On the Local Structure of Poisson Manifolds

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Rodrigo Baptista
Porto, 21.st of June 2023

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Abstract

We study Poisson geometry and several other related geometric structures, with particular interest in their local structure. It is well-known that understanding the local structure of manifolds endowed with certain structures allows us to better comprehend the geometry involved in said manifold, and, therefore, be better suited to tackle other problems concerning these geometric structures. Thus, it is important for any geometer to be well-acquainted, not only with these normal form results, but with the methods involved in their proofs as well.

In that perspective, we start by introducing the notions of tangent distributions in order to prove the Frobenius theorem in its local and global form and for both regular and singular distributions. Then, we proceed to study symplectic geometry and obtain Darboux's theorem, for which we prove the usual statement, using Moser's trick, as well as a version for foliated manifolds.

After these two structures, we are ready to understand Poisson manifolds. Being at the centre of the thesis, we explore in most detail the different ways we may introduce Poisson structures on a manifold and obtain Poisson and Hamiltonian automorphisms (as well as their infinitesimal counterparts) along with the characteristic distribution arising in a Poisson manifold. Then, we delve into the local structure of Poisson manifolds, starting with the symplectic foliation of Poisson manifolds and finishing with the statement and proofs of Weinstein's splitting theorem for both its simplest statement as well as the one around Poisson transversals.

Finally, we give an overview of Lie algebroids, Dirac manifolds and of a splitting theorem for anchored vector bundles, recently proposed in [1], which can be adapted for Lie algebroids, Dirac and Poisson structures as well as generalised complex manifolds.

Key-words: Poisson Geometry; Symplectic Geometry; Foliation Theory; Lie Algebras

Resumo

Estudamos geometria de Poisson e outras estruturas geométricas relacionadas, com interesse particular nas suas estruturas locais. É bem sabido que o conhecimento da estrutura local de variedades equipadas com certas estruturas permite-nos compreender melhor a geometria envolvida nessa variedade e, portanto, estar melhor preparado para tratar outros problemas que aparecem nessas estruras geométricas. Consequentemente, é importante que qualquer investigador que trabalhe em geometria tenha conhecimento não apenas destes resultados mas também dos métodos envolvidos nas suas demonstrações.

Nessa perspectiva, começamos por introduzir as noções de distribuições tangentes de forma a demonstrar o teorema de Frobenius na sua forma local e global para ambas distribuiçoes regulares e singulares. Seguidamente, procedemos ao estudo de geometria simpléctica e obtemos o teorema de Darboux, para o qual provamos o enunciado usual, usando o truque de Moser, bem como uma versão para variedades foliadas.

Após estas duas estruturas, estamos preparados para tratar variedades de Poisson. Estando no centro da presente tese, exploramos com maior detalhe as differentes formas de introduzir estruturas de Poisson numa variedade, e obtemos os automorfismos de Poisson e Hamiltonianos (tal como as suas versões infinitesimais) assim como a distribuição característica de uma variedade de Poisson. Em seguida, estudamos a estrutura local de variedades de Poisson, começando pela foliação simpléctica de variedades de Poisson e terminando com o enunciado e demonstrações do teorema de separação de Weinstein para as sua versão mais simples e também perto de transversais de Poisson.

Finalmente, damos uma visão geral de algebróides de Lie, variedades de Dirac e de teoremas de separação para fibrados vectoriais ancorados proposto recentemente em [1], que pode ser adaptado para algebróides de Lie, estruturas de Dirac e Poisson e ainda para variedades complexas generalizadas.

Palavras-chave: Geometria de Poisson; Geometria Simpléctica; Teoria de Foliações; Álgebras de Lie

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Chapter 1

Preliminaries

Since the inception of differential geometry, the problem of understanding the local structure of manifolds has been at the centre of the field. Theorems that give some kind of local structure are usually called (local) normal form results and their study has given many other useful results describing important geometries. Examples of such theorems are Frobenius' theorem for tangent distributions (1877), which shows how a distribution whose sections form a Lie subalgebra defines a foliation of the base manifold, Darboux's theorem in symplectic geometry (1882), describing how, locally, any symplectic form has a canonical form, and the Newlander-Nirenberg theorem (1957), which provides complex coordinates in complex geometry.

One other example of a geometry where normal form results have been found is Poisson geometry. This geometry, which shall be the focus of the present thesis, has developed into an active research field in pure mathematics with plenty of applications to physics, and can be seen as an amalgam of foliation theory (partition into leaves), symplectic geometry (along the leaves), and Lie theory (transverse to the leaves). The most notable normal form results for Poisson geometry are Weinstein's splitting theorem (1983), showing how every Poisson bivector at a point can be written as a sum of a non-degenerate Poisson structure and a residual Poisson structure vanishing at the target point and also Conn's theorem (1985) which shows that, under certain conditions, Poisson bivectors vanishing at a point are equivalent to the linear Poisson structure on the dual of the isotropy Lie algebra.

