

Passive Vibration Control of Composite Structures with Viscoelastic Damping Patches

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Abstract

The ever-increasing interest towards lightweight structures has significantly reshaped the landscape of research and design of mechanical systems over recent decades. This transformative influence is particularly evident in vibration control, where the susceptibility of light structures to higher vibration amplitudes becomes pronounced, particularly when made with composite materials.

Notably, the strategic use of viscoelastic damping materials emerges as a robust and established method for effective vibration damping. This approach has found systematic applications across diverse industries, including automotive, railway, and aerospace, dating back to the early years of the second half of the previous century. The efficacy of viscoelastic damping materials in mitigating vibrations is especially crucial in these sectors, where precision and performance are paramount. As the trend towards lightweight structures continues to shape engineering, its influence extends beyond the structural components and permeates the methodologies employed for viscoelastic damping treatments. The emphasis has shifted towards the localised application of viscoelastic damping treatments, ensuring a notable damping capacity without unwarranted mass increases. This strategic placement might enhance the overall efficiency of the damping process but also align with the overarching goal of maintaining structural integrity while achieving optimal vibration control in the face of evolving engineering demands.

The present work proposes a numerical framework to investigate viscoelastic damping patches, adopting the appropriate mechanical models to build a reliable and efficient numerical tool based on the Finite Element Method to handle inhomogeneous complex composite structures, including viscoelastic materials. Based on a complete set of parametric analyses, one develops a groundwork for a comprehensive understanding of the damping mechanisms involving viscoelastic damping materials in general and damping patches in particular.

Consistently with the ever-increasing demand for ultra-efficient and lightweight structures, one proposes some guidelines for structural vibration control covering the design, analysis and application of emerging passive damping technologies for composite laminates, including interleaved damping layers and patches, extending the numerical tools developed for a comprehensive understanding of the damping mechanisms involving viscoelastic damping materials.

This work also addresses a thorough experimental campaign of testing and analysis of composite laminates with interleaved damping materials with a detailed comparison with numerical results to investigate the proposed approach of embedding viscoelastic damping patches.

Keywords: beams, plates, shells, laminates, composite materials, vibration control, damping, viscoelastic materials, patch, finite element method, modal analysis, experimental testing.

Resumo

O interesse cada vez maior por estruturas leves tem moldado significativamente a investigação e projeto de sistemas mecânicos nas últimas décadas. Esta influência transformadora é notória no controlo de vibrações, onde a suscetibilidade destas estruturas a maiores amplitudes de vibração se torna evidente, particularmente quando fabricadas com materiais compósitos.

O uso estratégico de materiais viscoelásticos surge como um método robusto e estabelecido para um amortecimento eficaz de vibrações. Esta abordagem tem encontrado aplicações sistemáticas em diversas indústrias, incluindo a automóvel, a ferroviária e a aeroespacial, e que remontam aos primeiros anos da segunda metade do século passado. A eficácia de materiais viscoelásticos na atenuação de vibrações é especialmente crucial nestes sectores, onde a precisão e o desempenho são fundamentais. Como a tendência para estruturas leves continua a moldar a engenharia, a sua influência estende-se para além dos componentes estruturais e permeia as metodologias empregues nos tratamentos de amortecimento. A ênfase mudou para a aplicação localizada de tratamentos de amortecimento viscoelástico, assegurando uma maior eficiência sem aumentos de massa injustificados. Esta colocação estratégica pode aumentar a eficiência global, mas também se alinha com o objetivo de manter a integridade estrutural e, ao mesmo tempo, conseguir um controlo ótimo de vibrações face à evolução das exigências da engenharia.

O presente trabalho propõe um conjunto de soluções numéricas para a investigação de materiais de amortecimento viscoelásticos, adoptando modelos mecânicos apropriados para construir uma ferramenta fiável e eficiente baseada no Método dos Elementos Finitos para lidar com estruturas compósitas complexas não-homogêneas. Recorrendo a análises paramétricas, desenvolve-se uma base para uma compreensão abrangente dos mecanismos de amortecimento que envolvem materiais viscoelásticos em geral e tratamentos locais de amortecimento em particular.

Em coerência com a procura crescente por estruturas ultra-eficientes e leves, propõem-se algumas orientações para o controlo de vibrações, abrangendo o projeto, a análise e a aplicação de tecnologias emergentes de amortecimento passivo para laminados compósitos, incluindo tratamentos locais de amortecimento na sequência de empilhamento, alargando as ferramentas numéricas desenvolvidas para uma compreensão abrangente dos mecanismos de amortecimento.

Este trabalho contempla uma campanha experimental exaustiva de ensaios em laminados compósitos com materiais de amortecimento, bem como uma comparação com resultados numéricos para avaliar a abordagem proposta de incorporação de tratamentos locais de amortecimento.

Palavras-Chave: vigas, placas, cascas, laminados, materiais compósitos, controlo de vibrações, amortecimento, viscoelasticidade, elementos finitos, análise modal, ensaios experimentais.

“Il faut imaginer Sisyphe heureux.”
Albert Camus

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Acronyms

ADF Anelastic Displacement Fields.

ATF Augmenting Thermodynamic Fields.

BEM Boundary Element Method.

CBT Classical Beam Theory.

CE Complex Eigensolution.

CETRIB Laboratory of Tribology, Vibrations and Industrial Maintenance.

CFRP Carbon Fibre Reinforced Plastic.

CLPT Classical Laminate Plate Theory.

CMA Complex Modulus Approach.

COMAC Coordinate Modal Assurance Criterion.

CPT Classical Plate Theory.

CPT(C) Conforming Classical Plate Theory.

CPT(N) Non-Conforming Classical Plate Theory.

DFR Direct Frequency Response.

DKQ Discrete Kirchhoff Quadrilateral.

DMTA Dynamic Mechanical Thermal Analysis.

DRL Drilling Degree of Freedom.

ESLT Equivalent Single Layer Theory.

FDM Fractional Derivatives Model.

FDM Finite Difference Method.

FEM Finite Element Method.

FLD Free Layer Damping.

FoZZ First-Order Zig-Zag Theory.

FRF Frequency Response Function.

FSDT First-Order Shear Deformation Theory.

FTR Fast Test Reactor.

FVM Finite Volume Method.

GHM Golla-Hughes-McTavish.

GLWT General Layer-Wise Theory.

GMM Generalised Maxwell Model.

HSDT High-Order Shear Deformation Theory.

ICE Iterative Complex Eigensolution.

IDKQ Improved Discrete Kirchhoff Quadrilateral.

IMSE Iterative Modal Strain Energy.

INEGI Institute of Science and Innovation in Mechanical and Industrial Engineering.

LAC Local Assurance Criterion.

LWT Layer-Wise Theory.

MAC Modal Assurance Criterion.

MSE Modal Strain Energy.

NASA National Aeronautics and Space Administration.

PCLD Passive Constrained Layer Damping.

RAF Royal Air Force.

RKU Ross-Kerwin-Ungar.

RRM Rayleigh-Ritz Method.

ToZZ Third-Order Zig-Zag Theory.

TSDT Third-Order Shear Deformation Theory.

UCLD Unconstrained Layer Damping.

VDP Viscoelastic Damping Pocket.

VEM Viscoelastic Materials.

ZZT Zig-Zag Theory.

Part I

Introduction

Introduction

1.1 | Damping Panel Vibrations

William Swallow [1939] was the inventor of a patent for a British car body manufacturing company named Pressed Steel Company Limited, claiming an improved method of damping panel vibrations. The appended abstract stated that one could damp a sheet metal panel by attaching “*at least one narrow strip of a material such as rubber, rubber solution, felt, tar, asphalt, or other elastic or plastic material having substantial internal friction*”. Besides, “*a thin, preferably sheet-metal strip*” is attached to the rubber-like narrow strip “*so as to be wholly or substantially incapable of extension or contraction*”. The assumption of a “*material having substantial internal friction*” constrained by “*a thin, preferably sheet-metal strip*” was the forerunner of using Viscoelastic Materials (VEM) as a damping mechanism. Viscoelastic materials are capable of converting mechanical energy into heat when subjected to an external load and undergo deformation - the “*substantial internal friction*” pursued by William Swallow.

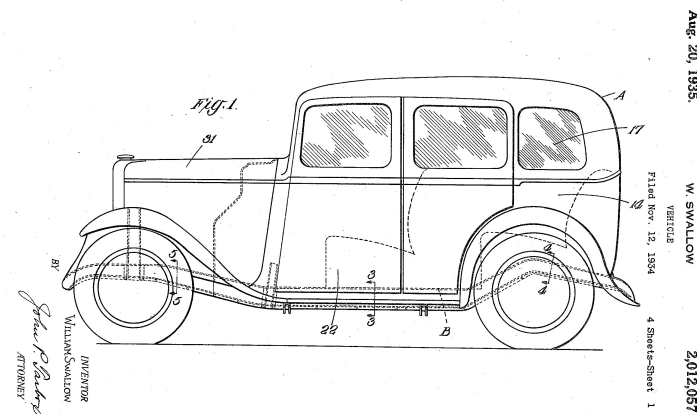


Figure 1.1 Vehicle car body designed and patented by William Swallow [1935]

The topic of vibration control using damping materials started to gain an interest in the early 1900s for a reason. The large structures and machines until the 19th century had fewer vibration problems, mainly because they were so massive and stiff that would require an enormous amount of energy to excite the first resonances. After the industrial revolution and the advent of the transportation industry powered or propelled by fossil fuels, there was an increasing demand for lightweight and flexible structures to produce faster vehicles, which gave rise to vibration problems due to lower resonant frequencies. The outbreak of World War II boosted the investigations, promoting a massive resource allocation and the corresponding developments in the area of damping materials.

Despite the first experimental observation of material damping attributed to Coulomb [1784] in the 1780s, the pioneering scientific works following a systematic approach to the problem establishing the foundations of this field started at the beginning of the 20th century with studies such as the one of Hopkinson and Williams [1912]. The subject was the rate of energy dissipation by elastic hysteresis, or internal molecular friction when steel undergoes cyclical stress variations. The temperature difference between distinct points of a steel bar determines the energy dissipation, proving the thermomechanical mechanism associated with this phenomenon. Following this work, Rowett and Hopkinson [1914] developed a further study of this problem.

It comes as no surprise that contributions as the one of Foppl [1936] pointing out that *“the great and increasing importance of the damping capacity of materials in many different fields of engineering is illustrated by reference to aerial cables, aeroplane wings and propellers, crankshafts of internal combustion engines, steam turbine blades, and welded versus riveted construction”*, came during the same decade as the work of William Swallow. Zener [1937], for instance, promoted the understanding of internal friction from thermodynamical considerations. Interestingly, Leaderman [1939] published one of the first works regarding these issues concerning a polymer, particularly bakelite, in the same year as William Swallow patented his technology.

As previously mentioned, the outbreak of World War II boosted the research effort in this area during the 1940s, but not necessarily the number of publications since the results were under military secrecy. Even so, valuable works like the one of Dillon et al. [1944] show that the damping capacity of polymers attracted research efforts. In the decades that followed, the mechanical behaviour of polymeric materials and their use as damping treatments have been subject to investigation. At the time, researchers and engineers presented the first damping treatments as we know them today. Later in this work, one discusses in great detail the seminal works describing the first analyses of layered damping treatments.

During the last half-century, damping materials have found increasing use in a wide range of engineering applications in different ways. The method pioneered by William Swallow is currently applied in the automotive, railway, naval, aircraft and aerospace industries, playing a vital role in achieving their high standard requirements, far beyond today from the purely structural ones, seeking technological innovation and competitive advantage. Oberst [1968] recalls that the fatigue life of metal panels used near the outer walls of jets, for example, increases as a result of damping. In addition to vibration, noise reduction is also essential for different applications in these industries, and VEM are a resourceful solution.

1.2 | The Nature of Damping

The core of vibration theory is oscillation and the continuous interchange of potential and kinetic energies. Energy dissipation, or damping, is of crucial importance to remove part of the mechanical energy, reducing the oscillation amplitude and consequently increasing the life cycle of machines and structures. Damping greatly influences the vibration level of a dynamic system

or structure. When vibration occurs close to resonant frequencies, where inertial and elastic forces cancel each other, damping forces govern the system response, and one can only reduce it by increasing damping. A convenient characterisation of the damping phenomenon is critical for vibration control.

While the elastic potential energy of a structure depends on its strain field, and so the capacity of storing and restoring energy, the kinetic energy relates to its velocity field, and both are relatively easy to sense, even though kinetic energy is much easier to handle [Pavic, 2005]. Damping mechanisms, even for the most elementary examples, are different in nature and challenging comprehension since it is impossible to derive them from “first principles” [Adhikari, 2013]. Despite being familiar to anyone as part of our daily life, the ring of a crystal goblet helps to distinguish the low damping of fine crystal from the high damping of the ordinary product [Maringer, 1966] no one has derived fundamental laws of damping yet, since it is a phenomenon which may arise from completely different sources. Considering the simple case of a cantilever aluminium beam in bending, it is possible to identify three damping mechanisms, to mention just a few; the internal material damping of aluminium, acoustic radiation damping due to sound radiation and friction damping at the support. Each of these phenomena is complex and generally depends upon many variables, making it cumbersome to address. While one could attribute energy losses due to sound radiation to the motion of the structure, thus to its kinetic energy, internal material damping “seems physically associated with potential energy [Pavic, 2005]. For convenience, since Lord Rayleigh and his seminal work [Rayleigh and Lindsay, 1945], researchers and engineers extensively use viscous damping as a measure of the energy dissipation rate and account for all the phenomena at stake.

1.2.1 | Damping Mechanisms

As a mechanism of energy dissipation, the damping phenomenon is somehow easy to sense. However, as mentioned before, it could be linked to totally different sources, which are complex, giving rise to modelling difficulties for researchers and engineers. It is possible to identify distinct damping mechanisms in many engineering problems.

1.2.1.1 | Viscous Damping

Viscous damping is the most well-known damping mechanism. An introductory course on mechanical vibrations deals with this kind of damping since the resulting equations of motion are simple, linear and analytical solutions are readily available. The modelling of such a mechanism is repeatedly associated with the classical dashpot device, representing the resistance to motion by viscous friction of a fluid passing through a piston inside a cylinder, creating an opposing force. The shock absorbers of a car suspension are examples of such devices. The straightforwardness of this damping mechanism is perhaps unique, and no other phenomenon associated with damping is simpler.

1.2.1.2 | Acoustic Radiation Damping

The displacement of an elastic medium like air by a vibrating structure such as a sheet metal panel promotes the propagation of sound waves. The required energy to create these pressure waves is energy dissipated from the vibrating structure.

1.2.1.3 | Friction Damping

Friction damping is one of the most demanding damping mechanisms to characterise. Due to micro slip friction, this phenomenon may occur in mechanical joints, for example. Regardless of the joint, which usually might be, among others, welded, bolted, riveted or adhesively bonded, mechanical joints have considerable effects on the dynamic behaviour of structures, and joint damping is relevant in studying the dynamics of mechanical systems.

1.2.1.4 | Internal Material Damping

Recalling the words of William Swallow [1939] at the beginning of this work, a sheet metal panel can be damped attaching an “*elastic or plastic material, having substantial internal friction*”. In the words of William Swallow, “*internal friction*” is the mechanism responsible for the energy dissipated by the material when subjected to cyclical stress variations. In Section 1.1, one discussed the pioneering works regarding this mechanism since internal material damping is the damping mechanism of interest for the present work.

The standard damping measures usually compare the lost energy in a cycle with the elastic or kinetic energy. So damping is defined as a relative measure, essentially due to its complex nature and origin. Most of the damping mechanisms in vibrating structures are neither the well-established linear viscous nor the linear hysteretic damping models associated with internal material damping. Nevertheless, it is possible to describe the energy dissipated per harmonic cycle when the structure is vibrating at mode Φ_j with generalised amplitude $|q_j|$ by

$$\delta W_j = \pi H_j |q_j|^2 \quad (1.1)$$

where H_j is the linear-hysteretic damping coefficient. The maximum amount of potential energy stored in the structure in the course of the cycle is written as

$$U_{j,\max} = \frac{1}{2} K_j |q_j|^2 \quad (1.2)$$

The ratio between these quantities is

$$\frac{\delta W_j}{U_{j,\max}} = 2\pi \frac{H_j}{K_j} = 2\pi \eta_j \quad (1.3)$$

where η_j is the hysteretic loss factor of the mode and is a convenient non-dimensional measure of the modal damping of a structure. As can be seen from Equation 1.3, it is amplitude and frequency independent. The modal loss factor can be rewritten as

$$\eta_j = \frac{1}{2\pi} \frac{\text{energy dissipated per cycle}}{\text{maximum stored energy during the cycle}} \quad (1.4)$$

For displacement dependent damping, Equation 1.4 still holds. The energy dissipated per cycle may be represented, in a general form, as

$$\delta W_j = \pi h_j |q_j|^n \quad (1.5)$$

where n is usually greater than 2 and h_j is the constant of proportionality. For such systems, vibrating harmonically in a given mode and at a given displacement amplitude, Equation 1.4 represents the *equivalent linear loss factor* and is defined as the loss factor of the structure with linear damping, dissipating the same amount of energy per cycle like the actual non-linear mechanisms, when the structure vibrates in the same mode and with the same amplitude Mead [1998].

An energy approach such as the one of Equation 1.4 is particularly useful to evaluate the overall loss factor of a structure with different materials of different loss factors. The loss factor calculation by the energy equation has become known as the Modal Strain Energy Method [Johnson and Kienholz, 1982]. Firstly, one computes the undamped modes of the structure, and from those, one finds the maximum strain energy, $\delta U_{j,i}$, stored in each of the N structural components ($i = 1 \dots N$) during one displacement cycle of the required mode, j . The energy dissipated by component i in the course of the cycle is $2\pi\eta_i\delta U_{j,i}$, η_i being the component loss factor. Considering Equation 1.4 one obtains

$$\eta_j = \frac{1}{2\pi} \frac{\text{energy dissipated per cycle}}{\text{maximum stored energy during the cycle}} = \frac{\sum_{i=1}^N \eta_i \delta U_{j,i}}{\sum_{i=1}^N \delta U_{j,i}} \quad (1.6)$$

Equation 1.6 is only valid for structures whose undamped modes are representative of the damped structure.

1.2.2 | Passive and Active Vibration Damping

The current interest in damping design, as previously mentioned, evolved from the need for energy dissipation in lightweight and flexible structures where dissipative mechanisms are not available. There are two major approaches in vibration damping design; passive and active vibration damping techniques. The solution proposed by William Swallow [1939] is a classic example of a passive damping technology. “*A material such as rubber, rubber solution, felt, tar, asphalt, or other elastic or plastic material having substantial internal friction*” is a passive element that doesn’t require any form of external stimulus to perform accordingly to its purpose. Such materials or devices with some damping capacity incorporated during the design process remove part of the vibratory energy due to their nature. From well-known classical spring-mass systems used as vibration absorbers present in an endless number of applications to the simple springs working as vibration isolators are but passive design techniques. On the other hand, active vibration damping techniques require sensor elements to monitor different physical states of the structure under control and appropriately designed actuators to counter-act the vibratory motion of the system. This process might involve a high degree of electronic integration, requiring controllers to perform logic operations and actively accomplish the external stimuli interacting with the structure. Smart materials can convert energy between different physical domains, and researchers and engineers often study them as prime examples of materials associated with active damping techniques. Piezoelectric materials are perhaps the most known materials in this field due to the coupling capabilities between mechanical and electrical domains.

It comes as no surprise that one should only consider active damping design when it outperforms passive design. The costs associated with additional active elements like transducers and controllers are the price to pay for higher adaptability and controllability. Nevertheless, active systems have a high probability of failure, which entails the design of a passive backup system to prevent catastrophic failure or even to ensure control stability. This approach is known as hybrid design, a strategy combining the best of both worlds, the mechanical robustness of the passive techniques with the adaptability and controllability of the active design.

Naturally, as a relatively recent research topic, active design of damping technologies has been drawing the attention of the scientific community mainly because of the high level of technology involved, covering different areas like mechanical and electrical engineering, electronics or even bioengineering as a truly multi-physics approach and a hot topic of current research. Even so,

after roughly sixty years of research in passive damping technologies, there are still open questions to answer. Passive vibration technologies are truly robust techniques and often outperform active technologies at high-frequency vibrations.

1.3 | Viscoelastic Damping Mechanisms

The increasing interest in the passive damping technologies developed during the 1950s led these techniques to the non-commercial aerospace industry in the early 1960s [Rao, 2003] and consequently triggered the scientific community's enthusiasm for the subject. Ungar and Jr. [1964] studied damping in plates due to thickness deformations in attached viscoelastic layers, a configuration currently designed as Free Layer Damping (FLD) or Unconstrained Layer Damping (UCLD) and represented in Figure 1.2a. Accordingly to the authors, "*one may readily visualize that flexure of the plate will produce extensional strains in the viscoelastic layer in directions parallel to the plate surface*". Earlier, Kerwin [1959] investigated the damping of flexural waves by a viscoelastic layer constrained by "*a thin, preferably sheet-metal strip*", as stated by William Swallow, and established the first steps for the theoretical analysis of what is currently designed as Passive Constrained Layer Damping (PCLD) and represented in Figure 1.2b. Differently from an FLD configuration, Kerwin postulated "*that damping by a constrained damping layer is attributable to the shear motion of the damping layer*". DiTaranto [1965] and later Mead and Markus [1969] extended the work developed by Kerwin. Focused on three-layer damped beams (PCLD configuration), the authors derived the differential equation of motion of these structures, which are also considered reference works in the field. Also, during the 1960s, Oberst [1968] pointed out the potential interest of the subject in noise reduction.

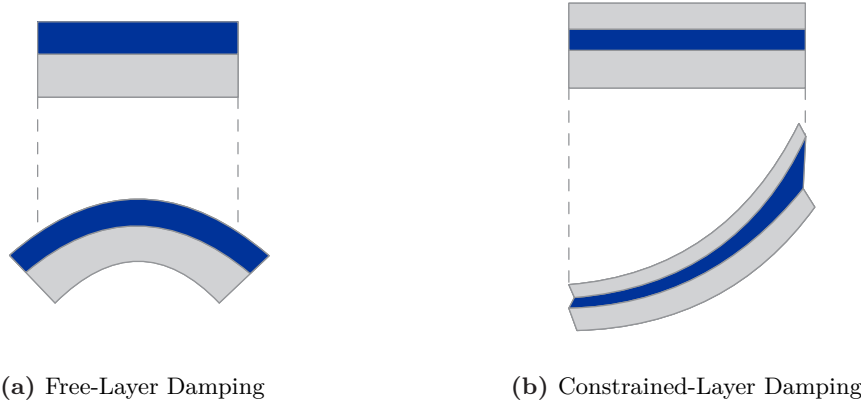


Figure 1.2 Basic configurations for passive layer damping. Adapted from Nakra [1998]

So the 1960s were an intensive and productive decade in the study of viscoelastic materials as a damping mechanism establishing the basis for the following developments and maximising damping efficiency became a new focus of interest following a decade of research concerning the proper modelling of viscoelastic structures such as beams or plates. Apart from the mechanical properties and damping capacities of different viscoelastic materials, geometric variables such as thickness ratios are of utmost importance to maximise damping, whose design is an essential task for engineers. Grootenhuis [1970] performed one of the first studies in this area, pointing out some interesting design considerations respecting mostly thicknesses. In his work, Grootenhuis showed how inefficient an FLD configuration is compared to a PCLD one. While in an FLD damping capacity comes from a higher viscoelastic thickness and shear modulus, one notices different

nuances in PCLD configurations where different thickness ratios or stacking arrangements could improve damping. Some years later, Rao [1978] attempted to design an optimal three-layer damped beam subjected to constraints to establish some design guides concerning the use of viscoelastic materials. Even though D. K. Rao did not reach definitive conclusions, he paved the way for systematic studies in optimal damping design using VEM.

1.4 | Industrial Applications of Viscoelastic Materials

Following the pioneering scientific research described in the previous section were some top industries that took advantage of viscoelastic materials to improve performance. One of the primary reasons for the immediate establishment of this technology is its ease of use, making it extremely valuable for different applications. In the following, one addresses some noticeable industrial applications of viscoelastic materials beyond the obvious car industry, which soon exploited the pioneering work of William Swallow [1939].

1.4.1 | Aircraft Industry

The military, and consequently the aircraft industry, earned a substantial resource allocation during two critical events, World War II and Cold War. As mentioned before, these efforts contributed to considerable developments in this field. Jones [1996] points out the fundamental achievements of the *Air Force Materials Laboratory at Wright-Patterson Air Force Base* regarding damping technologies. A remarkable example of viscoelastic damping treatments in this field is the “*multiple layer constrained damping treatment to control airflow induced vibrations in a weapons dispenser on the F-4 aircraft during the Vietnam War*”. The excessive vibration of the structure was so severe and the failure rate so high that “*a dispenser barely lasted a single mission*” [Jones and Trapp, 1971]. Afterwards, the damping treatment has proven to be a success increasing the service life five times [Jones and Trapp, 1971]. Additional remarkable works of the same group include the “*multiple layer high temperature bondable damping wrap for control of airflow induced vibrations of the inlet guide vanes (IGVs) of the TF-30 jet engine*” [Jones, 1996].

1.4.2 | Naval Industry

In 1967, two new ships started to cross the Irish Sea from Liverpool to Belfast as a car ferry service. Soon, the noisy passenger berths became a problem, and Turner [1969] bonded a damping material to the mild steel floor and walls of a store area with high vibration levels. Turner [1969] found that noise from the engine room was insignificant compared to that of the propellor blades. Despite the considerable amount of damping material attached to the panels, improvements in sound and vibration levels were unsatisfactory. Sound pressure levels inclusively increased at particular frequencies considering some rooms. Price and Crocker [1970] critically received some choices made by the author, but as Turner [1969] noted, the effectiveness of a damping treatment judiciously applied during the ship construction would improve results.

Sailing from sea to river, published results did not differ substantially. Some years later, Buiten [1972] studied the noise reduction due to damping material on a cruise ship from the Rhine River. An elasto-dissipative material was sprayed to the lower side of two decks near an engine room, covering a total area of about 400 m². Buiten [1972] presented in his work a view from a deckhead after the elasto-dissipative material application, and this is one of the most

impressive photographs from early applications of damping treatments in large structures. The rough finishing of the sprayed damping treatment discloses a high thickness variation reported to vary from 1 to 10 mm. Buiten [1972] took measurements during different stages, from the slip-way to a floating condition, but also before and after finishing the ship with vinyl covering, asphaltic floating floor and hair felt carpet tiles applied in different room floors. Despite some differences in the excitation mechanism between the recording at separate stages, the most curious result was the considerable influence of the completion of the accommodation to effectively reduce the sound pressure level compared to the elasto-dissipative layers, which had a reducing impact of 7.5 dB in sound pressure level at the considered frequency range. A remarkable effect pointed out by Buiten [1972] is that the decrease of acceleration level at points with applied elasto-dissipative layer is smaller than at positions outside the region of the elasto-dissipative layers. The author explains that the damping layer has a negligible influence on reducing the noise annoyance in the investigated cabin and concludes that applying similar damping materials in steel ships if noise reduction is hardly worthwhile. Later, Nilsson [1978] assumed that damping layers have only a local effect on the attenuation of structure-borne sound, but it is crucial to mention that Buiten [1972] worked with UCLD layers, which underperform PCLD layers. Sound and vibration suppression should be an integral aspect of the design process of the structure, considered as a whole and not as an 'add-on'. Today, it is possible to achieve better results, particularly at lower frequencies, commonly generated by engines and propellers, when considering integrated PCLD design.

1.4.3 | Railway Industry

Fan et al. [2009] experimentally accessed the influence of viscoelastic materials on vibration and noise reduction inside luxury sleeper carriages. The authors justified the work by considering the trend towards lighter trains and the consequent increase in structure-borne sound. The authors applied three viscoelastic materials developed by them in strategic locations. Using bitumen-based and butyl rubber damping layers with 3 mm thickness on the steel panel under the car floor and the upper surface of the floor panel as well as on the sidewall, the authors aimed to "isolate the transmission of vibration from the bogie frame to the car floor". They applied each of these materials in different carriages. In addition, sprayed a water-based damping material onto the "interior surfaces of the car body" for both carriages and considered a control carriage without damping materials in the experiments. Higher vibration amplitudes and noise levels were measured right above the bogies with increasing amplitudes as the train speed increased. As expected, the authors have found that vibration and noise reduction depend upon the material, location and train velocity. Nevertheless, Fan et al. [2009] measured consistent vibration and noise reduction in one specific carriage (with bitumen-based damping material). *"In the same measuring case, the damping treatment of both bitumen-based and butyl rubber damping sheets have been found to reduce vibration levels in the whole frequency range, to a different degree, at every measurement location within the carriage."*

1.4.4 | Gears

The use of viscoelastic materials as a damping mechanism is not restricted to large structures such as the panels used in the fuselage of an aircraft, the hull of a ship or a train carriage. For example, it is possible to consider its use in smaller mechanical components such as gears.

Slattery et al. [1989] introduced viscoelastic materials in gear systems to reduce the transmitted vibration to supporting structures. The inventors simply bonded a viscoelastic sheet to the outer surface of the side plates, which connect the hub and rim, upon which attached a constraining layer. The damping system covering 50% of the side plate's area is circumferentially applied and radially restrained by rings extending axially from the side plates.

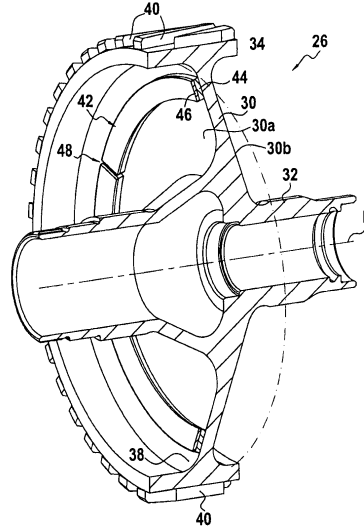


Figure 1.3 Gearwheel with viscoelastic damping patches [Marsaudon and Cutuli, 2014].

Marsaudon and Cutuli [2014] submitted a patent focusing on the particular case of gearwheels for gearboxes of turbine engines, presenting a detailed discussion of how high-speed gearwheels could be damped to prevent high vibration amplitudes and consequently damage. Regarding Figure 1.3, the inventors used a damper device (42) consisting of an annular shape patch made of a viscoelastic layer (44) directly attached to the inner surface of the gear web (30a) and constrained by a backing layer (46). A particular feature of such a device is that it can be advantageously split, according to inventors, over its entire radial length and thickness (48) to improve vibration absorption. Besides the considerable vibration reduction attainable with such a device, one of the most important results is the weight reduction of about 20% for the considered gear due to an unnecessarily thick rim, web and hub specifically designed for low vibration amplitudes.

The patent application of Marsaudon and Cutuli [2014] highlights an important practical issue, the local nature of viscoelastic damping treatments in complex structures. A damping treatment covering the total surface of the web would be not only infeasible but perhaps less effective.

1.4.5 | Wind Turbine Blades

Global decarbonisation efforts towards a low-carbon economy have increased the deployment of renewable energy systems such as wind turbines. Engineers have been stretching the dimensions of the blades out to their limits considering economic efficiency purposes, which leads to weight increase and consequently higher loads, thereby increasing the total cost of such structures. Sievers and Mullings [2015] addressed the issue of reducing the edgewise loads (parallel to the blade chord) by increasing damping with embedded viscoelastic materials, enabling larger blades

with lower mass requirements.

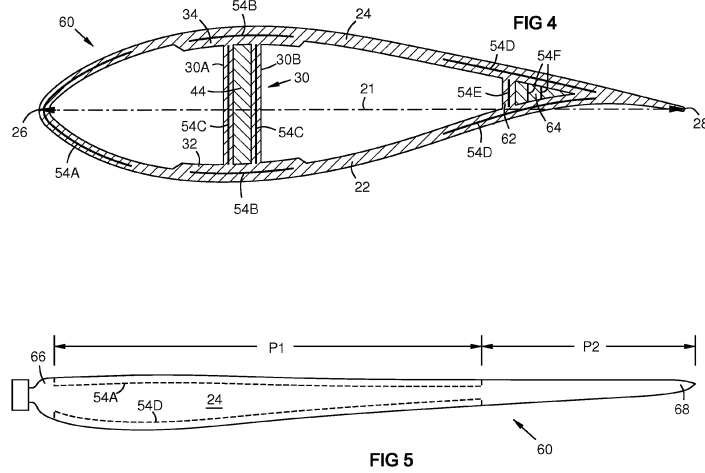


Figure 1.4 Wind turbine blade with viscoelastic material [Sievers and Mullings, 2015].

Figure 1.4 illustrates a patented blade and the corresponding cross-sectional view. The inventors embedded a viscoelastic damping material between sheets of glass fibre during the lay-up process. According to the inventors, it is possible to select several locations to achieve damping of edgewise oscillations. Those with "significant operational shear" are preferable, and Figure 1.4 depicts a possible configuration with embedded viscoelastic material at positions 54A-F. In addition to the local inserts of viscoelastic material along the cross-section, the tip region does not include dissipative material, and the blade span is not entirely damped.

According to preliminary results by Sievers and Mullings [2015], the reported technology enables a 10-30% load reduction resulting in cost savings and making possible even larger blades for improved efficiency of energy production.

1.5 | Composite Materials

The increasing use of structural composite materials in the last decades is deeply connected to the outbreak of World War II, similarly to what happened with damping materials, and the military industry has led the research efforts in this field for many years. Nowadays, commercial aircraft design and aerospace industries have established themselves as the leading research areas where structural composite materials are part of everyday life. For many years, companies have been restricting structural composite materials in the automotive sector to low production volumes like racing cars or unconventional parts of luxury vehicles. The latest efforts attempt to reduce their cost, which has been unaffordable for passenger car manufacturers. In fact, for many years, structural composites like CFRP found their application restricted to high-performance applications requiring low weight, the competitive edge for those seeking faster machines with improved vehicle dynamics. Substituting steel for carbon fibre components results in 60 % weight savings but at a twenty times higher cost. In the same way, one can replace aluminium parts for carbon fibre components with 30 % weight savings, but at ten times higher cost [Crow, 2015]. Besides, considering the trend to reduce the global carbon footprint, there is a genuine effort to employ natural fibres and recyclable resins and reduce the costs of composite structural parts by using them, for example, in the production series of passenger car vehicles.

Structural composite laminates consist of multiple fibre-reinforced laminae where, usually, fibres have very high Young's moduli in the longitudinal direction but lower ones in the transverse direction due to their particular structure of long fibres of tiny diameters. A polymeric resin matrix of relatively low modulus embodies these fibres, protecting them and giving shape to the system. Engineers design each lamina with appropriate volume fractions of each component, depending on the application and manufacturing process. While fibres are responsible for the final stiffness of the composite laminate, the polymeric resin matrix is also of great importance to other mechanical properties, including damping. Due to their polymeric nature, most resins have some damping capacity by converting vibrational energy into heat, and some resins have enhanced damping properties [Mead, 1998]. Thermoplastic polymers show high dissipation compared to thermoset ones, but weak adhesion properties restrict their use [Chung, 2003]. However, one can witness progress in thermoplastic resins with promising results. [Duc et al., 2014] studied the reinforcement of thermoset and thermoplastic resins with natural flax fibres and obtained the best compromise between stiffness and damping with flax fibre reinforced with semicrystalline polylactic acid (PLA), which has the advantage that it can be produced from renewable resources and is biodegradable. Damping of composite materials can be several orders of magnitude higher than that of traditional metallic ones [Treviso et al., 2015]. Apart from the intrinsic properties of each constituent (fibres and matrix), their interaction plays a critical role in the damping mechanism. The weaker the bond between fibres and matrix, the higher the energy dissipation due to the relative motion of fibres, leading to poor load transmission and strength loss [Nelson and Hancock, 1978].

Therefore, design requirements for high stiffness usually conflict with those for high damping capacity. Stiffness and damping capacity are essentially two conflicting needs. As the volume fraction of fibres increases to improve the laminate's stiffness, its damping capacity drops since the volume fraction of the resin matrix decreases. Like other parameters, the damping capacity of composite materials is also dependent on fibre orientation, ply stack up, fibre packing array, and weave pattern in woven composites [Bednarczyk et al., 2016]. Because of trade-offs like this one and the need for new lightweight materials with improved mechanical properties, researchers and engineers have developed hybrid composites. The term "hybrid composite" refers to a matrix containing at least two types of reinforcements [Swolfs et al., 2014]. Figure 1.5 shows the three main hybrid configurations.

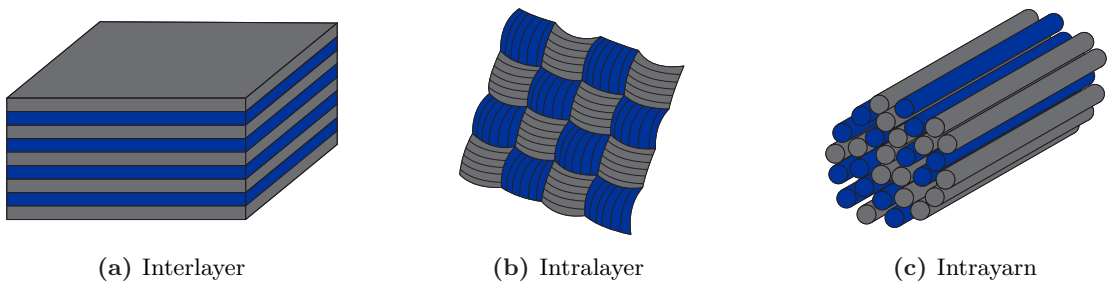


Figure 1.5 Three main hybrid configurations. Adapted from Swolfs et al. [2014]

In general, the aim of embedding two fibre types is to keep up the advantages of both fibres and suppress some disadvantages. For instance, as Swolfs et al. [2014] point out, replacing carbon fibres in the middle of a laminate with cheaper glass fibres can significantly reduce costs while maintaining practically unaffected flexural properties. One can divide the main hybrid configurations in Figure 1.5 into three. Figure 1.5a presents the layers of two fibre types or materials stacked onto each other. It is the simplest and cheapest method for manufacturing a hybrid composite, and it is possible to interleave a damping layer among the stiff carbon fibres, for

example. Figure 1.5b shows the intralayer hybrid, where one can mix two fibre types within the layers and co-woven different yarns into a fabric, for example. Combining stiff and higher damping capacity fibres could reduce the drawback of improving damping at the expense of stiffness loss. Finally, as depicted in Figure 1.5c, one can also mix the two fibre types on the fibre level, resulting in a hybrid intrayarn. It is possible to obtain more complex composites by combining these three configurations. It is possible to obtain more complex composites by combining these three configurations. One of the most interesting examples in the literature regarding hybridisation for high stiffness and damping capacity is the work developed by Alberts and Xia [1995], which proposed a new composite damping material consisting of a viscoelastic matrix and high elastic modulus fibre inclusions used as a surface treatment for passive vibration control. Such an approach provided a substantial increase in damping and a decrease in weight when compared with conventionally constrained viscoelastic damping treatments. In this work, one focuses on interlayer hybridisation for improved damping capacity of laminated composites. Later in this work, one discusses in great detail the seminal works proposing interlayer hybridisation with viscoelastic damping layers.

1.5.1 | Flax Fibres

Flax plays a role in the history of composite materials. A composite material with flax fibres and phenolic resin, developed in 1936 by Norman de Bruyne at Aero Research, Gordon Aerolite®, was the first fibre-reinforced plastic. This new material had half the specific density of a duralumin sheet. During World War II, fearing a shortage in the aluminium supply, prototypes of the Royal Air Force (RAF) iconic Supermarine Spitfire cockpit and fuselage were produced using Gordon Aerolite® [Ebnesajjad, 2010]. In the end, there was no need to replace aluminium for Gordon Aerolite in the Spitfire production during the conflict, but two other military aircraft used the material in their tailplanes and wing spars; Miles Magister trainers and Bristol Blenheim bombers, respectively [Glancey, 2008].

Nowadays, flax fibres' specific properties are comparable to those of glass [Yan et al., 2014]. Due to their availability and properties, researchers targeted plant fibres like flax, hemp, and jute as possible substitutes for classic synthetic fibres like E-glass. In fact, due to the lower density of the flax, some authors have stated that the specific longitudinal stiffness of unidirectional flax composites is higher. Specific bending stiffness of unidirectional flax composites outperforms steel and aluminium [Pil et al., 2016]. Despite their limited use, mainly in non-structural components, some researchers and engineers considered their application in structural composites [Shah et al., 2013]. However, it is not straightforward to determine mechanical properties like the stiffness of natural fibres. The main reason is that, contrary to the synthetic ones, there is no such thing as one natural fibre. Natural fibres are composed of several elementary fibres glued together [Pil et al., 2016]. Moreover, flax fibre is about three times more expensive than E-glass fibre, but as Pil et al. [2016] points out, the cost difference between glass and carbon fibres lowered from a factor of fifty in the early nineties to about ten in 2015.

Considering the replacement of E-glass reinforcement fibres in small wind turbine blades, Shah et al. [2013] manufactured two prototypes in a polyester resin, one using conventional E-glass fibres and another using flax fibres as reinforcement. The authors performed static flap-wise tests on both blades, and the conclusion is clear, flax is a suitable structural replacement for E-glass. Despite the lower flexural stiffness, which led to higher tip displacements, the flax reinforced specimen withstands static loads under general and worst-case conditions saving 10 % of the total E-glass sample weight.

In addition, and significant in the scope of the present work, one of the most unique and

attractive properties of flax fibres is their excellent vibration damping capacity. Their complex microstructure of weakly bonded elementary fibres and crystalline microfibrils embedded in an amorphous matrix enhance energy dissipation properties [Pil et al., 2016]. Researchers have been exploring the hybridisation of carbon and flax fibre composites for improved mechanical performance, taking advantage of the carbon stiffness and flax damping, for example [Flynn et al., 2016, Rueppel et al., 2017]. Rahman [2021] extensively reviewed mechanical and damping performances of flax fibre composites.

Last but not least, natural fibres play a decisive role in the global decarbonisation efforts towards a low-carbon economy. During production, “almost a third of these carbon fibre sheets end up on factory floors” Harris. The use of natural fibres like flax in structural composites is a feasible alternative to synthetic fibres Shah et al. [2013]. From production to end-of-life of the composites, flax fibres reveal a much lower impact than synthetic fibres like carbon or glass. From being less toxic and needing less water during their production to having several recycling options at their end-of-life stage, these natural fibres are a step towards the production of more sustainable structural materials [Pil et al., 2016].

1.5.2 | Woven Fabrics

Woven fabrics combine 0° and 90° fibres, or warp and weft, commonly used in the textile industry, to produce an in-plane balanced woven laminate. Figure 1.6 presents the basic configurations of woven fabrics based on weave patterns - warp tow coloured blue, and weft tow coloured grey. In plain weave, Figure 1.6a, each warp tow passes alternately under and over each weft tow producing a symmetric fabric with good stability. However, it is difficult to drape, and the high level of fibre crimp promotes relatively low mechanical properties compared with the other weave styles. Figure 1.6b shows the twill weave pattern, where one or more warp tows alternately weave over and under two or more weft tows regularly. Therefore, it is possible to recognise the effect of a straight or broken diagonal in the fabric. Consequently, this pattern offers superior wet out and drape over the plain weave with reduced crimp. The fabric also has a smoother surface and slightly higher mechanical properties. Satin weaves, represented in Figure 1.6c, are fundamentally modified twill weaves with fewer warp and weft intersections. Satin weaves are very smooth, have good wet out and offer a high degree of drape. Besides, the low crimp gives good mechanical properties. However, the asymmetry causes one face of the fabric to have fibre running predominantly in the warp direction, while the other face has fibres running predominantly in the weft direction. Therefore, one must take care when considering laminates with multiple layers of these fabrics to ensure that there are no pre-stresses.

Therefore, regardless of the pattern, conventional weaves do not have a homogeneous fibre distribution at the ply level - concentrated fibres form undulated yarns (bundles of thousands of interlocked fibres). These yarns cross under and over each other in several weave patterns, and consequently, one can not ensure homogeneously distributed fibres and matrices in the ply. Contrary to the unidirectional fabrics, where fibres and matrices form a homogeneously distributed ply, woven fabrics contain resin-rich zones around the yarns, which affect the mechanical properties of the composite - Figure 1.7a. However, it is possible to overcome some of these drawbacks with the spread tow thin-ply technology, which can produce composites in which the laminate consists of particularly flat plies, which enables dry ply weights of less than 100 g m^{-2} - Figure 1.7b. In spread tow fabrics, the filaments laid out next to each other create a ribbon which eases incredibly flat profiles eliminating the texturing of conventional woven fabrics.

Woven fabrics are easier to handle and shape, exhibit improved impact resistance, superior out-of-plane stiffness and balanced in-plane mechanical properties compared to unidirectional

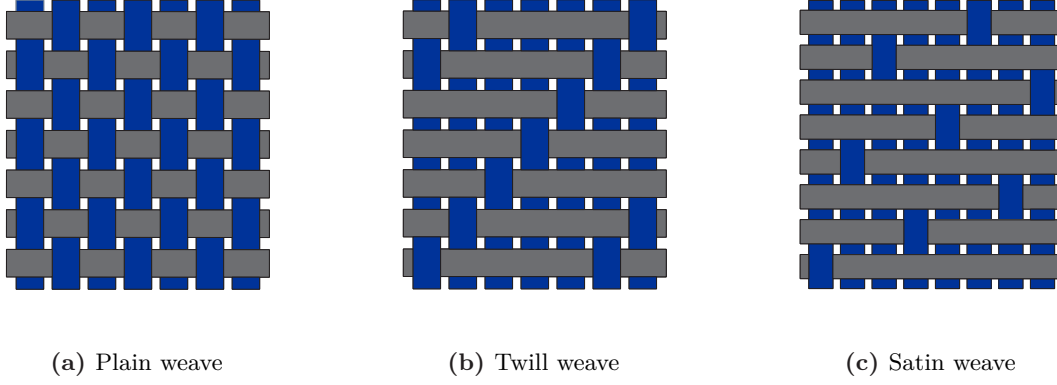
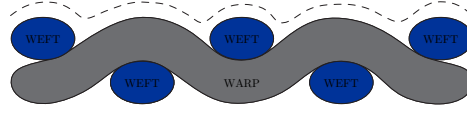


Figure 1.6 Basic classification of woven fabrics based on weave patterns



(a) Conventional woven fabric - Adapted from: Carbon Fiber - A Lesson in Composites



(b) Spread-Tow woven fabric - Adapted from: Carbon Fiber - A Lesson in Composites

Figure 1.7 Surface characteristics of conventional and spread-tow woven fabrics

fabrics. In addition, and recalling intralayer hybridisation in Figure 1.5b, woven fabrics can also be used as a way to increase damping in laminated composites.

1.6 | Motivation

The trend towards lightweight materials had a significant impact on passive damping design. On the one hand, lighter structures are prone to higher vibration amplitudes, which are readily transmitted, especially in composite materials. Therefore, research and development of innovative technologies for vibration damping are critical. On the other hand, attaching one or more layers of damping material to a host structure will increase the overall mass and limit it can be valuable.

Nokes and Nelson [1968] were among the first researchers to study the problem of a partially covered structure. According to Marcelin et al. [1992], the main conclusion of Nokes and Nelson [1968] work was the possible uselessness of covering an entire beam for optimal damping. This remarkable result was previously attributed to possible experimental errors by some researchers since it counter-intuitively indicates that higher damping is possible with less damping material. The lack of theoretical background to explain such a result has inspired some researchers to explore this issue. Based on theoretical and experimental results, Nokes and Nelson [1968] show that the loss factor is higher for partial coverage for stiff viscoelastic layers. Later in this work, one addresses the detailed history of this topic and the most recent developments.

1.6.1 | Partially Distributed Damping in Composite Structures

In classical damping treatments, viscoelastic layers take part in a stacking sequence of composite laminae to increase damping. Figure 1.8 shows a different approach, with partially distributed viscoelastic material within specific layers - damping patches. The primary purpose is to improve the damping capacity of composite structures without degrading excessively stiffness. The use of viscoelastic materials in a partial configuration has been successively reported as an efficient passive vibration control technique when applied as a coating treatment, whether in a free or constrained layer configuration. It is suggested in this work the numerical and experimental study of partially distributed viscoelastic damping treatments for composite structures in an integrated arrangement.

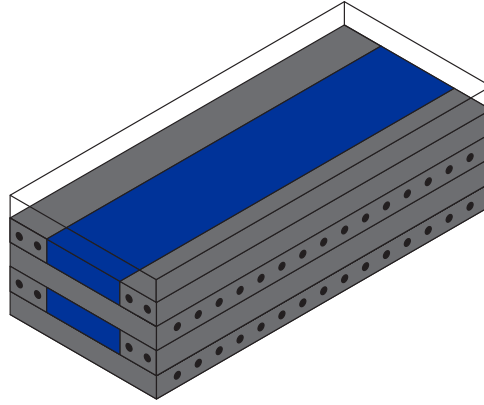


Figure 1.8 Composite laminate with viscoelastic layers partially distributed

Figure 1.9 presents detailed views of two different alternatives to a composite lamina made of a conventional composite material (coloured grey) and viscoelastic material (coloured blue). Figure 1.9a depicts a possible configuration where two strips of composite material surround a central viscoelastic strip along the lamina's, possibly decreasing the negative impact of damping material on stiffness. Similarly, Figure 1.9b represents a lamina of composite material with two interleaved viscoelastic damping patches.

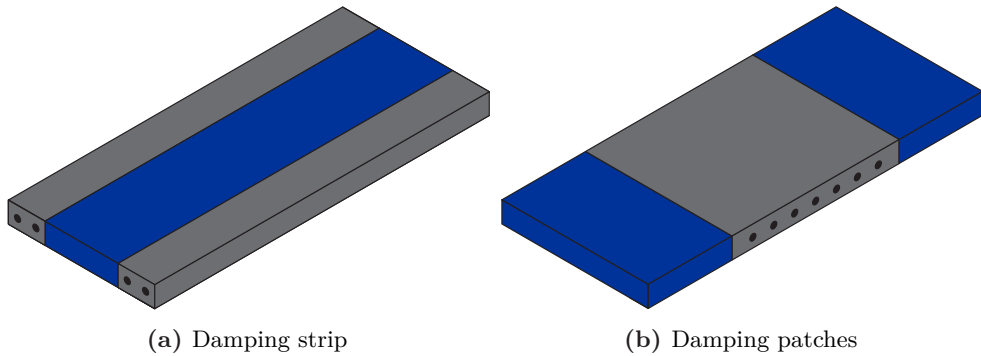


Figure 1.9 Detailed view of partial viscoelastic layers for composite laminates

The main concern about the lamina configurations represented in Figure 1.9 is their feasibility and manufacturing process. Is it feasible to have different materials at the same lamina/layer? Some stacking sequences are preferable from the design or manufacturing point of view. For example, one should avoid promoting the fibres' discontinuity in composite structures since it decreases the global stiffness of the system. On the other hand, temperature influences the

mechanical behaviour of viscoelastic materials but is essential to the curing process of composite materials in most manufacturing processes.

Even though there are possible benefits of interleaving a viscoelastic layer, Treviso et al. [2015] point out that it is not a simple procedure. Mainly because, for some viscoelastic materials, co-curing is mandatory. The different thermal behaviours of the resin matrix and the viscoelastic layers may give rise to manufacturing problems. To overcome this issue, SMAC-Group Montblanc Technologies developed a patented product called SMACWRAP ST[®]. It is a viscoelastic material that withstands composite curing, including the autoclave process, and it is possible to embed it into composite structures. The adoption of VEM as a damping mechanism has proved its efficiency in several industries during the last decades. A reasonable amount of research has already been done, mainly concerning the appropriate modelling of structures integrating VEM. However, one still requires accurate and efficient mechanical models for computational modelling of inhomogeneous multilayered structures.

Structures including VEM patches are typical examples of inhomogeneous multilayered structures. Researchers and engineers need suitable mechanical models and efficient numerical methods to study these complex systems. Studies involving Equivalent Single Layer Theory (ESLT) are ready to easily accomplish a wide range of studies due to the low computational complexity but fail to capture the mechanical phenomena from layer to layer appropriately. On the other hand, Layer-Wise Theory (LWT) models capture the intrinsic laminar structure of multilayered systems by accurately modelling the mechanical behaviour of each layer. However, these models bring some difficulties regarding computational efficiency since the degrees of freedom increase with the number of layers. In addition, these methods raise some issues when dealing with structures considering regions with a different number of layers, as PCLD patches, and therefore might be computationally cumbersome. The present work deals with a deformation theory involving a hybrid approach to the modelling of multilayered structures. This theory combines the advantages of both ESLT and LWT models by appropriately modelling the contribution of each layer, only requiring the generalised displacements of a simple ESLT.

The FEM is the numerical method implemented in this work, and the main advantages of this method are addressed for different studies to exploit their maximum potential. The primary purpose is to study viscoelastic damping patches, and FEM offers different approaches depending on the required analysis. One of the primary advantages is that, since it is element-based, it is not limited to regular shaped structures and is suitable for irregular geometries and boundary conditions. Taking the advantages further, one extends the plate theory to a flat shell element to deal with multilayered inhomogeneous shell structures embedding viscoelastic damping patches.

A great deal of work is still missing concerning the use of VEM damping patches. The significant number of published research during the last years regarding the optimisation of viscoelastic treatments shows the current interest in this topic. Optimisation studies of damping capacity with integrated viscoelastic materials aiming for low added mass or structural modification have been performed and reported in the literature. There is no dispute that those analyses have great importance and possibly lead to optimal solutions for particular and well-designed problems. However, one must understand the limitations of such approaches and what one can learn from them. For example, when one performs a thickness optimisation of a damping layer of a particular structure for maximum damping capacity, the algorithm delivers a final optimised solution according to the constraints expressed by the designer. One performs most optimisation studies in this area by applying evolutionary algorithms like black boxes with no physical meaning or understanding of the solution for general principles enabling one to formulate fundamental laws about the problem. When some variable changes, one requires a new optimisation study. The definition of appropriately damping guidelines for partially covered structures or structures with integrated damping treatments could be of prime importance for engineering applications.

As previously discussed, it is common knowledge that transverse shear strain within the viscoelastic layer is the fundamental mechanism responsible for damping. However, the transverse shear strain pattern of VEM patches and integrated damping treatments is not easy to address, and this is a subject of active research. Optimum damping here can be a challenging task.

1.7 | Objectives

As extensively reviewed in the present introduction, the interest in high-performance structures, in general, and damping technologies, in particular, has been mainly demonstrated by public and private companies from high-performance industries such as the transportation industry at the beginning of the twentieth century and the aerospace sector, in particular, more recently. The latest efforts related to the decarbonisation of the transport industry and the ever-increasing share of renewable energies in the energy market led to an investment in high-performance structures and systems demanding a high degree of performance and reliability, triggering the developments and applications of the field. Thus, and despite the applications and achievements of passive damping technologies in the past half-century, there is still room for a better comprehension of this technology, and the ever-increasing range of industries that may benefit from their use to improve performance justifies and endorses the relevance of this work. In particular, the more recent developments in composite materials and their increasing use in the high-performance industries mentioned before but also in more traditional ones leave an open field for exploration of the use of damping technologies in composite structures, and this work aims to make a small contribution in this area.

In the last decades, composite structures started to receive increasing attention from the sound and vibration-related communities for the adequate modelling and design of these lighter and stiffer structures to make them more efficient and reliable and reduce their cost. Nowadays, scientists and engineers face the requirement for developing refined and more accurate multi-physics models of those layered structures, which involves the development of adequate damping solutions and their numerical integration in those models. Such requirements push the complexity of the existing mechanical models and extend the boundaries existing nowadays grounded on the ever-increasing processing capabilities of modern computers.

Therefore, more generally, the objective is to contribute to the successful application and widespread use of damping technologies, aiming to develop some guidelines for structural vibration control covering the design, analysis and application of emerging passive damping technologies for composite structures and focusing on the following main research topics in particular.

Following a general introduction to the subject from a technological perspective with some industrial applications of viscoelastic damping materials to introduce and frame the discussion from a broad point of view, the present work aims to develop and implement a laminate composite shell finite element to assemble a general modelling tool able to numerically compute complex inhomogeneous laminated composite shell structures including local inserts of damping material with a high degree of accuracy. The mechanical model, built upon a laminated composite plate theory based on warping functions merging the advantages of LWT formulations, such as the high degree of accuracy with the fixed number of degrees of freedom of the ESLT approaches, thus reducing the model complexity, could enable its extension to more complex structures such as laminated shells and provide a refined and accurate model of intricate layered designs to researchers and engineers. Due to the local discretisation of the entire domain, which is the foundation of the FEM, and the advantages that such formulation adds to the modelling of inhomogeneous complex structures, the objective is to focus on the developments performed in this work in the FEM.

As previously mentioned, despite the studies conducted in the past half-century regarding passive damping technologies, there is still room for a better comprehension of this subject, and one of the main objectives of the present work is to fully comprehend the damping mechanisms behind improved damping performances of partially distributed damping treatments and establish design guidelines for the optimal design of structures with viscoelastic damping treatments. Firstly, one aims to focus on traditional surface damping treatments and to set up the appropriate metrics to correctly evaluate and comprehend the damping mechanisms responsible for improved damping performances. In addition, and considering the developments in composite materials and their increasing use in the high-performance industries mentioned before, one also intends to extend those metrics to correctly evaluate and comprehend the damping mechanisms responsible for improved damping performances in laminated composite plates with interleaved damping layers but also with interleaved damping patches, one of the main goals of the present work.

The previously defined objectives rely on the appropriate numerical tools that enable the corresponding analysis of the obtained results. However, despite the valuable insights and guidelines that a suitable computational model provides, the experimental testing and analysis of the proposed theories and the validation of the numerical results are of capital relevance to strengthen the conclusions achieved. Therefore, the closing objective of the present work is to perform an extensive experimental campaign involving the mechanical characterisation of viscoelastic damping materials and their integration in the innovative proposed design solutions to pursue the motivations of this thesis, the study of composite laminates with interleaved viscoelastic damping material, including viscoelastic damping patches.

Lastly, the final and general objective of this thesis is to contribute to the widespread use and application of damping technologies, aiming to develop theoretical but also technical, experimental, and practical knowledge and structural vibration control skills in classical and emerging sound and vibration-related technologies such as the emerging passive damping technologies for composite structures.

1.8 | Layout

To pursue the previously established objectives, this document includes five parts divided into ten chapters summarised below.

The first part, Part I - *Introduction*, opens the present work, including the *Introduction* in Chapter 1, where one outlines the motivation and objectives, giving some background about damping panel vibrations and the nature of damping. Discussing the main damping mechanisms, one focuses on passive and active vibration damping approaches and the details of the viscoelastic damping operation. The first chapter follows with a technological perspective of the subject with some industrial applications of damping materials, hoping to introduce the reader to the state of the art and leave a detailed review with more scientific details for the following parts of the work. One also addresses composite materials in Chapter 1, namely the main hybridisation techniques and how to add damping with an interlayer hybrid configuration. The chapter wraps up with the leading motivation for the present work, partially distributed damping in composite structures. The first part also includes Chapter 2, *Viscoelastic Materials*, where one introduces the basic concepts of viscoelastic materials, including their mechanical characterisation, selection and numerical modelling and implementation, a crucial step of this work.

Part II - *Mechanical Model* includes the *Laminated Composite Plate Theory* in Chapter 3 with the introduction of the analytical composite plate theory based on warping functions considered in this work, the *Numerical Methods* used to implement it in Chapter 4, and the *Numerical Results* used to validate the present work with published literature in Chapter 5. Firstly, Chapter 3

fully details the mechanical model of the *Laminated Composite Plate Theory*, including the warping functions for different deformation theories, merging the advantages of LWT and ESLT approaches. Then, Chapter 4 - *Numerical Methods* presents the RRM and the FEM, the two methods implemented here to solve the equilibrium equations derived in the previous chapter. Due to its nature of locally discretised elements and the advantages that such formulation adds to the modelling of inhomogeneous complex structures, one focuses on the main developments in the FEM. Furthermore, one presents in this chapter the facet-shell finite element for laminated composite shells proposed in this work. Part II includes Chapter 5, *Numerical Results*, where one compares and validates the proposed numerical tools for isotropic plates and composite plates, but also sandwich shells, including composite facings with internal damping, thus performing the numerical validation for the relevant applications of the proposed finite element.

The *Parametric Analysis of Damping Patches*, Part III, covers the complete set of numerical studies involving the evaluation of damping performance realised in the context of this work. *Beam and Plate Structures*, Chapter 6, presents the results of damping efficiency of partially covered isotropic beams and the performance of segmented constraining layers. One discusses the main mechanisms responsible for improved damping features and the proposed new metrics to comprehend and evaluate damping performance, taking advantage of the implemented mechanical model. Chapter 7, *Laminated Composite Plates*, presents all the numerical results concerning the evaluation of the damping performance of laminated composite plates with interleaved damping layers, but also laminated composite plates with interleaved damping patches, the leading motivation and objective of the present work. The proposed new metrics for the comprehension and evaluation of the main mechanisms responsible for improved damping performances find a *raison d'être* in laminated composite plates, and Chapter 7 fully details their relevance.

Part IV, *Experimental Testing and Analysis*, comprises all the experimental work performed in the present work, including the *Mechanical Characterisation of VEM* in Chapter 8 and the analysis of *Composite Laminates with Interleaved Viscoelastic Materials* in Chapter 9. The complete experimental testing and comparison with numerical results presented in the latter is only possible with the careful work carried out and described in the former. The *Mechanical Characterisation of VEM* in Chapter 8 includes the proposed experimental procedure to obtain the experimental transmissibility frequency response functions in a contactless manner and the corresponding material properties identification via numerical fitting with transmissibility frequency response functions computed with the FEM using the Local Amplitude Criterion. Using different manufacturing approaches, one identified the material properties of two viscoelastic materials used in the experimental work developed here, Smacwrap ST and Kraibon. Furthermore, Chapter 9 details the experimental developments carried out to accomplish one of the main objectives and motivations of this thesis, the study of composite laminates with interleaved viscoelastic material, including viscoelastic damping patches. This chapter describes the manufacturing process used to produce the complete set of specimens used to evaluate the damping performance of different configurations of laminated composite plates with interleaved damping layers and patches using the two viscoelastic materials whose properties one identified in Chapter 8. In addition, one describes the experimental procedure to obtain the experimental frequency response functions used to compare the experimental results with the numerical ones computed with the FEM and consequently evaluate the damping performance of different configurations.

Finally, the *Conclusion*, Part V, closes the work developed in this thesis with the eponymous Chapter 10, where one summarises the principal contributions of the present work, discussing the most significant aspects and paving the way to further developments in this field.

As damping technologies continue to evolve and spread over more and more applications, relevant research topics must be addressed for a successful implementation, especially in the emerging fields related to the decarbonisation of the transport industry and renewable energies.

1. Introduction

The main objective of this document is to contribute to a successful application and widespread use of damping technologies following the layout previously presented.

Viscoelastic Materials

2.1 | Introduction

Vibration damping design is increasingly a fundamental step of any project involving structures and machines. Damping consists of removing part of the vibrational energy and converting it into other forms of energy, like heat, for instance. Consequently, the physical mechanisms involving this subject span many disciplines, and one can find a vast body of literature. Nevertheless, the main interest of this work is the use of viscoelastic materials as a damping mechanism.

Viscoelastic materials used in damping treatments are usually polymeric-based materials. Unlike metallic materials, they have a considerable loss factor responsible for the elliptic curve of the hysteresis cycle, providing a remarkable damping capacity that depends, essentially, on the temperature and frequency of deformation. These materials consist of long reticulated molecular chains whose relative motion, developed during the cyclic deformation of the material, induces the energy dissipation in the form of heat, which, if efficiently dissipated to the exterior, allows an efficient form of damping [Moreira, 2004].

Viscoelastic treatments are highly efficient damping solutions for light structures, but their design and analysis are challenging [Vasques, 2008]. One of the difficulties comes from frequency and temperature dependence. In this chapter, one discusses the constitutive properties of general viscoelastic materials to understand the behaviour of these complex materials.

2.2 | Materials Characterisation

Materials for which the relationship between stress and strain depends on time are called viscoelastic [Hazard, 2006]. In essence, a viscoelastic material exhibits a tension state dependent on the whole strain history experienced [Mastroddi et al., 2017]. These are rubber-like polymers composed of intertwined and cross-linked molecular chains, and during deformation, the internal molecular interactions give rise to macroscopic properties such as energy dissipation. This mechanism is the basis of the viscoelastic damping treatments. As already stated, the constitutive

behaviour of viscoelastic materials depends upon the working temperature and frequency, but also upon the amplitude and type of excitation [Moreira, 2004]. In general, the type of excitation and corresponding amplitude is of reduced importance when compared to frequency and temperature influence. Still, one can consider most viscoelastic materials applied in damping treatments homogeneous since their properties are identical throughout the volume of the material and isotropic, i.e., their properties are the same in any direction.

2.2.1 | Temperature Dependence

The operating temperature of damping treatments based on viscoelastic materials can reach extreme values, from -40°C , or even less, up to 150°C , depending on the application [Jones, 2001]. Since the damping capacity and stiffness of a viscoelastic material depend on its transition temperature, T_g , the effectiveness of the damping treatment depends strongly on the relationship between the operating temperature and this transition temperature of the material. The transition temperature is the temperature at which the viscoelastic material changes from its glassy state, a material with high stiffness or storage modulus, to its amorphous or rubbery state, a material with low stiffness, and at which the material shows its maximum loss factor, which is a measure of the damping capacity - Figure 2.1, adapted from Nashif et al. [1985].

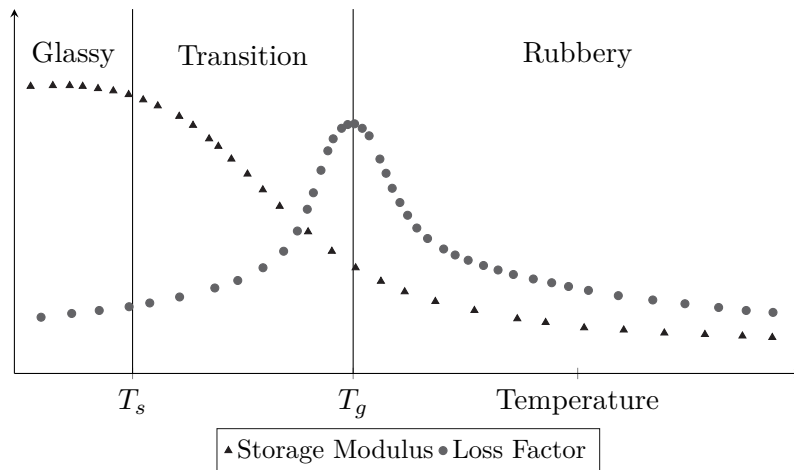


Figure 2.1 Temperature dependence of the storage modulus and loss factor

In applications subjected to high-temperature gradients, it is mandatory to develop and apply suitable construction techniques, like multi-layer and multi-material treatments, which extend the efficient temperature range of the viscoelastic treatment.

2.2.2 | Frequency Dependence

In addition to temperature, frequency plays a decisive role in the mechanical characterisation of viscoelastic materials - Figure 2.2, adapted from Nashif et al. [1985]. As the excitation frequency increases, the stiffness of viscoelastic materials increases, although with moderate behaviour in both glassy and rubbery regions [Nashif et al., 1985].

When deforming a viscoelastic material, one disturbs the long molecular chains which constitute the viscoelastic material from their initial equilibrium position, interacting with each other as a reaction to the imposed deformation. In response to this perturbation, the molecular

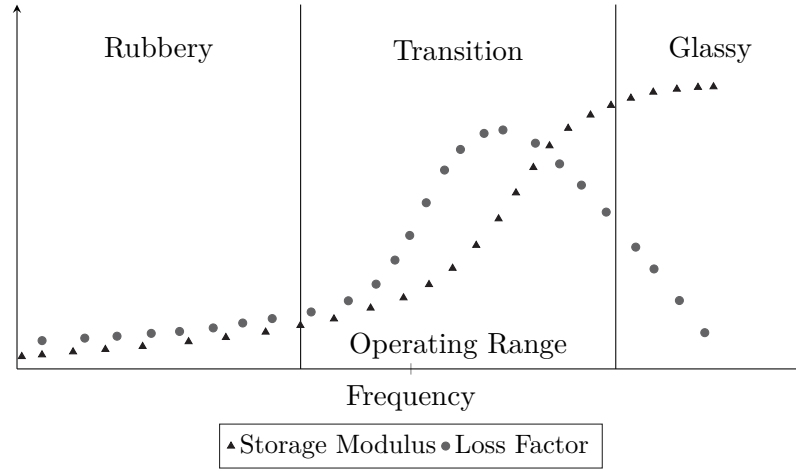


Figure 2.2 Frequency dependence of the storage modulus and loss factor

chains rearrange themselves to reach a new equilibrium position. If the imposed deformation is cyclic of harmonic type, the molecular chains reach a new state of dynamic equilibrium, responding in tune with the excitation. Thus, when critically excited at a high frequency, a viscoelastic material shows a high stiffness, having a lower stiffness when the solicitation has a lower frequency.

2.2.3 | Response to Harmonic Stress Loading

Considering a one-dimensional harmonic load of frequency ω and assuming linear viscoelasticity, one can write the following stress field

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad (2.1)$$

It is possible to verify that by subjecting a viscoelastic material to harmonic loading, the resulting deformation is also harmonic, presenting the same loading frequency. However, contrary to elastic materials, there is a significant relative mismatch between the strain obtained and the stress imposed by loading - Figure 2.3.

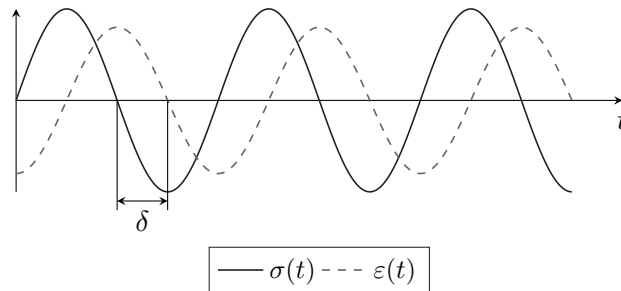


Figure 2.3 Excitation and harmonic response of a viscoelastic material. Adapted from Moreira [2004]

The strain is not in phase with the stress but lags behind δ , the phase angle, and takes the form

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t - \delta) \quad (2.2)$$

The lag δ is an indicator of the damping capacity of the viscoelastic material - the larger the lag, the greater the damping. The representation of the stress as a function of the strain in Figure 2.4, usually known as the hysteresis cycle, presents the shape of an ellipse, whose angle formed by its major axis and the strain axis is proportional to the stiffness of the material, W is the stored elastic energy per cycle, and the ratio between the minor axis and the major axis is an indicator of its damping capacity. If the damping is null, i.e. if the material exhibits an elastic behaviour, the hysteresis cycle takes the shape of a straight line, whose slope determines its stiffness. The ellipse area, ΔW , is the dissipated energy per cycle and is a measure of the material's damping capacity.

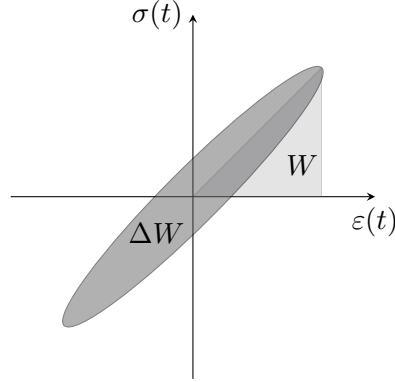


Figure 2.4 Hysteretic cycle of a viscoelastic material

A common way to represent the damping capacity of the material in structural dynamics is the loss factor η , a quantity equal to $\tan(\delta)$ that is popular in polymer engineering.

2.2.4 | Complex Modulus Approach

One can define the relation between stress and strain by the Complex Modulus Approach (CMA). For isotropic and homogeneous materials, the complex modulus is described by a complex stiffness modulus $\tilde{E}$, and a complex Poisson's ratio $\tilde{\nu}$ [Hazard, 2006]. However, the experimental measurement of these two quantities is not easy, and generally, one considers the Poisson's ratio not frequency-dependent. Generalising the stress and strain equations, one can write

$$\sigma(t) = \sigma_0 e^{j\omega t} \quad (2.3)$$

$$\varepsilon(t) = \varepsilon_0 e^{j\omega t - \delta} \quad (2.4)$$

In the frequency domain, one can write the relation between stress and strain as follows

$$\sigma(j\omega) = \tilde{E}(j\omega)\varepsilon(j\omega) = [E'(\omega) + jE''(\omega)]\varepsilon(j\omega) = E'(\omega)[1 + j\eta(\omega)]\varepsilon(j\omega) \quad (2.5)$$

where $\tilde{E}(j\omega)$ is the complex extensional modulus, $E'(\omega)$, the real part of the complex modulus, is the storage modulus and represents the material's capacity to store elastic energy and $E''(\omega)$, the imaginary part of the complex modulus, is the loss modulus and represents the material's capacity to dissipate energy. Consequently, $\eta(\omega)$ is the loss factor mentioned before and can be written as

$$\eta(\omega) = \frac{E''(\omega)}{E'(\omega)} = \tan(\delta(\omega)) \quad (2.6)$$

Figure 2.5 shows the representation of these variables on the complex plane. The loss factor is a relation between imaginary and real parts of the complex modulus, and one can view it as a measure of the dissipated energy, imaginary part, as a function of the stored one, real part.

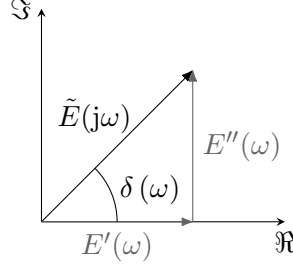


Figure 2.5 Complex modulus on the complex plane

Similarly, one can define a complex shear modulus considering shear stress and strain

$$\tau(j\omega) = \tilde{G}(j\omega)\gamma(j\omega) = [G'(\omega) + jG''(\omega)] \gamma(j\omega) = G'(\omega) [1 + j\eta(\omega)] \gamma(j\omega) \quad (2.7)$$

2.3 | Numerical Modelling and Implementation

The modelling and implementation of viscoelastic materials in numerical methods such as the finite element method is one of the most challenging tasks when considering these materials. As previously mentioned, due to their nature, the mechanical properties of these materials depend upon frequency, working temperature, amplitude and type of excitation. Developing a feasible model considering all of these factors is cumbersome. Even correctly considering the influence of one of these parameters on the mechanical properties of viscoelastic materials is something that has been a matter of a large amount of research by many authors. Nevertheless, the frequency and temperature dependence are the most important ones regarding the mechanical properties of viscoelastic materials. Researchers have focused their work on the influence of these two factors on the complex modulus.

As previously discussed, the damping capacity of a viscoelastic material is closely related to its transition temperature, where the loss factor reaches a maximum, so the working temperature is of great importance to the designer. Besides, there is a broad range of working temperatures at which these materials can operate since many applications in different industries require damping. However, the usual approach is to characterise the mechanical properties of viscoelastic materials for a wide range of temperatures, assuming isothermal conditions for different temperatures of interest in the simulations. Despite being an approximation, one must only consider the assumptions which enable one to work with isothermal conditions but never forget the damping mechanism associated with these materials and the conversion of mechanical energy into heat. Therefore, it is crucial to guarantee an appropriate thermal energy dissipation to ensure the isothermal condition. Consequently, in this work, one considers the frequency dependence of viscoelastic materials assuming isothermal conditions in the numerical modelling.

2.3.1 | Viscoelastic Constitutive Modelling

Modelling static and dynamic problems of structures using viscoelastic materials requires the development of reliable models to describe their behaviour accurately. As mentioned in

the current chapter, the constitutive behaviour of viscoelastic materials depends, above all else, upon frequency, working temperature, amplitude and type of excitation. However, in this work, one focuses on the numerical modelling of viscoelastic materials considering only the frequency dependence at isothermal conditions. Researchers and engineers have developed several mechanical models based on the combination of springs and viscous elements. The Maxwell and Kelvin-Voigt models are examples of such models. Even though these were of great interest in the past, writing the constitutive behaviour of viscoelastic materials in the form of a differential equation becomes arduous labour as the models get more complicated and require higher-order differential equations. It is usual to apply parametric analytical models defined by adjusting their parameters to the experimental results obtained in the characterisation of the material under analysis. Among these models, due to their representativeness and wide application, one highlights the Generalised Maxwell Model (GMM), the Golla-Hughes-McTavish (GHM) model [McTavish and Hughes, 1993], the Anelastic Displacement Fields (ADF) model [Lesieutre and Bianchini, 1995] and the Fractional Derivatives Model (FDM) [Pritz, 1996].

2.3.1.1 | Generalised Maxwell Model - GMM

The Voigt-Kelvin or GMM describes the frequency dependence of the complex modulus through simple rheologic models where there is a series of Maxwell elements in parallel.

$$\tilde{G}(j\omega) = G_0 \left(1 + \sum_{i=1}^n \frac{j\omega\tau_i\gamma_i}{j\omega\tau_i + 1} \right) \quad (2.8)$$

where G_0 is the low frequency or static shear modulus, and each mini-oscillator term involves two positive real constant parameters, τ_i and γ_i , defined by adjustment to the experimental results. However, these models usually require a high number of terms to achieve a good representation of the viscoelastic material behaviour.

2.3.1.2 | Golla-Hughes-McTavish Model - GHM

Originally developed as a tool for damping analysis in flexible space structures, the GHM model applies to any system incorporating viscoelastic materials [McTavish and Hughes, 1993]. It represents the frequency variation of the viscoelastic material modulus as a series of *mini-oscillator* terms or internal variables. Therefore, one can define the shear modulus of viscoelastic materials in the Laplace domain as

$$s\tilde{G}(s) = G_0 [1 + h(s)] = G_0 \left(1 + \sum_{i=1}^n \hat{\alpha}_i \frac{s^2 + 2\hat{\zeta}_i\hat{\omega}_i s}{s^2 + 2\hat{\zeta}_i\hat{\omega}_i s + \hat{\omega}_i^2} \right) \quad (2.9)$$

where G_0 is the low frequency or static shear modulus and each mini-oscillator term involves three positive real constant parameters, $\hat{\alpha}_i$, $\hat{\zeta}_i$ and $\hat{\omega}_i$, defined by adjustment to the experimental results.

The GHM model necessarily generates a dissipative component proportional to the velocity terms of the equation of motion [Moreira, 2004]. As a result of the inclusion of dissipative variables, the size of the final problem is larger than the original one. The problem formulated with recourse to the GHM model is twice the original size, at least, corresponding to the application of only one oscillator in the description of the behaviour of the viscoelastic material.

2.3.1.3 | Anelastic Displacement Fields Model - ADF

The ADF model, proposed by Lesieutre and Bianchini [1995] and based on the predecessor model Augmenting Thermodynamic Fields (ATF) [Lesieutre, 1992, Lesieutre and Mingori, 1990],

represents the complex modulus of the viscoelastic material through a series of terms of the form

$$\tilde{G}(j\omega) = G_0 \left(1 + \sum_{i=1}^n \Delta_i \frac{\omega^2 + j\omega\Omega_i}{\omega^2 + \Omega_i^2} \right) \quad (2.10)$$

where G_0 represents the low frequency or static modulus, as in the GHM model, and the terms Δ_i and Ω_i represent the resistance to relaxation, and the inverse of the characteristic relaxation time at constant strain, respectively. These scalar parameters characterise the behaviour of the viscoelastic material with frequency, and one can obtain them by fitting the model to experimental results.

Similarly to other internal variable models such as the GHM, one of the main disadvantages of the ADF model is that it requires the use of additional degrees of freedom leading to the definition of augmented systems of equations [Vasques et al., 2010].

2.3.1.4 | Fractional Derivative Model

The use of fractional calculus introduced a considerable simplification in viscoelastic constitutive modelling. In fractional derivative models, one writes the viscoelastic behaviour in the form of a differential equation where the time derivative order is not an integer number [Vasques et al., 2010]. The fractional derivative model with four parameters describes the behaviour of a viscoelastic system in a broad frequency range, being a sort of interpolation between the viscous and the elastic behaviour of the material [Pritz, 1996]. The differential equation of the four-parameter fractional derivative model describes the relationship between the stress tensor and the strain tensor in the following manner

$$\sigma(t) + \tau^\alpha \frac{d^\alpha \sigma(t)}{dt^\alpha} = E_0 \varepsilon(t) + E_\infty \tau^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha} \quad (2.11)$$

where E_0 , E_∞ , τ and α are the four parameters of the model. E_0 is the static modulus of elasticity or dynamic modulus at low frequencies, E_∞ is the limit value of the dynamic modulus at high frequencies, and τ represents the relaxation time of the material [Pritz, 1996]. The parameter α is relative to the order of the derivative, and since it is a fractional derivative, its value varies between $0 < \alpha < 1$. According to Riemann-Liouville definition, one can define the α^{th} order derivative of a function $f(t)$ as [Bagley and Torvik, 1985]

$$\frac{d^\alpha f(t)}{dt^\alpha} = f^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \right], \quad 0 < \alpha < 1 \quad (2.12)$$

The Gamma function $\Gamma(1-\alpha)$ can be written as

$$\Gamma(1-\alpha) = \int_0^\infty t^{-\alpha} e^{-t} dt \quad (2.13)$$

One must write the mathematical form of the fractional derivative model in the frequency domain. As such, Equation 2.11 can be re-written considering that one can replace the derivative term with a $(j\omega)^\alpha$ term in the frequency domain [Pritz, 1996]

$$\sigma(j\omega) = \frac{E_0 + E_\infty (j\omega\tau)^\alpha}{1 + (j\omega\tau)^\alpha} \varepsilon(j\omega) \quad (2.14)$$

Therefore, one can define the complex modulus as

$$\tilde{E}(j\omega) = \frac{E_0 + E_\infty (j\omega\tau)^\alpha}{1 + (j\omega\tau)^\alpha} \quad (2.15)$$

It is also possible to write an equivalent expression for the complex shear modulus

$$\tilde{G}(j\omega) = \frac{G_0 + G_\infty (j\omega\tau)^\alpha}{1 + (j\omega\tau)^\alpha} \quad (2.16)$$

The four-parameter fractional derivative model is simple and allows one to describe the dynamic behaviour of viscoelastic materials in a wide frequency range accurately. In this work, one determines the model parameters using a curve-fitting process for different viscoelastic materials. Therefore, the fact that it only requires four parameters to describe the variations of the complex modulus is practical, as the optimisation process becomes computationally efficient.

2.3.2 | Direct Frequency Response (DRF)

One considers in this work a frequency-dependent viscoelastic damping model through a CMA assuming isothermal conditions. It is possible to generalise the equations of motion of a discrete system with n degrees of freedom considering the CMA as

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \tilde{\mathbf{K}}(j\omega) \mathbf{u}(t) = \mathbf{f}(t) \quad (2.17)$$

where matrices $\mathbf{M}$, $\mathbf{C}$ and $\tilde{\mathbf{K}}(j\omega)$ represent the mass matrix, viscous damping matrix and complex stiffness matrix, respectively. Vectors $\mathbf{u}(t)$ and $\mathbf{f}(t)$ represent the response and force vectors, respectively. It is possible to divide the complex stiffness matrix in one elastic component and one viscoelastic counterpart

$$\tilde{\mathbf{K}}(j\omega) = \mathbf{K}_e + \mathbf{K}_v(j\omega) \quad (2.18)$$

where $\mathbf{K}_e$, which is real and constant, represents the elastic stiffness matrix including all the purely elastic elements of a given model, while $\mathbf{K}_v(j\omega)$, which is complex and frequency-dependent, represents the viscoelastic stiffness matrix. The stiffness matrix depends upon the mechanical properties of the materials, and so the viscoelastic stiffness matrix depends, in the most general case, upon the extensional $\tilde{E}(j\omega)$ and shear modulus $\tilde{G}(j\omega)$ of the viscoelastic material. It is well known the relation between the extensional and shear modulus for linear, homogeneous and isotropic materials through the Poisson's ratio. For viscoelastic materials, one can write

$$\tilde{G}(j\omega) = \frac{\tilde{E}(j\omega)}{2[1 + \tilde{\nu}(j\omega)]} \quad (2.19)$$

In fact, the extensional and shear loss factors are not the same since the Poisson's ratio is also frequency-dependent. However, many authors have been working on assuming a frequency-independent Poisson's ratio, which leads to similar shear and extensional loss factors. In addition, this assumption enables one to write the viscoelastic stiffness matrix as

$$\mathbf{K}_v(j\omega) = \tilde{G}(j\omega) \mathbf{K}_v \text{ or } \mathbf{K}_v(j\omega) = \tilde{E}(j\omega) \mathbf{K}_v \quad (2.20)$$

The main advantage of a frequency-independent Poisson's ratio is that one can establish a factorised real and constant valued viscoelastic stiffness matrix $\mathbf{K}_v$, calculated for a unitary shear or extensional modulus. Considering stationary harmonic regime with a harmonic excitation with the following form

$$\mathbf{f}(t) = \mathbf{F}(\omega) e^{j\omega t} \quad (2.21)$$

and a stationary harmonic response of the type

$$\mathbf{u}(t) = \mathbf{U}(j\omega) e^{j\omega t} \quad (2.22)$$

one can re-write Equation 2.17 in the following form

$$\left(-\omega^2 \mathbf{M} + j\omega \mathbf{D} + \left(\mathbf{K}_e + \tilde{E}(j\omega) \mathbf{K}_v \right) \right) \mathbf{U}(j\omega) = \mathbf{F}(\omega) \quad (2.23)$$

The CMA is an efficient method for Direct Frequency Response (DFR), yet limited to harmonic excitations. One solves the system of equations 2.23 at discrete frequency values updating the complex modulus of the viscoelastic material. Therefore, one can use the CMA to extract complex modes through an iterative updating process.

2.3.2.1 | Modal Strain Energy (MSE)

For lightly damped structures, it is possible to obtain an approximation to the modal loss factors by assuming that the natural modes obtained from the undamped system are representative of the damped system. Based on this assumption, Johnson and Kienholz [1982] proposed a direct method for the finite element prediction of the modal damping ratio in structures with viscoelastic damping layers, the Modal Strain Energy (MSE) approach. One can define the modal loss factors by the ratio between the loss and storage parts of the modal strain energy and thereby are an excellent mechanical indicator of the damping efficiency.

$$\eta_r = \frac{\{\phi\}_r^T \Im[\tilde{\mathbf{K}}(j\omega)] \{\phi\}_r}{\{\phi\}_r^T \Re[\tilde{\mathbf{K}}(j\omega)] \{\phi\}_r} \quad (2.24)$$

where $\{\phi\}_r$ is the mode shape of a generic mode r , obtained by solving the following generalised eigenvalue problem

$$\Re[\tilde{\mathbf{K}}(j\omega)] \{\phi\}_r = \omega_r^2 \mathbf{M} \{\phi\}_r \quad (2.25)$$

The relation expressed in Equation 2.24 represents the ratio between dissipated energy, proportional to the imaginary part of the complex stiffness matrix, and the stored energy, corresponding to the real part of the stiffness matrix. However, as can be seen in Equation 2.25, it is assumed that the natural modes obtained from the undamped system are representative of the damped system. This assumption is only valid for lightly damped structures since the coupling between modes is small due to the low damping, which couldn't be the case of viscoelastically damped systems [Vasques, 2008]. Alternatively, in a more realistic approach, one can solve the eigenvalue problem considering the frequency dependence independently under an iterative procedure by updating the stiffness matrix - Iterative Modal Strain Energy (IMSE).

2.3.2.2 | Iterative Modal Strain Energy (IMSE)

Researchers have been working with a modification to the original MSE algorithm proposed to introduce the stiffness dependence of the damped structural system through an iterative procedure to consider the variation of the storage and loss moduli with frequency - IMSE. Consequently, one performs an iterative calculation of the real (undamped) eigensolution using the continuously iteratively updated real and imaginary parts of the stiffness matrices using the viscoelastic material properties at the frequency of the natural mode determined in the previous iteration. Based on the real eigensolution performed using finite element analysis, which is the initial solution, since one verifies the convergence of the undamped natural frequency of the natural mode under analysis, one determines the correspondent modal loss factor using the same methodology of the original MSE method. One can find some examples of the use of the IMSE in the

context of finite element models in the literature [Zhang and Chen, 2006]. On the following, one presents a schematic procedure describing the IMSE approach based on the work of Vasques [2008].

IMSE Algorithm

Step 1. Eigensolution with $\omega = \omega^0$

$$\Re[\tilde{\mathbf{K}}(j\omega^0)]\{\phi\}_r = (\omega_r^0)^2 \mathbf{M}(\omega^0)\{\phi\}_r \quad (2.26)$$

Step 2. Loop for each eigenpair $(\omega_r^0, \{\phi\}_r)$ with $r = 1, \dots, p$

i. Initial value

$$\omega_r^i = \omega_r^0 \quad (2.27)$$

ii. Iterative loop for each natural frequency and mode shape

- Eigensolution $(\omega_s^0, \{\phi\}_s)$ with $s = 1, \dots, r$

$$\Re[\tilde{\mathbf{K}}(j\omega_t^i)]\{\phi\}_s = (\omega_s^{i+1})^2 \mathbf{M}(\omega_r^i)\{\phi\}_s \quad (2.28)$$

- Iterated natural frequency

$$\begin{cases} \omega_s^{i+1} : \text{rejected,} & \text{for } s < r \\ \omega_r^{i+1} = \sqrt{(\omega_s^{i+1})^2}, & \text{for } s = r \end{cases} \quad (2.29)$$

- Convergence condition test

$$\Delta_\omega = \frac{|\omega_r^{i+1} - \omega_r^i|}{\omega_r^{i+1}} \leq \Delta_{\max} \quad (2.30)$$

iii. Modal loss factor

$$\eta_r = \frac{\{\phi\}_r^T \Im[\tilde{\mathbf{K}}(j\omega_r^{i+1})]\{\phi\}_r}{\{\phi\}_r^T \Re[\tilde{\mathbf{K}}(j\omega_r^{i+1})]\{\phi\}_r} \quad (2.31)$$

2.3.2.3 | Iterative Complex Eigensolution (ICE)

Vasques [2008] proposed an alternative Iterative Complex Eigensolution (ICE) algorithm. This approach does not estimate the modal loss factor in the same way as the MSE, but it uses the determined complex eigenvalue to calculate the modal loss factor. However, the repeated complex eigensolutions before achieving convergence have the drawback of requiring a higher computational effort. The ICE method successively updates the complex eigenvalues and eigenvectors until convergence based on the real eigensolution performed using finite element analysis. Compared with conventional methods, the ICE method considerably improves the numerical accuracy while maintaining the computational efficiency and simplicity required for practical applications. One presents in the following a schematic procedure describing the ICE approach based on the work of Vasques [2008].

ICE Algorithm

Step 1. Eigensolution with $\omega = \omega^0$

$$\Re[\tilde{\mathbf{K}}(j\omega^0)]\{\phi\}_r = (\omega_r^0)^2 \mathbf{M}(\omega^0)\{\phi\}_r \quad (2.32)$$

Step 2. Loop for each eigenpair $(\lambda_r, \{\phi\}_r)$ with $r = 1, \dots, p$

i. Initial value

$$\omega_r^i = \omega_r^0 \quad (2.33)$$

ii. Iterative loop for each complex eigensolution and eigenvector

- Complex eigensolution $(\lambda_s, \{\phi\}_s)$ with $s = 1, \dots, r$

$$\tilde{\mathbf{K}}(j\omega_r^i)\{\phi\}_s = \lambda_s^{i+1} \mathbf{M}(\omega_r^i)\{\phi\}_s \quad (2.34)$$

- Iterated damped natural frequency

$$\begin{cases} \omega_s^{i+1} : \text{rejected,} & \text{for } s < r \\ \omega_r^{i+1} = \sqrt{\Re(\lambda_s^{i+1})}, & \text{for } s = r \end{cases} \quad (2.35)$$

- Convergence condition test

$$\Delta_\omega = \frac{|\omega_r^{i+1} - \omega_r^i|}{\omega_r^{i+1}} \leq \Delta_{\max} \quad (2.36)$$

iii. Modal loss factor

$$\eta_r = \frac{\Im(\lambda_r^{i+1})}{\Re(\lambda_r^{i+1})} \quad (2.37)$$

2.4 | Material Selection

Temperature is an important environmental factor influencing the mechanical properties of damping materials, as reported in Figure 2.1, and one can identify three main areas. The first region is the glassy state, where the material has the highest storage modulus but low damping capacity since that loss modulus is small. In the transition region, the material changes from a glassy state to a rubbery state and the material modulus decreases with increasing temperature. In this region, usually, the loss factor achieves its maximum. In the rubbery state, storage modulus and loss factor assume low values and vary slowly with temperature. Several authors define a fourth region, the flow region, where the loss factor reaches very high levels while material storage modulus decreases [Rao, 2003].

Apart from temperature, vibration frequency has a meaningful effect on the complex modulus of viscoelastic materials and, consequently, on damping capacity. Storage modulus and loss factor dependence with frequency for a typical high-damping material show that the effect of increasing

temperature on the storage modulus is similar to that of reducing the frequency for materials without flow region.

Thorough knowledge and understanding of the constitutive behaviour of viscoelastic materials are crucial to making an informed decision when choosing a viscoelastic material for a given application. Therefore, the loss factor peak should ideally occur at the standard system operating temperature. Preferably, the selected material should also have a broad loss factor peak to exhibit valuable damping properties for a wide range of temperatures. The mechanical properties of viscoelastic materials are also affected by environmental factors such as humidity, which consequently limit their efficiency. These factors introduce additional modelling difficulties to the dynamic behaviour of damped structures. For this reason, one assumes isothermal conditions and considers only the frequency dependence.

The temperature-frequency superposition principle forms the basis for reducing the three-dimensional relationship between the storage modulus (or loss factor), frequency and temperature to a two-dimensional one. It involves the reduced frequency or the reduced temperature, which combines the effects of frequency and temperature by shift factors. One does not deal with this subject in this work, but one suggests some references regarding this topic [Hazard, 2006, Moreira, 2004, Vasques, 2008].

Part II

Mechanical Model

Laminated Composite Plate Theory

3.1 | Introduction

The numerical modelling of structures has been a long-standing topic in research and one of the most important subjects since the classical works of Euler [1744] and Bernoulli [1751], who established the foundations of the most well-known beam theory. Researchers have since worked on mechanical models for beams and other more geometrically complex structures such as plates and shells. Love and Darwin [1888], using the assumptions proposed by Kirchhoff [1850], established the extension of Euler [1744]-Bernoulli [1751]'s beam theory to plates. The Classical Beam Theory (CBT) of Euler [1744]-Bernoulli [1751] and the Classical Plate Theory (CPT) of Kirchhoff [1850]-Love and Darwin [1888], as commonly known, are appropriate for thin isotropic beams and plates, since established that straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation. However, Timoshenko [1921] studied the vibration of beams and suggested that transverse straight lines before deformation will still be straight after deformation but not necessarily normal to the mid-surface. Mindlin [1951], who proposed a similar theory to that of Reissner [1945], extended the assumptions of Timoshenko [1921] to plates. Timoshenko [1921] or Mindlin [1951] theories are each referred to as a First-Order Shear Deformation Theory (FSDT) since there is a linear variation of the in-plane displacements across the thickness, accounting for both shear deformation and rotary inertia since they are intended for thick beams and plates, respectively. Naturally, theories based on high-order variations of the in-plane displacements were also proposed and are known as High-Order Shear Deformation Theory (HSDT). Lo et al. [1977] presented a unified approach for HSDT but the Third-Order Shear Deformation Theory (TSDT) proposed by Vlasov [1957] and generalised by Reddy [1984] is perhaps the most well known HSDT. The mechanical modelling of these structures is inherently associated with their geometry and constitutive materials, but the increasing use of sandwich and laminated composite structures has set new challenges for researchers and engineers. In fact, the basic assumption of CPT and FSDT that straight lines normal to the mid-surface remain straight after deformation does not hold for layered structures with very different mechanical properties. As pointed out by Liu and Li [1996], although the

in-plane displacements should be continuous over the laminate interfaces when perfectly bonded, their first derivatives, i.e. extensional strains, are discontinuous due to the abrupt change of material properties. Certainly, three-dimensional elasticity solutions associated with numerical methods such as the FEM are often a rigorous procedure to model these structures since such solutions do not rely upon geometric or material simplifications. However, the computational cost and intricate formulations associated with three-dimensional elasticity solutions for multi-layer inhomogeneous structures have become prohibitive for many applications. Therefore, researchers and engineers have been improving classical beam and plate theories mentioned previously, well suited for homogeneous structures, to capture the complex mechanical phenomena arising within structures with very different materials.

Over the years, researchers have been developing different displacement-based formulations. One can divide these theories into three main groups - Khandan et al. [2012] gives a recent review of laminated composite plate theories. The first group includes formulations expressed as an ESLT. Due to their simplicity, one can use CPT, FSDT and HSDT to model composite structures, but the innate multi-layer structure replaced by a single-layer with equivalent material properties inevitably reduces accuracy. Despite the inaccurate distribution of the transverse shear strain through the thickness, the interest in these theories comes from their straightforward implementation and a fixed number of kinematic variables, making this approach computationally efficient. By contrast, deformation theories involving multi-layer hypotheses can accurately capture the mechanical behaviour layer by layer and correctly account for normal and transverse shear strain variation. A typical example of such an approach is the LWT model, though the number of kinematic variables increases proportionally with the number of layers and, consequently, the computational effort. In a third group are theories dealing with multi-layer approaches of the hybrid type. The primary purpose of such hypotheses is to keep the advantages of a purely multi-layer deformation theory while maintaining the number of kinematic variables of a single-layer approach providing a good compromise between accuracy and computational time. Zig-Zag Theory (ZZT) models are the most standard approach of this kind. In fact, one can also see ZZT as a subgroup of the General Layer-Wise Theory (GLWT) in which the number of kinematic variables is independent of the number of layers. Carrera [2003] gives a detailed review of these theories and, as pointed out by Di Capua and Oñate [2015], most of the early ZZT enforce the continuity of the transverse shear stresses across the laminate thickness to obtain the zigzag function and ensure a fixed number of kinematic variables while capturing the mechanical behaviour of each layer.

3.2 | Plate Theory Based on Warping Functions

Loredo and Castel [2013] presented a plate model for static and dynamic analysis of inhomogeneous anisotropic multilayered plates, which one can define as a multi-layer approach of the hybrid type, based on previous works led by Woodcock [Woodcock and Nicolas, 1995, Woodcock, 2008]. In Loredo and Castel [2013] words, the theory is a reformulation of Woodcock [2008] work. In fact, this reformulation has led the model to a broader level by explicitly defining the displacement field, which eases the implementation and links with other plate theories. Following recent trends of improving the transverse shear strain variations through the thickness, particularly on account of the increase of inhomogeneous structures as composite laminates, Loredo and Castel [2013] use transverse warping functions to account for transverse shear strain variations through the thickness while keeping the number of kinematic variables. The main advantage of this model is that different plate theories, including ESLT or ZZT, can be formulated by properly defining the transverse warping functions. Touratier [1991] and Idlbi et al. [1997]

have already presented a unified formulation for symmetric orthotropic laminates based on a shear function describing the transverse shear strain across the thickness and accounting for interlayer continuity both for displacements and transverse shear stresses. Later, Loredó and Castel [2014] proposed enhanced transverse warping functions issued from three-dimensional elasticity equations to obtain the exact transverse shear stresses through the thickness of laminates with general lamination schemes.

A rectangular composite plate with PCLD patches is a particular case of an inhomogeneous multilayered structure with a different number of layers within the domain. Nevertheless, Loredó and Castel [2013] showed that this model has proven to be very efficient for modelling PCLD patches, and one follows their work here. However, regarding the mechanical model, the main goal of the present work is to derive a finite element based on it and extend it to a facet shell finite element. The main advantage of a unified mechanical model coupled with a facet shell element, which can formulate different plate theories, is to model complex composite inhomogeneous shell structures with PCLD efficiently.

In the present work, one slightly adapts Loredó and Castel [2013] notation by avoiding Einstein's summation convention, which allows a great level of concision even though it is not always easy to figure out. The theory unfolds here in a way that any reader familiar with classical plate theories can readily understand and implement. Figure 3.1 presents the fundamental geometric parameters of a laminate with n layers. Parameter z^ℓ measures the distance between layer's ℓ mid-plane and the reference plane, and h^ℓ measures its height. One identifies each interface with a ζ label. The theory embraces five generalised displacements, labelled with the superscript 0, which is a tag for the reference plane ($z = 0$). Whatever the reference plane, one fully defines the displacement field by the generalised displacements of such layer and the kinematic relations through the thickness, built up using four transverse warping functions $\varphi_{xx}(z)$, $\varphi_{xy}(z)$, $\varphi_{yx}(z)$ and $\varphi_{yy}(z)$.

3.3 | Displacement Field

The kinematic relations between generalised displacements establish the following displacement field

$$\begin{aligned} u(x, y, z, t) &= u^0(x, y, t) - z \frac{\partial w^0(x, y, t)}{\partial x} + \varphi_{xx}(z) \gamma_{xz}^0(x, y, t) + \varphi_{xy}(z) \gamma_{yz}^0(x, y, t) \\ v(x, y, z, t) &= v^0(x, y, t) - z \frac{\partial w^0(x, y, t)}{\partial y} + \varphi_{yx}(z) \gamma_{xz}^0(x, y, t) + \varphi_{yy}(z) \gamma_{yz}^0(x, y, t) \\ w(x, y, z, t) &= w^0(x, y, t) \end{aligned} \quad (3.1)$$

where the exponent 0 refers to the reference layer ($z = 0$). Transverse shear strains are described through the thickness using the warping functions' derivatives which enable one to link transverse shear between the reference layer and some point across the thickness. Considering the warping functions' derivative as independent of the coordinates x and y , one can write

$$\begin{aligned} \gamma_{xz}(x, y, z) &= \frac{d\varphi_{xx}(z)}{dz} \gamma_{xz}^0(x, y) + \frac{d\varphi_{xy}(z)}{dz} \gamma_{yz}^0(x, y) \\ \gamma_{yz}(x, y, z) &= \frac{d\varphi_{yx}(z)}{dz} \gamma_{xz}^0(x, y) + \frac{d\varphi_{yy}(z)}{dz} \gamma_{yz}^0(x, y) \end{aligned} \quad (3.2)$$

It is worth noting that for the reference plane, whatever the warping function, one assumes

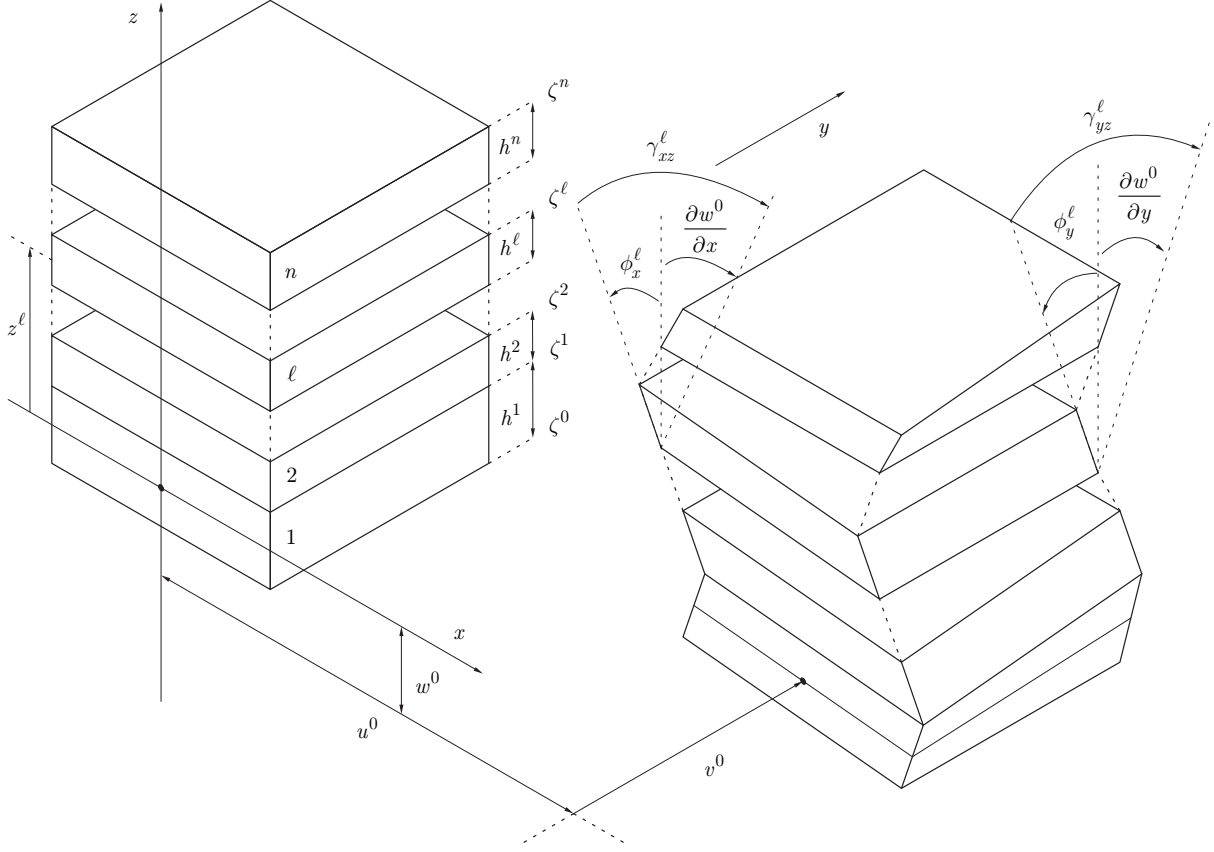


Figure 3.1 Laminate geometric properties and generalised displacements

$$\varphi(z = 0) = 0 \quad (3.3)$$

The displacement field at a given point in the structure is a function of five generalised kinematic variables - $u^0(x, y)$, $v^0(x, y)$, $w^0(x, y)$, $\gamma_{xz}^0(x, y)$ and $\gamma_{yz}^0(x, y)$. Note that the form of the displacement field expressed in Equation 3.1 allows the reduction of the 3-D problem to one expressing the deformation at the reference plane $z = 0$. Once the reference plane displacements are known, the displacements of an arbitrary point $P(x, y, z)$ can be determined using Equation 3.1. Henceforth, for the sake of simplicity, one implicitly assumes the (x, y, t) dependence.

3.4 | Strain Field

The strains are written according to the previous displacement field using the well-known linear strain-displacement relations given by

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (3.4)$$

Therefore, the strain field takes the form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ 2\kappa_{xy}^0 \end{Bmatrix} + \begin{bmatrix} \varphi_{xx}(z) & 0 & 0 & \varphi_{xy}(z) \\ 0 & \varphi_{yy}(z) & \varphi_{yx}(z) & 0 \\ \varphi_{yx}(z) & \varphi_{xy}(z) & \varphi_{xx}(z) & \varphi_{yy}(z) \end{bmatrix} \begin{Bmatrix} \Gamma_{xx}^0 \\ \Gamma_{yy}^0 \\ \Gamma_{xy}^0 \\ \Gamma_{yx}^0 \end{Bmatrix} \quad (3.5)$$

with separate transverse shear strains for the sake of clarity

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{d\varphi_{xx}}{dz}(z) & \frac{d\varphi_{xy}}{dz}(z) \\ \frac{d\varphi_{yx}}{dz}(z) & \frac{d\varphi_{yy}}{dz}(z) \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} \quad (3.6)$$

One writes the strain field in a vector form with separate strain orders. Vector quantities $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\Gamma}$ are specified in the following

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ 2\varepsilon_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u^0}{\partial x} \\ \frac{\partial v^0}{\partial y} \\ \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \end{Bmatrix}; \quad \boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ 2\kappa_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w^0}{\partial x^2} \\ -\frac{\partial^2 w^0}{\partial y^2} \\ -2\frac{\partial^2 w^0}{\partial x \partial y} \end{Bmatrix} \quad (3.7)$$

$$\boldsymbol{\Gamma} = \begin{Bmatrix} \Gamma_{xx}^0 \\ \Gamma_{yy}^0 \\ \Gamma_{xy}^0 \\ \Gamma_{yx}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \gamma_{xz}^0}{\partial x} \\ \frac{\partial \gamma_{yz}^0}{\partial y} \\ \frac{\partial \gamma_{xz}^0}{\partial y} \\ \frac{\partial \gamma_{yz}^0}{\partial x} \end{Bmatrix} \quad (3.8)$$

Following a detailed analysis, it is worth noting that vector quantities $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ represent, respectively, the well-known *membrane strains* and *curvatures* in classical laminate theories. To that extent, vector $\boldsymbol{\Gamma}$ quantities could be seen as *higher-order curvatures*.

3.5 | Lamina Constitutive Relations

The linear constitutive relations for an arbitrary orthotropic lamina in the principal material coordinates, Figure 3.2, and assuming a state of *generalised plane stress* are written in the following form [Reddy, 1997]

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}; \quad \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \quad (3.9)$$

where quantities Q_{ij} , called plane stress-reduced stiffnesses, are given by

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}; \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{66} = G_{12}; \quad Q_{44} = G_{23}; \quad Q_{55} = G_{13} \quad (3.10)$$

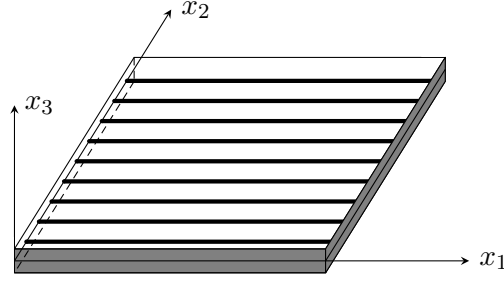


Figure 3.2 Unidirectional fibre-reinforced composite layer with the material coordinate system (x_1, x_2, x_3)

It is essential to convert the stress components expressed in the material (lamina) coordinate system, subscript m, to those of the problem (laminate) coordinate system, subscript p - Figure 3.3.

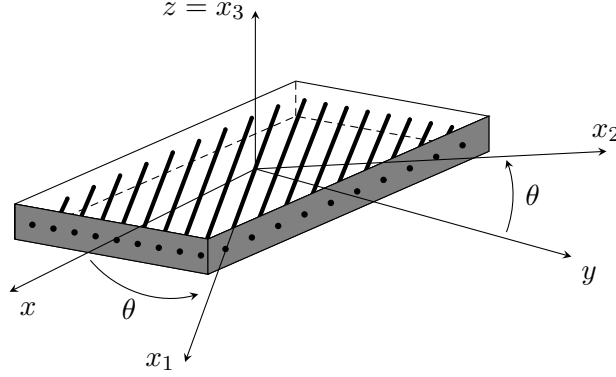


Figure 3.3 Lamina with material and problem coordinate systems

In this regard, one carries out the following transformation

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yx} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & -\sin 2\theta & 0 & 0 \\ \sin^2 \theta & \cos^2 \theta & \sin 2\theta & 0 & 0 \\ \sin \theta \cos \theta & -\sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & \sin \theta \\ 0 & 0 & 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix} \quad (3.11)$$

where θ represents the orientation angle. Briefly, one can write the preceding transformation in the following form

$$\sigma_p = \mathbf{T} \sigma_m \quad (3.12)$$

Considering the existing stress-strain relation, one can re-write the previous expression

$$\sigma_p = \mathbf{T} \sigma_m = \mathbf{T} \mathbf{Q}_m \epsilon_m = \mathbf{T} \mathbf{Q}_m \mathbf{T}^T \epsilon_p \quad (3.13)$$

Consequently, one can write the transformed stress-strain relations of an orthotropic lamina as follows

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}; \quad \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (3.14)$$

where $\bar{\mathbf{Q}}_p$ is a transformed plane stress-reduced stiffness matrix written in the form

$$\bar{\mathbf{Q}}_p = \mathbf{T} \mathbf{Q}_m \mathbf{T}^T \quad (3.15)$$

3.6 | Strain Energy

Considering the strain field defined in Equation 3.5 and the stress-strain relation presented in Equation 3.14, one can easily obtain the strain energy surface density (strain energy per unit area) with the following expression

$$\Pi^U(x, y) = \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dz \quad (3.16)$$

Developing the strain field one can write

$$\begin{aligned} \Pi^U(x, y) = \frac{1}{2} \int_{\zeta_0}^{\zeta_n} & \left[(\varepsilon_{xx}^0 + z\kappa_{xx}^0 + \varphi_{xx}\Gamma_{xx}^0 + \varphi_{xy}\Gamma_{yx}^0) \sigma_{xx} \right. \\ & + (\varepsilon_{yy}^0 + z\kappa_{yy}^0 + \varphi_{yy}\Gamma_{yy}^0 + \varphi_{yx}\Gamma_{xy}^0) \sigma_{yy} \\ & + (2\varepsilon_{xy}^0 + 2z\kappa_{xy}^0 + \varphi_{yx}\Gamma_{xx}^0 + \varphi_{xy}\Gamma_{yy}^0 + \varphi_{xx}\Gamma_{xy}^0 + \varphi_{yy}\Gamma_{yx}^0) \tau_{xy} \\ & \left. + \left(\frac{d\varphi_{xx}}{dz} \gamma_{xz}^0 + \frac{d\varphi_{xy}}{dz} \gamma_{yz}^0 \right) \tau_{xz} + \left(\frac{d\varphi_{yx}}{dz} \gamma_{xz}^0 + \frac{d\varphi_{yy}}{dz} \gamma_{yz}^0 \right) \tau_{yz} \right] dz \end{aligned} \quad (3.17)$$

Through-the-thickness integration provides the following expression

$$\begin{aligned} \Pi^U(x, y) = \frac{1}{2} & \left[\varepsilon_{xx}^0 N_{xx} + \varepsilon_{yy}^0 N_{yy} + 2\varepsilon_{xy}^0 N_{xy} - \kappa_{xx}^0 M_{xx} - \kappa_{yy}^0 M_{yy} - 2\kappa_{xy}^0 M_{xy} \right. \\ & \left. + \Gamma_{xx}^0 P_{xx} + \Gamma_{yy}^0 P_{yy} + \Gamma_{xy}^0 P_{xy} + \Gamma_{yx}^0 P_{yx} + \gamma_{xz}^0 \hat{Q}_x + \gamma_{yz}^0 \hat{Q}_y \right] \end{aligned} \quad (3.18)$$

Re-arranged in a vector notation, one can write quantities $\mathbf{N}$, $\mathbf{M}$, $\mathbf{P}$ and $\hat{\mathbf{Q}}$ in the form

$$\mathbf{N} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{\zeta_0}^{\zeta_n} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz; \quad \mathbf{M} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{\zeta_0}^{\zeta_n} z \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz \quad (3.19)$$

$$\mathbf{P} = \begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \\ P_{yx} \end{Bmatrix} = \int_{\zeta_0}^{\zeta_n} \begin{bmatrix} \varphi_{xx} & 0 & \varphi_{yx} \\ 0 & \varphi_{yy} & \varphi_{xy} \\ 0 & \varphi_{yx} & \varphi_{xx} \\ \varphi_{xy} & 0 & \varphi_{yy} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz \quad (3.20)$$

$$\hat{\mathbf{Q}} = \begin{Bmatrix} \hat{Q}_x \\ \hat{Q}_y \end{Bmatrix} = \int_{\zeta_0}^{\zeta_n} \begin{bmatrix} \frac{d\varphi_{xx}}{dz} & \frac{d\varphi_{yx}}{dz} \\ \frac{d\varphi_{xy}}{dz} & \frac{d\varphi_{yy}}{dz} \end{bmatrix} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz \quad (3.21)$$

Similarly to other deformation theories, the $\mathbf{N}$ components are known as the in-plane *force resultants*, $\mathbf{M}$ components as the *moment resultants*, $\hat{\mathbf{Q}}$ as the *transverse force resultants* and following the same reasoning $\mathbf{P}$ components can be seen as *moment resultants* of higher-order. Using the constitutive relations, one can write the stress components as

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{P} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{F} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\Gamma} \end{Bmatrix}; \quad \hat{\mathbf{Q}} = \mathbf{H}\boldsymbol{\gamma} \quad (3.22)$$

Therefore, force and moment resultants relate to the strain components of different orders through the following quantities

$$\mathbf{A} = \int_{\zeta_0}^{\zeta_n} \bar{\mathbf{Q}}_{1,2,6} dz; \quad \mathbf{B} = \int_{\zeta_0}^{\zeta_n} z \bar{\mathbf{Q}}_{1,2,6} dz; \quad \mathbf{D} = \int_{\zeta_0}^{\zeta_n} z^2 \bar{\mathbf{Q}}_{1,2,6} dz \quad (3.23)$$

$$\mathbf{E} = \int_{\zeta_0}^{\zeta_n} \bar{\mathbf{Q}}_{1,2,6} \begin{bmatrix} \varphi_{xx} & 0 & 0 & \varphi_{xy} \\ 0 & \varphi_{yy} & \varphi_{yx} & 0 \\ \varphi_{yx} & \varphi_{xy} & \varphi_{xx} & \varphi_{yy} \end{bmatrix} dz \quad (3.24)$$

$$\mathbf{F} = \int_{\zeta_0}^{\zeta_n} z \bar{\mathbf{Q}}_{1,2,6} \begin{bmatrix} \varphi_{xx} & 0 & 0 & \varphi_{xy} \\ 0 & \varphi_{yy} & \varphi_{yx} & 0 \\ \varphi_{yx} & \varphi_{xy} & \varphi_{xx} & \varphi_{yy} \end{bmatrix} dz \quad (3.25)$$

$$\mathbf{G} = \int_{\zeta_0}^{\zeta_n} \begin{bmatrix} \varphi_{xx} & 0 & 0 & \varphi_{xy} \\ 0 & \varphi_{yy} & \varphi_{yx} & 0 \\ \varphi_{yx} & \varphi_{xy} & \varphi_{xx} & \varphi_{yy} \end{bmatrix}^T \bar{\mathbf{Q}}_{1,2,6} \begin{bmatrix} \varphi_{xx} & 0 & 0 & \varphi_{xy} \\ 0 & \varphi_{yy} & \varphi_{yx} & 0 \\ \varphi_{yx} & \varphi_{xy} & \varphi_{xx} & \varphi_{yy} \end{bmatrix} dz \quad (3.26)$$

$$\mathbf{H} = \int_{\zeta_0}^{\zeta_n} \begin{bmatrix} \frac{d\varphi_{xx}}{dz} & \frac{d\varphi_{xy}}{dz} \\ \frac{d\varphi_{yx}}{dz} & \frac{d\varphi_{yy}}{dz} \end{bmatrix}^T \bar{\mathbf{Q}}_{5,4} \begin{bmatrix} \frac{d\varphi_{xx}}{dz} & \frac{d\varphi_{xy}}{dz} \\ \frac{d\varphi_{yx}}{dz} & \frac{d\varphi_{yy}}{dz} \end{bmatrix} dz \quad (3.27)$$

Some of these quantities are also known from other plate theories - $\mathbf{A}$ represents the *extensional stiffnesses*, $\mathbf{D}$ expresses the *bending stiffnesses* and $\mathbf{B}$ represents the *bending-extensional stiffnesses*. Besides, following the same reasoning, $\mathbf{G}$ can be seen as *higher-order bending stiffnesses*, $\mathbf{F}$ as the *higher-order bending-bending coupling stiffnesses* and $\mathbf{E}$ as the *higher-order bending-extensional coupling stiffnesses*. Expressing the relation between transverse force resultants and shear strains, $\mathbf{H}$ components can be read as *shear stiffnesses*.

3.7 | Kinetic Energy

The following expression represents the kinetic energy surface density (kinetic energy per unit area) of the laminated plate

$$\Pi^K(x, y) = \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \dot{\mathbf{u}}^T \rho \dot{\mathbf{u}} dz \quad (3.28)$$

where ρ represents density and $\dot{\mathbf{u}}$ the velocity field, obtained with a simple time derivative of the displacement field presented in Equation 3.1. By doing so, one can write

$$\begin{aligned} \Pi^K(x, y) = \frac{1}{2} \int_{\zeta_0}^{\zeta_n} \rho \left[\left(\dot{u}^0 - z \frac{\partial \dot{w}^0}{\partial x} + \varphi_{xx} \dot{\gamma}_{xz}^0 + \varphi_{xy} \dot{\gamma}_{yz}^0 \right)^2 \right. \\ \left. + \left(\dot{v}^0 - z \frac{\partial \dot{w}^0}{\partial y} + \varphi_{yx} \dot{\gamma}_{xz}^0 + \varphi_{yy} \dot{\gamma}_{yz}^0 \right)^2 + (\dot{w}^0)^2 \right] dz \end{aligned} \quad (3.29)$$

Through-the-thickness integration provides the following expression

$$\begin{aligned} \Pi^K(x, y) = \frac{1}{2} \left[R \left((\dot{u}^0)^2 + (\dot{v}^0)^2 + (\dot{w}^0)^2 \right) - 2S \left(\dot{u}^0 \frac{\partial \dot{w}^0}{\partial x} - \dot{v}^0 \frac{\partial \dot{w}^0}{\partial y} \right) \right. \\ + T \left(\frac{\partial \dot{w}^0}{\partial x} \frac{\partial \dot{w}^0}{\partial x} + \frac{\partial \dot{w}^0}{\partial y} \frac{\partial \dot{w}^0}{\partial y} \right) + 2 \{ \dot{u}^0 \quad \dot{v}^0 \} \mathbf{U} \{ \dot{\gamma}_{xz}^0 \quad \dot{\gamma}_{yz}^0 \}^T \\ \left. - 2 \{ \frac{\partial \dot{w}^0}{\partial x} \quad \frac{\partial \dot{w}^0}{\partial y} \} \mathbf{V} \{ \dot{\gamma}_{xz}^0 \quad \dot{\gamma}_{yz}^0 \}^T + \{ \dot{\gamma}_{xz}^0 \quad \dot{\gamma}_{yz}^0 \} \mathbf{W} \{ \dot{\gamma}_{xz}^0 \quad \dot{\gamma}_{yz}^0 \}^T \right] \end{aligned} \quad (3.30)$$

Following the same procedure used for the strain energy surface density, one can define the quantities presented in Equation 3.30 as

$$R = \int_{\zeta_0}^{\zeta_n} \rho dz; \quad S = \int_{\zeta_0}^{\zeta_n} \rho z dz; \quad T = \int_{\zeta_0}^{\zeta_n} \rho z^2 dz \quad (3.31)$$

$$\mathbf{U} = \int_{\zeta_0}^{\zeta_n} \rho \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{bmatrix} dz; \quad \mathbf{V} = \int_{\zeta_0}^{\zeta_n} \rho z \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{bmatrix} dz \quad (3.32)$$

$$\mathbf{W} = \int_{\zeta_0}^{\zeta_n} \rho \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{bmatrix}^T \begin{bmatrix} \varphi_{xx} & \varphi_{xy} \\ \varphi_{yx} & \varphi_{yy} \end{bmatrix} dz \quad (3.33)$$

Once again, R , S and T are the well known *mass moments of inertia* present in other deformation theories. $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ components are *higher-order mass moments of inertia*.

3.8 | Equilibrium Equations

It is possible to formulate the dynamic equilibrium equations performing a force equilibrium in the x, y and z -directions, considering an infinitesimal volume

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= \rho \ddot{u} \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= \rho \ddot{v} \end{aligned}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \ddot{w} \quad (3.34)$$

Integrating Equations 3.34 through-the-thickness results in the following

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + [\tau_{xz}]_{\zeta_0}^{\zeta_n} &= R\ddot{u}^0 - S \frac{\partial \ddot{w}^0}{\partial x} + U_{11}\ddot{\gamma}_{xz}^0 + U_{12}\ddot{\gamma}_{yz}^0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + [\tau_{yz}]_{\zeta_0}^{\zeta_n} &= R\ddot{v}^0 - S \frac{\partial \ddot{w}^0}{\partial y} + U_{21}\ddot{\gamma}_{xz}^0 + U_{22}\ddot{\gamma}_{yz}^0 \\ \frac{\partial \hat{Q}_x^c}{\partial x} + \frac{\partial \hat{Q}_y^c}{\partial y} + [\sigma_{zz}]_{\zeta_0}^{\zeta_n} &= R\ddot{w}^0 \end{aligned} \quad (3.35)$$

where $\hat{Q}^c$ are the classical shear forces. Likewise, one performs weighted integrations through the thickness with z , φ_{xx} , φ_{xy} , φ_{yx} and φ_{yy} as weight functions

$$\begin{aligned} \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + [z\tau_{xz}]_{\zeta_0}^{\zeta_n} - \hat{Q}_x^c &= S\ddot{u}^0 - T \frac{\partial \ddot{w}^0}{\partial x} + V_{11}\ddot{\gamma}_{xz}^0 + V_{12}\ddot{\gamma}_{yz}^0 \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + [z\tau_{yz}]_{\zeta_0}^{\zeta_n} - \hat{Q}_y^c &= S\ddot{v}^0 - T \frac{\partial \ddot{w}^0}{\partial y} + V_{21}\ddot{\gamma}_{xz}^0 + V_{22}\ddot{\gamma}_{yz}^0 \\ \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + [\varphi_{xx}\tau_{xz}]_{\zeta_0}^{\zeta_n} + [\varphi_{yx}\tau_{yz}]_{\zeta_0}^{\zeta_n} - \hat{Q}_x &= U_{11}\ddot{u}^0 + U_{21}\ddot{v}^0 - V_{11} \frac{\partial \ddot{w}^0}{\partial x} + V_{21} \frac{\partial \ddot{w}^0}{\partial y} \\ &\quad + W_{11}\ddot{\gamma}_{xz}^0 + W_{12}\ddot{\gamma}_{yz}^0 \\ \frac{\partial P_{yx}}{\partial x} + \frac{\partial P_{yy}}{\partial y} + [\varphi_{xy}\tau_{xz}]_{\zeta_0}^{\zeta_n} + [\varphi_{yy}\tau_{yz}]_{\zeta_0}^{\zeta_n} - \hat{Q}_y &= U_{12}\ddot{u}^0 + U_{22}\ddot{v}^0 - V_{12} \frac{\partial \ddot{w}^0}{\partial x} + V_{22} \frac{\partial \ddot{w}^0}{\partial y} \\ &\quad + W_{21}\ddot{\gamma}_{xz}^0 + W_{22}\ddot{\gamma}_{yz}^0 \end{aligned} \quad (3.36)$$

Assuming that transverse loading is given by $q = [\sigma_{zz}]_{\zeta_0}^{\zeta_n}$ and no tangential forces are present at both surfaces - $\tau_{xz}(\zeta^0) = \tau_{xz}(\zeta^n) = \tau_{yz}(\zeta^0) = \tau_{yz}(\zeta^n) = 0$, the final equilibrium equations are written as

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= R\ddot{u}^0 - S \frac{\partial \ddot{w}^0}{\partial x} + U_{11}\ddot{\gamma}_{xz}^0 + U_{12}\ddot{\gamma}_{yz}^0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} &= R\ddot{v}^0 - S \frac{\partial \ddot{w}^0}{\partial y} + U_{21}\ddot{\gamma}_{xz}^0 + U_{22}\ddot{\gamma}_{yz}^0 \\ \frac{\partial^2 M_{xx}}{\partial x^2} + 2 \frac{\partial M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x, y) &= S \left(\frac{\partial \ddot{u}^0}{\partial x} + \frac{\partial \ddot{v}^0}{\partial y} \right) + R\ddot{w}^0 - T \left(\frac{\partial^2 \ddot{w}^0}{\partial x^2} + \frac{\partial^2 \ddot{w}^0}{\partial y^2} \right) \\ &\quad + V_{11} \frac{\partial \ddot{\gamma}_{xz}^0}{\partial x} + V_{12} \frac{\partial \ddot{\gamma}_{yz}^0}{\partial x} + V_{21} \frac{\partial \ddot{\gamma}_{xz}^0}{\partial y} + V_{22} \frac{\partial \ddot{\gamma}_{yz}^0}{\partial y} \\ \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} - \hat{Q}_x &= U_{11}\ddot{u}^0 + U_{21}\ddot{v}^0 - V_{11} \frac{\partial \ddot{w}^0}{\partial x} + V_{21} \frac{\partial \ddot{w}^0}{\partial y} + W_{11}\ddot{\gamma}_{xz}^0 + W_{12}\ddot{\gamma}_{yz}^0 \\ \frac{\partial P_{yx}}{\partial x} + \frac{\partial P_{yy}}{\partial y} - \hat{Q}_y &= U_{12}\ddot{u}^0 + U_{22}\ddot{v}^0 - V_{12} \frac{\partial \ddot{w}^0}{\partial x} + V_{22} \frac{\partial \ddot{w}^0}{\partial y} + W_{21}\ddot{\gamma}_{xz}^0 + W_{22}\ddot{\gamma}_{yz}^0 \end{aligned} \quad (3.37)$$

3.9 | Warping Functions

As previously discussed, one can establish the variation of the transverse shear strains through the thickness using transverse warping functions, also known as shear functions. The main advantage of this model is the ability to set up these variations accordingly to the problem under analysis without changing anything within the basic formulation. With a simple step, one can change from an ESLT based on the Classical Laminate Plate Theory (CLPT), FSDT, or even the TSDT. In the following, one presents the appropriate use of transverse warping functions to describe some of these theories.

3.9.1 | CLPT - Kirchhoff-Love's Model

Kirchhoff [1850]-Love and Darwin [1888]'s deformation theory, also known as the CLPT, is the simplest ESLT and only suitable for thin homogeneous structures since it does not consider transverse shear strain through the thickness. Consequently, all warping functions are null.

$$\varphi_{xx}(z) = \varphi_{yy}(z) = \varphi_{xy}(z) = \varphi_{yx}(z) = 0 \quad (3.38)$$

3.9.2 | FSDT - Reissner-Mindlin's Model

In the FSDT, the deformation theory based on the works of authors like Reissner [1945] and Mindlin [1951], transverse shear strain is constant through the thickness. Consequently, the theory does not verify the transverse stress continuity at the layers' interfaces, and one can write the warping functions as follows

$$\varphi_{xx}(z) = \varphi_{yy}(z) = z; \quad \varphi_{xy}(z) = \varphi_{yx}(z) = 0 \quad (3.39)$$

3.9.3 | TSDT - Vlasov-Reddy's Model

The TSDT of Vlasov [1957] and Reddy [1984] is a TSDT that considers the in-plane displacements as a cubic expansion of the thickness coordinate. Consequently, one of the main advantages of this deformation theory is the quadratic variation of the transverse shear stress. Nevertheless, it is an ESLT, and one can write the warping functions for this theory as

$$\varphi_{xx}(z) = \varphi_{yy}(z) = C_1 z + C_2 z^2 + C_3 z^3; \quad \varphi_{xy}(z) = \varphi_{yx}(z) = 0 \quad (3.40)$$

One derives constants C_1 , C_2 and C_3 imposing $C_1 = 1$ and two conditions issued from the equilibrium equations, free shear stress at both top and bottom surfaces

$$\tau_{xz} \left(z^1 - \frac{h^1}{2} \right) = \tau_{xz} \left(z^n + \frac{h^n}{2} \right) = 0 \quad (3.41)$$

or

$$\tau_{yz} \left(z^1 - \frac{h^1}{2} \right) = \tau_{yz} \left(z^n + \frac{h^n}{2} \right) = 0 \quad (3.42)$$

From Equations 3.41 and 3.42 results

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2\left(z^1 - \frac{h^1}{2}\right) & 3\left(z^1 - \frac{h^1}{2}\right)^2 \\ 1 & 2\left(z^n + \frac{h^n}{2}\right) & 3\left(z^n + \frac{h^n}{2}\right)^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad (3.43)$$

For a single layer plate with the reference plane, $z = 0$, at the mid-plane, one obtains the well-known $C_2 = 0$ and $C_3 = -4/3h^2$.

3.9.4 | FoZZ - Sun-Whitney-Woodcock's Model

Woodcock [2008] studied the problem of free vibrations of advanced anisotropic multilayered structures proposing a different approach to the classical equivalent single and multi-layer theories, which one can define as a multi-layer approach of the hybrid type based on the classification presented before. First presented by Sun and Whitney [1973] for cross-ply laminates and generalised by Woodcock [2008] for general lamination sequences, the main advantage of this model is to keep the benefits of a purely multi-layer deformation theory while maintaining the number of kinematic variables of a single-layer approach, providing a good compromise between accuracy and computational time - Loredo and Castel [2014] called it First-Order Zig-Zag Theory (FoZZ). The procedure establishes a set of interface conditions between multiple layers to reduce the generalised displacements to those of the reference plane. Therefore, one describes the displacement field appropriately for each layer dealing with a fixed number of generalised displacements. At the first level, one assumes displacement's continuity between two consecutive layers

$$\begin{aligned} u\left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2}\right) &= u\left(x, y, z^{\ell} + \frac{h^{\ell}}{2}\right) \\ v\left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2}\right) &= v\left(x, y, z^{\ell} + \frac{h^{\ell}}{2}\right) \end{aligned} \quad (3.44)$$

Considering the displacement field defined in Equation 3.1 one can write

$$\begin{aligned} &\varphi_{xx}\left(z^{\ell+1} - \frac{h^{\ell+1}}{2}\right)\gamma_{xz}^0(x, y) + \varphi_{xy}\left(z^{\ell+1} - \frac{h^{\ell+1}}{2}\right)\gamma_{yz}^0(x, y) \\ &= \varphi_{xx}\left(z^{\ell} + \frac{h^{\ell}}{2}\right)\gamma_{xz}^0(x, y) + \varphi_{xy}\left(z^{\ell} + \frac{h^{\ell}}{2}\right)\gamma_{yz}^0(x, y) \end{aligned} \quad (3.45)$$

and also

$$\begin{aligned} &\varphi_{yx}\left(z^{\ell+1} - \frac{h^{\ell+1}}{2}\right)\gamma_{xz}^0(x, y) + \varphi_{yy}\left(z^{\ell+1} - \frac{h^{\ell+1}}{2}\right)\gamma_{yz}^0(x, y) \\ &= \varphi_{yx}\left(z^{\ell} + \frac{h^{\ell}}{2}\right)\gamma_{xz}^0(x, y) + \varphi_{yy}\left(z^{\ell} + \frac{h^{\ell}}{2}\right)\gamma_{yz}^0(x, y) \end{aligned} \quad (3.46)$$

which in turn leads to the following continuity at each interface

$$\varphi_{xx}\left(z^{\ell+1} - \frac{h^{\ell+1}}{2}\right) = \varphi_{xx}\left(z^{\ell} + \frac{h^{\ell}}{2}\right)$$

$$\begin{aligned}
\varphi_{xy} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{xy} \left(z^\ell + \frac{h^\ell}{2} \right) \\
\varphi_{yx} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{yx} \left(z^\ell + \frac{h^\ell}{2} \right) \\
\varphi_{yy} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{yy} \left(z^\ell + \frac{h^\ell}{2} \right)
\end{aligned} \tag{3.47}$$

Assuming that one can define warping functions by parts, i.e. layerwise, one can write

$$\begin{aligned}
\varphi_{xx}^{\ell+1}(\zeta^\ell) &= \varphi_{xx}^\ell(\zeta^\ell) \\
\varphi_{xy}^{\ell+1}(\zeta^\ell) &= \varphi_{xy}^\ell(\zeta^\ell) \\
\varphi_{yx}^{\ell+1}(\zeta^\ell) &= \varphi_{yx}^\ell(\zeta^\ell) \\
\varphi_{yy}^{\ell+1}(\zeta^\ell) &= \varphi_{yy}^\ell(\zeta^\ell)
\end{aligned} \tag{3.48}$$

Besides, one also ensures transverse shear stresses' continuity at each interface

$$\begin{aligned}
\tau_{xz} \left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \tau_{xz} \left(x, y, z^\ell + \frac{h^\ell}{2} \right) \\
\tau_{yz} \left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \tau_{yz} \left(x, y, z^\ell + \frac{h^\ell}{2} \right)
\end{aligned} \tag{3.49}$$

Considering the stress-strain relations previously defined, one can write

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) \\ \gamma_{yz} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^\ell & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & \bar{Q}_{44}^\ell \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \left(z^\ell + \frac{h^\ell}{2} \right) \\ \gamma_{yz} \left(z^\ell + \frac{h^\ell}{2} \right) \end{Bmatrix} \tag{3.50}$$

Substituting Equation 3.6 in the previous equality and recalling that derivatives of warping functions are constant across each layer's thickness

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{bmatrix} \frac{d\varphi_{xx}}{dz}(z^{\ell+1}) & \frac{d\varphi_{xy}}{dz}(z^{\ell+1}) \\ \frac{d\varphi_{yx}}{dz}(z^{\ell+1}) & \frac{d\varphi_{yy}}{dz}(z^{\ell+1}) \end{bmatrix} = \begin{bmatrix} \bar{Q}_{55}^\ell & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & \bar{Q}_{44}^\ell \end{bmatrix} \begin{bmatrix} \frac{d\varphi_{xx}}{dz}(z^\ell) & \frac{d\varphi_{xy}}{dz}(z^\ell) \\ \frac{d\varphi_{yx}}{dz}(z^\ell) & \frac{d\varphi_{yy}}{dz}(z^\ell) \end{bmatrix} \tag{3.51}$$

or, in a compact form, considering that one can define warping functions by parts, i.e. layerwise

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{bmatrix} \frac{d\varphi_{xx}^{\ell+1}}{dz} & \frac{d\varphi_{xy}^{\ell+1}}{dz} \\ \frac{d\varphi_{yx}^{\ell+1}}{dz} & \frac{d\varphi_{yy}^{\ell+1}}{dz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{55}^\ell & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & \bar{Q}_{44}^\ell \end{bmatrix} \begin{bmatrix} \frac{d\varphi_{xx}^\ell}{dz} & \frac{d\varphi_{xy}^\ell}{dz} \\ \frac{d\varphi_{yx}^\ell}{dz} & \frac{d\varphi_{yy}^\ell}{dz} \end{bmatrix} \tag{3.52}$$

One can obtain a particular solution to this system of equations as follows

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} \frac{d\varphi_{xx}^{\ell+1}}{dz} \\ \frac{d\varphi_{xy}^{\ell+1}}{dz} \\ \frac{d\varphi_{yx}^{\ell+1}}{dz} \\ \frac{d\varphi_{yy}^{\ell+1}}{dz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^{\ell} & 0 & \bar{Q}_{45}^{\ell} & 0 \\ 0 & \bar{Q}_{55}^{\ell} & 0 & \bar{Q}_{45}^{\ell} \\ \bar{Q}_{45}^{\ell} & 0 & \bar{Q}_{44}^{\ell} & 0 \\ 0 & \bar{Q}_{45}^{\ell} & 0 & \bar{Q}_{44}^{\ell} \end{bmatrix} \begin{Bmatrix} \frac{d\varphi_{xx}^{\ell}}{dz} \\ \frac{d\varphi_{xy}^{\ell}}{dz} \\ \frac{d\varphi_{yx}^{\ell}}{dz} \\ \frac{d\varphi_{yy}^{\ell}}{dz} \end{Bmatrix} \quad (3.53)$$

which leads to the following

$$\begin{Bmatrix} \frac{d\varphi_{xx}^{\ell+1}}{dz} \\ \frac{d\varphi_{xy}^{\ell+1}}{dz} \\ \frac{d\varphi_{yx}^{\ell+1}}{dz} \\ \frac{d\varphi_{yy}^{\ell+1}}{dz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{Q}_{55}^{\ell} & 0 & \bar{Q}_{45}^{\ell} & 0 \\ 0 & \bar{Q}_{55}^{\ell} & 0 & \bar{Q}_{45}^{\ell} \\ \bar{Q}_{45}^{\ell} & 0 & \bar{Q}_{44}^{\ell} & 0 \\ 0 & \bar{Q}_{45}^{\ell} & 0 & \bar{Q}_{44}^{\ell} \end{bmatrix} \begin{Bmatrix} \frac{d\varphi_{xx}^{\ell}}{dz} \\ \frac{d\varphi_{xy}^{\ell}}{dz} \\ \frac{d\varphi_{yx}^{\ell}}{dz} \\ \frac{d\varphi_{yy}^{\ell}}{dz} \end{Bmatrix} \quad (3.54)$$

Consequently, one can write warping functions' derivatives across the thickness of a generic layer ℓ , knowing those of a consecutive layer and the corresponding material properties of both. A detailed analysis of Equation 3.54 shows that one can establish such a relation by recurrence between any two layers. Therefore, one can write

$$\begin{Bmatrix} \frac{d\varphi_{xx}^{\ell}}{dz} \\ \frac{d\varphi_{xy}^{\ell}}{dz} \\ \frac{d\varphi_{yx}^{\ell}}{dz} \\ \frac{d\varphi_{yy}^{\ell}}{dz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^{\ell} & 0 & \bar{Q}_{45}^{\ell} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell} \\ \bar{Q}_{45}^{\ell} & 0 & \bar{Q}_{44}^{\ell} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell} \end{bmatrix}^{-1} \begin{bmatrix} \bar{Q}_{55}^0 & 0 & \bar{Q}_{45}^0 & 0 \\ 0 & \bar{Q}_{55}^0 & 0 & \bar{Q}_{45}^0 \\ \bar{Q}_{45}^0 & 0 & \bar{Q}_{44}^0 & 0 \\ 0 & \bar{Q}_{45}^0 & 0 & \bar{Q}_{44}^{\ell} \end{bmatrix} \begin{Bmatrix} \frac{d\varphi_{xx}^0}{dz} \\ \frac{d\varphi_{xy}^0}{dz} \\ \frac{d\varphi_{yx}^0}{dz} \\ \frac{d\varphi_{yy}^0}{dz} \end{Bmatrix} \quad (3.55)$$

Therefore, one can write warping functions' derivatives across the thickness of a generic layer ℓ as a function of its material properties and those of the reference layer. Finally, one writes warping functions for a generic layer ℓ as

$$\begin{Bmatrix} \varphi_{xx}^{\ell}(z) \\ \varphi_{xy}^{\ell}(z) \\ \varphi_{yx}^{\ell}(z) \\ \varphi_{yy}^{\ell}(z) \end{Bmatrix} = \begin{Bmatrix} \frac{d\varphi_{xx}^{\ell}}{dz} \\ \frac{d\varphi_{xy}^{\ell}}{dz} \\ \frac{d\varphi_{yx}^{\ell}}{dz} \\ \frac{d\varphi_{yy}^{\ell}}{dz} \end{Bmatrix} z + \begin{Bmatrix} A_0^{\ell} \\ B_0^{\ell} \\ C_0^{\ell} \\ D_0^{\ell} \end{Bmatrix} \quad (3.56)$$

One computes constants A_0^{ℓ} , B_0^{ℓ} , C_0^{ℓ} and D_0^{ℓ} in Equation 3.56 using Equation 3.47. Besides, this theory assumes a FSDT for each layer, and consequently, the reference layer assumes the

warping functions defined in Equation 3.39. Alternatively, assuming that one can define warping functions layerwise, one can establish the warping functions of a generic layer ℓ as

$$\begin{pmatrix} \varphi_{xx}^\ell(z) \\ \varphi_{xy}^\ell(z) \\ \varphi_{yx}^\ell(z) \\ \varphi_{yy}^\ell(z) \end{pmatrix} = \begin{pmatrix} A_0^\ell \\ B_0^\ell \\ C_0^\ell \\ D_0^\ell \end{pmatrix} + \begin{pmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{pmatrix} z \quad (3.57)$$

where

$$\begin{pmatrix} \frac{d\varphi_{xx}^\ell}{dz} \\ \frac{d\varphi_{xy}^\ell}{dz} \\ \frac{d\varphi_{yx}^\ell}{dz} \\ \frac{d\varphi_{yy}^\ell}{dz} \end{pmatrix} = \begin{pmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{pmatrix} \quad (3.58)$$

Therefore, the number of unknowns is 8ℓ , and one can compute a linear system of equations to find them.

Single-Layer

For a single-layer structure, this theory assumes an FSDT, and consequently, the unique layer takes the warping functions defined in Equation 3.39. One can compute it in a matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} A_0^1 \\ A_1^1 \\ C_0^1 \\ C_1^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} B_0^1 \\ B_1^1 \\ D_0^1 \\ D_1^1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (3.59)$$

By solving these two linear systems of equations, one obtains the dependent variables for all warping functions of a single layer structure. Although this is not the only way to group the eight equations that establish the eight dependent variables required, this arrangement is convenient for other configurations, as discussed later in this work. However, it is possible to notice at the outset that the coefficient matrix is the same for both systems of equations and only the independent vector changes.

Multi-Layer

For multi-layer structures, the number of dependent variables increases with the number of layers - eight new unknowns per layer, as reported in Equation 3.57. Therefore, one also requires eight equations per each new layer. One can establish four equations using the displacement continuity at each interface, as expressed in Equations 3.44, leading to the final form of Equations 3.48. Substituting the layerwise warping functions of a generic layer ℓ described in the system of Equations 3.57 in Equations 3.48, one obtains four new equations coupling the four unknowns of a generic layer $\ell + 1$ with those of the preceding layer ℓ .

$$\begin{Bmatrix} A_0^{\ell+1} \\ B_0^{\ell+1} \\ C_0^{\ell+1} \\ D_0^{\ell+1} \end{Bmatrix} + \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} \zeta^\ell = \begin{Bmatrix} A_0^\ell \\ B_0^\ell \\ C_0^\ell \\ D_0^\ell \end{Bmatrix} + \begin{Bmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{Bmatrix} \zeta^\ell \quad (3.60)$$

which leads to the following four equations per each new layer

$$\begin{Bmatrix} A_0^{\ell+1} \\ B_0^{\ell+1} \\ C_0^{\ell+1} \\ D_0^{\ell+1} \end{Bmatrix} - \begin{Bmatrix} A_0^\ell \\ B_0^\ell \\ C_0^\ell \\ D_0^\ell \end{Bmatrix} + \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} \zeta^\ell - \begin{Bmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{Bmatrix} \zeta^\ell = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.61)$$

The remaining four equations for each new layer come from the shear stress continuity assumed at each interface, as established in Equation 3.52 and recalled in the following.

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{bmatrix} A_1^{\ell+1} & B_1^{\ell+1} \\ C_1^{\ell+1} & D_1^{\ell+1} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{55}^\ell & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & \bar{Q}_{44}^\ell \end{bmatrix} \begin{bmatrix} A_1^\ell & B_1^\ell \\ C_1^\ell & D_1^\ell \end{bmatrix} \quad (3.62)$$

One can obtain a particular solution to this system of equations as follows

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell & 0 \\ 0 & \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell & 0 \\ 0 & \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell \end{bmatrix} \begin{Bmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{Bmatrix} \quad (3.63)$$

which leads to the following four equations per each new layer

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} - \begin{bmatrix} \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell & 0 \\ 0 & \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell & 0 \\ 0 & \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell \end{bmatrix} \begin{Bmatrix} A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.64)$$

Thus, Equations 3.61 and 3.64 establish the eight equations required for the warping functions of each new layer. For a two-layer laminate, for example, and assuming the first layer as the reference one, one can establish two linear systems of equations to compute the sixteen dependent variables required to settle the four warping functions as follows

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 1 & -\zeta^1 & \zeta^1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -\zeta^1 & \zeta^1 \\
0 & 0 & -\bar{Q}_{55}^1 & \bar{Q}_{55}^2 & 0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 \\
0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 & 0 & 0 & -\bar{Q}_{44}^1 & \bar{Q}_{44}^2
\end{bmatrix}
\begin{Bmatrix}
A_0^1 \\ A_0^2 \\ A_1^1 \\ A_1^2 \\ C_0^1 \\ C_0^2 \\ C_1^1 \\ C_1^2
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\quad (3.65)$$

and

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 1 & -\zeta^1 & \zeta^1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -\zeta^1 & \zeta^1 \\
0 & 0 & -\bar{Q}_{55}^1 & \bar{Q}_{55}^2 & 0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 \\
0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 & 0 & 0 & -\bar{Q}_{44}^1 & \bar{Q}_{44}^2
\end{bmatrix}
\begin{Bmatrix}
B_0^1 \\ B_0^2 \\ B_1^1 \\ B_1^2 \\ D_0^1 \\ D_0^2 \\ D_1^1 \\ D_1^2
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\quad (3.66)$$

Following and adapting the systems of Equations 3.59 by adding eight new equations, it is possible to notice that, once again, the coefficient matrix is the same for both algebraic systems of equations and only the independent vector changes. For any new layer, it is possible to adapt the preceding systems of equations by adding eight new equations, as described in Equations 3.61 and 3.64.

3.9.5 | ToZZ - Cho-Parmertert-Kim's Model

Cho and Parmertert [1993], and later Kim and Cho [2007], developed a composite plate theory for general lamination configurations obtained by superimposing a cubic varying displacement field on a Zig-Zag linearly variable displacement. Based on the classification presented earlier, one can define it as a multi-layer approach of the hybrid type since the theory has the same number kinematic variables as an FSDT, independent of the number of layers. Similarly to the FoZZ, the displacement satisfies transverse shear stress continuity conditions at the interface between layers, but shear-free surface conditions as well - Loredo and Castel [2014] called it Third-Order Zig-Zag Theory (ToZZ). For a generic layer ℓ , one can write the warping functions based on a ToZZ theory as

$$\begin{Bmatrix}
\varphi_{xx}^\ell(z) \\ \varphi_{xy}^\ell(z) \\ \varphi_{yx}^\ell(z) \\ \varphi_{yy}^\ell(z)
\end{Bmatrix}
=
\begin{Bmatrix}
A_0^\ell \\ B_0^\ell \\ C_0^\ell \\ D_0^\ell
\end{Bmatrix}
+
\begin{Bmatrix}
A_1^\ell \\ B_1^\ell \\ C_1^\ell \\ D_1^\ell
\end{Bmatrix}
z
+
\begin{Bmatrix}
A_2 \\ B_2 \\ C_2 \\ D_2
\end{Bmatrix}
z^2
+
\begin{Bmatrix}
A_3 \\ B_3 \\ C_3 \\ D_3
\end{Bmatrix}
z^3
\quad (3.67)$$

3. Laminated Composite Plate Theory

Therefore, considering four warping functions, the total number of dependent variables is $8\ell + 8$. It is worth noticing that, similarly to the FoZZ, there are eight dependent variables per layer, plus eight new dependent variables for this ToZZ, regardless of the number of layers. The unknown variables dependent on the quadratic and cubic terms of the warping functions do not depend upon the number of layers, and one can compute them with the shear-free surface conditions, as discussed later in this subsection. Similarly to the FoZZ theory, the displacement's continuity at each interface leads to the following continuity

$$\begin{aligned}\varphi_{xx}^{\ell+1} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{xx}^{\ell} \left(z^{\ell} + \frac{h^{\ell}}{2} \right) \\ \varphi_{xy}^{\ell+1} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{xy}^{\ell} \left(z^{\ell} + \frac{h^{\ell}}{2} \right) \\ \varphi_{yx}^{\ell+1} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{yx}^{\ell} \left(z^{\ell} + \frac{h^{\ell}}{2} \right) \\ \varphi_{yy}^{\ell+1} \left(z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \varphi_{yy}^{\ell} \left(z^{\ell} + \frac{h^{\ell}}{2} \right)\end{aligned}\tag{3.68}$$

Substituting Equations 3.67 in the previous equalities, one can write

$$\begin{aligned}\begin{Bmatrix} A_0^{\ell+1} \\ B_0^{\ell+1} \\ C_0^{\ell+1} \\ D_0^{\ell+1} \end{Bmatrix} + \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} \zeta^{\ell} + \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} (\zeta^{\ell})^2 + \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix} (\zeta^{\ell})^3 = \\ \begin{Bmatrix} A_0^{\ell} \\ B_0^{\ell} \\ C_0^{\ell} \\ D_0^{\ell} \end{Bmatrix} + \begin{Bmatrix} A_1^{\ell} \\ B_1^{\ell} \\ C_1^{\ell} \\ D_1^{\ell} \end{Bmatrix} \zeta^{\ell} + \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} (\zeta^{\ell})^2 + \begin{Bmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{Bmatrix} (\zeta^{\ell})^3\end{aligned}\tag{3.69}$$

which leads to the same four equations per each new layer as the FoZZ, written in the form

$$\begin{Bmatrix} A_0^{\ell+1} \\ B_0^{\ell+1} \\ C_0^{\ell+1} \\ D_0^{\ell+1} \end{Bmatrix} - \begin{Bmatrix} A_0^{\ell} \\ B_0^{\ell} \\ C_0^{\ell} \\ D_0^{\ell} \end{Bmatrix} + \begin{Bmatrix} A_1^{\ell+1} \\ B_1^{\ell+1} \\ C_1^{\ell+1} \\ D_1^{\ell+1} \end{Bmatrix} \zeta^{\ell} - \begin{Bmatrix} A_1^{\ell} \\ B_1^{\ell} \\ C_1^{\ell} \\ D_1^{\ell} \end{Bmatrix} \zeta^{\ell} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\tag{3.70}$$

Besides, similarly to the FoZZ, ToZZ also ensures the continuity of the transverse shear stresses at each interface

$$\begin{aligned}\tau_{xz} \left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \tau_{xz} \left(x, y, z^{\ell} + \frac{h^{\ell}}{2} \right) \\ \tau_{yz} \left(x, y, z^{\ell+1} - \frac{h^{\ell+1}}{2} \right) &= \tau_{yz} \left(x, y, z^{\ell} + \frac{h^{\ell}}{2} \right)\end{aligned}\tag{3.71}$$

which, recalling Equations 3.51 from the FoZZ, leads to the following

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^{\ell+1} & 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^{\ell+1} \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^{\ell+1} & 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^{\ell+1} \end{bmatrix} = \\ \begin{bmatrix} \bar{Q}_{55}^\ell & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & \bar{Q}_{44}^\ell \end{bmatrix} \begin{bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^\ell & 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^\ell \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^\ell & 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^\ell \end{bmatrix} \quad (3.72)$$

One can obtain a particular solution to this system of equations as follows

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^{\ell+1} \\ 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^{\ell+1} \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^{\ell+1} \\ 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^{\ell+1} \end{Bmatrix} = \\ \begin{bmatrix} \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell & 0 \\ 0 & \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell & 0 \\ 0 & \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^\ell \\ 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^\ell \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^\ell \\ 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^\ell \end{Bmatrix} \quad (3.73)$$

which leads to the following four equations per each new layer

$$\begin{bmatrix} \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} & 0 \\ 0 & \bar{Q}_{55}^{\ell+1} & 0 & \bar{Q}_{45}^{\ell+1} \\ \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} & 0 \\ 0 & \bar{Q}_{45}^{\ell+1} & 0 & \bar{Q}_{44}^{\ell+1} \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^{\ell+1} \\ 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^{\ell+1} \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^{\ell+1} \\ 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^{\ell+1} \end{Bmatrix} - \\ \begin{bmatrix} \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell & 0 \\ 0 & \bar{Q}_{55}^\ell & 0 & \bar{Q}_{45}^\ell \\ \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell & 0 \\ 0 & \bar{Q}_{45}^\ell & 0 & \bar{Q}_{44}^\ell \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^\ell)^2 + 2A_2 \zeta^\ell + A_1^\ell \\ 3B_3 (\zeta^\ell)^2 + 2B_2 \zeta^\ell + B_1^\ell \\ 3C_3 (\zeta^\ell)^2 + 2C_2 \zeta^\ell + C_1^\ell \\ 3D_3 (\zeta^\ell)^2 + 2D_2 \zeta^\ell + D_1^\ell \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.74)$$

In addition to the transverse shear stress continuity conditions at the interface between layers, ToZZ also ensures shear-free surface conditions. Thus

$$\begin{aligned} \tau_{xz} \left(x, y, z^1 - \frac{h^1}{2} \right) &= \tau_{yz} \left(x, y, z^1 - \frac{h^1}{2} \right) = 0 \\ \tau_{xz} \left(x, y, z^n + \frac{h^n}{2} \right) &= \tau_{yz} \left(x, y, z^n + \frac{h^n}{2} \right) = 0 \end{aligned} \quad (3.75)$$

Recalling Equations 3.49, 3.50, 3.51 and 3.53, it is possible to establish eight new equations for determining the eight new dependent variables of the ToZZ

$$\begin{bmatrix} \bar{Q}_{55}^1 & 0 & \bar{Q}_{45}^1 & 0 \\ 0 & \bar{Q}_{55}^1 & 0 & \bar{Q}_{45}^1 \\ \bar{Q}_{45}^1 & 0 & \bar{Q}_{44}^1 & 0 \\ 0 & \bar{Q}_{45}^1 & 0 & \bar{Q}_{44}^1 \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^0)^2 + 2A_2 \zeta^0 + A_1^1 \\ 3B_3 (\zeta^0)^2 + 2B_2 \zeta^0 + B_1^1 \\ 3C_3 (\zeta^0)^2 + 2C_2 \zeta^0 + C_1^1 \\ 3D_3 (\zeta^0)^2 + 2D_2 \zeta^0 + D_1^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.76)$$

and

$$\begin{bmatrix} \bar{Q}_{55}^n & 0 & \bar{Q}_{45}^n & 0 \\ 0 & \bar{Q}_{55}^n & 0 & \bar{Q}_{45}^n \\ \bar{Q}_{45}^n & 0 & \bar{Q}_{44}^n & 0 \\ 0 & \bar{Q}_{45}^n & 0 & \bar{Q}_{44}^n \end{bmatrix} \begin{Bmatrix} 3A_3 (\zeta^n)^2 + 2A_2 \zeta^n + A_1^n \\ 3B_3 (\zeta^n)^2 + 2B_2 \zeta^n + B_1^n \\ 3C_3 (\zeta^n)^2 + 2C_2 \zeta^n + C_1^n \\ 3D_3 (\zeta^n)^2 + 2D_2 \zeta^n + D_1^n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.77)$$

which leads to

$$\begin{Bmatrix} 3A_3 (\zeta^0)^2 + 2A_2 \zeta^0 + A_1^1 \\ 3B_3 (\zeta^0)^2 + 2B_2 \zeta^0 + B_1^1 \\ 3C_3 (\zeta^0)^2 + 2C_2 \zeta^0 + C_1^1 \\ 3D_3 (\zeta^0)^2 + 2D_2 \zeta^0 + D_1^1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}; \quad \begin{Bmatrix} 3A_3 (\zeta^n)^2 + 2A_2 \zeta^n + A_1^n \\ 3B_3 (\zeta^n)^2 + 2B_2 \zeta^n + B_1^n \\ 3C_3 (\zeta^n)^2 + 2C_2 \zeta^n + C_1^n \\ 3D_3 (\zeta^n)^2 + 2D_2 \zeta^n + D_1^n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.78)$$

Therefore, Equations 3.70, 3.74 and 3.78 establish the $8\ell + 8$ required conditions to compute a linear system of equations to find the dependent variables and write the warping functions.

Single-Layer

For a single-layer structure, this theory assumes a TSDT, and consequently, the unique layer takes the warping functions defined in Equation 3.40. One can compute it in a matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\zeta^0 & 3(\zeta^0)^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\zeta^1 & 3(\zeta^1)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\zeta^0 & 3(\zeta^0)^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\zeta^1 & 3(\zeta^1)^2 \end{bmatrix} \begin{Bmatrix} A_0^1 \\ A_1^1 \\ A_2 \\ A_3 \\ C_0^1 \\ C_1^1 \\ C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.79)$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\zeta^0 & 3(\zeta^0)^2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2\zeta^1 & 3(\zeta^1)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\zeta^0 & 3(\zeta^0)^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\zeta^1 & 3(\zeta^1)^2 \end{bmatrix} \begin{Bmatrix} B_0^1 \\ B_1^1 \\ B_2 \\ B_3 \\ D_0^1 \\ D_1^1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \quad (3.80)$$

Once again, by solving these two linear systems of equations, one obtains the dependent variables for all warping functions of a single layer structure. This way, it is possible to notice that the coefficient matrix is the same for both systems of equations and only the independent vector changes.

Multi-Layer

For multi-layer structures, and similarly to the FoZZ, the number of dependent variables increases with the number of layers - eight new unknowns per layer, as reported in Equation 3.67. Therefore, for each new layer, one requires Equations 3.70 and 3.74 to establish the displacement and shear stress continuity at each interface to couple the eight new unknowns of a generic layer $\ell + 1$ with those of the preceding layer ℓ . For a two-layer laminate, for example, and assuming the first layer as the reference one, one can establish two linear systems of equations to compute the twenty-four unknowns required to settle the four warping functions as follows

$$\Upsilon \{A_0^1, A_0^2, A_1^1, A_1^2, A_2, A_3, C_0^1, C_0^2, C_1^2, C_2, C_3\}^T = \{0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^T \quad (3.81)$$

and

$$\Upsilon \{B_0^1, B_0^2, B_1^1, B_1^2, B_2, B_3, D_0^1, D_0^2, D_1^2, D_2, D_3\}^T = \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0\}^T \quad (3.82)$$

with

$$\Upsilon = [\Upsilon_1, \Upsilon_2] \quad (3.83)$$

and

$$\Upsilon_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2\zeta^0 & 3(\zeta^0)^2 \\ 0 & 0 & 0 & 1 & 2\zeta^2 & 3(\zeta^2)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -\zeta^1 & \zeta^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{Q}_{55}^1 & \bar{Q}_{55}^2 & 2\zeta^1 \bar{Q}_{55}^{21} & 3(\zeta^1)^2 \bar{Q}_{55}^{21} \\ 0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 & 2\zeta^1 \bar{Q}_{45}^{21} & 3(\zeta^1)^2 \bar{Q}_{45}^{21} \end{bmatrix} \quad (3.84)$$

and

$$\Upsilon_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2\zeta^0 & 3(\zeta^0)^2 \\ 0 & 0 & 0 & 1 & 2\zeta^2 & 3(\zeta^2)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -\zeta^1 & \zeta^1 & 0 & 0 \\ 0 & 0 & -\bar{Q}_{45}^1 & \bar{Q}_{45}^2 & 2\zeta^1 \bar{Q}_{45}^{21} & 3(\zeta^1)^2 \bar{Q}_{45}^{21} \\ 0 & 0 & -\bar{Q}_{44}^1 & \bar{Q}_{44}^2 & 2\zeta^1 \bar{Q}_{44}^{21} & 3(\zeta^1)^2 \bar{Q}_{44}^{21} \end{bmatrix} \quad (3.85)$$

where

$$\bar{Q}_{55}^{21} = \bar{Q}_{55}^2 - \bar{Q}_{55}^1, \quad \bar{Q}_{45}^{21} = \bar{Q}_{45}^2 - \bar{Q}_{45}^1 \quad \text{and} \quad \bar{Q}_{44}^{21} = \bar{Q}_{44}^2 - \bar{Q}_{44}^1$$

Following and adapting the systems of Equations 3.79 and 3.80 by adding eight new equations, it is possible to notice that, once again, the coefficient matrix is the same for both algebraic systems of equations and only the independent vector changes. For any new layer, it is possible to adapt the preceding systems of equations by adding eight new equations, as described in Equations 3.70 and 3.74.

Numerical Methods

4.1 | Introduction

The essential work of an engineer is to represent the physical world with a physical model and transcribe it into mathematical language. One calls the analytical description of physical phenomena a mathematical model, which one develops using assumptions. In structural mechanics, for example, these assumptions may concern how the structure behaves under given circumstances using appropriate axioms or laws governing the deformable bodies. Therefore, it is possible to describe a physical model in terms of algebraic, differential, or integral equations based upon the laws of physics and relating all the involved phenomena. This approach has been the foundation for virtually any phenomenon in physics since Newton, particularly in mechanics.

With the advent of artificial intelligence, particularly machine learning, researchers have been dismissing this classical approach involving a physical model written in mathematical language in terms of algebraic, differential, or integral equations and, recurring to the explosive growth in computing resources, use data-driven techniques, enhancing the numerical modelling in computational mechanics with success across diverse scientific fields, as complex fluid dynamics simulations, for example. Nevertheless, for complex physical systems, the data acquisition cost is prohibitive, and one inevitably faces the challenge of drawing conclusions and making decisions with partial information [Raissi et al., 2017]. The vast majority of state of the art machine learning techniques lack robustness and fail to provide guarantees of convergence with limited data. However, more recently, the emerging “physics-informed neural network” appeared to be a capable and promising simulation approach trained to solve supervised learning tasks while respecting any given law of physics described by general nonlinear partial differential equations [Raissi et al., 2017]. Consequently, to represent the physical world with a physical model and transcribe it into mathematical language is fundamental, even for the most advanced algorithms.

In this work, one follows the classical approach of representing a physical problem with a physical model and transcribing it into mathematical language. The laminated composite plate theory presented in the previous chapter is a physical model written in mathematical language, a

mathematical model involving partial differential equations. As commonly known, exact solutions for plate structures are only possible for some particular cases, as for plates with all edges simply supported with Navier’s method or with a pair of opposite simply supported edges with Levy’s solution Timoshenko and Woinowsky-Krieger [1959]. Therefore, general procedures to model plate-like structures considering different geometries and boundary conditions are of great importance. As Bertrand Russell put it, “*although this may seem a paradox, all exact science is dominated by the idea of approximation*” [Russell, 2001], and one needs a numerical method to solve the partial differential equations that describe the physical model presented in this work.

The extraordinary development of computers in recent decades has made possible the evolution and generalised application of numerical methods and so the solution of numerous hitherto unsolvable problems and allowed the development of new formulations and new solution algorithms for many others. Invariably, these methods require a discretisation of the continuous model, representing the unknown function or functions from a finite set of parameters to be determined. Thus, one transforms a formulation in differential equations to the solution of equations or systems of algebraic equations, linear or nonlinear, for the resolution of those parameters. In some cases, one obtains approximate values of the solution at some points of the domain, and in other cases, one seeks approximate functions of the solution. This growing and vast body of knowledge involving the development of mathematical models and the corresponding numerical simulations is commonly known as computational mechanics [Reddy, 2006].

Of all these methods, the Finite Difference Method (FDM), the FEM and the Finite Volume Method (FVM) can be considered the most important due to their application and widespread use. The FDM is still the preferred method in fields mainly related to fluid mechanics or even associated with the FEM for integration in the time variable. Preferably, the FVM also finds interest in applications in computational fluid mechanics. There are also other methods, such as the Boundary Element Method (BEM), a numerical computational technique of solving linear partial differential equations formulated as integral equations, or, more recently, powerful mesh-free methods. While the discretisation in the previous methods requires some underlying computational mesh, which could become rather difficult for some applications or problems in three dimensions, mesh-free methods overcome some of these difficulties. But the FEM is undoubtedly the most powerful and versatile, reason why it has concentrated in the last decades the attention of most of the scientific community that develops its activity in computational mechanics, and so it is the case of the present work.

In the case of vibration analysis of regular structures, the RRM is one of the most important and used numerical methods. It is based on minimal, general assumptions and produces optimal solutions regarding the approximation properties of the underlying trial spaces [Yserentant, 2013]. The RRM is a variational method to solve the eigenvalue problem for elliptic differential operators, that is, to compute their eigenvalues and the corresponding eigenfunctions [Yserentant, 2013], enabling one to reduce an infinite number of degrees of freedom of a system into a finite number. The method relies on the approximation of the possible deformation shapes of the system, following the basic idea beyond Rayleigh’s principle.

In this work, one presents the formulation of the RRM applied to the laminated composite plate theory presented in the previous chapter, similarly to the work developed by Castel [2013]. As previously mentioned, the RRM produces optimal solutions relying on the approximations of the possible deformation shapes of the system - Figure 4.1a. However, despite all the robustness of the method, its use is constrained to applications with regular beams, plates or shells once one defines the trial functions in the structure domain, as represented in Figure 4.1a. To overcome this issue, one resorts to the FEM to, for the first time, generalise the application of the laminated plate theory presented in this work to structures with complex geometries, including shells. The FEM is a technique for solving partial differential equations by first discretising these equations

in their space dimensions. Contrary to the RRM, the discretisation is carried out locally over small regions of simple but arbitrary shape (the finite elements) - Figure 4.1b. The result is a system of algebraic equations relating the input at specified points in the elements (the nodes) to the output of these same points [Smith et al., 2013].

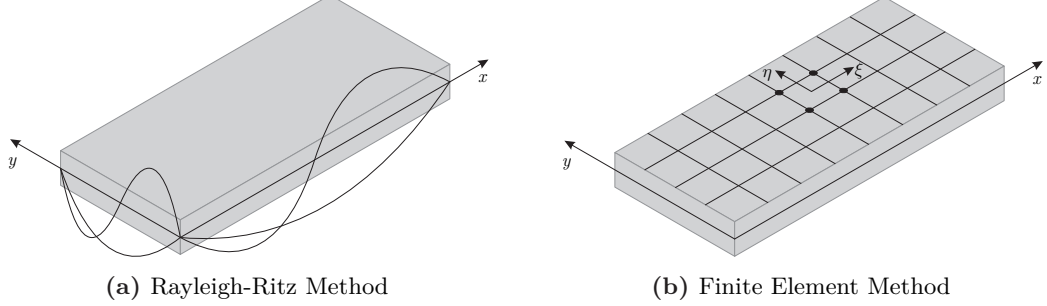


Figure 4.1 Numerical methods and their discretisation concepts

4.2 | Rayleigh-Ritz Method

The RRM has been widely used to study problems involving plate-like structures due to its conceptual simplicity, flexibility, high reliability and computational efficiency [Dozio, 2011]. It is conceptually simple once one can approximate any unknown variable by a linear combination of assumed functions, also called admissible functions, trial functions or basis functions ϕ_m parameterised by unknown coefficients δ_m

$$\Phi \simeq \sum_{m=1}^M \delta_m \phi_m \quad (4.1)$$

One can obtain the δ_m coefficients from the functional minimisation problem, stated as follows

$$\frac{\partial \mathcal{F}(\Phi)}{\partial \delta_m} = 0 \quad (4.2)$$

Linear independence of the mathematically complete set is crucial to achieving convergence. Nevertheless, admissible functions are not required to satisfy the natural boundary conditions. The function set is the core of the RRM since it critically affects computational efficiency and the numerical stability of the solution.

4.2.1 | Energy Formulation

Based on the principle of energy conservation, the RRM approximates an unknown variable, as defined in Equation 4.1, by a functional minimisation which needs to be specified, as reported in Equation 4.2. The functional under analysis here is the following energy potential

$$\mathcal{F}(\mathbf{U}) = \int_S (\mathcal{L} + W^{\text{ext}}) dS \quad (4.3)$$

where the Lagrangian $\mathcal{L}$ represents kinetic and potential energy surface densities

$$\mathcal{F}(\mathbf{U}) = \int_S (\Pi^K(x, y) - \Pi^U(x, y) + W^{\text{ext}}) dS \quad (4.4)$$

One derived the potential and kinetic energy surface densities in Chapter 3, resulting in Equations 3.18 and 3.30, respectively. Assuming that one approximates generalised displacements by admissible functions, one applies Lagrange's equation to minimise the mechanical energy

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{U}_i} \right) - \frac{\partial \mathcal{L}}{\partial U_i} = 0 \quad (4.5)$$

where U_i are the components of the unknown vector $\mathbf{U}$, or more comprehensively, the δ_m coefficients presented in Equation 4.1

$$\mathbf{U} = \{u^{11}(t), v^{11}(t), w^{11}(t), \gamma_{xz}^{11}(t), \gamma_{yz}^{11}(t), u^{12}(t), \dots, \gamma_{yz}^{mn}(t)\}^T \quad (4.6)$$

Considering that kinetic energy only depends on velocities and potential energy is a function of generalised displacements, Lagrange's equation can be re-written as follows

$$\frac{d}{dt} \left(\frac{\partial \Pi^K(x, y)}{\partial \dot{U}_i} \right) - \frac{\partial \Pi^U(x, y)}{\partial U_i} = 0 \quad (4.7)$$

Therefore, the kinetic energy can be written introducing a mass matrix

$$\Pi^K = \frac{1}{2} \dot{\mathbf{U}}^T \mathbf{M} \dot{\mathbf{U}} \quad (4.8)$$

Similarly, the same applies for potential energy written as a stiffness matrix function

$$\Pi^U = \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} \quad (4.9)$$

4.2.2 | Basis Functions

Beslin and Nicolas [1997] presented a hierarchical function set to predict the flexural motion of plate-like structures in the medium frequency range. The proposed trigonometric set proved to be more efficient than the classical hierarchical polynomials, being able to be used at very high orders with a low number of basic operations. Besides, this trigonometric set presented a better convergence rate than polynomial sets when predicting high-order modes with any boundary conditions. Working within the framework of the p-version of the FEM, Beslin and Nicolas [1997] proposed the following set

$$\phi_m(\xi) = \sin(a_m \xi + b_m) \sin(c_m \xi + d_m) \quad (4.10)$$

where m stands for the function order in ξ -direction. The main advantage of such a set is its extent regarding boundary conditions. Coefficients a_m , b_m , c_m and d_m , listed in Table 4.1, are defined in a way that enables the appropriate use of any classical boundary condition.

Figure 4.2 presents the first eight hierarchical trigonometric functions. The first four functions let one choose any classical boundary conditions with a proper combination of admissible functions. For example, $\phi_1(\xi)$ allows free deflection at the boundary $\xi = -1$; consequently, if one requires no displacement for that boundary, that function must be excluded from the set. Similarly, $\phi_2(\xi)$ enables rotation at $\xi = -1$, and if, on the one hand, is essential for a simply-support case, one must discard it for a clamped one. Moreover, from the 5th to the m^{th} function, only the order is upgraded since they do not influence boundary conditions. One must consider the entire set to study a rectangular plate with all edges free. One must define a similar function set for the η -direction, and one writes each degree of freedom as

Table 4.1 Hierarchical trigonometric set proposed by Beslin and Nicolas [1997]

Order	Coefficients			
	a_m	b_m	c_m	d_m
1	$\pi/4$	$3\pi/4$	$\pi/4$	$3\pi/4$
2	$\pi/4$	$3\pi/4$	$-\pi/2$	$-3\pi/2$
3	$\pi/4$	$-3\pi/4$	$\pi/4$	$-3\pi/4$
4	$\pi/4$	$-3\pi/4$	$\pi/2$	$-3\pi/2$
$m > 4$	$\pi/2(m-4)$	$\pi/2(m-4)$	$\pi/2$	$\pi/2$

$$\Phi(\xi, \eta) \simeq \sum_{m=1}^M \sum_{n=1}^N \delta_{mn} \phi_m(\xi) \phi_n(\eta) \quad (4.11)$$

Dozio [2011] evaluated the numerical stability of Beslin and Nicolas [1997] set considering a classical Kirchhoff plate theory and confirmed its excellent stability for medium-to-high frequency prediction. Given such promising results, Dozio [2011] proposed the association of this set with advanced deformation theories to predict medium-to-high frequencies.

4.2.3 | Numerical Implementation

The numerical implementation of the mass and stiffness matrices is of utmost importance to achieve the required computational efficiency in a formulation of this kind. In this work, one extends Dozio [2011] systematic implementation of a numerical procedure to compute mass and stiffness matrices for a Kirchhoff plate theory. Only considering bending stiffnesses, one writes the corresponding potential strain energy as

$$\begin{aligned} \Pi_b^U = \frac{1}{2} \int_S \left[D_{11} (\kappa_{xx}^0)^2 + 2D_{12} \kappa_{xx}^0 \kappa_{yy}^0 + D_{22} (\kappa_{yy}^0)^2 \right. \\ \left. + 4D_{16} \kappa_{xx}^0 \kappa_{xy}^0 + 4D_{26} \kappa_{yy}^0 \kappa_{xy}^0 + 4D_{66} (\kappa_{xy}^0)^2 \right] dS \end{aligned} \quad (4.12)$$

where D_{ij} are the bending stiffness components of matrix $\mathbf{D}$, defined in Equation 3.23. Substituting the expressions for curvatures defined in Equation 3.7

$$\begin{aligned} \Pi_b^U = \frac{1}{2} \int_S \left[D_{11} \left(\frac{\partial^2 w^0}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w^0}{\partial x^2} \frac{\partial^2 w^0}{\partial y^2} + D_{22} \left(\frac{\partial^2 w^0}{\partial y^2} \right)^2 \right. \\ \left. + 4D_{16} \frac{\partial^2 w^0}{\partial x^2} \frac{\partial^2 w^0}{\partial x \partial y} + 4D_{26} \frac{\partial^2 w^0}{\partial y^2} \frac{\partial^2 w^0}{\partial x \partial y} + 4D_{66} \left(\frac{\partial^2 w^0}{\partial x \partial y} \right)^2 \right] dS \end{aligned} \quad (4.13)$$

Similarly, one also considers the transverse velocities to write the corresponding part of the kinetic energy

$$\Pi_b^K = \frac{1}{2} \int_S \left[R (\dot{w}^0)^2 + T \left(\frac{\partial \dot{w}^0}{\partial x} \frac{\partial \dot{w}^0}{\partial x} + \frac{\partial \dot{w}^0}{\partial y} \frac{\partial \dot{w}^0}{\partial y} \right) \right] dS \quad (4.14)$$

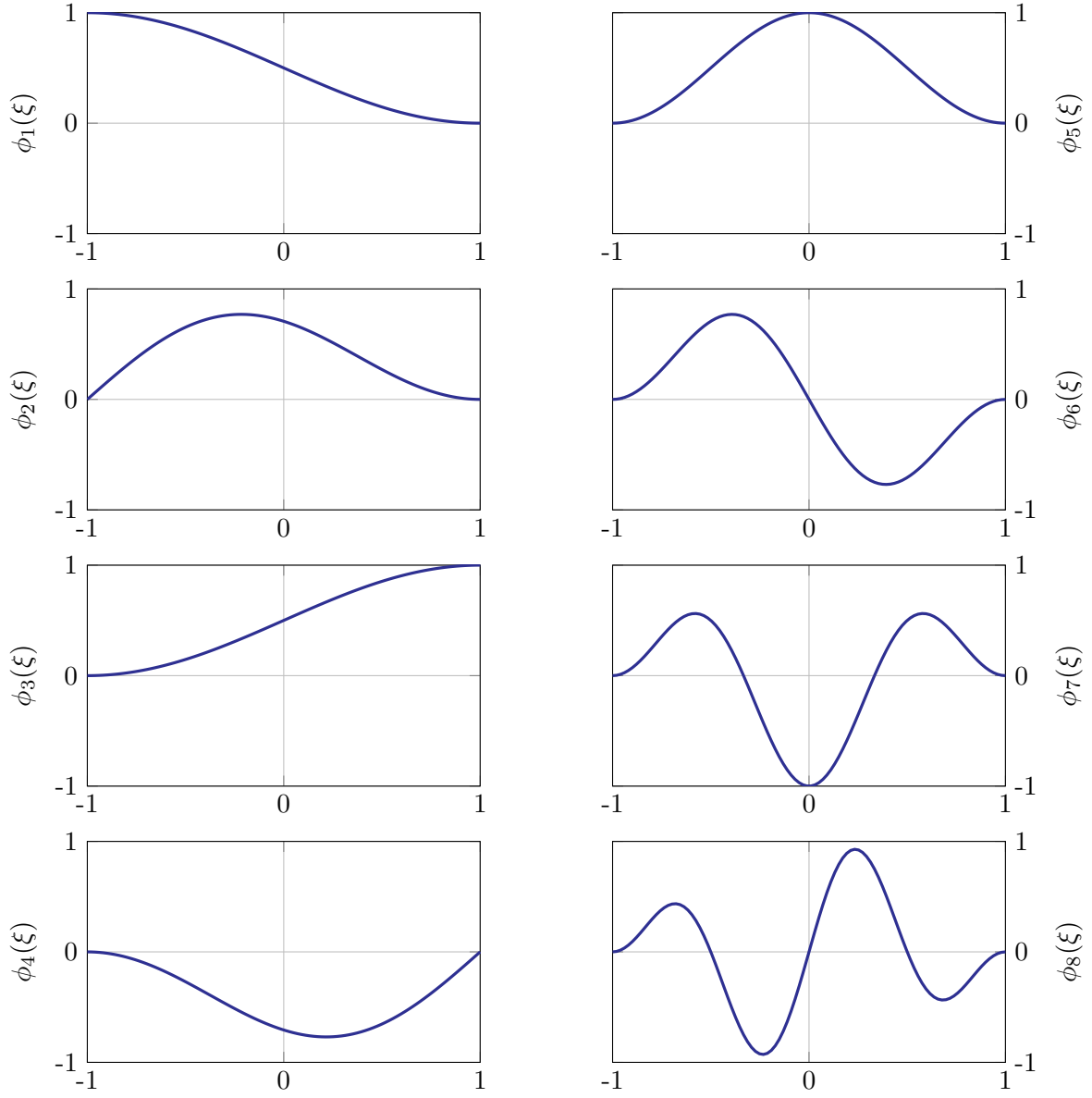


Figure 4.2 Hierarchical trigonometric set proposed by Beslin and Nicolas [1997]

Taking the following non-dimensional coordinates

$$\xi = \frac{2x}{a}; \quad \eta = \frac{2y}{b} \quad (4.15)$$

where a and b are the plate side lengths, the potential bending strain energy can be re-written as follows

$$\begin{aligned} \Pi_b^U = & \frac{1}{2} \frac{4b}{a^3} \int_{-1}^1 \int_{-1}^1 \left[D_{11} \left(\frac{\partial^2 w^0}{\partial \xi^2} \right)^2 + 2D_{12} \left(\frac{a}{b} \right)^2 \frac{\partial^2 w^0}{\partial \xi^2} \frac{\partial^2 w^0}{\partial \eta^2} + D_{22} \left(\frac{a}{b} \right)^4 \left(\frac{\partial^2 w^0}{\partial \eta^2} \right)^2 \right. \\ & \left. + 4D_{16} \left(\frac{a}{b} \right) \frac{\partial^2 w^0}{\partial \xi^2} \frac{\partial^2 w^0}{\partial \xi \partial \eta} + 4D_{26} \left(\frac{a}{b} \right)^3 \frac{\partial^2 w^0}{\partial \eta^2} \frac{\partial^2 w^0}{\partial \xi \partial \eta} + 4D_{66} \left(\frac{a}{b} \right)^2 \left(\frac{\partial^2 w^0}{\partial \xi \partial \eta} \right)^2 \right] d\xi d\eta \quad (4.16) \end{aligned}$$

Furthermore, kinetic energy is given by

$$\Pi_b^K = \frac{1}{2} \frac{ab}{4} \int_{-1}^1 \int_{-1}^1 \left[R (\dot{w}^0)^2 + 4T \left(\frac{b}{a} \right) \frac{\partial \dot{w}^0}{\partial \xi} \frac{\partial \dot{w}^0}{\partial \xi} + 4T \left(\frac{a}{b} \right) \frac{\partial \dot{w}^0}{\partial \eta} \frac{\partial \dot{w}^0}{\partial \eta} \right] d\xi d\eta \quad (4.17)$$

Following Equation 4.11, one approximates the transverse displacement by the following summation

$$w^0(\xi, \eta, t) \simeq \sum_{m=1}^M \sum_{n=1}^N w_{mn}(t) \phi_m(\xi) \phi_n(\eta) \quad (4.18)$$

Applying Lagrange's equation to both potentials and assuming a harmonically vibrating plate with amplitude $w(\xi, \eta)$ and frequency ω yields the following eigenvalue equation

$$\sum_{r=1}^M \sum_{s=1}^N [\mathbf{K}_{mnrs}^b - \omega^2 \mathbf{M}_{mnrs}^b] w_{rs} \text{ with } (m = 1, \dots, M; n = 1, \dots, N) \quad (4.19)$$

One obtains the stiffness matrix with the following computation

$$\begin{aligned} \mathbf{K}_{mnrs}^b = \frac{4b}{a^3} & \left[D_{11} I_{mr}^{22} I_{ns}^{00} + 2D_{12} \left(\frac{a}{b} \right)^2 (I_{mr}^{20} I_{ns}^{02} + I_{mr}^{02} I_{ns}^{20}) + D_{22} \left(\frac{a}{b} \right)^4 I_{mr}^{00} I_{ns}^{22} \right. \\ & \left. + 4D_{16} \left(\frac{a}{b} \right) I_{mr}^{21} I_{ns}^{01} + 4D_{26} \left(\frac{a}{b} \right)^3 I_{mr}^{01} I_{ns}^{21} + 4D_{66} \left(\frac{a}{b} \right)^2 I_{mr}^{11} I_{ns}^{11} \right] \end{aligned} \quad (4.20)$$

Correspondingly, one obtains the mass matrix with an identical procedure

$$\mathbf{M}_{mnrs}^b = \frac{ab}{4} \left[R I_{mr}^{00} I_{ns}^{00} + 4T \left(\frac{b}{a} \right) I_{mr}^{11} I_{ns}^{00} + 4T \left(\frac{a}{b} \right) I_{mr}^{00} I_{ns}^{11} \right]$$

One must emphasise that one only considers terms involving transverse displacement (not including couplings) to illustrate the numerical procedure. One identifies matrices $\mathbf{K}$ and $\mathbf{M}$ with the superscript b , a label for bending, associated with the transverse displacement w^0 . Following Dozio [2011] notation, I_{mr} and I_{ns} are square matrices defined by the following integrals

$$I_{mr}^{\alpha\beta} = \int_{-1}^{+1} \frac{d^\alpha \phi_m}{d\xi^\alpha} \frac{d^\beta \phi_r}{d\xi^\beta} d\xi \quad (4.21)$$

$$I_{ns}^{\alpha\beta} = \int_{-1}^{+1} \frac{d^\alpha \phi_n}{d\eta^\alpha} \frac{d^\beta \phi_s}{d\eta^\beta} d\eta \quad (4.22)$$

where α and β represent the derivative order. One of the most important advantages of the trigonometric set used in this work is made clear at this stage. Dozio [2011] showed that the integrals defined in Equations 4.21 and 4.22, involving four trigonometric functions, could be easily reduced to simple algebraic operations which do not increase with the order. In fact, most of the time, such integrals assume null values leading to regularly structured sparse matrices consisting of only three diagonals for $m, r > 4$ or $n, s > 4$.

4.3 | Finite Element Method

The FEM is undoubtedly the most used numerical method by the scientific community that develops its activity in computational mechanics due to its power and versatility. Contrary to the RRM, which can only deal with regular shaped structures, the FEM is appropriate to model structures with complex geometries by discretising the entire domain locally with the finite elements but also offers different possibilities regarding the boundary conditions. Besides, its element-based approach can also bring advantages when dealing with PCLD patches with complex geometries.

4.3.1 | Displacement Field

The displacement field defined in Equation 5.15 can be re-written in a matrix form as

$$\mathbf{u} = \hat{\mathcal{N}} \mathbf{d} \quad (4.23)$$

where vector $\mathbf{d}$ can be defined as

$$\mathbf{d} = \{u^0, v^0, w^0, \gamma_{xz}^0, \gamma_{yz}^0\}^T \quad (4.24)$$

and represents the generalised displacements of the reference plane. Matrix $\hat{\mathcal{N}}$ takes the following form

$$\hat{\mathcal{N}} = \begin{bmatrix} 1 & 0 & -z \frac{\partial}{\partial x} & \varphi_{xx}(z) & \varphi_{xy}(z) \\ 0 & 1 & -z \frac{\partial}{\partial y} & \varphi_{yx}(z) & \varphi_{yy}(z) \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (4.25)$$

and establishes the link between the generalised displacements of the reference plane and those of an arbitrary layer.

4.3.2 | Strain Matrix

The strain field defined in Equation 3.5 can also take a matrix form applying a differential operator

$$\boldsymbol{\varepsilon} = \mathcal{L} \mathbf{u} \quad (4.26)$$

where $\mathcal{L}$ can be defined as following

$$\mathcal{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}^T \quad (4.27)$$

Equation 4.26 can be re-arranged using Equation 4.23, where the displacement field $\mathbf{u}$ is defined as a function of the generalised displacements

$$\boldsymbol{\varepsilon} = \hat{\mathcal{B}} \mathbf{d} \quad (4.28)$$

Matrix $\hat{\mathcal{B}}$ is a strain matrix defined as

$$\hat{\mathcal{B}} = \mathcal{L} \hat{\mathcal{N}} \quad (4.29)$$

For the sake of clarity, one can divide the strain field defined in Equation 4.28 defining membrane, bending, higher-order bending and shear strains, respectively

$$\boldsymbol{\varepsilon} = \{\boldsymbol{\varepsilon}_m, \boldsymbol{\varepsilon}_b, \boldsymbol{\varepsilon}_{sd}, \boldsymbol{\varepsilon}_s\}^T \quad (4.30)$$

which consequently leads to four different strain matrices

$$\hat{\mathcal{B}} = [\hat{\mathcal{B}}_m, \hat{\mathcal{B}}_b, \hat{\mathcal{B}}_{sd}, \hat{\mathcal{B}}_s]^T \quad (4.31)$$

One can compute the multiple strain matrices as follows

$$\hat{\mathcal{B}}_m = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix}; \hat{\mathcal{B}}_b = \begin{bmatrix} 0 & 0 & -z \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & -z \frac{\partial^2}{\partial y^2} & 0 & 0 \\ 0 & 0 & -2z \frac{\partial^2}{\partial x \partial y} & 0 & 0 \end{bmatrix} \quad (4.32)$$

$$\hat{\mathcal{B}}_{sd} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} \varphi_{xx} & \frac{\partial}{\partial x} \varphi_{xy} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} \varphi_{yx} & \frac{\partial}{\partial y} \varphi_{yy} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} \varphi_{xx} + \frac{\partial}{\partial x} \varphi_{yx} & \frac{\partial}{\partial y} \varphi_{xy} + \frac{\partial}{\partial x} \varphi_{yy} \end{bmatrix} \quad (4.33)$$

$$\hat{\mathcal{B}}_s = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial \varphi_{xx}}{\partial z} & \frac{\partial \varphi_{xy}}{\partial z} \\ 0 & 0 & 0 & \frac{\partial \varphi_{yx}}{\partial z} & \frac{\partial \varphi_{yy}}{\partial z} \end{bmatrix} \quad (4.34)$$

4.3.3 | Strain Energy

The total potential energy driven by the stress field of the existing structure takes the form

$$\Pi^U = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \mathcal{D} \boldsymbol{\varepsilon} d\Omega \quad (4.35)$$

Equation 4.28 defines the strain field $\boldsymbol{\varepsilon}$. Additionally, $\mathcal{D}$ is a partitioned elasticity matrix defined as follows

$$\mathcal{D} = \begin{bmatrix} \bar{\mathbf{Q}}_{1,2,6} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ & \bar{\mathbf{Q}}_{1,2,6} & \mathbf{0} & \mathbf{0} \\ & & \bar{\mathbf{Q}}_{1,2,6} & \mathbf{0} \\ \text{sym.} & & & \bar{\mathbf{Q}}_{5,4} \end{bmatrix} \quad (4.36)$$

Substituting Equation 4.28 in the previous equation one can write

$$\Pi^U = \frac{1}{2} \mathbf{d}^T \left(\int_{\Omega} \hat{\mathbf{B}}^T \mathcal{D} \hat{\mathbf{B}} d\Omega \right) \mathbf{d} \quad (4.37)$$

which one can reduce to the straightforward expression

$$\Pi^U = \frac{1}{2} \mathbf{d}^T \mathcal{K} \mathbf{d} \quad (4.38)$$

Matrix $\mathcal{K}$ represents the well-known stiffness matrix

$$\mathcal{K} = \int_{\Omega} \hat{\mathbf{B}}^T \mathcal{D} \hat{\mathbf{B}} d\Omega \quad (4.39)$$

Performing a thickness integration and recalling the definition presented in Equation 3.22, one can re-write the stiffness matrix as

$$\mathcal{K} = \int_S \mathcal{B}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{F} & \mathbf{0} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} \end{bmatrix} \mathcal{B} dS \quad (4.40)$$

This implies a re-formulation of the strain matrices previously defined as the result of the thickness integration of warping functions and corresponding derivatives

$$\mathcal{B}_m = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix}; \mathcal{B}_b = \begin{bmatrix} 0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 & 0 \\ 0 & 0 & -2\frac{\partial^2}{\partial x \partial y} & 0 & 0 \end{bmatrix} \quad (4.41)$$

$$\mathcal{B}_{sd} = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial}{\partial x} \end{bmatrix}; \mathcal{B}_s = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.42)$$

which one can assemble in the following manner

$$\mathcal{B} = [\mathcal{B}_m, \mathcal{B}_b, \mathcal{B}_{sd}, \mathcal{B}_s]^T \quad (4.43)$$

The entire computation of matrix $\mathcal{K}$ can be performed, as a summation of multiple sub-matrices as shown in the following

$$\mathcal{K} = \int_S \left(\mathcal{B}_m^T \mathbf{A} \mathcal{B}_m + \mathcal{B}_b^T \mathbf{B} \mathcal{B}_b + \mathcal{B}_m^T \mathbf{B}^T \mathcal{B}_b + \mathcal{B}_b^T \mathbf{D} \mathcal{B}_b + \mathcal{B}_{sd}^T \mathbf{E} \mathcal{B}_m + \right.$$

$$+\mathcal{B}_m^T \mathbf{E}^T \mathcal{B}_{sd} + \mathcal{B}_{sd}^T \mathbf{F} \mathcal{B}_b + \mathcal{B}_b^T \mathbf{F}^T \mathcal{B}_{sd} + \mathcal{B}_{sd}^T \mathbf{G} \mathcal{B}_{sd} + \mathcal{B}_s^T \mathbf{H} \mathcal{B}_s) dS \quad (4.44)$$

4.3.4 | Kinetic Energy

The total kinetic energy, written as a function of the velocity field, takes the form

$$\Pi^K = \frac{1}{2} \int_{\Omega} \dot{\mathbf{u}}^T \rho \dot{\mathbf{u}} d\Omega \quad (4.45)$$

Similarly to the displacement field, one obtains the velocity field in a matrix form using Equation 4.23 and a simple time derivative, resulting in

$$\Pi^K = \frac{1}{2} \int_{\Omega} \dot{\mathbf{d}}^T \hat{\mathcal{N}}^T \rho \hat{\mathcal{N}} \dot{\mathbf{d}} d\Omega \quad (4.46)$$

Performing a thickness integration one obtains

$$\Pi^K = \frac{1}{2} \dot{\mathbf{d}}^T \left(\int_S \mathcal{N}^T \mathcal{J} \mathcal{N} dS \right) \dot{\mathbf{d}} \quad (4.47)$$

This implies a re-formulation of the previously defined matrix $\hat{\mathcal{N}}$ as the result of the thickness integration of the warping functions. Besides, matrix $\mathcal{N}$ can be partitioned into different sub-matrices accordingly to the corresponding degrees of freedom

$$\mathcal{N} = [\mathcal{N}_m, \mathcal{N}_b, \mathcal{N}_s]^T \quad (4.48)$$

Explicitly, one can write the sub-matrices as

$$\mathcal{N}_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \mathcal{N}_b = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 \end{bmatrix} \quad (4.49)$$

$$\mathcal{N}_s = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.50)$$

which leads to the condensed form of writing the kinetic energy, where matrix $\mathcal{M}$ is the well-known mass matrix

$$\Pi^K = \frac{1}{2} \dot{\mathbf{d}}^T \mathcal{M} \dot{\mathbf{d}} \quad (4.51)$$

One can readily define the mass matrix as

$$\mathcal{M} = \int_S \mathcal{N}^T \mathcal{J} \mathcal{N} dS \quad (4.52)$$

Matrix $\mathcal{J}$ groups the mass moments of inertia and reads

$$\mathcal{J} = \begin{bmatrix} R & 0 & 0 & S & 0 & U_{11} & U_{12} \\ 0 & R & 0 & 0 & S & U_{21} & U_{22} \\ 0 & 0 & R & 0 & 0 & 0 & 0 \\ S & 0 & 0 & T & 0 & V_{11} & V_{12} \\ 0 & S & 0 & 0 & T & V_{11} & V_{12} \\ U_{11} & U_{12} & 0 & V_{11} & V_{12} & W_{11} & W_{12} \\ U_{21} & U_{22} & 0 & V_{21} & V_{22} & W_{21} & W_{22} \end{bmatrix} \quad (4.53)$$

4.3.5 | Global-Local Interpolation

The discretisation in the FEM is carried out locally over the finite elements, and in this work, one adapts the stiffness and mass matrices to a 4-node finite element, as represented in Figure 4.3. With structure meshes defined in the global (x, y, z) Cartesian coordinate system, one obtains the stiffness and mass matrices performing numerical integrations in the (ξ, η, ζ) local coordinate system.

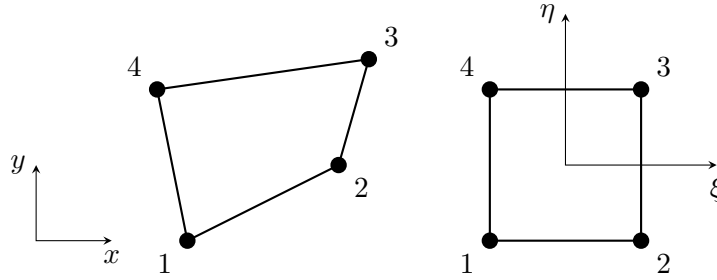


Figure 4.3 4-node element - global and local coordinates

One can establish the transformation between global and local coordinates with the following expressions

$$x(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta) x_i \quad (4.54)$$

$$y(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta) y_i \quad (4.55)$$

where (x_i, y_i) are the cartesian coordinates of the element nodes and $N_i(\xi, \eta)$ the bi-linear shape functions, defined as

$$N_i = \frac{1}{4} (1 + \xi_i \xi) (1 + \eta_i \eta) \quad (4.56)$$

Discrete values ξ_i and η_i simplify the previous expression and represent the nodal coordinates, made clear in Table 4.2.

In addition, one can write the generalised displacement vector, defined in Equation 4.24, on an element level as

$$\mathbf{d}^e = \{u^0, v^0, w^0, \gamma_{xz}^0, \gamma_{yz}^0\}^T \quad (4.57)$$

Table 4.2 Nodal parameters

Node	1	2	3	4
ξ_i	-1	1	1	-1
η_i	-1	-1	1	1

All these variables are defined in terms of nodal displacements through shape functions $\mathbf{N}$, as follows

$$\mathbf{d}^e = \mathbf{N} \mathbf{d}_i^e \quad (4.58)$$

Matrix $\mathbf{N}$ groups all four different shape function matrices, one per node, and $\mathbf{d}_i^e$ represents the set of all nodal displacements. Both are listed below

$$\mathbf{N} = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4] \quad (4.59)$$

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i^{C^1} & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (4.60)$$

where N_i are the bi-linear shape functions defined in Equation 4.67. Besides, one can group the nodal displacements as follows

$$\mathbf{d}_i^e = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4\}^T \quad (4.61)$$

In addition, one should focus on the third row of matrix $\mathbf{N}_i$. A closer look into the displacement field defined in Equation 3.1 shows that the boundary conditions involve specifying both the transverse displacement w and the first derivatives of w , similarly to the Euler [1744]-Bernoulli [1751]'s beam theory or the CPT of Kirchhoff [1850]-Love and Darwin [1888]. Therefore, the finite element interpolation of w must ensure that w and its first derivatives are continuous across the interelement boundaries in CPT elements. Finite elements requiring continuity of w and its first derivatives are C^1 elements, and there are some C^1 rectangular plate bending elements in the literature. However, as Reddy [2006] notes, there are some comments regarding the interelement continuity of w and its first derivatives for a four-node rectangular element with a 12-term polynomial approximation of w - w , $\partial w/\partial x$ and $\partial w/\partial y$ at each node. Reddy [2006] points out that w varies cubically along any line $x = \text{constant}$ or $y = \text{constant}$, and, along a given side, there are two nodes and two values (w and its first derivative) per node to define the cubic variation uniquely. Consequently, w is uniquely defined along the element boundary and is continuous along interelement boundaries. The first derivative $\partial w/\partial x$, for example, on a line $x = \text{constant}$, also varies cubically in y along the side. Since only two values of $\partial w/\partial x$ are available on a given side, since there are two nodes, the cubic variation cannot be uniquely defined, and the normal slope continuity is not satisfied. Moreover, $\partial^2 w/\partial x \partial y$ is not single-valued at the corner points of the element. Elements that violate any of the continuity conditions are *nonconforming*.

4.3.6 | CPT(N) - Melosh-Zienkiewicz-Cheung Bending Element

Based on the work of Melosh [1963] and Zienkiewicz and Cheung [1964], there is a nonconforming rectangular element with 12 degrees of freedom with w , $\partial w/\partial x$ and $\partial w/\partial y$ as nodal variables at each node. The normal derivatives vary cubically along an edge, but there are only two values available at the edge. Consequently, there is no interelement continuity of the normal derivatives between elements at the same edge. Considering the plate theory presented in this work and this Non-Conforming Classical Plate Theory (CPT(N)), one requires 7 degrees of freedom per node. The nodal variables are

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i/\partial x, \partial w^i/\partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.62)$$

Therefore, each element involves 28 degrees of freedom. Membrane and shear strain degrees of freedom do not require C^1 continuity, and thus the standard bi-linear shape functions, also used for geometric interpolation, are appropriate to interpolate the corresponding degrees of freedom. Considering the CPT(N), one interpolates the degree of freedom w^0 as follows

$$w^0(\xi, \eta) = \sum_{i=1}^4 w^i N_i^{\text{NC1}} + \sum_{i=1}^4 \frac{\partial w^i}{\partial \xi} N_i^{\text{NC2}} + \sum_{i=1}^4 \frac{\partial w^i}{\partial \eta} N_i^{\text{NC3}} \quad (4.63)$$

$$\begin{aligned} N_i^{\text{NC1}} &= \frac{1}{8} (1 + \xi \xi_i) (1 + \eta \eta_i) (2 + \xi \xi_i + \eta \eta_i - \xi^2 - \eta^2) \\ N_i^{\text{NC2}} &= \frac{1}{8} \xi_i (\xi \xi_i - 1) (1 + \eta \eta_i) (1 + \xi \xi_i)^2 \\ N_i^{\text{NC3}} &= \frac{1}{8} \eta_i (\eta \eta_i - 1) (1 + \xi \xi_i) (1 + \eta \eta_i)^2 \end{aligned} \quad (4.64)$$

where the superscript NC stands for nonconforming interpolation. Figure 4.4 illustrates the first three shape functions for each degree of freedom ($w^1, \partial w^1/\partial \xi, \partial w^1/\partial \eta$). At the node 1 coordinates, N_1^{NC1} assumes the value one, whereas the remaining are zero. One also depicts the first bi-linear shape function for comparison purposes. Despite being nonconforming, the element is known for its good results.

4.3.7 | CPT(C) - Bogner-Fox-Schmit Bending Element

Bogner et al. [1965] presented a *conforming* element with w , $\partial w/\partial x$, $\partial w/\partial y$ and $\partial^2 w/\partial x \partial y$ as nodal variables at each node. Consequently, considering the plate theory presented in this work and this Conforming Classical Plate Theory (CPT(C)), one requires 8 degrees of freedom per node. The nodal variables are

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i/\partial x, \partial w^i/\partial y, \partial^2 w^i/\partial x \partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.65)$$

Consequently, each element involves 32 degrees of freedom, which is somehow cumbersome to compute. However, one should note that this number is independent of the number of layers since the complete set of equations is written exclusively with the generalised displacements of the reference plane. Considering the CPT(C), one interpolates the degree of freedom w^0 using interpolation functions obtained from the tensor products of the one-dimensional Hermite cubic polynomials [Reddy, 2006]. Thus, one interpolates the degree of freedom w^0 with C^1 continuity. This element is also presented by Castel [2013] but, to the best author's knowledge, with no

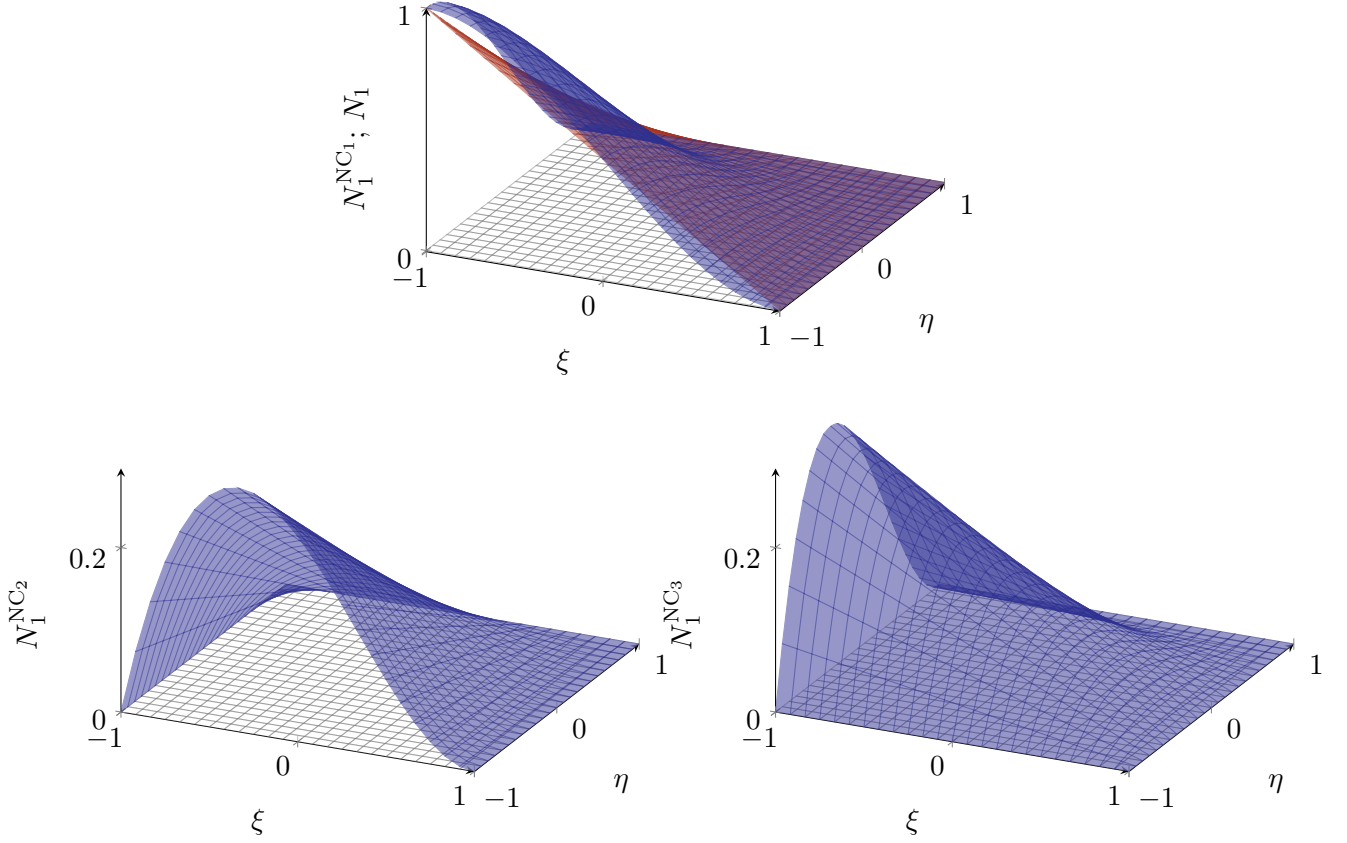


Figure 4.4 CPT(N) element shape functions for the first node

published results. The superscript H stands for Hermitian interpolation and one presents such interpolation as follows

$$w^0(\xi, \eta) = \sum_{i=1}^4 w^i N_i^{H1} + \sum_{i=1}^4 \frac{\partial w^i}{\partial \xi} N_i^{H2} + \sum_{i=1}^4 \frac{\partial w^i}{\partial \eta} N_i^{H3} + \sum_{i=1}^4 \frac{\partial^2 w^i}{\partial \xi \partial \eta} N_i^{H4} \quad (4.66)$$

$$\begin{aligned} N_i^{H1} &= \frac{1}{16} (\xi + \xi_i)^2 (\xi \xi_i - 2) (\eta + \eta_i)^2 (\eta \eta_i - 2) \\ N_i^{H2} &= \frac{1}{16} \xi_i (\xi + \xi_i)^2 (1 - \xi \xi_i) (\eta + \eta_i)^2 (\eta \eta_i - 2) \\ N_i^{H3} &= \frac{1}{16} \eta_i (\xi + \xi_i)^2 (\xi \xi_i - 2) (\eta + \eta_i)^2 (1 - \eta \eta_i) \\ N_i^{H4} &= \frac{1}{16} \xi_i \eta_i (\xi + \xi_i)^2 (1 - \xi \xi_i) (\eta + \eta_i)^2 (1 - \eta \eta_i) \end{aligned} \quad (4.67)$$

Figure 4.5 presents the first four shape functions for each degree of freedom (w^1 , $\partial w^1/\partial \xi$, $\partial w^1/\partial \eta$ and $\partial^2 w^1/\partial \xi \partial \eta$). At the node 1 coordinates, N_1^{H1} assumes the value one, whereas the remaining are zero. One also depicts the first bi-linear shape function for comparison purposes.

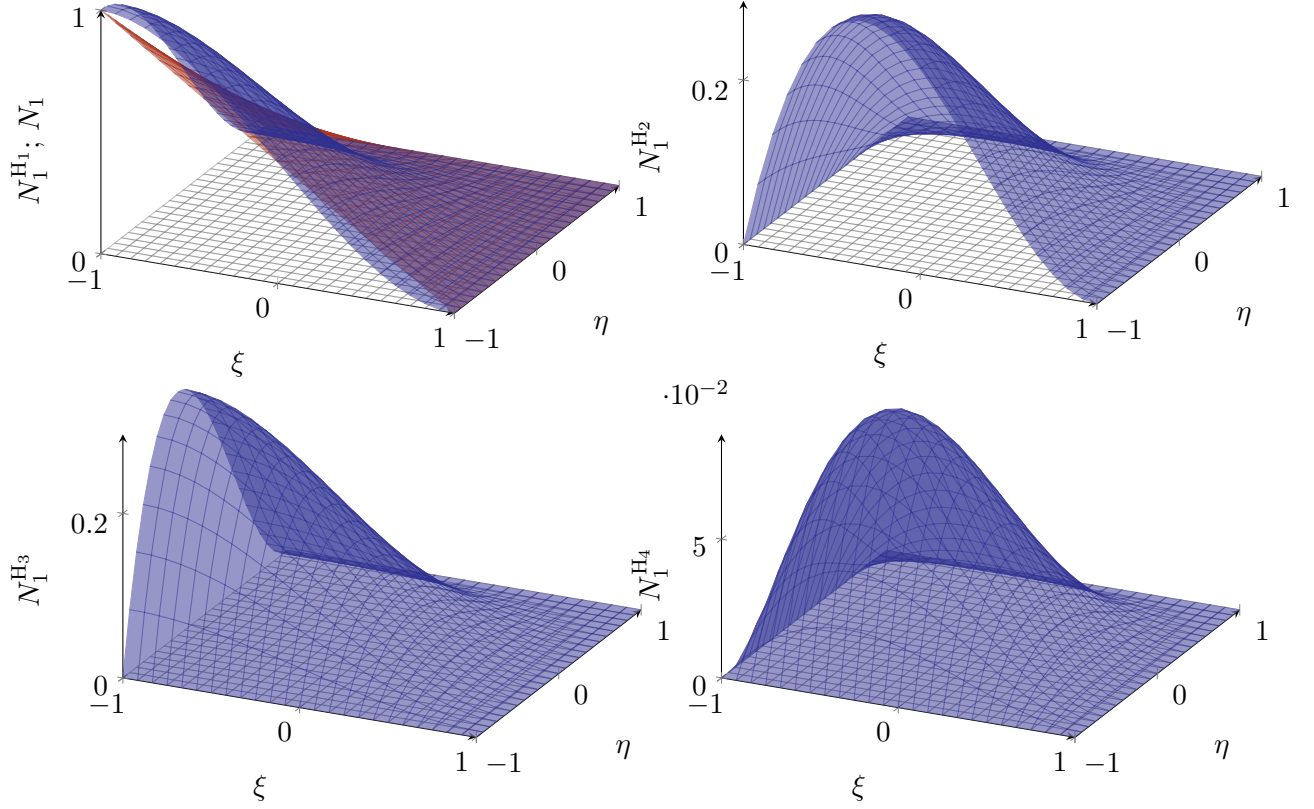


Figure 4.5 Shape functions for the first node

4.3.8 | DKQ - Batoz-Tahar Bending Element

As discussed before, the difficulty in deriving satisfactory elements for thin plates is mainly due to the Kirchhoff plate theory requiring C^1 continuity for the transverse displacement w , while thick Reissner [1945] and Mindlin [1951] plates require only C^0 continuity. To overcome this difficulty, Batoz and Tahar [1982] presented a 4-node rectangular element with 12 degrees of freedom developed for thin plates - the Discrete Kirchhoff Quadrilateral (DKQ). Based on a generalisation of the efficient and reliable triangular Discrete Kirchhoff Triangular element presented by J. L. Batoz in previous publications, the authors derived this element using the so-called discrete Kirchhoff technique. β_x and β_y are the rotations of the normal to the undeformed middle surface in the $x-z$ and $y-z$ planes, respectively, and the potential elastic bending energy depends only upon β_x and β_y with a C^0 continuity required. Then, it is necessary to relate the rotations β_x and β_y to the transverse displacement w in such a way that the final element has the characteristics of a Kirchhoff-type element. In the formulation of the DKQ element, one defines β_x and β_y by incomplete cubic polynomial expressions as follows

$$\beta_x = \sum_{i=1}^8 N_i^S \beta_{x_i}, \quad \beta_y = \sum_{i=1}^8 N_i^S \beta_{y_i} \quad (4.68)$$

where shape functions $N_i^S(\xi, \eta)$ are those of the 8-node Serendipity element, defined as

$$N_i^S = \frac{1}{4} (1 + \xi \xi_i) (1 + \eta \eta_i) (\xi \xi_i + \eta \eta_i - 1), \text{ for } i = 1, 2, 3, 4$$

$$\begin{aligned}
N_i^S &= \frac{1}{2} (1 - \xi^2) (1 + \eta \eta_i), \text{ for } i = 5, 7 \\
N_i^S &= \frac{1}{2} (1 + \xi \xi_i) (1 - \eta^2), \text{ for } i = 6, 8
\end{aligned} \tag{4.69}$$

Batoz and Tahar [1982] assume β_x and β_y as transitory variables, defined at the corner and mid-nodes of the quadrilateral element. To ensure the Kirchhoff assumptions, one must assume

$$\beta_{x_i} + \frac{\partial w}{\partial x_i} = 0 \text{ and } \beta_{y_i} + \frac{\partial w}{\partial y_i} = 0 \tag{4.70}$$

at the corner nodes and

$$\beta_{s_k} + \frac{\partial w}{\partial s_k} = 0 \tag{4.71}$$

at the mid-side nodes, where s represents the coordinate along the element side. Figure 4.6 presents the detailed geometry of the DKQ element.

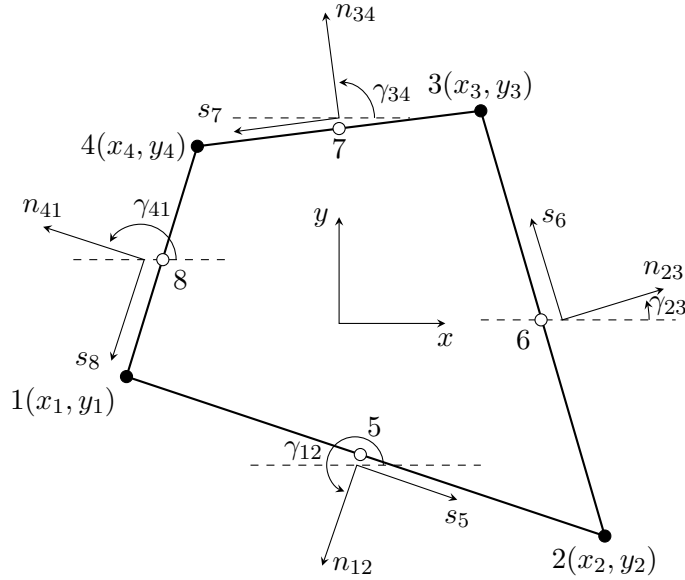


Figure 4.6 Geometry of the DKQ element. Adapted from Batoz and Tahar [1982]

$\partial w / \partial s_k$ is the derivative of the transverse displacement w with respect to s at the mid-node k , defined as

$$\frac{\partial w}{\partial s_k} = -\frac{3}{2\ell_{ij}} (w_i - w_j) - \frac{1}{4} \left(\frac{\partial w}{\partial s_i} + \frac{\partial w}{\partial s_j} \right) \tag{4.72}$$

where $k = 5, 6, 7, 8$ is the mid-node of the sides $ij = 12, 23, 34, 41$, respectively. ℓ_{ij} represents the length of the side ij . On the other hand, β_n varies linearly along the element sides

$$\beta_{n_k} = \frac{1}{2} (\beta_{n_i} + \beta_{n_j}) = -\frac{1}{2} \left(\frac{\partial w}{\partial n_i} + \frac{\partial w}{\partial n_j} \right) \tag{4.73}$$

As noted by Batoz and Tahar [1982], w is not defined inside the element but varies independently along the element sides. The nodal variable w emerges from Equation 4.72 at the four

corner nodes of the element. Besides, one satisfies the Kirchhoff assumptions along the entire boundary since $\partial w/\partial s$ and β_s are quadratic expressions along the element sides.

The nodal variables of the DKQ element combined with the laminated plate theory presented in this work are

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i/\partial x, \partial w^i/\partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.74)$$

At this point, for the sake of simplicity, considering just the nodal variables associated with plate bending of one element

$$\{\mathbf{d}_i^e\}_b = \{w^1, \partial w^1/\partial x, \partial w^1/\partial y, \dots, w^4, \partial w^4/\partial x, \partial w^4/\partial y\}^T \quad (4.75)$$

It is possible to write explicit expressions for the rotations β_x and β_y of a general quadrilateral in terms of the nodal variables defined in Equation 4.75 as

$$\begin{aligned} \beta_x &= [H_1^x(\xi, \eta), \dots, H_{12}^x(\xi, \eta)] \{\mathbf{d}_i^e\}_b \\ \beta_y &= [H_1^y(\xi, \eta), \dots, H_{12}^y(\xi, \eta)] \{\mathbf{d}_i^e\}_b \end{aligned} \quad (4.76)$$

with

$$\begin{aligned} H_1^x &= \frac{3}{2} (a_5 N_5^S - a_8 N_8^S) & H_1^y &= \frac{3}{2} (d_5 N_5^S - d_8 N_8^S) \\ H_2^x &= -N_1^S + b_5 N_5^S + b_8 N_8^S & H_2^y &= H_3^x \\ H_3^x &= c_5 N_5^S + c_8 N_8^S & H_3^y &= -N_1^S + e_5 N_5^S + e_8 N_8^S \\ H_4^x &= \frac{3}{2} (a_6 N_6^S - a_5 N_5^S) & H_4^y &= \frac{3}{2} (d_6 N_6^S - d_5 N_5^S) \\ H_5^x &= -N_2^S + b_6 N_6^S + b_5 N_5^S & H_5^y &= H_6^x \\ H_6^x &= c_6 N_6^S + c_5 N_5^S & H_6^y &= -N_2^S + e_6 N_6^S + e_5 N_5^S \\ H_7^x &= \frac{3}{2} (a_7 N_7^S - a_6 N_6^S) & H_7^y &= \frac{3}{2} (d_7 N_7^S - d_6 N_6^S) \\ H_8^x &= -N_3^S + b_7 N_7^S + b_6 N_6^S & H_8^y &= H_9^x \\ H_9^x &= c_7 N_7^S + c_6 N_6^S & H_9^y &= -N_3^S + e_7 N_7^S + e_6 N_6^S \\ H_{10}^x &= \frac{3}{2} (a_8 N_8^S - a_7 N_7^S) & H_{10}^y &= \frac{3}{2} (d_8 N_8^S - d_7 N_7^S) \\ H_{11}^x &= -N_4^S + b_8 N_8^S + b_7 N_7^S & H_{11}^y &= H_{12}^x \\ H_{12}^x &= c_8 N_8^S + c_7 N_7^S & H_{12}^y &= -N_4^S + e_8 N_8^S + e_7 N_7^S \end{aligned} \quad \begin{matrix} (4.77) \\ (4.78) \end{matrix}$$

where

$$a_k = -\frac{x_{ij}}{\ell_{ij}^2}, \quad b_k = \frac{1}{4} \frac{x_{ij}^2 - 2y_{ij}^2}{\ell_{ij}^2}, \quad c_k = \frac{3}{4} \frac{x_{ij}y_{ij}}{\ell_{ij}^2}, \quad d_k = -\frac{y_{ij}}{\ell_{ij}^2} \quad \text{and} \quad e_k = \frac{1}{4} \frac{y_{ij}^2 - 2x_{ij}^2}{\ell_{ij}^2} \quad (4.79)$$

with $k = 5, 6, 7, 8$ for the sides $ij = 12, 23, 34, 41$. Besides,

$$x_{ij} = x_i - x_j, y_{ij} = y_i - y_j \text{ and } \ell_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2} \quad (4.80)$$

Finally, one can compute the curvature vector $\boldsymbol{\kappa}$ of Equation 3.7 using the DKQ element as

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ 2\kappa_{xy}^0 \end{Bmatrix} = \mathcal{B}_b \{\mathbf{d}_i^e\}_b \quad (4.81)$$

where

$$\mathcal{B}_b = [\mathcal{B}_b^1, \mathcal{B}_b^2, \dots, \mathcal{B}_b^{12}] \quad (4.82)$$

and

$$\mathcal{B}_b^i = \begin{Bmatrix} \frac{\partial H_i^x}{\partial x} \\ \frac{\partial H_i^y}{\partial y} \\ \frac{\partial H_i^x}{\partial y} + \frac{\partial H_i^y}{\partial x} \end{Bmatrix} \quad (4.83)$$

4.3.9 | IDKQ - Jeyachandrabose-Kirkhope-Meektscho Bending Element

Later, Jeyachandrabose et al. [1987] developed an Improved Discrete Kirchhoff Quadrilateral (IDKQ) using smoothed shape functions based on previous work for triangular thin-plate bending plate elements [Jeyachandrabose and Kirkhope, 1986]. The formulation of the IDKQ element follows that of the DKQ element, with a difference in the application of the Kirchhoff hypothesis. For the DKQ, one only imposes the Kirchhoff hypothesis at the nodes on the element boundary. For the IDKQ, one also applies it to the element's interior. Consequently, Jeyachandrabose et al. [1987] considered the nine-node Lagrange element instead of the eight-noded serendipity element used by Batoz and Tahar [1982]. In the formulation of the IDKQ element, one defines β_x and β_y as follows

$$\beta_x = \sum_{i=1}^9 \bar{N}_i^L \beta_{x_i}, \quad \beta_y = \sum_{i=1}^9 \bar{N}_i^L \beta_{y_i} \quad (4.84)$$

where shape functions $\bar{N}_i^L(\xi, \eta)$ are the substitute shape functions derived as the least-squares polynomial versions of the original shape functions, $N_i^L(\xi, \eta)$, of the 9-node Lagrange element, defined as

$$\begin{aligned} \bar{N}_i^L &= -\frac{1}{36} + \frac{1}{12} (\xi^2 + \eta^2) + \frac{1}{4} \xi \xi_i \eta \eta_i + \frac{1}{4} (\xi^2 \eta \eta_i + \xi \xi_i \eta^2), \text{ for } i = 1, 2, 3, 4 \\ \bar{N}_i^L &= \frac{1}{18} + \frac{1}{2} \eta \eta_i - \frac{1}{6} \xi^2 + \frac{1}{3} \eta^2 - \frac{1}{2} \xi^2 \eta \eta_i, \text{ for } i = 5, 7 \\ \bar{N}_i^L &= \frac{1}{18} + \frac{1}{2} \xi \xi_i + \frac{1}{3} \xi^2 - \frac{1}{6} \eta^2 - \frac{1}{2} \xi \xi_i \eta^2, \text{ for } i = 6, 8 \\ \bar{N}_i^L &= \frac{8}{9} - \frac{2}{3} (\xi^2 + \eta^2), \text{ for } i = 9 \end{aligned} \quad (4.85)$$

Jeyachandrabose et al. [1987] presented the method of deriving $\bar{N}_i^L(\xi, \eta)$ from the nine-node Lagrange element - $N_i^L(\xi, \eta)$. According to the authors, previous studies improved the DKQ element using the $N_i^L(\xi, \eta)$ shape functions, but the stress predictions were poor and the element thus neglected. Similarly to Batoz and Tahar [1982], Jeyachandrabose et al. [1987] impose the Kirchhoff hypothesis at the corner nodes and at the interior node as

$$\beta_{x_i} = -\frac{\partial w}{\partial x_i} \text{ and } \beta_{y_i} = -\frac{\partial w}{\partial y_i} \quad (4.86)$$

and

$$\beta_{s_k} = -\frac{\partial w}{\partial s_k} \quad (4.87)$$

at the mid-side nodes, where s represents the coordinate along the element side. On the other hand, β_n varies linearly along the element sides

$$\beta_{n_k} = \frac{1}{2} (\beta_{n_i} + \beta_{n_j}) = -\frac{1}{2} \left(\frac{\partial w}{\partial n_i} + \frac{\partial w}{\partial n_j} \right) \quad (4.88)$$

Figure 4.7 presents the detailed geometry of the IDKQ element.

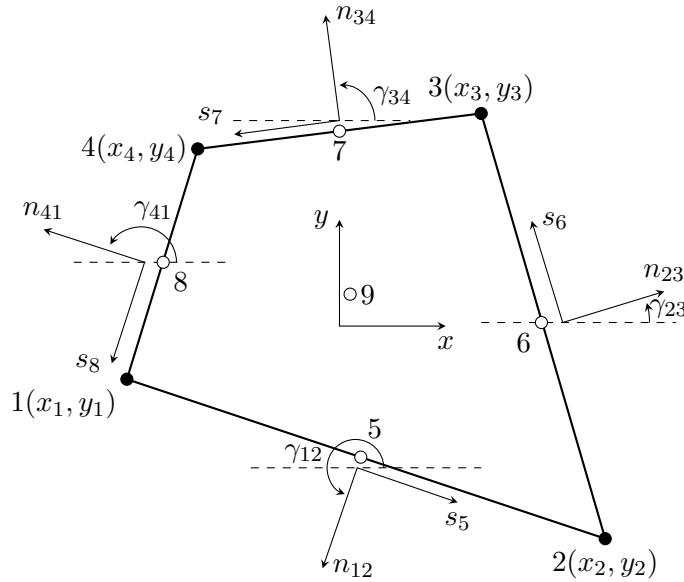


Figure 4.7 Geometry of the IDKQ element. Adapted from Jeyachandrabose et al. [1987]

The nodal variables of the IDKQ element combined with the laminated plate theory presented in this work are

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i / \partial x, \partial w^i / \partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.89)$$

At this point, for the sake of simplicity, considering just the nodal variables associated with plate bending of one element

$$\{\mathbf{d}_i^e\}_b = \{w^1, \partial w^1 / \partial x, \partial w^1 / \partial y, \dots, w^4, \partial w^4 / \partial x, \partial w^4 / \partial y\}^T \quad (4.90)$$

It is possible to write explicit expressions for the rotations β_x and β_y of a general quadrilateral in terms of the nodal variables defined in Equation 4.90 as

$$\begin{aligned}\beta_x &= [G_1(\xi, \eta), \dots, G_{12}(\xi, \eta)] \{\mathbf{d}_i^e\}_b \\ \beta_y &= [H_1(\xi, \eta), \dots, H_{12}(\xi, \eta)] \{\mathbf{d}_i^e\}_b\end{aligned}\tag{4.91}$$

with

$$\begin{aligned}G_1 &= \frac{p_1}{4} \bar{N}_5^L - \frac{p_4}{4} \bar{N}_8^L + \frac{3}{2\Delta} y_{42} \bar{N}_9^L \\ G_2 &= -\bar{N}_1^L + \frac{s_1 - 2}{4} \bar{N}_5^L + \frac{s_4 - 2}{4} \bar{N}_8^L + \frac{y_{42}}{4\Delta} (x_{21} + x_{41}) \bar{N}_9^L \\ G_3 &= -\frac{q_1}{4} \bar{N}_5^L - \frac{q_4}{4} \bar{N}_8^L + \frac{y_{42}}{4\Delta} (y_{21} + y_{41}) \bar{N}_9^L \\ G_4 &= -\frac{p_1}{4} \bar{N}_5^L + \frac{p_2}{4} \bar{N}_6^L + \frac{3}{2\Delta} y_{13} \bar{N}_9^L \\ G_5 &= -\bar{N}_2^L + \frac{s_1 - 2}{4} \bar{N}_5^L + \frac{s_2 - 1}{4} \bar{N}_6^L + \frac{y_{13}}{4\Delta} (x_{12} + x_{32}) \bar{N}_9^L \\ G_6 &= -\frac{q_1}{4} \bar{N}_5^L - \frac{q_2}{4} \bar{N}_6^L + \frac{y_{13}}{4\Delta} (y_{12} + y_{32}) \bar{N}_9^L \\ G_7 &= -\frac{p_2}{4} \bar{N}_6^L + \frac{p_3}{4} \bar{N}_7^L + \frac{3}{2\Delta} y_{24} \bar{N}_9^L \\ G_8 &= -\bar{N}_3^L + \frac{s_2 - 2}{4} \bar{N}_6^L + \frac{s_3 - 2}{4} \bar{N}_7^L + \frac{y_{24}}{4\Delta} (x_{23} + x_{43}) \bar{N}_9^L \\ G_9 &= -\frac{q_2}{4} \bar{N}_6^L - \frac{q_3}{4} \bar{N}_7^L + \frac{y_{24}}{4\Delta} (y_{23} + y_{43}) \bar{N}_9^L \\ G_{10} &= \frac{p_3}{4} \bar{N}_7^L + \frac{p_4}{4} \bar{N}_8^L + \frac{3}{2\Delta} y_{31} \bar{N}_9^L \\ G_{11} &= -\bar{N}_4^L + \frac{s_3 - 2}{4} \bar{N}_7^L + \frac{s_4 - 2}{4} \bar{N}_8^L + \frac{y_{31}}{4\Delta} (x_{14} + x_{34}) \bar{N}_9^L \\ G_{12} &= -\frac{q_3}{4} \bar{N}_7^L - \frac{q_4}{4} \bar{N}_8^L + \frac{y_{31}}{4\Delta} (y_{14} + y_{34}) \bar{N}_9^L\end{aligned}\tag{4.92}$$

and

$$\begin{aligned}H_1 &= -\frac{t_1}{4} \bar{N}_5^L + \frac{t_4}{4} \bar{N}_8^L + \frac{3}{2\Delta} x_{24} \bar{N}_9^L \\ H_2 &= -\frac{q_1}{4} \bar{N}_5^L - \frac{q_4}{4} \bar{N}_8^L + \frac{x_{24}}{4\Delta} (x_{21} + x_{41}) \bar{N}_9^L \\ H_3 &= -\bar{N}_1^L + \frac{r_1 - 2}{4} \bar{N}_5^L + \frac{r_4 - 2}{4} \bar{N}_8^L + \frac{x_{24}}{4\Delta} (y_{21} + y_{41}) \bar{N}_9^L \\ H_4 &= \frac{t_1}{4} \bar{N}_5^L - \frac{t_2}{4} \bar{N}_6^L + \frac{3}{2\Delta} x_{31} \bar{N}_9^L \\ H_5 &= -\frac{q_1}{4} \bar{N}_5^L - \frac{q_2}{4} \bar{N}_6^L + \frac{x_{31}}{4\Delta} (x_{12} + x_{32}) \bar{N}_9^L \\ H_6 &= -\bar{N}_2^L + \frac{r_1 - 2}{4} \bar{N}_5^L + \frac{r_2 - 2}{4} \bar{N}_6^L + \frac{x_{31}}{4\Delta} (y_{12} + y_{32}) \bar{N}_9^L\end{aligned}$$

$$\begin{aligned}
 H_7 &= \frac{t_2}{4} \bar{N}_6^L - \frac{t_3}{4} \bar{N}_7^L + \frac{3}{2\Delta} x_{42} \bar{N}_9^L \\
 H_8 &= -\frac{q_2}{4} \bar{N}_6^L - \frac{q_3}{4} \bar{N}_7^L + \frac{x_{42}}{4\Delta} (x_{23} + x_{43}) \bar{N}_9^L \\
 H_9 &= -\bar{N}_3^L + \frac{r_2 - 2}{4} \bar{N}_6^L + \frac{r_3 - 2}{4} \bar{N}_7^L + \frac{x_{42}}{4\Delta} (y_{23} + y_{43}) \bar{N}_9^L \\
 H_{10} &= \frac{t_3}{4} \bar{N}_7^L - \frac{t_4}{4} \bar{N}_8^L + \frac{3}{2\Delta} x_{13} \bar{N}_9^L \\
 H_{11} &= -\frac{q_3}{4} \bar{N}_7^L - \frac{q_4}{4} \bar{N}_8^L + \frac{x_{13}}{4\Delta} (x_{14} + x_{34}) \bar{N}_9^L \\
 H_{12} &= -\bar{N}_4^L + \frac{r_3 - 2}{4} \bar{N}_7^L + \frac{r_4 - 2}{4} \bar{N}_8^L + \frac{x_{13}}{4\Delta} (y_{14} + y_{34}) \bar{N}_9^L
 \end{aligned} \tag{4.93}$$

where

$$x_{ij} = x_i - x_j, y_{ij} = y_i - y_j \text{ and } \Delta = (x_1 y_{24} + x_2 y_{31} + x_3 y_{42} + x_4 y_{13}) \tag{4.94}$$

Besides, for $i = 1, \dots, 4$

$$p_i = \frac{6c_i}{b_i^2 + c_i^2}, q_i = \frac{3b_i c_i}{b_i^2 + c_i^2}, r_i = \frac{3b_i^2}{b_i^2 + c_i^2}, s_i = \frac{3c_i^2}{b_i^2 + c_i^2} \text{ and } t_i = \frac{6b_i}{b_i^2 + c_i^2} \tag{4.95}$$

with

$$b_1 = y_{12}, b_2 = y_{23}, b_3 = y_{34}, b_4 = y_{41}, c_1 = x_{21}, c_2 = x_{32}, c_3 = x_{43} \text{ and } c_4 = x_{14} \tag{4.96}$$

Finally, and similarly to the DKQ element, one can compute the curvature vector $\boldsymbol{\kappa}$ of Equation 3.7 using the IDKQ element as

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_{xx}^0 \\ \kappa_{yy}^0 \\ 2\kappa_{xy}^0 \end{Bmatrix} = \mathcal{B}_b \{\mathbf{d}_i^e\}_b \tag{4.97}$$

where

$$\mathcal{B}_b = [\mathcal{B}_b^1, \mathcal{B}_b^2, \dots, \mathcal{B}_b^{12}] \tag{4.98}$$

and

$$\mathcal{B}_b^i = \begin{Bmatrix} \frac{\partial G_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \\ \frac{\partial G_i}{\partial y} + \frac{\partial H_i}{\partial x} \end{Bmatrix} \tag{4.99}$$

According to the authors, this element exhibits improved performance over the DKQ element in predicting displacements and stresses.

4.3.10 | Elemental Stiffness Matrix

Following a proper interpolation of all degrees of freedom, one can establish an elemental strain matrix based on Equation 4.28, an essential step towards the definition of an elemental stiffness matrix

$$\boldsymbol{\varepsilon}^e = \mathcal{B} \mathbf{d}^e \quad (4.100)$$

Recalling the transformation between generalised and nodal displacements given in Equation 4.58, one can write

$$\boldsymbol{\varepsilon}^e = \mathcal{B} \mathbf{N} \mathbf{d}_i^e = \mathbf{B}^e \mathbf{d}_i^e \quad (4.101)$$

where $\mathbf{B}^e$ represents the elemental strain matrix. Recovering the previous definition of stiffness matrix and taking it to an elemental level

$$\mathbf{k}^e = \int_S \mathbf{N}^T \mathcal{B}^T \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{F} & \mathbf{0} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} \end{bmatrix} \mathcal{B} \mathbf{N} dS \quad (4.102)$$

one obtains the elemental stiffness matrix defined using the elemental strain matrix

$$\mathbf{k}^e = \int_S \mathbf{B}^{eT} \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{E} & \mathbf{0} \\ \mathbf{B}^T & \mathbf{D} & \mathbf{F} & \mathbf{0} \\ \mathbf{E}^T & \mathbf{F}^T & \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H} \end{bmatrix} \mathbf{B}^e dS \quad (4.103)$$

Recalling the assemblage presented in Equation 4.43, one can extend the elemental strain matrix to the four nodes as follows

$$\mathbf{B}_m^e = [\mathbf{B}_m^1, \mathbf{B}_m^2, \mathbf{B}_m^3, \mathbf{B}_m^4] \quad (4.104)$$

On the following, one arranges the complete set of elemental strain sub-matrices. Still, one must recall the nodal variables considering the different bending formulations. The nodal variables considering the CPT(N), DKQ and IDKQ formulations are

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i / \partial x, \partial w^i / \partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.105)$$

Thus, there are seven degrees of freedom per node and twenty-eight per element. On the other hand, the CPT(C) formulation requires one degree of freedom extra per node, as written on the following

$$\mathbf{d}_i = \{u^i, v^i, w^i, \partial w^i / \partial x, \partial w^i / \partial y, \partial^2 w^i / \partial x \partial y, \gamma_{xz}^i, \gamma_{yz}^i\}^T \quad (4.106)$$

4.3.10.1 | Membrane Strain Matrix

Considering the CPT(N), DKQ and IDKQ formulations, one can arrange the elemental membrane strain sub-matrix as

$$\mathbf{B}_m^i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.107)$$

For the CPT(C) formulation, one adds one null column between the fifth and sixth columns to account for the extra degree of freedom - $\partial^2 w^i / \partial x \partial y$.

4.3.10.2 | Bending Strain Matrix

As previously discussed, the bending formulation depends on the finite element interpolation of w since one must ensure that w and its first derivatives are continuous across the interelement boundaries. Considering the formulations presented before, one explicitly shows the bending strain sub-matrices considered in this work.

CPT(N) Element

$$\mathbf{B}_b^i = \begin{bmatrix} 0 & 0 & -\frac{\partial^2 N_i^{NC_1}}{\partial x^2} & -\frac{\partial^2 N_i^{NC_2}}{\partial x^2} & -\frac{\partial^2 N_i^{NC_3}}{\partial x^2} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2 N_i^{NC_1}}{\partial y^2} & -\frac{\partial^2 N_i^{NC_2}}{\partial y^2} & -\frac{\partial^2 N_i^{NC_3}}{\partial y^2} & 0 & 0 \\ 0 & 0 & -2\frac{\partial^2 N_i^{NC_1}}{\partial x \partial y} & -2\frac{\partial^2 N_i^{NC_2}}{\partial x \partial y} & -2\frac{\partial^2 N_i^{NC_3}}{\partial x \partial y} & 0 & 0 \end{bmatrix} \quad (4.108)$$

CPT(C) Element

$$\mathbf{B}_b^i = \begin{bmatrix} 0 & 0 & -\frac{\partial^2 N_i^{H_1}}{\partial x^2} & -\frac{\partial^2 N_i^{H_2}}{\partial x^2} & -\frac{\partial^2 N_i^{H_3}}{\partial x^2} & -\frac{\partial^2 N_i^{H_4}}{\partial x^2} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2 N_i^{H_1}}{\partial y^2} & -\frac{\partial^2 N_i^{H_2}}{\partial y^2} & -\frac{\partial^2 N_i^{H_3}}{\partial y^2} & -\frac{\partial^2 N_i^{H_4}}{\partial y^2} & 0 & 0 \\ 0 & 0 & -2\frac{\partial^2 N_i^{H_1}}{\partial x \partial y} & -2\frac{\partial^2 N_i^{H_2}}{\partial x \partial y} & -2\frac{\partial^2 N_i^{H_3}}{\partial x \partial y} & -2\frac{\partial^2 N_i^{H_4}}{\partial x \partial y} & 0 & 0 \end{bmatrix} \quad (4.109)$$

DKQ Element

$$\mathbf{B}_b^i = \begin{bmatrix} 0 & 0 & \frac{\partial H_{3i-2}^x}{\partial x} & \frac{\partial H_{3i-1}^x}{\partial x} & \frac{\partial H_{3i}^x}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial H_{3i-2}^y}{\partial y} & \frac{\partial H_{3i-1}^y}{\partial y} & \frac{\partial H_{3i}^y}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial H_{3i-2}^x}{\partial y} + \frac{\partial H_{3i-2}^y}{\partial x} & \frac{\partial H_{3i-1}^x}{\partial y} + \frac{\partial H_{3i-1}^y}{\partial x} & \frac{\partial H_{3i}^x}{\partial y} + \frac{\partial H_{3i}^y}{\partial x} & 0 & 0 \end{bmatrix} \quad (4.110)$$

IDKQ Element

$$\mathbf{B}_b^i = \begin{bmatrix} 0 & 0 & \frac{\partial G_{3i-2}}{\partial x} & \frac{\partial G_{3i-1}}{\partial x} & \frac{\partial G_{3i}}{\partial x} & 0 & 0 \\ 0 & 0 & \frac{\partial H_{3i-2}}{\partial y} & \frac{\partial H_{3i-1}}{\partial y} & \frac{\partial H_{3i}}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial G_{3i-2}}{\partial y} + \frac{\partial H_{3i-2}}{\partial x} & \frac{\partial G_{3i-1}}{\partial y} + \frac{\partial H_{3i-1}}{\partial x} & \frac{\partial G_{3i}}{\partial y} + \frac{\partial H_{3i}}{\partial x} & 0 & 0 \end{bmatrix} \quad (4.111)$$

4.3.10.3 | Higher-Order Bending Strain Matrix

For the higher-order bending strain matrix, and considering the CPT(N), DKQ and IDKQ formulations, one can arrange the corresponding sub-matrix as

$$\mathbf{B}_{sd}^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y} \\ 0 & 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (4.112)$$

Once again, for the CPT(C) formulation, one adds one null column between the fifth and sixth columns to account for the extra degree of freedom of this formulation.

4.3.10.4 | Shear Strain Matrix

Finally, for the shear strain matrix, and considering the CPT(N), DKQ and IDKQ formulations, one can write the corresponding sub-matrix as

$$\mathbf{B}_s^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (4.113)$$

with one more null column between the fifth and sixth columns when considering the CPT(C) formulation to account for the extra degree of freedom.

4.3.11 | Elemental Mass Matrix

Similarly, one also extends the mass matrix to an elemental level using the transformation presented in Equation 4.58. Recalling the expression for the kinetic energy

$$\Pi^K = \frac{1}{2} \dot{\mathbf{d}}^T \mathcal{M} \dot{\mathbf{d}} \quad (4.114)$$

Reducing it to an elemental level is to write

$$\Pi_e^K = \frac{1}{2} \dot{\mathbf{d}}_e^T \mathbf{N}^T \mathcal{M} \mathbf{N} \dot{\mathbf{d}}_e \quad (4.115)$$

Therefore, one can define the elemental mass matrix as

$$\mathbf{m}^e = \int_S \mathbf{N}^T \mathcal{N}^T \mathcal{J} \mathcal{N} \mathbf{N} dS \quad (4.116)$$

Or more clearly

$$\mathbf{m}^e = \int_S \hat{\mathbf{N}}^T \mathcal{J} \hat{\mathbf{N}} dS \quad (4.117)$$

The matrix $\hat{\mathbf{N}}$ is a set of sub-matrices assembled as follows

$$\hat{\mathbf{N}} = [\hat{\mathbf{N}}_m^e, \hat{\mathbf{N}}_b^e, \hat{\mathbf{N}}_s^e]^T \quad (4.118)$$

Together with a four node extension as the following example

$$\hat{\mathbf{N}}_m^e = [\hat{\mathbf{N}}_m^1, \hat{\mathbf{N}}_m^2, \hat{\mathbf{N}}_m^3, \hat{\mathbf{N}}_m^4] \quad (4.119)$$

The same is valid for all components. One clearly presents the elemental sub-matrices hereafter.

4.3.11.1 | Membrane Mass Matrix

Considering the CPT(N), DKQ and IDKQ formulations, one can arrange the elemental membrane mass sub-matrix as

$$\hat{\mathbf{N}}_m^i = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.120)$$

For the CPT(C) formulation, one adds one null column between the fifth and sixth columns to account for the extra degree of freedom - $\partial^2 w^i / \partial x \partial y$.

4.3.11.2 | Bending Mass Matrix

As previously discussed, the bending formulation depends on the finite element interpolation of w since one must ensure that w and its first derivatives are continuous across the interelement boundaries. Therefore this influences the bending mass matrices. A closer look at the matrix $\mathcal{J}$ in 4.53, which groups all the mass moments of inertia, shows two terms in the main diagonal directly related to bending, R and T , both defined in 3.31. Therefore, the FEM discretisation involves two bending mass sub-matrices, one involving the mass moment of inertia R , $\hat{\mathbf{N}}_{bR}^i$, and another one involving the mass moment of inertia T , $\hat{\mathbf{N}}_{bT}^i$. Besides, as previously discussed, DKQ and IDKQ elements use auxiliary variables to impose the Kirchhoff hypothesis and circumvent the C^1 continuity of the transverse displacement w while making it impossible to write it explicitly using the shape functions of these elements. To overcome it, some authors use consistent mass matrices, for instance. Here, one uses the mass matrix of the CPT(N) element for all the elements with seven degrees of freedom per node. Considering the formulations presented before, one explicitly shows the bending mass sub-matrices considered in this work.

CPT(N), DKQ and IDKQ Elements

$$\hat{\mathbf{N}}_{bR}^i = \begin{bmatrix} 0 & 0 & N_i^{\text{NC}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i^{\text{NC}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i^{\text{NC}_3} & 0 & 0 \end{bmatrix} \quad (4.121)$$

$$\hat{\mathbf{N}}_{bT}^i = \begin{bmatrix} 0 & 0 & -\frac{\partial N_i^{\text{NC}_1}}{\partial x} & -\frac{\partial N_i^{\text{NC}_2}}{\partial x} & -\frac{\partial N_i^{\text{NC}_3}}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{\partial N_i^{\text{NC}_1}}{\partial y} & -\frac{\partial N_i^{\text{NC}_2}}{\partial y} & -\frac{\partial N_i^{\text{NC}_3}}{\partial y} & 0 & 0 \end{bmatrix} \quad (4.122)$$

CPT(C) Element

$$\hat{\mathbf{N}}_{bR}^i = \begin{bmatrix} 0 & 0 & N_i^{\text{H}_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i^{\text{H}_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i^{\text{H}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_i^{\text{H}_4} & 0 & 0 \end{bmatrix} \quad (4.123)$$

$$\hat{\mathbf{N}}_{bT}^i = \begin{bmatrix} 0 & 0 & -\frac{\partial N_i^{\text{H}_1}}{\partial x} & -\frac{\partial N_i^{\text{H}_2}}{\partial x} & -\frac{\partial N_i^{\text{H}_3}}{\partial x} & -\frac{\partial N_i^{\text{H}_4}}{\partial x} & 0 & 0 \\ 0 & 0 & -\frac{\partial N_i^{\text{H}_1}}{\partial y} & -\frac{\partial N_i^{\text{H}_2}}{\partial y} & -\frac{\partial N_i^{\text{H}_3}}{\partial y} & -\frac{\partial N_i^{\text{H}_4}}{\partial y} & 0 & 0 \end{bmatrix} \quad (4.124)$$

4.3.11.3 | Shear Mass Matrix

Considering the CPT(N), DKQ and IDKQ formulations, one can arrange the elemental shear mass sub-matrix as

$$\hat{\mathbf{N}}_s^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (4.125)$$

For the CPT(C) formulation, one adds one null column between the fifth and sixth columns to account for the extra degree of freedom - $\partial^2 w^i / \partial x \partial y$.

4.3.12 | Jacobian Transformation

The stiffness and mass matrices computation require a double integral, defined for convenience at an elemental level, within the local coordinate system (ξ, η) . However, integral expressions hold derivatives expressed in the global coordinate system, defined for convenience at an elemental level and therefore within the local coordinate system (ξ, η) . However, integrate expressions hold derivatives expressed in the global coordinate system of shape functions defined in the elements natural coordinates (ξ, η) . Computing global derivatives using locally defined shape functions is thus mandatory, and one can perform it using the global-local geometric transformation presented in Equations 4.54 and 4.55. Therefore, one can define a Jacobian matrix $\mathbf{J}$ as follows

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (4.126)$$

Using the interpolation functions one can re-write

$$\mathbf{J} = \begin{bmatrix} \sum_{i=1}^4 x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^4 y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^4 x_i \frac{\partial N_i}{\partial \eta} & \sum_{i=1}^4 y_i \frac{\partial N_i}{\partial \eta} \end{bmatrix} \quad (4.127)$$

At this point, one can perform the global derivatives of shape functions using the chain rule and the Jacobian definition as follows

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x} \\ \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y} \end{pmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} \quad (4.128)$$

where $\mathbf{J}^{-1}$ is the inverse Jacobian. Besides, the bending strain sub-matrices of the CPT(N) and CPT(C) elements require the computation of the second derivatives of the shape functions - see 4.108 and 4.109 . Adopting the Jacobian transformation used for the first derivatives, one can write

$$\begin{aligned} \frac{\partial^2 N_i}{\partial \xi^2} &= \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial \xi} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} \right) \\ &= \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial \xi} \right) \frac{\partial N_i}{\partial x} + \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \xi} + \frac{\partial}{\partial \xi} \left(\frac{\partial y}{\partial \xi} \right) \frac{\partial N_i}{\partial y} \\ &= \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \xi} + \frac{\partial^2 x}{\partial \xi^2} \frac{\partial N_i}{\partial x} + \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \xi} + \frac{\partial^2 y}{\partial \xi^2} \frac{\partial N_i}{\partial y} \\ &= \frac{\partial^2 N_i}{\partial x^2} \left(\frac{\partial x}{\partial \xi} \right)^2 + 2 \frac{\partial^2 N_i}{\partial x \partial y} \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} \right) + \frac{\partial^2 x}{\partial \xi^2} \frac{\partial N_i}{\partial x} + \frac{\partial^2 N_i}{\partial y^2} \left(\frac{\partial y}{\partial \xi} \right)^2 + \frac{\partial^2 y}{\partial \xi^2} \frac{\partial N_i}{\partial y} \end{aligned} \quad (4.129)$$

Similarly,

$$\begin{aligned} \frac{\partial^2 N_i}{\partial \eta^2} &= \frac{\partial}{\partial \eta} \left(\frac{\partial N_i}{\partial \eta} \right) = \frac{\partial}{\partial \eta} \left(\frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} \right) \\ &= \frac{\partial}{\partial \eta} \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial \eta} \left(\frac{\partial x}{\partial \eta} \right) \frac{\partial N_i}{\partial x} + \frac{\partial}{\partial \eta} \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \eta} + \frac{\partial}{\partial \eta} \left(\frac{\partial y}{\partial \eta} \right) \frac{\partial N_i}{\partial y} \\ &= \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \eta} \right) \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \eta} + \frac{\partial^2 x}{\partial \eta^2} \frac{\partial N_i}{\partial x} + \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \eta} \right) \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \frac{\partial N_i}{\partial y} \\ &= \frac{\partial^2 N_i}{\partial x^2} \left(\frac{\partial x}{\partial \eta} \right)^2 + 2 \frac{\partial^2 N_i}{\partial x \partial y} \left(\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} \right) + \frac{\partial^2 x}{\partial \eta^2} \frac{\partial N_i}{\partial x} + \frac{\partial^2 N_i}{\partial y^2} \left(\frac{\partial y}{\partial \eta} \right)^2 + \frac{\partial^2 y}{\partial \eta^2} \frac{\partial N_i}{\partial y} \end{aligned} \quad (4.130)$$

Following the same reasoning for the cross partial derivative, one can establish

$$\begin{aligned} \frac{\partial^2 N_i}{\partial \xi \partial \eta} &= \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial \eta} \right) = \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} \right) \\ &= \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \eta} + \frac{\partial}{\partial \xi} \left(\frac{\partial x}{\partial \eta} \right) \frac{\partial N_i}{\partial x} + \frac{\partial}{\partial \xi} \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \eta} + \frac{\partial}{\partial \xi} \left(\frac{\partial y}{\partial \eta} \right) \frac{\partial N_i}{\partial y} \\ &= \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial N_i}{\partial x} \right) \frac{\partial x}{\partial \eta} + \frac{\partial^2 x}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial x} + \left(\frac{\partial}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial N_i}{\partial y} \right) \frac{\partial y}{\partial \eta} + \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial y} \\ &= \frac{\partial^2 N_i}{\partial x^2} \left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} \right) + \frac{\partial^2 N_i}{\partial x \partial y} \left(\frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} \right) + \frac{\partial^2 x}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial x} + \frac{\partial^2 N_i}{\partial y^2} \left(\frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} \right) + \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial y} \end{aligned} \quad (4.131)$$

One can write the preceding expressions in a matrix form, as shown in the following

$$\begin{Bmatrix} \frac{\partial^2 N_i}{\partial \xi^2} \\ \frac{\partial^2 N_i}{\partial \eta^2} \\ \frac{\partial^2 N_i}{\partial \xi \partial \eta} \end{Bmatrix} = \begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 & \left(\frac{\partial y}{\partial \xi}\right)^2 & 2\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} \\ \left(\frac{\partial x}{\partial \eta}\right)^2 & \left(\frac{\partial y}{\partial \eta}\right)^2 & 2\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ \frac{\partial^2 N_i}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} \frac{\partial^2 x}{\partial \xi^2} & \frac{\partial^2 y}{\partial \xi^2} \\ \frac{\partial^2 x}{\partial \eta^2} & \frac{\partial^2 y}{\partial \eta^2} \\ \frac{\partial^2 x}{\partial \xi \partial \eta} & \frac{\partial^2 y}{\partial \xi \partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \quad (4.132)$$

where the square matrix $\bar{\mathbf{J}}$ is given by

$$\bar{\mathbf{J}} = \begin{bmatrix} \left(\frac{\partial x}{\partial \xi}\right)^2 & \left(\frac{\partial y}{\partial \xi}\right)^2 & 2\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi} \\ \left(\frac{\partial x}{\partial \eta}\right)^2 & \left(\frac{\partial y}{\partial \eta}\right)^2 & 2\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} & \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \end{bmatrix} \quad (4.133)$$

Thus enabling one to compute the global second derivatives as a function of the local ones, defined in the (ξ, η) coordinate system

$$\begin{Bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \\ \frac{\partial^2 N_i}{\partial x \partial y} \end{Bmatrix} = \bar{\mathbf{J}}^{-1} \left\{ \begin{Bmatrix} \frac{\partial^2 N_i}{\partial \xi^2} \\ \frac{\partial^2 N_i}{\partial \eta^2} \\ \frac{\partial^2 N_i}{\partial \xi \partial \eta} \end{Bmatrix} - \begin{bmatrix} \frac{\partial^2 x}{\partial \xi^2} & \frac{\partial^2 y}{\partial \xi^2} \\ \frac{\partial^2 x}{\partial \eta^2} & \frac{\partial^2 y}{\partial \eta^2} \\ \frac{\partial^2 x}{\partial \xi \partial \eta} & \frac{\partial^2 y}{\partial \xi \partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \right\} \quad (4.134)$$

Besides, the coordinate transformation requires the definition of the integration domain in natural coordinates using the Jacobian determinant

$$dS = dx dy = \det(\mathbf{J}) d\xi d\eta \quad (4.135)$$

Accordingly, one performs the double integral of a function $f(x, y)$ at the element level in the following manner

$$\int_{S^e} f(x, y) dS^e = \int_{-1}^1 \int_{-1}^1 f(\xi, \eta) \det(\mathbf{J}) d\xi d\eta \quad (4.136)$$

The Gauss-Legendre numeric integration is also a common practice in the FEM. Usually, the integration domain of such a method considers the limits $[-1, 1]$ for both axes, and one approximates the integral by a weighted sum of function values at specified points within the domain as

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \sum_{g=1}^{n_g} w_g f(\xi_g, \eta_g) \quad (4.137)$$

where n_g is the number of Gauss points, and w_g is the weight factor of each Gauss point. One can exactly integrate a polynomial of degree p by considering

$$n_g = \frac{1}{2}(p + 1) \quad (4.138)$$

When $p + 1$ is odd, one should consider the nearest larger integer. Considering the element formulations presented before, one has to integrate polynomial integrands of different orders. Table 4.3 shows Gauss points, coordinates and the corresponding weight factors.

Table 4.3 Gauss-Legendre integration

Gauss Points	Coordinates	Weights
n_g	ξ_g, η_g	w_g
1	0	2
2	$\pm\sqrt{\frac{1}{3}}$	1
3	0	$\frac{8}{9}$
	$\pm\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18 + \sqrt{30}}{36}$
	$\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	$\frac{18 - \sqrt{30}}{36}$

4.4 | Facet-Shell Finite Element

In this work, one extends the plate element formulation considered in the previous sections to three-dimensional structures, namely shells. With this approach, the main goal is to expand the advantages of this plate deformation theory, namely the multi-layer mechanical behaviour with the degrees of freedom of single-layer approaches, to more complex structures like shells, providing a good compromise between accuracy and computational time. This advantage is of the utmost relevance to the automotive, railway, naval, aircraft or aerospace industries, for example, where researchers and engineers design complex structures every day and require efficient and reliable mechanical models. In fact, a shell element developed for the computational modelling of inhomogeneous multilayered composite shell structures, able to capture the mechanical behaviour of partial damping treatments, is one of the main contributions of the present work.

Therefore, one expresses the three-dimensional facet shell element in a global Cartesian frame (X, Y, Z) , Figure 4.8a, with its stiffness and mass matrices computed in a local Cartesian frame (x, y, z) , Figure 4.8b, whose $x - y$ plane is coplanar with the element face. Similar to the base plate, this element assumes a square configuration with each side equal to twice the unit when expressed in the local natural coordinates.

The unit vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$, the direction cosines, define the element facet plane in the global coordinate system, and one can obtain them with the global coordinates of the element nodes $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_4$. For example, it is possible to obtain $\hat{\mathbf{n}}_1$ using the coordinates of the four nodes as

$$\hat{\mathbf{n}}_1 = \frac{(\mathbf{P}_2 - \mathbf{P}_1) + (\mathbf{P}_3 - \mathbf{P}_4)}{\|(\mathbf{P}_2 - \mathbf{P}_1) + (\mathbf{P}_3 - \mathbf{P}_4)\|} \quad (4.139)$$

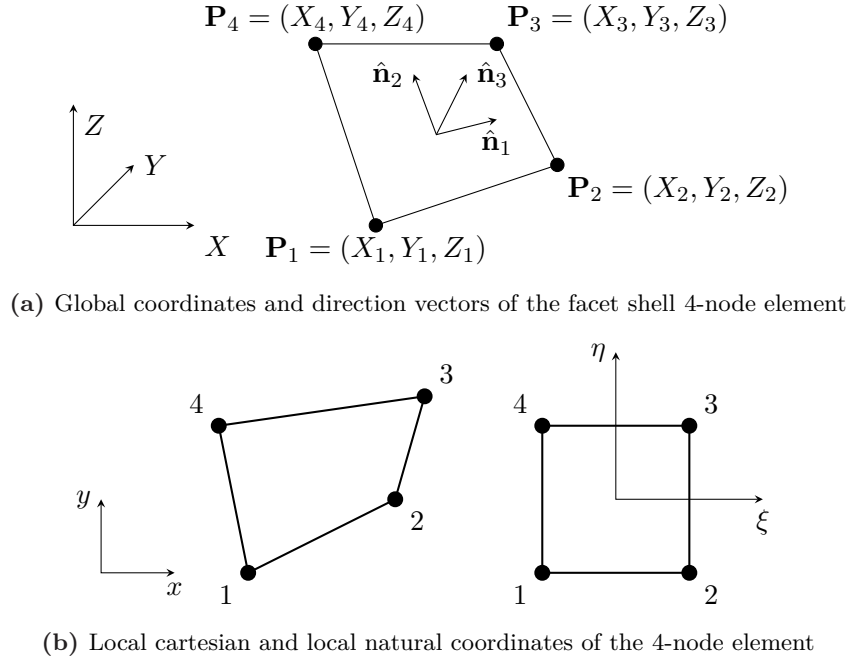


Figure 4.8 Facet shell 4-node element

Likewise, one can obtain the normal unit vector, $\hat{\mathbf{n}}_3$, using the unit vector $\hat{\mathbf{n}}_1$ and a coplanar vector as follows

$$\hat{\mathbf{n}}_3 = \frac{\hat{\mathbf{n}}_1 \times (\mathbf{P}_4 - \mathbf{P}_1)}{\|\hat{\mathbf{n}}_1 \times (\mathbf{P}_4 - \mathbf{P}_1)\|} \quad (4.140)$$

Finally, one can compute the third direction using the previous unit vectors

$$\hat{\mathbf{n}}_2 = \frac{\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_1}{\|\hat{\mathbf{n}}_3 \times \hat{\mathbf{n}}_1\|} \quad (4.141)$$

Before assembling the global stiffness and mass matrices, one transforms the respective elemental matrices into the global frame of the structure. This transformation, performed through a transformation matrix $\mathbf{T}$, depends on the director cosines between the two frames as follows

$$\mathbf{T}_{3 \times 3} = \begin{bmatrix} \hat{\mathbf{n}}_1 \\ \hat{\mathbf{n}}_2 \\ \hat{\mathbf{n}}_3 \end{bmatrix} \quad (4.142)$$

which allows the transformation of the stiffness and mass matrices to the global frame, operating on 3×3 sub-matrices corresponding to the translation or rotation degrees of freedom according to x , y and z . Therefore, one can compute the global stiffness sub-matrices as

$$\mathbf{k}_{XYZ}^e = \mathbf{T}^T \mathbf{k}_{xyz}^e \mathbf{T} \quad (4.143)$$

and the global mass sub-matrices as

$$\mathbf{m}_{XYZ}^e = \mathbf{T}^T \mathbf{m}_{xyz}^e \mathbf{T} \quad (4.144)$$

The main difficulty when using plate elements to model three-dimensional geometries like shells in a facet shell element configuration is the correct assembly of the in-plane rotations.

Similarly to the classical plate elements, there is no out-plane rotation degree of freedom, also called drill rotation, in the displacement field of the plate formulation - see Equation 4.24. This introduces an issue during the assembly process leading to ill-conditioned matrices [Clough and Johnson, 1971, Cook, 1987, Hughes and Brezzi, 1989]. Moreira and Dias Rodrigues [2011] describe, in detail, the main approaches expressed in literature to overcome such a problem. From the Clough and Johnson [1971] solutions to overcome the ill-conditioning problem resulting from the null stiffness associated with the drill rotation, eliminating the null columns and rows of the stiffness matrix associated with these degrees of freedom to the enforcement of a fictitious stiffness value for all the drilling degrees of freedom. Cook [1993] explored this concept by applying it to a triangular shell finite element with a fictitious drilling degree-of-freedom stiffness matrix, defined with a scaling constant and setting the relative magnitude of the drilling stiffness within the membrane stiffness. Similarly, Zienkiewicz and Taylor [2000] established a relation between the fictitious drilling stiffness terms and the bending stiffness. Such scaling procedures can lead to erroneous results due to numerical inconsistencies like membrane locking [Choi and Lee, 2003, Macneal and Harder, 1988], where the stiffness coupling between the membrane and bending stiffness tends to lock the in-plane rotations [Cook, 1994]. As Moreira and Dias Rodrigues [2011] pointed out, one can mitigate this locking effect when elements tend to be more coplanar, reducing the mesh size or defining small scaling parameters.

Nevertheless, despite being an ingenious way of three-dimensional modelling of geometries with out-plane rotation degrees of freedom, this approach has no physical meaning, and the procedure could lead to incorrect results. Later, Allman [1984] established the foundations for formulations with intrinsic drilling stiffness with a concept based on an equivalent quadratic finite element with only translation degrees of freedom - Figure 4.9.

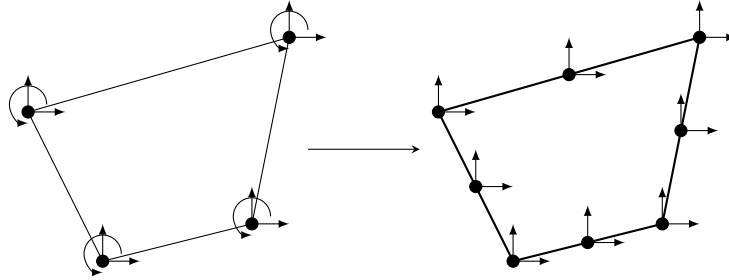


Figure 4.9 Allman's concept

Then, Cook [1986] extended this approach to a four-node quadrangular finite element relating the original degrees of freedom to an equivalent quadratic finite element using a projection matrix as follows

$$\{u_1, v_1, \dots, u_8, v_8\}^T = \mathbf{P}^D \{u_1, v_1, \theta_1^z, \dots, u_4, v_4, \theta_4^z\}^T \quad (4.145)$$

which one then applies to the stiffness matrix associated with only translation degrees of freedom to derive the low-order quadrilateral stiffness matrix

$$\mathbf{k}^{(4)} = \mathbf{P}^{D^T} \mathbf{k}^{(8)} \mathbf{P}^D \quad (4.146)$$

Macneal and Harder [1988] extended Cook [1986]'s concept by defining the stiffness matrix using the strain matrix of the four-node quadrilateral, obtaining inspiring results. Alternatively, Allman [1988] presented a low-order quadrilateral finite element with drilling stiffness employing the superposition of four membrane triangles with out-plane rotations. Nevertheless, these

elements rely on a *free-formulation*, where one introduces the out-plane rotation through the base displacement field improvement.

More recently, Hughes and Brezzi [1989] derived a variational formulation with independent interpolation of the drilling stiffness, extended by Ibrahimbegovic et al. [1990] to a quadrilateral finite element based on Allman [1984]s concept, developing a reliable finite element with a noteworthy performance. Nevertheless, the required 3×3 quadrature scheme is computationally expensive due to the finite element formulation as a cubic eight-node quadrilateral with an additional degree of freedom. Following the same variational formulation, Kouhia [1996] proposed a membrane finite element with out-plane rotation, formulated considering a modification of the element displacement field, introducing a non-conforming and independent interpolation of these rotation degrees of freedom. Moreira and Dias Rodrigues [2011] followed this concept, defining the out-plane rotation degrees of freedom at the corner nodes of the quadrilateral element. In this work, one follows Moreira and Dias Rodrigues [2011]’s approach to extend the plate element formulation considered in the previous sections to facet shells.

4.4.1 | Facet-Shell Finite Element Formulation

Following Kouhia [1996], Moreira and Dias Rodrigues [2011] presented a membrane element with out-plane rotation degrees of freedom by modifying the interpolation of the element displacement field by including a non-conformal and independent interpolation of the referred rotation degrees of freedom

$$\mathbf{u}_m^e = \sum_{i=1}^4 N_i \{\mathbf{d}_i^e\}_m + N_5 \frac{L_\xi}{8} (\theta_{\xi-}^z - \theta_{\xi+}^z) + N_6 \frac{L_\eta}{8} (\theta_{\eta+}^z - \theta_{\eta-}^z) \quad (4.147)$$

with $\{\mathbf{d}_i^e\}_m$ being the in-plane translation degrees of freedom at each node of the finite element, $\{u_i, v_i\}$, and N_i the bilinear shape functions presented earlier. The parameters L_ξ and L_η correspond to the distances between the opposite midline nodes along ξ and η natural coordinate system axes, respectively - Figure 4.10. The unit vectors $\hat{n}_\xi$ and $\hat{n}_\eta$ represent the normals to the lines connecting these same nodes.

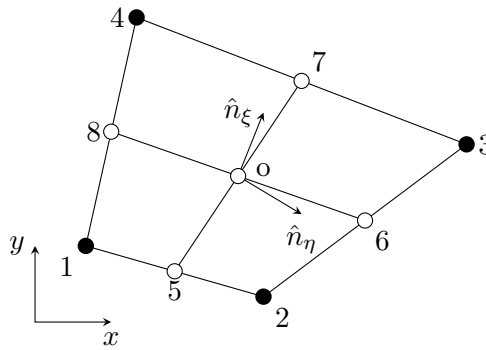


Figure 4.10 Element with out-plane rotation. Adapted from Moreira and Dias Rodrigues [2011]

The shape functions N_5 and N_6 are non-conforming interpolators of the mid-side out-plane rotations $\theta_{\xi-}^z$, $\theta_{\xi+}^z$, $\theta_{\eta-}^z$ and $\theta_{\eta+}^z$, and can be written in the form

$$N_5 = 1 - \xi^2, \quad N_6 = 1 - \eta^2 \quad (4.148)$$

In addition, Moreira and Dias Rodrigues [2011] proposed a projection operator to determine the mid-side rotations $\theta_{\xi^-}^z$, $\theta_{\xi^+}^z$, $\theta_{\eta^-}^z$ and $\theta_{\eta^+}^z$ from the corner out-plane rotation degrees of freedom. Based on the application of a conforming interpolation through the conventional bilinear shape functions, this projection operator is

$$\theta^z(\xi, \eta) = \sum_{i=1}^4 N_i(\xi, \eta) \theta_i^z \quad (4.149)$$

Therefore, one can compute the rotations at the dummy nodes using a projection matrix of the form

$$\begin{Bmatrix} \theta_{\eta^-}^z \\ \theta_{\xi^+}^z \\ \theta_{\eta^+}^z \\ \theta_{\xi^-}^z \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1^z \\ \theta_2^z \\ \theta_3^z \\ \theta_4^z \end{Bmatrix} \quad (4.150)$$

which leads to the relative rotation vector, defined as

$$\begin{Bmatrix} \theta_{\xi^-}^z - \theta_{\xi^+}^z \\ \theta_{\eta^+}^z - \theta_{\eta^-}^z \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_1^z \\ \theta_2^z \\ \theta_3^z \\ \theta_4^z \end{Bmatrix} \quad (4.151)$$

It is possible to write the projection matrix defined in Equation 4.151 from the nodal index matrix in Table 4.2. Considering the displacement field defined in Equation 4.147, one can obtain the strain matrix for this element by applying the differential operator described in Equation 3.4. Thus, the components of this strain matrix are of the form

$$\frac{\partial u}{\partial x} = \sum_{i=1}^4 \frac{\partial N_i}{\partial x} u_i + \sum_{i=1}^4 \frac{1}{16} \left(-L_{\xi} \hat{n}_{\xi}^x \frac{\partial N_5}{\partial x} \xi_i + L_{\eta} \hat{n}_{\eta}^x \frac{\partial N_6}{\partial x} \eta_i \right) \theta_i^z \quad (4.152a)$$

$$\frac{\partial u}{\partial y} = \sum_{i=1}^4 \frac{\partial N_i}{\partial y} u_i + \sum_{i=1}^4 \frac{1}{16} \left(-L_{\xi} \hat{n}_{\xi}^y \frac{\partial N_5}{\partial y} \xi_i + L_{\eta} \hat{n}_{\eta}^y \frac{\partial N_6}{\partial y} \eta_i \right) \theta_i^z \quad (4.152b)$$

$$\frac{\partial v}{\partial x} = \sum_{i=1}^4 \frac{\partial N_i}{\partial x} v_i + \sum_{i=1}^4 \frac{1}{16} \left(-L_{\xi} \hat{n}_{\xi}^y \frac{\partial N_5}{\partial x} \xi_i + L_{\eta} \hat{n}_{\eta}^y \frac{\partial N_6}{\partial x} \eta_i \right) \theta_i^z \quad (4.152c)$$

$$\frac{\partial v}{\partial y} = \sum_{i=1}^4 \frac{\partial N_i}{\partial y} v_i + \sum_{i=1}^4 \frac{1}{16} \left(-L_{\xi} \hat{n}_{\xi}^x \frac{\partial N_5}{\partial y} \xi_i + L_{\eta} \hat{n}_{\eta}^x \frac{\partial N_6}{\partial y} \eta_i \right) \theta_i^z \quad (4.152d)$$

where ξ_i and η_i are the nodal indices defined in Table 4.2.

4.4.2 | Stabilisation

Moreira and Dias Rodrigues [2011] acknowledge that the proposed formulation for the out-plane rotational degrees of freedom in the membrane formulation produces a rank deficiency in the stiffness matrix and consequently leads to the emergence of a spurious deformation mode with

equal rotational components and null translational ones, as described by Macneal and Harder [1988] or Cook [1986]

$$\theta_1^z = \theta_2^z = \theta_3^z = \theta_4^z \quad (4.153)$$

Allman [1984], Cook [1986] and Cook [1987] suppressed this spurious mode by restricting the out-plane rotation degree of freedom at a specific node. However, as Moreira and Dias Rodrigues [2011] pointed out, although it is a valid and effective solution, the degree of freedom constraint is not an intrinsic form of stabilisation of the associated finite element, requiring user intervention. Thus, Allman [1988] proposed a way to stabilise the spurious mode in the quadrilateral element with out-plane rotational degrees of freedom through the internal constraint of the central node of the element, forcing the equality between this central degree of freedom and the average of the nodal rotations.

With the same intent, Cook [1993] applied a stabilisation mechanism to the triangular element based on the addition of an energy term associated with the elastic constraint between the average of the nodal rotations and the average of the rotations of the sides of the triangle, calculated by the corresponding nodal translations. Therefore, one can extend this stabilisation process to the quadrilateral element by superposing four triangles Cook [1994]. Macneal and Harder [1988] proposed a similar solution for stabilisation, which allows, on the one hand, to relate the nodal rotations to the out-plane rotation of the element, and, on the other hand, avoid the spurious mode under analysis. Thus, one adds an energy term to stabilise the finite element in the form

$$\Pi^{D1} = \frac{1}{2} A^e h G \gamma \left(\theta_0^z - \frac{1}{n} \sum_{i=1}^n \theta_i^z \right)^2 \quad (4.154)$$

where A^e , h and G represent the finite element area, its thickness and the shear modulus of the material, respectively, and the parameter γ is a dimensionless constant, usually of small value - 10^{-6} [Cook et al., 2001, Macneal and Harder, 1988]. This energy term allows applying an elastic constraint between the average of the nodal rotations, θ_i^z , with the real rotation at the centre of the element, defined by

$$\theta_0^z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_0 \quad (4.155)$$

In the end, one accomplishes the addition of the energy term described in Equation 4.154 by correcting the membrane stiffness matrix of the element by the addition of an associated matrix of the form

$$\mathbf{k}^{D1} = \gamma A^e h G \mathbf{Q}^T \mathbf{Q} \quad (4.156)$$

where the matrix $\mathbf{Q}$ is given by the following expression

$$\mathbf{Q} \mathbf{d}_m^{\theta^z} = \theta_0^z - \frac{1}{n} \sum_{i=1}^n \theta_i^z \quad (4.157)$$

with $\mathbf{d}_m^{\theta^z}$ being the in-plane translation together with drill degrees of freedom at each node of the finite element $\{u, v, \theta^z\}^T$.

Moreira and Dias Rodrigues [2011] also called attention to the fact that Macneal and Harder [1988] reported the emergence of a second spurious mode whose shape depends on the element configuration. Still, for a rectangular shape, this spurious mode relates to alternating rotations, defined as

$$\theta_1^z = -\theta_2^z = \theta_3^z = -\theta_4^z \quad (4.158)$$

in addition to non-zero translational components. Following Macneal and Harder [1988], Moreira and Dias Rodrigues [2011] proposed an additional stabilisation matrix, defined as

$$\mathbf{k}^{D2} = \phi A^e h G \mathbf{R}^T \mathbf{R} \quad (4.159)$$

where ϕ is a dimensionless constant, assuming a low value (10^{-6} [Macneal and Harder, 1988]), and $\mathbf{R}$ is a matrix defined as

$$\mathbf{R} \mathbf{d}_m^{\theta^z} = \frac{1}{4} (\theta_1^z - \theta_2^z + \theta_3^z - \theta_4^z) \quad (4.160)$$

It is also important to mention that, according to Macneal and Harder [1988] and Clough and Johnson [1971], one should not include inertia terms associated with drill degrees of freedom in the formulation of the mass matrix; consequently, one avoids the associated spurious modes [Macneal and Harder, 1988]. Moreira and Dias Rodrigues [2011] presented this general and direct stabilisation mechanism to the quadrilateral element, and one follows the same approach in this work.

4.5 | Equations of Motion

Regardless of the numerical method, whether using the RRM, the FEM or any other procedure, the ultimate goal is to reduce the problem to the well-known system of equations of motion

$$\mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{K} \mathbf{u}(t) = \mathbf{f}(t) \quad (4.161)$$

Considering the use of viscoelastic materials

$$\mathbf{M}(\omega) \ddot{\mathbf{u}}(t) + \tilde{\mathbf{K}}(\omega) \mathbf{u}(t) = \mathbf{f}(t) \quad (4.162)$$

where $\mathbf{M}$ is the frequency dependent mass matrix and $\tilde{\mathbf{K}}$ the complex stiffness matrix which is frequency dependent due to the mechanical characterisation of viscoelastic materials. The $\mathbf{u}(t)$ and $\mathbf{f}(t)$ vectors are the system response and external force vector, respectively. For a harmonic excitation of frequency ω , one can represent the force vector in a complex form given by

$$\mathbf{f}(t) = \mathbf{F}(\omega) e^{j\omega t} \quad (4.163)$$

Consequently, the system response is a harmonic function with amplitude $\mathbf{u}(\omega)$ and frequency ω . However, due to the introduced damping, the response shows a phase lag δ

$$\mathbf{u}(t) = \mathbf{U}(\omega) e^{(j\omega t - \delta)} = \tilde{\mathbf{U}}(\omega) e^{j\omega t} \quad (4.164)$$

Substituting Equations 4.163, 4.164 and the respective time derivative in Equation 4.162, one obtains a system of algebraic equations

$$\left[\tilde{\mathbf{K}}(\omega) - \omega^2 \mathbf{M}(\omega) \right] \tilde{\mathbf{U}}(\omega) = \mathbf{F}(\omega) \quad (4.165)$$

One can obtain the response vector $\tilde{\mathbf{U}}(\omega)$ by

$$\tilde{\mathbf{U}}(\omega) = \left[\tilde{\mathbf{K}}(\omega) - \omega^2 \mathbf{M}(\omega) \right]^{-1} \mathbf{F}(\omega) \quad (4.166)$$

The system response depends upon the frequency and mechanical properties, and it can be re-written as

$$\tilde{\mathbf{U}}(\omega) = [\alpha(\omega)]\mathbf{F}(\omega) \quad (4.167)$$

One can obtain the Frequency Response Function (FRF) of a column k of the receptance matrix $[\alpha(\omega)]$ by

$$\alpha_{jk}(\omega) = \frac{\tilde{U}_j(\omega)}{F_k}; F_r = 0, r = 1, \dots, n \wedge r \neq k \quad (4.168)$$

where $\tilde{U}_j(\omega)$ represents the amplitude and phase of the system response in the degree-of-freedom j when applying an excitation F_k at the degree of freedom k . The system response can also be computed as velocity or acceleration, replacing the displacement response $\tilde{\mathbf{U}}(\omega)$ by velocity $j\omega\tilde{\mathbf{U}}(\omega)$, or acceleration $-\omega^2\tilde{\mathbf{U}}(\omega)$ [He and Fu, 2001]. Additionally, the mobility FRF takes the form

$$Y_{jk}(\omega) = j\omega \frac{\tilde{U}_j(\omega)}{F_k}; F_r = 0, r = 1, \dots, n \wedge r \neq k \quad (4.169)$$

which taken into account Equation 4.168 can also be written as

$$Y_{jk}(\omega) = j\omega\alpha_{jk}(\omega) \quad (4.170)$$

At last, the accelerance FRF takes the following form

$$\mathcal{A}_{jk}(\omega) = -\omega^2 \frac{\tilde{U}_j(\omega)}{F_k}; F_r = 0, r = 1, \dots, n \wedge r \neq k \quad (4.171)$$

which taken into account Equation 4.168 can also be written as

$$A_{jk}(\omega) = -\omega^2\alpha_{jk}(\omega) \quad (4.172)$$

Numerical Validation Results

5.1 | Introduction

One of the main contributions of the present work is the extension of Loredó and Castel [2013] model to inhomogeneous anisotropic multilayered shells, so the mechanical model must be evaluated, together with the corresponding numerical methods, by comparing the obtained results with those available in the literature. In the first place, one compares both the numerical methods implemented in the present work, the RRM and the FEM, with Loredó and Castel [2013] results using Navier's procedure, which provides exact solutions for simply supported plates. The main focus of such analysis is to compare both numerical methods and draw conclusions regarding their performance. To this end, one investigates the convergence of results for square isotropic plates using the two numerical methods presented and addresses the main differences between warping functions, i.e. deformation theories.

Regardless of the inherent merits of each procedure, the FEM has an advantage that is difficult to overcome by other methods, namely its elementary nature of sub-division of the domain, which is suitable for complex inhomogeneous structures with intricate geometries. Since one of the main goals of the present work is the extension of Loredó and Castel [2013] model to more complex geometries such as inhomogeneous anisotropic multilayered shells, one follows in this chapter with the evaluation of the implemented model considering composite plates but establishing comparisons with the FEM only. Considering a different work from Loredó and Castel [2014], one evaluates the presented implementation with the FEM considering different types of composite plates to handle all kinds of arrangements - symmetric and anti-symmetric stacking sequences, including cross-ply and angle-ply laminates.

Later in this chapter, one follows with the evaluation of the implemented mechanical model by performing a set of benchmark tests to assess the performance of the implemented facet shell finite element, firstly with patch tests concerning the membrane component of the proposed finite element, namely the out-of-plane rotation and the corresponding drilling degrees of freedom. Then, one carries out the performance of the facet shell element with both bending and shear

components coupled with the membrane one using well-known patch tests such as the Scordelis-Lo roof, the pinched cylinder with diaphragm supports or the pinched hemisphere with hole.

Finally, the present chapter closes with the validation of the mechanical model for sandwich shells, including composite sandwich shells with viscoelastic damping cores and composite facings with internal damping, assuming different loss factors for each elastic property. Thus, the main goal of the following chapter is to prove the robustness and feasibility of the mechanical model presented in this work, namely the extension to inhomogeneous composite shell structures with internal material damping of the plate model presented by Loredo and Castel [2013].

5.2 | Isotropic Plates

In their work, Loredo and Castel [2013] presented a simple three-layered isotropic square plate of two different materials to launch the study of their plate model for static and dynamic analysis of inhomogeneous anisotropic multilayered plates. The outer layers are of the same material, with Young's modulus E_s , while the core layer has Young's modulus E_c . All three layers are of the same thickness ($h_1 = h_2 = h_3 = h/3$), sharing the same density ($\rho_s = \rho_c$) and Poisson's ratio ($\nu_s = \nu_c = 0.33$). The square plate has a side length of a , and the length-to-thickness ratio is $a/h = 400$. Table 5.1 presents the first dimensionless natural frequency considering different deformation theories (FSDT, TSDT, FoZZ and ToZZ) and Young's modulus ratios between the outer and inner layers using the RRM. The dimensionless natural frequencies are given by

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_s}} \quad (5.1)$$

First and foremost, the results presented by Loredo and Castel [2013], shown in the last column of Table 5.1, are worth mentioning. The differences between an FSDT and FoZZ model for a laminate with outer and inner layers of the same material are incomprehensible. For such a laminate, the warping functions for both deformation theories are the same, i.e. there should be no differences in natural frequencies, as it happens in the presented results obtained with the RRM when the outer and inner layers share the same Young's modulus. Nevertheless, the results show convergence up to the fourth decimal place, even when the set is just 30 functions in each direction. Even so, the most noticeable aspect in Table 5.1 is Young's modulus ratio and its influence on the obtained results with different deformation theories. As this ratio increases, the outer layers become stiffer when compared with the core layer, and the difference between an ESLT like the FSDT or the TSDT and the FoZZ or the ToZZ models also increases. Indeed, an ESLT can not capture the mechanical behaviour of each lamina, contrary to a ZZT, as explained earlier in this work. Such behaviour is indispensable when dealing with different materials in the same laminate, as is the case of laminates with damping materials, the subject of this work. Loredo and Castel [2013] compared their results with 3D FEM solutions and found that the FoZZ model is accurate, particularly for structures exhibiting strong modulus ratios between layers, which is the case of composite structures with embedded viscoelastic layers, for example. Even so, unsurprisingly, there are no significant differences between the first and third-order deformation theories since the thin plate has an aspect ratio of $a/h = 400$.

To illustrate the differences between an ESLT and a ZZT, Figure 5.1 shows the warping functions for the plate theories analysed in this work considering different Young's modulus ratios. As expected, recalling the formulations presented in Chapter 3, one can verify that an ESLT as the FSDT or the TSDT is based on purely geometric concepts and is independent of the Young's modulus ratio. On the other hand, the warping functions of a ZZT as the FoZZ or the ToZZ depend on the material properties, namely the Young's modulus ratio between materials, as

Table 5.1 Isotropic plate - comparison with Navier's procedure using the RRM

Model	Basis Functions				Navier ¹
	20 × 20	30 × 30	40 × 40	50 × 50	
$E_s/E_c = 10^0$					
FSDT	6.03629	6.03626	6.03625	6.03625	6.03628 ¹
TSDT	6.03628	6.03624	6.03623	6.03623	-
FoZZ	6.03629	6.03626	6.03625	6.03625	6.03631 ¹
ToZZ	6.03628	6.03624	6.03623	6.03623	-
$E_s/E_c = 10^2$					
FSDT	5.92456	5.92452	5.92451	5.92451	5.92454 ¹
TSDT	5.92444	5.92441	5.92440	5.92440	-
FoZZ	5.91906	5.91904	5.91903	5.91903	5.91909 ¹
ToZZ	5.91903	5.91901	5.91900	5.91900	-
$E_s/E_c = 10^4$					
FSDT	5.92343	5.92339	5.92339	5.92338	5.92341 ¹
TSDT	5.92331	5.92328	5.92327	5.92327	-
FoZZ	5.43938	5.43938	5.43937	5.43937	5.43943 ¹
ToZZ	5.43936	5.43936	5.43935	5.43935	-
$E_s/E_c = 10^6$					
FSDT	5.92342	5.92338	5.92338	5.92337	5.92340 ¹
TSDT	5.92330	5.92327	5.92326	5.92326	-
FoZZ	2.05147	2.05145	2.05144	2.05144	2.05146 ¹
ToZZ	2.05147	2.05145	2.05144	2.05144	-

¹ Loredo and Castel [2013]

proved by the different slopes of the warping functions for different layers; the core layer becomes softer and softer as the Young's modulus ratio increases and the slopes between the outer and inner layers become increasingly contrasting. One might recall Equation 3.2 to link the warping functions derivatives, or slopes, with the laminate transverse shear strain through the thickness. Therefore, only plate theories whose warping functions present a layerwise behaviour can capture the mechanical behaviour of each layer individually.

In addition, Tables 5.2 and 5.3 present the same comparison, but one considers here the FEM. The main difference to the RRM is the convergence rate. Even when considering a refined regular mesh of 80 elements in each direction, the convergence with Navier's procedure is just up to the second decimal place and one needs a even refined mesh to improve convergence. All the conclusions drawn from the results obtained with the RRM regarding the mechanical models are valid here since higher Young's modulus ratios lead to the inapplicability of an ESLT. Besides, there are no worth mentioning differences between finite elements apart from the tiny higher convergence rate of the CPT(N) and DKQ elements which seems not relevant enough to justify the improved performance of one element over the others. Nonetheless, an essential difference between the two numerical methods is evident at this stage. The convergence rate in the RRM is high, and one achieves convergence with fewer degrees of freedom. This fact is worth mentioning if one recalls that for a basis with 30 functions in each direction in the RRM, the problem dimension is 4500 degrees of freedom ($5 \times 30 \times 30$).

