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Abstract

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On the Energy of gravitational waves in a De Sitter background

by José Queirós

This thesis is a small step forward in the complicated endeavour of building a complete framework isolating in a full non-linear way the gravitational degrees of freedom of an asymptotically De Sitter spacetime. We study the much simpler problem of a linearized gravitational field in a De Sitter background, computing its energy flux on the cosmological horizon and relating it to the quadrupole moments of a source described by a sufficiently general energy-momentum tensor.

We start by reviewing a framework in chapter 2, the covariant phase space formalism following [1, 2], since it gives us a method to define the energy and its flux, deriving formulas for conserved charges generated by asymptotic symetries and respective fluxes at a spacetime boundary. In Chapters 3 and 4, we review the work of Chandrasekaran et al who in [3] applied the work of Wald and Zoupas [2] to a general null surface, and derived expressions for charges and fluxes in terms of intrinsic objects on the null surface.

Since we'll be interested in the weak field regime, where we take perturbations to De Sitter, in chapter 5, we review the fundamentals of De Sitter spacetime, and use these to, in chapter 6, study linear perturbations to De Sitter. We compute the energy of our solutions in the cosmological horizon, and relate them to the quadrupole moments of the source, generalizing the Einstein's Quadrupole formula.

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Resumo

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Mestrado em Física

Sobre o fluxo de energia de ondas gravitacionais num background de De Sitter

por José Queirós

Esta tese representa um pequeno avanço na complicada tarefa de construir um formalism que isole de forma não linear os graus de liberdade do campo gravitacional num espaço-tempo assintoticamente De Sitter. Estudamos o problema mais simples de um campo gravitacional linearizado num background de De Sitter, calculamos o seu fluxo de energia no horizonte cosmológico e relacionamos com os momentos do quadrupolo de uma fonte descrita por um tensor de energia-momento suficientemente geral.

Começamos por rever no capítulo 2, o formalismo do espaço de fase covariante seguindo [1, 2], uma vez que nos proporciona um método para definir a energia e o seu fluxo, derivando fórmulas para cargas conservadas geradas por simetrias assintóticas e respetivos fluxos numa fronteira do espaço-tempo. Nos Capítulos 3 e 4, revimos o trabalho de Chandrasekaran et al, que em [3] aplicou o trabalho de Wald e Zoupas [2] a uma superfície nula, e obteve expressões para cargas e fluxos em termos de objetos intrínsecos na superfície nula.

Dado que estamos interessados no regime de campo fraco, onde efetuamos perturbações em De Sitter, no capítulo 5, revimos alguns conceitos importantes do espaço-tempo de De Sitter e usamo-los para, no capítulo 6, estudar perturbações lineares em De Sitter. Calculamos a energia das nossas soluções no horizonte cosmológico e relacionamo-las com os momentos de quadrupolo da fonte, generalizando a fórmula do Quadrupolo de Einstein.

Contents

Chapter 1

Introduction

It is an established fact since [4] was published, that the expansion of the universe is accelerating, which can be best modeled by a positive cosmological constant Λ. Motivated by this, in recent years, the study of gravitational waves in the presence of a positive Λ has received a lot of attention.

Gravitational waves were first predicted in 1916 by Einstein, who was able to isolate the gravitational degrees of freedom in the weak field approximation and derive the quadrupole formula, relating the energy of these waves to the dynamics of the source. There was, for many years, a lot of debate concerning the physics of these results: were these just a coordinate artifact and a product of the linear regime, or did they indeed describe physical waves? The debate was put to an end in the 1950s when Trautman [5] obtained expressions for energy and and momentum of gravitational waves in full non-linear General Relativity and in the early 1960's with the pioneering work of Bondi et al [6]. They studied the asymptotics of a metric in full non-linear GR as it approached null infinity, and were able to isolate the gravitational degrees of freedom via a second rank, trace free, transverse tensor field N_{ab}, which provided an invariant characterization of the gravitational degrees of freedom. Later, Penrose and Newman [7] extended their work, using conformal techniques. Both of these works provide nowadays the theoretical foundation for computations in the study of gravitational waves in Numerical Relativity.

Independently, in the 1980s a sympletic approach to GR was applied to the study of gravitational waves, and in [8] the results of Trautmann, Bondi and others were recovered. Later, Wald and Zoupas in [2] provided a general algorithm to compute fluxes of charges at an asymptotic boundary for diffeomorphism covariant theories in the context of the covariant phase space formalism, and by carefully choosing the sympletic potential Θ, also recovered the results in earlier works.

These works are valid in the context of asymptotically flat spacetimes, where no cosmological constant is present. Since observations [4] have established the existence of a positive Λ, it is important to construct a similar framework in this context. Recently, in a series of papers [9–11], Ashtekar and collaborators started the endeavor of trying to extend the previous framework to asymptotically de Sitter spacetimes, but have encountered some fundamental difficulties. In asymptotically flat space, one is able to define energy via a charge generated by an asymptotic symmetry which is timelike in a neighborhood of null infinity $\mathcal{I}$, but in asymptotically de Sitter spacetimes null infinity is a space-like hypersurface, and hence all symmetry vector fields will be spacelike in a neighborhood of $\mathcal{I}$; this implies that not only is it not obvious how to define energy but also that any definition will not be bounded by below. Also, in the case $\Lambda = 0$, infinity $\mathcal{I}$ has some structure which is preserved for all asyntoptically flat spacetimes, and the group of transformations which preserves this structure (BMS group) can be used to define charges at infinity. Here $\mathcal{I}$ has virtually no universal structure, and the asymptotic symmetry group is the full diffeomorphism group [9]. In [11] a more restrictive boundary condition was studied but it was shown to be too restrictive. In subsequent papers gravitational radiation was studied in the weak field regime; expressions for fluxes in the presence of Λ were proposed and a quadrupole formula was derived. Different arguments, proposals, definitions and ideas have been been brought out in the last few years by many authors, namely [12, 13].

In [10, 11] it was suggested that a more appropriate arena for these calculations might be the cosmological horizon $\mathcal{H}$ of De Sitter, since it is a null surface and for realistic physical situations, the flux on $\mathcal{H}$ should equal the flux on $\mathcal{I}$. The goal of this thesis is to generalize the quadrupole formula in the cosmological horizon of a De Sitter background. Recently, this calculation was done in [14], and for that reason our aim is to reproduce and clarify the results of [14].

We will start by briefly introducing linearized gravity without a cosmological constant. A much more detailed discussion will be made in chapter 6, when we treat the case of a linearized perturbation in a de Sitter background. Here our aim is to give a very short review of some of the standard results, to give motivation for what follows. A more thorough treatment of this subject can be for example found in [15] and [16].

1.1 Linearized gravity without a cosmological constant

When gravity is very weak, it can be useful to regard spacetime as a perturbation to Minkowski spacetime. Here, we assume that the spacetime manifold is such that one can find a set of 'almost inertial' coordinates $\{x^a\}$ in which the metric can be written as a small deviation h_{ab} to Minkowski

$$g_{ab} = \eta_{ab} + h_{ab} \tag{1.1}$$

Since we don't have any natural metric from which to measure the smallness of h_{ab}, here we assume that the components of h_{ab} are much smaller than 1 in this almost inertial frame. In this context the Einstein equations are given by

$$G^{(1)}_{ab}[h] = -\frac{1}{2}\partial^c\partial_c\bar{h}_{ab} + \partial^c\partial_{(b}\bar{h}_{a)c} - \frac{1}{2}\eta_{ab}\partial^c\partial^d\bar{h}_{cd} = 8\pi T_{ab} \tag{1.2}$$

where it was defined the trace reversed perturbation $\bar{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h$, and $G^{(1)}_{ab}$ is only first order in $\bar{h}_{ab}$. Since GR is a diffeomorphism invariant theory, a space time and a stress energy tensor (M, g, T) are physically equivalent to another $(M, \phi_* g, \phi_* T)$ if $\phi : M \to M$ is a diffeomorphism. In the context of Linearized gravity this leads to a gauge freedom: two perturbations h and $h + \mathcal{L}_\xi \eta$ are physically equivalent perturbations. This freedom can be used to simplify the Einstein equations, in particular one can always find some ξ so that the corresponding gauge transformed h obeys the gauge condition $\partial^a\bar{h}_{ab} = 0$. This is the Lorentz gauge, and in this gauge the Einstein's equations simplify to

$$\partial^c\partial_c\bar{h}_{ab} = -16\pi T_{ab} \tag{1.3}$$

In vacuum, the linearized Einstein's equations reduce to a source-free wave equation, which means the theory admits gravitational waves. An arbitrary well behaved solution to the vacuum Einstein equations will be a superposition of plane waves

$$\bar{h}_{ab} = H_{ab}e^{ik\cdot x} \tag{1.4}$$

where H_{ab} is a constant tensor field which describes the polarization of the waves. The vacuum Einstein's equations restrict k^a to be null, which implies that gravitational waves travel at the speed of light relative to the background Minkowski field, and the Lorentz gauge restricts the waves to be transverse. The remaining gauge freedom can be used to show that H_{ab} only has two independent components, which is a way of seeing that the gravitational field has two degrees of freedom.

1.2 The Quadrupole Formula

In order for these wave like solutions to be produced, some strong gravity phenomena is needed, where the linear approximation is not valid, and the full nonlinear Einstein equations must be used. It is however still instructive to study the problem of a linear perturbation sourced by some energy momentum tensor T_{ab} .

We then return to equation (1.3). Since each component of h satisfies a sourced scalar wave equation, its solution can be given by the same retarded Green's function we know for a scalar field in electromagnetism

$$\bar{h}_{ab}(x,t) = 4\int d^3x' \frac{T_{ab}(t-|x-x'|,x')}{|x-x'|} \tag{1.5}$$

We assume that the matter is confined in a compact region near the origin of size $d \ll r$ and that it moves non-relativistically. Then far from the source we have

$$\bar{h}_{ab}(x,t) \approx \frac{4}{r}\int d^3x' T_{ab}(x', t_{\text{ret}}) \tag{1.6}$$

where $t_{\text{ret}} = t - r$. At linear order, $\bar{h}_{00}$ and $\bar{h}_{0i}$ are related to the energy and momentum of the source which will be conserved (at linear order only) and hence do not capture the dynamics we are interested in. Let's take a look at $\bar{h}_{ij}$

$$\begin{aligned} \int d^3x' T^{ij} &= \int d^3x' [\partial_k(T^{ik}x^j) - (\partial_k T^{ik})x^j] \\ &= \partial_0 \int d^3x' \partial_0 T^{i0} x^j \end{aligned} \tag{1.7}$$

where we dropped a surface term a used the conservation of T^{ab}. Symmetrizing now gives

$$\begin{aligned} \int d^3x' T^{ij} &= \partial_0 \int d^3x' \frac{1}{2}[\partial_k(T^{0k}x^i x^j) - (\partial_k T^{0k})x^i x^j] \\ &= \frac{1}{2}\partial_0^2 Q^{ij} \end{aligned} \tag{1.8}$$

where again we dropped a surface term a used the conservation of T^{ab}. In the end, we defined the quadrupole moment of the source

$$Q^{ij} = \int d^3x' T_{00} x^i x^j \tag{1.9}$$

which is the second moment of the energy density (the zeroth being the total energy, and the first the energy dipole). We hence see that gravitational waves will appear when Q_{ij} has a time dependence.

Now, since it is expected that the source will lose energy, and the gravitational waves should carry such energy to null infinity, we move on to try to compute such flux. Such calculation is more subtle than one might assume. Since in GR the dynamical field is g itself, one could expect the energy density of the gravitational field to be related to some quadratic function of first derivatives of g. But we know, that at any point, one can always find coordinates for which these vanish. As we'll see, one can define gravitational energy, but not in a local sense. Here, we will follow a very short and direct path to compute the energy. Its not the most rigorous, but it will be enough for the purposes of this section.

To do this we go to second order in perturbation theory. At second order $\eta + h$ fails to satisfy the Einstein equation, hence we consider

$$g_{ab} = \eta_{ab} + h_{ab} + h^{(2)}_{ab} \tag{1.10}$$

where the components of $h^{(2)}$ will be $\mathcal{O}(\epsilon^2)$ if we assume the components of h to be $\mathcal{O}(\epsilon)$. In second order the Einstein tensor will be composed of terms linear in h and $h^{(2)}$ and a term quadratic in h

$$G_{ab}[g] = G^{(1)}_{ab}[h] + G^{(1)}_{ab}[h^{(2)}] + G^{(2)}_{ab}[h] \tag{1.11}$$

We assume that no matter is present. This implies that $G^{(1)}_{ab}[h] = 0$ as we've seen before and we are left with

$$G^{(1)}_{ab}[h^{(2)}] = 8\pi t_{ab}[h] \tag{1.12}$$

where

$$t_{ab}[h] \equiv -\frac{1}{8\pi} G^{(2)}_{ab}[h] \tag{1.13}$$

In fact, the contracted Bianchi identity $g^{ac}\nabla_c G_{ab}$ at first order, and the equations of motion $G^{(1)} = 0$ immediately imply the conservation of t_{ab}

$$\partial^a t_{ab} = 0 \tag{1.14}$$

Hence, we have a symmetric tensor (in the background spacetime) that is quadratic in h, conserved, and that sources the Einstein equation for the second order perturbation. This would be a very nice candidate for a local energy momentum tensor for the gravitational field, but unfortunately it it is not gauge invariant. This is again a way of seeing the impossibility of defining a local notion of energy in GR.

However, one can define the total energy on a spacelike hypersurface $\sum$

$$E = \int d^3x t_{00} \tag{1.15}$$

and this is in fact gauge invariant, provided the field h and its derivatives decay sufficiently rapidly as $r \to \infty$ (asymptotic flatness), and we restrict to gauge transformations which obey the same boundary conditions.

In [15], it was shown that if we average t_{ab} in a certain way, over a region much larger than the wavelength of the gravitational wave, we obtain a gauge invariant object. They used this object to define the flux of energy and the power of the gravitational wave. Here we take a more direct approach, we consider

$$F = -\int_S t_{a0} dS^a \tag{1.16}$$

In a time-dependent spacetime which is stationary initially and at later stages, this is gauge invariant and naturally defines the flux of energy of gravitational waves. Here S is a timelike surface bounded by two asymptotically null surfaces assumed to be in the future and past stationary regions, and where asymptotic flatness is again assumed. Using 1.13and 1.16 and after a very lengthy calculation one obtains

$$F = \frac{1}{5} \int dt \sum_{ij} (\dddot{q}_{ij})^2 (t_{\mathrm{ret}}) \tag{1.17}$$

where $q_{ij} = Q_{ij} - \frac{1}{3}\delta_{ij} Q$ is the trace-free quadrupole moment tensor. This is the quadrupole formula for energy lost via gravitational waves by a time-varying quadrupole.The main goal of these thesis will be to derive a generalization of this formula to the case of a perturbation to de Sitter spacetime.

Chapter 2

Covariant phase space for field theories

In diffeomorphism covariant theories such as GR, there is no local notion of stress-energy tensor for the gravitational field. Nevertheless, in asymptotically flat spacetimes charges associated with asymptotic symmetries have been defined at spatial and null infinity [6, 17]. Most recent approaches to the problem of defining energy in asymptotically de Sitter spacetimes use results from the so called covariant phase space formalism. This chapter is a gentle introduction to some of these results, following [1, 2]

2.1 Hamiltonian formalism

In classical mechanics, the phase space is a 2-dimensional differential manifold parameterized by position and momentum coordinates (q_i, p_i) with $(i = 0, ..., d-1)$. The Hamiltonian is a scalar function on this manifold that generates dynamics via the Hamilton's equations (indices will be omitted for simplicity of exposition)

$$\dot{q} = \frac{\partial H}{\partial p} \tag{2.1}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{2.2}$$

There is a geometric manner to present this subject which will allows us to generalize to field theories. The idea is that phase space can be characterized as a sympletic manifold, that is a manifold equipped with a closed, non-degenerate two form ω, which we call a sympletic form. Given this structure, one is able to produce a vector field X_H out of H, and as we will see, particles just follow the flow of X_H. Any two-form provides a map from vectors to one forms, via the interior product $X_H \cdot \omega$. Since the two form is assumed to be degenerate, this map has an inverse. We can then take a function H, build a one form out of it dH and use this inverse to construct a vector field

$$X_H \cdot \omega = -dH \tag{2.3}$$

If we introduce local coordinates $x = (q, p)$, and invert the previous expression, one obtains the equations for the internal curves of X_H

$$\frac{dx^\mu}{dt} = X^\mu_H = \omega^{\mu\nu}\partial_\nu H \tag{2.4}$$

Hamilton's equations are recovered if we take $\omega = dp \wedge dq$. Given two functions on M one can play the same game and recover the usual Poisson bracket

$$\{f, g\} = \omega(X_g, X_f) = \omega^{-1}(df, dg) \tag{2.5}$$

in terms of which we have the time evolution

$$\dot{f} = \{f, H\} = X_H(f) = \mathcal{L}_{X_H} f \tag{2.6}$$

It is clear why the form is required to be degenerate. The requirement that it is a closed form ensures that the poisson bracket is invariant under the Hamiltonian flow, or equivalently that the form itself is invariant

$$\begin{aligned} &= X_H \cdot d\omega + d(X \cdot \omega) \\ &= X_H \cdot d\omega - d^2 H \\ &= X_H \cdot d\omega \end{aligned} \tag{2.7}$$

which vanishes for arbitrary H if $d\omega = 0$.

Before going further to the fields, a few important remarks are in order.

- The phase space of a dynamical system is usually seen as the set of initial data on a time slice, which makes it a non-covariant construction. Fortunately, there is an isomorphism between these to the space of solutions. We have just seen that the sympletic form is invariant under the flow of the Hamiltonian, hence the sympletic structure that we pullback to space of solutions is independent of the choice of initial time slice. Being also a manifold, the space of solutions is in fact a sympletic manifold, one which we will call the covariant phase space of the system. This is the construction that we will be able to apply also for fields.

- In this new setting, an important question arises, how do we talk about dynamics in this construction? If a point in the phase space is one allowed history, what does 'time translation' look like in here? This notion can be introduced as follows: given a solution with some in initial data at t_1, we map it to another solution whose initial data at t_0 is the same as the initial data of the previous solution at t_1.

2.2 The pre-sympletic form

We consider a d-dimensional differential manifold M, with boundary ∂M and some dynamical fields defined on it which we collectively denote as ϕ (background fields γ are also allowed, the Lagrangian can in general depend on them, but its dependence on them will be neglected for simplicity). The boundary of the manifold ∂M will in general be a number of different components $\partial M = \bigcup_i B_i$, where B_i may be at a finite location for example a cosmological horizon (our case of interest), or an asymptotic boundary, in which case our manifold of interest will be the conformal completion of some physical spacetime.

We define a space of configurations $\mathcal{C}$ (an infinite dimensional manifold) of dynamical fields on M that satisfy suitable smoothness and boundary conditions on each B_i and their intersections. These will depend on the specific theories, and for now we only assume that these were specified.

We assume that the dynamics of the theory are specified by a Lagrangian d-form L which depends locally on the fields $\phi = (\phi, \gamma)$.

Consider variations of the fields $\phi \to \phi + \delta\phi$ that obey the boundary conditions. These will induce a variation on the Lagrangian

$$\delta L = E^a(\phi)\delta\phi^a + d\theta(\phi^a, \delta\phi^a) \tag{2.8}$$

where E is a d-form on spacetime that implies the equations of motion $E = 0$. The (d-1) form θ arises in the usual integration by parts, and is required to be constructed locally and covariantly out of ϕ and $\delta\phi$.

Before going further, we should take a detour to discuss some notation. Consider a smooth one-parameter family of solutions ϕ_λ that obey the boundary conditions; a variation is usually seen as $\delta\phi \equiv \partial_\lambda \phi_\lambda|_{\lambda=0}$. Similarly, given a function L on $\mathcal{C}$, its variation is just

$$\partial_\lambda L|_{\lambda=0} = \frac{\delta L}{\delta \phi} \partial_\lambda \phi_\lambda|_{\lambda=0} \tag{2.9}$$

Variations $\delta\phi^a$ can then be seen as components of elements X of the tangent space $T_\phi\mathcal{C}$. The action of the tangent vector X on functions $f : \mathcal{C} \to \mathbb{R}$ is just

$$X(f) = \int \varepsilon X^a(\phi, x) \frac{\delta f}{\delta \phi^a} \tag{2.10}$$

What about the dual space $(T_\phi\mathcal{C})^*$? Elements of the dual space will be one-forms that map X to the real numbers. In order to describe its elements, we would like to see the quantity $\delta\phi$ not as a variation, but as coordinate differential, so that δ is now an exterior derivative on forms on $\mathcal{C}$. Its action on tangent vectors X is given by

$$\delta\phi^b \left(\int \varepsilon X^a(\phi, x') \frac{\delta f}{\delta \phi^a}\right) = X^b(\phi, x) \tag{2.11}$$

If we wish to convert $\delta\phi$ back to a variation we can act with it on a vector whose components X^a are the desired variation. A 1-form in the configuration space can in general be written

$$\omega = \int \varepsilon \omega_a \delta \phi^a \tag{2.12}$$

Note that the potential θ, being a function of $X^a = \delta\phi^a$ can be seen as a 1-form on $\mathcal{C}$. The Lagrangian L is a scalar function on $\mathcal{C}$ and hence in this new way, δL is a one-form. We can recover the old variation as the action of a vector field on it $X(L) = X \cdot \delta L$.

To set up the Hamiltonian formalism we now introduce the pre-phase space and the pre-sympletic form. We define the pre-phase space $\widetilde{\mathcal{F}}$ as the elements of the configuration space $\mathcal{C}$ that also obey the equations of motion $E(\phi) = 0$. As we will see later, for theories with continuous local symmetries this space is still too 'big' for our purposes. We now define the pre-sympletic current on $\widetilde{\mathcal{F}}$ as

$$\omega = \delta\theta|_{\widetilde{\mathcal{F}}} \tag{2.13}$$

By construction, it is a closed 2-form on $\widetilde{\mathcal{F}}$, and we can show that is also also a closed (d-1) form on the manifold M

$$\begin{aligned} d\omega &= d(\delta\theta) \\ &= \delta(\delta L - E\delta\phi) \\ &= \delta^2 L - \delta E \wedge \delta\phi \\ &= 0 \end{aligned} \tag{2.14}$$

since $[\delta, d] = 0$, $E|_{\widetilde{\mathcal{F}}} = 0$ and δ^2 is an identity. Hence the pre-sympletic current, is also a conserved current on spacetime. Its action on variations $X^a = \delta_1\phi^a$ and $Y^a = \delta_1\phi^a$ is

$$\begin{aligned} \omega(X, Y) =& \delta\theta(X, Y) \\ =& X(\theta(Y)) - Y(\theta(X)) + [X, Y] \\ =& X(\theta(Y)) - Y(\theta(X)) \end{aligned} \tag{2.15}$$

In the last equality, we followed the practice adopted in the literature and consider only vector fields X that commute with any vector field Y on the tangent space. We then recover the usual definition of the pre-sympletic current.

Note that elements X of the tangent to $\widetilde{\mathcal{F}}$ will satisfy the linearized field equations. We can see this as follows: if $X^a = \delta\phi^a$ is tangent to $\widetilde{\mathcal{F}}$, then there exists a one-parameter family of solutions ϕ_λ in $\widetilde{\mathcal{F}}$ such that $\delta\phi^a = \partial_\lambda\phi|_{\lambda=0}$. Since $\phi_\lambda \in \widetilde{\mathcal{F}}$, we have $E(\phi_\lambda) = 0$ for all λ, so that $dE(\phi_\lambda)/d\lambda|_{\lambda=0} = F(\delta\phi) = 0$ which are the linearized equations of motion for $X^a = \delta\phi^a$.

