

Pushing the boundaries of cool-dwarf asteroseismology with ESPRESSO

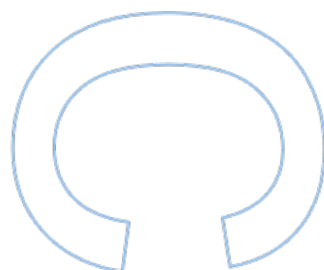
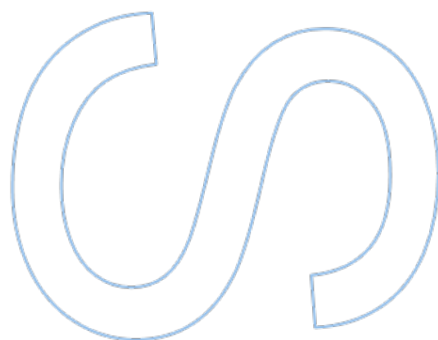
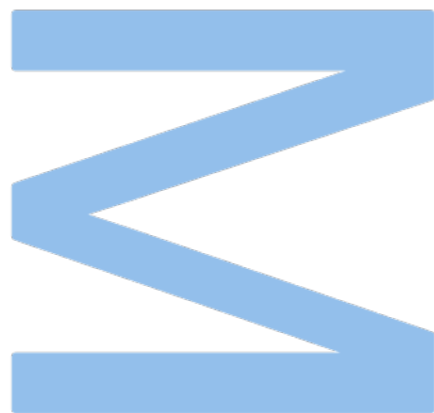
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Resumo

A asterossismologia tornou-se uma forma cada vez mais importante de determinar as propriedades globais das estrelas e investigar a sua estrutura interna, tanto através de avanços no lado observacional como teórico. Espectrógrafos de alta precisão, como o ESPRESSO, estenderam a asterossismologia para estrelas anãs frias de tipo K, onde nos concentraremos em ϵ Indi, a estrela mais fria na sequência principal para a qual foram detectados modos de oscilação até à data. Os nossos principais objetivos são estudar o termo de superfície, um erro sistemático nas frequências de oscilação entre modelos e observações, e avaliar os métodos atualmente usados para corrigi-lo, assim como investigar a física interna e as propriedades asterossísmicas de estrelas do tipo K por meio de modelação estelar.

Para fazer isso criámos mais de 2000 trajetos evolutivos estelares até ao final da sequência principal usando o código de evolução estelar MESA, calculando os modos asterossísmicos para cada modelo com o código GYRE. Na grelha de modelos variámos a massa inicial de $0.6M_{\odot}$ a $0.85M_{\odot}$, sendo $M_{\odot}$ a massa solar, o valor da fração metálica inicial, Z , de 0.005 a 0.015, e o *mixing length parameter*, que descreve a eficiência da convecção, de 1.2 a 2.4. Em seguida, realizámos otimização de modelos usando o programa AIMS, comparando a nossa grelha de modelos com as observações para ϵ Indi, e utilizando as diferentes opções para a correção do termo de superfície disponíveis.

Neste trabalho criámos diagramas C–D e ϵ para estrelas de tipo K, juntamente com diagramas das relações M – R e M – L . Representámos a dependência com a frequência da grande e pequena separação de frequência e testámos a compatibilidade de diferentes métodos para calculá-las. Na secção final, apresentámos as propriedades de ϵ Indi que determinámos através da otimização de modelos, como a massa e idade. Também descrevemos o termo de superfície e os resultados e características específicas de cada uma das correções ao termo de superfície que usámos.

Chegámos à conclusão que as diferentes correções do termos de superfície fornecem resultados compatíveis para ϵ Indi, mesmo se usarmos métodos diferentes para pesar observáveis clássicas e sísmicas na otimização dos modelos. Apesar disso, observámos que as correções ao termo de superfície tem dificuldade em modelar estrelas com um elevado grau de dispersão em termos da sua dependência com a ordem radial. Mostrámos que a escolha da atmosfera juntamente com o nível de eficiência convectiva têm um forte impacto nas frequências dos modos, conseguindo demonstrar assim como a

modelação asterossísmica deve levá-los em consideração ao realizar a otimização de modelos. Também mostrámos como os parâmetros asterossísmicos, por exemplo a separação de frequência grande e pequena, têm grande potencial para restringir a idade e a massa de uma estrela, o que revela o potencial da asterossismologia para o estudo de estrelas do tipo K, à medida que novas detecções de oscilações ficam disponíveis.

Palavras-chave: Asterossismologia - Estrelas: pequena massa - Termo de superfície - Estrelas: individual: ϵ Indi A - Astrofísica - técnicas: Velocidade radial

Abstract

Asteroseismology has become an increasingly relevant way to determine the global properties of stars and peer into their inner structure, both through advances in the observational and theoretical side. High-precision spectrographs, such as ESPRESSO, have extended asteroseismology to cool K-type dwarf stars, where we will focus on ϵ Indi, the coolest main-sequence star for which oscillation modes have been detected until now. Our main goals are studying the surface term, a systematic error in oscillation frequencies between models and observations, and evaluating the currently used methods for correcting it, along with investigating the inner physics and asteroseismic properties of K-type stars through stellar modelling.

To do this we created over 2000 stellar tracks until the end of the main-sequence using the stellar evolution code MESA, computing the asteroseismic modes for each model with GYRE. In the grid we varied the initial mass from $0.6M_{\odot}$ to $0.85M_{\odot}$, with $M_{\odot}$ being the mass of the Sun, the value of the initial metal fraction, Z , from 0.005 to 0.015, and the mixing-length parameter, describing the efficiency of convection, from 1.2 to 2.4. We then performed model optimization using AIMS, comparing our grid of models with the observations for ϵ Indi, utilizing the different options for the surface-term correction available in it.

In this work we have created C–D and ϵ diagrams for K-type stars, along with diagrams of the M – R and M – L relationships. We have represented the frequency dependency of the large frequency separation and the small frequency separation, and tested the compatibility of different methods for calculating them. In the final section, we show the properties of ϵ Indi we determined through model optimization, such as its mass and age. We also represented the surface term and the unique results and features of each of the surface-term corrections we used.

We have been able to conclude that the different surface-term corrections give compatible results for ϵ Indi, even if we use different methods for weighting the classical and seismic observables in model optimization. In spite of this, we observed that the surface-term corrections will have difficulties in stars whose mode frequencies show a high degree of scatter in their radial order dependency. To show that the choice of the atmosphere along with the level of convective efficiency have a strong impact on the mode frequencies, thereby showing how asteroseismic modelling has to take them into account when performing model optimization. We have also shown how asteroseismic parameters such as the large and small frequency separation have great potential for constraining the age

and mass of a star, which displays the potential of asteroseismology for the study of K-type stars as further mode detections occur in the future.

Keywords: Asteroseismology - Stars: low-mass - Surface-term - Stars: individual: ϵ Indi
A - Astrophysics - techniques: radial velocities

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List of Abbreviations

CoRoT Convection, Rotation and planetary Transits

TESS Transiting Exoplanet Survey Satellite

ESPRESSO Echelle SPectrograph for Rocky Exoplanet and Stable Spectroscopic Observations

MESA Modules for Experiments in Stellar Astrophysics

AIMS Asteroseismic Inference on a Massive Scale

ZAMS Zero age main-sequence

TAMS Terminal age main-sequence

EOS Equation of state

PMS Pre main-sequence

MS Main-sequence

HR Hertzsprung–Russell

ESA European Space Agency

NASA National Aeronautics and Space Administration

VLT Very Large Telescope

Hipparcos High Precision PARallax COLlecting Satellite

HARPS High Accuracy Radial velocity Planet Searcher

UVES Ultraviolet and Visual Echelle Spectrograph

1. Introduction

Asteroseismology is a growing field in astrophysics propelled by the ever increasing number of stars for which asteroseismic modes have been detected, through data collected from space-based missions such as CoRoT (Convection, Rotation and planetary Transits; France/ESA) [1], Kepler (NASA) [2], and TESS (Transiting Exoplanet Survey Satellite; NASA) [3], which used photometry to detect stellar oscillations by analysing the variations in the star's photon flux. Asteroseismology permits us to determine with high precision stellar features, such as age, mass, and radius, and allows us to have a more comprehensive view into the star's interior [4]. In spite of this success, asteroseismic observation of solar-like oscillators, where oscillations are stochastically excited in the near-surface convection zone of stars, has been limited to particular types of stars due to the difficulty in mode detection, mainly main-sequence stars at least as hot and massive as the Sun and red giants [5]. That is why the discovery of modes in ϵ Indi A (hereafter, ϵ Indi), a K-type dwarf that is much cooler and less luminous than the Sun, is an important discovery [6] [7]. This is the case both for asteroseismology, by extending it to a new stellar regime, and for astrophysics in general, as new insights into stellar interior physics might arise from it. Our work will thus be focused on using the recent data collected for ϵ Indi by ESPRESSO (Échelle SPectrograph for Rocky Exoplanet and Stable Spectroscopic Observations) [8], a ground-based spectrograph mounted on the Very Large Telescope (VLT). This shift to radial velocity instead of photometry measurements is important, given that in radial velocity there will be a much lower amount of noise due to granulation, i.e., surface patterns of convection, which is one of the main noise sources for the detection of p-mode frequencies [9]. We aim at analysing its asteroseismic behaviour, at determining whether asteroseismic modelling methods applied to other types of stars can also be applied in this new regime, and at gaining more precise and accurate knowledge of the features of ϵ Indi as a whole, such as its age and mass. In conjunction with this, we will use stellar modelling and analysis software, namely, MESA (Modules for Experiments in Stellar Astrophysics) [10], GYRE [11], and AIMS (Asteroseismic Inference on a Massive Scale) [12] to complement what we know from the observations and work towards creating a more accurate model for ϵ Indi.

The main issue we will focus on is the surface term, which is a systematic error that has been found between the model frequencies of oscillation modes and the observed

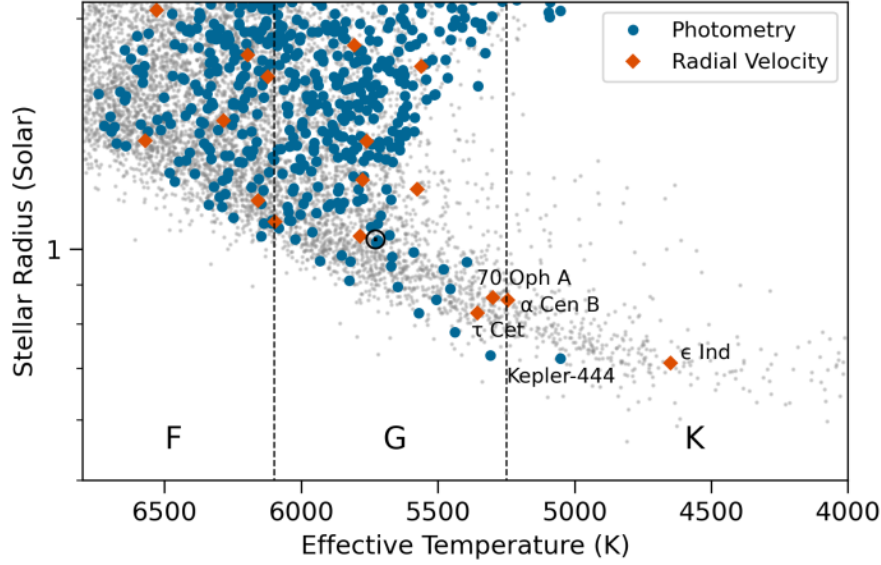


Figure 1.1: Stellar radius-effective temperature diagram containing stars with seismic detections in photometry (blue circles) and radial velocity (red diamonds). The background stellar sample (grey dots) is taken from the TESS Input Catalog [3]. From [6].

frequency spectrum across a variety of stars [13]. This error is assumed to be due to poor modelling of the near-surface layers of the star, and is of importance, since it diminishes our ability to use asteroseismic data to obtain precise knowledge of stellar properties [14]. Several methods have been devised to try to correct for the surface term in stellar models [15]. We plan to apply them to ϵ Indi in order to assess their performance and discover if any of them might be more suited in stars similar to ϵ Indi. Moreover, we set to find the optimal stellar model for ϵ Indi, which will provide us information on how the surface term depends on features such as the efficiency of convection, the initial stellar mass, and metallicity, along with precise estimates of the star's global parameters.

K-type stars are by themselves a very interesting research topic, possessing characteristic properties such as deep convective envelopes [16], long lifespans [17] and colder surface temperatures, that increase the wavelength of their light, leading them to also be called orange-dwarf stars. Figure. 1.1 permits us to see how far we have gone into this regime with ϵ Indi, and the potential for further asteroseismic observations in the "void" between ϵ Indi and stars such as Kepler-444 and α Cen B. The study of K-type stars also intersects with another growing area of study at the moment, which is exoplanet research, due to the fact that they have been identified as prime targets in the search for potentially habitable worlds [18]. ϵ Indi is a very good example of this, as it hosts a brown dwarf binary in a wide orbit, ϵ Indi Ba and ϵ Indi Bb, along with a cold Jupiter, ϵ Indi Ab. K-type stars are also seen as valuable for studying the chemical evolution of the Galaxy [19], due to having

Table 1.1: Properties of ϵ Indi. Information on the radius, effective temperature, and luminosity has been taken from [27]; mass from [28]; and distance, magnitude, and metallicity from [29].

Distance (pc)	Magnitude	T_{eff} (K)	Luminosity ($L_{\odot}$)	Metallicity (dex)	Radius ($R_{\odot}$)	Mass ($M_{\odot}$)
3.64	4.69	4649 ± 84	0.21 ± 0.02	-0.17 ± 0.05	0.711 ± 0.005	0.762 ± 0.038

almost the same metal abundances as in their origin and very long lifespans.

ϵ Indi itself has been studied for a long time, with measurements of its proper motion as early as 1847 [20], due to the fact that it is one of the closest stars to Earth at 3.64 pc, possessing an apparent magnitude of 4.69 that allows it to be visible to the naked eye in the southern hemisphere. These features, along with others that will be used later on in this work, are described in Table. 1.1. In it, besides the low value of T_{eff} we already mentioned in Fig. 1.1, we see how ϵ Indi is a metal poor star with a luminosity about 5 times lower than the Sun, and a radius and mass at 71% and 76% of the solar value respectively.

The determination of these parameters for ϵ Indi has come from several previous studies of it, using different observational data and methods for their determination. ϵ Indi has been observed using space-based missions such as Hipparcos (High Precision PARallax COLlecting Satellite) [21], GAIA [22] and TESS [23], along with ground-based missions such as HARPS (High Accuracy Radial velocity Planet Searcher) and UVES (Ultraviolet and Visual Echelle Spectrograph) [7]. When it comes to the modelling of ϵ Indi we will mention the work of [24] and [25] that used the R_{HK} index, which measures the emission in the calcium H and K bands relative to the total stellar flux, and rotation to determine the age of ϵ Indi, along with [26] that studied both ϵ Indi and its exoplanet companion ϵ Indi Ab. All of the previous effort dedicated to observing and modelling ϵ Indi shows how it is a star of high scientific interest, providing further justification for the work we intend to do here.

To achieve these goals, we plan to create a multidimensional grid of stellar models using MESA, calculate the frequencies of their oscillation modes using GYRE, and compare their properties to observations using AIMS, which optimizes this process for large sets of models. The grid will contain different values for the star's initial mass, metal mass fraction, and mixing length parameter (which determines the convective efficiency in the model), and different types of models for the stellar atmosphere will also be studied. We will then perform a detailed diagnosis of the models to assess how the star's asteroseismic features and internal structure vary across the parameter space of the grid. In the model

optimization stage, we will find the best-fitting stellar model of ϵ Indi, calculate the surface term for our grid of models, and apply the several methods for its correction.

Throughout this work, we hope to be able to add to the understanding of how the surface term depends on stellar properties and to gain insight into the best ways for correcting it in our case study. Through the stellar modelling component of the project, we intend to determine with high precision the global parameters of ϵ Indi as well as its internal structure, so as to shed light on the causes for the known discrepancy between the modelled and observed values of the radius and effective temperature of K dwarfs [28].

Our work will be divided in the following way. In Chapter 2 we present the fundamental concepts, equations and results in the field of stellar structure and evolution necessary for understanding our work, while in Chapter 3 the same is done for Asteroseismology. In Chapter 4 we will give an introduction to the software used for the creation of our grid of models, built with the goal of modelling ϵ Indi, which are called MESA and GYRE, along with explaining the properties of the grid itself. We will also do a study of the impact the choice of stellar atmosphere has on the models. In Chapter 5 we perform a detailed study of the grid of models created in the previous chapter, dividing it into the study of the global stellar parameters, inner structure and asteroseismic properties. Chapter 6 is where we perform model optimization using AIMS, and investigate the surface term along with the impact of the different surface-term corrections. Chapter 7 shows our conclusions and possible future work connected to what was done here.

2. Stellar structure and evolution

In this chapter we want to lay the theoretical foundations for the work that will be done in the subsequent chapters, by describing the essential concepts and equations in stellar structure and evolution. This chapter was mainly based on [30] and [31].

2.1. Fundamental equations of stellar structure

The fundamental equations in stellar structure and evolution are expressions of conservation laws, namely, the conservation of mass, momentum and energy, and are derived from the equations of Hydrodynamics with simplifying assumptions. Alongside the equations expressing conservation laws, we will also need to describe how energy is generated and transported in a star, how chemical abundances change with time and an equation of state relating the thermodynamic quantities of a star. Boundary conditions at the center and surface of the model will then need to be established for solving the set of differential equations describing the star.

The simplifying assumptions from which we start are that stars possess spherical symmetry, which means that instead of having 3D models with coordinates (r, θ, ϕ) they will solely depend on the radial coordinate r , and that there are no electromagnetic forces, so that the only active forces are due to the pressure gradient and gravity. Rotation and magnetic fields will break the spherical symmetry of a star, and the stellar oscillations in asteroseismology will also have non-radial components, which is why these assumptions are only approximate.

Another important assumption is that stars are black bodies, which means that the spectral density of their radiation will follow Planck's law and that we can relate their total luminosity L , the amount of energy per unit time flowing out of the star, to its radius R and effective temperature T_{eff} by the Stefan–Boltzmann law,

$$L = 4\pi \sigma_{\text{SB}} R^2 T_{\text{eff}}^4, \quad (2.1)$$

where σ_{SB} is the Stefan–Boltzmann constant. The effective temperature can be considered the temperature at the stellar photosphere, the region where energy is radiated out from the star, although later on we will give a more precise definition of it.

The four basic differential equations of stellar structure are the mass conservation equation,

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2, \quad (2.2)$$

the equation of hydrostatic equilibrium, which expresses the conservation of momentum,

$$\frac{\partial P}{\partial r} = -\frac{Gm\rho}{r^2}, \quad (2.3)$$

the equation describing energy production and expressing the conservation of energy,

$$\frac{\partial L_r}{\partial m} = \epsilon + \epsilon_g - \epsilon_\nu, \quad (2.4)$$

and the equation describing the transport of energy along with the temperature stratification in the star,

$$\frac{\partial T}{\partial r} = -\frac{Gm\rho T}{r^2 P} \nabla. \quad (2.5)$$

In these equations $m(r, t)$ is the mass contained within a sphere of radius r at instant t , ρ is the density, P is the pressure, T is the temperature and L_r is the luminosity, the amount of energy flowing through a shell at radius r per unit time as a result of generated energy. ϵ is the energy released per unit mass and second by nuclear fusion, called the emissivity, ϵ_ν the energy lost by production of neutrinos and ϵ_g the energy released by gravitational contraction or spent by gravitational expansion. ∇ is the temperature gradient, defined as $\nabla = \frac{\partial \log T}{\partial \log P} = \frac{P}{T} \frac{\partial T}{\partial P}$.

In these equations r and t are the independent variables and (m, P, L_r, T) are what we want to know by solving the equations. ρ will be determined by the equation of state, ∇ by describing the energy transport mechanisms in the star and ϵ by describing energy production through nuclear fusion.

The evolution and spatial profile of the chemical abundances of the star due to nuclear reactions is described by the following equation,

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \left(\sum_j r_{ji} - \sum_k r_{ik} \right), \quad (2.6)$$

where $X_i(r, t)$ is the mass fraction of element i at radius r and time t , m_i the mass of the nucleus of isotope i , r_{ji} the rate at which element i is produced from element j and r_{ik} the rate at which element i is converted into element k . The rates of nuclear reactions r_{ji} and r_{ik} form part of the microphysics that will need to be input into the models.

In order to describe the effect of gravity we will also need the Poisson equation, which relates the gravitational potential ϕ to the density, and whose general form is

$$\nabla^2 \phi = 4\pi G \rho, \quad (2.7)$$

and which considering spherical symmetry, becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho. \quad (2.8)$$

Integrating this equation in r permits us to obtain the definition of the gravitational acceleration in the star,

$$g = \frac{\partial \phi}{\partial r} = \frac{Gm}{r^2}, \quad (2.9)$$

whose value at the surface is known as the surface gravity.

2.2. Energy transport

To describe how energy is transported from the center to the surface of the star we will need to define ∇ in a more concrete way, according to the energy transport mechanisms active in the star, which are radiation, conduction and convection. This leads to defining the flux of energy $F(r)$, the amount of energy that passes through a shell of radius r per unit time, as

$$F = F_{\text{rad}} + F_{\text{cond}} + F_{\text{conv}}. \quad (2.10)$$

Assuming that the energy transport in a star is a diffusive process, then we have that the flux can be written in the form of a heat equation as

$$F = -K \frac{dT}{dr}, \quad (2.11)$$

where K is the thermal conductivity, that can be written in a general way as $K = \frac{1}{3} v l c_V$, with v being the average velocity of the particles, C_V the specific heat per unit volume and l their mean free path. It is v and l that will then have to be defined uniquely for radiation, conduction and convection.

Through this definition we can then write the temperature gradient as,

$$\nabla = \frac{L_r}{4\pi G m} \frac{P}{\rho T} \frac{1}{K}. \quad (2.12)$$

2.2.1 Radiative transport and conduction

In the case of radiative transport the energy is transported by photons, so that $v = c$, where c is the speed of light, and the mean free path of photons is defined as

$$l_{\text{photon}} = \frac{1}{\kappa_{\text{photon}} \rho}, \quad (2.13)$$

with κ_{photon} being the opacity, that describes how much a photon tends to be absorbed by its environment.

Given that for a photon gas the energy per unit volume, u , is $u = aT^4 \Rightarrow c_V = \frac{du}{dT} = 4aT^3$, where a is the radiation constant, and using the above definition for the mean free path, we will have that

$$K_{\text{rad}} = \frac{4ac}{3} \frac{T^3}{\kappa_{\text{photon}} \rho}. \quad (2.14)$$

From a formal point of view there is no difference between the treatment of photons and other particles, so that conduction will be described in the same manner as radiation, with the opacity including contributions from other particles besides photons,

$$\frac{1}{\kappa} = \frac{1}{\kappa_{\text{photons}}} + \frac{1}{\kappa_{\text{particles}}}. \quad (2.15)$$

Inserting the expression for the radiative thermal conductivity, in Eq.(2.14), into Eq.(2.12), we get that the radiative temperature gradient ∇_{rad} , the temperature gradient in a region

if energy was fully transported by radiation and gas particles, is

$$\nabla_{\text{rad}} = \frac{3}{16\pi acG} \frac{\kappa P L_r}{m T^4} . \quad (2.16)$$

The opacity is a function that still needs to be defined, and that depends both on the frequency ν of a photon and on the local conditions of a star, the local temperature and density. In order to simplify the description of the opacity it is typically considered its averaged value over frequency, which is called the Rosseland mean opacity,

$$\frac{1}{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu} , \quad (2.17)$$

where B_ν is the Planck function.

The type of process through which photons are absorbed will affect the equation for the opacity, with these relations taking the approximate form of,

$$\kappa = \kappa_0 \rho^\alpha T^{-\beta} , \quad (2.18)$$

where κ_0 is a parameter that will depend on the chemical composition of the star, and α and β positive-value parameters that depend on the properties of the star, which for the case of a solar-like star are $\alpha \simeq 1$ and $\beta \simeq \frac{7}{2}$.

2.2.2 Convective transport of energy

Convection is more difficult to model than radiation due to the fact that it is an inherently 3D phenomena, while our models are 1D, and its timescale is much faster than that of stellar evolution, minutes and hours instead of millions or billions of years, thus making it impossible to evolve convection and the star as a whole at the same time. To get over these difficulties approximations are used, with one of the most common methods for modelling convection, and which will be used in subsequent chapters, being mixing-length theory (MLT) [32].

Transport of energy by convection occurs when radiation is not capable of transporting all of the required energy. We can get a condition for the existence of convection by considering a perturbation in a small parcel of stellar gas over a distance Δr . A region will then

be convectively unstable if, after the perturbation, the perturbed blob of gas experiences a force which continues to push it away from its initial position, thereby making it continue to move across the star. Starting from this scenario, and by analysing the density difference $D\rho$ between the original and perturbed position of the gas, we obtain the Ledoux criterion for convective instability,

$$\nabla_{\text{rad}} \geq \nabla_{\text{ad}} + \frac{\varphi}{\delta} \nabla_{\mu} \Rightarrow \text{convection}, \quad (2.19)$$

where ∇_{ad} is the adiabatic gradient, $\varphi = (\frac{\partial \log \rho}{\partial \log \mu})_{P,T}$, $\delta = (\frac{\partial \log \rho}{\partial \log T})_{P,\mu}$ and $\nabla_{\mu} = \frac{d \log \mu}{d \log P}$, with μ being the mean mass per particle.

According to Eq.(2.19) a star will be more convectively stable if μ tends to increase with depth, as $\nabla_{\mu} > 0$ in that case, thereby increasing the threshold for the value of ∇_{rad} . If, however, we perform the approximation of assuming a constant chemical composition, $\nabla_{\mu} = 0$, then we obtain the Schwarzschild criterion,

$$\nabla_{\text{rad}} \geq \nabla_{\text{ad}} \Rightarrow \text{convection}. \quad (2.20)$$

We see from the above conditions that convection occurs in regions where ∇_{rad} is very high, which according to Eq.(2.16), will be cold and dense regions with a high opacity or regions where the luminosity is very high. The first is what happens in the envelope of the Sun and dwarf stars, such as ϵ Indi, while the latter occurs in the core of high-mass stars where an enormous amount of energy is produced.

Now we need to find an expression for the convective flux, F_{conv} . To do this we go back to the scenario of the perturbed parcel of gas, and we note that its energy excess compared to its environment per unit mass will be $\Delta u = c_p \Delta T$, where c_p is the specific heat per unit mass at constant pressure. Now we need to find an expression for the temperature difference between the parcel of gas and its environment, which can be shown to be of the form,

$$\Delta T = T \frac{l_{\text{MLT}}}{H_P} (\nabla - \nabla_{\text{ad}}), \quad (2.21)$$

where l_{MLT} is the mixing length, the average distance that will be traveled due to convection, and $H_P = P(-\frac{dP}{dr})^{-1}$ is the pressure scale height, that is the height that a parcel of

gas will have to travel until the pressure decreases by a factor of e .

We can write the convective flux as the velocity of the parcel of gas multiplied by the amount of energy it carries per unit volume,

$$F_{\text{conv}} = v_{\text{conv}} \rho \Delta u = v_{\text{conv}} \rho c_p \Delta T = v_{\text{conv}} \rho T \frac{l_{\text{mixing}}}{H_p} (\nabla - \nabla_{\text{ad}}), \quad (2.22)$$

and now we need to be able to estimate the convective velocity v_{conv} .

Assuming that the parcel of gas has a constant acceleration, we have that $l_{\text{mixing}} = \frac{1}{2}at^2$ and $v_{\text{conv}} = \frac{l_{\text{mixing}}}{t}$. The acceleration due to the buoyancy force can be written as $a = -g \frac{\Delta \rho}{\rho} \approx g \frac{\Delta T}{T}$, where the approximation is exact for the case of an ideal gas $P \propto \rho T$, and assuming that $\Delta P = 0$, i.e, the pressure of the moving gas remains in equilibrium with the environment. Substituting a in the equation for the mixing-length parameter we can get an expression for t , which we can then use to obtain the convective velocity,

$$v_{\text{conv}} = \sqrt{\frac{1}{2} l_{\text{mixing}} \frac{g \Delta T}{T}} = \sqrt{\frac{1}{2} l_{\text{mixing}}^2 \frac{g (\nabla - \nabla_{\text{ad}})}{H_p}}. \quad (2.23)$$

Substituting v_{conv} into Eq.(2.22) we obtain the convective flux,

$$F_{\text{conv}} = \rho c_p T \left(\frac{l_{\text{mixing}}}{H_p} \right)^2 \sqrt{\frac{1}{2} g H_p (\nabla - \nabla_{\text{ad}})^{\frac{3}{2}}}. \quad (2.24)$$

We do not know how to properly model l_{MLT} , however, given that convection is not one-dimensional, so we adopt a parameterization of the mixing length,

$$l_{\text{MLT}} = \alpha_{\text{MLT}} H_p, \quad (2.25)$$

where we introduce the mixing-length parameter α_{MLT} , a free parameter that needs to be calibrated for each particular star.

Using this parameterization of the mixing-length and Eq.(2.3) in the form $\frac{dP}{dr} = -g\rho$ to write $H_p = P(-\frac{dP}{dr})^{-1} = \frac{P}{g\rho}$, the equation for the convective flux becomes

$$F_{\text{conv}} = c_p T (\alpha_{\text{MLT}})^2 \sqrt{\frac{1}{2} P \rho (\nabla - \nabla_{\text{ad}})^{\frac{3}{2}}}. \quad (2.26)$$

From Eq.(2.26) we can see that the higher the value of α_{MLT} , the more efficient convection will be in transporting energy. We can also see that convection becomes more efficient in super-adiabatic regions, i.e, $\nabla > \nabla_{\text{ad}}$, and less effective in cooler and less dense regions, with these two effects happening at the same time in the surface layers of a star, as will be seen in Chapter 5.

2.3. Energy production

A star is continuously losing energy at its surface through radiation, which means that for maintaining an equilibrium configuration and having a long lifetime as a very luminous object, it will have to produce its own energy. Stars have two main ways of producing energy, the first being through gravitational contraction, and the second through nuclear fusion, with the phase of stellar evolution where it lives from the fusion of Hydrogen being called the main-sequence.

Through the Virial theorem we can related the gravitational energy of a star V to its internal energy U by $V + \alpha U = 0$, where α is a constant that will depend on the specific type of gas in the star. The total energy of the star will be $E = V + U$, while the gravitational energy can be written as

$$V = - \int_0^M \frac{Gm^2}{r} dm = - \frac{GM^2}{R} \int_0^M \left(\frac{m}{M} \right) \left(\frac{R}{r} \right) d(m/M) = -f \frac{GM^2}{R}, \quad (2.27)$$

where f is the value of the integral, which will be close to 1.

