Chern-Simons and Holographic Gravity Theories

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This thesis represents the culmination of my five-years long journey through higher education. It was not an easy path to traverse. Indeed, it was not anything remotely close to a simple journey, but rather one plagued with hardships, doubts and struggles.
Yet, despite all the rough moments, theoretical physics was a goal of mine since since I first had the subject back in 7th grade and one I do not regret pursuing. It challenged me beyond any expectations, but it also brought me joy and fulfillment.

Any measure of success is due, in large part, to the support and care of the people around me. Without them, this work would not exist.

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Abstract

Faculdade de Ciências da Universidade do Porto

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Chern-Simons and Holographic Gravity Theories

by Guilherme Moreira

In this thesis we study a three dimensional conformal field theory based on a SU(N) Chern-Simons gauge field coupled to fermionic matter transforming in the adjoint representation of the gauge group. We address the question of whether this simple model in the 't Hooft limit exhibits features expected of a holographic dual of a theory of Einstein gravity in AdS_4.

The theory studied in this thesis is non-supersymmetric and contains a large N parameter that provides a limit in which we perform a perturbative expansion in order to analyse the spectrum of conformal dimensions of local operators. Concretely, we focus on computing the 2-loop anomalous dimension of the simplest single-trace scalar operator given by $\mathcal{O}_0(x) = \mathrm{Tr}\left[\bar{\Psi}(x)\Psi(x)\right]$, successfully verifying it obeys the behaviour conjectured for an unprotected operator in a theory with a dual.

This calculation is done in the framework of the AdS/CFT correspondence, which we briefly explain. We motivate our work by reiterating the relevance of this duality and how it demands that the conformal dimension of operators behave.

We explain what a CFT is and the restrictions conformal symmetry imposes on the correlation functions of these quantum field theories.

We then go over Chern-Simons theory and explore the constraints that must be imposed on its level for the theory to be gauge-invariant. Afterwards, we follow with an explanation of how the Large-N limit can be used to simplify the calculations and with a derivation of

the Feynman rules used in our computations.

The computation of anomalous dimension is revisited through the familiar ϕ^4-theory. After explaining the approach that will be followed, we compute the anomalous dimension of $\mathcal{O}_0(x)$ at 1-loop, concluding it is 0. We then carry out the computation at 2-loops, going over the IBP Reduction method and Asymptotic Expansion technique as tools for evaluating the required Feynman integrals.

Lastly, we attempt to validate the result by verifying that the conserved current $\mathcal{O}_1^{\mu}(x) = \mathrm{Tr}\left[\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\right]$ does not gain anomalous dimension. We successfully check this holds at 1-loop.

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Resumo

Faculdade de Ciências da Universidade do Porto

Departamento de Física e Astronomia

Mestrado em Física

Chern Simons e Teorias Holográficas de Gravidade

por Guilherme Moreira

Nesta tese estudamos uma teoria de campo comforme baseada num campo de gauge de Chern-Simons com simetria SU(N), acoplada a matéria fermiónica que se transforma na representação adjunta do grupo de gauge. Investigamos se este modelo simples exibe, no limite de 't Hooft, o comportarmento esperado de uma teoria cujo dual é uma teoria de gravidade de Einstein em AdS_4.
A teoria nesta tese é não-supersimétrica e é dotada de um parâmetro N grande que possibilita uma análise perturbativa com o intuito de analisar o espetro de dimensões anómalas de operadores locais. Concretamente, focámo-nos em calcular a dimensão anómala a 2-loops do operador escalar mais simples, $\mathcal{O}_0(x) = \mathrm{Tr}\left[\bar{\Psi}(x)\Psi(x)\right]$, tendo verificado com sucesso que esta se comporta da forma esperada para um operador não-protegido numa teoria com dual.

O cálculo é feito no contexto da correspondência AdS/CFT, que começamos por explicar brevemente. Motivamos o trabalho reiterando a importância desta dualidade e a forma como esta constringe o comportamento da dimensão conforme dos operadores.

Introduzimos ainda teorias de Chern-Simons e explorámos as condição a impôr ao nível da teoria, κ, para que a teoria seja invariante sob transformações de gauge. De seguida, explicamos como a expansão em $1/N$ simplifica os cálculos e derivamos as regras de Feynman que iremos usar.

O cálculo de dimensões anómalas é revisitado no caso da familiar teoria ϕ^4. Após explicar a abordagem, a dimensão anómala é determinada para o operador $\mathcal{O}_0(x)$ a 1-loop,

concluíndo-se que é 0. No cálculo a 2-loops introduzimos o método da redução por integração por partes e o método da expansão assimptótica como técnicas para avaliar os integrais de Feynman.
Verificámos que o operador $\mathcal{O}_0(x)$ ganha dimensão anómala a 2-loops, como esperado.

Por fim, tentámos validar o resultado, verificando que a corrente conservada $\mathcal{O}_1^\mu(x) = \mathrm{Tr}\left[\bar{\Psi}(x)\gamma^\mu\Psi(x)\right]$ não ganha dimensão anómala. Conseguimos confirmar que isto se mantém a 1-loop.

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Glossary

GR	General Relativity
QFT	Quantum Field Theory
QED	Quantum Electrodynamics
AdS	Anti-de Sitter
SYM	Supersymmetric Yang-Mills
CFT	Conformal Field Theory
RG	Renormalization Group
IBP	Integration By Parts
SCT	Special Conformal Transformations

This thesis is dedicated to my family, and in particular to my dad who sadly passed away before I could share it with him.

Chapter 1

Introduction

Modern physics is a fascinating subject, capable of describing the most fundamental of interactions with great precision. This possibility has allowed us to understand the Universe at a deep level, being able to describe the most diverse systems around us. There is physics to describe the behaviour and interactions of the minute elementary particles that form the building blocks of everything that surrounds us and there is physics to describe the attractive force between masses at astronomical scales.

Satisfying as it is to claim having such vast knowledge of the Universe, the previous sentence reflects one pressing tension lying at the heart of physics. Namely, the fact that there are two distinct ”physics” describing the four fundamental forces, with our theory of gravity being incompatible with the theory for the remaining three.

The theory of General Relativity (GR), developed by Albert Einstein (1879-1955), describes how masses and energy warp the space-time background and how, in turn, the curve of the background conditions the movement of bodies. It is this curving of space-time that we understand gravity to be. This theory, while mathematically demanding, paints a conceptually simple picture of the way masses and energy interact through gravity.
This theory has had extensive experimental validation, namely in being able to accurately describe the precession of Mercury, gravitational lensing and, more recently with the confirmation of the existence of gravitational waves [1].

On the other hand, the electromagnetic, weak and strong interactions are described using the framework of Quantum Field Theory (QFT), which is, in itself, no less of a success story. QFT is a paradigm in which particles are thought of as excitations of fundamental fields that permeate the entire Universe and its predictions have been experimentally verified time and time again. Quantum Electrodynamics (QED), for instance, the QFT that describes the electromagnetic interaction has predicted with extremely high accuracy the anomalous magnetic moment of the electron, the lack of mass of photons, amongst other high-precision results.

Nonetheless, despite both General Relativity and QFT being extremely successful frameworks, they are, at hearth, fundamentally incompatible. To understand this, one only has to remark that General Relativity is a classical theory. There is no quantization of space-time. Furthermore, in the Standard Model (the theory that describes fundamental particles) forces are the result of local exchanges of virtual bosons and no evidence for a gravity-carrying particle (the so called graviton) has been found.

Therefore, physics finds itself with a theory that describes really well the smallest of systems and another incompatible yet equally well proven theory that describes the largest of systems. Reconciling these two, and developing a theory of quantum gravity, would be quite remarkable. Countless physicists have attempted to solve this problem and there have been some approaches to a unifying theory, of which, a very promising one is String Theory [2]. It is precisely in the context of string theory that the AdS/CFT correspondence was first discussed, as will be explained below.

1.1 The AdS-CFT Correspondence

String theory, as is the case with QFT, is not so much a theory in itself, but rather a paradigm physicists use to construct their models of reality. In String Theory, particles are thought of as small vibrating strings, that can either be closed or open, and it by construction a quantum theory of gravity. Besides these 1d strings, which are the most fundamental structures of the Universe according to String Theory, there also exist some higher-dimensional objects, the so-called Dp-Branes. These are hyper-planes with p space-like dimensions where open strings can end. By studying a system of N D3-Branes in Type-IIB string theory Maldacena, in what is now a famous paper [3], conjectured a duality between string theory in $AdS_5 \times \mathbb{S}^5$ and $\mathcal{N} = 4$ SYM with $SU(N)$ symmetry. This observation came about from studying the low energy limit of this system and noticing that both theories have the same isometries.

By AdS_5 it is meant the 5-dimensional space with negative curvature known as Anti-de Sitter space, corresponding to the maximally symmetric solution to Einstein's equations with a cosmological constant. $\mathcal{N} = 4$ SYM is a supersymmetric (meaning it has an additional space-time symmetry between bosons and fermions) version of the usual Yang-Mills theory which is, in itself, a non-abelian gauge theory with $SU(N)$ symmetry.
In $\mathcal{N} = 4$ SYM, the $\mathcal{N} = 4$ part is related to the number of supercharges. This is a conformally invariant field theory, meaning it not only has the usual symmetries of the Poincaré group (which are translations, rotations and boosts) it is also endowed with scale invariance and is therefore invariant under the conformal group $SO(d, 1)$.

Thus, Maldacena conjectured a duality between theories of quantum gravity in the bulk of AdS space and conformally invariant gauge theories existing in the boundary of that space. This aspect is in itself remarkable. The fact that the gauge theory exists in the boundary of the space where the gravity theory is defined, means that it has dimension one less than that of the theory in the bulk, while being, nonetheless, equivalent to it. This is precisely the holographic principle, the idea that information about the space can be encoded in its boundary. It gets its name from the technique through which a three dimensional image can be encoded in a 2d surface using light.
This is indeed a powerful tool at our disposal, in the sense that it is possible to study a d-dimension quantum theory of gravity by focusing on a (d-1)-dimensional Conformal Field Theory (CFT) which is, in principle, simpler to study.

This thesis seeks to explore this possibility. We know that we live in a 4-dimensional Universe, meaning that a theory of quantum gravity that reflects physical reality is a 4d theory. Hence, in light of the AdS/CFT correspondence, its dual must be a conformal field theory in 3 dimensions. The simplest possible such theory is a Chern-Simons theory, by virtue of it being a topological field theory. For this reason, we will focus on studying this theory since, if it has a dual theory of gravity in AdS_4, then that would be a good candidate for a theory of Einstein gravity in this space.

Nonetheless, it is important to note that not all CFTs have a corresponding dual theory of gravity in AdS. The duality conjectured by Maldacena, and further developed in the years since, imposes some conditions that the CFT must verify in order to admit a bulk dual.
Namely, there are two necessary conditions (which are also thought to be sufficient) for this duality to be verified. In the next few paragraphs we seek to justify in an intuitive manner why it is important that these features are present in the CFT.

In the low energy limit, it is possible to treat the AdS space as a background with small excitations, corresponding to the weakly coupled gravitons. In this limit, correlation functions of a QFT on the AdS boundary factorize as is expect of correlation functions in the large N expansion of a CFT, provided the identification

$$N^2 \sim \left(\frac{R}{l_P}\right)^{d-1} \tag{1.1}$$

is made, where R corresponds to the AdS radius and l_P to the Planck scale [4]. The idea that $\frac{R}{l_P} \gg 1$ is precisely the statement that the theory is weakly coupled.

The AdS/CFT correspondence is, in essence, the assertion of an equivalence between theories of gravity in d-dimensional AdS space-time and (d-1)-dimensional conformal field theories. These equivalence is established through the existence of an "AdS/CFT dictionary" that provides a translation between a given object in one of the theories to the corresponding dual in the other theory. For instance, it is through this "dictionary" that the stress-tensor of the CFT is matched with the graviton in the theory of quantum gravity.
Likewise, the mass of fields in the AdS side of the duality is related to the conformal dimension (often called the scaling dimension) of operators in the CFT, which informs how the correlation functions of the theory behave under scaling transformations [3].
This dimension, in a free theory, can be read straightforwardly from the action by dimensional analysis and is known as the bare dimension. On the other hand, when considering an interacting theory the dimension of the operator will receive quantum corrections. The difference between this dimension and the bare dimension is what is called the anomalous dimension. This will be discussed more clearly when covering CFTs in Chapter 2. The relevant point here is that single-trace primary operators (those that are annihilated by

the generator of special conformal transformations as will be discussed in Chapter 2) with scaling dimension Δ match to weakly coupled fields in AdS with mass $m \sim \frac{\Delta}{R}$. Thus, as argued in [4], if we are seeking a theory of Einstein gravity in AdS that has no other low-energy excitations, the required mass-gap to achieve this must translate in a gap in the anomalous dimension on the CFT side of the duality. Furthermore, since the stress-energy tensor in the CFT is in correspondence with the graviton in the theory of quantum gravity, the previous reasoning means that is expected that the anomalous dimension of all operators other than the stress tensor grow with the increasing coupling.
Thus, as mentioned in [5], it is expected that the CFT dual of a quantum theory of gravity is a CFT that admits a large N expansion and also one where the anomalous dimension of all operators other than the stress-energy tensor (which is said to be protected) grows parametrically large. It should be noted that conserved currents in the CFT are operators that are protected by symmetry and, therefore, should not gain anomalous dimension either.

This conjecture has been verified in some cases, namely for $\mathcal{N} = 4$ SYM. For instance, when considering the Konishi operator in this theory, the authors in [6] found the results shown in figure 1.1. These results were obtained by using integrability techniques. This

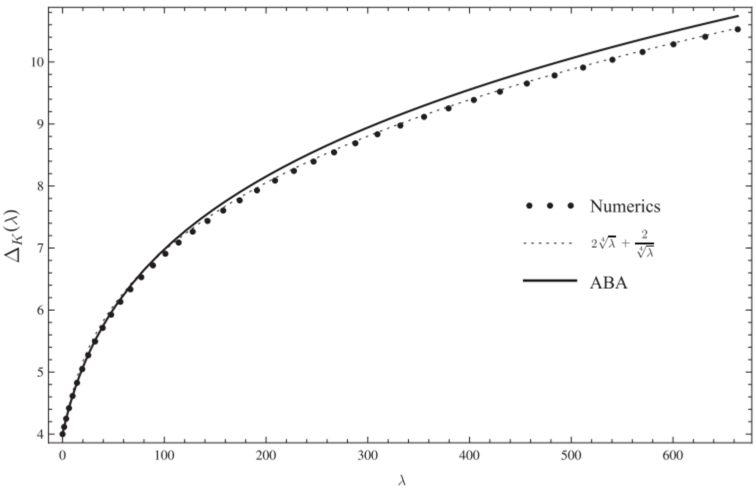


Figure 1.1: Plot of dimension of the Konishi operator in relation to the t'Hooft coupling. Numerical values are compared with the asymptotic Bethe ansatz (ABA) and to a best fit curve. Taken from reference [6]

differs from the approach that will be used in this thesis, which relies on the study of the divergences that appear when studying the 2-point function of the operator, nonetheless, the key behaviour is the same: the dimension of the operator (which for the Konishi operator starts at 4) grows with the increasing coupling.

To sum up, by looking for an increase in the anomalous dimension of operators other than the stress-tensor, it will be possible to gain evidence in favor or against this CFT having a dual theory of gravity in AdS. If there is such a dual, the anomalous dimension must grow with the coupling. Furthermore, by studying a CFT in 3d we seek to gain insights into the possibility of it admitting a dual that matches Einstein gravity in AdS_4. In this thesis, the focus is in the large N limit of a Chern-Simons theory coupled to fermionic matter in the adjoint representation of the $SU(N)$ group and, in particular, in computing the anomalous dimension of the single-trace operator $\mathcal{O}(x) = \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]$. Similar theories have been studied before, namely in [7] and [8].
The authors in [7] studied the large N-limit of 2 Chern-Simons gauge fields coupled to a scalar field that transforms in the bi-fundamental representation of $SU_k(N) \times SU_{-k}(M)$. The authors in [8], on the other hand, sought to couple a bi-fundamental fermion to a $U_{k_N}(N) \times U_{k_M}(M)$ Chern-Simons theory and compute the anomalous dimensions of operators. Despite these theories not being dual to Einstein theory, they provide interesting cases to consider and provide valuable insights.

Having described with broad strokes what we hope to accomplish with this work, we go over the layout of this thesis in the next section.

1.2 Outline

We opted to structure this thesis in such a way as to allow the reader to refresh the key concepts required to follow the calculations and arguments presented in each chapter.

For that reason, we start with, in Chapter 2, with a brief summary of the main results concerning CFTs. We explore the consequences of the additional symmetries that these theories possess in terms of the algebra of the conformal group, and the way this constrains the correlation functions of these type of theories. In particular, it is shown how the embedding-space formalism [9] allows for the computation of the 2, 3 and 4-point functions of scalar operators. Furthermore, we introduce the Operator Product Expansion

(OPE) as a powerful tool in the study of CFTs and as a backbone of the Conformal Bootstrap approach.

In Chapter 3 we introduce the Chern-Simons Lagrangian both for an abelian and non-abelian theory. We explore the Large N expansion as a framework that greatly simplifies perturbative calculations and illustrate its relevance by looking at the well-studied example of $\mathcal{N} = 4$ SYM. We also list and compute the Feynman rules needed to carry out the computations in this thesis.

Chapter 4 contains the bulk of the work for this thesis. Having established the required fundamentals in the previous chapters, we now seek to compute the anomalous dimension of the single trace operator $\mathcal{O}(x) = \text{Tr}[\bar{\Psi}\Psi](x)$. We start by reviewing how this computation can be done in a scalar theory with a quartic interaction, obtaining the known result for the anomalous dimension of this operator.
Afterwards, we carry out the computation for the operator we are interested in, including the different techniques and approaches we used to obtain a result. These include a brief explanation of the Integration-By-Parts (IBP) identities, which allow us to reduce the number of integrals to compute, as well as the Asymptotic-Expansion technique for studying the divergent behaviour of the integrals.
We obtain an expression for the divergence of the 2-point function of $\mathcal{O}(x) = \text{Tr}[\bar{\Psi}\Psi](x)$ at 2-loops, which allowed for the extraction of the anomalous dimension of this operator to 2-loops.

Lastly, we conclude the work with a summary of the results obtained and a brief discussion on the work that remains to be done.

Chapter 2

Conformal Field Theory

Quantum Field Theory (QFT) is a paradigm wherein all particles and their interactions are modelled through quantum fields that permeate all space. This tool has proved itself to be remarkably successful in describing physical reality. In a QFT, a particle is interpreted as an excitation of fundamental fields and they interact through mean of force-mediating particles, the so called gauge bosons.

By construction, QFT incorporates the principles of special relativity. Examples of QFTs are Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), both extremely successful theories with extensive experimental validation.
That these theories are in accordance with special relativity means that they are invariant under the Poincaré group, the group made up of all Lorentz transformations. These correspond to rotations, boosts and translations, which act on the coordinate vectors according to

$$x^{\mu} \mapsto x\prime^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}, \tag{2.1}$$

where Λ is a matrix with determinant 1 and a is some 4-vector. In equation (2.1), as in subsequent cases (unless explicitly stated otherwise), we are making use of Einstein's summation convention, meaning we are summing over repeated indices.

Transformations that belong to the Poincaré group are those that leave the flat space-time metric invariant. Yet, it is possible to construct theories that go beyond these symmetries and, in particular, theories that are also invariant under dilations and special conformal transformations (which will be discussed shortly). These last two inclusions are clearly not in the Poincaré group, as they do indeed change the metric, but they have the notable feature of leaving the angle between any 2 lines unchanged (such that 2 lines intersecting at right angles will still verify this property after the coordinate transformation). Poincaré transformations alongside dilations (scale transformations) and special conformal transformations (SCT) make up the conformal group and a QFT that is invariant under said group is said to be a Conformal Field Theory (CFT).

The importance of CFTs cannot be overstated. A CFT is a scale invariant theory and, therefore, it looks the same at all energy scales. In particular, one can study the Renormalization Group (RG) flow of a theory in order to understand how the couplings behave at different energy scales. This behaviour is mathematically encoded in the beta function, a relation between the coupling parameter g, and the energy scale Λ, given in (2.2) as:

$$\beta(g) = \Lambda \frac{dg}{d\Lambda} = \frac{\partial g}{\partial \log \Lambda}. \tag{2.2}$$

Fixed points of the beta function, $\beta(g^*) = 0$, occur when the coupling is the same at all energy scales. Therefore, at these points, the theory is invariant under scale transformations and is a CFT.

Furthermore, there are multiple important physical systems that exhibit this scale invariance, namely, the 3-dimensional Ising model as well as water near criticality.

2.1 Conformal Transformations

To study a CFT is to study its symmetries. Hence, understanding a CFT relies in knowing the algebra of the conformal group, that is, knowing of its generators and the commutation relations they verify.

This knowledge is key to understand how correlation functions transform, as well as to study one of the main properties of a CFT, the operator product expansion (OPE) that will be discussed later on.

To further this point, I present in the following pages a derivation of these results, along the lines of what is done in [10].

The fact that the laws of physics are invariant under diffeomorphisms provides a way to determine the generators of the conformal group. To make this derivation more pedagogical we start by analysing the generators of the Poincaré group, before imposing conformal symmetry.

The most general infinitesimal transformations ($|\epsilon| \ll 1$) is of the form:

$$x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x). \tag{2.3}$$

To study the different types of infinitesimal transformations one can have, it is useful to consider the metric tensor. The metric provides a way of measuring distances:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.4}$$

All observers must agree on the measure of distances, thus:

$$\begin{aligned} &g'_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} dx^\mu dx^\nu \\ &g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} = (\delta^\rho_\mu - \partial_\mu \epsilon^\rho)(\delta^\sigma_\nu - \partial_\nu \epsilon^\sigma) g_{\rho\sigma} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \end{aligned}$$

If 2 observers agree on the metric then the most general form of the infinitesimal parameter can be written in terms of an antisymmetric matrix $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0 \implies \epsilon^\mu = a^\mu + \omega^\mu_\nu x^\nu. \tag{2.5}$$

This is the result we have in theories that are invariant under the Poincaré group. There are infinitesimal translations, as well as infinitesimal Lorentz transformations.

It is also possible that 2 observers disagree on the metric, but only by a scale factor, that is $g'_{\nu\mu} \propto g_{\mu\nu}$. In this situation:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\lambda g_{\mu\nu} \implies \epsilon^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu \tag{2.6}$$

The transformation introduced by this relaxation of constraints on the transformation of the metric is a scale transformation (dilation):

$$x^\mu \mapsto x'^\mu = (1+\lambda) x^\mu. \tag{2.7}$$

Lastly, it can also happen that the 2 different observers not only disagree on the metric by a scale factor, but that they do so by a scale factor that is dependent on the point in space-time they are considering. If so, we have:

$$g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu} \qquad , \Omega(x) = e^{-2\sigma(x)} \approx 1 - 2\sigma(x) \tag{2.8}$$

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\sigma(x) g_{\mu\nu}. \tag{2.9}$$

Equation (2.9) is extremely powerful. It constrains the type of infinitesimal transformations that are compatible with the symmetries of the conformal group, and can be used to determine its generators in the manner described below.