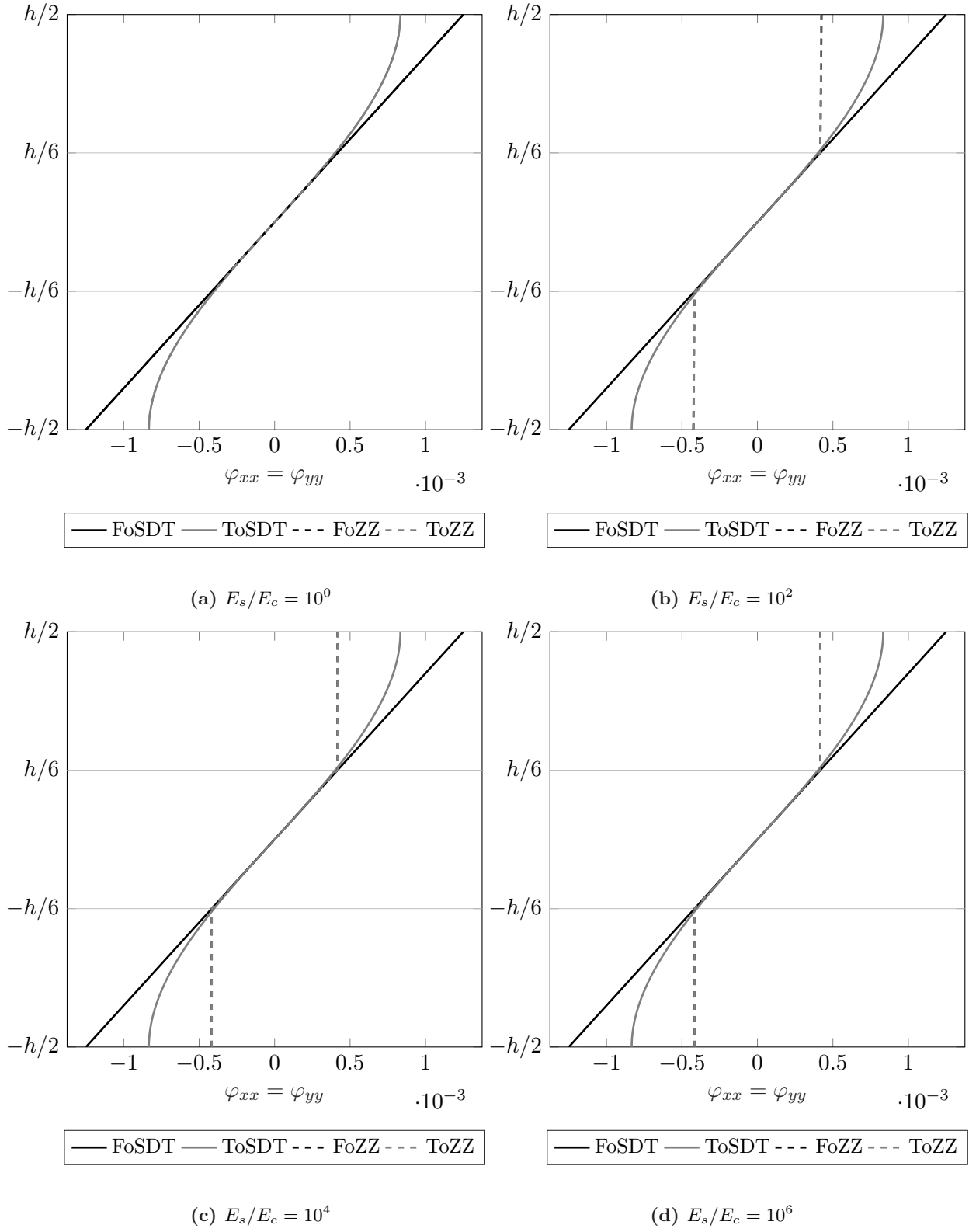


Figure 5.1 Warping functions $\varphi_{xx} = \varphi_{yy}$ of the isotropic plate with $a/h = 400$

Table 5.2 Isotropic plate - comparison with Navier's procedure using the FEM - 1

Model	Element	Mesh				Navier ¹
		20 × 20	40 × 40	80 × 80	160 × 160	
		$E_s/E_c = 10^0$				
FSDT	CPT(N)	6.04176	6.03763	6.03659	6.03633	6.03628 ¹
	CPT(C)	6.05001	6.03969	6.03711	6.03646	
	DKQ	6.03864	6.03685	6.03640	6.03628	
	IDKQ	6.04600	6.03869	6.03686	6.03640	
TSDT	CPT(N)	6.04174	6.03761	6.03657	6.03631	-
	CPT(C)	6.04999	6.03967	6.03709	6.03644	
	DKQ	6.03862	6.03683	6.03638	6.03627	
	IDKQ	6.04598	6.03867	6.03684	6.03638	
FoZZ	CPT(N)	6.04176	6.03763	6.03659	6.03633	6.03631 ¹
	CPT(C)	6.05001	6.03969	6.03711	6.03646	
	DKQ	6.03864	6.03685	6.03640	6.03628	
	IDKQ	6.04600	6.03869	6.03686	6.03640	
ToZZ	CPT(N)	6.04174	6.03761	6.03657	6.03631	-
	CPT(C)	6.04999	6.03967	6.03709	6.03644	
	DKQ	6.03862	6.03683	6.03638	6.03627	
	IDKQ	6.04598	6.03867	6.03684	6.03638	
		$E_s/E_c = 10^2$				
FSDT	CPT(N)	5.92992	5.92587	5.92485	5.92459	5.92454 ¹
	CPT(C)	5.93802	5.92789	5.92536	5.92472	
	DKQ	5.92686	5.92510	5.92466	5.92455	
	IDKQ	5.93408	5.92691	5.92511	5.92466	
TSDT	CPT(N)	5.92981	5.92575	5.92474	5.92448	-
	CPT(C)	5.93791	5.92778	5.92524	5.92461	
	DKQ	5.92674	5.92499	5.92455	5.92443	
	IDKQ	5.93397	5.92679	5.92500	5.92455	
FoZZ	CPT(N)	5.92443	5.92038	5.91937	5.91911	5.91909 ¹
	CPT(C)	5.93253	5.92241	5.91987	5.91924	
	DKQ	5.92137	5.91962	5.91918	5.91907	
	IDKQ	5.92859	5.92142	5.91963	5.91918	
ToZZ	CPT(N)	5.92440	5.92036	5.91934	5.91908	-
	CPT(C)	5.93250	5.92238	5.91984	5.91921	
	DKQ	5.92134	5.91959	5.91915	5.91904	
	IDKQ	5.92857	5.92139	5.91960	5.91915	

¹ Loredo and Castel [2013]

5. Numerical Validation Results

In the FEM, considering a regular mesh with 80 elements in each direction and the CPT(C) element is of 52 488 degrees of freedom ($8 \times (80 + 1) \times (80 + 1)$), and considering the CPT(N), DKQ or IDKQ element is of 45 927 degrees of freedom ($7 \times (80 + 1) \times (80 + 1)$). The benefits of the FEM are of critical importance and are extremely useful for this work, but the convergence rate is the price to pay.

Table 5.3 Isotropic plate - comparison with Navier's procedure using the FEM - 2

Model	Element	Mesh				Navier ¹
		20 × 20	40 × 40	80 × 80	160 × 160	
		$E_s/E_c = 10^4$				
FSDT	CPT(N)	5.92879	5.92474	5.92372	5.92347	5.92341 ¹
	CPT(C)	5.93689	5.92676	5.92423	5.92359	
	DKQ	5.92573	5.92397	5.92353	5.92342	
	IDKQ	5.93295	5.92578	5.92398	5.92353	
TSDT	CPT(N)	5.92867	5.92462	5.92361	5.92335	-
	CPT(C)	5.93677	5.92665	5.92411	5.92348	
	DKQ	5.92561	5.92386	5.92341	5.92330	
	IDKQ	5.93284	5.92566	5.92387	5.92342	
FoZZ	CPT(N)	5.44438	5.44063	5.43969	5.43945	5.43943 ¹
	CPT(C)	5.45182	5.44249	5.44015	5.43957	
	DKQ	5.44139	5.43988	5.43950	5.43940	
	IDKQ	5.44811	5.44156	5.43992	5.43951	
ToZZ	CPT(N)	5.44436	5.44061	5.43967	5.43943	-
	CPT(C)	5.45180	5.44247	5.44013	5.43955	
	DKQ	5.44137	5.43986	5.43948	5.43938	
	IDKQ	5.44809	5.44154	5.43990	5.43949	
		$E_s/E_c = 10^6$				
FSDT	CPT(N)	5.92878	5.92473	5.92371	5.92345	5.92340 ¹
	CPT(C)	5.93688	5.92675	5.92422	5.92358	
	DKQ	5.92572	5.92396	5.92352	5.92341	
	IDKQ	5.93294	5.92577	5.92397	5.92352	
TSDT	CPT(N)	5.92866	5.92461	5.92359	5.92334	-
	CPT(C)	5.93676	5.92664	5.92410	5.92347	
	DKQ	5.92560	5.92385	5.92340	5.92329	
	IDKQ	5.93283	5.92565	5.92385	5.92340	
FoZZ	CPT(N)	2.05636	2.05267	2.05175	2.05152	2.05146 ¹
	CPT(C)	2.05918	2.05337	2.05192	2.05156	
	DKQ	2.05270	2.05176	2.05152	2.05146	
	IDKQ	2.05638	2.05267	2.05175	2.05152	
ToZZ	CPT(N)	2.05636	2.05267	2.05175	2.05152	-
	CPT(C)	2.05918	2.05337	2.05192	2.05156	
	DKQ	2.05270	2.05176	2.05152	2.05146	
	IDKQ	2.05638	2.05267	2.05175	2.05152	

¹ Loredo and Castel [2013]

5.3 | Composite Plates

In a different work, Loredo and Castel [2014] presented results involving a similar mechanical model with extended capabilities to obtain the transverse warping functions from three-dimensional elasticity equations and improve the range of applications of the mechanical model. The authors considered composite materials in different stacking sequences to validate and evaluate the proposed procedures. One uses those test cases in this section to validate the mechanical model for dynamic analysis of inhomogeneous anisotropic multilayered plates using transverse warping functions. To evaluate the numerical implementation of the presented theory in its full extension, one considers composite materials in different stacking sequences to analyse symmetric and antisymmetric laminates, including cross-ply and angle-ply configurations.

As noted, Loredo and Castel [2014] proposed two methods to obtain the transverse warping functions from three-dimensional elasticity equations, which allows one to analyse and compare the results obtained with plate deformation theories, either ESLT or ZZT, with 3D solutions and to know the advantages and disadvantages of each one. One refers in this work to the model using warping functions issued from transverse shear stresses of analytical solutions as 3D-WF and to the model using warping functions computed from iterative integration of equilibrium equations as Ite-WF. Besides, the authors compared the results of each test case with exact solutions obtained with an algorithm which is a generalisation for general lamination sequences of existing algorithms for cross-ply and antisymmetric angle-ply lamination sequences, as described by Loredo [2014], and one refers to this model as Exa. Loredo and Castel [2014] have also extended the limits of the length-to-thickness ratio in their study to $a/h = 4$ and $a/h = 2$, values far beyond what one can consider a plate structure, to assess the capabilities of the method to obtain the transverse warping functions from three-dimensional elasticity equations. However, all models assume the generalised plane stress assumption and do not also consider the normal strain ε_{zz} in their formulation, and it is well known that these assumptions are valid for plates with high length-to-thickness ratios and made of similar materials. Therefore, the authors have also compared their results with another exact solution, Exa² reference here, computed for a virtual laminate with modified stiffness values to fit the generalised plane stress assumption and force $\varepsilon_{zz} \approx 0$.

One follows the same approach in this section of pushing the boundaries of the length-to-thickness ratio to evaluate the performance of the numerical implementation, particularly the FEM and the corresponding elements implemented in this work, and investigate their performance under demanding requirements. In this section, one performs a numerical comparison considering four laminates, including a sandwich plate. The analysis only considers two materials, a composite ply and a honeycomb core material used in the sandwich panel, and Table 5.4 list their properties.

Table 5.4 Material properties of the composite ply and honeycomb core

	E_1	E_2	G_{12}	G_{13}	G_{23}	ν_{12}	ρ
Composite ply (p)	$25E_2^p$	10^6	$0.5E_2^p$	$0.5E_2^p$	$0.2E_2^p$	0.25	1500
Core material (c)	E_2^c	4×10^4	$0.4E_2^c$	$1.5E_2^c$	$1.5E_2^c$	0.25	100

Loredo and Castel [2014] assessed the theory performance with Navier's procedure and compared three dimensionless quantities: deflections, natural frequencies and shear stresses. As previously discussed, the main goal of the present work is to extend this model to a facet shell theory with a FEM implementation. Consequently, the main focus in this section is to compare the different finite elements implemented in this work with Navier's procedure and not the theory itself, so one focuses our analysis on one of these dimensionless quantities, particularly

the natural frequencies, given by the following expression

$$\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\frac{\rho^{\text{ref}}}{E_2^{\text{ref}}}} \quad (5.2)$$

where one considers E_2^{ref} and ρ^{ref} as values of the core material for the sandwich panel and as values of the composite ply for the other laminates.

5.3.1 | Square [0/c/0] Sandwich Plate

In the first place, one considers a sandwich panel with an orthotropic composite material at the outer layers and a honeycomb-type material at the core - the square sandwich panel consists of three layers, two outer composite layers of thicknesses $h_1 = h_3 = 0.1h$ and a core layer of thickness $h_2 = 0.8h$. Considering this configuration, one simulates a high stiffness variation through the thickness to analyse the applicability of different deformation theories. Table 5.5 presents the first dimensionless natural frequencies considering distinct deformation theories and length-to-thickness ratios using the RRM and a basis of 30 functions in each direction. Additionally, one shows the results obtained by Loredo and Castel [2014] using Navier's procedure for comparison purposes.

Table 5.5 Square [0/c/0] sandwich plate - comparison with Navier's procedure using the RRM

Model	Method	a/h			
		2	4	10	100
FSDT	RRM	6.9339	12.233	21.096	27.275
	Navier¹	6.9339	12.233	21.096	27.275
TSDT	RRM	5.4307	9.4350	17.716	27.172
	Navier¹	5.4307	9.4350	17.716	27.172
FoZZ	RRM	5.2900	9.1084	17.185	27.149
	Navier¹	5.2900	9.1084	17.185	27.149
ToZZ	RRM	5.3281	9.1152	17.168	27.148
	Navier¹	5.3281	9.1152	17.168	27.148
3D-WF ¹	Navier¹	5.2730	9.0843	17.143	27.147
Ite-WF ¹	Navier¹	5.2703	9.0840	17.143	27.147
Exa ²¹	Navier¹	5.2702	9.0840	17.143	27.147
Exa ¹	Navier¹	5.3389	9.1452	17.177	27.148

¹ Loredo and Castel [2014]

Considering just 30 functions in each direction, the obtained results are precisely those given by Loredo and Castel [2014] using Navier's procedure. Once again, the results obtained with any ESLT differ from those using a ZZT, particularly as the length-to-thickness ratio decreases and the laminate thickens, proving the inadequacy of these theories to model multi-layer structures of different materials like sandwich panels. Furthermore, despite the expected differences, it is

interesting to note the remarkable performance of any ZZT compared with the results obtained through procedures based on 3D formulations, even for small length-to-thickness ratios.

In addition, one performed the same analysis considering the FEM and a regular mesh with 120 elements along each direction, and Table 5.6 shows the results. As described in Chapter 4, the overall FEM formulation relies on different base elements for each stiffness component; as the bi-linear shape functions discretise the membrane degrees of freedom, one needs higher-order shape functions like those of the CPT elements to model the bending component. Consequently, the complete formulation resembles a jigsaw puzzle of sub-elements, so one needs to validate this implementation with challenging conditions, namely low length-to-thickness ratios.

Table 5.6 Square [0/c/0] sandwich plate - comparison with Navier's procedure using the FEM

Model	Element	a/h			
		2	4	10	100
FSDT	CPT(N)	6.9340	12.234	21.096	27.275
	CPT(C)	6.9343	12.234	21.097	27.277
	DKQ	-	12.233	21.096	27.275
	IDKQ	6.9341	12.234	21.097	27.276
	Navier¹	6.9339	12.233	21.096	27.275
TSDT	CPT(N)	5.4308	9.4352	17.717	27.172
	CPT(C)	5.4310	9.4357	17.718	27.174
	DKQ	5.4306	9.4351	17.717	27.172
	IDKQ	5.4309	9.4354	17.717	27.173
	Navier¹	5.4307	9.4350	17.716	27.172
FoZZ	CPT(N)	5.2901	9.1086	17.185	27.149
	CPT(C)	5.2904	9.1090	17.186	27.151
	DKQ	5.2899	9.1085	17.185	27.149
	IDKQ	5.2902	9.1088	17.186	27.150
	Navier¹	5.2900	9.1084	17.185	27.149
ToZZ	CPT(N)	5.3100	9.1026	17.159	27.148
	CPT(C)	5.3102	9.1030	17.160	27.149
	DKQ	5.3098	9.1024	17.159	27.148
	IDKQ	5.3100	9.1028	17.159	27.149
	Navier¹	5.3281	9.1152	17.168	27.148
3D-WF ¹	Navier¹	5.2730	9.0843	17.143	27.147
Ite-WF ¹	Navier¹	5.2703	9.0840	17.143	27.147
Exa ²¹	Navier¹	5.2702	9.0840	17.143	27.147
Exa ¹	Navier¹	5.3389	9.1452	17.177	27.148

¹ Loredo and Castel [2014]

Therefore, it is pertinent to discuss the results presented in Table 5.6, analysing the accuracy of the FEM and the performance of each element under the challenging conditions of low length-to-thickness ratios. As previously noted with the isotropic plate, convergence using the FEM requires more degrees of freedom, so it is difficult to achieve, reaching the results obtained with

Navier's procedure up to two decimal places using 120 elements along each direction for FSDT, TSDT and FoZZ models. Somewhat unexpectedly, the results for the ToZZ theory tend to slightly underestimate those obtained with Navier's procedure, particularly for low length-to-thickness ratios, but always with relative differences below 0.34 % and reaching the results obtained with Navier's method up to one decimal place. Thus, these results help to demonstrate the robustness of the FEM, even for challenging structures with low length-to-thickness ratios, merging versatility and accuracy in a single method.

The conclusions drawn from the previous results using the RRM regarding deformation theories are still valid here; the ZZT models are the only ones able to capture the mechanical behaviour of a laminate with different materials with different stiffness. As already discussed, this is particularly important for laminates with viscoelastic layers, the case under study in this work, since the difference in stiffness between such materials and those usually used in structural applications is of many orders of magnitude. Differences between deformation theories are of great importance for this work. From the preceding example, one can notice that the ZZT models offer completely different results when considering a high stiffness variation across the thickness and offer a high level of accuracy compared with 3D solutions, as one can notice in Tables 5.5-5.6.

Furthermore, similarly to the example involving isotropic plates, there are no relevant differences between finite elements, apart from the results with the CPT(N) and DKQ elements, which seem slightly close to those obtained with Navier's procedure but not relevant enough to justify some improved performance of one implementation over the others. However, there are other significant issues that one must consider in the assessment of finite element performance. For a low length-to-thickness ratio of $a/h = 2$ and the FSDT model, the DKQ element presents hundreds of spurious modes before the first normal mode, i.e. despite being able to provide an accurate result, its performance is truly inefficient.

To illustrate the differences between deformation theories, Figure 5.2 depicts the warping functions φ_{xx} and φ_{yy} considering a square sandwich panel with a length-to-thickness ratio of $a/h = 4$. As clearly shown in Figure 5.2, and as expected, while the warping functions of an ESLT as the FSDT or the TSDT are continuous across the laminate's thickness, this is not the case for those of a ZZT as the FoZZ or the ToZZ since that depends on the material properties, namely Young's modulus ratio between materials, as proved by the contrasting slopes of these warping functions for different layers. Once again, the displacement field of any ESLT is a purely geometrical concept and does not consider the mechanical behaviour of each layer independently.

To better assess the differences between deformation theories and the performance of the FEM implementation presented in this work, Figure 5.3 depicts the normalised transverse shear stresses of the first mode at two points, coordinates $(b/2, 0)$ and $(0, a/2)$, using the CPT(C) element and considering the challenging case of a square sandwich panel with a length-to-thickness ratio of $a/h = 4$. These normalised transverse shear stresses, computed with the first mode shape results, provide insightful information regarding the performance of each deformation theory at the stress level. The normalisation considers the highest value of transverse shear stress at these two points, taking it as a reference. Figures 5.3a and 5.3b show the normalised transverse shear stresses obtained directly from the shear strain degrees of freedom (see Equations 3.6 and 3.14), and Figures 5.3c and 5.3d present the normalised transverse shear stresses obtained by integration of the equilibrium equations (see Equation 3.34).

Figures 5.3a and 5.3b clearly show the differences between an ESLT and a ZZT at the transverse shear stress level. An ESLT such as the FSDT or the TSDT does not rely on the transverse shear stress continuity, contrary to any ZZT like those presented in this work, either the FoZZ or the ToZZ. As already discussed, the transverse shear stress continuity emerges from the 3D equilibrium solutions and is indispensable when dealing with different materials in the same laminate, as is the case of laminates with damping materials.

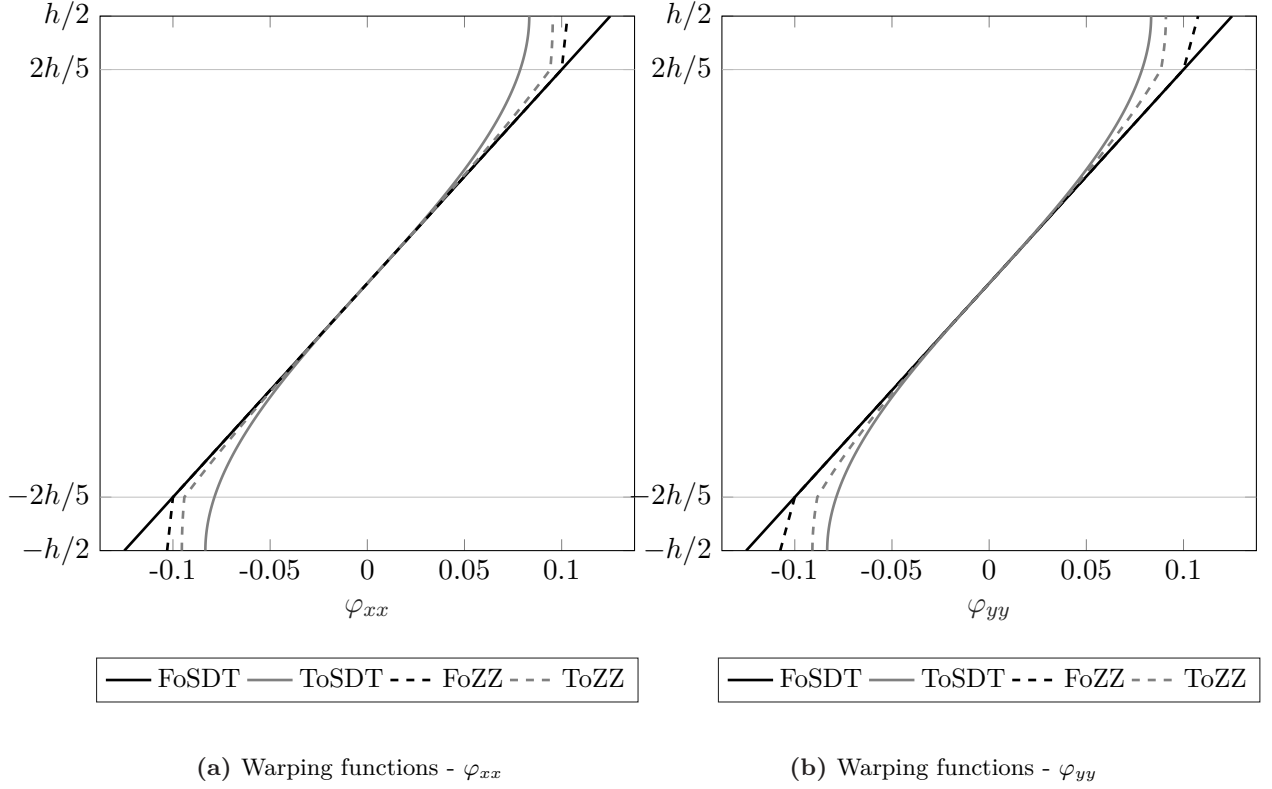


Figure 5.2 Warping functions of the square $[0/c/0]$ sandwich plate with $a/h = 4$

The significance of the transverse shear stress continuity is evident when considering the transverse shear stresses obtained by integration of the equilibrium equations - Figures 5.3c and 5.3d. Regardless of the deformation theory, transverse shear stresses are continuous through the thickness. Besides, transverse shear stresses are also null at the bottom and top free surfaces, as expected. At this point, it is relevant to highlight the main difference between the FoZZ or the ToZZ theories at this level, namely the ability of the ToZZ theory, contrary to the FoZZ one, to ensure null transverse shear stresses at both bottom and top surfaces - see Figures 5.3a and 5.3b.

Moreover, for a length-to-thickness ratio of $a/h = 4$, Table 5.6 shows similar results with both ZYT models; one can not distinguish them at the transverse shear stresses obtained from the integration of the equilibrium equations - Figures 5.3c and 5.3d.

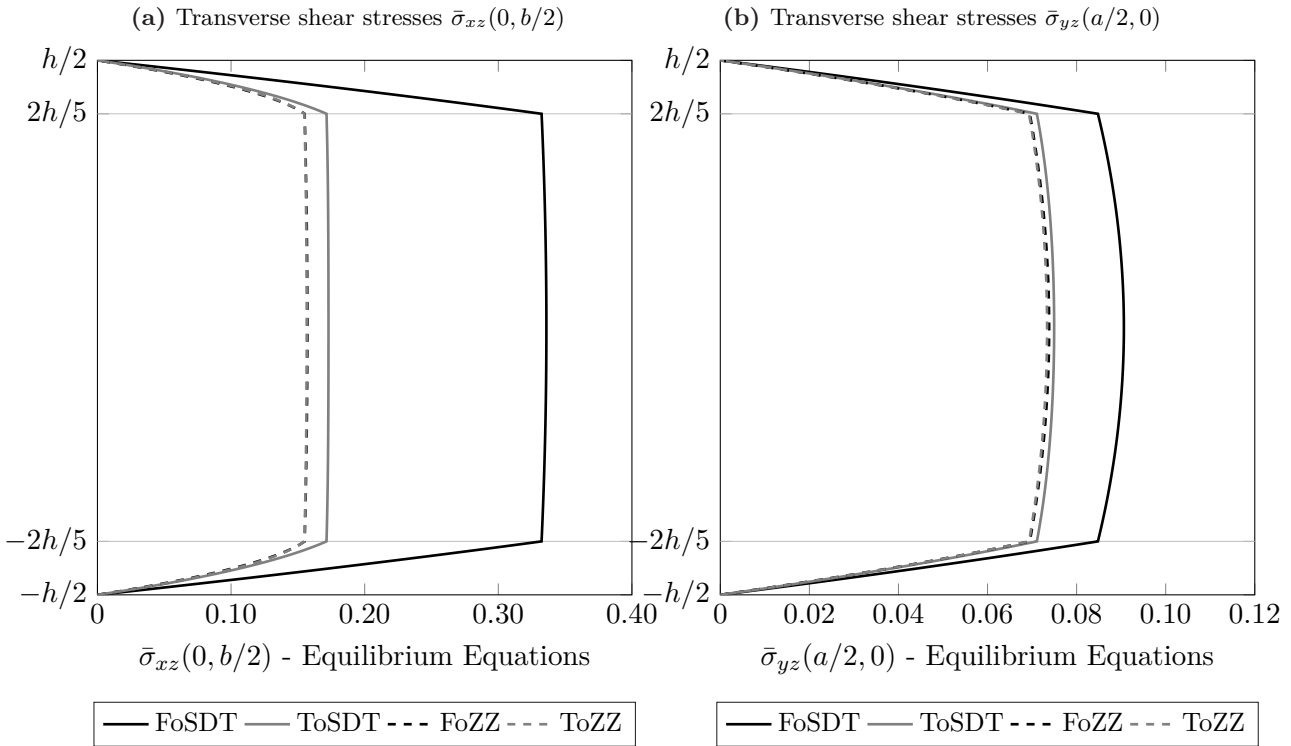
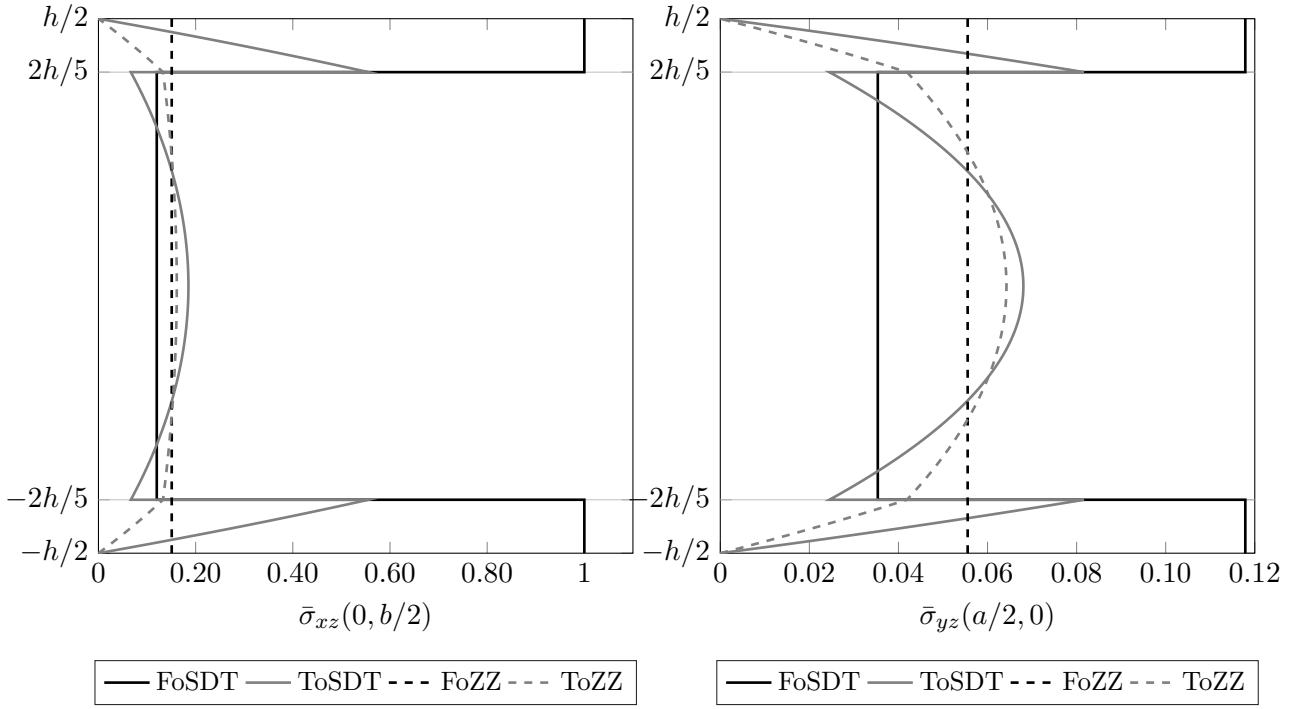


Figure 5.3 Normalised transverse shear stresses of the square $[0/c/0]$ sandwich plate with $a/h = 4$

5.3.2 | Rectangular [0/90/0] Cross-Ply Composite Plate

In the following sections, one examines the performance of the FEM implementation of the presented mechanical model only considering laminates with composite plies in the stacking sequence. One previously concluded that the RRM offers a great convergence rate; however, the FEM, also able to provide excellent results, is more flexible and suitable to more complex geometries like inhomogeneous shell structures with partial damping treatments with viscoelastic materials embedded into the composite layers.

For the first test case, one considers a rectangular plate ($b = 3a$) with three composite plies of equal thickness and a [0/90/0] stacking sequence. Table 5.7 presents the first dimensionless natural frequencies considering the deformation theories and length-to-thickness ratios studied in this work using the FEM and a regular mesh of 120 elements along each direction.

Table 5.7 Rectangular [0/90/0] composite plate - comparison with Navier's procedure using the FEM

Model	Element	a/h			
		2	4	10	100
FSDT	CPT(N)	3.8633	6.9504	11.497	14.060
	CPT(C)	3.8635	6.9508	11.498	14.060
	DKQ	-	6.9504	11.497	14.060
	IDKQ	3.8634	6.9506	11.498	14.060
	Navier¹	3.8633	6.9503	11.497	14.059
TSDT	CPT(N)	3.5336	6.1332	10.750	14.044
	CPT(C)	3.5338	6.1335	10.751	14.044
	DKQ	3.5335	6.1332	10.750	14.044
	IDKQ	3.5337	6.1334	10.751	14.044
	Navier¹	3.5335	6.1331	10.750	14.043
FoZZ	CPT(N)	3.5485	6.0461	10.635	14.041
	CPT(C)	3.5487	6.0464	10.636	14.042
	DKQ	3.5485	6.0461	10.635	14.041
	IDKQ	3.5486	6.0463	10.636	14.041
	Navier¹	3.5485	6.0461	10.635	14.041
ToZZ	CPT(N)	3.8399	6.0012	10.411	14.034
	CPT(C)	3.8401	6.0015	10.411	14.035
	DKQ	3.8399	6.0012	10.411	14.034
	IDKQ	3.8400	6.0014	10.411	14.035
	Navier¹	3.8845	6.0265	10.418	14.034
3D-WF ¹	Navier¹	3.4196	5.9084	10.405	14.034
Ite-WF ¹	Navier¹	3.4186	5.9084	10.405	14.034
Exa ²¹	Navier¹	3.4185	5.9083	10.405	14.034
Exa ¹	Navier¹	3.4197	5.9100	10.408	14.034

¹ Loredo and Castel [2014]

Similarly, one shows the results obtained by Loredo and Castel [2014] using Navier's procedure

for comparison purposes. The performance of the FEM is remarkable, achieving the results obtained with Navier's method up to two, and sometimes three, decimal places for all deformation theories. The only exception is the ToZZ model for low length-to-thickness ratios of $a/h = 4$ and $a/h = 2$; even so, the maximum relative difference is less than 1.15% for such challenging test cases. Besides, there is no such thing in the real world as a composite laminate with a $[0/90/0]$ stacking sequence and a thickness that is half of its length. As previously discussed, such demanding conditions imposed on the simulation enable one to validate the FEM implementation. Once again, there are no relevant differences between results considering different finite elements. However, one should highlight the repeated inefficiency of the DKQ element for the low length-to-thickness ratio of $a/h = 2$, considering an FSDT model, presenting hundreds of spurious modes before the first normal mode.

Regarding the deformation theories, as expected, any ZZT tends to present better results as the length-to-thickness ratio decreases. However, the TSDT produces satisfactory results with relatively simple warping functions. Considering all deformation theories, Figure 5.4 depicts the corresponding warping functions for $a/h = 4$.

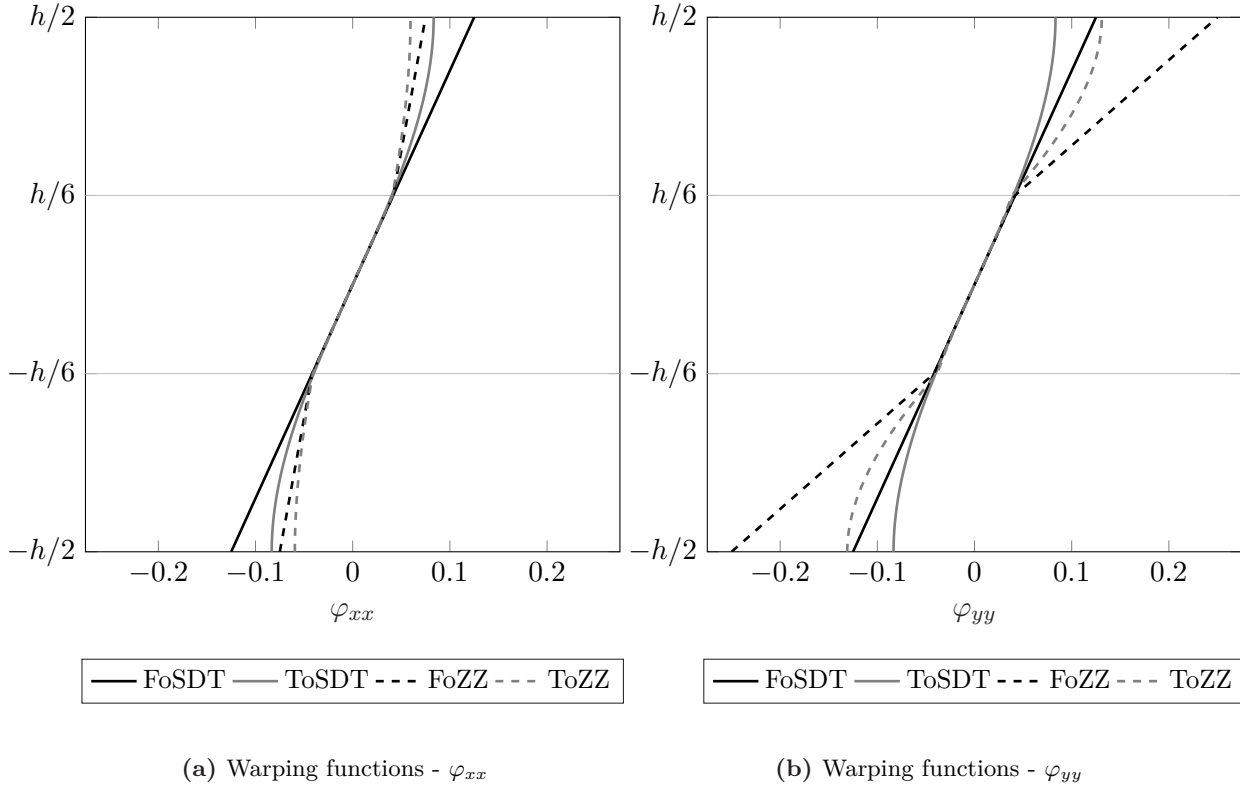


Figure 5.4 Warping functions of the rectangular $[0/90/0]$ composite plate with $a/h = 4$

Following the same reasoning of the previous study, Figure 5.5 depicts the normalised transverse shear stresses of the first mode at two points, coordinates $(b/2, 0)$ and $(0, a/2)$, using the CPT(C) element and considering a length-to-thickness ratio of $a/h = 4$. Both normalised transverse shear stresses obtained directly from the shear strain degrees of freedom, Figures 5.5a and 5.5b, and those obtained by integration of the equilibrium equations, Figures 5.5c and 5.5d, are relevant to discuss the performance of the deformation theories and the corresponding FEM implementation.

Once more, Figures 5.5a and 5.5b highlight the main differences between an ESLT and a ZZT at the transverse shear stress level. As already discussed, the transverse shear stress continuity

emerges from the 3D equilibrium equations, as depicted in Figures 5.5c and 5.5d, where one can understand the reliable results of the TSDT deformation theory for this laminate, providing an adequate representation of the transverse shear stresses compared to any ZZT.

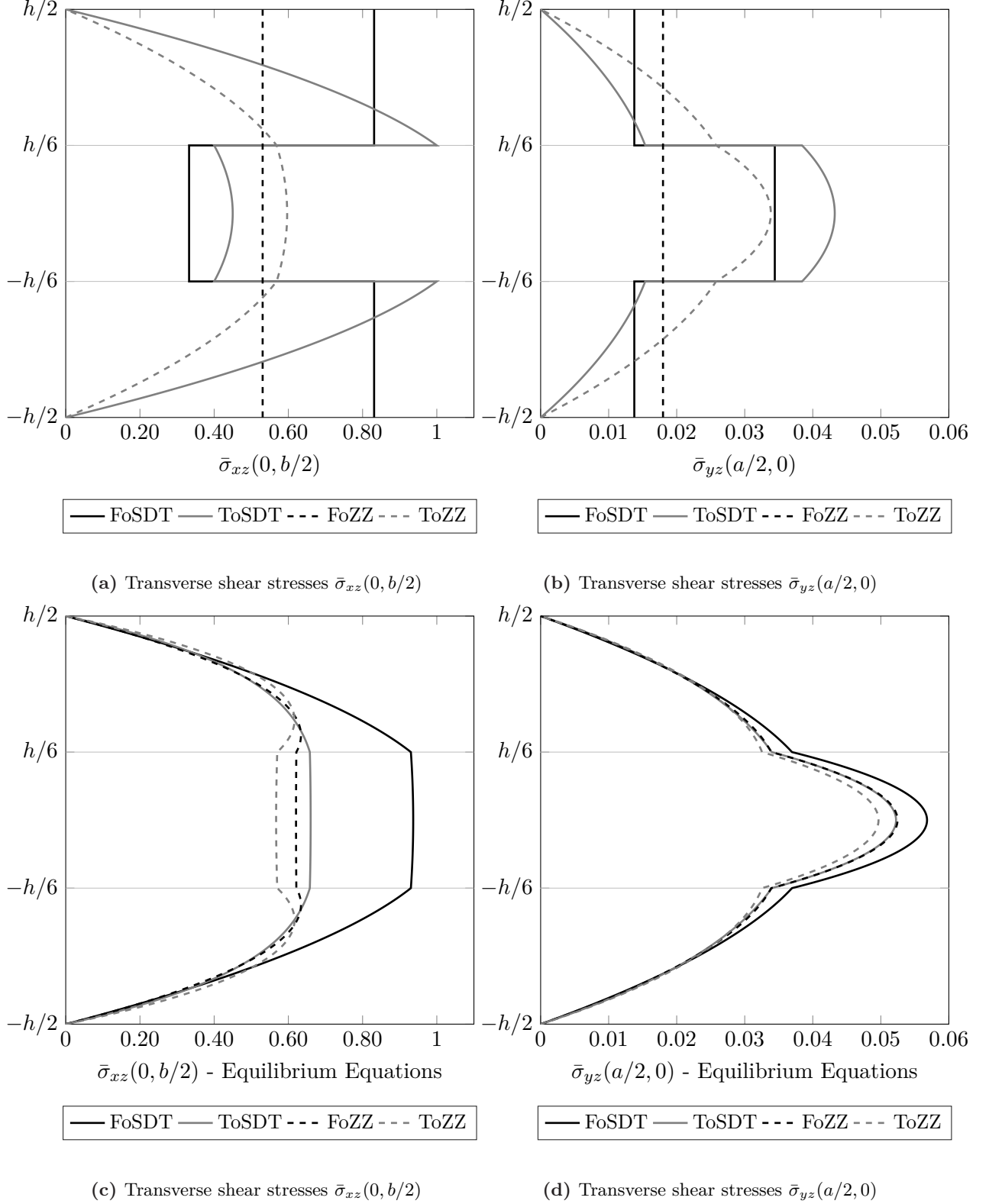


Figure 5.5 Normalised transverse shear stresses of the rectangular $[0/90/0]$ composite plate with $a/h = 4$

5.3.3 | Square [-15/15] Antisymmetric Angle-Ply Composite Plate

Following two examples of symmetric laminates, one considers an antisymmetric angle-ply square plate with two layers of equal thickness and a [-15/15] stacking sequence in the following section. Table 5.8 presents the first dimensionless natural frequencies considering the deformation theories and length-to-thickness ratios studied in this work using the FEM and a regular mesh of 120 elements along each direction, including the results obtained by Loredo and Castel [2014] using Navier's procedure for comparison purposes.

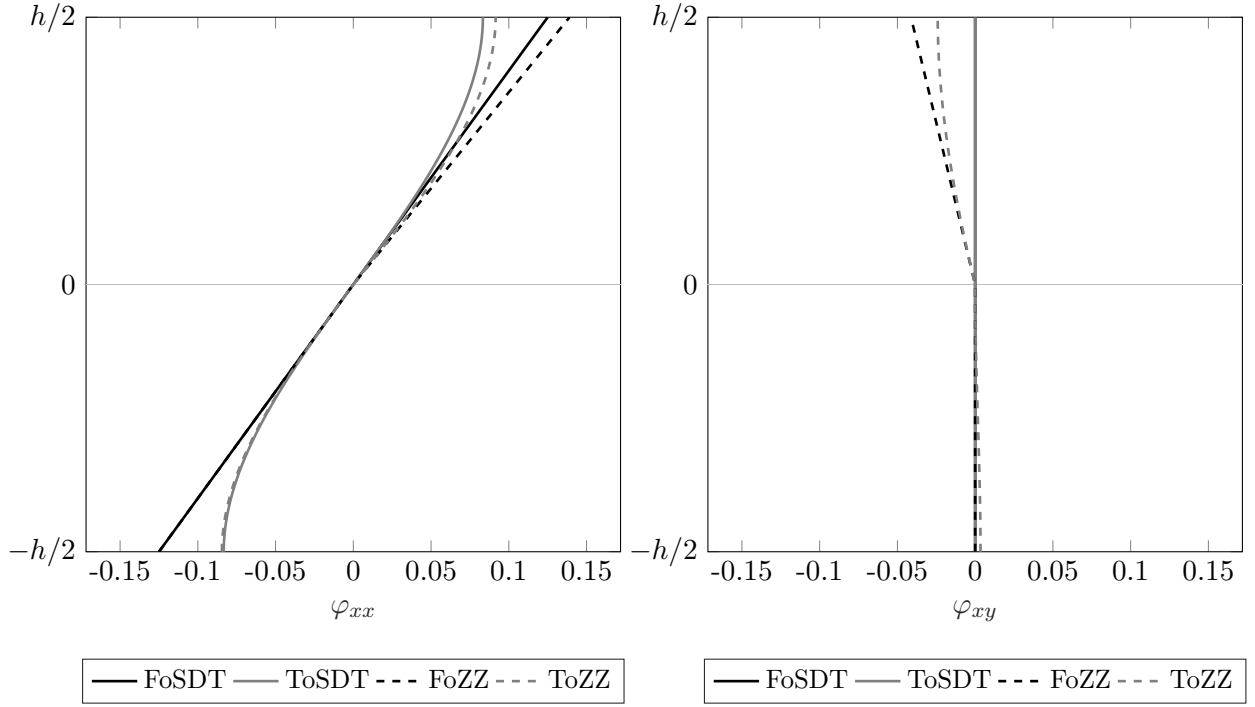
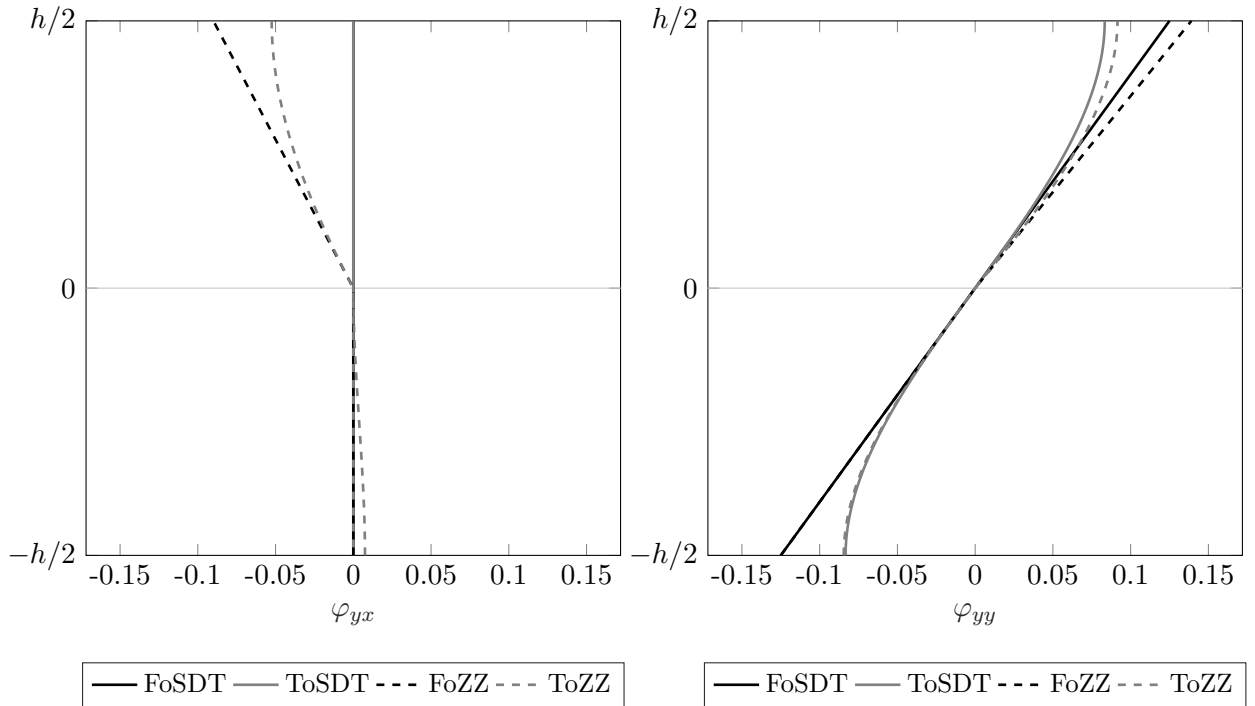
Table 5.8 Square [-15/15] composite plate - comparison with Navier's procedure using the FEM

Model	Element	a/h			
		2	4	10	100
FSDT	CPT(N)	4.7208	7.7955	11.218	12.677
	CPT(C)	4.7210	7.7958	11.219	12.678
	DKQ	-	7.7953	11.218	12.677
	IDKQ	4.7208	7.7957	11.219	12.678
	Navier¹	4.7206	7.7953	11.218	12.677
TSDT	CPT(N)	4.7614	7.6326	11.093	12.675
	CPT(C)	4.7616	7.6329	11.094	12.676
	DKQ	4.7611	7.6324	11.093	12.675
	IDKQ	4.7614	7.6327	11.093	12.676
	Navier¹	4.7611	7.6323	11.093	12.675
FoZZ	CPT(N)	5.0427	7.9475	11.227	12.675
	CPT(C)	5.0425	7.9474	11.228	12.676
	DKQ	5.0409	7.9457	11.226	12.675
	IDKQ	5.0426	7.9478	11.228	12.676
	Navier¹	4.9993	7.9690	11.275	12.678
ToZZ	CPT(N)	5.0549	7.7576	11.094	12.673
	CPT(C)	5.0549	7.7578	11.095	12.673
	DKQ	5.0540	7.7568	11.094	12.673
	IDKQ	5.0549	7.7579	11.095	12.673
	Navier¹	4.9478	7.7273	11.116	12.675
3D-WF ¹	Navier¹	4.4826	7.4376	11.012	12.674
Ite-WF ¹	Navier¹	4.4706	7.4354	11.012	12.674
Exa ²¹	Navier¹	4.4660	7.4324	11.012	12.674
Exa ¹	Navier¹	4.4695	7.4548	11.026	12.674

¹ Loredo and Castel [2014]

It is worth to comment the results presented in Table 5.8 considering the warping functions depicted in Figure 5.6 for $a/h = 4$, together with the normalised transverse shear stresses of the first mode at two points, coordinates $(b/2, 0)$ and $(0, a/2)$, using the CPT(C) element and considering the same length-to-thickness ratio in Figures 5.7 and 5.8. In Figure 5.6, one can notice, for the first time, the presence of non-null warping functions φ_{xy} and φ_{yx} when considering

a ZZT.

(a) Warping functions - φ_{xx} (b) Warping functions - φ_{xy} (c) Warping functions - φ_{yx} (d) Warping functions - φ_{yy} **Figure 5.6** Warping functions of the square $[-15/15]$ composite plate with $a/h = 4$

These non-null warping functions influence how the model couples the displacement field in the

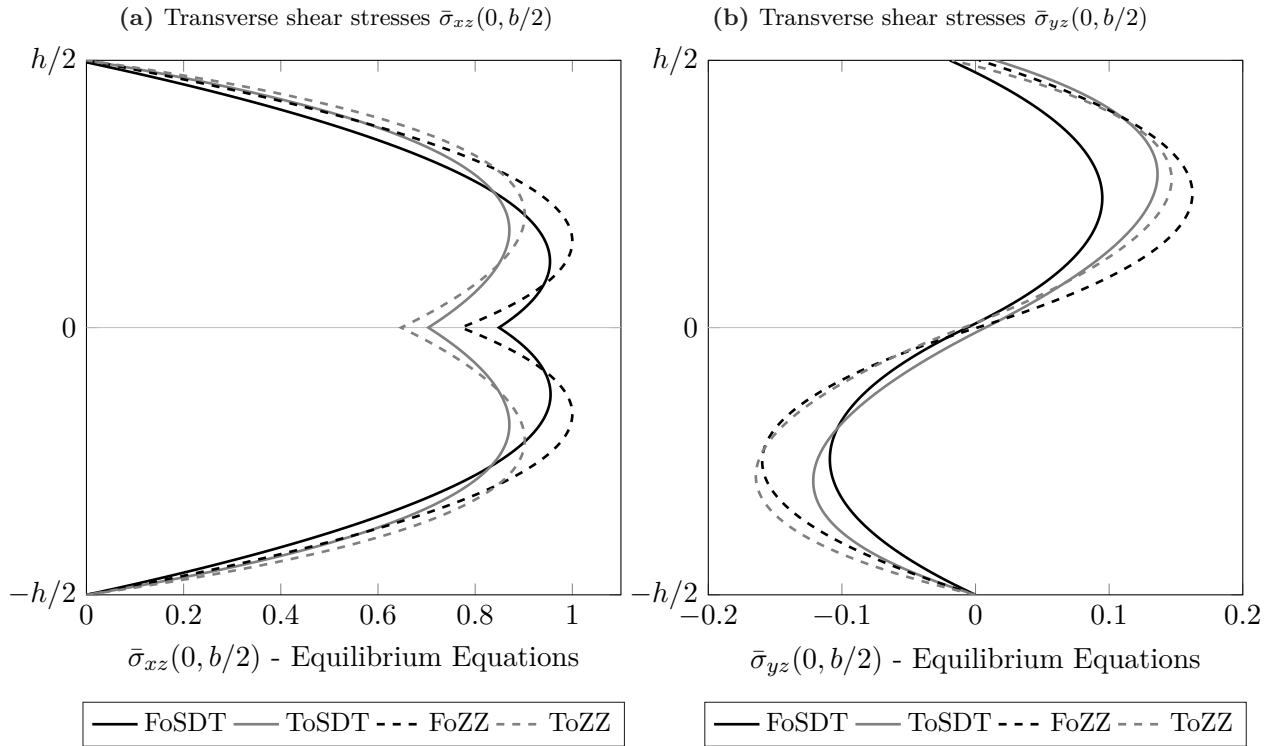
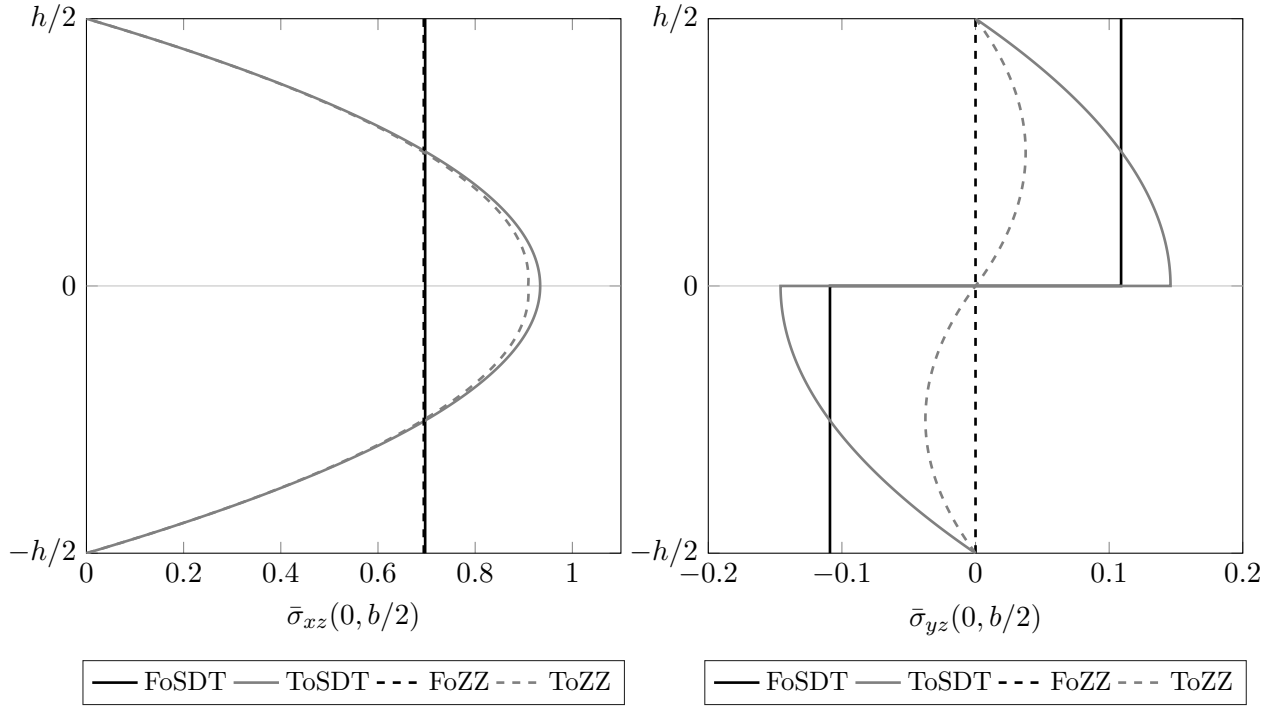


Figure 5.7 Normalised transverse shear stresses of the square $[-15/15]$ composite plate with $a/h = 4 - 1$

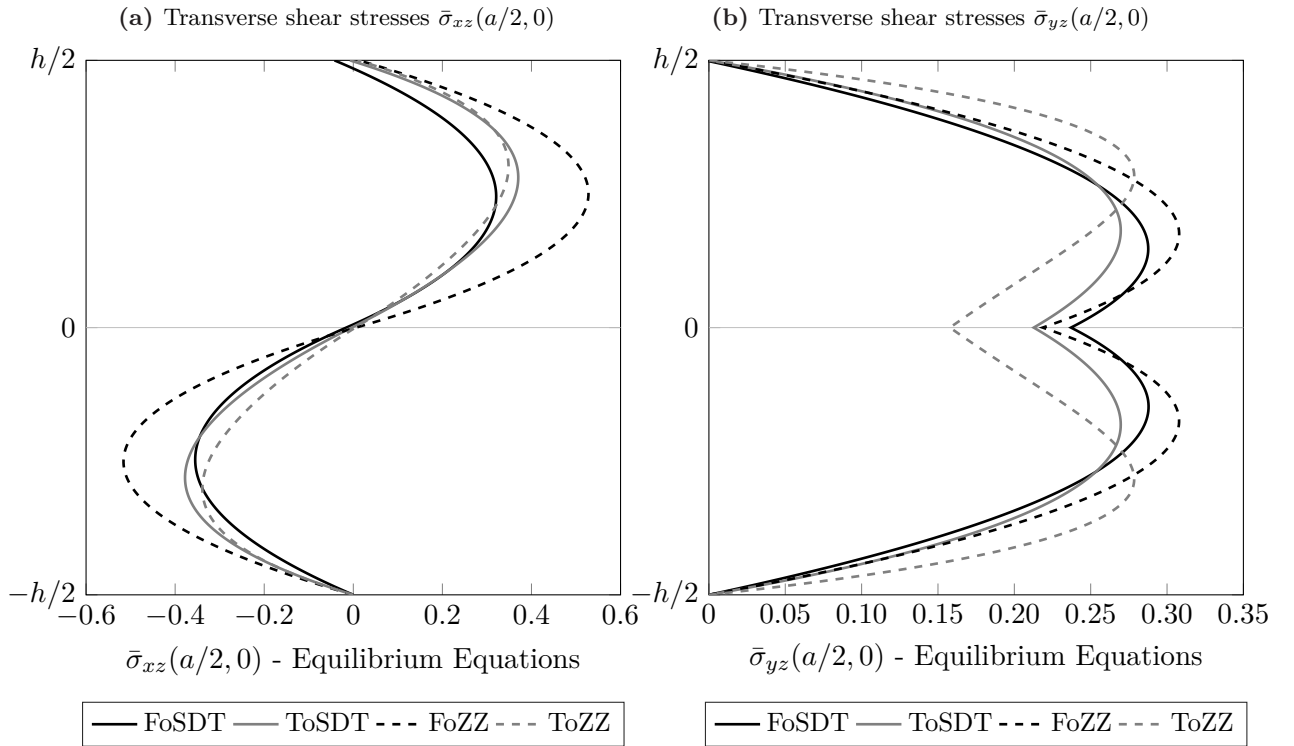
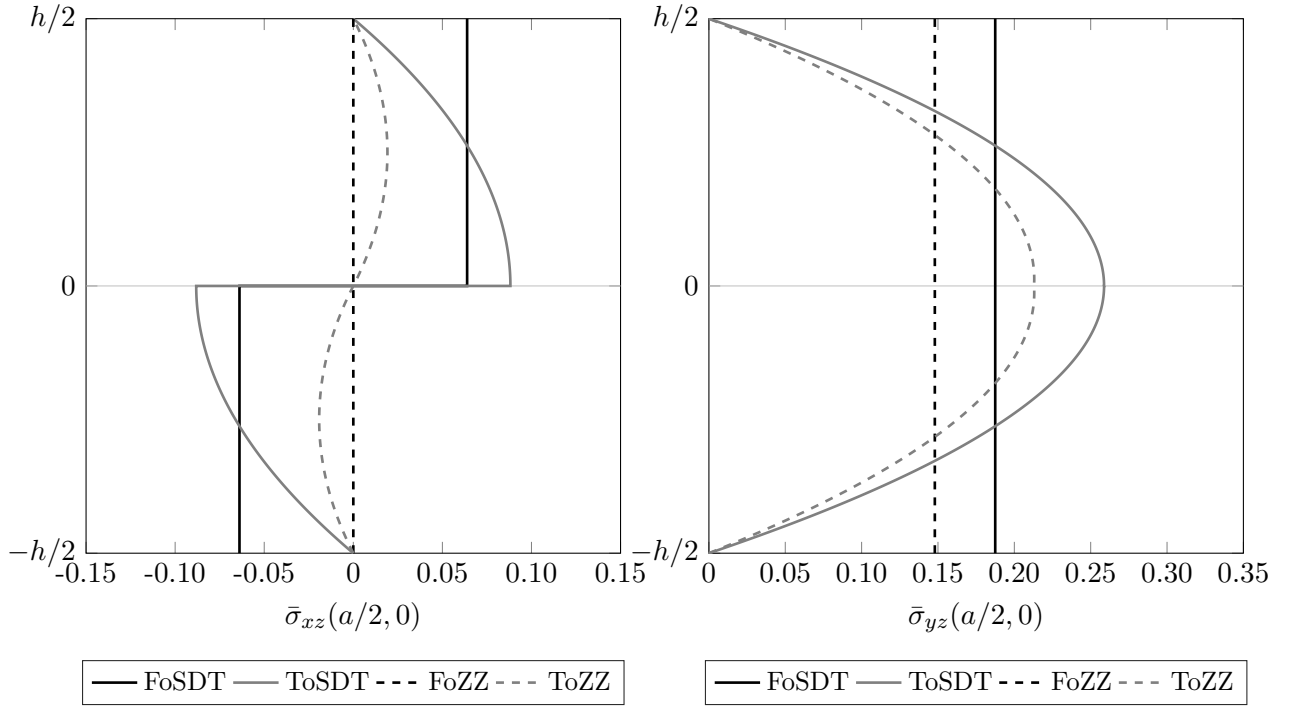


Figure 5.8 Normalised transverse shear stresses of the square $[-15/15]$ composite plate with $a/h = 4 - 2$

x and y direction (see Equation 5.15) and might help to explain the results in Table 5.8. Analysing both ESLT models, the performance of the FEM is remarkable, reaching the results obtained with Navier's procedure up to two and three decimal places. Regarding the ZZT deformation theories, both FoZZ and ToZZ models show excellent results for high length-to-thickness ratios. Still, the performance of the FEM implementation drops slightly for low length-to-thickness ratios. Still, the relative differences are below 1 %, except for a length-to-thickness of $a/h = 2$ and considering the ToZZ model, whose relative difference between the FEM implementation and Navier's procedure is 2.16 % for such a challenging test case. Once again, it is worth emphasizing that no one would imagine a square [-15/15] composite laminate whose thickness is half its length in a real-world application. Still, this procedure tests the boundaries of the FEM implementation and enables one to validate the implemented numerical method. Regarding this particular topic, only the DKQ element shows its expected inability for an FSDT model considering low length-to-thickness ratios of $a/h = 2$.

Additionally, Figures 5.7 and 5.8 depict the normalised transverse shear stresses of the first mode at two points, coordinates $(b/2, 0)$ and $(0, a/2)$, and one can notice that contrary to the previous examples, $\bar{\sigma}_{yz}(0, b/2)$ and $\bar{\sigma}_{xz}(a/2, 0)$ are no longer null. Figures 5.7a, 5.7b, 5.8a and 5.8b show the normalised transverse shear stresses obtained directly from the shear strain degrees of freedom and exhibit the contrast between an ESLT and a ZZT. Still, Figures 5.7c, 5.7d, 5.8c and 5.8d show the normalised transverse shear stresses obtained by integration of the equilibrium equations and expose the inadequacy of the presented deformation theories to represent the transverse shear stresses through the thickness and explain the differences in Table 5.8 when comparing the obtained results with exact solutions for low length-to-thickness ratios.

5.3.4 | Square [0/30/0] Composite Plate

Lastly, one considers a symmetric angle-ply square plate with three layers of equal thickness and a [0/30/0] stacking sequence. Table 5.9 presents the first dimensionless natural frequencies, considering the deformation theories and length-to-thickness ratios studied in this work using the FEM and a regular mesh of 120 elements along each direction. Once again, Loredó and Castel [2014] results using Navier's procedure are also available for comparison purposes.

It is worth mentioning that such configuration does not involve simplifications of the stiffness matrix, i.e. there are no null terms in the stiffness matrix. Therefore, this configuration poses a challenging task to the FEM implementation, and one considers the following results as a final step to validate the numerical implementation.

Even considering such challenging conditions of no simplifications of the stiffness matrix and recalling that the FEM implementation relies on different formulations for each stiffness component, the performance of each element is noteworthy. Considering distinct deformation theories and length-to-thickness ratios, the relative difference between the proposed FEM implementation and Navier's procedure is well below 0.39 % for the FoZZ theory and a length-to-thickness $a/h = 4$ and below 0.05 % for the vast majority of cases. Once more, the only exceptions are the ZZT models for a length-to-thickness $a/h = 2$, the most challenging test. Nevertheless, there is a relative difference of only 1.26 % for the FoZZ and 1.32 % for the ToZZ, a remarkable result for such a borderline case. As expected, only the DKQ element shows its inefficiency for an FSDT model considering low length-to-thickness ratios of $a/h = 2$.

In addition, Figure 5.9 depicts the warping functions for an $a/h = 4$ length-to-thickness ratio; once again, one can notice non-null warping functions for the ZZT models. One previously discussed the influence of non-null warping functions in displacement field coupling and the corresponding challenges that this poses to the numerical method. Nevertheless, the performance

of the FEM implementation enables one to validate it.

At the end of this section, one presents the normalised transverse shear stresses of the first mode again at two points, coordinates $(b/2, 0)$ and $(0, a/2)$, using the CPT(C) element and considering the demanding case of an $a/h = 4$ length-to-thickness ratio at Figures 5.10 and 5.11, including the normalised transverse shear stresses obtained directly from the shear strain degrees of freedom, Figures 5.10a, 5.10b, 5.11a and 5.11b, and those obtained by integration of the equilibrium equations, Figures 5.10c, 5.10d, 5.11c and 5.11d. There is no novelty that Figures 5.10a, 5.10b, 5.11a and 5.11b illustrate the differences between an ESLT and a ZZT at the transverse shear stress level, namely the transverse shear stress continuity, emerging from the 3D equilibrium equations, as one can notice in Figures 5.10c, 5.10d, 5.11c and 5.11d. Interestingly, the transverse shear stress recovery from the third-order models, i.e. TSDT and ToZZ models, result in similar representations of stress distribution through the thickness, explaining the similar results presented in Table 5.9.

Table 5.9 Square [0/30/0] composite plate - comparison with Navier's procedure using the FEM

Element		a/h			
		2	4	10	100
FSDT	CPT(N)	4.7177	8.1201	12.843	15.360
	CPT(C)	4.7179	8.1205	12.844	15.360
	DKQ	-	8.1201	12.843	15.360
	IDKQ	4.7178	8.1204	12.844	15.360
	Navier¹	4.7242	8.1258	12.850	15.367
TSDT	CPT(N)	4.5043	7.5692	12.370	15.350
	CPT(C)	4.5045	7.5696	12.371	15.351
	DKQ	4.5042	7.5691	12.370	15.350
	IDKQ	4.5044	7.5694	12.371	15.350
	Navier¹	4.4994	7.5603	12.370	15.358
FoZZ	CPT(N)	4.6100	7.9127	12.626	15.354
	CPT(C)	4.6101	7.9129	12.627	15.355
	DKQ	4.6096	7.9123	12.626	15.354
	IDKQ	4.6100	7.9129	12.626	15.354
	Navier¹	4.6683	7.8819	12.625	15.363
ToZZ	CPT(N)	4.6353	7.5390	12.304	15.347
	CPT(C)	4.6354	7.5393	12.304	15.348
	DKQ	4.6350	7.5388	12.303	15.347
	IDKQ	4.6356	7.5394	12.304	15.348
	Navier¹	4.5753	7.5351	12.299	15.356
Navier¹		4.4067	7.4846	12.287	15.356
Navier¹		4.3977	7.4841	12.287	15.356
Navier¹		4.3960	7.4836	12.287	15.356
Navier¹		4.4064	7.4961	12.295	15.356

¹ Loredo and Castel [2013]

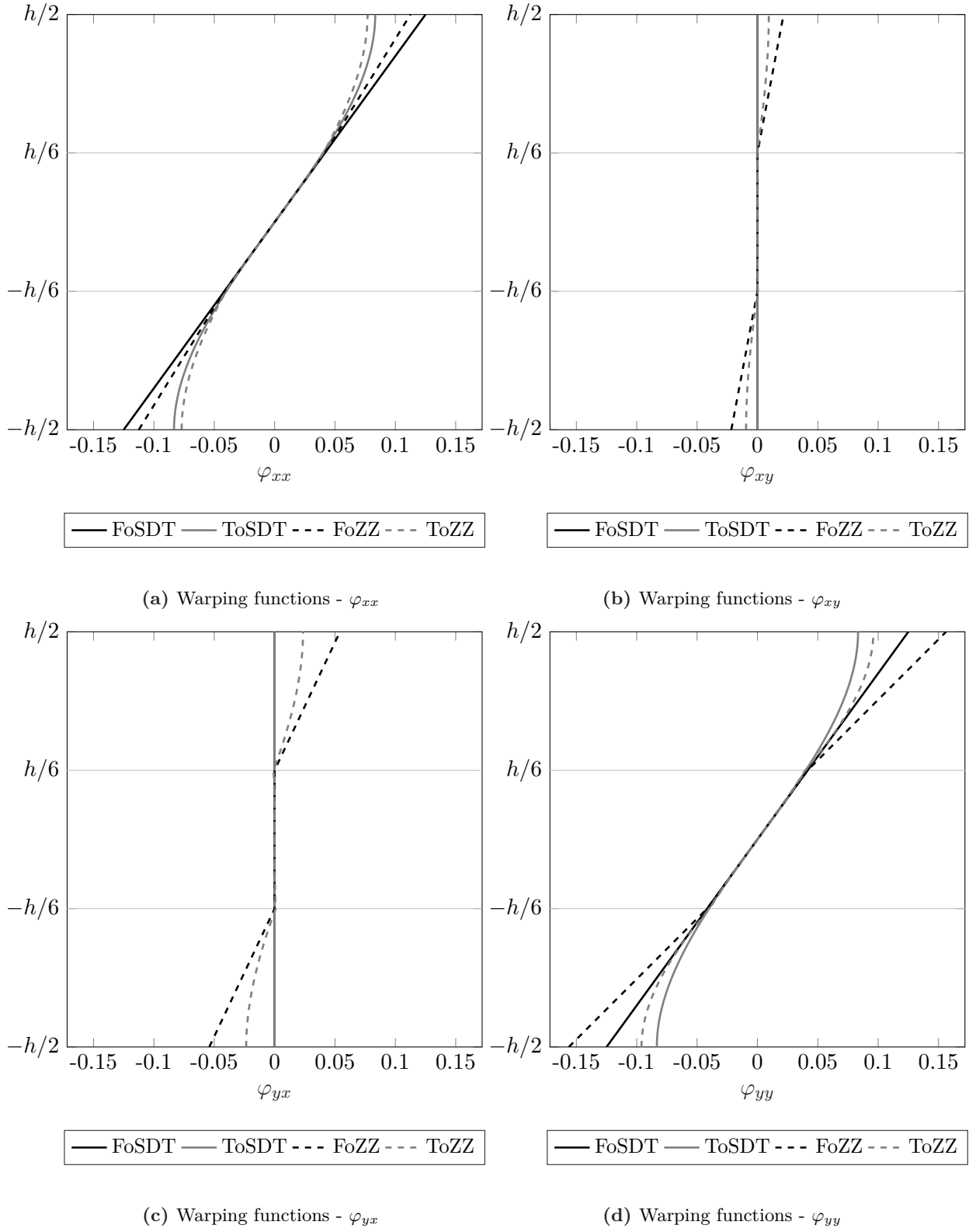


Figure 5.9 Warping functions of the square $[0/30/0]$ composite plate with $a/h = 4$

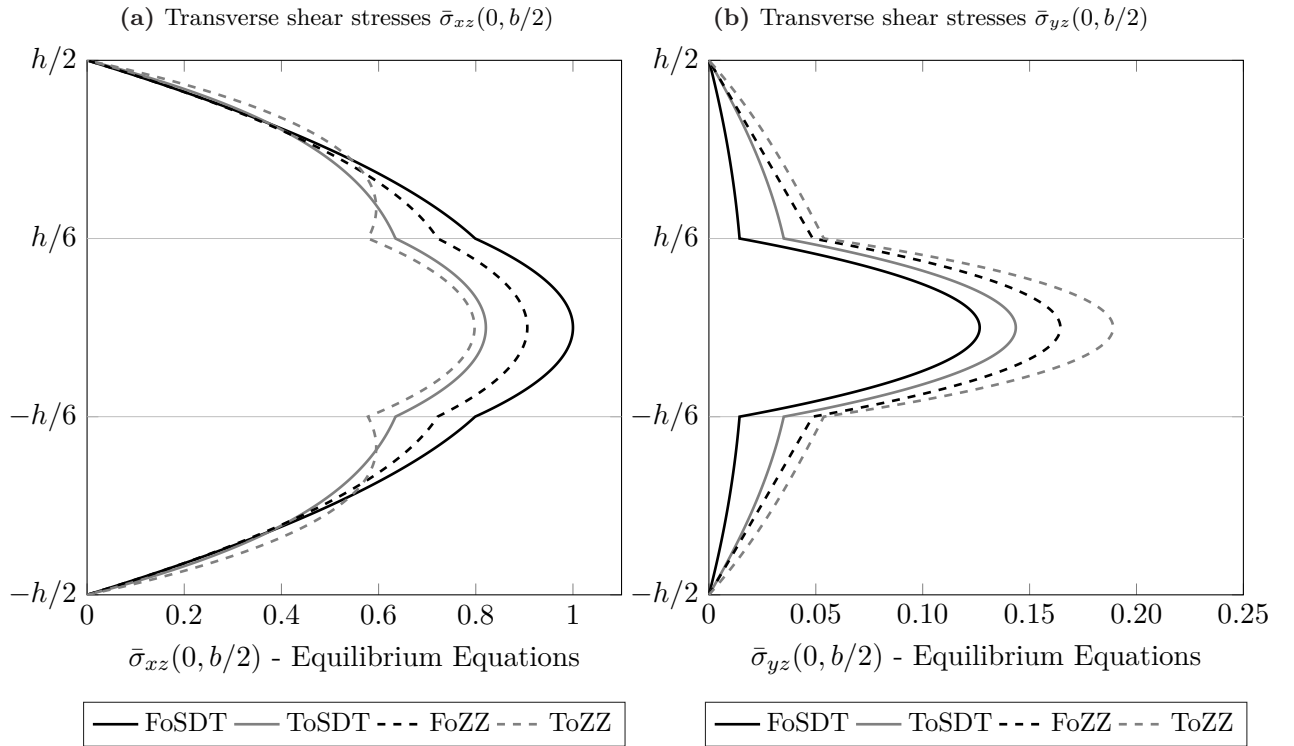
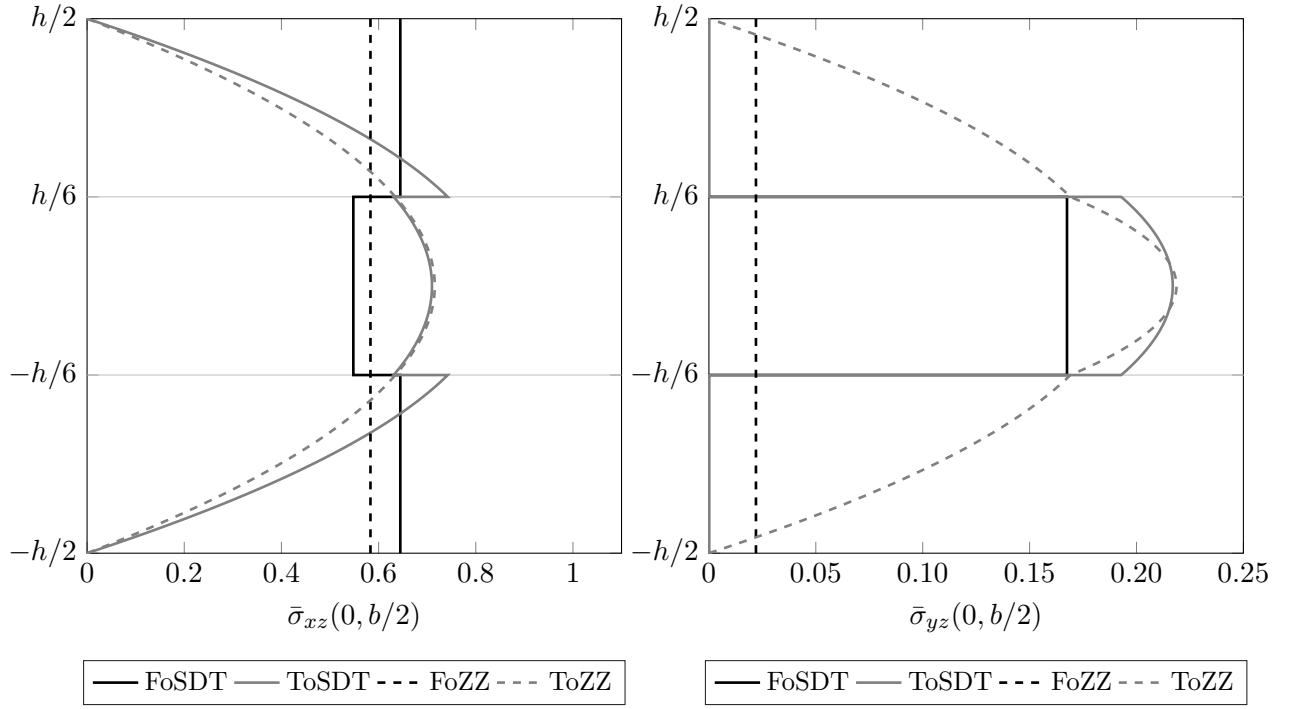


Figure 5.10 Normalised transverse shear stresses of the square $[0/30/0]$ composite plate with $a/h = 4 - 1$

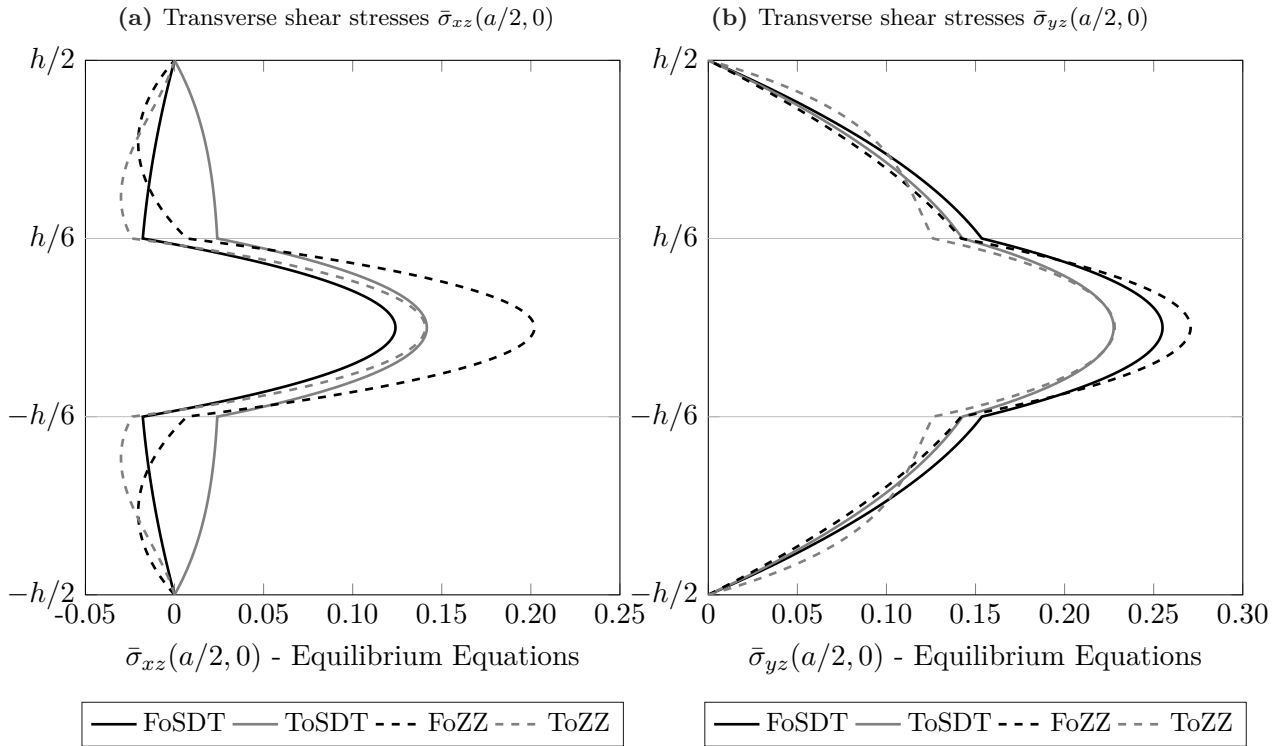
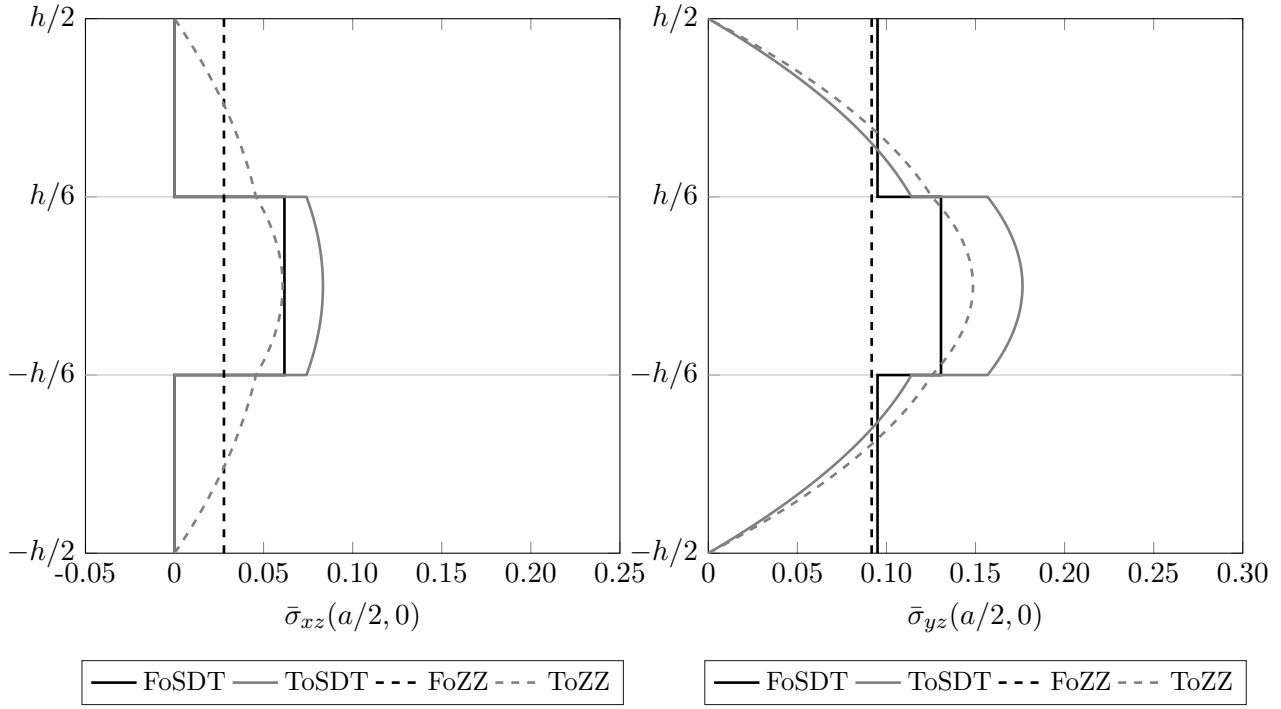


Figure 5.11 Normalised transverse shear stresses of the square $[0/30/0]$ composite plate with $a/h = 4 - 2$

5.4 | Facet Shell Element

In this section, one performs a set of benchmark tests to assess the performance of the implemented facet shell finite element. Initially, the patch tests concern the membrane component of the proposed finite element, namely the out-of-plane rotation and the corresponding drilling degrees of freedom. Introducing the drilling degree of freedom at each node, based on the work of Moreira and Dias Rodrigues [2011], as presented earlier, improves the performance of the bilinear quadrilateral membrane element. Here, one compares the performance of this improved membrane element with the original isoparametric four-node element labelled as the Q4DRL and Q4 versions of the element, respectively. One compares these elements, particularly the Q4DRL version, with those from other authors available in published research by fully describing their merits and shortcomings for different benchmark structures. Subsequently, one accomplishes the performance of the proper facet shell element with bending and shear components coupled with the membrane one. As previously reported, one implemented several versions for the bending part of the finite element in this work, and one tests all of them in the facet shell finite element. The acronym for the bending component (CPT(N), CPT(C), DKQ or IDKQ) labels the facet shell element, and one adds the Drilling Degree of Freedom (DRL) suffix when the formulation has drilling degrees of freedom. Following the standard practice of published research in this field, one presents the geometric and mechanical properties of the structures as non-dimensional parameters and the corresponding results listed coherently.

5.4.1 | Skewed Membrane - One Element Test

Firstly, one focuses on the well-known one-element test [Taylor et al., 1986]. As pointed out by Ibrahimbegovic et al. [1990], this test is well-designed to detect the presence of spurious modes associated with constant values of the nodal rotations which might exist. The skewed element is fixed with a minimum number of constraints and loaded with two equal loads, which subject it to a uniform stress field - see Figure 5.12. Ibrahimbegovic et al. [1990] and Zhu and Zacharia [1996] point out that formulations with an Allman [1988] type interpolation field do not pass this element test; consequently, it is valuable to ensure that the implemented formulation passes the test. Table 5.10 lists the geometric and mechanical properties of the element, and the analytical horizontal displacements at the free nodes are

$$u_2 = \frac{10}{3h}, \quad u_3 = \frac{2}{h} \quad (5.3)$$

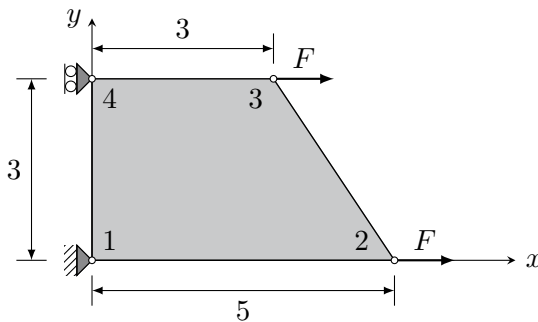


Table 5.10 Geometric and mechanical properties

Thickness (h)	0.01/0.1/1.0
Young's Modulus (E)	1000
Poisson's Ratio (ν)	0.25
Point Load (F)	1000

Figure 5.12 Skewed element

The finite element with drilling degrees of freedom implemented in this work passes the

skewed element test when the stabilisation operators proposed by Moreira and Dias Rodrigues [2011] to prevent spurious modes are present. As reported by the authors, the stiffness matrix is rank deficient when there are no stabilisation operators, having full rank when including them. Following the original proposal by Cook [1987], if one restrains one nodal out-of-plane rotation, one also avoids spurious modes.

5.4.2 | Simple Beam - Higher-Order Patch Test

Ibrahimbegovic et al. [1990] proposed a patch test consisting of a simple beam subjected to a pure bending state to assess the performance of membrane elements, particularly those with drilling degrees of freedom - Figure 5.13. One restrains a minimum number of constraints, with no restriction on the drilling degrees of freedom. Then, one considers two load cases at the same nodes, consisting of a unit couple applied at the free ends and a concentrated moment applied directly at the drilling degrees of freedom. For comparison purposes, one considers two meshes of six elements, a regular mesh and a distorted one, to assess the element performance with irregular meshes. Table 5.11 lists the geometric and mechanical properties of the beam.

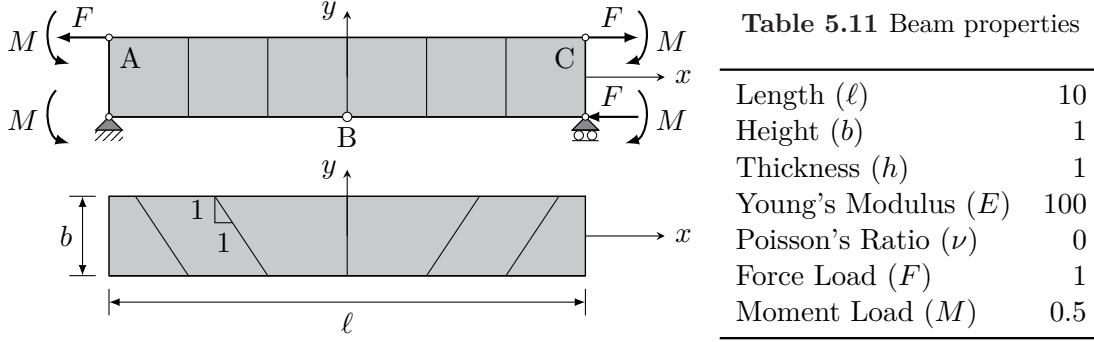


Figure 5.13 Simple beam in pure bending

Table 5.12 presents the results obtained with the implemented membrane element compared with some reference works. From the Euler-Bernoulli beam theory, one knows the reference value of 1.5 for the mid-span vertical displacement and -0.6 for the end rotation at point C . Besides, Figure 5.14 shows the normal strain distribution in the domain according to the Euler-Bernoulli beam theory.

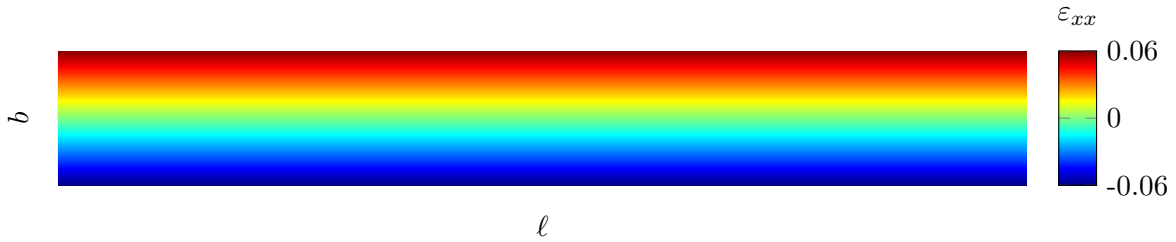


Figure 5.14 Normal strain distribution in pure bending

The extensional strain does not vary along the beam's length but varies linearly along the beam's thickness. Recalling Lagrange's interpolation for the displacement field of the bilinear quadrilateral element Q4 in terms of generalised coordinates

$$u(x, y) = a_1 + a_2x + a_3y + a_4xy$$

$$v(x, y) = a_5 + a_6x + a_7y + a_8xy \quad (5.4)$$

the extensional strains are given by

$$\varepsilon_{xx} = \frac{\partial u(x, y)}{\partial x} = a_2 + a_4y \quad (5.5a)$$

$$\varepsilon_{yy} = \frac{\partial v(x, y)}{\partial y} = a_7 + a_8x \quad (5.5b)$$

and the shear strain

$$\gamma_{xy} = \frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} = a_3 + a_4x + a_6 + a_8y \quad (5.6)$$

From Equation 5.5a, it turns out that the Q4 element manages to represent a linearly varying extensional strain along y since $a_4 \neq 0$ but, consequently, $a_4 \neq 0$ implies $\gamma_{xy} \neq 0$ as shown in Equation 5.6, which is inaccurate for the beam in pure bending under analysis. Therefore, the Q4 element presents a spurious shear strain in pure bending, which makes it stiffer and underestimates displacements. Figure 5.15 shows a graphical representation of the difference between the bilinear quadrilateral element in pure bending, where the sides remain straight, and the bilinear quadrilateral element with drilling degrees of freedom which can represent out-of-plane rotation.

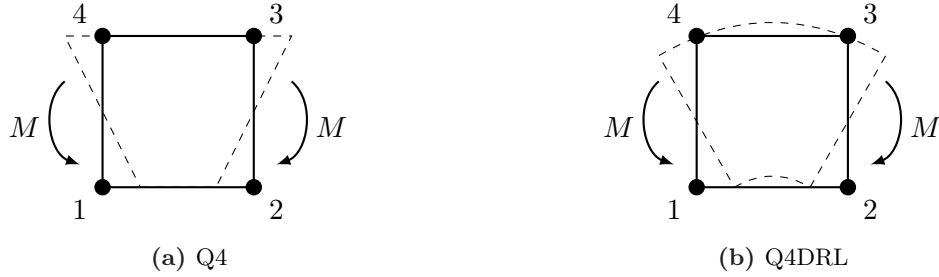


Figure 5.15 Membrane elements in pure bending

Results from Table 5.12 prove these different behaviours between elements. The numerical results obtained with the Q4DRL formulation match the exact solution for regular meshes, and for both loads, except for the end rotation with a moment load directly applied. Different authors explain this outcome since a moment load at a drilling degree of freedom can not be considered a consistent load. The bilinear quadrilateral element underestimates the displacements due to spurious shear strain. In addition, considering the distorted mesh, there is a lower performance, but the accuracy is still good. Finite elements established on displacement-based formulations involve the inversion of the Jacobian matrix and therefore are sensitive to mesh distortions. Once the Q4DRL element implemented in this work is that proposed by Moreira and Dias Rodrigues [2011], with no surprise, the results for this test are equivalent.

Table 5.12 Simple beam results and comparison with reference works

Reference	Vertical Displacement	End Rotation	Horizontal Displacement
	v_B	θ_C	u_A
Regular Mesh			
Force Load			
Ibrahimbegovic et al. [1990]	1.5	−0.6	-
Pimpinelli [2004]	1.50	−0.6	−0.60
Moreira and Dias Rodrigues [2011]	1.50	−0.60	−0.60
Wang et al. [2016]	1.50	−0.6	−0.60
Q4DRL	1.50	−0.60	−0.60
Q4	0.63	-	−0.25
Moment Load			
Ibrahimbegovic et al. [1990]	1.5	−0.620 70	-
Pimpinelli [2004]	1.51	−1.44	−0.60
Moreira and Dias Rodrigues [2011]	1.50	−1.03	0.60
Q4DRL	1.50	−1.03	0.60
Distorted Mesh			
Force Load			
Ibrahimbegovic et al. [1990]	1.140 45	−0.572 47	-
Pimpinelli [2004]	1.39	−0.54	−0.56
Moreira and Dias Rodrigues [2011]	1.26	−0.54	−0.51
Wang et al. [2016]	1.518	−0.65	−0.599
Q4DRL	1.26	−0.54	−0.51
Q4	0.25	-	−0.12
Moment Load			
Ibrahimbegovic et al. [1990]	1.392	−0.495 08	-
Pimpinelli [2004]	1.39	−1.28	−0.56
Moreira and Dias Rodrigues [2011]	1.35	−0.84	−0.54
Q4DRL	1.35	−0.84	−0.54

5.4.3 | Cantilever Beam

A cantilever beam loaded at its end is a well-established test for comparing finite element results with exact solutions available in the literature [Timoshenko and Goodier, 1969], especially to evaluate the performance of plane-stress elements when predominantly subjected to in-plane bending behaviour - see Figure 5.16. Table 5.13 lists the geometric and mechanical properties of the beam.

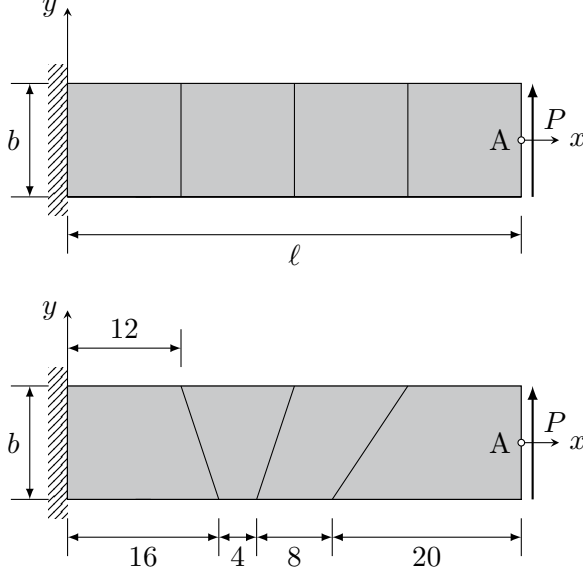


Table 5.13 Cantilever beam parameters

Length (ℓ)	48.0
Height (b)	12.0
Thickness (h)	1.0
Young's Modulus (E)	30.0×10^3
Poisson's Ratio (ν)	0.25
Load (P)	40.0

Figure 5.16 Cantilever beam

To ensure the appropriate shear stress distribution according to the elasticity theory [Timoshenko and Goodier, 1969],

$$\tau_{xy} = \frac{P}{2I} \left(\frac{b^2}{4} - y^2 \right) \quad (5.7)$$

one applies the load parabolically at the free end. Many authors use this hypothesis to compare numerical solutions obtained with the FEM with exact solutions derived from the 2D elasticity theory. It is the case of Allman [1984], which cites the work of Olson and Bearden [1979] and compares the tip displacement of the cantilever beam at point A with the exact value of 0.3558. Neither work explain the background for this value. However, Bergan and Felippa [1985] present an expression for the tip displacement of the cantilever beam derived from the 2D elasticity theory with the same result

$$v(x = \ell, y = 0) = \frac{P\ell^3}{3EI} + (4 + 5\nu) \frac{P\ell}{2Eb} = 0.35583 \quad (5.8)$$

where the $P\ell^3/3EI$ term corresponds to the classical tip displacement of a cantilever beam due to pure bending, and the remaining one accounts for the shear deformation. However, there is an inconsistency between such result and the parameters presented in Table 5.13. The correct result for this expression is

$$v(x = \ell, y = 0) = \frac{P\ell^3}{3EI} + (4 + 5\nu) \frac{P\ell}{2Eb} = 0.35533 \quad (5.9)$$

Probably, the result presented by Allman [1984] and Bergan and Felippa [1985] has the same basis. Following works like the one of Allman [1988] used the same “exact” solution. Cook [1986] and Macneal and Harder [1988] compared their numerical solutions with the reference value of 0.3573 for the tip displacement of the cantilever beam, including 0.0160 due to shear deformation. Therefore, the total displacement sums up the classical tip displacement due to pure bending ($P\ell^3/3EI$) to the small quantity due to shear deformation

$$v(x = \ell, y = 0) = \frac{P\ell^3}{3EI} + 0.0160 = 0.3413 + 0.0160 = 0.3573 \quad (5.10)$$

Neither work explain the basis for the shear deformation part, and if one compares it with Equation 5.9, where one can also divide the total tip displacement into two parts (pure bending plus shear deformation), one can notice that the shear deformation part is exactly 0.0140 for the parameters presented in Table 5.13, resulting in the total displacement of Equation 5.9.

Later, Ibrahimbegovic et al. [1990] also studied this configuration as a test problem for a finite element with drilling degrees of freedom. Citing Timoshenko and Goodier [1969], they present the elasticity solution for the tip displacement of the cantilever beam exactly as Equation 5.9. Consistently, Kouhia [1996], Zhu and Zacharia [1996] and Pimpinelli [2004], among others, used the same reference value. Interestingly, the boundary conditions at the built-in section considered in these works are slightly different from the zero displacement condition of the previous studies. As Augarde and Deeks [2008] pointed out, there are potential sources of error when using the elasticity solution from Timoshenko and Goodier [1969] and comparing it to particular boundary conditions. For completeness, one derives the elasticity solution presented earlier in Equation 5.9.

Such differences and missing references are incompatible with the “exact” solution statement since the results are non-reproducible and vary between authors. Another critical feature of this problem is the boundary conditions at the built-in section. All the cited works considered zero displacements along any direction at the built-in section, but this is incompatible with the expression presented by Bergan and Felippa [1985]. For the sake of completeness, one derives in this work the exact solution of the displacement field for the cantilever beam of Figure 5.16. Considering the cantilever beam of unit thickness in Figure 5.16, the well-known normal stress distribution is

$$\sigma_{xx} = \frac{Py}{I} (x - \ell) \quad (5.11)$$

For the case of two-dimensional stress, recalling the equilibrium equation,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (5.12)$$

it is simple to derive the shear stress distribution in the cantilever beam by integration

$$\tau_{xy} = \frac{P}{2I} \left(\frac{b^2}{4} - y^2 \right) \quad (5.13)$$

Extensional and shear strains can be written as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\sigma_{xx}}{E} = \frac{Py}{EI} (x - \ell) \quad (5.14a)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -\frac{\nu \sigma_{xx}}{E} = -\frac{\nu Py}{EI} (x - \ell) \quad (5.14b)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = \frac{P}{2IG} \left(\frac{b^2}{4} - y^2 \right) \quad (5.14c)$$

where $u(x, y)$ and $v(x, y)$ represent the displacement along x and y directions, respectively. One can derive the displacement field by integration of the extensional strains,

$$u(x, y) = -\frac{P\ell yx}{EI} + \frac{Px^2y}{2EI} + f(y) \quad (5.15a)$$

$$v(x, y) = \frac{\nu P\ell y^2}{2EI} - \frac{\nu Pxy^2}{2EI} + g(x) \quad (5.15b)$$

and re-writing the shear strain using this displacement field, one can write

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{P}{2IG} \left(\frac{b^2}{4} - y^2 \right) \quad (5.16)$$

$$-\frac{P\ell x}{EI} + \frac{Px^2}{2EI} + \frac{\partial f(y)}{\partial y} - \frac{\nu Py^2}{2EI} + \frac{\partial g(x)}{\partial x} = \frac{P}{2IG} \left(\frac{b^2}{4} - y^2 \right) \quad (5.17)$$

One can also rearrange the previous equation considering the x and y dependent terms, but also the constant term as

$$\overbrace{-\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} + \frac{\partial f(y)}{\partial y}}^{F(y)} - \overbrace{\frac{P\ell x}{EI} + \frac{Px^2}{2EI} + \frac{\partial g(x)}{\partial x}}^{G(x)} = \overbrace{\frac{Pb^2}{8IG}}^K \quad (5.18)$$

As Timoshenko and Goodier [1969] pointed out, $F(y)$ and $G(x)$ must be some constants d and e , respectively. Otherwise, the sum of both $G(x)$ and $F(y)$ functions would not be constant, as it emerges from Equation 5.18

$$d + e = \frac{Pb^2}{8IG} \quad (5.19)$$

Thereby,

$$\frac{\partial f(y)}{\partial y} = \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG} + d \quad (5.20)$$

$$\frac{\partial g(x)}{\partial x} = \frac{P\ell x}{EI} - \frac{Px^2}{2EI} + e \quad (5.21)$$

$$f(y) = \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + dy + g \quad (5.22)$$

$$g(x) = \frac{P\ell x}{EI} - \frac{Px^2}{2EI} + ex + h \quad (5.23)$$

Integrating both functions and substituting them into the displacement field written in Equation 5.15

$$u(x, y) = -\frac{P\ell yx}{EI} + \frac{Px^2y}{2EI} + \frac{\nu Py^3}{6EI} - \frac{Py^3}{6IG} + dy + g \quad (5.24a)$$

$$v(x, y) = \frac{\nu P\ell y^2}{2EI} - \frac{\nu Pxy^2}{2EI} + \frac{P\ell x}{EI} - \frac{Px^2}{2EI} + ex + h \quad (5.24b)$$

5. Numerical Validation Results

Considering the relation between constants d and e from Equation 5.19, one requires three prescribed boundary conditions to define the displacement field. Both displacements $u(x, y)$ and $v(x, y)$ must be null at the root section. Taking the neutral axis as a reference point

$$u(x = 0, y = 0) = 0 \implies g = 0 \quad (5.25a)$$

$$v(x = 0, y = 0) = 0 \implies h = 0 \quad (5.25b)$$

Additionally, one requires a third boundary condition which one can realise in various ways, such as a zero slope condition at the root section. Alternatively, the displacement along the x direction at the top and bottom of the root section might be assumed null

$$u(x = 0, y = \pm b/2) = 0 \implies d = \frac{Pb^2}{24EI} (2 + \nu) \quad (5.26)$$

Substituting into Equation 5.19

$$e = \frac{Pb^2}{24EI} (4 + 5\nu) \quad (5.27)$$

Finally, one can write the displacement field of the cantilever beam as

$$u(x, y) = \frac{Py}{6EI} \left[(3x - 6\ell)x + (2 + \nu) \left(\frac{b^2}{4} - y^2 \right) \right] \quad (5.28a)$$

$$v(x, y) = \frac{P}{6EI} \left[3\nu y^2 (\ell - x) + \frac{b^2 x}{4} (4 + 5\nu) + (3\ell - x)x^2 \right] \quad (5.28b)$$

Following Timoshenko and Goodier [1969] procedure, this exact solution derives from the boundary conditions established in Equations 5.25 and 5.26. If one ignores those boundary conditions, Equations 5.28 are no longer valid. Many authors overlook this in many publications. Therefore, it is difficult to be sure if the comparisons between different formulations consider the same boundary conditions, especially considering that boundary conditions of Equations 5.25 and 5.26 do not correctly model fixed supports in FEM analyses. Still, Equation 5.9 gives the correct tip displacement of the cantilever beam. Figure 5.17 depicts a proper cantilever beam model with the known exact solution.

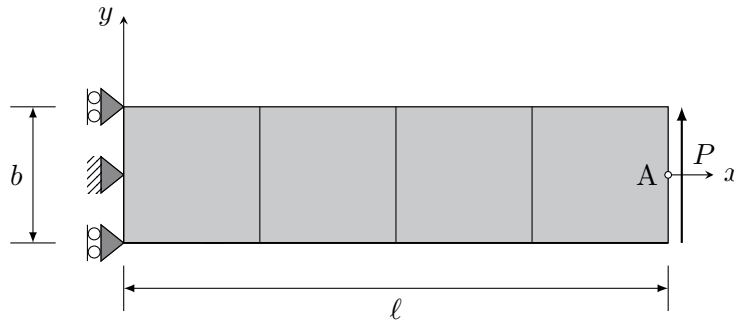


Figure 5.17 Cantilever beam model with known exact solution

A thorough comparison of FEM results with exact solutions requires a complete understanding of the assumptions in which the mathematical model lies. The boundary conditions at the root section are of primary importance when comparing the results of the presented finite element with the known exact solution. The finite element model created for this comparison

considers prescribed displacements at each node of the root section given by the exact solution of Equation 5.28

$$u(x = 0, y) = \frac{Py(2 + \nu)}{6EI} \left(\frac{b^2}{4} - y^2 \right) \quad (5.29a)$$

$$v(x = 0, y) = \frac{Py^2\nu\ell}{2EI} \quad (5.29b)$$

Moreover, one should apply the vertical load as shear traction varying parabolically to properly model the prescribed shear stress distribution presented in Equation 5.13. Olson and Bearden [1979] proposed a possible solution to approximate the nodal forces of the finite element model to accomplish the parabolically varying shear load, and one follows the same approach here. Table 5.14 shows the normalised results of the vertical displacement at point A for the present work compared to some reference works found in the literature.

Table 5.14 Vertical displacement at point A (Normalised results with respect to exact solution)

Reference	Regular			Irregular
	4×1	8×2	16×4	4×1
Allman [1984]	0.759	0.918	0.977	-
Bergan and Felippa [1985]	0.969	0.996	0.999	-
Cook [1986]	0.924	-	-	0.951
Allman [1988]	0.852	0.955	0.988	-
Macneal and Harder [1988]	0.963	-	-	0.892
Ibrahimbegovic et al. [1990]	0.970	0.986	0.997	0.863
Ibrahimbegović and Frey [1993]	0.983	0.992	1.000	0.932
Kouhia [1996]	0.927	0.979	0.999	0.770
Zhu and Zacharia [1996]	0.924	0.986	1.000	1.008
Pimpinelli [2004]	0.983	0.992	0.999	0.968
Moreira and Dias Rodrigues [2011]	0.924	0.974	0.993	0.932
Q4DRL	0.930	0.982	0.996	0.935
Q4	0.688	0.898	0.972	0.605

Table 5.14 shows both versions of the membrane element, with and without drilling degrees of freedom, presented in this work for comparison purposes. The performance of the implemented drilling degree of freedom is good and follows other reference works. Figure 5.18 depicts the extensional strain distribution in the cantilever beam under analysis, and one can notice

$$\varepsilon_{xx} \propto x \quad (5.30)$$

However, as shown in Equation 5.5a,

$$\varepsilon_{xx} \neq f(x) \quad (5.31)$$

for each bilinear quadrilateral element. Consequently, the drilling degree of freedom improves accuracy significantly, particularly for coarser meshes, by capturing the out-of-plane rotation.

The noticeable performances of the elements proposed by Ibrahimbegović and Frey [1993] or Pimpinelli [2004], for example, reflect the use of a set of incompatible modes to enhance the displacement field of the membrane element or an element with an assumed strain field. Apart from the boundary conditions at the built-in section, one can not find any other reasoning to

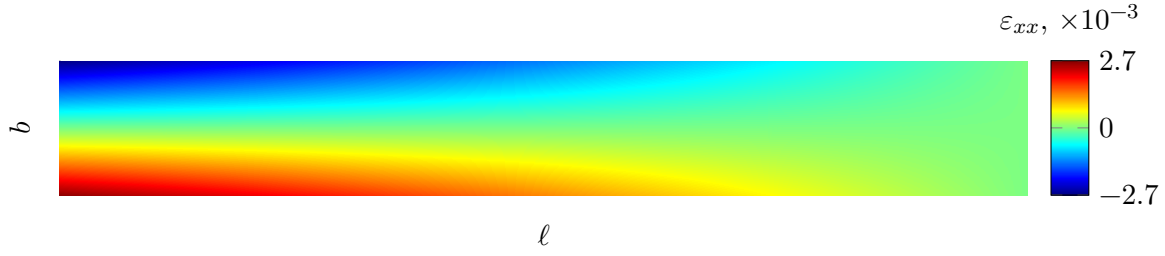


Figure 5.18 Normal strain distribution in cantilever beam

explain the differences between the present work and that of Moreira and Dias Rodrigues [2011], and here one explains why one can not constrain all the translations at the built-in section.

5.4.4 | Cook's Beam

Cook [1974] proposed a short distorted cantilever beam subjected to a distributed load along its end specially designed to assess the effect of the finite element distortion - Figure 5.19. Table 5.16 shows the geometric and mechanical properties of Cook's skew beam.

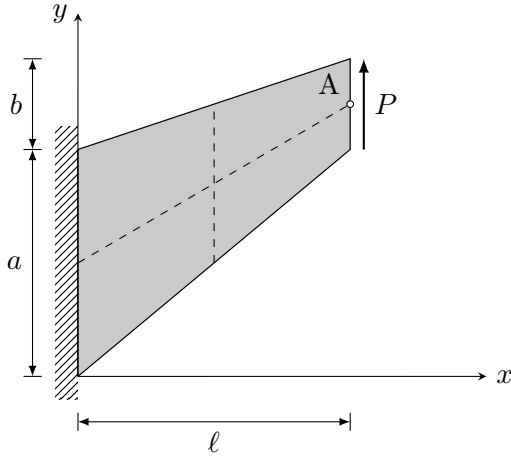


Table 5.15 Cook's beam parameters

Length (ℓ)	48.0
Height (a)	44.0
Height (b)	16.0
Thickness (h)	1.0
Young's Modulus (E)	1.0
Poisson's Ratio (ν)	1/3
Load (P)	1.0

Figure 5.19 Cook's skew beam

As shown in the previous example, analytical solutions are only available for specific structures under particular conditions, and there is no exact solution for Cook's beam vertical displacement at point A. After Bergan and Felippa [1985] and their convergent numerical analysis using a 32×32 finite element mesh, many authors considered 23.91 as the vertical reference displacement for point A. However, Yuqiu and Yin [1994] showed a detailed convergent numerical analysis with a new reference value of 23.96 for the vertical displacement of point A using a finer mesh of 64×64 elements, and some authors have been using this new reference value for comparison purposes. One assumes in this work the reference value of 23.96 for the vertical displacement of point A and the restraining of the drilling degrees of freedom at the root section, something that is not always explicit in other works and weakens the comparison. Table 5.16 presents the numerical results obtained with the implemented element compared to those of other researchers.

The element proves its accuracy, and the vertical tip displacement converges to the reference value proposed by Yuqiu and Yin [1994]. In comparison, the bilinear quadrilateral element Q4 show the need to account for the membrane rotation, as explained in the previous examples, especially for coarse meshes. It is important to note that the excellent results obtained by Cen

Table 5.16 Vertical displacement at point A (Normalised results with respect to 23.96)

Reference	Mesh					
	2×2	4×4	8×8	16×16	32×32	64×64
Bergan and Felippa [1985]	0.850	0.936	0.977	0.998	-	-
Cook [1987]	0.898	0.967	-	-	-	-
Allman [1988]	0.846	0.951	0.983	-	-	-
Ibrahimbegovic et al. [1990]	0.863	0.960	0.988	-	-	-
Yuqiu and Yin [1994]	0.828	0.937	0.979	0.994	0.998	1.000
Geyer and Groenwold [2002]	0.899	0.962	0.989	-	-	-
Pimpinelli [2004]	0.877	0.960	0.988	-	-	-
Choi et al. [2006]	0.941	0.978	0.993	0.997	-	-
Moreira and Dias Rodrigues [2011]	0.928	0.957	0.986	-	-	-
Cen et al. [2011]	0.941	0.978	0.993	0.997	-	-
Wang et al. [2016]	0.941	0.978	0.993	0.997	-	-
Q4DRL	0.928	0.968	0.989	0.996	0.999	1.000
Q4	0.492	0.762	0.920	0.977	0.996	0.998

et al. [2011] and Wang et al. [2016], for example, are explained by their formulations based on assumed initial stresses, which are different from the usual displacement-based models. There is no Jacobi inverse in integration formulae for such formulations, and the element shape might be either convex or concave and consequently immune to distorted meshes.

5.4.5 | MacNeal's Thin Cantilever Beam

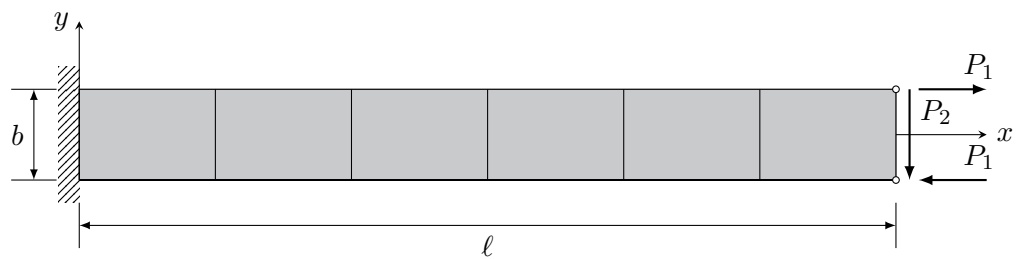
MacNeal and Harder [1985] extended a frequently used test problem consisting of a straight cantilever beam (see Figure 5.20a) using irregular meshes - trapezoidal and parallelogram elements represented in Figures 5.20b and 5.20c, respectively. The test can assess the element sensitivity to different mesh distortions apart from the high length-to-height aspect ratio, which is a demanding test for membrane elements. Table 5.17 lists the geometric and mechanical properties of the cantilever beam.

Table 5.17 Macneal's thin cantilever beam properties

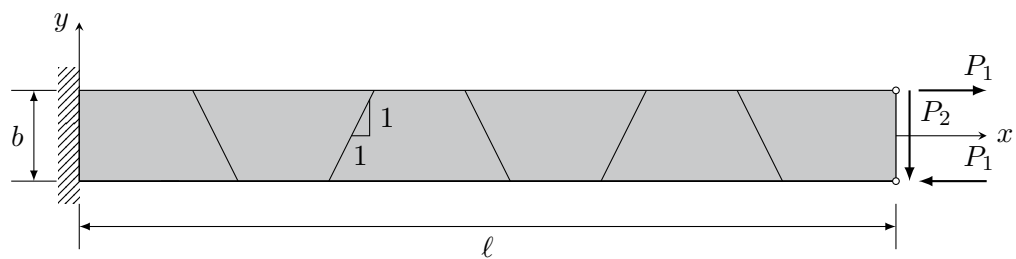
Length (ℓ)	6.0
Height (b)	0.2
Thickness (h)	0.1
Young's Modulus (E)	1×10^7
Poisson's Ratio (ν)	0.3
Load (P_1)	1.0
Load (P_2)	1.0

As most of the researchers performing this test, one considers two load cases: pure bending (P_1) and transverse linear bending (P_2). According to Wang et al. [2016], reference solutions for the tip deflection are -0.0054 and -0.1081 , respectively. Table 5.18 presents the obtained results with the implemented element compared to reference works.

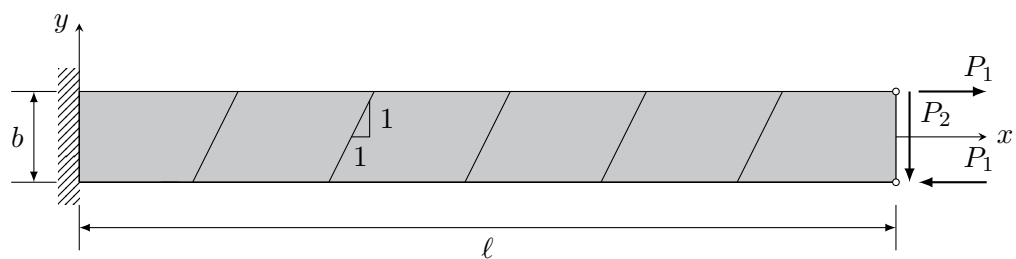
Table 5.18 shows that the present non-conforming element has acceptable accuracy for regular



(a) Mesh A



(b) Mesh B



(c) Mesh C

Figure 5.20 MacNeal's thin cantilever beam with regular and distorted meshes

Table 5.18 Vertical tip displacements (Normalised results with respect to -0.0054 and -0.1081)

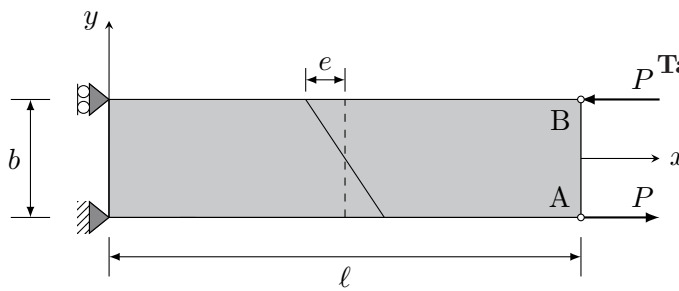
Reference	Load P_1			Load P_2		
	A	B	C	A	B	C
Taylor et al. [1976] ¹	1.000	0.037	0.722	0.993	0.623	0.044
Pian and Sumihara [1984] ¹	1.0	0.167	0.852	0.993	0.221	0.798
MacNeal and Harder [1985]	-	-	-	0.904	0.071	0.080
Wang et al. [2016]	1.0	1.0	1.0	0.993	0.988	0.972
Q4DRL	0.910	0.097	0.680	0.904	0.103	0.603
Q4	0.093	0.022	0.031	0.093	0.027	0.034

¹ Wang et al. [2016]

meshes but performs worse with irregular meshes, especially for the trapezoidal meshes of this case. Still, the performance of the non-conforming element with drilling degrees of freedom used in this work is incomparably superior to the classic isoparametric membrane four-node element Q4. Elements with formulations based on assumed initial stresses like the one of Wang et al. [2016] are different from the usual displacement-based models, there is no Jacobi inverse in integration formulae, and the element shape might be convex or concave and consequently immune to distorted meshes.

5.4.6 | Cantilever Beam with Distortion Parameter

Figure 5.21 presents a cantilever beam subjected to an end moment meshed with two elements containing a distortion parameter - e . Increasing e distorts the mesh severely, and this is another established test to assess the element's sensitivity to mesh distortion. Both elements always have the same area, and the limit case consists of two triangles. Table 5.21 lists the geometric and mechanical properties of the beam. Table 5.20 presents the displacements at the tip section (Points A and B) using the finite element implemented in this work compared with reference results and some published works.

**Table 5.19** Thick curved beam parameters

Lenght (ℓ)	10.0
Height (b)	2.0
Thickness (h)	1.0
Young's Modulus (E)	1500.0
Poisson's Ratio (ν)	0.25
Load (P)	1000.0

Figure 5.21 Cantilever beam with distorted element

As expected, the element performs poorly as the distortion parameter increases. In Sections 5.4.4 and 5.4.5, one explained that formulations based on assumed initial stresses perform better than displacement-based formulations for distorted meshes and Cen et al. [2011] and Wang et al. [2016] works are an example. It is worth addressing these issues even knowing that the Q4DRL membrane element was not derived originally to deal with highly distorted meshes with few elements, and these test cases prove it. Nevertheless, the non-conforming element with drilling degrees of freedom used in this work clearly outperforms the conventional four-node

isoparametric membrane element, which is even more sensitive to mesh distortion.

Table 5.20 Displacements at tip section of the two element cantilever beam with distortion parameter

	Distortion Parameter, e						Reference
	0	1	2	3	4	5	
Allman [1988] ¹							
v_B	93.8	90.7	56.7	31.8	17.9	-	100
Cen et al. [2011]							
u_A	20	20	20	20	20	20	20
u_B	-20	-20	-20	-20	-20	-20	-20
v_A	100	99.47	95.95	87.14	71.87	50	100
v_B	100	100	100	100	100	100	100
Wang et al. [2016]							
u_A	20	20	20	20	20	20	20
u_B	-20	-20	-20	-20	-20	-20	-20
v_A	100	99.47	95.95	87.14	71.87	100	100
v_B	100	100	100	100	100	100	100
Q4DRL							
u_A	18.750	16.411	11.707	7.260	4.647	3.212	20
u_B	-18.750	-16.411	-11.707	-7.260	-4.647	-3.212	-20
v_A	93.749	85.864	63.923	41.418	27.848	20.442	100
v_B	93.749	82.054	58.536	36.298	23.233	16.061	100
Q4							
u_A	5.608	2.826	1.942	1.661	1.441	1.228	20
u_B	-5.608	-2.826	-1.942	-1.661	-1.441	-1.228	-20
v_A	28.037	14.444	9.820	8.299	7.145	6.064	100
v_B	28.037	14.129	9.707	8.305	7.204	6.139	100

¹ Cen et al. [2011]

5.4.7 | Thick Curved Beam

Curvature is one of the most relevant topics when one analyses shell structures using flat facet finite elements. Too often, one requires a considerable number of elements not only to achieve convergence but also to capture the actual curvature of the structure since the facet is flat. A similar phenomenon occurs for curved membranes meshed with quadrilateral elements, and one must represent a curve with straight lines. One considers a thick curved beam clamped at one end and subjected to a shear load at the free end - see Figure 5.22. Table 5.21 presents the geometric and mechanical properties of the structure.

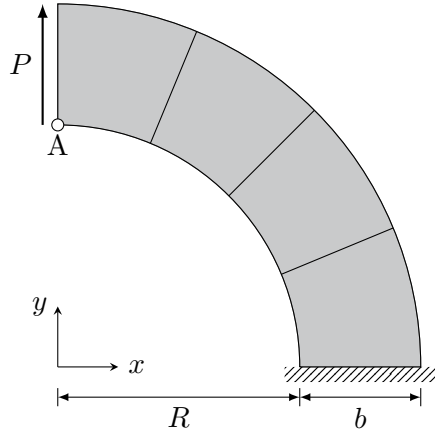


Figure 5.22 Thick curved beam

Table 5.21 Thick curved beam parameters

Radius (R)	10.0
Width (b)	5.0
Thickness (h)	1.0
Young's Modulus (E)	1000.0
Poisson's Ratio (ν)	0
Load (P)	600.0

Table 5.22 lists the results of the vertical deflection at point A, comparing the present work with other published research. Taylor et al. [1976] and their QM6 element were among the first approaches to improve the elements from the isoparametric family and their low accuracy for distorted meshes. Pian and Sumihara [1984] proposed finite elements based on assumed stress to increase the accuracy and efficiency of simulations. Recent works like those of Rezaiee-Pajand and Karkon [2013] and Wang et al. [2016] show membrane elements with great performances. Rezaiee-Pajand and Karkon [2013] developed a four-node stress membrane element with drilling degrees of freedom where the stress field derives from analytical solutions of compatibility equations. Elements based on these formulations are less sensitive to mesh distortions and present high accuracy and robustness for a wide range of benchmark tests, as pointed out previously. Wang et al. [2016] also presented an element based on assumed initial stresses, which proved to be immune to distorted meshes.

Table 5.22 Vertical displacement at point A (Normalised results with respect to 90.10)

Reference	Mesh							
	1×2	1×3	1×4	2×8	4×16	16×64	32×128	64×256
Taylor et al. [1976] ¹	-	-	0.928	-	-	-	-	-
Pian and Sumihara [1984] ¹	-	-	0.939	-	-	-	-	-
R.-Pajand and Karkon [2013]	0.987	0.989	0.995	1.000	-	-	-	-
Wang et al. [2016]	0.987	0.989	0.997	1.000	-	-	-	-
Q4DRL	0.743	0.928	0.968	0.986	0.999	1.004	1.004	1.004
Q4	0.251	0.484	0.643	0.880	0.969	1.002	1.003	1.004

¹ Wang et al. [2016]

Researchers use the 90.10 reference value for the vertical displacement at point A. The membrane element with the drilling degree of freedom used in this work shows good convergence even though the elements of Rezaiee-Pajand and Karkon [2013] and Wang et al. [2016] converge more quickly. As previously mentioned, based on assumed stresses, these elements are less sensitive to mesh distortion and perform better for fewer elements. It is worth noting that both the simple bilinear four-node Q4 and the Q4DLR converge to a slightly higher value than the reference one.

5.4.8 | Thin Curved Beam

In this section, one considers another benchmark test, as represented in Figure 5.23. It is a similar structure to the curved beam of the previous test but thinner, which is a demanding evaluation for a membrane element since the configuration would be better described by beam elements - compared with the thick beam, the elements become narrower. Table 5.23 lists the geometric and mechanical properties of the structure, and Table 5.24 presents the normalised results for the tip displacement at point A, which researchers point out as -0.0886 .

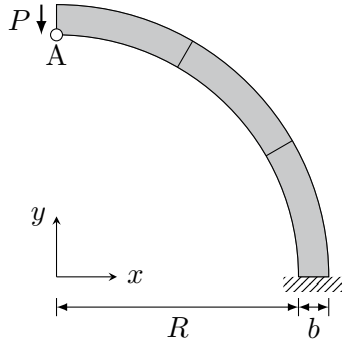


Figure 5.23 Thin curved beam

Table 5.23 Thin curved beam parameters

Radius (R)	4.12
Width (b)	0.20
Thickness (h)	0.10
Young's Modulus (E)	10×10^6
Poisson's Ratio (ν)	0.25
Load (P)	1.00

Table 5.24 Vertical displacement at point A (Normalised results with respect to -0.0886)

Reference	Mesh							
	1×2	1×3	1×6	2×12	4×24	8×48	16×96	32×192
Allman [1988] ¹	-		0.875	0.986	0.994	-	-	-
R.-Pajand and Karkon [2013]	1.052	0.984	0.993	0.998	0.999	-	-	-
Wang et al. [2016]	1.052	1.016	0.993	0.998	0.999	-	-	-
Q4DRL	0.108	0.426	0.876	0.974	0.992	0.998	0.999	0.999
Q4	-	0.017	0.072	0.244	0.564	0.837	0.954	0.986

¹ Wang et al. [2016]

Despite the demanding test, one achieves convergence without major problems. Rezaiee-Pajand and Karkon [2013] and Wang et al. [2016] achieve convergence for fewer elements on account of the characteristics of their formulations, as explained earlier. It is clear from this example that the simple isoparametric membrane element Q4 faces convergence problems for demanding structures.

5.4.9 | Scordelis-Lo Roof

The membrane element Q4DRL tested in the previous sections, coupled with bending and shear components, complete the facet shell finite element proposed in this work. In the following, the main focus is the numerical validation of the facet shell finite element in its distinct versions, whose differences relate to the bending component and drill degree of freedom. The first benchmark test considers the work of Scordelis and Lo [1964], who published a reference example to test finite shell elements. Figure 5.24 presents the roof-type structure proposed by Scordelis and Lo [1964], subjected to its weight and supported at both curved edges by rigid diaphragm boundary conditions. Table 5.25 presents the geometric and mechanical properties of the struc-

ture, which one might simulate using only one-quarter of the roof due to the symmetry conditions.

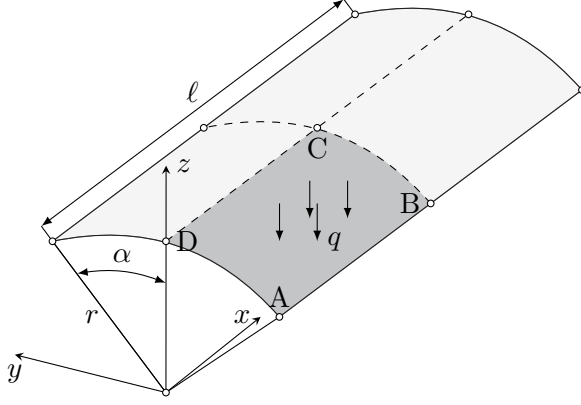


Figure 5.24 Scordelis-Lo roof

Table 5.25 Scordelis-Lo roof parameters

Length (ℓ)	50.0
Radius (r)	25.0
Angle (α)	40°
Thickness (h)	0.25
Young's Modulus (E)	4.32×10^8
Poisson's Ratio (ν)	0.0
Distributed Load (q)	90.0

Both membrane and bending deformations contribute significantly to the strain energy of the structure [MacNeal and Harder, 1985], which makes it suitable to represent complex membrane deformation states since a substantial part of the strain energy is the membrane strain energy [Belytschko et al., 1985]. Researchers frequently use the vertical displacement at the midpoint of the free edge as the test result - w_B . Scordelis and Lo [1964] pointed out the reference value of -0.3086 for the vertical displacement of point B, but some works, like MacNeal and Harder [1985], converge to a slightly smaller value, $w_B = -0.3024$.

Table 5.26 presents the test results obtained with the proposed formulations considering different deformation theories for a convergence study. The analysis compares the conventional finite elements with and without drilling degrees of freedom presented in this work. In addition, one considers reference works using facet shell elements ([Allman, 1994], [Moreira and Dias Rodrigues, 2011], [Shang et al., 2016]), but also curved shells for comparison purposes [Dvorkin et al., 1984]. Similarly to other works, one considers uniform meshes of equal-sized elements, and the mesh size in Table 5.26 refers to the number of divisions of one-quarter of the shell.

Finite elements with or without drilling degrees of freedom converge to the same value for refined meshes, as Table 5.26 shows. However, asymptotic convergence is distinctive for fine meshes. The structure becomes softer with mesh refinement considering the conventional finite elements, i. e. without drilling degrees of freedom, whereas using drilling degrees of freedom, the flexible shell for coarser meshes becomes stiffer with mesh refinement. Most importantly, considering drilling stiffness, one achieves convergence with fewer elements, increasing efficiency. Furthermore, the performance of the proposed implementation is comparable to those of the most recent formulations published in other works. Similarly to previous analyses considering plates, there are no relevant differences between formulations of the bending component. Different finite elements converge at the same rate to the same result; the only remarkable distinction is that CPT(C) formulations have one extra degree of freedom per node, which could be important when considering shell structures with several elements. Another worth mentioning issue is whether one assumes an FSDT or TSDT; both models tend to result in the same vertical displacement, slightly above those using the CPT, as expected from a thin shell structure.

Finally, according to the results obtained, one must conclude that the different versions of the proposed facet shell element pass this benchmark test and ensure high performances.

Table 5.26 Vertical displacement at point B (Normalised results with respect to $w_B = -0.3024$)

Reference		Mesh						
		2×2	4×4	8×8	16×16	32×32	64×64	128×128
Dvorkin et al. [1984]		-	-	0.981	-	-	-	-
MacNeal and Harder [1985]		1.376	1.050	1.008	-	-	-	-
Allman [1994]		-	0.911	0.967	0.991	-	-	-
Moreira and Rodrigues [2011]		0.906	0.775	0.913	-	-	-	-
Shang et al. [2016]		-	1.037	1.003	0.996	0.995	-	-
CPT	CPT(N)	1.182	0.939	0.971	0.987	0.992	0.993	0.994
	CPT(C)	1.143	0.932	0.969	0.987	0.992	0.993	0.994
	DKQ	1.211	0.944	0.972	0.988	0.992	0.993	0.994
	IDKQ	1.168	0.936	0.970	0.987	0.992	0.993	0.994
	CPT(N)DRL	1.355	1.042	1.003	0.996	0.994	0.994	0.994
	CPT(C)DRL	1.306	1.033	1.001	0.996	0.994	0.994	0.994
	DKQDRL	1.392	1.048	1.005	0.996	0.995	0.994	0.994
	IDKQDRL	1.335	1.039	1.003	0.996	0.994	0.994	0.994
FSDT	CPT(N)	1.182	0.939	0.971	0.988	0.993	0.995	0.996
	CPT(C)	1.143	0.932	0.969	0.988	0.993	0.995	0.996
	DKQ	1.211	0.944	0.973	0.988	0.993	0.995	0.996
	IDKQ	1.168	0.936	0.971	0.988	0.993	0.995	0.996
	CPT(N)DRL	1.356	1.042	1.004	0.997	0.996	0.996	0.996
	CPT(C)DRL	1.307	1.034	1.002	0.996	0.995	0.996	0.996
	DKQDRL	1.392	1.048	1.005	0.997	0.996	0.996	0.996
	IDKQDRL	1.335	1.039	1.003	0.997	0.995	0.996	0.996
TSDT	CPT(N)	1.182	0.939	0.971	0.988	0.994	0.995	0.996
	CPT(C)	1.143	0.932	0.969	0.988	0.993	0.995	0.996
	DKQ	1.211	0.944	0.973	0.989	0.994	0.996	0.996
	IDKQ	1.168	0.936	0.971	0.988	0.993	0.995	0.996
	CPT(N)DRL	1.356	1.042	1.004	0.997	0.996	0.996	0.996
	CPT(C)DRL	1.307	1.034	1.002	0.996	0.996	0.996	0.996
	DKQDRL	1.392	1.048	1.005	0.997	0.996	0.996	0.996
	IDKQDRL	1.335	1.039	1.003	0.997	0.996	0.996	0.996

5.4.10 | Pinched Cylinder with Diaphragm Supports

Figure 5.25 presents another well-known benchmark test for convergence analysis of finite shell elements. It is a cylinder with rigid diaphragm supports at both ends and subjected to a pair of opposite point loads applied at its mid-span. The symmetry conditions make it possible to analyse only one-eighth of the cylinder. Table 5.27 presents the geometric and mechanical properties of the cylinder.

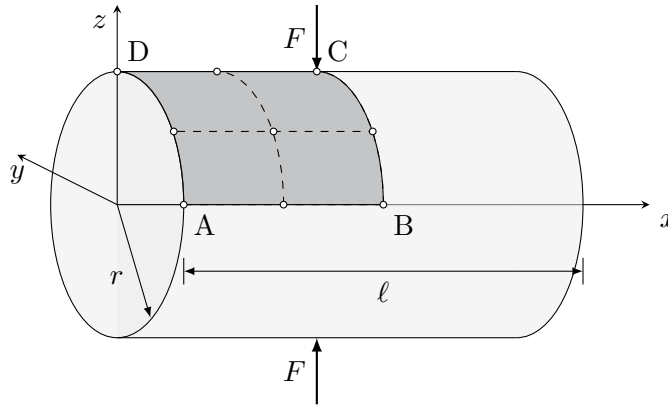


Table 5.27 Pinched cylinder parameters

Length (ℓ)	600
Radius (r)	300
Thickness (h)	3.0
Young's Modulus (E)	3.0×10^6
Poisson's Ratio (ν)	0.3
Point Load (F)	1.0

Figure 5.25 Pinched cylinder

Allman [1994] noted that the results for a similar cylinder to that of Figure 5.25 but with free ends follow the same behaviour as a flat plate problem since it mainly involves inextensional deformation as the strain energy comes from bending and virtually no membrane strain energy. However, as Allman [1994] put it, the analysis of curved shells presents a more demanding challenge than the analysis of flat plates. Therefore, the presence of rigid diaphragm boundary conditions at both ends imposes the membrane strain energy contribution into the strain energy, even if it is smaller when compared to the bending part.

Table 5.28 presents the normalised results of the vertical displacement at load point C using the facet shell finite elements proposed in this work, with and without drilling degrees of freedom, compared to some reference works. Belytschko et al. [1985] considered the result of 1.8248×10^{-5} for the radial displacement at the load points. As previously mentioned, the presence of rigid diaphragm boundary conditions at both ends enforces extensional strain, and the noticeable effect is the significant displacement variation over a narrow zone around the loaded points [Ibrahimbegović and Frey, 1994]. Consequently, one expects weak results for coarse meshes, which improve with finer meshes around the loaded node. Thus, when comparing meshes with an equal number of elements but different from one another, those with high-order interpolations of the displacement field would perform better. It is the case of CPT(C) and IDKQ elements, as shown in Table 5.28. The influence of the drilling stiffness on the results follows the same trends as the previous example and enables one to achieve convergence with fewer elements. As expected from a thin shell structure, higher-order deformation theories like the FSDT or TSDT tend to result in the same vertical displacement, marginally above those using the CPT.

In conclusion, the different versions of the proposed facet shell element pass another benchmark test with remarkable performances showing that the implementation can manage complex deformation states.

Table 5.28 Vertical displacement at point C (Normalised results with respect to $w_C = -1.8248 \times 10^{-5}$)

Reference		Mesh					
		4×4	8×8	16×16	32×32	64×64	128×128
Allman [1994]		0.599	0.909	0.973	-	-	-
Ibrahimbegović and Frey [1994]		0.370	0.737	0.934	-	-	-
Kouhia [1996]		1.100	1.500	1.357	1.298	-	-
Pimpinelli [2004]		0.626	0.938	1.094	-	-	-
Moreira and Dias Rodrigues [2011]		0.348	0.732	0.926	0.988	-	-
Shang et al. [2016]		0.629	0.920	0.992	-	-	-
CPT	CPT(N)	0.612	0.933	1.005	1.007	1.005	1.004
	CPT(C)	0.603	0.911	0.987	0.998	1.000	1.001
	DKQ	0.616	0.942	1.013	1.010	1.007	1.005
	IDKQ	0.610	0.927	0.999	1.003	1.003	1.003
	CPT(N)DRL	0.630	0.941	1.007	1.007	1.005	1.004
	CPT(C)DRL	0.621	0.919	0.989	0.999	1.001	1.001
	DKQDRL	0.634	0.950	1.016	1.011	1.007	1.005
	IDKQDRL	0.625	0.932	0.999	1.003	1.003	1.003
FSDT	CPT(N)	0.613	0.934	1.008	1.011	1.010	1.010
	CPT(C)	0.604	0.912	0.989	1.002	1.006	1.007
	DKQ	0.616	0.943	1.016	1.015	1.012	1.011
	IDKQ	0.610	0.928	1.001	1.007	1.008	1.009
	CPT(N)DRL	0.631	0.942	1.010	1.011	1.011	1.010
	CPT(C)DRL	0.621	0.920	0.991	1.003	1.006	1.007
	DKQDRL	0.634	0.951	1.018	1.015	1.013	1.012
	IDKQDRL	0.625	0.933	1.001	1.007	1.008	1.008
TSDT	CPT(N)	0.613	0.934	1.008	1.012	1.011	1.011
	CPT(C)	0.604	0.912	0.989	1.003	1.006	1.008
	DKQ	0.616	0.943	1.017	1.016	1.013	1.013
	IDKQ	0.610	0.928	1.001	1.008	1.009	1.010
	CPT(N)DRL	0.631	0.942	1.010	1.012	1.012	1.011
	CPT(C)DRL	0.621	0.920	0.991	1.004	1.007	1.008
	DKQDRL	0.634	0.951	1.019	1.016	1.014	1.013
	IDKQDRL	0.625	0.933	1.002	1.008	1.009	1.010

5.4.11 | Pinched Hemisphere with Hole

The following benchmark test, proposed originally by MacNeal and Harder [1985], is a well-designed test to assess the performance of facet shell finite elements. It consists of a hemisphere with a hole subjected to diametrical pinching and pulling loads, as depicted in Figure 5.26. This doubly-curved shell problem constitutes a very severe test for flat facet finite elements since there is a poor geometrical representation of the curved shell by the flat elements for coarse meshes [Allman, 1994]. This problem involves mainly inextensional bending deformation, and the result for the displacement at the load points along the load direction is 0.094 [MacNeal and Harder, 1985]. The symmetry conditions make it possible to analyse only one-quarter of the hemisphere.

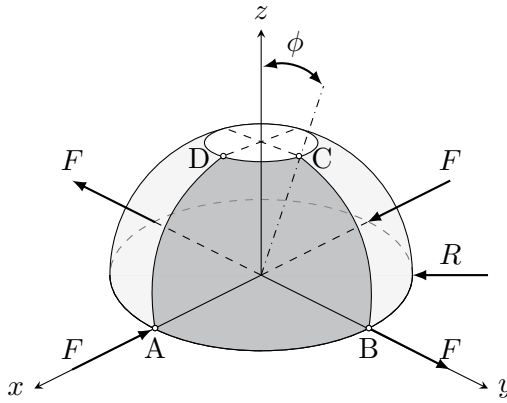


Table 5.29 Pinched hemisphere parameters

Radius (R)	10.0
Thickness (h)	0.04
Angle (ϕ)	18°
Young's Modulus (E)	6.825×10^7
Poisson's Ratio (ν)	0.3
Point Load (F)	2.0

Figure 5.26 Pinched hemisphere with hole

Table 5.30 presents the results obtained for the horizontal displacement along the x -axis at point A compared with some reference works and considering uniform meshes of regular elements. The mesh size in Table 5.30 refers to the number of divisions in one-quarter of the hemisphere.

As previously mentioned, the doubly-curved shell creates a severe test for flat facet finite elements due to the geometrical representation of the structure, particularly for coarser meshes. Table 5.30 shows the poor results when considering a small number of elements. Apart from this specific detail, the results follow the same trend as the previous studies and finite elements with or without drilling degrees of freedom converge to the same value for refined meshes. Besides, there are no relevant differences between formulations of the bending component, and different finite elements converge to the same result at the same rate.

However, it is relevant to point out that one obtained the results for the non-conforming and conforming classical bending components, CPT(N) and CPT(C), considering a reduced integration of the corresponding sub-matrices - two Gauss points along each direction. The results considering a complete integration suggest some locking effect that artificially increases the structure's stiffness. On the other hand, DKQ and IDKQ formulations do not present this behaviour, proving higher robustness for these cases.

Regarding deformation theories, this benchmark test confirms previous results with higher-order models like FSDT or TSDT, resulting in the same displacement, slightly above those using the CPT, as expected from a thin shell structure. Ultimately, it is possible to conclude that the proposed facet finite elements pass the classical benchmark tests available in the literature with excellent performances and thus constitute a robust and efficient alternative in the simulation of shell structures.

Table 5.30 Horizontal displacement at point A (Normalised results with respect to $u_A = -0.094$)

Reference		Mesh						
		2×2	4×4	8×8	16×16	32×32	64×64	128×128
MacNeal and Harder [1985]		0.972	1.024	1.005	-	-	-	-
Allman [1994]		0.054	0.143	0.616	0.948	-	-	-
Ibrahimbegović and Frey [1994]		-	0.931	0.997	0.995	-	-	-
Pimpinelli [2004]		0.369	0.806	0.986	0.994	-	-	-
Moreira and Rodrigues [2011]		0.872	0.990	0.987	0.990	-	-	-
Shang et al. [2016]		0.516	0.951	0.982	-	-	-	-
CPT	CPT(N)	0.938	1.018	1.000	0.991	0.993	0.994	0.995
	CPT(C)	1.027	1.034	1.003	0.991	0.993	0.994	0.995
	DKQ	1.020	1.042	1.008	0.993	0.993	0.994	0.995
	IDKQ	1.020	1.032	1.003	0.992	0.993	0.994	0.995
	CPT(N)DRL	0.637	0.829	0.980	0.991	0.994	0.995	0.995
	CPT(C)DRL	0.569	0.827	0.982	0.992	0.994	0.995	0.995
	DKQDRL	0.657	0.717	0.968	0.992	0.994	0.995	0.995
	IDKQDRL	0.561	0.699	0.963	0.991	0.994	0.995	0.995
FSDT	CPT(N)	0.938	1.018	1.000	0.991	0.993	0.996	0.997
	CPT(C)	1.027	1.034	1.003	0.992	0.994	0.996	0.997
	DKQ	1.020	1.042	1.008	0.993	0.994	0.995	0.997
	IDKQ	1.020	1.032	1.003	0.992	0.994	0.995	0.997
	CPT(N)DRL	0.637	0.829	0.980	0.992	0.994	0.996	0.997
	CPT(C)DRL	0.569	0.827	0.980	0.992	0.994	0.996	0.997
	DKQDRL	0.657	0.717	0.968	0.993	0.995	0.995	0.997
	IDKQDRL	0.561	0.699	0.963	0.991	0.995	0.995	0.997
TSDT	CPT(N)	0.938	1.018	1.000	0.991	0.993	0.996	0.997
	CPT(C)	1.027	1.034	1.003	0.992	0.994	0.996	0.997
	DKQ	1.020	1.042	1.008	0.994	0.994	0.995	0.997
	IDKQ	1.020	1.032	1.003	0.992	0.994	0.995	0.997
	CPT(N)DRL	0.637	0.829	0.980	0.992	0.994	0.996	0.997
	CPT(C)DRL	0.569	0.827	0.982	0.992	0.995	0.996	0.997
	DKQDRL	0.657	0.717	0.968	0.993	0.995	0.995	0.997
	IDKQDRL	0.561	0.699	0.963	0.991	0.995	0.995	0.997

5.5 | Sandwich Shells

The study of shells is of great relevance in engineering structures such as the ones used in naval or aerospace industries, and vibration control using damping materials in marine crafts or aircraft, for example, is inseparable from sandwich shells. Therefore, it is worth establishing mechanical models to assess the damping capacity of sandwich shell structures to control high amplitudes of vibration. These models must include the classical constrained layer damping treatment, and the mechanical model associated with the sandwich shell must be able to take into account the shear deformation pattern of the viscoelastic core to capture the dominant damping mechanism. The extension of the multi-layer approach of the hybrid type using warping functions proposed in this work, which can model different deformation theories from the CPT to higher-order ZZT formulations, is of great relevance from the author's perspective. For example, the capacity to model highly inhomogeneous complex structures involving shell elements and partial damping treatments with viscoelastic materials embedded in composite layers is a significant advantage of the present mechanical model. Such models are sparse or incomplete in the literature and constantly complex when accessible, involving complex geometry definitions and many degrees of freedom to capture the deformation pattern of each layer. Figure 5.27 shows a schematic representation of a cylindrical sandwich shell of radius r , length ℓ , and elastic facings with thickness h_f and a viscoelastic core with thickness h_c .

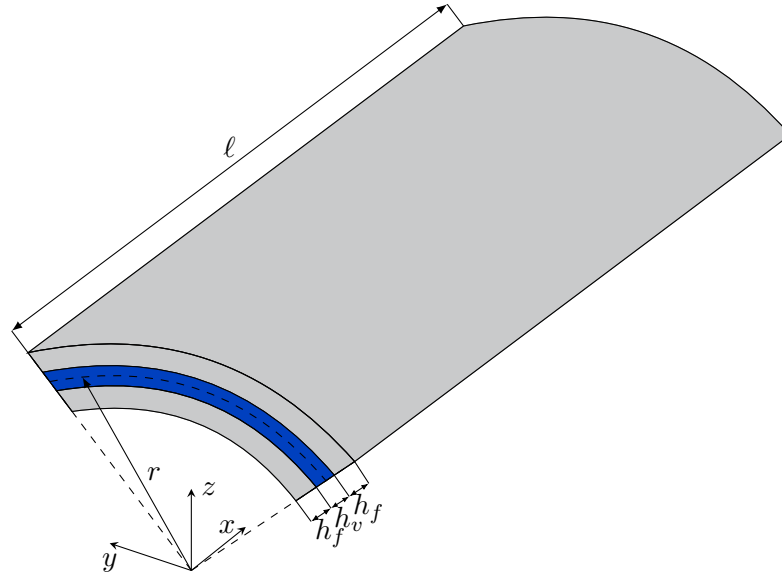


Figure 5.27 Sandwich cylindrical shell

Pan [1969] was among the pioneer researchers studying the vibrations of sandwich shells with a viscoelastic core, extending the work of DiTaranto [1965] to the axisymmetrical vibration of a finite-length circular cylindrical shell with Passive Constrained Layer Damping (PCLD) under the sponsorship of the National Aeronautics and Space Administration (NASA). In the words of Pan [1969], “*the study of the dynamic characteristics of a composite circular hollow cylinder with elastic and viscoelastic layers has a direct and important bearing on the structural problems of missiles and rockets*”. Later, Ramesh and Ganesan [1994a] used a three-node finite element with seven degrees of freedom per node, including separate rotations for the viscoelastic core and elastic facings, to study the vibration and damping characteristics of three-layered cylindrical shells. At the same time, Ramesh and Ganesan [1994b] extended the mechanical model to

orthotropic materials and studied sandwich shells with composite facings, namely graphite/epoxy, with internal material damping. Following, Ramesh and Ganesan [1995] investigated cylindrical shells with constrained damping treatments, including the influence of extensional effects in the core, comparing three deformation theories and their impact on the loss factor estimation.

Ramasamy and Ganesan [1999] studied the fluid effect on the vibration and damping analysis of fluid-filled orthotropic cylindrical shells with constrained viscoelastic damping and, recently, Yang et al. [2015] studied the vibration and damping of thick sandwich cylindrical shells with a viscoelastic core using an FSDT and new displacement field expression based on a standard Fourier series and auxiliary functions expansion to improve convergence.

In this section, the focus relies on the performance comparison of the flat facet shell finite element proposed in this work with the results obtained by some of the cited researchers.

5.5.1 | Bare Cylindrical Shell

Firstly, one studies the facet finite element model for a bare cylindrical shell by comparing its natural frequencies with those reported in the literature. An interesting example is the case of the Fast Test Reactor (FTR) thermal liner in the U.S.A., a thin circular cylindrical shell inside a reactor vessel Chung [1981]. Table 5.31 presents the geometric and mechanical properties of the structure, clamped at the bottom and free at the top - CF boundary conditions.

Table 5.31 Bare cylindrical shell geometric and mechanical properties

ℓ	r	h	E	ν	ρ
511.12 mm	216.2 mm	1.5 mm	183 GPa	0.3	7492 kg m ⁻³

Table 5.32 shows the natural frequencies of the bare circular cylindrical shell representing the FTR obtained with the proposed facet shell finite element compared with other published research. Chung [1981] studied this circular cylindrical shell using a solution procedure based on Sander's shell equations. Contrary to published research, Chung [1981] used simple Fourier series along the axial direction rather than exponential functions leading to a low-order problem which simplified the solution. Besides, Chung [1981] expanded the modal displacements along the circumferential direction using the Fourier series. As pointed out by the author, he considered an upper limit of fifty terms for the Fourier expansion. Djoudi and Bahai [2004] proposed a strain-based element for cylindrical shell structures and investigated the FTR example. The strain-based approach enables one to improve the displacement field with higher-order polynomials without increasing the degrees of freedom, and many authors regularly apply it to shell structures. Table 5.33 lists both outcomes of Djoudi and Bahai [2004], NASTRAN and strain-based element results obtained considering ten elements in the axial direction and nine elements over one-quarter of the shell circumferentially. Mohammadi and Sedaghati [2012], studying the linear and non-linear vibration analysis of cylindrical sandwich shells with constrained viscoelastic layers, reported the natural frequencies of the FTR example using a semi-analytical finite element method in which discretised the displacements in the axial direction using three-noded elements but expanded the displacements in the circumferential direction using a Fourier series.

Figure 5.28 shows a side view of the first ten mode shapes of the cylindrical bare shell. Modes with one half-wave in the axial direction ($m = 1$) are predominant, but it is worth mentioning that there is no explicit link between higher circumferential modes and higher natural frequencies. The relation between bending and extensional deformations might explain such results. In particular, extensional deformation is predominant for lower circumferential modes, which explains why

Table 5.32 Natural frequencies (Hz) of the bare circular cylindrical shell

Mode n	Reference ¹	NASTRAN ²	Reference ²	Reference ³	IDKQDRL
$m = 1$					
1	855.10	-	-	857.5	855.84
2	403.72	410.1	403.91	405.6	404.12
3	223.34	232.2	224.79	225.1	223.41
4	171.77	180.5	172.40	174.6	171.83
5	199.16	206.2	203.37	203.6	199.98
6	268.86	275.5	274.04	274.9	270.94
7	361.92	370.1	369.80	369.8	365.53
8	472.54	483.5	482.96	482.5	477.84
9	599.03	614.0	611.42	611.1	606.12
10	740.00	-	755.14	755.4	749.76
$m = 2$					
1	2318.98	-	-	-	2320.72
2	1437.11	-	1446.56	1440.5	1438.32
3	928.28	943.2	940.20	932.9	928.67
4	644.48	671.4	647.25	649.4	643.51
5	494.69	529.3	509.11	500.0	491.63
6	442.00	478.0	442.16	448.6	436.80
7	464.59	496.9	477.32	473.1	458.40
8	539.45	567.2	542.69	550.0	533.76
9	648.34	673.2	656.06	661.2	644.24
10	781.15	-	788.39	796.5	779.35
$m = 3$					
2	2487.60	-	2530.79	2491.30	2488.67
3	1834.82	-	1876.80	1841.50	1835.70
4	1367.64	-	1412.72	1376.10	1367.12
5	1057.12	-	1103.56	1066.30	1053.97
6	864.82	-	875.29	874.70	857.75
7	767.65	-	789.50	795.97	755.94
8	750.67	-	748.35	763.30	734.87
9	798.18	-	809.48	812.90	779.98
10	893.60	-	884.60	910.80	874.89

¹ Chung [1981]² Djoudi and Bahai [2004]³ Mohammadi and Sedaghati [2012]

mode (1,3) has a higher frequency than mode (1,5), for example.

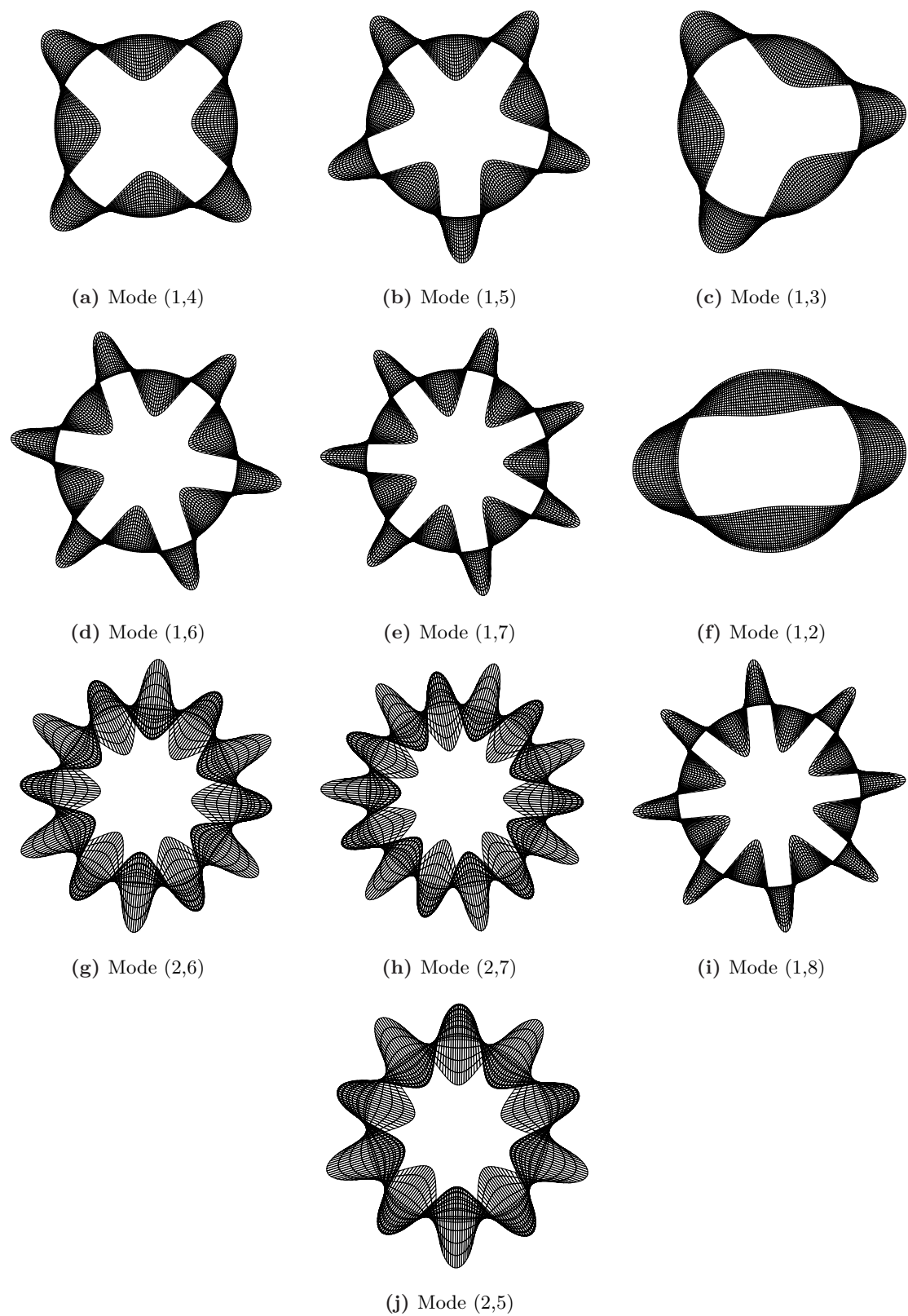


Figure 5.28 First ten mode shapes of the bare cylindrical shell with CF boundary conditions

Table 5.32 shows a good agreement between the presented results and those reported by the cited authors. The presented results, obtained with the IDKQDRL element considering an FSDT and 64 elements over the axial direction and 256 elements over one-quarter of the shell circumferentially, are in close agreement with those of Chung [1981], which is the best-known solution. For $m = 1$ (one half-wave in the axial direction), there is a perfect match between both works for lower n 's (wave number in the circumferential direction). As the wave number in the circumferential direction increases, Chung [1981] results show slightly lower natural frequencies due to the Fourier expansion of the displacements along this direction. For higher half-waves in the axial direction, the close agreement between the results obtained with the proposed flat facet finite element and those of Chung [1981] remains. It is worth mentioning the fact that for $m > 2$ and $n \geq 4$, the natural frequencies presented with the proposed element are lower than those of Chung [1981] - the difference increases as the half-wave in the axial direction increases; this might be due to the number of terms used by Chung [1981] in the axial direction and the high number of elements considered for this analysis. In this regard, an important point is the limited number of real applications that one can study with the semi-analytical solutions of the cited works. Too often, these procedures limit the analysis to circular cylindrical shells. The proposed flat facet finite element might be used in many applications, even if one has to increase the mesh refinement. Even when compared to semi-analytical solutions, the proposed formulation exhibits remarkable results.

5.5.2 | Short Sandwich Composite Shell

The main goal of this section is to compare the performance of the facet shell finite element proposed in this work with published research considering a short cantilever sandwich cylindrical shell with composite facings over a viscoelastic core. Table 5.33 shows the geometric properties of the short cylindrical shell, and Tables 5.34 and 5.35 list the material properties of both composite facings and core material, respectively. One assumes that the properties of the damping material do not vary with frequency or temperature.

Table 5.33 Short shell geometric properties

ℓ	r_v	h_v
12.2 m	18.29 m	12.7 mm

Table 5.34 Material properties of the short shell facings

	E_1	E_2	$G_{12} = G_{13} = G_{23}$	$\nu_{12} = \nu_{13} = \nu_{23}$	ρ
Kevlar/Epoxy	76 GPa	5.5 GPa	2.3 GPa	0.34	1460 kg m ⁻³

Table 5.35 Material properties of the short shell viscoelastic core

	E	η	ν	ρ
PVC	23 MPa	0.34	0.34	1340 kg m ⁻³

To evaluate the performance of the proposed formulation in comparison with other published research, one considers a thin ($h_v/h_f = 1$) and a thick ($h_v/h_f = 5$) sandwich shell and different

orientations of the composite facings - $[0^\circ/\text{Core}/0^\circ]$ and $[90^\circ/\text{Core}/90^\circ]$. These examples enable one to compare the present work with published research regarding composite sandwich shells and, taking advantage of the proposed finite element capabilities, investigate the influence of the deformation theory on the natural frequencies and damping capacity. Using the IDKQDRL element and considering 32 elements along the axial direction and 128 elements circumferentially over one-quarter of the shell, one evaluates the proposed element and its suitability to model sandwich shells with composite facings and viscoelastic core.

Tables 5.36 and 5.37 present the first fifteen circumferential natural frequencies and modal loss factors of the $[0^\circ/\text{Core}/0^\circ]$ sandwich shell for thin and thick configurations, respectively. As explained in the previous example, it is worth noting that higher circumferential modes do not directly correspond to higher natural frequencies, which depend on the contribution between bending and extensional deformations. For short shells, in particular, extensional deformation is predominant for lower circumferential modes, and this explains the ever-decreasing frequencies from mode (1,1) to mode (1,11) of the short thin shell with $[0^\circ/\text{Core}/0^\circ]$ stacking sequence. Another consequence is that while the natural frequencies of lower circumferential modes decrease when the core is thicker, natural frequencies of higher circumferential modes increase since the distance to the neutral axis of the stiffer layers increases. On the other hand, damping capacity depends on the circumferential mode - higher circumferential modes imply higher bending deformation and anti-nodes in phase opposition improve the shear strain at the damping material. Comparing the obtained results with those of Ramasamy and Ganesan [1999] and Yang et al. [2015], there is an excellent match between natural frequencies and modal loss factors. For the thin shell with $[0^\circ/\text{Core}/0^\circ]$ stacking sequence, there is an almost perfect agreement between natural frequencies. It is worth mentioning that frequencies obtained with the present finite element are slightly lower for higher circumferential modes and slightly higher for lower circumferential modes. In addition, there is no noticeable difference between a FoZZ and a ToZZ and between the MSE and the Complex Eigensolution (CE) - the shell is thin and low-damped. However, analysing modal loss factors of the thin $[0^\circ/\text{Core}/0^\circ]$ shell, the CE presents lower values than the MSE and closer to those of Yang et al. [2015], for example.

Both Ramasamy and Ganesan [1999] and Yang et al. [2015] use the CE to estimate natural frequencies and modal loss factors since it is based on the complex eigensolution and does not take the normal modes of the undamped system as representative of the damped structure, which is the case of the MSE approach. As for natural frequencies, the obtained modal loss factors with the CE approach are slightly lower for higher circumferential modes and slightly higher for lower circumferential modes. One notices the same trend for the natural frequencies of the thick shell but with somewhat higher discrepancies - lower natural frequencies for higher circumferential modes and slightly higher ones for the first five lower circumferential modes. However, modal loss factors obtained with the implemented finite element are always marginally lower than those presented by Ramasamy and Ganesan [1999]. It is worth mentioning that using an MSE or a CE approach is more important than considering a higher-order deformation theory, regardless if it is a thin or thick shell.

Similarly, Tables 5.38 and 5.39 present the first fifteen circumferential natural frequencies and modal loss factors of the $[90^\circ/\text{Core}/90^\circ]$ sandwich shell for thin and thick configurations, respectively. Both natural frequencies and modal loss factors increase significantly for higher circumferential modes for this stacking sequence, which is not surprising since the material direction of higher tensile modulus goes along with the circumferential direction - higher stiffness and higher stresses along this direction. Therefore, the $[90^\circ/\text{Core}/90^\circ]$ stacking sequence is preferable for this short shell, whether for its higher natural frequencies or higher damping capacity. Higher circumferential modes are highly damped, and the corresponding natural frequencies and modal loss factors estimated with MSE or a CE approaches show noticeable

differences, which consistently are higher for the thick shell. Once again, this example shows that using an MSE or CE approach becomes more influential than considering a higher-order deformation theory, even when one deals with the thick shell. Discrepancies between the obtained results and reference works are slight and worth mentioning just for higher circumferential modes.

Table 5.36 Natural frequencies (Hz) of a sandwich cylindrical short shell - $[0^\circ/\text{Core}/0^\circ]$

Mode n	Reference ¹	Reference ²	FoZZ		ToZZ	
			MSE	CE	MSE	CE
$h_v/h_f = 1$						
1	11.0980	11.0942	11.0971	11.0973	11.0971	11.0973
2	8.0516	8.0450	8.0501	8.0505	8.0501	8.0505
3	6.0940	6.0846	6.0913	6.0919	6.0913	6.0919
4	4.8252	4.8133	4.8212	4.8219	4.8212	4.8219
5	3.9623	3.9485	3.9570	3.9578	3.9570	3.9578
6	3.3505	3.3350	3.3433	3.3442	3.3433	3.3442
7	2.9054	2.8892	2.8964	2.8973	2.8964	2.8973
8	2.5825	2.5660	2.5710	2.5720	2.5710	2.5720
9	2.3578	2.3415	2.3432	2.3443	2.3432	2.3443
10	2.2191	2.2036	2.2010	2.2022	2.2009	2.2022
11	2.1595	2.1453	2.1378	2.1392	2.1378	2.1392
12	2.1735	2.1609	2.1458	2.1502	2.1458	2.1502
13	2.2544	2.2436	2.2267	2.2289	2.2267	2.2289
14	2.3942	2.3850	2.3645	2.3673	2.3645	2.3673
15	2.5843	2.5765	2.5536	2.5572	2.5535	2.5571
$h_v/h_f = 5$						
1	7.3886	-	7.4474	7.4484	7.4474	7.4484
2	5.3839	-	5.4252	5.4267	5.4252	5.4267
3	4.1060	-	4.1344	4.1364	4.1344	4.1364
4	3.2992	-	3.3095	3.3119	3.3095	3.3119
5	2.7624	-	2.7682	2.7710	2.7682	2.7710
6	2.4258	-	2.4181	2.4214	2.4181	2.4214
7	2.2407	-	2.2173	2.2210	2.2173	2.2210
8	2.1878	-	2.1475	2.1519	2.1475	2.1519
9	2.2534	-	2.1968	2.2023	2.1968	2.2023
10	2.4213	-	2.3511	2.3583	2.3511	2.3583
11	2.6735	-	2.5932	2.6027	2.5932	2.6027
12	2.9934	-	2.9064	2.9190	2.9064	2.9190
13	3.3674	-	3.2769	3.2934	3.2769	3.2933
14	3.7854	-	3.6943	3.7154	3.6942	3.7153
15	4.2400	-	4.1509	4.1774	4.1508	4.1773

¹ Ramasamy and Ganesan [1999]

² Yang et al. [2015]

Table 5.37 Modal loss factors ($\eta(\%)$) of a sandwich cylindrical short shell - $[0^\circ/\text{Core}/0^\circ]$

Mode n	Reference ¹	Reference ²	FoZZ		ToZZ	
			MSE	CE	MSE	CE
$h_v/h_f = 1$						
1	0.0902	0.0887	0.0906	0.0895	0.0906	0.0896
2	0.1111	0.1073	0.1126	0.1104	0.1128	0.1105
3	0.1455	0.1388	0.1490	0.1450	0.1493	0.1453
4	0.1898	0.1801	0.1965	0.1903	0.1971	0.1908
5	0.2398	0.2277	0.2515	0.2425	0.2522	0.2432
6	0.2919	0.2786	0.3105	0.2984	0.3115	0.2993
7	0.3437	0.3305	0.3711	0.3557	0.3724	0.3568
8	0.3947	0.3829	0.4327	0.4135	0.4342	0.4148
9	0.4476	0.4382	0.4927	0.4730	0.4986	0.4746
10	0.5084	0.5021	0.5672	0.5376	0.5695	0.5395
11	0.5850	0.5817	0.6495	0.6121	0.6522	0.6143
12	0.6835	0.6826	0.7477	0.7000	0.7510	0.7028
13	0.8051	0.8058	0.8624	0.8024	0.8666	0.8058
14	0.9463	0.9478	0.9915	0.9174	0.9965	0.9214
15	1.1010	1.1036	1.1312	1.0420	1.1371	1.0467
$h_v/h_f = 5$						
1	0.5002	-	0.4897	0.4843	0.4898	0.4844
2	0.6450	-	0.6330	0.6224	0.6332	0.6226
3	0.8696	-	0.8519	0.8331	0.8524	0.8335
4	1.1481	-	1.1172	1.0871	1.1179	1.0877
5	1.4508	-	1.3926	1.3484	1.3935	1.3493
6	1.7535	-	1.6448	1.5834	1.6461	1.5845
7	2.0473	-	1.8579	1.7745	1.8595	1.7759
8	2.3476	-	2.0507	1.9387	2.0526	1.9403
9	2.6847	-	2.2735	2.1251	2.2758	2.1270
10	3.0802	-	2.5711	2.3798	2.5740	2.3822
11	3.5342	-	2.9549	2.7169	2.9585	2.7197
12	4.0382	-	3.4104	3.1241	3.4147	3.1275
13	4.5766	-	3.9168	3.5823	3.9218	3.5863
14	5.1405	-	4.4575	4.0757	4.4632	4.0803
15	5.7231	-	5.0214	4.5934	5.0278	4.5986

¹ Ramasamy and Ganesan [1999]² Yang et al. [2015]

Table 5.38 Natural frequencies (Hz) of a sandwich cylindrical short shell - [90°/Core/90°]

Mode n	Reference ¹	Reference ²	FoZZ		ToZZ	
			MSE	CE	MSE	CE
$h_v/h_f = 1$						
1	14.4570	-	14.4442	14.4442	14.4442	14.4442
2	8.8090	-	8.7833	8.7834	8.7833	8.7834
3	5.8589	-	5.8304	5.8306	5.8304	5.8306
4	4.1738	-	4.1484	4.1486	4.1484	4.1486
5	3.1772	-	3.1567	3.1573	3.1567	3.1573
6	2.6519	-	2.6363	2.6382	2.6362	2.6382
7	2.5070	-	2.4964	2.5013	2.4963	2.5012
8	2.6632	-	2.6587	2.6682	2.6585	2.6681
9	3.0291	-	3.0323	3.0478	3.0320	3.0477
10	3.5330	-	3.5430	3.5658	3.5425	3.5655
11	4.1191	-	4.1445	4.1758	4.1439	4.1754
12	4.7700	-	4.8109	4.8516	4.8100	4.8511
13	5.4673	-	5.5270	5.5782	5.5259	5.5775
14	6.2017	-	6.2836	6.3460	6.2822	6.3451
15	6.9665	-	7.0744	7.1486	7.0727	7.1475
$h_v/h_f = 5$						
1	9.6601	9.6485	9.6519	9.6521	9.6519	9.6521
2	5.8965	5.8778	5.8803	5.8806	5.8803	5.8806
3	3.9580	3.9389	3.9395	3.9401	3.9395	3.9401
4	2.9487	2.9324	2.9303	2.9328	2.9303	2.9328
5	2.5585	2.5471	2.5455	2.5538	2.5455	2.5537
6	2.6427	2.6347	2.6431	2.6618	2.6430	2.6617
7	3.0346	3.0269	3.0567	3.0880	3.0565	3.0878
8	3.5913	3.5811	3.6421	3.6863	3.6418	3.6861
9	4.2305	4.2160	4.3166	4.3730	4.3162	4.3727
10	4.9118	4.8917	5.0389	5.1064	5.0384	5.1060
11	5.6151	5.5884	5.7882	5.8654	5.7876	5.8649
12	6.3303	6.2962	6.5536	6.6389	6.5528	6.6383
13	7.0518	7.0096	7.3287	7.4207	7.3278	7.4199
14	7.7767	7.7261	8.1100	8.2071	8.1090	8.2063
15	8.5022	8.4440	8.8952	8.9964	8.8941	8.9954

¹ Ramasamy and Ganesan [1999]² Yang et al. [2015]

Table 5.39 Modal loss factors ($\eta(\%)$) of a sandwich cylindrical short shell - $[90^\circ/\text{Core}/90^\circ]$

Mode n	Reference ¹	Reference ²	FoZZ		ToZZ	
			MSE	CE	MSE	CE
$h_v/h_f = 1$						
1	0.0623	-	0.0640	0.0639	0.0640	0.0639
2	0.0664	-	0.0721	0.0717	0.0721	0.0717
3	0.0722	-	0.0830	0.0821	0.0831	0.0821
4	0.0931	-	0.1083	0.1055	0.1085	0.1057
5	0.2035	-	0.2177	0.2053	0.2186	0.2060
6	0.6137	-	0.6085	0.5612	0.6117	0.5637
7	1.5389	-	1.4812	1.3568	1.4894	1.3634
8	2.8261	-	2.6904	2.4614	2.7056	2.4735
9	4.1383	-	3.9229	3.5918	3.9450	3.6096
10	5.3384	-	5.0552	4.6374	5.0833	4.6602
11	6.4344	-	6.0968	5.6074	6.1303	5.6347
12	7.4570	-	7.0770	6.5282	7.1153	6.5595
13	8.4253	-	8.0137	7.4155	8.0562	7.4506
14	9.3478	-	8.9139	8.2752	8.9603	8.3137
15	10.2270	-	9.7791	9.1078	9.8289	9.1496
$h_v/h_f = 5$						
1	0.3148	0.3202	0.3214	0.3208	0.3214	0.3209
2	0.3485	0.3666	0.3696	0.3676	0.3697	0.3677
3	0.4353	0.4692	0.4667	0.4585	0.4668	0.4587
4	0.9630	1.0110	0.9648	0.9166	0.9656	0.9172
5	2.9660	3.0227	2.8625	2.6724	2.8657	2.6751
6	6.5308	6.5859	6.2857	5.8571	6.2932	5.8633
7	10.1440	10.2021	9.7906	9.1460	9.8022	9.1558
8	13.0790	13.1559	12.6756	11.8950	12.6903	11.9076
9	15.4170	15.5242	15.0117	14.1618	15.0287	14.1766
10	17.3310	17.4746	16.9564	16.0811	16.9751	16.0975
11	18.9340	19.1157	18.6094	17.7366	18.6293	17.7543
12	20.2910	20.5125	20.0292	19.1767	20.0500	19.1955
13	21.4480	21.7074	21.2540	20.4329	21.2755	20.4523
14	22.4350	22.7312	22.3119	21.5285	22.3338	21.5486
15	23.2410	23.6083	23.2254	22.4831	23.2477	22.5036

¹ Ramasamy and Ganesan [1999]² Yang et al. [2015]

5.5.3 | Sandwich Shell with Composite Facings with Internal Damping

The importance of composite materials in many high-performance industries has been extensively discussed in this work. The damping capabilities of these materials are of great importance to the applications of such industries, and one should consider the damping phenomenon within these materials during the design process. Here, one presents the results of a cylindrical sandwich shell with composite facings with internal material damping using the proposed facet shell finite element compared with other models found in published research. Table 5.40 presents the geometric properties of the studied cylindrical shell, which can be found in previous works by Ramesh and Ganesan [1994b], Ramasamy and Ganesan [1999] and Yang et al. [2015], and Tables 5.41 and 5.42 present the material properties of both composite facings and core material, respectively. One assumes that the properties of the damping material do not vary with frequency or temperature.

Table 5.40 Shell geometric properties

L	R_v	h
0.1 m	0.1 m	0.1 mm

The h parameter denotes the entire thickness of the composite shell, core plus facings, assuming that both facings and core have equal thickness - $h_v/h_f = 1$.

Table 5.41 Composite shell facings material properties

Graphite/Epoxy				
E_1 /GPa	E_2 /GPa	$G_{12} = G_{13} = G_{23}$ /GPa	$\nu_{12} = \nu_{13} = \nu_{23}$	ρ /kg m ⁻³
184.5 (1+0.000716j)	10.91 (1+0.00671j)	7.31 (1+0.0112j)	0.28	1600

Table 5.42 Shell core material properties

	E	η	ν	ρ
PVC	23 MPa	0.34	0.34	1340 kg m ⁻³

Tables 5.43 and 5.44 compare the non-dimensional frequencies and loss factors of the sandwich shell fixed at one end and free at the other with a [90°/Core/90°] stacking sequence - CF boundary conditions. The following expression gives the non-dimensional natural frequency

$$\Omega = \frac{\rho_f h_f R \omega^2}{E_1} \quad (5.32)$$

One should note that this is a demanding comparison for the facet shell finite element. Ramesh and Ganesan [1994b] used a two-noded finite element adopting Fourier expansions in the circumferential direction, and Ramasamy and Ganesan [1999] applied a Fourier expansion for the circumferential direction in a three-noded isoparametric finite element. Likewise, Yang et al. [2015] express the displacements in the form of an improved Fourier series and the RRM procedure. These numerical procedures, well designed for cylindrical shells only, together with the Fourier expansions, namely in the circumferential direction, grant enhanced convergence of the results. In the following, one discusses the present results obtained with the CE approach

Table 5.43 Frequency parameter (Ω) of a sandwich cylindrical shell - $[90^\circ/\text{Core}/90^\circ]$

Mode n	Reference ¹	Reference ²	Reference ³	IDKQDRL
$m = 1$				
1	9.30×10^{-5}	9.52×10^{-5}	9.54×10^{-5}	9.60×10^{-5}
2	2.80×10^{-5}	2.93×10^{-5}	2.93×10^{-5}	2.95×10^{-5}
3	1.10×10^{-5}	1.12×10^{-5}	1.12×10^{-5}	1.13×10^{-5}
4	6.50×10^{-6}	6.66×10^{-6}	6.57×10^{-6}	6.77×10^{-6}
$m = 2$				
1	3.50×10^{-4}	3.70×10^{-4}	3.72×10^{-4}	3.74×10^{-4}
2	2.30×10^{-4}	2.39×10^{-4}	2.39×10^{-4}	2.40×10^{-4}
3	1.20×10^{-4}	1.19×10^{-4}	1.18×10^{-4}	1.18×10^{-4}
4	6.40×10^{-4}	6.59×10^{-4}	6.48×10^{-4}	6.48×10^{-4}

¹ Ramesh and Ganesan [1994b]² Ramasamy and Ganesan [1999]³ Yang et al. [2015]

based on the IDKQDRL element considering a FoZZ and 64 elements over the axial direction and 256 elements over one-quarter of the shell circumferentially. As seen in Tables 5.43 and 5.44, the results in the present study are in excellent agreement with those found in reference works. Modal loss factors, in particular, show an almost perfect match with the most recent published work, which is remarkable since the composite facings have internal material damping with different material loss factors for each direction.

Table 5.44 Modal loss factors ($\eta(\%)$) of a sandwich cylindrical shell - $[90^\circ/\text{Core}/90^\circ]$

Mode n	Reference ¹	Reference ²	Reference ³	IDKQDRL
$m = 1$				
1	1.05	1.03	1.03	1.04
2	1.00	1.02	1.02	1.02
3	1.90	1.97	1.99	1.94
4	7.70	6.60	6.69	6.71
$m = 2$				
1	0.80	0.79	0.79	0.80
2	0.97	0.97	0.97	0.97
3	1.20	1.18	1.19	1.17
4	1.80	1.77	1.77	1.74

¹ Ramesh and Ganesan [1994b]² Ramasamy and Ganesan [1999]³ Yang et al. [2015]

Part III

Parametric Analysis of Damping Patches

Beam and Plate Structures

6.1 | Introduction

The study of partial damping treatments is a subject of great importance within the field of passive vibration control because of its wide range of applications, especially considering the possibility of achieving a significant damping capacity using less material. Researchers have been studying this subject in the last decades, considering analytical models and imposing some simplifications, which restricted the analysis to a few realistic cases, followed by refined deformation theories and suitable numerical methods. This section describes the state of the art summary, reporting the purposes of this work regarding this topic.

Pulgrano [1962] pioneered the study of partially covered structures by focusing on the fundamental mode of one-dimensional simply supported and clamped ended plates. Considering thin treatment applications, the author, somewhat unexpectedly, found that, for some shear parameters, damping increases rather than decreases as the coverage reduces. Nokes and Nelson [1968] also studied partially covered structures with constrained viscoelastic damping. Focused on partially covered symmetric beams with symmetric boundary conditions, the authors have concluded that partial coverage could promote higher damping for some specific cases. According to the authors, this effect is particularly noticeable for stiffer viscoelastic layers due to the higher coupling between the main beam and the constraining layer. Therefore, the shear strain at the viscoelastic layer increases for less coverage due to the stretching of the constraining layer. Experimental work supported the results. A couple of years later, Plunkett and Lee [1970] followed a slightly different approach. The authors studied the length optimisation for constrained viscoelastic layer damping. Even though the study was not about partial coverage with viscoelastic damping treatments, the authors discuss a method for increasing damping by cutting the constrained layer into appropriate lengths.

Interested in the optimum distribution of additive damping, Lundén [1979] studied beam structures with spaced free damping layers. Focused on reducing resonant vibration using a total amount of distributed damping material, the author concluded that the ideal location for damping material should be at the centre of a spring-supported symmetric beam if only a limited

amount of material is available. Later, following the same principles, Lundén [1980] extended his work to the optimum damping distribution in vibrating frames. The author claims that the best place for damping material is at sections with high bending moment amplitudes, with laboratory experiments supporting the theoretical results.

Later, Lekszycki and Olhoff [1981] derived a general set of equations governing the optimal design of one-dimensional structures comprising a non-uniform elastic (or viscoelastic) core covered by non-uniform viscoelastic layers. Adopting the minimum dynamic response as the design criterion of harmonically excited one-dimensional structures, the authors optimised the non-uniform viscoelastic thickness distribution for different boundary conditions.

Following a similar approach to that of Lundén [1979], Yildiz and Stevens [1985] optimised the thickness distribution of unconstrained viscoelastic treatments for plates focusing on the first mode of vibration. Considering a viscoelastic damping treatment with full coverage but varying thickness, the author states that the central part of the plate is the most important one concerning damping. The optimisation technique considered both simply supported and edge fixed plates, resulting in higher values of viscoelastic material thickness in the central part of the plate. Besides, the loss factor might increase compared with treatments with uniform thickness distribution.

Pierson [1986] extended the work of Lekszycki and Olhoff [1981] by simultaneously optimising the core thickness and cover layers distribution for minimum dynamic compliance using an optimal control approach. For all the cases considered, the authors have found that the optimal mass distributions are similar and should be at the ends of the beams.

Lall et al. [1987] were among the pioneer researchers in partially covered plates with constrained viscoelastic damping. Adopting the RRM, the authors performed parametric studies, including the effect of the shear modulus, patch position and size, and the impact of thickness variation on a simply supported plate. The authors examined different patch locations, as shown in Figure 6.1, and concluded that it is possible to achieve better damping performances with a square patch located at one corner of the plate (Position 1), enabling a higher modal system loss factor for the first mode when compared to other locations, such as the central region (Position 13). Viscoelastic shear modulus and loss factor are considered frequency independent in this study. Moreover, it shows that the maximum attainable loss factor for different modes with partial coverage is higher at specific locations when compared with a fully covered plate. An interesting result is also the effect of the patch length, which might change the best patch location. According to the authors, increasing the constraining layer thickness provides higher shear strains in the viscoelastic core, resulting in higher modal loss factors for the first mode. However, this effect is not linear, and one achieves a maximum damping capacity for all the considered positions. Once again, covering the corner location results in higher modal loss factors than other patch locations. These results are somewhat unanticipated considering the works of Lundén [1979] and Yildiz and Stevens [1985] on additive damping, which concluded that the central part of the beam or plate is the most important one and damping material should be there. These differences may occur due to different viscoelastic damping mechanisms. While Lundén [1979] and Yildiz and Stevens [1985] considered an FLD configuration, Lall et al. [1987] addressed the problem following a PCLD approach. Still, the latest study lacks explanations or comments regarding such differences. Lall et al. [1988] have also studied partially covered sandwich beams using different mathematical formulations. However, the focus was on differences between mathematical formulations rather than parametric analyses with partially covered simply supported beams.

Focusing on laminated composite beams with passive constrained damping treatments, Mantena et al. [1991] studied the optimal length of damping tape numerically and experimentally. The authors show by measurements and predictions that there is an optimum damping tape distribution for maximum damping for a cantilever beam vibrating in a mode. The authors have

strained damping treatments for beams and plates, assuming a constant volume of damping material as a design constraint. The authors have found that ABAQUS plate elements should be used with care for structures with low aspect ratios and performed the optimisation using quadratic continuum elements. Considering the height of the damping layer at each element as the design variables and the peak displacement as a measure of damping performance, Lumsdaine and Scott [1998] have found that, for small values of damping layer thicknesses, all of the damping material tends to move to the centre of the simply supported beam, or plate, at the first mode. Interestingly, the improvement in the peak displacement for a thick beam reached its maximum for 30 % of added damping material compared to the base layer volume and then decreased as the damping layer volume increased, proving the existence of optimal solutions with partially covered structures in unconstrained configurations. Furthermore, the authors have confirmed that assuming constant viscoelastic properties results in slight differences in the optimisation, especially for thin damping layers.

Fully aware of the importance of patch damping design, Kung and Singh [1999] performed parametric studies on the best patch position and size. Based on the eigenfunctions of an undamped beam and using the CBT, the authors derived expressions for the natural frequencies and modal loss factors of these structures with constrained patch elements, aiming at reducing the computational complexity to have efficient and suitable methods for rapid design studies. Considering a patch performance index based on the displacement second derivative maxima, the authors concluded that, for a simply supported beam and considering its second mode, damping patches at the anti-nodes leads to high modal damping in agreement with the empirical design concept earlier stated by Cheng and Lapointe [1995]. However, the authors have shown the case dependence of this criterion when considering a cantilevered beam. The issue with a patch performance index based on displacement second derivative, related to the curvature and thus to the local bending moment, is its inherent local nature. This specific criterion neglects, for instance, the patch length, an important design parameter.

Nakra [2000] presented multi-parameter optimisation studies in structures incorporating viscoelastic damping, studying three different patch locations (Positions 1, 3 and 13 in Figure 6.1) in a simply supported plate covering 40 % of the base plate area whose added mass was equal to that of the fully covered case. Considering a constant loss factor for the viscoelastic material, Nakra [2000] shows that one can obtain a higher system loss factor with partial coverage at a suitable location than that of a fully covered case. For instance, the author achieves a higher system loss factor when the patch is at Position 3, even higher than for a fully covered plate. Subsequently, in a design sensitivity analysis for a cantilever beam, Nakra [2000] found that the core thickness of a patch near the clamped region is the most sensitive parameter regarding the system loss factor. Similarly, Chen and Huang [2002] searched for the optimum position of PCLD treatments for vibration suppression of simply supported isotropic plates. The authors concluded that the positioning of PCLD treatments should be close to the middle of an edge of the structure (Position 3 in Figure 6.1) to maximise the first mode damping ratio when the covered area is limited. Additionally, it approaches the corner location (Position 1 in Figure 6.1) as the covered area increases.

Focused on the vibration control of cylindrical shells, Zheng et al. [2005] performed optimisation studies focusing on the structural volume displacement decrease of a simply supported cylinder excited by a broadband transverse force. They note that PCLD patches increase their coverage in the axial direction and distribute over the entire cylinder surface. Besides, rectangular patches with a broad axial-to-angular length ratio improve damping.

Kristensen et al. [2008] studied the effect of constrained layer damping, bearing in mind the leaf spring application. Using a Kelvin-Voigt viscoelastic material model, the authors have concluded that for a constant thickness of the constraining layer, the beam's loss factor is almost

constant for different thicknesses of the rubber layer. Once the constraining layer masters the imposed shear deformation on the rubber layer, the loss factor increases monotonically with its thickness. Concerned about real applications where it may be inappropriate to apply loads directly on the damping treatment, the authors also studied two types of discontinuous layers, or patches, by filling the untreated region with composite material or leaving it unfilled. Although discouraged by the results showing a decrease in damping compared with continuous treatments, Kristensen et al. [2008] showed that filled gaps with a stiffer material increase the damping effect. Butler and Oyadiji [2008] followed a different approach and worked on the concept of Viscoelastic Damping Pocket (VDP). Contrary to the classical PCLD patches, VDP is a concept where designed-in viscoelastic layers are part of the host structure, leading to a higher damping capacity but a significant reduction in the structural stiffness. Using the ABAQUS FEM package, the authors tried to establish a compromise between added damping and loss of structural stiffness by a proper design of VDP.

Alvelid [2008] minimised the transverse vibration levels of a plate by optimizing the position and shape of the attached PCLD. Considering a square plate subjected to a transverse harmonic vibration at its sides, the author found an optimal location for a fixed amount of PCLD at the centre of the structure. Likewise, the study shows that, for low order modes, one can reduce the vibration level up to 18 dB by covering less than 30 % of the surface with damping treatments. Chia et al. [2009] developed strategies for using cellular automata algorithms to place PCLD on vibrating structures based on the concept of surface strain energy density and concluded that viscoelastic material should be in higher strain energy density regions. They followed the approach experimentally using a curved composite panel. Following the work of Kristensen et al. [2008], Le Maoût et al. [2011a] investigated the optimisation of sandwich plates with discontinuous elastomer patches. Comparing three different structures, one with continuous viscoelastic layers (Reference), a second one with periodic circular elastomer patches embedded in constraining composite layers (Plate 1) and another one with patches surrounded by unfilled gaps (Plate 2), the authors used the FEM package ABAQUS, together with the linear-search algorithm from MATLAB, to perform a constrained optimisation of the modal loss factors within a frequency range. Results showed that Plate 1 has the best damping capacity. The central claim by the authors is that the use of viscoelastic patches is efficient for structures globally optimised *a priori*, since damping properties of patched plates are about ten times greater than for classical damped plates in this study.

Zoghaib and Mattei [2014] presented a parametric study on vibration and noise reduction of plates with local constraint elastomer treatments. Besides the reminder that thin and stiff bonding layers have a powerful influence on damping, the authors developed a series of parametric studies to define the optimal patch location and size. Carefully, Zoghaib and Mattei [2014] pointed out that the displacement maximum is not entirely satisfactory as the best patch location. In contrast, the authors followed the criterion, pointed out by Chia et al. [2009], of using displacement second derivative and its maxima since it relates with the curvature and thus with the local bending moment. The keyword here is local. Therefore, the criterion does not consider the patch size. Xu et al. [2015] studied the optimal design of viscoelastic damping structures using a multi-objective genetic algorithm. The authors studied a simple cantilever beam and investigated the effect of added mass on the first modal loss factor, concluding that damping material should be near the root, where modal strain energy reaches the highest value.

Montemurro et al. [2015] established the problem of maximizing the damping capacity of composite laminated plates by considering a discontinuous aperiodic distribution of viscoelastic material, namely elastomer patches bonded to a structure in an FLD configuration. Following a two-level strategy and using the FEM package ANSYS, the authors performed a global optimisation, maximizing the first modal loss factors of two different structures using a genetic

algorithm. First, a laminated plate with fixed geometric and mechanical properties was at the basis of optimisation of damping material addition, namely the number of patches and three design variables for each patch, including position, diameter and thickness. Considering two different cases, a cantilever plate and a simply supported one, the authors found a different optimum number of patches for each one, but both converge to a solution where patches are near the clamped root section or where the plate is supported, where the strains are superior. One should note that FLD is the configuration under analysis in this study, and extensional strain is the main contribution to the damping mechanism. The authors also performed a second-level optimisation, where the design of the elastic and viscoelastic parts of the structure took place, and with this a priori design strategy, the optimal number of viscoelastic patches increases, and thus the damping capacity. Using a thin plate theory model, Zheng et al. [2016] performed a topology optimisation of PCLD patches to minimise sound radiation at low frequencies. The significant contribution of this work is to show the differences in optimizing damping or sound radiation. Considering the (3,1) mode of a thin steel plate, for example, the authors achieved different covered areas, mainly because one should also address the volume velocity of the structure when optimizing sound radiation.

Recently, Khalfi and Ross [2016] presented an interesting parametric study for sandwich panels with PCLD patches, studying the influence of thickness and patch positioning on natural frequencies and loss factors for different configurations. It addressed a valuable three-dimensional representation of the loss factor as a function of patch positioning for the first mode and concluded that maximum damping occurs when the patch is at the corners of the plate, even though one can achieve good performance when positioned at the centre. In the following, the authors conclude that, except for the corners, modal loss factors follow the deformation mode, emphasizing the result with a similar representation for the mode (2,1). A striking result confirmed by Khalfi and Ross [2016] is that one can achieve higher modal loss factors for partial treatments when compared with those of fully covered plates. Besides, even disregarding the influence of the boundary conditions, the proposed representation of the shear angle along the plate length is a relevant resource to foresee the best patch position.

From this state of the art summary, it is clear that the study of viscoelastic damping patches is an active research area, albeit with sparse studies, which still do not provide in-depth knowledge of this phenomenon. Most of the time, researchers used simplified theories because of the problem's complexity, which may have led to unsatisfactory results. Likewise, conflicting results achieved by different authors without plausible explanations for such differences prove the need for ongoing research.

6.1.1 | Damping Efficiency of Partially Covered Structures

Nashif et al. [1985] states that “*in engineering practice, controlling vibration and noise in structures and machines is partly art and partly science*” because, usually, it is difficult to have in-depth knowledge of all the involved phenomena. The primary goal when using viscoelastic damping materials is, for sure, to maximise the damping efficiency of a given treatment. However, the damping efficiency of a viscoelastic treatment depends on many parameters, even for basic structures such as beams. These parameters are the geometrical ones, like the length or width of a host structure, the thickness of each layer, or mechanical ones, like Young's modulus, shear modulus or material density. If one considers partial damping treatments, the number of independent variables increases even more; the number of treatments and their corresponding length, width and thickness, for example, impact the damping efficiency. The statement of Nashif et al. [1985] becomes intelligible considering all these factors regarding the laborious task of

optimal damping design. It is something like that cartoon where the dike is leaking by so many holes that patching them all is mission impossible; to plug one hole means to unplug the other.

There are two main approaches to the optimal design of viscoelastic damping treatments. On the one hand, to perform discrete calculations of damping properties for different values of a given parameter in a strategy that one may call parametric analysis of damping efficiency. On the other hand, one may search for optimal damping efficiency using optimisation algorithms. These two approaches have advantages and disadvantages, and it is interesting to discuss them here and explain the choice in the context of this work. Performing a parametric analysis computing the damping properties of a structure as a function of a given parameter is computationally expensive. It includes a complex process of matrices assemblage and resolution of a non-linear eigenvalues problem, continuously repeated for each parameter value. Besides, this approach does not necessarily lead to the optimal solution. However, it enables a comprehensible understanding of some parameters' influence on the damping capacity of the structure. Optimization techniques, by contrast, are a powerful tool to design structures optimally, considering a given objective without necessarily performing an endless number of computations. One can divide optimisation algorithms into two classes, gradient-based optimisation algorithms and evolutionary algorithms. In simple terms, optimisation algorithms based on gradients imply derivatives, whether analytical or numerically computed, and evolutionary algorithms are population-based techniques inspired by biological evolution where the fittest individuals survive and reproduce to achieve the optimal solution. Evolutionary algorithms, namely genetic algorithms, have been widely used in many engineering problems considering their robustness and ability to achieve optimal solutions, especially for highly complex structures involving many design variables. These techniques are indispensable, and their use will increase further due to their applications in artificial intelligence, but the scope of this work is slightly different.

In most cases, these powerful algorithms are black boxes that can provide a possible optimal solution to a given complex problem but offer the engineer or researcher little help in identifying the mechanisms which lead to the desired damping capacity and ways to apply the same principles elsewhere. The main objective of this work is to study the damping efficiency of partially covered structures aiming to achieve a comprehensive knowledge of appropriate design guidelines.

In the following sections, one presents a set of parametric analyses to evaluate the damping efficiency of partially covered beams and plates. The primary goal is to set up some design guidelines for the optimal design of structures with viscoelastic damping treatments. Teng and Hu [2001] somehow pioneered the parametric studies of the damping efficiency of viscoelastic damping treatments using the Ross-Kerwin-Ungar (RKU) equations. As Sher and Moreira [2013] later proved, the authors, identifying that symmetric sandwich laminates are preferable to asymmetrical ones, stated incompletely that damping increases when the viscoelastic layer thickens. Sher and Moreira [2013] proved that the viscoelastic layer has two distinct effects on the shear strain energy in their dimensionless parametric analysis of constrained damping treatments. On the one hand, higher volumes of damping material increase the potential to dissipate mechanical energy but change the transverse shear strain field simultaneously, which might perish for thicker viscoelastic layers.

Apart from the parametric study exploring the effect of different design parameters such as the thickness/length ratio, constraining layer thickness, material modulus, natural mode or boundary conditions, the work of Sher and Moreira [2013] considers dimensionless variables, promoting general guidelines for the design of viscoelastic treatments. In this work, following the same approach, one extends it to partial damping treatments and developing a comprehensive understanding of design guidelines.

6.2 | Partially Covered Beams

In the following sections, one performs a set of parametric analyses focusing on partially covered beams considering different boundary conditions with the geometric and mechanical properties listed in Tables 6.1 and 6.2, respectively. Variables h_d and h_c represent the thickness of the damping and constraining layers, respectively. Considering a FEM mesh of 100 elements along the length and the ToZZ, one performs a modal analysis with the CE to obtain the natural frequencies, mode shapes and modal loss factors to study the damping capacity of partial damping treatments on beams.

Table 6.1 Geometric properties of the cantilever beam

Length	Width	Thickness		
		Base	Damping	Constraining
ℓ	b	h	h_d	h_c
0.3 m	0.02 m	0.01ℓ	$0.02h$	h

Table 6.2 Material properties of the cantilever beam

	Aluminium	Damping Layer
Young's Modulus	70 GPa	1 MPa
Poisson's Ratio	0.32	0.49
Density	2710 kg m^{-3}	1140 kg m^{-3}
Loss Factor	-	0.3

6.3 | Damping Efficiency of Partially Covered Cantilever Beams

One considers in this section a cantilever beam with the geometric and mechanical properties listed in Tables 6.1 and 6.2, respectively. Focusing on the influence of the patch positioning and covered length, the main goal is to evaluate the performance of damping patches to better understand the damping mechanisms promoting optimum damping capacities to derive some guidelines for structural vibration control.

6.3.1 | Influence of Patch Positioning

To evaluate the effect of a viscoelastic damping patch, one performs a parametric analysis considering three different patch positions: a constrained viscoelastic patch placed at the tip of the cantilever beam, a patch positioned at the beam's mid-length and the last case considering a patch located at the clamped region - Figures 6.2a, 6.2b and 6.2c, respectively.

Figure 6.3 depicts the influence of the covered length on the patch-damping performance for the three different configurations. One evaluates the patch performance considering a specific modal loss factor, given by the first modal loss factor for a patch size and location divided by the first modal loss factor of the structure when considering a fully covered beam. Therefore, one considers the damping capacity by taking the fully covered plate as the reference case.

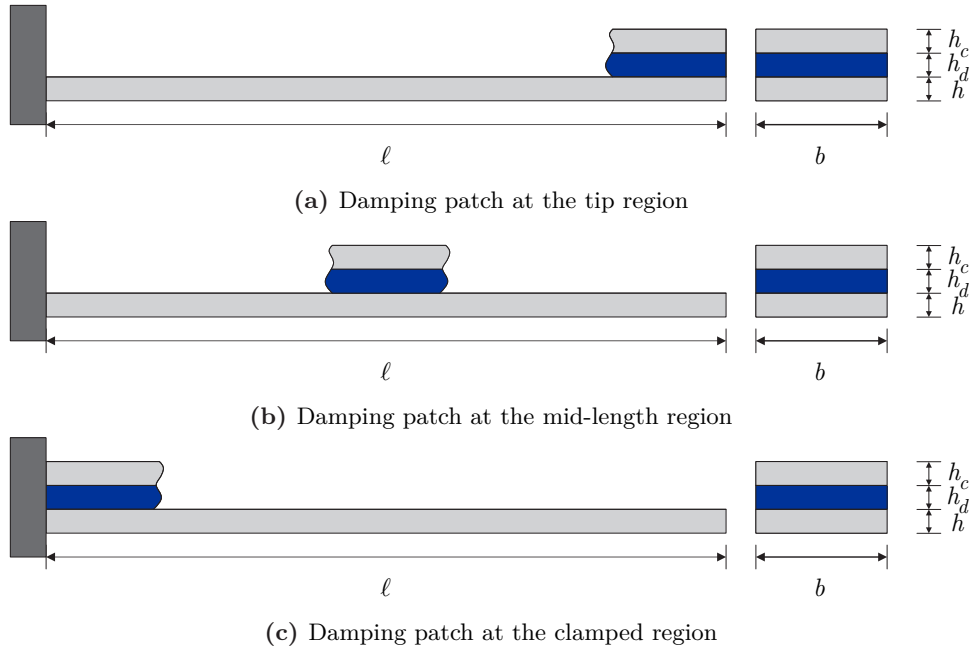
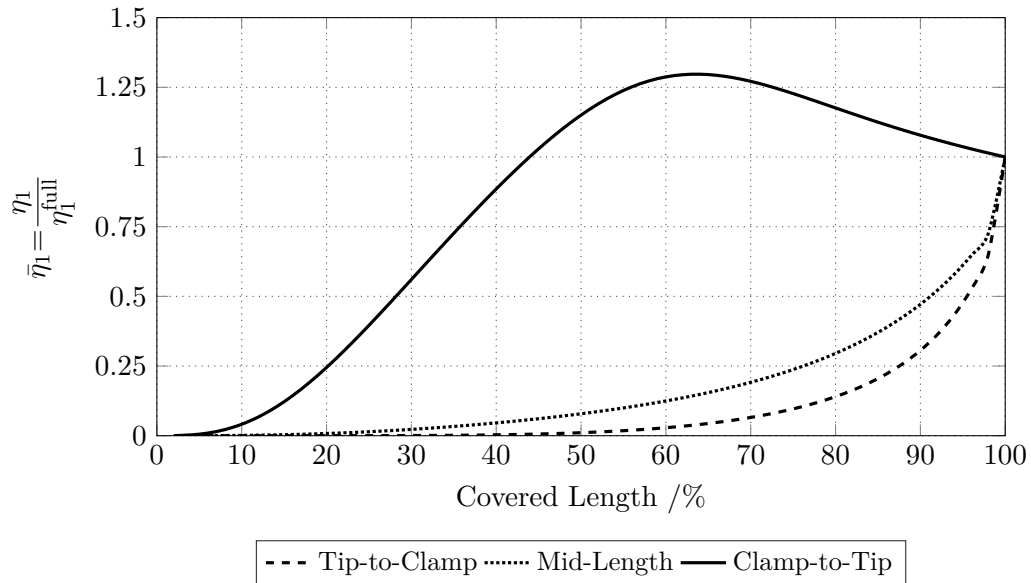


Figure 6.2 Three damping patch positions


 Figure 6.3 1st specific modal loss factor for different positions and sizes

The differences between the three configurations are noticeable. The damping capacity for a patch at the beam's tip and increasing towards the clamped section is inefficient, particularly for small lengths. There is a slight improvement in the modal loss factor for a patch at the beam's mid-length, regardless of the damping patch extent. However, a damping efficiency of half that of a fully covered beam is only achievable with a 90 % covered length. On the other hand, the first modal loss factor of a cantilever beam with a viscoelastic damping patch at the clamped region is quite higher, even considering relatively small covered lengths. One can notice that the corresponding modal loss factor with a covered area of just 50 % is higher than that of a fully covered beam. Further, there is an optimum covered length at which the modal loss factor is the highest one. This result proves the possible uselessness of integral damping treatments, at least for specific modes.

The structural modification induced by a constrained viscoelastic damping patch is also a matter of great importance for the proper design of passive damping. Figure 6.4 depicts the variation of the first natural frequency due to different patch locations and covered lengths.

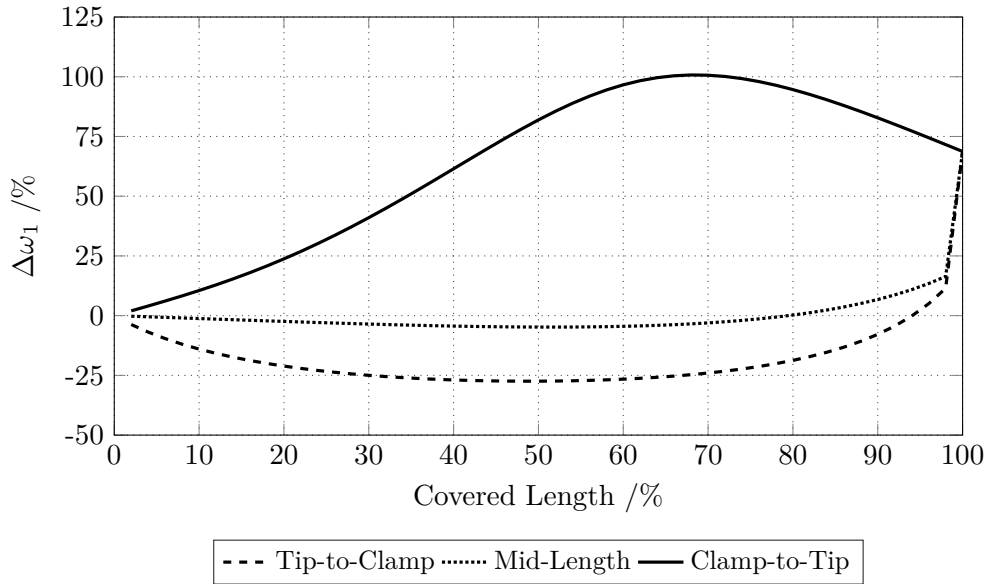
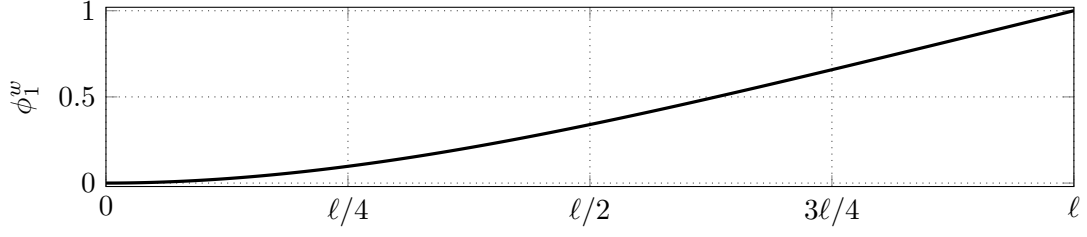
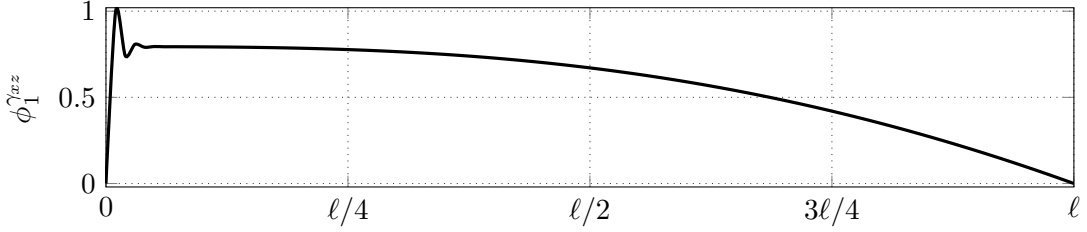


Figure 6.4 1st natural frequency variation for different patch positions and sizes

While a damping patch at the beam's tip decreases the first natural frequency, a patch at the clamped section increases it. The relative influence of added mass or stiffness sets the variation of the first natural frequency. From Figure 6.4, it is reasonable to admit that, for a patch at the mid-length, the opposed influence of added mass and stiffness cancel each other since the variation of the first natural frequency is negligible. One remarkable aspect is that both the maxima of the modal loss factor and the natural frequency occur for practically the same covered length between 60 and 70 %.

Figure 6.5 depicts the cantilever beam's first transverse displacement mode shape without damping treatment, considering displacement normalisation. One of the primary purposes of this work is to realise the influence of viscoelastic patches on damping and their appropriate design. Since the transverse shear strain within the viscoelastic damping layer is the main responsible for the damping mechanism in PCLD configurations, Figure 6.6 illustrates the first mode shape associated with the γ_{xz} degrees of freedom of an uncovered beam. Therefore, one can see this mode shape as a strain mode shape associated with shear strain γ_{xz} at the reference plane, as Figure 6.5 represents the transverse displacement mode shape.

Strain mode shapes, as depicted in Figure 6.6, can be a helpful tool to understand the


 Figure 6.5 1st transverse displacement mode shape of the bare cantilever beam

 Figure 6.6 1st shear strain mode shape of the bare cantilever beam

obtained results of modal loss factors represented in Figure 6.3. For the bare beam in Figure 6.6, soon after the boundary at the clamped section, where there is a singularity for γ_{xz} since it is null because of the imposed displacement, the strain mode shape reaches its peak and then decreases until the free boundary. As expected in a beam with such a configuration, transverse shear strain is maximum at the clamped region. Still, Figure 6.6 represents the transverse shear strain pattern reduced to the reference plane of the bare beam, and damping capacity comes from the transverse shear strain in the damping layer. It is valuable to understand the shear pattern for different damping treatments and explain the maximum loss factor for a 65 % covered beam, as depicted in Figure 6.3. To that purpose, Figure 6.7 represents the influence of the covered length on the first transverse shear strain mode shape of a cantilever beam with a patch treatment of different sizes at its tip.

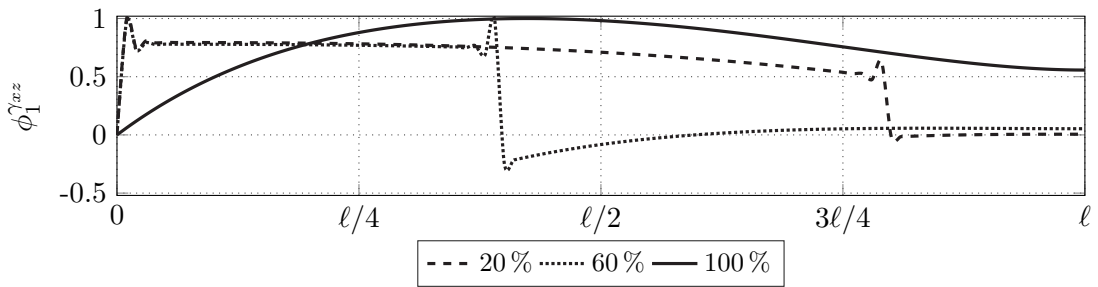


Figure 6.7 Shear strain mode shapes for a tip patch with different sizes

Figure 6.7 depicts three cases considering a covered length of 20 %, 60 % and 100 %. The influence of a patch treatment on the strain mode shape is notorious, especially in the first two cases. The patch positioning is noticeable because of the flat regions at the tip created by the damping treatments, which follow a boundary region where one can notice a sharp change in the strain pattern. However, comparing mode shapes represented with displacement normalisation might not be appropriate for analysing different treatments. The displacement normalisation is particular, and one gets the transverse shear strain pattern for each configuration from Figure 6.7, but one can not compare levels of modal strain between different damping

treatments. Alternatively, mass normalisation of the mode shapes is global, one uses the mass properties of the system to perform the normalisation, and one can compare levels of modal strain between different configurations. Figure 6.8 presents the mass normalised transverse shear strain mode shapes of a cantilever beam with damping patches of different sizes at its tip.

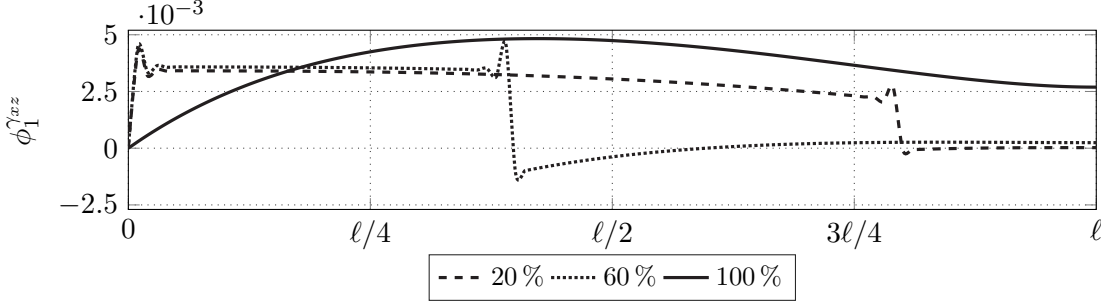


Figure 6.8 Mass normalised shear strain mode shapes for a tip patch with different sizes

Figure 6.8 shows no difference between the maximum and mass normalised mode shapes. The patch positioning is clear from the flat regions at the tip created by the damping treatments, but the relative differences between maximum levels of modal strain are identical. A similar pattern is discernible in Figure 6.9, which depicts the influence of covered length on the normalised transverse shear strain mode shape for a centre patch treatment considering equal covered lengths. In a readily comprehensible form, it shifted the flat regions to the centre, thus creating two sharp transitions which illustrate the abrupt change in the beam's geometry and material distribution.

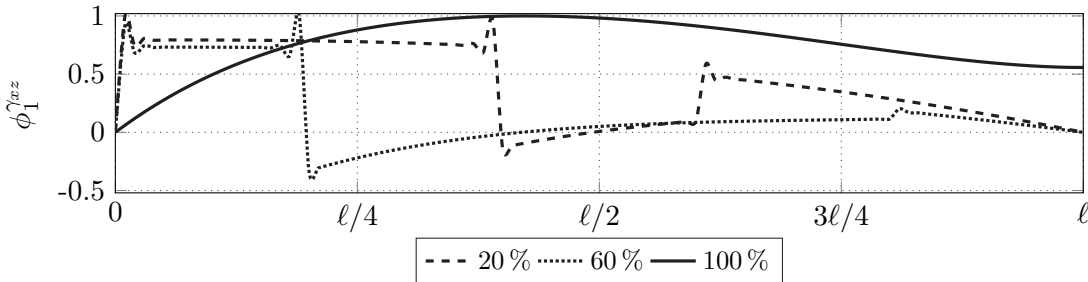


Figure 6.9 Shear strain mode shapes for a mid-length patch with different sizes

Figure 6.10 presents the mass normalised transverse shear strain mode shapes, and it is worth noting the fact that the maximum value of modal transverse shear strain is higher for the 60 % covered length patch damping treatment, though the modal transverse shear strain is relatively small at the region where the damping treatment is applied. This information is invisible with maximum normalised mode shapes, so it is decisive to consider mass normalised mode shapes. Nevertheless, the flat regions show little or no transverse shear strain at the damping patches' locations, which explains the poor damping capacity of these damping treatments, as represented in Figure 6.3.

Contrary to tip and mid-length damping treatments, a patch at the clamped root of the beam does not create a flat region in the transverse shear strain mode shape, as illustrated in Figure 6.11. The patch positioning is detectable only by the sharp transition in the strain pattern, but the transverse shear strain within the damping patch increases with the covered length. One should also note differences for different sizes. For a 20 % covered length, there is an abrupt transition at the edge of the damping treatment, which decreases until the 60 % covered length, where it is less blunt.

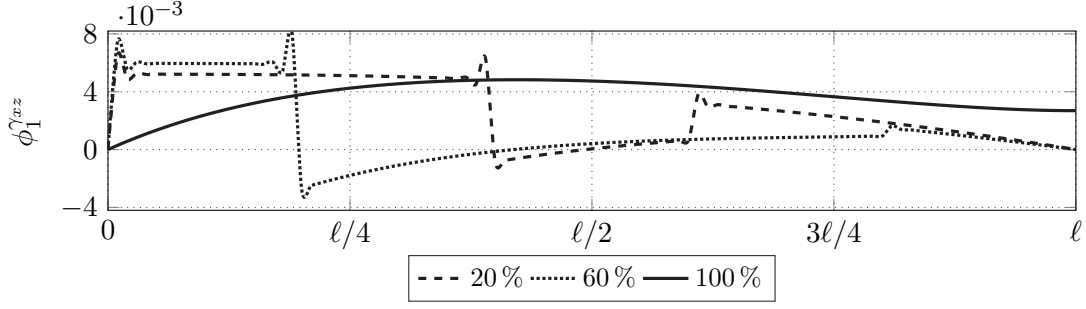


Figure 6.10 Mass normalised shear strain mode shapes for a mid-length patch with different sizes

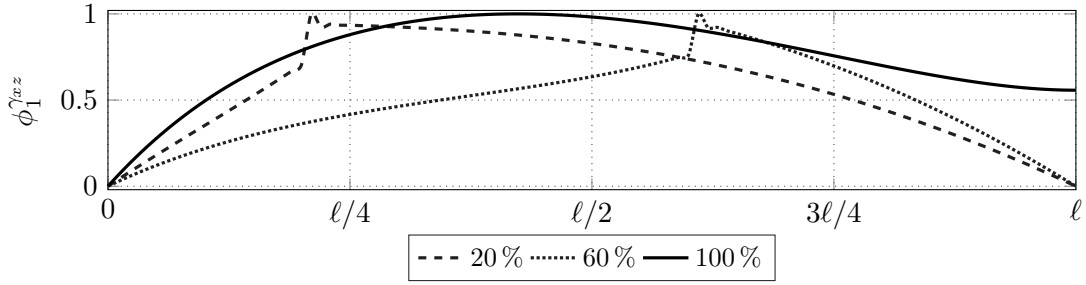


Figure 6.11 Shear strain mode shapes for a clamp patch with different sizes

To evaluate these patterns, one should closely consider the specific modal loss factor variation represented in Figure 6.3. As previously stated, the maximum modal loss factor for a cantilever beam with the geometric and material properties of Tables 6.1 and 6.2, and a viscoelastic damping patch at the clamped region, occurs for a covered length at around 65 %. Nevertheless, maximum normalised transverse strain mode shapes do not entirely explain this phenomenon. As done with the previous examples, Figure 6.12 presents the mass normalised strain mode shapes for the cantilever beam with patch treatments at the clamped section.

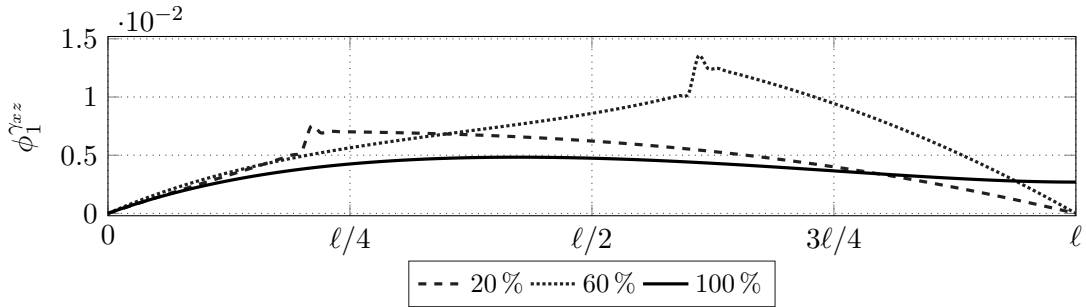


Figure 6.12 Mass normalised shear strain mode shapes for a clamp patch with different sizes

The modal transverse shear strain patterns depicted in Figure 6.12 are self-explanatory. A covered length of 60 % presents a higher modal shear strain than a fully covered beam, which explains the superior performance of this configuration. There is such a value of modal transverse shear strain that, even considering just half the damping material, improves damping capacity over the fully covered beam.

During this work, the transverse shear strain has been considered the mechanism responsible for the damping capacity of constrained layer treatments. It is a well-established concept among researchers, but one can prove it illustratively. Figure 6.13 presents the influence of the covered length on the modal loss factor of the first mode for a clamped region treatment in the same way

as Figure 6.3, using the MSE approach. In the MSE approach, one considers the mode shapes of the undamped system and the imaginary and real parts of the stiffness matrix to compute the ratio between loss and storage parts of the modal strain energy - Equation 6.1.

$$\eta_r = \frac{\{\phi\}_r^T \Im[\tilde{\mathbf{K}}] \{\phi\}_r}{\{\phi\}_r^T \Re[\tilde{\mathbf{K}}] \{\phi\}_r} \quad (6.1)$$

As the computation of the loss part of the modal strain energy only considers the transverse shear component of the stiffness matrix, as expressed in Equation 6.2, the modal loss factor represents the damping capacity associated with transverse shear strain energy.

$$\eta_r^s = \frac{\{\phi\}_r^T \Im[\tilde{\mathbf{K}}_s] \{\phi\}_r}{\{\phi\}_r^T \Re[\tilde{\mathbf{K}}] \{\phi\}_r} \quad (6.2)$$

Figure 6.13 presents the specific modal loss factors computed using both approaches, and the results are conclusive. Both curves are indistinguishable, proving that the damping capacity of a constrained layer damping treatment is only associated with the transverse shear component of the modal strain energy.

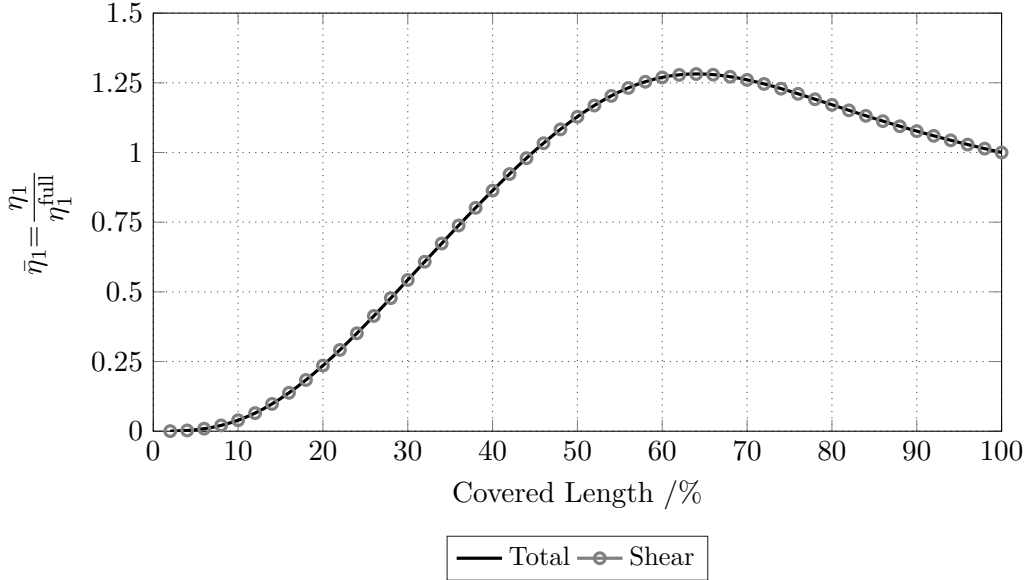


Figure 6.13 1st specific modal loss factor for a clamp patch treatment - total and shear component

This result proves the significance of the modal transverse shear strain on the damping capacity of constrained layer structures and reinforces the importance of transverse shear strain mode shapes as a tool to predict and design damping treatments for damped systems. However, transverse shear strain mode shapes associated with the γ_{xz} degree of freedom of the reference plane sum up all the information of a multilayered structure to just one point through the thickness. Taking advantage of the hybrid multi-layer theory implemented in this work, which accurately captures the layer-by-layer mechanical behaviour while keeping the number of kinematic variables of a single-layer approach, it is relatively simple to compute the modal transverse shear strain through-the-thickness using the mode shapes associated with γ_{xz} degrees of freedom, presenting a detailed view of the strain mode shapes. Figure 6.14 shows the through-the-thickness view of the first transverse shear strain mode shape of the bare cantilever beam considered in the previous examples and already presented in Figure 6.6 without the thickness variation. A through-the-thickness representation of the transverse shear strain mode shape provides more

information about the pattern occurring in a specific mode, presenting a clear insight regarding the possible best solutions for damping treatments. Apart from the maximum transverse shear strain near the clamped region and null transverse shear strain at the free edge, Figure 6.14 shows null transverse shear strain at both top and bottom free surfaces, as prescribed by the ToZZ, and maximum shear strain at the neutral axis, as already known from static analyses.

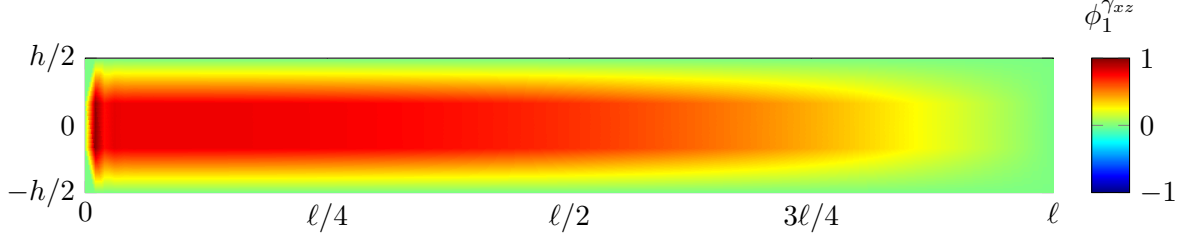


Figure 6.14 1st shear strain mode shape of a cantilever bare beam

Given this, it is intuitive to think of a transverse shear stress mode shape as the modal transverse shear strain multiplied by the material shear modulus point-by-point. Moreover, one can use modal transverse shear strain and stress to compute the modal transverse shear strain energy per unit volume and present a tool for an in-depth analysis of patch damping treatments. Figures 6.15 and 6.16 depict the bare cantilever beam's first transverse shear stress and strain energy density, respectively.

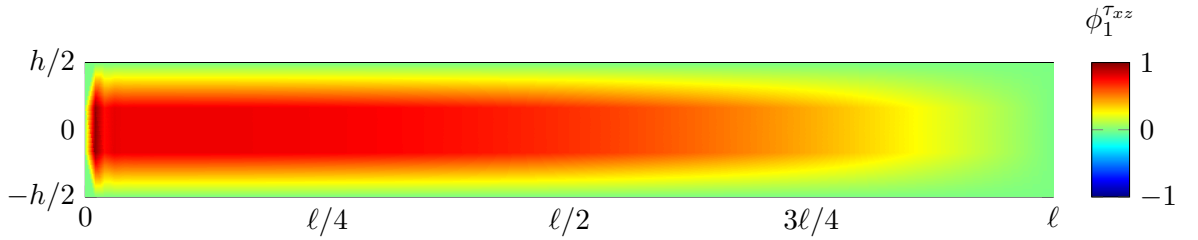


Figure 6.15 1st shear stress mode shape of a cantilever bare beam

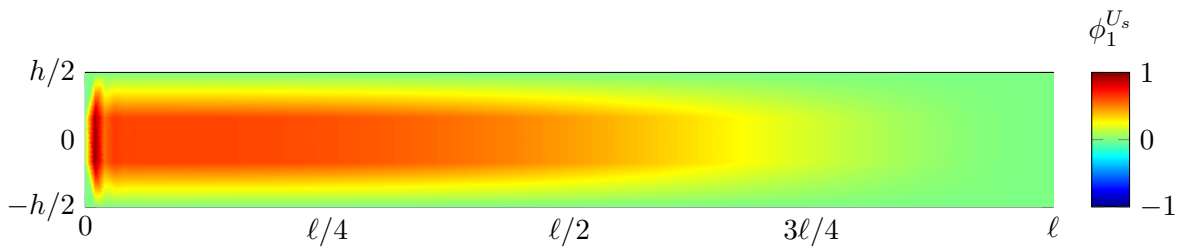


Figure 6.16 1st shear strain energy density mode shape of a cantilever bare beam

The maximum normalised transverse shear stress mode shape is naturally indistinguishable from the transverse shear strain mode shape for a single material layer, as is the case of a bare cantilever beam. Nevertheless, there are minor differences in the mode shape pattern of the transverse shear strain energy density, as it fades away until the null transverse shear strain energy density at the free end earlier than the modal transverse shear strain or stress. One should compare these mode shapes, associated with modal transverse shear within the structure, with the obtained results for the optimal length of a damping treatment at the clamped region. Transverse shear strain near the free end of the bare beam tends to zero, so it is reasonable to

suppose that adding damping material in this region would be useless. Besides, the obtained results inclusively show that it could be disadvantageous.

Transverse shear strain within the damping layer is essential to promote damping but is the extensional strain, also known as normal strain, of the stiffer constraining layer that imposes shear on the softer damping layer. Following the same reasoning, it is possible to compute the extensional strain mode shapes using Equation 3.5 and the corresponding degrees of freedom. Figure 6.17 presents the first extensional strain mode shape of the single material cantilever bare beam. The extensional strain mode shape pattern could be a helpful tool for understanding through-the-thickness variations of the in-plane displacements and extensional strain patterns of the constraining layer. As expected, the extensional strain is higher at the clamped section near the top and bottom free surfaces.

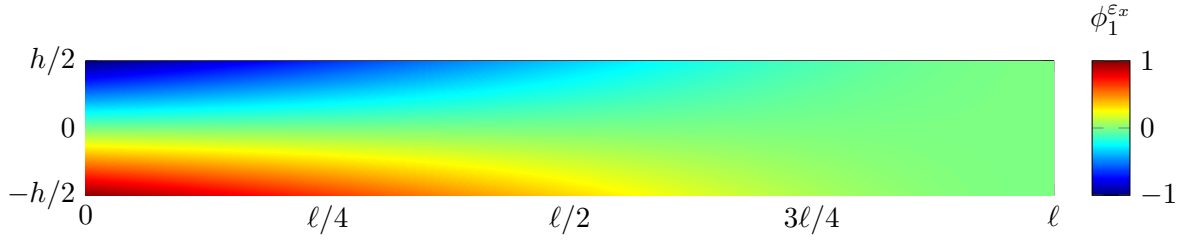


Figure 6.17 1st normal strain mode shape of a cantilever bare beam

Furthermore, one can also compute extensional stress and strain energy per unit volume mode shapes, as represented in Figures 6.18 and 6.19, respectively.

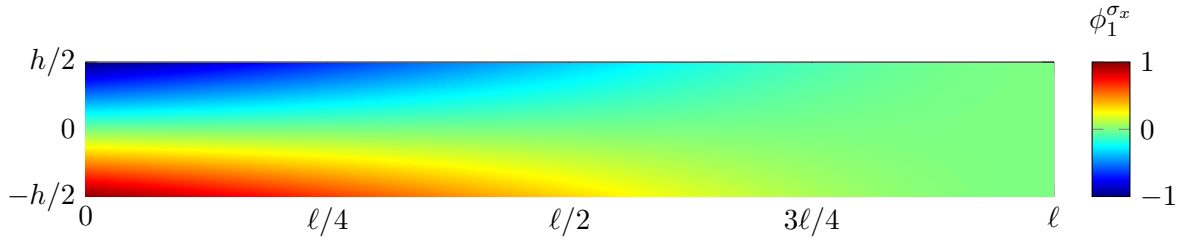


Figure 6.18 1st normal stress mode shape of a cantilever bare beam

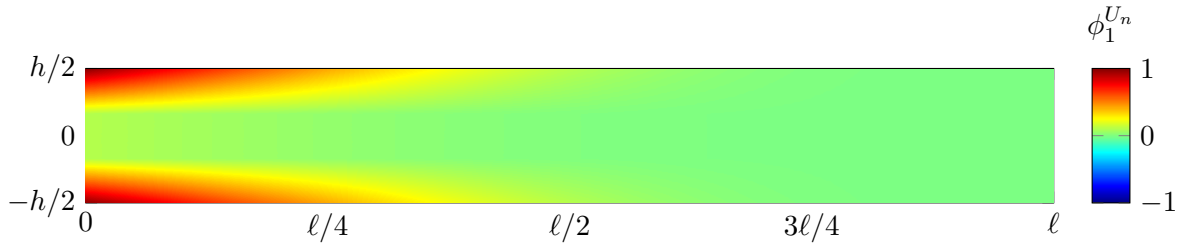


Figure 6.19 1st normal strain energy density mode shape of a cantilever bare beam

As previously noted, there is a singularity for γ_{xz} right after the boundary at the clamped section because it is zero due to the imposed null in-plane displacement. Once the transverse shear strain is a degree of freedom, the present model can not correctly represent transverse shear strains and stresses at clamped supports. Transverse shear strain is immediately corrected at the following nodes, as shown in Figure 6.6, but there is always a perturbation at clamped supports for models requiring C^1 continuity. To overcome this issue, one can calculate transverse shear

stresses from equilibrium equations - see Equations 3.34. It is possible to compute transverse shear stress by a through-the-thickness integration of the extensional stress variation. Figure 6.20 shows the first transverse shear stress mode shape obtained from equilibrium equations. The same pattern of Figure 6.15 is discernible, obtained there from the kinematic variables, but the stress recovery from equilibrium equations enables a correct representation of the transverse shear stress at clamped supports.

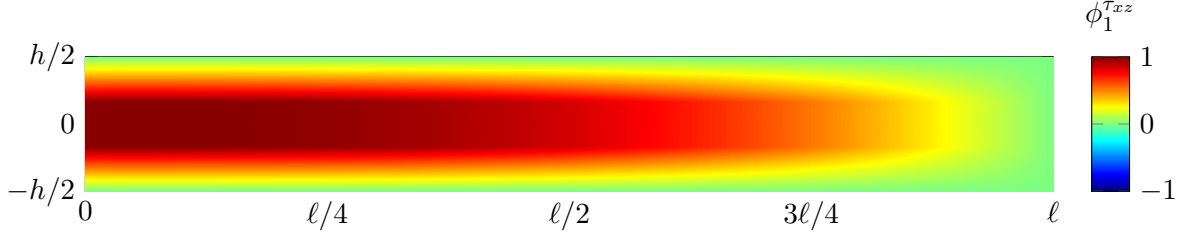


Figure 6.20 1st shear stress mode shape from equilibrium equations

Similarly, one can also compute transverse shear strain and transverse shear strain energy density mode shapes from transverse shear stress, recovered from equilibrium equations, obtaining a correct representation of these quantities at clamped supports - Figures 6.21 and 6.22, respectively.

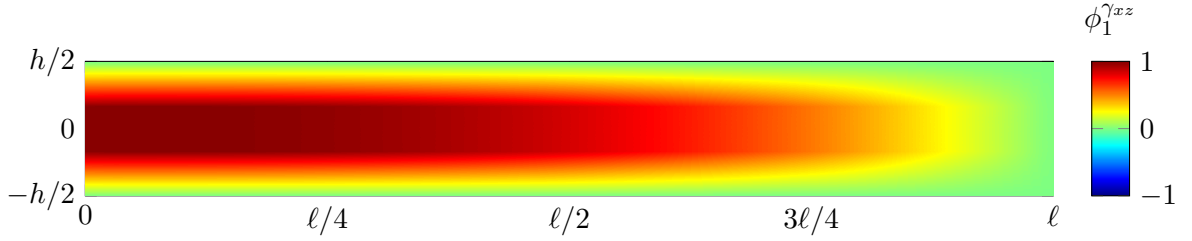


Figure 6.21 1st shear strain mode shape from equilibrium equations

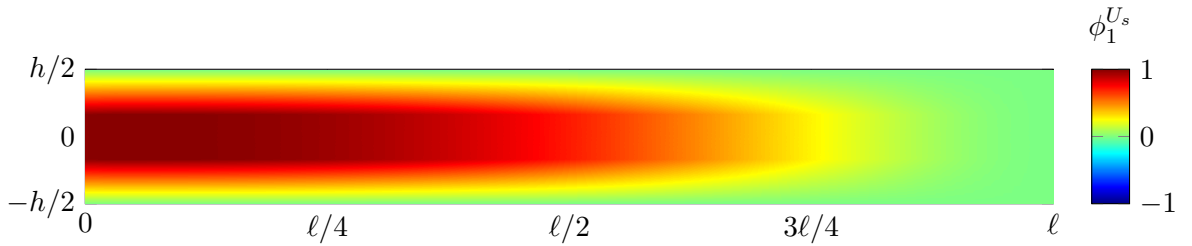


Figure 6.22 1st shear strain energy density mode shape from equilibrium equations

The main goal of this set of representations is to achieve a comprehensive understanding of the damping mechanisms associated with PCLD patches. Figures 6.23 and 6.24 show the first mass normalised shear strain mode shapes of the cantilever beam with full coverage and damping patch, respectively. Figure 6.24 presents three damping patch treatments; clamp, mid-length and tip patches. These representations disclose the modal shear strain pattern of the structure in the first mode and help to explain the superior damping performance of the damping patch at the clamped region. As expected, it is clear from Figures 6.23 and 6.24 that modal transverse shear strain is higher in the damping layer due to its lower shear modulus. A zoomed view of the damping layer represented in each figure helps to explore the transverse shear strain

pattern within this layer. By doing so, it is intelligible the superior damping performance of the damping patch at the clamped region, inclusively over the fully covered treatment, since the modal transverse shear strain at the viscoelastic layer with this configuration is higher. In particular, one can notice that the maximum modal transverse shear strain of the damping patch at the clamped region doubles that of the fully covered configuration. Besides, the expected weak performance of the other damping treatments at the first mode of the cantilever beam is visible.

Moreover, Figures 6.25 and 6.26 disclose the first transverse shear stress mode shapes for the same damping treatments. Apart from the noticeable transverse shear stress continuity between consecutive layers, an assumption of the present mechanical model, it is possible to verify higher levels of modal transverse shear stresses at the damping layer for the clamp patch.

The transverse shear strain energy density mode shapes are even more self-evident in Figures 6.27 and 6.28. Comparing the modal transverse shear strain energy density for different configurations, including a fully covered beam, one can notice higher values at the damping layer of a clamp patch treatment. There is little or no transverse shear strain energy density at the damping layer for the mid-length and tip patch treatments.

As previously mentioned, extensional strains are essential for understanding the damping mechanism of constrained layer damping. As the constraining layer imposes a transverse shear strain on the damping layer, it is valuable to study these modal strain patterns to understand the results of damping efficiency, such as those presented in Figure 6.3. Figures 6.29 and 6.30 present the first extensional strain mode shapes with full and partial damping treatments, respectively. The analysis of these patterns shows a high concentration of modal extensional strain at the edges of the constraining layers because of a significant change in geometry, which inclusively conceals the extensional strain pattern of the entire structure. Analysing Equations 3.5 and 3.8, one can remember the dependency of extensional strain upon membrane and bending components, but also on higher-order bending, which is a function of the shear strain variation - Equation 3.8. From Figures 6.7 to 6.12, which present the mode shapes of transverse shear strain at the reference plane, it is clear the already mentioned sudden variation of the transverse shear strain mode shape between consecutive nodes at the limits of the patches. Numerical computation of those variations explains the high levels of modal extensional strain noticeable in Figures 6.29 and 6.30. Therefore, Figures 6.29 to 6.30 represent the modal higher-order bending strains. In addition, Figures 6.31 and 6.32 present the first extensional modal strain with only membrane and bending components to overcome this issue of the mechanical model. It is worth analysing membrane and bending strain patterns of the different configurations of damping treatments. Full coverage and clamp patch treatments in Figures 6.31 and 6.32a, respectively, help to explain the superior damping performance of those treatments. The constraining layers of those treatments experience higher extensional strains (membrane plus bending), justifying the higher transverse shear strain states of the corresponding damping layers. Notably, the extensional strain (membrane plus bending) at the constraining layer of the partial clamp patch treatment are higher than those in the fully covered treatment. There is a low extensional strain state at the constraining layers of the mid-length and tip treatments.

Figures 6.33 and 6.34 show the corresponding membrane and bending stress mode shapes, and it is worth noting that there is no extensional stress continuity between consecutive layers. At the same time, membrane plus bending modal stresses at the constraining layers show the same patterns already noticed for the corresponding modal strains. Following the same reasoning, Figures 6.35 and 6.36 present the extensional and bending strain energy from where it is possible to draw similar conclusions.

As previously shown, it is possible to recover transverse shear stresses from extensional stresses using the equilibrium equations. Figures 6.37 and 6.38 represent the transverse shear stress mode shapes recovered from the membrane and bending stresses of the total, clamp,

mid-length, and tip treatments. Even so, recovered modal transverse shear stresses are higher at the edges of the constraining layers of partial treatments. Nevertheless, the corresponding transverse shear strain mode shapes in Figures 6.39 and 6.40 show the higher modal strain rates at the damping layers and explain the superior damping performance of a patch treatment at the clamp region. The transverse shear strain energy density modal patterns in Figures 6.41 and 6.42 support the same conclusion.

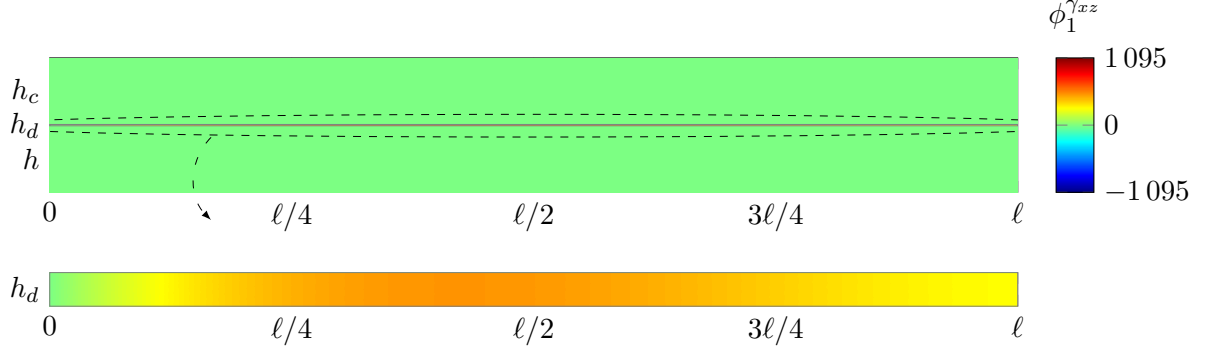


Figure 6.23 1st shear strain mode shape of a cantilever beam with full coverage

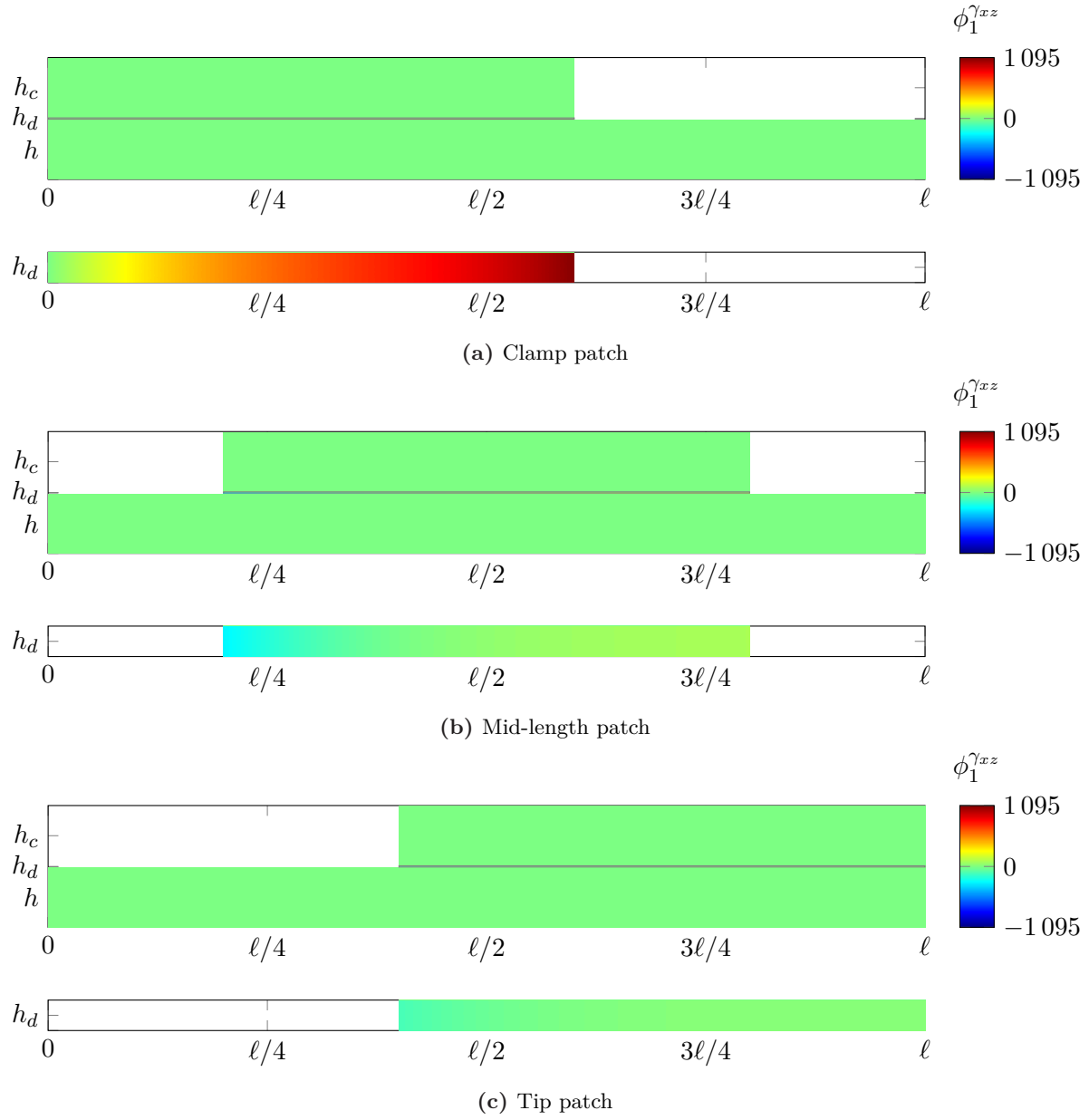


Figure 6.24 1st shear strain mode shapes of a cantilever beam with damping patch

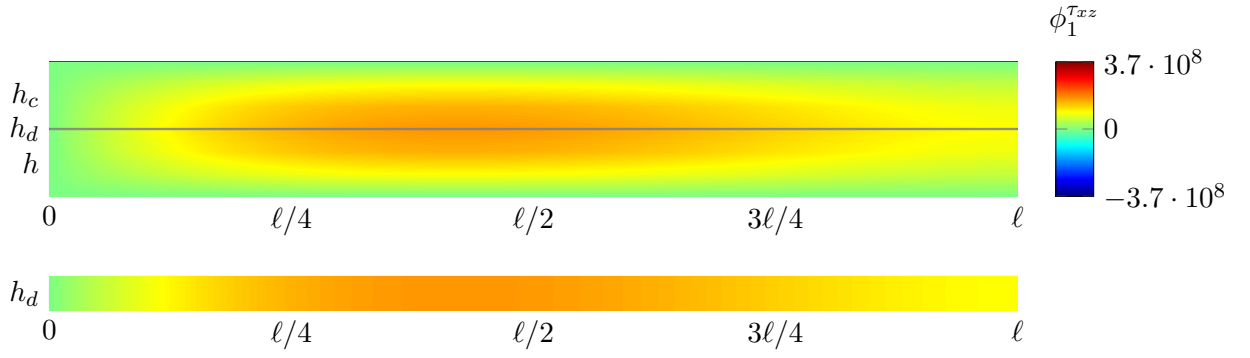
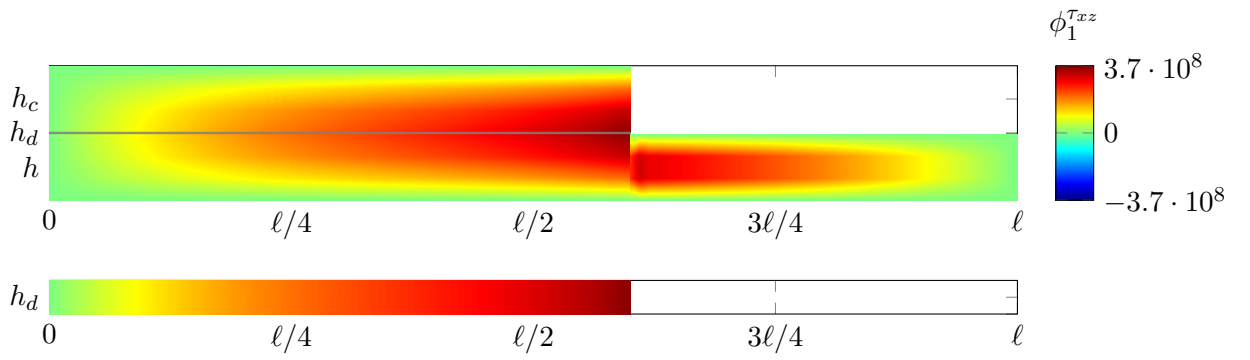
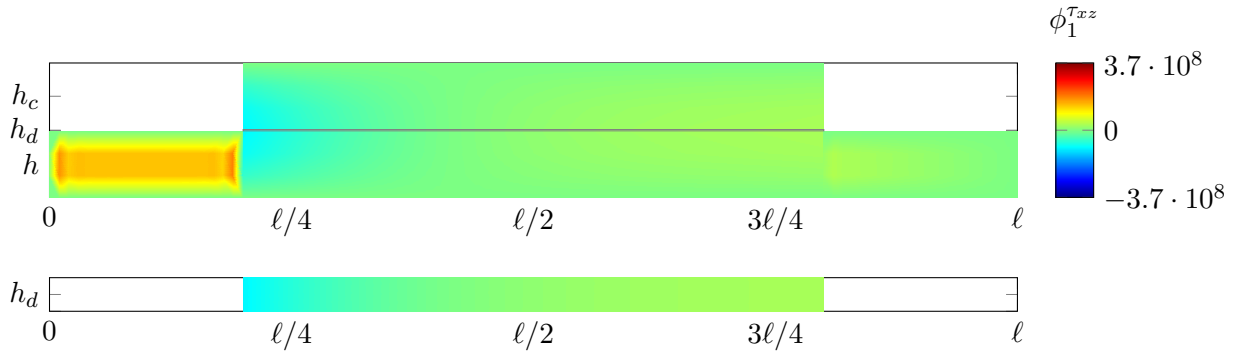


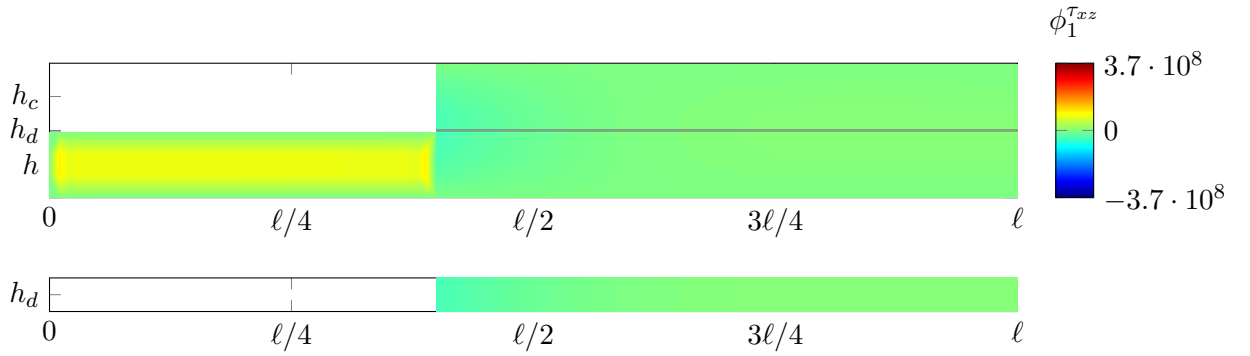
Figure 6.25 1st shear stress mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.26 1st shear stress mode shapes of a cantilever beam with damping patch

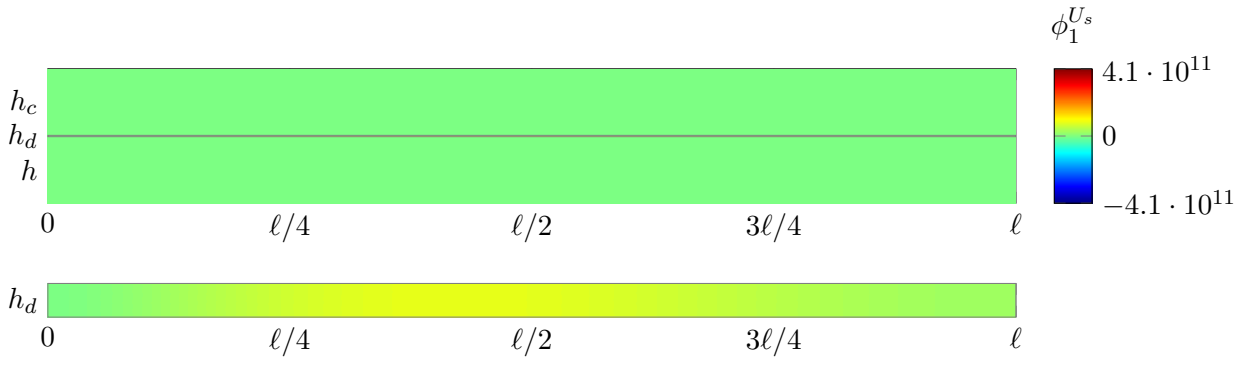
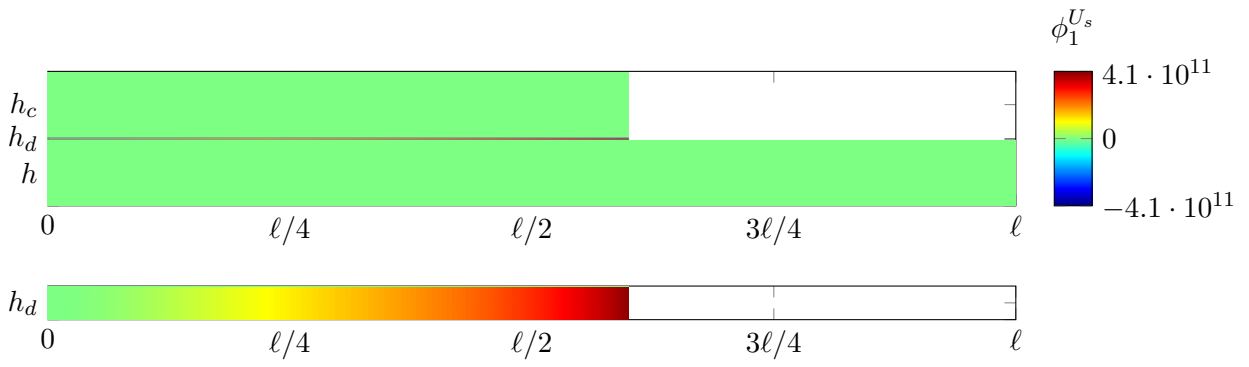
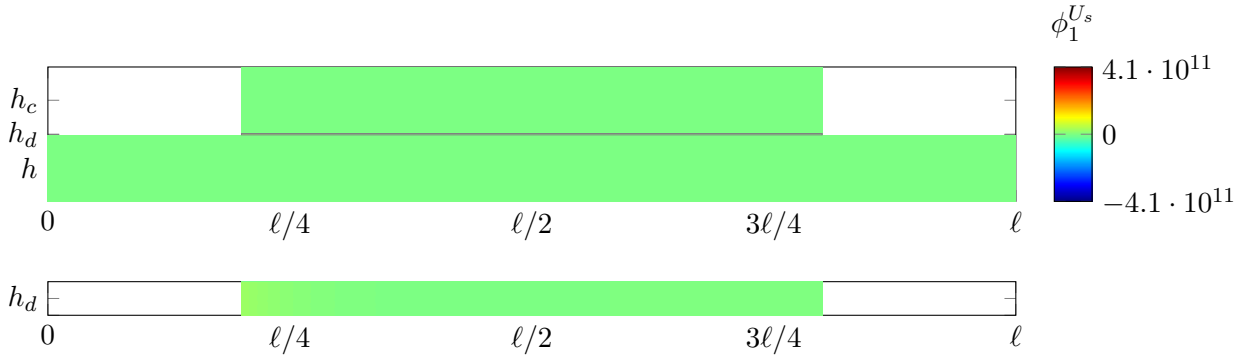


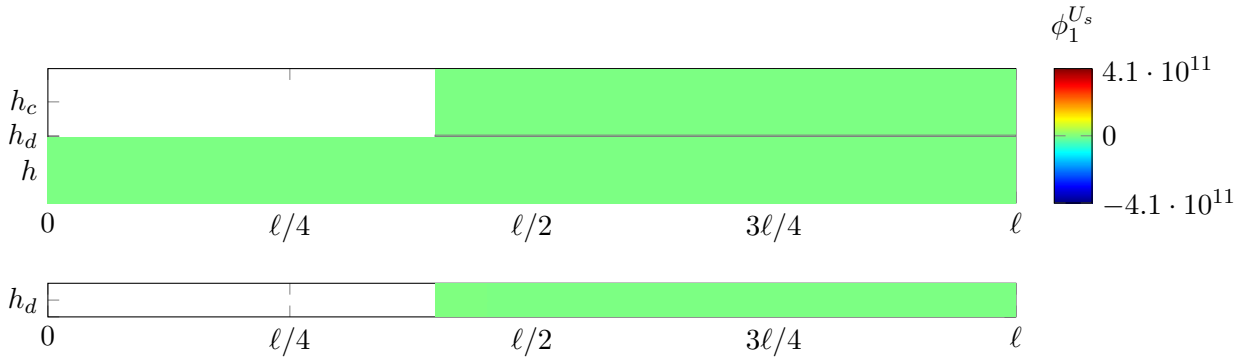
Figure 6.27 1st shear strain energy density mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.28 1st shear strain energy density mode shapes of a cantilever beam with damping patch

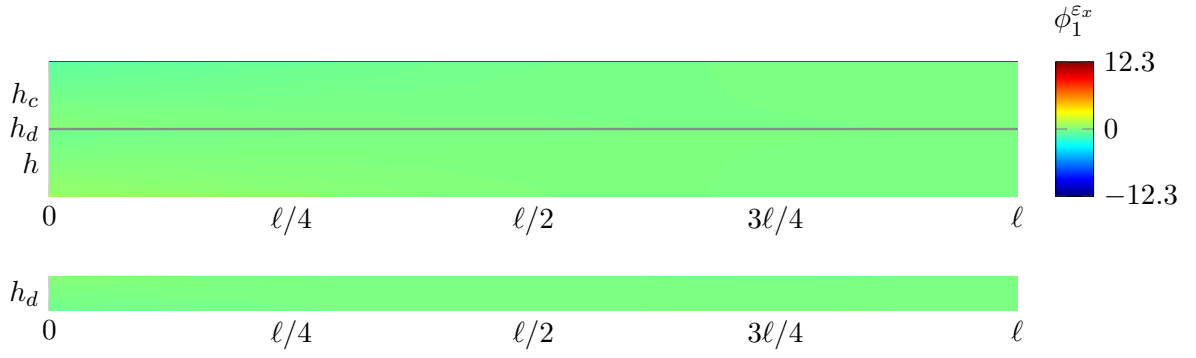
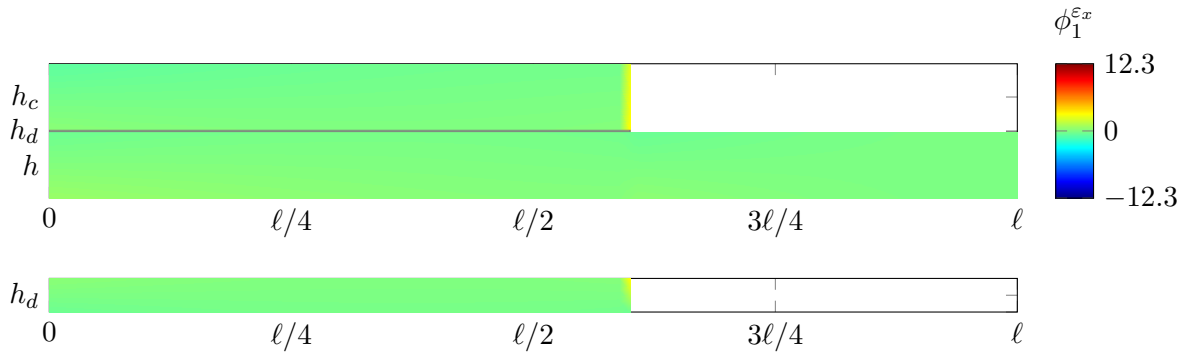
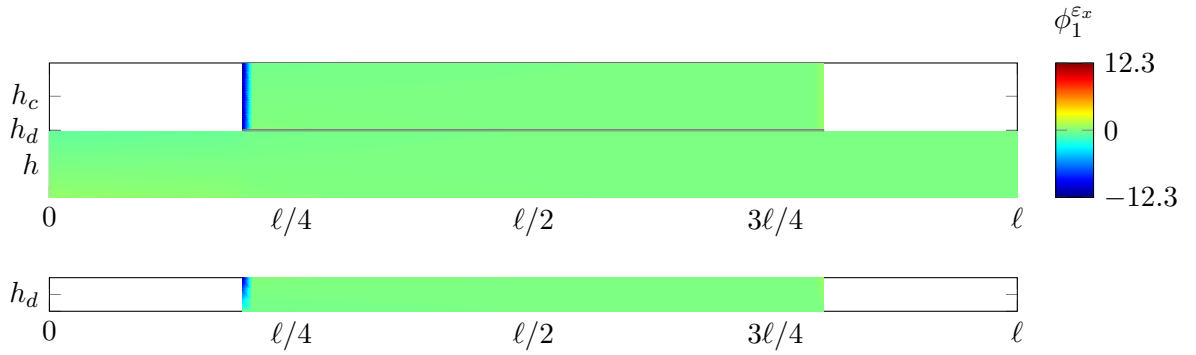


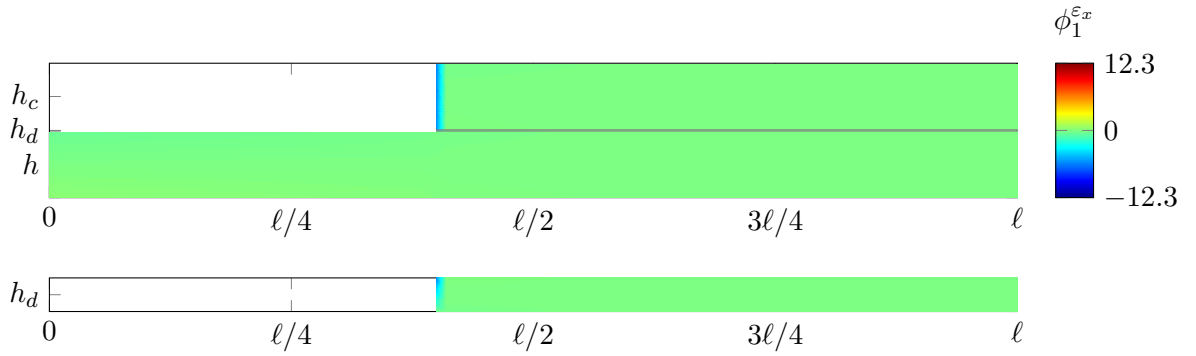
Figure 6.29 1st normal strain mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.30 1st normal strain mode shapes of a cantilever beam with damping patch

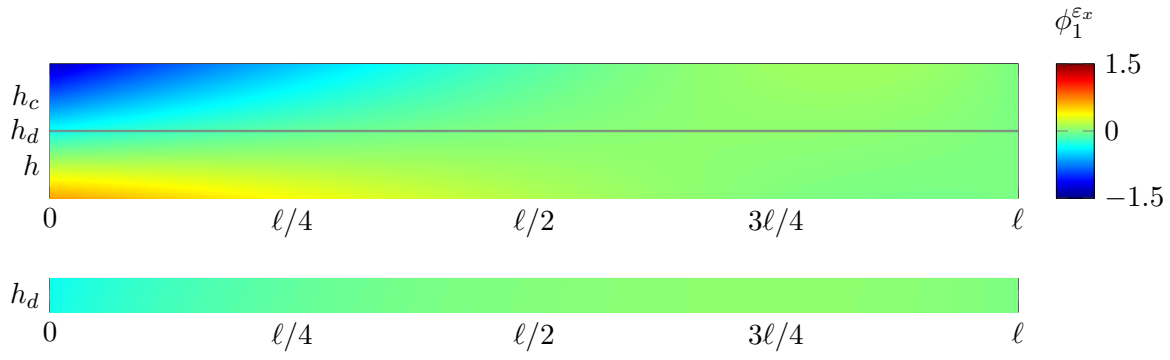
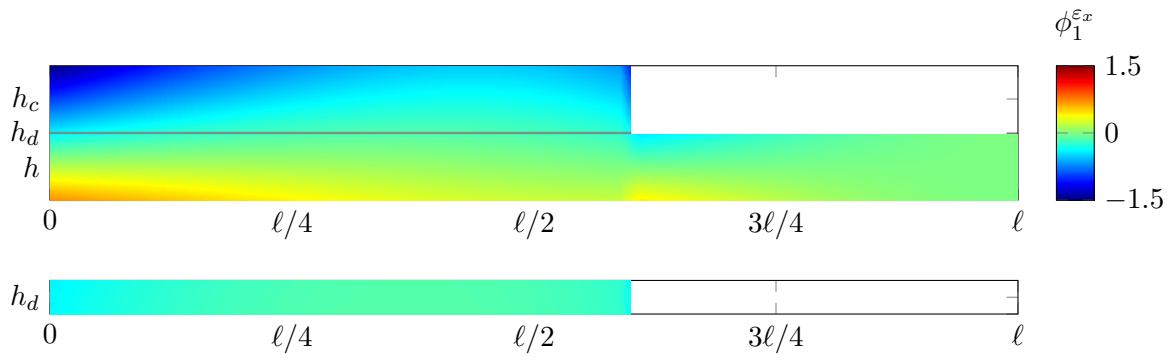
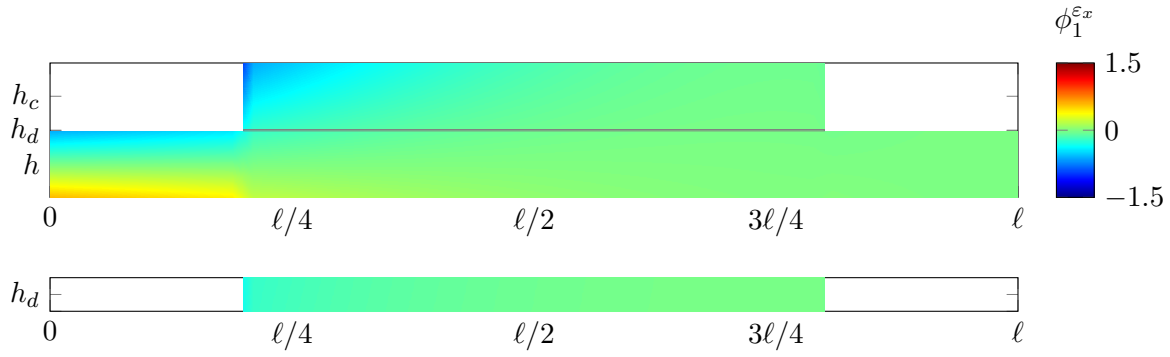


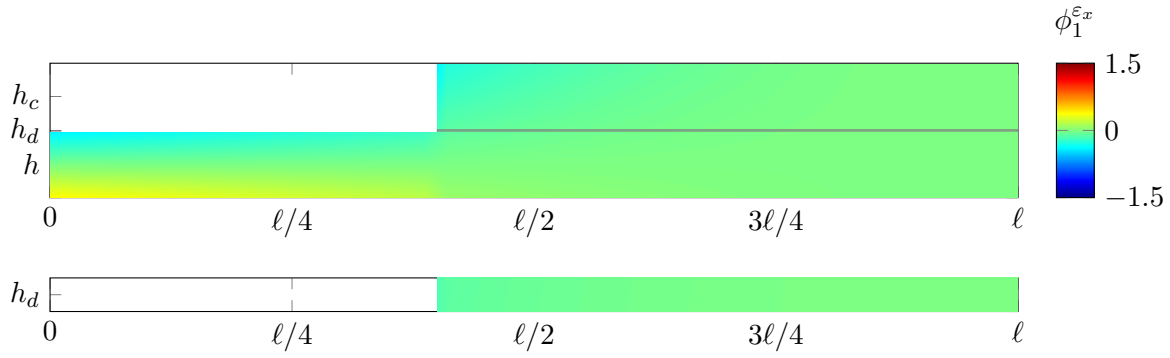
Figure 6.31 1st normal strain mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.32 1st normal strain mode shapes of a cantilever beam with damping patch

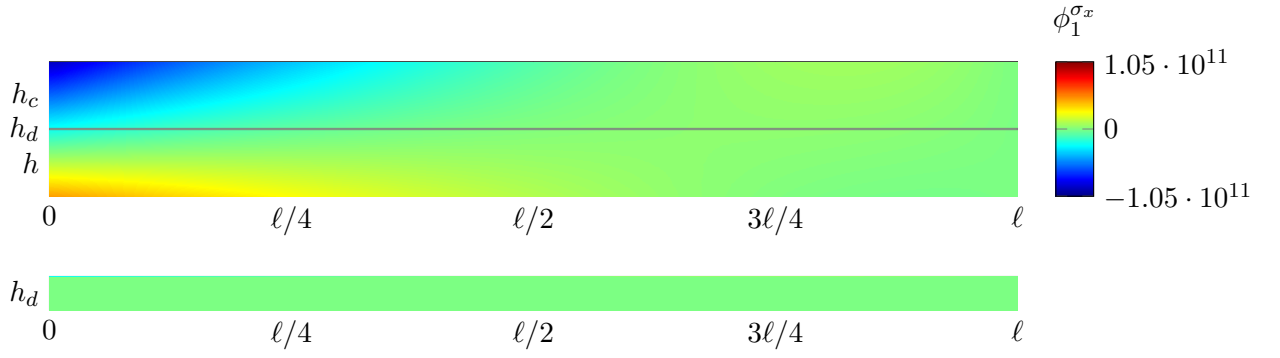
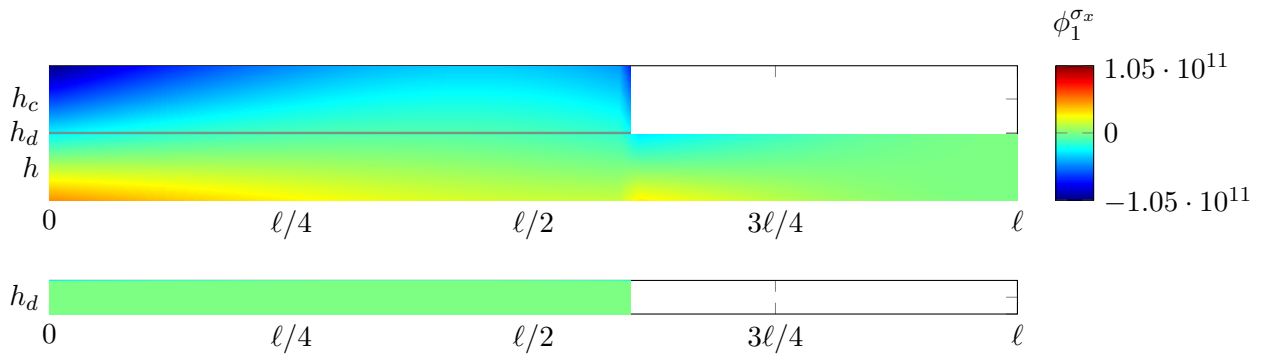
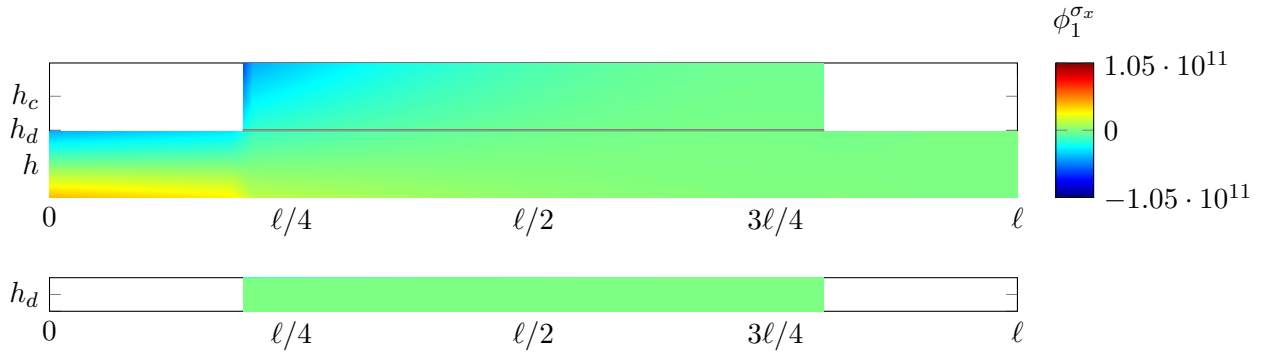


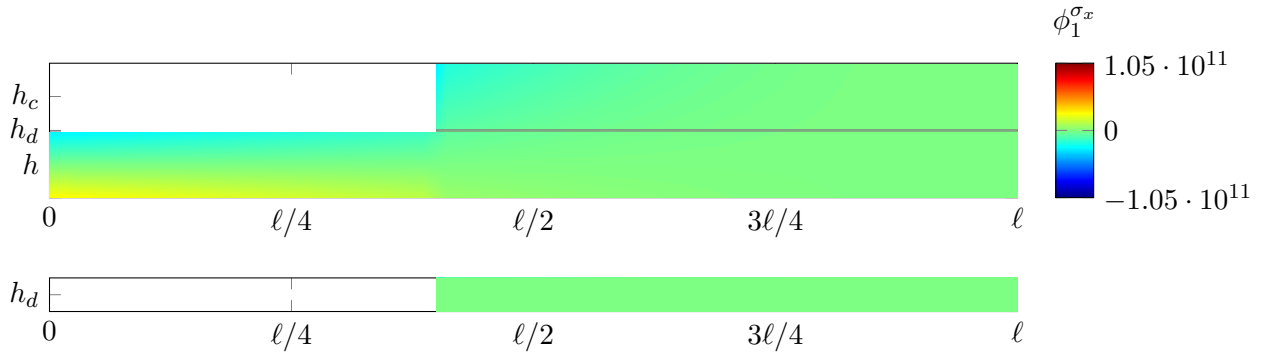
Figure 6.33 1st normal stress mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.34 1st normal stress mode shapes of a cantilever beam with damping patch

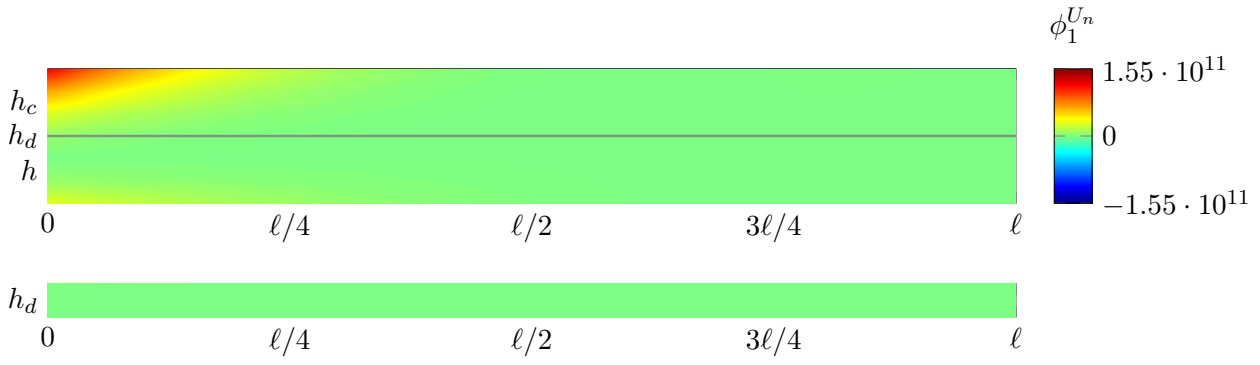
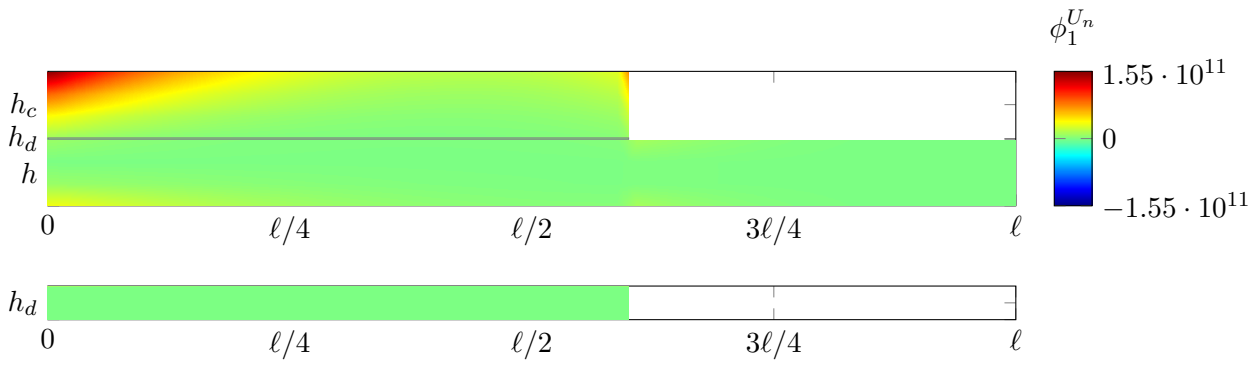
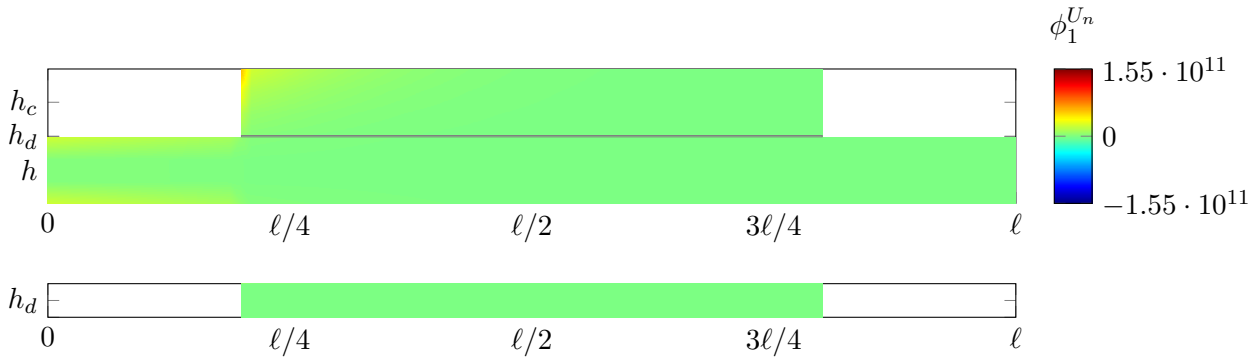


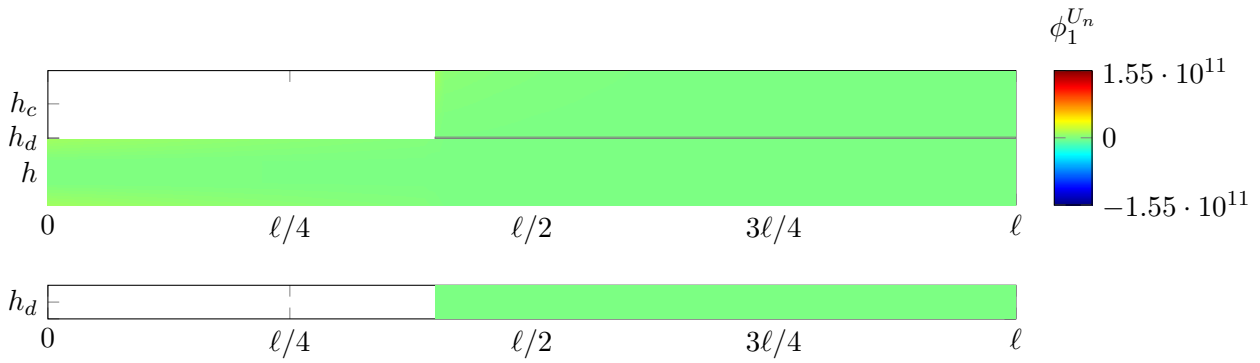
Figure 6.35 1st normal strain energy density mode shape of a cantilever beam with full coverage



(a) Clamp patch



(b) Mid-length patch



(c) Tip patch

Figure 6.36 1st normal strain energy density mode shapes of a cantilever beam with damping patch

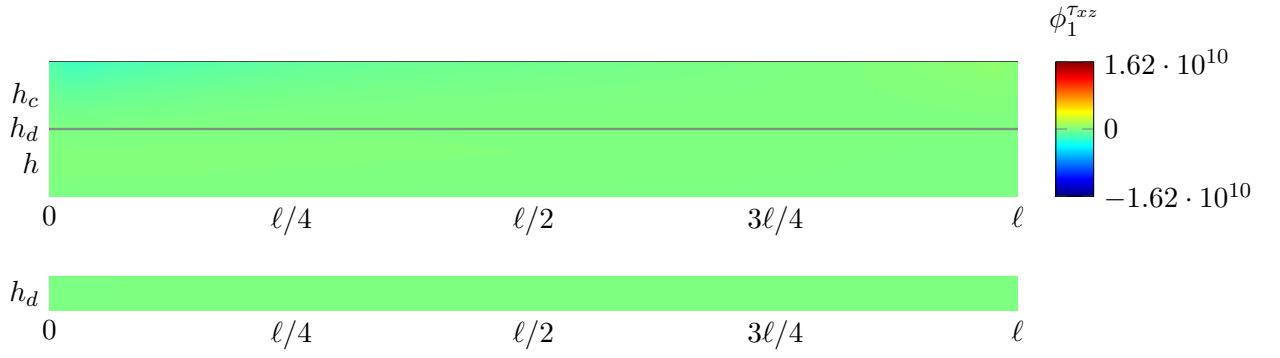
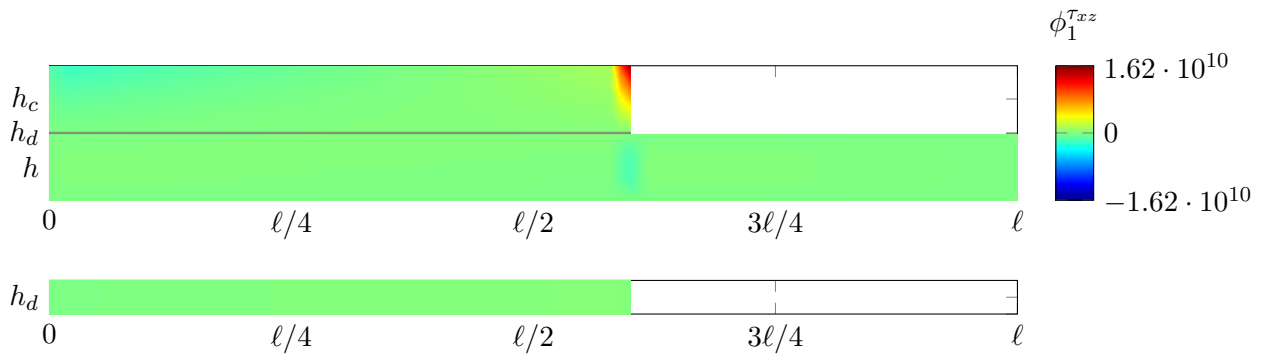
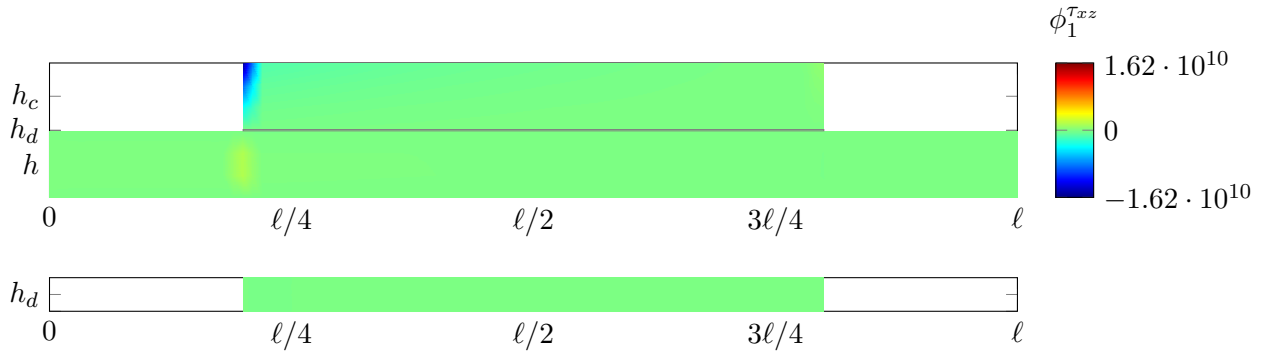


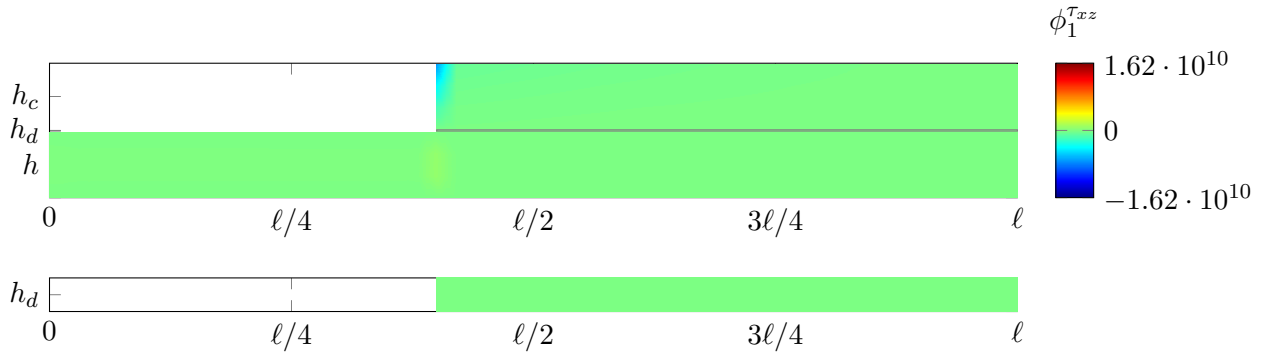
Figure 6.37 1st shear stress mode shape of a cantilever beam with full coverage - equilibrium



(a) Clamp patch - equilibrium equations



(b) Mid-length patch - equilibrium equations



(c) Tip patch - equilibrium equations

Figure 6.38 1st shear stress mode shapes of a cantilever beam with damping patch

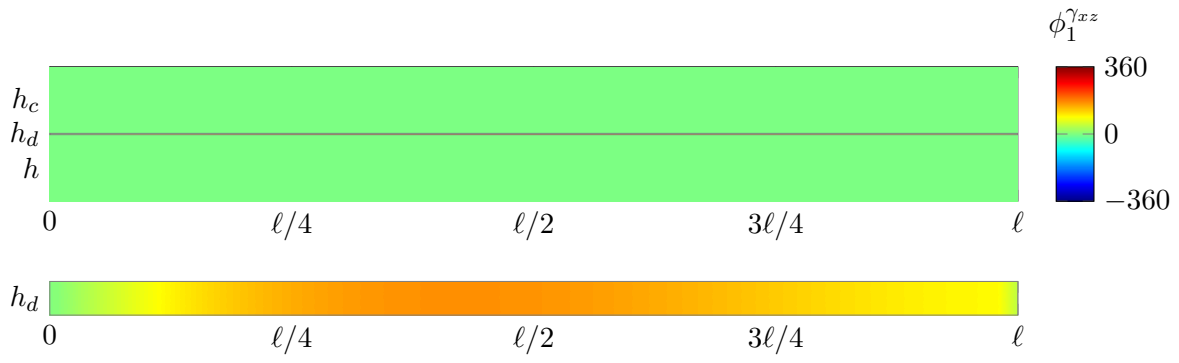
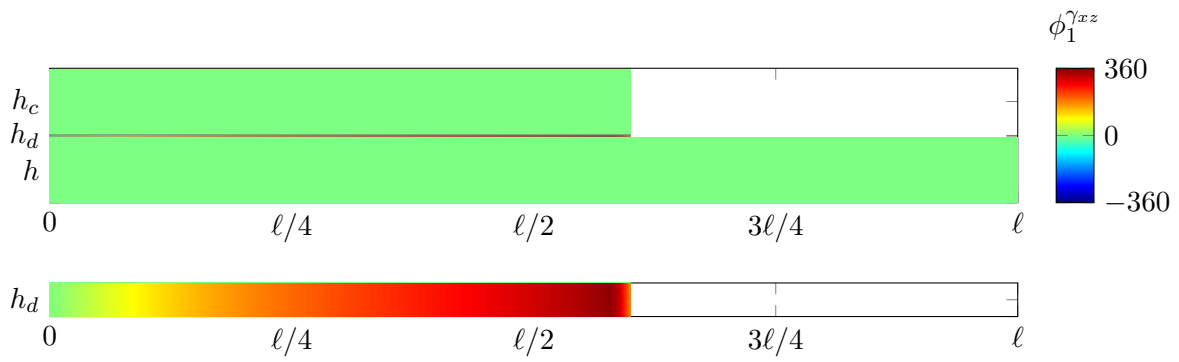
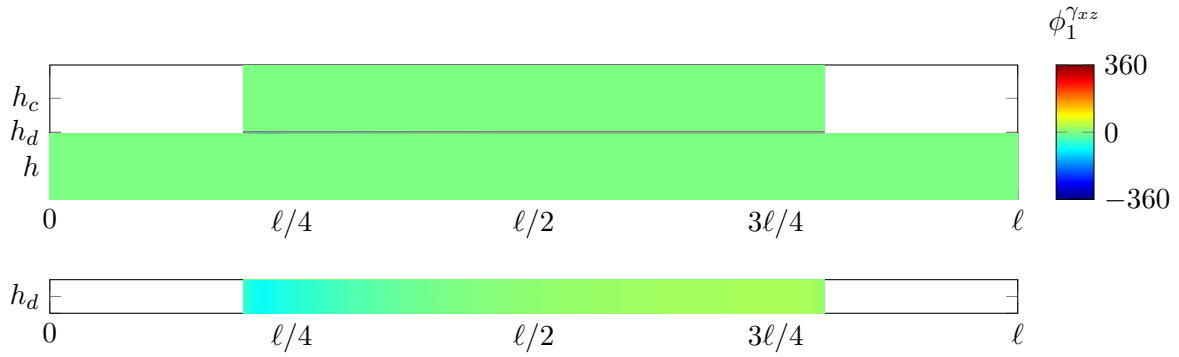


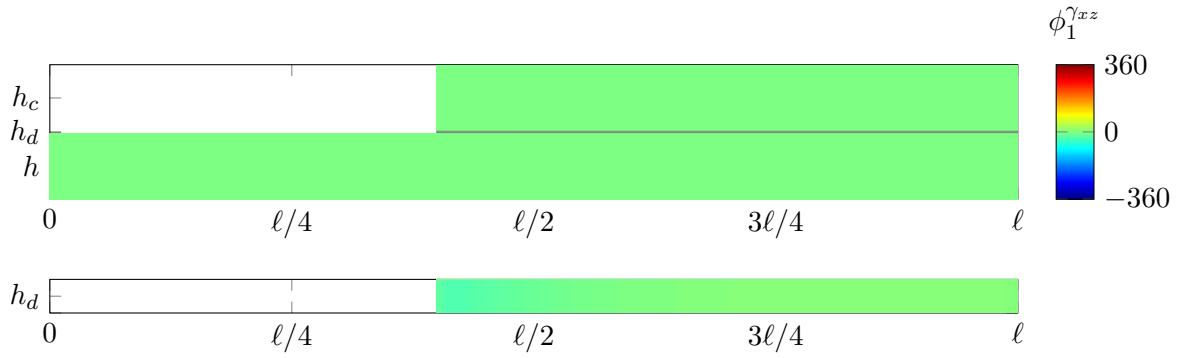
Figure 6.39 1st shear strain mode shape of a cantilever beam with full coverage - equilibrium



(a) Clamp patch - equilibrium equations



(b) Mid-length patch - equilibrium equations



(c) Tip patch - equilibrium equations

Figure 6.40 1st shear strain mode shapes of a cantilever beam with damping patch

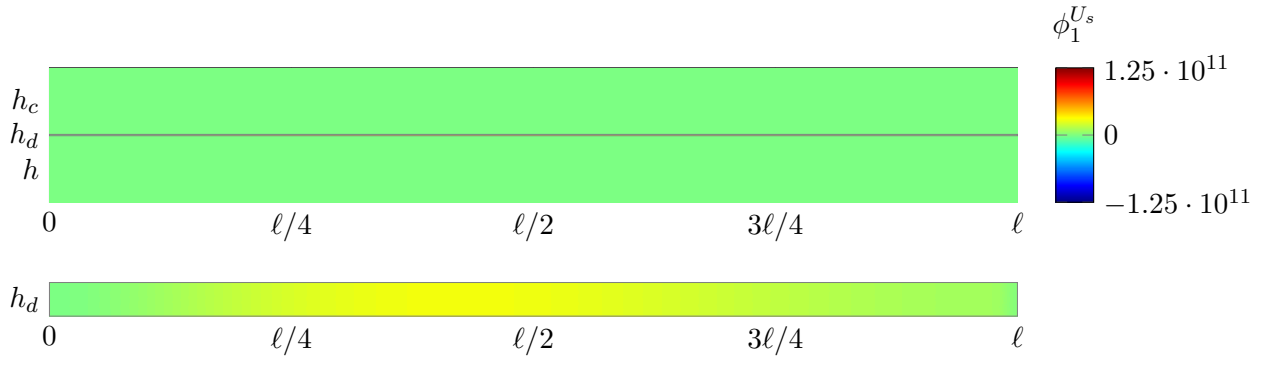
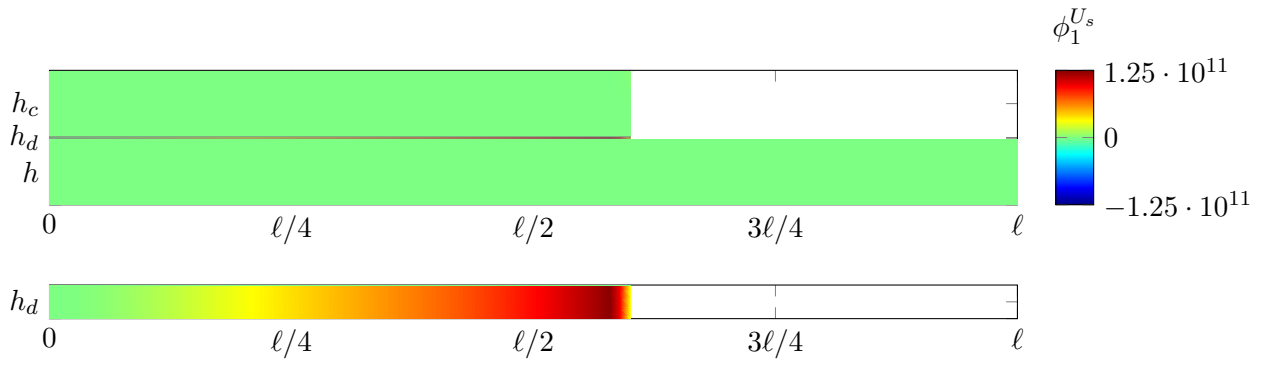
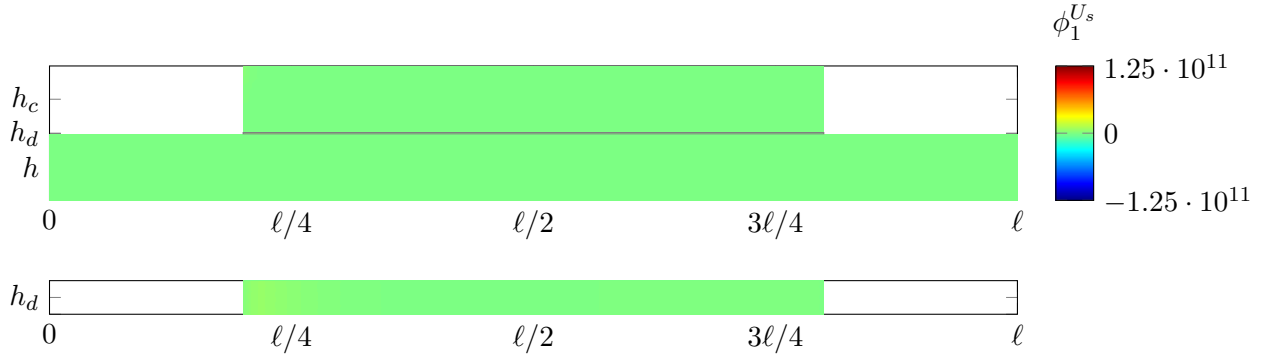


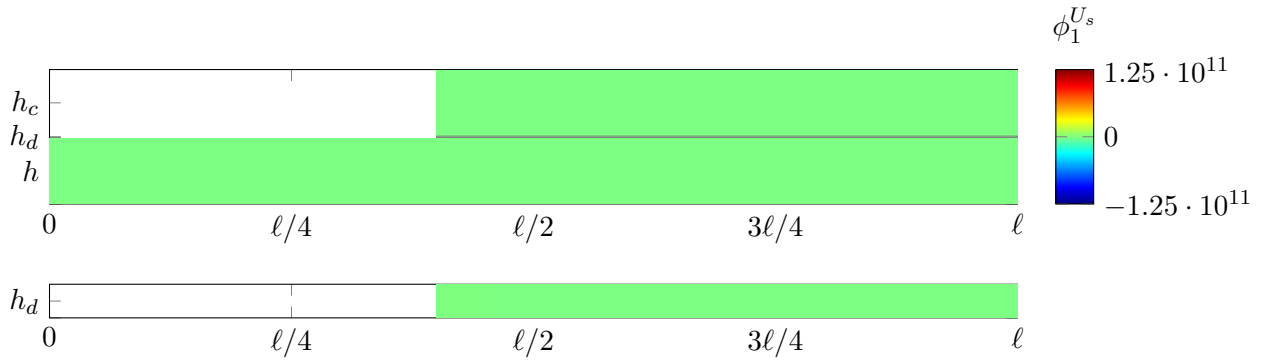
Figure 6.41 1st shear strain energy density mode shape of a cantilever beam with full coverage - equilibrium



(a) Clamp patch - equilibrium equations



(b) Mid-length patch - equilibrium equations



(c) Tip patch - equilibrium equations

Figure 6.42 1st shear strain energy density mode shapes of a cantilever beam with damping patch

6.3.2 | Influence of Covered Length

This section deals with the study of partially covered beams. One considers a dimensionless analysis of an aluminium beam treated with a damping patch with a viscoelastic layer constrained by another aluminium layer to evaluate the influence of the covered length on the damping capacity. Tables 6.1 and 6.2 show the geometric and material properties of the beam layers. First, one considers a beam with clamped-free boundary conditions in a symmetric configuration ($h_c = h$) - Figure 6.43.