Consider now a slice Σ, that is a closed, embedded (d-1) dimensional submanifold of M. We define the pre-sympletic form Ω_Σ on $\widetilde{\mathcal{F}}$ as

$$\widetilde{\Omega}_\Sigma = \int_\Sigma \omega \tag{2.16}$$

Consider now a spacetime volume V bounded by two slices Σ_1 and Σ_2. Using Stokes theorem we have

$$\begin{aligned} 0 &= \int_V d\omega \\ &= \int_{\Sigma_1} \omega - \int_{\Sigma_2} \omega + \int_{B_{12}} \omega \end{aligned} \tag{2.17}$$

where B_{12} is a subset of one of the boundary elements bounded by $\partial\Sigma_1$ and $\partial\Sigma_2$. This shows that in general $\widetilde{\Omega}_\Sigma$ will depend on Σ. Now, if the slices are compact or are required to be Cauchy surfaces so that $B_{12} \subset i^0$ then $\widetilde{\Omega}_\Sigma$ will not depend on Σ provided suitable asymptotic conditions are imposed on the fields.

2.3 The sympletic form and the covariant phase space

For theories with continuous local symmetries, a problem will arise. The redundancy in the description of the solutions appears as a degeneracy of the pre-sympletic form. This is solved by quotienting $\widetilde{\mathcal{F}}$ by the action of the group of zero modes of the pre-sympletic form.

Take two vector fields $\tilde{X}, \tilde{Y}$ for which $\tilde{X} \cdot \tilde{\Omega} = \tilde{Y} \cdot \tilde{\Omega} = 0$; its commutator $[\tilde{X}, \tilde{Y}]$ also annihilates $\tilde{\Omega}$

$$\begin{aligned} \mathcal{L}_{\tilde{X}} \tilde{Y} \cdot \tilde{\Omega} &= \mathcal{L}_{\tilde{X}} (\tilde{Y} \cdot \tilde{\Omega}) - \tilde{Y} \cdot \mathcal{L}_{\tilde{X}} \tilde{\Omega} \\ &= -\tilde{Y} \cdot (\tilde{X} \cdot \delta\tilde{\Omega} + \delta(\tilde{X} \cdot \tilde{\Omega})) \\ &= 0 \end{aligned} \tag{2.18}$$

which shows that the zero modes form a Lie algebra. Frobenius's theorem then implies that the zero modes are jointly tangent to a set of submanifolds which foliate $\widetilde{\mathcal{F}}$. These are just the orbits of the degeneracy group $\tilde{G}$ whose Lie algebra are just the zero modes of $\tilde{\Omega}$. The covariant phase space is then defined as the quotient

$$\mathcal{F} = \widetilde{\mathcal{F}} / \tilde{G} \tag{2.19}$$

We now need to define a sympletic form Ω in $\mathcal{F}$ and show that it is in fact non-degenerate. For this purpose, let $\pi : \tilde{\mathcal{F}} \to \mathcal{F}$ be the map that projects each point $\tilde{p}$ in $\tilde{\mathcal{F}}$ to its $\tilde{G}$-orbit, p a point in $\mathcal{F}$ and vectors X, Y in $T_p\mathcal{F}$. It is always possible to find a point $\tilde{p}$ in $\tilde{\mathcal{F}}$ and vectors $\tilde{X}$, $\tilde{Y}$ in $T_{\tilde{p}}\tilde{\mathcal{F}}$ such that X, Y are the pushforwards of $\tilde{X}$, $\tilde{Y}$ by π. We define the sympletic form Ω for such that

$$\Omega(X, Y) \equiv \tilde{\Omega}(\tilde{X}, \tilde{Y}) \tag{2.20}$$

Is it well defined? Well, take two vectors $\tilde{X}$, $\tilde{X}'$ in $T_{\tilde{p}}\tilde{\mathcal{F}}$ that both project to the same $X \in T_p\mathcal{F}$; by construction they can only differ by a zero mode of $\tilde{\Omega}$, hence not affecting the definition of Ω. Also, since for some $\tilde{g} \in \tilde{G}$ we have $\pi \circ \tilde{g} = \pi$, and any two points $\tilde{p}, \tilde{p'} \in \tilde{\mathcal{F}}$ which project to the same $p \in \mathcal{F}$ should be related by the action of one such $\tilde{g}$, we have that the pushforward $\tilde{X}'$ of $\tilde{X}$ by $\tilde{g}$ should map via pushforward by π to the same X. The form Ω is then well defined.

Is it non-degenerate? Well, assume that for some $p \in \mathcal{F}$ there is some $X \neq 0$ in $T_p\mathcal{F}$ that annihilates Ω. This vector is the pushforward of some other $\tilde{X} \in T_{\tilde{p}}\tilde{\mathcal{F}}$ which annihilates $\tilde{\Omega}$. Extending $\tilde{X}$ throughout $\tilde{\mathcal{F}}$ we get a vector field that annihilates $\tilde{\Omega}$ and hence by construction its pushforward via π should vanish, contracting our assumption. We see that our definition is in fact unique and non-degenerate, hence defining a sympetic form on $\mathcal{F}$.

2.4 Symmetries and diffeomorphism covariance

The formalism we are developing is especially useful for theories whose dynamics are invariant under at least some subgroup of spacetime diffeomorphisms. These diffeomorphisms will be described on phase space by some vector field X that will generate the flow for the dynamical fields.

We define a continuous symmetry of a Lagrangian field theory to be a vector field X tangent to the configuration space $\mathcal{C}$, such that

$$X \cdot \delta L = d\alpha \tag{2.21}$$

where α is a (d-1) form locally constructed out of the fields ϕ. Note that two Lagrangians are equivalent if they differ by some boundary term $d\alpha$, and we are essentially saying that the vector field X is a symmetry if its action on the Lagrangian leaves it invariant up to these boundary terms.

Consider now infinitesimal diffeomorphisms $M \to M$, that preserve the boundary ∂M. These will be generated by a vector field ξ, and induce variations on the dynamical fields $\phi \to \phi + \delta_\xi\phi$ with $\delta_\xi\phi = \mathcal{L}_\xi\phi$ (which is the usual Lie derivative on a manifold M). We call X_ξ the vector field on $T\mathcal{C}$ whose components are $\mathcal{L}_\xi\phi$, and that generates the ξ flow in phase space

$$\delta_\xi\phi = \mathcal{L}_\xi\phi = X_\xi \cdot \delta\phi = X_\xi(\phi) = \mathcal{L}_{X_\xi}\phi \tag{2.22}$$

In order for X_ξ to be a continuous symmetry, that is in order for it to leave the Lagrangian invariant, it should map solutions to solutions. Hence, let $D(B)$ to be the group of diffeomorphisms in M that preserve the boundary B and map $\mathcal{F}$ to $\mathcal{F}$, and

$D_0(B)$ the group of diffeomorphisms that are the identity on B; we define the symmetry group G as

$$G = D(G)/D_0(G) \tag{2.23}$$

The Lie algebra g of this group is obtained by modding out from the set of representatives of infinitesimal diffeomoprhisms ξ the ones which correspond to the trivial diffeomorphisms (whose charges vanish). An example of this, is the well known BMS algebra at null infinity.

Note that if $\xi \in G$, one can show, using the same argument we used before for tangent vectors to $\mathcal{F}$, that $\mathcal{L}_\xi \phi$ obeys the linearized equations of motion and hence X_ξ is tangent to $\mathcal{F}$.

As previously mentioned, a vector field X_ξ will evolve the dynamical fields, but this is not the case for a general infinitesimal diffeomorphism represented by ξ on M; these will also act on the background fields.

Consider a configuration space tensor σ that is also a spacetime tensor locally constructed out of dynamical and backgrouNd fields; one says that σ is covariant under ξ if

$$\mathcal{L}_{X_\xi} \phi = \mathcal{L}_\xi \phi \tag{2.24}$$

The Lagrangian will be covariant under ξ; if ξ is either a symmetry of the background fields that is if $\mathcal{L}_\xi \gamma = 0$, or the Lagrangian is built in such a way that any background fields enter in combinations that are invariant under ξ. An extreme case is if the Lagrangian does not depend in any background fields, like for example the Einstein-Hilbert Lagrangian. In fact, it was shown in [18] that this is the only way for a Lagrangian to be covariant under arbitrary diffeomorphisms.

2.5 Existence of an Hamiltonian

In the first section we tried to construct a vector field out of an Hamiltonian on phase space. Now, we need to do the exact opposite: given some representative ξ of an asymptotic symmetry, we would like to construct the corresponding Hamiltonian H_ξ.

In other words, given the pre-sympletic form $\widetilde{\Omega}$ on pre-space $\widetilde{\mathcal{F}}$, we would like to find a function such that

$$\delta H_\xi = -X_\xi \cdot \widetilde{\Omega} \tag{2.25}$$

Since for any zero mode $\widetilde{X}$ of $\widetilde{\Omega}$ we have

$$\widetilde{X} \cdot \delta H_\xi = \widetilde{\Omega}(\widetilde{X}, X_\xi) = 0 \tag{2.26}$$

the function H_ξ will be a well defined one on the phase space. This is the analogue of Hamilton's equations on the covariant phase space; if such function exists one $\mathcal{F}$ we will call it the Hamiltonian H_ξ. Applied to a variation $\delta\phi \in T_\phi\mathcal{F}$ we read

$$\delta H_\xi = \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) = \int_\Sigma \omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) \tag{2.27}$$

Suppose that such function exists, then assuming that $\mathcal{F}$ is connected, its value on $\mathcal{F}$ will be uniquely determined by 2.27 up to an arbitrary constant, that can be fixed by requiring that H_ξ vanishes for some reference solution, for example Minkowski spacetime.

It is convenient to define the Noether current

$$j_\xi = X_\xi \cdot \theta - \xi \cdot L \tag{2.28}$$

This is a scalar function on $\mathcal{C}$ and a (d-1)-form on M. Note that

$$\begin{aligned} dj_\xi &= d(X_\xi \cdot \theta) - d(\xi \cdot L) \\ &= X_\xi \cdot (\delta L - E\delta\phi) - \mathcal{L}_\xi L \\ &= \delta_\xi L - \mathcal{L}_\xi L - E\mathcal{L}_\xi\phi \\ &= -E\mathcal{L}_\xi\phi \end{aligned} \tag{2.29}$$

and we see that, when the equations of motion of the theory are satisfied, j_ξ is a closed form. Assume, for now that the Lagrangian is covariant under arbitrary diffeomorphisms (hence prohibiting any background fields); that is if f is a diffeomorphism $f : M \to M$ then $L(f^*\phi) = f^*L(\phi)$. Theories arising from these Lagrangians will be diffeomorphism covariant.

It can be shown (see [19]) that for diffeomorphism covariant Lagrangians, j_ξ is exact as well. That is, there is a Noether potential Q_ξ so that

$$j_\xi = dQ_\xi \tag{2.30}$$

If the equations of motion of the theory are not satisfied, it was shown that that there is a d-form C so that

$$j_\xi = dQ_\xi + \xi \cdot C \tag{2.31}$$

In fact, $C = 0$ will be equal to zero whenever $E = 0$; it corresponds to the constraints of the theory. Let's now look at the variation of j_ξ

$$\begin{aligned} \delta j_\xi &= \delta(X_\xi \cdot \theta) - \delta(\xi \cdot L) \\ &= L_{X_\xi}\theta - X_\xi \cdot \delta\theta - \xi \cdot E\delta\phi - \xi \cdot d\theta \\ &= -X_\xi \cdot \omega + d(\xi \cdot \theta) \end{aligned} \tag{2.32}$$

Hence, we get

$$\begin{aligned} \delta H_\xi &= -\int_\Sigma X_\xi \cdot \omega \\ &= \int_\Sigma \delta j_\xi - \int_{\partial\Sigma} (\xi \cdot \theta) \end{aligned} \tag{2.33}$$

If we assume a diffeomorphism covariant Lagrangian, we'll have

$$\delta H_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) \tag{2.34}$$

Hence, we see that if H_ξ exists, when evaluated on $\mathcal{F}$ it is only a surface term. A necessary condition for existence of H_ξ immediately arises: since $\delta^2 = 0$ is an identity

$$\begin{aligned} 0 &= \delta^2 H_\xi \\ &= -\int_{\partial\Sigma} \xi \cdot \omega \end{aligned} \tag{2.35}$$

Hence, a solution will only exist, if the previous integral vanishes. In fact, it was shown in [2] that it is also a sufficient condition for the existence of H_ξ. If the integrand vanishes on $\partial\Sigma$, the condition will automatically be satisfied. This can happen in two ways:

- either the pullback of ω to $\partial\Sigma$ vanishes
- or ξ is tangent to $\partial\Sigma$, so that the pullback of $\xi \cdot \omega$ to $\partial\Sigma$ will be zero

2.6 Generalized Wald-Zoupas charges

As we've seen in the last section, the consistency condition may not always be satisfied. so that an Hamiltonian cannot be defined. In fact, this will be true, in many interesting situations. In this section, we wish to give a prescription to define conserved quantities even when no Hamiltonian can be defined.

First of all, some terminology and assumptions should be clarified. We assume a manifold M with boundary $\partial M = B$. If no boundary exists, a boundary B may be attached by conformal compactification, so that $M \bigcup B$ is a manifold with boundary. The boundary conditions will be specified by requiring certain limiting behavior of the dynamical fields ϕ as one approaches B. In general, $M \bigcup B$ will be equipped with additional structure, that will enter into the specification of the limiting behavior of the fields, and will be part of the definition of $\mathcal{C}$ and $\mathcal{F}$. We further assume:

- $\mathcal{C}$ has been defined so that in $\mathcal{F}$, ω extends continuously to B
- slices Σ extend smoothly to B and intersect B in a $(d-2)$-dimensional submanifold $\partial\Sigma$, that we will call a cross-section of B

Consider now an infinitesimal asymptotic symmetry, represented by a vector field ξ, and a slice Σ. Two situations will arise.

- The pullback of ω to B vanishes, in which case an Hamiltonian exists. In this situation, one can show that the H_ξ is in fact a conserved quantity, that is, it doesn't depend on the choice of slice Σ. Consider two slices Σ_1 and Σ_2 which intersect B in two cross-sections $\partial\Sigma_1$ and $\partial\Sigma_2$ that bound a region $B_{12} \subset B$. We have

$$\begin{aligned} \delta H_\xi|_{\Sigma_1} - \delta H_\xi|_{\Sigma_2} &= (\int_{\partial\Sigma_1} - \int_{\partial\Sigma_2})(\delta Q_\xi - \xi \cdot \theta) \\ &= \int_{B_{12}} d(\delta Q_\xi - \xi \cdot \theta) \\ &= -\int_{B_{12}} X_\xi \cdot \omega \end{aligned} \tag{2.36}$$

which shows that ∂H_ξ is independent of the cross-section. If a reference solution is chosen so that H_ξ is zero for any cross-section, then H_ξ will also be independent of the choice of cross-section.

- The pullback of ω to B is nonzero, so that in general H_ξ cannot be defined. The exception will be when ξ is tangent to $\partial\Sigma$. In this case, even tough an Hamiltonian can be defined, it will not be conserved since

$$\delta H_\xi|_{\Sigma_1} - \delta H_\xi|_{\Sigma_2} = -\int_{B_{12}} X_\xi \cdot \omega \tag{2.37}$$

will not vanish in general.

One now wishes to provide a prescription to define charges in this second situation. We do that by trying to find a term that compensates the offending term in the consistency equation (2.35).

Consider the pullback $\tilde{\omega}$ of ω to B. Let $\tilde{\theta}$ be a potential for this sympletic current so that on B we have $\tilde{\omega} = \delta\tilde{\theta}$. This potential is required to be locally constructed from the fields in B and to be independent of the choices of universal structure on B [2]. Note that a potential $\tilde{\theta}$ will be defined up to $\tilde{\theta} \to \tilde{\theta} + \delta W$ for some 3-form suitably constructed from the fields on B. We then define a conserved quantity $\mathcal{H}_\xi$ by

$$\delta\mathcal{H}_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) + \int_{\partial\Sigma} \xi \cdot \tilde{\theta} \tag{2.38}$$

It is easy to check that this quantity satisfies the consistency condition (2.35), and hence there will be some function $\mathcal{H}_\xi$ that defines a 'conserved quantity'. It will not actually be conserved since

$$\begin{aligned}
\delta\mathcal{H}_\xi|_{\Sigma_1} - \delta\mathcal{H}_\xi|_{\Sigma_2} &= (\int_{\partial\Sigma_1} - \int_{\partial\Sigma_2})(\delta Q_\xi - \xi \cdot \theta + \xi \cdot \tilde{\theta}) \\
&= \int_{B_{12}} d(\delta Q_\xi - \xi \cdot \theta + \xi \cdot \tilde{\theta}) \\
&= \int_{B_{12}} (-X_\xi \cdot \omega + d(\xi \cdot \tilde{\theta})) = \int_{B_{12}} \delta f_\xi
\end{aligned} \tag{2.39}$$

which in general does not vanish. This is to be expected due to the possible presence of radiation at B. Hence, we will have a (3-form) f_ξ at B that describes the flux of $\mathcal{H}_\xi$ on B.

Note however, that the potential is not uniquely defined. We can fix it by demanding that f_ξ vanishes when no radiation is present. This is expected to happen when ϕ is a stationary solution, that is, when there is a nonzero infinitesimal asymptotic symmetry represented by an exact symmetry t^a which is timelike in a neighborhood of B. We then require that f_ξ vanishes for any asymptotic symmetry ξ^a on a stationary solution.

To see what condition on $\tilde{\theta}$ will ensure that this happens, we first note

$$\begin{aligned}\delta f_\xi &= - X_\xi \cdot \tilde{\omega} + d(\xi \cdot \tilde{\theta}) \\ &= - X_\xi \cdot \tilde{\omega} + L_\xi \theta' - \xi \cdot d\tilde{\theta} \\ &= - X_\xi \cdot \tilde{\omega} + L_\xi \tilde{\theta}\end{aligned} \tag{2.40}$$

where we used that $\xi \cdot d\tilde{\theta} = 0$ since ξ is tangent to B. Now, since we assume $\tilde{\theta}$ to be covariant, we have

$$L_\xi \tilde{\theta} = L_{X_\xi} \tilde{\theta} = \delta(X_\xi \cdot \tilde{\theta}) + X_\xi \cdot \delta\tilde{\theta} \tag{2.41}$$

to finally get

$$\delta f_\xi = \delta(X_\xi \cdot \tilde{\theta}) \tag{2.42}$$

Note that while defining $\mathcal{H}_\xi$ we assumed that there was some reference solution ϕ_0 on which $\mathcal{H}_\xi$ should vanish for all cross-sections; this immediately implies that f_ξ also vanishes on this reference solution. We assume that this solution is stationary, and impose the requirement that $\tilde{\theta}$ vanishes on stationary solutions. Since both $\tilde{\theta}$ and f_ξ vanish on ϕ_0 we get

$$f_\xi = X_\xi \cdot \tilde{\theta} \tag{2.43}$$

The requirement that $\tilde{\theta}$ vanishes on all stationary solutions, then immediately implies the flux f_ξ is zero as desired.

A quick summary of what we've done:

- given a representative ξ of an asymptotic symmetry, we defined a charge $\mathcal{H}_\xi$ on a cross-section of S of B as a solution of

$$\delta\mathcal{H}_\xi = \int_{S_1} (\delta Q_\xi - \xi \cdot \theta + \xi \cdot \tilde{\theta}) \tag{2.44}$$

with $\mathcal{H}_\xi$ defined by requiring that it vanishes on a reference solution (that is assumed to be stationary) on all cross-sections S and for all ξ

- Also, we required that $\tilde{\theta}$ vanishes on all stationary solutions, fixing its ambiguity. The charge $\mathcal{H}_\xi$ will then not be conserved, since in general there will be non-zero flux of $\mathcal{H}_\xi$ on B

$$\mathcal{H}_\xi(S_1) - \mathcal{H}_\xi(S_2) = \int_{\Delta B} f_\xi = F_\xi \tag{2.45}$$

A very well known example of spacetimes where the consistency condition (2.35) is not satisfied is that of asyntotically flat spacetimes when gravitational radiation is present at $\mathcal{I}^+$. In [2] it was shown that the above prescription gives the flux

$$f_\xi = X_\xi \cdot \tilde{\theta} = -\frac{1}{32\pi} N_{ab}\chi^{ab}\tilde{\epsilon} \tag{2.46}$$

where N_{ab} is the News tensor which encodes the gravitational degrees of freedom, $\tilde{\epsilon}$ is the volume form on $\mathcal{I}^+$ and $\chi_{ab} = \Omega\mathcal{L}_\xi g_{ab}$ where Ω is the conformal factor chosen in the conformal compactification of M and g_{ab} is the metric on M. The given formula for the flux was shown to match the one derived previously in [8].

2.7 Theories with a background (non-dynamical) metric

In the analysis of the last few sections, we assumed that the Lagrangian was diffeomorphism covariant, which implied that background fields χ were forbidden in the Lagrangian. In this section will derive a few important results in the presence of a background metric g.

We then assume a Lagrangian dependent on some dynamical fields ϕ and non-dynamical metric g. We will for now assume that the metric is dynamical so that a general variation of the Lagrangian is of the form

$$\delta L = (E_g)^{ab}\delta g_{ab} + E_\phi \delta\phi + d\theta \tag{2.47}$$

We define a symmetric stress-energy tensor of the fields ϕ by

$$\epsilon\frac{1}{2}T^{ab} = (E_g)^{ab} \tag{2.48}$$

Let's now look at the behavior of the Lagrangian under the action of an infinitesimal diffeomorphism ξ

$$\begin{aligned} X_\xi \cdot L &= (E_g)^{ab}\delta_\xi g_{ab} + E_\phi\delta_\xi\phi + d(X_\xi\cdot\theta) \\ &= \epsilon\frac{1}{2}T^{ab}\mathcal{L}_\xi g_{ab} + E_\phi\delta_\xi\phi + d(X_\xi\cdot\theta) \\ &= \epsilon T^{ab}\nabla_a\xi_b + E_\phi\delta_\xi\phi + d(X_\xi\cdot\theta) \end{aligned} \tag{2.49}$$

We can fix the metric and impose the matter field equations E_ϕ so that we are now on-shell in the original theory. We have

$$\begin{aligned} X_\xi \cdot L &= \epsilon T^{ab} \nabla_a \xi_b + d(X_\xi \cdot \theta) \\ &= \epsilon(\nabla_a T^{ab} \xi_b - \nabla_a (T^{ab} \xi_b) + d(X_\xi \cdot \theta) \\ &= \epsilon \nabla_a T^{ab} \xi_b + d(X_\xi \cdot \theta - k_\xi \cdot \epsilon) \end{aligned} \tag{2.50}$$

where we've defined $k^a_\xi \equiv \nabla_a T^{ab} \xi_b$ and noted that for a vector field S we have $d(S \cdot \epsilon) = \nabla_a S^a \epsilon$. We've defined previously a symmetry as a vector field X that left the Lagrangian invariant up to boundary terms

$$X \cdot L = d\alpha \tag{2.51}$$

Since ξ is arbitrary, it follows that

$$\nabla_a T^{ab} = 0 \tag{2.52}$$

We've hence, shown that covariance of the Lagrangian implies the conservation of the stress-energy tensor. Again, if we allow the metric g to be varied, the Noether current j_ξ will no longer be conserved when the $\phi-$field equations are obeyed since

$$\begin{aligned} dj_\xi &= -\epsilon \frac{1}{2} T^{ab} \mathcal{L}_\xi g_{ab} \\ &= -\epsilon T^{ab} \nabla_a \xi_b \\ &= -d(k_\xi \cdot \epsilon) + \epsilon \nabla_a T^{ab} \xi_b \end{aligned} \tag{2.53}$$

By inspection of the last equality, we see that $\epsilon \nabla_a T^{ab} \xi_b$ should be exact for arbitrary ξ. But this is only possible unless

$$\nabla_a T^{ab} = 0 \tag{2.54}$$

and we again recover the conservation of the stress energy tensor. Note that from this, (2.53) yields

$$d(j_\xi + k_\xi \cdot \epsilon) = 0 \tag{2.55}$$

from which it follows that

$$j_\xi + k_\xi \cdot \epsilon = dK_\xi \tag{2.56}$$

for some 2-form K_ξ locally constructed out of g_{ab}, ϕ, ξ and their derivatives. In other words, we've shown that, apart from a surface term K_ξ, the Noether current j_ξ is just $-T^{ab}\xi_b$.