In other geometries related to the the ones mentioned previously, such as Lie algebroids, Dirac structures or complex generalised structures, some normal form results are also known, however much is still to be understood in these and more general geometries. Regardless, all of the results above give a new perspective of the specific geometry they target and allow for new geometry to be developed, thus justifying the intense work put into finding more local normal forms for these different structures. In this thesis, we give a first introduction to some of the most notable normal form results as well as give an overview of a recent splitting theorem for anchored vector bundles introduced by Bursztyn, et al. [1].

In this first chapter we introduce the basic concepts required for a smoother description of later chapters. Therefore, in the first section, we quickly go through the basic notions about differential manifolds in order to also fix the notation used throughout the rest of the thesis. In the second section, we discuss fibre bundles, particularly the ones with vector space structures on their fibres, i.e. vector bundles, their homogeneity structures as well as double vector bundles, which are vector bundles with vector bundles as their base manifolds. Section 1.3 gives a brief overview of the theory of Lie algebras, with particular emphasis on the theory of graded algebras and graded Lie algebras as they form a central role in most geometric descriptions. The fourth and last section deals with the extension of vector fields to anti-symmetric contravariant tensors, the so-called multivector fields, and the extension of the Lie bracket to such a space in order to endow this collection of multivector fields with a graded Lie algebra structure essential when studying Poisson manifolds.

1.1 Brief Overview of Concepts in Differential Manifolds

In this first section we introduce the elementary concepts involved in differential geometry and, in doing so, we shall follow very closely the similar treatment done in [2]. Moreover, we shall assume the reader is familiarised with the most basic notions of differential geometry so that the following section is simply a review of certain notion in order to fix notation that will be used in subsequent chapters.

From now on, we will always assume that M is a smooth dimensional manifold and, unless stated otherwise, shall be of dimension $m \in \mathbb{Z}_{\geqslant 0}$. We shall denote a chart by (U, φ) with $U \subseteq M$ open and $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^m$ a diffeomorphism. Sometimes, in order to emphasise the local coordinates, we may write a chart instead as $(U, \varphi = (x_1, \ldots, x_m))$.

Differential Maps on a Manifold

From this we can define, differentiability at a point in the manifold:

Definition 1.1.1. A function $f : M \to \mathbb{R}^m$ is said to be *differentiable at* $p \in M$ if, for some chart (φ, U) at p, $f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$.

We state, as usual, that f is differentiable in M if it is differentiable for all $p \in M$. Also, we can easily extend this definition for any two manifolds M and N and a map $f : M \to N$, by considering charts (φ, U) and (ψ, V) in M and N, respectively, and letting $\psi \circ f \circ \varphi^{-1}$ be differentiable at $\varphi(p)$. We call a map *smooth* when it is infinitely differentiable and denote the set of all smooth maps as $C^\infty(M, N)$ and, in particular, $C^\infty(M) \stackrel{\text{def}}{=} C^\infty(M, \mathbb{R})$.

Remark 1.1.2. The introduction of smooth maps on manifolds allows us to define the category of smooth manifolds, **Man** whose objects are smooth manifolds and whose objects are smooth maps, so $\mathrm{Hom}_{\mathbf{Man}}(M,N) = C^\infty(M,N)$.

Remark 1.1.3. For smooth functions, i.e. elements of $C^\infty(M)$, given the algebraic structure of the codomain, we can define the addition and multiplication of functions point-wise, which gives a ring structure (in fact it gives an $\mathbb{R}$-algebra structure) to $C^\infty(M)$. As we shall later see, this ring structure turns out to be quite important when dealing with vector fields and differential forms. For the reader more familiarised with Algebraic Geometry, $C^\infty(\cdot,\mathbb{R})$ for a manifold M is an example of a sheaf.

The definition of tangent vectors follows directly from the definition of a differentiable map:

Definition 1.1.4. Let M be a smooth manifold, $p \in M$ and $\gamma :]-\epsilon,\epsilon[\to M$ be a smooth map such that $\gamma(0) = p$. Construct an equivalence class $\sim$ on the set of these smooth maps γ on M passing through p at $t = 0$ as follows

$$\gamma \sim \delta \iff D(\varphi \circ \gamma)(0) = D(\varphi \circ \delta)(0),$$

for given smooth curves γ, δ on M passing through p at $t = 0$ and a given chart φ on M around p, where D is Euler's notation for differentiation in $\mathbb{R}$. A *Tangent Vector at p* is an equivalence class $[\gamma]_p$ resulting from this equivalence relation.