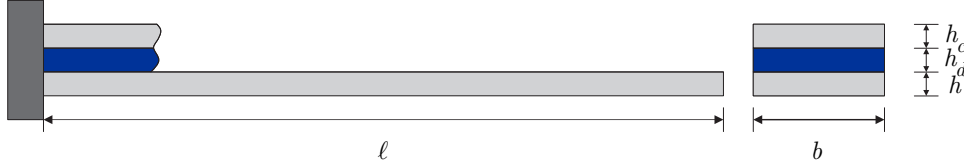


Figure 6.43 Damping patch at the clamped region

Considering a FEM mesh of 100 elements along the length of the beam and the ToZZ, one performs a modal analysis with the CE to obtain the first three transverse displacement mode shapes of the cantilever beam with no damping treatment, as depicted in Figure 6.44 with displacement normalisation. Besides, Figure 6.45 presents the first three shear strain mode shapes to understand the transverse shear pattern at the cantilever beam.

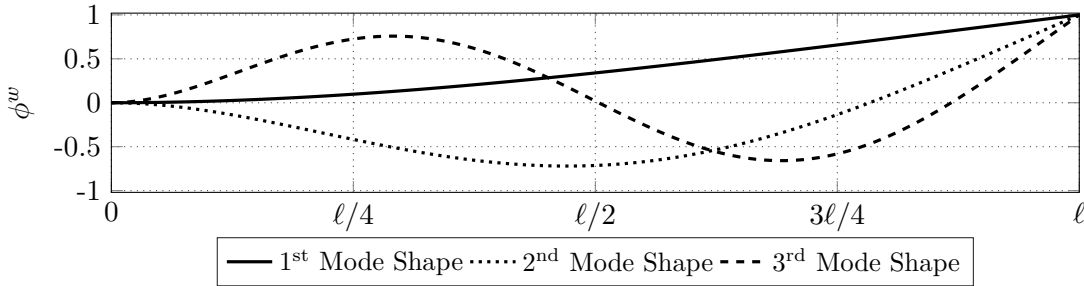


Figure 6.44 Transverse displacement mode shapes of the cantilever bare beam

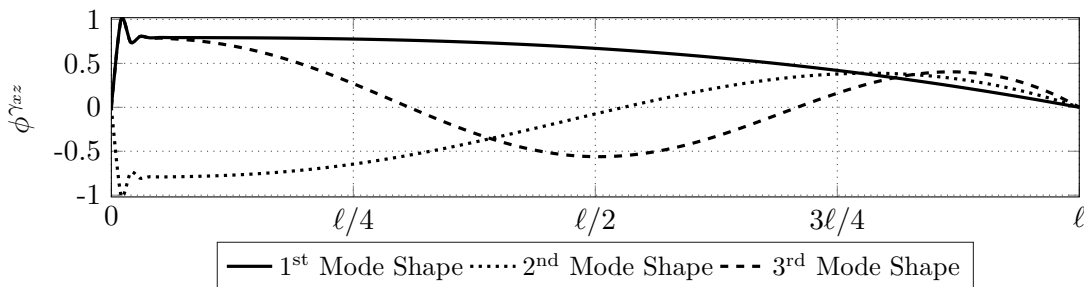
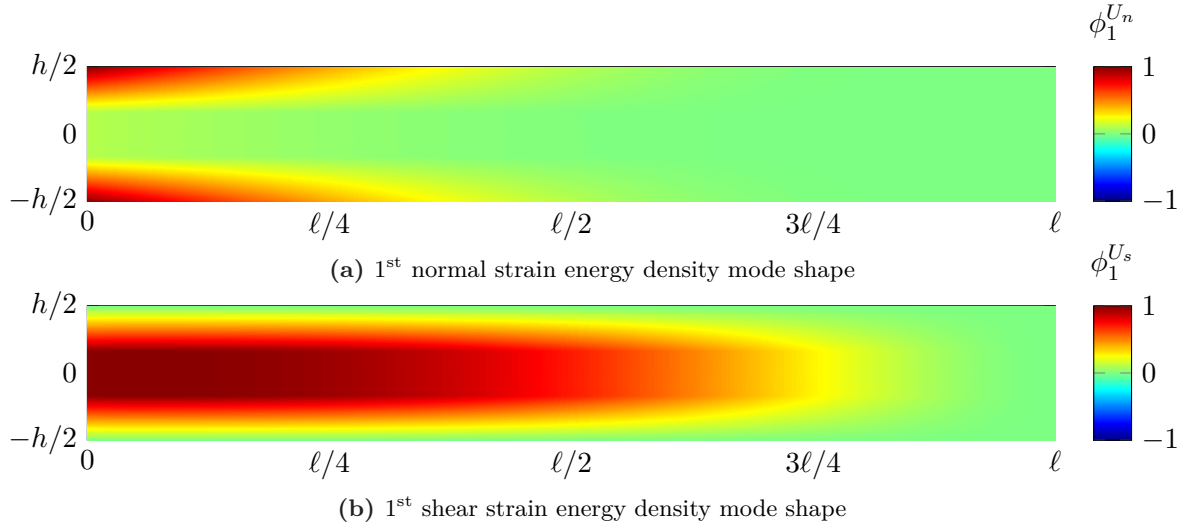
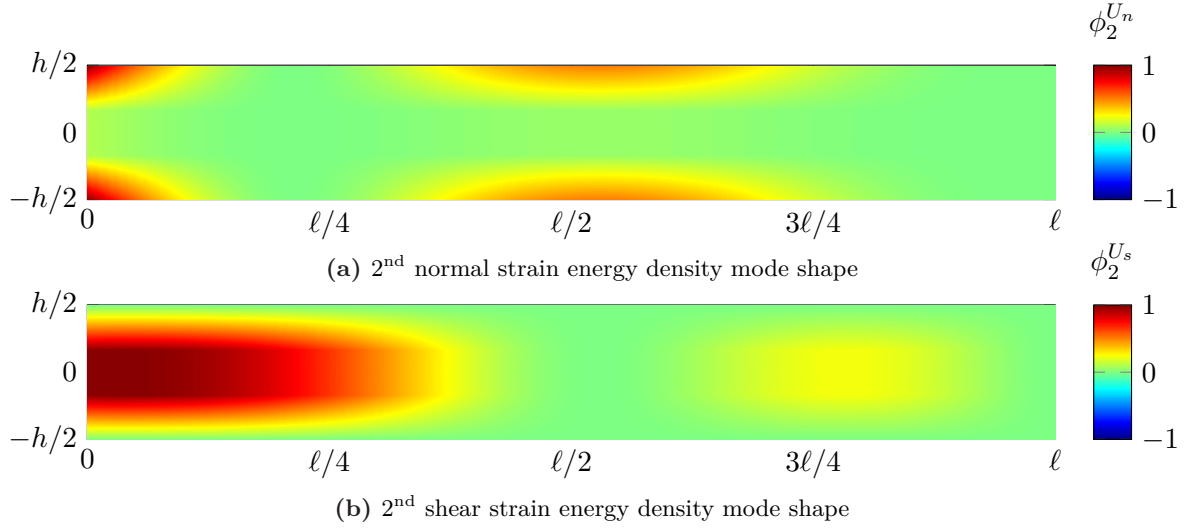
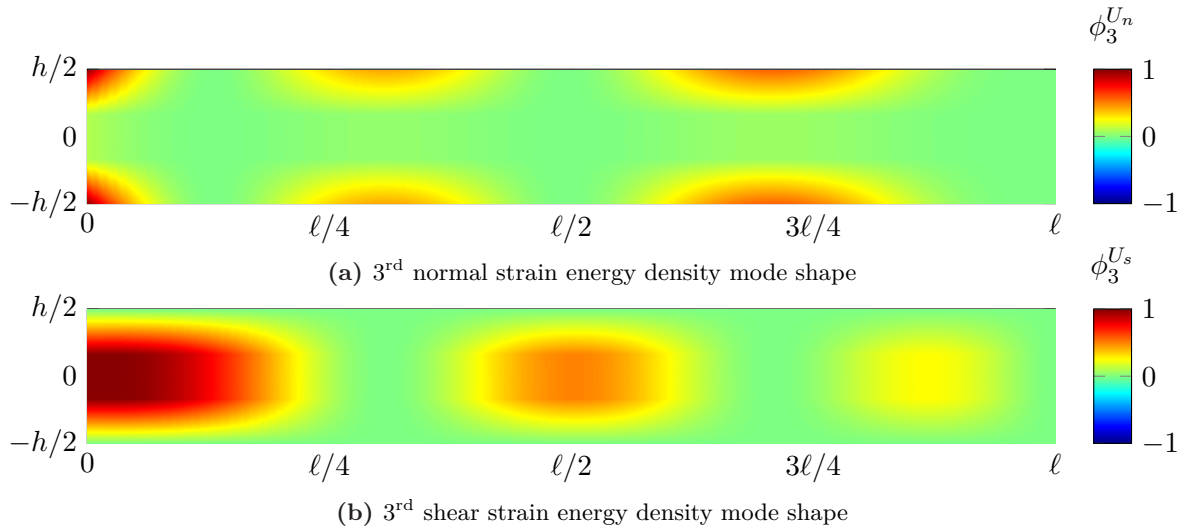


Figure 6.45 Shear strain mode shapes of the cantilever bare beam

Figures 6.46a, 6.47a and 6.48a depict, in a through-the-thickness representation, the first three extensional strain energy density mode shapes of the cantilever bare beam. Moreover, and worth analysing simultaneously, Figures 6.46b, 6.47b and 6.48b present an analogous view of the first three transverse shear strain energy density mode shapes.


 Figure 6.46 1st mode shapes of a cantilever bare beam

 Figure 6.47 2nd mode shapes of a cantilever bare beam

 Figure 6.48 3rd mode shapes of a cantilever bare beam

6.3.2.1 | Reference Beam - $h = 0.01\ell$

Figure 6.49 presents the influence of the covered length on the damping capacity of the first three specific modal loss factors, i.e. the modal loss factors relative to the material loss factor, for different thicknesses of the damping layer. It is noticeable that there is an optimum covered length to achieve the maximum damping capacity in the first mode for a specific damping layer thickness. This optimum covered length is a function of the damping layer thickness; beams with thinner damping layers have maximum damping efficiency with shorter treatments, increasing for thick damping layers. Besides, these results suggest a maximum attainable damping capacity for the first mode, achievable with partial coverage. The maximum specific modal loss factor decreases for damping layers thicker than $0.04h$. Even for full coverage, there is a threshold for maximum damping capacity. This increasing-decreasing effect on the damping capacity is noticeable for thicker damping layers. One can notice that the thicker the damping layer, the higher the damping efficiency, but there is a threshold, after which damping efficiency decreases. Besides, after this threshold, one can only achieve the maximum damping capacity with fully covered base beams. Sher and Moreira [2013] identified an effect when dealing with thin viscoelastic layers on fully covered beams and proved that peak damping performances do not need thick damping layers. The authors showed that increasing damping layer thickness might not increase damping since the shear strain at the viscoelastic layer falls after the mentioned threshold. That explains the increasing-decreasing behaviour of the damping capacity, and Figure 6.49 shows that this effect also influences the optimum covered length.

Damping efficiency curves for higher modes show a somewhat unexpected behaviour for the damping layer thicknesses under analysis. Thicker damping layers mean less damping capacity. Even for the second and third modes, maximum damping capacity occurs with partial coverage when damping layers are thinner. There is also a worth mentioning aspect of the higher modal loss factors. It is clear, considering Figures 6.47, 6.48 and 6.49, the locally decreased performances of the specific modal loss factors when the edge of the damping treatment lies right above the anti-nodes of the extensional strain energy density (nodes of transverse shear strain energy density) of the bare beam. The specific modal loss factors alternately increase and decrease as the edge of the damping treatment passes, respectively, through the nodes and anti-nodes of the extensional strain energy density of the bare beam. From these results, one may conclude that it is desirable to cover regions with a higher extensional strain energy density of the bare beam and avoid the edges of the damping treatment over these areas. The reason is straightforward: elastic layers must undergo extensional strain to impose shear on the damping layer.

6.3.2.2 | Thin Beam - $h = 0.005\ell$

Figure 6.50 illustrates the influence of the covered length of a thin base beam on the damping capacity for different thicknesses of the damping layer, presenting the first three specific modal loss factors. Analysing the first specific modal loss factor for a thin cantilever beam, it is noticeable that the thicker the damping layer, the higher the damping efficiency. In fact, one obtains the maximum damping capacity for this specific mode with partial coverage for the damping layers under analysis. The increasing-decreasing effect noticed for the reference beam does not occur for the damping layers under analysis. It is possible to achieve a higher damping capacity with the same damping material with thinner base beams. It is worth mentioning that a damping layer of $0.02h$ at the thin beam has the same thickness as a damping layer of $0.01h$ at the reference beam since the thin beam has half the base beam thickness of the reference beam. Later in this section, one compares thin and thick base beams considering a constant damping layer thickness for detailed comprehension.

Damping efficiency curves for the second and third specific modal loss factors are worth

analysing. The increasing-decreasing damping effect, already noticed for the first mode of the reference beam, is present here. Besides, maximum damping efficiency occurs with partially covered lengths for thin damping layers and fully covered lengths for thick damping layers. Both second and third modal loss factors present local maxima of damping efficiency with partially covered structures, apart from the global maximum, and the number of these local maxima is proportional to the order of the mode. However, the most significant aspect of these modes is the existence of a turning point of damping layer thickness, for which the increase of the damping layer decreases the damping efficiency, regardless of the covered length. The most obvious example is, perhaps, the third modal loss factor, as this turning point occurs for a thinner damping layer. These results prove how disadvantageous blindly adding damping material could be, specifically for higher modes. As seen here, adding damping material is far from increasing damping in the structure.

6.3.2.3 | Thick Beam - $h = 0.02\ell$

Figure 6.51 shows the influence of the covered length on the damping capacity for a thicker beam. Once again, thinner damping layers show a maximum damping efficiency for partial coverage of the base beam. The increasing-decreasing effect of damping efficiency is noticeable for this thicker base beam; thicker damping layers shortly decrease the damping performance for the first mode. Considering the damping layers under analysis, thicker damping layers decrease damping efficiency for the second and third modes. For thicker damping layers, damping efficiency reduces so much that it is difficult to figure out the local maxima of the specific modal loss factors.

Regardless of the base and constraining layer thicknesses, improved damping capacity with tiny covered lengths requires thinner damping layers. A small damping patch with a thick damping layer decouples the constraining layer from the host structure, and the damping layer does not experience a high level of transverse shear strain. Coupling the damping patch to the host beam to impose transverse shear strain on the damping layer requires thinner damping layer thicknesses.

6.3.2.4 | Constant Damping Layer Thickness

The previous examples compare the influence of the damping layer thickness on the damping efficiency for different base beams. However, it might be interesting to reuse the same information to understand the effect of the same amount of damping material with a constant damping layer thickness on different base beams. For this purpose, Figures 6.52 and 6.53 present the first specific modal loss factor for damping layers of $h_d = 0.0001\ell$ and $h_d = 0.0016\ell$, respectively. One reuses the curves of the previous examples to compare the same reference, thin and thick base beams with symmetric configurations of damping treatments.

Figures 6.52 and 6.53 show a significant result. For thin damping layers in Figure 6.52, thicker base beams promote more damping capacity, while for thick damping layers in Figure 6.53, thin base beams improve damping compared to thick base beams.

6.3.2.5 | Non-symmetric Configurations

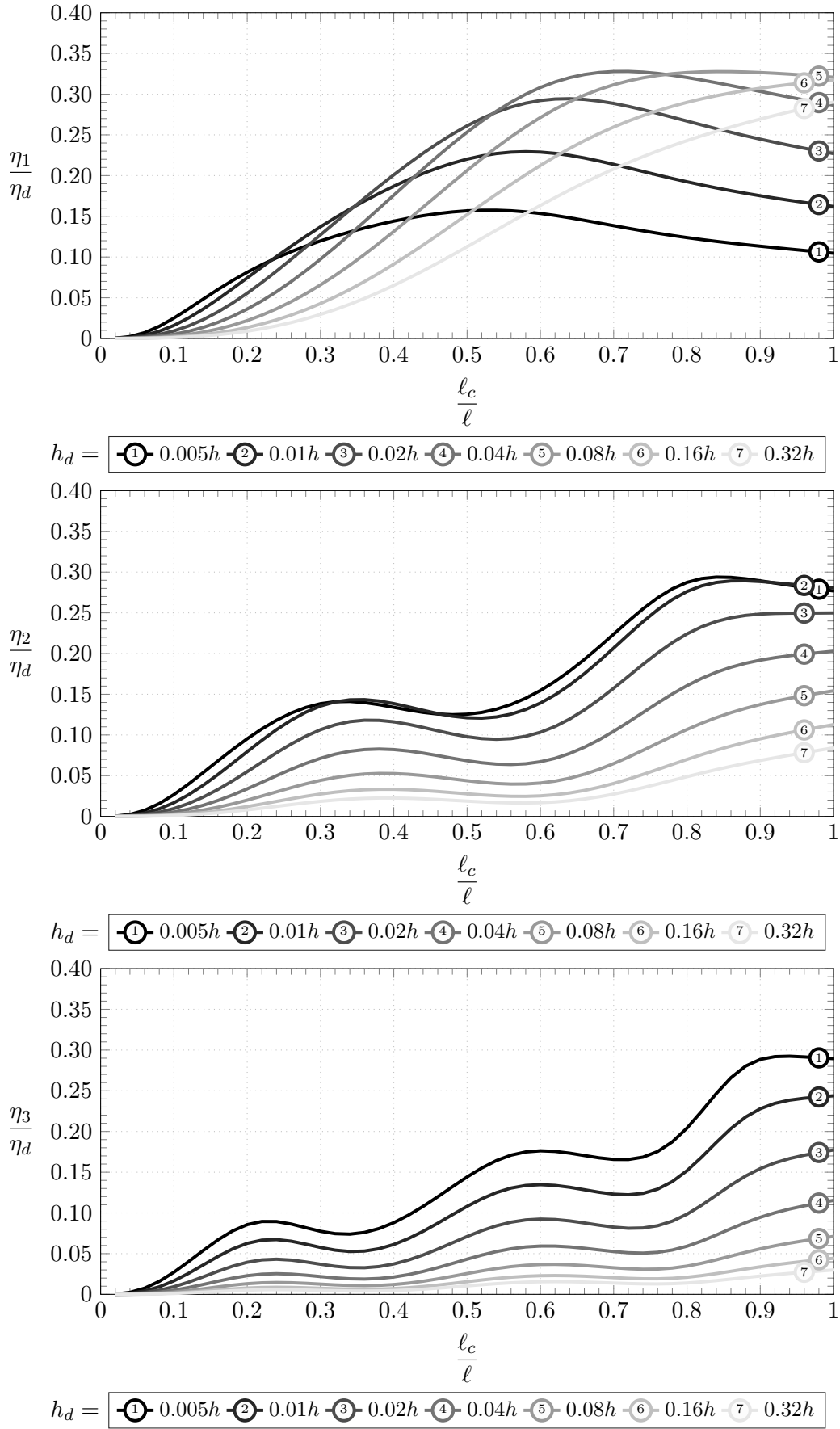
One loses symmetry when the constraining layer is thinner than the base beam. The symmetric configurations of the previous examples are also known as integrated damping treatments since the damping layer lies between two layers of equal thickness. In this section, one studies the influence of the constraining layer thickness on the optimal covered length of a cantilever beam. Considering the reference beam of the previous studies, $h = 0.01\ell$, Figures 6.54 and 6.55 present the first three specific modal loss factors for two thicknesses of the constraining layer - $h_c = 0.5h$ and $h_c = 0.1h$, respectively.

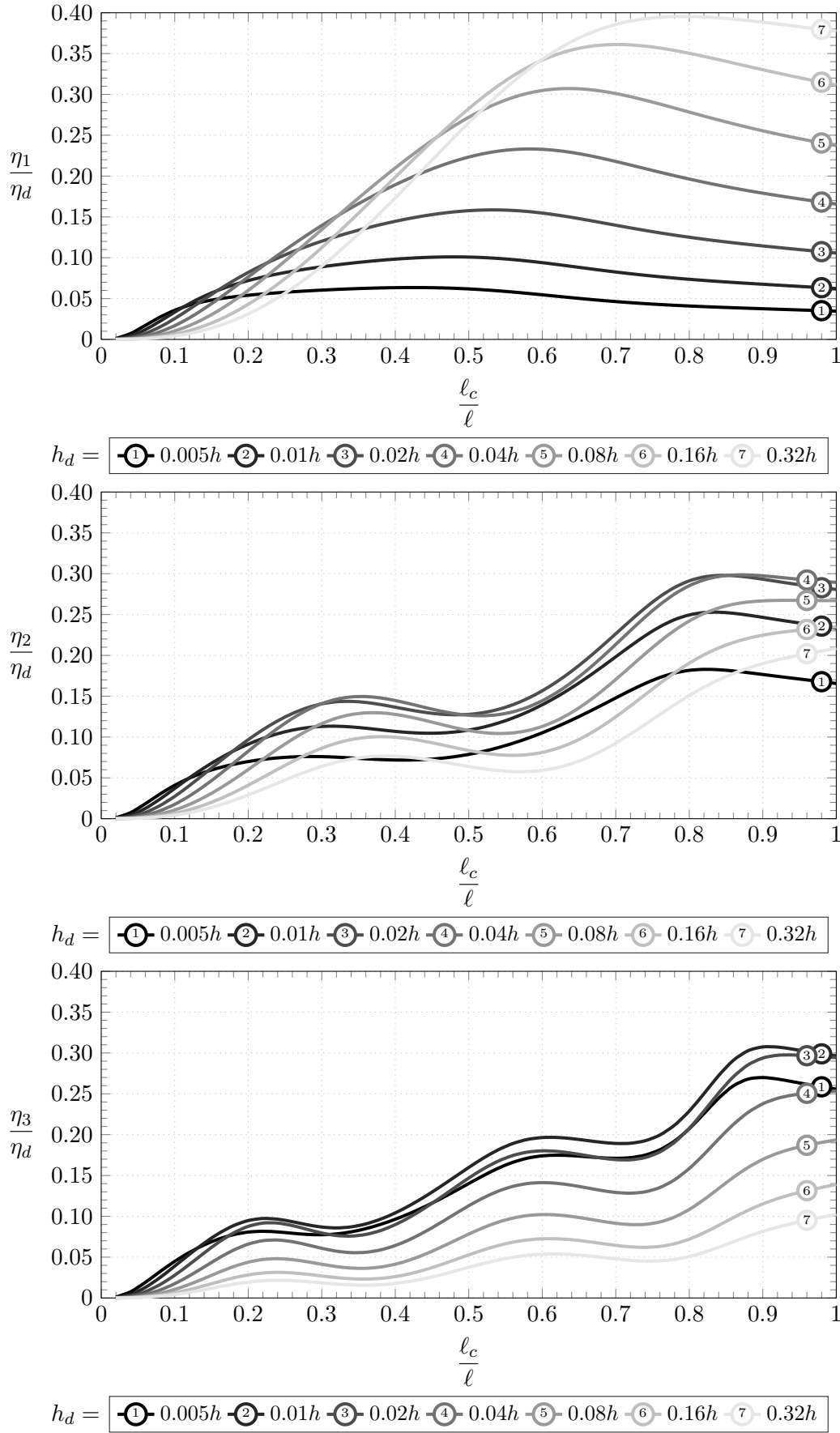
Both figures help to highlight crucial aspects regarding the partial damping of a cantilever beam. The most noticeable issue is the lower performance of the damping treatments as the constraining layer gets thinner. It is an expected result considering the results of other researchers when studying full-covered beams; thinner constraining layers impose less shear strain on the damping layer. However, analysing the first specific modal loss factor, it is worth mentioning that maximum damping efficiency occurs for smaller covered lengths as the constraining layer gets thinner, though damping capacity decreases. Still, there is maximum damping efficiency with partial coverage for a particular thickness of the damping layer, and one can notice the increasing-decreasing damping effect for the first and second modal loss factors of the beam with $h_c = 0.5h$ and the second and third modal loss factors of the beam with $h_c = 0.1h$.

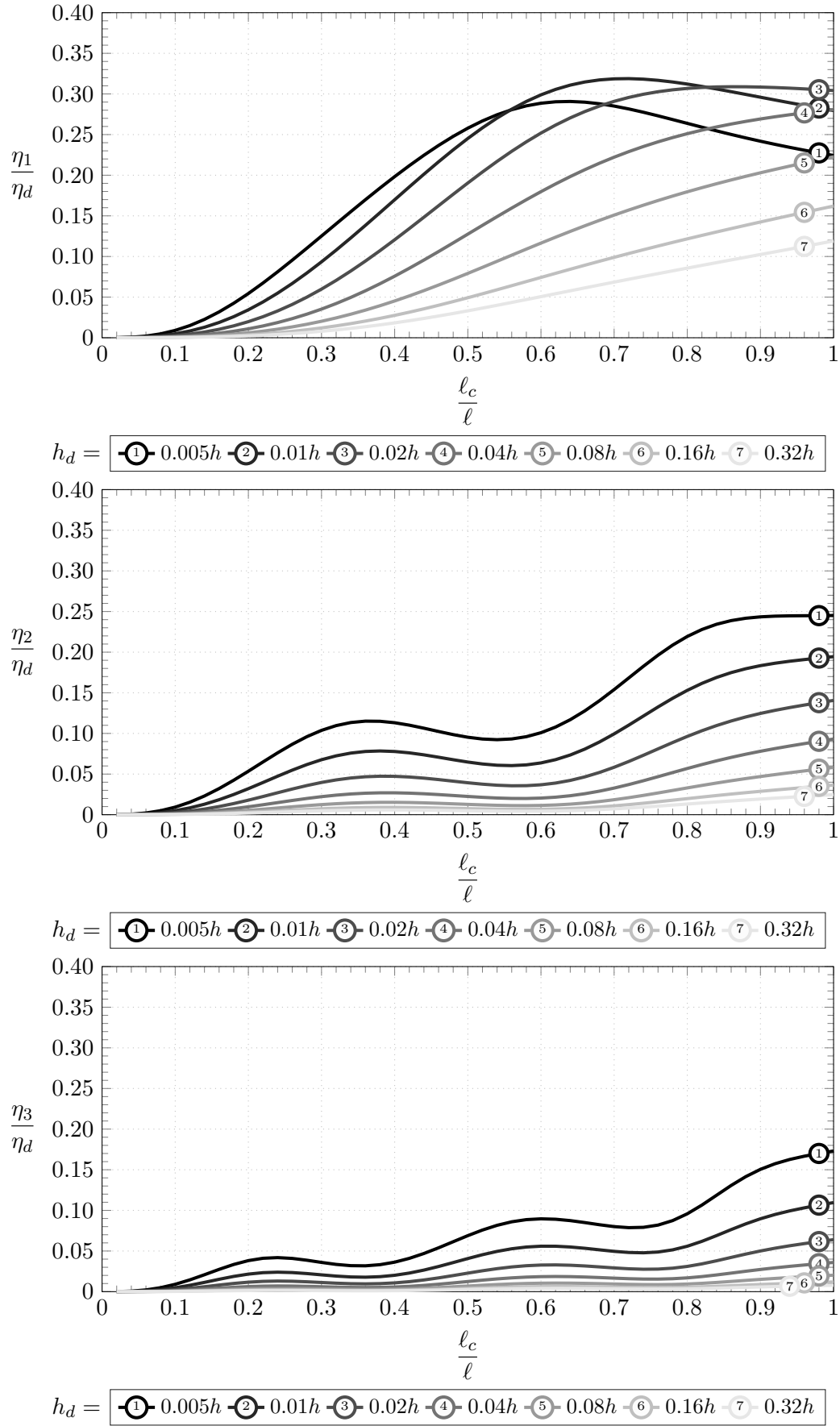
6.3.2.6 | Influence of Material Properties

The material properties of the base, damping and constraining layers have a central role in the damping efficiency of partial damping treatments. Since damping capacity comes from the transverse shear strain within a soft damping layer constrained by a stiff layer, apart from the thicknesses, the material properties influence the damping efficiency of a damping treatment. To illustrate this influence, Figures 6.56 to 6.58 depict the first three specific modal loss factors as a function of the covered length for different ratios of material properties representing the relation between the stiffness of the base and constraining layers and the storage modulus of the damping layer - E_c/E_d . The purpose of the modulus ratio E_c/E_d is to express not only the influence of different materials on the damping capacity of partially covered structures but also the temperature and frequency effect on the storage modulus of viscoelastic damping materials. Figure 6.56 presents the influence of the modulus ratio on a reference symmetric beam with $h = 0.01\ell$ and a constant damping layer of $h_d = 0.01h$. The first specific system modal loss factor increases with the modulus ratio. Higher modulus ratios increase damping efficiency and push the optimum covered length closer to the end of the beam. The second and third modal loss factors show the increasing-decreasing effect on the damping capacity as the modulus ratio increases, proving the existence of an optimum modulus ratio for optimum damping performance. There is also an optimum partially covered length for specific modulus ratios.

Figure 6.57 shows the influence of the modulus ratio on thinner beams. Thinner base and constraining layers cannot impose high transverse shear strain rates on the damping layer as thicker ones, so one needs higher modulus ratios to increase damping capacity. On the contrary, thicker base and constraining layers need lower modulus ratios for optimum damping performances, as one can notice from Figure 6.58. Regardless of the base and constraining layers thicknesses, one requires lower modulus ratios for better damping capacity with small covered lengths. A small damping patch with a high modulus ratio decouples the constraining layer from the host structure, and the damping layer does not experience high levels of transverse shear strain. Consequently, one requires lower modulus ratios for minor differences between the layers to couple the damping patch with the host beam and impose transverse shear strain on the damping layer.


 Figure 6.49 Damping efficiency curves, $h = 0.01\ell$, $h_c = h$

Figure 6.50 Damping efficiency curves, $h = 0.005\ell$, $h_c = h$


 Figure 6.51 Damping efficiency curves, $h = 0.02\ell$, $h_c = h$

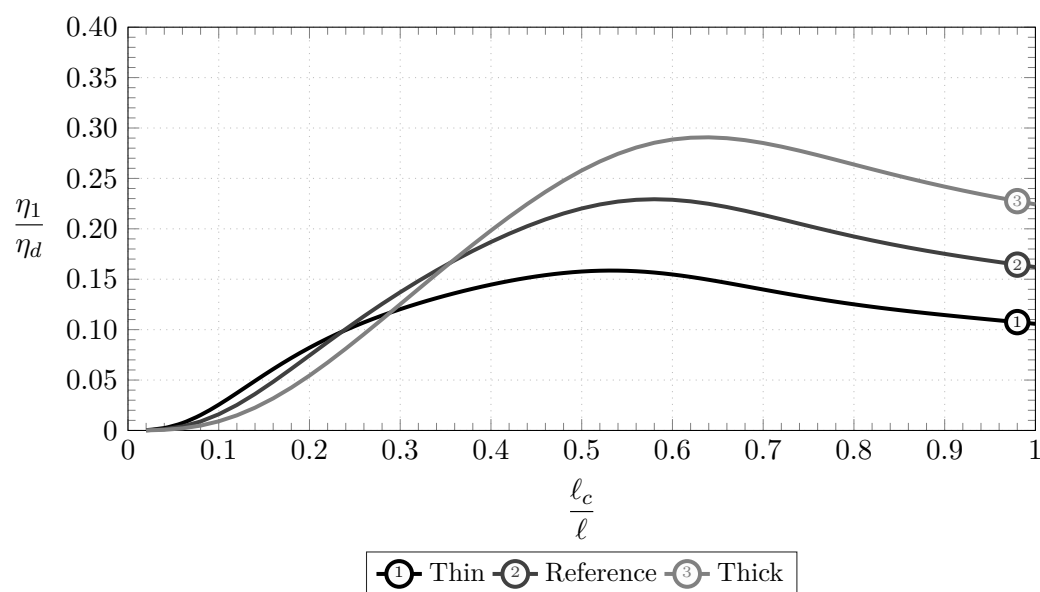


Figure 6.52 Damping efficiency curves, $h_d = 0.0001\ell$, $h_c = h$

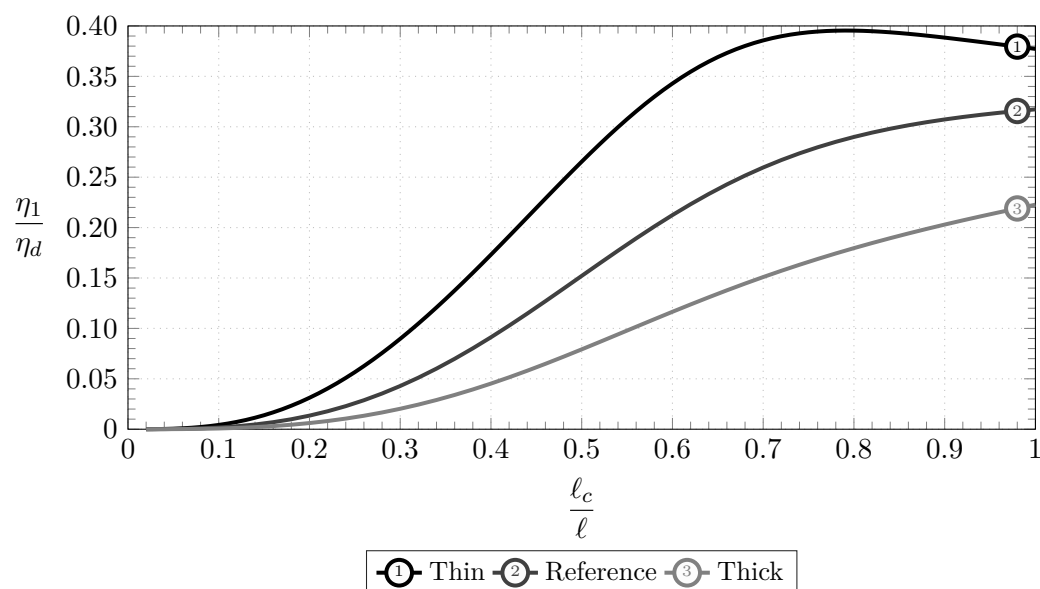
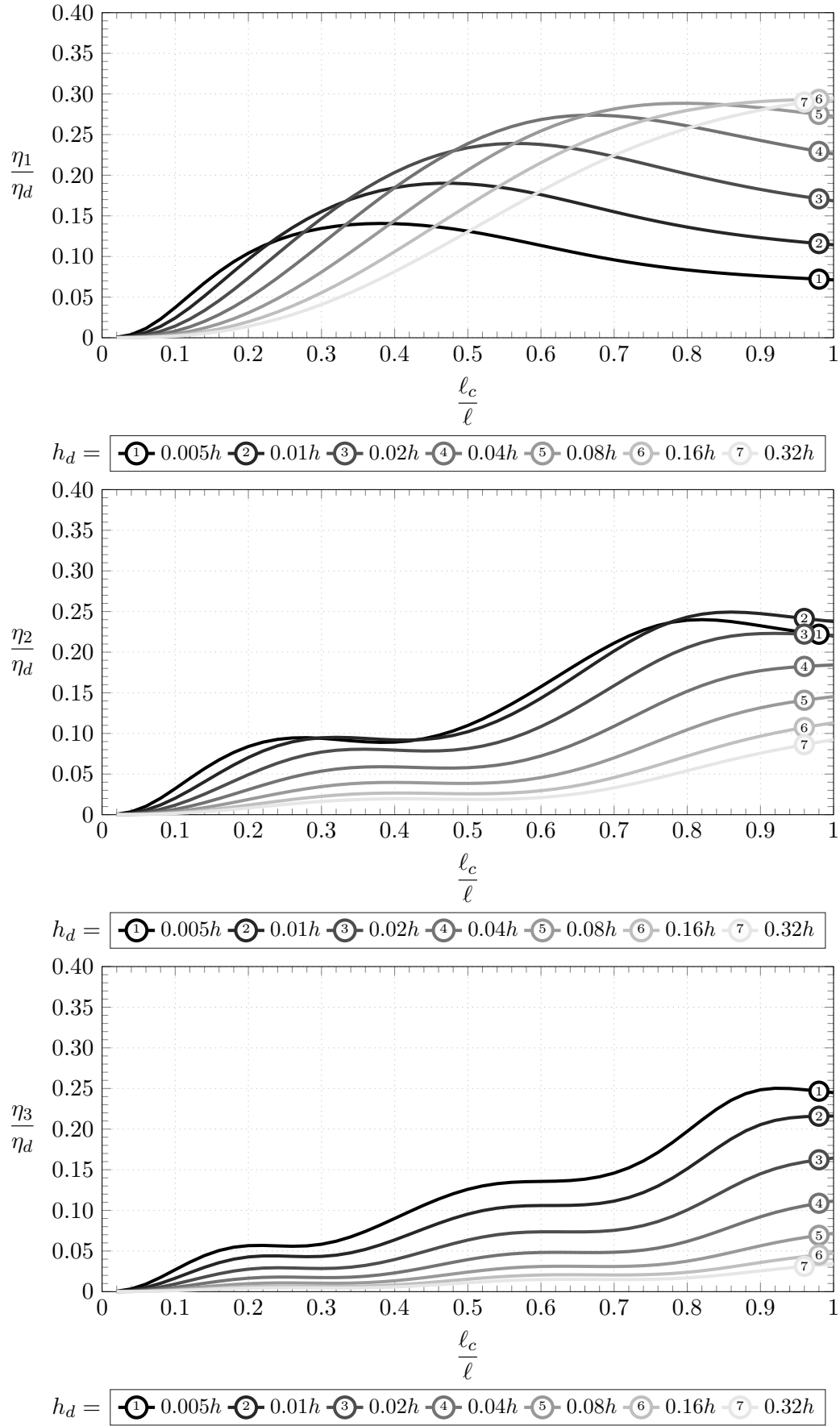
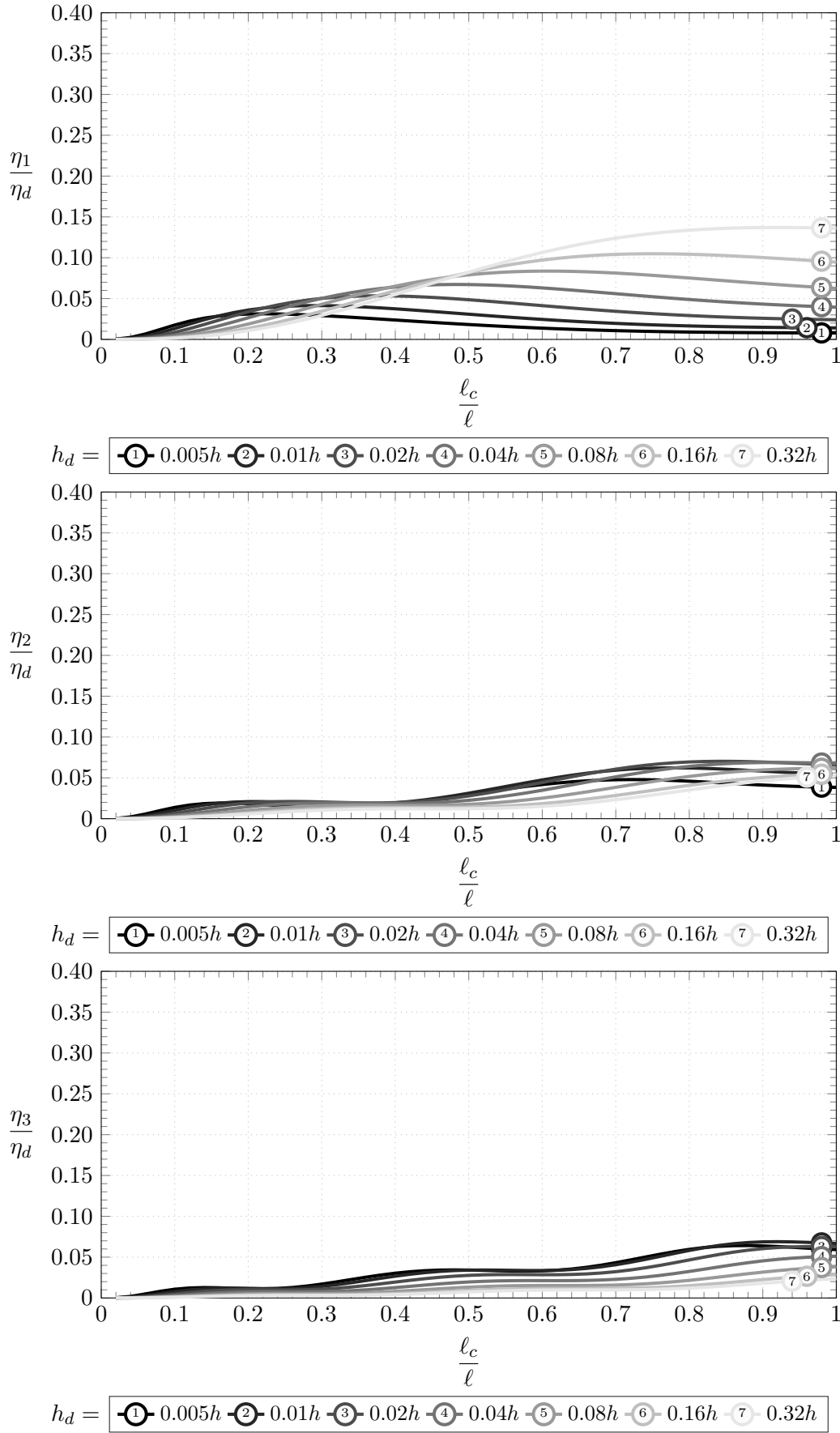
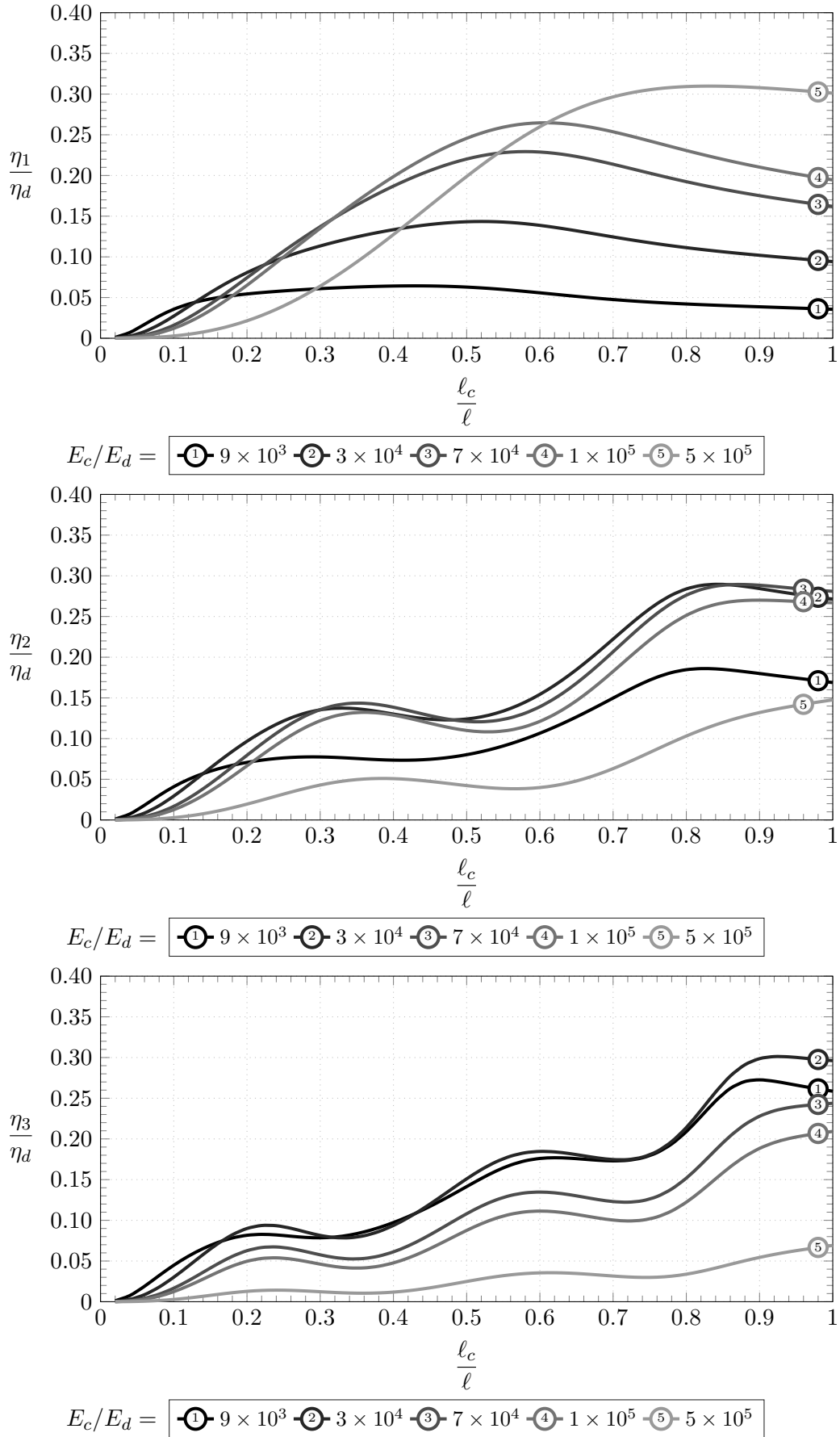


Figure 6.53 Damping efficiency curves, $h_d = 0.0016\ell$, $h_c = h$


 Figure 6.54 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.5h$

Figure 6.55 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.1h$


 Figure 6.56 Damping efficiency curves, $h = 0.01\ell$, $h_d = 0.01h$, $h_c = h$

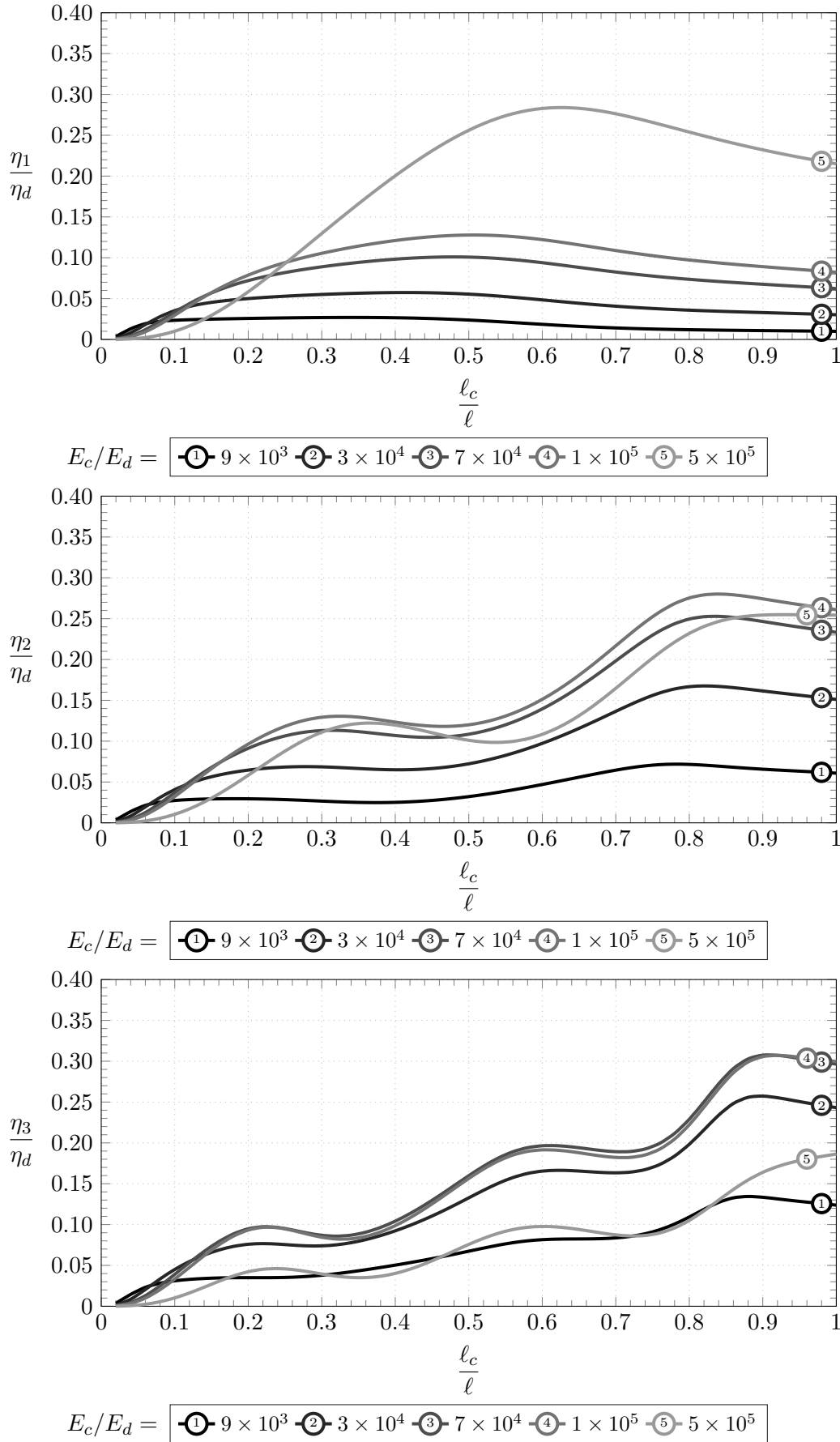
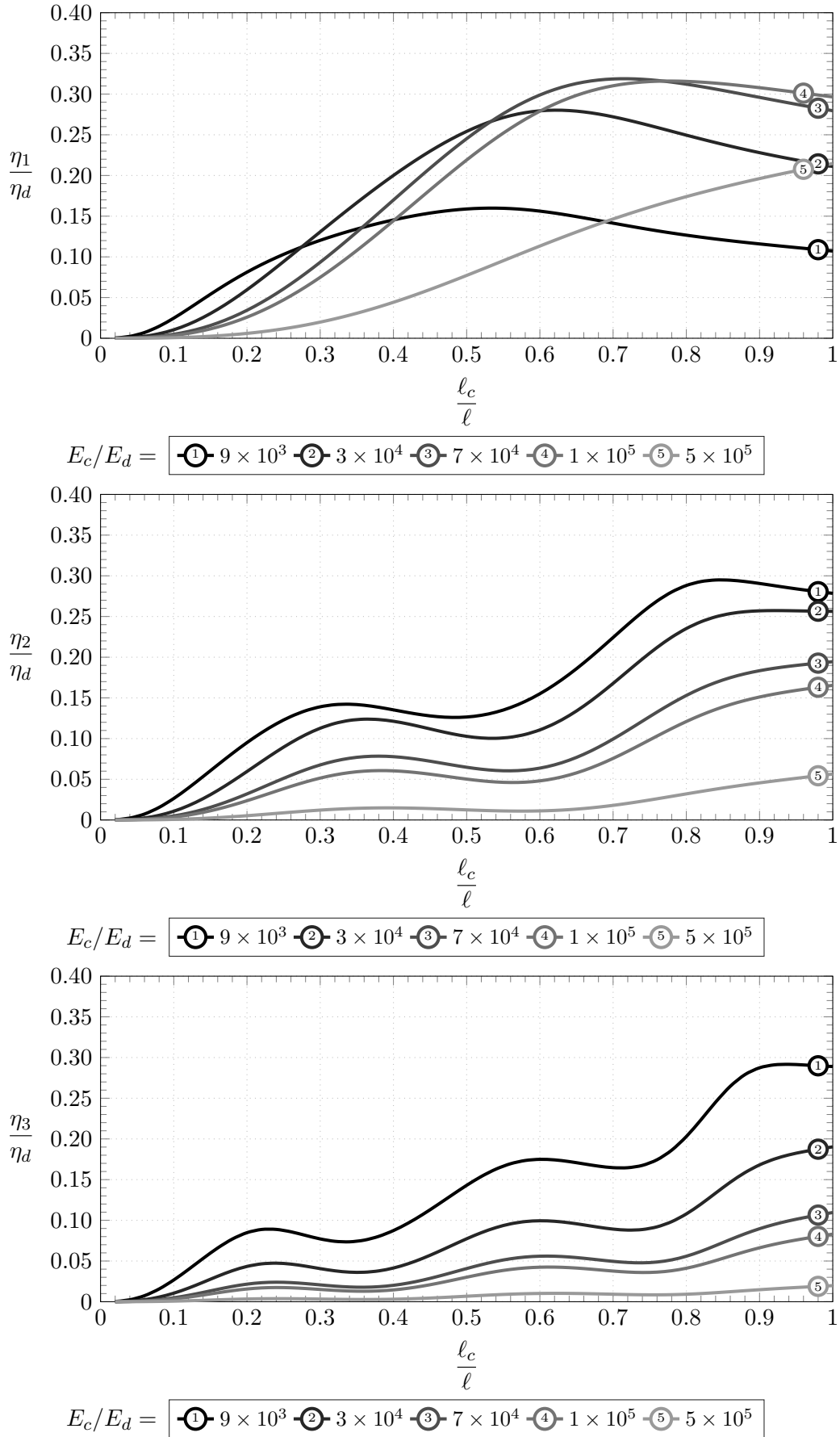


Figure 6.57 Damping efficiency curves, $h = 0.005\ell$, $h_d = 0.01h$, $h_c = h$


 Figure 6.58 Damping efficiency curves, $h = 0.02\ell$, $h_d = 0.01h$, $h_c = h$

6.4 | Damping Efficiency of Partially Covered Free-Free Beams

As boundary conditions play a decisive role in the mechanical behaviour of structures, it is important to study their influence on the damping efficiency of partial treatments. As reported in the previous example, a damping patch near the clamped section improves damping capacity by subjecting the damping layer to higher shear strain. This section deals with a free-free beam, as represented in Figure 6.59.

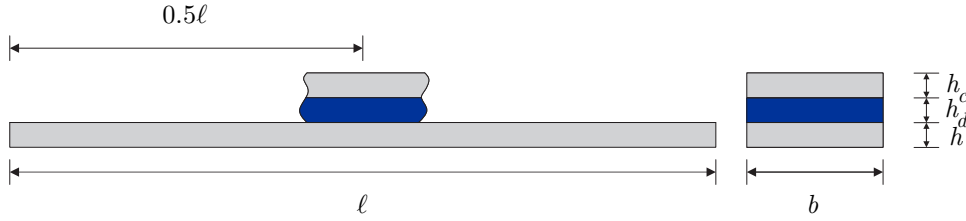


Figure 6.59 Central patch treatment on a free-free beam

6.4.1 | Influence of Covered Length

To evaluate the influence of the covered length on the damping capacity, one considers a dimensionless analysis of an aluminium beam treated with a central damping patch as a function of damping layer thickness and for three different beams, similar to the former example of the cantilever beam. Geometric and mechanical properties are those of Tables 6.1 and 6.2.

Considering a FEM mesh of 100 elements along the length and the ToZZ, one performs a modal analysis with the CE to obtain the first three transverse displacement mode shapes of the free-free beam, with no damping treatment, as depicted in Figure 6.60 with displacement normalization. Besides, Figure 6.61 presents the first three transverse shear strain mode shapes to understand the shear pattern at the free-free beam.

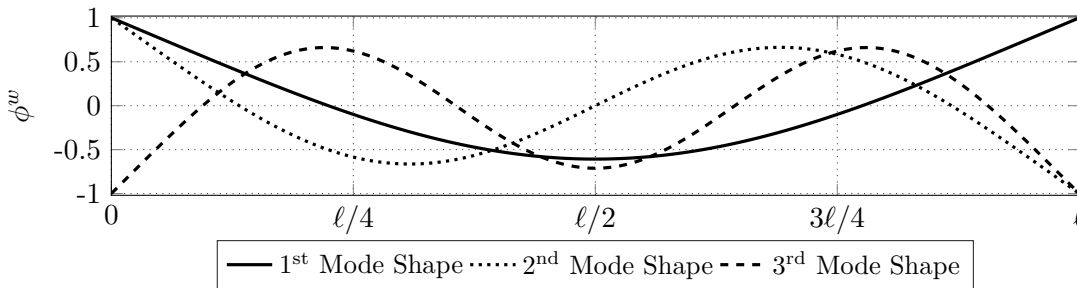


Figure 6.60 Transverse displacement mode shapes of the free-free bare beam

Figures 6.62a, 6.63a and 6.64a present the first three extensional strain energy mode shapes of the free-free bare beam in a through-the-thickness representation. In addition, and worth analysing simultaneously, Figures 6.62b, 6.63b and 6.64b present an analogous view of the first three transverse shear strain energy mode shapes.

To evaluate the influence of the covered length in the damping capacity, one considers a dimensionless analysis of an aluminium beam with length ℓ (0.3 m) and width b (0.02 m), covered by a centre damping patch with a viscoelastic layer constrained by another aluminium layer. Firstly, one considers the symmetric configuration ($h_c = h$) - Figure 6.59.

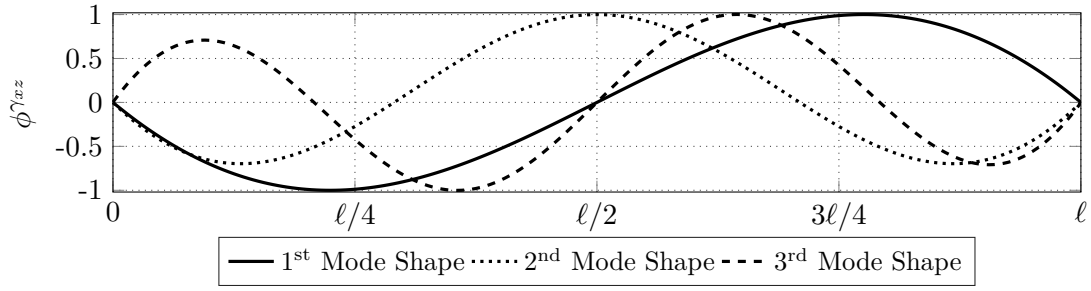
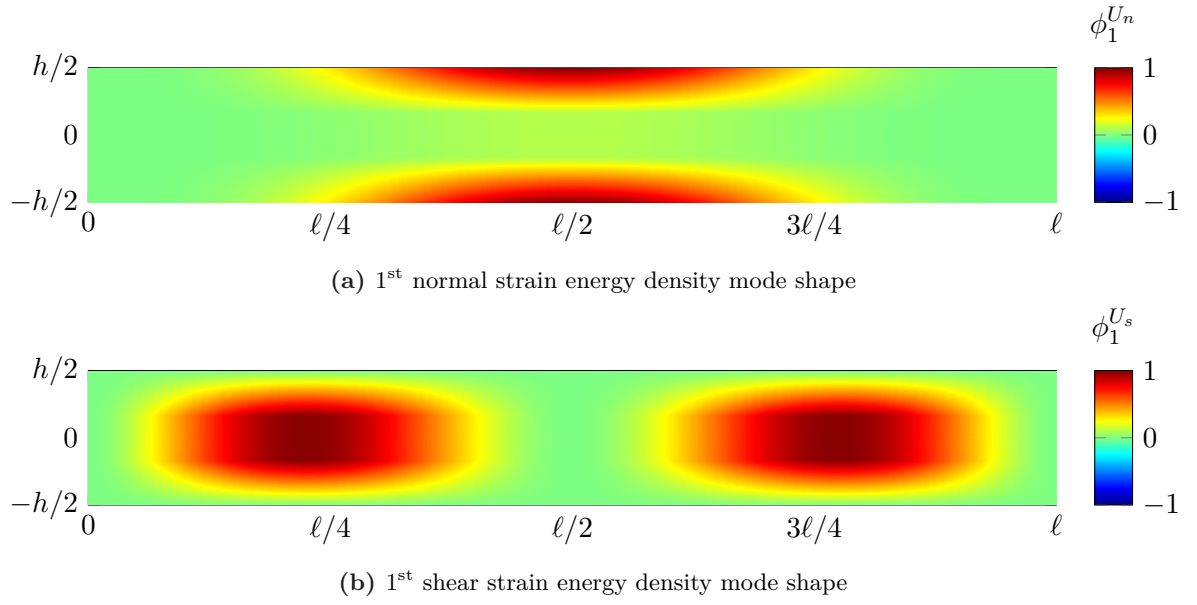
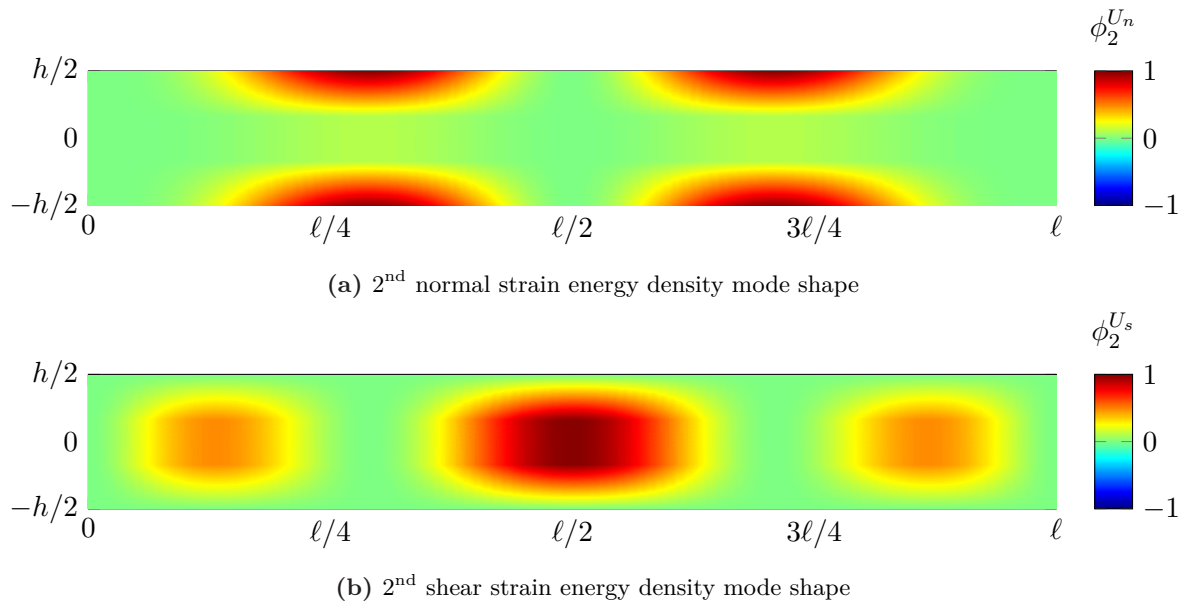


Figure 6.61 Shear strain mode shapes of the free-free bare beam


 Figure 6.62 1st mode shapes of a free-free bare beam

 Figure 6.63 2nd mode shapes of a free-free bare beam

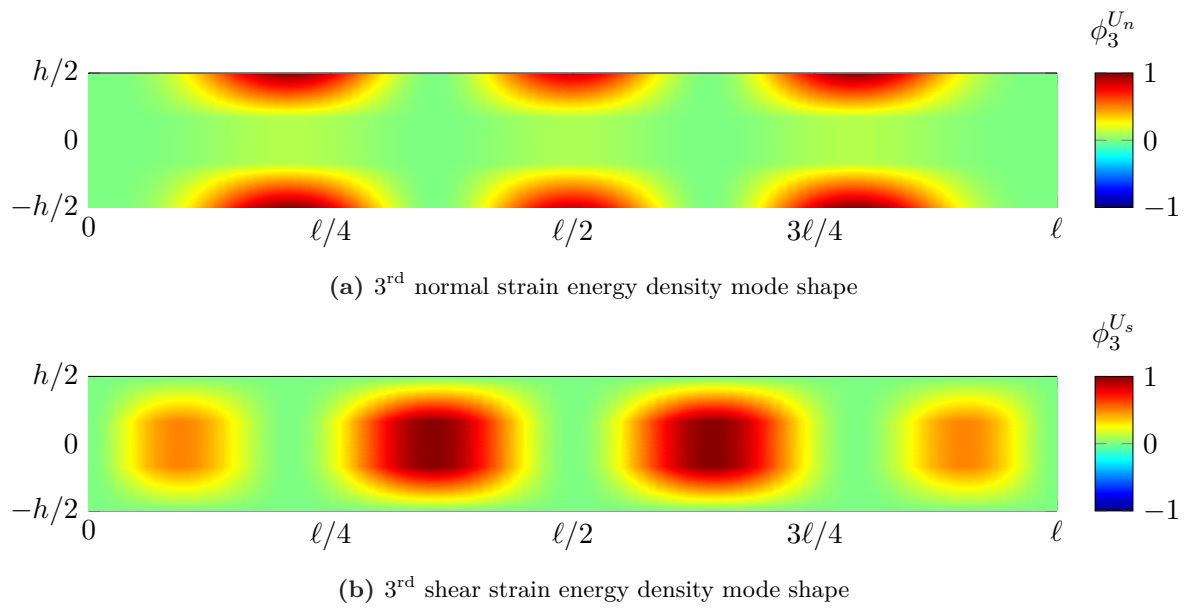


Figure 6.64 3rd mode shapes of a free-free bare beam

6.4.1.1 | Reference Beam - $h = 0.01\ell$

Figure 6.65 presents the influence of the covered length on the damping capacity of the first three specific modal loss factors. First, there is an optimum covered length to achieve the maximum damping capacity at the first mode for a given damping layer thickness, as already noticed for a cantilever beam. These results show that this phenomenon does not depend upon specific boundary conditions. Curves of damping efficiency present the same trends as those of a cantilever beam with a damping patch at the clamped region. The optimum covered length depends on damping layer thickness: beams with thinner damping layers have a maximum damping efficiency with less covered length, which increases for thick damping layers. The increasing-decreasing effect of the damping capacity is noticeable as the thickness of the damping layer gets thick, and one can only achieve the maximum damping capacity with full coverage when considering thicker damping layers.

Damping efficiency curves for the second mode also show maximum damping capacity with partial coverage considering thin damping layers. Besides, as previously noticed, thicker damping layers means less damping capacity for higher modes.

Lastly, it is worth pointing out significant differences between odd and even modes of a central patched free-free beam for small patches. For covered lengths lower than 30 %, and by analysing the second specific modal loss factor, it is possible to notice that there is almost no damping, in contrast with odd modes for which damping capacity is similar for a central patch with 30 % covered length. Figures 6.62a to 6.64a are essential to understand this phenomenon. Odd extensional strain energy density mode shapes present anti-nodes at the beam's half-length, the reference position for the central patch, as even extensional strain energy density mode shapes have nodes at that location. Extensional strains are fundamental to understanding the damping mechanism of constrained layer damping. For the damping layer to experience shear, the constraining layer must undergo cyclic extensional strain. Positioning a damping patch at high extensional strain energy density locations will strain the constraining layer (stiffer layer) and promote shear at the damping layer (softer layer). Besides, in contrast to the cantilever beam example, the second specific modal loss factor does not present local minima, and the third specific modal loss factor has only one. One can explain this because the edges of a central patch increasing towards both ends of the beam pass through no nodes of extensional strain energy density at the second mode and just one (one each edge simultaneously) at the third mode. That highlights the importance of completely covering the regions with a higher extensional strain energy density of the bare beam and avoiding the edges of the damping treatment over these areas. As mentioned, the elastic layers must undergo extensional strain to impose shear on the damping layer.

6.4.1.2 | Thin Beam - $h = 0.005\ell$

Figure 6.66 depicts, for a thin beam, the influence of the covered length on the damping capacity for different thicknesses of the damping layer, presenting the first three specific modal loss factors. One can re-state all the relevant conclusions drawn before, from the maximum damping capacity for partially covered lengths to the increasing-decreasing effect and low damping for central patches at even modes. However, it is interesting to note that, similarly to cantilever beams, thin free-free beams with thin damping layers highlight the possibility of maximum damping efficiency with partial treatments even for higher modes.

6.4.1.3 | Thick Beam - $h = 0.02\ell$

Figure 6.67 shows the influence of the covered length on the damping capacity for a thicker host beam. Repeatedly, thicker damping layers show poor damping efficiency on beams with

thick bases, inclusively suppressing the possibility of optimum damping capacity with partially covered lengths. The second and third specific modal loss factors omit the damping efficiency curves for thicker damping layers since they change the corresponding mode shapes and inhibit an appropriate comparison.

For the free-free beam, regardless of the base and constraining layers' thickness, improved damping capacity requires thinner damping layers with tiny covered lengths. As previously explained, small damping patches with a high damping layer thickness decouple the constraining and host layers, and the damping layer does not experience a high level of shear strain.

6.4.1.4 | Constant Damping Layer Thickness

Figures 6.68 and 6.69 present the first specific modal loss factor for constant damping layers of $h_d = 0.0001\ell$ and $h_d = 0.0016\ell$, respectively, considering the reference, thin and thick host beams in a symmetric configuration. The leading purpose is to highlight the influence of different base beams when damped by the same amount of damping material. Figure 1 presents an increasing-decreasing effect on damping capacity for a thin damping layer as the thickness of the base and constraining layers increase. Still, thinner base beams show an optimum damping efficiency for smaller covered lengths. On the other hand, thicker damping layers need thinner base beams to achieve better damping performances.

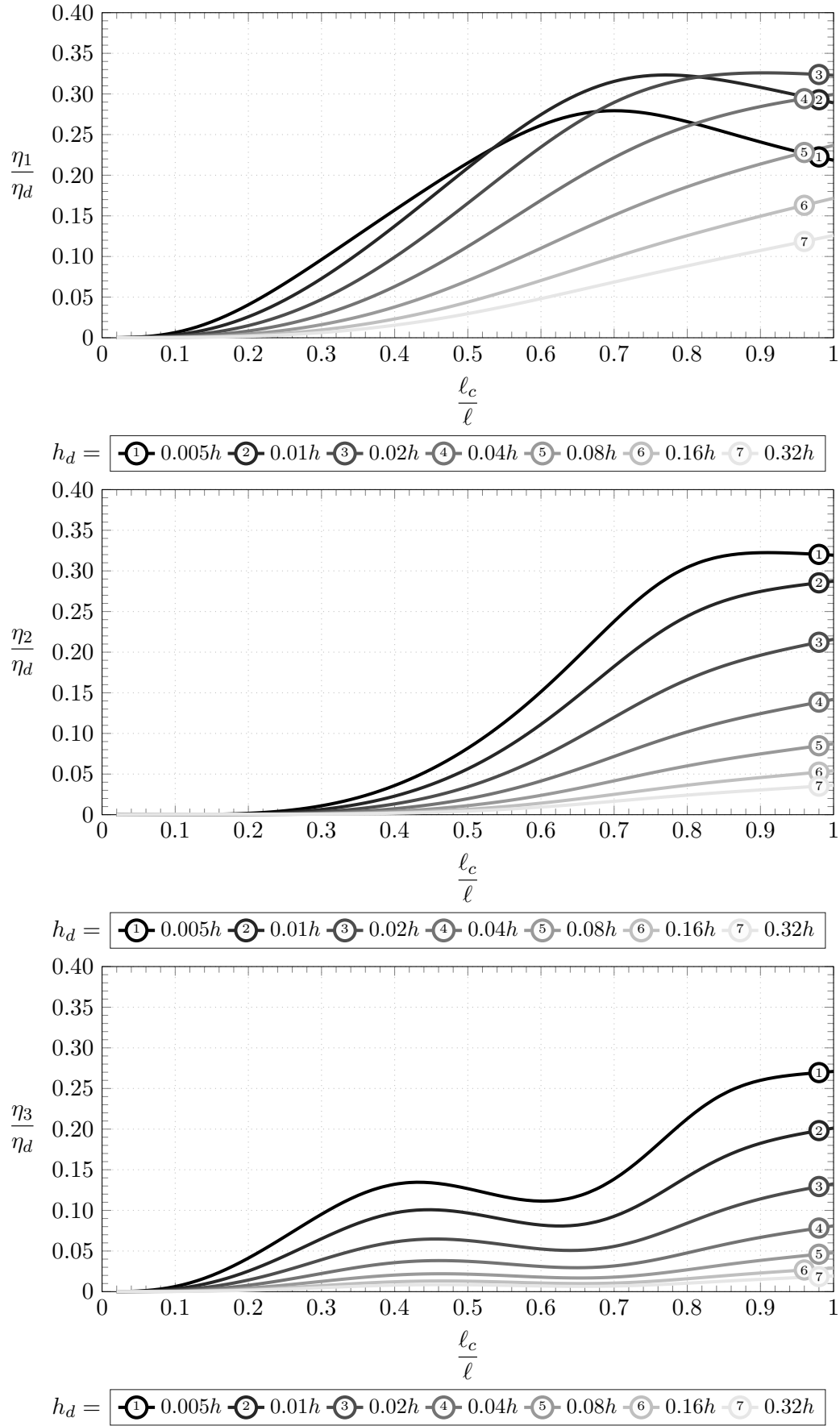
6.4.1.5 | Non-symmetric Configurations

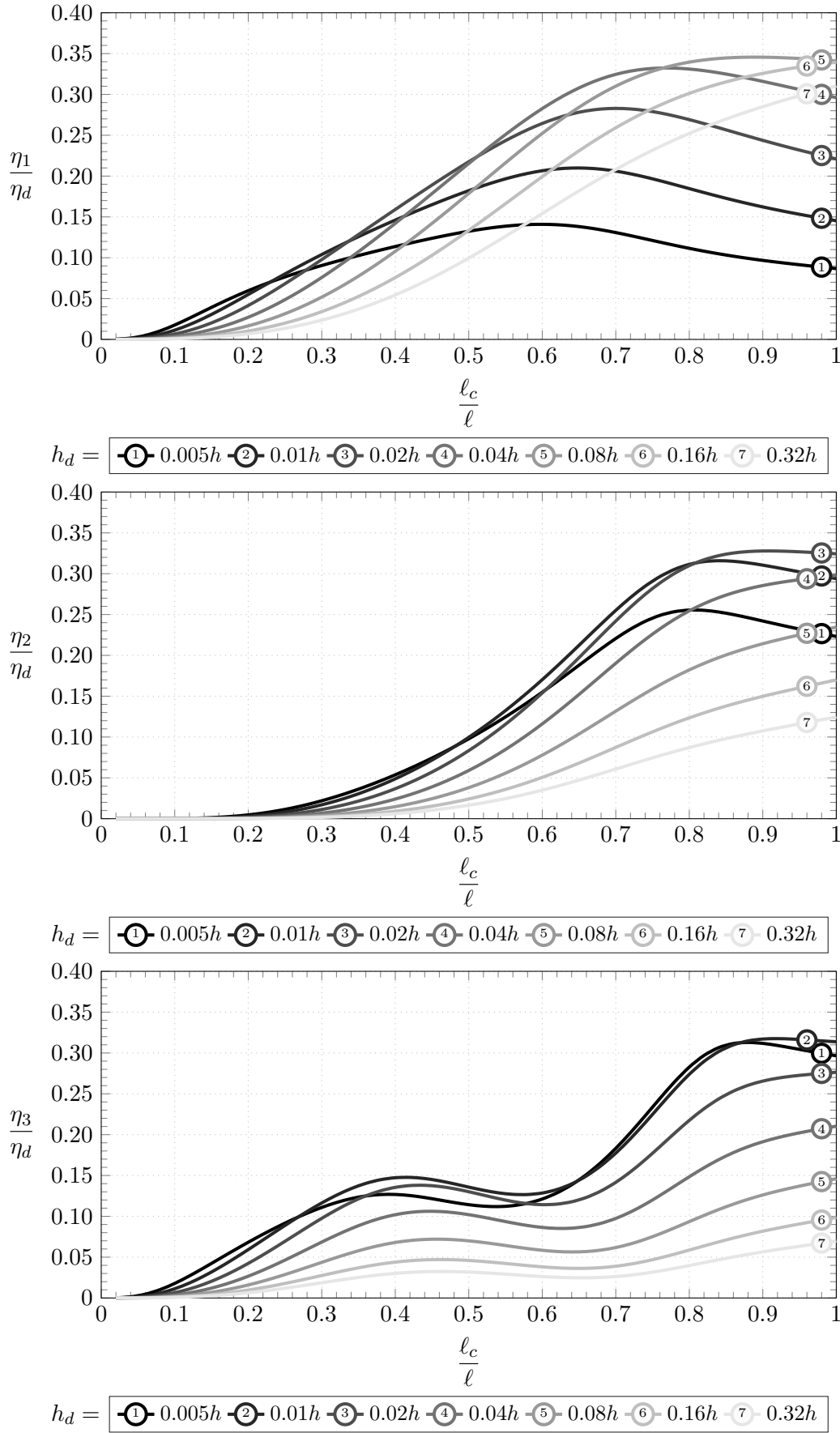
In non-symmetric configurations, where the base and constraining layers have different thicknesses, the damping layer no longer experiences higher modal transverse shear strains as in symmetric ones. In fact, as the difference between the base and constraining layer thicknesses increases, damping capacity drops. One can see this in Figures 6.70 and 6.71, depicting the influence of the covered length on the first three specific modal loss factors for $h_c = 0.5h$ and $h_c = 0.1h$, respectively. Nevertheless, it is possible to achieve optimal damping performances with partially covered beams, and Figures 6.70 and 6.71 show that this optimum covered length for a given damping layer thickness decreases as the constraining layer thickness decreases.

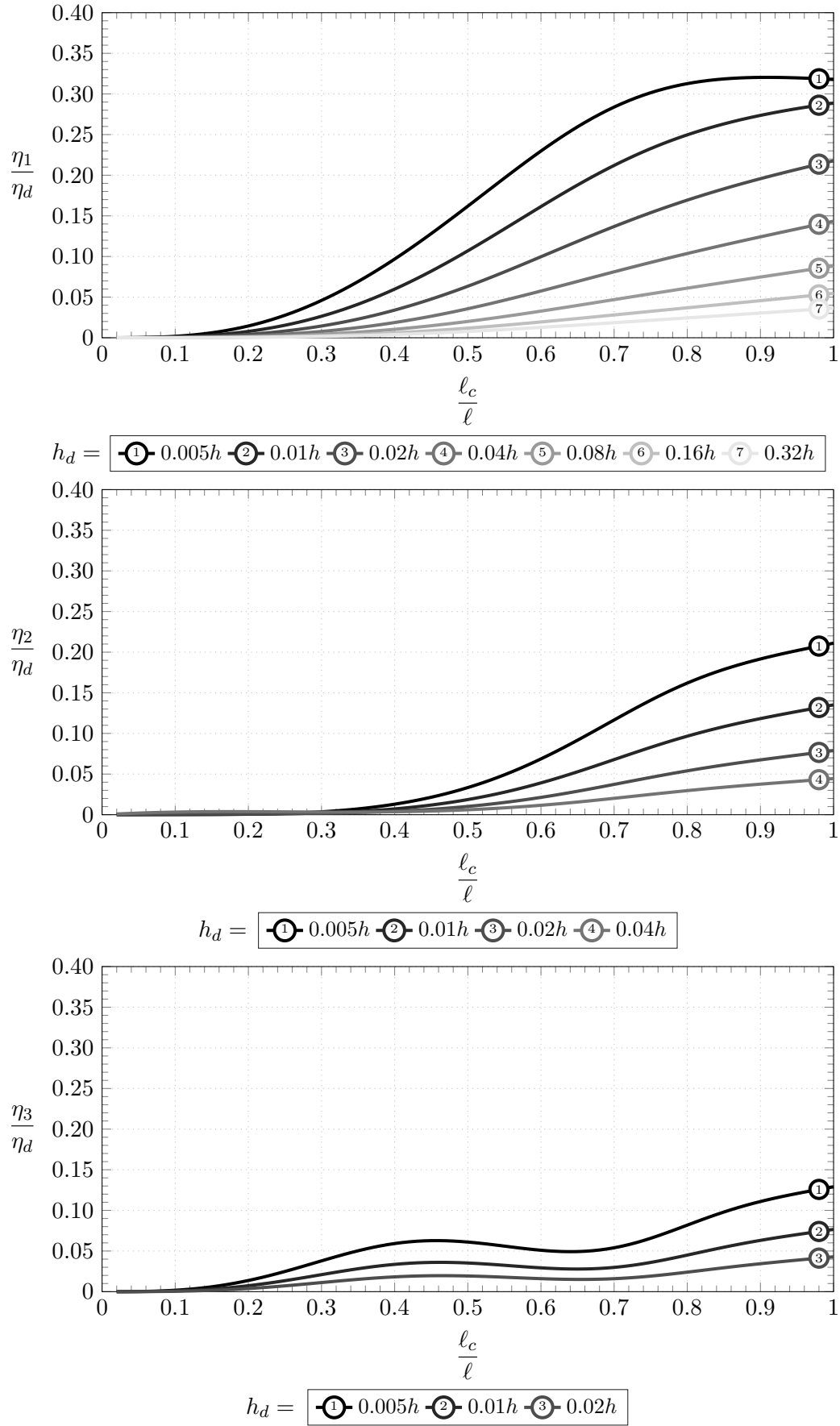
6.4.1.6 | Influence of Material Properties

To explore the importance of the material properties on the damping capacity of partially covered free-free beams, Figure 6.72 presents the influence of the modulus ratio on a symmetric reference beam with $h = 0.01\ell$ and a constant damping layer thickness of $h_d = 0.01h$. For long-covered lengths, the first specific modal loss factor increases with the modulus ratio and decreases with increasing modulus ratios for small-covered lengths. There is an increasing-decreasing effect and a turning point of modulus ratio for long-covered lengths, after which damping capacity decreases for higher modulus ratios. The second and third specific modal loss factors also show the increasing-decreasing effect and the turning point of the modulus ratio, after which damping capacity decreases for any covered length.

One requires higher modulus ratios to improve damping capacity when dealing with thinner base beams - Figure 6.73. The turning point of the increasing-decreasing behaviour of damping capacity occurs for higher modulus ratios either for the first or higher specific modal loss factors. For thicker beams, the referred turning point occurs at lower ratios, and the optimum length of damping patches increases - Figure 6.74. A thick beam is stiffer than a thin one, considering the same materials, so higher modulus ratios tend to decouple the base and constraining layers, decreasing the capacity of the constraining layer to impose shear on the damping material.


 Figure 6.65 Damping efficiency curves, $h = 0.01\ell$, $h_c = h$

Figure 6.66 Damping efficiency curves, $h = 0.005\ell$, $h_c = h$


 Figure 6.67 Damping efficiency curves, $h = 0.02\ell$, $h_c = h$

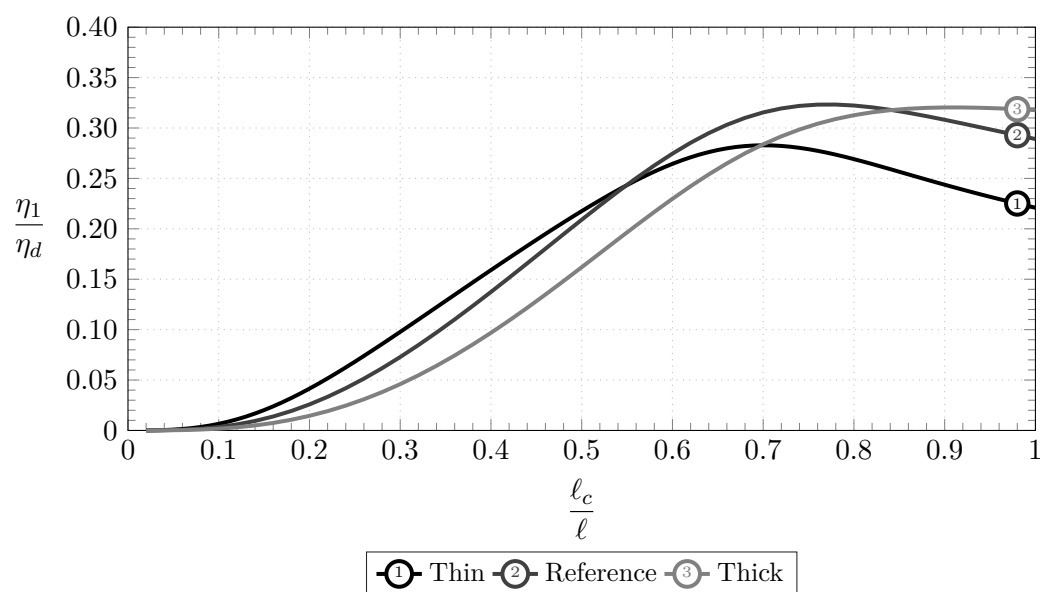


Figure 6.68 Damping efficiency curves, $h_d = 0.0001\ell$, $h_c = h$

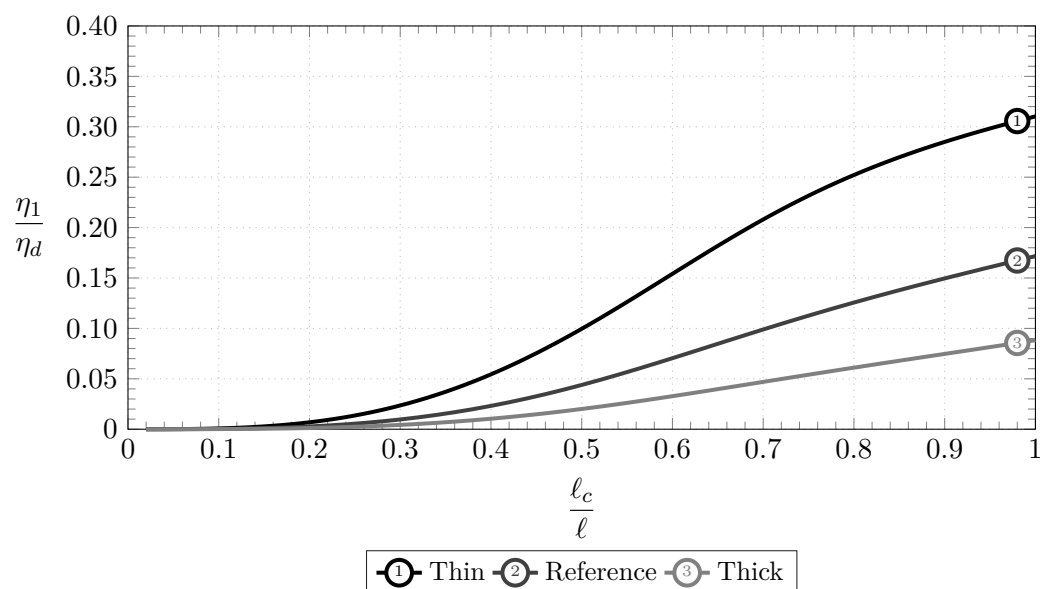
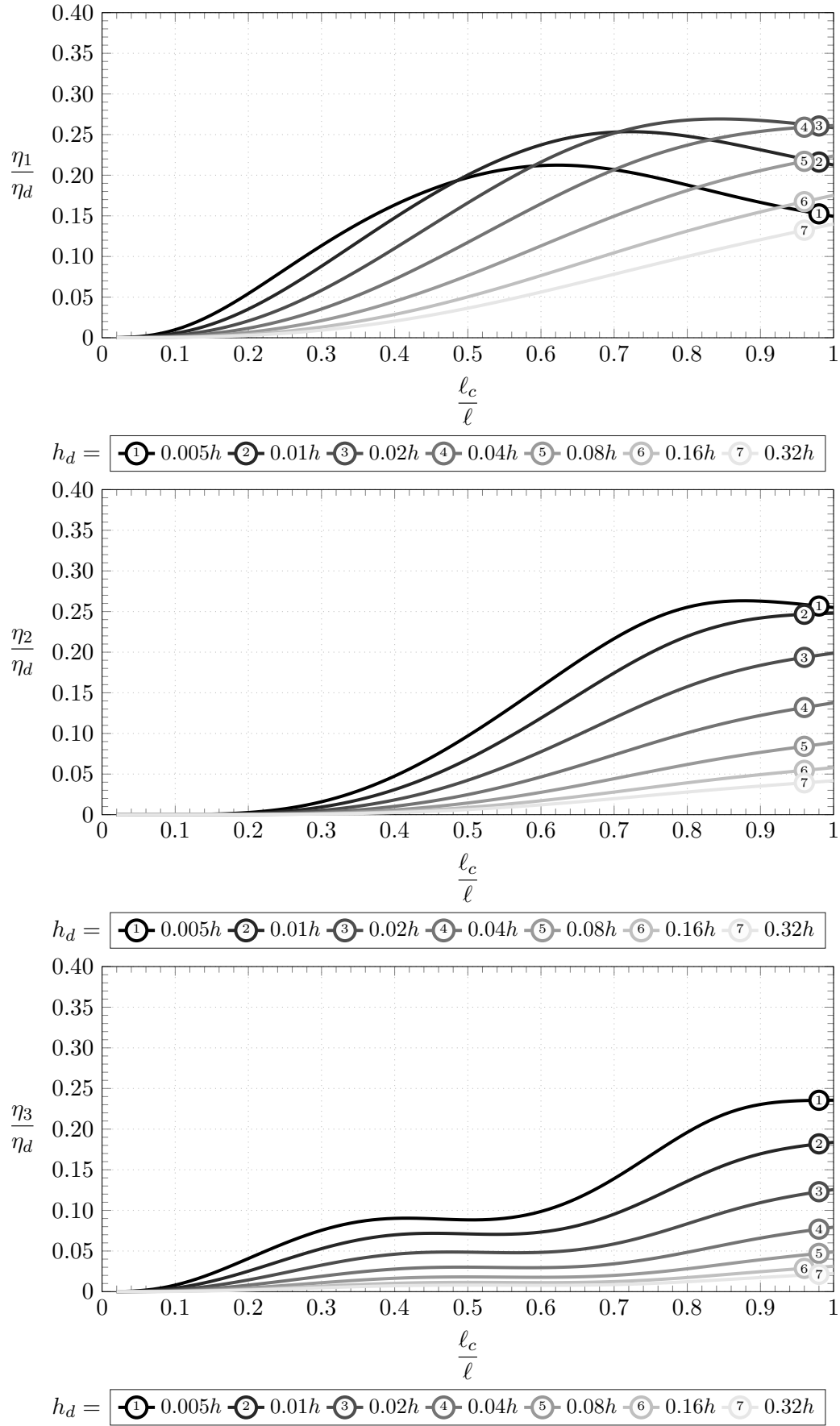
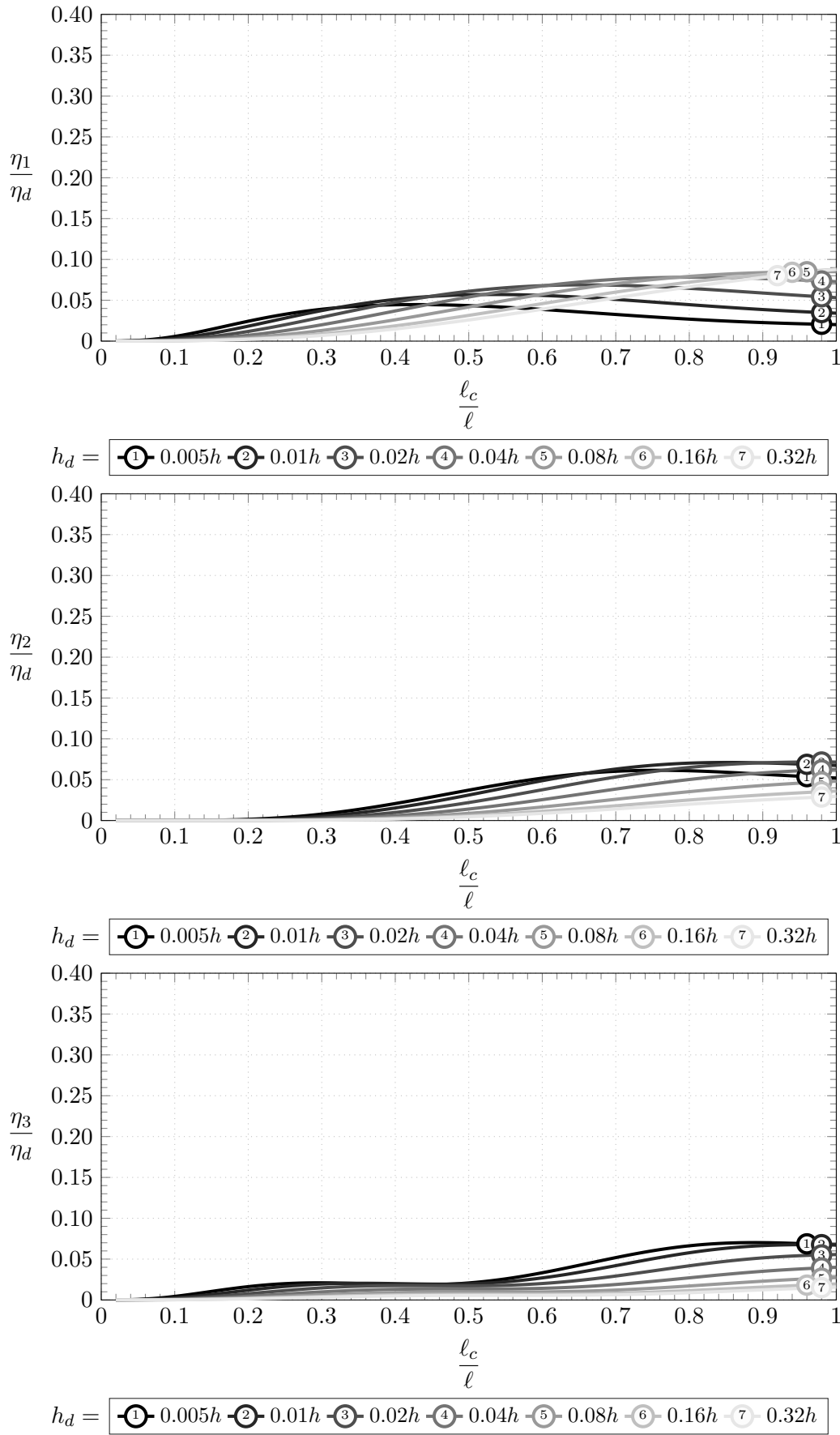
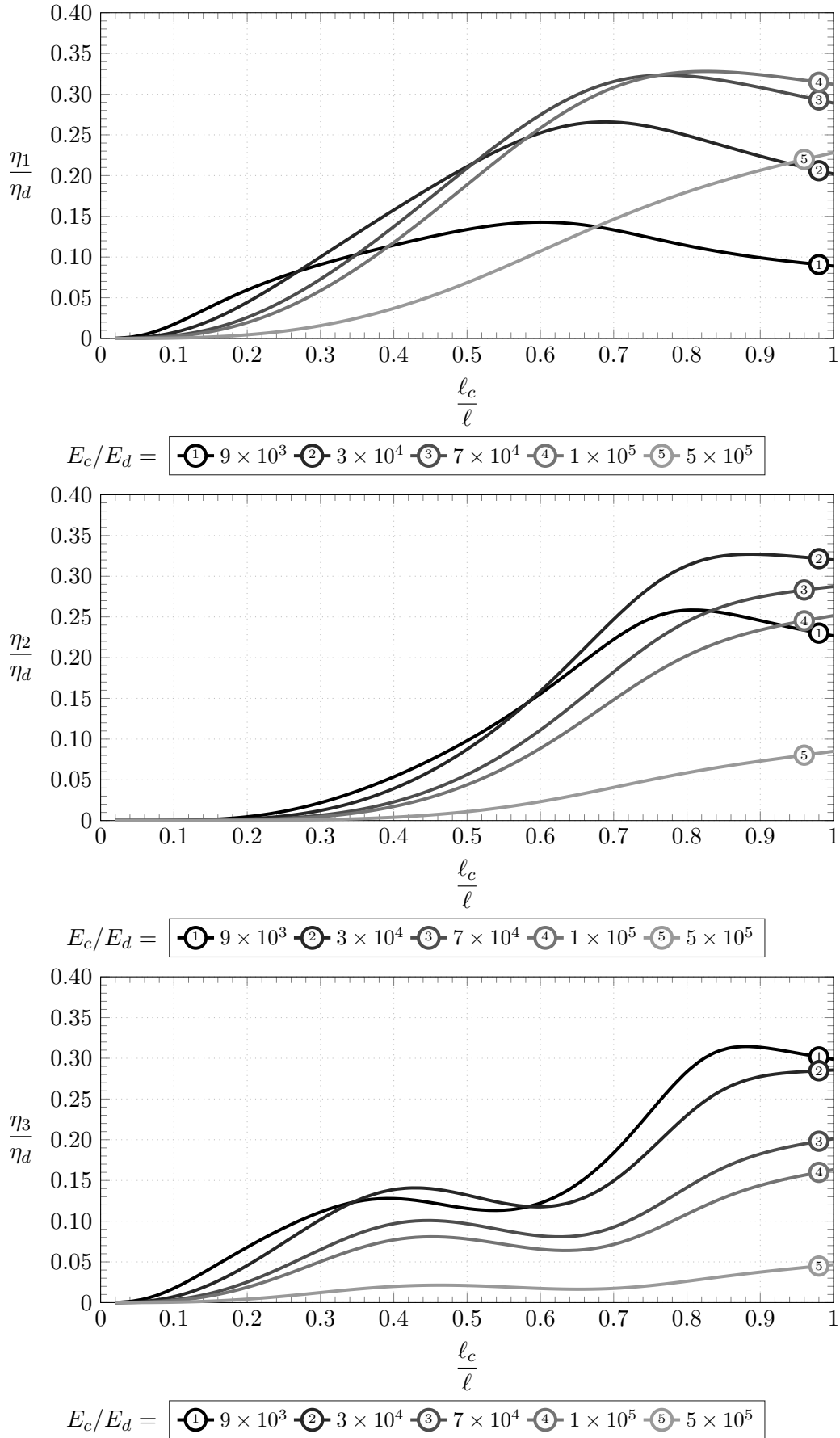


Figure 6.69 Damping efficiency curves, $h_d = 0.0016\ell$, $h_c = h$


 Figure 6.70 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.5h$

Figure 6.71 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.1h$


 Figure 6.72 Damping efficiency curves, $h = 0.01\ell$, $h_d = 0.01h$, $h_c = h$

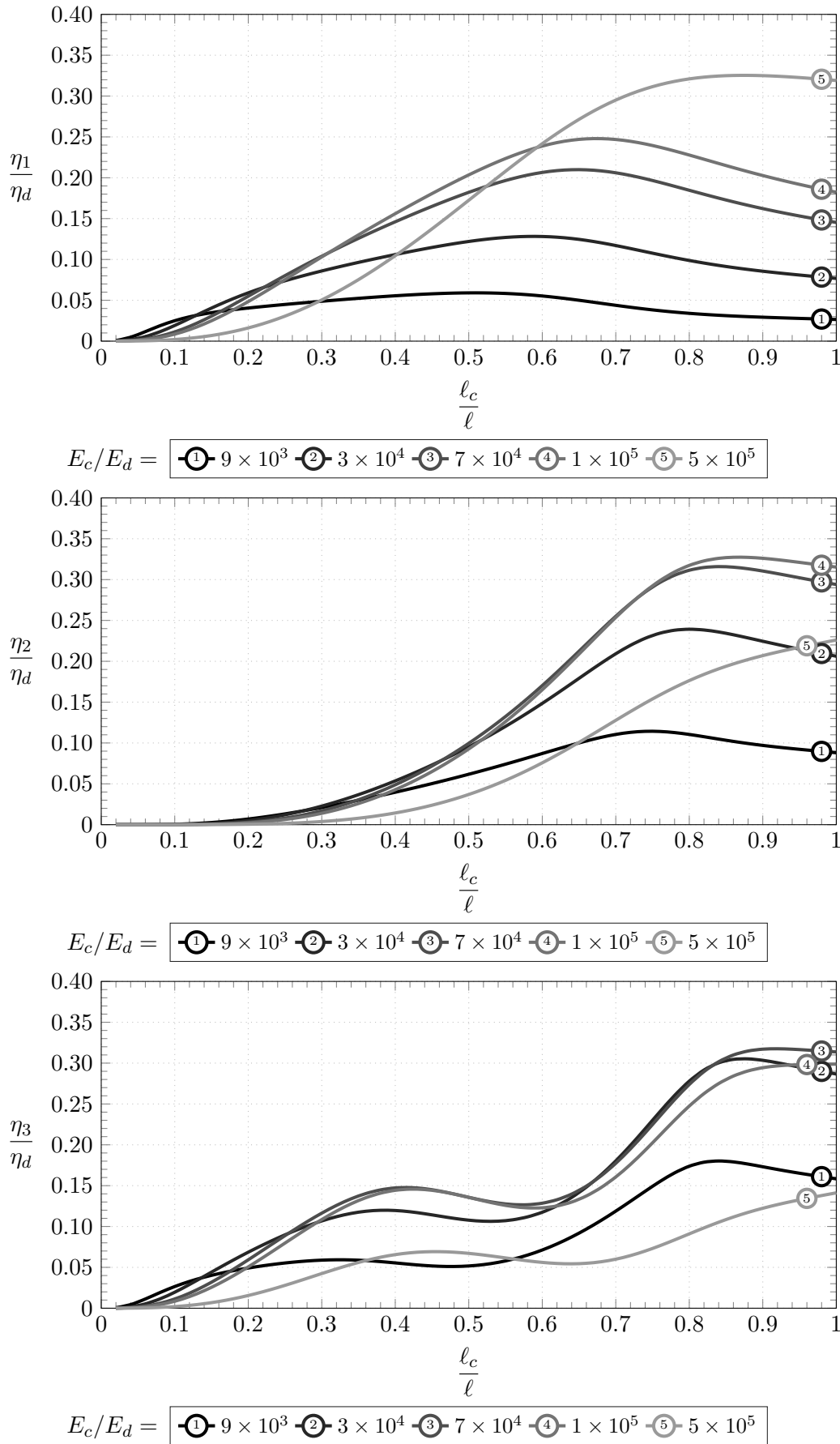
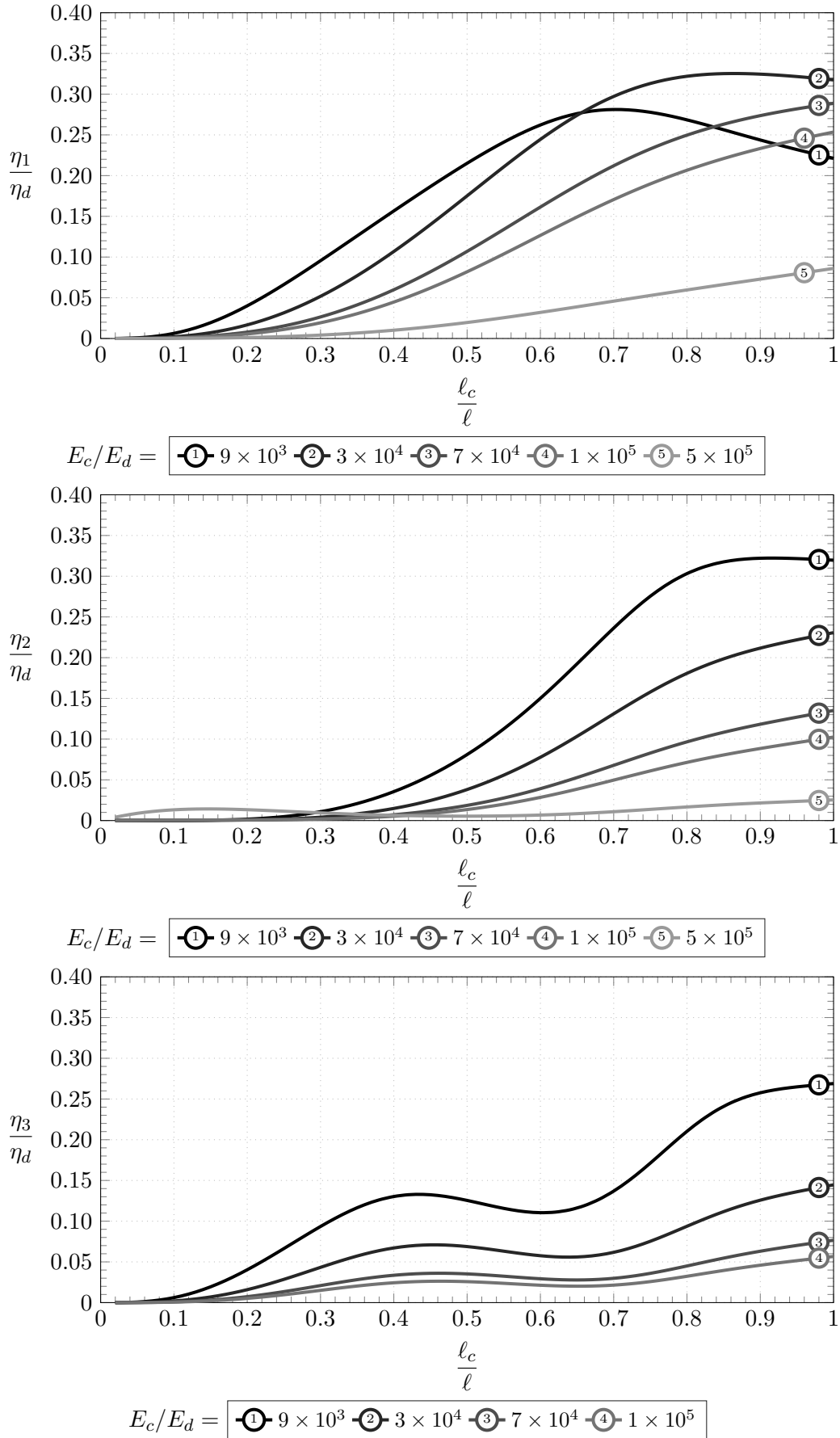


Figure 6.73 Damping efficiency curves, $h = 0.005\ell$, $h_d = 0.01h$, $h_c = h$


 Figure 6.74 Damping efficiency curves, $h = 0.02\ell$, $h_d = 0.01h$, $h_c = h$

6.5 | Damping Efficiency of Segmented Constraining Layers

Some authors have discussed cutting the constraining layer into appropriate lengths to increase transverse shear strain locally for increased damping performance. As mentioned in Section 6.1, Plunkett and Lee [1970] were among the pioneers of this concept studying, for example, the optimum distance between equidistant cuts of constrained viscoelastic treatments in beams. In addition, Plunkett and Lee [1970] studied the inclusion of multiple constrained viscoelastic layers to increase the treatment's damping capacity. Torvik and Strickland [1972] extended the work of Plunkett and Lee [1970] to plates by determining the energy dissipated by layers of viscoelastic material with unanchored constraining layers in a similar configuration. The authors calculated the total damping considering the energy dissipated by a characteristic element and then integrated it over the plate surface. Some recent works such as that of Lepoittevin and Kress [2010] have been performing optimisation studies concerning this technique, but in the author's perspective, these optimisation studies, even though being essential for important conclusions, tend to pass over important information regarding the mechanics of the damping treatment and do not allow to draw broad conclusions for other complex structures.

6.5.1 | Cantilever Beams

This section deals with the parametric study of cantilever beams with segmented constraining layers, as represented in Figure 6.75. It is important to emphasise that preliminary studies regarding segmented damping treatments proved that cutting the constraining and damping layers or just the constraining layer is the same regarding the treatment's damping capacity. Unlike other studies, one follows an approach considering just the cut of the constraining layer to prove the equivalent effectiveness of the damping treatment. To evaluate the influence of the cut position on the damping capacity, one considers a dimensionless analysis of an aluminium beam covered by a damping layer constrained by a segmented aluminium layer. One also considers the same geometric and material properties, listed in Tables 6.1 and 6.2, and a symmetric configuration ($h_c = h$) - Figure 6.75.



Figure 6.75 Cantilever beam with segmented constraining layer

Considering a FEM mesh of 100 elements along the length and the ToZZ, one performs the segmented constraining layer modelling by considering just two layers (base plus damping) at one specific element, thus defining the cutting position. Once the cut has the exact length of one element, for the example under analysis, one considers the cut with a constant length of 0.01ℓ . One performs a modal analysis considering the CE to estimate the modal loss factors.

6.5.1.1 | Reference Beam - $h = 0.01\ell$

Figure 6.76 presents the first three specific modal loss factors as a function of the cut position for different thicknesses of the damping layer. It is worth noting that one can significantly

improve damping capacity with segmented constraining layers. For the first specific modal loss factor, there is an optimum position for the cut section of the constraining layer as a function of the damping layer thickness. As the damping layer thickness increases, the optimal section for segmentation comes close to the free end section of the beam. Compared with thinner damping layers, as thickness increases, a cut in the constraining layer near the clamped section decreases the efficiency. One can notice here the increasing-decreasing effect of damping capacity already discussed for partially covered beams; there is a turning point of damping layer thickness after which it is not just worthless but prejudicial to cut the constraining layer. Moreover, it is crucial to emphasise that the optimal cut position for the thinner damping layers is the worst position for the thicker ones.

Higher specific modal loss factors are also worth analysing. Thinner damping layers improve the damping capacity by cutting the constraining layer, but there is a threshold after which to cut the constraining layer only degrades the treatment efficiency. This result proves the importance of parametric studies to understand that the difference between optimum and poor solutions is often small. In particular, a cut right after the beam's half-length degrades the damping capacity at the second specific modal loss factor, and a cut at one-third or three-quarters of the beam's length degrades the treatment efficiency for the third specific modal loss factor, regardless of the damping layer thickness.

To better understand these phenomena, Figure 6.77 presents the first three transverse mode shapes of the cantilever beam with full treatment, considering a damping layer thickness of $0.005h$. The cut of the constraining layer near the anti-nodes directly relates to degraded damping performances. In addition, Figure 6.78 presents the first three extensional strain mode shapes. Figures 6.79 and 6.80 present the first three extensional stress and strain energy density mode shapes, respectively. Extensional strain, stress or strain energy density mode shapes openly show that cuts of the constraining layer at sections of maximum extensional strain lead to the worst damping performances. The extensional strain of the constraining layer promotes the transverse shear strain of the damping layer, as previously discussed. Consequently, one degrades the damping performance by cutting the constraining layer at sections with higher extensional strains, breaking the capacity to impose shear on the damping layer. That is clear for the second mode when considering a cut near the beam's half-length and for the third mode when considering cuts right at one-third or three-quarters of the beam's length, sections of a maximum extensional strain of the constraining layer.

Figures 6.81, 6.82 and 6.83 present the first three shear strain, stress and strain energy density mode shapes obtained from equilibrium equations, respectively. As expected, transverse shear strain at the soft damping layer is high, but it is interesting to note how the nodes of transverse shear strain, or stress, match the anti-nodes of the extensional counterpart.

To comprehend the mechanics behind the improved performance of segmented damping treatments, Figure 6.84 presents the first transverse shear strain mode shapes for fully covered beams, patched beams at the clamped region, and beams with segmented constraining layers. Figures 6.85 and 6.86 present the transverse shear stress and strain energy density mode shapes. The patched treatment covers 0.3ℓ of the structure, and the segmentation of the constraining layer occurs precisely at 0.3ℓ since it is the optimised position for a cut considering a damping layer with $h_d = 0.005h$ - Figure 6.76. The patch treatment provides a comparative example of the segmented constraining layer case. Figure 6.76 highlights that it is possible to increase the damping capacity of a damping treatment with the segmentation of the constraining layer; the optimum position of the cut is at 0.3ℓ for a damping layer of $h_d = 0.005h$. Figure 6.84 explains this result, showing the increased transverse shear strain at the damping layer created by the segmentation of the constraining layer. Transverse shear stress and shear strain energy density mode shapes represented in Figures 6.85 and 6.86 show that the damping layer of the segmented

treatment presents a higher shear strain/stress state than a full coverage or patch treatment.

As stated before, the extensional strain of the constraining layer plays a decisive role in its capacity to impose transverse shear strain on the damping layer. Figure 6.87 presents the extensional strain mode shapes for the full coverage, patch and segmented treatments, only considering membrane and bending deformations. Unsurprisingly, the segmented constraining layer experiences higher extensional strain when compared to fully covered or patched structures. Figure 6.88, with the extensional stress mode shapes, as well as Figure 6.89, with the extensional strain energy density mode shapes, emphasise the higher extensional strain/stress state at the constraining layer of the segmented treatment, being responsible for the higher transverse shear strain state of the damping layer and consequently the superior damping performance.

6.5.1.2 | Thin Beam - $h = 0.005\ell$

Similarly to partially covered beams, one also considers a symmetric beam with a thinner base and constraining layers. Figure 6.90 presents the same trends as those of the reference beam. However, one can notice slight differences for the optimum positions of the constraining layer cuts, and the damping capacity for higher damping layer thicknesses increases. Nevertheless, it is worth mentioning that the maximum damping capacity for the second mode, considering either the reference or the thin beam, is similar and occurs roughly for the same amount of damping material.

At this point, one should notice the exception that breaks the rule mentioned for the reference beam that one must avoid cuts of the constraining layer at sections of maximum extensional strain. For the second mode of the thin base beam with thinner damping layers, $h_d = 0.005h$, the cut near the section with the maximum extensional strain improves the specific modal loss factor. This result explains the rule of thumb raised by Lepoittevin and Kress [2010], recommending segmentation of the damping treatment at sections with the maximum bending moment, a conflicting conclusion with the results presented in this work. Lepoittevin and Kress [2010] performed the optimisation of the segmentation sections considering constant thicknesses of either base, damping and constraining layers, as well as different mechanical properties, which may well have led to a particular conclusion. The optimum position for segmentation is not always the section with the maximum bending moment, as this is just the case for particular cases such as thin base beams with thin damping layers.

6.5.1.3 | Thick Beam - $h = 0.02\ell$

Figure 6.91 presents the damping efficiency of segmented thick constraining layers in symmetric configurations. For the first specific modal loss factor, the optimum cut positions for thin damping layers are closer to the beam's free end when compared to the reference beam. Once again, when the damping layer thickness exceeds the threshold of the increasing-decreasing behaviour, segmentation is worthless and downgrades the damping capacity. For higher specific modal loss factors, there is no real benefit from the segmentation of the constraining layer, and there is a section, virtually independent of the damping layer thickness, at which the damping capacity decreases drastically. Figure 6.91 does not consider thicker damping layers, as the mode shape changes, rendering the comparison meaningless.

6.5.1.4 | Non-symmetric Configurations

In the case of non-symmetric configurations, as represented in Figure 6.92 for segmented constraining layers with half the thickness of the base layer, the influence of the cut positioning on the damping capacity is similar to the symmetric reference beam for the three specific modal

loss factors. The main difference, as previously noticed for partial damping treatments, is that thinner constraining layers have limited capacity to impose shear strain in the damping layer, and the overall damping capacity decreases.

Considering even thinner constraining layers, $h_c = 0.01h$, the capacity to impose shear on the damping layer drops significantly, even with similar trends - Figure 6.93. Still, it is worth emphasizing an aspect of the second specific modal loss factors, zoomed in for a better analysis. Despite the degraded damping performance, and in contrast to thicker damping layers, it is possible to increase the damping capacity by segmenting the constraining layer at sections of higher extensional strain when the damping layer is thin.

6.5.1.5 | Comparison with Partial Coverage

The comparison between segmented constraining layers and partial treatments is worth mentioning. Figure 6.94 shows the overlapping of both configurations for a symmetric reference beam with damping layers of different thickness - dashed lines for segmented constraining layers, marked with the letter S, and continuous ones for partial damping treatments, marked with the letter P. Focusing on the first specific modal loss factor and thinner damping layers, the superior performance of a segmented constraining layer treatment near the clamped section comparing to a partial treatment is unquestionable, but so the additional weight.

However, as the length of the patch and the cut of the constraining layer move towards the beam's free end, there is a "focal" point after which the first specific modal loss factors of both configurations are equivalent. The weight of the free-end side treatment in segmented treatments is needless. There is no shear strain in the damping layer at the free-end side after the cut of the constraining layer. Moreover, as the damping layer thickness increases and reaches the already mentioned turning point of the increasing-decreasing damping effect, differences between segmented and partial damping treatments become negligible and damping less efficient.

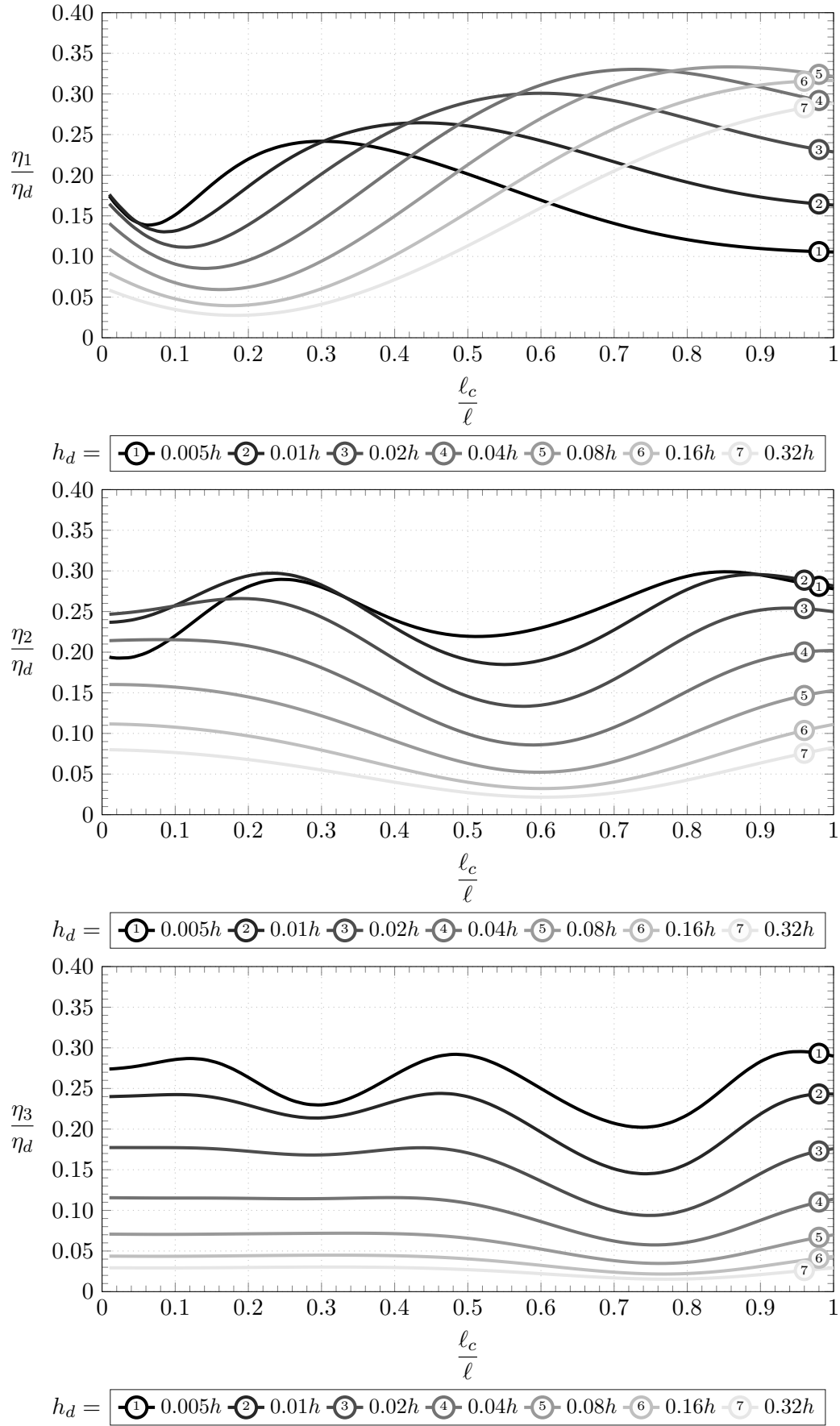
Higher specific modal loss factors in Figure 6.94 highlight the differences between both configurations for thin damping layers and smaller covered lengths but also show the poor damping performance of thick damping layers.

6.5.1.6 | Influence of Material Properties

To investigate the influence of the material properties on the damping capacity of segmented treatments, Figure 6.95 presents the influence of the modulus ratio on a symmetric reference beam with $h = 0.01\ell$ and a damping layer of $h_d = 0.01h$. The first specific modal loss factor shows that there is an optimum position for the cut section of the constraining layer as a function of the modulus ratio. As the modulus ratio increases and so does the difference in stiffness between both materials, the optimal section for segmentation comes close to the free end section of the beam. By contrast, as the modulus ratio increases, a cut in the constraining layer near the clamped region decreases the efficiency. Besides, these results show that segmentation of the constraining layer is more effective as the modulus ratio decreases. As the difference in stiffness between both materials decreases, there is less shear in the damping layer and damping capacity, but the coupling between the base and constraining layers increases, and segmentation has a higher impact in promoting shear. Moreover, one can notice for higher specific modal loss factors the increasing-decreasing behaviour of damping capacity; there is an optimum value of modulus ratio for improved damping capacity. Furthermore, the second and third specific modal loss factors show, once again, the decreased damping performance when segmentation occurs at sections of maximum extensional strain energy density except for the second modal loss factor for the lowest modulus ratio.

Figure 6.96 presents the influence of the modulus ratio on the damping capacity for thinner beams, and the specific modal loss factors show roughly the same trends as the reference beam. However, there are some aspects to point out. The overall damping capacity of the first modal loss factor decreases, but segmentation becomes more effective in this mode for thinner beams. In addition, the exception to the rule that one must avoid segmentation at the sections of higher extensional modal strain occurs for different modulus ratios in the second mode and the lowest modulus ratio in the third mode.

In contrast, Figure 6.97 shows the influence of the modulus ratio on the damping capacity for thicker beams. Segmentation becomes less effective, and, once again, these damping efficiency curves show that segmenting the constraining layer at sections of higher modal extensional strain is prejudicial.


 Figure 6.76 Damping efficiency curves, $h = 0.01\ell$, $h_c = h$

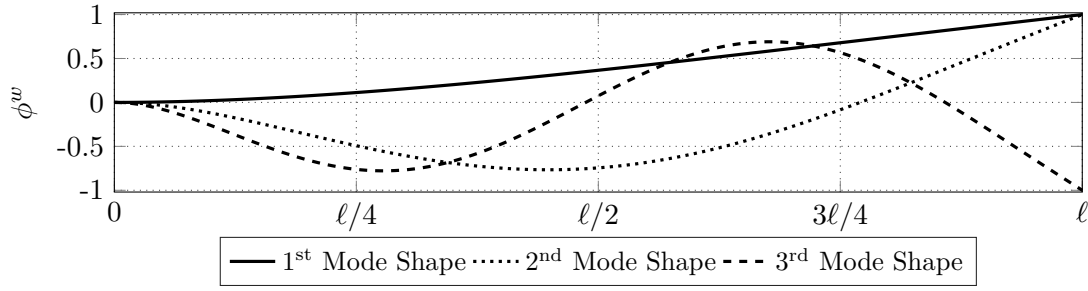
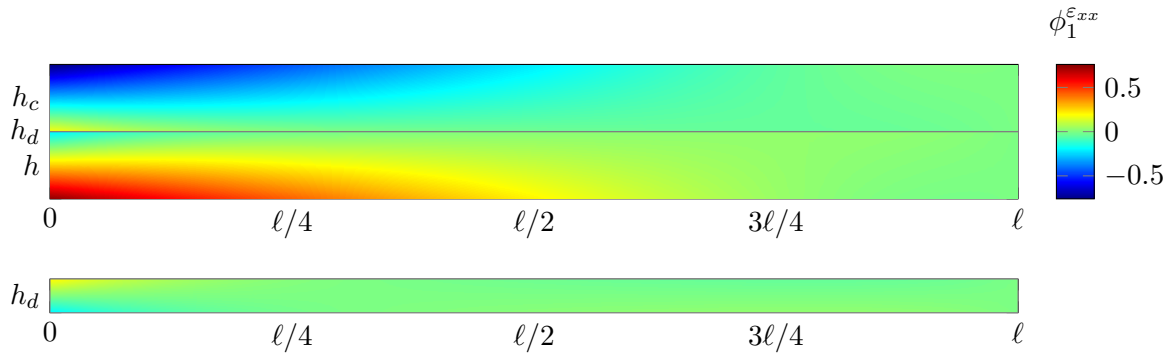
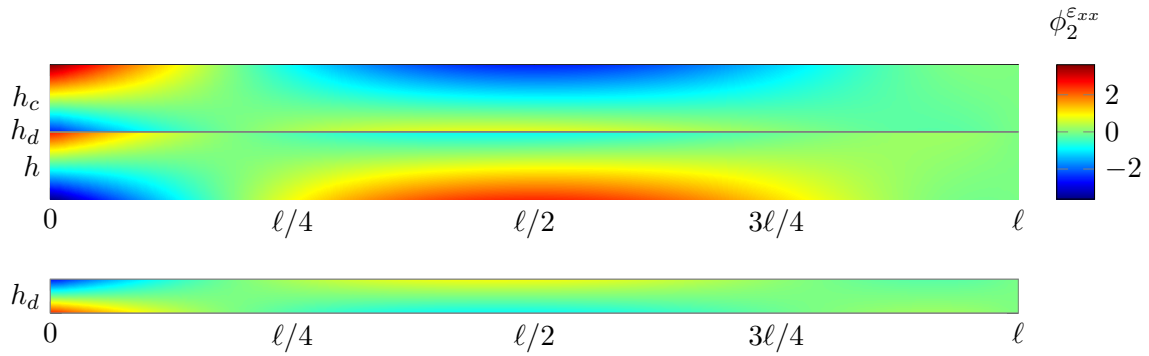


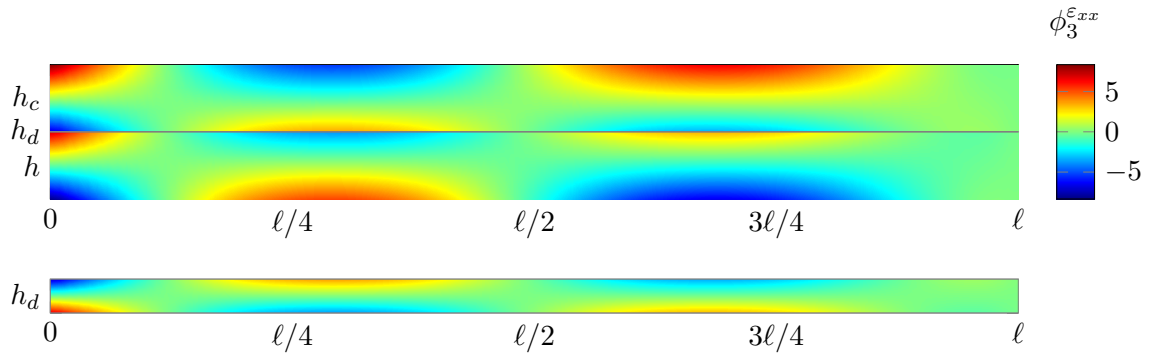
Figure 6.77 First three transverse displacement mode shapes of the cantilever beam with full treatment



(a) 1st normal strain mode shape



(b) 2nd normal strain mode shape



(c) 3rd normal strain mode shape

Figure 6.78 Normal strain mode shapes of a cantilever beam with full coverage

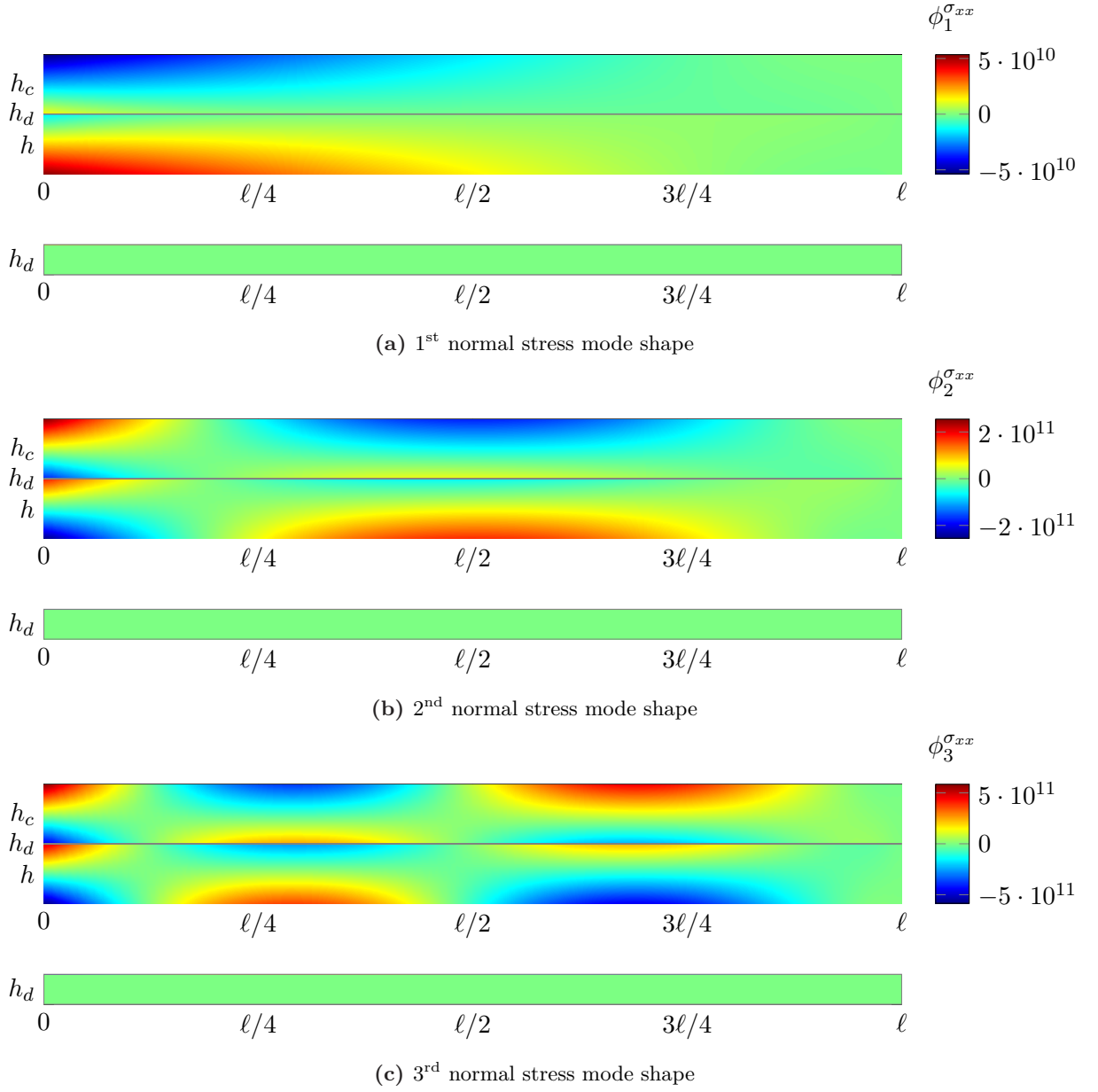


Figure 6.79 Normal stress mode shapes of a cantilever beam with full coverage

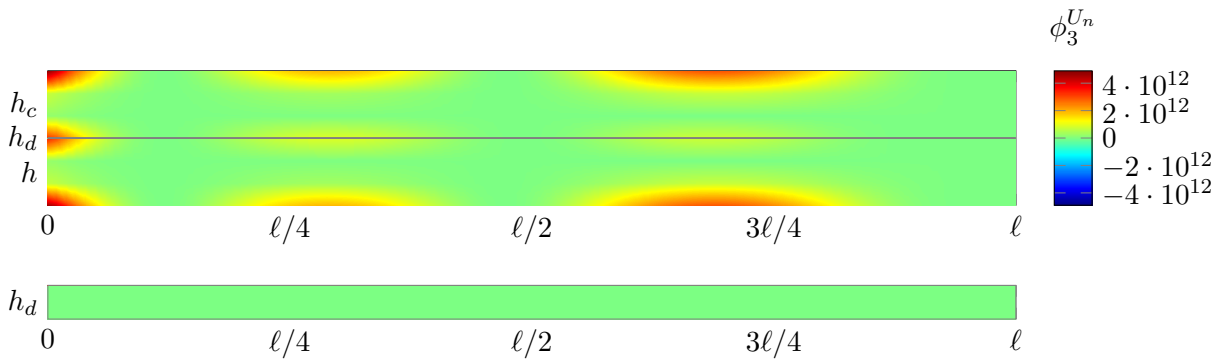
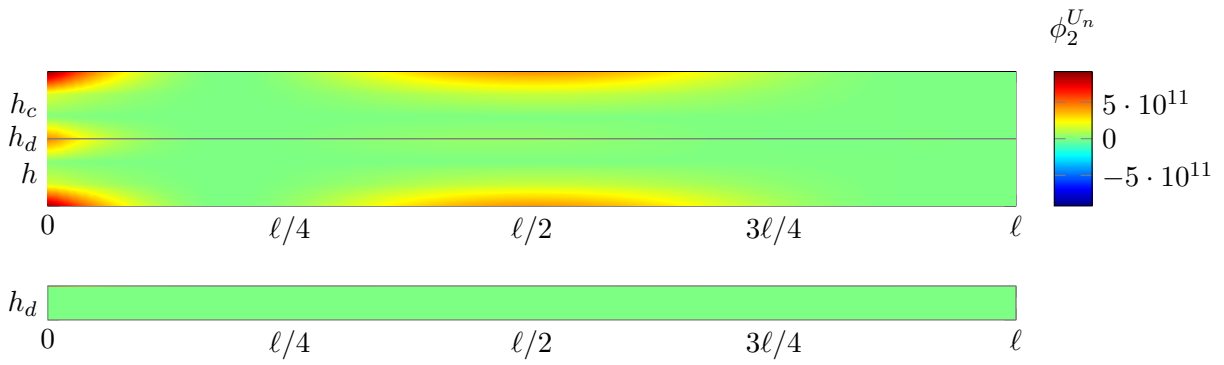
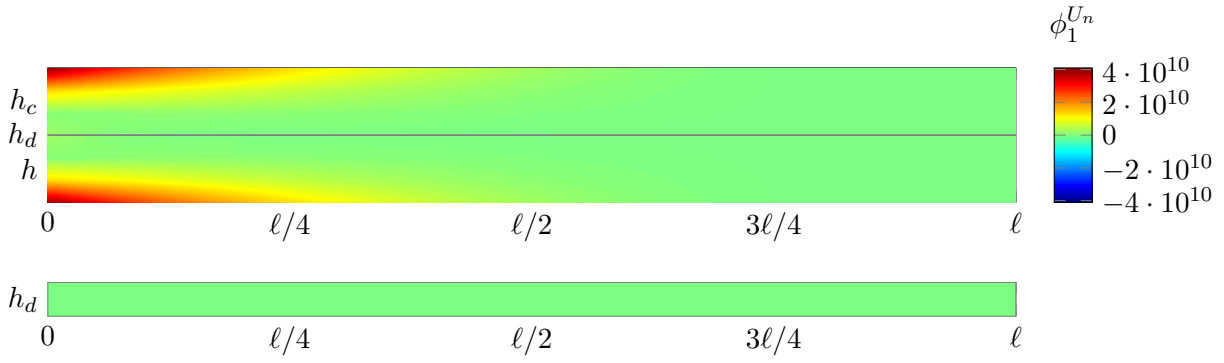


Figure 6.80 Normal strain energy density mode shapes of a cantilever beam with full coverage

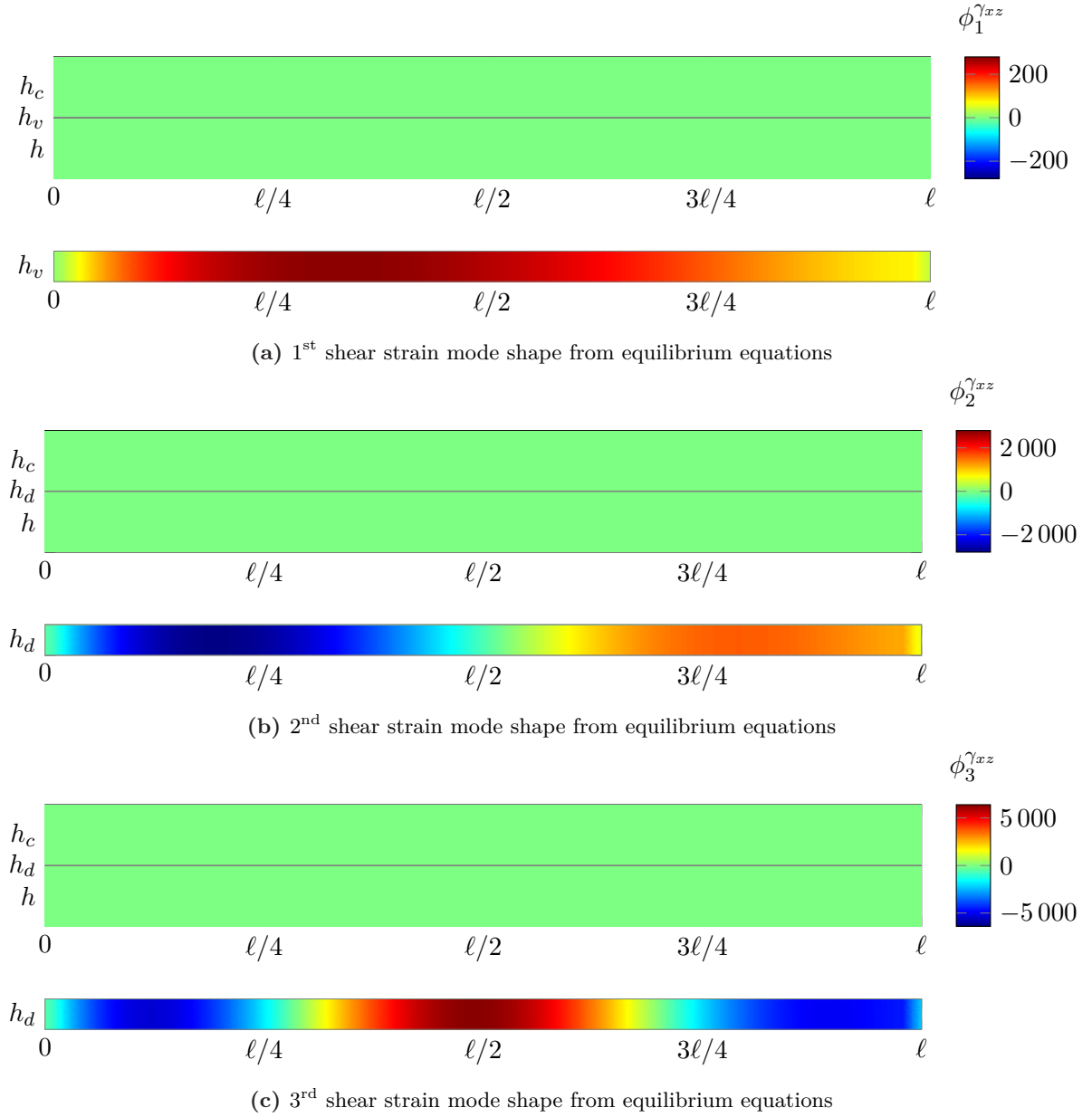


Figure 6.81 Shear strain mode shapes of a cantilever beam with full coverage

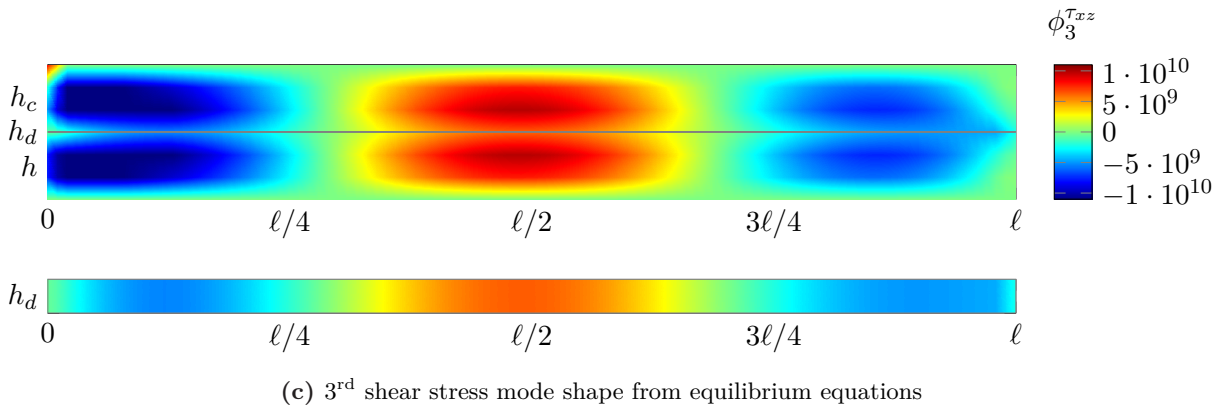
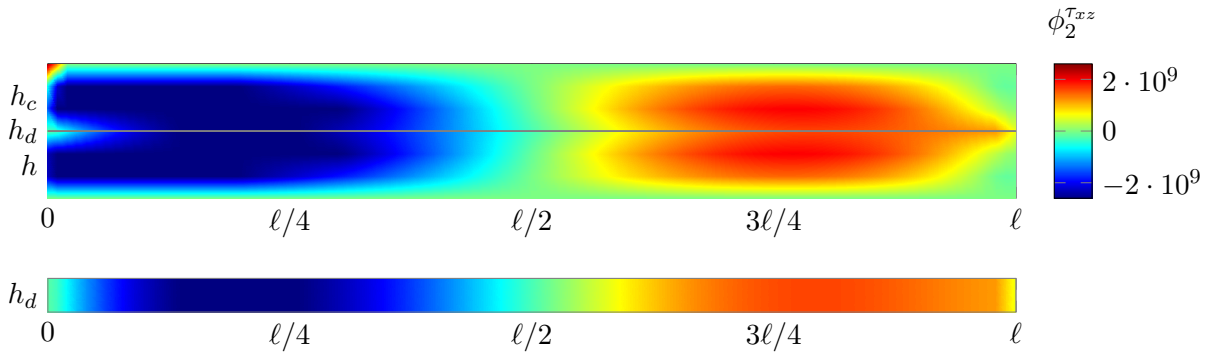
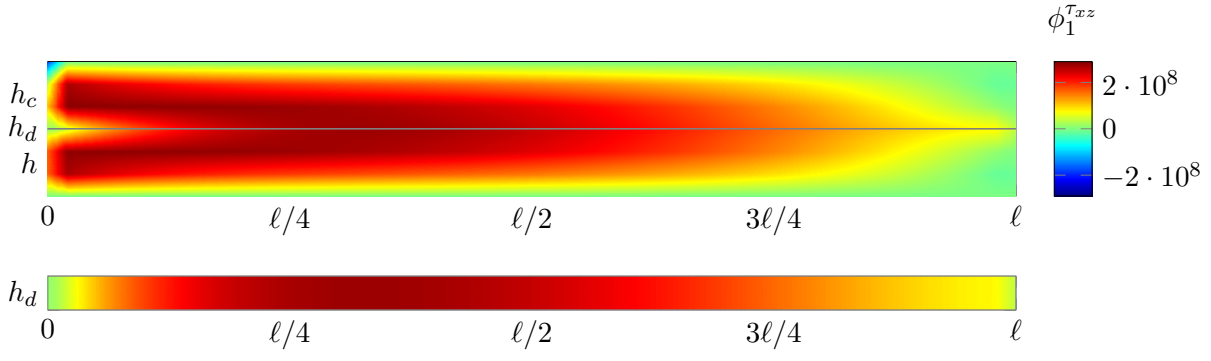


Figure 6.82 Shear stress mode shapes of a cantilever beam with full coverage

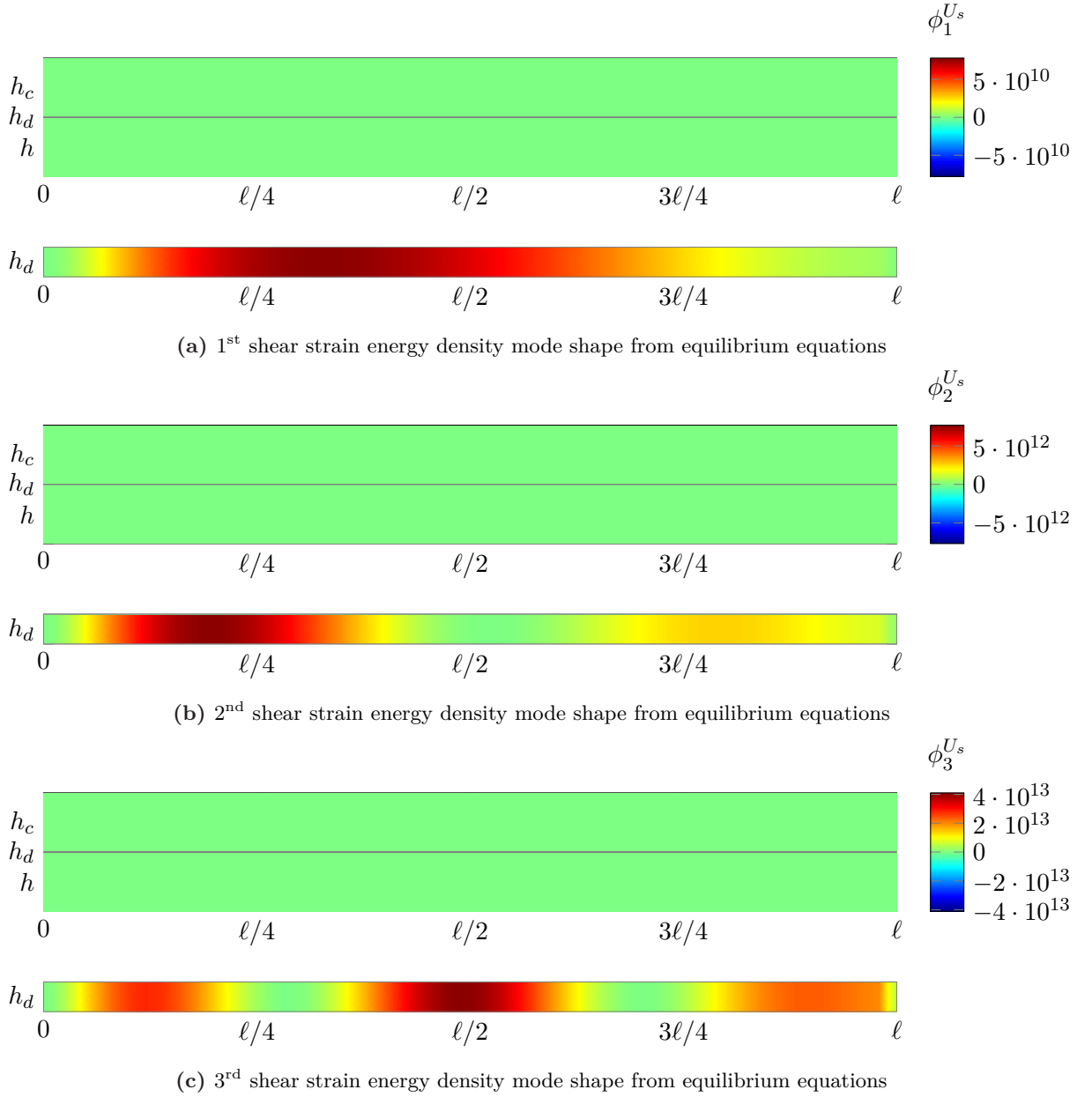


Figure 6.83 Shear strain energy density mode shapes of a cantilever beam with full coverage

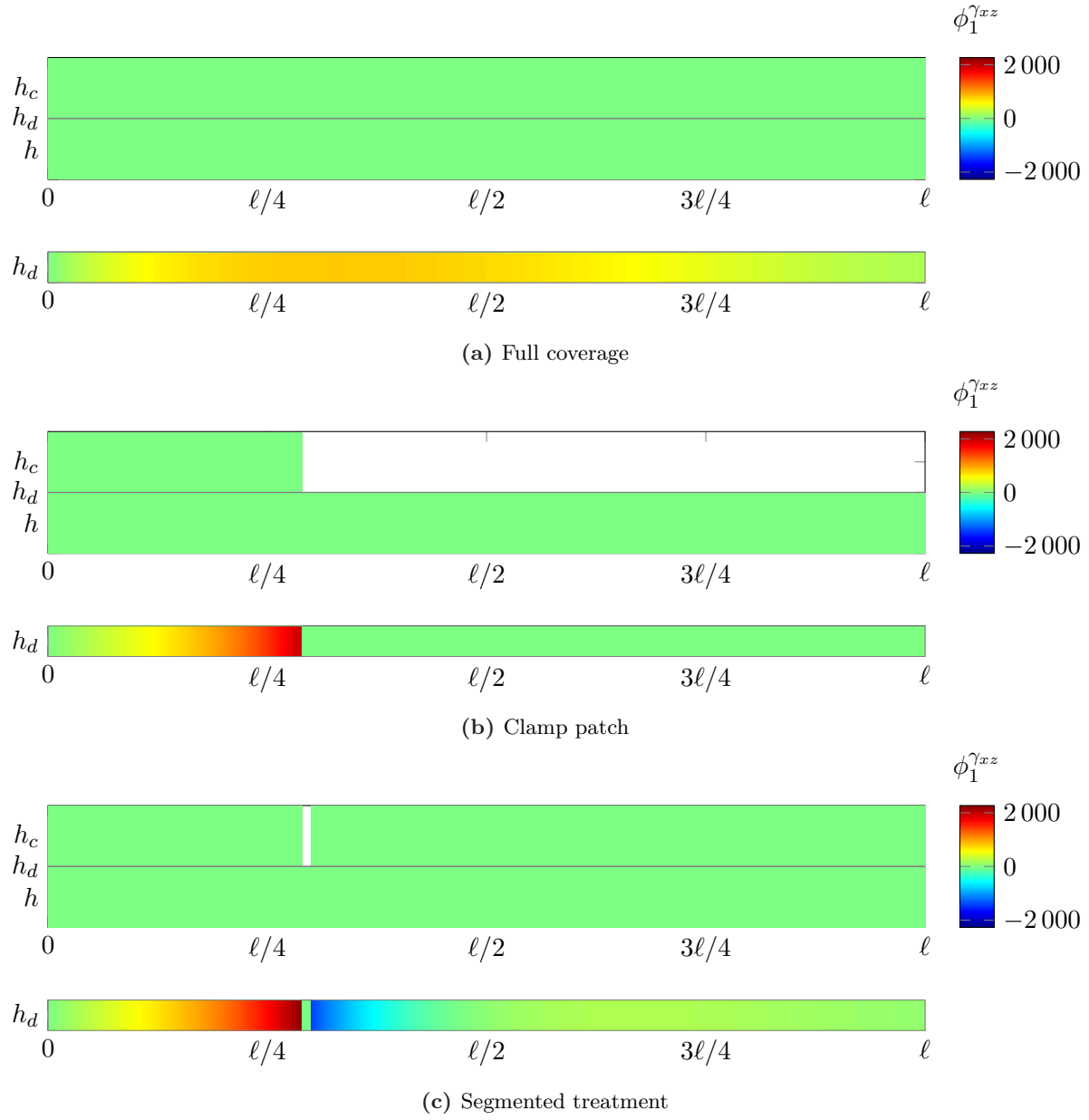


Figure 6.84 1st shear strain mode shapes of a cantilever beam with different treatments

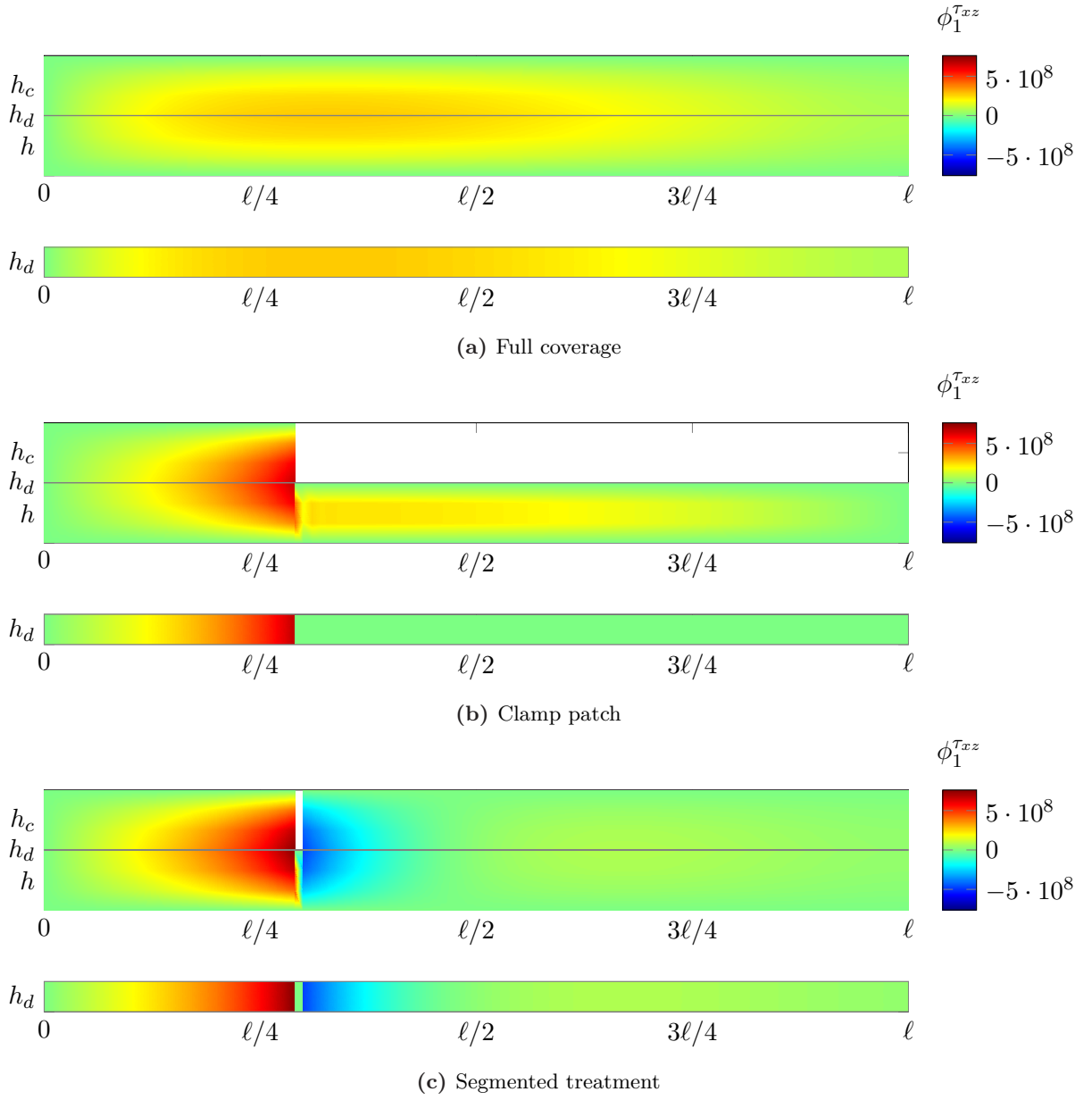


Figure 6.85 1st shear stress mode shapes of a cantilever beam with different treatments

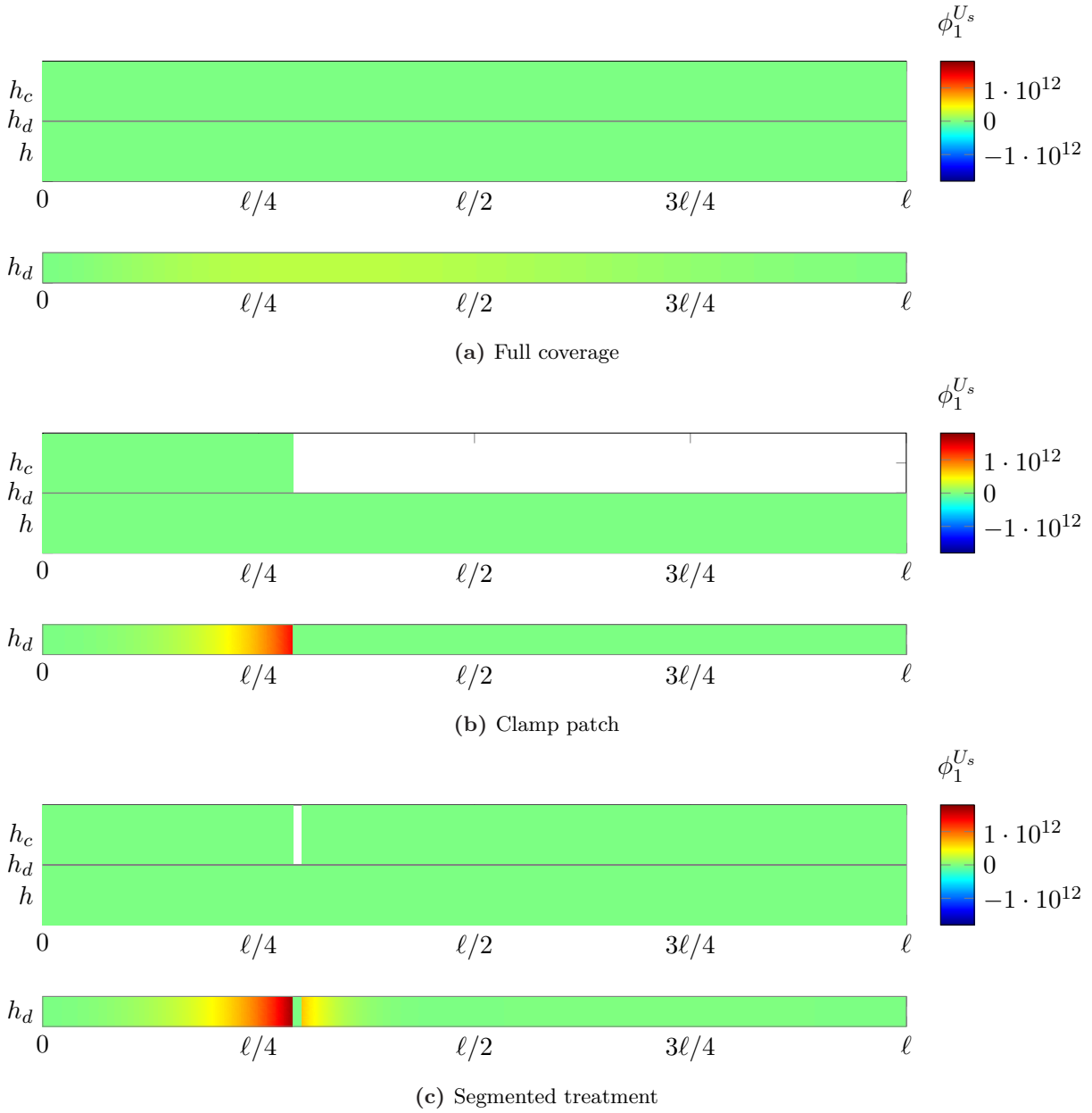


Figure 6.86 1st shear strain energy density mode shapes of a cantilever beam with different treatments

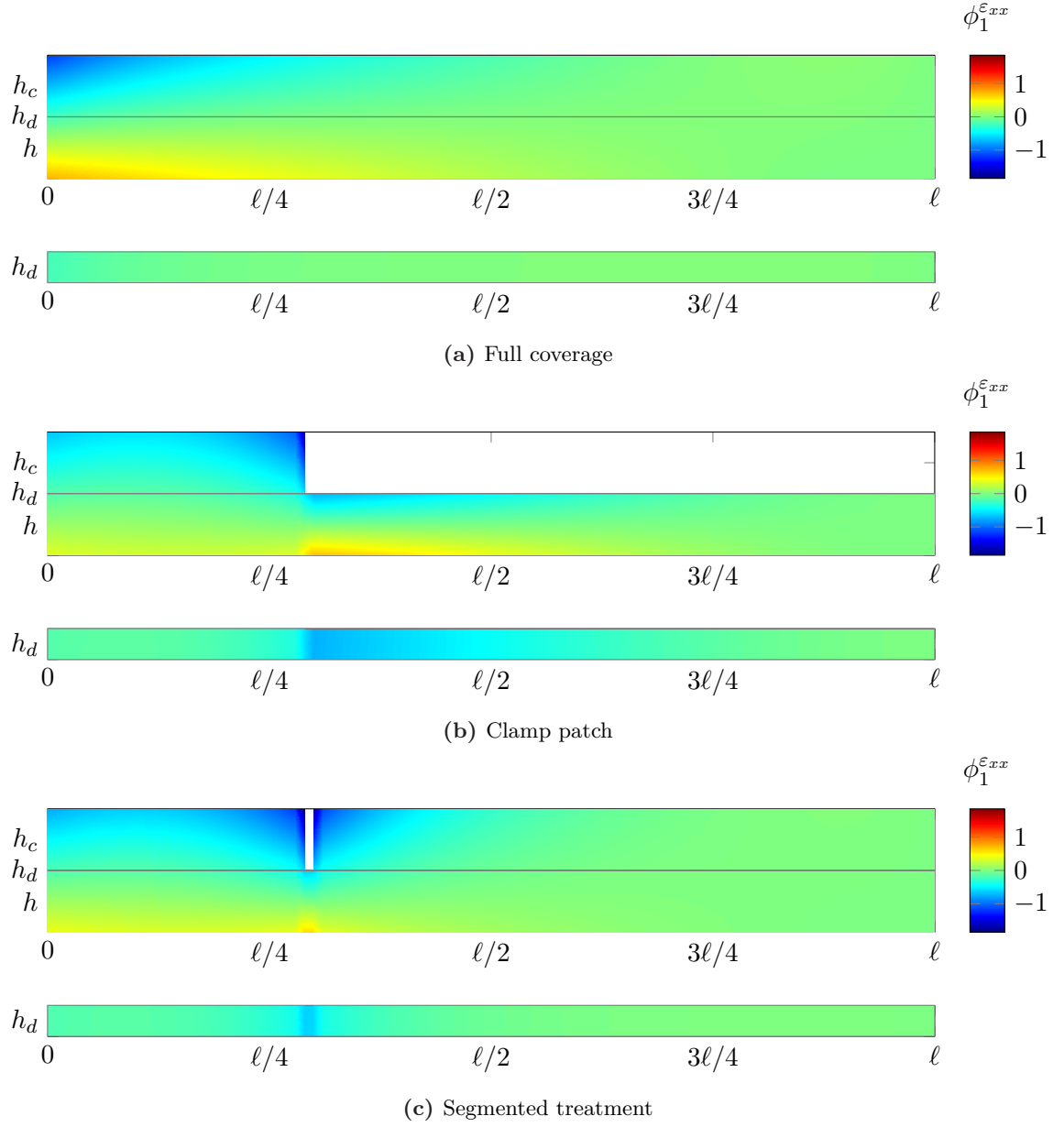


Figure 6.87 1st normal strain mode shapes of a cantilever beam with different treatments

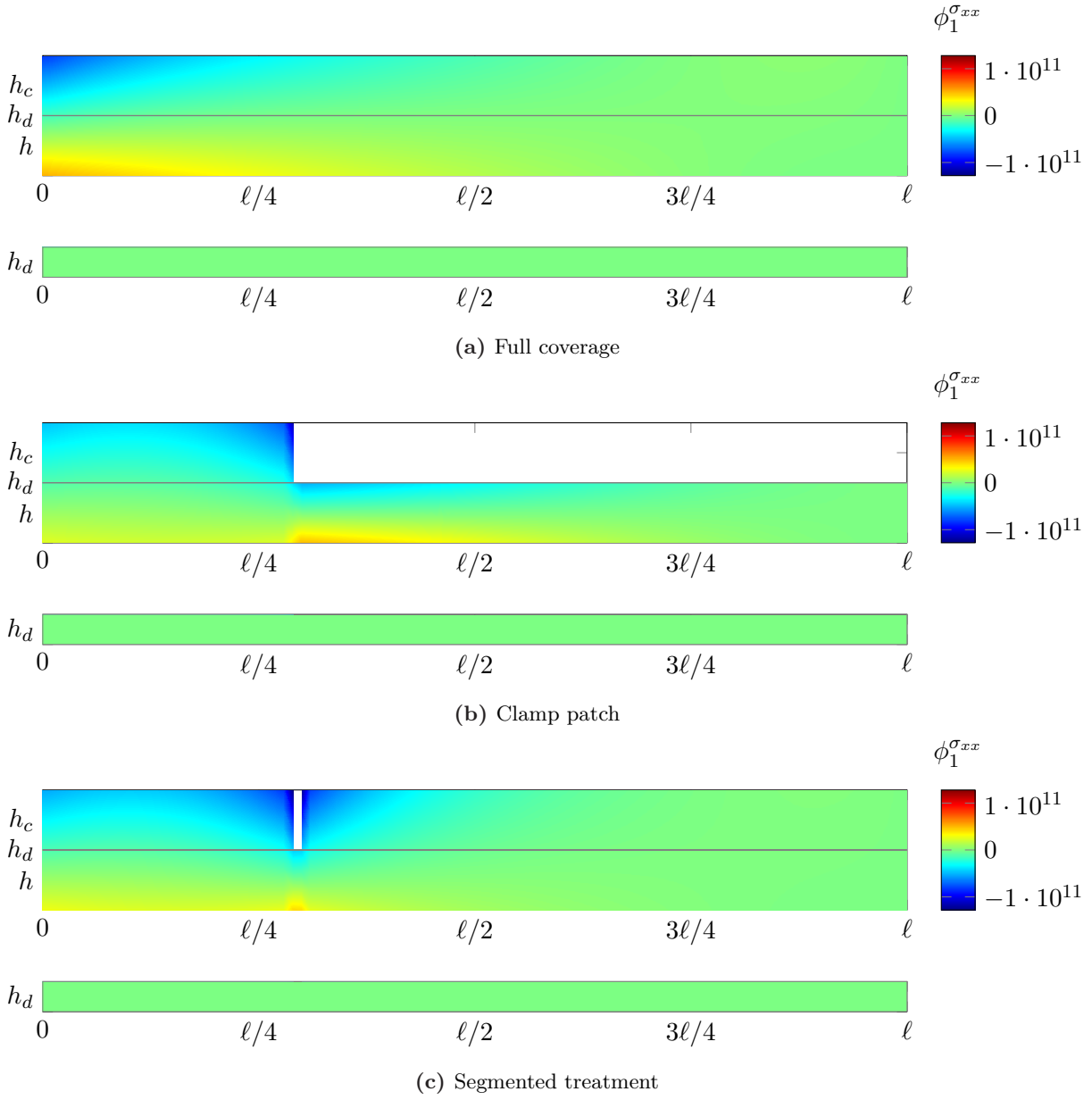


Figure 6.88 1st normal stress mode shapes of a cantilever beam with different treatments

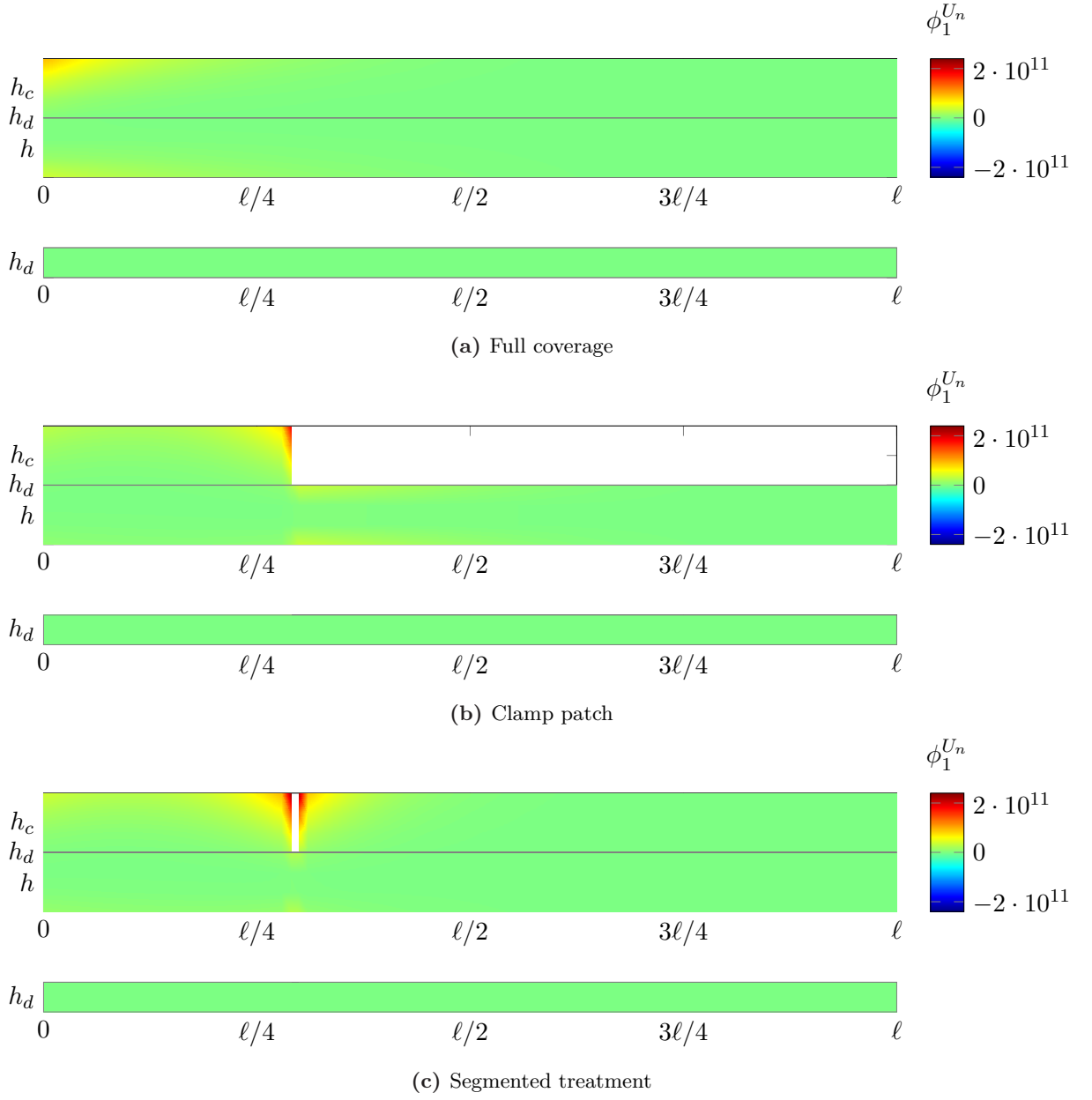
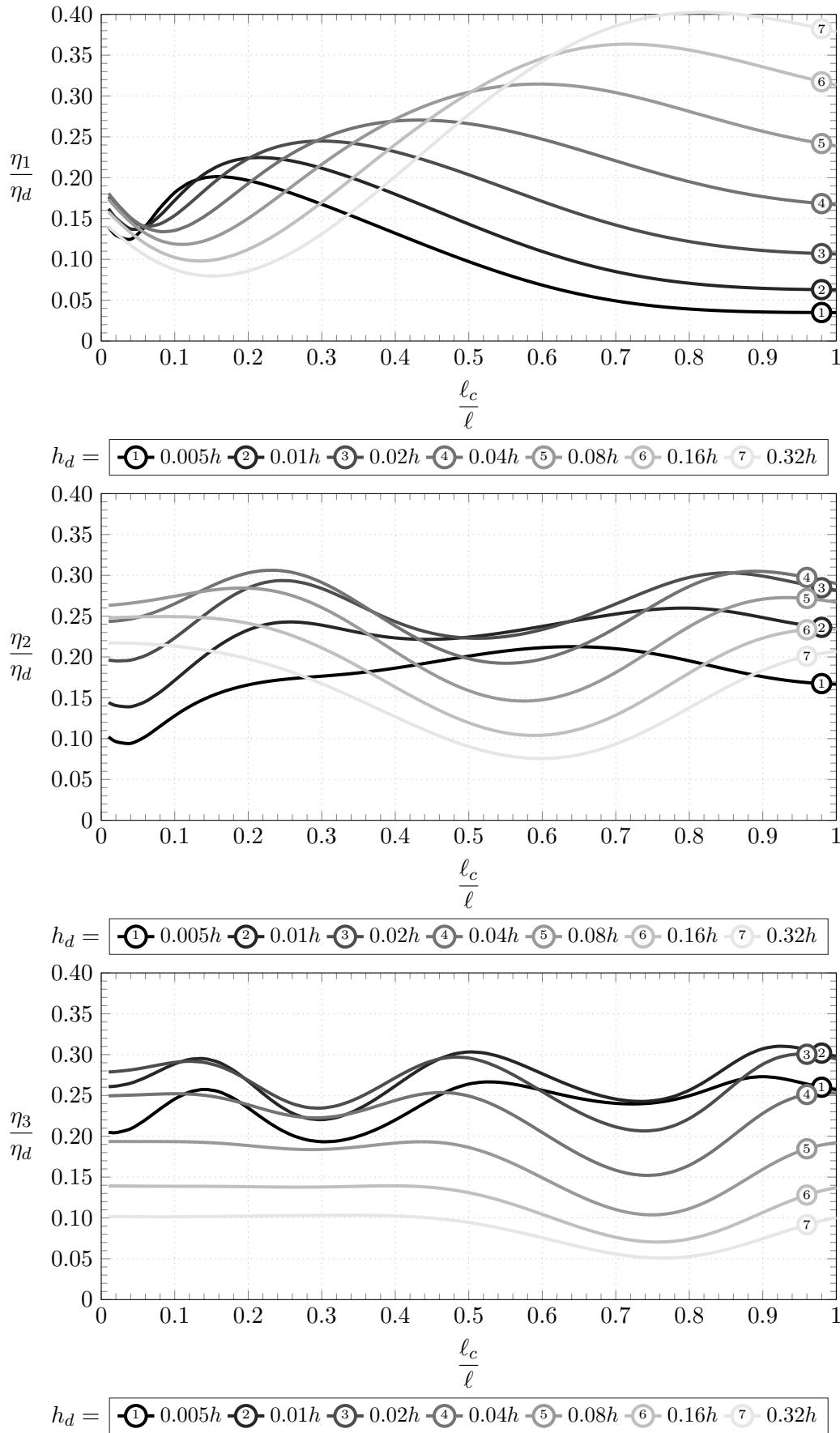
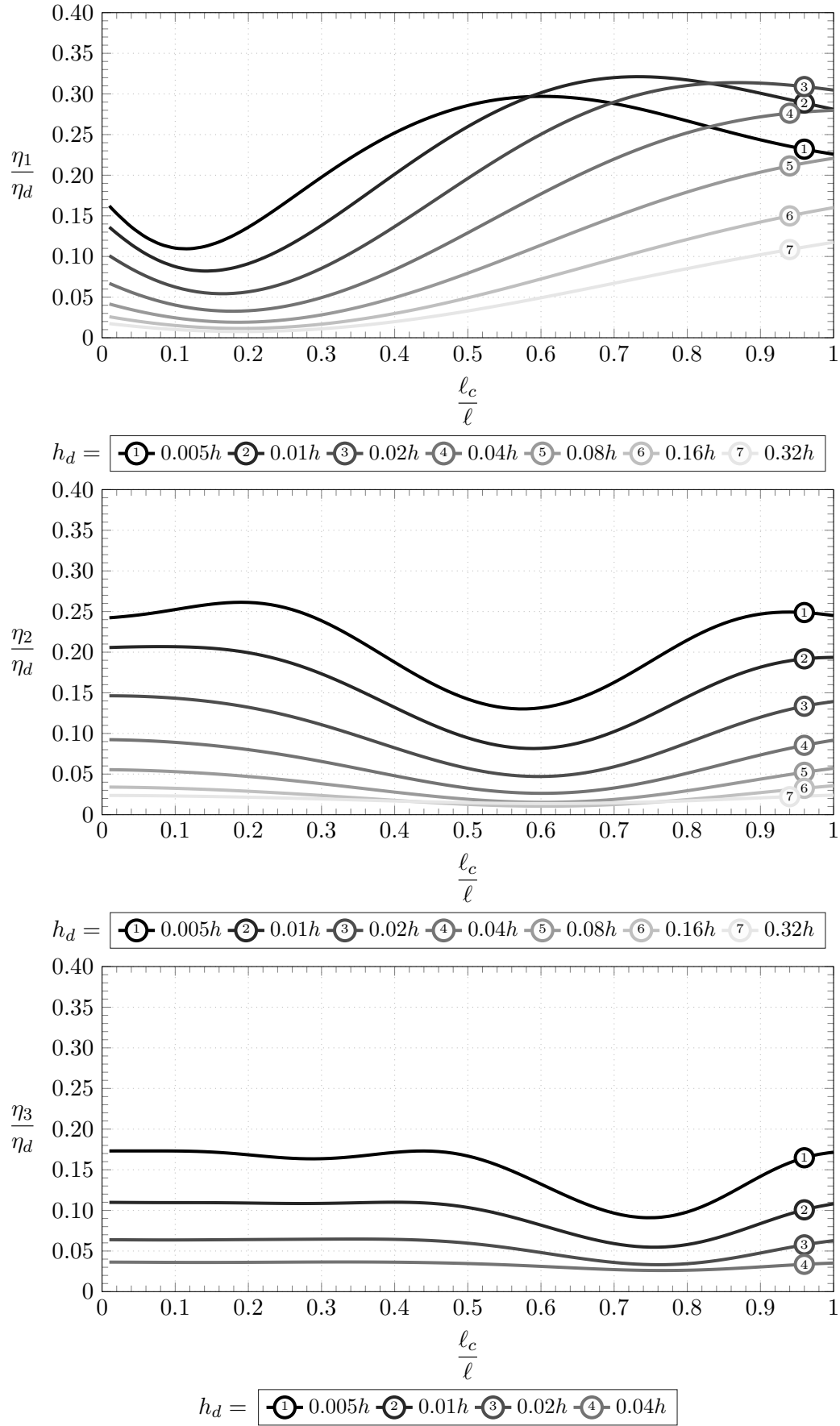
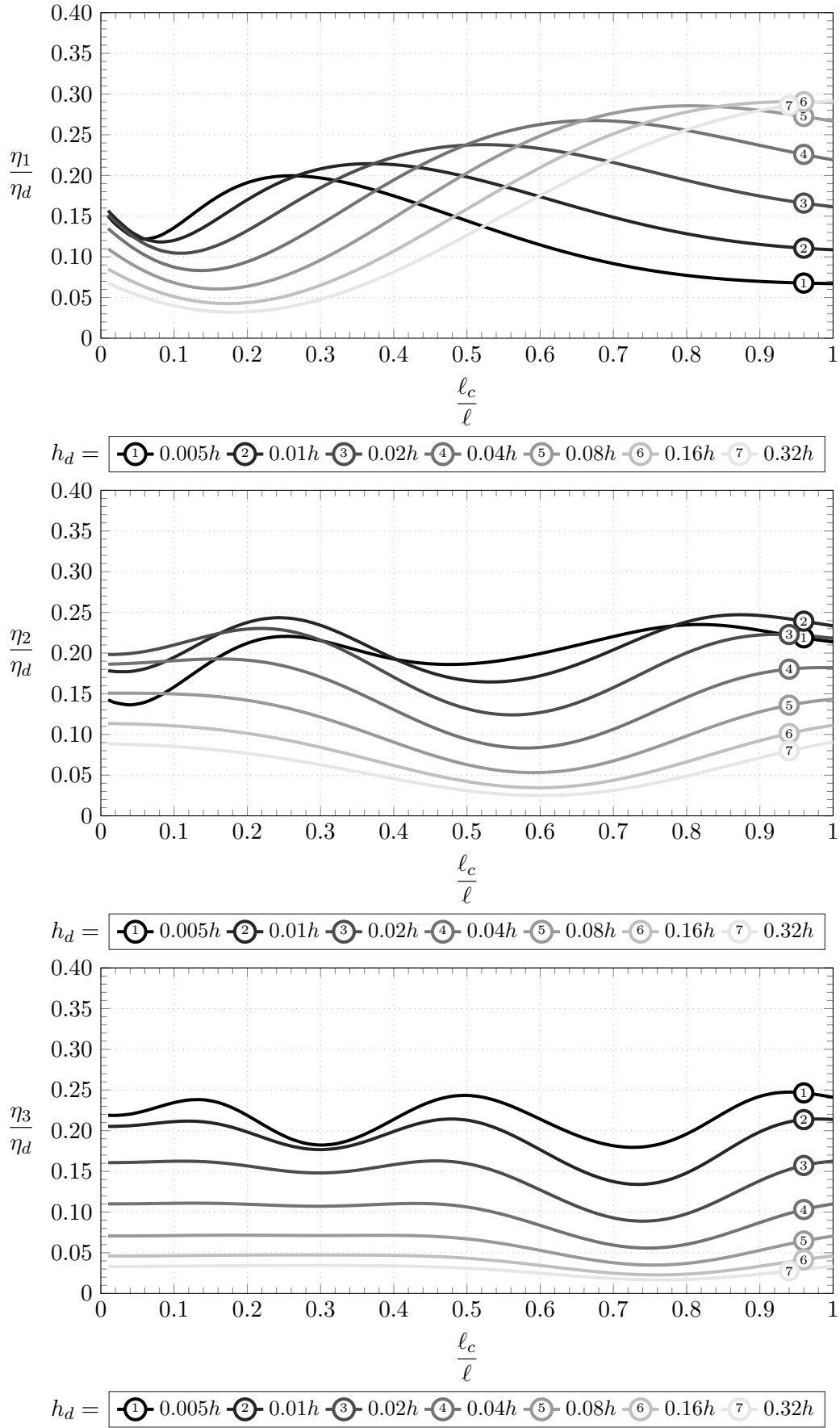
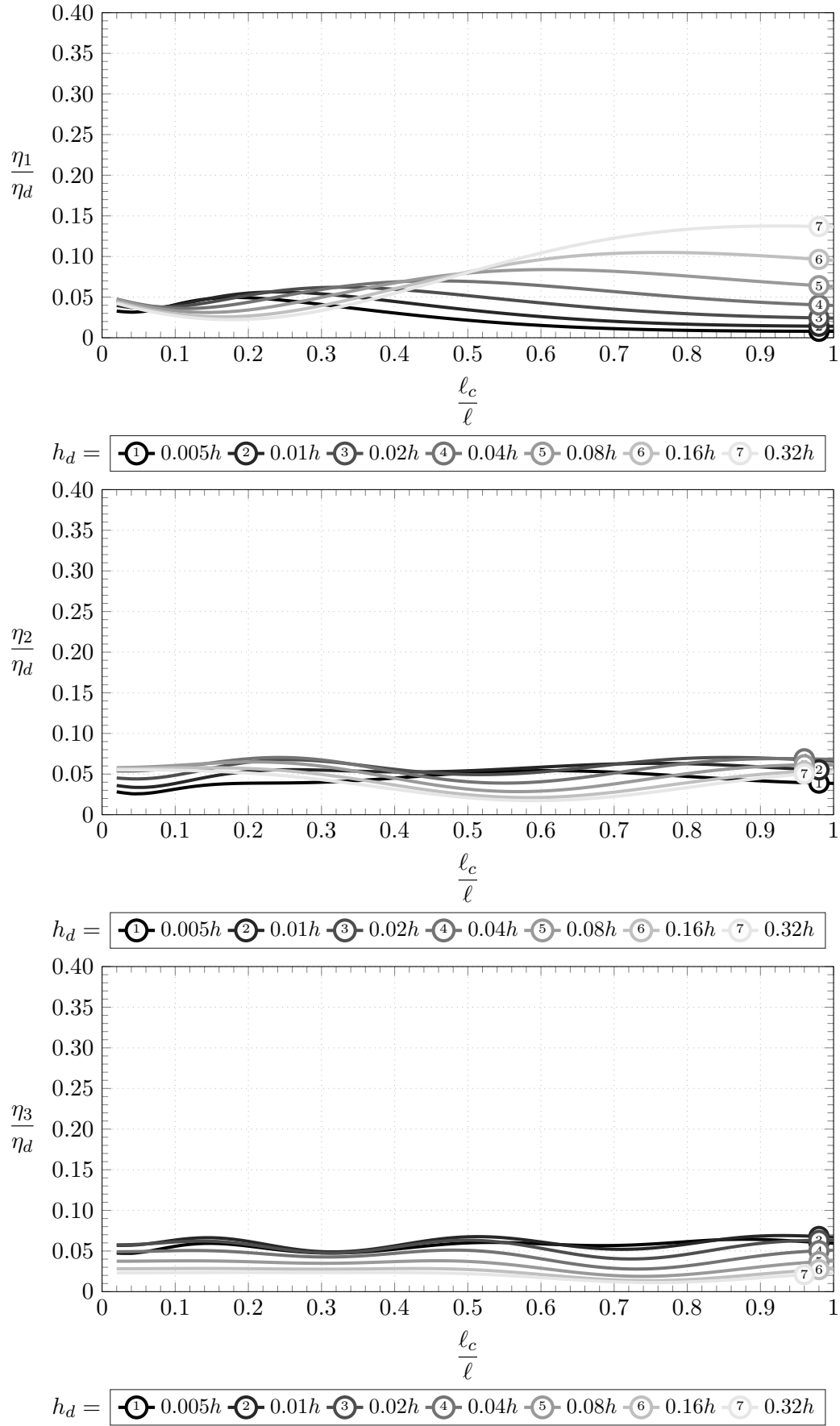


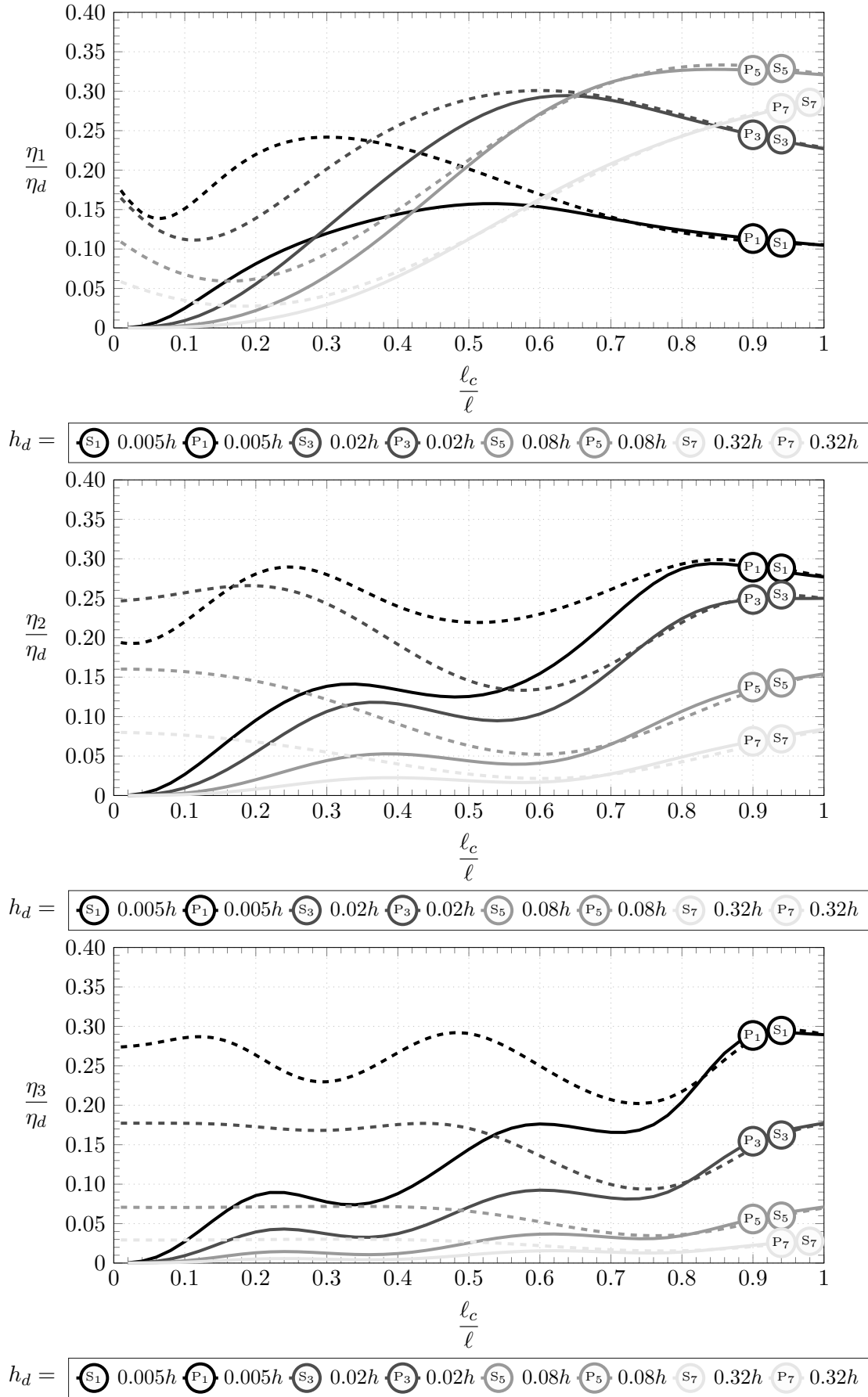
Figure 6.89 1st normal strain energy density mode shapes of a cantilever beam with different treatments

Figure 6.90 Damping efficiency curves, $h = 0.005\ell$, $h_c = h$


 Figure 6.91 Damping efficiency curves, $h = 0.02\ell$, $h_c = h$

Figure 6.92 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.5h$


 Figure 6.93 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.1h$

Figure 6.94 Damping efficiency curves, $h = 0.01\ell$, $h_c = h$

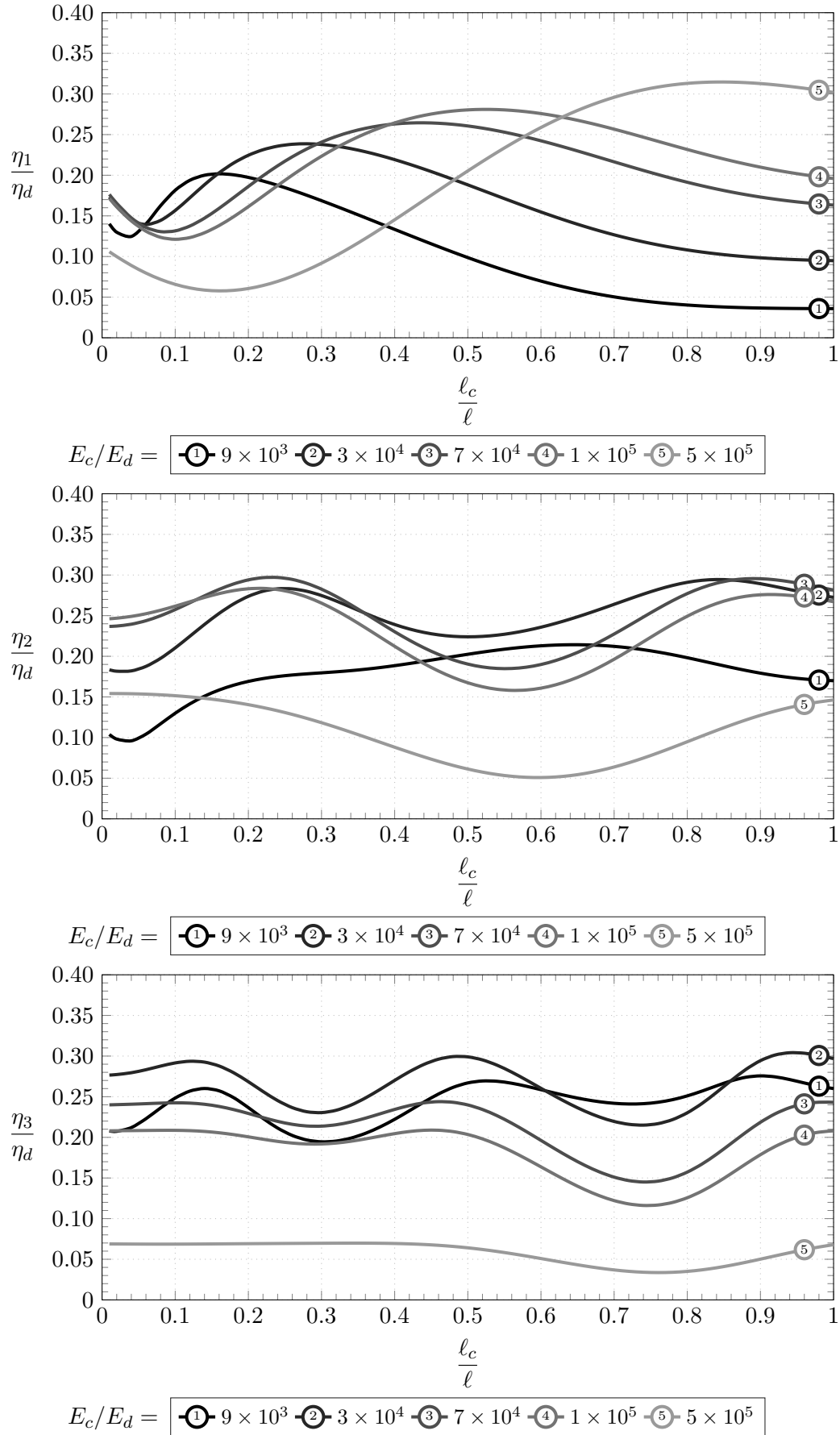


Figure 6.95 Damping efficiency curves, $h = 0.01\ell$, $h_d = 0.01h$, $h_c = h$

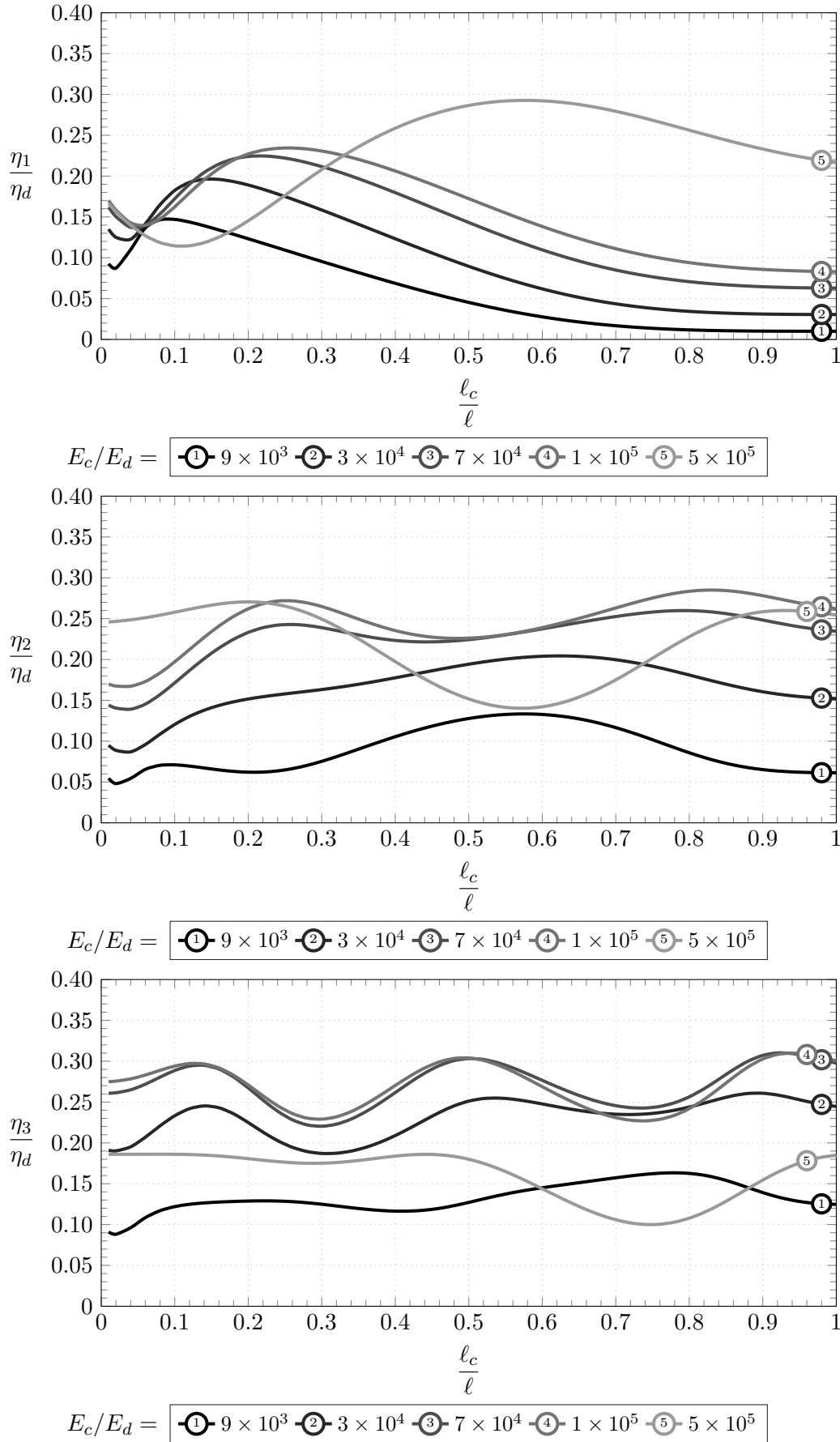
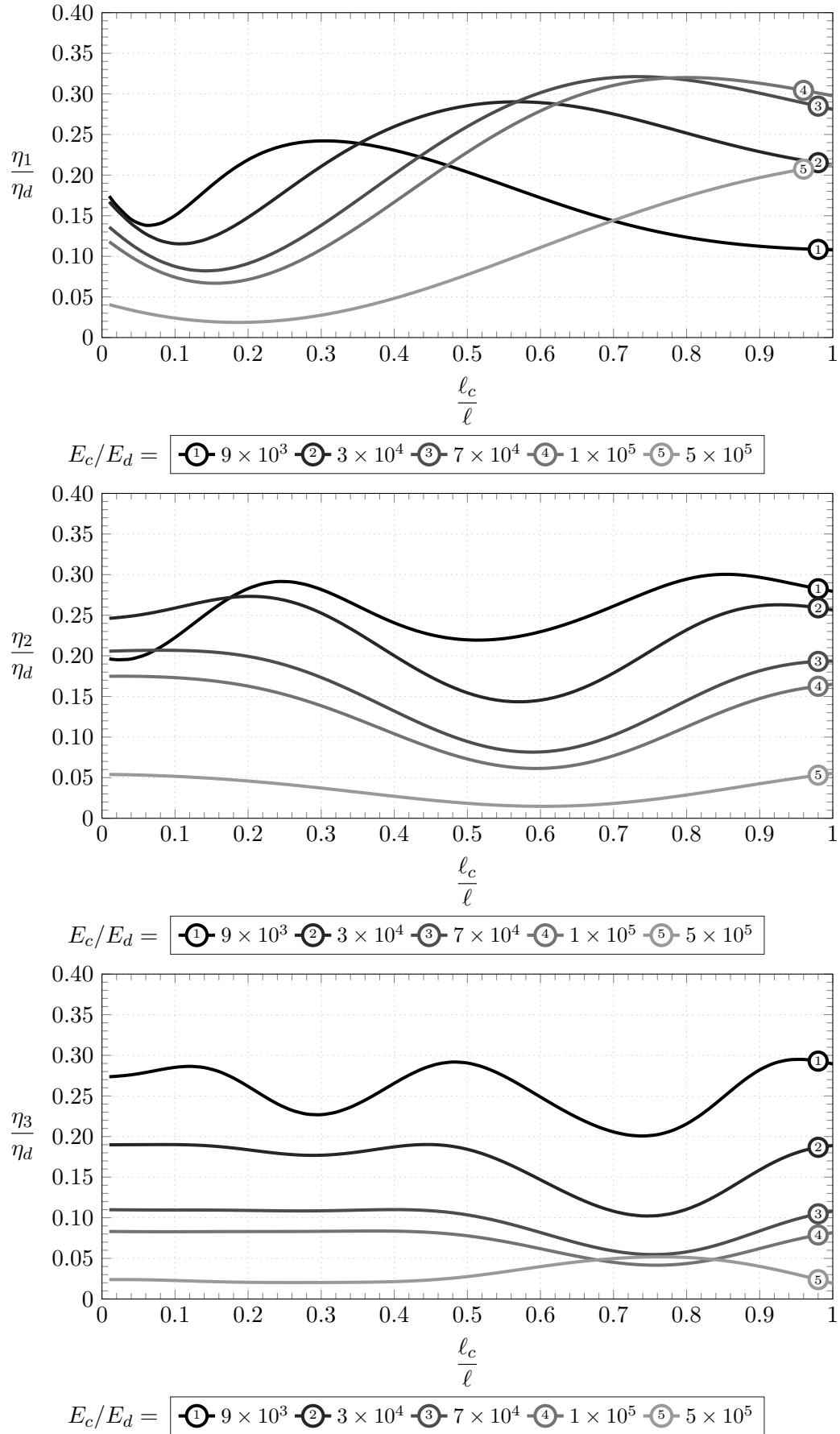


Figure 6.96 Damping efficiency curves, $h = 0.005\ell$, $h_d = 0.01h$, $h_c = h$


 Figure 6.97 Damping efficiency curves, $h = 0.02\ell$, $h_d = 0.01h$, $h_c = h$

6.5.2 | Free-Free Beams

The study of a free-free beam completes a more comprehensive study on the mechanical influence of segmented constraining layers and their impact on damping capacity. This section deals with the parametric study of free-free beams with segmented constraining layers, as represented in Figure 6.98. One evaluates a beam with symmetric configuration considering the same geometric and material properties listed in Tables 6.1 and 6.2, - Figure 6.98.

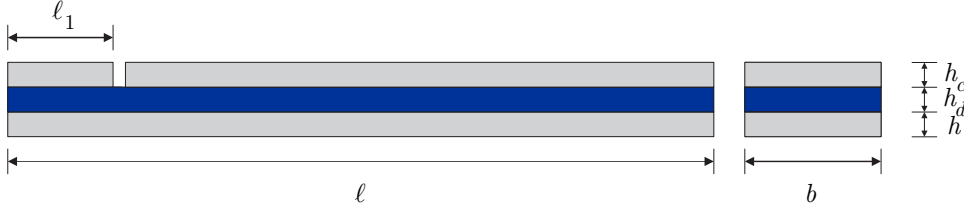


Figure 6.98 Free-free beam with segmented constraining layer

Considering an odd FEM mesh of 101 elements along the length of the beam to ensure a central element at the beam's centre and the ToZZ, one performs the segmented constraining layer modelling by considering just two layers (base plus damping) at one specific element, thus defining the cutting position. Once the cut has the exact length of an element, for the example under analysis, one considers a constant cut length of 0.0099ℓ . One performs a modal analysis with the CE to obtain the modal loss factors.

6.5.2.1 | Reference Beam - $h = 0.01\ell$

Figures 6.62a, 6.63a and 6.64a present the first three extensional strain energy free-free bare beam's mode shapes in a through-the-thickness representation. In addition, and worth analysing simultaneously, Figures 6.62b, 6.63b and 6.64b present an analogous view of the first three transverse shear strain energy mode shapes.

Figure 6.99 presents the first three specific modal loss factors of the free-free damped beam as a function of the cut positioning in the constraining layer. As expected, there is symmetry in the specific modal loss factors between the left and right-hand sides of the beam due to its own geometric and material symmetry. At this point, it is worth remembering the shear and extensional strain energy density mode shapes of the bare free-free beam in Figures 6.62b to 6.64b and 6.62a to 6.64a. Figure 6.99 validates, for all the specific modal loss factors represented, that segmenting the constraining layer at sections of maximum modal extensional strain energy is troublesome. Focusing the analysis on the first specific modal loss factor, damping capacity decreases markedly for a cut at the central region of the constraining layer where the extensional strain energy density is maximum for the first mode shape of a bare beam - Figure 6.62a. By cutting the constraining layer at such a point, one breaks the extensional strain of the constraining layer, thus not enforcing the shear strain of the damping layer. For thinner damping layers, it is possible to increase the damping capacity of the first mode by moving forward the cut, closer to the beam's free ends, as the damping layer gets thicker, although the increasing-decreasing damping effect of the damping layer thickness sets a threshold. Consequently, adding damping material or cutting the constraining layer, particularly at the beam's centre, is worthless. The second and third specific modal loss factors firmly support the fact that cutting the constraining layer at sections with the maximum modal extensional strain energy density of the bare beam sharply decreases damping capacity - see Figures 6.63a and 6.64a.

Figure 6.100 presents the first three transverse displacement free-free beam's mode shapes with full coverage, considering a damping layer thickness of $0.01h$. Similarly to the cantilever beam, it is unquestionable that the cut of the constraining layer near the extensional strain anti-nodes relates to degraded damping performances. In addition, Figure 6.101 presents the first three extensional strain mode shapes and Figures 6.102 and 6.103 present the first three extensional stress and strain energy density mode shapes, respectively. Once again, similar to the cantilever beam, extensional strain, stress, or strain energy density mode shapes show that the worst damping performances relate to cuts of the constraining layer at sections of maximum extensional strain. As previously noted, the reason is that the extensional strain of the constraining layer handles the imposition of the transverse shear strain to the damping layer. When cutting the constraining layer at sections of higher extensional strain, one degrades the damping performance by breaking its capacity to impose shear on the damping layer. It is clear for any mode when considering a cut near the anti-nodes of the respective mode, sections with the maximum extensional strain of the constraining layer.

Figures 6.104, 6.105 and 6.106 present the first three transverse shear strain, stress and strain energy density mode shapes obtained from equilibrium equations. As expected, transverse shear strain at the soft damping layer is high, but it is interesting to note, once again, how the nodes of transverse shear strain, or stress, match the anti-nodes of the extensional counterpart.

Repeatedly, to have a better insight into the mechanics behind the improved performance of segmented damping treatments, Figures 6.107 and 6.108 present the first transverse shear strain mode shapes for fully covered beams, mid-length patched beams, and beams with segmented constraining layers, respectively. Figures 6.109 to 6.110 and 6.111 to 6.112 present the corresponding transverse shear stress and transverse shear strain energy density mode shapes, respectively. The patched treatment covers 0.7ℓ of the beam, and the segmentation of the constraining layer occurs at 0.15ℓ for one cut since it is in the optimised position, considering a damping layer with $h_d = 0.01h$ - Figure 6.99. For comparison purposes, one considers the extensional shear strain, stress and strain energy density mode shapes of a free-free beam with two symmetric cuts. The patch treatment also provides a comparative example for the segmented constraining layer cases. Figure 6.99 shows that it is possible to increase the damping capacity of a damping treatment with the segmentation of the constraining layer for some damping layer thicknesses; the optimum position of the cut is near 0.15ℓ , or 0.85ℓ , for a damping layer of $h_d = 0.01h$.

Figures 6.108b and 6.108c explain the obtained results, showing the increased transverse shear strain state at the damping layer created by the segmentation of the constraining layer. Transverse shear stress mode shapes presented in Figures 6.109 and 6.110, together with transverse shear strain energy density mode shapes in Figures 6.111 and 6.112, show the higher levels of shear strain/stress state of the damping layer of the segmented treatment when compared to a full coverage one. Nevertheless, the beam with the centre patch presents the highest levels of shear strain energy and consequently improved damping performance.

As previously mentioned, the extensional strain of the constraining layer plays a decisive role in its capacity to impose shear strain on the damping layer. Figures 6.113 and 6.114 present the extensional strain mode shapes for the full coverage, patch, and segmented treatments only considering membrane and bending deformations. Unsurprisingly, the centre patch treatment experiences higher extensional strain when compared to fully covered or segmented structures. Nevertheless, it is worth emphasising the capacity to increase the extensional strain levels by cutting the constraining layer. Figures 6.115 and 6.116 with the extensional stress mode shapes, as well as Figures 6.117 and 6.118 with the extensional strain energy density mode shapes, emphasise the higher extensional strain/stress values at the constraining layer of the centre patch treatment, responsible for the higher transverse shear strain state of the damping layer and consequently its superior damping performance.

6.5.2.2 | Thin Beam - $h = 0.005\ell$

Considering thinner symmetric beams, Figure 6.119, it is worth noting the exception to the rule that one must avoid cuts of the constraining layer at sections of maximum extensional strain. For thin damping layers, the first specific modal loss factor increases by cutting the constraining layer at the mid-length of the beam. Nevertheless, as the damping layer marginally increases, a cut of the constraining layer at the region of maximum modal extensional strain energy promptly reduces the damping capacity of the treatment. Once more, after the threshold of the increasing-decreasing behaviour, to cut the constraining layer is to break the ability of the damping layer to strain and consequently dissipate vibrational energy.

The second and third specific modal loss factors follow similar trends to those of the reference beam, proving the segmentation ineffectiveness in sections with high extensional modal strain energy. Curiously, by cutting the constraining layer at regions of maximum modal transverse shear strain energy of the bare beam, the damping capacity increases.

6.5.2.3 | Thick Beam - $h = 0.02\ell$

When dealing with thick base beams and consequently with thick constraining layers in symmetric configurations, it is noticeable that damping capacity decreases with cuts of the constraining layer at sections of maximum modal extensional strain energy even more markedly - see Figure 6.120.

6.5.2.4 | Non-symmetric Configurations

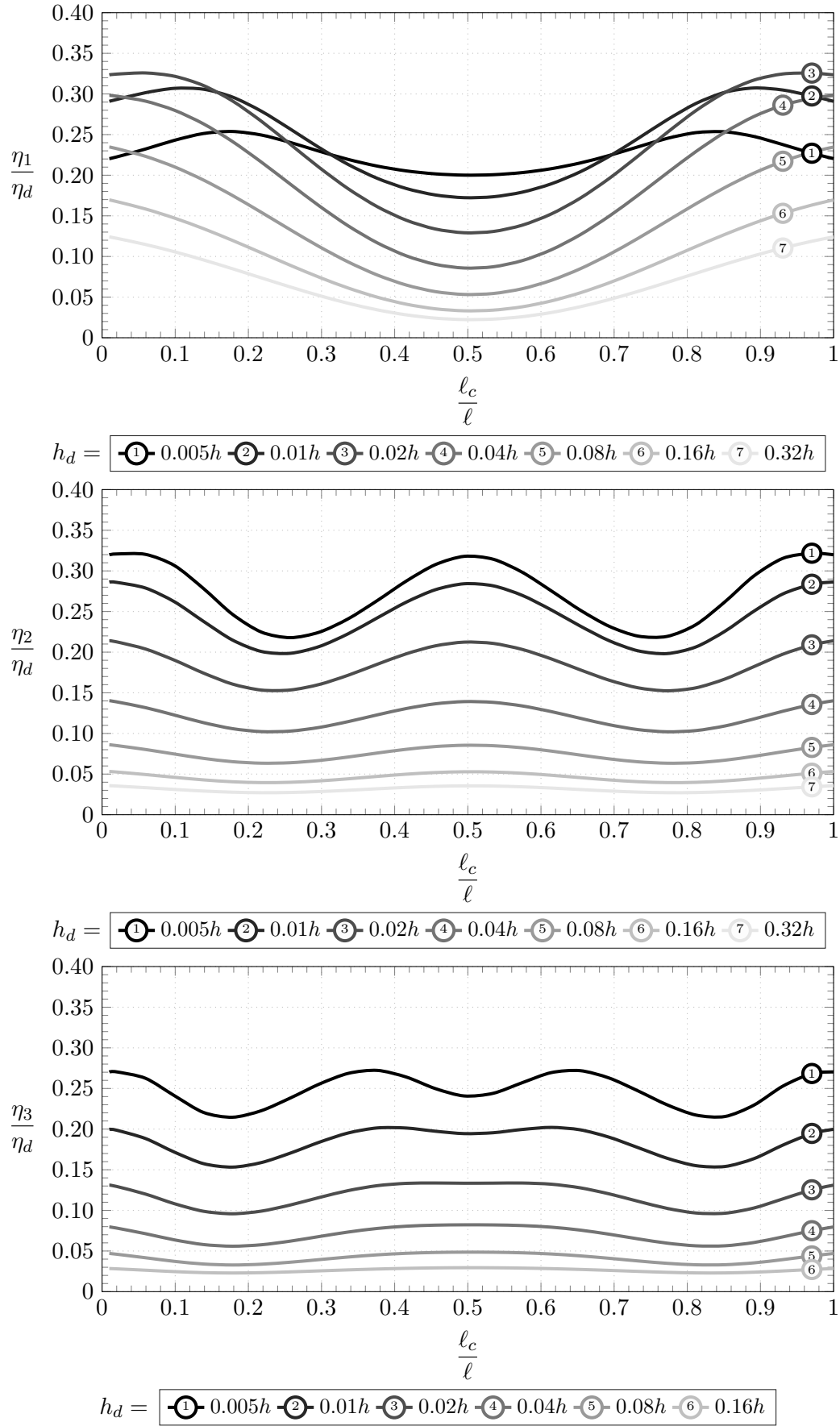
For non-symmetric configurations with a constraining layer with half the thickness of the base layer, as represented in Figure 6.121, it is interesting to compare it with the symmetric arrangement with the same base layer thickness, as depicted in Figure 6.99. One can notice the same patterns of the symmetric configuration in Figure 6.99 but with a shift in the specific modal loss factors - the non-symmetric arrangement presents lower damping capacity.

Considering even thinner constraining layers, $h_c = 0.01h$, the capacity to impose shear on the damping layer drops significantly - Figure 6.122.

6.5.2.5 | Influence of Material Properties

To explore the influence of the material properties on the damping capacity of segmented treatments for free-free beams, Figure 6.123 presents the influence of the modulus ratio on a symmetric reference beam with $h = 0.01\ell$ and a damping layer of $h_d = 0.01h$. The first specific modal loss factor highlights the importance of the parametric analysis for this study. For low modulus ratios, the stiffness difference between both materials is lower, and segmentation of the constraining layer at the centre of the beam increases the damping capacity. But, as the modulus ratio increases due to a higher working temperature, for example, a cut at this section drastically decreases the damping capacity.

Figure 6.124 presents the influence of the modulus ratio on the damping capacity for thin beams. The improved damping performance with the segmentation of the constraining layer at the centre of the beam is visible for more modulus ratios and not just the lower one. In contrast, Figure 6.125 shows the influence of the modulus ratio on the damping capacity for thicker beams. Segmentation becomes less effective for thick beams, and, once again, these damping efficiency curves show that segmenting the constraining layer at sections of higher modal extensional strain is prejudicial.


 Figure 6.99 Damping efficiency curves, $h = 0.01\ell$, $h_c = h$

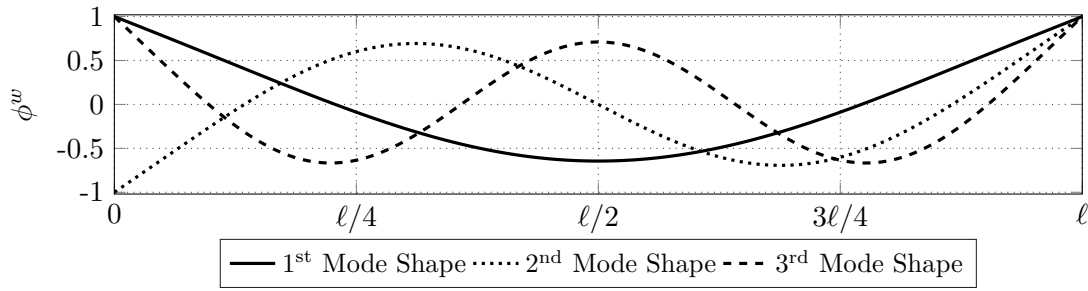


Figure 6.100 First three transverse displacement mode shapes of the free-free beam with full treatment

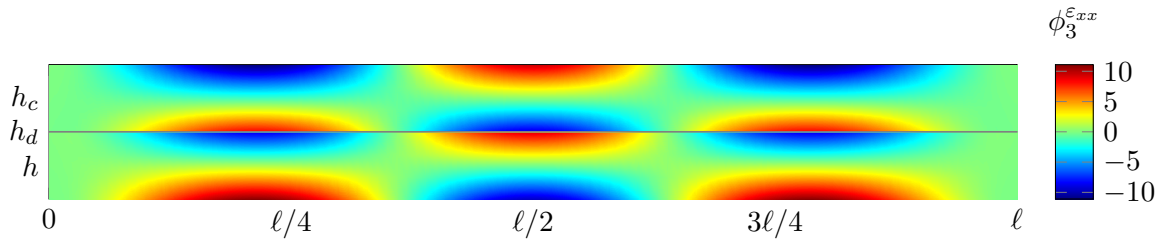
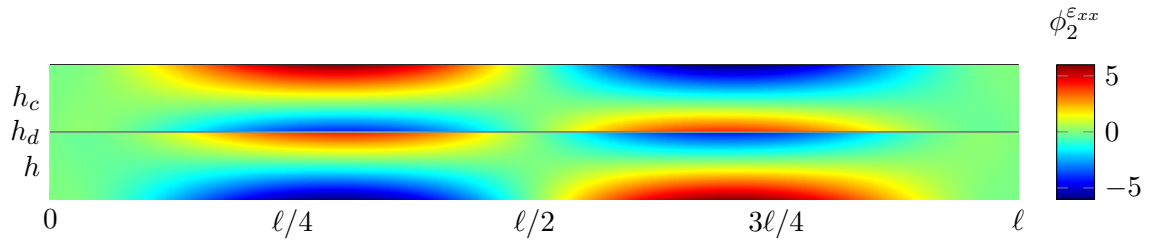
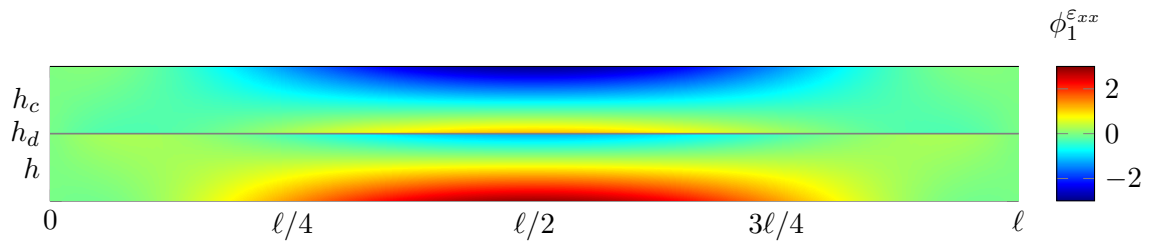


Figure 6.101 Normal strain mode shapes of a free-free beam with full coverage

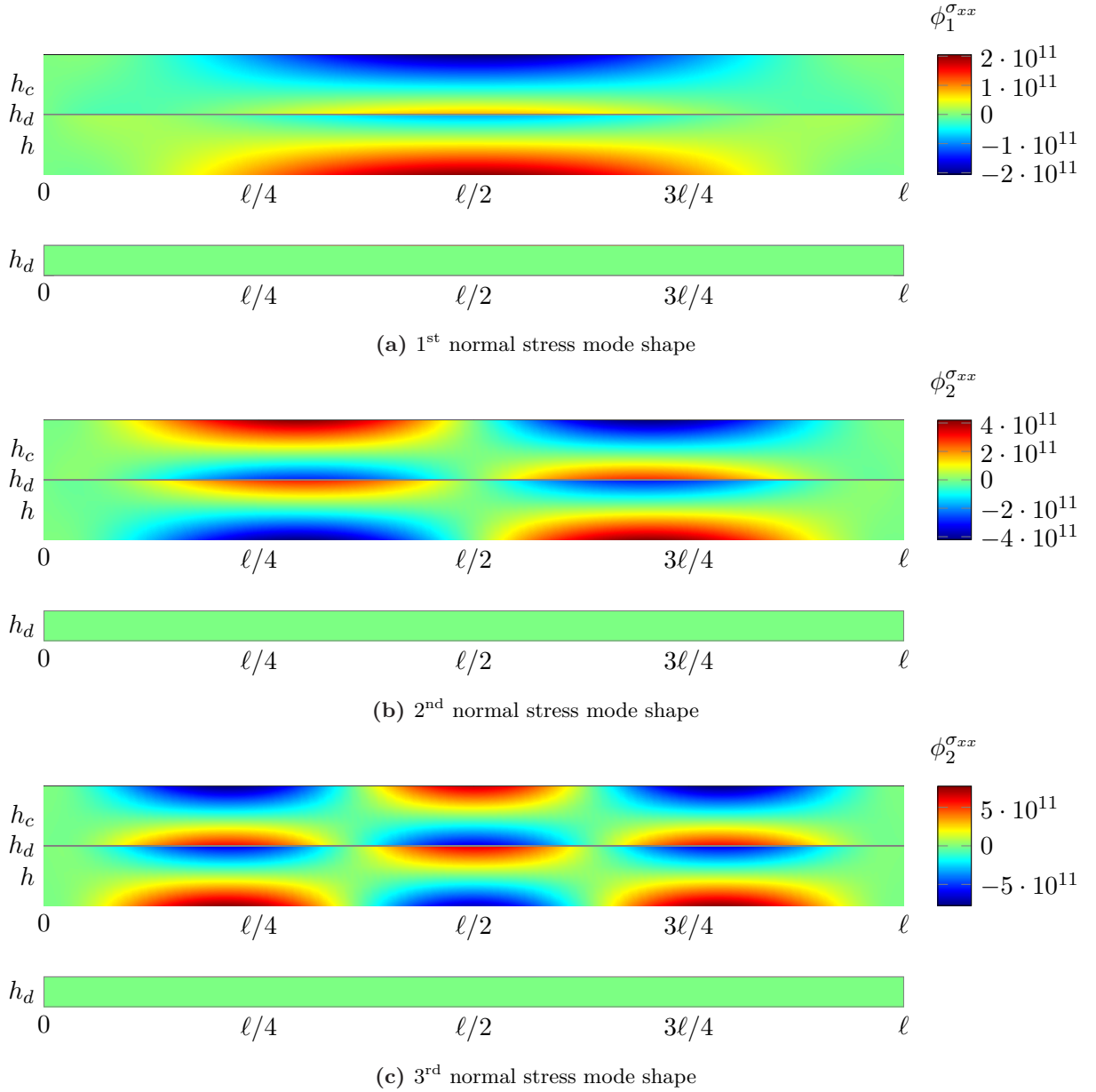


Figure 6.102 Normal stress mode shapes of a free-free beam with full coverage

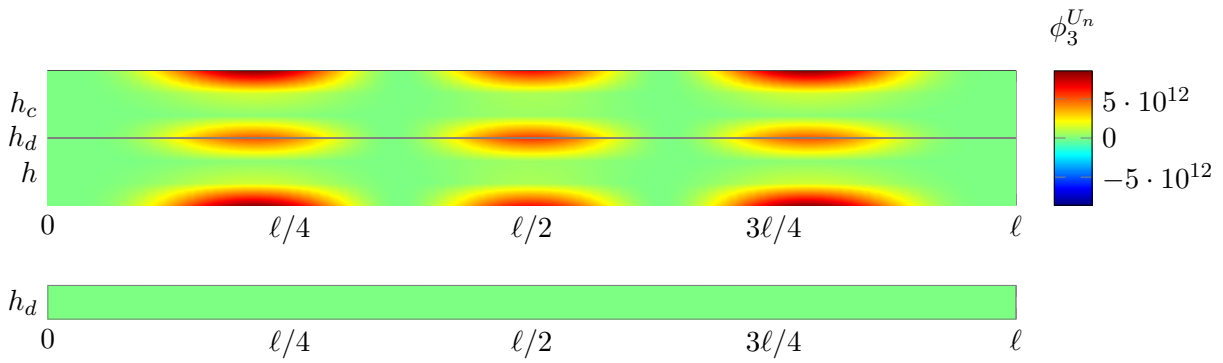
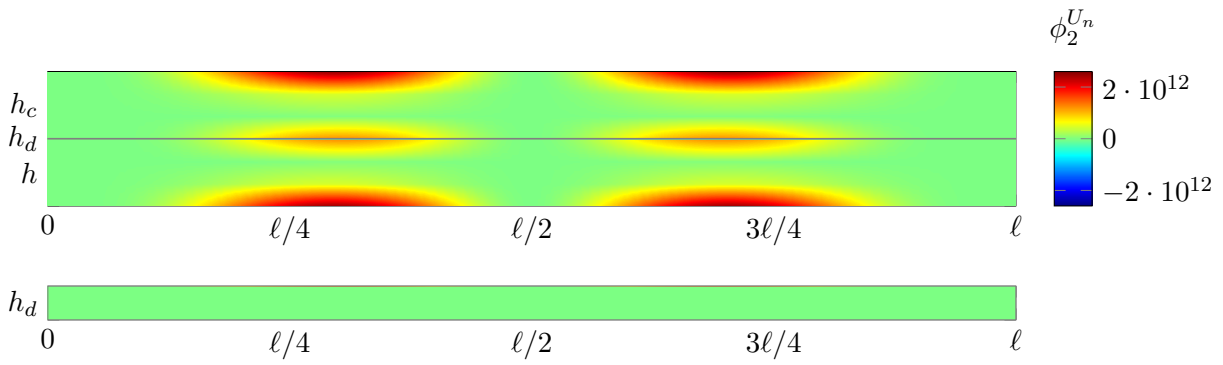
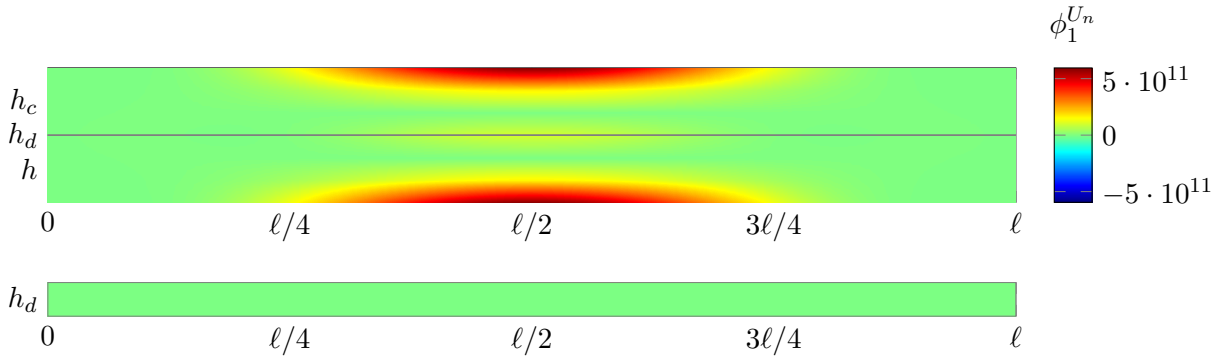


Figure 6.103 Normal strain energy density mode shapes of a free-free beam with full coverage

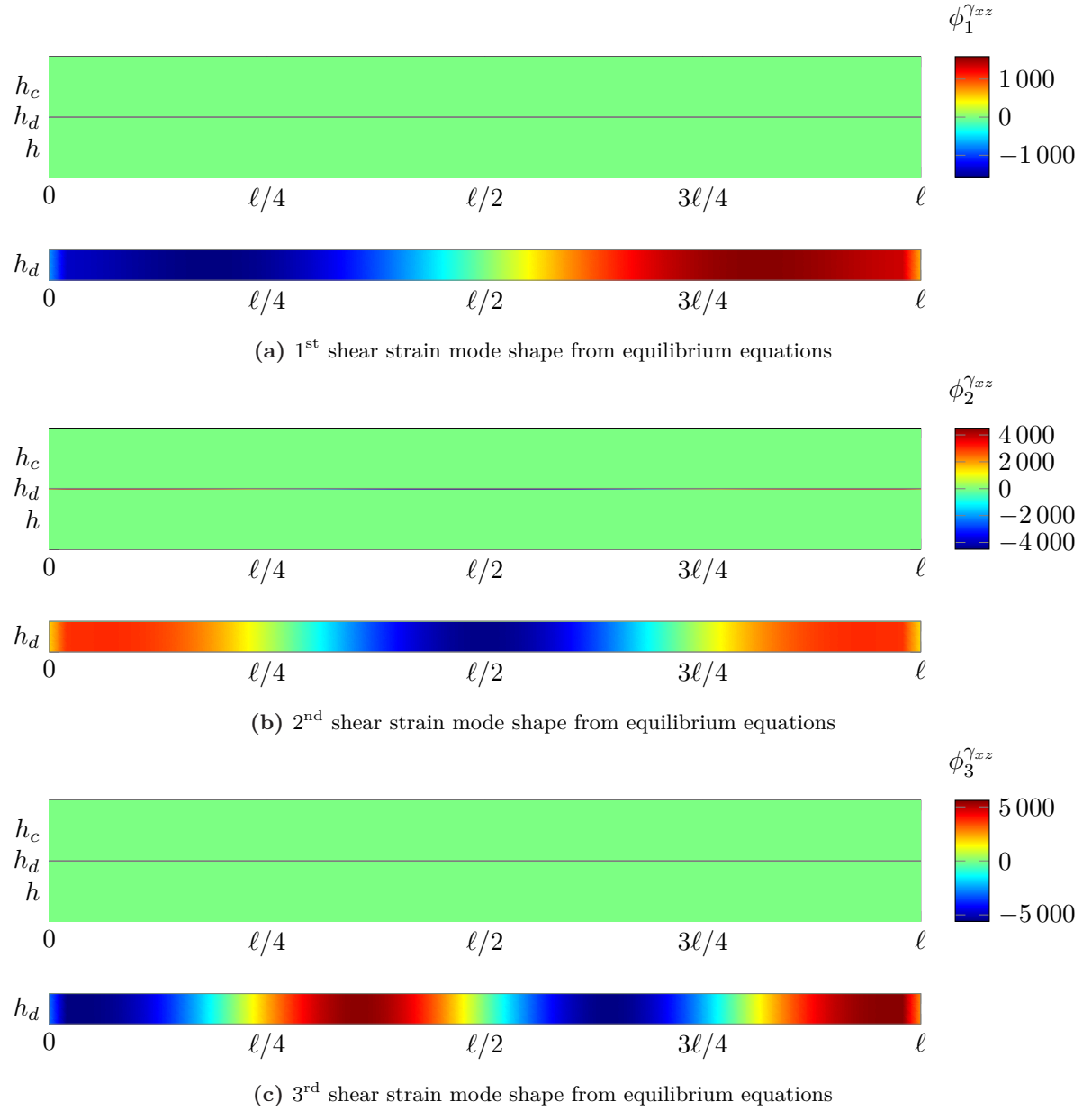


Figure 6.104 Shear strain mode shapes of a free-free beam with full coverage

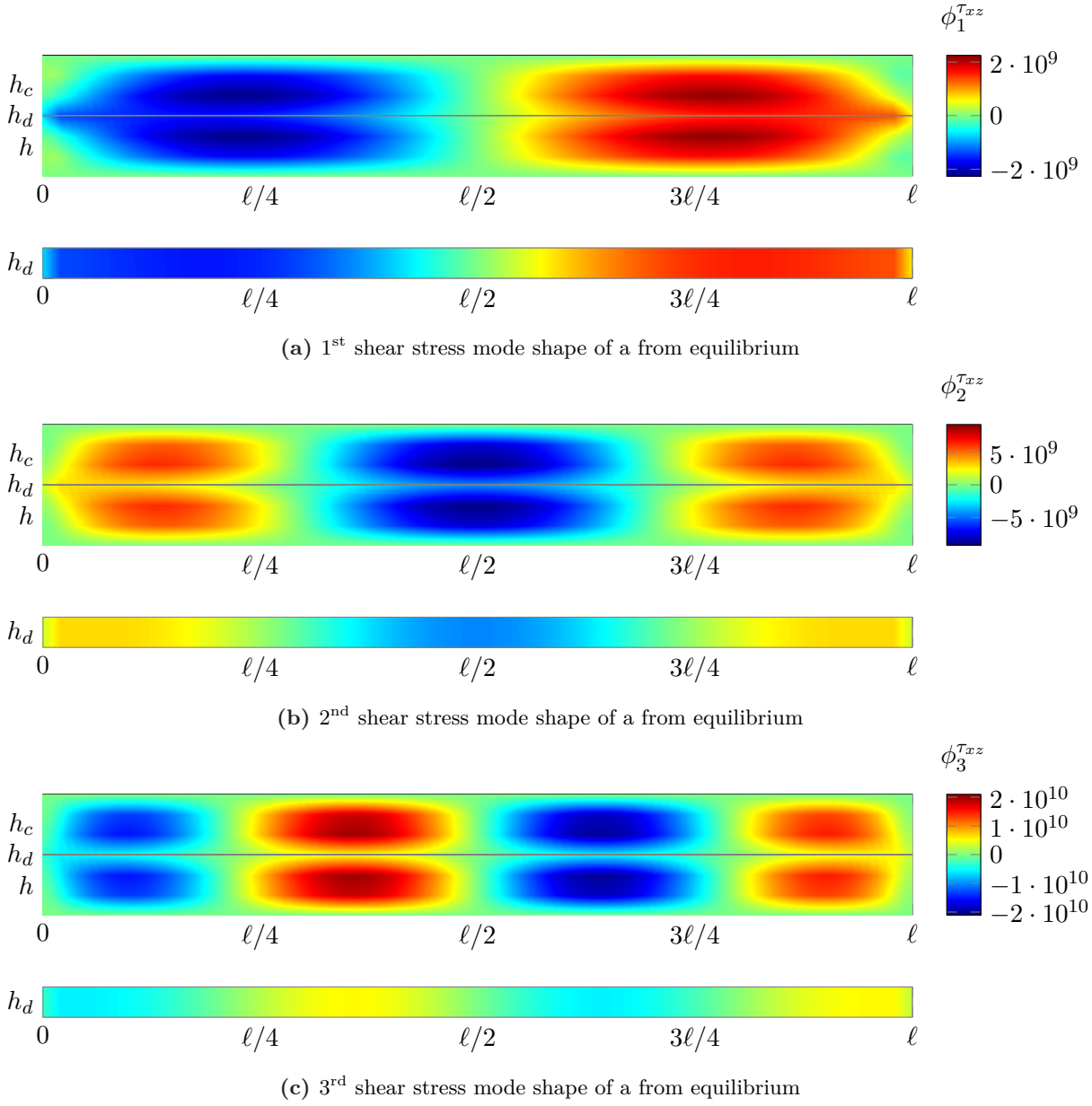


Figure 6.105 Shear stress mode shapes of a free-free beam with full coverage

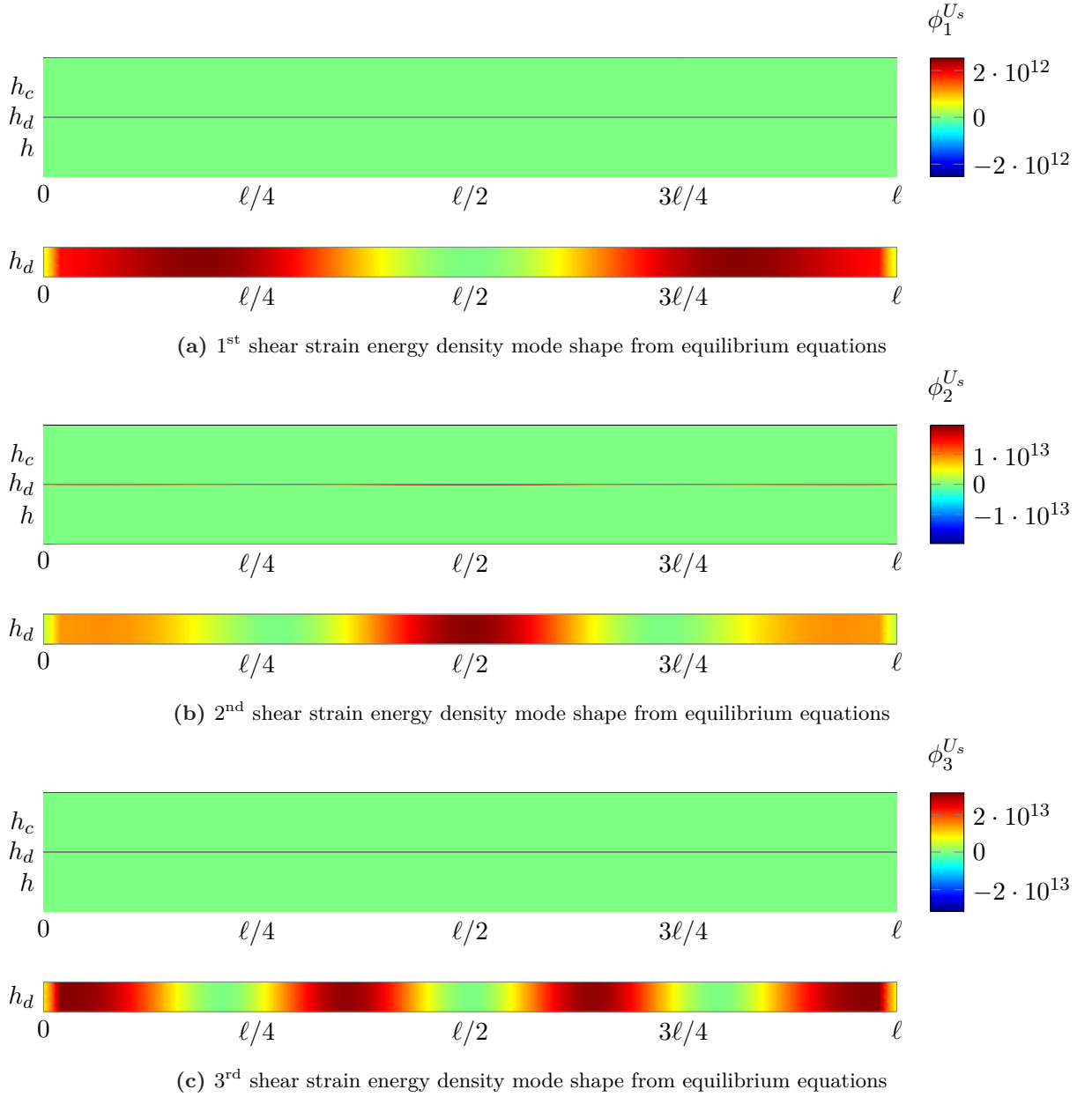


Figure 6.106 Shear strain energy density mode shapes of a free-free beam with full coverage

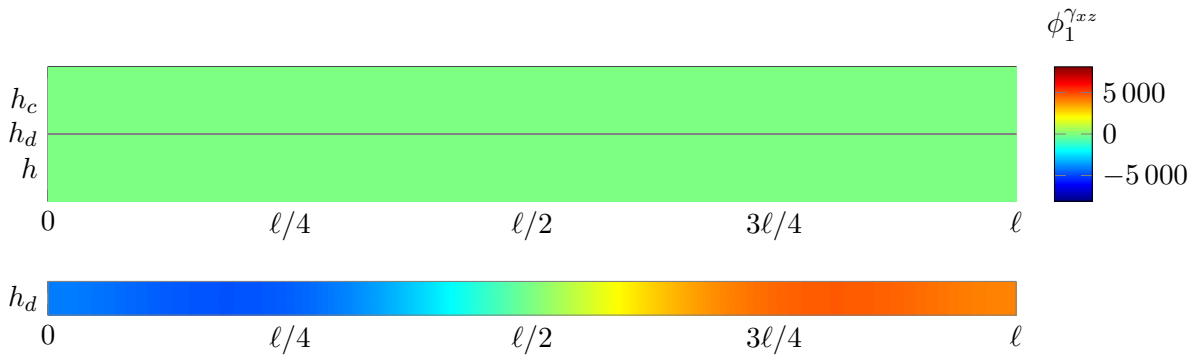
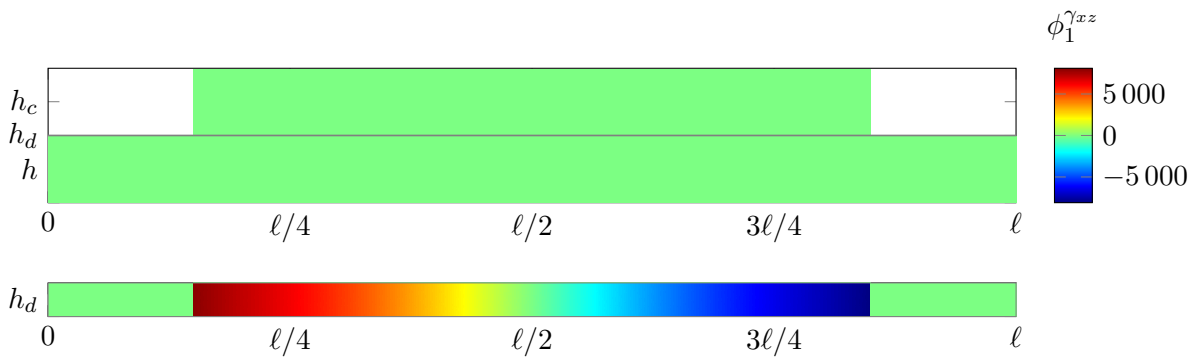
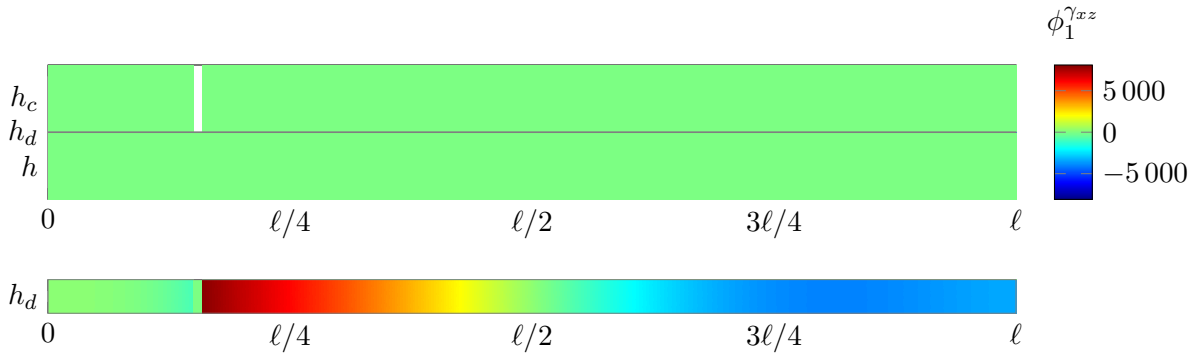


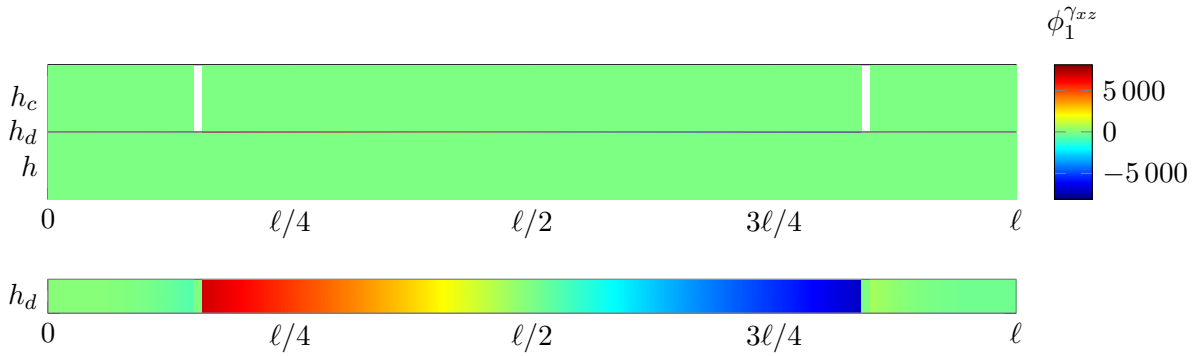
Figure 6.107 1st shear strain mode shape of a free-free beam with full coverage



(a) Clamp patch



(b) Segmented treatment with one cut



(c) Segmented treatment with two cuts

Figure 6.108 1st shear strain mode shapes of a free-free beam with different treatments

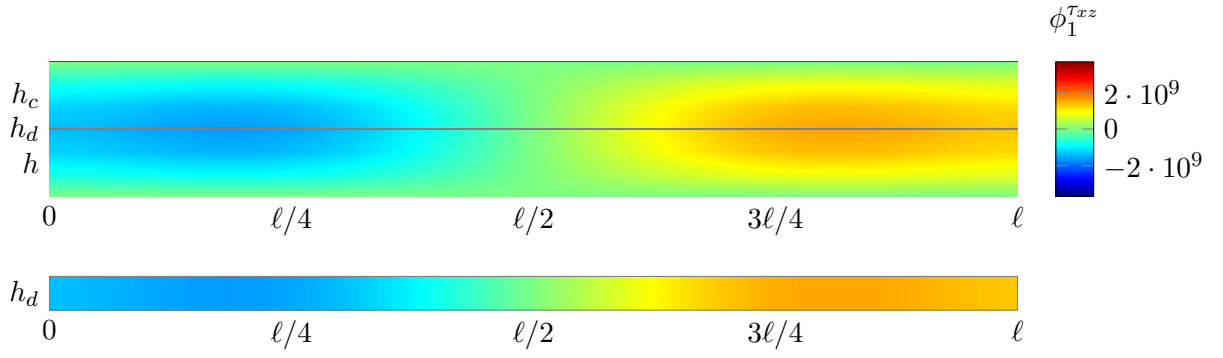


Figure 6.109 1st shear stress mode shape of a free-free beam with full coverage

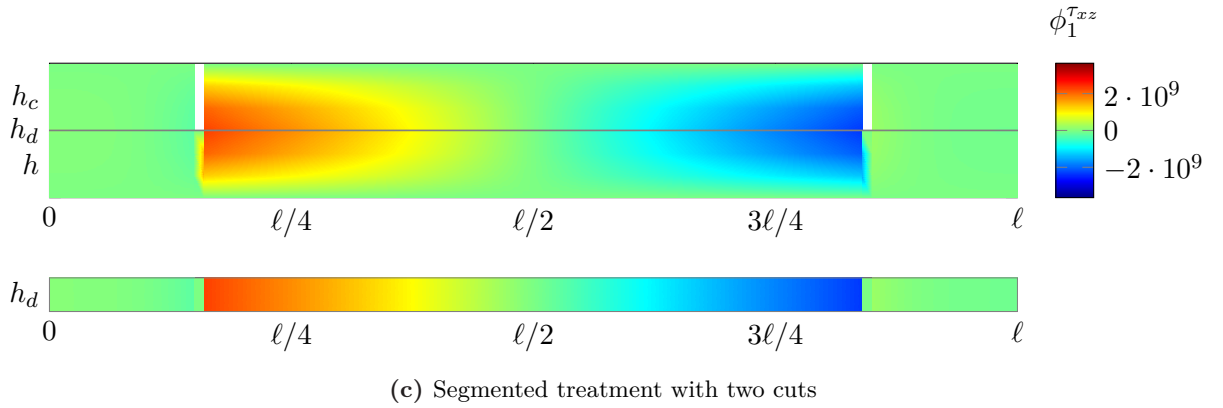
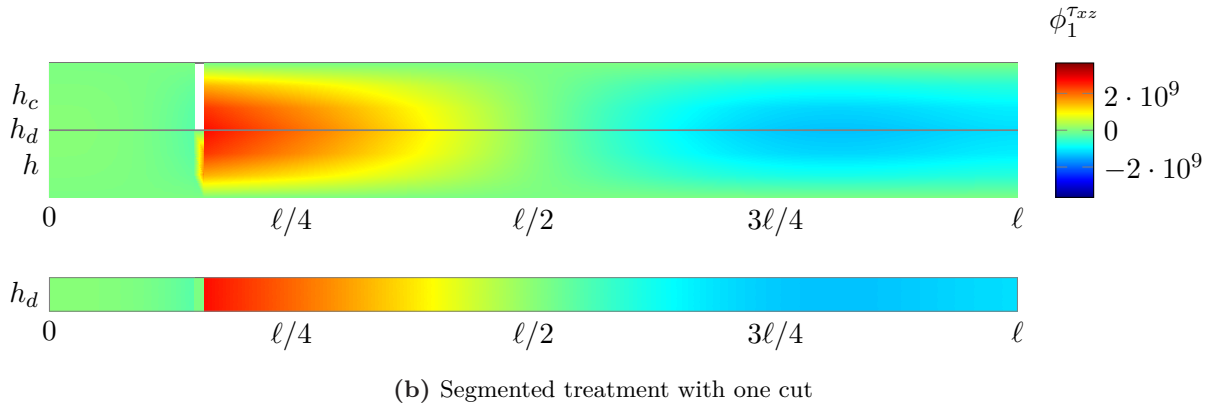
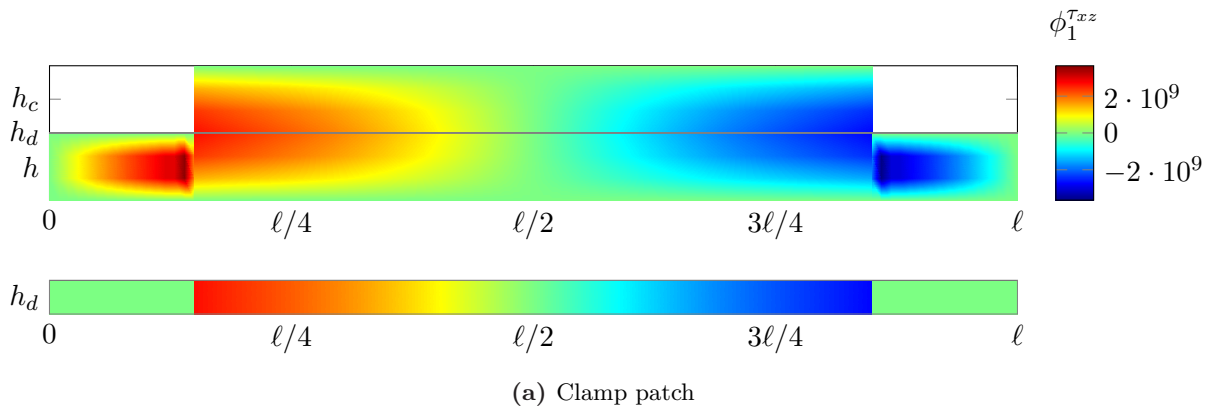


Figure 6.110 1st shear stress mode shapes of a free-free beam with different treatments

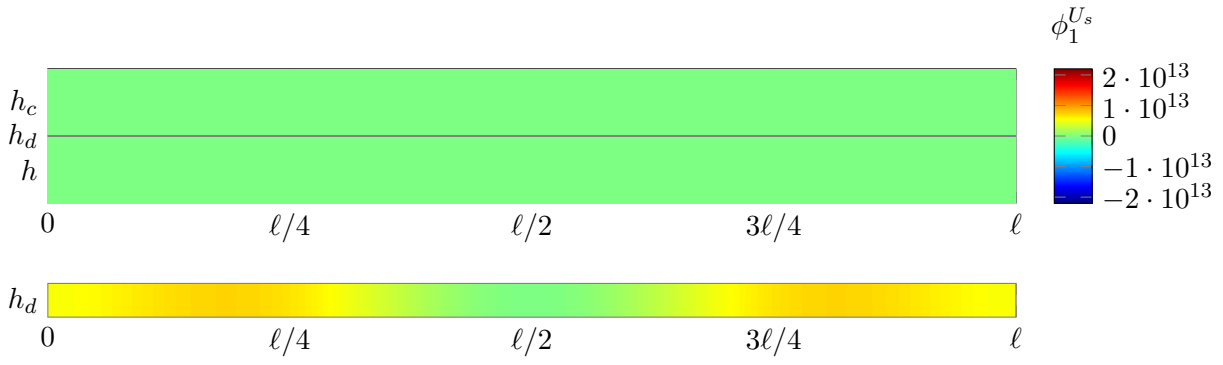
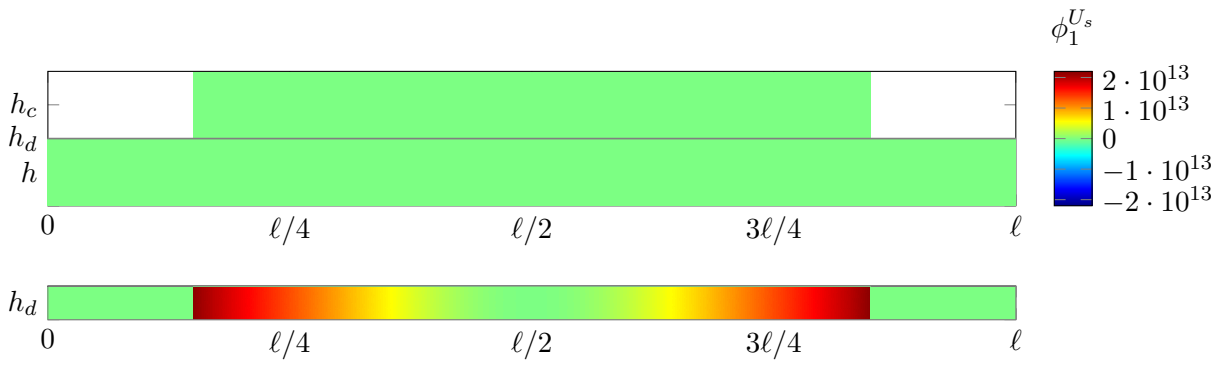
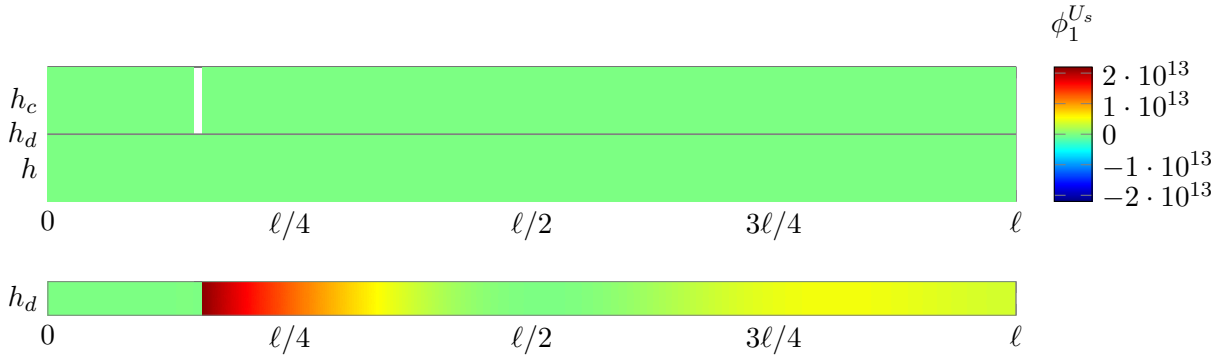


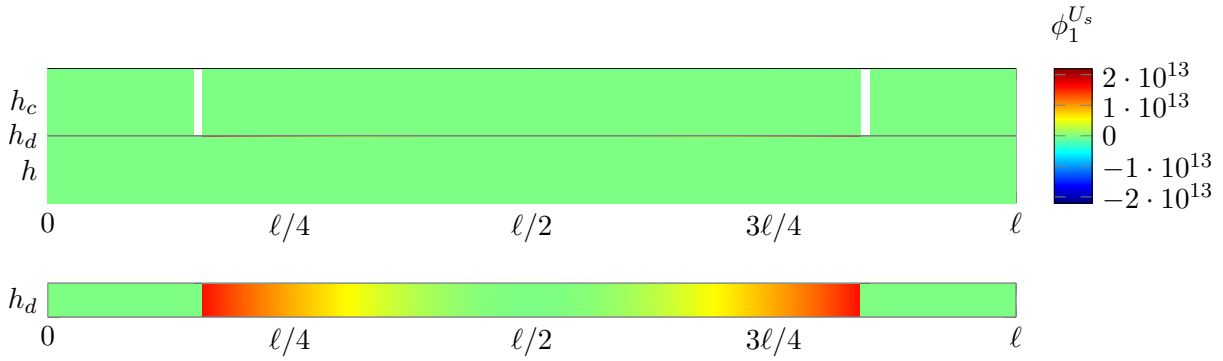
Figure 6.111 1st shear strain energy mode shape of a free-free beam with full coverage



(a) Clamp patch

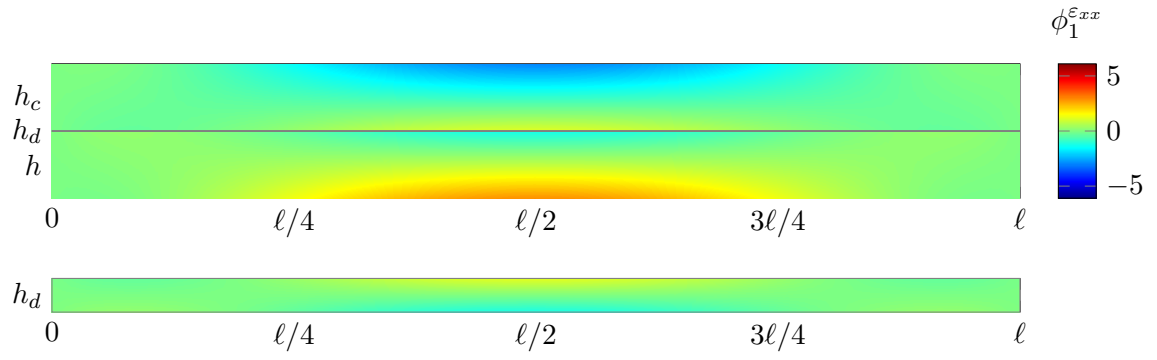
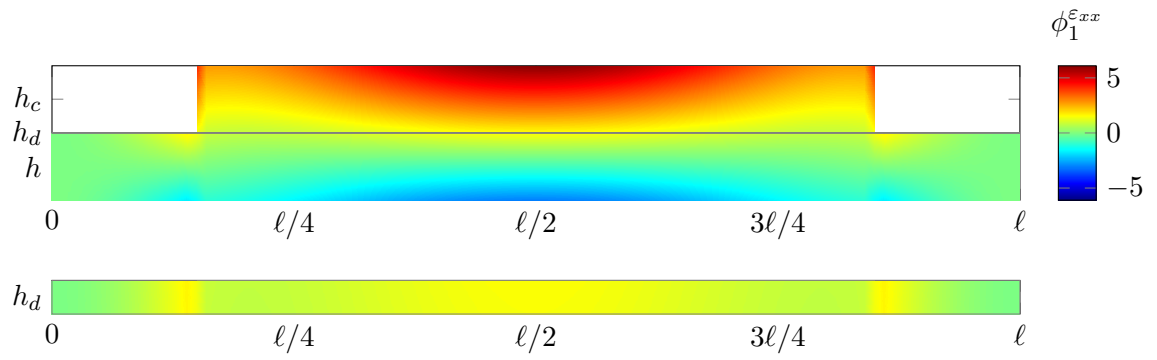


(b) Segmented treatment with one cut

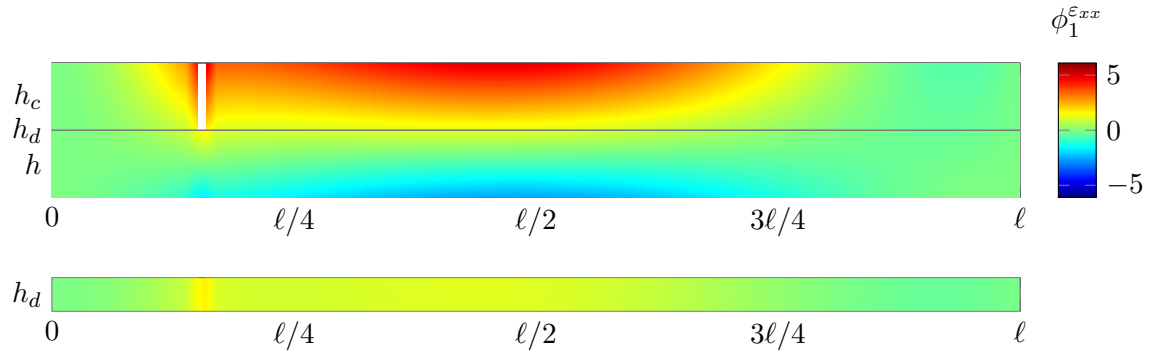


(c) Segmented treatment with two cuts

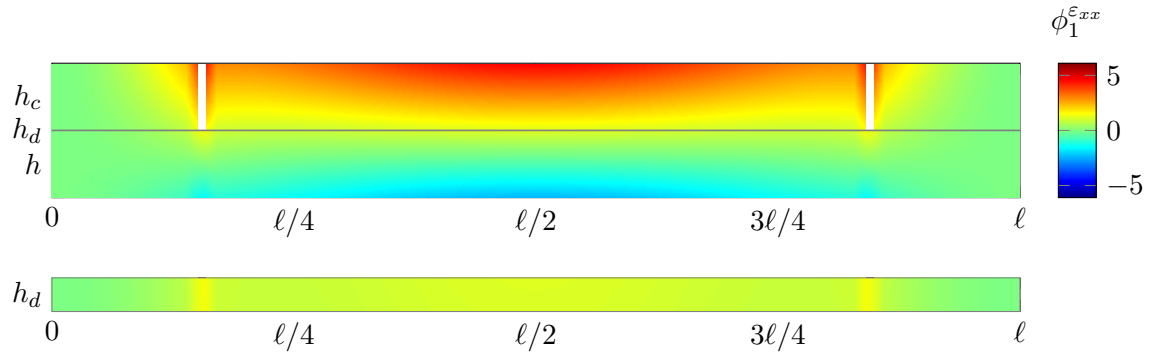
Figure 6.112 1st shear strain energy mode shapes of a free-free beam with different treatments


 Figure 6.113 1st normal strain mode shape of a free-free beam with full coverage


(a) Clamp patch



(b) Segmented treatment with one cut



(c) Segmented treatment with two cuts

 Figure 6.114 1st normal strain mode shapes of a free-free beam with different treatments

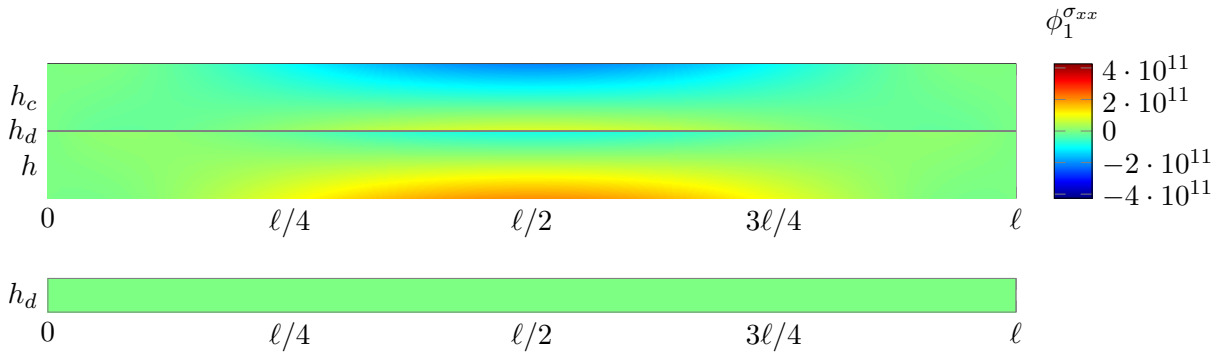
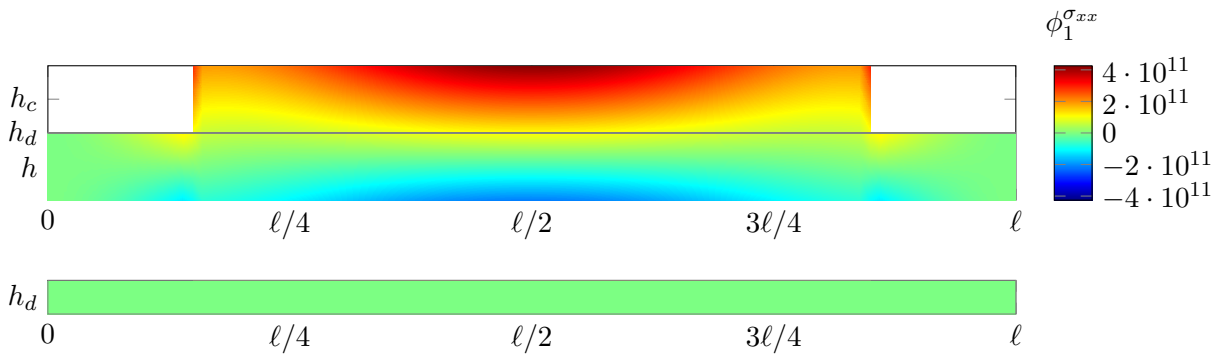
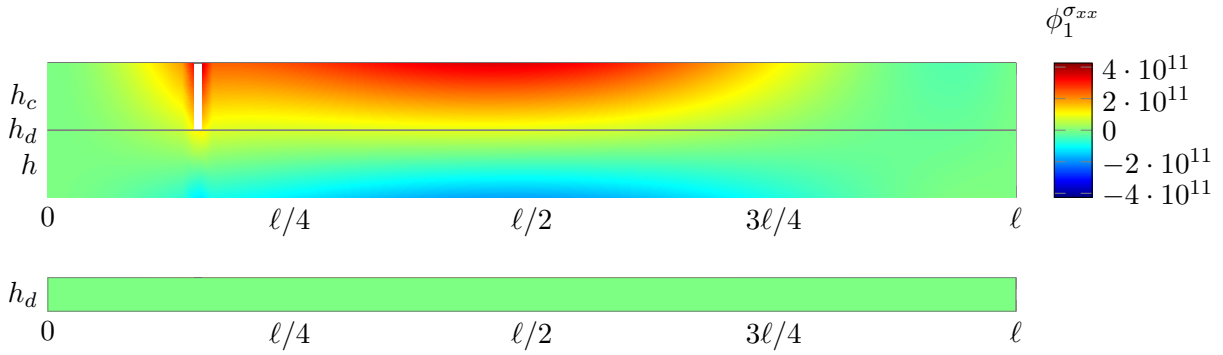


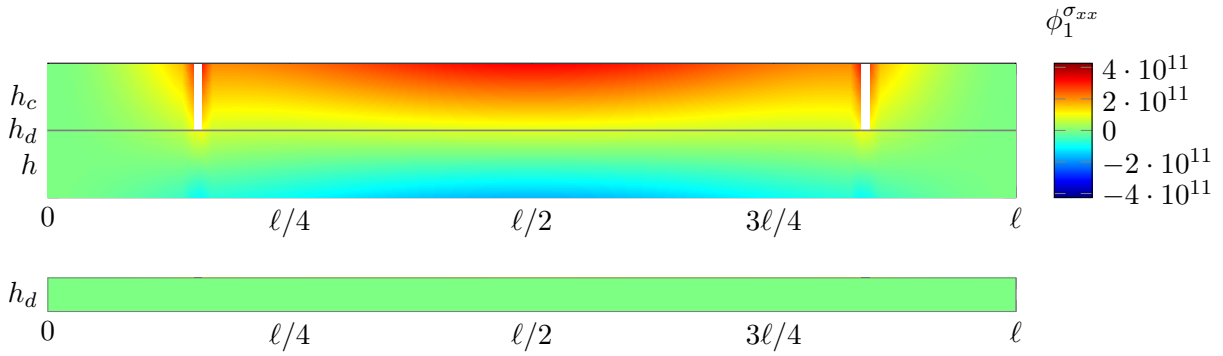
Figure 6.115 1st normal stress mode shape of a free-free beam with full coverage



(a) Clamp patch



(b) Segmented treatment with one cut



(c) Segmented treatment with two cuts

Figure 6.116 1st normal stress mode shapes of a free-free beam with different treatments

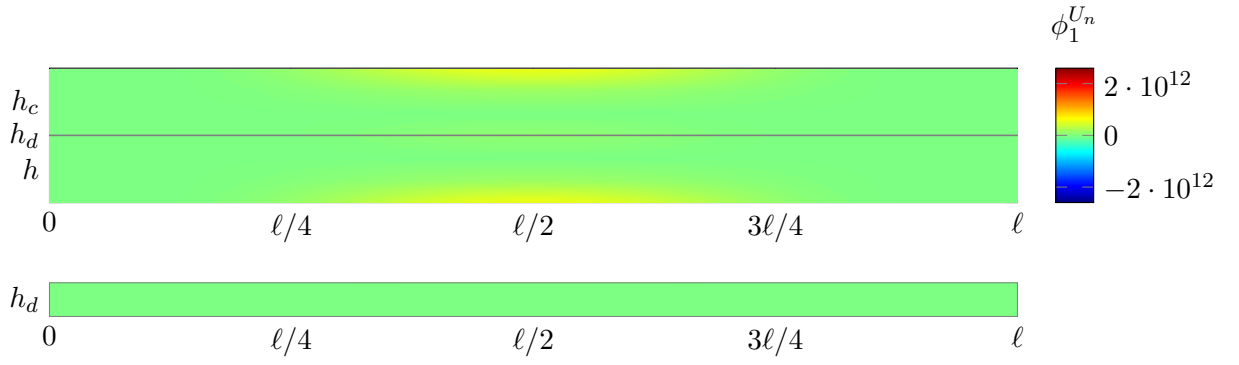


Figure 6.117 1st normal strain energy mode shape of a free-free beam with full coverage

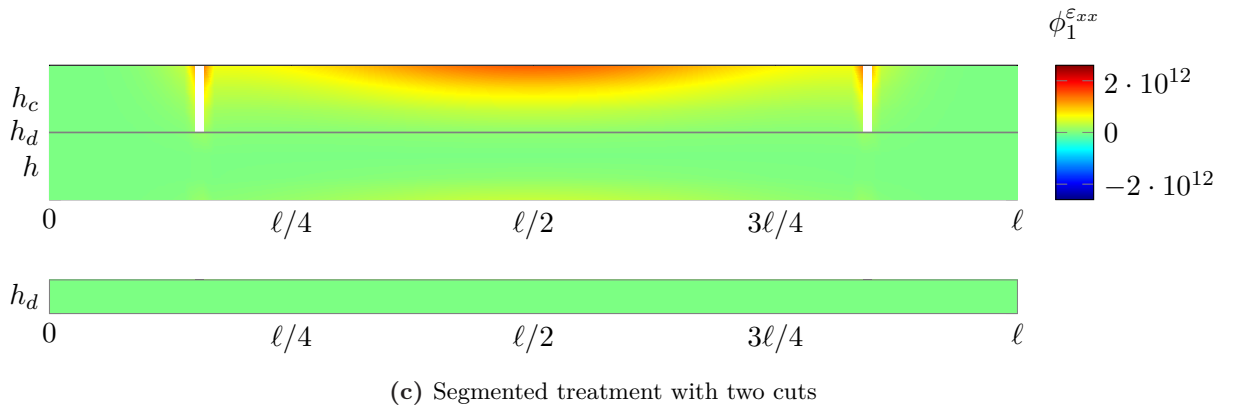
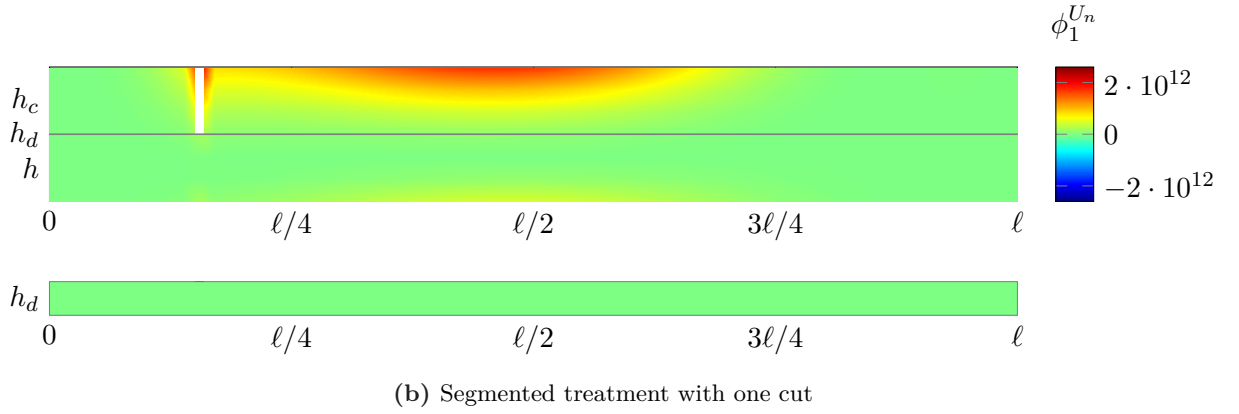
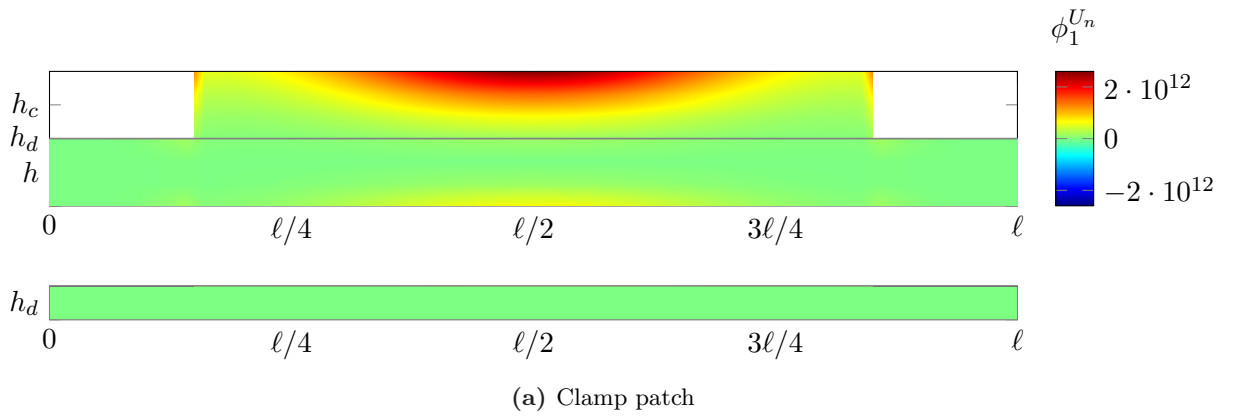
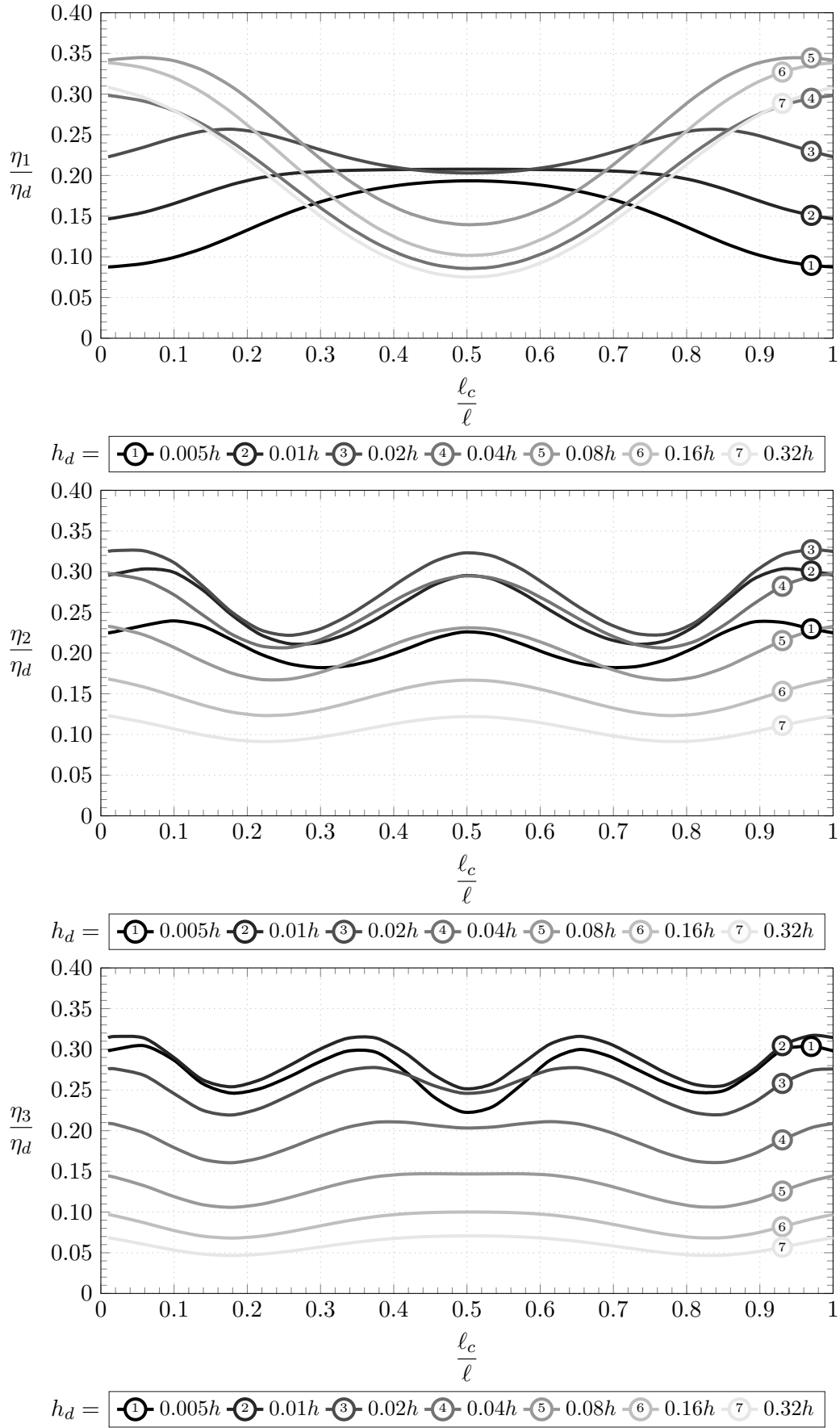
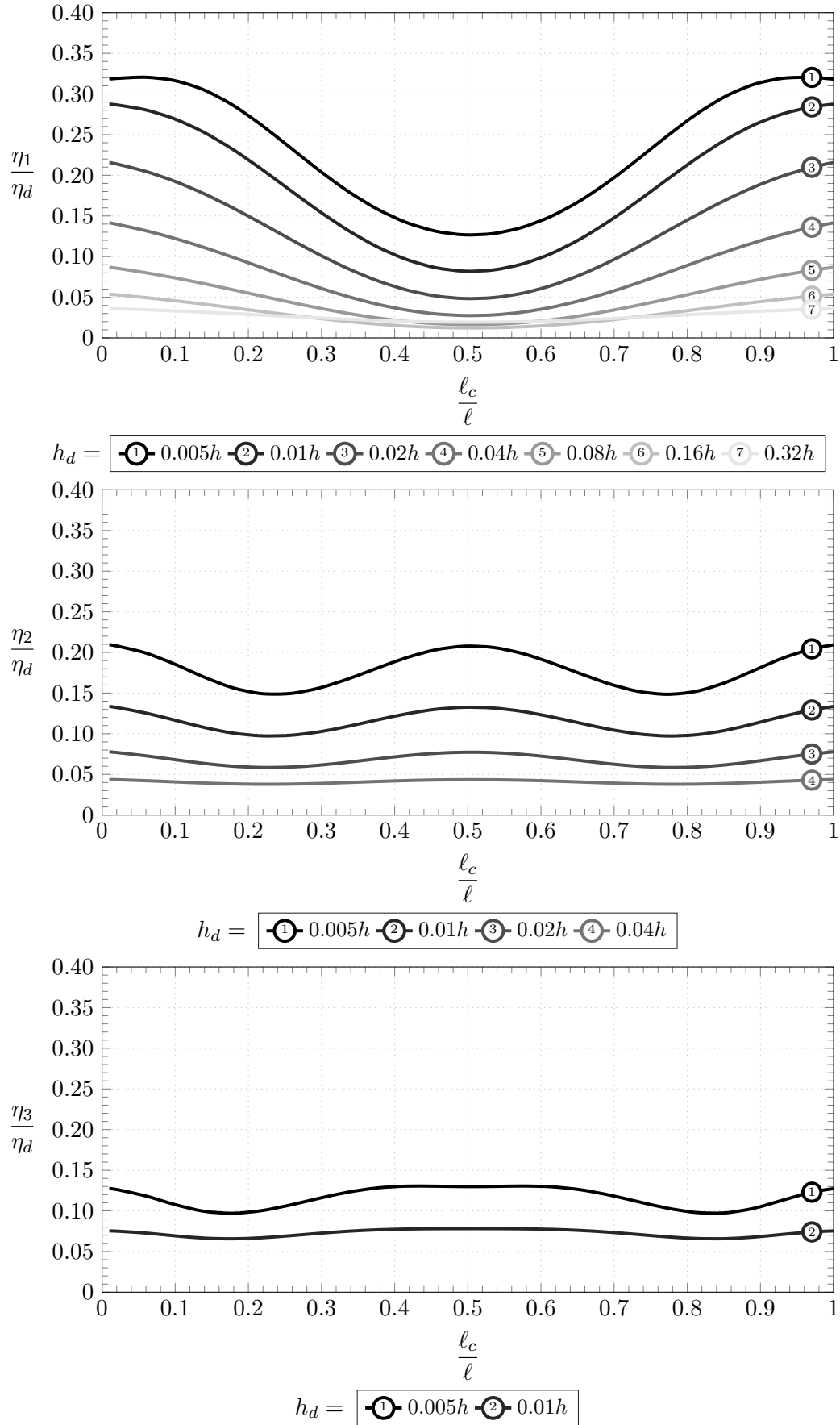
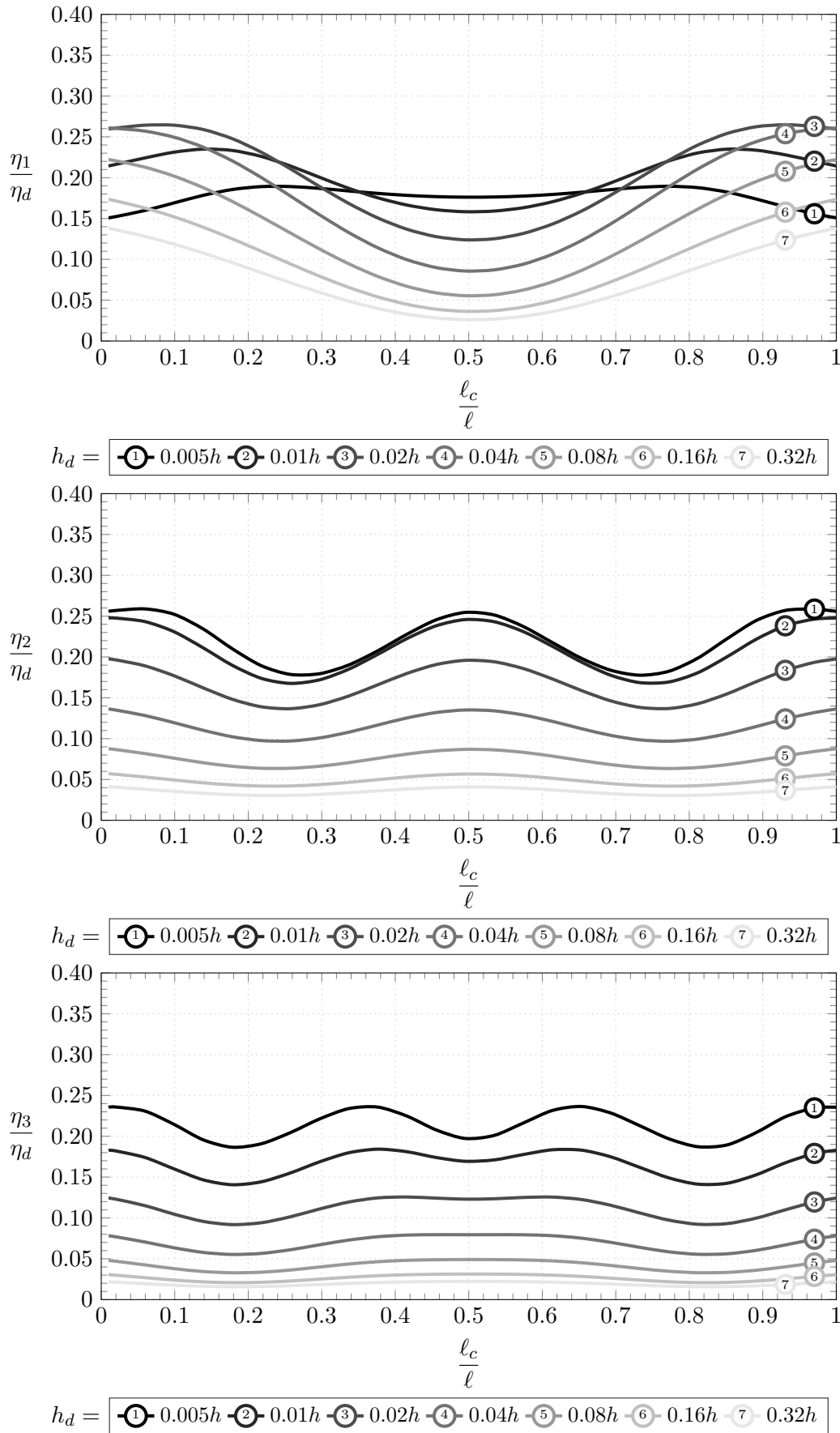
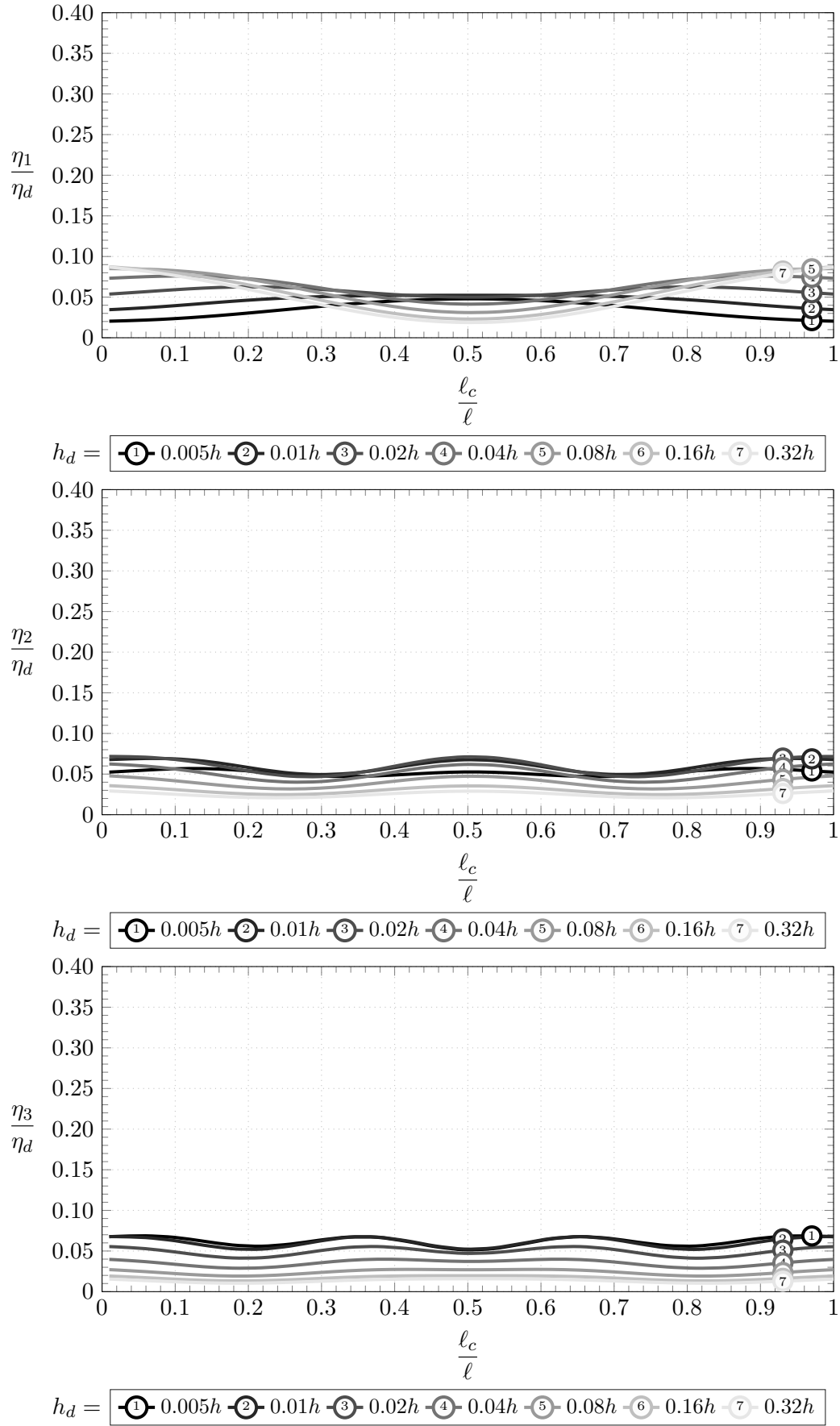


Figure 6.118 1st normal strain energy density mode shapes of a free-free beam with different treatments

Figure 6.119 Damping efficiency curves, $h = 0.005\ell$, $h_c = h$


 Figure 6.120 Damping efficiency curves, $h = 0.02\ell$, $h_c = h$

**Figure 6.121** Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.5h$


 Figure 6.122 Damping efficiency curves, $h = 0.01\ell$, $h_c = 0.1h$

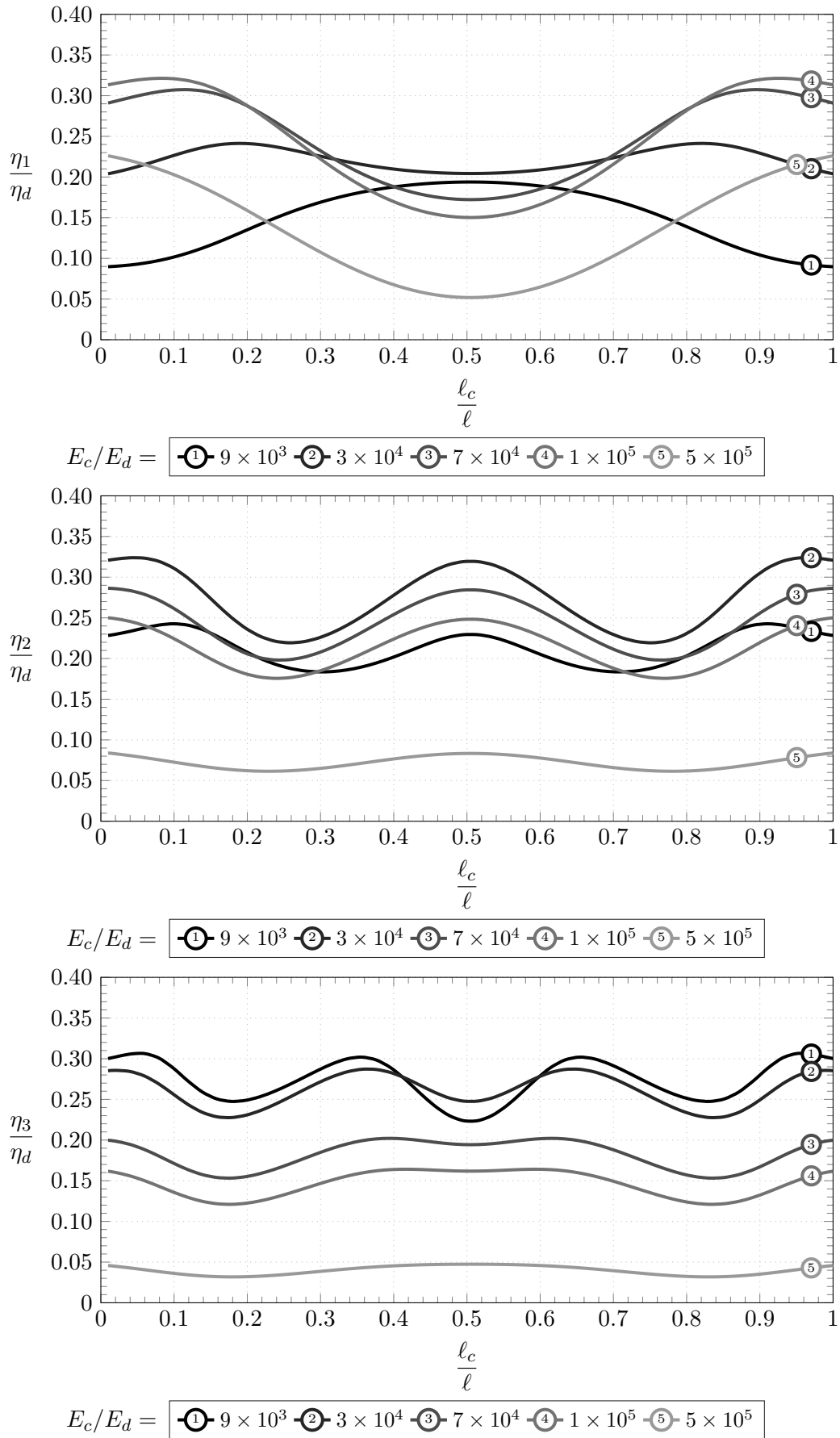
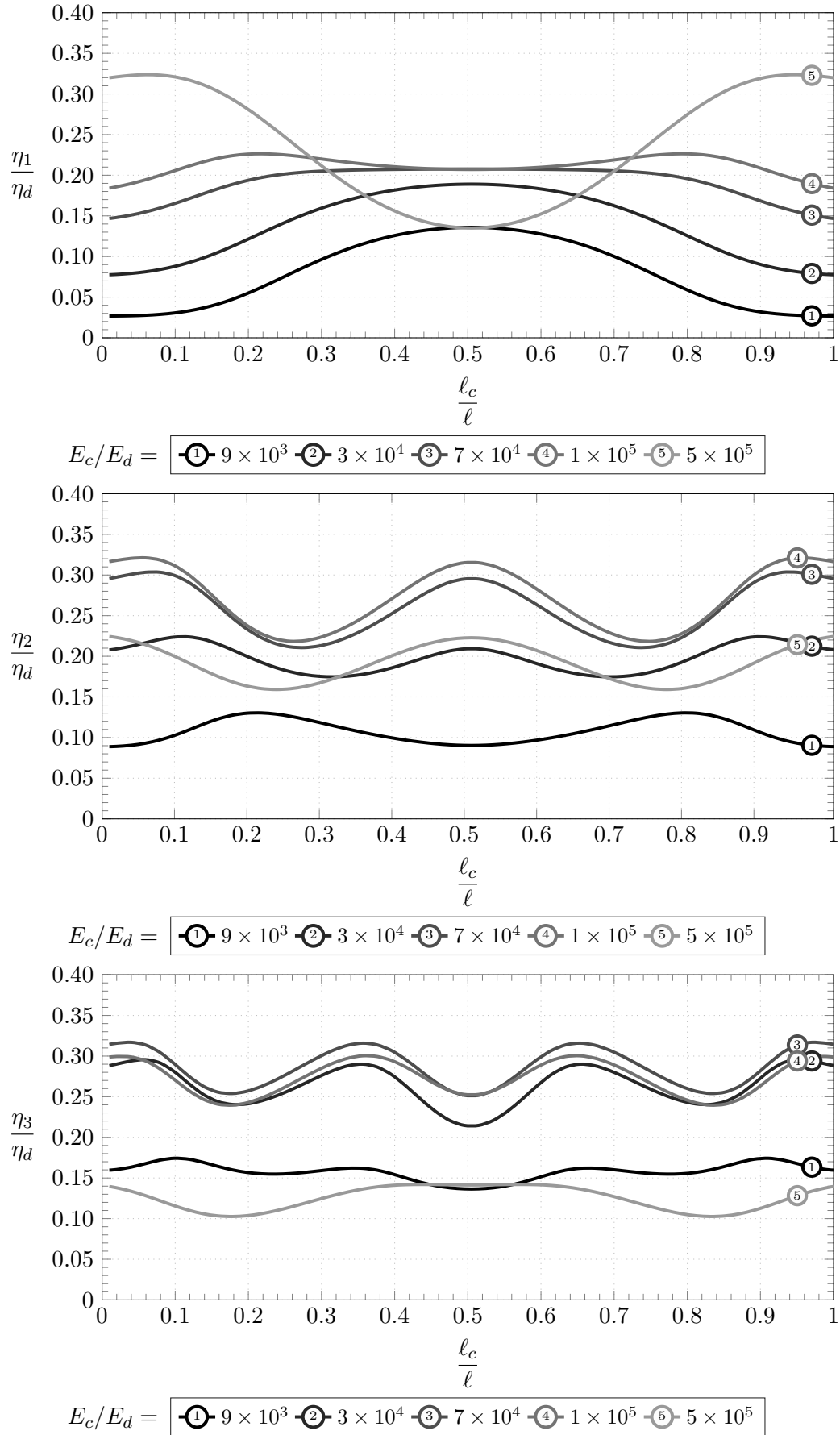


Figure 6.123 Damping efficiency curves, $h = 0.01\ell$, $h_d = 0.01h$, $h_c = h$


 Figure 6.124 Damping efficiency curves, $h = 0.005\ell$, $h_d = 0.01h$, $h_c = h$

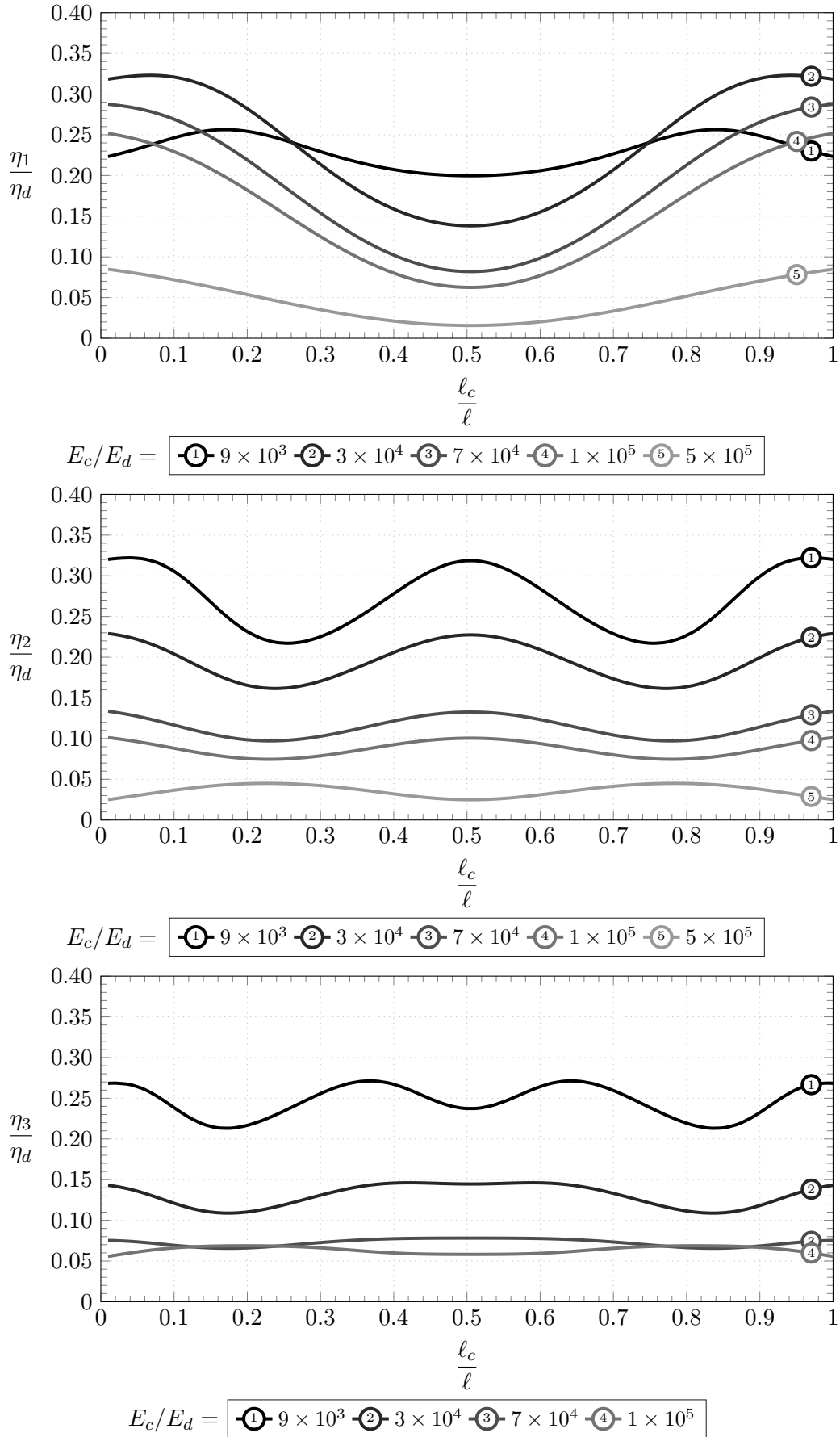


Figure 6.125 Damping efficiency curves, $h = 0.02\ell$, $h_d = 0.01h$, $h_c = h$

6.6 | Partially Covered Plates

In the following sections, one extends the study of partially covered structures to plates. The uselessness of full coverage in passive constrained layer damping, already investigated for beams and noticed by some authors, like Lall et al. [1987] for plates, is the primary motivation for this study. Besides, Lall et al. [1987] restricted their investigation to square patches, which might not be the optimum shapes for improved damping efficiency and did not thoroughly explain the superior performance of some locations over others, presenting somewhat unexpected results considering some previous works like that of Yildiz and Stevens [1985] on additive damping. Yildiz and Stevens [1985] concluded that the central part of a simply supported plate is the most important one, and one should place damping material there; Lall et al. [1987] concluded that it is possible to achieve better damping performances with a square patch at one corner of the plate. One plausible explanation for these differences is the adoption of different damping mechanisms; while Yildiz and Stevens [1985] considered an FLD configuration, Lall et al. [1987] addressed the problem by considering a PCLD approach. In the previous chapter, one identified that the geometric and mechanical properties of the structure have critical importance on the performance of damping layer treatments. Therefore, it would not be surprising that different geometric and mechanical properties could influence the best positions for partial damping treatments. The primary motivation for this section is to understand the basic mechanisms explaining the improved damping performances of some treatments over others, including either the modal patterns of the structure or the geometric and mechanical properties.