As we've seen above, to a slice Σ and a symmetry X_ξ is associated an Hamiltonian H_ξ that solves

$$\delta H_\xi = \int_\Sigma \delta j_\xi - \int_{\partial\Sigma} (\xi \cdot \theta) \tag{2.57}$$

Note that such an Hamiltonian exists if we can find an (n-1) form B such that

$$\delta \int_{\partial\Sigma} (\xi \cdot B) = \int_{\partial\Sigma} (\xi \cdot \theta) \tag{2.58}$$

and

$$H_\xi = \int_\Sigma j_\xi - \int_{\partial\Sigma} (\xi \cdot B) \tag{2.59}$$

We see that apart from a surface term (which will vanish for the most commonly considered theories) the Hamiltonian is the integral of j_ξ over a slice Σ. Note that since we now longer have $j_\xi = dQ_\xi$, the Hamiltonian will not be a surface integral, and it will depend on representative ξ of the infinitesimal symmetry X_ξ. Also, since j_ξ is no longer closed, H_ξ will depend on the choice of slice Σ. Using (2.56) we have

$$\begin{aligned} H_\xi &= -\int_\Sigma k_\xi \cdot \epsilon + \int_{\partial\Sigma} (K_\xi - \xi \cdot B) \\ &= \int_\Sigma \tilde{\epsilon} T_{ab} n^b \xi^b + \int_{\partial\Sigma} (K_\xi - \xi \cdot B) \end{aligned} \tag{2.60}$$

where n is the future directed unit normal to Σ and $\tilde{\epsilon} = n \cdot \epsilon$ is the volume form induced on Σ by M. In particular if $\xi \equiv t$ is timelike, we define the canonical energy as $\mathcal{E} = H_t$, and see that apart from possible contributions at 'infinity', the canonical energy is the usual formula for the energy of some field ϕ.

Now, if ξ is an isometry of the background metric g, j_ξ is closed and so is $k_\xi \cdot \epsilon$. Then, in the case where Σ is a Cauchy surface, the fields ϕ decay sufficiently rapidly to infinity, and $B|_{\partial\Sigma} = 0$ (as it usually happens); H_ξ will not depend on Σ.

Suppose now that g is stationary (there is a timelike Killing vector), and consider a one-parameter family of solutions ϕ_λ in the phase space $\mathcal{F}$ such that ϕ_0 is also stationary ($\mathcal{L}_t \phi_0 = 0$ for the timelike Killing field t^a).

Let $\mathcal{E}_\lambda$ be the canonical energy on the solutions ϕ_λ. Consider now its derivative $d/d\lambda = Y_\lambda$

$$\begin{aligned} \frac{d\mathcal{E}_\lambda}{d\lambda} = Y_\lambda(\mathcal{E}_\lambda) = Y_\lambda \cdot \delta\mathcal{E}_\lambda &= -(X_t \cdot \Omega)(Y_\lambda) \\ &= \Omega_{\phi_\lambda}(\frac{d\phi_\lambda}{d\lambda}, \mathcal{L}_t\phi_\lambda) \end{aligned} \tag{2.61}$$

Since we assumed that $\mathcal{L}_t\phi_0 = 0$ we have

$$\frac{d}{d\lambda}\mathcal{E}|_{\lambda=0} = Y \cdot \delta\mathcal{E} = 0 \tag{2.62}$$

Consider now the second derivative of $\mathcal{E}$ with respect to λ. A nonzero contribution will occur only when the derivative acts on $\mathcal{L}_t\phi_\lambda$. Hence we find

$$\frac{d^2}{d\lambda^2}\mathcal{E}|_{\lambda=0} = \frac{1}{2}\Omega_\phi(\delta\phi, \mathcal{L}_t\delta\phi) \tag{2.63}$$

We showed that given a 1-parameter family of solutions ϕ_λ with $\mathcal{L}_t\phi_0 = 0$, the second order change to the energy of ϕ due to a perturbation $\delta\phi$ is just (up to a factor of 1/2) the integral of the sympletic form acting on $\delta\phi$ and $\mathcal{L}_t\delta\phi$.

In particular one can establish the following result. Let g be a non-dynamical spacetime metric so that $\mathcal{L}_t g = 0$, and g_λ a 1-parameter family of metrics with $g_{\lambda=0} = g$. As we've shown in section 2.2, δg will be a solution of the linearized field equations. Given a slice Σ, the energy of a perturbation δg on Σ, to first order in δg is

$$\mathcal{E}_{\delta g} = \frac{1}{2}\int_\Sigma \omega(\delta g, \mathcal{L}_t\delta g) \tag{2.64}$$

where ω is the pre-sympletic current of the original non-perturbed theory.

Chapter 3

Geometry of null hypersurfaces

In the last chapter, given some dynamical fields defined on a manifold M, we learned how to compute the flux of energy of these fields, which is defined on the boundary ∂M of the manifold. In this thesis we will use that same prescription to compute the flux of gravitational fields on a null surface (the cosmological horizon) in a de Sitter spacetime. In this chapter we will review some of the fundamentals of the geometry of null surfaces, following mainly [3].

As we've discussed, charges will in general be associated with asymptotic symmetries. In asymptotical flat spacetimes (spacetimes that 'look like' Minkowski at infinity), asymptotic symmetries at null infinity can be introduced as vectors fields ξ^a such that Killing's equation is approximately satisfied as one approaches null infinity. This gives rise to the asymptotic symmetric group, the BMS group, which is universal in the sense that it is shared by all asymptotical flat spacetimes. Here, we are interested not in asymptotical flat spacetimes, but in spacetimes with a general null boundary. Hence, instead of the BMS group, we will find the symmetry group of a null hypersurface; not surprisingly its structure will actually be similar to that of the BMS group, and will provide us with the symmetries which will generate our charges.

3.1 Basic definitions

Given a d-dimensional spacetime (M, g_{ab}), an hypersurface can be defined as the image of the embedding $\Pi : \mathcal{N} \to M$, where $\mathcal{N}$ is a $(d-1)$-dim manifold. Vectors in $\mathcal{N}$ can be mapped to vectors in M via push-forward Π_*. Similarly, forms in M are mapped to forms in $\mathcal{N}$ via pull-back Π^*, in particular the spacetime M will induce a metric q on $\mathcal{N}$, $q = \mathit{\Pi}^* g$.

We define a null hypersurface, as an hypersurface $\mathcal{N}$ whose normal l^a is null, that is $g_{ab}l^a l^b|_{\mathcal{N}} = 0$. Locally, an embedded hypersurface can be described as the zero sets of a scalar function f on M, which implies that $l_a = e^{\rho} df_a$ where $l_a = g_{ab}l^b$. Note the the null vector l^a is defined only on the null surface $\mathcal{N}$. The vector field l can be extended in a neighborhood of $\mathcal{N}$ in an arbitrary way; we define this extension by considering $\mathcal{N}$ as an element of a foliation $\mathcal{N}_f$ of M which are level sets of f, so that l is defined to be null and normal to each $\mathcal{N}_f$, and ρ is now also a scalar field on an open region of M around $\mathcal{N}$. This allows us to define the covariant derivative of l.

Now, one can show that the null hypersurface is ruled by null geodesics, that is the integral curves of l are null geodesics. First note that

$$dl = d\rho \wedge l \tag{3.1}$$

which can be seen by applying the exterior derivative on $l = e^{\rho} df$. Now, since $g_{ab}l^a l^b = 0$ we have

$$l \cdot dl = (l \cdot d\rho) l \tag{3.2}$$

Noting that

$$(l \cdot dl)_a = l^b \nabla_b l_a - l^b \nabla_a l_b = l^b \nabla_b l_a \tag{3.3}$$

where we used $l^b \nabla_a l_b = 1/2 \nabla_a (l^b l_b) = 0$; we get the geodesic equation for a non-affinelly parameterized geodesic

$$l^a \nabla_a l_b = k l_b \tag{3.4}$$

with $k \equiv l \cdot d\rho$, the non-affinity coefficient.

Note that a choice of null normal l_a is not unique, a rescaling of the following form gives a new null vector

$$l_a \to e^{\sigma} l_a \tag{3.5}$$

where σ is smooth function on M. Under a rescaling the non-affinity transforms as

$$k \to e^{\sigma}(k + \mathcal{L}_l \sigma) \tag{3.6}$$

We will say that a function that transforms as $f \to e^{-n\sigma} f$ under a rescaling of the null normals, has scaling weight n.

The tangent space $T(\mathcal{N})$ will be those vectors that satisfy $v^a l_b = 0$. Since l^a lies in this subspace, we identify it with a vector field l^i on $\mathcal{N}$. Covectors ω_a on M are mapped to covectors ω_i on $\mathcal{N}$ via pullback

$$\omega_a \to \omega_i = \Pi_i^a \omega_a \tag{3.7}$$

The pullback of the normal l_a is identically null since $l_i v^i = 0$ for all v^i

$$l_a \to l_i = \Pi_i^a l_a = 0 \tag{3.8}$$

3.2 Geometric fields on a null surface

The metric g on M induces a metric q on $\mathcal{N}$ via the pullback, which in local coordinates is just

$$q_{ij} = \Pi_i^a \Pi_j^b g_{ab} \tag{3.9}$$

The pullback of $l_a = g_{ab} l^b$ is

$$l_i = q_{ij} l^j = 0 \tag{3.10}$$

which shows that l^i is an eigenvector of the induced metric with eigenvalue zero, and the metric q_{ij} has signature $(0, +, +)$. We denote by W_p the two-dimensional subspace of $T_p(\mathcal{N})^*$ consisting of covectors ω_i that satisfy $\omega_i l^i = 0$. Tensors built on W_p will be denoted by indices A, B. Note that q_{ij} can be regarded as a tensor on $W_p \otimes W_p$, which we write as q_{AB}. Its inverse is written as q^{AB}, and we'll use these to raise and lower capital letter indices.

The bending of $\mathcal{N}$ in M is described by the Weingarten map. Consider the map $\mathcal{K}(X) = X \cdot \nabla l$, which is a map between vectors on $T(M)$ to vectors on $T(M)$; and its pullback to $T(\mathcal{N})$

$$(\Pi^*\mathcal{K})(X) = \mathcal{K}(\Pi_* X) = X^i \Pi_i^a \nabla_a l \tag{3.11}$$

In local coordinates, we write its components as $\mathcal{K}_i^a$. Note that $X^i\mathcal{K}_i^a$ is a vector field on $\mathcal{N}$ since

$$l_a \mathcal{K}_i^a = l_a \Pi_i^b \nabla_b l^a = \Pi_i^b \nabla_b (l^a l_a)/2 = 0 \tag{3.12}$$

where we used $l_a l^a|_{\mathcal{N}} = 0$ and the fact that the derivative is along the surface. Hence $\mathcal{K}$ maps vector fields on $\mathcal{N}$ to vector fields on $\mathcal{N}$, and it is an intrinsic tensor that we write in local coordinates as $\mathcal{K}_i^j$. This is the Weingarten map and it associates with each vector tangent to $\mathcal{N}$ the variation of the normal along that vector, with respect to the spacetime connection.

It follows immediately that

$$l^i \mathcal{K}_i^j = \kappa l^j \tag{3.13}$$

that is, the non-affinity coefficient κ is an eigenvalue of the Weingarten map.

We now define a map on $TM \otimes TM$

$$K(X,Y) = (\nabla_X l)(Y) = X^a Y^b \nabla_a l_b = X^a Y^b K_{ab} \tag{3.14}$$

where we've defined its components K_{ab}. Its pullback to $\mathcal{N}$ $K_{ij} = \Pi_i^a \Pi_j^b \nabla_a l_b$ is the second fundamental form of $\mathcal{N}$.

Since l_a is hypersurface-orthogonal it follows from Frobenius theorem that $\nabla_{[a} l_{b]} = l_{[a}\omega_{b]}$ for some 1-form ω in $\mathcal{N}$. Hence, even though K_{ab} is not symmetric, its pullback to $\mathcal{N}$ is, since

$$\begin{aligned} K_{[ij]} &= \Pi_i^a \Pi_j^b \nabla_{[a} l_{b]} \\ &= l_{[i}\omega_{j]} \\ &= 0 \end{aligned} \tag{3.15}$$

where the last equality follows since $l_i = \Pi_i^a l_a = 0$. Also, note that $K(X,Y) = q(X, \mathcal{K}(Y))$. Hence, in local coordinates we'll have

$$K_{ik} = q_{ij} \mathcal{K}_i^j \tag{3.16}$$

Similarly from the pullback of $l^a \nabla_a l_b = \kappa l_b$ it follows that K_{ij} is degenerate since

$$l^i K_{ij} = 0 \tag{3.17}$$

We see that K_{ij} lies in the subspace $W_p \otimes W_p$ and so we can write its components as K_{AB}. We can uniquely decompose the second fundamental form as

$$K_{AB} = \frac{1}{2}\theta q_{AB} + \sigma_{AB} \tag{3.18}$$

where θ is called the expansion and is the trace of K_{AB} and σ_{AB} is the shear, and is the traceless part of K_{AB}. There is another nice way to compute the second fundamental form. We first note that $\mathcal{L}_l g_{ab} = 2\nabla_{(a} l_{b)}$, then taking the pullback to $\mathcal{N}$ and noting that the pullback commutes with the Lie derivative we obtain

$$K_{ij} = \frac{1}{2}\mathcal{L}_l q_{ij} \tag{3.19}$$

We will say that a region of a null surface is stationary if there is a choice of normal vector T which satisfies the Killing equation on the surface and to first order in deviations from the surface, that is

$$\mathcal{L}_T g_{ab}|_{\mathcal{N}} = 0 \tag{3.20}$$

$$\nabla_c(\mathcal{L}_T g_{ab})|_{\mathcal{N}} = 0 \tag{3.21}$$

We will call the non-affinity of T as κ_T, the surface gravity. Taking the pullback to $\mathcal{N}$ of the previous equations and using the fact that the pullback commutes with the Lie derivate we have

$$K_{ij} = 0 \tag{3.22}$$

that is, the surface is shear free and expansion free.

3.3 Universal intrinsic structure of a null hypersurface

Consider a manifold $\mathcal{N}$ which is equipped with a smooth, nowhere vanishing vector field l^i and a smooth function κ. We define the base space $\mathcal{Z}$ as the space of integral curves of l^i, and assume that $\mathcal{N}$ is diffeomorphic to $\mathcal{Z} \times \mathbb{R}$. We will say that two pairs (l^i, κ) are equivalent if they are related by

$$\begin{aligned} l^i &\to e^{\sigma} l^i \\ \kappa &\to e^{\sigma}(\kappa + \mathcal{L}_l \sigma) \end{aligned} \tag{3.23}$$

where σ is a smooth function on $\mathcal{N}$. This leads to the equivalence class $\mathfrak{u} = \{l^i, \kappa\}$ of pairs related by (3.23), which we call the intrinsic structure of $\mathcal{N}$.

Suppose now that the surface $\mathcal{N}$ is part of the boundary of a spacetime (M, g). The spacetime will then induce a structure $\mathfrak{u}$ in the following way: the metric g will determine l^i by raising the index of a choice of normal covector; κ is just the non-affinity of l^i as defined in the previous section.

The intrinsic structure determines a foliation and a coordinate system on $\mathcal{N}$ as follows:

- Pick a cross-section S of $\mathcal{N}$
- Pick a member of $\mathfrak{u}$ for which κ vanishes, $(l^i, \kappa = 0)$
- Lie drag S along integral curves of l^i to get a foliation of $\mathcal{N}$
- Take a function $u : \mathcal{N} \to \mathbb{R}$, which satisfies $u|_S = 0$ and $\mathcal{L}_l u = 1$; the function u will be constant on the foliation we just defined and we'll have $l^i = (\partial_u)^i$
- Pick any coordinates θ^A on S and extend them to $\mathcal{N}$ requiring that $\mathcal{L}_l \theta^A = 0$

$\{u, \theta^A\}$ is then a coordinate system on $\mathcal{N}$ adapted to the foliation $\{S\}$. We will say that an intrinsic structure is complete if the generators l^i are complete, that is if their affine parameters can be extended arbitrarily. In the coordinate system we just discussed, this means that the range of u is $(-\infty, \infty)$. The equivalence class $\mathfrak{u}$ is intrinsic in the sense that if we have two different complete intrinsic structures, they should be diffeomorphic.

3.4 Symmetry group of an intrinsic structure

The symmetry group $G_{\mathfrak{u}}$ of a complete intrinsic structure is the group of diffeomorphisms $\varphi : \mathcal{N} \to \mathcal{N}$ which send an element of $\mathfrak{u}$ to another one. Hence, $\varphi \in G_{\mathfrak{u}}$ iff for any choice of representative, there is a smooth function β_φ so that

$$\begin{aligned} (\varphi)_* l^i &= e^{\beta_\varphi} l^i \\ (\varphi)_* \kappa &= e^{\beta_\varphi}(\kappa + \mathcal{L}_l \beta_\varphi) \end{aligned} \tag{3.24}$$

In fact, if this holds for one choice of representative (l^i, κ) it will hold for any $(l^{i'}, \kappa')$ with $\beta_\varphi(e^\sigma l) = \beta_\varphi(l) + (\varphi)_*\sigma - \sigma$ where σ is such that $l' = e^\sigma l$.

The Lie algebra $\mathfrak{g}_u$ of G_u are the vector fields ξ on $\mathcal{N}$ whose flow gives the diffeomorphisms φ. Their Lie derivative on the pair (l, κ) can be worked out explicitly, and gives

$$\begin{aligned} \mathcal{L}_\xi l &= \beta_\xi l \\ \mathcal{L}_\xi \kappa &= \beta_\xi + \mathcal{L}_l \beta \end{aligned} \tag{3.25}$$

where β_φ is another function on $\mathcal{N}$. Its dependence on the normal l is

$$\beta_\xi(e^\sigma l) = \beta_\xi(l) + \mathcal{L}_\xi \sigma \tag{3.26}$$

Pick now a representative $(l^i, \kappa = 0)$ and coordinates $x^i = (u, \theta^A) : \mathcal{N} \to U \subset R^3$ for which $l^i = (\partial_u)^i$. We consider diffeomorphisms φ that map a representative $l^i = (\partial_u)^i$ to a representative $l^i = (\partial_{\bar{u}})^i$. The action of such diffeomorphism on the coordinates can be given in general by $(\varphi^* x)^i = \bar{x}^i$ where $\bar{x}^i$ are given by

$$\begin{aligned} \bar{u}(u, \theta^A) &= \alpha(\theta^A) + e^{-\beta(\theta^A)} u \\ \bar{\theta}^A(u, \theta^B) &= \bar{\theta}^A(\theta^B) \end{aligned} \tag{3.27}$$

The symmetry algebra of these transformations, are solutions of (3.24) and its explicit form can be worked out quite easily by linearizing the transformation (3.27), which yields the vector field

$$\xi = [\alpha(\theta^A) - \beta(\theta^A) u]\partial_u + X^A(\theta^B)\partial_A \tag{3.28}$$

for arbitrary $X^A(\theta^B)$.

The group is parameterized by the three functions $\alpha, \beta, \bar{\theta}$, and its subgroups can be characterized as follows:

1. $\theta^A = \bar{\theta}^A, \alpha = 0, \beta = 0$ are the diffeomorphisms on the base space $\mathcal{Z}$. In most cases, $\mathcal{Z}$ will be diffeomorphic to a S^2

2. $\theta^A = \theta^A, \alpha = \alpha(\theta^A), \beta = \beta(\theta^A)$ which are reparametrizations of the null generators and are called supertranslations (analog to the supertranslations of the BMS group at null infinity)

3. $\theta^A = \theta^A, \alpha(\theta^A), \beta = 0$ are the affine supertranslations since they are just angle dependent displacements in affine parameter

4. $\theta^A = \theta^A, \alpha = 0, \beta(\theta^A)$ are angle dependent rescaling of the affine parameter

The Lie algebra $\mathfrak{g}_u$ was already defined. It inherits the Lie bracket structure of the space of vector fields on $\mathcal{N}$. From the definition of G_u as a subgroup of the group of diffeomoprhisms in $\mathcal{N}$, it follows that $\mathfrak{g}_\mathfrak{u}$ is closed under the Lie bracket. In [3] its structure was worked out in detail. Because its derivation is out of the scope of this thesis, we will only briefly describe it.

In [3],it was argued that

$$\mathfrak{g}_u \cong \mathrm{diff}(\mathcal{Z}) \ltimes (\mathfrak{b} \ltimes \mathfrak{s}_0) \tag{3.29}$$

where diff($\mathcal{Z}$) is the algebra of linearized diffeomorphism of $\mathcal{Z}$; $\mathfrak{s}_0$ is an abelian algebra of linearized affine supertranslations with its elements satisfying $\chi^i = fl^i$ and $\mathcal{L}_l f + \kappa f = 0$; finally $\mathfrak{b}$ is an abelian algebra of linearized rescaling and its elements $\chi^i = fl^i$ with $\mathcal{L}_l(\mathcal{L}_l f + \kappa f) = 0$. The semi-direct product $\mathfrak{b} \ltimes \mathfrak{s}_0 \cong \mathfrak{s}$ is the algebra of linearized supertranslations.

Chapter 4

General Relativity with a null boundary

After discussing charges on covariant phase spaces and a bit of geometry of null surfaces, now it's time to discuss the case of General relativity with a null boundary.

We will consider a manifold M, for which a null hypersurface $\mathcal{N}$ is a portion of its boundary. The pre-phase space $\tilde{\mathcal{F}}$ is the space of smooth metrics g on M for which the boundary $\mathcal{N}$ is null, and the induced boundary structure is complete. After we get rid of the diffeomorphism redundancy we discussed before, we get our covariant phase space $\mathcal{F}$. In [3] this was done in detail, and it was shown that the symmetry group associated with this phase space coincide with the one of the universal intrinsic structure of $\mathcal{N}$.

We will end this chapter with a formula for the flux corresponding to a generalized charge due to a perturbation at linear order in a null boundary.

4.1 Lagrangian, pre-sympletic and Noether potentials in General Relativity

Given a Lagrangian d-form L, all the quantities we introduced in chapter 2 can be easily computed by varying L. Here, we will review a method to do so, and derive expressions for all the relevant quantities.

