

# Clustering of rare events and other limiting properties of chaotic dynamics

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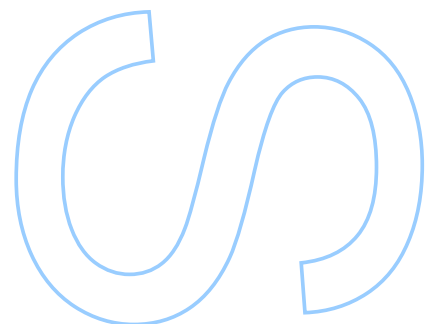
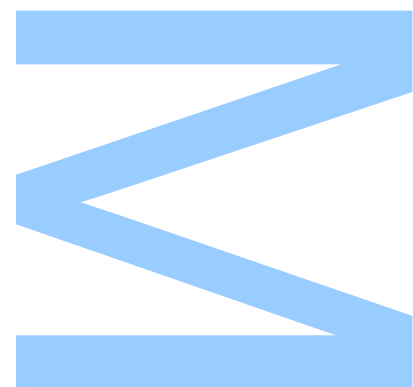
2023

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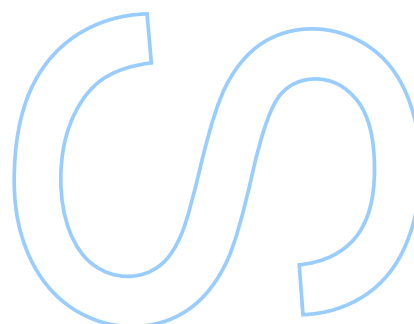
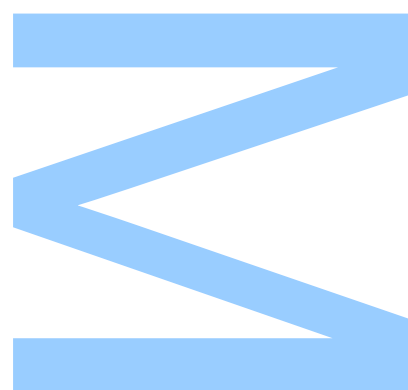
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## Acknowledgements

I would like to express my sincere gratitude

To God, because He gave me strength and courage to surpass all the challenges;

To my supervisor Jorge Freitas, for the wisdom and patience shared through this year, that were crucial to my academic grow and completion of this thesis;

To my co-supervisor Romain Aimino, for the support and understanding that he had with me in the preparation of this work;

To my beloved father Roberto, my caring and generous mother Angélica, my dear sister Isabella, my extraordinary brother Gabriel, for being with me when I most needed to reach this stage of my life, giving care and support in these difficult years that I was in the university, encouraged me and did not let me give up;

To my nephews who cheered me up on the weekends;

To my grandmother Quirina, aunts Adriana and Jaciara who gave me moral support;

To my uncle Hélio Luiz Simonetti, who encouraged me to take this course and supported me throughout it;

To my friends at FCUP, without them I would not have made it through these last few years. In particular, to Rui Miranda and Rodrigo Pereira who that lightened the moments when I was at university;

To the teachers I had, who through their teachings allowed me to complete this work.

Thank you very much for everything,  
without you I would not have been able to complete this thesis.

## Resumo

Este trabalho visa estudar *clustering* de eventos raros de um sistema dinâmico. Este estudo explica dois métodos muito úteis para o entendimento do comportamento de eventos raros. A teoria do valor extremos e a convergência de processos pontuais de eventos raros, que pode ser aplicado em um sistema dinâmico, com boas propriedades da mistura.

Em relação a estrutura de trabalho, nós apresentamos algumas suposições sobre o sistema e a função observável, que se satisfeitas garantem as condições suficientes para o uso dos métodos mencionados.

Em relação às aplicações dos métodos, nós focamo-nos em sistemas com expansão uniforme no intervalo  $[0, 1]$ , e estudamos a variação do índice extremal, que mede a intensidade de clustering, assim como a distribuição da multiplicidade de processos pontuais de eventos raros associados.

**Palavras-chave:** Sistemas Dinâmicos, Teoria do Valor Extremo, Processo de Ponto de Eventos Raros

## Abstract

This work aimed to study the clustering of rare events for dynamical systems. The study explains two useful methods for the understanding of the behaviour of rare events. The extreme value theory and the convergence of the rare event point process, which can be applied to a dynamical system, as long as the system has nice mixing properties.

Regarding the framework, we present some assumptions about the system and the observable, that if satisfied then the sufficient conditions to use the methods mentioned hold.

Regarding the application of the methods, we focused mainly on uniformly expanding maps on the interval  $[0, 1]$ , for which we studied the variation of the extremal index, which quantifies the degree of clustering, as well as the multiplicity distribution of associated rare events point processes.

**Keywords:** Dynamical system, Extreme Value Theory, Rare Events Point Process

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## Introduction

In this work we will study the behavior of rare events and other limiting properties of chaotic dynamics, using Extreme Value Theory (EVT), and Rare Events Point Processes (REPP). Extreme Value Theory is a branch of statistics that focuses on the analysis and modeling of extreme events or rare occurrences, in areas such as finance, insurance, engineering, environmental risk assessment, and climatology. Its ability to characterize, quantify and predict extreme events makes it a valuable tool for understanding rare phenomena and managing associated risks. Its origins can be traced back to the late 19th century when the investigation of statistical properties of extreme values began. Rare Events Point Processes focus on modeling and analyzing the occurrence of rare events in time or space. REPP are used in finance, insurance, telecommunications, and network analysis, thus the need to model and understand rare events has driven the development of sophisticated REPP models. This has a more recent origin, since the study started around the 20th century. Although, EVT and REPP both study rare events, EVT focuses on extreme values and the tail behavior of distributions, used to estimate the probabilities of extreme events and assess risks associated with them, whereas REPP are concerned with modeling and analyzing the occurrence of rare events, focusing on understanding the patterns, intensities, and dependencies of rare events within a point process framework.

We can see many types of extreme events by observing the world, for example in the study of geophysics events, such as earthquakes, volcanoes and avalanches, this natural phenomena can cause large damage to the population. Therefore, with the purpose of helping to study this unwanted scenarios and how frequently will these situations repeat themselves in the future, we use mathematics, more precisely, we will use dynamical systems.

In dynamical systems theory the focus is on understanding and analyzing the behavior of complex systems over time rather than studying individual states, allowing us to make predictions about the future. The behavior of a dynamical system can range from stable and predictable to chaotic and highly sensitive to initial conditions. To clarify, stable systems tend to return to their original state over time, while unstable systems can exhibit significant deviations or diverge entirely. We will study a stochastic process

generated by the orbit of a dynamical system that is evaluated by a given observable function. Although, we can apply EVT and REPP to dynamical systems, these need to have some properties and satisfy some conditions, which is one of the focus of this thesis. In this setting, an extreme or rare event happens when the orbit of a point enters in a region of the phase space that has small measure. These events may appear more than one time in a short amount of time, or not, leading to the other main subject of this thesis: clustering of rare events. By cluster we mean a concentration of rare events. We have the goal to understand the frequency of clusters of rare events and if they have a pattern.

Finally, for a stochastic process arising from a dynamical system where the dynamics is a uniformly expanding map and an observable that allows to compute easily a preimage of a ball, we show how to estimate a parameter and a distribution. The extremal index used in EVT and in the convergence of REPP, and the multiplicity distribution which describes the REPP. For this purpose, we develop expressions for the extremal index that only depend on the sizes of the intervals that indicate high observations. For the multiplicity distribution we obtained an expression which depends again on the sizes of the intervals determining rare events.

This work was based on the book "Extremes and Recurrence in Dynamical Systems" (see [11]), and the article "Clustering of extreme events created by multiple correlated maxima" (see [3]).

The text is structured as follows:

- In section 1, we have a brief introduction to dynamical systems and present the framework that we will be working throughout this study of clusters.
- In section 2, we explain with detail the Extreme Value Theory, which is a way to study the behavior of extreme events.
- In section 3, we describe a more sophisticated method, using point processes.
- In section 4, we use a coding in order to understand better the stochastic processes studied, taken from the article [1].
- In section 5, we describe and prove an assumption on the dynamical systems to

be possible to apply the theories presented.

- In sections 6 and 7, we demonstrate how to compute the extremal index and the clustering distribution, respectively.

## 1 Dynamical systems

Dynamical systems is an area of mathematics that studies how certain systems change over time. These systems are characterized by their initial condition, which has the relevant information needed to predict the future behavior of the system, and their time evolution through a set of rules or equations that describe how the system's state changes over time. The time can be discrete or continuous, however, we will only work in discrete time. A **deterministic dynamical system** is a quadruple  $(\mathcal{X}, \mathcal{B}, \mu, T)$ , where:

- $\mathcal{X}$  is a topological phase space;
- $\mathcal{B}$  is an  $\sigma$ -algebra of  $\mathcal{X}$ ;
- $\mu$  is the probability measure;
- $T : \mathcal{X} \rightarrow \mathcal{X}$  is measurable function.

In this work we will study deterministic discrete time dynamical systems in which:  $\mathcal{X}$  is a topological space, with the norm induced topology;  $\mathcal{B}$  is the Borel  $\sigma$ -algebra, i.e., the  $\sigma$ -algebra generated by the open sets of  $\mathcal{X}$ ; and  $\mu$  is a  $T$ -invariant, this means that

$$\mu(T^{-1}(A)) = \mu(A), \quad \text{for all } A \in \mathcal{B},$$

where  $T^{-1}(A)$  denotes the set of points mapped into  $A$ , we can also say that  $T$  preserves  $\mu$ .

In the phase space  $\mathcal{X}$ , every point  $x \in \mathcal{X}$  represents a possible state of the system, which transitions to  $T(x)$  after one unit of time. We want to study the long-term behavior of the evolution of time starting at an initial state  $x \in \mathcal{X}$ , i.e., the sequence of states  $x, T(x), T^2(x), \dots, T^n(x), \dots$ , which is named the **orbit** of  $x$ , where

$$T^n(x) = \underbrace{T \circ T \circ \dots \circ T}_{n \text{ times}}(x)$$

is the  $n$ th-iteration of  $T$ .

We also consider that this dynamical system is **ergodic** with respect to  $\mu$ , this means, for every  $A \in \mathcal{B}$  such that  $T^{-1}(A) = A$  then either  $\mu(A) = 0$  or  $\mu(A) = 1$ , in other words,

a system is ergodic if it is not decomposable into two parts (detected by the measure  $\mu$ ) that do not interact with each other.

## 1.1 Observable

In applications, the actual states  $T^n(x) \in \mathcal{X}$  are often not observable, instead we investigate using a real-valued function  $\varphi : \mathcal{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , called **observable**. Therefore, we observe the time evolution of successive iterations of  $T$ , evaluated by  $\varphi$ , that is, the stochastic process  $X_0, X_1, \dots$ , where

$$X_n = \varphi \circ T^n, \quad \forall n \in \mathbb{N}_0, \quad (2)$$

this means, at time  $n$  we observe the value  $\varphi \circ T^n$ . As we know, a stochastic process is a family of random variables,  $(X_i)_{i \in \mathbb{N}}$ . A random variable,  $X_i$ , is associated to a distribution function,  $F$ , defined by

$$F(x) = \mu(X_i \leq x), \quad \forall x \in \mathbb{R} \cup \{\pm\infty\}.$$

Similarly, for a finite collection of random variables,  $X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_j}$ , we have the joint distribution function,  $F_j$ , defined by

$$F_{X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_j}}(x) = \mu(X_{i_1} \leq x, X_{i_2} \leq x, X_{i_3} \leq x, \dots, X_{i_j} \leq x),$$

for all  $x \in \mathbb{R} \cup \{\pm\infty\}$ .

An important feature of this process, due to the invariance of  $\mu$ , is that it is **stationary**, this means that the joint distribution function of any finite collection of random variables in the process is equal to the joint distribution function of that collection of random variables displaced by the same time period, that is, for any  $k, t \in \mathbb{N}$  and  $i_1 < i_2 < \dots < i_k \in \mathbb{N}$ ,  $F_{X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_j}}(x)$  and  $F_{X_{i_1+t}, X_{i_2+t}, X_{i_3+t}, \dots, X_{i_j+t}}(x)$  are the same.

Regarding the observable, we consider that  $\varphi$  achieves  $N$  global maxima at the points  $\xi_1, \dots, \xi_N \in \mathcal{X}$ , which are finite, and for each  $\xi_i$ ,  $\varphi$  is defined on a neighbourhood of  $\xi_i$  by

$$\varphi(x) = g_i(\text{dist}(x, \xi_i)) \quad (3)$$

where  $g_i : [0, +\infty) \rightarrow \mathbb{R} \cup \{\infty\}$  is strictly decreasing bijection on a neighbourhood of 0. Moreover, we assume that  $\varphi$  has the following property:

**(R1)**

Let  $u_F := \varphi(\xi_i)$  ( $i \in \{1, \dots, N\}$ ), we allow  $u_F = +\infty$ .

For  $u$  sufficiently close to  $u_F$ ,

$$U(u) := \{x \in \mathcal{X} : \varphi(x) > u\} = \{X_0 > u\}, \quad (4)$$

corresponds to  $U(u) = \bigcup_{i=1}^N B_{g_i^{-1}(u)}(\xi_i)$ , such that the  $N$  balls are disjoint. Moreover, the quantity  $\mu(U(u))$ , as a function of  $u$ , varies continuously on a neighbourhood of  $u_F$ .

We will be particularly interested in the extremal behaviour of the process  $X_0, X_1, \dots$  which is associated to the observation of abnormal values (classically taken as large values) referred to as exceedances  $\{X_i > u\}$  of an high threshold  $u$ . The choice of the high threshold  $u$  depends on the setting of the events that we are studying.

## 2 Extreme Value Theory

An approach to examine the behaviour of extreme events is by studying the statistical properties of extreme values using Extreme Value Theory, which aids in risk assessment. This theory uses the process of partial maxima  $(M_n)_{n \in \mathbb{N}}$ , where

$$M_n := \max\{X_0, \dots, X_{n-1}\}.$$

Hence, by knowing if  $M_n$  is less or equal to a threshold  $u$  allows us to conclude that there are no occurrences of values surpassing  $u$  until time  $n - 1$ . This process is also a stochastic process, which we want to find the asymptotic distribution (also called limiting distribution).

### Definition 2.0.1

We say that we have an **Extreme Value Law** (EVL) for  $M_n$  if there is a non-degenerate distribution function  $H : \mathbb{R} \rightarrow [0, 1]$  with  $H(0) = 0, \forall \tau > 0$ , there exists a normalized sequence  $u_n = u_n(\tau)$ ,  $n = 1, 2, \dots$ , such that

$$n\mu(X_0 > u_n) \xrightarrow{n \rightarrow \infty} \tau \quad (5)$$

and for which the following holds:

$$\mu(M_n \leq u_n) \xrightarrow{n \rightarrow \infty} \bar{H}(\tau), \quad (6)$$

for the points of continuity of  $H(\tau)$  and  $\bar{H}(\tau) = 1 - H(\tau)$ .

*Remark 2.0.1.* In the case that  $X_0, X_1, \dots$  is independent and identically distributed (i.i.d.) and the sequence  $(u_n)_{n \in \mathbb{N}}$  of the form  $u_n = y/a_n + b_n$ ,  $y \in \mathbb{R}$  satisfies (5), we

have

$$\begin{aligned}
 \mu(M_n \leq u) &= \mu(X_0 \leq u, \dots, X_n \leq u) \\
 &= \mu(X_0 \leq u) \cdots \mu(X_n \leq u) \\
 &= (\mu(X_0 \leq u))^n \\
 &= (1 - \mu(X_0 > u))^n \sim {}^1 \left(1 - \frac{\tau}{n}\right)^n \rightarrow e^{-\tau}
 \end{aligned}$$

Which means  $\bar{H}(\tau) = e^{-\tau}$ .

*Remark 2.0.2.* The use of  $u_n$  in the form  $u_n = y/a_n + b_n$ , where  $y \in \mathbb{R}$  it is useful, hence we can write (6) as

$$\mu(a_n(M_n - b_n) \leq y) \xrightarrow{n \rightarrow \infty} \bar{H}(\tau).$$

For example, considering exponential distribution with parameter 1, we have

$$F(x) = 1 - e^{-x}, \quad x > 0.$$

For  $\tau > 0$ , we may consider  $u_n$  such that  $n(1 - F(u_n)) = \tau$ , that is,  $u_n = \log n - \log \tau$ . Thus,

$$\mu(M_n - \log n \leq -\log \tau) \rightarrow e^{-\tau}.$$

Putting  $\tau = e^{-y}$ , we obtain

$$\mu(M_n - \log n \leq y) \rightarrow e^{-e^{-y}}.$$

So, in this case  $a_n = 1$  and  $b_n = \log n$ .

## 2.1 EVL for stationary sequences and dependence conditions

In the setting that we will work the stochastic processes, unlike the i.i.d. case, may have memory, this leads to the clustering effect. By cluster we are referring to the concentration of exceedance of a high threshold, in a reduced time gap. Therefore, in view of the EVL definition, in this section we will introduce two conditions for the process  $X_0, X_1, \dots$  that can hold in the presence or absence of clusters. Before, we introduce new events.

---

<sup>1</sup>Where, for positive  $f$  and  $g$ ,  $f(x) \sim g(x)$  as  $x \rightarrow x_0$ , means that  $\frac{f(x)}{g(x)} \rightarrow 1$  as  $x \rightarrow x_0$ .

Let

$$\mathcal{A}_q(u) := U(u) \cap \bigcap_{i=1}^q T^{-i}(U(u)^c) = \{X_0 > u, X_1 \leq u, \dots, X_q \leq u\}, \quad (7)$$

this set corresponds to occurrence of a rare event at time 0 and not having it happen again in a interval for the size  $q$ . Note that  $\mathcal{A}_0(u) = U(u)$ , and given a sequence of threshold  $u_n = u_n(\tau)$ ,  $\mathcal{A}_{q,n} := \mathcal{A}_q(u_n)$  and  $U_n := U(u_n)$ . Hence, we can "measure" the proportion of points in  $U_n$  that does not come back again, and define this value as:

$$\theta_n := \frac{\mu(\mathcal{A}_{q,n})}{\mu(U_n)}. \quad (8)$$

Let  $A \in \mathcal{B}$  be an event, for  $t \leq 0$  and  $\ell \leq 0$ , we define

$$\mathcal{W}_{t,\ell}(A) = \bigcap_{i=\lfloor t \rfloor}^{\lfloor t \rfloor + \max\{\lfloor \ell \rfloor - 1, 0\}} T^{-i}(A^c)^2, \quad (9)$$

this means that  $A$  does not occur during  $[t, t + \ell)$ . It is interesting to note that:

$$\begin{aligned} T^{-1}(\mathcal{W}_{0,\ell}(A)) &= T^{-1} \left( \bigcap_{i=0}^{\max\{\lfloor \ell \rfloor - 1, 0\}} T^{-i}(A^c) \right) \\ &= \bigcap_{i=0}^{\max\{\lfloor \ell \rfloor - 1, 0\}} T^{-i-1}(A^c) \\ &= \bigcap_{i=0}^{\max\{\lfloor \ell \rfloor - 1, 0\}} T^{-(i+1)}(A^c) \\ &= \bigcap_{j=1}^{\max\{\lfloor \ell \rfloor - 1, 0\}} T^{-j}(A^c) \\ &= \mathcal{W}_{1,\ell}(A). \end{aligned}$$

Futhermore, by induction, we get that  $\mathcal{W}_{t,\ell}(A) = T^{-t}(\mathcal{W}_{0,\ell}(A))$ .

Now we will introduce the first condition.

---

<sup>2</sup>  $A^c$  is the notation of the complement of  $A$ , i.e.,  $A^c := \mathcal{X} \setminus A$ .

### Condition $\mathcal{D}_q(u_n)$

We say that the condition  $\mathcal{D}_q(u_n)$  holds for the sequence  $X_0, X_1, \dots$  if for every  $\ell, t, n \in \mathbb{N}$ ,

$$|\mu(\mathcal{A}_{q,n} \cap \mathcal{W}_{t,\ell}) - \mu(\mathcal{A}_{q,n})\mu(\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n}))| \leq \gamma(q, n, t) \quad (10)$$

where  $\gamma(q, n, t)$  is decreasing in  $t$  for each  $n$  and, there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $t_n = o(n)$  and  $n\gamma(q, n, t_n) \xrightarrow{n \rightarrow \infty} 0$ .

This condition imposes a certain asymptotic independence, in other words, if we observe no exceedance after a cluster during  $q$  steps, then the time for the next exceedance is basically an independent event.

The next condition serves to estimate the probability of not having another rare event in the interval with size  $q$ .

### Condition $\mathcal{D}'_q(u_n)$

We say that the condition  $\mathcal{D}'_q(u_n)$  holds for the sequence  $X_0, X_1, \dots$  if there exists a sequence  $(k_n)_{n \in \mathbb{N}}$  such that

$$k_n \rightarrow \infty \text{ and } k_n t_n = o(n), \quad (11)$$

where for some  $q \in \mathbb{N}$ , the sequence  $(t_n)_{n \in \mathbb{N}}$ , from the condition  $\mathcal{D}_q(u_n)$ , is such that:

$$\lim_{n \rightarrow \infty} n \sum_{j=q+1}^{\lfloor n/k_n \rfloor - 1} \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) = 0 \quad (12)$$

**Remark 2.1.1.** Note that, if the condition  $\mathcal{D}'_q(u_n)$  holds for a given  $q = q_0 \in \mathbb{N}_0$ , then also holds for any  $q > q_0$ .

When both conditions hold, although we cannot guarantee that the limit (6) exists, if it does, we know what type of distribution to expect.

### Theorem 2.1.1

Let  $X_0, X_1, \dots$  be a stationary stochastic process and  $(u_n)_{n \in \mathbb{N}}$  a sequence

satisfying (5), for some  $\tau > 0$ . Assume that the conditions  $\mathcal{D}(u_n)$  and  $\mathcal{D}'(u_n)$  hold for  $q \in \mathbb{N}_0$ , and  $(t_n)_{n \in \mathbb{N}}$  and  $(k_n)_{n \in \mathbb{N}}$  are the sequences in those conditions. Moreover, assume that the limit  $\theta = \lim_{n \rightarrow \infty} \theta_n$  exists, then

$$\lim_{n \rightarrow \infty} \mu(M_n \leq u_n) = e^{-\theta\tau} \quad (13)$$

**Proof.** This theorem is a direct consequence of [11, Corollary 4.1.5.], which asserts that

$$\begin{aligned} |\mu(M_n \leq u_n) - e^{-\theta\tau}| &\leq C \left[ k_n t_n \frac{\tau}{n} + n\gamma(q, n, t_n) \right. \\ &\quad + n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) \\ &\quad \left. + e^{-\theta\tau} \left( |\tau - n\mu(U_n)| + \frac{\tau^2}{k_n} + |\theta_n - \theta| \tau \right) + q\mu(U_n \setminus \mathcal{A}_{q,n}) \right], \end{aligned}$$

We will show that the limit, as  $n \rightarrow \infty$ , of the right side is zero.

The first and third term converges to zero, because the condition  $\mathcal{D}'_q(u_n)$  have sequences  $(t_n)_{n \in \mathbb{N}}$  and  $(k_n)_{n \in \mathbb{N}}$  from the conditions, are such that  $k_n t_n = o(n)$ , so

$$k_n t_n \frac{\tau}{n} = \frac{k_n t_n}{n} \tau \xrightarrow{n \rightarrow \infty} 0.$$

and from (12),

$$n \sum_{j=1}^{\lfloor n/k_n \rfloor} \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) \xrightarrow{n \rightarrow \infty} 0.$$

The second term converges, by the condition  $\mathcal{D}_q(u_n)$  gives exactly the fact that  $n\gamma(q, n, t_n) \xrightarrow{n \rightarrow \infty} 0$ . The fifth term goes to zero since  $U_n$  decrease as  $n \rightarrow \infty$ .

Finally, the fourth term converges, since from (5) we have that

$$|\tau - n\mu(U_n)| \xrightarrow{n \rightarrow \infty} 0;$$

(11) implies  $\frac{\tau^2}{k_n}$  converges to 0;  $\lim_{n \rightarrow \infty} |\theta_n - \theta| = 0$  by hypothesis.  $\square$

The main difference between (13) and the asymptotic distribution of the case i.i.d.

on the Remark 2.0.1 is the parameter  $\theta$ , which quantifies the intensity of the cluster.

**Definition 2.1.2**

We say that  $X_0, X_1, \dots$  has **Extremal Index** (EI)  $0 \leq \theta \leq 1$  if we have an EVL for  $M_n$  with  $\bar{H}(\tau) = e^{-\theta\tau}, \forall \tau > 0$ .

The extremal index,  $\theta = 1$  means that the process have practically no memory and almost independence, leading to absence of clustering. While,  $\theta$  close to 0 means that the process have a large amount of memory, thus we expect intensive clustering.

The computation of the EI for different cases will be done in section 6, using the limit that appears in the Theorem 2.1.1, that is

$$\theta = \lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \frac{\mu(\mathcal{A}_{q,n})}{\mu(U_n)}. \quad (14)$$

## 2.2 Candidate for $q$

To find  $q$  that  $\bar{D}'_q(u_n)$  holds, the following procedure has proved to be very useful.