By contracting the indices in equation (2.9) and denoting the space-time dimension by d ($g_{\mu\nu} g^{\mu\nu} = d$):

$$\begin{aligned} g^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = 2\sigma(x) g_{\mu\nu} g^{\mu\nu} \implies 2\partial_\mu \epsilon^\mu = 2\sigma(x) d \\ \implies \partial_\mu \epsilon^\mu = d \cdot \sigma(x) \end{aligned} \tag{2.10}$$

Acting on equation (2.9) with ∂_ν and using the result in (2.10) yields:

$$\begin{aligned} \partial^\nu \partial_\mu \epsilon_\nu + \partial^\nu \partial_\nu \epsilon_\mu = 2\partial^\nu \sigma(x) g_{\mu\nu} \implies \partial_\nu \partial_\mu \epsilon^\nu + \partial^2 \epsilon_\mu = 2\partial_\mu \sigma(x) \implies \\ \implies \partial^2 \epsilon_\mu = (2-d) \partial_\mu \sigma(x) \end{aligned} \tag{2.11}$$

Now, acting with ∂^μ on (2.11) and using (2.10):

$$\begin{aligned} \partial^\mu \partial^2 \epsilon_\mu = (2-d) \partial^2 \sigma \implies \partial^2 (d \cdot \sigma(x)) = (2-d) \sigma(x) \implies \\ \implies (d-1) \partial^2 \sigma(x) = 0 \end{aligned} \tag{2.12}$$

Lastly, acting with ∂_ν on (2.11):

$$\begin{aligned} \partial^2 \epsilon_\mu = (2-d) \partial_\mu \sigma(x) \implies \partial_\nu \partial^2 \epsilon_\mu = (2-d) \partial_\nu \partial_\mu \sigma(x) \implies \\ \implies g_{\mu\nu} \partial_\nu \partial^2 \epsilon^\nu = (2-d) \partial_\nu \partial_\mu \sigma \implies g_{\mu\nu} \partial^2 (d\sigma(x)) = (2-d) \partial_\nu \partial_\mu \sigma(x) \implies \\ \implies (2-d) \partial_\nu \partial_\mu \sigma(x) = g_{\mu\nu} \partial^2 \sigma(x) \end{aligned} \tag{2.13}$$

Equation (2.13) allows us to derive some important results concerning the most general form of the infinitesimal transformations:

- For $d > 1$: Equations (2.12) and (2.13) require that:

$$(2-d) \partial_\nu \partial_\mu \sigma(x) = 0 \tag{2.14}$$

- For $d > 2$: Equation (2.14) requires that $\partial_\mu \partial_\nu \sigma(x) = 0$.
  The solution to this equation is of the form : $\sigma(x) = \lambda + 2b \cdot x$, which, when combined with (2.9), yields:

$$\epsilon^\mu = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu. \tag{2.15}$$

The case of $d > 2$ is the relevant one here. Chern-Simons theories are theories in (2+1) dimensions. Therefore, equation (2.15) gives the most general form of an infinitesimal transformation. Understanding the behaviour of infinitesimal transformations is fundamental, as any finite transformation can be thought of as an infinite succession of infinitesimal steps.

As mentioned before, the conformal transformations form a group. This conformal group is characterized by its generators (the generators of infinitesimal transformations) as well as an algebra (the commutation relations between the generators G). As usual, if G describes an infinitesimal transformation then $e^{i\theta G}$ is the finite transformation with parameter θ. For instance, in the case of an infinitesimal translation (with generator P) we have

$$f(x) \xrightarrow{P} f(x') = f(x+a) \approx f(x) + a^\mu \partial_\mu f(x) \approx e^{-ia_\mu P^\mu} f(x),$$

thus identifying $P_\mu = i\partial_\mu$ as the generator of translations. Repeating this procedure with all the generators it's possible to arrive at:

Transformation	Infinitesimal Generator
Translations	$P_\mu = i\partial_\mu$
Lorentz Transformation	$M^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$
Scale Transformations	$D = ix^\mu \partial_\mu$
Special Conformal Transformations (SCT)	$K^\mu = i(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu)$

There are:

- d translations : one in each direction of the d-dimensional space-time
- $\frac{d(d-1)}{2}$ Lorentz transformations : $M^{\mu\nu}$ is a $d \times d$ antisymmetric matrix
- 1 scale transformation
- d SCT

This means that the conformal group in d-dimensions is of dimension $\frac{(d+1)(d+2)}{2}$. The group algebra is given by the following commutation relations, that can be easily verified using the representation of the generators as differential operators given above:

$$\begin{aligned}
&[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}) \\
&[M^{\mu\nu}, P^{\rho}] = -i(g^{\mu\rho}P^{\nu} - g^{\nu\rho}P^{\mu}) \qquad [M^{\mu\nu}, K^{\rho}] = -i(g^{\mu\rho}K^{\nu} - g^{\nu\rho}K^{\mu}) \\
&[D, P^{\mu}] = -iP^{\mu} \qquad [D, K^{\mu}] = iK^{\mu} \qquad [P^{\mu}, K^{\nu}] = 2i(g^{\mu\nu}D - M^{\mu\nu}).
\end{aligned} \tag{2.16}$$

An important feature that becomes evident from these relations is the fact that D, P^{μ} and K^{μ} verify commutations relations similar to those of the ladder operators for the quantum harmonic oscillator, meaning P raises the eigenvalue of D, while K decreases it.

Furthermore, this allows for the organization of operators by the way they transform under the symmetries of the group. In particular, it becomes natural to use operators that diagonalize the dilation operator.

$$[D, \phi(0)] = -i\Delta\phi(0). \tag{2.17}$$

Equation (2.17) defines what is known as the dimension of the operator, Δ, the eigenvalue of the dilation operator.

These commutation relations also dictate how operators away from the origin transform:

$$\begin{aligned}
&[P^{\mu}, \phi(x)] = i\partial^{\mu}\phi(x) \\
&[M^{\mu\nu}, \phi^{a}(x)] = -i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})\phi^{a}(x) + i(S^{\mu\nu})^{a}_{b}\phi^{b}(0) \\
&[D, \phi(x)] = -i(x^{\mu}\partial_{\mu} + \Delta)\phi(x) \\
&[K^{\mu}, \phi(x)] = -i(2x^{\mu}x^{\nu}\partial_{\nu} - x^{2}\partial^{\mu} + 2\Delta x^{\mu} - 2S^{\mu\nu}x_{\nu})\phi(x).
\end{aligned} \tag{2.18}$$

In particular, these relations determine the 2-point correlation function up to a multiplicative constant:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C}{((x_1 - x_2)^2)^{\frac{\Delta_1+\Delta_2}{2}}}. \tag{2.19}$$

Equation (2.19) implies that $\Delta > 0$. If $\Delta < 0$, then that would mean that points very far apart would be strongly correlated which is undesirable and nonphysical. This lower bound on the dimension of an operator, as well as the action of K as a lowering operator, means that there must be some operators that are annihilated by K:

$$[K^{\mu}, \phi(0)] = 0. \tag{2.20}$$

These are said to be primary operators and all others can be obtained from them by acting with P, as P is the operator that increases the conformal dimension. For this reason, operators obtained by taking derivatives of the primaries are called descendants and it is enough to focus on the primaries. Let us add that the invariance of a two point function under Special Conformal Transformations (SCT) requires it to be orthogonal, in the sense that two point functions of operators with different dimensions vanish.

This formulation is entirely non-perturbative and can, therefore, be used to describe a variety of physical systems.

For a free theory the spectrum (the set of eigenvalues of D) is highly degenerate and can be read through dimensional analysis, yielding what we call the bare dimensions, Δ_i^0. For an interacting theory (with some coupling g) this is not the case. The dimensions are given by the bare dimensions plus some "corrections" that depend on g, $\Delta_i = \Delta_i^0 + \delta\Delta_i(g)$, and that are not necessarily small.

2.2 Conformal Correlation Functions

Due to the fact that the d-dimensional conformal group has dimension $\frac{(d+1)(d+2)}{2}$, which is precisely the dimension of $\mathrm{SO}(d+1,1)$, alongside the commutation relations of the generators given in (2.16), it is possible to conclude that both of these groups are isomorphic. The isomorphism between $\mathrm{SO}(d+1,1)$, with generators $J^{\mu\nu}$, and the conformal group in d-dimensions is made clear by the following identification,

$$\begin{aligned} M^{\mu\nu} &= J^{\mu\nu} \\ P^{\mu} &= J^{\mu,d+1} + J^{\mu,d+2} \\ K^{\mu} &= J^{\mu,d+1} - J^{\mu,d+2} \\ D &= J^{d+1,d+2}, \end{aligned} \tag{2.21}$$

and provides an easier way of constructing conformally invariant correlation functions, by going to the so-called Embedding Space [9].

The Embedding Space is the (d+2)-dimensional space where conformal transformations act linearly (2.22), this making it preferable for computing the correlators.

$$[J^{MN}, J^{RS}] = -i(\eta^{MR}J^{NS} - \eta^{MS}J^{NR} - \eta^{NR}J^{MS} + \eta^{NS}J^{MR}) \tag{2.22}$$

To move from $x^M \in \mathbb{R}^{d+1,1}$ to $x^\mu \in \mathbf{R}^d$ we make use of the following precepts:

1. Restrict the analysis to the future light-cone $X^2 = 0$ and $X^{d+2} > 0$

2. Identify 2 points of the lightcone related by a scale transformation

In the section of the cone under consideration, the following correspondence is made:

$$X^\mu = x^\mu \qquad X^{d+1} = \frac{1-x^2}{2} \qquad X^{d+2} = \frac{1-x^2}{2}. \tag{2.23}$$

In order to obtain the transformation rules for x^μ we:

1. Map x^μ to the lightcone
2. Apply the transformation on X^M (it is just a Lorentz transformation: $X^M \mapsto X'^M = \Lambda^M_N X^N$)
3. Perform a rescaling $X'^M \mapsto \lambda(X')X'^M$
4. Identify $x'^\mu \mapsto \lambda(x')x'^\mu$

Local operators are also lifted into the embedding space and from them it is required they obey the following two properties [9]:

$$\begin{aligned} &\phi(\lambda X) = \lambda^{-\Delta}\phi(X) && \text{(Homogeneity)} \\ &X^{A_1}\phi_{A_1 A_2 \ldots A_s}(X) = 0 && \text{(Transversality)} \end{aligned}. \tag{2.24}$$

The transversality property concerns only operators with spin. These operators in embedding space are related to their physical-space version through:

$$\phi_{\alpha_1\alpha_2\ldots\alpha_s}(x) = \frac{\partial X^{A_1}}{\partial x^{\alpha_1}}\frac{\partial X^{A_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial X^{A_s}}{\partial x^{\alpha_s}}\phi_{A_1 A_2 \ldots A_s}(X). \tag{2.25}$$

The correlation functions must depend on Lorentz-invariant quantities in the embedding space. This means they can only depend on scalar products $X_i \cdot X_j$. Since $X_i^2 = 0$ on the null cone, the only scalar products that contribute to the correlation functions are those with $i \neq j$.

Since scalar fields are lifted to the null-cone in such a way as to preserve the homogeneity property in (2.24) then the correlation functions obey:

$$\langle \phi_1(X_1)...\phi_N(X_N)\rangle \mapsto \lambda(X_1)^{-\Delta_1} \ldots \lambda(X_N)^{-\Delta_N} \langle \phi_1(X_1) \ldots \phi_N(X_N)\rangle\,. \tag{2.26}$$

Knowing this, the 2-point function must verify $\langle \phi_1(X_1)\phi_2(X_2)\rangle \propto (X_1 \cdot X_2)^{-\Delta}$. This fact, alongside the known correspondence between points in embedding space and physical space, implies that the 2-point function is given by:

$$\begin{aligned}
X_1 \cdot X_2 &= x_1^{\mu} x_{2\mu} + \left(\frac{1-x_1^2}{2}\right)\left(\frac{1-x_2^2}{2}\right) - \left(\frac{1-x_1^2}{2}\right)\left(\frac{1-x_2^2}{2}\right) = \\
&= x_1 \cdot x_2 - \frac{1}{2}(x_1^2 x_2^2) = \frac{1}{2}(x_1 - x_2)^2 \implies \\
&\implies \langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{\left[(x_1-x_2)^2\right]^{\Delta}}
\end{aligned} \quad . \tag{2.27}$$

2.2.1 3-point Correlation Function

The procedure discussed previously also allows for the computation of higher point correlation functions. As with the 2-point function, the 3-point correlator is also constrained by conformal symmetry. Yet, while the 2-point function is entirely determined by conformal symmetry, the 3-point function can only be fixed up to a multiplicative constant.
Consider the 3-point function $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle$, where the 3 operators may have distinct scaling dimensions: Δ_1, Δ_2 and Δ_3. This function may only depend on the invariants $X_1 \cdot X_2$, $X_1 \cdot X_3$ and $X_2 \cdot X_3$.

As with the 2-point function, the dependence of the correlator on these invariants must be of the form:

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle \propto (X_1 \cdot X_2)^{\alpha_{12}} (X_1 \cdot X_3)^{\alpha_{13}} (X_2 \cdot X_3)^{\alpha_{23}} . \tag{2.28}$$

Now, using the scaling properties of the correlation function

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle \mapsto \lambda_1^{\alpha_{12}+\alpha_{13}} \lambda_2^{\alpha_{12}+\alpha_{23}} \lambda_3^{\alpha_{13}+\alpha_{23}} \langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle , \tag{2.29}$$

we obtain the following system of equations:

$$\begin{cases} \alpha_{12} + \alpha_{13} = -\Delta_1 \\ \alpha_{12} + \alpha_{23} = -\Delta_2 \\ \alpha_{13} + \alpha_{23} = -\Delta_3 \end{cases} \implies \begin{cases} \alpha_{12} = \frac{\Delta_1+\Delta_2-\Delta_3}{2} \equiv \Delta_{123} \\ \alpha_{13} = \frac{\Delta_1+\Delta_3-\Delta_2}{2} \equiv \Delta_{132} \\ \alpha_{23} = \frac{\Delta_2+\Delta_3-\Delta_1}{2} \equiv \Delta_{231} \end{cases} . \tag{2.30}$$

Thus:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle \, \frac{\lambda_{123}}{(x_{12}^2)^{\Delta_{123}}(x_{13}^2)^{\Delta_{132}}(x_{23}^2)^{\Delta_{231}}} \, , \qquad \begin{aligned} x_{ij} &= |x_i - x_j| \\ \Delta_{ijk} &= \frac{\Delta_i + \Delta_j - \Delta_k}{2} \end{aligned} . \tag{2.31}$$

2.2.2 4-point Function

As a last example, we can look at the 4-point function of 4 identical scalar operators: $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$. In this context it is convenient to define 2 invariant quantities, named conformal cross-ratios and defined as:

$$\begin{aligned} u &= \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} \\ v &= \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} \end{aligned}. \tag{2.32}$$

The most general 4-point function is of the form:

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\phi_4(X_4)\rangle \propto \frac{g(u,v)}{(X_1 \cdot X_2)^\Delta (X_3 \cdot X_4)^\Delta} \tag{2.33}$$

or, in Euclidean coordinates:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{g(u,v)}{(x_{12}^2 x_{34}^2)^\Delta} \;, \qquad \begin{aligned} u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ v &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \end{aligned}. \tag{2.34}$$

When working with the conformal cross-sections it is convenient to parameterize them in terms of complex numbers $z, \bar{z}$:

$$u = z\bar{z} \qquad v = (1-z)(1-\bar{z}), \tag{2.35}$$

One important aspect relating to 4-point functions is the fact that they exhibit crossing symmetry. This results from the fact that the 4-point function is invariant under pairwise permutations of the position of the operators, yielding the so-called crossing equation:

$$v^\Delta g(u,v) = u^\Delta g(v,u) \tag{2.36}$$

Crossing symmetry is, alongside the operator product expansion (OPE), one of the fundamental principles behind the bootstrap program briefly discussed in section 2.3.

2.2.3 Correlation Function of Operators With Spin

The embedding-space formalism was useful in determining the 3-point and 4-point correlation functions of a scalar operator, in the previous sections. These very same results could, however, be obtained equally as easily by making use of the symmetries of the conformal group in regular space. The true usefulness of this approach is in computing the correlation function of operators with spin.

As an example lets consider the case of a spin-1 operator, $\mathcal{O}^{\alpha_1}(x)$, for which the 2-point correlation function is an object with two open indices. In embedding-space the correlator can only depend on the Lorentz-invariant quantity $X_1 \cdot X_2$ and all terms must have two open indices. Consequently, the most general form of this 2-point function is

$$\langle \mathcal{O}^{A_1}(X_1)\mathcal{O}^{A_2}(X_2)\rangle = \frac{1}{(X_1 \cdot X_2)^a}\left[b\delta^{A_1 A_2} + c\frac{X_1^{A_1}X_2^{A_2}}{X_1 \cdot X_2} + d\frac{X_1^{A_2}X_2^{A_1}}{X_1 \cdot X_2}\right], \tag{2.37}$$

where we extracted a pre-factor of $(X_1 \cdot X_2)^{-a}$ and wrote all terms inside the parenthesis such that they all had weight 0 both in X_1 and X_2.
The fact that all the weight in X_1 and X_2 makes it easy to use the homogeneity property in (2.26) to conclude:

$$\lambda_1^{\Delta_1}\lambda_2^{\Delta_2} = (\lambda_1\lambda_2)^{-a} \implies a = \Delta_1 = \Delta_2 \equiv \Delta. \tag{2.38}$$

The requirement that the operator in embedding space must be transverse fixes the remaining degrees o freedom. In particular, since equation (2.25) implies that the term proportional to c projects to 0 and has no expression on the physical correlator, the contraction of (2.41) with $X_1^{A_1}$:

$$bX_1^{A_2} + dX_1^{A_2} = 0 \implies b + d = 0. \tag{2.39}$$

Therefore, the 2-point function of a spin-1 operator with conformal dimension Δ is, in embedding space:

$$\langle \mathcal{O}^{A_1}(X_1)\mathcal{O}^{A_2}(X_2)\rangle = \frac{C}{(X_1 \cdot X_2)^{\Delta}}\left[\delta^{A_1 A_2} - \frac{X_1^{A_2}X_2^{A_1}}{X_1 \cdot X_2}\right]. \tag{2.40}$$

Equivalently, using the correspondence in (2.23) and the projection matrices in (2.25), the physical-space correlation function can be expressed as:

$$\langle \mathcal{O}^{\alpha_1}(x_1)\mathcal{O}^{\alpha_2}(x_2)\rangle = \frac{c}{(x_{12}^2)^{\Delta}}\left[\delta^{\alpha_1\alpha_2} - 2\frac{x_{12}^{\alpha_1}x_{12}^{\alpha_2}}{x_{12}^2}\right]. \tag{2.41}$$

Similar approaches can be used to compute the correlators of operators with arbitrary spin, as illustrated in [9].

2.3 Operator Product Expansion (OPE)

When discussing the structure of 4-point functions in a CFT, the concept of Operator Product Expansion (OPE) was fist mentioned. In a CFT, the OPE is the analogous to the multipole expansion in a classical field theory. The OPE relies on the fact that when 2 operators are sufficiently close they can be written as a sum of terms involving a single operator

$$\phi_1(x)\phi_2(y) \mapsto \sum_i f_i(x-y)\phi_i(y).$$

This is analogous to the multipole expansion in the sense that when we are considering the effects of a localized distribution in a faraway point, all points in said distribution are comparatively close. Consequently, it is possible to expand the distribution in a series of terms with increasingly smaller contribution.

This is particularly useful in a CFT, since it allows us to rewrite a n-point correlation function in terms of a (n-1)-point correlation function. In fact, this procedure can be iterated all the way down up to the 2-point correlator which is, as discussed previously, entirely determined by conformal symmetry.

For instance, the correlation function of 3 scalar operators can be written as:

$$\langle \phi_1(x)\phi_2(0)\phi_3(z)\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}} \mathcal{C}_{\mathcal{O}}(x,\partial_y) \langle \mathcal{O}(y=0)\phi_3(z)\rangle . \tag{2.42}$$

Using the fact that 2-point functions are diagonal, the only non-zero contribution to the sum happens when the operator we are summing over corresponds to the third operator in the original 3-point function. As such:

$$\langle \phi_1(x)\phi_2(0)\phi_3(z)\rangle = \lambda_{\phi_3} \mathcal{C}_{\phi_3}(x,\partial_y) \langle \phi_3(y=0)\phi_3(z)\rangle . \tag{2.43}$$

Chapter 3

Chern-Simons Theory

In this section, we focus on discussing Chern-Simons theories, following the notes in reference [11]. These lecture notes are extremely vast in scope, but omit some of the derivations that we deem important to this thesis, namely that of Appendix A.

Chern-Simons theory is a type of gauge theory defined on a topological 3-manifold meaning that similarly to other gauge theories, for instance Yang-Mills, this theory includes a a gauge field. A Chern-Simons theory is defined in terms of a Lie Group and a constant κ, known as the level of the theory, that acts as a multiplicative constant in the action.

A Chern-Simons theory obeys the usual requirements of a gauge theory in the sense that it is a local theory with both Lorentz invariance and gauge invariance. Its Lagrangian is given by

$$\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - A_\mu J^\mu, \tag{3.1}$$

where J^μ is a matter current that is conserved ($\partial_\mu J^\mu = 0$).

Even though the Lagrangian in (3.1) does not depend explicitly on the Field Strength, $F_{\mu\nu}$, which is a gauge invariant quantity, it is in itself gauge invariant in the sense that under a gauge transformation it changes by a total derivative (3.2). That is, under a transformation $A_\mu \mapsto A'_\mu \equiv A_\mu + \partial_\mu \alpha$, the Lagrangian changes by some quantity $\delta\mathcal{L}_{CS} \equiv \mathcal{L'}_{CS} - \mathcal{L}_{CS}$, given by:

$$
\begin{aligned}
\mathcal{L}'_{CS} &= \frac{\kappa}{2}\epsilon^{\mu\nu\rho}(A_\mu + \partial_\mu\alpha)\partial_\nu(A_\rho + \partial_\rho\alpha) - (A_\mu + \partial_\mu\alpha)J^\mu = \\
&= \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu\partial_\rho\alpha + \partial_\mu\alpha\partial_\nu A_\rho + \partial_\mu\alpha\partial_\nu\partial_\rho\alpha\right) - \left(\partial_\mu\alpha\right)J^\mu = \\
&= \mathcal{L}_{CS} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu\partial_\rho\alpha + \partial_\mu\alpha\partial_\nu A_\rho\right) - \left(\partial_\mu\alpha\right)J^\mu + \mathcal{O}(\alpha^2) = \\
&= \mathcal{L}_{CS} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\partial_\mu\alpha\partial_\nu A_\rho - \left(\partial_\mu\alpha\right)J^\mu + \mathcal{O}(\alpha^2)
\end{aligned}
. \tag{3.2}
$$

In the previous calculation, one of the terms was set to 0 since $\partial_\mu\partial_\nu$ is symmetric and it's being contracted with the antisymmetric tensor $\epsilon^{\mu\nu\rho}$.

Furthermore, when considering the action we are taking an integral of the Lagrangian. Performing an integration by parts the derivative of $\partial_\mu\Lambda$ can be moved to J^μ which is 0 due to J being a conserved current. Equation (3.2) implies then, as claimed, that the gauge transformation shifts the Lagrangian by a total derivative:

$$
\delta\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\partial_\mu\left(\alpha\partial_\nu A_\rho\right). \tag{3.3}
$$

This gauge invariant theory can then be coupled to a fermion field. Therefore, it is important to understand the behaviour of fermions in (2+1)-dimensions. In particular, the Dirac equation for a spinor coupled to a gauge-field is given as

$$
\left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m\right)\Psi = 0, \tag{3.4}
$$

where the gamma matrices give the representation of the Clifford Algebra, characterized by the following anti-commutation relation:

$$
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{3.5}
$$

Following reference [12], the gamma matrices in 2+1 Minkowski space-time with signature $(-,+,+)$ can be expressed in terms of the usual Pauli matrices:

$$
\begin{aligned}
\gamma^0 &\equiv i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\gamma^1 &\equiv \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\gamma^2 &\equiv \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{3.6}
$$

obeying

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho. \tag{3.7}$$

These matrices verify some important relations that will be used later on. In particular, we call attention to the following identities, valid in $d = 3$ dimensions:

$$\begin{aligned} \mathrm{Tr}[\gamma^\mu] &= 0 \\ \mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 2g^{\mu\nu} \end{aligned}. \tag{3.8}$$

These few relations are key to the calculations that will be carried out in this thesis, since they allow us to compute the trace of a product of any number of gamma matrices in an iterative manner. Concretely, the trace of a product of n matrices is given as the sum of the trace of a product of $n-2$ and $n-1$ gamma matrices.

This can be easily implemented in MATHEMATICA, providing an immediate way to compute the trace of a product of ten or even twelve gamma matrices. To achieve this, the function `tracos` was constructed in such a way as to receive any number of inputs, corresponding to the indices of the Gamma matrices in the appropriate order. This functions is defined, as suggested previously, in a recursive manner, meaning that when it is required to evaluate the trace of a product of ten Gamma matrices, we merely call `tracos` with ten arguments, defined as

```
tracos[α1_,α2_,α3_,α4_,α5_,α6_, α7_,α8_,α9_,α10_] =
    δ[α1,α2]*tracos[α3,α4,α5,α6,α7,α8,α9,α10] -
    leviCivitaϵ[α1,α2,β1]*tracos[β1,α3,α4,α5,α6,α7,α8,α9,α10]
    // Expand // # //. ruledim &
```

Listing 3.1: Function to compute the trace of a product of 10 Gamma matrices

which in itself calls for the trace of eight and nine Gamma matrices.