One performs a parametric analysis considering different patch positions to evaluate the effect of a viscoelastic damping patch. Figure 6.126 illustrates the numbered locations proposed by Lall et al. [1987] for square patches. One considers the three most relevant positions considered by Lall et al. [1987] in their study - Position 1, Position 3 and Position 13. Apart from being well-defined positions like the corner, mid-side length and central area of the plate, respectively, these positions allow a comprehensive study of the influence of the covered area since one can perform a parametric analysis with covered areas ranging from zero to full, keeping the square patches. Nevertheless, square patches offer a limited analysis of the damping efficiency of a patch treatment since this shape may not be the one providing optimal efficiency. Therefore, Figure 6.127 presents the non-square patch treatments proposed in this study; the L-contour patch treatment in Figure 6.127a, the Cross-Shape patch treatment in Figure 6.127b, and the Contour patch treatment in Figure 6.127c. The L-Contour patch treatment has equal-sized arms, and one can see it as the counterpart treatment of the square patch at Position 1 since the non-patched area with this treatment is a square at the corner. The Cross-Shape treatment also has equal-sized arms, performing a Greek cross and covering the nodal lines of the second and third transverse displacement mode shapes, an important feature to study the influence of damping patches preferably covering these regions. Finally, one also considers a Contour patch treatment with equal-sized strips of damping treatment covering all the edges of the plate and leaving a non-patched central square area, making this treatment the counterpart of the square patch at Position 13.

Taking advantage of the mechanical model implemented in this work, namely its ease of use concerning passive constrained damping patches, it is possible to execute a simple routine to perform a set of parametric studies regarding the damping efficiency of partial damping treatments other than square patches. Following the same reasoning of the previous chapter, one studies the damping efficiency of partially covered plates using parametric analyses for the optimal design of such structures. The main goal is to explore the effect of different design parameters such as the covered area by a patch, thickness/side length ratio, constraining layer thickness, material modulus, natural mode or boundary conditions on the damping efficiency following the

Table 6.3 Geometric properties of the plate

Length	Width	Thickness		
		Base	Damping	Constraining
a	b	h	h_d	h_c
0.3 m	0.3 m	-	-	$0.5h$

Table 6.4 Material properties of the plate

	Aluminium	Damping Layer
Young's Modulus	70 GPa	1 MPa
Poisson's Ratio	0.32	0.49
Density	2710 kg m^{-3}	1140 kg m^{-3}
Loss Factor	-	0.3

6.7 | Damping Efficiency of Partially Covered SSSS Plates

In this section, one considers the influence of different parameters of a damping patch, such as the patch position, covered area, thickness/side length ratio, constraining layer thickness and material modulus on the damping performance of a simply supported square plate.

6.7.1 | Influence of Covered Area

Here, one performs a comparative analysis regarding the damping efficiency of constrained partial damping treatments considering the influence of the covered area. Considering a reference base plate, the main goal is to identify the impact of the covered area on the damping performance of different partial damping treatments.

6.7.1.1 | Reference Plate - $h = 0.01a$

Figure 6.128 presents the influence of the covered area by a patch at Position 1 on the damping capacity using the first three specific modal loss factors for different thicknesses of the damping layer. Following the same reasoning of the previous chapter, one considers the thickness of the damping layer as a function of the base layer thickness. It is noticeable that the damping capacity of the first mode depends on the covered area, increasing almost steadily with the amount of damping material. Nevertheless, these results suggest a maximum attainable damping capacity for the first mode accomplished with full coverage. For damping layers thicker than $0.01h$, the highest specific modal loss factor decreases, proving the existence of a threshold for maximum damping capacity. It is the increasing-decreasing effect of the damping capacity already studied in the previous chapter for the case of beams. As discussed earlier, there is more damping material with thicker damping layers, but shear strain at the viscoelastic layer drops.

Damping efficiency curves for higher modes show that thick damping layers mean less damping capacity for the damping layer thicknesses under analysis. Presumably, the threshold value of damping layer thickness for the increasing-decreasing behaviour of the damping capacity occurs for even thinner damping layers. Following the same trends of the first specific modal loss factor, the second and third specific modal loss factors increase steadily with the covered area. The

noticeable sharp jumps of the second and third specific modal loss factors for small covered areas occur due to the switch between the second and third mode shapes during the parametric analyses since these two modes of the symmetric bare square plate have the same frequency. As the parametric studies consider different covered areas, there is a switch between the mode shapes of the corresponding second and third modes, explaining the visible sharp jumps. It is worth mentioning that the unexpected rise of the second specific modal loss factor matches the sudden drop of the third specific modal loss factor. A square damping patch at Position 1 promotes higher damping capacity in the third mode.

Figure 6.129 presents the same damping efficiency curves for a square patch at Position 3. There are no significant differences compared with the results for Position 1. The threshold of the increasing-decreasing behaviour of the first specific modal loss factor occurs for the same damping layer thickness, presumably thinner for the higher modes. The sharp jumps of the second and third specific modal loss factors are also present for a square patch at this position.

Subsequently, Figure 6.130 shows the influence of the covered area and different damping layer thicknesses on the damping efficiency curves considering a square patch at the central Position 13. There are noticeable differences between a square patch at this position and the previous ones; the increase of the first specific modal loss factor with the covered area is no longer steady. The growth rate of damping capacity is higher for small covered areas and lower as the covered area increases. Such effect is evident for the damping layer thicknesses promoting higher damping capacities, proving the potential to achieve higher damping with partial coverage. In fact, for the thinner damping layers, partial treatments between 50 % and 100 % covered areas promote higher damping capacities with bounded differences between covered areas. However, the damping layer thickness threshold for the increasing-decreasing behaviour of damping capacity is the same. Higher specific modal loss factors present the same trends as the previous positions with thicker damping layers, promoting less damping. The structure holds symmetry in both directions for a square patch at the central Position 13, and the second and third specific modal loss factors present the same results. It is relevant to point out the poor damping efficiency and its low growth rate for small covered areas, which proves that the plate's central area is not the best location for improved damping capacity of the second and third modes.

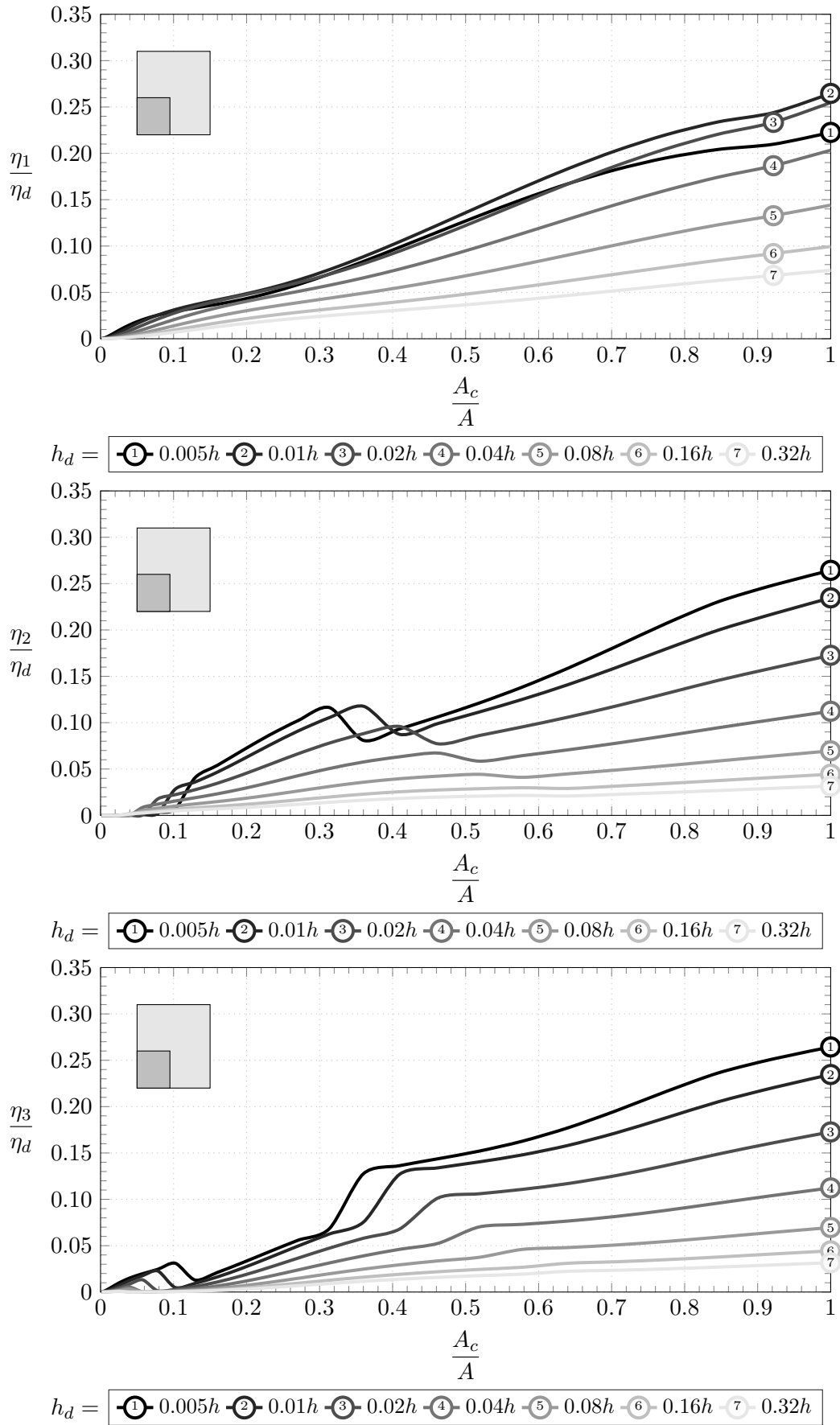
Figure 6.131 presents the damping efficiency curves for the L-Contour treatment, the first non-square patch treatment considered in this work. The results are noticeable, and the lower damping capacity of this treatment for partial coverage is indisputable either for the first or higher specific modal loss factors. Naturally, the damping efficiency for fully covered plates is the same for every treatment, and the threshold of the increasing-decreasing behaviour holds. However, the relatively poor performance of the L-Contour treatment compared with other positions is clear.

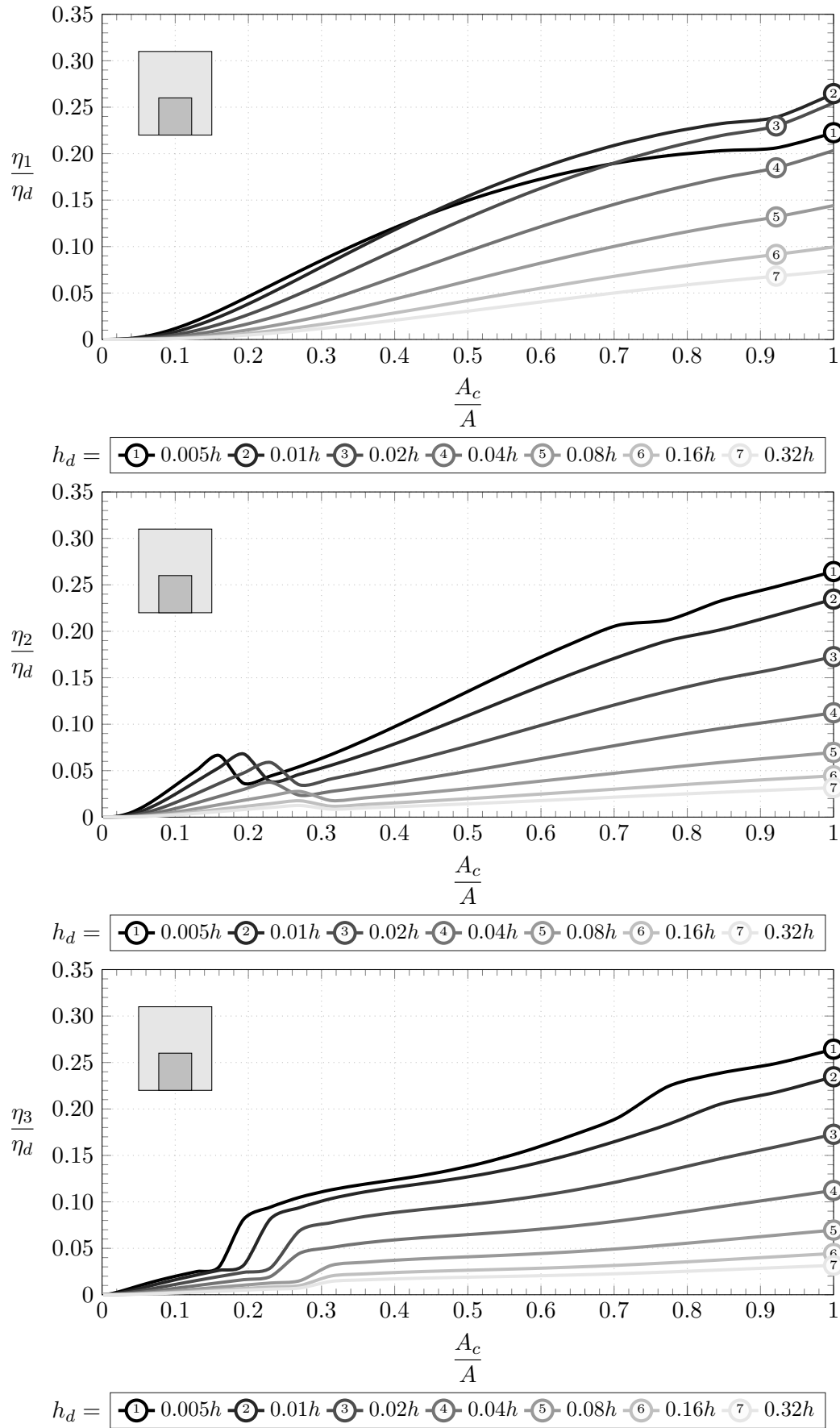
In Figure 6.132, one can investigate the damping efficiency curves for the Cross-Shape treatment. Contrary to the L-Contour treatment, a damping patch with this shape ensures a significant damping capacity, even for small covered areas. This also holds for the higher specific modal loss factors, for which small covered areas promote a significant damping capacity, contrary to the previous damping treatments. Repeatedly, the threshold value of the damping layer thickness for the increasing-decreasing behaviour of damping capacity is the same, and higher specific modal loss factors present the same trends of the previous positions with thicker damping layers, promoting less damping. Similarly to the square patch at the central Position 13, the structure holds its symmetry in both directions, and the second and third specific modal loss factors present the same results.

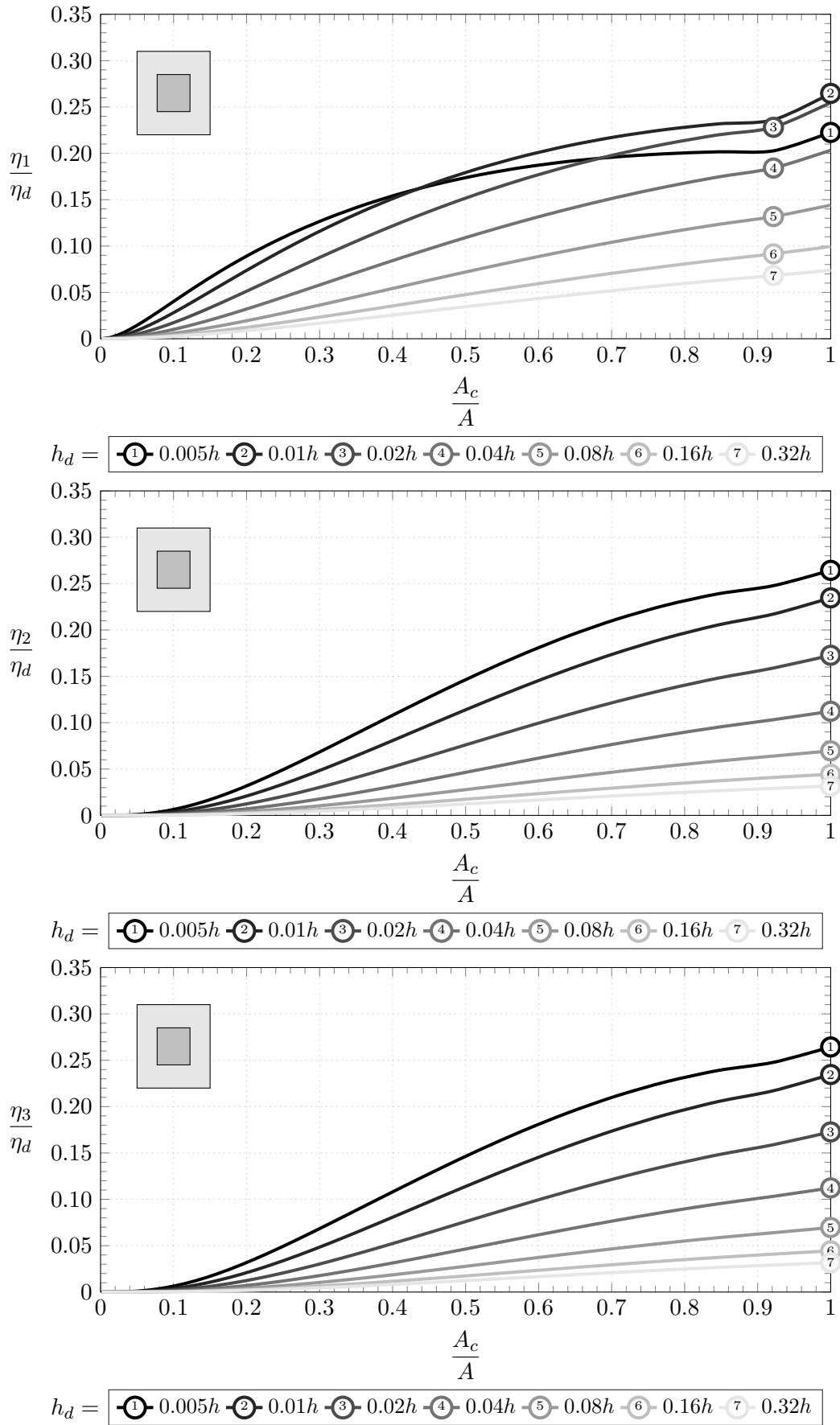
Finally, Figure 6.133 depicts the damping efficiency curves for the Contour treatment. These are the most distinctive damping efficiency curves among all the considered treatments. As it is clear from Figure 6.133, the Contour treatment promotes poor damping capacities for all the

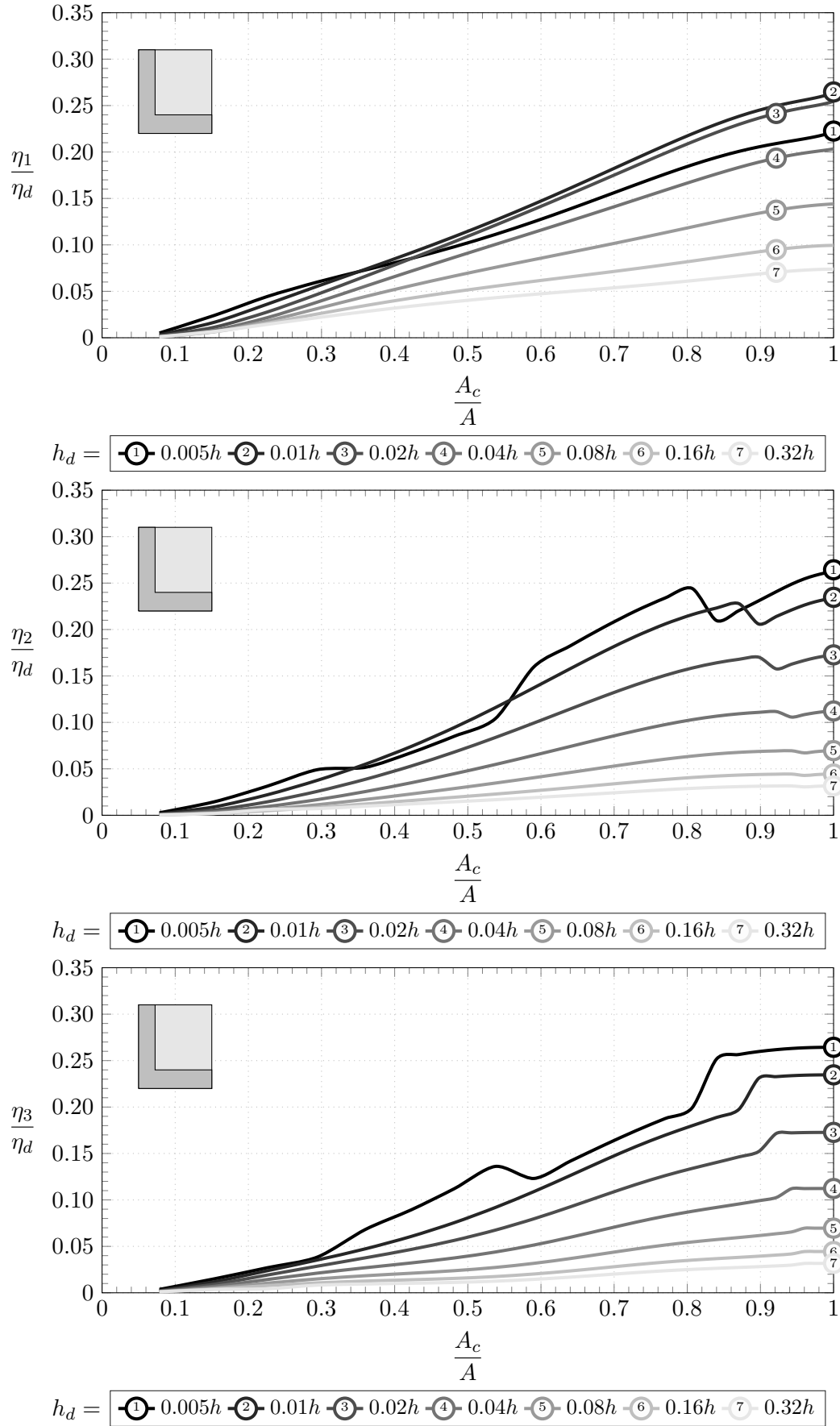
modes studied. It is just for large covered areas, with the damping treatment increasing towards the centre of the plate, that damping capacity increases. These results prove the inefficiency of the Contour treatment to raise the damping capacity of a simply supported plate.

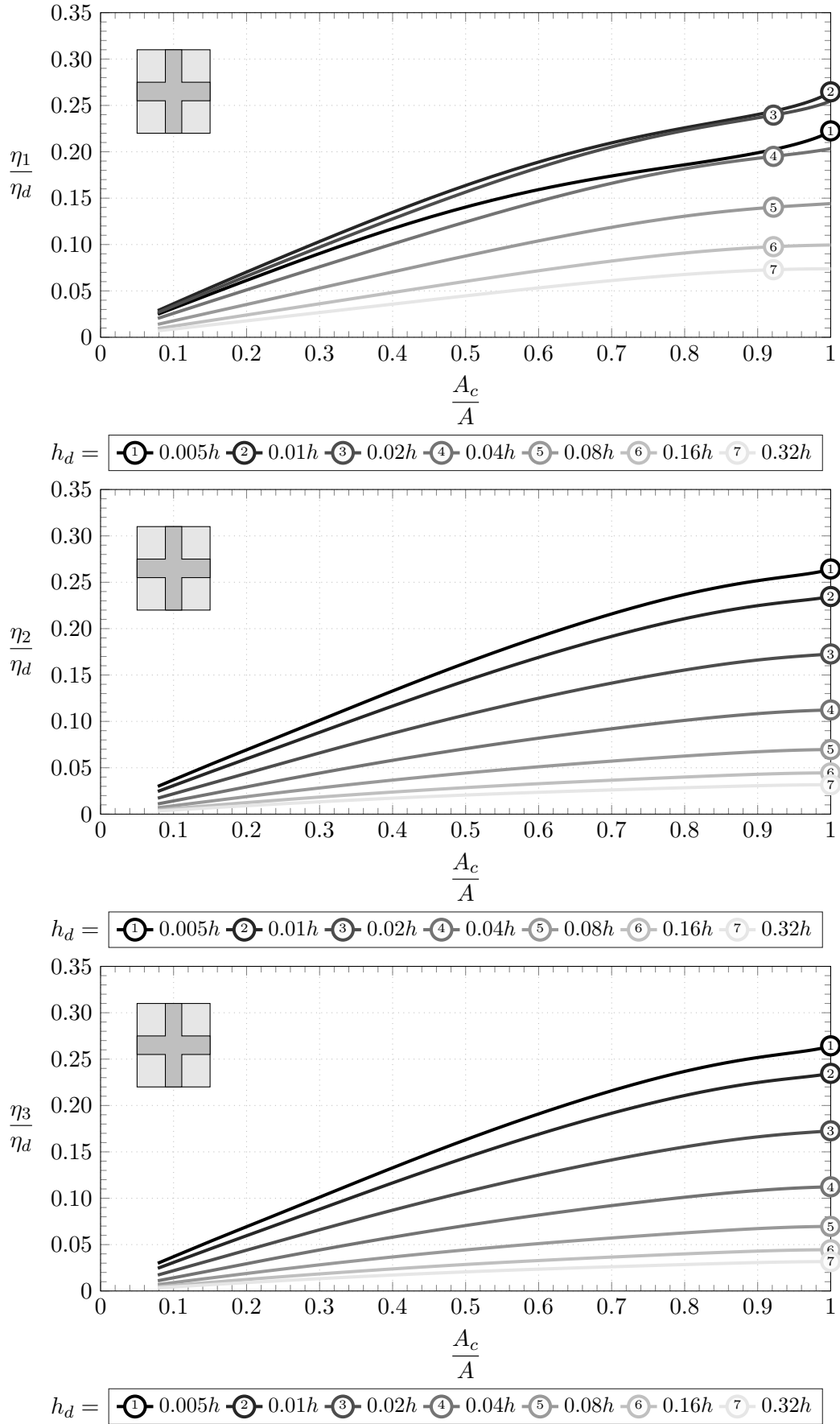
These results show the influence of the covered area and damping layer thickness on the damping capacity for the first three modes of vibration considering different partial constrained damping treatments. The conclusions are significant; one can identify the threshold of the increasing-decreasing behaviour and distinguish damping efficiency patterns between treatments. However, it is difficult to perform a detailed comparative analysis between positions and conclude about the optimal solutions for increased damping performance. In the following section, one studies the influence of the patch position considering a comparative analysis between the different damping treatments.

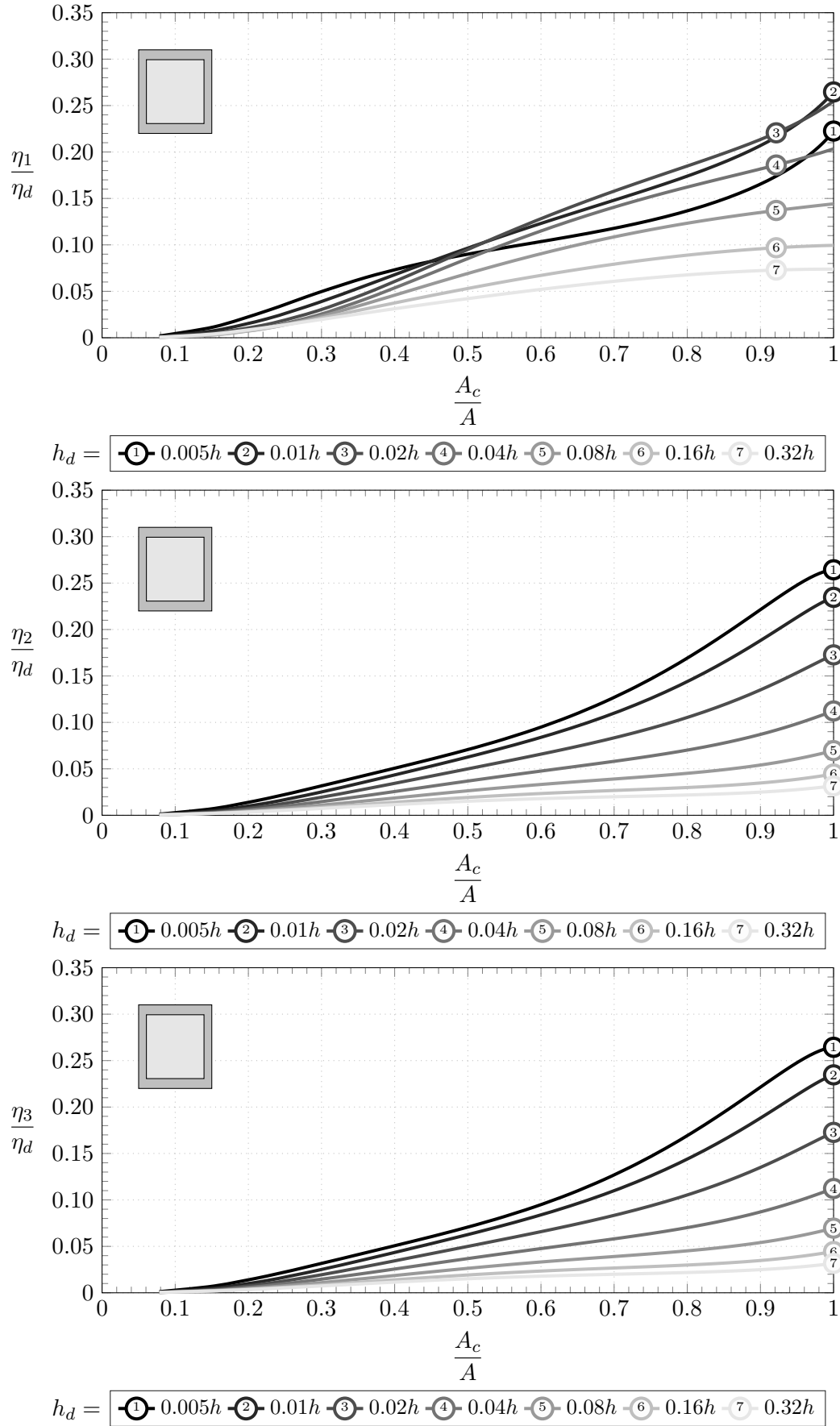

 Figure 6.128 Damping efficiency curves for Position 1, $h = 0.01a$, $h_c = 0.5h$

Figure 6.129 Damping efficiency curves for Position 3, $h = 0.01a$, $h_c = 0.5h$

Figure 6.130 Damping efficiency curves for Position 13, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.131 Damping efficiency curves for L-Contour treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.132 Damping efficiency curves for Cross-Shape treatment, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.133 Damping efficiency curves for Contour treatment, $h = 0.01a$, $h_c = 0.5h$

6.7.2 | Influence of Patch Positioning

In this section, one performs a comparative analysis regarding the damping efficiency of constrained partial damping treatments. The primary goal is to identify the best positions for increased damping performance and understand the mechanisms which lead to improved damping capacity. Taking this into account and following the conclusions of the previous chapter regarding the study of constrained layer damping treatments in beams, it is important to analyse the mode shapes of the plate to understand the obtained results and the most appropriate areas for partial damping treatments. Figure 6.134 illustrates the well-known transverse displacement mode shapes of the first three normal modes of the simply supported square reference plate considering a constrained layer damping treatment with full coverage and a damping layer thickness at the threshold value for the increasing-decreasing behaviour of damping capacity at the first mode of vibration - $h = 0.01a$, $h_d = 0.01h$ and $h_c = 0.5h$. In the previous chapter, one also discussed the importance of the extensional strain of the constraining layer as a mechanism to impose shear on the damping layer. Consequently, Figure 6.135 presents the curvature mode shapes computed in this work using the finite element deformation matrices to understand the bending pattern of the plate and the preferential positions for constrained damping treatments. Furthermore, mode shapes associated with the shear strain degrees of freedom γ_{xz} and γ_{yz} might be important to understand the modal shear strain pattern of the plate and figure out important aspects of the vibration energy dissipation - see Figure 6.136.

6.7.2.1 | Reference Plate - $h = 0.01a$

As previously discussed, it is relevant to perform a detailed comparative analysis between different patch positions and conclude regarding the optimal solutions for increased damping performance. In this subsection, one recovers the previous results to perform a comparative analysis between the different positions considered in this work for partial damping treatment. Figure 6.137 presents the first specific modal loss factor for different patch positions and covered areas simultaneously. Moreover, one studies the damping efficiency of the various patch positions considering three different damping layer thicknesses - $h_d = 0.005h$, $h_d = 0.01h$ and $h_d = 0.32h$. One deals with the thinnest and thickest damping layer thicknesses under analysis in this study, as well as the threshold value of damping layer thickness for the increasing-decreasing behaviour of damping capacity at the first vibration mode.

Figure 6.137a shows the damping efficiency curves for the first specific modal loss factor considering the thinnest damping layer thickness under analysis - $h_d = 0.005h$. It is worth mentioning the significant damping capacity achievable with this damping layer thickness. However, the most noteworthy feature is to compare damping patches considering the same covered area. One can clearly distinguish the damping efficiency curves between two reference positions; the Contour treatment and the Position 13 treatment. The square patch at the central position (Position 13) presents the best damping efficiency for almost the entire domain of the covered area. On the other hand, the Contour treatment exhibits the lowest damping capacity. The differences between these two counterpart treatments are significant; for 60% covered area, the damping treatment at the centre of the plate nearly doubles the damping capacity of the Contour treatment. Following these trends, the L-Contour treatment presents a slightly better damping capacity than the Contour treatment but emphasises the inefficiency of damping treatments at the edges of a simply supported plate. The Cross-Shape treatment promotes a significant damping capacity for small covered areas but loses efficiency compared to a square patch at Position 3 as the covered area increases. It is interesting to note that a square patch at Position 3 markedly increases its performance as it covers the central area of the

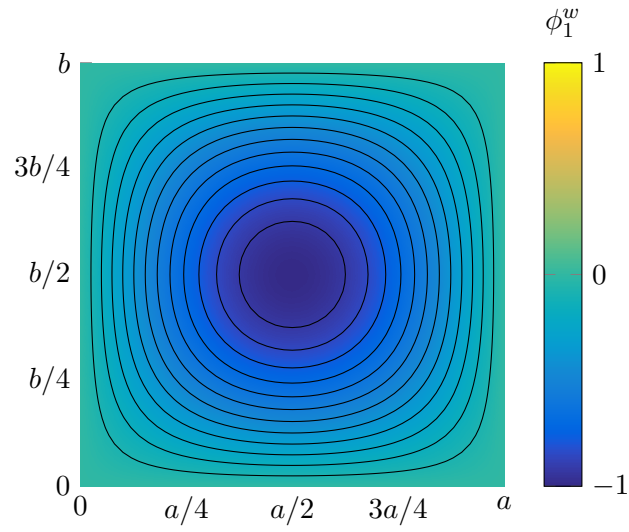
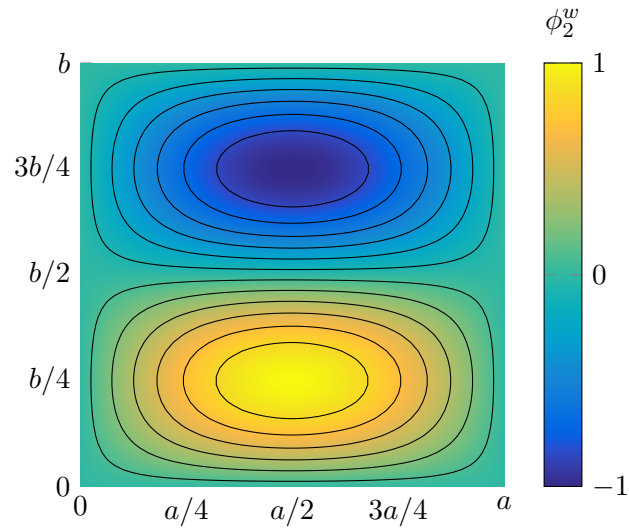
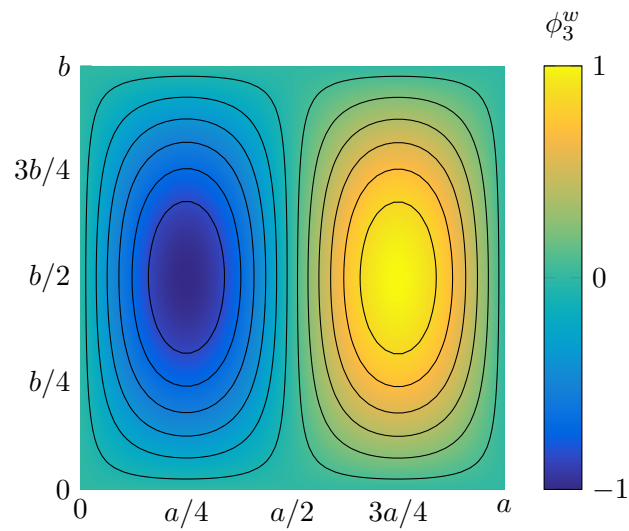

 (a) 1st transverse displacement mode shape

 (b) 2nd transverse displacement mode shape

 (c) 3rd transverse displacement mode shape

Figure 6.134 Transverse displacement mode shapes of a simply supported plate with damping treatment

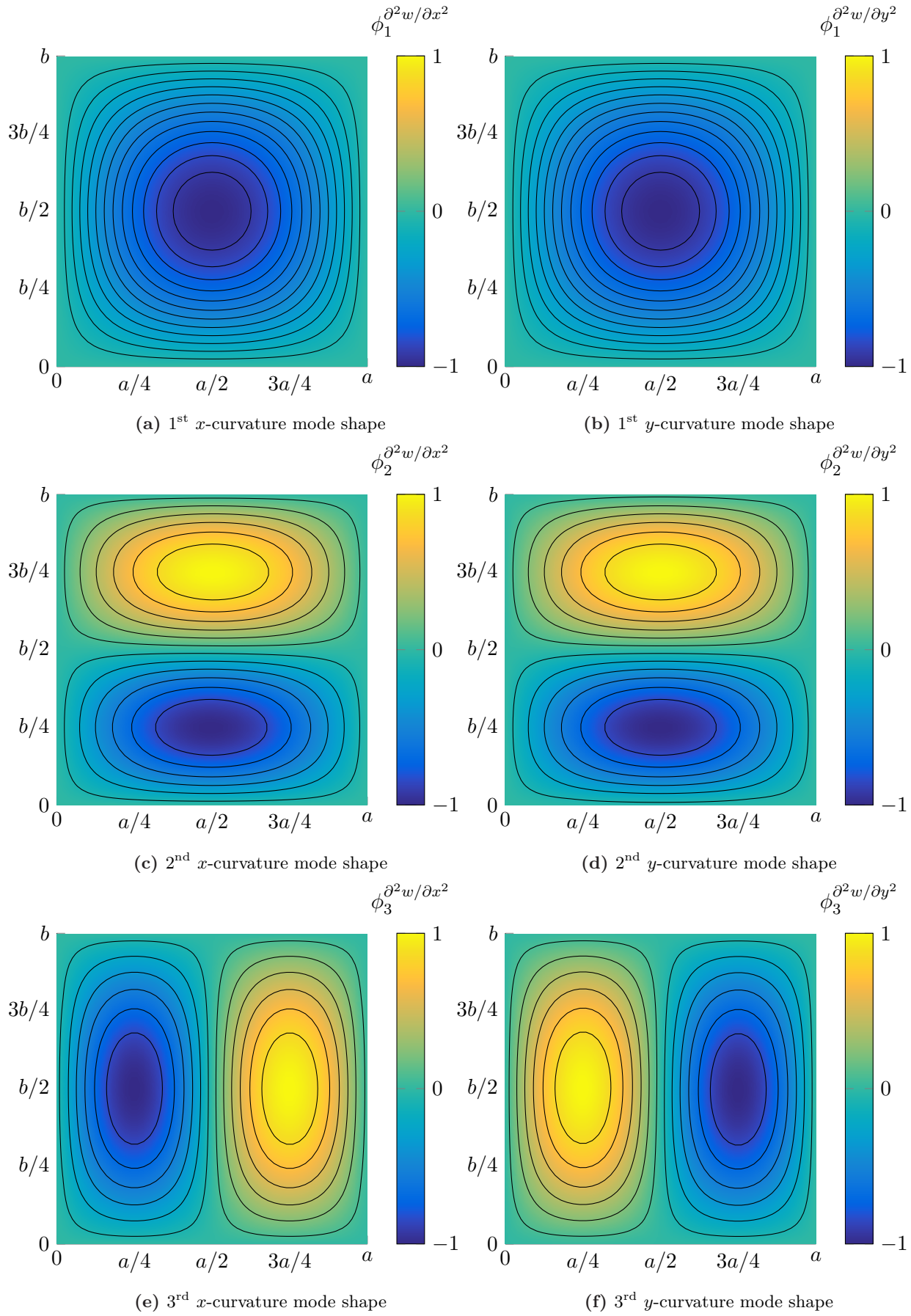


Figure 6.135 Curvature mode shapes of a simply supported plate with damping treatment

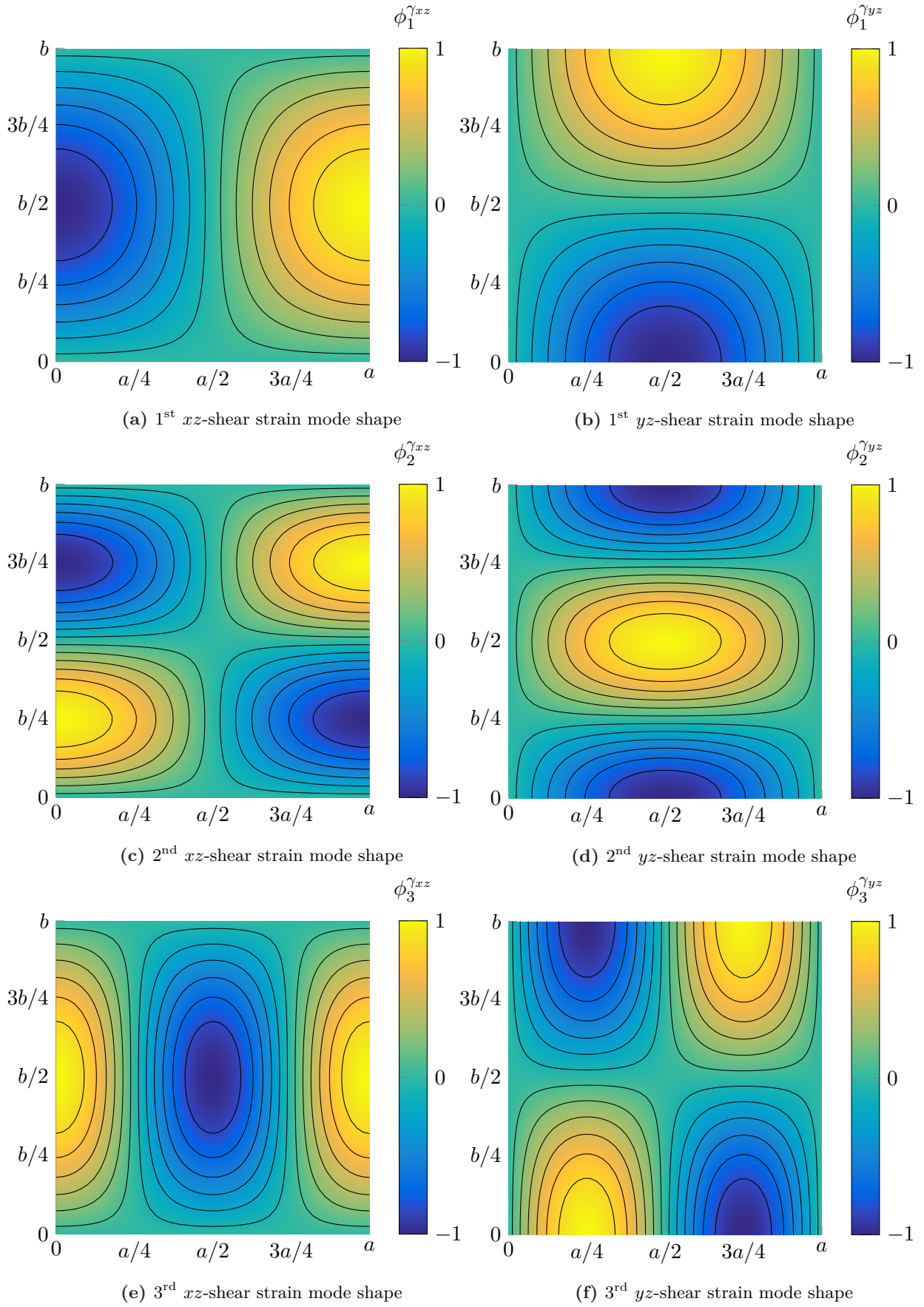
**Figure 6.136** Shear strain mode shapes of a simply supported plate with damping treatment

plate. Likewise, except for the Contour treatment, all treatments increase their performance as the covered area increases towards the centre of the plate. For a covered area of at least 80 %, there is no significant difference between all the damping treatments under analysis except for the Contour treatment.

In Figure 6.137b, one can compare the damping efficiency of different positions of damping treatments for the first mode of vibration, considering the threshold value for the increasing-decreasing behaviour of damping capacity - $h_d = 0.01h$. The results are similar to those considering the thinnest damping layer discussed above, with slightly improved performances. The patch at the central Position 13 still provides the best damping efficiency for almost the entire domain of the covered area in contrast with the Contour treatment, which once more presents the lowest damping capacity. Nevertheless, it is worth noting the improved damping performance of the Cross-Shape treatment for this damping layer thickness, a damping capacity close to that of the patch at the central Position 13. Except for tiny covered areas, the patch at Position 3 provides a better damping capacity than the one at Position 1, probably because a patch at Position 3 covers a larger portion of the central area as the covered area increases. Recalling Figure 6.135, one can analyse the x and y -curvature of the fully covered plate at the first mode of vibration and identify the maximum curvature for both directions at the centre of the plate, and so bending deformation of the constraining layer. As discussed in the previous chapter, it is crucial to ensure the extensional deformation of the constraining layer as a mechanism to impose shear on the damping layer. Consequently, damping treatments covering the central area of the plate promote better damping performances as the constraining layer experiences higher states of extensional deformation and so imposes major levels of shear strain on the damping layer. In contrast, treatments covering preferentially the contour of the plate, where the curvature and bending deformation are null, present limited damping performances.

At this point, it is mandatory to support the previous comments by justifying the obtained results based on the insights drawn from the preceding section regarding the partial damping of beams. Figures 6.138 and 6.139 present the first mass normalised shear strain mode shapes of simply supported plates with a 50 % covered area considering the different treatments under analysis in this work. The geometric and mechanical properties of the plate are those considered for the damping efficiency curves of the reference plate with a damping layer thickness of $h_d = 0.01h$. Transverse shear strain mode shapes in both directions, $\phi_1^{\gamma_{xz}}$ and $\phi_1^{\gamma_{yz}}$, disclose a pattern; it is possible to identify the corresponding damping treatment without reading the caption. It is so because the maximum values of modal shear strain occur at the edges of the damping treatment, where there are sharp transitions in the modal shear strain levels, illustrating the abrupt change in geometry and material distribution. These patterns follow the expected behaviour considering the results from the previous chapter, including the flat regions corresponding to the patched area. Such results prove the importance of the edges of the treatment regarding their damping capacity. Considering a 50 % covered area, Figure 6.137b shows the best damping treatments in the following order: Position 13, Cross-Shape treatment, Position 3, Position 1, L-Contour treatment and Contour treatment. Despite the valuable insights, it is not possible to justify this classification only by taking into account the transverse shear strain mode shapes.

For simply supported plates with a 50 % covered area considering the different treatments under analysis in this work and the valuable representations of the through-the-thickness mode shapes considered in the previous chapter, one follows here the same approach to explain the mechanisms leading to the improved damping performances of some treatments over the others. Figures 6.140 - 6.141 and 6.142 - 6.143 depict the through-the-thickness views of the first transverse shear strain energy density mode shapes, $\phi_1^{U_{xz}}$ and $\phi_1^{U_{yz}}$, respectively, considering section cuts along with both directions. As previously mentioned, Figure 6.135 shows that the

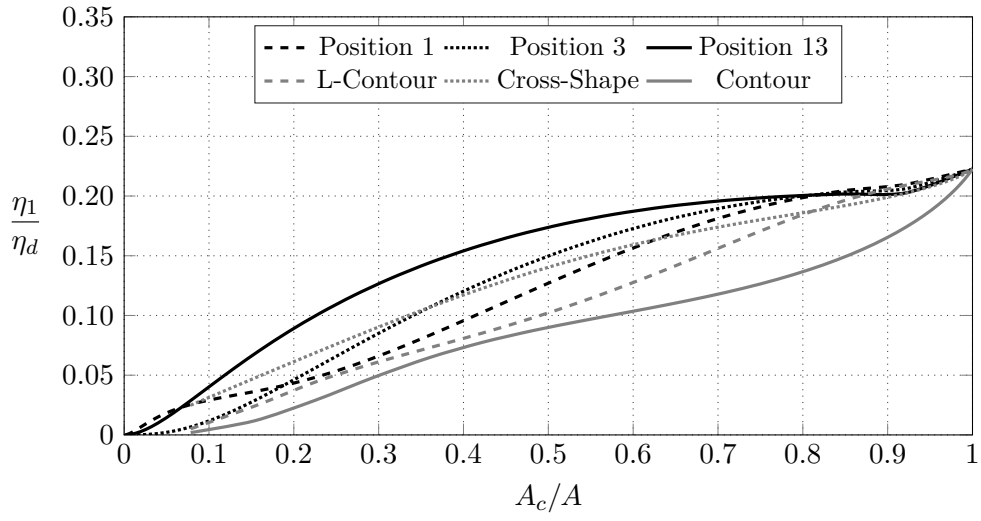
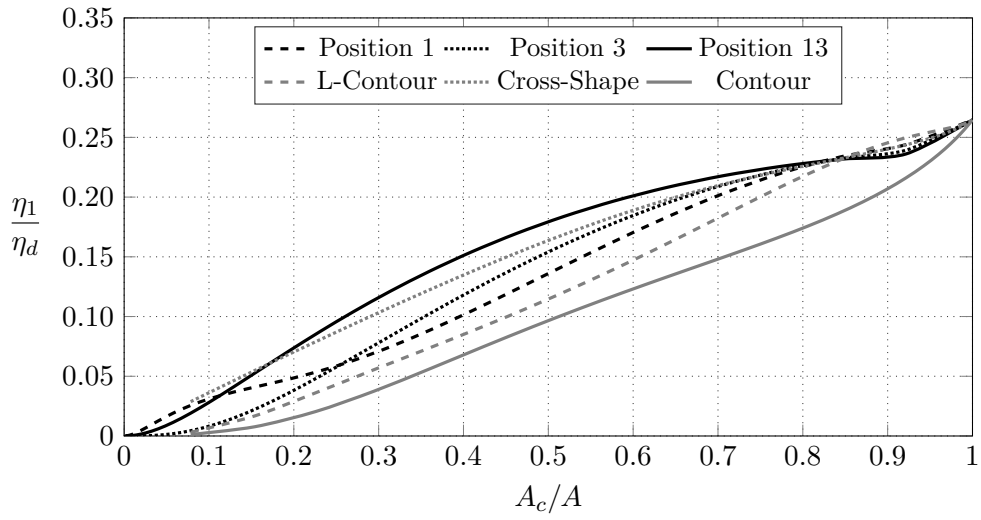
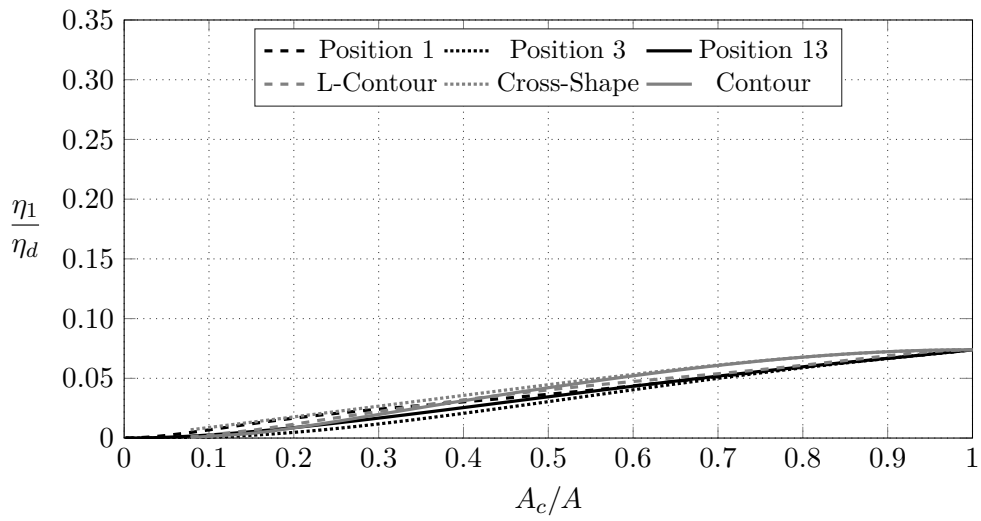
first x and y -curvature mode shapes present a similar pattern to the first transverse displacement mode shapes, with zero curvature at the edges and maximum curvatures at the centre of the plate. One expects higher levels of bending deformation of the constraining layer at the centre of the plate, which is crucial to ensure the extensional deformation of the constraining layer as a mechanism to impose shear on the damping layer. Thus, one considers a section cut at $b/2$ to represent the through-the-thickness view of the first transverse shear strain energy density mode shape $\phi_1^{U_s^{xz}}$, and a section cut at $a/2$ to represent the through-the-thickness view of the first transverse shear strain energy density mode shape $\phi_1^{U_s^{yz}}$. Both sections cut the plate in half in both directions. The results are obvious, and the edges of the damping treatments play a decisive role in the damping efficiency of a given treatment. From Figures 6.140 - 6.141 and 6.142 - 6.143, it is possible to notice the higher levels of modal transverse shear strain energy density at the edges of the damping treatments. But more interestingly, taking both directions into account, it is worth noting the higher levels of modal shear strain energy density of the damping layer at Position 13, explaining its superior performance. Besides, consistently, the Cross-Shape treatment and the square patch at Position 3 also present high levels of modal shear strain energy density at the damping layer. By contrast, Figures 6.140 - 6.141 and 6.142 - 6.143 also disclose the low shear strain energy and so the damping capacity of the contour treatments.

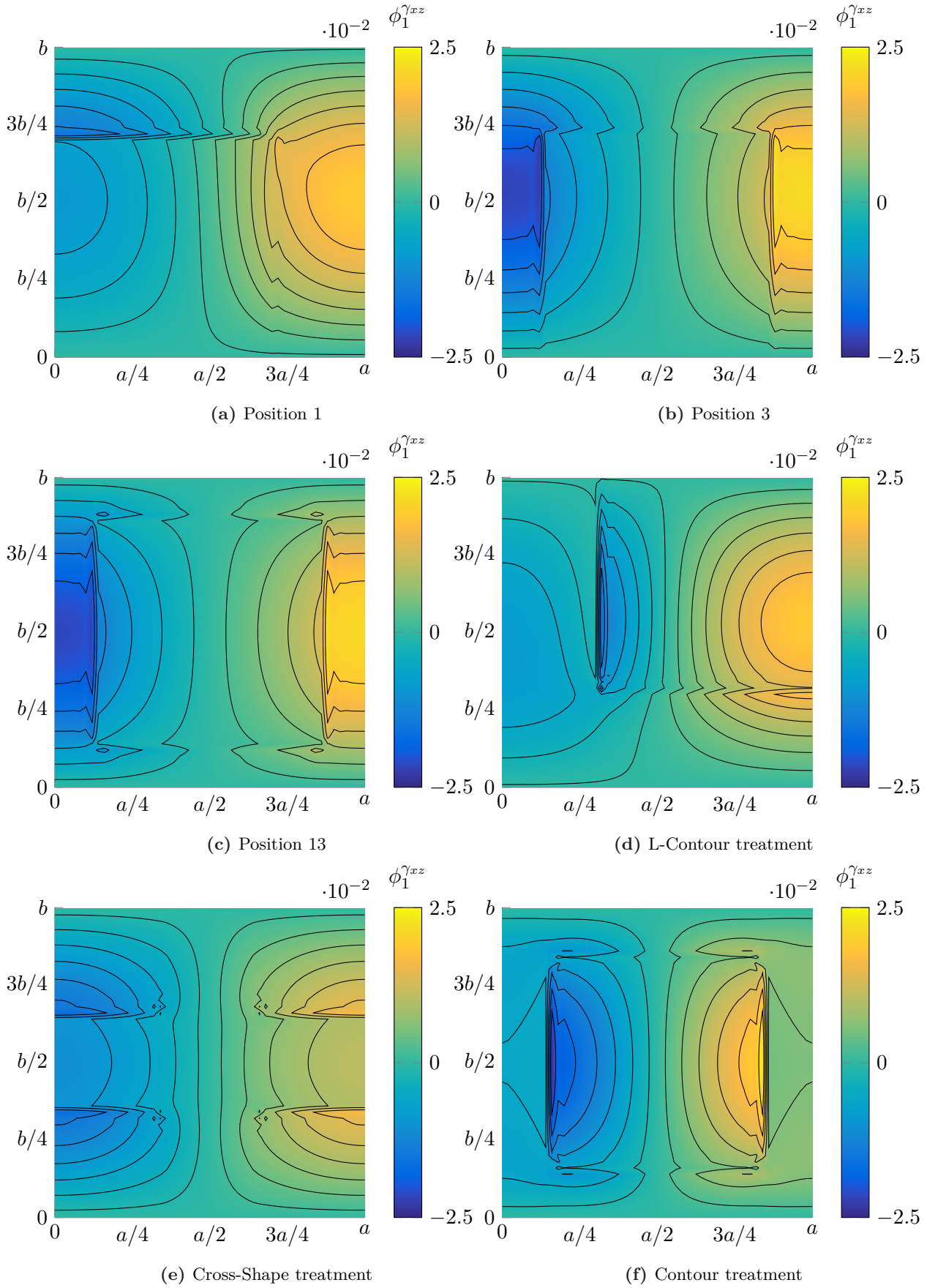
In the previous chapter, one discussed the extensional deformation of the constraining layer as the mechanism responsible for imposing transverse shear strain on the damping layer. It is important to discuss and extend it to plates. Figures 6.144 - 6.145 and 6.146 - 6.147 depict the through-the-thickness views of the first membrane plus bending strain energy density mode shapes, $\phi_1^{U_n^{xx}}$ and $\phi_1^{U_n^{yy}}$, respectively, considering the same section cuts along both directions. Once again, the results are noteworthy. Taking both directions into account, Figures 6.144a and 6.146a show the highest levels of membrane plus bending strain energy density at the constraining layer of the square patch at Position 1. However, this high membrane plus bending strain energy density is restricted to a small area of the constraining layer near the edge of the damping treatment. Focusing on Figures 6.144c and 6.146c, it is noticeable not the highest levels of membrane plus bending strain energy density are restricted to a limited area, but high levels of membrane plus bending strain energy density along the entire constraining layer section. These results prove the need for high levels of extensional deformation at the constraining layer for high levels of shear strain at the damping layer and, consequently, improved damping performances. To support this evidence, Figures 6.144 - 6.145 and 6.146 - 6.147 also highlight the absence of membrane plus bending strain energy at the constraining layers of the contour treatments. Consequently, there is a limited capacity to impose transverse shear strain on the damping layer, and the damping performance is not impressive.

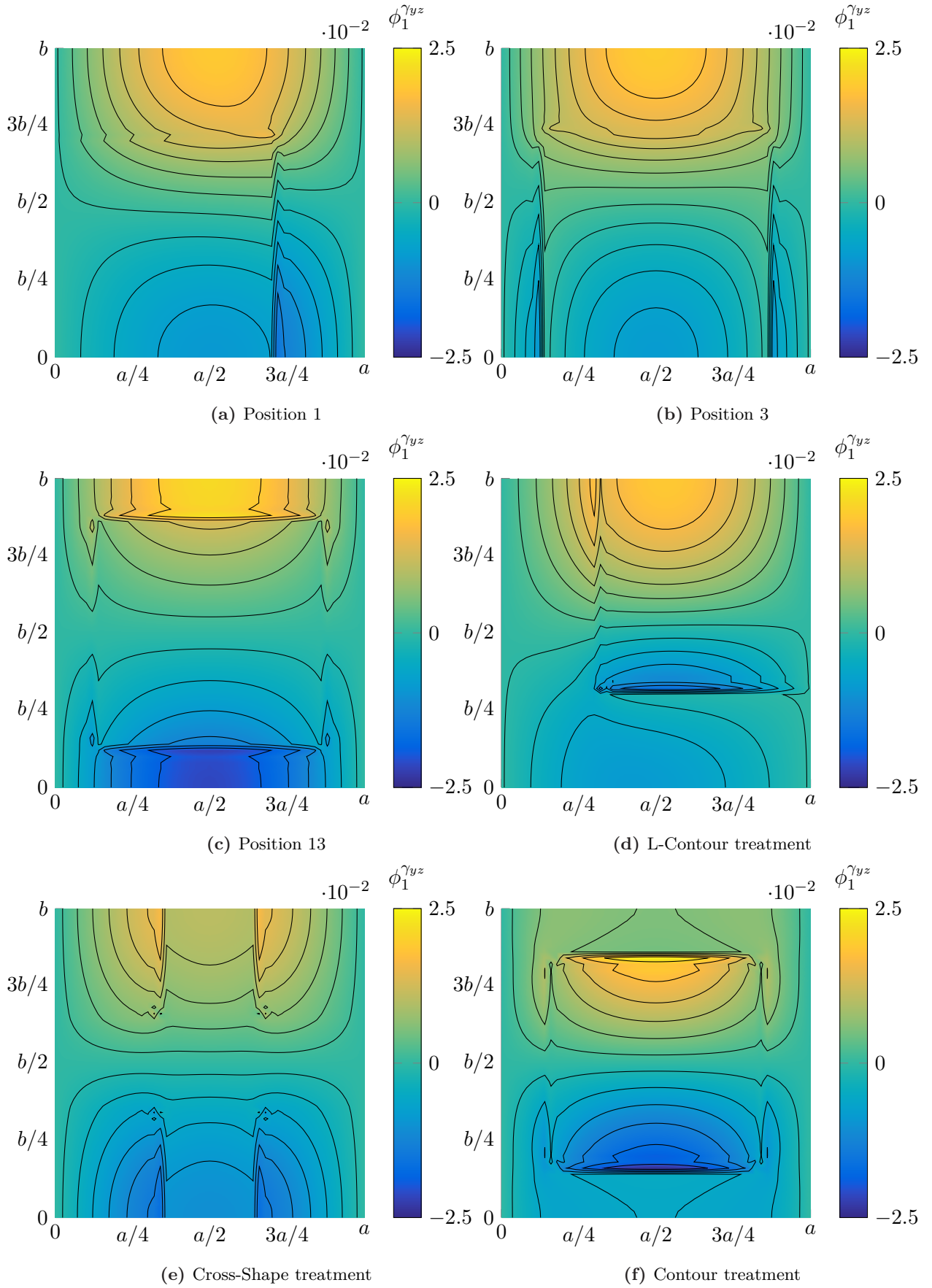
Following the parametric analysis, Figure 6.137c depicts the same damping efficiency curves for the first specific modal loss factor considering the thickest damping layer under analysis in this study - $h_d = 0.32h$. The results are simple; this higher damping layer thickness decreases the damping performance remarkably, and the differences between distinct damping treatment positions are negligible. Thicker damping layers decrease not only the damping performance but also cancel any improved damping performance of a given position over the others.

In this subsection, one also compares the damping efficiency curves for higher modes of vibration. Considering the interchange between mode shapes of the second and third modes of vibration for different covered areas, as previously described, it is appropriate to compare different positions analysing both modal loss factors simultaneously. Figures 6.148 and 6.149 present the second and third specific modal loss factors, respectively, comparing different positions for partial damping treatments. For both modes of vibration, one compares the damping efficiency curves considering the thinnest and thickest damping layers under analysis in this study. As discussed in the previous subsection, the threshold value of damping layer thickness for the

increasing-decreasing behaviour of damping capacity at the second and third specific modal loss factors occurs for even thinner damping layers than those considered here. Consequently, thicker damping layers mean less damping capacity. Considering the damping efficiency curves for the thinnest damping layer in Figures 6.148a and 6.149a, it is possible to identify significant differences regarding the best patch positions compared to the first specific modal loss factor. Similarly, the Contour treatment still provides the lowest damping capacity at these modes of vibration explained by the zero curvature for both directions at the edges of the plate in Figure 6.135. However, the central patch at Position 13 no longer exhibits the best damping performance. Apart from local optima for small covered areas attainable with a square patch at Position 1 or Position 3, the Cross-Shape treatment presents the best overall damping performance at the second and third modes of vibration. Once again, Figure 6.135 could help to explain the mechanisms behind these results. The curvature mode shapes follow the same pattern as the transverse displacement mode shapes for both directions, with a nodal line separating two equal areas with maximum curvature at the centre and rotating ninety degrees between the second and third mode shapes. Each arm of the Cross-Shape treatment covers both anti-nodes of the second and third curvature mode shapes, promoting higher levels of the extensional deformation of the constraining layer and so transverse shear strain of the damping layer. The damping efficiency curves for the thickest damping layer in Figures 6.148b and 6.149b show, once again, that thicker damping layers above the threshold value of the increasing-decreasing behaviour not only decrease the damping performance but cancel any improved damping performance of a given position over the others.

(a) $h_d = 0.005h$ (b) $h_d = 0.01h$ (c) $h_d = 0.32h$ **Figure 6.137** 1st modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.138 1st xz -shear strain mode shapes of 50% covered plates


 Figure 6.139 1st yz -shear strain mode shapes of 50% covered plates

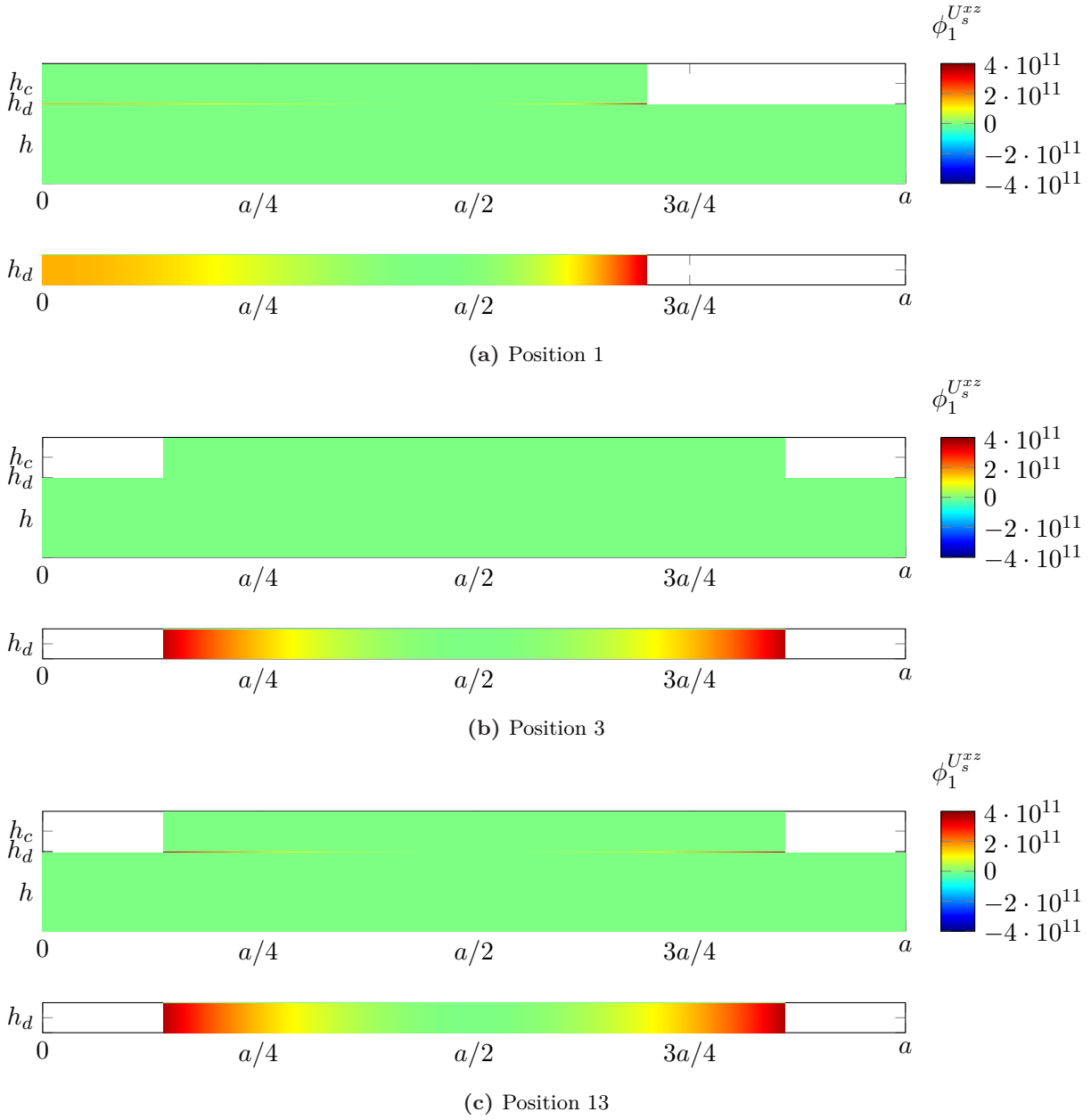


Figure 6.140 1st xz -shear strain energy density mode shapes - section cut at $b/2$

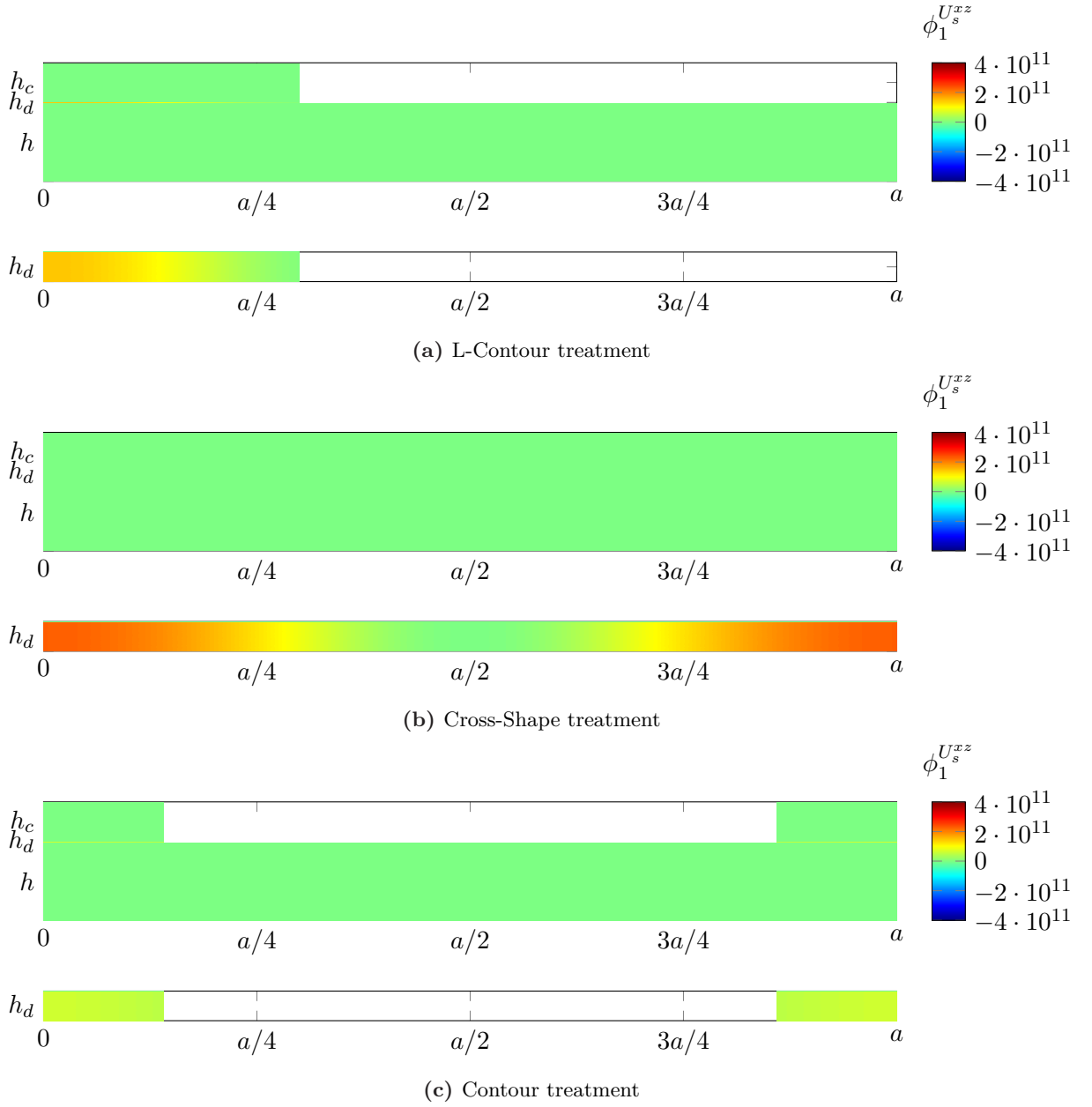


Figure 6.141 1st xz -shear strain energy density mode shapes - section cut at $b/2$

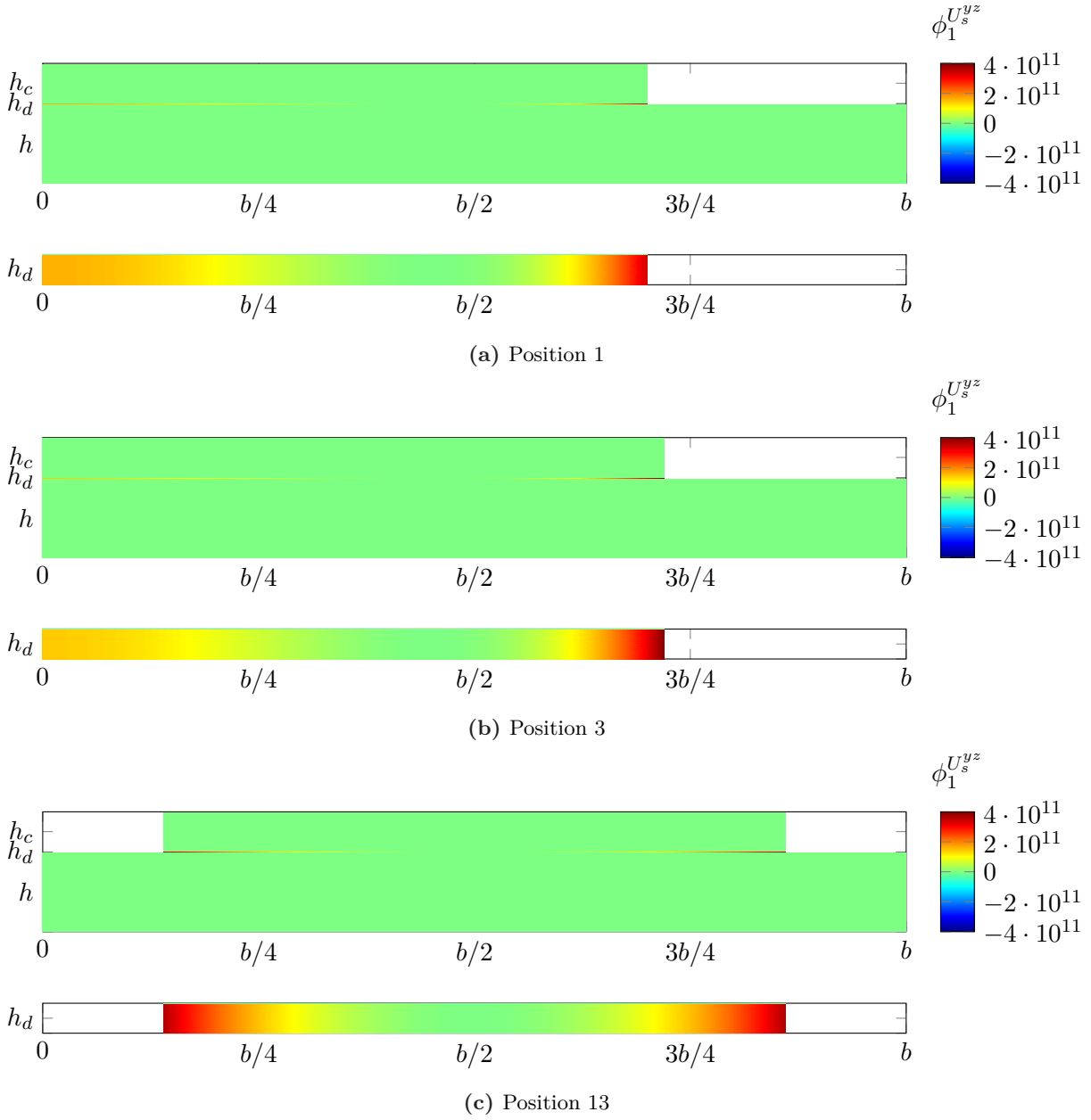


Figure 6.142 1st yz -shear strain energy density mode shapes - section cut at $a/2$

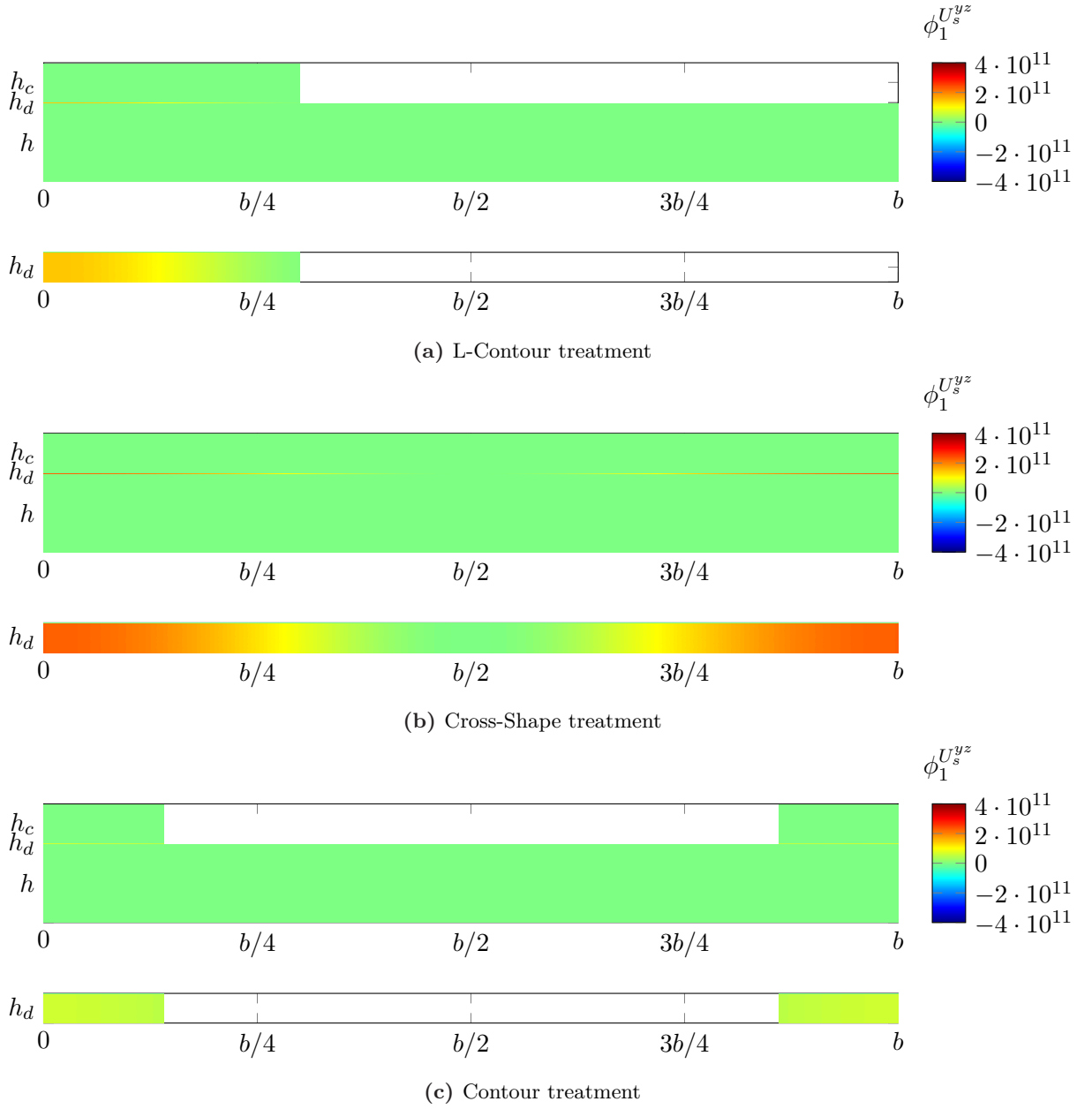


Figure 6.143 1st yz -shear strain energy density mode shapes - section cut at $a/2$

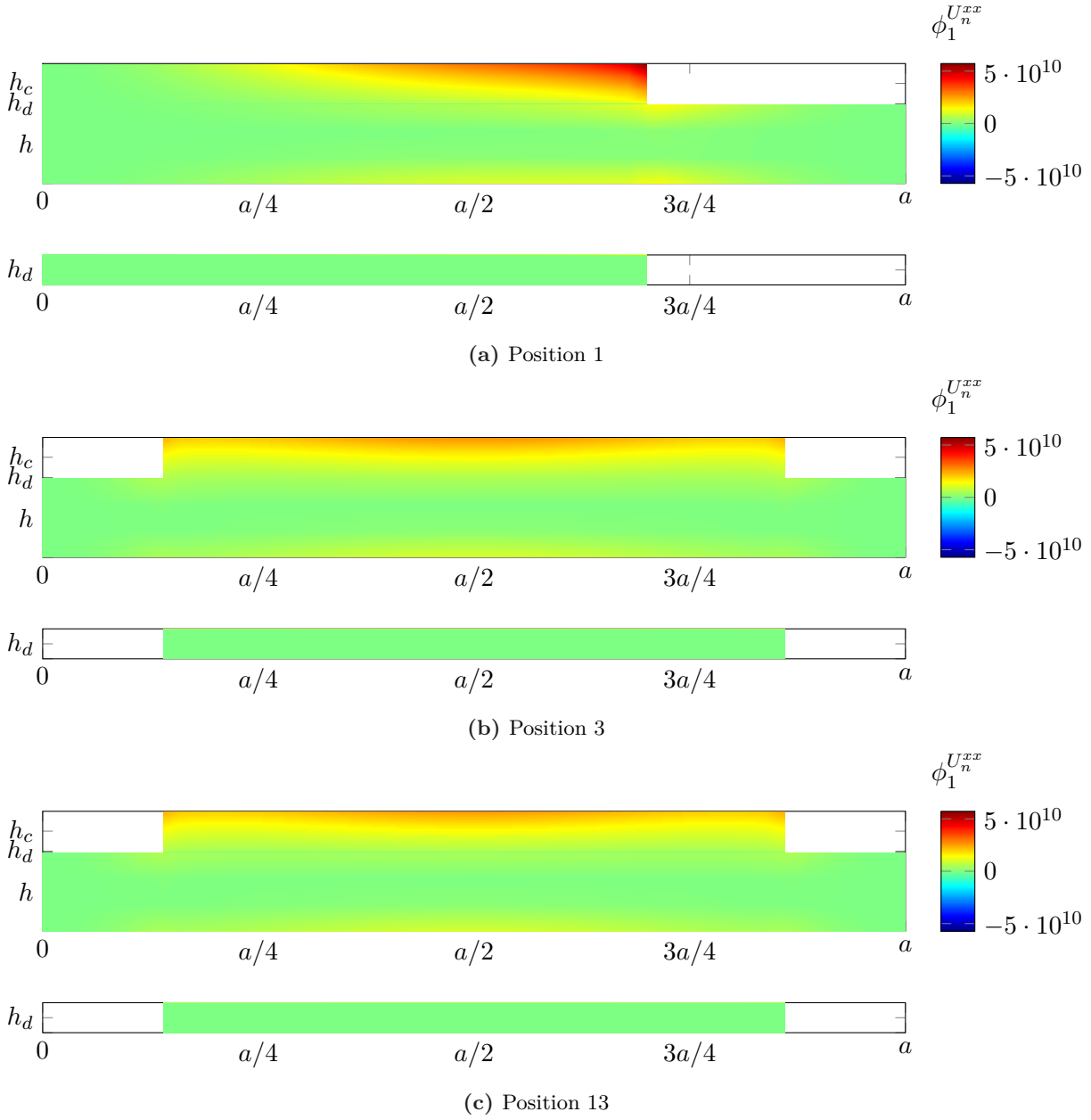


Figure 6.144 1st xx -membrane plus bending strain energy density mode shapes - section cut at $b/2$

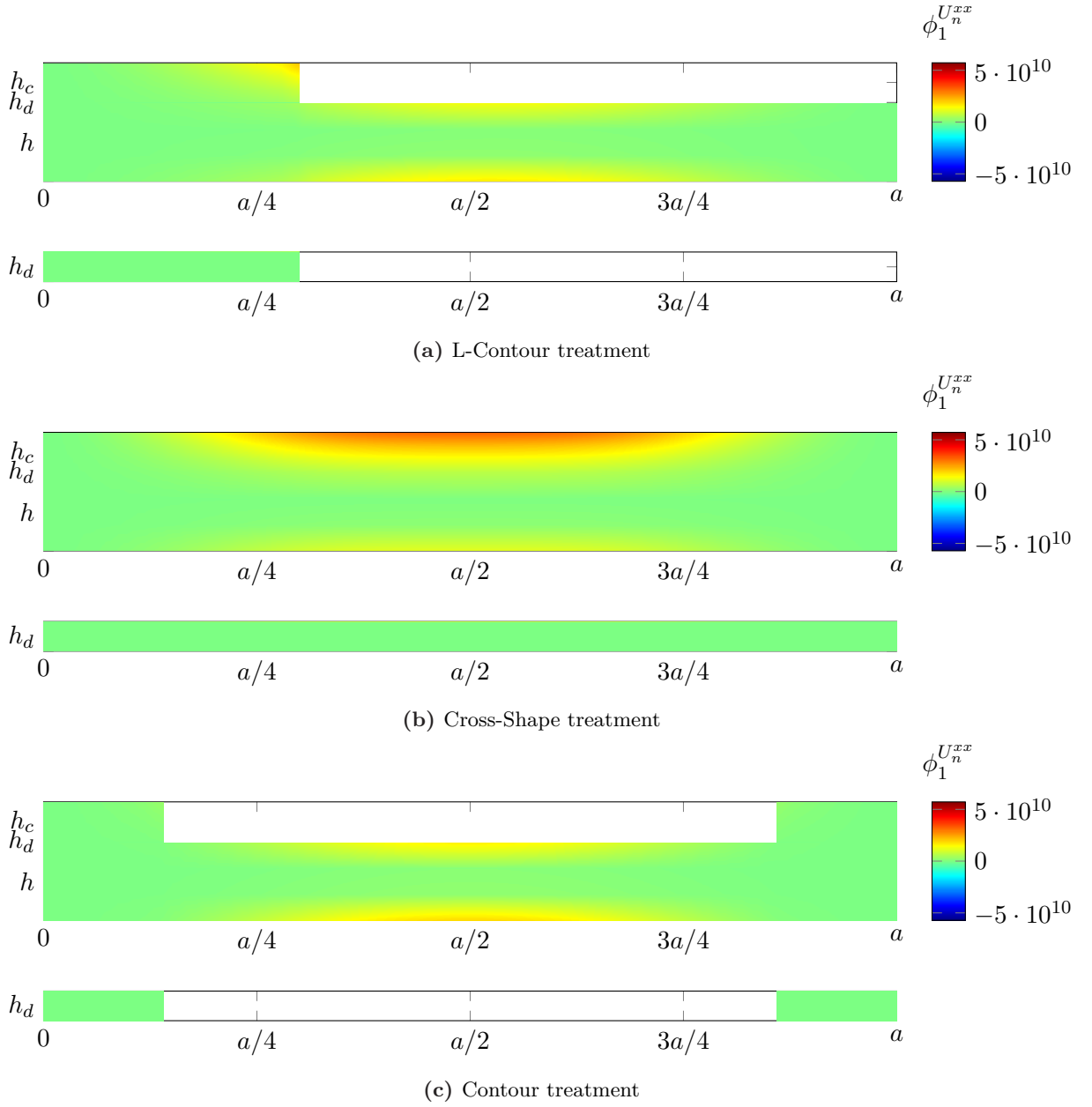


Figure 6.145 1st xx -membrane plus bending strain energy density mode shapes - section cut at $b/2$

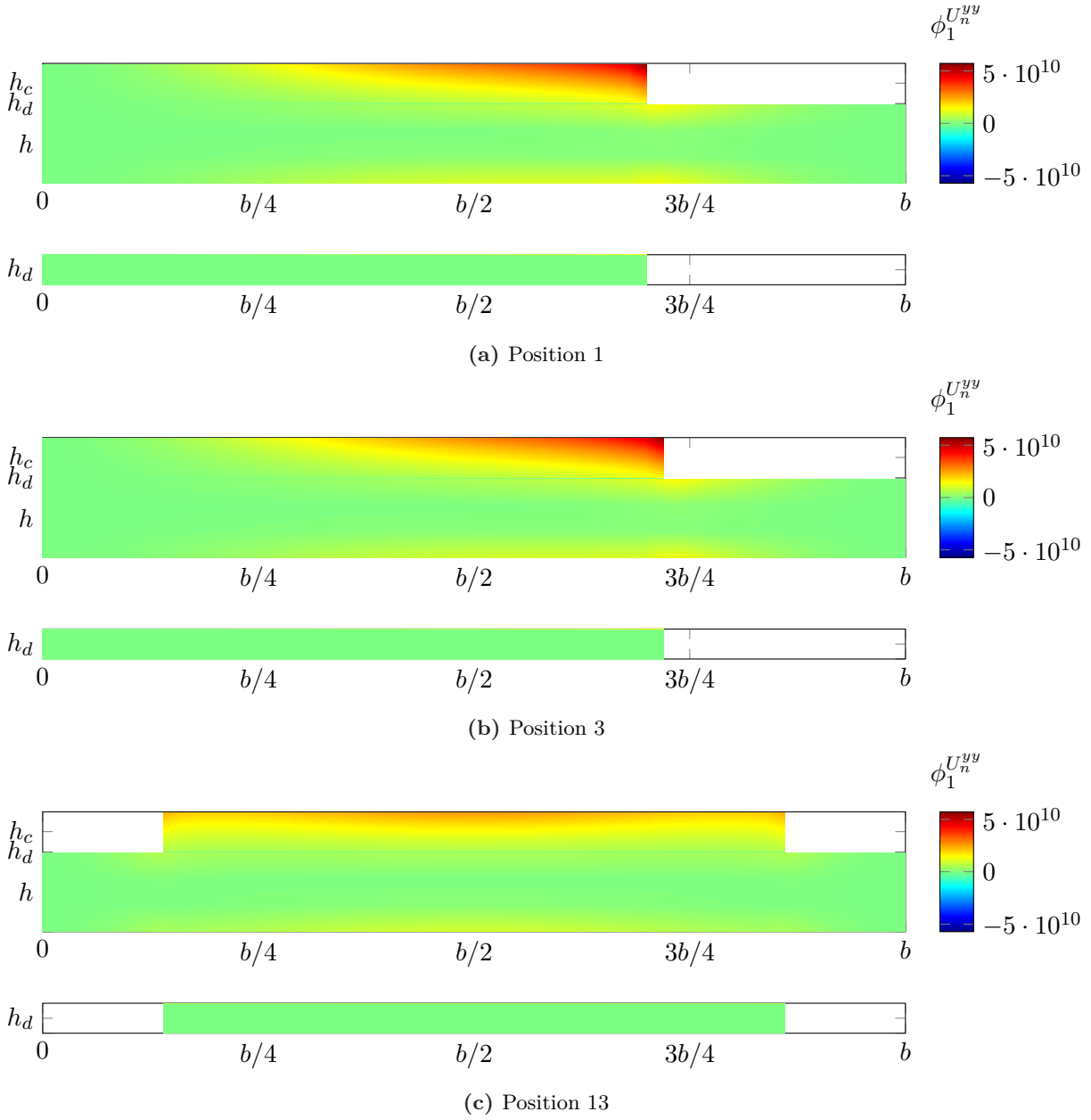


Figure 6.146 1st yy -membrane plus bending strain energy density mode shapes - section cut at $a/2$

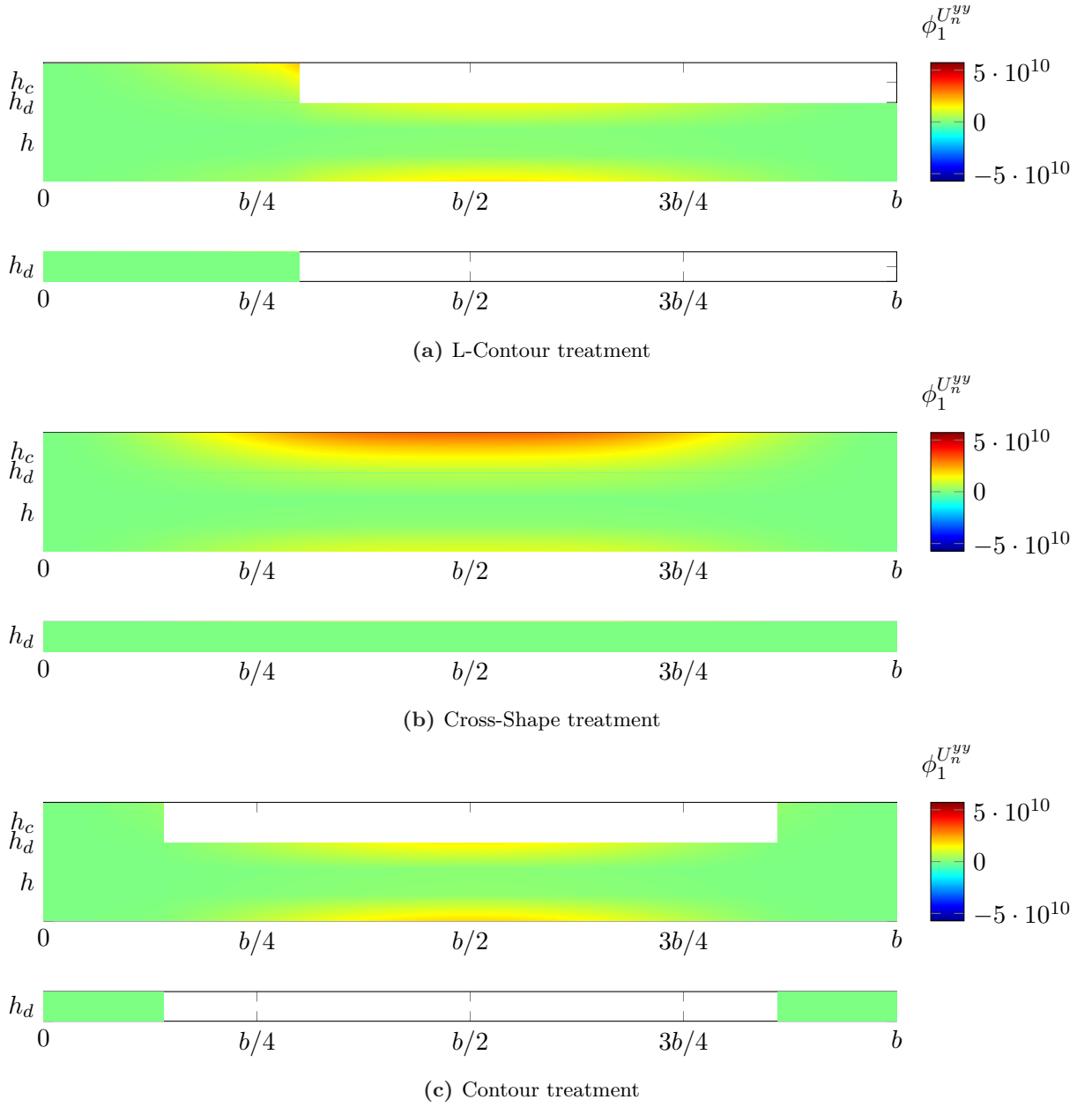
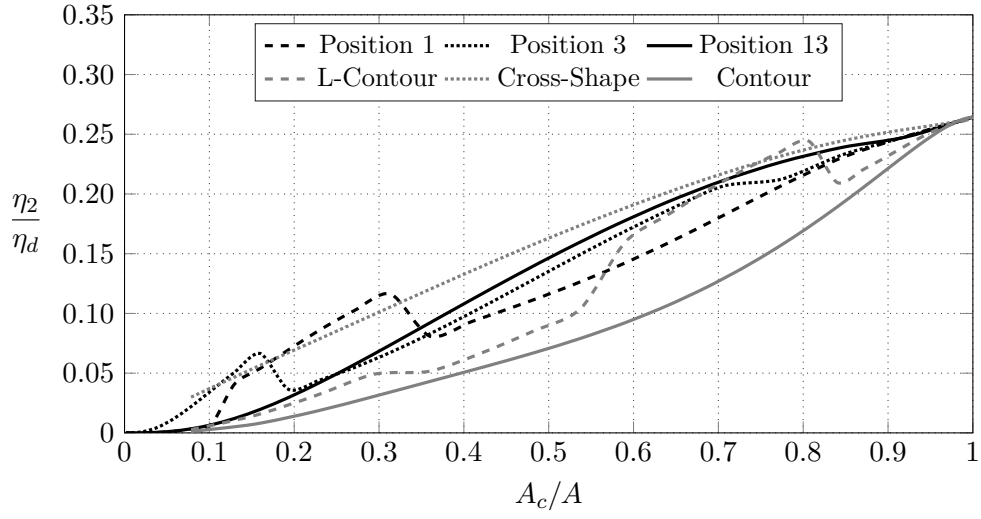
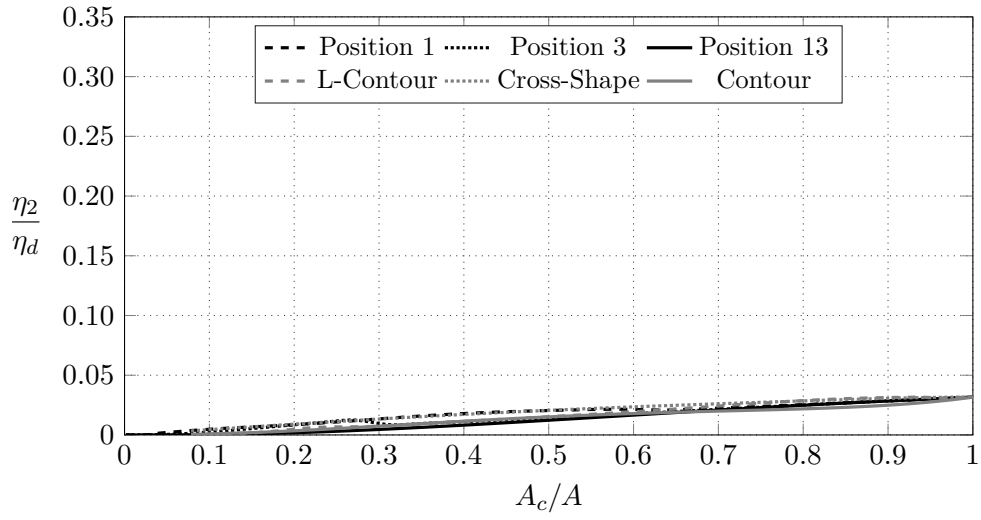
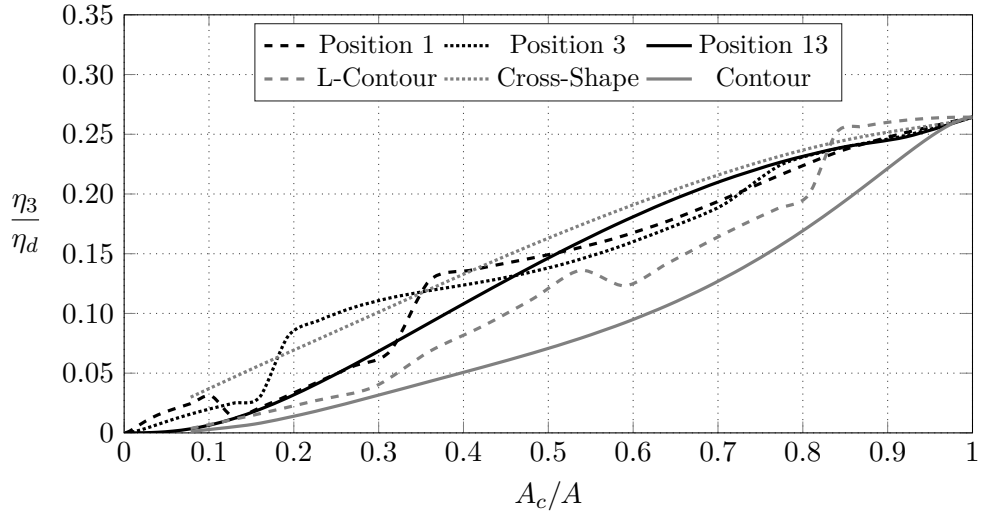
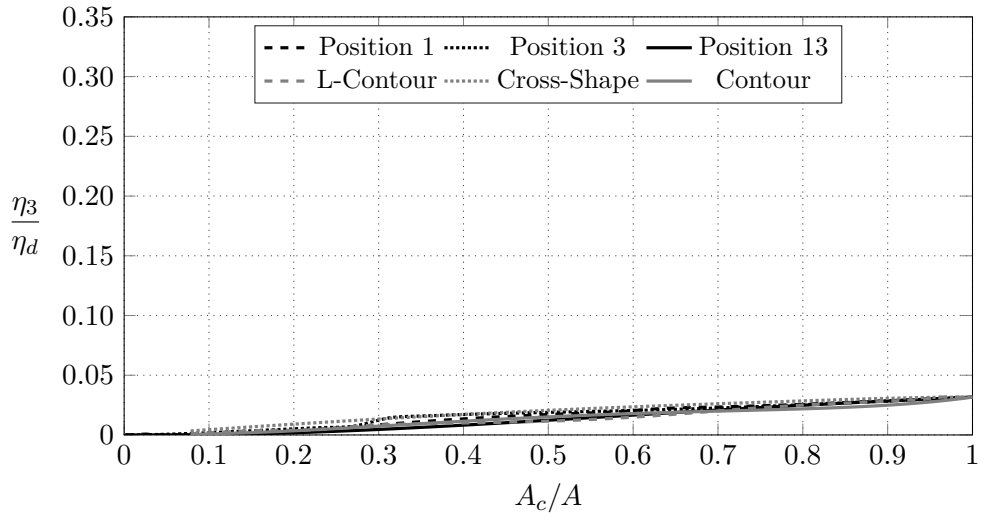


Figure 6.147 1st yy -membrane plus bending strain energy density mode shapes - section cut at $a/2$

(a) $h_d = 0.005h$ (b) $h_d = 0.32h$ **Figure 6.148** 2nd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.32h$ **Figure 6.149** 3rd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

6.7.2.2 | Thin Plate - $h = 0.005a$

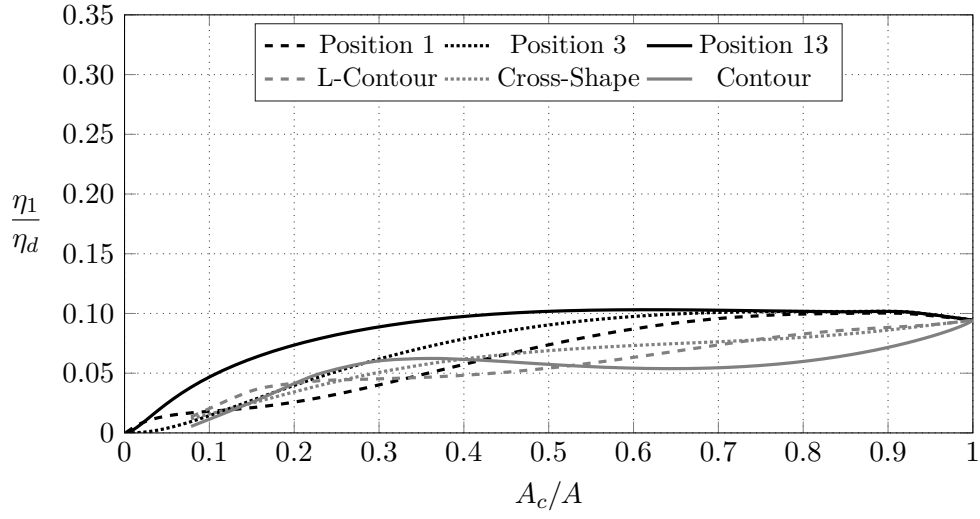
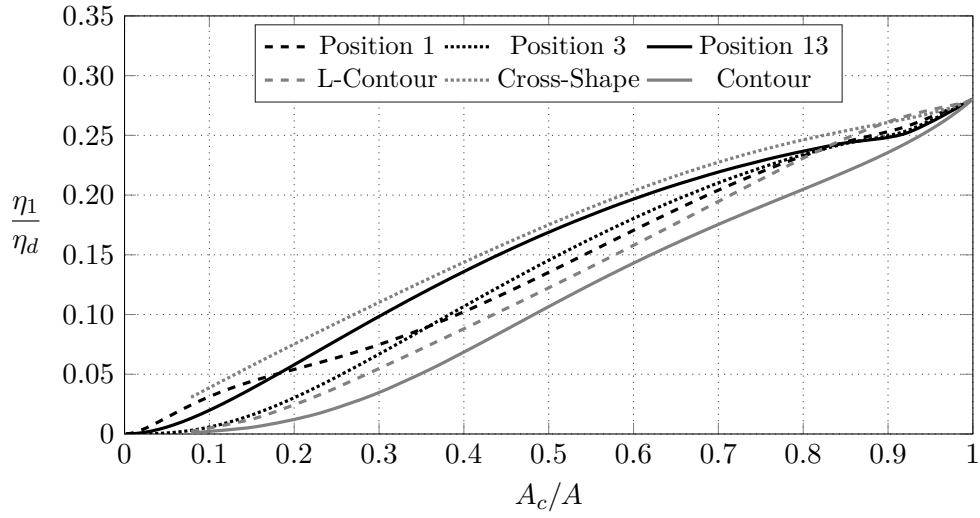
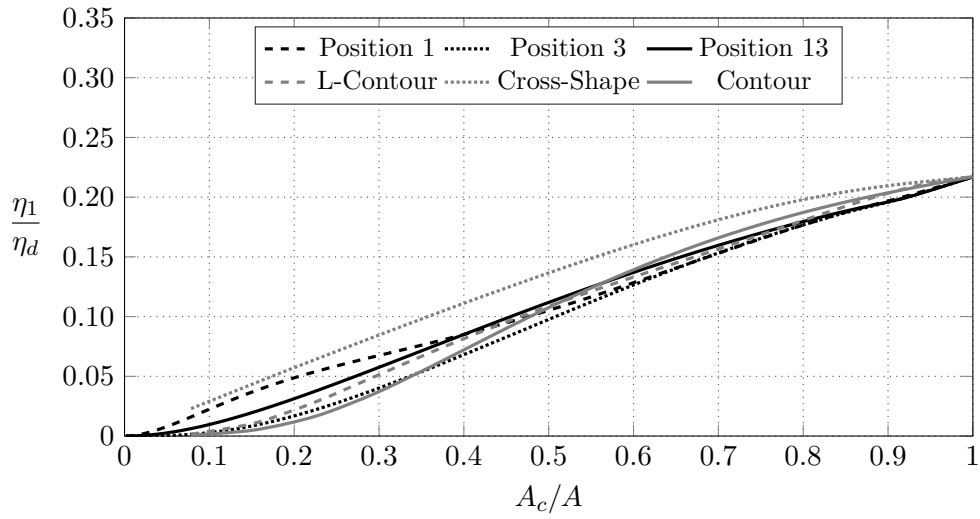
Considering a thin plate, $h = 0.005a$, Figure 6.150 depicts similar damping efficiency curves for the first specific modal loss factor. Following the same reasoning of the previous chapter, in this section, one studies the damping efficiency of partially covered plates, also considering a thinner base plate. Besides, one studies the damping efficiency of the different patch positions, taking three different damping layer thicknesses into account - $h_d = 0.005h$, $h_d = 0.08h$ and $h_d = 0.32h$. Once again, one considers the thinnest and thickest damping layers under analysis in this study, together with the threshold value of damping layer thickness for the increasing-decreasing behaviour of damping capacity at the first mode of vibration.

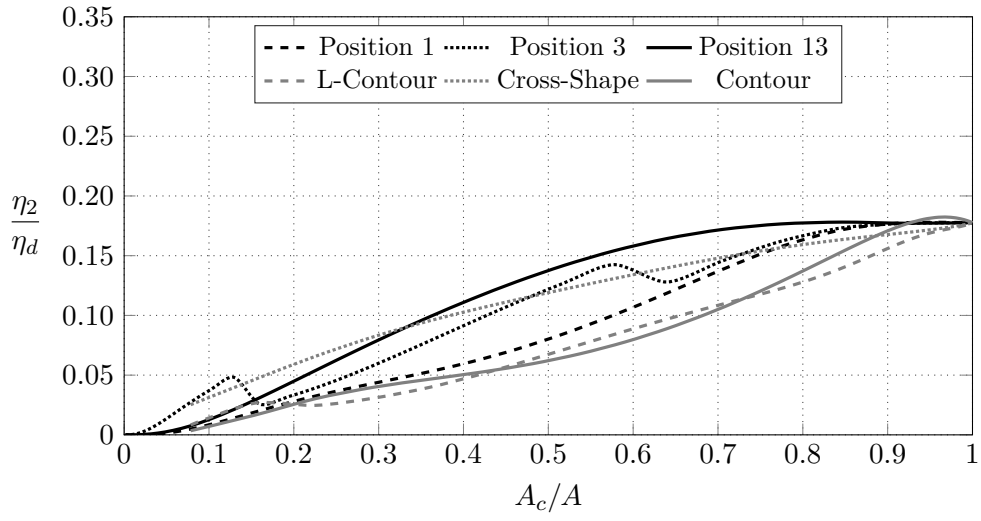
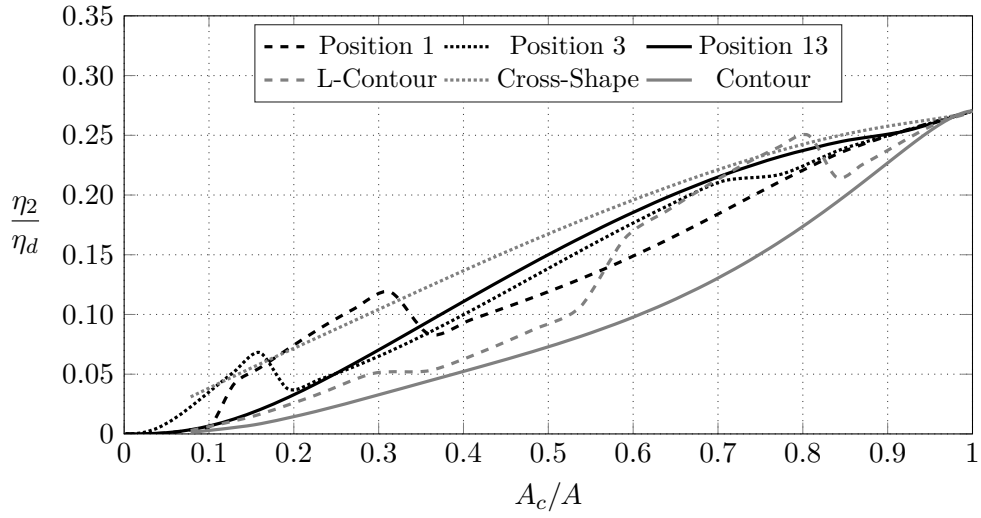
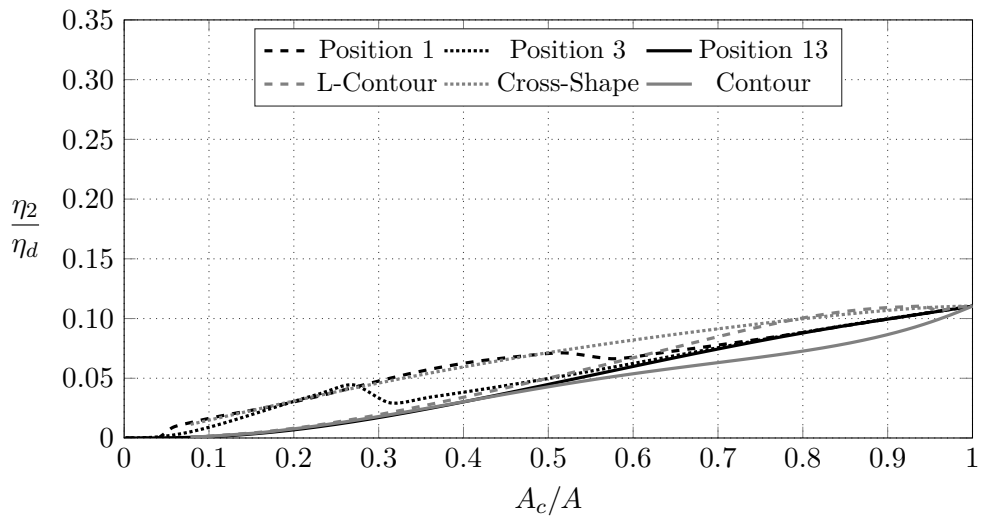
In the first place, Figure 6.150a shows the damping efficiency curves for the thinnest damping layer thickness under analysis - $h_d = 0.005h$. Firstly, it is worth emphasizing the limited damping capacity of thin damping layers when applied over thin base plates. Nevertheless, one should discuss some noteworthy features; the square patch at Position 13 presents the best damping capacity of all positions, but beyond that, one achieves the best damping capacity with a partially covered area. Unexpectedly, the Contour treatment reveals a considerable damping capacity for small covered areas but loses its marginal efficiency as the covered area increases, following the trend of the reference plate.

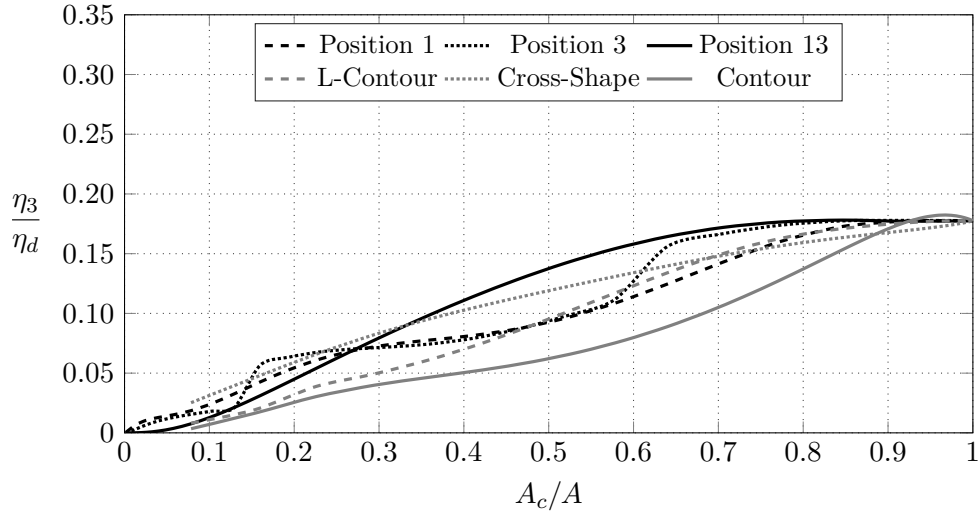
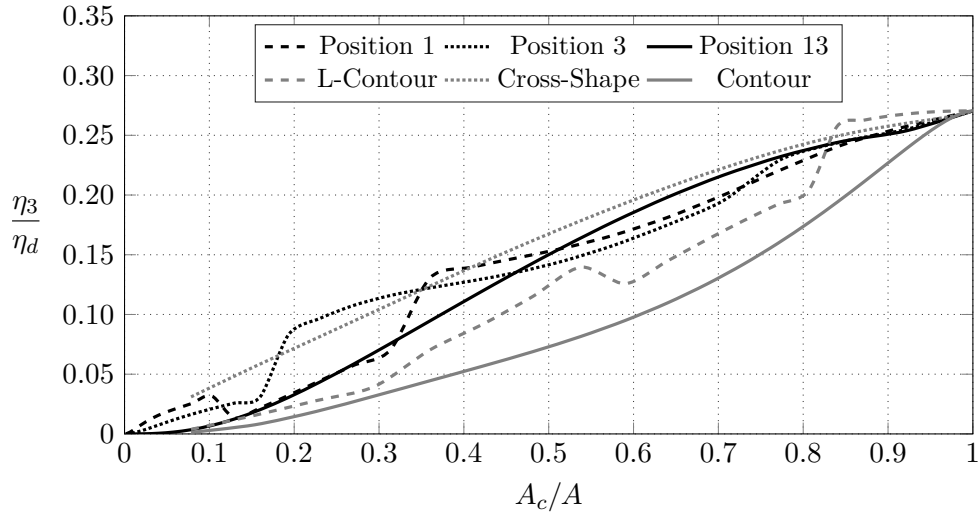
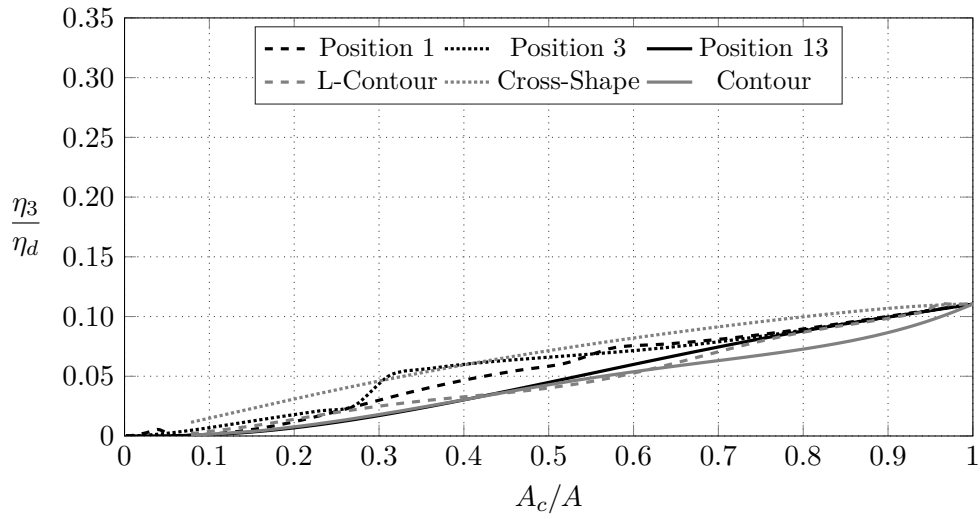
Additionally, Figure 6.150b presents the damping efficiency curves for the threshold value of the increasing-decreasing behaviour of damping capacity - $h_d = 0.08h$. Following the trend of the reference plate, the Contour patch is the most ineffective damping treatment for the reasons previously mentioned in this work. In contrast, the central patch at Position 13 also presents a distinguished damping capacity, but it is interesting to realise that it is no longer the best one. For a thin base plate, the Cross-Shape treatment exhibits the best damping performance at the first mode of vibration, regardless of the covered area. As previously discussed, the first curvature mode shapes in Figure 6.135 highlight the anti-nodes of curvature at the centre of the plate, and each arm of this treatment crosses the central area of the structure. It is a critical result; one proves that the thickness of the base structure changes the optimal shape of a damping treatment, and analysing its mode shapes is mandatory, but not everything. By increasing the damping layer thickness beyond the threshold value, the efficiency of any damping treatment decreases, and the relative difference between them narrows. Figure 6.150c shows this trend, but one can still notice the superior damping performance of the Cross-Shape treatment.

Figures 6.151 and 6.152 present the damping efficiency curves for the second and third specific modal loss factors, respectively. Following the same reasoning of the former examples, one studies the same damping layer thicknesses - the thinnest and thickest damping layers, as well as the threshold value for the increasing-decreasing behaviour of damping capacity at the second and third modes of vibration. The main conclusions regarding the damping performances of the different treatments are similar to those discussed for the first specific modal loss factor. The Contour treatment introduces less damping capacity at the second and third modes when comparing the different positions, regardless of the damping layer thickness. Besides, for the thinnest damping layer and as the covered area exceeds a given value, the square patch at Position 13 outperforms the other treatments and inclusively achieves the best damping capacity for a partially covered area. Analysing Figures 6.151b and 6.152b, which consider the threshold value for the increasing-decreasing behaviour of $h_d = 0.02h$, the conclusions are slightly different but similar to those considering a reference base plate thickness. Except for some improved performances for small covered areas when considering a square patch at Position 1 or Position 3, the Cross-Shape treatment introduces the highest overall damping performance at both the second and third modes. Despite the reduced damping performance when compared with thinner thicknesses, the thickest damping layer example in Figures 6.151c and 6.152c holds the superior performance of the Cross-Shape treatment. The improved efficiency of the Cross-Shape treatment

for the second and third modes of vibration results from each branch of the Cross-Shape treatment covering an anti-node of the second and third curvature mode shapes, promoting high levels of extensional deformation of the constraining layer and, consequently, the transverse shear strain of the damping layer, as previously discussed.

(a) $h_d = 0.005h$ (b) $h_d = 0.08h$ (c) $h_d = 0.32h$ **Figure 6.150** 1st modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

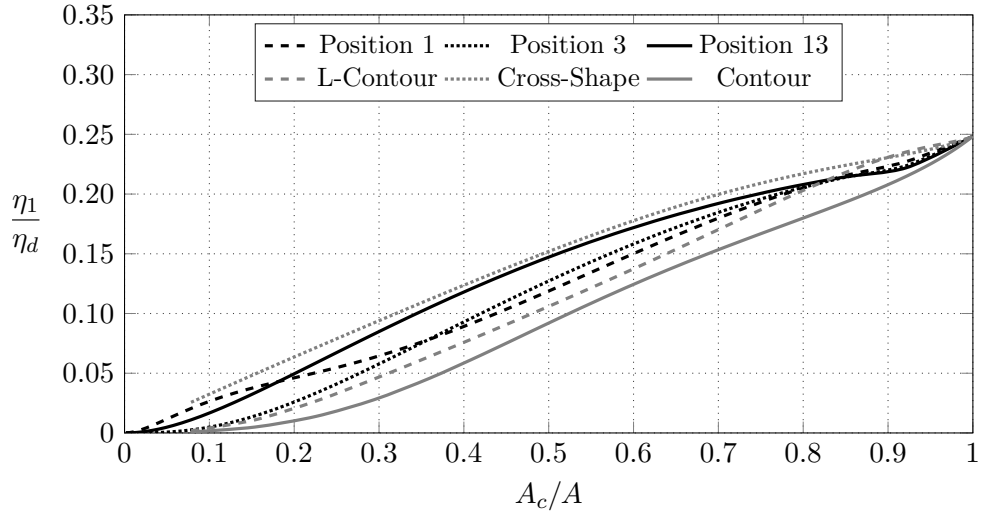
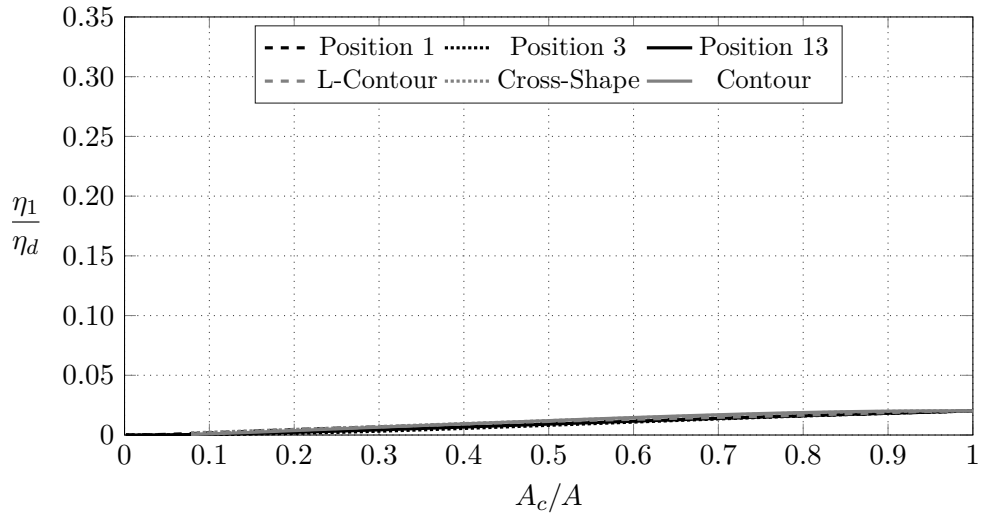
(a) $h_d = 0.005h$ (b) $h_d = 0.02h$ (c) $h_d = 0.32h$ **Figure 6.151** 2nd modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

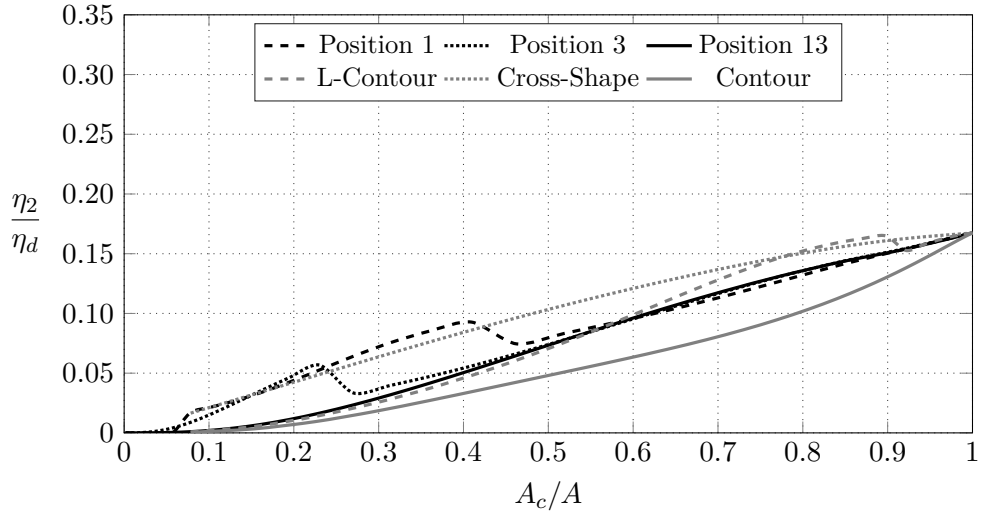
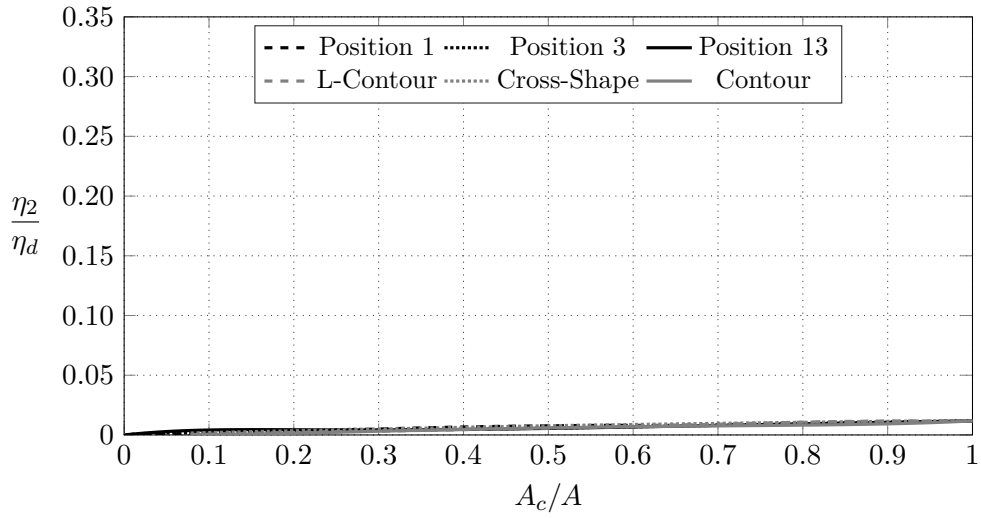
(a) $h_d = 0.005h$ (b) $h_d = 0.02h$ (c) $h_d = 0.32h$ **Figure 6.152** 3rd modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

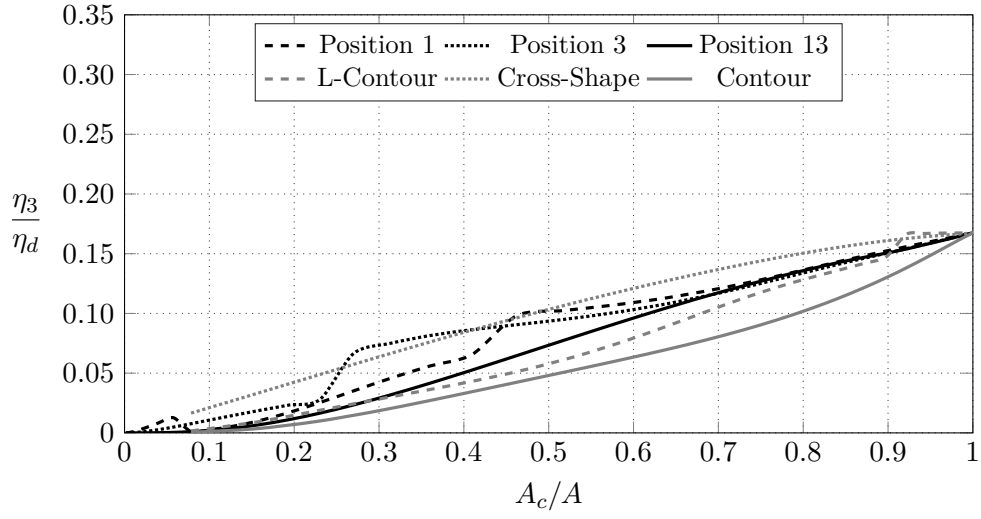
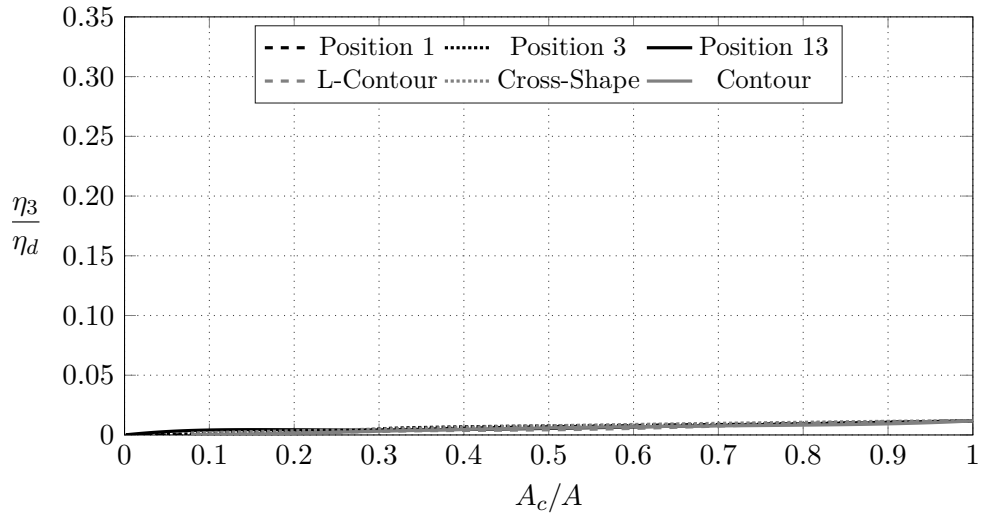
6.7.2.3 | Thick Plate - $h = 0.02a$

Focusing on a thicker base plate for comparing the different patch positions, Figure 6.153 shows the damping efficiency curves of the first specific modal loss factor. Repeatedly, one also considers the thinnest and thickest damping layer thickness under analysis in this work. A preliminary study considering this range of damping layer thicknesses shows that thicker damping layers decrease the damping performance because the threshold value for the increasing-decreasing behaviour of the damping capacity occurs for even thinner damping layers. Thus, the results are simple to discuss. For thin damping layers, as reported in Figure 6.153a, the Cross-Shape treatment exhibits the best damping capacity at the first mode of vibration regardless of the covered area, followed by the square patch at Position 13. Treatments covering preferentially the edges of the plate, Contour or L-Contour patch treatments, follow the expected performance and present limited damping capacities. Thicker damping layers follow the already noticed patterns, decreasing the damping performance remarkably and cancelling any improved damping performance of a given position over the others - Figure 6.153b.

Figures 6.154 and 6.155 show the damping efficiency curves for the second and third specific modal loss factors, respectively, considering in both cases the thinnest and thickest damping layers. The main conclusions for this thick base plate do not differ from those reported for the reference and thin ones. For both the second and third modes of vibration, the Cross-Shape treatment reveals its superior overall damping performance for thin damping layers despite some specific improved performances of Position 1 and Position 3 for small covered areas - Figures 6.154a and 6.155a. As expected, the Contour treatment presents a lower damping capacity. Thicker damping layers consistently follow a recurring pattern; meaningless damping performances with no difference between the various patch positions - Figures 6.154b and 6.155b. In the previous chapter, when studying the damping efficiency of partially covered beams, one discussed the mechanism behind the reduced damping performance of structures covered by thick damping layers. Damping patches with thick damping layers decouple the constraining layer from the base structure, and the damping layer does not experience a high level of transverse shear strain.

(a) $h_d = 0.005h$ (b) $h_d = 0.32h$ **Figure 6.153** 1st modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.16h$ **Figure 6.154** 2nd modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.16h$ **Figure 6.155** 3rd modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

6.7.3 | Influence of Material Properties

Apart from the thickness of the damping layer, the material properties of the base, damping and constraining layers also play a crucial role in the damping efficiency of partial damping treatments. As discussed in this work several times, damping capacity derives from the shear strain experienced by the damping layer when constrained by a stiff layer. Therefore, the material properties influence the damping efficiency of a damping treatment decisively. To understand this influence, Figures 6.156 to 6.161 present, for the patch positions under analysis in this work, the first three specific modal loss factors for different ratios of material properties representing the relation between Young's modulus of the base and constraining layers and the storage modulus of the damping layer - E_c/E_d . One already discussed this modulus ratio in the previous chapter for partially covered beams. One considers for this analysis a reference base plate with $h = 0.01a$ and a constant damping layer of $h_d = 0.01h$.

Figure 6.156 presents the influence of the covered area by a patch at Position 1 on the damping capacity of the first three specific modal loss factors for different modulus ratios. As expected, the most noticeable feature is the damping efficiency dependence on the covered area. However, the increase of damping capacity with the covered area is not steady for all modulus ratios. In particular, for low modulus ratios, the materials of both base/constraining and damping layers have comparable stiffness, and one achieves the optimum damping capacity for any of the first three modes with a partially covered area. Moreover, it is noticeable that there is a maximum attainable damping capacity for any of the first three modes accomplished with full coverage. For the first specific modal loss factor, damping capacity increases for modulus ratios up to 7×10^4 , gradually decreasing afterwards. As previously discussed, when studying the influence of the damping layer thickness, one can also notice the increasing-decreasing behaviour of damping capacity for different modulus ratios. The results show an optimum modulus ratio promoting coupling between base and constraining layers, leading to enhanced transverse shear strain at the damping layer and improved damping performances. The optimum damping capacity occurs for a lower modulus ratio for the second and third specific modal loss factors.

Following the study of different damping treatment positions, Figure 6.157 depicts the influence of the covered area by a patch at Position 3 on the damping capacity of the first three specific modal loss factors for different modulus ratios. The threshold values of the modulus ratio for the increasing-decreasing behaviour are the same regardless of the damping treatment position. Besides, similarly to Position 1, one achieves the optimum damping capacity for any of the first three modes with a partially covered area considering low modulus ratios. The main difference is that this optimum for the first specific modal loss factor occurs for a smaller covered area with a patch at Position 3 since a patch at this position covers the central area of the plate, where curvatures reach their maximum for lower covered areas.

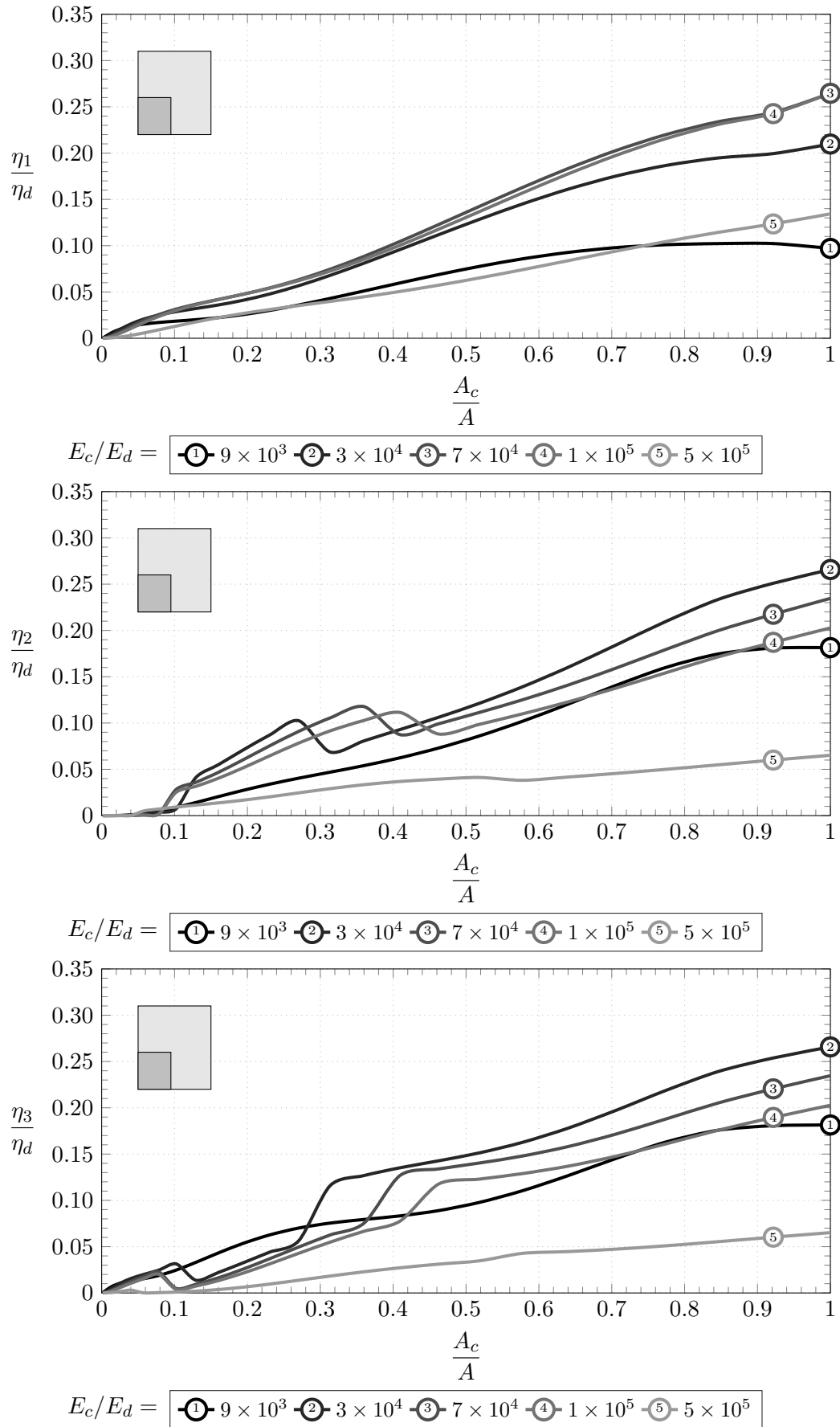
Concerning a damping treatment at the central Position 13, one can notice the same threshold values of the modulus ratio for the increasing-decreasing behaviour of damping capacity - Figure 6.158. Nevertheless, it is worth mentioning the higher damping performance of a patch in this position for small covered areas, considering the first mode of vibration. This feature discussed during the study of the influence of the damping layer thickness has to do with the superior curvatures at the centre of the plate at this mode of vibration, leading to the higher extensional deformation of the constraining layer and consequently, the shear strain of the damping layer. The second and third specific modal loss factors show a result that was already noticeable for the previous positions, but it is clear here. For higher modulus ratios, the stiffness of both elastic and damping materials is quite different, and the damping capacity drops remarkably, regardless of the covered area. This result shows that when the stiffness of the elastic material is far greater than that of the damping layer, promoting the decoupling of the

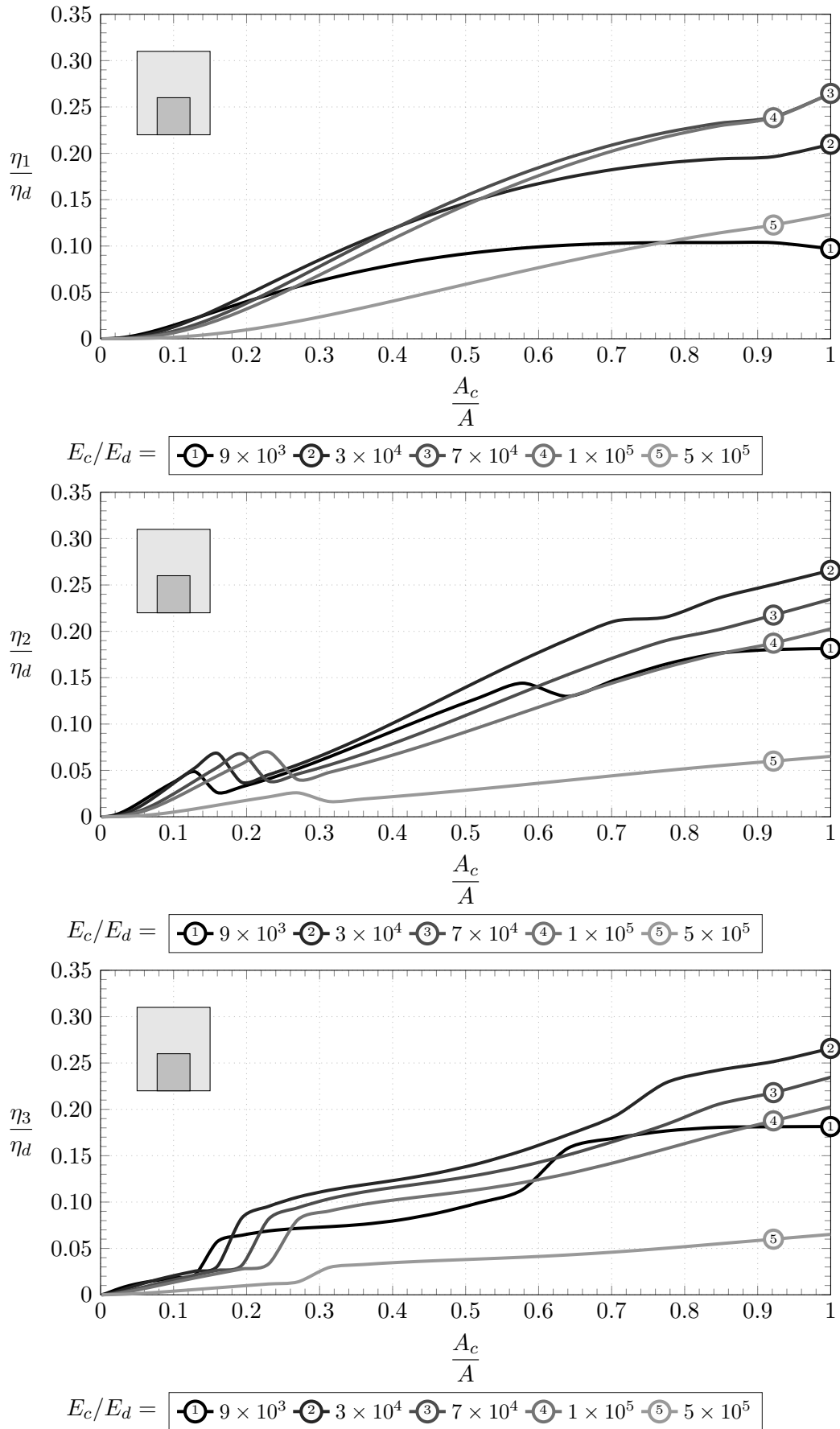
constraining layer from the host structure, the damping treatment loses efficiency.

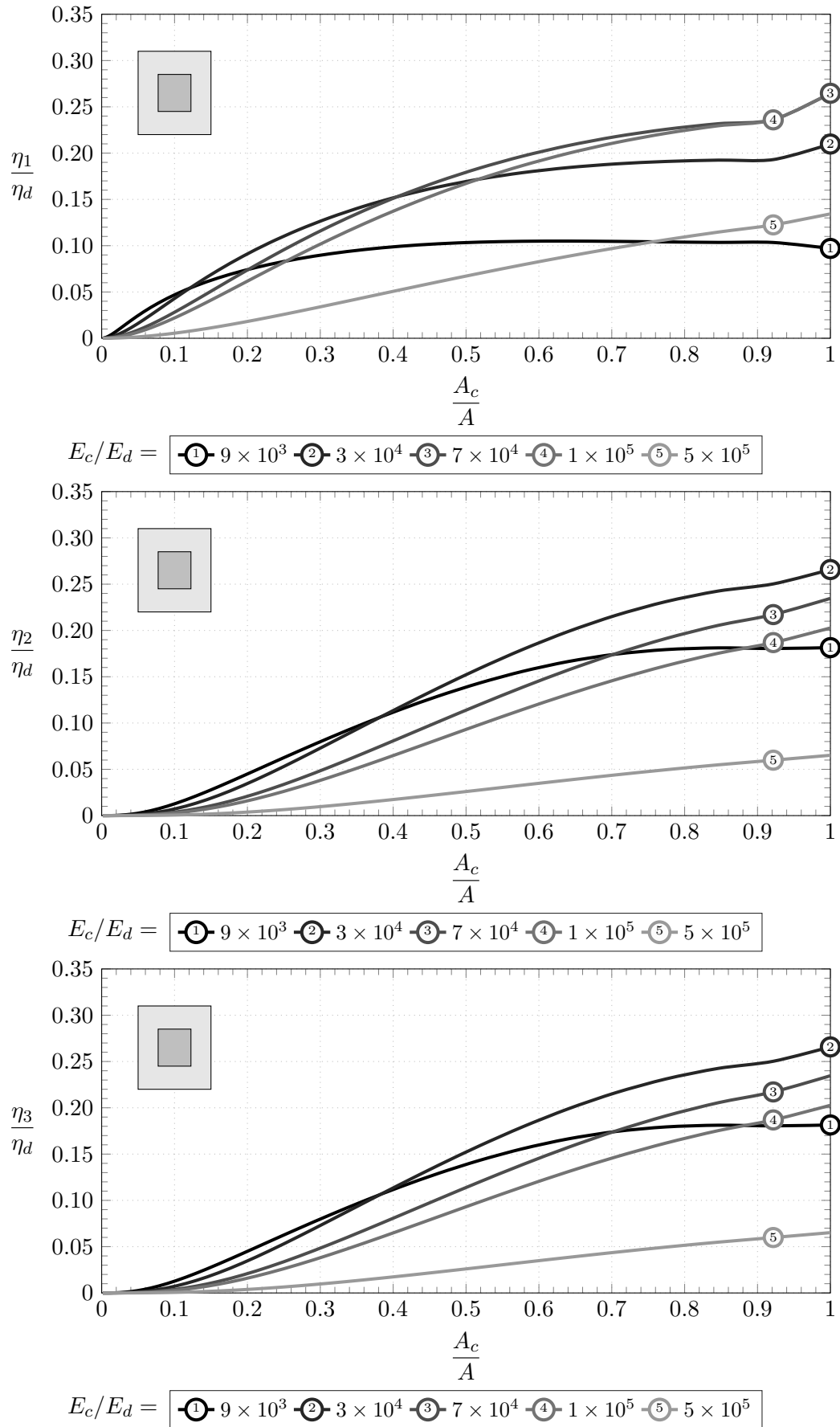
Figure 6.159 shows the damping efficiency curves for the non-square L-Contour treatment. As previously discussed, partial treatments preferentially covering the boundaries of a simply supported plate promote limited damping capacities for small covered areas. As expected, the threshold values of the increasing-decreasing behaviour of damping capacity are the same but are noticeable in the relatively poor performance of the L-Contour treatment compared to the other positions.

The results are slightly different regarding the Cross-Shape treatment and its damping efficiency curves represented in Figure 6.160. As expected, the threshold values of the increasing-decreasing behaviour of damping capacity keep the same measure, but damping capacity increases for small covered areas.

The results considering the Contour damping patch treatment and different modulus ratios, represented in Figure 6.161, highlight the outcomes already noticed for this damping pattern. It includes the noticeable low damping capacity for small covered areas and the maintenance of the threshold values for the increasing-decreasing effect of damping capacity.

Figure 6.156 Damping efficiency curves for Position 1, $h = 0.01a$, $h_c = 0.5h$

Figure 6.157 Damping efficiency curves for Position 3, $h = 0.01a$, $h_c = 0.5h$

Figure 6.158 Damping efficiency curves for Position 13, $h = 0.01a$, $h_c = 0.5h$

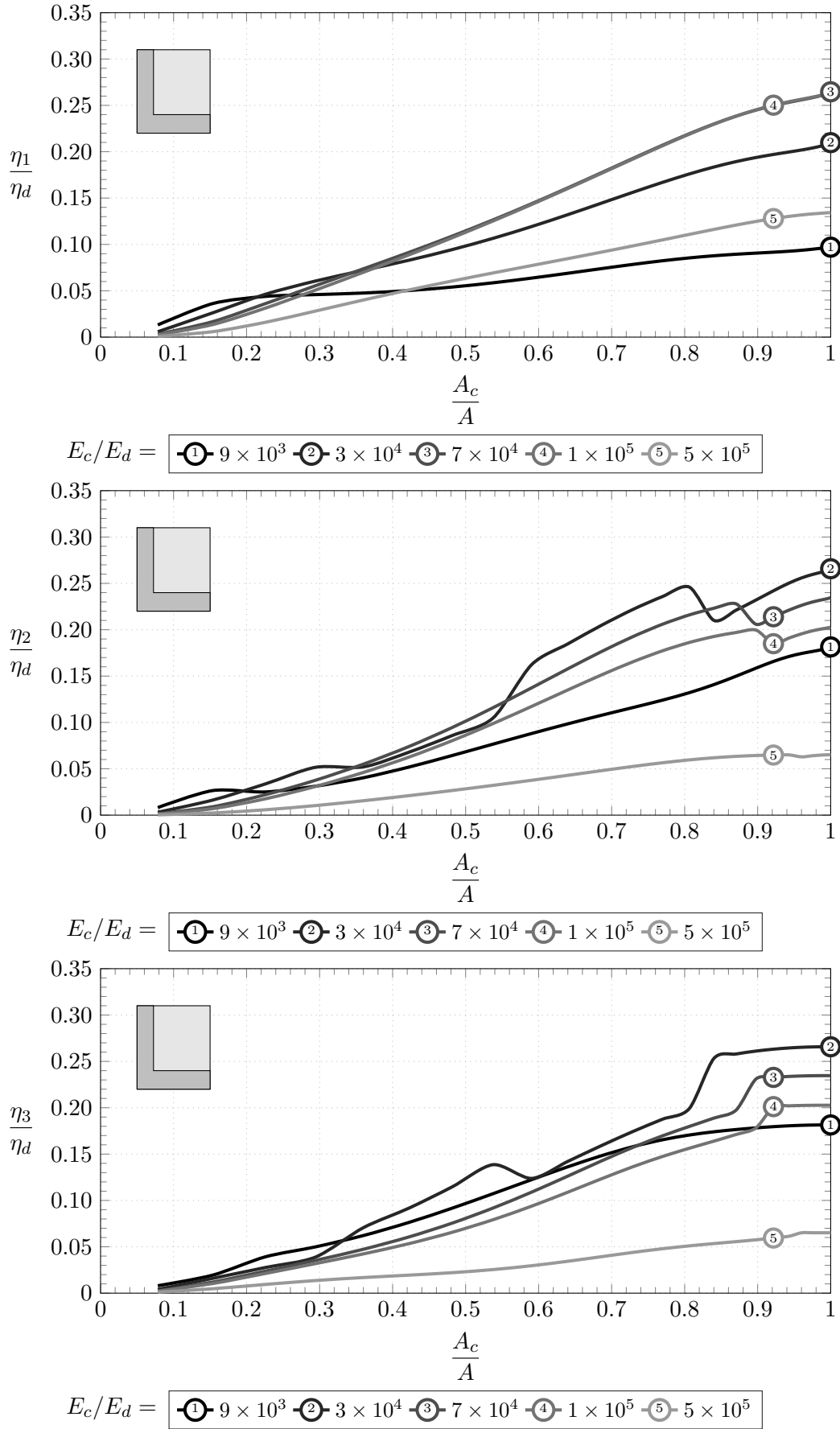
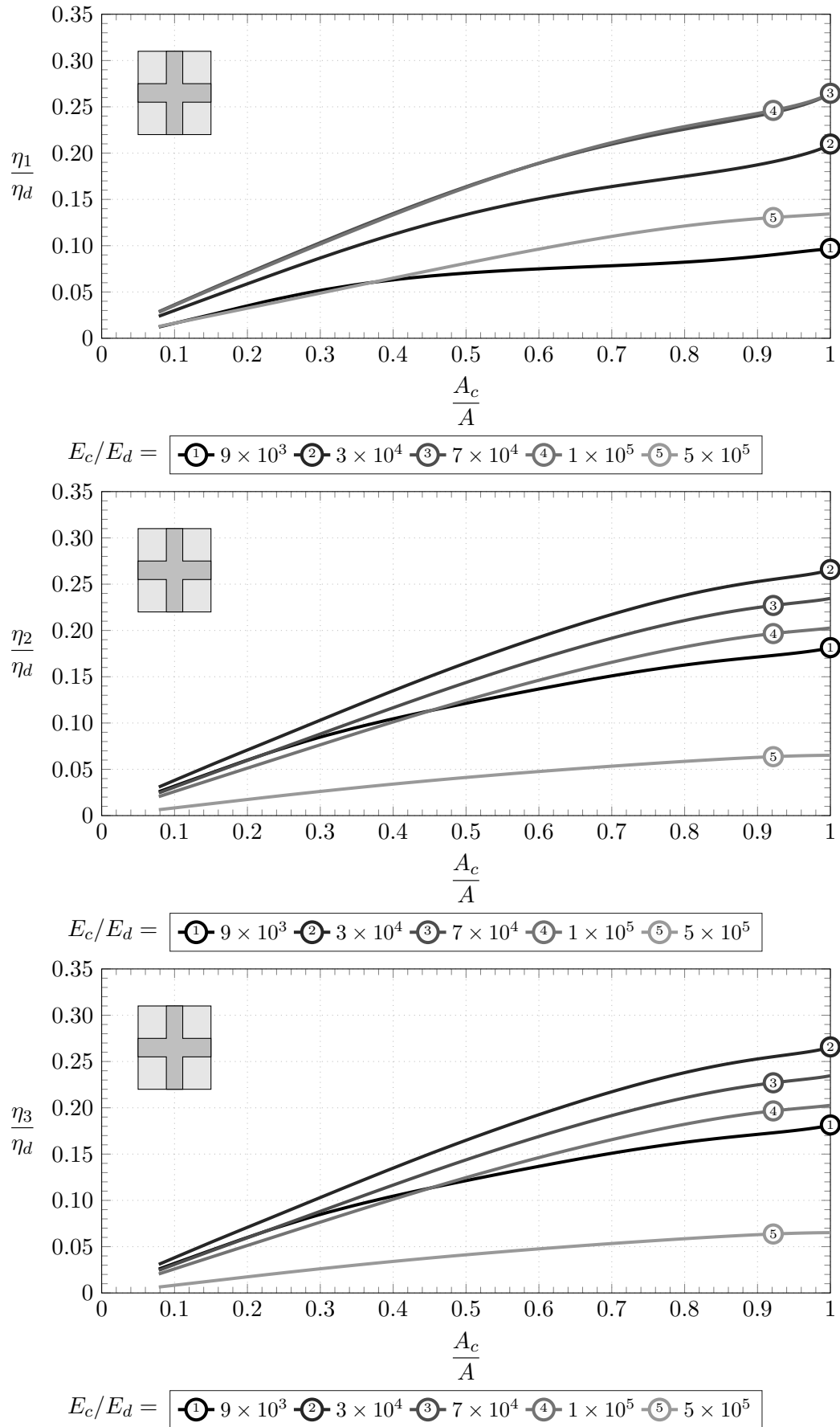
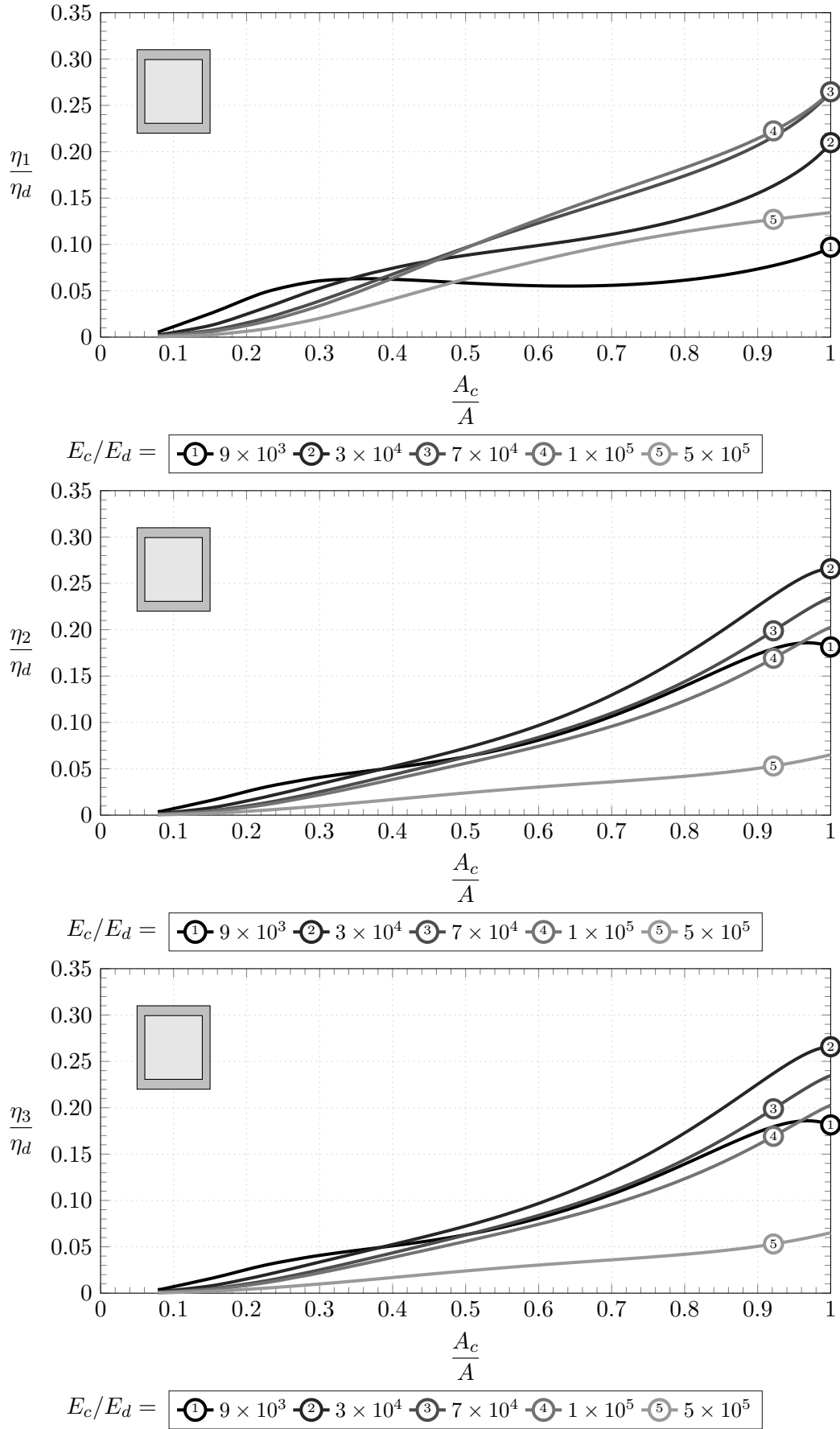


Figure 6.159 Damping efficiency curves for L-Contour treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.160 Damping efficiency curves for Cross-Shape treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.161 Damping efficiency curves for Contour treatment, $h = 0.01a$, $h_c = 0.5h$

6.7.3.1 | Influence of Patch Positioning

In this section, one performs a comparative analysis between different patch positions, considering distinct material properties and the corresponding damping efficiency. The primary goal is to identify the patches with increased damping performance, considering the influence of the material properties, and assess the mechanisms which lead to the improved damping capacity. Once again, this study needs to consider the well-known transverse displacement mode shapes of the first three normal modes of the simply supported square reference plate considering a constrained layer damping treatment with full coverage represented in Figure 6.134 - $h = 0.01a$, $h_d = 0.01h$ and $h_c = 0.5h$. Besides, one also considers the curvature mode shapes presented in Figure 6.135 to understand the bending pattern of the plate and the preferential positions for constrained damping treatments, as well as the mode shapes associated with the shear strain degrees of freedom γ_{xz} and γ_{yz} in Figure 6.136.

As already mentioned, it is relevant to perform a detailed comparative analysis between patch positions and here, one reworks the data of the previous section for a better understanding of the best positions for improved damping performances. Figure 6.162 presents the first specific modal loss factor considering different patch positions and covered areas simultaneously. Furthermore, one studies the damping efficiency of the various patch positions considering three different modulus ratios - $E_c/E_d = 9 \times 10^3$, $E_c/E_d = 7 \times 10^4$ and $E_c/E_d = 5 \times 10^5$. In effect, one considers the lowest and highest modulus ratio under analysis in this study, together with the threshold value of the modulus ratio for the increasing-decreasing behaviour of damping capacity at the first mode of vibration.

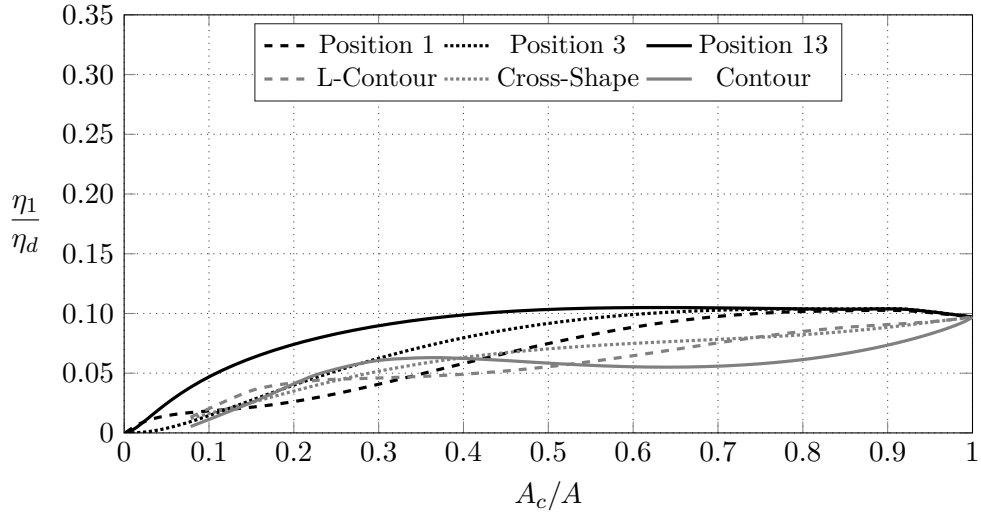
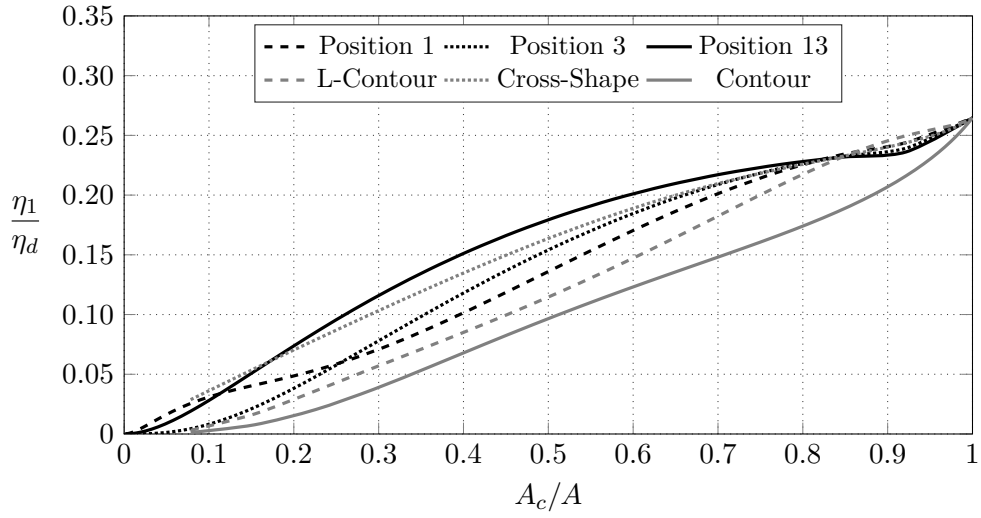
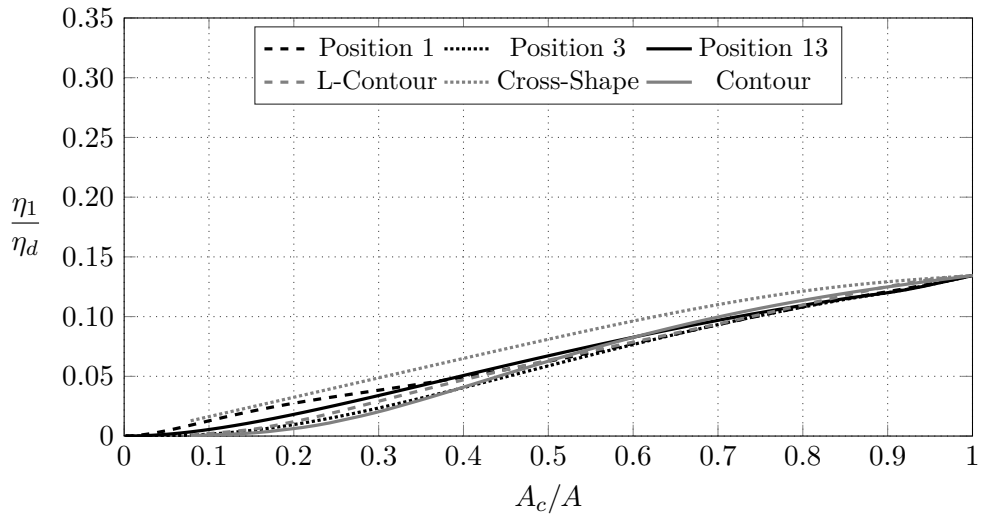
Figure 6.162a presents the damping efficiency curves for the first specific modal loss factor considering the lowest modulus ratio under analysis - $E_c/E_d = 9 \times 10^3$. When both constraining and damping layers have comparable stiffness, the constraining layer can no longer impose a significant transverse shear strain on the damping layer, limiting the damping capacity. Nevertheless, the square patch at Position 13 exhibits a distinctive damping efficiency, including an optimal performance with partial coverage. There is no significant distinction between the other configurations because of the unremarkable damping capacity achievable with this modulus ratio, but the contour treatments show the worst performances as the covered area increases.

On the other hand, Figure 6.162b presents the damping efficiency of different positions of damping treatments for the first mode of vibration and considering the threshold value for the increasing-decreasing behaviour of damping capacity - $E_c/E_d = 7 \times 10^4$. There is a noteworthy improvement in damping capacity, considering this modulus ratio. Such difference between the constraining and damping layers' stiffness promotes the optimal shear of the damping layer and consequently improves damping capacity. Remarkably, the square patch at Position 13 still has the best damping capacity for almost the entire domain of the covered area, followed soon by the Cross-Shape treatment. Unsurprisingly, damping treatments covering the contour areas of the simply supported plate present the worst damping performances.

Additionally, Figure 6.162c demonstrates that damping capacity decreases when the stiffness of the constraining layer is far higher than that of the damping layer. For a higher modulus ratio, $E_c/E_d = 5 \times 10^5$, damping efficiency drops significantly, and there is no significant distinction between different positions. As previously discussed, when considering thicker damping layers, a high modulus ratio decouples the base and constraining layers, disabling the ability to impose shear on the damping layer. Nevertheless, the Cross-Shape treatment offers a narrow margin of damping capacity.

In this subsection, one also studies the damping efficiency curves for higher modes of vibration. Once again, considering the interchange between mode shapes of the second and third modes of vibration for different covered areas as previously discussed, one analyses both specific modal loss factors simultaneously. Figures 6.163a and 6.164a present the second and third specific modal

loss factors, respectively, comparing different positions for partial damping treatments and the lowest modulus ratio under analysis in this study - $E_c/E_d = 9 \times 10^3$. There are no significant differences between the best patch positions compared with the first specific modal loss factor for the lowest modulus ratio. For almost the entire domain of the covered area, the square patch at Position 13 exhibits the best damping performance for the second and third modes of vibration, which inclusively one achieves with a partially covered area. Predictably, the contour treatments still present the worst damping performances explained by the nil curvature for both directions at the edges of the plate in Figure 6.135. For the second and third specific modal loss factors, one also establishes the comparison between damping efficiency curves in Figures 6.163b and 6.164b, respectively, for the threshold value of modulus ratio - $E_c/E_d = 3 \times 10^4$. Interestingly, though, similarly to the results regarding the threshold value of damping layer thickness discussed before, the Cross-Shape treatment presents the overall best damping performance at the second and third modes of vibration, followed by the square patch at Position 13. Moreover, Figures 6.163c and 6.164c prove beyond all doubt that a higher modulus ratios decouple the constraining and base layers and undermine the damping capacity of the treatment. As one can notice, for the second and third modes of vibration, there are no significant differences between any treatment, and the amount of damping capacity is residual.

(a) $E_c/E_d = 9 \times 10^3$ (b) $E_c/E_d = 7 \times 10^4$ (c) $E_c/E_d = 5 \times 10^5$ **Figure 6.162** 1st modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

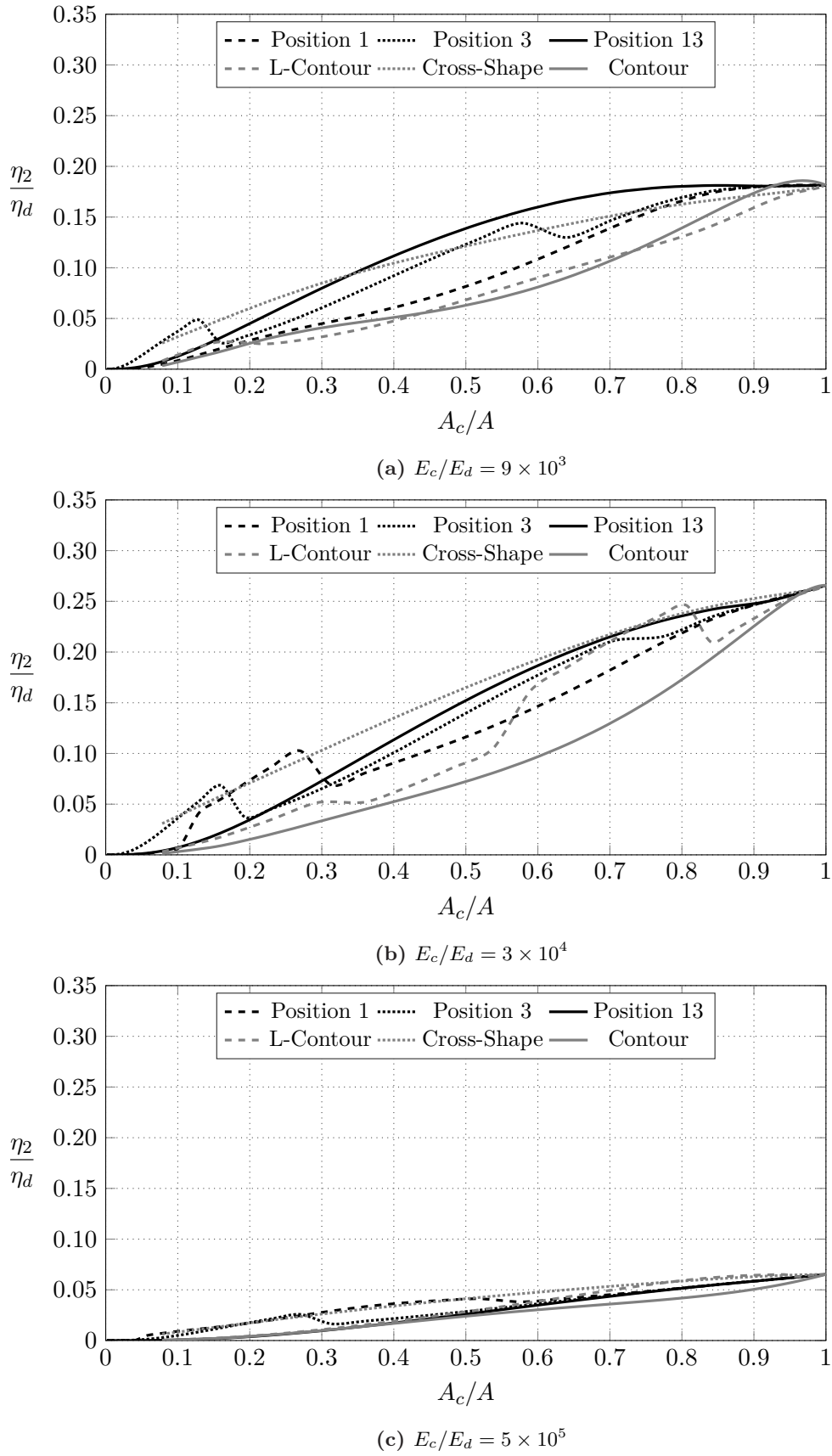
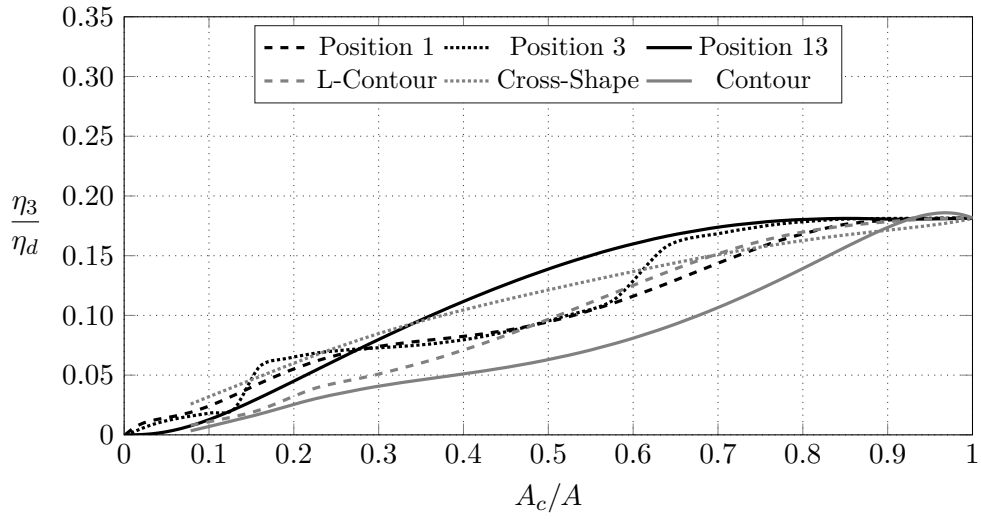
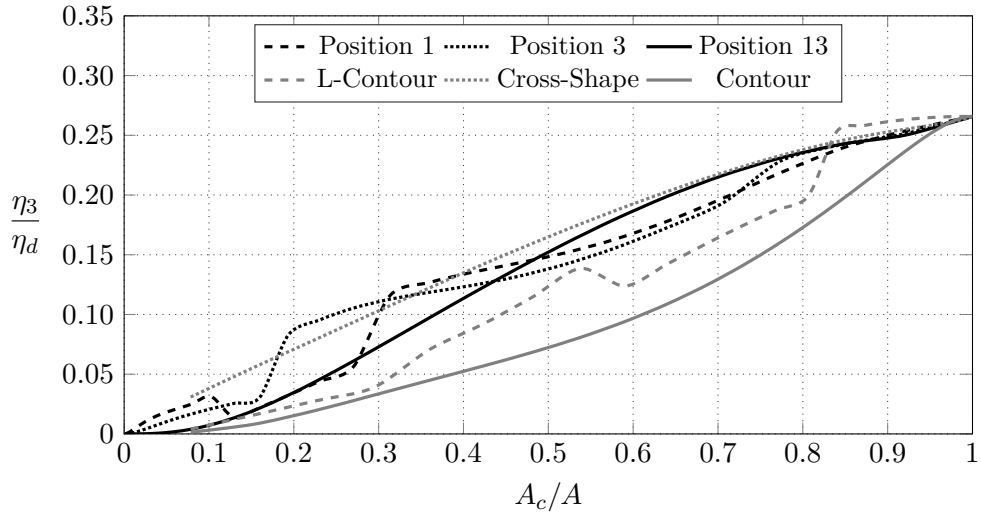
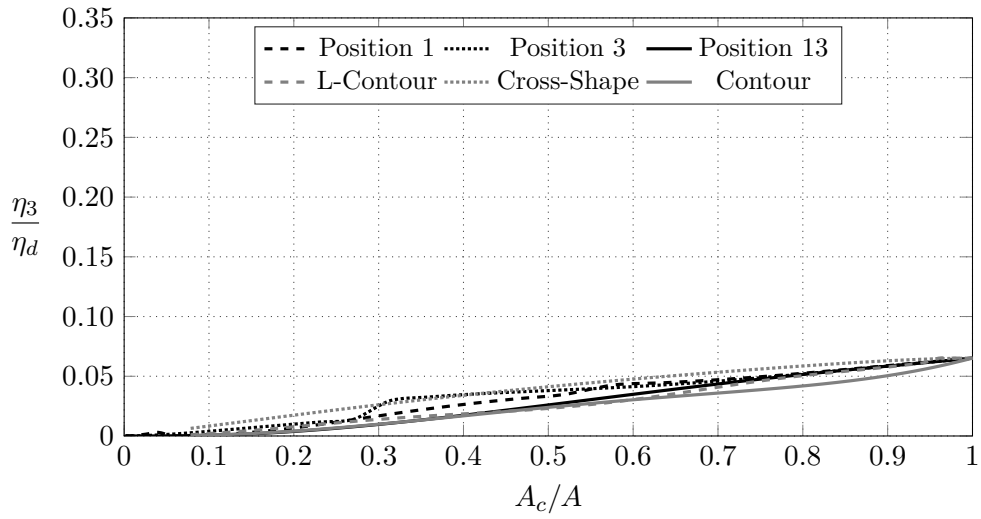


Figure 6.163 2nd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

(a) $E_c/E_d = 9 \times 10^3$ (b) $E_c/E_d = 3 \times 10^4$ (c) $E_c/E_d = 5 \times 10^5$ **Figure 6.164** 3rd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

6.8 | Damping Efficiency of Partially Covered FFFF Plates

The influence of the base, damping and constraining layer thicknesses, together with the mechanical properties of the different materials on the damping efficiency, is decisive, but so are the boundary conditions. As discussed in the previous example, the positioning of a damping patch promotes different damping capacities depending on the potential to impose transverse shear strain on the damping layer. Consistently, patch treatments covering regions with high curvatures promote the extensional deformation of the constraining layer and, consequently, the transverse shear strain of the damping layer, which proved to improve the damping efficiency of the simply supported plate. In the following, one considers the influence of different parameters of a damping patch, such as the patch position, covered area, thickness/side length ratio, constraining layer thickness and material modulus on a square plate with all free edges in this section.

6.8.1 | Influence of Covered Area

In this section, one performs a comparative analysis regarding the damping efficiency of constrained partial damping treatments, considering the influence of the covered area. Considering a reference base plate, the main goal is to identify the impact of the covered area on the damping performance of different partial damping treatments.

6.8.1.1 | Reference Plate - $h = 0.01a$

The study of the influence of the covered area on the damping efficiency of a free plate starts with Figure 6.165, where one can analyse the impact of a square patch at Position 1 on the damping capacity using the first three specific modal loss factors for different thicknesses of the damping layer. Following the dimensionless approach of the previous examples, one considers the thickness of the damping layer as a function of the base layer thickness. It is worth mentioning that the damping capacity of the first vibration mode increases with the covered area, but not steadily. More importantly, it has limits. For thinner damping layers, the first specific modal loss factor increases up to a certain limit with a partially covered area, and the further increase of damping material becomes useless. Besides, there is a maximum achievable damping capacity for the first mode of vibration accomplished with full coverage. The overall damping capacity increases with damping layer thickness until $0.04h$, with a subsequent decrease in damping efficiency for thicker damping materials. This increasing-decreasing effect of the damping capacity, already identified in the previous examples, also proves the existence of a threshold for maximum damping capacity for a free plate. Thicker damping layers do not necessarily increase their shear strain levels. Still, for a free panel, the crucial distinction compared to the simply supported case is the maximum damping capacity achievable with a partially covered area for some damping layer thicknesses.

In addition, one analyses the damping efficiency for higher modes. It is worth mentioning the same value as the threshold of the increasing-decreasing behaviour of damping capacity for the second specific modal loss factor. Remarkably, the maximum damping capacity at the second vibration mode is higher than that of the first one, but similarly, for thinner damping layers, the corresponding specific modal loss factor increases up to a certain limit with a partially covered area. The main difference is that the additional increase in damping material, more than useless, is prejudicial. Considering the third specific modal loss factor, one can notice a thinner damping layer thickness as the threshold of the increasing-decreasing behaviour of damping capacity and its rapid decay as the damping layer gets thick. Nevertheless, it is possible to attain a remarkable damping capacity for this mode, inclusively for a partially covered area.

Considering a square patch at Position 3, Figure 6.166 depicts the damping efficiency curves for the first three vibration modes. At first sight, it is hard to distinguish the main differences to the patch at Position 1. Maximum damping capacities for thinner damping layers occur with partially covered areas for all the considered vibration modes, and the threshold values of damping layer thickness for the increasing-decreasing behaviour are those of the previous example. Perhaps the most noteworthy difference when considering a square patch at Position 3 is the slightly improved performance of damping capacity for small covered areas.

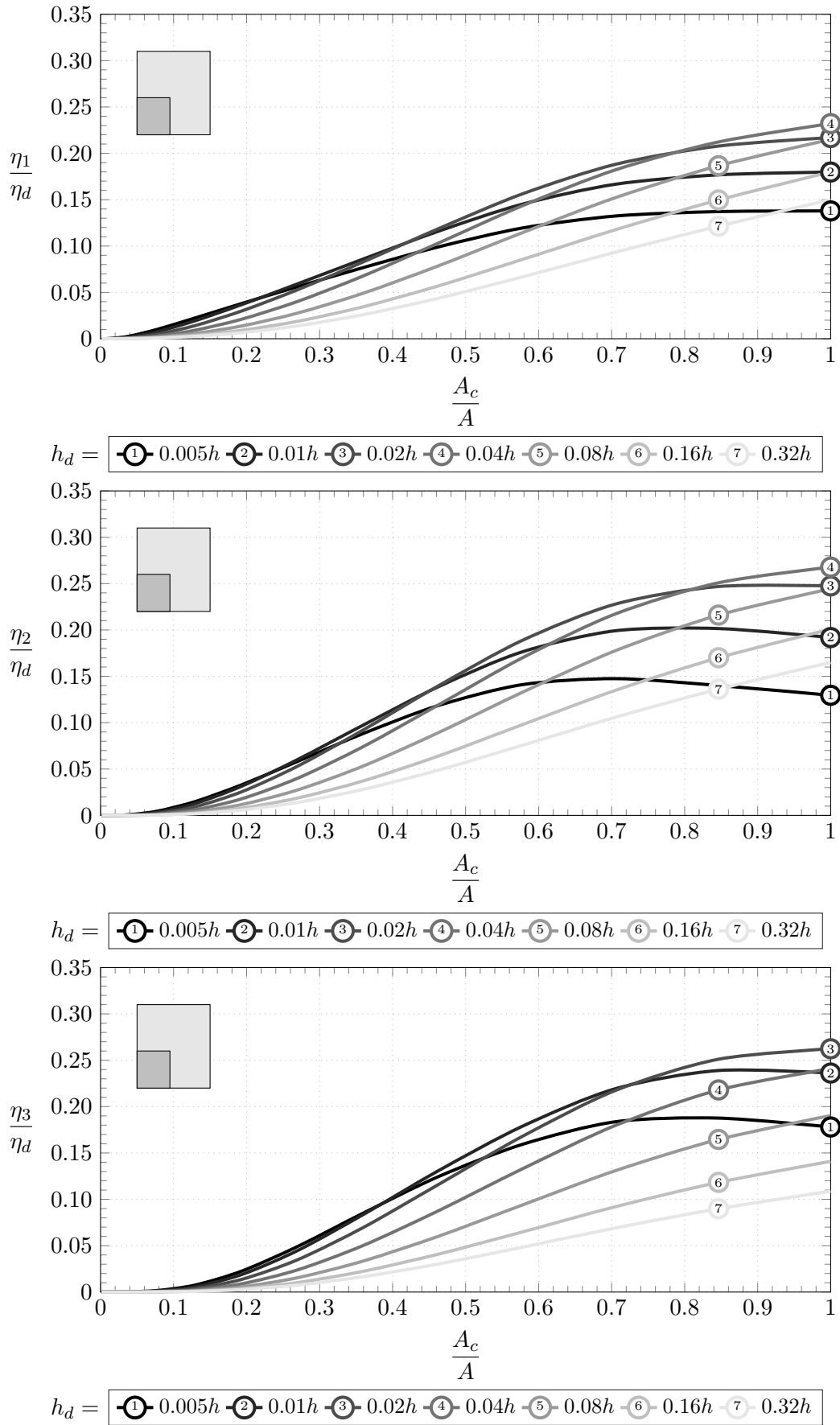
Figure 6.167 presents the damping efficiency curves for different covered areas and damping layer thicknesses considering a square patch at Position 13. The results for small covered areas are noticeably distinct compared to the previous positions. There is a quick increase in specific damping capacity as soon as the covered area increases, regardless of the mode of vibration. For thinner damping layers, the first specific modal loss factor increases and peaks with a partially covered area smaller than the previous positions. Even though the threshold of damping layer thickness for the increasing-decreasing behaviour holds the same value, it is possible to achieve better damping performances with the square patch at Position 13 for the same covered area. For higher vibration modes, the maximum damping capacity considering thinner damping layers occurs for smaller covered regions than those of the previous positions, proving the benefits of the square patch at the central Position 13 for partially covered areas.

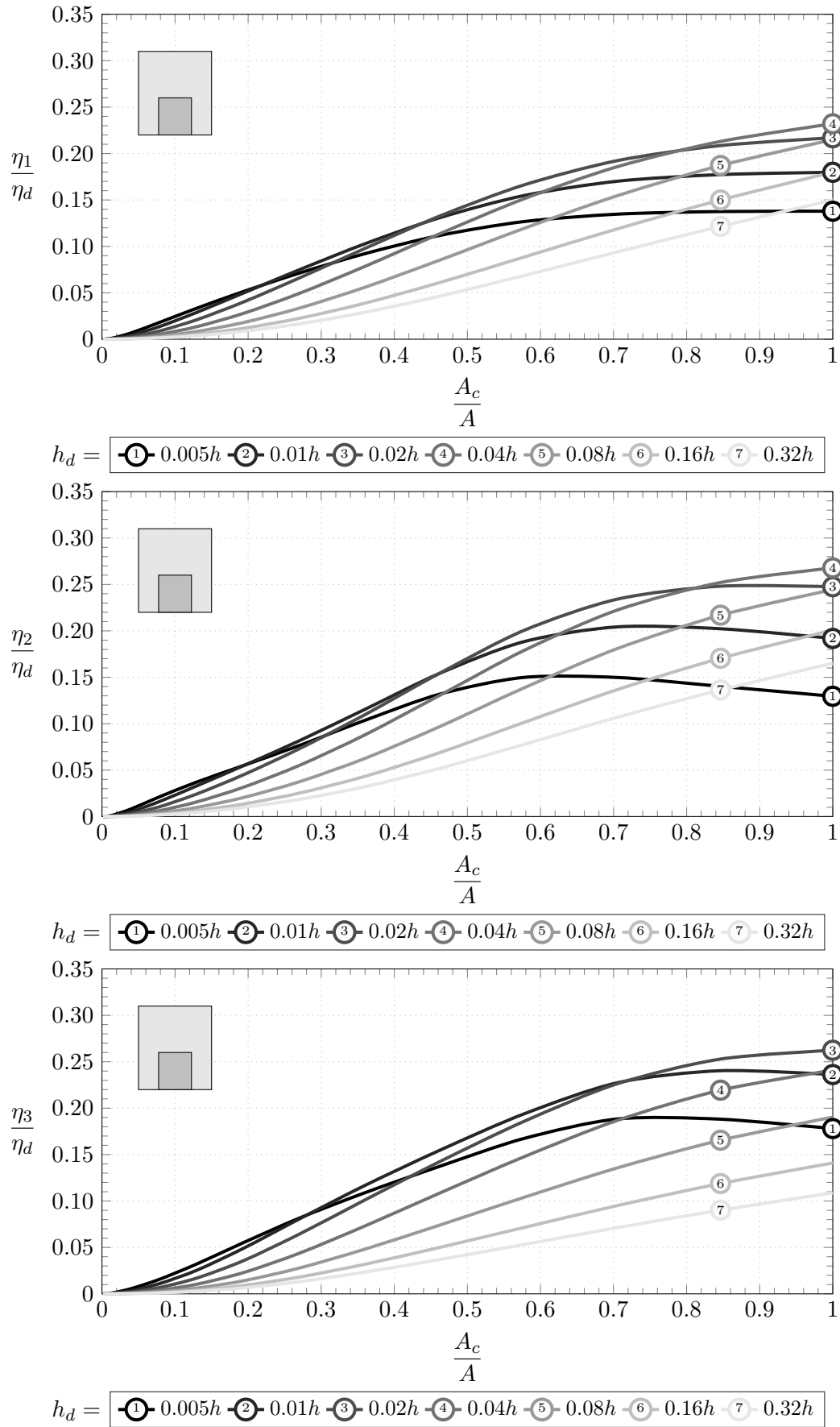
Subsequently, one also explores the non-square damping patches and their damping efficiency, considering different covered areas and damping layer thicknesses. It is the case of Figure 6.168 and the corresponding first three specific modal loss factors for the L-Contour treatment. The evident decrease in damping capacity for partial coverage is the most noticeable feature of these damping efficiency curves. There is no exception between the different modes of vibration regarding this trend, and the less damping capacity for partial coverage with the L-Contour treatment holds for the three specific modal loss factors. As expected, the damping efficiency for complete covered areas is the same for any configuration and the threshold value for the increasing-decreasing behaviour of damping capacity. Similarly to the simply supported case, the L-Contour treatment exhibits a weak damping performance compared to the square patches.

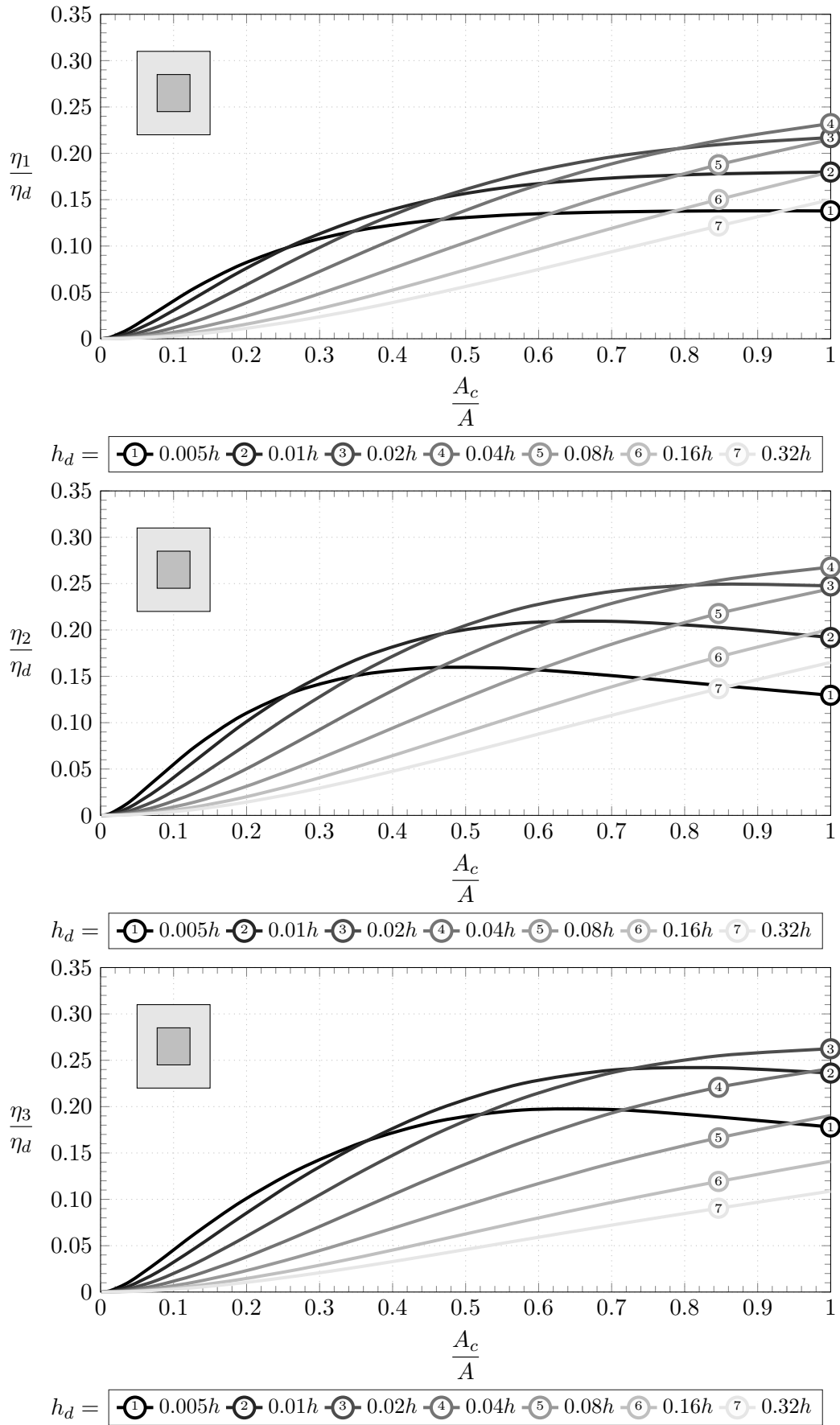
The study of the damping efficiency curves considering non-square patches extends to the Cross-Shape treatment in Figure 6.169. In contrast with the simply supported example, for which the Cross-Shape treatment presented interesting results for small covered areas, the damping capacity using smaller amounts of damping material is not promising. However, as the covered area increases, and in contrast with the square patches considered before, one can achieve the maximum damping capacity at the first and third vibration modes with partial coverage of the base plate. It is a remarkable result, proving the existence of optimal solutions of damping capacity with partially covered areas. Repeatedly, the threshold values of damping layer thickness hold and consequently, the Cross-Shape treatment proves to be the one with better damping performance, at least for the first and third vibration modes.

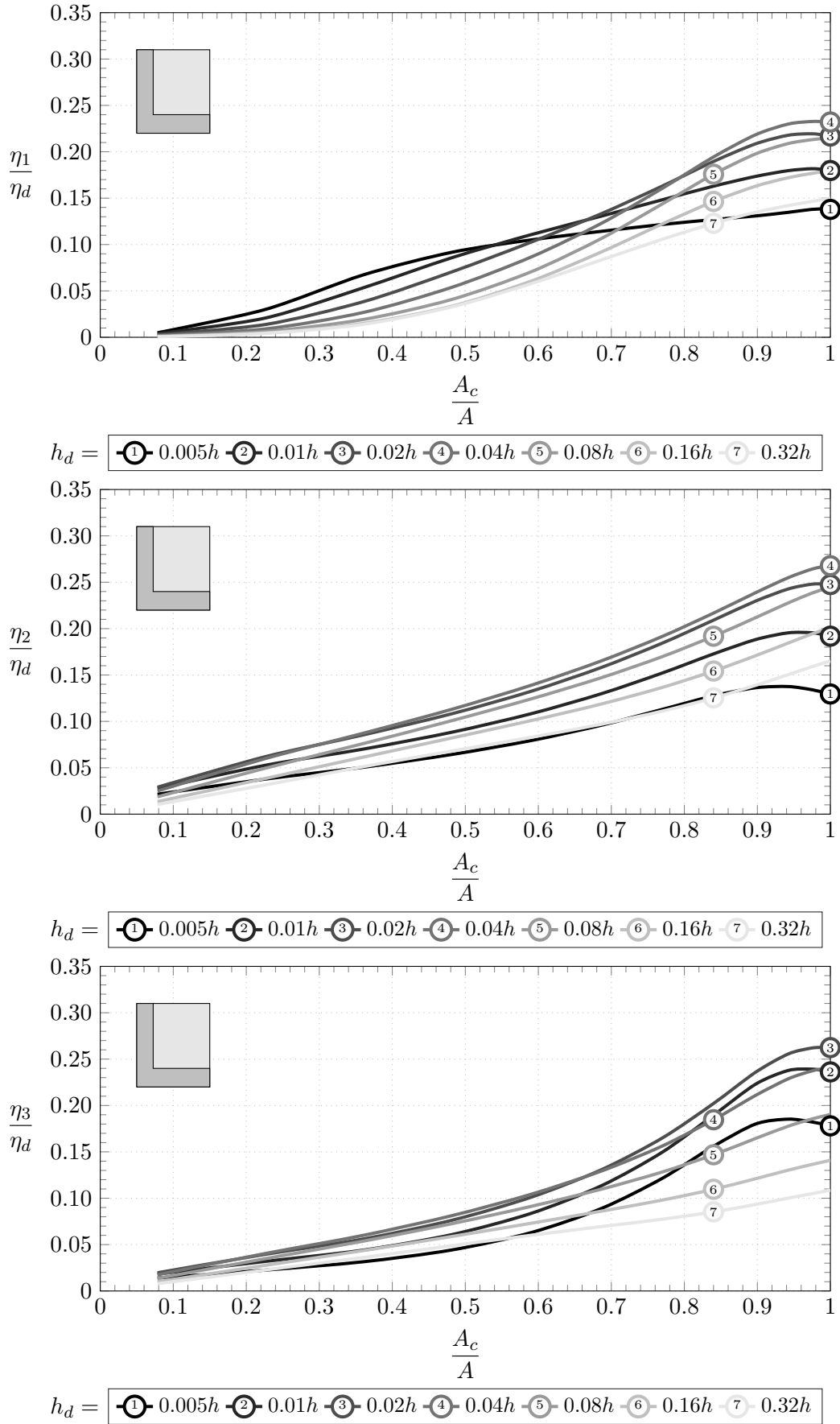
Finally, Figure 6.170 presents the damping efficiency curves for Contour treatment. The results for the free plate follow the trends of the simply supported example with low damping capacities for any of the considered vibration modes. The damping efficiency is comparable with other patches for fully covered plates. These results confirm the inefficiency of the Contour treatment in terms of damping capacity for a free panel.

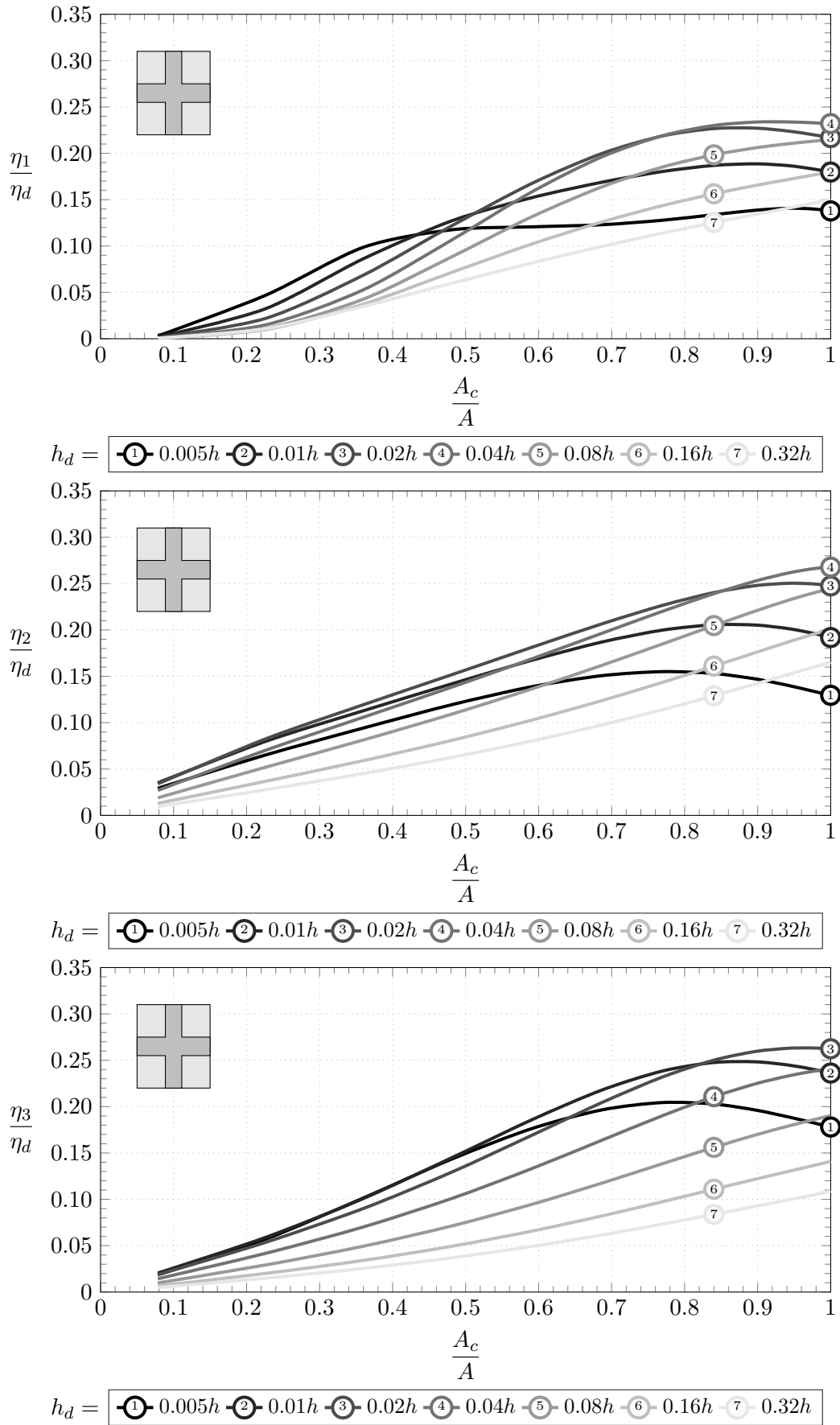
The results highlight the efficiency of different damping treatments considering the influence of the covered area and damping layer thickness for the first three modes of an FFFF plate. One determined and discussed the thresholds of the increasing-decreasing behaviour and assessed the main patterns of different damping patches and their damping performances. However, in the following section, one analyses the influence of the patch position, considering a comparative study between damping treatments to perform a detailed comparison between different configurations.

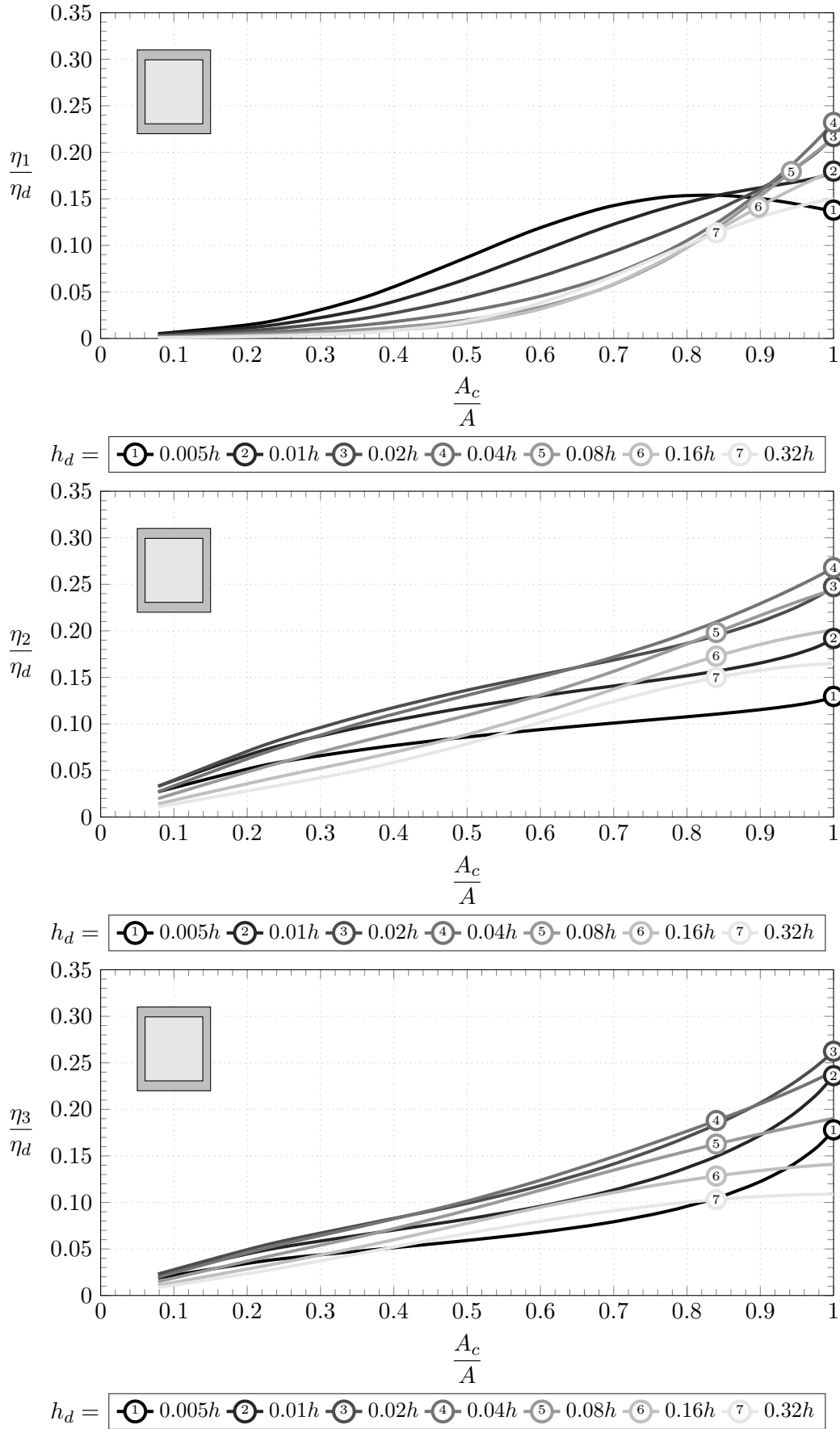

 Figure 6.165 Damping efficiency curves for Position 1, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.166 Damping efficiency curves for Position 3, $h = 0.01a$, $h_c = 0.5h$

Figure 6.167 Damping efficiency curves for Position 13, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.168 Damping efficiency curves for L-Contour treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.169 Damping efficiency curves for Cross-Shape treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.170 Damping efficiency curves for Contour treatment, $h = 0.01a$, $h_c = 0.5h$

6.8.2 | Influence of Patch Positioning

Following the same reasoning of the simply supported plate, in this section, one performs a comparative analysis between different positions of damping treatments concerning their damping efficiency. It is crucial to perform a detailed analysis determining the positions promoting improved damping performances and prove the main conclusions drawn in the previous example for the simply supported plate regarding the basic mechanisms leading to such enhanced results. Thus, one also investigates the mode shapes of the free plate to understand the obtained results and the most suitable areas for partial damping. Figure 6.171 depicts the transverse displacement mode shapes of the first three normal modes of the free square reference plate considering a constrained layer damping treatment with full coverage and a damping layer thickness at the threshold value for the increasing-decreasing behaviour of damping capacity at the first and second modes of vibration - $h = 0.01a$, $h_d = 0.04h$ and $h_c = 0.5h$. During this work, one discussed the importance of the extensional deformation of the constraining layer as a mechanism for imposing the transverse shear strain of the damping layer. Taking this into account, Figure 6.172 shows the curvature mode shapes to figure out the bending pattern of the free plate and the helpful positions for constrained damping treatments. Besides, Figure 6.173 presents the shear strain mode shapes γ_{xz} and γ_{yz} , crucial to comprehending the modal shear strain pattern of the plate and understanding the mechanisms of the energy dissipation.

6.8.2.1 | Reference Plate - $h = 0.01a$

The previous study of the damping efficiency of different partial damping treatments does not allow for a comprehensive comparative analysis between different positions and their improved damping performances. Recovering those results, Figure 6.174 depicts the first specific modal loss factor considering different patch positions and covered areas simultaneously. One performs the comparison by studying three different damping layer thicknesses - $h_d = 0.005h$, $h_d = 0.04h$ and $h_d = 0.32h$. Following the same reasoning of the previous studies, one considers the thinnest and thickest damping layers together with the threshold value of damping layer thickness for the increasing-decreasing behaviour of damping capacity for the first vibration mode.

Specifically, Figure 6.174a presents the damping efficiency curves of the first specific modal loss factor for the thinnest damping layer - $h_d = 0.005h$. The results are straightforward, there is a meaningful damping capacity, and the square patch at Position 13 provides the best damping capacity for a wide range of covered areas, which inclusively peaks for a relatively small partially covered area. However, quite surprisingly, the Contour treatment presents the best damping capacity, achievable with a covered area of roughly 80 %, after being the worst possible solution when considering small patches. Then, the Cross-Shape treatment and the square patch at Position 3 introduce better damping capacity than the square patch at Position 1 and the L-Contour treatment.

Subsequently, Figure 6.174b depicts the damping efficiency of different positions for the first mode of vibration, considering the threshold value for the increasing-decreasing behaviour of damping capacity - $h_d = 0.04h$. It is possible to distinguish two reference positions - the square patch at Position 13, with the best damping capacity for a wide range of covered areas, and the Contour treatment, with the worst damping efficiency considering any covered area. Other square patches introduce significant damping capacities, but one achieves the optimum damping capacity at this mode of vibration with the Cross-Shape treatment and a partially covered area. The preceding proves the possible inefficiency of fully covered plates for improved damping performances. Even so, the former results raise some issues. On the one hand, the Contour treatment presents conflicting results for different covered areas when considering the thinnest

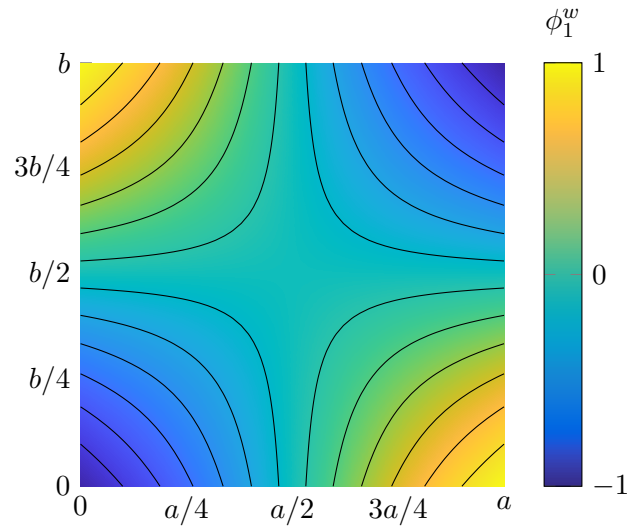
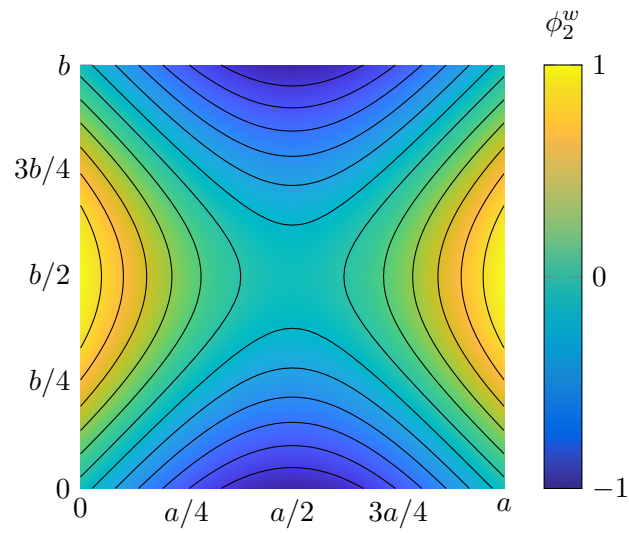
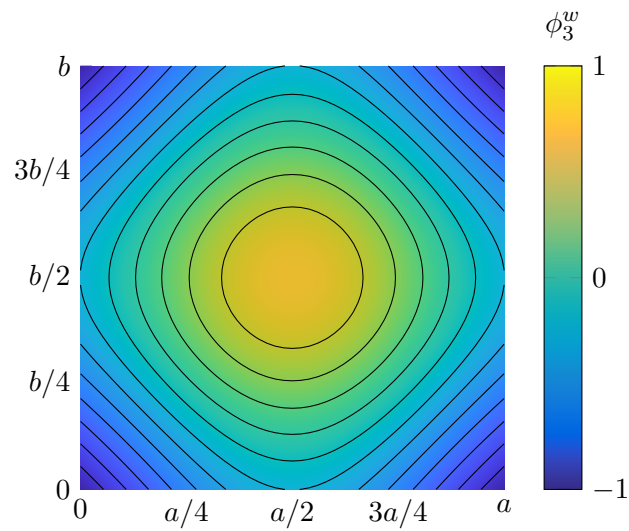

 (a) 1st transverse displacement mode shape

 (b) 2nd transverse displacement mode shape

 (c) 3rd transverse displacement mode shape

Figure 6.171 Transverse displacement mode shapes of a free plate with damping treatment

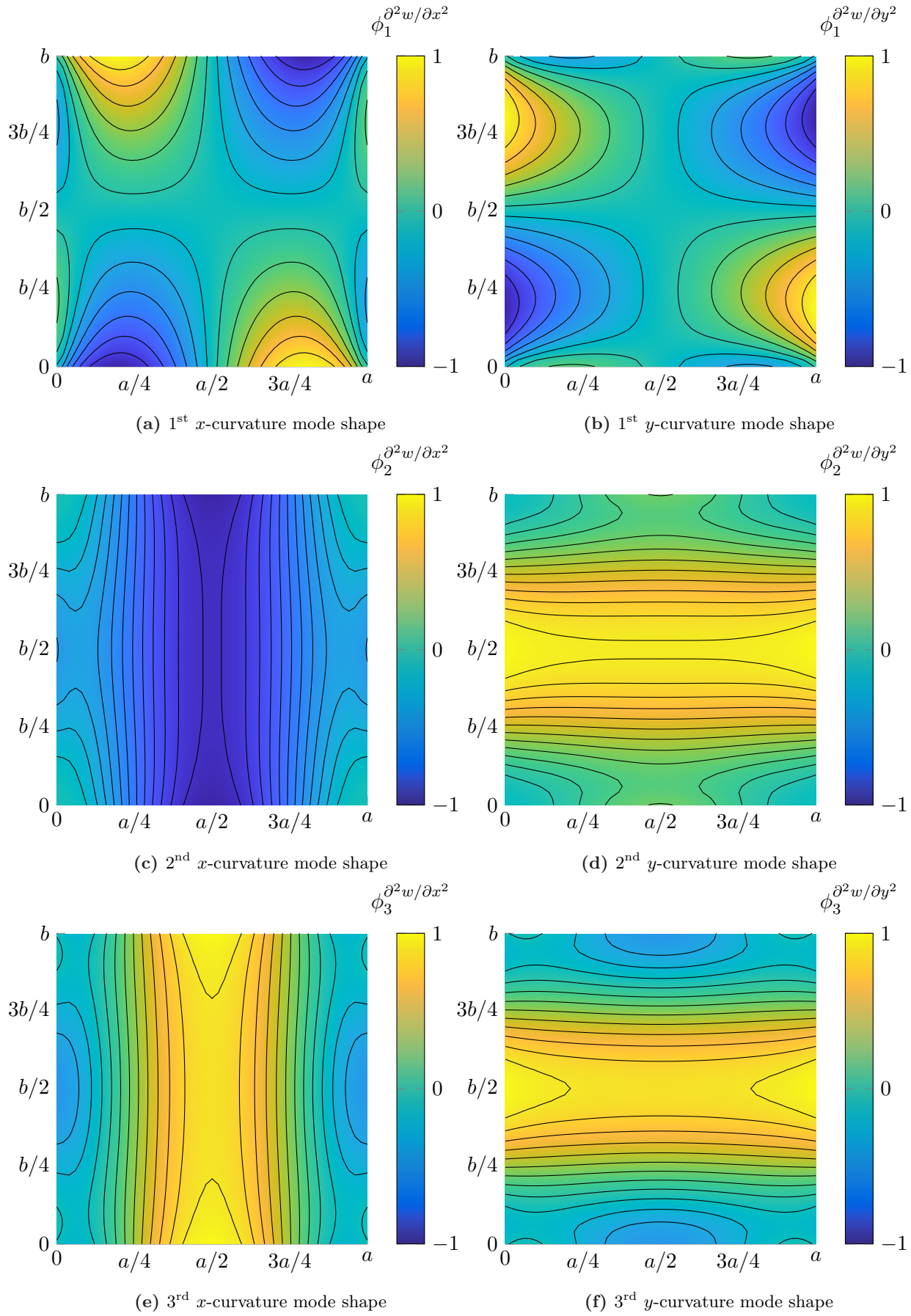


Figure 6.172 Curvature mode shapes of a free plate with damping treatment

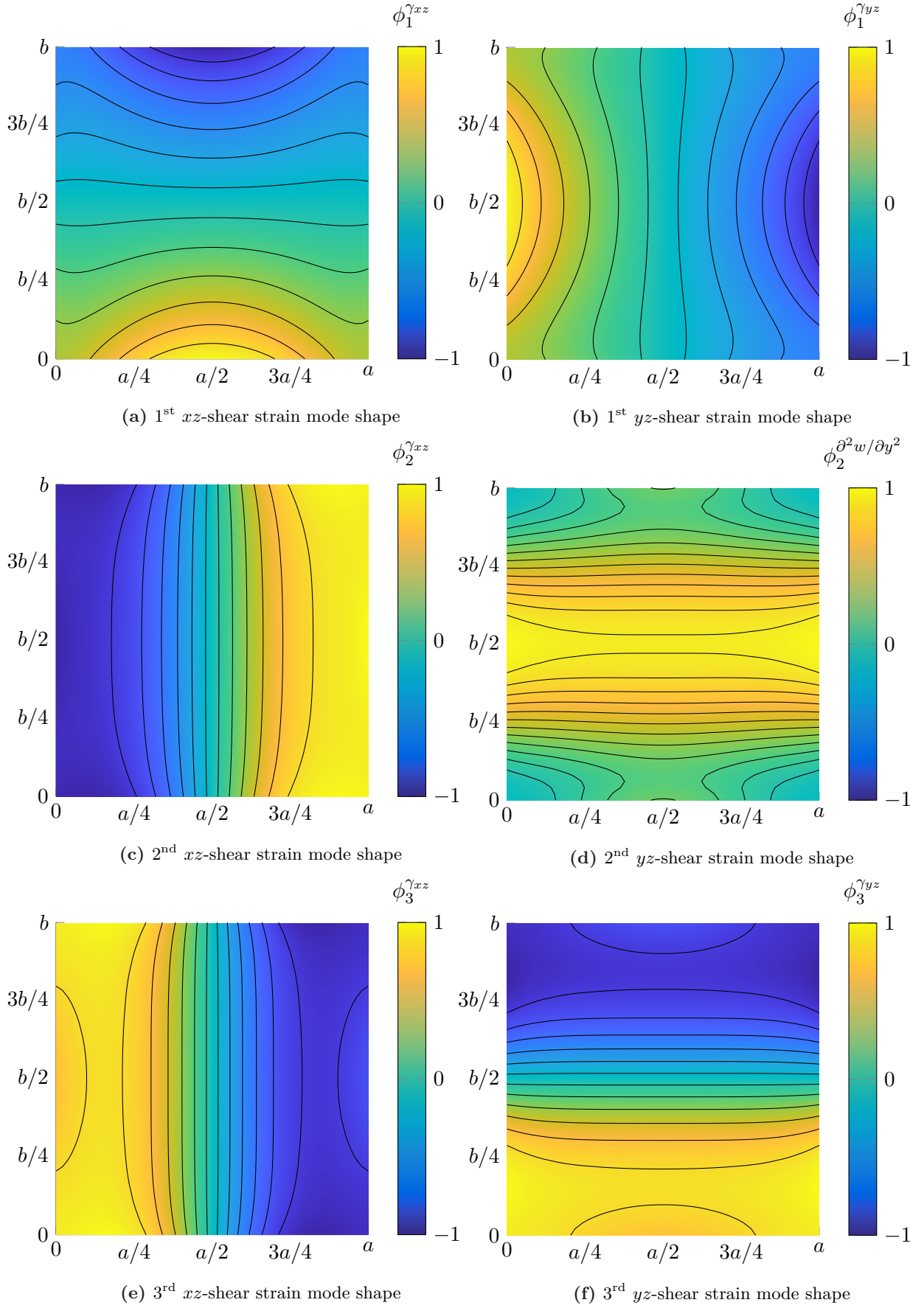


Figure 6.173 Shear strain mode shapes of a free plate with damping treatment

damping layer. On the other hand, the Contour treatment shows low efficiency regardless of the covered area for the threshold value of damping layer thickness. In more detail, the L-Contour treatment also presents weak damping performances for a wide range of covered areas. It is challenging to explain the poor damping capacities of the contour treatments using the same reasoning as the previous examples. Recalling Figure 6.172, one can analyse the x and y -curvature of the fully covered plate at the first mode of vibration and identify the maximum curvature for both directions at the edges of the plate, and so bending deformation of the constraining layer. Investigating the mode shapes of the base or fully covered plates might not be enough to design efficient patch treatments. Besides, these results highlight the importance of the parametric analyses on the appropriate design of partial damping treatments.

It is of crucial importance to understand these results. Following the same reasoning of the previous section, Figures 6.175 and 6.176 present the first mass normalised shear strain mode shapes of free plates with an 80 % covered area considering the different treatments under analysis in this work. The geometric and mechanical properties of the plate are those considered for the damping efficiency curves of the reference plate with a damping layer thickness of $h_d = 0.04h$. Transverse shear strain mode shapes at both directions, $\phi_1^{\gamma_{xz}}$ and $\phi_1^{\gamma_{yz}}$, present no valuable information about the damping treatments and their effectiveness. These mode shapes present no more information for any damping treatment than flat regions for almost the entire domain with maximum values at the edges of the plate. There are no insights to withdraw from the transverse shear strain mode shapes of plates with free boundary conditions and different damping treatments.

Considering an 80 % covered area, Figure 6.174b shows the best damping treatments in the following order: Cross-Shape treatment, Position 13, Position 3, Position 1, L-Contour treatment and Contour treatment. To explain such sorting, one considers the through-the-thickness mode shapes to clarify the mechanisms leading to the improved damping performances of some treatments over the others for free plates with an 80 % covered area considering the different treatments under analysis in this work. Figures 6.177 - 6.178 and 6.179 - 6.180 illustrate the through-the-thickness views of the first transverse shear strain energy density mode shapes, $\phi_1^{U_{s^{xz}}}$ and $\phi_1^{U_{s^{yz}}}$, respectively, considering section cuts along both directions. These section cuts rely on the expected higher levels of bending deformation of the constraining layer at the edges of the plate, which is crucial to ensure the extensional deformation of the constraining layer as a mechanism to impose shear to the damping layer - see Figure 6.172. Therefore, one considers a section cut at $b/12.5$ to represent the through-the-thickness view of the first transverse shear strain energy density mode shape $\phi_1^{U_{s^{xz}}}$, and a section cut at $a/12.5$ to represent the through-the-thickness view of the first transverse shear strain energy density mode shape $\phi_1^{U_{s^{yz}}}$. One considers the section cuts at a narrow distance from the edges and not at the edges to ensure that every through-the-thickness view includes a damping treatment. The transverse shear strain energy density mode shapes at both directions highlight and help to explain the best damping performances of some treatments over others. Modal transverse shear strain energy density at the damping layer of the Cross-Shape treatment and square patch at Position 13 is higher than any other treatment, helping to explain the superior damping performance of such treatments, especially the Cross-Shape treatment. As expected, the modal transverse shear strain energy density levels of the contour treatments are low.

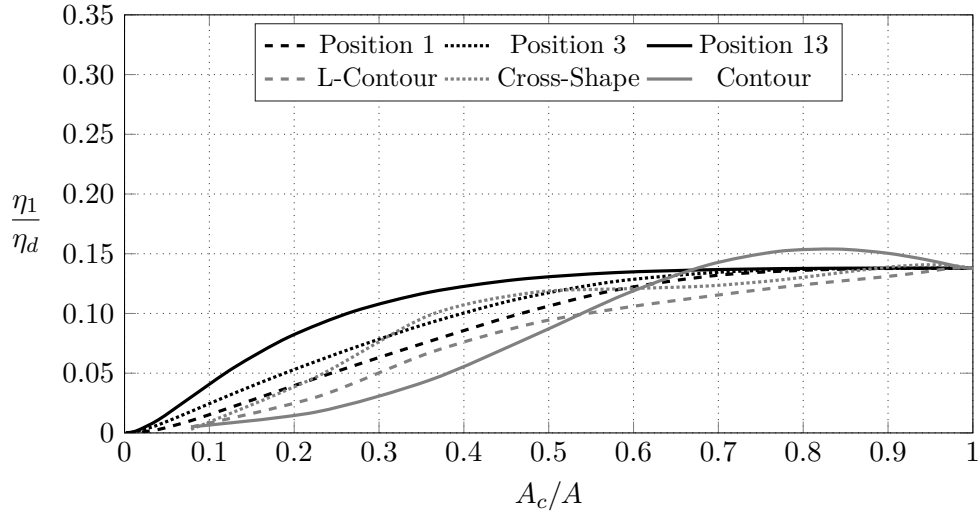
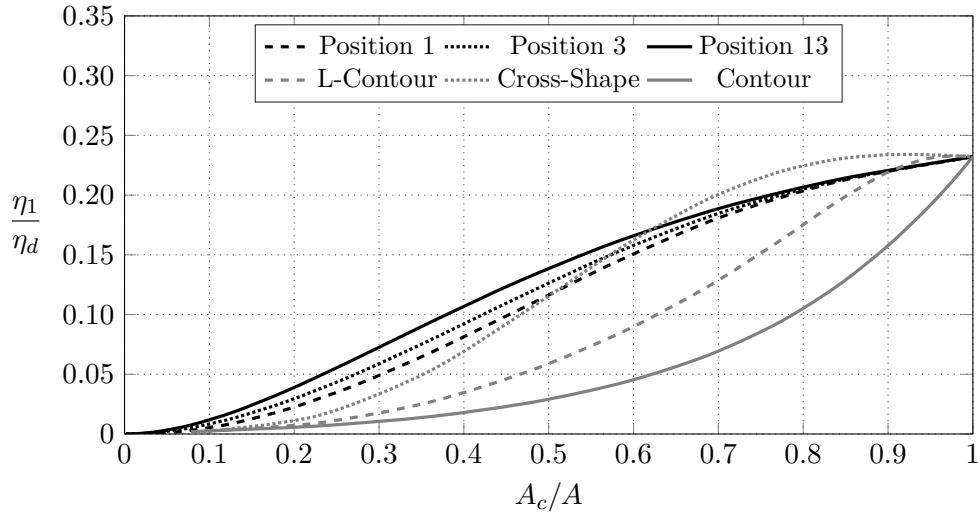
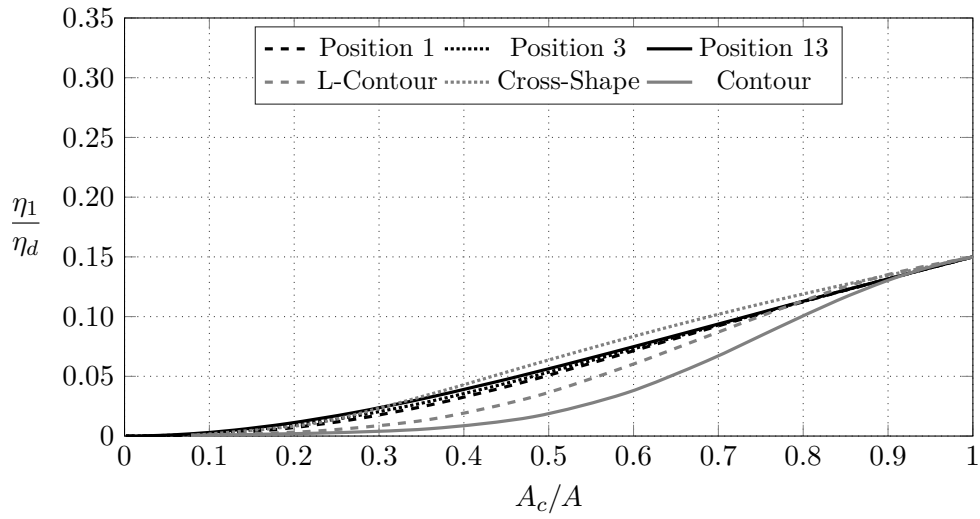
As previously discussed, the extensional deformation of the constraining layer plays a decisive role as the primary mechanism responsible for imposing transverse shear strain on the damping layer. Figures 6.181 - 6.182 and 6.183 - 6.184 present the through-the-thickness views of the first membrane plus bending strain energy density mode shapes, $\phi_1^{U_n^{xx}}$ and $\phi_1^{U_n^{yy}}$, respectively, considering the same section cuts along both directions. Unfortunately, the results are inconclusive. There are no significant differences in modal extensional deformation between damping treatments

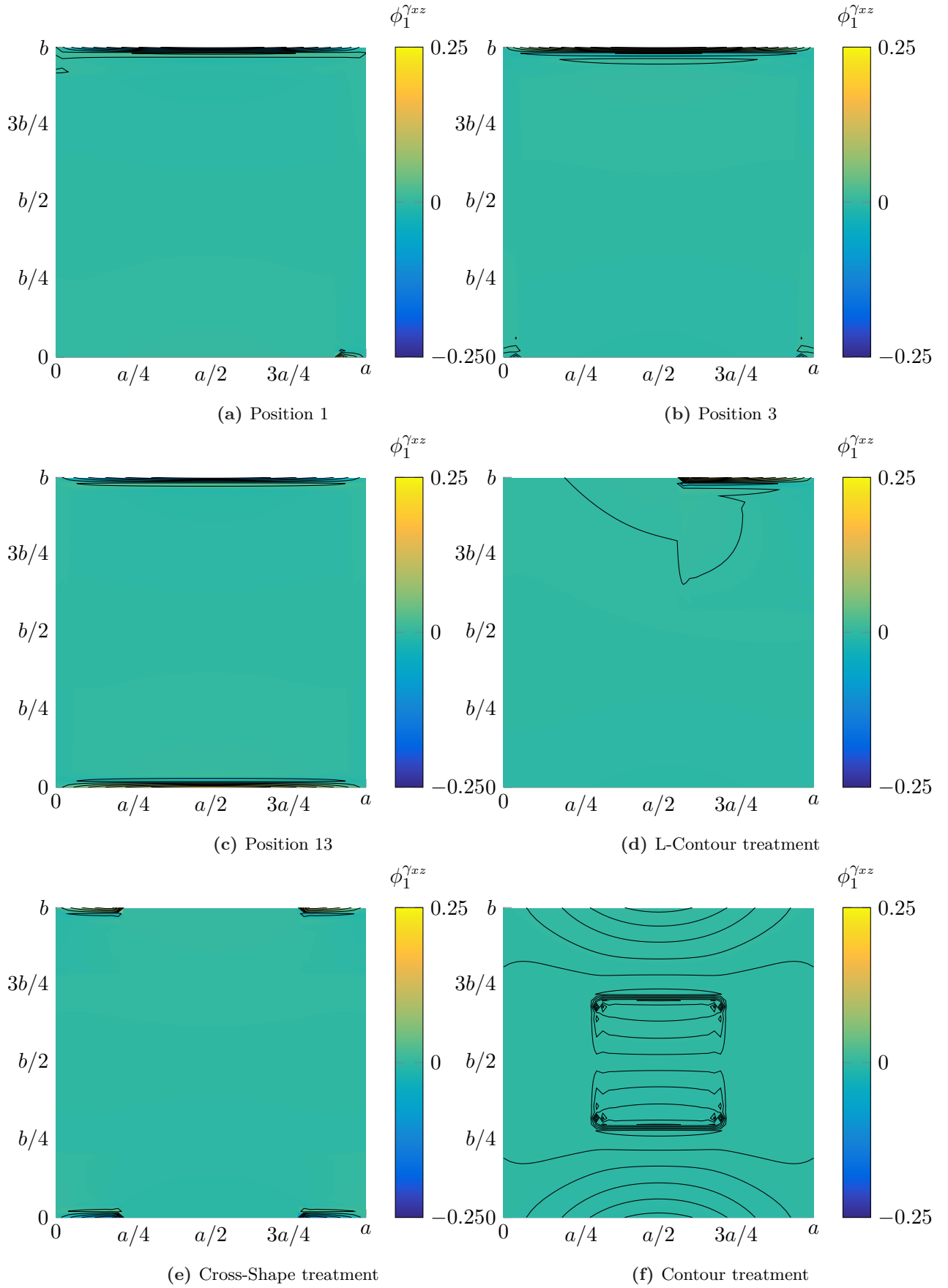
that justify the distinct damping performances. But one should be careful here. Unlike the simply supported plate, the section cuts are not coincident with symmetry planes, raising some issues. For example, if the section cuts followed exactly along the edges, where curvatures are maximum and so higher levels of bending deformation of the constraining layer, one would not include damping treatments in the through-the-thickness representation of the square patches at Position 1, Position 3 and Position 13, while the Cross-Shape treatment would present high levels of extensional deformation. This example proves that one must be careful when selecting and analysing through-the-thickness views of section cuts. The visual information is impressive, but one has to consider the constraints. To be more specific, integrating the modal membrane plus bending strain energy density over the volume of the constraining layer would give a precise numerical value of the extensional modal deformation energy of the constraining layer. Here is a suggestion for future work.

Following the parametric analysis, Figure 6.174c shows the result of increasing the damping layer thickness beyond the threshold value - $h_d = 0.32h$. By decoupling the base and constraining layers, transverse shear strain levels at the damping layer drop, so the treatment loses efficiency. As reported previously, thicker damping layers also break any improved performance of some positions over others, and damping efficiency becomes almost linearly dependent on the amount of damping material.

Following the same approach as the previous examples, one also performs a detailed comparison between the damping efficiency of different positions for higher vibration modes. As previously reported, the threshold of damping layer thickness for the increasing-decreasing behaviour of damping capacity at the second vibration mode holds the same value as the first one, so one considers the same thicknesses in the following analysis with a partially covered area of roughly 45 %. Other square patches and the Cross-Shape treatment also introduce a significant damping capacity, but the contour treatments, once again, show limited damping efficiencies. The trend holds for the threshold damping layer thickness value - see Figure 6.185b. Even though one only achieves the optimum damping capacity with fully covered plates, the square patch at Position 13 keeps the best performance, and the contour treatments, the worst ones, as the covered area increases. Following the reasoning of the previous examples and the curvature mode shapes of Figure 6.172, it is simpler to understand the results for the second mode of vibration based on the maximum curvature for both directions along each mid-side length of the plate, and so bending deformation of the constraining layer. Then, Figure 6.185c proves that thicker damping layers decouple the base and constraining layers and degrade the damping capacity.

Lastly, Figure 6.186 shows the third specific modal loss factor considering different patch positions and covered areas simultaneously. Following the same reasoning as the previous studies, one studied three damping layer thicknesses - $h_d = 0.005h$, $h_d = 0.02h$ and $h_d = 0.32h$. One considers the thinnest and thickest damping layers, together with the threshold value of damping layer thickness for the increasing-decreasing behaviour of damping capacity at the third mode of vibration. Figure 6.172 shows that similarly to the second mode of vibration, maximum curvature for both directions at the third mode is along each mid-side length of the plate. Unsurprisingly, the trends of the former mode of vibration remain valid here, and from an overall perspective, the square patch at the central Position 13 presents the best damping efficiency. For thinner damping layers in Figure 6.186a, the Cross-Shape treatment exhibits the best damping capacity by slightly exceeding the square patch at Position 13. On the other hand, the contour treatments consistently show weak damping performances, either for the thinnest or optimum damping layer thicknesses. As expected, Figure 6.186 shows that thicker damping layers reduce the damping capacity, after the threshold values, for the third mode of vibration too.

(a) $h_d = 0.005h$ (b) $h_d = 0.04h$ (c) $h_d = 0.32h$ **Figure 6.174** 1st modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$


 Figure 6.175 1st xz -shear strain mode shapes of 80% covered plates

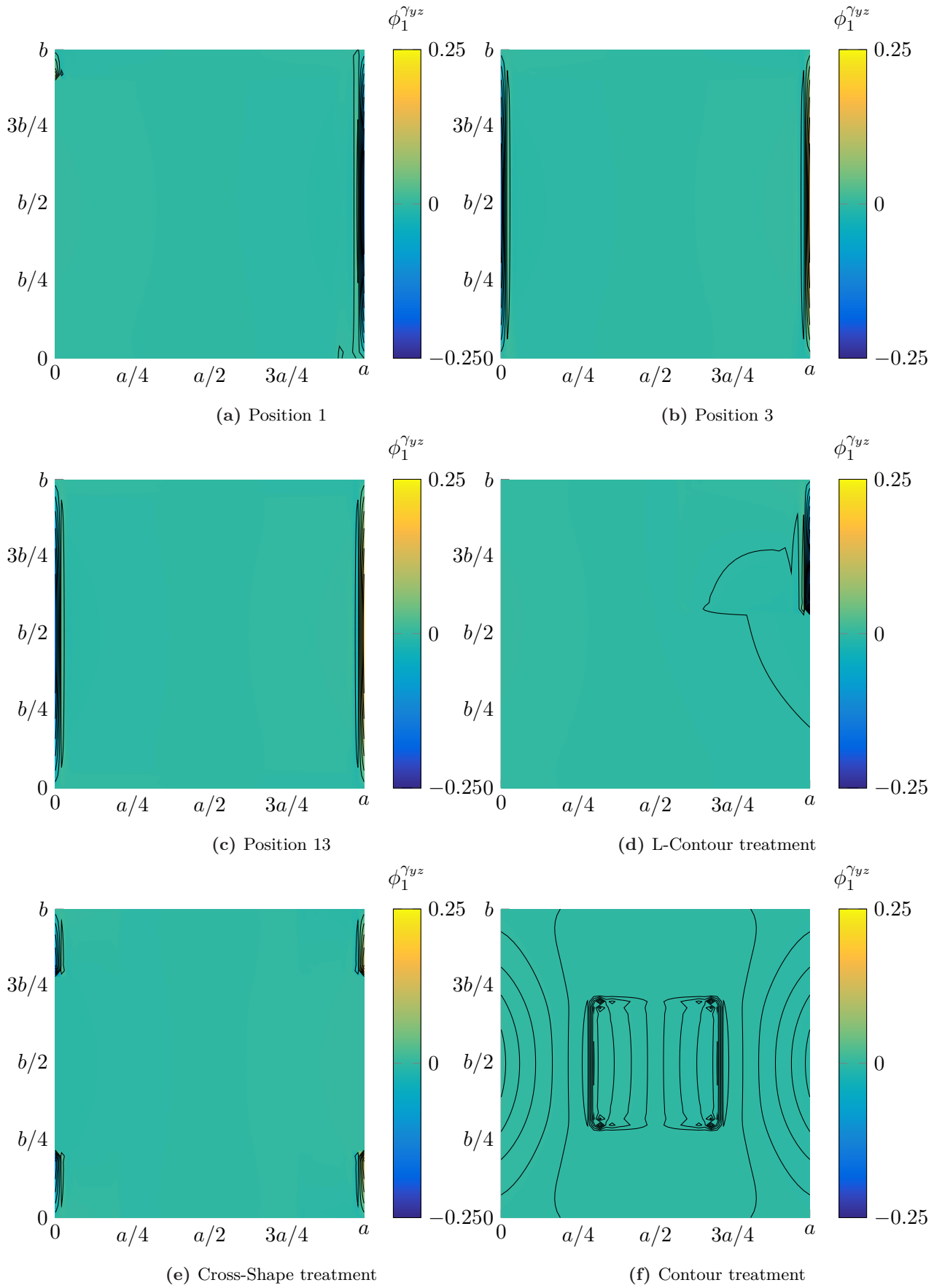


Figure 6.176 1st yz -shear strain mode shapes of 80 % covered plates

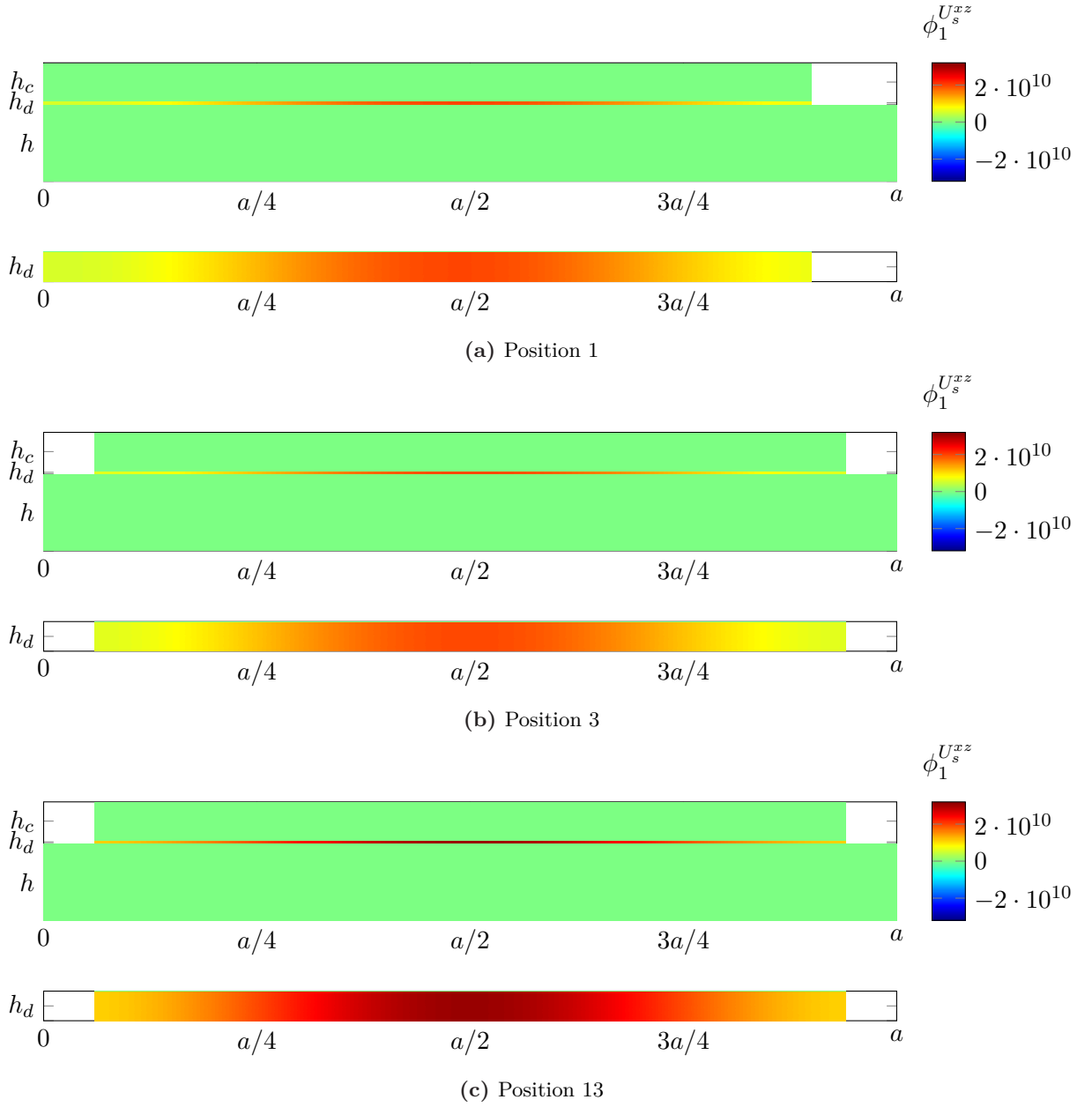


Figure 6.177 1st xz -shear strain energy density mode shapes - section cut at $b/12.5$

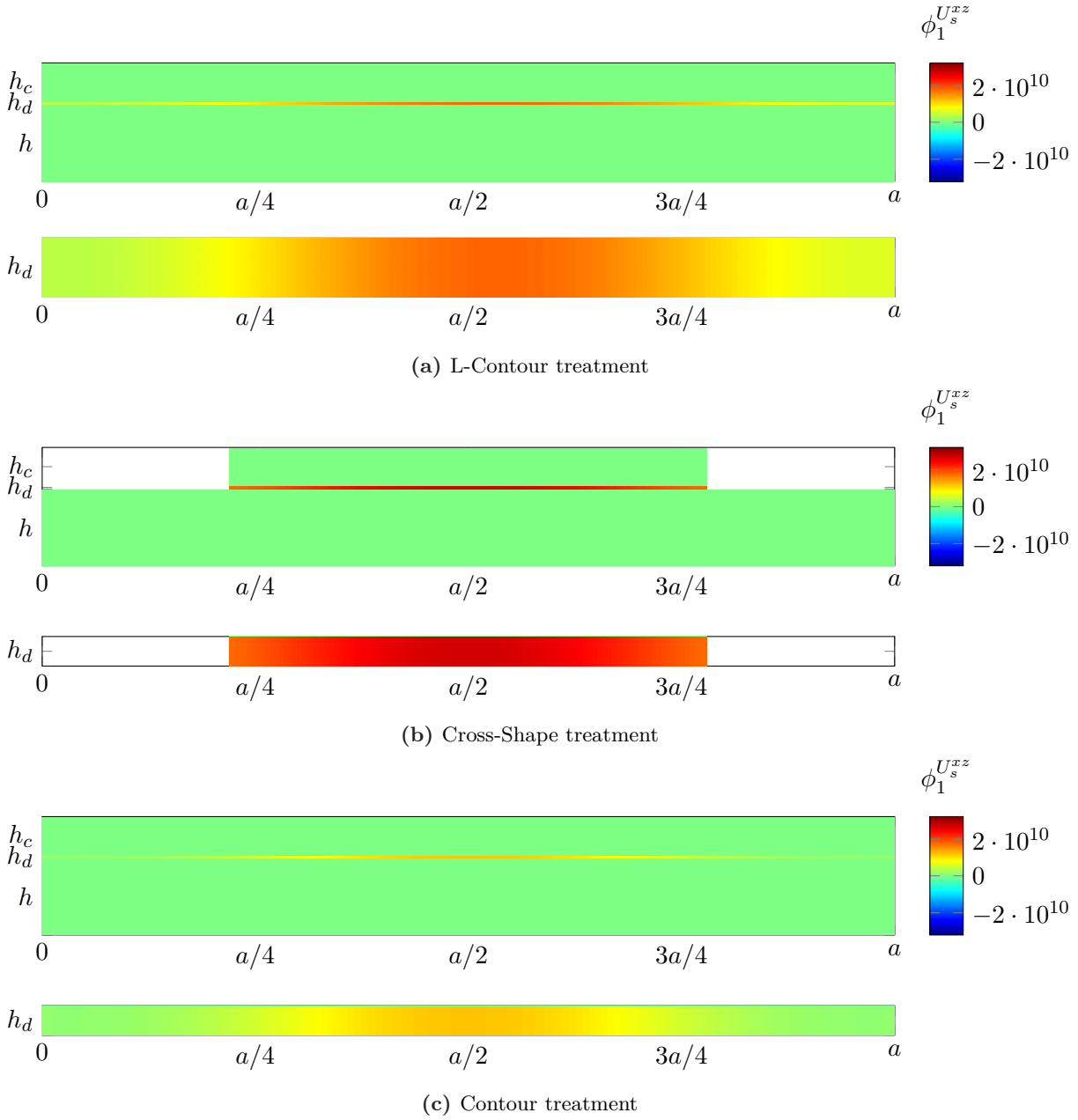


Figure 6.178 1st xz -shear strain energy density mode shapes - section cut at $b/12.5$

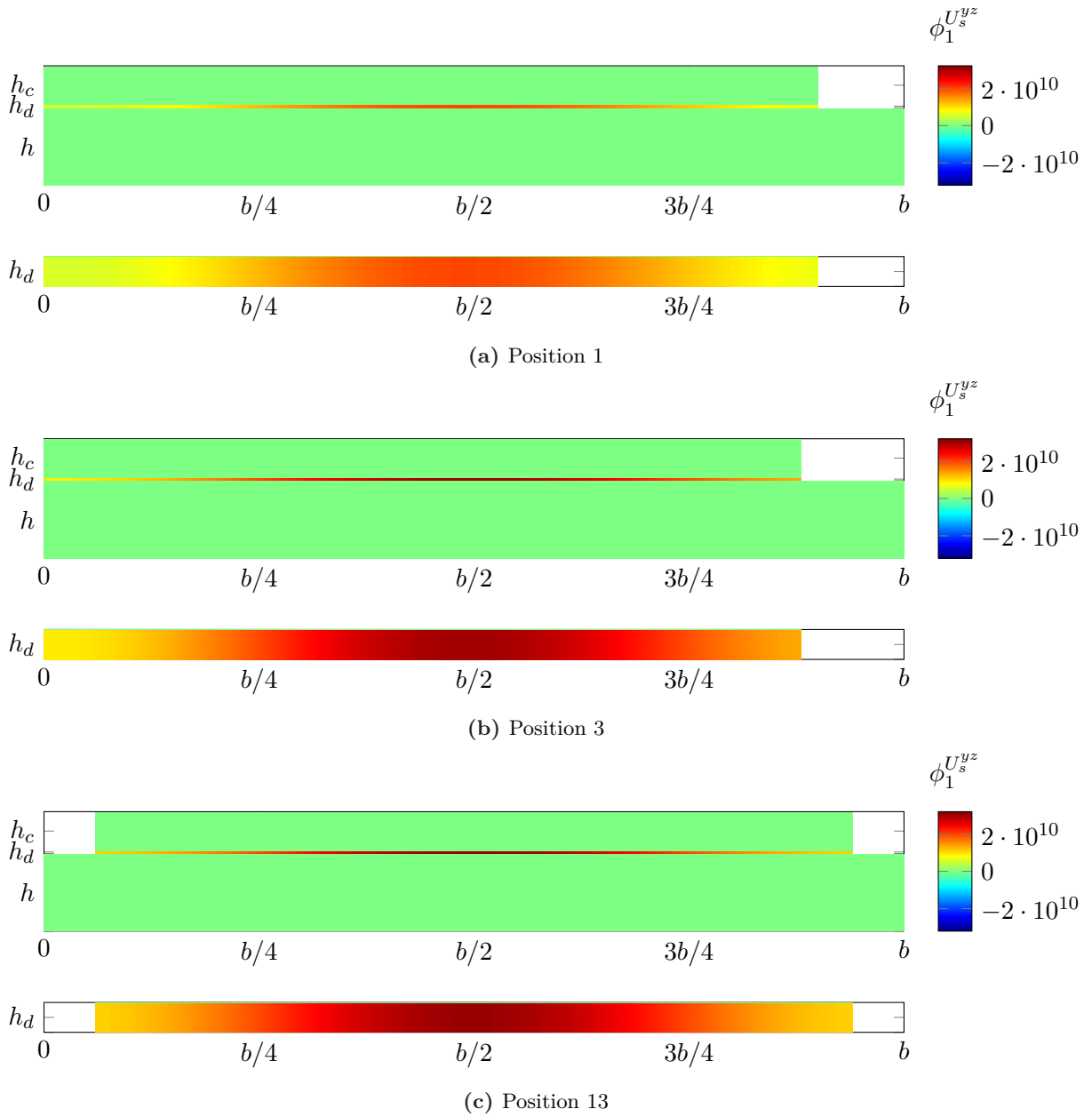


Figure 6.179 1st yz -shear strain energy density mode shapes - section cut at $a/12.5$

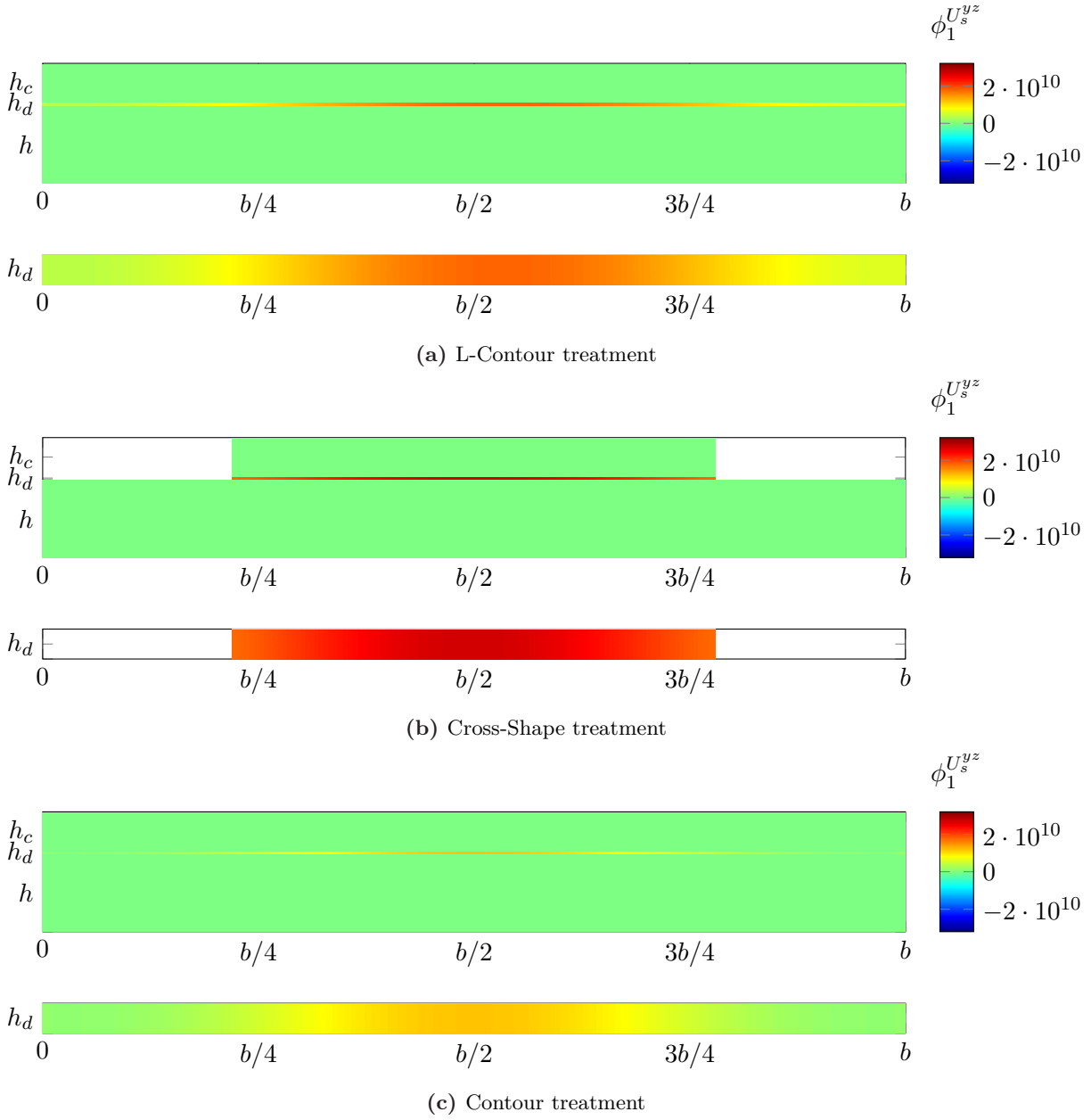


Figure 6.180 1st yz -shear strain energy density mode shapes - section cut at $a/12.5$

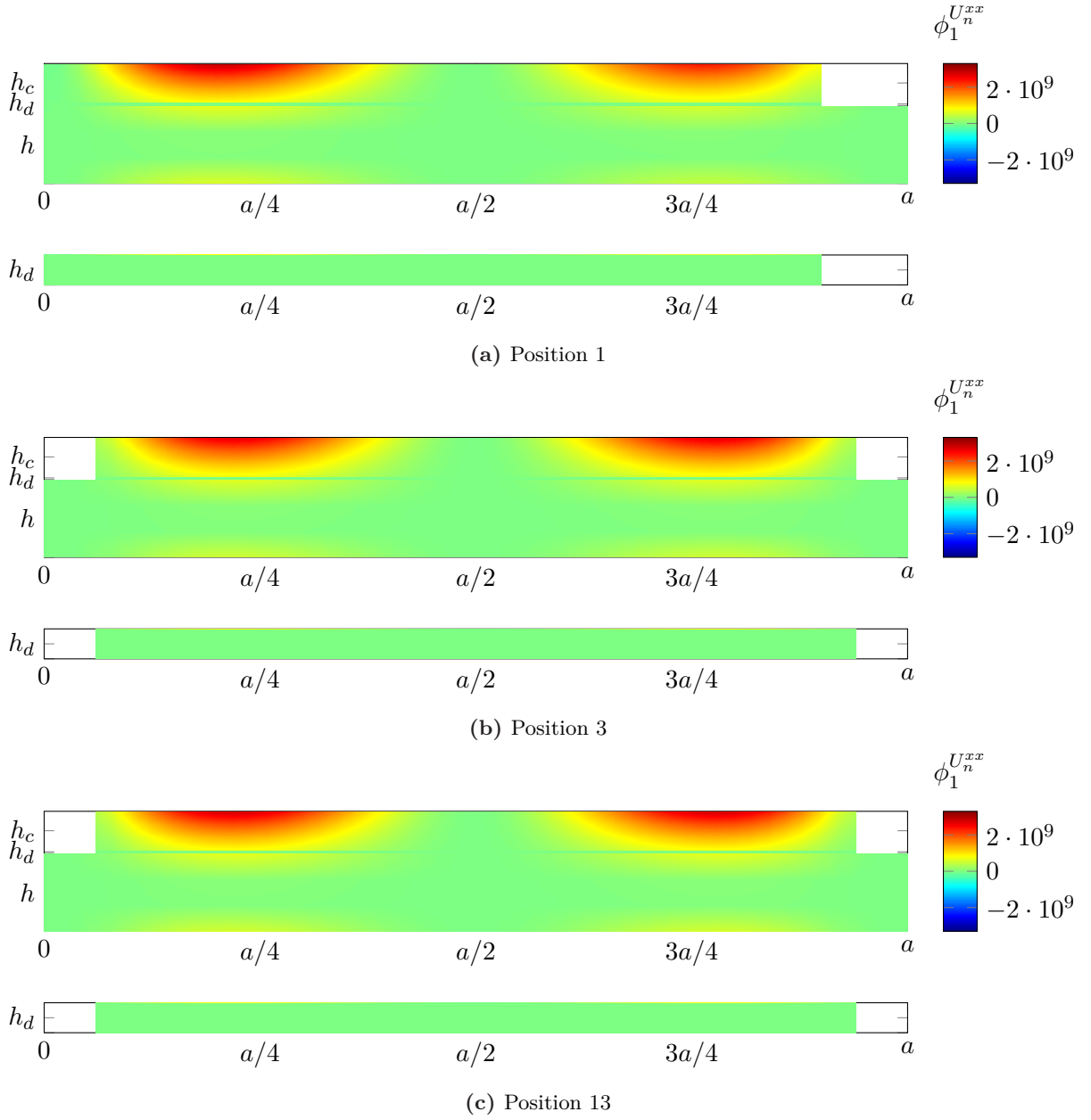


Figure 6.181 1st xx -membrane plus bending strain energy density mode shapes - section cut at $b/12.5$

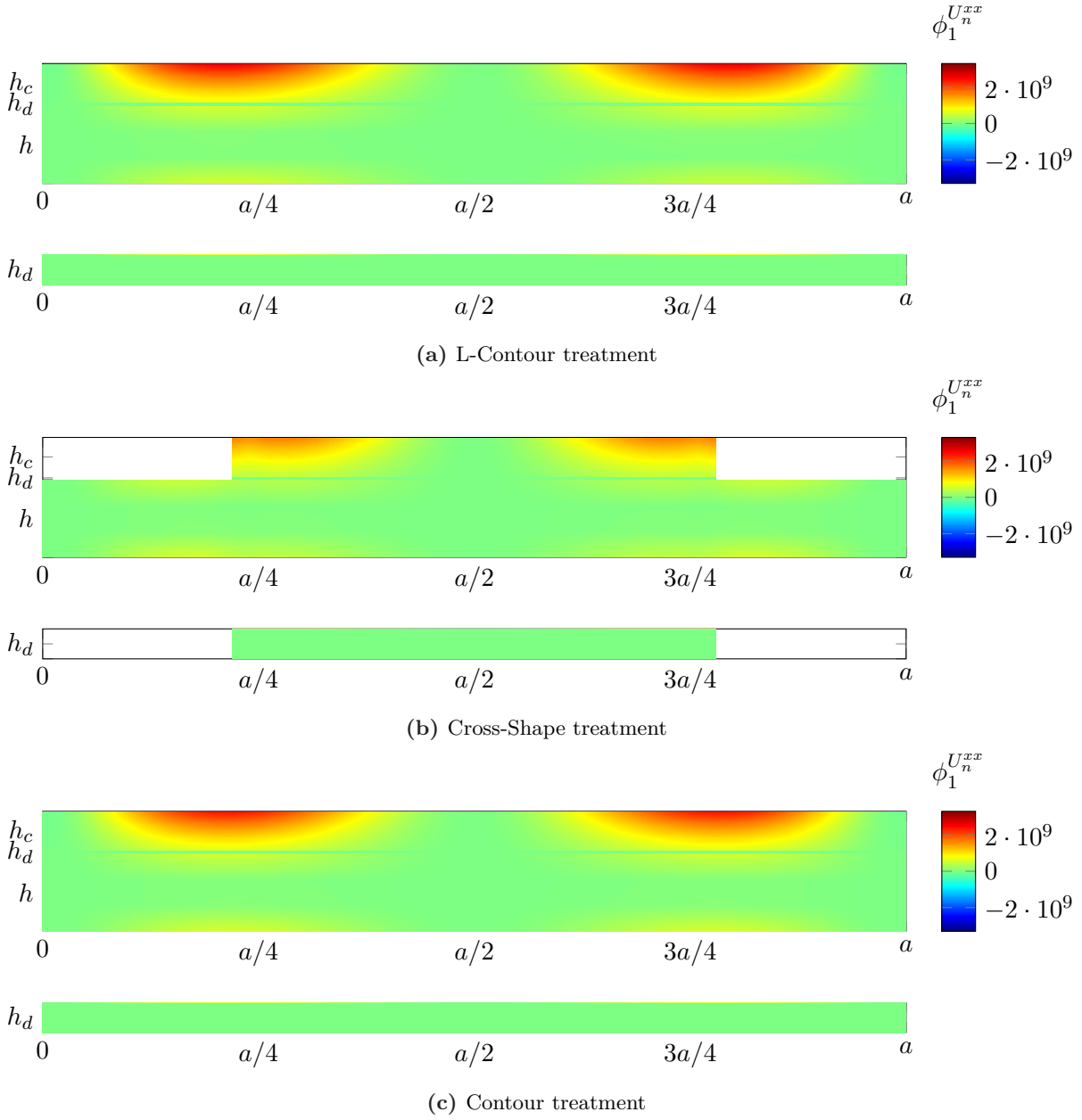


Figure 6.182 1st xx -membrane plus bending strain energy density mode shapes - section cut at $b/12.5$

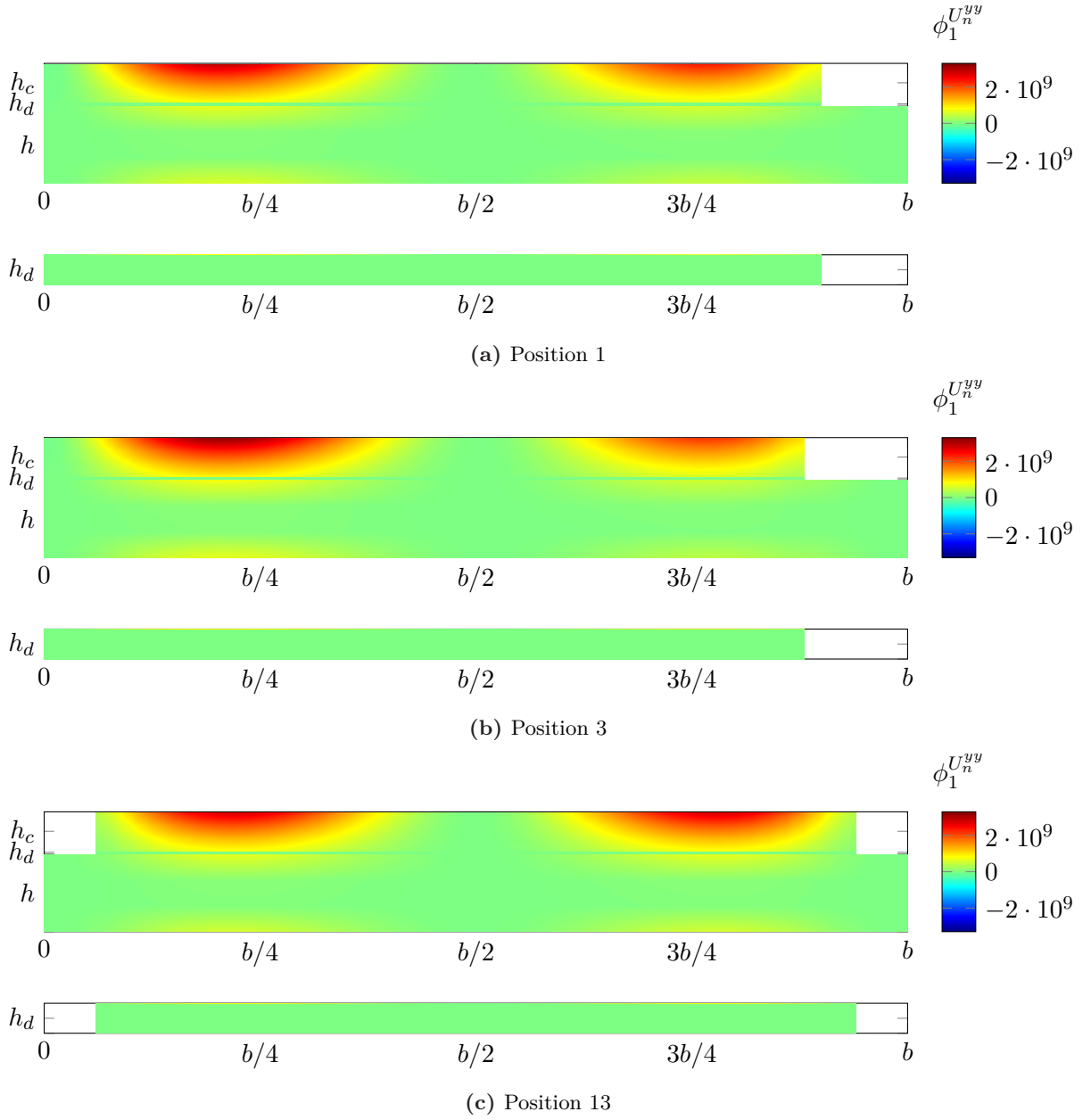


Figure 6.183 1st yy -membrane plus bending strain energy density mode shapes - section cut at $a/12.5$

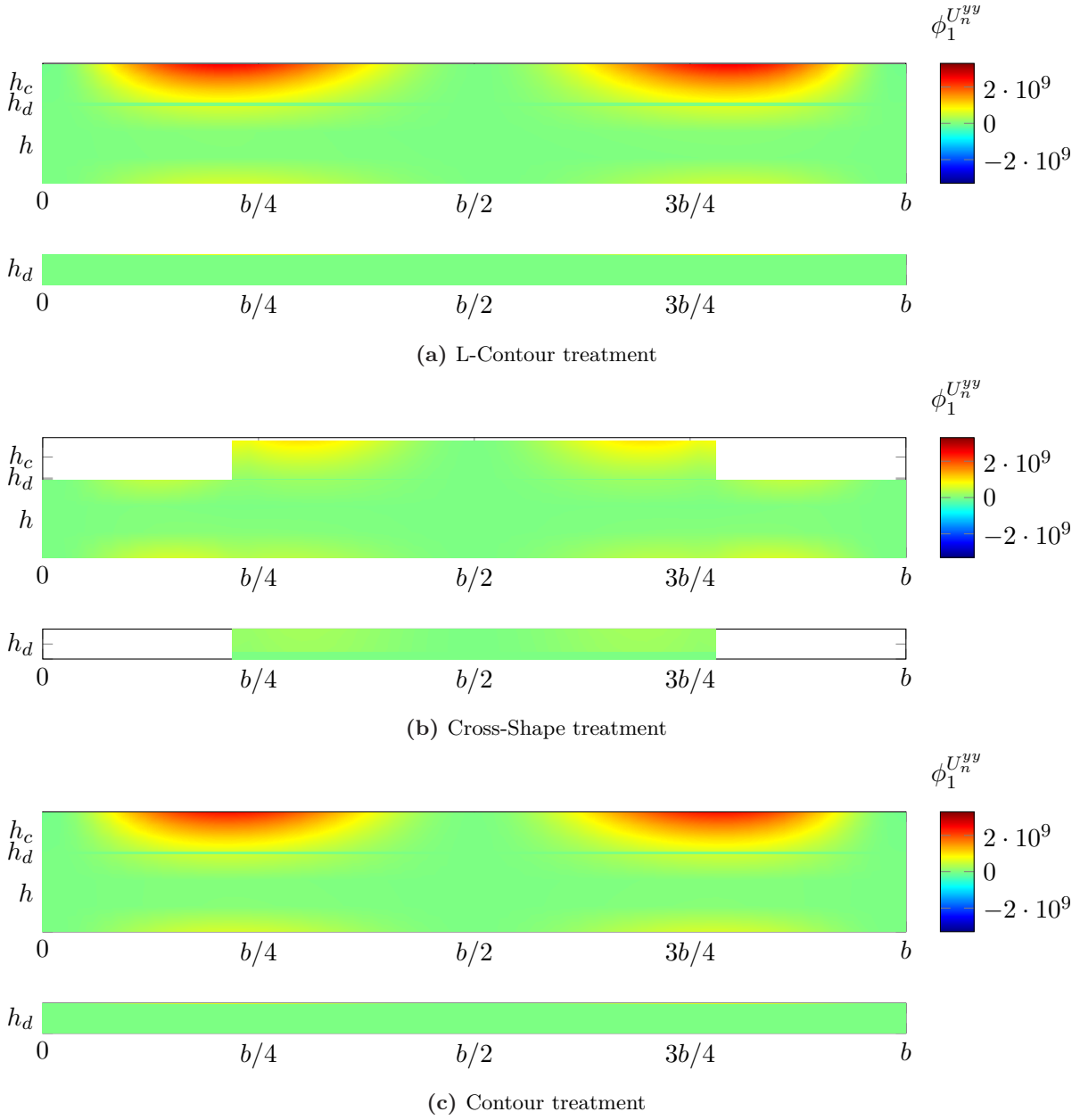
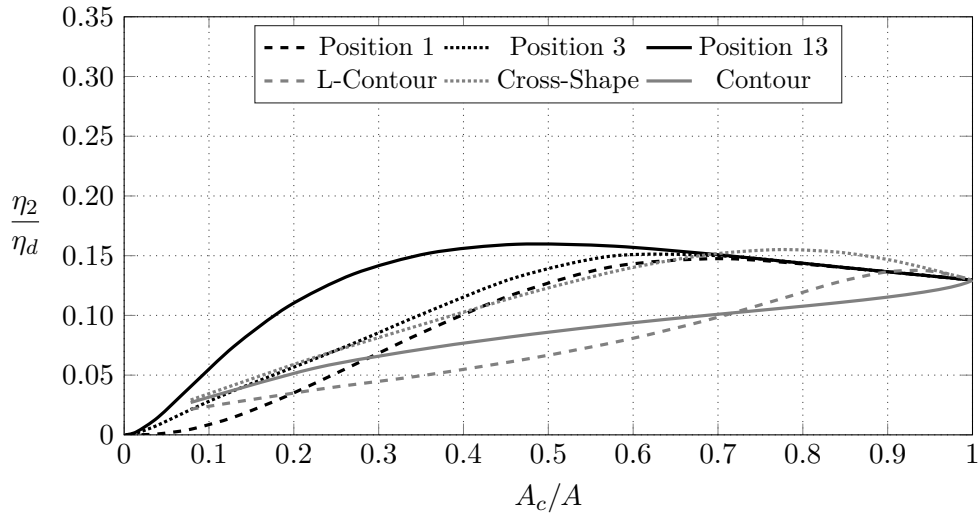
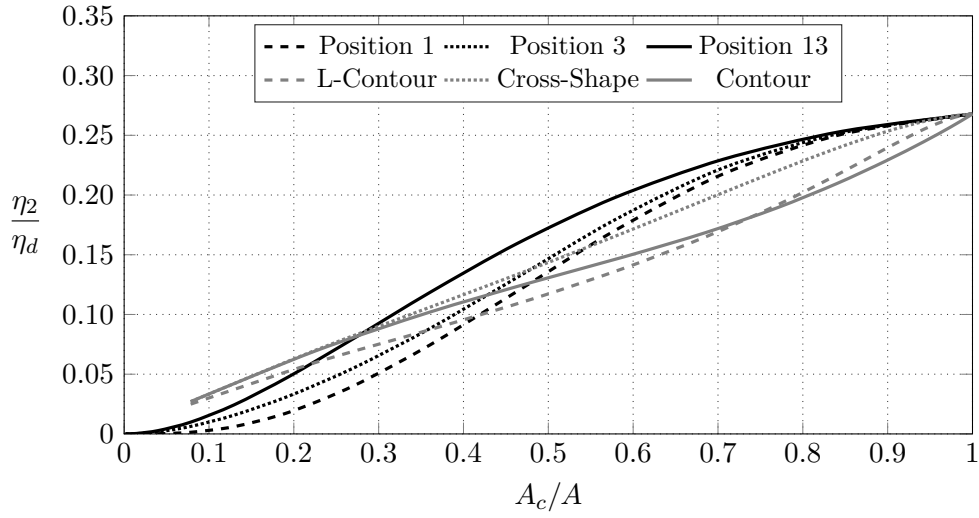
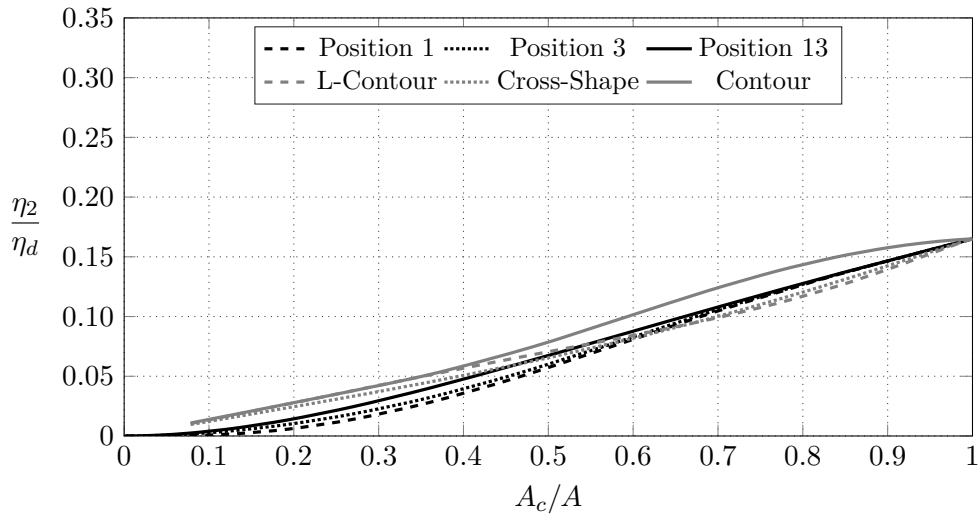
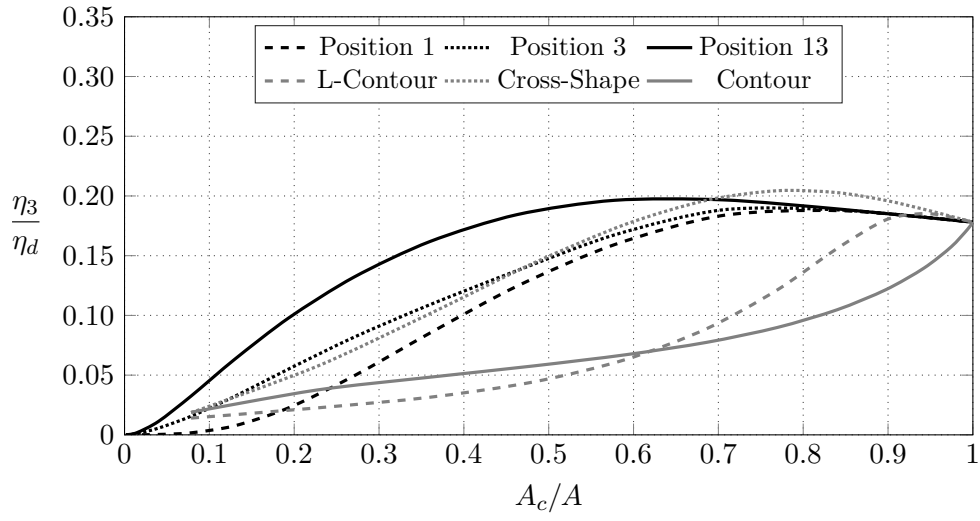
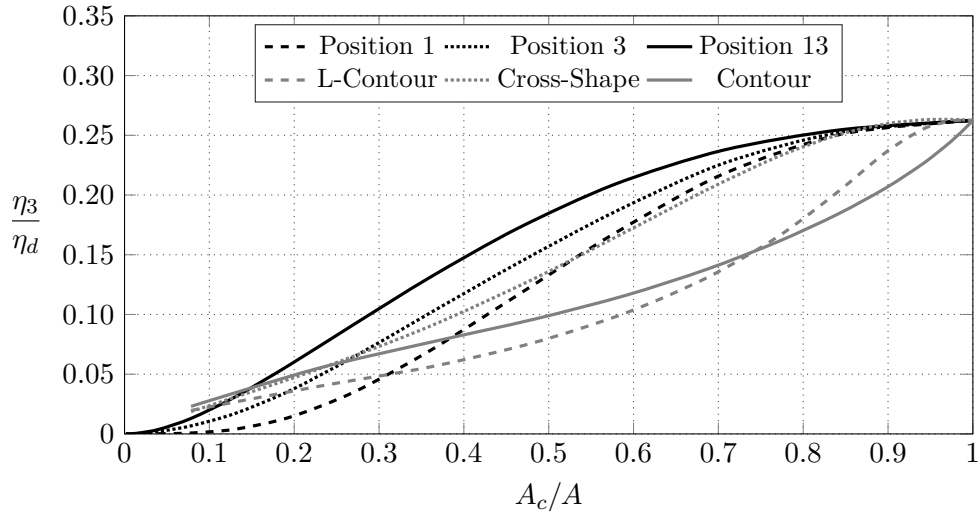
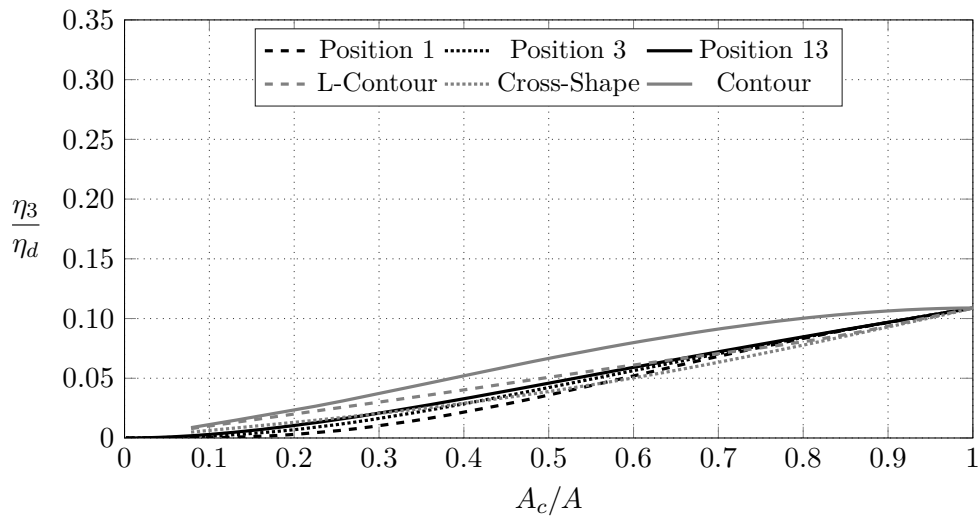


Figure 6.184 1st yy -membrane plus bending strain energy density mode shapes - section cut at $a/12.5$

(a) $h_d = 0.005h$ (b) $h_d = 0.04h$ (c) $h_d = 0.32h$ **Figure 6.185** 2nd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.02h$ (c) $h_d = 0.32h$ **Figure 6.186** 3rd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

6.8.2.2 | Thin Plate - $h = 0.005a$

The damping efficiency of partially covered structures depends on the position of the damping material and the amount of damping material itself, though it is not a linear dependence. However, the properties of the host structure influence the optimal design of partially covered structures. In this section, considering plates with free edges, one compares the optimal patch positions for improved damping performances. Figure 6.150 shows the comparison between different positions of the first specific modal loss factor for an SSSS thin plate - $h = 0.005a$. Considering the same specific damping layer thicknesses of the previous examples, one deals with the thinnest and thickest damping layers considered in this work - $h_d = 0.005h$ and $h_d = 0.32h$. A preliminary analysis indicates an ever-increasing growth of damping capacity as the thickness of the damping layer extends from $h_d = 0.005h$ to $h_d = 0.32h$. Certainly, the threshold value of damping layer thickness for thin base plates occurs for thicker layers than those considered here. Interestingly, the results for the first specific modal loss factor of a thin base plate are similar to those of the reference plate. Figure 6.187a proves it for thin damping layers. Despite the low damping capacity attainable with such thickness, the square patch at the central Position 13 and the Cross-Shape treatment provide the best damping capacities considering small covered areas. As reported for the reference plate, the Contour treatment shows conflicting results, with weak damping performances for small covered areas and the best ones as the covered area increases. On the other hand, Figure 6.187b details the damping efficiency curves for the thickest damping layer. The damping capacity increases remarkably, and the square patch at Position 13, together with the Cross-Shape treatment, presents the best damping capacity. Different from thin damping layers, contour treatments perform poorly for thick ones. As already discussed for the reference plate, it is challenging to explain these conflicting results of the best position for partial damping based on Figure 6.172 and the curvatures of the fully covered plate. It is noticeable that mode shapes of the base or fully covered plates might not be enough to design efficient patch treatments.