In an ideal gas, where $\alpha = 2$, we can then use the Virial theorem to write the total energy of a star as,

$$E = -\frac{f}{2} \frac{GM^2}{R}. \quad (2.28)$$

A good way of evaluating the time that a star can live from a certain energy source is dividing the total amount of energy that a star has available through that process by the luminosity, since the luminosity is the amount of energy per unit time that a star will be losing. Doing this for a star that is living solely of its gravitational and internal energy we get the Kelvin—Helmholtz timescale,

$$t_{\text{kh}} = \frac{|E|}{L} = 1.5 \times 10^7 f \left(\frac{M}{M_{\odot}} \right)^2 \left(\frac{R_{\odot}}{R} \right) \left(\frac{L_{\odot}}{L} \right) \text{ years}. \quad (2.29)$$

Table 2.1: Diagram of the pp chain. H is the element Hydrogen, H_e is Helium, D is Deuterium, L_i is Lithium, B_e is Beryllium, e⁺ stands for positrons, γ for photons and ν_e for the electron neutrino.

pp I	pp II	pp III
${}^1\text{H} + {}^1\text{H} \longrightarrow {}^2\text{D} + \text{e}^+ + \nu_e$		
${}^2\text{D} + {}^1\text{H} \longrightarrow {}^3\text{He} + \gamma$		
${}^4\text{He} + {}^4\text{He} \longrightarrow {}^8\text{Be}$	${}^3\text{He} + {}^4\text{He} \longrightarrow {}^7\text{Be} + \gamma$	
	${}^7\text{Be} + \text{e}^- \longrightarrow {}^7\text{Li} + \nu_e$	${}^7\text{Li} + {}^1\text{H} \longrightarrow {}^4\text{He} + {}^4\text{He}$
	${}^7\text{Li} + {}^1\text{H} \longrightarrow {}^8\text{Be}$	${}^8\text{Be} \longrightarrow {}^4\text{He} + {}^4\text{He}$
		${}^8\text{Be} \longrightarrow {}^4\text{He} + {}^4\text{He}$

Table 2.2: Diagram of the CNO cycle. C is the element Carbon, N is Nitrogen, O is Oxygen and F is Fluorine. The CN cycle is thus named due to the fact that it contains no stable isotopes of Oxygen, while the ON cycle begins from the last reaction of the CN cycle only that now producing Oxygen and a photon instead of Carbon and Helium.

CN cycle	ON cycle
${}^{12}\text{C} + {}^1\text{H} \longrightarrow {}^{13}\text{N} + \gamma$	${}^{15}\text{N} + {}^1\text{H} \longrightarrow {}^{16}\text{O} + \gamma$
${}^{13}\text{N} \longrightarrow {}^{13}\text{C} + \text{e}^+ + \nu_e$	${}^{16}\text{O} + {}^1\text{H} \longrightarrow {}^{17}\text{F} + \gamma$
${}^{13}\text{C} + {}^1\text{H} \longrightarrow {}^{14}\text{N} + \gamma$	${}^{17}\text{F} \longrightarrow {}^{17}\text{O} + \text{e}^+ + \nu_e$
${}^{14}\text{N} + {}^1\text{H} \longrightarrow {}^{15}\text{O} + \gamma$	${}^{17}\text{O} \longrightarrow {}^{14}\text{N} + {}^4\text{He}$
${}^{15}\text{O} \longrightarrow {}^{15}\text{N} + \text{e}^+ + \nu_e$	
${}^{15}\text{N} + {}^1\text{H} \longrightarrow {}^{12}\text{C} + {}^4\text{He}$	

We know that the lifetime of stars, especially less massive stars such as ε Indi, can go well above 1 billion years or even 10 billion years, so it is obvious that stars cannot live only from gravitational contraction, needing to transition to nuclear fusion. t_{kh} characterises the first stage of stellar evolution, the pre-main-sequence, that will be described more fully later on, where the star is continuously contracting until its central temperature becomes high enough to begin fusing Hydrogen.

The process of gravitational contraction continues in the pre-main-sequence until the central temperature starts allowing the fusion of Hydrogen, at around 2×10^7 K. The chain of reactions that is available for stars at lower temperatures is the pp chain, shown in Table 2.1, while at higher temperatures the CNO cycle becomes possible, shown in Table 2.2.

It is important to note the fact that the emissivity behaves differently if the star is using the pp chain or the CNO cycle. In general, the emissivity of a star can be approximated by

$$\epsilon \approx \epsilon_0 \rho T^\eta, \quad (2.30)$$

where η is a parameter that depends on the type of reaction, and will be larger the heavier and more complex are the elements involved in the reaction, such that

$$\text{pp chain} \Rightarrow 4 \leq \eta \leq 5 \quad \text{CNO cycle} \Rightarrow 13 \leq \eta \leq 20. \quad (2.31)$$

This means that in stars using the CNO cycle there is a much higher dependence of the emissivity on the central temperature than when using the pp chain. Not only that, the production of energy will be much higher. The CNO cycle is available only for the most massive stars, however, being a null or small portion of the energy produced in M, K and G-type stars, since they do not have enough mass to heat the core to the required temperatures.

We can now estimate, as we did for gravitational contraction, how long can a star live from the fusion of Hydrogen. To do this we have to take into account the mass fraction of Hydrogen in the star, X , the efficiency of nuclear fusion, f , and the total amount of mass in the core $M_{\text{core}} = \alpha M$. We have then that

$$\tau_{\text{ms}} = \frac{f \cdot X \cdot \alpha M c^2}{L} = 7.7 \times 10^{10} \left[\alpha \left(\frac{M}{M_{\odot}} \right) \left(\frac{L_{\odot}}{L} \right) \right] \text{ years}, \quad (2.32)$$

where in the final step we introduced the values $X \approx 0.73$ and $f \approx 0.0075$.

For a star such as ϵ Indi where $M \approx 0.75 M_{\odot}$ and $L \approx 0.22 L_{\odot}$, and using $\alpha = 0.2$, we would get a lifetime in the main-sequence of $\tau_{\text{ms}} \approx 55 \text{ Gyr}$, far beyond the age of the universe, which means that ϵ Indi has not yet left the main-sequence.

2.4. Equation of state

The equation of state relates the three major thermodynamical quantities in a star, ρ , P and T , which we can write in a general form as $P = P(r, \rho, T, Y, Z)$, where we are taking into account that the equation of state can differ between different depths of the star, and that it can depend on the chemical composition. The type of equation of state to be used will depend primarily on ρ and T , which means that we need to describe the different kinds of effects that need to be taken into account in the equation of state over the density and temperature ranges that stars explore.

In Table 2.3 we see some of the main types of equations of state important in stellar physics, where m_u is the atomic mass unit, m_e the electron mass, h the Planck constant,

μ_e the mean molecular weight for free electrons and R_{gas} is the gas constant, while in Fig. 2.1 are shown the regions where each one applies. In the previous sections we have at times assumed that the star is made of an ideal gas, and we can see from Fig. 2.1 that this is a pretty good assumption for solar-type stars, which live in the region of intermediate values of ρ and T . In this region the stellar gas can be well approximated as not having inter-particle interactions, or more precisely, with their kinetic energy being much higher than their interaction energy, while in other regions their interactions need to be taken into account.

As we move towards higher temperature the density of photons will increase proportionally to T^3 , and the pressure exerted by electro-magnetic waves will become increasingly important until the point where it is much higher than the ideal gas pressure. This limit will be seen especially in the core of very massive stars, that can reach large values of the central temperature, although every star will have a mix of both radiation pressure and ideal gas pressure.

On the other hand, as we move towards higher density and lower temperatures, the equation of state becomes dominated by electron degeneracy pressure, the pressure due to the fact that electrons are fermions that cannot occupy the same quantum state. At lower temperatures the available momentum states for electrons decrease, while with higher densities the number of electrons increases, thus forcing them to go to states of higher momentum due to the Pauli exclusion principle and creating electron degeneracy pressure. With high enough density the electron speed increases so much that they become relativistic, introducing another change in the equation of state. Electron degeneracy pressure will become important in less massive M-type stars, brown dwarfs and white dwarfs, being able to counteract the contracting tendency of gravity even as the star gets colder and colder, due to the fact that the equation of state for a degenerate gas is independent of temperature.

Still other effects need to be taken into account for a fuller description of the equation of state, such as the fraction of ionization in a gas through the Saha equation, the effect of the Coulomb interactions between electrons and ions, which become more important as we move towards lower masses, and the impact of pair production, the reaction where a photon turns into an electron and a positron. While we do not discuss them here, they are included in the more advanced equations of state that are used for stellar modelling.

Table 2.3: Principal equations of state present in stars, along with the conditions where they become prevalent.

Ideal gas	$P = \frac{R_{\text{gas}}}{\mu} \rho T$	Intermediate values of ρ and T
Radiation pressure	$P = \frac{1}{3} a T^4$	High values of T
Non-relativistic degenerate gas	$P = \frac{h^2}{20m_e m_u^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \left(\frac{\rho}{\mu_e}\right)^{5/3}$	High ρ and low T
Relativistic degenerate gas	$P = \frac{hc}{8m_u^{4/3}} \left(\frac{3}{\pi}\right)^{1/3} \left(\frac{\rho}{\mu_e}\right)^{4/3}$	Very high ρ

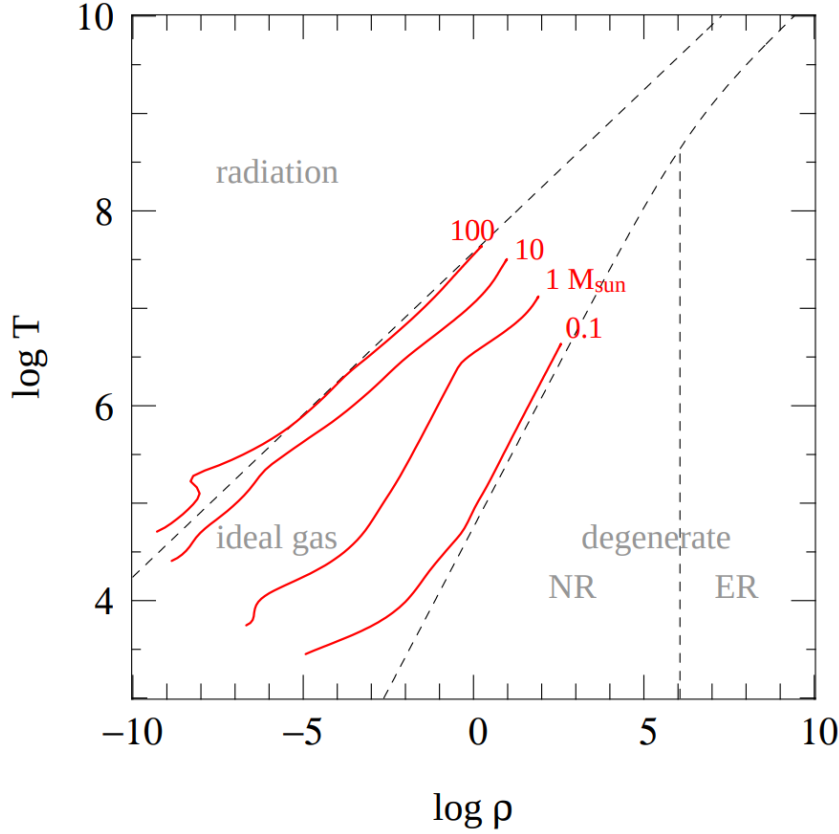


Figure 2.1: $\log(T)$ - $\log(\rho)$ diagram showing the conditions where each equation of state applies. From [30].

2.5. Stellar atmospheres and boundary conditions

The last component that is needed in order to solve the equations for stellar structure and evolution, Eq.(2.2) to Eq.(2.6), are the center and surface boundary conditions of the model.

At the center we can establish that

$$r = 0 \Rightarrow m = 0 \text{ and } L_r = 0, \quad (2.33)$$

which are logical consequences of how we defined these variables, or else we would have infinite density and infinite energy production at the center.

For the temperature and pressure we do not have any prior conditions that we can apply at the center, so their boundary conditions must be on the surface. It is this need to define the surface boundary conditions for temperature and pressure that leads us to the subject of stellar atmospheric models.

The simplest possible condition would be to set $P = 0$ and $T = 0$ at the surface of the star, but this is not realistic as even in the interstellar medium there is a finite value of density and temperature. Another option is defining the surface of the star as being on the photosphere, which is mathematically defined as the point at which the optical depth, $\tau = \int_R^\infty \kappa \rho dr$, reaches a certain value, that will depend on the model used. This is why the effective temperature is sometimes defined as the temperature at which $\tau = \frac{2}{3}$.

The ideal option is to solve Eq.(2.3) and the radiative transfer equation at the surface of the star, that describes how radiation at each frequency is absorbed, emitted and scattered as it passes through the surface. For a one-dimensional planar atmosphere, where we imagine the surface of the star as if it were flat, we have that

$$\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda, \quad (2.34)$$

where I_λ is the intensity of radiation at wavelength λ and S_λ is the source function, which can be interpreted as the number of photons emitted per optical depth interval at wavelength λ .

The equation can be solved using the integrating factor e^{τ_λ} , leading to the solution,

$$I_\lambda(\tau_\lambda) = I_{\lambda,0}e^{-\tau_\lambda} + S_\lambda(1 - e^{-\tau_\lambda}), \quad (2.35)$$

where it is considered that an optically transparent layer of the atmosphere is one with $\tau_\lambda < 1$, while an optically opaque layer has $\tau_\lambda \geq 1$, and in the optically opaque limit we can write,

$$\tau_\lambda \gg 1 \Rightarrow e^{-\tau_\lambda} \approx 0 \Rightarrow I_\lambda \approx S_\lambda, \quad (2.36)$$

which is the case where radiation comes solely from emission, as in a black body.

If we assume that the optical depth does not depend on the wavelength, $\tau_\lambda = \tau$, which is called the grey atmosphere approximation, we are able to obtain that $S(\tau)$ and the mean

energy intensity over solid angle $J(\tau)$ are equal, $S(\tau) = J(\tau)$, and the following relation between $J(\tau)$ and the energy flux $F(\tau)$:

$$J(\tau) = \frac{3}{4\pi} F(\tau) \tau + \frac{2}{3}. \quad (2.37)$$

Given that for local radiative equilibrium, which means that the amount of energy entering the atmosphere is the same as the energy leaving it, the source function is given by the Planck function $S(\tau) = \frac{\sigma_{\text{SB}} T^4}{\pi}$, and that by the definition of T_{eff} we have that $F(\tau) = \sigma_{\text{SB}} T_{\text{eff}}^4$, we can obtain a solution for the temperature in terms of optical depth as

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + \frac{2}{3} \right). \quad (2.38)$$

Equation. (2.38) gives us the Eddington atmosphere, which is still commonly used today. The Eddington atmosphere gives very good results for the optically opaque layers of the atmosphere, and performs more poorly in the more superficial layers, such as the Chromosphere and the Corona.

In a more general case, we get solutions that have a function $q(\tau)$, called the Hopf function, substituting the $\frac{2}{3}$ in the previous relation,

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 \left(\tau + q(\tau) \right), \quad (2.39)$$

whose impact we will study in Chapter 4.

For a more in depth study of this topic, starting from first principles, one can read [33] along with [34], and to see how one can derive $T(\tau)$ relations for non-grey atmosphere from the properties of stellar models one can read [35].

2.6. Stellar evolution

In stellar modelling we want to be able to describe how a star evolves over time, and how it transitions through very distinct phases of evolution, which give us the different types of stars we see in the universe. We want to be able to know from how a star is formed starting with disorganized matter in interstellar space, to how it will end if it had enough time to go through all of its evolution.

The first phase of stellar evolution is called the Hayashi track, in which the cloud of interstellar gas from which the star will form is continuously contracting, not yet producing any energy by nuclear fusion. The low temperature makes the star opaque to radiation, so that energy transport is fully done by convection. The effective temperature during this phase remains almost constant, while the luminosity decreases substantially due to the smaller radius, which is described by the following scaling relation, applicable in this phase of evolution,

$$T_{\text{eff}} \propto \frac{L^{1/20}}{M^{1/5}}, \quad (2.40)$$

where we see the very weak dependence of the effective temperature on the luminosity.

As the released gravitational energy gets turned into internal energy, in accordance with the Virial theorem, the core of the star begins to heat up until the point where it stops being convective. The main energy transport mechanism becomes radiative transport, marking the entrance of the star into the Henyey track.

In the Henyey track the stellar radius keeps contracting, but since the star is now radiative more energy from the core can reach the surface, and T_{eff} begins to increase rapidly, leading the luminosity to increase as well. During this phase the star is able to perform the fusion of Deuterium, although this process is not very significant or long lasting due to the small initial mass fraction of Deuterium in the star forming material. The Henyey track ends when the core reaches temperatures high enough to be able to fuse Hydrogen, thus beginning the main-sequence, the longest phase in stellar evolution, and the one which we observe the most in the universe.

In Fig. 2.2 are represented the evolution of the stellar radius, luminosity, effective temperature and central temperature over the pre-main-sequence. We can see how less massive stars take much longer to complete the pre-main-sequence, which will also occur in the main-sequence, a fact that can be understood from Eq.(2.29) as being due to their much smaller radius and luminosity. The Hayashi track is the phase where the radius is decreasing almost vertically, like the luminosity, while the Henyey track begins when we see the luminosity and effective temperature begin to increase more rapidly. In the Henyey track we even see some bumps in the radius, that should correspond to the period in which deuterium is being burned. The central temperature continuously increases, as expected, due to the energy from gravitational contraction.

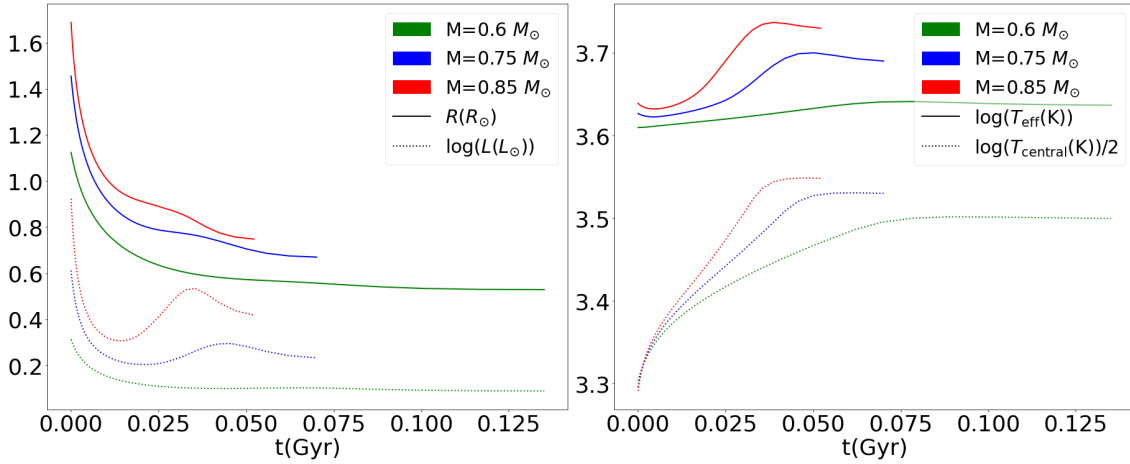


Figure 2.2: Left panel: Evolution of the radius and luminosity in the pre-main-sequence for three different masses with the metal abundance fraction $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: Evolution of the effective temperature and central temperature in the pre-main-sequence for three different masses with $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$.

In the main-sequence a star lives from the fusion of ^1H into ^4He , as we saw in the section about energy production. The stellar radius begins to expand instead of contract, and the luminosity tends to increase, while the effective temperature only increases slightly. This phase will end when the star runs out of Hydrogen in the core to fuse, at which point it will enter the sub-giant phase. The properties of stars in the main-sequence will be described in detail in the following chapters, as it is the evolutionary phase most relevant to our work.

In the subsequent phases of evolution stars will attempt to move on to using Helium, and even heavier elements, as the reactants in nuclear fusion. Whether this is possible or not will depend on the mass of the star, as lower-mass stars will not be capable of reaching high enough central temperatures to fuse heavier elements.

The transition to fusing Helium involves a phase where the stellar envelope strongly expands while the core contracts and becomes degenerate, called the red-giant phase. At the end of the red giant phase, if the star has $M < 2M_\odot$, happens the Helium flash, where huge amounts of energy are produced in a short time by fusing Helium into Carbon, removing the degeneracy in the core and beginning the horizontal branch, a more stable phase, where stars live from the fusion of Helium. When a star runs out of Helium and cannot move on towards fusing heavier elements, the outer layers of the star will expand again and move into interstellar space, while the core contracts until it stabilizes through electron degeneracy pressure, thus becoming a white dwarf.

For a more detailed description of these later phases of evolution we recommend reading [30].

3. Asteroseismology

Asteroseismology is the branch of Astrophysics that studies pulsations in stars, and is associated to Helioseismology, which is the study of pulsations in the Sun. Asteroseismology attempts to answer questions such as how stellar pulsations are formed, what are their properties, how do they vary between different types of star, and use our understanding of them to more properly constrain the interior physics and global properties of stars. The pulsations can be characterized as being the contraction and expansion of the stellar fluid that arise as the star attempts to maintain equilibrium, and affecting specific regions of a star.

Stellar pulsations are detected through periodic variations in the luminosity or radial velocity of a star, and can have a large range of possible amplitudes, that will correspond to different excitation mechanisms for the pulsations, and different types of stars. We will focus on solar-like pulsators, which have small-amplitude oscillations that are stochastically excited by turbulence in the star's convective envelope. The Sun is an example of a star exhibiting this type of oscillation, along with many other stars in the main-sequence, subgiant and red-giant phases of evolution, most of them discovered only recently due to the level of precision required to detect solar-like oscillations.

For the purposes of asteroseismic modelling, these oscillations are described as linear and adiabatic. They are linear because their velocity amplitudes, i.e., the velocity of the stellar fluid perturbed by the oscillations, is much smaller than the speed of sound in the surface layers of the star, and adiabatic because their periods are in the order of minutes while the timescale of thermal transport is much higher, thereby not allowing time for heat exchange.

In this section we will review the mathematical formalism used to model this type of oscillation, define the concepts in asteroseismology relevant to our work and describe some of the main observed features of solar-like oscillation spectra.

3.1. Equations for adiabatic stellar oscillations

To describe stellar oscillations we need to start from the 3D equations of fluid dynamics, given that the oscillations will be perturbations to the 1D equilibrium structure that was

described in the previous section. So we start from the equation for the conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \quad (3.1)$$

the conservation of momentum,

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla P + \rho \nabla \Phi, \quad (3.2)$$

the Poisson equation, Eq.(2.7), and the heat equation, which for an ideal gas in the adiabatic limit is,

$$\frac{dP}{dt} = c_s^2 \frac{d\rho}{dt}, \quad (3.3)$$

where $c_s = \sqrt{\frac{\Gamma_1 P}{\rho}}$ is the sound speed, and $\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho} \right)_{\text{ad}}$ is an adiabatic index determined by the equation of state.

To obtain the equations for stellar oscillations we consider linear perturbations to P , ρ , Φ and $\vec{v}$ of the form

$$P(\vec{r}, t) = P_0(r) + P_1(\vec{r}, t), \quad (3.4)$$

where the equilibrium value P_0 is spherically symmetric and static, while the perturbation is much smaller than P_0 , $\frac{P_1}{P_0} \ll 1$, does not have to be spherically symmetric, and depends on time.

After inserting the perturbation into Eq. (3.1) to Eq. (3.4) we will obtain a new set of equations for the perturbed quantities, where we can decompose the different quantities into a tangential and radial component. A very important result is the fact that a solution for the tangential component can be found analytically, with it being the spherical harmonics Y_l^m , also found in the solution for the Hydrogen atom in quantum mechanics. Another important result is that the time-dependent part of the equations can be separated from the spatial part, and written in terms of $e^{-i\omega t}$, where ω can be both real or complex, taking into account solutions that are propagating or that decay. This allows us to write any of the perturbed quantities in the form

$$P_1(r, \theta, \phi, t) = P_1(r) Y_l^m(\theta, \phi) e^{-i\omega t}, \quad (3.5)$$

where we write the pressure only as an example.

Substituting these solutions into the perturbed equations one obtains equations solely for the radial components in the perturbations,

$$\frac{d\tilde{\xi}_r}{dr} = - \left(\frac{2}{r} + \frac{1}{\Gamma_1 P} \frac{dP}{dr} \right) \tilde{\xi}_r + \frac{1}{\rho c_s^2} \left(\frac{S_l^2}{\omega^2} - 1 \right) P_1 - \frac{l(l+1)}{\omega^2 r^2} \Phi_1, \quad (3.6)$$

$$\frac{dP_1}{dr} = \rho(\omega^2 - N^2) \tilde{\xi}_r + \frac{1}{\Gamma_1 P} \frac{dP}{dr} P_1 + \rho \frac{d\Phi_1}{dr}, \quad (3.7)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi_1}{dr} \right) = -4\pi G \left(\frac{P_1}{c_s^2} + \frac{\rho \tilde{\xi}_r}{g} N^2 \right) + \frac{l(l+1)}{r^2} \Phi_1, \quad (3.8)$$

where $\tilde{\xi}_r$ is the displacement vector of the perturbation, $S_l^2 = \frac{l(l+1)c_s^2}{r^2}$ is the Lamb frequency and $N^2 = g \left(\frac{1}{\Gamma_1 P} \frac{dP}{dr} - \frac{1}{\rho} \frac{d\rho}{dr} \right)$ is the square of the Brunt—Väisälä or buoyancy frequency. These frequencies are relevant for describing the propagation of the modes, along with the acoustic cut-off frequency, which will be described later.

The three above equations form a set of coupled differential equations and are an eigenvalue problem where the eigenvalue is ω , with ω being the mode frequencies, the frequencies in which the star oscillates. Each value of ω will be characterized by a radial order n and angular degree l , with a $2l + 1$ degeneracy in m , due to the fact that we start from a spherically symmetric equilibrium structure. This degeneracy in m will, however, be lifted in the presence of rotation or magnetic fields, and a dependency on m has been observed in actual stars.

From now on we will use ν to denote the linear frequency of the modes, with $\nu = \frac{\omega}{2\pi}$, as it is customary to present the value of mode frequencies in Hz instead of radians per second.

For a complete derivation of these results one can see [4].

3.2. Main types and properties of stellar oscillations

In the previous section we mentioned that the perturbations could have propagating or decaying solutions, but at an observational level, we will only be able to observe a mode if it is propagating at the surface. So it is important to know the regions inside a star where a mode will be propagating or decaying. The following conditions establish whether a mode is propagating or not, and permit us to distinguish between the two main types of modes in solar-like oscillators, p and g modes,

$$\omega^2 < S_l^2 \text{ and } \omega^2 < N^2 \Rightarrow \text{Propagating g-mode,} \quad (3.9)$$

$$\omega^2 > S_l^2 \text{ and } \omega^2 > N^2 \Rightarrow \text{Propagating p mode.} \quad (3.10)$$

g modes propagate in the inner regions of the star, which makes them unobservable, and are maintained by gravity acting on density fluctuations. The fact that $N^2 < 0$ in convective regions means that these modes can only propagate in convectively stable regions, which is why they cannot occur at the surface of solar-like oscillators. They are always non-radial modes, with a value of l different from 0.

p modes are propagating close to the surface and can reach variable depths, being the type of modes we actually observe in solar-like main-sequence stars. They can propagate in both convectively stable and unstable regions, and are maintained by the pressure gradient. p modes can have both $l = 0$ and $l > 0$, and typically we label p modes as having $n > 0$, while g modes have $n < 0$.

There is another type of mode, called mixed-modes, which can exhibit p-mode features at the surface and g-mode features at the core, but since this type of mode only appears in subgiant and red-giant stars we will not focus on it here.

In the left panel of Fig. 3.1 we show what is called a propagation diagram, where the Lamb frequency and Brunt–Väisälä frequency are plotted over the depth of the star, and its purpose is to assess the regions of the star where a mode is propagating. Through it we can see that higher frequency and lower- l p modes can propagate to deeper regions in the star, and that the g modes are trapped in the central regions.

For p modes we can obtain an important equation that permits us to calculate the frequencies of the modes in the asymptotic limit, high-radial order n and low-angular degree

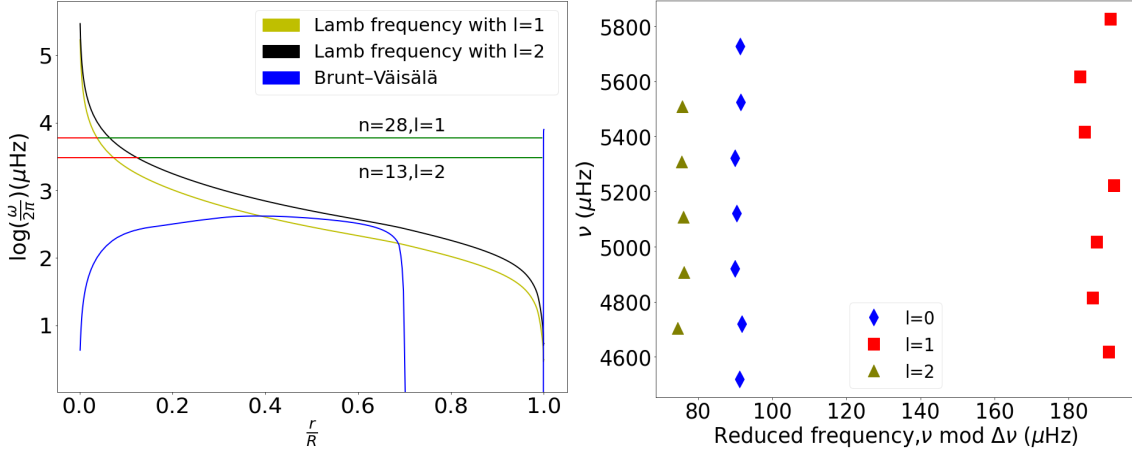


Figure 3.1: Left panel: Propagation diagram for a star with $M = 0.75M_{\odot}$, $Z = 0.01$, $\alpha_{\text{MLT}} = 1.8$ at 4.15 Gyr. In it we represent the propagating regions for a $n = 28, l = 1$ and a $n = 13, l = 2$ mode as green, while the evanescent regions are in red. Right panel: Échelle diagram for ϵ Indi.

l , which is called the asymptotic relation,

$$\nu_{nl} \simeq \Delta\nu \left(n + \frac{l}{2} + \varepsilon \right) - \delta\nu_{0l}, \quad (3.11)$$

where we are introduced to the large frequency separation $\Delta\nu = \nu_{n+1,0} - \nu_{n,0}$, the small frequency separation $\delta\nu_{l,l+2} = \nu_{nl} - \nu_{n-1,l+2}$ and the dimensionless offset ε , that will depend on the properties of the reflective layers of the star. In the asymptotic relation it is typically used an average value of the large and small frequency separation over n , and we will see that these frequency separations are very meaningful, as they are related to specific properties of the stellar structure.