**Definition 2.2.1**

Consider  $A \in \mathcal{B}$ , we define the **first hitting time** function  $r_A : \mathcal{X} \rightarrow \mathbb{N} \cup \{+\infty\}$  by

$$r_A(x) := \inf\{n \in \mathbb{N} : T^n(x) \in A\},$$

which corresponds to the first time that the orbit of  $x$  hits  $A$ . If  $x$  never enters  $A$  under iteration by  $T$ , then  $r_A(x) = +\infty$ .

The restriction of  $r_A$  to  $A$  is called the **first return time** function to  $A$ , denoted by  $R(A)$ , that is

$$R(A) = \min_{x \in A} r_A(x).$$

Note that, almost every point of  $A$  will return to  $A$  infinitely often, this is guaranteed by the the following theorem.

### Theorem 2.2.2: Poincaré's Recurrence Theorem

If  $(\mathcal{X}, \mathcal{B}, \mu)$  is a measure space,  $T : \mathcal{X} \rightarrow \mathcal{X}$  preserves  $\mu$ , then for any  $A \in \mathcal{B}$  with positive measure  $\mu(A) > 0$ ,  $\mu$ -almost every point  $x \in A$ , the orbit returns to  $A$  infinitely often, that is, the set of points  $x \in A$  which the orbit returns to  $A$  only finitely many times has measure zero.

In addition, as the system is ergodic with respect to  $\mu$  and  $\mu(A) > 0$ , the orbit of almost every point will hit  $A$  at some point in time.

### Lemma 2.2.3: Kac's Lemma

Let  $(\mathcal{X}, \mathcal{B}, \mu)$  be a probability space, and let  $T$  be an ergodic measure that preserves  $\mu$ . Given a set  $A$  of measure  $\mu(A) > 0$ , the orbit of almost every point of  $A$  eventually returns to  $A$  and the expected time of the first recurrence is  $1/\mu(A)$ .

Moreover, by the Kac's Lemma, at least in average,  $r_A(x)$  should go to  $\infty$  as  $\mu(A)$  goes to zero.

Therefore, a natural candidate to try to show the validity of  $\mathbb{D}'_q(u_n)$  is

$$q := \min\{j \in \mathbb{N}_0 : \lim_{n \rightarrow \infty} R(\mathcal{A}_{j,n}) = \infty\}. \quad (15)$$

In other words,  $q$  is the non-negative integer that the measure  $\mu(\mathcal{A}_{q,n})$  is very small, meaning it will not return to  $\mathcal{A}_{q,n}$  in a long time.

### 3 Point Process of rare events

In this section we will present other way to study rare events, using rare events point process (REPP). Different from the previous section, the purpose of this method is to analyze and model the occurrence of extreme events within the stochastic process, by examining the patterns of occurrence and timing of this rare events. We start to build the point process which counts the times that a certain threshold is exceeded, and we try to see the convergence of this point process, aiming to understand the long-term properties.

#### 3.1 Definition of point processes

Let  $E \subset \mathbb{R}^d$ ,  $d \in \mathbb{N}$ , be a locally compact topological space with countable basis and let  $\mathcal{E}$  be its Borel  $\sigma$ -algebra.

Recall the Dirac measure  $\delta_x$  at  $x \in E$ :

$$\delta_x(A) = \mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \quad A \in \mathcal{E}.$$

A **point measure** on  $E$  is a measure  $m$  of the following form:

$$m(A) = \sum_{i=1}^{\infty} \delta_{x_i}(A) = \#\{i \geq 1 : x_i \in A\}, \quad x_i \in E, \quad A \in \mathcal{E}$$

and  $m(K) < \infty$ , for all compact sets  $K \in E$ , in other words,  $m$  is Radon meaning the measure of compact sets is always finite.

Let  $M_p(E)$  be the space of all point measures on  $E$  and let  $\mathcal{M}_p(E)$  be the smallest  $\sigma$ -algebra containing the sets

$$\{m \in M_p(E) : m(A) \in B\}$$

for all  $A \in E$  and all Borel sets  $B \subset [0, \infty)$ , i.e., it is the smallest  $\sigma$ -field making the maps  $m \rightarrow m(A)$  measurable for all  $A \in E$ . We endow  $M_p(E)$  with the vague topology so that it becomes a complete, separable metric space.

### Definition 3.1.1

A **point process**  $N$  on  $E$  is a measurable map from a probability space  $\Omega$  to  $M_p(E)$  equipped with its  $\sigma$ -algebra  $\mathcal{M}_p(E)$ . This means, a point process  $N$  is a random function which assumes point measures as values:  $N(\cdot, \omega) = m(\cdot)$  is a point measure for each  $\omega \in \Omega$ . In particular,  $N(K) < \infty$  for compact sets  $K \subset E$ .

Actually, the point process  $N$  can be seen as a collection  $(N(A))_{A \in E}$  of the random variables  $N(A)$  with values in  $\{0, 1, \dots, \infty\}$ , as the next lemma asserts.

### Lemma 3.1.2 A point process as a collection of random counting variables

The mapping  $N$  from  $(\Omega, \mathcal{F})$  to  $(M_p(E), \mathcal{M}_p(E))$  is a point process on  $E$  if and only if, for every  $A \in E$ ,  $N(A)$  is a random variable with values in  $\{0, 1, \dots, \infty\}$  such that  $N(A) < \infty$  for compact  $A \subset E$ .

**Proof.** See [12, p. 217] □

Moreover, consider  $m = \sum_{i=1}^{\infty} \delta_{x_i}$  to be a point measure on  $E$ , and let  $(y_i)_{i \in \mathbb{N}}$  be a subsequence of  $(x_i)_{i \in \mathbb{N}}$  containing all mutually distinct values  $x_i$  (with no repeats). The **multiplicity** of  $y_i$  is the number  $n_i = \#\{j \geq 1 : y_i = x_j\}$ . This allow us to rewrite  $m = \sum_{i=1}^{\infty} n_i \delta_{y_i}$ .

Furthermore, if  $n_i = 1$  for all  $i$ , then  $m$  is called a **simple point measure**. Similary, a point process  $N$  is called **simple point process**, when the realizations of  $N$  are simple point measures with probability 1.

On the other hand, consider a point process  $N = \sum_{i=1}^{\infty} \delta_{X_i}$ , where  $(X_i)_{i \in \mathbb{N}}$  is a sequence of  $E$ -valued random vectors. The point process  $N$  is simple if all points  $X_i$ ,  $i = 1, 2, \dots$ , are distinct with probability 1.

## 3.2 Rare Events Point Processes

In this section, we will define and study the convergence of a point process of rare events. Let  $E = (0, 1]$  be the state space, endowed with the Borel  $\sigma$ -algebra  $\mathcal{E}$ , for

all point processes in  $(N_n)_{n \in \mathbb{N}}$ , and the normalising sequence of thresholds  $(u_n(\tau))_{n \in \mathbb{N}}$  that satisfies (5).

**Definition 3.2.1**

The **rare events point process** corresponding to the threshold  $u_n$  is given by

$$N_n(A) = \sum_{i=1}^n \delta_{n^{-1}i}(A) \mathbb{1}_{\{X_i > u_n\}}, \quad A \in \mathcal{E}. \quad (16)$$

To put it another way, for a sample of size  $n$ ,  $N_n$  counts the number of excedences of the threshold  $u_n$  by the sequence  $X_1, \dots, X_n$ , that occur in the normalized time gap  $nA$ .

To clarify take the whole interval  $(0, 1]$ . Thus

$$\begin{aligned} N_n(0, 1] &= \#\{i : 0 < n^{-1}i \leq 1 \text{ and } X_i > u_n\} \\ &= \#\{i \leq n : X_i > u_n\} \end{aligned}$$

Due to the fact that exists only a finitely number of points  $n^{-1}i$ ,  $i = 1, \dots, n$ , which are separated from each other,  $N_n$  is a simple point process. Now observe that:

$$\begin{aligned} \{N_n(0, 1] = 0\} &= \{\#\{i \leq n : X_i > u_n\} = 0\} \\ &= \{\text{None of the } X_i\text{'s, } i \leq n, \text{ exceeds } u_n\} \\ &= \{\max(X_1, \dots, X_n) \leq u_n\}, \\ &= \{M_n \leq u_n\}. \end{aligned}$$

With this, we can see the connection with the EVT. Furthermore, let  $X_{(n-k+1)}$  denote the  $k$ th largest order statistic of the sample  $X_1, \dots, X_n$ . Then

$$\begin{aligned} \{N_n(0, 1] < k\} &= \{\#\{i \leq n : X_i > u_n\} < k\} \\ &= \{\text{Fewer than } k \text{ among the } X_i\text{'s, } i \leq n, \text{ exceed } u_n\} \\ &= \{\text{The order statistic } X_{(n-k+1)} \text{ does not exceed } u_n\} \\ &= \{X_{(n-k+1)} \leq u_n\}. \end{aligned}$$

### 3.3 Convergence of REPP

#### Definition 3.3.1: Weak convergence of point processes

Let  $N, N_1, N_2, \dots$ , be point processes on the state space  $E \subset \mathbb{R}^d$  equipped with the  $\sigma$ -field  $\mathcal{E}$  of its Borel sets. We say that the sequence of point processes  $(N_n)_{n \in \mathbb{N}}$  converges weakly to the point process  $N$  in  $M_p(E)$  (we write  $N_n \xrightarrow{d} N$ ) if

$$(N_n(A_1), \dots, N_n(A_m)) \xrightarrow{d} (N(A_1), \dots, N(A_m)), \quad m \geq 1, A_1, \dots, A_m \in \mathcal{E}$$

is satisfied for all possible choices of bounded sets  $A_i \in E$  satisfying  $P(N(\partial A_i) = 0) = 1, i = 1, \dots, m, m \geq 1$ , where  $\partial A$  denotes the boundary of  $A$  in  $E$ .

#### 3.3.1 Absence of Clusters

The classical setting of stationary stochastic processes, in the absence of clustering, these processes converge to a homogeneous Poisson process of intensity  $\tau$ . This homogeneous Poisson process has an expected value given by  $\mathbb{E}(N(a, b]) = (b - a)\tau$ , and let  $B$  be an union of disjoint intervals of  $E$ , the probability of the random variable  $N(B)$  being equal to 0 is

$$\mu(N(B) = 0) = e^{-\tau|B|}, \quad (17)$$

where  $|(a, b]| = b - a, 0 < a < b \leq 1$  (for more detail about Poisson process see [12]).

In order to prove this convergence we need the following theorem:

#### Theorem 3.3.1.1

Let  $N, N_1, N_2, \dots$ , be point processes on  $E = (a, b] \subset \mathbb{R}$  and assume that  $N$  is simple. Suppose the following two conditions hold:

$$E(N_n(c, d]) \rightarrow E(N(c, d]), \quad a < c < d \leq b, \quad (18)$$

$$\mu(N_n(B) = 0) \rightarrow \mu(N(B) = 0), \quad (19)$$

for any union  $B = \bigcup_{i=1}^k (c_i, d_i]$  of disjoint intervals  $(c_i, d_i] \subset (a, b]$  with  $\min_{i=1, \dots, k} c_i > a, k \geq 1$ . Then  $N_n \xrightarrow{d} N$  in  $M_\mu(E)$ .

Consider that  $u_n$  satisfies (5). Assume that  $X_0, X_1, \dots$  are i.i.d.

For  $(a, b] \subset (0, 1]$  the random variable

$$N_n(a, b] = \sum_{i=1}^n \delta_{n^{-1}i}(a, b] \mathbb{1}_{\{X_i > u_n\}} = \sum_{i: na < i \leq nb} \mathbb{1}_{\{X_i > u_n\}}$$

is  $\text{Bin}(n(a, b], \mu(X_0 > u_n))$  distributed, that is, binomially distributed, where

$$n(a, b] = \#\{1 \leq i \leq n : na < i \leq nb\},$$

with expected value  $\mathbb{E}(N_n(a, b]) = n(a, b] \mu(X_0 > u_n)$ .

Then, by the condition (5),

$$E(N_n(a, b]) = n(a, b] \mu(X_0 > u_n) \sim (n(b-a)) \mu(X_0 > u_n) \sim (b-a) \tau = E(N(a, b]),$$

which proves condition (18) of Kallenberg's Theorem.

It is yet to be demonstrated the second condition (19). Firstly, note that for any  $0 < a < b \leq 1$ ,

$$\mu(N_n(a, b] = 0) = \mu\left(\max_{na < i \leq nb} X_i \leq u_n\right) \tag{i}$$

$$= (\mu(X_0 \leq u_n))^{n(a, b]} \sim (\mu(X_0 \leq u_n))^{n(b-a)} \tag{ii}$$

$$= e^{(n(b-a)) \log(\mu(X_0 \leq u_n))} \tag{iii}$$

$$= e^{(n(b-a)) \log(1 - \mu(X_0 > u_n))} \tag{iv}$$

$$= e^{(n(b-a))(-\mu(X_0 > u_n)(1+o(1)))} \tag{v}$$

$$= e^{-n\mu(X_0 > u_n)(1+o(1))(b-a)} \tag{vi}$$

$$\rightarrow e^{-\tau(b-a)} \quad \text{as } n \rightarrow +\infty. \tag{vii}$$

In the (v) step we used the Taylor expansion  $\log(1+x) = x(1+o(1))$  as  $x \rightarrow 0$  and the fact that  $n\mu(X_0 > u_n) \rightarrow \tau$  implies  $\mu(X_0 > u_n) \rightarrow 0$ . In the (vi) we used again the fact that  $n\mu(X_0 > u_n) \rightarrow \tau$  and that  $o(1) \xrightarrow{n \rightarrow +\infty} 0$ .

Due to the fact that the random variables  $X_j$  are independent and the sets  $(c_i, d_i] \subset (0, 1]$ ,  $c_i > 0$ , which constitute the union  $B$  in (19), are disjoint, the counting variables  $N_n(c_i, d_i]$  are independent. Hence,

$$\begin{aligned} \mu(N_n(B) = 0) &= \mu(N_n(c_i, d_i] = 0, i = 1, \dots, k) \\ &= \prod_{i=1}^k \mu(N_n(c_i, d_i] = 0) \\ &\rightarrow \prod_{i=1}^k e^{-\tau(d_i - c_i)} \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

The limit under consideration is due to the multiple use of the fact proven before, which is,  $\mu(N_n(a, b] = 0) \rightarrow e^{-\tau(b-a)}$ .

Moreover, by the Poisson property of  $N$ , (17),

$$\mu(N(B) = 0) = e^{-\tau|B|} = e^{-\tau \sum_{i=1}^k (d_i - c_i)} = \prod_{i=1}^k e^{-\tau(d_i - c_i)}.$$

Therefore, by the Kallenberg's Theorem,  $N_n \xrightarrow{d} N$  as we wanted to demonstrate.

### 3.3.2 Presence of Clusters

Under the presence of clustering, the convergence of the REPP is more complicated. However, before, we introduce some useful notation, for  $\kappa \geq 0$ , consider:

$$U^{(0)}(u) := U(u) \text{ and } \mathcal{A}_q^{(0)}(u) := \mathcal{A}_q(u);$$

$$U^{(\kappa)}(u) := U^{(\kappa-1)}(u) \setminus \mathcal{A}_q^{(\kappa-1)}(u),$$

$$\mathcal{A}_q^{(\kappa)}(u) := U^{(\kappa)}(u) \cap \bigcap_{i=1}^q T^{-i}(U^{(\kappa)}(u)^c) = U^{(\kappa)}(u) \setminus U^{(\kappa-1)}(A).$$

The event  $U^{(\kappa)}(u)$  corresponds to observing  $U(u)$  at time 0 and then observing  $U(u)$  for at least  $\kappa$  times, so that the waiting time between two observations of  $U(u)$  is at most  $q$ . And the event  $\mathcal{A}_q^{(\kappa)}(u) = U^{(\kappa)}(u) \setminus U^{(\kappa-1)}(A)$  corresponds to observing  $U(u)$  exactly  $\kappa + 1$  times within no more than  $q$  units of time between one and the next observation of  $U(u)$ , this means, in particular, that the  $\kappa + 2$ th observation of  $U(u)$  occurs at least  $q + 1$  iterations after the  $\kappa + 1$ th observation of  $U(u)$ . Also, as previously, for a sequence

of threshold  $(u_n)_{n \in \mathbb{N}_0}$ , let  $U_n^{(\kappa)} := U^{(\kappa)}(u_n)$ ,  $\mathcal{A}_{q,n}^{(\kappa)} = \mathcal{A}_q^{(\kappa)}(u_n)$ .

It is certain that under some conditions, the weak limit  $N$  is a compound Poisson process with intensity  $\eta$  and multiplicity distribution given by  $\pi(\kappa) = \mu(D_1 = \kappa)$ , which can be described in the following way:

$$N = \sum_{i=1}^{\infty} D_i \delta_{T_i},$$

where

- $W_1, W_2, \dots$  is an i.i.d. sequence of exponentially distributed random variables with mean  $1/\eta > 0$ , that is,  $W_i \sim \text{Exp}(\eta)$ ;
- $T_i = \sum_{j=1}^i W_j$ ;
- $D_1, D_2, \dots$  be an i.i.d. sequence of positive integer valued random variables independent of  $T_1, T_2, \dots$ .

We usually interpret  $T_i$  as the time of appearance of the  $i$ th cluster and  $D_i$  the respective size.

Recall that  $\mu(M_n \leq u_n) = \mu(N_n(0, 1] = 0)$ , then the limiting distribution will be  $\mu(N([0, 1) = 0)$ . Which we can interpret as the first time that a cluster appears is greater than 1, thus obtaining:

$$\mu(N([0, 1) = 0) = \mu(W_1 > 1) = e^{-\eta}.$$

Observe that in most situations  $\mathbb{E}(N_n([0, 1))) = \tau$  then  $\eta = \tau/E(D_1)$ . Normally, we have that  $\theta^{-1} = E(D_1)$ , where  $\theta$  is the EI.

With the purpose of obtaining the convergence of the point processes described above, we will use the condition  $\mathbb{D}'_q(u_n)$  and a new condition.

This new condition that will substitute the condition  $\mathbb{D}_q(u_n)$ , is a sort of mixing condition specially designed for this extreme analysis with applications to dynamically generated stochastic processes.

### Condition $\mathcal{D}_q(u_n)^*$

We say that  $\mathcal{D}_q(u_n)^*$  holds for the sequence  $X_0, X_1, X_2, \dots$  if for any integers  $t, \kappa_1, \dots, \kappa_\varsigma, n$  and any  $J = \cup_{i=2}^\varsigma I_j \in \mathcal{R}$  with  $\inf\{x : x \in J\} \geq t$ ,

$$\left| \mu \left( \mathcal{A}_{q,n}^{(\kappa_1)} \cap \left( \cap_{j=2}^\varsigma N_{u_n}(I_j) = \kappa_j \right) \right) - \mu \left( \mathcal{A}_{q,n}^{(\kappa_1)} \right) \mu \left( \cap_{j=2}^\varsigma N_{u_n}(I_j) = \kappa_j \right) \right| \leq \gamma(n, t),$$

where for each  $n$  we have that  $\gamma(n, t)$  is nonincreasing in  $t$  and  $n\gamma(n, t_n) \rightarrow 0$  as  $n \rightarrow \infty$ , for some sequence  $t_n = o(n)$ .

Thus, we have the following theorem, as stated in the article [1].

### Theorem 3.3.2.1

Let  $X_0, X_1, \dots$  satisfy conditions  $\mathcal{D}_q(u_n)^*$  and  $\mathcal{D}'_q(u_n)$ , where  $(u_n)_{n \in \mathbb{N}}$  is such that (5) holds. Assume that the limit  $\theta = \lim_{n \rightarrow \infty} \theta_n$  exists, where  $\theta_n$  is as in (8) and moreover that for each  $\kappa \in \mathbb{N}$ , the following limit also exists

$$\pi(\kappa) := \lim_{n \rightarrow \infty} \pi_n(\kappa) = \lim_{n \rightarrow \infty} \frac{(\mu(\mathcal{A}_{q,n}^{(\kappa-1)}) - \mu(\mathcal{A}_{q,n}^{(\kappa)}))}{\mu(\mathcal{A}_{q,n}^{(0)})}. \quad (20)$$

Then the REPP  $N_n$  converges in distribution to a compound Poisson process with intensity  $\theta\tau$  and multiplicity distribution  $\pi(\kappa)$  given by (20).

## 4 Binary coding of rare events for general stationary stochastic processes

In this section, we will use the method described in the article [1] to interpret the study of rare events problem. This method allows a different perspective, facilitates the computation of the set  $\mathcal{A}_q^{(\kappa)}$ ,  $q, \kappa \in \mathbb{N}$ . We will put it into practice in section 7.

The stationary stochastic process  $X_0, X_1, \dots$  is identified with the following coordinate-variable process on  $(\mathcal{R}^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0}, \mu)$ , where  $\mathcal{R} = \mathbb{R}^d$  and  $\mathcal{B}^{\mathbb{N}_0}$  is the  $\sigma$ -field generated by the coordinate functions  $V_n : \mathcal{R}^{\mathbb{N}_0} \rightarrow \mathcal{R}$ , with  $V_n(x_0, x_1, \dots) = x_n$ , for  $n \in \mathbb{N}_0$ , so that there is a natural measurable map, the shift operator,  $\sigma : \mathcal{R}^{\mathbb{N}_0} \rightarrow \mathcal{R}^{\mathbb{N}_0}$ , given by  $\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$ .

Due to the fact that the process is stationary,  $\mu$  is  $\sigma$ -invariant. Observe that

$$V_{i-1} \circ \sigma = V_i, \forall i \in \mathbb{N}.$$

Also  $V_i = V_0 \circ \sigma^i$ , for all  $i \in \mathbb{N}_0$ , where  $\sigma^i$  denotes the  $i$ -fold composition of  $\sigma$ , with the convention that  $\sigma^0$  denotes the identity map on  $\mathcal{R}^{\mathbb{N}_0}$ .

Accordingly with the notation above we can rewrite the already known sets,

$$U(u) = V_0^{-1}([u, +\infty)) \text{ and } \mathcal{A}_q(u) = U(u) \cap \bigcap_{i=1}^q \sigma^{-i}((U(u))^c),$$

for a given  $q \in \mathbb{N}$ . Also, for  $\kappa \in \mathbb{N}$ ,

$$U^{(\kappa)}(u) = U^{(\kappa-1)}(u) \setminus \mathcal{A}_q^{(\kappa-1)}(u) \text{ and } \mathcal{A}_q^{(\kappa)}(A) = U^{(\kappa)} \cap \bigcap_{i=1}^q \sigma^{-i}((U^{(\kappa)}(u))^c).$$

Now, let us define

$$\begin{aligned} h : \quad \mathcal{R}^{\mathbb{N}_0} &\rightarrow \{0, 1\}^{\mathbb{N}_0} \\ \underline{x} = (x_0, x_1, \dots) &\mapsto \underline{\omega} = \omega_0 \omega_1 \dots \end{aligned}$$

such that, for each  $n \in \mathbb{N}_0$ ,  $\omega_n = \tilde{V}_n(h(\underline{x})) = 1$  if  $x_n \in [u_n, +\infty)$  and  $\omega_n = \tilde{V}_n(h(\underline{x})) = 0$

otherwise, where  $\tilde{V} = V|_{\{0,1\}^{\mathbb{N}_0}}$ . Let  $J$  be an interval contained in  $[0, \infty)$ . We can write

$$\mathcal{W}_J(A) = \bigcap_{i \in J \cap \mathbb{N}_0} \sigma^{-i}(V_0^{-1}(A^c)),$$

meaning if  $\underline{x} \in \mathcal{W}_J(A)$ ,  $h(\underline{x})$  has a block of consecutive 0 that correspond to the observations in  $J \cap \mathbb{N}_0$ .