The object `leviCivitaϵ[α1,α2,β1]` remains at this point undefined, but will later on be converted to `LeviCivitaϵ[α1,α2,β1]`, which is endowed with the properties of the Levi-Civita symbol. The reason we are not yet using this second object when computing the traces is to avoid contractions between Levi-Civita symbols, which would result in much larger expression given in terms of Kronecker deltas.

Furthermore, `ruledim` corresponds to a list of rules that implement the contractions between the delta functions, as shown in 3.2.

```
ruledim = {δ[a1_,a1_]->dim, δ[a1_,a2_]^2->dim,
    δ[a1_,a2_] a3_ :> (a3 /. a1 -> a2) /;Not@FreeQ[a3, a1]};
```

Listing 3.2: Rules for implementing contraction of Kronecker deltas. The last rule replaces a1 with a2 in the term a3 if it is present

In order to optimize the code and avoid it taking too long to run, we opted to define only the traces of more than nine matrices this way. When dealing with a smaller number, it is possible to merely copy and paste the result (which was nonetheless computed using this iterative approach) as we are dealing with a much more manageable expression. *
This will be required since we will be carrying out the computation of the anomalous dimension to 2-loops, and each fermion propagator will introduce one more gamma matrix (as will each vertex between fermions and gauge-field).

Alongside the identities in (3.7) and (3.8) concerning the spinor structure of the theory, it will also be required to consider the color structure. This is a consequence of studying not an abelian Chern-Simons theory, but rather a non-abelian Chern-Simons theory. In this situation, the fields are given as

$$A_\mu = A_\mu^A T^A, \tag{3.9}$$

where T^A is an element of the gauge group ($SU(N)$ in our case) verifying:

$$\begin{aligned}
&\left[T^A, T^B\right] = i f^{ABC} T^C \\
&\mathrm{Tr}[T^A] = 0 \\
&\mathrm{Tr}[T^A T^B] = \frac{\delta^{AB}}{2} \\
&\mathrm{Tr}\left[T^A\left[T^B, T^C\right]\right] = \frac{i}{2} f^{ABC}.
\end{aligned} \tag{3.10}$$

The object f^{ABC} is called the group structure constant. Since all the fields being considered are in the adjoint representation, the elements of the group can be thought of as $N^2 \times N^2$ antisymmetric matrices given in terms of these structure constants as:

$$(T^A)_{BC} = -i f_{ABC}. \tag{3.11}$$

*This comes as a consequence of the fact that the number of terms in the expression grows very fast with the number of indices. For instance, four Gamma matrices result in an expression with 3 terms but when dealing with ten of them, one has 321 terms. If we consider the trace of 12 matrices (the most that will be required) we are already dealing with almost 1500 terms, each of which is a product of 6 Kronecker deltas

The Lagrangian of a non-abelian Chern-Simons theory involves a term of the form

$$S_{CS} = \frac{\kappa}{4\pi} \int d^3x \epsilon^{\mu\nu\rho} Tr \left[A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right]. \tag{3.12}$$

Under a gauge transformation,

$$A_\mu \mapsto A_\mu^g \equiv g^{-1} A_\mu g - i g^{-1} \partial_\mu g, \tag{3.13}$$

the variation of the action can be computed (Appendix A) to yield

$$\delta \mathcal{L}_{CS} = \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \mathrm{Tr} \left[-i\partial_\nu \left(A_\mu g \partial_\rho g^{-1} \right) + \frac{1}{3} g^{-1} \partial_\mu g g^{-1} \partial_\nu g^{-1} \partial_\rho g \right]. \tag{3.14}$$

In order to ensure the proper quantization of the theory it is required that the level, κ, must be an integer [11].

This happens because even though the first term is a total derivative and therefore does not contribute to the variation of the action, the second term does not vanish in the same way. This would appear to indicate that the Chern-Simons actions is actually not gauge-invariant, which would be indeed problematic. However, the quantity

$$w(g) = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho} \int d^3x g^{-1} \partial_\mu g g^{-1} \partial_\nu g^{-1} \partial_\rho g \quad , \tag{3.15}$$

the so-called winding number, can be shown to be an integer [11]. The implication of this is that, despite the action itself not being gauge-invariant, when we are performing calculations we are considering an exponential e^{iS}, which under a gauge transformation transforms into

$$e^{iS} \mapsto e^{iS} e^{2\pi i \cdot \kappa w(g)}. \tag{3.16}$$

This equation implies that, so long as the level of the theory, κ, is a discrete parameter, then the amplitude e^{iS}, corresponding to the weight in the path integral (4.8), is gauge-invariant.

Having established that it possible to construct a gauge-invariant theory with a Chern-Simons gauge field, the next natural step is to include matter. Concretely, it is possible to couple a fermion transforming in the adjoint representation of the gauge group, $SU(N)$ in our case, to the gauge-field . Using minimal coupling this corresponds to including in the action a term of the following form [12]:

$$S_F = i \int d^3x Tr \left[\bar{\Psi} \not{D} \Psi \right], \qquad \not{D}\Psi = \gamma^\mu D_\mu \Psi = \gamma^\mu (\partial_\mu \Psi + i[A_\mu, \Psi]), \tag{3.17}$$

where the γ matrices in three dimensions are, as mentioned previously, given in terms of the Pauli matrices (3.6).

Our interest lies in perturbatively computing the anomalous dimension. This calculation will be carried out in the large N limit and requires an understanding of it as well as of the Feynman rules that will be used.

3.1 Large N limit

The Large N expansion is a framework introduced by t'Hooft [13] in the context of the study of strong interactions in a gauge theory with $U(N)$ symmetry. Even though it originates from the study of QCD, it has been shown to be relevant beyond that first application, being extremely pertinent in the study of gauge theories.

Furthermore, as was discussed in the beginning of this thesis, the large N expansion is a required feature of a CFT with a dual theory of gravity in the bulk of AdS space. In this context, the large N limit has an important role in a perturbative analysis of the CFT. Since we will be carrying out such a perturbative calculation in this work, it is useful to first explore this limit in the framework of a more familiar theory, as is the case of the Yang-Mills theory.

3.1.1 Large N Limit in QCD

QCD is a Yang-Mills theory with $SU(3)$ symmetry. Even though it is clear that 3 is not quite the same as infinity, this expansion has proved powerful, even in this instance, in greatly simplifying the computations. The action is

$$S_{YM} = -\frac{1}{2g^2}\int d^4x \mathrm{Tr}\left[F^{\mu\nu}F_{\mu\nu}\right], \tag{3.18}$$

where $F_{\mu\nu}$ is the usual Field Strength Tensor.

Following the ideas of t'Hooft, it is possible to introduce a parameter

$$\lambda = g^2 N \tag{3.19}$$

that is kept fixed as $N \to \infty$ and makes it so that the action can be rewritten as

$$S_{YM} = -\frac{N}{2\lambda}\int d^4x \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]. \tag{3.20}$$

The most important feature of taking this limit is that it results in a rearrangement of the Feynman diagrams contributing at each order in perturbation theory. This rearrangement is the result of different diagrams depending on different powers of N (diagrams with the same number of vertices can have a different dependence of N). Therefore, when taking the large N limit, some diagrrams will be suppressed.

The propagator of the gauge field $\left(A_\mu\right)^i_j$, where $i, j = 1, 2, ..., N$ for a $SU(N)$ theory, is given in terms of the propagator for a single gauge field, $\Delta_{\mu\nu}$, by:

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y)\rangle = \Delta_{\mu\nu}(x-y)\left(\delta^i_l \delta^k_j - \frac{1}{N}\delta^i_j \delta^k_l\right). \tag{3.21}$$

Neglecting subdominant terms in N, what remains is:

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y)\rangle = \Delta_{\mu\nu}(x-y)\delta^i_l \delta^k_j. \tag{3.22}$$

Since the gauge field has 2 indices, it is useful to introduce a new notation for the Feynman diagrams. Instead of the usual curly line, the gauge field can be represent by 2 straight lines, pointing in opposite directions. This depicts the fact that the adjoint representation is given by the direct product of the fundamental and antifundamental representations.

The scaling properties of the propagator can be read from equation (3.20). Since the propagator is merely the inverse of the quadratic term in the action, then it must go as $\frac{\lambda}{N}$.

Applying this notation to the vacuum bubbles in QCD illustrates the usefulness of this expansion.

The fact that the action contains a cubic vertex, coming from the term $ig\partial^\nu A^\mu \left[A_\mu, A_\nu\right]$ gives rise to a vacuum bubble with 2 vertices, coming from the contractions between 2 terms that are the trace of a product of 3 gauge fields, that is:

$$\mathrm{Tr}[A^a A^b A^c] \cdot \mathrm{Tr}[A^d A^e A^f].$$

This expression can be rewritten by introducing explicitly the indices of the matrix representation, yielding:

$$A^{i_1}_{j_1} A^{i_2}_{j_2} A^{i_3}_{j_3} \cdot A^{k_1}_{l_1} A^{k_2}_{l_2} A^{k_3}_{l_3} \cdot \delta^{i_1}_{j_3} \delta^{k_1}_{l_3} \tag{3.23}$$

The 2 delta terms in the previous equation come from explicitly writing the traces in Einstein's notation.
Since there are three fields being contracted with 3 other fields, there could be up to six possible Wick contractions. Yet, due to the cyclic property of the trace, there are only two distinct contractions that can be made. Including the additional delta factors coming from matrix multiplications, the resulting expressions are:

$$\langle A^{i_1}_{j_1} A^{k_1}_{l_1}\rangle \langle A^{i_2}_{j_2} A^{k_2}_{l_2}\rangle \langle A^{i_3}_{j_3} A^{k_3}_{l_3}\rangle \delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{i_1}_{j_3}\delta^{k_1}_{l_3} \tag{3.24}$$

$$\langle A^{i_1}_{j_1} A^{k_1}_{l_1}\rangle \langle A^{i_2}_{j_2} A^{k_3}_{l_3}\rangle \langle A^{i_3}_{j_3} A^{k_2}_{l_2}\rangle \delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{i_1}_{j_3}\delta^{k_1}_{l_3}. \tag{3.25}$$

Now, making use of equation (3.22), the color contribution of equation (3.24) is

$$\begin{aligned}
&\delta^{k_1}_{j_1}\delta^{i_1}_{l_1}\delta^{k_2}_{j_2}\delta^{i_2}_{l_2}\delta^{k_3}_{j_3}\delta^{i_3}_{l_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{i_1}_{j_3}\delta^{k_1}_{l_3} = \delta^{k_1 i_2}\delta^{i_1 k_2}\delta^{i_3 k_2}\delta^{i_2 k_3}\delta^{i_1 k_3}\delta^{i_3 k_1} = \\
&= \delta^{k_1 k_3}\delta^{k_2 k_3}\delta^{k_2 k_1} = \delta^{k_2 k_3}\delta^{k_2 k_3} = N.
\end{aligned} \tag{3.26}$$

Likewise, equation (3.25) gives rise to a contribution:

$$\begin{aligned}
&\delta^{k_1}_{j_1}\delta^{i_1}_{l_1}\delta^{k_3}_{j_2}\delta^{i_2}_{l_3}\delta^{k_2}_{j_3}\delta^{i_3}_{l_2}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{k_2}_{l_1}\delta^{k_3}_{l_2}\delta^{i_1}_{j_3}\delta^{k_1}_{l_3} = \delta^{k_1 i_2}\delta^{i_1 k_2}\delta^{i_3 k_3}\delta^{i_2 k_1}\delta^{i_1 k_2}\delta^{i_3 k_3} = \\
&= \left(\delta^{i_2 k_1}\right)^2 \left(\delta^{i_1 k_2}\right)^2 \left(\delta^{i_3 k_3}\right)^2 = N^3.
\end{aligned} \tag{3.27}$$

To these contributions correspond the Feynman diagrams in double-line notation given in figures 3.1 and 3.2. The first factor comes from having three propagators and the second

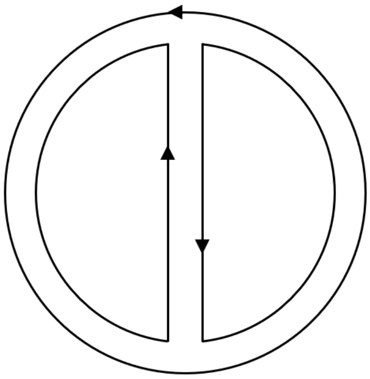


Figure 3.1: Correction of order λ to the vacuum energy. This diagram scales as $\left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 = \lambda N^2$

from the two vertices. The last factor comes from the contractions of the three indices, represented by the arrows. It is equal to the number of faces in the diagram and corresponds precisely to the factor computed previously and given in (3.27). This diagram, despite also being of order λ, is of order 0 in N. As a result, when taking the limit of large

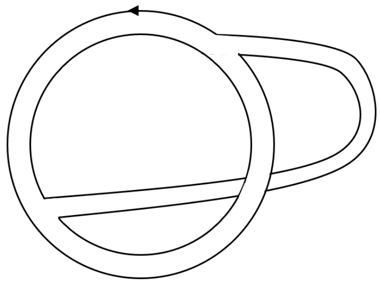


Figure 3.2: Correction of order λ to the vacuum energy (non-planar diagram). This diagram scales as $\left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N = \lambda$

N this diagram gives a smaller correction.

This computations elucidate a general result. Diagrams that remain relevant in the large N limit are called planar diagrams since they are those that can be drawn in the plane or the surface of a sphere. Diagrams that cannot be inscribed on a sphere (like this last diagram, which can be drawn on a torus) are subdominant in the large N limit.

3.2 Feynman Rules

A gauge-invariant action was constructed from the field A_μ^A. In order to work with this action it is first necessary to fix the gauge and the Feynman gauge [14] is one possible option. This choice is realized by adding to the action a term of the form

$$S_{GF} = \frac{k}{4\pi} \int d^3x \xi Tr[(\partial_\mu A^\mu)^2], \tag{3.28}$$

according to the Fadeev-Popov's prescription and then sending $\xi \to \infty$.
This procedure also requires introducing anti-commutating fields, $\bar{c}$ and c, the so called ghosts, whose action is given in [15] as

$$S_{ghost} = \int d^3x Tr[\bar{c}(-\partial^\mu D_\mu)c]. \tag{3.29}$$

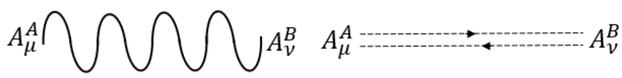
The Feynman rules, derived in Appendix B, are summarized here. By introducing a new normalization for the gauge field, achieved by absorbing a factor of $\sqrt{\kappa}$ into the definition of A, and defining

$$g = \frac{1}{\sqrt{\kappa}}, \tag{3.30}$$

such that the covariant derivative becomes $D_\mu = \partial_\mu + ig[A, \bullet]$, the action for the theory becomes:

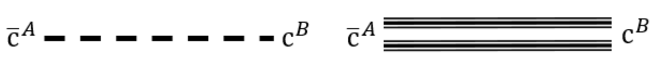
$$\begin{aligned} S = \frac{1}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\mathrm{Tr}[A_\mu\partial_\nu A_\rho + \frac{2ig}{3}A_\mu A_\nu A_\rho] + i\int d^3x \mathrm{Tr}[\bar{\Psi}\not{D}\Psi] + \\ + \int d^3x \mathrm{Tr}[\bar{c}(-\partial^\mu D_\mu)c] + \frac{1}{4\pi}\int d^3x \xi \mathrm{Tr}[(\partial A)^2] \end{aligned} . \tag{3.31}$$

The propagators for the gauge field, for the ghosts and for the fermions, given in equations (B.9), (B.10) and (B.12) are shown in the figures below.



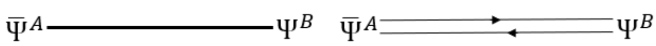
$$\langle A^A_\mu(p)A^B_\nu(-q)\rangle = -4\pi\delta^{AB}\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}\delta(p-q)$$

Figure 3.3: Propagator for the gauge field in the standard form, as well as in double-line notation



$$\langle \bar{c}^A(p)\partial^\nu c^B(-q)\rangle = \delta^{AB}\frac{p^\nu}{p^2}\delta(p-q)$$

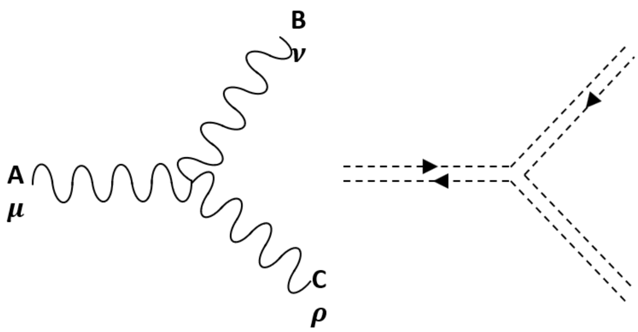
Figure 3.4: Propagator for the ghosts in the standard form, as well as in double-line notation



$$\langle \bar{\Psi}^A(p)\Psi^B(-q)\rangle = \delta^{AB}i\frac{\gamma\cdot p}{p^2}\delta(p-q)$$

Figure 3.5: Propagator for the fermions in the standard form, as well as in double-line notation

The results that were obtained for the interaction vertices, also derived in Appendix B, are summarized in the figures that follow.



$$\equiv \frac{-ig\epsilon^{\mu\nu\rho}}{24\pi}f^{ABC}$$

Figure 3.6: Interaction vertex between 3 gluons

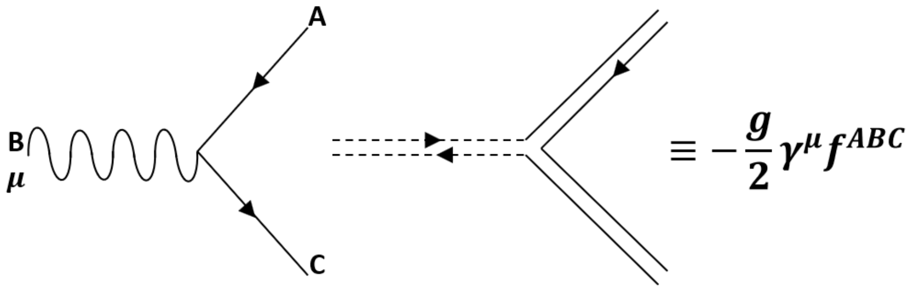


Figure 3.7: Interaction vertex between the gauge field and the fermion

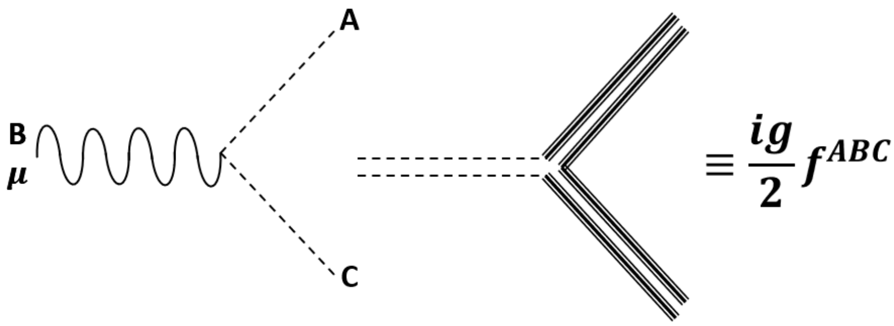


Figure 3.8: Interaction vertex between the gauge field and the Fadeev-Popov's ghosts

By combining these vertices and propagators in the appropriate way it is possible to construct all planar diagrams contributing to the correlation function we are computing, as will be done in the following chapter.

Chapter 4

Anomalous Dimension of $\mathcal{O}_0(x) = \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]$

The goal of this chapter is to compute the anomalous dimension of the single-trace operator $\mathcal{O}_0(x) = \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]$.

As mentioned previously, this calculation initiates the study of the spectrum of conformal dimensions of local operators. Specifically, we are investigating which operators within this spectrum remain protected, i.e., do not acquire anomalous dimensions. In order for the theory to admit a dual in AdS, the expectation is that only the stress energy tensor (and its composites) remain protected while all others acquire a dimension that grows with the marginal coupling.

Since the operator $\mathcal{O}_0(x)$ is the simplest operator in the theory, it presents itself as a natural first step for our investigation.

It should be noted that we are dealing with a composite operator, in the sense that it is made up of elementary fields and, as such, its renormalization requires renormalizing each of the fields comprising it, allowing for the possibility of mixing. This is reflected in a relation between bare and renormalized fields given through the so called Mixing Matrix defined as [16]:

$$\mathcal{O}^M_{bare} = \sum_N Z^{MN}(\mu)\mathcal{O}^N_{ren}(\mu), \tag{4.1}$$

where μ is some energy scale.

By noting that the left side of the equation is scale independent, and taking a derivative with respect to $\ln(\mu)$, one obtains the anomalous dimension as[16]:

$$\gamma(\mu) = Z^{-1}(\mu)\frac{dZ(\mu)}{d\ln\mu}. \tag{4.2}$$

In terms of the renormalized operators, the 2-point function is not only finite, but also has the form expected of a correlation function in a CFT, where the dimension corresponds to the classical dimension plus the anomalous one. Since, in dimensional regularization (wherein we will be evaluating the integrals in $d = 3 - \epsilon$ dimensions), the anomalous dimension appears as the coefficient of a $\frac{1}{\epsilon}$ pole, the study of this 2-point functions provides a way to read the anomalous dimension of the operator.

Concretely, by taking one of the operators in the 2-point function as being a renormalized probe, it is possible to extract the anomalous dimension directly by looking at the $\frac{1}{\epsilon}$ contribution.

To illustrate and clarify the procedure that will be used, we will start by studying a much simpler theory: the ϕ^4-theory.

4.1 Anomalous dimension in ϕ^4-theory

The ϕ^4-theory, that is the theory of a scalar field with a quartic interaction constitutes an easy example through which to illustrate the main ideas of this thesis.

The action for a real massless scalar is given by

$$S_\phi = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4!}\phi^4\right), \tag{4.3}$$

meaning that the only propagator in this theory is that of a scalar field, which in momentum space corresponds to

$$\Delta_F(p) = \frac{i}{p^2}. \tag{4.4}$$

Furthermore, ϕ^4-theory is a renormalizable field theory, in the sense that it is possible to renormalize the coupling in such a way as to avoid the infinities that plague calculations in QFTs in general. These UV divergences are associated to high-momentum contributions (hence UV) and in the Wilsonian approach to the renormalization group, they can be taken care off through means of introducing an energy cutoff and integrating out the high-energy contributions.

The flow along the renormalization group (often called RG flow) corresponds to the evolution of the couplings as one integrates out the high energy modes, and to which is associated a beta function, as defined in (2.2).
For the ϕ^4-theory in d-dimensions, the beta function to two-loop order is given in [15] as

$$\beta_{\phi^4}(\lambda) = -(4-d)\lambda + \frac{3\lambda^2}{16\pi^2} = \frac{3\lambda^2}{16\pi^2} - \lambda\epsilon, \tag{4.5}$$

where $\epsilon = 4 - d$. This tells us that the theory has two stationary points which, recalling Section 2, correspond to zeros of the beta function. One is the usual Gaussian fixed point where $\lambda = 0$ and the theory is free. The second corresponds to the Wilson-Fisher fixed point, where the coupling takes the value of $\lambda = \frac{16\pi^2}{3}\epsilon$.

The reason we chose this theory as an example is not only because it simple and has been thoroughly studied, but also because it has some interesting results already at the 1-loop level, unlike Chern-Simons theory. In Chern-Simons theory, the scalar operator we are concerned with, only gains anomalous dimension at 2-loops.
In this situation, however, the operator

$$\mathcal{O}_\phi(x) = \phi^2(x) \tag{4.6}$$

gains anomalous dimension at 1-loop, which can be seen from its two-point function:

$$G_\phi(x_1, x_2) = \langle \mathcal{O}_\phi(x_1)\mathcal{O}_\phi(x_2)\rangle = \langle \phi(x_1)\phi(x_1)\phi(x_2)\phi(x_2)\rangle \,. \tag{4.7}$$

The correlation function can be computed by making use of the path integral formalism, which tells us that the n-point function is given by

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle = \frac{\int D\phi \; \phi(x_1)\phi(x_2)...\phi(x_n)e^{iS_\phi}}{\int D\phi e^{iS_\phi}}, \tag{4.8}$$

as shown in [15]. The normalization factor in the denominator makes it so that disconnected diagrams do not contribute to the correlation function. Consequently, the correlator can be computed just from the connected Feynman diagrams.