Equation. (3.11) suggests that the frequencies of the modes grow linearly with n , and the échelle diagram is meant to help us visualize this. In it we divide the modes into chunks of length $\Delta\nu$ and then stack them on top of each other in ascending order, which corresponds to plotting ν_{nl} on the y-axis and $\nu_{nl} \bmod \Delta\nu$ in the x-axis. This makes it so that modes of different l should form different vertical ridges in the diagram. The right panel of Fig. 3.1 is the échelle diagram for ϵ Indi, where we can see an overall linear trend, although with a noticeable curvature also present. This curvature is due to the fact that $\Delta\nu$ and ε will depend on the frequency of the mode, which will be analysed in Chapter 5. The curvature itself can provide us meaningful information about the star, regarding sharp discontinuities in a star's inner structure, such as the base of the convective envelope and the second Helium ionization zone.

The large frequency separation is one of the simplest parameters to observe, but it can also be related to the inner structure of the star through the equation,

$$\Delta\nu \approx \left(2 \int_0^R \frac{dr}{c_s} \right)^{-1}, \quad (3.12)$$

so that $\Delta\nu$ can be considered as the inverse of the sound travel time across the star to the center and back.

Through dimensional analysis we can obtain another important equation for calculating $\Delta\nu$. From Eq. (3.12) we see that

$$\Delta\nu \propto \frac{c_s}{R}, \quad (3.13)$$

and given that $c_s = \sqrt{\frac{\Gamma_1 P}{\rho}} \propto \sqrt{\frac{P}{\rho}}$ and (via Eq.(2.3)) $P \propto \frac{M\rho}{R}$ and $\rho \propto \frac{M}{R^3}$, one gets

$$\Delta\nu \propto \frac{\sqrt{\frac{M\rho}{\rho R}}}{R} \propto \sqrt{\frac{M}{R^3}} \propto \sqrt{\rho}, \quad (3.14)$$

which means $\Delta\nu$ is also related to the mean density of a star.

The small frequency separation $\delta\nu_{l,l+2}$ can be related to the inner structure of the star through the gradient of the sound speed,

$$\delta\nu_{l,l+2}(n) \approx -(4l+6) \frac{\Delta\nu}{4\pi^2\nu_{nl}} \int_0^R \frac{dc_s}{dr} \frac{dr}{r}, \quad (3.15)$$

where we can also see that $\delta\nu_{l,l+2}$ depends on the frequency. We will see in Chapter 5 how $\delta\nu_{02}$ can provide a good diagnostic of the age, as its variation over time will be linked to the changes in chemical abundance happening in the core.

The small frequency separation $\delta\nu_{01}$ is also used as a way to study the inner structure of a star,

$$\delta\nu_{01}(n) = \nu_{n,0} - \frac{1}{2}(\nu_{n-1,1} + \nu_{n,1}). \quad (3.16)$$

Another important feature of a mode is the amplitude with which it oscillates. Higher-amplitude modes will be easier to observe in a star, with typically several radial orders

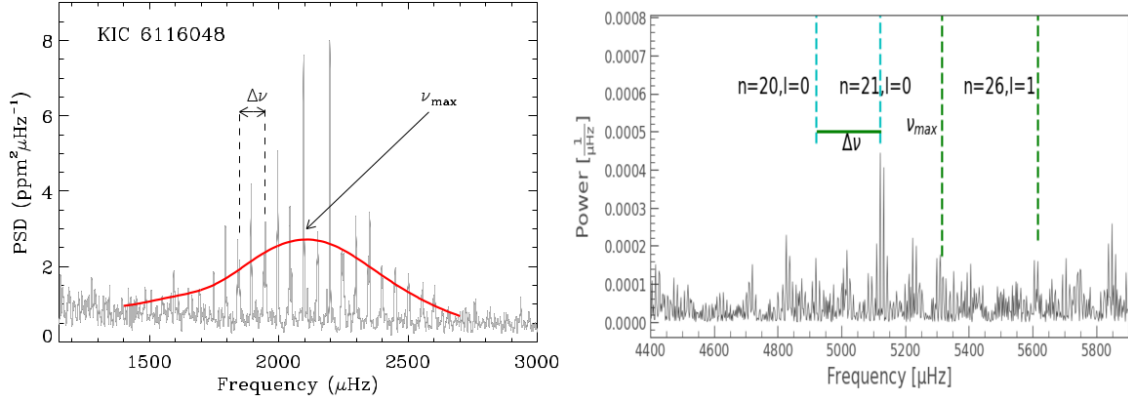


Figure 3.2: Left panel: Frequency power spectrum for KIC 6116048. From [36]. Right panel: Frequency power spectrum for ϵ Indi. The first was obtained from multi-month photometry using Kepler, while the latter from a 6 day radial velocity campaign using ESPRESSO.

and the lower angular degrees being observable, and it has been found that the frequency power spectrum of solar-like oscillators has a Gaussian shape, of which we show examples in Fig. 3.2. The frequency at which is the peak of the Gaussian envelope is called $\nu_{\max}$.

It has been empirically determined that mode amplitudes in Doppler velocity obey a scaling relation of the form

$$v = v_{\odot} \left(\frac{L/L_{\odot}}{M/M_{\odot}} \right)^s = v_{\odot} \left(\frac{T_{\text{eff}}^4/T_{\text{eff},\odot}^4}{g/g_{\odot}} \right)^s, \quad (3.17)$$

where s is a parameter obtained by fitting to observations, taking values in the range 0.5–1. For K-type stars this will result in oscillation amplitudes much lower than for the Sun, since their mass is around $0.8M_{\odot}$, while their luminosity can be several times lower than the Sun's. For our case, we would expect ϵ Indi to have mode amplitudes of around 5 cm/s, much lower than the solar value of around 23.6 cm/s [37], and than the sensitivities of previous-generation spectrographs that were of a few cm/s. This provides the reason for why it had been very difficult to detect modes in stars as cold as ϵ Indi.

For $\nu_{\max}$ there is another important scaling relation. It is related to the acoustic cut-off frequency, which is the maximum frequency at which p modes can oscillate, and that for an ideal gas is given by

$$\nu_{\text{ac}} = \frac{c_s g \rho}{2P}, \quad (3.18)$$

although this is a particular case of a more general definition of the acoustic cut-off frequency.

It is assumed that $\nu_{\max} \propto \nu_{\text{ac}}$, and since $\nu_{\text{ac}} \propto \frac{c_s g \rho}{P} \propto \frac{g}{\sqrt{T_{\text{eff}}}}$, where we used the ideal gas equation in Table 2.3, we have that

$$\nu_{\max} \propto \frac{g}{\sqrt{T_{\text{eff}}}} \propto \frac{M}{R^2 \sqrt{T_{\text{eff}}}}. \quad (3.19)$$

It is worth noting that we can adapt both Eq. (3.14) and Eq. (3.19) to calculate the actual values of $\Delta\nu$ and $\nu_{\max}$ by dividing them by the values for the Sun, using the fact that we know the global parameters and asteroseismic properties of the Sun with great precision. They can also be inverted to obtain formulas for calculating the mass and radius of a star from $\Delta\nu$ and $\nu_{\max}$. Eq. (3.19) is also very useful due to the fact that most stellar oscillation codes cannot model the amplitude of the modes, and so we use the scaling relation for $\nu_{\max}$ in order to know the radial orders where we expect the modes to be observable.

The final feature of the modes we will discuss here is the mode inertia $I_{n,l}$, which is defined by,

$$I_{n,l} = \int_V \rho \vec{\xi} \cdot \vec{\xi} d^3r, \quad (3.20)$$

in analogy to the moment of inertia in classical mechanics.

Low-angular degree and low-frequency modes have higher inertia than high-angular degree and high-frequency modes, and low-inertia modes are more sensitive to perturbations to the structure of the star. This can offer us an idea as to why the surface term increases with frequency, and we will see that the mode inertia will be taken into account in the surface-term correction.

3.3. The surface term in the frequencies

The surface term is a systematic error that has been found between model and observed frequencies. It tends to increase with frequency and is almost independent of mode degree, l [14]. The surface term arises from the improper modelling of the near-surface layers of a star, with the main difficulties encountered being associated with the modelling of convection, the atmospheric model, and the uncertainty in the microphysics (e.g., low-temperature opacities) [36]. Convection is already a complex phenomenon to model in

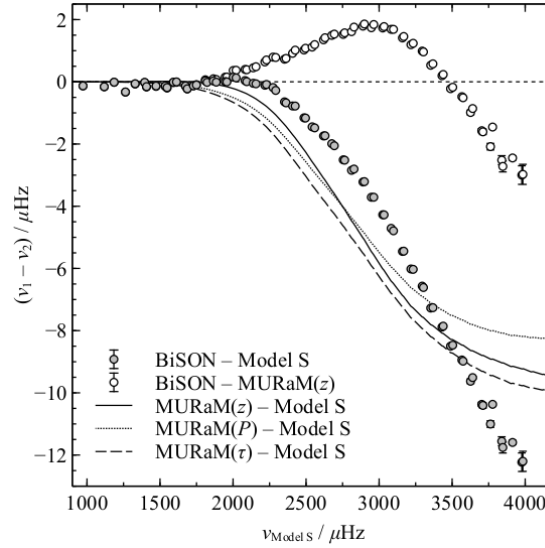


Figure 3.3: Frequency differences between BiSON observations and various types of solar models. In the figure the subscript 1 indicates that the mode frequency comes from observations, while the subscript 2 indicates that it comes from models. From [14].

the interior of a star, as we are trying to model an intrinsically 3D process by means of a 1D formulation, but it strays even further in the near-surface layers where the temperature gradient becomes superadiabatic and high convective velocities are present [13]. Adding to this, there are the issues related to how we model stellar oscillations where the assumption of adiabatic perturbations tends to fail, and to the effects of turbulent pressure not being taken into account [38].

Figure. 3.3 shows the shape of the surface term for different models of the Sun. We can see how it tends to increase with frequency, has the shape of a smooth continuous function, and the difference is due to the models overestimating the frequencies of the modes, although its exact shape will change from model to model. The reason for why the surface term tends to increase with frequency is that higher frequency modes, besides propagating deeper into the star, as we saw in Fig. 3.1, are also able to propagate into shallower layers of the atmosphere than low frequency modes, which implies that they will more affected by improper modelling at the superficial layers of stars.

The surface term is a complicating factor in asteroseismic modelling given that it can lead to inaccurate results when comparing models and observations. The magnitude of the surface term will typically be multiple times that of the observational uncertainty in a mode frequency, that are between $0.2\mu\text{Hz}$ to $2\mu\text{Hz}$, making it very significant. It can also hide subtler effects on the mode frequencies, and that are actually due to the inner structure of the star.

To try to fix some of these issues there have been proposed several different forms for correcting it. The main method used to solve this issue has been to try to fit various types of functions containing free parameters to model the surface term. The first type of function is motivated by the fact that the curve of the surface term in Fig. 3.3 seems to be a polynomial. This is formulated in [39] as

$$\delta\nu(\nu) = a \left(\frac{\nu_{\text{mod}}}{\nu_{\text{ref}}} \right)^b, \quad (3.21)$$

where a and b are the parameters to be adjusted, and ν_{ref} is typically taken as being ν_{max} .

The next function takes into account the fact that the dispersion of the surface term in terms of l tends to decrease when dividing by the mode inertia, which is defined in Eq. (3.20), and was first presented in [40]:

$$\delta\nu(\nu) = \frac{a_3}{I_{nl}} \left(\frac{\nu}{\nu_{\text{ac}}} \right)^3 + \frac{a_{-1}}{I_{nl}} \left(\frac{\nu}{\nu_{\text{ac}}} \right)^{-1}, \quad (3.22)$$

where a_3 and a_{-1} are the free parameters.

The justification for the cubic and inverse term in Eq. (3.22) comes from considering perturbations near the surface under a variational principle and seeing the impact they have on the displacement ξ [41]. From this analysis it can be concluded that a perturbation caused by a magnetic field and affecting the sound speed would create a shift proportional to ν^3/I , while perturbations due to convection and affecting the pressure scale height would create a shift proportional to $\frac{1}{\nu I}$.

In [42] they use a modified Lorentzian function instead of a polynomial, in order to take better into account how the derivative of the surface term changes across the frequencies. The function they use is,

$$\frac{\delta\nu(\nu)}{\nu_{\text{max}}} = \alpha \left[1 - \frac{1}{1 + \left(\frac{\nu}{\nu_{\text{max}}} \right)^\beta} \right], \quad (3.23)$$

with α and β being free parameters. We can also notice that in the low-frequency range the function will tend towards Eq. (3.21), because what the authors in [42] wanted to change was the behaviour of the surface term correction at higher frequencies.

Another method has been to calculate the surface term for an arbitrary star by scaling the solar surface term, which is called the SsS-1 method [13]:

$$\delta\nu(\nu) = a \frac{\Delta\nu_{\text{mod}}}{\Delta\nu_{\odot}} \delta\nu_{\odot}(\nu_{\text{mod}}), \quad (3.24)$$

where a is a free parameter and $\delta\nu_{\odot}(\nu_{\text{mod}})$ is the value of the surface term for the Sun at ν_{mod} . The reason for doing this is that we know the surface term for the Sun quite well, although the applicability of this method will decrease for stars whose properties differ considerably from the Sun.

These are some of the main ways that have been used for correcting the surface term, and the first three are especially important for us because they will be the ones we will apply and evaluate when we perform model optimization. These methods are based mainly on the Sun, opening the question of how well will they perform for a star like ϵ Indi, which is why testing them here is important in order to assess if they can be extended to cooler stars, and whether new types of functional models for the surface term might be needed.

4. Model Grid calculation

The collection of new observational data and their analysis must be accompanied by an equivalent effort in stellar modelling, so that we can make full use of the data towards acquiring more meaningful insights. To do this we need to create large sets of models directed towards the properties of the stars we have observed and want to study, encompassing both the uncertainty in what we know about them and the ambiguity introduced by having free parameters in the models.

In this chapter we will describe the software that was used to create a grid of evolutionary tracks directed towards the properties of ϵ Indi, along with the oscillation modes for each model in the tracks, the features of the grid, why they are suited towards ϵ Indi and what we intend to study through the grid we have created.

4.1. Stellar structure and evolution code – MESA

MESA (Modules for Experiments in Stellar Astrophysics) [10] is an open source stellar modelling software coded in FORTRAN 95 which solves the equations of stellar structure and evolution presented in Chapter 2, thereby permitting us to simulate the properties of a star as it evolves across its lifespan. MESA provides solvers for both non-stiff and stiff differential equations, using explicit Runge–Kutta integrators of order 5 and 8 for the former and linearly implicit Runge–Kutta up to order 4 for the latter, which allows it to solve systems of equations that are more unstable [43]. For root finding algorithms it uses a Newton–Raphson method, permitting it to solve non-linear and multidimensional systems of differential-algebraic equations. It was launched in 2010 and has been in continuous development ever since.

It is split into several modules, such as `star` to evolve a star, `eos` to simulate the equation of state, `kap` for the opacities among others, each dealing with part of the required tasks in the simulation. Each of these modules is highly customisable and permits us to choose the initial properties of the simulated star (such as picking its initial mass and chemical composition), the physics included in the simulation (choices in the equation of state, diffusion, convection, microphysics, opacities among others), the type of information which it outputs, including both how global properties evolve through time and how features vary spatially across the star for a particular model, and the rate with which it outputs information and creates models. It also allows us to control which phases of a star’s evolution will

Table 4.1: Time step and stopping conditions for the grid of stellar tracks.

	Criteria	
Stage	Time Step	Stopping Condition
Pre-MS	—	$\frac{L_{nuc}}{L_{\star}} = 0.99$
MS	$\Delta t < 0.6 \times 10^8 \text{ yr}$	$X_{\text{center}} < 1 \times 10^{-7}$

be simulated by picking a stopping condition denoting the end of a phase and the beginning of another, such as a condition for the end of the main-sequence and the beginning of the subgiant phase. All of these features can be chosen through the inlist files which MESA reads as it begins to run, whose commands can be found in the default files that come with MESA as it is installed or in the documentation. To create our grid we began from the inlists provided in the 1M-pre-ms-to-wd test suite which comes with MESA, and that we adapted towards our purposes.

In our grid we run the evolutionary tracks only until the end of the main sequence, saving data regarding both the pre-main-sequence and the main-sequence. This is justified by the fact that previous studies of ϵ Indi have shown it to be a somewhat young star, with an age between 2.5 Gyr and 4 Gyr [25], and our estimate for the time it would take for it to finish the main-sequence in Chapter 2 makes it impossible for it to already be in the subgiant phase.

Table 4.1 shows the criteria for the time step and stopping condition used in the tracks. The stopping condition for the pre-main sequence signals the point when nuclear fusion begins to be the main energy producing mechanism in the star, instead of gravitational contraction, which is the main energy source in the pre-main-sequence. The time step for the main sequence is chosen so that we have a good coverage of points throughout all of it, and the stopping condition signals the point when the star has burned all of the Hydrogen in its core, forcing it to move into another energy production mechanism. It is worth noting that we have picked a very low value for X_{center} in the stopping condition, which makes it possible that we might already begin to see some features of the subgiant phase towards the end of our tracks.

The parameters we will vary in our grid are the initial mass of the star, the initial Metal fraction, Z , and the value of the mixing-length parameter α_{MLT} , where we will use the TDC (Time-dependent Convection) option to model convection, which is based on the Kuhfuss

Table 4.2: Range of the initial mass in solar mass units, initial Metal fraction and α_{MLT} used in the main grid of stellar tracks. The values of ϵ Indi are shown for comparison.

	Range	Step	ϵ Indi values
M_{initial}	0.6-0.85 $M_{\odot}$	0.01	0.762 ± 0.038 [28]
Z_{initial}	0.005-0.015	0.001	0.0089 ± 0.001 [29]
α_{MLT}	1.2-2.4	0.2	—

(1986) model [44]. For the metal fraction of the Sun we use the value from [45], where $Z_{\odot} = 0.0142$ for the star as a whole and $Z_{\text{surf},\odot} = 0.0134$ at the surface. We should note that in Astrophysics the word metal means every element heavier than Helium.

The range of the parameter space, shown in Table 4.2, is chosen so as to contain the known values for ϵ Indi within, and to be much wider than the uncertainty in those values, so as to not cut the probability distributions for the models in our grid when we do model optimization. The large range in the grid will also allows us to do a more extended study of stellar models in K-type stars, that will be shown in Chapter 5. The small step size was chosen so that we could have low uncertainties in the results obtained from model optimization.

The value for the initial Helium fraction, Y , was chosen in each track by using the Helium enrichment law,

$$Y = \left(\frac{\Delta Y}{\Delta Z} \right) Z + Y_0, \quad (4.1)$$

where $Y_0 = 0.2485$ is the primordial Helium abundance [46], the fraction of Helium at the Big Bang nucleosynthesis, and $\frac{\Delta Y}{\Delta Z}$ is the Helium enrichment ratio for which we chose the value $\frac{\Delta Y}{\Delta Z} = 1.2$. We decided to use the Helium enrichment law in order not to have Y as a free parameter in the models, and the value of $\frac{\Delta Y}{\Delta Z} = 1.2$ was chosen based on [47], where it was used for a stellar grid with similar properties.

MESA picks the equation of state for a model based on its values of density and temperature. Given the low mass of our models we expect them to have high density and low temperature, which, according to the left panel of Fig. 4.1, would mean that their equation of state would tend to be `OPAL` or a mixture of it with `FreeEos`. In the right panel of Fig. 4.1 we compare the HR diagram for a track in our grid with a track where only the `OPAL` equation of state is active, and a track where only the `FreeEos` is active. We can clearly see how the track in our grid more closely follows the `OPAL` equation of state, especially in the pre-main-sequence and early main sequence.

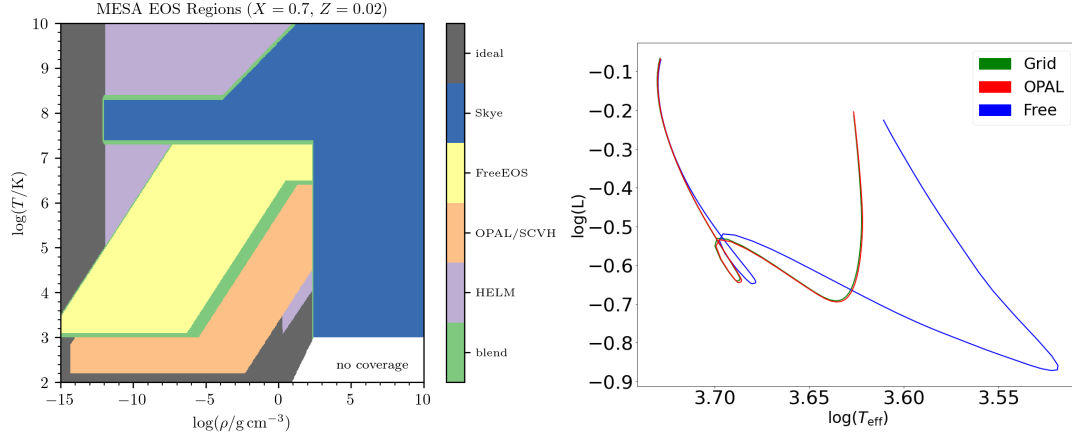


Figure 4.1: Left panel: $\rho - T$ diagram showing the regions where each equation of state applies in the latest version of MESA. From [48]. Right panel: HR diagram comparing a track in our grid with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ to tracks with the same parameters but the OPAL or FreeEos equation of state.

Gravitational settling, which is a form of atomic diffusion, is a chemical transport mechanism whereby heavier elements in a star will tend to fall towards its center, while lighter elements will rise to the surface. In the case of main sequence stars, it will be Hydrogen rising to the surface and Helium along with metals falling to the center. Our grid includes diffusion since it is an important chemical transport mechanism and not taking it into account would affect the main-sequence lifetime, the surface chemical composition and the effective temperature [49]. Other aspects that can be influenced by diffusion are the location of the base of the convective envelope, and with regard to asteroseismology, [50] has found that including diffusion could lower the value of the large frequency separation and the small frequency separation $\delta\nu_{02}$.

For the opacities we use the OPAL opacity tables from [51] at the higher temperature regime and the tables by [52] at lower temperatures, which correspond to choosing the options a09, a09-co and lowT-fa05-a09p in the kap module, with the base initial Metal fraction for the opacity table being the same as the initial Metal fraction for the track. We did not include effects such as convective overshoot, rotation or solar wind in the models, as adding them would be computationally expensive and have a small impact for the kind of star we are modelling.

4.1.1 Atmospheric models in MESA

We will now give attention to the question of the impact that the atmosphere has in the models. To do this we have constructed smaller grids of stellar tracks, where the only

difference with respect to the main grid is the atmospheric model used.

This study is important given that one of our main goals is to study the surface term, and the poor modelling of the stellar atmosphere is one of its main causes [38]. Another reason is that, since we are going to perform model optimization using only one of the atmospheres, doing this study now will permits us to see later on how the results of the model optimization could have been different if we had used another atmosphere in the grid. Finally, it will also allow us to see the impact that the choice of the atmosphere can have on the $M-R$ and $M-T_{\text{eff}}$ relations for K-type stars, another topic that we intended to study.

MESA implements the atmosphere using the `atm` module. The first method that it uses is through direct integration of the hydrostatic balance equation [10],

$$\frac{dP_{\text{gas}}}{d\tau} = \frac{g}{\kappa} - \frac{a}{3} \frac{dT^4}{d\tau}, \quad (4.2)$$

where the opacities are calculated at each point of the atmosphere, according to local temperature and pressure conditions, and $\frac{dT^4}{d\tau}$ is defined using one of the $T(\tau)$ relations that will be described next. Eq. (4.2) is solved using a minimum value of τ for the atmosphere of $\tau = 10^{-5}$.

The second option is using pre-computed tables, with the atmosphere being interpolated from these tables according to the specific properties of the model. In regions that are not within the limits of the tables, MESA will resort to using the $T(\tau)$ relation once again.

We studied all of the available options in MESA for atmospheric models, using both atmospheres given through $T(\tau)$ relations, and atmospheres calculated from tables. The atmospheres using $T(\tau)$ relations follow the general form [43],

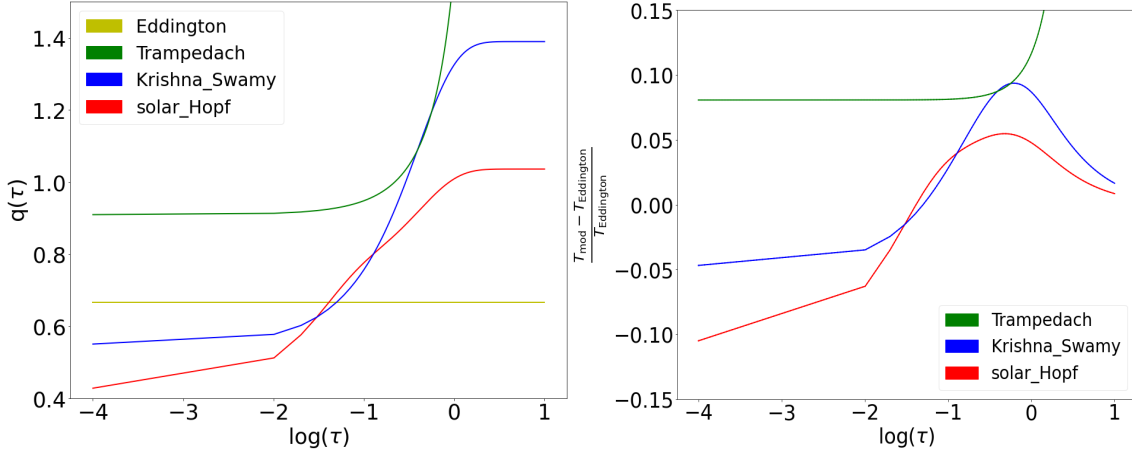
$$T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 [\tau + q(\tau)], \quad (4.3)$$

where $q(\tau)$ is the Hopf function, which is what will change in the different $T(\tau)$ relations, and that can be derived either theoretically or through empirical fits. The different options that were used for $q(\tau)$ are in Table 4.3, where the Eddington relation is the one that was used for the main grid.

Before comparing the impact that each atmosphere has on the models, we can already

Table 4.3: $T(\tau)$ relations available in MESA.

Eddington (from [53])	$q(\tau) = \frac{2}{3}$
Trampedach (from [54])	$q(\tau) = 0.6887 + 0.668(\tau - 0.1148) + 0.7657e^{\frac{\tau - 0.9262}{0.76578}}$
Krishna Swamy (from [55])	$q(\tau) = 1.39 - 0.815e^{-2.54\tau} - 0.025e^{-30\tau}$
Solar-Hopf (from [43])	$q(\tau) = 1.0361 - 0.3134e^{-2.448\tau} - 0.2959e^{-30\tau}$


 Figure 4.2: (a) $q(\tau)$ relations available in MESA (b) Relative difference in temperature between the other atmospheric models and Eddington.

study the differences in temperature between the $T(\tau)$ relations in Table 4.3. This will enable us to form prior expectations regarding which atmospheres will differ the most from using Eq. (2.38), along with the magnitude of the differences.

What we can conclude from Fig. 4.2 is that the Trampedach atmosphere should be the one causing the largest difference between models, and that this difference should be towards higher values of T_{eff} by about 10%. The Krishna-Swamy and Solar-Hopf atmospheres are much more similar to each other and to the Eddington relation than to Trampedach, which is justified by the fact that Solar-Hopf and Krishna-Swamy use the same function just with different parameters. A similar graph describing these relations can also be found in [54].

The tabulated atmosphere used for comparison with our grid is called Photosphere in the following graphs, which uses the PHOENIX model atmospheres from [56] and the models by [57]. The tables cover the parameter space in our grid, given that they include entries for values of $\log(Z/Z_{\odot})$ from -4 to 0.5 and $\log(g)$ from -0.5 to 5.5 in steps of 0.5 dex along with T_{eff} from 2000 K to 50000 K in steps of 250 K. The use of tabulated atmospheres instead of $T(\tau)$ relations can potentially be more accurate, given that the tables will have

values specifically for the temperature ranges of each type of star, while the above $T(\tau)$ relations were created or adjusted according to the atmosphere of the Sun.

In Fig. 4.3, Fig. 4.4 and Fig. 4.5 we show the relative differences between the radius, effective temperature and the mode frequencies using the atmospheres denoted in the graphs and the Eddington atmosphere, with the x-axis being the scaled time, i.e, the the age of a model divided by the age of the oldest model in the track. For the figures regarding the mode frequencies, they were done by calculating the differences between modes of same n and l , followed by doing an weighted average over the modes using a Gaussian distribution, a procedure that will be explained in more detail in Chapter 5.

Some common patterns can be seen in all of these figures, such as the fact that the Trampedach atmosphere results in the largest differences from Eddington, followed by solar-Hopf, Krishna-Swamy and the Photosphere tabulated atmosphere. Another common aspect is that the impact of changing the atmosphere depends on the stellar properties, with this impact being larger for lower-mass stars, higher values of Z and lower values of α_{MLT} . Finally, we can see that the differences tend to decrease as we move towards older ages, while the star is within the main-sequence.

The fact that Trampedach has the largest differences compared to the others is in line with what we expect from the left panel of Fig. 4.2 and Fig. 4.3, but what is more surprising is the fact that the tabulated atmosphere is the closest one to the Eddington atmosphere. We expect the tabulated atmosphere to be better adjusted towards the type of star we are studying, given that its properties are adapted to the value of T_{eff} and $\log(g)$ of each model, so it is unexpected to see it being more similar to the simplest atmospheric model.

The dependencies on mass, age and α_{MLT} that we see in Fig. 4.3, Fig. 4.4 and Fig. 4.5 suggest that it is as we move towards lower temperatures that the impact of the choice of the atmosphere increases. One possible reason for this might be the higher complexity of the atmospheres of cooler stars, where the presence of granulation [9] and molecules [58] start to become more important, leading to the grey atmosphere approximation becoming unrealistic. Our results are in agreement with [59], which also observed the fact that it is in lower-mass stars that the choice of the atmosphere has a larger impact and reached similar values for the difference in the effective temperature when changing from the Photosphere atmosphere to the Eddington relation. [60] have also studied the impact of using the Eddington or the Krishna-Swamy relation for the atmosphere, and saw that it results in differences of about 7.7% for Z and 2.5% for the age in model optimization,

along with obtaining better results for the fitting of the binary α Centauri A and B using the Eddington instead of the Krishna-Swamy relation, providing some justification for using the former in our grid.