The Einstein-Hilbert Lagrangian is

$$L = R\epsilon \tag{4.1}$$

where R is the Ricci scalar. Consider a one parameter family of metrics g_λ in the configuration space $\mathcal{C}$, with $g_0 \equiv g$. A variation $\delta \equiv \partial_\lambda|_{\lambda=0}$ of L is

$$\delta L = \delta R \epsilon + R \delta \epsilon \tag{4.2}$$

In order for us to compute the variation of the Ricci scalar, we will consider two affine connections ∇ and $\tilde{\nabla}$ on M and the map

$$C(X, Y) = (\tilde{\nabla}_X - \nabla_X)(Y) \tag{4.3}$$

which is a map from $T_pM \times T_pM$ to T_pM. Linearity in X follows from the definition of a connection. For the linearity in Y note

$$\begin{aligned} C(X, gY + fZ) &= (\tilde{\nabla}_X - \nabla_X)(gY + fZ) \\ &= (\tilde{\nabla}_X - \nabla_X)(gY) + (\tilde{\nabla}_X - \nabla_X)(fZ) \\ &= g(\tilde{\nabla}_X - \nabla_X)(Y) + f(\tilde{\nabla}_X - \nabla_X)(Z) \end{aligned} \tag{4.4}$$

Hence, C is a $(2, 1)$ tensor. If both connections are torsion-free we'll have

$$\nabla_X Y - \nabla_Y X = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X, Y] \tag{4.5}$$

and hence

$$\begin{aligned} 0 &= (\nabla_X Y - \nabla_Y X) - (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) \\ &= C(X, Y) - C(Y, X) \end{aligned} \tag{4.6}$$

which implies that C is symmetric. Its components are defined by $C(\partial_a, \partial_b) = C^c_{ab}\partial_c = (\tilde{\Gamma}^c_{ab} - \Gamma^c_{ab})\partial_c$ and we write

$$\tilde{\nabla}_a Y^b = \nabla_a Y^b + C^b_{ac} Y^c \tag{4.7}$$

We may also define another map $\tilde{C}$

$$\tilde{C}(X, \alpha) = (\tilde{\nabla}_X - \nabla_X)(\alpha) \tag{4.8}$$

which similarly is a $(1,2)$ tensor. It can easily be checked that its components are $\tilde{C}^c_{ab} = -C^c_{ab}$. For any tensor, the relation between the two connections can be easily derived using the Leibniz rule. For example

$$\tilde{\nabla}_a T^b_{cd} = \nabla T^b_{cd} + C^b_{af} T^f_{cd} - C^f_{ac} T^b_{fd} - C^f_{ad} T^b_{cf} \tag{4.9}$$

Assume now that $\tilde{\nabla}$ is compatible with g so that $\tilde{\nabla} g = 0$. Then we have

$$\begin{aligned} 0 &= \tilde{\nabla}_a g_{bc} \\ &= \nabla_a g_{bc} - C^f_{ab} g_{fc} - C^f_{ac} g_{bf} \end{aligned} \tag{4.10}$$

By subtracting this from its two cyclic permutations, one can easily obtain

$$C^c_{ab} = \frac{1}{2} g^{cf} (\nabla_a g_{bf} + \nabla_b g_{af} - \nabla_f g_{ab}) \tag{4.11}$$

In order for us to compare the Riemann tensors of the two connections note

$$\begin{aligned} \tilde{\nabla}_a \tilde{\nabla}_b Y^c &= \tilde{\nabla}_a Y^b = \tilde{\nabla}_a (\nabla_b Y^c + C^c_{bf} Y^f) \\ &= \nabla_a \nabla_b Y^c + C^f_{ab} \nabla_f Y^c - C^c_{af} \nabla_b Y^f \\ &+ (\nabla_a C^c_{bf}) Y^f + C^c_{bf} (\nabla_a Y^f) + C^c_{ae} C^e_{bf} Y^f - C^e_{ab} C^c_{ef} Y^f \end{aligned} \tag{4.12}$$

so that

$$\begin{aligned} \tilde{R}^c_{abd} Y^d &= (\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a Y^c) Y^c \\ &= (R^c_{abd} + \nabla_a C^c_{bf} - \nabla_b C^c_{af} + C^c_{ae} C^e_{bf} - C^c_{be} C^e_{af}) Y^f \end{aligned} \tag{4.13}$$

Note that the usual expressions for the Christoffell symbols and the Riemann tensor are recovered when $\nabla_a = \partial_a$.

The variation of the Ricci scalar can now be easily computed. First, note

$$\begin{aligned} \delta C^c_{ab} &= \frac{1}{2} g^{cf} (\nabla_a \delta g_{bf} + \nabla_b \delta g_{af} - \nabla_f \delta g_{ab}) \\ &= \frac{1}{2} (\nabla_a \delta g^c_b + \nabla_b \delta g^c_a - \nabla^c \delta g_{ab}) \end{aligned} \tag{4.14}$$

where we used the fact that all indices are raised and lowered using g. Note that δg_{ab} is the variation of g_{ab}, and we can use g to raise its indices, but we need to be careful. For example, one might naively think that $\delta g^{ab} = g^{ac}g^{bd}\delta g_{cd}$ is the variation of g^{cd} but since $g_{ab}g^{bc} = \delta^c_a$ one has

$$\delta(g_{ab})g^{bc} = -g_{ab}\delta(g^{bc}) \tag{4.15}$$

so that

$$\delta(g^{cd}) = -g^{bc}g^{ad}\delta g_{ab} = -\delta g^{cd} \tag{4.16}$$

The variation of the Riemann tensor is just

$$\delta R^c_{abd} = \nabla_a \delta C^c_{bf} - \nabla_b \delta C^c_{af} \tag{4.17}$$

The variation of the Ricci tensor follows by contracting indices

$$\begin{aligned}\delta R_{ab} &= \nabla_f \delta C^f_{ab} - \nabla_a \delta C^f_{fb} \\ &= \frac{1}{2}\nabla_f(\nabla_a \delta g^f_b + \nabla_b \delta g^f_a - \nabla^f \delta g_{ab}) - \frac{1}{2}\nabla_a \nabla_b \delta g^f_f\end{aligned} \tag{4.18}$$

The Ricci scalar can then be varied:

$$\begin{aligned}\delta R = \delta(g^{ab}R_{ab}) &= g^{ab}\delta R_{ab} + \delta g^{ab} R_{ab} \\ &= \nabla_a \Theta^a - \delta g_{ab} R^{ab}\end{aligned} \tag{4.19}$$

where we've defined $\Theta^a = g^{ab}\nabla^c \delta g_{bc} - g^{cb}\nabla^a \delta g_{cb}$, and used $-g^{bc}g^{ad}\delta g_{ab} = \delta g^{cd}$. Using the well known result $\delta\epsilon = \frac{1}{2}g^{ab}\delta g_{ab}\epsilon$, one concludes

$$\begin{aligned}\delta L &= \delta R\epsilon + R\delta\epsilon \\ &= \epsilon(-G^{ab}\delta g_{ab} + \nabla_a \Theta^a) \\ &= E^{ab}(g)\delta g_{ab} + \nabla_a \Theta^a \epsilon\end{aligned} \tag{4.20}$$

where G^{ab} is the Einstein tensor. Now, note that if $\theta = \Theta \cdot \epsilon$, one can show that $d\theta = \nabla_a \Theta^a \epsilon$. Hence

$$\delta L = E^{ab}(g)\delta g_{ab} + d\theta(g, \delta g) \tag{4.21}$$

with

$$\theta = \Theta \cdot \epsilon \tag{4.22}$$

We have then obtained an expression for the pre-sympletic potential θ. The Noether current j_ξ was defined as $j_\xi = X_\xi \cdot \theta - \xi \cdot L$, and using the previous result one has

$$j_\xi = J_\xi \cdot \epsilon \tag{4.23}$$

with

$$J^a_\xi = \nabla_b \nabla^{[b}\xi^{a]} - E^{ab}\xi_b \tag{4.24}$$

As we've noted previously, on pre-phase space where $E^{ab} = 0$ there is a Noether potential so that $j_\xi = dQ_\xi$. Indeed, note that for any 2-form s one has

$$d * s = S \cdot \epsilon \tag{4.25}$$

with

$$S^a = \nabla_b s^{ab} \tag{4.26}$$

Hence, one immediately obtains

$$Q_\xi = - * d\xi \tag{4.27}$$

with components

$$(Q_\xi)_{ab} = -\epsilon_{abcd}\nabla^c \xi^d \tag{4.28}$$

4.2 Noether potential on a null surface

In order for us to compute the charges H_ξ and their fluxes F_ξ, one needs to compute first the Noether potential Q_ξ and its variation δQ_ξ on a null surface $\mathcal{N}$. Its pullback to $\mathcal{N}$ is given by

$$\begin{aligned}(Q_\xi)_{ij} &= -\Pi^a_i\Pi^b_j\epsilon_{abcd}\nabla^c\xi^d \\ &= -\Pi^a_i\Pi^b_j[\tilde{\epsilon}_{abc}l_d - \tilde{\epsilon}_{dab}l_c + \tilde{\epsilon}_{cda}l_b - \tilde{\epsilon}_{bcd}l_a]\nabla^c\xi^d \\ &= -\Pi^a_i\Pi^b_j\tilde{\epsilon}_{abc}q^c\end{aligned} \tag{4.29}$$

where we used $\Pi^a_i l_a = 0$ and defined $q^c = 2l_d\nabla^{[c}\xi^{d]}$. Note that we have $l_c q^c = 2l_c l_d\nabla^{[c}\xi^{d]} = 0$, since the contraction of anti-symmetric and symmetric indices yields zero, and then $q^c \in T\mathcal{N}$. Hence, one should have

$$(Q_\xi)_{ij} = -\tilde{\epsilon}_{ijk}q^k \tag{4.30}$$

The vector q^c can be rewritten as

$$\begin{aligned}q^c &= l_d\nabla^c\xi^d - l_d\nabla^d\xi^c \\ &= \mathcal{L}_\xi l^c + g^{ac}\mathcal{L}_\xi l_a - 2\xi^d\nabla_d l^c \\ &= 2\beta_\xi l^c - 2\xi^d\nabla_d l^c\end{aligned} \tag{4.31}$$

where we've used the definition of β. Now, its straightforward to write Q_ξ in terms of intrinsic objects

$$(Q_\xi)_{ij} = 2\tilde{\epsilon}_{ijk}(\xi^p\mathcal{K}^k_p - \beta_\xi l^k) \tag{4.32}$$

Note that Q_ξ is invariant under the scaling transformation $l \to e^\sigma l$.

We now turn to the computation of the variation of the Noether potential. In order for us to do so, one needs to know $\delta\tilde{\epsilon}$ and $\delta\mathcal{K}$, since

$$(\delta Q_\xi)_{ij} = 2\delta\tilde{\epsilon}_{ijk}(\xi^p\mathcal{K}^k_p - \beta_\xi l^k) + 2\tilde{\epsilon}_{ijk}(\xi^p\delta\mathcal{K}^k_p) \tag{4.33}$$

We already know that $\delta\epsilon = \frac{1}{2}\epsilon g^{ab}\delta g_{ab}$, so that $\delta\tilde{\epsilon} = \frac{1}{2}\tilde{\epsilon}\delta q$ with $\delta q = q^{AB}\delta q_{AB} = h$. From now on, we'll denote δg and δq as h. Hence

$$(\delta Q_\xi)_{ij} = \tilde{\epsilon}_{ijk}(h\xi^p\mathcal{K}^k_p - h\beta_\xi l^k + 2\xi^p\delta\mathcal{K}^k_p) \tag{4.34}$$

Let's now look at the variation of $\mathcal{K}^j_i$. In order to define it, we first introduced a map $\mathcal{K} : TM \to TM$ with tensor components $\mathcal{K}^b_a = \nabla_a l^b$, and noticed that the pull back $\Pi^a_i\mathcal{K}^b_a$

defined a map $T\mathcal{N} \to T\mathcal{N}$ and hence the tensor with intrinsic coordinates $\mathcal{K}_i^j$. Hence we need to compute the variation of $\Pi_i^a \mathcal{K}_a^b$.

Notice that, given two connections $\tilde{\nabla}$ and ∇ we'll have

$$\tilde{\mathcal{K}_a^b} = \mathcal{K}_a^b + C_{ac}^b l^c \tag{4.35}$$

so that $\Pi_i^a \delta\mathcal{K}_a^b$ can be seen immediately from δC_{ab}^c

$$\begin{aligned} \Pi_i^a \delta\mathcal{K}_a^b &= \Pi_i^a \delta C_{ab}^b l^c \\ &= \frac{1}{2}\Pi_i^a(\nabla_a h_c^b + \nabla_c h_a^b - \nabla^b h_{ac}) l^c \end{aligned} \tag{4.36}$$

The first and last terms can be rewritten using the definition of the 1-form Γ, which was defined as

$d(h_a) = l \wedge \Gamma$ with $h_a = l_c h_a^c$. Note

$$\begin{aligned} \nabla_a h_c^b - \nabla^b h_{ac} &= \nabla_a h^b - \nabla^b h_a + h_a^c \nabla^b l_c - h^{bc}\nabla_a l_c \\ &= 2g^{bc} l_{[a}\Gamma_{c]} + h_a^c \nabla^b l_c - h^{bc}\nabla_a l_c \\ &= l_a \Gamma^b - \Gamma_a l^b + h_a^c \nabla^b l_c - h^{bc}\nabla_a l_c \end{aligned} \tag{4.37}$$

The second term can be rewritten noticing $\mathcal{L}_l h_a^b = l^c \nabla_c h_a^b + h_c^b \nabla_a l^c - h_a^c \nabla_c l^b$. Since $l_i = 0$ we get

$$2\Pi_i^a \delta\mathcal{K}_a^b = \Pi_i^a(-\Gamma_a l^b + \mathcal{L}_l h_a^b + 2h_a^c \nabla_c l^b - 2h_c^b \nabla_a l^c) \tag{4.38}$$

Notice that every term is orthogonal to l_b so that they give rise to tensors intrinsic to $\mathcal{N}$. Hence

$$\delta\mathcal{K}_i^j = -\frac{1}{2}\Gamma_i l^j + \frac{1}{2}\mathcal{L}_l h_i^j + h_i^k \mathcal{K}_k^j - h_k^j \mathcal{K}_i^k \tag{4.39}$$

This finishes the job and we finally obtain a full expression for the variation of the pullback of the Noether potential 2-form

$$(\delta Q_\xi)_{ij} = \tilde{\epsilon}_{ijk}[h\xi^p \mathcal{K}_p^k - h\beta_\xi l^k + 2\xi^p(-\frac{1}{2}\Gamma_i l^j + \frac{1}{2}\mathcal{L}_l h_i^j + h_i^k \mathcal{K}_k^j - h_k^j \mathcal{K}_i^k)] \tag{4.40}$$

4.3 Hamiltonian

As we've seen previously, to a generator ξ of an asymptotic symmetry, and a slice Σ, it is associated an Hamiltonian H_ξ that we've shown to be the solution of

$$\delta H_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) \tag{4.41}$$

We now assume that a cross-section S of $\mathcal{N}$ is a component of the boundary $\partial\Sigma$ so that

$$\delta H_\xi = \int_S (\delta Q_\xi - \xi \cdot \theta) + (...) \tag{4.42}$$

where $(...)$ are the contributions from the other components of $\partial\Sigma$. We've already computed the pullback of δQ_ξ to $\mathcal{N}$, and the remaining contribution is from $\xi^c \theta_{cab}$ where θ_{abc} is given by

$$\theta_{abc} = \epsilon_{abcd}\Theta^d \tag{4.43}$$

with $\Theta^d = g^{ef}\nabla^d h_{ef} - g^{de}\nabla^f h_{ef}$. The pullback of this to $\mathcal{N}$ is

$$\begin{aligned} \Pi_i^a \Pi_j^b \Pi_k^c \theta_{abc} &= \Pi_i^a \Pi_j^b \Pi_k^c \epsilon_{abcd}\Theta^d \\ &= \Pi_i^a \Pi_j^b \Pi_k^c \Pi_k^c \tilde{\epsilon}_{[abc} l_{d]}\Theta^d \\ &= \tilde{\epsilon}_{ijk} l_d \Theta^d \end{aligned} \tag{4.44}$$

since $l_i = 0$. Since $-l^f \nabla_e h_f^e = h_f^e \nabla_e l^f$ and using the definition of the Weingarten map we get

$$\begin{aligned} \theta &= \tilde{\epsilon}[\mathcal{L}_l h + h_i^j \mathcal{K}_j^i] \\ &= \tilde{\epsilon}[\mathcal{L}_l h + h_{AB}(\frac{1}{2}\theta q^{AB} + \sigma^{AB})] \end{aligned} \tag{4.45}$$

Combining everything we get a full expression for the component in $\mathcal{N}$ of the variation of the Hamiltonian. In general, given the conditions prescribed on the boundary, there might be no solution to (4.42). This can be seen explicitly by computing the pre-sympletic form explicitly and showing that the

$$\int_S \xi \cdot \omega = 0 \tag{4.46}$$

may not be satisfied in general. This was done in [3].

4.4 Generalized charges and corresponding fluxes

Since, as we've just seen an Hamiltonian H_ξ may not exist, we now turn to the calculation of the generalized charges $\mathcal{H}_\xi$ and respective fluxes F_ξ. To this end one needs to compute a pre-sympletic potential $\tilde{\theta}$ for the pullback of ω to $\mathcal{N}$. This potential is not unique, and we required that it should be such that the flux vanishes for stationary solutions. In general, a choice of a potential can be parameterized as

$$\tilde{\theta} = (\Pi^* \theta) - \delta\alpha \tag{4.47}$$

where α is a 3-form on $\mathcal{N}$ locally constructed from the pullback of the fields to $\mathcal{N}$. Its components are

$$\tilde{\theta}_{ijk} = \theta_{ijk} - \delta\alpha_{ijk} \tag{4.48}$$

With this choice of θ', $\mathcal{H}_\xi$ takes the form

$$\mathcal{H}_\xi = \int_S (Q_\xi - \xi \cdot \alpha) \tag{4.49}$$

up to an overall constant of integration on phase space, that can be fixed by fixing the value of $\mathcal{H}_\xi$ on a reference solution.

As previously mentioned, we say that a solution g is stationary, when there is a nonzero infinitesimal asymptotic symmetry represented by an exact symmetry T which is timelike in a neighborhood of B. This implies that T satisfies Killing's equation on $\mathcal{N}$. We've also showed that if ξ represents an asymptotic boundary symmetry, we have $\xi_a l^a$. Hence, from our previous discussion this implies that $\mathcal{N}$ is shear and expansion free on a stationary solution. Hence, since $f_\xi = X_\xi \cdot \tilde{\theta}$, the requirement of vanishing flux on stationary solutions amounts to the requirement that $\tilde{\theta} = 0$ when $\theta = \sigma_{AB} = 0$. Note that $\alpha_{ijk} = \tilde{\epsilon}_{ijk}\alpha$ for some function on $\mathcal{N}$, and then

$$\tilde{\theta} = \tilde{\epsilon}[\mathcal{L}_l h + h_{AB}(\frac{1}{2}\theta q^{AB} + \sigma^{AB}) - \delta\alpha] \tag{4.50}$$

We are then looking for a choice of a function α locally constructed from the fields so that $\delta\alpha = \mathcal{L}_l h + f(\theta, \sigma_{ab})$ for some function f which vanishes when $\theta = \sigma = 0$. We'll choose

$$\alpha = 2\theta\tilde{\epsilon} \tag{4.51}$$

and show that in fact it fits our requirements. One easily obtains $\delta\theta = \frac{1}{2}\mathcal{L}_l h$ and

$$\delta\alpha = \tilde{\epsilon}(\mathcal{L}_l h + h\theta) \tag{4.52}$$

so that

$$\begin{aligned}\tilde{\theta} &= \tilde{\epsilon} h_{AB}(\frac{1}{2}\theta q^{AB} + \sigma^{AB}) \\ &= \tilde{\epsilon}(h_i^j \mathcal{K}_j^i - h\theta)\end{aligned} \tag{4.53}$$

which vanishes when $\theta = \sigma_{AB} = 0$. With this choice of α, $\mathcal{H}_\xi$ becomes

$$\mathcal{H}_\xi = \int_S h_\xi \cdot \tilde{\epsilon} \tag{4.54}$$

with

$$h_\xi = \mathcal{K}(\xi) - \theta\xi - \beta_\xi l \tag{4.55}$$

One can show explicitly that $h'_\xi \cdot \tilde{\epsilon}$ is invariant under a rescaling of the null normal. We now compute $f_\xi = X_\xi \cdot \tilde{\theta} = \tilde{\theta}(\mathcal{L}_\xi g_{ab})$. Note that

$$\begin{aligned}\tilde{\theta} &= \tilde{\epsilon}(h_i^j \mathcal{K}_j^i - h\theta)) \\ &= \tilde{\epsilon} h_{ab}(\nabla^a l^b - g^{ab}\theta)\end{aligned} \tag{4.56}$$

so that $h_{ab} = \mathcal{L}_\xi g_{ab} = \nabla_{(a}\xi_{b)}$ implies

$$f_\xi = \frac{1}{8\pi}\tilde{\epsilon}\nabla_a\xi_b(\nabla^{(a}l^{b)} - g^{ab}\theta) \tag{4.57}$$

One can similarly compute f_ξ by noting that $f_\xi = d(h_\xi \cdot \tilde{\epsilon}) = \tilde{\epsilon} D_i h^i_\xi$ or

$$f_\xi = \frac{1}{8\pi}\tilde{\epsilon} D_p(\xi^m \mathcal{K}^p_m - \theta\xi^p - \beta_\xi l^p) \tag{4.58}$$

4.5 Flux for a supertranslation

Take a now a supertranslation $\xi = fl$, which are the ones we will be interested in. The flux F_ξ is then given by

$$
\begin{aligned}
F_\xi &= \frac{1}{8\pi}\int \tilde{\epsilon} D_p(\xi^m \mathcal{K}^p_m - \theta\xi^p - \beta_\xi l^p) \\
&= \frac{1}{8\pi}\int \tilde{\epsilon} D_p(f\kappa - \theta f + \mathcal{L}_l f)l^p \\
&= \frac{1}{8\pi}\int \tilde{\epsilon} f(\kappa\theta - \theta^2 - \mathcal{L}_l\theta) \\
&= \frac{1}{8\pi}\int \tilde{\epsilon} f(\sigma_{AB}\sigma^{AB} - \frac{1}{2}\theta^2)
\end{aligned}
\tag{4.59}
$$

where in the last equality we used the Raychaduri's equation.

Consider now a 1-parameter family of metrics $g_\lambda \in \mathcal{F}$ (that depend smoothly on λ), so that $g_0 = g$. In particular we will be interested in the case of a spacetime M and a metric g_0 for which the boundary of interest is expansion and shear free. A Taylor expansion around $\lambda = 0$ gives

$$
g_{ab}(\lambda) = g_{ab} + \lambda h_{ab} \tag{4.60}
$$

where $\delta g_{ab} = dg_{ab}/d\lambda|_{\lambda=0}$ is a first order perturbation. Charges will also depend smoothly on λ so that

$$
f_\xi(\lambda) = f_\xi + \lambda\delta f_\xi + \frac{1}{2}\lambda^2\delta f_\xi \tag{4.61}
$$

The first variation is trivially zero if both zero if θ and σ_{AB} vanish. The second variation is

$$
\delta^2 f_\xi = \frac{1}{8\pi}\epsilon[2\delta\sigma_{AB}\delta\sigma^{AB} - \delta\theta\delta\theta] \tag{4.62}
$$

The variation of θ is

$$
\delta\theta = \frac{1}{2}\mathcal{L}_l h = \frac{1}{2}\mathcal{L}_\xi h_{AB} q^{AB} \tag{4.63}
$$

and the variation of σ_{AB} is

$$
\begin{aligned}
\delta\sigma_{ij} &= \delta K_{ij} - \frac{1}{2}\delta\theta q_{ij} \\
&= \Pi^a_i \Pi^b_j \delta K_{ab} - \frac{1}{2}(\mathcal{L}_\xi h) q_{ij} \\
&= \mathcal{L}_\xi h_{ij} - \frac{1}{2}(\mathcal{L}_\xi h) q_{ij}
\end{aligned}
\tag{4.64}
$$

In terms of these we have the final result

$$\frac{1}{2}\delta^2 f_\xi = \frac{1}{32\pi}\epsilon q^{AB} q^{CD}(\mathcal{L}_\xi h_{AC}\mathcal{L}_\xi h_{BD} - \mathcal{L}_\xi h_{AD}\mathcal{L}_\xi h_{BC}) \tag{4.65}$$

which describes the flux corresponding to a generalized charge due to a perturbation δg in a null surface. For the purpose of these thesis, this is the main result of this chapter. In chapter 6, we will use it to compute the flux of energy of gravitational waves in the cosmological horizon of a de Sitter background.

Chapter 5

De Sitter space

With the machinery developed in the last chapters, we now turn to the discussion of the main setting of our problem: de Sitter spacetime. We will construct de Sitter spacetime, and describe its main features. We will discuss several coordinate systems which will help uncover many of the different masks of de Sitter, and also gain some intuition on them.