As suggested in [17] when studying scalar operators in $\mathcal{N} = 4$ SYM, the anomalous dimension can be extracted by taking one of the operators as being already renormalized and considering it a probe to extract the divergent behaviour from the non-renormalized operator. The fields of the renormalized operator are applied at different points. In the free theory case this corresponds to

$$G_{\phi}^{(0)}(x,y_1,y_2) = \langle \phi(x)\phi(x)\phi(y_1)\phi(y_2)\rangle = 2\,\langle \phi(x)\phi(y_1)\rangle\,\langle \phi(x)\phi(y_2)\rangle\,. \tag{4.9}$$

By going to momentum space and making use of (4.4), while at the same time taking the Fourier transform of the external points y_1 and y_2, one arrives at the following correlator:

$$\begin{aligned} G_{\phi}^{(0)}(x,p,k) &= 2\int d^d y_1 d^d y_2 e^{-ipy_1}e^{-iky_2}\int \frac{d^d q_1 d^d q_2}{(2\pi)^{2d}}\Delta_F(q_1)\Delta_F(q_2)e^{iq_1(x-y_1)}e^{iq_2(x-y_2)} = \\ &= 2\int d^d q_1 d^d q_2 \Delta_F(q_1)\Delta_F(q_2)\delta(p+q_1)\delta(p+q_2)e^{i(q_1+q_2)x} = \\ &= -\frac{2}{p^2k^2}e^{-i(p+k)x}. \end{aligned} \tag{4.10}$$

In the first step of the previous calculation, the terms in y_1 and y_2 were grouped together and then integrated over. The result is the appearance of the Dirac-delta functions in the second line.

In terms of Feynman diagrams, this contribution corresponds to figure 4.1. This procedure

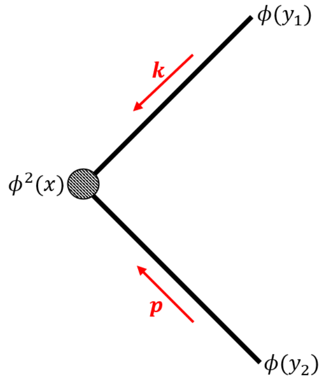


Figure 4.1: 2-point function for the operator ϕ^2 in the ϕ^4 theory

can be carried out to the next order in perturbation theory, by retaining higher-order terms in the expansion of the exponential of the action in equation (4.8). Doing so yields

$$G_{\phi}^{(1)}(x,y_1,y_2) = \left(-i\frac{\lambda}{4!}\right)\int d^d z_1\,\langle \phi^2(x)\phi(y_1)\phi(y_2)\phi^4(z_1)\rangle \quad . \tag{4.11}$$

Performing all possible Wick contractions and discarding those that correspond to disconnected diagrams leaves 3 contributions remaining. These all come with a combinatorics factor of 24 that cancels the 4! appearing in the denominator:

$$G_{\phi}^{(1)}(x,y_1,y_2) = -i\lambda \int d^d z_1 \Big(\langle\phi(x)\phi(z)\rangle \langle\phi(z)\phi(z)\rangle \langle\phi(z)\phi(y_1)\rangle \langle\phi(x)\phi(y_2)\rangle + \\ + \langle\phi(x)\phi(z)\rangle \langle\phi(z)\phi(z)\rangle \langle\phi(z)\phi(y_2)\rangle \langle\phi(x)\phi(y_1)\rangle + \\ + \langle\phi(x)\phi(z)\rangle^2 \langle\phi(z)\phi(y_1)\rangle \langle\phi(z)\phi(y_2)\rangle \Big). \tag{4.12}$$

Schematically, these terms correspond to the Feynman diagrams in figure 4.2. As was

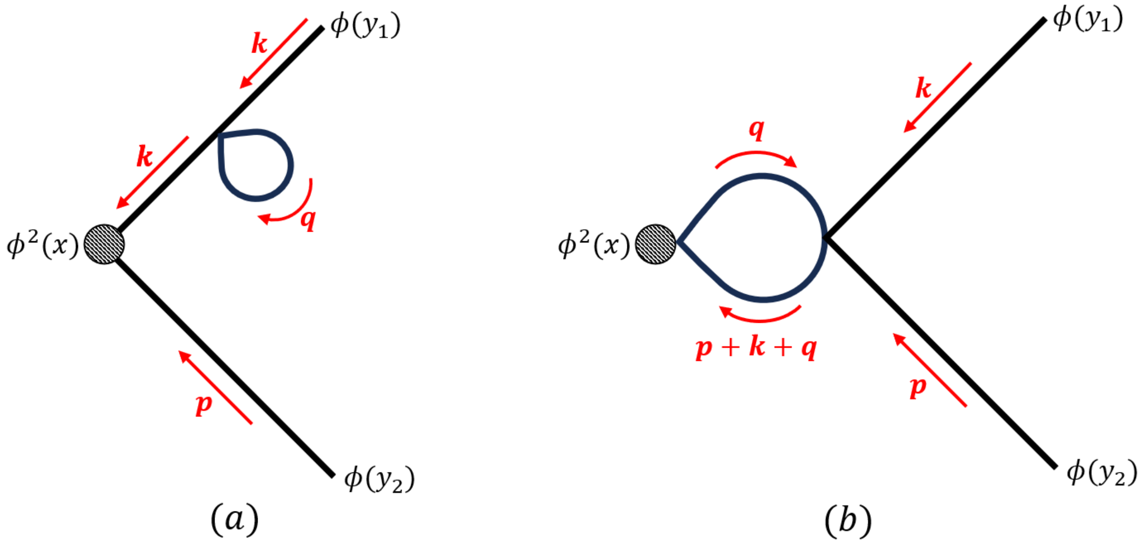


Figure 4.2: 1-loop connected diagrams contributing to the 2-point function. There is an analogous diagram to (a), with the interaction vertex in the other propagator

the case when considering the free-theory it is now possible to introduce the momentum representation of the propagators and take the Fourier transform of the external points.

$$G_{\phi}^{(1)}(x,p,k) = -i\lambda \int d^d z_1 d^d y_1 d^d y_2 \int \frac{d^d q_1 ... d^d q_4}{(2\pi)^{4d}} \Delta_F(q_1)...\Delta_F(q_4) \Big[\\ e^{i(q_3-q_1)z} e^{i(q_1+q_4)x} \left(e^{-i(q_3+p)y_1} e^{-i(q_4+k)y_2} + e^{-i(q_3+k)y_2} e^{-i(q_4+p)y_1} \right) + \\ + e^{i(q_1+q_2)x} e^{i(q_3+q_4-q_1-q_2)z} e^{-i(p+q_3)y_1} e^{-i(k+q_4)y_2} \Big]. \tag{4.13}$$

Integrating over the spatial coordinates introduces Dirac-delta functions, yielding:

$$G_{\phi}^{(1)}(x,p,k) = -i\lambda \int \frac{d^d q_1 ... d^d q_4}{(2\pi)^{4d}} \Delta_F(q_1)...\Delta_F(q_4) \Big[e^{i(q_1+q_4)x} \delta(q_3-q_1) \\ \big(\delta(q_3+p)\delta(q_4+k) + \delta(q_3+k)\delta(q_4+p)\big) + \\ + \delta(q_3+q_4-q_1-q_2)\delta(p+q_3)\delta(k+q_4) e^{i(q_1+q_2)x} \Big]. \tag{4.14}$$

The presence of the delta functions means that some of the integrals over the momenta are trivial. Performing these integrations leaves a single undetermined momentum, over which we are summing. Renaming this momentum as q, the resulting contribution is given as

$$G_{\phi}^{(1)}(x,p,k) = -i\lambda \frac{e^{-i(p+k)x}}{p^2k^2}\left[\left(\frac{1}{p^2}+\frac{1}{k^2}\right)\int \frac{d^dq}{2\pi^d}\frac{1}{q^2} + \int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2(q+p+k)^2}\right]. \tag{4.15}$$

The regularization scheme chosen was dimensional regularization. From the more detailed computation in Appendix C, it is clear that the first integral is immediately 0*. The second integral is given in terms of a G-function (C.11), such that the result corresponds to:

$$G_{\phi}^{(1)}(x,p,k) = -\lambda \frac{-e^{i(p+k)x}}{k^2p^2}\frac{G(1,1)}{[(p+k)^2]^{2-\frac{d}{2}}}. \tag{4.16}$$

The anomalous dimension of this operator can be read by performing and ϵ-expansion in $d = 4-\epsilon$ dimensions, and looking at the pole in ϵ. Expanding the G-function in ϵ results in

$$G_{\phi}^{(1)}(x,p,k) = -\lambda \frac{e^{i(p+k)x}}{k^2p^2}\frac{1}{8\pi^2\epsilon} + \mathcal{O}(\epsilon^0), \tag{4.17}$$

meaning that the correlation function of the operator ϕ^2 up to 1-loop, is given as:

$$G_{\phi}^{1-\text{loop}}(x,p,k) = -2\frac{e^{i(p+k)x}}{k^2p^2}\left(1+\frac{\lambda}{16\pi^2\epsilon}\right) + \mathcal{O}(\lambda^2), \tag{4.18}$$

Equation (4.18) implies that the anomalous dimension, γ_{ϕ^2}, is given by

$$\gamma_{\phi^2} = \frac{\lambda}{16\pi^2} + \mathcal{O}(\lambda^2), \tag{4.19}$$

a result that is in accordance with the known value for the dimension of this operator [15].

Having established in the previous sections the required theoretical fundamentals to understand how to work with Chern-Simons theory it is now the time to shift attention to the main focus of the work, that is, the computation of the anomalous dimensions of the single trace operator $\mathcal{O}_0(x) = \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]$. The anomalous dimension corresponds to a correction to the bare dimension of the operator and it appears in the 2-point function given in equation (2.27).

*By introducing a Schwinger parameter, the integral $\int \frac{d^dq}{q^2}$ becomes $\int d^dq \int_0^\infty du e^{-uq^2} \propto \int_0^\infty du u^{-d/2}$. By noticing that the contribution on the region from 0 to 1 cancels out the contribution from 1 to ∞ it becomes clear this integral is null. Thus, as argued in [18], this type of scaleless integral is 0 in dimensional regularization

As was done previously for the operator ϕ^2 in the theory of a massless scalar with a quartic interaction, so too in this instance can the anomalous dimension be extracted from the correlator

$$\begin{aligned} G_0(x,y_1,y_2) &= \langle \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]\mathrm{Tr}[\bar{\Psi}(y_1)\Psi(y_2)]\rangle = \\ &= \frac{\delta^{AD}\delta^{BC}}{4}\langle \bar{\Psi}^A(x)\Psi^D(x)\bar{\Psi}^C(y_1)\Psi^B(y_2)\rangle\,. \end{aligned} \tag{4.20}$$

In the last equation, the trace identity given in (3.10) was used.
This correlation function can be evaluated at different orders in perturbation theory, by lowering the appropriate interaction terms in the action.

In a free theory, this is straightforward to compute. The result is merely given by the Wick contractions of the fields, as shown in figure 4.3, and corresponding to equation (4.21):

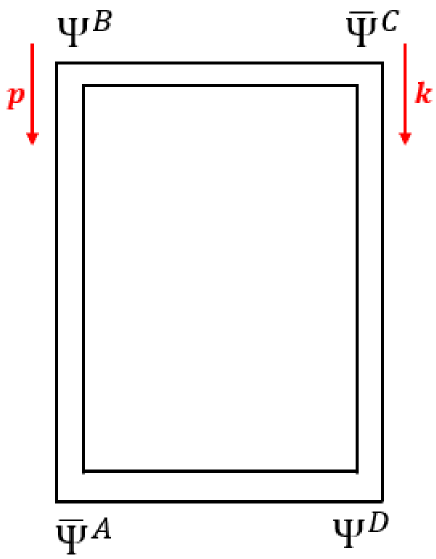


Figure 4.3: 2-point function in a free theory

$$G_0^{(0)}(x,y_1,y_2) = -\frac{\delta^{AD}\delta^{BC}}{4}\langle \bar{\Psi}^A(x)\Psi^B(y_2)\rangle\langle \bar{\Psi}^C(y_1)\Psi^D(x)\rangle\,. \tag{4.21}$$

Since at higher orders it will be more convenient to work in momentum space since the computations are more straightforward, we repeat what was just done before in ϕ^4 and introduce the necessary momentum representations. The propagator in position space can be rewritten as the Fourier transform of the propagator in momentum space, yielding

$$G_0^{(0)}(x,y_1,y_2) = -\frac{\delta^{AD}\delta^{BC}}{4}\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}S_F^{AB}(p_1)e^{ip_1(x-y_2)}S_F^{CD}(p_2)e^{ip2(y_1-x)}, \tag{4.22}$$

where $S_F(p)$ corresponds to the fermion propagator with momentum p. In order to obtain the momentum-space correlation function at tree-level, $G_0^{(0)}(x,p,k)$, we should, at this point, also take the Fourier transform of the probes, as in (4.23).
This approach has the advantage of avoiding the two integrals in (4.22).

$$\begin{aligned}
&G_0^{(0)}(x,p,k) = \int d^dy_1 d^dy_2 G^{(0)}(x,y_1,y_2) e^{-ipy_1} e^{-iky_2} = \\
&= -\frac{\delta^{AD}\delta^{BC}}{4} \int d^dy_1 d^dy_2 \int \frac{d^dp_1}{(2\pi)^d} \int \frac{d^dp_2}{(2\pi)^d} S_F^{AB}(p_1) S_F^{CD}(p_2) e^{ip_1(x-y_2)} e^{ip_2(y_1-x)} e^{-ipy_1} e^{-iky_2} = \\
&= -\frac{\delta^{AD}\delta^{BC}}{4} \int \frac{d^dp_1}{(2\pi)^d} \int \frac{d^dp_2}{(2\pi)^d} S_F^{AB}(p_1) S_F^{CD}(p_2) (2\pi)^{2d} \delta(p-p_2)\delta(k+p_1) e^{-i(p_2-p_1)x} = \\
&= -\frac{\delta^{AD}\delta^{BC}}{4} S_F^{AB}(-k) S_F^{CD}(p) e^{-i(p+k)x} = \frac{\delta^{AD}\delta^{BC}\delta^{AB}\delta^{CD}}{4} \frac{\gamma_{ab}\cdot k}{k^2} \frac{\gamma_{ba}\cdot p}{p^2} e^{-i(p+k)x} = \\
&= \frac{N^2}{4} \mathrm{Tr}[\gamma^\mu \gamma^\nu] \frac{k_\mu p_\nu}{k^2 p^2} e^{-i(p+k)x} = \frac{N^2}{4} 2g^{\mu\nu} \frac{k_\mu p_\nu}{k^2 p^2} e^{-i(p+k)x} = \\
&= \frac{N^2}{2} \frac{k\cdot p}{k^2 p^2} e^{-i(p+k)x}
\end{aligned} \tag{4.23}$$

To obtain the result in (4.23), the definitions of the propagators computed in the previous section were used, as well as the expression for the trace of a product of two gamma matrices given in (3.8).

4.2 Correlation function at 1 loop

It is possible to compute the correlation function at the next order in perturbation theory. This contribution results from lowering the interaction terms twice. It is necessary to have two vertices to obtain a non-zero contribution since the gauge fields appears an odd number of times in each one and cannot, therefore, be contracted.
Furthermore, since the ghosts are only coupled to the gauge fields and the vertices appearing at 1-loop must necessarily involve the fermions, it is immediate to conclude that there will not be any contribution involving the ghosts at this order. The very same argument applies to the vertex that involves three gluons.

Consequently, there is only one interaction (appearing twice) that must be considered at this order. This single interaction results, in the planar limit, in the two Feynman diagrams shown in figure 4.4. These 2 diagram are analogous to the ones in figure 4.2, concerning the ϕ^4 theory, with the difference that here we are dealing with a cubic interaction and so the gluon propagator must be between 2 different points (and also why we must go to second order perturbation theory). For each vertex being lowered, there will

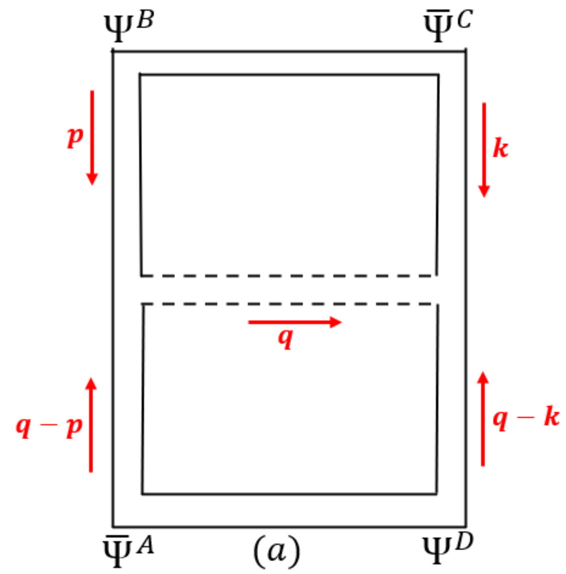


Figure 4.4: Diagrams contributing to the 1-loop 2-point correlation function. There is also a contribution similar to diagram (b) with the self-energy correction coming from the right propagator

be an extra integral over the point of insertion of the interaction:

$$
\begin{aligned}
G_0^{(1)}(x,y_1,y_2) &= \left\langle \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]\mathrm{Tr}[\bar{\Psi}(y_1)\Psi(y_2)] \left(\frac{g f^{EFG}}{2} \gamma^\mu \int d^3 z_1 \bar{\Psi}^E(z_1) A_\mu^F(z_1) \Psi^G(z_1) \right)^2 \right\rangle = \\
&= \frac{\delta^{AD}\delta^{BC}}{4} \frac{f^{EFG} f^{HIJ} g^2}{4} \gamma^\mu \gamma^\nu \int d^3 z_1 d^3 z_2 \times \langle A_\mu^F(z_1) A_\mu^I(z_2) \rangle \\
&\quad \times \left\langle \bar{\Psi}^A(x)\Psi^D(x)\bar{\Psi}^C(y_1)\Psi^B(y_2)\bar{\Psi}^E(z_1)\Psi^G(z_1)\bar{\Psi}^H(z_2)\Psi^J(z_2) \right\rangle .
\end{aligned}
\tag{4.24}
$$

In equation (4.24), it was possible to separate the gauge fields from fermions since these different fields commute with each other (but not amongst fields of the same type).
It is also important to keep in mind that there are spinor indices that are being summed over and are being omitted. These indices appear in all fermions, as well as in the gamma matrices.
Evaluating the contribution in equation (4.24) requires both performing the Wick contractions, as well as the integration over the internal vertices. As was done in the free-theory case, it is possible to take the Fourier transform of the propagators as well as of the 2 external points y_1 and y_2.

This process was implemented on `MATHEMATICA` in the way described below.

First, a function we called `ContrA` was created in order to perform the Wick contractions of the gauge-fields. This function acts by receiving as an input a product of fields and outputting an expression given as a product of propagators of said fields.

This function was constructed and defined through the code in 4.1. It takes the first field in the input and groups it with another one (resulting in a propagator involving these two fields), removing both from the initial product that it received. This means that assuming we started with a product of $2k$ fields, we would have at this point a propagator between two of those fields (one of which being the first one in the input) times a product of $2k-2$ fields.

```
Clear[ContrA]
ContrA[a1_Times] := Sum[PropA[a1[[1]], a1[[kk]]] ContrA[
    Times @@ (Drop[Drop[List @@ a1, {kk}], {1}])],
    {kk, 2, Length[a1]}]
ContrA[1] = 1;
ContrA[a_ + b_] := ContrA[a] + ContrA[b]
ContrA[a_?NumericQ b_] := a ContrA[b]
```

Listing 4.1: Function for performing the Wick contractions of the gauge fields

By contracting the first field with each of the others in the input we obtain a sum of $2k-1$ terms of the previous type. Then, to each of those terms we apply the function `ContrA` once again, removing from each of them two other fields and forming a new propagator. This process is then iterated until the function `ContrA` is acting on 1, at which point it outputs 1.

The last two lines of the code ensure that the function `ContrA` acts on the sum by acting on each of its terms and also that numerical constants are factored out.

An example of the action of the function `ContrA` is given in 4.2, where the input corresponds to a product of four gauge-fields. The notation being used is such that the gauge-fields are themselves functions of 3 variables which, in order, stand for the color-index, the space-time index and the position they are applied at.

```
In[1]:=
    ContrA[𝔸[E][μ][x]𝔸[F][ν][y]𝔸[G][ρ][z]𝔸[H][σ][w]]

Out[1]=
    PropA[𝔸[E][μ][x],𝔸[H][σ][w]] PropA[𝔸[F][ν][y],𝔸[G][ρ][z]]+
    PropA[𝔸[E][μ][x],𝔸[G][ρ][z]] PropA[𝔸[F][ν][y],𝔸[H][σ][w]]+
    PropA[𝔸[E][μ][x],𝔸[F][ν][y]] PropA[𝔸[G][ρ][z],𝔸[H][σ][w]]
```

Listing 4.2: Example of a Wick contraction of the gauge fields

In order to perform the Wick contractions of the fermions, an analogous function can be used. The only subtle point that must be taken into account is the fact that fermions anti-commute. Consequently, each time it is necessary to switch two fermions, we will gain a minus sign. In terms of implementing this in MATHEMATICA, the only alteration that must be made is the inclusion of a factor of $(-1)^k$ when constructing the propagator, as indicated in (4.3).

```
    Clear[ContrΨ]
    ContrΨ[a1_CenterDot] := Sum[PropΨ[a1[[1]], a1[[kk]]]
        (-1)^kk ContrΨ[CenterDot @@ (Drop[Drop[List @@ a1,
        {kk}], {1}])],{kk, 2, Length[a1]}]
    ContrΨ[CenteDot @@ {}] = 1;
    ContrΨ[a_ + b_] := ContrΨ[a] + ContrΨ[b]
    ContrΨ[a_?NumericQ b_] := a ContrΨ[b]
```

Listing 4.3: Function for performing the Wick contractions of the fermions

We opted to use `CenterDot` instead of the usual product when working with fermions since this builtin operation in MATHEMATICA is not commutative. This is an important distinction since fermions anti-commute and therefore it is important to keep track of the order they appear when performing the Wick contractions. This fact is a subtlety that does not matter when working with the gauge-fields but that must be considered both for the fermions and the ghosts and that was taken into account when constructing our code.

With all Wick contractions done it is then possible to substitute the propagators. At this point we are still working in position space because this allows us to convert each of the terms in the 1-loop contribution into a graph and then use `MATHEMATICA` builtin functions to eliminate disconnected graphs. Therefore, there will not be any disconnected Feynman diagrams erroneously being considered.

Having removed the disconnected terms we then take the necessary Fourier transforms, similarly to what was done in the free-theory case, to obtain the correlator in momentum space. This will be shown more clearly for the 2-loop analysis, since that is the most relevant one, but is in essence, a repeat of what was done for the massless scalar field.

Throughout all this process we are keeping track of the spinor indices in order to know in which order to perform the trace of the gamma matrices appearing in our expression.

The result is given as a sum of integrals over an internal momentum which we call q, as well as of 2 external momentum, p and k. At 1-loop, all contributions are variations of a few different structures presented in (4.26). For ease of reading, we introduce the following notation for a common pre-factor:

$$T^{\mu_3}(k,p) = g^2 N^4 \pi e^{i(k+p)x} \epsilon^{\mu_1\mu_2\mu_3} k_{\mu_2} p_{\mu_1}. \tag{4.25}$$

This allows us to express the 1-loop contributions as:

$$\begin{aligned}
T_1 &= \frac{T^{\mu_3}(k,p)}{2k^2}\left(\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}}{q^2(k-q)^2(k+p-q)^2} + p \leftrightarrow k\right)\\
T_2 &= \frac{T^{\mu_3}(k,p)}{2p^2}\left(\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}}{q^2(p-q)^2(k+p-q)^2} + p \leftrightarrow k\right)\\
T_3 &= \frac{T^{\mu_3}(k,p)}{k^2p^2}\left(\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}}{(p-q)^2(k+p-q)^2} + p \leftrightarrow k\right)\\
T_4 &= \frac{T^{\mu_3}(k,p)}{2k^2p^2}\left(\int \frac{d^dq}{(2\pi)^d}\frac{(k-q)^2 q_{\mu_3}}{q^2(p-q)^2(k+p-q)^2} + p \leftrightarrow k\right)\\
T_5 &= \frac{(k\cdot p)T^{\mu_3}(k,p)}{k^2p^2}\left(\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}}{q^2(p-q)^2(k+p-q)^2} + p \leftrightarrow k\right)
\end{aligned} \quad . \tag{4.26}$$

In the previous equation, $p \leftrightarrow k$ stands for an equal integral with the external momenta exchanged. That these contributions come in pairs is in accordance with our expectations, since the correlation function should be invariant under permutation of the external momenta. This fact will end up not mattering since the overall contribution is 0.