We can conclude from this exercise that the choice of the atmosphere has a significant impact on the radius, effective temperature and mode frequencies of a star, at the level of respectively 7%, 5% and 10%, for the mass range of ϵ Indi. The differences in radius and effective temperature are significant due to the fact that they are already high enough to surpass the typical values for the error between models and observations in K-type stars [16] [28]. The uncertainties in the frequencies for ϵ Indi are of only about 1 μ Hz [6], thus making the impact on the mode frequencies significant. This suggests that it would be important to create a stellar grid where besides the parameters that we already vary, i.e., M , Z and α_{MLT} , the atmosphere should also be varied, although we did not perform this task due to the time it would take to construct such a grid.

Using the scaling relation for $\Delta\nu$ and ν_{max} , Eq. (3.14) and Eq. (3.19), we get that a 5% shift in radius and effective temperature will cause a 10% shift in the values of $\Delta\nu$ and ν_{max} , which is consistent with our results for the impact of the choice of the atmospheric model on the mode frequencies. We can understand why a 5% shift in radius causes a larger shift in mode frequencies from the fact that $\Delta\nu \propto R^{-3/2}$ and $\nu_{\text{max}} \propto R^{-2}$, which also explains why models with a smaller radius than Eddington in Fig. (4.3) have larger mode frequencies in Fig. (4.5).

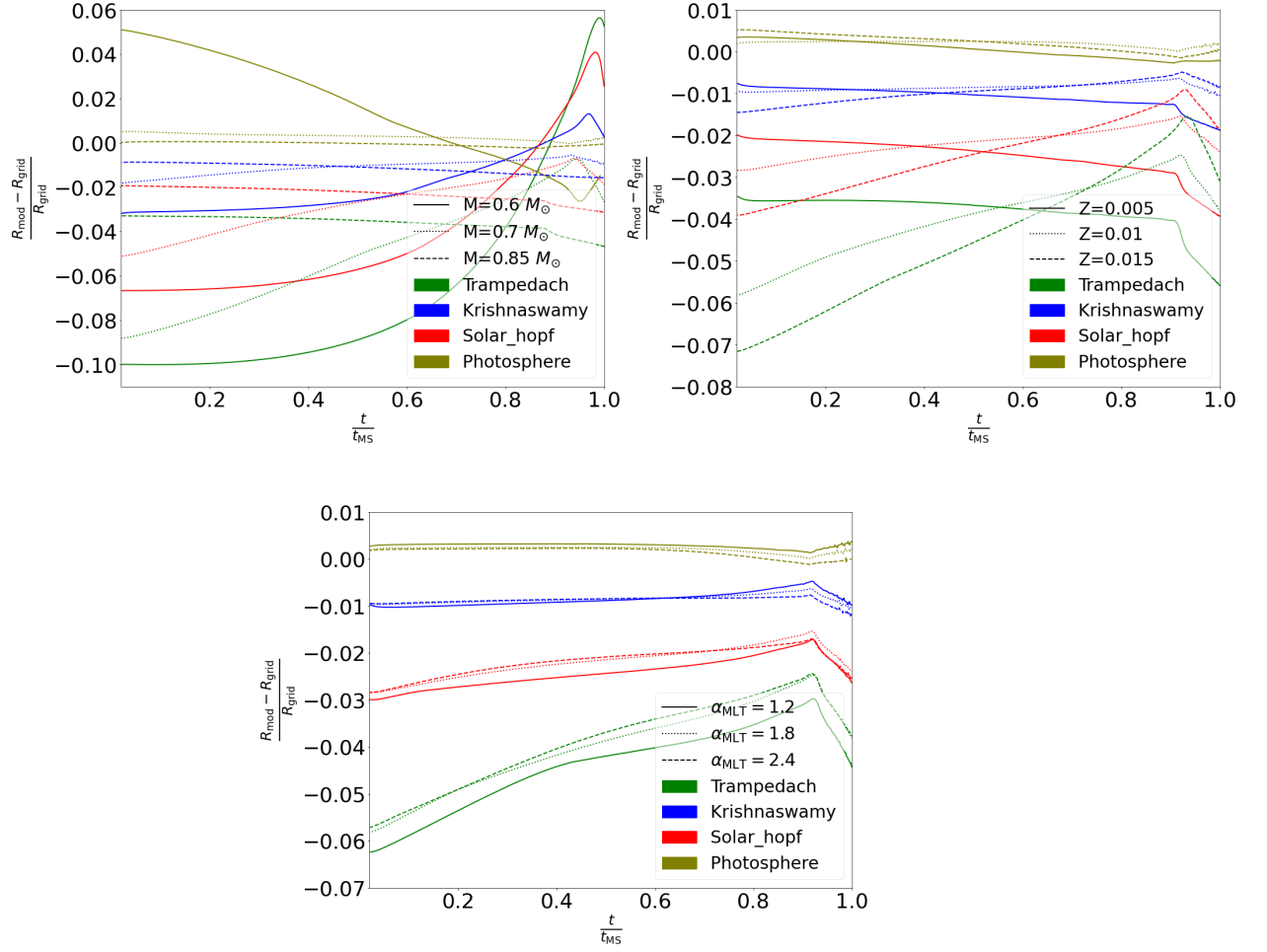


Figure 4.3: Relative difference in radius between the several atmospheric models and that used in the main grid (Eddington) for different M in the top left panel, different Z in the top right panel and different α_{MLT} in the bottom panel.

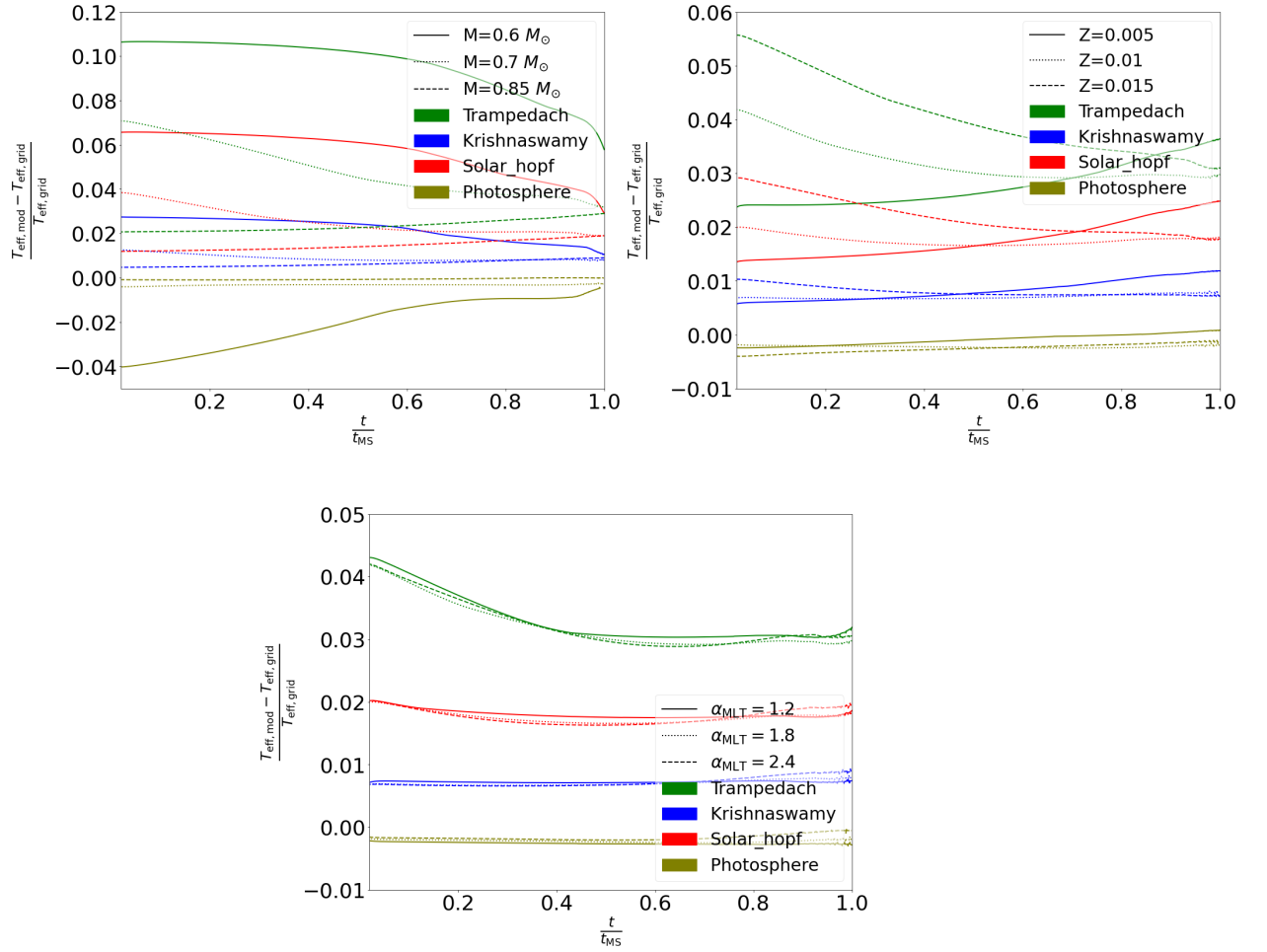


Figure 4.4: Relative difference in T_{eff} between the several atmospheric models and that used in the main grid (Eddington) for different masses in the top left panel, different Z in the top right panel and different α_{MLT} in the bottom panel.

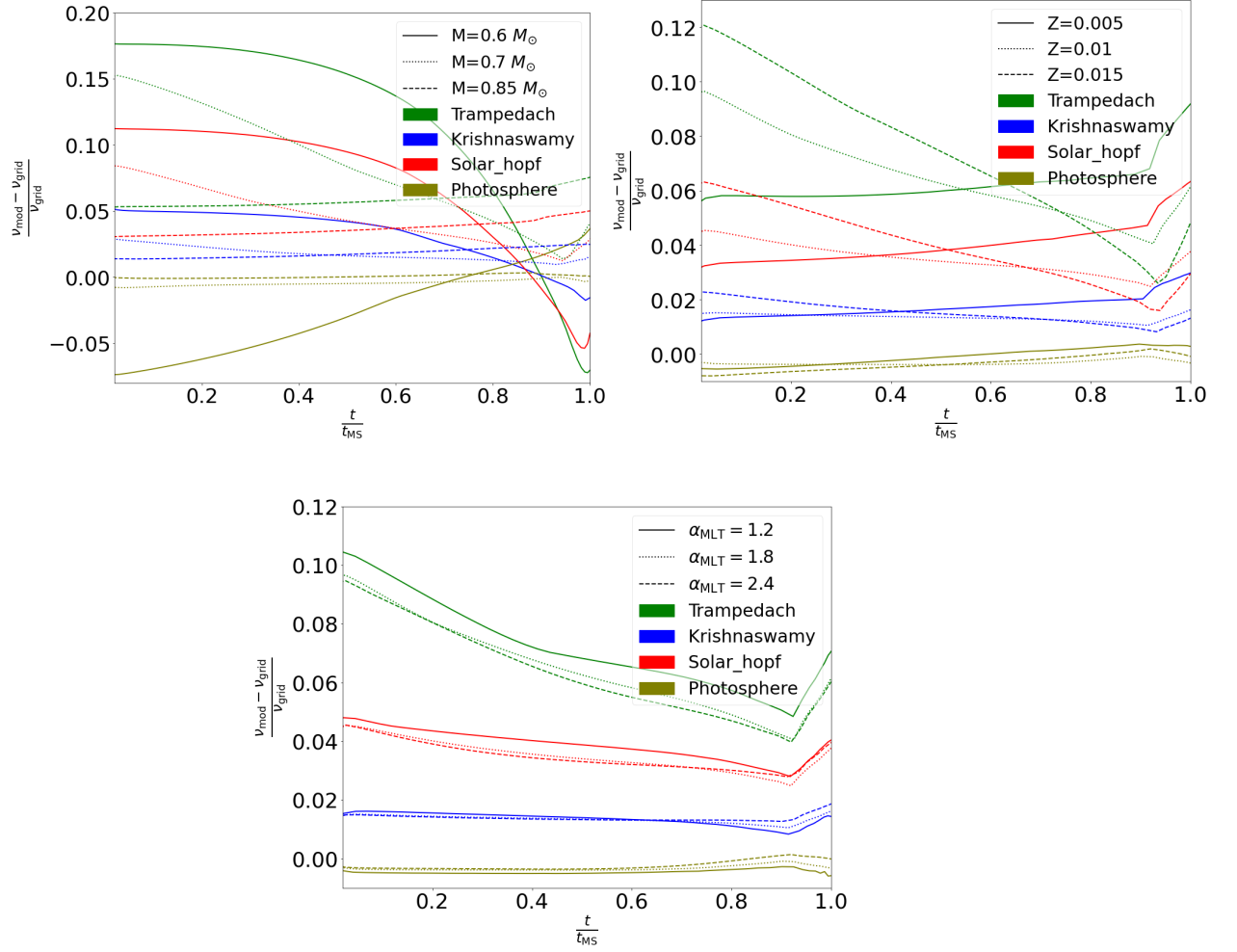


Figure 4.5: Relative difference in $\nu_{n,0}$ between the several atmospheric models and that used in the main grid (Eddington) for different masses in the top left panel, different Z in the top right panel and different α_{MLT} in the bottom panel.

4.2. Stellar oscillation code – GYRE

GYRE [11] is an open source stellar oscillation software coded in FORTRAN 2008 that calculates the properties of the oscillation modes corresponding to a stellar model. It does this by solving the stellar pulsation equations, both for the adiabatic and non-adiabatic case, starting from the data output by a stellar modelling software, such as MESA.

The stellar pulsation equations can be seen as an eigenvalue problem with boundary conditions at the center and surface of a star, where the oscillation frequencies ν are the eigenvalues, and to solve them GYRE uses a Magnus Multiple Shooting scheme which combines aspects of relaxation schemes and shooting schemes, that were the previously used methods to solve them. Through this method, finding the eigenfrequencies ν amounts to finding the roots of the function $D(\nu) = \det(S[\nu])$ where S is related to the form and values of the stellar pulsation equations, and which can then enable us to obtain the eigenfunctions of the problem. This method has some advantages over shooting schemes, such as the fact that it can handle situations where an "exponential dichotomy" appears, i.e, when the boundary value problem has components with both exponential growth and decay. The quality of this method, and of GYRE for asteroseismic modelling, is shown by how it can reproduce results of previously used stellar oscillation codes and be computationally efficient [11].

We have used GYRE for all of the models in the grid created using MESA, calculating the frequency ν of each mode, its radial order n and the mode inertia I_{nl} for modes of angular degree l up to $l = 2$. Just like MESA, GYRE is also customizable in what it outputs and the modes which it calculates, where here we have decided to only compute modes until $l = 2$ given that these are the only ones observed until now for ϵ Indi [6]. In the left panel of Fig. 4.6 there is an example of the modes calculated by GYRE for a specific model. We can see there how the frequencies of the modes are related to n and l in accordance with the asymptotic relation, Eq. (3.11), with a linear dependence on n . In the right panel of Fig. 4.6, an échelle diagram, we note how the modes tend to be in vertical ridges, although with a significant amount of curvature, which are due to the frequency variations in $\Delta\nu$ and ϵ . The small value of $\delta\nu_{02}$ is also apparent looking at the right panel of Fig. 4.6, which is why it is called the small frequency separation.

GYRE has different boundary conditions to pick from, based on different articles and formulations of the stellar pulsation equations, and the one that we have used to calculate

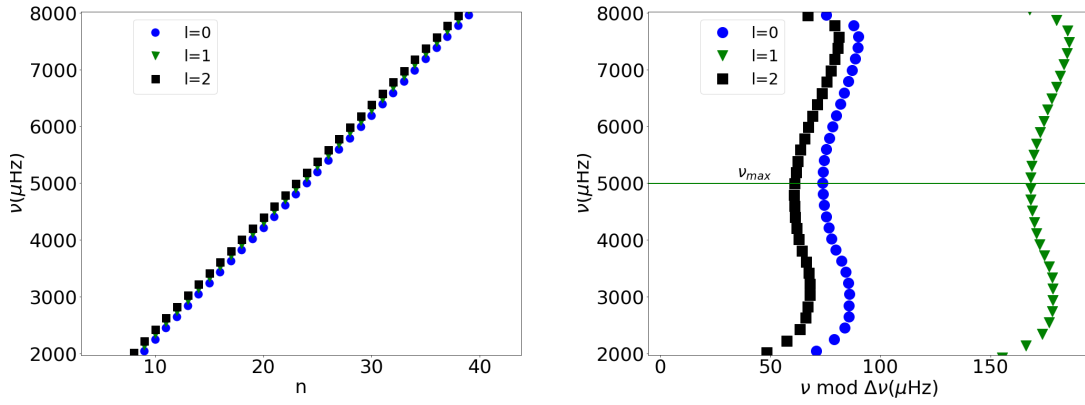


Figure 4.6: Left panel: Frequencies computed by GYRE for a model with $M = 0.75M_{\odot}$, $Z = 0.01$, $\alpha_{\text{MLT}} = 1.8$ at an age of 5.35 Gyr. Right panel: Échelle diagram corresponding to the model.

the grid is JCD, which is inspired by Jørgen Christensen-Dalsgaard and the stellar oscillation code ADIPLS [61]. Given that we intend to study the surface term, however, and the boundary condition is directly related to the modelling of the surface, we decided that it would be interesting to see how changing the boundary condition would impact the mode frequencies.

To do this we recalculated the asteroseismic modes in the track with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ using the other boundary conditions offered by GYRE, which are called VACUUM [11], UNNO [62], DZIEM [63] and Isothermal [61]. The VACUUM boundary condition assumes that the pressure and density of the star vanish at the surface, and the condition for the temperature is obtained from perturbing the $T(\tau)$ relation for an Eddington atmosphere, Eq. (2.38), while the DZIEM boundary condition is similar to VACCUM except that it supposes a polytropic atmosphere, where $P \propto \rho^{1+1/n}$, instead of Eq. (2.38). In the JCD, UNNO and Isothermal boundary conditions the values of the pressure are obtained by assuming an isothermal atmosphere, and the density by considering a vacuum solution for the gravitational potential outside the star. The difference between Isothermal and JCD along with UNNO, is that it not does not permit the existence of modes with frequencies higher than the acoustic cut-off frequency.

We then computed the difference between the mode frequencies in the original model and the models with the differing boundary conditions. In Fig. 4.7 and Fig. 4.8 we present some results of this exercise, from which we can observe that DZIEM and VACUUM are the most differing boundary conditions, but even there the differences are very small, being

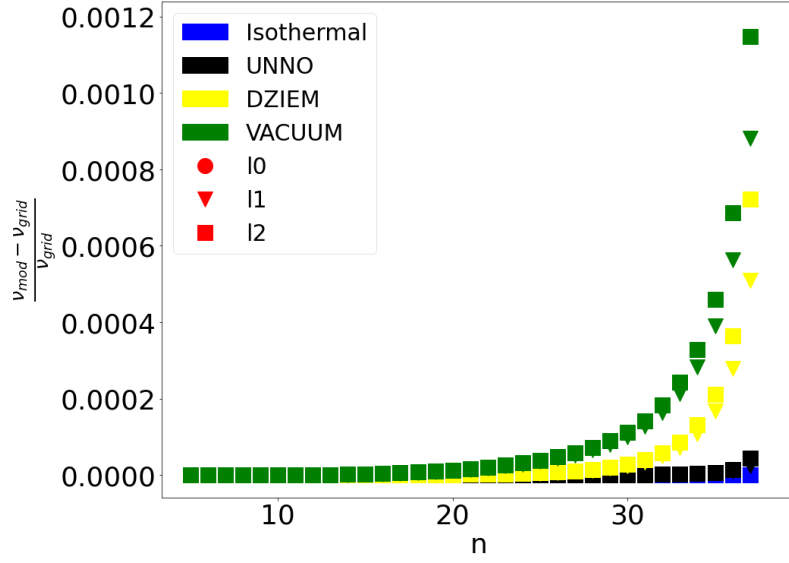


Figure 4.7: Fractional error between the frequencies of the modes calculated through the other boundary conditions offered by GYRE and the boundary condition adopted in the grid for a model with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ at an age of 5.35 Gyr.

only of about 0.004%, while the typical mode frequencies uncertainties for ϵ Indi are of about 0.02%. Fig. 4.7 permits us to see that the difference is larger at higher radial orders, which should be related to the fact that higher frequency modes are more affected by the surface, as described in Chapter 3. Fig. 4.8 shows that the difference has a small increase throughout the main-sequence. From these results, we can conclude that the difference between the boundary conditions is not large enough to have a significant impact in the calculation of asteroseismic parameters or in the results of the model optimization.

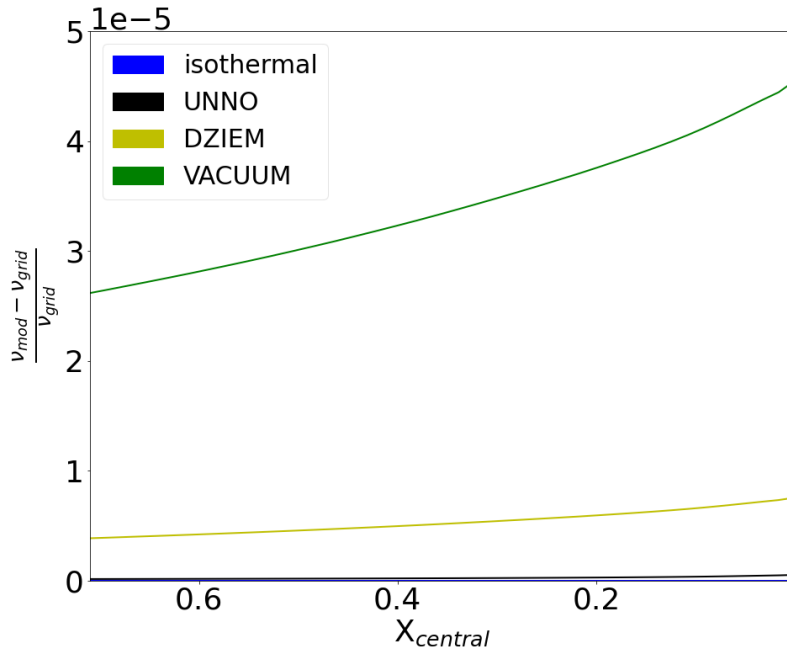


Figure 4.8: Evolution of the average fractional error on the frequencies for the $l = 0$ modes of a stellar track with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{MLT} = 1.8$.

5. Model Grid diagnosis

With the grid of stellar evolution tracks created in Chapter 4 we will now focus on one of our main research goals, the study of the asteroseismic properties of orange-dwarfs similar to ϵ Indi. In spite of our work being focused on ϵ Indi in terms of observational data, we would not be using the full potential of the grid if we limited ourselves to using it for model optimization without before doing a deep study of its properties. It is such a study that allows us to understand the properties of K-type stars near the spectral type of ϵ Indi, thus moving our work to the wider context of cool-dwarf asteroseismology. As more observational datum are made available for K-type stars, such a study will be helpful to have a more solid understanding of what we expect to see in early to mid K dwarfs. In the following paragraphs we will provide a summary of what we intend to study in this chapter.

Asteroseismology offers us the best way to infer a star's interior structure, but to achieve that we need to have a good prior knowledge of the inner structure of the models we have created. So, in this chapter, we intend to study the inner properties of the models in our grid, such as the length of the convection zone, the energy transport gradients, the Brunt–Väisälä and Lamb frequencies, and the sound speed among other aspects. We will also look at the changes and discontinuities that occur at the near-surface layers with particular attention, given that they can point to the causes of the surface term.

Since asteroseismic data for K-type stars have until now been unavailable, our grid can be of use to study the expected asteroseismic properties of stars of this type. To do this we will calculate all of the most important asteroseismic parameters for it, such as the large frequency separation $\Delta\nu$, the small frequency separations $\delta\nu_{01}$ and $\delta\nu_{02}$ along with the parameter ε in the asymptotic relation, calculating both their average and frequency-dependent values. We will also see how they change with a star's evolution by plotting C-D and ε diagrams, thus performing a similar kind of study to [64] and [65]. It is worth noting that all of these parameters have been determined in [6] for ϵ Indi.

Both in the study of the inner structure and the asteroseismic properties of the grid, we will attempt to investigate how their features are correlated with the grid parameters, namely, M , Z and α_{MLT} , justifying these correlations through what we know about the physics of K-type stars.

5.1. Global stellar parameters

We will begin by studying how the atmospheric parameters of a star vary across the grid, features such as the luminosity, effective temperature and radius, that can be estimated through methods such as interferometry, spectroscopy and the study of binaries. The radius is of particular interest given that for ϵ Indi we have a very precise measurement using interferometry [66], with an uncertainty of less than 1%, making it possible to study the systematic discrepancy between models and observations in K-type stars [28] for the radius and effective temperature.

We begin by creating Hertzsprung–Russell (HR) diagrams for our grid in Fig. 5.1, the left panel to study the impact of mass and the right panel the impact of Z and α_{MLT} . In the left panel of Fig. 5.1 we can clearly see the Hayashi track to the right, where the effective temperature is almost constant while the luminosity rapidly decreases, followed by the Henyey track where the star begins to have a radiative core and the contraction of the radius causes the effective temperature to increase.

In the left panel of Fig. 5.1 we can observe how increasing the mass leads to higher values for L and T_{eff} throughout the whole evolution. This is expected from the mass-luminosity relation for main-sequence stars, $L \propto M^{3.5}$ [31], and the fact that higher-mass stars have a higher amount of gravitational energy they can convert into thermal energy. In the right panel of Fig. 5.1 we see how increasing the metal mass fraction leads to lower values for both L and T_{eff} , while increasing α_{MLT} leads to higher values for T_{eff} without changing the luminosity. The reason for this is that higher Z will increase the number of molecular compounds in the atmosphere [58], leading to a higher opacity that decreases conductivity, as per Eq. (2.14). Higher values of α_{MLT} will increase the effective temperature due to the fact that more energy can flow to the surface, so that for the luminosity to remain nearly constant the stellar radius will have to contract as α_{MLT} increases, which can be seen in the left panel of Fig. 5.3.

In Fig. 5.2 and Fig. 5.3 we find described how the lifespan in the main-sequence, the luminosity, the radius and the logarithm of the surface gravity vary across the mass range in our grid.

The left panel of Fig. 5.2 clearly shows that higher-mass stars evolve much faster, with a star with $0.85M_{\odot}$ spending about a third of the time in the main-sequence than a star with $0.6M_{\odot}$. This behaviour can be explained by the approximate expression for the lifetime of a star in the main-sequence, Eq. (2.32), in which, although τ_{MS} is proportional to the

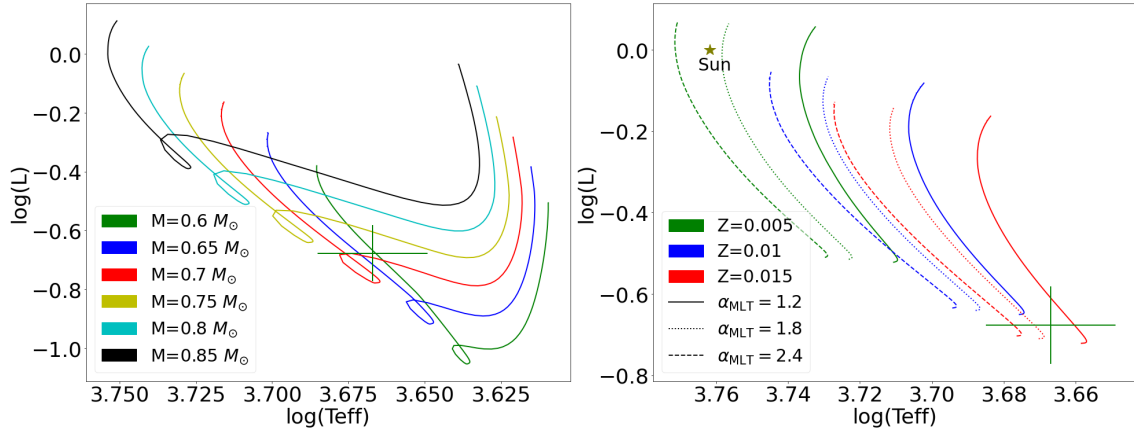


Figure 5.1: Left panel: HR diagram, including the pre-main-sequence, for tracks with varying mass, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: HR diagram of the main-sequence for stars with $M = 0.75 M_{\odot}$ and varying Z and α_{MLT} . The cross represents the values for ϵ Indi.

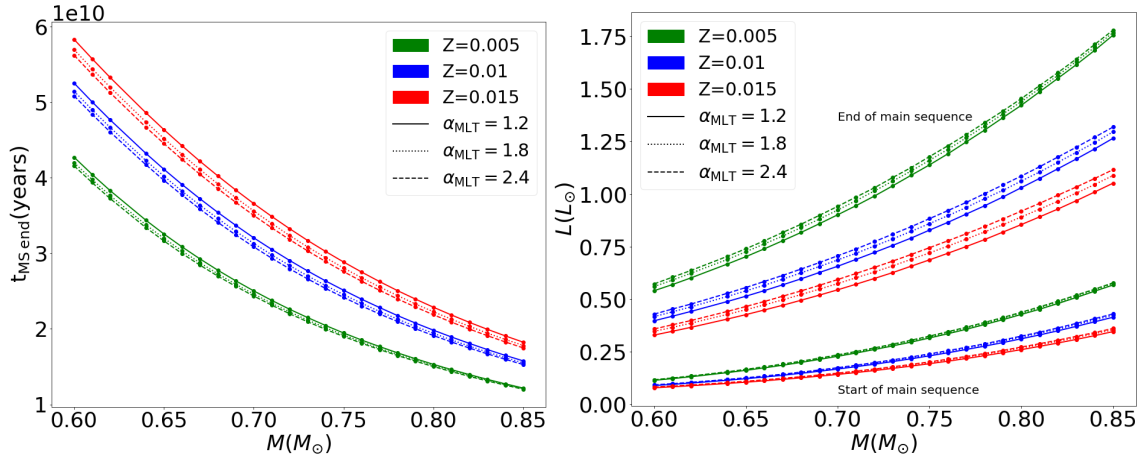


Figure 5.2: Left panel: Lifespan in the main-sequence. Right panel: Mass-Luminosity relationship at the beginning and end of the main-sequence.

mass, the strong increase in luminosity towards higher masses, as can be seen in the right panel of Fig. 5.2, compensates this fact. We also see a significant dependence on Z , where higher values lead to increased lifetime, along with a weak decrease of the lifetime with α_{MLT} that is more significant towards lower masses. The increase in lifespan with Z can be explained through the right panel of Fig. 5.2, where we can see that higher values of Z significantly decrease the luminosity, thus increasing the lifespan. The stronger dependence on α_{MLT} for lower masses might be due to the fact that they have a larger convective envelope. It is worth noting the very high main-sequence lifetimes of K-type stars, surpassing the age of the universe, which is the reason why we do not expect to see this type of stars already in the subgiant branch.

