrac{\alpha\rho}{\gamma} - \ln T\right) - \alpha\rho + w^{**}(Y^*(z)) - t_{\mathrm{f}}|z - Y^*(z)| + \bar{c}_0 - t_{\mathrm{s}}|z| - \bar{U}\right]\lambda^{**}(z) \\
&\quad + \left[\frac{\alpha N}{\sigma - 1}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w^{**}(z)L\right]\mu^{**}(z)\Bigg\}dz - \int_{\mathcal{B}}\alpha(1-\rho)\lambda^{**}(z)dz \\
&\quad - \int_{\mathcal{B}}\frac{\alpha N}{\sigma(\sigma-1)L}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}}L\mu^{**}(z)dz \\
= &\underbrace{\int_{\mathcal{B}} R^{**}(z)\left[\lambda^{**}(z) + \mu^{**}(z)\right]dz}_{\text{TDR}} - \underbrace{\left[gN + \int_{\mathcal{B}} s(z;\hat{p})L\mu^{**}(z)dz\right]}_{\text{TS}}.
\end{aligned}$$

Proof of Proposition 4.5 On the one hand, the maximum bid rents provided by households and firms read as (39) and (40), respectively, replacing $\hat{p}$ by $\frac{\gamma}{\rho}$ and g by $g(z)$. We take into account the commuting correspondence Y^* from Lemma 2.1. On the other hand, the bid rents from households and firms through the competitive market should be

$$\Psi_{\mathrm{h}}^{\mathrm{eq},2}(z) = \alpha\left(\ln\tfrac{\alpha\rho}{\gamma} - \ln T\right) - \alpha + w(Y^*(z)) - \rho\, t_{\mathrm{f}}|z - Y^*(z)| + \bar{c}_0 - \rho\, t_{\mathrm{s}}|z| - \bar{U},$$

$$\Psi_{\mathrm{f}}^{\mathrm{eq},2}(z) = \frac{\alpha N}{\sigma}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w(z)L.$$

By confronting the values of land, we must have $\Psi_{\mathrm{h}}^{\mathrm{eq},2}(z) = \hat{\Psi}_{\mathrm{h}}(z)$ and $\Psi_{\mathrm{f}}^{\mathrm{eq},2}(z) = \hat{\Psi}_{\mathrm{f}}(z)$. Therefore,

$$\begin{aligned}
w(Y^*(z)) - \rho\, t_{\mathrm{f}}|z - Y^*(z)| - \rho\, t_{\mathrm{s}}|z| &= \hat{w}(Y^*(z)) - t_{\mathrm{f}}|z - Y^*(z)| + g(z) - t_{\mathrm{s}}|z| \\
-w(z) &= -\left[\hat{w}(z) - s(z;p)\right],
\end{aligned}$$

which gives

$$g(z) + s(Y^*(z);\hat{p}) = (1-\rho)t_{\mathrm{f}}|z - Y^*(z)| + (1-\rho)t_{\mathrm{s}}|z|.$$

The proof of Theorem 3.5 reveals that $\rho\left[\Psi_{\mathrm{f}}^{\mathrm{opt},2} - \Psi_{\mathrm{h}}^{\mathrm{opt},2}\right] = \Psi_{\mathrm{f}}^{\mathrm{eq},2} - \Psi_{\mathrm{h}}^{\mathrm{eq},2}$. When the planner subsidises travel costs, the rent for land ends up being partially supported by the former. The proposed bid rents at each location z in a socially optimal city are $\Psi_{\mathrm{h}}^{\mathrm{opt},2}(z)$ and $\Psi_{\mathrm{f}}^{\mathrm{opt},2}(z)$. Households and firms can effectively pay $\Psi_{\mathrm{h}}^{\mathrm{eq},2}(z)$ and $\Psi_{\mathrm{f}}^{\mathrm{eq},2}(z)$ to reproduce the optimal urban structure as an equilibrium outcome. The maximised net surplus related to the second-best optimum is thus given by

$$
\begin{aligned}
\mathscr{S} = \int_{\mathcal{B}} \Bigg\{ & \left[\alpha \left(\ln \tfrac{\alpha\rho}{\gamma} - \ln T\right) - \alpha + w(Y^*(z)) - \rho\, t_{\mathrm{f}}|z - Y^*(z)| + \overline{c}_0 - \rho\, t_{\mathrm{s}}|z| - \bar{U}\right] \lambda^{**}(z) \\
& + \left[\frac{\alpha N}{\sigma}\frac{\mathrm{e}^{(1-\sigma)\tau|z|}}{T^{1-\sigma}} - w(z)L\right] \mu^{**}(z) \Bigg\} dz \\
& - \int_{\mathcal{B}} \left[(1-\rho)t_{\mathrm{f}}|z - Y^*(z)| + (1-\rho)t_{\mathrm{s}}|z|\right] \lambda^{**}(z) dz \\
= & \underbrace{\int_{\mathcal{B}} R(z) \left[\lambda^{**}(z) + \mu^{**}(z)\right] dz}_{\text{TDR}} - \underbrace{\int_{\mathcal{B}} g(z)\lambda^{**}(z) dz}_{\text{TS}}\,.
\end{aligned}
$$