5.1 De Sitter as an embedding

The D-dimensional De Sitter spacetime dS_D is the maximally symmetric solution to Einstein's equations

$$G_{ab} + \Lambda g_{ab} = 0 \tag{5.1}$$

A maximally-symmetric space is a space with a maximal number of Killing vectors, which in a D-dim manifold is $D(D+1)/2$. Locally, a maximally symmetric space is fully specified by its scalar curvature R which is constant, and its Riemann tensor has the simple form

$$R_{abcd} = \frac{R}{D(D-1)}(g_{ac}g_{bd} - g_{da}g_{bc}) \tag{5.2}$$

Anticipating the definition of l as the de Sitter radius we will have

$$R_{ab} = \frac{(D-1)}{l^2}g_{ab}; \quad R = \frac{D(D-1)}{l^2} \tag{5.3}$$

D-dimensional de Sitter can be viewed as a timelike hyperboloid, embedded in a (D+1)-dim Minkowski spacetime $\mathcal{M}_{1,D}$ with metric

$$\eta = \eta_{AB} dx^A dx^B = -dX_0^2 + \sum_{i=1}^{D} dX_i^2 \tag{5.4}$$

where η_{AB} is the $(D+1)$-dim Minkowski metric. and $A, B = 0, ..., D$. In $\mathcal{M}_{1,D}$, de Sitter is the defined as the hypersurface $\mathcal{H}_l$

$$\eta_{AB} X^A X^B = -X_0^2 + \sum_{i=1}^{D} X_i^2 = l^2 \tag{5.5}$$

where l is the de Sitter radius introduced earlier. Given this description, it's easy to find coordinates for dS_D. It's clear that cross-sections of dS_D at constant X_0 are spheres S^{D-1}. The hyperboloid can be parametrized by

$$X_0 = l \sinh(t/l), \quad X_i = l \cosh(t/l) z_i \tag{5.6}$$

where $i = 1, ..., D$ and z_i are constrained to the unit sphere S_1^{D-1}, $\sum_i {z_i}^2 = 1$. Coordinates on the sphere can be built in the usual way $(\theta_1, ..., \theta_{D-1})$

$$\begin{aligned} z_1 &= \cos\theta_1 \\ z_2 &= \sin\theta_1 \cos\theta_2 \\ &\vdots \\ z_{D-1} &= \sin\theta_1 \dots \sin\theta_{D-2} \cos\theta_{D-1} \\ z_{D-1} &= \sin\theta_1 \dots \sin\theta_{D-2} \sin\theta_{D-1} \end{aligned}$$

The de Sitter metric is then the pull back of $\bar{g}$ to $\mathcal{H}_l$

$$g = -dt^2 + l^2 \cosh^2(t/l) d\Omega_{D-1}^2 \tag{5.7}$$

where $d\Omega_{D-1}^2$ is the metric on a unit sphere S_1^{D-1} in coordinates $(\theta_1, ..., \theta_{D-1})$. These are called global coordinates since they cover the entire de Sitter space. In these coordinates, de Sitter can be seen as a contracting and expanding sphere.

One can compute the Riemann tensor, Ricci tensor and Ricci scalar and recover the formulas introduced before. By plugging them in the Einstein's equations one gets

$$\Lambda = \frac{(D-1)(D-2)}{2l^2} \tag{5.8}$$

Since a maximally symmetric space is uniquely defined by the constant value of R, the metric we introduced describes in fact the geometry of de Sitter.

5.2 Isometries of de Sitter

In this embedding, the isometries of de Sitter can be seen as the isometries of $\mathcal{M}_{1,D}$ that preserve $\mathcal{H}_l$. By our knowledge of Minkowski spacetime, we know that the group of these isometries is just the Lorentz group $SO(D,1)$ of the embedding in the (D+1)-dim space. The $D(D+1)/2$ generators of $SO(D,1)$ can be written as

$$J_{AB} = \eta_{AC} X^C \partial_B - \eta_{BC} X^C \partial_A \tag{5.9}$$

with $J_{AB} = -J_{BA}$. Its Lie algebra is

$$[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} + \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} \tag{5.10}$$

These $D(D+1)/2$ generators naturally decompose in $D(D-1)/2$ rotations

$$J_{ij} = X_i \partial_j - X_j \partial_i \tag{5.11}$$

and D boosts

$$K_i = J_{0i} = X_0 \partial_i + X_i \partial_0 \tag{5.12}$$

5.3 Penrose Diagram

In order for us to better understand the causal structure of dS_D we will build its Penrose diagram. A Penrose diagram is a compactified representation of a spacetime as a bounded subset of R^2 endowed with a Lorentzian metric, so that light rays travel at 45º. 'Infinity' will be at finite coordinates and hence part of the boundary of this region. In order for us to do so, one needs to design a new coordinate system that

1. covers the whole spacetime
2. makes massless particles travel at 45º (that is, makes null geodesics where we fix all but one timelike and spacelike coordinates travel at 45º)

3. has finite range, so that one is able to fit it inside R^2

Note that given a spacetime (M, g) one can define a new spacetime $(M, \tilde{g})$ where $\bar{g} = \Omega^2 g$ and Ω is a smooth positive function on M. Given two vectors X, Y it's easy to see that $g(X, Y)$ and $\tilde{g}(X, Y)$ have the same sign; in particular null curves on (M, g) will also be null curves on $(M, \bar{g})$ (similarly for spacelike or timelike curves). Then, these spacetimes will have the same light cones, and hence the same causal structure. Noticing this, we only need to find coordinates for which the metric is conformally related to one where we know that light rays travel at 45º .

Notice that in general, by using coordinates where 'infinity' is at finite coordinates, the metric g will be singular at these coordinates, and we will need $\Omega \to 0$ so that $\Omega^2 g$ is regular at 'infinity'. Then the spacetime $(M, \bar{g})$ will be extendible, that is it will be isometric to a proper subset of a larger spacetime $(\bar{M}, \bar{g})$ where $\partial\bar{M}$ corresponds to 'infinity' in M. We call $(\bar{M}, \bar{g})$ the conformal compactification of (M, g). Our conformal diagram will be the projection of $(\bar{M}, \bar{g})$ to a plane.

The global coordinates that we introduced in the last section fit the first requirement. In order to achieve the other two note that we can expand the spherical part of the metric as

$$g = -dt^2 + l^2 \cosh^2(t/l)(d\theta_1^2 + \sin^2\theta_{D-2} d\Omega_{D-2}^2) \tag{5.13}$$

Any θ_1 but $\theta_1 = 0$ and $\theta_1 = \pi$ are $(D-2)$-spheres. These two values of θ_1 are just points, and we call them the north and south pole respectively. Note that null geodesics in the (t, θ_1) plane (geodesics where we fix all other θ_i's) do not travel at 45º .

We wish to make a transformation $t \to \sigma(t)$ that transform the metric as

$$-dt^2 + l^2 \cosh^2(t/l) d\theta_1^2 = \Omega^{-2}(\sigma)[-d\sigma^2 + d\theta_1^2] \tag{5.14}$$

This can happen if we make the transformation

$$dt = \Omega^{-1} d\sigma$$

with $\Omega^{-1} = l\cosh(t/l)$ and solution

$$\cos(\sigma) = 1/\cosh(t/l) \tag{5.15}$$

This accomplishes our second requirement. For the last one, note that $t \in (-\infty, \infty)$ corresponds to $\sigma \in (-\pi/2, \pi/2)$, so that it is also automatically satisfied.

The metric in the new coordinates is then

$$g = \Omega^{-2}(\sigma)[-d\sigma^2 + d\theta_1^2 + \sin^2\theta_1 d\Omega_{D-2}^2] \tag{5.16}$$

with $\Omega(\sigma) = \frac{\cos(\sigma)}{l}$. Note that this metric is in fact singular for $\sigma \in \{-\pi/2, \pi/2\}$, but $\bar{g} = \Omega^2 g$ is not, and $(\bar{M}, \bar{g})$ is a viable conformal compactification of our spacetime. Then our conformal diagram is just the projection of $(\bar{M}, \bar{g})$ to the plane (σ, θ_1). A few comments about it

- It's a square in the (σ, θ_1) plane, since σ and θ_1 have coordinate range π
- Slices of constant $\sigma$are the constant S_{D-1}'s
- Points in the diagram are S_{D-2}'s. The exception are the points $\sigma \in \{-\pi/2, \pi/2\}$, (right and left boundaries) which are the poles of the S_{D-1}'s
- The top and bottom boundaries are just past and future conformal infinities $\mathcal{I}^{\pm}$. Every causal curve reaches it but it takes infinite affine parameter to do so
- Null geodesics restricted to the (t, θ) plane on the physical spacetime will travel at 45º . Null geodesics with nontrivial motion on the S_{D-2}'s (other spherical coordinates not fixed along the motion) will not
- No single observer can access the entire spacetime. Take for example an observer at the north pole: this observer will not be able to receive signals from points outside the region $\mathcal{O}^-$ (figure 5.1). The boundary of this region, the line that stretches from the south pole at $\mathcal{I}^-$ to the north pole at $\mathcal{I}^+$, is called the event horizon of this observer. It is similar to the event horizon of a black hole, since signals beyond it will not be able to reach her. However, it is an observer dependent horizon, other observer will have different horizons. We call them cosmological horizons.
- Similarly, this observer at the north pole will only able to send signals to another half of the spacetime $\mathcal{O}^+$. The boundary of this region, the line that stretches from the north pole at $\mathcal{I}^-$ to the south pole at $\mathcal{I}^+$, is called the particle horizon of this observer.

- The intersection of these two regions $\mathcal{O}^+ \cap \mathcal{O}^-$ is called the (northern) causal diamond. It is the region fully accessible to the observer at the north pole, one is able to send a signal to someone in this region and receive it before we reach $\mathcal{I}^+$. Note that the causal diamond of an observer at the south pole is causally disconnected from the northern diamond.

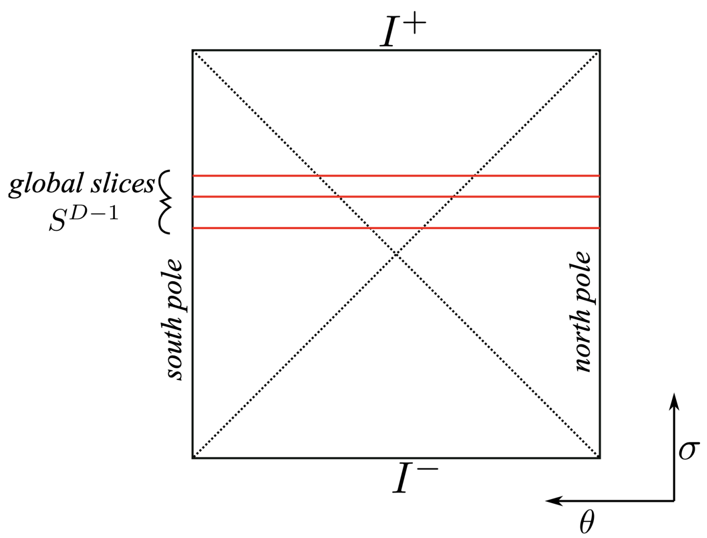

Figure 5.1: Conformal diagram of De Sitter. Diagram taken from [20]

5.4 Other coordinates on de Sitter

We introduced de Sitter as an hyperboloid on an higher dimension Minkowski space. We used this embedding to introduce a set of coordinates that cover the whole de Sitter space. In these coordinates, de Sitter is seen as an expanding and contracting S_{D-1}. Given the high degree of symmetry of de Sitter, different sets of coordinates, and hence different foliations will uncover very different perspectives on it. We'll introduce two of them, which will cover different parts of de Sitter.

5.4.1 Flat slicing

Again, we use the embedding on $\mathcal{M}_{1,D}$. In terms of the coordinates X_A we define the coordinates (t, x_i) as

$$X_0 = l\sinh(t/l) + \frac{r^2}{2l}e^{t/l} \quad X_1 = l\cosh(t/l) - \tfrac{r^2}{2l}e^{t/l} \quad X_i = e^{t/l}x_i$$

with $i = 2, ..., D$ and $r^2 = \sum_i x_i^2$. One can verify that in fact $\eta_{AB}X^A X^B = l^2$ as required. In the new coordinates the metric of de Sitter is just

$$g = -dt^2 + e^{2t/l}dx^2 \tag{5.17}$$

where $dx^2 = \sum dx_i^2$ is just the flat metric of R^{D-1} in the coordinates x_i. It's clear that this t is a different time from the time used in the global coordinates. These are the planar coordinates of de Sitter, or the flat slicing coordinates.

Note that

$$X_0 + X_1 = l(\cosh(t/l) + \sinh(t/l)) > 0 \tag{5.18}$$

for all t.Since we have in global coordinates

$$X_0 = l\sinh(t/l), \quad X_1 = l\cosh(t/l)\cos\theta_1 \tag{5.19}$$

$X_O = -X_1$ will be in conformal coordinates

$$\sin\sigma = -\cos\theta_1 \quad \Rightarrow \quad \sigma = -(\theta_! - \pi/2)$$

This is the diagonal line that goes from the north pole at $\mathcal{I}^+$ to the south pole at $\mathcal{I}^-$. Hence we see that these coordinates only cover the upper triangle (figure 5.1). Similar coordinates can be chosen to cover the lower triangle. Since flat slicing covers only a part of dS_D, the spacetime described by these coordinates (M, g) will be past geodesically incomplete, since all timelike geodesics will exit this region in the past, in finite affine parameter (unless we restrict them to sit at the south pole).

In order for us to relate these coordinates to the global ones we introduces before let's write the metric as

$$g = -dt^2 + e^{2t/l}(dr^2 + r^2 d\Omega_{D-2}^2) \tag{5.20}$$

where we put spherical coordinates on the flat slices. From this it's clear that we need to map $(t_g, \theta_!)$ to (t, r). Using the definitions of both of these coordinates in terms of the embedding coordinates X_A, with a lit bit of algebra, we get

$$e^{t/l} = \cos\theta_1 \cosh(t_g/l) + \sinh(t_g/l), \qquad \frac{l}{r} e^{t/l} = \sin\theta_1 \cosh(t_g/l) \tag{5.21}$$

Note in particular that for large times one has $t \sim t_g$ and $\cosh t/l \sim e^{t/l}$.

It will be also useful to transform the flat slicing coordinates (t, x_i) to conformal flat slicing coordinates (η, x_i). Again these will be defined so that

$$\begin{aligned} g &= -dt^2 + e^{2t/l} dx^2 \\ &= \Omega(t)^{-2}(-d\eta + dx^2) \end{aligned} \tag{5.22}$$

where η was defined by $\Omega^{-1} d\eta = dt$ with $\Omega^{-1} = e^{t/l}$. We get

$$\frac{\eta}{l} = -\Omega(t) \tag{5.23}$$

so that $t \in (-\infty, \infty)$ corresponds to $\eta \in (-\infty, 0)$. The metric in these coordinates is

$$g = (l/\eta)^2(-d\eta^2 + dx^2) \tag{5.24}$$

5.4.2 Flat vs global in cosmology

In an introductory cosmology course one showed that an homogeneous and isotropic universe could be described by a FRW metric

$$g_{FRW} = -dt^2 + a(t)^2 d\Sigma^2 \tag{5.25}$$

where $a(t)$ is a scale factor, and $d\Sigma^2$ is the metric for a maximally symmetric (D-1)-dimensional slice Σ which could be closed with positive curvature (S_{D-1}), flat with zero curvature (R^{D-1}) or open with negative curvature (hyperbolas H_{D-1}).

In global coordinates we wrote the metric of dS_D as

$$g = -dt^2 + l^2 \cosh^2(t/l) d\Omega^2_{D-1} \tag{5.26}$$

It's clear that in this coordinates dS_D can be seen as closed FRW universe with $a(t) = l\cosh(t/l)$ and $\Sigma = S_{D-1}$; as we've mentioned, it describes an exponentially decreasing S_{D-1} until $t = 0$ which then expands.

But also note that in flat slicing we got

$$g = -dt^2 + e^{2t/l}dx^2 \tag{5.27}$$

which describes an expanding flat FRW universe with $\Sigma = R^{D-1}$ and $a(t) = e^{t/l}$.

In Cosmology one usually assumes that matter has a preferred rest frame - the average rest frame of the galaxies - so that one assumes all the matter to be comoving in some fixed coordinate. In the case of global de Sitter we have an universe where matter is comoving at fixed θ coordinates, where proper distances first shrink and then expand. In the flat slicing de Sitter, matter is comoving at a fixed x coordinate, and proper distances are expanding exponentially.

Even tough one hasn't included any matter in the Einstein's equations, this will be a good approximation when the energy density of the cosmological constant dominates over all the other components.

Note that we got two different FRW universes from the same de Sitter spacetime. This happens because de Sitter has no preferred coordinate system and hence no preferred rest frame. However, in the context of cosmology, matter will set an average rest frame. When we specify different coordinates, one is setting a different rest frame for the matter, and hence describing different universes. This amounts to a choice of different initial conditions on the matter distribution. Note however that for asymptotic times, as the sphere becomes larger, the two spaces become the same.

5.4.3 Static coordinates

Has we've seen de Sitter we'll have a bunch of timelike isometries, but in the coordinates we've seen the metric was always dependent on the timelike coordinate. One wishes to construct a coordinate system where the timelike coordinate does not appear in the metric. This amounts to picking a coordinate system where ∂_t is proportional to one of the timelike isometries, for example K_1

$$\partial_t = \frac{\partial X_0}{\partial t}\partial_0 + \frac{\partial X_1}{\partial t}\partial_1 \propto K_1 = X_0\partial_1 + X_1\partial_0 \tag{5.28}$$

Take for example Rindler (ρ, t) coordinates in the (X_0, X_1) plane

$$X_0 = \rho \sinh(t/l) \quad X_1 = \rho \cosh(t/l) \tag{5.29}$$

with $\rho > 0$ which covers $X_1 > 0$. In these coordinates we have

$$g = -\,(\frac{\rho}{l})^2 dt^2 + d\rho^2 \tag{5.30}$$

and using (5.28) and (5.29) we have

$$\partial_t = \frac{1}{l} K_1 \tag{5.31}$$

One can generalize this to build coordinates on the hyperbola $\eta_{AB} X^A X^B = l^2$. Take the following parametrization of the hyperbola inspired by the Rindler coordinates

$$X_0 = \sqrt{l^2 - r^2} \sinh(t/l) \quad X_1 = \sqrt{l^2 - r^2} \cosh(t/l) \quad X_i = r z_i$$

in terms of coordinates (t, r, z_i) with $i = 2, ..., D$, $r < l$ and $\sum_i z_i^2 = 1$. In terms of these coordinates, the de Sitter metric is just

$$g = -(1 - r^2/l^2) dt^2 + \frac{dr^2}{1 - r^2/l^2} + r^2 d\Omega^2_{D-2} \tag{5.32}$$

The metric is independent of t as we intended. In fact, one can repeat the steps that let to (5.31) to get $\partial_t = \frac{1}{l} K_1$. These are then called the static patch coordinates.Note that the metric is singular when $r \to l$. What happens when $r \to l$? Looking at the embedding coordinates one might assume that $X^0, X^4 \to 0$, but we can do a little bit better. Close to l we write $r = l(1 - \epsilon^2/2)$ with $\epsilon \ll 1$ so that $r \to l$ is equivalent to $\epsilon \to 0$ and X^A becomes

$$X_0 \sim \epsilon \sinh(t/l) \qquad X_1 \sim \epsilon \cosh(t/l)$$

The limit $\epsilon \to 0$ can now be done while fixing X^A if we also send $t \to \pm\infty$ and keep $\epsilon e^{\pm t/l}$ finite. Hence we see that we can identify the surface $r = l$ with $X_0 = \pm X_1$, which corresponds to the boundary of the southern causal diamond. Hence this set of coordinates only cover the causal diamond of an observer at the south pole; a similar coordinate system can be designed to cover the northern causal diamond as well as the future and past causal diamonds. In fact note that when $r \to l$ we have

$$\partial_t \propto \partial_0 - \partial_1 \tag{5.33}$$

which implies that it is null in the horizon. In fact it will be normal to it. Hence we see that the cosmological horizon of an observer sitting at the north pole is a bifurcate Killing horizon for ∂_t and the bifurcation sphere is S_{D-2} at the middle of the Penrose diagram. One could use this timelike Killing vector to define a notion of energy on dS_D, but unfortunately the situation is a bit more tricky. One can see explicitly that K_1 is timelike in the southern and northern causal diamonds but spacelike in the future and past causal diamonds. Crucially it will be spacelike in a neighborhood of $\mathcal{I}^+$. In fact any Killing field will since they should be tangent to it, and $\mathcal{I}$ in de Sitter is spacelike. More generally there is no asymptotic symmetry which is timelike in a neighborhood of $\mathcal{I}$.

Chapter 6

Linearized gravity with a cosmological constant

In this chapter, we will use all the machinery developed so far to try and compute the flux of energy of gravitational waves in the cosmological horizon of a de Sitter background in the linear approximation, and generalize the quadrupole formula of [10] to the cosmological horizon.

6.1 De Sitter in a Poincaré Patch

When a cosmological constant is not present, gravitational waves can be studied as linear perturbations to Minkowski spacetime as we've seen in 1.2. They are solutions to the linearized Einstein equations. In the presence of a cosmological constant, one wishes to study linear perturbations to de Sitter spacetime. Since the de Sitter causal structure is quite different from the one of Minkowski, a few considerations need to be made.

Consider a situation where matter fields are bounded to some region $\sum$, with fixed physical size. What we mean by this is that given some some foliation of de Sitter in spacelike hypersurfaces (for example the S_{D-1} spheres in the global chart or the R^{D-1} in flat slicing) the matter fields should be bounded by some constant proper distance. If this is the case, region Σ will have future and past endpoints in $i^{\pm}$ (see Figure 6.1), since the spacelike hypersurfaces will be expanding (or contracting in the limit to $\mathcal{I}^{-}$). To see this just take conformal coordinates, and notice that in order for proper distance to be preserved as $\sigma \to \pm\pi/2$ ($t \to \pm\infty$) we need $\theta \to 0$.

Note also that, as we've discussed earlier, in contrast to Minkowski, in de Sitter there is an observer dependent horizon. In particular in Minkowski $E^+(i^-)$ is the whole Minkowski space, but in de Sitter it is only $\mathcal{O}^+$, which we call the Poincaré Patch.

Hence, in order to study the radiation emitted by matter confined to the the region $\sum$, we can restrict ourselves to this region, since any observer whose wordline is confined outside this region will never be able to measure the radiation.

Following the remarks made in the end of chapter 5, we know that there is no asymptotic symmetry which is timelike in a neighborhood of $\mathcal{I}$, which makes a definition of energy in its neighborhood slightly sloppy. Note that in the linear case, if we impose no incoming radiation across $E^+(i^-)$, the flux across $E^-(i^+)$ will equal that across $\mathcal{I}$. Also, the time translation Killing field we will consider is future pointing and timelike inside the region bounded by these surfaces, and null on both surfaces. This seems to mimic the behavior of the time translation field on Minkowski spacetime. For these reasons, and since $E^-(i^+)$ is null and could be taken to be in the 'far zone', it was suggested in [10, 11] that its analog in the non-linear theory could be the appropriate arena to study gravitational waves.

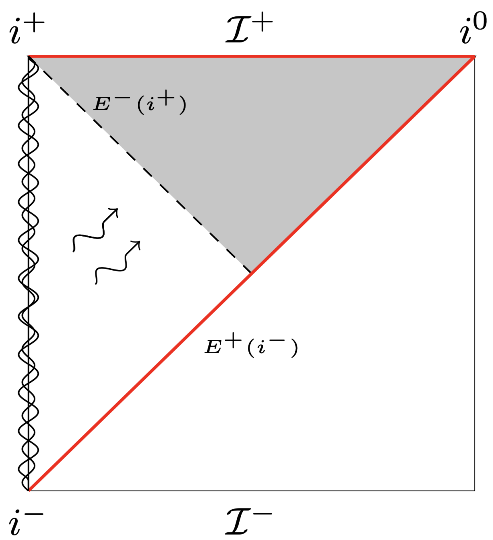

Figure 6.1: Conformal diagram of De Sitter. Diagram taken from [10]

In $\mathcal{O}^+$, we already know that the metric can be written in flat coordinates as

$$g = -dt^2 + e^{2t/l}dx^2 \tag{6.1}$$

or in conformal coordinates

$$g = (\frac{l}{\eta})^2(-d\eta^2 + dx^2) \tag{6.2}$$

We will be mostly using the conformal metric, but notice that these coordinates are not well suited for $\Lambda \to 0$ $(1/l \to 0)$ limit. To do that, we need to go back to flat slicing. Hence, when making comparisons with Minkowski, we should use the differential structure induced by (t, x).

As discussed before, the flat slicing metric is past geodesically incomplete, since all timelike geodesics will exit this region in the past, in finite affine parameter (unless we restrict them to sit at the south pole). Because of this, even though this metric admits 10 Killing vectors locally, only 7 preserve the boundary of the Poincaré Patch and are hence global exact symmetries of $(\mathcal{O}^+, g)$: 3 spatial translations T_i, 3 spatial rotations R_i and a dilation $T = -H[\eta\partial_\eta + \delta^{ij}x^i\partial_i]$.