From (4.26), it is easy to see that all these terms will give a null contribution to the correlator. This result is immediate since all integrals contain a single space-time index in the variable of integration which, after integrating, will be either in k or p. As $T^{\mu_3}(k,p)$ contains a Levi-Civita symbol being contracted with p and k, the fact that the new index will be attributed to one of these two vectors means that there is an even tensor-structure being contracted with the antisymmetric ϵ-symbol and consequently the result is 0.

In order to reduce the overall expression to these few contributions we made repeated use of the antisymmetry of the Levi-Civita symbol. This was particularly useful to eliminate integrals with two of the indices $\{\mu_1,\ \mu_2,\ \mu_3\}$ in the numerator of the integral, as is the case in:

$$T^{\mu_3}(k,p)\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_1}q_{\mu_2}}{q^2(p-q)^2}.$$

Alongside this simplifications it was also useful to directly compute a few of the integrals, which immediately cancel. For instance, the integral in

$$T_6 = \frac{e^{i(k+p)x}g^2\pi N^4}{2k^4p^2}\epsilon_{\mu_1\mu_2\mu_4}k^{\mu_2}k^{\mu_3}p^{\mu_1}\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}q^{\mu_4}}{q^2(k+q)^2} \tag{4.27}$$

is, in its most general form, equal to

$$\int \frac{d^dq}{(2\pi)^d}\frac{q_{\mu_3}q^{\mu_4}}{q^2(k+q)^2} = f_1(k)\delta^{\mu_4}_{\mu_3} + f_2(k)k_{\mu_3}k^{\mu_4}. \tag{4.28}$$

In the previous equation, the unknown functions $f_1(k)$ and $f_2(k)$ are functions of only k, with the appropriate dimension such that the overall integral goes as k^2. Regardless of the specificity of these functions, the tensor structure of the integral means that when contracted with the Levi-Civita symbol, the contribution of the term in (4.27) is 0.

This analysis allows us to conclude that the correlation function in (4.20) does not receive corrections at 1-loop and, consequently, there is no contribution to the anomalous dimension at 1-loop. This very particular result comes as a consequence of the symmetries of the Lagrangian. The existence of the antisymmetric Levi-Civita tensor results in the fact that all contributions at 1-loop cancel out. This is the typical result for Chern-Simons theories [19].

4.3 Correlation function at 2 loops

After verifying that our operator does not gain anomalous dimension at 1-loop, the natural next step is to carry out the computation at the next order in perturbation theory. For the same reason that lowering a single vertex yields a null contribution, so too the diagrams with 3 vertices evaluate to zero.

Lowering a total of four vertices of interaction in the action yields the 2-loops contribution. At this order it is expected that all three vertices listed in chapter 3 are involved in the diagrams contributing to the correlator.
Evaluating these diagrams is the focus of the next section, where, beyond listing the contributing diagrams, the computation of an explicit example is carried out.

4.3.1 2-loop Feynman Diagrams

As happened when studying this correlator at 1-loop, there exist self-energy type diagrams contributing, namely those in figure 4.5. Alongside these three, there are three other iden-

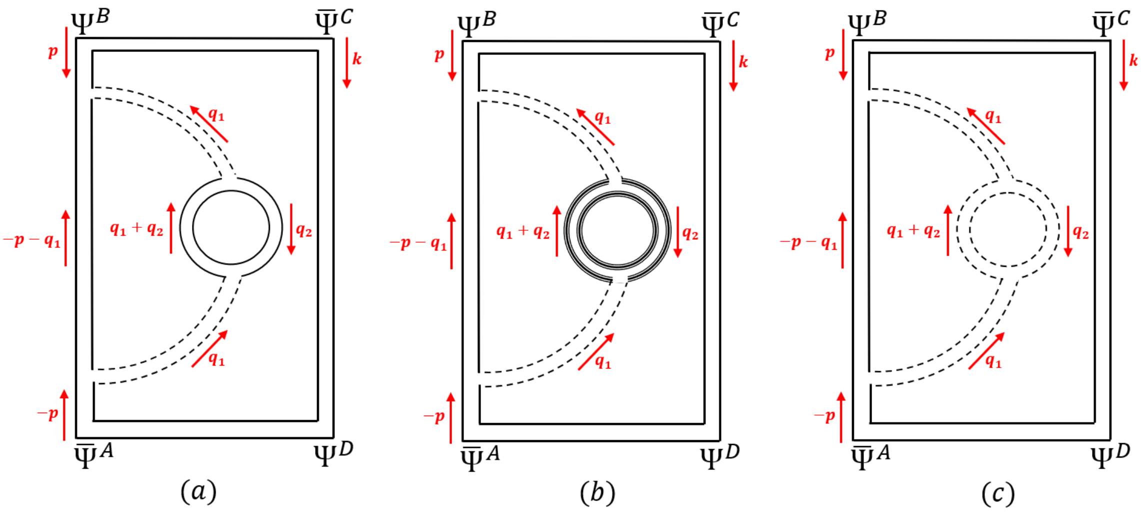


Figure 4.5: Self-energy type diagrams

tical diagrams with the self-energy correction in the other propagator. The diagrams in this figure have the same momentum structure, but with different interaction vertices and propagators, as a consequence of considering different terms when expanding the exponential of the action in equation (4.8).
Their contribution can be computed using the Feynman rules derived in Appendix B.

These diagrams come with an appropriate combinatoric factor, resulting from the different permutations of the fields. For instance, looking at a particular contracting (figure 4.6) of the type yielding diagram (a) in figure 4.5, there is a contribution at 2-loops, $G_0^{(2.1)}(x, y_1, y_2)$:

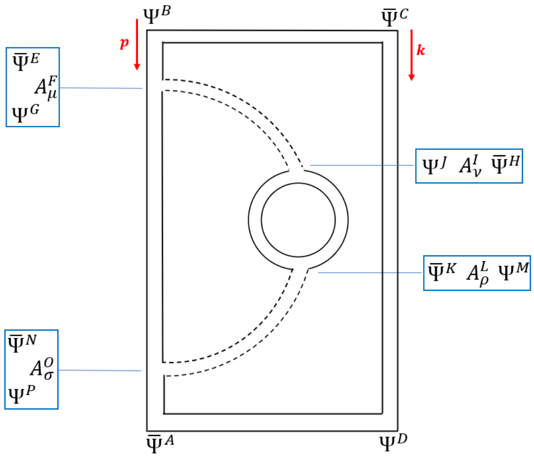


Figure 4.6: A specific contraction yielding diagram (a) in 4.5

$$
\begin{aligned}
G_0^{(2.1)}(x, y_1, y_2) = \frac{\delta^{AD}\delta^{BC}}{4}\frac{g^4}{16}\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}\gamma^{\rho}_{ef}\gamma^{\sigma}_{gh} f^{EFG} f^{HIJ} f^{KLM} f^{NOP} \int d^d z_1 ... d^d z_4 \Big[\\
\Delta^{FI}_{\mu\nu}(z_1 - z_2)\Delta^{OL}_{\sigma\rho}(z_4 - z_3) S_F^{AP}(x - z_4)_{ih} S_F^{NG}(z_4 - z_1)_{gb} \times \\
\times S_F^{EB}(z_1 - y_2)_{aj} S_F^{CD}(y_1 - x)_{ji} S_F^{HM}(z_2 - z_3)_{cf} S_F^{KJ}(z_3 - z_2)_{ed} \Big],
\end{aligned}
\tag{4.29}
$$

where $\Delta^{FI}_{\mu\nu}(z_1 - z_2)$ corresponds to the gauge propagator between the points z_1 and z_2, with color contribution δ^{FI} and two open indices, μ and ν. Similarly, $S_F^{AP}(x - z_4)_{ih}$ is the fermion propagator with spinor indices i and h, between the points x and z_4 and where A and P are the color indices.

Taking the Fourier transform of the external points y_1 and y_2 and introducing the momentum-space representation of the propagators gives an expression similar to 4.22. If we extract the color contribution and perform the contractions of the color indices, a factor of $\frac{N^4}{4}$ appears. Integrating then over the position-space coordinates $y_1, y_2, z_1, ..., z_4$ results in a contribution:

$$G_0^{(2.1)}(x,p,k) = \frac{N^4 g^4}{64}\gamma^\mu_{ab}\gamma^\nu_{cd}\gamma^\rho_{ef}\gamma^\sigma_{gh}\int \frac{d^d q_1 ... d^d q_8}{(2\pi)^{2d}} e^{i(q_6-q_3)x}\delta(p-q_6)\delta(k+q_5)$$
$$\times\,\delta(q_4-q_1-q_5)\delta(q_1-q_7+q_8)\delta(q_2+q_7-q_8)\delta(q_3-q_2-q_4) \quad . \tag{4.30}$$
$$\times\,\Delta_{\mu\nu}(q_1)\Delta_{\sigma\rho}(q_2)S_F(q_3)_{ih}S_F(q_4)_{gb}S_F(q_5)_{aj}S_F(q_6)_{ji}S_F(q_7)_{cf}S_F(q_8)_{ed}$$

The factor of $\frac{1}{(2\pi)^{2d}}$ results from taking the Fourier representation of the propagators (there are eight propagators and each of which comes with $\int \frac{d^d q_i}{(2\pi)^d}$) and from performing the integrals over the six spatial coordinates which yield $(2\pi)^d\delta(\sum_j p_j)$.

Performing the trivial integrals over q_3, q_5, q_6 and q_7, this equation can be expressed as:

$$G_0^{(2.1)}(x,p,k) = \frac{N^4 g^4}{64}\gamma^\mu_{ab}\gamma^\nu_{cd}\gamma^\rho_{ef}\gamma^\sigma_{gh}\int \frac{d^d q_1 d^d q_2 d^d q_4 d^d q_8}{(2\pi)^{2d}} e^{i(p-q_2-q_4)x}\delta(q_4-q_1+k)$$
$$\times\,\delta(q_2+q_1)\Delta_{\mu\nu}(q_1)\Delta_{\sigma\rho}(q_2)S_F(q_2+q_4)_{ih}S_F(q_4)_{gb}S_F(-k)_{aj}$$
$$\times\,S_F(p)_{ji}S_F(q_1+q_8)_{cf}S_F(q_8)_{ed} = \quad . \tag{4.31}$$
$$= \frac{N^4 g^4}{64}\gamma^\mu_{ab}\gamma^\nu_{cd}\gamma^\rho_{ef}\gamma^\sigma_{gh}e^{i(p+k)x}\int \frac{d^d q_1 d^d q_8}{(2\pi)^{2d}}\Delta_{\mu\nu}(q_1)\Delta_{\sigma\rho}(-q_1)$$
$$\times\,S_F(-k)_{ih}S_F(q_1-k)_{gb}S_F(-k)_{aj}S_F(p)_{ji}S_F(q_1+q_8)_{cf}S_F(q_8)_{ed}$$

In the last line the integration over both q_2 and q_4 was performed, eliminating all delta-functions in the expression and leaving only two integrals, over q_1 and q_8, remaining.
Renaming the integration variables as q_1 and q_2, and taking into account that there are extra γ matrices coming from each fermion propagator, as well as an extra Levi-Civita symbol from each of the gauge propagators, the resulting contribution can be written as:

$$G_0^{(2.1)}(x,p,k) = -\frac{\pi^2 N^4 g^4}{4}e^{i(k+p)x}\epsilon_{\mu\nu\beta_1}\epsilon_{\mu\nu\beta_2}\mathrm{Tr}[\gamma_\mu\gamma_{\alpha_2}\gamma_\sigma\gamma_{\alpha_1}\gamma_{\alpha_4}\gamma_{\alpha_3}]\mathrm{Tr}[\gamma_{\alpha_5}\gamma_\nu\gamma_{\alpha_6}\gamma_\rho]\times$$
$$\times\,\frac{k^{\alpha_1}k^{\alpha_3}p^{\alpha_4}}{k^4p^2}\int\frac{d^d q_1 d^d q_2}{(2\pi)^{2d}}\frac{q_1^{\beta_1}q_1^{\beta_2}(q_1-k)^{\alpha_2}(q_1+q_2)^{\alpha_5}q_2^{\alpha_6}}{q_1^4(q_1-k)^2(q_1+q_2)^2q_2^2} \quad . \tag{4.32}$$

There are two traces of products of gamma matrices that come about from contracting the spinor indices. Furthermore, all indices are fully contracted and it is possible to expand both the integrand and the contribution of the γ-matrices in such a way as to obtain only scalar integrals.
As the large number of figures in this section indicates, the 2-loop computation is quite more arduous than the 1-loop computation. This happens since, at this order, there are six extra fields coming from the additional two vertices. Consequently, the correlation

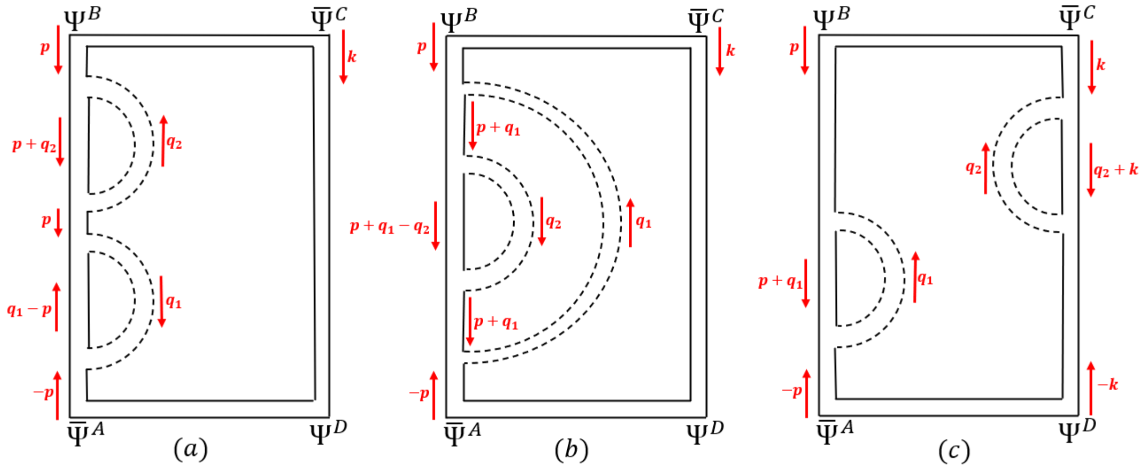


Figure 4.7: Self-energy type diagrams. There are identical diagrams with the self energy correction in the other propagator.

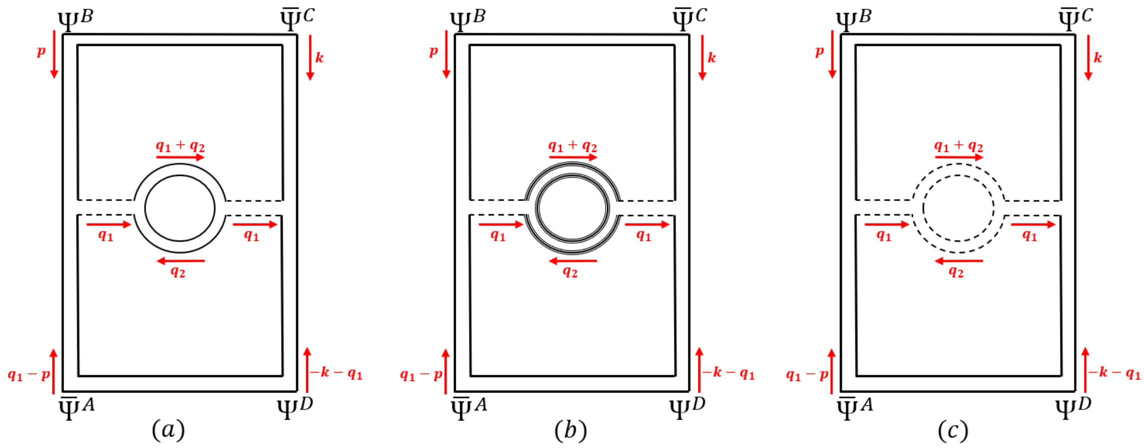


Figure 4.8: Diagrams with the exchange of a gluon between the two propagators

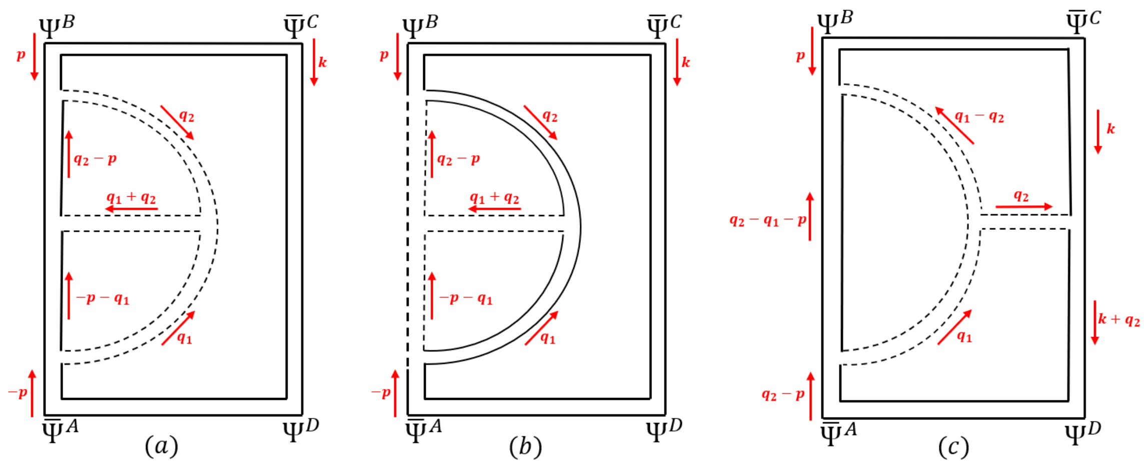


Figure 4.9: Diagrams that contain a 3-gluon vertex

function being computed results from the Wick contractions of the sixteen fields involved, which means there are 16! (roughly 2×10^{13}) possible terms. Some of those contractions

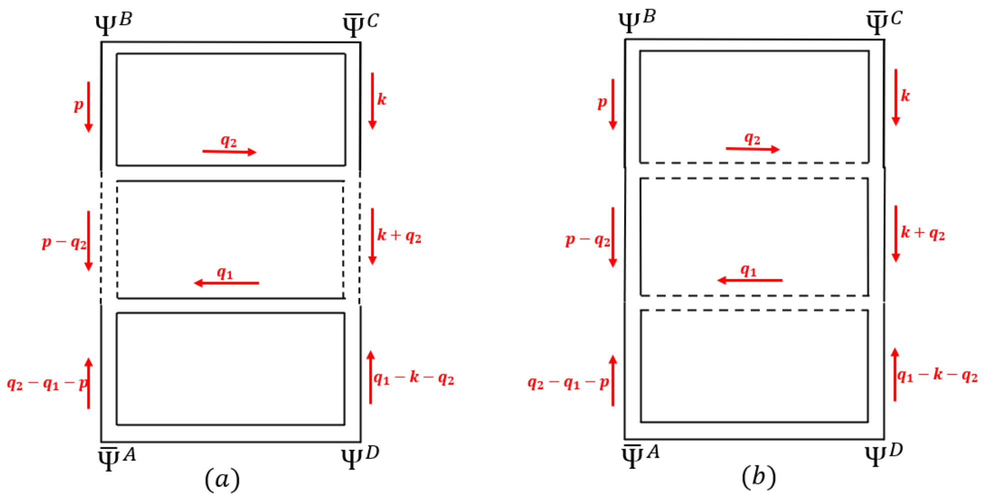


Figure 4.10: Diagrams (a) corresponds to the exchange of two gluons. Diagram (b) is a rotated version of diagram (a)

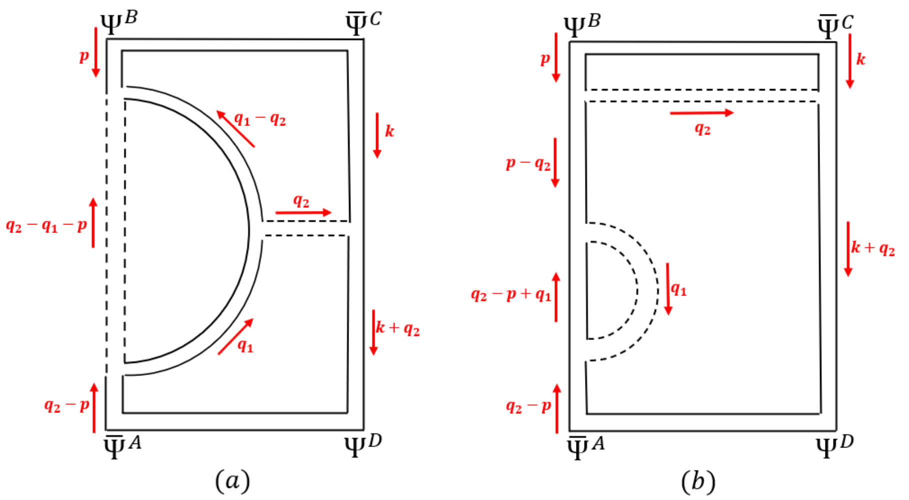


Figure 4.11: More 2-loop diagrams

correspond to either disconnected diagrams or non-planar diagrams, both of which can be discarded since the first do not contribute to the correlation function and the second are subdominant in the large N limit.

This process was automated in `MATHEMATICA` as was the case for the 1-loop calculation. By first separating the different types of fields before performing the Wick contractions it was ensured that the contractions involved at most twelve fields. Even taking this into account, we are still dealing with massive expressions, far beyond what could be handled if we were done the computations by hand.

Furthermore, since the correlation function involves a large number of fermion fields, there is an extra layer of complexity in keeping track of the minus sign each time we need to exchange two of these fields. In this, `Mathematica` proved itself invaluable.

The code used followed the same approach as for the 1-loop calculations, requiring only some modifications to optimize it to deal with the far greater number of terms. As mentioned previously, the fact that we are dealing with a large number of Gamma matrices means that there are many more terms.
However, the main issue we faced, arose after having fully contracted all indices, at the point the resulting expression depended only on scalar integrals. This in itself is already different from what happened at 1-loop, where the variables of integration appeared in the numerator with indices that were being contracted with the Levi-Civita symbol. In this instance, all integrals are scalar integrals in 2 distinct variables of integration. That is, all terms in our expression involve four momentum variables, namely the two external momenta p and k and the two integration variables q_1 and q_2.

In order to make progress in the calculation we look towards the method of Integration By Parts (IBP). This is the standard approach to compute multi-loop Feynman integrals and was, consequently, the tool we immediately tried to deploy. It relies on using the IBP identities to determine the master integrals, who get their name from the fact that they are a basis in which all others can be expressed as. In other words, the IBP Reduction procedure allows us to rewrite all the integrals we encountered (and there were over 1400 different ones, after expanding all the dot products in the numerator) as a linear combination of a few master integrals.

This method is described in greater detail in the next section, wherein we give an example of its use in a more familiar situation.

4.3.2 IBP Reduction

The IBP Reduction procedure is a powerful tool at our disposal. It is described in this section, following the explanations in the great textbook by Vladimir A. Smirnov [18].

The integrals we are evaluating are integrals over two undetermined momenta of products of scalar propagators (each with a momentum that is a linear combination of the four momenta involved in the expression):

$$I(p,k) = \int d^d q_1 \int d^d q_2 \prod_{j=1}^{L} \Delta_j(p_j). \tag{4.33}$$

The upper limit in the product corresponds to the number of terms in the denominator, which varies for each integral and. Furthermore, as mentioned, $p_j \equiv p_j(p, k, q_1, q_2)$.

The IBP reduction is a method of evaluating multi-loop Feynman integrals and relies on the property that the derivative of an integral with respect to a momentum is the same as the integral of the derivative [18]. In other words,

$$\int d^d q_1 \int d^d q_2 \frac{\partial}{\partial q_i^\mu} \prod_{j=1}^{L} \Delta_j(p_j) = 0 \qquad i = 1, 2. \tag{4.34}$$

By using this property of dimensionally regularized Feynman integrals, it is possible to obtain a recursive relation between integrals that can be iterated until they are all written in terms of a smaller number of integrals, which are, in principle, easier to compute.