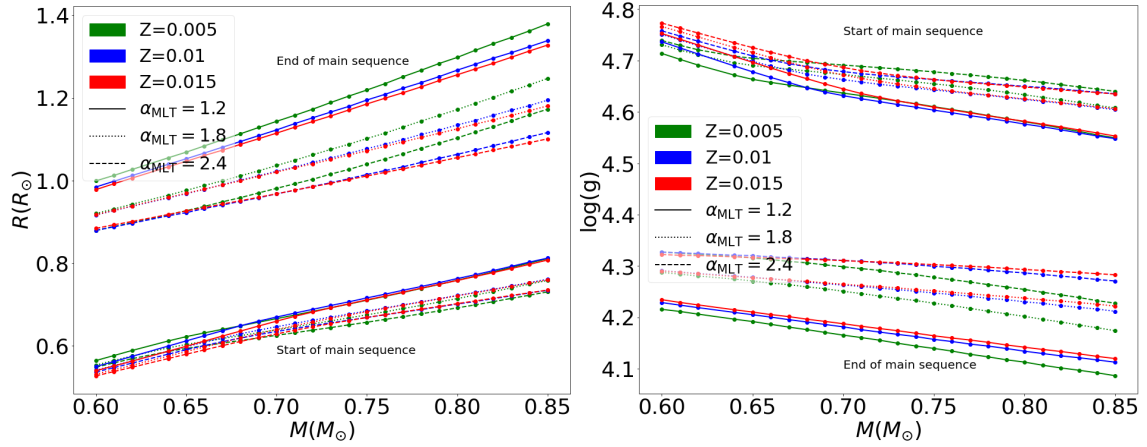


Figure 5.3: Left panel: Mass-radius relationship at the beginning and end of the main-sequence. Right panel: Mass- $\log(g)$ relationship at the beginning and end of the main-sequence.

In the left panel of Fig. 5.3 we can see a nearly linear relationship between mass and radius, but with distinct behaviours for the metal mass fraction and α_{MLT} at the different masses and phases of evolution, with them having a stronger impact towards older ages and higher masses. The dependence on mass is in line with the fact that lower-mass stars tend to be more dense, while the dependence on α_{MLT} shows how more efficient convection will contract the stellar radius. The M - $\log(g)$ diagram, the right panel in Fig. 5.3, shows opposite behaviours to those of the M - R diagram, which is due to the fact that $g \propto \frac{M}{R^2}$, i.e, the surface of larger stars will be less gravitationally bound to the center.

5.2. Inner structure

One of the benefits we get from stellar modelling is that through it we can obtain detailed portrayals of the deeper layers of a star, while most types of observational data will only tell us about the superficial layers. Asteroseismology already allows us to go deeper with more detail, especially if we use asteroseismic inversion techniques [67]. So, to see how stellar properties change with depth, we will study the radial profiles of models in our grid along with the parameters related to the central layers of a star.

In Fig. 5.4 we see $\log T_{\text{central}}$ vs $\log \rho_{\text{central}}$ diagrams, including only the main-sequence in the right panel of Fig. 5.4. The graphs are to be read from the bottom left to the upper right in terms of time, with the stellar core becoming increasingly hotter and denser as the star evolves. In the pre-main-sequence this is due to the fact that the star is undergoing gravitational contraction until its central temperature becomes high enough for nuclear fusion to become possible, while in the main-sequence it is due to the fact that the core

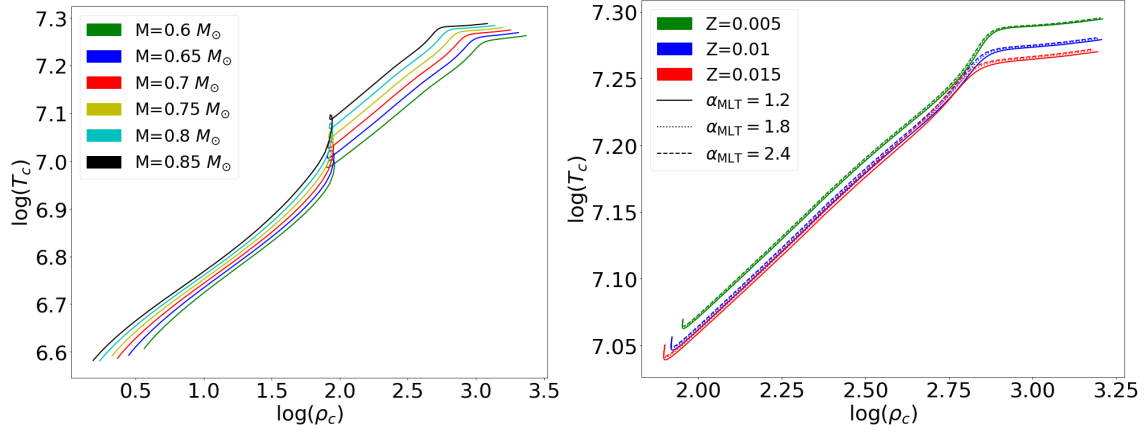


Figure 5.4: Left panel: $\rho_{\text{central}}-T_{\text{central}}$ diagram for tracks of varying mass, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$, including the pre-main-sequence. Right panel: $\rho_{\text{central}}-T_{\text{central}}$ diagram in the main-sequence for tracks with $M = 0.75 M_\odot$ and varying Z and α_{MLT} .

becomes heavier as Hydrogen is turned into Helium and the central temperature needs to increase to counterbalance the higher pressure.

Changing the grid parameters has a significant impact, with higher mass leading to an increase in central temperature, while increasing Z decreases central temperature and α_{MLT} has a very small impact. The increase in central temperature with mass is due to the fact that higher-mass stars will have a higher amount of gravitational energy to convert into thermal energy, thereby enabling them to reach higher temperatures.

The sound speed is an important quantity to study, since it is not only related to the thermodynamical properties of a star, but also to its asteroseismic features, such as, for example, $\Delta\nu$ and $\delta\nu_{02}$ in Eq. (3.12) and Eq. (3.15). In the left panel of Fig. 5.5 we study the impact of the mass and the phase of evolution on the sound speed, while in the right panel of Fig. 5.5 we look at the impact of Z and α_{MLT} .

A behaviour which is common to both the earlier and more advanced phases of evolution is that lower-mass stars have a lower sound speed than higher-mass stars in the core. This could be due to the fact that lower-mass stars have denser cores and lower pressure, which will lower c_s , since $c_s \propto \sqrt{\frac{P}{\rho}}$, in the case of an ideal gas. Another interesting aspect is that in the later phases of evolution c_s stops being monotonic, instead having a small decrease in the region of the core at a fractional radius of about 0.1. This behaviour is also observed in the results of [68], with the cause being the difference between the chemical composition of the core and that of the rest of the star, with the former dominated by Helium instead of Hydrogen towards the end of the main-sequence.

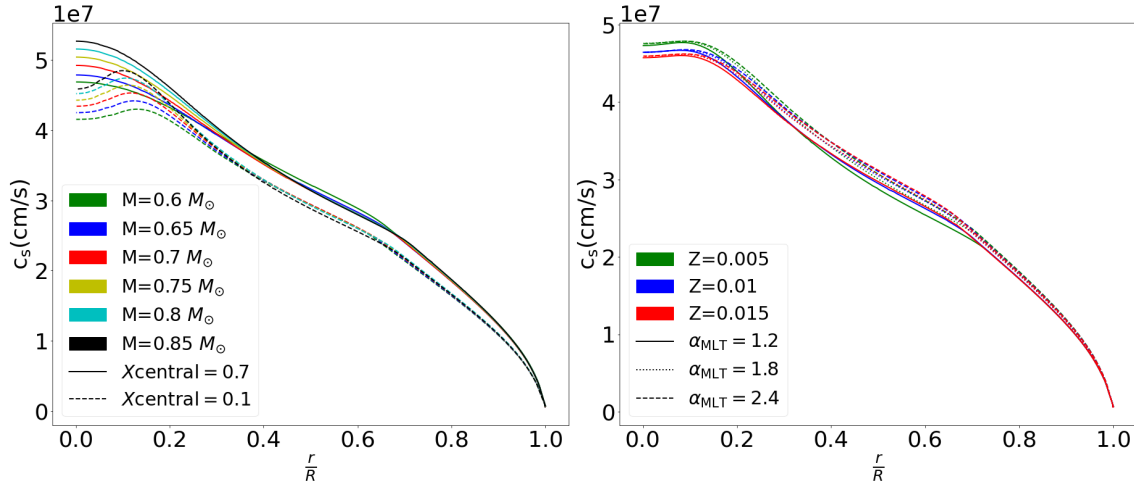


Figure 5.5: Left panel: Sound speed for different masses and phases of evolution. Right panel: Sound speed at $X_{\text{central}} \approx 0.35$ for different Z and α_{MLT} .

We see a much weaker dependence of the sound speed when it comes to Z and α_{MLT} , with the former having more impact in the region of the core, while the latter towards the more superficial layers and the transition region between the convective envelope and the radiative interior of the star at a fractional radius of about 0.7. This sensitivity of the core to Z might be connected to the behaviours observed in [68] and [69], where changing the opacity and the central Hydrogen abundance resulted in significant differences in the value of c_s in the core.

In Fig. 5.6 are represented some of the most important thermodynamical quantities to understand the energy transport inside a star, namely, the adiabatic, thermal and radiative temperature gradients, which tell us about how temperature is stratified in a star. These temperature gradients will also provide information about the radiative and convective zones of a star, which we will look at more directly in Fig. 5.7.

The adiabatic temperature gradient is constant throughout most of the star, only lowering from the value of $\frac{2}{5}$ in the near-surface layers and increasing back to that value at the atmosphere of the star. This behaviour is due to the degree of ionization of the atoms in the star. For a fully-ionized ideal gas we have that $\nabla_{\text{ad}} = \frac{2}{5}$, which corresponds to the inner regions of the star, where the high temperature is able to ionize every chemical element. Since the temperature and density decrease as we move to the surface, the degree of ionization stops being complete, and for an ideal gas, this will lead ∇_{ad} to decrease, going back to $\nabla_{\text{ad}} = \frac{2}{5}$ when the gas becomes fully neutral.

In the inset of Fig. 5.6 we can see a region very near the surface with a drop on ∇_{ad} ,

where we find some degree of dependence of the drop on the mass and Z of the model. From the left panel of Fig. 5.6 we see that the drop increases as we lower the mass, while from the right panel of Fig. 5.6 we can observe that the drop is more pronounced and moves closer to the surface at higher Z . A possible reason for the drop might be due to the ionization of molecular compounds, which are more present in lower-mass and more metallic stars.

The radiative temperature gradient shows the temperature stratification required if all of the energy of the star was to be transported through radiation, and up to a fractional radius of 0.65 it overlaps with the thermal temperature gradient, the actual temperature stratification of the star. It is around this point that the radiative temperature gradient begins to increase dramatically, which means that we have reached the convective envelope of the star, in accordance with the Schwarzschild criterion, Eq. (2.20). The reason for this increase is that $\nabla_{\text{rad}} \propto \kappa$, and the opacity tends to increase with decreasing temperature, as per Eq. (2.18).

The convective flux of a star will be larger the more the thermal temperature gradient is super-adiabatic, given that $F_{\text{conv}} \propto \rho T (\nabla - \nabla_{\text{ad}})^{3/2}$ (cf. Eq. 2.26). We can see this in Fig. 5.6, as the thermal temperature gradient follows the radiative temperature gradient until the beginning of the convective envelope, following then the adiabatic temperature gradient. Very near the surface this behaviour breaks down, however, where the thermal temperature gradient begins to follow the radiative temperature gradient in a region where $\nabla > \nabla_{\text{ad}}$. This is because convection can be active as a mixing mechanism while not being efficient at transporting energy, which is what happens near the surface due to the fact that ρT becomes much smaller, thereby decreasing F_{conv} .

This analysis of the energy transport gradients now permits us to understand the results of Fig. 5.7, showing the convective envelope as a fraction of the total radius of a star. We find the same relations that could already be seen in Fig. 5.6, such as the fact that the convective envelope becomes larger at lower masses along with higher values of Z and α_{MLT} . The increase in the extent of the convective envelope towards lower masses has to do with their very large values for the opacities, making radiation an inefficient energy transport mechanism, given that the opacity will increase as stars become denser and colder. Our grid is still far from the fully convective limit though, which is at around $0.3M_{\odot}$ or $0.35M_{\odot}$ [70]. The behaviour with α_{MLT} has to do with what we saw in the left panel of Fig. 5.3, where higher values of α_{MLT} lead to a substantial decrease in the radius, thereby

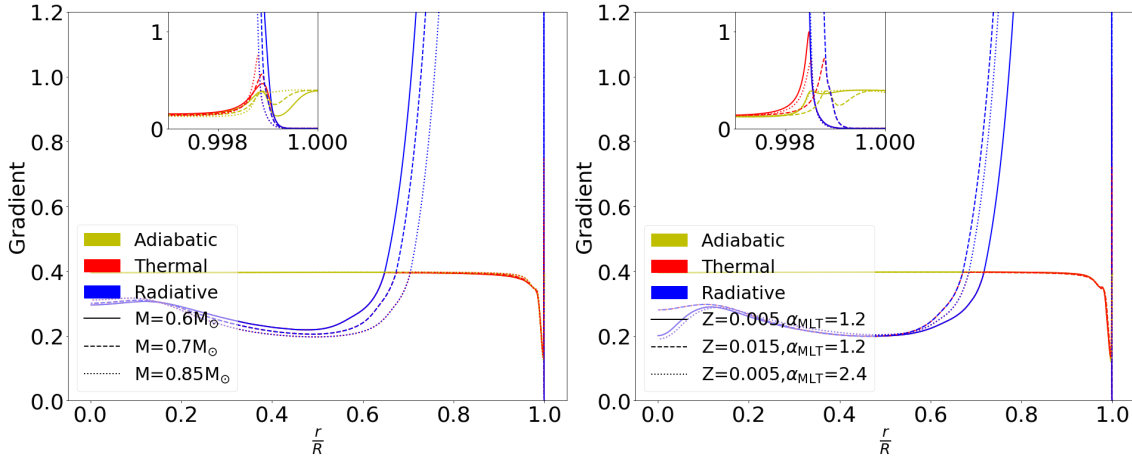


Figure 5.6: Left panel: Adiabatic, thermal and radiative temperature gradient for three models with different masses at $X_{\text{central}} \approx 0.35$. Right panel: Adiabatic, thermal and radiative temperature gradient for three models with different Z and α_{MLT} at $X_{\text{central}} \approx 0.1$.

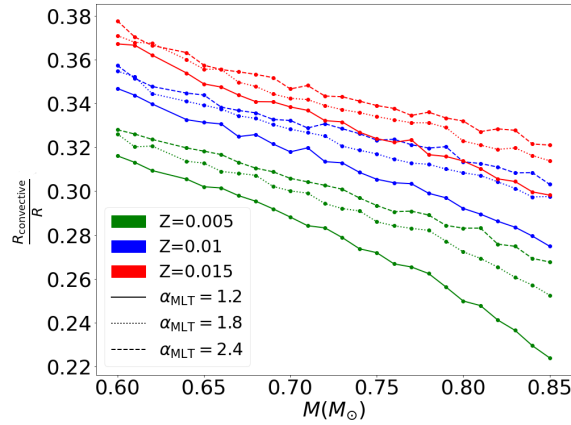


Figure 5.7: Length of the convective envelope as a fraction of total radius of a star at $X_{\text{central}} \approx 0.35$.

making the star denser and more opaque.

The left and right panel of Fig. 5.8 are propagation diagrams, which show how the Lamb frequency and the Brunt–Väisälä frequency change across the interior of the star. They are highly relevant in the context of asteroseismology, since by using them we can visualize the propagation cavities for the oscillatory modes. This is because p modes are propagating when their frequency is higher than both the Lamb and Brunt–Väisälä frequencies, which in Fig. 5.8 correspond to the horizontal lines being above both the blue and the yellow or red curves. They also give us information regarding the energy transport in stars, since the Brunt–Väisälä frequency is only defined in radiative regions.

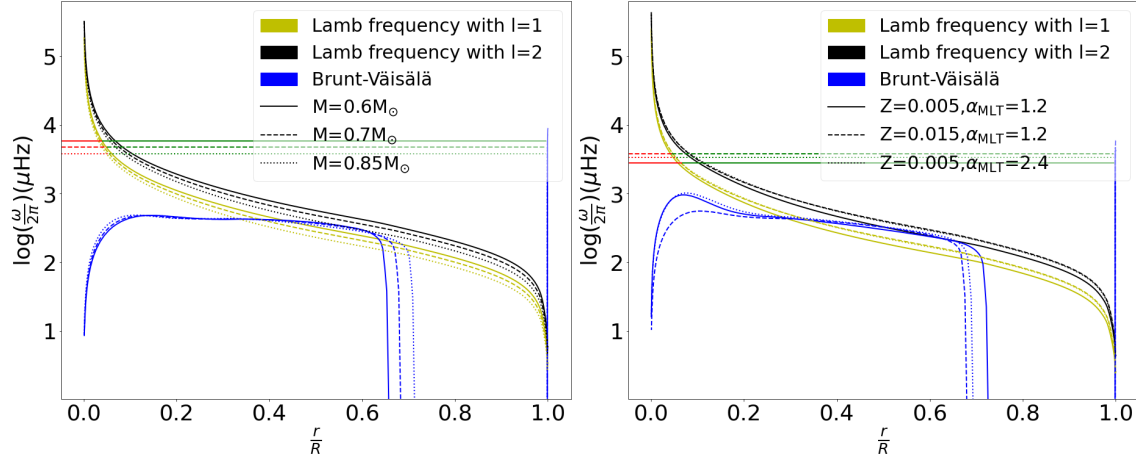


Figure 5.8: Left panel: Propagation diagram for different masses at $X_{\text{central}} \approx 0.35$. Right panel: Propagation diagram for models with different Z and α_{MLT} at $X_{\text{central}} \approx 0.1$. In both panels the horizontal lines represent the values of ν_{max} for the models, with propagating regions for an $l = 1$ mode being shown in green while evanescent regions in red.

We can see how in Fig. 5.8 the Brunt–Väisälä frequency becomes undefined at a fractional radius of about 0.65, corresponding to our expectations for the size of the convective envelope. The lower-mass stars have larger convective envelopes, which is why their Brunt–Väisälä frequency becomes undefined at a smaller radius. We also see a divergence in the value of the Brunt–Väisälä frequency at the surface, which corresponds to the radiative region in the photosphere. The right panel of Fig. 5.8 permits us to observe how α_{MLT} will have a small impact on the core of the star, affecting primarily the size of the convective envelope, while Z will have a strong impact both on the properties of the core and the size of the convective envelope.

The Lamb frequency increases monotonically as we move towards the center, in line with the fact that the sound speed also increases monotonically with depth. This is why higher-frequency modes are able to propagate into deeper regions, as the lower-frequency modes will become smaller than the Lamb frequency at higher values of r . The Lamb frequency does not have a very strong dependence on the grid parameters, with the most impactful one being the mass, where a lower value of the mass leads to higher values of the Lamb frequency.

5.3. Asteroseismic properties

To study the asteroseismic properties of the models in the grid it is meaningful to also analyse global seismic parameters and frequency separations instead of just the mode frequencies, given that we can track their evolution along the main sequence and compare

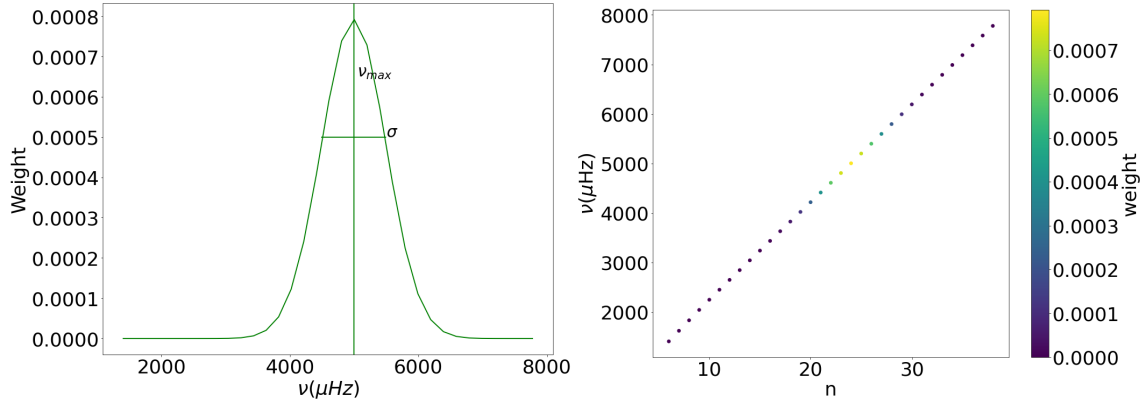


Figure 5.9: Left panel: Weight function for a model with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ at an age of 5.35 Gyr for $l = 0$ modes, although it would be similar for non-radial modes. Right panel: Frequencies of the $l = 0$ modes with the colour bar denoting their weights.

our results to those of other types of stars. The individual frequencies will also suffer from the fact that they contain an error due to the surface term, that we do not correct for in this chapter.

In order to calculate the global seismic parameters we need to take into account all of the modes in a model. Each mode will not contribute equally, however, since we will assign to each of them a unique weight. The weights are described by a normal distribution centered at ν_{max} , that we calculate through the scaling relation in Eq. (3.19), and the standard deviation is given by

$$\sigma = \frac{0.66\nu_{\text{max}}^{0.88}}{2\sqrt{2\ln(2)}}, \quad (5.1)$$

which has previously been used in [71] and [72]. The reason for attributing these weights is that we want to model the observed amplitudes of the modes, which approximately follow a Gaussian centered at ν_{max} , in order to give more importance to the modes that would actually be detectable in a real star. The left panel of Fig. 5.9 shows an example of one of these weight distributions and the right panel of Fig. 5.9 the radial orders that are given more importance.

5.3.1 $\Delta\nu$ and ε

To calculate the average value for $\Delta\nu_l$ and ε we used a similar method to [64] and [65], which is based on the properties of the asymptotic relation, Eq. (3.11).

We perform a weighted linear fit to $\nu_{nl}(n)$ where, as we can see from Eq. (3.11), the gradient will be $\Delta\nu_l$ and the intercept $\varepsilon\Delta\nu_l$, thereby allowing to calculate $\Delta\nu_l$ and ε from the coefficients of the fit. The adopted weights are obtained as illustrated in the right panel of Fig. 5.9, with the modes whose frequency is above the acoustic cut-off frequency not being included in the fit.

To determine the values for $\Delta\nu_{nl}$ we use the same method as for $\Delta\nu_l$, only that now we do the linear fit using only 2 modes, without weighting them, and then we calculate ε_{nl} using $\Delta\nu_{nl}$.

To test that our method for calculating $\Delta\nu_l$ is valid, and to compare it to methods for calculating $\Delta\nu_l$ that do not require modelling the mode frequencies, we have computed the relative difference between the value of $\Delta\nu_0$ we obtained from GYRE with what is obtained using the scaling relation for $\Delta\nu$, Eq. (3.14). We also did the same for the value of $\Delta\nu$ calculated through the acoustic radius, Eq. (3.12).

Figure. 5.10 studies the dependence on mass, while Fig. 5.11 show the dependence on Z and α_{MLT} . The acoustic radius tends to give values about 4% higher than those from GYRE, with the difference increasing for higher masses and lower metallicities. These results are in agreement with [73] and [74], who found similar values for the difference between fitted and acoustic values for $\Delta\nu_0$. One of the causes for Eq. (3.12) giving a significant error is the ambiguity in defining what is the upper limit in the integral, i.e, whether it is the photospheric radius or it needs to be extended beyond that [73].

Based on Fig. 5.10 and the left panel of Fig. 5.11, we can say that the value for $\Delta\nu_0$ calculated through the scaling relation is more precise than through the acoustic radius, and precision increases for stars with masses closer to the Sun, which is expected since the scaling relation was established based on solar-like stars. We find the same kind of behaviour in [64], that even proposed introducing an effective temperature dependence on the scaling relation which would improve the results for hotter and higher-mass stars.

Besides the method for calculating ε described above, we have also tried the method from [75], which fixes the value of $\Delta\nu$ instead of assuming that it also varies with frequency, and calculates ε through

$$\varepsilon_{nl} = \varepsilon_l(\nu_{nl}) = \frac{\nu_{nl}}{\Delta\nu} - n - \frac{l}{2}, \quad (5.2)$$

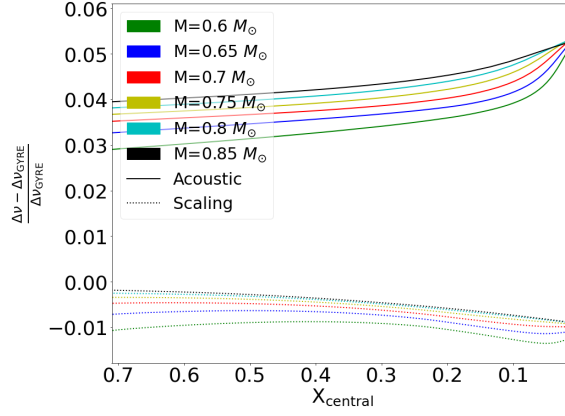


Figure 5.10: Relative error in $\Delta\nu_0$ between its value calculated through GYRE and the scaling relation or the acoustic radius for varying mass, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$

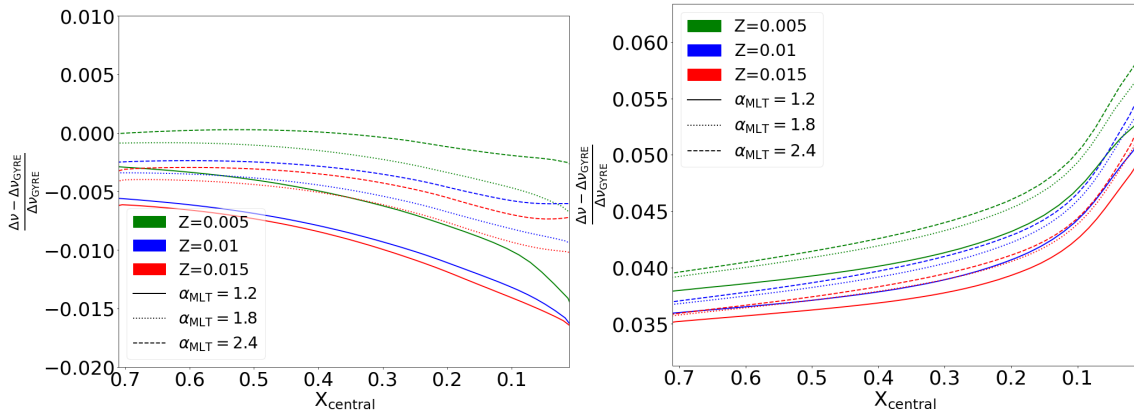


Figure 5.11: Left panel: Relative error in $\Delta\nu_0$ between its value calculated through GYRE and the scaling relation for $M = 0.75M_\odot$ and varying Z and α_{MLT} . Right panel: Relative error in $\Delta\nu_0$ between its value calculated through GYRE and the acoustic radius for $M = 0.75M_\odot$ and varying Z and α_{MLT} .

where the average value of $\Delta\nu$ is used. We will denote the results using this method when plotting by the name "lan", since the author of [75] is named Ian Roxburgh.

We want to see whether the average value of ε obtained through both methods is consistent, and whether the frequency dependence of ε is different in the two methods. In the left panel of Fig. 5.12 we see the relative difference in the value of ε calculated through the two methods. The difference has a maximum of 0.3%, without a strong dependence on l , increasing slightly for higher-mass stars and towards the end of the main sequence. From this we can conclude that there is not any meaningful difference in using one or the other method to calculate the average value of ε , given that the experimental uncertainties connected to it are usually much larger than 0.3%. For example, in the case of ϵ Indi the uncertainty in ε is of 1.3% [6], while in [64] it is of around 5% for most stars.

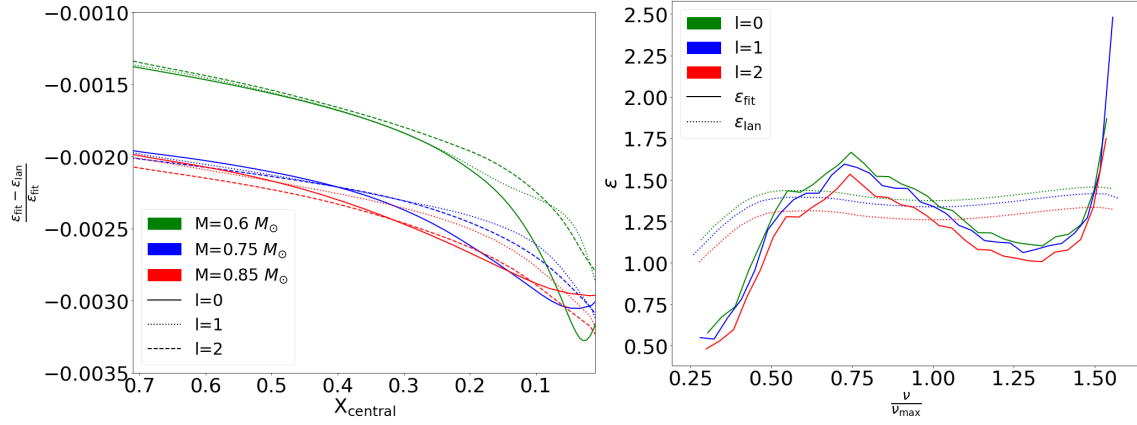


Figure 5.12: Left panel: Relative difference between the value of ϵ calculated through fitting the frequencies and through the method of [75]. Right panel: Value of ϵ over as a function of the normalized frequencies for a model with $M = 0.75 M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ at an age of 5.35 Gyr.

In the right panel of Fig. 5.12 it is shown how the value of ϵ changes with frequency using both methods, where we can see some clear differences, with the method of [75] creating a much smoother and less varying function. The reason for this might be that, since in our method we use a value of $\Delta\nu$ also varying with frequency to calculate $\epsilon(\nu)$, we introduce another cause for fluctuations that is not present in the method of [76].

In Fig. 5.13 we present what is called an ϵ diagram, where ϵ is plotted in the x-axis and $\Delta\nu$ is plotted in the y-axis. In both figures we calculated ϵ through the linear interpolation method. The timeline in the diagrams is from top to bottom, since $\Delta\nu$ will always decrease throughout the main sequence due to the fact that $\Delta\nu \propto \sqrt{\langle\rho\rangle}$ and $\langle\rho\rangle$ is decreasing as the stellar radius increases. It is also clear from the left panel of Fig. 5.13 that the lower-mass models have higher values of $\Delta\nu$, in line with the fact that lower-mass stars are denser.