Note that if we take the limit as the mass goes to zero of the dilation T we get the time translation in the Schwarzschild de Sitter spacetime. Also, the limit $\Omega \to 0$ takes the dilation to time translation in Minkowski. For these reasons, and since it is future pointing and timelike inside the southern causal diamond, we will refer to T as the time translation. These Killing vectors form a Lie algebra, the Lie algebra of symmetries of the poincare patch $\mathcal{O}^+$.

6.2 Linearized theory

In order for us to study gravitational perturbations of de Sitter spacetime, we consider a 1-parameter family of metrics g_λ

$$\begin{aligned} g_\lambda &= g + \delta g \lambda \\ &= g + h_\lambda \end{aligned} \tag{6.3}$$

with $g_0 = g$ and where g is the de Sitter metric, $\delta g = \frac{dg}{d\lambda}|_{\lambda=0}$ and $\lambda \ll 1$ since we are interested in metrics close to the de Sitter metric (small perturbations). We omit the dependence on λ for simplicity of notation and will write in general

$$\bar{g}_{ab} = g_{ab} + h_{ab} \tag{6.4}$$

The components of h are of $O(\varepsilon)$ where $\varepsilon \ll 1$, and h will be treated as a first order quantity. Indices will be raised and lowered using g and we'll have

$$\bar{g}^{ab} = g^{ab} - h^{ab} \tag{6.5}$$

where $h^{ab} = g^{ac}g^{bd}h_{cd}$. The physical spacetime $(M, \bar{g})$ can then be described in terms of a background spacetime with a dynamical field on it (M, g, h).

As we've mentioned before, GR is a diffeomorphism invariant theory. This means that a space time and a stress energy tensor (M, g, T) is physically equivalent to another $(M, \phi_* g, \phi_* T)$ if $\phi : M \to M$ is a diffeomorphism. Now we are restricting our attention to metrics of the form (6.4), and are hence interested in diffeomorphisms that preserve this form. Given a 1-parameter family of diffeomorphisms ϕ_t generated by a vector field X, for $t \ll 1$ we'll have for any tensor T

$$\begin{aligned}(\phi_{-t})_* T &= T + t\mathcal{L}_X T \\ &= T + \mathcal{L}_\xi T\end{aligned} \tag{6.6}$$

where $\xi = tX$ is assumed to be a first order quantity. In particular for the metric we have

$$(\phi_{-t})_* \bar{g} = g + h + \mathcal{L}_\xi g \tag{6.7}$$

where $\mathcal{L}_\xi h$ was neglected since it is a higher order quantity. Since $(\phi_{-t})_* \bar{g}$ and $\bar{g}$ are physically equivalent, we see that linearized gravity has a gauge freedom: two perturbations h and $h + \mathcal{L}_\xi g$ will describe the same physics. Note also that any spacetime tensor $T_\lambda = T + \lambda \delta T$ which vanishes in the unperturbed theory will be gauge invariant to first order since

$$(\phi_{-t})_* T_\lambda = T_\lambda + \mathcal{L}_\xi T = T_\lambda \tag{6.8}$$

The metric $\bar{g} = g_\lambda$ will be a solution of Einstein equations

$$G_{ab}(\lambda) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda) \tag{6.9}$$

and hence perturbation $h = \lambda \delta g$ will be a solution of the linearized Einstein equations

$$\delta G_{ab} + \Lambda \delta g_{ab} = 8\pi \delta T_{ab}$$
$$\delta R_{ab} - \frac{1}{2}(R\delta g_{ab} + \delta R g_{ab}) + \Lambda \delta g_{ab} = 8\pi T_{ab}$$

Note that since $T_{ab}(0) = 0$ we will just denote δT_{ab} by T_{ab}. We can use the results from section 4.1 to write

$$-\frac{1}{2}(D_d D^d h_{ab} + D_a D_b h) + D^a D_{(a} h_{b)d} -$$
$$-\frac{1}{2} g_{ab}(-D_a D^a h + D^c D^d h_{cd} - \Lambda h) = 8\pi T_{ab}$$

where D is the connection of the background metric g. The field equations simplify considerably if one considers the trace reversed perturbations $\bar{h}_{ab} = h_{ab} - \frac{1}{2} g_{ab} \bar{h}_{ab}$ with $\bar{h} = -h$

$$-\frac{1}{2}(D_d D^d \bar{h}_{ab} + g_{ab} D^c D^d \bar{h}_{cd}) + D^d D_{(a} \bar{h}_{b)d} - \Lambda \bar{h}_{ab} = 8\pi T_{ab} \tag{6.10}$$

6.3 Solving the Einstein equations

Now, pick a coordinate system for $\mathcal{O}^+$. We take the conformal flat coordinates introduced before so that

$$\begin{aligned} \bar{g}_{ab} &= g_{ab} + h_{ab} \\ &= (\frac{l}{\eta})^2 (\eta_{ab} + \gamma_{ab}) \end{aligned} \tag{6.11}$$

where we introduced $\gamma = a(\eta)^{-2} h$ with $a(\eta) = l/\eta$. Consider now the coordinate basis $(\partial_\eta, \partial_i)$. The ∂_i will be tangent to the slices of constant η, and ∂_η will be orthogonal. Since $g(\partial_\eta, \partial_\eta) = -(l/\eta)^2$, $n = -(\eta/l)\partial_\eta$ is the future pointing unit normal to these slices (note that η is negative in $\mathcal{O}^+$).

6.3.1 Source-free Einstein equations

As noted before, there is a gauge freedom in linearized gravity. One can use this freedom to impose conditions on the perturbations h that simplify the equations. In Minkowski spacetime its common to impose the Lorentz and radiation gauge where $\partial^a \bar{h}_{ab} = 0$, $\eta^{ab}\bar{h}_{ab} = 0$ and $\bar{h}_{0a} = 0$. In this case, these will be similarly useful.

It's easy to note that (6.10) simplifies to

$$\nabla^c \nabla_c \bar{h}_{ab} - 2H^2 \bar{h}_{ab} = 0 \tag{6.12}$$

if one imposes

$$\nabla^a \bar{h}_{ab} = 0 \tag{6.13}$$

This can always be done, since one can pick ξ_a so that $\nabla^a \nabla_b \xi_a = -\nabla^a \bar{h}_{ab}$ (which can be solved using a Green function) and we get the gauge (6.13). Note that (6.13) does not restrict completely our gauge freedom, one can further gauge transform $\bar{h}$ provided that (6.13) is still satisfied. One can do so by choosing an additional ξ_a that satisfies $\nabla^a \nabla_b \xi_a = 0$, which gives us four additional residual gauge transformations to work with. With this one can further impose

$$g^{ab}\bar{h}_{ab} \quad \partial^a_\eta \bar{h}_{ab} = 0$$

which is the natural generalization of the Lorentz and radiation gauge. We started with a field $\bar{h}_{ab}$ with 10 independent components; the condition (6.13) gave us 4 restrictions, and we further imposed 4 residual transformations, ending up with a field $\bar{h}$ with 2 independent components, describing the polarization components of the gravitational field. In terms of the field γ_{ab}, (6.12) becomes even simpler. In fact, the gauge conditions will be

$$D^a \gamma_{ab} = 0 \quad \gamma_{ab}\partial^a_\eta = 0 \quad \gamma_{ab}\eta^{ab} = 0$$

where D is the connection compatible to η_{ab}; and the Einstein equations simplify to

$$D_a D^a \gamma_{ab} - 2\frac{a'(\eta)}{a(\eta)}\gamma'_{ab} = D_a D^a \gamma_{ab} + \frac{2}{\eta}\gamma'_{ab} = 0 \tag{6.14}$$

with $D_a D^a = \eta^{ab} D_a D_b$ and $f' = \partial^a_\eta D_a f$.

These are just the equations satisfied by a linear perturbation h in Minkowski space, with the exception of the term $\frac{2}{\eta}\gamma'_{ab}$. General solutions on $\mathcal{O}^+$ can be easily found in Fourier space (see for example [10]).

6.3.2 Einstein equations with a source

If a nonzero energy momentum tensor is considered, the previous gauge choice is also possible, but not as useful (see [21]). A generalization of the Lorentz gauge will be considered

$$\nabla^a \bar{h}_{ab} = B_b \tag{6.15}$$

with $B_b = \frac{2}{l} n^a \bar{h}_{ab}$. Note that such a choice is also always possible, provided that we pick ξ_a

so that $\nabla^a \nabla_b \xi_a = -\nabla^a \bar{h}_{ab} + B_b$. If we take the field $\bar{\chi}_{ab} = a^{-2} \bar{h}_{ab}$ and decompose it in components adapted to the cosmological slices $\eta = \text{const}$ the Einstein equations simplify significantly. Take

$$\tilde{\chi} = (\eta^a \eta^b + \mathring{q}^{ab}) \chi_{ab} \quad \chi_a = \eta^c \mathring{q}_a^c \chi_{bc} \quad \chi_{ab} = \mathring{q}_a^c \mathring{q}_b^d \chi_{dc}$$

$$\tilde{\mathcal{T}} = (\eta^a \eta^b + \mathring{q}^{ab}) \mathcal{T}_{ab} \quad \mathcal{T}_a = \eta^c \mathring{q}_a^c \mathcal{T}_{bc} \quad \mathcal{T}_{ab} = \mathring{q}_a^c \mathring{q}_b^d \mathcal{T}_{dc}$$

where $\mathring{q}$ is the pullback of the metric η_{ab} to the cosmological slices. Note that in the previous decomposition, one took the pullback of $a^{-2} g = \eta$ and not $g$itself. In fact, like in the previous case, the Einstein equations are more easily written in terms of the derivative operators of η_{ab}. Hence, the fields χ_{ab} are thought to be living in a spacetime $(\mathcal{O}^+, \eta_{ab})$. In the gauge we introduced, and using this decomposition, the Einstein equations are split as follows (see [21])

$$\begin{aligned} \mathring{D}^2 (\frac{1}{\eta} \tilde{\chi}) &= -\frac{16\pi}{\eta} \tilde{\mathcal{T}} \\ \mathring{D}^2 (\frac{1}{\eta} \chi_a) &= -\frac{16\pi}{\eta} \mathcal{T}_a \\ (\mathring{D}^2 + \frac{2}{\eta} \partial_\eta) \chi_{ab} &= -16\pi \mathcal{T}_{ab} \end{aligned} \tag{6.16}$$

where again $\mathring{D}$ is the compatible with η_{ab}, $\mathring{D}^2 = \eta^{ab} \mathring{D}_a \mathring{D}_b$, and the gauge condition (6.15) becomes

$$\mathring{D}^a \chi_{ab} = \partial_\eta \chi_b - \frac{2}{\eta}\chi_b \tag{6.17}$$
$$\mathring{D}^a \chi_a = \partial_\eta(\tilde{\chi} - \chi) - \frac{1}{\eta}\tilde{\chi}$$

In the foliation of $(\mathcal{O}^+, \eta_{ab})$ induced by the coordinates $(\eta, \mathbf{x})$, it can be seen explicitly that $(-l/4\eta)\tilde{\chi}$ is the perturbed lapse function and $(l/\eta)^2 q^{ab}\chi_b$ is the perturbed shift field. These are just gauge fields, that depend on the specific foliation considered and hence do not contain any physical information (for an introduction to the 3+1 formalism in General Relativity and further information on these objects please consult [22]). We thus see that the physical degrees of freedom are only encoded in the spatial projection χ_{ab}.

Now, we wish to obtain solutions to the previous equations. To do so, we introduce the Green's function $G(x, x')$ of the operator $\mathring{D}^2$ which obeys

$$\mathring{D}^2 G(x, x') = \delta(x - x') \tag{6.18}$$

Given $G(x, x')$, solutions to the first of the equations in (6.16) are written as

$$\tilde{\chi}(x) = -\eta \int \mathring{\epsilon} G(x, x') \frac{\mathcal{T}}{\eta'} \tag{6.19}$$

Since, we are studying solutions on $(\mathcal{O}^+, \eta_{ab})$ full translational invariance is assumed, which implies $G(x, x') = G(x - x')$. In the physical context we are interested, radiation is not expected to be present at $E^+(i^-)$, since it would only be possible if emitted by events outside of $\mathcal{O}^+$. Hence, the retarded green's function provides us the correct boundary conditions. In fact its solution is already well known from Electromagnetism

$$G_r(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\delta(\eta - \eta' - |\mathbf{x} - \mathbf{x}'|) \tag{6.20}$$

The solutions are then written

$$\tilde{\chi}(\eta, \mathbf{x}) = 4\eta \int_\Sigma \frac{\mathring{\epsilon}}{|\mathbf{x} - \mathbf{x}'|} \frac{\tilde{\mathcal{T}}(\eta_r, \mathbf{x}')}{\eta_r}$$
$$\chi_{\bar{a}}(\eta, \mathbf{x}) = 4\eta \int \frac{\mathring{\epsilon}}{|\mathbf{x} - \mathbf{x}'|} \frac{\tilde{\mathcal{T}}a(\eta_r, \mathbf{x}')}{\eta_r}$$

where $\eta_r = \eta - |x - x'|$. Note however, that this solution is only valid for the cartesian components $\chi_{\bar{a}}$ of χ_a. In ([21]) the Green's function satisfying

$$(\mathring{D}^2 + \frac{2}{\eta}\partial_\eta)G(x,x') = -(\frac{\eta}{l})^2\delta(x-x') \tag{6.21}$$

was computed, using methods from Quantum field theory. Given this solution, $\chi_{\bar{a}\bar{b}}$ follows by

$$\chi_{\bar{a}\bar{b}}(x) = \int d^4x' G(x,x')(\frac{l}{\eta'})^2 \mathcal{T}_{\bar{a}\bar{b}} \tag{6.22}$$

where there we again have to define a function to impose the boundary conditions. The retarded green's function is given

$$G_r(x,x') = \frac{\eta\eta'}{4\pi l^2|\mathbf{x}-\mathbf{x}'|}\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|) + \frac{1}{4\pi l^2}\theta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|) \tag{6.23}$$

where θ is the usual step function. (6.23) can be simplified using the following identity

$$\begin{aligned} 4\pi(\frac{l}{\eta'})^2 G_r(x,x') =& (\frac{1}{|\mathbf{x}-\mathbf{x}'|}\frac{\eta}{\eta'})\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|) + \frac{1}{\eta'^2}\theta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|) \\ =& \frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|) - \frac{\partial}{\partial\eta'}(\frac{1}{\eta'}\theta(\eta-\eta'-|\mathbf{x}-\mathbf{x}'|)) \end{aligned} \tag{6.24}$$

With this we have, for the cartesian components of χ_{ab}

$$\chi_{\bar{a}\bar{b}}(\eta,\mathbf{x}) = 4\int_\Sigma \mathring{\epsilon}\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_r,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} + 4\int_{-\infty}^{\eta_r} d\eta' \frac{1}{\eta'}\partial_{\eta'}\int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}}(\eta',\mathbf{x}') \tag{6.25}$$

where we integrated by parts and discarded the boundary terms. Here, we see that contrary to Minkowski, at first order the perturbation also has a term dependent on the entire history of the source up to η_r. This term which we call a tail term, is the result of the back-scattering of the perturbation by the curvature of the background. Even though it's absent at first order, it also appears at higher orders in Minkowski.

6.4 Approximations

One now wishes to look at this solution in a region near $\mathcal{H}$. In order to proceed, one will need to sharpen and clarify our notion of isolated system. As noted before, the physical size of the region $\sum$ will be assumed to be bounded by some distance d on all slices (which implies that the source will intersect $\mathcal{I}^+$ at i^+). We also assume that this region is very small compared to a cosmological scale $D \ll l$. We will also need to assume stationarity of the source as it approaches $i^{\pm}$; this can be done by assuming some fall-off of $\mathcal{L}_T T_{ab}$, but to avoid convergence issues one can use the stronger requirement that $\mathcal{L}_T T_{ab}$ should vanish outside some finite interval $[\eta_1, \eta_2]$.

Consider a cosmological slice, and pick cartesian coordinates on it such that the source is at the origin. The sharp term can be treated just like in Minkowski. Since the source is bounded to a region of proper size D and coordinate size $d = D/a$, far from the source we have $r \equiv |\mathbf{x}| \gg |\mathbf{x}'| \sim d$. So we can expand

$$|\mathbf{x} - \mathbf{x}'|^2 = r^2 - 2\mathbf{x}\cdot\mathbf{x}' + \mathbf{x}'^2 = r^2(1 - \frac{2}{r}\hat{\mathbf{x}}\cdot\mathbf{x}' + \mathcal{O}(d^2/r^2)) \tag{6.26}$$

and

$$|\mathbf{x} - \mathbf{x}'| = r - \hat{\mathbf{x}}\cdot\mathbf{x}' + \mathcal{O}(d^2/r^2) \quad \Rightarrow \quad \frac{1}{|x - x'|} = \frac{1}{r} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{r^2}$$

Inside the integral one can expand around $x' = 0$

$$\begin{aligned}\frac{\mathcal{T}_{ab}(\eta_{\text{ret}} - \hat{\mathbf{x}}\cdot\mathbf{x}', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} =& (\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') + \hat{\mathbf{x}}\cdot\mathbf{x}'\partial_\eta \mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}'))(\frac{1}{r} + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{r^2}) + \mathcal{O}(d^2/r^2)\\ =& \frac{\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}')}{r} + \hat{\mathbf{x}}\cdot\mathbf{x}''\frac{\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}')}{r^2} + \hat{\mathbf{x}}\cdot\mathbf{x}'\frac{\partial_\eta \mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}')}{r} + \mathcal{O}(d^2/r^2)\end{aligned} \tag{6.27}$$

where $\eta_{\text{ret}} = \eta - r$ and

$$4\int_\Sigma \mathring{\epsilon}\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_r, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \frac{4}{r}\int_\Sigma \mathring{\epsilon}(\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') + \frac{\hat{\mathbf{x}}\cdot\mathbf{x}'}{r}\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') + \hat{\mathbf{x}}\cdot\mathbf{x}'\partial_\eta \mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}')) \tag{6.28}$$

Now, let's look at each term. Let τ denote the time scale (measured in the η coordinate) on which $\mathcal{T}_{ab}$ is varying so that $\partial_\eta \mathcal{T}_{ab} \sim \mathcal{T}_{ab}/\tau$. Note that d is also the coordinate time it takes for light to transverse the source so that $\frac{d}{\tau} \sim v$ where v is the velocity of the source. Hence we have

$$4\int_\Sigma \mathring{\epsilon}\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_r, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \sim \frac{4}{r}\int_\Sigma \mathring{\epsilon}\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}')(1 + \frac{d}{r} + \frac{d}{\tau}) \tag{6.29}$$

and

$$\frac{d}{r} = -\frac{D}{l}\frac{\eta_R}{r} = \frac{D}{l}(1 - \frac{\eta}{r}) \tag{6.30}$$

Note that if we were interested in the behavior of the source near $\mathcal{I}$, in Minkowski we would just take the limit of high r. Take any r, one can choose a cosmological slice such that $(1 - \frac{\eta}{r})$ is arbitrarily close to 1, so that has $\frac{d}{r} \sim \frac{D}{l}$. In the present case, we are interested in the approach to the cosmological horizon at $\eta = -r$, so we immediately have $\frac{d}{r} \sim \frac{D}{l}$ which we assumed much smaller than one. We will also assume that $v \ll 1$ that is the source moves non-relativistically. Hence, the second and third terms can be neglected and

$$4\int_{\Sigma} \mathring{\epsilon} \frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_r, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \frac{4}{r}\int_{\Sigma} \mathring{\epsilon}\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') \tag{6.31}$$

One can repeat the same procedure for the tail term which will end up in the replacement $\eta_r \to \eta_{\text{ret}}$ and we have

$$\chi_{ab}(\eta, \mathbf{x}) \approx \frac{4}{r}\int_{\Sigma} \mathring{\epsilon}\mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') + 4\int_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'}\partial_{\eta'}\int_{\Sigma} \mathring{\epsilon}\mathcal{T}_{ab}(\eta', \mathbf{x}') \tag{6.32}$$

6.5 Quadrupole moments

The goal of this section is to relate the perturbation χ_{ab} with certain properties of the source, in particular we want to relate it to multipole moments of the source. First we need to discuss the equation of the conservation of the energy momentum tensor in the de Sitter background, which will be used extensively. At first order in the perturbation we will have

$$\nabla^a T_{ab} = 0 \tag{6.33}$$

where T_{ab} is the Energy momentum tensor that produces a first order perturbation.

We use planar coordinates (t, x) and project (6.33) along ∂_t and $\mathring{q}_a^b$ where $\mathring{q}$ is the spatial metric induced on a slice of constant t.

The time component of the conservation equation then reads

$$g^{ca}\nabla_c T_{a0} = -\partial_0 T_{00} + e^{-\frac{2t}{l}}\partial^i T_{i0} - \frac{1}{l}(3T_{00} + e^{-2Ht}\sum_i T_{ii}) \tag{6.34}$$

The spatial component reads

$$g^{ca}\nabla_c T_{ai} = -\partial_0 T_{0i} + e^{-\frac{2t}{l}}\partial^j T_{ji} - \frac{1}{l}(3T_{0i}) \tag{6.35}$$

And defining the matter density and pressures via

$$\rho = T_{ab}n^a n^b \quad \text{and} \quad p^i = T_{ab}\partial^a x^i \partial^b x^i \tag{6.36}$$

it is easy to check that the previous two expressions can then be written in a coordinate invariant fashion as

$$\begin{aligned} \partial_t \rho - e^{-3Ht}\mathring{D}^a \mathcal{T}_a + H(3\rho + \sum_i p_i) = 0 \\ \partial_t \mathcal{T}_a - e^{-Ht}\mathring{D}^b \mathcal{T}_{ab} + 2H\mathcal{T}_a = 0 \end{aligned} \tag{6.37}$$

where $\mathring{D}_a$ is the derivative operative compatible with flat spatial metric $\mathring{q}_{ab}$. Note that in this chart, these equations reduce to their Minkowskian analogue for $\Lambda \to 0$.

Quadrupole moments are defined for p and ρ_i similarly as

$$Q_{ab} = \int_\Sigma \epsilon \rho \bar{x}_a \bar{x}_b \tag{6.38}$$

$$\tilde{Q}_{ab} = \int_\Sigma \epsilon (\sum_i p_i) \bar{x}_a \bar{x}_b \tag{6.39}$$

where $\bar{x}_a = a(t)x_a$. These should only depend on the slice considered and on the physical geometry of the source.