One also considers the damping efficiency at higher vibration modes for thin base plates. Figures 6.188a and 6.188b show the damping capacity for the thinnest and thickest damping layers considered in this work, respectively. The threshold value at the second vibration mode is the same as the first one, and consequently, damping capacity increases with thicker damping layers. In Figure 6.188a, it is noticeable that thin damping layers on thin base structures provide reduced damping capacities. However, thicker damping layers improve significantly energy dissipation - Figure 6.188b. Either for thin or thick damping layers, the square patch at Position 13 presents the best performance, followed by the Cross-Shape treatment, similar to the previous trend already noticed for the reference plate. Consistently, the improved performance of the square patch at Position 13 holds for the third specific modal loss factor - Figure 6.189. For this mode, one can identify the threshold value of damping layer thickness within the range considered throughout this work, and so one compares the efficiency curves considering three specific damping layer thicknesses - $h_d = 0.005h$, $h_d = 0.16h$ and $h_d = 0.32h$. Figure 6.189a highlights the inefficiency of thin damping layers on thin base structures despite the better capacity of the square patch at Position 13 and the Cross-Shape treatment over the contour treatments, which perform poorly. One improves the energy dissipation by increasing the damping layer thickness up to $h_d = 0.16h$, the threshold value of the increasing-decreasing behaviour at this mode, and the square patch at Position 13 still shows the best performance - Figure 6.189b. In Figure 6.189c, despite the decrease when one doubles the damping layer thickness, it is possible to notice no significant differences worth commenting on. The results for the second and third vibration modes are consistent with the mechanisms described in the previous studies. Figure 6.172 shows that maximum curvature for both directions at these modes is along each mid-side length of the plate, promoting the extensional deformation of the constraining layer

at such regions. Noticeably, the square patch at the central Position 13 and the Cross-Shape treatment benefit from such a feature.

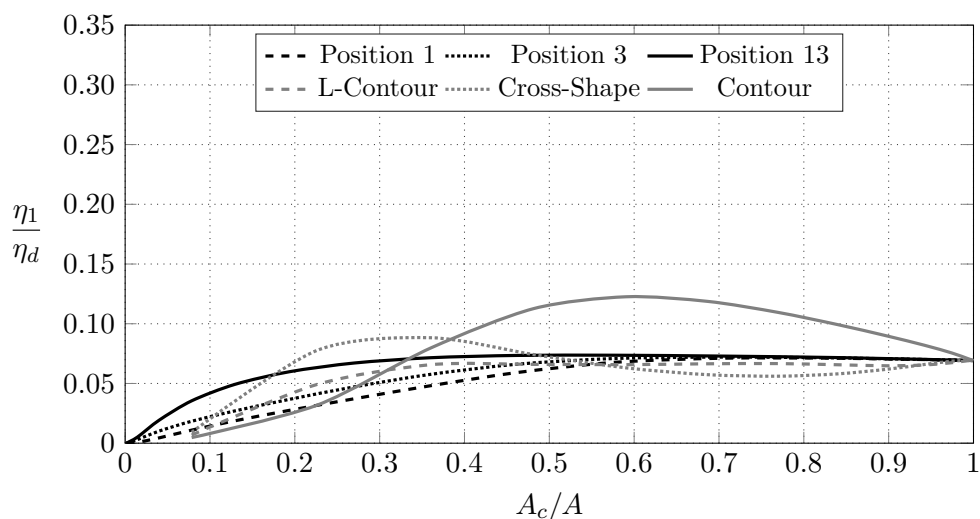
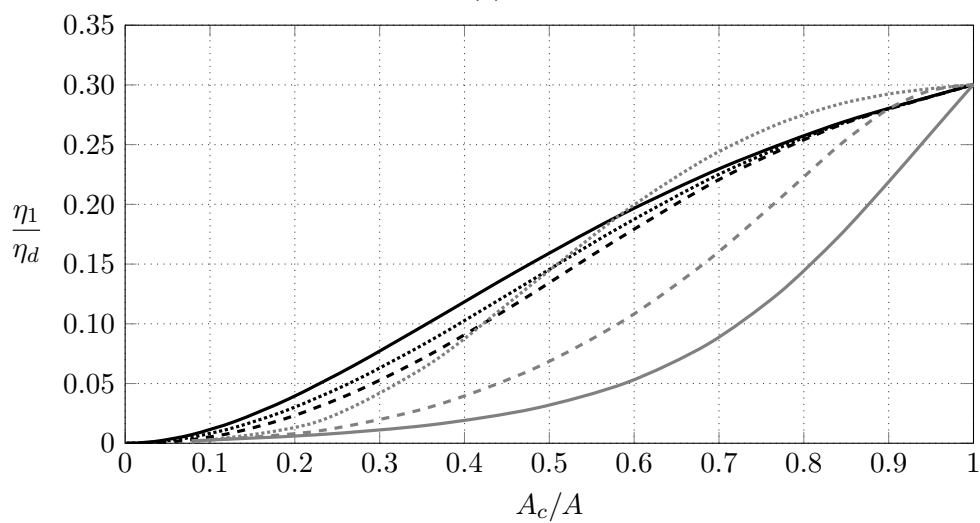
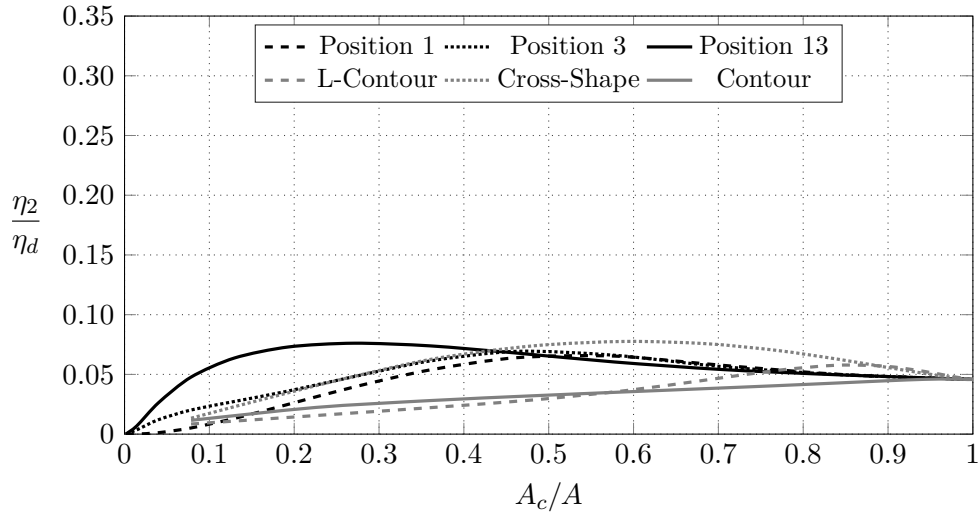
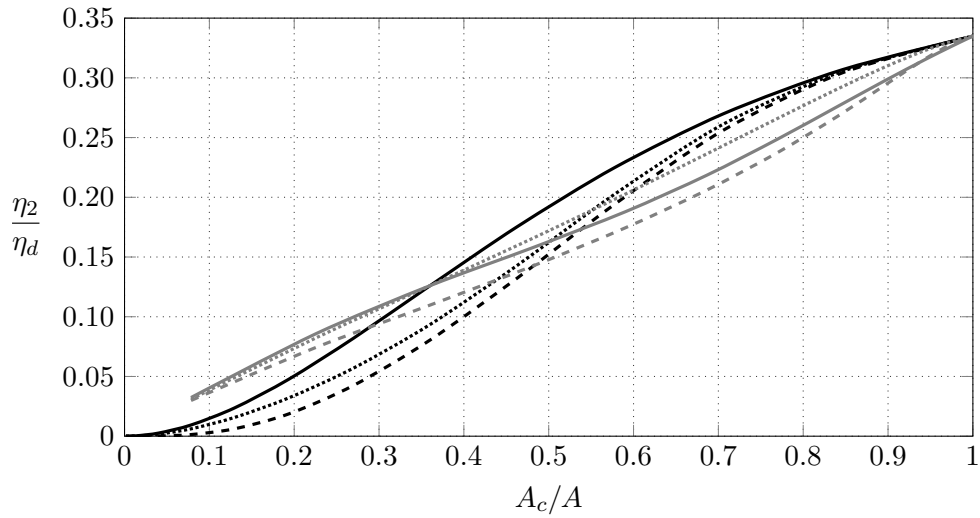

 (a) $h_d = 0.005h$

 (b) $h_d = 0.32h$

 Figure 6.187 1st modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

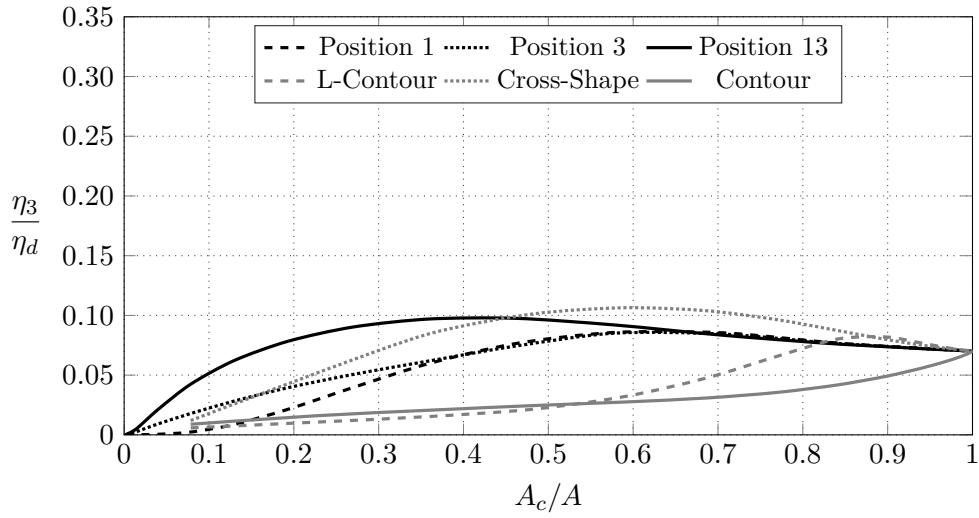
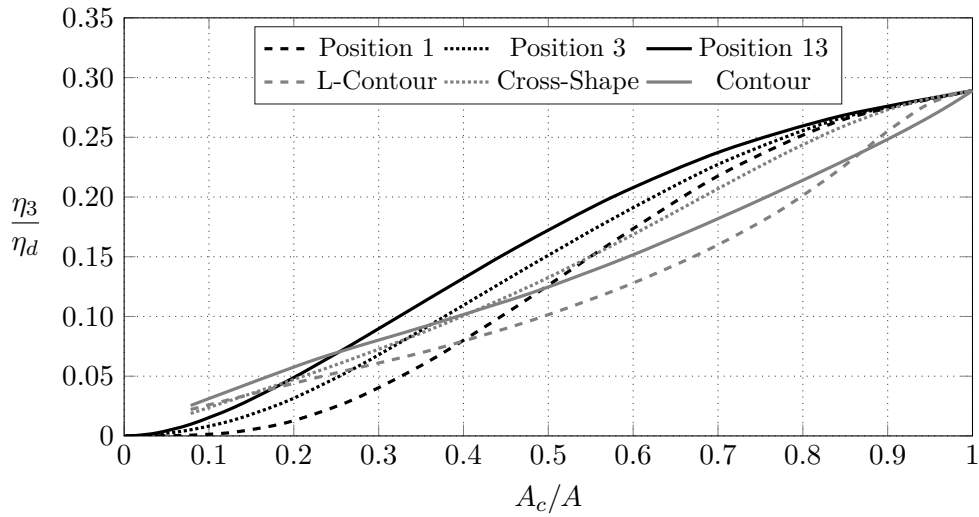
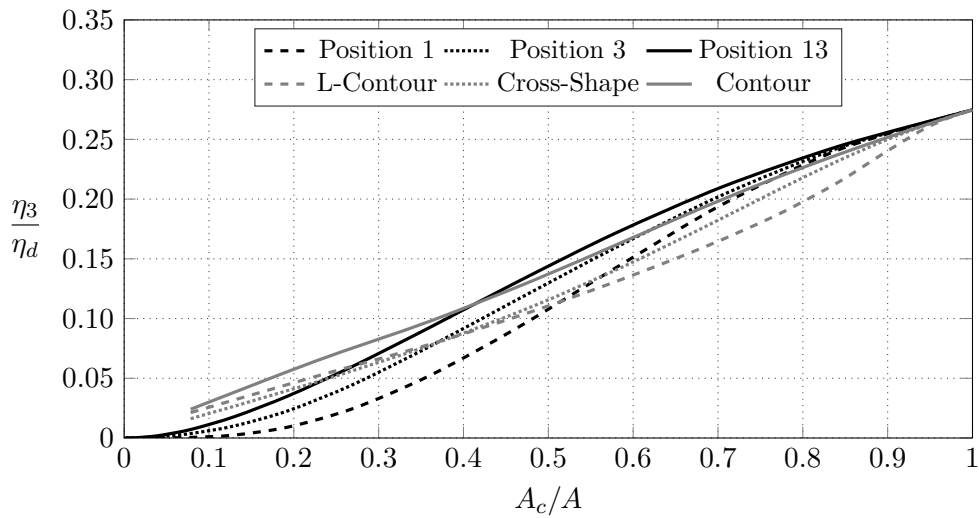


(a) $h_d = 0.005h$



(b) $h_d = 0.32h$

Figure 6.188 2nd modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.16h$ (c) $h_d = 0.32h$ **Figure 6.189** 3rd modal loss factor for different patch positions and covered areas, $h = 0.005a$, $h_c = 0.5h$

6.8.2.3 | Thick Plate - $h = 0.02a$

Lastly, one focuses on the damping efficiency of different partial damping treatments considering thicker base plates. Figure 6.190 shows the damping efficiency curves of the first specific modal loss factor comparing different patch positions. Repeatedly, one considers the thinnest and thickest damping layer thicknesses together with the threshold value for the increasing-decreasing behaviour. This threshold value occurs for thin damping layers, $h_d = 0.01h$ for the first and second vibration modes, demonstrating that thick base structures require thin damping layers for improved performances, contrary to thin base structures, which require thick damping layers. There are no significant differences between the thinnest damping layer and the threshold value of damping layer thickness in Figures 6.190a and 6.190b, respectively. The square patch at Position 13 presents an efficient damping capacity for a wide range of covered areas for thin damping layers, but one achieves the optimum damping capacity with the Cross-Shape treatment and a partially covered area. Similar to the reference base plate, one proves the usefulness of fully covered free plates for thicker base plates. Besides, and unsurprisingly, considering the reference and thin base plates, the contour treatments exhibit limited capacities for the first vibration mode. Figure 6.190c demonstrates the disadvantageous approach of increasing the damping layer thickness beyond the threshold value of the increasing-decreasing behaviour.

Figure 6.191 extends the detailed comparison between different patch positions to the second vibration mode. The threshold value for the increasing-decreasing behaviour at this mode occurs for the same damping layer thickness of the first one - $h_d = 0.01h$. Additionally, one also considers in this section the thinnest and thickest damping layer thicknesses under analysis in this work. The obtained results are clear - there are no significant differences between the thinnest and the threshold value of damping layer thickness in Figures 6.191a and 6.191b, respectively. The square patch at Position 13 provides the best damping efficiency at this vibration mode, noticeable for both thin damping layers - $h_d = 0.005h$ and $h_d = 0.01h$. Apart from the consistent results of the contour treatments disclosing limited damping capacities, there are no other worth mentioning outcomes for the best damping layer thicknesses. It is therefore not surprising that thicker damping layers decouple the base and constraining layers, decreasing the shear strain levels at the dissipative material and thus dropping the damping capacity as represented in Figure 6.190c. As one considers higher-order modes, the damping layer thickness threshold for optimum damping decreases. Figure 6.192 shows the parametric analysis for the third mode of vibration, considering the thinnest and thickest damping layers of the range under study in this work, and thicker damping layers inevitably decrease damping capacity. Reasonably, the threshold value of damping layer thickness for the third vibration mode occurs for even thinner damping layers. Nevertheless, Figure 6.192a depicts the excellent damping performance of the thinnest damping layer with $h_d = 0.005h$. Following the expected trend, the square patch at Position 13 keeps the status of the best damping patch position, with the contour treatments as the worst ones. The specific modal loss factors for the thickest damping layer in Figure 6.192b prove once again how disadvantageous thicker damping layers might be for the overall damping performance.

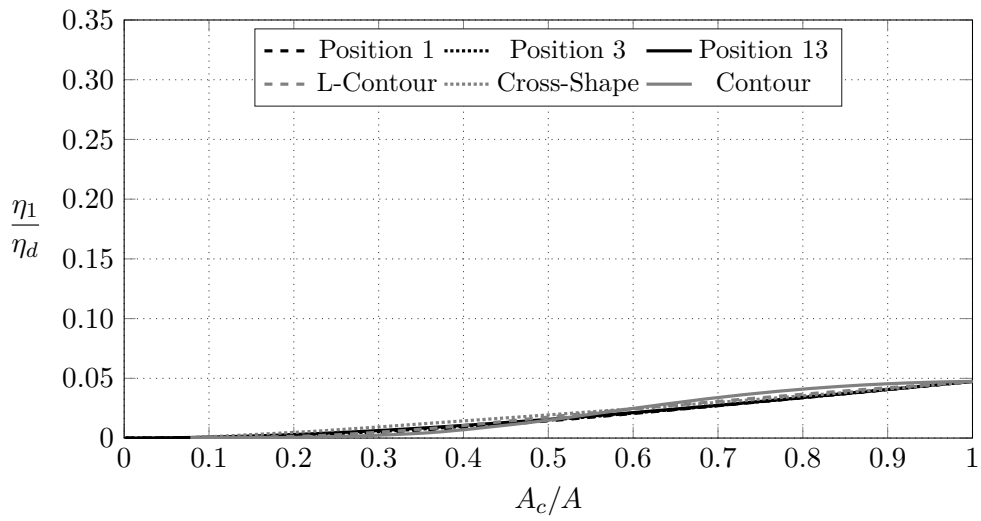
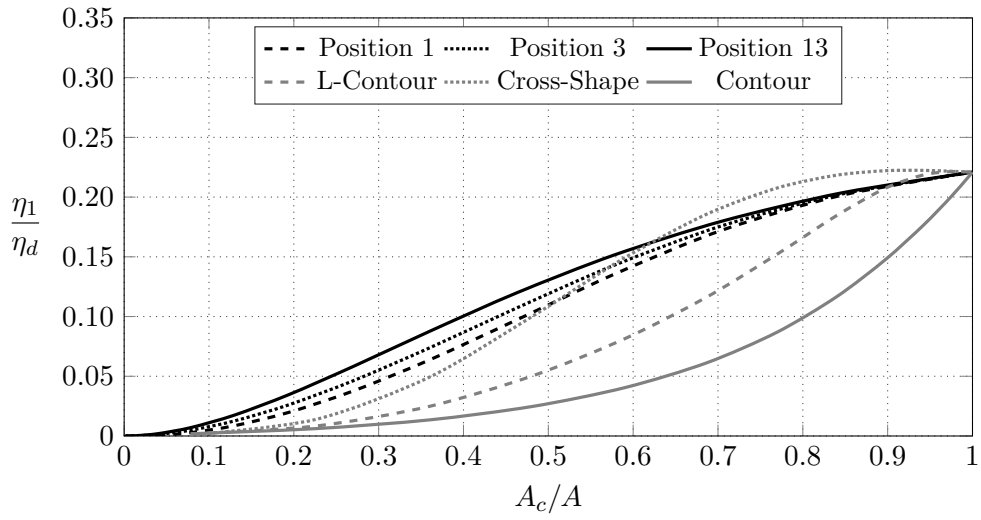
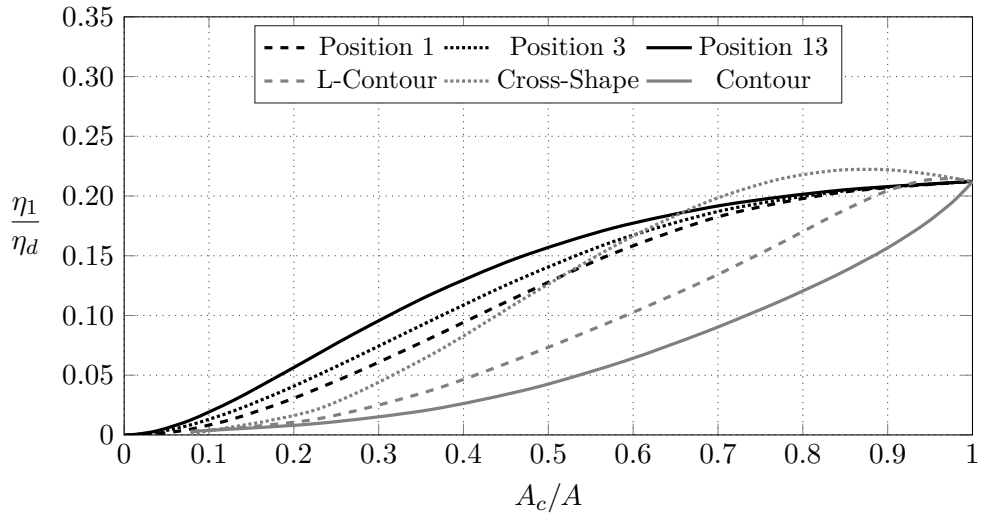
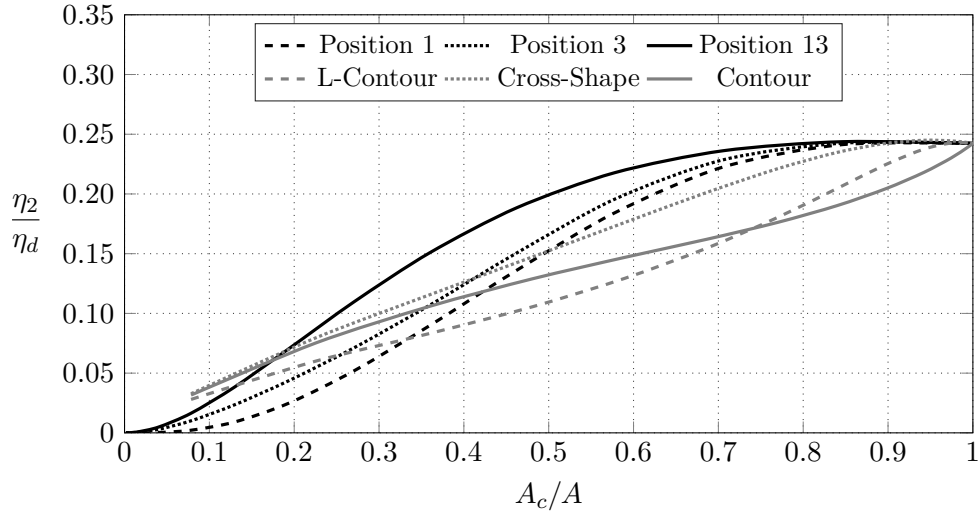
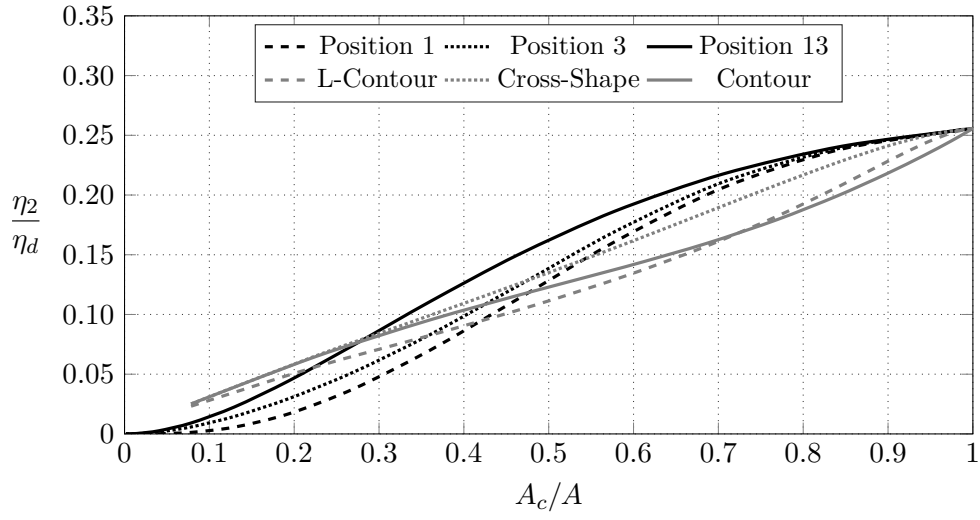
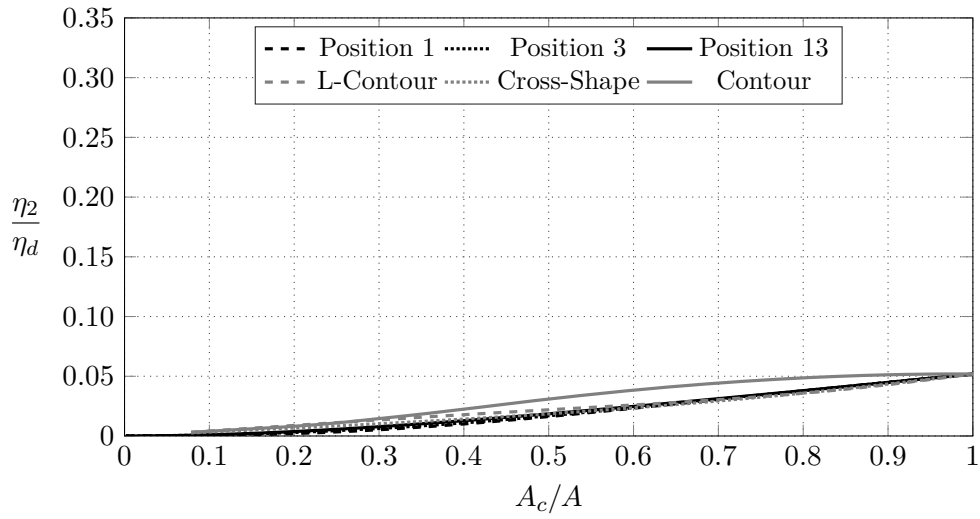
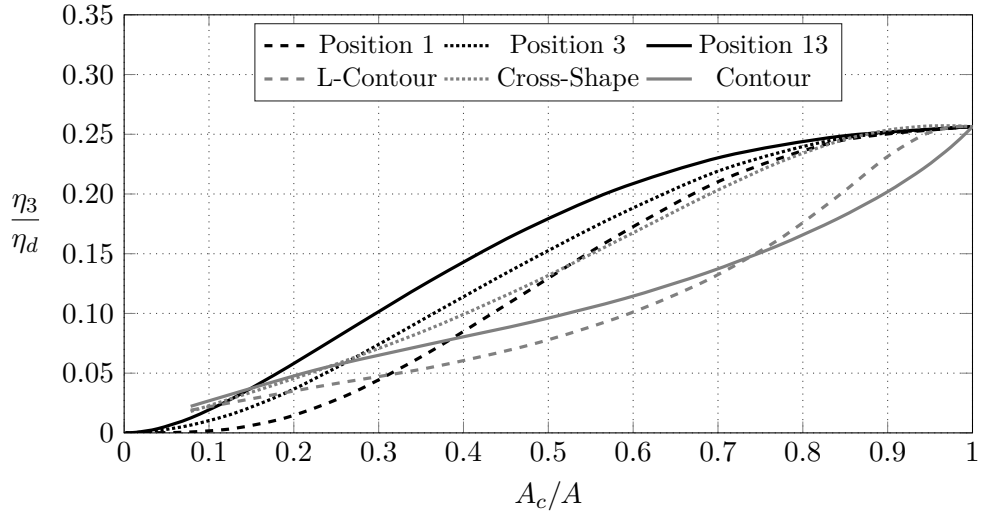
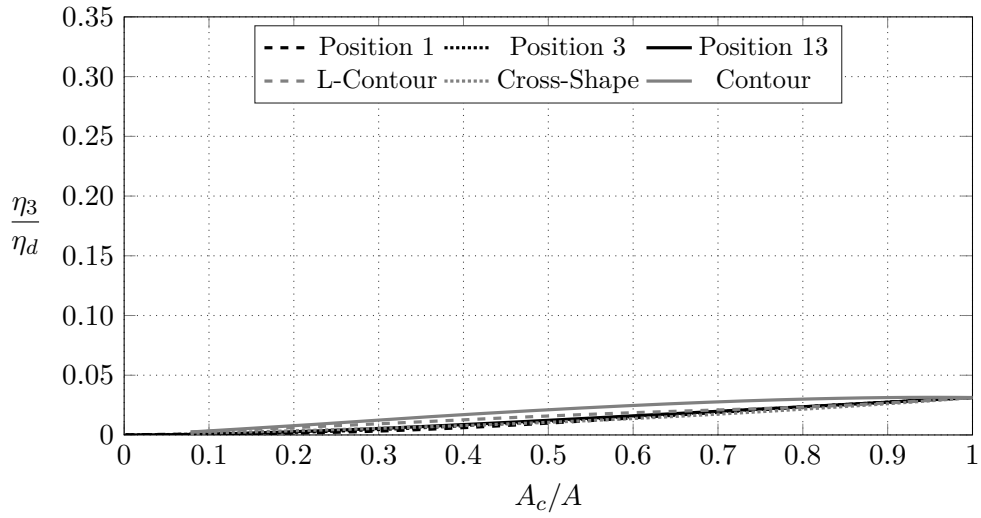


Figure 6.190 1st modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.01h$ (c) $h_d = 0.32h$ **Figure 6.191** 2nd modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

(a) $h_d = 0.005h$ (b) $h_d = 0.32h$ **Figure 6.192** 3rd modal loss factor for different patch positions and covered areas, $h = 0.02a$, $h_c = 0.5h$

6.8.3 | Influence of Material Properties

Following a similar analysis to those performed earlier in this work for other examples, one considers in this section the influence of the material properties of the base, damping and constraining layers on the damping efficiency of partial damping treatments applied to free plates. The material properties decisively influence the ability of the constraining layer to impose shear on the constrained damping material. To illustrate this influence, Figures 6.193 to 6.198 show the first three specific modal loss factors for different ratios of material properties considering the patch positions under analysis in this work. The modulus ratio, previously defined, represents the relation between Young's modulus of the base and constraining layers and the storage modulus of the damping material - E_c/E_d . One considers for this analysis a reference base plate with $h = 0.01a$ and a constant damping layer of $h_d = 0.04h$.

First, Figure 6.193 displays the influence of the covered area by a square patch at Position 1 on the damping capacity of the first three specific modal loss factors considering different modulus ratios. The growth rate of damping capacity with the covered area depends on the modulus ratio. Namely, when the materials of both base and constraining layers and the damping material have comparable stiffness, the modulus ratio is low, and the optimum damping capacity occurs for partially covered areas. As the modulus ratio increases and one considers softer damping materials or stiffer base and constraining layers, the respective optimum damping capacity increases but occurs for increasingly larger covered areas, which applies to the first three specific modal loss factors. However, the damping capacity does not increase indefinitely, and there is a maximum attainable damping capacity for any of the first three modes. For the first two vibration modes, the corresponding specific modal loss factors increase to the threshold value around 7×10^4 , then gradually decrease. The optimum damping capacity for these first two modes occurs with full coverage. When considering higher modes of vibration, such as the third one, for example, it is possible to realise that the corresponding specific modal loss factors increase up to the threshold value around 3×10^4 , proving that optimum damping efficiency for higher modes requires lower modulus ratios. These results prove the existence of an optimum modulus ratio promoting the perfect coupling between base and constraining layers, which leads to enhanced transverse shear strain at the damping material and improved damping performance.

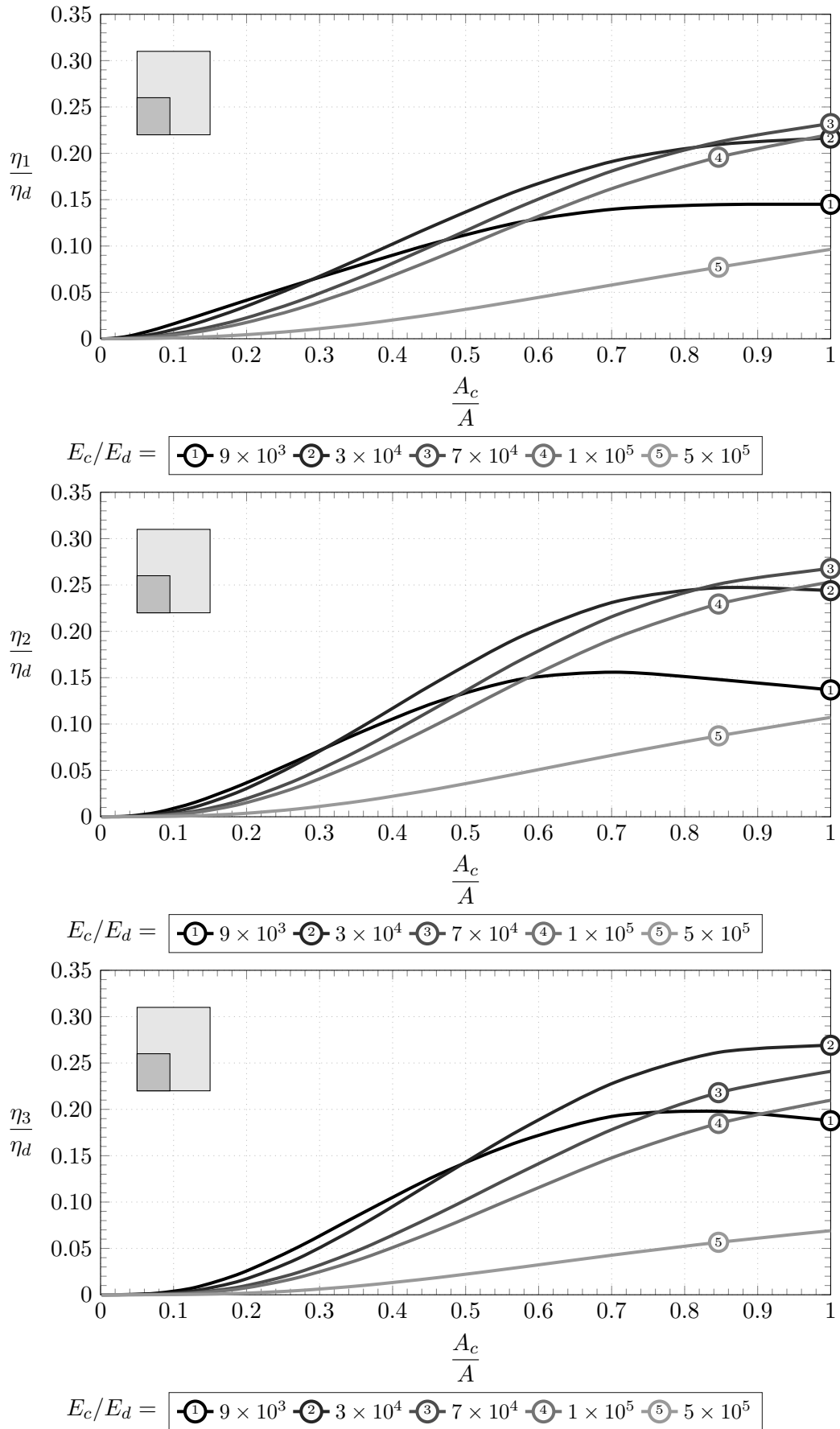
Subsequently, Figure 6.194 presents the same damping efficiency curves for a square patch at Position 3. There are no noticeable differences in the damping capacity of a square patch at Position 1; the threshold values of the modulus ratio for the increasing-decreasing behaviour are the same, and the optimum damping capacity occurs for partially covered areas when considering low modulus ratios. The most noteworthy difference is the better performance of the square patch at Position 3 for lower covered areas, consequently leading to optimum damping capacities with lower covered areas.

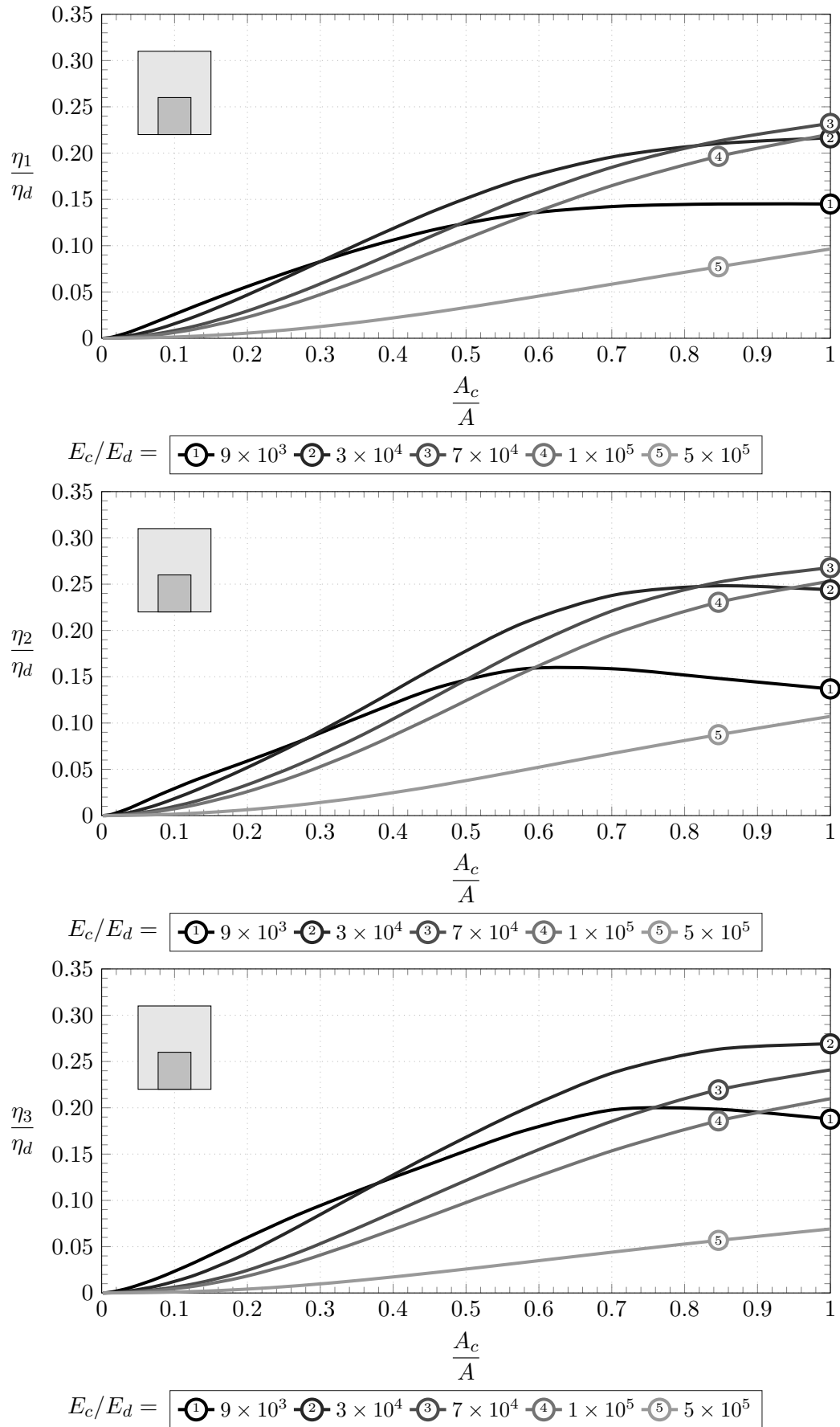
Previous results show that the square patch at the central Position 13 is crucial to this study. Figure 6.195 presents the damping efficiency curves as a function of the covered area and for different modulus ratios. Once again, the threshold values of the modulus ratio for the increasing-decreasing behaviour of damping capacity are the same, proving that similar results apply regardless of the damping treatment position. However, the enhanced performance of a patch at this position for small covered areas is indisputable. That is a remarkable feature of the square patch at Position 13, already discussed in previous analyses; the constraining layer at this position experiences higher extensional deformation, leading to the higher shear strain of the damping layer. Similarly to the previous positions, higher modulus ratios decrease the damping capacity at any vibration mode. The stiffness of both elastic and damping materials is significantly different, promoting the decoupling between the base and constraining layers and the consequent inability to impose shear on the damping material.

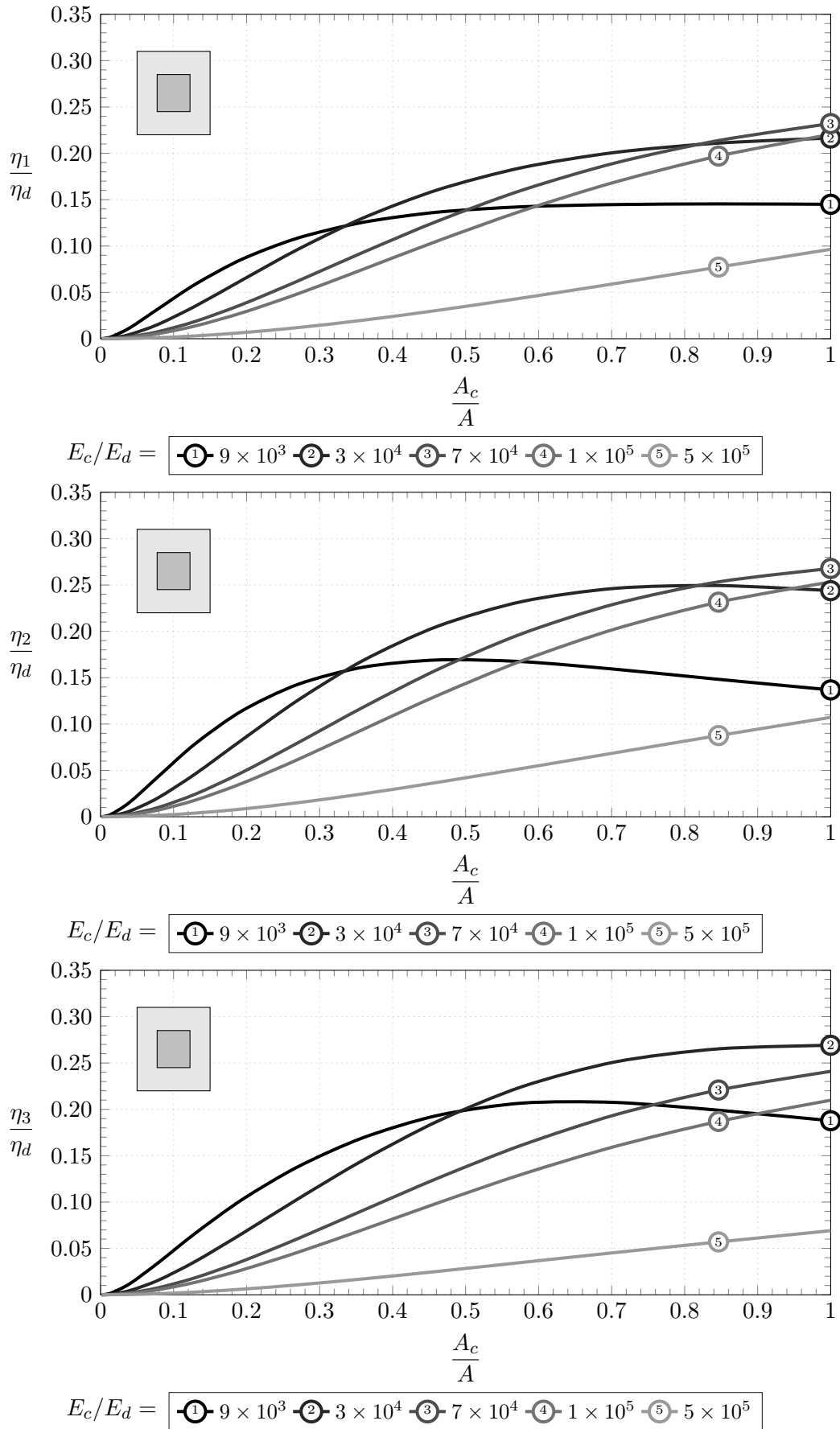
Apart from the square patches at different positions, one also considers non-square configurations such as the L-Contour treatment and its damping efficiency curves in Figure 6.196 to study its damping capacity. Previous studies considering the influence of the damping layer thickness reported weak damping performances for treatments covering the contour of the free plate preferentially. The results presented in Figure 6.196 highlight the same patterns of limited damping capacities for small covered areas, even though the threshold values of the increasing-decreasing behaviour of damping capacity and the damping capacity for fully covered plates, still holds.

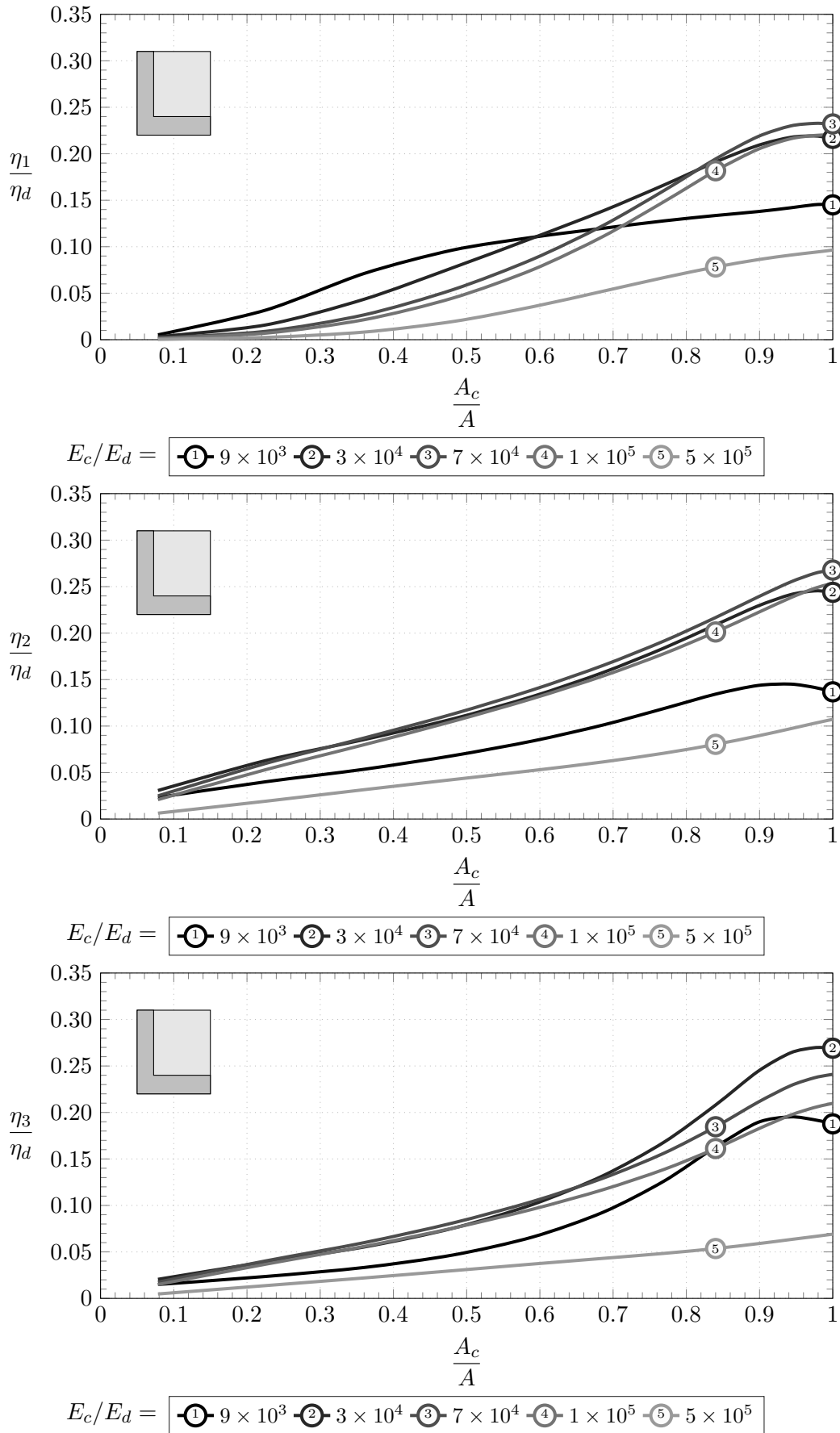
The non-square Cross-Shape treatment performs differently, as one can note from its damping efficiency curves represented in Figure 6.197. In particular, it is worth mentioning a clear improvement in damping capacity for small covered areas compared with the L-Contour treatment. Unsurprisingly, the threshold values of the increasing-decreasing behaviour of damping capacity keep the same magnitude.

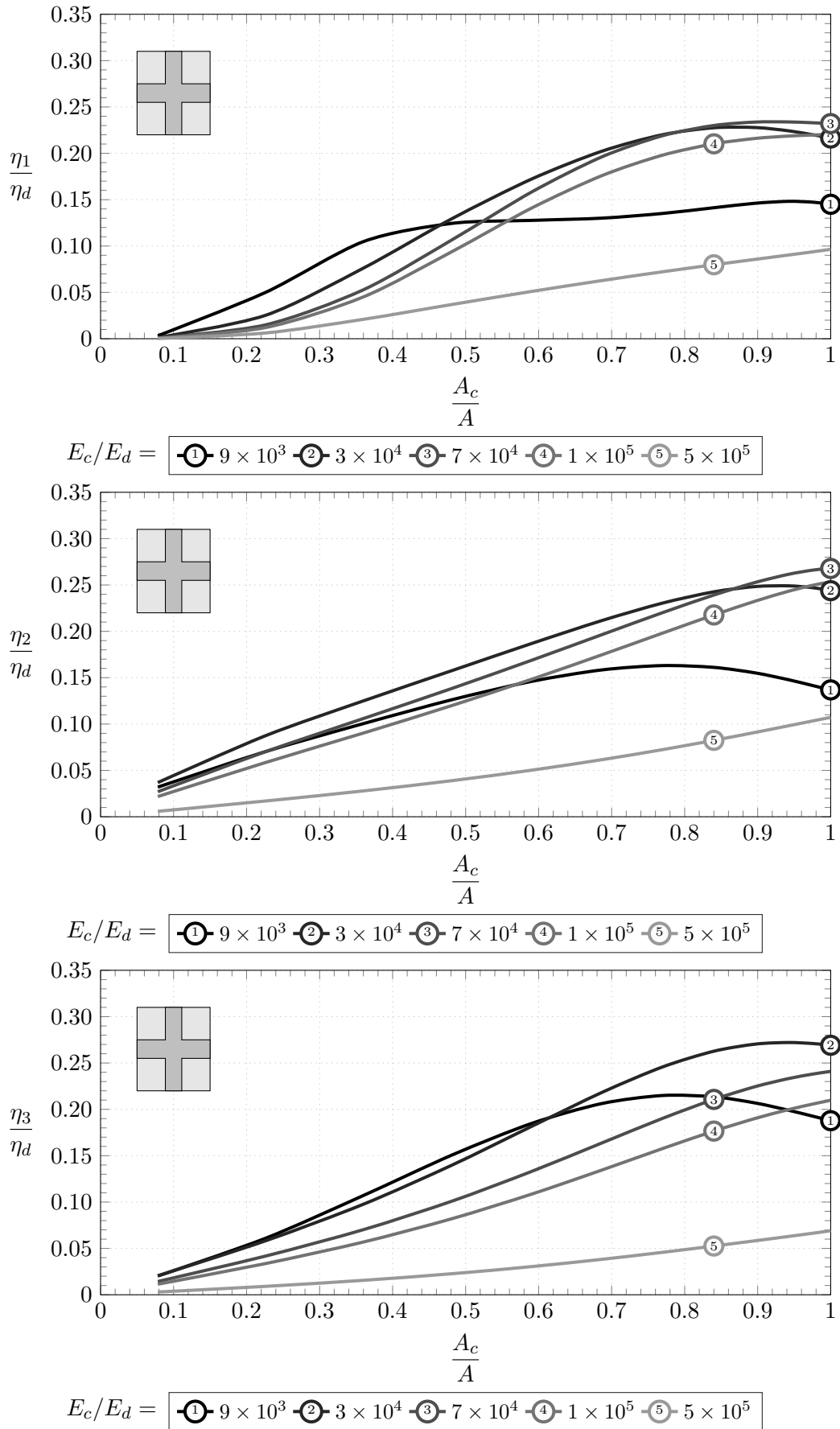
Figure 6.198 presents the damping efficiency curves of the Contour treatment for different modulus ratios. The results follow the expected pattern for this damping treatment, including the weakest damping capacity for small covered areas. The damping capacities of the first three vibration modes only become substantial as the covered area increases and tends to the centre of the plate. Besides, the threshold values for the increasing-decreasing effect of damping capacity are the same.

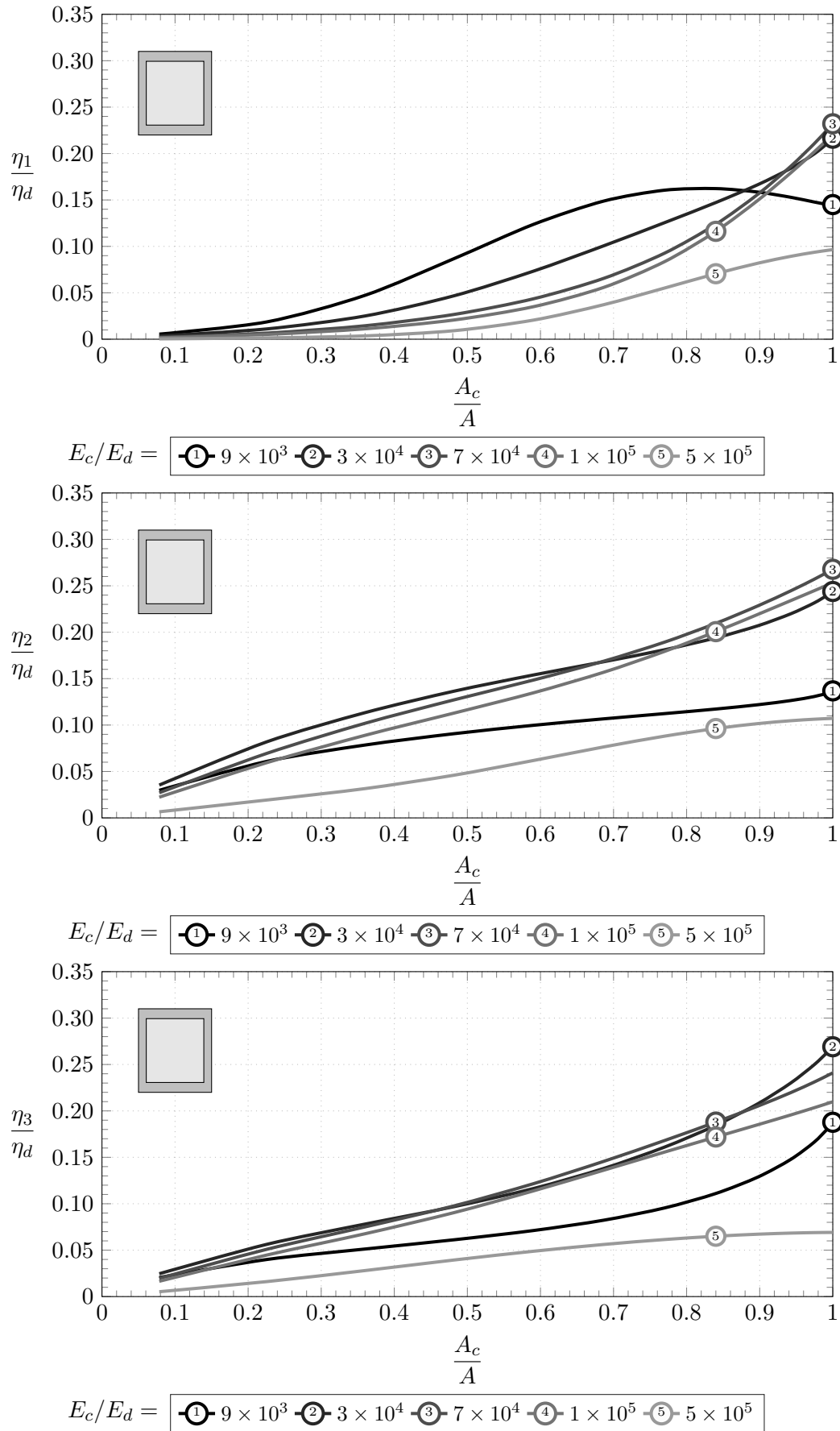
Figure 6.193 Damping efficiency curves for Position 1, $h = 0.01a$, $h_c = 0.5h$

Figure 6.194 Damping efficiency curves for Position 3, $h = 0.01a$, $h_c = 0.5h$

Figure 6.195 Damping efficiency curves for Position 13, $h = 0.01a$, $h_c = 0.5h$

Figure 6.196 Damping efficiency curves for L-Contour treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.197 Damping efficiency curves for Cross-Shape treatment, $h = 0.01a$, $h_c = 0.5h$

Figure 6.198 Damping efficiency curves for Contour treatment, $h = 0.01a$, $h_c = 0.5h$

6.8.3.1 | Influence of Patch Positioning

In the following, one performs a detailed comparative analysis regarding the damping efficiency of the different patch positions and considers distinct material properties. The previous damping efficiency curves for different positions do not enable a comprehensive comparison between different configurations and thus identify the best positions for improved damping performances. The goal is to evaluate the fundamental mechanisms responsible for such increased damping efficiency to identify design guidelines for complex structures. With this in mind, it is important to remember the transverse displacement mode shapes of the first three normal modes of the free square reference plate considering a constrained layer damping treatment with full coverage represented in Figure 6.171 - $h = 0.01a$, $h_d = 0.04h$ and $h_c = 0.5h$. Moreover, it is crucial to consider the curvature mode shapes presented in Figure 6.172 to assess the bending pattern of the plate and the preferential positions for constrained damping treatments, as well as the mode shapes associated with the shear strain degrees of freedom γ_{xz} and γ_{yz} in Figure 6.173.

For this detailed comparative analysis, one exploits the damping efficiency curves of the previous section regarding the influence of the modulus ratio in each particular position and adapt them for a comprehensive understanding of the best positions for improved damping performances. Figure 6.199 details the first specific modal loss factor considering different patch positions and covered areas simultaneously. In addition, one considers for this analysis of the damping efficiency of the different patch positions three modulus ratios - $E_c/E_d = 9 \times 10^3$, $E_c/E_d = 7 \times 10^4$ and $E_c/E_d = 5 \times 10^5$. One considers the lowest and highest modulus ratio with the threshold value of the modulus ratio for the increasing-decreasing behaviour of damping capacity at the first vibration mode.

Figure 6.199a opens this analysis of the damping efficiency curves for the first specific modal loss factor, considering the lowest modulus ratio under investigation - $E_c/E_d = 9 \times 10^3$. The constraining and damping layers have comparable stiffness for low modulus ratios, but there is a meaningful damping efficiency. Undoubtedly, the square patch at Position 13 presents the best damping capacity for a wide range of covered areas, peaking for nearly half of the covered area. Nevertheless, the Contour treatment provides the worst damping performance for small covered areas and exceeds all other treatments as the covered area increases. These results fit perfectly with the former detailed comparison for the reference plate between different positions for the thinnest damping layer, inclusively the other patch treatments, with the Cross-Shape treatment and the square patch at Position 3 introducing better damping capacity than the square patch at Position 1 and the L-Contour treatment. Low modulus ratios and thinner damping layers develop similar damping efficiency outcomes. That is not an unexpected result, considering that both cases tend to couple the base and constraining layers more effectively.

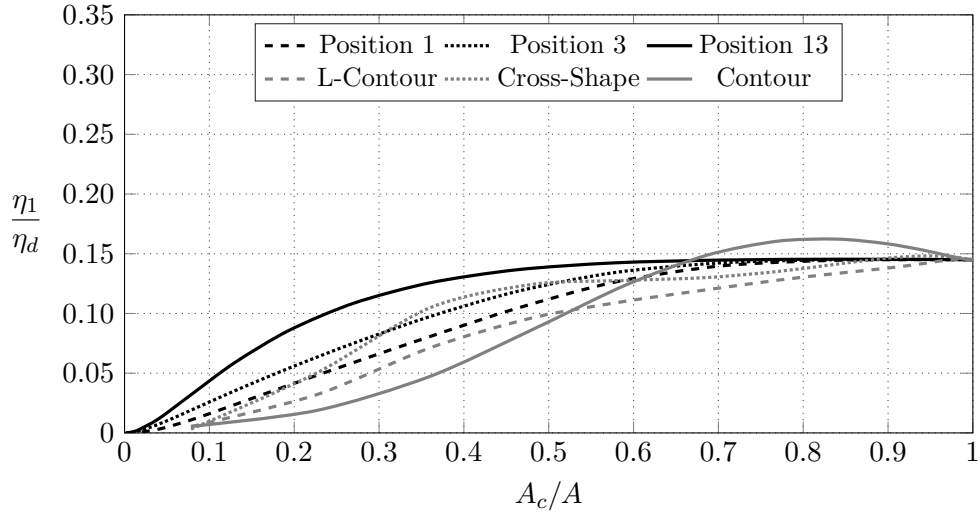
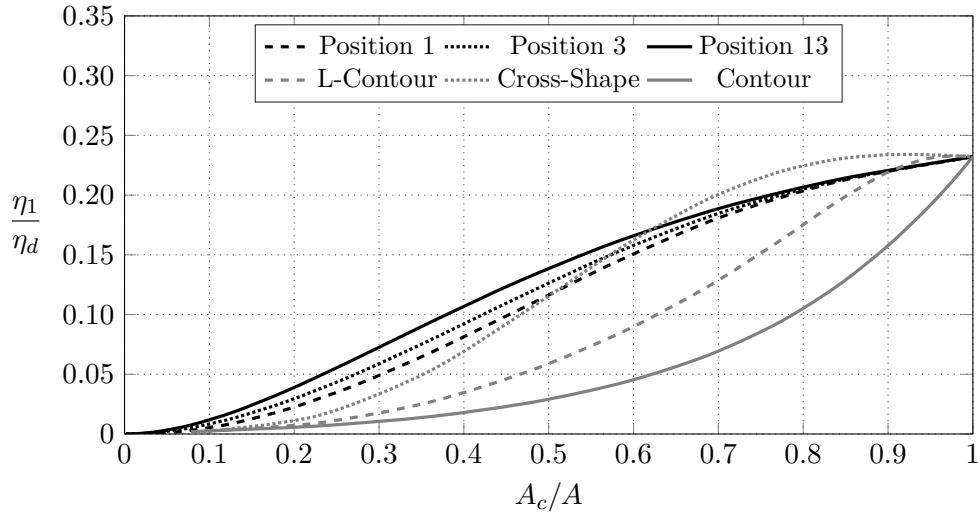
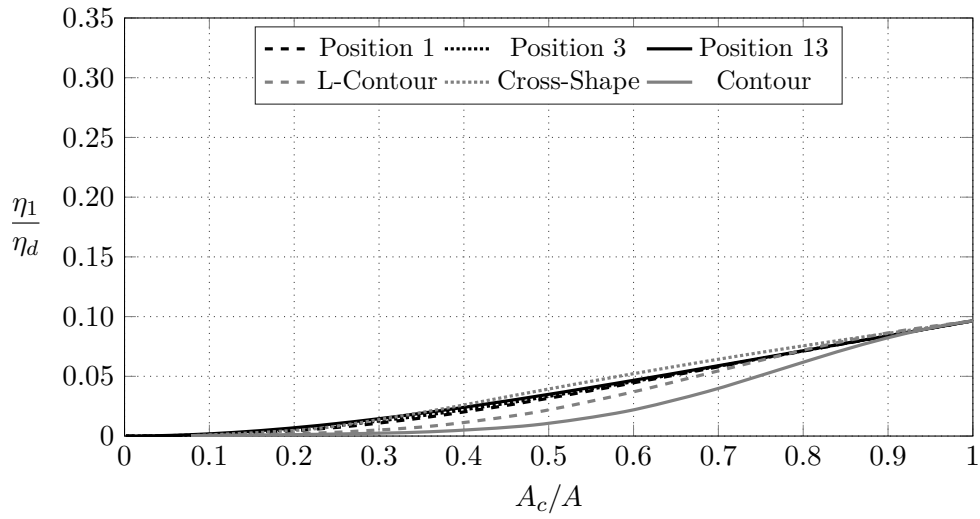
Subsequently, Figure 6.199b shows the damping efficiency curves of different positions of damping treatments for the first mode of vibration and considering the threshold value for the increasing-decreasing behaviour of damping capacity - $E_c/E_d = 7 \times 10^4$. With such geometric and mechanical properties, these damping efficiency curves are those of Figure 6.174b, where one studied the influence of the patch positioning considering different damping layer thicknesses and a modulus ratio of $E_c/E_d = 7 \times 10^4$. Here, one emphasises the reference positions of the square patch at Position 13, with the best damping capacity for a wide range of covered areas and the Contour treatment with the worst damping efficiency considering any covered area. Nevertheless, the Cross-Shape treatment proves the possible uselessness of fully covered plates for specific configurations since one achieves the optimum damping capacity at this mode of vibration with this treatment and a partially covered area.

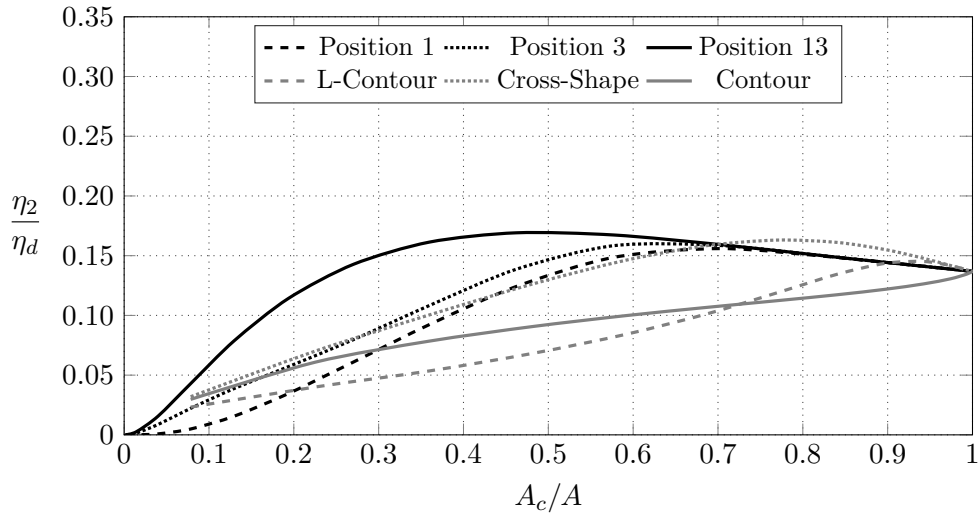
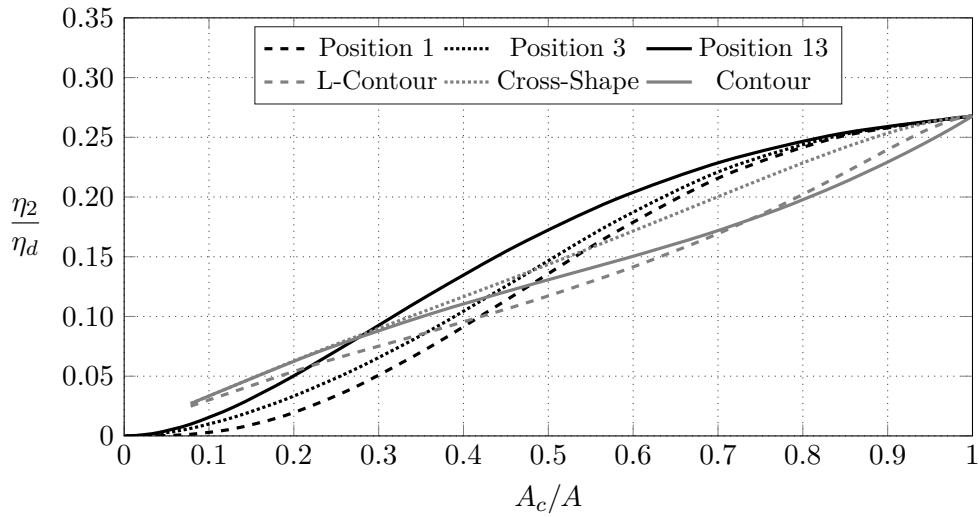
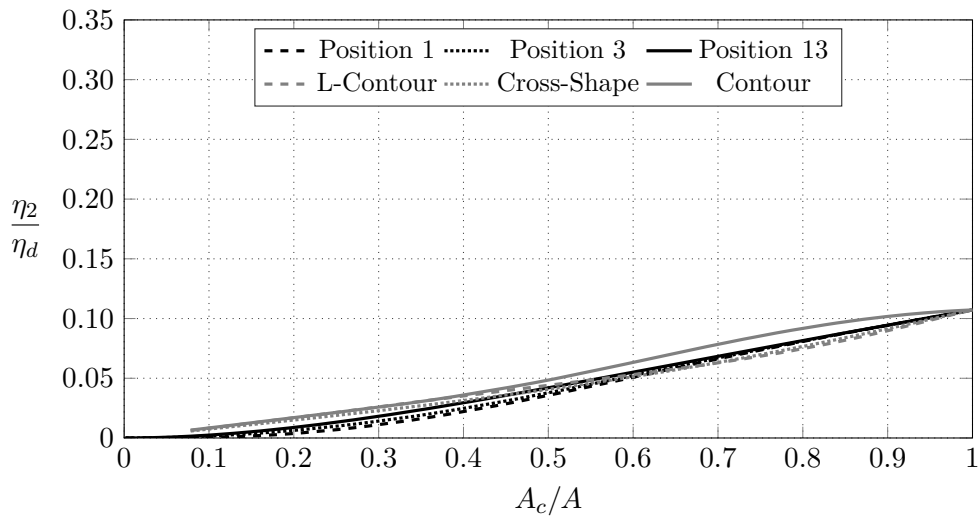
Below, Figure 6.199c proves undoubtedly the drop in damping efficiency when the modulus ratio exceeds the threshold value of the increasing-decreasing behaviour. For a higher modulus ratio, $E_c/E_d = 5 \times 10^5$, there is a remarkable distinction between the stiffness of the damping and

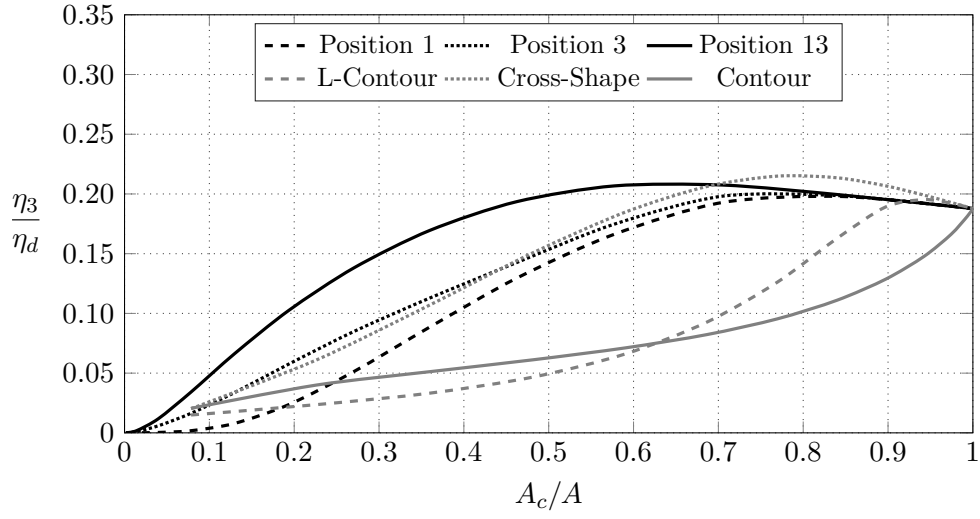
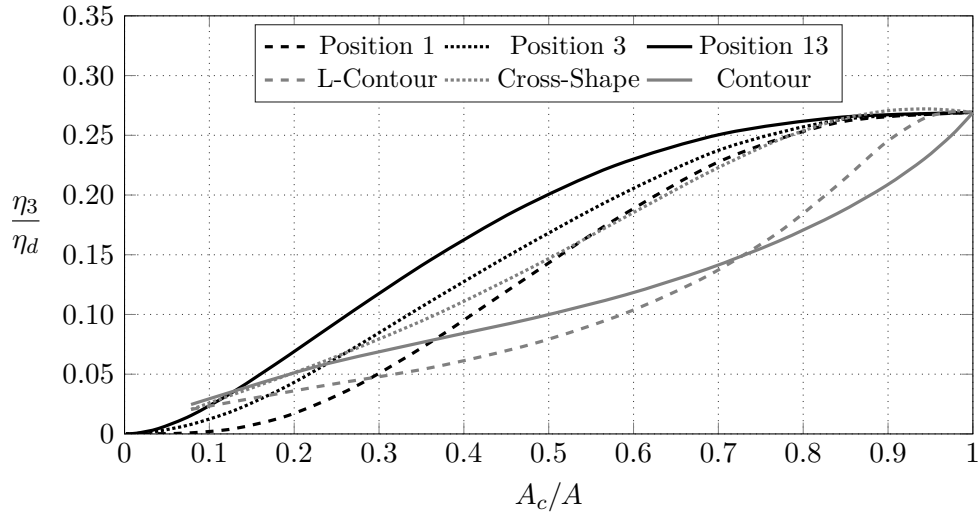
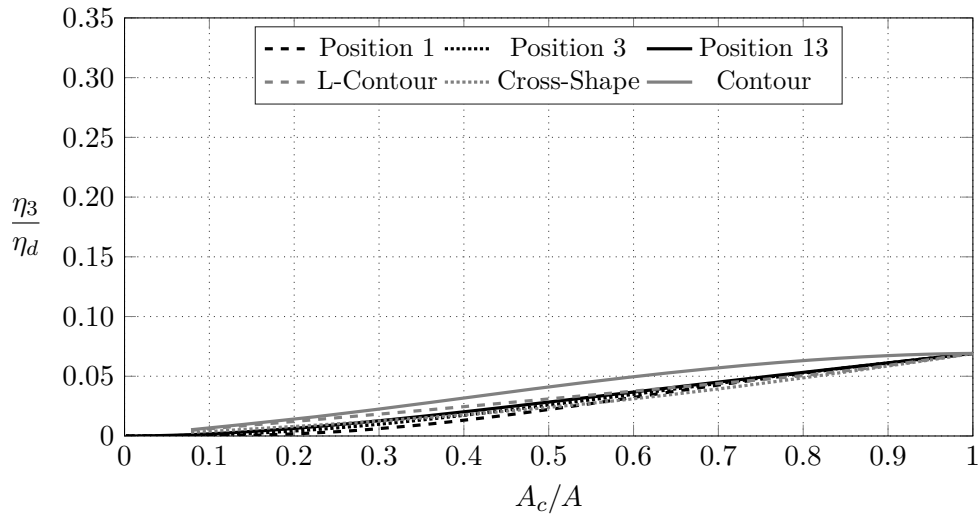
elastic layers, promoting the decoupling between the base and constraining layers and disabling the ability to impose shear to the damping layer. Indeed, any damping treatment presents limited damping capacities, and there are no noteworthy differences between the different damping treatments.

Following the same approach as previous analyses, one performs a detailed comparison regarding the damping capacity at higher vibration modes. Figure 6.200 presents the damping efficiency curves considering the second specific modal loss factor for different damping positions and modulus ratios. As previously discussed, the threshold of the modulus ratio for the increasing-decreasing behaviour of damping capacity at the second vibration mode holds the same value as the first one, so one considers the same modulus ratios in the following analysis. First, Figure 6.200a depicts the progress of the damping capacity with the covered area for a low modulus ratio of $E_c/E_d = 9 \times 10^3$. As discussed earlier, low modulus ratios have a similar effect to thin damping layers and tend to couple the base and constraining layers. As anticipated, both cases present similar results, namely, the square patch exceeding the damping performance of any other treatment peaking with a partially covered area of roughly 45%. The remaining square patches and the Cross-Shape treatment introduce a significant damping capacity as well, but the contour treatments, once again, show limited damping capacity. The main trends are still valid for the threshold value of the modulus ratio in Figure 6.200b. More importantly, considering these geometric and mechanical properties, these damping efficiency curves are those of Figure 6.185b, where one studied the influence of the patch positioning considering different damping layer thicknesses and a modulus ratio of $E_c/E_d = 7 \times 10^4$. Next, Figure 6.199c shows, once more, the inability of structures with high modulus ratios to provide significant damping capacities due to the base and constraining layers decoupling.

Finally, Figure 6.201 finishes this study by considering the third specific modal loss factor for different damping treatments and modulus ratios. As usual, one considers three modulus ratios, the lowest and highest modulus ratio under analysis in this study, as well as the threshold value of modulus ratio for the increasing-decreasing behaviour of damping capacity at the third mode of vibration - $E_c/E_d = 3 \times 10^4$. One should remember that Figure 6.172 depicts, for the third mode of vibration and both directions, similar maximum curvatures to those of the second mode of vibration. Given this, it comes as no surprise that the results of damping efficiency for the third mode of vibration follow the same trends as the second one. Namely, the square patch at Position 13 provides the best overall damping capacity for both the lowest and threshold modulus ratios. Nevertheless, and similarly to the thinnest damping layer in the study considering different damping layer thicknesses, the Cross-Shape treatment exceeds by a small amount, and for a restricted range of covered areas, the square patch at Position 13 for the lowest modulus ratio in Figure 6.201a. Consistently, the contour treatments present the already expected limited damping capacities. Lastly, Figure 6.201c shows that even for higher-order modes, higher modulus ratios decouple the elastic layers, limiting the shear strain at the damping layer and consequently damping capacity.

(a) $E_c/E_d = 9 \times 10^3$ (b) $E_c/E_d = 7 \times 10^4$ (c) $E_c/E_d = 5 \times 10^5$ **Figure 6.199** 1st modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

(a) $E_c/E_d = 9 \times 10^3$ (b) $E_c/E_d = 7 \times 10^4$ (c) $E_c/E_d = 5 \times 10^5$ **Figure 6.200** 2nd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

(a) $E_c/E_d = 9 \times 10^3$ (b) $E_c/E_d = 3 \times 10^4$ (c) $E_c/E_d = 5 \times 10^5$ **Figure 6.201** 3rd modal loss factor for different patch positions and covered areas, $h = 0.01a$, $h_c = 0.5h$

Composite Laminate Structures

7.1 | Introduction

In the last chapter, one studied passive damping solutions as surface treatments for vibration control of structures comprising isotropic materials. Nevertheless, viscoelastic materials have also seen their use extended to applications where composite materials are increasingly significant. Due to their excellent specific mechanical properties, these materials find widespread use in the automotive, railway or aerospace industries, where vibration control is mandatory.

Structural composite laminates consist of multiple fibre-reinforced laminae where, usually, fibres have very high elastic moduli in the longitudinal direction but lower ones in the transverse direction due to their particular structure of long fibres of tiny diameters. A polymeric resin matrix of relatively low modulus incorporates these fibres, and its function is to give shape and protect those fibres. Each lamina includes appropriately designed volume fractions of fibres and resin depending on the application and manufacturing process. While fibres are the constituent responsible for the final stiffness of the composite laminate, the polymeric resin matrix is also of great importance to other mechanical properties like damping. Due to their polymeric nature, most resins have some damping capacity by converting vibrational energy into heat, and some specially designed resins enhance damping [Mead, 1998]. According to different studies, thermoplastic polymers show high dissipation levels compared to thermoset ones, though one usually disregards them due to their weak adhesion properties [Chung, 2003]. However, one can witness progress in thermoplastic resins with promising results. Duc et al. [2014] studied the reinforcement of thermoset and thermoplastic resins with natural flax fibres and obtained the best compromise between stiffness and damping with flax fibre reinforced with semicrystalline polylactic acid (PLA), which has the advantage that it can be produced from renewable resources and is biodegradable. The damping of composite materials can be several orders of magnitude higher than traditional metallic ones [Treviso et al., 2015]. Apart from the intrinsic properties of each constituent (fibre and matrix), their interaction plays a leading role in the damping mechanism. The weaker the bond between fibres and matrix, the higher the energy dissipation due to the relative motion of fibres, but this leads to limited load transmission and strength

loss [Nelson and Hancock, 1978]. Therefore, the design requirements for high stiffness often conflict with those for high damping capacity. Stiffness and damping capacity are two conflicting needs; as the volume fraction of fibres increases to improve the laminates stiffness, its damping capacity decreases since the volume fraction of the resin matrix is consequently lower.

The design of appropriate stacking sequences for several aligned fibre laminae is a mandatory step of every laminate composite project. The composite laminate is suitable to be generally loaded and to offer the required mechanical properties through the proper arrangement of each lamina material and the corresponding alignment. For example, if made of fire-resistant material, the outer layers work as a shield protector for the laminate structure. One might follow a similar strategy for damping enhancement by adding viscoelastic layers to the stacking sequence to dissipate part of the vibrational energy in the form of heat. Figure 7.1 shows a conceptual example representing a laminate with composite layers, coloured grey, and viscoelastic material coloured blue.

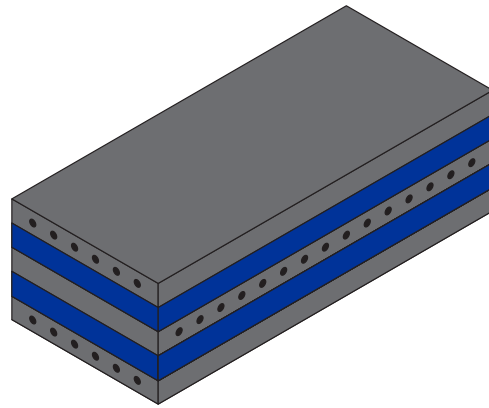


Figure 7.1 Schematic representation of a laminated composite with interleaved damping layers

7.1.1 | Laminated Composites with Interleaved Damping Layers

Mukhopadhyay and Kingsbury [1976] pioneered the study of the dynamic response of rectangular sandwich plates with viscoelastic cores and generally orthotropic facings. Considering the case of simply supported edges, the authors have studied the orientation of the facings and its influence on the maximum lateral deflection, the rate of dissipation of energy and the total strain energy of the core, but also the fundamental natural frequency. Since the authors generally studied orthotropic facings, as the plate undergoes flexural deformation, the facings deform with normal strain, but each facing also undergoes shear deformation in its plane. Mukhopadhyay and Kingsbury [1976] concluded that an additional in-plane shear component is present in the core, which would not occur in a conventional sandwich structure. Consequently, as the lateral response increases with the increasing elastic axis orientation of the facing, the energy stored in the core due to in-plane shear also increases.

Aware of the loss of stiffness that a soft damping layer can bring to a composite structure, Moser and Lumassegger [1988] tried to increase the damping of flexural vibrations of laminated composite beam structures by incorporating soft intermediate damping plies with a minimum reduction of stiffness. By appropriately choosing a damping ply with a higher shear modulus, the authors found that one can hold down the stiffness loss by about 20 %, even though one reduces the maximum loss factor for the first mode. Interestingly, for higher vibration modes, it is possible to achieve higher modal loss factors than those with the choice of the shear modulus

for maximum damping of the first mode. Following the work of Moser and Lumassegger [1988], Barrett [1991] reported the results of a parametric study that examined the effect of damping applications on the static and dynamic response of composite structures. The designer faces significant stiffness penalties through damped composites, which confirmed previous works, although it is possible to control static stiffness loss by thinning the damping layer. However, if a designer accepts a moderate reduction in static stiffness, it can substantially improve dynamic stiffness with off-the-shelf materials and proven technological procedures.

In a landmark work, Saravanos and Pereira [1992] studied the effects of interleaving damping layers on the dynamic response of composite plates since the combination of composite materials with interlaminar damping layers offers the advantages of high damping, damping tailoring, good mechanical properties, and low weight addition. Besides, Saravanos and Pereira [1992] noted that interlaminar damping is highly compatible with laminated configurations of composite structures and their fabrication techniques. Based on a discrete layer laminate theory, the authors concluded that composite plates with high anisotropy variations between the restraining sublaminates presented the potential for higher damping than geometrically equivalent aluminium plates. In addition, Saravanos and Pereira [1992] investigated the effects of the thickness aspect ratio, damping layer thickness, and composite fibre volume ratio on the modal damping, natural frequency, and static deflection and validated the methodology with measured and predicted frequency response functions of T300/934 graphite/epoxy plates with damping layers.

Interested in the influence of damping layers on the stiffness and strength of composite laminates, Fujimoto and Tamura [1993] investigated unidirectional carbon/epoxy prepreg sheets and a polyethylene-based damping material sheet used as an interleaf and discussed the interleaving effect on tensile and compression strength. In addition to the fact that CFRP/damping-material laminates have 5 to 50 times higher loss factors when compared with conventional CFRP values, tensile test results for quasi-isotropic laminate-based composites indicated that the damping material interleaf is effective for suppressing delamination and multiple splitting in some cases. On the other hand, and as proposed by other authors, the flexural modulus for these laminates showed slightly lower values than those obtained with conventional composite laminates. For some configurations, the interleaf may promote fatigue resistance, but it is not very impressive for increasing the compression strength.

Rao and He [1993] performed a parametric study to investigate the variation of dynamic stiffness and modal loss factor of composite beam structures with structural parameters like the ply orientations and the thickness of the damping layers and laminates, operating temperature, and damping material properties. In addition, the authors presented the design of composite beams for maximum damping capacity and compared some experimental results with the theory. Rao and He [1993] drew attention to the fact that for multiple damping layer composite beams, the thicknesses of the constraining layers and the corresponding positioning of the damping layers in the stacking sequence highly influence the beam's dynamic stiffness and damping capacity. By investigating the variation of damping capacity and dynamic stiffness of single, double and triple damping layer composite beams, the authors noted that it is nearly impossible to enhance damping without some stiffness penalties.

Interested in the finite element analysis of vibration damping of laminated composites, Rikards [1993] presented a study comparing modal loss factors calculated in two ways: the complex frequency and the energy method. The authors developed special laminated super beam and plate elements to establish a feasible model that surpassed the differences encountered in studying the dynamics of laminated structures with viscoelastic damping layers. Later, Liao et al. [1994] investigated the vibration-damping behaviour of carbon fibre-epoxy composite beams interleaved with damping layers at the mid-plane and proved that the effectiveness of the damping treatment increases significantly with the flexural modulus of the stiff outer layers.

In an early study considering the damping of composite shells with embedded viscoelastic layers, Zhou and Rao [1996] discussed the results of a parametric study with single and double-interleaved composite tubes, including material and geometric factors. Once again, similarly to previous works involving beam and plate structures, the main concern is optimising damping benefits with acceptable stiffness reduction.

Following previous work, Rao et al. [1997] explored the damping of composite beams embedded with three different viscoelastic materials using the finite element based MSE. The authors compared the results with previous work and found excellent agreement at higher temperatures, even though some minor differences exist at lower temperatures. More recently, Yim et al. [2003] explored the damping behaviour of a 0° laminated sandwich composite beam interleaved with a viscoelastic layer. As expected, results demonstrated the capacity of laminated sandwich composites with embedded viscoelastic layers to considerably enhance structural damping.

Suzuki et al. [2003] developed a multilayered and higher-order-deformable finite element for free vibration analysis of laminated plates with constrained viscoelastic interlayers. Expectedly, higher-order theories give better results than the conventional ESLT or FSDT theories for flexural vibration analysis and damping characterisation of multilayer-deformable laminates due to their intrinsic multilayered deformations in the through-the-thickness direction. The work analyses five quasi-isotropic laminated coupons with interleaved viscoelastic layers in different stacking sequences. In addition to the FEM, the authors performed experimental measurements to obtain the dynamic properties of several fabricated laminated coupons and sandwich panels. Comparisons of the numerical results with the obtained experimental results show good agreement in the natural frequencies assessment but poor accuracy in damping estimation.

Kishi et al. [2004] investigated and characterised the damping properties of carbon fibre-reinforced interleaved epoxy composites using several thermoplastic-elastomer films, such as polyurethane elastomers, polyethylene-based ionomers and polyamide elastomers as the interleaving materials. Interleave fibre-reinforced composites with damping materials has a high impact on the damping properties of the final structure. As expected, damping depended not only on the viscoelastic properties of the interleaved polymer material but also on the arrangements of the reinforcing carbon fibre in the laminates, which controlled the stiffness of the intralaminar zone and the strain of the interleaf films.

Investigating the damping of unidirectional glass fibre composites with interleaved viscoelastic layers, Berthelot and Sefrani [2006] obtained good agreement between numerical and experimental results. The study considers the effects of the thickness, Young's modulus and the damping of a viscoelastic layer interleaved in the middle plane of laminates, in addition to the outcomes of two symmetric viscoelastic layers closer to the laminates skins. According to the authors, damping increases when the loss factor of the viscoelastic layer increases or when Young's modulus decreases. As pointed out by other authors, transverse shear strain energy is the main responsible for damping when interleaving viscoelastic layers in composite structures. The presented results show that damping is notably lower for viscoelastic layers positioned out of the middle plane when compared with laminates with interleaved layers.

Zhang and Chen [2006] considers the contribution of energy dissipation due to fibre-reinforced composites by integrating a 3-D FEM/strain energy approach to predict the modal loss factor of laminated fibre-reinforced composite structures embedded with viscoelastic damping layers. The authors focused on parametric damping studies to investigate the effect of damping of fibre-reinforced composites, compliant layering (the inner layers adjacent to the damping layers of the constraining laminates) and location of viscoelastic layers along the thickness direction on the modal loss factor and frequency of damped composite beams. Results compared well with previous studies, namely with the work of Suzuki et al. [2003]. In the conclusions of their work, Zhang and Chen [2006] advocate an appropriate number of compliant layers and the increase of

the ply angle of compliant layers to improve the flexural modal loss factor. The enhancing effect of compliant layering is significant for higher flexural modes. Besides, the isolated placement of damping layers in the vicinity of the mid-plane of the beam is beneficial to improving the damping characteristics of the structure.

Pan and Zhang [2009] proposed a new method for damping estimation in co-cured composite laminates with embedded viscoelastic layers based on mode superposition and MSE. The calculated damping value is not the modal loss factor but a combination of damping from the contributing modes. To test the new method, the authors compared the results with experimental work performed by Dynamic Mechanical Thermal Analysis (DMTA), and the obtained loss factor agrees well with the experimental one. For Pan and Zhang [2009], one can apply the proposed method to any viscoelastic damping structure. In the same year, with a new finite element model for anisotropic laminated plate structures with viscoelastic core and anisotropic face layers and considering a layerwise mixed approach, Araújo et al. [2009] performed constrained optimisation for the maximisation of modal loss factors. Using gradient-based optimisation, associated with the developed mechanical model, and single and multiobjective optimisation based on genetic algorithms using an alternative ABAQUS finite element model, the optimisation problem searched for the maximisation of modal loss factors considering thicknesses and elastic ply angles of laminated face layers as design variables. Based on the results for a simply supported sandwich beam and a simply supported sandwich plate, the authors claim that the proposed technique can improve modal loss factors substantially. Later, Araújo et al. [2010] focused on the optimal design and parameter estimation of frequency-dependent viscoelastic laminated sandwich composite plates. Considering, once again, viscoelastic frequency-dependent cores and laminated anisotropic face layers, the authors performed constrained optimisation of passive damping for the maximisation of modal loss factors with thickness and elastic ply angles of laminated face layers as design variables.

As polymeric materials, viscoelastic layers have low stiffness, and the global stiffness of laminate structures embodying these materials drops in the name of damping capacity. Careful design of the stacking sequence can smooth this effect. Still, it is currently a problem. For example, Le Maoût et al. [2011b] studied the geometric and material design of hybrid elastomer and composite sandwich plates by enhancing the damping capabilities of a structure without degrading its stiffness and increasing the total mass too much. The main goal was to optimise the number of layers and the position of the viscoelastic layers in the stacking sequence of a composite laminate. The authors also studied the optimisation of fibre orientations and layer thicknesses, coupling MATLAB and ABAQUS software to perform both minimisation and finite element problems. The obtained results depend highly on the minimisation constraints, and the authors acknowledge that the minimisation problem may converge to local minima.

Akoussan et al. [2014] investigated the vibration of orthotropic multilayer sandwich structures with viscoelastic cores by developing a finite element model derived from a classical zigzag model with shear deformation in the viscoelastic layer. The work aims to find the best orientation of face layers, which can give the optimal damping properties of the structure.

Hamdaoui et al. [2015] took an interest in the optimal design of frequency-dependent three-layered rectangular composite beams for low mass and high damping. Considering the thicknesses of the viscoelastic and elastic layers as the design variables and keeping the beam thickness constant and its maximal deflection under its weight limited to its thickness, the approach consisted of a multi-objective optimisation based on evolutionary algorithms to find the Pareto surface that minimises mass and maximises the damping. The authors showed that the best compromise depends on the weight given to each criterion. Moreover, the computational burden associated with the optimisation process is relatively high, suggesting efficient surrogate models to reduce it.

In a two-paper study, Madeira et al. [2015a,b] proposed multiobjective approaches for passive damping optimisation and vibration control of sandwich structures. The authors proposed a constrained optimisation for maximising modal loss factors and minimising the weight of sandwich beams and plates with elastic laminated constraining layers and a viscoelastic core, with layer thickness, material and laminate plies orientation angles as design variables. Claiming more reliable solutions than those available in the literature and obtained by a genetic algorithm approach, the authors presented multiple Pareto fronts, valuable for a compromise between weight reduction and damping maximisation.

Martone et al. [2016] worked on a simplified approach to model the damping behaviour of interleaved carbon fibre laminates, proposing a semi-analytical model based on the FSDT and accounting for the out-of-plane strain contributions to the energy dissipation. In addition, the authors performed experimental research fabricating two different interleaved architectures, single and double interleaved composites, and tested them to evaluate the effect of the damping layers distribution within the stacking sequence. The analysis of the loss factor of both configurations confirms that one obtains the higher enhancement of damping performance with the single interleaved arrangement due to the viscoelastic layer positioning at the neutral axis, where transverse shear strains are high.

Following previous work, Akoussan et al. [2016] performed a sensitivity analysis of the damping properties of viscoelastic composite structures according to the layers' thicknesses. Solving the eigenvalue problem continuously with a numerical method coupling asymptotic numerical method, homotopy technique, continuation method and automatic differentiation by writing the damping properties of the structure as a high-order function of the thickness allows getting the gradients and high-order derivatives of the loss factor and the damping frequency of the system efficiently, which are highly desirable for optimisation needs. Considering two cases, the authors investigate the variation of only the thickness of the core layer and the thickness variation of each layer, keeping the total thickness of the structure constant. Results showed that the damping relates to the viscoelastic material, the fibre orientation of the laminate faces, the boundary conditions and the structure dimensions. The method allows identifying the exact value of the core thickness for optimal damping of the structure, for example, at a low computation cost.

In the footsteps of previous works like that of Hamdaoui et al. [2015], Xu et al. [2016] consider the design problem of simultaneously maximising structural modal damping and minimising weight in composites structures with integral viscoelastic layers using a layerwise finite element model and a multi-objective optimisation problem solved by an evolutionary algorithm. The work demonstrated that the inserting position and the material type are two important design parameters, showing Pareto-optimal design solutions with the laminate stacking sequence, damping inserting position and thickness as design variables, and the most appropriate frequency-dependent viscoelastic damping material type and its geometric parameter determined simultaneously.

The study of laminated composites with interleaved damping layers is an active topic of research and development with growing interest. As shown above, the optimal design of viscoelastic damping treatments usually follows two main approaches. Some authors use optimisation algorithms to search for optimal damping efficiency of viscoelastic materials embedded in laminated composites. On the other hand, one might follow the approach assumed throughout this work, considering parametric analyses by performing discrete calculations of damping properties for different parameters. As already discussed in Chapter 6, there are advantages and disadvantages in both approaches and optimisation algorithms are of the utmost importance in this world of complex structures and big data. However, as already mentioned, these powerful algorithms often are black boxes providing optimal solutions to complex problems, although giving the designer, engineer or researcher little help in identifying the mechanisms which lead to the desired damping capacity and ways to apply the same principles elsewhere. The main goal is to

perform parametric analyses regarding damping materials in laminated composite structures and understand the appropriate design guidelines.

7.1.2 | Laminated Composites with Interleaved Damping Patches

The classical method of embedding viscoelastic layers into the laminate stacking sequence to improve damping capacity might undermine the laminate's stiffness and mechanical strength beyond weight increase. To overcome this, Fotsing et al. [2012] performed an experimental study using partial viscoelastic layers interleaved between carbon-epoxy layers in a beam structure with a honeycomb core. The authors studied the position of a damping patch along the beam and concluded that patches close to the clamped region and over the mode shape nodes proved efficient in improving damping capacity. Furthermore, the authors considered multiple patch-damping treatments and confirmed the previous conclusions for single treatments. In addition, Fotsing et al. [2012] also proved the uselessness of full coverage treatments for some configurations. As demonstrated in this work for structures comprising isotropic materials, it is possible to achieve the same or even more damping capacity with less added mass favouring areas promoting high shear strain in the viscoelastic layers, which occur near the clamped area when considering a cantilever beam, for example.

Differently from the classical damping treatments, where viscoelastic layers take part in a stacking sequence of composite laminae to increase damping, Figure 7.2 shows a different approach with viscoelastic material partially distributed within specific layers. The main goal is to improve the damping capacity of laminated composites without degrading stiffness or mechanical strength excessively or increasing the mass unnecessarily. The use of viscoelastic materials in a partial configuration has been successively reported as an efficient method of passive vibration control when applied as a coating treatment, whether in a free or constrained layer configuration. One suggests in this work the numerical and experimental investigation of partially distributed viscoelastic damping treatments for laminated composite plates in an integrated arrangement.

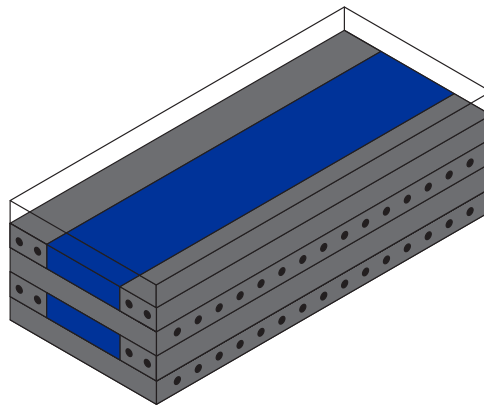


Figure 7.2 Laminated composite with interleaved damping patches

Figure 7.3 presents detailed views of two alternatives to a composite lamina made of a conventional composite material (coloured grey) and viscoelastic material (coloured blue). Figure 7.3a depicts a possible configuration where two strips of composite material surround a central viscoelastic strip along the lamina's length, aiming to decrease the negative impact of damping material on stiffness. Similarly, Figure 7.3b represents a lamina of composite material with two viscoelastic damping pieces.

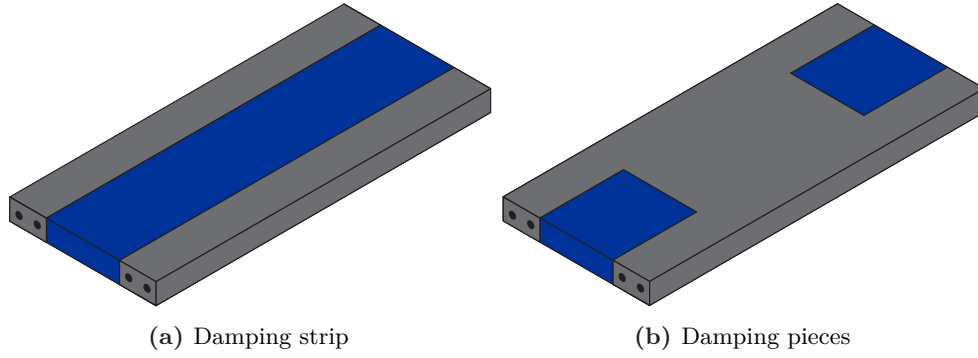


Figure 7.3 Detailed view of partially distributed damping material for laminated composites

The main concern about the lamina configurations depicted in Figure 7.3 is the manufacturing process and the corresponding feasibility. Is it feasible to have a lamina made of different materials? Some arrangements are preferable regarding the design or manufacturing point of view. Promote the fibre discontinuity, for example, should be avoided in laminated composite structures since it decreases the global stiffness of the system. On the other hand, temperature greatly influences viscoelastic materials, which is essential to the curing process of composite materials in most manufacturing processes.

Despite the possible benefits of interleaving a viscoelastic layer, Treviso et al. [2015] warn that this is not a simple procedure since co-curing is necessary. The different thermal behaviours of the resin matrix and the viscoelastic layer may pose manufacturing problems. To overcome this issue, SMAC-Group Montblanc Technologies developed a patented product called Smacwrap ST, claiming to be a viscoelastic material that withstands composite curing, including the autoclave process, and one can embed it into composite structures. SMAC-Group Montblanc Technologies provided material samples for the experimental testing reported and discussed later in this work. In this chapter, beyond the parametric study of viscoelastic materials interleaved in composite plates, one studies the damping efficiency of partially distributed treatments, like those of Figure 7.3, for the optimal design of composite structures.

7.2 | Laminated Plates with Interleaved Damping Layers

This section deals with the parametric study of the damping efficiency of interleaved viscoelastic layers in laminated composite plates. One considers a dimensionless analysis of six stacking sequences to evaluate the influence of the viscoelastic layer thickness on the damping capacity. Figure 7.86 depicts the reference stacking sequence of a symmetric laminate consisting of 0° and 90° oriented layers - $[90^\circ/0^\circ/90^\circ]_S$.

Table 7.1 presents the material properties of the unidirectional carbon fabrics considered in the present study. One studies a square plate with side length a and six layers of equal thickness - a total laminate thickness of h . In addition, Table 7.2 exhibits the mechanical properties of the frequency and temperature-independent viscoelastic layers considered in this section.

Figure 7.5 shows six different stacking sequences with interleaved viscoelastic damping layers. One considers any possible configuration considering symmetric laminates with six composite plies. Figure 7.5a shows a single interleaved viscoelastic layer at the core of the laminate. In addition, Figures 7.5b and 7.5c present two possible double interleaved configurations - a stacking sequence with the viscoelastic layers closer to the core and another with the viscoelastic damping layers closer to the skins, respectively. One follows the same reasoning with triple interleaved

90°
0°
90°
90°
0°
90°

Figure 7.4 Reference stacking sequence**Table 7.1** Reference material properties of the Hexcel 8552-IM7 UD Carbon Prepregs

Tensile Modulus GPa		Poisson		Shear Modulus GPa		Density kg m ⁻³
E_{11}	E_{22}	ν_{12}, ν_{13}	ν_{23}	G_{12}, G_{13}	G_{23}	ρ
161	11.38	0.32	0.45	5.2	3.9	1570

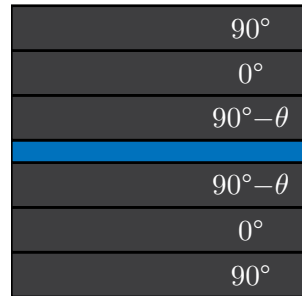
Table 7.2 Material properties of the viscoelastic layers

Young's Modulus MPa	Poisson	Density kg m ⁻³	Loss Factor
1	0.49	1140	0.3

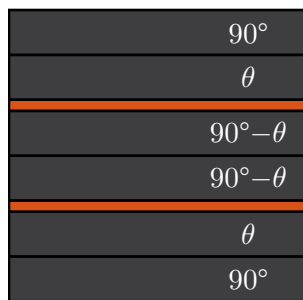
symmetric stacking sequences; one with the viscoelastic damping layers closer to the core, another with one layer at the core of the laminate and the others closer to the skins - Figures 7.5d and 7.5e, respectively. At last, one considers a quintuple interleaved stacking sequence with a viscoelastic layer between any two composite layers, as represented in Figure 7.5f.

These arrangements cover the range of possible symmetric configurations for a six-layer reference laminate, including viscoelastic layers within the stacking sequence. The main goal is to compare different positions by performing parametric analyses and using the same amount of damping material. The focus of such a study is to understand the preferential stacking sequences, whether it is preferable to have more or less damping layers for the same amount of damping material and whether they should be closer to the core or the skins. Figure 7.5 shows another meaningful design variable. Composite plies adjacent to damping layers, usually known as compliant layers, present variable orientation angles represented by tags θ and $90^\circ - \theta$. Therefore, it is possible to assess the influence of the orientation angle on the damping capacity and stiffness of the laminate.

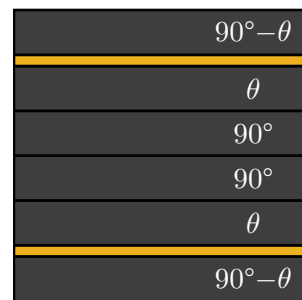
In this section, one performs different parametric analyses considering a regular FEM mesh of 30 CPT(C) elements along each side of the plate and the ToZZ. One obtains the natural frequencies, mode shapes and modal loss factors by performing a modal analysis with the MSE to study the damping capacity of composite plates with interleaved viscoelastic damping layers.



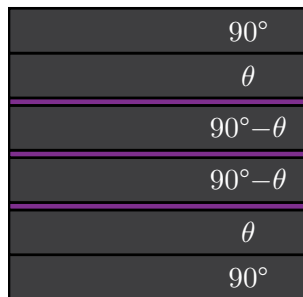
(a) Single (S)



(b) Double-Core (D-C)



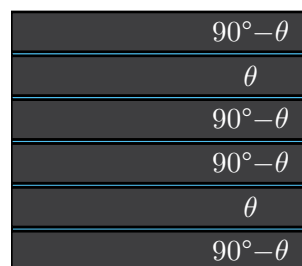
(c) Double-Skins (D-S)



(d) Triple-Core (T-C)



(e) Triple-Skins (T-S)



(f) Quintuple (Q)

Figure 7.5 Interleaved stacking sequences

7.2.1 | Damping Efficiency of Interleaved SSSS Laminated Plates

In the following, one studies the damping efficiency of interleaved viscoelastic layers considering simply supported laminated composite plates. One focuses the analysis on the influence of the position and thickness of the viscoelastic layers on the damping capacity, considering the six stacking sequences presented before. The main goal is to compare different stacking sequences with interleaved damping layers considering the same volume of damping material. Therefore, to study the influence of the damping layers' position, h_d describes the total thickness of the damping material regardless of the number of damping layers. When comparing a single interleaved configuration with a double interleaved one, for example, the thickness of each lamina in the double interleaved stacking sequence has half the total thickness. The following studies consider $\theta = 0$.

7.2.1.1 | Influence of Damping Layer Thickness and Position

Following the conclusions of previous chapters, it is relevant to analyse the plate mode shapes to understand the damping performance of damping treatments. Figure 7.6 presents the familiar transverse displacement mode shapes of the first six normal modes of the simply supported square laminated composite plate considering the reference stacking sequence of Figure 7.86 and an aspect ratio $a/h = 100$. Previously, one also discussed the importance of the mode shapes associated with the transverse shear strain degrees of freedom γ_{xz} and γ_{yz} to understand the modal transverse shear strain pattern, and so Figures 7.7 and 7.8 show, respectively, the first six ones for the same reference laminated composite simply supported plate.

Furthermore, Figures 7.9 to 7.11 display the through-the-thickness extensional, transverse shear and in-plane shear density mode shapes for the first normal mode and different section cuts, respectively. Namely, sections at which the extensional and transverse shear strain energy densities are higher - for $y = b/2$ and $x = a/2$. Figures 7.6a, 7.7a and 7.8a highlight these section cuts along the transverse displacement and transverse shear strain pattern mode shapes.

Figure 7.9 depicts the first extensional strain, stress and energy density mode shapes along both x , section cut at $y = b/2$, and y , section cut at $x = a/2$, directions. The section cuts displayed in Figure 7.6a help address the through-the-thickness patterns of extensional strain, stress and energy density. Besides, considering the extensional strain patterns $\phi_1^{\varepsilon_x}$ and $\phi_1^{\varepsilon_y}$ depicted in Figures 7.9a and 7.9b exposing the extensional strain continuity through-the-thickness, it is relevant to discuss the through-the-thickness distribution of extensional stress and strain energy density. As previously reported, one can easily distinguish higher extensional stress and strain energy density along the x -direction within layers oriented at 0° . On the other hand, one can identify higher extensional stress and strain energy density along the y -direction within the 90° oriented ones. Once again, and as previously discussed, this pattern depends on the material properties presented in Table 7.1, namely that $E_{11} > E_{22}$. As discussed in previous chapters, it is crucial to comprehend the extensional strain/stress distribution to evaluate damping capacity by the ability of the stiffer layers to impose transverse shear strain on damping layers.

Besides, Figure 7.10 illustrates the through-the-thickness distribution of the first transverse shear strain, stress and energy density mode shapes at the same section cuts. The patterns follow the expected behaviour for both sections, namely the shear stress continuity, the higher levels closer to the neutral axis of the structure and the nodal line for $y = b/2$ and $x = a/2$ accordingly to Figures 7.7a and 7.8a. Additionally, Figure 7.11 displays the through-the-thickness distribution of the first in-plane shear strain, stress and energy density mode shapes. The in-plane shear strain, stress and strain energy density are null for both section cuts.

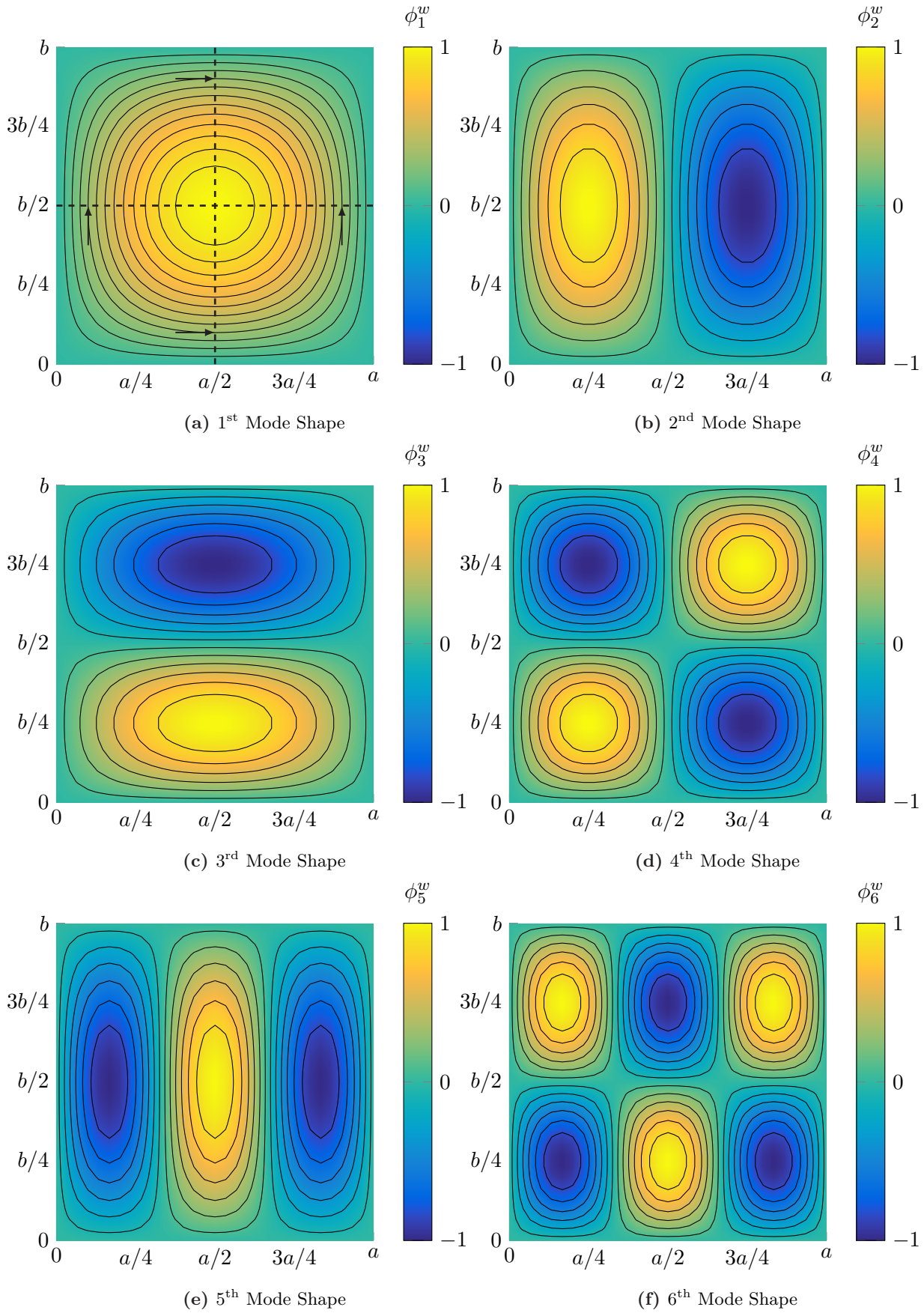


Figure 7.6 Transverse displacement mode shapes of the laminated composite square SSSS plate

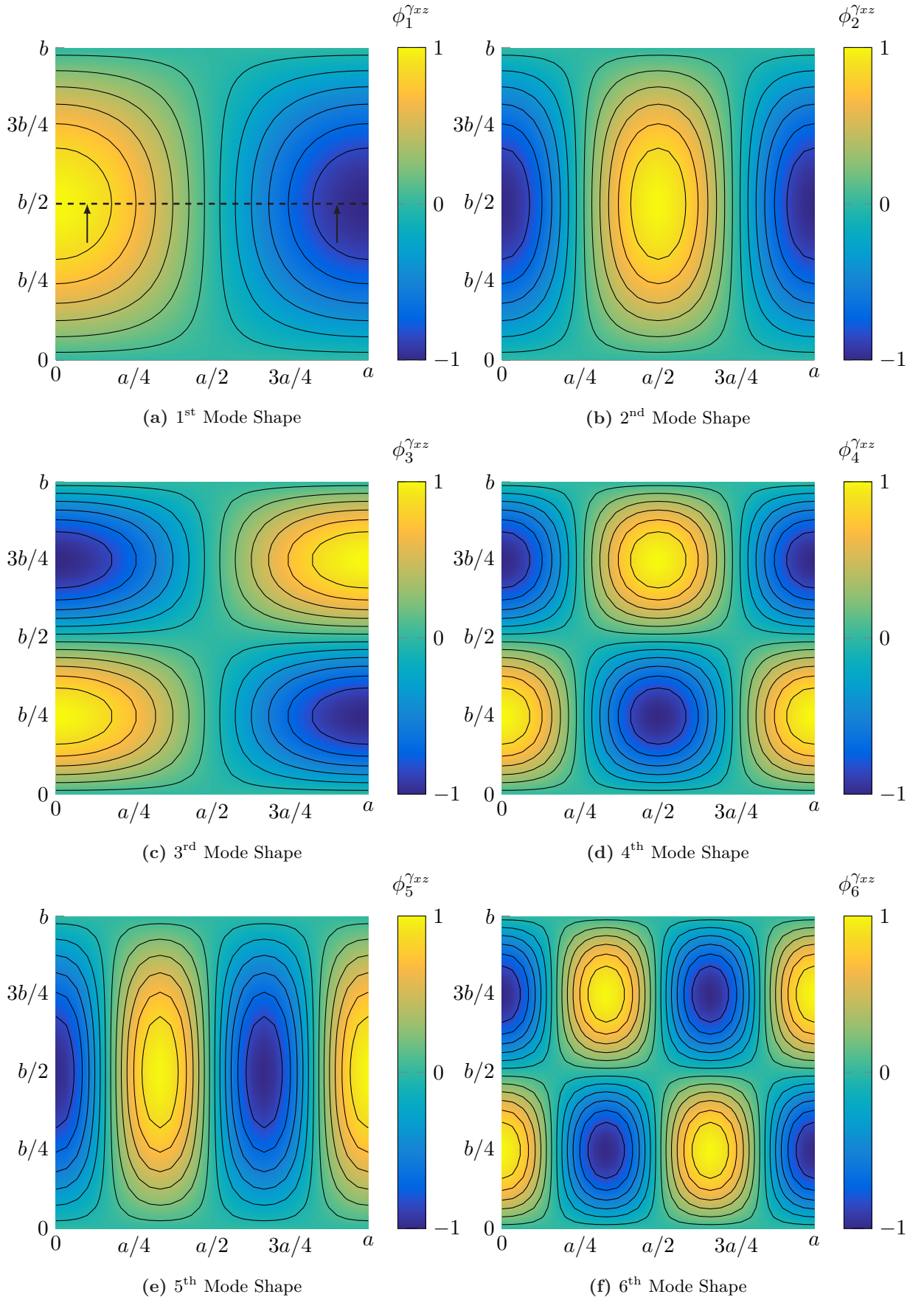


Figure 7.7 Transverse shear strain mode shapes, $\phi^{\gamma_{xz}}$, of the laminated composite square SSSS plate

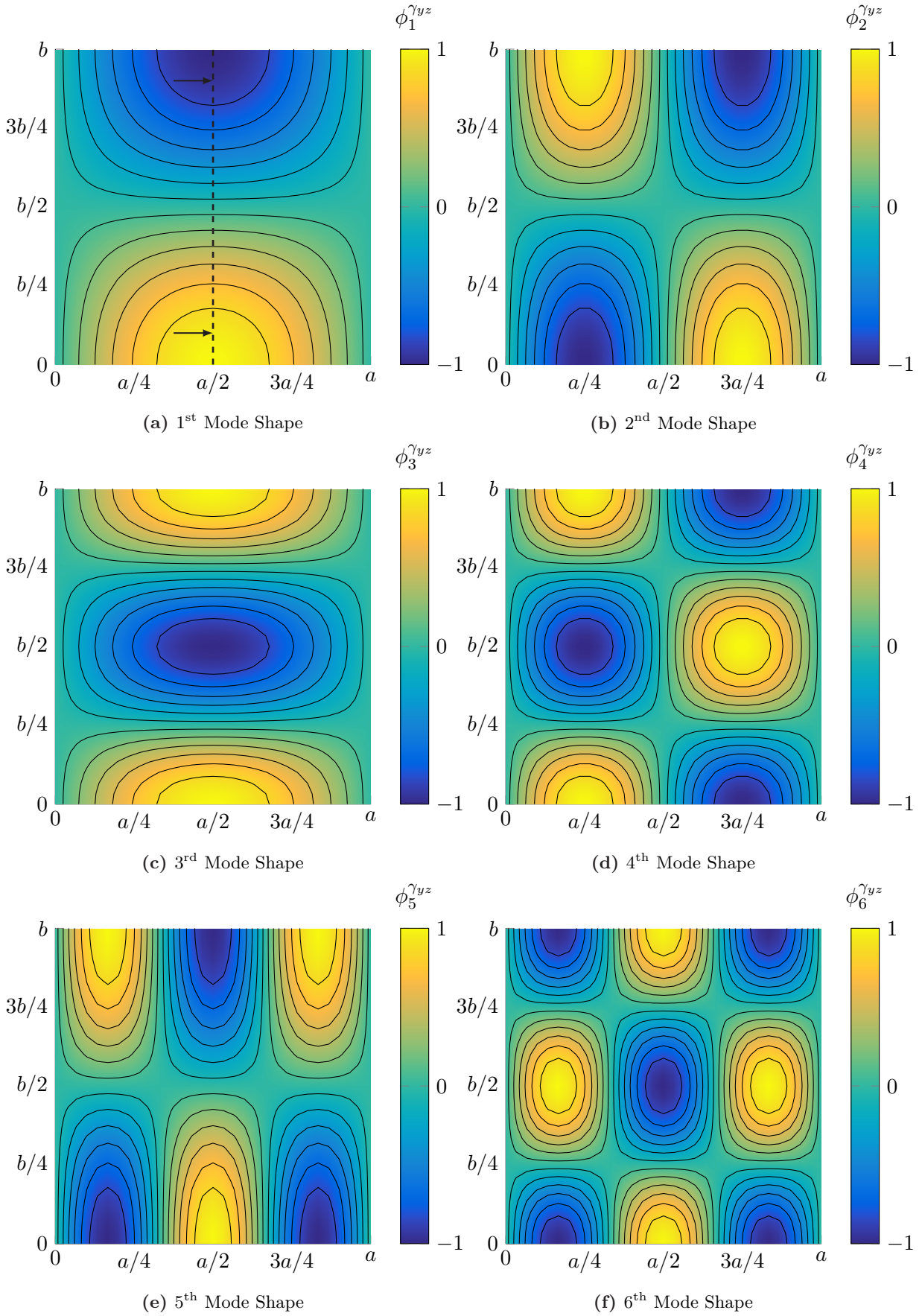


Figure 7.8 Transverse shear strain mode shapes, $\phi^{\gamma_{yz}}$, of the laminated composite square SSSS plate

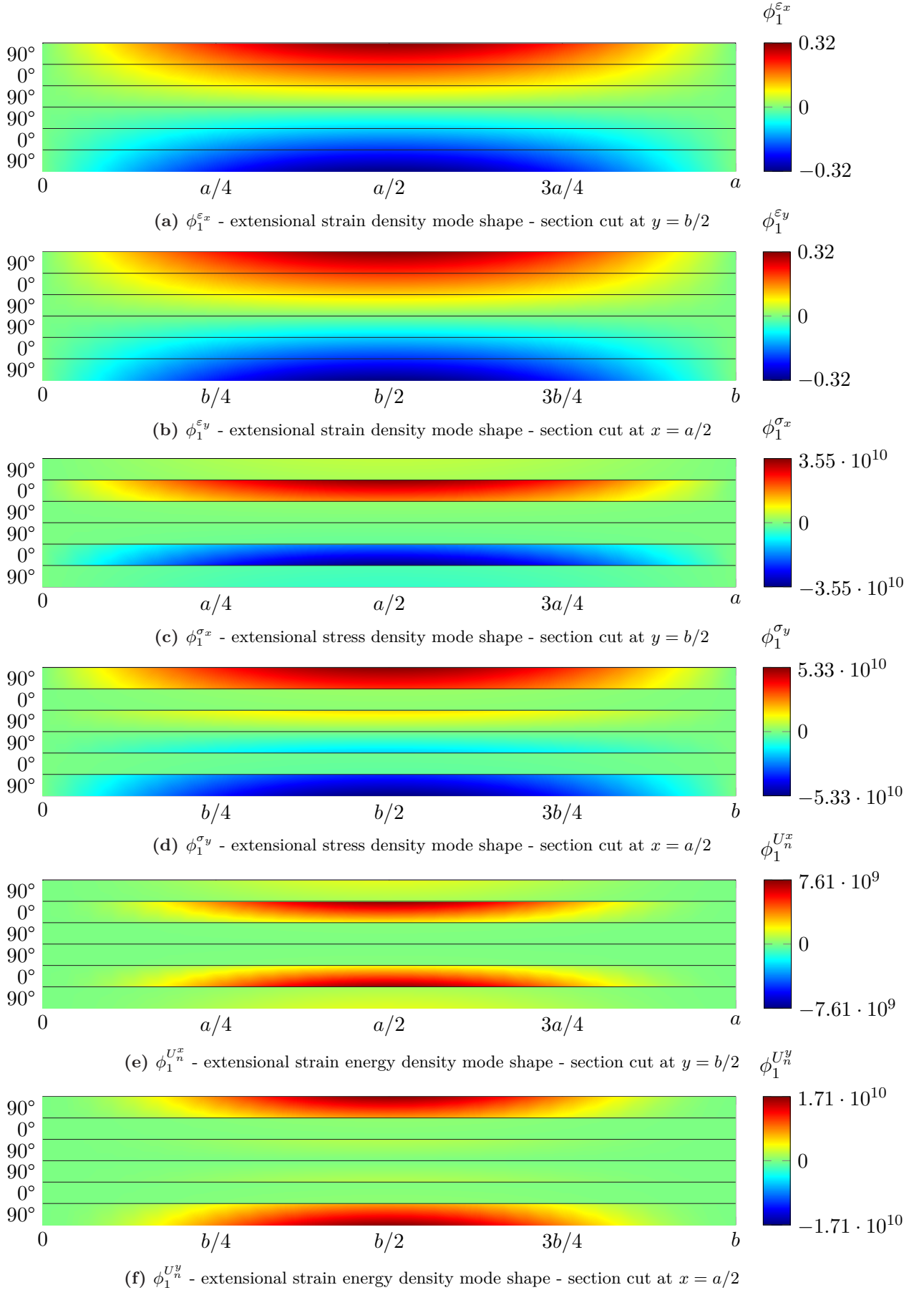


Figure 7.9 First extensional density mode shapes of the laminated composite square SSSS plate

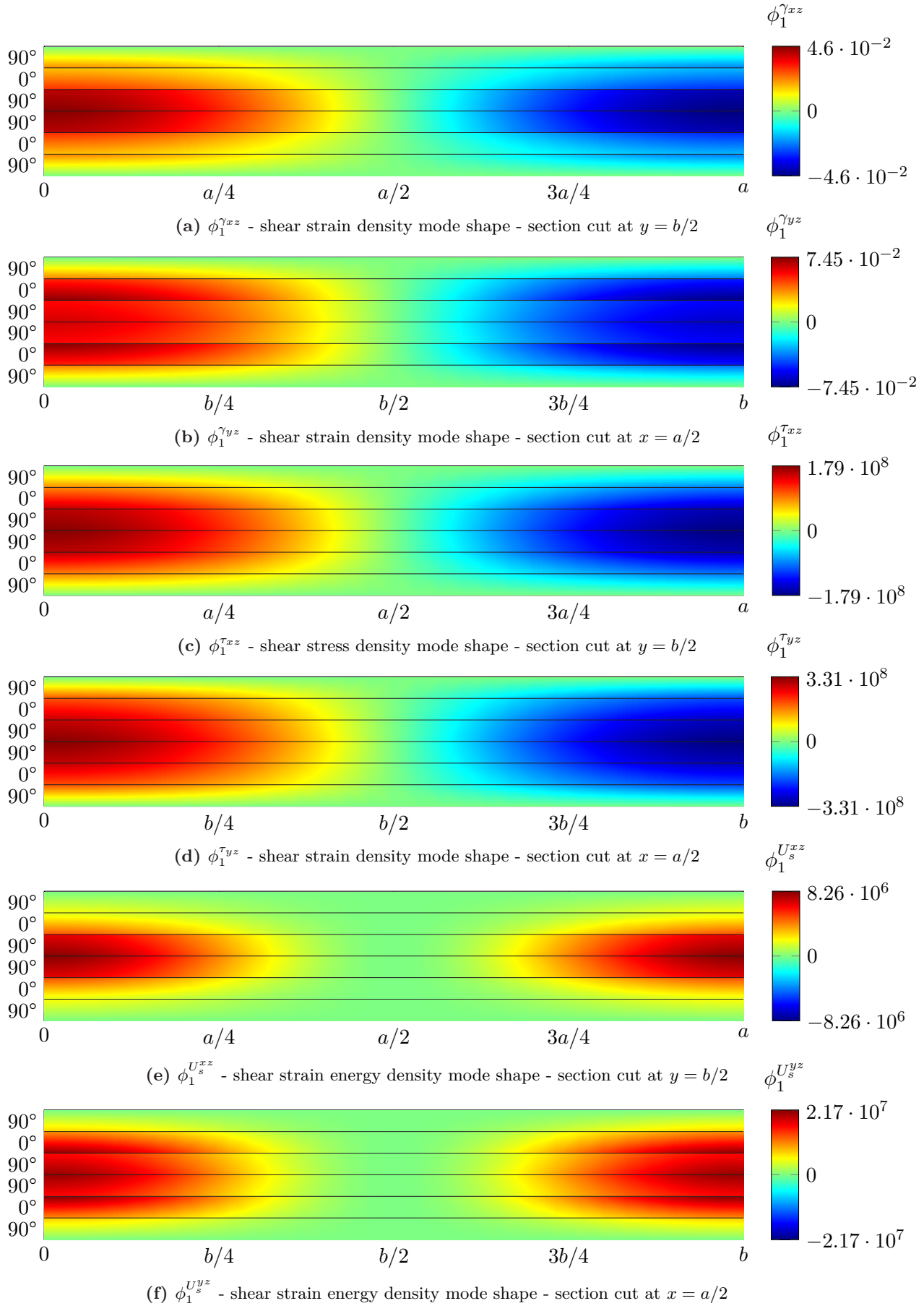


Figure 7.10 First shear density mode shapes of the laminated composite square SSSS plate

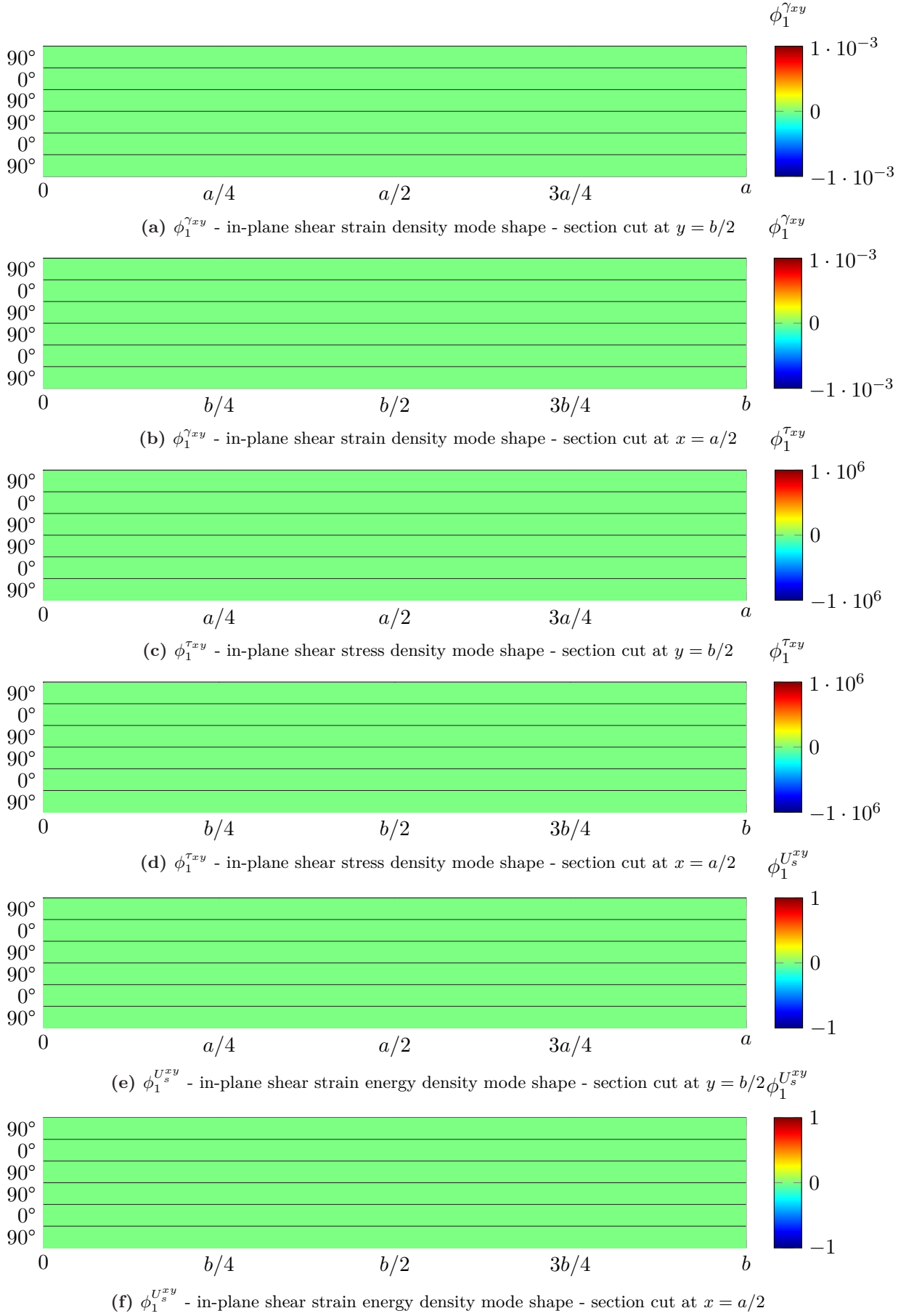


Figure 7.11 First in-plane shear density mode shapes of the laminated composite square SSSS plate

Focusing on the analysis of the influence of thickness and position of the viscoelastic layers on the damping capacity of laminated plates, Figure 7.12 exhibits the specific modal loss factors of the first three normal modes as a function of the damping layer thickness for the different stacking sequences under analysis; Single (S), Double-Core (D-C), Double-Skins (D-S), Triple-Core (T-C), Triple-Skins (T-S) and Quintuple (Q). One considers the modal loss factor as a function of the material loss factor, η_n/η_d , and damping layer thickness as a function of the total laminate thickness without damping layers, h_d/h . In particular, Figures 7.12a, 7.12c and 7.12e feature the corresponding specific modal loss factors for reference laminated composite plates with an aspect ratio $a/h = 100$ and, side-by-side, Figures 7.12b, 7.12d and 7.12f for thicker laminated composite plates with an aspect ratio $a/h = 50$.

For the first modal loss factor of a reference laminated composite plate, Figure 7.12a, it is possible to define three ranges of thickness ratio h_d/h , matching the three decades represented. The modal loss factor varies almost linearly in the first decade for thinner damping layer thicknesses, and it is clear that damping capacity depends almost exclusively on the amount of damping material regardless of the stacking configuration (log scale for the thickness variable). But then, the Single (S) and both stacking sequences with damping layers at the skins lose their ability to improve damping capacity as the damping layer thicknesses increase. In the last decade of dimensionless damping layer thickness, i. e., for thicker damping layers, there is a plateau of modal loss factor, and increasing damping material no longer increases damping capacity at this mode. Indeed, the modal loss factor can even decrease with increasing damping layer thicknesses. The Single (S) and double interleaved configurations with damping layers closer to the skins, Double-Skins (D-S), show the worst damping capacity of all stacking sequences for the same volume of damping material. On the other hand, interleaved stacking sequences with damping layers close to the core, Double-Core (D-C) and Triple-Core (T-C), and particularly the Quintuple (Q) configuration, show the best overall damping capacities. It is important to note that core configurations with more than one layer are preferable for higher damping capacity than skins configurations. As previously discussed, the transverse shear strain of the damping layer is the primary mechanism responsible for damping capacity, and often, many authors report that the damping layer should be positioned at the neutral axis to maximise damping. When dealing with laminated composite materials, such requirements may be insufficient, as the damping capacity of the Single (S) interleaved configuration shows. Nevertheless, skin configurations show low damping capacities for the entire range of damping layer thickness.

For thicker laminated composite plates with an aspect ratio $a/h = 50$, Figure 7.12b presents the same specific modal loss factors. There are significant differences in damping efficiency between the reference and a thicker laminated composite plate with interleaved viscoelastic layers. For thicker laminated composite plates, it is possible to identify a maximum attainable damping capacity for different stacking sequences. Until then, the modal loss factor increases approximately linearly (log scale for the thickness variable), with core configurations offering the best damping capacities. After the maximum attainable modal loss factor, damping capacity drops markedly, and more damping material decreases damping capacity. The Quintuple (Q) configuration presents the best overall damping capacity. On the other hand, despite the poor performance of the stacking sequences considering damping layers close to the skins, the single-layer arrangement shows a weaker damping performance for thicker laminated composite plates, especially for thicker damping layers. A soft single-layer damping treatment at the core of the laminate decouples both halves of the CFRP, and so their ability to impose transverse shear strain to the viscoelastic layer, as previously discussed for patch treatments in structures comprising isotropic materials.

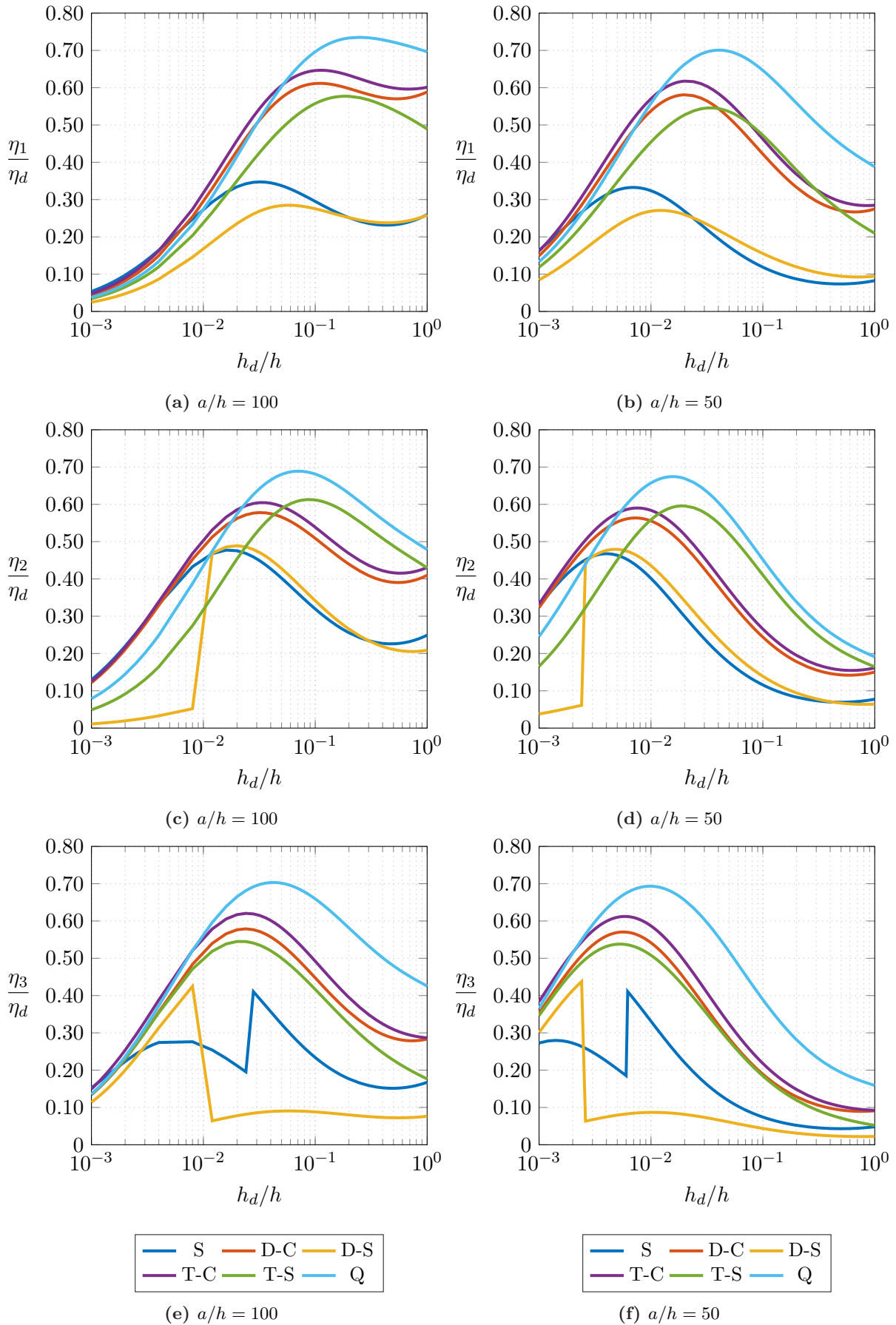
Figures 7.12c and 7.12d present the same specific modal loss factors for the second vibration mode of reference laminated composite plates with an aspect ratio $a/h = 100$ and thicker

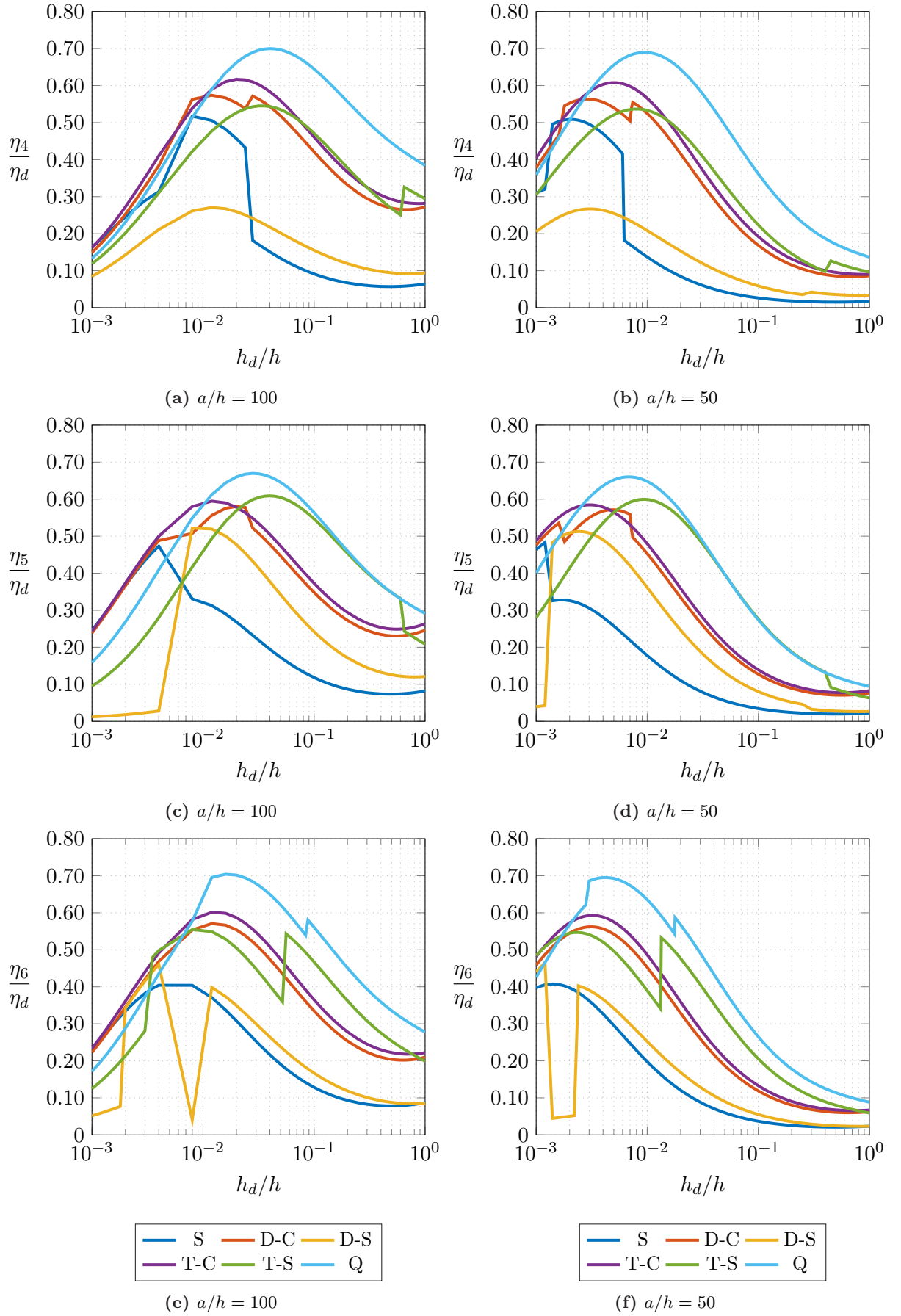
laminated composite plates with an aspect ratio $a/h = 50$, respectively. For both aspect ratios, it is possible to notice the same trends as for the first modal loss factor considering a thicker laminated composite plate - the modal loss factor increases with increasing damping layer thicknesses for thin damping layers, reaches a global maximum damping capacity, and the damping capacity sharply decreases. It is possible to identify a maximum attainable damping capacity for different stacking sequences, and Core configurations, together with the Quintuple (Q) stacking sequence, perform better than Single (S) and Skins stacking sequences. Exceptionally, the Triple-Skins (T-S) arrangement improves its damping performance for thicker damping layers. Once again, the Quintuple (Q) configuration presents the best overall damping capacity, which shows that this stacking sequence is consistently the best regarding maximum damping capacity. The main difference between the reference laminated square composite plate with an aspect ratio $a/h = 100$ and the thicker one with an aspect ratio $a/h = 50$ is that the maximum modal loss factor for a thicker laminate occurs for a lower specific damping layer thickness. Everything else is similar; it is clear that there is a maximum attainable damping capacity for any stacking sequence with interleaved damping layers, after which there is a decrease in damping capacity.

One presents the corresponding specific modal loss factors for the third mode of the reference and thicker laminated composite plates in Figures 7.12e and 7.12f, respectively. Unsurprisingly, one can identify the same trends as the second specific modal loss factor for both aspect ratios, but it is crucial to recall Figures 7.6b and 7.6c illustrating the same mode shapes, rotated by 90° , for both modes. Besides, there is a remarkable feature of the Double-Skins (D-S) configuration in these two modes (and the Single (S) configuration for the second mode) - i.e. the noticeable sharp jumps in the specific modal loss factor. This phenomenon occurs due to the already discussed switch between mode shapes during the parametric analyses, the second and third mode shapes in this case, and it is clear considering Figures 7.12c to 7.12f. As the parametric studies for the Double-Skins (D-S) configuration consider thicker damping layers, there is a switch between the mode shapes of the corresponding second and third modes, explaining the visible sharp jumps. It is worth mentioning that the unexpected rises of the second specific modal loss factor match the sudden drops of the third specific modal loss factor and vice-versa.

For the fourth, fifth and sixth normal modes, Figure 7.13 shows the same specific modal loss factors as a function of the dimensionless damping layer thickness for the stacking sequences considered in this work. Similarly, one presents the specific modal loss factors for reference laminated composite plates with an aspect ratio $a/h = 100$ and for thicker laminated composite plates with an aspect ratio $a/h = 50$ side-by-side. It is possible to distinguish the same trends presented for lower modes, namely for the second and third ones. More importantly, global maxima of modal loss factor for any configuration. Even so, thicker laminated composite plates present global maxima of damping capacity for thinner specific damping layers. Increasing the damping layer thickness beyond those maxima drops the damping capacity significantly, and more damping material means less damping capacity. Generally, and similarly to the first three modal loss factors, configurations with damping layers close to the core show better damping performances compared with stacking sequences involving damping layers interleaved close to the laminate skins. The Quintuple (Q) interleaved arrangement needs a large amount of damping material to be efficient but presents the best overall modal loss factors.

Earlier in this work, when dealing with isotropic beams and plates covered with viscoelastic damping patches, one demonstrated that modal strain, stress and strain energy density mode shapes are valuable tools to explain the damping performance of a given damping treatment. These tools are even more helpful for laminated composite plates due to the complexity of the problem, i.e. the large number of design variables under consideration.

**Figure 7.12** Influence of damping layer thickness and position on modal loss factors

**Figure 7.13** Influence of damping layer thickness and position on modal loss factors

Considering different interleaved configurations, Figures 7.14 to 7.19 present the through-the-thickness modal transverse shear, extensional and in-plane shear strain energy density mode shapes for the first normal mode. Exploring damping layers in print poses challenges, making the digital version of this document the ideal choice for readers seeking a detailed, zoomed-in view. One considers for these representations a thick laminated composite plate with an aspect ratio $a/h = 50$ and a thickness ratio $h_d/h = 4 \times 10^{-2}$, which is the configuration presenting the best overall damping capacity - see Figure 7.12b. In ascending order of modal loss factor: Single (S), Double-Skins (D-S), Double-Core (D-C) and Triple-Skins (T-S) *ex aequo*, Triple-Core (T-C) and Quintuple (Q) interleaved configuration. One considers the same section cuts $y = b/2$ and $x = a/2$, sections of maximum modal extensional and transverse shear strain energy densities.

Figures 7.14 and 7.15 exhibit the modal transverse shear strain energy density mode shapes $\phi_1^{U_{sz}^{xz}}$ and $\phi_1^{U_{sz}^{yz}}$, respectively, for all interleaved configurations and in ascending order of modal loss factor. These patterns give a reason for the damping capacity of each stacking sequence. The Quintuple (Q) interleaved configuration presents the highest levels of modal transverse shear strain energy density, and the Single (S) interleaved arrangement shows the lowest ones. One considers the same colourmap maxima to help the evaluation. Much more than the total modal transverse shear strain energy, through-the-thickness patterns are helpful representations that help to interpret the damping mechanism of laminated composites with interleaved damping layers. Figures 7.14 and 7.15 show that layers close to the laminate core present high levels of modal transverse shear strain energy density, which explains the superior performance of core stacking sequences, together with the Quintuple (Q) configuration. Given this, one may ask why the Single (S) interleaved damping layer does not present the best damping capacity since there is just one thick core layer. As previously discussed, when dealing with damping patches on isotropic beams and plates, there is a turning point after which thick soft layers tend to decouple the stiff layers and so their ability to impose shear. In the Single (S) stacking sequence, both halves of the laminated composite work as the base and constraining sub-laminates. One should note that, for thinner damping layers, this arrangement presents the best results for different vibration modes. However, there is the previously mentioned threshold of damping layer thickness, and damping capacity can even drop drastically with thicker layers - see Figure 7.12b.

To evaluate the ability of the stiff composite layers to impose transverse shear strain on the soft damping layers, Figures 7.16 and 7.17 present the through-the-thickness modal extensional strain energy densities $\phi_1^{U_n^{xx}}$ and $\phi_1^{U_n^{yy}}$, respectively. The results are straightforward for both modal strain energy densities, regardless of the stacking sequence - 0° oriented composite layers present the highest modal extensional strain energy density $\phi_1^{U_n^x}$, while the outer 90° oriented ones exhibit the highest levels of modal extensional strain energy density $\phi_1^{U_n^{yy}}$. As expected, the highest levels of $\phi_1^{U_n^{xx}}$ arise in the stiffer layers along the x -direction, the fibre direction of such laminas, while the highest levels of $\phi_1^{U_n^{yy}}$ appear in the outer stiffer layers along the y -direction, the fibre direction of such laminas. It is worth discussing the Single (S) configuration to highlight the importance of through-the-thickness modal extensional strain energy densities. Layers with high levels of modal extensional strain energy density are those closer to the laminate skins and far away from the neutral axis of the laminated composite. Thus, the ability to impose shear on the soft damping layer in the middle of the laminate is limited. On the other hand, the Quintuple (Q) configuration includes soft damping layers between stiff composite layers with high levels of modal extensional strain energy, promoting transverse shear strain and consequently damping.

Figures 7.18 and 7.19 show the through-the-thickness distribution of the modal in-plane shear strain energy density associated with the first normal mode. As previously discussed, the concept of a through-the-thickness representation for an in-plane variable is somewhat unusual. Besides, for the section cuts considered in this analysis, it is not possible to draw extensive conclusions since the modal in-plane shear strain energy density is null at such section cuts.

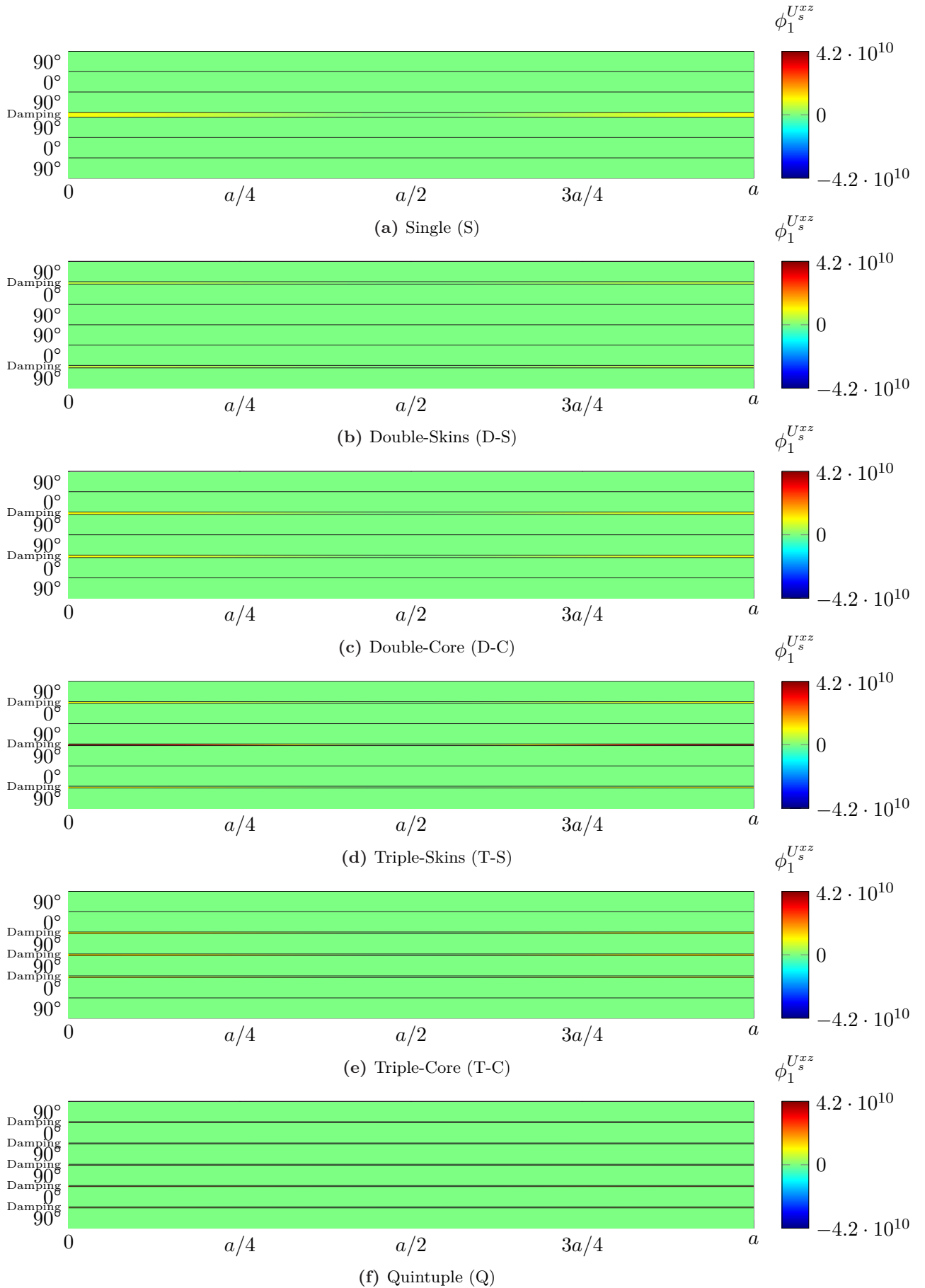


Figure 7.14 $\phi_1^{U_s^{xz}}$ - shear strain energy density mode shape - section cut at $y = b/2$

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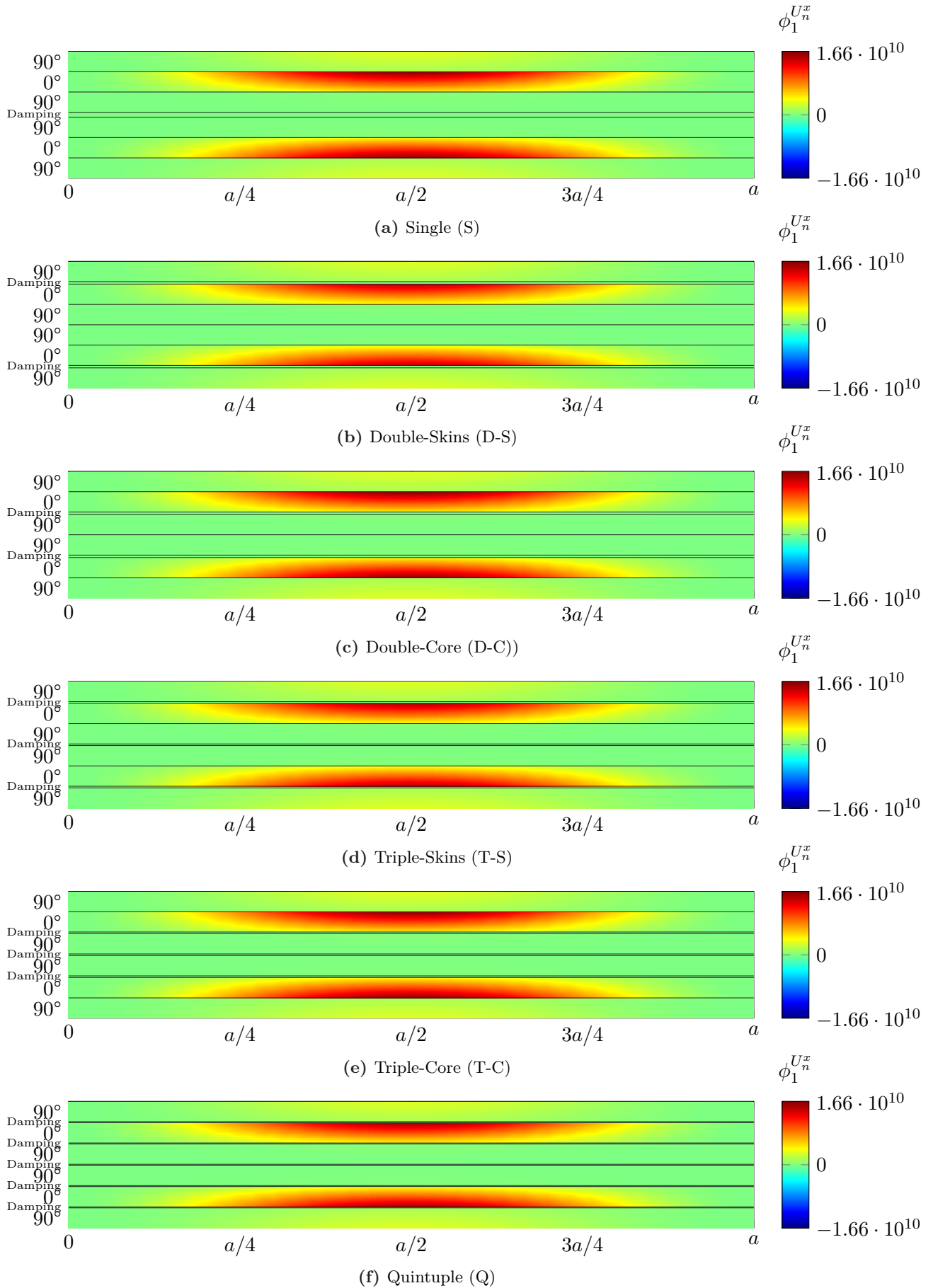


Figure 7.16 $\phi_1^{U_n^x}$ - extensional strain energy density mode shape - section cut at $y = b/2$

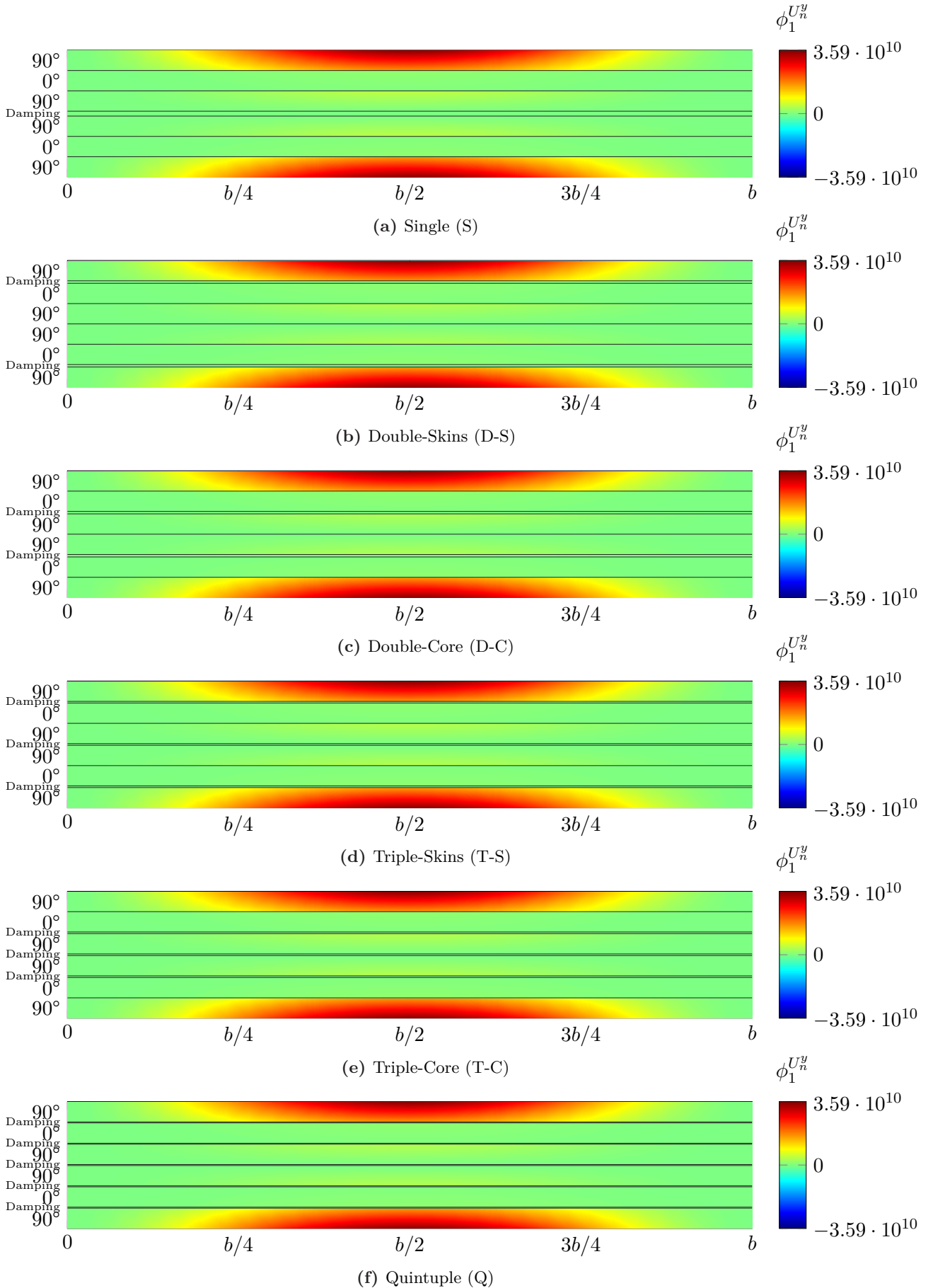


Figure 7.17 $\phi_1^{U_n^y}$ - extensional strain energy density mode shape - section cut at $x = a/2$

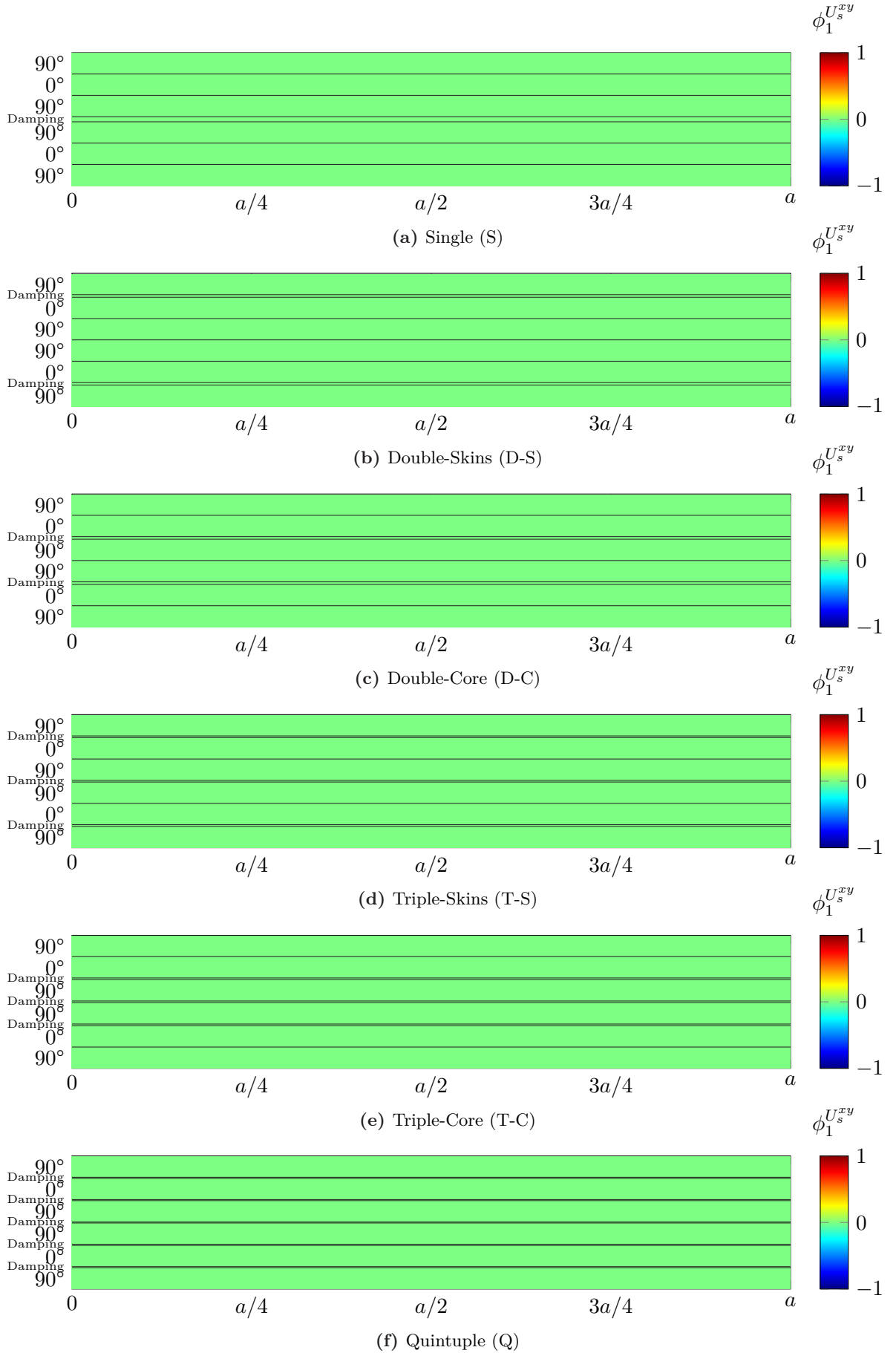


Figure 7.18 $\phi_1^{U_s^{xy}}$ - in-plane shear strain density mode shape - section cut at $y = b/2$

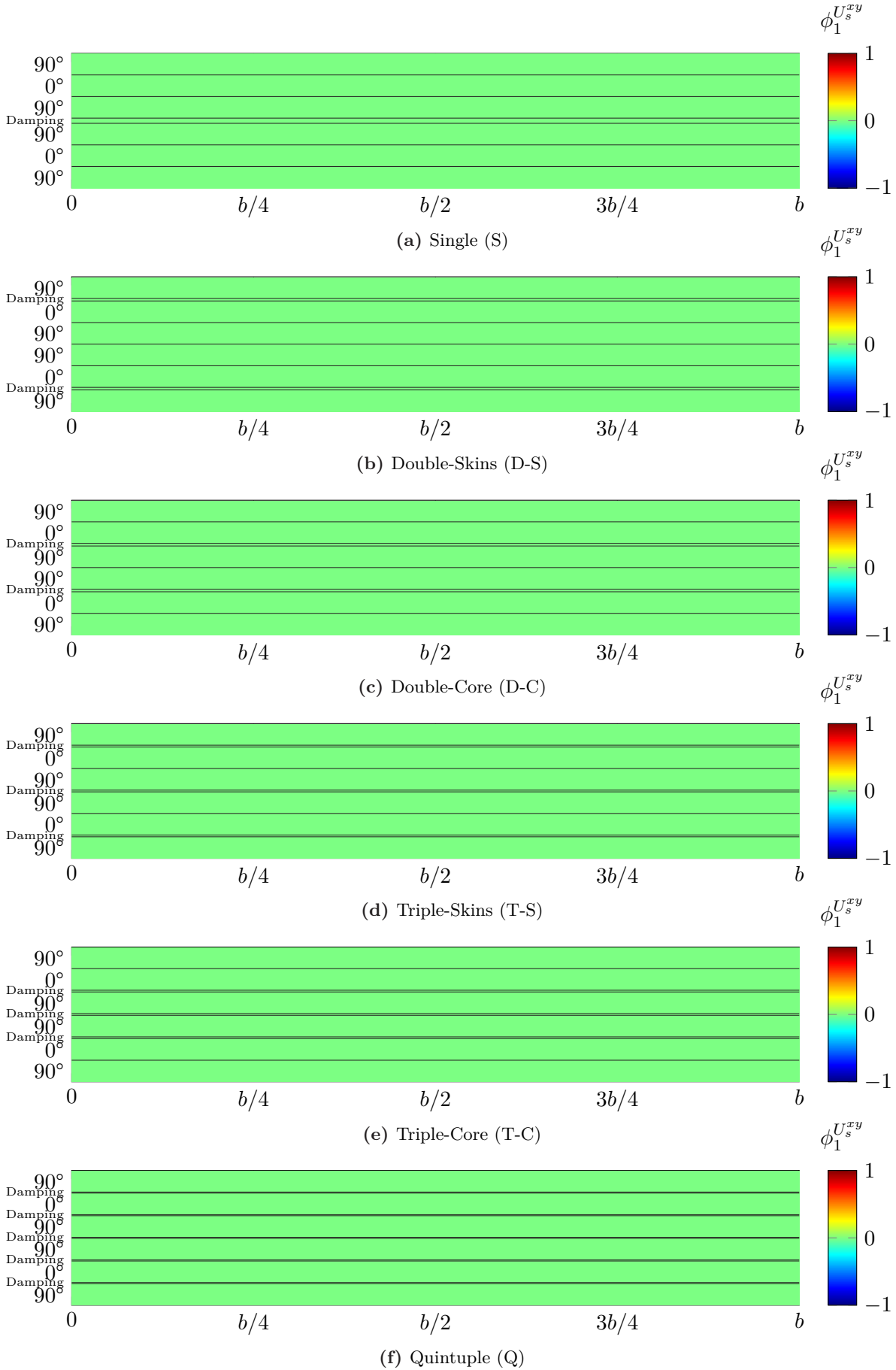


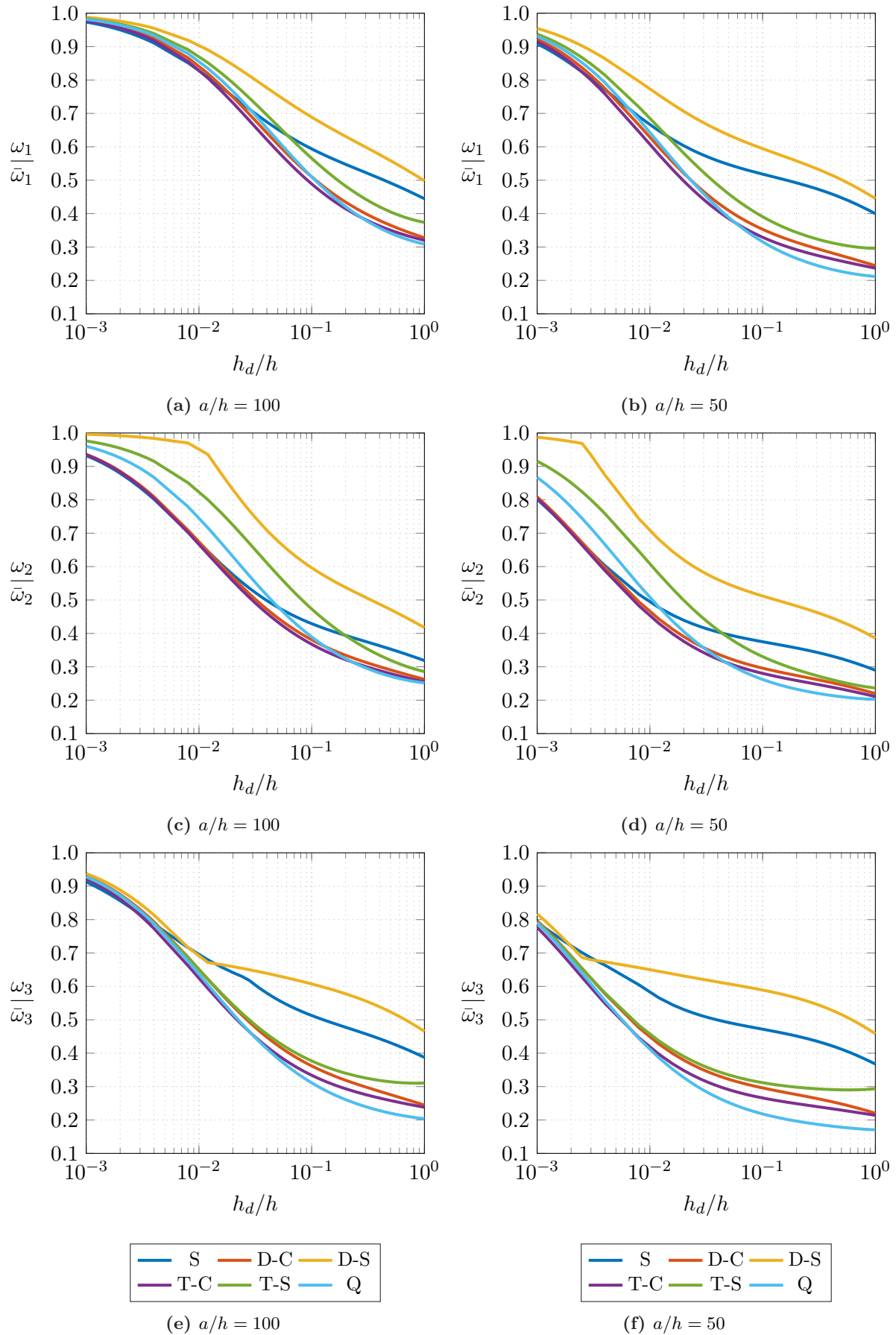
Figure 7.19 $\phi_1^{U_s^{xy}}$ - in-plane shear strain energy density mode shape - section cut at $x = a/2$

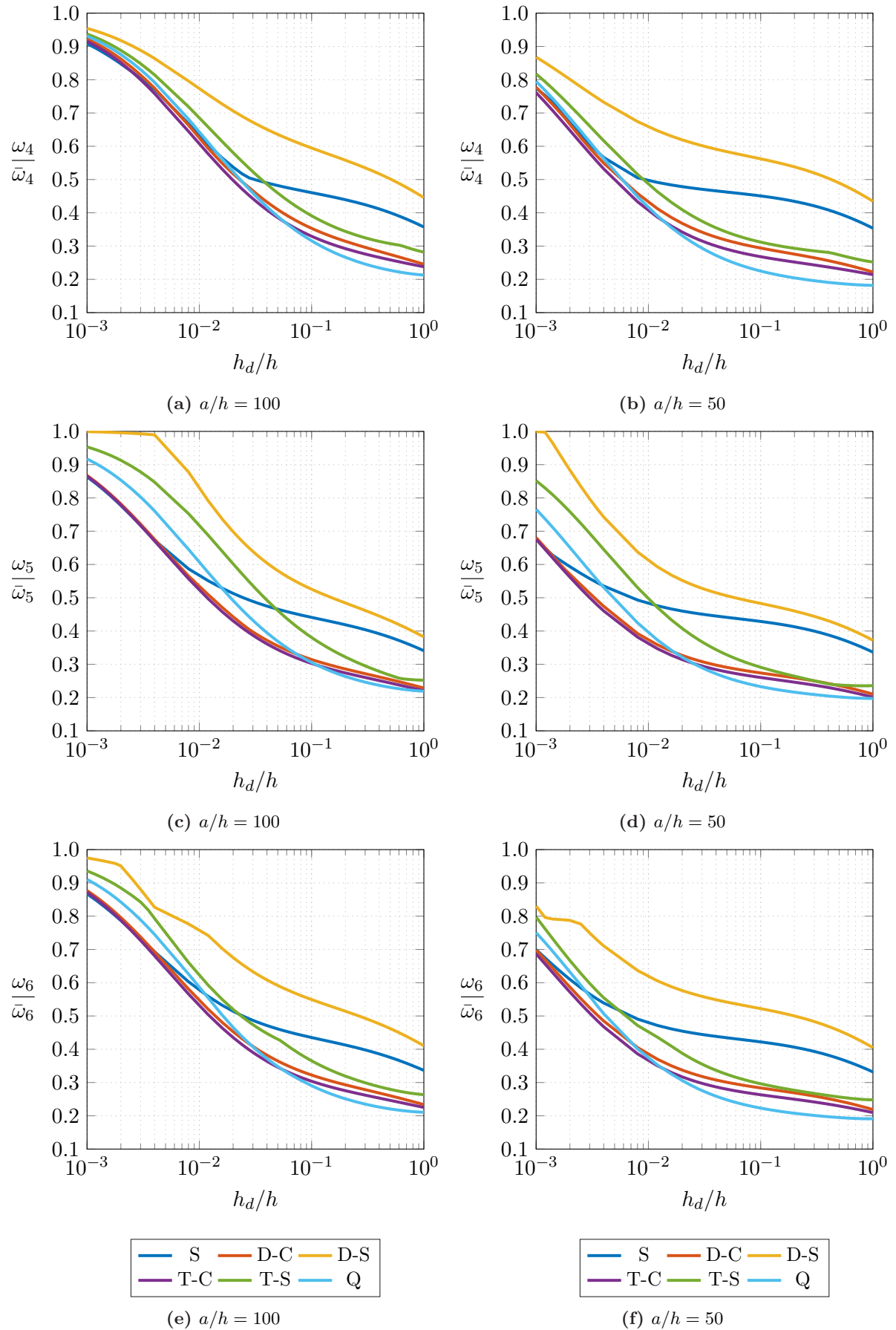
Design requirements for high damping capacity generally contrast with those for high stiffness, as extensively reported in subsection 7.1.1. Often, stiffness and damping are two conflicting needs, and the design of appropriate stacking sequences for laminated composites with interleaved damping layers is a mandatory step. To study the influence of interleaved soft damping layers on the natural frequencies of laminated composite plates, Figures 7.20 and 7.21 present the first six dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ as a function of the dimensionless damping layer thickness h_d/h for the reference and thick composite plates with aspect ratios of $a/h = 100$ and $a/h = 50$, side-by-side. Dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ represent the ratio between the natural frequencies of the laminated composite with interleaved damping layer(s) and the natural frequencies of the reference six-layer composite plate.

For the first three natural frequencies in Figure 7.20, natural frequencies decrease remarkably as the damping layer thickness increases. The Double-Skins (D-S) arrangement stands out as the stacking sequence with less natural frequency variation, i.e., for the same amount of soft damping material, interleaving two layers close to the skins of the laminate promotes a stiffness loss, even though at a smaller extent. Unsurprisingly, one discussed this configuration when analysing the modal loss factors of different stacking sequences in Figure 7.12, and the Double-Skins (D-S) arrangement presented one of the worst damping performances. As previously discussed, design requirements for high damping capacity and stiffness conflict in most cases.

Nonetheless, this rule of thumb is not linear and if one considers the thick laminated composite plate with an aspect ratio $a/h = 50$ and a thickness ratio $h_d/h = 4 \times 10^{-2}$, which is the configuration presenting the best overall damping capacity for the first natural mode studied before one can order the stacking sequences in ascending order of the first natural frequency as Triple-Core (T-C), Quintuple (Q), Double-Core (D-C), Triple-Skins (T-S), Single (S) and Double-Skins (D-S). By taking a closer look, one can conclude that it is not the reverse order of ascending first modal loss factor for the same thick laminated composite plate, which is Single (S), Double-Skins (D-S), Double-Core (D-C) and Triple-Skins (T-S) *ex aequo*, Triple-Core (T-C) and Quintuple (Q). This result proves that it is possible to optimise damping capacity and stiffness within certain limits.

Figure 7.21 presents the fourth, fifth and sixth dimensionless natural frequencies, which, as expected, drop significantly as the damping layer thicknesses increase. As expected, the Double-Skins (D-S) and Single (S) configurations stand out from the remaining ones, presenting lower natural frequency variation since those are the stacking sequences promoting less damping capacity for these vibration modes. On the whole, there are no significant differences between reference and thick laminates, but the natural frequencies of thicker laminates tend to drop more steeply for higher damping layer thicknesses.

**Figure 7.20** Influence of damping layer thickness and position on natural frequencies

**Figure 7.21** Influence of damping layer thickness and position on natural frequencies

7.2.1.2 | Influence of Ply Angle

In the previous sections, one considered the six symmetric stacking sequences presented in Figures 7.5a to 7.5f. These arrangements cover the range of possible symmetric configurations for a six-layer reference laminate, including viscoelastic layers within the stacking sequence. Stiff composite plies adjacent to soft damping layers, compliant layers, present variable orientation angles represented by tags θ and $90^\circ - \theta$. In this section, one studies the influence of the ply-angle (θ) on the damping capacity and stiffness of the laminate. Figures 7.22 to 7.33 show, successively, the first six modal loss factors as a function of the dimensionless damping layer thickness h_d/h and ply angle θ for the six stacking sequences presented in this chapter and considering the reference and thick composite base plates with an aspect ratio $a/h = 100$ and $a/h = 50$, side-by-side.

For the first modal loss factor of the reference and thick simply supported composite plates, Figures 7.22 and 7.23 show that damping capacity increases markedly with increasing thickness, a wider variation than with varying the ply angle orientation. Incidentally, for the configurations with damping layers close to the skins (D-S and T-S), just as in the Quintuple (Q) arrangement, the ply angle θ has practically no influence on the first modal loss factor, even though it slowly decreases from 0° to 45° , after which symmetrically increases until 90° . For Double-Core (D-C) and Triple-Core (T-C) stacking sequences, 0° oriented ply angles present better damping performances. As previously concluded, the Quintuple (Q) arrangement exhibits the highest modal loss factor for the reference and thick simply supported laminated composite plates.

Similar conclusions apply to higher normal modes. Previously, when analysing Figures 7.12 and 7.13, one drew attention to the fact that there is a maximum damping capacity for reference and thick composite plates, after which the modal loss factor drops significantly. For the second modal loss factor, for example, Figures 7.24 and 7.25 show the same behaviour for the reference and thick composite plates but show that angle ply orientation plays a relevant role. For the stacking sequences promoting higher damping capacities, it is possible to distinguish $\theta = 0^\circ$ as the best orientation angle. Increasing the ply angle θ for the same damping layer thickness decreases the modal loss factor.

Moreover, one can notice sharp jumps in the specific modal loss factors for higher normal modes. This phenomenon occurs due to the already discussed switch between mode shapes during the parametric analyses. Primarily, this phenomenon arises for the Single (S) and Double-Skins (D-S) stacking sequences, where a thick and soft damping layer decouples the stiff layers of the laminate. In general, the ply angle has some influence for higher vibration modes, but the damping layer thickness and stacking sequence command the damping capacity of the laminate.

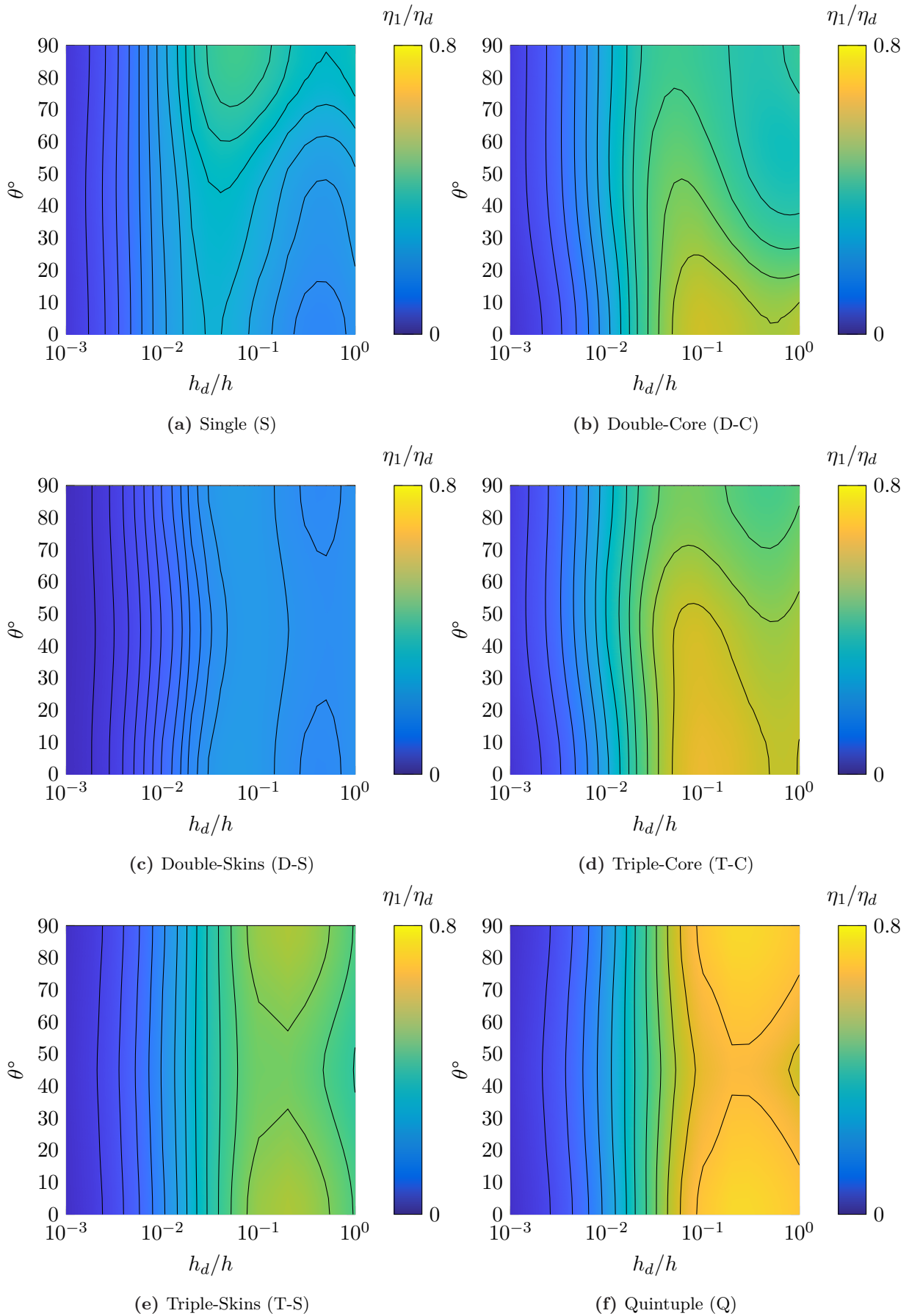


Figure 7.22 Influence of damping layer thickness and ply angle on the 1st modal loss factor, $a/h = 100$

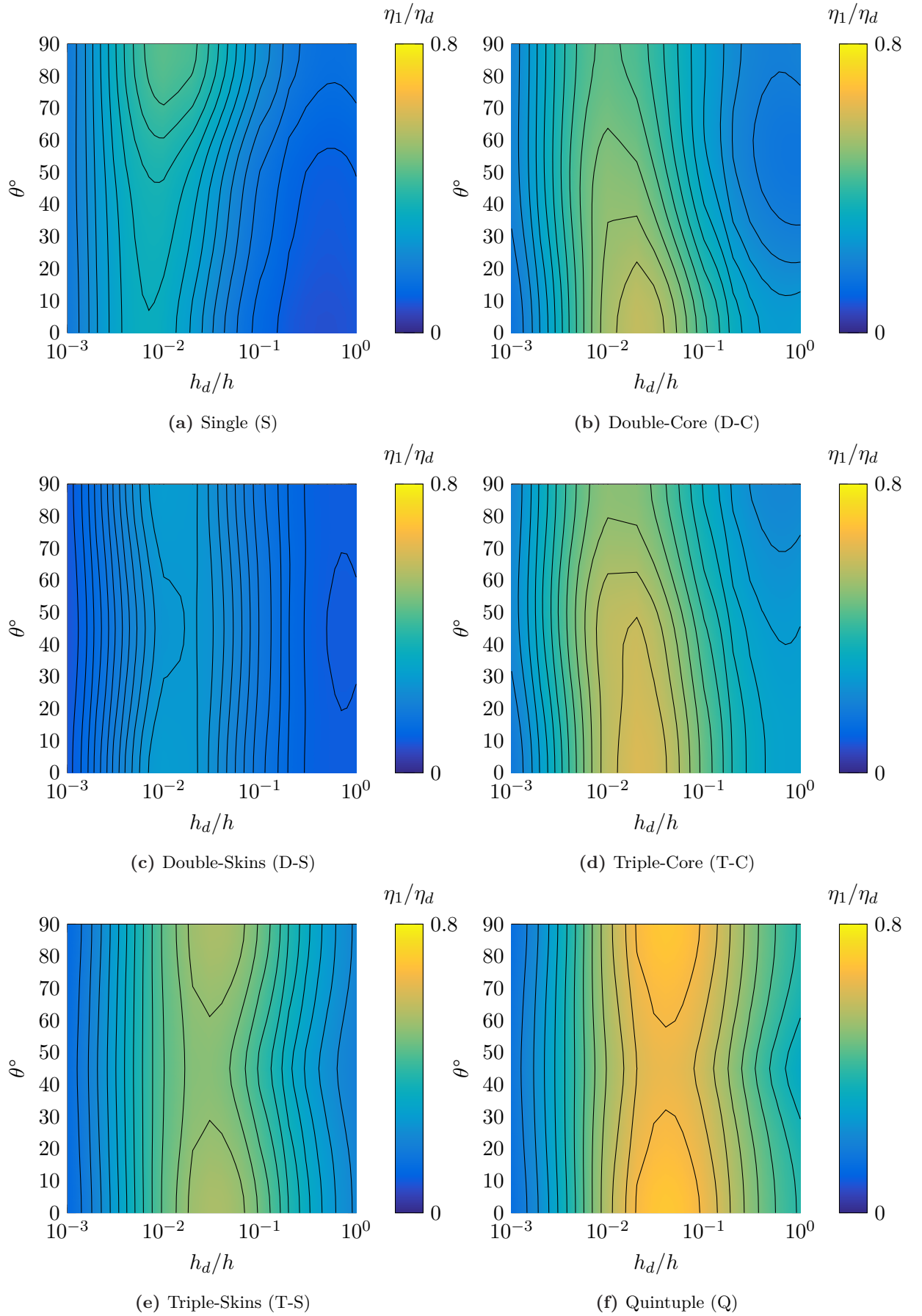


Figure 7.23 Influence of damping layer thickness and ply angle on the 1st modal loss factor, $a/h = 50$

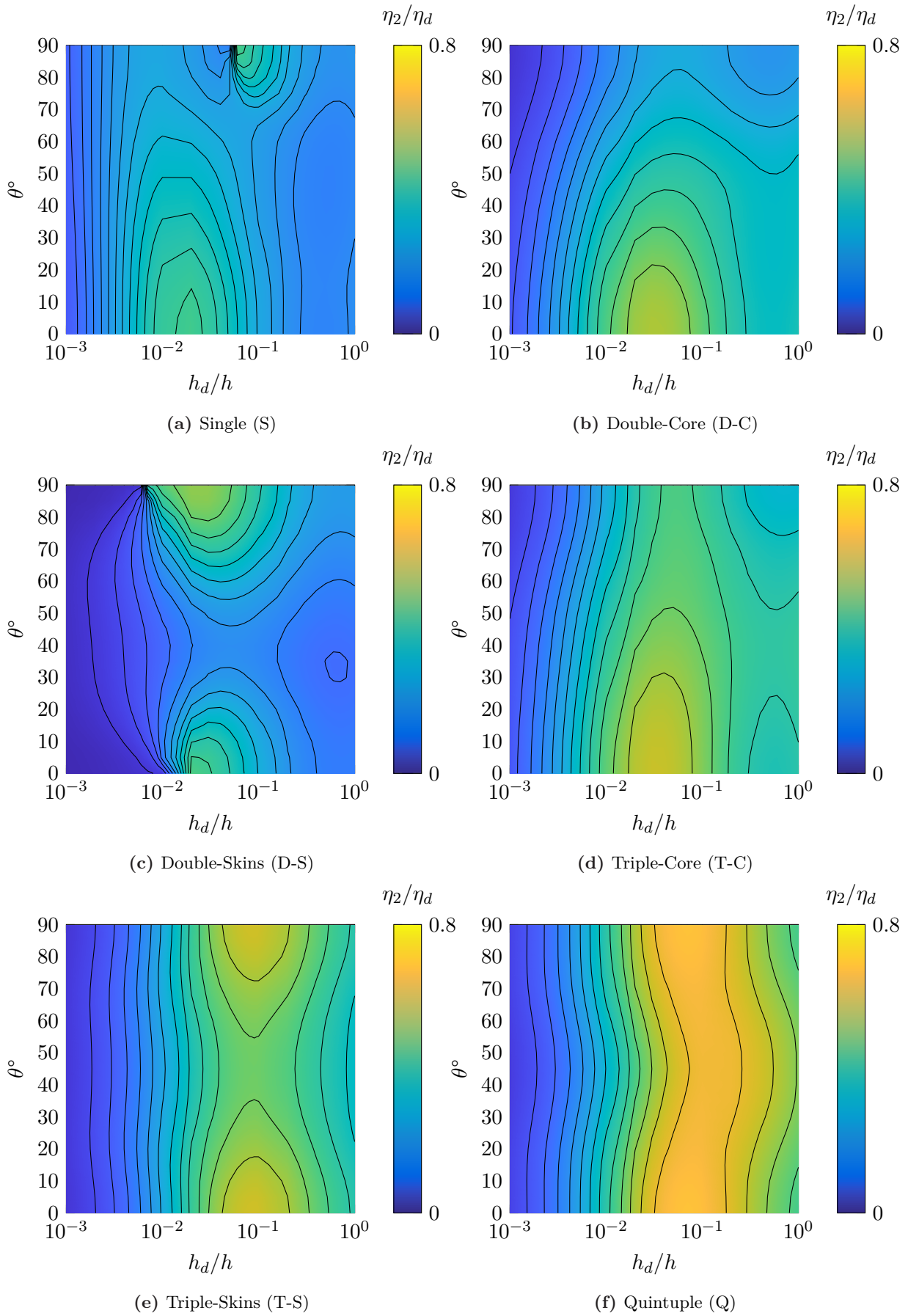


Figure 7.24 Influence of damping layer thickness and ply angle on the 2nd modal loss factor, $a/h = 100$

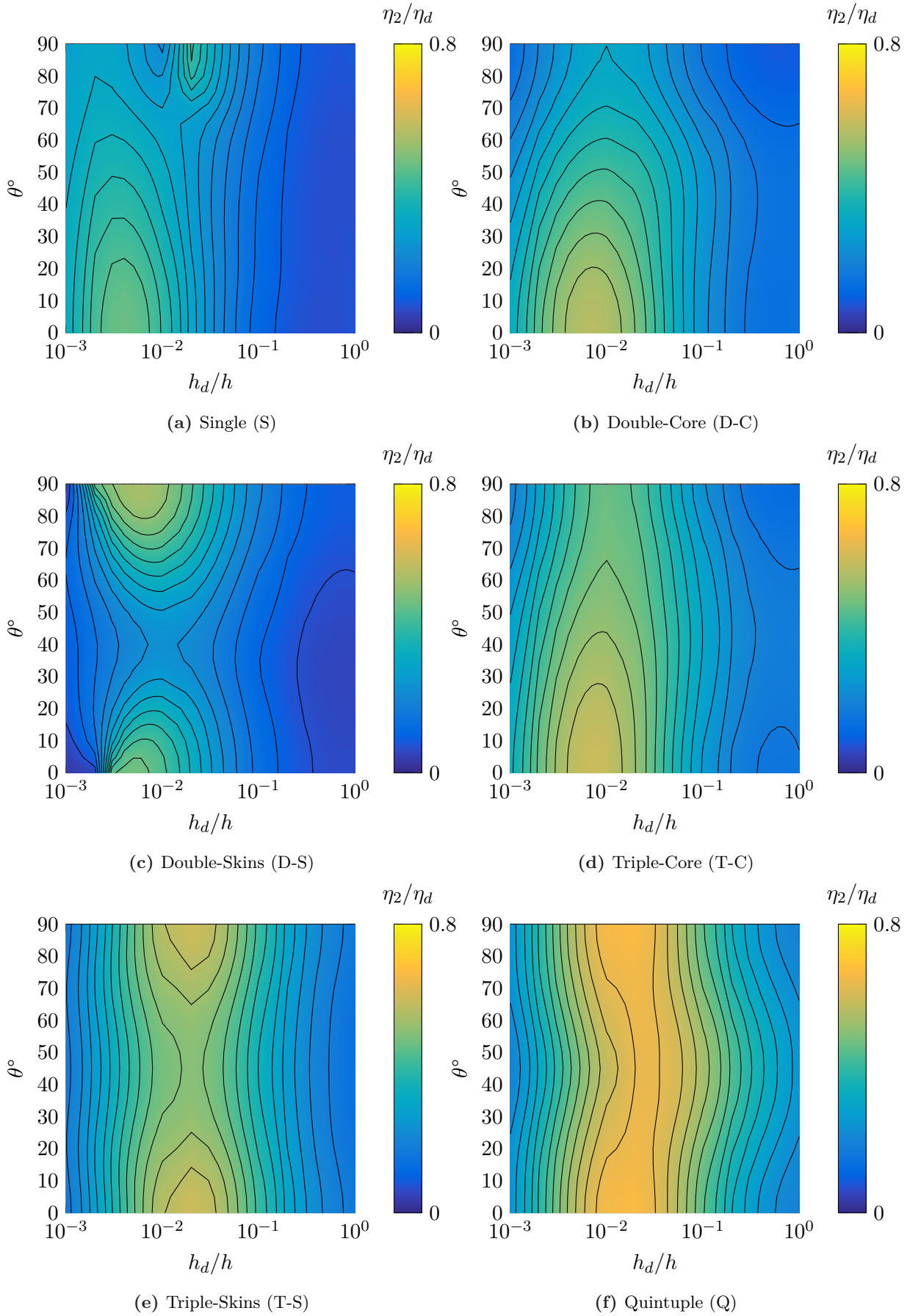


Figure 7.25 Influence of damping layer thickness and ply angle on the 2nd modal loss factor, $a/h = 50$

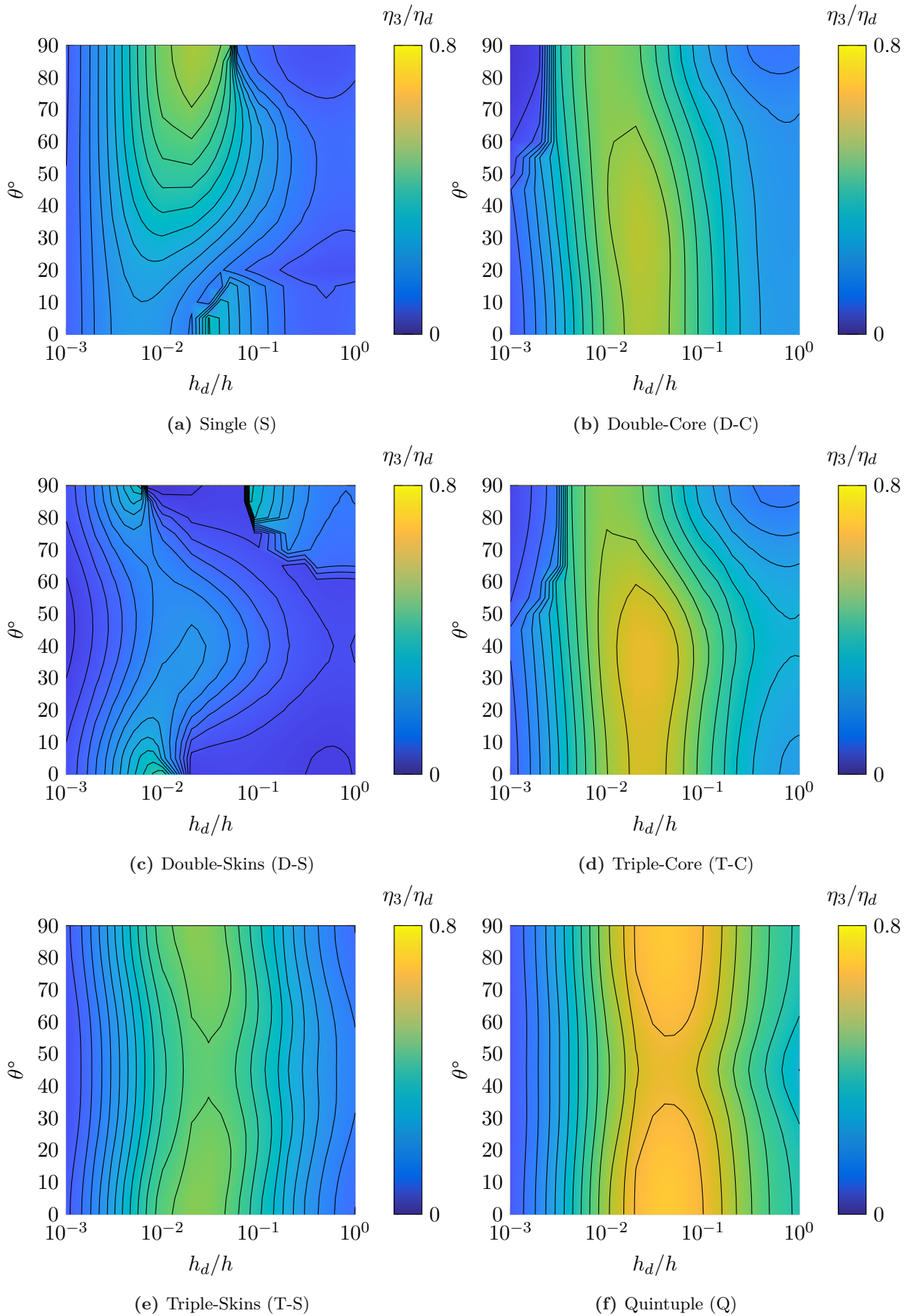


Figure 7.26 Influence of damping layer thickness and ply angle on the 3rd modal loss factor, $a/h = 100$

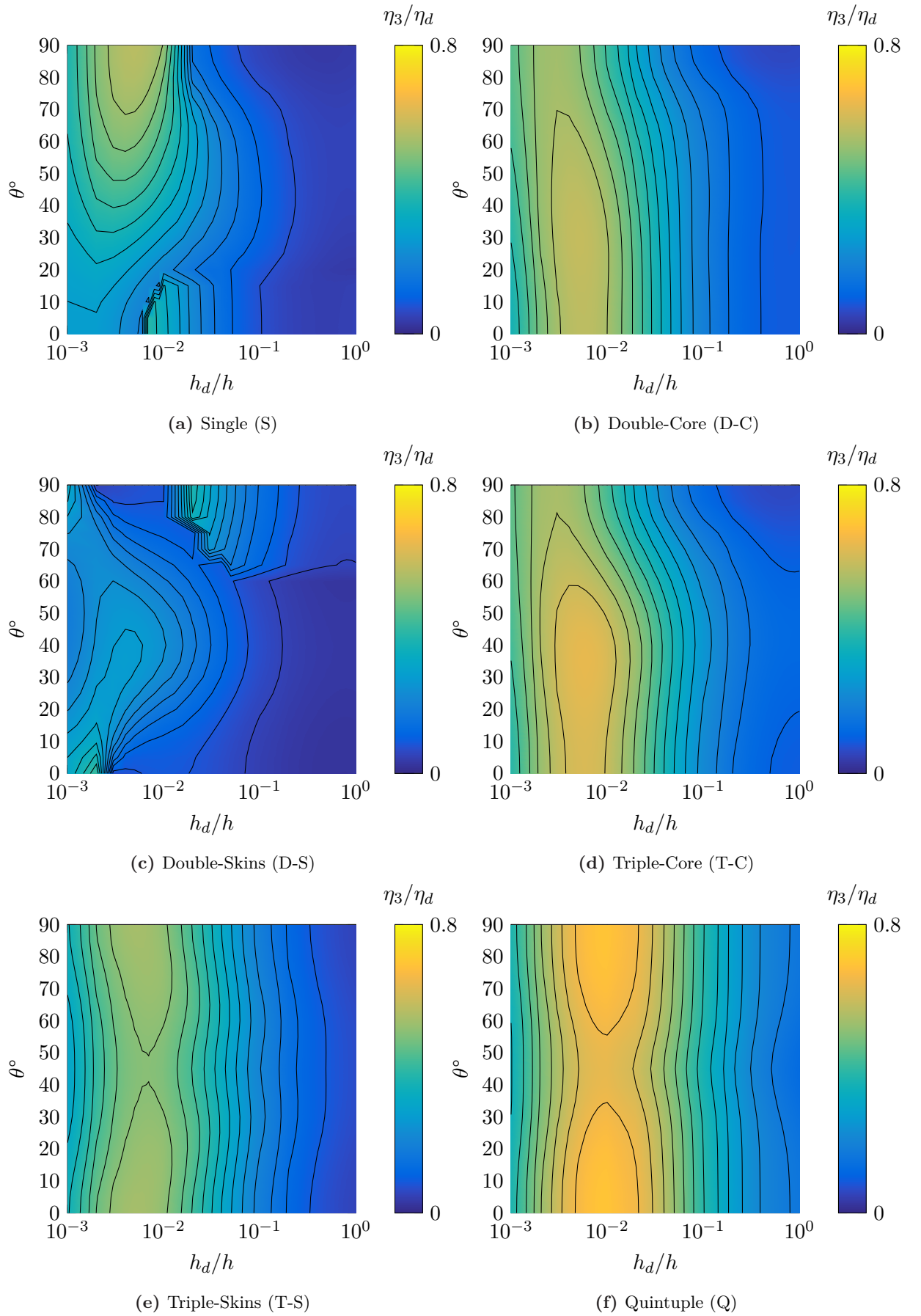


Figure 7.27 Influence of damping layer thickness and ply angle on the 3rd modal loss factor, $a/h = 50$

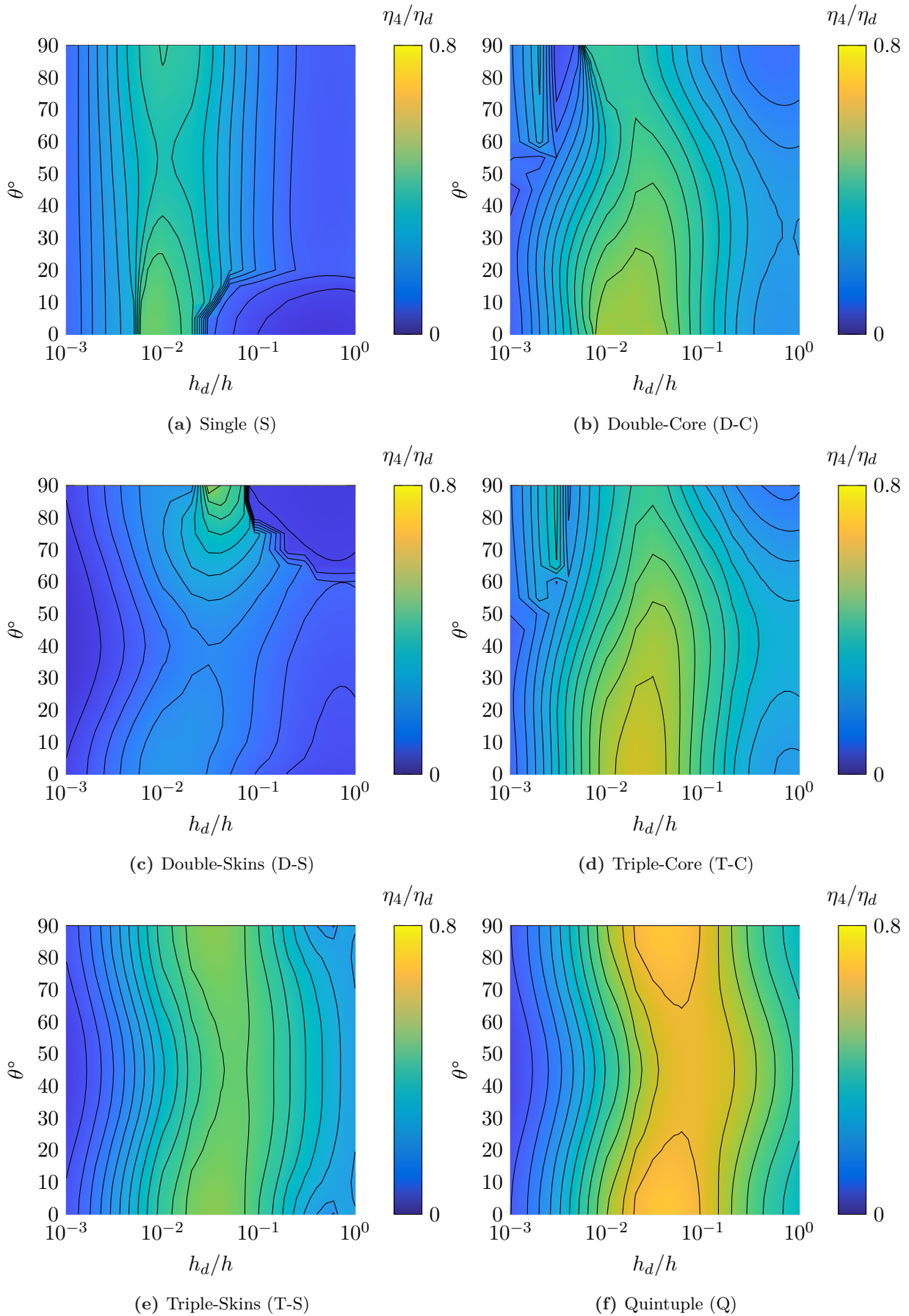


Figure 7.28 Influence of damping layer thickness and ply angle on the 4th modal loss factor, $a/h = 100$

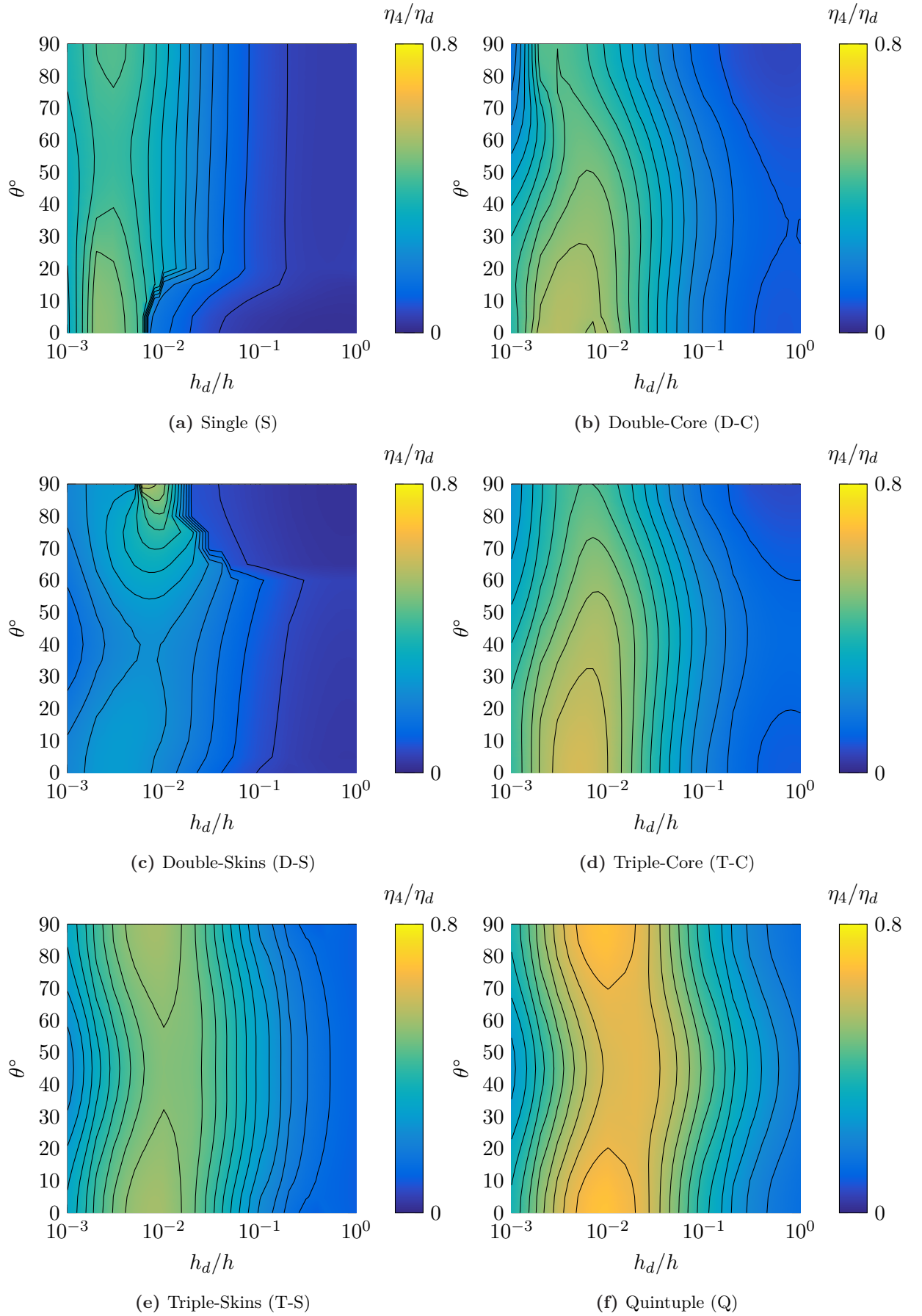


Figure 7.29 Influence of damping layer thickness and ply angle on the 4th modal loss factor, $a/h = 50$

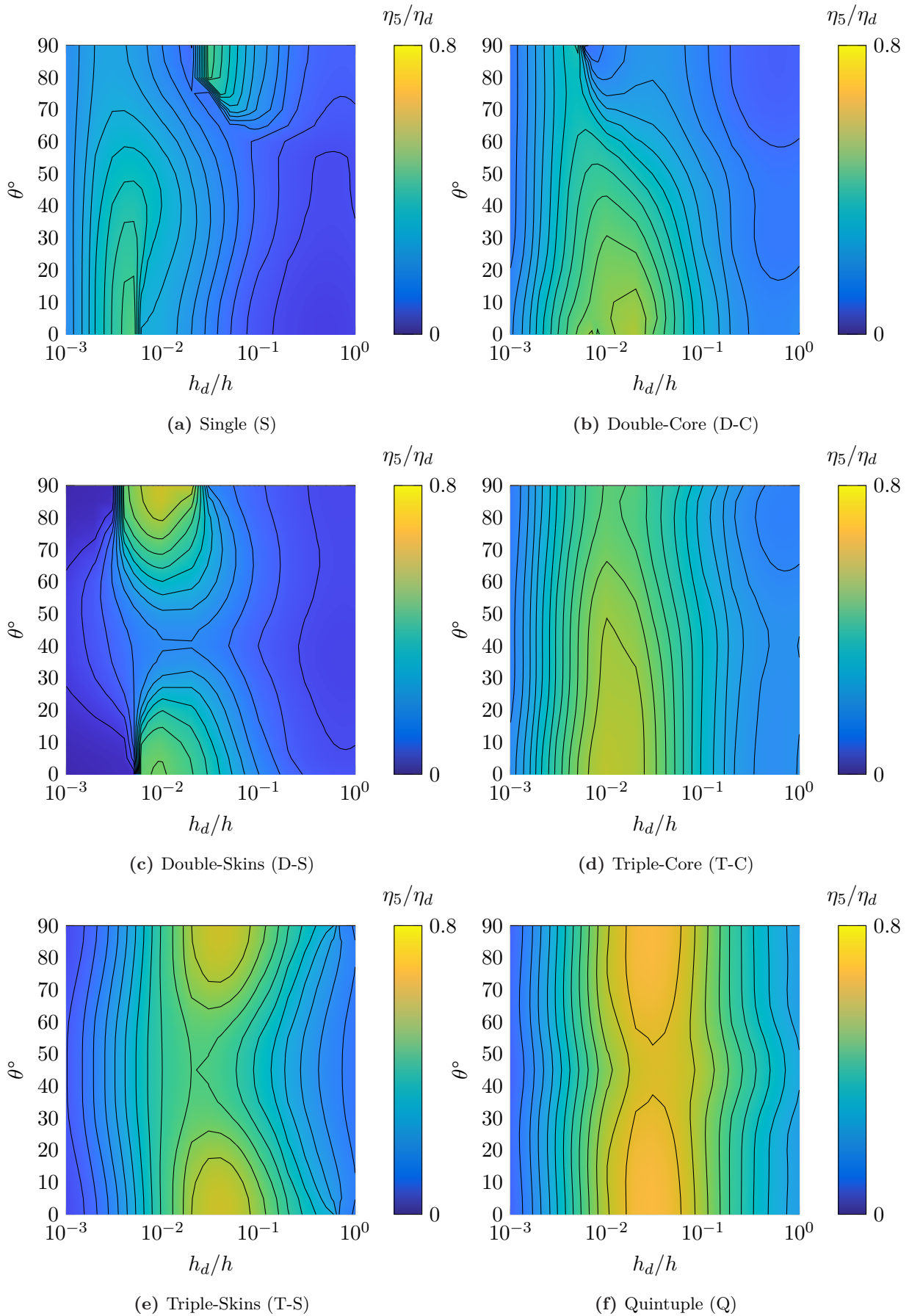


Figure 7.30 Influence of damping layer thickness and ply angle on the 5th modal loss factor, $a/h = 100$

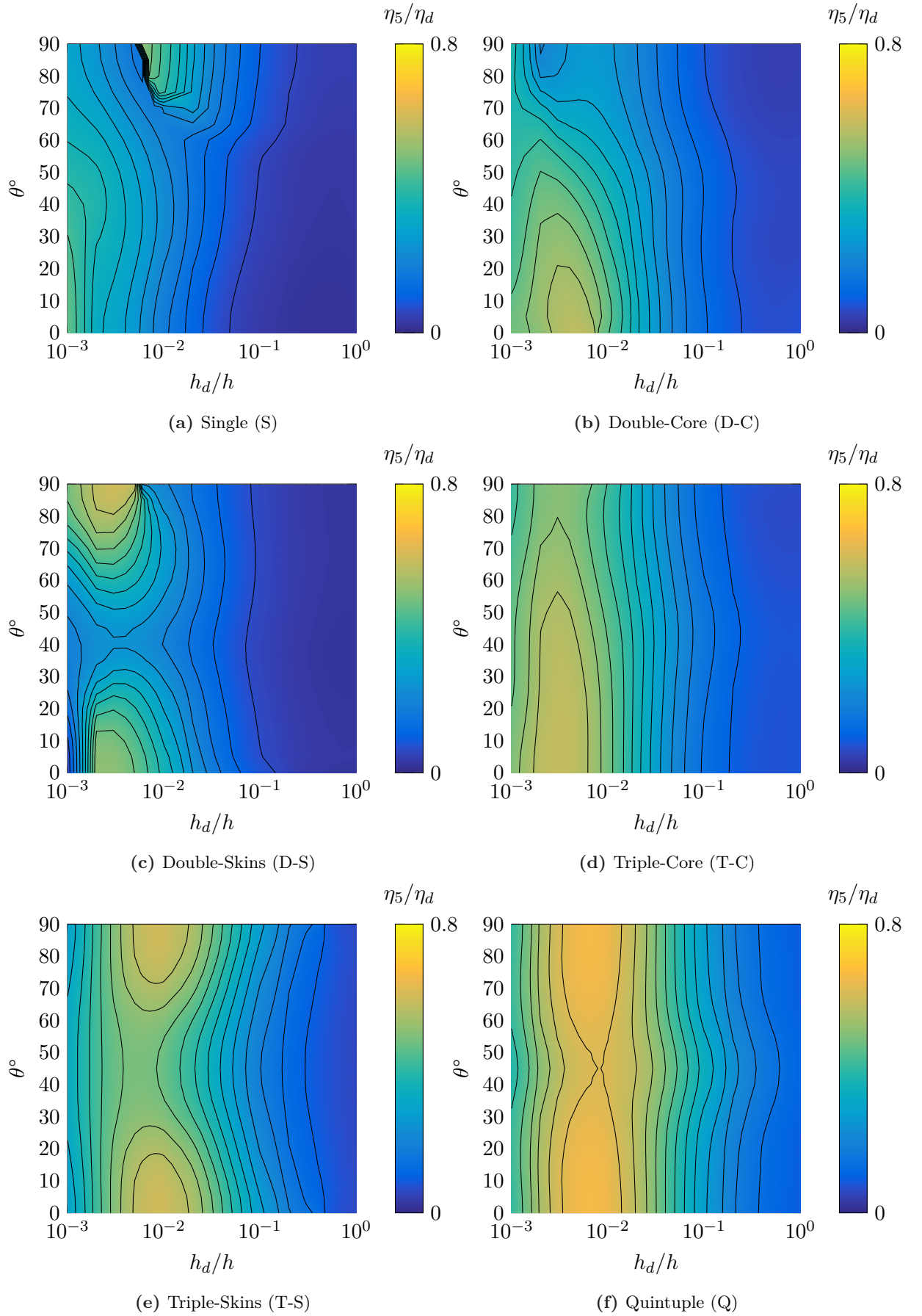


Figure 7.31 Influence of damping layer thickness and ply angle on the 5th modal loss factor, $a/h = 50$

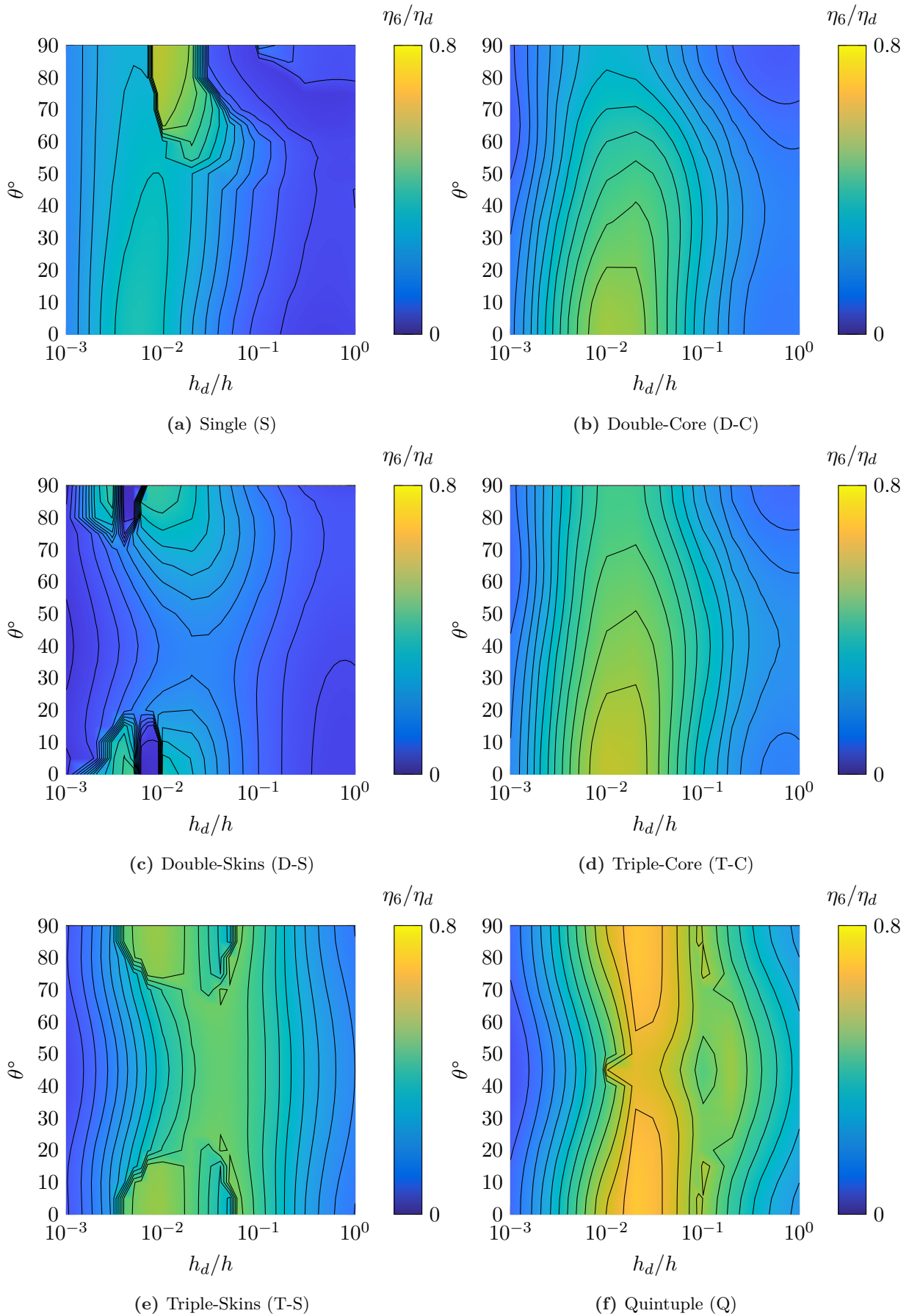


Figure 7.32 Influence of damping layer thickness and ply angle on the 6th modal loss factor, $a/h = 100$

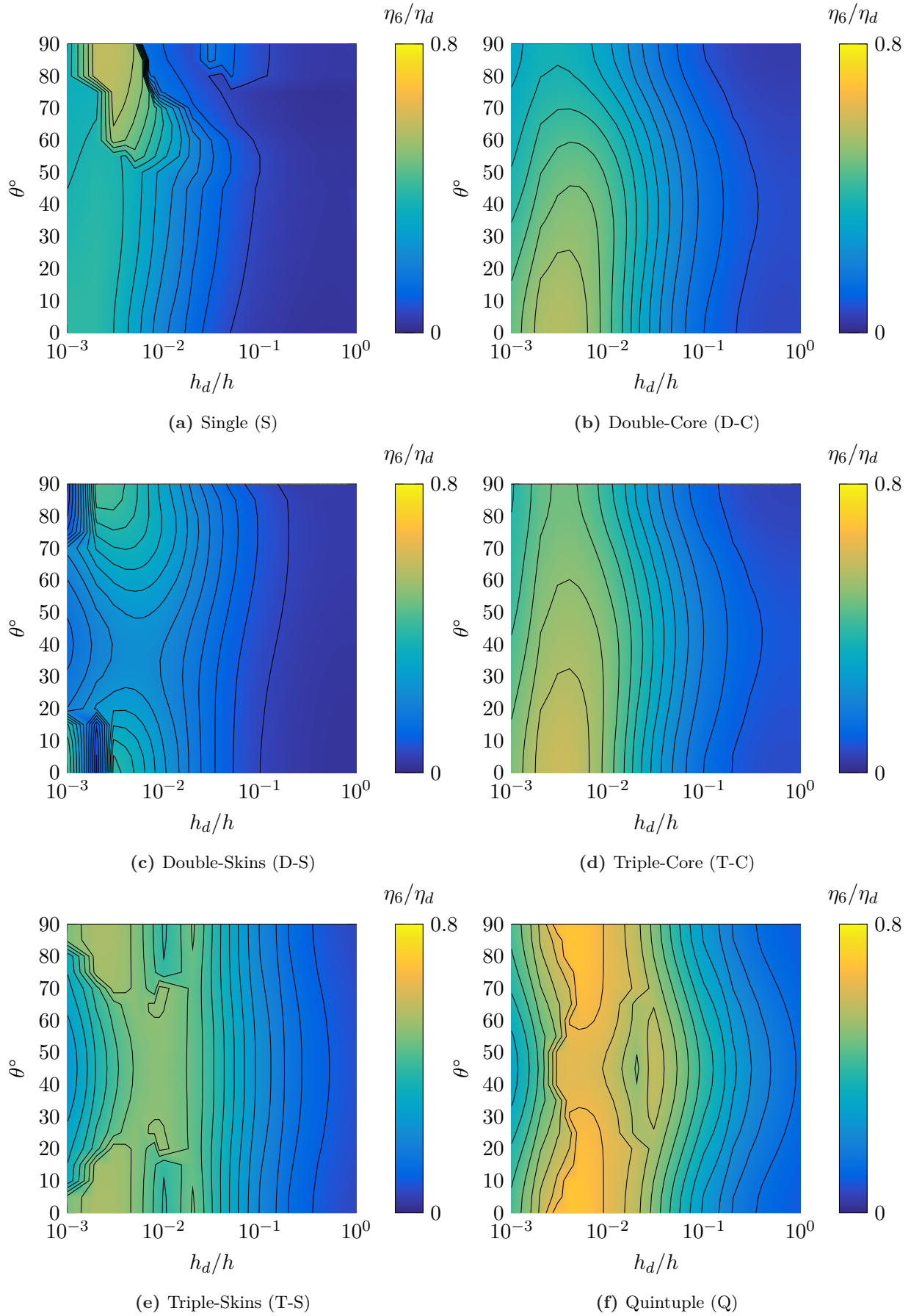


Figure 7.33 Influence of damping layer thickness and ply angle on the 6th modal loss factor, $a/h = 50$

Earlier in this work, one mentioned that design requirements for high damping capacity and stiffness often conflict. To study the influence of the ply angle on the natural frequencies of laminated composite plates with interleaved soft damping layers, Figures 7.34 to 7.45 present the first six dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ as a function of the ply angle θ and the dimensionless damping layer thickness h_d/h for the reference and thick composite plates with aspect ratios of $a/h = 100$ and $a/h = 50$, side-by-side. Dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ represent the ratio between the natural frequencies of the laminated composite with interleaved damping layer(s) and the natural frequencies of the reference six-layer composite plate.

For the first mode, Figures 7.34 and 7.35, the natural frequency decreases significantly with increasing damping layer thickness, as expected. The ply angle influences some stacking sequences, particularly for the reference plate and thin damping layers, and the first dimensionless natural frequency factor increases as the ply angle increases from 0° to 45° .

Figures 7.36 and 7.37 exhibit the second dimensionless natural frequency for reference and thick composite plates, respectively, and show that the ply angle θ has the exact opposite effect compared to the first dimensionless natural frequency, i.e., for some stacking sequences, the natural frequency decreases as the ply angle increases from 0° to 45° , specifically for thin damping layers. These results prove that, even though the natural frequency decreases with increasing damping layer thickness, there is room for optimisation.

For higher normal modes, ply angle orientation has limited influence on the natural frequency of reference and thick simply supported composite plates with interleaved damping layers. Specific damping layer thickness is the main parameter influencing natural frequency, which strongly decreases with increasing damping layer thickness.

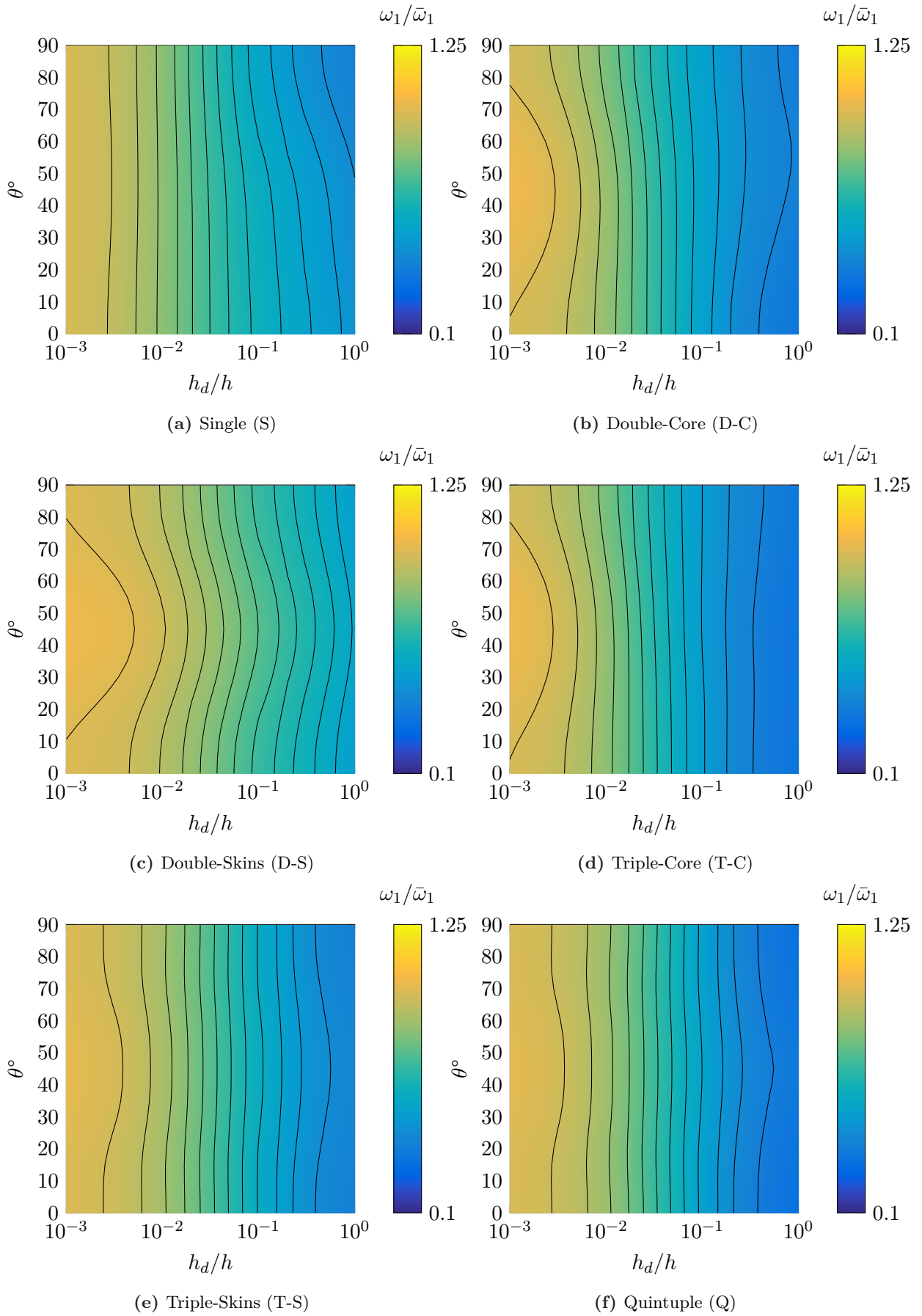


Figure 7.34 Influence of damping layer thickness and ply angle on the 1st natural frequency, $a/h = 100$

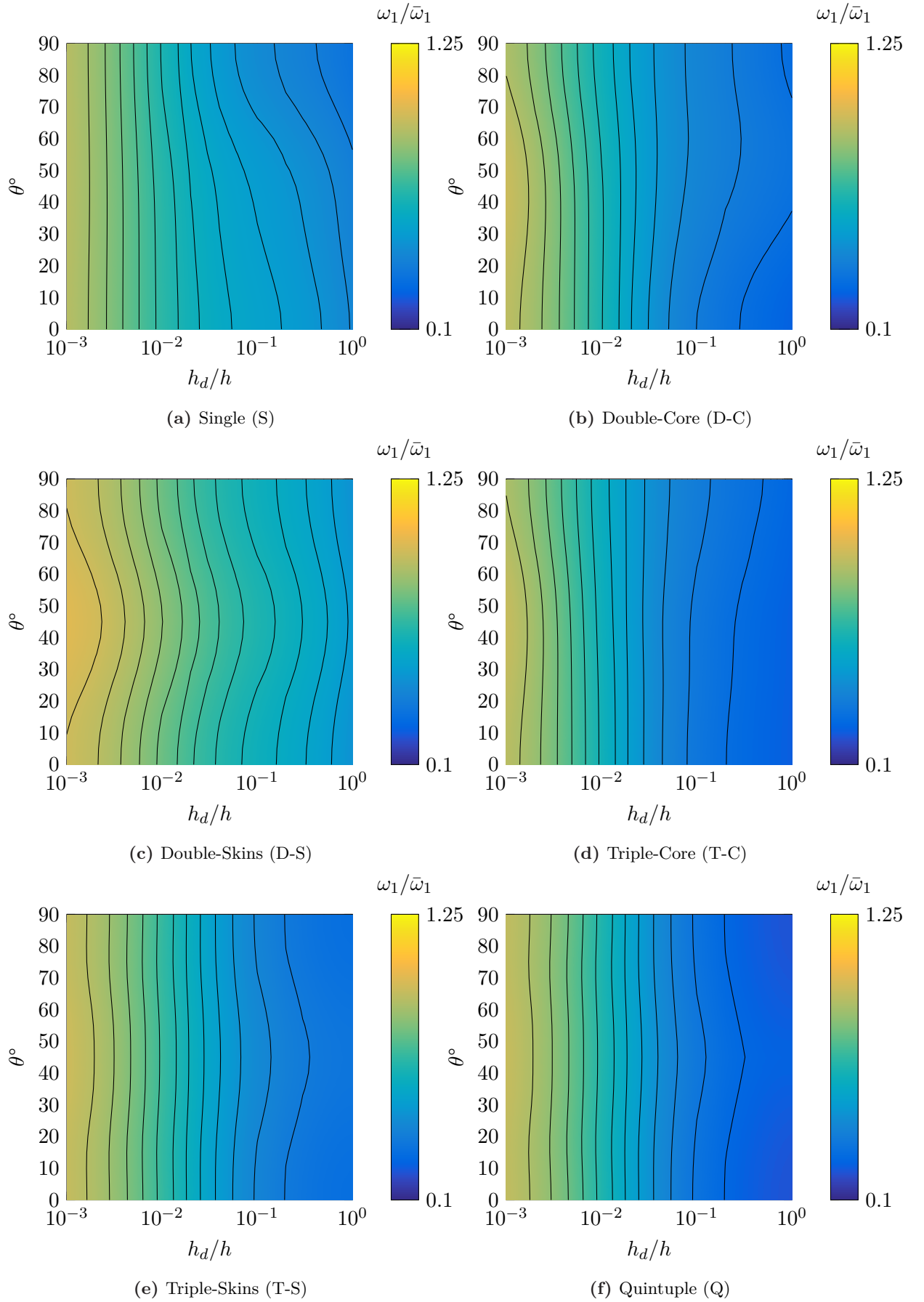


Figure 7.35 Influence of damping layer thickness and ply angle on the 1st natural frequency, $a/h = 50$

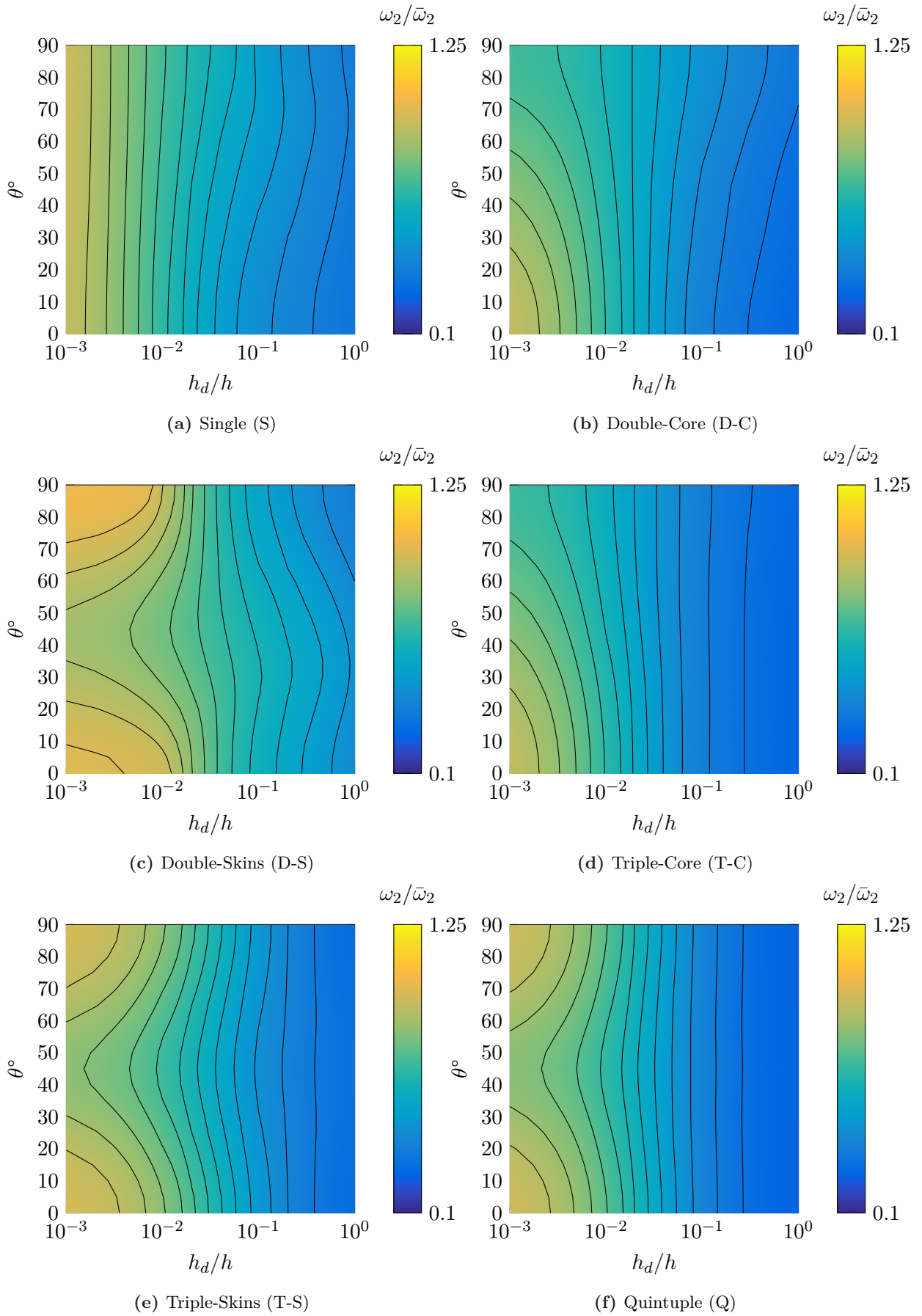


Figure 7.36 Influence of damping layer thickness and ply angle on the 2nd natural frequency, $a/h = 100$

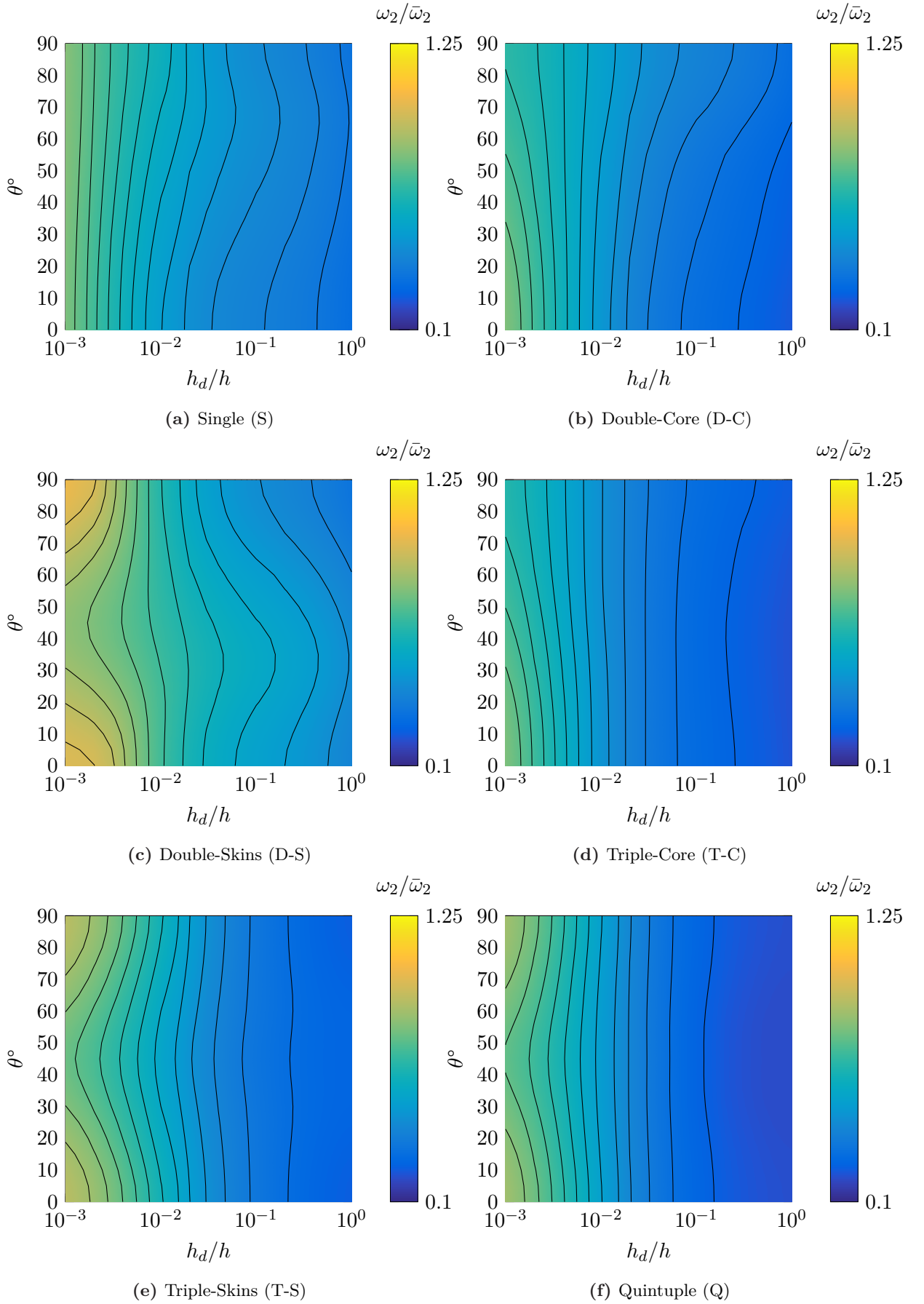


Figure 7.37 Influence of damping layer thickness and ply angle on the 2nd natural frequency, $a/h = 50$

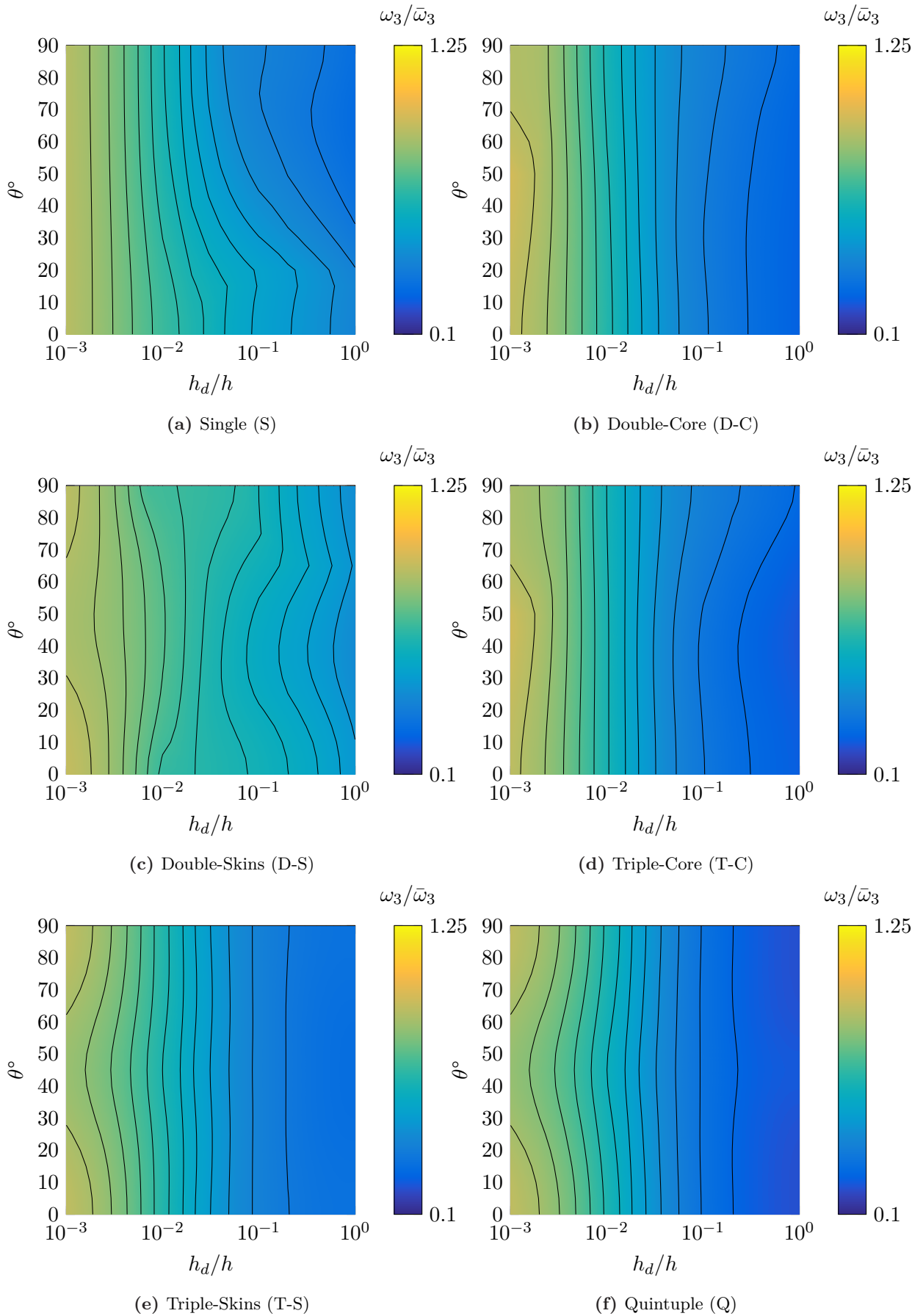


Figure 7.38 Influence of damping layer thickness and ply angle on the 3rd natural frequency, $a/h = 100$

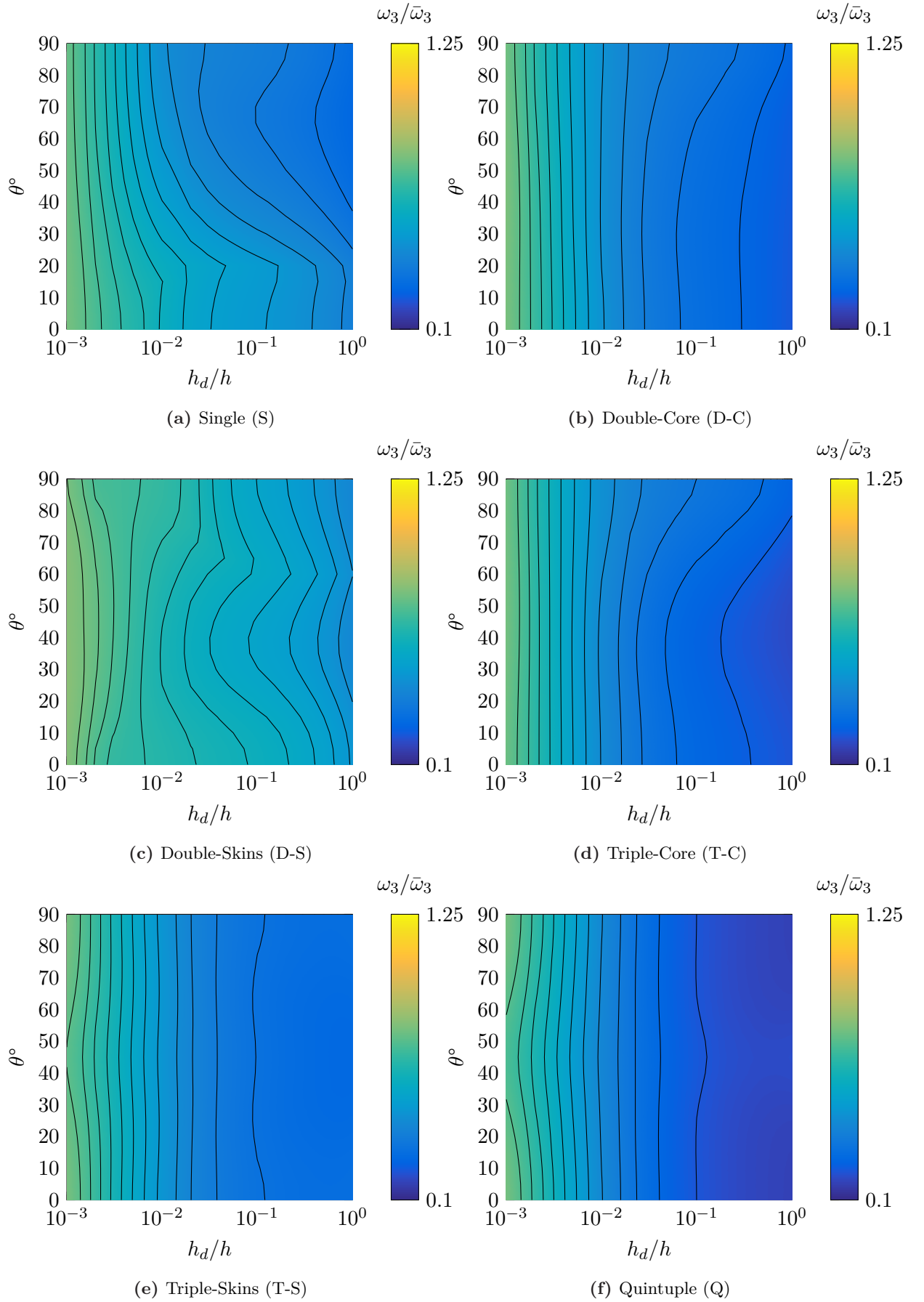


Figure 7.39 Influence of damping layer thickness and ply angle on the 3rd natural frequency, $a/h = 50$

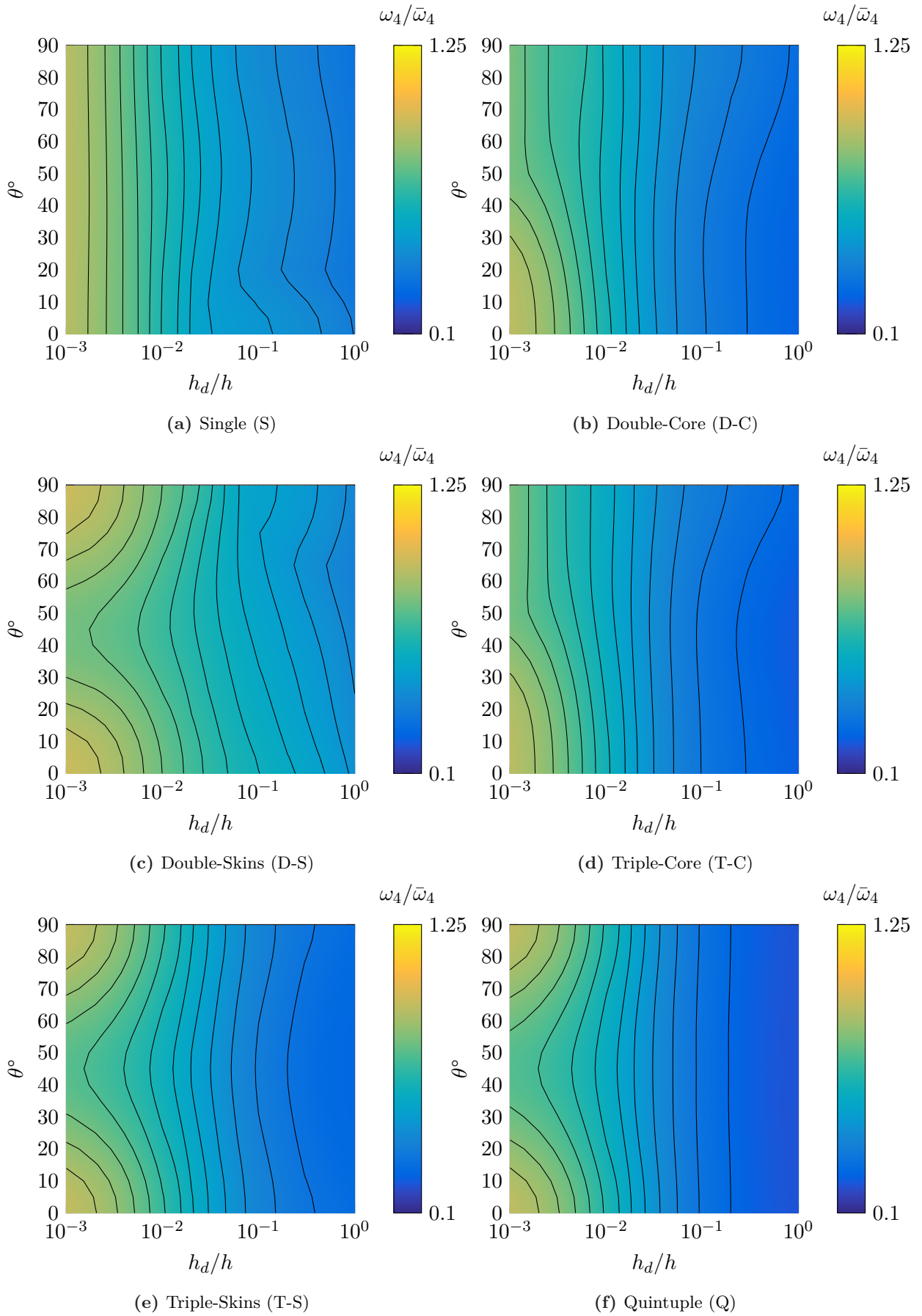


Figure 7.40 Influence of damping layer thickness and ply angle on the 4th natural frequency, $a/h = 100$

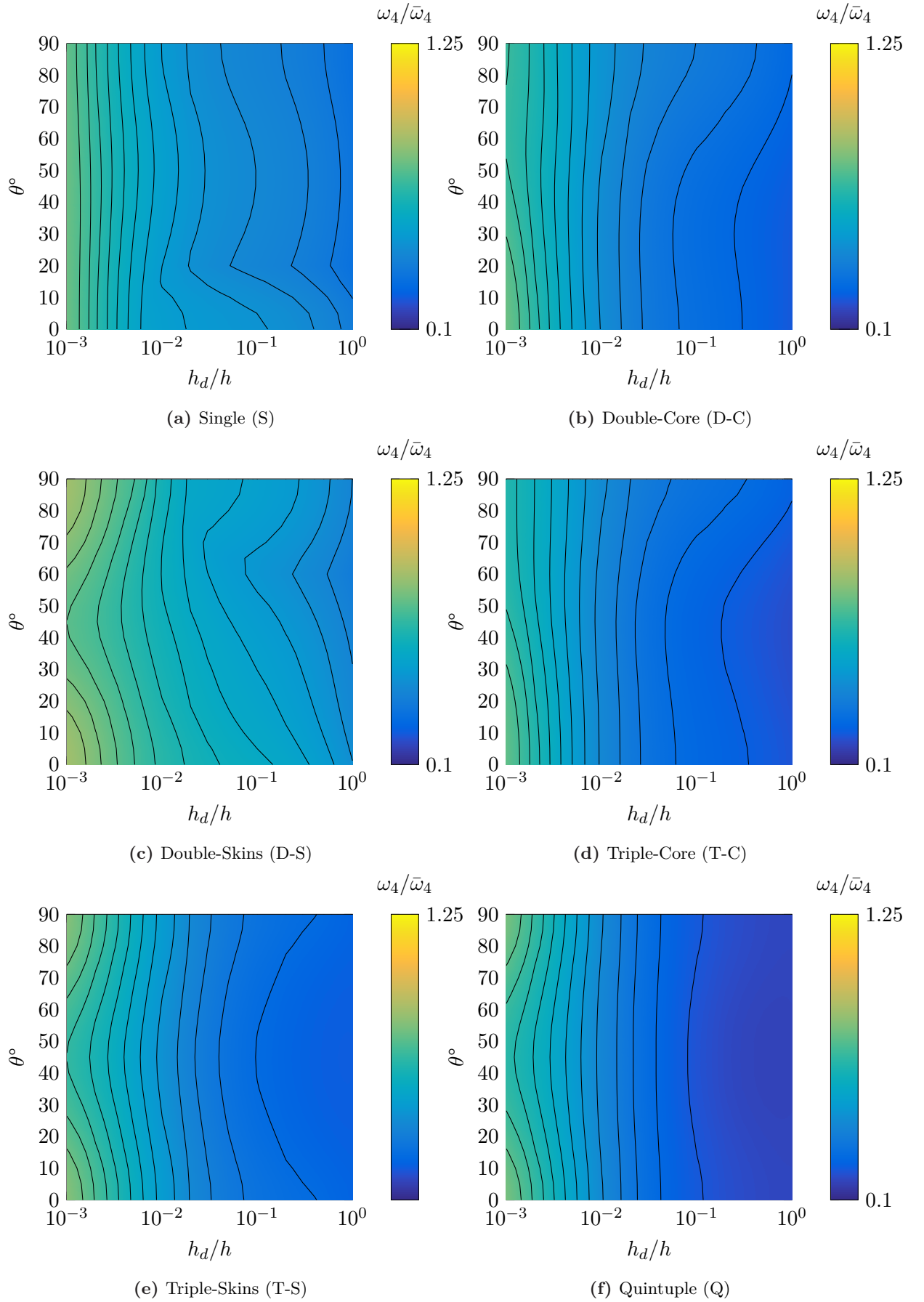


Figure 7.41 Influence of damping layer thickness and ply angle on the 4th natural frequency, $a/h = 50$

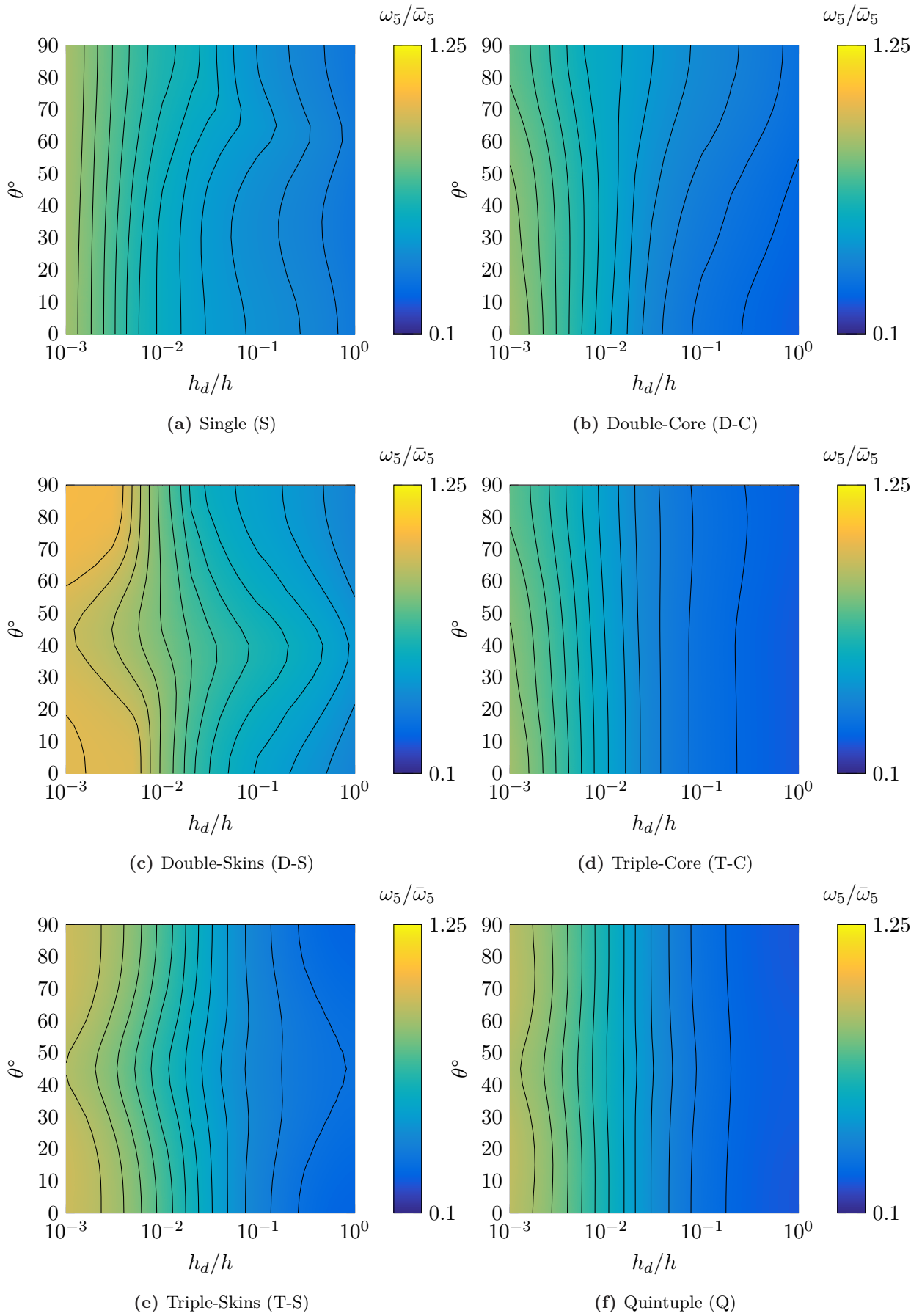


Figure 7.42 Influence of damping layer thickness and ply angle on the 5th natural frequency, $a/h = 100$

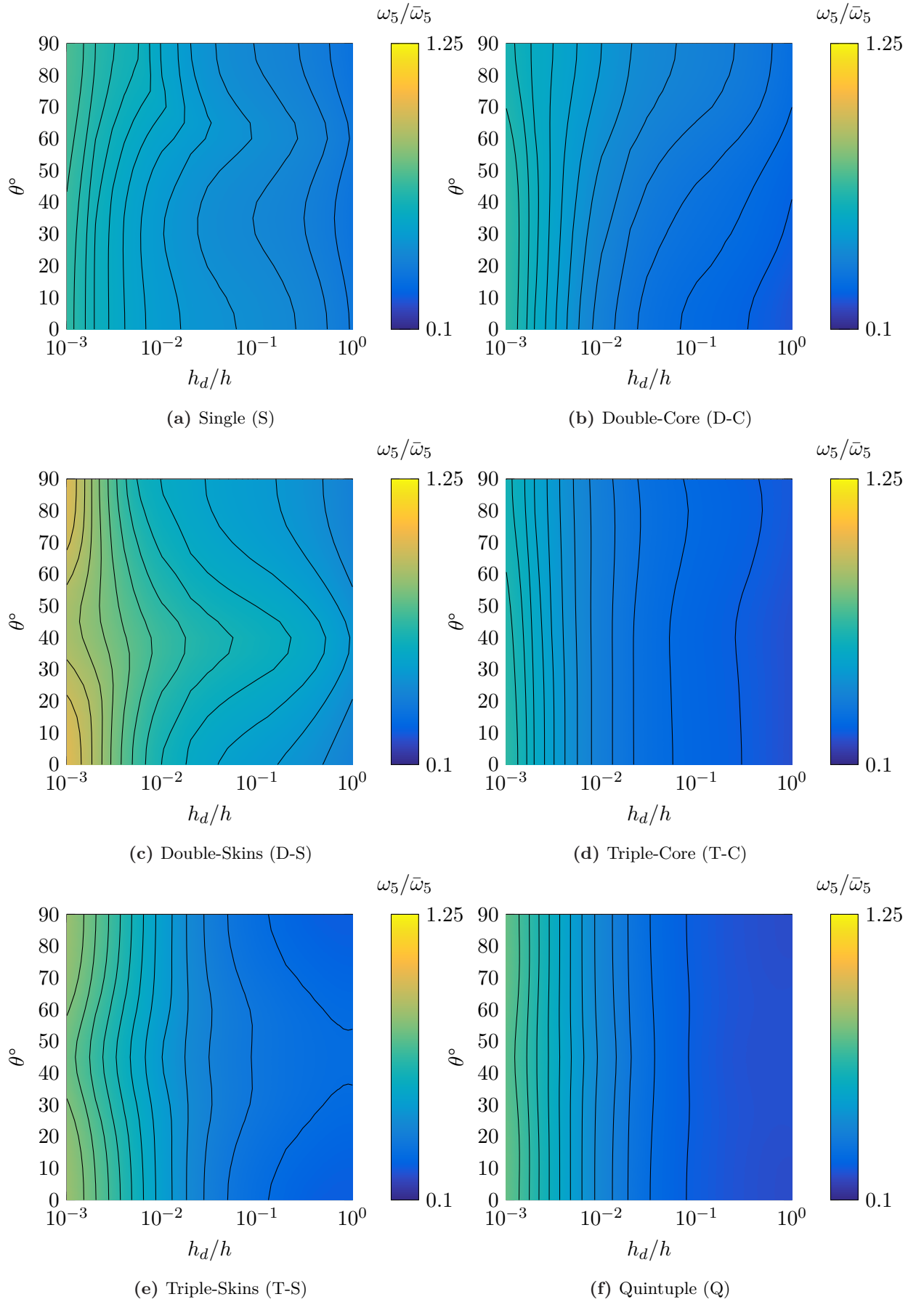


Figure 7.43 Influence of damping layer thickness and ply angle on the 5th natural frequency, $a/h = 50$

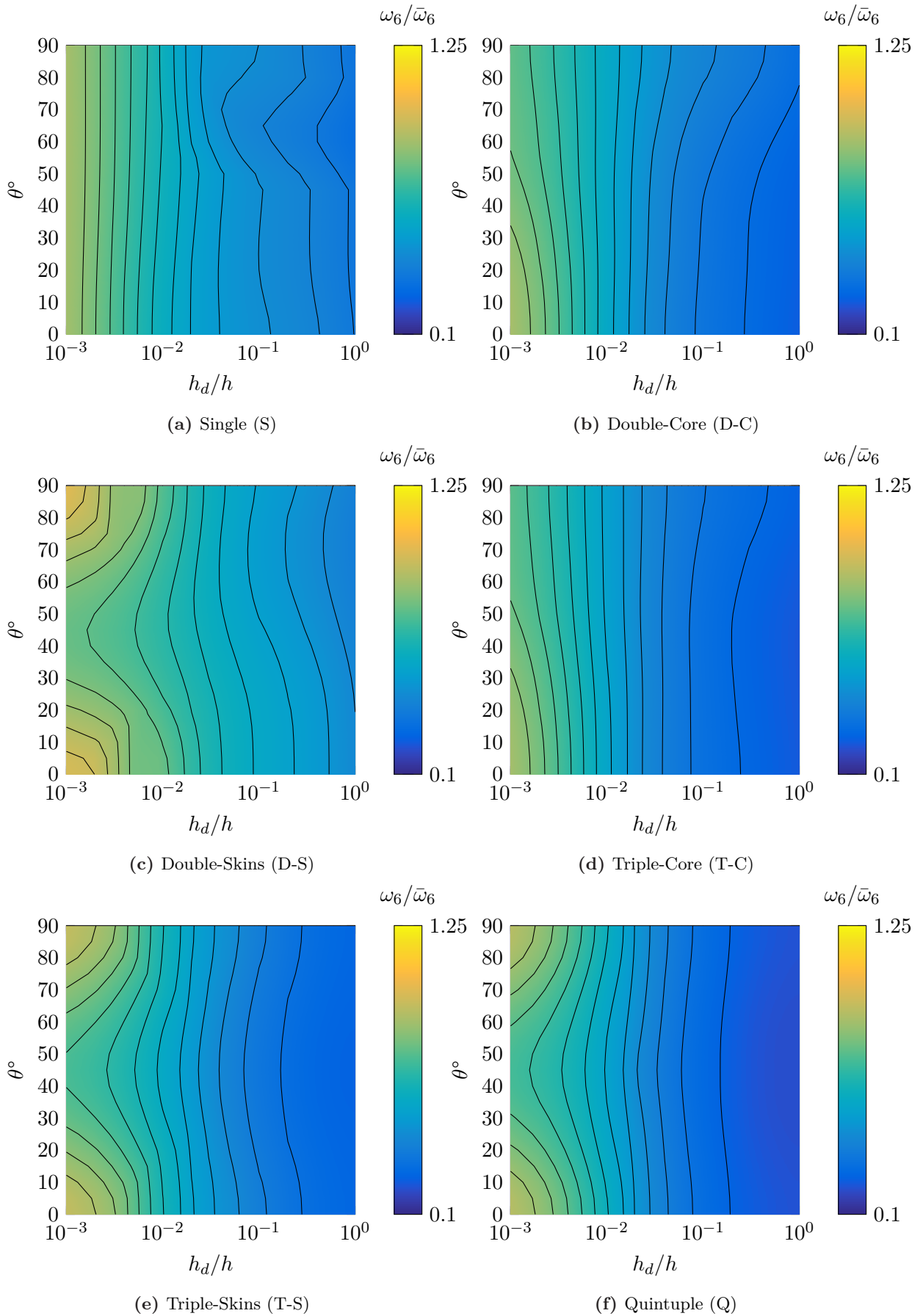


Figure 7.44 Influence of damping layer thickness and ply angle on the 6th natural frequency, $a/h = 100$

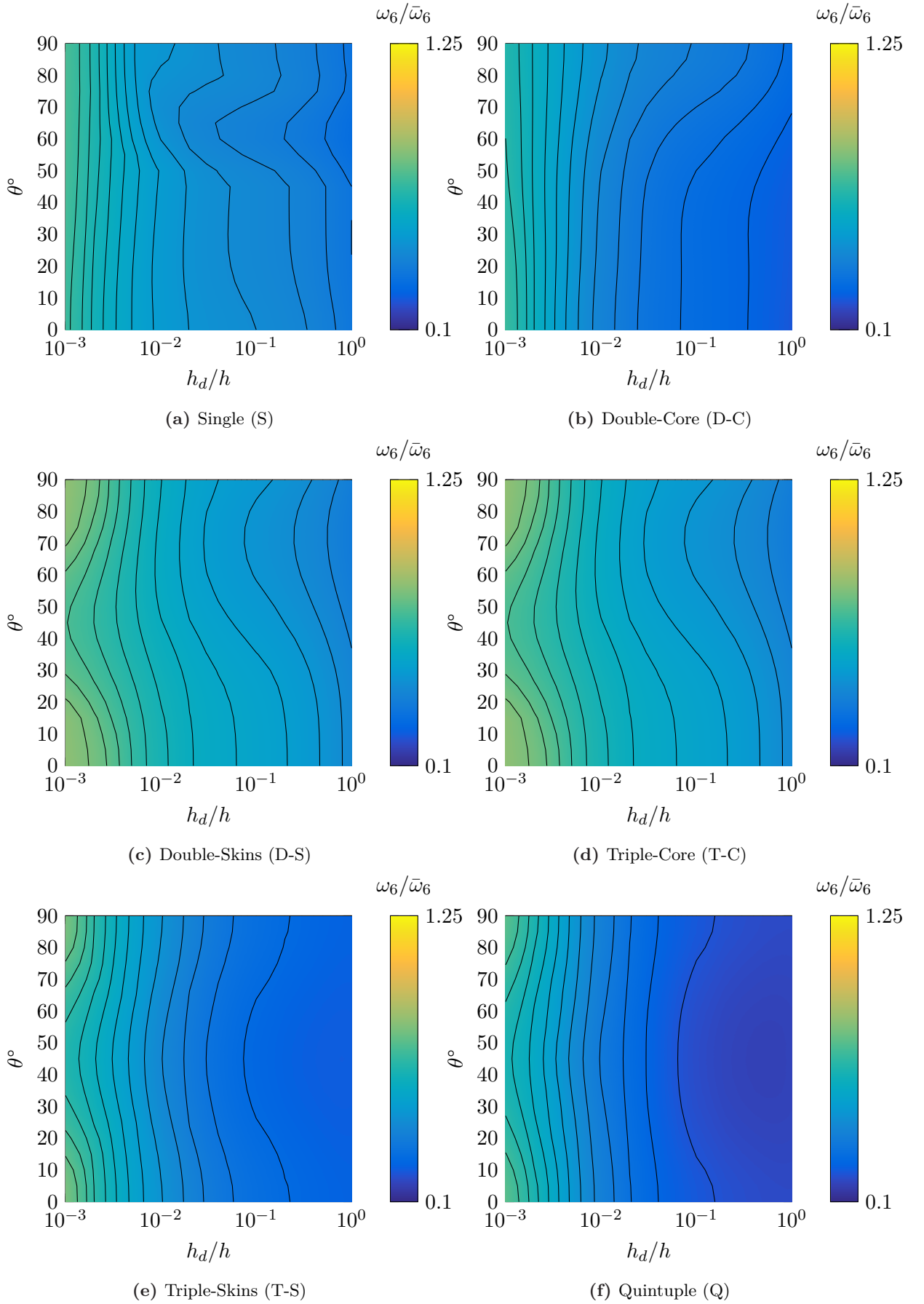


Figure 7.45 Influence of damping layer thickness and ply angle on the 6th natural frequency, $a/h = 50$

7.2.2 | Damping Efficiency of Interleaved FFFF Laminated Plates

In the following, considering that all edges are free and following the same reasoning of the previous section, one studies the damping efficiency of laminated composite plates with interleaved viscoelastic layers. Considering the six stacking sequences presented before, one discusses the influence of damping layers' position and thickness on the damping capacity of laminated composite plates. As boundary conditions play a decisive role in the mechanical response of mechanical structures, it is important to study their influence on damping efficiency and ensure that the conclusions stated above do not depend on local conditions and apply to structures with different boundary conditions. The following studies consider $\theta = 0$.