The behaviour of ϵ is complex, it decreases during the first part of the main-sequence, only to begin increasing again in a later phase. In [64] and [65] we see this kind of behaviour for higher-mass stars, while at lower mass ϵ tends to remain nearly constant. It is also worth noting how the tracks are located to the left of the value of ϵ Indi, a behaviour also seen in [64] and [65], which is due to the surface term introducing an error in the modelled values of ϵ , thereby showing the importance of applying a surface-term correction.

In [75] and [76] it is described how the value of ϵ can be divided into a part that depends on the inner structure of the star, while the other is dependent on the near-surface layers. The right panel of Fig. 5.13 illustrates this by showing how varying Z and α_{MLT} , the former affecting both the inner and surface layers while the latter affecting mostly the near-surface

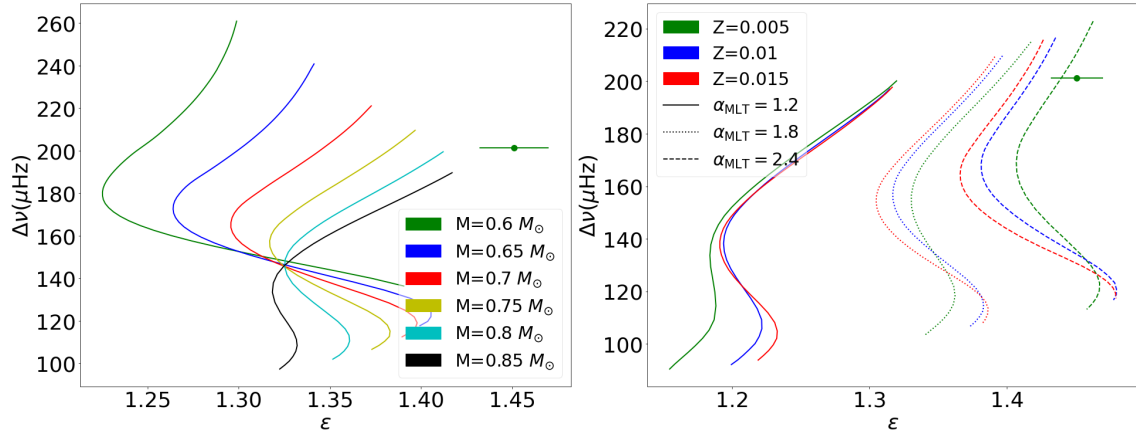


Figure 5.13: Left panel: ϵ diagram for varying masses at $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: ϵ diagram for $M = 0.75 M_\odot$ at varying Z and α_{MLT} . In both diagrams the green line and point indicate the values of ϵ and $\Delta\nu$ for ϵ Indi. The uncertainty on $\Delta\nu$ is so small that it can not be seen in the graph.

layers, has a strong impact on the value of ϵ . We find that higher values of α_{MLT} cause a significant increase in ϵ , while the dependence on Z changes depending on the age and α_{MLT} . This shift in ϵ due to α_{MLT} might be due to the fact that α_{MLT} has a strong impact on the stellar radius, affecting the point at which the modes are reflected.

To conclude this section, we study how $\Delta\nu$ varies with frequency. In the left panel of Fig. 5.14 we represent the shape of the frequency profile at different phases of evolution and l , while in the right panel of Fig. 5.14 for different Z and α_{MLT} . In both panels we plot normalized versions of $\Delta\nu_l(\nu)$ and ν , divided by the average value of $\Delta\nu$ and ν_{max} , respectively, so that we are able to compare the profiles of models with different values of $\Delta\nu$. What we can conclude from both panels is that the general behaviour of $\Delta\nu_l(\nu)$ does not depend strongly on any of the parameters mentioned before, not having significant changes when we change l and X_{central} in the left panel of Fig. 5.14 or Z and α_{MLT} in its right panel. This shape is in agreement with the results of [77] and [78], if we take into account that the latter is only focusing on the frequencies around ν_{max} , where $\Delta\nu_l(\nu)$ increases.

From the asymptotic relation we would expect $\Delta\nu$ to be constant over frequency, but this is not so due to the fact that each mode is sensitive to different parts of the star and due to acoustic glitches, which are sudden discontinuities in a star's inner structure. [77] and [79] mention the base of the convective envelope and the second Helium ionization zone as acoustic glitches that lead to a frequency dependence of $\Delta\nu$, introducing an oscillatory behaviour. The study of this oscillatory component is important, as it can give us

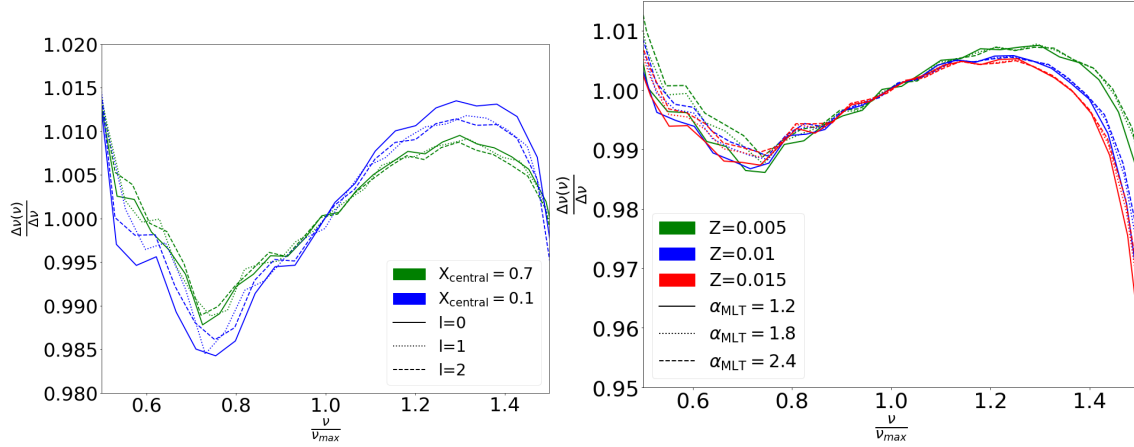


Figure 5.14: Left panel: $\Delta\nu$ as a function of frequency for a track with $M = 0.75M_{\odot}$, $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$ at different phases along the main sequence and for different l . Right panel: $\Delta\nu_0$ as a function of frequency for models with $M = 0.65M_{\odot}$.

information about the depth and width of the acoustic glitches.

5.3.2 $\delta\nu_{02}$ and $\delta\nu_{01}$

To determine the value of $\delta\nu_{02}(\nu)$ and $\delta\nu_{01}(\nu)$ we used their definition, Eq. (3.15) and Eq. (3.16), respectively, then assigned as their value the average spacing of the modes involved in the calculation. When computing the average separation we performed a weighted average of $\delta\nu_{02}(\nu)$ and $\delta\nu_{01}(\nu)$, using the same weights as for $\Delta\nu$ and ε , which we described in section. 5.3.

The C–D diagram, named after Christensen-Dalsgaard, and sometimes called the “Asteroseismic HR diagram”, plots the small frequency separation $\delta\nu_{02}$ against the large frequency separation $\Delta\nu$, and it has been used to determine stellar properties such as the mass and age. In Fig. 5.15 we display examples for models in our grid. We also present in Fig. 5.16 the evolution of $\delta\nu_{02}$, in order to gain insight regarding how it is affected by the grid parameters.

We see that $\delta\nu_{02}$ and $\Delta\nu$ follow a linear relationship for most of the track, a relation which was already noted by [80] and [81] using real data from Kepler. Just like $\Delta\nu$, $\delta\nu_{02}$ tends to decrease with time, which can be understood through the fact that $\delta\nu_{02} \propto \frac{dc_s}{dr}$, and the gradient of the sound speed will decrease due to the steeper chemical gradient as more Helium is produced in the core. In some cases its values even become negative, which [82] attributes to a helium-rich core. Towards the end of the main sequence this behaviour changes, however, as $\delta\nu_{02}$ begins to increase while $\Delta\nu$ decreases, a behaviour which is

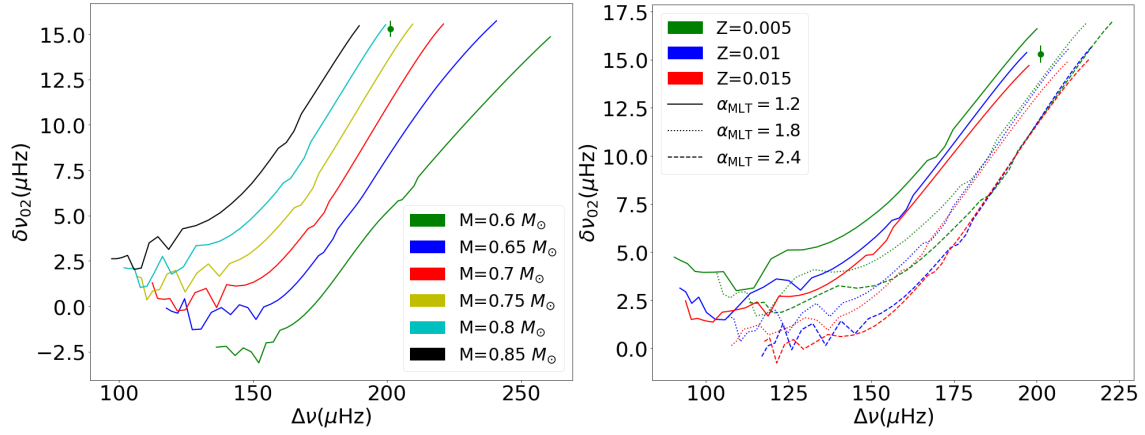


Figure 5.15: Left panel: C-D diagram for varying masses at $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: C-D diagram for $M = 0.75 M_{\odot}$ at varying Z and α_{MLT} . In both panels the green line and point indicate the values of $\delta\nu_{02}$ and $\Delta\nu$ for ϵ Indi, with the uncertainty on $\Delta\nu$ being too small to be seen.

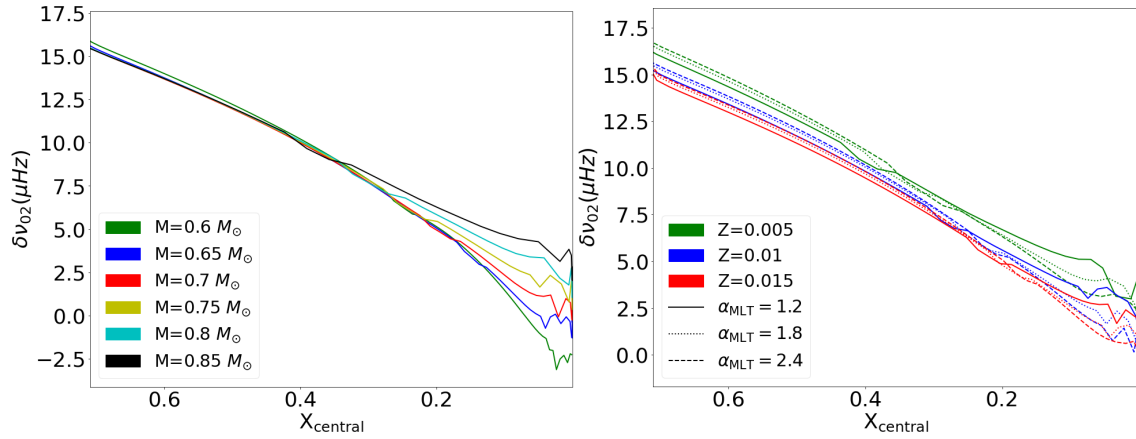


Figure 5.16: Left panel: Evolution of $\delta\nu_{02}$ along the main sequence with $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: Evolution of $\delta\nu_{02}$ along the main sequence with $M = 0.75 M_{\odot}$ and varying Z along with α_{MLT} .

also found in [83] and [64]. This diminishes the predictive power of $\delta\nu_{02}$ for age towards the end of the main sequence, as a one-to-one correspondence between $\delta\nu_{02}$ and age along a particular track no longer exists at that point. The results from [50] do not show this increasing trend in the value of $\delta\nu_{02}$ towards the end of the main sequence, but this is most likely due to the fact that they do not run their tracks up to evolutionary phases as advanced as we do, since with $X_{\text{central}} = 10^{-7}$ being our stopping condition we will already see signs of entering into the subgiant branch.

We now proceed with the study of the frequency dependence of $\delta\nu_{02}$. In the left panel of Fig. 5.17 we want to see how the frequency profile changes with mass and age, while in the right panel of Fig. 5.17 on Z and α_{MLT} . The universal feature that we find across all models

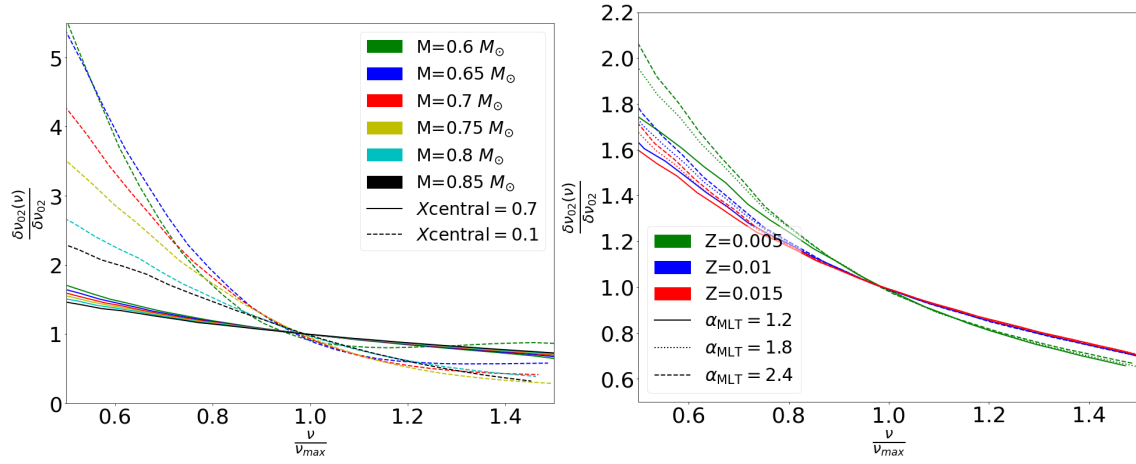


Figure 5.17: Left panel: $\delta\nu_{02}(\nu)$ with $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: $\delta\nu_{02}(\nu)$ with $M=0.75M_{\odot}$ and varying Z along with α_{MLT} .

is the decrease in $\delta\nu_{02}$ as we go towards higher frequencies, with an approximately linear shape. We find the same behaviour in the results of [64], [77] and [84], with the differences being with regard to the value of the slope and whether the shape of the function begins to transition to a more exponential shape. From our results in the left panel of Fig. 5.17 we can clearly see how that for more advanced ages and higher masses the slope of $\delta\nu_{02}(\nu)$ is much lower, and from the right panel of Fig. 5.16 that the slope decreases at lower metallicities. Another common aspect is that the departure from linearity is more noticeable as we move away from $\nu_{\max}$, especially towards lower frequencies.

From Eq. (3.15) we see that $\delta\nu_{02} \propto \frac{1}{\nu}$, which explains why $\delta\nu_{02}$ decreases with frequency. The reason for why the magnitude of the slope in $\delta\nu_{02}(\nu)$ is much higher as we move towards older ages is that higher-frequency modes are more sensitive to the core, and the differences between the properties of the core and the surface will become larger due to nuclear fusion and diffusion.

We will now look at $\delta\nu_{01}$. In Fig. 5.18 we see how it evolves along the main sequence. What we can notice is that the shape of the curves is very similar to what we saw for $\delta\nu_{02}$ in Fig. 5.16, only that the magnitude is divided 3. This is not a coincidence, since we know that in the asymptotic limit $\delta\nu_{01} = \frac{1}{3}\delta\nu_{02}$ [85].

The study of $\delta\nu_{01}(\nu)$ is done in Fig. 5.19. Once again we can see some similarities to $\delta\nu_{02}$ with a decreasing trend as we move towards higher frequencies, being approximately linear near $\nu_{\max}$. Also like for $\delta\nu_{02}$, the slope is larger as the star becomes older and less metallic.

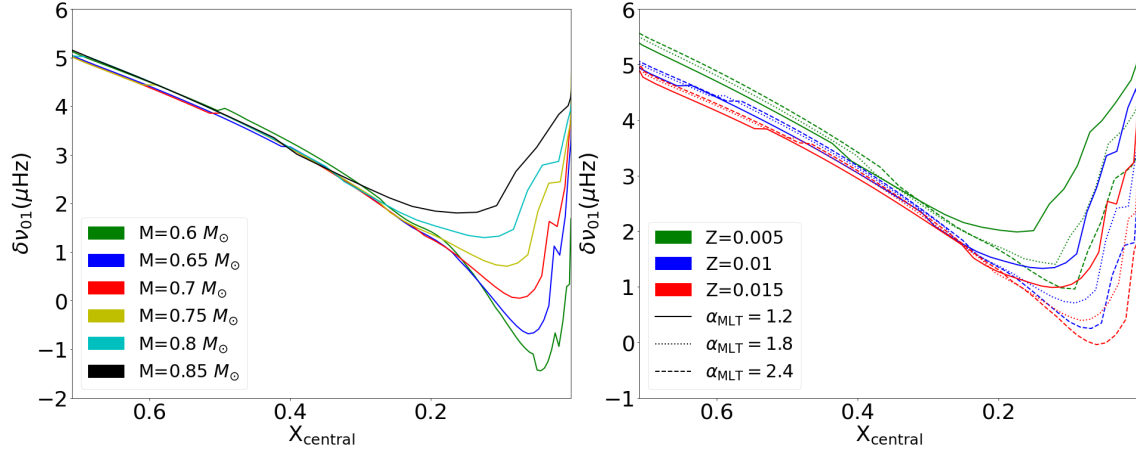


Figure 5.18: Left panel: Evolution of $\delta\nu_{01}$ along the main sequence with $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: Evolution of $\delta\nu_{01}$ along the main sequence with $M = 0.75 M_{\odot}$ and varying Z along with α_{MLT} .

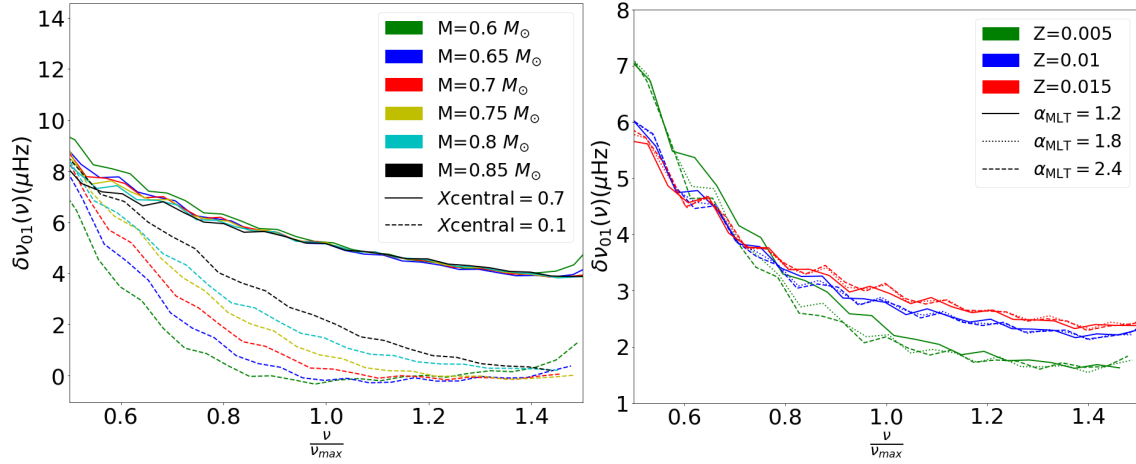


Figure 5.19: Left panel: $\delta\nu_{01}(\nu)$ with $Z = 0.01$ and $\alpha_{\text{MLT}} = 1.8$. Right panel: $\delta\nu_{01}(\nu)$ with $M = 0.75 M_{\odot}$ and varying Z along with α_{MLT} .

Our results are similar to those of [86], which we would be able to see more clearly if we had plotted $\delta\nu_{01}(\nu) \cdot \nu$ as they did. They point to an interesting property of $\delta\nu_{01}(\nu)$, which is the small oscillatory modulation in its value, that in our case can be seen more clearly in the left panel of Fig. 5.19, and that is not present in $\delta\nu_{02}(\nu)$. In [86] they conclude that the modulation is due to the sharp change of acoustic variables at the base of the convective envelope, with the value of the oscillation period being related to its acoustic radius. This motivates the use of $\delta\nu_{01}(\nu)$ as a probe of the inner structure of a star.

6. Results from model optimization

In this chapter we compare our grid of stellar tracks to the observational data for ϵ Indi, using both classical and seismic observations. The comparison is done through an optimization procedure, which tries to minimize the difference between the parameters of the models and the observational constraints of ϵ Indi. This is done with two major research goals in mind.

The first, more directly connected to the process of optimization itself, is to determine in a precise way the global properties of ϵ Indi, most importantly its mass, age and value of α_{MLT} . These can help us situating ϵ Indi within our grid, so that we can apply what we learned in Chapter 5 about the inner structure of K-type stars to ϵ Indi. It also enables us to see the importance of using seismic observational constraints.

The second is the study of the surface term and its correction. We will want to assess its magnitude and shape over the observed radial orders, along with an evaluation of the quality of the different available corrections.

The task of model optimization and implementation of the surface-term correction will be done using a program called AIMS (Asteroseismic Inference on a Massive Scale) [12], which is why we will begin this chapter by describing it, followed by our results and analysis.

6.1. Stellar optimization software – AIMS

AIMS is a program written in `python` whose main goal is to estimate stellar parameters with robust error bars and intervals, requiring for this a pre-computed grid of models, from MESA and GYRE or other stellar modelling software, and a list of observational constraints. It is a free and open-source project with a modular approach, in order to facilitate contributions from different people [87].

The estimation of stellar parameters is done through Bayesian statistics, and using Monte Carlo Markov Chain (MCMC) methods to search for optimal models within the grid along with calculating the probability distribution functions of the stellar parameters. The MCMC methods are based on the `python` package `emcee.py` [88], and AIMS allows for the customization of how the process is done by allowing the choice of the number of walkers

used, the number of starting points from which it will begin searching for optimal models, the “temperature”, with higher temperatures meaning that the walkers will explore a broader range in parameter space, and the number of steps of the walkers. The walkers can be initialized either through the prior probability distribution of the parameter space, or according to a tight ball, where AIMS performs an initial scan of the grid and then starts the walkers near the first estimate for the optimal model [87]. It was the latter option that was used in this work.

The way that the Bayesian statistics algorithm works is that AIMS tries to find the probability of a set of stellar parameters Θ taking into account the observational constraints I , which is given by the posterior probability function $p(\Theta|I)$. It can then be calculated using Bayes’ theorem by

$$p(\Theta|I) \propto p(I|\Theta) p(\Theta), \quad (6.1)$$

where $p(\Theta)$ will be the priors and $p(I|\Theta)$ the likelihood function.

The priors are set in the `AIMS-configure.py` file, with the possibility of using uniform, Gaussian or truncated Gaussian priors. In our work the only prior we set was relative to the age, so that we would not constrain the freedom of AIMS in exploring the parameter space, where we used a uniform prior between 0 and the age of the universe.

The likelihood function is given through a reduced chi-square statistic,

$$p(I|\Theta) = \frac{1}{(2\pi)^{1/2}\sqrt{C}} \cdot e^{-\chi^2/2}, \quad (6.2)$$

where C is the covariance matrix of the observed parameters and χ^2 is given by

$$\chi^2 = \sum_{i=1}^N \frac{(I_i - \theta_i)^2}{\sigma_i^2}, \quad (6.3)$$

with I_i being an observed value, θ_i a modelled value and σ_i the corresponding observational uncertainty.

Another important feature of AIMS is model interpolation, which permits AIMS to fill some of the gaps in the grid of models that is used for optimization. AIMS has the ability to do both interpolation along a stellar track, which means interpolating between models of different

Table 6.1: Options for the surface correction in AIMS. The first column shows the name and reference in which it first appeared, the second column shows the way it is implemented in AIMS and the third column shows the free parameters in the correction.

Ball2014 (from [40])	$\delta\nu = \nu_{\text{char}} a_3 \frac{\nu_{\text{norm}}^3}{I}$	a_3
Ball2014-2 (from [40])	$\delta\nu = \nu_{\text{char}} \left(a_3 \frac{\nu_{\text{norm}}^3}{I} + a_{-1} \frac{\nu_{\text{norm}}^{-1}}{I} \right)$	a_3 and a_{-1}
Sonoi2015 (from [42])	$\delta\nu = \nu_{\text{char}} \alpha \left(1 - \frac{1}{1 + (\nu/\nu_{\text{max}})^4} \right)$	α
Kjeldsen2008 (from [90])	$\delta\nu = \nu_{\text{char}} a_{\text{surf}} (\nu_{\text{norm}})^{4.9}$	a_{surf}

ages, and interpolation across different stellar tracks, creating models with parameters that were not in the original grid. It is able to interpolate both the mode frequencies and the classical variables of the models [89].

AIMS has the option of implementing a surface-term correction, which it will calculate and apply to each of the modes in every model. They are computed in terms of the normalized frequencies $\nu_{\text{norm}} = \frac{\nu}{\frac{10^6}{2\pi} \sqrt{\frac{GM}{R^3}}}$, where we will call $\nu_{\text{char}} = \frac{10^6}{2\pi} \sqrt{\frac{GM}{R^3}}$, the characteristic frequency of a model in μHz . These different options are shown in Table 6.1.

The surface corrections in Table 6.1 are the functions that we saw in Eq.(3.22), Eq.(3.21) and Eq.(3.23), with some minor changes in how the modes are normalized. The parameters a_3 , a_{-1} , α and a_{surf} will be fitted by AIMS as it performs model optimization. There was also the option for letting b in Eq. (3.21) and β in Eq. (3.23) as free parameters, but we found that this did not give good results, possibly due to there not being enough observed modes in ϵ Indi. AIMS chose values of nearly 0 for b and β and with a very high uncertainty, which corresponds to applying a constant surface correction across all modes.

AIMS also allows choosing the way in which the seismic and classical variables are weighted. In the "Absolute" option the weight of each mode frequency is equal to that of a classical variable, while in the "Relative" option we can choose the importance we want to give to the classical and seismic variables as a whole. In this work we used both options, with the Relative option set to give equal weight to the modes and the classical variables, to see if it had an impact on the results. The correct way of weighting the classical and seismic variables is still a topic of debate [60].

What AIMS will output after finishing a run are the probability distribution functions for each of the parameters, the average and median of these distributions along with the

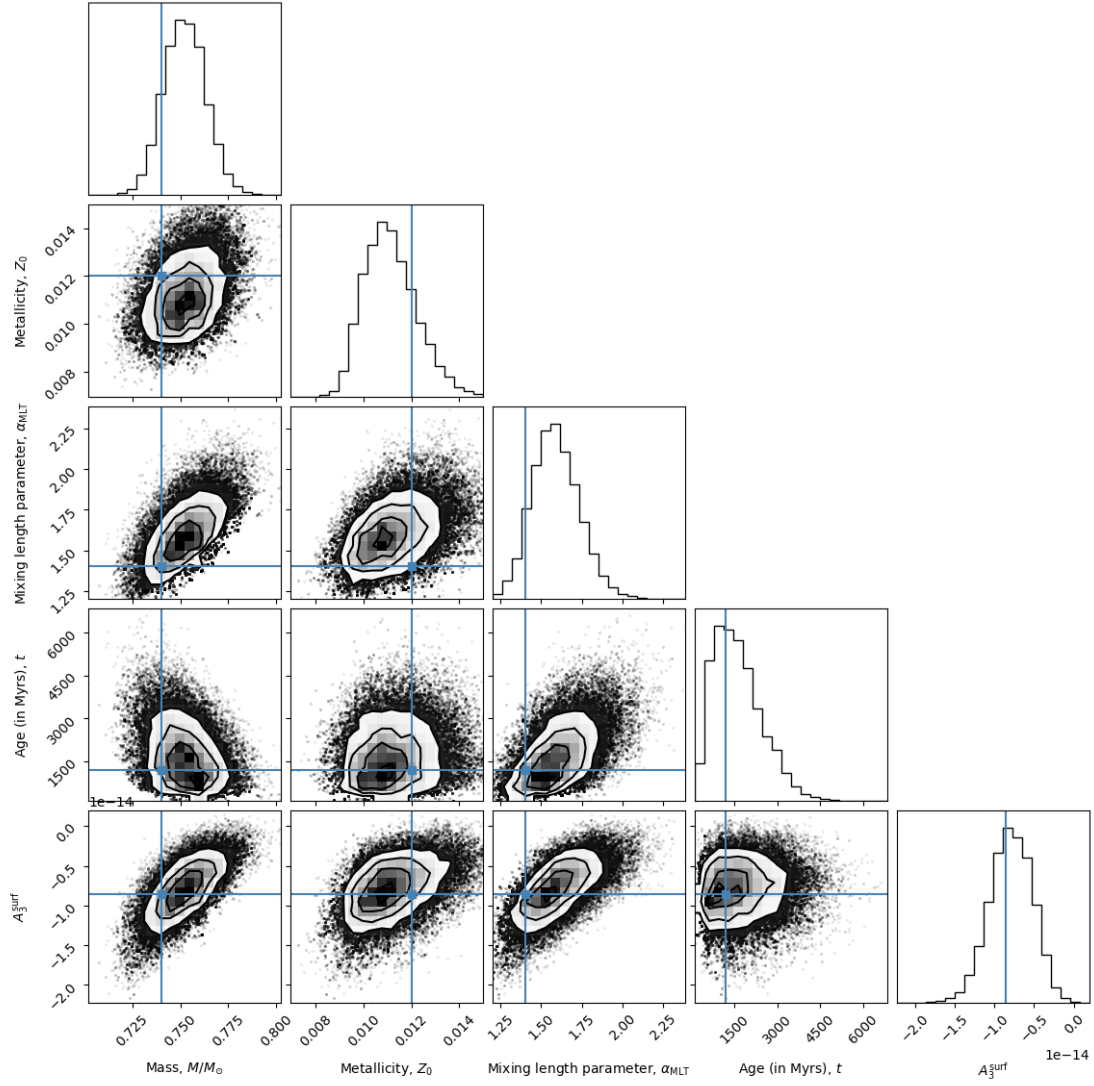


Figure 6.1: Correlation matrix and probability distributions for mass, Z , α_{MLT} , age and a_3 using the Ball2014 correction.

corresponding standard deviation and percentiles, the correlation between parameters, such as in Fig. 6.1, the échelle diagram and features of the best models. It is these results that we will describe in the following section.