Collecting all these tangent vectors at a point p yields a rather important space:

Definition 1.1.5. The *Tangent Space at p* is defined as the set of all the tangent vectors at this point:

$$T_pM \overset{\text{def}}{=} \big\{[\gamma]_p \,\big|\, \gamma :]-\epsilon,\epsilon[\to M,\ \gamma(0) = p\big\}. \tag{1.1}$$

This tangent space is in fact a vector space over the reals, as the following proposition shows:

Proposition 1.1.6. *Given a manifold M and a point in the manifold, $p \in M$, the tangent space at p, T_pM, has the structure of a real vector space.*

Proof. In order to endow T_pM with a canonical structure of a real vector space, we need to make use of the vector structure of $\mathbb{R}^m$. Let $\varphi : U \to \varphi(U)$ be a chart centred at p, with $U \subseteq M$ a neighbourhood of p. Then, φ determines the following map

$$\begin{aligned} \varphi_*(p) : T_pU &\longrightarrow \mathbb{R}^m \\ [\gamma]_p &\longmapsto D(\varphi \circ \gamma)(0) \end{aligned}$$

To conclude the result it is enough to show that $\varphi_*(p)$ is bijective since, then, the vector structure of T_pM is inherited from the one in $\mathbb{R}^m$ and this structure is well-defined because, given another chart (V,ψ), the coordinate changes $\psi\circ\varphi^{-1} : \varphi(U\cap V)\to\psi(U\cap V)$ are diffeomorphisms between open subsets of $\mathbb{R}^m$. The injectivity comes directly from the definition of the tangent space T_pM. As for the surjectivity, notice that equivalence classes, for any $v\in\mathbb{R}^m$, we may consider the curve $\gamma :]-\epsilon,\epsilon[\to M$ defined by $\gamma(t)=\varphi^{-1}(tv)$. Then, $\gamma(0)=p$ and

$$\varphi_*(p)([\gamma]_p) = D(\varphi\circ\varphi^{-1}\circ\mu_v)(0) = D\mu_v(0) = v,$$

where $\mu_v : \mathbb{R}\to\mathbb{R}^m$ is given by $\mu_v(t)=tv$. Therefore, $\varphi_*(p)$ is also surjective. Q.E.D.

Remark 1.1.7. Notice that any finite dimensional real vector space V has a natural structure of a smooth manifold. In fact, chosen a basis $\{e_1,\ldots,e_n\}$ ($n=\dim(V)$) of V, the associated coordinates $\{x_1,\ldots,x_n\}$ (so that $x_i\left(\sum\limits_{j=1}^{m} v_je_j\right)=v_i$, for all $i,j=1,\ldots,n$) give a smooth (global) chart $(V,(x_1,\ldots,x_n))$ of V. Thus, the tangent space T_pM also has an induced smooth manifold structure.

For a differentiable map $f : M\to N$ between manifolds M and N, the derivative map is defined as:

Definition 1.1.8. If $f : M\to N$ is differentiable, then the *derivative of f at p* is the map

$$\begin{aligned} f_*(p) : T_pM &\longrightarrow T_{f(p)}N \\ [\gamma]_p &\longmapsto [f\circ\gamma]_{f(p)} \end{aligned}. \tag{1.2}$$

Remark 1.1.9. Throughout the text, we shall make use of different notations for the derivative of a map $f : M\to N$ at a point $p\in M$, which we now explain:

- $f_*(p)$ is the standard notation used and it is also used to denote the pushforward of vector fields (when there is no mention of the point p);
- $(\mathrm{d}f)_p$ is used whenever we want to emphasise that the derivative can be seen as an element of the dual of the tangent space at p, T_p^*M;
- T_pf is used whenever we intend to emphasise the functorial properties of the derivative as well as to emphasise the fact that the derivative map is a vector bundle morphism covering the map f (notions to be defined in Section 1.2).

One can check that the definition of the derivative map is well-defined and consistent with the Chain Rule. In fact, given a point $p\in M$ and $f : M\to N$ and $g : N\to P$ maps between smooth manifolds differentiable at p and $f(p)$, respectively, we have $g\circ f : M\to P$ and

$$\begin{aligned}(g\circ f)_*(p)([\gamma]_p) &= [(g\circ f)\circ\gamma]_{(g\circ f)(p)} = [g\circ(f\circ\gamma)]_{g(f(p))}\\ &= g_*(f(p))([f\circ\gamma]_{f(p)}) = \big(g_*(f(p))\circ f_*(p)\big)([\gamma]_p),\end{aligned}$$

which can be written compactly as

$$(g\circ f)_*(p) = g_*(f(p))\circ f_*(p). \tag{1.3}$$

Furthermore, it is easily shown that, for a given (smooth) map $f : M \to N$, the derivative $f_*(p) : T_pM \to T_{f(p)}N$ is linear (cf. [2, Proposition 2.2.1]).