Hence, if  $(x_0, x_1, \dots) \in \mathcal{R}^{\mathbb{N}_0}$  is a realisation of  $X_0, X_1, \dots$ , then the beginning of cluster and the ending of cluster can be easily identified in  $h(x_0, x_1, \dots)$  by the appearance of a block of at least  $q$  consecutive 0. Therefore we can describe the events  $\mathcal{A}_{q,n}^{(\kappa)}$ ,  $\kappa \in \mathbb{N}_0$  using binary strings, first, for  $q, \kappa \in \mathbb{N}$  fixed, let

$$W_q(\kappa) = \left\{ \bar{\omega} \in \bigcup_{i=\kappa}^{q(\kappa-1)+1} \{0,1\}^i : \tilde{V}_0(\bar{\omega}) = \tilde{V}_{|\bar{\omega}|-1}(\bar{\omega}) = 1, \sum_{i=0}^{|\bar{\omega}|-1} \tilde{V}_0(\sigma^i(\bar{\omega})) = \kappa, \right. \\ \left. \sigma^i(\bar{\omega}) \in \bigcup_{j=0}^{q-1} V_j^{-1}(1), \forall i = 0, \dots, |\bar{\omega}| - 1 \right\},$$

where  $|\bar{\omega}|$  is the length of the finite string  $\bar{\omega}$ . This represents the set of finite strings of 0's and 1's such that each string starts and ends with a 1, has exactly  $\kappa$  1's, which are separated by at most  $q - 1$  0's, that is, there is no block of  $q$  or more consecutive 0 in the string. Accordingly, we have:

$$\mathcal{A}_{q,n}^{(\kappa)} = h^{-1}(\{\underline{\omega} \in \{0,1\}^{\mathbb{N}_0} : \underline{\omega} = \bar{\omega} \underbrace{0 \dots 0}_{q \text{ symbols}} \dots, \text{ for some } \bar{\omega} \in W_q^{(\kappa+1)}\}).$$

In order to simplify, let  $\ell \in \mathbb{N}$ ,  $\varsigma \in \{0,1\}^\ell$  and define the cylinder

$$C(\varsigma) = \{\underline{\omega} \in \{0,1\}^{\mathbb{N}} : \underline{\omega} = \varsigma \underline{\omega}^*, \text{ for some } \underline{\omega}^* \in \{0,1\}^{\mathbb{N}}\},$$

corresponding to the binary words which start with the string  $\varsigma$ . Then,

$$\mathcal{A}_{q,n}^{(\kappa)} = h^{-1}(\{C(\bar{\omega} \underbrace{0 \dots 0}_{q \text{ symbols}}), \bar{\omega} \in W_q^{(\kappa+1)}\}). \quad (21)$$

For example, take  $q = 2$ :

$$\mathcal{A}_{2,n}^{(0)} = h^{-1}(C(100)) = \{X_0 > u_n, X_1 \leq u_n, X_2 \leq u_n\}$$

$$\begin{aligned} \mathcal{A}_{2,n}^{(1)} = h^{-1}(C(1100) \cup C(10100)) &= \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n\} \cup \\ &\cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 \leq u_n, X_4 \leq u_n\} \end{aligned}$$

$$\mathcal{A}_{2,n}^{(2)} = h^{-1}(C(11100) \cup C(110100) \cup C(101100) \cup C(1010100))$$

$$\begin{aligned} &= \{X_0 > u_n, X_1 > u_n, X_2 > u_n, X_3 \leq u_n, X_4 \leq u_n\} \\ &\cup \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 > u_n, X_4 \leq u_n, X_5 \leq u_n\} \\ &\cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 > u_n, X_4 \leq u_n, X_5 \leq u_n\} \\ &\cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 \leq u_n, X_4 > u_n, X_5 \leq u_n, X_6 \leq u_n\} \end{aligned}$$

## 5 Application to Dynamical Systems

In this section we will see some examples of dynamical systems for which we can use the extreme value theory or rare events point process to study their properties. As we have seen, we need to check the conditions  $\mathbb{D}_q(u_n) / \mathbb{D}_q(u_n)^*$  and  $\mathbb{D}'_q(u_n)$ . However these are not easy to verify directly, therefore we will see which properties of the system allow us to check them.

### 5.1 Assumptions on the system

We assume that the dynamical system has decay of correlations against  $\mathbb{L}^1$  observables, which means:

#### Definition 5.1.1

Let  $\mathcal{C}_1, \mathcal{C}_2$  denote Banach spaces of real valued measurable functions defined on  $\mathcal{X}$ . We denote the **correlation** of non-zero functions  $\phi \in \mathcal{C}_1$  and  $\psi \in \mathcal{C}_2$  with respect to the invariant measure  $\mu$  as

$$\text{Cor}_\mu(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1} \|\psi\|_{\mathcal{C}_2}} \left| \int \phi(\psi \circ T^n) d\mu - \int \phi d\mu \int \psi d\mu \right|.$$

We say that we have **decay of correlations**, with respect to the measure  $\mu$ , for observables in  $\mathcal{C}_1$  **against** observables in  $\mathcal{C}_2$  if, for every  $\phi \in \mathcal{C}_1$  and every  $\psi \in \mathcal{C}_2$  we have

$$\text{Cor}_\mu(\phi, \psi, n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

We say that we have **decay of correlations against  $\mathbb{L}^1$  observables** whenever this holds for  $\mathcal{C}_2 = \mathbb{L}^1(\mu)$  and  $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int |\psi| d\mu$ .

#### Assumption A

Let  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a system with summable decay of correlations against  $\mathbb{L}^1$  observables, i.e., for all  $\varphi \in \mathcal{C}_1$  and  $\psi \in \mathbb{L}^1$ , then  $\text{Cor}(\varphi, \psi, n) \leq \rho_n$ , with  $\sum_{n \geq \mathbb{N}} \rho_n < \infty$ .

Moreover, we assume that there exists  $C > 0$  such that  $\forall n \in \mathbb{N}$ , we have  $\mathbb{1}_{\mathcal{A}_n}, \mathbb{1}_{\mathcal{A}^{(\kappa)}} \in \mathcal{C}_1$  and  $\|\mathbb{1}_{\mathcal{A}_n}\|_{\mathcal{C}_1}, \|\mathbb{1}_{\mathcal{A}^{(\kappa)}}\|_{\mathcal{C}_1} \leq C$ .

## 5.2 Verification of the conditions

By assuming that Assumption A is satisfied, it is now easy to prove that the conditions  $\mathbb{D}_q(u_n)$ ,  $\mathbb{D}_q(u_n)^*$  and  $\mathbb{D}'_q(u_n)$  hold.

**5.2.1. Verification of condition  $\mathbb{D}_q(u_n)$ .** Taking  $\phi = \mathbb{1}_{\mathcal{A}_{q,n}}$  and  $\psi = \mathbb{1}_{\mathcal{W}_{t,\ell}(\mathcal{A}_{q,n})}$  in Definition 5.1.1, there exists  $D > 0$ , so that for any positive numbers  $\ell$  and  $t$  we have

$$\begin{aligned}
 & |\mu(\mathcal{A}_{q,n} \cap \mathcal{W}_{t,\ell}(\mathcal{A}_{q,n})) - \mu(\mathcal{A}_{q,n})\mu(\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n}))| = \\
 & = \left| \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n} \cap \mathcal{W}_{t,\ell}(\mathcal{A}_{q,n})} d\mu - \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} d\mu \int_{\mathcal{X}} \mathbb{1}_{\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n})} d\mu \right| \\
 & = \left| \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} \cdot \mathbb{1}_{\mathcal{W}_{t,\ell}(\mathcal{A}_{q,n})} d\mu - \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} d\mu \int_{\mathcal{X}} \mathbb{1}_{\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n})} d\mu \right| \\
 & = \left| \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} \cdot \mathbb{1}_{T^{-t}(\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n}))} d\mu - \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} d\mu \int_{\mathcal{X}} \mathbb{1}_{\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n})} d\mu \right| \\
 & = \left| \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} \cdot \left( \mathbb{1}_{\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n})} \circ T^t \right) d\mu - \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} d\mu \int_{\mathcal{X}} \mathbb{1}_{\mathcal{W}_{0,\ell}(\mathcal{A}_{q,n})} d\mu \right| \\
 & = \left| \int_{\mathcal{X}} \phi \cdot (\psi \circ T^t) d\mu - \int_{\mathcal{X}} \phi d\mu \int_{\mathcal{X}} \psi d\mu \right| \\
 & = \|\phi\|_{\mathcal{C}_1} \|\psi\|_{L^1(\mu)} \text{Cor}_\mu(\phi, \psi, t) \\
 & \leq D\rho_t.
 \end{aligned}$$

Then, Condition  $\mathbb{D}_q(u_n)$  follows by taking  $(t_n)_{n \in \mathbb{N}}$  such that  $t_n = o(n)$  and  $\lim_{n \rightarrow \infty} n\rho_{t_n} = 0$ .

**5.2.2. Verification of condition  $\mathbb{D}_q(u_n)^*$ .** Taking  $\phi = \mathbb{1}_{\mathcal{A}_{q,n}^{(\kappa)}}$  and  $\psi = \mathbb{1}_{N_{u_n}(I_j)=\kappa_j}$  in Definition 5.1.1, and using the same arguments as in the previous verification, we are able to check  $\mathbb{D}_q(u_n)^*$ .

**5.2.3. Verification of condition  $\mathbb{D}'_q(u_n)$ .** Taking  $\phi = \psi = \mathbb{1}_{\mathcal{A}_{q,n}}$  in Definition 5.1.1 we obtain:

$$\begin{aligned}
 \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) &= \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})} d\mu \\
 &= \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} \cdot (\mathbb{1}_{T^{-j}(\mathcal{A}_{q,n})}) d\mu \\
 &= \int_{\mathcal{X}} \mathbb{1}_{\mathcal{A}_{q,n}} \cdot (\mathbb{1}_{\mathcal{A}_{q,n}} \circ T^j) d\mu \\
 &= \int_{\mathcal{X}} \phi \cdot (\phi \circ T^j) d\mu \\
 &= \left( \int_{\mathcal{X}} \phi d\mu \right)^2 + \|\phi\|_{C_1} \|\phi\|_{L^1(\mu)} \text{Cor}(\phi, \phi, j) \\
 &\leq (\mu(\mathcal{A}_{q,n}))^2 + \|\mathbb{1}_{\mathcal{A}_{q,n}}\|_{C_1} \mu(\mathcal{A}_{q,n}) \rho_j \\
 &\leq (\mu(\mathcal{A}_{q,n}))^2 + C \mu(\mathcal{A}_{q,n}) \rho_j.
 \end{aligned}$$

Let  $R_n := R(\mathcal{A}_{q,n})$ ,  $t_n$  be as above and take  $(k_n)_{n \in \mathbb{N}}$  as in (11). Recalling that  $\lim_{n \rightarrow \infty} n\mu(U_n) = \tau$  and that there exists some constant  $D > 0$ , such that

$$\begin{aligned}
 n \sum_{j=q+1}^{\lfloor n/k_n \rfloor} \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) &= n \sum_{j=R_n}^{\lfloor n/k_n \rfloor} \mu(\mathcal{A}_{q,n} \cap T^{-j}(\mathcal{A}_{q,n})) \\
 &= n \sum_{j=R_n}^{\lfloor n/k_n \rfloor} \left( (\mu(\mathcal{A}_{q,n}))^2 + C \mu(\mathcal{A}_{q,n}) \rho_j \right) \\
 &\leq n \left\lfloor \frac{n}{k_n} \right\rfloor (\mu(\mathcal{A}_{q,n}))^2 + n C \mu(\mathcal{A}_{q,n}) \sum_{j=R_n}^{\lfloor n/k_n \rfloor} \rho_j \\
 &\leq \frac{(n\mu(\mathcal{A}_{q,n}))^2}{k_n} + n C \mu(\mathcal{A}_{q,n}) \sum_{j=R_n}^{\infty} \rho_j \\
 &\leq \frac{(n\mu(U_n))^2}{k_n} + C n \mu(U_n) \sum_{j=R_n}^{\infty} \rho_j \\
 &\leq D \left( \frac{\tau^2}{k_n} + \tau \sum_{j=R_n}^{\infty} \rho_j \right) \xrightarrow{n \rightarrow \infty} 0,
 \end{aligned}$$

if we check that  $\lim_{n \rightarrow \infty} R_n = +\infty$ , then the estimate works.

### 5.3 Frobenius-Perron Operators

In this section we define the Frobenius-Perron operator and present its basic properties. This operator will be very helpful for the next sections, in order to prove decay of correlations for the given examples.

#### Definition 5.3.1

Let  $(\mathcal{X}, \mathcal{B}, \mu)$  a probability measure space, where  $\mathcal{B}$  is a Borel  $\sigma$ -algebra of the topological space  $\mathcal{X}$ , and  $T : \mathcal{X} \rightarrow \mathcal{X}$  is a measurable map (not necessarily preserving  $\mu$ ). Let  $T$  be *nonsingular* with respect to  $\mu$ , which means that  $T_*\nu \ll \mu$  whenever  $\nu \ll \mu$ . We define the **Frobenius-Perron operator**  $P_{T,\mu} : \mathbb{L}^1(\mu) \rightarrow \mathbb{L}^1(\mu)$  as follows:

For any  $\varphi \in \mathbb{L}^1(\mu)$ ,  $P_{T,\mu}\varphi$  is the unique (up to a.e. equivalence) function in  $\mathbb{L}^1(\mu)$  such that

$$\int_A P_{T,\mu}\varphi d\mu = \int_{T^{-1}(A)} \varphi d\mu \quad (22)$$

for any  $A \in \mathcal{B}$ . Whenever it is clear which map  $T$  is being used, we will drop the  $T$  and just write  $P_\mu$ .

The validity of this definition, i.e., the existence and the uniqueness of  $P_{T,\mu}\varphi$ , follows by the Radon-Nikodym Theorem.

#### Theorem 5.3.2: Radon-Nikodym

Given two  $\sigma$ -finite measure,  $\mu, \nu$ , in  $(\Omega, \mathcal{B})$ , if  $\nu \ll \mu$  then exists a unique  $\varphi \in \{\varphi : \mathcal{X} \rightarrow \mathbb{R}_0^+ : \varphi \text{ is measurable}\} \cap \mathbb{L}^1(\mu)$  such that  $\nu(A) = \int_A \varphi d\mu$  for all  $A \in \mathcal{B}$ .

#### Definition 5.3.3

The function  $\varphi$  given in the previous theorem is usually referred to as the **Radon-Nikodym derivative** of  $\nu$  with respect to  $\mu$  and denoted by

$$\frac{d\nu}{d\mu} := \varphi.$$

Futhermore, other used definition for this operator is the conditional expectation.

**Definition 5.3.4**

Let  $(\mathcal{X}, \mathcal{B}, \mu)$  be a probability space and let  $\mathcal{C} \subset \mathcal{B}$  be a sub  $\sigma$ -algebra. For  $\varphi \in \mathbb{L}^1(\mu)$ , we define the **conditional expectation** of  $\varphi$  with respect to  $\mathcal{C}$  as the Radon-Nikodym derivative of the measure  $\varphi\mu|_{\mathcal{C}}$  with respect to the measure  $\mu|_{\mathcal{C}}$ , and denote it by  $\mathbb{E}(\varphi|\mathcal{C})$ :

$$\mathbb{E}(\varphi|\mathcal{C}) = \frac{d(\varphi\mu|_{\mathcal{C}})}{d(\mu|_{\mathcal{C}})}.$$

**Theorem 5.3.5**

For a function  $\psi \in \mathbb{L}^1(\mu)$ , we have  $\psi = \mathbb{E}(\varphi|\mathcal{C})$  if and only if

$$\int_A \psi d\mu = \int_A \varphi d\mu$$

for any  $A \in \mathcal{C}$ .

Consider the Frobenius-Perron operator  $P_\mu$  as in the Definition 5.3.1, we will now see some useful properties of this operator.

(P1) **Linearity:**  $P_\mu : \mathbb{L}^1 \rightarrow \mathbb{L}^1$  is a linear operator.

**Proof.** Let  $A \subset \mathcal{X}$  be measurable and let  $\alpha, \beta$  be constants. Then, if  $\varphi, \psi \in \mathbb{L}^1(\mu)$ ,

$$\begin{aligned} \int_A P_\mu(\alpha\varphi + \beta\psi)d\mu &= \int_{T^{-1}A} (\alpha\varphi + \beta\psi)d\mu \\ &= \alpha \int_{T^{-1}A} \varphi d\mu + \beta \int_{T^{-1}A} \psi d\mu \\ &= \alpha \int_A P_\mu\varphi d\mu + \beta \int_A P_\mu\psi d\mu \\ &= \int_A (\alpha P_\mu\varphi + \beta P_\mu\psi)d\mu. \end{aligned}$$

Since this is true for any measurable set  $A$ ,

$$P_\mu(\alpha\varphi + \beta g) = \alpha P_\mu\varphi + \beta P_\mu g \quad \text{a.e.}$$

□

(P2) **Positivity:** Let  $\varphi \in \mathbb{L}^1$  and assume  $\varphi \geq 0$ . Then  $P_\mu\varphi \geq 0$ .

**Proof.** Let  $\varphi \geq 0$ , for  $A \in \mathcal{B}$ ,

$$\int_A P_\mu\varphi d\mu = \int_{T^{-1}(A)} \varphi d\mu \geq 0.$$

Since  $A \in \mathcal{B}$  is arbitrary,  $P_\mu\varphi \geq 0$ .

□

(P3) **Preservation of Integrals:**

$$\int_{\mathcal{X}} P_\mu\varphi d\mu = \int_{\mathcal{X}} \varphi d\mu.$$

**Proof.** Since

$$\int_{\mathcal{X}} P_\mu\varphi d\mu = \int_{T^{-1}(\mathcal{X})} \varphi d\mu = \int_{\mathcal{X}} \varphi d\mu,$$

and  $T^{-1}(\mathcal{X}) = \mathcal{X}$ , the result follows.

□

(P4) **Contraction Property:**  $P_\mu : \mathbb{L}^1 \rightarrow \mathbb{L}^1$  is a contraction, i.e.,  $\|P_\mu\varphi\|_1 \leq \|\varphi\|_1$  for any  $\varphi \in \mathbb{L}^1$ .

**Proof.** Consider  $\varphi \in \mathbb{L}^1$ . Let  $\varphi^+ = \max(\varphi, 0)$  and  $\varphi^- = -\min(0, \varphi)$ . Then  $\varphi^+, \varphi^- \in \mathbb{L}^1$ ,  $\varphi = \varphi^+ - \varphi^-$  and  $|\varphi| = \varphi^+ + \varphi^-$ . By the linearity of  $P_\mu\varphi$ , (P1), we have

$$P_\mu\varphi = P_\mu(\varphi^+ - \varphi^-) = P_\mu\varphi^+ - P_\mu\varphi^-.$$

Hence,

$$|P_\mu\varphi| \leq |P_\mu\varphi^+| + |P_\mu\varphi^-| = P_\mu\varphi^+ + P_\mu\varphi^- = P_\mu|\varphi|,$$

where we have used (P2), and

$$\|P_\mu \varphi\|_1 = \int_{\mathcal{X}} |P_\mu \varphi| d\mu \leq \int_{\mathcal{X}} P_\mu |\varphi| d\mu = \int_{\mathcal{X}} |\varphi| d\mu = \|\varphi\|_1,$$

where we have applied (P3).  $\square$

(P5) **Composition Property:** Let  $T : \mathcal{X} \rightarrow \mathcal{X}$  and  $S : \mathcal{X} \rightarrow \mathcal{X}$  be nonsingular with respect to  $\mu$ . Then  $P_{T \circ S, \mu} \varphi = P_{T, \mu} \circ P_{S, \mu} \varphi$ . In particular,  $P_{T^n, \mu} \varphi = P_{T, \mu}^n \varphi$ .

**Proof.** Let  $\varphi \in \mathbb{L}^1$  and define the measure  $\nu$  by

$$\nu(A) = \int_{(T \circ S)^{-1}(A)} \varphi d\mu.$$

Since  $T$  and  $S$  are nonsingular,  $\nu \ll \mu$  and there exists a function  $P_{T \circ S} \varphi$  such that

$$\nu(A) = \int_A P_{T \circ S, \mu} \varphi d\mu.$$

Now

$$\int_A P_{T \circ S, \mu} \varphi d\mu = \int_{(T \circ S)^{-1}(A)} \varphi d\mu = \int_{S^{-1}(T^{-1}A)} \varphi d\mu$$

and

$$\int_A P_{T, \mu} (P_{S, \mu} \varphi) d\mu = \int_{T^{-1}A} P_{S, \mu} \varphi d\mu = \int_{S^{-1}(T^{-1}A)} \varphi d\mu.$$

Hence  $P_{T \circ S, \mu} \varphi = P_{T, \mu} P_{S, \mu} \varphi$  a.e. By induction, it follows that  $P_{T^n, \mu} \varphi = P_{T, \mu}^n \varphi$  a.e.  $\square$

(P6) **Adjoint Property:** If  $\varphi \in \mathbb{L}^1$  and  $\psi \in \mathbb{L}^\infty$ , then

$$\int_{\mathcal{X}} (P_\mu \varphi) \cdot \psi d\mu = \int_{\mathcal{X}} \varphi \cdot (\psi \circ T) d\mu.$$

**Proof.** Let  $A$  be a measurable subset of  $\mathcal{X}$  and let  $\psi = \mathbb{1}_A$ . Then the left hand side is

$$\int_{\mathcal{X}} (P_\mu \varphi) \cdot \mathbb{1}_A d\mu = \int_A P_\mu \varphi d\mu = \int_{T^{-1}A} \varphi d\mu,$$

and the right hand side is

$$\int_{\mathcal{X}} \varphi \cdot (\mathbb{1}_A \circ T) d\mu = \int_{\mathcal{X}} \varphi \cdot \mathbb{1}_{T^{-1}A} d\mu = \int_{T^{-1}A} \varphi d\mu.$$

Hence, this propriety is verified for indicator functions. Since the linear combinations of indicator functions are dense in  $\mathbb{L}^\infty$ , then this property holds for all  $\varphi \in \mathbb{L}^1$  and  $\psi \in \mathbb{L}^\infty$ .  $\square$

Moreover, if  $\mu$  is  $T$ -invariant, then for all  $\varphi \in \mathbb{L}^1(\mu)$  we have:

$$(P7) \quad \int_A P_\mu(\varphi \circ T) d\mu = \int_A \varphi d\mu$$

**Proof.**

$$\int_A P_\mu(\varphi \circ T) d\mu = \int_{T^{-1}(A)} \varphi \circ T d\mu \quad (i)$$

$$= \int_A \varphi d(T_*\mu) \quad (ii)$$

$$= \int_A \varphi d\mu \quad (iii)$$

To go from (ii) to (iii) we used the fact that  $\mu$  is  $T$ -invariant.  $\square$

(P8)  $(P_\mu^n \varphi) \circ T^n = \mathbb{E}(\varphi | T^{-n}(\mathcal{B}))$  for all  $n \geq 1$  a.e.

**Proof.** First consider  $n = 1$ , since  $(P_\mu \varphi) \circ T$  is obviously  $T^{-1}\mathcal{B}$ -measurable, we can apply Theorem 5.3.5. Let  $A = T^{-1}(B)$ ,  $B \in \mathcal{B}$ . Then

$$\begin{aligned} \int_A (P_\mu \varphi) \circ T d\mu &= \int_{T^{-1}B} (P_\mu \varphi) \circ T d\mu \\ &= \int_B P_\mu \varphi d(T_*\mu) \\ &= \int_B P_\mu \varphi d\mu \\ &= \int_{T^{-1}B} \varphi d\mu = \int_A \varphi d\mu. \end{aligned}$$

By induction, it follows that  $\int_A (P_\mu \varphi) \circ T^n d\mu = \int_A \varphi d\mu$ , with  $A = T^{-n}(B)$ .

Therefore,  $(P_\mu^n \varphi) \circ T^n = \mathbb{E}(\varphi | T^{-n}(\mathcal{B}))$ , for all  $n \geq 1$ , a.e.  $\square$

(P9)  $P_\mu$  is a contraction on any space  $\mathbb{L}(\mu)^p$ ,  $1 \leq p \leq +\infty$ , i.e.,  $\|P_\mu \varphi\|_p \leq \|\varphi\|_p$  whenever  $\varphi \in \mathbb{L}^p(\mu)$  for some  $1 \leq p \leq \infty$ .

**Proof.** Let  $1 \leq p < +\infty$ . Then

$$\begin{aligned} (\|P_\mu \varphi\|_p)^p &= \int |P_\mu \varphi|^p d\mu \\ &= \int |(P_\mu \varphi) \circ T|^p d\mu \\ &= \int |\mathbb{E}(\varphi | T^{-1}(\mathcal{B}))|^p d\mu \\ &\leq \int \mathbb{E}(|\varphi|^p | T^{-1}(\mathcal{B})) d\mu \\ &= \int |\varphi|^p d\mu = (\|\varphi\|_p)^p \end{aligned}$$

thus  $\|P_\mu \varphi\|_p \leq \|\varphi\|_p$ . Let  $p = +\infty$ . Then

$$\begin{aligned} \|P_\mu \varphi\|_\infty &= \text{esssup} |P_\mu \varphi| \\ &= \text{esssup} |(P_\mu \varphi) \circ T| \\ &= \text{esssup} |\mathbb{E}(\varphi | T^{-1}(\mathcal{B}))| \\ &\leq \text{esssup} |\varphi|. \end{aligned}$$

$\square$

## 5.4 Application examples

In this work we will mainly study dynamical systems  $(\mathcal{X}, \mathcal{B}, \mu, T)$ , where  $\mathcal{X} = [0, 1]$ ,  $\mathcal{B}$  is the Borel  $\sigma$ -algebra of  $\mathcal{X}$ ,  $\mu$  is the Lebesgue measure, and the measurable function  $T(x) = a \cdot x \mod 1$  with  $a \in \mathbb{N}_{\geq 1}$ . This type of systems have summable decay of correlation, which we will prove directly in this section.

### Proposition 5.4.1

Consider the system given by  $T(x) = 2x \mod 1$ . Consider the Banach space of the continuous, real-valued function on  $\mathcal{X}$  that are Lipschitz, i.e.,  $\varphi \in \text{Lip} \iff$

$|\varphi(x) - \varphi(y)| \leq K|x - y|$ ,  $x, y \in \mathcal{X}$ , where  $K$  is a constant, equipped with the norm  $\|\varphi\|_{\text{Lip}} = \|\varphi\|_{\infty} + \text{Lip}(\varphi)$ , where  $\text{Lip}(\varphi) = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$ . Then this system has summable decay of correlation in Lip against  $\mathbb{L}^1(\mu)$ .