As an example we can consider an integral that will appear often in our calculations. It is the 1-loop integral given by

$$I(\alpha, \beta) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^\alpha [(q-p)^2]^\beta}, \tag{4.35}$$

that is computed through means of the introduction of Schwinger parameter in Appendix C. Despite the fact that the result of (4.35) is known in terms of a G-function, it is, nonetheless, a simple example through which this method can be illustrated.

Using the IBP identity given in (4.34), which in this situation corresponds to taking a derivative in q, and contracting with q^μ yields:

$$\int \frac{d^d q}{(2\pi)^d} \frac{\partial}{\partial q^\mu} \frac{q^\mu}{(q^2)^\alpha [(q-p)^2]^\beta} = 0. \tag{4.36}$$

Taking the derivative of the integrand in the previous equation and then contracting all indices it is possible to obtain an equation that involves the original integral in (4.35), given as:

$$d \cdot I(\alpha, \beta) - 2\alpha I(\alpha, \beta) + 2\beta \int \frac{d^d q}{(2\pi)^d} \frac{q \cdot (p-q)}{(q^2)^\alpha [(q-p)^2]^{\beta+1}} = 0. \tag{4.37}$$

Now, expanding the dot product in the integral and using $p \cdot q = \frac{1}{2}\left(q^2 + p^2 - (q-p)^2\right)$, results in

$$d \cdot I(\alpha, \beta) - 2\alpha I(\alpha, \beta) - \beta I(\alpha - 1, \beta + 1) + \beta p^2 I(\alpha, \beta + 1) - \beta I(\alpha, \beta) = 0. \tag{4.38}$$

Solving for $I(\alpha, \beta+1)$ and renaming $\beta \to \beta - 1$, the following equation (valid for $\beta > 1$) arises:

$$I(\alpha, \beta) = \frac{d - 2\alpha - \beta + 1}{\beta - 1} I(\alpha, \beta - 1) - \frac{1}{p^2} I(\alpha - 1, \beta) \tag{4.39}$$

A simple change of variables in (4.35) makes it clear that $I(\alpha, \beta) = I(\beta, \alpha)$. Therefore, switching $\alpha \leftrightarrow \beta$ in the first term of the right-hand-side, and then setting the second argument to 1 results in:

$$I(\alpha, 1) = -\frac{d - \alpha - 1}{(\alpha - 1)q^2} I(\alpha - 1, 1). \tag{4.40}$$

Equation (4.40), alongside the symmetry property of $I(\alpha, \beta)$ shows that all integrals of the form of equation (4.35) can be determined in terms of $I(1,1)$, up to a function of α, β, d and q that can be determined recursively. For this reason it is said that $I(1,1)$ is a master integral (and the only one) for this family of integrals.

The fact that it is possible to greatly reduce the number of integrals that require explicit computation by solving some algebraic equations is a massive simplification of the problem. These principles are implemented in `MATHEMATICA` through the `LiteRed` package [20], which has been used extensively and with great success.

The main issue that arose when deploying this method, came from the fact that the different integrals were written in terms of different basis. By basis it should be understood the set of momenta that appear in the propagators for each integral. This can be noticed by looking at the figures in section 4.3.1.
To counteract this problem, and to be able to use `LiteRed`, it was first necessary to write all integrals in terms of a common basis. The chosen basis was

$$\{q_1,\ q_2,\ p - q_1,\ p - q_2,\ k - q_1,\ k - q_2,\ q_2 - q_1\}, \tag{4.41}$$

and to each integral was applied an appropriate change of variables to express it using this basis.

This allowed us to reduce the number of integrals to the twenty-two scalar integrals in Appendix D. Consequently, the 2-loop contribution to the 2-point function of the single-trace operator $\mathcal{O}(x) = \mathrm{Tr}[\bar{\Psi}(x)\Psi(x)]$ has been entirely expressed in terms of these few integrals.

However, in order to determine the anomalous dimension of this operator, there is no need to fully compute the integrals. It is possible to determine the anomalous dimension merely by looking at the divergent part of these integrals in ϵ, since dimensional regularization is being used and the integrals are being computed in $d = 3 - \epsilon$ dimensions.

Some of the integrals, namely the integrals M_1, M_2, M_3, M_6, M_7, M_8, M_{10}, M_{11} and M_{12}, can be computed straightforwardly in terms of G-functions (C.11). This still leaves a total of thirteen integrals that are not so easy to compute in $d = 3 - \epsilon$ dimensions, in particular those that involve 3 terms with the same variable of integration in the denominator. Regarding these integrals, their divergent behaviour can be studied by making use of the asymptotic expansion technique, as explained in the following section.

4.3.3 Asymptotic Expansion

The variables of integration q_1 and q_2 cover the entire space-time. Thus, if considering the limit where one of the external momenta is very small, and we will be taking $p \to 0$, then the domain of integration is naturally divided into 4 regions:

- Region 1: $q_1 \sim p$ and $q_2 \sim p$
- Region 2: $q_1 \sim p$ and $q_2 \not\sim p$
- Region 3: $q_1 \not\sim p$ and $q_2 \sim p$
- Region 4: $q_1 \not\sim p$ and $q_2 \not\sim p$

A this point is then possible to evaluate each integral in each of the 4 regions, where they greatly simplify. This happens because in different regions the propagators in the denominator can be expanded, resulting in an integral that can be evaluated in dimensional regularization. This method is illustrated below, by applying it to the master integral

$$M_{20} = \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 q_2^2 (k - q_1)^2 (p - q_2)^2 (q_2 - q_1)^2}. \tag{4.42}$$

For instance in region 1 we have $(k-q_1)^2 \sim k^2$ since q_1 is of the order of p and p is considered very small. Applying this logic to all four regions yields:

$$\begin{aligned}
M_{20}^{(1)} &= \frac{1}{k^2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2q_2^2(p-q_2)^2(q_2-q_1)^2} = \\
&= \frac{G(1,1)}{k^2}\int \frac{d^dq_2}{(2\pi)^d}\frac{1}{(q_2^2)^{3-\frac{d}{2}}(p-q_2)^2} = \frac{G(1,1)G(1,3-\frac{d}{2})}{k^2(p^2)^{4-d}} \\
M_{20}^{(2)} &= \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2(k)^2(q_2^2)^3} = 0 \\
M_{20}^{(3)} &= \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{(q_1^2)^2q_2^2(k-q_1)^2(p-q_2)^2} = \frac{G(1,1)G(1,2)}{(k^2)^{3-\frac{d}{2}}(p^2)^{2-\frac{d}{2}}} \\
M_{20}^{(4)} &= \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2(q_2^2)^2(k-q_1)^2(q_2-q_1)^2} = \\
&= G(1,2)\int \frac{d^dq_1}{(2\pi)^d}\frac{1}{(q_1^2)^{3-\frac{d}{2}}(k-q_1)^2} = \frac{G(1,2)G(1,3-\frac{d}{2})}{(k^2)^{4-d}}
\end{aligned} \tag{4.43}$$

Combining all contributions results in

$$M_{20} = -\frac{2\pi}{\epsilon\left(k^2p^2\right)} + O\left(\epsilon^0\right), \tag{4.44}$$

and consequently the contribution of this master integral is of the following form:

$$M_{20} = \frac{m_{20}(k,p)}{\epsilon}. \tag{4.45}$$

This method was then systematized in `MATHEMATICA`, applying it to all the master integrals.

What was found was that the master integrals M_4, M_6, M_7, M_8, M_{10}, M_{11}, M_{12}, M_{13}, M_{14}, M_{16}, M_{19}, M_{21} and M_{22} have no divergences in ϵ and will not, therefore, contribute to the anomalous dimension.

On the other hand, the divergent part of M_1, M_2 and M_3 can be computed without relying in the asymptotic expansion approach since these integrals are merely products of G-functions with the appropriate momentum structure. Their contribution at order $\frac{1}{\epsilon}$ is $\frac{2\pi}{\epsilon}$. It is equal for all three integrals, as expected, since they have the same structure, changing only by permutations of the external momenta.

Lastly, the master integrals M_5, M_9, M_{15}, M_{17}, cannot be evaluated exactly. However, the asymptotic expansion technology allowed us to conclude that they do indeed diverge at order $\frac{1}{\epsilon}$.

It should be mentioned that for the integral M_{17}, when considering the region where both q_1 and q_2 are much larger than p, what was found was that the resulting integral, given in equation (4.46), was not one of the master integrals and could not be evaluated in terms of G-functions. It could, however, be reduced to master integrals,

$$\begin{aligned} M_{17}^{(4)} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 (k-q_1)^2 (k-q_2)^2 q_2^2 (q_2-q_1)^2} = \\ &= \frac{2(3d-10)(3d-8)M_2}{(d-4)^2 (k^2)^2} - \frac{2(d-3)M_{10}}{(d-4)k^2}, \end{aligned} \tag{4.46}$$

whose divergences are known.

Table 4.1 contains the contributions of each of the master integrals at order $\frac{1}{\epsilon}$. This ap-

Integral	Contribution at order ϵ^{-1}
M_1, M_2, M_3	$\frac{2\pi}{\epsilon}$
M_4,M_6,M_7,M_8,M_{10},M_{11},M_{12},M_{13},M_{14},M_{16},M_{19},M_{21}, M_{22}	0
M_5,M_9	$\frac{m_1}{\epsilon}$
M_{18},M_{20}	$\frac{m_2}{\epsilon}$
M_{15}	$\frac{m_3}{\epsilon}$
M_{17}	$\frac{m_4}{\epsilon}$

Table 4.1: Divergent part of the Master Integrals

proach allows the overall 2-loop contribution to be re-expressed in terms of a few unknown functions, in the hope that all terms involving them cancel out at order $\frac{1}{\epsilon}$. This, however, did not happen. In fact, not only did these functions appear in the end contribution, but, furthermore, there was an unexpected pole in $\frac{1}{\epsilon^2}$. This last fact in particular was an indicator that there was some unidentified symmetry relation between these functions that could lead to some cancellations.

Due to time constrains, we were unable to keep pursuing this approach and instead opted to try another method. Nonetheless, we recognize the importance of performing checks to validate our results and of recovering the same results through different procedures. For this reason, we hope to return to this point at a future date.

However, we considered that there was another avenue that we could take and that could prove itself more fruitful. We proceed to describe below the process through which we ended up being able to obtain the anomalous dimension at 2-loops.

Since we are taking one of operators to be a probe, it is possible to adjust their momenta without compromising the divergent behaviour. Consequently, by taking $p \to k$ after performing all contractions, we greatly reduce the number of integrals that require evaluation. In fact, in this limit, all integrals can be evaluated in terms of two master integrals which cannot involve more than two terms in the denominator for each integration and can, therefore, be evaluated through G-functions.

The common basis, in which all these integrals can be expressed, corresponds to

$$\{q_1,\ q_2,\ k-q_1,\ k-q_2,\ q_2-q_1\} \tag{4.47}$$

and the two master integrals are

$$\begin{aligned} I_1 &= \int \frac{d^dq_1 d^dq_2}{(2\pi)^{2d}} \frac{1}{q_1^2(k-q_1)^2 q_2^2(k-q_2)^2} = \left(\frac{G(1,1)}{(k^2)^{2-\frac{d}{2}}}\right)^2 \\ I_2 &= \int \frac{d^dq_1 d^dq_2}{(2\pi)^{2d}} \frac{1}{q_2^2(k-q_1)^2(q_2-q_1)^2} = \frac{G(1,1)G(1,2-\frac{d}{2})}{(k^2)^{3-d}} \end{aligned} . \tag{4.48}$$

In this limit, not only does the pole in $\frac{1}{\epsilon^2}$ disappear, but it was also possible to determine the $\frac{1}{\epsilon}$ coefficient of the 2-loop two point function as

$$G_0^{(2)}(x,k,k) = \frac{N^2 e^{2ikx}}{2k^2} \frac{401 N^2 g^4}{3456\epsilon} + \mathcal{O}(\epsilon^0). \tag{4.49}$$

It is also possible to take this very same limit in the free-theory contribution in (4.23). At the same time, recalling that $g^2 = \frac{1}{\kappa}$ and the t'Hooft parameter is defined as $\lambda = \frac{N}{\kappa}$, then the correlation function becomes, up to 2 loops::

$$G_0^{(2)}(x,k,k) = \frac{N^2 e^{2ikx}}{2k^2}\left(1 + \frac{401\lambda^2}{3456\epsilon} + ...\right) + \mathcal{O}(\lambda^3). \tag{4.50}$$

In equation (4.50), "..." stands for the remaining terms in the series in ϵ. This equation allows us to conclude that the single-trace operator $\mathrm{Tr}[\bar{\Psi}\Psi](x)$ does indeed gain anomalous dimension and that it is given by:

$$\gamma_{\mathcal{O}_0} = \frac{401\lambda^2}{3456} + \mathcal{O}(\lambda^3). \tag{4.51}$$

Chapter 5

Conserved Current $\mathrm{Tr}[\bar{\Psi}\gamma^{\mu}\Psi]$

A good consistency check for the approach we used in the previous section comes from studying the anomalous dimension of the spin-1 operator $\mathcal{O}_1^{\mu}(x) = Tr[\bar{\Psi}(x)\gamma^{\mu}\Psi(x)]$. This is a conserved current of the theory, protected under symmetry, and should not gain anomalous dimension, as we will discuss in this chapter.

Thus, the goal in this chapter is computing the anomalous dimension of the conserved current $\mathcal{O}_1^{\mu}(x) = Tr[\bar{\Psi}(x)\gamma^{\mu}\Psi(x)]$, which we expect to be zero. However, in this case we will need to use a different approach than the one used for the scalar operator we considered previously. This will be explained later as we do the required computations.

We start by looking at the 2-point function of $\mathcal{O}_1^{\mu}(x)$, which takes the form

$$\begin{aligned} G_1^{\mu\nu\,(0)}(x,y) &= \langle \mathrm{Tr}[\bar{\Psi}_a(x)\gamma^{\mu}_{ab}\Psi_b(x)]\mathrm{Tr}[\bar{\Psi}_c(y)\gamma^{\nu}_{cd}\Psi_d(y)]\rangle = \\ &= \frac{\delta^{AD}\delta^{BC}}{4}\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}\,\langle \bar{\Psi}^A_a(x)\Psi^D_b(x)\bar{\Psi}^C_c(y)\Psi^B_d(y)\rangle = \\ &= -\frac{\delta^{AD}\delta^{BC}}{4}\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}\,\langle \bar{\Psi}^A_a(x)\Psi^B_d(y)\rangle\,\langle \bar{\Psi}^C_c(y)\Psi^D_b(x)\rangle\,. \end{aligned} \tag{5.1}$$

Once again, introducing the Fourier representation of the propagators while, simultaneous, taking the Fourier transform of the external points y_1 and y_2, yields a contribution:

$$\begin{aligned} G_1^{(0)}(x,p,k) &= -\frac{\delta^{AD}\delta^{BC}}{4}\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}\int d^dq_1d^dq_2\delta(p-q_2)\delta(k+q_1)S_F^{AB}(q_1)_{ad}S_F^{CD}(q_2)_{cd}e^{i(q_2-q_1)x} = \\ &= \frac{\delta^{AD}\delta^{BC}\delta^{AB}\delta^{CD}}{4}\gamma^{\mu}_{ab}\gamma^{\nu}_{cd}\frac{\gamma^{\alpha_1}_{da}k_{\alpha_1}}{k^2}\frac{\gamma^{\alpha_2}_{bc}p_{\alpha_2}}{p^2}e^{i(p+k)x} = \\ &= \frac{N^2e^{-i(p+k)x}}{2k^2p^2}\left(k^{\mu}p^{\nu}+k^{\nu}p^{\mu}-k\cdot p\delta^{\mu\nu}\right) \end{aligned} \tag{5.2}$$

Repeating the same trick that was used in the previous chapter and sending $p \mapsto -k$, this correlator can be written as:

$$G_1^{(0)}(x,-k,k) = \frac{N^2}{2k^4}(2k^\mu k^\nu - k^2\delta^{\mu\nu}) \tag{5.3}$$

The next step is to carry out the computation to higher orders in perturbation theory and check that this operator does indeed not receive any corrections to its conformal dimension. The 1-loop calculation is very similar to what we had in the previous Chapter, nonetheless, its computation is quite more challenging since the introductions of these two extra gamma matrices means that we are dealing with traces of up to twelve gamma matrices. This is much more demanding from a computational point of view, and reflected in a significantly longer time for the code in `MATHEMATICA` to finish running.

Furthermore, while in the previous situation all indices should be contracted and the Levi-Civita symbol provided a nice symmetry argument to discard most contributions, in this case there will still be 2 open indices left by the end of the calculation. Nonetheless, in a similar way to what we did previously, it is possible to explore the symmetries of the integrals themselves.

For instance, it is possible to decompose more complicated integrals in terms of simpler ones, as is the case with:

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\frac{(k-q)^2q^\mu q^\nu}{q^4(k+q)^2} = 2\int \frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{q^2(k+q)^2} + 2k^2\int \frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{q^4(k+q)^2} + \\ + \int \frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{q^4}.\end{aligned} \tag{5.4}$$

Afterwards, each of these integrals can be looked at with more detail. For instance, the last integral in the right-hand-side is immediately zero in dimensional regularization. The first integral, on the other hand, can be computed by first proposing an ansatz of the form:

$$I_1^{\mu\nu}(k) \equiv \int \frac{d^dq}{(2\pi)^d}\frac{q^\mu q^\nu}{q^2(k+q)^2} = g_1(k)k^\mu k^\nu + \delta^{\mu\nu}g_2(k). \tag{5.5}$$

By contracting the previous integral with a Kronecker delta function, $\delta^{\mu\nu}$, or with 2 external vectors $k^\mu k^\nu$, it is possible to obtain 2 equations that fix the unknown functions.

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\frac{1}{(k+q)^2} = 0 = g_1(k)k^2 + dg_2(k) \\ \int \frac{d^dq}{(2\pi)^d}\frac{(k\cdot q)^2}{q^2(k+q)^2} = g_1(k)k^4 + g_2(k)k^2\end{aligned} \tag{5.6}$$

Expanding the second integral in (5.6) by using $2k \cdot q = k^2 + q^2 - (k-q)^2$, allows us to obtain an equation for the two unknown functions in terms of the G-functions. Solving for g_1 and g_2, the resulting contribution becomes.

$$\int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{q^2(k+q)^2} = \frac{d-1}{4d} \frac{G(1,1)}{(k^2)^{2-d/2}} \left(k^\mu k^\nu - \frac{k^2}{d}\delta^{\mu\nu} \right). \tag{5.7}$$

A similar calculation can be done for the remaining contribution. Repeating this procedure to all integrals it is possible to compute the overall correction to the 2-point function at the 1-loop level.

What was found was that all contributions cancel out and, therefore, there is no anomalous dimension at 1-loop.

Having obtained the expected result it is still possible to verify if this is maintained at 2-loops, as predicted. This is done in exactly the same way, merely by including higher order terms in the expansion of the action.

It is at this point that we find the trace of twelve gamma matrices, coming from the diagrams that involve 4 $\bar{\Psi}A\Psi$ vertices, which while computational demanding are still doable.

We were able to use all the techniques deployed previously to carry out this computation. Concretely, by expanding the integrals as much as possible and exploring their symmetry properties, it was possible to reduce the number of integrals contributing to the correlator. Having performed as many as these simplifications as possible, the result is a sum of terms with two open indices, μ and ν.

Since our interest lies merely in checking if the operator gains anomalous dimension, it is not necessary to actually compute all the integrals with open indices. Because we know that the correlation function is of the form

$$G_1^{\mu\nu\ (2)} = h_1(k)k^\mu k^\nu + h_2(k)\delta^{\mu\nu}, \tag{5.8}$$

we know that if we contract it with a Kronecker delta the resulting function should be given by:

$$\delta_{\mu\nu}G_1^{\mu\nu(2)} = k^2 h_1(k) + d h_2(k). \tag{5.9}$$

This contraction does not affect the functions $h_1(k)$ and $h_2(k)$, meaning that any divergence present in the 2-loop correlation function should still be present in this object. The advantage of this approach is that we obtain an expression entirely in terms of scalar integrals, all of which can be reduced to the master integrals in (4.48) since the basis in which the integrals are expressed is the very same one as in (4.47).

What was found was that (5.9) is given as

$$\delta_{\mu\nu} G_1^{\mu\nu(2)} = \frac{\pi^2 g^4 \left(5G(1,1) + \frac{41}{6} G\left(1, \frac{1}{2}\right)\right) G(1,1)}{36k^2}. \tag{5.10}$$

The issue here is that the function $G\left(1, \frac{1}{2}\right)$ diverges in $d = 3 - \epsilon$ dimensions as $\frac{1}{2\pi^2\epsilon}$, which would lead to the conclusion that this conserved current gains anomalous dimension at 2-loops. This result is clearly wrong and means there are some corrections to the code or to the necessity of including some improvement terms in the current.

5.1 Method of Differential Equations

To check if this was indeed the case, we opted to try an approach along the lines of what was done in [21], that relies on knowing the equations of motion of the fermion fields.

This method is implemented in position space, meaning that it is useful to introduce at this point the position-space version of the fermion 2-point function. This result can be obtained by reversing the Fourier transform in the momentum-space propagator:

$$\begin{aligned}
\langle \bar{\Psi}^A(x) \Psi^B(0) \rangle &= i\delta^{AB} \gamma_\alpha \int \frac{d^3q}{(2\pi)^3} \frac{q^\alpha}{q^2} e^{-iqx} = \\
&= i\delta^{AB} \gamma_\alpha \frac{1}{-i} \frac{\partial}{\partial x^\alpha} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} e^{-iqx} = \\
&= -i\delta^{AB} \frac{\gamma_\alpha x^\alpha}{|x|^3}.
\end{aligned} \tag{5.11}$$

In the previous equation we used the fact that the second integral corresponds to the propagator of a free scalar field, whose 2-point function is completely fixed by conformal symmetry.

Substituting this expression in (5.1), means that the 2.point function can be expressed in position space as:

$$\begin{aligned}
G_1^{\mu\nu\ (0)}(x,y) &= \frac{N^2}{4} \mathrm{Tr}[\gamma^\mu \gamma^{\alpha_1} \gamma^\nu \gamma^{\alpha_2}] \frac{(x-y)_{\alpha_2} (y-x)_{\alpha_1}}{\left((x-y)^2\right)^3} = \\
&= \frac{N^2/2}{((x-y)^2)^2} \left(\delta_{\mu\nu} - 2 \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right).
\end{aligned} \tag{5.12}$$

The first aspect of note is that the correlation function in equation (5.12) has the expected structure of a 2-point function of a spin-1 operator given in (2.41). This allows us to read the dimension of the operator $\mathcal{O}_1^\mu(x)$ as $\Delta_{\mathcal{O}_1} = 2$, which once again is the expected result for a spin-1 conserved current.*

The ingenuity of the method exemplified in [21] is that it allows us to read the dimension of the operator merely by acting with a derivative on the 2-point function. Concretely, because $\mathcal{O}_1^\mu(x)$ is a conserved current, we have that $\partial_\mu \mathcal{O}_1^\mu(x) = 0$ and, consequently, acting with this same derivative on the correlator in (2.41) imposes a condition on the conformal dimension. If we use the appropriate coefficient $c = \frac{N^2}{2}$ in (2.41) and, for simplicity, set $x_1 \equiv x$ and $x_2 = 0$ then:

$$\begin{aligned} \partial_\mu \langle \mathcal{O}_1^\mu(x)\mathcal{O}_1^\nu(0)\rangle = 0 = \frac{N^2}{2x^{2\Delta}}\partial_\mu\left[\delta^{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}\right] = \\ = -N^2(\Delta-2)\frac{x^\nu}{(x^2)^{\Delta+1}} = 0 \end{aligned} \tag{5.13}$$

Since the previous equation must be verified for any x, then it is immediate to conclude that $\Delta = 2$.