Definition 1.1.10. The *Tangent Bundle of a smooth manifold* M is defined as the union of all tangent spaces of M:

$$TM = \coprod_{p\in M} T_pM, \tag{1.4}$$

where $\coprod$ represents the disjoint union. The *canonical projection of the tangent bundle* is the map $\tau_M : TM \to M$ which, to every element $v \in TM$ associates the point $p \in M$ such that $v \in T_pM$.

Thus, given two manifolds M, N and smooth map $f : M \to N$, the derivative map is such that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d, "\tau_M"'] & TN \arrow[d, "\tau_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Because of this property, we say that $Tf = f_*$ is a bundle morphism covering f (this shall be seen with more detail when we study vector bundles).

Remark 1.1.11. For the reader more well versed with category theory, by defining the following two categories:

- the category of pointed manifolds $\mathbf{Man}_*$ whose objects are pairs (M, p) where M is a smooth manifold and $p \in M$ is a point and whose morphisms between objects (M, p) and (N, q) are smooth maps $f \in C^\infty(M, N)$ such that $f(p) = q$;
- the category of real vector fields $\mathbf{Vect}_\mathbb{R}$ whose objects are real vector fields and whose morphisms are linear maps;

then we may define the tangent functor:

$$\begin{aligned} T : \mathbf{Man}_* &\longrightarrow \mathbf{Vect}_\mathbb{R}\\ (M,p) &\xmapsto{\text{Ob}} T_pM\\ f &\xmapsto{\text{Mor}} T_pf = f_*(p)\end{aligned}$$

where the second line denotes the assignment of objects and the third line the assignment of morphisms.

One very important result regarding this tangent bundle is the fact that it inherits a smooth manifold structure from M:

Proposition 1.1.12. *Given a (smooth) manifold M, the tangent bundle TM of M is a smooth manifold.*

Proof. Firstly, the necessary topological properties are directly inherited from the manifold M as long as we find suitable charts for TM. With respect to these charts, we shall define them as follows: let $\varphi : U \to V$ be a chart for M for open sets $U \subseteq M$ and $V \subseteq \mathbb{R}^m$. Then, define a map Φ by

$$\begin{aligned} \Phi : TM\big|_U &\longrightarrow V \times \mathbb{R}^m \\ [\gamma]_p &\longmapsto \big(\varphi(p), \varphi_*(p)([\gamma]_p)\big), \end{aligned} \tag{$*$}$$

where $TM\big|_U = \coprod_{p \in U} T_pM$. We shall show that Φ can be charts for TM. For that purpose, we show that the coordinate change built from these maps is smooth. Let $\psi : U' \to V'$ be another chart and $\Psi : TM\big|_{U'} \to V' \times \mathbb{R}^m$ the associated map similar to the definition of $(*)$. Then the coordinate change $\Theta = \Psi \circ \Phi^{-1} : \varphi(U \cap U') \times \mathbb{R}^m \to \psi(U \cap U') \times \mathbb{R}^m$, has the following expression:

$$\Theta(q, v) = \Psi \circ \Phi^{-1}(q, v) = (\theta(q), \theta_*(q)(v)), \text{ for all } (q, v) \in \varphi(U \cap U') \times \mathbb{R}^m,$$

where $\theta = \psi \circ \varphi^{-1} : \varphi(U \cap U' \to \psi(U \cap U')$ is the coordinate change in M. Since θ is smooth, so is Θ. Q.E.D.

For any point p of a manifold M, we shall also define the cotangent space as:

Definition 1.1.13. The *Cotangent Space* T_p^*M to M at p is the dual of the tangent space to M at p, T_pM, i.e. the cotangent space is

$$T_p^*M \overset{\text{def}}{=} \big\{\alpha : T_pM \to \mathbb{R} \,\big|\, \alpha \text{ is linear}\big\}. \tag{1.5}$$

Similarly, we define the *Cotangent Bundle of M* as

$$T^*M \overset{\text{def}}{=} \coprod_{p \in M} T_p^*M \tag{1.6}$$

and the *canonical projection of the cotangent bundle* as the map $\hat{\tau}_M : T^*M \to M$ which, to every element $\alpha \in T^*M$ associates the point $p \in M$ such that $\alpha \in T_p^*M$.