**Proof.** To prove the decay of correlation, first we will compute the Frobenius-Perron operator,  $P_{\mu}\varphi = P_{T,\mu}\varphi$ :

$$\begin{aligned}
 \int_{[a,b]} P_{\mu}\varphi(x) d\mu(x) &= \int_{T^{-1}([a,b])} \varphi(x) d\mu(x) \\
 &= \int_{[\frac{a}{2}, \frac{b}{2}] \cup [\frac{a+1}{2}, \frac{b+1}{2}]} \varphi(x) d\mu(x) \\
 &= \int_{\frac{a}{2}}^{\frac{b}{2}} \varphi(x) d\mu(x) + \int_{\frac{a+1}{2}}^{\frac{b+1}{2}} \varphi(x) d\mu(x) \\
 &= \int_a^b \varphi\left(\frac{x}{2}\right) \cdot \frac{1}{2} d\mu(x) + \int_a^b \varphi\left(\frac{x+1}{2}\right) \cdot \frac{1}{2} d\mu(x) \\
 &= \int_a^b \frac{1}{2} \cdot \varphi\left(\frac{x}{2}\right) + \frac{1}{2} \cdot \varphi\left(\frac{x+1}{2}\right) d\mu(x) \\
 &= \int_a^b \frac{1}{2} \cdot \left( \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) \right) d\mu(x),
 \end{aligned}$$

for  $a, b \in \mathcal{X}$ ,  $a < b$ . Since this is true for every interval inside  $\mathcal{X}$ , which are the basis for the open sets, then  $P_{\mu}\varphi = \frac{1}{2} \left( \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) \right)$ .

Now, take  $\varphi \in \text{Lip}$  and assume with no loss of generality that  $\int \varphi d\mu = 0$ . Hence, we have that

$$\left\| P_{\mu}^n \varphi - \int \varphi d\mu \right\|_{\infty} = \left\| P_{\mu}^n \left( \varphi - \int \varphi d\mu \right) \right\|_{\infty} \quad (\text{i})$$

$$\leq \left\| \varphi - \int \varphi d\mu \right\|_{\infty} = \|\varphi\|_{\infty} \quad (\text{ii})$$

We use the linearity of the Frobenius-Perron operator, (P1), in (i), and to pass to

(ii) we use:

$$\begin{aligned}
 \|P_m^n \varphi\|_\infty &= \|P_\mu(P_\mu^{n-1} \varphi)\|_\infty && \text{(applying (P9))} \\
 &\leq \|P_m^{n-1} \varphi\|_\infty \\
 &\dots \\
 &\leq \|P_\mu \varphi\|_\infty \\
 &\leq \|\varphi\|_\infty.
 \end{aligned}$$

Furthermore, we get:

$$\begin{aligned}
 Lip(P_\mu \varphi) &= Lip\left(\frac{1}{2} \cdot \left(\varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right)\right)\right) \\
 &= \sup_{x \neq y} \frac{\left|\frac{1}{2} \cdot \left(\varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right)\right) - \frac{1}{2} \cdot \left(\varphi\left(\frac{y}{2}\right) + \varphi\left(\frac{y+1}{2}\right)\right)\right|}{|x - y|} \\
 &= \sup_{x \neq y} \frac{1}{2} \cdot \frac{\left|\varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{y}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{y+1}{2}\right)\right|}{|x - y|} \\
 &\leq \sup_{x \neq y} \frac{1}{2} \cdot \frac{\left|\varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{y}{2}\right)\right| + \left|\varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{y+1}{2}\right)\right|}{|x - y|} \\
 &\leq \frac{1}{2} \left( \sup_{x \neq y} \frac{\left|\varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{y}{2}\right)\right|}{|x - y|} + \sup_{x \neq y} \frac{\left|\varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{y+1}{2}\right)\right|}{|x - y|} \right)
 \end{aligned}$$

by changing variables,  $z = \frac{x}{2}$ ,  $w = \frac{y}{2}$  and  $z' = \frac{x+1}{2}$ ,  $w' = \frac{y+1}{2}$ , we get

$$\begin{aligned}
 &= \frac{1}{2} \left( \sup_{z \neq w} \frac{|\varphi(z) - \varphi(w)|}{|2z - 2w|} + \sup_{z' \neq w'} \frac{|\varphi(z') - \varphi(w')|}{|2z' - 1 - (2w' - 1)|} \right) \\
 &= \frac{1}{2} \left( \frac{1}{2} \cdot \sup_{z \neq w} \frac{|\varphi(z) - \varphi(w)|}{|z - w|} + \frac{1}{2} \cdot \sup_{z' \neq w'} \frac{|\varphi(z') - \varphi(w')|}{|z' - w'|} \right) \\
 &= \frac{1}{2} \left( \frac{1}{2} Lip(\varphi) + \frac{1}{2} Lip(\varphi) \right) \\
 &= \frac{1}{2} Lip(\varphi),
 \end{aligned}$$

and

$$Lip(P_\mu^2 \varphi) = Lip(P_\mu(P_\mu \varphi)) \leq \frac{1}{2} Lip(P_\mu \varphi) \leq \frac{1}{2^2} Lip(\varphi).$$

Thus, by induction, for  $n \in \mathbb{N}_0$ :

$$\text{Lip}(P_\mu^n \varphi) \leq \frac{1}{2^n} \text{Lip}(\varphi)$$

Hence, for all  $\varphi \in \text{Lip}$

$$\left\| P_\mu^n \varphi - \int_{\mathcal{X}} \varphi \, d\mu \right\|_\infty = \left\| P_\mu^n \varphi - \int_{\mathcal{X}} P_\mu^n \varphi \, d\mu \right\|_\infty \quad \text{by (P3)}$$

Using the Mean Value Theorem:  $\exists x^* \in \mathcal{X} : \int_{\mathcal{X}} P_\mu^n \varphi \, d\mu = P_\mu^n \varphi(x^*) \mu(\mathcal{X})$ , and  $\mu(\mathcal{X}) = 1$ ,

$$\begin{aligned} &= \left\| P_\mu^n \varphi - P_\mu^n \varphi(x^*) \right\|_\infty \\ &= \sup_{x \in [0,1]} |P_\mu^n \varphi - P_\mu^n \varphi(x^*)| \\ &\leq \text{Lip}(P_\mu^n \varphi) \cdot |x - x^*| \\ &\leq \text{Lip}(P_\mu^n \varphi) \quad \text{because } |x - x^*| \leq 1 \\ &\leq 2^{-n} \text{Lip}(\varphi) \leq 2^{-n} \|\varphi\|_{\text{Lip}}. \end{aligned}$$

Now we have the inequality needed to prove the decay of correlation. Let  $\phi \in \text{Lip}$ ,  $\psi \in \mathbb{L}^1(\mu)$ :

$$\text{Cor}(\phi, \psi, n) = \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \left| \int \phi(\psi \circ T^n) \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right|$$

using the adjoint property, (P6)

$$\begin{aligned} &= \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \left| \int \psi \cdot P_\mu^n \phi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right| \\ &= \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \left| \int \psi \cdot P_\mu^n \phi \, d\mu - \int \left( \int \phi \, d\mu \right) \psi \, d\mu \right| \\ &= \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \left| \int \psi \cdot P_\mu^n \phi - \left( \int \phi \, d\mu \right) \psi \, d\mu \right| \\ &= \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \left| \int \psi \cdot \left( P_\mu^n \phi - \int \phi \, d\mu \right) \, d\mu \right| \\ &\leq \frac{1}{\|\phi\|_{\text{Lip}} \|\psi\|_1} \cdot \int \left| \psi \cdot \left( P_\mu^n \phi - \int \phi \, d\mu \right) \right| \, d\mu \end{aligned}$$

by Hölder's Inequality,

$$\begin{aligned} &\leq \frac{1}{\|\phi\|_{\text{Lip}}\|\psi\|_1} \cdot \|\psi\|_1 \cdot \left\| P_\mu^n \phi - \int \phi d\mu \right\|_\infty \\ &\leq \frac{1}{\|\phi\|_{\text{Lip}}} \cdot 2^{-n} \|\phi\|_{\text{Lip}} \\ &= 2^{-n} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

Although we prove just for  $T(x) = a \cdot x \bmod 1$  with  $a = 2$ , we can adapt this proof for  $a \in \mathbb{N}_{\geq 2}$ . Moreover, in [5, Ch. 4.3] it is given an explicit expression for the Frobenius-Perron operator for this case:

$$P_{T,\mu}\varphi(x) = \sum_{z \in \{T^{-1}(x)\}} \frac{\varphi(z)}{|T'(z)|}.$$

These are very specific examples, we now will see more general examples, which also have summable decay of correlation against  $\mathbb{L}^1$ , provided in article [2].

#### Theorem 5.4.2

Let  $\mathcal{X}$  be a measurable space,  $m$  a reference probability measure on a  $\sigma$ -algebra  $\mathcal{B}$ , and let  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a measurable map. Consider the usual Perron-Frobenius operator  $P_m : \mathbb{L}^1(m) \rightarrow \mathbb{L}^1(m)$ , defined in 5.3.1. Assume that there is a seminorm  $|\cdot|_{\mathcal{Y}}$  on  $\mathbb{L}^1(m)$  such that:

(1)  $\mathcal{Y} = \{\varphi \in \mathbb{L}^1(m) : |\varphi|_{\mathcal{Y}} < \infty\}$  is a Banach space with the norm

$$\|\cdot\|_{\mathcal{Y}} = |\cdot|_{\mathcal{Y}} + \|\cdot\|_{\mathbb{L}^1(m)};$$

(2)  $\mathcal{Y}$  is adapted to  $\mathbb{L}^1(m)$ : the inclusion  $\mathcal{Y} \hookrightarrow \mathbb{L}^1(m)$  is compact;

(3)  $P_m(\mathcal{Y}) \subset \mathcal{Y}$  and  $P_m|_{\mathcal{Y}}$  is bounded with respect to the norm  $\|\cdot\|_{\mathcal{Y}}$ ;

- (4) Lasota–Yorke inequality holds: there are  $n_0 \geq 1$ ,  $0 < \alpha < 1$  and  $\beta > 0$  such that

$$|P_m^{n_0} \varphi|_{\mathcal{Y}} \leq \alpha |\varphi|_{\mathcal{Y}} + \beta \|\varphi\|_{L^1(m)}, \quad \forall \varphi \in \mathcal{B};$$

- (5)  $\mathcal{Y}$  is a Banach algebra with the norm  $\|\cdot\|_{\mathcal{Y}}$ ; in particular, there is  $C > 0$  such that

$$\|\varphi\psi\|_{\mathcal{Y}} \leq C \|\varphi\|_{\mathcal{Y}} \|\psi\|_{\mathcal{Y}}, \quad \forall \varphi, \psi \in \mathcal{Y};$$

- (6)  $\mathcal{Y}$  is continuously injected in  $L^\infty(m)$ : there exists a constant  $C' > 0$  such that

$$\|\varphi\|_{L^\infty(m)} \leq C' \|\varphi\|_{\mathcal{Y}}, \quad \forall \varphi \in \mathcal{Y}.$$

Then  $T$  admits an absolutely continuous invariant probability measure (acip)  $\mu$ .

Assuming moreover that  $d\mu/dm$  is uniformly bounded away from zero, then  $T$  exhibits exponential (and thus in particular, summable) decay of correlations for observables in  $\mathcal{Y}$  against  $\mathbb{L}^1(\mu)$ .

**Proof.** Under conditions (1)-(4) we can apply the Ionescu-Tulcea–Marinescu Theorem [15] (in [7] it was given a simple proof of the ITM Theorem with estimate of the essential spectral radius) which asserts that the operator  $P_m$  is quasi-compact and this implies the existence of an invariant probability measure  $\mu$  for the map  $T$  which is absolutely continuous with respect to  $m$  on  $\mathcal{X}$  and with density  $h \in \mathcal{Y}$ . Furthermore, suppose that  $\mathcal{X}$  is the only mixing component for  $T$ .

The iterates of the Frobenius-Perron operator have the following spectral decomposition:

$$P_m^n = \Pi + Q^n, \tag{23}$$

where  $\Pi$  projects  $\varphi \in \mathcal{Y}$  into the fixed points of  $P_m$ ,

$$\Pi(\varphi) = h \int \varphi dm, \tag{24}$$

and the linear operator  $Q$  verifies

$$\|Q^n(\varphi)\|_{\mathcal{Y}} \leq C'' q^n \|\varphi\|_{\mathcal{Y}}, \quad (25)$$

where  $C'' > 0$  and  $0 < q < 1$  are constants depending on  $T$ .

Let  $\varphi \in \mathcal{Y}$  and consider with no loss of generality that  $\int \varphi d\mu = 0$ , which also can be written as  $\int \varphi h dm = 0$ , where  $h = d\mu/dm$ . Thus, for any  $\psi \in \mathbb{L}^1(m)$  we obtain:

$$\left| \int \varphi(\psi \circ T^n) d\mu \right| = \left| \int \varphi h(\psi \circ T^n) dm \right| \quad (i)$$

$$= \left| \int \psi P_m^n(\varphi h) dm \right| \quad (ii)$$

$$= \left| \int \psi (\Pi(\varphi h) + Q^n(\varphi h)) dm \right| \quad (iii)$$

$$= \left| \int \psi Q^n(\varphi h) dm \right| \quad (iv)$$

$$\leq \int |\psi Q^n(\varphi h)| dm \quad (v)$$

by Hölder's Inequality,

$$\leq \|\psi\|_{\mathbb{L}^1(m)} \|Q^n(\varphi h)\|_{\mathbb{L}^\infty(m)} \quad (vi)$$

$$\leq C' \|\psi\|_{\mathbb{L}^1(m)} \|Q^n(\varphi h)\|_{\mathcal{Y}} \quad (vii)$$

$$\leq C' C'' q^n \|\psi\|_{\mathbb{L}^1(m)} \|h\varphi\|_{\mathcal{Y}} \quad (viii)$$

$$\leq C C' C'' q^n \|\psi\|_{\mathbb{L}^1(m)} \|h\|_{\mathcal{Y}} \|\varphi\|_{\mathcal{Y}}. \quad (ix)$$

Where in (ii) step we use the fact that  $h \in \mathcal{Y} \Rightarrow \varphi h \in \mathcal{Y}$ , by property (5), and the adjoint property, (P6). The equality (iii), is obtained directly by (23), and (iv), is achieved by using (24) and  $\int \varphi h dm = 0$ . The inequality (vii) is due to the property (6) and (viii) from (25). Finally, in the last set we use (5).

Recalling that by assumption there is  $c > 0$  such that  $h \geq c$ , it then follows that

$$\|\psi\|_{\mathbb{L}^1(m)} = \int |\psi| dm = \int \frac{|\psi|}{h} d\mu \leq \int \frac{|\psi|}{c} d\mu = \frac{1}{c} \|\psi\|_{\mathbb{L}^1(\mu)}.$$

Therefore, we have

$$\begin{aligned}\text{Cor}_\mu(\varphi, \psi, n) &= \frac{1}{\|\varphi\|_{\mathcal{Y}} \|\psi\|_{\mathbb{L}^1(\mu)}} \left| \int \varphi(\psi \circ T^n) d\mu - \int \varphi d\mu \int \psi d\mu \right| \\ &= \frac{1}{\|\varphi\|_{\mathcal{Y}} \|\psi\|_{\mathbb{L}^1(\mu)}} \left| \int \varphi(\psi \circ T^n) d\mu \right| \quad \left( \text{because } \int \varphi d\mu = 0 \right) \\ &\leq \frac{1}{c} C C' C'' \|h\|_{\mathcal{Y}} q^n.\end{aligned}$$

Finally, observe that  $L^p(\mu) \subset L^p(m)$  for any  $1 \leq p \leq \infty$ , as long as  $d\mu/dm$  is uniformly bounded away from zero.  $\square$

Now we will see some explicit examples of dynamical systems that we may use this theorem.

#### 5.4.3. Uniformly expanding maps on the unit interval (See [5]).

For the first example let  $\mathcal{X} = [0, 1]$  and  $T$  be an  $C^1$  piecewise uniformly expanding, i.e., there exist a partition  $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, \dots, k\}$  of  $\mathcal{X}$  such that  $T|_{I_i}$  is  $C^1$ , and  $|T'_i(x)| \geq \alpha > 1$  for any  $i$  and for all  $x \in (a_{i-1}, a_i)$ ; and  $|T'(x)|^{-1}$  is a function of bounded variation, where  $T'(x)$  is the appropriate one-sided derivative at the endpoints of  $\mathcal{P}$  (see [5, Chapter 5.2] for a more detail). Furthermore, verifying the Adler property

$$\sup_{A \in \mathcal{P}} \sup_{x \in A} \frac{|T''(x)|}{(T'(x))^2} < \infty.$$

Consider  $\mathcal{Y}$  as the space of functions  $\varphi$  on the interval with bounded total variation  $V_0^1(\varphi)$ , that is, the space  $BV([0, 1])$ . The corresponding Banach norm will be given by

$$\|\varphi\|_{BV} = \inf_{\phi = \varphi \text{ a.e.}} V_0^1 \phi + \|\varphi\|_{\mathbb{L}^1(m)}.$$

this norm is adapted to  $\mathbb{L}^1(m)$ . Moreover with this construction of a Banach space the Lasota–Yorke inequality holds and the space is an algebra.

For the second part of the theorem, when it is a Markovian case, i.e., the images under  $T$  of the elements in  $\mathcal{P}$  coincide with the whole space  $[0, 1]$ , the density  $d\mu/dm$  is bounded from below by a strictly positive constant. Otherwise, in non-Markovian cases, generally, the density is positive whenever the support of the density is the whole interval. By support, we mean,  $\text{supp } f = \{x : f(x) \neq 0\}$ . Figure 1 shows two examples

of uniformly expanding maps, 1.a is Markovian and 1.b is non-Markovian.

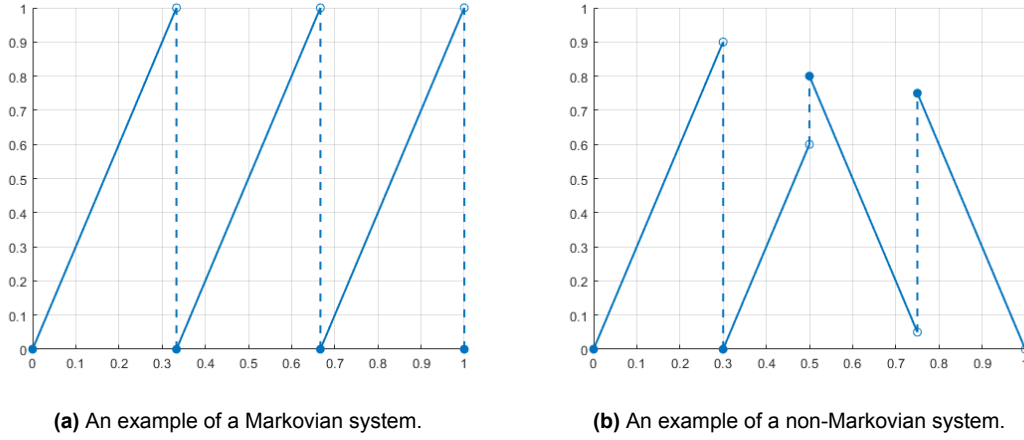


**Figure 1:** Examples of uniformly expanding maps on the unit interval

#### 5.4.4. Markov maps (see [5]).

Let  $\mathcal{X} = [a, b]$  and  $\mathcal{P}$  be a measurable partition of  $\mathcal{X}$ . Let  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a measurable map such that  $T|_A$ , for all  $A \in \mathcal{P}$ , is a homeomorphism from  $A$  onto some connected union of intervals of  $\mathcal{P}$ . Consider that  $\mathcal{X}$  is endowed with a probability measure  $m$  on a  $\sigma$ -algebra  $\mathcal{B}$ , where  $\mathcal{B}$  is generated by  $\mathcal{P}$  under  $T$ , that is,  $\mathcal{B} = \sigma(\bigvee_{n=0}^{\infty} T^{-n}(\mathcal{P}))$ <sup>3</sup>. We also assume that  $T|_A$  is nonsingular for all  $A \in \mathcal{P}$ .

Consider the inverse branches  $g_{A,n} : T^n(A) \rightarrow A$  for all  $A \in \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P})$  and the Radon–Nykodym derivatives  $\rho_{A,n} = dm \circ g_{A,n} / dm$ . Moreover, suppose the following properties:

- **Mixing:**  $\forall A, B \in \mathcal{P}, \exists n_0 \geq 0 : T^n(A) \supset B, \forall n \geq n_0$ ;
- **Big images:**  $\inf_{A \in \mathcal{P}} m(T(A)) > 0$ ;
- **Bounded distortion:**  $\exists C > 0, \forall n \geq 1, \forall A \in \bigvee_{j=0}^{n-1} T^{-j}(\mathcal{P}), \forall x, y \in T^n(A) :$

$$\left| \frac{\rho_{A,n}(x)}{\rho_{A,n}(y)} - 1 \right| \leq C \lambda^{s(x,y)},$$

where  $\lambda$  is some real number in  $(0, 1)$  and  $s(x, y)$  is given by the maximum interger such that  $T^i(x)$  and  $T^i(y)$  belong to the same element of the partition  $\mathcal{P}$ , which is

<sup>3</sup> $\sigma(\mathcal{B})$  stands for the  $\sigma$ -algebra generated by  $\mathcal{B}$ .

called the separation time.

Now, take  $\mathcal{Y}$  as the functional space of piecewise Lipschitz functions,  $\mathcal{L}$ , defined in the following way:  $\varphi : \mathcal{X} \rightarrow \mathbb{R}$  is Lipschitz on the set  $A \subset \mathcal{X}$  if the following seminorm is finite:

$$D_A(\varphi) := \sup_{x,y \in A} \frac{|\varphi(x) - \varphi(y)|}{\lambda^{s(x,y)}} < \infty.$$

Let  $\hat{\mathcal{P}}$  be the partition such that  $\sigma(T(\mathcal{P})) = \sigma(\hat{\mathcal{P}})$ , we define  $D_{\hat{\mathcal{P}}}\varphi = \sup_{A \in \hat{\mathcal{P}}} D_A\varphi$ . Lastly, consider

$$\mathcal{L} = \{\varphi \in \mathbb{L}^\infty(m) : D_{\hat{\mathcal{P}}}\varphi < \infty\},$$

equipped with the norm

$$\|\varphi\|_{\mathcal{L}} := \|\varphi\|_{\mathbb{L}^\infty(m)} + D_{\hat{\mathcal{P}}}\varphi.$$

This norm is adapted to  $\mathbb{L}^1(m)$ .

With  $\mathcal{Y} = \mathcal{L}$  the Perron–Frobenius operator verifies the Lasota–Yorke inequality and the density of the invariant measure will be  $m$ –almost everywhere bounded away from zero.

#### 5.4.5. Higher dimensional piecewise expanding maps studied by Saussol in [14].

In this example we will see the multidimensional piecewise uniformly expanding maps, using the definition proposed by Saussol [14].

Let  $\mathcal{X}$  be a compact subset of  $\mathbb{R}^N$  with  $\overline{\text{int}(\mathcal{X})} = \mathcal{X}$  and  $T : \mathcal{X} \rightarrow \mathcal{X}$ . Assume that there exist at most countably many disjoint open sets  $U_i$  such that  $m(\mathcal{X} \setminus \bigcup_{i=1}^\infty U_i) = 0$ , where  $m$  denotes Lebesgue measure in the Borel sets of  $\mathbb{R}^N$ . Furthermore, consider that there are open sets  $\tilde{U}_i \supset \bar{U}_i$  and  $C^{1+\alpha}$  maps  $T_i : \tilde{U}_i \rightarrow \mathbb{R}^N$  such that for some constants  $c, \epsilon_1 > 0$  and  $0 < \alpha < 1$ :

- (1)  $T_i|_{U_i} = T|_{U_i}$  and  $T_i(\tilde{U}_i) \supset B_{\epsilon_1}(T(U_i))$  for each  $i$ ;
- (2) for all  $i$  and  $x, y \in T(U_i)$  with  $d(x, y) \leq \epsilon_1$  (where  $d$  is the Euclidean distance in  $\mathbb{R}^N$ ),

$$|\det DT_i^{-1}(x) - \det DT_i^{-1}(y)| \leq c |\det DT_i^{-1}(x)| d(x, y)^\alpha;$$

(3) there exists  $s = s(T) < 1$  such that

$$\sup_i \sup_{x \in T_i(\tilde{U}_i)} \|DT_i^{-1}(x)\| < s;$$

(4) each  $\partial U_i$  is a codimension one embedded compact  $C^1$  submanifold and

$$s^\alpha + \frac{4s}{1-s} Y(T) \frac{\gamma_{N-1}}{\gamma_N} < 1, \quad (26)$$

where  $Y(T) = \sup_x \sum_i \# \{\text{smooth pieces intersecting } \partial U_i \text{ containing } x\}$  and  $\gamma_N$  is the volume of the unit ball in  $\mathbb{R}^N$ .

With these assumptions the Perron-Frobenius operator is well defined in  $\mathbb{L}^1(m)$ , and is given by

$$P_m \varphi = \sum_i (|\det DT|^{-1} \varphi) \circ T_i^{-1} \mathbb{1}_{T(U_i)}.$$

Moreover, given  $\epsilon_0 > 0$  and  $0 < \alpha < 1$ , consider the  $\alpha$ -seminorm of  $\varphi \in \mathbb{L}^1(m)$ :

$$|\varphi|_\alpha = \sup_{0 < \epsilon \leq \epsilon_0} \epsilon^{-\alpha} \int_{\mathbb{R}^N} \text{osc}(\varphi, B_\epsilon(x)) dm(x),$$

where the oscillation of  $\varphi$  over  $\Omega$ , a given Borel set of  $\mathcal{X}$ , is defined as

$$\text{osc}(\varphi, \Omega) := \text{esssup} \varphi|_\Omega - \text{essinf} \varphi|_\Omega.$$

Therefore, let the set  $\mathcal{Y}$  be the Banach space  $V_\alpha$ , defined by

$$V_\alpha = \{\varphi \in \mathbb{L}^1(m) : |\varphi|_\alpha < \infty\},$$

endowed with the norm

$$\|\cdot\|_\alpha = \|\cdot\|_{\mathbb{L}^1(m)} + |\cdot|_\alpha.$$

It is important to remark that this space does not depend on the choice of  $\epsilon_0$  and according to [9, Theorem 1.13], the unit ball in  $V_\alpha$  is compact in  $\mathbb{L}^1(m)$ . Furthermore, the assumptions (1)–(4) allow us to verify the Lasota–Yorke inequality when the Frobenius-Perron operator is applied to functions belonging to the space  $V_\alpha$ .