7.2.2.1 | Influence of Damping Layer Thickness and Position

Following the conclusions of previous chapters, it is relevant to analyse the mode shapes of the free plate to understand the damping performance of damping treatments. Figure 7.46 presents the well-known transverse displacement mode shapes of the first six normal modes of the free square laminated composite plate considering the reference stacking sequence of Figure 7.86 and an aspect ratio a/h of 100. In previous chapters, one also discussed the importance of the mode shapes associated with the shear strain degrees of freedom γ_{xz} and γ_{yz} to understand the modal shear strain pattern, and so Figures 7.47 and 7.48 present, respectively, the first six ones for the same reference laminated composite free plate.

Additionally, Figures 7.49 to 7.51 show the through-the-thickness representations of the extensional, transverse shear, and in-plane shear density mode shapes, respectively, for the third normal mode and different section cuts. Namely, for $y = 0$ and $x = 0$, sections where the modal extensional and transverse shear strain energy densities are highest.

Figure 7.49 depicts the third extensional strain, stress and energy density mode shapes along both x , section cut at $y = 0$, and y , section cut at $x = 0$, directions. One can readily address the through-the-thickness patterns of modal extensional strain, stress and energy density considering the transverse displacement mode shape of the third normal mode in Figure 7.46c. Additionally, considering the modal extensional strain patterns $\phi_3^{\varepsilon_x}$ and $\phi_3^{\varepsilon_y}$ depicted in Figures 7.49a and 7.49b, it is worth discussing the through-the-thickness distribution of modal extensional stress and energy density. One can notice higher modal extensional stress and energy density along the x -direction within layers oriented at 0° . On the other hand, one can detect higher modal extensional stress and energy density along the y -direction within the 90° oriented ones. One can easily explain this pattern considering the material properties, namely the fact that $E_{11} > E_{22}$. As discussed in previous chapters, it is decisive to understand the modal extensional strain/stress distribution to evaluate the capacity of the stiffer layers to impose transverse shear strain on damping layers and consequently improve damping. Due to their intrinsic structure of multiple plies, this is crucial in laminates to provide valuable information to the designer.

In parallel, Figure 7.50 illustrates the through-the-thickness distribution of the third shear strain, stress and energy density mode shapes at the same section cuts. The patterns follow the expected behaviour for both sections, namely the shear stress continuity and the higher levels closer to the neutral axis of the structure. Moreover, Figure 7.51 shows the through-the-thickness distribution of the third in-plane shear strain, stress and energy density mode shapes, once again, at the same section cuts. The concept of a through-the-thickness representation for an in-plane variable is somewhat unusual. However, following this approach, one can have a representation of the in-plane shear strain, stress or strain energy density for each layer at a given section. For both section cuts and following the modal pattern of the third normal mode, the in-plane shear strain energy density is higher at the outer layers.

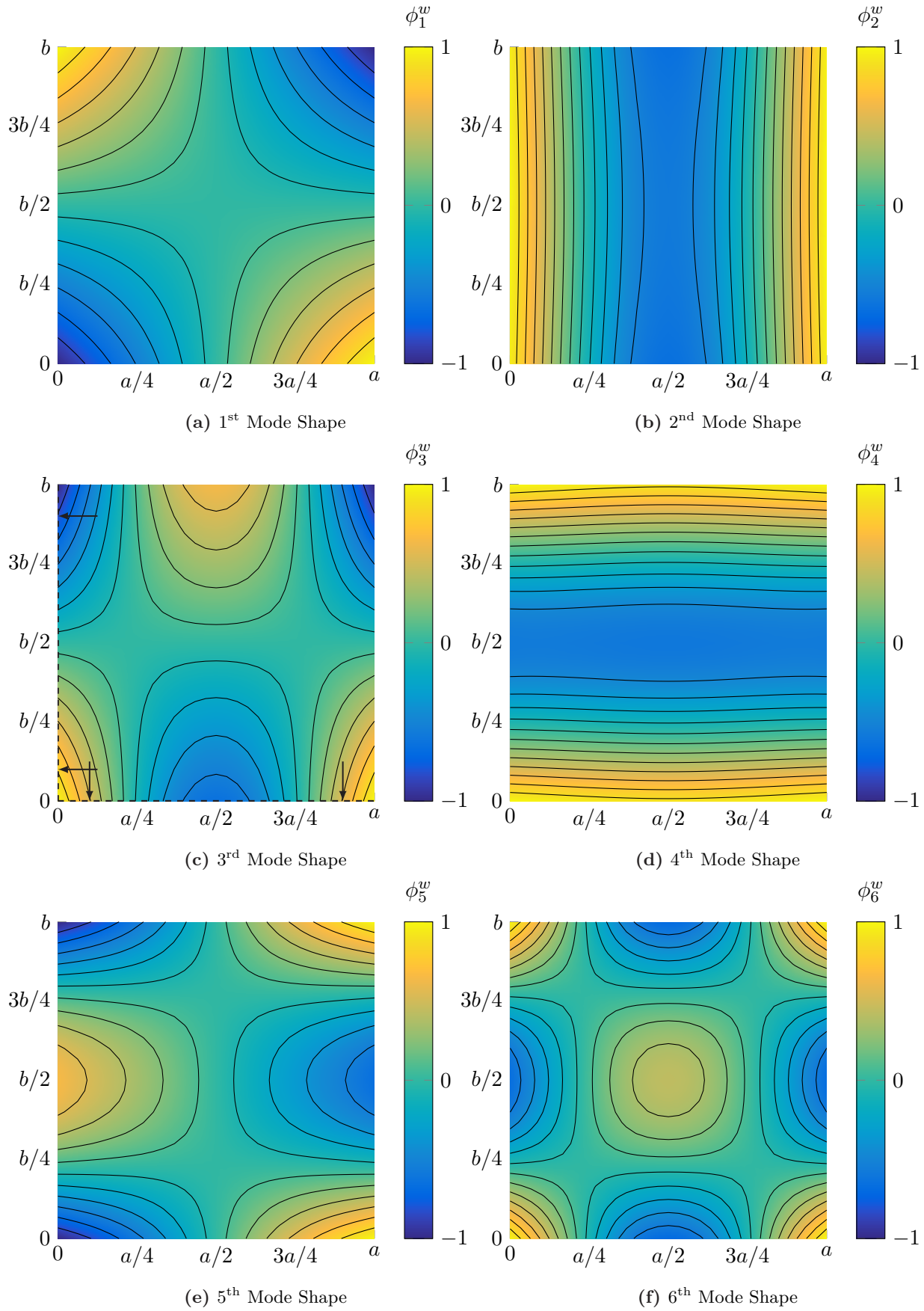


Figure 7.46 Transverse displacement mode shapes of the laminated composite square FFFF plate

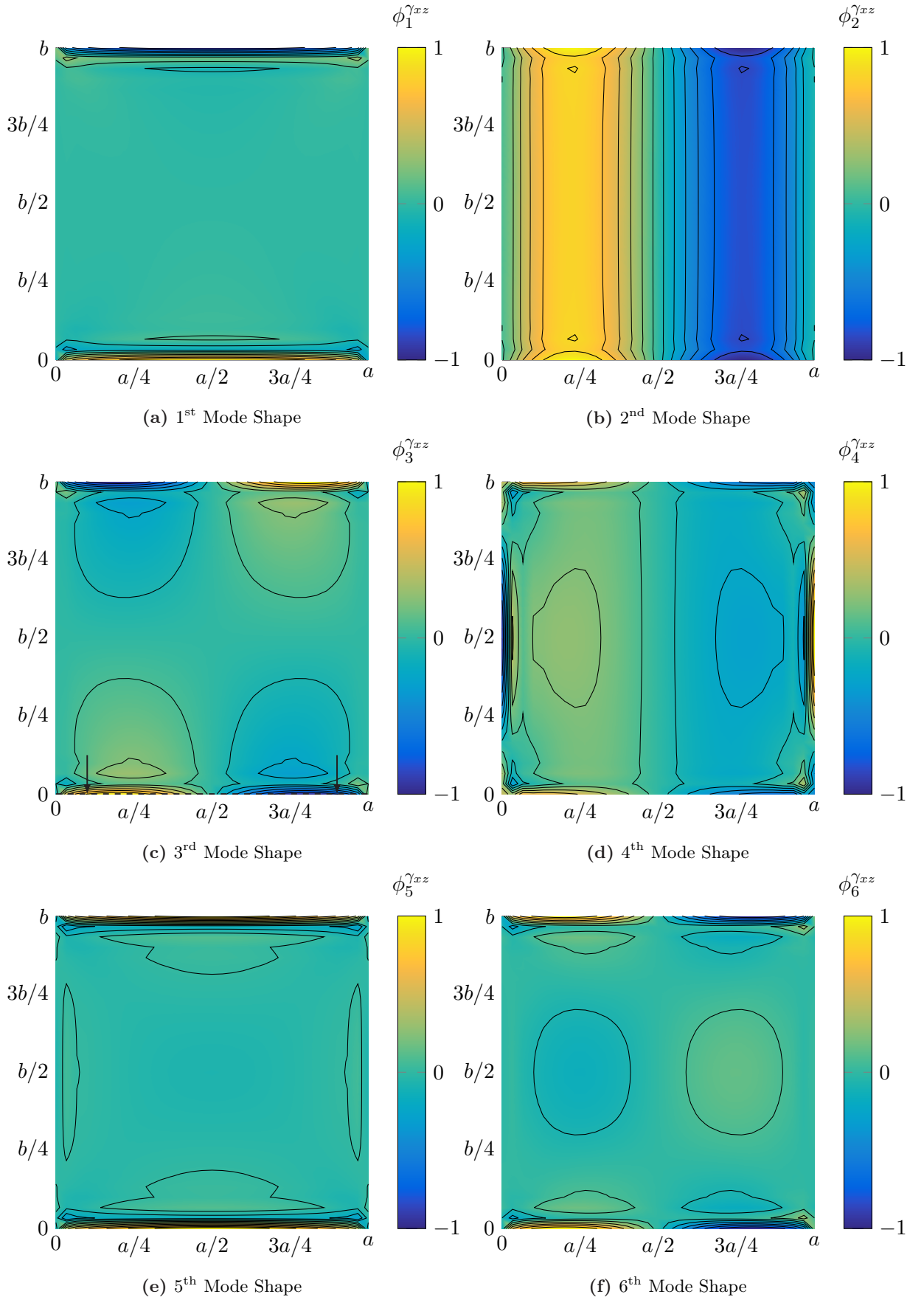


Figure 7.47 Transverse shear strain mode shapes, $\phi^{\gamma_{xz}}$, of the laminated composite square FFFF plate

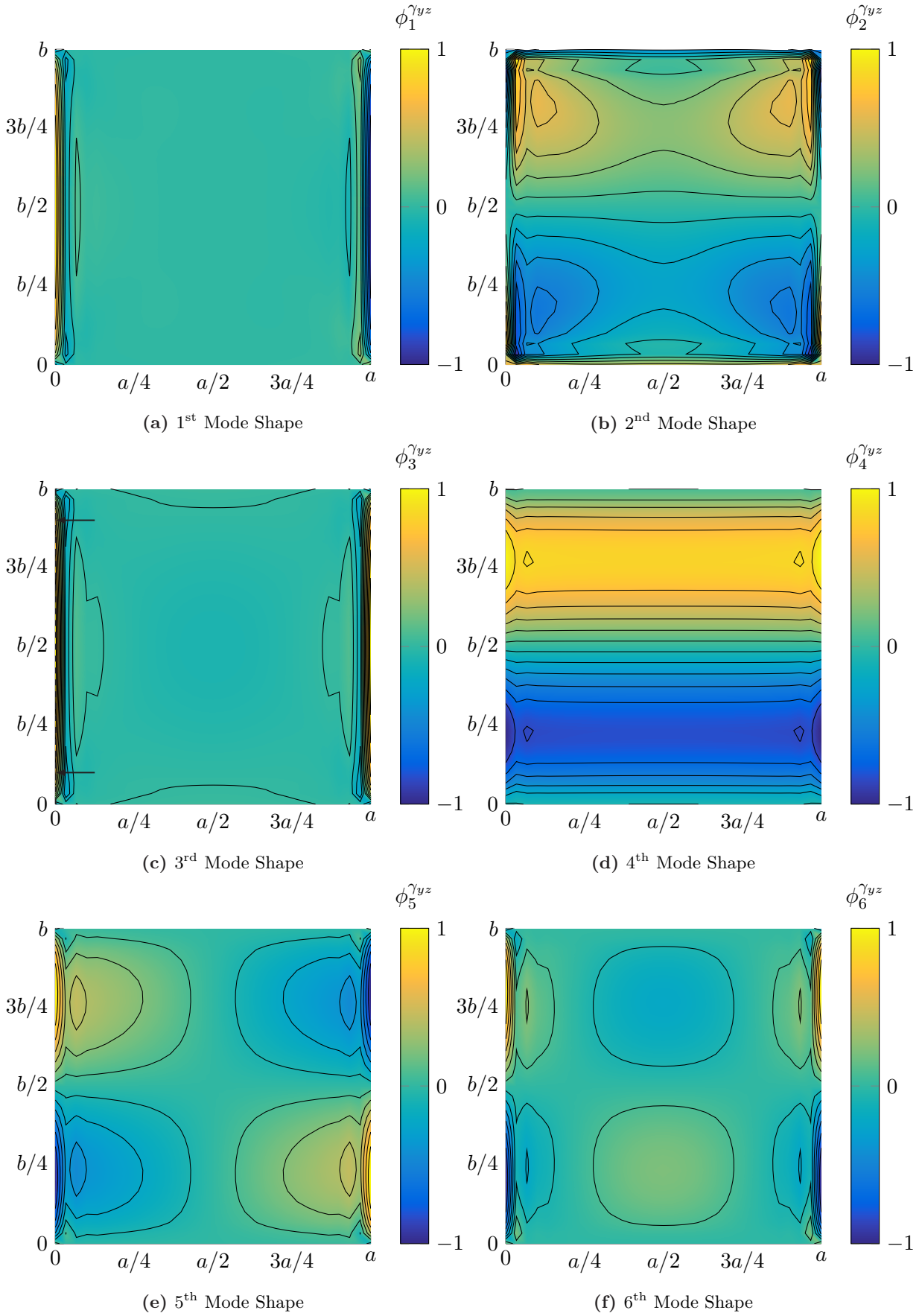


Figure 7.48 Transverse shear strain mode shapes, $\phi^{\gamma_{yz}}$, of the laminated composite square FFFF plate

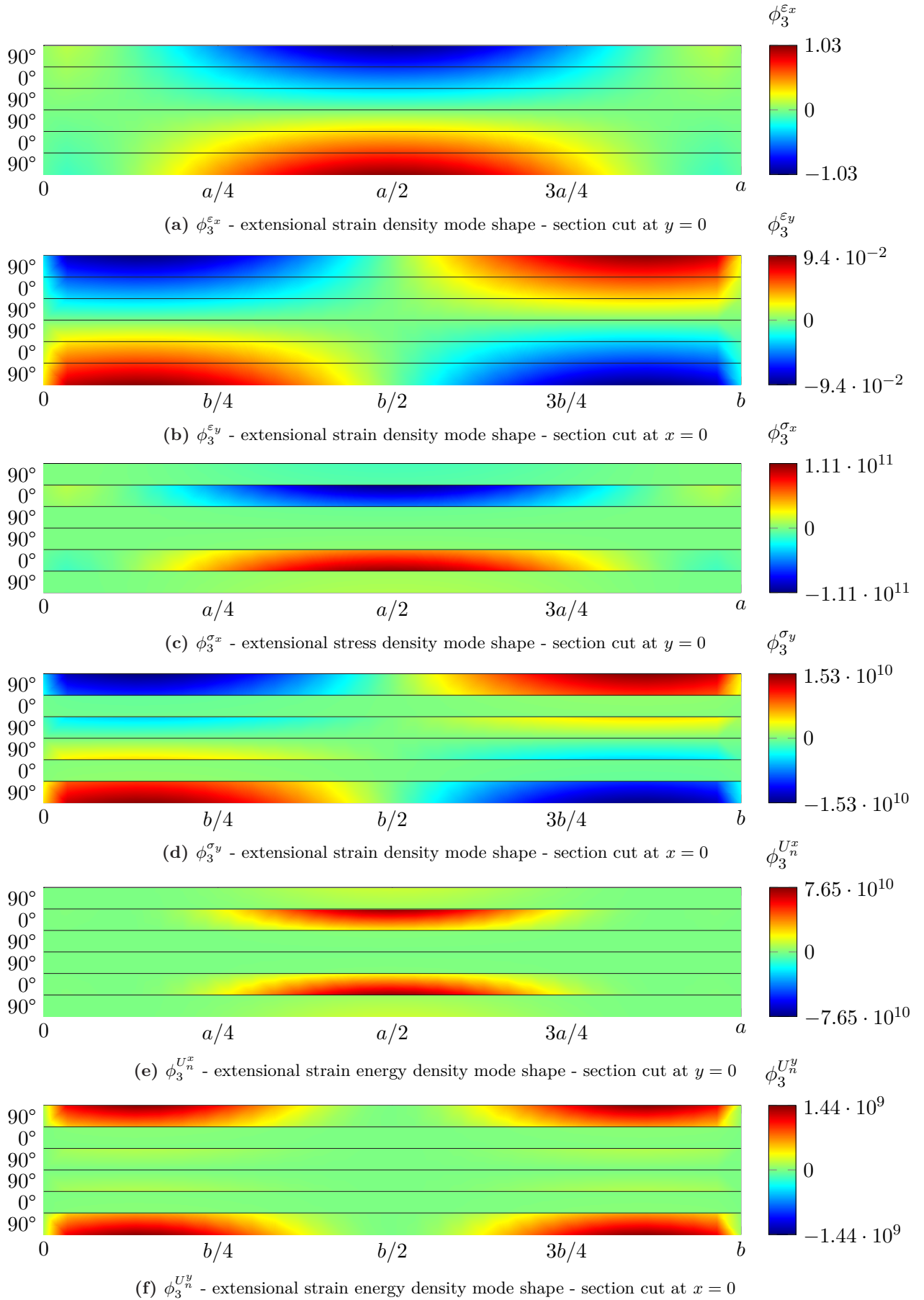


Figure 7.49 Third extensional density mode shapes of the laminated composite square FFFF plate

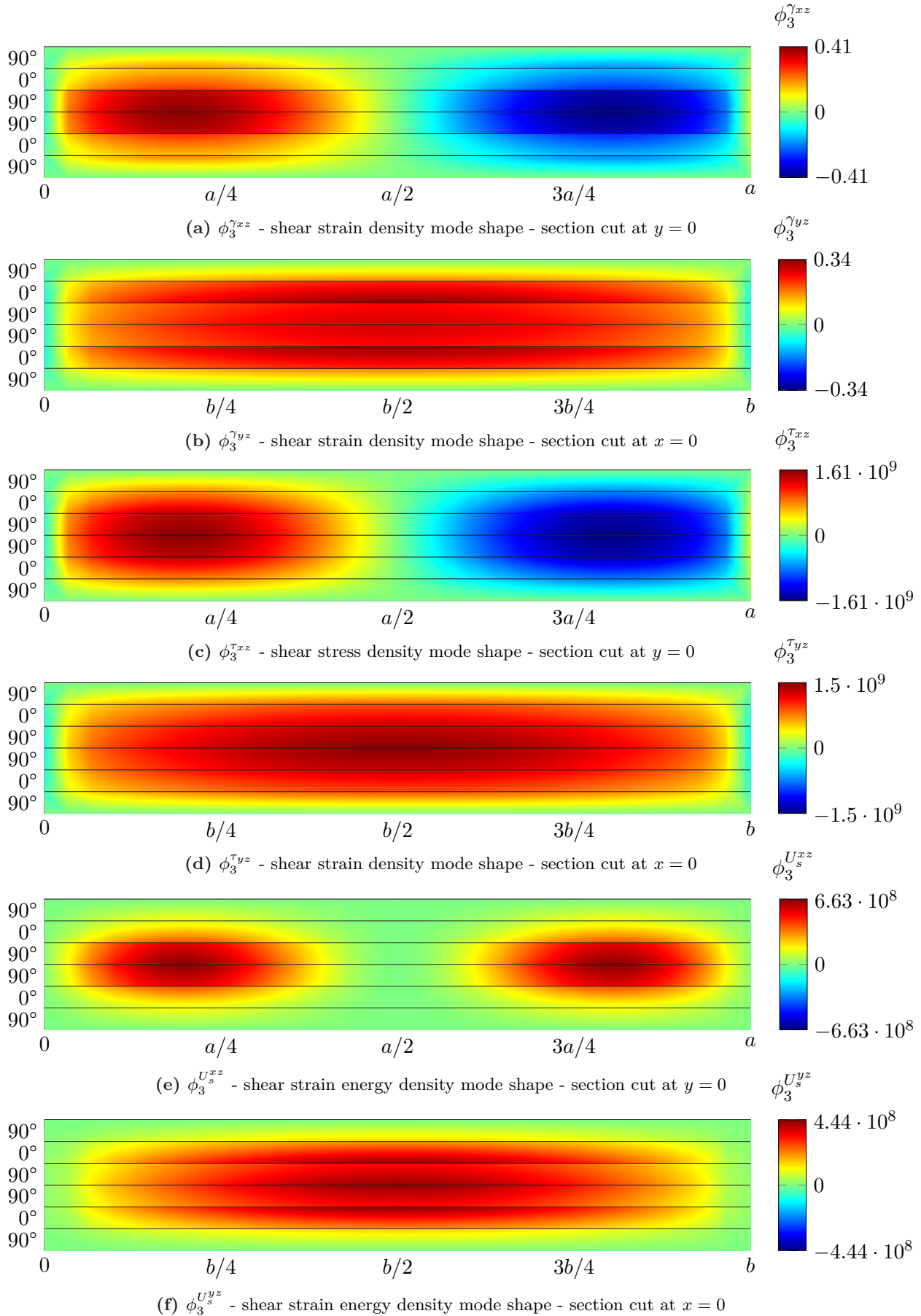


Figure 7.50 Third shear density mode shapes of the laminated composite square FFFF plate

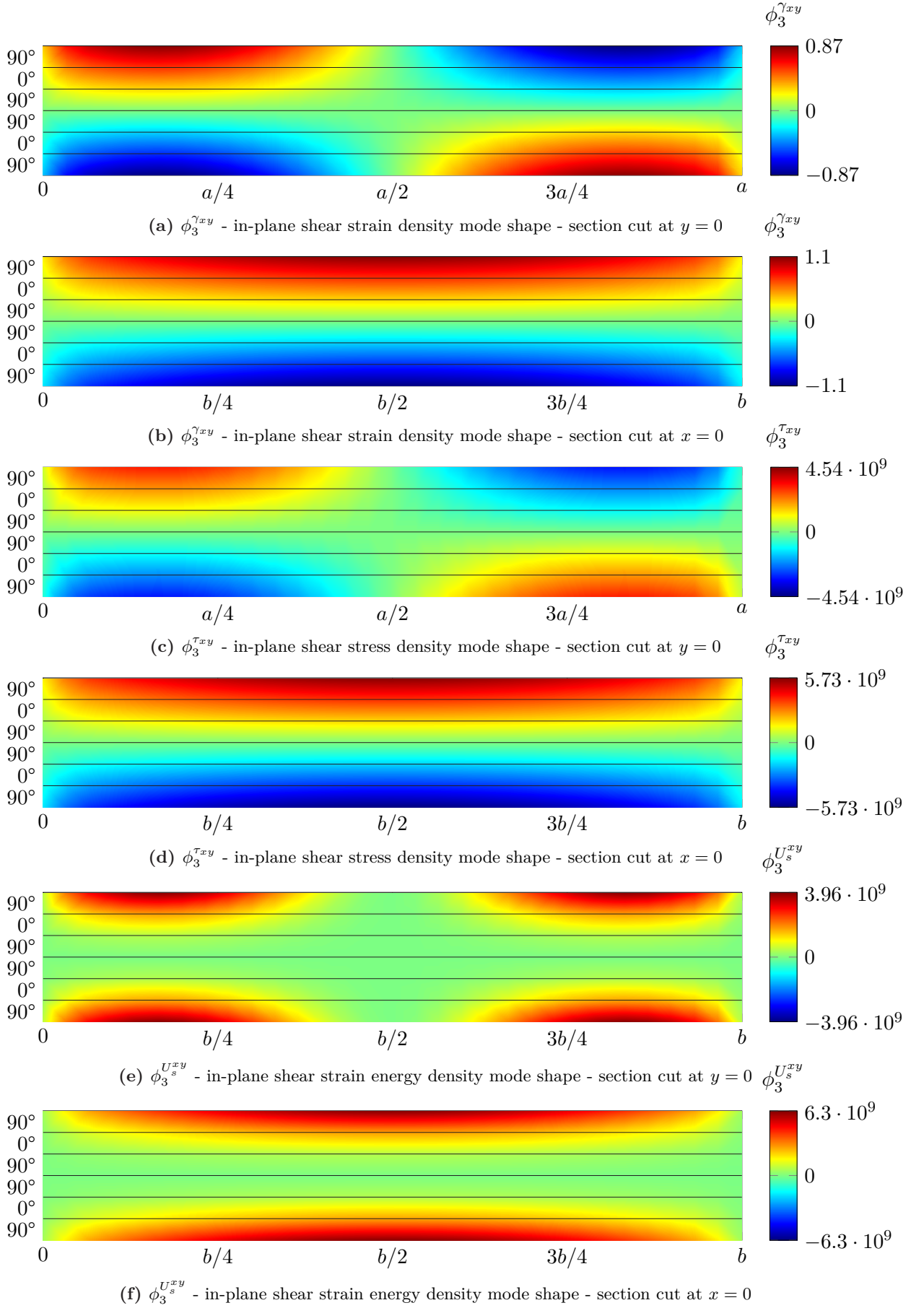


Figure 7.51 Third in-plane shear density mode shapes of the laminated composite square FFFF plate

To illustrate the influence of damping layers' thickness and position on free laminated composite plates, Figure 7.52 presents the specific modal loss factors of the first three normal modes as a function of the damping layer thickness for the different stacking sequences under analysis; Single (S), Double-Core (D-C), Double-Skins (D-S), Triple-Core (T-C), Triple-Skins (T-S) and Quintuple (Q). One considers modal loss factors as a function of the material loss factor, η_n/η_d , and damping layer thickness as a function of the total laminate thickness without damping layers, h_d/h . Figures 7.52a, 7.52c and 7.52e illustrate the corresponding specific modal loss factor for reference laminated composite plates with an aspect ratio $a/h = 100$ and, side-by-side, Figures 7.52b, 7.52d and 7.52f for thicker laminated composite plates with $a/h = 50$.

For the first modal loss factor of a reference laminated composite plate, Figure 7.52a, one might define three ranges of thickness ratio h_d/h , matching the three decades represented; the modal loss factor varies almost linearly in each decade with higher slopes for higher thicknesses (log scale for the thickness variable). For the first mode, it is clear that damping capacity depends almost exclusively on damping material volume regardless of the stacking configuration. Focusing on the logarithmic scale of the thickness ratio, double the length in the corresponding axis represents a damping layer ten times higher or lower. Undoubtedly, the double interleaved configuration with damping layers close to the skins, Double-Skins (D-S), shows the worst damping capacity of all configurations for the same amount of damping material. On the other hand, the Triple-Core (T-C) configuration presents the best damping capacity for the first mode. In addition, regardless of the specific damping capacity of each configuration, all of them show the same trend. However, there are slight differences between configurations. Double-Core (D-C), Quintuple (Q), and Triple-Core (T-C) configurations show the best damping capacities, while Single (S), Triple-Skins (T-S) and Double-Skins (D-S) well below show the worst damping capacities. It is interesting to note that core configurations with more than one layer are better for higher damping capacity when compared with skins configurations but also with the single interleaved one. As previously stated, the modal transverse shear strain of the damping layer is the primary mechanism responsible for damping capacity, and often, authors report that the damping layer should be positioned at the neutral axis to maximise damping. In laminated composite materials, such requirements may be insufficient, as the damping capacity of the single interleaved configuration shows. Furthermore, these results show that these conclusions are independent of the boundary conditions of the laminated plate. Nevertheless, skins' configurations exhibit the lowest damping capacities for the entire range of damping layer thickness.

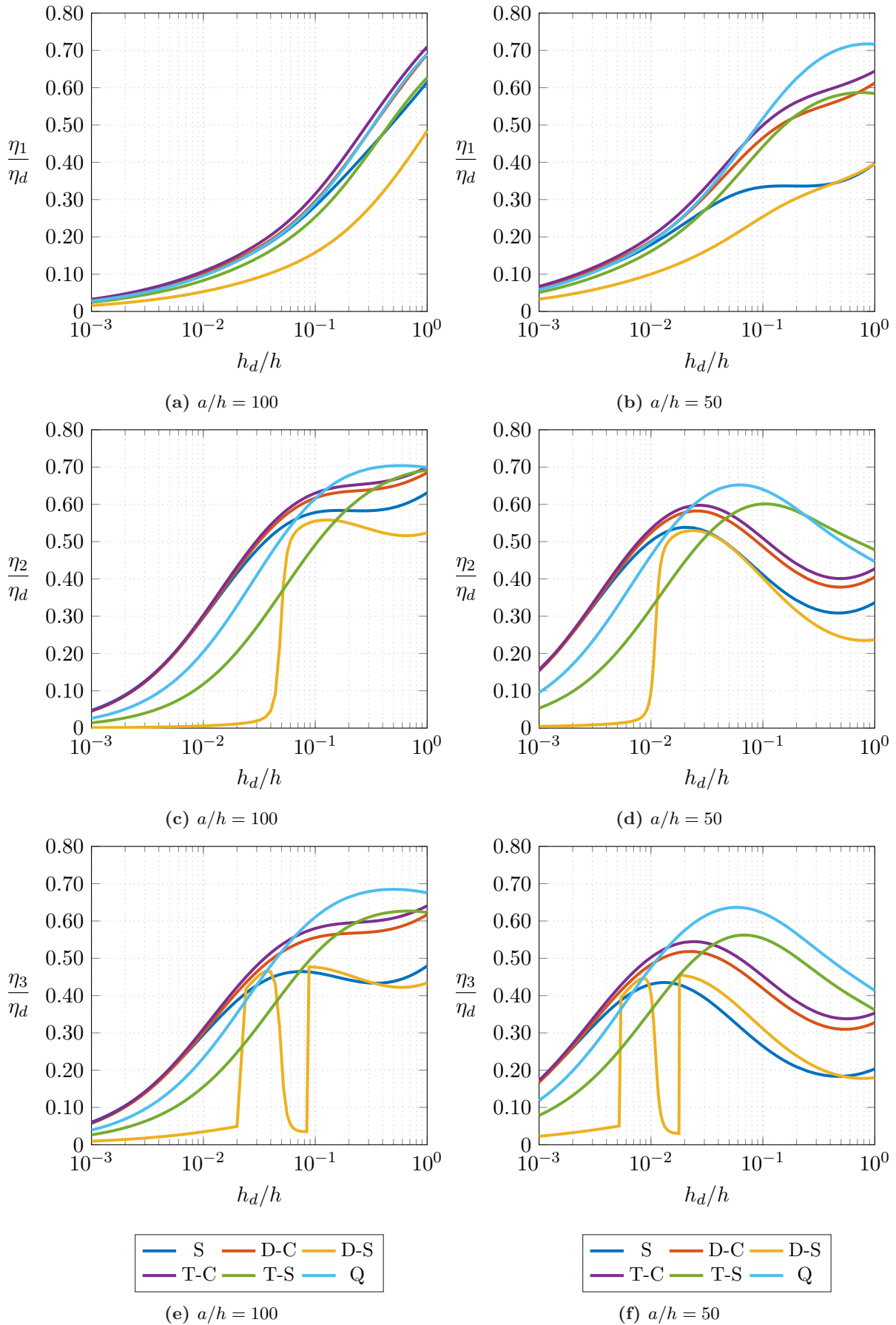
Similarly, for thicker laminate composite plates, Figure 7.52b presents the specific modal loss factor of the first mode as a function of the damping layer thickness for the different stacking sequences under analysis. There are considerable differences in damping efficiency when comparing a reference and a thicker laminated composite plate with interleaved viscoelastic layers. For the latter, one might define two ranges of thickness ratio, h_d/h , matching the first two decades represented and where the specific modal loss factor varies almost linearly in each decade with higher slopes for higher thicknesses. However, it is easy to identify a plateau region in the last decade, i.e. for thicker damping layers, nearly regardless of the stacking sequence. Besides, it is possible to distinguish a maximum attainable damping capacity when considering specific stacking sequences. For a wide range of damping layer thicknesses, the Triple-Core (T-C) stacking sequence offers the best damping capacity for thicker FFFF laminated composite plates, as is the case for thinner ones. However, the Quintuple (Q) configuration presents the best damping capacity for higher amounts of viscoelastic material. On the other hand, despite the poor performance of the stacking sequences considering damping layers close to the skins, the single-layer arrangement shows a weak damping performance for thicker laminated composite plates, especially for thicker damping layers. A soft single-layer damping treatment at the core of the laminate decouples both halves of the CFRP and their ability to impose transverse

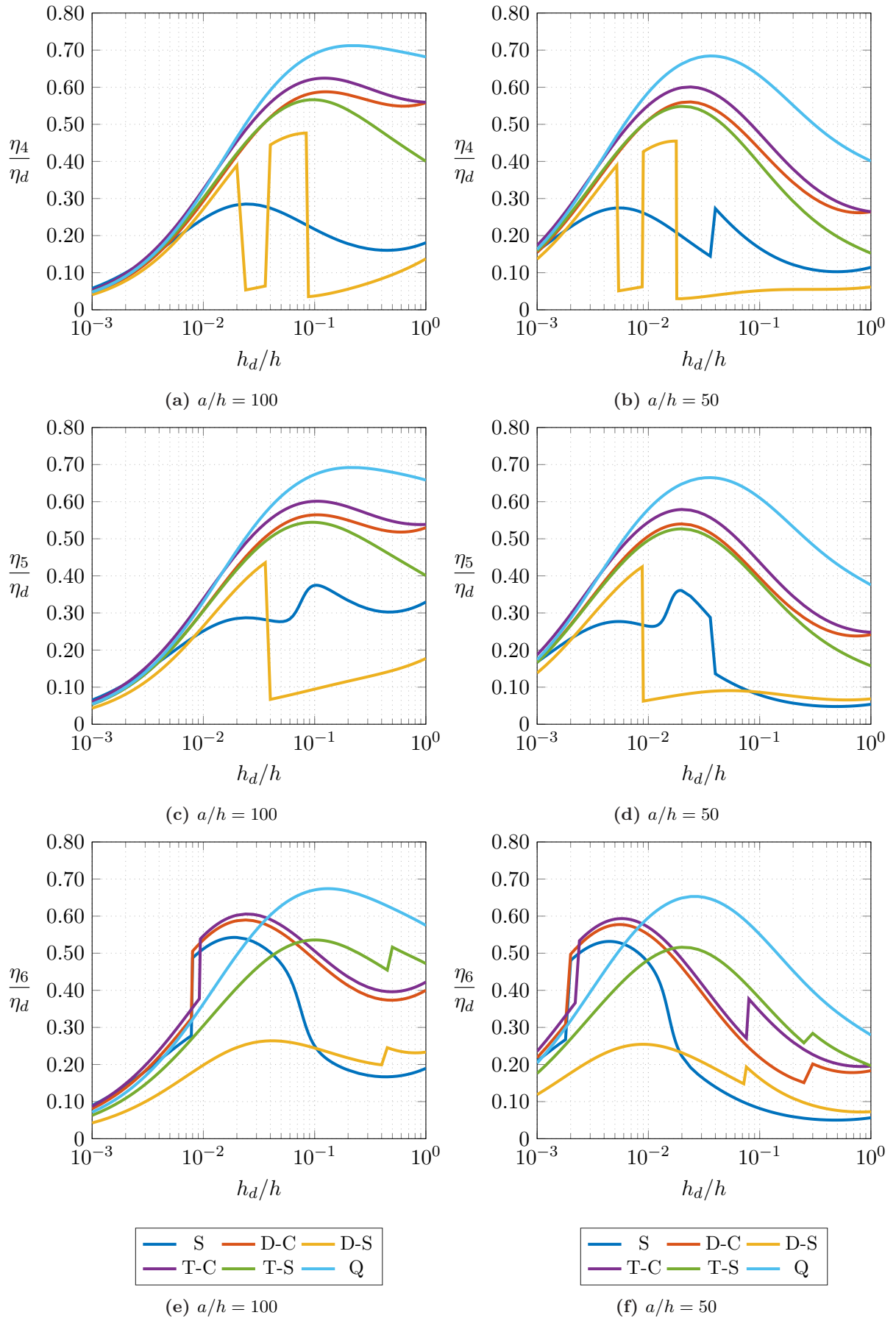
shear strain to the viscoelastic layer, as previously discussed for patch treatments in structures comprising isotropic materials and simply supported laminated composite plates.

For the second vibration mode, Figures 7.52c and 7.52d present the same specific modal loss factors for laminated composite plates with an aspect ratio $a/h = 100$ and thicker laminated composite plates with an aspect ratio $a/h = 50$, respectively. Overall, and similarly to the first modal loss factor, configurations with damping layers close to the skins show poor damping performances compared to stacking sequences involving damping layers interleaved close to the laminate core. The Quintuple (Q) interleaved arrangement needs a large amount of damping material to be efficient. Reviewing Figure 7.52c for the reference laminated square composite plate with an aspect ratio $a/h = 100$, and excluding the Double-Skins (D-S) configuration, the damping capacity of any stacking sequence increases with increasing thickness ratios, at least for low levels. Nevertheless, there is a plateau in damping capacity and increasing the amount of damping material does not increase damping significantly. There was no plateau for the first mode, at least for thicknesses considered physically relevant. This result highlights that one may find global maxima of damping capacity or, at least, local maxima for some configurations at higher vibration modes for laminated composite plates with free edges, proving that this phenomenon does not depend on boundary conditions. Sometimes, more damping material even means less damping capacity. For thicker laminates with an aspect ratio $a/h = 50$, Figure 7.52d presents the specific second modal loss factor as a function of the thickness ratio h_d/h . Here, it is clear that there is a maximum attainable damping capacity for any stacking sequence with interleaved damping layers, after which there is a decrease in damping capacity. Unsurprisingly, for thinner damping layers, stacking sequences with interleaved damping layers close to the laminate core present better damping performances over stacking sequences with damping layers close to the skins. However, the Triple-Skins (T-S) configuration shows a somewhat unexpectedly improved damping performance for thicker damping layers. Still, one should remember that similar damping performances are possible with less damping material with different configurations. The Quintuple (Q) configuration shows the best overall damping performance.

It is possible to notice the same trends for the third specific modal loss factors for both aspect ratios in Figures 7.52e and 7.52f. Nevertheless, despite the same patterns and except for the Quintuple (Q) configuration, it is interesting to note a decrease in the damping capacity for the other stacking sequences for the third vibration mode. Besides, there is a remarkable feature of the Double-Skins (D-S) configuration - i.e. the noticeable sharp jumps in the specific modal loss factor. This phenomenon occurs due to the already discussed switch between mode shapes during the parametric analyses, the third and fourth mode shapes in this case, and it is clear considering Figures 7.53a and 7.53b, which present the same specific modal loss factors for the fourth mode of vibration. As the parametric studies for the Double-Skins (D-S) configuration consider thick damping layers, there is a switch between the mode shapes of the corresponding third and fourth normal modes, explaining the visible sharp jumps. It is worth mentioning that the unexpected rises of the third specific modal loss factor match the sudden drops of the fourth specific modal loss factor and vice-versa.

Figure 7.53 shows the same specific modal loss factors for the fourth, fifth and sixth normal modes as a function of the damping layer thickness for the stacking sequences under analysis. Similarly, one presents the specific modal loss factors for reference laminated composite plates with an aspect ratio of $a/h = 100$ and for thicker laminated composite plates with an aspect ratio of $a/h = 50$ side-by-side. For the fourth, fifth and sixth modal loss factors, one can notice the same trends between reference and thicker laminated composite plates; it is possible to identify the global maxima of modal loss factors for almost any configuration.

**Figure 7.52** Influence of damping layer thickness and position on modal loss factors

**Figure 7.53** Influence of damping layer thickness and position on modal loss factors

Even so, thicker laminated composite plates present global maxima of damping capacity for thinner damping layers. Increasing the damping layer thickness beyond those maxima drops the damping capacity significantly, and more damping material means less damping capacity. Generally, and similarly to the first three specific modal loss factors, configurations with damping layers close to the core perform better than stacking sequences involving damping layers interleaved close to the laminate skins. The Quintuple (Q) interleaved arrangement needs a large amount of damping material to be efficient but presents the best overall modal loss factors, proving its efficiency regardless of the boundary conditions. Additionally, compared to the first three vibration modes, the Triple-Skins (T-S) configuration exhibits a remarkable performance for higher ones. Despite being lower, the damping capacity of this arrangement presents damping capacities comparable to those of the core configurations, and one should remember that this stacking sequence includes a damping layer at the core of the laminate.

Previously, when dealing with isotropic beams and plates covered with viscoelastic damping patches or simply supported laminated composite plates, one demonstrated that strain, stress and strain energy density mode shapes are valuable tools to explain the damping performance of a given damping treatment. Figures 7.54 to 7.59 present the through-the-thickness representations of the transverse shear, extensional and in-plane shear strain energy density mode shapes for the third normal mode and different interleaved configurations. Given the challenge of effectively presenting thin damping layers in print, readers will find a more detailed perspective by referring to the digital version of this document, offering a zoomed-in view. One considers for these representations a thick laminated composite plate with an aspect ratio $a/h = 50$ and a thickness ratio $h_d/h = 6 \times 10^{-2}$, which is the configuration presenting the best overall damping capacity - see Figure 7.52f. In ascending order of specific modal loss factor: Single (S), Double-Skins (D-S), Double-Core (D-C), Triple-Core (T-C), Triple-Skins (T-S) and Quintuple (Q) interleaved configuration. One considers the section cuts $y = 0$ and $x = 0$, sections of maximum modal extensional and transverse shear strain energy densities.

Figures 7.54 and 7.55 exhibit shear strain energy density mode shapes $\phi_3^{U_s^{xz}}$ and $\phi_3^{U_s^{yz}}$, respectively, for all interleaved configurations and in ascending order of modal loss factor. These patterns give a reason for the damping capacity of each stacking sequence, despite the somewhat confusing patterns of the Double-Skins (D-S) configuration, presenting high levels of modal shear strain $\phi_3^{U_s^{yz}}$ but almost null modal shear strain $\phi_3^{U_s^{xz}}$. Much more than the total modal shear strain energy, through-the-thickness patterns are useful widgets to interpret the damping mechanism of laminated composites with interleaved damping layers, and this is particularly noticeable comparing the Triple-Skins (T-S) and the Quintuple (Q) configurations, those presenting the highest levels of modal loss factor in this case. The Triple-Skins (T-S) arrangement ensures the second-best damping capacity. Still, it is possible to identify in Figures 7.54 and 7.55 that the core layer of this configuration is the one subjected to high levels of modal shear strain energy density, being responsible for the improved damping capacity of the Triple-Skins (T-S) configuration. Through-the-thickness patterns of the Quintuple (Q) configuration support the same reasoning because core layers are those exposed to higher levels of modal shear strain energy density. Given this, one may ask why the Single (S) interleaved damping layer does not present the best damping capacity since there is just one thick core layer. As previously discussed, when dealing with damping patches on isotropic beams and plates, there is a turning point after which thick soft layers tend to decouple the stiff layers and so their ability to impose shear. In the Single (S) stacking sequence, both halves of the laminated composite work as a base and constraining sub-laminates. One should note that, for thinner damping layers, this arrangement presents the best results for different vibration modes. However, there is the previously mentioned threshold of damping layer thickness, and damping capacity might even drastically drop with thicker viscoelastic layers - see Figure 7.52f.

To evaluate the ability of the stiff composite layers to impose transverse shear strain on the soft damping layers, Figures 7.56 and 7.57 present the through-the-thickness modal extensional strain energy densities $\phi_3^{U_n^x}$ and $\phi_3^{U_n^y}$, respectively. The results are straightforward for both modal strain energy densities, and for each stacking sequence - 0° oriented composite layers present the highest modal extensional strain energy density $\phi_3^{U_n^x}$, while the outer 90° oriented ones exhibit the highest levels of modal extensional strain energy density $\phi_3^{U_n^y}$. As expected, the highest levels of $\phi_3^{U_n^x}$ arise in the stiffer layers along the x -direction, the fibre direction of such laminas, while the highest levels of $\phi_3^{U_n^y}$ appear in the outer stiffer layers along the y -direction, the fibre direction of such laminas.

Single (S) configuration justifies some attention to discuss the importance of through-the-thickness modal extensional strain energy densities. Layers with high levels of modal extensional strain energy density are those closer to the laminate skins and far away from the neutral axis of the laminated composite. Thus, the ability to impose transverse shear on the soft damping layer in the middle of the laminate is limited. Moreover, Figure 7.57a shows low modal extensional strain energy density $\phi_3^{U_n^y}$ for the Single (S) stacking sequence. On the other hand, the Quintuple (Q) configuration includes soft damping layers between stiff composite layers with high levels of modal extensional strain energy, promoting transverse shear strain and damping.

Figures 7.58 and 7.59 show the through-the-thickness distribution of the modal in-plane shear strain energy density associated with the third normal mode. As previously discussed, the concept of a through-the-thickness representation for an in-plane variable is somewhat unusual, but one can conclude that the modal in-plane shear strain energy density is higher for the skin layers of the laminate.

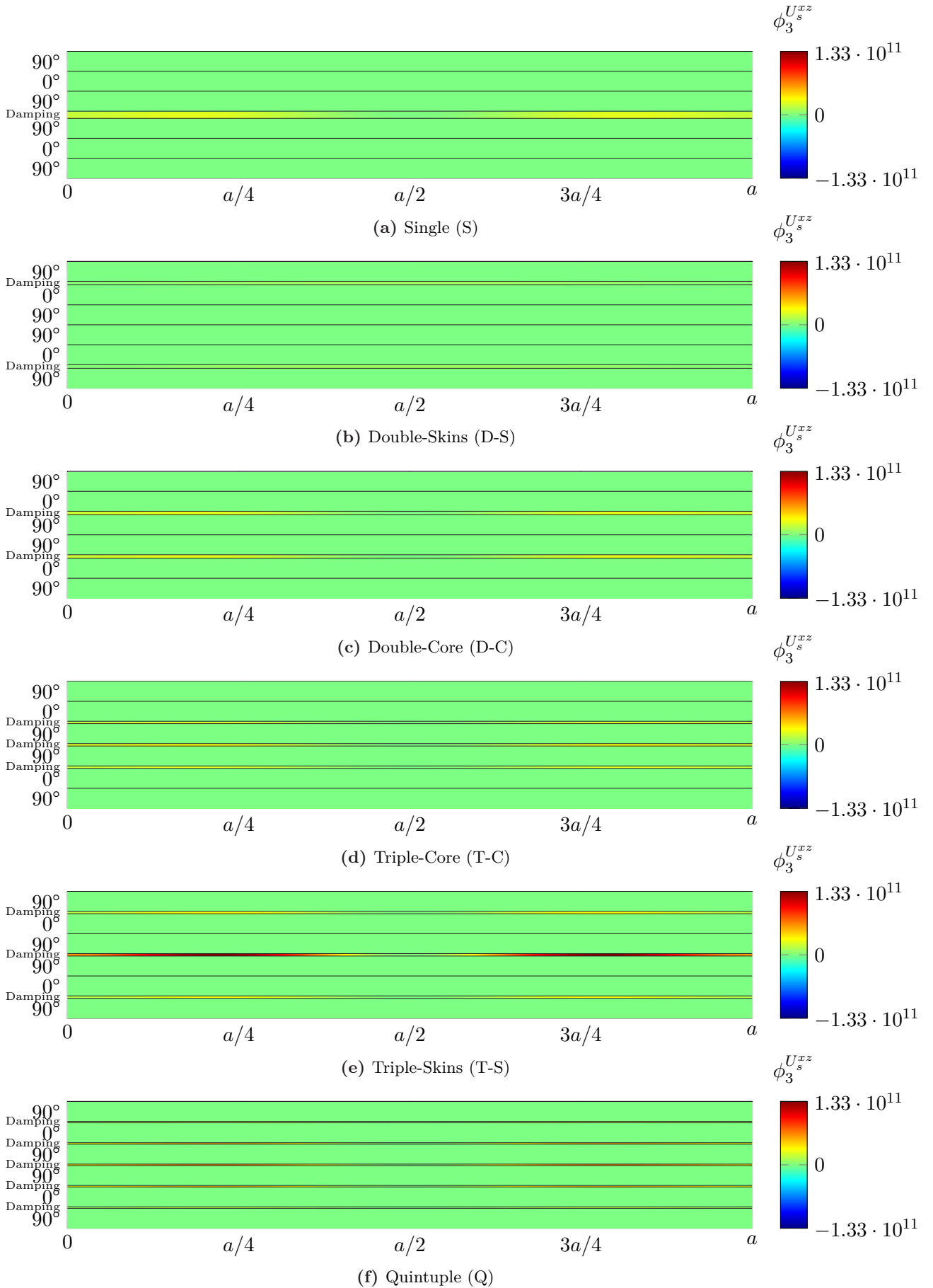


Figure 7.54 $\phi_3^{U^{xz}}$ - shear strain energy density mode shape - section cut at $y = 0$

Figure 7.55 $\phi_3^{U^{yz}}$ - shear strain energy density mode shape - section cut at $x = 0$

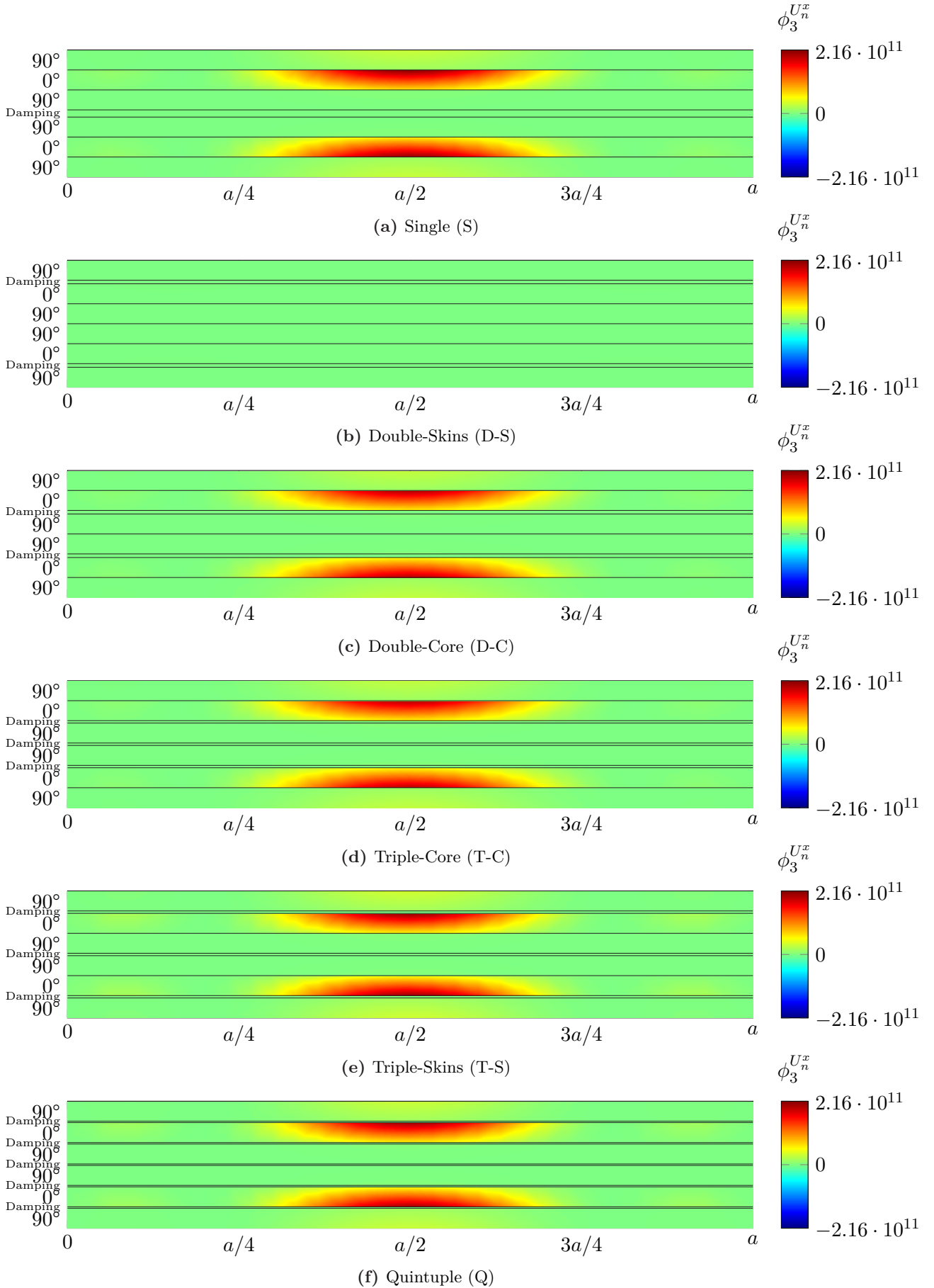


Figure 7.56 $\phi_3^{U_n^x}$ - extensional strain energy density mode shape - section cut at $y = 0$

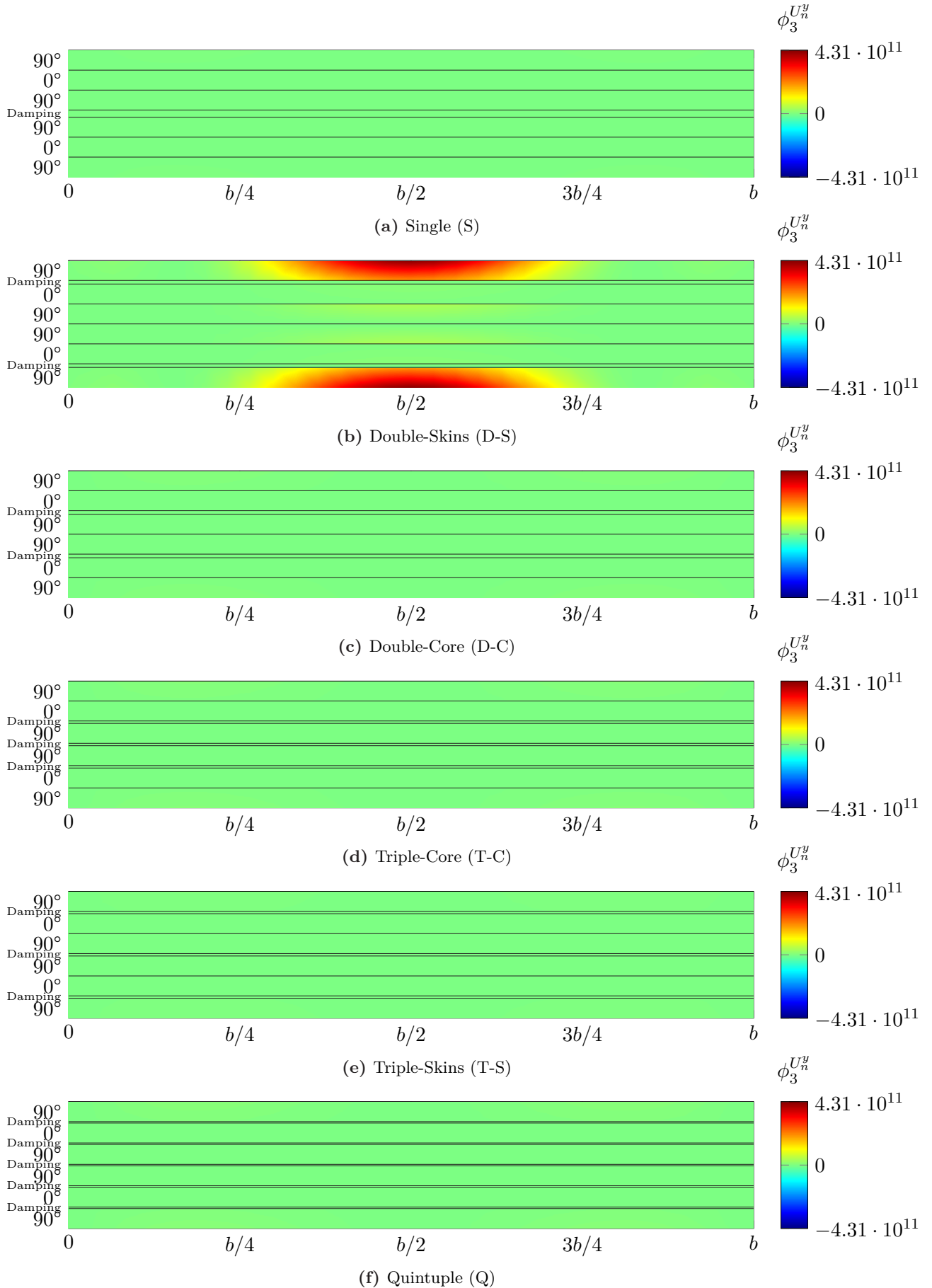


Figure 7.57 $\phi_3^{U_n^y}$ - extensional strain energy density mode shape - section cut at $x = 0$

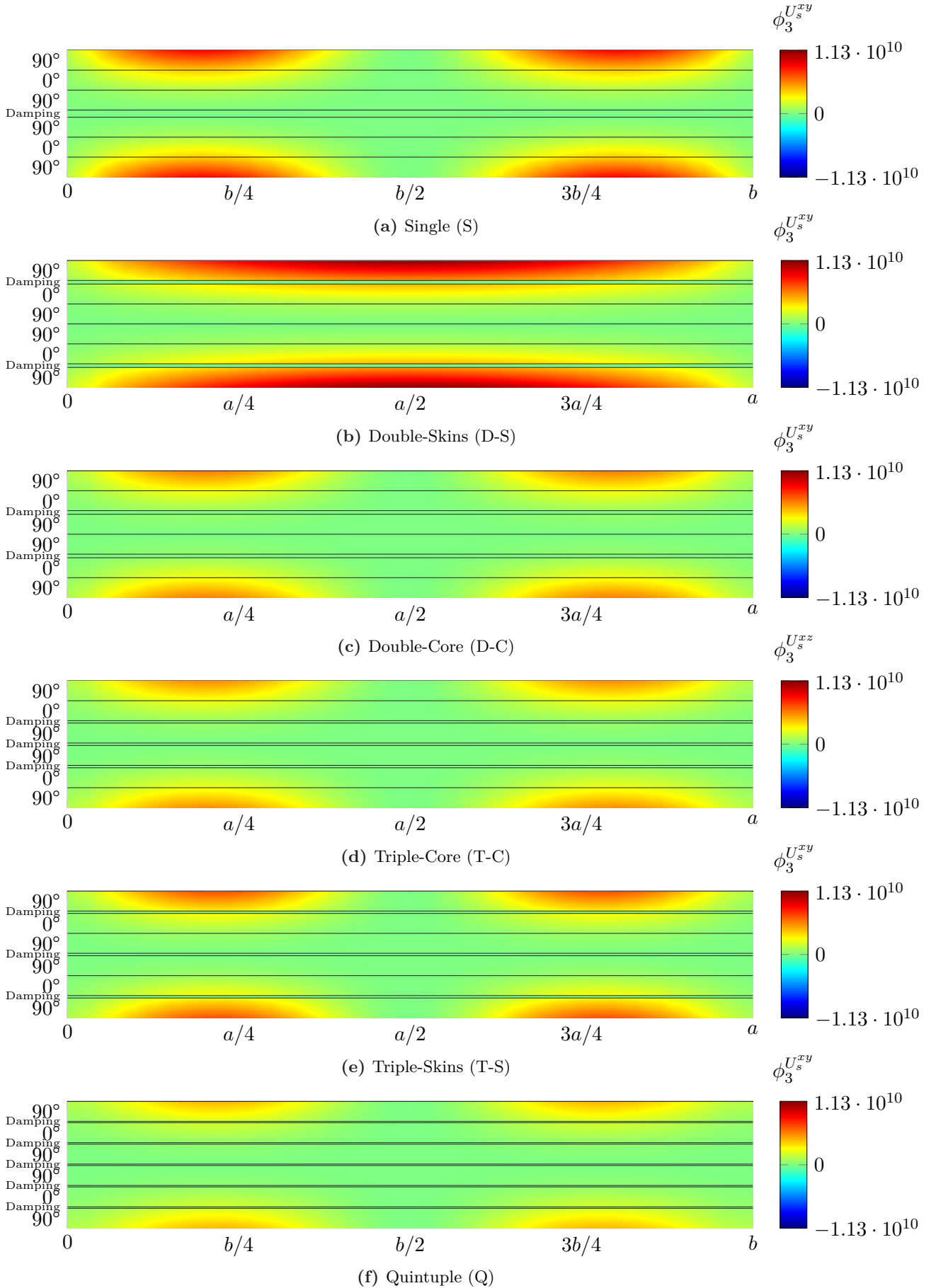


Figure 7.58 $\phi_3^{U_s^{xy}}$ - in-plane shear strain density mode shape - section cut at $y = 0$

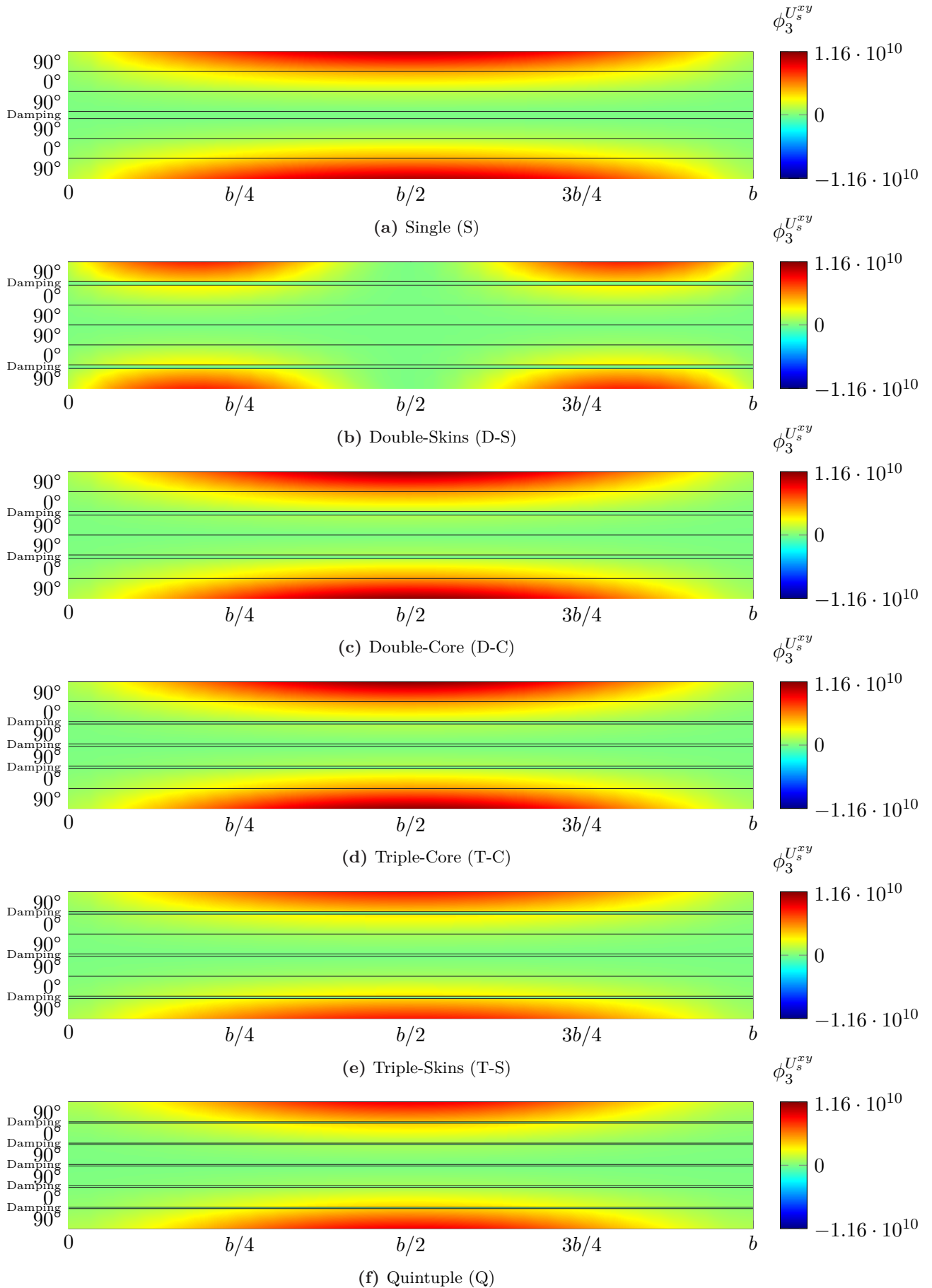


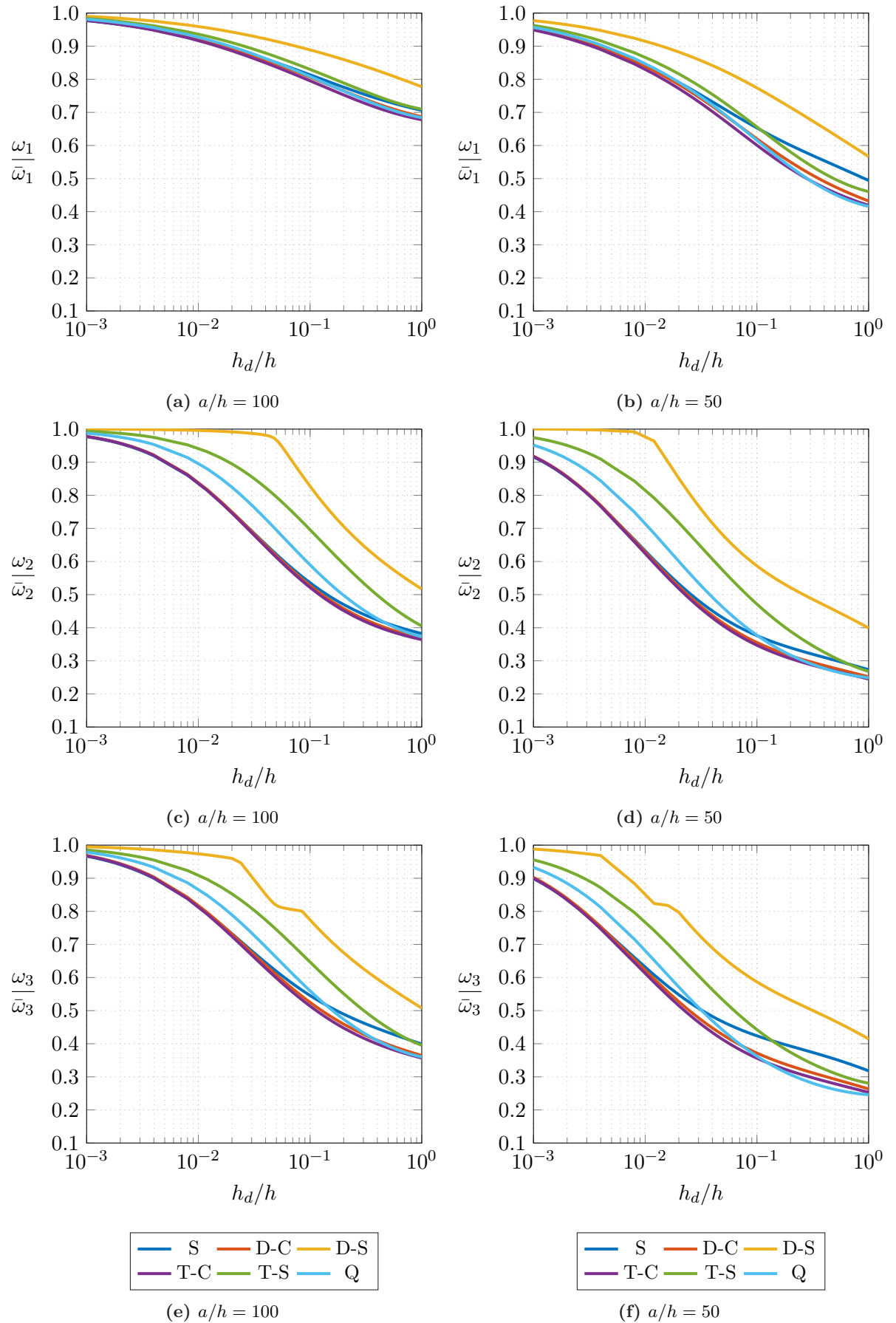
Figure 7.59 $\phi_3^{U_s^{xy}}$ - in-plane shear strain density mode shape - section cut at $x = 0$

As extensively discussed in subsection 7.1.1 and for the simply supported laminated composite plate, design requirements for high damping capacity often conflict with those for high stiffness. To study the influence of interleaved soft damping layers on the natural frequencies of laminated composite plates, Figures 7.60 and 7.61 present the first six dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ as a function of the dimensionless damping layer thickness h_d/h for the reference and thick composite plates with aspect ratios of $a/h = 100$ and $a/h = 50$, side-by-side. Dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ represent the ratio between the natural frequencies of the laminated composite with interleaved damping layer(s) and the natural frequencies of the reference six-layer composite plate.

For the first three natural frequencies in Figure 7.60, natural frequencies drop significantly as the damping layer thicknesses increase. The Double-Skins (D-S) arrangement stands out as the one presenting less natural frequency variation, i.e., for the same amount of soft damping material, interleaving two layers close to the skins of the laminate promotes a stiffness loss, even though at a smaller extent. Unsurprisingly, one discussed this configuration when analysing the modal loss factors of different stacking sequences in Figure 7.52, and the Double-Skins (D-S) arrangement presented one of the worst damping performances. As previously discussed, design requirements for high damping capacity and stiffness conflict in most cases.

Nonetheless, this general rule is not linear, as previously reported for the simply supported laminated composite plate. Considering the thick laminated composite plate with an aspect ratio $a/h = 50$ and a thickness ratio $h_d/h = 6 \times 10^{-2}$, which is the configuration presenting the best overall damping capacity for the third normal mode studied before, one can order the stacking sequences in increasing order of the third natural frequency as Triple-Core (T-C), Double-Core (D-C), Quintuple (Q), Single (S), Triple-Skins (T-S) and Double-Skins (D-S). By taking a closer look, one can conclude that it is not the reverse order of increasing third modal loss factor for the same thick laminated composite plate, which is Single (S), Double-Skins (D-S), Double-Core (D-C), Triple-Core (T-C), Triple-Skins (T-S) and Quintuple (Q). For example, the Triple-Skins (T-S) arrangement presents the second-best modal loss factor and the second-highest natural frequency for the same amount of damping material. This result proves that there is room for optimisation.

Figure 7.61 presents the fourth, fifth and sixth dimensionless natural frequencies, which, as expected, drop significantly as the damping layer thicknesses increase. As expected, the Double-Skins (D-S) and Single (S) configurations stand out from the remaining ones, presenting less natural frequency variation since those are the stacking sequences promoting less damping capacity for these vibration modes. Generally, there are no significant differences between reference and thick laminates, but the natural frequencies of thicker laminates tend to drop more steeply for higher damping layer thicknesses.

**Figure 7.60** Influence of damping layer thickness and position on natural frequencies

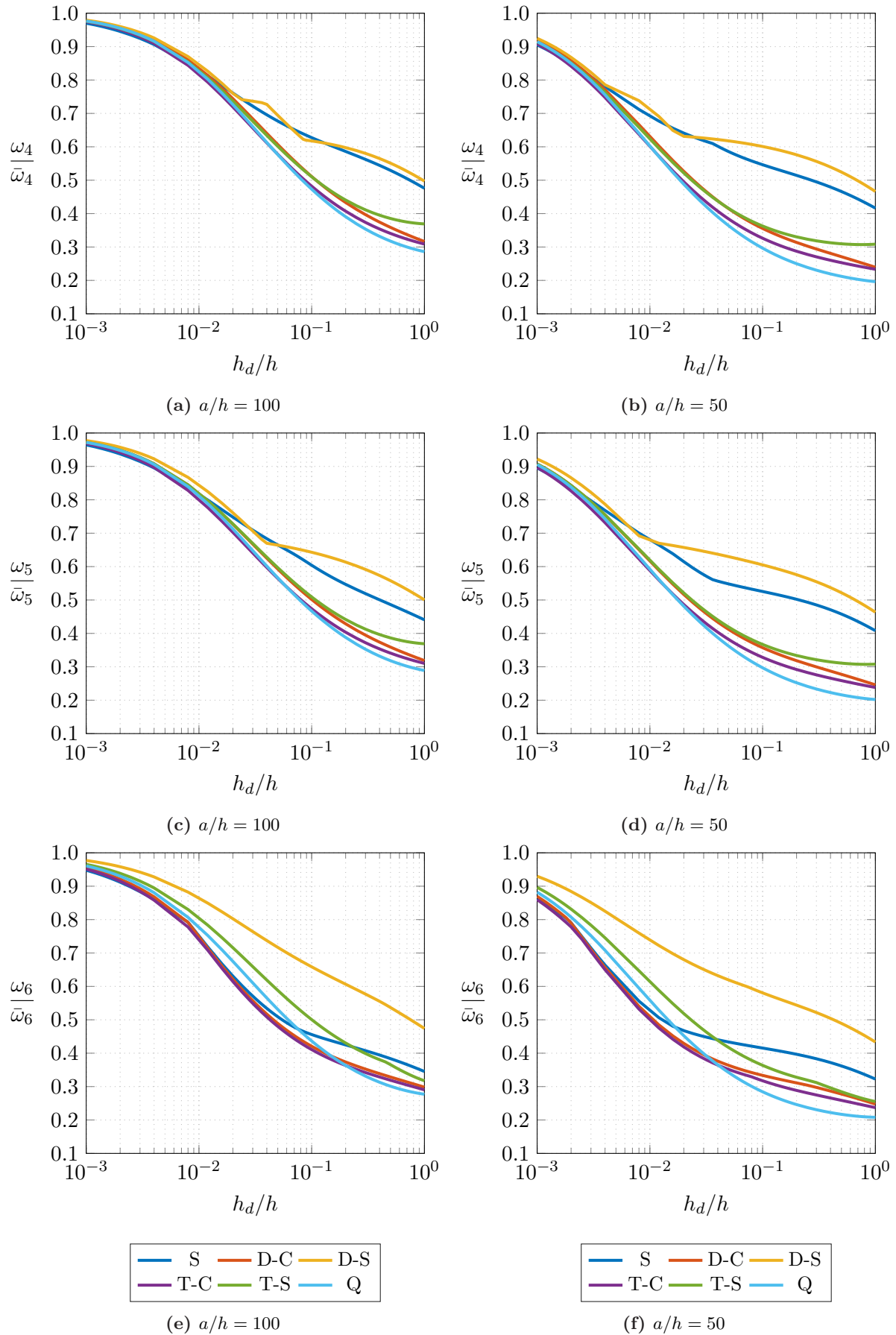


Figure 7.61 Influence of damping layer thickness and position on natural frequencies

7.2.2.2 | Influence of Ply Angle

In the previous sections, one considered the six symmetric stacking sequences presented in Figures 7.5a to 7.5f. These arrangements cover the range of possible symmetric configurations for a six-layer reference laminate, including damping layers within the stacking sequence. Compliant layers, composite plies adjacent to soft damping layers, present variable orientation angles, represented by tags θ and $90^\circ - \theta$. In this section, one studies the influence of the ply-angle (θ) on the damping capacity and dynamic stiffness of the laminated composite plate. Figures 7.62 to 7.73 show, consecutively, the first six specific modal loss factors as a function of the dimensionless damping layer thickness h_d/h and ply angle θ for the six stacking sequences under analysis in this chapter and considering the reference and thick composite base plates with an aspect ratio $a/h = 100$ and $a/h = 50$, side-by-side.

For the first specific modal loss factor of the reference and thick composite base plates, Figures 7.62 and 7.63 show that damping capacity is almost exclusively a function of the damping layer thickness, and the ply angle of compliant layers has practically no influence. For the reference plate, Double-Core (D-C), Triple-Core (T-C), and Quintuple (Q) configurations present the best damping capacities, as already seen in Figure 7.52a. Likewise, the Quintuple (Q) arrangement exhibits the highest modal loss factor for the thick composite plate. For higher modes, damping capacity no longer varies linearly with damping layer thickness, and the discussion demands some detail. Previously, when analysing Figures 7.52 and 7.53, one drew attention to the fact that there is a maximum damping capacity for thick composite plates, after which the modal loss factor dramatically drops. For the second specific modal loss factor, for example, Figures 7.64 and 7.65 show the same pattern for thick composite plates but prove that ply angle plays an important role. For the stacking sequences presenting higher damping capacities for the reference composite plate, it is possible to distinguish $\theta = 0^\circ$ as the best orientation angle for the compliant layers. Increasing the ply angle θ for the same damping layer thickness decreases the modal loss factor. For the Triple-Skins (T-S) and Quintuple (Q) stacking sequences, the modal loss factor increases symmetrically again from $\theta = 45^\circ$ to $\theta = 90^\circ$. The influence of ply orientation for thick composite plates is similar, as depicted in Figure 7.65. Interestingly, the maximum damping capacity for the Quintuple (Q) arrangement occurs for $\theta = 45^\circ$.

Besides, for higher vibration modes, one can notice sharp jumps in the specific modal loss factors. This phenomenon occurs due to the already discussed switch between mode shapes during the parametric analyses. Primarily, and similarly to the simply supported laminated composite plate, this phenomenon arises for the Single (S) and Double-Skins (D-S) stacking sequences, where a thick and soft damping layer decouples the stiff layers of the laminate. In general, the ply angle has some influence on higher vibration modes, but the damping layer thickness and stacking sequence command the damping capacity of the laminate.

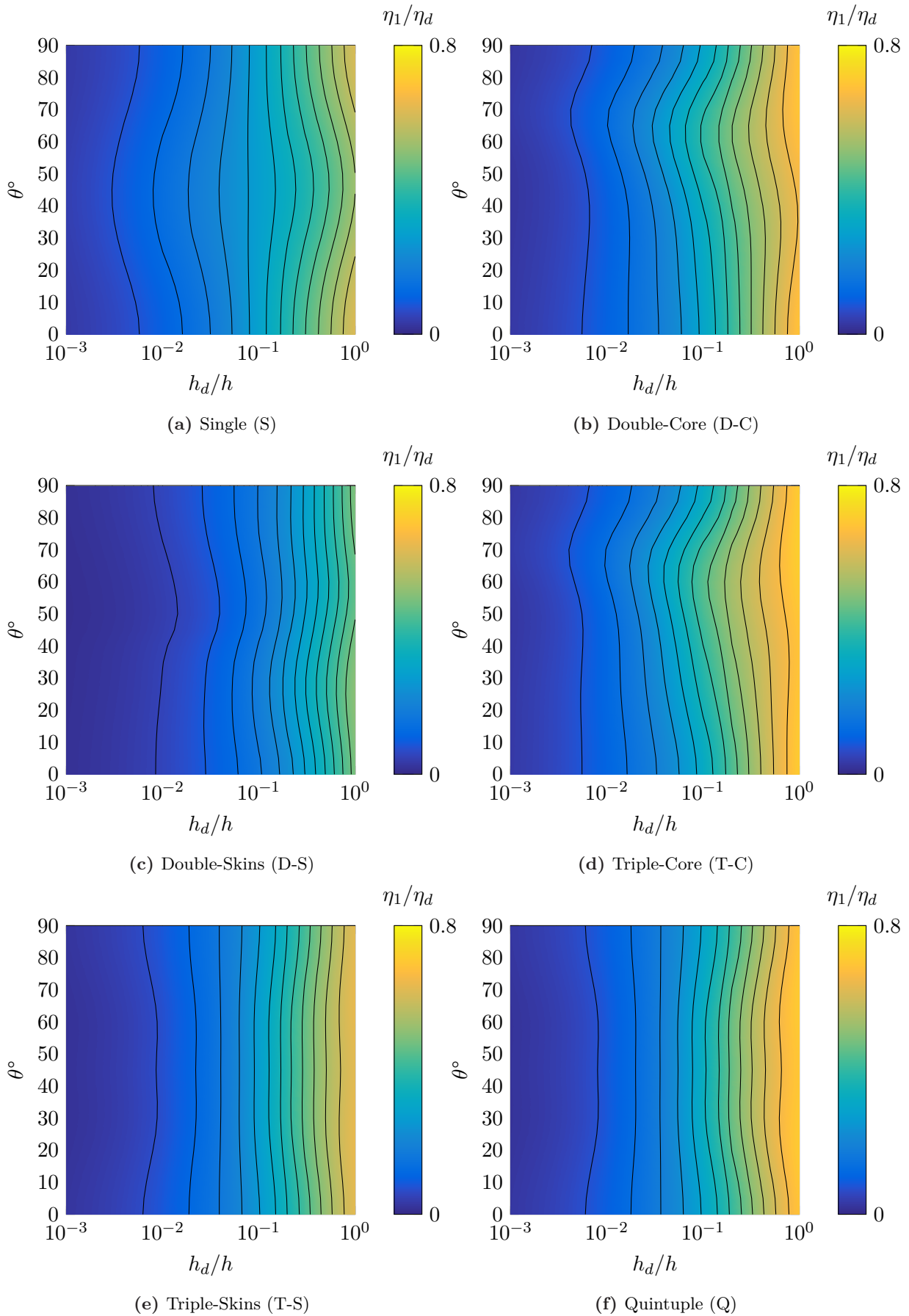


Figure 7.62 Influence of damping layer thickness and ply angle on the 1st modal loss factor, $a/h = 100$

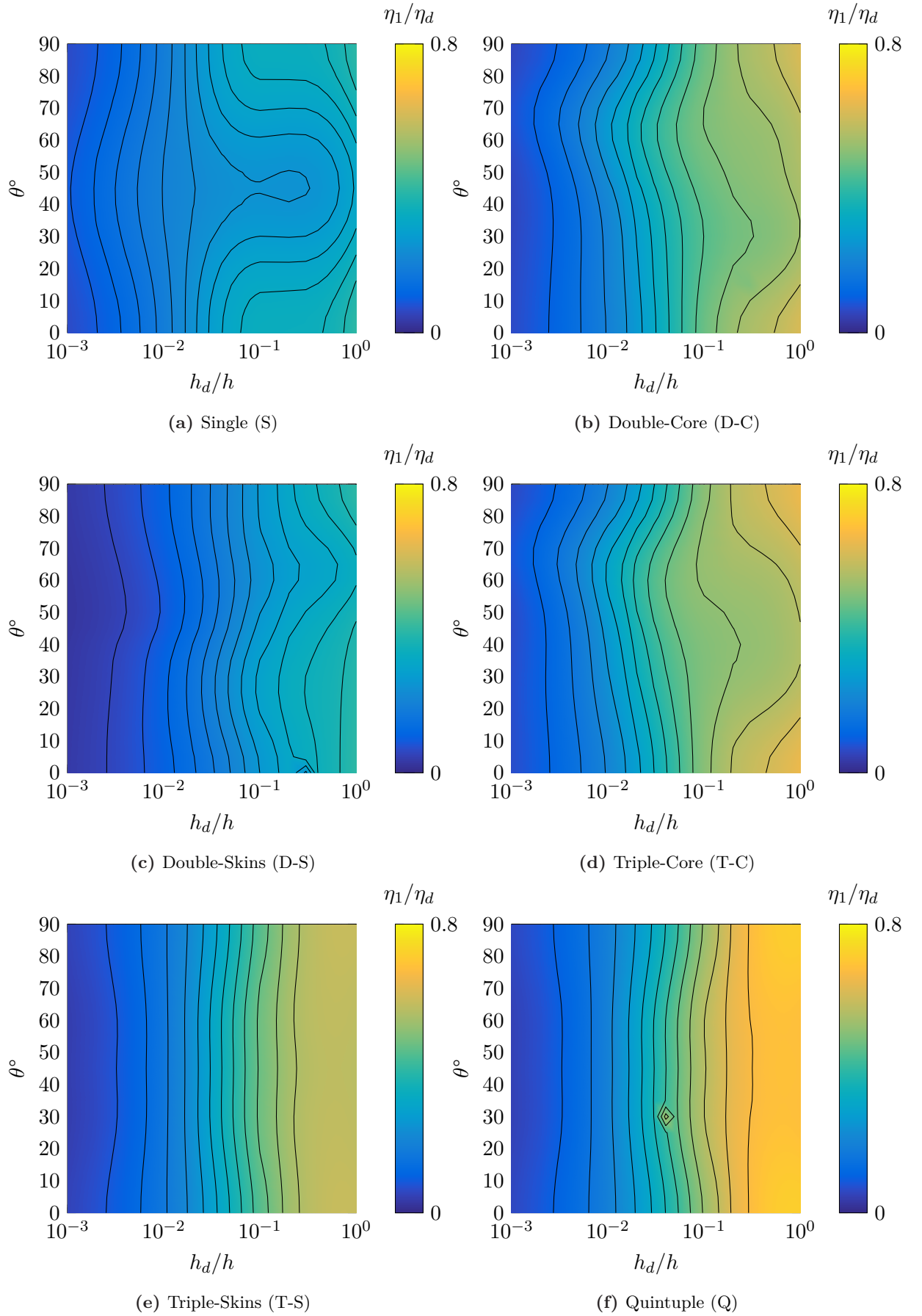


Figure 7.63 Influence of damping layer thickness and ply angle on the 1st modal loss factor, $a/h = 50$

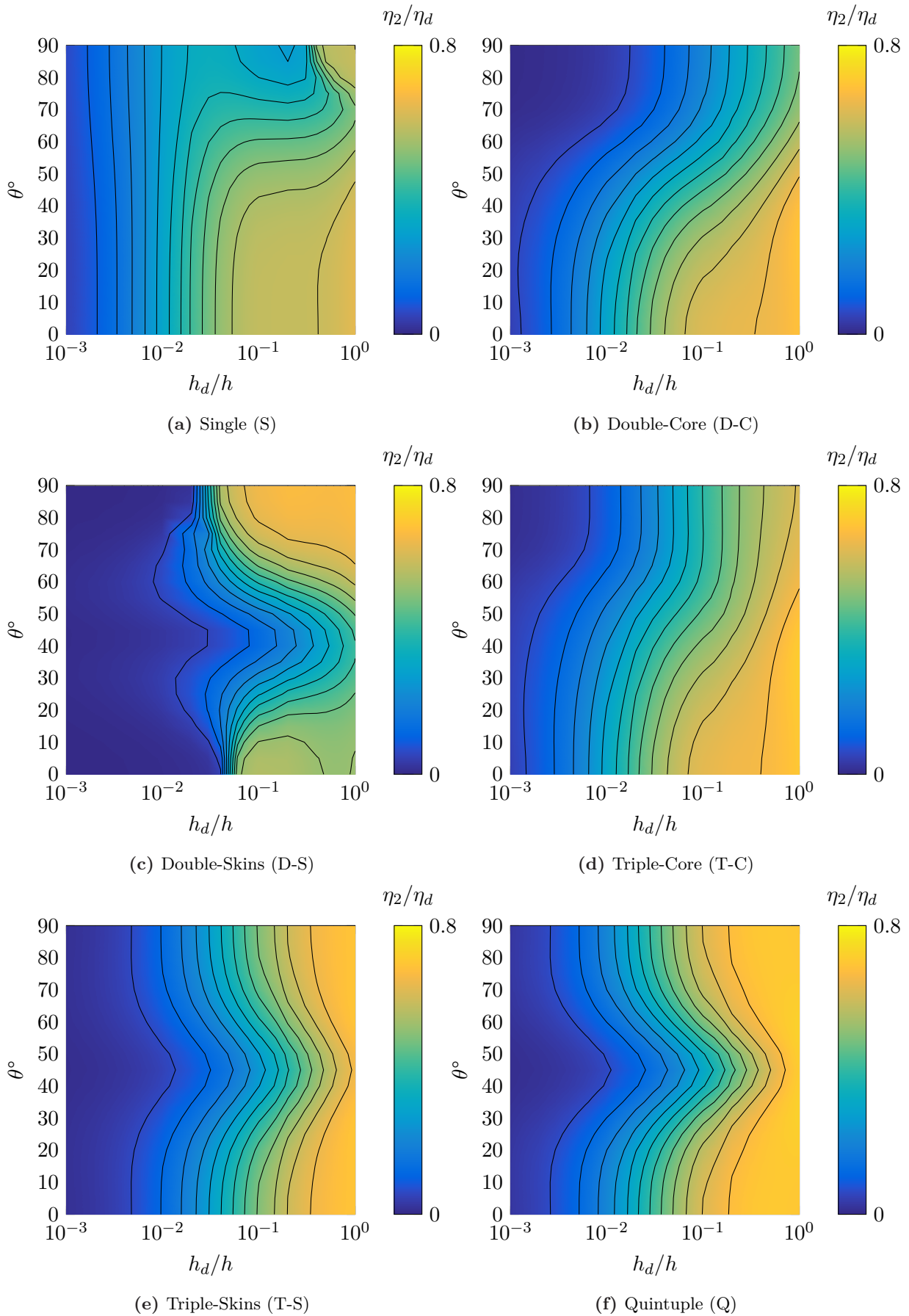


Figure 7.64 Influence of damping layer thickness and ply angle on the 2nd modal loss factor, $a/h = 100$

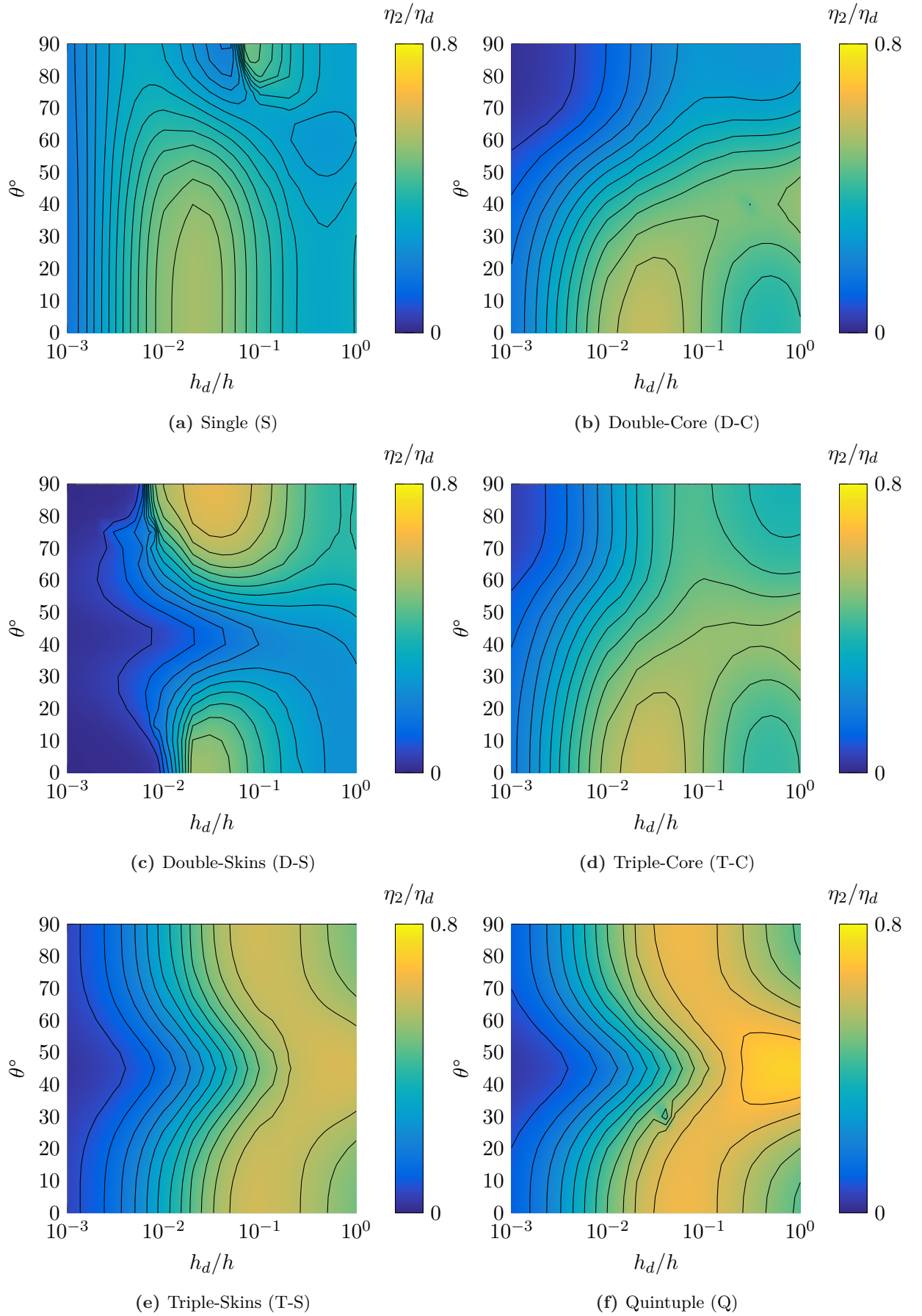


Figure 7.65 Influence of damping layer thickness and ply angle on the 2nd modal loss factor, $a/h = 50$

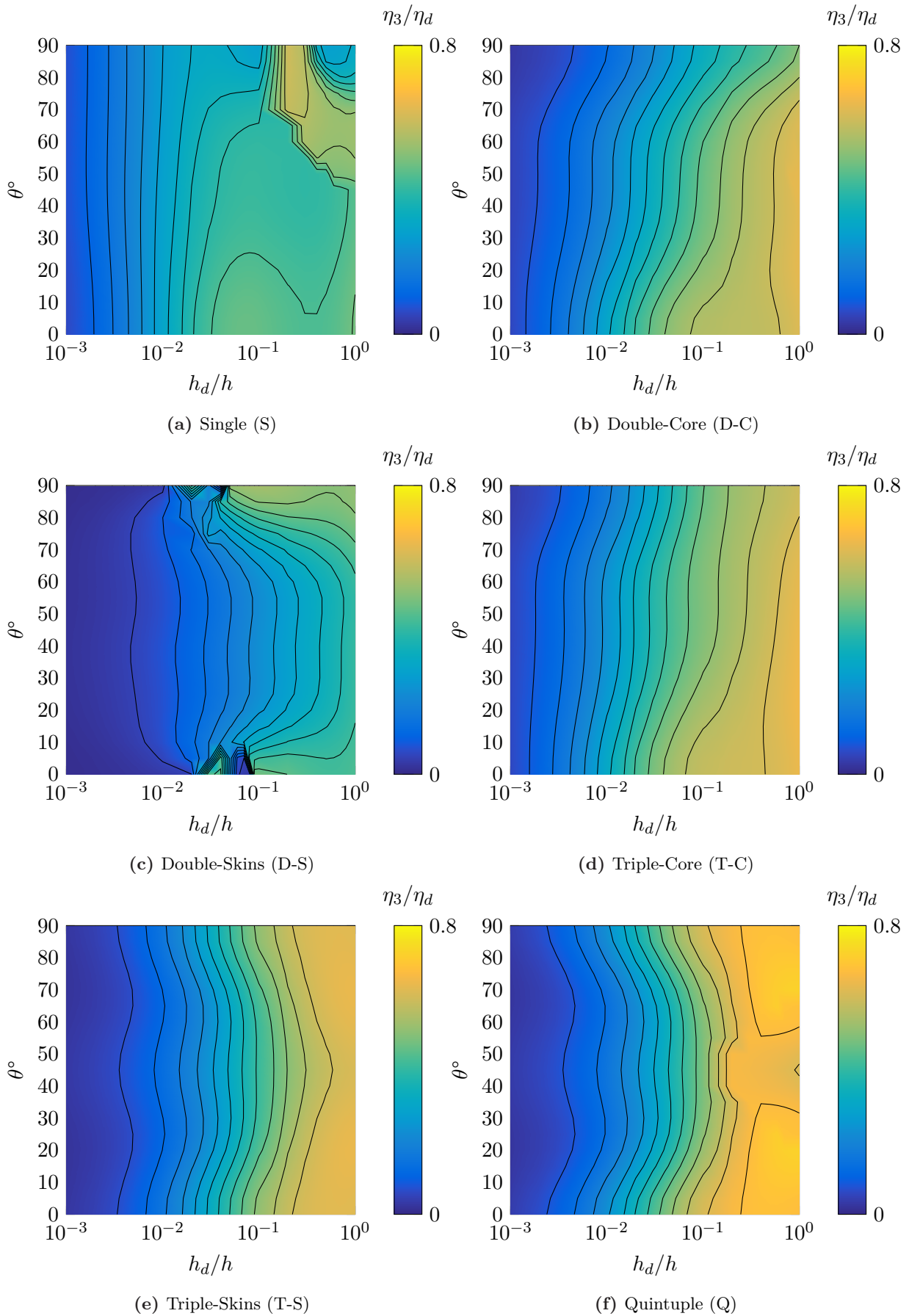


Figure 7.66 Influence of damping layer thickness and ply angle on the 3rd modal loss factor, $a/h = 100$

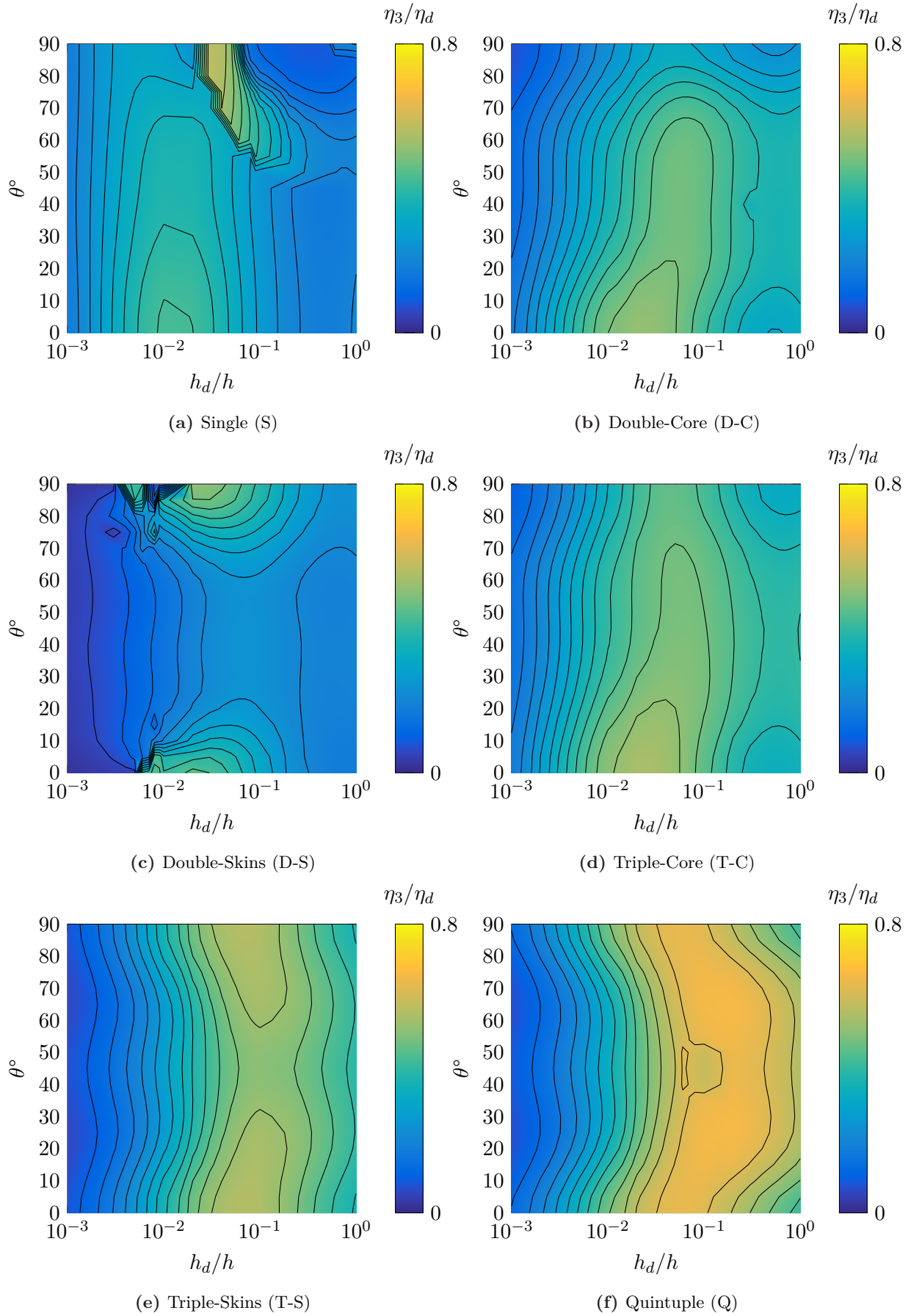


Figure 7.67 Influence of damping layer thickness and ply angle on the 3rd modal loss factor, $a/h = 50$

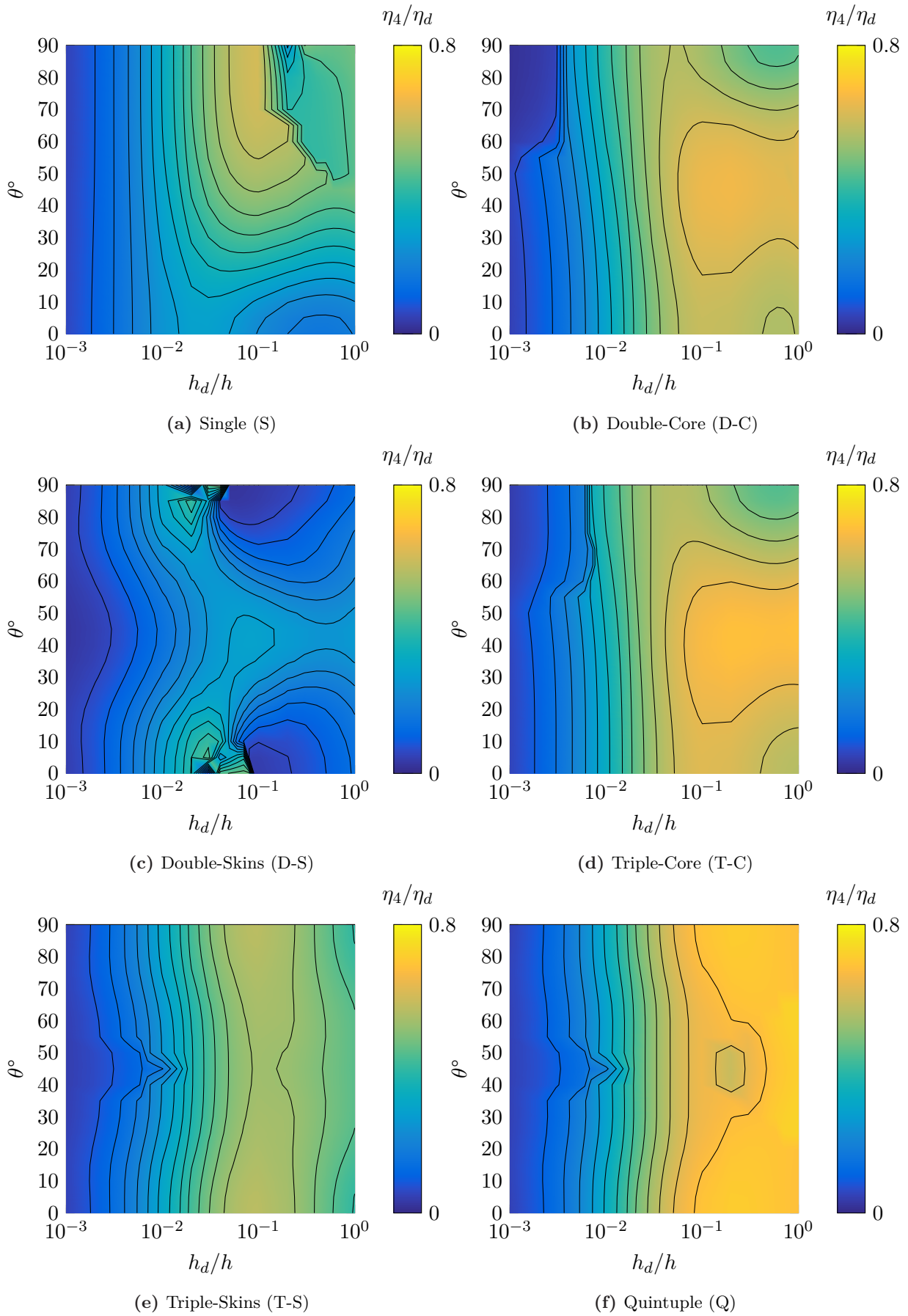


Figure 7.68 Influence of damping layer thickness and ply angle on the 4th modal loss factor, $a/h = 100$

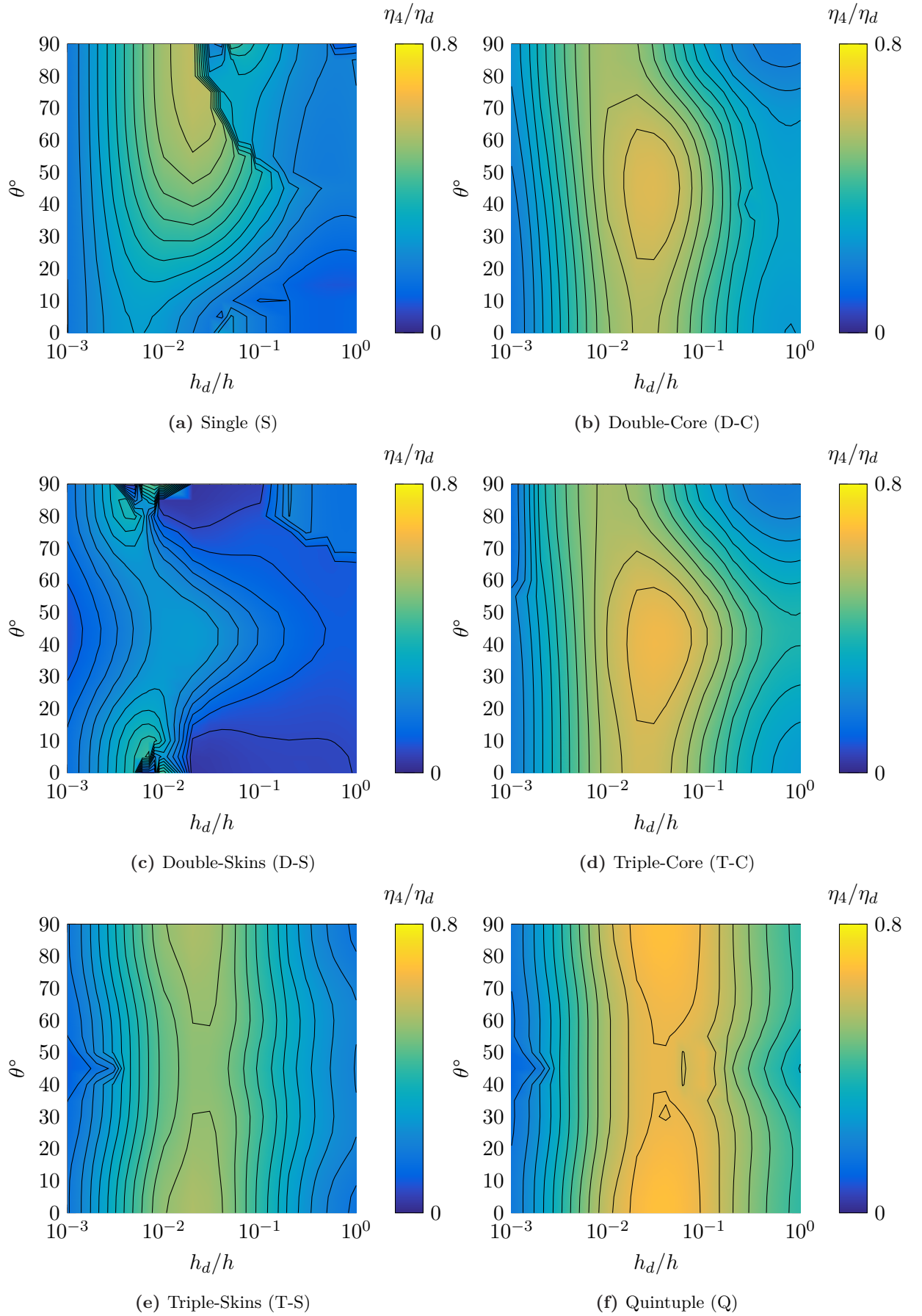


Figure 7.69 Influence of damping layer thickness and ply angle on the 4th modal loss factor, $a/h = 50$

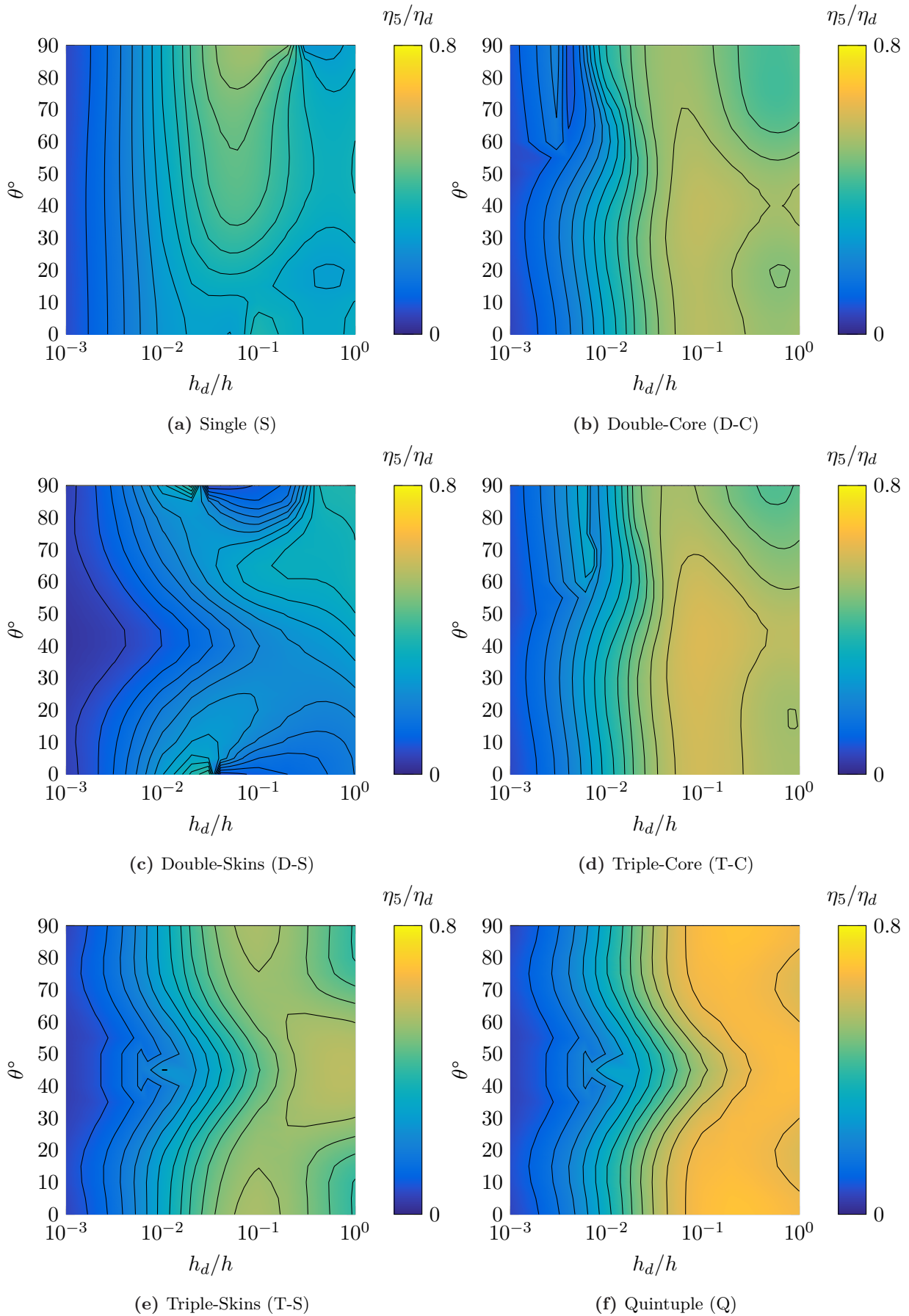


Figure 7.70 Influence of damping layer thickness and ply angle on the 5th modal loss factor, $a/h = 100$

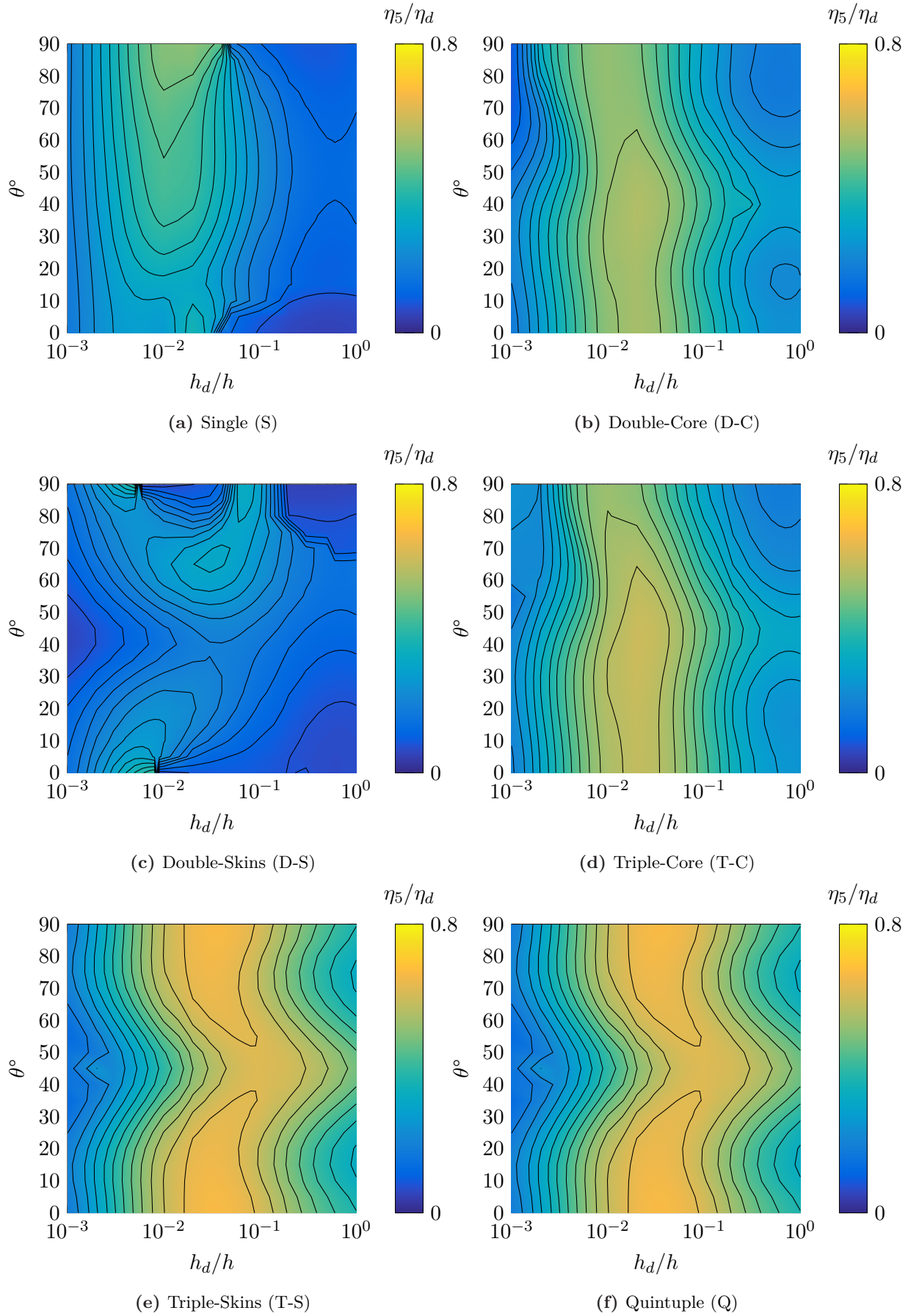


Figure 7.71 Influence of damping layer thickness and ply angle on the 5th modal loss factor, $a/h = 50$

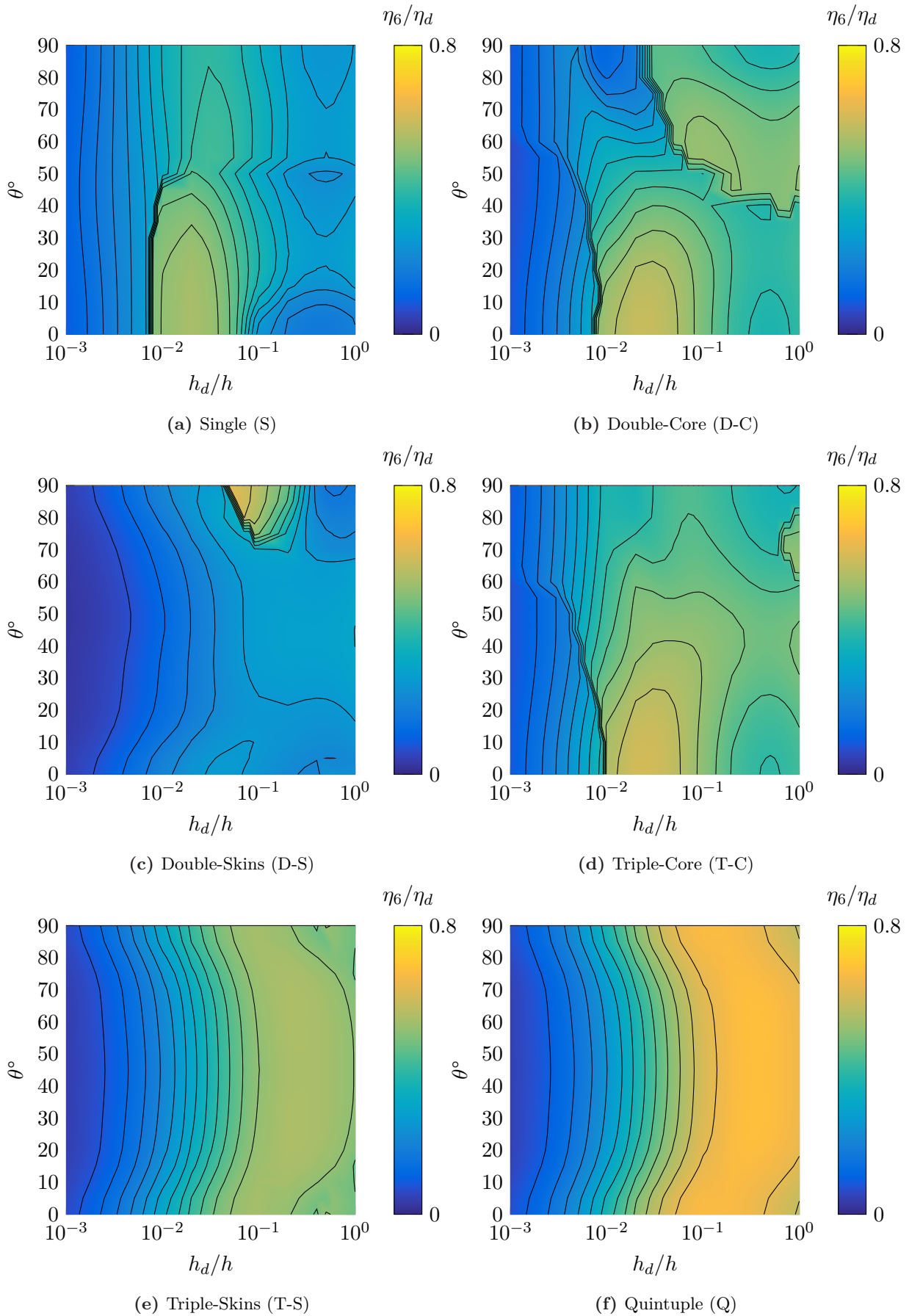


Figure 7.72 Influence of damping layer thickness and ply angle on the 6th modal loss factor, $a/h = 100$

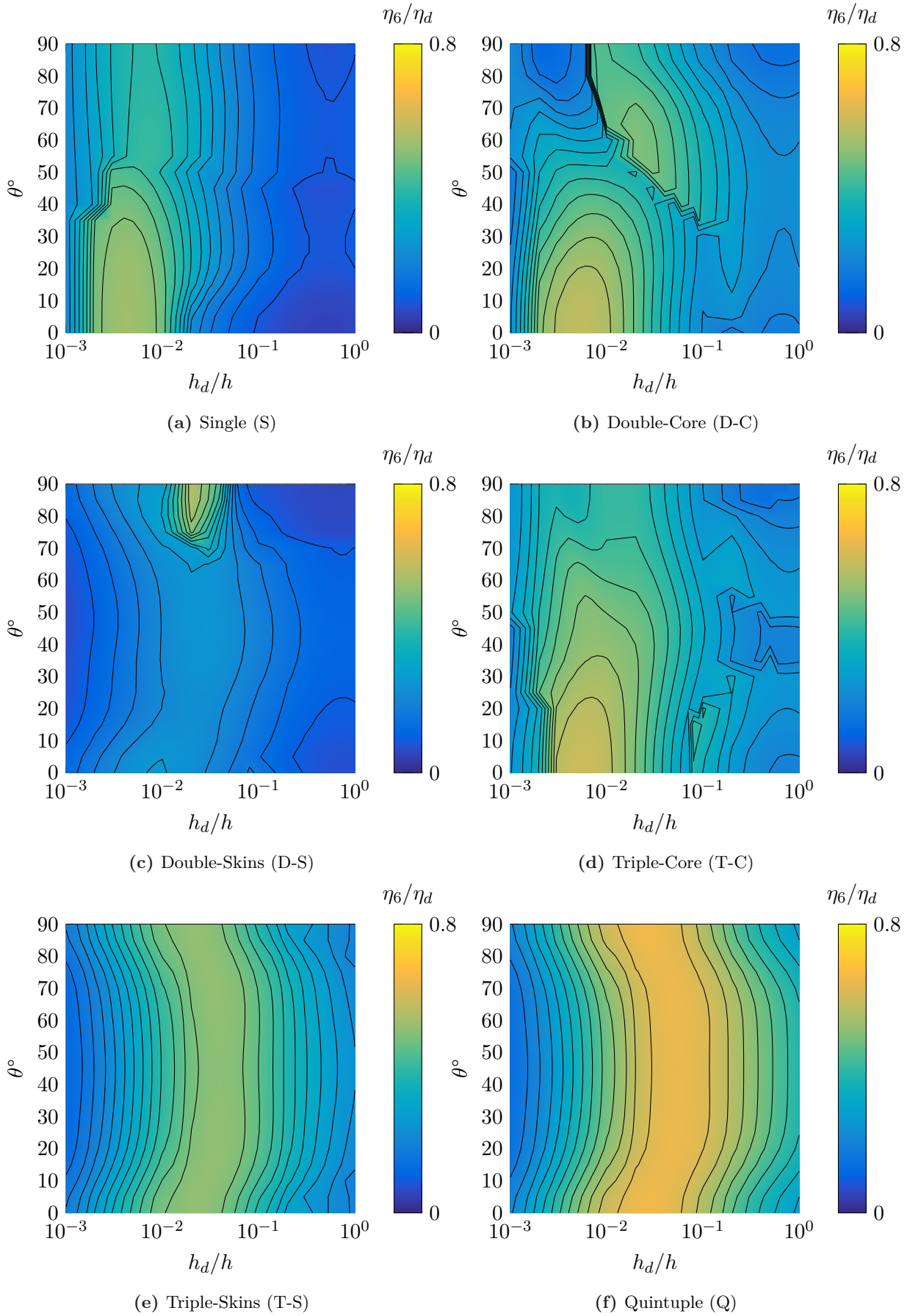


Figure 7.73 Influence of damping layer thickness and ply angle on the 6th modal loss factor, $a/h = 50$

Throughout the work, one mentioned that design requirements for high damping capacity often clash with those for high stiffness. To study the influence of the ply angle on the natural frequencies of laminated composite plates with interleaved soft damping layers, Figures 7.74 to 7.85 present the first six dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ as a function of the ply angle θ and the dimensionless damping layer thickness h_d/h for the reference and thick composite plates with aspect ratios of $a/h = 100$ and $a/h = 50$, side-by-side. Dimensionless natural frequencies $\omega_n/\bar{\omega}_n$ represent the ratio between the natural frequencies of the laminated composite with interleaved damping layer(s) and those of the reference six-layer composite plate.

For the first normal mode, Figures 7.74 and 7.75, the ply angle has a strong influence on the natural frequency; regardless of the stacking sequence, the first natural frequency increases for θ around 45° . As expected, the laminate stiffness decreases with increasing damping layer thickness, and so does the natural frequency. Figures 7.76 and 7.77 exhibit the second dimensionless natural frequency for reference and thick base composite plates, respectively, and show that the ply angle can no longer increase the natural frequency compared to the base plate. As previously noted, for this mode, the Triple-Skins (T-S) and Quintuple (Q) configurations present the best overall damping capacities, mainly for $\theta = 0^\circ$ or $\theta = 90^\circ$. It is important to emphasise in Figures 7.76 and 7.77 that higher natural frequencies also arise for $\theta = 0^\circ$ or $\theta = 90^\circ$. This result proves that even though the natural frequency decreases with increasing damping layer thickness, there is room for optimisation and control of the stiffness penalties.

For higher normal modes, the influence of the compliant ply angle and damping layer thickness on the dimensionless natural frequencies is similar to that of the second normal mode; higher natural frequencies for $\theta = 0^\circ$ or $\theta = 90^\circ$ and thinner damping layers.

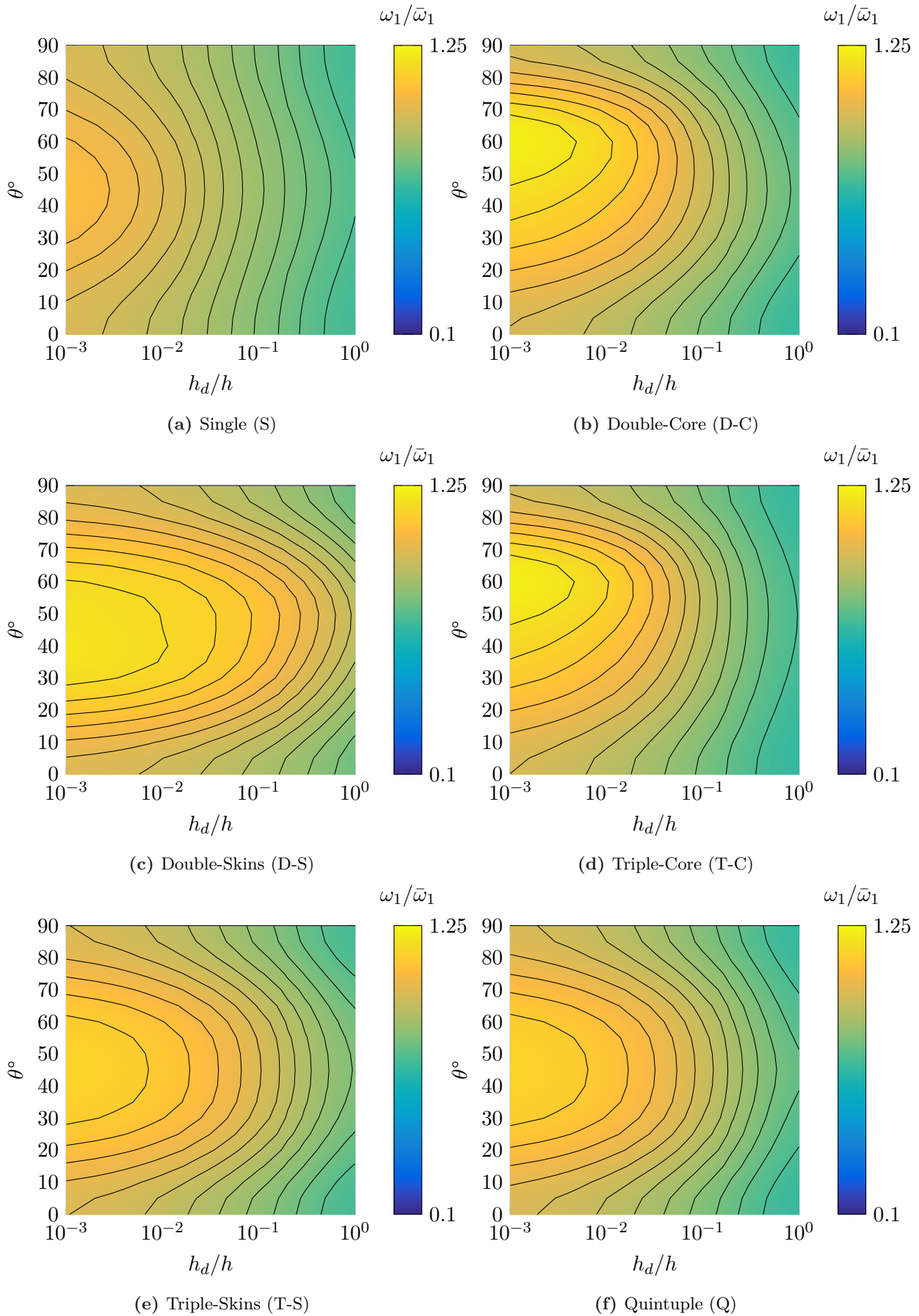


Figure 7.74 Influence of damping layer thickness and ply angle on the 1st natural frequency, $a/h = 100$

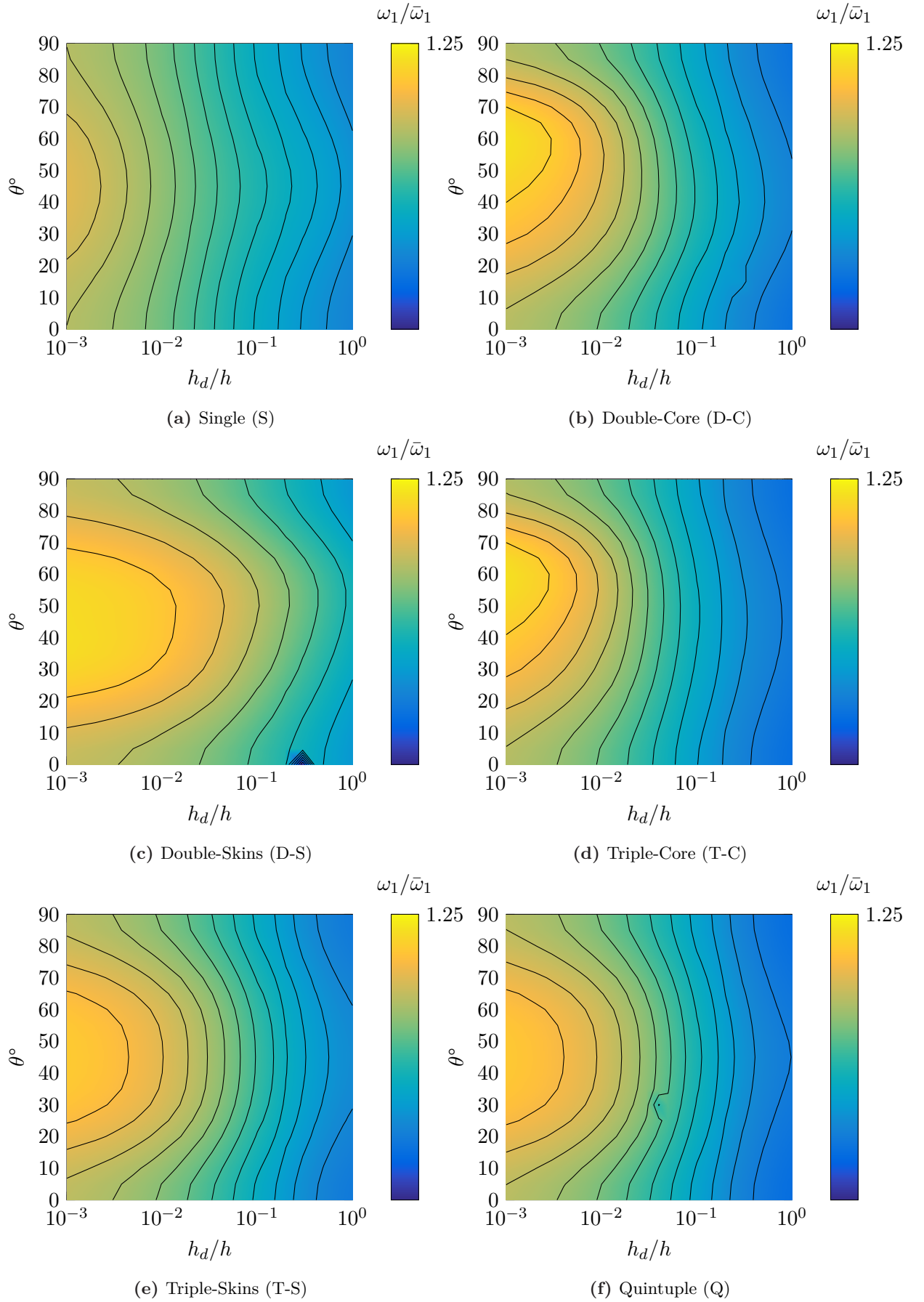


Figure 7.75 Influence of damping layer thickness and ply angle on the 1st natural frequency, $a/h = 50$

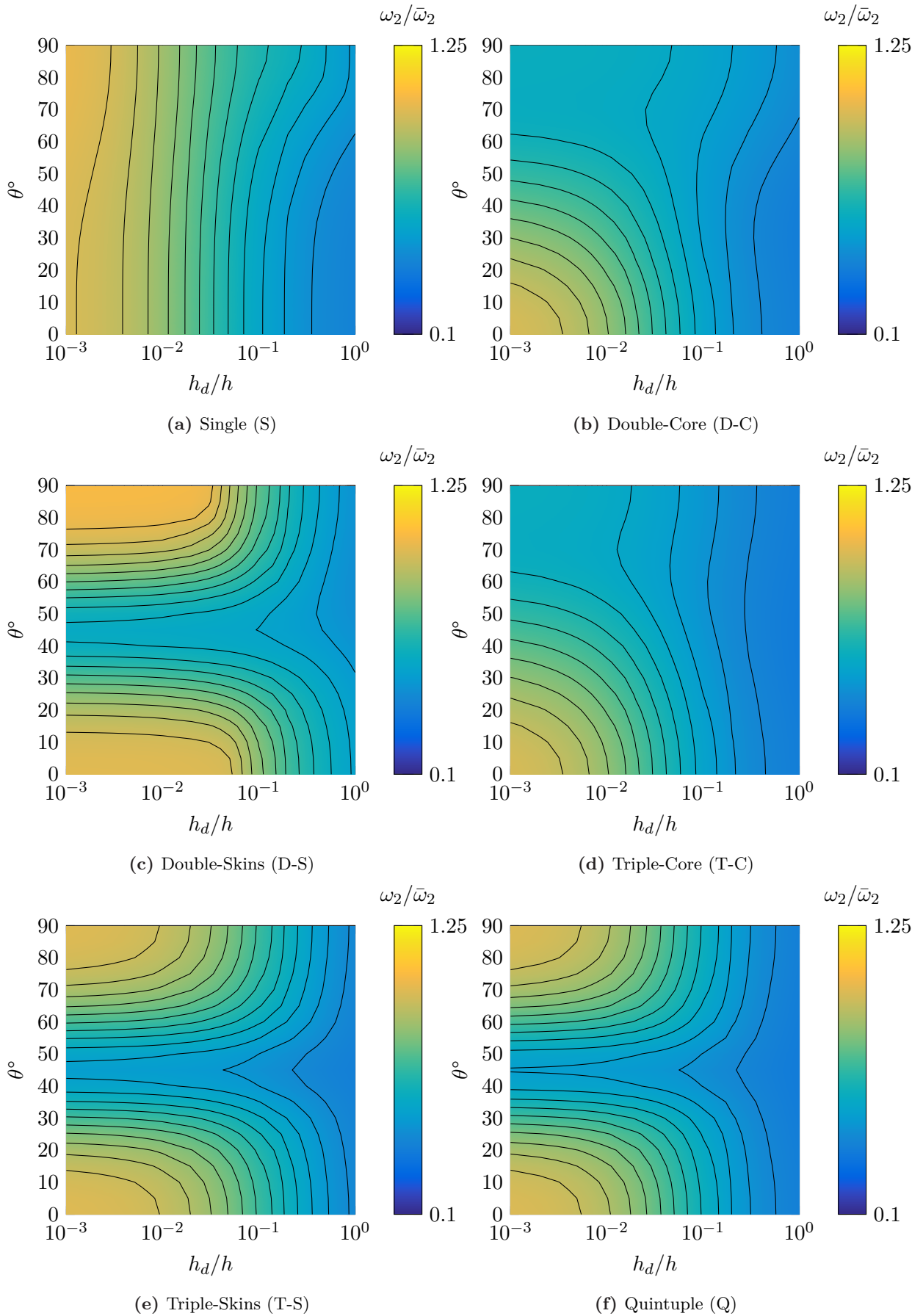


Figure 7.76 Influence of damping layer thickness and ply angle on the 2nd natural frequency, $a/h = 100$

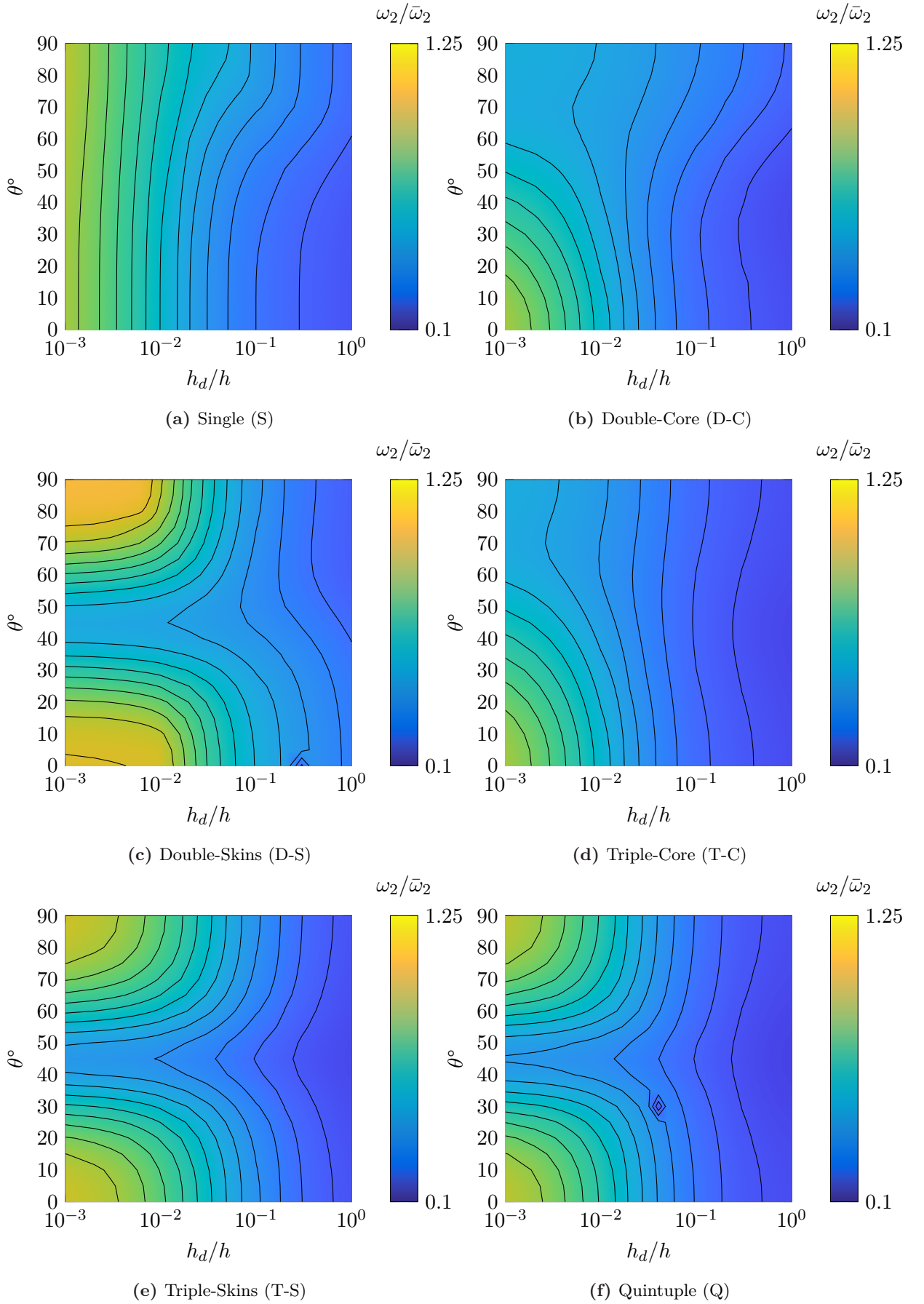


Figure 7.77 Influence of damping layer thickness and ply angle on the 2nd natural frequency, $a/h = 50$

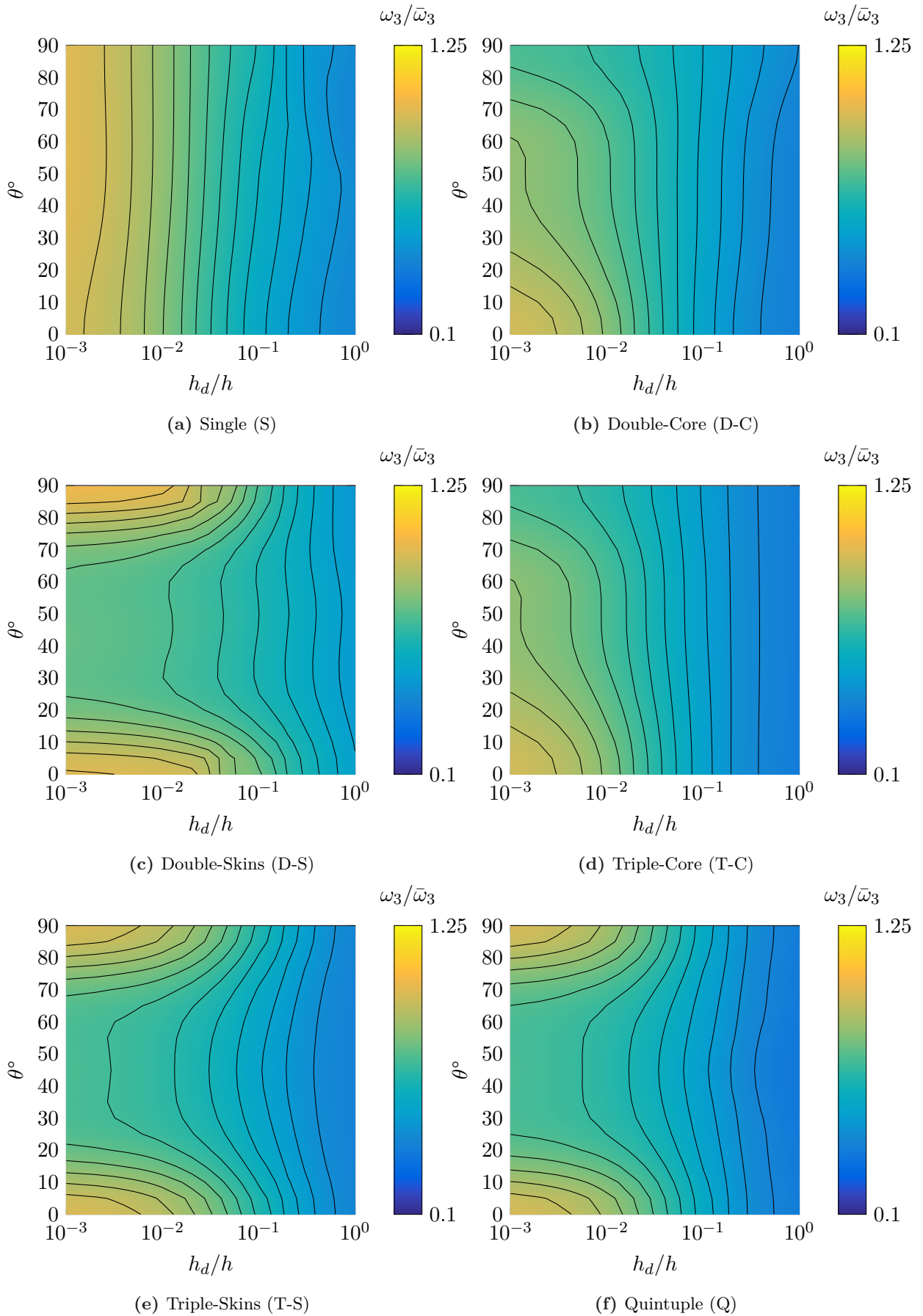


Figure 7.78 Influence of damping layer thickness and ply angle on the 3rd natural frequency, $a/h = 100$

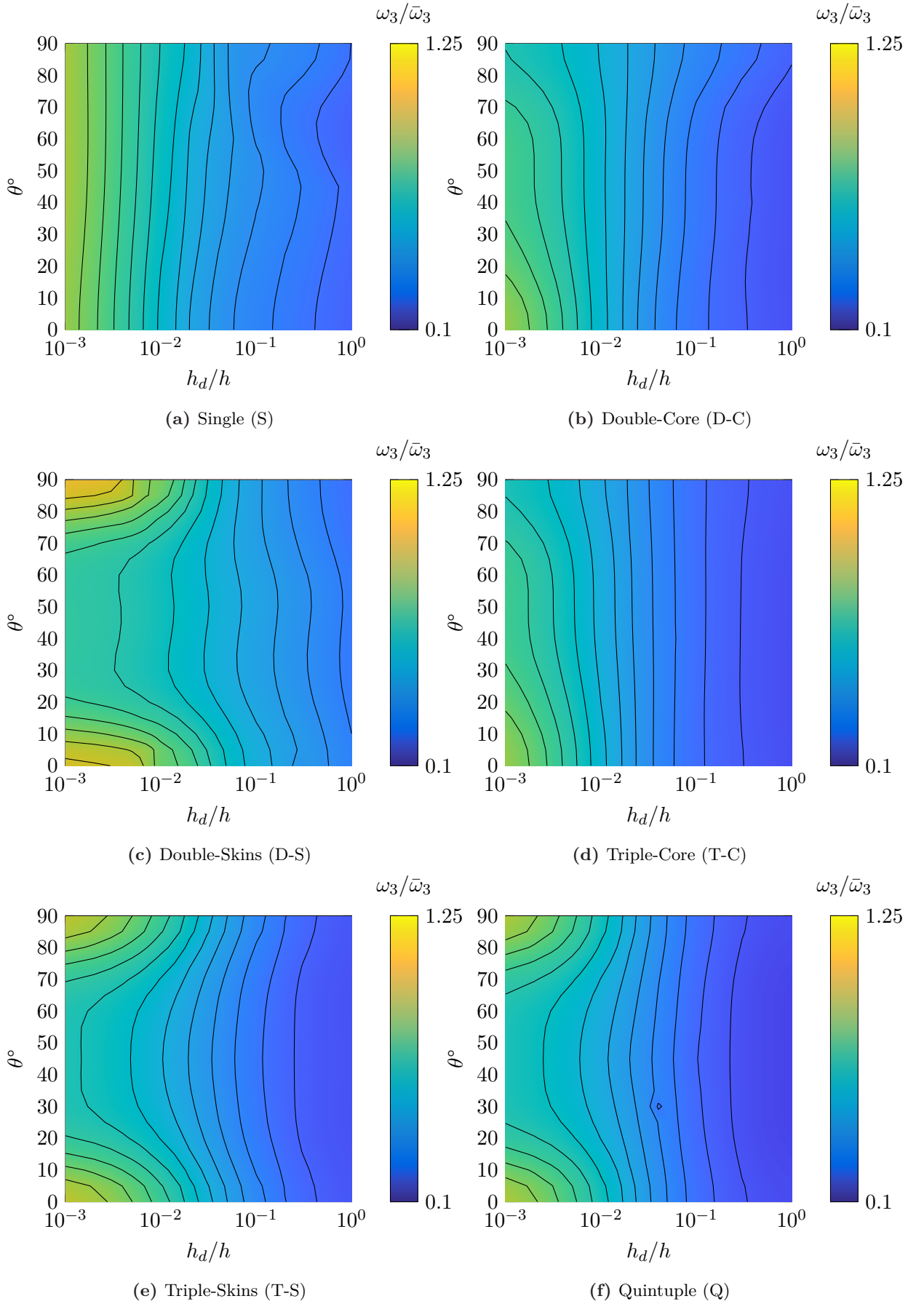


Figure 7.79 Influence of damping layer thickness and ply angle on the 3rd natural frequency, $a/h = 50$

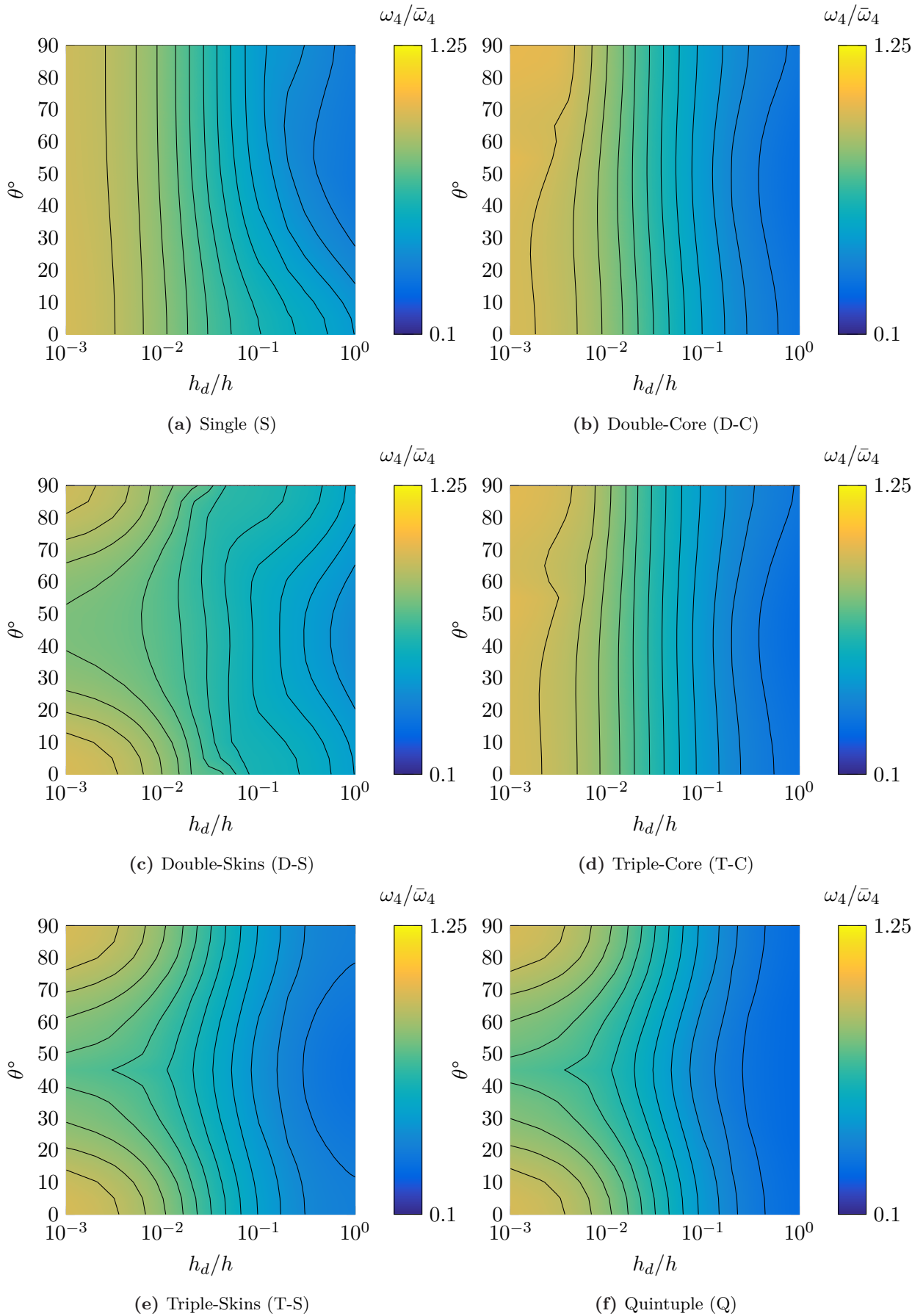


Figure 7.80 Influence of damping layer thickness and ply angle on the 4th natural frequency, $a/h = 100$

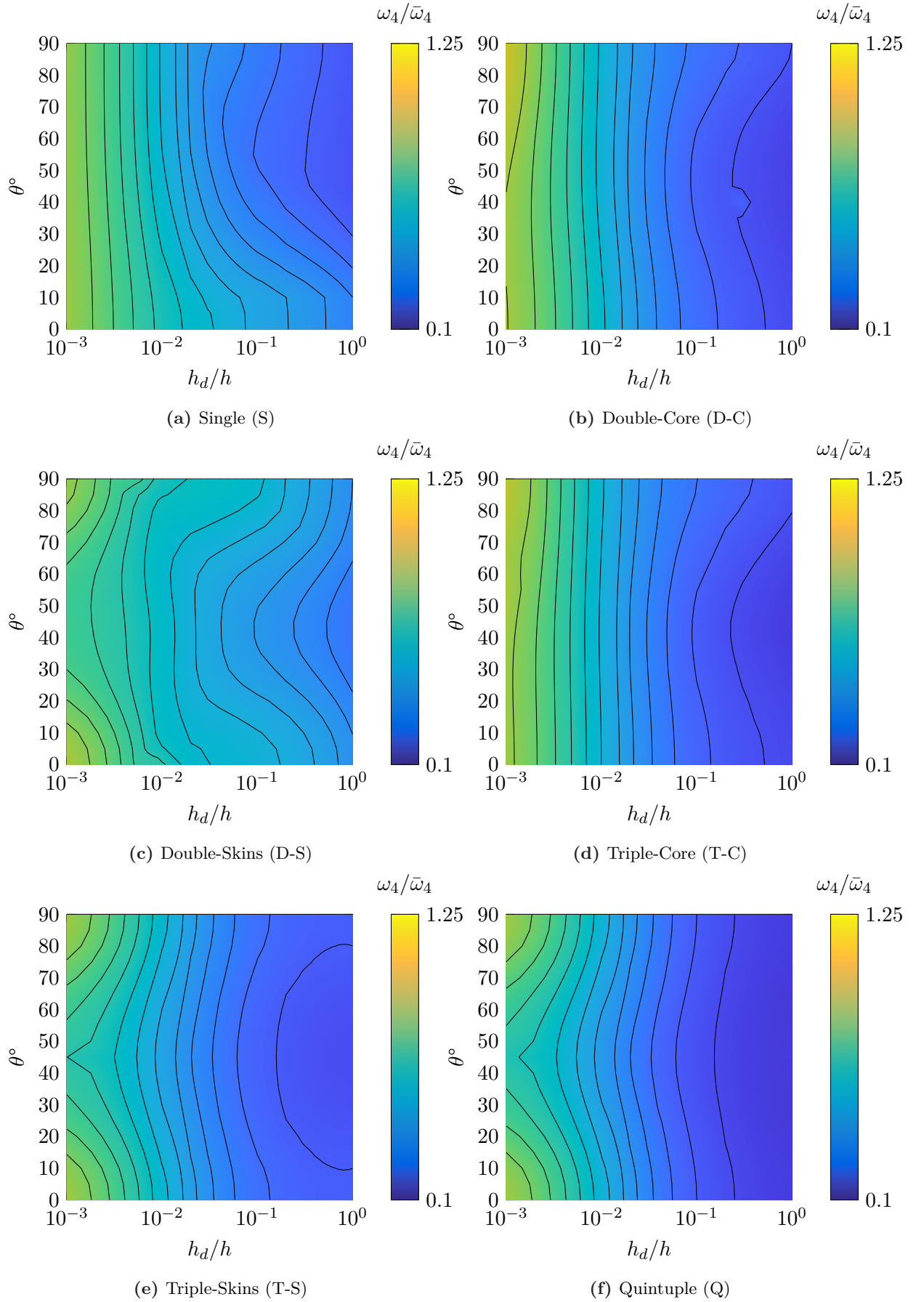


Figure 7.81 Influence of damping layer thickness and ply angle on the 4th natural frequency, $a/h = 50$

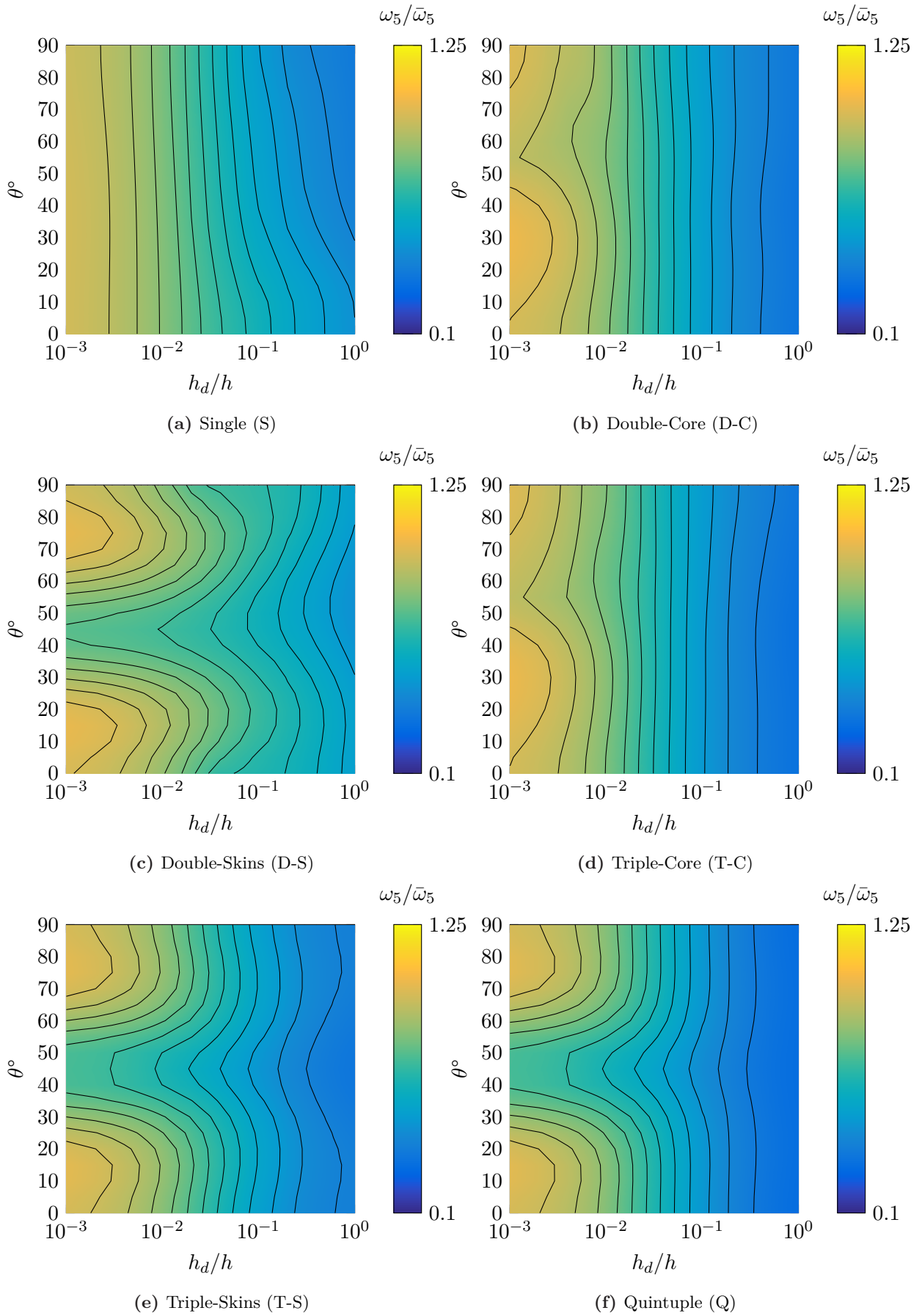


Figure 7.82 Influence of damping layer thickness and ply angle on the 5th natural frequency, $a/h = 100$

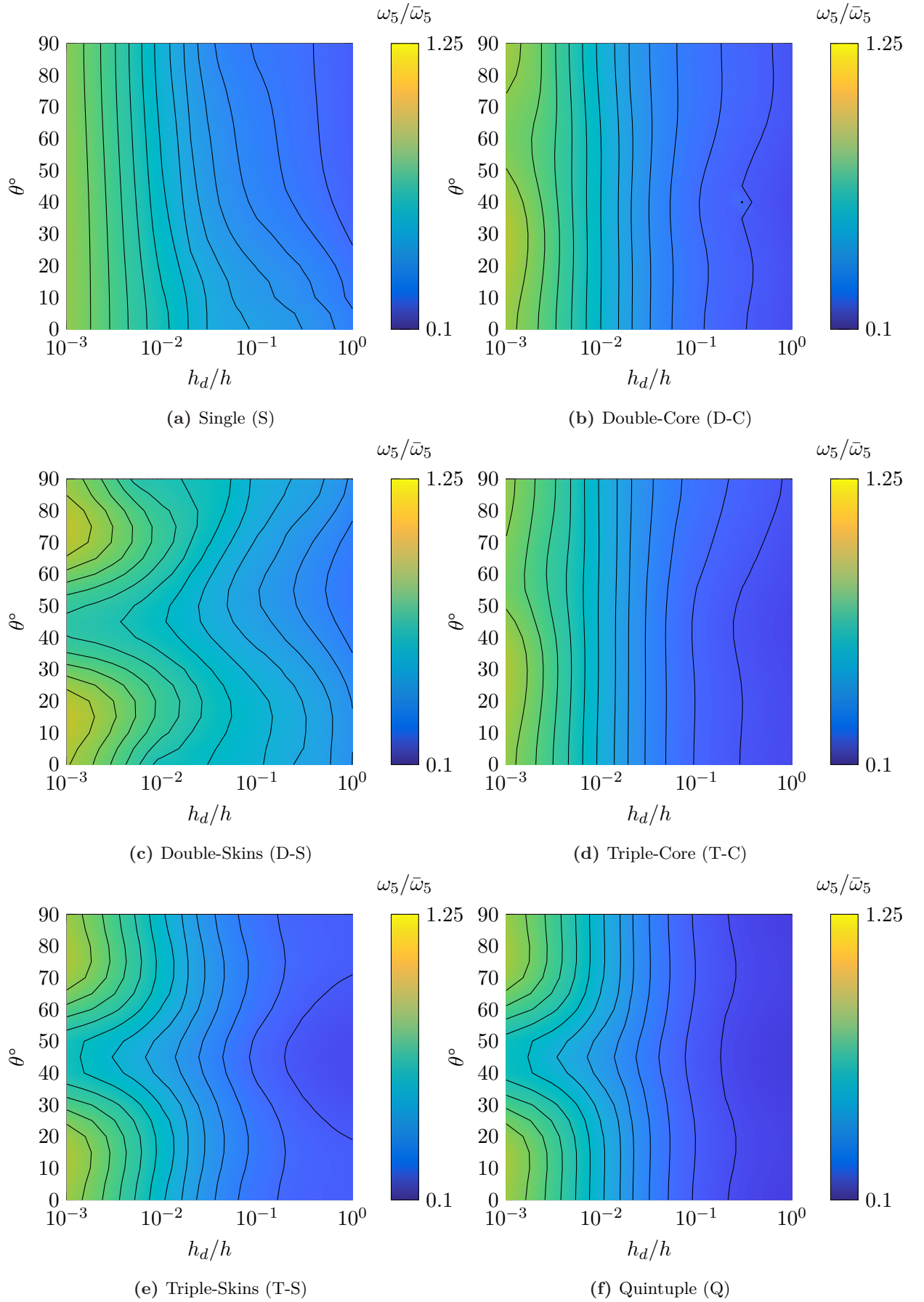


Figure 7.83 Influence of damping layer thickness and ply angle on the 5th natural frequency, $a/h = 50$

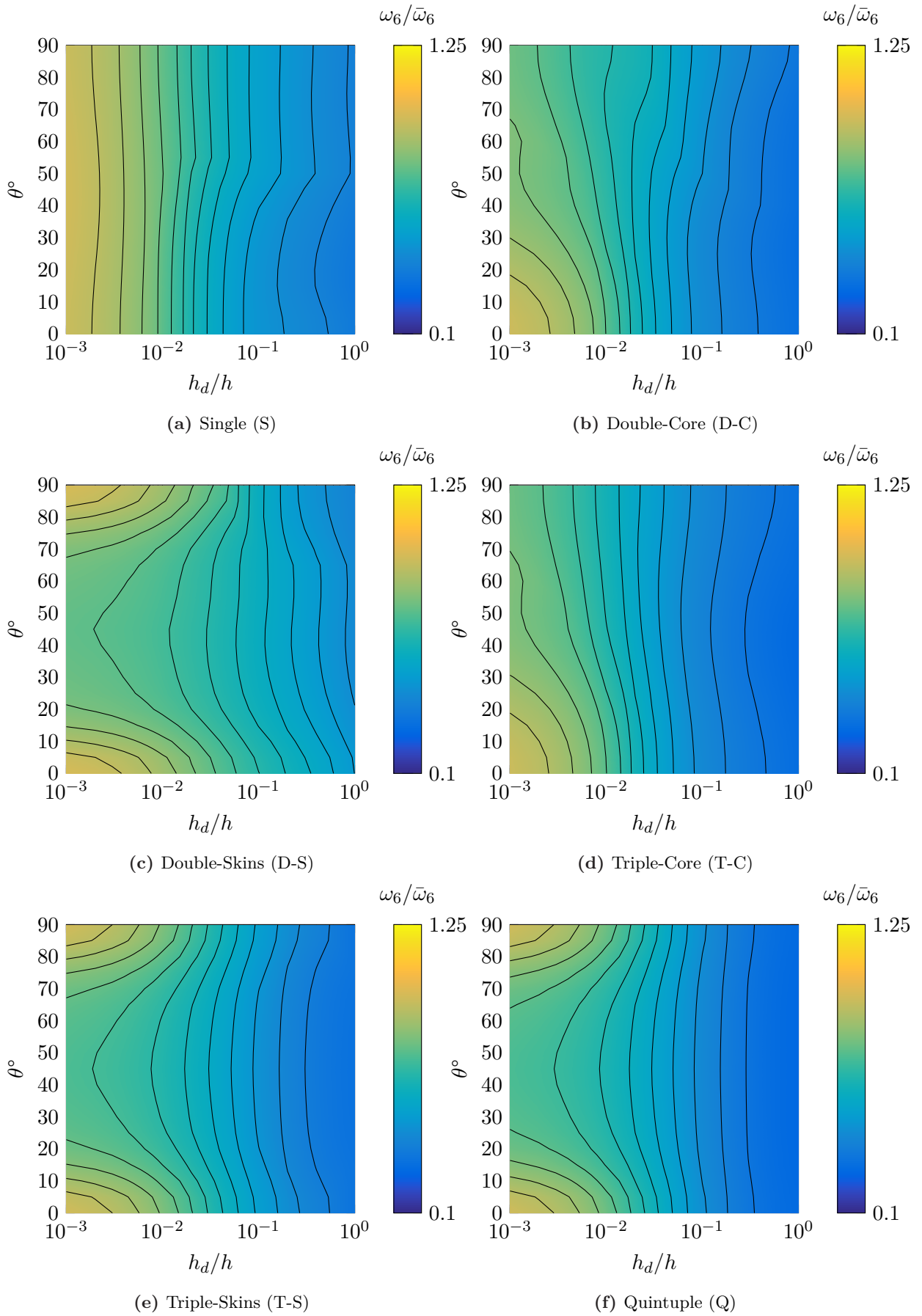


Figure 7.84 Influence of damping layer thickness and ply angle on the 6th natural frequency, $a/h = 100$

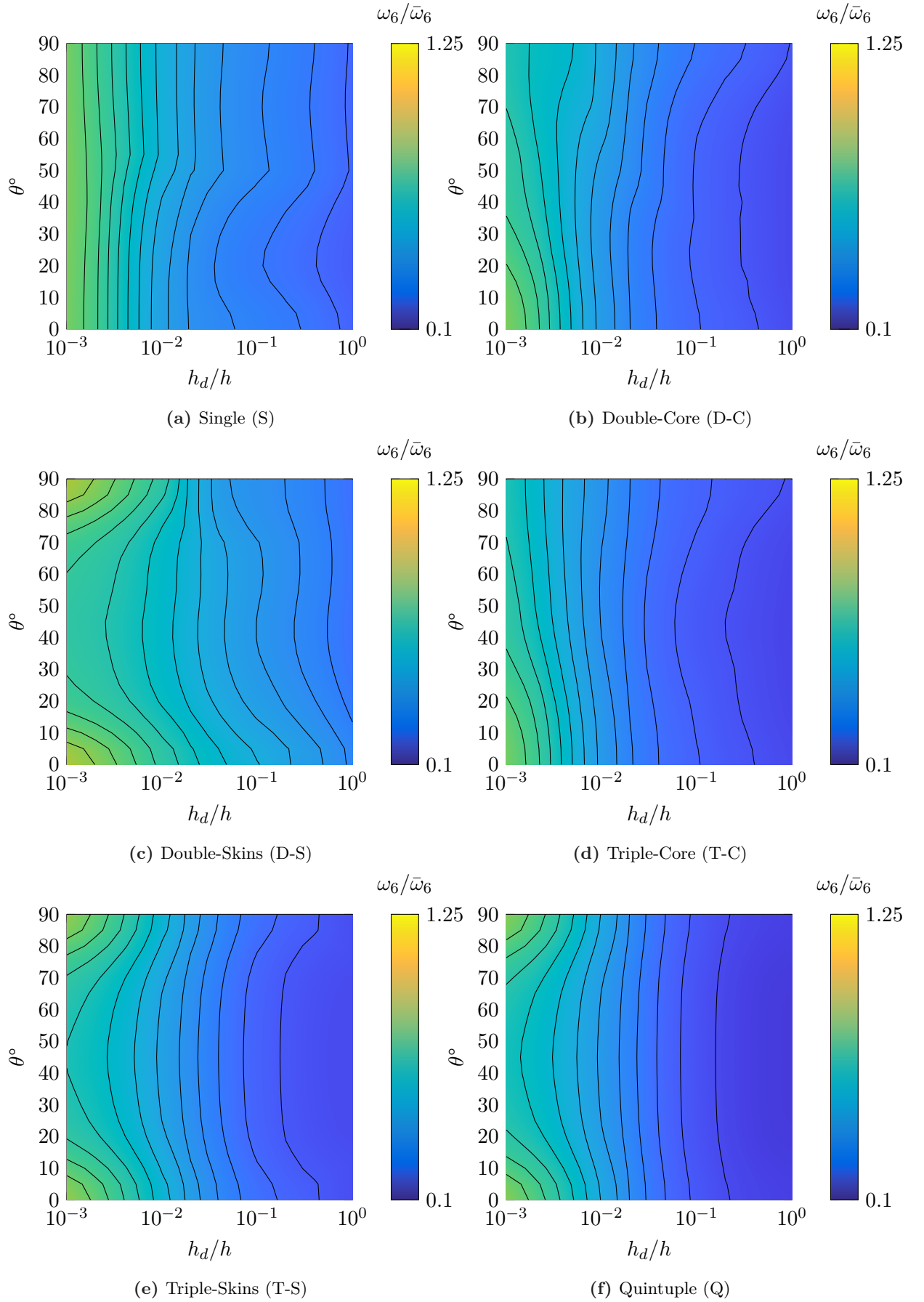


Figure 7.85 Influence of damping layer thickness and ply angle on the 6th natural frequency, $a/h = 50$

7.3 | Laminated Plates with Interleaved Damping Patches

In the previous section, one analysed and discussed the classical method of embedding damping layers into the stacking sequence of laminated composites to improve damping capacity. However, as extensively demonstrated, embedding soft layers into a fibre-reinforced plastic laminate may excessively undermine the structure's stiffness. In addition, one can also compromise the mechanical strength, and the increase in weight may be excessive.

In Section 7.1.2, alternatively to the classical damping treatments with interleaved damping layers in the stacking sequence, one presented a different approach with viscoelastic materials partially distributed in specific layers in a configuration one can refer to as interleaved damping patches. As extensively reported, the ultimate purpose is to improve the damping capacity of laminated composites without excessively degrading the stiffness or mechanical strength or unnecessarily increasing the structure's weight. It has been extensively demonstrated throughout this work that damping patches of viscoelastic materials are an efficient method of passive vibration control when applied as a surface treatment. Here, one suggests the numerical investigation of viscoelastic damping patches for laminated composite plates interleaved in the stacking sequence.

This section addresses the parametric study of the damping efficiency of interleaved damping patches in laminated composite plates. One considers a dimensionless analysis of eight stacking sequences to evaluate the influence of damping patches on the damping capacity. Figure 7.86 depicts the reference stacking sequence of a symmetric laminate consisting of 0° and 90° oriented layers, $[90^\circ/0^\circ/90^\circ]_S$, the same reference stacking sequence of the previous section.

Adopting the same mechanical properties for the unidirectional carbon fabrics as those considered in the previous section and presented in Table 7.1, one studies a square plate with side length a and six layers of equal thickness and a total laminate thickness of h . Similarly, Table 7.2 exhibits the mechanical properties of the frequency and temperature-independent damping layers considered in the previous section.

90°
0°
90°
90°
0°
90°

Figure 7.86 Reference stacking sequence

Figure 7.87 shows four different stacking sequences with interleaved damping patches, and it is possible to distinguish the adaptation of two configurations presented in the previous study - the Single (S) and Double-Skins (D-S) stacking sequences. In the former section, one considered six arrangements, but there were two main classes of interleaved damping sequences; the Core and Skins configurations. Here, one extends the elementary forms of Core and Skins configurations, Single (S) and Double-Skins (D-S), respectively, to arrangements with interleaved damping patches. Figures 7.87a and 7.87b depict the extensions of the Single (S) interleaved configuration to two versions with interleaved damping patches - a Centre-to-Edge run of the damping patch, Figure 7.87a, and an Edge-to-Centre run - Figure 7.87b. Similarly, Figures 7.87c and 7.87d represent the extensions of the Double-Skins (D-S) interleaved configuration to two versions with

interleaved damping patches - a Centre-to-Edge run of the damping patch, Figure 7.87c, and an Edge-to-Centre run - Figure 7.87d.

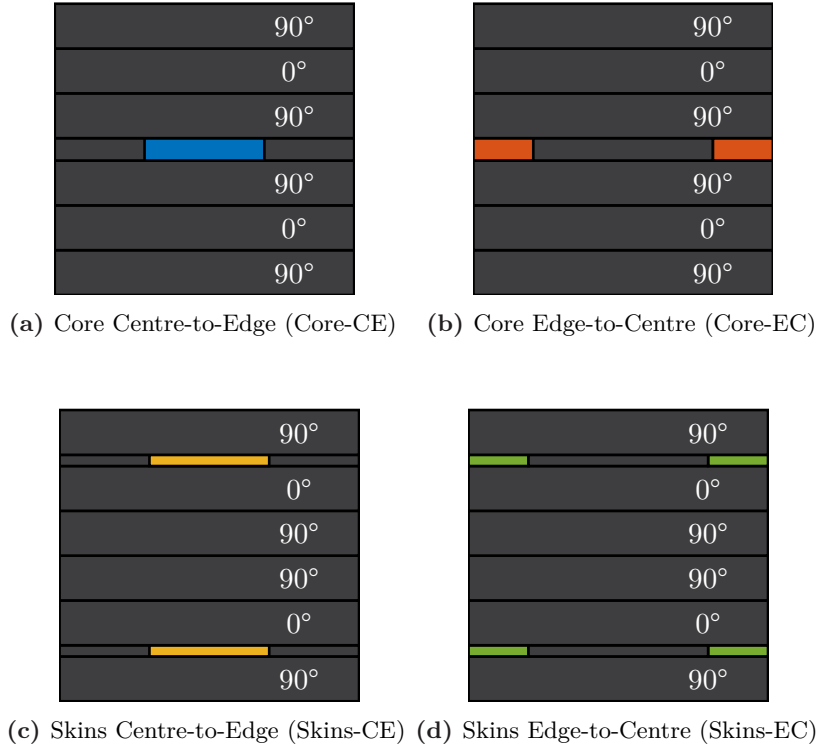


Figure 7.87 Configurations of interleaved damping patches

These arrangements cover a significant range of symmetric configurations for a six-layer reference laminate with interleaved damping patches, including core and skins configurations. The main goal is to compare different stacking sequences, including interleaved damping patches, by performing parametric analyses and using the same amount of damping material. It is important to note that each patch in the Double-Skins (D-S) configuration has half the thickness of each patch in the Single (S) arrangement, with a total damping layer thickness of h_d . Besides, each patch of the Edge-to-Core (EC) configurations has half the width of each patch of the Centre-to-Edge (CE) configurations, as detailed further in this section. Similarly to the previous section, the focus is to understand the preferential stacking sequences, whether it is preferable to have more or less interleaved damping patches for the same amount of damping material and whether they should be closer to the core or the skins, depending on the vibration mode and boundary conditions, for example.

Figure 7.87 shows four different stacking sequences with interleaved damping patches, but one previously reported the parametric analysis of eight different configurations. One must consider the orientation of the interleaved damping patches along with the 0° or 90° CFRP layers. Figure 7.88 presents the alternatives - Figures 7.88a and 7.88b show the width of the damping patch oriented with the 0° CFRP layers, Configuration 0, and Figures 7.88c and 7.88d depict the width of the damping patch oriented with the 90° CFRP layers, Configuration 90. Figure 7.88 highlights the width of each interleaved damping patch, with the Edge-to-Core (EC) configurations having half the width of each patch of the Centre-to-Edge (CE) configurations. In addition, it is worth emphasising that one always considers the CFRP patches represented in Figures 7.87 and 7.88 oriented at 0° .

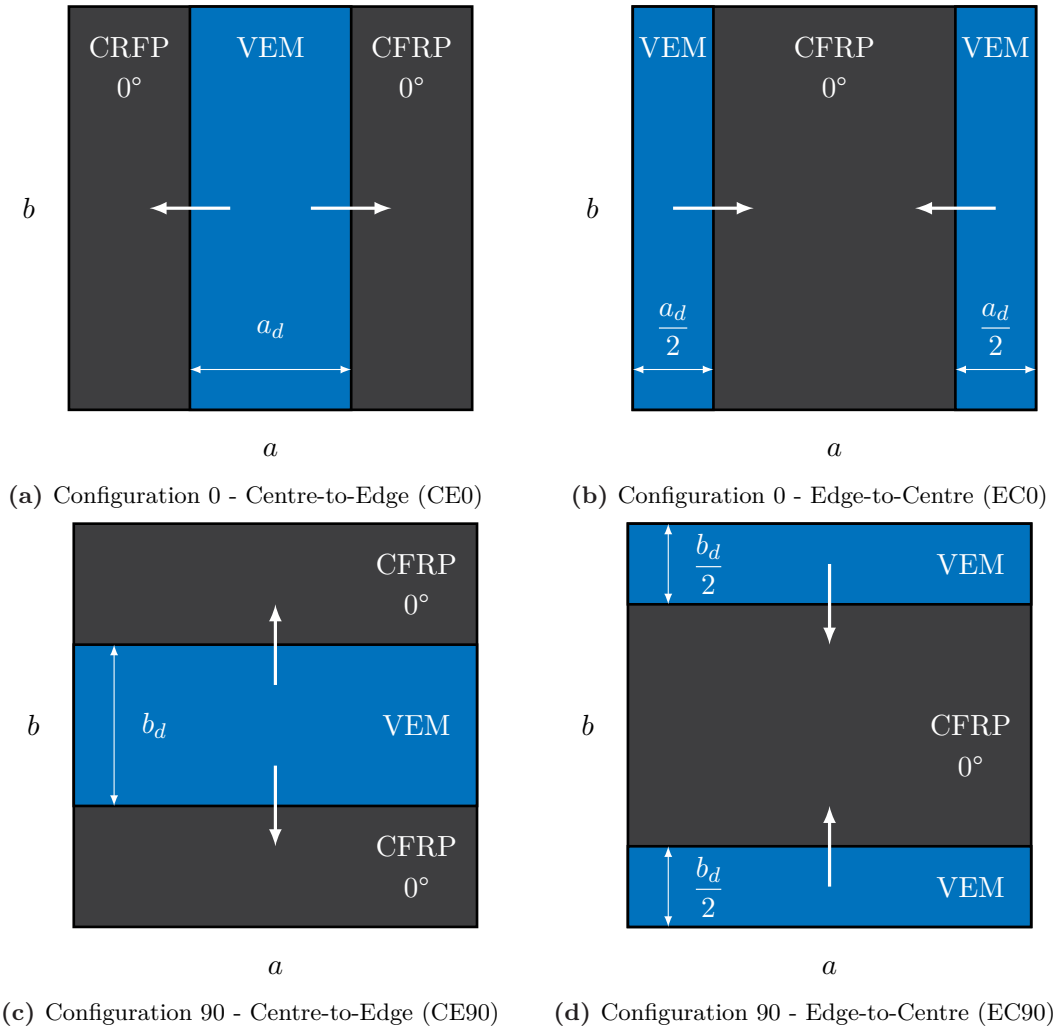


Figure 7.88 Orientation of the interleaved damping patches

In this section, similarly to the previous ones, one performs different parametric analyses considering a regular FEM mesh of 30 CPT(C) elements along each side of the plate and the ToZZ. One obtains the natural frequencies, mode shapes and modal loss factors by performing a modal analysis with the MSE to study the damping capacity of composite plates with interleaved damping patches.

7.3.1 | Damping Efficiency of SSSS Laminates with Interleaved Patches

In the following, one studies the damping efficiency of interleaved damping patches considering simply supported laminated composite plates. One focuses the analysis on the influence of the damping patch width, thickness and position on the damping capacity, considering the eight stacking sequences presented before. The main goal is to compare different configurations with interleaved damping patches considering the same volume of damping material. Therefore, to study the influence of damping patch position, h_d describes the total thickness of the damping material regardless of the number of damping layers, and a_d expresses the width of the damping patch, varying from $a_d/a = 0$ to $a_d/a = 1$ ($a_d = b_d$ in square plates). When comparing the single interleaved configuration with the double interleaved one, the thickness of each patch in the

double interleaved stacking sequence is half the total thickness. On the other hand, each patch of an Edge-to-Centre configuration has half the width of a Centre-to-Edge patch. One considers different damping layer thicknesses to comprehend their influence on the damping efficiency, considering the total damping patch thickness, h_d , as a function of one reference layer's thickness, h_ℓ . The total laminate thickness of the reference stacking sequence in Figure 7.86 is $h = 6h_\ell$.

7.3.1.1 | Influence of Interleaved Damping Patch Width, Thickness and Position

Following the same reasoning of previous chapters, it is relevant to analyse the laminated plate mode shapes to understand the damping performance of damping treatments. Since the reference stacking sequence is the same as the one considered in the previous section, one recalls Figure 7.6 presenting the familiar and already discussed transverse displacement mode shapes of the first six normal modes of the simply supported square laminated composite plate considering the reference stacking sequence of Figure 7.86 and an aspect ratio $a/h = 100$. Previously, one also discussed the importance of the mode shapes associated with the transverse shear strain degrees of freedom γ_{xz} and γ_{yz} to understand the modal transverse shear strain pattern, and so Figures 7.7 and 7.8 showed, respectively, the first six ones for the same reference laminated composite simply supported plate.

Furthermore, one also draws attention to Figures 7.9 to 7.11 displaying the through-the-thickness extensional, transverse shear and in-plane shear strain density mode shapes for the first normal mode and different section cuts, respectively. Namely, sections at which the extensional and transverse shear strain energy densities are the highest ones - for $y = b/2$ and $x = a/2$. Figures 7.6a, 7.7a and 7.8a highlight these section cuts along the transverse displacement and transverse shear strain pattern mode shapes, respectively. One extensively discussed all these through-the-thickness density mode shapes in Section 7.2.1.1.

One perhaps should acknowledge that it is not easy to analyse the influence of the interleaved damping patch width, thickness, and, above all, position on the damping efficiency of laminated composite plates. Figures 7.87 and 7.88 show the eight different configurations under analysis here, and this is just a short measure of the number of variables at stake in this analysis. Similarly to previous studies in this work, one must also consider the aspect ratio of the laminate a/h , for example. Therefore, and without surprise, the obtained results require a demanding read.

The first analysis addresses the influence of the interleaved damping patch width, thickness and position on the damping efficiency for reference laminated plates with an aspect ratio of $a/h = 100$. Figures 7.89 to 7.92 exhibit the specific modal loss factors of the first three normal modes as a function of the damping patch layer thickness and width for the different stacking sequences under analysis - Core-CE, Core-EC, Skins-CE and Skins-EC, respectively. Once again, following the same approach of the previous studies in this work, one considers more than just one mode to generalise the conclusions one can draw from such an approach. The solid curves represent Configuration 0, and the dashed curves represent Configuration 90 - see Figure 7.88. One considers four different damping patch layer thicknesses to comprehend their influence on the damping efficiency - $h_d = h_\ell/64$, $h_d = h_\ell/16$, $h_d = h_\ell/4$ and $h_d = h_\ell$. The progression is obvious and covers a wide range of possible damping patch layer thicknesses. Still, it is worth emphasising that $h_d = h_\ell/64$ is an extremely thin damping layer thickness, difficult to conceive in practice, but it helps to understand the influence of the damping patch layer thickness on the damping efficiency.

At this point, it is worth commenting on the approach followed to present the results regarding the damping efficiency curves of the different stacking sequences under analysis. Tested in a first approach, including all the stacking sequences in one figure to compare the influence of the damping patch position has proved counter-effective because one compares eight different stacking sequences (four damping patch positions times two configurations) for each damping layer

thickness and the figure quickly became unreadable. Thus, one presents in Figures 7.89 to 7.92 the specific modal loss factors of the first three normal modes as a function of the damping patch layer thickness and width for the different stacking sequences under analysis - Core-CE, Core-EC, Skins-CE and Skins-EC, respectively, and side-by-side. For example, presenting Figures 7.89 and 7.90 side-by-side, one can easily compare the influence of the Centre-to-Edge or Edge-to-Centre run of the damping patch in Core configurations in addition to the damping layer thickness and width influence - the same applies for the Skins configurations in Figures 7.91 and 7.92. This way, it is possible to compare the damping patch layer thickness and width for the different stacking sequences comprehensibly. Besides, there is an extra feature to help the plot's reading. The specific modal loss factor plots for the Core-EC stacking sequence in Figure 7.90 include the overall best damping performance curves of the Core-CE stacking sequence to compare the Core configurations directly. The same applies to the Skins-CE stacking sequence plots in Figure 7.91, and thus, one can directly compare the performance between Centre-to-Edge runs in Core and Skins configurations. Lastly, the damping efficiency curves of the Skins-EC stacking sequence in Figure 7.92 include the best overall performances of the Core and Skins-CE configurations, helping to identify the preferential arrangements in damping capacity.

As expected, some of the more general conclusions addressed in this work when dealing with surface-damping treatments and interleaved damping layers are still valid for interleaved damping patches. Figures 7.89 to 7.92 clearly show the increasing-decreasing effect of the damping layer thickness on the damping capacity. Even for laminated plates with interleaved damping patches, there is a threshold of damping layer thickness beyond which further increases reduce the material's damping capacity. As already discussed, thick damping layers decouple the elastic stiff ones of the laminate and, consequently, their ability to impose shear on the damping material. Moreover, it is worth drawing attention to the mode dependence of the optimum damping layer thickness; overall, optimum damping layer thickness decreases for higher-order modes.

It is possible to draw some conclusions regarding interleaved damping patches in laminated plates focusing on the damping performances of core configurations in Figures 7.89 and 7.90. Overall, core configurations with the damping patch running from the centre to the edges (CE), Figure 7.89, promote more damping than those with the damping patch running from the edges to the centre (EC), Figure 7.90, and this is particularly noticeable for the first mode, as the Core-EC configurations show low damping performance, regardless of the damping patch thickness. In this regard, one should recall Figure 7.9 and the section cuts of the first extensional strain energy density mode shapes of the simply supported reference composite plate, highlighting the greater extensional strain energy density at the centre rather than the edges.

Nevertheless, one of the most noteworthy results is the optimum damping performance of interleaved damping patches over interleaved damping layers for some damping layer thicknesses. This significant result proves the importance of considering interleaved damping patches in laminated composites, as it is possible to achieve improved damping performances with less damping material for specific damping layer thicknesses. Similarly to surface-damping treatments, and as previously discussed in this work, using partial damping treatments could be beneficial for some configurations, and interleaved damping patches in laminated composites are not an exception. Also, in this case, Core-CE configurations appear to be preferable since Edge-to-Centre treatments need higher amounts of damping material to be efficient, i.e. the damping material must reach the centre of the laminate.

Regarding the comparison between performances of Configurations 0 and 90 in core damping patches, Figures 7.89 and 7.90 show that damping layer thickness and position (CE or EC) are far more determinant to the interleaved damping patch performance than the orientation of the different configurations. Nevertheless, a closer analysis of the minor differences between orientations discloses the mode dependence of the preferable orientation for a given configuration.

Figures 7.91 and 7.92 complete the analysis of reference laminated composite plates with interleaved damping patches, presenting the specific modal loss factors of the first three normal modes for configurations with interleaved damping patches closer to the skins. To help with this analysis, one added the best damping performances of the core configurations for each mode to these charts. Consistently with the results obtained with the core configurations, it is clear from Figures 7.91 and 7.92 that Centre-to-Edge dispositions of the interleaved damping material are preferable to Edge-to-Centre ones. The consistency of these results helps to draw some robust guidelines for interleaved damping patches in laminated plates.

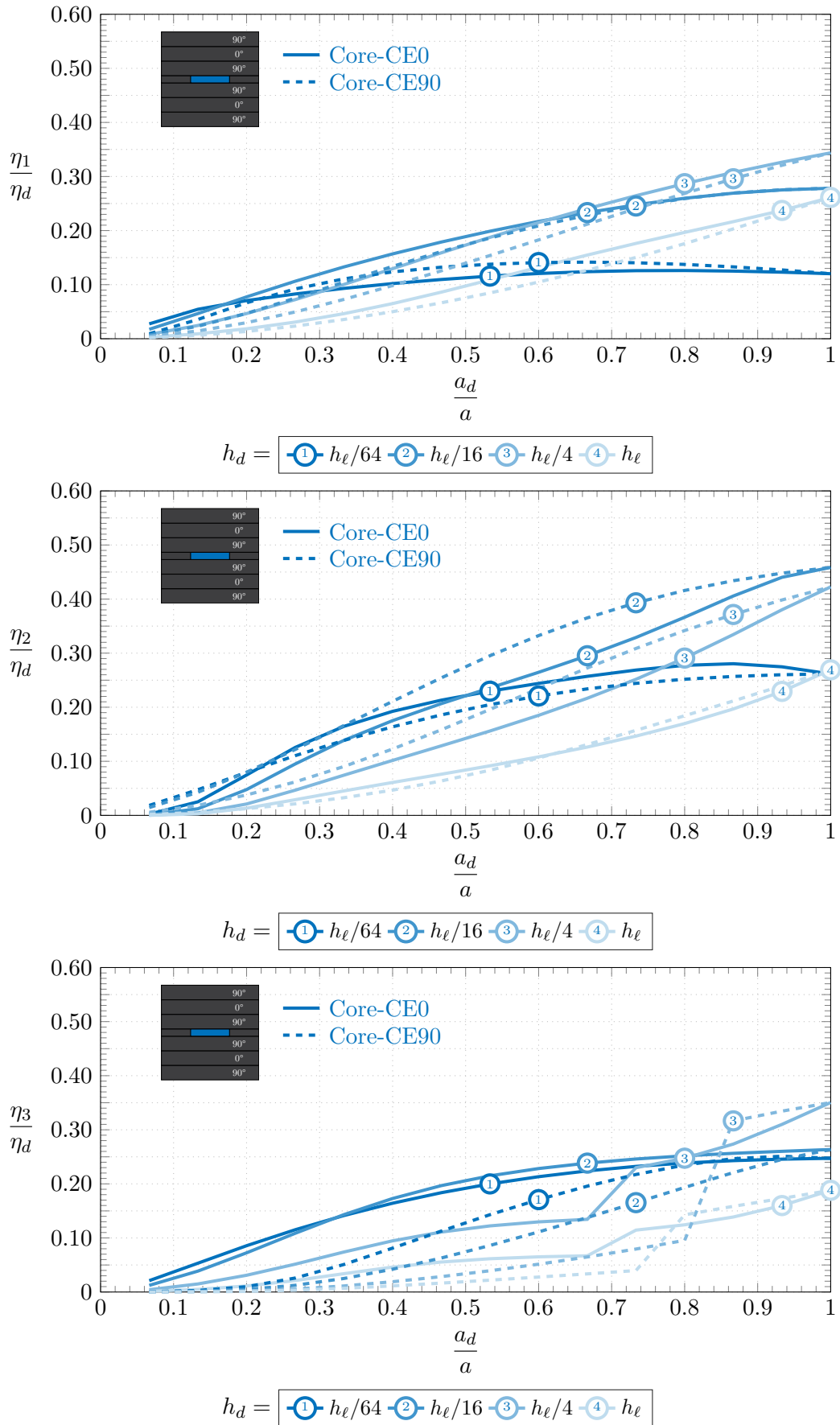
Nevertheless, there are some noteworthy differences in the damping performances of Core and Skins configurations. Generally speaking, and despite the remarkable performance of the Skins-CE configurations for some damping layer thicknesses, core configurations show superior damping capacities for the first and second modes. Somewhat unexpectedly, the Skins-CE0 configuration shows an outstanding damping capacity for the third mode, surpassing all other arrangements, including core configurations.

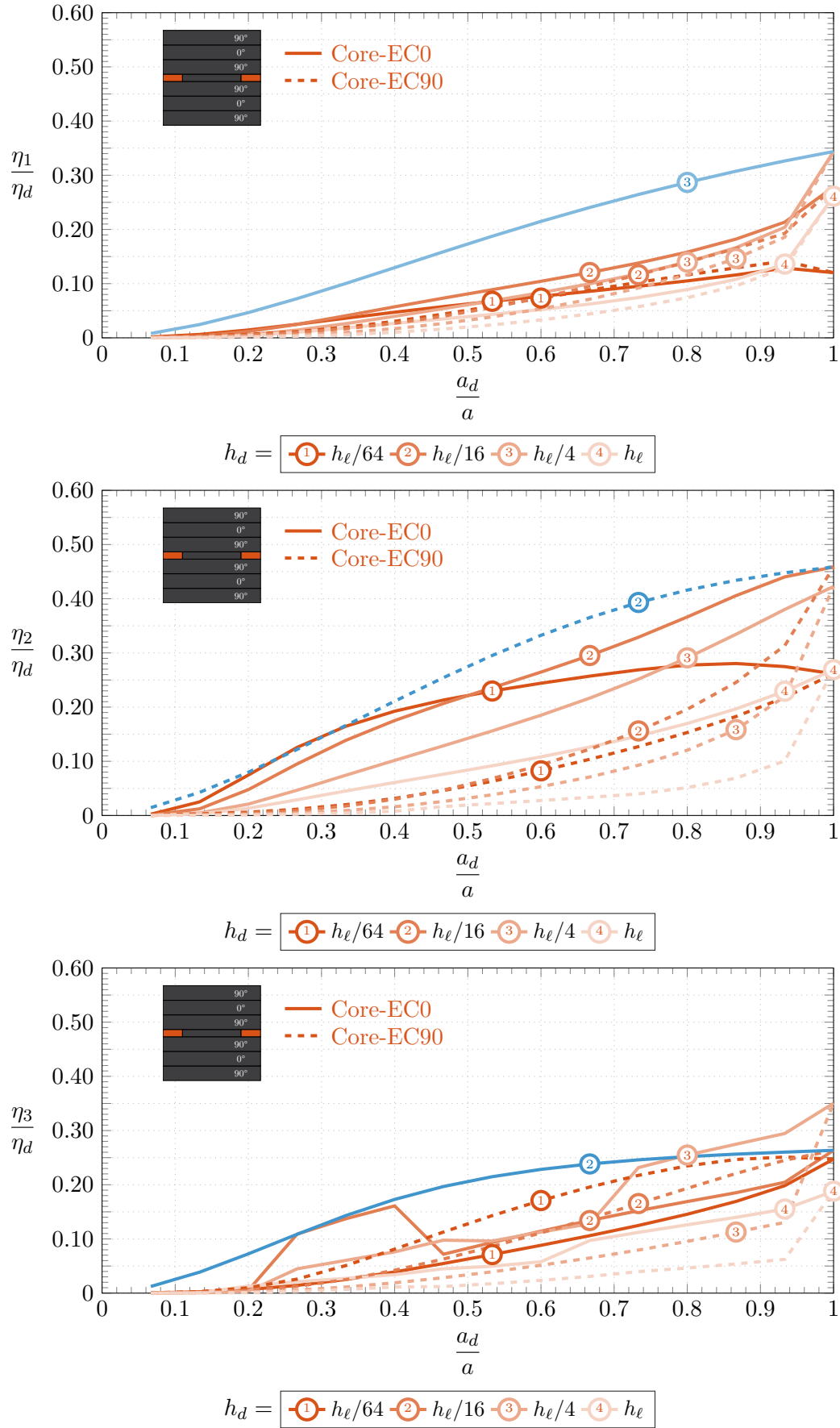
Even so, analysing the damping performance of Skins configurations in Figures 7.91 and 7.92, it is possible to conclude that these arrangements need thick damping layers to accomplish noticeable damping capacities. For thin interleaved damping patches close to the skins, the top and bottom layers of the laminate can not impose significant transverse shear strain on thin damping patches, and the corresponding damping capacity is negligible. This inability is particularly evident for the Skins-EC configurations, which combine the positioning of the damping material close to the skins with the material running from the edges to the centre, dispositions that are not the most efficient regarding damping capacity.

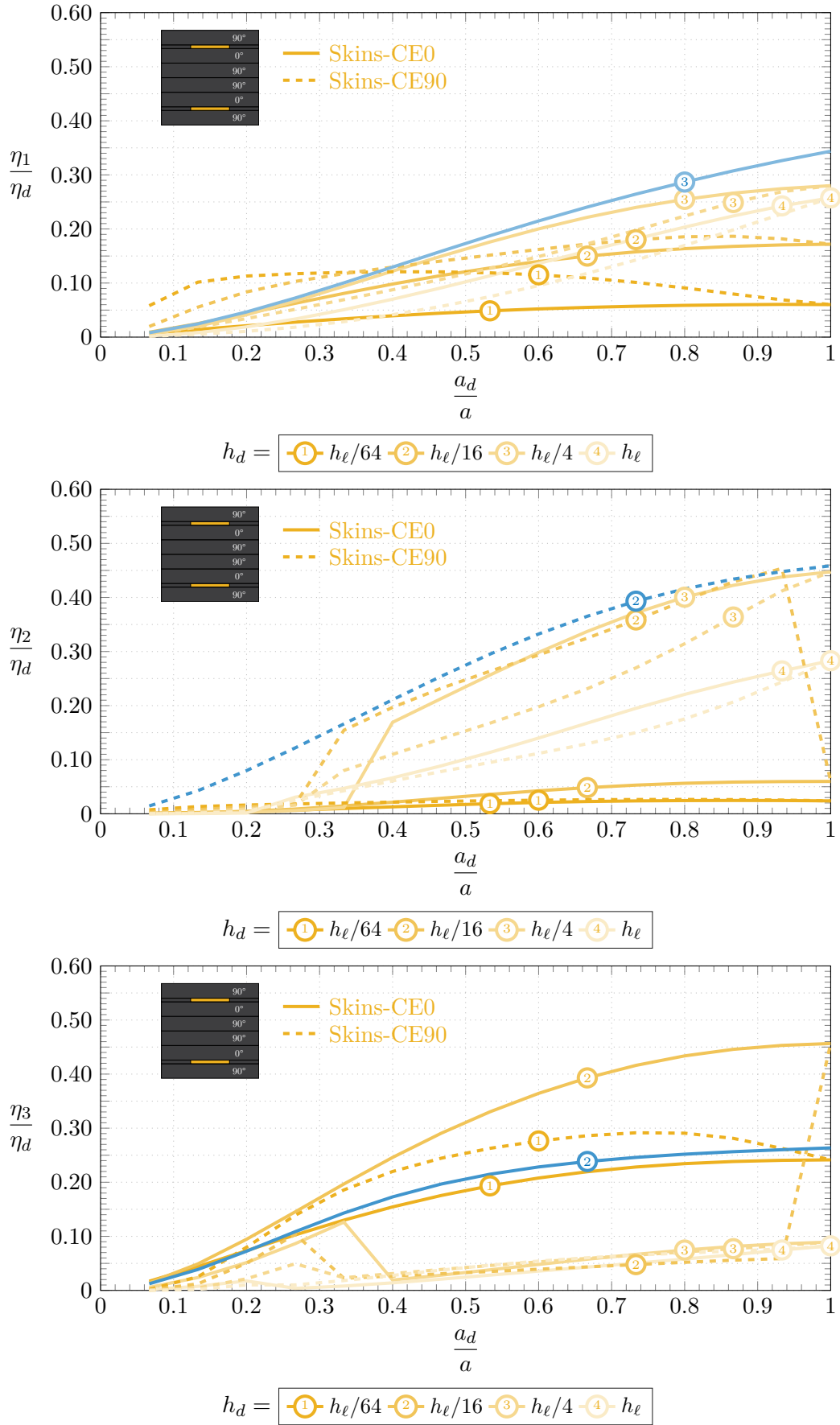
Focusing on the analysis of the influence of thickness and position of the interleaved damping patches on the damping capacity of laminated plates, Figures 7.93 to 7.100 exhibit, side-by-side for a better reading, the specific modal loss factors of the first three normal modes as a function of the damping patch width for reference laminated plates with an aspect ratio $a/h = 100$, and for thicker laminated plates with an aspect ratio $a/h = 50$. Following the same reasoning of the previous studies and for comparison purposes, one considers damping layer thicknesses as a function of the thickness of each elastic layer h_ℓ for both aspect ratios. The increasing-decreasing effect of the damping layer thickness, thoroughly discussed in this work, finds another demonstration of its impact and results in these representations. The most noticeable difference between reference and thick laminated plates in Figures 7.93 to 7.100, either for Core or Skins configurations, is the decreasing effect on the damping capacity of the damping layer thickness increase. For thick laminated plates with an aspect ratio $a/h = 50$, one also considers thick damping layers and increasing damping layer thickness decreases damping capacity.

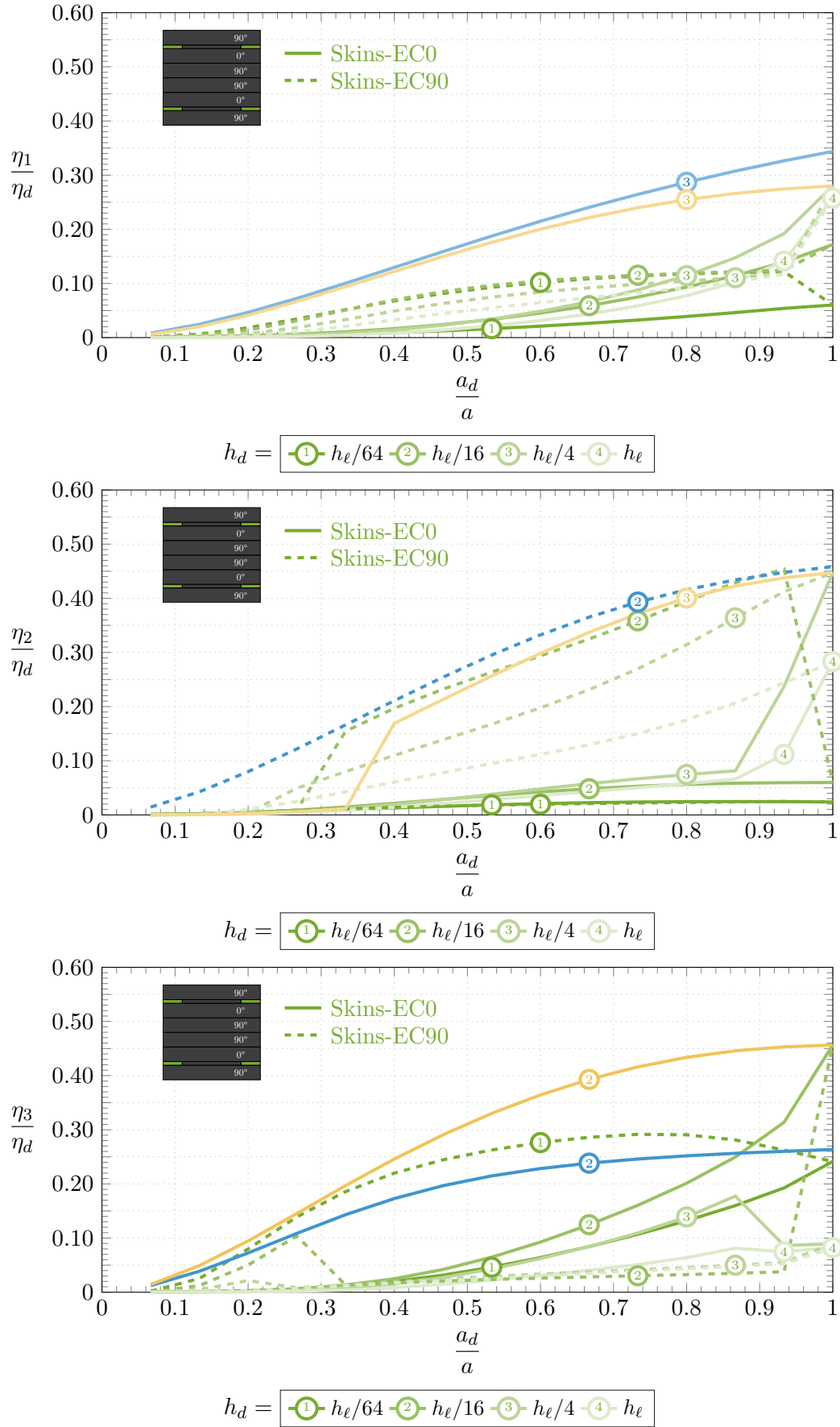
Therefore, thick laminated composite plates show improved damping performances with thinner damping layers, as shown by Core and Skins configurations in Figures 7.94, 7.96, 7.98 and 7.100. As extensively discussed throughout this work, thick damping layers tend to decouple the elastic stiff ones and, consequently, their ability to impose shear on the damping material; for the most part, this effect is even more evident for thick laminate plates.

Moreover, the trends identified before for reference laminated plates with an aspect ratio $a/h = 100$ and the main differences between Core and Skins configurations as well as Centre-to-Edge (CE) or Edge-to-Centre (EC) runs of the interleaved damping patches are still applicable for thick laminated composite plates.

Figure 7.89 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.90 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.91 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.92 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 100$

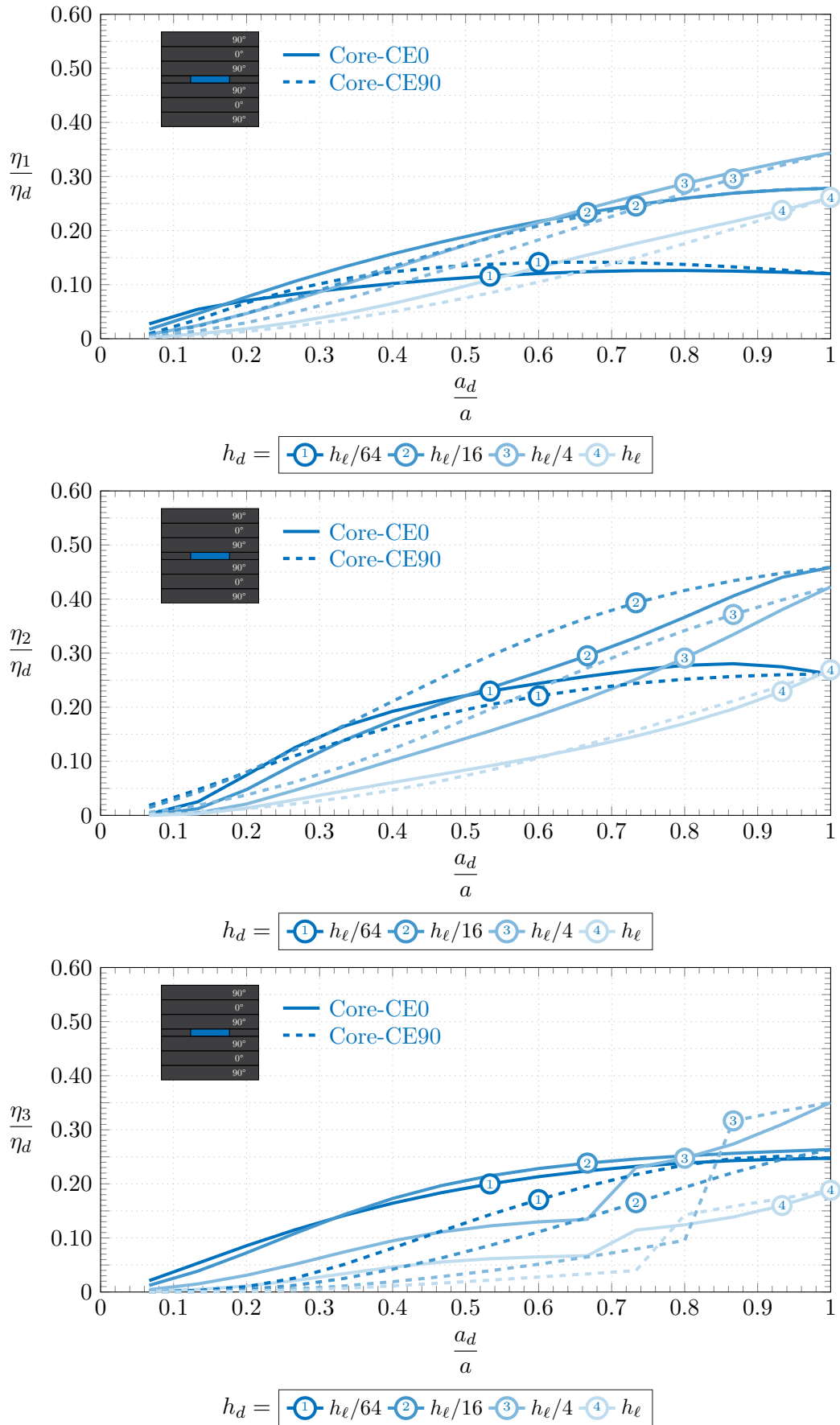
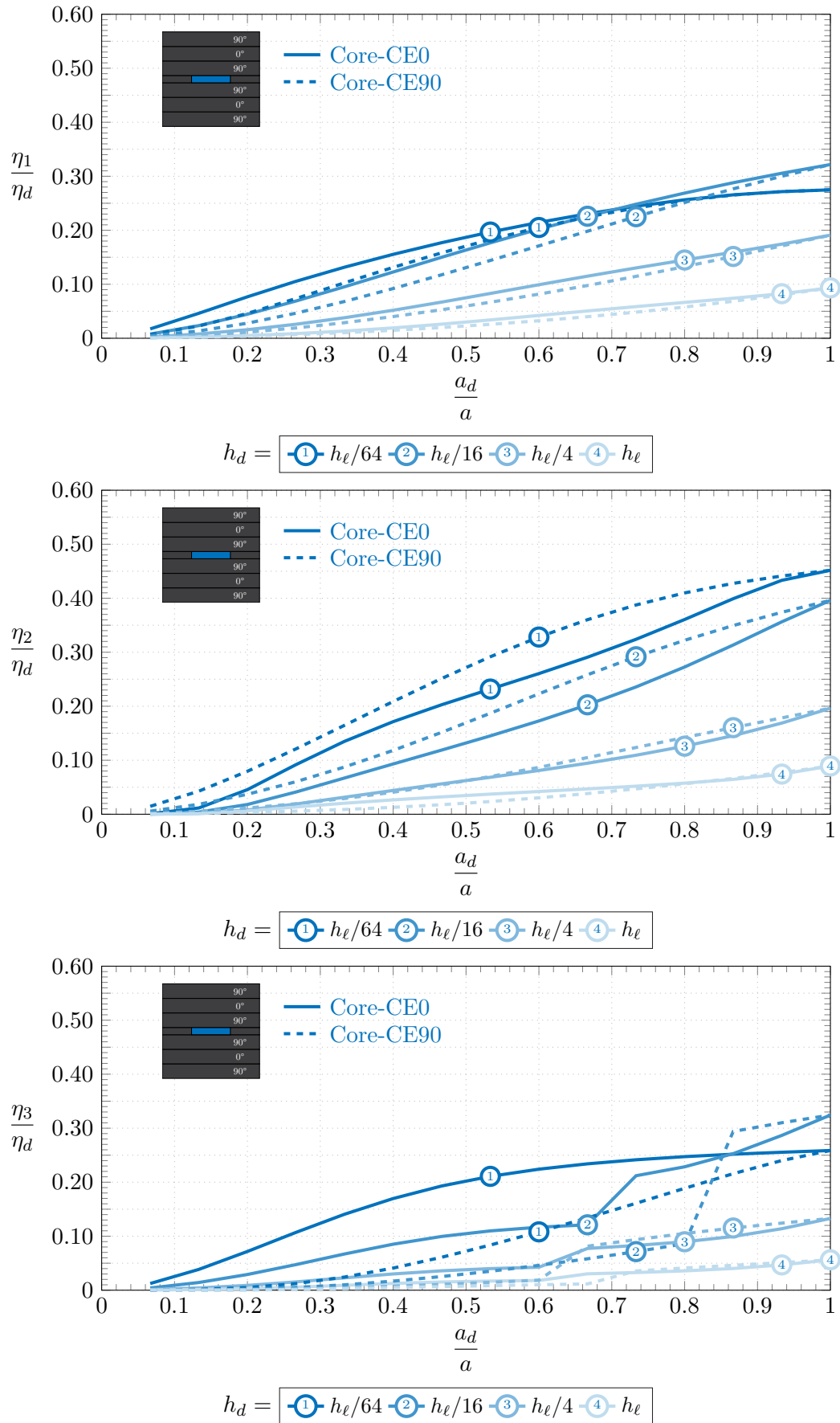
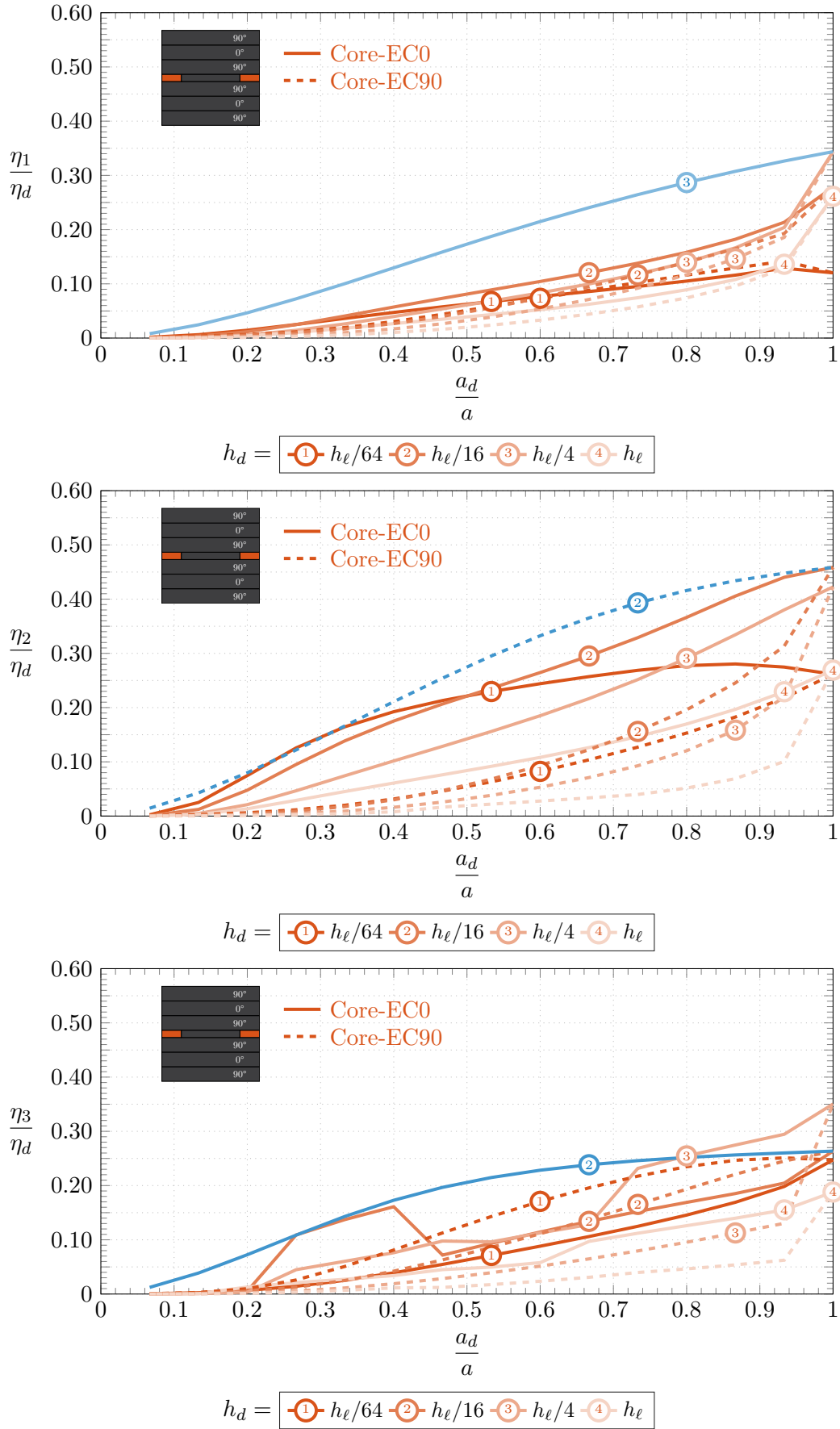
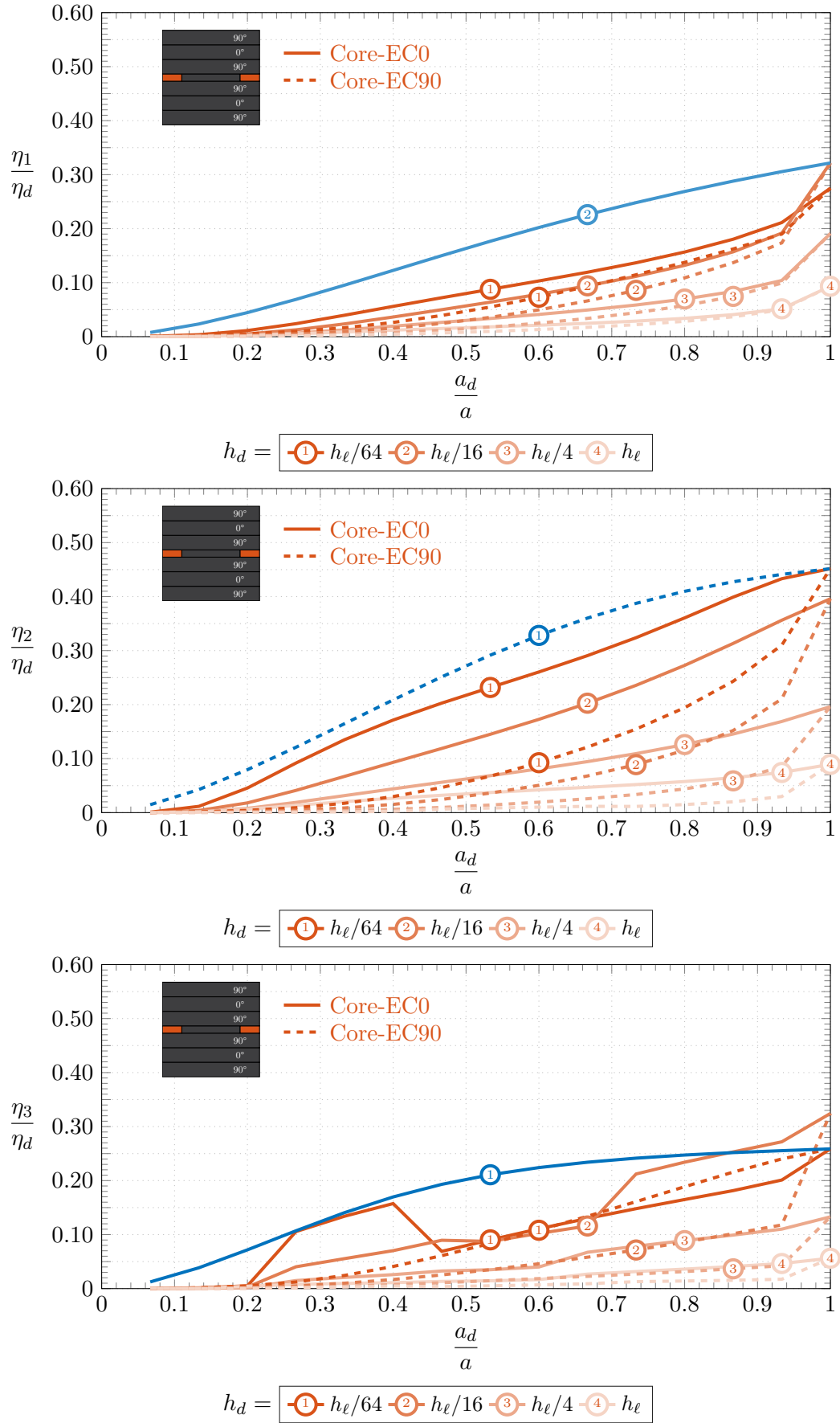
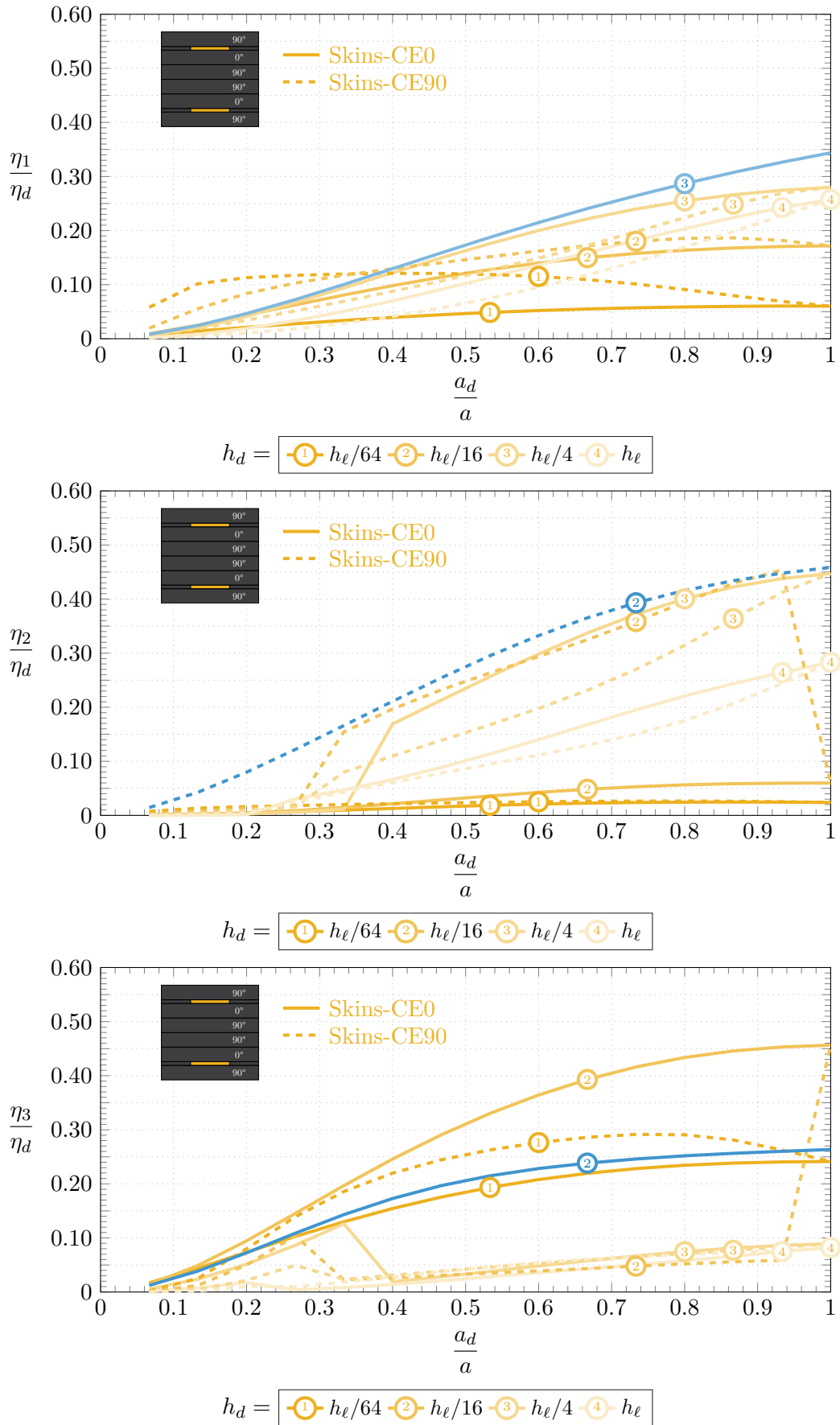


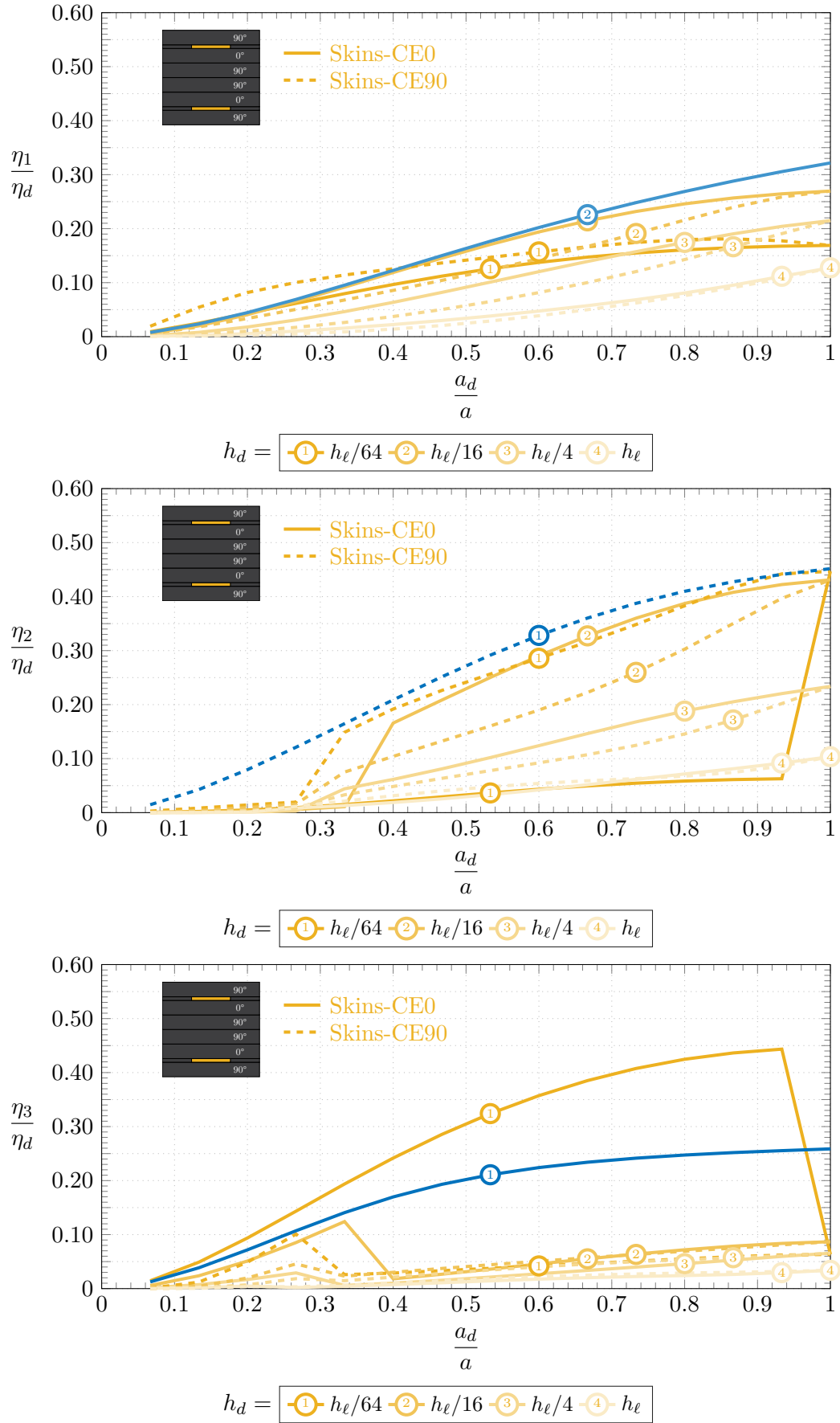
Figure 7.93 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 100$

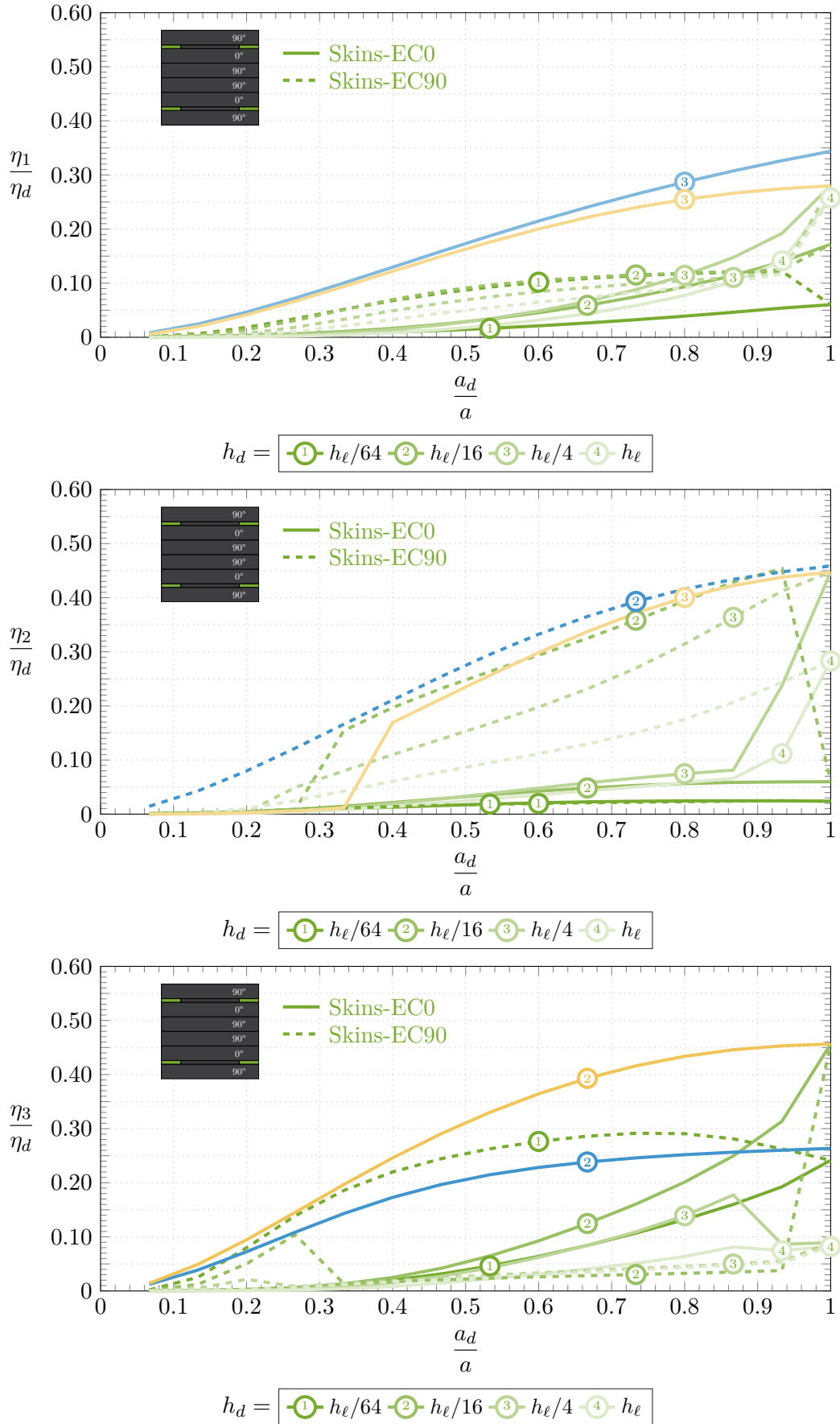
Figure 7.94 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 50$

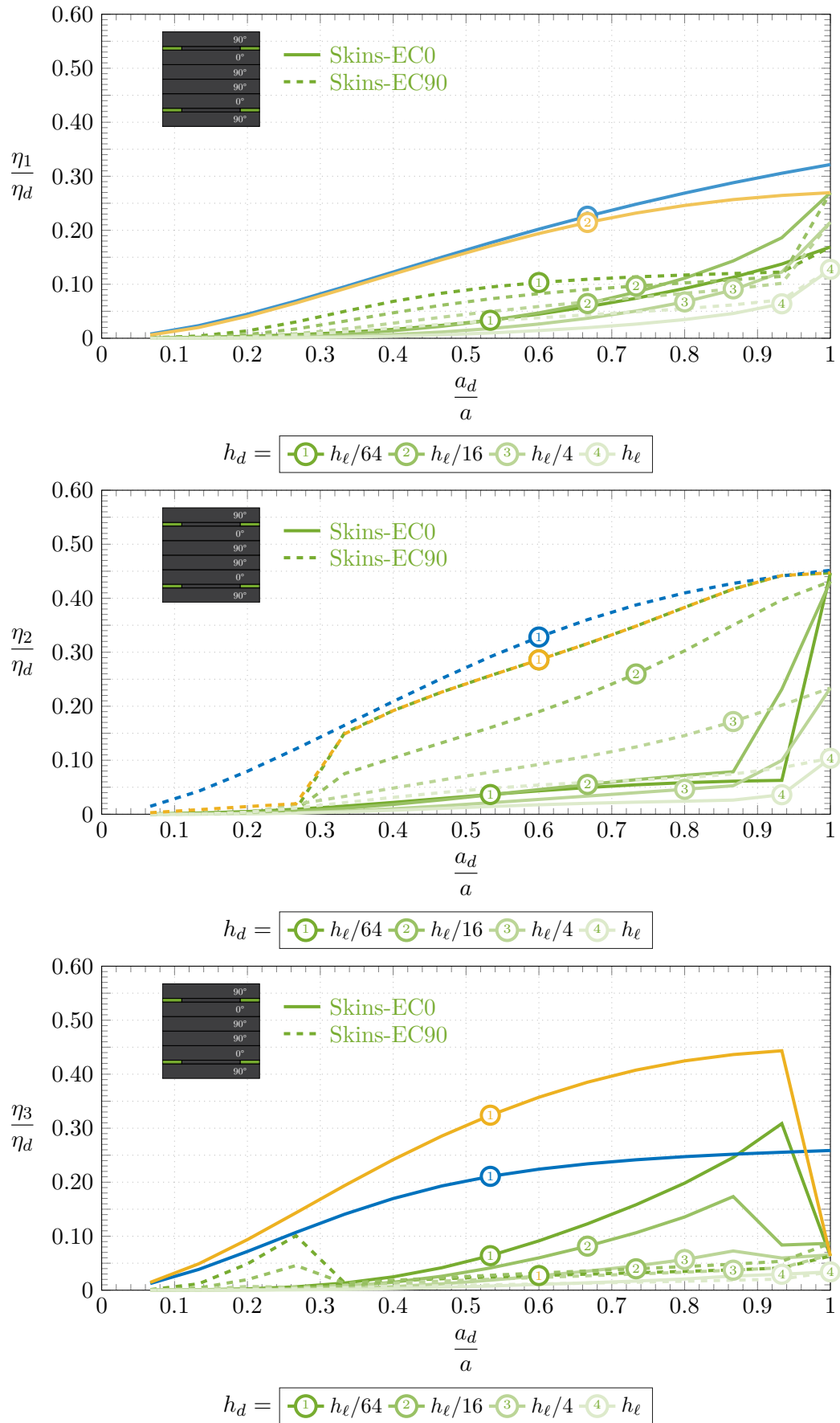
Figure 7.95 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.96 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 50$

Figure 7.97 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.98 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 50$

Figure 7.99 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.100 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 50$

Based on the configurations of interleaved damping patches investigated in this work, Figures 7.101 to 7.108 present the through-the-thickness modal transverse shear and extensional strain energy density mode shapes for the first normal mode. One considers for these representations a laminated composite plate with an aspect ratio $a/h = 100$ and a damping patch with width $a_d = a/3$ and thickness $h_d = h_\ell/4$, which is the threshold damping layer thickness for the increasing-decreasing behaviour of damping capacity. Once again, one examines the same section cuts analysed before, $y = b/2$ and $x = a/2$, sections of maximum modal extensional and transverse shear strain energy densities.

Figures 7.101 - 7.102 and 7.103 - 7.104 show the through-the-thickness modal transverse shear strain energy density mode shapes $\phi_1^{U_s^{xz}}$ and $\phi_1^{U_s^{yz}}$, respectively, for all configurations with interleaved damping patches. As expected, consistently with all the results achieved throughout this work, the patterns help to visually explain the damping capacity results. Core-CE0 and Core-EC0 configurations present the highest $\phi_1^{U_s^{xz}}$ modal transverse shear strain energy density. Indeed, a Core-CE0 configuration with the patch width and thickness considered here presents the best damping capacity for this mode. In addition, Skins-CE90 and Skins-EC90 configurations also feature high levels of $\phi_1^{U_s^{yz}}$ modal transverse shear strain energy density. Figure 7.91 shows that the Skins-CE90 arrangement has a damping capacity comparable to that of the Core-CE0 configuration for the patch width and thickness considered in this analysis. As already discussed, beyond the cumulative modal transverse shear strain energy, the representations of through-the-thickness patterns prove valuable for interpreting the damping mechanisms in composite laminates featuring interleaved damping patches. This kind of tool is even more relevant in the context of interleaved damping patches, where the complexity of the structure does not help to extract conclusions easily. Nevertheless, patches close to the centre rather than the edges generally perform better in terms of damping capacity for the structure analysed here. Once again, one must not forget Figure 7.9 with the first extensional strain density mode shapes of the laminated composite reference plate highlight higher extensional strain at the centre rather than the edges of the simply supported plate.

Specifically for assessing the capacity of the stiff composite layers to induce transverse shear strain on the soft damping layers, Figures 7.105- 7.106 and 7.107- 7.108 present the through-the-thickness modal extensional strain energy densities $\phi_1^{U_n^{xx}}$ and $\phi_1^{U_n^{yy}}$, respectively. The outcomes are clear and uncomplicated for both modal strain energy densities. Those configurations presenting the best damping capacity are those with some stiff layers subjected to higher modal extensional strain energy densities. In particular, Core-CE0 with the highest levels of modal extensional strain energy density $\phi_1^{U_n^{xx}}$, and Skins-CE90 and Skins-EC90 with the highest levels of modal extensional strain energy density $\phi_1^{U_n^{yy}}$. In addition, modal extensional strain energy densities $\phi_1^{U_n^{xx}}$ and $\phi_1^{U_n^{yy}}$ are consistent with the orientation of the stiff layers - $\phi_1^{U_n^{xx}}$ highlighted at 0° oriented stiff layers and $\phi_1^{U_n^{yy}}$ highlighted at 90° oriented ones.

However, the through-the-thickness representations of the modal extensional strain energy densities are not symmetrical with respect to the laminate mid-plane. At this point, it is important to emphasise some premises of numerical analysis conducted here. To establish the same reference plane for all the configurations, one considered the bottom layer of the laminate to be the reference layer and its middle plane the neutral axis. Due to the complexity of the inhomogeneous structures, one defines some common ground to establish a reliable comparison. Therefore, all the configurations have the same reference plane. Numerically, this breaks the symmetry of the warping functions and, consequently, the possible symmetry of the through-the-thickness modal extensional strain energy densities. Although this is not the case when considering interleaved damping layers, the interleaved damping patches introduce heterogeneity into the structure, leading to highly concentrated modal extensional strain energy densities.

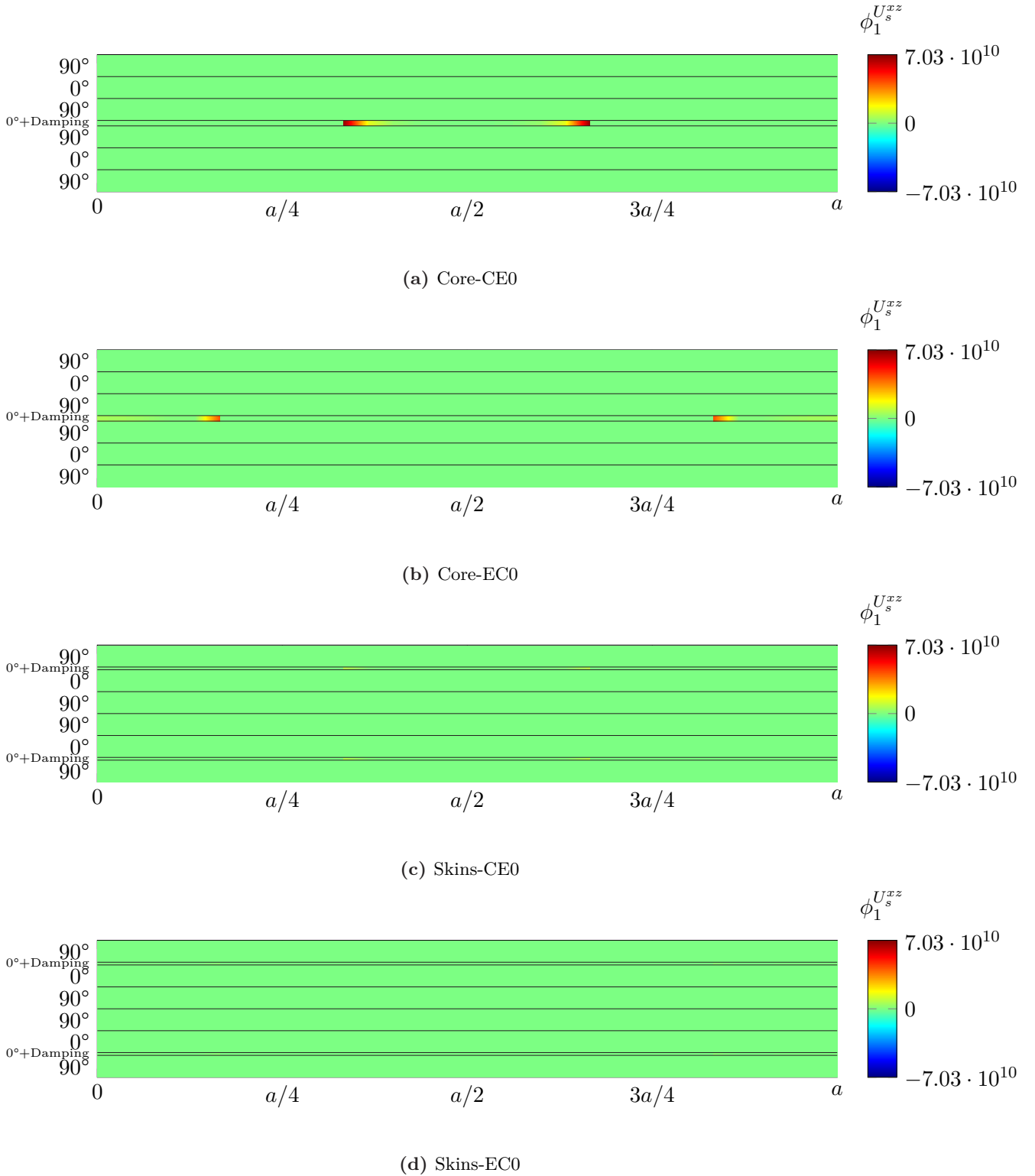


Figure 7.101 $\phi_1^{U^{xz}}$ - shear strain energy density mode shape - section cut at $y = b/2$

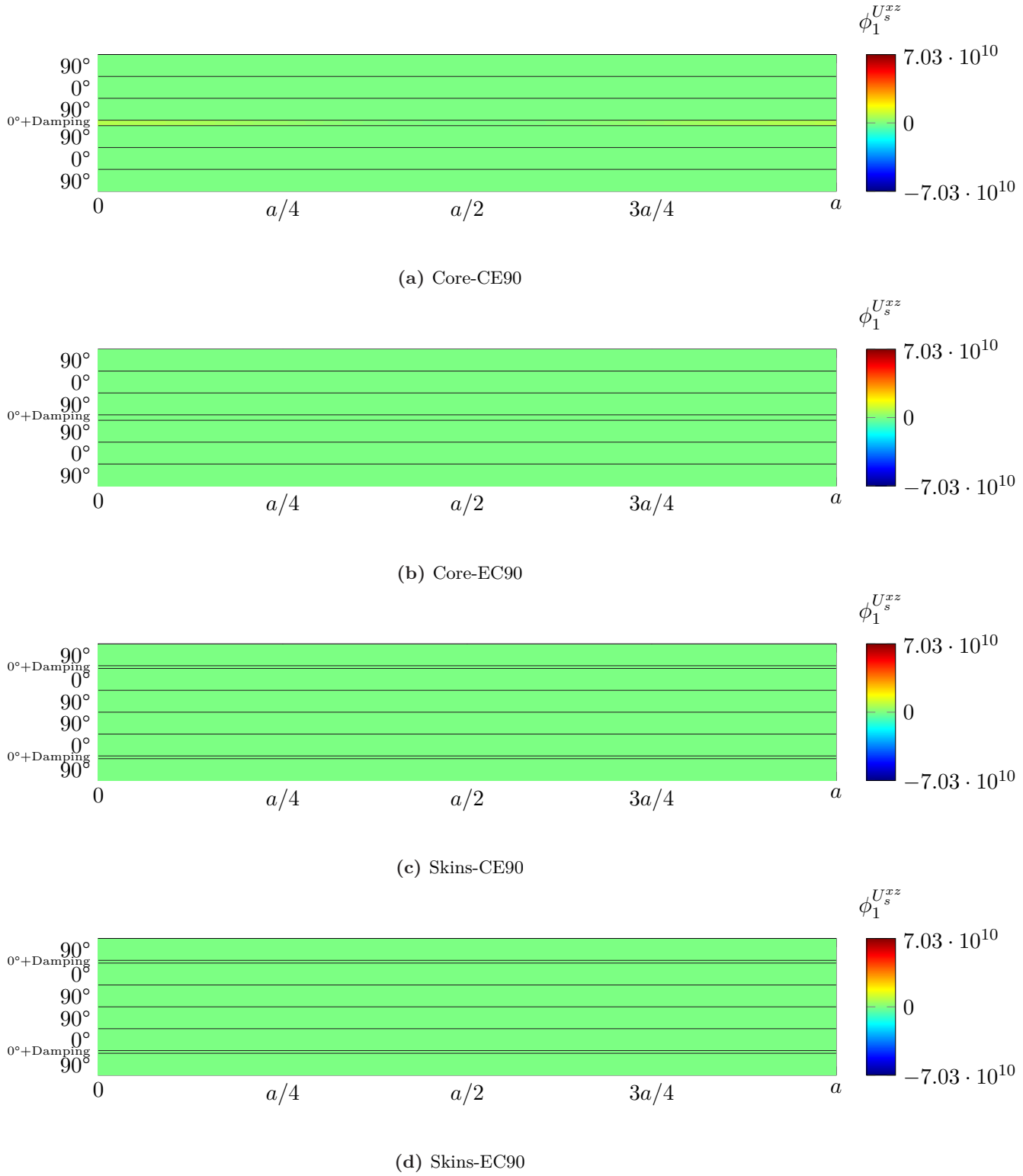
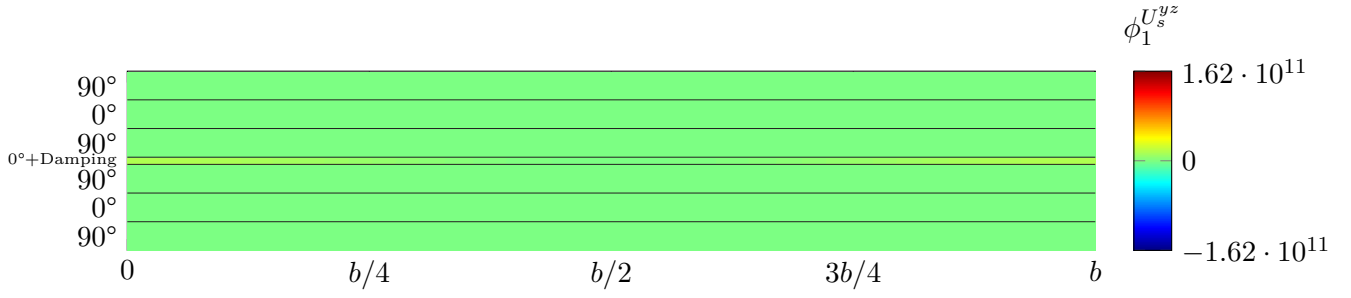
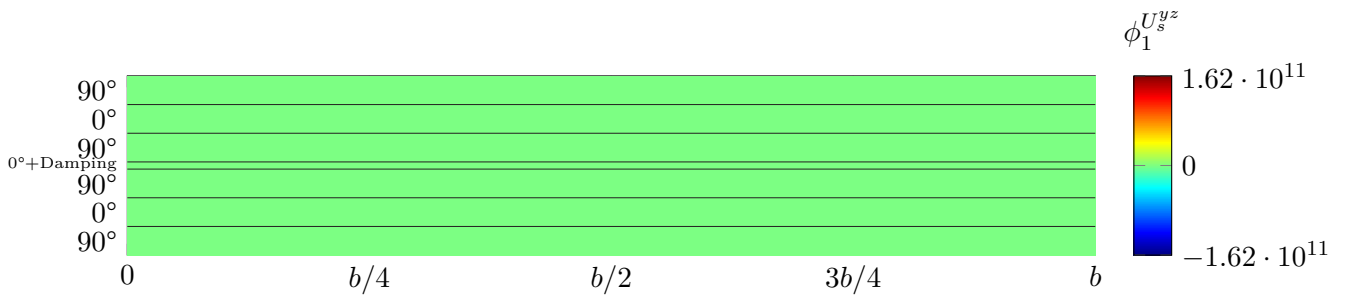


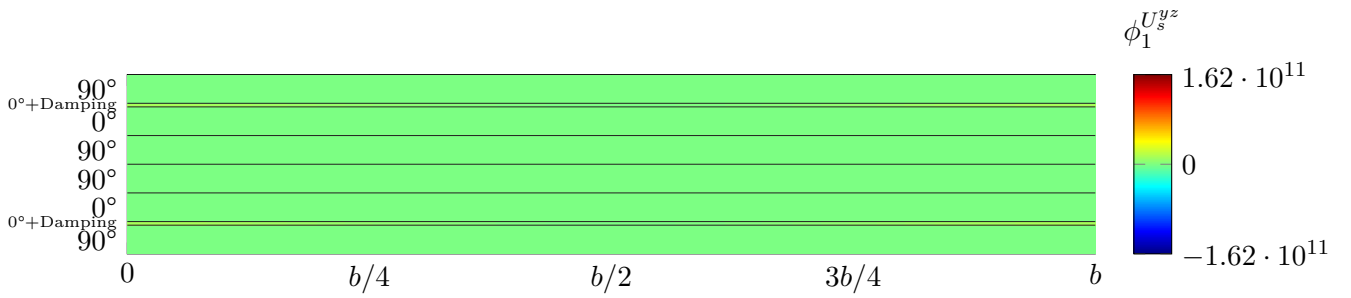
Figure 7.102 $\phi_1^{U_s^{xz}}$ - shear strain energy density mode shape - section cut at $y = b/2$



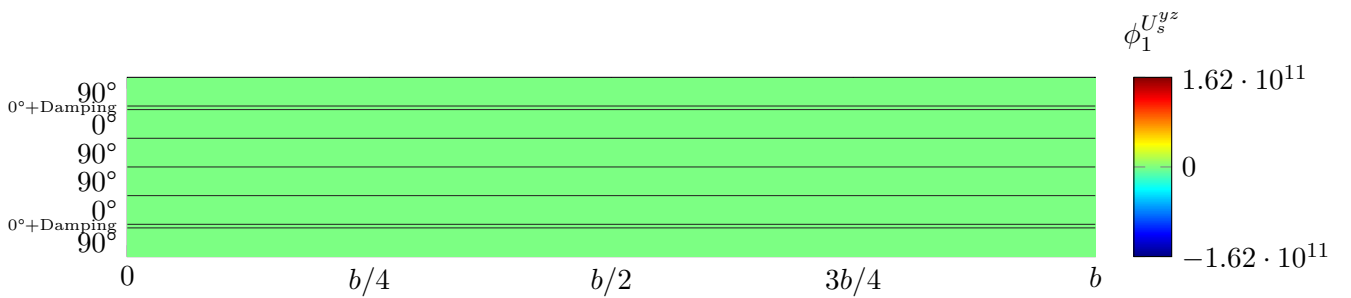
(a) Core-CE0



(b) Core-EC0



(c) Skins-CE0



(d) Skins-EC0

Figure 7.103 $\phi_1^{U_s^{yz}}$ - shear strain energy density mode shape - section cut at $x = a/2$

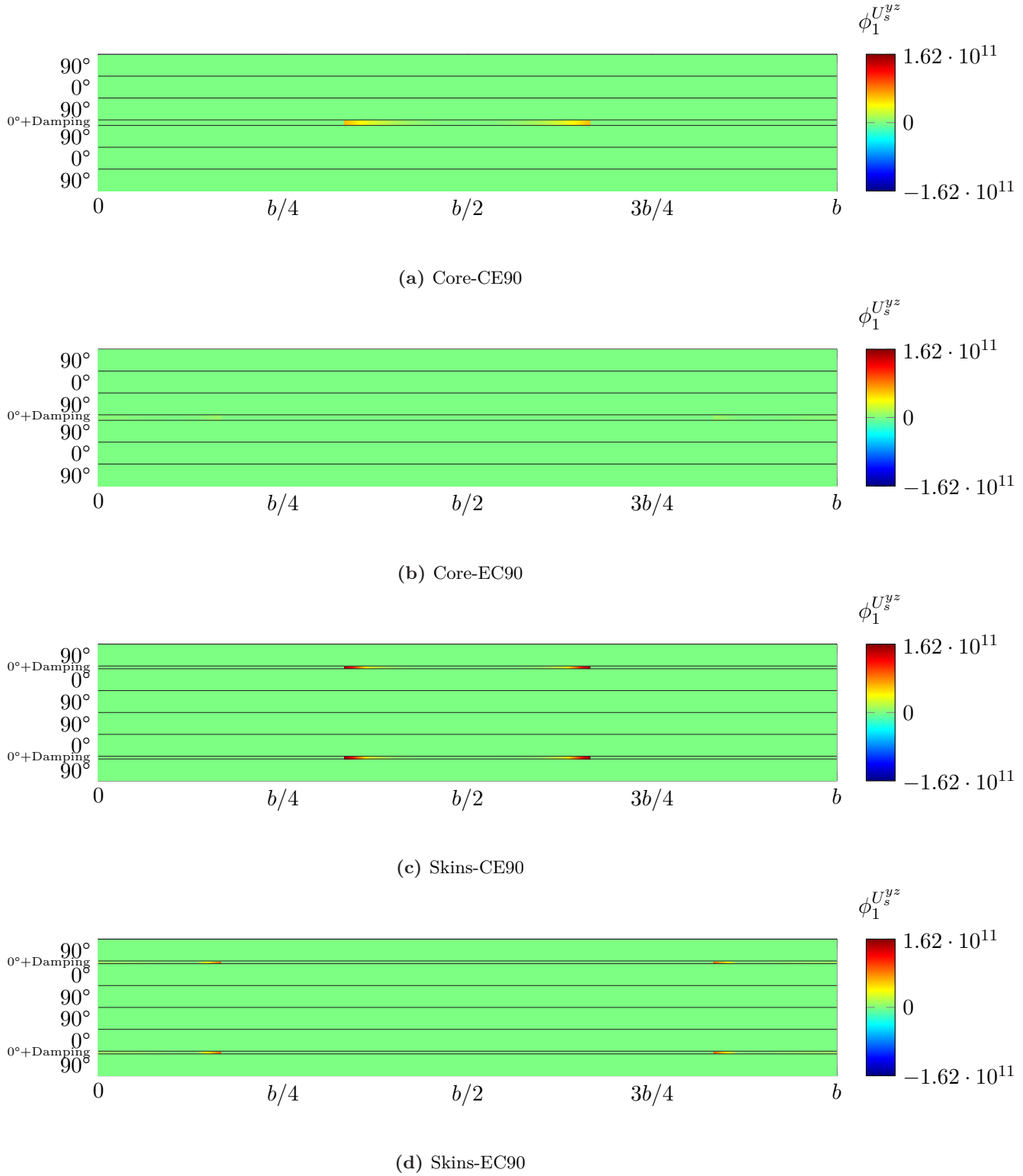


Figure 7.104 $\phi_1^{U_s^{yz}}$ - shear strain energy density mode shape - section cut at $x = a/2$

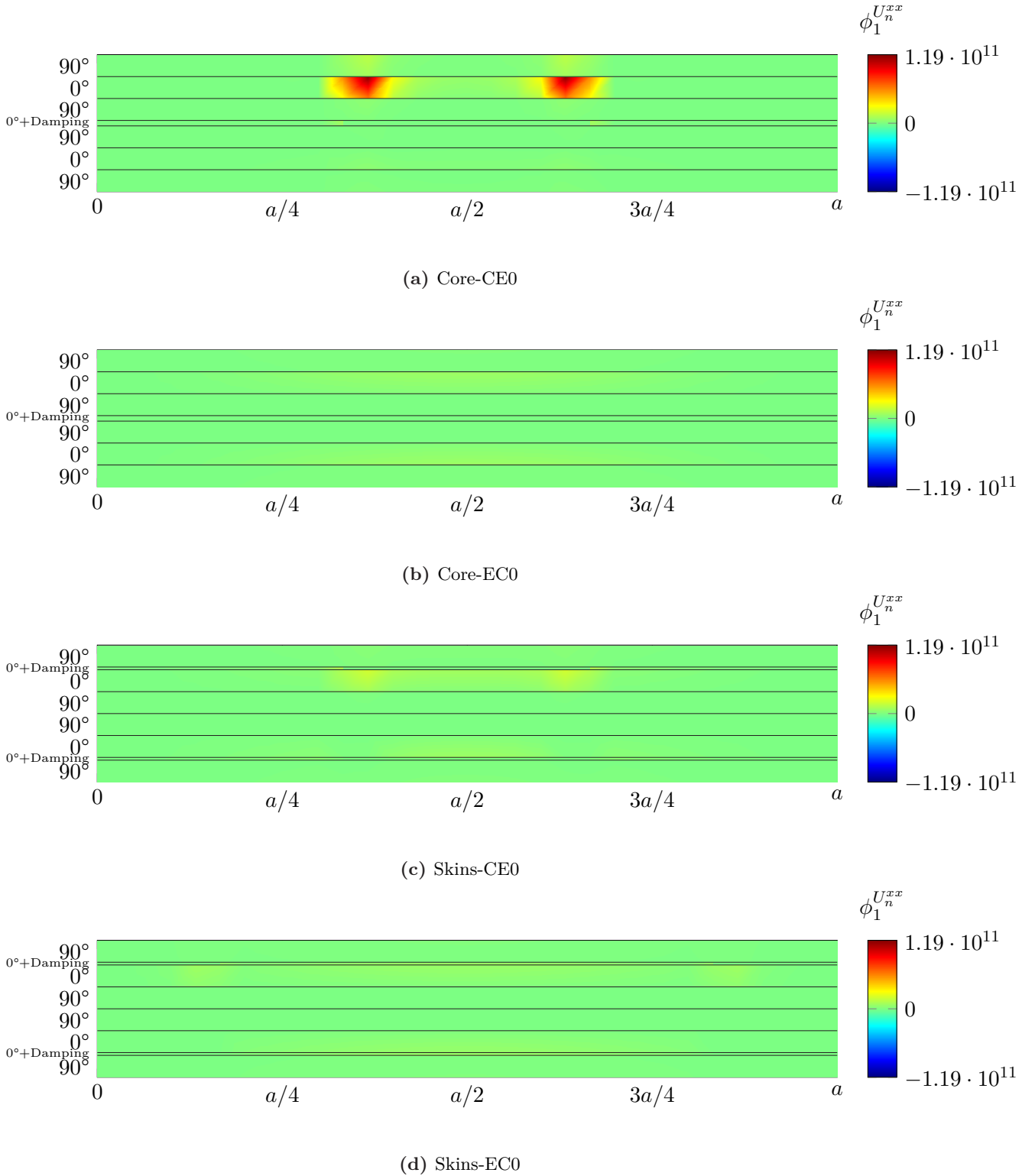
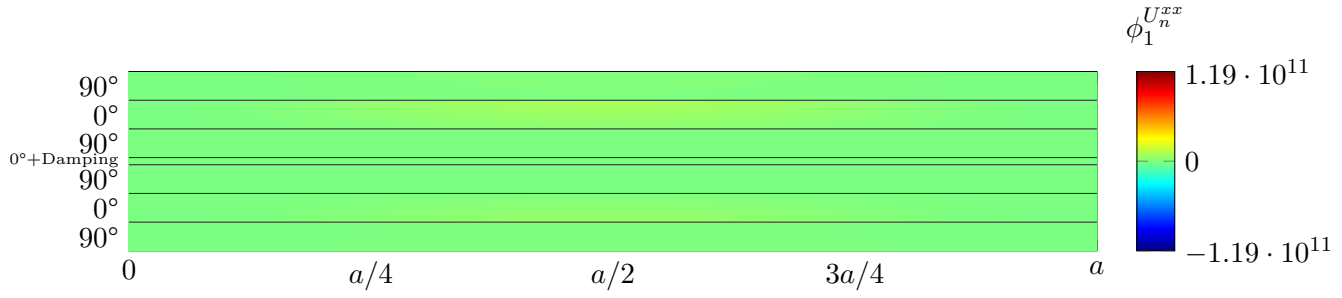
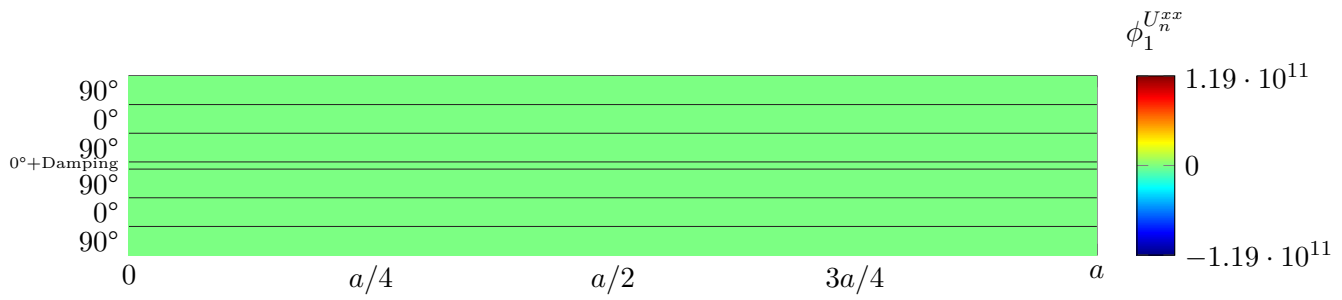


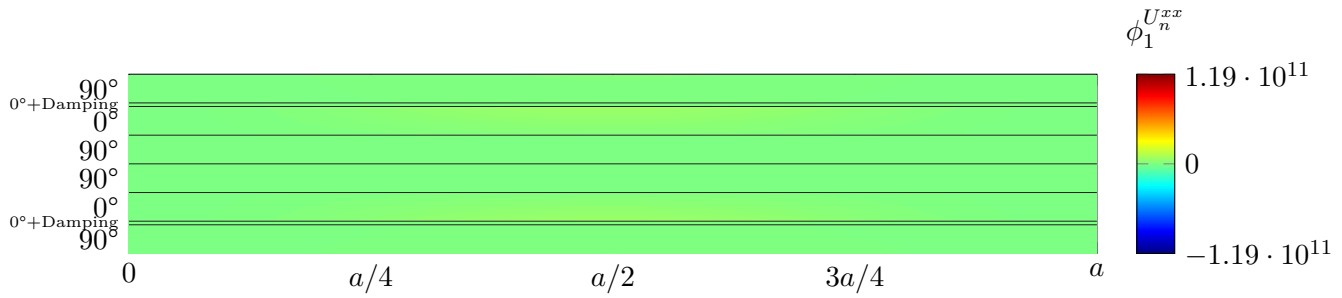
Figure 7.105 $\phi_1^{U_n^{xx}}$ - extensional strain energy density mode shape - section cut at $y = b/2$



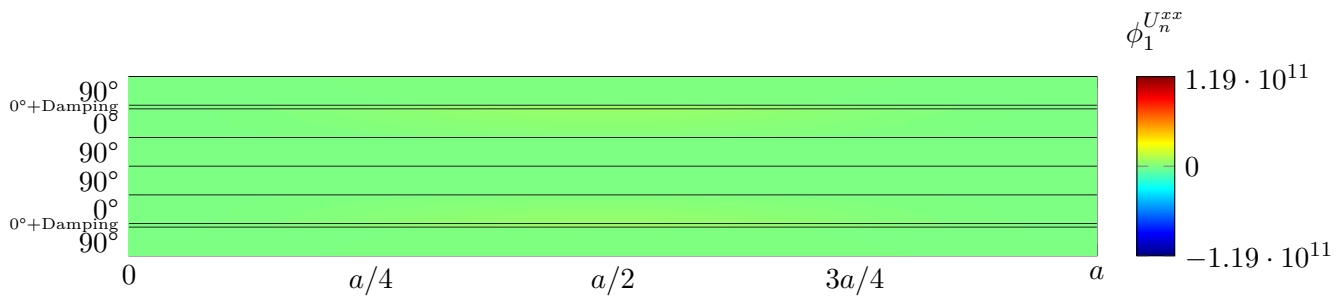
(a) Core-CE90



(b) Core-EC90



(c) Skins-CE90



(d) Skins-EC90

Figure 7.106 $\phi_1^{U_n^{xx}}$ - extensional strain energy density mode shape - section cut at $y = b/2$

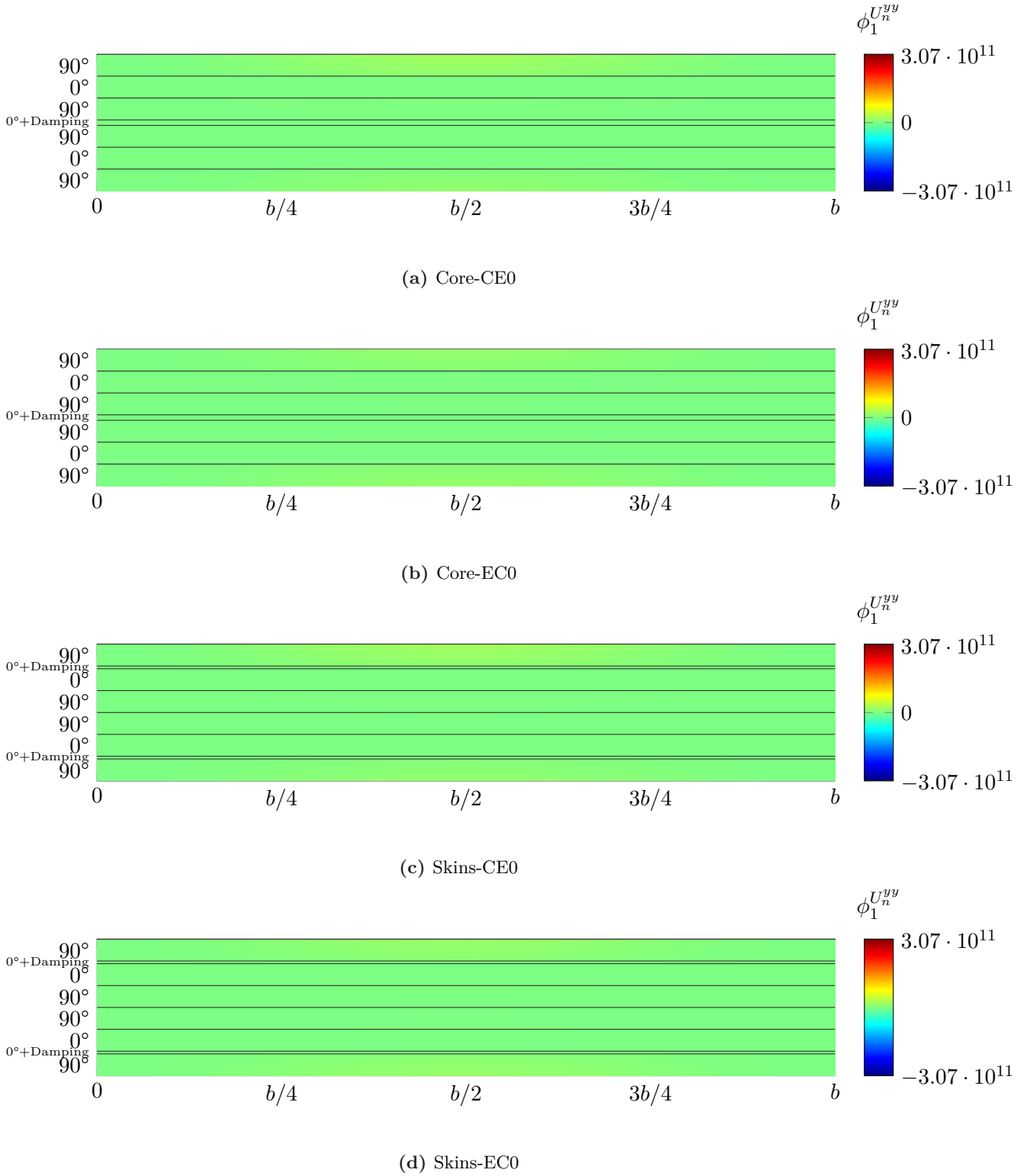


Figure 7.107 $\phi_1^{U_n^{yy}}$ - extensional strain energy density mode shape - section cut at $x = a/2$

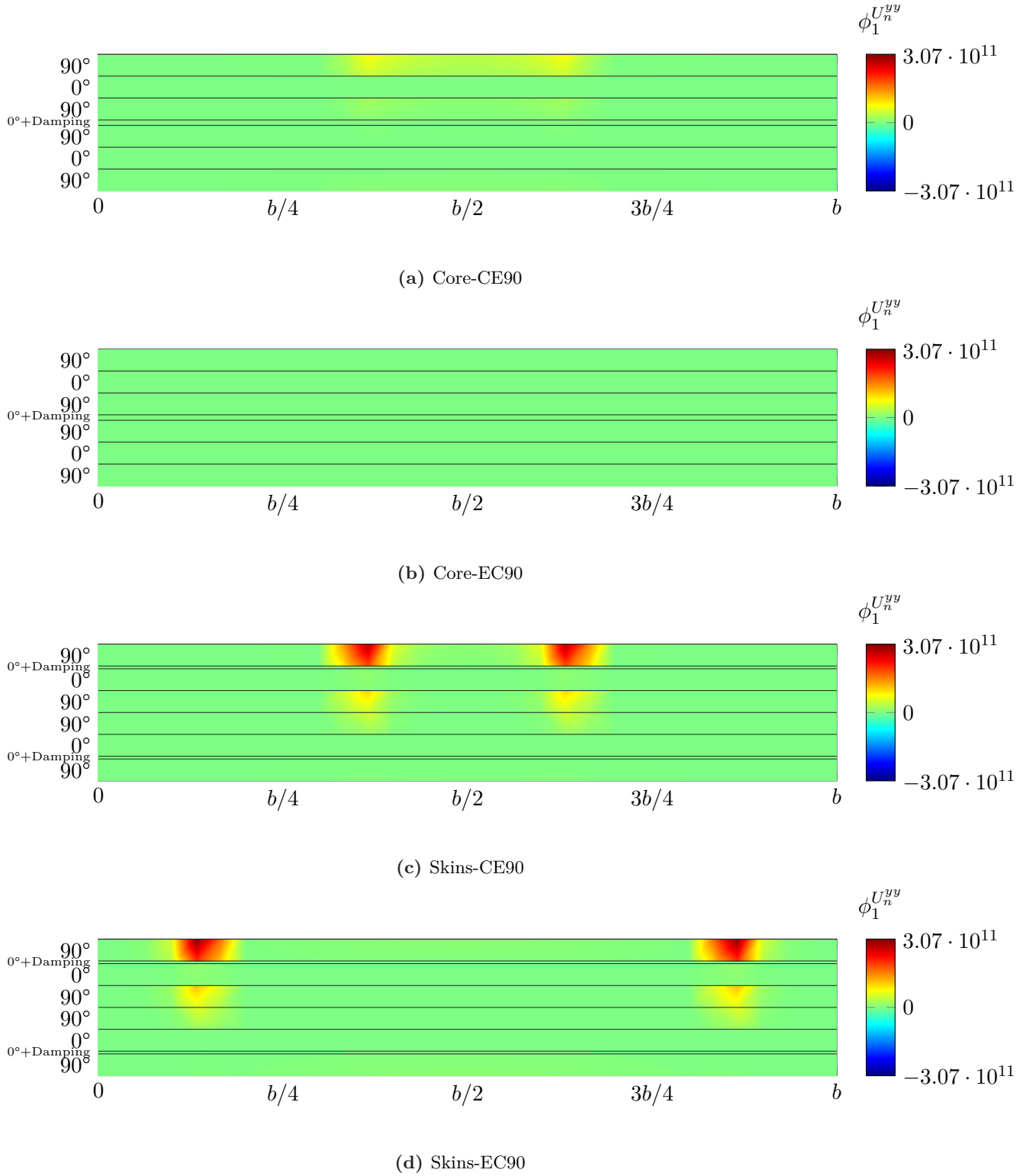


Figure 7.108 $\phi_1^{U_n^{yy}}$ - extensional strain energy density mode shape - section cut at $x = a/2$

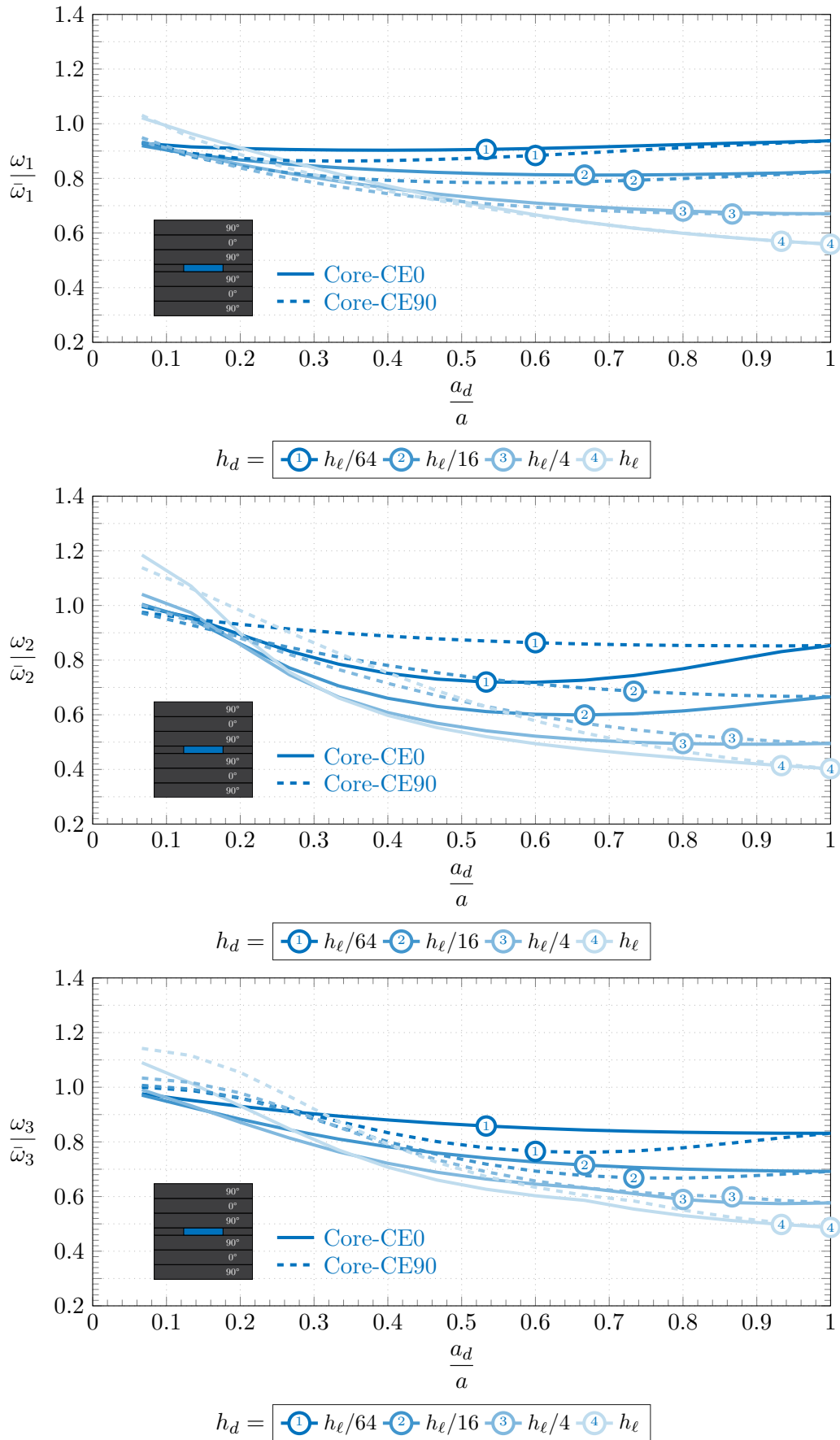
One of the main motivations for the proposed approach of interleaved damping patches is an already discussed topic throughout this work. Design requirements for high damping capacity and stiffness often conflict. To study the influence of the damping patch width, thickness and position on the natural frequencies of composite laminates with interleaved damping patches, Figures 7.109 to 7.112 present the first three dimensionless natural frequencies, $\omega_n/\bar{\omega}_n$, as a function of the patch width, a_d/a , and dimensionless damping layer thickness, h_d/h_ℓ . As a reminder, dimensionless natural frequencies represent the ratio between the natural frequencies of the composite laminate with interleaved damping patches and the natural frequencies of the reference six-layer composite plate.

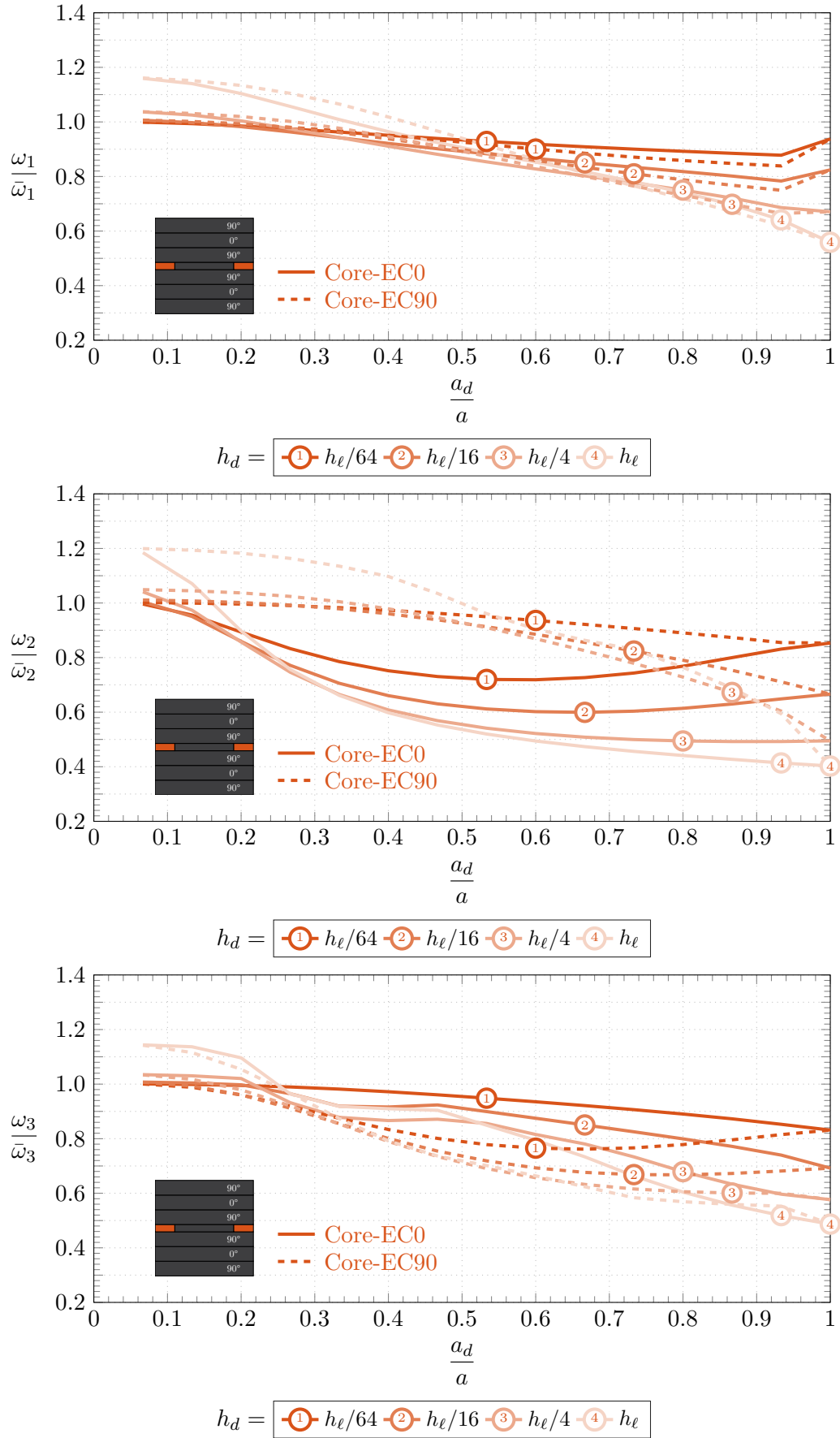
Overall, analysing Figures 7.109 to 7.112, the conflict between damping capacity and stiffness is noticeable. Configurations presenting improved damping performances are the ones that generally show the biggest decreases in natural frequencies and stiffness, with larger damping patches. The comparison between Core-CE and Core-EC configurations for the first natural frequency, for example, is a prime example. Core-CE configurations disclosed a considerable damping capacity for small interleaved damping patches in contrast to the Core-EC arrangements. Here, the dimensional natural frequency variation shows the mirror image - Core-CE configurations present higher decreases for interleaved damping patches with small widths. This result proves that the well-known conflict between damping capacity and stiffness is present even for interleaved damping patches.

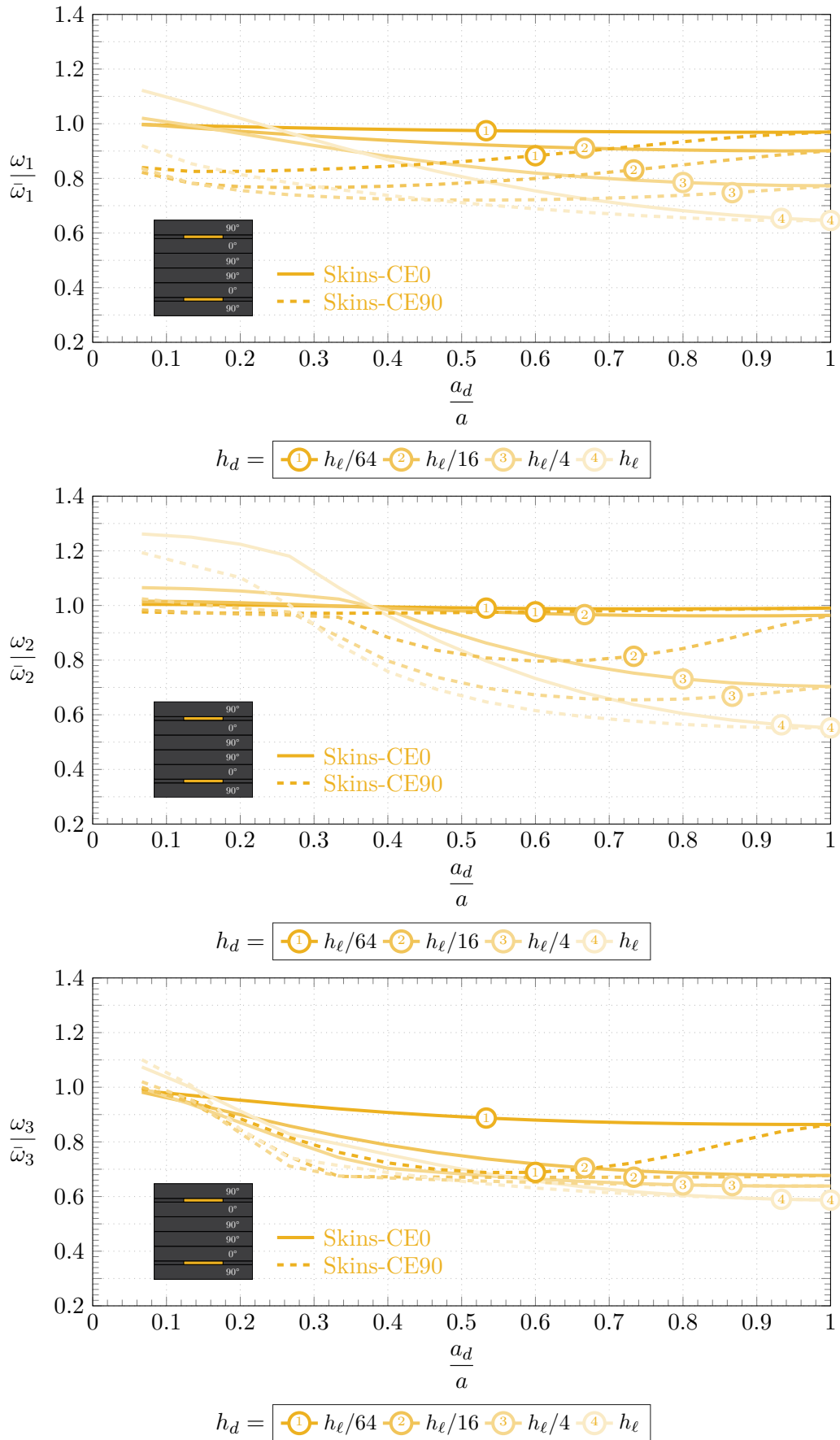
Another worth mentioning aspect is that, for some configurations, there is a minimum dimensionless natural frequency attainable for a given structure, i.e., increasing the damping patch width beyond this threshold value does not necessarily decrease the natural frequency, proving that there is room for optimisation at this level.