One now wishes to write the integral $\int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}}$ in terms of $Q^{(\rho)}_{\bar{a}\bar{b}}$ and$Q^{(p)}_{\bar{a}\bar{b}}$. First note that

$$\begin{aligned} \int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}} &= \int_\Sigma \mathring{\epsilon}[\mathring{D}^c(\mathcal{T}_{\bar{c}\bar{a}}x_{\bar{b}}) - (\mathring{D}^c\mathcal{T}_{\bar{c}\bar{a}})x_{\bar{b}}] \\ &= -\int_\Sigma \mathring{\epsilon}(\mathring{D}^{\bar{c}}\mathcal{T}_{\bar{c}\bar{a}})x_{\bar{b}} \end{aligned} \tag{6.40}$$

where we used the vanishing of the boundary term that arises using Stokes Theorem, since the stress-energy tensor has compact spatial support.

$$\begin{aligned}\int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}} &= -\int_\Sigma \mathring{\epsilon}e^{Ht}(\partial_t + 2H)\mathcal{T}_{(\bar{a}}x_{\bar{b})} \\ &= -\frac{1}{2}\int_\Sigma \mathring{\epsilon}e^{Ht}(\partial_t + 2H)[\mathring{D}^{\bar{c}}(\mathcal{T}_{\bar{c}}x_{\bar{a}}x_{\bar{b}}) - (\mathring{D}^{\bar{c}}\mathcal{T}_{\bar{c}})x_{\bar{a}}x_{\bar{b}}] \\ &= \frac{1}{2}\int_\Sigma \mathring{\epsilon}e^{Ht}(\partial_t + 2H)(\mathring{D}^{\bar{c}}\mathcal{T}_{\bar{c}})x_{\bar{a}}x_{\bar{b}}\end{aligned} \tag{6.41}$$

where we've used the spatial component of the conservation equation and dropped the boundary term using the same argument. Now, we use the time component, and get an expression for $\int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}}$ solely in terms of ρ and $\sum_i p_i$:

$$\int_\Sigma \mathring{\epsilon}\mathcal{T}_{\bar{a}\bar{b}} = \frac{1}{2}\int_\Sigma \mathring{\epsilon}e^{4Ht}[\partial_t^2\rho + H\partial_t(8\rho + \sum_i p_i) + 5H^2(3\rho + \sum_i p_i)]x_{\bar{a}}x_{\bar{b}} \tag{6.42}$$

In terms of which we have

$$\int_\Sigma \mathring{\epsilon}(x')\mathcal{T}_{\bar{a}\bar{b}}(t,x') = \frac{1}{2a(t)}[\partial_t^2 Q^{(\rho)}_{\bar{a}\bar{b}} - 2H\partial_t Q^{(\rho)}_{\bar{a}\bar{b}} + H\partial_t Q^{(p)}_{\bar{a}\bar{b}}](t) \tag{6.43}$$

Here, the tensors components are written in the basis of the cartesian chart. In a general chart we have

$$e^{\bar{a}}_a e^{\bar{b}}_b \int_\Sigma \mathring{\epsilon}(x')\mathcal{T}_{\bar{a}\bar{b}}(t,x') = \frac{1}{2a(t)}[\partial_t^2 Q^{(\rho)}_{ab} - 2H\partial_t Q^{(\rho)}_{ab} + H\partial_t Q^{(p)}_{ab}](t) \tag{6.44}$$

where $e^{\bar{a}}_a$ are the basis 1-forms in cartesian coordinates. Note also that given any tensor field S_{ab} and $T = -H(\eta\partial_\eta + x\partial_x) = \partial_t - Hx\partial_x$ we have

$$\begin{aligned}\mathcal{L}_T S_{ab} &= T^c\partial_c S_{ab} - 2HS_{ab} \\ &= T^c\mathring{D}_c S_{ab} - 2HS_{ab} \\ &= \partial_t Q_{ab} - 2HQ_{ab}\end{aligned} \tag{6.45}$$

In the first equality we used flat conformal coordinates (η, x), the second equality follows trivially, but being a tensorial equality in (M_P^+, η) it should be valid in any set of coordinates. The last equality follows from the fact that $Q^{(i)}_{ab}$ are only a function of the slice $\Sigma_t \equiv \Sigma_\eta$. It follows

$$e^a_{\bar{a}} e^a_{\bar{b}} \int\limits_{\Sigma} \mathring{\epsilon} \mathcal{T}_{\bar{a}\bar{b}} = \frac{1}{a}[\mathcal{L}_T \mathcal{L}_T Q_{ab} + 2H\mathcal{L}_T Q_{ab} + H\mathcal{L}_T \tilde{Q}_{ab} + 2H^2 \tilde{Q}_{ab}] \tag{6.46}$$

The solution can then be written as

$$\begin{aligned} \chi_{ab}(\eta, \mathbf{x}) &\approx \frac{4}{r} \int_{\Sigma} \mathring{\epsilon} \mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') + 4 \int\limits_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \int_{\Sigma} \mathring{\epsilon} \mathcal{T}_{ab}(\eta_{\text{ret}}, \mathbf{x}') \\ &= \frac{2H}{r} \eta_{\text{ret}} [\mathcal{L}_T \mathcal{L}_T Q_{ab} + 2H\mathcal{L}_T Q_{ab} + H\mathcal{L}_T \tilde{Q}_{ab} + 2H^2 \tilde{Q}_{ab}](\eta_{\text{ret}}) \\ &+ 2 \int\limits_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'} \partial_{\eta'} \frac{1}{a(\eta')} [\mathcal{L}_T \mathcal{L}_T Q_{ab} + 2H\mathcal{L}_T Q_{ab} + H\mathcal{L}_T \tilde{Q}_{ab} + 2H^2 \tilde{Q}_{ab}](\eta') \end{aligned} \tag{6.47}$$

6.6 Energy on the horizon

Now, one turns to the main computation of these thesis: the computation of the flux of a gravitational wave in a de Sitter background and its dependence on the quadrupole moments.

With (6.47), we succeeded in relating a perturbation χ_{ab} with the quadrupole moments (as defined in (6.38,6.39)); to finish the job one now wishes to use our work in chapter 4, where we've computed the flux of a generalized charge in a null surface due to a perturbation h_{ab}.

Before we proceed, note that the null surface in question in chapter 4, was the boundary of a spacetime M; hence going forward we will keep only a subset $\hat{\mathcal{O}}$ of the Poincaré Patch $\mathcal{O}^+$, the lower triangle (the southern causal diamond) in figure 6.1, so that $\partial\hat{\mathcal{O}}$ is $\mathcal{H}$. Our framework is then a spacetime $(\hat{\mathcal{O}}, \bar{g})$, where $\bar{g}$ is the unperturbed de Sitter metric. In our formalism of chapters 2, 3 and 4, the energy was defined as a charge $\mathcal{H}_\xi$ generated by a timelike asymptotic symmetry ξ. The flux of $\mathcal{H}_\xi$ on the boundary could then be computed using the Wald-Zoupas perscription. In our situation, we are presented with 7 exact symmetries of $(\mathcal{O}^+, \bar{g})$, where we singled out the dilation T for being timelike in the southern causal diamond and going to the time translation in the Minkowski limit. In fact, T is null on the cosmological horizon and hence is also an asymptotic symmetry of $(\hat{\mathcal{O}}, \bar{g})$ (in the language we introduced in chapter 3, it is easy to see that it is also an affine supertranslation).

As expected, since $\mathcal{H}$ is expansion and shear free, the flux of energy F_T vanishes in our smaller de Sitter $(\hat{\mathcal{O}}, \bar{g})$, which makes sense as we do not have any gravitational radiation.

The situation should change when we perturbe $\bar{g}$ as we did in this chapter, and since we bump straight into section 4.5 assumptions, we should be able to use (4.65) without too much trouble.

Note however that $\mathcal{H}$ does not necessarily remain null relative to the perturbed geometry. Fortunately, the condition (6.15) we imposed earlier didn't exhaust our gauge freedom; one can still chose an additional ξ_a satisfying $\nabla^a \nabla_b \xi_a = 0$, which gives us four additional residual gauge transformations to work with, and impose the additional gauge condition

$$T \cdot h \hat{=} 0 \tag{6.48}$$

which does the job.

A direct calculation gives, using (4.65)

$$F_T = \frac{1}{16\pi} \int_{\mathcal{H}} \epsilon q^{\theta\theta} q^{\varphi\varphi} [(T \cdot \partial h_{\theta\varphi})^2 - T \cdot \partial h_{\theta\theta} T \cdot \partial h_{\varphi\varphi}] \tag{6.49}$$

and after imposing 6.48, we get

$$\begin{aligned} F_T &= \frac{1}{16\pi} \int_{\mathcal{H}} \epsilon q^{\theta\theta} q^{\varphi\varphi} [(T \cdot \partial (h_{\theta\varphi} + \mathcal{L}_\xi g_{\theta\varphi}))^2 - T \cdot \partial (h_{\theta\theta} + \mathcal{L}_\xi g_{\theta\theta}) T \cdot \partial (h_{\varphi\varphi} + \mathcal{L}_\xi g_{\varphi\varphi})] \\ &= \frac{1}{16\pi} \int_{\mathcal{H}} \epsilon q^{\theta\theta} q^{\varphi\varphi} [(T \cdot \partial h_{\theta\varphi})^2 - (T \cdot \partial h_{\theta\theta})(T \cdot \partial h_{\varphi\varphi})] \\ &+ \frac{1}{16\pi} \int_{\mathcal{H}} \epsilon q^{\theta\theta} q^{\varphi\varphi} [2(T \cdot \partial h_{\theta\varphi})(T \cdot \partial \mathcal{L}_\xi g_{\theta\varphi}) - (T \cdot \partial h_{\theta\theta})(T \cdot \partial \mathcal{L}_\xi g_{\varphi\varphi}) \\ &- (T \cdot \partial h_{\varphi\varphi})(T \cdot \partial \mathcal{L}_\xi g_{\theta\theta})] \\ &= \int_{\mathcal{H}} (f_0^{abcd}(T \cdot \partial h) \dot{\chi}_{ab} \dot{\chi}_{cd} + f_1^{abcd}(T \cdot \partial h, T \cdot \partial \mathcal{L}_\xi g) \dot{\chi}_{ab} \chi_{cd}) \end{aligned} \tag{6.50}$$

Imposing (6.48) allowed us to work with (4.65) but it also made the resulting computation much more cumbersome. We'll continue the computation only to leading order in Λ. As we saw in the previous section, one knows how to relate χ_{ij} with the quadrupole moments, but not the time components $\chi_{\eta\eta}$. In the next few sections we will proceed first by using the gauge conditions we imposed earlier (6.17) to relate $\chi_{\eta\eta}$ to the quadrupole moments (to leading order in Λ); after that we will solve (6.48) explicitly to express $\mathcal{L}_\xi g_{AB}$ in terms of $\chi_{ij}$and hence the quadrupole moments (to leading order in Λ).

6.7 Time components

First, we take a closer look at the gauge conditions (6.17). We have

$$\begin{aligned}\mathring{D}^a \chi_{ab} &= \partial_\eta \chi_b - \frac{2}{\eta}\chi_b \\ \mathring{D}^a \chi_a &= \partial_\eta(\tilde{\chi} - \chi) - \frac{1}{\eta}\tilde{\chi}\end{aligned} \tag{6.51}$$

with $\tilde{\chi} = \bar{\chi}_{\eta\eta} + \sum_i \bar{\chi}_{ii}$ and $\chi = \sum_i \bar{\chi}_{ii}$, so that

$$\partial_\eta \bar{\chi}_{\eta\eta} - \frac{1}{\eta}\bar{\chi}_{\eta\eta} = \mathring{D}^a \chi_a + \frac{1}{\eta}\chi \tag{6.52}$$

which is a first order linear differential equation of the form $\partial_\eta f + P(\eta) f = Q(\eta)$ and can be solved by

$$\begin{aligned}\bar{\chi}_{\eta\eta} &= \eta[\int^\eta \frac{d\eta'}{\eta'}(\mathring{D}^a \chi_a + \frac{\chi}{\eta'}) + C'] \\ &= (-H\eta)[\int^t dt'(\mathring{D}^a \chi_a + \frac{\chi}{\eta(t')}) + C]\end{aligned} \tag{6.53}$$

Similarly we have

$$\begin{aligned}\chi_a &= \eta^2(\int^\eta \frac{d\eta'}{\eta'^2}\mathring{D}^a \chi_{ab} + B') \\ &= (H\eta)^2(\int^t \frac{dt'}{(-H\eta(t'))}\mathring{D}^a \chi_{ab} + B)\end{aligned} \tag{6.54}$$

Note that we imposed the vanishing of both $\tilde{\chi}$ and χ_{ab} (and hence χ) on $E^+(i^-)$, which implies the vanishing of both C and B.

We have

$$\bar{\chi}_{\eta\eta} = (-H\eta)\int^t dt'[(H\eta(t'))^2(\int^{t'} \frac{dt''}{(-H\eta(t''))}\mathring{D}^a \mathring{D}^b \chi_{ab}) + \frac{\chi}{\eta(t')}] \tag{6.55}$$

Note that we have $\chi_{ab} = g(Q_{ab}(\eta_{\text{ret}})) = f(Q_{ab}(t_{\text{ret}}))$ with $\eta_{\text{ret}} = \eta - r$ and $t_{\text{ret}} = -\frac{1}{H}\ln(-H\eta_{\text{ret}}))$. It is hence convenient to compute the following derivatives

$$
\begin{aligned}
&\partial_t[Q_{ab}(t_{\text{ret}})] - (\partial_t Q_{ab})(t_{\text{ret}})\frac{e^{-Ht}}{H\eta_{\text{ret}}} \\
&\partial_r[Q_{ab}(t_{\text{ret}})] = \frac{(\partial_t Q_{ab})(t_{\text{ret}})}{H\eta_{\text{ret}}} \\
&\mathring{D}^a[Q_{ab}(t_{\text{ret}})] = \frac{(\partial_t Q_{ab})(t_{\text{ret}})}{H\eta_{\text{ret}}}\hat{x}^b = (\partial_t Q_{ab})(t_{\text{ret}})d^a
\end{aligned} \tag{6.56}
$$

where $d^a \equiv \frac{\hat{x}^a}{H\eta_{\text{ret}}}$ and $\hat{x}^a = x^a/r$. For our main computation, we will only need $(T\cdot\partial)\bar{\chi}_{\eta\eta}$ which is given by

$$
\begin{aligned}
(T\cdot\partial)\bar{\chi}_{\eta\eta} = {} & H^2\eta\int^t dt'[(H\eta(t'))^2(\int^{t'}\frac{dt''}{(-H\eta(t''))}\mathring{D}^a\mathring{D}^b\chi_{ab}) + \frac{\chi}{\eta(t')}] \\
& - (H\eta)^3\int^{t'}\frac{dt''}{(-H\eta(t''))}\mathring{D}^a\mathring{D}^b\chi_{ab} - H\chi \\
& - (H\eta)^2\int^t dt'[(H\eta(t'))^2(\int^{t'}\frac{dt''}{(-H\eta(t''))}\partial_r\mathring{D}^a\mathring{D}^b\chi_{ab}) + \frac{\partial_r\chi}{\eta(t')}]
\end{aligned} \tag{6.57}
$$

The plan is now to compute $\mathring{D}^a\mathring{D}^b\chi_{ab}$ and $\partial_r\mathring{D}^a\mathring{D}^b\chi_{ab}$ in terms of the quadrupole moments up to $\mathcal{O}(H^2)$, and a direct but cumbersome computation will yield $\bar{\chi}_{\eta\eta}$. We start with

$$
\begin{aligned}
\chi_{ab} &= \frac{2}{r}(-H\eta(t_{\text{ret}}))A_{ab}(t_{\text{ret}}) + 2\int^{\eta_{\text{ret}}}\frac{d\eta'}{\eta'}\partial_{\eta'}(-H\eta' A_{ab}(\eta')) \\
&= \frac{2}{r}(-H\eta(t))A_{ab}(t_{\text{ret}}) + 2H^2\int^{t_{\text{ret}}} dt' H A_{ab}(t') \\
&\sim \frac{2}{r}(-H\eta(t))A_{ab}(t_{\text{ret}}) + 2H^2\partial_t Q^{(\rho)}_{ab}(t_{\text{ret}})
\end{aligned} \tag{6.58}
$$

where $A_{ab} = \partial_t^2 Q^{(\rho)}_{ab} - 2H\partial_t Q^{(\rho)}_{ab} + H\partial_t Q^{(p)}_{ab}$. We ought to ignore in the future terms of higher order than $\mathcal{O}(H^2)$. Since in the horizon we have $r = e^{-Ht}/H$ we ignore terms of order $H^n r^{-m}$ with $(n+m)\geq 3$. Now note that $\mathring{D}^b x^a = \tilde{\delta}^{ab}$ and

$$
\mathring{D}^b r^{-n} d^a = \frac{1}{r^{n+1}}\frac{\tilde{\delta}^{ab}}{H\eta_{\text{ret}}} + \frac{H}{r^n}d^a d^b - \frac{n+1}{r^{n+1}}d^a\hat{x}^b = \mathcal{O}(H^{1+n}) \tag{6.59}
$$

since $d^a \sim \hat{x}^a \sim \mathcal{O}(H^0)$. We get

$$
\begin{aligned}
\mathring{D}^a\chi_{ab} =& -\frac{2}{r^2}\hat{x}^a e^{-Ht}[A_{ab}(t)](t_{\text{ret}}) + \frac{2}{r}d^a e^{-Ht}[\partial_t A_{ab}(t)](t_{\text{ret}}) + 2H^2 d^a(\partial_t^2 Q_{ab}^{(\rho)})(t_{\text{ret}}) \\
\sim& -\frac{2}{r^2}\hat{x}^a e^{-Ht}[\partial_t^2 Q_{ab}^{(\rho)}(t)](t_{\text{ret}}) + \frac{2}{r}d^a e^{-Ht}[\partial_t A_{ab}(t)](t_{\text{ret}}) + 2H^2 d^a(\partial_t^2 Q_{ab}^{(\rho)})(t_{\text{ret}})
\end{aligned}
\tag{6.60}
$$

and

$$
\begin{aligned}
\mathring{D}^b\mathring{D}^a\chi_{ab} \sim& 2\left(\frac{1}{r^2}\frac{\tilde{\delta}^{ab}}{H\eta_{\text{ret}}} + \frac{H}{r}d^a d^b - \frac{3}{r^2}d^a\hat{x}^b\right)e^{-Ht}[\partial_t^3 Q_{ab}^{(\rho)}](t_{\text{ret}}) \\
&+ \frac{2}{r}d^a d^b e^{-Ht}[\partial_t^2 A_{ab}](t_{\text{ret}}) + 2H^2 d^a d^b[\partial_t^3 Q_{ab}^{(\rho)}](t_{\text{ret}})
\end{aligned}
\tag{6.61}
$$

and finally

$$
\begin{aligned}
\partial_r\mathring{D}^b\mathring{D}^a\chi_{ab} \sim& 2\left(\frac{1}{r^2}\frac{\tilde{\delta}^{ab}}{(H\eta_{\text{ret}})^2} + \frac{H}{r}\frac{d^a d^b}{H\eta_{\text{ret}}} - \frac{3}{r^2}d^a d^b\right)e^{-Ht}[\partial_t^4 Q_{ab}^{(\rho)}](t_{\text{ret}}) \\
&+ 2H^2\frac{d^a d^b}{H\eta_{\text{ret}}}[\partial_t^4 Q_{ab}^{(\rho)}](t_{\text{ret}}) - \frac{2}{r^2}d^a d^b e^{-Ht}[\partial_t^2 A_{ab}](t_{\text{ret}}) \\
&+ \frac{4H}{r}\frac{d^a d^b}{H\eta_{\text{ret}}}e^{-Ht}[\partial_t^2 A_{ab}](t_{\text{ret}}) + \frac{2}{r}\frac{d^a d^b}{H\eta_{\text{ret}}}e^{-Ht}[\partial_t^3 A_{ab}](t_{\text{ret}})
\end{aligned}
\tag{6.62}
$$

where we've made extensive use of (6.56) and neglected higher order terms. To conclude the computation one will need to integrate these terms. For that purpose we will use the following

$$
\begin{aligned}
\int^t dt' A_n^{ab}(\partial^n Q_{ab})(t'_{\text{ret}}) =& \int^t dt' A_n^{ab}\partial(\partial^{n-1}Q_{ab}(t'_{\text{ret}}))(-H\eta'_{\text{ret}}e^{Ht'}) \\
=& -e^{Ht'}H\eta_{\text{ret}}A_n^{ab}\partial^{n-1}Q_{ab}(t'_{\text{ret}}) \\
&+ \int^t \partial_t(e^{2Ht'}H\eta_{\text{ret}}A_n^{ab})\partial^{n-1}Q_{ab}(t'_{\text{ret}})
\end{aligned}
\tag{6.63}
$$

where again we used (6.56) and integrated by parts. Note that in general the derivative in the second term will increase the order of H by one. One also needs the trace of the perturbation itself which in general will be multiplied by a term of $\mathcal{O}(H)$ and we will have $\mathcal{O}(H)\chi \sim \mathcal{O}(H)\frac{2}{r}e^{-Ht}\partial_t^2 Q^{(\rho)}$.

The first line will be

$$
\begin{aligned}
-He^{-Ht}\int^{t} dt'[e^{-2Ht'}(\int^{t'} dt''e^{Ht''}\mathring{D}^a\mathring{D}^b\chi_{ab}) - He^{Ht'}\chi] &\sim -e^{-Ht}\frac{2H}{r}\hat{x}^a\hat{x}^b\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) \\
&\hat{=} - 2H^2\hat{x}^a\hat{x}^b[\partial_t^2Q_{ab}^{(\rho)}](t_{\text{ret}})
\end{aligned}
\tag{6.64}
$$

The second line will be just

$$
\begin{aligned}
e^{-3Ht}\int^{t'} dt'e^{Ht'}\mathring{D}^a\mathring{D}^b\chi_{ab} - H\chi \sim& -2(\frac{1}{r^2}\tilde{\delta}^{ab} + \frac{H}{r}d^a\hat{x}^b - \frac{3}{r^2}\hat{x}^a\hat{x}^b)e^{-2Ht}\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) \\
&- \frac{2}{r}d^a\hat{x}^be^{-2Ht}[\partial_tA_{ab}](t_{\text{ret}}) - 2H^2d^a\hat{x}^be^{-Ht}[\partial_t^2Q_{ab}^{(\rho)}](t_{\text{ret}}) \\
&- 2H(2\frac{\hat{x}^a\hat{x}^b}{r} + H\hat{x}^ad^b)e^{2Ht}\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) - \frac{2H}{r}e^{-Ht}\partial_t^2Q^{(\rho)} \\
\hat{=}&(-4H^2\tilde{\delta}^{ab} + 7H^2\hat{x}^a\hat{x}^b)\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) + H\hat{x}^a\hat{x}^b[\partial_tA_{ab}](t_{\text{ret}})
\end{aligned}
\tag{6.65}
$$

and the third

$$
\begin{aligned}
&- e^{-2Ht}\int^{t} dt'[e^{-2Ht'}(\int^{t'} dt''e^{Ht''}\partial_r\mathring{D}^b\mathring{D}^a\chi_{ab}) + \frac{\partial_r\chi}{\eta(t')}] \\
&\sim -2e^{-2Ht}((\frac{1}{r^2} + \frac{H}{r})\tilde{\delta}^{ab} + (H^2e^{Ht} + \frac{H}{r})\hat{x}^ad^b - \frac{3}{r^2}\hat{x}^a\hat{x}^b)\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) \\
&+ 2e^{-2Ht}(\frac{1}{r^2}\hat{x}^a\hat{x}^b - \frac{H}{r}\hat{x}^a\hat{x}^be^{Ht} + \frac{2H}{r}\hat{x}^ad^be^{Ht})A_{ab}(t_{\text{ret}}) \\
&- \frac{2}{r}\hat{x}^ad^be^{-2Ht}\partial_tA_{ab}(t_{\text{ret}}) \\
&\hat{=}(-4H^2\tilde{\delta}^{ab} + 5H^2\hat{x}^a\hat{x}^b)\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) + H\hat{x}^a\hat{x}^b\partial_tA_{ab}(t_{\text{ret}})
\end{aligned}
\tag{6.66}
$$

Finally, adding all the terms we get

$$
(T\cdot\partial)\bar{\chi}_{\eta\eta} = (-8H^2\tilde{\delta}^{ab} + 10H^2\hat{x}^a\hat{x}^b)\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}}) + 2H\hat{x}^a\hat{x}^b\partial_tA_{ab}(t_{\text{ret}}) \tag{6.67}
$$

and also

$$
\begin{aligned}
\chi_a &= e^{-2Ht}(\int^{t} dt'e^{Ht'}\mathring{D}^a\chi_{ab}) \\
&\sim -2H\hat{x}^b\partial_t^2Q_{ab}^{(\rho)}(t_{\text{ret}})
\end{aligned}
\tag{6.68}
$$

6.7.1 Gauge Fix

To impose the gauge fix (6.48), we use a new set of null coordinates $v = \eta - r$ and $u = \eta + r$. In these coordinates the metric is

$$g = \frac{1}{H^2(u+v)^2}(-4dudv + (v-u)^2 d\Omega^2) \tag{6.69}$$

with $v = 0$ the horizon. Note also that $T = -(u\partial_u + v\partial_v)$ and $T|_{\mathcal{H}} = -u\partial_u$ so that

$$T \cdot h \hat{=} 0 \rightarrow (h_{ua} + \mathcal{L}_\xi g_{ua})_{a \neq v} = 0 \tag{6.70}$$