We just saw the case where the expansion is uniform, for the other situation we

recommend the paper [8], that studies this case.

## 6 Computing the value of the Extremal Index

In this section we consider the dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, T)$  described in the first section with decay of correlation against  $\mathbb{L}^1$  observables. An observable  $\varphi$  with  $N$  maxima,  $\xi_i$ ,  $i \in \{1, \dots, N\}$ , which belong to the same orbit, i.e., for  $i \in \{1, \dots, N\}$ ,  $m_i \in \mathbb{N}_0$  we have

$$m_1 = 0 \text{ and } m_i < m_j, \forall i < j, \text{ such that } \xi_i = T^{m_i}(\xi_1). \quad (27)$$

Consider the stochastic process  $X_0, X_1, \dots$  as in (1). Furthermore, we will calculate the value of EI for different cases using (14), therefore it is needed to calculate  $\mu(\mathcal{A}_{q,n})$  and  $\mu(U_n)$ .

For this purpose, we will introduce a new notation:  $U_n(\xi_i)$  is the ball from  $U_n$  centered in  $\xi_i$  and  $\mathcal{A}_{q,n}(\xi_i)$  is the set of points which start in  $U_n(\xi_i)$  and in the following  $q$  iterations do not come back to  $U_n$ , in short,  $U_n = \bigcup_{i=1}^N U_n(\xi_i)$  and  $\mathcal{A}_{q,n} = \bigcup_{i=1}^N \mathcal{A}_{q,n}(\xi_i)$ .

Moreover, since the preimages of the balls  $U_n(\xi_i)$  by  $T$  may have different shapes and sizes, meaning that, even if  $T^{m_j-m_i}(U_n(\xi_j)) \cap U_n(\xi_i) \neq \emptyset$ , it is not clear if one set contains the other or not. Hence we assume that for  $1 \leq i < j \leq N$ ,

**(R3)**

$$U_n^{(0)}(\xi_i) \subseteq T^{-(m_j-m_i)}(U_n^{(0)}(\xi_j)) \text{ or } T^{-(m_j-m_i)}(U_n^{(0)}(\xi_j)) \cap \partial U_n^{(0)}(\xi_i)^4 = \emptyset.$$

So this condition means that, or  $U_n^{(0)}(\xi_i)$  is contained in  $T^{-(m_j-m_i)}(U_n^{(0)}(\xi_j))$  or the part of  $T^{-(m_j-m_i)}(U_n^{(0)}(\xi_j))$  which intersects  $U_n^{(0)}(\xi_i)$  is in its interior.

### 6.1 Multiple maxima in a periodic orbit

First case will be the maxima  $\xi_1, \dots, \xi_N$  belonging to the same periodic orbit with prime period  $p$ , i.e., there exists  $m_i \in \mathbb{N}_0$  : such as in (27), with

$$0 = m_1 < m_2 < \dots < m_N \leq p-1, \quad (28)$$

and we assume that (R3) is satisfied.

<sup>4</sup> $\partial A$ ,  $A \in \mathcal{B}$ , is the notation for the border of  $A$ .

We begin by choosing the value of  $q$  for which the condition  $\mathcal{D}'_q(u_n)$  can be checked. From the result computations in the verification of condition  $\mathcal{D}'_q(u_n)$  in section 5.2, we need to verify that  $R(\mathcal{A}_{q,n}) \rightarrow \infty$ , as  $n \rightarrow \infty$  for the chosen  $q$ . Since is a periodic orbit a acceptable option would be  $q = p$ .

**Lemma 6.1.1**

Assume that  $\xi_1$  is a repelling periodic point of prime period  $p$ . Let  $\xi_1, \dots, \xi_N$  be as in (28). Let  $q = p$ , then  $\lim_{n \rightarrow \infty} R(\mathcal{A}_{q,n}) = +\infty$ .

**Proof.** Let  $f := T^p$ , then every  $\xi_i$  is a fixed point of  $f$ .

Since  $\xi_1$  is a repelling periodic point, there exist a neighbourhood of  $\xi_i$ ,  $W(\xi_i)$  such that  $f|_{W(\xi_i)} : W(\xi_i) \rightarrow f(W(\xi_i))$  is a diffeomorphism.

The Hartman-Grobman theorem asserts that there exist a neighbourhood  $\widetilde{W}(0)$  of  $0 \in T_{\xi_1}\mathcal{X}$  and a homeomorphism  $H : \widetilde{W}(0) \rightarrow W(\xi_1)$  such that

$$H \circ Df(\xi_1) = f \circ H.$$

For the reason of  $\widetilde{W}(0)$  being an open set, there exists  $r > 0$  so that  $B_r(0) \subset \widetilde{W}(0)$ , where  $B_r(0)$  is the ball of radius  $r$ . Let  $x \in B_r(\xi_i)$  and  $\epsilon = \text{dist}(x, \xi_i)$ . By hypothesis, the point  $\xi_i$  is repelling periodic and it implies that there exists  $\lambda > 1$  such that the diameter of  $(Df(\xi_1))^k(B_\epsilon(\xi_i))$  is at least  $\lambda^k \epsilon$ .

Thus, to obtain  $\lambda^k > r$ , we must have that  $k > \frac{\log(r) - \log(\epsilon)}{\log(\lambda)}$ . Therefore,

$$Df(\xi_1)^k(B_\epsilon(\xi_i)) \subset B_r(\xi_i), \text{ for } k \leq \frac{\log(r) - \log(\epsilon)}{\log(\lambda)}.$$

Furthermore, for all  $j \in \{1, \dots, N\}$ , let  $W(\xi_j)$  be an open neighbourhood of  $\xi_j$  such that  $T|_{W(\xi_j)} : W(\xi_j) \rightarrow W(T^{m_j+1}(\xi_1))$  is a diffeomorphism and the Hartman-Grobman theorem holds.

Moreover, let  $n \in \mathbb{N}$  such that  $U_n(\xi_j) \subset W(\xi_j)$ , for each  $j \in \{1, \dots, N\}$ . By definition of  $\mathcal{A}_{q,n}$ , we know that for  $x \in \mathcal{A}_{q,n}$ , if  $f^k(x) \in B_r(\xi_i)$  then  $f^k(x) \notin \mathcal{A}_{q,n}$ .

Let  $\epsilon_n$  be the diameter of  $\mathcal{A}_{q,n}$ . Hence, we have that

$$k \leq \frac{\log(r) - \log(\epsilon_n)}{\log(\lambda)} \Rightarrow f^k(x) \notin \mathcal{A}_{q,n}.$$

Since  $\frac{\log(r) - \log(\epsilon_n)}{\log(\lambda)} \rightarrow \infty$ , as  $n \rightarrow \infty$ , therefore  $\lim_{n \rightarrow \infty} R(\mathcal{A}_{q,n}) = \infty$ .  $\square$

To compute  $\mathcal{A}_{q,n}$ , it is needed to calculate  $\mathcal{A}_{q,n}(\xi_i) = U_n(\xi_i) \cap \bigcap_{k=1}^q T^{-k}((U_n)^c)$  for each  $i \in \{1, \dots, N\}$ . In particular, considering the Lemma 6.1.1, we have that  $q = p$ , verifies the condition  $\mathcal{D}'_q(u_n)$ . Since, for  $i, j, k \in \{1, \dots, N\}$ ,  $i < j$  and  $k \neq m_j - m_i$ ,

$$U_n(\xi_i) \cap T^{-k}(U_n(\xi_j)) = \emptyset, \text{ then } U_n(\xi_i) \cap T^{-k}((U_n(\xi_j))^c) = U_n(\xi_i),$$

similarly when  $j < i$  with  $k \neq p - (m_i - m_j)$ . Hence, to know  $\mathcal{A}_{q,n}(\xi_i)$ , we only need to compute the intersections that may change the set, which are

$$U_n(\xi_i) \cap T^{-(m_j - m_i)}((U_n(\xi_j))^c), \text{ for } i < j \leq N$$

and

$$U_n(\xi_i) \cap T^{-(p - (m_i - m_j))}((U_n(\xi_j))^c), \text{ for every } i < j \leq N,$$

thus their intersection will be the set  $\mathcal{A}_{q,n}(\xi_i)$ . Recall that, since we assume that (R3) is satisfied, we only have two options for each of these intersections:

$$U_n(\xi_i) \subseteq T^{-k}((U_n(\xi_j))) \text{ or } \partial U_n(\xi_i) \cap T^{-k}((U_n(\xi_j))) = \emptyset,$$

with  $k$  depending on  $i$  and  $j$ .

Therefore, consider  $i \in \{1, \dots, N\}$  and  $i < j < N$ , if  $U_n(\xi_i) \subseteq T^{-(m_j - m_i)}(U_n(\xi_j))$  then we get

$$U_n(\xi_i) \cap T^{-(m_j - m_i)}((U_n(\xi_j))^c) = \emptyset,$$

otherwise,

$$U_n(\xi_i) \cap T^{-(m_j - m_i)}((U_n(\xi_j))^c) = U_n(\xi_i) \setminus T^{-(m_j - m_i)}(U_n(\xi_j)).$$

Similarly, for  $0 \leq j \leq i$ , if  $U_n(\xi_i) \subseteq T^{-(p-(m_i-m_j))}(U_n(\xi_j))$  then we get

$$U_n(\xi_i) \cap T^{-(p-(m_i-m_j))}((U_n(\xi_j))^c) = \emptyset,$$

if not,

$$U_n(\xi_i) \cap T^{-(p-(m_i-m_j))}((U_n(\xi_j))^c) = U_n(\xi_i) \setminus T^{-(p-(m_i-m_j))}(U_n(\xi_j)).$$

Observe that, if exists at least one  $j \in \{1, \dots, N\}$  that the intersestion is empty, then  $\mathcal{A}_{q,n}(\xi_i) = \emptyset$ .

In particular, if the preimage of a ball by  $T$  is a union of balls, then, for  $n$  sufficiently large, the only parts of the preimage  $T^{-k}(U_n), k \in \{0, \dots, q\}$  that is relevant to compute  $\mathcal{A}_{q,n}$  are the balls whose center is one of the maxima. Leading to the conclusion, that  $\mathcal{A}_{q,n}(\xi_i), i \in \{1, \dots, N\}$  is the smaller of the preimages cited above.

### 6.1.2 Example 1.

Consider the dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, T)$ , where the phase space  $\mathcal{X} = [0, 1)$ , equipped with the usual topology,  $\mathcal{B}$  the Borel  $\sigma$ -algebra of  $\mathcal{X}$ ,  $\mu$  is the Lesbegue measure, and let

$$T(x) = 2x \mod 1 \tag{29}$$

Since  $T^{-1}(B_r(c)) = B_{\frac{r}{2}}(\frac{c}{2}) \cup B_{\frac{r}{2}}(\frac{c}{2} + \frac{1}{2})$ ,  $r \in \mathbb{R}$ ,  $c \in \mathcal{X}$ , i.e., the preimage of a ball by  $T$  is a union of balls, therefore (R3) holds, and we can use the method described.

Also observe that  $\mu(B_r(c)) = 2r = \mu(T^{-1}(B_r(c)))$ , thus  $\mu$  is a  $T$ -invariant probability measure.

#### 6.1.2.1 Example 1.1, $N = 2$ .

First we choose the observable

$$\varphi(x) = -\log |\xi_1 - x| \mathbb{1}_{W_1} - \log |\alpha^{-1}(\xi_2 - x)| \mathbb{1}_{W_2}, \alpha \geq 0, \tag{30}$$

where  $W_1, W_2$  are neighborhood of  $\xi_1$  and  $\xi_2$ , respectively,  $\varphi$  achives 2 global

maxima at  $\xi_1$  and  $\xi_2$ . Consider that  $\xi_1 = T^2(\xi_1)$ ,  $\xi_2 = T(\xi_1)$ , i.e., the 2 maxima of the  $\varphi$  belongs to the same periodic orbit with prime period 2.

For  $n$  sufficiently large,  $\{X_0 > u_n\} = (\xi_1 - e^{-u_n}, \xi_1 + e^{-u_n}) \cup (\xi_2 - \alpha e^{-u_n}, \xi_2 + \alpha e^{-u_n})$ . Let  $r_n = e^{-u_n}$ , then we can write  $U_n(\xi_1) = B_{r_n}(\xi_1)$  and  $U_n(\xi_2) = B_{\alpha r_n}(\xi_2)$ , so  $U_n = B_{r_n}(\xi_1) \cup B_{\alpha r_n}(\xi_2)$ .

Now to compute  $\mathcal{A}_{q,n} = \mathcal{A}_{q,n}(\xi_1) \cup \mathcal{A}_{q,n}(\xi_2)$ , we will consider all the different cases, but first, notice that we cannot have  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$  and  $U_n(\xi_2) \subseteq T^{-1}(U_n(\xi_1))$  simultaneously, because

$$U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2)) \Rightarrow (T^{-1}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$$

which contradicts the second hypothesis.

Hence, whenever  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$ , we have  $(T^{-1}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ . From this we can conclude that

$$Leb(U_n(\xi_1)) = 2r_n \leq Leb(T^{-1}(U_n(\xi_2)) \cap U_n) = 2 \times 0.5\alpha r_n \iff 2 \leq \alpha,$$

and

$$Leb(T^{-1}(U_n(\xi_1)) \cap U_n) = 2 \times 0.5r_n < Leb(U_n(\xi_2)) = 2 \times \alpha r_n \iff 0.5 < \alpha.$$

In this case, as we have  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$ , then  $\mathcal{A}_{q,n}(\xi_1) = \emptyset$ , and because

$$T^{-1}(U_n(\xi_1)) \cap U_n = B_{0.5r_n}(\xi_2) \subset B_{0.25\alpha r_n}(\xi_2) = T^{-2}(U_n(\xi_2)) \cap U_n,$$

we have  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{-2}(U_n(\xi_2))$ .

Therefore we can calculate the EI using the Theorem 2.1.1:

$$\theta = \lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \frac{Leb(\mathcal{A}_{q,n})}{Leb(U_n)} = \lim_{n \rightarrow \infty} \frac{2\alpha r_n - 2 \times 0.25\alpha r_n}{2r_n + 2\alpha r_n} = \frac{0.75\alpha}{1 + \alpha}, \text{ when } \alpha \geq 2.$$

Now if  $(T^{-1}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$  and  $U_n(\xi_2) \subseteq T^{-1}(U_n(\xi_1))$ , then  $\alpha r_n \leq 0.5r_n$ . Since  $\mathcal{A}_{q,n}(\xi_2) = \emptyset$ , we follow the same argument as before, obtaining  $\mathcal{A}_{q,n} =$

$U_n(\xi_1) \setminus T^{-2}(U_n(\xi_1))$  and

$$\theta = \frac{0.75}{1+\alpha}, \quad \text{when } 0 \leq \alpha \leq 0.5.$$

Finally,  $(T^{-1}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$  and  $(T^{-1}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ , from the first hypothesis we get  $\alpha < 2$ , and from the second,  $0.5 < \alpha$ . In this case none of the  $\mathcal{A}_{q,n}(\xi_i)$ ,  $i \in \{1, 2\}$  are non-empty sets,  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-1}(U_n(\xi_2))$  and  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{-1}(U_n(\xi_1))$ , given that

$$T^{-2}(U_n(\xi_1)) \cap U_n = B_{0.25r_n}(\xi_1) \subset B_{0.5\alpha r_n}(\xi_2) = T^{-1}(U_n(\xi_2)) \cap U_n$$

and  $T^{-2}(U_n(\xi_2)) \subset T^{-1}(U_n(\xi_2))$ , respectively. Thus,

$$\begin{aligned} \theta &= \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} = \lim_{n \rightarrow \infty} \frac{2\alpha r_n - 2 \times 0.5\alpha r_n + 2\alpha r_n - 2 \times 0.5r_n}{2r_n + 2\alpha r_n} \\ &= \frac{(1+\alpha)(1-0.5)}{1+\alpha} = 0.5 \end{aligned}$$

when  $0.5 < \alpha < 2$ .

Therefore, we get the following expression for  $\theta$ , whose dependence on  $\alpha$  is depicted in the Figure 2.

$$\theta = \begin{cases} \frac{0.75}{1+\alpha}, & 0 \leq \alpha \leq 0.5 \\ 0.5, & 0.5 < \alpha < 2 \\ \frac{0.75\alpha}{1+\alpha}, & 2 \leq \alpha \end{cases}.$$

Note that  $\lim_{\alpha \rightarrow +\infty} \frac{0.75\alpha}{1+\alpha} = \lim_{\alpha \rightarrow +\infty} \frac{0.75}{1+\frac{1}{\alpha}} = 0.75$ , which is also the value for  $\theta$  when  $\alpha$  is zero.

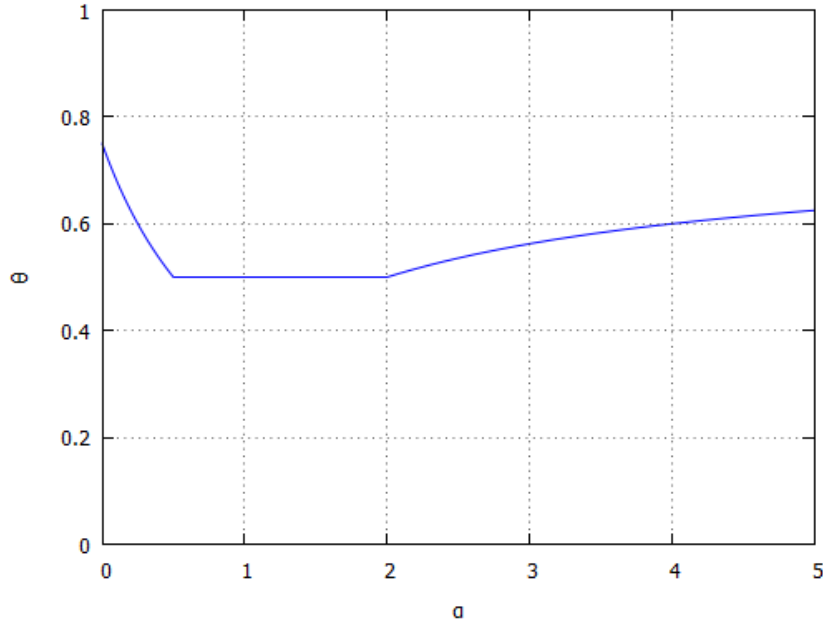


**Figure 2:** Graph of the values of  $\theta$  for  $T(x) = 2x \mod 1$  and two maxima along a periodic orbit.

### 6.1.2.2 Example 1.2, $N = 3$ .

Let us consider the  $T$  from the previous example and  $\varphi$ , an observable

$$\varphi(x) = -\log |\xi_1 - x| \mathbb{1}_{W_1} - \log |\alpha^{-1}(\xi_2 - x)| \mathbb{1}_{W_2} - \log |\beta^{-1}(\xi_3 - x)| \mathbb{1}_{W_3}, \quad (31)$$

for  $\alpha \geq 0, \beta \geq 0$ , where  $W_1, W_2, W_3$  are, respectively, neighborhood of  $\xi_1, \xi_2, \xi_3$ , which are the 3 maxima over a periodic orbit of prime period  $p$ , i.e.,  $\xi_1 = T^{m_1}(\xi_1), \xi_2 = T^{m_2}(\xi_1), \xi_3 = T^{m_3}(\xi_1)$ . Recall that  $0 = m_1 < m_2 < m_3 < p$ .

Regarding  $U_n = (\xi_1 - e^{-u_n}, \xi_1 + e^{-u_n}) \cup (\xi_2 - \alpha e^{-u_n}, \xi_2 + \alpha e^{-u_n}) \cup (\xi_3 - \beta e^{-u_n}, \xi_3 + \beta e^{-u_n})$ , the ball centered at  $\xi_1$  have a radius of  $r_n = e^{-u_n}$ , and  $\alpha$  and  $\beta$  are the proportions of the radius of the balls centered at  $\xi_2$  and  $\xi_3$ , respectively, when compared with  $r_n$ .

We now have more preimages to compare to know  $\mathcal{A}_{q,n}$ .

The first case that we will study, is when  $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2)), U_n(\xi_1) \subseteq T^{-m_3}(U_n(\xi_3))$  and  $U_n(\xi_2) \subseteq T^{-(m_3-m_2)}(U_n(\xi_3))$ .

To compute  $\mathcal{A}_{q,n}(\xi_1)$ , note that because  $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$  the intersection  $U_n(\xi_1) \cap T^{-m_2}((U_n(\xi_2))^c)$  is empty, therefore  $\mathcal{A}_{q,n}(\xi_1) = \emptyset$ ;  $\mathcal{A}_{q,n}(\xi_2)$  also has an

empty intersestion,  $U_n(\xi_2) \cap T^{-(m_3-m_2)}((U_n(\xi_3))^c)$ , so  $\mathcal{A}_{q,n}(\xi_2) = \emptyset$ ; for  $\mathcal{A}_{q,n}(\xi_3)$  we have that  $U_n(\xi_1) \subset T^{-m_3}(U_n(\xi_3))$  and  $U_n(\xi_2) \subset T^{-m_3}(U_n(\xi_3))$ , hence

$$\begin{aligned} \mathcal{A}_{q,n}(\xi_3) &= (U_n(\xi_3) \setminus T^{-(p-m_3)}(U_n(\xi_1))) \cap (U_n(\xi_3) \setminus T^{-(p-(m_3-m_2))}(U_n(\xi_2))) \cap \\ &\quad \cap (U_n(\xi_3) \setminus T^{-p}(U_n(\xi_3))) \end{aligned}$$

is not empty. To know which one is smaller, we need to check the bigger radius of  $T^{-p}(U_n(\xi_3))$ ,  $T^{-(p-m_3)}(U_n(\xi_1))$  and  $T^{-(p-(m_3-m_2))}(U_n(\xi_2))$  intersected with  $U_n$ , respectively,  $0.5^p \beta r_n$ ,  $0.5^{p-m_3} r_n$  and  $0.5^{p-(m_3-m_2)} \alpha r_n$ . Using the fact that  $1 \leq 0.5^{m_3} \beta$  and  $\alpha \leq 0.5^{m_3-m_2} \beta$ , we obtain that

$$r_n \leq 0.5^{m_3} \beta r_n \iff 0.5^{-m_3} r_n \leq \beta r_n \iff 0.5^{p-m_3} r_n \leq 0.5^p \beta r_n$$

and

$$\alpha r_n \leq 0.5^{m_3-m_2} \beta r_n \iff 0.5^{-(m_3-m_2)} \alpha r_n \leq \beta r_n \iff 0.5^{p-(m_3-m_2)} \alpha r_n \leq 0.5^p \beta r_n.$$

Therefore  $\mathcal{A}_{q,n}(\xi_3) = U_n(\xi_3) \setminus T^{-p}(U_n(\xi_3))$ .

So  $\mathcal{A}_{q,n} = U_n(\xi_3) \setminus T^{-p}(U_n(\xi_3))$ , then

$$\theta = \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} = \lim_{n \rightarrow \infty} \frac{2\beta r_n - 2 \times 0.5^p \beta r_n}{2r_n + 2\alpha r_n + 2\beta r_n} = \frac{(1 - 0.5^p)\beta}{1 + \alpha + \beta}.$$

Actually, when  $\mathcal{A}_{q,n}(\xi_1)$  and  $\mathcal{A}_{q,n}(\xi_2)$  still are an empty set, and since the proportion of the third ball with the first two will not change, the  $\theta$  will have the same formula. The cases are:

- $(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ ,  $U_n(\xi_1) \subseteq T^{-m_3}(U_n(\xi_3))$  and  $U_n(\xi_2) \subseteq T^{-(m_3-m_2)}(U_n(\xi_3))$ ;
- $U_n(\xi_2) \subseteq T^{-(p-m_2)}(U_n(\xi_1))$  and  $U_n(\xi_1) \subseteq T^{-m_3}(U_n(\xi_3))$ ;
- $(T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ ,  $(T^{-m_3}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_1)$  and  $U_n(\xi_2) \subseteq T^{-(m_3-m_2)}(U_n(\xi_3))$ .