6.2. Optimized properties of ϵ Indi

As observational constraints for AIMS we used the asteroseismic modes found in [6], the effective temperature and interferometric radius from [27] and the metallicity from [29]. We did not use information regarding the mass and the age since these were the main properties that we wanted to determine, nor the luminosity since it would be correlated with the radius and effective temperature via Eq. (2.1). The results obtained with AIMS for

Table 6.2: Average value and error bar (standard deviation) of the optimized parameters for the Ball2014 and Ball2014-2 surface corrections.

	Ball2014-A	Ball2014-R	Ball2014-2-A	Ball2014-2-R
Mass($M_{\odot}$)	0.752 ± 0.011	0.753 ± 0.011	0.752 ± 0.011	0.75 ± 0.011
Z	0.0111 ± 0.001	0.0110 ± 0.0011	0.0111 ± 0.0011	0.0107 ± 0.001
α_{MLT}	1.58 ± 0.14	1.57 ± 0.17	1.59 ± 0.15	1.47 ± 0.16
Age(My)	1617 ± 830	1524 ± 1145	1674 ± 833	1402 ± 1177
Radius($R_{\odot}$)	0.698 ± 0.0035	0.699 ± 0.0036	0.698 ± 0.0036	0.700 ± 0.0037
$T_{\text{eff}}(\text{K})$	4804 ± 69	4811 ± 70	4805 ± 71	4786 ± 68

Table 6.3: Average value and error bar (standard deviation) of the optimized parameters for the Kjeldsen2008 and Sonoi2015 surface corrections.

	Kjeldsen2008-A	Kjeldsen2008-R	Sonoi2015-A	Sonoi2015-R
Mass($M_{\odot}$)	0.754 ± 0.011	0.754 ± 0.011	0.755 ± 0.011	0.755 ± 0.011
Z	0.0112 ± 0.0011	0.011 ± 0.0011	0.0113 ± 0.0011	0.0114 ± 0.0011
α_{MLT}	1.62 ± 0.14	1.58 ± 0.17	1.64 ± 0.14	1.61 ± 0.18
Age(My)	1635 ± 828	1490 ± 1125	1645 ± 818	1532 ± 1114
Radius($R_{\odot}$)	0.699 ± 0.0034	0.699 ± 0.0035	0.698 ± 0.0036	0.699 ± 0.0037
$T_{\text{eff}}(\text{K})$	4818 ± 67	4815 ± 68	4820 ± 69	4823 ± 71

the grid parameters are shown in Table 6.2 and Table 6.3, where the A and R refer to whether absolute or relative weighting was used, as described in the previous section.

One of the important conclusions that we can arrive at from Table 6.2 and Table 6.3 is that the different surface corrections give consistent results in model optimization. The differences in the values are all well within the error bars, with the largest divergence from the others being for the Ball2014-2-R run. We can also see that using Absolute or relative weighting gives compatible results, which makes our results more robust, since the choice of using one or the other method introduces some ambiguity to the optimization procedure. Nevertheless, we can see a clear trend that using relative weighting tends to decrease the values for α_{MLT} and the age, while increasing their uncertainty. Given that we have 19 observed modes and 3 classical variables, using relative weighting will give more importance to the classical variables than when using absolute weighting, which suggests that it is the value of one of the classical variables that is producing this effect. The most likely cause is the radius, since it is classical observable with the highest level of precision, with an uncertainty of 0.7%, while the uncertainty in the effective temperature is of 1.8%.

The value for the mass is in close agreement with that of [28], while for the age, radius and T_{eff} there are differences with respect to some of the values in the literature. With regard to the age we are obtaining lower values for ϵ Indi than those of [25], $3.5^{+0.8}_{-1.0}$ Gyr,

although they used a very different method to ours, based on the activity indices of the star. Different authors have determined lower ages for ϵ Indi, however, such as [24], that gave an age between 1 Gyr and 2.7 Gyr using a combination of different methods to obtain it. In [6], through inspection of a C-D diagram, they also estimate ϵ Indi to be quite young, younger than 4 Gyr. They also use a C-D diagram to estimate the mass of ϵ Indi as being around $0.75M_{\odot}$, showing the potential of the type of diagram displayed in Chapter 5 to determine stellar properties.

Regarding the radius and effective temperature, in spite of using them as observational constraints, we obtained a radius smaller by about $0.01R_{\odot}$ and a higher T_{eff} by about 150 K, although their uncertainties still overlap. This is in line with what is discussed in [28] and [59], that in K-type stars models tend to underestimate the stellar radius by about 5% to 15% and overestimate T_{eff} by 3-5%, although we have errors of about 1.7% for the radius and 3.2% for T_{eff} .

The uncertainties for the mass, Z , T_{eff} and the radius were all quite low, under 2%, while for α_{MLT} and the age these were of around 9% and 50%, respectively, when using absolute weighing. The low density of points in the grid for α_{MLT} is the most likely causes for its high uncertainty, that would have been reduced if we had used a step size of 0.1 instead of 0.2. The uncertainty for the age is quite high, but we note that it is difficult to obtain uncertainties for the age lower than 20% [91], and that this is even more difficult for stars near the beginning of the main sequence. Another factor that limited the precision on age is that, while during the main sequence we saved a model at every 60 million years, the profiles with mode frequencies were saved only at intervals of 10 models, so that there was a gap of 600 million years between each profile with mode frequencies.

In order to focus more on what the mode frequencies themselves tell us about the star and to assess the impact of the radius constraint, we decided to run AIMS without using the radius as an observational constraint. The results are shown in Table 6.4, where absolute weighting was used.

We notice how ignoring the radius constraint has resulted in lowering the mass, α_{MLT} , the radius and T_{eff} by significant amounts, while strongly increasing both the value and the uncertainty on the age. This implies that the radius is an important parameter in constraining the age of the star, and that its value pushes the star towards being younger in comparison to the mode frequencies. This is what we already saw in Table 6.2 and Table 6.3 where the relative weighting resulted in younger values for the age. We can

Table 6.4: Optimization results without using the radius as an observational constraint and with absolute weighting.

	Ball2014	Ball2014-2	Kjeldsen2008	Sonoi2015
Mass($M_{\odot}$)	0.720 ± 0.013	0.719 ± 0.013	0.724 ± 0.012	0.719 ± 0.014
Z	0.010 ± 0.001	0.010 ± 0.001	0.010 ± 0.001	0.010 ± 0.001
α_{MLT}	1.36 ± 0.12	1.36 ± 0.12	1.39 ± 0.13	1.39 ± 0.12
Age(Myrs)	2809 ± 1157	2858 ± 1204	2568 ± 1087	2896 ± 1182
Radius($R_{\odot}$)	0.687 ± 0.004	0.687 ± 0.004	0.689 ± 0.004	0.686 ± 0.005
T_{eff} (K)	4674 ± 71	4669 ± 70	4693 ± 72	4675 ± 73

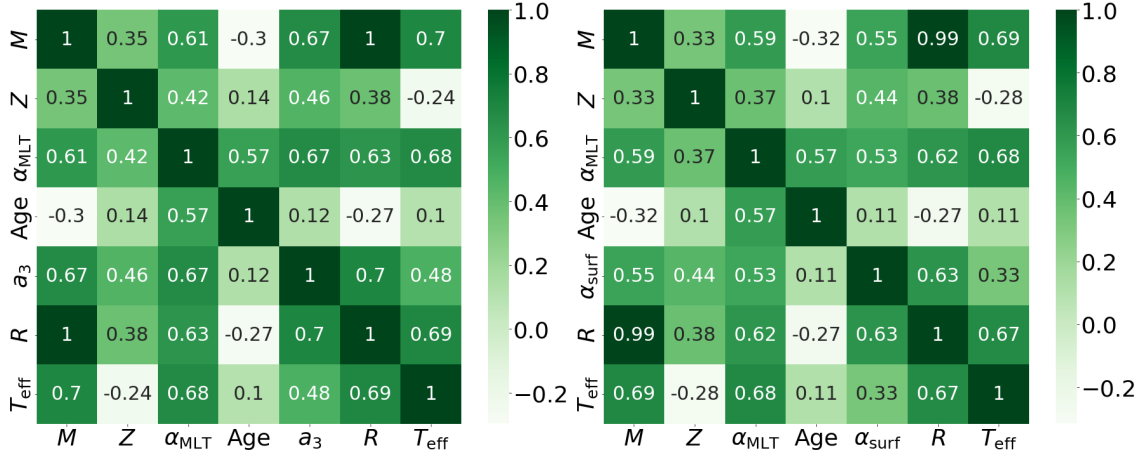


Figure 6.2: Left panel: Correlation matrix for the Ball2014 surface correction using absolute weighting. Right panel: Correlation matrix for the Sonoi2015 surface correction using absolute weighting.

understand this based on the results of the previous chapter, where we saw that R had a monotonic increase throughout the main sequence, thus making it a good parameter to constrain age, along with the left panel of Fig. 5.3, where we see that ϵ Indi is much closer to the $M - R$ relation at the beginning rather than the end of the main sequence.

In Fig. 6.2 we find the correlation matrix for the Ball2014 and Sonoi2015 surface corrections. The mass is strongly correlated with α_{MLT} , with a possible reason for this being that a higher mass will increase the radius while higher α_{MLT} will decrease it, so that both will need to change in the same direction if we want the radius to remain constant. The age is negatively correlated with mass and positively correlated with α_{MLT} , since increasing α_{MLT} will contract the radius, thereby requiring the star to spend more time in the main sequence to reach the radius of ϵ Indi. The mass has a correlation of 1 with the radius, which is in line with what we see in the left panel of Fig. 5.3, where the radius always increases with mass.

The fitted parameters in the surface corrections a_3 and α all show positive correlations with

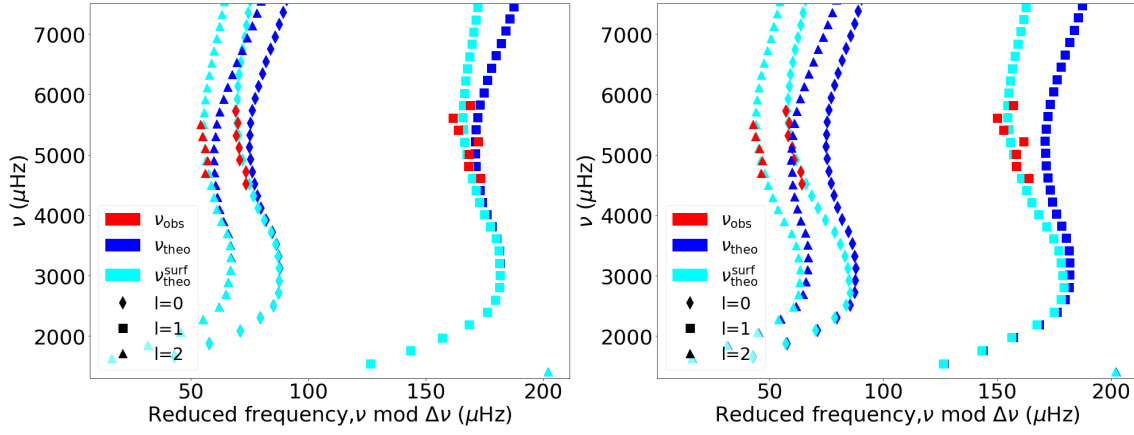


Figure 6.3: Échelle diagram for the best model using the Ball2014 surface correction in the left panel and Sono12015 in the right panel. ν_{obs} are the observed mode frequencies for ϵ Indi, from [6], ν_{theo} are the mode frequencies computed by GYRE and $\nu_{\text{theo}}^{\text{surf}}$ are the ν_{theo} mode frequencies with the surface-term correction applied by AIMS.

mass, Z and α_{MLT} , which, given that they are directly proportional to the magnitude of the correction, indicates that it would be a star with a higher mass, Z and convective efficiency that would have a larger surface correction. The surface term increasing towards larger masses is in line with the results of [14], which indicated that hotter main-sequence stars have a larger surface term.

A way of studying how the surface term behaves is by analysing the échelle diagrams of the best models, which are shown in Figs. 6.3 and 6.4. We notice how in all cases the surface-term correction begins nearly at 0 for low frequencies and then increases, which mirrors the behaviour observed in [13] and [38] for the Sun and other stars. An interesting aspect to notice in the échelle diagram is the scatter in the $l = 1$ modes of ϵ Indi, which we had already pointed out in the right panel of Fig. 3.1, making these difficult to fit for the models. Given that ϵ Indi is too young to have mixed modes and that the $l = 3$ ridge is supposed to be further away, a possible explanation given in [6] for this behaviour is the presence of a strong dipole magnetic field.

All of the surface corrections have a negative value, with the one that stands out being Sono12015, having a larger magnitude than the other surface corrections, which will be analysed closely in the following section where we will deal more directly with the surface term.

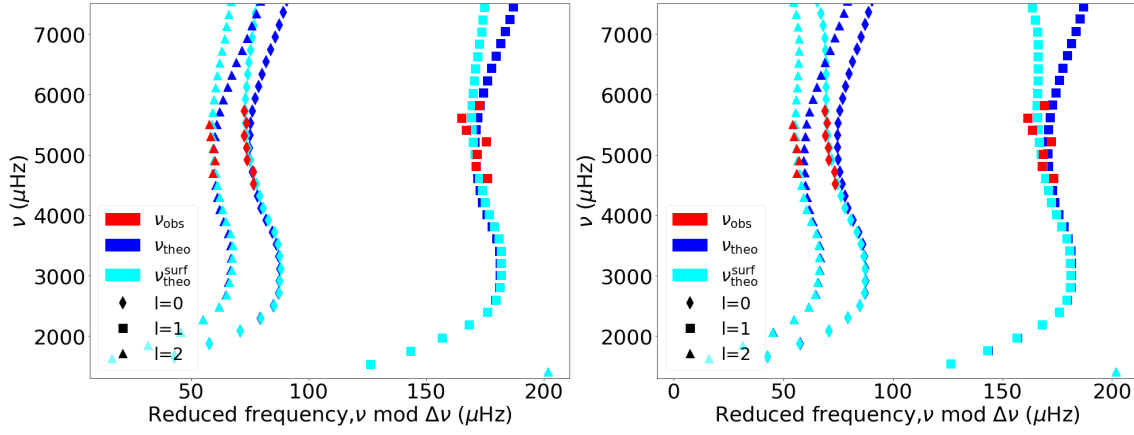


Figure 6.4: Échelle diagram for the best model using the Ball2014-2 surface correction in the left panel and Kjeldsen2008 in the right panel. ν_{obs} are the observed mode frequencies for ϵ Indi, from [6], ν_{theo} are the mode frequencies computed by GYRE and $\nu_{\text{theo}}^{\text{surf}}$ are the ν_{theo} mode frequencies with the surface-term correction applied by AIMS.

6.3. Evaluation of the surface term

In this section we want to characterize the surface term more directly for ϵ Indi and find a way to evaluate the quality of the surface-term corrections used in the previous section. AIMS does not provide a way to do this evaluation, since it does not give us any index to compare the quality of the fit between different runs. So we added the surface correction to the models in our grid and computed directly the χ^2 values between the corrected and observed frequencies using the equation:

$$\chi^2 = \frac{1}{n - m} \sum_i \frac{(\nu_i^{\text{obs}} - \nu_i^{\text{corrected}})^2}{\sigma_i^2}, \quad (6.4)$$

where in our case $n = 19$, since there are 19 observed modes, and m , which is the number of free parameters in the models, will be 1 for the Ball2014, Kjeldsen2008 and Sonoi2015 corrections and 2 for Ball2014-2.

Before analyzing more closely the surface correction, however, we want to know the shape of the surface term itself, i.e, the difference in mode frequencies between the models computed by GYRE or through interpolation and the observations. To do this we plotted the surface term for the best models chosen by AIMS in the left panel of Fig. 6.5 and in the right panel for three representative models in our grid with properties within the range of ϵ Indi, Model A with $M = 0.75M_{\odot}$, $Z = 0.011$, $\alpha_{\text{MLT}} = 1.6$ and an age of 1.76 Gyr, Model B with $M = 0.74M_{\odot}$, $Z = 0.012$, $\alpha_{\text{MLT}} = 1.4$ and an age of 1.19 Gyr and Model C with $M = 0.76M_{\odot}$, $Z = 0.01$, $\alpha_{\text{MLT}} = 1.6$ and an age of 1.149 Gyr.

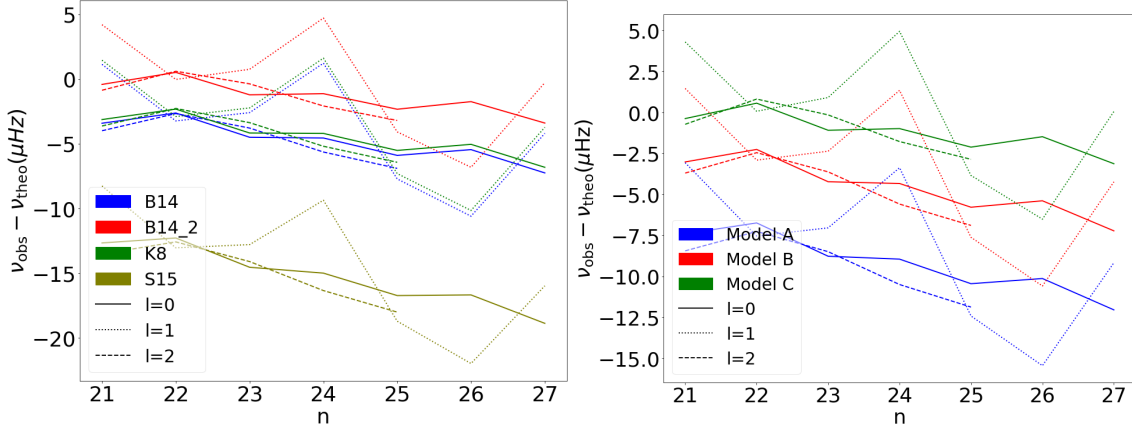


Figure 6.5: Surface term for the best models from AIMS in the left panel and for three representative models from the grid in the right panel.

We can see that the surface-term tends to increase with larger n and to have similar values for different l , which is the expected behaviour as we had already pointed out previously, although to perform a complete study of how the surface term in ϵ Indi changes with n we would need to have more observed radial orders. For $l = 1$ it is less monotonic than for the other angular degrees, with the cause being the scatter in the $l = 1$ modes for ϵ Indi that we observed in the right panel of Fig. 3.1. The shape of the surface term is always the same, as all of the models have a similar dependence with n , in accordance with Eq. (3.11). The magnitude of the error itself has similar values to those found in [13] and [38], that are between $5\mu\text{Hz}$ and $-30\mu\text{Hz}$.

In Fig. 6.6 and Fig. 6.7 we show some of the results from computing the χ^2 values for the corrected mode frequencies in our grid. The goal is to find a surface correction that consistently gives higher χ^2 values, indicating that it performed worse than the others.

The general shape for these curves is that they tend to have a sharp minimum and then to rise rapidly, with the minimum corresponding to the best value of the age for the properties of that evolutionary track. We can see in the left panel of Fig. 6.6 that each value of the mass has a minimum at a different age, with lower mass stars having the minimum at higher ages, which is expected as lower-mass stars evolve more slowly and since we saw that the mass is negatively correlated with the age in Fig. 6.2. For $M = 0.75M_{\odot}$ the minimum is at around 1.6 Gyr, which is what we expect from the results in the previous section.

In the right panel of Fig. 6.6 we see that the different values of Z have a minimum at the same age, while in Fig. 6.7 there is a strong impact, with higher α_{MLT} leading to higher

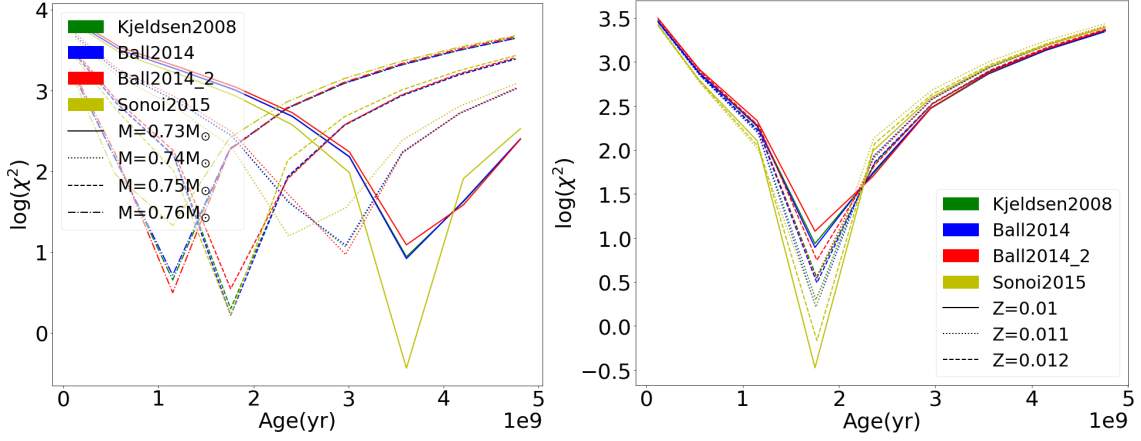


Figure 6.6: Value of $\log(\chi^2)$, see Eq.(6.4), for the different surface corrections at varying mass in the left panel and varying Z in the right panel.

ages. This correlation with age was already found in the previous section, and shows how altering convection in the models has a strong impact on the results from asteroseismic modelling and optimization.

In Fig. 6.6 and Fig. 6.7 we can observe that the different surface corrections generally follow a similar type of behaviour, with Sonoi2015 standing out. Looking at the functional forms of the surface correction in Table 6.1 we can understand this, given that Sonoi2015 uses a modified Lorentzian function while Kjeldsen2008, Ball2014 and Ball2014-2 all use polynomials, making them more similar to each other. The pattern seems to be that the χ^2 values for the different surface corrections tend to be closer at $M = 0.75M_{\odot}$, $Z = 0.011$ and $\alpha = 1.6$, the values that are more representative of ϵ Indi, and as we move away from this Sonoi2015 begins to show an erratic behaviour, both towards having higher and lower values of χ^2 .

Another interesting aspect is that Ball2014-2 does not show a clear improvement over Ball2014, in spite of having two free parameters to fit instead of one. One of the reasons is that in Eq. (6.4) we penalize Ball2014-2 for having two free parameters by increasing the value of m , and another reason might be that the a_3 term is more relevant for ϵ Indi than the a_{-1} term. In [40] they mention that the term proportional to ν^3 is expected to take into account the effect of magnetic fields in the sound speed, while the term proportional to ν^{-1} would take into account the effect of the description of convection on the pressure scale height, so that a higher importance of the ν^3 term could be indicative of a strong effect of the magnetic fields on the surface term.

In Fig. 6.8 we show the values for the surface correction over n , in the left panel for the

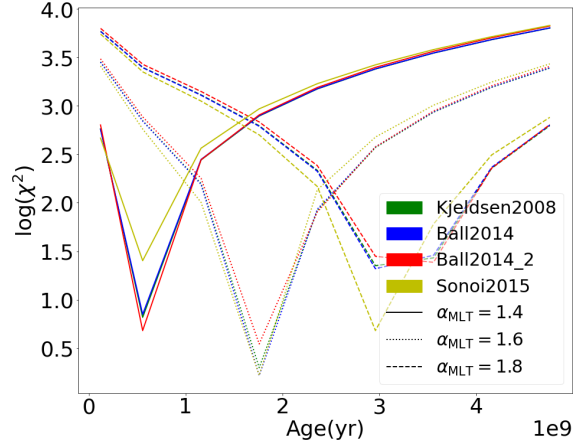


Figure 6.7: Value of $\log(\chi^2)$, see Eq.(6.4), for the different surface corrections at varying α_{MLT} .

best models chosen by AIMS, which come from the results in Section 6.2, while in the right panel the curves represent the average values for the surface correction in the grid models with $\chi^2 < 10$, based on the procedure described in the beginning of Section 6.3. The shaded regions in the right panel of Fig. 6.8 represent the standard deviation around the average value.

Here we can confirm what we already saw in the right panel of Fig. 6.3, that the Sonoi2015 correction has much larger values than the other corrections, especially at the radial orders with observed modes. This also explains why Sonoi2015 would have χ^2 values different from the others, as Sonoi2015 will get better χ^2 values at models with a higher surface term. Ball2014-2 is the only surface correction that takes positive values, due to the fact that the a_{-1} parameter is positive, while in the other surface corrections the multiplicative parameters are always negative. In general, we can say that the shapes of these surface corrections are in accordance with the results of [14], [38] and [59].

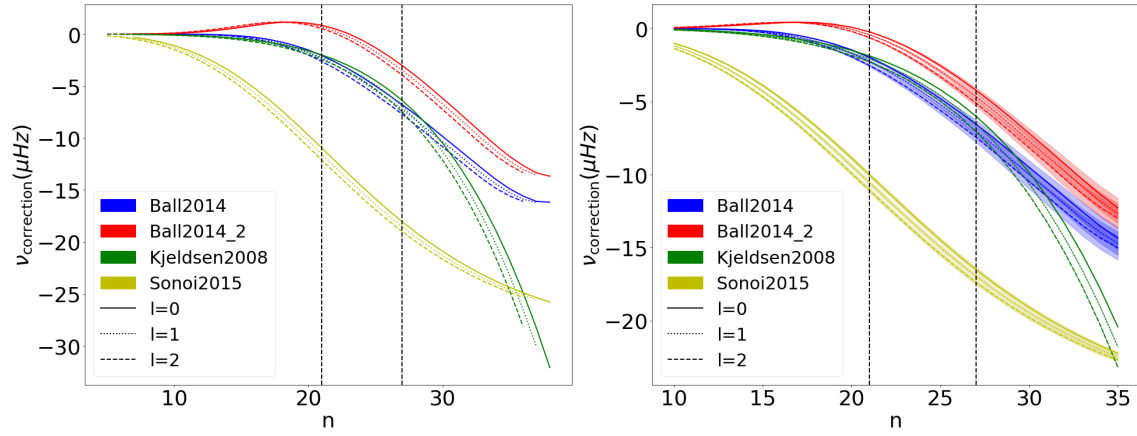


Figure 6.8: Left panel: Surface corrections for the best models chosen by AIMS. Right panel: Average value of the surface correction for the best models from our grid, with the shaded regions representing the standard deviation. The vertical lines represent the radial orders where modes were detected for ϵ Indi.

7. Conclusions and future work

In this work we have combined the use of the most recently available asteroseismic data with advanced stellar modelling tools in order to characterize the properties of the cold dwarf star ϵ Indi along with the asteroseismic properties and inner structure of the K-type stars more similar to it. Furthermore, we have determined the surface-term for ϵ Indi, evaluated the impact of the surface-term correction in model optimization, compared the quality of the different surface term corrections used and did a deep analysis of one of the aspects in the model physics that is a cause of the surface term, the modelling of the atmosphere.

We were able to demonstrate that the choice of the atmospheric model has a significant impact on the properties of the models, causing maximum shifts of about 10% in radius and effective temperature along with 17% differences in the mode frequencies. We also concluded that the impact of the atmospheric model depends on the properties of the star we are modelling, in particular on the mass and age, with higher-mass and older models having differences that were less than half those found in $0.6M_{\odot}$ models at the beginning of the main sequence. In the mass range encompassing ϵ Indi the maximum differences were of about 7% in radius, 6% in effective temperature and 12% in the mode frequencies. Overall, we found that it is as the surface of the star is colder that the impact of the choice of the atmospheric model increases.

A similar, though less exhaustive, analysis of the impact of the boundary conditions when modelling the mode frequencies was performed, with the average differences reaching a maximum of only 0.004%, with the larger impact being towards older ages. The difference also increases with n , in line with the frequency dependence of the surface-term. Since the value of the shift induced by the boundary condition is smaller than the typical observational uncertainty in the modes, we can conclude that it does not have a significant impact on the asteroseismic properties of the models or when performing model optimization.

Regarding the M – R relationship in K-type stars, we saw how the relationship obtained has a linear shape and is very dependent on the evolutionary phase of the star, with an increase in the radius of around 50% between the beginning and the end of the main sequence. We were also able to determine that at older ages the gradient of the M – R

relationship increases along with the impact of the convective efficiency, with higher convective efficiency contracting the radius by about 15% to 20% at the end of the main sequence and only about 3% to 8% at the beginning. Therefore, to create an M - R relationship using models to compare with observed stars we need to match the ages of the models with those of the stars, and take into account the ambiguity introduced by the parameterization of the convective efficiency in the models.

For one of our main goals, the study of the inner structure and asteroseismic properties of K-type stars, there are several interesting conclusions we can point to. We found that the relative size of the convective envelope, the size of the convective envelope divided by the radius, significantly decreases with mass and increases with Z , with the decrease being of about 10% to 40% as we move from $0.6M_{\odot}$ to $0.85M_{\odot}$ and with a 10% increase as we go from $Z = 0.005$ to $Z = 0.015$. We also saw how nuclear fusion has a strong impact on the sound speed, causing it to have a decrease in the core at the end of the main sequence, which is the kind of change in structure that might impact the small frequency separations of a star. This is supported by the propagation diagrams in Fig. 3.1 and Fig. 5.8, where mode frequencies around $\nu_{\max}$ could go down to a fractional radius of 0.1.

Our method for calculating $\Delta\nu$ through performing a linear interpolation of mode frequencies, based on the asymptotic relation, showed maximum differences of 1.5% with respect to the value of $\Delta\nu$ calculated through a scaling relation, Eq. (3.14), and 5% with respect to the value calculated through the acoustic radius, Eq. (3.12). Both of these differences are already significant, given that observational errors in $\Delta\nu$ can be as low as 0.2%. We also saw how our method for computing ε gave results compatible with [75], with maximum differences of about 0.3% in the average value, although with more divergent results in how ε varies with frequency. Both methods showed similar general trends, although ours had higher variations with frequency, which means that it could be more sensitive to the impact of the inner structure on the frequency dependence, such as the impact of acoustic glitches.

Through the creation of C-D diagrams and ε diagrams we have been able to discern that both $\delta\nu_{02}$ and $\Delta\nu$ can act as good constraints on the stellar age, given that both decrease monotonically as the star evolves along the main sequence, and that $\Delta\nu$ can also serve as a strong constraint on the mass. For both it was also possible to separate tracks of different Z and α_{MLT} . We also show that ε can be a useful observational parameter for constraining the convective efficiency of the models, although the stellar tracks in the

ε diagrams were far away from the location of ϵ Indi, thus showing the impact that the surface term has on ε .