This condition implies a differential equation for the components of ξ

$$\begin{aligned} h_{uu} - \frac{4}{(Hu)^2}\partial_u \xi^v = 0 \\ h_{u\theta} + \frac{1}{(Hu)^2}(u^2 \partial_u \xi^\theta - 2\partial_\theta \xi^v) = 0 \\ h_{u\varphi} + \frac{1}{(Hu)^2}(u^2 \sin^2\theta \partial_u \xi^\varphi - 2\partial_\varphi \xi^v) = 0 \end{aligned} \tag{6.71}$$

which can be solved by

$$\begin{aligned} \xi^v &= \frac{1}{4}H^2 \int^u w^2 h_{uu} dw \\ \xi^\theta &= \int^u (\frac{2}{w^2}\partial_\theta \xi^v - H^2 h_{u\theta})dw \\ \xi^\varphi &= \frac{1}{\sin^2\theta}\int^u (\frac{2}{w^2}\partial_\varphi \xi^v - H^2 h_{u\varphi})dw \end{aligned} \tag{6.72}$$

Now, our goal is to express $T \cdot \partial(\mathcal{L}_\xi g_{AB})$ in therms of χ_{ij}. First note that $\bar{h}_{ab} = h_{ab} - \frac{1}{2}g_{ab}h \overset{a\neq v}{\rightarrow} \bar{h}_{ua} = h_{ua}$, and $\bar{h}_{uc} = (\frac{1}{\eta H})^2 \Lambda^a_u \Lambda^b_c \bar{\chi}_{ab}$

$$
\begin{aligned}
h_{uu} &= (\frac{1}{\eta H})^2(\chi_{\eta\eta} + \hat{x}^i\hat{x}^j\chi_{ij} - \hat{x}^i\chi_{\eta i}) \\
&\sim (\hat{x}^i\hat{x}^j\chi_{ij}) \\
h_{u\theta} &= r(\frac{1}{\eta H})^2(-2\hat{x}^i\partial_\theta\hat{x}^j\chi_{ij} + 2\partial_\theta\hat{x}^i\chi_{\eta i}) \\
&\sim \frac{1}{H}(-\hat{x}^i\partial_\theta\hat{x}^j\chi_{ij}) \\
h_{u\varphi} &= r(\frac{1}{\eta H})^2(-2\hat{x}^i\partial_\varphi\hat{x}^j\chi_{ij} + 2\partial_\varphi\hat{x}^i\chi_{\eta i}) \\
&\sim \frac{1}{H}[-\hat{x}^i\partial_\varphi\hat{x}^j\chi_{ij}]
\end{aligned}
\tag{6.73}
$$

where the computation was done only to leading order in H. Let's focus first focus on the $\theta\theta$ component

$$
\mathcal{L}_\xi g_{\theta\theta} = (2z + 2Hz^2)\xi^t + 2z^2\partial_\theta\xi^\theta \tag{6.74}
$$

where $z^n \equiv r^n e^{nHt}$, $z^n|_{\mathcal{H}} = H^{-n}$ and $T \cdot \partial(z^n) = 0$. We have

$$
T \cdot \partial(\mathcal{L}_\xi g_{\theta\theta}) = \frac{e^{Ht}}{2}(2z + 2Hz^2)(H + T \cdot \partial)\xi^v + 2z^2(T \cdot \partial\partial_\theta\xi^\theta) \tag{6.75}
$$

and

$$
\begin{aligned}
T \cdot \partial(\mathcal{L}_\xi g_{\theta\theta})|_{\mathcal{H}} &= \frac{8}{H}e^{-2Ht}h_{uu} + 4e^{Ht}\xi^v + 2e^{Ht}\partial_\theta^2\xi^v - 8e^{-Ht}\partial_\theta h_{u\theta} \\
&\approx \frac{8}{H}h_{uu} - 8\partial_\theta h_{u\theta} = \frac{8}{H}(\hat{x}^i\hat{x}^j + \partial_\theta(\hat{x}^i\partial_\theta\hat{x}^j))\chi_{ij}
\end{aligned}
\tag{6.76}
$$

The component $\theta\varphi$ follows similarly

$$
T \cdot \partial(\mathcal{L}_\xi g_{\theta\varphi}) = h_{\theta\theta} + z^2\partial_\varphi\xi^\theta + z^2\sin^2\theta\partial_\theta\xi^\varphi \tag{6.77}
$$

and

$$
\begin{aligned}
T \cdot \partial(\mathcal{L}_\xi g_{\theta\varphi})|_{\mathcal{H}} &= 2e^{Ht}\partial_\varphi\partial_\theta\xi^v - 2e^{-Ht}(\partial_\varphi h_{u\theta} + \partial_\theta h_{u\varphi}) \\
&\approx -2(\partial_\varphi h_{u\theta} + \partial_\theta h_{u\varphi}) = \frac{2}{H}(\partial_\varphi(\hat{x}^i\partial_\theta\hat{x}^j) + \partial_\theta(\hat{x}^i\partial_\varphi\hat{x}^j))\chi_{ij}
\end{aligned}
\tag{6.78}
$$

Finally, we have for the $\varphi\varphi$ component

$$T \cdot \partial(\mathcal{L}_\xi g_{\varphi\varphi}) = h_{\theta\theta} + (2z + 2Hz^2)\xi^t \sin^2\theta\xi^t + 2z^2 \sin\theta\cos\theta\xi^\theta + 2z^2 \sin^2\theta\partial_\varphi\xi^\varphi \quad (6.79)$$

and

$$\begin{aligned} T \cdot \partial(\mathcal{L}_\xi g_{\varphi\varphi})|_{\mathcal{H}} &= \sin^2\theta(\frac{8}{H}e^{-2Ht}h_{uu} + 4e^{-Ht}\xi^v) + 4\sin\theta\cos\theta(e^{Ht}\partial_\theta\xi^v - 2e^{-Ht}h_{u\theta}) \\ &+ 4(e^{Ht}\partial_\varphi^2\xi^v - 2e^{-Ht}\partial_\varphi h_{u\varphi}) \\ &\approx \sin^2\theta\frac{8}{H}h_{uu} - 8\sin\theta\cos\theta h_{u\theta} - 8\partial_\varphi h_{u\varphi} \\ &= \frac{8}{H}(\sin^2\theta\hat{x}^i\hat{x}^j + 8\sin\theta\cos\theta\hat{x}^i\partial_\theta\hat{x}^j + 8\partial_\varphi(\hat{x}^i\partial_\varphi\hat{x}^j)) \end{aligned}$$

6.7.2 Quadrupole Formula

Now, one needs to express $T \cdot \partial h_{AB}$ in terms of χ_{ij}. We have

$$\begin{aligned} h_{AB} &= (\frac{1}{\eta H})^2\Lambda^i_A\Lambda^j_B[\chi_{ij} - \frac{1}{2}\delta_{ij}(-\chi_{\eta\eta} + \sum_i \chi_{ii})] \\ &= \frac{1}{2}(\frac{1}{\eta H})^2[\chi_{\eta\eta}(\sum_i \Lambda^i_A\Lambda^i_B) + \sum_i \chi_{ii}(\Lambda^i_A\Lambda^i_B - \sum_{j\neq i}\Lambda^j_A\Lambda^j_B) + 2\sum_{i\neq j}\chi_{ij}(\Lambda^i_A\Lambda^j_B)] \quad (6.80) \\ &= (h_{AB})^{ab}\chi_{ab} \end{aligned}$$

where $\Lambda^i_A = r\partial_A\hat{x}^i$. Denoting $T \cdot \partial h \equiv \dot{h}$ we have

$$\dot{h}_{AB} = (h_{AB})^{ab}\dot{\chi}_{ab} \quad (6.81)$$

since $(h_{AB})^{ab}$ is of the form $f(\frac{r}{\eta H})g(x^A)$ and $T \cdot \partial(h_{AB})^{ab} = 0$.

Finally, we are ready to compute the flux of energy of gravitational waves in the cosmological horizon. The final calculation is a bit long, but quite direct. First we look at the first part of the flux formula

$$\begin{aligned} \int_{\mathcal{H}}(f_0^{abcd}(T \cdot \partial h)\dot{\chi}_{ab}\dot{\chi}_{cd} &= \frac{1}{48H^2}\int dt(\sum_i \dot{\chi}_{ii}^2 + 2\sum_{i\neq j}(2\dot{\chi}_{ij}^2 - \dot{\chi}_{ii}\dot{\chi}_{jj}) \\ &- \frac{1}{64\pi H^2}\int dt\int d\varphi\int d\theta\sin\theta\dot{\chi}_{\eta\eta}(\dot{\chi}_{\eta\eta} - 2\hat{x}^i\hat{x}^j\dot{\chi}_{ij}) \quad (6.82) \end{aligned}$$

where we used the fact that $\chi_{ij} \equiv \chi_{ij}(t_{\rm ret})$, and all the remaining angular integrals were done. Note that the integral containing $\dot{\chi}_{\eta\eta}$ cannot be done immediately since these will depend on x^A. The second part is

$$\int_{\mathcal{H}} (f_1^{abcd}(T\cdot\partial h, T\cdot\partial\mathcal{L}_\xi g)\dot{\chi}_{ab}\chi_{cd}) = \int_{\mathcal{H}} [f_1^{ijkp}(T\cdot\partial h, T\cdot\partial\mathcal{L}_\xi g)) \\ - \hat{x}^i\hat{x}^j f_1^{\eta\eta kp}(T\cdot\partial h, T\cdot\partial\mathcal{L}_\xi g)]\dot{\chi}_{ij}\chi_{pk} \tag{6.83}$$

$$= \frac{4}{15H}\int dt\{2\sum_i \chi_{ii}\dot{\chi}_{ii} + 6\sum_{i\neq j}(\chi_{ij}\dot{\chi}_{ij} - \chi_{ii}\dot{\chi}_{jj})\} \tag{6.84}$$

where again we used the fact that $\chi_{ij} \equiv \chi_{ij}(t_{\rm ret})$ and $\dot{\chi}_{\eta\eta} = \hat{x}^i\hat{x}^j\dot{\chi}_{ij} + \mathcal{O}(H^2)$.From these results, the quadrupole nature of gravitational waves becomes obvious, since it is now a matter of substitution (using (6.47) and (6.67)) to get

$$F_T = \frac{2}{15}\int dt\{\sum_i(\dddot{Q}_{ii}^{(\rho)})^2 + 3\sum_{i\neq j}(\dddot{Q}_{ij}^{(\rho)})^2 - \sum_{i\neq j}\dddot{Q}_{ii}^{(\rho)}\dddot{Q}_{jj}^{(\rho)}) \\ - H[2\sum_i \dddot{Q}_{ii}^{(\rho)}(\ddot{Q}_{ii}^{(\rho)} + 7\ddot{Q}_{ii}^{(p)}) + \sum_{i\neq j}(6\dddot{Q}_{ij}^{(\rho)}(\ddot{Q}_{ij}^{(\rho)} + 7\ddot{Q}_{ij}^{(p)}) - \dddot{Q}_{ii}^{(\rho)}(\ddot{Q}_{jj}^{(\rho)} + 7\ddot{Q}_{jj}^{(p)}))]\} \tag{6.85}$$

which can be put in a much simpler form if we introduce a trace-free quadrupole moment tensor

$$q_{ab}^{(i)} = Q_{ab}^{(i)} - \frac{1}{2}\mathring{q}_{ab}Q^{(i)} \tag{6.86}$$

in terms of which the flux of energy can be simply written as

$$F_T = \frac{1}{5}\int dt[\sum_{ij}[(\dddot{q}_{ij}^{(\rho)})^2 + 2H\dddot{q}_{ij}^{(\rho)}(\ddot{q}_{ij}^{(\rho)} + 7\ddot{q}_{ij}^{(p)})] \tag{6.87}$$

This is the proposed generalization of the Einstein quadrupole formula to first order in $H = \sqrt{\Lambda}$, found in [14]. Note that in deriving these results, planar coordinates were used, and hence the limit $\Lambda \to 0$ is trivial, and we get the Einstein quadrupole formula (1.17).

A few comments about these results are in order. We note that even tough corrections from the cosmological constant appear, it is still quadrupolar like in Minkowski. Contrary to the Einstein formula, contributions from the pressure quadrupole moment appear. Note also that even though the perturbations themselves have sharp propagation already at first order, this dependency disappears in the flux formula (that's not the case for higher orders).

The Einstein formula has received extensive validity from observation with very big accuracy. Given how small Λ is assumed to be, one may suppose that the corrections can be neglected. For most applications this is probably correct, but corrections can accumulate over cosmological distances. For example consider the merger of two supermassive black holes in the center of two different galaxies. Since the time scale of such event is cosmological, corrections can become significant. Its important to note though that such events will not be detected in any foreseeable future.

Chapter 7

Conclusions and Outlook

One of the main goals of this line of research is to be able to define Bondi type charges and respective fluxes in full non-linear General Relativity with a positive Λ at $\mathcal{I}$. Note that in the linear case, since there is no incoming radiation across $E^+(i^-)$, the flux across $E^-(i^+)$ will equal that across $\mathcal{I}$. Also the time translation killing field considered is future pointing and timelike inside the region bounded by these surfaces, and null on both surfaces. This seems to mimick the behaviour of the time translation field on Minkowski spacetime. For these reasons, and since $E^-(i^+)$ is null and could be taken to be in the 'far zone', it was suggested in [10, 11] that its analog in the non-linear theory could be the apropriate arena to study gravitational waves. In this thesis we proposed to extend the works on [10, 11] in the weak field regime to the cosmological horizon $E^-(i^+)$.

In order to do so, we started by reviewing the covariant phase space formalism following [1, 2]. The formalism is quite general and gives a prescription to compute charges and their fluxes in a boudary, for any symmetry vector field. We continued by reviewing the work of Chandrasekaran et al who in [3] applied the work of Wald and Zoupas [2] to a general null surface, and derived expressions for charges and fluxes in terms of intrinsic objects on the null surface. With expressions for the energy and its flux on $E^-(i^+)$ at hand, we generalized the work of [11] to the computation of the energy flux of gravitational waves in the cosmological horizon of a De Sitter background, and related this flux to the quadrupole moments of a source.

To this end, we considerd a dynamic T_{ab} sourcing the Einstein's equations with positive Λ. We studied the linear case, and constructed retarded solutions which obeyd our boundary conditions. To apply the work of [11], some work had to be done. First, since only the space components of our solutions relate to the quadrupoles, we used the gauge conditions we had imposed to relate the space and time components, which appear in our flux formula. Also, the formalism of Chandrasekaran et al is valid only when the boundary of our spacetime is null, hence a gauge fix had to be introduced, in order for us to ensure that the cosmological horizon remained null even after perturbed. With these issues behind our backs, we computed the generalized quadrupole formula to first order in $\sqrt{\Lambda}$. We took the limit to Minkowski and obtained the well known Einstein's Quadrupole formula.

Appendix A

Additional structures on a null surface

The volume form $\epsilon_{\mathcal{N}}$ of a non-null hypersurface $\mathcal{N}$ is defined as the unique 3-form on $\mathcal{N}$ which gives 1 when applied to an orthonormal basis of $T_p\mathcal{N}$. Given a volume form ϵ of M, and a unit normal vector N, one can define the form

$$\tilde{\epsilon} = N \cdot \epsilon \tag{A.1}$$

Given any orthonormal basis $\{X\}_p$ of $T_p\mathcal{N}$ with the right orientation; $\{N; X\}_p$ will constitute an orthonormal basis of T_pM, so that $\tilde{\epsilon}$ will be 1 when applied to it. This is valid for any ponit p, and hence

$$\tilde{\epsilon} = N \cdot \epsilon \tag{A.2}$$

is the unique volume form induced by M. Note also that, given a volume form $\tilde{\epsilon}$ on $\mathcal{N}$ a volume form on M can be constructed by

$$\epsilon = N \wedge \tilde{\epsilon} \tag{A.3}$$

Consider now, the case where $\mathcal{N}$ is null. First,notice that there is no unit normal, but in fact a class of null normals. Secondly, given a choice of null normal l, the pull back of $\tilde{\epsilon} = l \cdot \epsilon$ is immediately zero since l is tangent to $\mathcal{N}$. We consifer then a class of forms that satisfy $\epsilon = l \wedge \tilde{\epsilon}$. A choice of $\tilde{\epsilon}$ is not unique, since $\tilde{\epsilon} + \alpha$ where α is a 3-form such that $l \wedge \alpha = 0|_{\mathcal{N}}$ would also be permissible. Notice however that their pullback to $\mathcal{N}$ will be unique. In coordinates it is given by

$$\epsilon_{ijk} = \Pi_i^a \Pi_j^b \Pi_k^c \tilde{\epsilon}_{abc} \tag{A.4}$$

One can see explicitly that this choice is the one that allows to prove the divergence theorem. We also define the antisymetric tensor so that

$$\tilde{\epsilon}_{a_1 \dots a_n} \tilde{\epsilon}^{a_1 \dots a_n} = n! \tag{A.5}$$

and

$$\tilde{\epsilon}^{a_1 \dots a_j a_{j+1} \dots a_n} \tilde{\epsilon}_{a_1 \dots a_j b_{j+1} \dots b_n} = (n-j)! j! \delta^{[a_{j+1}}_{bj+1} \dots \delta^{a_n]}_{b_n} \tag{A.6}$$

with $n = 3$.

Given a non-degenerate metric g, we know that there is a unique connection connection D which is compatible with g , that is $Dg = 0$;we call it Levi-Civita connection. Since the induced metric q to $\mathcal{N}$ is degenerate, there will be a class of compatible connections. Given a vector field X in M, and a connection ∇, one might define a connection in $\mathcal{N}$, by projection the (1,1) tensor ∇X, but since $\mathcal{N}$ is null, even tough one is able to define a projector, such map depends on a choice of auxiliary vector field which is also not unique.

However, one knows how to define exterior derivatives of forms in $\mathcal{N}$, and one can use it to define a divergence operator. With a volume form, one can define the following maps $* : \Omega^r(M) \to \Omega'^{n-r}(M)$

$$(*S)_{a_1 \dots a_{n-r}} = \frac{1}{r!} \tilde{\epsilon}_{a_1 \dots a_{n-r} b_1 \dots b_r} S^{b_1 \dots b_r} \tag{A.7}$$

and $* : \Omega'^r(M) \to \Omega^{n-r}(M)$

$$(*\omega)^{a_1 \dots a_{n-r}} = \frac{1}{r!} \tilde{\epsilon}^{a_1 \dots a_{n-r} b_1 \dots b_r} \omega_{b_1 \dots b_r} \tag{A.8}$$

A divergence operator can then be defined as

$$\begin{aligned}
D_i X^i &=: *d(*X) \\
&= \frac{1}{2} \tilde{\epsilon}^{ijk} D_{[i} (\tilde{\epsilon}_{jk]m} X^m) \\
&= \frac{1}{2} \tilde{\epsilon}^{ijk} D_i (\tilde{\epsilon}_{jkm} X^m)
\end{aligned}$$

where D is any connection on $\mathcal{N}$. This obviously independent of the choice of D, and in fact we have

$$\begin{aligned}\frac{1}{2}\tilde{\epsilon}^{ijk}D_i(\tilde{\epsilon}_{jkm}X^m) &= \frac{1}{2}\tilde{\epsilon}^{ijk}\tilde{\epsilon}_{jkm}D_iX^m \\ &= D_iX^i\end{aligned}$$

Bibliography

[1] D. Harlow and J. Wu, "Covariant phase space with boundaries," *Journal of High Energy Physics*, vol. 2020, no. 10, Oct. 2020. [Cited on pages v, vii, 7, and 81.]

[2] R. M. Wald and A. Zoupas, "General definition of conserved quantities in general relativity and other theories of gravity," *Physical Review D*, vol. 61, no. 8, p. 084027, Mar. 2000. [Cited on pages v, vii, 1, 7, 16, 18, 20, and 81.]

[3] V. Chandrasekaran, E. E. Flanagan, and K. Prabhu, "Symmetries and charges of general relativity at null boundaries," *Journal of High Energy Physics*, vol. 2018, no. 11, Nov. 2018. [Cited on pages v, vii, 25, 32, 33, 40, and 81.]

[4] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B. P. Schmidt, R. A. Schommer, R. C. Smith, J. Spyromilio, C. Stubbs, N. B. Suntzeff, and J. Tonry, "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant," *The Astronomical Journal*, vol. 116, no. 3, p. 1009, September 1998. [Online]. Available: https://dx.doi.org/10.1086/300499 [Cited on pages 1 and 2.]

[5] A. Trautman, "Radiation and Boundary Conditions in the Theory of Gravitation," *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.*, vol. 6, no. 6, pp. 407–412, 1958. [Cited on page 1.]

[6] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 269, no. 1336, pp. 21–52, 1962. [Online]. Available: http://www.jstor.org/stable/2414436 [Cited on pages 1 and 7.]

[7] R. Penrose, "Zero rest mass fields including gravitation: Asymptotic behavior," *Proc. Roy. Soc. Lond. A*, vol. 28, p. 15, 1965. [Cited on page 1.]

[8] A. Ashtekar and M. Streubel, "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity," *Proc. Roy. Soc. Lond. A*, vol. 376, pp. 585–607, 1981. [Cited on pages 1 and 20.]

[9] A. Ashtekar, B. Bonga, and A. Kesavan, "Asymptotics with a positive cosmological constant: I. basic framework," *Classical and Quantum Gravity*, vol. 32, no. 2, p. 025004, Dec. 2014. [Cited on page 2.]

[10] ——, "Asymptotics with a positive cosmological constant. III. The quadrupole formula," *Physical Review D*, vol. 92, no. 10, p. 104032, Nov. 2015. [Cited on pages 2, 57, 58, 62, and 81.]

[11] ——, "Asymptotics with a positive cosmological constant. II. Linear fields on de Sitter spacetime," *Physical Review D*, vol. 92, no. 4, p. 044011, Aug. 2015. [Cited on pages 2, 58, and 81.]

[12] M. Kolanowski and J. Lewandowski, "Energy of gravitational radiation in the de Sitter universe at I+ and at a horizon," *Physical Review D*, vol. 102, no. 12, p. 124052, Dec. 2020. [Cited on page 2.]

[13] P. T. Chrusciel, S. J. Hoque, and T. Smolka, "Energy of weak gravitational waves in spacetimes with a positive cosmological constant," *Physical Review D*, vol. 103, no. 6, p. 064008, Mar. 2021. [Cited on page 2.]

[14] D. Dobkowski-Rylko and J. Lewandowski, "Generalization of the quadrupole formula for the energy of gravitational radiation in de Sitter spacetime," *Physical Review D*, vol. 106, no. 4, p. 044047, Aug. 2022. [Cited on pages 2 and 79.]

[15] H. Reall, "General Relativity," University of Cambridge, Mathematical Tripos Part III. [Cited on pages 2 and 6.]

[16] R. M. Wald, *General relativity.* Chicago: Univ. of Chicago Press, 2009. [Cited on page 2.]

[17] R. Arnowitt, S. Deser, and C. W. Misner, "The Dynamics of General Relativity," *"Gravitation: an introduction to current research", Louis Witten ed. (Wiley 1962), chapter 7, pp 227–265*, vol. 40, no. 9, pp. 1997–2027, Aug. 2004. [Cited on page 7.]

[18] V. Iyer and R. M. Wald, "Comparison of the Noether charge and Euclidean methods for computing the entropy of stationary black holes," *Physical Review D*, vol. 52, no. 8, pp. 4430–4439, Oct. 1995. [Cited on page 14.]

[19] ——, "Some properties of the Noether charge and a proposal for dynamical black hole entropy," *Physical Review D*, vol. 50, no. 2, pp. 846–864, Jul. 1994. [Cited on page 15.]

[20] T. Hartman, "Lecture notes on classical de Sitter space," Cornell University. [Online]. Available: http://www.hartmanhep.net/GR2017/deSitterLectures-1.pdf [Cited on page 50.]

[21] H. J. de Vega, J. Ramirez, and N. Sanchez, "Generation of gravitational waves by generic sources in de Sitter spacetime," *Physical Review D*, vol. 60, no. 4, p. 044007, Jul. 1999. [Cited on pages 63 and 64.]

[22] E. Gourgoulhon, *3+1 Formalism in General Relativity*, ser. Lecture Notes in Physics Ser. Berlin, Heidelberg: Springer Berlin / Heidelberg, 2012, no. v.846, description based on publisher supplied metadata and other sources. [Cited on page 64.]