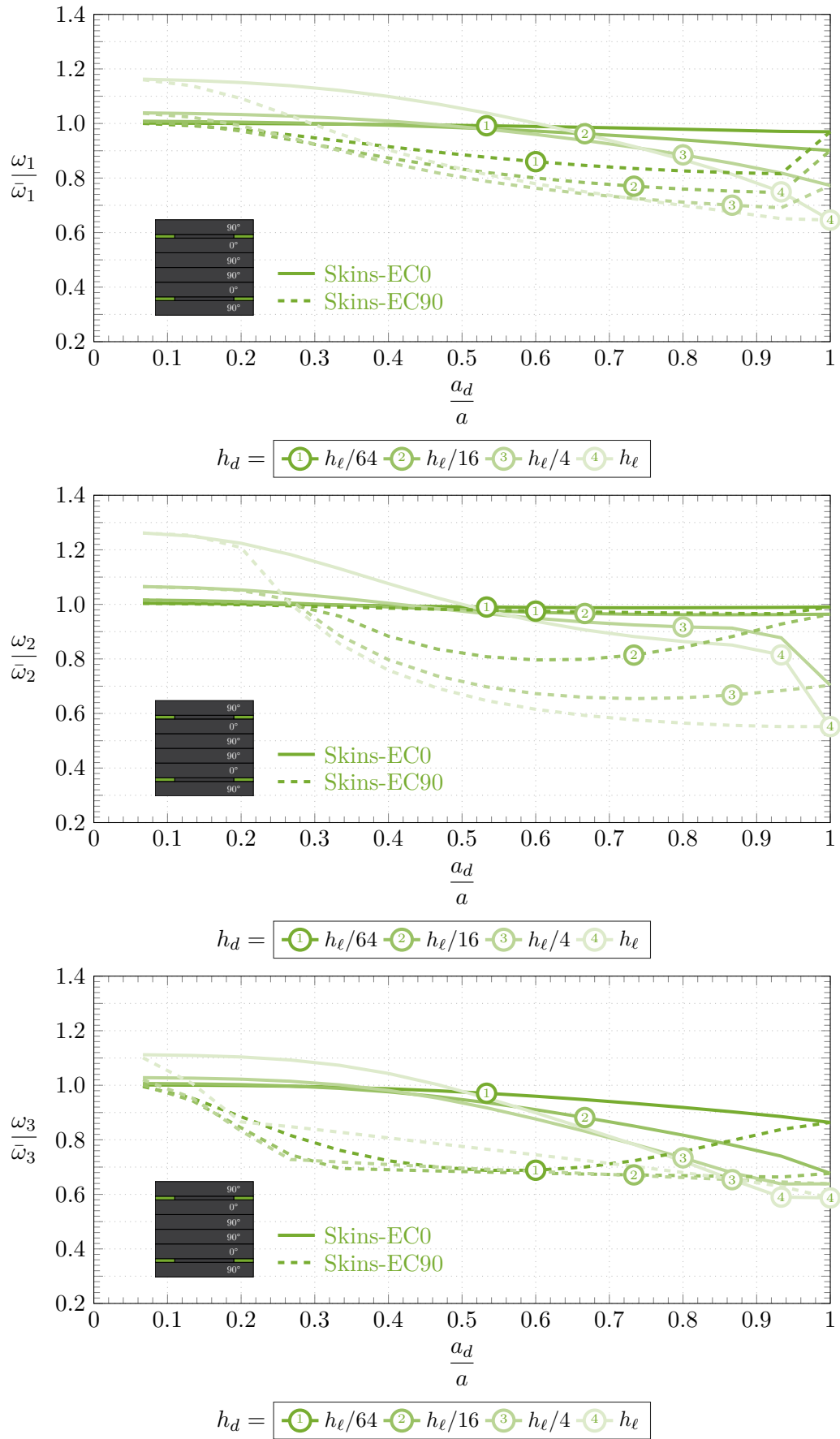
Nevertheless, the most common trend between configurations is the influence of the damping layer thickness on the dimensionless natural frequencies. For thicker damping patches, the decrease in stiffness with the increase of the damping patch width is more marked. This result shows that there is room for optimisation since it is possible to achieve an efficient damping capacity for a given damping patch thickness without unnecessarily decreasing the stiffness. In other words, there is no increasing-decreasing effect for the dimensionless natural frequency as for the modal loss factor.

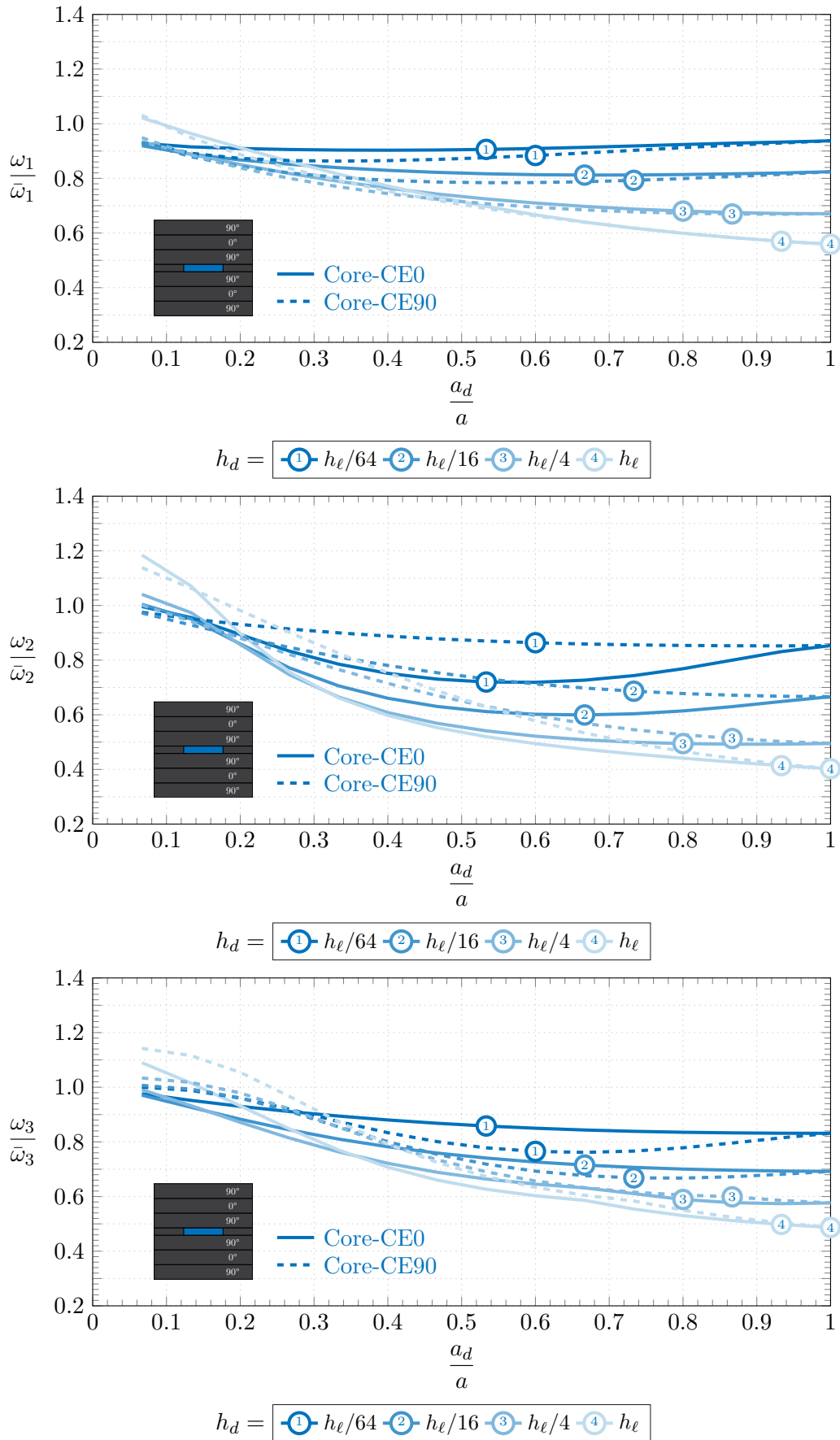
In addition, Figures 7.113 to 7.120 feature the dimensionless natural frequency variation for the same configurations, but presenting the results for reference laminated composite plates with an aspect ratio $a/h = 100$ and for thicker laminated composite plates with an aspect ratio $a/h = 50$, side-by-side. Somewhat monotonously, the main trends for the dimensionless natural frequency variation remain when one considers thicker laminates. The only meaningful difference is the higher decrease in natural frequencies for thicker laminates, depicting the expected result of the impact of a larger soft damping layer embedded in thick composite laminates.

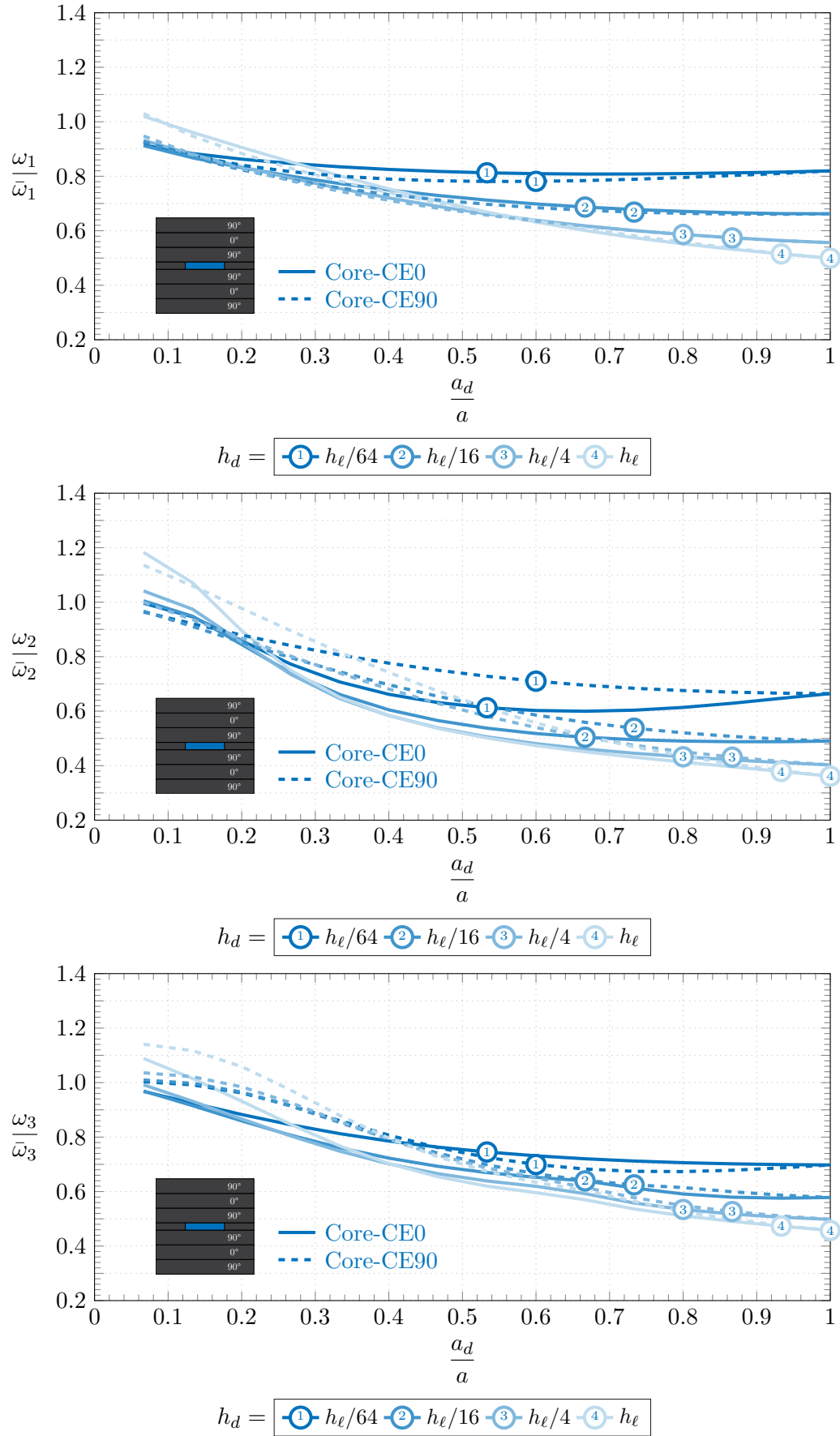
Figure 7.109 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 100$

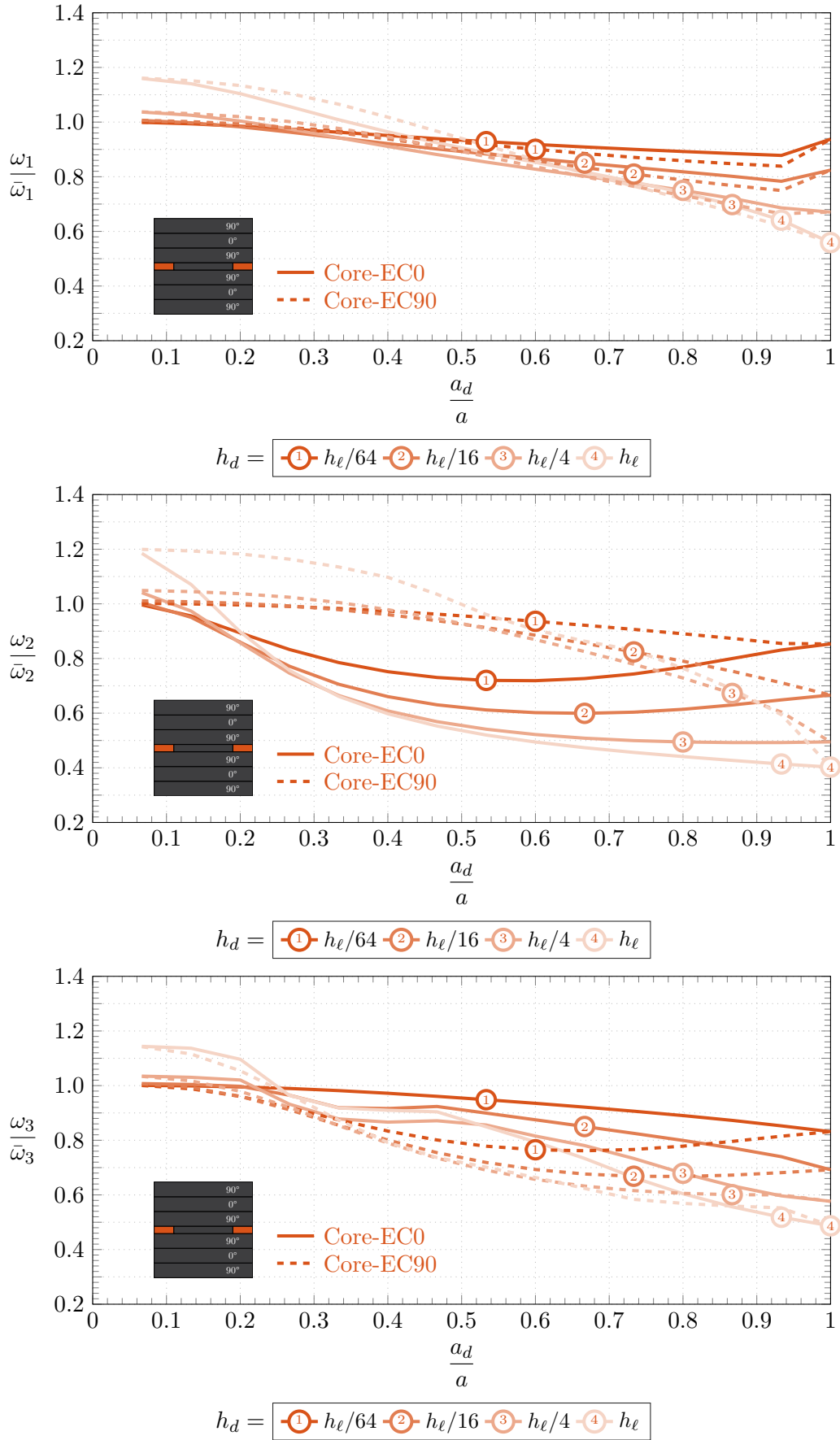
Figure 7.110 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 100$

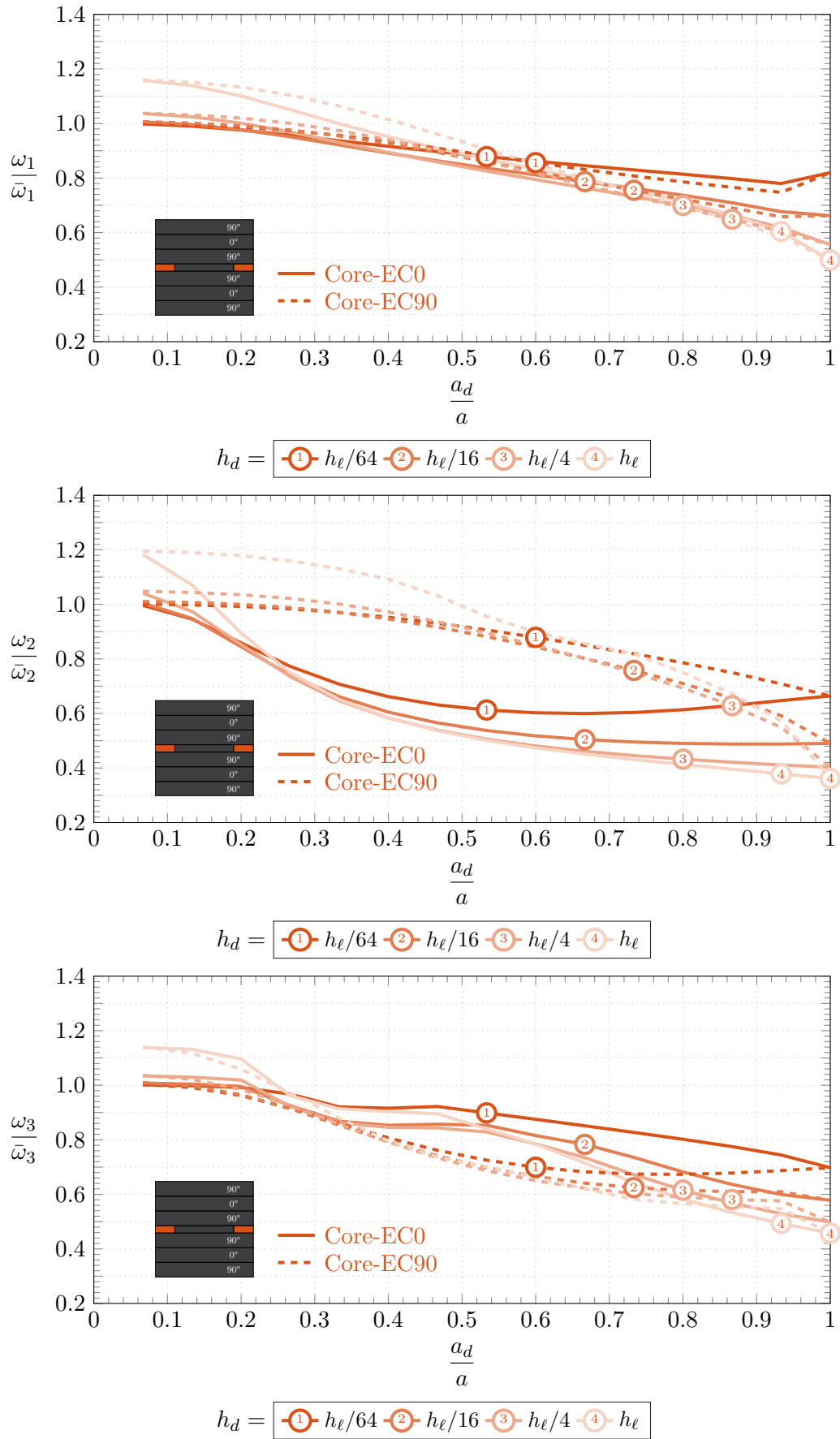
Figure 7.111 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 100$

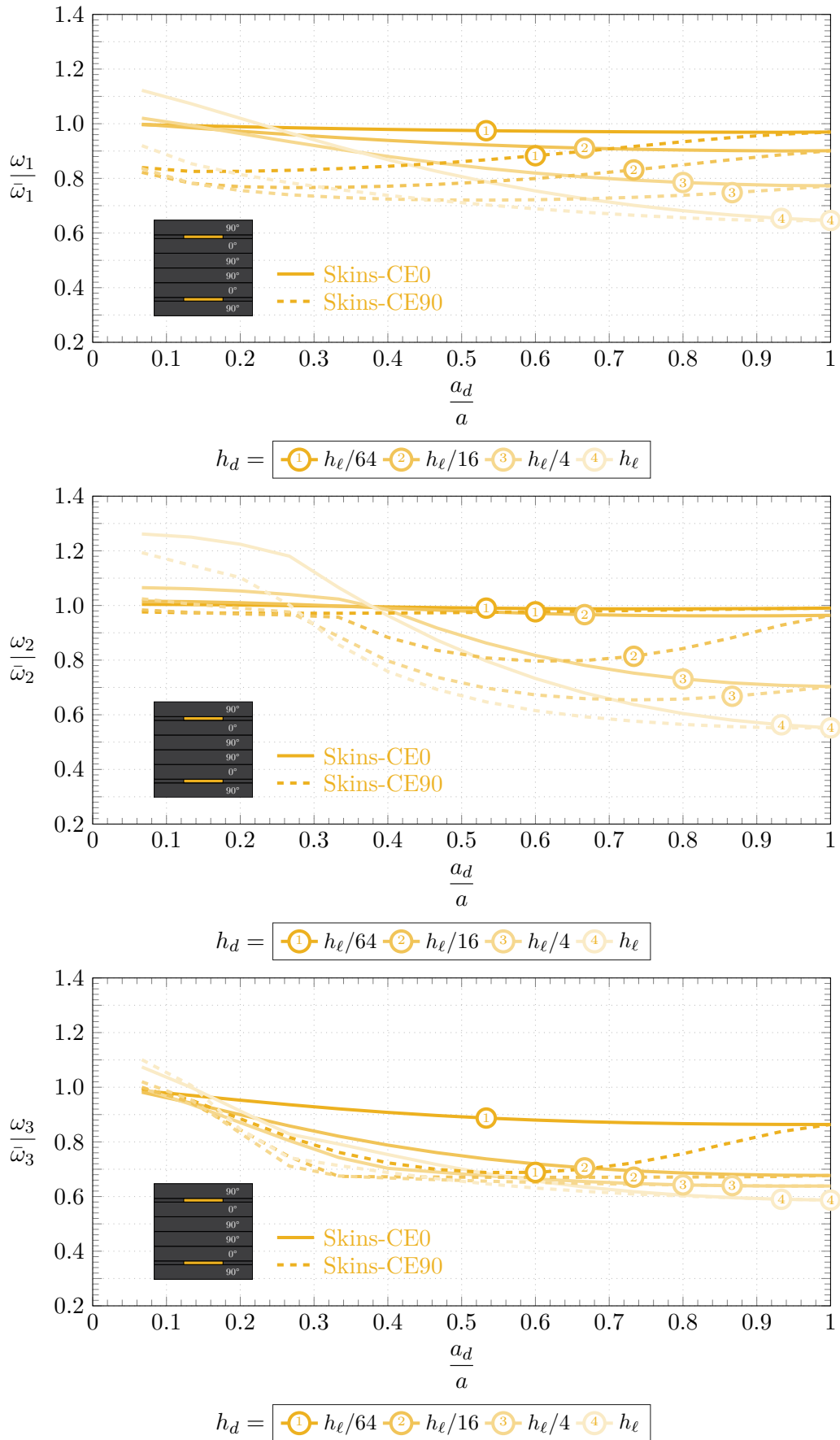
Figure 7.112 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 100$

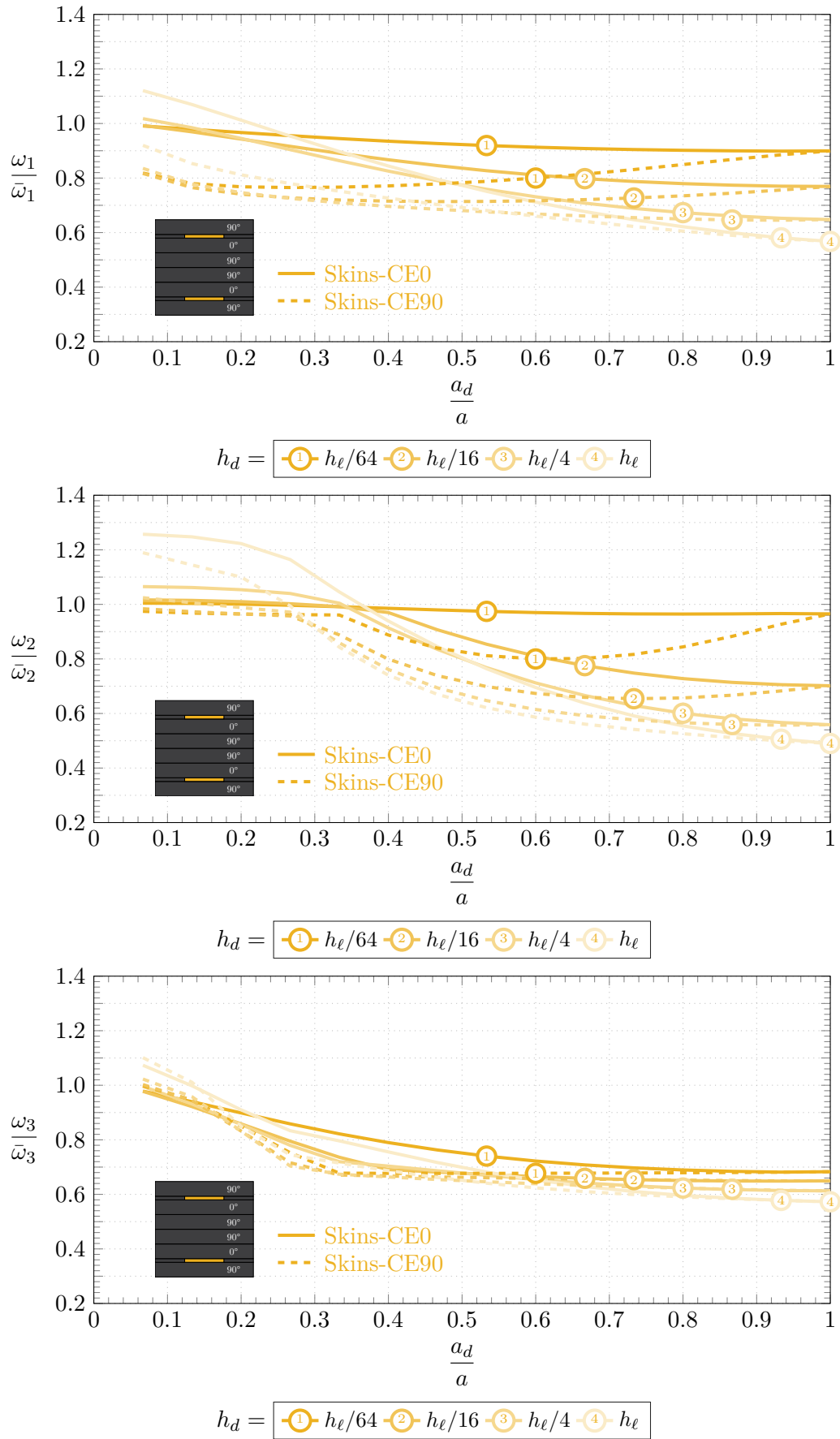
Figure 7.113 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 100$

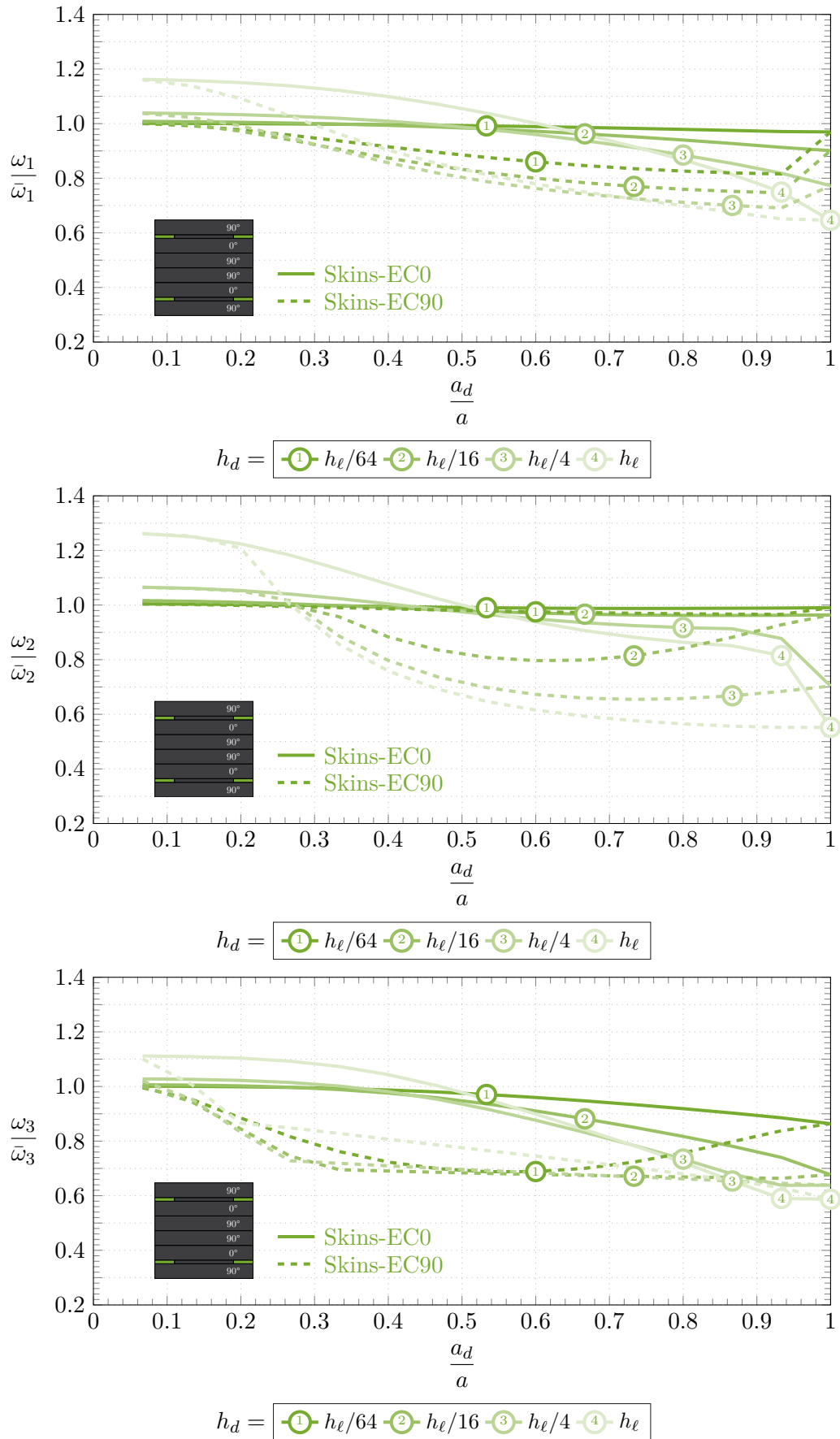
Figure 7.114 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 50$

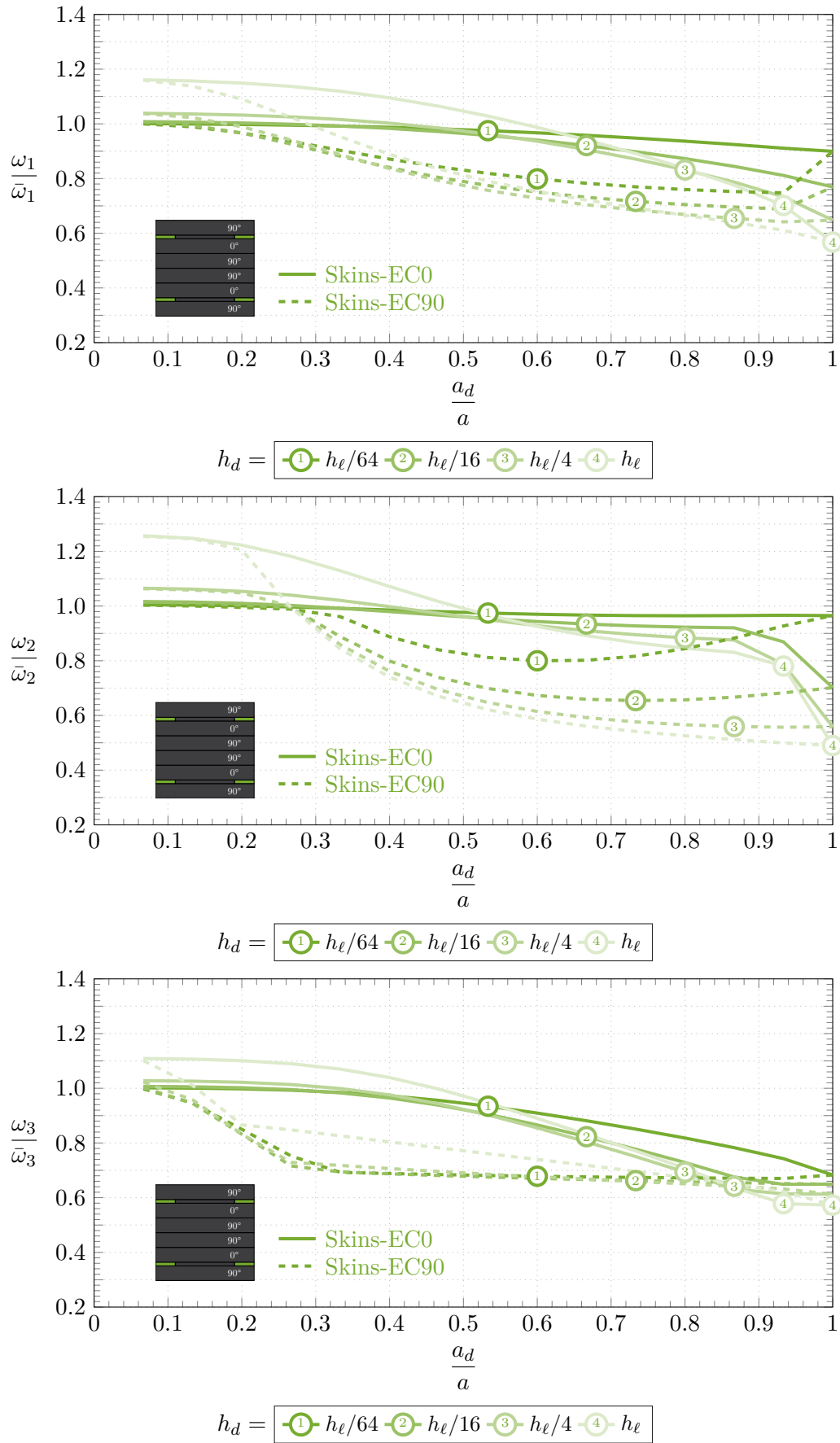
Figure 7.115 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.116 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 50$

Figure 7.117 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.118 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 50$


 Figure 7.119 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.120 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 50$

7.3.2 | Damping Efficiency of FFFF Laminates with Interleaved Patches

Given the crucial impact of boundary conditions on structural mechanics, and as extensively investigated and reported in this work, examining their influence on the efficiency of interleaved damping patches in composite laminates becomes imperative. It is relevant to emphasise that one of the main goals of the present work is to draw general conclusions and design guidelines for the use of damping patches, so it is decisive to explore a representative set of examples.

In the following, one studies the damping efficiency of interleaved damping patches considering laminated composite plates with all free edges. Following the procedure of the previous section, one focuses on the influence of the damping patch width, thickness and position on the damping capacity, considering the eight stacking sequences presented before. Once again, the primary objective is to evaluate configurations featuring interleaved damping patches while maintaining an equal volume of damping material. Consequently, in exploring the impact of damping patch placement as in the previous section, h_d denotes the overall thickness of the damping material, independent of the damping layers in the stacking sequence, while a_d represents the width of the damping patch, varying from $a_d/a = 0$ to $a_d/a = 1$ ($a_d = b_d$ in square plates). As a reminder, when comparing the single interleaved configuration with the double interleaved one, the thickness of each patch in the double interleaved stacking sequence is half the total thickness. On the other hand, each patch of an Edge-to-Centre configuration has half the width of a Centre-to-Edge patch. Exploring various thicknesses of damping layers helps in understanding their effect on damping efficiency. This analysis involves assessing the overall thickness of damping patches, denoted as h_d , in relation to the thickness of a single reference layer, represented as h_ℓ . The total laminate thickness of the reference stacking sequence in Figure 7.86 is $h = 6h_\ell$.

7.3.2.1 | Influence of Interleaved Damping Patch Width, Thickness and Position

Consistent with earlier discussions, a crucial aspect involves analysing the mode shapes of laminated plates to gauge the effectiveness of damping treatments. Retaining the same reference stacking sequence from the previous section, one recalls Figure 7.46. This figure showcases the transverse displacement mode shapes for the first six normal modes of a laminated composite plate with all free edges, maintaining an aspect ratio of $a/h = 100$, mirroring our previous analysis. Previously, one also discussed the importance of the mode shapes associated with the transverse shear strain degrees of freedom γ_{xz} and γ_{yz} to understand the modal transverse shear strain pattern, and so Figures 7.47 and 7.48 showed, respectively, the first six ones for the same reference laminated composite plate.

Consistently with previous analyses, one draws attention to Figures 7.49 to 7.51 displaying the through-the-thickness extensional, transverse shear and in-plane shear strain density mode shapes for the third normal mode and different section cuts, respectively. In particular, sections for $y = 0$ and $x = 0$ are worth analysing since the extensional and transverse shear strain energy densities are the highest. Figures 7.46a, 7.47a and 7.48a highlight these section cuts along the transverse displacement and transverse shear strain pattern mode shapes, respectively. One extensively discussed all these through-the-thickness density mode shapes in Section 7.2.2.1.

Recognising the complexity of analysing the impact of interleaved damping patch width, thickness, and positioning on the damping efficacy within laminated composite plates is crucial. One recalls Figures 7.87 and 7.88 displaying the eight distinct configurations under examination, offering a glimpse into the multitude of variables involved in this study, as already pointed out in the previous section. Similarly, as seen in earlier investigations within this study, the aspect ratio of the laminate, a/h , remains a significant consideration. Consequently, the attained results demand careful and thorough scrutiny, considering the intricate interplay of these variables.

The initial analysis focuses on how width, thickness, and interleaved damping patches' position impact damping efficiency within reference laminated plates with an aspect ratio of $a/h = 100$. Figures 7.121 to 7.124 present the specific modal loss factors for the initial three normal modes, showcasing their variation concerning damping patch layer thickness and width across different stacking sequences - Core-CE, Core-EC, Skins-CE, and Skins-EC, respectively. Employing a methodology consistent with previous studies in this work, one considers multiple modes to establish comprehensive conclusions. In order to ensure consistency, the solid curves represent Configuration 0, while the dashed curves depict Configuration 90 (refer to Figure 7.88). Assessing four distinct damping patch layer thicknesses ($h_d = h_\ell/64$, $h_d = h_\ell/16$, $h_d = h_\ell/4$, and $h_d = h_\ell$) aids in understanding their impact on damping efficiency. This range covers a substantial span of potential damping patch layer thicknesses; however, it's essential to note that $h_d = h_\ell/64$ signifies an exceedingly thin layer, challenging to implement practically. Nevertheless, its inclusion contributes significantly to understanding the relationship between damping patch layer thickness and damping efficiency.

In similar fashion, one presents in Figures 7.121 to 7.124 the specific modal loss factors of the first three normal modes as a function of the damping patch layer thickness and width for the different stacking sequences under analysis - Core-CE, Core-EC, Skins-CE and Skins-EC, respectively, and side-by-side to comprehensively compare the thickness and width of damping patch layers across various stacking sequences. Following the same reasoning of the previous section, the specific modal loss factor plots depicted in Figure 7.122 for the Core-EC stacking sequence comprise the most effective damping performance curves obtained within the Core-CE stacking sequence, allowing for a direct comparison between Core configurations. Likewise, the Skins-CE stacking sequence results in Figure 7.123 enable a direct assessment between Centre-to-Edge runs in both Core and Skins configurations. Lastly, the damping efficiency curves of the Skins-EC stacking sequence in Figure 7.124 highlight the superior overall performances of the Core and Skins-CE configurations, thus enhancing the reading experience of the plots and thus enabling the identification of preferred arrangements in terms of damping capacity.

It comes as no surprise the consistency with prior findings concerning surface-damping treatments, interleaved damping layers and interleaved damping patches in simply supported composite plates. Illustrated in Figures 7.121 to 7.124, the impact of varying damping layer thicknesses on damping capacity is clear, revealing an upper limit consistent with the increasing-decreasing effect already discussed. Obviously, the composite laminates with all free edges are no exception, there exists a critical threshold for damping layer thickness. Beyond this threshold, further increases in thickness lead to a decline in the material's damping capacity. As previously discussed, thicker damping layers disengage the laminate's elastic components, reducing their ability to induce shear on the damping material. Still, it is worth to emphasise some noteworthy differences to the simply supported case. In particular, the threshold of the increasing-decreasing effect occurs for thicker damping layers in composite plates with all free edges and one needs thicker damping layers to obtain comparable damping capacities.

Core configurations in Figures 7.121 and 7.122 help establish some conclusions about the damping performance of interleaved damping patches in laminated plates and the influence of boundary conditions. Compared overall, and similarly to the simply supported case, configurations featuring the damping patch extending from the centre to the edges (CE), as illustrated in Figure 7.121, exhibit superior damping performance compared to those running from the edges to the centre (EC), depicted in Figure 7.122. This difference is particularly noticeable for damping patches with smaller widths and even more clearly for the second natural mode, at least for configurations oriented with the 0° CFRP layers - Configuration 0. Indeed, one of the most noteworthy results is the difference between damping performances for Configuration 0 and Configuration 90 for the Core-CE at the second mode. Undoubtedly, Configuration 0 promotes

better damping capacity, regardless of the damping layer thickness. At this point, it is worth comparing Figures 7.88a and 7.46b to comprehend the superior damping performance of the Centre-to-Edge running of the damping patch at the second natural mode. The corresponding displacement mode shape perfectly matches the geometry of a patch running from the centre to the edges (CE), thus promoting the transverse shear strain of the damping patch. Contrary to the simply supported example, where damping layer thickness and position (CE or EC) are far more determinant to the interleaved damping patch performance than the orientation of the different configurations, the composite plate with all free edges proves the importance of considering various boundary conditions to establish more general conclusions.

Similarly to the simply supported case, it is worth noting that Core-CE configurations emerge as preferable, given that Edge-to-Centre treatments demand large quantities of damping material for improved capacity. Still, a standout outcome is the superior damping performance of some configurations with interleaved damping patches compared to interleaved damping layers, particularly for Core-CE0 configuration. This noteworthy finding underscores the significance of integrating interleaved damping patches in laminated composites, showcasing the potential for enhanced damping with reduced material usage.

Figures 7.123 and 7.124 extend the analysis of reference laminated composite plates featuring interleaved damping patches, showcasing the specific modal loss factors for the first three normal modes in configurations with damping patches positioned closer to the skins. In aid of this analysis, the optimal damping performances of the core configurations for each mode have been included in these charts. Generally, core configurations are preferable to achieve optimal damping performances, with the exception of specific cases for the Skins-CE configuration. Nevertheless, the clearest result is that Centre-to-Edge arrangements are preferable to Edge-to-Centre ones, either for Core or Skins configurations, due to the ability of the stiff composite layers of the all-free edges plate to impose transverse shear strain on the damping patches.

Having analysed the damping performance of Skins configurations in Figures 7.123 and 7.124, and taking into account previous analyses, it becomes evident that, generally, these configurations require thick damping layers to achieve noticeable damping capacities. Thin interleaved damping patches positioned near the skins lack the ability to impose substantial transverse shear strain on thin damping patches, resulting in negligible damping capacity. This limitation is particularly noticeable for the Skins-EC configurations, which combine the placement of damping material near the skins along with the material running from the edges to the centre. These arrangements are not optimal in terms of damping capacity efficiency.

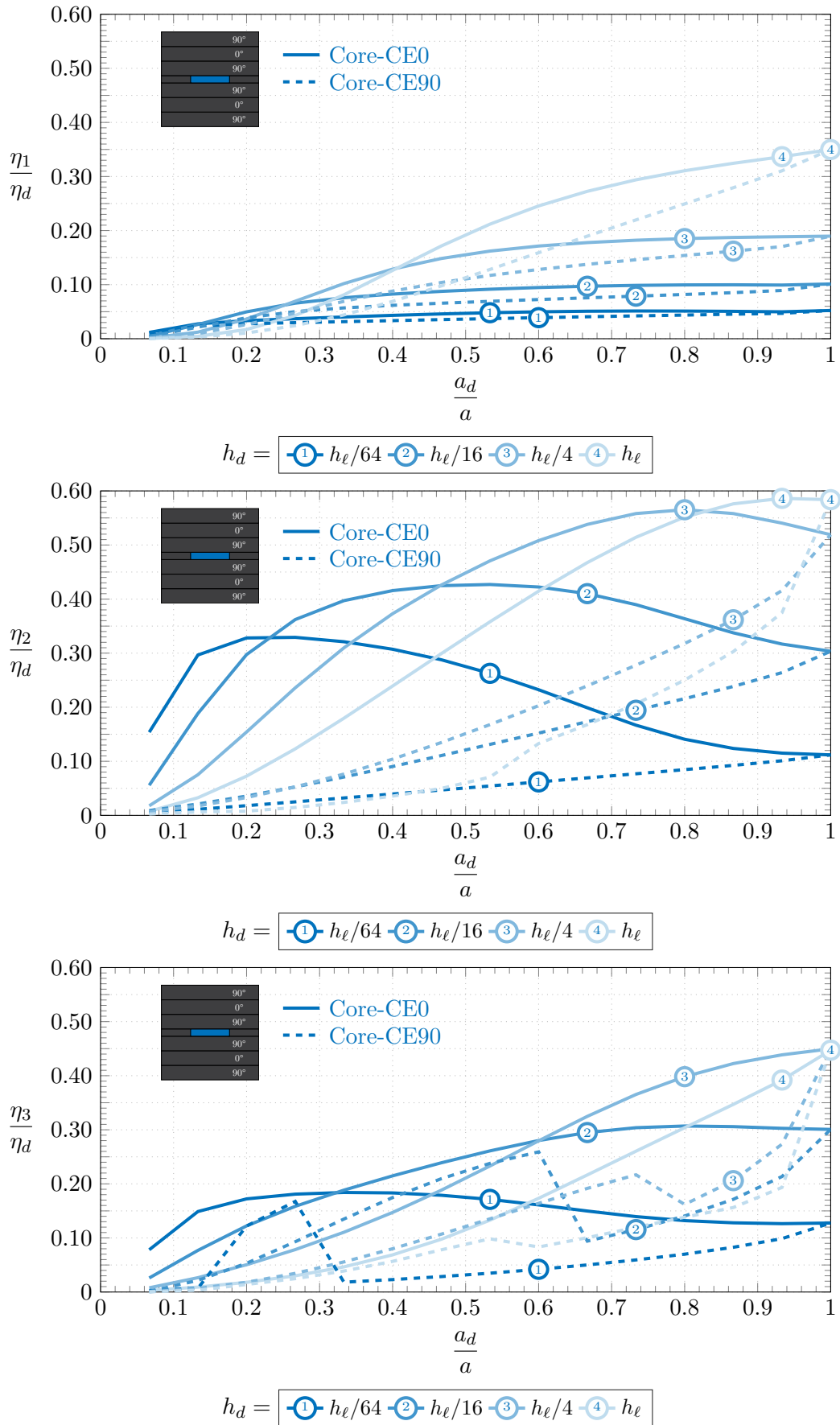
In delving into the analysis of how the thickness and position of interleaved damping patches influence the damping capacity of laminated plates, Figures 7.125 to 7.132 present a side-by-side display for enhanced clarity of the specific modal loss factors of the first three normal modes as a function of the damping patch width for both reference laminated plates with an aspect ratio of $a/h = 100$ and thicker laminated plates with an aspect ratio of $a/h = 50$.

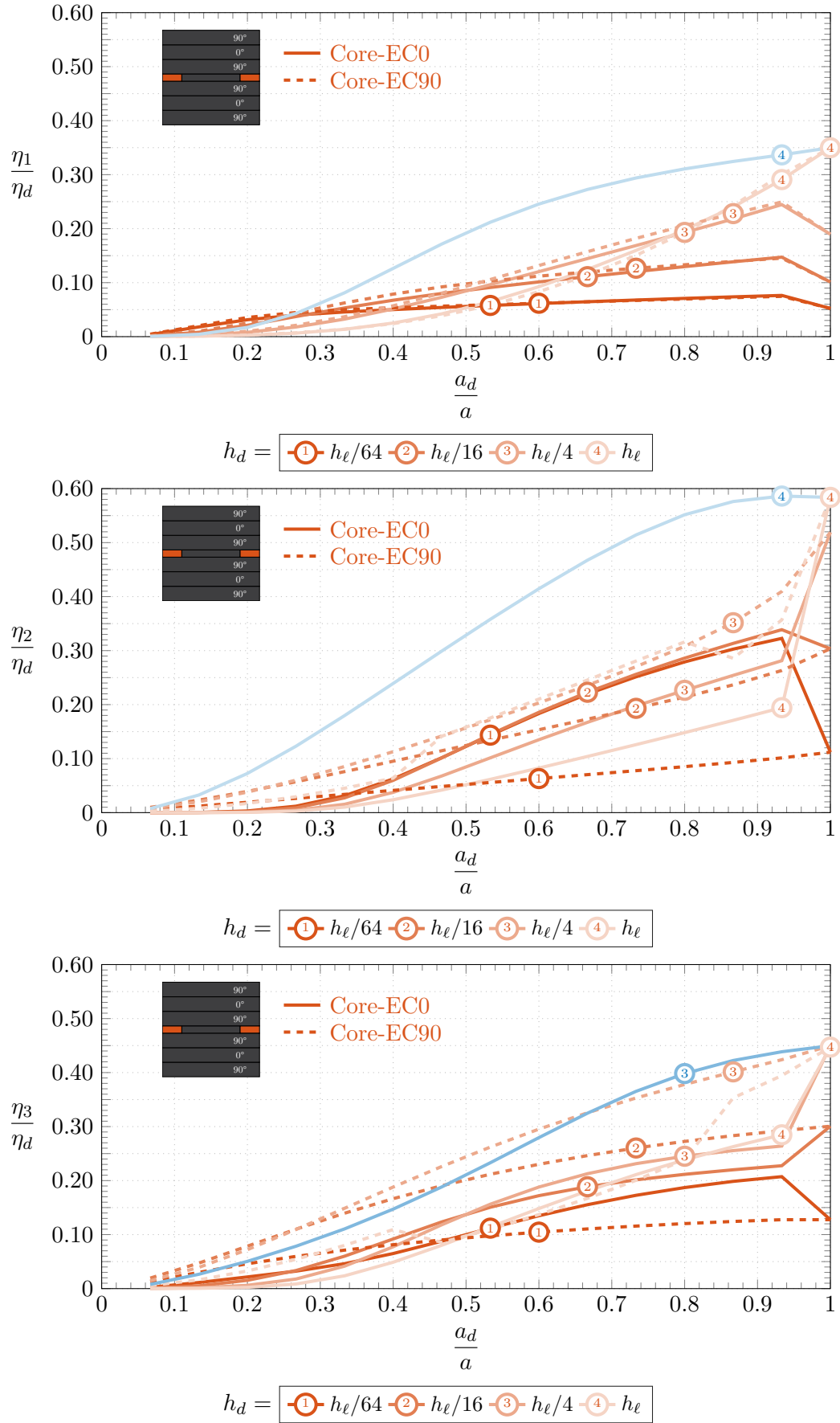
Adhering to the same rationale employed in prior studies and for ease of comparison, damping layer thicknesses are considered relative to the thickness of each elastic layer h_ℓ for both aspect ratios. The well-explored effect of increasing and decreasing effect of damping layer thickness in damping, thoroughly discussed in this work, finds further illustration and validation in these representations. Notably, the most pronounced distinction between reference and thick laminated plates in Figures 7.125 to 7.132, for both Core and Skins configurations, lies in the diminishing impact on damping capacity with increasing damping layer thickness.

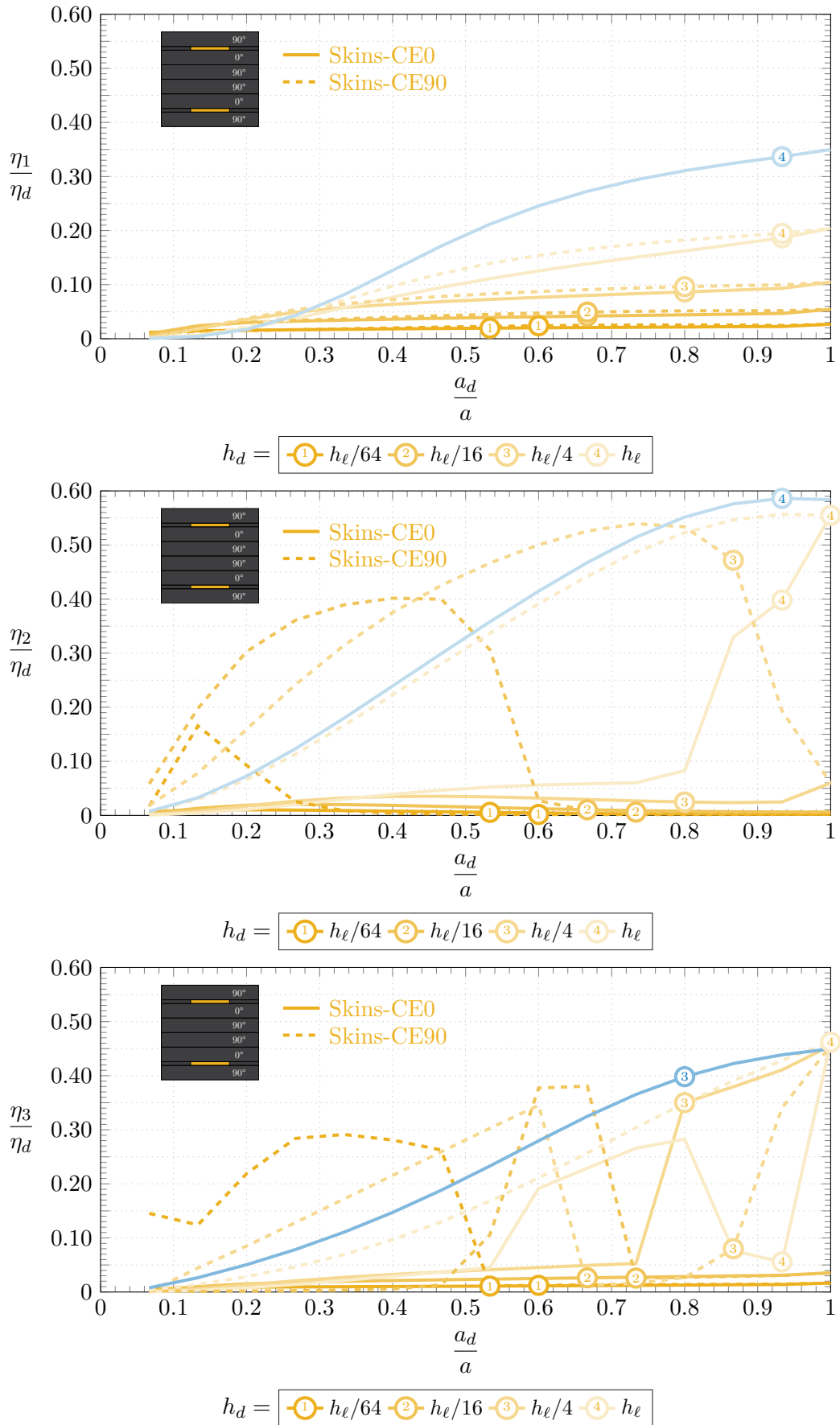
There is no news in thick laminated composite plates exhibiting enhanced damping performance with thinner damping layers, as illustrated by the Core and Skins configurations in Figures 7.126, 7.128, 7.130, and 7.132. Consistent with the thorough discussions throughout this work, the tendency of thick damping layers to decouple from the elastic stiff layers becomes

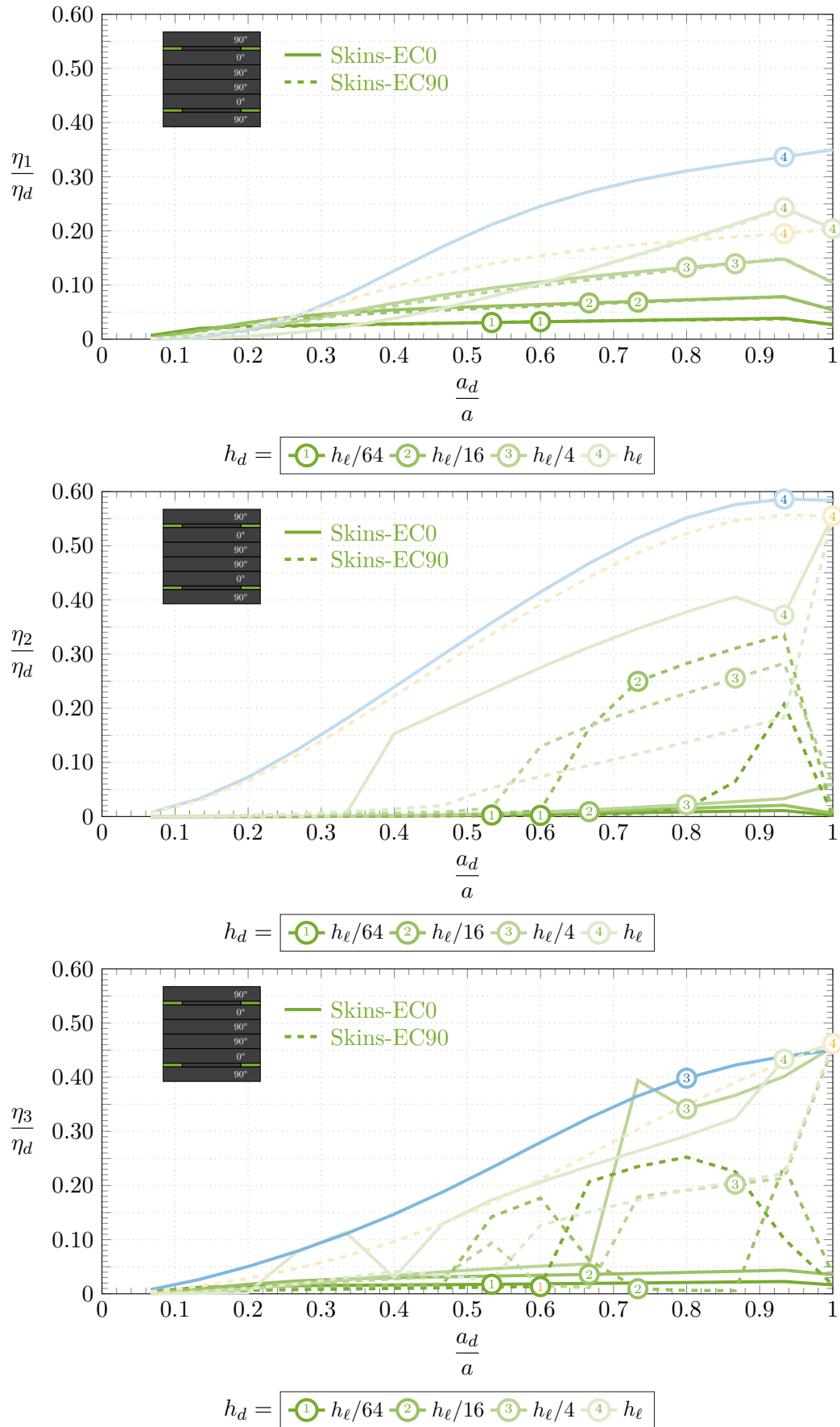
apparent, diminishing their capacity to impose shear on the damping material. Importantly, this decoupling effect is even more pronounced in the context of thick laminate plates.

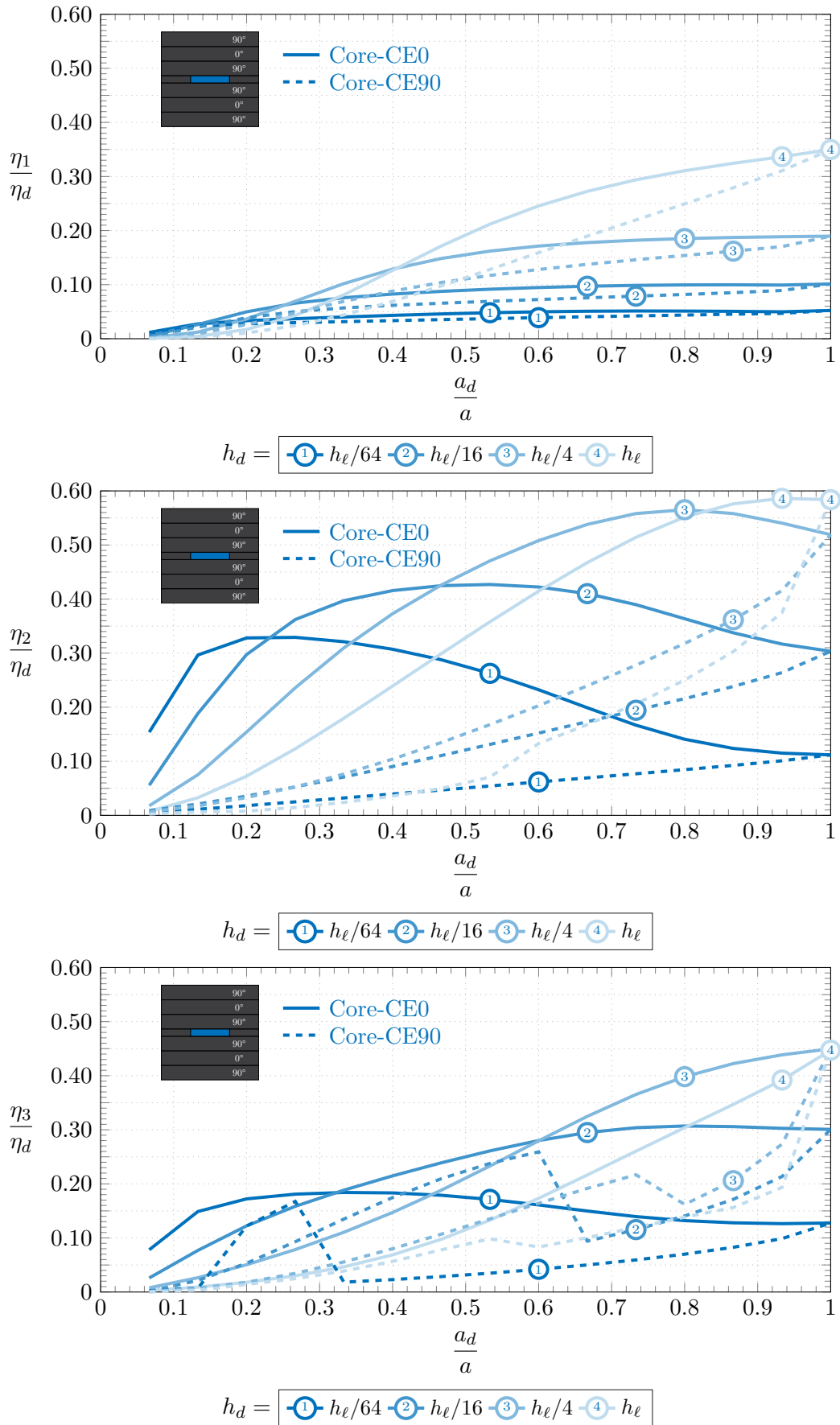
In fact, the trends previously observed in reference laminated plates with an aspect ratio of $a/h = 100$, along with the key distinctions between Core and Skins configurations, as well as Centre-to-Edge (CE) or Edge-to-Centre (EC) orientations of interleaved damping patches, persist in the context of thick laminated composite plates.

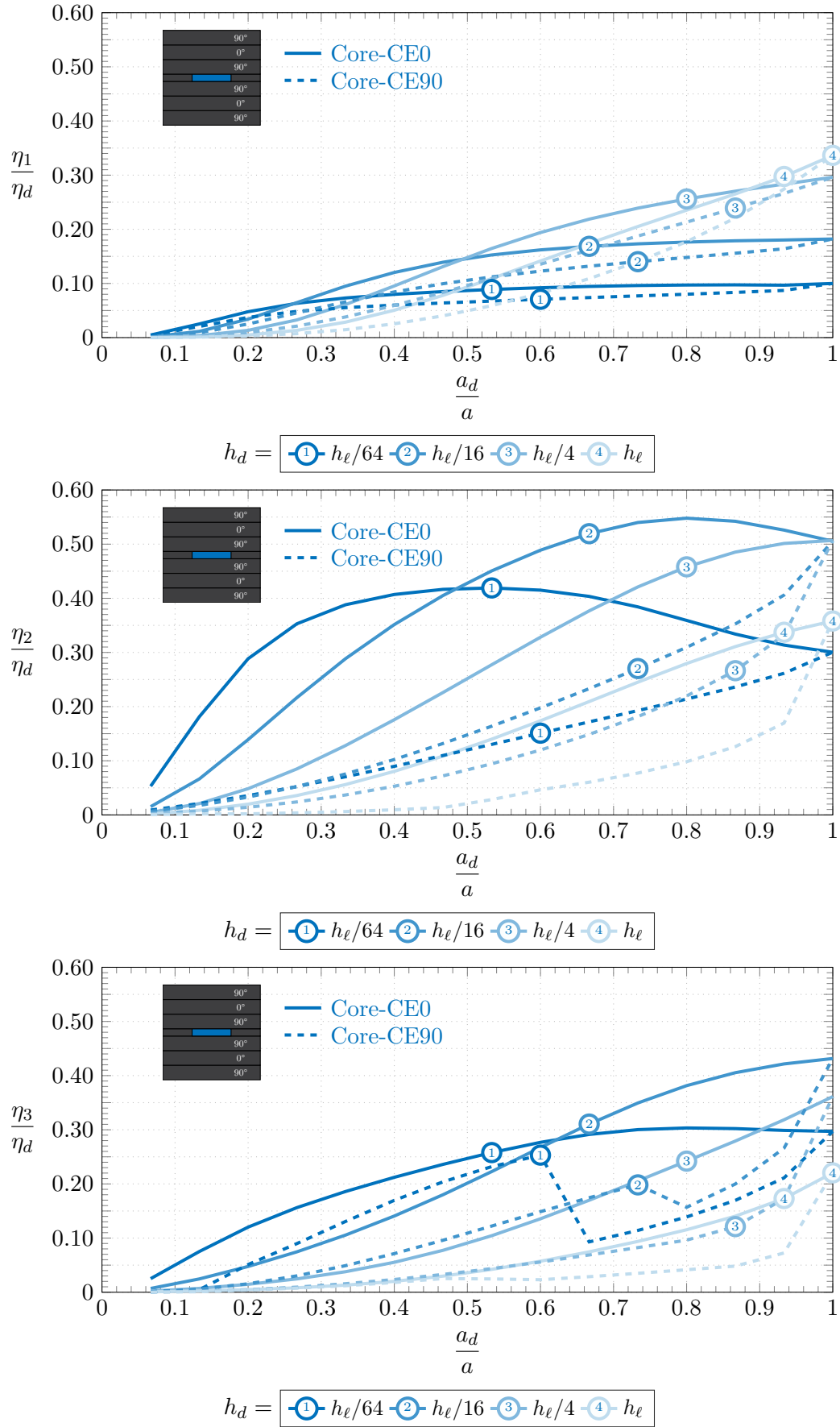
Figure 7.121 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 100$

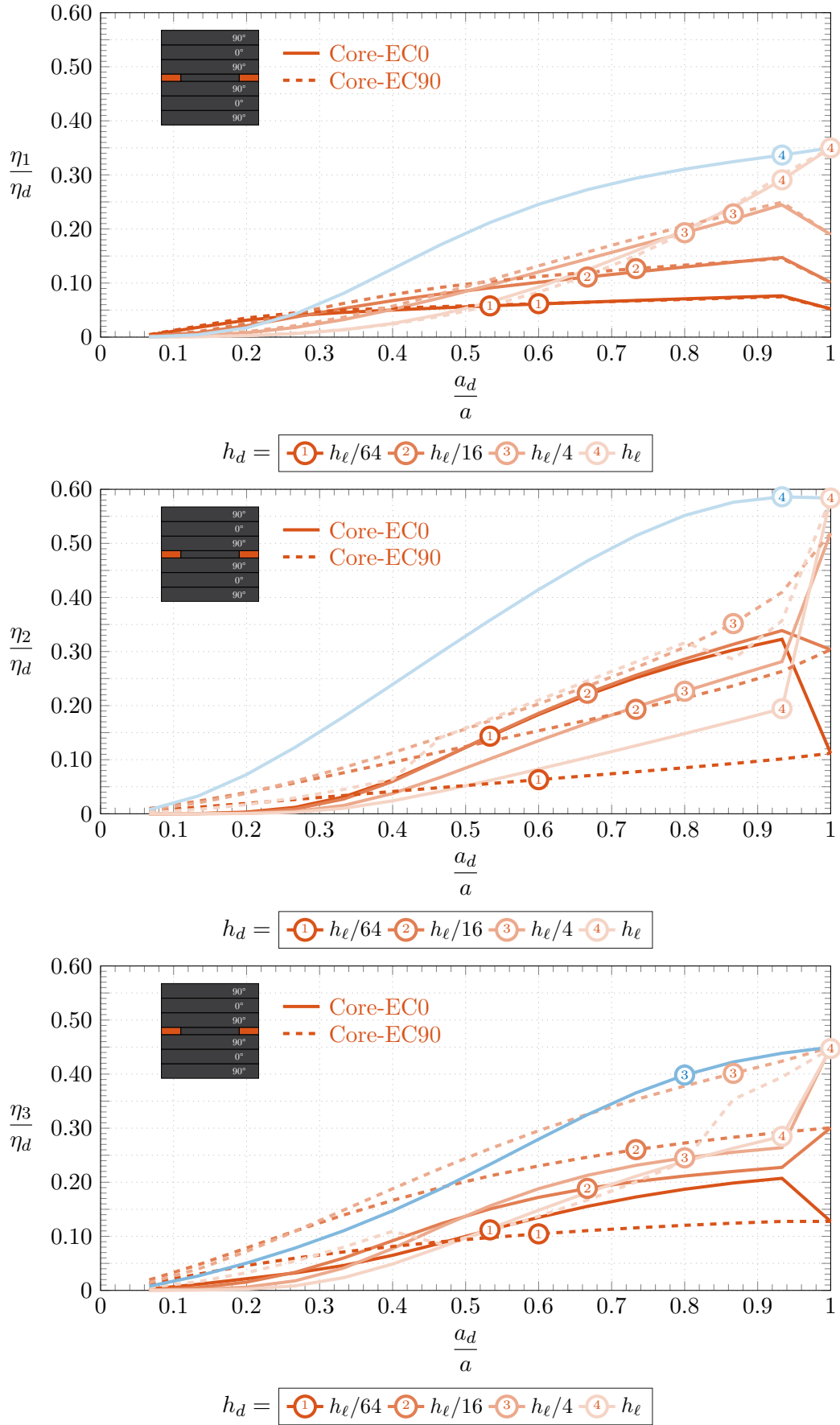
Figure 7.122 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 100$

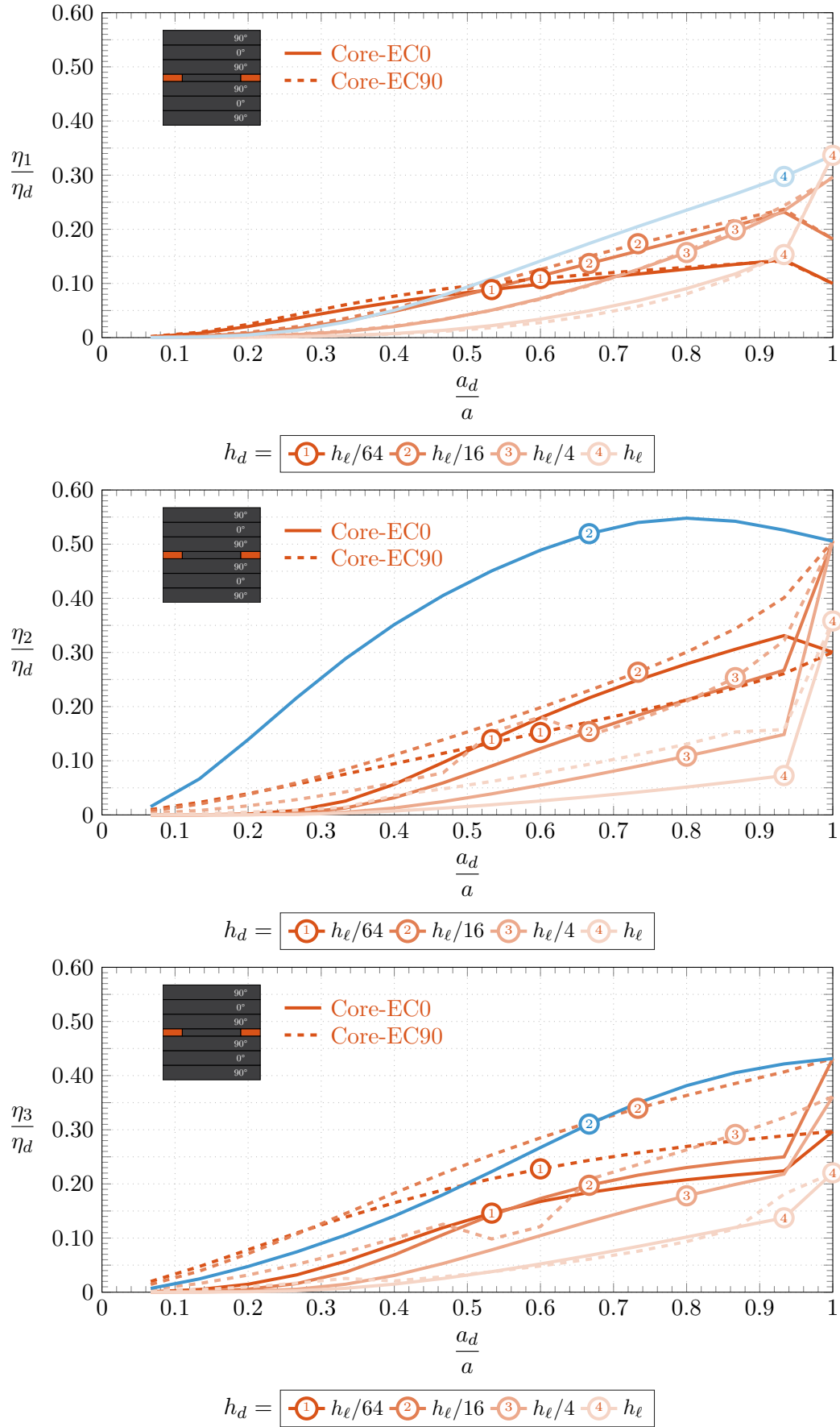
Figure 7.123 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 100$

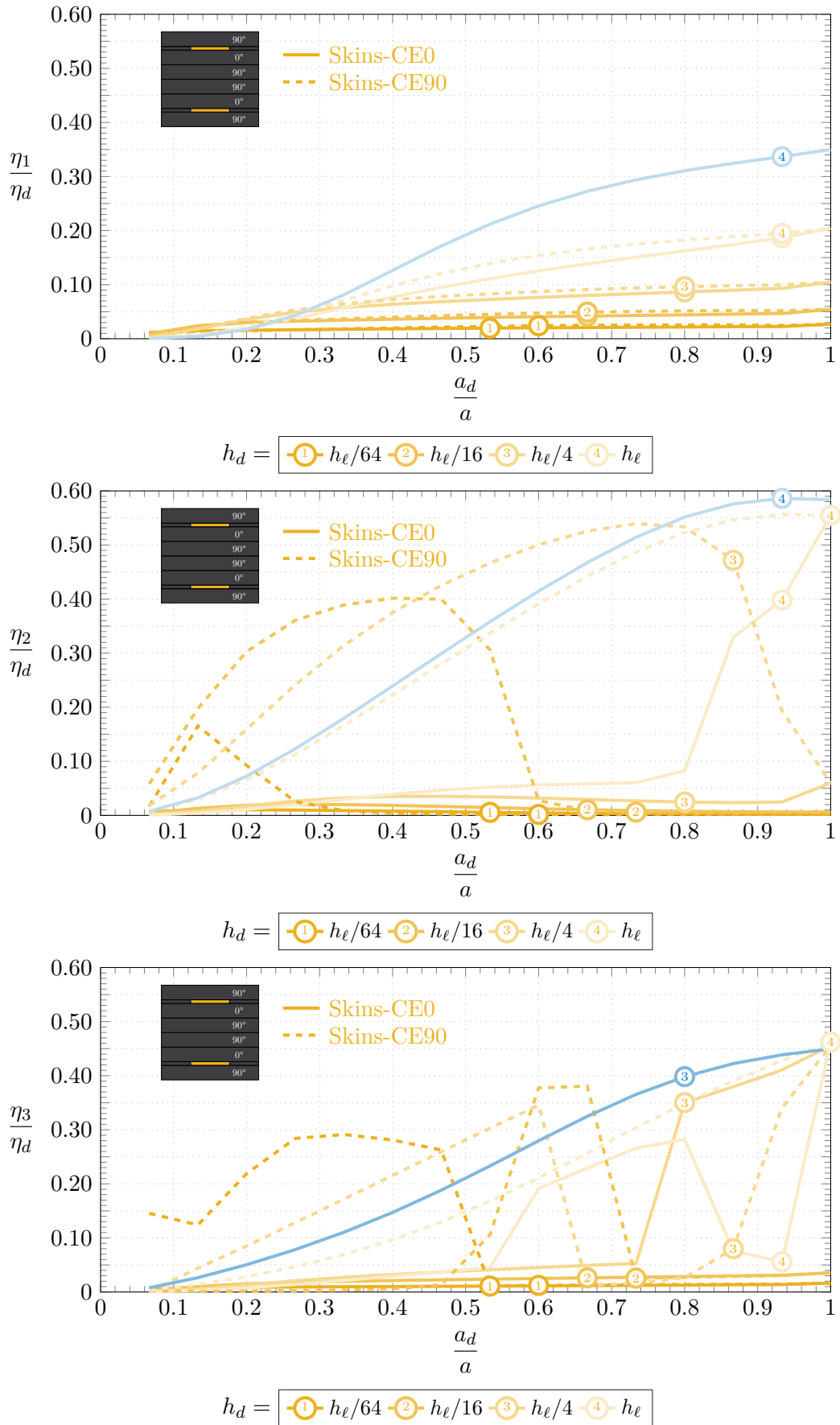
Figure 7.124 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 100$

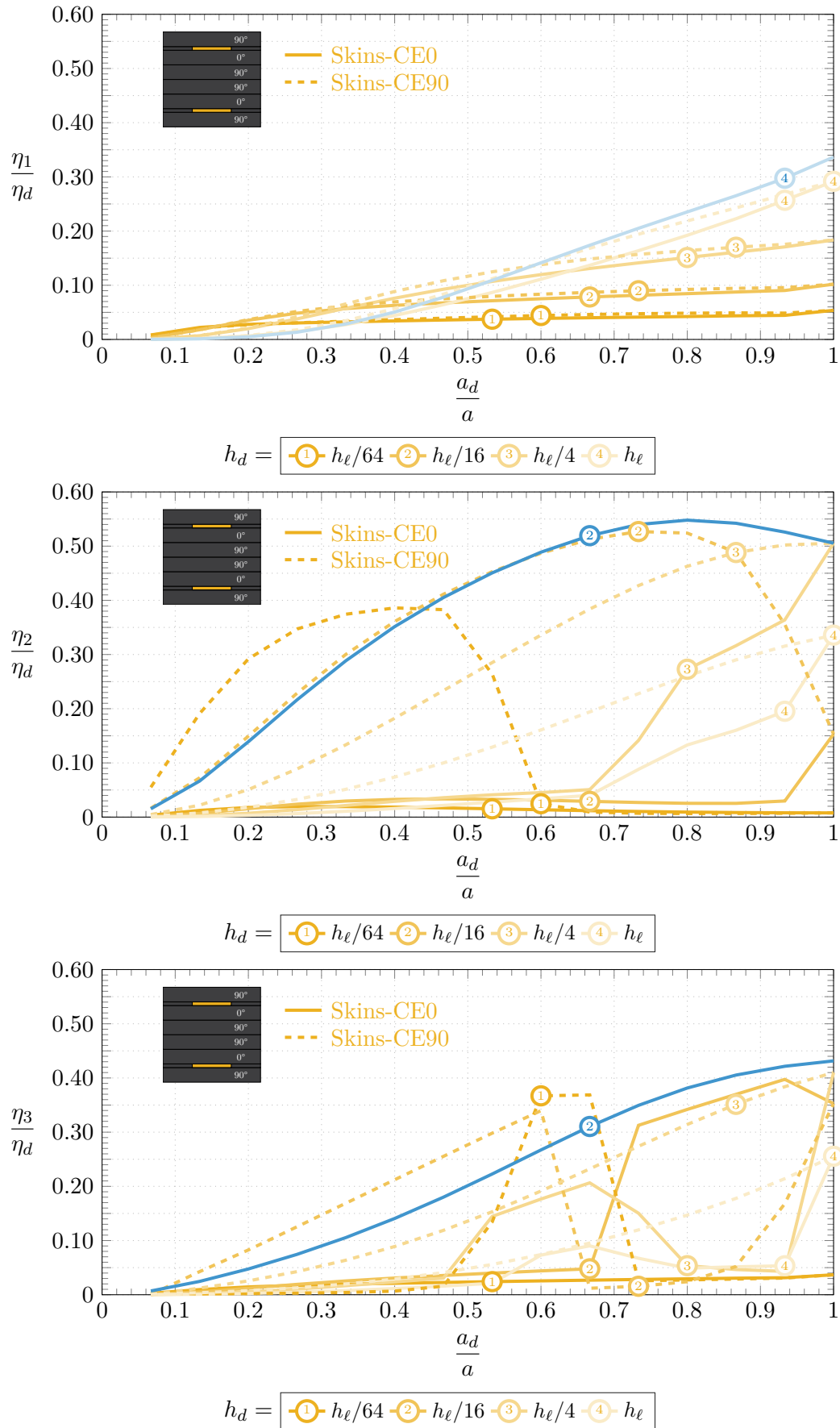
Figure 7.125 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 100$

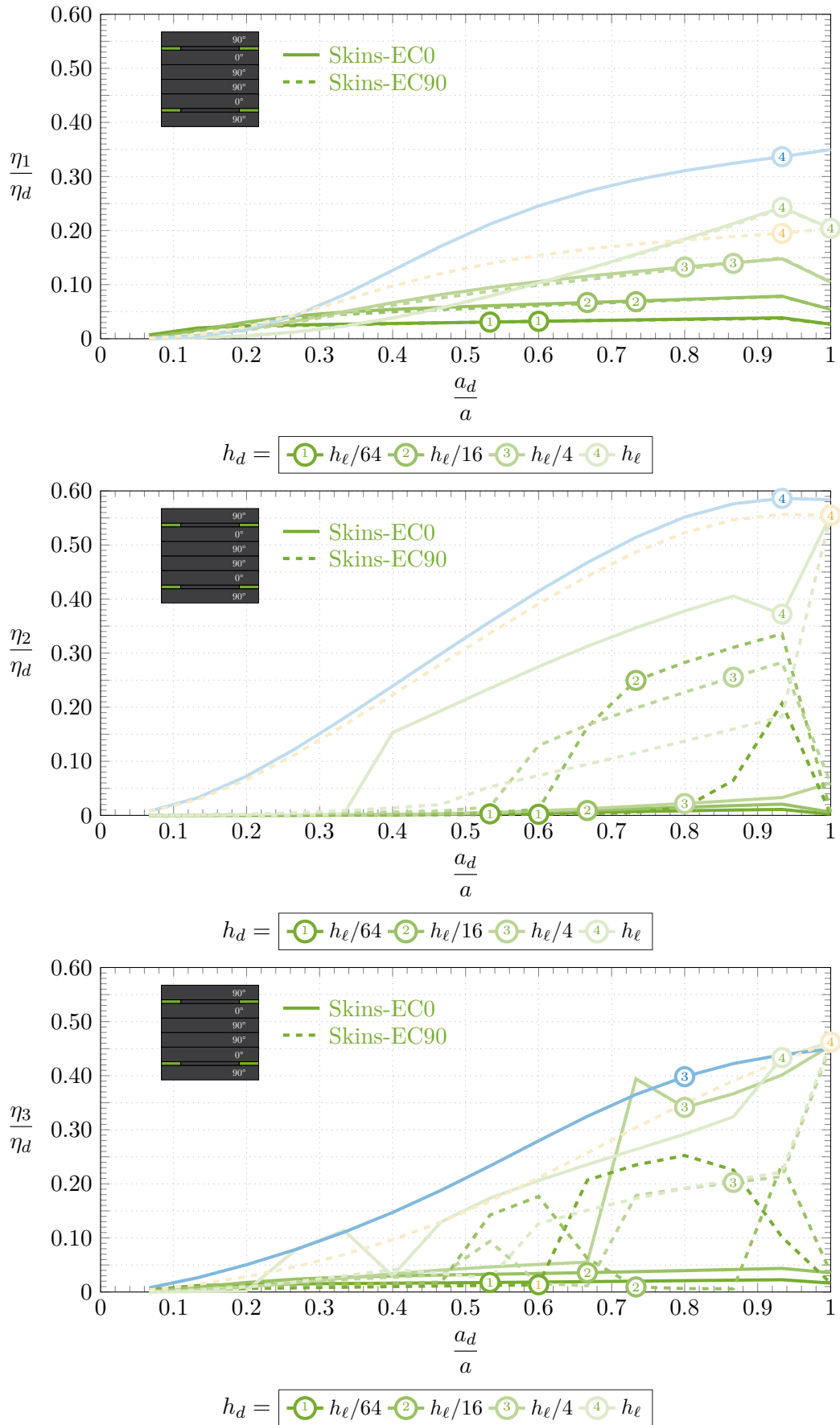
Figure 7.126 Influence of damping Core-CE patch width and thickness on modal loss factors, $a/h = 50$

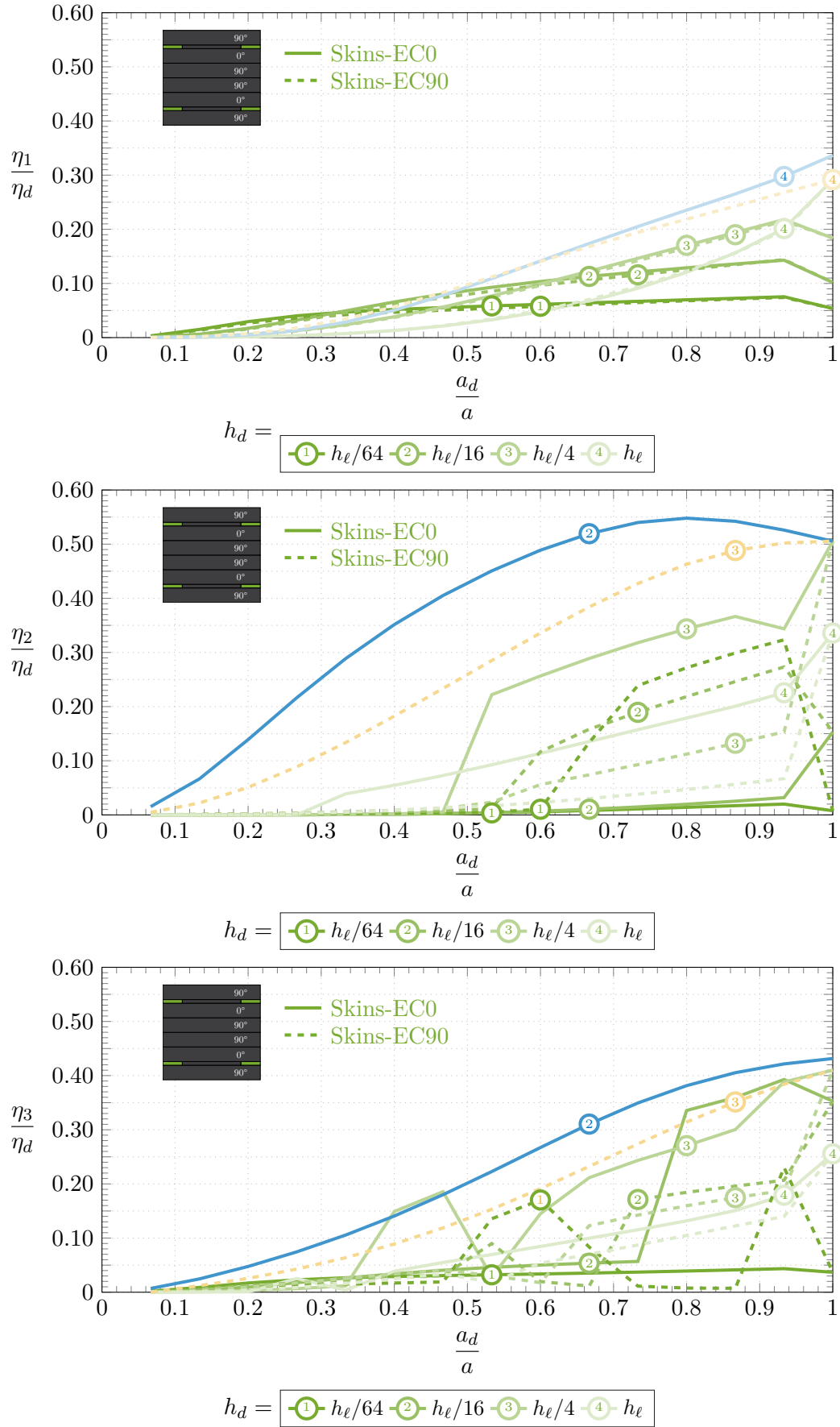
Figure 7.127 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.128 Influence of damping Core-EC patch width and thickness on modal loss factors, $a/h = 50$

Figure 7.129 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 100$

Figure 7.130 Influence of damping Skins-CE patch width and thickness on modal loss factors, $a/h = 50$

Figure 7.131 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 100$

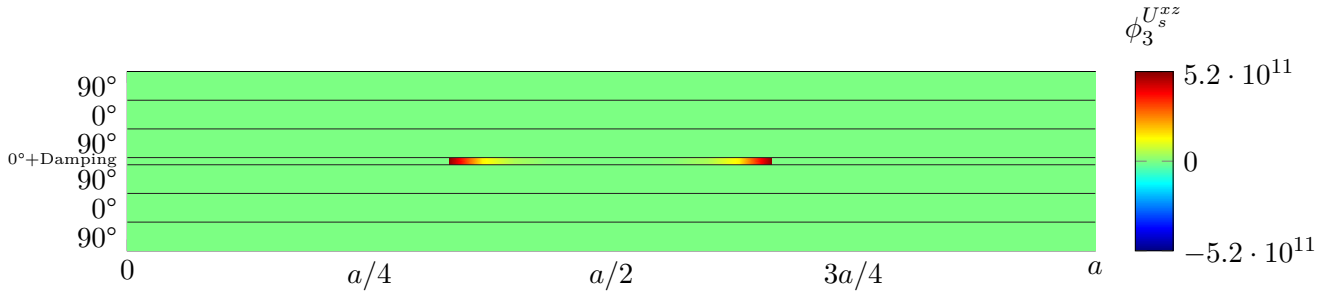
Figure 7.132 Influence of damping Skins-EC patch width and thickness on modal loss factors, $a/h = 50$

Following the same approach of the previous section, Figures 7.133 to 7.140 present the through-the-thickness modal transverse shear and extensional strain energy density mode shapes for the third normal mode. Similarly, one advance with a laminated composite plate with an aspect ratio $a/h = 100$ and a damping patch with width $a_d = a/3$ and thickness $h_d = h_\ell/4$. Here, one examines the same section cuts analysed before for the damping efficiency of FFFF interleaved laminated plates, $y = 0$ and $x = 0$, sections of maximum modal extensional and transverse shear strain energy densities at the third vibration mode.

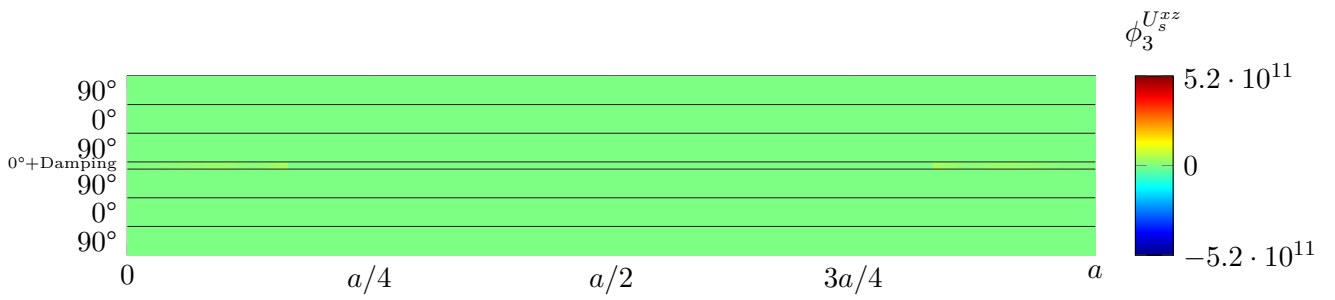
Figures 7.133 - 7.134 and 7.135 - 7.136 show the through-the-thickness modal transverse shear strain energy density mode shapes $\phi_3^{U_s^{xz}}$ and $\phi_3^{U_s^{yz}}$, respectively, for all configurations with interleaved damping patches. At this point, considering all the work discussed before, there is no doubt about the expected results regarding through-the-thickness modal transverse shear strain energy density mode shapes. It is true that damping efficiency curves for the third vibration mode are not easy to discern due to the constant switches between mode shapes, but the general guidelines of interleaved damping patches are already known. Generally speaking, Core configurations present the highest levels of modal transverse shear strain energy density, but there is an outlier. Through-the-thickness modal transverse shear strain energy density mode shape $\phi_3^{U_s^{yz}}$ for the Skins-EC0 configuration reveals a relevant transverse shear strain energy density at the damping patch. However, it is worth recalling that these are through-the-thickness modal transverse shear strain energy densities and proved to be a helpful indicator to disclose the damping mechanisms associated with damping patches but do not represent the cumulative transverse shear strain energy. High transverse shear strain energy densities at a specific section cut do not necessarily mean optimum damping performances, particularly in complex structures, and this emphasises the need to implement numerical tools to compute the cumulative strain energies.

To complete the analysis, Figures 7.137- 7.138 and 7.139- 7.140 present the through-the-thickness modal extensional strain energy densities $\phi_3^{U_n^{xx}}$ and $\phi_3^{U_n^{yy}}$, respectively. The distribution of modal extensional strain energy density follows the expected pattern, with the stiff layers subjected to higher modal extensional strain energy densities being those promoting the modal transverse shear strain energy on the damping patches and so improved damping performances. The distribution of $\phi_3^{U_n^{xx}}$ and $\phi_3^{U_n^{yy}}$ modal extensional strain energy densities is consistent with the orientation of the stiff layers.

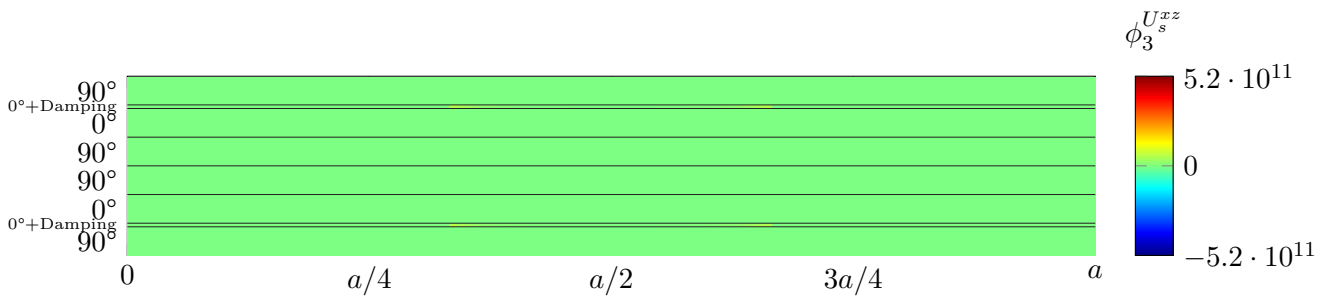
Once again, the through-the-thickness representations of modal extensional strain energy densities are asymmetrical relative to the laminate mid-plane. However, as previously discussed, one considered the bottom layer of the laminate as the reference layer, and its middle plane was deemed the neutral axis. Consequently, all configurations share the same reference plane. Numerically, this disrupts the symmetry of the warping functions and, by extension, the potential symmetry of through-the-thickness modal extensional strain energy densities. While this is not the case when considering interleaved damping layers, interleaved damping patches add heterogeneity to the structure, resulting in highly concentrated modal extensional strain energy densities.



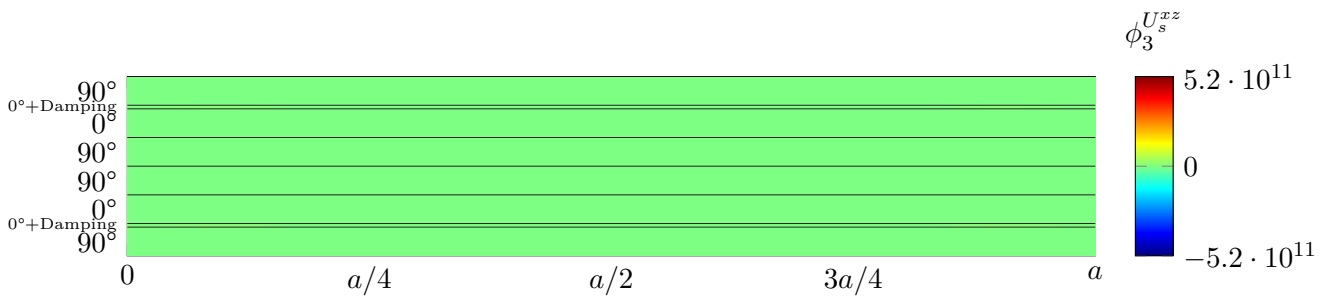
(a) Core-CE0



(b) Core-EC0



(c) Skins-CE0



(d) Skins-EC0

Figure 7.133 $\phi_3^{U^{xz}}$ - shear strain energy density mode shape - section cut at $y = 0$

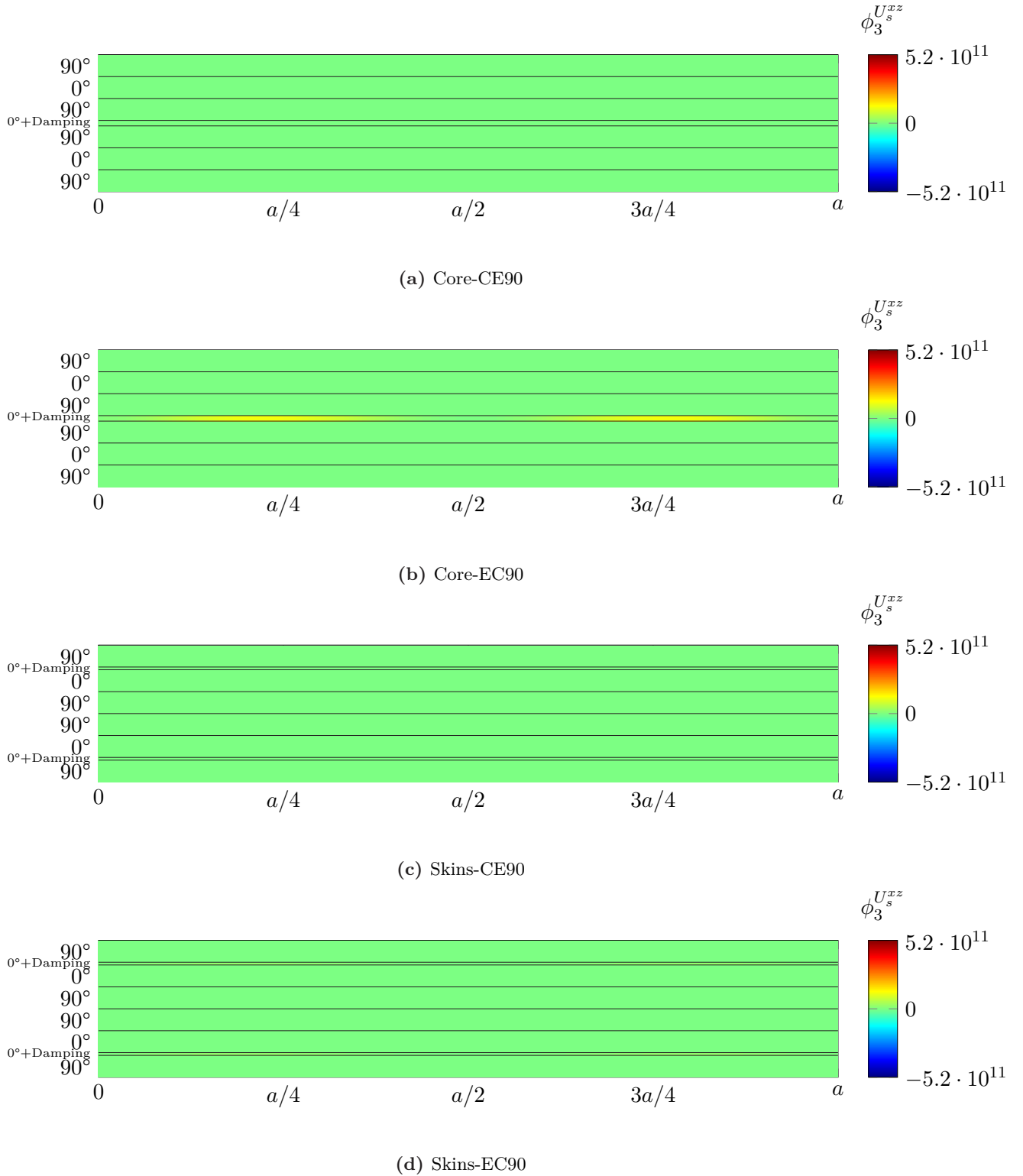


Figure 7.134 $\phi_3^{U^{xz}}$ - shear strain energy density mode shape - section cut at $y = 0$

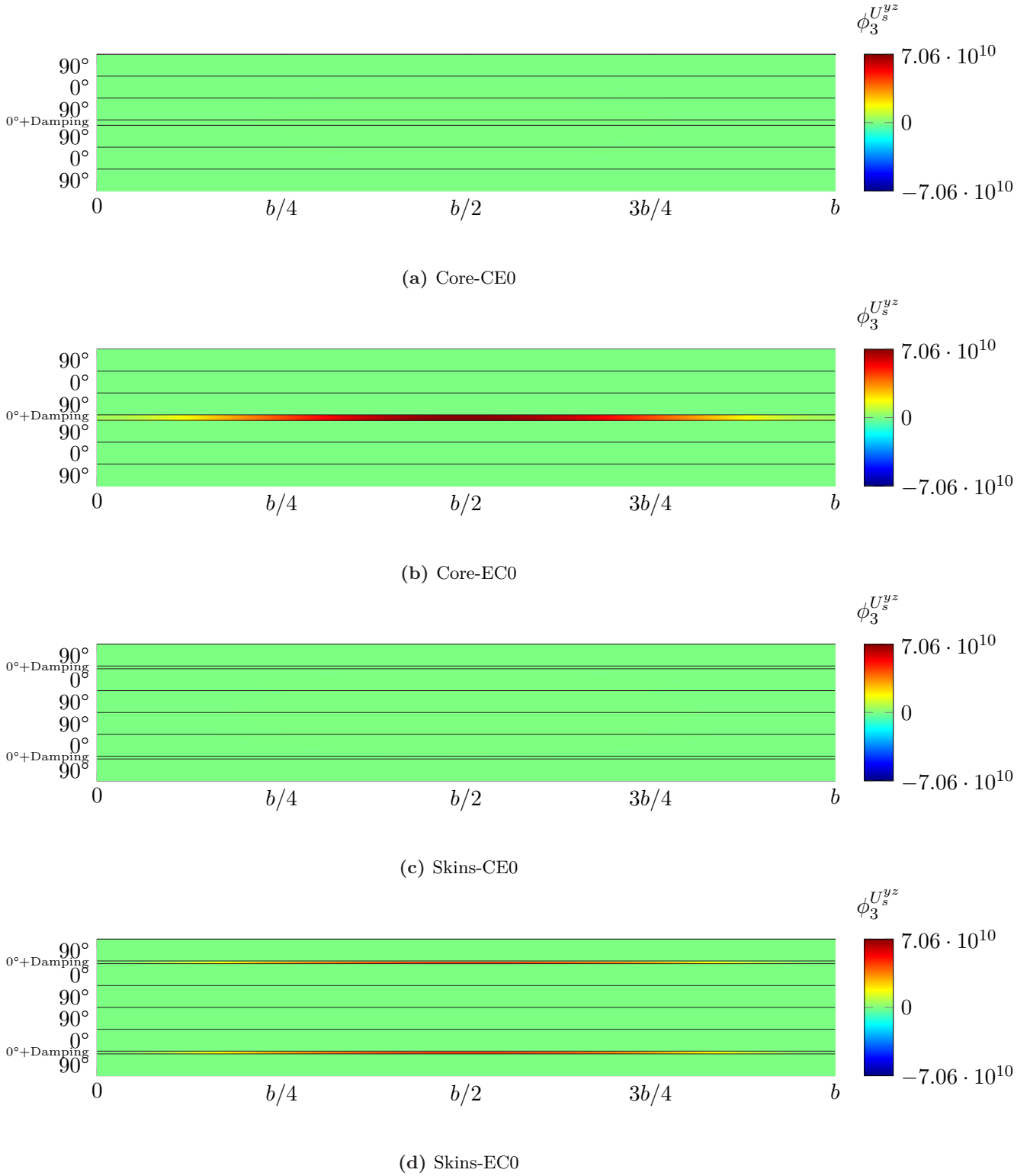


Figure 7.135 $\phi_3^{U_s^{yz}}$ - shear strain energy density mode shape - section cut at $x = 0$

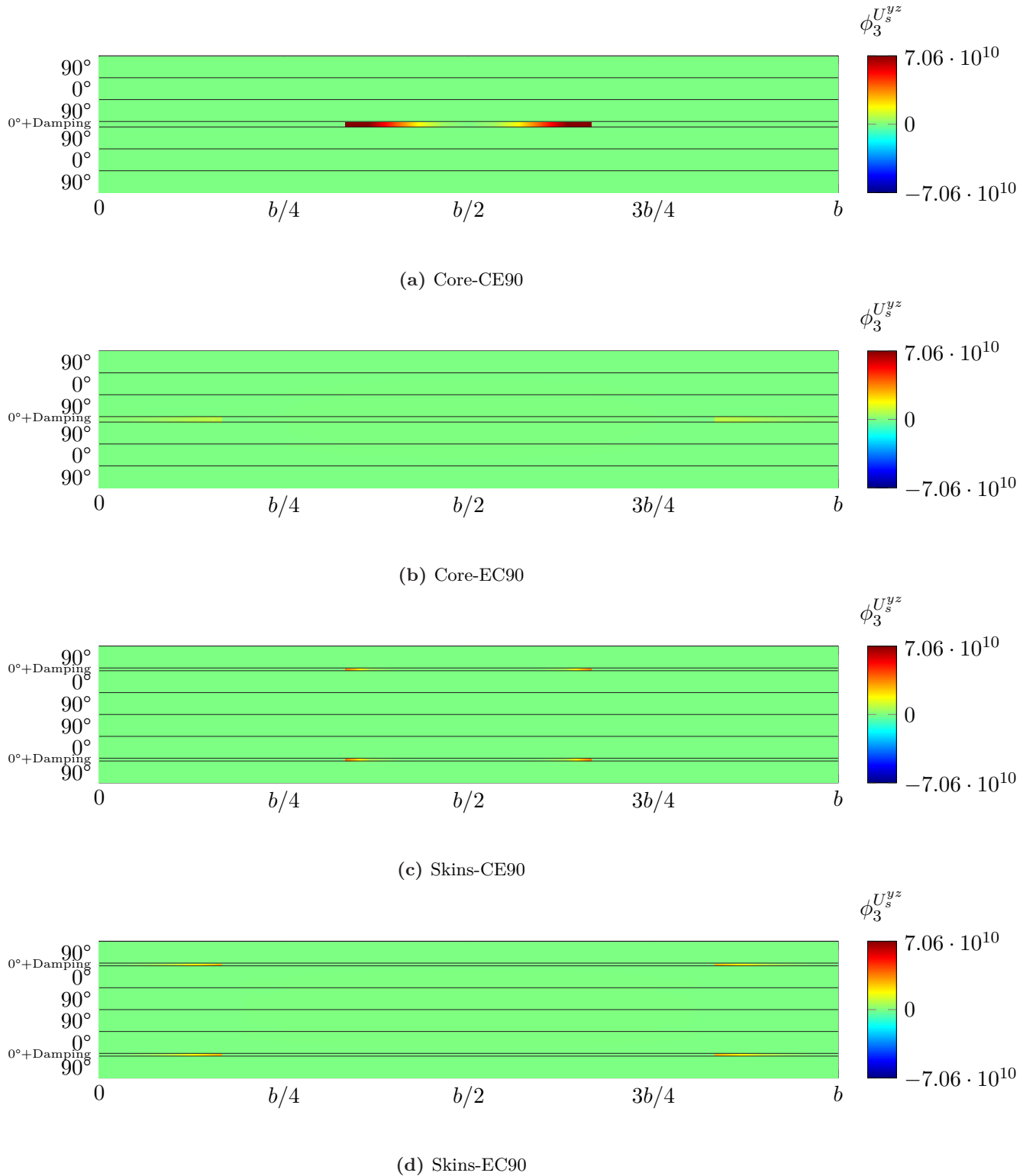
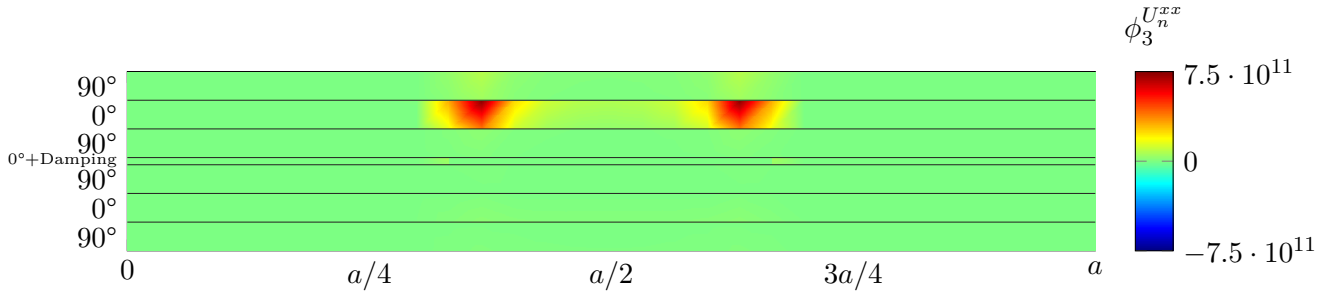
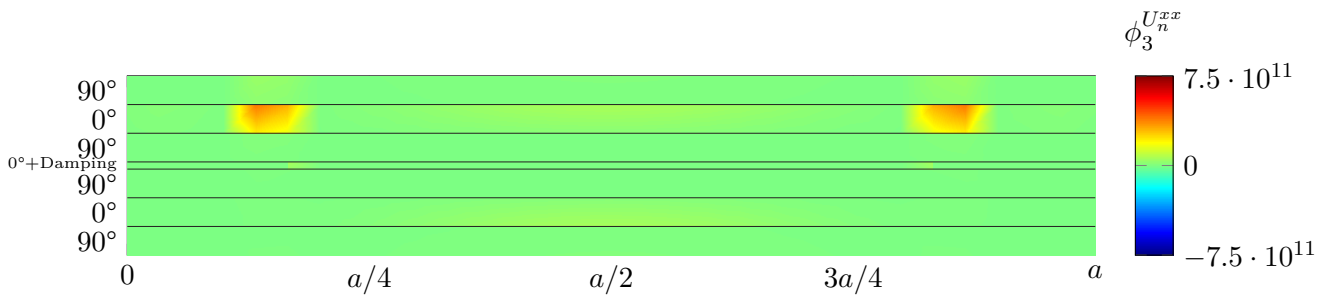


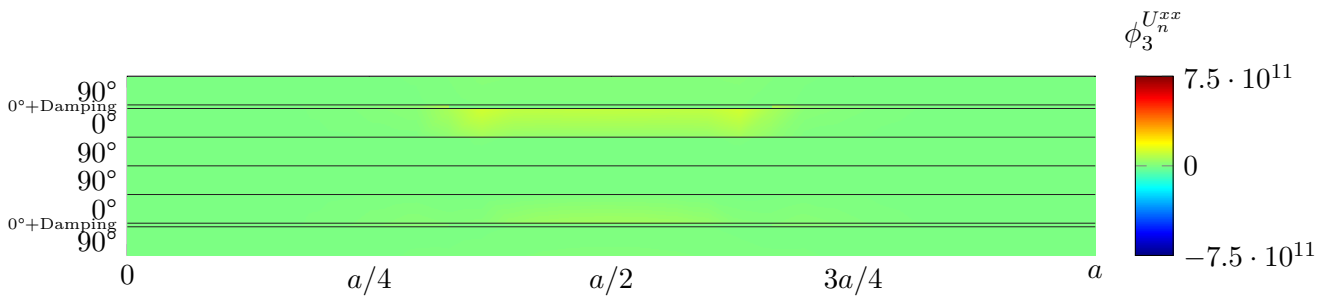
Figure 7.136 $\phi_3^{U_s^{yz}}$ - shear strain energy density mode shape - section cut at $x = 0$



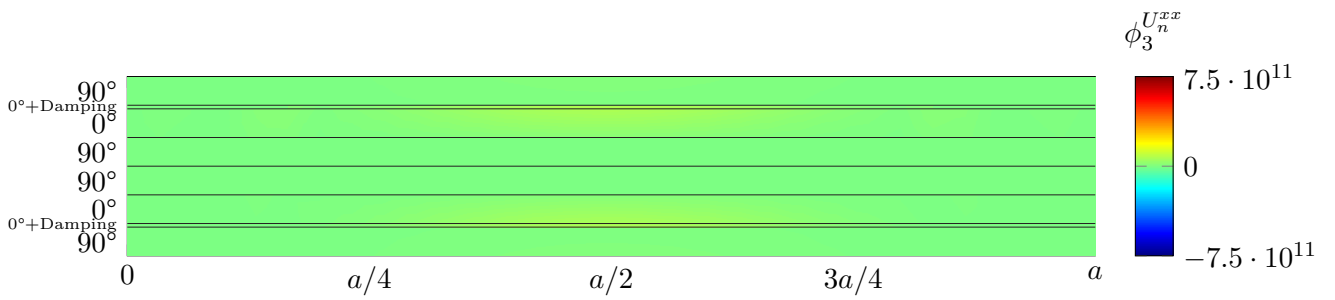
(a) Core-CE0



(b) Core-EC0



(c) Skins-CE0



(d) Skins-EC0

Figure 7.137 $\phi_3^{U_n^{xx}}$ - extensional strain energy density mode shape - section cut at $y = 0$

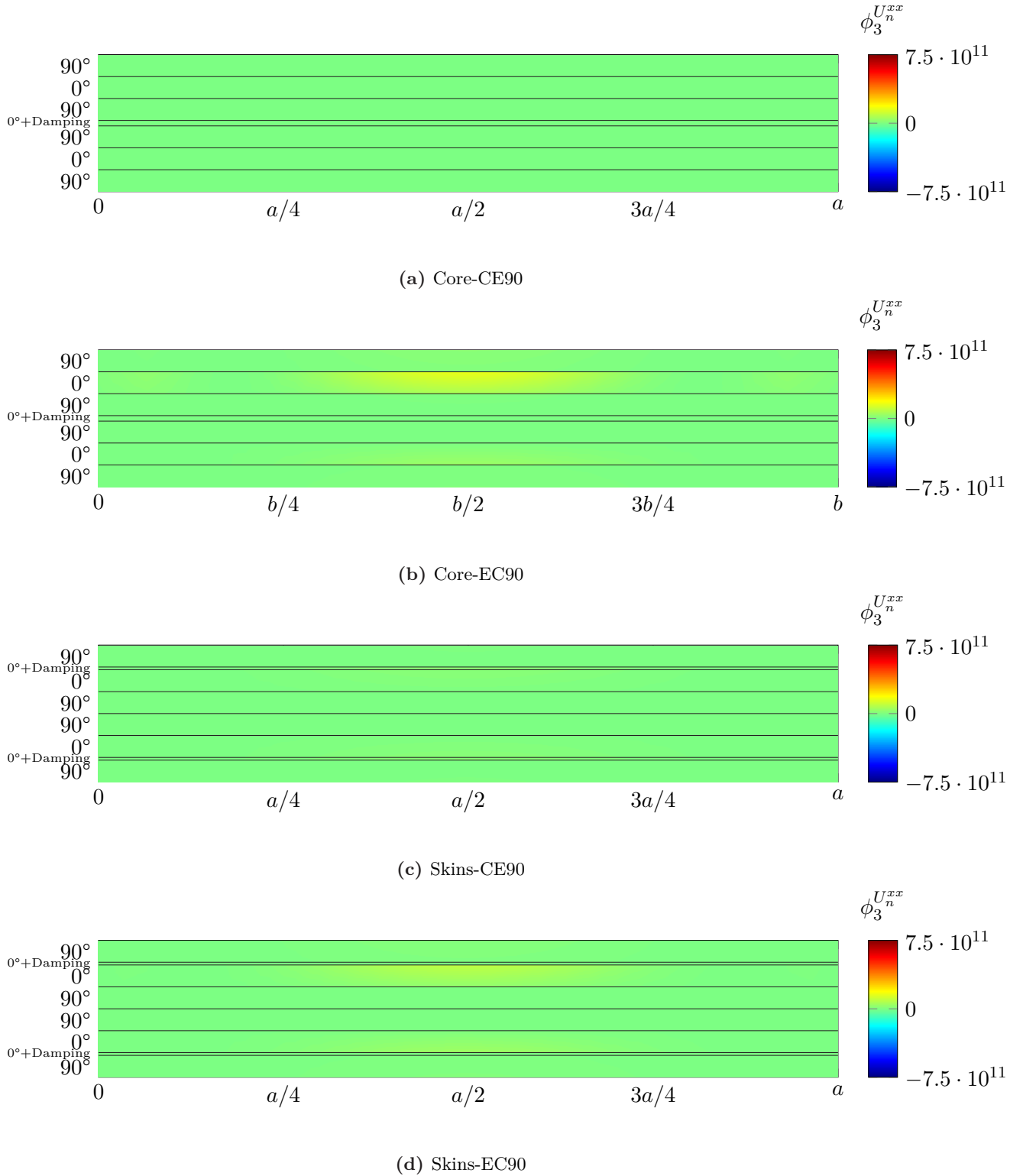


Figure 7.138 $\phi_3^{U_n^{xx}}$ - extensional strain energy density mode shape - section cut at $y = 0$

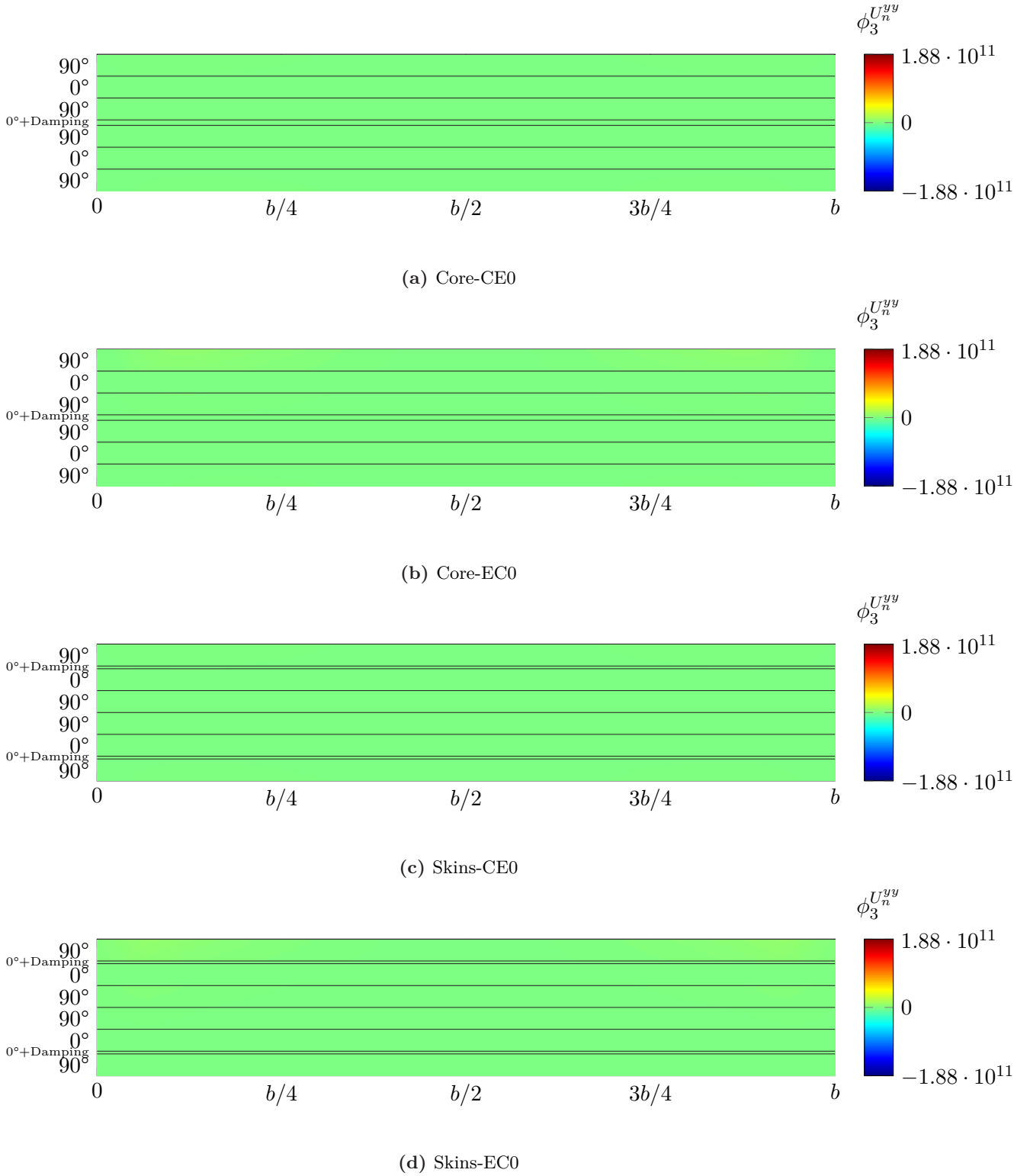


Figure 7.139 $\phi_3^{U_n^{yy}}$ - extensional strain energy density mode shape - section cut at $x = 0$

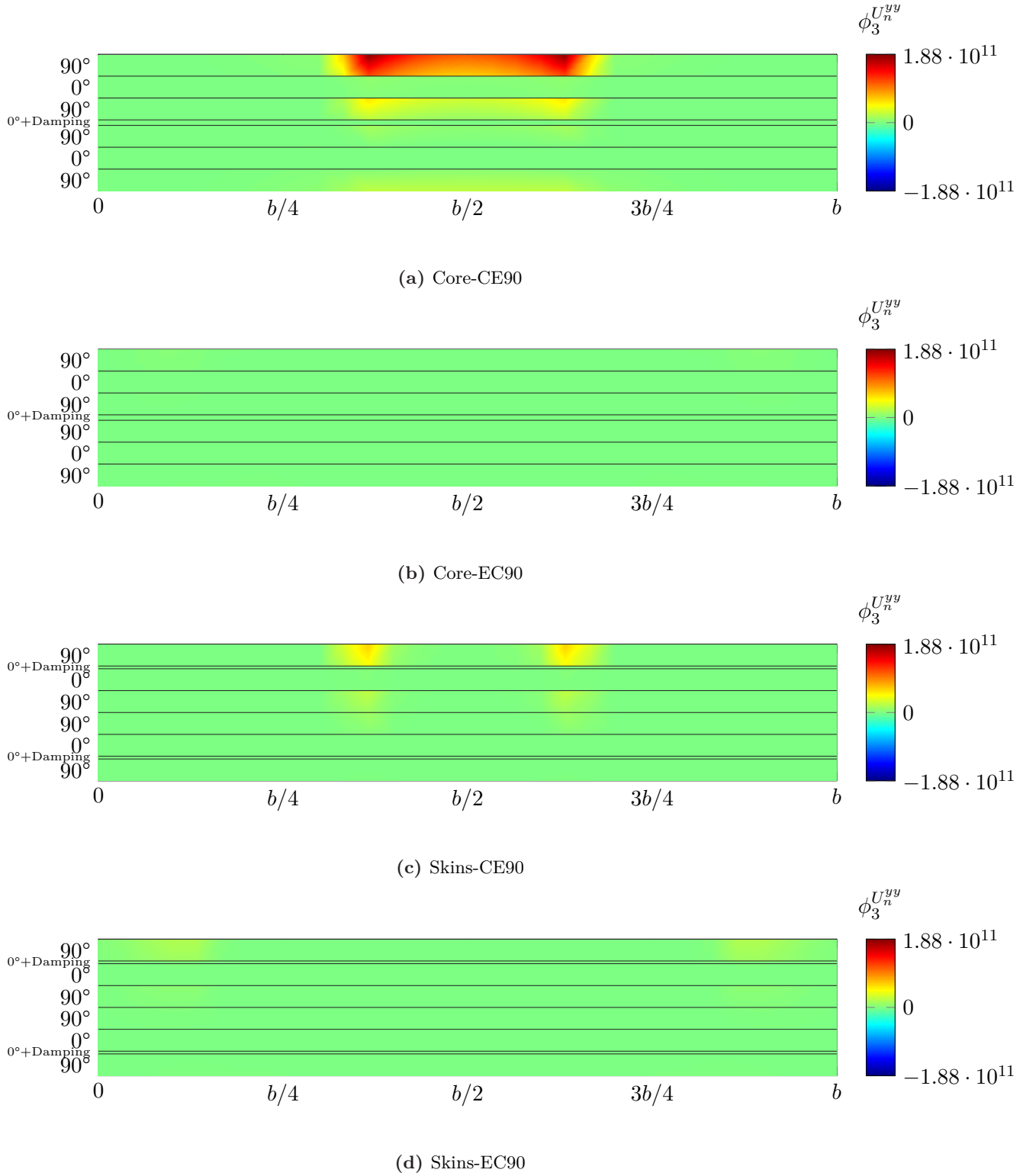


Figure 7.140 $\phi_3^{U_n^{yy}}$ - extensional strain energy density mode shape - section cut at $x = 0$

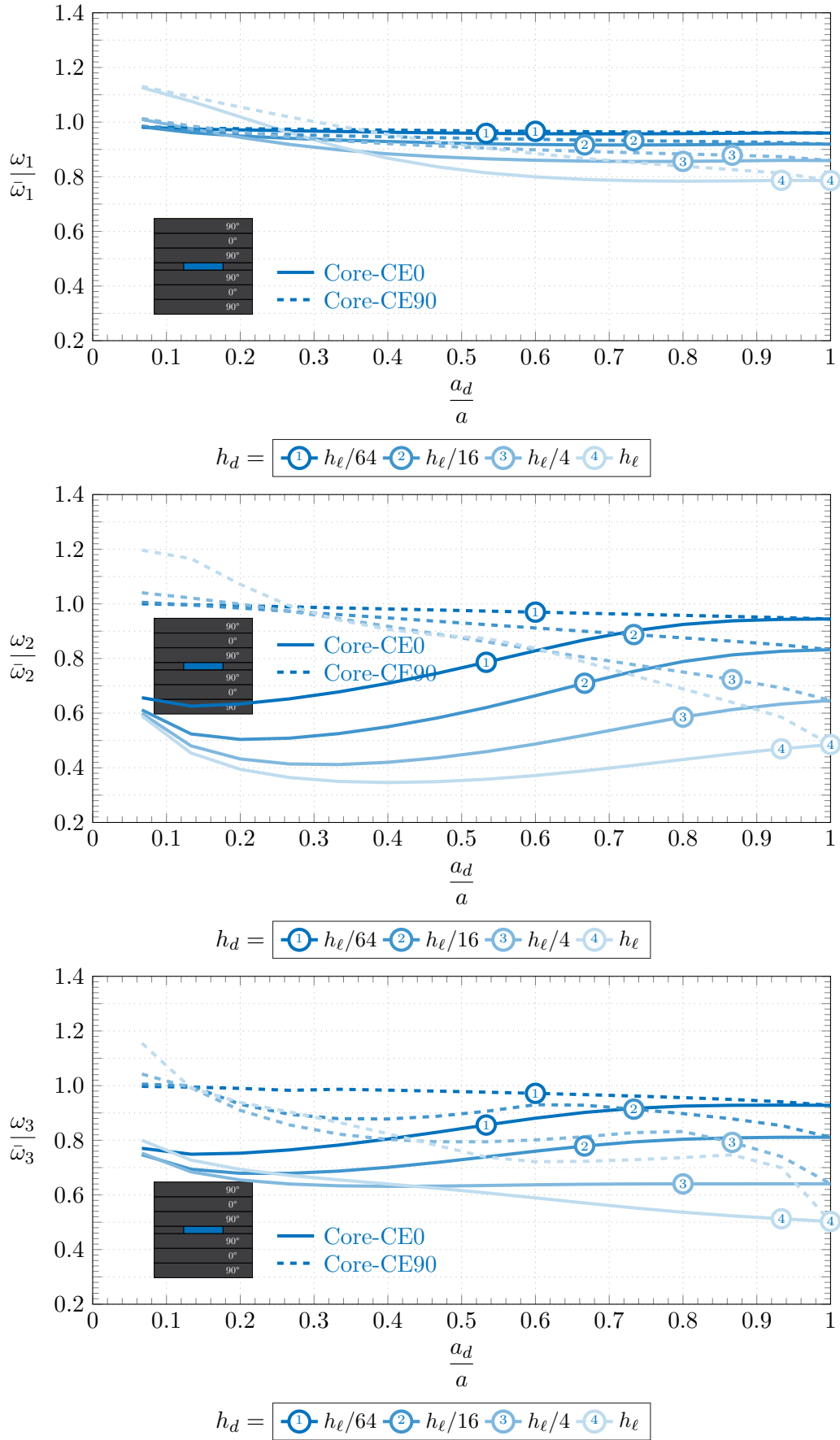
As the study of the damping efficiency and the corresponding conflict with stiffness is ever-present throughout this work, one concludes the analysis of FFFF composite laminates with interleaved damping patches studying the influence of the damping patch width, thickness and position on the natural frequencies. Figures 7.141 to 7.144 present the first three dimensionless natural frequencies, $\omega_n/\bar{\omega}_n$, as a function of the patch width, a_d/a , and dimensionless damping layer thickness, h_d/h_ℓ .

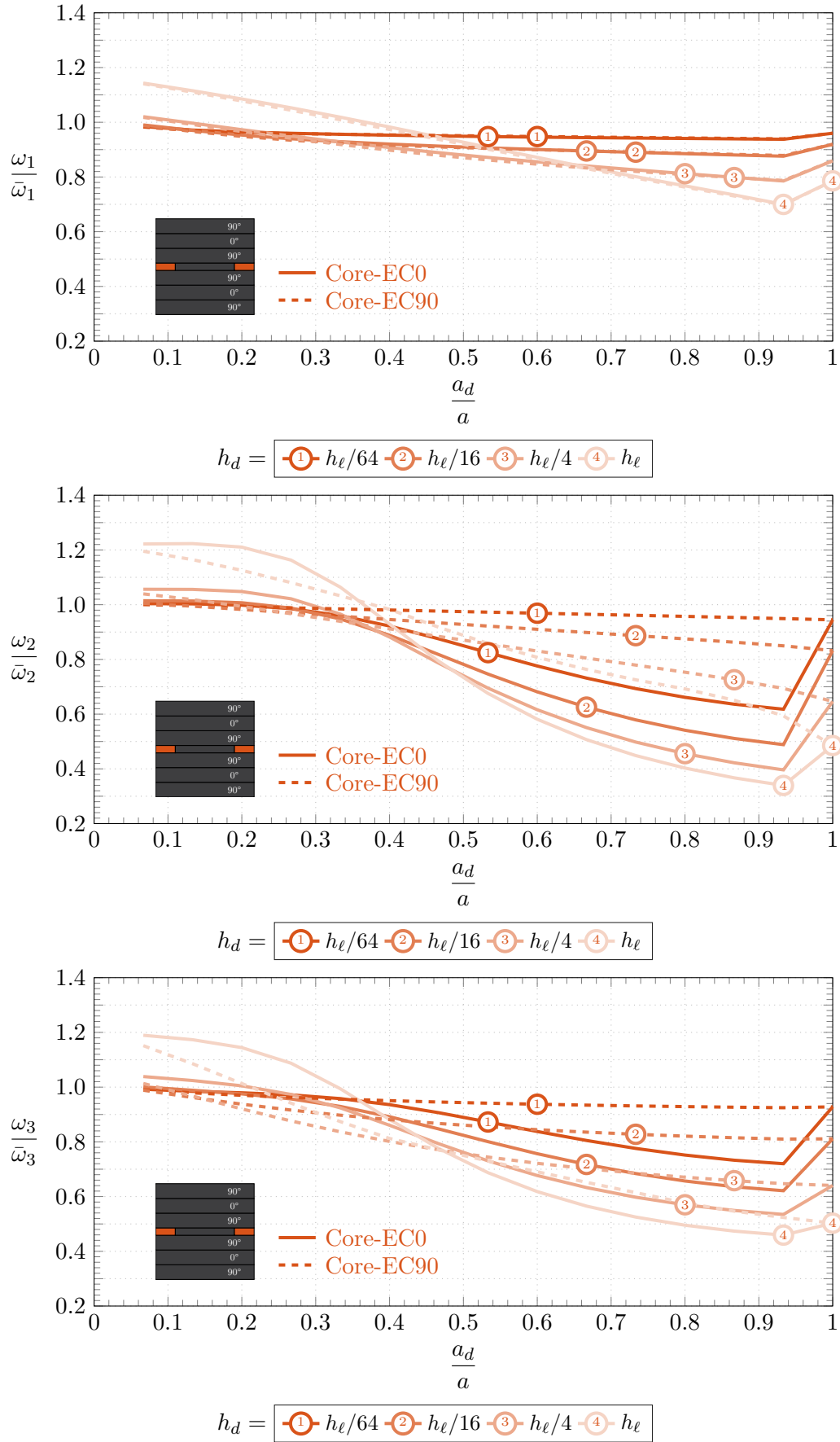
The relevance of studying different boundary conditions lies in the fact that one can draw comprehensive conclusions not limited to simple cases. Figures 7.141 to 7.144 prove that the conflict between damping capacity and stiffness is a general conclusion, regardless of the boundary conditions. This clash is even more evident when analysing the second natural frequency variation of the Core-CE and Core-EC configurations. Configurations exhibiting enhanced damping performances are typically associated with notable reductions in both natural frequencies and stiffness - Core-CE.

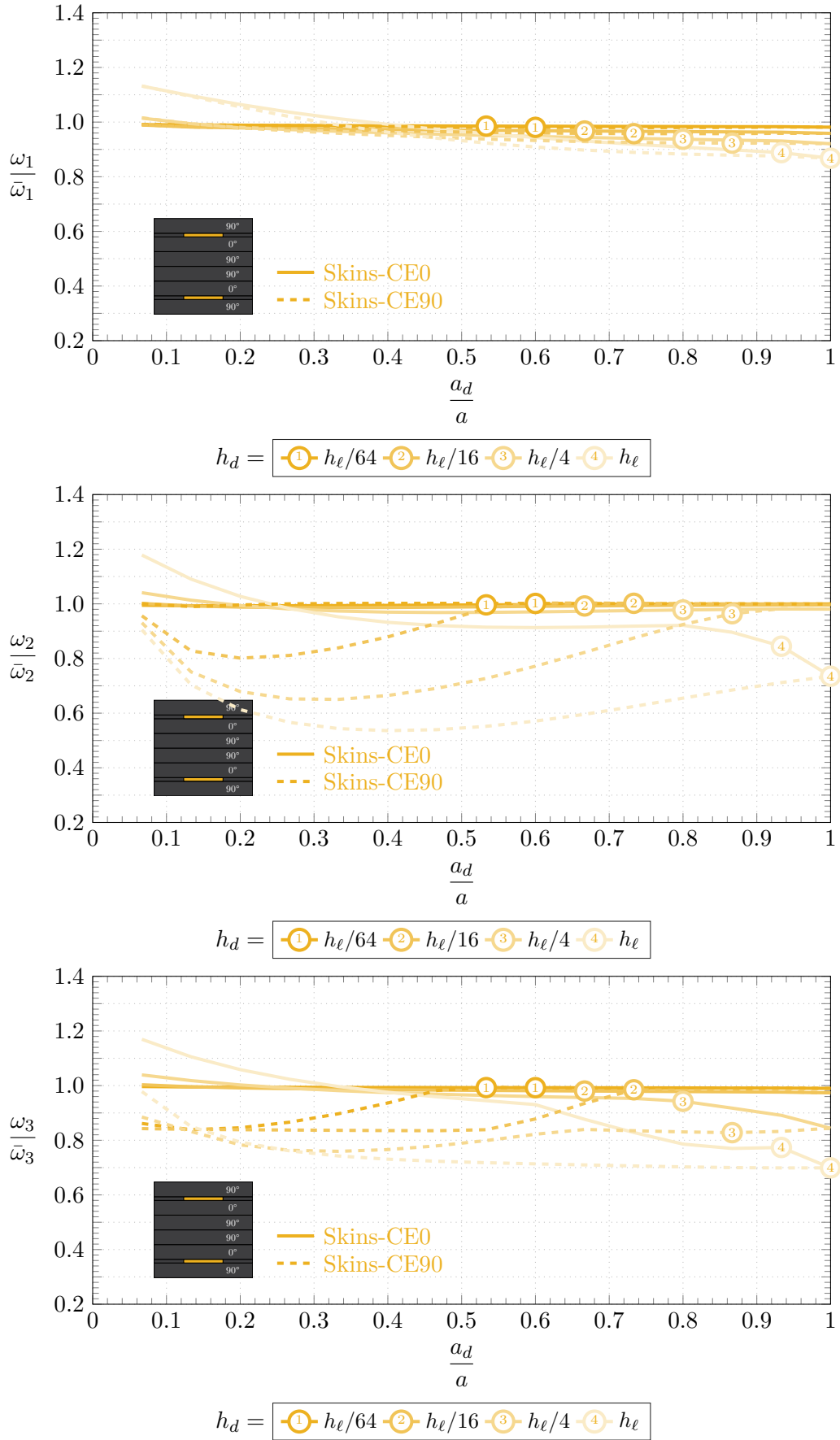
Once again, there exists a minimum dimensionless natural frequency achievable for a given structure. In other words, increasing the damping patch width beyond this threshold value may not necessarily result in a natural frequency decrease. This outcome underscores the potential for optimisation at this level and proves the universality of this conclusion.

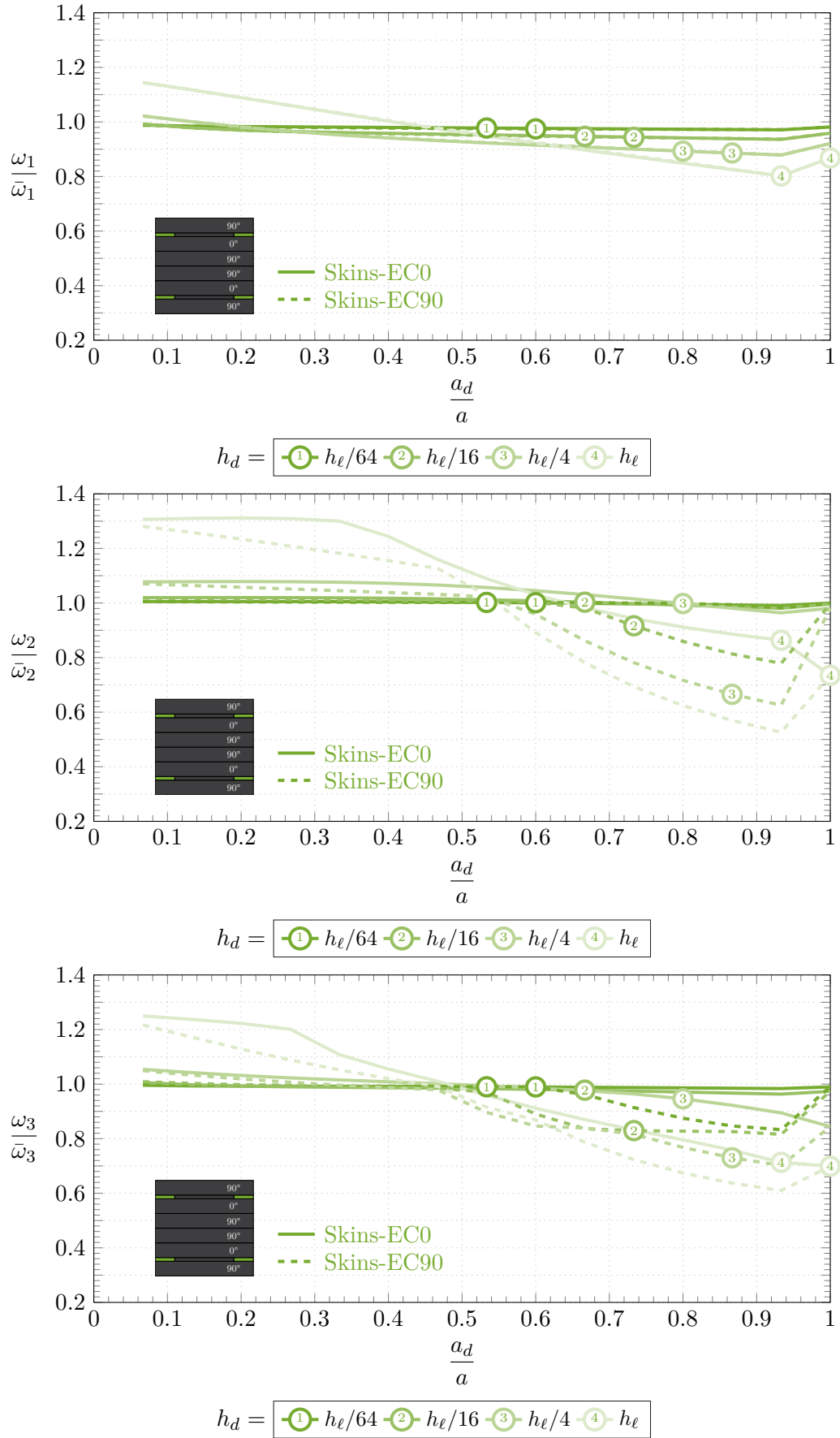
In addition, the influence of the damping layer thickness on the dimensionless natural frequencies remains straightforward. Generally, thicker damping patches decrease the laminate stiffness. One already discussed that such a result leaves the door open for optimisation since it is possible to achieve an efficient damping capacity without unnecessarily decreasing the stiffness for a given damping patch thickness.

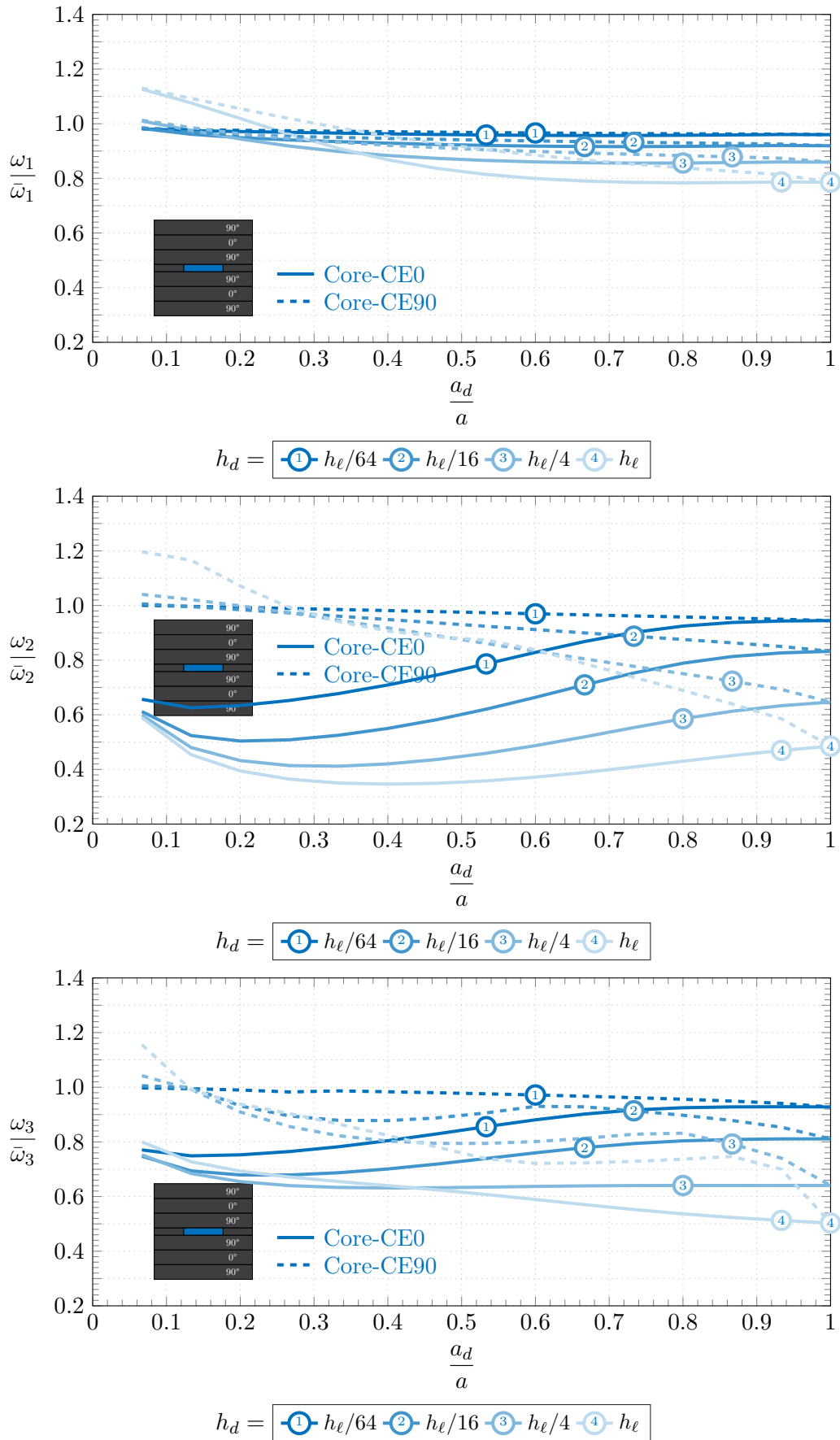
Furthermore, the dimensionless natural frequency variation for the same configurations, but for reference laminated composite plates with an aspect ratio $a/h = 100$ and for thicker laminated composite plates with an aspect ratio $a/h = 50$, side-by-side, are depicted in Figures 7.145 to 7.152. Once again, consistent with prior results, the primary trends in the variation of dimensionless natural frequencies persist when examining thicker laminates, and the only notable distinction is the tendency to a decrease in natural frequencies for thicker laminates, illustrating the anticipated outcome of the influence of a larger soft damping layer integrated into thick composite laminates.

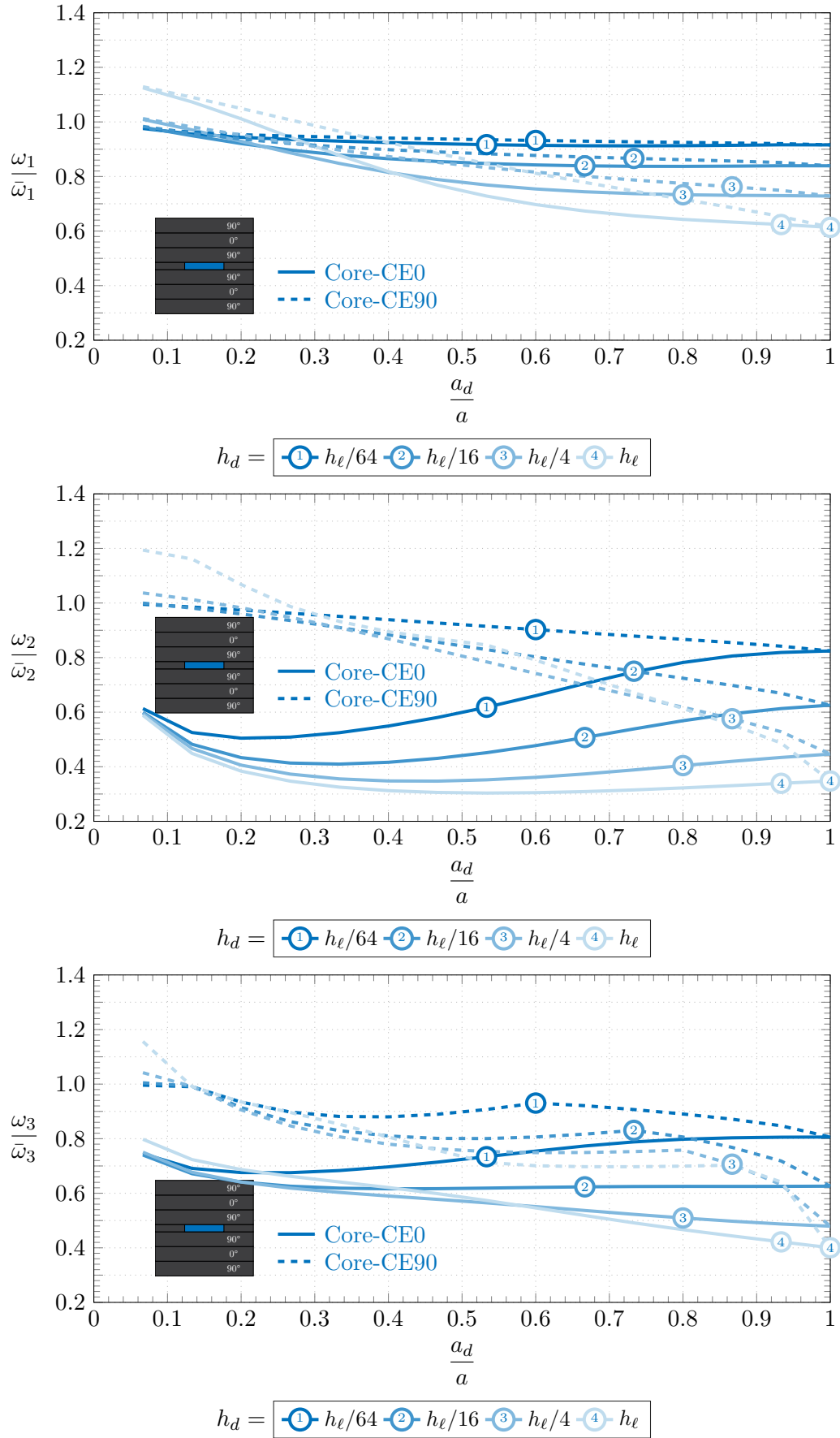
Figure 7.141 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 100$

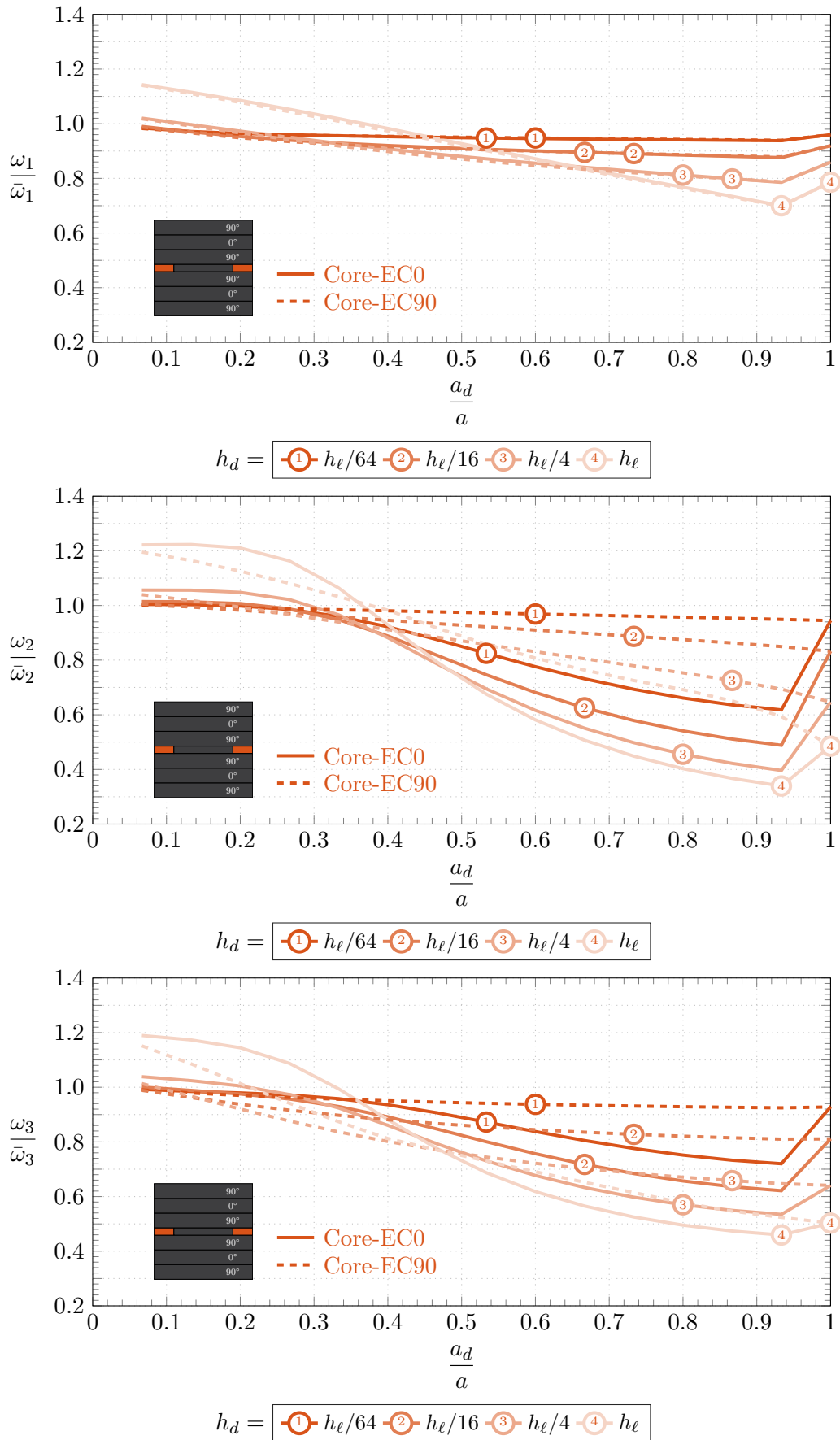
Figure 7.142 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 100$

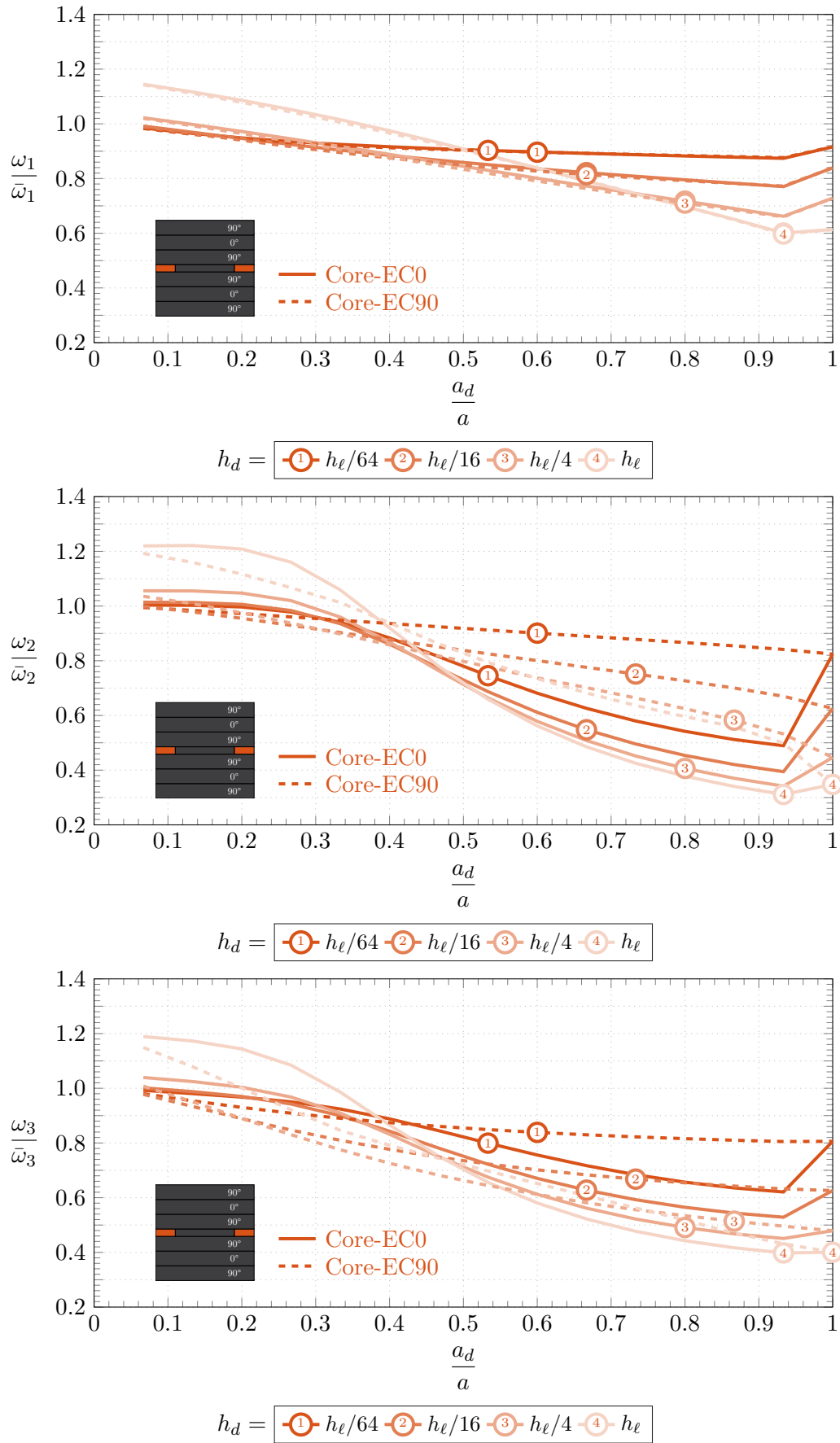
Figure 7.143 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 100$

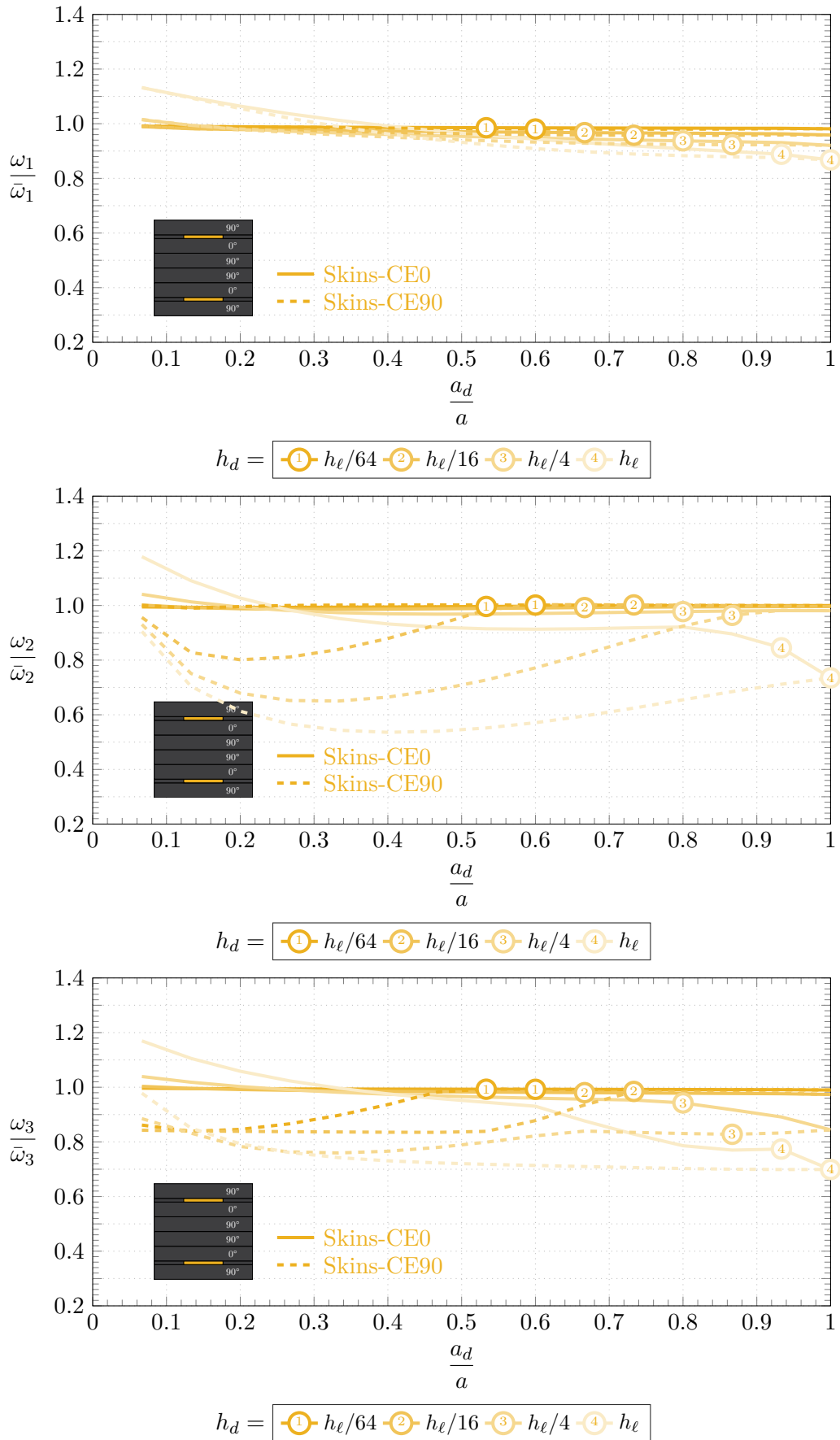
Figure 7.144 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 100$

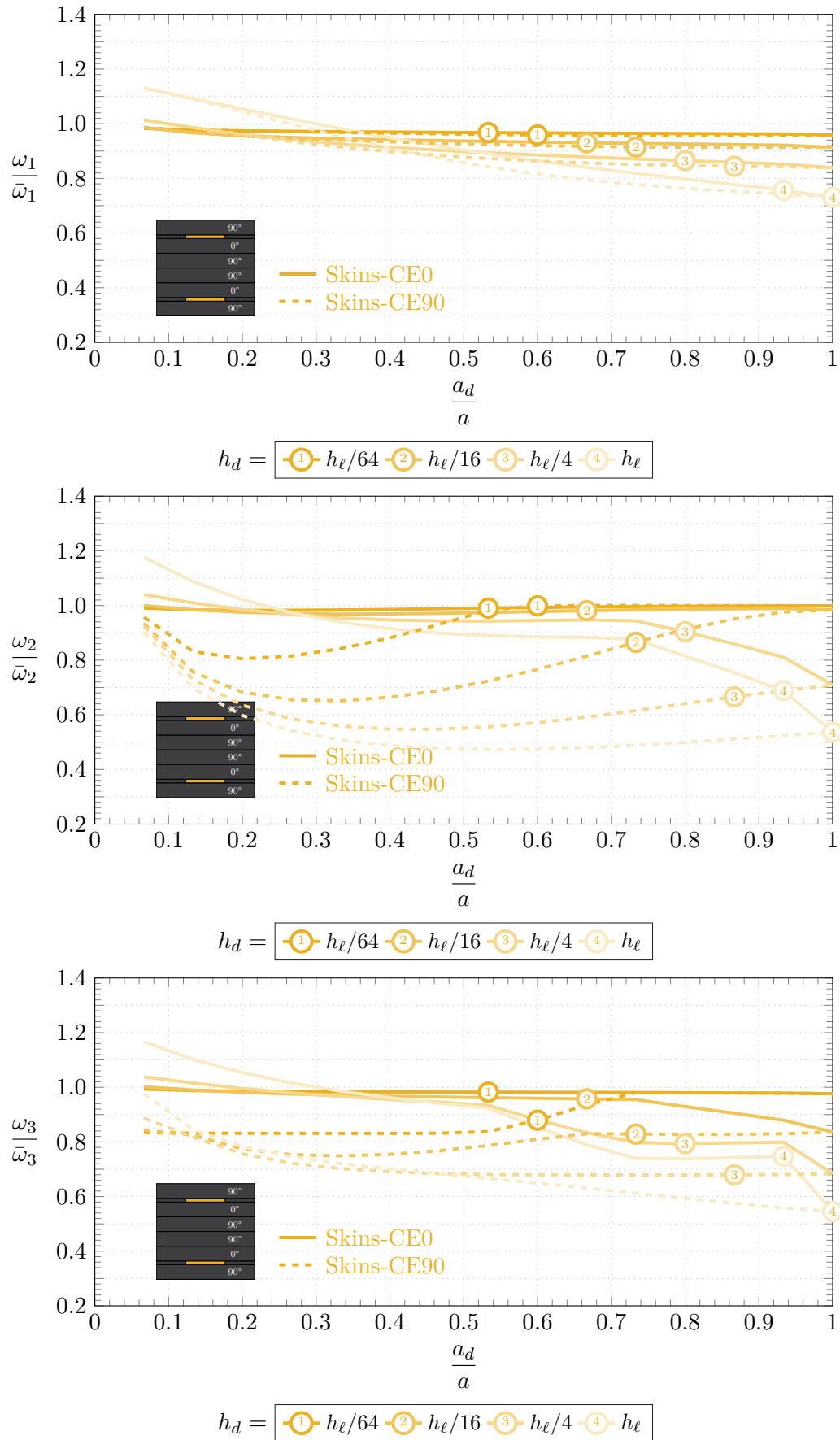
Figure 7.145 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 100$

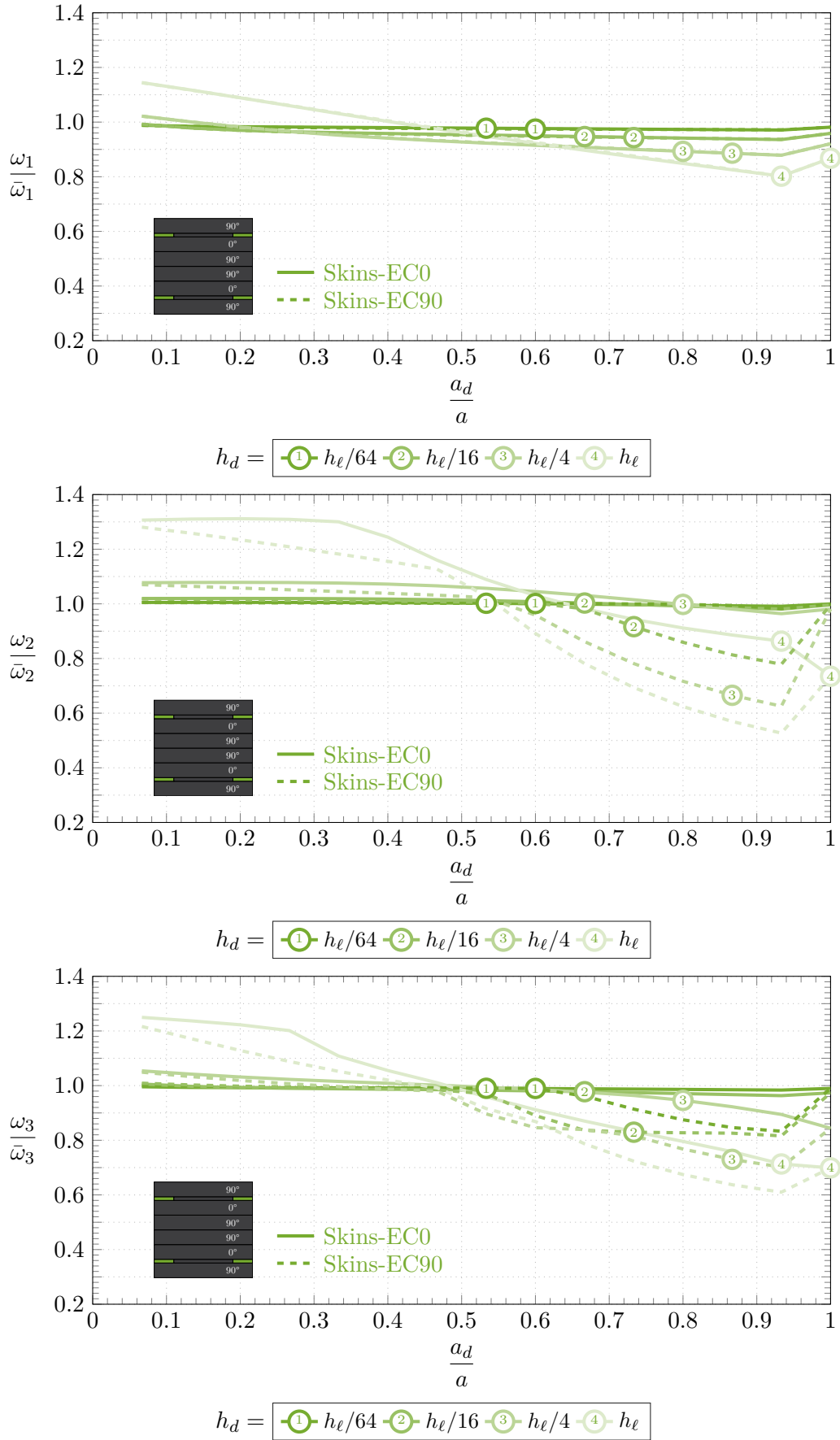
Figure 7.146 Influence of damping Core-CE patch width and thickness on natural frequencies, $a/h = 50$

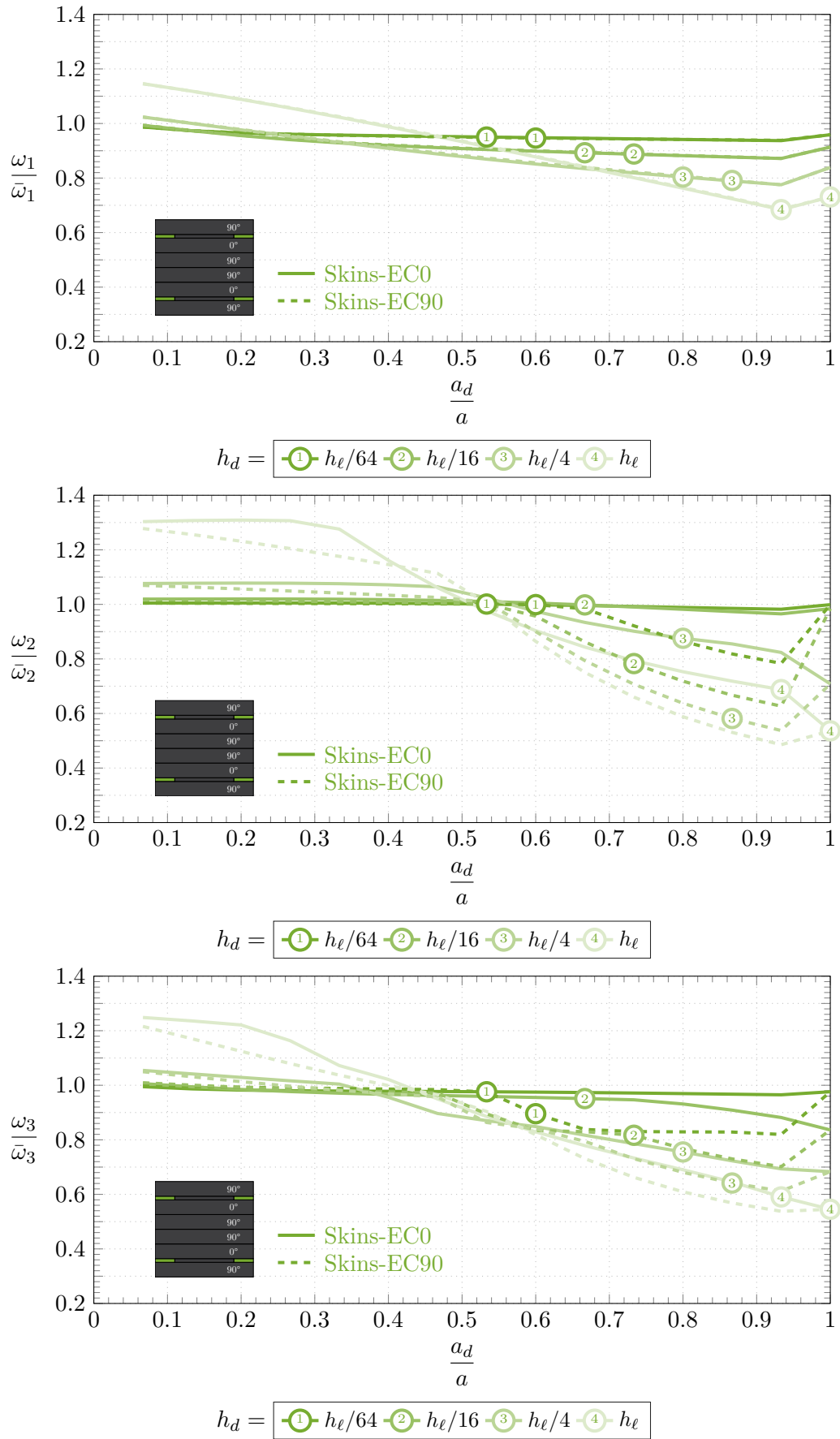
Figure 7.147 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.148 Influence of damping Core-EC patch width and thickness on natural frequencies, $a/h = 50$

Figure 7.149 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.150 Influence of damping Skins-CE patch width and thickness on natural frequencies, $a/h = 50$

Figure 7.151 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 100$

Figure 7.152 Influence of damping Skins-EC patch width and thickness on natural frequencies, $a/h = 50$

Part IV

Experimental Testing and Analysis

Mechanical Characterisation of Viscoelastic Materials

8.1 | Introduction

The ASTM E756-05 [2010] standard procedure is a widely known method for material characterisation of vibration-damping materials: the material loss factor, η , and storage modulus, E , or the shear modulus, G , can be estimated by measuring the resonant frequencies and modal loss factors of specially designed and tested specimens. This procedure ensures the appropriate estimation of mechanical properties as long as the experimental setup does not influence those same properties. For example, excitation and measuring devices might interfere with mechanical properties, namely damping capacity and mass, particularly for small specimens. Wojtowicki et al. [2004] overcome the technical difficulties of the ASTM E756-05 [2010] procedure using a double-sized free-free beam excited in its centre, acquiring “*a frequency response function measured between the imposed velocity at the centre (measured with an accelerometer) and an arbitrary point on the beam (measured with a laser vibrometer)*” - or transmissibility function.

8.2 | Transmissibility Functions

This section describes the proposed experimental procedure for identifying the dynamic properties of damping materials, including composite materials. In particular, the primary purpose is to measure their damping capacity. The main concern is to establish an experimental procedure with a contactless measuring method to minimise possible disturbances, which may lead to erroneous results, based on the work of Wojtowicki et al. [2004]. ASTM E756-05 [2010] assumes that one must use non-contacting transducers. However, one doesn't need an excitation force transducer in the procedure described below.

8.2.1 | Apparatus

Figure 8.1 illustrates the proposed experimental procedure scheme - the signal analyser unit (1) manages the whole setup: it generates and sends an excitation signal to the shaker amplifier (2), which channels the amplified signal to the electromagnetic shaker (3). The test specimen (4) is in a cantilever configuration, fixed to a clamping bracket connected to the electromagnetic shaker (3), and an accelerometer (5) measures the imposed acceleration, $\ddot{s}(t)$, at the fixed support (see Figure 8.2) - the signal analyser unit (1) records this signal. Besides, the laser vibrometer sensor head (6) measures the specimen's velocity $\dot{u}(t)$ at a given section, sending the signal to the vibrometer controller (7), which in turn delivers the response velocity to the signal analyser unit (1). Therefore, the signal analyser unit (1) receives the imposed acceleration and measured velocity, and one can manage the signal analyser unit data via computer (8).

Considering both signals, imposed acceleration and measured velocity, one can perform the required integrations via signal analyser unit and define a displacement transmissibility frequency response function: the ratio between the displacement response spectrum at a given section, U_r , and the displacement spectrum at the excitation section, S , defined as

$$T_{rs}(\omega) = \frac{U_r}{S} \quad (8.1)$$

8.2.2 | Specimens

The specimen configuration may vary depending on the testing material. ASTM E756-05 [2010] proposes four different test specimen configurations - uniform, Oberst, modified Oberst and sandwich specimens. Uniform beams are compatible with self-supporting damping materials, but one can use them for base beams. When measuring the vibration-damping properties of composite materials, uniform beams are also appropriate. Oberst and modified Oberst beams are compatible with stiff damping materials since the primary damping mechanism is the extensional deformation of the damping layer. On the other hand, sandwich specimens are suitable for soft damping materials since shear is the relevant mechanism responsible for energy dissipation.

8.2.3 | Experimental Procedure

One attaches the test specimen to the rigid support with a tightening torque of 10 N m, randomly exciting it in a frequency range of interest depending on the testing material (For fibre-reinforced plastics, the tightening torque should not exceed 5 N m). One performs each measurement in the linear regime to keep the linear viscoelastic theory valid. Figure 8.3 depicts a typical transmissibility frequency response function of a low-damping material in a frequency range between 0 and 800 Hz with one Hertz resolution. Typically, for damping estimation, one requires higher resolution near resonances since slight frequency deviations cause considerable differences in amplitude. Thus, several resonant modes of the test specimen should be measured using a higher resolution for consistent damping estimation, namely modal loss factors. Figure 8.4 depicts a transmissibility frequency response function near the first resonant mode with a 31.3 mHz resolution.

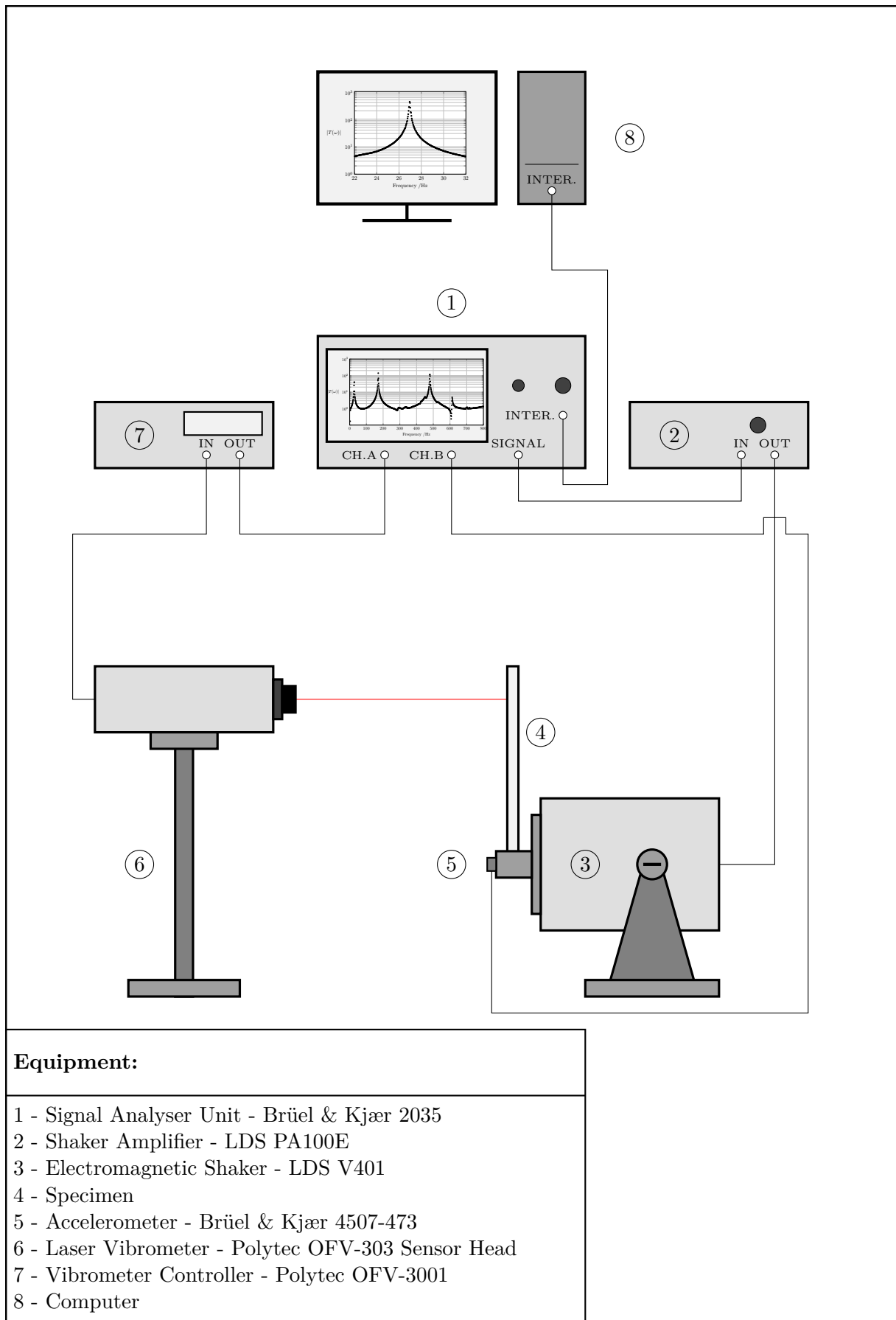
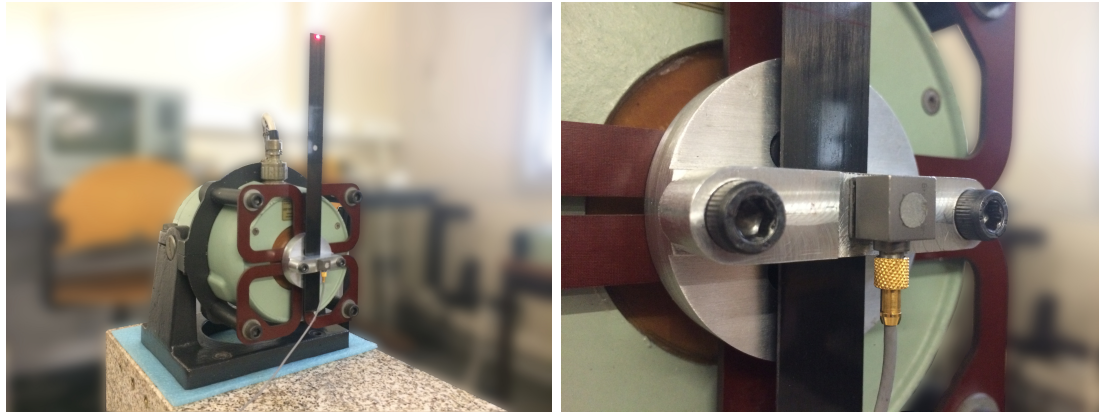


Figure 8.1 Measurement setup



(a) Shaker and specimen assembly

(b) Accelerometer detail

Figure 8.2 Shaker, specimen and accelerometer positioning

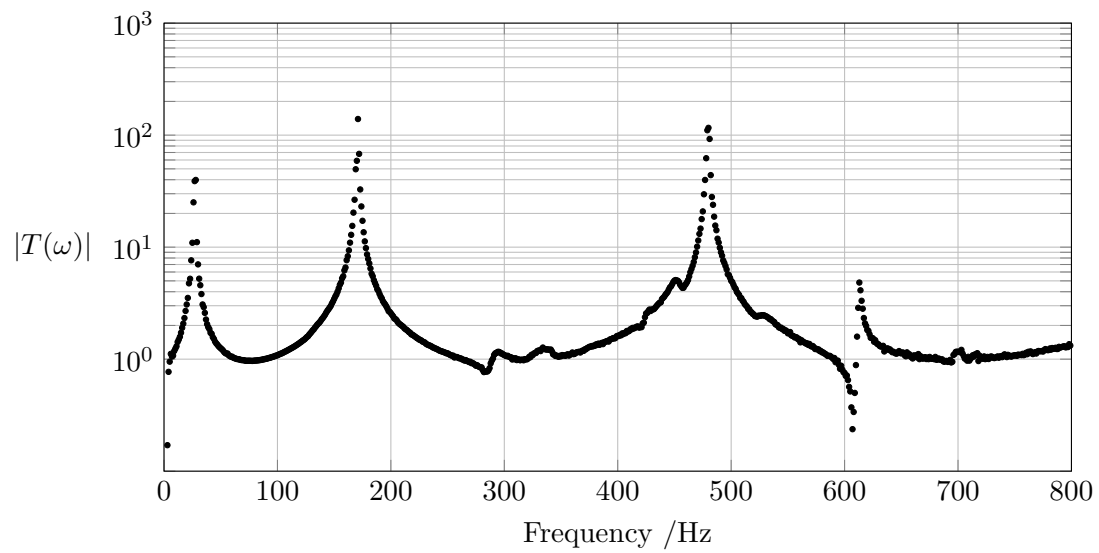


Figure 8.3 Typical transmissibility function of a low damping material

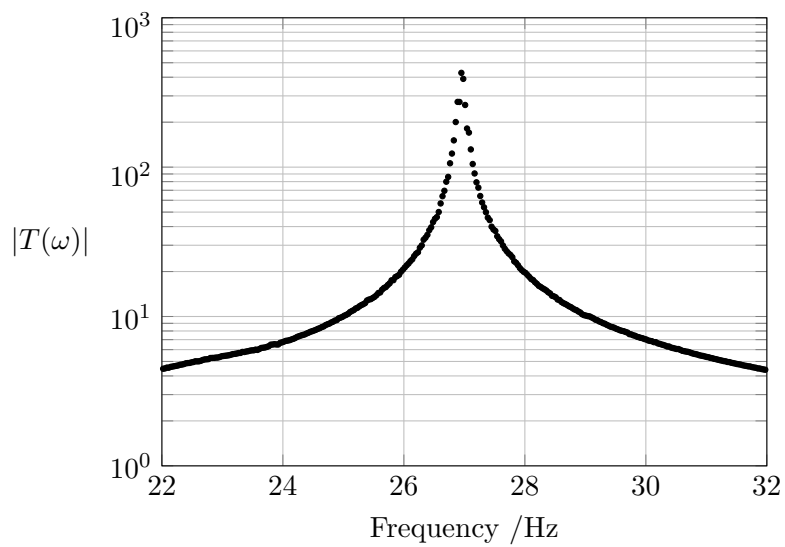


Figure 8.4 Transmissibility function near the first resonance

8.2.4 | Numerical Transmissibility

Recalling the system of equations of motion derived before in Equation (4.162), one can write the semi-discrete equations for specimens with viscoelastic materials as follows

$$\mathbf{M}(\omega)\ddot{\mathbf{u}}(t) + \tilde{\mathbf{K}}(\omega)\mathbf{u}(t) = \mathbf{f}(t) \quad (8.2)$$

where $\mathbf{M}(\omega)$ is the mass matrix, and $\tilde{\mathbf{K}}(\omega)$ is the complex stiffness matrix. Both are frequency-dependent due to the mechanical characterisation of viscoelastic materials. The $\mathbf{u}(t)$ and $\mathbf{f}(t)$ are the system response and external force vectors, respectively. Considering a passive excitation through an imposed harmonic motion $\mathbf{s}(t)$ and omitting the frequency dependence for the sake of simplicity, one can re-write the system of equations of motion (8.2) as follows

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{su} \\ \mathbf{M}_{us} & \mathbf{M}_{uu} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{s}}(t) \\ \ddot{\mathbf{u}}(t) \end{Bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ss} & \tilde{\mathbf{K}}_{su} \\ \tilde{\mathbf{K}}_{us} & \tilde{\mathbf{K}}_{uu} \end{bmatrix} \begin{Bmatrix} \mathbf{s}(t) \\ \mathbf{u}(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{Bmatrix} \quad (8.3)$$

For an imposed harmonic displacement so that

$$\mathbf{s}(t) = \mathbf{S}e^{j\omega t} \quad (8.4)$$

the stationary response and transmitted force can be written as

$$\mathbf{u}(t) = \mathbf{U}e^{j\omega t} \quad (8.5)$$

$$\mathbf{f}(t) = \mathbf{F}e^{j\omega t} \quad (8.6)$$

Substituting the previous assumptions in Equation 8.3 one can write

$$\begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{su} \\ \mathbf{M}_{us} & \mathbf{M}_{uu} \end{bmatrix} \begin{Bmatrix} -\omega^2 \mathbf{S}e^{j\omega t} \\ -\omega^2 \mathbf{U}e^{j\omega t} \end{Bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ss} & \tilde{\mathbf{K}}_{su} \\ \tilde{\mathbf{K}}_{us} & \tilde{\mathbf{K}}_{uu} \end{bmatrix} \begin{Bmatrix} \mathbf{S}e^{j\omega t} \\ \mathbf{U}e^{j\omega t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}e^{j\omega t} \\ \mathbf{0} \end{Bmatrix} \quad (8.7)$$

Re-arranging

$$\left[-\omega^2 \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{su} \\ \mathbf{M}_{us} & \mathbf{M}_{uu} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ss} & \tilde{\mathbf{K}}_{su} \\ \tilde{\mathbf{K}}_{us} & \tilde{\mathbf{K}}_{uu} \end{bmatrix} \right] \begin{Bmatrix} \mathbf{S} \\ \mathbf{U} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \end{Bmatrix} \quad (8.8)$$

The former system of equations can be written separately as

$$-\omega^2 \mathbf{M}_{ss} \mathbf{S} - \omega^2 \mathbf{M}_{su} \mathbf{U} + \tilde{\mathbf{K}}_{ss} \mathbf{S} + \tilde{\mathbf{K}}_{su} \mathbf{U} = \mathbf{F} \quad (8.9a)$$

$$-\omega^2 \mathbf{M}_{us} \mathbf{S} - \omega^2 \mathbf{M}_{uu} \mathbf{U} + \tilde{\mathbf{K}}_{us} \mathbf{S} + \tilde{\mathbf{K}}_{uu} \mathbf{U} = \mathbf{0} \quad (8.9b)$$

Equation (8.9b) establishes a linear system of algebraic equations

$$(\tilde{\mathbf{K}}_{uu} - \omega^2 \mathbf{M}_{uu}) \mathbf{U} = -(\tilde{\mathbf{K}}_{us} - \omega^2 \mathbf{M}_{us}) \mathbf{S} \quad (8.10)$$

whose solution $\mathbf{U}$ is a phasor representing the magnitude and phase of each degree of freedom

$$\mathbf{U} = [U_1, U_2, \dots, U_n]^T \quad (8.11)$$

One can also calculate the transmitted force using Equation (8.9a) by

$$\mathbf{F} = (\tilde{\mathbf{K}}_{su} - \omega^2 \mathbf{M}_{su}) \mathbf{U} + (\tilde{\mathbf{K}}_{ss} - \omega^2 \mathbf{M}_{ss}) \mathbf{S} \quad (8.12)$$

The numerical transmissibility function $T_{rs}(\omega)$ between the degree of freedom r and the imposed displacement is given by

$$T_{rs}(\omega) = \frac{U_r}{S} \quad (8.13)$$

8.2.5 | Material Properties Identification

There are different methods in the literature for modal loss factor estimation, including the half-power bandwidth or circular curve fitting of the Nyquist curve. However, a noteworthy feature of the transmissibility frequency response function arises at this point. While the ASTM E756-05 [2010] estimates the damping properties using equations based on the sinusoidal expansion for the mode shapes of vibration, inverse methods like curve fitting of transmissibility function generated by a finite element model identify the material properties with no further assumptions beyond those of the mechanical model used in the finite element formulation.

Fitting a numerical transmissibility frequency response function to an experimental one is to optimise an objective function, somehow related to the difference between both functions, searching for the mechanical properties correlating both functions. This objective function might vary, and minimising the difference between experimental and numerical resonant frequencies might be sufficient for storage modulus identification but inadequate for loss factor evaluation. Such an approach requires an efficient updating procedure since trial and error is time-consuming and the global minimum isn't assured. Some researchers often use the minimisation of an objective function defined by the differences between real and imaginary parts of the numerical and experimental transmissibility at some control frequencies ω_k . Figures 8.5 and 8.6 show numerically fitted transmissibility frequency response functions used for properties identification.

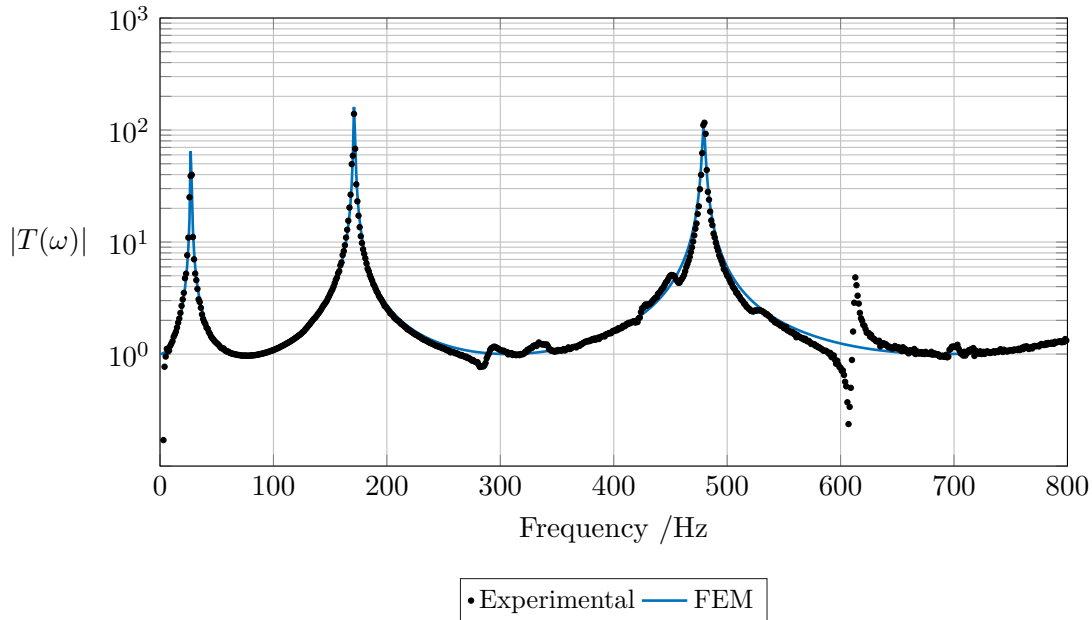


Figure 8.5 Numerically fitted transmissibility frequency response function

Kim and Lee [2009] proposed a procedure to identify the fractional-derivative-model parameters of viscoelastic materials based on two steps - a resonant-frequency alignment followed by an amplitude alignment between simulated and measured response functions with gradient-based

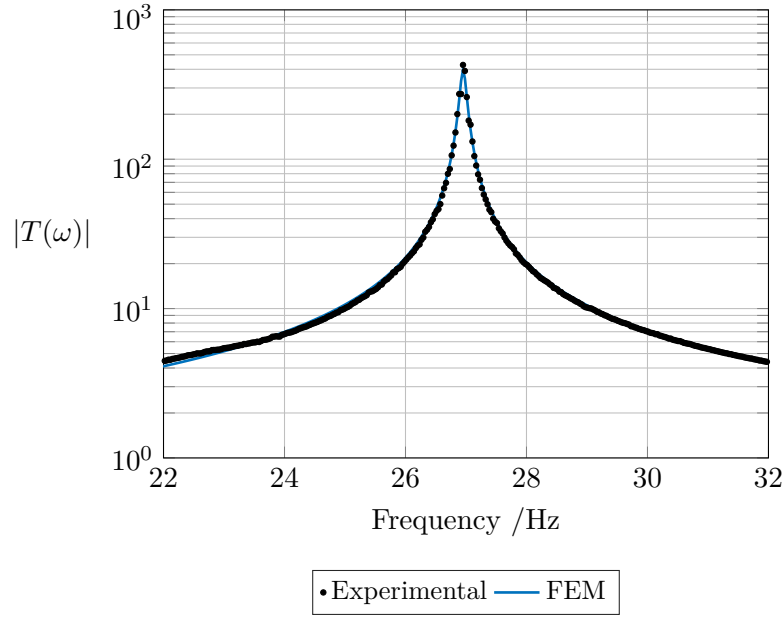


Figure 8.6 Numerically fitted transmissibility frequency response function near resonance

optimisation algorithms. The primary purpose is to optimise a global shape in the first step, followed by a fine adjustment in the second stage using the least-square error. According to the authors, the proposed method can reduce the required measurements to identify the material properties, being the curve-fitting process sped up due to the two-step approach.

It is possible to assess the correlation between measured and simulated frequency response functions using frequency-domain correlation tools available in the literature. Many of these indicators rely on modal correlation indicators such as Modal Assurance Criterion (MAC) or Coordinate Modal Assurance Criterion (COMAC). Zang et al. [2001] highlighted the most relevant ones, and one uses the Local Assurance Criterion (LAC) in this work.

8.2.5.1 | Local Amplitude Criterion (LAC)

The Local Amplitude Criterion is a local indicator where one can distinguish between frequency ranges with poor and good correlation of single frequency response functions [Zang et al., 2001]. One can define this correlation criterion as

$$\text{LAC}_{j\ell}(\omega) = \frac{2 \left| \{H_{j\ell}^X(\omega)\}^H \{H_{j\ell}^A(\omega)\} \right|}{\{H_{j\ell}^X(\omega)\}^H \{H_{j\ell}^X(\omega)\} + \{H_{j\ell}^A(\omega)\}^H \{H_{j\ell}^A(\omega)\}} \quad (8.14)$$

where $\{H_{j\ell}^X(\omega)\}$ and $\{H_{j\ell}^A(\omega)\}$ are the measured and simulated frequency response functions between the degrees of freedom j and ℓ , respectively. Therefore, one obtains a real-valued frequency-dependent function whose values range from 0 (no correlation) to 1 (perfect correlation).

8.2.5.2 | Objective Function

In this work, one proposes an objective function for the correlation-updating process of a simulated frequency response function and a measured one to identify the mechanical properties of materials. One can define an objective function based on the LAC criterium as the summation of the differences to one at each frequency value.

$$\min \sum_{k=1}^N (1 - \text{LAC}_{j\ell}(\omega_k)) \quad (8.15)$$

One should note that the LAC is a unit value constant function for perfectly correlated functions, and therefore, the objective function ranges from 0 (perfect correlation) to the dimension of the vector space (no correlation). The proposed objective function proved its efficiency and reliability as the core of the correlation-updating process, independently of the test specimen under consideration. It identified elastic and viscoelastic material properties with low or high damping capacity, proving its robustness. During the correlation-updating process, simulated and measured frequency response functions began to correlate better at the resonant frequencies. Naturally, as slight deviations in frequency cause significant differences in amplitude near resonances, such frequency bands promote considerable variations of the objective function, and both frequency responses tend to align their resonances in the first place. Then, the correlation updating process forces an amplitude alignment of both frequency response functions, and consequently, this is a two-step procedure per se requiring only the curves and not some parameter identification and respective comparison.

The optimisation procedures conducted in this work use different algorithms from the MATLAB Optimization Toolbox. One comment those relevant to this work on below.

- *fminsearch*

This minimisation function is an unconstrained direct search method based on the Nelder-Mead algorithm and does not require any derivative information.

- *fmincon*

A constrained gradient-based minimisation function is available with multiple algorithms. The default Interior Point Algorithm computes the gradient vector and Hessian matrix numerically using finite difference methods.

- *patternsearch*

Direct search minimisation algorithm using pattern search with an adaptive mesh. This function enables one to introduce linear equality or inequality constraints.

8.3 | Viscoelastic Materials

In order to identify the mechanical properties of the viscoelastic materials used in this work, one manufactured two sets of specimens. In the first step, one manufactured specimens using Smacwrap ST with a 0.1 mm thickness using different adhesives - Procedure A. The main goal is to study the influence of adhesives on the identified mechanical properties of viscoelastic materials. In a second step, one manufactured a set of specimens using Smacwrap ST with a 0.2 mm thickness and Kraibon with a 0.5 mm thickness to identify the material properties using more test pieces including different thicknesses - Procedure B.

8.3.1 | Procedure A

For this procedure, one manufactured four test specimens, including two control specimens (Figures 8.7b and 8.9a) with aluminium layers bonded by two different adhesives - SR 1500, an

epoxy resin with excellent wet-out combined with SD 2505 hardener and SikaForce-7818 L7, a two-component polyurethane structural adhesive with non-sag behaviour for wind turbine bonding applications. Simultaneously, these two adhesives were used to bond the Smacwrap ST 0.1 mm viscoelastic material between two aluminium layers - Figures 8.7c and 8.9b. The control specimens were manufactured using 0.1 mm feeler gauge strips to set an equivalent adhesive thickness for the control specimens considering the 0.1 mm thickness of the viscoelastic material. The main objective of the control specimens is to identify the possible influence of the adhesive on the properties of the viscoelastic material identified. Manufacturing two control specimens with different adhesive materials might test the mechanical properties of such control specimens, verify possible differences between them, and help to draw conclusions to help with the VEM properties identification. It is well-known that the energy dissipation in viscoelastic materials in PCLD design is due to their shear deformation, and the adhesive layer must ensure the load transmission between the elastic material and the viscoelastic layer. In the case of slipping, for example, the imposed deformation from the base beam to the damping material may lead to a slight underestimation of the material properties. For ease of readability, one labelled control specimens with epoxy resin and polyurethane adhesive E01 and P01, respectively. Specimens with Smacwrap ST 0.1 mm viscoelastic material are labelled EV01 and PV01 according to the adhesive used to bond the different layers - epoxy resin and polyurethane adhesive, respectively.

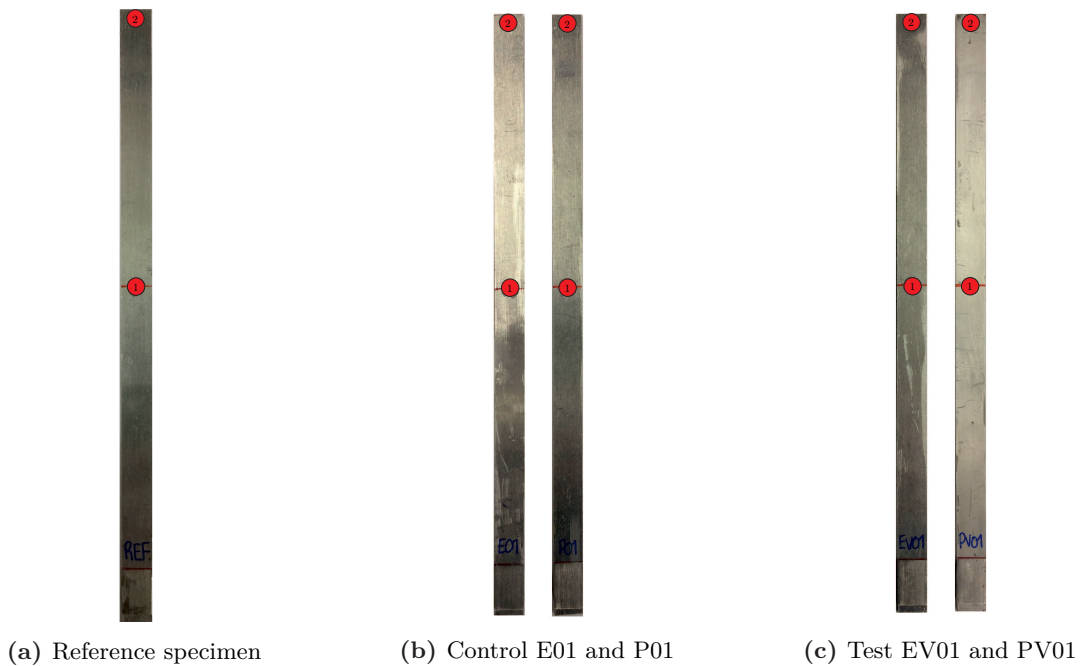


Figure 8.7 Reference and sandwich specimens for viscoelastic material characterisation

The aluminium specimens were sanded with a P220 grit sandpaper sheet and degreased with acetone. Besides, a release agent was applied on both surfaces of a steel mould, enabling the simultaneous production of the four specimens and ensuring easy post-cure handling - see mould in Figure 8.8. The curing process of the specimens was carried out at room temperature and compaction pressure of 2 MPa for 24 hours. Specimens rested for fourteen days at room temperature after demolding. Figure 8.7 shows a photograph of the test specimens. To characterise the mechanical properties of aluminium, one also considers a reference specimen consisting of a bare beam - Figure 8.7a.

Figure 8.9 depicts a schematic representation of the specimens used in this procedure and the respective geometric properties; free length, ℓ , base width, b , base thickness, h_b , adhesive

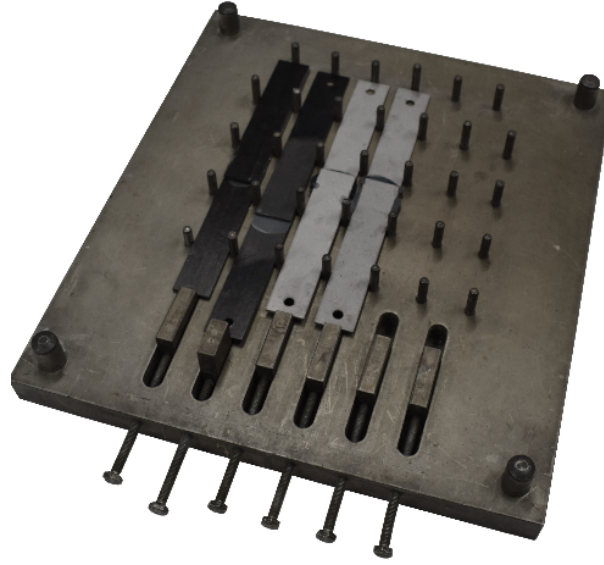


Figure 8.8 Mold for specimens fabrication - Photo courtesy of José Machado

thickness, h_a , viscoelastic thickness, h_v , and constraining layer thickness, h_c . Table 8.1 lists the geometric properties of the specimens. One should note that these values are within those recommended by ASTM E756-05 [2010].

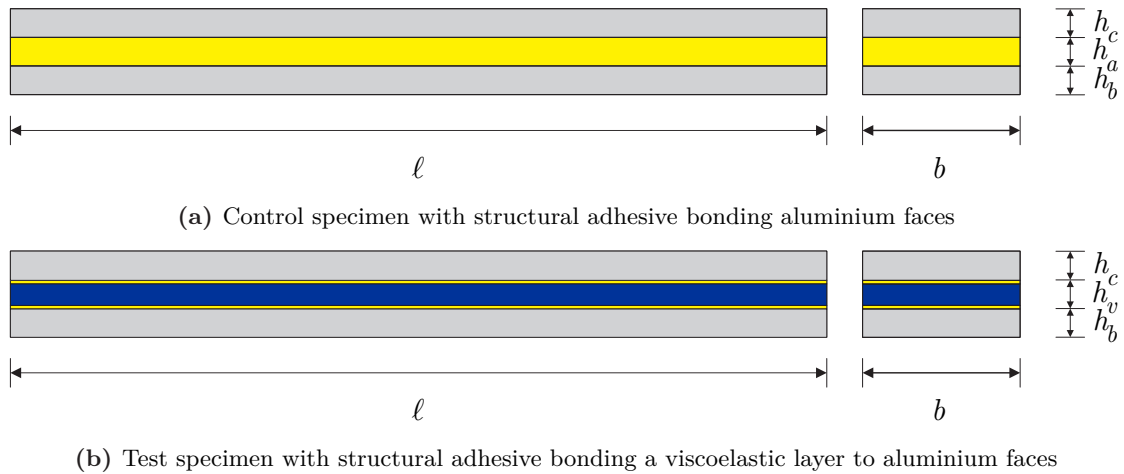


Figure 8.9 Schematic representation of the test specimens

Table 8.2 lists the measured mass of the test specimens and the corresponding theoretical values based on calculations using reference properties. One should expect slight differences between reference and current dimensions, namely on the thickness of the adhesive and viscoelastic material, so it is relevant to track the final mass to ensure reliability. Calculations for the theoretical values in specimens with viscoelastic materials neglect the mass of the adhesive layer. Table 8.2 shows marginal differences between expected and measured mass.

Table 8.1 Geometric properties of the test specimens - A

Length**	Width	Layer Thickness			
		Base	Adhesive	Viscoelastic	Constraining
ℓ	b	h_b	h_a	h_v	h_c
180	10	1	0.1	0.1	1

* values in mm

** effective length

Table 8.2 Final mass [g] of the test specimens - A

	Measured	Theoretical	Relative Difference
Reference	5.438	-	-
Control Specimens			
E01	10.658	10.885	2.13 %
P01	10.844	10.912	0.62 %
Viscoelastic Specimens			
EV01	10.866	10.925	0.55 %
PV01	10.868	10.925	0.53 %

Table 8.3 lists the mechanical properties of the materials used in this procedure based on data available from manufacturers.

Table 8.3 Reference material properties of the test specimens - A

	Young's Modulus E /GPa	Poisson ν	Density ρ /kg m ⁻³
Host Structure			
Aluminium	70	0.3	2715
Adhesives			
Sicommin SR1500 / SD2505	3.1	0.3	1095
SikaForce-7818 L7	2.5	0.3	1230
Viscoelastic Material			
Smacwrap ST 0.1	-	0.49	1300

8.3.1.1 | Reference Specimen

One tested a bare reference specimen following the experimental procedure described in the previous sections to identify the mechanical properties of the aluminium base and constraining layers. At this step, one measured and estimated via the FEM the transmissibility function at two different sections of the specimen - Section 1: Half-Length and Section 2: Tip - see Figure 8.7.

Figures 8.10 to 8.12 show the experimental and numerical transmissibility functions at the half-length and tip sections. One recorded these experimental transmissibilities considering

different frequency ranges but a fixed number of frequency points - 801. Consequently, numerical transmissibility functions fit the experimental curves considering an equivalent resolution. The primary intent is to understand the influence of the frequency range and resolution in Young's modulus and material loss factor estimation. Table 8.4 lists the optimised results for both properties considering the same measuring points but different frequency resolutions. Besides, Figure 8.13 shows the fitting process result near the first three resonances, using the same 801 frequency points, and Table 8.4 lists the results.

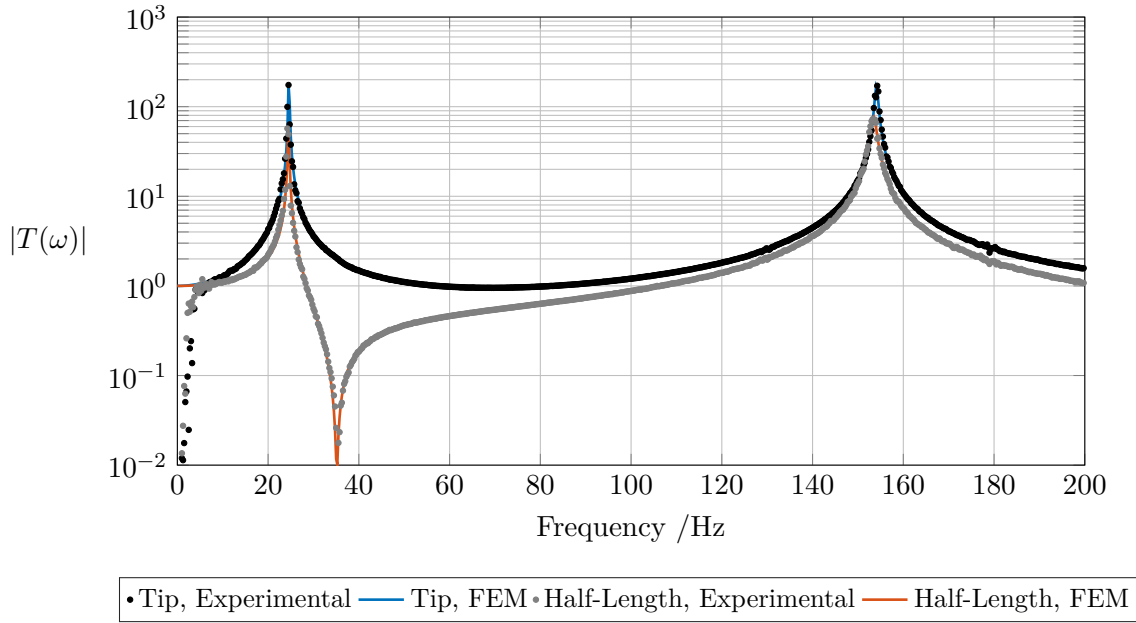


Figure 8.10 Transmissibilities of the reference aluminium specimen - 0.25 Hz resolution

Considering any frequency range, one can notice from Figures 8.10 to 8.13 that the optimisation procedure reached an almost perfect fit. Consider Table 8.4 in detail to discuss the results. At this point, one must signal two leading outcomes. The procedure estimates a higher Young's modulus when considering broader frequency ranges. The fitting process includes more resonances for extended frequency ranges, and the fit is more demanding since the resonant frequencies depend on Young's modulus. Therefore, the frequency range influences the estimation. Consistently, the independent fitting process of the first three resonances shows the same trend; higher Young's modulus estimations for higher-order modes.

The second main observation regarding Table 8.4 relates to the estimated material loss factor. When considering the half-length section, the procedure overestimates the material loss factor. Transmissibility functions have anti-resonances for the half-length section, and the dynamic stiffness is high at such regions, so the fitting process is demanding and damping overestimated. A closer look at the final values of the objective function shows the worst fittings for the half-length transmissibilities. For this reason, tip section measurements, for which only resonances occur, seem to be preferable for material loss factor estimations using transmissibility functions. Contrary to Young's modulus, the material loss factor estimation decreases with increasing frequency ranges. It is evident that the material loss factor estimation strongly depends on the number of measuring points near resonances, so, at first sight, the results presented above might be influenced by this. Higher frequency ranges result in fewer points near resonances, leading to the worst material loss factor estimations. However, once again, the material loss factor estimations using the frequency range near the first three resonances show the same trend - lower material loss factors for higher frequencies but similar values, concluding that a suitable material

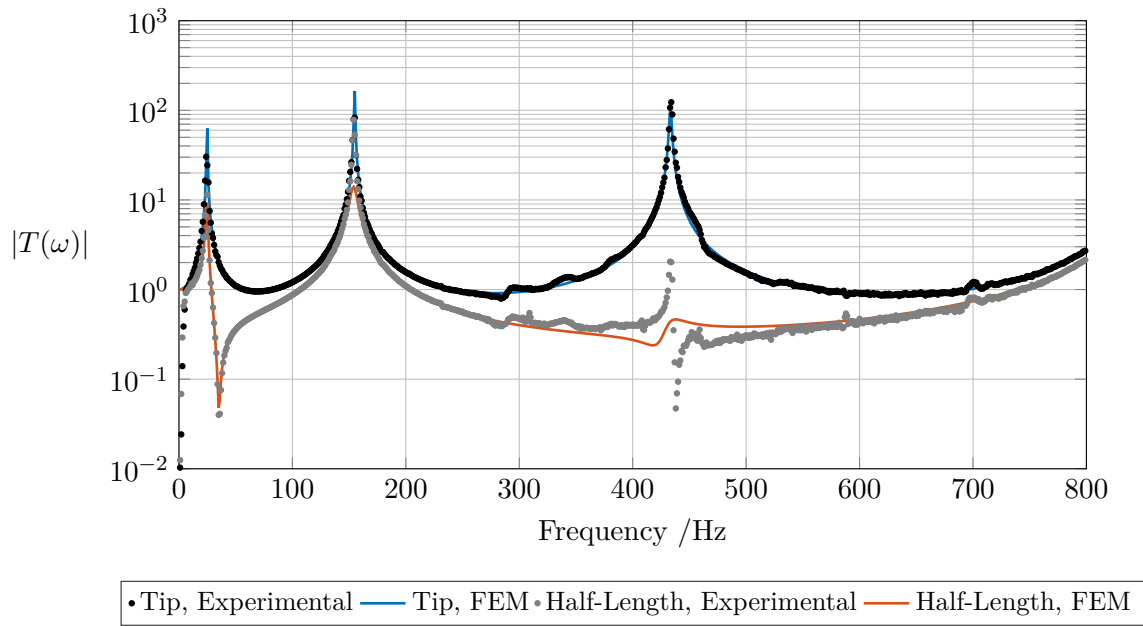


Figure 8.11 Transmissibilities of the reference aluminium specimen - 1 Hz resolution

loss factor estimation with tip measurements and the frequency range considered is possible.

Table 8.4 Optimisation results of the reference specimen

Range		Section	Young's Modulus	Loss Factor	Objective Function
			E /GPa	η	
			Initial Value		
			70	0.0020	
[0, 200] Hz		Tip	66.0420	0.0047	14.5764
		Half-Length	65.5202	0.0081	12.1315
[0, 800] Hz		Tip	66.6430	0.0040	7.8817
		Half-Length	65.8604	0.0443	26.3249
[0, 1600] Hz		Tip	67.1486	0.0025	6.9688
		Half-Length	66.8229	0.0078	48.9158
1st	Resonance	Tip	65.3044	0.0044	1.3296
2nd	Resonance	Tip	66.0921	0.0032	1.1497
3rd	Resonance	Tip	66.8873	0.0022	5.6317

For the following numerical studies with this reference aluminium material, one considers a 67 GPa Young's modulus since it is the estimation for broader frequency ranges, including more modes, so it is a more representative measure. Since the material loss factor of aluminium is low, one neglected it for the estimation of viscoelastic material properties.

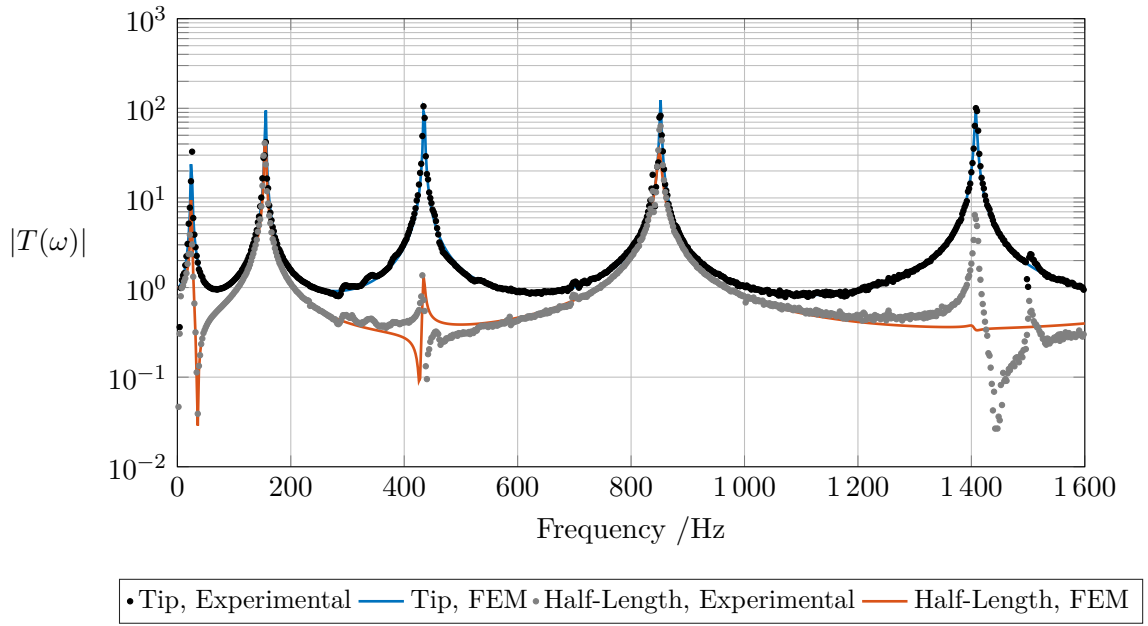


Figure 8.12 Transmissibilities of the reference aluminium specimen - 2 Hz resolution

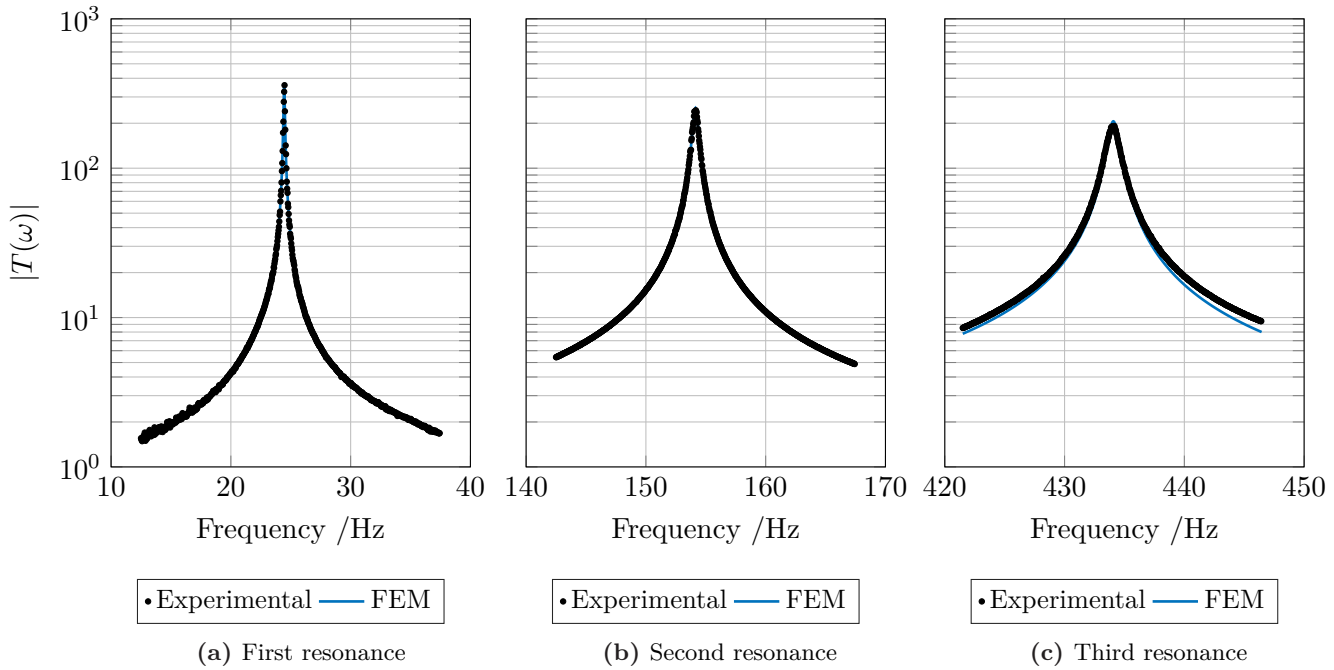


Figure 8.13 Transmissibilities of the reference specimen near the first three resonances

8.3.1.2 | Control Specimens: E01-P01

Figures 8.14 to 8.16 show the experimental and numerical transmissibility functions at the tip section of both control specimens for two frequency ranges and near the first three resonances. The first evidence is that the control specimens show different resonance values. In the first step, one identified the Young's modulus and material loss factor of both adhesives considering two aluminium layers bonded by a 0.1 mm thick adhesive layer. The numerical transmissibility function optimisation proved insensitive to the Young's modulus of such thin layers, particularly

considering that the Young's modulus of aluminium and adhesives are not so different (in the GPa range) - Table 8.3. Then, using the available reference Young's modulus in the material data sheets, the fitting process searched for the adhesive layer thickness. The natural frequencies of beams are a function of the square root of the 2nd moment of area, which varies cubically with the beam's thickness. Consequently, the resonant frequencies are more sensitive to the thickness of the layer than its Young's modulus. Tables 8.5 and 8.6 the optimised results for the control specimens E01 and P01, respectively.

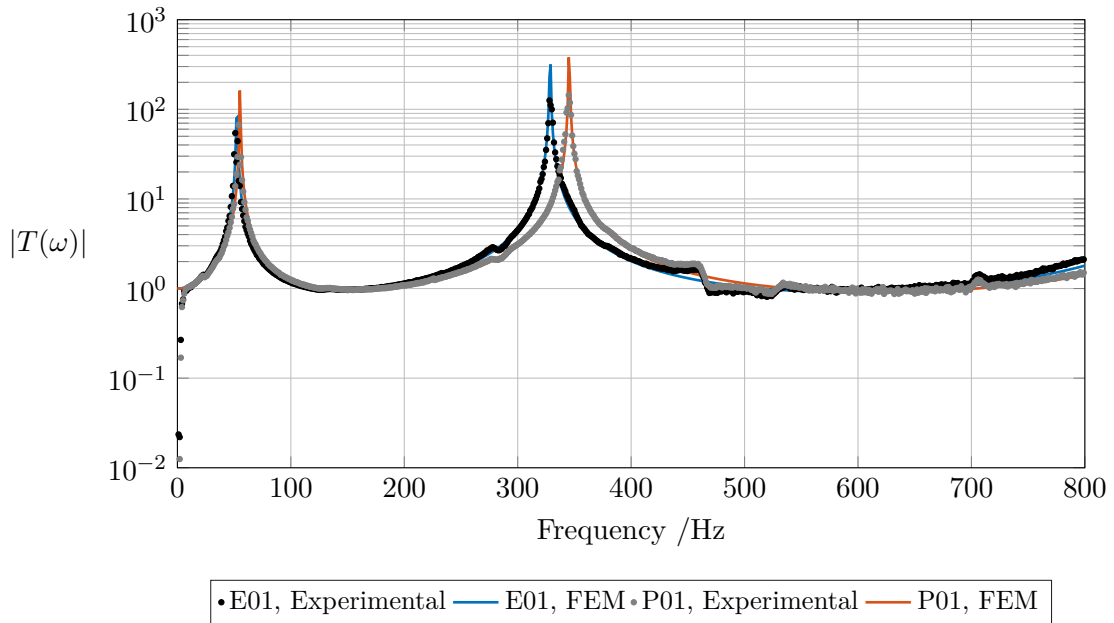


Figure 8.14 Transmissibilities of the control specimens - 1 Hz resolution

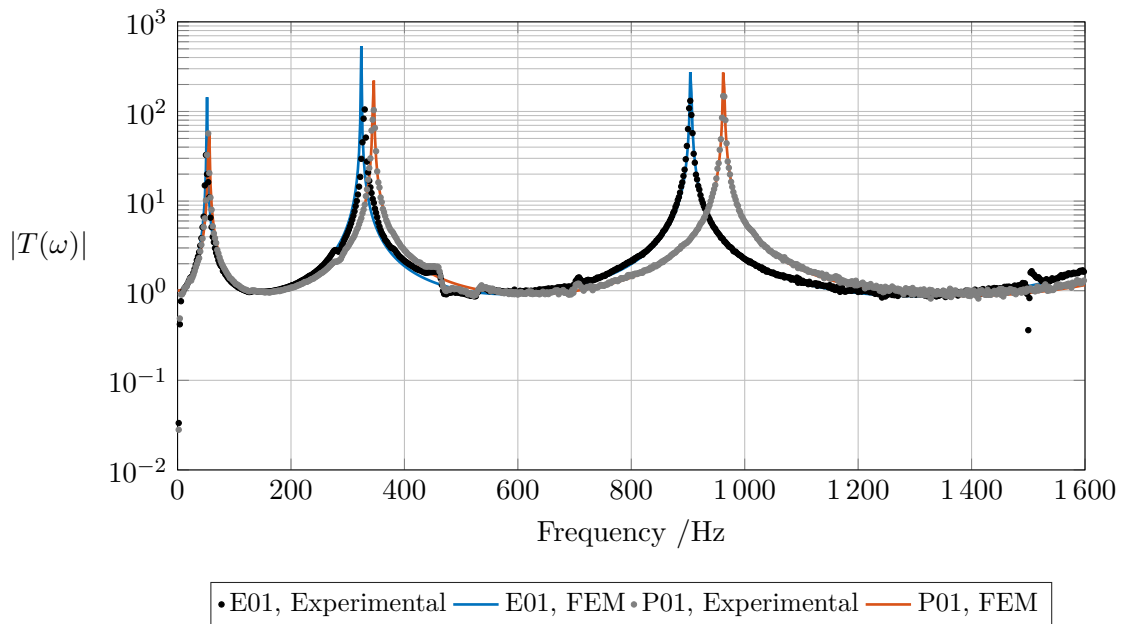


Figure 8.15 Transmissibilities of the control specimens - 2 Hz resolution

To examine the obtained results, one prepared a section cut of each specimen for microscopic

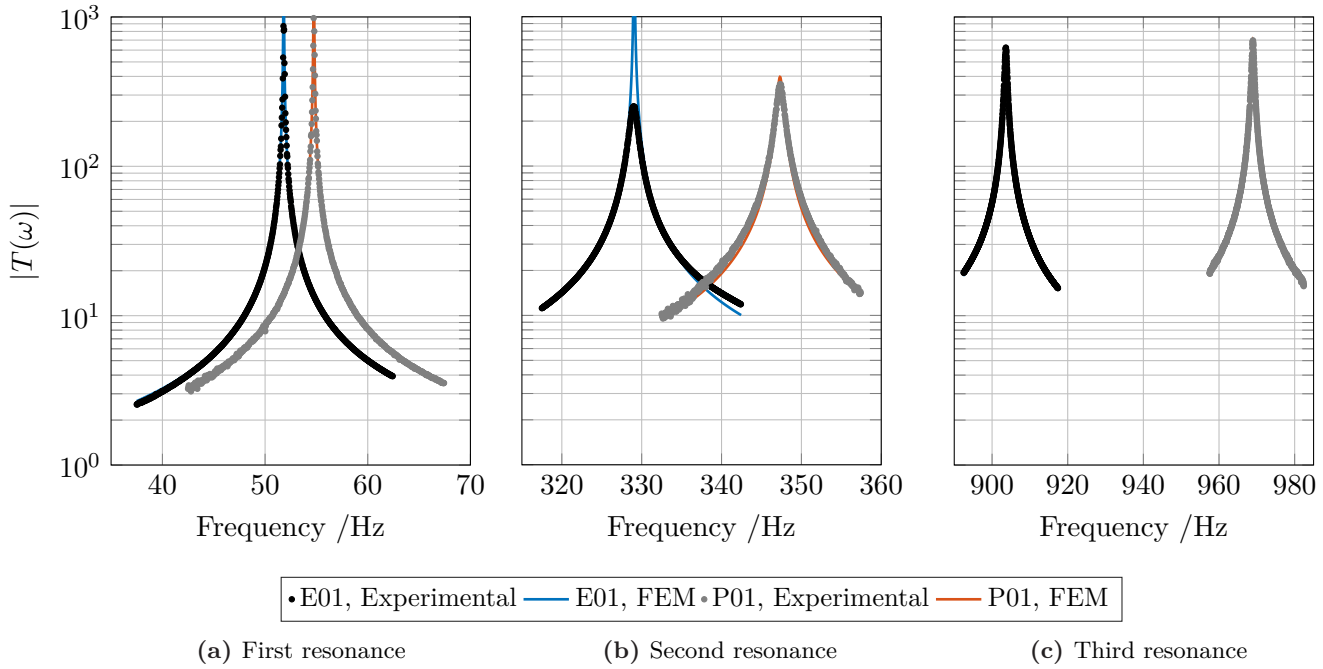


Figure 8.16 Transmissibilities of the control specimens near the first three resonances

analysis by consecutively polishing each sample with 180, 320 and 800 grit sandpaper and finally polishing with a polishing cloth and diamond suspensions of 3 and 1 μm after being embedded in a resin mount. Figure 8.17 shows a section cut of the E01 and P01 specimens with the same magnification and the corresponding measurement of the adhesive layer thickness. Appendix B shows several microscopic images of both specimens. The thickness of the adhesive layers observed at the microscope justifies the resonant frequencies observed in the transmissibility functions. For a single section cut of both samples, the P01 sample has a thicker adhesive layer (175 μm) than the E01 specimen (67 μm). Despite being manufactured to have two adhesive layers with equal thickness, the nature of each adhesive results in different final thicknesses. The higher viscosity of the polyurethane adhesive (P01 specimen) leads to a higher adhesive layer due to the higher resistance to flow with the applied pressure using the mould. In contrast, the epoxy resin in the E01 specimen flows freely away from the beam with the applied pressure due to its lower viscosity. Quite surprisingly, for such small thicknesses, with values in the order of microns, the fitting process estimated adhesive layer thicknesses very close to those observed at the microscope.

The optimised adhesive layer thickness for the E01 specimen presented in Table 8.5 varies between 66 μm and 95 μm considering different frequency ranges. Still, and considering the broader frequency range with more resonances, one estimates an adhesive thickness of 68 μm . For the P01 specimen results presented in Table 8.6, the adhesive layer thickness varies between 165 μm and 193 μm with a 181 μm estimated thickness when considering the broader frequency range.

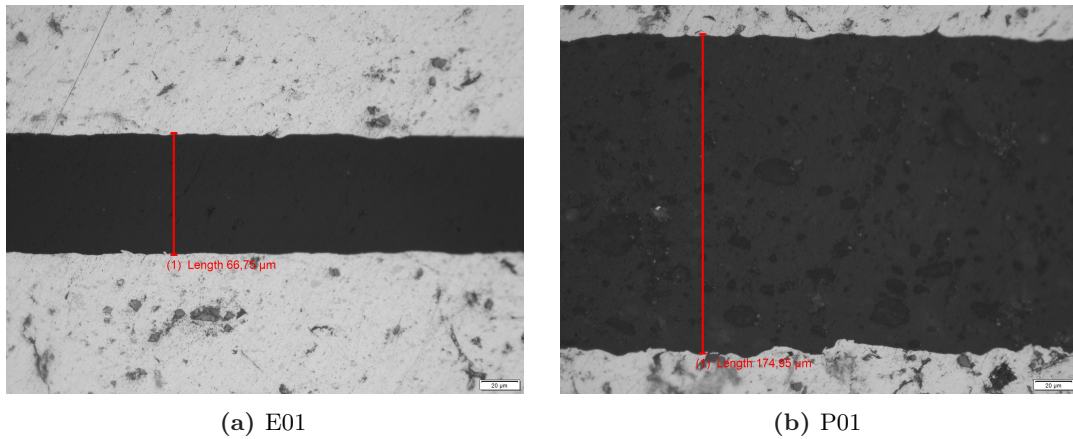


Figure 8.17 Cross sections of the control specimens with 50X magnification

Table 8.5 Optimisation results of the E01 control specimen

Range		Section	Adhesive Thickness	Loss Factor	Objective Function
			h_a / μm	η	
Initial Value					
			100	0.0020	
[0, 800] Hz		Tip	92.4508	0.0023	9.7811
[0, 1600] Hz		Tip	68.1903	0.0027	12.7353
1st	Resonance	Tip	70.8496	0.0023	2.2671
2nd	Resonance	Tip	94.8486	0.0079	16.1028
3rd	Resonance	Tip	66.0830	0.2184	1.3489

Table 8.6 Optimisation results of the P01 control specimen

Range		Section	Adhesive Thickness	Loss Factor	Objective Function
			h_a / μm	η	
Initial Value					
			100	0.0020	
[0, 800] Hz		Tip	181.4673	0.0022	9.2156
[0, 1600] Hz		Tip	181.4639	0.0023	6.6470
1st	Resonance	Tip	165.2246	0.0015	1.8754
2nd	Resonance	Tip	190.8992	0.4438	5.9105
3rd	Resonance	Tip	193.2739	0.0518	1.2106

8.3.1.3 | Viscoelastic Specimens: EV01 PV01

In the previous section, one proved that different adhesives might influence the dynamic behaviour of test specimens produced for material characterisation. Here, one followed the same approach and prepared section cuts of the viscoelastic specimens for microscopic analysis. Figure 8.18

shows a section view of the viscoelastic specimens EV01 and PV01, and it is interesting to notice a similar behaviour - polyurethane adhesive is thicker than epoxy.

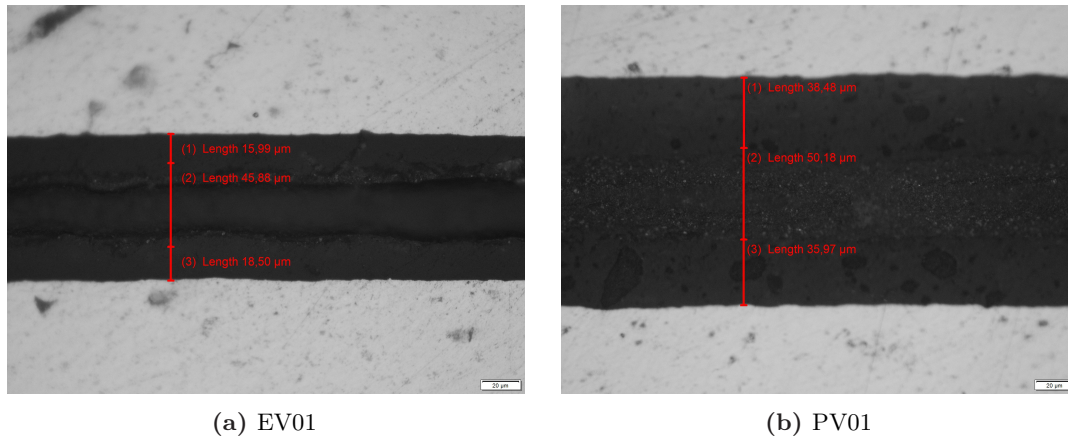


Figure 8.18 Cross sections of the EV01 and PV01 viscoelastic specimens with 50X magnification

Due to the careful polishing of the samples, one can clearly distinguish adhesive and viscoelastic layers in Figure 8.18. One must signal the 50 μm measurement of the viscoelastic layer thickness for both specimens, a value entirely below the listed one. Moreover, one records roughly double adhesive layer thickness for the PV01 compared with EV01. For the following numerical studies considering these specimens, one assumes a 50 μm viscoelastic layer thickness for both cases and adhesive layers of 15 μm for EV01 and 40 μm for PV01.

Figures 8.14 and 8.15 show the experimental and numerically fitted transmissibility functions considering two frequency ranges. The plot suggests higher resonant frequencies for the PV01 specimen, particularly the third resonance in Figure 8.15. Recalling that the PV01 sample has a higher adhesive thickness, such a result comes naturally but reminds one of the importance of considering the adhesive layers. Neglecting the existence of different adhesives would undoubtedly lead to an inaccurate estimation of viscoelastic material properties. Nevertheless, there is an excellent fit between experimental and numerical transmissibilities, specifically for the frequency range until 1600 Hz since it captures more resonances and is more representative.

Table 8.7 presents the final optimisation results of the four parameters of the fractional derivative model. These results show a slightly better fit for both specimens if one considers a broader frequency range. Besides, one can also notice that the four parameters for such a frequency range are within the same magnitude for both specimens, while there is a significant deviation considering a frequency range until 800 Hz. A narrow frequency range considering only two resonances is insufficient for proper material characterisation. Figure 8.21 depicts the final optimised viscoelastic properties for both specimens using the four parameters obtained with the broader frequency range. The optimised material properties for the PV01 test piece show higher storage modulus and material loss factor values. However, such differences are minor considering this kind of materials - a 15 % difference for the storage modulus and a 9 % difference for the material loss factor at 1600 Hz seems acceptable.

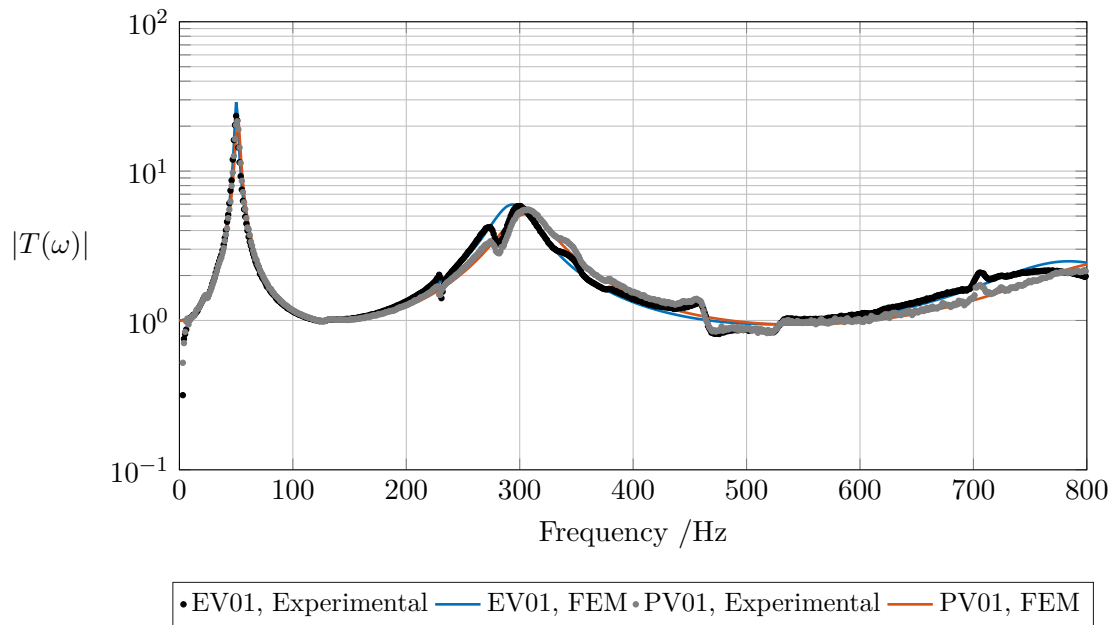


Figure 8.19 Transmissibilities of the EV01 and PV01 viscoelastic specimens - 1 Hz resolution

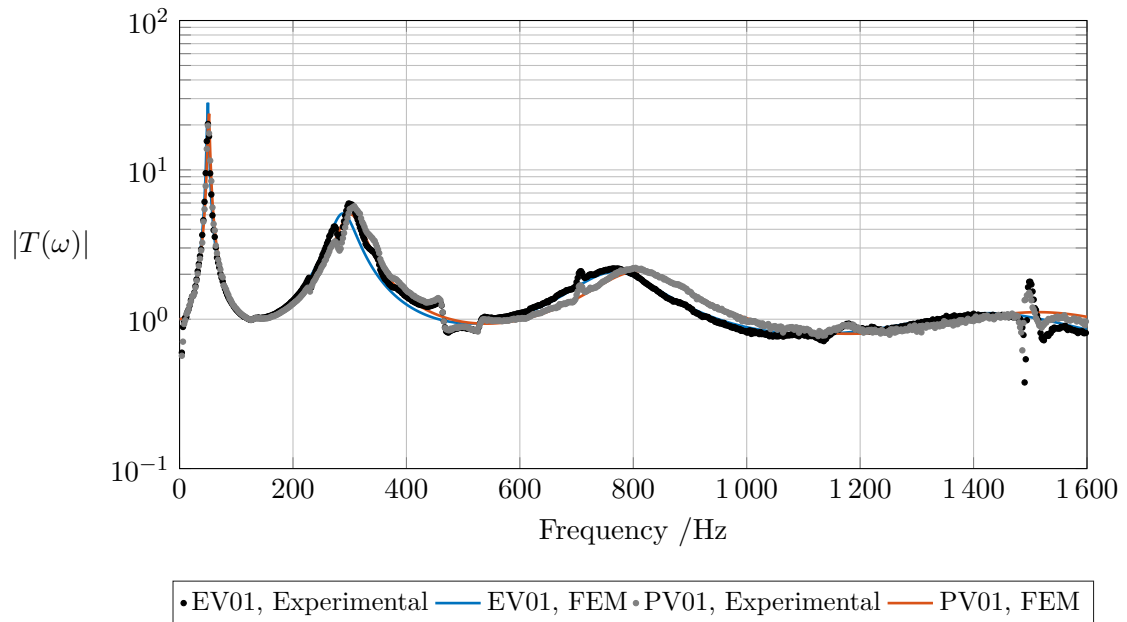


Figure 8.20 Transmissibilities of the EV01 and PV01 viscoelastic specimens - 2 Hz resolution

Table 8.7 Optimisation results of the viscoelastic specimens EV01 and PV01

Range	Specimen	E_0	E_∞	τ	α	Objective Function
Initial Value						
		1.0000×10^6	1.0000×10^9	1.0000×10^{-6}	0.5	
[0, 800] Hz	EV01	0.4197×10^6	0.4386×10^9	3.2396×10^{-7}	0.4883	7.4107
	PV01	1.4766×10^6	0.0702×10^9	4.6863×10^{-5}	0.7001	6.3856
[0, 1600] Hz	EV01	1.3632×10^6	1.5151×10^9	4.2749×10^{-8}	0.5457	7.2116
	PV01	0.9148×10^6	2.4330×10^9	3.1702×10^{-8}	0.5587	5.6063

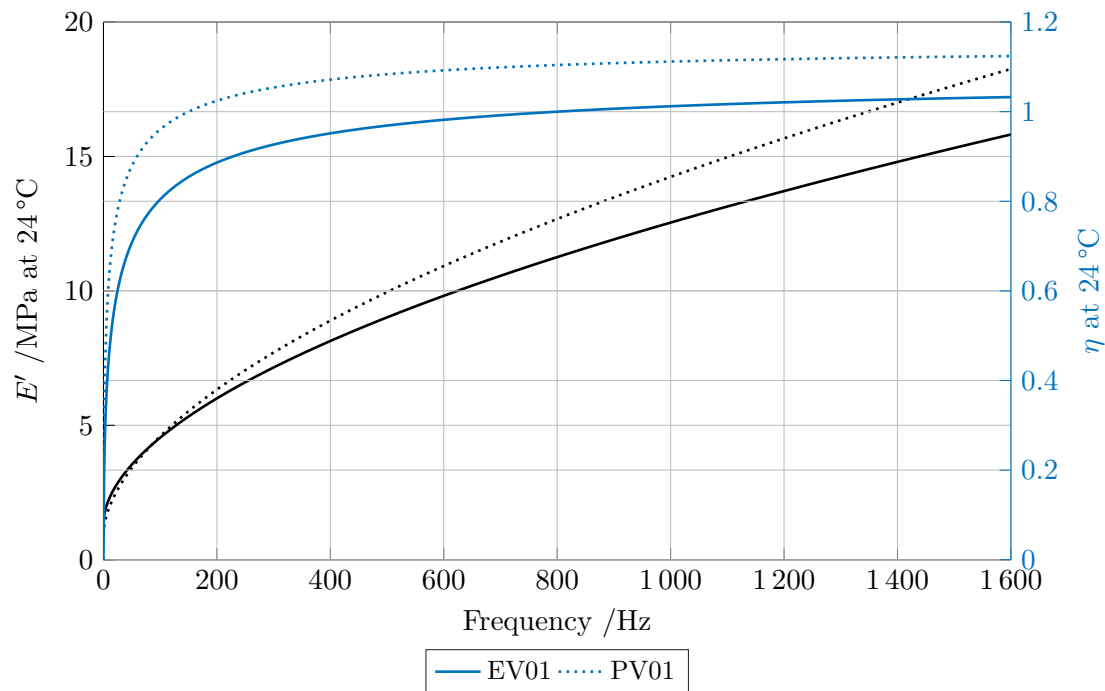


Figure 8.21 Identified viscoelastic material properties of EV01 and PV01 specimens

8.3.2 | Procedure B

For Procedure B, one manufactured two sets of specimens using aluminium layers of equal thickness with two bonded different viscoelastic materials, the Smacwrap ST used in Procedure A but a different thickness of 2 mm (S02) and Kraibon (K05). Here, the adhesive is Permabond ES560, a single-part epoxy adhesive with low viscosity. Each set of viscoelastic specimens has two samples - see Figure 8.23.

Aluminium specimens were sanded with a P220 grit sandpaper sheet and degreased with isopropyl. Contrary to the Smacwrap ST, which withstands autoclave curing but does not undergo a chemical reaction, Kraibon rubber must vulcanise at 150 °C. Consequently, one manufactured two plate specimens with a final cure process using an industrial autoclave with the cure cycle represented in Figure 8.22.

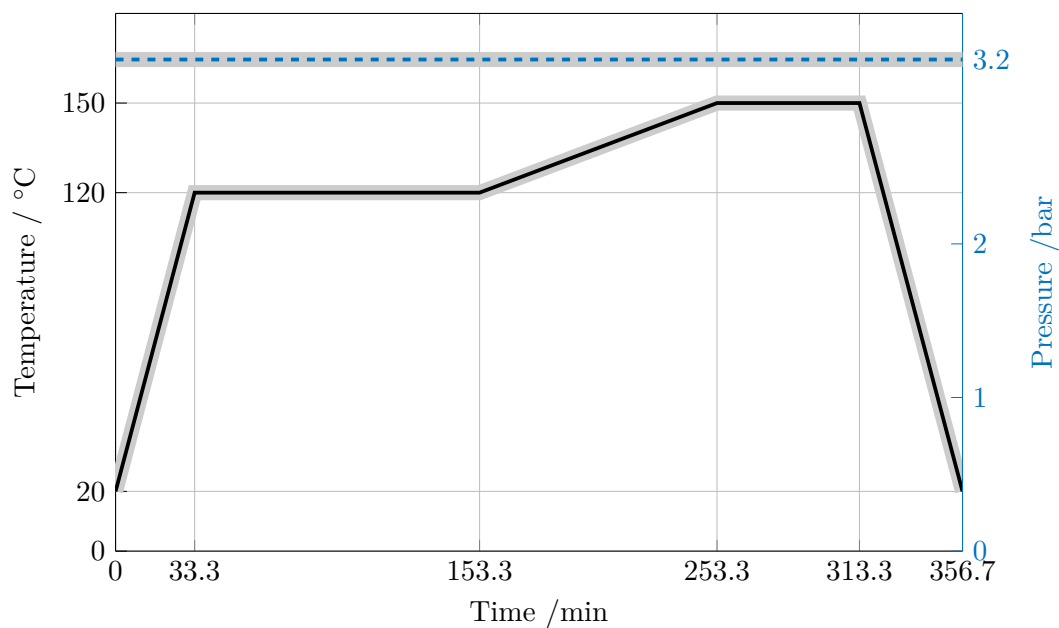


Figure 8.22 Cure cycle of the test specimens - B

Tables 8.8 and 8.9 present the mechanical and geometric properties of the materials used in this procedure for both sets of specimens.

Table 8.8 Reference material properties of the test specimens - B

	Young's Modulus E /GPa	Poisson ν	Density ρ /kg m ⁻³
Host Structure			
Aluminium 1050A H24	70	0.32	2708
Adhesive			
Permabond ES560	2.1	0.3	1200
Viscoelastic Materials			
Smacwrap ST 0.2	-	0.49	1300
Kraibon 0.5	-	0.49	1740

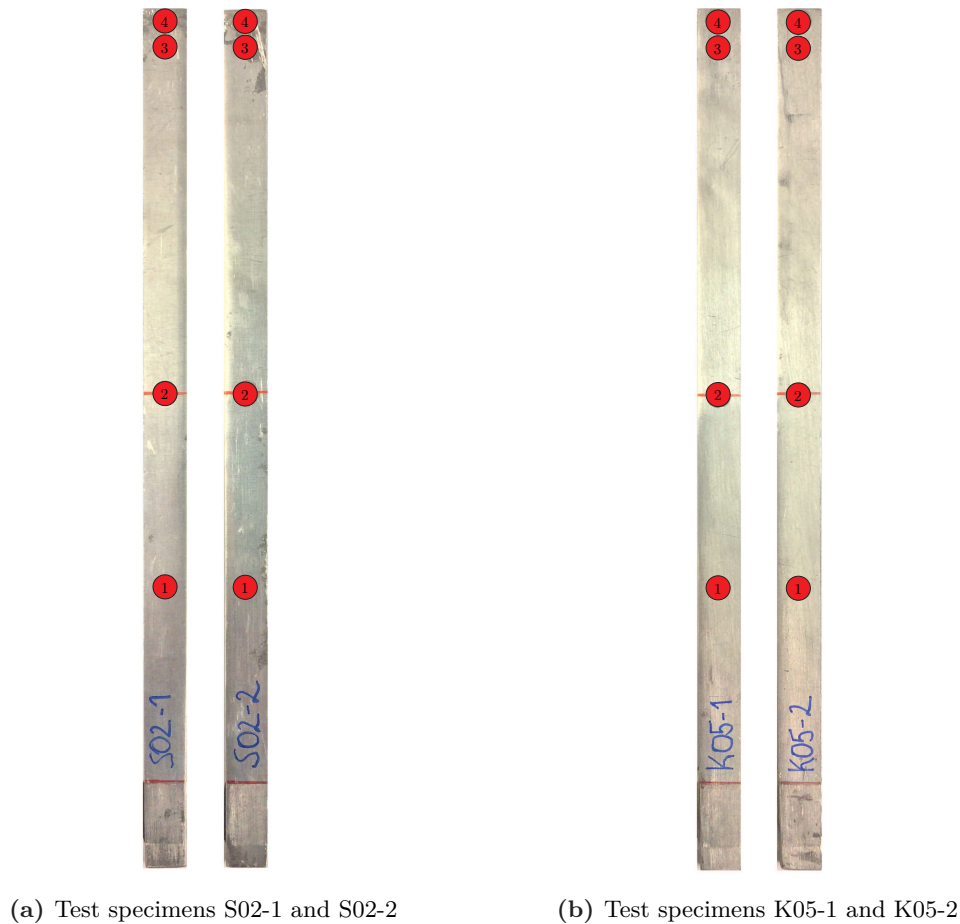


Figure 8.23 Sandwich specimens for viscoelastic material characterisation

Table 8.9 Geometric properties of the test specimens - B

Length	Width	Layer Thickness			
		Base	Smacwrap ST 0.2	Kraibon 0.5	Constraining
ℓ	b	h_b	h_v	h_v	h_c
180	10	1	0,2	0,5	1

* values in mm

Table 8.10 lists the measured mass of the test specimens and the corresponding theoretical values based on calculations using reference properties. Once again, one should expect slight differences between reference and current dimensions, so it is crucial to track the final weight to ensure reliability. Calculations for the theoretical values of the mass in specimens with viscoelastic materials neglect the mass of the adhesive layer. This inspection shows marginal differences between expected and measured masses, but one should emphasise the importance of considering more than one specimen per set since slight differences between them are always possible.

To identify the mechanical properties of the viscoelastic materials in this Procedure B, one considered the measured and estimated transmissibility functions at four different sections (see Figure 8.23) - Section 1: 45 mm, Section 2: 90 mm, Section 3: 171 mm and Section 4: 177.5 mm. In addition to the number of specimens of each set, the main goal is to consider more than two section measurements to obtain a reliable estimation of mechanical properties. The

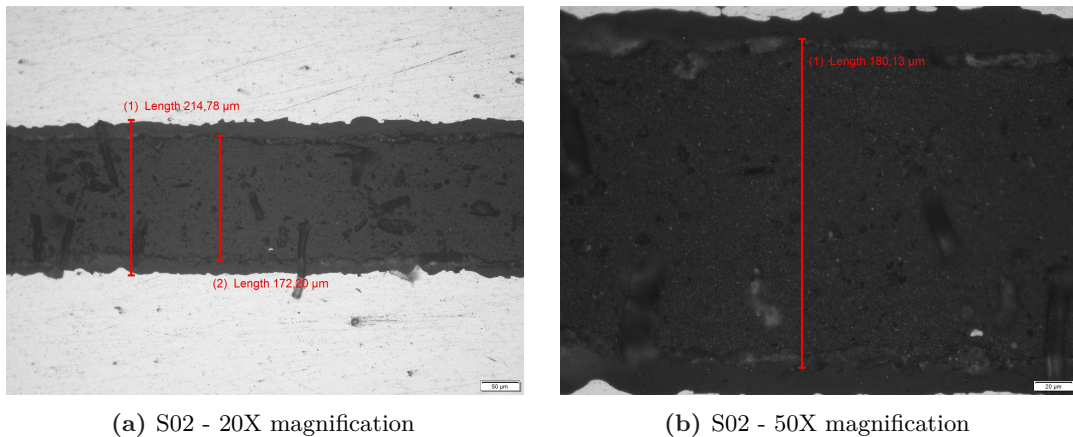
Table 8.10 Final mass [g] of the test specimens - B

	Measured	Theoretical	Relative Difference
Smacwrap ST			
1	11.308	11.352	0.39 %
2	11.285	11.352	0.59 %
Kraibon			
1	12.582	12.572	0.10 %
2	12.824	12.572	1.97 %

optimisation procedure considers the sum minimisation of the objective function presented earlier for each specimen section. Therefore, the fitting process relies on numerical transmissibility functions simulated with the same material properties for different transmissibility functions of two specimens. In Procedure B, one followed this global optimisation strategy for both sets of samples with distinct viscoelastic materials.

8.3.2.1 | Viscoelastic Specimens: S02

Similarly to Procedure A, one prepared a section cut of each set of specimens for microscopic analysis using the same technique. Figure 8.24 shows the damping and adhesive layers of the S02 samples in detail, including measurements. The thickness of the damping and adhesive layers are roughly 180 μm and 10 μm , respectively, which one considered in the FEM.

**Figure 8.24** Cross sections of the S02 viscoelastic specimen

Figures 8.25 to 8.28 show the experimental transmissibility functions of both specimens for each section in a frequency range between 0 and 2400 Hz with 1 Hz resolution and numerically fitted transmissibility functions. It is worth mentioning that the final viscoelastic properties used to represent the numerical transmissibility functions are the same for all figures since the global optimisation procedure described above considers the complete set of sections and specimens. Table 8.11 presents the final values from the optimisation procedure using the four-parameter fractional derivative model.

Despite the minimal deviations between specimens, Figures 8.25 to 8.28 show a near-perfect fit for any section. Interestingly, variations between experimental transmissibility functions are distinguishable for the tip measurements (Sections 3 and 4) and higher frequency ranges.

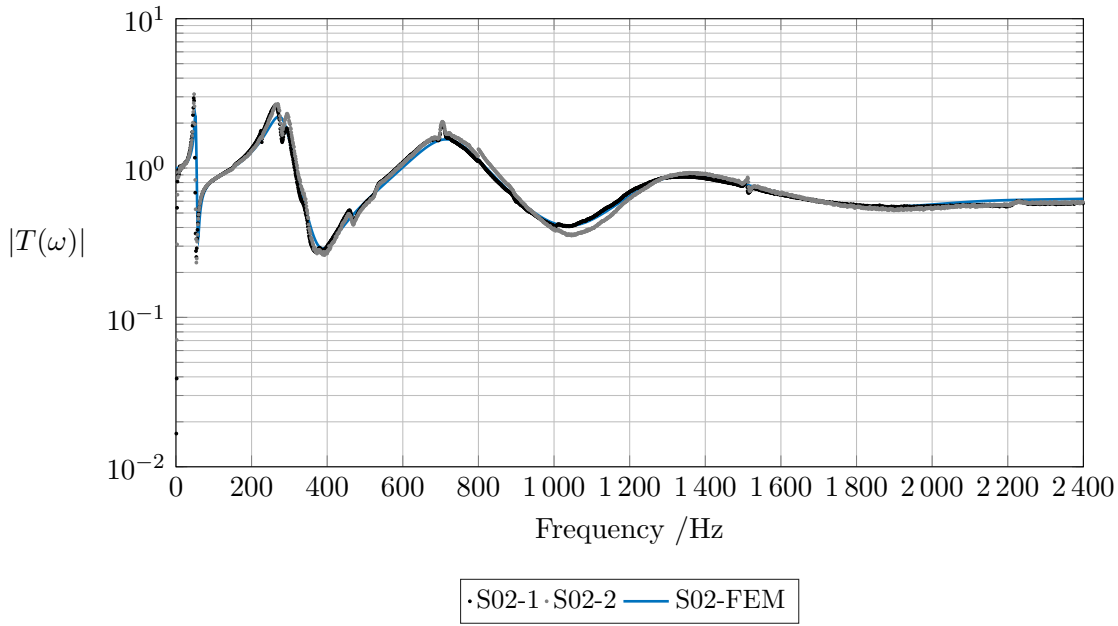


Figure 8.25 Transmissibilities of the S02 viscoelastic specimens - Section 1

It is even more interesting to note that the numerical transmissibility function obtained with optimised viscoelastic properties lies between experimental curves of both specimens, particularly for Section 3, as if it were an averaged transmissibility function. The four parameters of the fractional derivative model presented in Table 8.11 are within the same range of the parameters estimated for the same material in the previous procedure. Figure 8.29 shows the final properties compared to those in Procedure A.

Table 8.11 Optimisation results of the viscoelastic specimens S02

Range	E_0	E_∞	τ	α	Objective Function
Initial Value					
	1.0000×10^6	1.0000×10^9	1.0000×10^{-6}	0.5	
[0, 2400] Hz	1.5408×10^6	1.3907×10^9	5.0933×10^{-8}	0.4827	158.8350

However, one can notice a meaningful difference between the 0.2 mm and 0.1 mm thick viscoelastic layer. Procedure B led to an estimation of stiffer and lower damping material compared to Procedure A. Such a result raises some worth discussing questions. Do these differences come from the fact that different procedures are at stake and the autoclave conditions change the material properties, for example? Or does the thickness of the damping layer influence the estimated properties? At this point, it is not possible to definitively answer these questions within the scope of this work. Nonetheless, one can add that one embeds these materials in composite structures, typically manufactured with processes involving high temperatures and pressures. So, if the manufacturing process might change, even to a small extent, the estimated mechanical properties of this damping material, one considers the mechanical properties obtained in Procedure B.

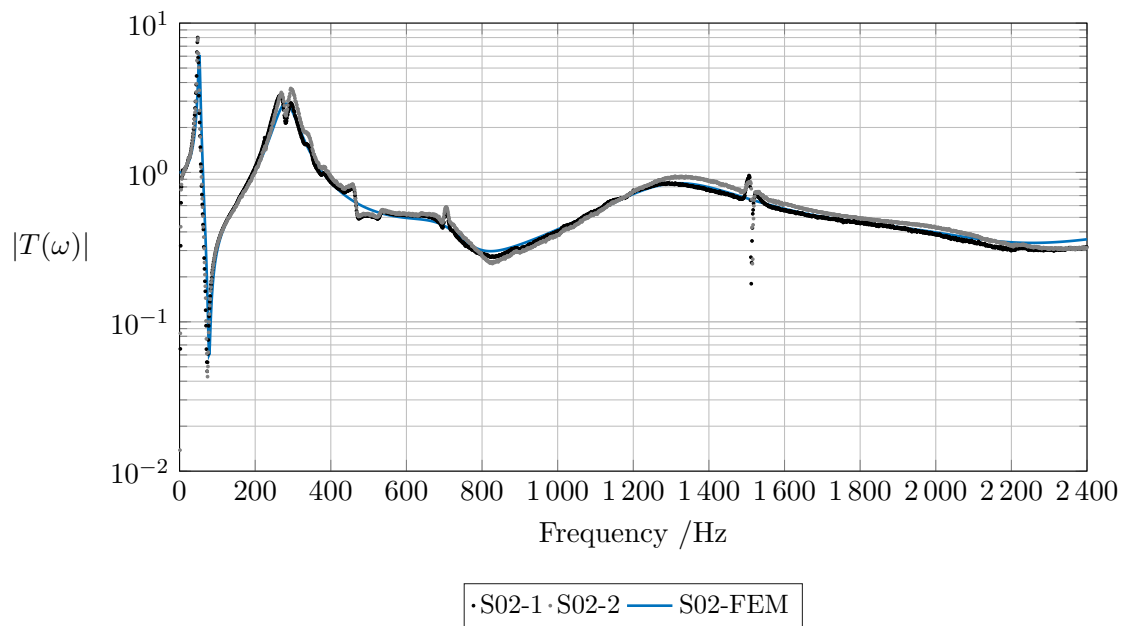


Figure 8.26 Transmissibilities of the S02 viscoelastic specimens - Section 2

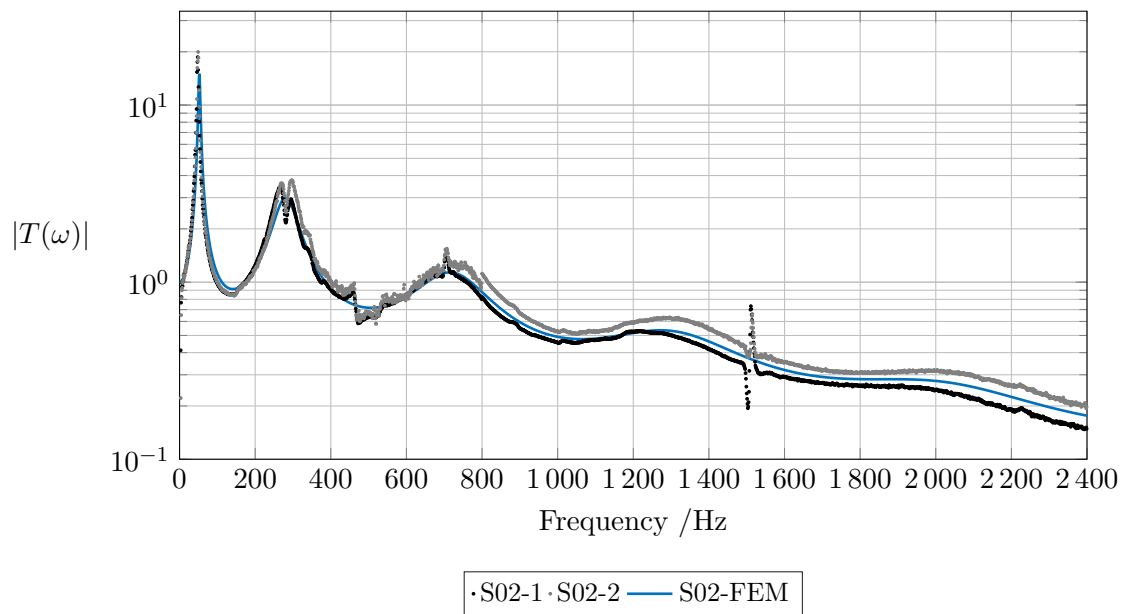


Figure 8.27 Transmissibilities of the S02 viscoelastic specimens - Section 3

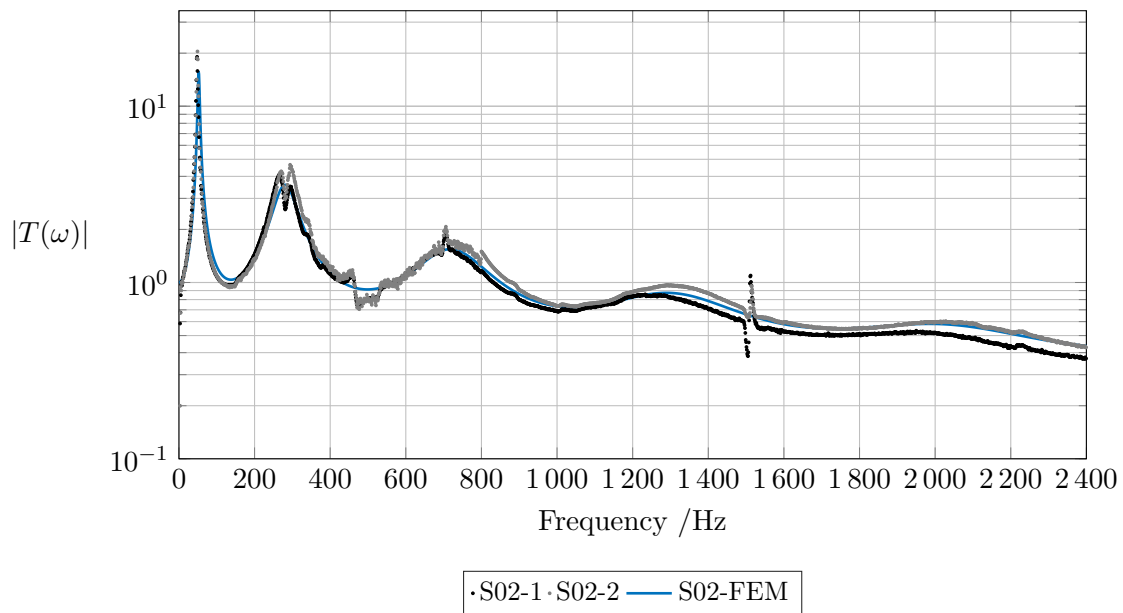


Figure 8.28 Transmissibilities of the S02 viscoelastic specimens - Section 4

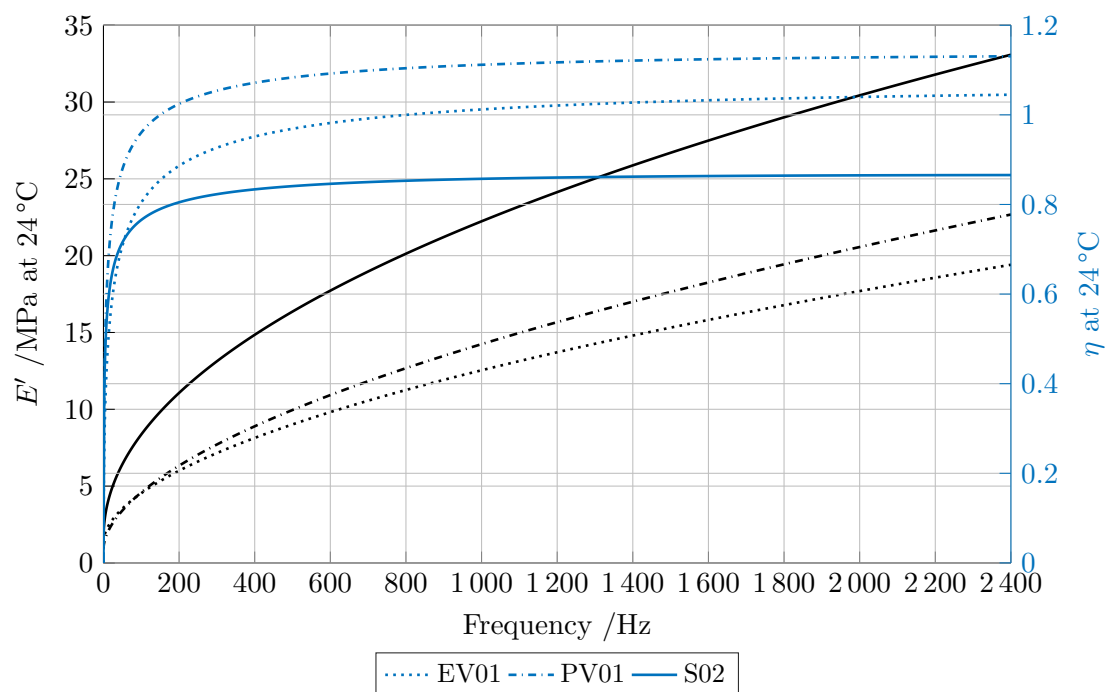


Figure 8.29 Identified viscoelastic material properties of S02 specimens

8.3.2.2 | Viscoelastic Specimens: K05

The manufacturing process of the specimens using Kraibon viscoelastic material followed the same conditions as previously detailed. Figure 8.30 shows microscope photographs with different magnifications of the damping layer. Measurements prove the 0.5 mm thickness of the damping layer but also show that one can not distinguish between damping and adhesive layers for this material. Presumably, the damping material accommodated the adhesive in its porous surface, probably designed to enhance adhesion, and the curing reaction merged both materials.

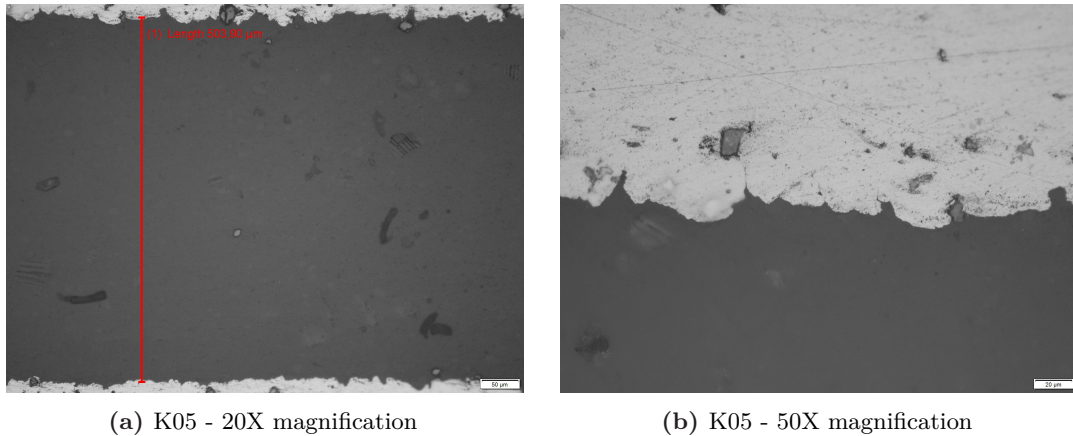


Figure 8.30 Cross sections of the K05 viscoelastic specimens

Figures 8.31 to 8.34 show the experimental and numerical fitted transmissibility functions for the considered sections. One can notice discrepancies between experimental transmissibilities for higher frequencies, particularly for third and fourth resonances. Considering that both specimens were manufactured using the same materials and subjected to the same manufacturing conditions, these results prove the need for several samples to ensure reliability. Nevertheless, the global optimisation procedure using four sections of two specimens proved its reliability for the Smacwrap ST material. Table 8.12 presents the optimised four parameters of the fractional derivative model. The high value of the objective function shows the not-so-good fit that one can notice in Figures 8.31 to 8.34. Apart from discrepancies between the experimental transmissibilities of both specimens, the numerically fitted transmissibilities tend to overestimate the third and fourth resonances, which probably leads to an overestimation of the material storage modulus - Figure 8.35.

Table 8.12 Optimisation results of the viscoelastic specimens K05

Range	E_0	E_∞	τ	α	Objective Function
Initial Value					
	1.0000×10^6	1.0000×10^9	1.0000×10^{-6}	0.5	
[0, 2400] Hz	4.8542×10^6	8.0067×10^9	2.3725×10^{-8}	0.3316	911.2703

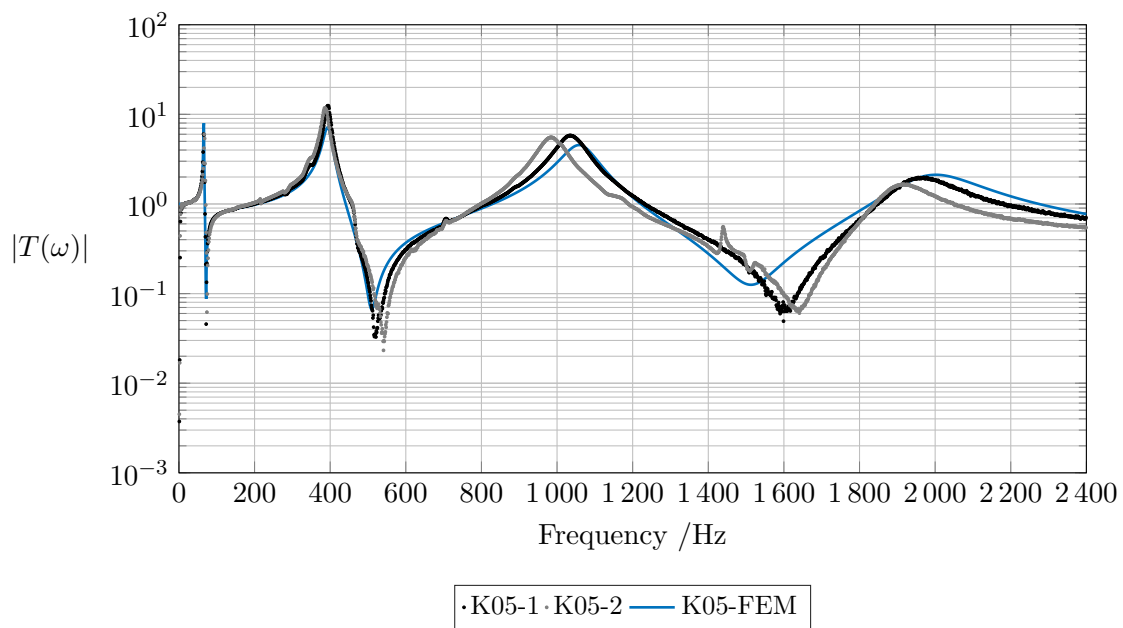


Figure 8.31 Transmissibilities of the K05 viscoelastic specimens - Section 1

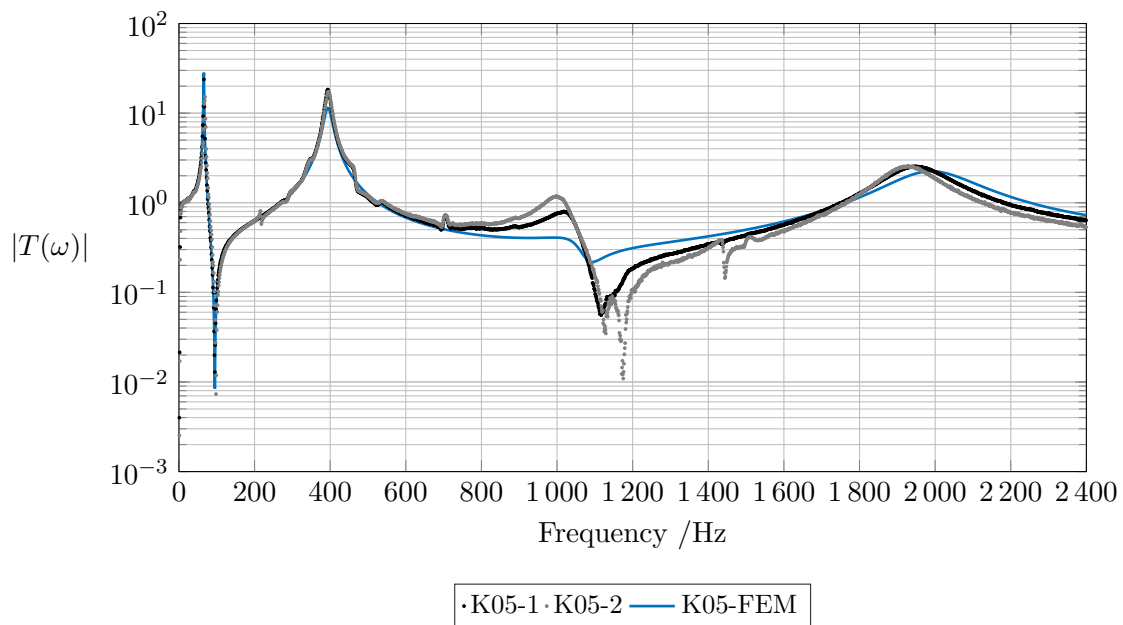


Figure 8.32 Transmissibilities of the K05 viscoelastic specimens - Section 2

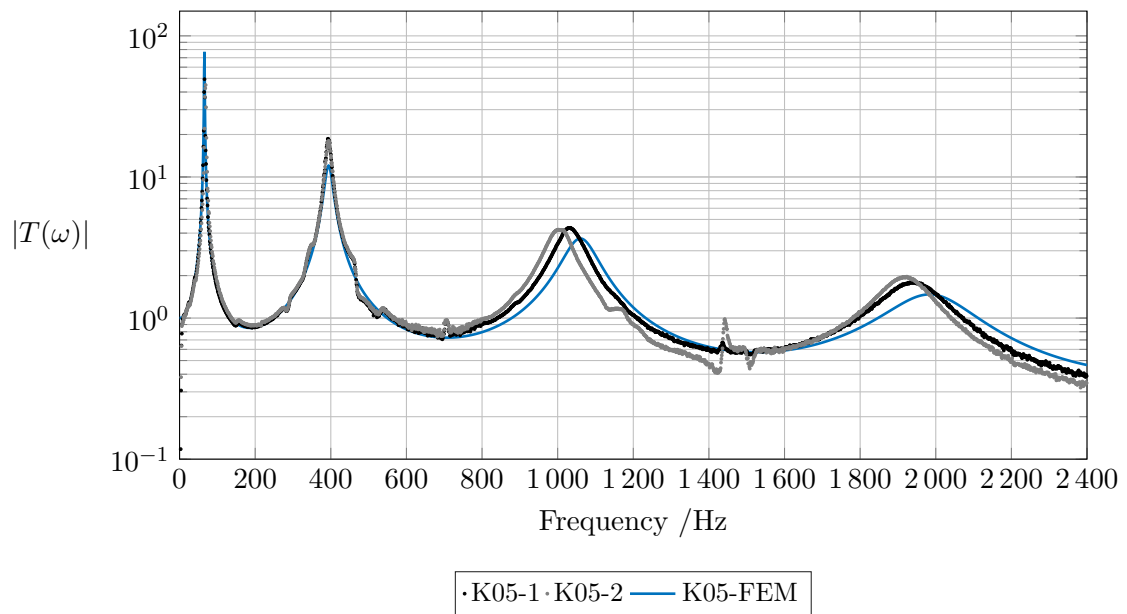


Figure 8.33 Transmissibilities of the K05 viscoelastic specimens - Section 3

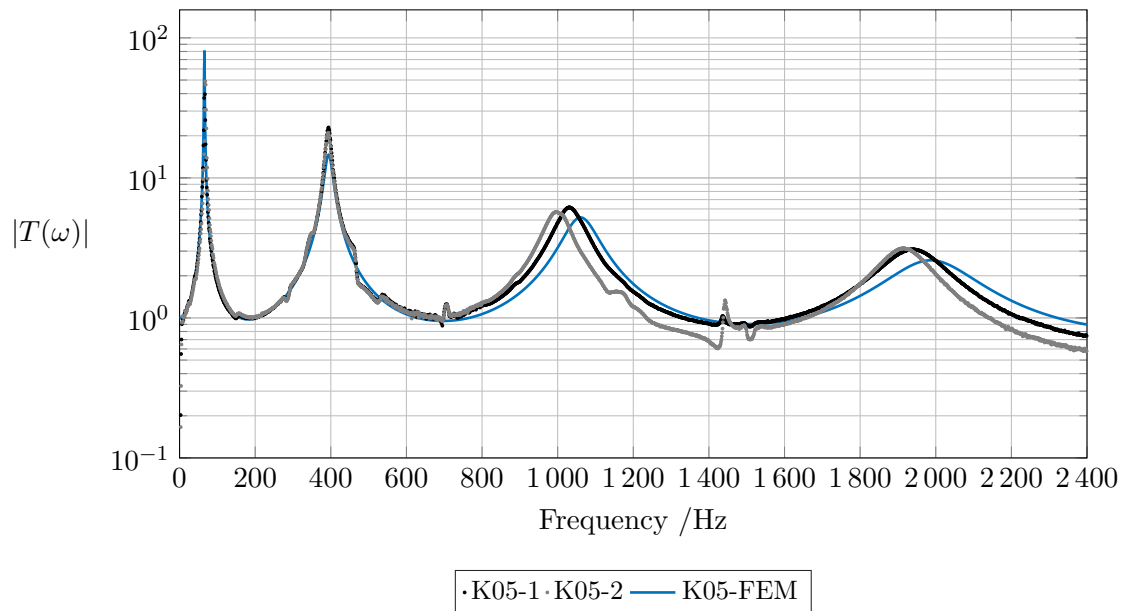


Figure 8.34 Transmissibilities of the K05 viscoelastic specimens - Section 4

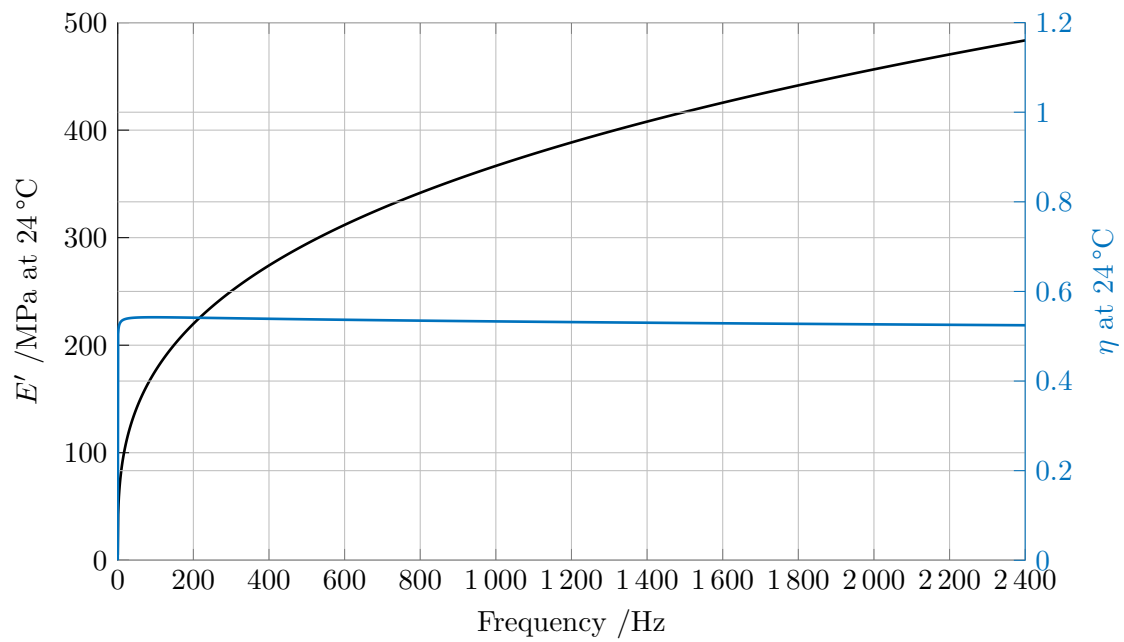


Figure 8.35 Identified viscoelastic material properties of K05 specimens

Composite Laminates with Interleaved Viscoelastic Materials

9.1 | Introduction

This chapter deals with the experimental campaign carried out involving the use of viscoelastic damping materials to enhance damping in composite laminates. In particular, one investigates the applicability of interleaving viscoelastic damping patches in the stacking sequence to increase damping without compromising stiffness. The viscoelastic materials studied in this work, Smacwrap ST and Kraibon, with the mechanical properties identified in Chapter 2, were embedded in composite laminate plates and tested experimentally. The material properties of the viscoelastic materials previously identified are valuable here to compare experimental and numerical investigations. The experimental procedure described in the following chapter involved manufacturing the composite laminate plates with interleaved viscoelastic materials, either in the classical configuration, interleaved viscoelastic layers within the stacking sequence, or partially distributed damping treatments as investigated in this work, as interleaved viscoelastic patches.

9.2 | Specimens and Manufacturing Process

The composite plate specimens manufactured for this experimental campaign used Hexcel 8552 IM7 unidirectional prepreg plies, manually stacked in the clean room of the INEGI. Figures 9.1 to 9.3 present the proposed stacking sequences for the experimental testing of composite laminates, including a reference plate specimen with only CFRP layers used for comparison purposes - see Figure 9.1. Additionally, one considers three plate specimens with interleaved viscoelastic materials in different arrangements using Smacwrap ST 0.1 mm, Smacwrap ST 0.2 mm and Kraibon - Figures 9.2a, 9.2b and 9.2c, respectively. Finally, Figures 9.3a to 9.3c depict three stacking sequences for composite plate specimens with interleaved damping patches, including the same viscoelastic damping materials.