Similarly to what was done in the case of the tangent bundle, for any smooth map $f : M \to N$ between manifolds M and N we may define a bundle map between the cotangent bundles:

Definition 1.1.14. The *cotangent lift of a local diffeomorphism* $f : M \to N$ *at* p is the map

$$\begin{aligned} T_p^* f : T_p^* M &\longrightarrow T_{f(p)}^* N \\ \alpha &\longmapsto \alpha \circ T_{f(p)} f^{-1} \end{aligned} \tag{1.7}$$

which is such that the following diagram commutes:

$$\begin{tikzcd} T^*M \arrow[r, "T^*f"] \arrow[d, "\hat{\tau}_M"'] & T^*N \arrow[d, "\hat{\tau}_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Remark 1.1.15. The requirement of $f : M \to N$ to be a local diffeomorphism in the previous definition is important only to guarantee that the inverse map of $f_*(p) : T_pM \to T_{f(p)}M$ exists.

Submanifolds of a Manifold

In the study of manifolds, it is often important to understand the properties of certain subsets. A special class of these sets are called submanifolds. Different types of submanifolds exist but, for us, the interesting ones appear from the image of certain maps:

Definition 1.1.16. Let M, N be smooth manifolds and $f : N \to M$ a smooth map.

1) f is called an *immersion* if, for every $p \in N$, $f_*(p) : T_pN \to T_{f(p)}M$ is injective. Then $f(N)$ or N are called an *immersed submanifold* on M;

2) f is said to be an *embedding* if f is an immersion and if $f : N \to f(N)$ is a homeomorphism. Then $f(N)$ or N is said to be an *(embedded) submanifold* on M.

Remark 1.1.17. In the literature, it is more common to simply call submanifolds to embedded submanifolds, since these are the most common examples and also sufficient to most purposes. We shall follow this nomenclature, only referencing the fact that a submanifold is embedded to emphasise the necessity for the distinction with immersed submanifolds.

Remark 1.1.18. It is common practice to describe immersed submanifolds in two ways: either by assuming that it is the image of an immersion $i : N \to M$ which is not necessarily injective or to assume that it is a subset $N \subseteq M$ equipped not necessarily with the subspace topology such that the inclusion $\iota : N \hookrightarrow M$ is an immersion (and, since it is the inclusion, injective). Clearly, both descriptions are entirely analogous so, for our purposes, we shall adopt the former description.

Example 1.1.19. The most usual examples of (embedded) submanifolds come from the pre-image of smooth maps $f : N \to M$ taking a regular value, i.e. a value $p \in M$ such that, for every $q \in f^{-1}(p)$, $\dim(f_*(q)(T_qN)) = \dim(M)$. A few examples of this construction are the following:

- Let $\|\cdot\| : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ represent the standard norm of $\mathbb{R}^{n+1}$. Then, $1 \in \mathbb{R}_{\geq 0}$ is a regular value (in fact every element of $\mathbb{R}_{>0}$ is a regular value), hence the pre-image is a submanifold which is, in fact, the n-sphere, S^n;

- Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries, $\det : M_n(\mathbb{R}) \to \mathbb{R}$ be the determinant map and $\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ be the *general linear group* of matrices. Then $\det : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ is a differentiable group homomorphism (here $\mathbb{R}^\times$ represents the multiplicative group of real numbers) and $1 \in \mathbb{R}$ is a real value. Thus,

$$\det{}^{-1}(1) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\} \overset{\text{def}}{=} \mathrm{SL}_n(\mathbb{R}),$$

  the *special linear group*, is an embedded submanifold of $\mathrm{GL}_n(\mathbb{R})$. ♮

Example 1.1.20. One other example of submanifolds are surfaces of revolution in $\mathbb{R}^3$. Consider (s, ϑ, z) to be the cylindrical coordinates on $\mathbb{R}^3$. Then, considering any curve C on $\mathbb{R}_{>0} \times \mathbb{R}$, we obtain a 2-dimensional submanifold

$$S_C = \{(s, \vartheta, z) \in \mathbb{R}^3 \mid \vartheta \in [0, 2\pi[\ ,\ (s, z) \in C\}.$$

To check that S_C is, in fact, a submanifold, let $\phi : \mathbb{R} \to \mathbb{R}_{>0} \times \mathbb{R}$ be such that $C = \mathrm{im}(\phi)$ and $\phi(t) = (\phi_s(t), \phi_z(t))$. Then, defining

$$\begin{aligned} \Phi : \mathbb{R} \times [0, 2\pi[\ &\longrightarrow \mathbb{R}^3 \\ (t, \vartheta) &\longmapsto (\phi_s(t), \vartheta, \phi_z(t)) \end{aligned}$$