We know that the previous result must hold in a free theory, but we can also analyze what happens when the coupling is non-zero. In order to repeat the previous calculation in an interacting theory, we first need to know the equations of motion for the spinor fields. These can be obtained by solving the Euler Lagrange equations,

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\bar{\Psi}^A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi}^A)} = 0 \\ \frac{\partial\mathcal{L}}{\partial\Psi^A} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^A)} = 0 \end{aligned} , \tag{5.14}$$

where $\mathcal{L}$ corresponds to the Lagrangian density for the fermions in (3.31), which can be expanded as:

$$\mathcal{L}_\Psi = \frac{\delta^{AB}}{2}\bar{\Psi}^A \partial\!\!\!/ \Psi^B - \frac{g}{2}f^{ABC}\bar{\Psi}^A A\!\!\!/^B\Psi^C. \tag{5.15}$$

The solutions to (5.14) are:

$$\begin{aligned} \partial_\mu\bar{\Psi}^C = -gf^{ABC}\bar{\Psi}^A A_\mu^B \\ \partial_\mu\Psi^A = gf^{ABC}A_\mu^B\Psi^C \end{aligned} , \tag{5.16}$$

*The bare dimension of a spin l operator is $\Delta_l = d + l - 2$. For a spin-1 operator in d= 3 dimensions this implies $\Delta = 2$

meaning that the divergence of our conserved current is, when the theory is not free is given by:

$$
\begin{aligned}
B(x) \equiv \partial_\mu \mathcal{O}_1^\mu(x) = \frac{\delta^{AB}}{2}\partial_\mu \left(\bar{\Psi}^A \gamma^\mu \Psi^B\right) &= \frac{\delta^{AB}}{2}\left[(\partial_\mu \bar{\Psi}^A)\gamma^\mu \Psi^B + \bar{\Psi}^A \not{\partial} \Psi^B\right] = \\
&= \frac{g}{2} f^{ADE}\left(-\bar{\Psi}^D \not{A}^E \Psi^A + \bar{\Psi}^A \not{A}^D \Psi^E\right) = 0.
\end{aligned}
\tag{5.17}
$$

In principle, when acting with a derivative twice on the correlation function one has:

$$
\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\left\langle \mathcal{O}^\mu(x)\mathcal{O}^\nu(y)\right\rangle = g^2 \left\langle B(x)B(y)\right\rangle, \tag{5.18}
$$

meaning a similar analysis to what was done in (5.13) could be performed to obtain information about the dimension of this operator at order $\mathcal{O}(g^2)$. The usefulness of this approach is that we are able to pick up a factor of g^2 without having to perform any integration, meaning that evaluating the tree-level correlator on the right hand-side would give information about the anomalous dimension of the current at 1-loop. However, since the operator $B(x)$ is identically 0, this is a fruitless endeavour and no additional information can be gathered from here.

Even though the method described in this section ended up not providing any help in our case, it should be noted that it remains an interesting tool at our disposal. It could prove itself particularly useful in determining the anomalous dimension of higher-spin conserved currents.

Form this further analysis we gather that there is some unidentified error in the code we implemented that is preventing the cancelling of the different contributions in (5.10). We reiterate that the calculation that is being performed is quite involved.

Not only are we performing the Wick contractions of sixteen fields (including multiple fermions, adding to the complexity), we are evaluating each of the 16! terms to discard the disconnected diagrams. The fact that the we are dealing with so many fermion propagators, means that there are multiple traces of Gamma matrices to be evaluated, further expanding the size of the expression we are dealing with.

Having to take the Fourier transforms not only of the propagators but also of the external points and having to perform a change of basis in most of the integrals encountered as well as then reducing them to Master Integrals, further adds to the intricacies of the calculation.

Obtaining a zero for the 2-loop anomalous dimension of the spin-1 conserved current would require that all terms cancel out. This means that any small error in any of the previous steps would compromise this result. Unfortunately, due to time constrains and to the complexity previously described, we were unable to successfully verify that the 2-loop anomalous dimension of $\mathcal{O}_1^\mu$ vanishes. Nonetheless, this is an important check and one that we hope to be able to revisit.

Chapter 6

Conclusion and Outlook

In this thesis we delved into the study of the AdS/CFT correspondence, which conjectures a duality between CFTs in d dimensions and theories of quantum gravity in (d+1)-dimensional AdS space. This conjecture, first proposed in 1997 by Maldacena, has been studied with great interest in the last decades in hopes of advancing our knowledge of quantum gravity. To add to the great body of work concerning this subject, we proposed ourselves to find a non-supersymmetric conformal field theory whose dual was a theory of Einstein gravity in AdS_4, as this most resembles our Universe. The fact that this theory of gravity exists in four dimensions immediately restricts us to studying a 3d CFT, the simplest of which is a Chern-Simons theory with fermionic matter transforming in the adjoint representation of the gauge group.

As conjectured in [4], in a CFT with a dual it is expected that the anomalous dimension of unprotected operators grow parametricaly large with the increasing coupling. Protected operators are objects such as conserved currents or the stress-tensor (which is dual to the graviton) that do not gain anomalous dimension. In particular, requiring that operators other than the stress tensor gain anomalous dimension when the coupling is strong (and a strongly coupled CFT corresponds to a weakly coupled theory of gravity) ensures that massive modes in the dual theory decouple, resulting in a theory with only gravity.

We focused on looking at the simplest operator likely to gain anomalous dimension, the single-trace scalar $\mathcal{O}_0 = \mathrm{Tr}\left[\bar{\Psi}(x)\Psi(x)\right]$. We successfully verified that this operator does not gain anomalous dimension at 1-loop, meeting our expectations from looking at the symmetries of the action for this theory. In fact, this is analogous to what happens in

ABJM, a supersymmetric CFT similar to Chern-Simons, where the anomalous dimension vanishes for odd powers of the coupling. The real test came from the 2-loop computation, through which we were able to determine that this operator does indeed gain anomalous dimension, finding a value of $\lambda_{\mathcal{O}_0} = \frac{401\lambda^2}{3456}$. This result means that, so far as this test is concerned, we are dealing with a good candidate for a theory with a dual theory of gravity.

As a consistency check, we looked at the conserved current $\mathcal{O}_1^{\mu}(x) = \mathrm{Tr}\left[\bar{\Psi}(x)\gamma^{\mu}\Psi(x)\right]$ and were able to verify that it does not gain anomalous dimension at 1-loop, as expected. The calculation at 2-loops has proved itself more challenging. Despite exploring both a repeat of the approach followed at 1-loop and a simpler method that relied on the equations of motion and our knowledge of conformal symmetry to extract the anomalous dimension without having to perform the integrations, we were unable to verify the result. We expect that with more time and a more extensive analysis we might resolve the issue preventing the canceling out of all contributions that would result in a vanishing anomalous dimension.

Nonetheless, from the result we obtained we have confidence and strong evidence that the CFT discussed throughout this thesis has a dual theory of gravity in light of the AdS/CFT correspondence.

In order to arrive at the previous conclusion, we had to carry out an intricate computation, seeing as our 2-loop result comes about from a perturbative expansion to the fourth order in the coupling. This means that, even after Fourier transforming to momentum space, we were dealing with two integrations for each of the 16! terms resulting from the Wick contractions of the fields appearing in the 2-point function.

Even after discarding disconnected and non-planar diagrams, we were left with the contributions of the 16 diagrams shown in Section 4.3. The contribution of each of these diagrams was evaluated by taking the trace over the large number of Gamma matrices multiplying them and then contracting all the indices, resulting in more than 1400 distinct scalar integrals.

In order to deal with this large number of integrals it was required introducing the concept of Master Integrals and of the Integration By Parts Reduction procedure, which allowed us to rewrite all contributions as linear combinations of a few integrals (the so called Master Integrals). These integrals were evaluated using dimensional regularization in $d \to 3 - \epsilon$ dimensions and by sending the external momenta (over which we were not integrating) to the same value. This allowed us to deal in a more efficient manner with the substantial number of integrals needing evaluation.

Despite having successfully achieved the goal we set at the beginning of this work, we recognize that these are but the first steps in a more in-depth analysis of this theory. In fact, it remains to be checked that the 2-loop anomalous dimension of the spin-1 conserved current vanishes, which would be a natural next step in our study.
Furthermore, beyond the single operator whose anomalous dimension we computed, it would be interesting to extend the calculation to other operators, and in particular to higher spin operators. Towards this, the method that relies on the equations of motion of the fields (discussed in section 5.1) might prove useful, providing a way to avoid the cumbersome computations inherent to a 2-loop analysis.
In particular, we would like to study the case of a spin-4 conserved current, as it has been shown in [22] that in a three dimensional CFT with such a higher-spin conserved current, there are infinite conserved currents whose correlators are those of a free-theory.

Furthermore, if we are able to successfully verify that all local operators other than the stress-tensor and multi-trace operators constructed from it gain anomalous dimension, this opens up the possibility of investigating the gravitational side and performing potential quantitative tests. One thing we expect though: the theory should not be purely Einstein since the CFT contains a marginal coupling, rather one might expect Einstein gravity coupled with a dilaton field.

Another interesting avenue to pursue is that of computing the beta function for this theory, which would allows us to determine its fixed points (where the theory is in fact conformal). This is a calculation that does not differ much from what was done in this work.

Lastly, we would like to remark on the possibility of investigating ressumation or pursuing a non-perturbative approach.

Appendix A

Gauge transformation in non-Abelian Chern-Simons theory

The goal of this section is to demonstrate equation (3.14). To do so, it will be necessary to make repeated use of the following identity:

$$\begin{aligned}\partial_\mu \left(g^{-1}g\right) = \partial_\mu g^{-1}g + g^{-1}\partial_\mu g \leftrightarrow 0 = \partial_\mu g^{-1}g + g^{-1}\partial_\mu g \leftrightarrow \\ \leftrightarrow \partial_\mu g^{-1}g = -g^{-1}\partial_\mu g \implies \\ \implies g\partial_\mu g^{-1} = -\partial_\mu g g^{-1}\end{aligned} \tag{A.1}$$

In the following derivation the prefactor of $\frac{\kappa}{4\pi}\epsilon_{\mu\nu\rho}$ is omitted for convenience and ease of reading. Focusing on the first term of equation (3.12) we have:

$$\begin{aligned}&\mathrm{Tr}\left[\left(g^{-1}A_\mu g - ig^{-1}\partial_\mu g\right)\partial_\nu\left(g^{-1}A_\rho g - ig^{-1}\partial_\rho g\right)\right] = \\ &= \mathrm{Tr}\Big[g^{-1}A_\mu g\partial_\nu\left(g^{-1}A_\rho g\right) - ig^{-1}A_\mu g\partial_\nu\left(g^{-1}\partial_\rho g\right) - \\ &\quad - ig^{-1}\partial_\mu g\partial_\nu\left(g^{-1}A_\rho g\right) - g^{-1}\partial_\mu g\partial_\nu\left(g^{-1}\partial_\rho g\right)\Big] = \\ &= \mathrm{Tr}\Big[g^{-1}A_\mu g\partial_\nu g^{-1}A_\rho g + g^{-1}A_\mu\partial_\nu(A_\rho g) - ig^{-1}A_\mu g\partial_\nu g^{-1}\partial_\rho g - \\ &\quad - ig^{-1}\partial_\mu g\partial_\nu g^{-1}A_\rho g - ig^{-1}\partial_\mu g g^{-1}\partial_\nu(A_\rho g) - g^{-1}\partial_\mu g\partial_\nu g^{-1}\partial_\rho g\Big]\end{aligned} \tag{A.2}$$

In the previous computation the terms with $\partial_\mu\partial_\nu g$ were set to 0 since the partial derivatives commute and are contracted with the antisymmetric tensor $\epsilon^{\mu\nu\rho}$.

At this point it is easier to simplify each of the terms separately. To do so will require using the results in (A.1), the cyclic property of the trace, as well as the symmetry properties of the Levi-Civita tensor.

It is important to keep in mind that each of the following terms is being contracted with a Levi-Civita symbol which allows us to rotate the indices (a factor of -1 being required for each permutation that takes the indices out of order).

1.
$$\begin{aligned}\mathrm{Tr}\left[g^{-1}A_\mu g\partial_\nu g^{-1}A_\rho g\right] = \mathrm{Tr}\left[A_\mu g\partial_\nu g^{-1}A_\rho\right] = \mathrm{Tr}\left[A_\rho A_\mu g\partial_\nu g^{-1}\right] =\\ = -\mathrm{Tr}\left[A_\mu A_\nu \partial_\rho g g^{-1}\right]\end{aligned} \tag{A.3}$$

2.
$$\begin{aligned}\mathrm{Tr}\left[g^{-1}A_\mu\partial_\nu(A_\rho g)\right] = \mathrm{Tr}\left[g^{-1}A_\mu A_\rho\partial_\nu g + g^{-1}A_\mu\partial_\nu A_\rho g\right] =\\ = -\mathrm{Tr}\left[A_\mu A_\nu\partial_\rho g g^{-1}\right] + \mathrm{Tr}\left[A_\mu\partial_\nu A_\rho\right]\end{aligned} \tag{A.4}$$

3.
$$\begin{aligned}\mathrm{Tr}\left[g^{-1}A_\mu g\partial_\nu g^{-1}\partial_\rho g\right] = \mathrm{Tr}\left[A_\mu g\partial_\nu g^{-1}\partial_\rho g g^{-1}\right] = -\mathrm{Tr}\left[A_\mu\partial_\nu g g^{-1}\partial_\rho g g^{-1}\right]\\ = \mathrm{Tr}\left[A_\mu\partial_\nu g\partial_\rho g^{-1}g g^{-1}\right] = \mathrm{Tr}\left[A_\mu\partial_\nu g\partial_\rho g^{-1}\right]\end{aligned} \tag{A.5}$$

4.
$$\begin{aligned}\mathrm{Tr}\left[g^{-1}\partial_\mu g\partial_\nu g^{-1}A_\rho g\right] = \mathrm{Tr}\left[\partial_\mu g\partial_\nu g^{-1}A_\rho\right] = \mathrm{Tr}\left[A_\rho\partial_\mu g\partial_\nu g^{-1}\right]\\ = \mathrm{Tr}\left[A_\mu\partial_\nu g\partial_\rho g^{-1}\right]\end{aligned} \tag{A.6}$$

5.
$$\begin{aligned}\mathrm{Tr}\left[g^{-1}\partial_\mu g g^{-1}\partial_\nu(A_\rho g)\right] = \mathrm{Tr}\left[\partial_\mu g g^{-1}\partial_\nu A_\rho + g^{-1}\partial_\mu g g^{-1}A_\rho\partial_\nu g\right] =\\ = \mathrm{Tr}\left[\partial_\nu A_\rho\partial_\mu g g^{-1} + g^{-1}\partial_\mu g g^{-1}A_\rho\partial_\nu g\right] =\\ = -\mathrm{Tr}\left[\partial_\nu A_\mu\partial_\rho g g^{-1}\right] - \mathrm{Tr}\left[g^{-1}\partial_\mu g g^{-1}A_\nu\partial_\rho g\right]\end{aligned} \tag{A.7}$$

Substituting the previous few results in (A.2), the end result is that the first term in (3.12) transforms, under a gauge transformation, into:

$$\mathrm{Tr}\left[A_\mu\partial_\nu A_\rho - 2A_\mu A_\nu\partial_\rho g g^{-1} - 3iA_\mu\partial_\nu g\partial_\rho g^{-1} + i\partial_\mu A_\nu\partial_\rho g g^{-1} - g^{-1}\partial_\mu g\partial_\nu g^{-1}\partial_\rho g\right] \tag{A.8}$$

The analysis for the second term (the one that is cubic in the gauge field) is very similar and is carried out bellow.

$$\begin{aligned}
&\frac{2i}{3}\mathrm{Tr}\left[\left(g^{-1}A_\mu g - ig^{-1}\partial_\mu g\right)\left(g^{-1}A_\nu g - ig^{-1}\partial_\nu g\right)\left(g^{-1}A_\rho g - ig^{-1}\partial_\rho g\right)\right] = \\
&= \frac{2i}{3}\mathrm{Tr}\left[\left(g^{-1}A_\mu g - ig^{-1}\partial_\mu g\right) \times \left(g^{-1}A_\nu A_\rho g - ig^{-1}A_\nu \partial_\rho g + \right.\right. \\
&\qquad \left.\left. - ig^{-1}\partial_\nu g g^{-1}A_\rho g - g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right)\right] = \\
&= \frac{2i}{3}\mathrm{Tr}\Big[g^{-1}A_\mu A_\nu A_\rho g - ig^{-1}A_\mu A_\nu g g^{-1}\partial_\rho g - ig^{-1}A_\mu \partial_\nu g g^{-1}A_\rho g - \\
&\qquad - g^{-1}A_\mu \partial_\nu g g^{-1}\partial_\rho g - ig^{-1}\partial_\mu g g^{-1}A_\nu A_\rho g - g^{-1}\partial_\mu g g^{-1}A_\nu \partial_\rho g - \\
&\qquad - g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}A_\rho g + ig^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\Big]
\end{aligned} \tag{A.9}$$

Looking at each term individually:

1.
$$\frac{2i}{3}\mathrm{Tr}\left[g^{-1}A_\mu A_\nu A_\rho g\right] = \frac{2i}{3}\mathrm{Tr}\left[A_\mu A_\nu A_\rho\right] \tag{A.10}$$

2.
$$\begin{aligned}
&\frac{2i}{3}\mathrm{Tr}\left[-g^{-1}A_\mu \partial_\nu g g^{-1}\partial_\rho g - g^{-1}\partial_\mu g g^{-1}A_\nu \partial_\rho g - g^{-1}\partial_\mu g g^{-1}A_\nu \partial_\rho g\right] = \\
&= -\frac{2i}{3}\mathrm{Tr}\left[A_\mu \partial_\nu g g^{-1}\partial_\rho g g^{-1} + \partial_\mu g g^{-1}A_\nu \partial_\rho g g^{-1} + \partial_\mu g g^{-1}A_\nu \partial_\rho g g^{-1}\right] = \\
&= -\frac{2i}{3}\mathrm{Tr}\left[A_\mu \partial_\nu g g^{-1}\partial_\rho g g^{-1} + A_\nu \partial_\rho g g^{-1}\partial_\mu g g^{-1} + A_\nu \partial_\rho g g^{-1}\partial_\mu g g^{-1}\right] = \\
&= \frac{2i}{3}\mathrm{Tr}\left[A_\mu \partial_\nu g \partial_\rho g^{-1} g g^{-1} + A_\nu \partial_\rho g \partial_\mu g^{-1} g g^{-1} + A_\nu \partial_\rho g \partial_\mu g^{-1} g g^{-1}\right] = \\
&= \frac{2i}{3}\mathrm{Tr}\left[A_\mu \partial_\nu g \partial_\rho g^{-1} + A_\mu \partial_\nu g \partial_\rho g^{-1} + A_\mu \partial_\nu g \partial_\rho g^{-1}\right] = \\
&= \mathrm{Tr}\left[2iA_\mu \partial_\nu g \partial_\rho g^{-1}\right]
\end{aligned} \tag{A.11}$$

3.
$$\begin{aligned}
&\frac{2i}{3}\mathrm{Tr}\left[ig^{-1}A_\mu A_\nu g g^{-1}\partial_\rho g + ig^{-1}A_\mu \partial_\nu g g^{-1}A_\rho g + ig^{-1}\partial_\mu g g^{-1}A_\nu A_\rho g\right] = \\
&= -\frac{2}{3}\mathrm{Tr}\left[A_\mu A_\nu \partial_\rho g g^{-1} + A_\rho A_\mu \partial_\nu g g^{-1} + A_\nu A_\rho \partial_\mu g g^{-1}\right] = \\
&= \mathrm{Tr}\left[-2A_\mu A_\nu \partial_\rho g g^{-1}\right]
\end{aligned} \tag{A.12}$$

Hence, the second term in the nonabelian Chern-Simons Lagrangian becomes, after a gauge transformation:

$$\mathrm{Tr}\left[\frac{2i}{3}A_\mu A_\nu A_\rho + 2A_\mu A_\nu \partial_\rho g g^{-1} + 2iA_\mu \partial_\nu g \partial_\rho g^{-1} - \frac{2}{3} g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g\right] \tag{A.13}$$

Thus, joining both terms, a gauge-transformation transforms the Chern-Simons Lagrangian according to $\mathcal{L}_{CS} \to \mathcal{L}_{CS} + \delta\mathcal{L}_{CS}$, where:

$$\begin{aligned}\delta\mathcal{L}_{CS} &= \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[-iA_\mu\partial_\nu g\partial_\rho g^{-1} - i\partial_\nu A_\mu \partial_\rho g g^{-1} + \frac{1}{3}g^{-1}\partial_\mu g g^{-1}\partial_\nu g^{-1}\partial_\rho g\right] = \\ &= \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[-i\partial_\nu\left(A_\mu g\partial_\rho g^{-1}\right) + \frac{1}{3}g^{-1}\partial_\mu g g^{-1}\partial_\nu g^{-1}\partial_\rho g\right].\end{aligned} \tag{A.14}$$

Appendix B

Derivation of the Feynman Rules

In the main thesis we carry out a perturbative analysis of Chern-Simons theory coupled to fermionic matter transforming in the adjoint representation. This approach requires expanding the exponential in the numerator of equation (4.8) as a power series in the coupling, κ. Doing so corresponds, diagrammatically, to including vertices and further propagators in the Feynman diagrams being considered, increasing their complexity.

In this appendix, we derive the Feynman rules required to evaluate those diagrams, namely the expressions corresponding to each vertex and propagator.

The full action we are considering is

$$\begin{aligned} S = \frac{\kappa}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}[A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho] + i\int d^3x\mathrm{Tr}[\bar{\Psi}\not{D}\Psi]+ \\ + \int d^3x\mathrm{Tr}[\bar{c}(-\partial^\mu D_\mu)c] + \frac{\kappa}{4\pi}\int d^3x\xi\mathrm{Tr}[(\partial A)^2], \end{aligned} \tag{B.1}$$

which includes the gauge-fixing term and where the covariant derivative is being contracted with a gamma matrix and is defined as:

$$D_\mu = \partial_\mu + i[A_\mu, \bullet]. \tag{B.2}$$

Since in the action above the normalization for the propagators is not the same, it is useful to absorb the level of theory in the definition of the gauge field. That is, by sending $\sqrt{\kappa}A \mapsto A$ while introducing a new parameter

$$g = \frac{1}{\sqrt{\kappa}}, \tag{B.3}$$

all interaction terms in the action become proportional to g while no propagators depend on it.

Rewriting the action in (B.1) in terms of the new coupling, we obtain:

$$\begin{aligned} S = \frac{1}{4\pi}\int d^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[A_\mu\partial_\nu A_\rho + \frac{2ig}{3}A_\mu A_\nu A_\rho\right] + i\int d^3x\mathrm{Tr}[\bar{\Psi}\not{D}\Psi]+ \\ + \int d^3x\mathrm{Tr}[\bar{c}(-\partial^\mu D_\mu)c] + \frac{1}{4\pi}\int d^3x\xi\mathrm{Tr}[(\partial A)^2], \end{aligned} \tag{B.4}$$

where the covariant derivative is given by

$$D_\mu = \partial_\mu + ig[A_\mu, \bullet]. \tag{B.5}$$

B.1 Gauge Propagator

The propagators are relatively straightforward to compute. They are simply given as the inverse of the quadratic term in the action, thus, one merely has to identify that term and then calculate its inverse. As a general rule, the calculations are easier in momentum space and, therefore, we also opt for this approach.