90°
0°
90°
90°
0°
90°

Figure 9.1 Stacking sequence of the reference CFRP plate specimen

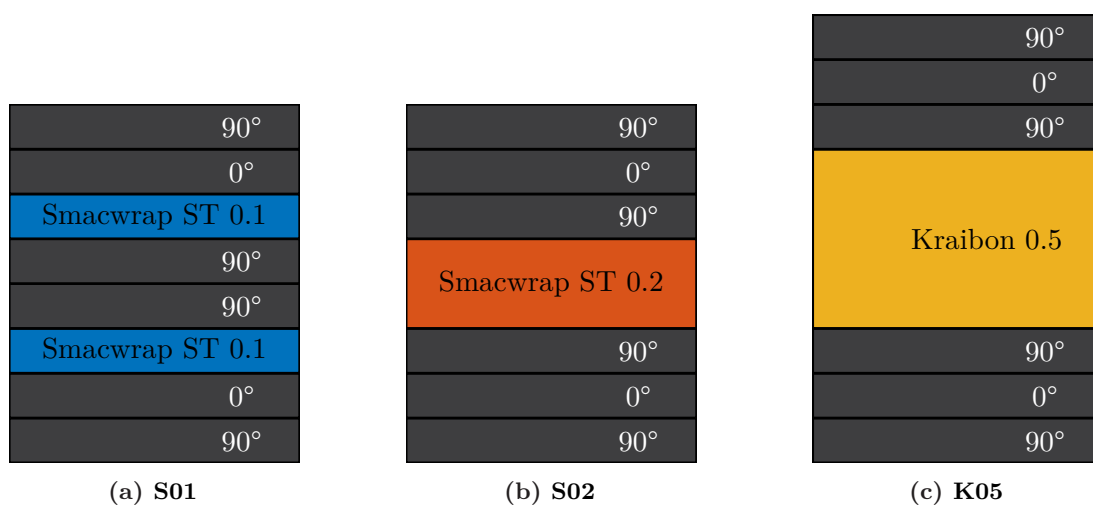


Figure 9.2 Stacking sequences of the CFRP plates with interleaved VEM damping layers

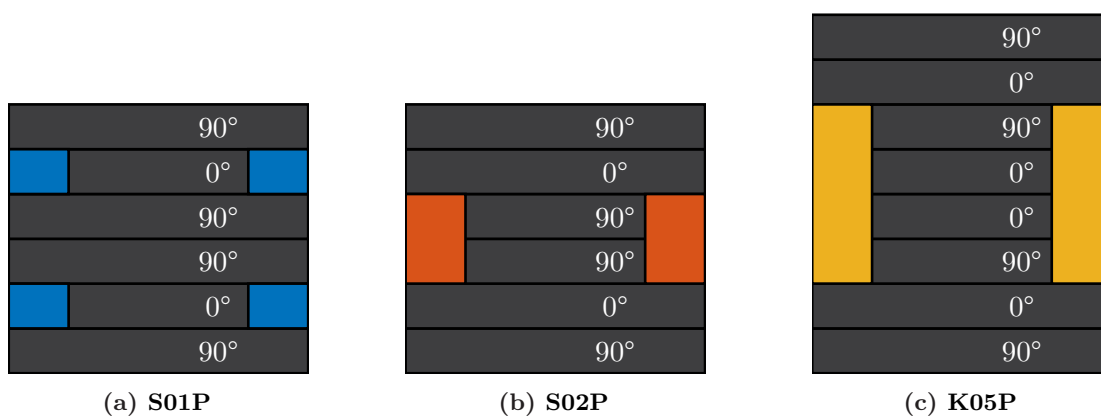


Figure 9.3 Stacking sequences of the CFRP plates with interleaved VEM damping patches

The proposed stacking sequences are symmetric and follow some worth-mentioning criteria. S01 and S02 specimens, Figures 9.2a and 9.2b, used the same material with different thicknesses and configurations to study the influence of the stacking sequence using the same amount of damping material. The K05 specimen uses the same configuration as the S02 specimen but with different damping material and thicknesses - see Figure 9.2c.

The manufacturing of specimens with partially distributed damping material followed some crucial concerns. The main interest was to ensure, as much as possible, a constant laminate thickness. Considering that each ply of the CFRP material is roughly 0.1 mm thick (measurements after the curing process registered an average thickness of 0.125 mm for each layer), the Smacwrap ST with 0.1 mm (S01P) was partially distributed together with one layer of CFRP and the Smacwrap ST with 0.2 mm (S02P) using two layers inserted between damping treatments - Figures 9.3a and 9.3b, respectively. Following the same reasoning, the partially distributed Kraibon specimen (K05P) includes four CFRP plies between damping treatments. At this point, one should note an important aspect regarding the timeline of this work. The manufacturing of the composite laminate specimens occurred before the microscopy analysis of the beam specimens discussed in Chapter 2, where one assessed the thicknesses of the viscoelastic materials. As previously proved, there are significant discrepancies in the thickness of the viscoelastic materials, and one should consider this issue in the results discussed later in the chapter. Furthermore, it is also important to mention that this experimental campaign occurred even before the published parametric analyses of the laminated composite plates with interleaved damping patches presented in the previous chapter. The proposed stacking sequences resemble the Edge-to-Centre configurations considered in the previous chapter, which proved to show worse results when compared to the Centre-to-Edge configurations. However, it is worth noting that practical considerations related to the manufacture of the specimens prevailed here. One favoured stacking sequences with interleaved damping patches requiring a single composite ply patch per layer to reduce the number of prepreg cuts, thus trying to reduce the probability of defects.

As previously reported, composite laminate plate specimens manufacturing used Hexcel 8552 IM7 unidirectional prepreg plies manually stacked in the cleanroom of INEGI. Figure 9.4 shows a photograph of the K05P coupon during the hand lay-up process. In particular, one can identify the Kraibon damping material partially distributed as two damping strips along the entire length of the composite laminate plate with a central ply of Hexcel 8552 IM7 unidirectional prepreg. The hand lay-up manufacture of composite laminates with partially distributed interleaved viscoelastic materials is a challenging procedure that requires diligence and caution. The plies of CFRP and VEM damping patches must be precisely cut and positioned, which is all the more difficult for thinner plies - the composite laminate plate with Smacwrap ST 0.1 mm interleaved damping patches was the most challenging specimen in this regard.

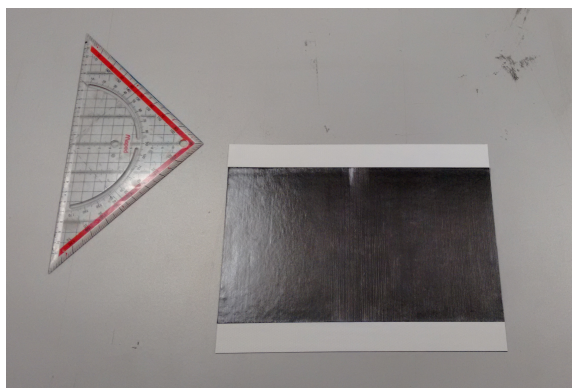


Figure 9.4 Hand lay-up process of the K05P specimen

When finally stacked, all seven specimens were carefully placed in the autoclave using a film release agent and a vacuum bag, Figures 9.5a and 9.5b, respectively. The final curing process occurred in the autoclave, with the cure cycle represented in Figure 9.6, and Table 9.1 presents the expected mechanical properties of the Hexcel 8552 IM7 UD plies. Besides, one considers the mechanical properties of the viscoelastic materials identified in the previous chapter to compare experimental and numerical results.

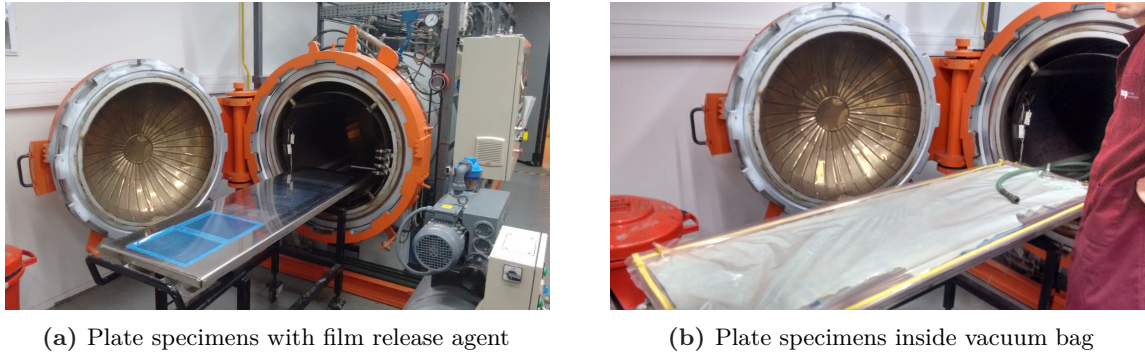


Figure 9.5 Preparation of the plate specimens for autoclave curing process

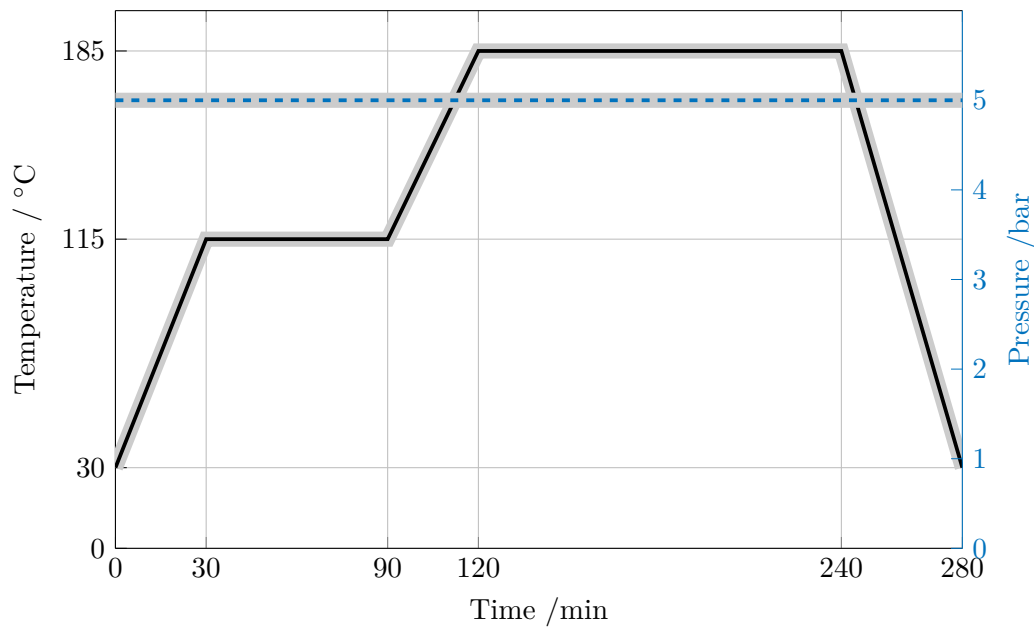


Figure 9.6 Curing cycle of the composite specimens

Table 9.1 Reference material properties of the Hexcel 8552-IM7 UD CFRP

Tensile Modulus GPa		Poisson			Shear Modulus GPa			Density kg.m^{-3}
E_{11}	E_{22}	ν_{12}	ν_{13}	ν_{23}	G_{12}	G_{13}	G_{23}	ρ
161	11.38	0.32	0.32	0.45	5.2	5.2	3.9	1570

9.3 | Experimental Procedure

For the experimental procedure, one suspended the specimens in a rigid frame using thin nylon wires to obtain the experimental FRF considering free boundary conditions. The nylon wires, bonded to the plate specimens using a cyanoacrylate adhesive, are 43 mm away from the plate sides, as represented in Figure 9.7. Figure 9.7 includes the plate's dimensions and the positioning of the interleaved damping patches (coloured grey). The regular centred grid of Figure 9.7 provided the basis for the 15 measuring points considered for the FRF measurements.

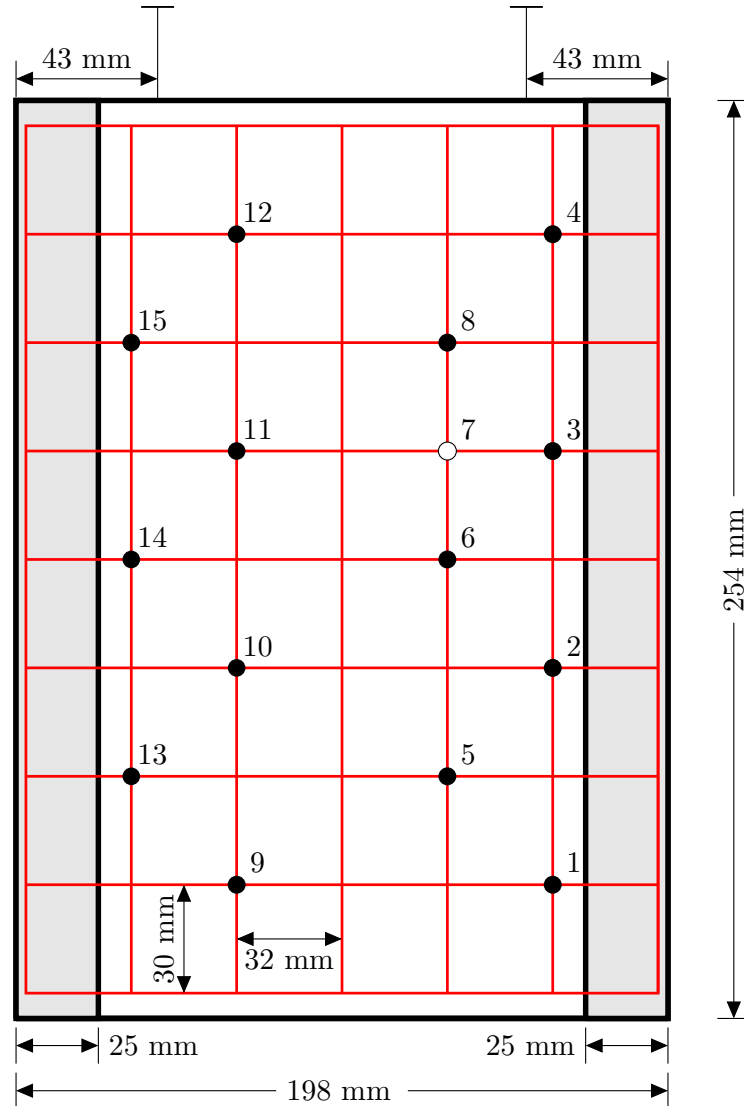


Figure 9.7 Plate specimens dimensions and measurement points over regular grid

In addition, Figure 9.8 illustrates the proposed experimental procedure scheme - the signal analyser unit (1) manages the whole setup: it generates and sends an excitation signal to the shaker amplifier (2), which channels the amplified signal over a frequency range between 0 and 400 Hz to the electromagnetic shaker (3). The force transducer (5), connected to the plate specimen using a screw bonded with cyanoacrylate adhesive, measures the excitation force, and the signal analyser unit (1) records this signal.

Besides, the laser vibrometer sensor head (6) measures the specimen's velocity at a given

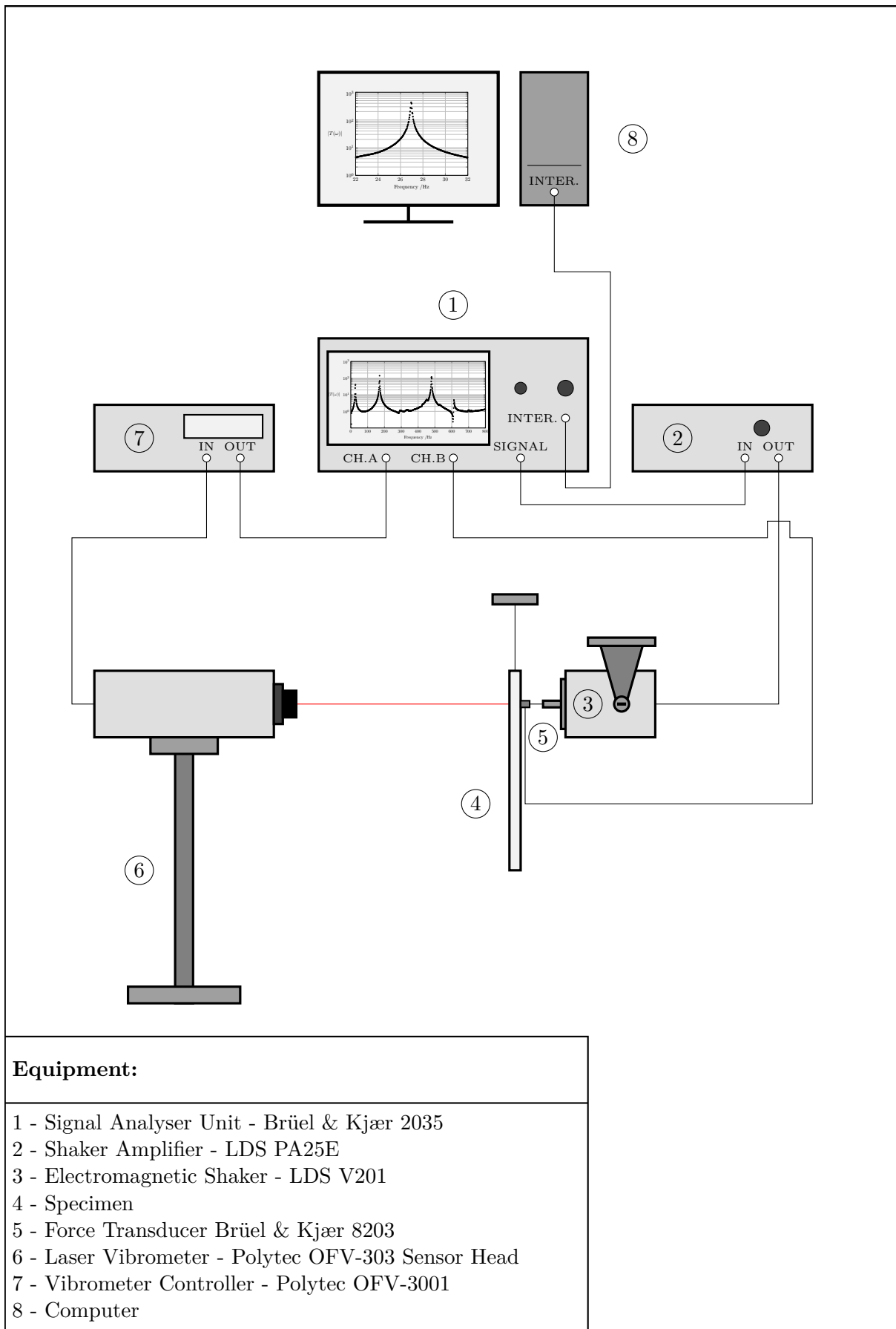
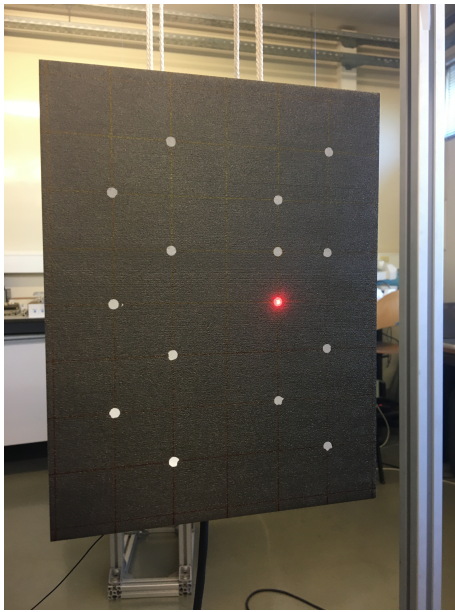


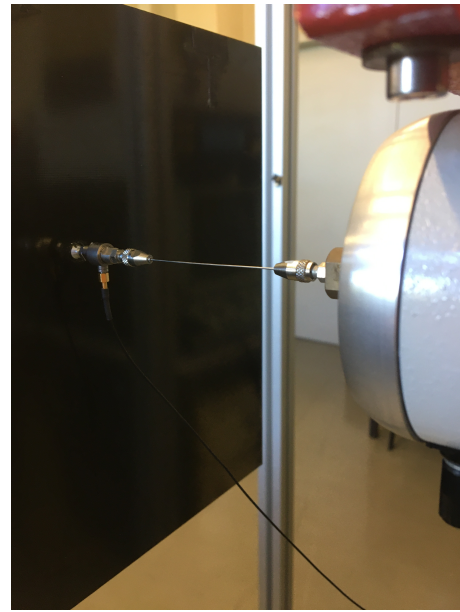
Figure 9.8 Measurement setup
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point, sending the signal to the vibrometer controller (7), which in turn delivers the response velocity to the signal analyser unit (1). Therefore, the signal analyser unit (1) receives the excitation force and measured velocity, and one can manage the signal analyser unit data via computer (8). One measured the FRF of the 15 points depicted in Figure 9.7, considering point 7 as the excitation point.

Additionally, for a thorough description of the experimental procedure, Figure 9.9 presents a detailed view of the measurement grid and excitation setup. In particular, Figure 9.9a exhibits a composite laminate specimen suspended in a rigid frame with the grid points visible, including the laser signal. Furthermore, Figure 9.9b highlights the excitation setup, including the electromagnetic shaker, rod and force transducer.



(a) Measurement grid and laser signal



(b) Electromagnetic shaker, rod and force transducer

Figure 9.9 Detailed view of the measurement grid and excitation setup

9.4 | Experimental Frequency Response Functions

The exploratory nature of the experimental campaign conducted in this study and the challenges that the proposed idea presents limited the number of specimens analysed. One considered one specimen for each configuration in Figure 9.3. The composite laminate manufacturing with thin damping materials posed challenges due to their heightened vulnerability to imperfections. Some specimens show some warping, specifically those with thin damping layers, S01 and S01P. In particular, the S01P specimen presents two stable configurations (bi-stable composite). Future studies with more specimens are recommended to conduct an experimental campaign with a heightened focus on refining the production process to address this issue. Despite such constraints, one aims to achieve valuable insights and provide a foundation for future research.

In the following sections, one aims to accomplish such meaningful conclusions by conducting a series of studies involving the direct comparison between experimental and numerical frequency response functions, but also detailed comparisons between different damping configurations involving CFRP with interleaved damping layers and interleaved damping patches.

9.4.1 | Comparison with Numerical Frequency Response Functions

In the evaluation of laminated composite plates with interleaved damping material, establishing a meaningful correspondence between experimental and numerical outcomes is enhanced when focusing on rigorous analysis of frequency response functions. The choice of frequency response functions over other standard procedures like experimental modal analysis, for example, becomes imperative due to the nuanced nature of modal parameters, which might not ease a precise comparison. This issue is particularly evident in composite laminates, where closely spaced resonances can obscure the accuracy of modal parameter matching. On the other hand, frequency response functions offer distinct advantages in such scenarios. Their inherent characteristics make them less susceptible to the influence of measurement noise and mitigate the impact of cumulative computational errors. Consequently, favouring frequency response functions ensures a more reliable and robust comparison between experimental and numerical results for laminated composite plates with embedded damping materials.

The comprehensive experimental campaign to study damping in composite laminates by embedding viscoelastic damping materials considered diverse materials sourced from various manufacturers. The acquisition of prior knowledge regarding the mechanical properties of these materials proved instrumental in enabling a meaningful comparison between experimental and numerical investigations. The inclusion of materials from different manufacturers added a layer of complexity and diversity to the study, allowing for a more robust evaluation of damping enhancements in composite laminates.

Figures 9.10 to 9.16 show the overlapping of experimental and numerical frequency response functions of the REF, S01, S02, K05, S01P, S02P and K05P, respectively, for comparison purposes. For each specimen, one presents the point mobility and one transfer mobility frequency response function to promote a better comparison between experimental and numerical results. One considers a frequency range between 0 and 400 Hz with a 0.5 Hz resolution.

Firstly, one compares the experimental and numerical frequency response functions of the composite laminates with interleaved damping layers - REF, S01, S02 and K05 specimens. Overall, thicker composite laminates, S02 and K05, Figures 9.12 and 9.13, respectively, present an almost perfect fit. Taking into account the number of variables at stake in such comparisons, from the material properties of the UD CFRP to the material properties of the viscoelastic damping materials estimated in the previous chapter and considering the manufacturing process itself, these results are nonetheless quite remarkable. On the other hand, thinner composite laminates, REF and S01, Figures 9.10 and 9.11, respectively, show significant discrepancies between the experimental and numerical frequency response functions. As previously reported, thinner specimens present some warping, thus making it difficult to compare the numerical and experimental results. Indeed, the consequences of the manufacturing process on thin composite laminates are probably what best explains the difference between a good fit and a less good fit. The geometric and mechanical properties of the CFRP plies are the same for all specimens, and so are the estimated mechanical properties of both viscoelastic damping materials. The comparison for S01 and S02 specimens considers the same mechanical properties, and both exhibit different fitness levels between experimental and numerical frequency response functions.

Manufacturing thin composite laminates presents several challenges due to their reduced thickness. This inherent thinness heightens the susceptibility to imperfections throughout the manufacturing process, and the impact of irregularities, such as voids or resin-rich areas, is magnified, posing a potential threat to the structural integrity and overall performance of the composite. Even minor imperfections can exert a disproportionate influence on the structural soundness of the composite. Delicacy becomes a defining characteristic of thin laminates, requiring meticulous handling across various manufacturing stages. Processes like cutting and shaping

demand heightened precision to avoid damage such as delamination. The challenge extends to achieving uniform resin distribution, wherein variations in thickness can result in uneven resin dispersion. Sensitivity to fluctuations in processing conditions, including temperature and pressure, is accentuated, and ensuring optimal conditions throughout the manufacturing process becomes paramount to prevent defects and secure the desired mechanical properties. One used fragile materials such as thin plies and damping patches to manufacture the specimens proposed in this work. Handling these materials without causing damage requires specialised techniques and equipment to reduce the risk of material failure. Thin laminates, more susceptible to distortions and deformations during processes like curing, require rigorous control measures essential for maintaining dimensional stability and meeting design specifications. Indeed, the manufacturing intricacies associated with thin composite laminates encompass an array of challenges, spanning handling complexities, resin distribution difficulties, and heightened sensitivity to processing conditions. Successfully navigating these challenges mandates the application of advanced manufacturing techniques, meticulous quality control, and a comprehensive understanding of material properties and behaviour at reduced thicknesses.

Embedding viscoelastic damping patches heightens this susceptibility to imperfections throughout the manufacturing process of thin composite laminates. Figures 9.14 to 9.16, presenting the comparisons between experimental and numerical frequency response functions of the specimens with interleaved damping patches S01P, S02P and K05P, prove just that. The overall fit quality between experimental and numerical frequency response functions is weaker when considering laminated composites with interleaved damping patches but improves with the laminate thickness. In a way, one could say that the fit quality improves as the thickness of the laminate with interleaved damping patches increases. The frequency response functions of the S02P specimen present a better fit than those of the S01P specimen, and those of the K05P specimen show the best fit between all the specimens with interleaved damping patches. The S01P specimen, in particular, presents two stable configurations with a considerable amount of warping due to the manufacturing issues discussed above. In addition, as proven by microscopic analysis in Appendix B, the final viscoelastic layer thicknesses are different from those listed by the suppliers, undermining the manufacturing process. The disparity in thicknesses makes it impossible to create a stacking sequence incorporating interleaved damping patches with consistently uniform thickness laminas. This misalignment adds to the structural intricacy, introducing supplementary variables and making the establishment of comparisons more intricate.

Amidst the intricate geometric complexities, it is essential to underscore that manufacturing composites with plies of different materials, often termed hybrid composites, introduces several challenges. The combination of diverse materials demands thoughtful consideration of their compatibility. Variances in thermal expansion coefficients, curing temperatures, and other material properties pose hurdles in achieving a seamless and bonded interface between the plies. Ensuring proper adhesion between dissimilar materials adds an extra layer of complexity to uphold the structural integrity of the composite. Establishing a robust bond at the interface between different plies may prove challenging when dealing with materials inherently resistant to bonding. Besides, different materials may exhibit distinct curing or setting rates, complicating the synchronisation of the manufacturing process. Plies with varying materials often showcase diverse mechanical properties, encompassing stiffness, strength, and thermal conductivity. Navigating these variations requires meticulous planning to obtain a well-balanced and optimised composite structure. The judicious selection of the arrangement and sequence of plies with different materials emerges as a pivotal decision in hybrid composites, requiring thorough analysis and optimisation during the design phase. Scaling from laboratory-scale manufacturing to large-scale production magnifies the inherent difficulties associated with hybrid composites. Ensuring consistent quality and performance across a large scale introduces additional challenges.

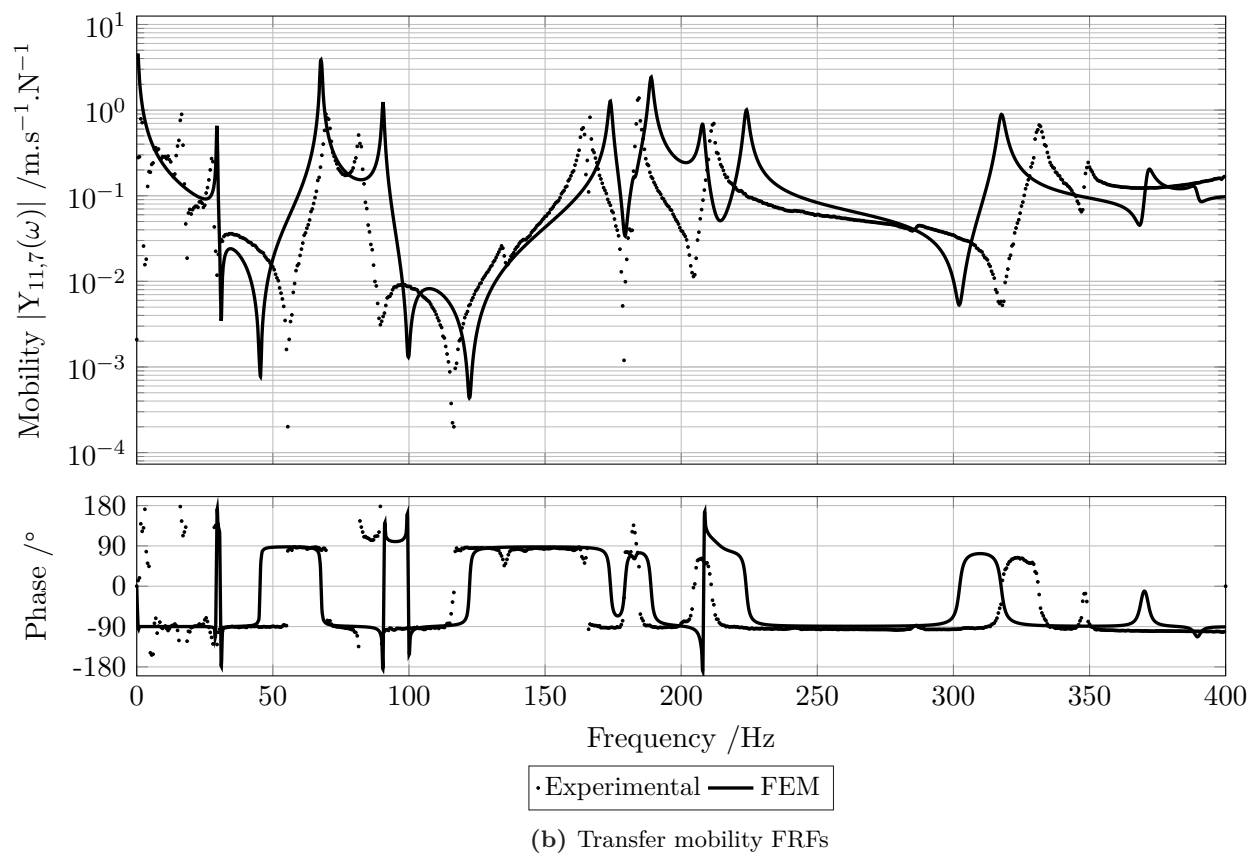
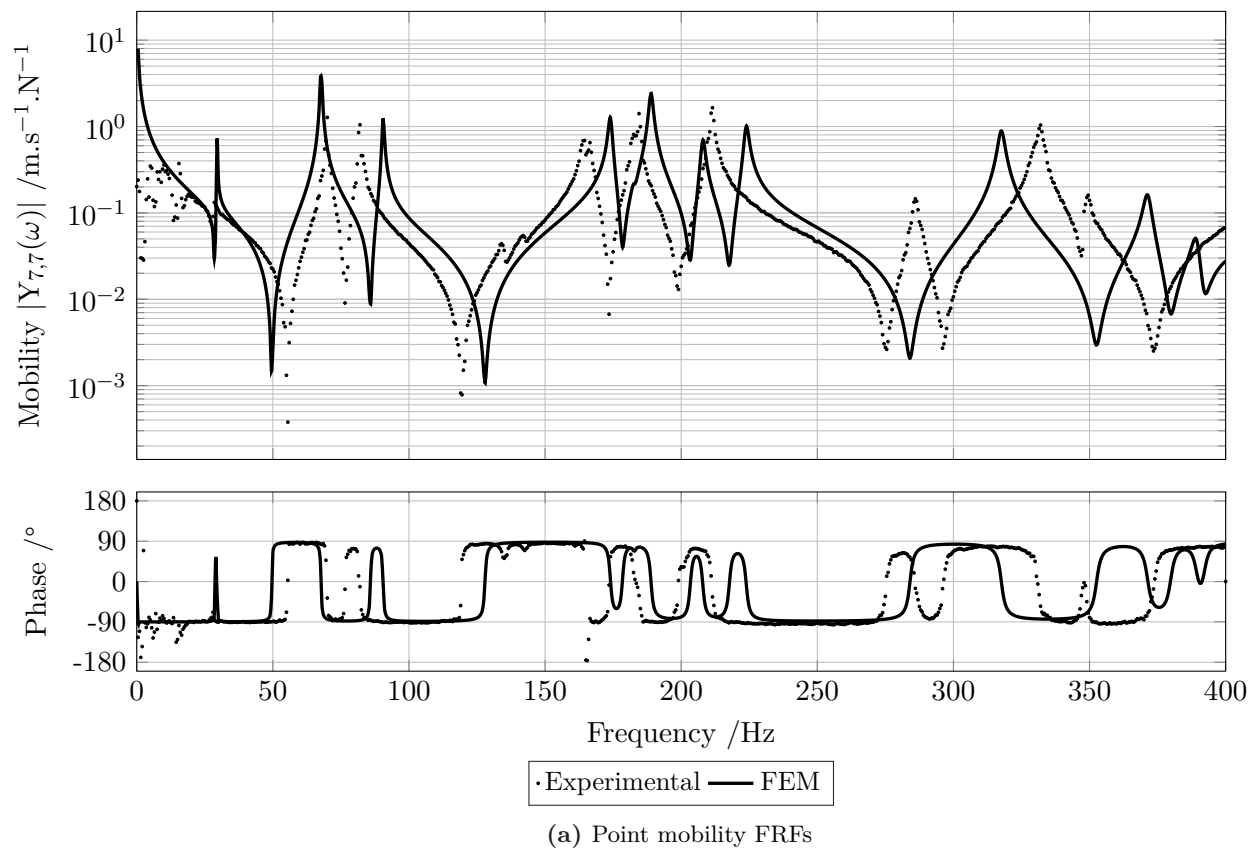
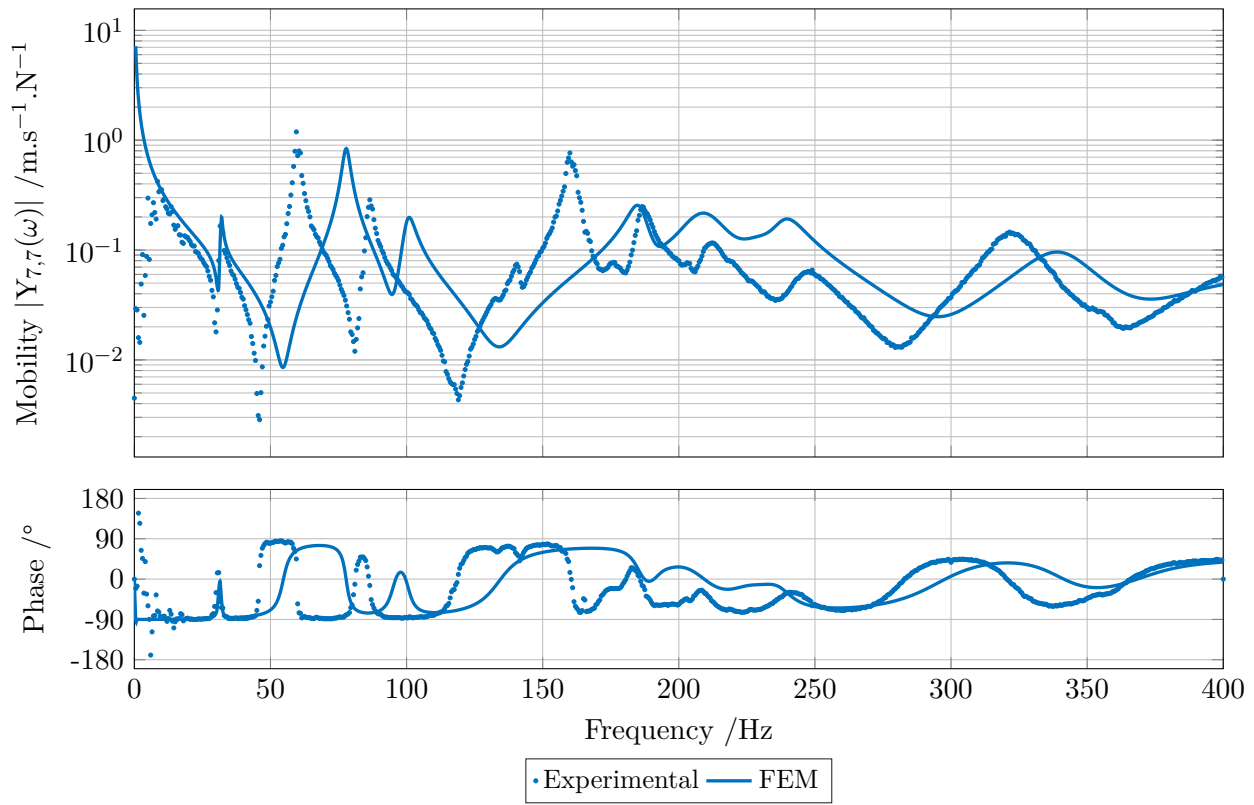
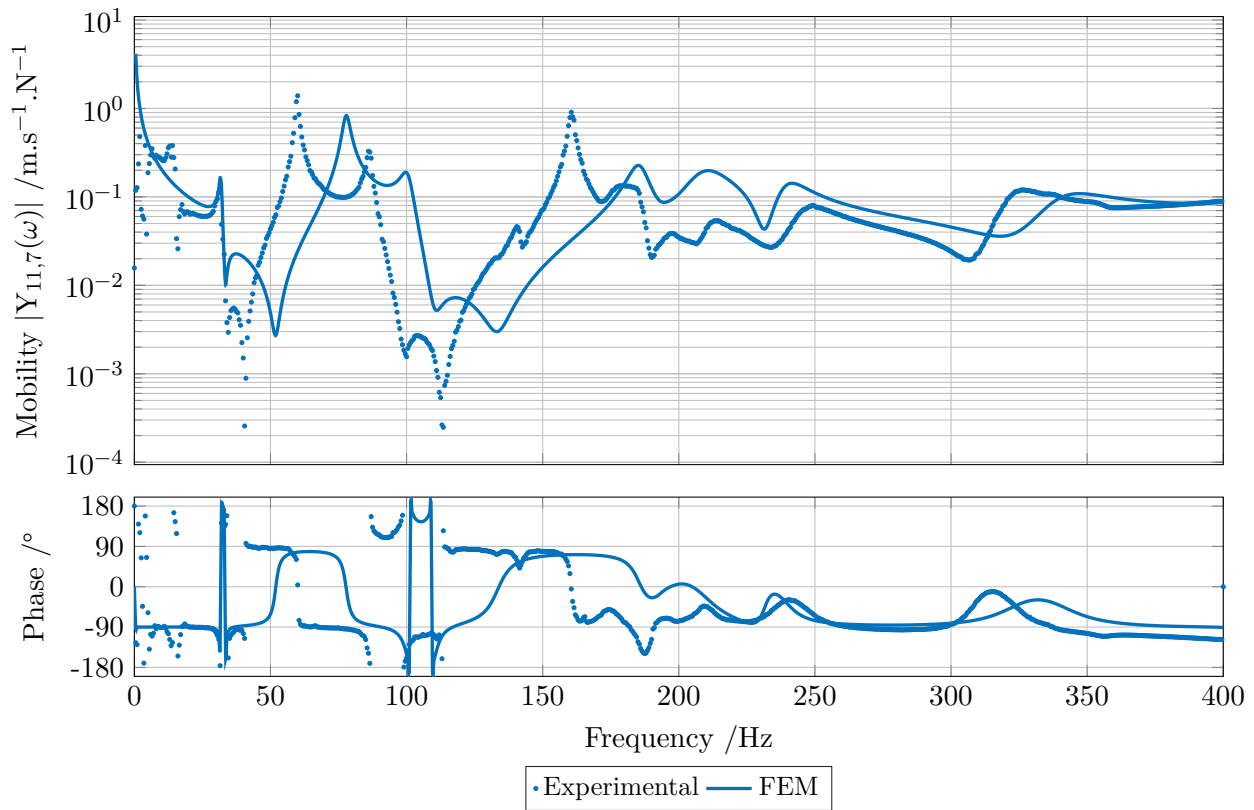


Figure 9.10 Mobility FRFs of the reference specimen

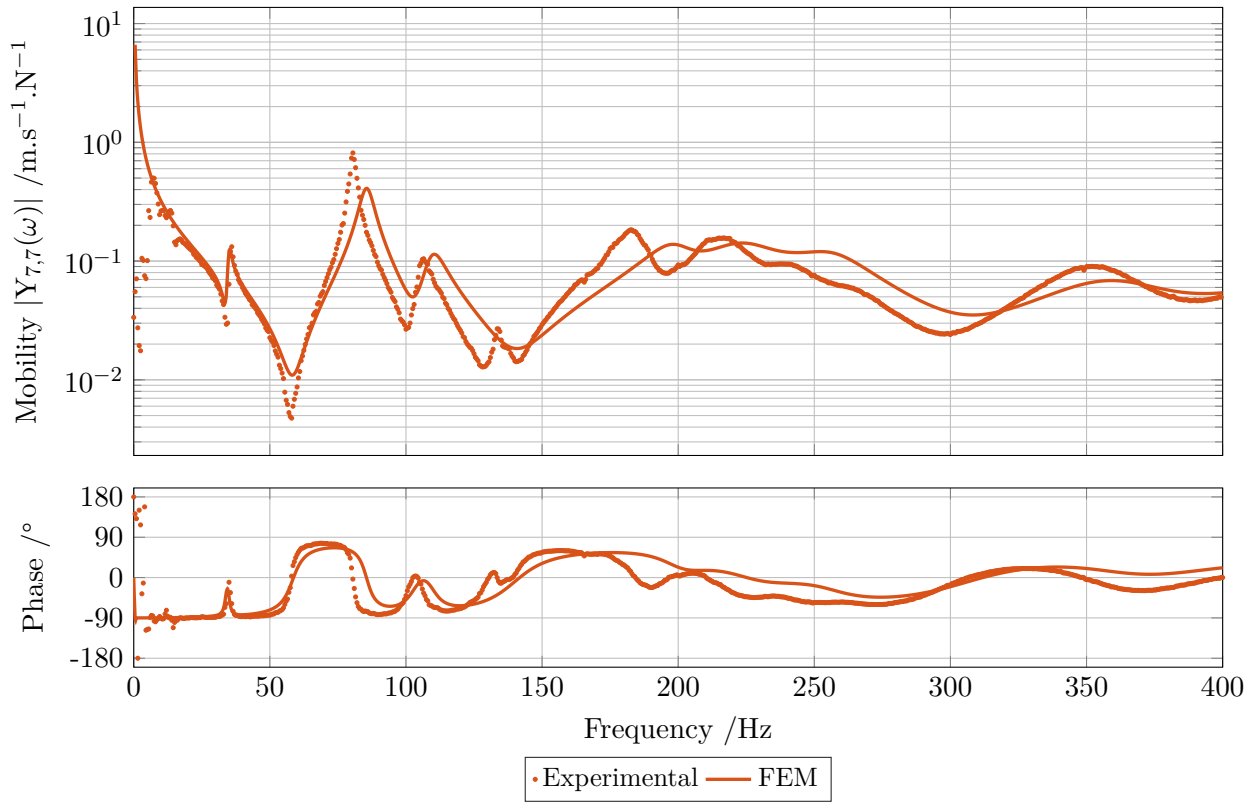


(a) Point mobility FRFs

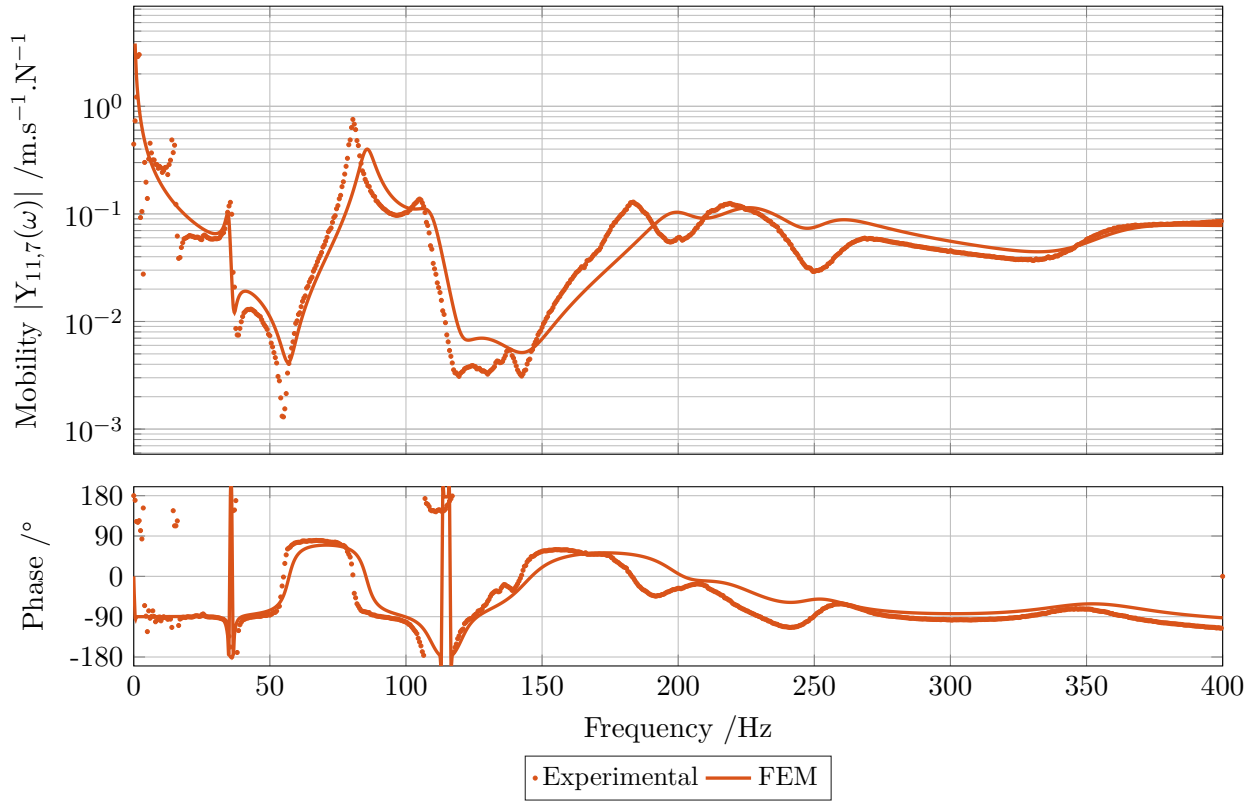


(b) Transfer mobility FRFs

Figure 9.11 Mobility FRFs of the S01 specimen

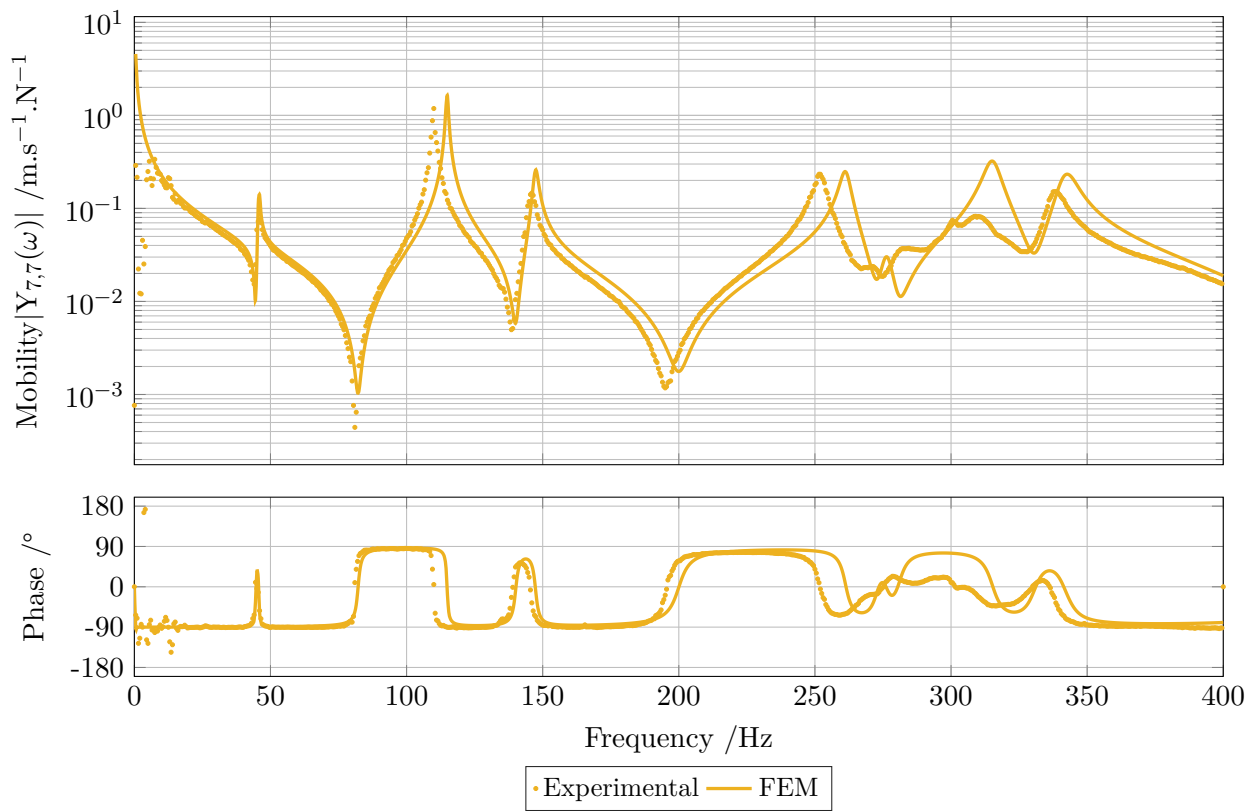


(a) Point mobility FRFs

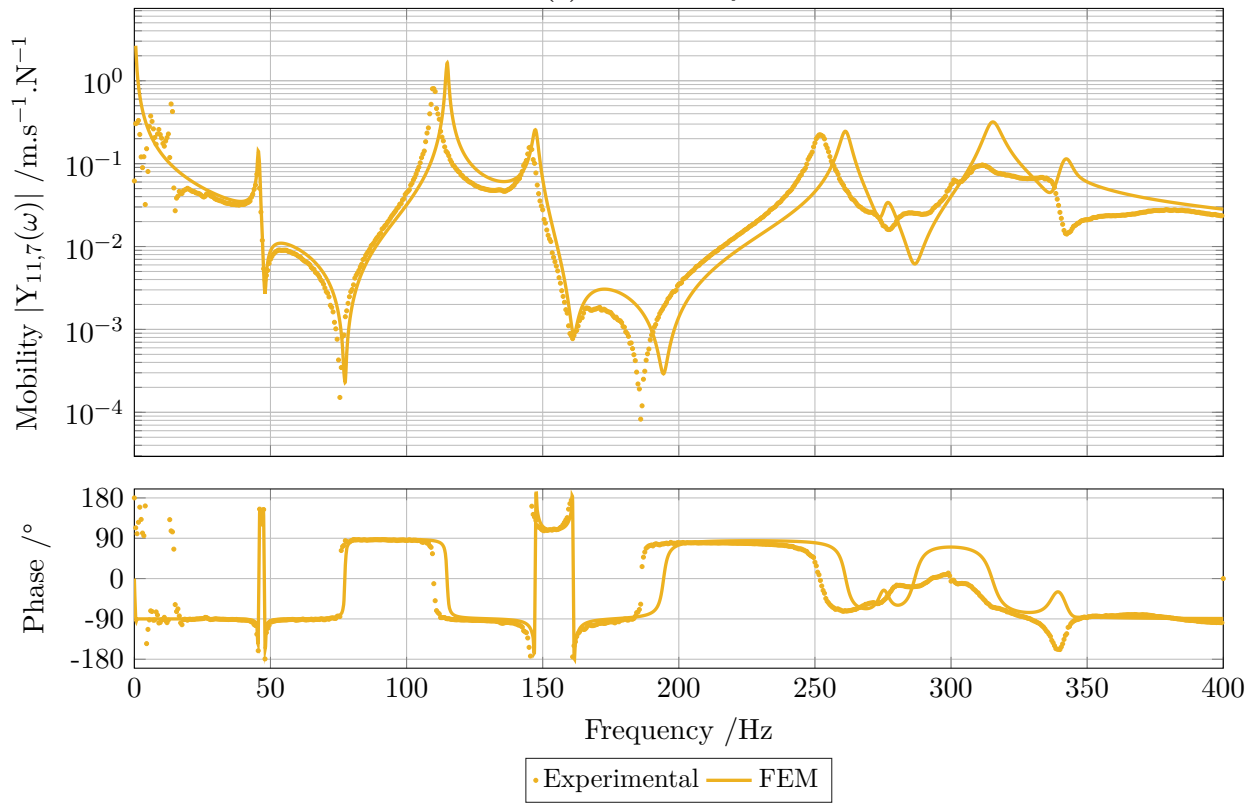


(b) Transfer mobility FRFs

Figure 9.12 Mobility FRFs of the S02 specimen



(a) Point mobility FRFs



(b) Transfer mobility FRFs

Figure 9.13 Mobility FRFs of the K05 specimen

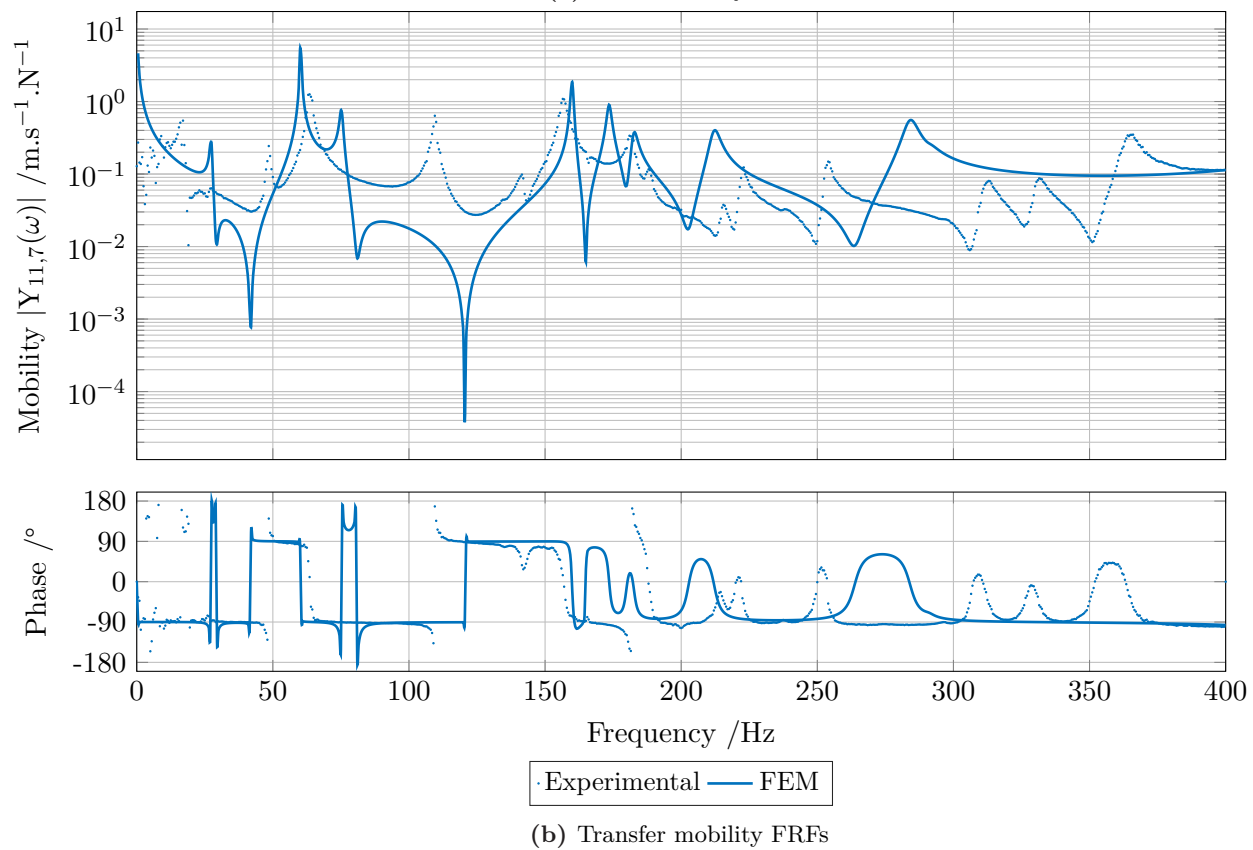
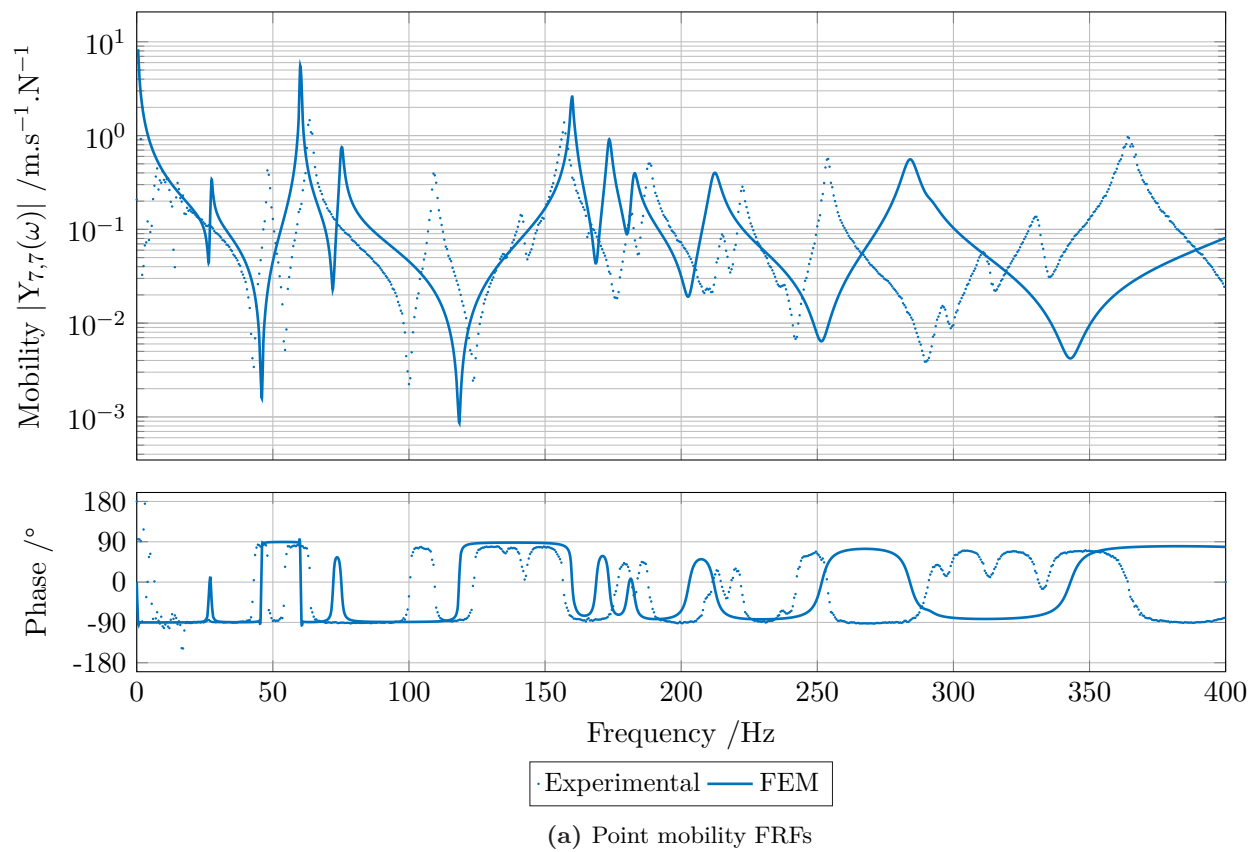
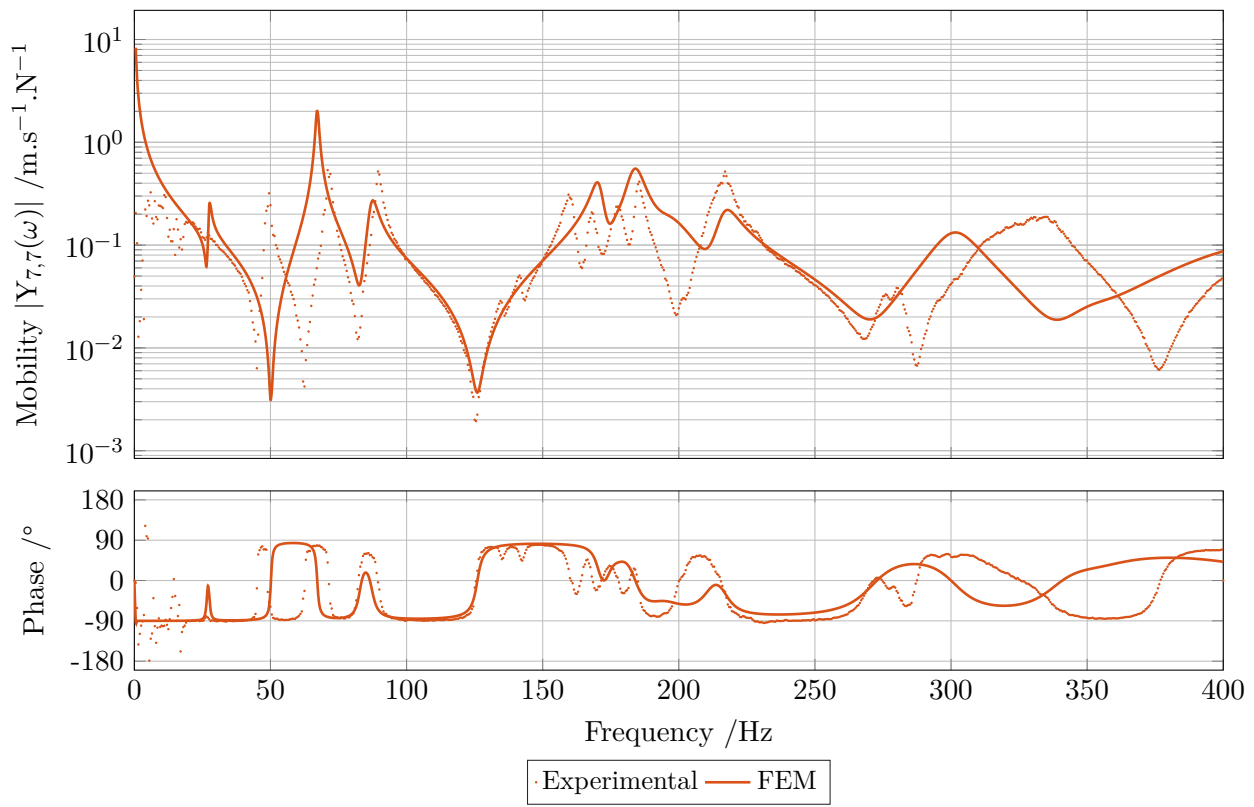
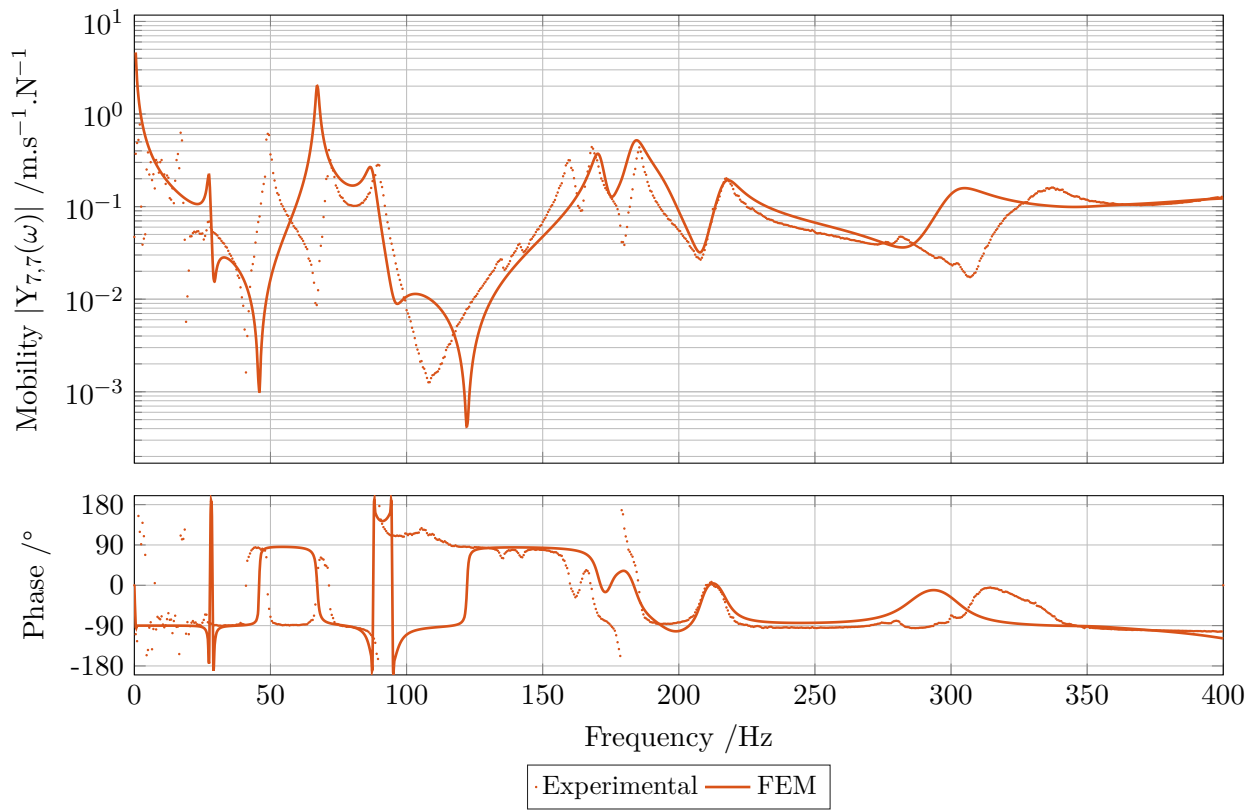


Figure 9.14 Mobility FRFs of the S01P specimen

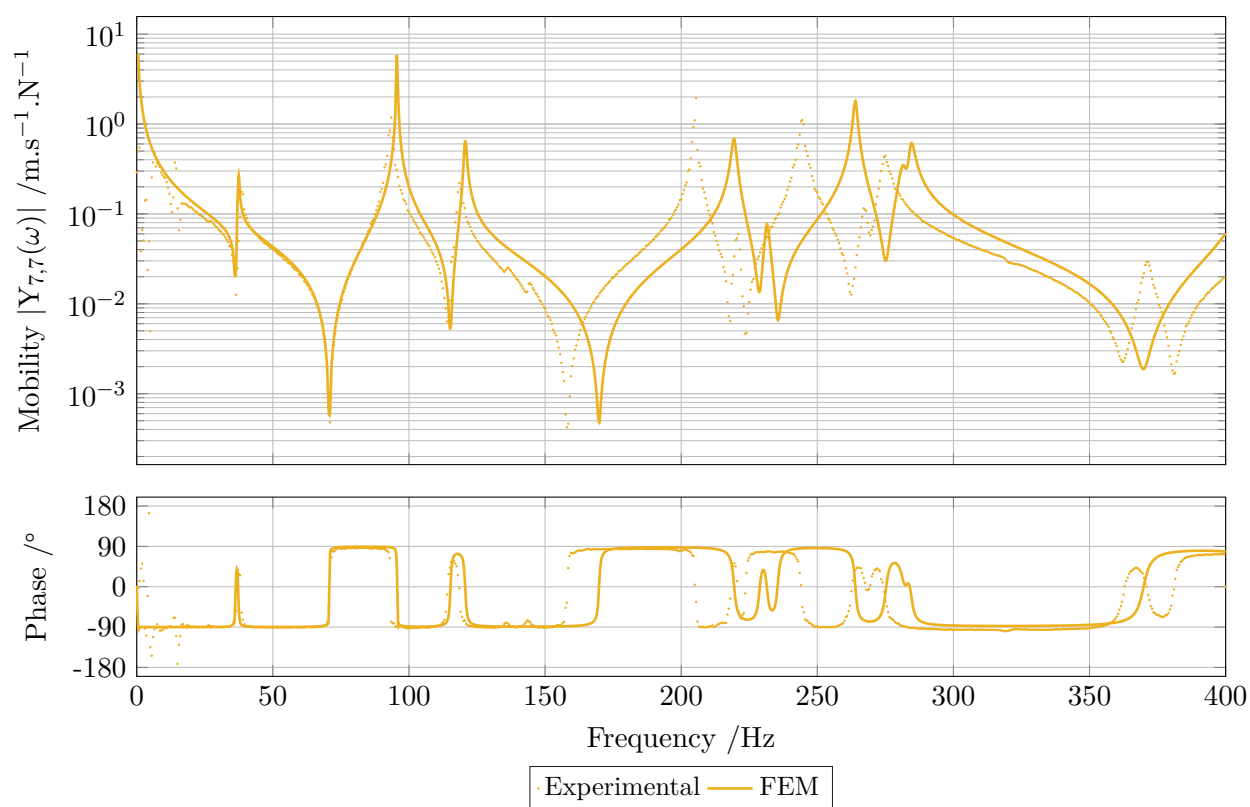


(a) Point mobility FRFs

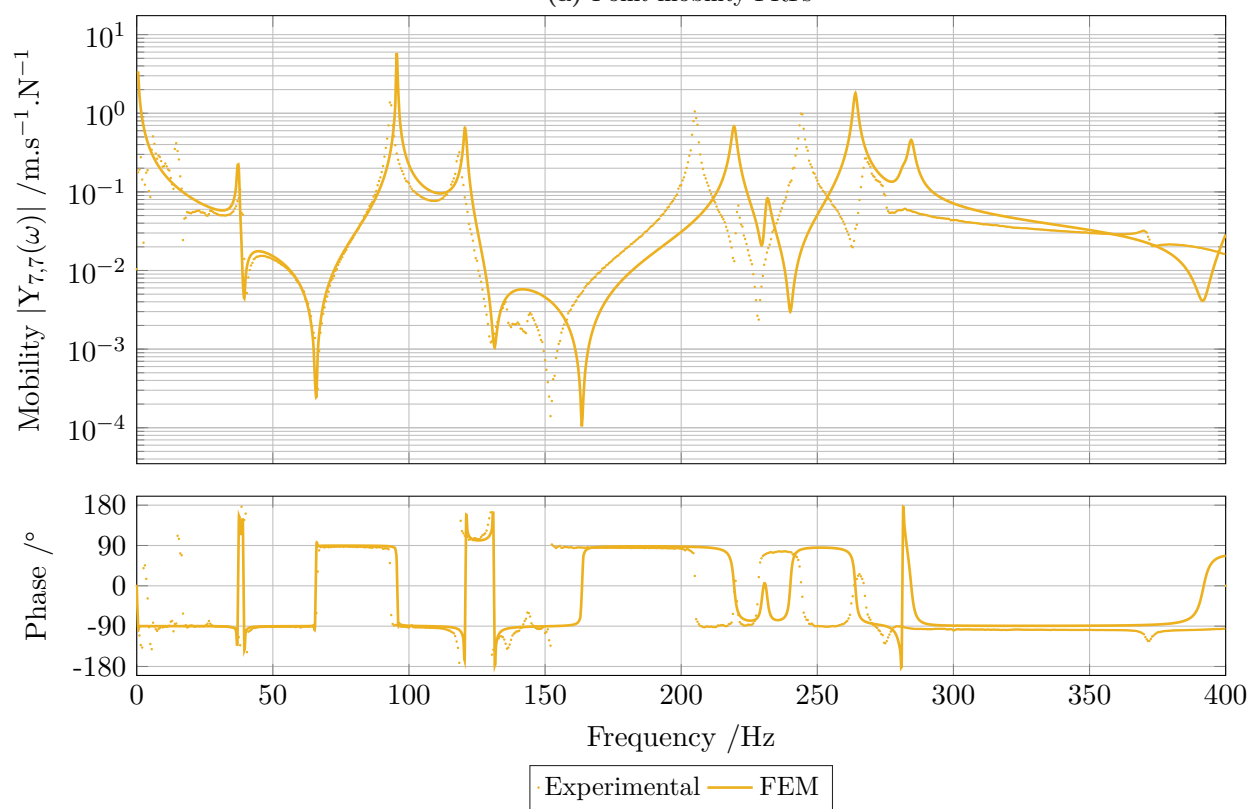


(b) Transfer mobility FRFs

Figure 9.15 Mobility FRFs of the S02P specimen



(a) Point mobility FRFs



(b) Transfer mobility FRFs

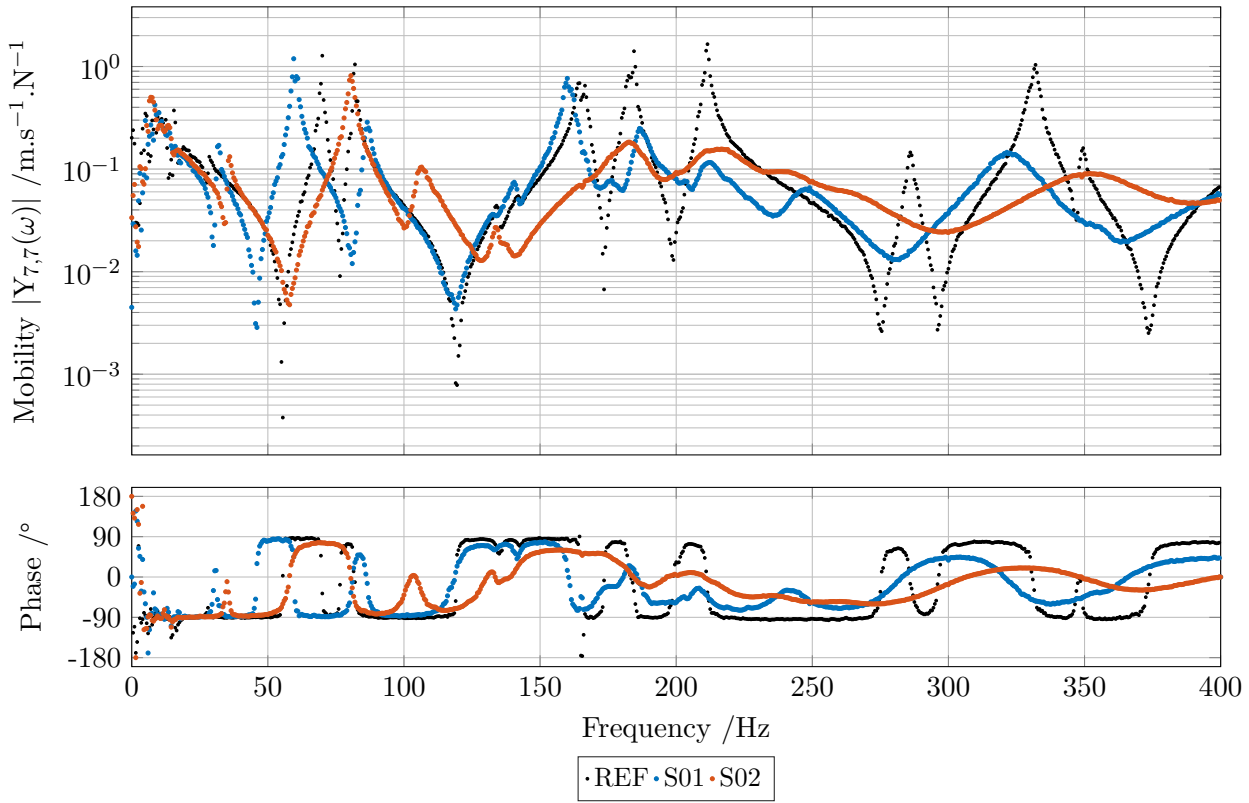
Figure 9.16 Mobility FRFs of the K05P specimen

The meticulous manufacturing of composites with plies of different materials demands unwavering attention to material compatibility, bonding, curing synchronisation, and quality control. Triumphant over these challenges requires a multidisciplinary approach, enlisting expertise in materials science, engineering, and advanced manufacturing techniques. However, and despite all the challenges in manufacturing hybrid composites, the comparison between experimental and numerical frequency response functions for the K05P specimen in Figure 9.16 shows a perfect fit for low frequencies and minor misalignments for higher frequencies. In fact, both damping materials considered in this experimental testing, Smacwrap ST and Kraibon, are viscoelastic materials specifically conceived to be embedded in composite materials. According to the manufacturers, Smacwrap ST is a viscoelastic material that withstands composite curing, including the autoclave process, without undergoing a chemical reaction, and, on the other hand, Kraibon rubber must vulcanise at 150 °C. This pilot experimental campaign suggests that the thickness of the laminated composite plays a decisive role when considering hybrid composites. The intricate geometric and manufacturing complexities are less prone to undermine the final properties of the structures for thicker laminates with interleaved damping patches, and it is possible to combine diverse materials more easily.

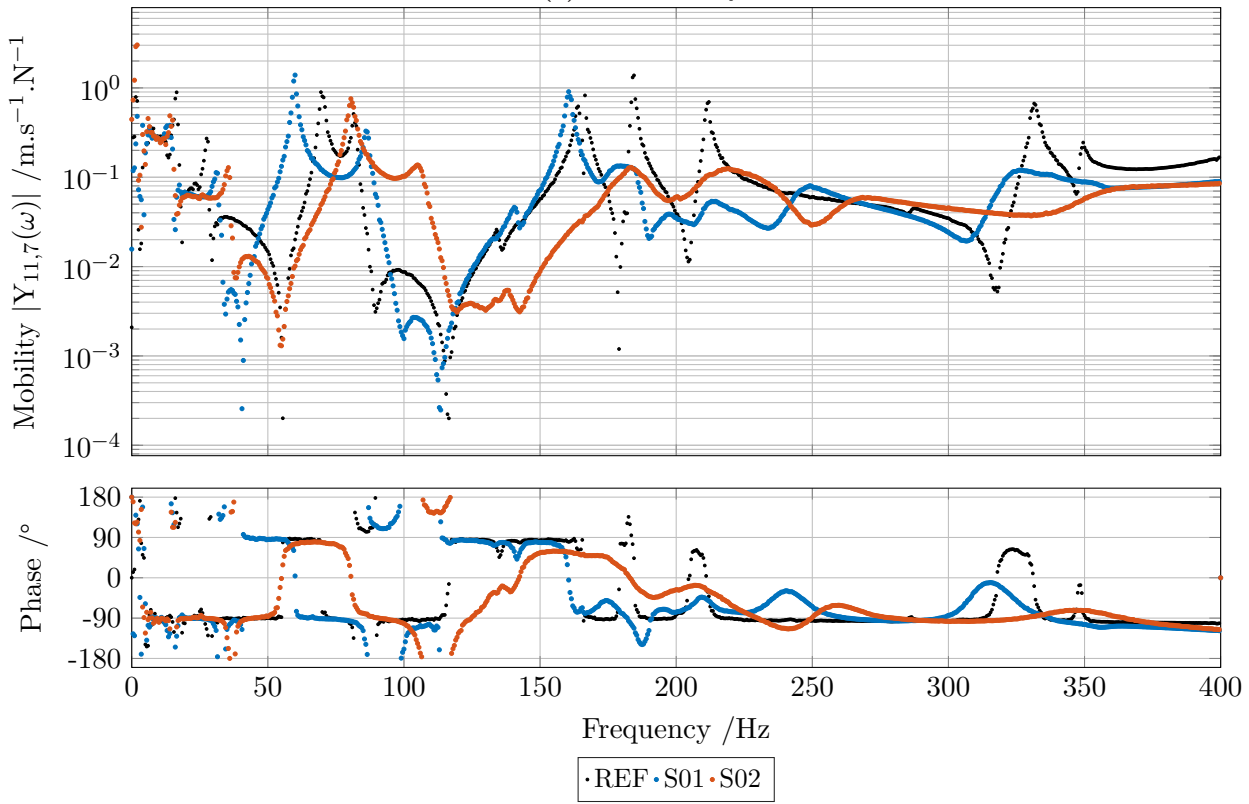
The difficulties posed by the manufacturing processes should not overlook the potential of the solution proposed in this work. The advantages presented by interleaved damping patches justify further research in this field, not least because one must bear in mind that manufacturing processes evolve and are often responsible for significant technological advances. The integration of 3D printing technologies with fibre-reinforced plastics, for instance, represents a groundbreaking synergy in advanced manufacturing. 3D printing, or additive manufacturing more generally, enables the layer-by-layer construction of intricate structures with high precision and design flexibility. When coupled with fibre-reinforced plastics, which typically involve incorporating strong and lightweight fibres such as carbon or glass into a polymer matrix, the resulting composites harness the strengths of both techniques. Thus, 3D printing allows for the precise placement of reinforcing fibres, optimising their orientation and distribution within the printed components. This flexibility not only enhances the structural integrity of the final product but also allows for tailoring mechanical properties to specific application requirements, including damping with interleaved damping patches. The synergy of 3D printing and fibre-reinforced plastics opens up new horizons in manufacturing, offering a pathway to complex, high-performance components with customised functionalities across diverse industries, including aerospace, automotive, and medical technology.

9.4.2 | CFRP with Interleaved Damping Layers

Beyond the comparison with numerical frequency response functions, the experimental testing conducted in this work aims to compare the different configurations experimentally. Figures 9.17 and 9.18 depict the mobility frequency response functions of the REF, S01 and S02 specimens and the REF and K05 specimens, respectively. The main goal is to compare different configurations of interleaved damping layers. The specimens S01 and S02 employed identical damping material with varying thicknesses. However, to investigate the impact of the stacking sequence while maintaining an equal quantity of damping material, one adjusted the configurations; specifically, one considered two layers of S01 to have the same theoretical mass as one layer of S02. The K05 specimen adopted the same configuration as S02 but used different damping materials and thicknesses, so it makes more sense to analyse it separately.

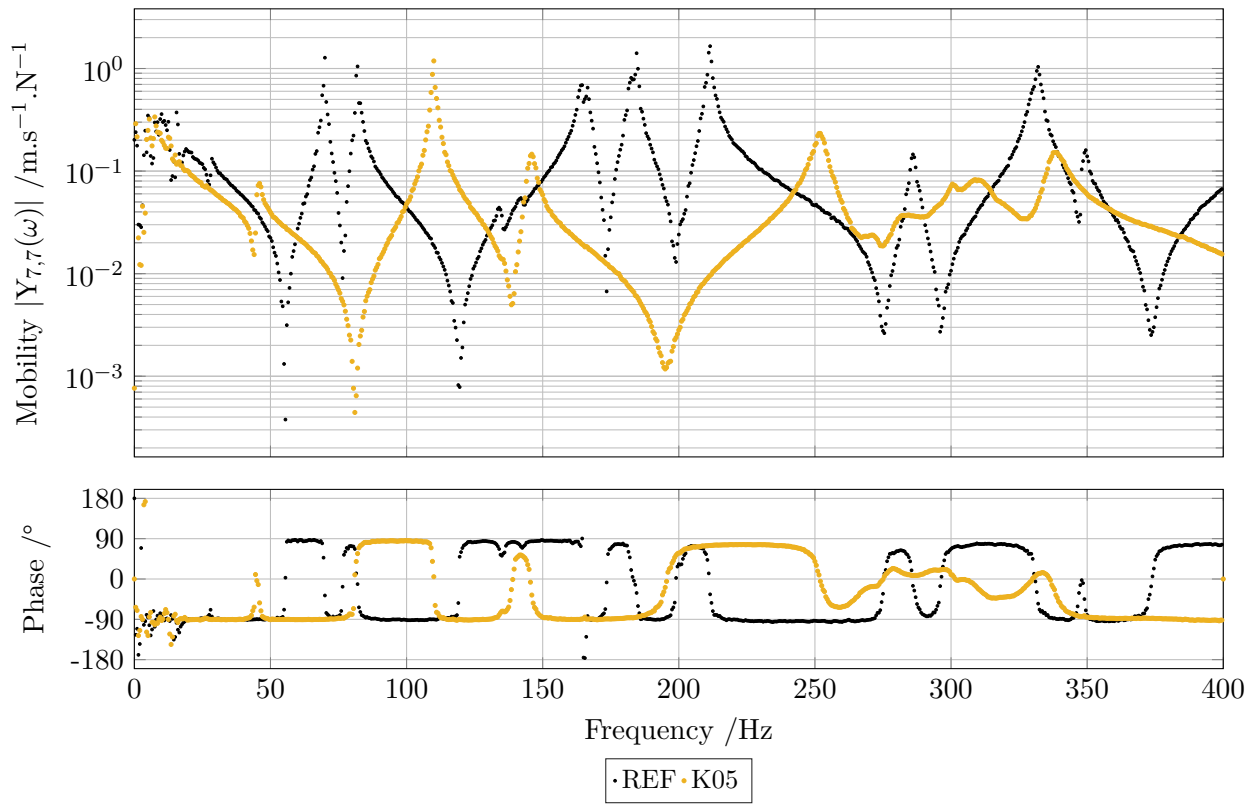


(a) Point mobility FRFs

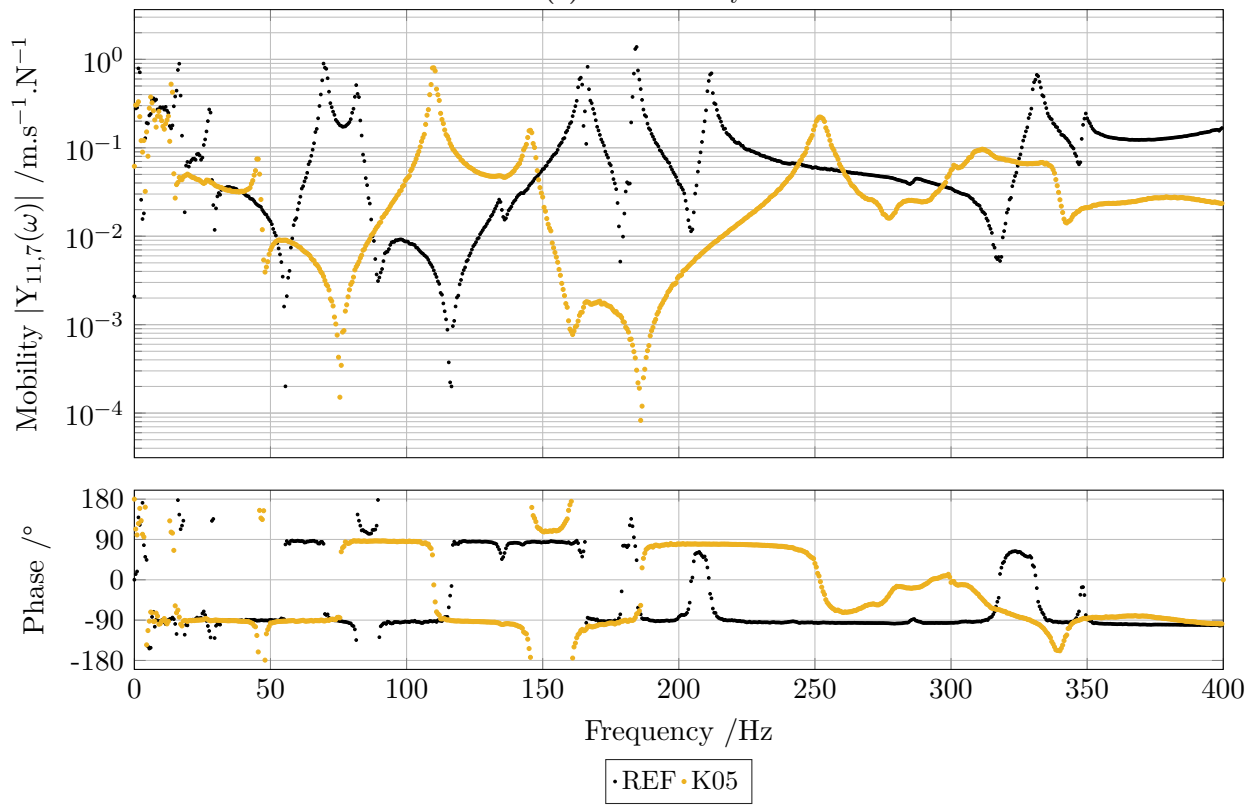


(b) Transfer mobility FRFs

Figure 9.17 Experimental Mobility FRFs of the REF, S01 and S02 specimens



(a) Point mobility FRFs



(b) Transfer mobility FRFs

Figure 9.18 Experimental Mobility FRFs of the REF and K05 specimens

Regrettably, as revealed by the microscopy analysis and discussed before, the thicknesses of the viscoelastic layers, particularly for the Smacwrap ST 0.1 case, deviated from the values provided by the suppliers. Such misalignment between thicknesses challenges the manufacturing process and the corresponding comparisons. Consistently with microscopic analysis of the viscoelastic layer thicknesses revealing that two layers of the Smacwrap ST 0.1 samples do not match one layer of the Smacwrap ST 0.2 samples, one expects the final thickness of the S02 specimen to be slightly higher. The frequency response functions in Figure 9.17 disclose a shift in the resonant frequencies of the S02 specimen comparatively to the REF and S01 specimens, which is in line with what one expects from a stiffer structure - higher resonant frequencies.

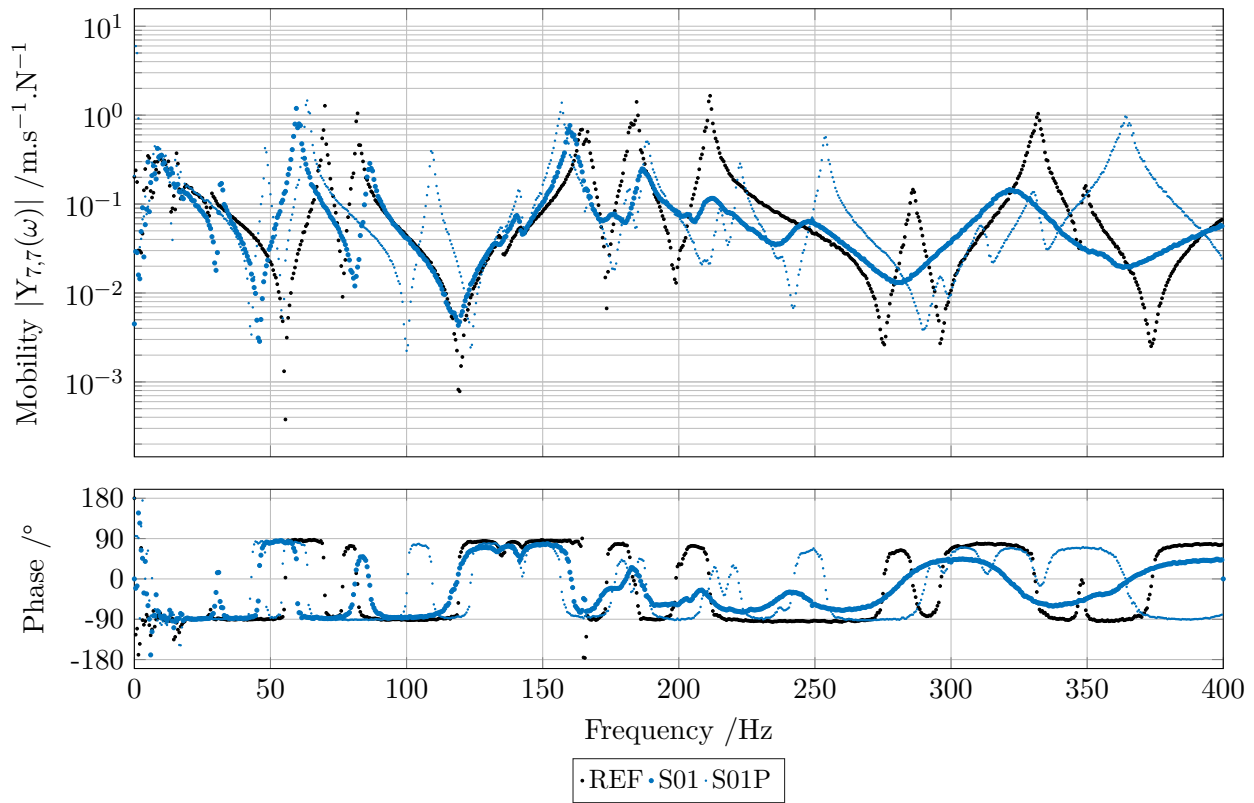
Still, it is possible to draw some conclusions regarding damping efficiency. Remarkably, both configurations with Smacwrap ST disclose a high damping capacity throughout the entire frequency range - Figure 9.17. Nevertheless, the frequency response functions highlight that the S02 specimen shows a slightly higher damping capacity, evident in the attenuation of the resonance peaks for the higher frequencies. Recalling Figure 9.2, the S02 configuration comprises a damping layer at the core of the composite laminate, whereas configuration S01 includes two layers with half the thickness, one lamina away from the core - Single and Double-Core configurations adopted in Chapter 7. Uncertainties regarding the viscoelastic layers' thickness do not help to establish unequivocal conclusions. Apparently, based on the microscopic analysis, the S02 specimen has more damping material than the S01, so it is not easy to infer whether the greater damping capacity is due to the specific configuration or to the fact that there is slightly more damping material.

Nonetheless, both configurations exhibit a remarkable damping capacity, which is not surprising given the results discussed in the parametric analyses. Both arrangements can be classified as Core configurations according to the categorisation proposed in Chapter 7, and as discussed then, the extensional strain of the outer layers promotes the transverse shear strain of the damping layers at the core, and consequently damping capacity.

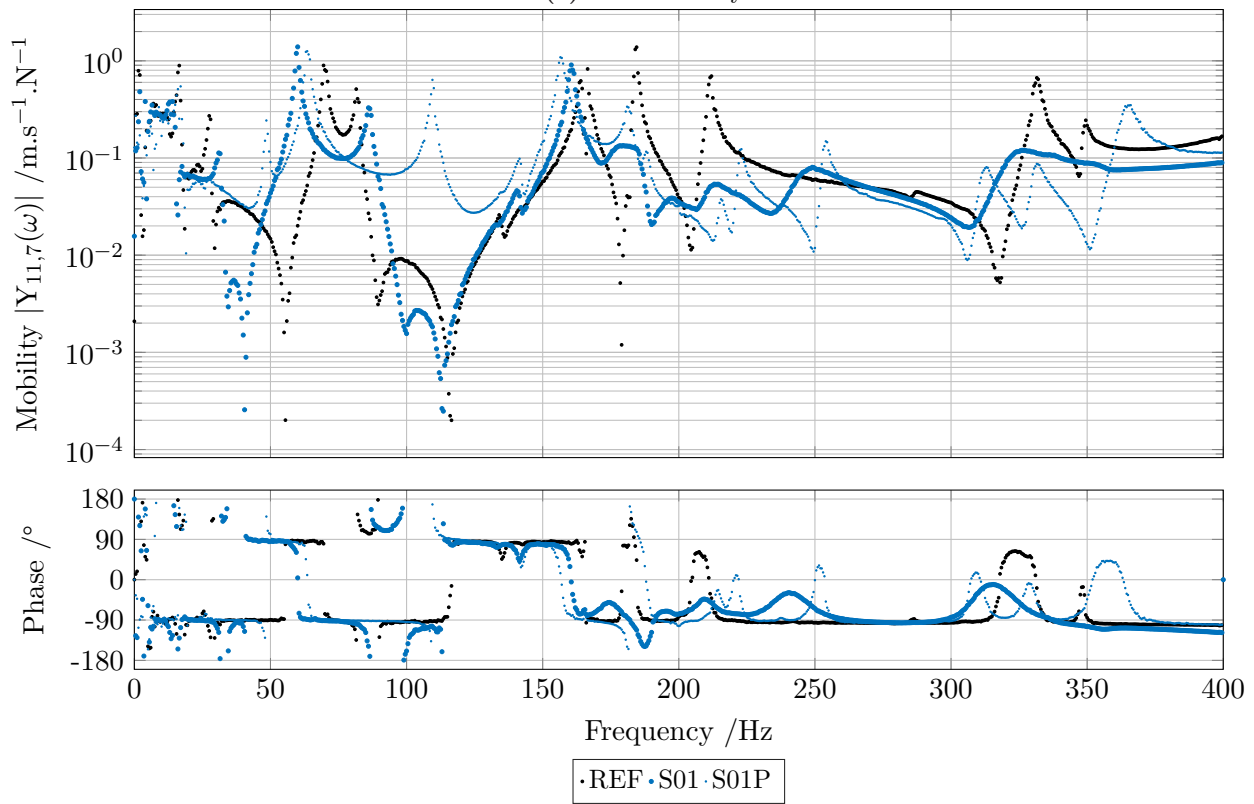
Furthermore, Figure 9.18 shows the mobility frequency response functions of the REF and K05 specimens. The increased thickness of the damping layer is discernible in the much more noticeable shift of the resonant frequencies, denoting an increased laminate stiffness due to increased thickness. The higher storage modulus of the Kraibon material, when compared to the Smacwrap ST, plays a role here, but not as significant. In addition, the damping capacity of the damping layer at the laminate core is not so remarkable, but one might distinguish two factors. On the one hand, the estimated loss factor of the Kraibon material in Chapter 8 is slightly lower than that estimated for the Smacwrap ST, but, on the other hand, it is worth to emphasise a previously discussed result. Thick damping layers might promote the decoupling of the laminate stiff layers, losing the ability to impose transverse shear strain on the damping layer and, consequently, damping efficiency.

9.4.3 | CFRP with Interleaved Damping Patches

Finally, one also compares the different configurations incorporating interleaved damping patches experimentally. Figures 9.19 to 9.21 depict the mobility frequency response functions of the REF, S01 and S01P specimens, the REF, S02 and S02P specimens and the REF, K05 and K05P specimens, respectively. The goal here is to develop a more comprehensive knowledge of the intricacies of interleaved damping patches by comparing them with more usual methodologies such as interleaved damping layers.

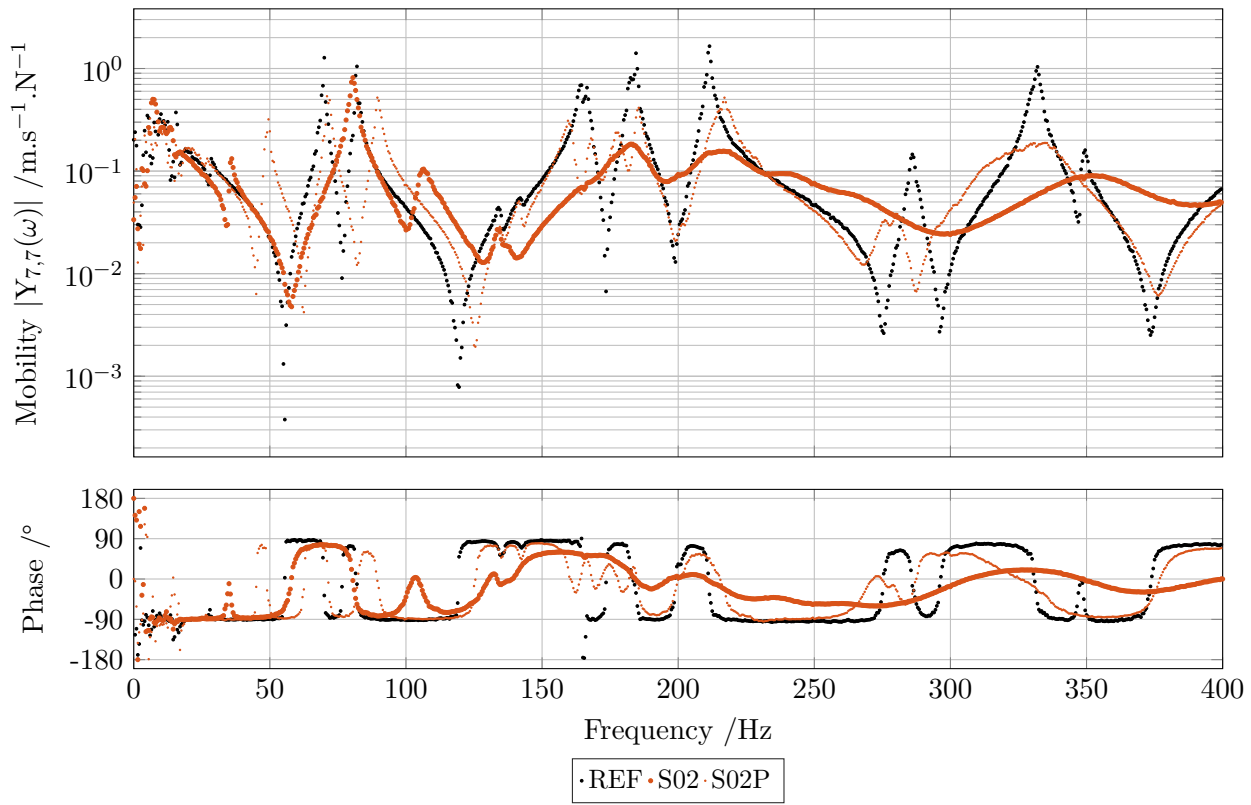


(a) Point mobility FRFs

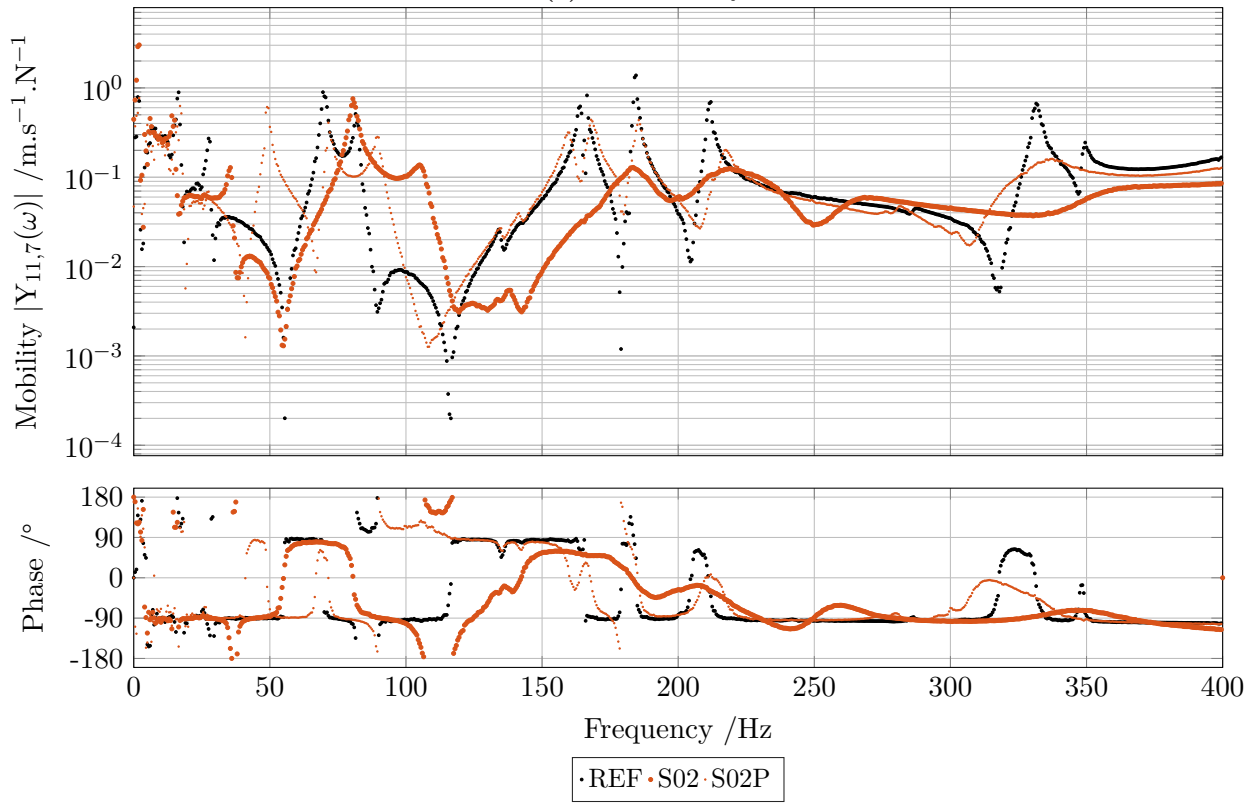


(b) Transfer mobility FRFs

Figure 9.19 Experimental Mobility FRFs of the REF, S01 and S01P specimens

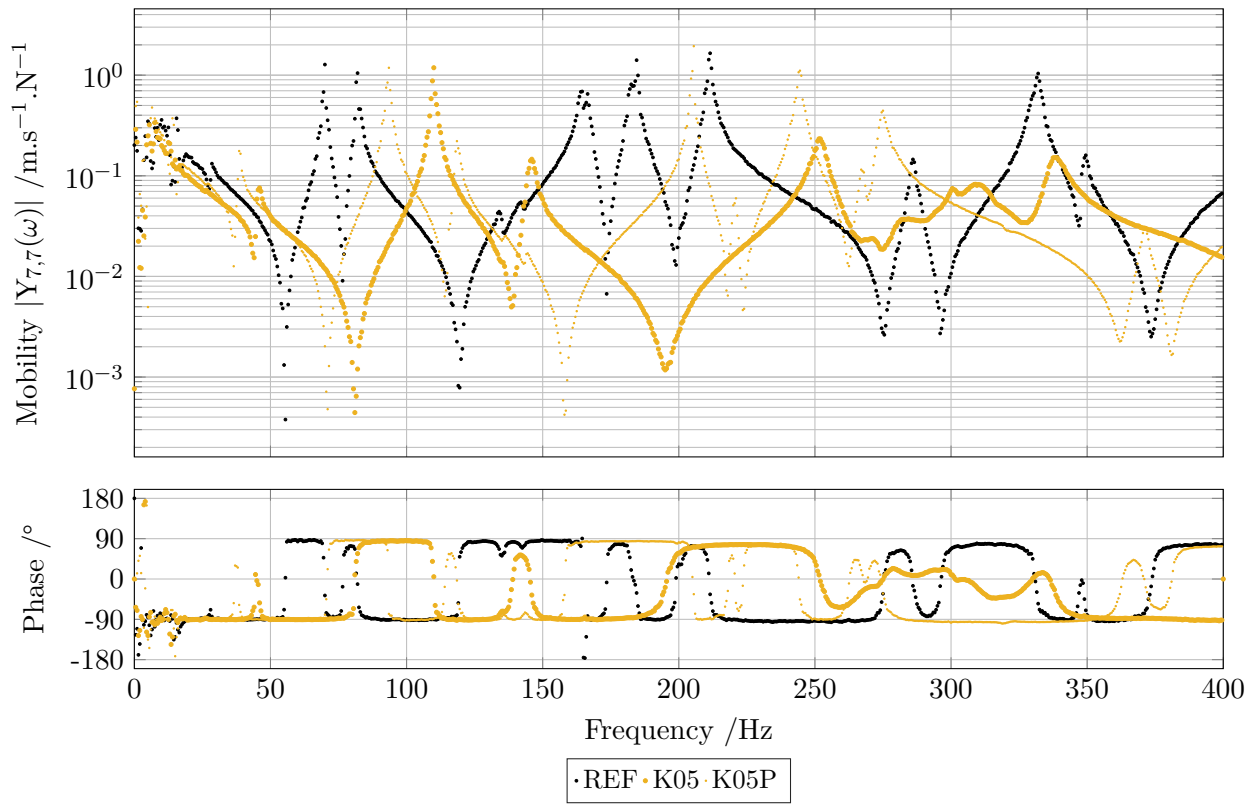


(a) Point mobility FRFs

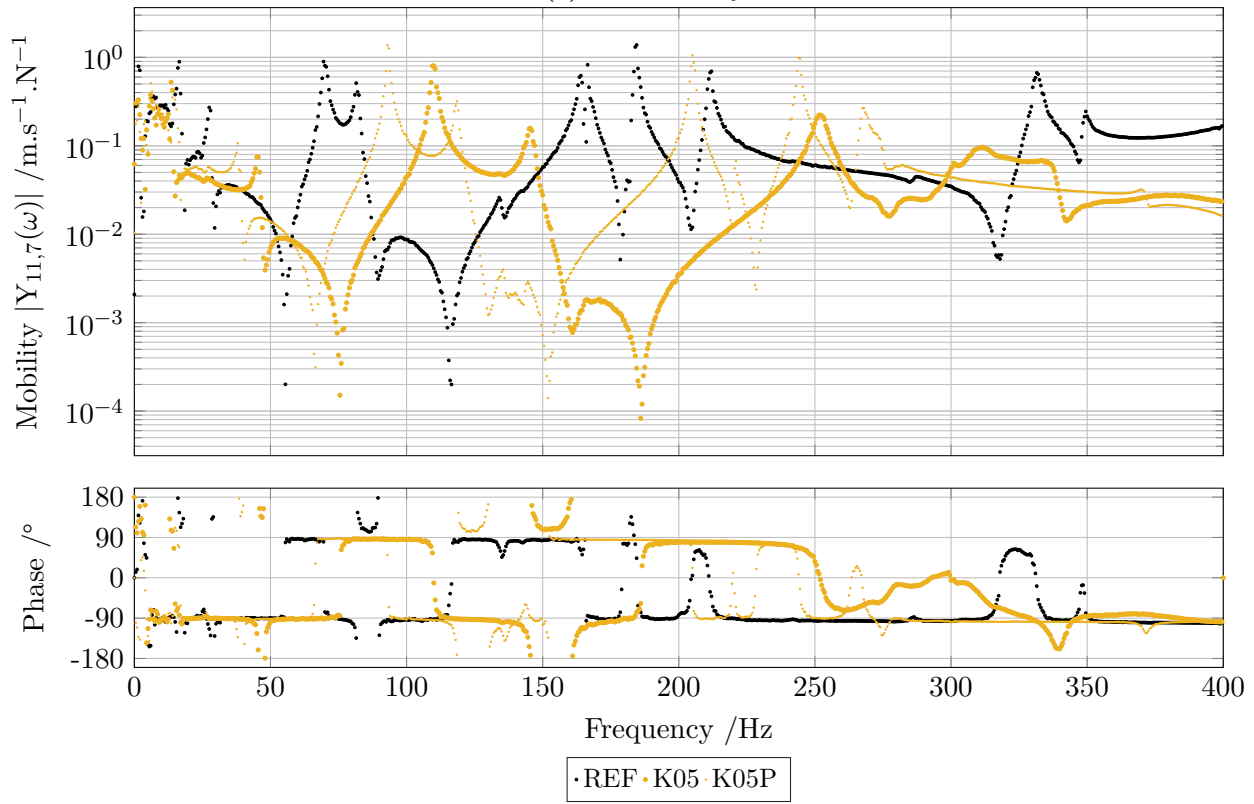


(b) Transfer mobility FRFs

Figure 9.20 Experimental Mobility FRFs of the REF, S02 and S02P specimens



(a) Point mobility FRFs



(b) Transfer mobility FRFs

Figure 9.21 Experimental Mobility FRFs of the REF, K05 and K05P specimens

As already discussed, embedding viscoelastic damping patches in thin composite laminates heightens the susceptibility to imperfections. The S01P specimen, in particular, shows significant warping, which makes it impossible to establish a proper comparison and draw reliable conclusions. The mobility frequency response functions of the S01P in Figure 9.19 show exactly this. While the frequency response function of the S01 specimen with two interleaved damping layers shows a clear relationship with the REF specimen and the damping capacity it can add, the frequency response function of the S01P specimen seems to be a frequency response function of a completely different structure. The warping contributes to the complexity of the structure, introducing additional variables and making it difficult to establish comparisons.

Nevertheless, despite the challenges involving manufacturing hybrid composites with damping patches, the results for thicker laminates are more encouraging. Figures 9.20 and 9.21 show the frequency response functions for the S02/S02P and K05/K05P specimens, respectively. As the manufacturing process for thicker laminates tends to be less prone to errors, the result is closer to the expected one. Comparing the configurations with interleaved damping patches and those with interleaved damping layers with the reference case, the first thing that stands out is the ability of the interleaved damping patches to introduce some damping to the reference structure. Undoubtedly, the damping capacity of an interleaved damping layer is far higher. Still, a small amount of damping material provides some damping, particularly at higher frequencies, and Figure 9.20 highlights this quite clearly. The S02P specimen reveals a high damping capacity at higher frequencies, significantly attenuating the resonances at a level comparable to the interleaved damping layer S02.

Manufacturing composite laminates with interleaved damping patches presents some challenges, but the possible outcomes are worth trying. The preliminary experimental testing conducted in this work proves that it is possible to follow this approach to improve the damping capacity of composite structures without increasing the mass unnecessarily. Hence, in forthcoming studies, it is suggested to conduct an experimental campaign to ensure better control over the production process with a higher number of specimens. The work of Correia [2021], using specimens from the current study, represents a step in this direction. As already discussed, the rise of new manufacturing technologies involving fibre-reinforced plastics is another step towards better control over the manufacturing process. The advantages of additive manufacturing ease the meticulous layer-by-layer assembly of intricate structures with exceptional precision and design adaptability, enabling the integration of viscoelastic damping patches into a composite laminate.

Part V

Conclusion

Conclusion

10.1 | Introduction

In this chapter, one extensively reviews the work conducted in this thesis on the use of passive damping technologies, with particular emphasis on composite structures, by confronting the motivations and objectives established in advance with the numerical and experimental results obtained, presenting the main conclusions and achievements, but also highlighting the unsuccessful results to improve the motivations and objectives of future works. The present chapter covers four sections, including this *Introduction*. Then, one presents an *Overview of the Dissertation* to review the whole work performed in the context of this work and quickly cover the main topics developed here to give the reader a quick insight into a document by commenting on its different parts. One also presents the main conclusions accomplished during the development of the present work together with the significant contributions to the field of viscoelastic damping technologies in the *Conclusion* section. Here, one also emphasises the primary difficulties and unsuccessful results to help the interested reader in future research. The *Novel and Key Contributions* emphasise and delineate the novel, pivotal aspects and contributions, which stand out prominently in a dedicated subsequent section, showcasing the distinctiveness and significance of the new findings and contributions brought in this comprehensive study. Finally, any earnest endeavour opens up new paths for further developments, and one compiles those open up in the present work involving passive damping technologies, particularly those involving composite structures, in *Suggestions for Further Research*.

10.2 | Overview of the Dissertation

Following the motivations and objectives defined for this dissertation and set out in advance, one carried out a study aiming to introduce, model, characterise, apply and develop passive damping technologies using viscoelastic damping materials with an emphasis on their partial application either as surface or integrated treatments, or both, into composite structures.

The primary advancements within this study revolve around the intricate mechanical modelling and the concurrent numerical methodologies specifically tailored to simulate the dynamic behaviour exhibited by structures featuring viscoelastic treatments. The initial focus, as detailed in Part I - *Introduction*, was to meticulously delve into the fundamental damping mechanisms and establish the motivation for passive damping technologies. This journey commenced by intricately dissecting composite materials, exploring their nuanced compositions and properties, and further delving into a comprehensive overview of hybrid configurations. This research formed the bedrock for the inter-layer hybrid configurations, meticulously designed to augment damping capacity through the strategic placement of interleaved viscoelastic damping patches. Moreover, the endeavour to establish a precise mechanical model for structures housing viscoelastic damping materials needs a thorough exploration encompassing their mechanical characterisation, discerning selection criteria, and the intricate process of numerical modelling and its implementation.

In Part II - *Mechanical Model*, one described the efforts to model structures comprising viscoelastic damping materials numerically. Considering the historical perspective of a long-standing research topic, one reviewed the most important categories of plate theories available in the literature. More specifically, one reviewed formulations expressed as an ESLT, considered due to their simplicity but with an inevitable reduced accuracy when modelling multi-layer structures as a single-layer with equivalent material properties, formulations involving multi-layer hypotheses which can accurately capture the mechanical behaviour layer by layer and correctly account for normal and transverse shear strain variations such as the LWT models but which are often computationally cumbersome due to the increase of kinematic variables with the number of layers, and theories dealing with multi-layer approaches of the hybrid type. The laminated composite plate theory based on warping functions considered here is a multi-layer approach of the hybrid type, keeps the advantages of a purely multi-layer deformation theory while maintaining the number of kinematic variables of a single-layer approach, providing a good compromise between accuracy and computational time. The warping functions enable the adoption of different deformation theories according to the application without any implication in the computational model size. One thoroughly addressed the numerical methods used to solve the derived equilibrium equations, emphasising the developments carried out with the FEM due to the advantages that a formulation considering a local discretisation over small regions of simple but arbitrary shape elements brings to the implementation of any mechanical model for inhomogeneous complex composite structures comprising viscoelastic damping materials, particularly when compared with the RRM which one also implemented in the context of this thesis. In addition, the FEM formulation enabled a simple but robust extension of the laminated composite plate theory to a laminated composite shell theory, one of the primary achievements of this work. In particular, one realised such extension with a facet shell finite element by assembling out-plane rotation degrees of freedom modifying the interpolation of the element displacement field and thus establishing a robust shell element developed for the computational modelling of inhomogeneous multilayered composite shell structures. One obtained a broad set of comparison and validation results to assess the advantages and drawbacks of the proposed finite element, including sandwich shells with composite facings with internal orthotropic damping, one of the most complex test cases available in the literature for passive damping technologies.

The complete set of numerical investigations regarding the analysis of the damping performance of viscoelastic damping treatments carried out in the context of this work has a dedicated part. Part III - *Parametric Analysis of Damping Patches* includes two relevant chapters, a first one covering all the developments and results concerning the use of viscoelastic damping technologies as a surface damping treatment applied to isotropic host structures such as beams and plates and a second one regarding the performance of laminated composite plates with interleaved damping layers and, more innovatively, with interleaved damping patches. Interested in the study

of partial damping treatments as a mechanism to improve the overall performance of structures for innovative industries, one reviewed the significant scientific works related to this topic and addressed the main arguments of this scientific community, proving there is still room for a better comprehension of this topic and the development of new metrics for an appropriate evaluation of damping mechanisms might benefit the ever-increasing range of industries using the technology. An example is the study of the damping efficiency of partially covered beams, which opens the numerical investigations part in this work, proving that it is still possible to develop new metrics for the evaluation of damping mechanisms by taking advantage of well-designed mechanical models and performing dimensionless parametric analyses to derive general design guidelines for improved damping performances. Additionally, the research concerning the damping efficiency of segmented constraining layers in beams highlighted the real influence of the constraining layer and the importance of well-designed parametric analyses, preferably with dimensionless parameters, to avoid hasty conclusions, admissible in the context of a complex problem with many variables, but which one can overcome. The extension of the developments to partially covered plates helped to strengthen and generalise the proposed metrics and the corresponding conclusions, enabling the development of design guidelines applicable to more complex structures. More innovatively, one investigated the damping performance of laminated composite plates with interleaved damping layers and interleaved damping patches. Covering the complete range of symmetric configurations for a six-layer reference laminate, including viscoelastic damping layers, one compared the positioning of the damping layer(s) in the stacking sequence by performing parametric analyses and employing the same amount of damping material focusing, primarily, on the preferential stacking sequences, whether it is preferable to have more or less damping layers for the same amount of damping material and whether they should be closer to the core or the skins. Taking advantage of the insights obtained with the interleaved damping layers approach, one followed the numerical studies investigating the proposed methodology of interleaved damping patches. One examined eight different stacking sequences with interleaved damping patches, evaluating the performance of each one and establishing design guidelines considering different approaches, such as whether the interleaved patch should run from the Centre-to-Edge or Edge-to-Centre of the laminated plate. As mentioned before, the proposed new metrics for the evaluation of the main mechanisms responsible for improved damping performances found a well-fit application in laminated composite plates.

Not least of all, one discussed the experimental work performed in this work, including that related to composite laminates with interleaved viscoelastic materials, but also the required mechanical characterisation of those viscoelastic materials in Part IV - *Experimental Testing and Analysis*. Regarding the latter, one presented the proposed experimental procedure to acquire the transmissibility frequency response functions with contactless sensors and the corresponding material properties identification via numerical fitting with transmissibility frequency response functions computed with the FEM using the Local Amplitude Criterion. The material properties identification developed here considered two viscoelastic materials, Smacwrap ST and Kraibon, and one manufactured different sets of specimens taking into account the distinct manufacture requirements; Smacwrap ST withstands composite curing, including autoclave process, while Kraibon must vulcanise at 150 °C. Both materials served as damping materials in the complete set of specimens manufactured and used to evaluate the damping performance of different configurations of laminated composite plates with interleaved damping layers and patches. Lastly, one detailed the experimental procedure carried out to obtain the experimental frequency response functions used to evaluate the damping performance of different configurations and to compare the experimental results with the numerical ones computed with the FEM.

Naturally, one discusses the main results, achievements and conclusions of the dissertation in this *Conclusion*, Part V.

10.3 | Conclusions

One can summarise the conclusions of this dissertation considering four main groups, following the contents of Parts I to IV. Concerning the *Introduction*, i.e. the state of the art and the basic concepts of viscoelastic materials, one can conclude the following:

- The use of damping materials in vibration control dates back to the beginning of the 20th century, and the developments in this area followed the significant events that impacted the history of the world in the last century. The evolution of the transport industry and the need to control vibrations have determined the course of this research area;
- Since vibration is an oscillation and the corresponding continuous interchange of potential and kinetic energies, energy dissipation, or damping, is of crucial importance to remove part of the mechanical energy and damping materials can dissipate such energy in the form of heat;
- Viscoelastic materials are capable of converting mechanical energy into heat when undergoing deformation and are used as damping materials in many industries as a passive damping design;
- Currently, there are passive and active approaches to the vibration-damping problem. Active vibration damping techniques require sensor elements to monitor the structure under control and well-designed actuators to reduce the vibratory motion of the system, so one only considers it when it outperforms passive design, which is robust and often outperforms active approaches at high frequencies;
- The main viscoelastic damping mechanisms are well established and documented in the literature but often overlooked when evaluating damping performances in optimisation studies, leading to misconceptions and conflicting conclusions in investigations dealing with traditional surface damping treatments;
- The increasing use of structural composite materials in high-performance industries raised an even bigger problem in vibration control due to the overall weight savings since light structures are prone to higher vibration amplitudes, which are readily transmitted;
- It is easy to think of the possible benefits of interleaving a viscoelastic layer with fibre-reinforced plastics due to their laminate nature, but there are manufacturing difficulties, mainly due to the cure requirements of traditional composites that jeopardise the viability of using traditional viscoelastic materials, designed for surface damping treatments in metallic structures, in laminated composite structures;
- More recently, some companies overcame this issue and developed patented viscoelastic materials that withstand composite curing, including the autoclave process, and it is possible to embed them into laminated composite structures;
- The mechanical properties of viscoelastic materials depend upon many factors, but temperature and frequency are the most relevant, and it is crucial to rely on suitable constitutive models. The numerical modelling of viscoelastic materials in numerical methods such as the FEM is one of the most challenging tasks when considering these materials. The numerical modelling of viscoelastic materials considering only the frequency dependence at isothermal conditions is one of the most common approaches;

- In particular, it is usual to apply parametric analytical models defined by adjusting their parameters to the experimental results obtained in the characterisation of the viscoelastic material under analysis. Among these models, due to their representativeness and wide application, one highlights the Generalised Maxwell Model (GMM), the Golla-Hughes-McTavish (GHM) model, the Anelastic Displacement Fields (ADF) model and the Fractional Derivatives Model (FDM);
- The Complex Modulus Approach (CMA) defines the relation between stress and strain in viscoelastic materials, and it is widely used with numerical methods such as the Finite Element Method (FEM) to compute the Direct Frequency Response (DFR) of structures comprising these materials. In particular, the Modal Strain Energy (MSE), Iterative Modal Strain Energy (IMSE) and Iterative Complex Eigensolution (ICE) are the most common approaches to extract complex modes through an iterative updating process using the Complex Modulus Approach (CMA).

Regarding the *Mechanical Model* in Part II, one should highlight the following conclusions:

- The increasing use of multi-layer structures comprising inhomogeneous complex materials has set new challenges in the mechanical modelling of such designs for researchers and engineers. On the one hand, the basic assumptions of classic formulations, such as the CPT or FSDT, do not hold for layered structures whose materials have very different mechanical properties. On the other hand, the computational cost and intricate formulations associated with three-dimensional elasticity solutions for multi-layer inhomogeneous structures are prohibitive in many applications;
- Researchers and engineers have been developing different displacement-based formulations to model fundamental multi-layer structural elements such as beams, plates and shells that one can divide into three main groups: Equivalent Single-Layer Theories (ESLT), Layer-Wise Theories (LWT) and multi-layer approaches of the hybrid type;
- Multi-layer approaches of the hybrid type try to keep the advantages of a purely multi-layer deformation theory while maintaining the number of kinematic variables of a single-layer theory, providing a good compromise between accuracy and computational time. Zig-Zag Theory (ZZT) models are the most standard approach of this kind;
- The plate theory based on warping functions presented by Loredó and Castel [2013] and adapted in this work is a multi-layer approach of the hybrid type, and the main advantage of this model is that it is possible to consider different plate theories within the same computational model, including ESLT or ZZT, by properly defining the transverse warping functions;
- Similarly to the plate theory itself, one readily computes the warping functions when defined in a matrix form, as demonstrated in this work, and helps other researchers to implement the mechanical model effortlessly and propose improved formulations;
- The current trend of artificial intelligence, particularly machine learning, dismisses this classical approach of representing the physical world with a physical model written in mathematical language and, recurring to the explosive growth in computing resources, uses data-driven techniques, enhancing numerical modelling in diverse scientific fields. However, most of the time, the data acquisition cost is prohibitive, and one inevitably faces the challenge of drawing conclusions and making decisions with partial information, and one still needs robust numerical methods to solve physical descriptions of the world;

- The Rayleigh-Ritz Method (RRM) and Finite Element Method (FEM) methods are two of the most important and used numerical methods for vibration analysis. The RRM is based on minimal assumptions and relies on the possible deformation shapes approximation, following the basic idea beyond Rayleigh's principle. On the other hand, the FEM is a technique for solving partial differential equations by first discretising these equations in their space dimensions. The RRM provides rapidly converging solutions but is constrained to regular shaped structures, while the local discretisation nature over small regions (finite elements) of the FEM enables the application of such a method to more complex geometries like inhomogeneous shells with PCLD patches;
- The plate theory based on warping functions adopted here requires the definition of the boundary conditions involving both the transverse displacement and its first derivatives, similarly to the Euler [1744]-Bernoulli [1751]'s beam theory or the CPT of Kirchhoff [1850]-Love and Darwin [1888]. The finite element interpolation of the transverse displacement must ensure that both transverse displacement and its first derivatives are continuous across the interelement boundaries, requiring C^1 plate bending elements. Based on previous works for simple bending plate elements, one proposed here a finite element formulation for the plate theory based on warping functions with the CPT(N), CPT(C), DKQ or IDKQ formulations for the bending component of the proposed element;
- Innovatively, one extended the adopted plate formulation to a facet-shell finite element by adding a drill degree of freedom and so widespread the modelling possibilities of a multi-layer approach of the hybrid type to complex inhomogeneous shells with PCLD patches, which is of the utmost relevance to the automotive, railway, naval, aircraft or aerospace industries, for instance, where researchers and engineers design complex structures every day and require efficient and reliable mechanical models;
- The numerical validation of the proposed formulations proved their reliability and efficiency for isotropic plates, sandwich panels, and composite plates, including symmetric, antisymmetric, cross-ply and angle-ply designs, but also sandwich shells, either bare cylindrical shells, short composite shells or sandwich shells with composite facings with orthotropic damping, one of the most complex test cases available in the literature for passive damping technologies;
- The RRM proved its faster convergence rate with fewer degrees of freedom to the analytical solutions when considering isotropic plates, but one directed efforts to the FEM due to its wide range of applicability to more complex geometries like multi-layer shells with PCLD patches;
- In the context of the present work, there are no worth mentioning differences between the implemented finite elements apart from tiny variations in the obtained results for specific cases, which seems not relevant enough to justify the improved performance of one formulation over the others;
- As expected, Equivalent Single-Layer Theories (ESLT) such as the FSDT or the TSDT proved their applicability for structures involving materials with identical mechanical properties but are obviously inefficient when modelling multilayered designs comprising materials of a very different nature like composite plates with CFRP layers and damping materials;
- As the mechanical properties of each lamina differ from each other, Zig-Zag Theories (ZZT) such as the FoZZ and ToZZ prove their efficiency in capturing the mechanical behaviour of each layer, which one promptly noticed by the warping of the warping functions;

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- In particular, differences between ESLT and ZZT are more notorious as the length-to-thickness ratio decreases, i.e. the laminate thickens, emphasising the inadequacy of ESLT to model multi-layer structures of different materials;
 - Following the validation procedure of the base plate theory, one compared the results of different test cases with formulations involving transverse warping functions computed from three-dimensional elasticity equations, extending the limits of the length-to-thickness ratio to $a/h = 4$ and $a/h = 2$, values far beyond what one can consider a plate structure, to assess the capabilities of the formulations. However, one emphasises that such limit cases are not part of the core applications for which the present model is intended;
 - The through-the-thickness representation of the transverse shear stresses highlights the differences between deformation theories, particularly compared to the transverse shear stresses obtained from the 3D equilibrium equations. An ESLT such as the FSDT or the TSDT does not rely on the transverse shear stress continuity, contrary to ZZT like those presented in this work, either the FoZZ or the ToZZ, and as emerges from the 3D equilibrium solutions regardless of the deformation theory;
 - The benchmark tests performed to assess the performance of the implemented facet shell finite element proved the robustness and reliability of the implemented formulation. Formulations including drill degrees of freedom show improved performance and better convergence rates than those without this out-of-plane rotation, mainly because finite elements established on displacement-based formulations involve the inversion of the Jacobian matrix and therefore are sensitive to mesh distortions;
 - The boundary conditions are an overlooked issue when comparing results between different works in benchmark tests. It is often difficult to be sure if the comparisons between different formulations consider the same boundary conditions, and there are potential sources of error when taking solutions from the literature without the proper framework. That is particularly evident in more complex structures involving a large number of degrees of freedom;
 - Facet shell elements, including the non-conforming CPT(N) and conforming CPT(C) bending components, require reduced integrations of the corresponding sub-matrices for some benchmark tests such as doubly-curved shells to perform accurately, suggesting some locking effect that artificially increases the structure's stiffness. On the other hand, elements including DKQ and IDKQ formulations do not present this behaviour, proving higher robustness for these cases;
 - The extension of the plate theory using warping functions to sandwich shells, enabling the modelling of different deformation theories, from the CPT to higher-order ZZT formulations, and with the capacity to model highly inhomogeneous complex shell structures with partial damping treatments with viscoelastic materials embedded in composite layers is a significant advantage of the present mechanical model. Such models are sparse or incomplete in literature and constantly complex when accessible, involving complex geometry definitions and many degrees of freedom to capture the deformation pattern of each layer;
 - There is an excellent agreement between the proposed facet shell finite element formulation and published results using different approaches, including analytical procedures and commercial software like NASTRAN. Analytical and semi-analytical methods show remarkable performance with few degrees of freedom but, too often, these procedures limit the analysis to circular cylindrical shells, and the proposed flat facet finite element might be used in

many applications involving complex inhomogeneous structures, even if one has to increase the mesh refinement;

- The proposed facet shell finite element also proves its excellent performance when modelling sandwich shells with orthotropic composite facings and viscoelastic cores. It is possible to investigate the influence of the deformation theory on the natural frequencies and damping capacity for thin and thick sandwich shells by taking advantage of the modelling capabilities involving viscoelastic damping materials;
- For short sandwich shells, damping capacity depends on the circumferential mode - higher circumferential modes imply higher bending deformation and anti-nodes in phase opposition improve the shear strain at the damping material;
- The proposed numerical model proved its robustness and versatility with a cylindrical sandwich shell comprising composite facings with orthotropic material damping and a viscoelastic core. The performance considering this demanding model for the facet shell finite element, particularly when compared with improved Fourier series and the RRM procedure, numerical procedures well designed for cylindrical shells only, proves the relevance of the suggested tool.

Fundamentally, one should highlight the following conclusions regarding the *Parametric Analysis of Damping Patches* in Part III:

- There is no doubt that the study of partial damping is a subject of great importance within vibration control. The wide range of applications that benefit from the possibility of achieving a significant damping capacity using less material is extensive, and researchers have been studying this subject in the last half-century;
- One started the parametric analysis with partially covered cantilever beams to better understand the influence of patch positioning, and there is no doubt that the damping capacity at the first mode of a cantilever beam with a viscoelastic damping patch at the clamped region is the highest one regardless of the covered length. For specific parameters, the modal loss factor at the first mode with a covered area of just 50 % is higher than that of a fully covered beam;
- There is an optimum covered length at which the modal loss factor is the highest one, proving the uselessness of integral damping treatments, at least for specific modes;
- The structural modification induced by a constrained viscoelastic damping patch is also relevant, and while a damping patch at the cantilever beam's tip decreases the first natural frequency, a patch at the clamped section increases it. The relative influence of added mass or stiffness sets the variation of the natural frequencies;
- One remarkable aspect is that both the maxima of the first modal loss factor and the first natural frequency occur for practically the same covered length between 60 and 70 %;
- Since the transverse shear strain within the viscoelastic damping layer is the primary responsible for the damping mechanism in PCLD configurations, transverse shear strain degrees of freedom and the corresponding mode shapes can be a helpful tool to predict the best patch location;
- For a cantilever beam, the transverse shear strain mode shape peaks at the clamped region and then decreases until the free boundary. However, one should be careful with

representations of the transverse shear strain pattern reduced to the reference plane of the bare beam since damping capacity comes from the transverse shear strain in the damping layer;

- Transverse shear strain mode shapes help to realise the patch positioning in covered beams. For low-performance treatments, one can notice flat regions (low transverse shear strain) created by the damping treatments and a boundary region where one can note a sharp change in the strain pattern;
- Mass normalised mode shapes are more suitable than displacement normalised mode shapes to distinguish patch-damping treatments and evaluate their relative performance. One can not compare levels of modal strain between different damping treatments with displacement normalised mode shapes. Preferably, mass normalisation of the mode shapes is global - one uses the mass properties of the system to perform the normalisation, and one can compare levels of modal strain between different configurations;
- Damping patches promoting higher damping performances than a fully covered beam present higher modal transverse shear strains;
- For PCLD patches, the modal loss factors computed with only the shear component of the stiffness matrix are indistinguishable from those obtained with the complete stiffness matrix, proving the significance of the modal transverse shear strain on the damping capacity of constrained layer structures and reinforcing the importance of transverse shear strain mode shapes as a tool to predict and design damping treatments for damped systems;
- Although relevant, transverse shear strain mode shapes associated with the transverse shear strain degree of freedom of the reference plane sum up all the information of a multilayered structure to just one point through-the-thickness, and one uses the potential of the proposed numerical model to compute the modal transverse shear strain through-the-thickness;
- A through-the-thickness representation of the transverse shear strain mode shape provides more information about the pattern occurring in a specific mode, presenting a clear insight regarding the possible best solutions for damping treatments;
- Since it is intuitive to think of a transverse shear stress mode shape as the modal transverse shear strain multiplied by the material shear modulus point-by-point, one used modal transverse shear strain and stress to compute the modal transverse shear strain energy per unit volume and presented a tool for an in-depth analysis of patch damping treatments;
- Transverse shear strain near the free end of a bare cantilever beam tends to zero, so it is reasonable to suppose that adding damping material in this region would be useless. Besides, the obtained results inclusively show that it could be disadvantageous;
- Transverse shear strain within the damping layer is essential to promote damping but is the extensional strain, also known as normal strain, of the stiffer constraining layer that imposes shear on the softer damping layer. One computed the extensional strain, stress and strain energy density using the corresponding degrees of freedom;
- Since the transverse shear strain is a degree of freedom, the present model can not correctly represent transverse shear strains and stresses at clamped supports. One can compute transverse shear stresses from equilibrium equations to overcome this issue. Similarly, one can also obtain transverse shear strain and transverse shear strain energy density mode shapes from transverse shear stress, recovered from equilibrium equations, generating a correct representation of these quantities at clamped supports;

- The main goal of such representations is to achieve a comprehensive understanding of the damping mechanisms associated with PCLD patches since they disclose the modal shear strain pattern of the structure and help to explain the superior damping performance of some configurations over the others;
- Zoomed views of the damping layers help to explore the transverse shear strain pattern within these layers and disclose the higher levels of transverse shear strain for a PCLD patch at the clamped region, for example. With such representations, the superior damping performance of the damping patch at the clamped area of a cantilever beam, inclusively over the fully covered treatment, is intelligible since the modal transverse shear strain within the viscoelastic layer is higher;
- In some cases, the maximum modal transverse shear strain of the damping patch at the clamped area doubles that of the fully covered configuration;
- Modal transverse shear stresses follow the same trend - damping patches with improved damping performances show damping layers subjected to higher modal transverse shear stresses;
- Transverse shear strain energy density mode shapes are even more self-evident. Higher levels of modal transverse shear strain energy density are distinguishable in damping patches with improved damping performances;
- Different authors point out that extensional strain/stress is essential for understanding the damping mechanism of constrained layer damping since the constraining layer imposes the transverse shear strain on the damping layer. This work confirms the allegation and shows that damping patches with improved damping performances show higher levels of extensional strain/stress at the constraining layers, inclusively when damping performance is better than those with fully covered treatments;
- More specifically, and regarding the influence of the covered length for a cantilever beam, there is an optimum covered length to achieve the maximum damping capacity in the first mode for a specific damping layer thickness. This optimum covered length is a function of the damping layer thickness - beams with thinner damping layers have maximum damping efficiency with shorter treatments, increasing for thick damping layers;
- There is an increasing-decreasing effect on the damping capacity as damping layers thicken. The thicker the damping layer, the higher the damping efficiency, but there is a threshold, after which damping efficiency decreases. After this threshold, one can only achieve the maximum damping capacity with fully covered base beams. Sher and Moreira [2013] identified this effect when dealing with thin viscoelastic layers on fully covered beams and proved that peak damping performances do not need thick damping layers;
- Increasing damping layer thickness does not necessarily increase damping since the shear strain at the viscoelastic layer might fall by doing so, explaining the increasing-decreasing behaviour of the damping capacity;
- Damping efficiency curves for higher modes show that improved damping performances require considerably thinner damping layers than those optimal for the first mode. Still, maximum damping capacity occurs with partial coverage when damping layers are thinner enough;

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- One noticed locally decreased performances of the specific modal loss factors for higher modes when the edge of the damping treatment lies right above the anti-nodes of the extensional strain energy density (nodes of transverse shear strain energy density) of the bare beam. The specific modal loss factors alternately increase and decrease as the edge of the damping treatment passes, respectively, through the nodes and anti-nodes of the extensional strain energy density of the bare beam. It is desirable to cover regions with a higher extensional strain energy density of the host beam and avoid the edges of the damping treatment over these areas as elastic layers must undergo extensional strain to impose shear on the damping layer;
 - It is possible to achieve a higher damping capacity with the same damping material with thinner base beams;
 - For thin base beams, maximum damping efficiency occurs with partially covered lengths for thin damping layers and fully covered lengths for thick damping layers. Besides, modal loss factors for higher modes present local maxima of damping efficiency with partially covered structures, apart from the global maximum, and the number of these local maxima is proportional to the order of the mode;
 - Just add damping material blindly could be highly ineffective, specifically for higher modes. Adding damping material is far from increasing damping in the structure;
 - For thicker beams, thicker damping layers quickly decrease the damping performance. For thicker damping layers, damping efficiency reduces so much that it is difficult to figure out the local maxima of the specific modal loss factors;
 - Regardless of the base and constraining layer thicknesses, improved damping capacity with tiny covered lengths requires thinner damping layers. A small damping patch with a thick damping layer decouples the constraining layer from the host structure, and the damping layer does not experience a high level of transverse shear strain. Coupling the damping patch to the host beam to impose transverse shear strain on the damping layer requires thinner damping layer thicknesses;
 - For constant thin damping layers, thicker base beams promote more damping capacity, while for thick damping layers, thinner base beams improve damping compared to thick base beams;
 - For non-symmetric configurations, with constraining layers thinner than the base beam, the damping performance of the damping treatments lowers as the constraining layer gets thinner - thinner constraining layers impose less transverse shear strain on the damping layer;
 - For non-symmetric configurations, maximum damping efficiency occurs for smaller covered lengths as the constraining layer gets thinner, though damping capacity decreases. Still, there is maximum damping efficiency with partial coverage for a particular thickness of the damping layer;
 - The modulus ratio expresses the ratio between the elastic and damping layers modules. Higher modulus ratios increase damping efficiency until a threshold and push the optimum covered length closer to the end of the beam;
 - Higher modal loss factors show the increasing-decreasing effect on the damping capacity as the modulus ratio increases, proving the existence of an optimum modulus ratio for

optimum damping performance. There is also an optimum partially covered length for specific modulus ratios;

- Thinner base and constraining layers cannot impose high transverse shear strain rates on the damping layer as thicker ones, so one needs higher modulus ratios to increase damping capacity. On the contrary, thicker base and constraining layers need lower modulus ratios for optimum damping performances;
- Regardless of the base and constraining layers thicknesses, improved damping capacity with small covered lengths requires lower modulus ratios. A small damping patch with a high modulus ratio decouples the constraining layer from the host structure, and the damping layer does not experience high levels of transverse shear strain. Consequently, one requires lower modulus ratios for minor differences between the layers to couple the damping patch with the host beam and impose transverse shear strain on the damping layer;
- For free-free beams with a central damping patch, there is also an optimum covered length to achieve the maximum damping capacity at the first mode for a given damping layer thickness, proving that this phenomenon does not depend upon specific boundary conditions, and damping efficiency presents the same trends as those of a cantilever beam with a damping patch at the clamped region;
- The optimum covered length depends on damping layer thickness - beams with thinner damping layers have a maximum damping efficiency with less covered length, which increases for thick damping layers. The increasing-decreasing effect of the damping capacity is noticeable as the thickness of the damping layer gets thick, and one can only achieve the maximum damping capacity with full coverage when considering thicker damping layers;
- Odd and even modes of a central patched free-free beam with small patches show relevant results, namely that a mid-length patch shows improved performances for odd modes. Odd extensional strain energy density mode shapes present anti-nodes at the beam's half-length, the reference position for the central patch, and even extensional strain energy density mode shapes have nodes at that location. At the same time, the extensional strain energy density mode shape anti-nodes match the transverse shear strain energy density mode shape nodes, proving that the commanding mechanism is the extensional strain of the constraining layer, something often ignored in many studies;
- Positioning a damping patch at high extensional strain energy density locations will strain the constraining layer (stiffer layer) and promote shear at the damping layer (softer layer). Quite often, researchers mention the higher transverse shear strain density locations as the preferential ones regarding the positioning of damping patches, but the free-free beam shows the missing part of the story;
- Some authors have discussed cutting the constraining layer into appropriate lengths to increase transverse shear strain locally for improved damping performances, and one studied such an approach in this work. It is worth noting that one can significantly improve damping capacity with segmented constraining layers, and there is an optimum position for the cut section of the constraining layer as a function of the damping layer thickness, for example;
- For cantilever beams, as the damping layer thickness increases, the optimal section for segmentation comes close to the free end section of the beam. Compared with thinner damping layers, as thickness increases, a cut in the constraining layer near the clamped region decreases the efficiency;

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- Similarly to partially covered beams, higher specific modal loss factors show that thinner damping layers improve the damping capacity by cutting the constraining layer, but there is a threshold after which to cut the constraining layer only degrades the treatment efficiency. This result proves the importance of parametric studies to understand that the difference between optimum and poor solutions is often small.
 - In particular, for a cantilever beam, a cut right after the beam's half-length degrades the damping capacity at the second specific modal loss factor, and a cut at one-third or three-quarters of the beam's length degrades the treatment efficiency for the third specific modal loss factor, regardless of the damping layer thickness, since cuts of the constraining layer at sections of maximum extensional strain lead to poor damping performances;
 - The extensional strain of the constraining layer plays a decisive role in its capacity to impose transverse shear strain on the damping layer, and the segmented constraining layer can experience high levels extensional strain when compared to fully covered or patched structures, similarly to patch treatments;
 - Thin cantilever beams with segmented constraining layers reveal the exception that breaks the rule that one must avoid cuts of the constraining layer at sections of maximum extensional strain. For the second mode of thin base beams with thin damping layers, a cut near the section with the maximum extensional strain improves the specific modal loss factor;
 - The previous result explains the rule of thumb raised by Lepoittevin and Kress [2010] recommending segmentation of the damping treatment at sections with the maximum bending moment, a conflicting conclusion with the results presented in this work;
 - Lepoittevin and Kress [2010] performed the optimisation of the segmentation sections considering constant thicknesses of either base, damping and constraining layers, as well as different mechanical properties, which may well have led to a particular conclusion;
 - The optimum position for segmentation is not always the section with the maximum bending moment, as this is just the case for particular cases such as thin base beams with thin damping layers;
 - When the damping layer thickness exceeds the threshold of the increasing-decreasing behaviour in damping capacity, segmentation is worthless and downgrades the damping performance;
 - For higher modes of thick base beams, there is no real benefit from the segmentation of the constraining layer, and there is a section, virtually independent of the damping layer thickness, at which the damping capacity decreases drastically;
 - There are no significant differences regarding segmentation for non-symmetric configurations, apart from the expected limited capacity to impose shear strain in the damping layer of thinner constraining layers;
 - Comparing segmented constrained layer with damping patches and focusing on the first specific modal loss factor and thinner damping layers, for example, the superior performance of segmented constraining layer treatments near the clamped section of a cantilever beam compared to a partial treatment is unquestionable, but so the additional weight;

- For cantilever beams, as the length of the damping patch increases or the cut of the constraining layer moves towards the free end of the beam, there is a focal point after which the first specific modal loss factors of both configurations are equivalent. The weight of the free-end side treatment in segmented treatments is needless. There is no shear strain in the damping layer at the free-end side after the cut of the constraining layer;
- There is an optimum position for the cut section of the constraining layer as a function of the modulus ratio. For the first mode of a cantilever beam, as the difference in stiffness between both elastic and damping materials increases, the optimal section for segmentation comes close to the free end section of the beam;
- Segmentation of the constraining layer is more effective as the modulus ratio decreases. As the difference in stiffness between both materials decreases, there is less shear in the damping layer and so damping capacity, but the coupling between base and constraining layers increases, and segmentation has a higher impact in promoting shear;
- Segmentation of free-free beams proves that segmenting the constraining layer at sections of maximum modal extensional strain energy is troublesome. Damping capacity decreases markedly for the first specific modal loss factor with a cut at the central region of the constraining layer where the extensional strain energy density is maximum;
- Considering thin symmetric free-free beams arises the exception to the rule that one must avoid cuts of the constraining layer at sections of maximum modal extensional strain. For thin damping layers, the first specific modal loss factor increases by cutting the constraining layer at the mid-length of the beam;
- When dealing with thick base free-free beams and consequently with thick constraining layers in symmetric configurations, it is noticeable that damping capacity decreases with cuts of the constraining layer at sections of maximum modal extensional strain energy even more markedly;
- The first specific modal loss factor of free-free beams highlights the importance of the parametric studies. For low modulus ratios, the stiffness difference between both materials is lower, and segmentation of the constraining layer at the centre of the beam (extensional strain energy anti-node) increases the damping capacity;
- For thin free-free beams, the improved damping performance with the segmentation of the constraining layer at the centre of the beam (extensional strain energy anti-node) is visible for more modulus ratios and not just the lower one. In contrast, segmentation becomes less effective for thick beams, and damping efficiency curves show that segmenting the constraining layer at sections of higher modal extensional strain is prejudicial;
- Extending the study of partially covered structures to plates, many authors restricted their investigation to square patches, which might not be the optimum shapes for improved damping efficiency and did not thoroughly explain the superior performance of some locations over others, presenting somewhat conflicting results;
- The study of partially covered beams proves that the geometric and mechanical properties of the structure have critical importance on the performance of damping layer treatments. Therefore, it is not surprising that different geometric and mechanical properties could influence the best positions for partial damping treatments in plates;

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- Taking advantage of the mechanical model implemented in this work, namely its ease of use concerning passive constrained damping patches, it is possible to execute a simple routine to perform a set of parametric studies regarding the damping efficiency of partial damping treatments other than square patches;
 - Following the same reasoning of the previous studies, one considered the influence of different parameters of a damping patch, such as the patch position, covered area, thickness/side length ratio, constraining layer thickness and material modulus on the damping performance of a simply supported and free square plates;
 - Damping capacity depends on the covered area, increasing almost steadily with the amount of damping material. Nevertheless, for virtually all positions, there is a maximum attainable damping capacity accomplished with full coverage of a simply supported plate. For thick damping layers, the highest specific modal loss factor decreases, proving the existence of a threshold for maximum damping capacity. It is the increasing-decreasing effect of the damping capacity already studied for the case of beams. As discussed earlier, there is more damping material with thicker damping layers, but shear strain at the viscoelastic layer drops;
 - Damping efficiency curves for higher modes show that thick damping layers mean less damping capacity, proving the thinner damping layer thickness threshold for the increasing-decreasing behaviour of the damping performance;
 - There are noticeable differences between the different positions of a square patch for a simply supported plate. The increase of the first specific modal loss factor with the covered area for a patch at the central region is not steady, as for other positions, and the growth rate of damping capacity is higher for small covered areas and lower as the covered area increases. Such effect is evident with optimum damping layer thicknesses, proving the potential to achieve higher damping with partial coverage for plates;
 - The low damping capacity of the L-Contour treatment for partial coverage of a simply supported plate is indisputable either for the first or higher specific modal loss factors;
 - Contrary to the L-Contour treatment, a damping patch with this shape ensures a significant damping capacity, even for small covered areas of a simply supported plate;
 - The Contour treatment promotes poor damping capacities for all the modes studied. It is just for large covered areas, with the damping treatment increasing towards the centre of the plate, that damping capacity increases, proving the inefficiency of the Contour treatment to raise the damping capacity of a simply supported plate;
 - One can clearly distinguish the damping efficiency curves between two reference positions for a simply supported plate - the Contour treatment and the Position 13 (central position) treatment. The square patch at the central position presents the best damping efficiency for almost the entire domain of the covered area. On the other hand, the Contour treatment exhibits the lowest damping capacity. The differences between these two counterpart treatments are significant - for 60 % covered area, the damping treatment at the centre of the plate nearly doubles the damping capacity of the Contour one;
 - As discussed for partially covered beams, it is crucial to ensure the extensional deformation of the constraining layer as a mechanism to impose shear on the damping layer. Consequently, damping treatments covering the central area of simply supported plates promote better damping performances as the constraining layer experiences higher states of extensional

deformation and so imposes major levels of shear strain on the damping layer. In contrast, treatments covering preferentially the contour of the plate, where the curvature and extensional deformation are null, present limited damping performances;

- Through-the-thickness representations of the transverse shear strain energy density mode shapes are clear - the edges of the damping treatments play a decisive role in the damping efficiency of a given treatment. It is possible to notice the higher levels of modal transverse shear strain energy density at the edges of the damping treatments for both directions of the simply supported plate and the higher levels of modal shear strain energy density of the damping layer at Position 13 (central position), explain its superior performance;
- A square damping patch at Position 1 (corner position) presents the highest extensional strain energy density level at the constraining layer. However, this high extensional strain energy density is restricted to a small area of the constraining layer near the edge of the damping treatment. For square damping patches at Positions 3 and 13 (mid-length and central positions, respectively), the high extensional strain energy density levels are not restricted to a limited area but evolve along the entire constraining layer section. These results prove the need for high levels of extensional strain at the constraining layer for high levels of shear strain at the damping layer and consequently improved damping performances;
- The damping performance decreases remarkably for higher damping layer thickness, and the differences between distinct damping treatment positions are negligible. Thicker damping layers cut down not only the damping performance but cancel any improved damping performance of a given position over the others;
- For higher vibration modes, thicker damping layers mean less damping capacity, as expected from the parametric studies considering patched beams;
- Interestingly, there are significant differences regarding the best patch positions when considering thin damping layers and higher modes. The Contour treatment still provides the lowest damping capacity at these vibration modes, explained by the zero curvature for both directions at the edges of the simply supported plate, but the central patch at Position 13 no longer exhibits the best damping performance. Apart from local optima for small covered areas attainable with a square patch at Position 1 (corner position) or Position 3 (mid-length position), the Cross-Shape treatment presents the best overall damping performance at the second and third modes of vibration;
- The curvature mode shapes follow the same pattern as the transverse displacement mode shapes for both directions of a simply supported plate, with a nodal line separating two equal areas with maximum curvature at the centre and rotating ninety degrees between the second and third mode shapes. Each arm of the Cross-Shape treatment covers both anti-nodes of the second and third curvature mode shapes, promoting higher levels of the extensional deformation of the constraining layer and the transverse shear strain of the damping layer;
- For thin base plates, and in the first place, it is worth emphasising the limited damping capacity of thin damping layers;
- Following the trend of the reference base plate, the Contour patch is the most ineffective damping treatment for thin base plates. In contrast, the central patch at Position 13 also presents a distinguished damping capacity, but it is worth to realise that it is no longer the

best one. For a thin base simply supported plate, the Cross-Shape treatment exhibits the best damping performance at the first mode of vibration, regardless of the covered area;

- The curvature mode shapes highlight the anti-nodes of curvature at the centre of the plate, and each arm of the Cross-Shape treatment crosses the central area of a simply supported plate. It is a critical result; one proves that the thickness of the base structure changes the optimal shape of a damping treatment, and analysing its mode shapes is mandatory, but not everything. By increasing the damping layer thickness beyond the threshold value, the efficiency of any damping treatment decreases, and the relative difference between them narrows;
- Unsurprisingly, when considering thick base plates, thicker damping layers decrease the damping performance since the increasing-decreasing behaviour of the damping capacity occurs for even thinner damping layers;
- For thin damping layers in thick base simply supported plates, the Cross-Shape treatment exhibits the best damping capacity at the first mode of vibration regardless of the covered area, followed by the square patch at the central Position 13. Treatments covering preferentially the edges of the plate, Contour or L-Contour patch treatments, follow the expected performance and present limited damping capacities;
- For both the second and third modes of vibration, the Cross-Shape treatment over thick base plates reveals its superior overall damping performance for thin damping layers despite some specific improved performances of Position 1 (corner position) and Position 3 (mid-length position) for small covered areas. Similarly, the Contour treatment presents a lower damping capacity. Thicker damping layers consistently follow a recurring pattern - meaningless damping performances with no difference between the various patch positions;
- When studying the damping efficiency of partially covered beams, one discussed the mechanism behind the reduced damping performance of structures covered by thick damping layers. Damping patches with thick damping layers decouple the constraining layer from the base structure, and the damping layer does not experience a high level of transverse shear strain;
- As already discussed, the material properties of the base, damping and constraining layers also play a crucial role in the damping efficiency of partial damping treatments. For partially covered simply supported plates and low modulus ratios, the materials of both base/constraining and damping layers have comparable stiffness, and one achieves the optimum damping capacity for any of the first three modes of vibration with a partially covered area;
- Similarly to the influence of the damping layer thickness, there is an increasing-decreasing behaviour of damping capacity for different modulus ratios and a maximum attainable damping capacity for any of the first three modes accomplished with thorough coverage of simply supported plates. The results show an optimum modulus ratio promoting coupling between base and constraining layers, leading to enhanced transverse shear strain at the damping layer and improved damping performances. The optimum damping capacity occurs for a lower modulus ratio for the second and third specific modal loss factors;
- The threshold values of the modulus ratio for the increasing-decreasing behaviour are the same regardless of the damping treatment position. For higher modulus ratios, the stiffness of both elastic and damping materials is quite different, and the damping capacity drops

remarkably, regardless of the covered area. This result shows that when the stiffness of the elastic material is far greater than that of the damping layer, promoting the decoupling of the constraining layer from the host structure, the damping treatment loses efficiency;

- Partial treatments preferentially covering the boundaries of a simply supported plate promote limited damping capacities for small covered areas. As expected, the threshold modulus ratio values of the increasing-decreasing behaviour of damping capacity are the same but are noticeable in the relatively poor performance of the L-Contour or Contour treatments compared to the other positions;
- When both constraining and damping layers have comparable stiffness, the constraining layer can no longer impose a significant transverse shear strain on the damping layer, limiting the damping capacity. Nevertheless, a square patch at Position 13 (central position) exhibits a distinctive damping efficiency, including an optimal performance with partial coverage for the first mode of vibration of a simply supported plate;
- For optimal modulus ratios, the square patch at the central Position 13 of a simply supported plate shows the best damping capacity for almost the entire domain of the covered area, followed soon by the Cross-Shape treatment;
- Damping capacity decreases when the stiffness of the constraining layer is far higher than that of the damping layer, and there is no significant distinction between different positions. As previously discussed, when considering thicker damping layers, a high modulus ratio decouples the base and constraining layers, disabling the ability to impose shear on the damping layer;
- There are no significant differences between the best patch positions between the first specific modal loss factor and the second and third one for the lowest modulus ratio. For almost the entire domain of the covered area, the square patch at the central Position 13 exhibits the best damping performance for the second and third modes of vibration, which inclusively one achieves with a partially covered area;
- For optimum modulus ratios, though, and similarly to the results regarding the threshold value of damping layer thickness discussed before, the Cross-Shape treatment presents the overall best damping performance at the second and third modes of vibration, followed by the square patch at the central Position 13;
- Higher modulus ratios decouple the constraining and base layers and undermine the damping capacity of the treatment. For the second and third modes of vibration, there are no significant differences between any treatment, and the amount of damping capacity is residual;
- Following the same reasoning of previous analyses, one extended the study of partially covered structures to free boundary plates to generalise the conclusions and prove the independence from specific boundary conditions;
- The crucial distinction between the results for a free panel compared to the simply supported one is the maximum damping capacity achievable with a partially covered area;
- Regarding the influence of the patch positioning with thin damping layers on free boundary plates, the results are straightforward. There is a meaningful damping capacity, and the square patch at central Position 13 provides the best damping capacity at the first vibration mode for a wide range of covered areas, which inclusively peaks for a relatively small

partially covered area. However, quite surprisingly, the Contour treatment presents the best damping capacity, achievable with a covered area of roughly 80 %, after being the worst possible solution when considering small patches;

- For optimum damping layer thicknesses, it is possible to distinguish two reference positions for free boundary plates - the square patch at Position 13, with the best damping capacity for a wide range of covered areas, and the Contour treatment, with the worst damping efficiency considering any covered area. Other square patches introduce significant damping capacities, but one achieves the optimum damping capacity at this mode of vibration with the Cross-Shape treatment and a partially covered area, proving the possible inefficiency of fully covered plates for improved damping performances;
- The Contour treatment presents conflicting results for different covered areas and damping layer thicknesses. It is challenging to explain the poor damping capacities of the contour treatments, and investigating the mode shapes of the base or fully covered plates might not be enough to design efficient patch treatments. Besides, these results highlight the importance of the parametric analyses on the appropriate design of partial damping treatments;
- Transverse shear strain mode shapes at both directions show no valuable insights about the damping treatments and their effectiveness, presenting no more information for any damping treatment than flat regions for almost the entire domain with maximum values at the edges of the free plate;
- On the other hand, through-the-thickness views of the transverse shear and extensional strain energy density mode shapes highlight and help to explain the best damping performances of some treatments over others;
- Nevertheless, one must be careful when selecting and analysing through-the-thickness views of section cuts. The visual information is impressive, but one has to consider the constraints, as the through-the-thickness section cuts of the first membrane plus bending strain energy density mode shapes for the free boundary plate shows;
- As expected, increasing the damping layer thickness beyond the threshold value decouples the base and constraining layers and transverse shear strain levels at the damping layer drop regardless of the boundary conditions;
- The free boundary plate confirms the already noticed conclusion of thinner optimum damping layer thicknesses for higher vibration modes;
- For thin base free plates, damping capacity increases as the thickness of the damping layer extends from $h_d = 0.005h$ to $h_d = 0.32h$. Certainly, the threshold value of damping layer thickness for thin base plates occurs for thicker layers than those considered here, but the main results for the first specific modal loss factor of a thin base plate are similar to those of the reference plate;
- Thin damping layers over thin free boundary host plates provide reduced damping capacities. However, thicker damping layers significantly improve energy dissipation. Either for thin or thick damping layers, the square patch at central Position 13 presents the best performance, followed by the Cross-Shape treatment, similar to the previous trend already noticed for the reference plate. Consistently, the improved performance of the square patch at the central Position 13 holds for the second and third specific modal loss factors;

- When considering thick base free plates, one proves that the threshold value of damping layer thickness occurs for thin damping layers for the first and second vibration modes, demonstrating that thick base structures require thin damping layers for improved performances, contrary to thin base structures, which require thick damping layers;
- The square patch at the central Position 13 presents an efficient damping capacity for a wide range of covered areas for thick free base plates and thin damping layers, but one achieves the optimum damping capacity with the Cross-Shape treatment and a partially covered area. Similar to previous examples, one proves the usefulness of fully covered free plates for thicker base plates. Besides, and unsurprisingly, considering the reference and thin base plates, the contour treatments exhibit limited capacities for the first vibration mode;
- Thick free base plates and higher vibration modes show the same trends presented before, proving the universality of the conclusions drawn in this work and their independence from boundary conditions;
- The material properties decisively influence the ability of the constraining layer to impose shear on the constrained damping material. Focusing on the material properties of free boundary plates, it turns out that the growth rate of damping capacity with the covered area depends on the modulus ratio. Namely, when the materials of both base and constraining layers and the damping material have comparable stiffness, the modulus ratio is low, and the optimum damping capacity occurs for partially covered areas. As the modulus ratio increases and one considers softer damping materials or stiffer base and constraining layers, the respective optimum damping capacity increases but occurs for increasingly larger covered areas, which applies to the first three specific modal loss factors;
- When considering higher modes of vibration, such as the third one, for example, it is possible to realise that the corresponding specific modal loss factors increase up to the modulus ratio threshold value around 3×10^4 , proving that optimum damping efficiency for higher modes requires lower modulus ratios. These results prove the existence of an optimum modulus ratio promoting the perfect coupling between base and constraining layers, which leads to enhanced transverse shear strain at the damping material and improved damping performance;
- The enhanced performance of a patch at the central Position 13 for small covered areas and low modulus ratios, in particular, is indisputable. That is a remarkable feature of the square patch at Position 13, already discussed in previous analyses and still valid for free boundary plates - the constraining layer at this position experiences higher extensional deformation, leading to the higher shear strain of the damping layer;
- Similarly to previous analyses, higher modulus ratios decrease the damping capacity at any vibration mode of free boundary plates, proving the universality of such a conclusion regardless of the boundary conditions. The stiffness of both elastic and damping materials is significantly different, promoting the decoupling between the base and constraining layers and the consequent inability to impose shear on the damping material;
- Free boundary plates with treatments covering the contour regions reported weak damping performances. Such results highlight the same patterns of limited damping capacities for small covered areas;
- When the constraining and damping layers have comparable stiffness, i.e. for low modulus ratios, there still is a meaningful damping efficiency. Undoubtedly, the square patch at

central Position 13 presents the best damping capacity for a wide range of covered areas, peaking for nearly half of the covered area. Nevertheless, the Contour treatment provides the worst damping performance for small covered areas and exceeds all other treatments as the covered area increases;

- Curiously, low modulus ratios and thinner damping layers develop similar damping efficiency outcomes. Still, it is not an unexpected result since both cases tend to couple the base and constraining layers more effectively;
- For optimum modulus ratios, the reference positions of the square patch at the central Position 13 at a free boundary plate show the best damping capacities for a wide range of covered areas and the Contour treatment with the worst damping efficiency considering any covered area. Nevertheless, the Cross-Shape treatment proves the possible uselessness of fully covered plates for specific configurations since one achieves the optimum damping capacity at the first vibration mode with a partially covered area;
- For higher modulus ratios, free boundary plates prove that there is a decoupling between the base and constraining layers, disabling the ability of the latter to impose shear on the former. Indeed, any damping treatment configuration presents limited damping capacities, and there are no noteworthy differences between the different damping treatments;
- Apart from the surface damping treatments for vibration control of structures comprising isotropic materials, viscoelastic materials have also seen their use extended to applications where composite materials find increasing applicability. Due to their excellent specific mechanical properties, these materials find widespread use in the automotive, railway or aerospace industries, where vibration control is mandatory;
- Composite materials exhibit damping levels that are several orders of magnitude higher than traditional metallic materials. In addition to the inherent properties of their constituent elements (fibres and matrix), the interaction between them plays a crucial role in the damping mechanism. The strength of the bond between fibres and matrix directly impacts the energy dissipation resulting from the relative movement of the fibres. However, a weaker bond compromises load transmission and can lead to a loss in strength;
- The design requirements for high stiffness frequently clash with those for high damping capacity. Stiffness and damping capacity represent two conflicting demands. As the volume fraction of fibres increases to enhance the laminate's stiffness, the corresponding reduction in the volume fraction of the resin matrix leads to a decrease in its damping capacity;
- Composite laminates with interleaved damping layers are an active research topic with increasing interest. The optimal design of viscoelastic damping treatments typically follows two main approaches: optimisation algorithms for optimal damping efficiency or parametric analyses to calculate damping properties using discrete calculations;
- Optimisation algorithms, for example, are of the utmost importance when dealing with complex structures and big data. However, these powerful algorithms often are black boxes providing optimal solutions to complex problems, although giving the designer, engineer or researcher little help in identifying the mechanisms which lead to the desired damping capacity and ways to apply the same principles elsewhere;
- The classical method of embedding viscoelastic damping layers into the laminate stacking sequence to improve damping capacity might undermine the laminate's stiffness and mechanical strength beyond weight increase, and to overcome this, authors like Fotsing

et al. [2012] proposed the use of partial viscoelastic layers interleaved between carbon-epoxy layers, experimentally studying the position of a damping patch along the beam and concluding that patches close to the clamped region and over the mode shape nodes proved efficient in improving damping capacity, confirming previous conclusions for isotropic materials, including the possible uselessness of full coverage treatments;

- Differently from the classical damping treatments, one suggested the numerical and experimental investigation of partially distributed viscoelastic damping treatments for laminated composite plates in an integrated arrangement in this work, with viscoelastic layers taking part in a stacking sequence of composite laminae to increase damping, partially distributed within specific layers;
- Firstly, studying the damping efficiency of interleaved viscoelastic layers in laminated composite plates parametrically, one considered a dimensionless analysis of six stacking sequences to evaluate the influence of the viscoelastic layer thickness on the damping capacity with arrangements covering the whole range of possible symmetric configurations for a six-layer reference laminate, including viscoelastic layers within the stacking sequence to compare different positions by performing parametric analyses and using the same amount of damping material;
- Mode shapes, and strain, stress and energy density mode shapes, in particular, show all their relevance in applications involving laminated composites;
- One can easily distinguish higher extensional stress and strain energy density along the different directions within layers depending on the specific mechanical properties, namely $E_{11} > E_{22}$, for example. As previously discussed, it is crucial to comprehend the extensional strain/stress distribution to evaluate damping capacity by the ability of the stiffer layers to impose transverse shear strain on damping layers;
- Regarding the influence of thickness and position of the interleaved viscoelastic layers on the damping capacity of laminated plates and considering the first modal loss factor of the simply supported six-layer reference laminated composite plate considered in this work, it is possible to conclude that the modal loss factor depends almost exclusively on the amount of damping material regardless of the stacking configuration;
- However, for thicker damping layers, the Single (S) and stacking sequences with damping layers at the skins lose their ability to improve damping capacity. Furthermore, there is a plateau of modal loss factor, and increasing damping material no longer increases damping capacity at the first vibration mode. The modal loss factor can even decrease with increasing damping layer thicknesses, and the Single (S) and double interleaved configurations with damping layers closer to the skins, Double-Skins (D-S), show the worst damping capacity of all stacking sequences for the same volume of damping material;
- In contrast, interleaved stacking sequences with damping layers close to the core, Double-Core (D-C) and Triple-Core (T-C), and particularly the Quintuple (Q) configuration, show the best overall damping capacities for the simply supported six-layer reference laminated composite plate;
- Core configurations with more than one layer are preferable for higher damping capacity than skins configurations. As already discussed, the transverse shear strain of the damping layer is the primary mechanism responsible for damping capacity, and often, many authors report that the damping layer should be positioned at the neutral axis to maximise damping.

When dealing with laminated composite materials, such requirements may be insufficient, as the damping capacity of the Single (S) interleaved configuration shows;

- For thicker laminated composite plates, $a/h = 50$, and thin damping layers, the first modal loss factor increases approximately linearly (log scale for the thickness variable), with core configurations offering the best damping capacities. After the maximum attainable modal loss factor, damping capacity drops markedly, and more damping material decreases damping capacity;
- The Quintuple (Q) configuration presents the best overall damping capacity at the first vibration mode for thicker laminated composite plates, $a/h = 50$. On the other hand, despite the poor performance of the stacking sequences considering damping layers close to the skins, the single-layer arrangement shows a weaker damping performance for thicker laminated composite plates, especially for thicker damping layers. A soft single-layer damping treatment at the core of the laminate decouples both halves of the CFRP, and so their ability to impose transverse shear strain to the viscoelastic layer, as previously discussed for patch treatments in structures comprising isotropic materials;
- Higher vibration modes show the same trends as those of the first vibration mode - the modal loss factor increases with increasing damping layer thicknesses for thin damping layers, reaches a global maximum damping capacity, and the damping capacity sharply decreases. It is possible to identify a maximum attainable damping capacity for different stacking sequences, and Core configurations, together with the Quintuple (Q) stacking sequence, perform better than Single (S) and Skins stacking sequences;
- The main difference between the reference laminated square composite plate with an aspect ratio $a/h = 100$ and a thicker one with an aspect ratio $a/h = 50$ is that the maximum modal loss factor for a thicker laminate occurs for a lower specific damping layer thickness;
- The modal strain, stress and strain energy density mode shapes proved to be valuable tools to explain the damping performance of a given damping treatment. These tools are even more helpful for laminated composite plates due to the complexity of the problem, i.e. the large number of design variables under consideration;
- Much more than the total modal transverse shear strain energy, through-the-thickness patterns are helpful representations that help to interpret the damping mechanism of laminated composites with interleaved damping layers;
- Layers close to the laminate core present high levels of modal transverse shear strain energy density, which explains the superior performance of core stacking sequences, together with the Quintuple (Q) configuration;
- Despite being just one thick core layer, the Single (S) interleaved damping layer does not present the best damping capacity. As previously discussed, when dealing with damping patches on isotropic beams and plates, there is a turning point after which thick soft layers tend to decouple the stiff layers and so their ability to impose shear. In the Single (S) stacking sequence, both halves of the laminated composite work as the base and constraining sub-laminates;
- Regarding the ability of the stiff composite layers to impose transverse shear strain on the soft damping layers, the results are straightforward for both modal strain energy densities, regardless of the stacking sequence - 0° oriented composite layers present the highest modal extensional strain energy density $\phi_1^{U_n^x}$, while the outer 90° oriented ones exhibit the highest

levels of modal extensional strain energy density $\phi_1^{U_n^y}$. As expected, the highest levels of $\phi_1^{U_n^x}$ arise in the stiffer layers along the x -direction, the fibre direction of such laminas, while the highest levels of $\phi_1^{U_n^y}$ appear in the outer stiffer layers along the y -direction, the fibre direction of such laminas;

- Layers with high levels of modal extensional strain energy density are those closer to the laminate skins and far away from the neutral axis of the laminated composite. Thus, the ability to impose shear on the soft damping layer in the middle of the laminate is limited. On the other hand, the Quintuple (Q) configuration includes soft damping layers between stiff composite layers with high levels of modal extensional strain energy, promoting transverse shear strain and consequently damping;
- Frequently, stiffness and damping are two conflicting needs, and the design of appropriate stacking sequences for laminated composites with interleaved damping layers is a mandatory step. For the first three natural frequencies in Figure 7.20, natural frequencies decrease remarkably as the damping layer thickness increases;
- The Double-Skins (D-S) arrangement stands out as the stacking sequence with less natural frequency variation, i.e., for the same amount of soft damping material, interleaving two layers close to the skins of the laminate promotes a stiffness loss, even though at a smaller extent. Unsurprisingly, one discussed this configuration when analysing the modal loss factors of different stacking sequences, and the Double-Skins (D-S) arrangement presented one of the worst damping performances. As previously discussed, design requirements for high damping capacity and stiffness conflict in most cases;
- Nonetheless, one can not set a rule of thumb, the relation between stiffness and damping is not linear here and it is possible to optimise damping capacity and stiffness within certain limits;
- In general, there aren't substantial distinctions between reference and thick laminates. However, the natural frequencies of thicker laminates have a tendency to decrease more rapidly with increasing thickness of the damping layer;
- For the first modal loss factor of the reference and thick simply supported composite plates, damping capacity increases markedly with increasing thickness, a wider variation than with varying the stiff ply angle orientation. Incidentally, for the configurations with damping layers close to the skins (D-S and T-S), just as in the Quintuple (Q) arrangement, the ply angle θ has practically no influence on the first modal loss factor, even though it slowly decreases from 0° to 45° , after which symmetrically increases until 90° ;
- For higher normal modes, and considering the stacking sequences promoting higher damping capacities, it is possible to distinguish $\theta = 0^\circ$ as the best orientation angle. Increasing the ply angle θ for the same damping layer thickness decreases the modal loss factor. In general, the ply angle has some influence for higher vibration modes, but the damping layer thickness and stacking sequence command the damping capacity of the laminate;
- The first natural frequency of the simply supported laminated composite plate decreases significantly with increasing damping layer thickness, as already mentioned, but the ply angle influences some stacking sequences, particularly for the reference plate and thin damping layer - the first dimensionless natural frequency factor increases as the ply angle increases from 0° to 45° ;

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- The ply angle θ has the exact opposite effect on the second dimensionless natural frequency for reference and thick composite plates compared to the first dimensionless natural frequency, i.e., for some stacking sequences, the natural frequency decreases as the ply angle increases from 0° to 45° , specifically for thin damping layers, proving that, even though the natural frequency decreases with increasing damping layer thickness, there is room for optimisation;
 - The ply angle orientation has limited influence on the natural frequencies of higher normal modes for reference and thick simply supported composite plates with interleaved damping layers;
 - As boundary conditions play a decisive role in the mechanical response of structures, one studied their influence on damping efficiency and ensured that the conclusions stated above do not depend on local conditions and apply to structures with different boundary conditions analysing the same laminated composite plate with all free edges;
 - As previously discussed, it is decisive to understand the modal extensional strain/stress distribution to evaluate the capacity of the stiffer layers to impose transverse shear strain on damping layers and consequently improve damping. Once again, one can notice higher modal extensional stress and energy density along the x -direction within layers oriented at 0° . On the other hand, one can detect higher modal extensional stress and energy density along the y -direction within the 90° oriented ones. Such a pattern is easily explained considering the material properties, namely the fact that $E_{11} > E_{22}$, proving the independence of boundary conditions;
 - For the first natural mode of the laminated composite reference plate with all free edges, damping capacity depends almost exclusively on damping material volume regardless of the stacking configuration;
 - Double-Core (D-C), Quintuple (Q), and Triple-Core (T-C) configurations show the best damping capacities, while Single (S), Triple-Skins (T-S) and Double-Skins (D-S) well below show the worst damping capacities for the first natural mode of a free edge laminated composite plate, proving that the main conclusions drawn here are independent of boundary conditions;
 - Core configurations with more than one layer are better for higher damping capacity when compared with skins configurations but also with the single interleaved one. As previously concluded, the modal transverse shear strain of the damping layer is the primary mechanism responsible for damping capacity, and often, many authors report that the damping layer should be positioned at the neutral axis to maximise damping. Here, one concludes that it is not always the case;
 - The single-layer arrangement shows an even weaker damping performance for thicker laminated composite plates, especially for thicker damping layers. A soft single-layer damping treatment at the core of the laminate decouples both halves of the CFRP and their ability to impose transverse shear strain to the viscoelastic layer, as previously concluded for patch treatments in structures comprising isotropic materials and simply supported laminated composite plates;
 - Overall, and similarly to the first modal loss factor, configurations with damping layers close to the skins show poor damping performances compared to stacking sequences involving damping layers interleaved close to the laminate core for the second vibration mode;

- The Quintuple (Q) interleaved arrangement needs a large amount of damping material to be efficient for higher vibration modes;
- Higher vibration modes show a plateau in damping capacity that is not present for the first vibration mode, at least for thicknesses considered physically relevant, highlighting the possible optimisation of damping capacity at higher vibration modes for reference laminated composite plates with free edges, proving that this phenomenon does not depend on boundary conditions. Sometimes, more damping material even means less damping capacity;
- Thicker laminates exhibit maximum attainable damping capacities for any stacking sequence with interleaved damping layers, after which there is a decrease in damping capacity;
- Laminated composite plates with all free edges prove the boundary conditions independence of transverse shear strain, extensional strain and in-plane shear strain energy density mode shapes as useful widgets to interpret the damping mechanism of laminated composites with interleaved damping layers;
- Single (S) interleaved damping layers do not present successful results regarding damping capacity for thick damping layers since it is just one thick core layer. As previously concluded, when dealing with damping patches on isotropic beams and plates, there is a turning point after which thick soft layers tend to decouple the stiff layers and so their ability to impose shear. In the Single (S) stacking sequence, both halves of the laminated composite work as a base and constraining sub-laminates;
- One should note that, for thinner damping layers, the Single (S) arrangement presents excellent results for different vibration modes. However, there is the previously mentioned threshold of damping layer thickness, and damping capacity might even drastically drop with thicker viscoelastic layers;
- Through-the-thickness representations show that layers with high levels of modal extensional strain energy density are those closer to the laminate skins and far away from the neutral axis of the laminated composite for thicker laminates with a thick, soft damping layer at the neutral axis - Single (S) configuration. Thus, the ability to impose transverse shear on the soft damping layer in the middle of the laminate is limited;
- The Quintuple (Q) arrangement, with soft damping layers between stiff composite layers with high levels of modal extensional strain energy promoting transverse shear strain and damping, presents a justified excellent damping performance;
- As previously concluded, design requirements for high damping capacity and stiffness conflict in most cases, as the laminated composite plate with all free edges confirms. The Double-Skins (D-S) arrangement stands out as the one presenting less natural frequency variation, i.e., for the same amount of soft damping material. Unsurprisingly, one discussed this configuration when analysing the modal loss factors of different stacking sequences, and the Double-Skins (D-S) arrangement presented one of the worst damping performances;
- Still, it is not possible to state general rules since there is not a linear relation between damping capacity and natural frequency, proving that there is room for optimisation;
- The laminated composite plate with all free edges also supports the conclusion that damping capacity is almost exclusively a function of the damping layer thickness, and the ply angle of compliant layers has practically no influence on the first modal loss factor;

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- For higher vibration modes, the ply angle plays a more significant role;
 - As expected, the laminate stiffness decreases with increasing damping layer thickness, and so does the natural frequency, but the ply angle has a strong influence on the first natural frequency, regardless of the stacking sequence. The first natural frequency increases for θ around 45° ;
 - For higher vibration modes, the ply angle can no longer increase the natural frequencies of a laminated composite plate in the same way;
 - The classical method of embedding soft damping layers into the stacking sequence of laminated composites to improve damping capacity may excessively undermine the structure's stiffness. One presented a different approach with viscoelastic materials partially distributed in specific layers in a configuration one can refer to as interleaved damping patches;
 - One performed a dimensionless analysis of eight stacking sequences to evaluate the influence of damping patches on the damping capacity of laminated plates, adapting two configurations considered in the previous studies, the Single (S) and Double-Skins (D-S) stacking sequences, proposing a Centre-to-Edge and Edge-to-Centre run of the damping patches and using the same amount of damping material;
 - As anticipated, several conclusions drawn in this study regarding surface-damping treatments and interleaved damping layers remain applicable to interleaved damping patches;
 - There is also a threshold of damping layer thickness for laminated plates with interleaved damping patches, beyond which further increases in thickness reduce the material's damping capacity. As already concluded, thick damping layers decouple the elastic stiff ones of the laminate and, consequently, their ability to impose shear on the damping material;
 - The mode dependence of the optimum damping layer thickness still holds for laminated plates with interleaved damping patches, and optimum damping layer thickness decreases for higher-order modes;
 - In general, for reference simply supported laminated plates, core configurations with the damping patch running from the centre to the edges (CE) promote more damping than those with the damping patch running from the edges to the centre (EC), and this is particularly noticeable for the first mode, as the Core-EC configurations show low damping performance, regardless of the damping patch thickness;
 - Among the standout findings, the optimal damping performance observed with interleaved damping patches surpasses that of interleaved damping layers for specific thicknesses. This compelling outcome underscores the significance of incorporating interleaved damping patches in laminated composites, showcasing the potential for enhanced damping performance with reduced material usage in specific damping layer thicknesses;
 - For some configurations, incorporating partial damping treatments proves advantageous, and this holds true for interleaved damping patches within laminated composites. Specifically, Core-CE configurations emerge as more favourable, opposing Edge-to-Centre treatments, as they demand large quantities of damping material to achieve efficiency, i.e. the damping material must reach the centre of the laminate;

- Damping layer thickness and position (CE or EC) are far more determinant to the interleaved damping patch performance than the orientation of the different configurations 0 and 90. Still, the minor differences between orientations disclose the mode dependence of the preferable orientation for a given configuration;
- Centre-to-Edge dispositions of the interleaved damping material are also preferable to Edge-to-Centre ones for configuration with interleaved damping patches close to the skins. These consistent outcomes provide a solid foundation for establishing robust guidelines concerning the application of interleaved damping patches in laminated plates;
- However, notable variations exist in the damping performances between Core and Skins configurations for simply supported laminate reference plates. While Skins-CE configurations exhibit remarkable performance for select damping layer thicknesses, core configurations demonstrate superior damping capacities for the first and second modes. Surprisingly, the Skins-CE0 configuration unexpectedly displays exceptional damping capacity for the third mode, outperforming all other arrangements, including core configurations;
- Skins configurations need thicker damping layers to achieve substantial damping capacities. When thin interleaved damping patches are placed near the skins, the top and bottom layers of the laminate exert minimal transverse shear strain on these thin damping patches, resulting in negligible damping capacity. This is notably pronounced in the Skins-EC configurations, where the arrangement situates the damping material near the skins and spans from the edges to the centre, configurations that prove less efficient in terms of enhancing damping capacity;
- The most noticeable difference between reference and thick laminated plates, either for Core or Skins configurations, is the decreasing effect on damping capacity of the increasing damping layer thickness. For thick laminated plates with an aspect ratio $a/h = 50$, one also considers thick damping layers and increasing damping layer thickness decreases damping capacity.
- Thick, simply supported, laminated plates highlight the decreasing effect of increasing damping layer thickness on damping capacity. For thick laminated plates with an aspect ratio $a/h = 50$, one also considers thick damping layers and increasing damping layer thickness decreases damping capacity;
- Thick laminated composite plates show improved damping performances with thin damping layers. As concluded in this work, thick damping layers often lead to a decoupling effect of the stiff elastic layers, reducing their capacity to exert shear forces on the damping material. This phenomenon is particularly pronounced in thick laminate plates;
- The main features of Core and Skins configurations as well as Centre-to-Edge (CE) or Edge-to-Centre (EC) runs of the interleaved damping patches are still applicable for thick laminated composite plates.

Throughout this work, one conducted a series of experimental validations involving specially developed viscoelastic damping materials for integration into laminated composites. Additionally, one investigated the feasibility of integrating partial damping treatments into laminated composite structures. Here, one highlights the following conclusions regarding the *Experimental Testing and Analysis* in Part IV:

- Initially, one performed the mechanical characterisation of the viscoelastic materials used in this work, Smacwrap ST and Kraibon, by adapting the ASTM E756-05 [2010] standard

procedure and establishing an experimental campaign with a contactless measuring method to minimise possible disturbances;

- One adapted different methods to estimate the mechanical properties of damping materials and proposed a new one based on the curve fitting of transmissibility functions generated by a finite element model to an experimental one by searching for the material model parameters;
- The objective function to optimise might vary, and optimising the difference between experimental and numerical resonant frequencies might be sufficient for storage modulus identification, for example, but inadequate for loss factor evaluation. Here, one proposed an objective function based on the Local Amplitude Criterion (LAC), a local indicator where one can distinguish between frequency ranges with poor and good correlation of single frequency response functions;
- The proposed objective function demonstrated its efficiency and reliability at the core of the correlation-updating process, regardless of the test specimen under examination. It successfully distinguished elastic and viscoelastic material properties with low and high damping capacities, proving its robustness;
- One manufactured two sets of specimens with viscoelastic materials to characterise their properties. Specimens using Smacwrap ST with a 0.1 mm thickness using different adhesives to study the influence of adhesives on the identified mechanical properties of viscoelastic materials and specimens using Smacwrap ST with a 0.2 mm thickness and Kraibon with a 0.5 mm thickness to identify the material properties using more test pieces including different thicknesses;
- The frequency range influences the estimation. The fitting process includes more resonances for higher frequency ranges, and the fit is more challenging since the resonant frequencies depend on Young's modulus;
- Transmissibility functions with anti-resonances, where the dynamic stiffness is high, present a fitting process that is more demanding, leading to a damping overestimation;
- Contrary to Young's modulus, the material loss factor estimation decreases with increasing frequency ranges. The material loss factor estimation strongly depends on the number of measuring points near resonances, and higher frequency ranges result in fewer points near resonances, leading to the worst material loss factor estimations;
- The fitting procedure estimated adhesive layer thicknesses very close to those observed at the microscope, proving its robustness even for small thicknesses with values in the order of microns;
- Different adhesives might influence the dynamic behaviour of test specimens produced for material characterisation;
- Microscopic measurements of the viscoelastic layer thicknesses disclosed thicknesses well below those listed by the suppliers;
- Failing to account for the adhesive thickness in the fitting process would inevitably result in an inaccurate estimation of viscoelastic material properties;
- A narrow frequency range considering only a few resonances is insufficient for proper material characterisation;

- Different manufacturing procedures led to an estimation of slightly different material properties. It was not possible to definitively explain such minor differences within the scope of this work;
- The experimental campaign involving viscoelastic damping materials to enhance damping in laminated composites considered different materials from different manufacturers. The mechanical properties of such materials, previously identified, were valuable to compare experimental and numerical investigations involving laminated composites with embedded damping layers and patches;
- One considered three laminated plate specimens with interleaved damping layers in different arrangements using Smacwrap ST 0.1 mm, Smacwrap ST 0.2 mm and Kraibon 0.5 mm embedded in the stacking sequences;
- Besides, and considering the same sample materials, one extended the experimental campaign to laminated composite plates with interleaved damping patches in an Edge-to-Centre configuration to simplify the manufacturing process;
- S01 and S02 specimens used the same damping material with different thicknesses, but one adapted the configurations to study the influence of the stacking sequence using the same amount of damping material - two layers of S01 have the same theoretical mass as one layer of S02. The K05 specimen uses the same configuration as the S02 specimen but with different damping material and thicknesses. Unfortunately, as proved by the microscopy analysis, the viscoelastic layer thicknesses disclosed thicknesses different from those listed by the suppliers, which jeopardised the manufacturing process;
- This misalignment between thicknesses makes it impossible to manufacture a stacking sequence involving interleaved damping patches with regular thickness laminas, contributing to the structural complexity, introducing additional variables and rendering comparisons more challenging to establish;
- The manufacturing of the laminated composite specimens occurred before the published parametric analyses of the laminated composite plates with interleaved damping patches presented in the previous chapter and considered only one specimen for each configuration since this is an exploratory study;
- Manufacturing laminated composites featuring interleaved damping patches through hand lay-up is a demanding process, requiring meticulous care and precision. It involves the intricate cutting and precise alignment of CFRP plies and VEM damping patches, which becomes notably more challenging when dealing with thinner plies;
- Truthfully, the S01 specimens, either with interleaved damping layers or interleaved damping patches, revealed many imperfections after manufacturing, presenting inclusively two stable configurations (bi-stable composite). Such deviations from the proposed configurations contributed to the poor fit between the experimental and numerical frequency response functions for S01 specimens;
- Thicker specimens with thick damping layers proved to be less prone to imperfections and so close to the proposed theoretical configurations, resulting in excellent fits between the experimental and numerical frequency response functions;
- For laminated composite plates, a comparison between experimental and numerical results benefits from a strict comparison between frequency response functions. Modal parameters

might not lead to a proper comparison since laminated composites tend to show closely spaced resonances, and frequency response functions are less sensitive to measurement noise and accumulation of computational errors.

10.4 | Novel and Key Contributions

The following section encapsulates the pivotal strides and innovative contributions outlined in this work. Throughout this comprehensive study, novel methodologies and distinctive approaches have been meticulously developed and presented. This segment is a testament to the advancements made in the field, emphasizing the unique contributions that push the boundaries of existing knowledge. From pioneering advancements in mechanical modelling to the strategic integration of viscoelastic damping treatments, each contribution represents a critical leap forward in understanding, methodology, and application within structural dynamics and damping technologies. This section aims to spotlight the main contributions, each adding a distinct layer to the collective understanding and furthering the discourse in this field. Here are a few aspects that, by their relative novelty character and significance, are worth highlighting:

- A grounded introduction to the damping of panel vibrations with a structured incursion into the nature of damping, with a particular focus on different damping mechanisms. The study of basic viscoelastic damping mechanisms and the main differences between passive and active vibration damping serve as the basis for the central proposal of this work - partially distributed damping in composite structures;
- The present work lays the groundwork for the developments in partial damping treatments, helping any interested reader to know more about a subject of great importance within the field of passive vibration control due to its wide range of applications, especially considering the possibility of achieving a significant damping capacity using less material. The research published in this field undergoes exhaustive investigation, presenting established findings and highlighting key disputes to pave the way for new research topics and relevant discoveries;
- Taking advantage of the laminated composite plate theory based on warping functions proposed by Loredó and Castel [2013], a multi-layer theory of the hybrid type keeping the advantages of a purely multi-layer deformation theory while maintaining the number of kinematic variables of a single-layer approach providing a good compromise between accuracy and computational time, one proposed the FEM implementation of such theory using different elements, benefiting from its appropriate nature to model structures with complex geometries by discretising the entire domain locally and offering contrasting possibilities regarding the boundary conditions. Its element-based approach can also bring advantages when dealing with PCLD patches with complex geometries;
- Extension of the laminated composite plate theory with warping functions to three-dimensional structures, namely shells, using the FEM. Therefore, one expanded the advantages of a versatile and robust plate deformation theory, namely the multi-layer mechanical behaviour with the degrees of freedom of single-layer approaches, to more complex structures like shells, providing a good compromise between accuracy and computational time. The significance of this advancement cannot be overstated, particularly in the automotive, railway, naval, aircraft, and aerospace industries. Daily, researchers and engineers grapple with the challenge of designing intricate structures, demanding accurate and effective mechanical models. Notably, a pivotal contribution of this work lies in this

specialised shell element tailored for the computational modelling of complex, inhomogeneous multilayered composite shell structures encapsulating the mechanical intricacies inherent in partial damping treatments;

- A set of dimensionless parametric analyses to evaluate and comprehend the damping efficiency of partial damping treatments, setting some design guidelines for the optimal design of structures with viscoelastic damping treatments combining different design parameters such as the thickness/length ratio, constraining layer thickness, material modulus, natural mode or boundary conditions in a fundamental perspective;
- Since the transverse shear strain within the viscoelastic damping layer is the main responsible for the damping mechanism in PCLD configurations, one proposed transverse shear strain mode shapes to comprehend the damping phenomena in partial damping treatments and realise the damping efficiency of different configurations. The laminated composite plate theory, as a multi-layer theory of the hybrid type with the kinematic variables of a single-layer approach, is relatively simple to compute the modal transverse shear strain through-the-thickness using the mode shapes associated with shear strain degrees of freedom, presenting a detailed view of the strain pattern at each mode, helping to interpret the basic design guidelines for the optimal design of structures with viscoelastic damping treatments;
- Based on the proposed metrics to evaluate and interpret the damping phenomena in partial damping treatments, one established a comprehensive understanding of the damping mechanisms that lead to improved damping performances of some configurations over others, including those with partial damping treatments in a general way;
- Dimensionless parametric analyses to evaluate and comprehend the damping efficiency of damping treatments in laminated plates with interleaved damping layers. By considering different arrangements covering the range of possible symmetric configurations for a six-layer reference laminate, including viscoelastic layers within the stacking sequence, one compared different stacking sequences using the same amount of damping material to evaluate the preferential stacking sequences, whether it is preferable to have more or less damping layers for the same amount of damping material and whether they should be closer to the core or the skins;
- One of the main contributions of this work to this field is the study of partially distributed viscoelastic damping treatments for composite structures in an integrated arrangement. The extension of the dimensionless parametric analyses of the damping efficiency of damping treatments to laminated plates with interleaved damping patches enables the first steps in this field and helps to explore the embedding of soft damping layers into a fibre-reinforced plastic laminate without excessively undermining the structure's stiffness;
- The experimental testing and analysis of laminated composites with embedded soft damping layers and patches closes the novel and key contributions with a significant input. The mechanical characterisation of VEM that withstand composite curing, including the autoclave process, and are able to be embedded into composite structures paves the way to the experimental campaign conducted to evaluate the damping performance of composite laminates with interleaved viscoelastic damping materials, including CFRP with interleaved damping patches.

10.5 | Suggestions for Further Research

The investigation outlined in this thesis sets the stage for a broad spectrum of future inquiries and possible advancements. As this study concludes, it opens numerous avenues for further exploration and expansion. This segment outlines possible trajectories and promising areas for continued investigation, building upon the foundations laid in this study. By identifying potential areas of enhancement, innovation, and refinement, this section aims to inspire and guide future research endeavours, fostering continued growth and evolution within the domain of passive vibration control of composite structures with viscoelastic damping patches. In general, one can emphasise the following main research topics:

- Concerning the mechanical model and the corresponding numerical implementation, it would be valuable to implement a FEM version of the warping functions computing via equilibrium equations, as proposed by Loredo and Castel [2014] and Loredo [2014]. In general, 3D solutions often lack practical relevance, as they are straightforward to compute for unrealistic problems yet challenging for real-world applications. However, a multi-layer of the hybrid type, merging the multi-layer mechanical behaviour with the degrees of freedom of single-layer approaches, enables a simple and robust computation of 3D solutions and might help to explore more complex inhomogeneous structures;
- The parametric analyses and experimental testing conducted for more basic structures should be extended to laminated composite shells, including shells with interleaved damping patches, taking advantage of the extension of the laminated composite plate theory with warping functions to a robust shell element developed for the computational modelling of inhomogeneous multilayered composite shell structures;
- The through-the-thickness representations of the modal transverse shear strain using the mode shapes associated with shear strain degrees of freedom are one of the most relevant contributions of this work. Nevertheless, as pointed out, one must be careful when selecting and analysing through-the-thickness views of section cuts. The visual information is impressive, but one has to consider the constraints. To be more specific, integrating the modal strain energy density over the volume of the structure would give a precise numerical value of the modal strain energy. The primary motivation of this work was to comprehend the basic damping mechanisms behind viscoelastic damping patches and derive some general design guidelines, but a more specific application of the metrics proposed here would benefit from a simple computation as this one;
- Laminated composite plates featuring interleaved damping layers and patches are areas poised for further development, and this work marks an initial step in that direction. Exploring dimensionless analyses with additional design parameters and intricate stacking sequences represents just a glimpse into the expansive possibilities awaiting exploration;
- Regarding the experimental testing and analysis conducted in this work, different manufacturing procedures led to an estimation of slightly different viscoelastic material properties. It was not possible to definitively explain such minor differences within the scope of this work, but minor differences in the adhesive thickness impact the numerical estimation, for example. The mechanical characterisation of material properties is a never-ending research topic, and damping materials are no exception;
- The experimental campaign carried out in this work was exploratory in nature and involved only a few specimens. However, it was possible to draw relevant conclusions that may

guide future works. Composite laminates with thin damping materials are challenging to produce since they are more susceptible to imperfections. For this reason, in future works, one proposes an experimental campaign favouring better control of the production process, considering more specimens and trying to separate the different resonances as much as possible to enable better comparison between vibration modes and their modal parameters. Correia [2021] work, in which the specimens produced in this study were used, is an attempt in this direction. This work shows that interleaved damping patches in composite laminates can be an effective alternative for optimised vibration control where there's no need to sacrifice the structure stiffness;

- As the field of structural engineering continues to evolve, future trends in partial damping treatments are anticipated to undergo a transformative shift, notably with the integration of 3D printing technology. The advent of 3D printing presents an unprecedented opportunity to revolutionise the fabrication and implementation of damping treatments within structural components. This innovative approach offers the prospect of tailored and intricate designs, enabling the creation of custom damping structures optimised for specific applications. Moreover, the scalability and precision inherent in 3D printing techniques pave the way for enhanced damping performance while minimizing material usage and weight. The seamless integration of viscoelastic materials into complex geometries made feasible by 3D printing heralds a new era of lightweight yet highly efficient damping solutions, promising to redefine the landscape of structural dynamics and vibration control in diverse industries.

Appendices



Experimental Apparatus

A.1 | Technical Specifications

Accelerometer BK4507-473

Manufacturer/Model	Brüel & Kjær / 4507
Sensitivity	$9.76 \times 10^{-3} \text{ V/ms}^{-2}$
Mass	4.752 g

Electromagnetic Shaker

Manufacturer/Model	Ling Dynamic Systems (LDS) / V401
Range	98 N @ 5 Hz to 9000 Hz
Mass (with trunnion)	14.1 kg (22.7 kg)

Laser Vibrometer

Manufacturer/Model	Polytec /OFV303
Sensitivity	$0.125 \text{ ms}^{-1}/\text{V}$

Shaker Amplifier LDS PA100E

Manufacturer/Model	Ling Dynamic Systems (LDS) / PA100E
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Signal Analyzer Unit

Manufacturer/Model	Brüel & Kjær / 2035
Input/Output	2 x 3019 (25 kHz)/1 x 3106 (25 kHz)

Vibrometer Controller

Manufacturer/Model	Polytec /OFV3001
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Microscopy Analysis

B.1 | Beam Specimen E01 - Cross Section

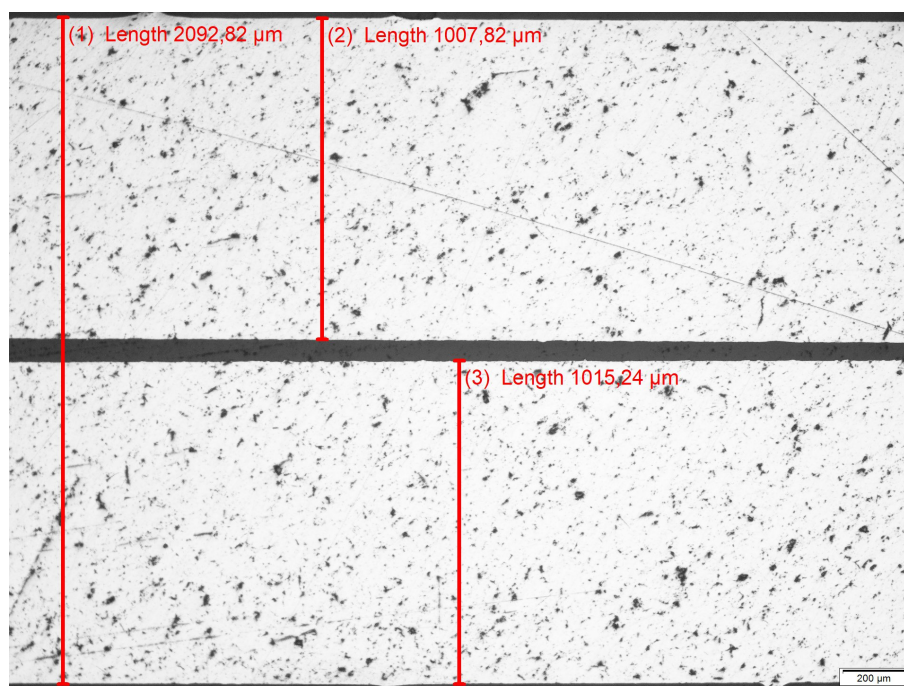


Figure B.1 Beam specimen E01 - 5x Magnification

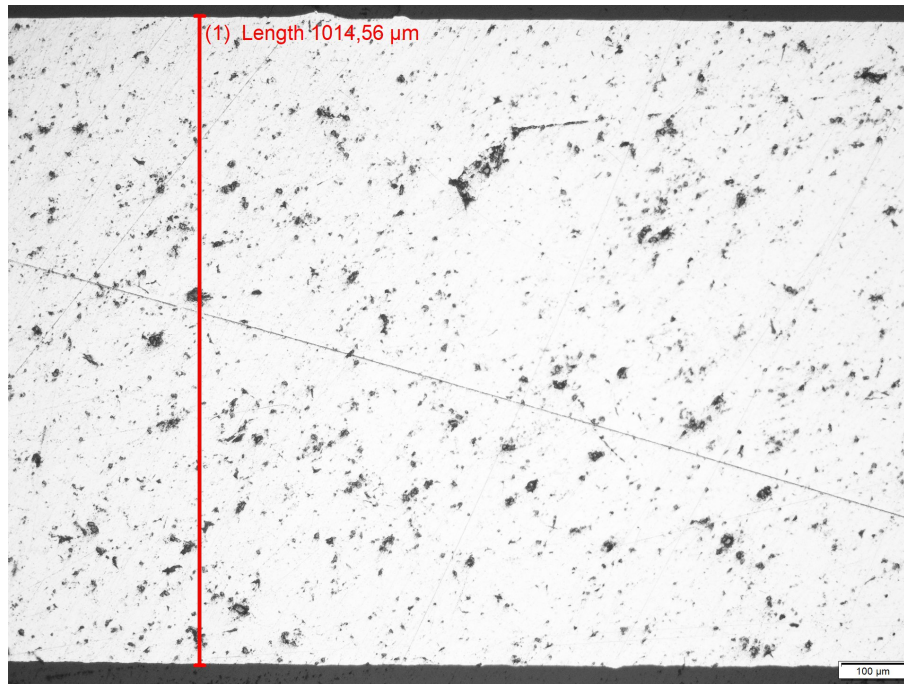


Figure B.2 Beam specimen E01 - constraining layer - 10x Magnification

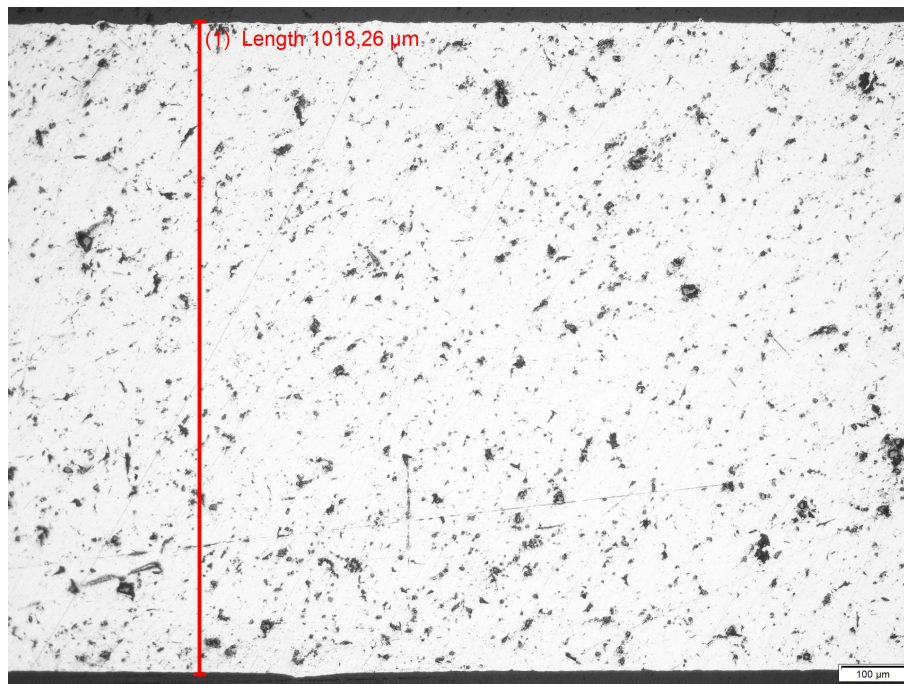


Figure B.3 Beam specimen E01 - base layer - 10x Magnification

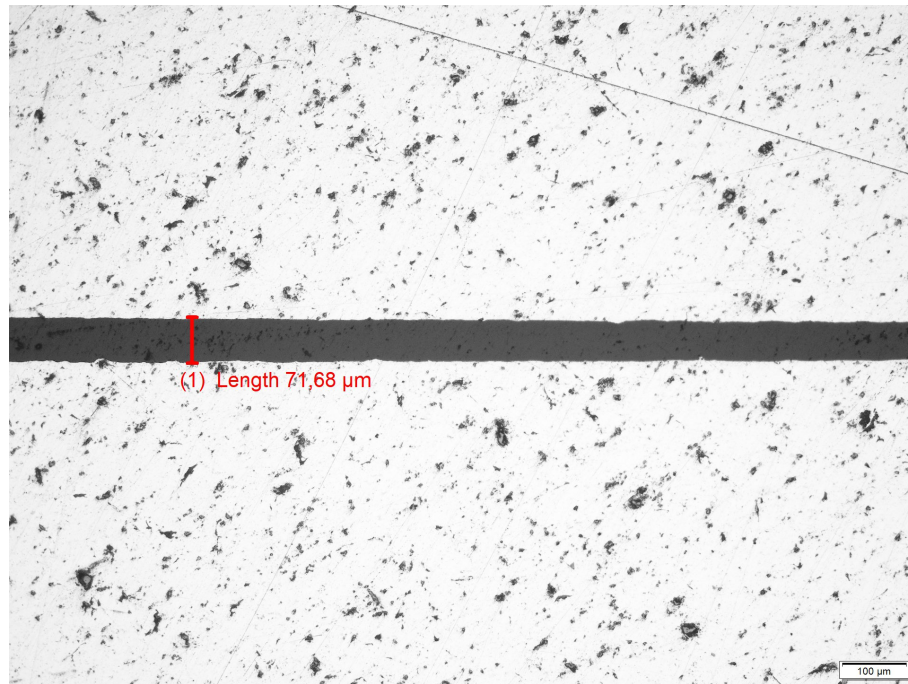


Figure B.4 Beam specimen E01 - adhesive layer - 10x Magnification

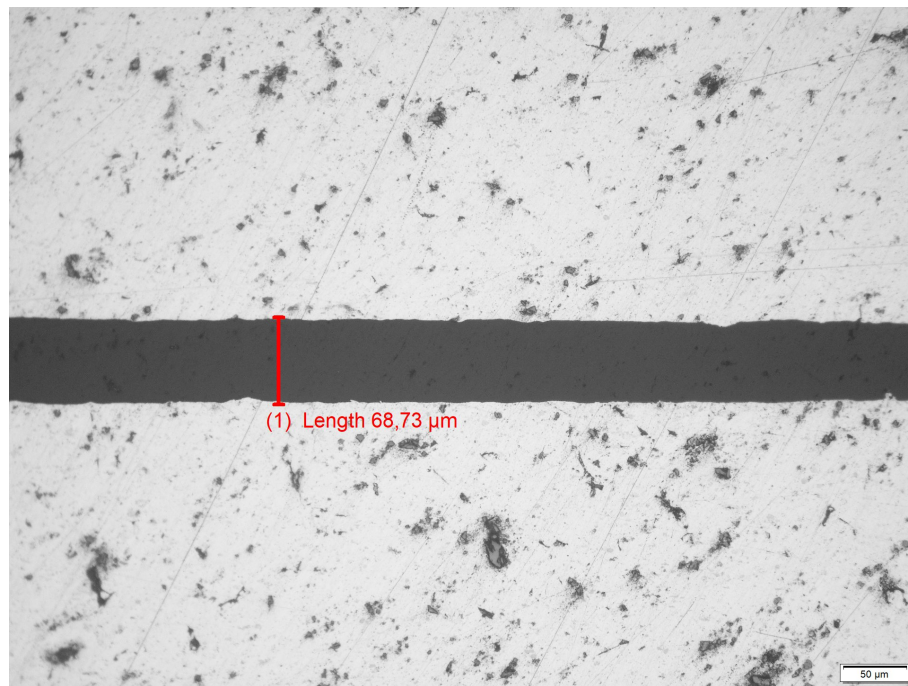


Figure B.5 Beam specimen E01 - adhesive layer - 20x Magnification

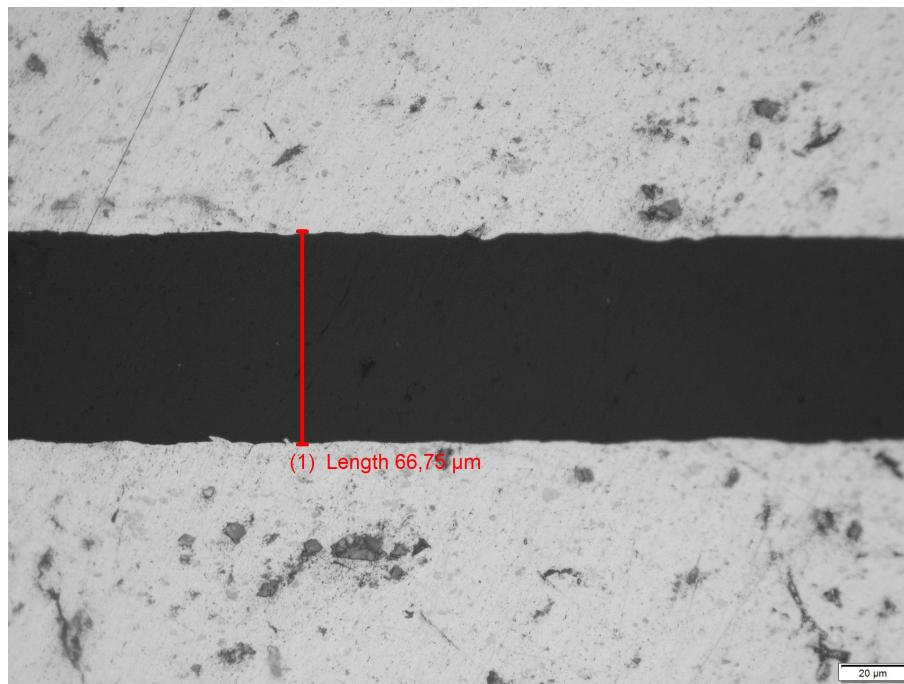


Figure B.6 Beam specimen E01 - adhesive layer - 50x Magnification

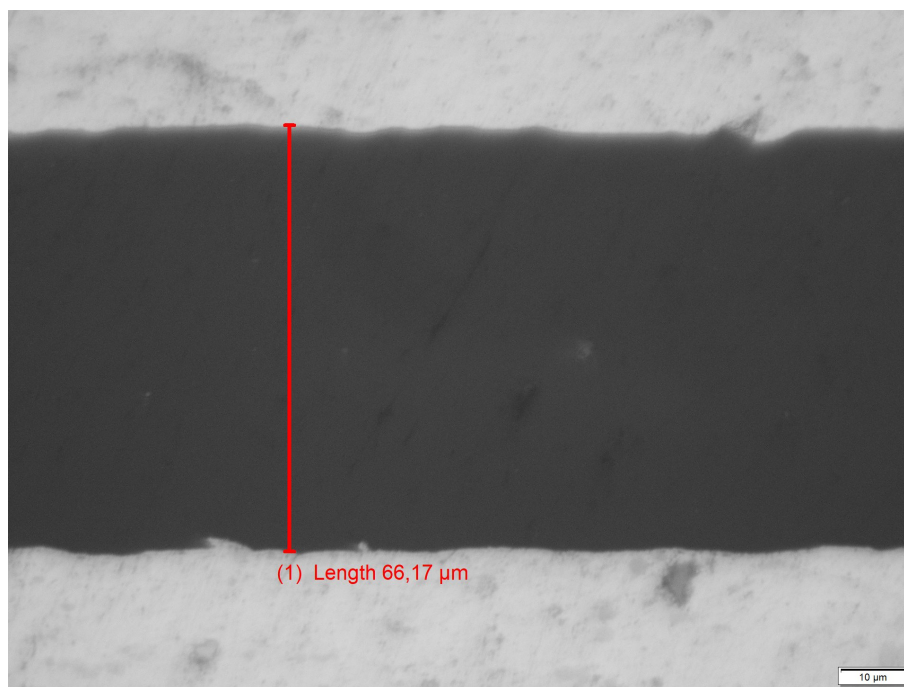


Figure B.7 Beam specimen E01 - adhesive layer - 100x Magnification

B.2 | Beam Specimen P01 - Cross Section

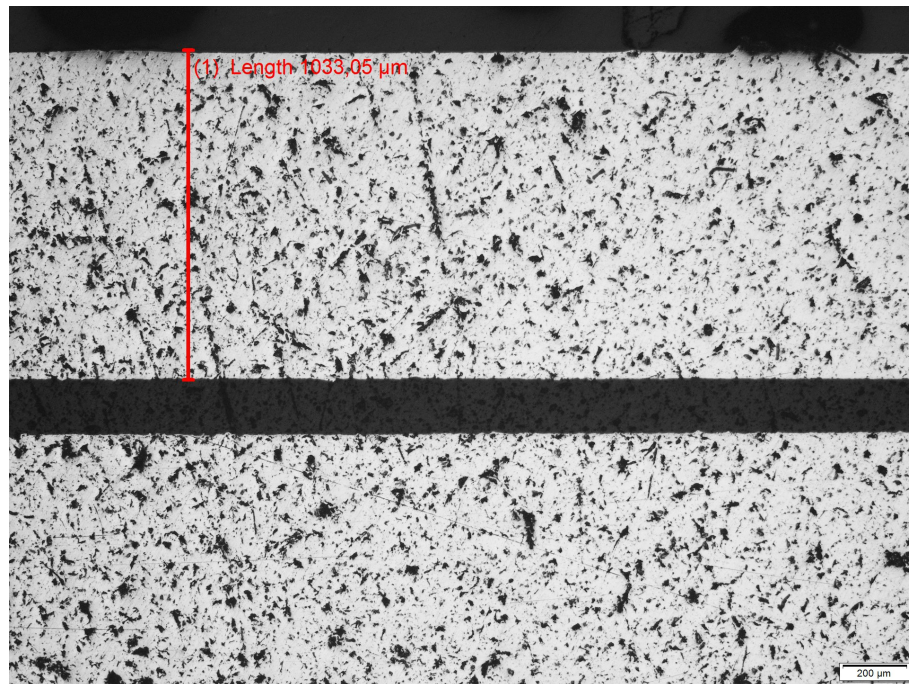


Figure B.8 Beam specimen P01 - constraining layer - 5x Magnification

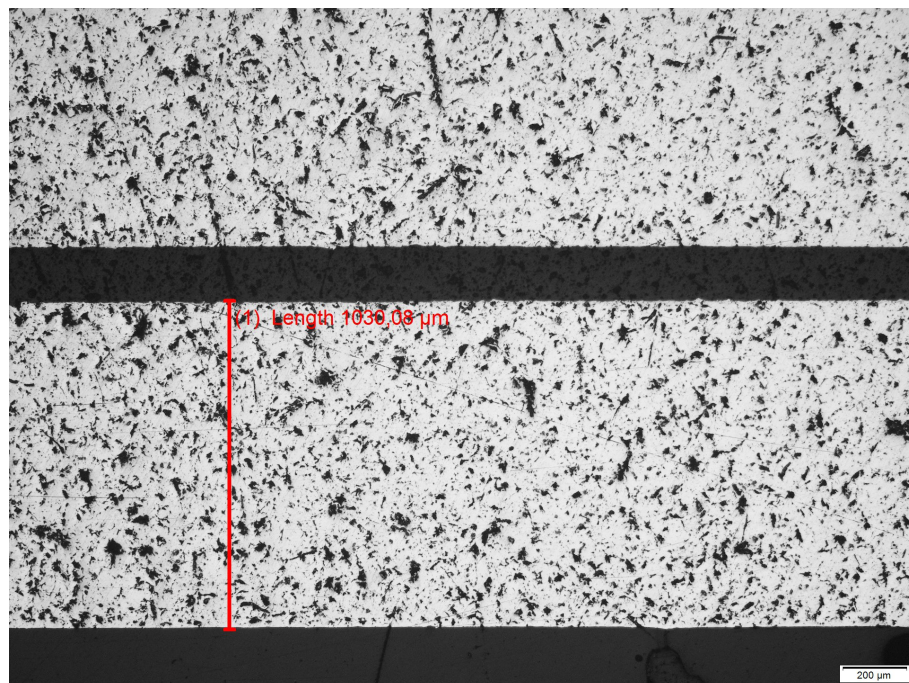


Figure B.9 Beam specimen P01 - base layer - 5x Magnification

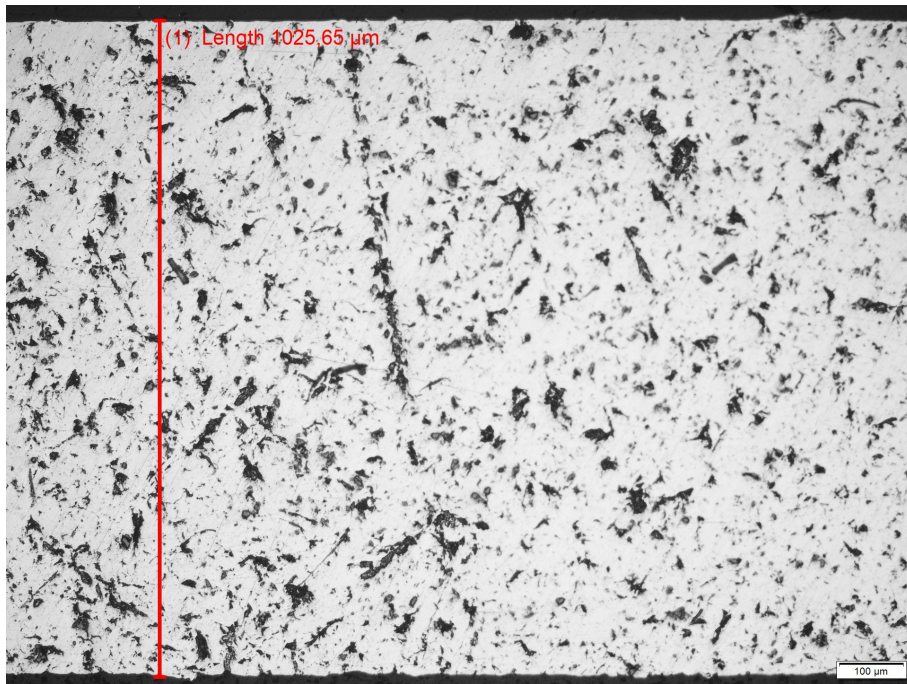


Figure B.10 Beam specimen P01 - constraining layer - 10x Magnification

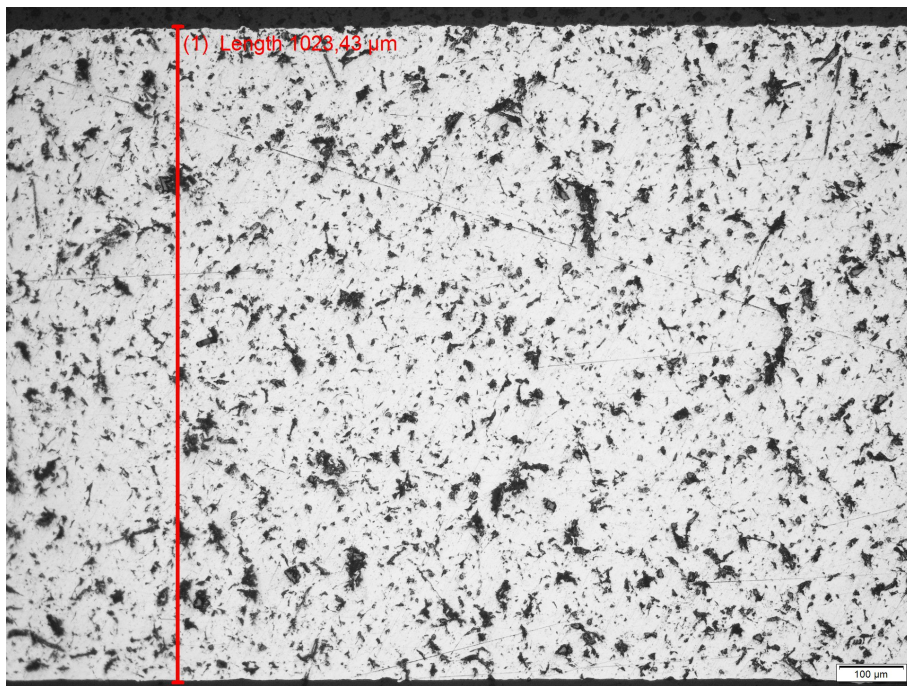


Figure B.11 Beam specimen P01 - base layer - 10x Magnification

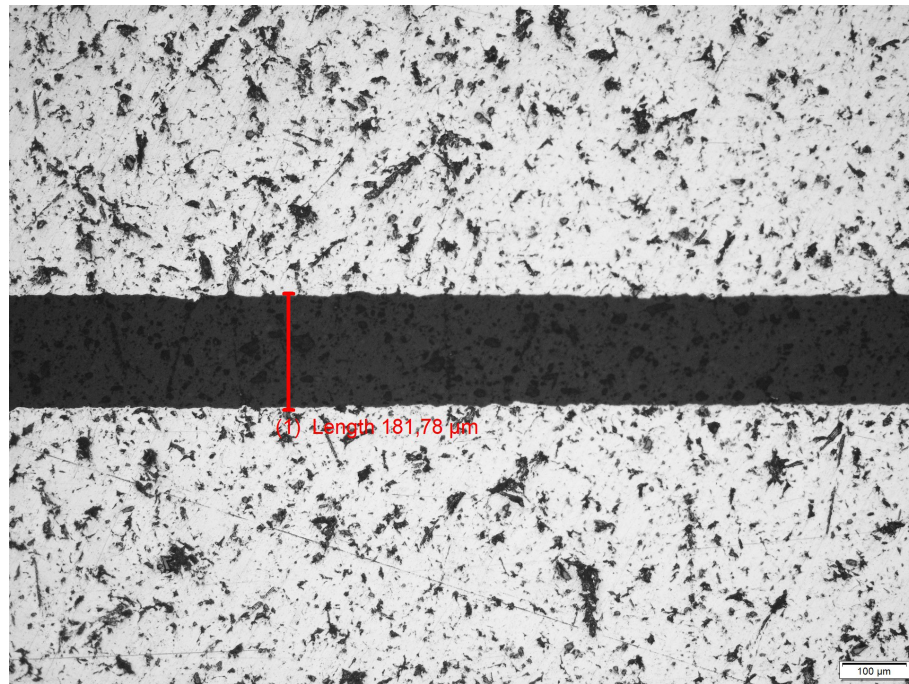


Figure B.12 Beam specimen P01 - adhesive layer - 10x Magnification

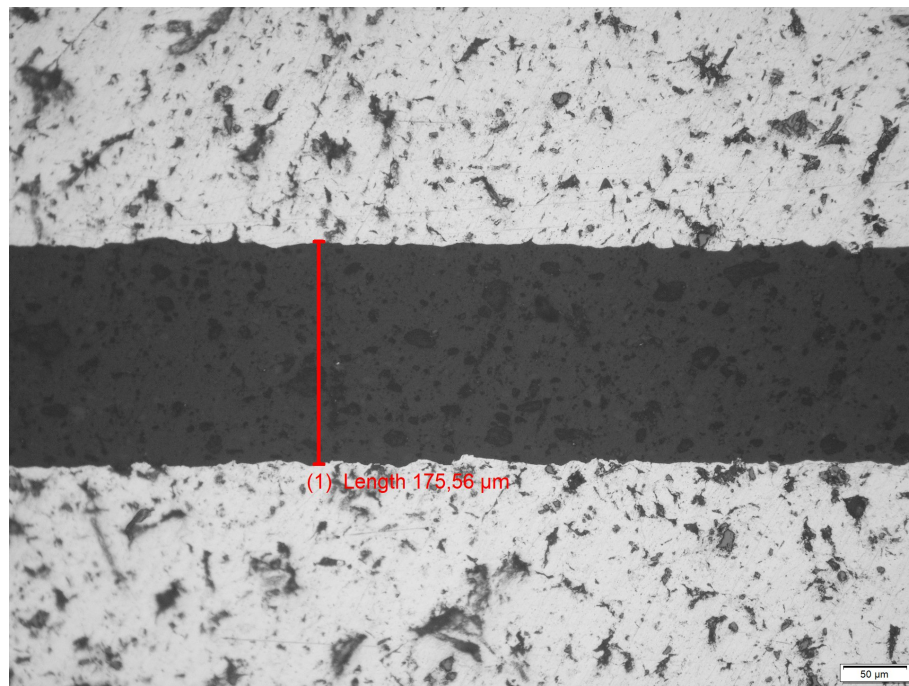


Figure B.13 Beam specimen P01 - adhesive layer - 20x Magnification

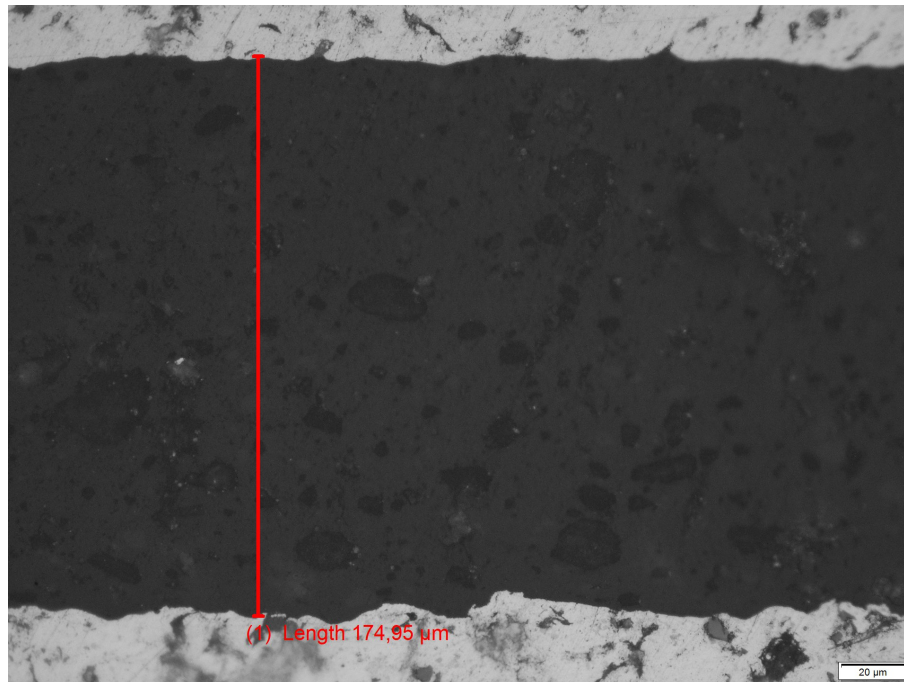


Figure B.14 Beam specimen P01 - adhesive layer - 50x Magnification

B.3 | Beam Specimen EV01 - Cross Section

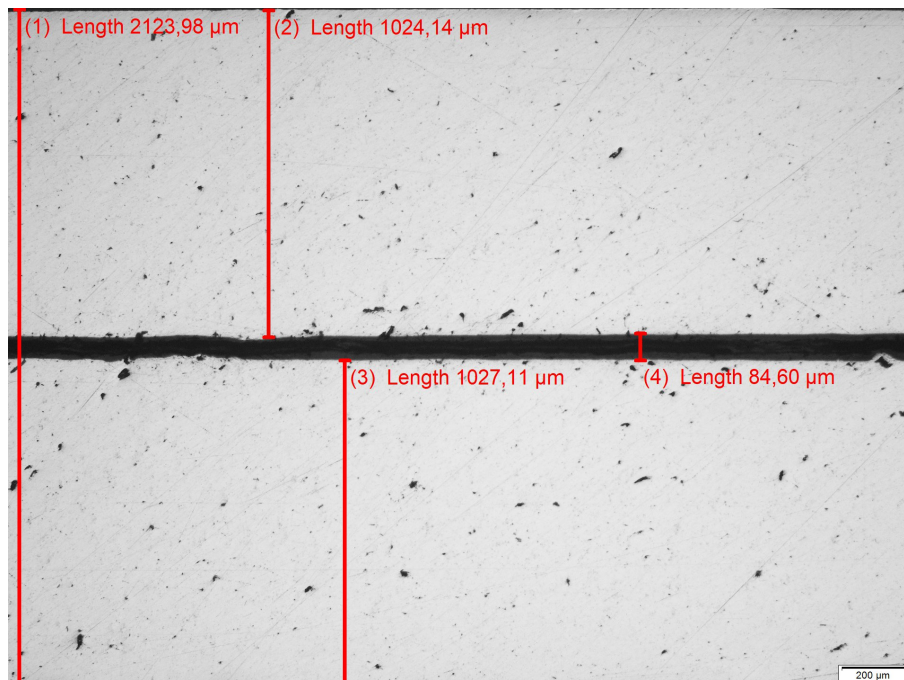


Figure B.15 Beam specimen EV01 - 5x Magnification

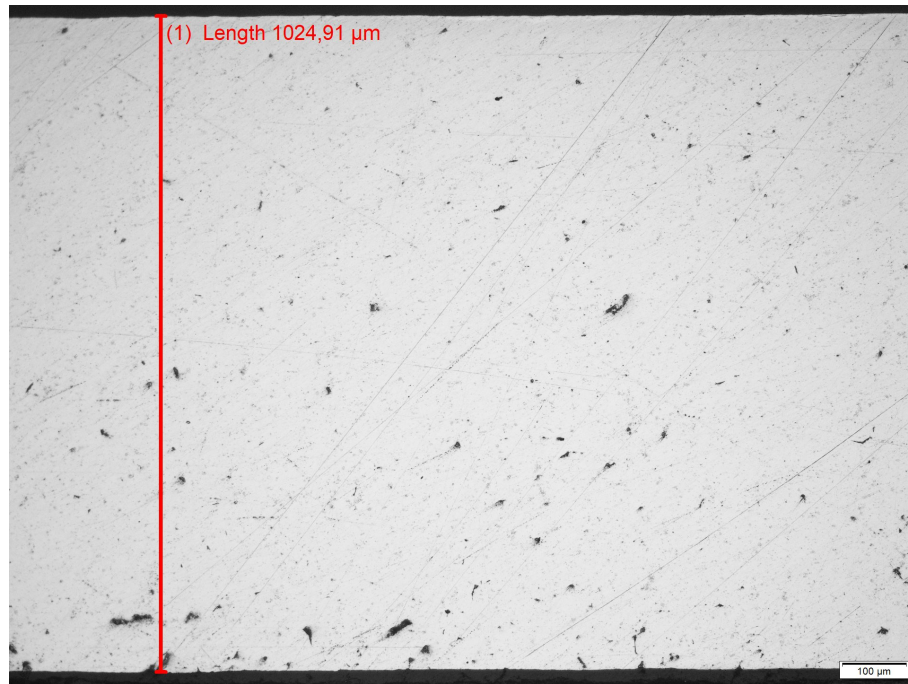


Figure B.16 Beam specimen EV01 - constraining layer - 10x Magnification

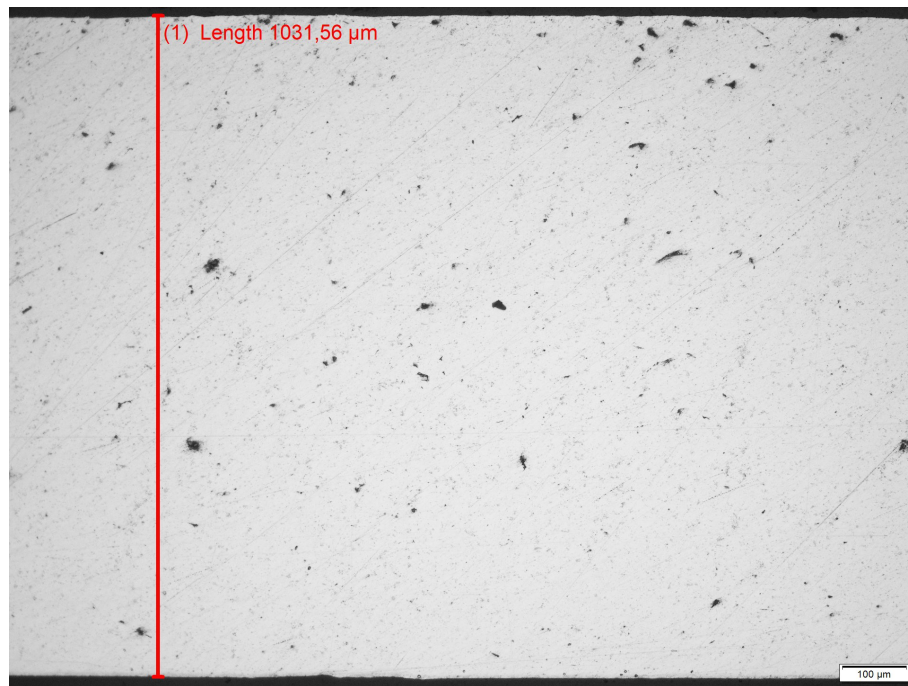


Figure B.17 Beam specimen EV01 - base layer - 10x Magnification

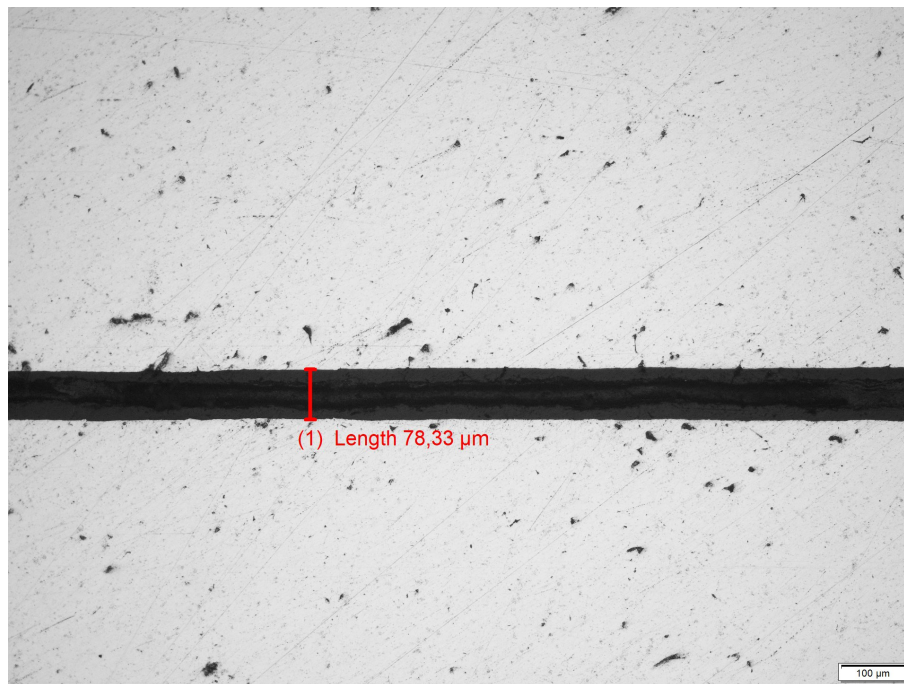


Figure B.18 Beam specimen EV01 - damping layer - 10x Magnification

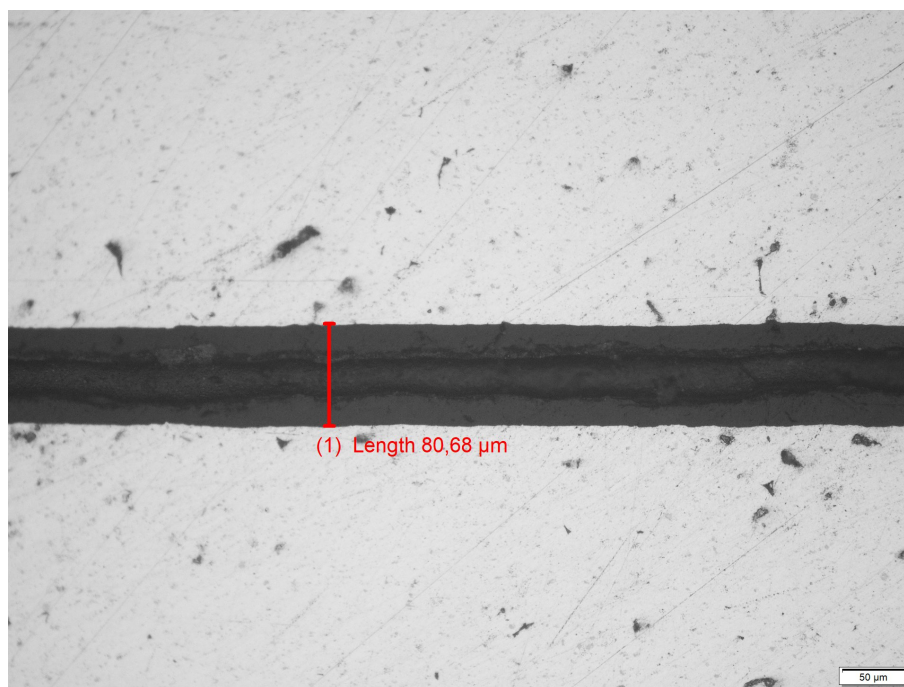


Figure B.19 Beam specimen EV01 - damping and adhesive layers - 20x Magnification

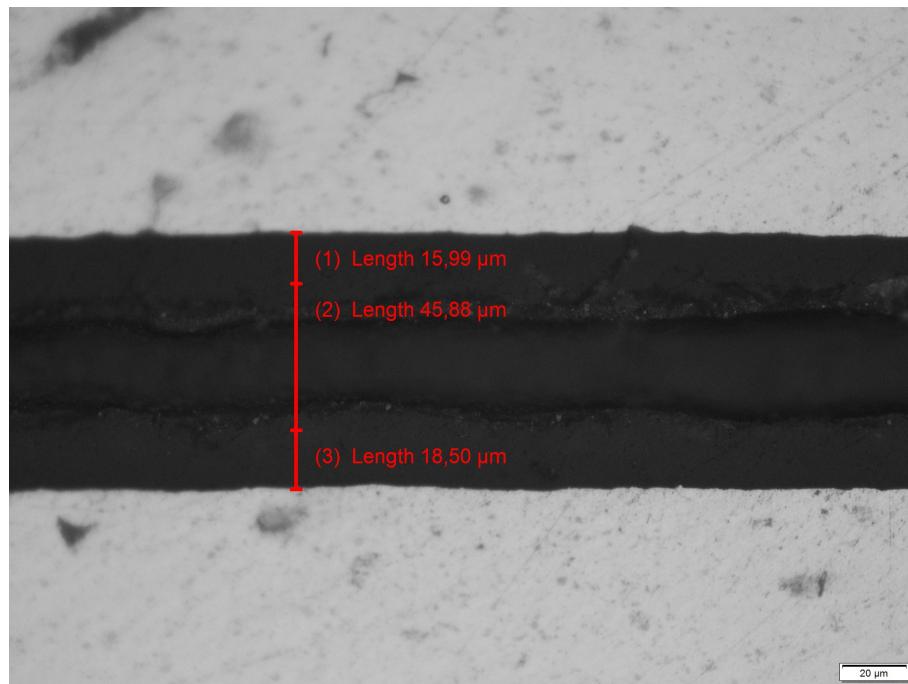


Figure B.20 Beam specimen EV01 - damping and adhesive layers - 50x Magnification

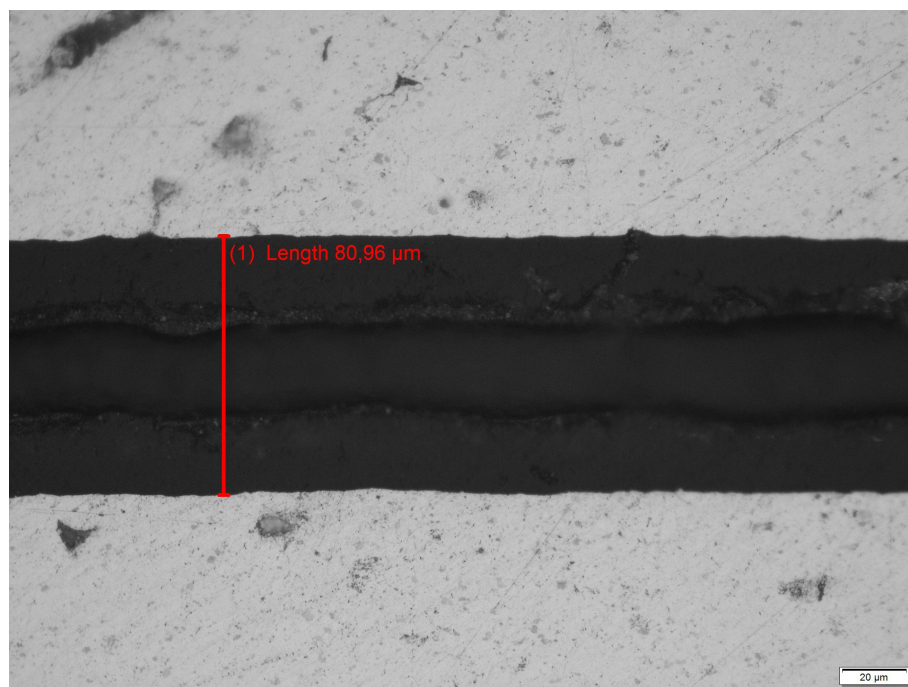


Figure B.21 Beam specimen EV01 - damping and adhesive layers - 50x Magnification

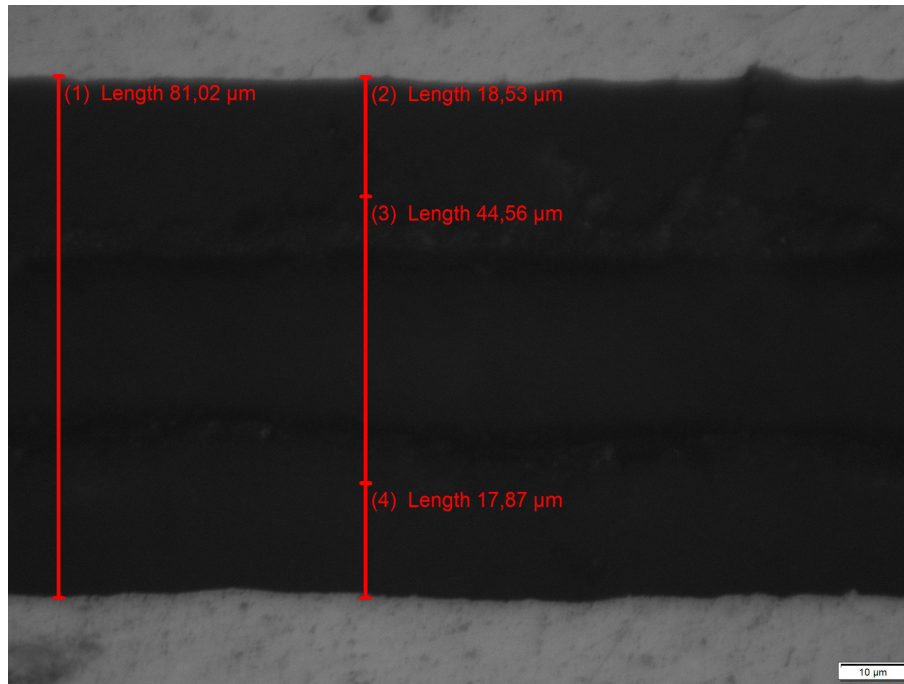


Figure B.22 Beam specimen EV01 - damping and adhesive layers - 100x Magnification

B.4 | Beam Specimen PV01 - Cross Section

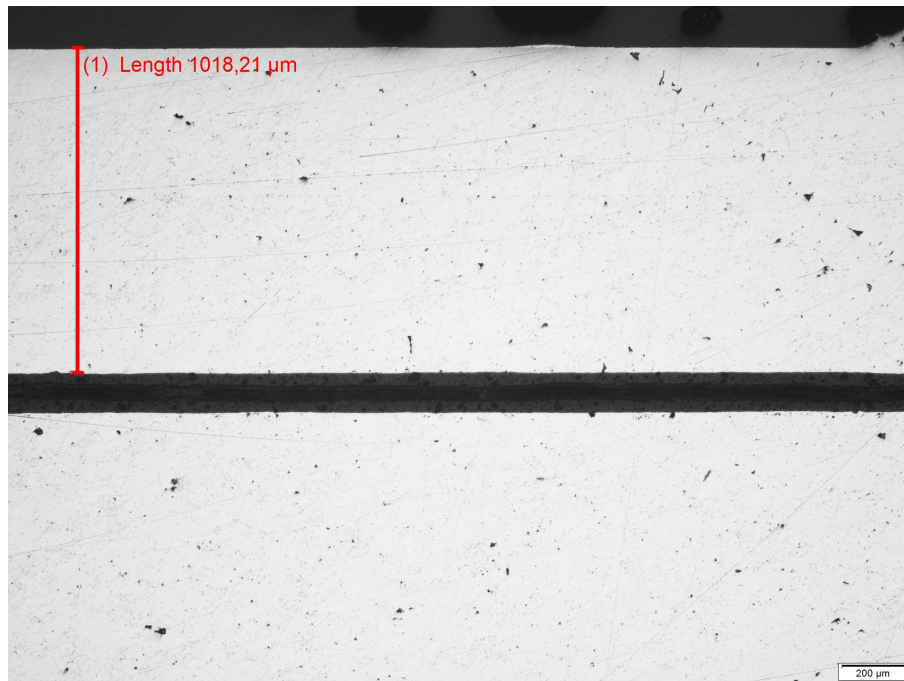


Figure B.23 Beam specimen PV01 - constraining layer - 5x Magnification

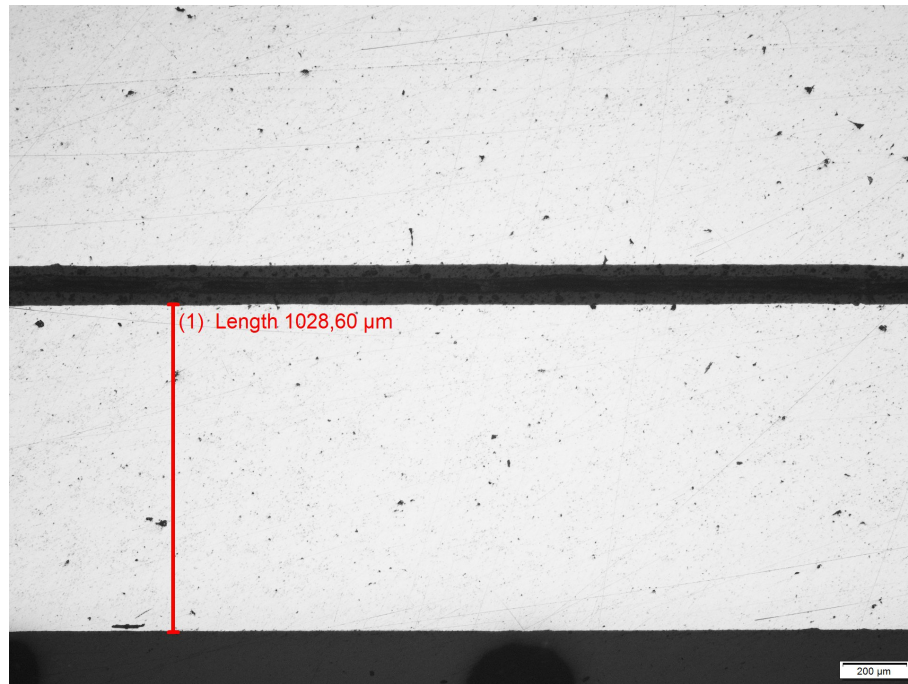


Figure B.24 Beam specimen PV01 - base layer - 5x Magnification

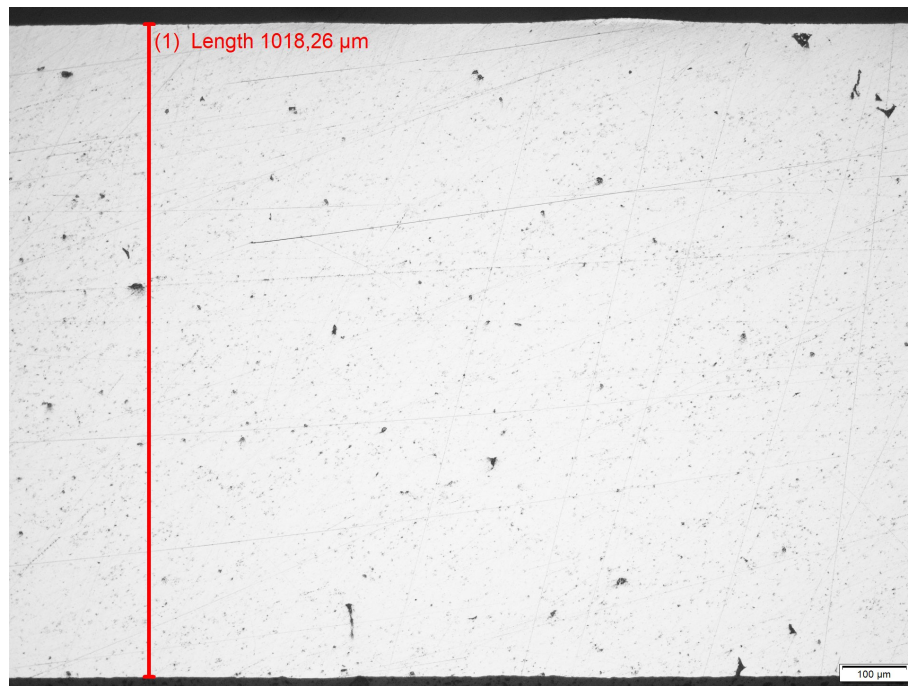


Figure B.25 Beam specimen PV01 - constraining layer - 10x Magnification

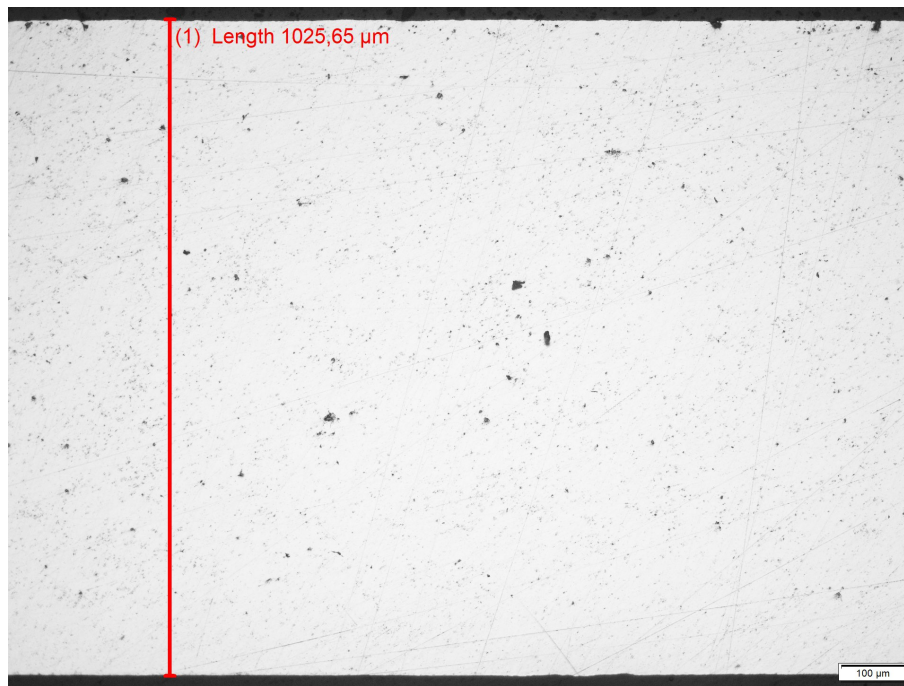


Figure B.26 Beam specimen PV01 - base layer - 10x Magnification

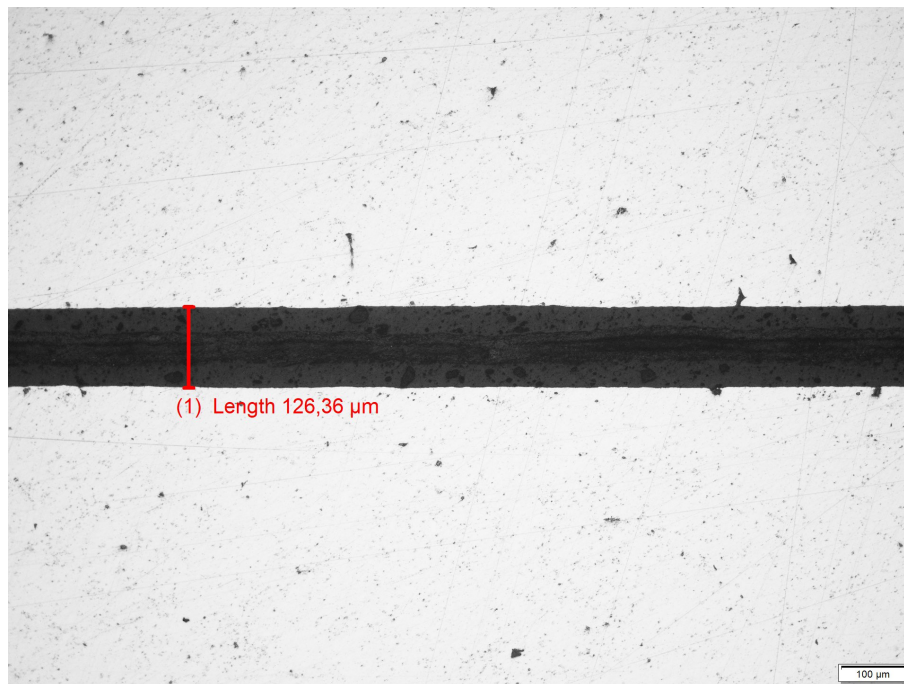


Figure B.27 Beam specimen PV01 - damping and adhesive layer - 10x Magnification

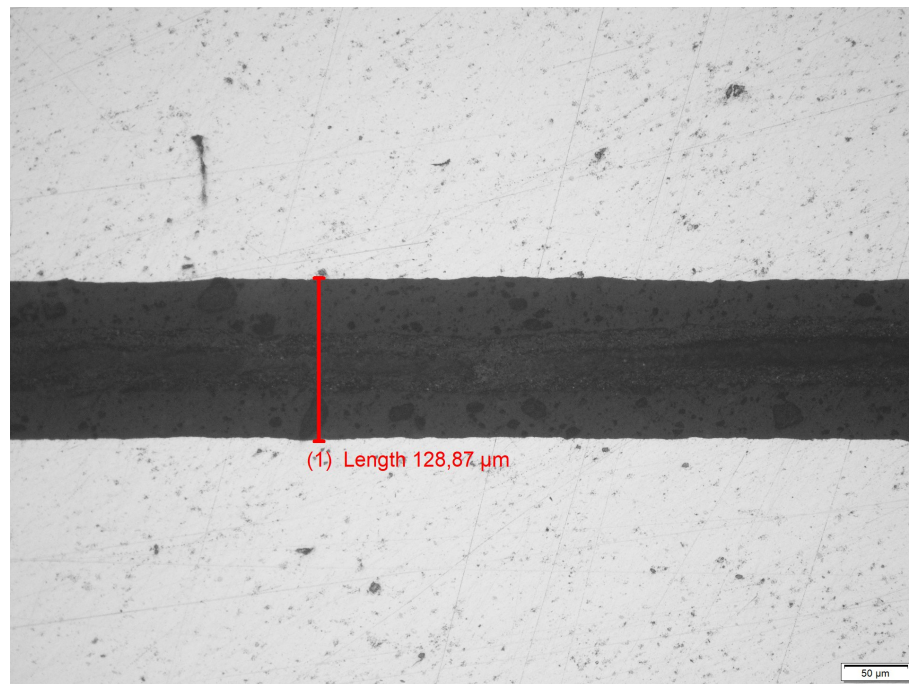


Figure B.28 Beam specimen PV01 - damping and adhesive layer - 20x Magnification

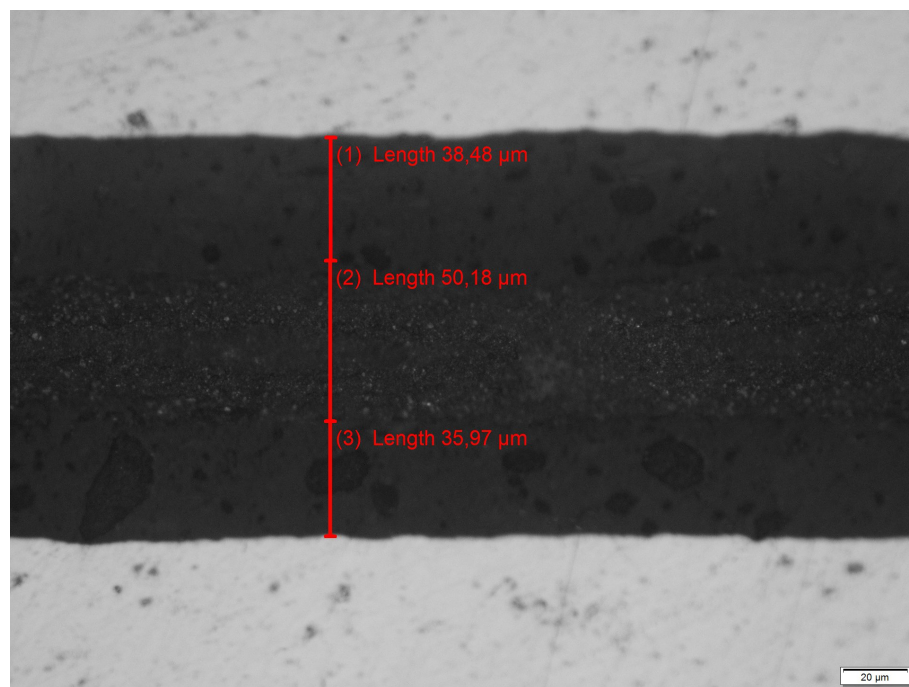


Figure B.29 Beam specimen PV01 - damping and adhesive layer - 50x Magnification

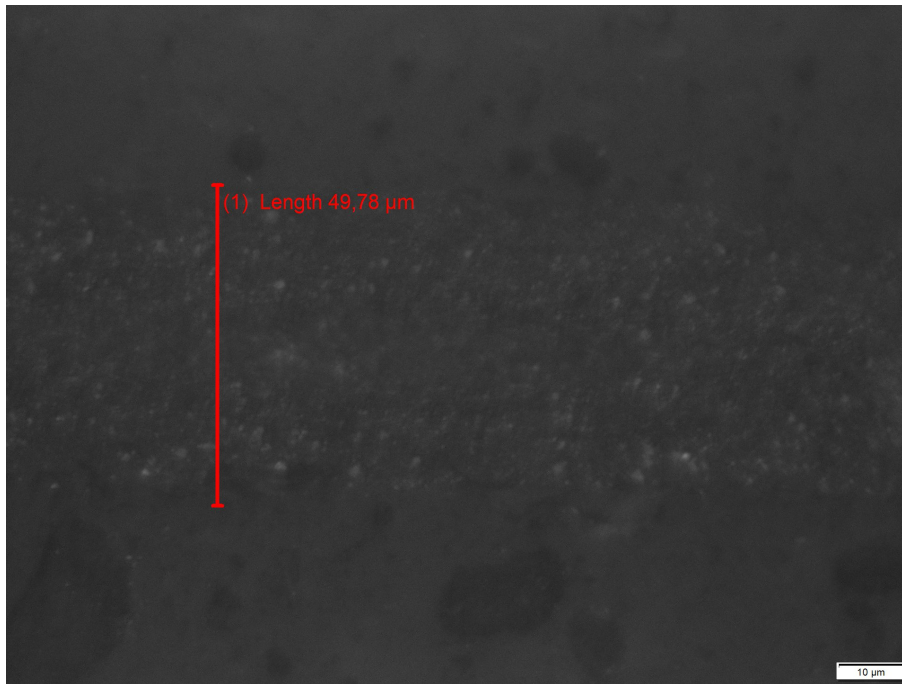


Figure B.30 Beam specimen PV01 - damping and adhesive layer - 100x Magnification

B.5 | Beam Specimen S02 - Cross Section

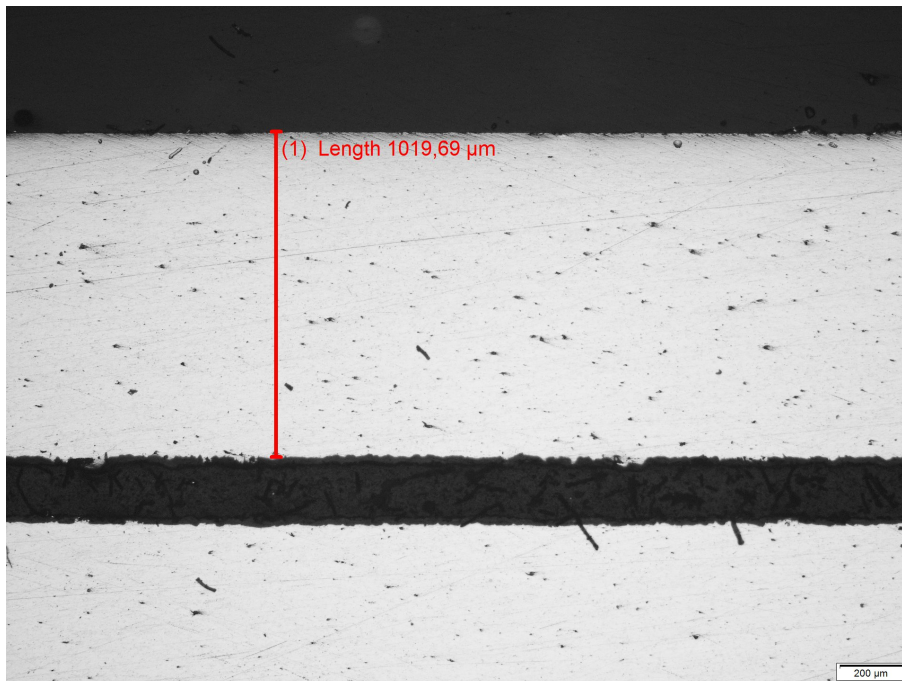


Figure B.31 Beam specimen S02 - constraining layer - 5x Magnification

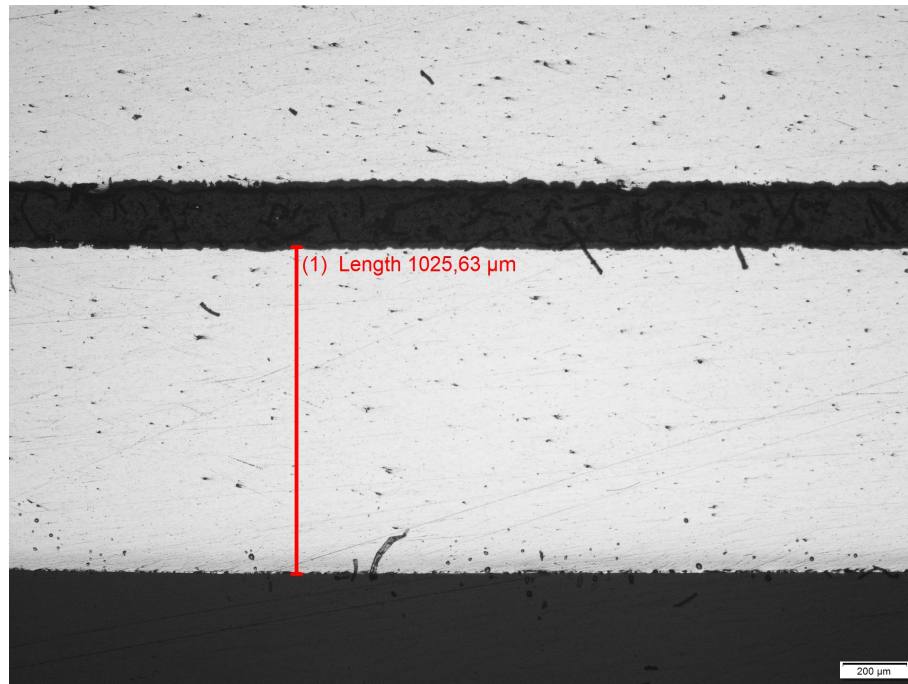


Figure B.32 Beam specimen S02 - base layer - 5x Magnification

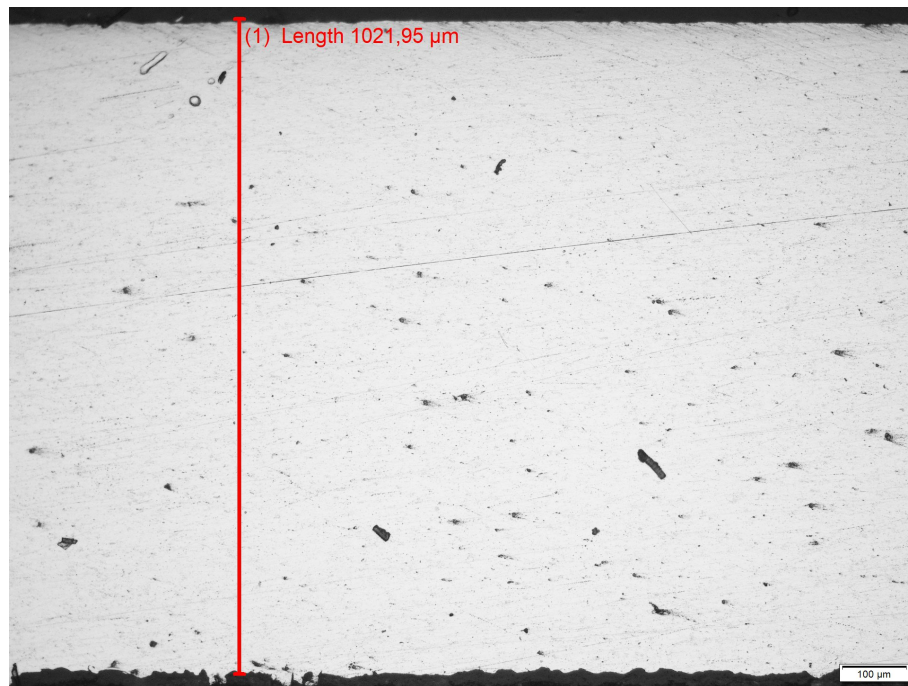


Figure B.33 Beam specimen S02 - constraining layer - 10x Magnification

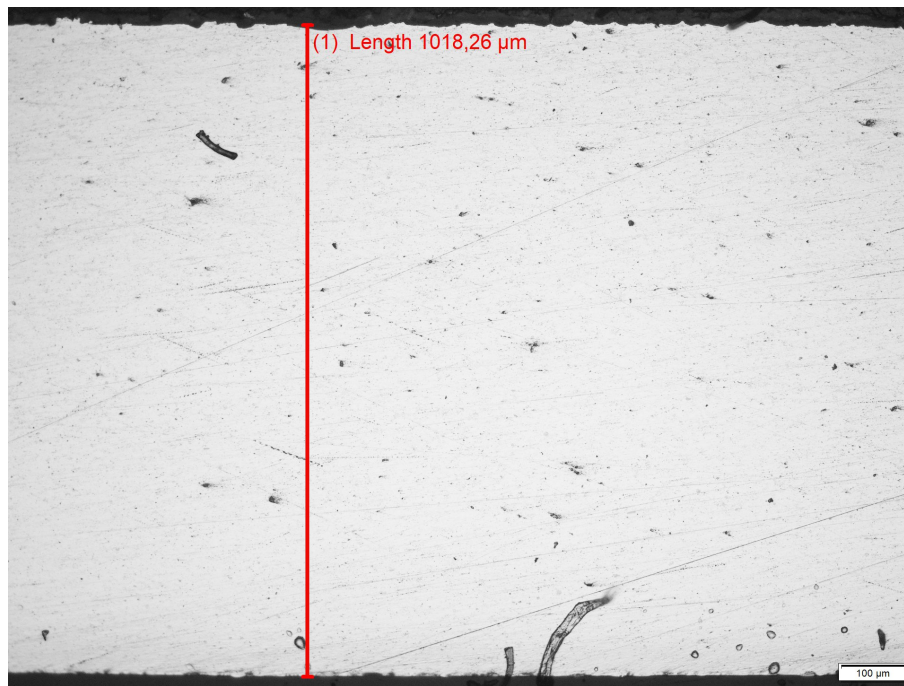


Figure B.34 Beam specimen S02 - base layer - 10x Magnification

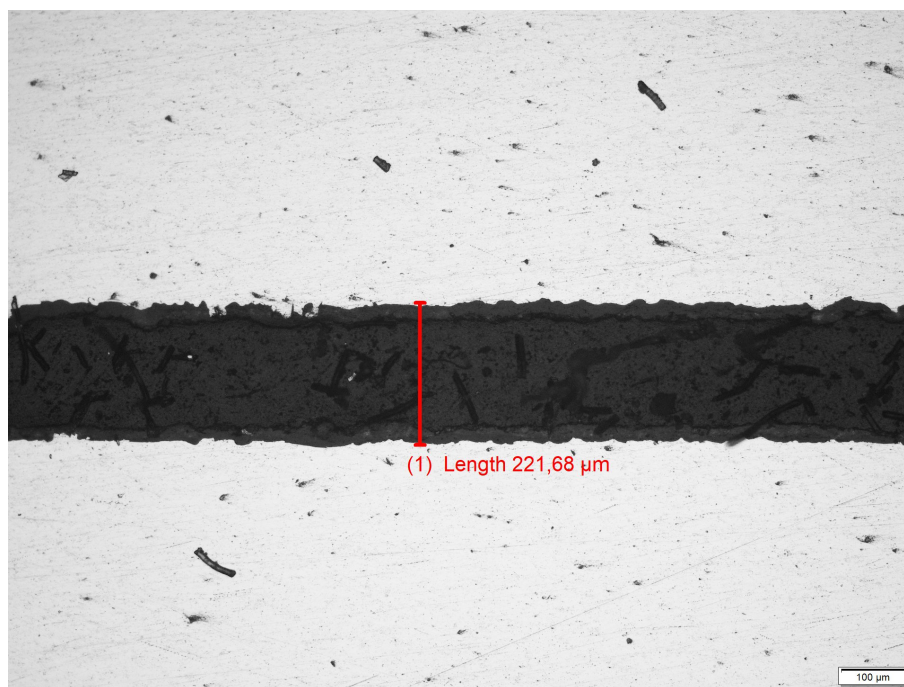


Figure B.35 Beam specimen S02 - damping and adhesive layer - 10x Magnification

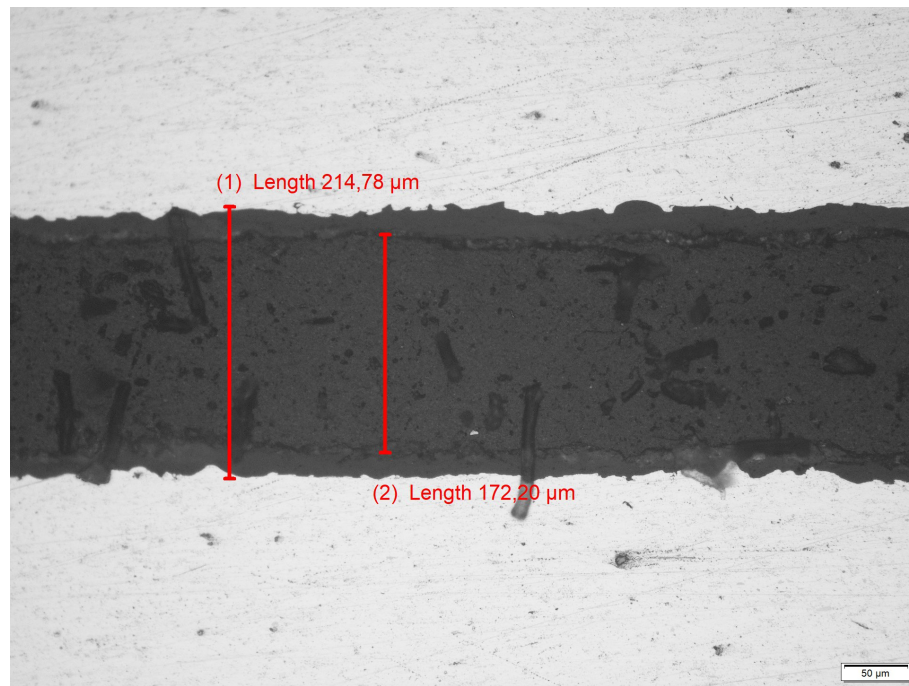


Figure B.36 Beam specimen S02 - damping and adhesive layer - 20x Magnification

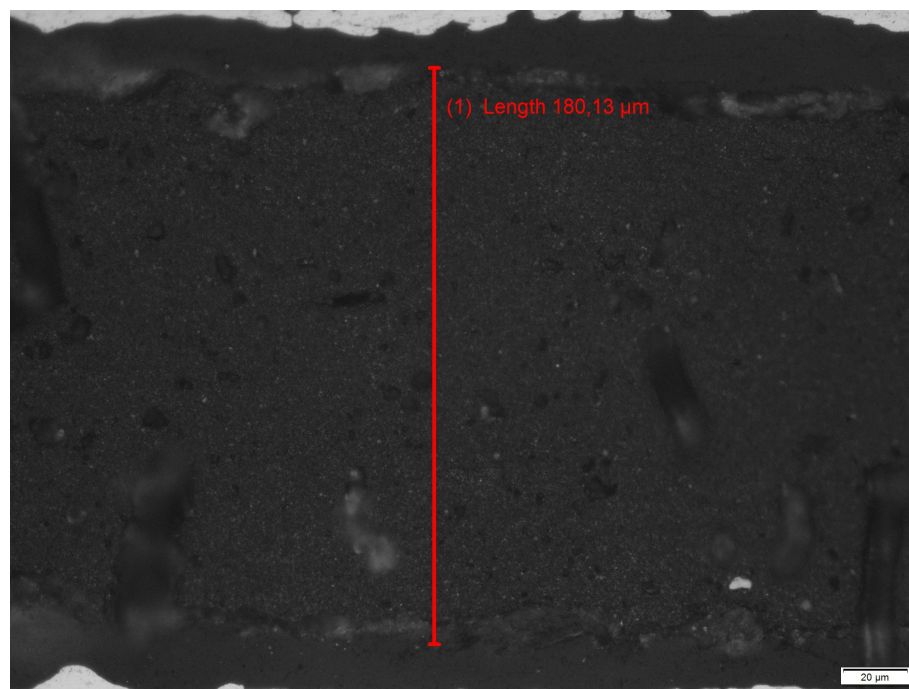


Figure B.37 Beam specimen S02 - damping and adhesive layer - 50x Magnification

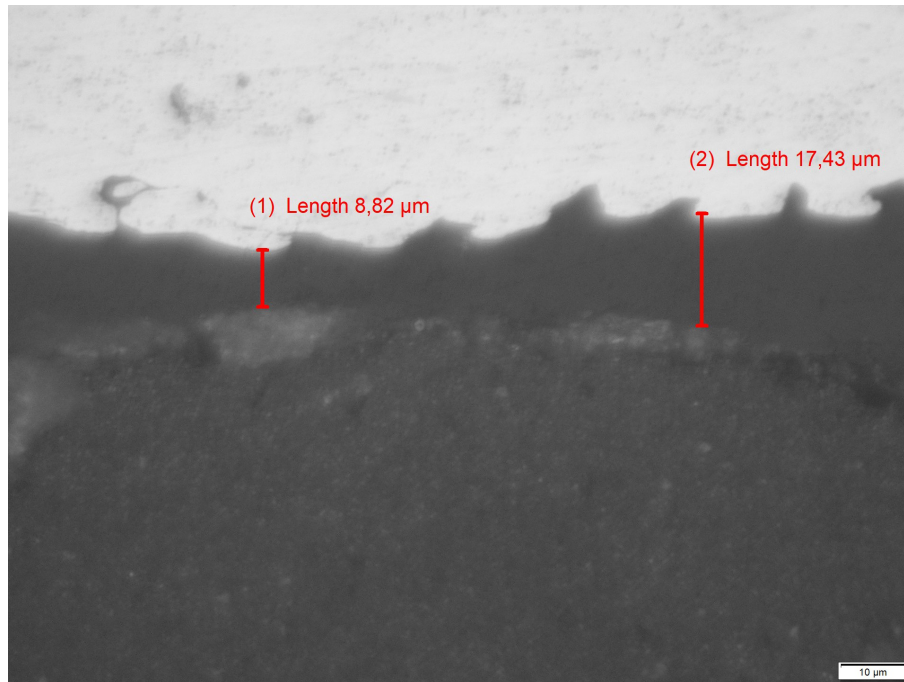


Figure B.38 Beam specimen S02 - damping and adhesive layer detail - 100x Magnification

B.6 | Beam Specimen K05 - Cross Section

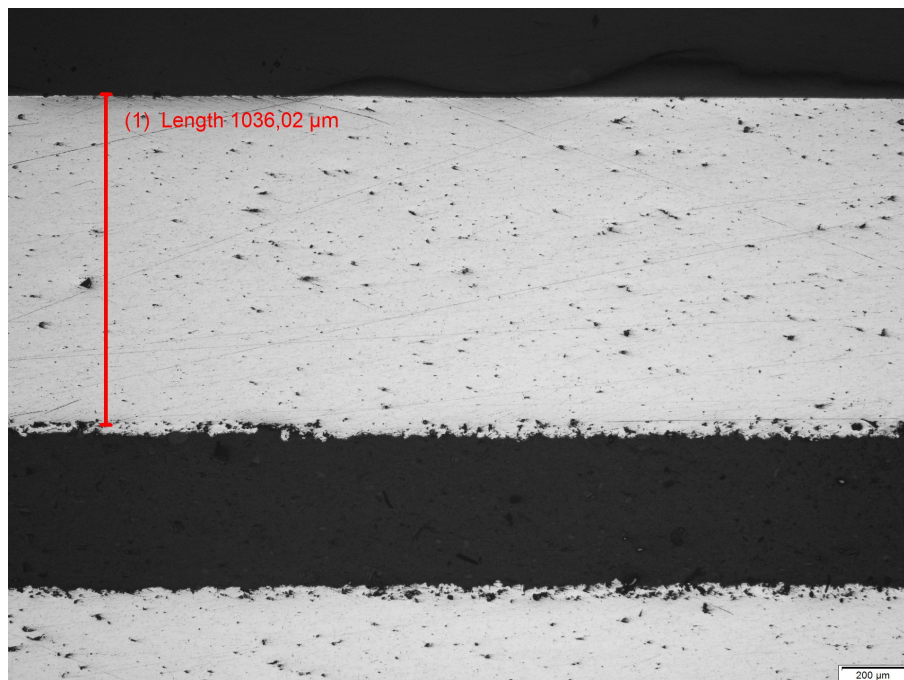


Figure B.39 Beam specimen K05 - constraining layer - 5x Magnification

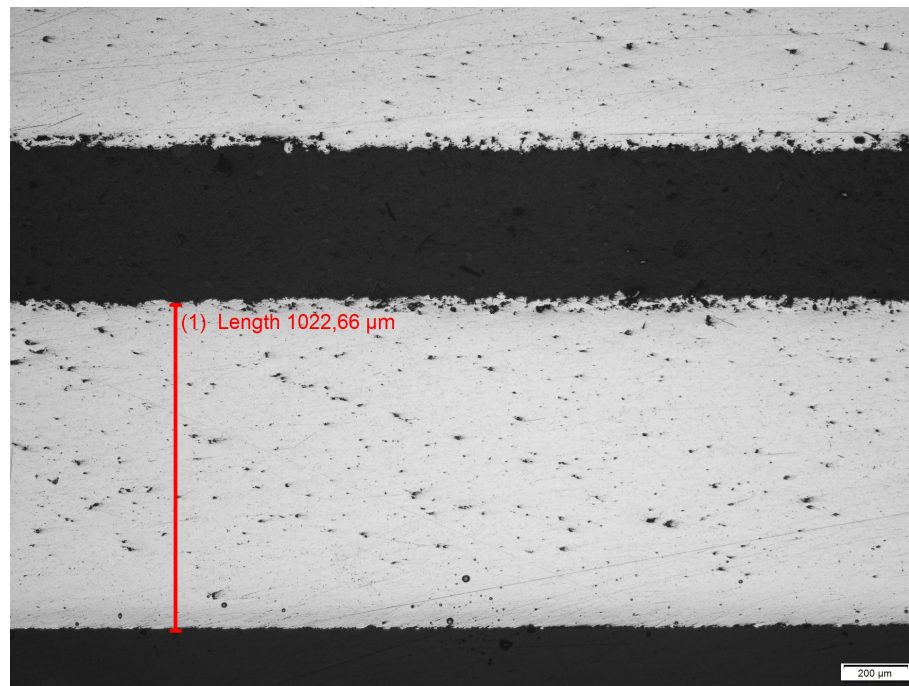


Figure B.40 Beam specimen K05 - base layer - 5x Magnification

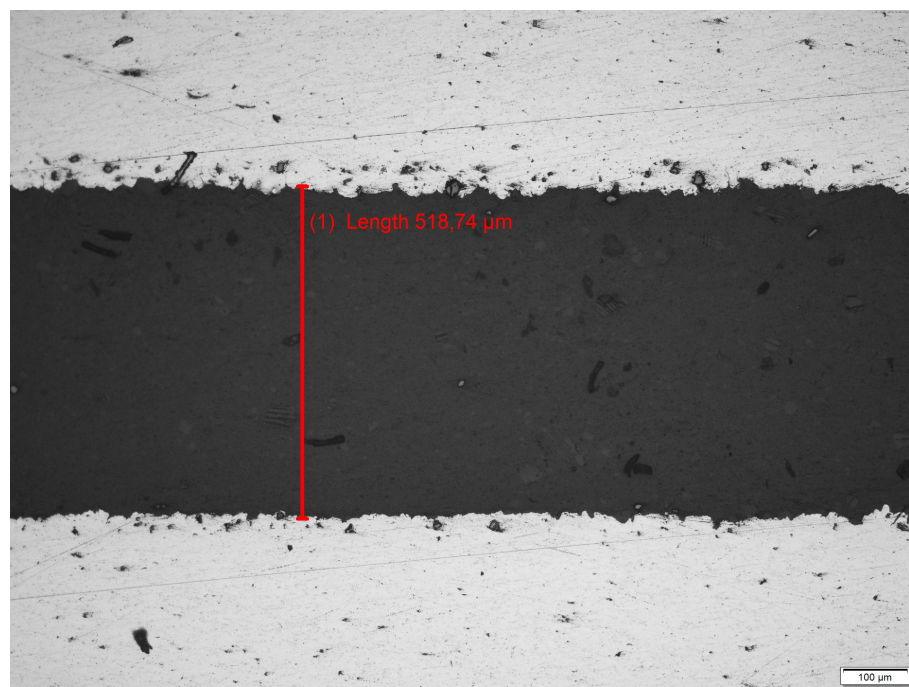


Figure B.41 Beam specimen K05 - damping and adhesive layer - 10x Magnification

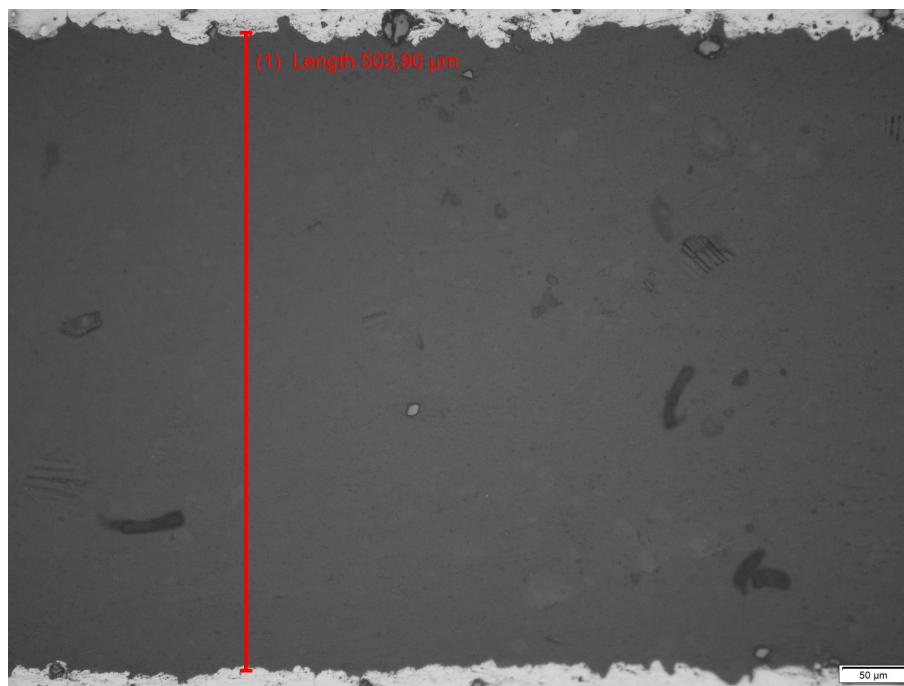


Figure B.42 Beam specimen K05 - damping and adhesive layer - 20x Magnification

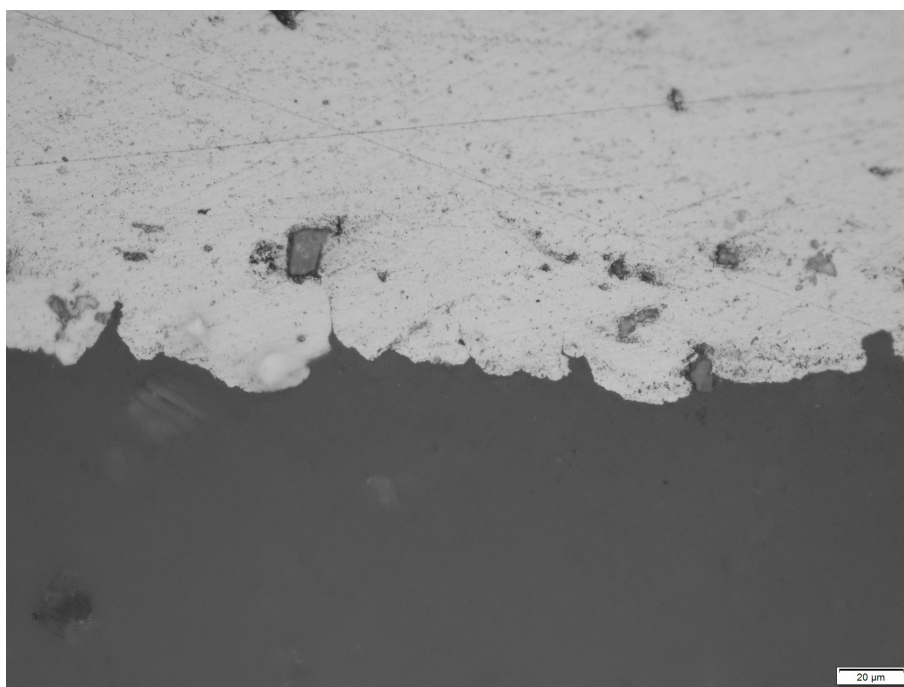


Figure B.43 Beam specimen K05 - damping and adhesive layer - 50x Magnification

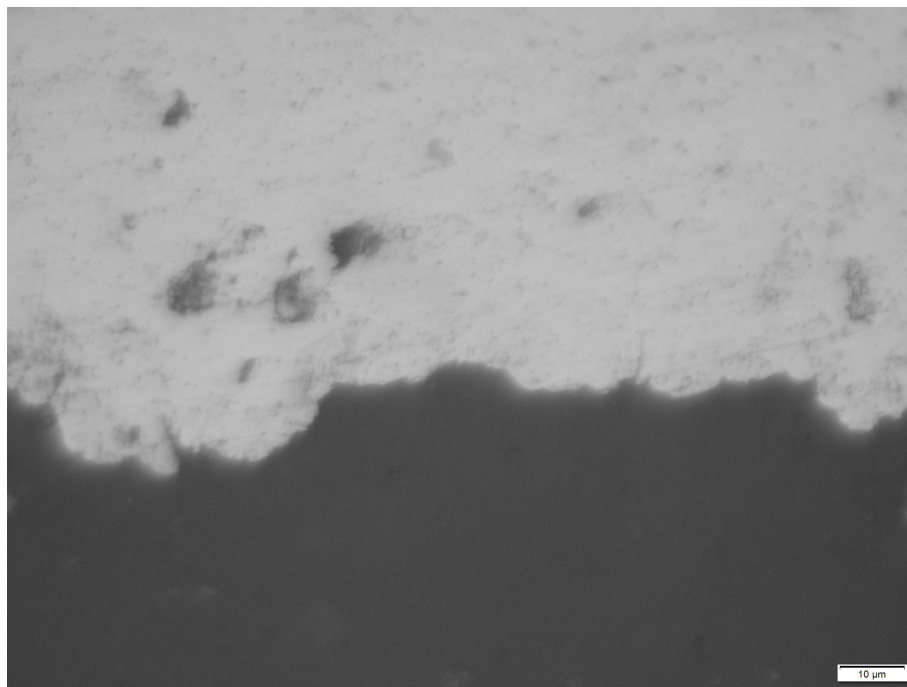


Figure B.44 Beam specimen K05 - damping and adhesive layer detail - 100x Magnification

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