we have that $S_C = \mathrm{im}(\Phi)$ and, clearly, Φ is a homeomorphism onto its image. Since $\Phi = \varsigma \circ (\phi \times \mathrm{id}_{[0,2\pi[})$, where $\varsigma : \mathbb{R}^3 \to \mathbb{R}^3$ switches the last two coordinates (so $\varsigma(s, z, \vartheta) = (s, \vartheta, z)$), the derivative map of Φ at a point $(t, \vartheta) \in \mathbb{R} \times [0, 2\pi[$ is

$$\Phi_*(t, \vartheta) = \varsigma_*(\phi(t), \vartheta) \circ (\phi_*(t) \times (\mathrm{id}_{[0,2\pi[})_*(\vartheta)) = \varsigma \circ (\phi_*(t) \times \mathrm{id}_{[0,2\pi[})$$

and, therefore, $\Phi_*(t, \vartheta)$ is injective. Thus, Φ is an embedding and, hence, S_C is an embedded submanifold. ♮

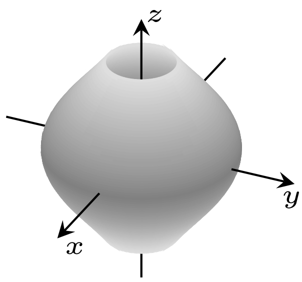


Figure 1.1: Solid of revolution with curve $C = \left\{(e^{-z^2}, 0, z) \in \mathbb{R}^3 \,\middle|\, z \in \mathbb{R}\right\}$.

Example 1.1.21. The usual example of an immersed manifold is the figure eight curve in $\mathbb{R}^2$ (Fig. 1.2). This curve can be described by $\gamma(t) = (\sin(2\pi t), \sin(4\pi t))$ with $t \in [0, 1]$, and, since $D\gamma(t) = (2\pi \sin(2\pi t), 4\pi \sin(4\pi t))$, which is injective for $t \in [0, 1]$, γ is an immersion and $\mathrm{im}(\gamma)$ is an immersed submanifold. ♮

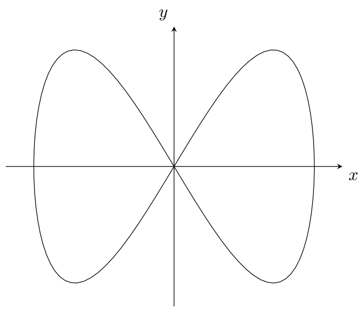


Figure 1.2: Figure eight curve

Example 1.1.22. Another example of an immersed submanifold is the dense curve in the torus $T^2 = S^1 \times S^1$. This curve is can be parameterised by $\gamma_\alpha(t) = (e^{it}, e^{i\alpha t})$. Then, $D\gamma_\alpha(t) = (ie^{it}, i\alpha e^{i\alpha t})$ is injective if and only if $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ (otherwise the curve closes and and repeats), and so, $\mathrm{im}(\gamma_\alpha)$ is an immersed submanifold. ♮

Definition 1.1.23. One says that two maps $f_1 : N_1 \to M$ and $f_2 : N_2 \to M$ are *transverse to each other*, and denotes it by $f_1 \pitchfork f_2$, if, for all $p_i \in N_i$ $(i = 1, 2)$, with $p = f_1(p_1) = f_2(p_2)$, one gets that

$$T_pM = (f_1)_*(p_1)(T_{p_1}N_1) + (f_2)_*(p_2)(T_{p_2}N_2). \tag{1.8}$$

Remark 1.1.24. With this definition is possible to introduce the notion of a map $f : L \to M$ transverse to an (immersed) submanifold N by saying that f and the immersion $i : N \to M$ are transverse (to each other). Furthermore, we say that two smooth submanifolds N_1, N_2 of M *intersect transversely* and represent $N_1 \pitchfork N_2$, if, the inclusion maps $\iota_i : N_i \hookrightarrow M$ $(i = 1, 2)$ are transverse, i.e. if, for every $p \in N_1 \cap N_2$, $T_pM = T_pN_1 + T_pN_2$.

Vector Fields and k-forms

Let $\tau_M : TM \to M$ and $\hat{\tau}_M : T^*M \to M$ be the fibre projections of TM and T^*M onto M, respectively. Then, we may define

Definition 1.1.25. A smooth map $X : M \to TM$ such that $\tau_M \circ X = \mathrm{id}_M$ is called a *vector field on M*. Similarly, a map $\omega : M \to T^*M$ such that $\hat{\tau}_M \circ \omega = \mathrm{id}_M$ is called a *covector field* or *1-form on M*.

Remark 1.1.26. With this definition, we assume that a vector field has to be smooth. However, sometimes it is also useful to consider a vector field to be a (in principle non--smooth) map $X : M \to TM$ such that $\tau_M \circ X = \mathrm{id}_M$. These non-smooth vector fields shall be called *rough vector fields*, following the same nomenclature used in [3].