Similarly, we can see that in the cases which  $U_n(\xi_3) \subseteq T^{-(p-(m_3-m_2))}(U_n(\xi_2))$

and  $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$  the only no-empty set is  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{-p}(U_n(\xi_2))$ , having

$$\theta = \frac{(1 - 0.5^p)\alpha}{1 + \alpha + \beta}.$$

Likewise, when  $U_n(\xi_2) \subseteq T^{-(p-m_2)}(U_n(\xi_1))$  and  $U_n(\xi_3) \subseteq T^{-(p-m_3)}(U_n(\xi_1))$ , the only set which is not empty is the  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-p}(U_n(\xi_1))$ , so

$$\theta = \frac{1 - 0.5^p}{1 + \alpha + \beta}.$$

Next, let us see the following case  $(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ ,  $(T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ ,  $(T^{-(p-m_3-m_2)}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_3)$  and  $U_n(\xi_3) \subseteq T^{-(p-m_3)}(U_n(\xi_1))$ . From this we get  $\mathcal{A}_{q,n}(\xi_3) = \emptyset$ , which is the only empty set. For  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \cap (T^{-p}(U_n(\xi_1))^c \cap T^{-m_2}(U_n(\xi_2))^c \cap T^{-m_3}(U_n(\xi_3))^c)$ , we need to check which is smaller from the preimages. To see it, observe that

$$0.5^{p-m_2} < \alpha \iff 0.5^p < 0.5^{m_2}\alpha, \beta \leq 0.5^{p-m_3} \iff 0.5^{m_3}\beta \leq 0.5^p < 0.5^{m_2}\alpha,$$

thus  $T^{-m_2}(U_n(\xi_2))^c \cap U_n(\xi_1)$  is the smaller, so  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-m_2}(U_n(\xi_2))$ . And for,  $\mathcal{A}_{q,n}(\xi_2)$ , we have that

$$0.5^{m_2}\alpha < 1 \iff \alpha < 0.5^{-m_2} \iff 0.5^p\alpha < 0.5^{p-m_2}$$

and

$$0.5^{m_3}\beta \leq 0.5^p \iff 0.5^{m_3-m_2}\beta \leq 0.5^{p-m_2},$$

then  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{p-m_2}(U_n(\xi_1))$ . Therefore,  $\mathcal{A}_{q,n} = U_n(\xi_1) \setminus T^{-m_2}(U_n(\xi_2)) \cup U_n(\xi_2) \setminus T^{p-m_2}(U_n(\xi_1))$  and

$$\theta = \lim_{n \rightarrow \infty} \frac{2r_n(1 - 0.5^{m_2}\alpha + \alpha - 0.5^{p-m_2})}{2r_n(1 + \alpha + \beta)} = \frac{1 + \alpha - (0.5^{p-m_2} + 0.5^{m_2}\alpha)}{1 + \alpha + \beta}.$$

Which is the same  $\theta$  for the cases that have

- $(T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ ,  $(T^{-(m_3-m_2)}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_2)$  and  $U_n(\xi_3) \subseteq T^{-(p-m_3)}(U_n(\xi_1))$ ;

- $(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$  and  $U_n(\xi_3) \subseteq T^{-(p-m_3)}(U_n(\xi_1))$ ;
- $(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ ,  $U_n(\xi_3) \subseteq T^{-(p-m_3)}(U_n(\xi_1))$ ,  $U_n(\xi_3) \subseteq T^{-(p-(m_3-m_2))}(U_n(\xi_2))$  and  $(T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$ .

Using similar arguments the cases  $U_n(\xi_2) \subseteq T^{-(m_3-m_2)}(U_n(\xi_3))$  and  $(T^{-m_3}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_1)$ ;  $U_n(\xi_2) \subseteq T^{-(m_3-m_2)}(U_n(\xi_3))$  and  $(T^{-(p-m_3)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_3)$  we obtain that  $\mathcal{A}_{q,n}(\xi_2) = \emptyset$ ,  $\mathcal{A}_{q,n} = U_n(\xi_1) \setminus T^{-m_3}(U_n(\xi_3)) \cup U_n(\xi_3) \setminus T^{p-m_3}(U_n(\xi_3))$  and

$$\theta = \frac{1 + \beta - (0.5^{p-m_3} + 0.5^{m_3}\beta)}{1 + \alpha + \beta}.$$

As well, when

- $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$ ,  $(T^{-m_3}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_1)$  and  $(T^{-(m_3-m_2)}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_2)$ ;
- $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$  and  $(T^{-(m_3-m_2)}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_2)$ ;
- $(T^{-(p-(m_3-m_2))}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_3)$  and  $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$ ;
- $(T^{-(p-m_3)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_3)$ ,  $(T^{-(p-(m_3-m_2))}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_3)$  and  $U_n(\xi_1) \subseteq T^{-m_2}(U_n(\xi_2))$ ,

we have  $\mathcal{A}_{q,n}(\xi_1) = \emptyset$ ,  $\mathcal{A}_{q,n} = U_n(\xi_2) \setminus T^{-(m_3-m_2)}(U_n(\xi_3)) \cup U_n(\xi_3) \setminus T^{-(p-(m_3-m_2))}(U_n(\xi_2))$  and

$$\theta = \frac{\alpha + \beta - (0.5^{p-(m_3-m_2)}\alpha + 0.5^{m_3-m_2}\beta)}{1 + \alpha + \beta}.$$

Finally consider the case when:

- $(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ ,  $(T^{-m_3}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_1)$ ,
- $(T^{-(m_3-m_2)}(U_n(\xi_3)) \cap U_n) \subset U_n(\xi_2)$ ,  $(T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_2)$  and
- $(T^{-(p-(m_3-m_2))}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_3)$ ,  $(T^{-(p-m_3)}(U_n(\xi_1)) \cap U_n) \subset U_n(\xi_3)$ ,

so none of the intersestions are empty. From the third condition we get  $0.5^{-(m_3-m_2)}\beta < \alpha$ , which implies that  $0.5^{-m_3}\beta < 0.5^{-m_2}\alpha$ , so the bigger preimage is

$(T^{-m_2}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ , then  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-m_2}(U_n(\xi_2))$ . For  $\mathcal{A}_{q,n}(\xi_2)$  we note that in this case, from the last condition

$$0.5^{-(p-m_3)} < \beta \iff 0.5^{-p} < 0.5^{-m_3} \beta \iff 0.5^{-(p-m_2)} < 0.5^{-(m_3-m_2)} \beta,$$

this means that  $T^{-(p-m_2)}(U_n(\xi_1)) \cap U_n \subset T^{-(m_3-m_2)}(U_n(\xi_3)) \cap U_n$ , therefore  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{-(m_3-m_2)}(U_n(\xi_3))$ . To compute  $\mathcal{A}_{q,n}(\xi_3)$ , observe that from the first condition

$$0.5^{-m_2} \alpha < 1 \iff 0.5^{m_3-m_2} \alpha < 0.5^{m_3} \iff 0.5^{-(p-(m_3-m_2))} \alpha < 0.5^{-(p-m_3)},$$

then  $\mathcal{A}_{q,n}(\xi_3) = U_n(\xi_3) \setminus T^{-(p-m_3)}(U_n(\xi_1))$ . Concluding, in this case  $\mathcal{A}_{q,n} = U_n(\xi_1) \setminus T^{-m_2}(U_n(\xi_2)) \cup U_n(\xi_2) \setminus T^{-(m_3-m_2)}(U_n(\xi_3)) \cup U_n(\xi_3) \setminus T^{-(p-m_3)}(U_n(\xi_1))$ .

Obtaining,

$$\begin{aligned} \theta &= \lim_{n \rightarrow \infty} \frac{2r_n \times (1 - 0.5^{-m_2} \alpha + \alpha - 0.5^{-(m_3-m_2)} \beta + \beta - 0.5^{-(p-m_3)})}{2r_n \times (1 + \alpha + \beta)} \\ &= \frac{(1 + \alpha + \beta) + (-0.5^{-m_2} \alpha - 0.5^{-(m_3-m_2)} \beta - 0.5^{-(p-m_3)})}{1 + \alpha + \beta} \\ &= 1 - \frac{0.5^{p-m_3} + 0.5^{m_2} \alpha + 0.5^{m_3-m_2} \beta}{1 + \alpha + \beta} \end{aligned}$$

This happens for the rest of the cases, which none of the preimages containing the  $U_n$ .

In conclusion, we obtain the expression below for the value of  $\theta$  in function of  $\alpha$ , which is represented by Figure 3.

$$\left\{ \begin{array}{l}
 \frac{(1-0.5^p)\beta}{1+\alpha+\beta}, 0 \leq \alpha \leq 0.5^{m_3-m_2}\beta \wedge 1 \leq 0.5^{m_3}\beta \\
 \frac{(1-0.5^p)\alpha}{1+\alpha+\beta}, 0 \leq \beta \leq 0.5^{p+m_2-m_3}\alpha \wedge 1 \leq 0.5^{m_2}\alpha \\
 \frac{1-0.5^p}{1+\alpha+\beta}, 0 \leq \alpha \leq 0.5^{p-m_2} \wedge \alpha \leq \beta \leq 0.5^{p-m_3} \vee 0 \leq \beta \leq 0.5^{p-m_3} \wedge \beta \leq \alpha \leq 0.5^{p-m_2} \\
 \frac{1+\alpha-(0.5^{p-m_2}+0.5^{m_2}\alpha)}{1+\alpha+\beta}, 0.5^{p-m_2} \leq \alpha \wedge 0.5^{m_3-m_2}\beta \leq \alpha \leq \beta \leq 0.5^{p-m_3} \vee \\
 0 \leq \beta \leq 0.5^{p-m_3} \wedge 0.5^{m_2}\alpha \leq 1 \leq \alpha \\
 \frac{1+\beta-(0.5^{p-m_3}+0.5^{m_3}\beta)}{1+\alpha+\beta}, 0 \leq \alpha \leq 1 \wedge \alpha \leq 0.5^{m_3-m_2}\beta \wedge 0.5^{m_3}\beta \leq 1 \leq \beta \vee \\
 0 \leq \alpha \leq 0.5^{m_3-m_2}\beta \wedge 0.5^{m_2}\alpha \leq 1 \wedge 0.5^{m_3}\beta \leq 1 \wedge 0.5^{p-m_3} \leq \beta \\
 \frac{\alpha+\beta-(0.5^{p+m_2-m_3}\alpha+0.5^{m_3-m_2}\beta)}{1+\alpha+\beta}, 1 \leq 0.5^{m_2}\alpha \wedge 0.5^{p-m_3} \leq \beta \leq \alpha \wedge 0.5^{p+m_2-m_3}\alpha \leq \beta \vee \\
 1 \leq 0.5^{m_2}\alpha \wedge 0.5^{m_3-m_2}\beta \leq \alpha \leq \beta \\
 1 - \frac{0.5^{p-m_3}+0.5^{m_2}\alpha+0.5^{m_3-m_2}\beta}{1+\alpha+\beta}, 0.5^{p-m_2} \leq \alpha \leq \beta \wedge 0.5^{m_3-m_2}\beta \leq \alpha \wedge 0.5^{p-m_3} \leq \beta \wedge \\
 0.5^{m_3}\beta \leq 1 \wedge 0.5^{m_2}\alpha \leq 1 \vee \\
 \beta \leq \alpha \wedge 0.5^{m_2}\alpha \leq 1 \leq \alpha \wedge 0.5^{p+m_2-m_3}\alpha \leq \beta \wedge \\
 0.5^{m_3}\beta \leq 1 \wedge 0.5^{p-m_3} \leq \beta
 \end{array} \right.$$

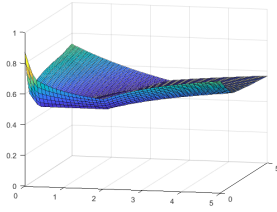
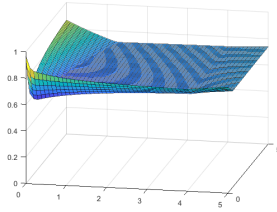
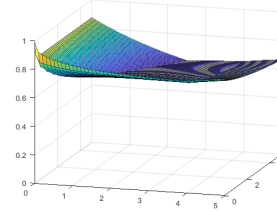
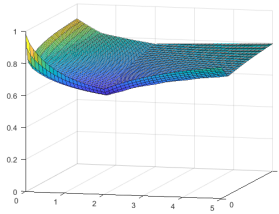
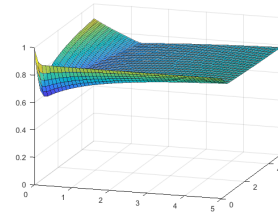

 (a)  $p = 3, m_2 = 1, m_3 = 2.$ 

 (b)  $p = 5, m_2 = 2, m_3 = 4.$ 

 (c)  $p = 6, m_2 = 2, m_3 = 3$ 

 (d)  $p = 6, m_2 = 1, m_3 = 4.$ 

 (e)  $p = 6, m_2 = 3, m_3 = 5.$ 

**Figure 3:** Graph of the values of  $\theta$  for  $T(x) = 2x \bmod 1$  and three maxima along a periodic orbit, with different periods.

As one can see, in both examples, the new expression for EI just depends on the radius of the balls centered at the maxima, i.e., the proportion  $\alpha$  on the first, and  $\alpha, \beta$  in the second. Therefore, we can apply the Theorem 2.1.1 for the stochastic process arising from the dynamical system, this means that the corresponding process of partial maxima has the asymptotic distribution equal to  $e^{-\theta\tau}$ .

Moreover, we can adapt the expression for any dynamical system of type  $a \cdot x \bmod 1$ ,  $a \in \mathbb{N}_{>1}$ , as we will see in the next example, and for a dynamical system that is diffeomorphic to that type of dynamic but with some restrictions, seen in the Example 3.

### 6.1.3 Example 2.

In this example we consider the dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, T)$ , where the phase space  $\mathcal{X}, \mathcal{B}, \mu$  are the same as before, but

$$T(x) = 3x \bmod 1 \quad (32)$$

Observe that,  $T^{-1}(B_r(c)) = B_{\frac{r}{3}}(\frac{c}{3}) \cup B_{\frac{r}{3}}(\frac{c}{3} + \frac{1}{3}) \cup B_{\frac{r}{3}}(\frac{c}{3} + \frac{2}{3})$ ,  $r \in \mathbb{R}$ ,  $c \in \mathcal{X}$ , so the preimage of a ball by  $T$  is a union of balls. Hence (R3) holds, and we can use the same procedure. Similary as the previous example,  $\mu$  is  $T$ -invariant.

Let,

$$\varphi(x) = |\xi_1 - x|^{-0.5} \mathbb{1}_{W_1} + \alpha |\xi_2 - x|^{-0.5} \mathbb{1}_{W_2}, \quad \alpha \geq 0, \quad (33)$$

be an observable with 2 maxima,  $\xi_1$  and  $\xi_2 = T(\xi_1)$ , in a periodic orbit with prime period 2, and  $W_1, W_2$  are the, respectively, neighborhood of  $\xi_1$  and  $\xi_2$ . Therefore, for  $n$  sufficiently large,  $U_n = (\xi_1 - u_n^{-2}, \xi_1 + u_n^{-2}) \cup (\xi_2 - \alpha u_n^{-2}, \xi_2 + \alpha u_n^{-2})$ , consider the ball centered over  $\xi_1$  having radius size  $r_n = u_n^{-2}$  and the one centered at  $\xi_2$  have a radius of  $\alpha r_n$ .

The case  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$  and  $U_n(\xi_2) \subseteq T^{-1}(U_n(\xi_1))$  is not possible for the same reason has in the Example 1.1.

For  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$  and  $T^{-1}(U_n(\xi_1)) \cap U_n \subset U_n(\xi_2)$ , we obtain  $3 \leq \alpha$ .

$\mathcal{A}_{q,n}(\xi_1) = \emptyset$ , and accordingly with

$$T^{-1}(U_n(\xi_1)) \cap U_n = B_{r_n/3}(\xi_2) \subset B_{\alpha r_n/9}(\xi_2) = T^{-2}(U_n(\xi_2)) \cap U_n,$$

$\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2) \setminus T^{-2}(U_n(\xi_2))$ . Thus, we can compute  $\theta$  with the theorem 2.1.1:

$$\begin{aligned} \theta &= \lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} \\ &= \lim_{n \rightarrow \infty} \frac{2\alpha r_n - 2 \times \alpha r_n/9}{2r_n + 2\alpha r_n} \\ &= \frac{(8/9)\alpha}{1 + \alpha}, \text{ when } 3 \leq \alpha. \end{aligned}$$

If  $T^{-1}(U_n(\xi_2)) \cap U_n \subset U_n(\xi_1)$  and  $U_n(\xi_2) \subseteq T^{-1}(U_n(\xi_1))$ , then  $\alpha r_n \leq r_n/3$ , with the same argument of the previous case we get,  $\mathcal{A}_{q,n}(\xi_2) = \emptyset$ ,  $\mathcal{A}_{q,n} = \mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-2}(U_n(\xi_1))$  and

$$\theta = \frac{8/9}{1 + \alpha}, \text{ when } 0 \leq \alpha \leq 1/3.$$

Next  $T^{-1}(U_n(\xi_2)) \cap U_n \subset U_n(\xi_1)$  and  $T^{-1}(U_n(\xi_1)) \cap U_n \subset U_n(\xi_2)$ , achieving  $1/3 < \alpha < 3$ . This case do not have empty set for  $\mathcal{A}_{q,n}(\xi_i)$ ,  $i \in \{1, 2\}$ , so  $\mathcal{A}_{q,n} = (U_n(\xi_1) \setminus T^{-1}(U_n(\xi_2))) \cup (U_n(\xi_2) \setminus T^{-1}(U_n(\xi_1)))$ , since

$$T^{-2}(U_n(\xi_1)) \cap U_n = B_{r_n/9}(\xi_1) \subset B_{\alpha r_n/3}(\xi_1) = T^{-1}(U_n(\xi_2)) \cap U_n.$$

Therefore,

$$\begin{aligned} \theta &= \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} \\ &= \lim_{n \rightarrow \infty} \frac{2r_n - 2 \times \frac{\alpha r_n}{3} + 2\alpha r_n - 2 \times \frac{r_n}{3}}{2r_n + 2\alpha r_n} \\ &= \frac{1 - \alpha/3 + \alpha - 1/3}{1 + \alpha} = \frac{\frac{2}{3}(1 + \alpha)}{1 + \alpha} = \frac{2}{3}, \text{ when } 1/3 \leq \alpha \leq 3. \end{aligned}$$

Accordingly, we achieve the following expression for  $\theta$  and the Figure 4:

$$\theta = \begin{cases} \frac{8/9}{1+\alpha}, & 0 \leq \alpha \leq 1/3 \\ 2/3, & 1/3 < \alpha < 3 \\ \frac{(8/9)\alpha}{1+\alpha}, & 3 \leq \alpha \end{cases}$$

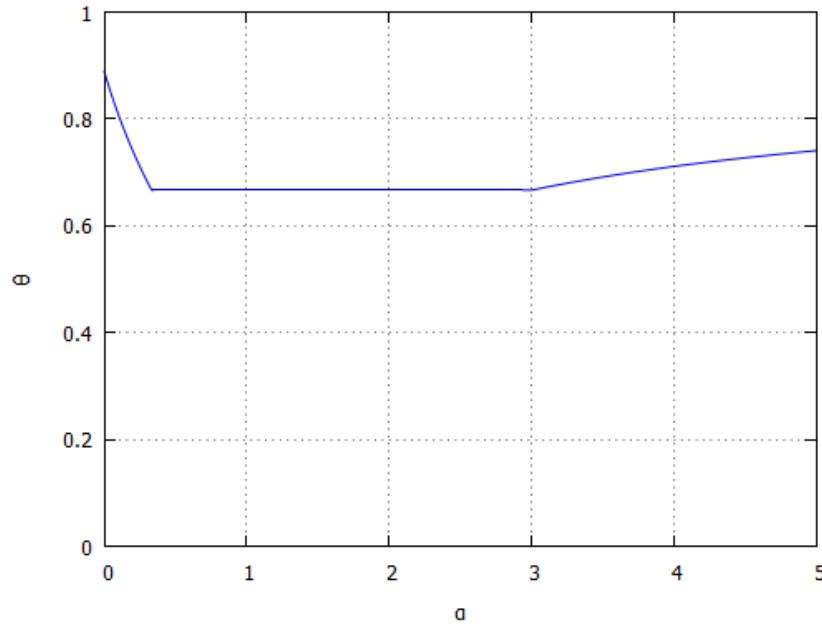


**Figure 4:** Graph of the values of  $\theta$  for  $T(x) = 3x \mod 1$  and two maxima along a periodic orbit.

Observe that, the limits for  $\alpha \rightarrow +\infty$  is the same as when  $\alpha = 0$ ,  $\lim_{\alpha \rightarrow +\infty} \frac{(8/9)\alpha}{1+\alpha} = \lim_{\alpha \rightarrow +\infty} \frac{8/9}{1+\frac{1}{\alpha}} = 8/9$ .

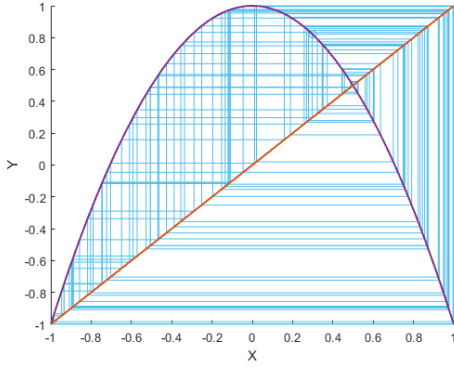
It is interesting to note the similarity between the Figure 2 and the Figure 4, that is, they are separated in three parts, where the first is decreasing, the second is constant and the third is increasing.

### 6.1.3 Example 3.

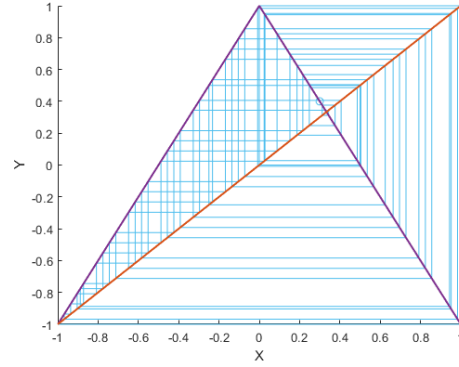
For this example, consider a different dynamic, let  $\mathcal{X} = [-1, 1]$ , with the Borel  $\sigma$ -algebra of  $\mathcal{X}$ ,  $\mathcal{B}$ , and the function  $T : \mathcal{X} \rightarrow \mathcal{X}$ , defined by

$$T(x) = 1 - 2x^2, \quad (34)$$

which is represented in Figure 5.a.



(a) Graph of  $T(x) = 1 - 2x^2$ .



(b) Graph of  $S(x) = 1 - 2|x|$ .

**Figure 5:** Graphs of  $T$  and  $S$ .

Unlike previous examples the pre-image by  $T$  is difficult to compute. For this reason, consider the dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, S)$ , where  $\mu$  is the Lebesgue measure, which is  $S$ -invariant and  $S : \mathcal{X} \rightarrow \mathcal{X}$ , such that  $S(x) = 1 - 2|x|$ , see Figure 5.b. Furthermore, consider the diffeomorphism  $\phi(x) = \frac{2}{\pi} \arcsin(x)$ , with the inverse  $\phi^{-1}(x) = \sin(\frac{\pi}{2}x)$ . Obtaining, the commutative diagram in the Figure 6, which means that  $\phi \circ T = S \circ \phi \iff T \circ \phi^{-1} = \phi^{-1} \circ S$ . This means that,  $T$  and  $S$  are topologically equivalent, so an periodic orbit in  $T$  is periodic in  $S$ , and vice-versa. Finally, let the probability measure of  $(\mathcal{X}, \mathcal{B}, T)$  be  $\tilde{\mu} = \phi_*^{-1}\mu$ , where

$$\phi_*^{-1}\mu = \frac{2}{\pi} \cdot \frac{1}{\sqrt{(1-x)(1+x)}}.$$

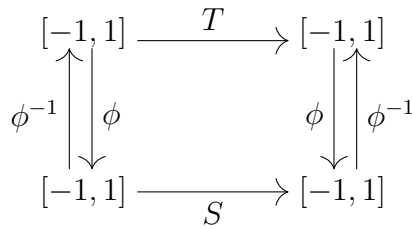


**Figure 6:** Commutative diagram.

The probability measure  $\tilde{\mu}$  is  $T$ -invariant because

$$\begin{aligned}
 T_*\tilde{\mu} &= T_*\phi_*^{-1}\mu \\
 &= (T \circ \phi^{-1})_*\mu = (\phi^{-1} \circ S)_*\mu \\
 &= \phi_*^{-1}S_*\mu \\
 &= \phi_*^{-1}\mu = \tilde{\mu}.
 \end{aligned}$$

To simplify the next calculations, note that, for  $r \in \mathbb{R}, c \in \mathcal{X}$ , consider the ball with radius  $r$  centered at  $c$  on  $(\mathcal{X}, \tilde{\mu})$  will be represented by  $\tilde{B}_r(c)$ . Therefore  $\phi(\tilde{B}_r(c)) = B_r(\phi(c))$  and  $\phi^{-1}(B_r(\phi(c))) = \tilde{B}_r(c)$ .