Rewriting the term in the action that involves two gauge fields, and including the gauge-fixing contribution, one finds that the quadratic term is given by

$$D_{\mu\rho} = \frac{i\delta^{AB}}{4\pi}\left(\epsilon_{\mu\nu\rho}p^\nu - \xi p_\mu p_\rho\right). \tag{B.6}$$

This operator can be read from:

$$\begin{aligned} &\frac{1}{4\pi}\int d^3x\left(\epsilon^{\mu\nu\rho}A^A_\mu\partial_\nu A^B_\rho - i\xi\partial^\mu A^A_\mu\partial_\rho A^B_\rho\right)\mathrm{Tr}[T^AT^B] = \\ &= \frac{1}{4\pi}\frac{\delta^{AB}}{2}\int d^3x\int\frac{d^3pd^3q}{(2\pi)^6}\left(\epsilon^{\mu\nu\rho}A^A_\mu(p)(iq_\nu)A^B_\rho(q) - i\xi p_\mu q_\rho A^A_\mu(p)A^B_\rho(q)\right)e^{i(p+q)x} = \\ &= \frac{1}{4\pi}\frac{\delta^{AB}}{2}\int\frac{d^3pd^3q}{(2\pi)^6}A^A_\mu(p)\left(i\epsilon^{\mu\nu\rho}q_\nu - i\xi p_\mu q_\rho\right)A^B_\rho(q)(2\pi)^3\delta^{(3)}(p+q) = \\ &= \frac{i}{4\pi}\frac{\delta^{AB}}{2}\int\frac{d^3p}{(2\pi)^3}A^A_\mu(p)\left(\epsilon^{\mu\nu\rho}p_\nu - \xi p_\mu p_\rho\right)A^B_\rho(-p) = \\ &= \frac{i}{4\pi}\frac{\delta^{AB}}{2}\int\frac{d^3p}{(2\pi)^3}A^A_\mu(p)\left(\epsilon^{\mu\nu\rho}p_\nu - \xi p_\mu p_\rho\right)A^B_\rho(-p) = \\ &= \frac{i}{4\pi}\frac{\delta^{AB}}{2}\int\frac{d^3p}{(2\pi)^3}A^A_\mu(p)\left(\epsilon^{\mu\nu\rho}p_\nu - \xi p_\mu p_\rho\right)A^B_\rho(-p). \end{aligned} \tag{B.7}$$

Now, to invert the quadratic term, we propose an ansatz and use the fact that the propagator multiplied by its inverse is the identity to fix the unknown coefficients. The most general form of the propagator is

$$\Pi_{\mu\nu} = a(p)\epsilon_{\mu\nu\rho}p^{\nu} + b(p)p_{\mu}p_{\nu}, \tag{B.8}$$

where $a(p)$ and $b(p)$ are undetermined functions with the appropriate weight in p, such that dimensional analysis is satisfied and the overall propagator scales as p^2. Requiring that the operators in equations (B.8) and (B.6) are the inverse of each other, that is $\Pi_{\mu\nu}D^{\nu\rho} = i\delta^{\rho}_{\mu}$, one finds:

$$\frac{i\delta^{AB}}{4\pi}\left[a(p)\epsilon_{\mu\alpha\nu}p^{\alpha} + b(p)p_{\mu}p_{\nu}\right]\left[\epsilon^{\nu\beta\rho}p_{\beta} - \xi p^{\nu}p^{\rho}\right] = i\delta^{\rho}_{\mu} \implies$$
$$\implies \begin{cases} a(p) = -\frac{4\pi\delta^{AB}}{p^2} \\ b(p) = \frac{a}{\xi p^2} \end{cases} .$$

We then take the usual choice of the Feynman gauge by sending $\xi \to \infty$. This is the most convenient choice since the propagator becomes much more simple, being given by:

$$\langle A^{A}_{\mu}(p)A^{B}_{\nu}(q)\rangle = -4\pi\delta^{AB}\epsilon_{\mu\nu\rho}\frac{p^{\rho}}{p^2}\delta(p+q) \tag{B.9}$$

B.2 Ghost Propagator

The propagator for the ghost field can be determined analogously, through a much more simple calculation. It is simpler for these fields since there is no necessity to implement any sort of gauge-fixing and since the quadratic term involves only a second derivative. In Fourier space, this second derivative corresponds to a momentum-squared factor, which is extremely easy to invert.

Consequently, the ghost propagator is given as

$$\langle \bar{c}^{A}(p)c^{C}(q)\rangle = \frac{i\delta^{AC}}{p^2}\delta(p+q). \tag{B.10}$$

B.3 Fermion Propagator

The only propagator yet to be determined is that corresponding to the fermion fields. This propagator is not much harder to compute than the previous one. The only subtlety that must be considered is the fact that the derivative is being contracted with a gamma matrix.

Thus, rewriting the quadratic term in momentum space:

$$\begin{aligned}
i\int d^3x \bar{\Psi}^A \partial\!\!\!/ \Psi^B &= \int d^3x \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\bar{\Psi}^A(q)e^{iqx}\partial\!\!\!/\Psi^B(p)e^{ipx}\mathrm{Tr}[T^AT^B] = \\
&= -\frac{\delta^{AB}}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\bar{\Psi}^A(q)p\!\!\!/\Psi^B(p)(2\pi^3)\delta^{(3)}(p+q) = \\
&= -\frac{\delta^{AB}}{2}\int \frac{d^3p}{(2\pi)^3}\bar{\Psi}^A(-p)p\!\!\!/\Psi^B(p) = -\frac{\delta^{AB}}{2}\int \frac{d^3p}{(2\pi)^3}\bar{\Psi}^A(-p)p\!\!\!/\Psi^B(p) = \\
&= \frac{\delta^{AB}}{2}\int \frac{d^3p}{(2\pi)^3}\bar{\Psi}^A(p)\left[\gamma^\mu p_\mu\right]\Psi^B(-p)
\end{aligned} \tag{B.11}$$

This term is straightforward to invert, resulting in

$$\langle\bar{\Psi}^A(p)\Psi^B(q)\rangle = \delta^{AB}i\frac{\gamma^\mu p_\mu}{p^2}\delta(p+q). \tag{B.12}$$

B.4 Interaction Vertices

Regarding the interaction vertices, these can be read directly from the action, by looking at the terms that involve the coupling between the fields and adding an extra factor of i. Since there is a cubic term in the gauge fields, a term with a gauge field and fermion/anti-fermion pair and also a term involving a gauge field and 2 ghosts, then this theory contains three vertices, corresponding to each of these three interactions.

By first looking at the term in the action that involves the three gauge fields, it is immediate to see that the cubic interaction vertex, appearing when the exponential in the path integral is expanded, corresponds to a multiplicative factor involving the coupling and a structure constant:

$$\begin{aligned}
&-\frac{g}{6\pi}\int d^3x\epsilon^{\mu\nu\rho}A_\mu^AA_\nu^BA_\rho^C\mathrm{Tr}[T^AT^BT^C] = \\
&= -\frac{ig}{24\pi}\epsilon^{\mu\nu\rho}f^{ABC}\int d^3xA_\mu^AA_\nu^BA_\rho^C
\end{aligned}$$

To obtain the previous result we made use of the last identity in (3.10), as well as the fact that it is possible to write the trace of three $SU(N)$ generators in the following manner:

$$\begin{aligned}
\mathrm{Tr}[T^AT^BT^C] = \mathrm{Tr}\left[\left[T^A,T^B\right]T^C\right] - \mathrm{Tr}[T^AT^BT^C] \implies \\
\implies \mathrm{Tr}[T^AT^BT^C] = \frac{i}{4}f^{ABC}.
\end{aligned} \tag{B.13}$$

In terms of Feynman diagrams, this interaction corresponds to the vertex in figure B.1. Likewise, looking at the coupling between the gauge field and the fermion, one finds the

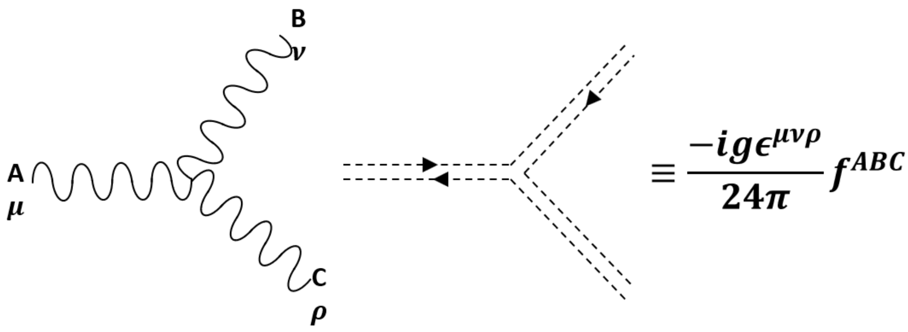


Figure B.1: Interaction vertex between 3 gluons

following term:

$$- g \int d^3x \bar{\Psi}^a_b \cdot i\gamma^\mu \left[A_\mu, \Psi\right]^b_a = ig \int d^3x \bar{\Psi}^A \gamma^\mu A^B_\mu \Psi^C Tr[T^A[T^B, T^C]] =$$
$$= -\frac{g}{2}\gamma^\mu f^{ABC} \int d^3x \bar{\Psi}^A A^B_\mu \Psi^C.$$

Lastly, only the vertex involving the ghosts remains to be determined. This interaction

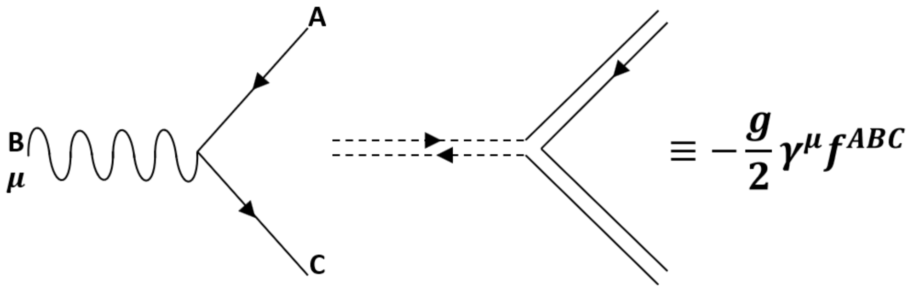


Figure B.2: Interaction vertex between the gauge field and the fermion

that can be computed from 3.29 but requires some extra care. First, taking the trace over the color indices results in:

$$\begin{aligned} &- ig \int d^3x Tr\left[\bar{c} \cdot i\partial^\mu [A_\mu, c]\right] = g \int d^3x \bar{c}^A \partial^\mu A^B_\mu c^C \mathrm{Tr}\left[T^A[T^B, T^C]\right] = \\ &= \frac{ig}{2} f^{ABC} \int d^3x Tr\left[\bar{c}^A (\partial^\mu A^B_\mu) c^C\right] + \frac{ig}{2} f^{ABC} \int d^3x Tr\left[\bar{c}^A A^B_\mu \partial^\mu c^C\right] \end{aligned} \tag{B.14}$$

There are two terms in the previous expression, only one of which is relevant. The term with the derivative acting on the gauge field will not contribute as will be argued next.

The derivative acting on the gauge field will still be present after performing the Wick contractions. Consequently, there will be a derivative acting on a gauge propagator in the following manner:

$$\begin{aligned}\partial_y^\nu \langle A_\mu^A(x) A_\nu^B(y)\rangle &= \partial_y^\nu \int \frac{d^3q}{(2\pi)^3}\Pi_{\mu\nu}^{AB}(q)e^{iq(x-y)} = \\ &= \int \frac{d^3q}{(2\pi)^3}(iq^\nu)4\pi\delta^{AB}\epsilon_{\mu\nu\rho}\frac{q^\rho}{q^2}e^{iq(x-y)} = \\ &= 0.\end{aligned} \tag{B.15}$$

In the previous equation $\Pi_{\mu\nu}^{AB}(q)$ stands for the gauge propagator with momentum q. In the last line we set the whole term to 0 since there is an even tensor structure, $q^\nu q^\rho$, being contracted with an odd one, the Levi-Civita symbol.

Regarding the second term in equation (B.14), the fact that this work is focused on correlators of fermionic operators (and consequently all the ghost involved come from this vertex), means that all correlators of ghosts appear with a derivative.
Thus, for our purposes, we can consider the following correlator:

$$\begin{aligned}\partial_y^\mu \langle \bar{c}(x)^A c^B(y)\rangle &= \int \frac{d^3q}{(2\pi)^3}\frac{i\delta^{AB}}{q^2}\partial_y^\mu e^{iq(x-y)} = \\ &= \int \frac{d^3q}{(2\pi)^3}\frac{-\delta^{AB}q^\nu}{q^2}e^{iq(x-y)},\end{aligned} \tag{B.16}$$

which is very similar to the fermion propagator, without a γ matrix (and consequently with an open index).
Using this as the propagator between the ghosts makes it so that the vertex is given as the following multiplicative factor: This yields the vertex shown in figure B.3

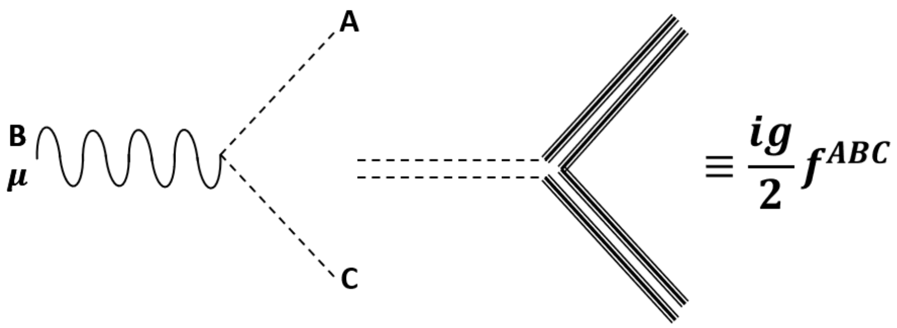


Figure B.3: Interaction vertex between the gauge field and the Fadeev-Popov's ghosts

Appendix C

Integrals in Dimensional Regularization

The computations in this thesis require the evaluation of scalar integrals of the following form:

$$I[p,\alpha,\beta,d] = \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^\alpha [(q-p)^2]^\beta}. \tag{C.1}$$

The regularization scheme chosen was dimensional regularization, meaning the integrals are evaluated in d-dimensions, where d is a continuous parameter. After performing the integration we take $d = 3 - \epsilon$, with epsilon being a small parameter.

Dimensional analysis fixes the structure of the integral in (C.1). Since there is a factor of q^d coming from the metric and a factor of $q^{-2\alpha-2\beta}$ from the denominator and since q is being integrated over such that the only remaining dimensional parameter is p, it is possible to conclude that the integral must be proportional to

$$\frac{1}{(p^2)^{\alpha+\beta-\frac{d}{2}}}.$$

In order to extract the appropriate coefficient (which is in itself a function of α, β and d), the integral must be computed exactly. The first step requires introducing 2 Schwinger parameters x and y to deploy the relation

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-xA} \tag{C.2}$$

twice.

$$\begin{aligned} I[p,\alpha,\beta,d] &= \frac{1}{(2\pi)^d \Gamma(\alpha)\Gamma(\beta)} \int d^d q \int_0^\infty dx \int_0^\infty dy x^{\alpha-1} y^{\beta-1} e^{-xq^2 - y(q-p)^2} = \\ &= \frac{1}{(2\pi)^d \Gamma(\alpha)\Gamma(\beta)} \int_0^\infty dx \int_0^\infty dy x^{\alpha-1} y^{\beta-1} e^{\frac{xy}{x+y}p^2} \int d^d q e^{-(x+y)(q-\frac{y}{x+y}p)^2} \end{aligned} \tag{C.3}$$

In (C.3) we made use of equation (C.2) twice and then proceeded to complete the square in the exponential. The resulting integral over q is the usual d-dimensional Gaussian integral and can, therefore, be easily done.

Afterwards, it is convenient to perform the change of variables

$$\begin{cases} x \to ab \\ y \to a(1-b) \end{cases}, \tag{C.4}$$

to which corresponds the Jacobian

$$J = \begin{vmatrix} b & a \\ 1-b & -a \end{vmatrix} = a. \tag{C.5}$$

This allows the integral to be rewritten in a far more convenient way, as given in equation (C.6).

$$\begin{aligned} I[p,\alpha,\beta,d] &= \frac{1}{(2\pi)^d\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty dx\int_0^\infty dy x^{\alpha-1}y^{\beta-1}e^{\frac{xy}{x+y}p^2}\left(\frac{\pi}{(x+y)}\right)^{\frac{d}{2}} = \\ &= \frac{1}{2^d\pi^{d/2}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 db\int_0^\infty daa\frac{a^{\alpha-1}b^{\alpha-1}a^{\beta-1}(1-b)^{\beta-1}}{a^{d/2}}e^{-b(1-b)p^2a} = \\ &= \frac{1}{2^d\pi^{d/2}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dbb^{\alpha-1}(1-b)^{\beta-1}\int_0^\infty daa^{\alpha+\beta-1-d/2}e^{-b(1-b)p^2a} \end{aligned} \tag{C.6}$$

A simple change of variables, $a \to u \equiv b(1-b)p^2a$, allows the integral over a to be performed, yielding both a Γ-function as well as the expected momentum structure. Afterwards the integral over b can be evaluated, giving a contribution that is written in terms of a Beta-function.*

$$\begin{aligned} I[p,\alpha,\beta,d] &= \frac{1}{2^d\pi^{d/2}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dbb^{\alpha-1}(1-b)^{\beta-1}\times \\ &\qquad\times\int_0^\infty du\frac{1}{b(1-b)p^2}\frac{u^{\alpha+\beta-1-d/2}}{[b(1-b)p^2]^{\alpha+\beta-1-d/2}}e^{-u} = \\ &= \frac{\Gamma[\alpha+\beta-d/2]}{2^d\pi^{d/2}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 dbb^{d/2-\beta-1}(1-b)^{d/2-\alpha-1}\frac{1}{(p^2)^{\alpha+\beta-d/2}} = \\ &= \frac{\Gamma[\alpha+\beta-d/2]}{2^d\pi^{d/2}\Gamma(\alpha)\Gamma(\beta)(p^2)^{\alpha+\beta-d/2-1}}B[d/2-\alpha,d/2-\beta] \end{aligned} \tag{C.7}$$

*$B(z_1,z_2) = \int_0^1 dxx^{z_1-1}(1-t)^{z_2-1}$, where z_1 and z_2 are two complex numbers

Lastly, knowing that the beta-function is related to the gamma-function by

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}, \tag{C.8}$$

the integral in (C.7) becomes:

$$I[p, \alpha, \beta, d] = \frac{\Gamma[\alpha + \beta - d/2]\Gamma[d/2 - \alpha]\Gamma[d/2 - \beta]}{2^d \pi^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma[d - \alpha - \beta](p^2)^{\alpha+\beta-d/2-1}}. \tag{C.9}$$

The part that is independent of the momentum is often designated by the G-function and is ubiquitous in computations in dimensional regularization. Therefore, we have:

$$I[p, \alpha, \beta, d] \equiv \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2)^\alpha \left[(q-p)^2\right]^\beta} = \frac{G(\alpha, \beta)}{(p^2)^{\alpha+\beta-\frac{d}{2}}}, \tag{C.10}$$

where

$$G(a, b) = \frac{\Gamma[\alpha + \beta - d/2]\Gamma[d/2 - \alpha]\Gamma[d/2 - \beta]}{2^d \pi^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma[d - \alpha - \beta]}. \tag{C.11}$$

Appendix D

List of 2-loop Master Integrals

The following twenty-two master integrals were obtained using the `LiteRed` package.

$$
\begin{aligned}
M_1 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(k-q_1)^2 (p-q_2)^2 (q_2-q_1)^2} \\
M_2 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (q_2-q_1)^2} \\
M_3 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (q_2-q_1)^2} \\
M_4 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (p-q_2)^2 (q_2-q_1)^2} \\
M_5 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_2)^2 (p-q_1)^2 (q_2-q_1)^2} \\
M_6 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{(k-q_1)^2 (k-q_2)^2 (p-q_1)^2 (p-q_2)^2} \\
M_7 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (k-q_2)^2 (p-q_1)^2} \\
M_8 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (p-q_1)^2 (p-q_2)^2} \\
M_9 &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^2 (p-q_1)^2 (q_2-q_1)^2} \\
M_{10} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 q_2^2 (k-q_1)^2 (k-q_2)^2} \\
M_{11} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 q_2^2 (k-q_1)^2 (p-q_2)^2} \\
M_{12} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_1^2 q_2^2 (p-q_1)^2 (p-q_2)^2} \\
M_{13} &= \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2 (k-q_1)^4 (p-q_2)^2 (q_2-q_1)^2}
\end{aligned}
\tag{D.1}
$$

$$M_{14} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_2^2(k-q_2)^2(p-q_1)^4(q_2-q_1)^2}$$
$$M_{15} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_2^2(k-q_1)^2(p-q_1)^4(q_2-q_1)^2}$$
$$M_{16} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_2^2(k-q_1)^2(k-q_2)^2(p-q_1)^2(p-q_2)^2}$$
$$M_{17} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_2^2(k-q_1)^2(k-q_2)^2(p-q_1)^2(q_2-q_1)^2}$$
$$M_{18} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_2^2(k-q_1)^2(p-q_1)^2(p-q_2)^2(q_2-q_1)^2}$$
$$M_{19} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2q_2^2(k-q_1)^2(k-q_2)^2(p-q_2)^2}$$
$$M_{20} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2q_2^2(k-q_1)^2(p-q_2)^2(q_2-q_1)^2}$$
$$M_{21} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2q_2^2(k-q_2)^2(p-q_1)^2(p-q_2)^2}$$
$$M_{22} = \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{1}{q_1^2q_2^2(k-q_1)^2(k-q_2)^2(p-q_1)^2(p-q_2)^2}$$

Bibliography

[1] B. P. Abbott et al., “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett., vol. 116, no. 6, p. 061102, 2016. [Cited on page 1.]

[2] D. Tong, “String Theory,” 1 2009. [Cited on page 2.]

[3] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys., vol. 2, pp. 231–252, 1998. [Cited on pages 2 and 4.]

[4] J. Penedones, “TASI lectures on AdS/CFT,” in Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings, 2017, pp. 75–136. [Cited on pages 4, 5, and 67.]

[5] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, “Holography from Conformal Field Theory,” JHEP, vol. 10, p. 079, 2009. [Cited on page 5.]

[6] N. Gromov, V. Kazakov, and P. Vieira, “Exact spectrum of planar $n = 4$ supersymmetric yang-mills theory: Konishi dimension at any coupling,” Phys. Rev. Lett., vol. 104, p. 211601, May 2010. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevLett.104.211601 [Cited on page 5.]

[7] S. Banerjee and Đ. Radičević, “Chern-simons theory coupled to bifundamental scalars,” Journal of High Energy Physics, vol. 2014, no. 6, 2014. [Online]. Available: https://doi.org/10.1007%2Fjhep06%282014%29168 [Cited on page 6.]

[8] V. Gurucharan and S. Prakash, “Anomalous dimensions in non-supersymmetric bifundamental chern-simons theories,” Journal of High Energy Physics, vol. 2014, no. 9, 2014. [Online]. Available: https://doi.org/10.1007%2Fjhep09%282014%29009 [Cited on page 6.]

[9] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, “Spinning Conformal Correlators,” JHEP, vol. 11, p. 071, 2011. [Cited on pages 6, 15, 16, and 19.]

[10] M. Gillioz, “Conformal field theory for particle physicists,” 7 2022. [Cited on page 10.]

[11] G. V. Dunne, “Aspects of Chern-Simons theory,” in Les Houches Summer School in Theoretical Physics, Session 69: Topological Aspects of Low-dimensional Systems, 7 1998. [Cited on pages 21 and 25.]

[12] D. Bak and S.-J. Rey, “Integrable Spin Chain in Superconformal Chern-Simons Theory,” JHEP, vol. 10, p. 053, 2008. [Cited on pages 22 and 25.]

[13] G. ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” Nucl. Phys. B, vol. 72, p. 461, 1974. [Cited on page 26.]

[14] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia, and X. Yin, “Chern-Simons Theory with Vector Fermion Matter,” Eur. Phys. J. C, vol. 72, p. 2112, 2012. [Cited on page 29.]

[15] M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory. Reading, USA: Addison-Wesley, 1995. [Cited on pages 29, 35, and 38.]

[16] M. Neubert, “Renormalization Theory and Effective Field Theories,” 1 2019. [Cited on pages 33 and 34.]

[17] J. A. Minahan and K. Zarembo, “The Bethe ansatz for N=4 superYang-Mills,” JHEP, vol. 03, p. 013, 2003. [Cited on page 35.]

[18] V. A. Smirnov, Feynman integral calculus. Berlin, Heidelberg: Springer, 2006. [Cited on pages 38, 51, and 52.]

[19] W. Chen, G. W. Semenoff, and Y.-S. Wu, “Two loop analysis of nonAbelian Chern-Simons theory,” Phys. Rev. D, vol. 46, pp. 5521–5539, 1992. [Cited on page 45.]

[20] R. N. Lee, “Presenting LiteRed: a tool for the Loop InTEgrals REDuction,” 12 2012. [Cited on page 53.]

[21] A. V. Belitsky, J. Henn, C. Jarczak, D. Mueller, and E. Sokatchev, “Anomalous dimensions of leading twist conformal operators,” Phys. Rev. D, vol. 77, p. 045029, 2008. [Cited on pages 62 and 63.]

[22] J. Maldacena and A. Zhiboedov, “Constraining Conformal Field Theories with A Higher Spin Symmetry,” J. Phys. A, vol. 46, p. 214011, 2013. [Cited on page 69.]