In the previous definition, id_M is the identity map on M. The respective sets of vector and covector fields are

$$\mathfrak{X}(M) \overset{\text{def}}{=} \{X : M \to TM \,|\, X \text{ is a vector field}\} \tag{1.9a}$$

$$\Omega^1(M) \overset{\text{def}}{=} \{\omega : M \to T^*M \,|\, \omega \text{ is a 1-form}\}. \tag{1.9b}$$

In order to simplify notation, we will write a vector field and a 1-form evaluated at a point p by notating said point in subscript, i.e. we shall notate X_p and ω_p instead of $X(p)$ and $\omega(p)$.

A rather important interpretation of a vector field comes from the algebraic notion of a derivation. Since the set of functions, $C^\infty(M)$ forms an associative real algebra, it makes sense to talk about the derivations of $C^\infty(M)$, $\mathrm{Der}(C^\infty(M))$, which are maps $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz rule, i.e. such that, for every $f, g \in C^\infty(M)$, $\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$. Then, we have the following result:

Proposition 1.1.27. *There is a canonical one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Derivations on } M \\ \mathcal{D} \in \mathrm{Der}(C^\infty(M)) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{Vector Fields on } M \\ X \in \mathfrak{X}(M) \end{array} \right\}, \tag{1.10a}$$

established by the relation:

$$\mathcal{D}(f)(p) = f_*(p)(X_p), \quad \textit{for all } f \in C^\infty(M) \textit{ and } p \in M. \tag{1.10b}$$

Proof. Firstly, let $X \in \mathfrak{X}(M)$ be a vector field and define the map $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ as per relation (1.10b). Then, we need to check that $\mathcal{D}$ is a derivation. For that, we take $f, g \in C^\infty(M)$ and $p \in M$ and notice that, given a curve $\gamma :]-\epsilon, \epsilon[\to M$ such that $\gamma(0) = p$

and $[\gamma]_p = X_p$, one gets that $(fg)\circ\gamma :]-\epsilon,\epsilon[\to \mathbb{R}$, thus

$$\begin{aligned}\mathcal{D}(fg)(p) &= (fg)_*(p)(X_p) = (fg)_*(p)([\gamma]_p) = [(fg)\circ\gamma]_{f(p)g(p)} = \mathcal{D}((fg)\circ\gamma)(0)\\ &= \mathcal{D}((f\circ\gamma)(g\circ\gamma))(0) = \mathcal{D}(f\circ\gamma)(0)\,(g\circ\gamma)(0) + (f\circ\gamma)(0)\,\mathcal{D}(g\circ\gamma)(0)\\ &= [f\circ\gamma]_{f(p)}g(p) + f(p)[g\circ\gamma]_{g(p)} = f_*(p)([\gamma]_p)g(p) + f(p)g_*(p)([\gamma]_p)\\ &= (f_*(p)(X_p)\,g(p) + f(p)\,g_*(p)(X_p) = \mathcal{D}f(p)\,g(p) + f(p)\,\mathcal{D}g(p)\end{aligned}$$

Hence, by relation (1.10b), we get

$$\begin{aligned}\mathcal{D}(fg)(p) &= (fg)_*(p)(X_p) = f_*(p)(X_p)g(p) + f(p)g_*(p)(X_p)\\ &= \mathcal{D}(f)(p)g(p) + f(p)\mathcal{D}(g)(p),\end{aligned}$$

i.e. $\mathcal{D}$ is a derivation of the algebra of smooth functions.

Conversely, let $\mathcal{D} \in \mathrm{Der}(C^\infty(M))$ and let us first assume that exists $X : M \to TM$ such that $\mathcal{D}(f)(p) = f_*(p)(X_p)$, for all $f \in C^\infty(M)$ and $p \in M$. We want to show that X defines a vector field on M. The smoothness of X is guaranteed by the smoothness of $\mathcal{D}(f)$ and of the linearity of the derivative map $f_*(p)$. Moreover, by relation (1.10b), the definition of X implies that $X_p \in T_pM$, thus it is clear that $(\tau_M \circ X)(p) = p$, i.e. X is a vector field. As for the existence, the interested reader can check, for instance, Ref. [3, Propostion 8.15] Q.E.D.

Remark 1.1.28. From this proposition, since there is a one-to-one correspondence between vector fields on M and derivations of the algebra of smooth functions of M, we shall sometimes consider $X \in \mathfrak{X}(M)$ as a derivation of the algebra of smooth function on M. In that case, we still denote said derivation by X and denote $X(f) \in C^\infty(M)$ the action of the derivation X on a function $f \in C^\infty(M)$.