Let the observable be as in (30), having  $N = 2$  in a periodic orbit,  $\xi_1 = T^2(\xi_1)$  and  $\xi_2 = T(\xi_1)$ . Also, for  $n$  sufficiently large, we can write  $U_n(\xi_1) = \tilde{B}_{r_n}(\xi_1)$  and  $U_n(\xi_2) = \tilde{B}_{\alpha r_n}(\xi_2)$ , with  $r_n = e^{-u_n}$ . Therefore,  $U_n = \tilde{B}_{r_n}(\xi_1) \cup \tilde{B}_{\alpha r_n}(\xi_2)$ , and that

$$\phi(U_n) = \phi(\tilde{B}_{r_n}(\xi_1) \cup \tilde{B}_{\alpha r_n}(\xi_2)) = \phi(\tilde{B}_{r_n}(\xi_1)) \cup \phi(\tilde{B}_{\alpha r_n}(\xi_2)) = B_{r_n}(\phi(\xi_1)) \cup B_{\alpha r_n}(\phi(\xi_2)),$$

leading to the following measure:

$$\begin{aligned}
 \tilde{\mu}(U_n) &= \mu(\phi(U_n)) \\
 &= \mu(B_{r_n}(\phi(\xi_1)) \cup B_{\alpha r_n}(\phi(\xi_2))) \\
 &= \mu(B_{r_n}(\phi(\xi_1))) + \mu(B_{\alpha r_n}(\phi(\xi_2))) \\
 &= 2 \times r_n + 2 \times \alpha r_n = 2r_n \times (1 + \alpha).
 \end{aligned}$$

To compute the set  $\mathcal{A}_{q,n}$ , note that,

$$\begin{aligned}
 T^{-1}(U_n) &= \phi^{-1} \circ \phi \circ T^{-1}(U_n) \\
 &= \phi^{-1} \circ S^{-1} \circ \phi(U_n) \\
 &= \phi^{-1} \circ S^{-1}(B_{r_n}(\phi(\xi_1)) \cup B_{\alpha r_n}(\phi(\xi_2))) \\
 &= \phi^{-1}(B_{\frac{r_n}{2}}(\phi(\xi_2)) \cup B_{\frac{r_n}{2}}(\cdot) \cup B_{\frac{\alpha r_n}{2}}(\phi(\xi_1)) \cup B_{\frac{\alpha r_n}{2}}(\cdot))^5 \\
 &= \tilde{B}_{\frac{r_n}{2}}(\xi_2) \cup \tilde{B}_{\frac{r_n}{2}}(\cdot) \cup \tilde{B}_{\frac{\alpha r_n}{2}}(\xi_1) \cup \tilde{B}_{\frac{\alpha r_n}{2}}(\cdot).
 \end{aligned}$$

By induction, we get that for  $i \in \mathbb{N}$ ,  $T^{-i}(U_n) = \tilde{B}_{\frac{\alpha r_n}{2^i}}(\xi_1) \cup \tilde{B}_{\frac{r_n}{2^i}}(\xi_2) \cup \dots$ . Therefore, for  $n$  sufficiently large, the condition (R3) is satisfied, thus we can use the same method as before.

For the first case we consider  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$  and  $T^{-1}(U_n(\xi_1)) \cap \partial U_n(\xi_2) = \emptyset$ , implies that  $1 \leq 0.5\alpha$  and  $0.5 < \alpha$ , so  $2 \leq \alpha$ . We achieve  $\mathcal{A}_{q,n}(\xi_1) = \emptyset$  and  $\mathcal{A}_{q,n}(\xi_2) = \tilde{B}_{\alpha r_n}(\xi_2) \setminus \tilde{B}_{\frac{\alpha r_n}{4}}(\xi_2)$ , thus  $\mathcal{A}_{q,n} = \tilde{B}_{\alpha r_n}(\xi_2) \setminus \tilde{B}_{\frac{\alpha r_n}{4}}(\xi_2)$  leading to the measure:

$$\begin{aligned} \tilde{\mu}(\mathcal{A}_{q,n}) &= \tilde{\mu}(\tilde{B}_{\alpha r_n}(\xi_2) \setminus \tilde{B}_{\frac{\alpha r_n}{4}}(\xi_2)) \\ &= \tilde{\mu}(\tilde{B}_{\alpha r_n}(\xi_2)) - \tilde{\mu}(\tilde{B}_{\frac{\alpha r_n}{4}}(\xi_2)) \\ &= \mu(B_{\alpha r_n}(\phi(\xi_2))) - \mu(B_{\frac{\alpha r_n}{4}}(\phi(\xi_2))) \\ &= 2 \times \left( \alpha r_n - \frac{\alpha r_n}{4} \right) = 2 \times \left( \frac{3}{4} \alpha r_n \right) = 2r_n \times 0.75\alpha. \end{aligned}$$

Using the limit in the Theorem 2.1.1, we get:

$$\theta = \lim_{n \rightarrow \infty} \frac{\tilde{\mu}(\mathcal{A}_{q,n})}{\tilde{\mu}(U_n)} = \lim_{n \rightarrow \infty} \frac{2r_n \times (0.75\alpha)}{2r_n \times (1 + \alpha)} = \frac{0.75\alpha}{1 + \alpha}, \quad 2 \leq \alpha.$$

Actually, we can note that now the computations are the same as in example Example 1.1. Therefore we obtain, that the sequence  $(M_n)_{n \in \mathbb{N}}$  as an EVL, since by the Theorem 2.1.1 we have that  $\lim_{n \rightarrow \infty} \tilde{\mu}(M_n \leq u_n) = e^{-\theta\tau}$ , where  $\theta$  is given by the following expression:

$$\theta = \begin{cases} \frac{0.75}{1+\alpha}, & 0 \leq \alpha \leq 0.5 \\ 0.5, & 0.5 < \alpha < 2 \\ \frac{0.75\alpha}{1+\alpha}, & 2 \leq \alpha. \end{cases}$$

It is intriguing to observe that in this example we obtain the exact same formula for the EI as in the Example 1.1!

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<sup>5</sup>As seen before, in the introduction of the section 6, the balls not centered at the maxima points of  $\varphi$  are not relevant for the computation, thus we represented the center by a dot.

## 6.2 Multiple maxima over non periodic orbits

In this section, we are going to study the case when  $\varphi$  has  $N$  maxima which belong to the same non-periodic orbit. As before, let  $\xi_1, \dots, \xi_N$  be these maxima, and  $m_i \in \mathbb{N}$ ,  $m_1 = 0$  and  $m_i < m_j$ ,  $\forall i < j$  such that  $\xi_i = T^{m_i}(\xi_1)$ , and we assume that (R3) holds.

Just like before, we start by choosing a value of  $q$  for which the condition  $\mathcal{D}'_q(u_n)$  can be checked.

### Lemma 6.2.1

Assume that  $T$  satisfies Assumption A and that  $\xi_1$  is a non-periodic point and that  $T$  is continuous in every point of the orbit of  $\xi_1$ . Let  $\xi_1, \dots, \xi_N$  be as in (27). Let  $q = m_N - m_1$ , then  $\lim_{n \rightarrow \infty} R(\mathcal{A}_{q,n}) = \infty$ .

**Proof.** Let  $q = m_N - m_1$ . Observe that, by definition of  $\mathcal{A}_{q,n}$ , we have that

$$T^j(\mathcal{A}_{q,n}) \cap \mathcal{A}_{q,n} \subset T^j(\mathcal{A}_{q,n}) \cap U_n = \emptyset, \forall j = 1, \dots, q.$$

Consequently, it remains to check for all  $j = q + 1, \dots, q + L$ , where  $L \in \mathbb{Z}$  is a given arbitrarily large integer, that the same holds true. For this purpose we will use the continuity of  $T$  over the orbit of the maxima. Let

$$\epsilon := \min_{1 \leq i \leq L, 1 \leq j \leq N} \text{dist}(T^i(\xi_N), \xi_j).$$

Using the continuity over the orbit of the maxima, in particular, of  $\xi_N$ , for every  $1 \leq i \leq L$ , consider the balls  $B_{\delta_i}(\xi_N)$  such that  $T^i(B_{\delta_i}(\xi_N)) \subset B_{\epsilon/2}(T^i(\xi_N))$  and consider the open set  $D_N = \bigcap_{i=1}^L B_{\delta_i}(\xi_N)$ .

Moreover, for each  $j = 1, \dots, N - 1$  let  $D_j$  be a ball around  $\xi_j$  so that  $T^{m_N - m_j}(D_j) \subset D_N$ .

Lastly, due to (R1), take  $n$  sufficiently large such that each  $U_n(\xi_i) \subset D_i \cap B_{\epsilon/2}(\xi_i)$ , then  $U_n \subset \bigcup_{i=1}^N D_i \cap B_{\epsilon/2}(\xi_i)$ .  $\square$

Now, for the calculation of the set  $\mathcal{A}_{q,n}$ , as before, we need to know  $\mathcal{A}_{q,n}(\xi_i)$ ,  $i \in \{1, \dots, N\}$ , this means we have to compute  $U_n(\xi_i) \cap \bigcap_{k=1}^q T^{-k}((U_n)^c)$ . Notice that,

because the orbit does not have a periodic behavior, for  $j \in \{1, \dots, N\}$  and  $j \leq i$ ,  $U_n(\xi_i) \cap \bigcap_{k=1}^q T^{-k}(U_n(\xi_j)) = \emptyset$ , so we just need to check for  $j > i$ , that is, if

$$U_n(\xi_i) \subseteq T^{-(m_j-m_i)}(U_n(\xi_j)), \text{ then } \mathcal{A}_{q,n}(\xi_i) = \emptyset,$$

else we get that

$$U_n(\xi_i) \setminus T^{-(m_j-m_i)}(U_n(\xi_j)).$$

Note that  $\mathcal{A}_{q,n}(\xi_N) = U_n(\xi_N)$ , because it does not have any  $j > N, j \in \{1, \dots, N\}$ .

#### 6.2.2 Example 4.

Consider the dynamical system described in the previous examples, with  $\mathcal{X} = [0, 1)$  equipped with the usual topology,  $\mathcal{B}$  the Borel  $\sigma$ -algebra and  $\mu$  is the Lesbegue measure. Let  $T$  be like in (32),  $\varphi$  as in (33), but the maxima  $\xi_1$  and  $\xi_2 = T(\xi_1)$  belongs to a non periodic orbit. This example, as we have seen, for  $n$  sufficiently large,  $r_n = u_n^{-2}$ ,  $U_n(\xi_1) = B_{r_n}(\xi_1)$  and  $U_n(\xi_2) = B_{\alpha r_n}(\xi_2)$ .

Observe that we just have to check the preimages related to  $\mathcal{A}_{q,n}(\xi_1)$ , because  $N = 2$ , so  $\mathcal{A}_{q,n}(\xi_2) = U_n(\xi_2)$ .

The first case will be when  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$ , we get  $3 \leq \alpha$  and  $\mathcal{A}_{q,n}(\xi_1) = \emptyset$ . Therefore,

$$\theta = \lim_{n \rightarrow \infty} \theta_n = \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} = \lim_{n \rightarrow \infty} \frac{2r_n \alpha}{2r_n + 2\alpha r_n} = \frac{\alpha}{1 + \alpha}$$

Otherwise, we obtain  $0 \leq \alpha < 3$  and  $\mathcal{A}_{q,n}(\xi_1) = U_n(\xi_1) \setminus T^{-1}(U_n(\xi_2))$ , thus  $\mathcal{A}_{q,n} = (U_n(\xi_1) \setminus T^{-1}(U_n(\xi_2))) \cup U_n(\xi_2)$ . Hence,

$$\theta = \lim_{n \rightarrow \infty} \frac{\text{Leb}(\mathcal{A}_{q,n})}{\text{Leb}(U_n)} = \lim_{n \rightarrow \infty} \frac{2r_n - 2 \times \alpha r_n / 3 + 2\alpha r_n}{2r_n + 2\alpha r_n} = \frac{1 + \frac{2}{3}\alpha}{1 + \alpha}$$

Finally, achieving the following expression of  $\theta$  which is represented in the Figure 7.

$$\theta = \begin{cases} \frac{1 + \frac{2}{3}\alpha}{1 + \alpha}, & 0 \leq \alpha < 3 \\ \frac{\alpha}{1 + \alpha}, & 3 \leq \alpha \end{cases}$$

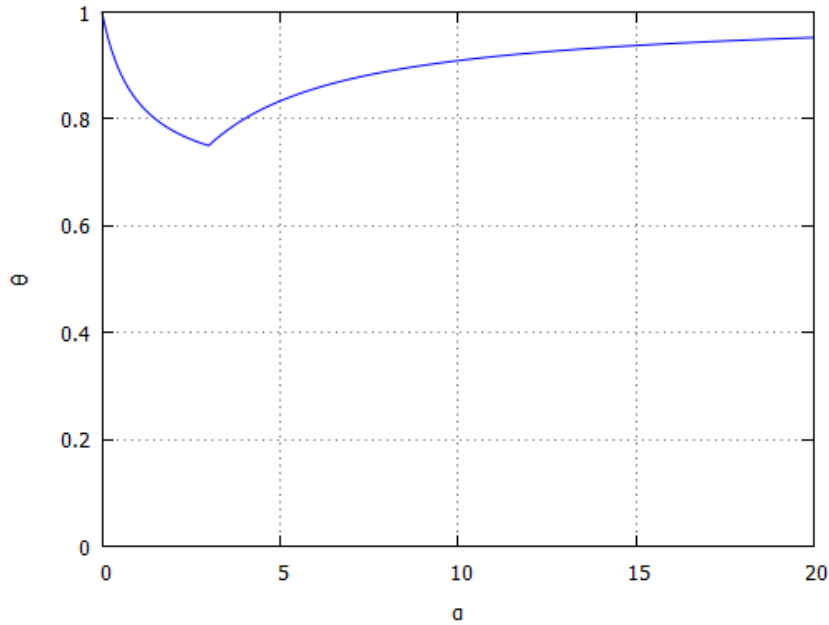


**Figure 7:** Graph of the values of  $\theta$  for  $T(x) = 3x \bmod 1$  and two maxima along a non-periodic orbit.

Observe that  $\frac{\alpha}{1+\alpha} \rightarrow 1$  as  $\alpha \rightarrow +\infty$ , as expected, since the orbit is non-periodic.

Therefore, by the Theorem 2.1.1, we get that  $\lim_{n \rightarrow \infty} \mu(M_n \leq u_n) = e^{-\theta\tau}$ , where  $\theta$  is given by the expression above.

Although this example is similar to the Example 2, the result is very different. The main difference between Figure 3 and Figure 7, is that there are just two parts in the case of the non-periodic orbit. On account of the fact that the two possible cases, in comparison to the Example 2, correspond to the increasing and decreasing parts, there is no constant part.

## 7 Computing the clustering distribution

This section, as in the previous, we consider the dynamical system  $(\mathcal{X}, \mathcal{B}, \mu, T)$  described in the first section, and an observable  $\varphi$  with  $N$  maxima,  $\xi_i, i \in \{1, \dots, N\}$ , which belongs to the same orbit as in (27). In this section we assume that the stochastic process  $X_0, X_1, \dots$ , as is described in (1), satisfies the conditions  $\mathbb{D}_q(u_n)^*$  and  $\mathbb{D}'_q(u_n)$ .

Next, we will compute the multiplicity distribution  $\pi(\kappa)$ ,  $\kappa \in \mathbb{N}$ , in several examples that we revisit, using the formula in Theorem 3.3.2.1 and the formalism introduced in Section 4.

### 7.0.1 Example 1.1 (Continuation).

In this example, as we have seen before, we consider three cases, and note that, the cases when  $0 \leq \alpha \leq 0.5$  and  $2 \leq \alpha$  are similar.

Consider  $0.5 \leq \alpha \leq 2$ , then  $T^{-1}(U_n(\xi_1)) \cap U_n = B_{\frac{r_n}{2}}(\xi_2) \subseteq B_{\alpha r_n}(\xi_2) = U_n(\xi_2)$  and  $T^{-1}(U_n(\xi_2)) \cap U_n = B_{\frac{\alpha r_n}{2}}(\xi_1) \subseteq B_{r_n}(\xi_1) = U_n(\xi_1)$ , so  $T^{-1}(U_n) \subseteq U_n$ , which means that  $q = 1$  and if a zero appears in the sequence, it is the end. So we get simply:

$$\mathcal{A}_{1,n}^{(0)} = h^{-1}(C(10)) = \{X_0 > u_n, X_1 \leq u_n\},$$

And, for  $\kappa \in \mathbb{N}$

$$\mathcal{A}_{1,n}^{(\kappa)} = h^{-1}C(\underbrace{1 \cdots 1}_{\kappa+1} 0) = \{X_0 > u_n, \dots, X_\kappa > u_n, X_{\kappa+1} \leq u_n\}.$$

It is easy to see that these sets correspond to  $T^{-\kappa}(U_n) \setminus T^{-(\kappa+1)}(U_n)$ , which means that  $\mathcal{A}_{1,n}^{(\kappa)} = (B_{r_n/2^\kappa}(\xi_2) \cup B_{\alpha r_n/2^\kappa}(\xi_1)) \setminus (B_{r_n/2^{\kappa+1}}(\xi_2) \cup B_{\alpha r_n/2^{\kappa+1}}(\xi_1))$ , then

$$\begin{aligned} \mu(\mathcal{A}_{q,n}^{(\kappa)}) &= 0.5^\kappa r_n - 0.5^{\kappa+1} r_n + 0.5^\kappa \alpha r_n - 0.5^{\kappa+1} \alpha r_n \\ &= 0.5^{\kappa+1} r_n + 0.5^{\kappa+1} \alpha r_n = 0.5^{\kappa+1} r_n (1 + \alpha). \end{aligned}$$

Now, we have everything needed to compute the multiplicity distribution  $\pi$  using

(20). For  $\kappa \in \mathbb{N}$

$$\begin{aligned}\pi(\kappa) &= \lim_{n \rightarrow \infty} \pi_n(\kappa) = \lim_{n \rightarrow \infty} \frac{\mu(\mathcal{A}_{q,n}^{(\kappa-1)}) - \mu(\mathcal{A}_{q,n}^{(\kappa)})}{\mu(\mathcal{A}_{q,n}^{(0)})} \\ &= \lim_{n \rightarrow \infty} \frac{0.5^\kappa r_n(1+\alpha) - 0.5^{\kappa+1} r_n(1+\alpha)}{0.5 r_n(1+\alpha)} \\ &= 0.5^{\kappa-1} - 0.5^\kappa = 0.5^\kappa.\end{aligned}$$

We consider now the case  $2 \leq \alpha$ . Recall that,  $q = 2$  and  $T(U_n(\xi_1)) \subset U_n(\xi_2)$ , then if  $x_\ell \in U_n(\xi_1)$ , for some  $\ell \in \mathbb{N}_0$ , then  $h(\dots, x_\ell, x_{\ell+1}, \dots) = \dots 11 \dots$ , or in other words, the sequence  $h(\dots, x_\ell, x_{\ell+1}, \dots) = \dots 10 \dots$ , means that the  $x_\ell$  belongs to  $U_n(\xi_2)$ . Hence when the cluster ends, the last interval from  $U_n$  that it enters is the one centred at  $\xi_2$ .

Observe that the sequence ' $\dots 1011 \dots$ ', in this case, is impossible to occur due to the fact that if we already removed the points in  $U_n(\xi_2)$  which go into to  $U_n(\xi_1)$ , they will not be back to that interval.

Therefore, we get

$$\mathcal{A}_{2,n}^{(0)} = h^{-1}(C(100)) = B_{\alpha r_n}(\xi_2) \setminus B_{\alpha r_n/4}(\xi_2)$$

and for  $\kappa \in \mathbb{N}$ :

$$\mathcal{A}_{2,n}^{(\kappa)} = h^{-1}(C(\underbrace{1 \dots 1}_{\kappa+1} 00) \cup C(\underbrace{1 \dots 1}_{\kappa} 0100) \cup \dots \cup C(\underbrace{10 \dots 101}_{\kappa+1 \text{ 1's e } \kappa \text{ 0's}} 00)).$$

Consider  $\kappa = 1$ , we get

$$\begin{aligned}\mathcal{A}_{2,n}^{(1)} &= h^{-1}(C(1100) \cup C(10100)) = \\ &= \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n\} \cup \\ &\cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 \leq u_n, X_4 \leq u_n\}.\end{aligned}$$

For the first event,  $\{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 \leq u_n\}$ , we have that  $X_1 \in U_n(\xi_2)$ , this means that  $X_0 \in U_n(\xi_1)$ ,  $X_2, X_3 \notin U_n$ . Therefore, the points which

satisfy these conditions belong to

$$\left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{4}}(\xi_1) \cup B_{\frac{\alpha r_n}{8}}(\xi_1) \right) = B_{r_n}(\xi_1) \setminus B_{\frac{\alpha r_n}{8}}(\xi_1).$$

For the second,  $X_1, X_3, X_4 \notin U_n$  and  $X_2 \in U_n(\xi_2) \Rightarrow X_0 \in U_n(\xi_2)$ , corresponding to

$$\begin{aligned} & \left( B_{\alpha r_n}(\xi_2) \cap B_{\frac{\alpha r_n}{4}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{r_n}{8}}(\xi_2) \cup B_{\frac{\alpha r_n}{16}}(\xi_2) \right) \\ &= B_{\frac{\alpha r_n}{4}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{8}}(\xi_2) \right). \end{aligned}$$

Having

$$\mathcal{A}_{2,n}^{(1)} = \left( B_{r_n}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^3}}(\xi_1) \right) \cup \left( B_{\frac{\alpha r_n}{2^2}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^4}}(\xi_2) \right) \right).$$

Assume  $\kappa = 2$ ,

$$\begin{aligned} \mathcal{A}_{2,n}^{(2)} &= h^{-1}(C(11100) \cup C(110100) \cup C(1010100)) = \\ &= \{X_0 > u_n, X_1 > u_n, X_2 > u_n, X_3 \leq u_n, X_4 \leq u_n\} \cup \\ &\quad \cup \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 > u_n, X_4 \leq u_n, X_5 \leq u_n\} \cup \\ &\quad \cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 \leq u_n, X_4 > u_n, X_5 \leq u_n, X_6 \leq u_n\}. \end{aligned}$$

Thus, the first event has  $X_3, X_4 \notin U_n$  and  $X_2 \in U_n(\xi_2)$ , which implies that  $X_1 \in U_n(\xi_1)$  and  $X_0 \in U_n(\xi_2)$ , then these points correspond to

$$\left( B_{\alpha r_n}(\xi_2) \cap B_{\frac{r_n}{2}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^2}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^3}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^4}}(\xi_2) \right) = B_{\frac{r_n}{2}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^4}}(\xi_2).$$

The second implies that  $X_2, X_4, X_5 \notin U_n$ ,  $X_3, X_1 \in U_n(\xi_2)$  and  $X_0 \in U_n(\xi_1)$ , from which we obtain

$$\begin{aligned} & \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2}}(\xi_1) \cap B_{\frac{\alpha r_n}{2^3}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{r_n}{2^4}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \\ &= \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^3}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right). \end{aligned}$$

For the third, we have  $X_1, X_3, X_5, X_6 \notin U_n$  and  $X_4, X_2, X_0 \in U_n(\xi_2)$ , leading to

$$\begin{aligned} & \left( B_{\alpha r_n}(\xi_2) \cap B_{\frac{\alpha r_n}{2^2}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^4}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{r_n}{2^3}}(\xi_2) \cup B_{\frac{r_n}{2^5}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^6}}(\xi_2) \right) \\ &= B_{\frac{\alpha r_n}{2^4}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^6}}(\xi_2) \right). \end{aligned}$$

Therefore we can write:

$$\begin{aligned} \mathcal{A}_{2,n}^{(2)} = & B_{\frac{r_n}{2}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^4}}(\xi_2) \cup \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^3}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \cup \\ & \cup B_{\frac{\alpha r_n}{2^4}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^6}}(\xi_2) \right). \end{aligned}$$

Now, let  $\kappa = 3$ ,

$$\begin{aligned} \mathcal{A}_{2,n}^{(3)} = & h^{-1}(C(111100) \cup C(1110100) \cup C(11010100) \cup C(101010100)) = \\ & = \{X_0 > u_n, X_1 > u_n, X_2 > u_n, X_3 > u_n, X_4 \leq u_n, X_5 \leq u_n\} \cup \\ & \cup \{X_0 > u_n, X_1 > u_n, X_2 > u_n, X_3 \leq u_n, X_4 > u_n, X_5 \leq u_n, X_6 \leq u_n\} \cup \\ & \cup \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n, X_3 > u_n, X_4 \leq u_n, X_5 > u_n, X_6 \leq u_n, \\ & \quad X_7 \leq u_n\} \cup \\ & \cup \{X_0 > u_n, X_1 \leq u_n, X_2 > u_n, X_3 \leq u_n, X_4 > u_n, X_5 \leq u_n, X_6 > u_n, \\ & \quad X_7 \leq u_n, X_8 \leq u_n\}. \end{aligned}$$

Using the same arguments as before, we achieve:

$$\begin{aligned} \mathcal{A}_{2,n}^{(3)} = & B_{\frac{r_n}{2^2}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^5}}(\xi_1) \cup \left( B_{\frac{r_n}{2}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^4}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^3}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^6}}(\xi_2) \right) \cup \\ & \cup \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^7}}(\xi_1) \right) \\ & \cup B_{\frac{\alpha r_n}{2^6}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^8}}(\xi_2) \right). \end{aligned}$$

By induction we have that, for  $k \in \mathbb{N}$ :

$$\mathcal{A}_{2,n}^{(2k)} = \left( B_{\frac{r_n}{2^{2k-1}}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^{2k}}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^{2k+1}}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^{2(k+1)}}}(\xi_2) \right) \cup \quad (i)$$

$$\cup \left( B_{\frac{r_n}{2^{2k-2}}}(\xi_1) \cap B_{\frac{\alpha r_n}{2^{2k+1}}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^{2k}}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{2(k+1)+1}}}(\xi_1) \right) \cup \quad (ii)$$

$$\cup \dots \cup \left( B_{\frac{r_n}{2}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^{4k-2}}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^3}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^{4k}}}(\xi_2) \right) \cup \quad (iii)$$

$$\cup \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^{4k-1}}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{4k+1}}}(\xi_1) \right); \quad (\text{iv})$$

And for  $k \in \mathbb{N}_0$ ,

$$\mathcal{A}_{2,n}^{(2k+1)} = \left( B_{\frac{r_n}{2^{2k}}}(\xi_1) \cap B_{\frac{\alpha r_n}{2^{2k+1}}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^{2k+2}}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{2(k+1)+1}}}(\xi_1) \right) \cup \quad (\text{i})$$

$$\cup \left( B_{\frac{r_n}{2^{2k-1}}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^{2k+2}}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^{2k+1}}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^{2(k+1)+2}}}(\xi_2) \right) \cup \quad (\text{ii})$$

$$\cup \dots \cup \left( B_{\frac{r_n}{2}}(\xi_2) \cap B_{\frac{\alpha r_n}{2^{4k}}}(\xi_2) \right) \setminus \left( B_{\frac{r_n}{2^3}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^{4k+2}}}(\xi_2) \right) \cup \quad (\text{iii})$$

$$\cup \left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^{4k+1}}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{4k+3}}}(\xi_1) \right) \cup \quad (\text{iv})$$

$$\cup B_{\frac{\alpha r_n}{2^{4k+2}}}(\xi_2) \setminus \left( B_{\frac{r_n}{2}}(\xi_2) \cup B_{\frac{\alpha r_n}{2^{4k+4}}}(\xi_2) \right). \quad (\text{v})$$

Observe that for  $\kappa$  odd or even, we get that the expression in (i) it is either non-empty or not depending on  $\alpha r_n \leq 2^3 r_n$  or not. For the second expression, the determining condition is  $2^3 r_n < \alpha r_n$ . Then the next condition is whether  $2^5 r_n < \alpha r_n$ . If it holds the result is an empty set, else it is non-empty; in the case that  $\alpha r_n \leq 2^3 r_n$ , it is also non-empty set. The third, is necessary to compare  $2^5 r_n$  with  $\alpha r_n$ , and with the same arguments, next we should know which one is smaller between  $2^7 r_n$  and  $\alpha r_n$ , obtaining similar conclusion, where the expression is not empty if  $2^5 < \alpha \leq 2^7$  or  $2^3 < \alpha \leq 2^5$ , and this is repeated successively.