Our model optimization results are in agreement with previous determinations of the mass and metal fraction of ϵ Indi, with a 1.5% lower value for the radius than that obtained through interferometry and a 3.5% higher effective temperature, even when we use the radius as an observational constraint. For the age, our results show ϵ Indi as being toward the younger side of previous determinations of its age, at around 1.6 Gyr. We found that including or not the radius as an observational constraint significantly impacts the age, with its value being at around 2.8 Gyr for the latter case.

Using different surface-term corrections and different weighting schemes between classical and seismic variables led to consistent results, i.e, within the uncertainties output by AIMS, thus showing that these choices do not introduce additional ambiguity in the model optimization procedure. The surface correction that showed the most dissimilar behaviour was *Sonoi2015*, being most suited for scenarios where the surface term has significant values even at low radial orders.

The shape of the surface term for ϵ Indi has an approximately monotonic behaviour for $l = 0$ and $l = 2$, but for $l = 1$ it has a more irregular pattern, which is hard to match for both the modelled oscillation frequencies and the surface-term corrections. Thus, we expect that for stars whose mode frequencies show a high degree of scatter or non-linear behaviour in terms of n , that both asteroseismic models and surface-term corrections will have difficulties in matching this behaviour, leading to systematically higher values for the surface term.

There are multiple possibilities for future work based on what was already done here. It would be important to do an extended stellar grid where we also varied the stellar atmosphere, according to the options described in Chapter 4, in which we had already suggested the importance of this task. We would use it to repeat the study of the grid done in Chapter 5 along with the determination of the properties of ϵ Indi and evaluation of the surface-term in Chapter 6, with the goal of seeing the impact of the choice of the atmospheric model. It would also be relevant to extend the mass and Z range of our grid, so that we could perform model optimization for hotter K-type stars in which asteroseismic modes have been detected, such as Kepler-444 and α Cen B.

Another addition would be to do an extensive study of the M - R and M - T_{eff} relations in

K-type stars, by comparing the relations obtained from our grid with observed values for K-type stars. These relations could then also be compared to previous results for the M - R and M - T_{eff} relations in the literature.

References

- [1] E. Michel, A. Baglin, M. Auvergne, C. Catala, R. Samadi, F. Baudin, T. Appourchaux, C. Barban, W. W. Weiss, G. Berthomieu, P. Boumier, M.-A. Dupret, R. A. Garcia, M. Fridlund, R. Garrido, M.-J. Goupil, H. Kjeldsen, Y. Lebreton, B. Mosser, A. Grotzsch-Noels, E. Janot-Pacheco, J. Provost, I. W. Roxburgh, A. Thoul, T. Toutain, D. Tiphène, S. Turck-Chieze, S. D. Vauclair, G. P. Vauclair, C. Aerts, G. Alecian, J. Ballot, S. Charpinet, A.-M. Hubert, F. Lignières, P. Mathias, M. J. P. F. G. Monteiro, C. Neiner, E. Poretti, J. Renan de Medeiros, I. Ribas, M. L. Rieutord, T. Roca Cortés, and K. Zwintz, “CoRoT Measures Solar-Like Oscillations and Granulation in Stars Hotter Than the Sun,” *Science*, vol. 322, no. 5901, p. 558, oct 2008. [Cited on page 1.]
- [2] J. Jackiewicz, “Solar-like oscillators in the kepler era: A review,” *Frontiers in Astronomy and Space Sciences*, vol. 7, 2021. [Online]. Available: <https://www.frontiersin.org/articles/10.3389/fspas.2020.595017> [Cited on page 1.]
- [3] E. Hatt, M. B. Nielsen, W. J. Chaplin, W. H. Ball, G. R. Davies, and B. et al, “Catalogue of solar-like oscillators observed by TESS in 120-s and 20-s cadence,” *Astronomy & Astrophysics*, vol. 669, p. A67, Jan. 2023. [Cited on pages ix, 1, and 2.]
- [4] S. Basu and W. J. Chaplin, *Asteroseismic Data Analysis: Foundations and Techniques*. Princeton University Press, 2017. [Online]. Available: <http://www.jstor.org/stable/j.ctt1vwmgmn> [Cited on pages 1 and 23.]
- [5] W. J. Chaplin and A. Miglio, “Asteroseismology of Solar-Type and Red-Giant Stars,” *Annual Review of Astronomy & Astrophysics*, vol. 51, no. 1, pp. 353–392, Aug. 2013. [Cited on page 1.]
- [6] T. L. Campante and K. et al, “Expanding the frontiers of cool-dwarf asteroseismology with ESPRESSO. Detection of solar-like oscillations in the K5 dwarf Epsilon Indi,” *Astronomy & Astrophysics*, vol. 683, p. L16, Mar. 2024. [Cited on pages ix, xi, 1, 2, 39, 43, 47, 58, 68, 70, 72, and 73.]
- [7] M. Lundkvist, H. Kjeldsen, and T. e. a. Bedding, “Low-amplitude solar-like oscillations in the k5 v star ϵ indi a,” *The Astrophysical Journal*, vol. 964, p. 110, 03 2024. [Cited on pages 1 and 3.]
- [8] F. Pepe, S. Cristiani, R. Rebolo, N. C. Santos, H. Dekker, A. Cabral, P. Di Marcantonio, and F. et al, “ESPRESSO at VLT. On-sky performance and first results,” *Astronomy & Astrophysics*, vol. 645, p. A96, Jan. 2021. [Cited on page 1.]
- [9] T. Kallinger, J. De Ridder, S. Hekker, S. Mathur, B. Mosser, M. Gruberbauer, R. A. García, C. Karoff, and J. Ballot, “The connection between stellar granulation and oscillation as seen by the Kepler mission,” *Astronomy & Astrophysics*, vol. 570, p. A41, Oct. 2014. [Cited on pages 1 and 38.]
- [10] B. Paxton, L. Bildsten, A. Dotter, F. Herwig, P. Lesaffre, and F. Timmes, “Modules for experiments in stellar astrophysics (mesa),” *The Astrophysical Journal Supplement Series*, vol. 192, no. 1, p. 3, Dec. 2010. [Online]. Available: <http://dx.doi.org/10.1088/0067-0049/192/1/3> [Cited on pages 1, 32, and 36.]
- [11] R. H. D. Townsend and S. A. Teitler, “Gyre: an open-source stellar oscillation code based on a new magnus multiple shooting scheme,” *Monthly Notices of the Royal Astronomical Society*, vol. 435, no. 4, p. 3406–3418, Sep. 2013. [Online]. Available: <http://dx.doi.org/10.1093/mnras/stt1533> [Cited on pages 1, 43, and 44.]
- [12] B. M. Rendle, G. Buldgen, A. Miglio, D. Reese, A. Noels, G. R. Davies, T. L. Campante, W. J. Chaplin, M. N. Lund, J. S. Kuszelewicz, L. J. A. Scott, R. Scuflaire, W. H. Ball, J. Smetana, and B. Nsamba, “AIMS - a new tool for stellar parameter determinations using asteroseismic constraints,” *Monthly Notices of the Royal Astronomical Society*, vol. 484, no. 1, pp. 771–786, Mar. 2019. [Cited on pages 1 and 65.]

- [13] J. R. Schmitt and S. Basu, "Modeling the asteroseismic surface term across the hr diagram," *The Astrophysical Journal*, vol. 808, no. 2, p. 123, Jul. 2015. [Online]. Available: <http://dx.doi.org/10.1088/0004-637X/808/2/123> [Cited on pages 2, 29, 30, 72, and 74.]
- [14] W. H. Ball, B. Beeck, R. H. Cameron, and L. Gizon, "MESA meets MURaM. Surface effects in main-sequence solar-like oscillators computed using three-dimensional radiation hydrodynamics simulations," *Astronomy & Astrophysics*, vol. 592, p. A159, Aug. 2016. [Cited on pages ix, 2, 28, 29, 72, and 76.]
- [15] Y. Li, T. R. Bedding, and S. et al, "A prescription for the asteroseismic surface correction," *Monthly Notices of the Royal Astronomical Society*, vol. 523, no. 1, pp. 916–927, jul 2023. [Cited on page 2.]
- [16] G. A. Feiden and B. Chaboyer, "Magnetic inhibition of convection and the fundamental properties of low-mass stars. i. stars with a radiative core," *The Astrophysical Journal*, vol. 779, no. 2, p. 183, Dec. 2013. [Online]. Available: <http://dx.doi.org/10.1088/0004-637X/779/2/183> [Cited on pages 2 and 39.]
- [17] S. G. Engle and E. F. Guinan, "Red dwarf stars: Ages, rotation, magnetic dynamo activity and the habitability of hosted planets," 2011. [Cited on page 2.]
- [18] J. Lillo-Box and Santos, "The kobe experiment: K-dwarfs orbited by habitable exoplanets. project goals, target selection, and stellar characterization," *Astronomy and Astrophysics*, vol. 667, Nov. 2022. [Cited on page 2.]
- [19] E. Kotoneva, C. Flynn, C. Chiappini, and F. Matteucci, "K dwarfs and the chemical evolution of the solar cylinder," *Monthly Notices of the Royal Astronomical Society*, vol. 336, no. 3, pp. 879–891, 11 2002. [Online]. Available: <https://doi.org/10.1046/j.1365-8711.2002.05825.x> [Cited on page 2.]
- [20] "Miscellanea," *Monthly Notices of the Royal Astronomical Society*, vol. 8, no. 1, pp. 16–16, 11 1847. [Online]. Available: <https://doi.org/10.1093/mnras/8.1.16> [Cited on page 3.]
- [21] M. A. C. Perryman and L. et al, "The HIPPARCOS Catalogue," *Astronomy & Astrophysics*, vol. 323, pp. L49–L52, Jul. 1997. [Cited on page 3.]
- [22] Gaia Collaboration, A. Vallenari, and B. et al, "Gaia Data Release 3. Summary of the content and survey properties," *Astronomy & Astrophysics*, vol. 674, p. A1, Jun. 2023. [Cited on page 3.]
- [23] K. G. Stassun and O. et al, "The Revised TESS Input Catalog and Candidate Target List," *The Astronomical Journal*, vol. 158, no. 4, p. 138, Oct. 2019. [Cited on page 3.]
- [24] R. Lachaume, C. Dominik, T. Lanz, and H. J. Habing, "Age determinations of main-sequence stars: combining different methods," *Astronomy & Astrophysics*, vol. 348, pp. 897–909, Aug. 1999. [Cited on pages 3 and 70.]
- [25] M. Chen, Y. Li, T. D. Brandt, T. J. Dupuy, C. V. Cardoso, and M. J. McCaughrean, "Precise Dynamical Masses of Epsilon Indi Ba and Bb: Evidence of Slowed Cooling at the L/T Transition," *The Astronomical Journal*, vol. 163, no. 6, p. 288, Jun. 2022. [Cited on pages 3, 33, and 69.]
- [26] F. Feng, G. Anglada-Escudé, M. Tuomi, H. R. A. Jones, J. Chanamé, P. R. Butler, and M. Janson, "Detection of the nearest Jupiter analogue in radial velocity and astrometry data," *Monthly Notices of the Royal Astronomical Society*, vol. 490, no. 4, pp. 5002–5016, Dec. 2019. [Cited on page 3.]
- [27] A. D. Rains, M. J. Ireland, T. R. White, L. Casagrande, and I. Karovicova, "Precision angular diameters for 16 southern stars with VLT/PIONIER," *Monthly Notices of the Royal Astronomical Society*, vol. 493, no. 2, pp. 2377–2394, 03 2020. [Online]. Available: <https://doi.org/10.1093/mnras/staa282> [Cited on pages viii, 3, and 68.]
- [28] B. O. Demory, D. Ségransan, T. Forveille, D. Queloz, X. Delfosse, and C. Perrier, "Mass-Radius relation of low-mass stars revisited with the VLT," in *15th Cambridge Workshop on Cool Stars, Stellar Systems, and the Sun*, ser. American Institute of Physics Conference Series, E. Stempels, Ed., vol. 1094. AIP, Feb. 2009, pp. 792–795. [Cited on pages viii, 3, 4, 34, 39, 48, 69, and 70.]

- [29] J. Gomes da Silva, N. C. Santos, V. Adibekyan, and S. et al, “Stellar chromospheric activity of 1674 FGK stars from the AMBRE-HARPS sample. I. A catalogue of homogeneous chromospheric activity,” *Astronomy & Astrophysics*, vol. 646, p. A77, Feb. 2021. [Cited on pages viii, 3, 34, and 68.]
- [30] O. Pols, *Stellar Structure and Evolution*. Astronomical Institute Utrecht, September 2011. [Cited on pages ix, 5, 16, and 20.]
- [31] M. J. P. F. G. Monteiro, “Estrutura e evolução estelar,” centro de Astrofísica da Universidade do Porto. [Cited on pages 5 and 48.]
- [32] E. Böhm-Vitense, “Über die Wasserstoffkonvektionszone in Sternen verschiedener Effektivtemperaturen und Leuchtkräfte. Mit 5 Textabbildungen,” *Zeitschrift für Astrophysik*, vol. 46, p. 108, Jan. 1958. [Cited on page 9.]
- [33] I. Hubeny, “Stellar atmospheres theory an introduction,” *Springer*, 2007. [Cited on page 18.]
- [34] J. J. F. G. A. L. Mário João Pires Fernandes Garcia Monteiro, Jorge Filipe S. Gameiro, “Astrofísica estelar,” centro de Astrofísica da Universidade do Porto. [Cited on page 18.]
- [35] R. Trampedach, R. F. Stein, J. Christensen-Dalsgaard, □Nordlund, and M. Asplund, “Improvements to stellar structure models, based on a grid of 3d convection simulations – i. $t(\tau)$ relations,” *Monthly Notices of the Royal Astronomical Society*, vol. 442, no. 1, p. 805–820, Jun. 2014. [Online]. Available: <http://dx.doi.org/10.1093/mnras/stu889> [Cited on page 18.]
- [36] S. Basu, “Global seismology of the sun,” *Living Reviews in Solar Physics*, vol. 13, 08 2016. [Cited on pages ix, 27, and 28.]
- [37] H. Kjeldsen and T. R. Bedding, “Amplitudes of stellar oscillations: the implications for asteroseismology.” *Astronomy & Astrophysics*, vol. 293, pp. 87–106, Jan. 1995. [Cited on page 27.]
- [38] W. H. Ball, “Surface effects in solar-like oscillators,” *EPJ Web of Conferences*, vol. 160, p. 02001, 2017. [Online]. Available: <http://dx.doi.org/10.1051/epjconf/201716002001> [Cited on pages 29, 36, 72, 74, and 76.]
- [39] H. Kjeldsen, T. R. Bedding, and J. Christensen-Dalsgaard, “Correcting Stellar Oscillation Frequencies for Near-Surface Effects,” *The Astrophysical Journal Letters*, vol. 683, no. 2, p. L175, Aug. 2008. [Cited on page 30.]
- [40] W. H. Ball and L. Gizon, “A new correction of stellar oscillation frequencies for near-surface effects,” *Astronomy amp; Astrophysics*, vol. 568, p. A123, Aug. 2014. [Online]. Available: <http://dx.doi.org/10.1051/0004-6361/201424325> [Cited on pages 30, 67, and 75.]
- [41] Y. Osaki and H. Shibahashi, *Progress of Seismology of the Sun and Stars*, 1990, vol. 367. [Cited on page 30.]
- [42] T. Sonoi, R. Samadi, K. Belkacem, H.-G. Ludwig, E. Caffau, and B. Mosser, “Surface-effect corrections for solar-like oscillations using 3d hydrodynamical simulations: I. adiabatic oscillations,” *Astronomy amp; Astrophysics*, vol. 583, p. A112, November 2015. [Online]. Available: <http://dx.doi.org/10.1051/0004-6361/201526838> [Cited on pages 30 and 67.]
- [43] B. Paxton, M. Cantiello, P. Arras, L. Bildsten, E. F. Brown, A. Dotter, C. Mankovich, M. H. Montgomery, D. Stello, F. X. Timmes, and R. Townsend, “Modules for experiments in stellar astrophysics (mesa): Planets, oscillations, rotation, and massive stars,” *The Astrophysical Journal Supplement Series*, vol. 208, no. 1, p. 4, Aug. 2013. [Online]. Available: <http://dx.doi.org/10.1088/0067-0049/208/1/4> [Cited on pages 32, 36, and 37.]
- [44] R. Kuhfuss, “A model for time-dependent turbulent convection,” *Astronomy & Astrophysics*, vol. 160, no. 1, pp. 116–120, May 1986. [Cited on page 34.]
- [45] M. Asplund, N. Grevesse, A. J. Sauval, and P. Scott, “The Chemical Composition of the Sun,” *Annual Review of Astronomy & Astrophysics*, vol. 47, no. 1, pp. 481–522, Sep. 2009. [Cited on page 34.]

- [46] R. H. Cyburt, B. D. Fields, and K. A. Olive, "Primordial nucleosynthesis in light of wmap," *Physics Letters B*, vol. 567, no. 3–4, p. 227–234, Aug. 2003. [Online]. Available: <http://dx.doi.org/10.1016/j.physletb.2003.06.026> [Cited on page 34.]
- [47] N. Moedas, D. Bossini, M. Deal, and B. Campilho, "Atomic Diffusion and Turbulent Mixing in Solar-Like Stars: Impact on the Fundamental Properties of FG-Type Stars," in *The 21st Cambridge Workshop on Cool Stars, Stellar Systems, and the Sun*, ser. Cambridge Workshop on Cool Stars, Stellar Systems, and the Sun, Dec. 2022, p. 75. [Cited on page 34.]
- [48] A. S. Jermyn, E. B. Bauer, J. Schwab, R. Farmer, W. H. Ball, E. P. Bellinger, A. Dotter, M. Joyce, P. Marchant, J. S. G. Mombarg, W. M. Wolf, T. L. Sunny Wong, G. C. Cinquegrana, E. Farrell, R. Smolec, A. Thoul, M. Cantiello, F. Herwig, O. Toloza, L. Bildsten, R. H. D. Townsend, and F. X. Timmes, "Modules for Experiments in Stellar Astrophysics (MESA): Time-dependent Convection, Energy Conservation, Automatic Differentiation, and Infrastructure," *The Astrophysical Journal Supplement*, vol. 265, no. 1, p. 15, Mar. 2023. [Cited on pages ix and 35.]
- [49] M. Deal, M. J. Goupil, J. P. Marques, D. R. Reese, and Y. Lebreton, "Chemical mixing in low mass stars. I. Rotation against atomic diffusion including radiative acceleration," *Astronomy & Astrophysics*, vol. 633, p. A23, Jan. 2020. [Cited on page 35.]
- [50] N. Gai, S. L. Bi, Y. K. Tang, and L. H. Li, "Asteroseismic study of helium and heavy element diffusion in solar-type stars," *Astronomy & Astrophysics*, vol. 508, no. 2, pp. 849–854, Dec. 2009. [Cited on pages 35 and 62.]
- [51] C. A. Iglesias and F. J. Rogers, "Updated Opal Opacities," *The Astrophysical Journal*, vol. 464, p. 943, Jun. 1996. [Cited on page 35.]
- [52] J. W. Ferguson, D. R. Alexander, F. Allard, T. Barman, J. G. Bodnarik, P. H. Hauschildt, A. Heffner-Wong, and A. Tamanai, "Low-Temperature Opacities," *The Astrophysical Journal*, vol. 623, no. 1, pp. 585–596, Apr. 2005. [Cited on page 35.]
- [53] A. S. Eddington, *The Internal Constitution of the Stars*. Cambridge University Press, 1926. [Cited on page 37.]
- [54] W. H. Ball, "A novel analytic atmospheric $t(\tau)$ relation for stellar models," *Research Notes of the AAS*, vol. 5, no. 1, p. 7, Jan. 2021. [Online]. Available: <http://dx.doi.org/10.3847/2515-5172/abd9cb> [Cited on page 37.]
- [55] K. S. Krishna Swamy, "Profiles of Strong Lines in K-Dwarfs," *The Astrophysical Journal*, vol. 145, p. 174, Jul. 1966. [Cited on page 37.]
- [56] P. H. Hauschildt, F. Allard, J. Ferguson, E. Baron, and D. R. Alexander, "The nextgen model atmosphere grid. ii. spherically symmetric model atmospheres for giant stars with effective temperatures between 3000 and 6800 k," *The Astrophysical Journal*, vol. 525, no. 2, p. 871, nov 1999. [Online]. Available: <https://dx.doi.org/10.1086/307954> [Cited on page 37.]
- [57] F. Castelli and R. L. Kurucz, "New Grids of ATLAS9 Model Atmospheres," in *Modelling of Stellar Atmospheres*, N. Piskunov, W. W. Weiss, and D. F. Gray, Eds., vol. 210, Jan. 2003, p. A20. [Cited on page 37.]
- [58] M. López-Morales, "On the Correlation between the Magnetic Activity Levels, Metallicities, and Radii of Low-Mass Stars," *The Astrophysical Journal*, vol. 660, no. 1, pp. 732–739, May 2007. [Cited on pages 38 and 48.]
- [59] F. Spada, P. Demarque, Y. C. Kim, and A. Sills, "The radius discrepancy in low-mass stars: Single versus binaries," *The Astrophysical Journal*, vol. 776, no. 2, p. 87, oct 2013. [Online]. Available: <https://dx.doi.org/10.1088/0004-637X/776/2/87> [Cited on pages 38, 70, and 76.]
- [60] M. Joyce and B. Chaboyer, "Classically and Asteroseismically Constrained 1D Stellar Evolution Models of α Centauri A and B Using Empirical Mixing Length Calibrations," *The Astrophysical Journal*, vol. 864, no. 1, p. 99, September 2018. [Cited on pages 38 and 67.]

- [61] J. Christensen-Dalsgaard, “Adiplus—the aarhus adiabatic oscillation package,” *Astrophysics and Space Science*, vol. 316, no. 1–4, p. 113–120, Jan. 2008. [Online]. Available: <http://dx.doi.org/10.1007/s10509-007-9689-z> [Cited on page 44.]
- [62] W. Unno, Y. Osaki, H. Ando, H. Saio, and H. Shibahashi, *Nonradial oscillations of stars*, 1989. [Cited on page 44.]
- [63] W. A. Dziembowski, “Nonradial Oscillations of Evolved Stars. I. Quasiadiabatic Approximation,” *Acta Astron*, vol. 21, pp. 289–306, Jan. 1971. [Cited on page 44.]
- [64] T. R. White, T. R. Bedding, D. Stello, J. Christensen-Dalsgaard, D. Huber, and H. Kjeldsen, “Calculating Asteroseismic Diagrams for Solar-like Oscillations,” *The Astrophysical Journal*, vol. 743, no. 2, p. 161, Dec. 2011. [Cited on pages 47, 56, 57, 58, 59, 62, and 63.]
- [65] T. R. White and B. et al, “Asteroseismic Diagrams from a Survey of Solar-like Oscillations with Kepler,” *The Astrophysical Journal Letters*, vol. 742, no. 1, p. L3, Nov. 2011. [Cited on pages 47, 56, and 59.]
- [66] A. D. Rains, M. J. Ireland, T. R. White, L. Casagrande, and I. Karovicova, “Precision angular diameters for 16 southern stars with VLT/PIONIER,” *Monthly Notices of the Royal Astronomical Society*, vol. 493, no. 2, pp. 2377–2394, 03 2020. [Online]. Available: <https://doi.org/10.1093/mnras/staa282> [Cited on page 48.]
- [67] L. Buchele, E. P. Bellinger, S. Hekker, S. Basu, W. Ball, and J. Christensen-Dalsgaard, “Asteroseismic inversions for internal sound speed profiles of main-sequence stars with radiative cores,” *The Astrophysical Journal*, vol. 961, no. 2, p. 198, Jan. 2024. [Online]. Available: <http://dx.doi.org/10.3847/1538-4357/ad1680> [Cited on page 50.]
- [68] L. Buchele, E. P. Bellinger, S. Hekker, S. Basu, W. Ball, and J. Christensen-Dalsgaard, “Asteroseismic Inversions for Internal Sound Speed Profiles of Main-sequence Stars with Radiative Cores,” *The Astrophysical Journal*, vol. 961, no. 2, p. 198, Feb. 2024. [Cited on pages 51 and 52.]
- [69] S. Turck-Chièze, S. Basu, G. Berthomieu, A. Bonanno, A. S. Brun, J. Christensen-Dalsgaard, M. Gabriel, P. Morel, J. Provost, S. Turcotte, and GOLF Team, “Sensitivity of the Sound Speed to the Physical Processes Included in the Standard Solar Model,” in *Structure and Dynamics of the Interior of the Sun and Sun-like Stars*, ser. ESA Special Publication, S. Korzenik, Ed., vol. 418, Jan. 1998, p. 555. [Cited on page 52.]
- [70] I. Baraffe and G. Chabrier, “A closer look at the transition between fully convective and partly radiative low-mass stars,” *Astronomy & Astrophysics*, vol. 619, p. A177, Nov. 2018. [Cited on page 53.]
- [71] L. Viani, S. Basu, E. Corsaro, W. Ball, and W. Chaplin, “Determining the best method of calculating the large frequency separation for stellar models,” *The Astrophysical Journal*, vol. 879, p. 33, 07 2019. [Cited on page 56.]
- [72] R. et al, “Determining stellar parameters of asteroseismic targets: going beyond the use of scaling relations,” *Monthly Notices of the Royal Astronomical Society*, vol. 467, no. 2, pp. 1433–1448, 01 2017. [Online]. Available: <https://doi.org/10.1093/mnras/stx120> [Cited on page 56.]
- [73] O. J. M. Joel and S. Basu, “Explaining deviations from the scaling relationship of the large frequency separation,” *The Astrophysical Journal*, vol. 870, no. 1, p. 41, Jan. 2019. [Online]. Available: <http://dx.doi.org/10.3847/1538-4357/aaf1b5> [Cited on page 57.]
- [74] D. Stello, W. J. Chaplin, S. Basu, Y. Elsworth, and T. R. Bedding, “The relation between $\Delta\nu$ and $v_{\max}$ for solar-like oscillations,” *Monthly Notices of the Royal Astronomical Society: Letters*, vol. 400, no. 1, p. L80–L84, Nov. 2009. [Online]. Available: <http://dx.doi.org/10.1111/j.1745-3933.2009.00767.x> [Cited on page 57.]
- [75] I. W. Roxburgh, “Asteroseismic model fitting by comparing $\epsilon_{n\ell}$ values,” *Astronomy & Astrophysics*, vol. 585, p. A63, Jan. 2016. [Cited on pages x, 57, 58, 59, and 79.]

- [76] I. W. Roxburgh, "Surface layer independent model fitting by phase matching: theory and application to HD 49933 and HD 177153 (aka Perky)," *Astronomy & Astrophysics*, vol. 574, p. A45, Jan. 2015. [Cited on page 59.]
- [77] M. J. P. F. G. Monteiro, J. Christensen-Dalsgaard, and M. J. Thompson, "Asteroseismic inference for solar-type stars," 2001. [Cited on pages 60 and 63.]
- [78] B. Mosser, E. Michel, K. Belkacem, M. J. Goupil, A. Baglin, C. Barban, J. Provost, R. Samadi, M. Auvergne, and C. Catala, "Asymptotic and measured large frequency separations," *Astronomy and Astrophysics*, vol. 550, p. A126, Feb. 2013. [Online]. Available: <http://dx.doi.org/10.1051/0004-6361/201220435> [Cited on page 60.]
- [79] S. Hekker, S. Basu, Y. Elsworth, and W. J. Chaplin, "Frequency dependence of the large frequency separation of solar-like oscillators: influence of the helium second-ionization zone," *Monthly Notices of the Royal Astronomical Society*, vol. 418, no. 1, pp. L119–L123, Nov. 2011. [Cited on page 60.]
- [80] T. R. Bedding and e. a. Huber, "Solar-like Oscillations in Low-luminosity Red Giants: First Results from Kepler," *The Astrophysical Journal Letters*, vol. 713, no. 2, pp. L176–L181, Apr. 2010. [Cited on page 61.]
- [81] D. Huber and B. et al, "Asteroseismology of Red Giants from the First Four Months of Kepler Data: Global Oscillation Parameters for 800 Stars," *The Astrophysical Journal*, vol. 723, no. 2, pp. 1607–1617, Nov. 2010. [Cited on page 61.]
- [82] M. Soriano and S. Vauclair, "Asteroseismology of solar-type stars: signatures of convective and/or helium cores," *Astronomy and Astrophysics*, vol. 488, no. 3, p. 975–985, Jul. 2008. [Online]. Available: <http://dx.doi.org/10.1051/0004-6361:200809977> [Cited on page 61.]
- [83] G. Valle, M. Dell'Omodarme, P. G. Prada Moroni, and S. Degl'Innocenti, "Relevance of the small frequency separation for asteroseismic stellar age, mass, and radius. A statistical investigation for main-sequence low-mass stars," *Astronomy & Astrophysics*, vol. 635, p. A77, Mar. 2020. [Cited on page 62.]
- [84] Y. Elsworth, R. Howe, G. R. Isaak, C. P. McLeod, and R. New, "Evidence from solar seismology against non-standard solar-core models," *Nature*, vol. 347, no. 6293, pp. 536–539, Oct. 1990. [Cited on page 63.]
- [85] W. J. Chaplin and A. Miglio, "Asteroseismology of solar-type and red-giant stars," *Annual Review of Astronomy and Astrophysics*, vol. 51, no. 1, p. 353–392, Aug. 2013. [Online]. Available: <http://dx.doi.org/10.1146/annurev-astro-082812-140938> [Cited on page 63.]
- [86] I. W. Roxburgh, "Small separations and phase shift differences of $\ell = 0, 1$ p-modes," *Astronomy & Astrophysics*, vol. 493, no. 1, pp. 185–191, Jan. 2009. [Cited on page 64.]
- [87] D. Reese, "AIMS Documentation," p. 62, 2020. [Cited on pages 65 and 66.]
- [88] D. Foreman-Mackey, D. W. Hogg, D. Lang, and J. Goodman, "emcee: The MCMC Hammer," *Astrophysics Source Code Library*, vol. 125, no. 925, p. 306, Mar. 2013. [Cited on page 65.]
- [89] D. Reese, "AIMS Overview," p. 30, 2020. [Cited on page 67.]
- [90] H. Kjeldsen, T. R. Bedding, T. Arentoft, R. P. Butler, T. H. Dall, C. Karoff, L. L. Kiss, C. G. Tinney, and W. J. Chaplin, "The amplitude of solar oscillations using stellar techniques," *The Astrophysical Journal*, vol. 682, no. 2, p. 1370, aug 2008. [Online]. Available: <https://dx.doi.org/10.1086/589142> [Cited on page 67.]
- [91] S. Moser, G. Valle, M. Dell'Omodarme, S. Degl'Innocenti, and P. G. Prada Moroni, "Uncertainties in asteroseismic grid-based estimates of the ages of halo stars," *Astronomy & Astrophysics*, vol. 671, p. A78, Mar. 2023. [Cited on page 70.]