As we can note, for example, if we want to know to compute  $\mathcal{A}_{2,n}^\kappa(\xi_1)$  with  $\kappa$  odd the expressions that we need to evaluate correspond to the odd ones (if  $\kappa$  is even, then we need the even ones), so for the first expression to be non-empty we need to have  $2^3 < \alpha \leq 2^5$  or  $2 < \alpha \leq 2^3$ , and for the third,  $2^7 < \alpha \leq 2^9$  or  $2^5 < \alpha \leq 2^7$ , this means that at least one of them must be an empty set, and this happens for all the other expressions, which means that in the end there will just be at most one non-empty expression for each point.

Due to the fact that  $\alpha > 2$ , sometimes  $\mathcal{A}_{2,n}^\kappa(\xi_1) = \emptyset$ , which does not happen for  $\mathcal{A}_{2,n}^\kappa(\xi_2)$ . For example, consider  $2 < \alpha \leq 2^3$ , in  $\mathcal{A}_{2,n}^{(3)}$ ,  $B_{\frac{r_n}{2^2}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^5}}(\xi_1) \neq \emptyset$  and

$$\left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^7}}(\xi_1) \right) = B_{\frac{\alpha r_n}{2^5}}(\xi_1) \setminus B_{\frac{r_n}{2^2}}(\xi_1) = \emptyset,$$

obtaining  $\mathcal{A}_{2,n}^{(3)}(\xi_1) = B_{\frac{r_n}{2^2}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^5}}(\xi_1)$ . Now, let  $2^3 < \alpha \leq 2^5$ , so  $B_{\frac{r_n}{2^2}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^5}}(\xi_1) = \emptyset$  and

$$\left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^7}}(\xi_1) \right) = B_{\frac{\alpha r_n}{2^5}}(\xi_1) \setminus B_{\frac{r_n}{2^2}}(\xi_1) \neq \emptyset,$$

then  $\mathcal{A}_{2,n}^{(3)}(\xi_1) = B_{\frac{\alpha r_n}{2^5}}(\xi_1) \setminus B_{\frac{r_n}{2^2}}(\xi_1)$  instead. However, if  $2^7 < \alpha \leq 2^9$ ,  $\mathcal{A}_{2,n}^{(3)}(\xi_1) = \emptyset$ , because  $B_{\frac{r_n}{2^2}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^5}}(\xi_1) = \emptyset$  and

$$\left( B_{r_n}(\xi_1) \cap B_{\frac{\alpha r_n}{2^5}}(\xi_1) \right) \setminus \left( B_{\frac{r_n}{2^2}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^7}}(\xi_1) \right) = B_{r_n}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^7}}(\xi_1) = \emptyset.$$

As we can see,  $\mathcal{A}_{2,n}^{(3)}(\xi_1)$  will be an empty set for  $\alpha > 2^7 = 2^{2 \times 3 + 1}$ , which happens for all  $\kappa \in \mathbb{N}$ ,  $\mathcal{A}_{2,n}^{(\kappa)}(\xi_1) = \emptyset$  if  $\alpha > 2^{2\kappa+1}$ .

Actually, it is better to split into further two cases, namely, consider  $b \in \mathbb{N}_0$ , when  $2^{4b+1} < \alpha \leq 2^{4b+3}$ , whose components of the set  $U_n(\xi_1)$  appears for the first time in  $\mathcal{A}_{2,n}^{(\kappa)}$  is with an odd  $\kappa$ , and  $2^{4b+3} < \alpha \leq 2^{4b+5}$ , whose the first appearance is with an even  $\kappa$ . For a better view of these arguments, see Figure 8.

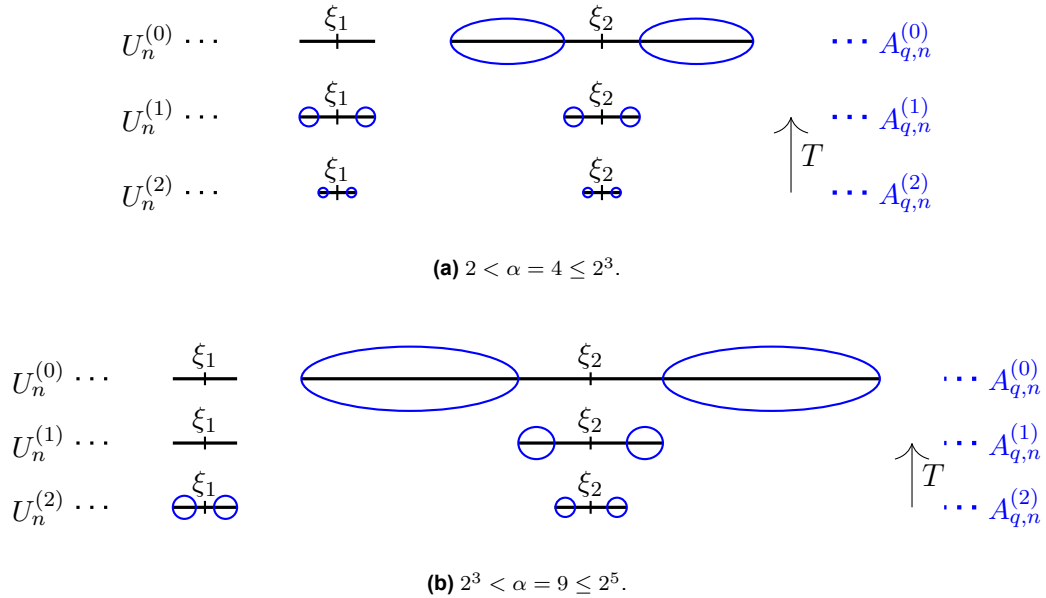


**Figure 8:** Set  $U_n^{(\kappa)}$  and  $\mathcal{A}_{2,n}^{(\kappa)}$ , for  $\kappa \in \{0, 1, 2\}$ .

Let us analyse the first case,  $2^{4b+1} < \alpha \leq 2^{4b+3}$ , for a  $b \in \mathbb{N}_0$ . Since  $2^{2\kappa+1} \leq$

$2^{2b+1} \iff \kappa \leq 2b$ , the set  $\mathcal{A}_{2,n}^{(\kappa)}(\xi_1)$  is empty. Therefore, for  $0 \leq k \leq 2b$ :

$$\mathcal{A}_{2,n}^{(k)} = B_{\frac{\alpha r_n}{2^{2k}}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^{2(k+1)}}}(\xi_2),$$

whose measure is

$$\mu\left(\mathcal{A}_{2,n}^{(k)}\right) = 2 \times \frac{\alpha r_n}{2^{2k}} - 2 \times \frac{\alpha r_n}{2^{2(k+1)}} = 2r_n \times \alpha(0.5^{2k} - 0.5^{2k+1}) = 2r_n \times \alpha(0.5^{2k+1}).$$

Thus, for  $\kappa$  even and  $\kappa > 2b$ , we have

$$\mathcal{A}_{2,n}^{(2k)} = B_{\frac{\alpha r_n}{2^{2k+1+2b}}}(\xi_1) \setminus B_{\frac{r_n}{2^{2k-2b}}}(\xi_1) \cup B_{\frac{r_n}{2^{2k-1-2b}}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^{2k+2+2b}}}(\xi_2), \quad k \geq b,$$

leading to

$$\begin{aligned} \mu\left(\mathcal{A}_{2,n}^{(2k)}\right) &= 2r_n \times (0.5^{2k+1+2b}\alpha - 0.5^{2k-2b} + 0.5^{2k-1-2b} - 0.5^{2k+2+2b}\alpha) \\ &= 2r_n \times (0.5^{2k-2b} + 0.5^{2k+2+2b}\alpha). \end{aligned}$$

For  $\kappa$  odd and  $\kappa > 2b$  we have,

$$\mathcal{A}_{2,n}^{(2k+1)} = B_{\frac{r_n}{2^{2k-2b}}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^{2k+3+2b}}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{2k+2+2b}}}(\xi_2) \setminus B_{\frac{r_n}{2^{2k+1-2b}}}(\xi_2), \quad k \geq b,$$

leading to

$$\begin{aligned} \mu\left(\mathcal{A}_{2,n}^{(2k+1)}\right) &= 2r_n \times (0.5^{2k-2b} - 0.5^{2k+3+2b}\alpha + 0.5^{2k+2+2b}\alpha - 0.5^{2k+1-2b}) \\ &= 2r_n \times (0.5^{2k+1-2b} + 0.5^{2k+3+2b}\alpha). \end{aligned}$$

Now, we have all the information needed to compute the multiplicity distribution, using (20). Consider  $\kappa \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$ :  $\pi(\kappa) = \lim_{n \rightarrow \infty} \pi_n(\kappa) = \lim_{n \rightarrow \infty} \frac{\mu(\mathcal{A}_{q,n}^{(\kappa-1)}) - \mu(\mathcal{A}_{q,n}^{(\kappa)})}{\mu(\mathcal{A}_{q,n}^{(0)})} =$

$$= \begin{cases} \lim_{n \rightarrow \infty} \frac{2r_n \times \alpha(0.5^{2\kappa-1} - 0.5^{2\kappa+1})}{2r_n \times \alpha(0.75)} = \frac{0.5^{2\kappa-1}(0.75)}{0.75} = 0.5^{2\kappa-1}, 1 \leq \kappa \leq 2b \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{4b+1}\alpha - 0.5 - 0.5^{4b+3}\alpha)}{2r_n \times \alpha(0.75)} = \frac{0.75 \times 0.5^{4b+1}\alpha - 0.5}{0.75\alpha} = 0.5^{4b+1} - \frac{2}{3\alpha}, \kappa = 2b+1 \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{2k-2b} + 0.5^{2k+2+2b}\alpha - 0.5^{2k+1-2b} - 0.5^{2k+3+2b}\alpha)}{2r_n \times \alpha(0.75)} = \\ = \frac{0.5^{2k+1-2b} + 0.5^{2k+3+2b}\alpha}{0.75\alpha}, \kappa > 2b \text{ and is odd} \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{2k-1-2b} + 0.5^{2k+1+2b}\alpha - 0.5^{2k-2b} - 0.5^{2k+2+2b}\alpha)}{2r_n \times \alpha(0.75)} = \\ = \frac{0.5^{2k-2b} + 0.5^{2k+2+2b}\alpha}{0.75\alpha}, \kappa > 2b \text{ and is even.} \end{cases}$$

Now, consider that  $2^{4b+3} < \alpha \leq 2^{4b+5}$  for  $b \in \mathbb{N}_0$ . Like in the previous case,  $U_n(\xi_1)$  will only appear for  $\kappa > 2b+1$ , because  $(U_n \cap T^{-(4b+3)}(U_n(\xi_2))) \subseteq U_n(\xi_1) \subseteq T^{-(4b+1)}(U_n(\xi_2))$ . Thus, for  $0 \leq k \leq 2b+1$ :

$$\mathcal{A}_{2,n}^{(k)} = B_{\frac{\alpha r_n}{2^{2k}}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^{2(\kappa+1)}}}(\xi_2),$$

whose measure is:  $\mu(\mathcal{A}_{2,n}^{(k)}) = 2r_n \times \alpha 0.5^{2k+1}$ .

Therefore, with the same arguments as before, for  $k > b$ :

$$\mathcal{A}_n^{(2k)} = B_{\frac{r_n}{2^{2k-2-2b}}}(\xi_1) \setminus B_{\frac{\alpha r_n}{2^{2k+3+2b}}}(\xi_1) \cup B_{\frac{\alpha r_n}{2^{2k+2+2b}}}(\xi_2) \setminus B_{\frac{r_n}{2^{2k-1-2b}}}(\xi_2),$$

$$\mathcal{A}_n^{(2k+1)} = B_{\frac{\alpha r_n}{2^{2k+3+2b}}}(\xi_1) \setminus B_{\frac{r_n}{2^{2k-2b}}}(\xi_1) \cup B_{\frac{r_n}{2^{2k-1-2b}}}(\xi_2) \setminus B_{\frac{\alpha r_n}{2^{2k+4+2b}}}(\xi_2),$$

leading to:

$$\begin{aligned} \mu(\mathcal{A}_n^{(2k)}) &= 2r_n \times (0.5^{2k-2-2b} - 0.5^{2k+3+2b}\alpha + 0.5^{2k+2+2b}\alpha - 0.5^{2k-1-2b}) \\ &= 2r_n \times (0.5^{2k-1-2b} + 0.5^{2k+3+2b}\alpha), \\ \mu(\mathcal{A}_n^{(2k+1)}) &= 2r_n \times (0.5^{2k+3+2b}\alpha - 0.5^{2k-2b} + 0.5^{2k-1-2b} - 0.5^{2k+4+2b}\alpha) \\ &= 2r_n \times (0.5^{2k-2b} + 0.5^{2k+4+2b}\alpha). \end{aligned}$$

Hence, the multiplicity distribution is

$$\left\{ \begin{array}{l} 0.5^{2\kappa-1}, 1 \leq \kappa \leq 2b \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{4b+3}\alpha - 0.5^1 - 0.5^{4b+5}\alpha)}{2r_n \times \alpha(0.75)} = \frac{0.75 \times 0.5^{4b+3}\alpha - 0.5}{0.75\alpha} = 0.5^{4b+3} - \frac{2}{3\alpha}, \kappa = 2b + 2 \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{2k-1-2b} + 0.5^{2k+3+2b}\alpha - 0.5^{2k-2b} - 0.5^{2k+4+2b}\alpha)}{2r_n \times 0.75\alpha} = \\ = \frac{0.5^{2k-2b} + 0.5^{2k+4+2b}\alpha}{0.75\alpha}, \kappa > 2b \text{ and is odd} \\ \lim_{n \rightarrow \infty} \frac{2r_n \times (0.5^{2k-2-2b} + 0.5^{2k+2+2b}\alpha - 0.5^{2k-1-2b} - 0.5^{2k+3+2b}\alpha)}{2r_n \times 0.75\alpha} = \\ = \frac{0.5^{2k-1-2b} + 0.5^{2k+3+2b}\alpha}{0.75\alpha}, \kappa > 2b \text{ and is even.} \end{array} \right.$$

For the case  $0 \leq \alpha \leq 0.5$ , mimicking the arguments above we can compute an expression for the  $\pi(\kappa)$ ,  $\kappa \in \mathbb{N}$ .

This is rather a complex case, considering that it is a periodic orbit, the multiplicity will never be zero. Due to the fact that the closer to the center, more likely to behave like the periodic point that is close to.

The next example is a simpler case with a non-periodic orbit.

#### 7.0.2 Example 4 (Continuation).

Recall that this example set  $\mathcal{A}_{q,n}^{(0)}$  depends on whether  $0 \leq \alpha < 3$  or  $\alpha \geq 3$ . Either way, because of the non-periodic behaviour and  $\xi_2 = T(\xi_1)$ ,  $q = 1$ . Using (21) with  $q = 1$ , we get:

$$\begin{aligned} \mathcal{A}_{1,n}^{(0)} &= h^{-1}(C(10)) = \{X_0 > u_n, X_1 \leq u_n\}, \\ \mathcal{A}_{1,n}^{(1)} &= h^{-1}(C(110)) = \{X_0 > u_n, X_1 > u_n, X_2 \leq u_n\}, \\ \mathcal{A}_{1,n}^{(2)} &= h^{-1}(C(1110)) = \{X_0 > u_n, X_1 > u_n, X_2 > u_n, X_3 \leq u_n\}, \\ &\dots \\ \mathcal{A}_{1,n}^{(\kappa)} &= h^{-1}(C(\underbrace{11\dots 1}_{\kappa+1}0)) = \{X_0 > u_n, X_1 > u_n, \dots, X_\kappa > u_n, X_{\kappa+1} \leq u_n\}. \end{aligned}$$

Observe that, the sets  $T^{-3}(U_n(\xi_2))$  and  $T^{-3}(U_n(\xi_1))$  do not intersect with the set  $U_n$ , therefore for the sequences ' $\dots 11\dots 10\dots$ ', with more than three successive

'1': are impossible to occur. This means that  $\mathcal{A}_{1,n}^{(\kappa)} = \emptyset$ , for  $\kappa \geq 2$ .

Consider  $\alpha \geq 3$ , having  $U_n(\xi_1) \subseteq T^{-1}(U_n(\xi_2))$ , this implies that if  $x_\ell \in U_n$ , for some  $\ell \in \mathbb{N}_0$ , and is such that  $h(\dots, x_\ell, x_{\ell+1}, \dots) = \dots 10 \dots$ , then  $x_\ell \in U_n(\xi_2)$ .

Therefore we get

$$\mathcal{A}_{1,n}^{(0)} = U_n(\xi_2) = B_{\alpha r_n}(\xi_2),$$

whose measure is  $\mu(\mathcal{A}_{1,n}^{(0)}) = 2\alpha r_n$ , and

$$\mathcal{A}_{1,n}^{(1)} = U_n(\xi_1) = B_{r_n}(\xi_1),$$

with measure  $\mu(\mathcal{A}_{1,n}^{(1)}) = 2r_n$ .

Thus, we already know the sets to compute the multiplicity distribution using (20),

$$\pi(\kappa) = \begin{cases} \lim_{n \rightarrow \infty} \frac{2\alpha r_n - 2r_n}{2\alpha r_n} = \frac{\alpha-1}{\alpha}, & \kappa = 1 \\ \lim_{n \rightarrow \infty} \frac{2r_n - 0}{2\alpha r_n} = \frac{1}{\alpha}, & \kappa = 2 \\ 0, & \kappa \geq 3 \end{cases}.$$

Now, let  $0 \leq \alpha < 3$ , so that  $(T^{-1}(U_n(\xi_2)) \cap U_n) \subset U_n(\xi_1)$ . This means that for  $h(\dots, x_\ell, x_{\ell+1}, \dots) = \dots 10 \dots$ ,  $x_\ell$  can belong to  $U_n(\xi_1)$ . This gives rise to:

$$\mathcal{A}_{1,n}^{(0)} = (U_n(\xi_1) \setminus T^{-1}(U_n(\xi_2))) \cup U_n(\xi_2) = (B_{r_n}(\xi_1) \setminus B_{\frac{\alpha r_n}{3}}(\xi_2)) \cup B_{\alpha r_n}(\xi_2),$$

with measure of  $\mu(\mathcal{A}_{1,n}^{(0)}) = 2r_n - \frac{2}{3}\alpha + 2\alpha r_n = 2r_n(1 + \frac{2}{3}\alpha)$ , and

$$\mathcal{A}_{1,n}^{(1)} = T^{-1}(U_n(\xi_2)) = B_{\frac{\alpha r_n}{3}}(\xi_1),$$

leading to the measure  $\mu(\mathcal{A}_{1,n}^{(1)}) = 2 \times \frac{\alpha r_n}{3}$ .

Therefore, we get the multiplicity distribution:

$$\pi(\kappa) = \begin{cases} \lim_{n \rightarrow \infty} \frac{2r_n(1 + \frac{2}{3}\alpha) - 2 \times \frac{\alpha r_n}{3}}{2r_n(1 + \frac{2}{3}\alpha)} = \frac{1+\alpha}{1+\frac{2}{3}\alpha} = \frac{3+\alpha}{3+2\alpha}, & \kappa = 1 \\ \lim_{n \rightarrow \infty} \frac{2 \times \frac{\alpha r_n}{3} - 0}{2r_n(1 + \frac{2}{3}\alpha)} = \frac{\frac{\alpha}{3}}{1+\frac{2}{3}\alpha} = \frac{\alpha}{3+2\alpha}, & \kappa = 2 \\ 0, & \kappa \geq 3 \end{cases}.$$

As expected, after  $\kappa$  exceeds  $N$  the multiplicity will be zero. In this case, since  $N = 2$ , after  $\kappa \geq 3$  we have  $\pi(\kappa) = 0$ . This is predictable, because it is a non-periodic orbit, thus after surpassing the last maximum is unlikely that to return to  $U_n$ .

## Conclusion

In this work we study the behaviour of the stochastic processes  $X_0, X_1, \dots$  generated by the time evolution of a dynamical system evaluated by an observable. To be more concrete, we studied the emergence of clusters of rare events. These clusters are identified by observing exceedances of a high thresholds. For that purpose, we saw conditions that allow us use Extreme Value Theory, in order to analyze the probability distribution of the process of partial maxima of  $X_0, X_1, \dots$ , and rare events point processes, used to model and analyse the occurrence of extreme events.

Extreme value theory gives us the limiting distribution function of the process of partial maxima  $(M_n)_{n \in \mathbb{N}}$ , originally for independent and identically distributed random variables. Namely, for a normalized sequence  $(u_n)_{n \in \mathbb{N}}$  such that  $n\mu(X_0 > u_n) \rightarrow \tau$  as  $n \rightarrow \infty$ , and  $\mu(M_n \leq u_n) \rightarrow e^{-\tau}$  as  $n \rightarrow \infty$ . If the stochastic processes  $X_0, X_1, \dots$  satisfies conditions  $\Delta_q(u_n)$  and  $\Delta'_q(u_n)$  we can recover the result from the i.i.d. case. Furthermore, whenever we have the presence of clusters, the limiting distribution is weighed by the extremal index,  $0 \leq \theta \leq 1$ , which is, in general, equal to the inverse of the average size of the clusters.

The rare event point processes, given a sample of size  $n$ , count the number of times that the threshold is surpassed. Therefore, its convergence gives us the information of the expected time between the occurrence of extreme events. Moreover, for a normalized sequence of threshold  $(u_n)_{n \in \mathbb{N}}$  such as in (5), the sequence  $(N_n)_{n \in \mathbb{N}}$ , in case of absence of clusters, converges to a homogeneous Poisson process of intensity  $\tau$ . Otherwise, the REPP, that is under the conditions  $\Delta_q(u_n)^*$  and  $\Delta'_q(u_n)$ , converges to a compound Poisson process with intensity  $\theta\tau$ , where  $\theta$  is the EI, and multiplicity distribution  $\pi(\kappa)$ .

As we could see, in the presence of clusters one needs to check if the conditions  $\Delta_q(u_n)/\Delta_q(u_n)^*$  and  $\Delta'_q(u_n)$  hold. Since the verification of those conditions is not always easy, we considered Assumption A, that implies those conditions. The main part of this assumption is that the system has summable decay of correlation against  $\mathbb{L}^1$  observables, furthermore, we show some examples of systems that have this property.

Moreover, considering systems satisfying the Assumption A, and observables were the maxima belongs to the same orbit, we compute the limits needed for the theorems

which give us the asymptotic distribution, that is, the  $\theta = \lim_{n \rightarrow \infty} \theta_n$  and  $\pi = \lim_{n \rightarrow \infty} \pi_n$ . We remark that in cases that the orbit is periodic, the EI is smaller comparing with the non-periodic case. Since the periodic behaviour creates multiple exceedances it is unlikely that  $\theta$  will be near 1, as we can see from the expression we developed, that only depends on the proportions of the radius between the balls that compose  $U_n$ . Furthermore, the clustering distribution will not be zero, whenever the orbit is periodic. On the other hand, for a non-periodic orbit, the cluster size will never be larger than the number of maxima. The expression we obtained, depends again on the proportions of the radius and  $\kappa$ .

In conclusion, whenever we have a dynamical systems satisfying Assumption A, and we have an observable that satisfies (R1) and (R3), we can study the behaviour of clustering of rare events using EVT and the convergence of REPP.

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