We will give more results about the set of vector fields in a subsequent paragraph.

Now, considering the set of 1-forms, from the interpretation of the cotangent bundle as the dual of the tangent bundle, we can, in fact, rethink our view of any 1-form ω as a map from vector fields to the ring of smooth functions $C^\infty(M)$

$$\begin{aligned}\omega : \mathfrak{X}(M) &\longrightarrow C^\infty(M)\\ X &\longmapsto [p \mapsto \omega_p(X_p)]\end{aligned}$$

Because each ω_p is $\mathbb{R}$-linear for each $p \in M$, we clearly obtain that a 1-form, in this view, must be $C^\infty(M)$-linear. Thus, the set of 1-forms can also be thought as

$$\Omega^1(M) = \big\{\omega : \mathfrak{X}(M) \to C^\infty(M) \,\big|\, \omega \text{ is } C^\infty(M)\text{-linear}\big\} \tag{1.11}$$

The definitions of covectors and 1-forms can be extended to arbitrary alternating[1] k--linear functions, whose set, defined at each point $p \in M$ is

$$\textstyle\bigwedge^k T_p^*M = \big\{\alpha : (T_pM)^{\times k} \to \mathbb{R} \,\big|\, \alpha \text{ is anti-symmetric and } \mathbb{R}\text{-multilinear}\big\}, \tag{1.12a}$$

[1] By alternating, we mean anti-symmetric in the exchange of any two arguments.

which can be extended to all points as

$$\bigwedge^k T^*M = \coprod_{p\in M} \bigwedge^k T_p^*M. \tag{1.12b}$$

Now, by following the same procedure that gave rise to the notion of a 1-form, we could define a k-form ϑ as a map $\vartheta : M \to \bigwedge^k T^*M$ such that $\eta \circ \vartheta = \mathrm{id}_M$ where $\eta : \bigwedge^k T^*M \to M$ sends $\alpha \in \bigwedge^k T_p^*M$ to the point $p \in M$. However, it is much simpler to do the same definition following instead the second view presented for 1-forms. In that perspective, we should take k-forms as maps $\vartheta : \mathfrak{X}(M)^{\times k} \to C^\infty(M)$ that are now $C^\infty(M)$-multilinear and anti-symmetric to follow the properties of $\bigwedge^k T^*M$. Thus, we define the collection of *differential k-forms* as

$$\Omega^k(M) \stackrel{\text{def}}{=} \left\{\vartheta : \mathfrak{X}(M)^{\times k} \to C^\infty(M) \,\middle|\, \vartheta \text{ is anti-symmetric and } C^\infty(M)\text{-multilinear}\right\}. \tag{1.13}$$

In fact, viewing the differential k-forms as these anti-symmetric, $C^\infty(M)$-multilinear maps allows us to define a module structure for $\Omega^k(M)$:

Proposition 1.1.29. *The collection of differential k-forms on a manifold M, $\Omega^k(M)$, has the structure of a $C^\infty(M)$-module.*

Proof. Since, for all $p \in M$, $\bigwedge^k T_p^*M$ has the structure of a real vector space (it is, in fact, inherited from the real vector space structure of T_pM), we define the addition of k-forms pointwise[2], i.e. for $\omega, \theta \in \Omega^k(M)$, $(\omega + \theta)_p = \omega_p + \theta_p$. Additionally, we define the following operation

$$\begin{aligned} \varrho : C^\infty(M) \times \Omega^k(M) &\longrightarrow \Omega^k(M) \\ (f, \omega) &\longmapsto [p \mapsto f(p)\,\omega_p] \end{aligned} \tag{$*$}$$

Again, from the real vector space structure of T_pM for all $p \in M$, we can easily check that the operation $(*)$ obeys the necessary properties to endow $\Omega^k(M)$ with the structure of a $C^\infty(M)$-module:

$$\begin{aligned}
(\varrho(fg,\omega))(p) &= (fg)(p)\omega_p = f(p)g(p)\omega_p = f(p)(\varrho(g,\omega))(p) \\
&= (\varrho(f, \varrho(g,\omega)))(p) \\
(\varrho(f+g,\omega))(p) &= (f+g)(p)\omega_p = (f(p)+g(p))\omega_p = f(p)\omega_p + g(p)\omega_p \\
&= (\varrho(f,\omega))(p) + (\varrho(g,\omega))(p) \\
(\varrho(f,\omega+\theta))(p) &= f(p)(\omega+\theta)_p = f(p)(\omega_p+\theta_p) = f(p)\omega_p + f(p)\theta_p \\
&= (\varrho(f,\omega))(p) + (\varrho(f,\theta))(p) \\
(\varrho(\mathbf{1},\omega))(p) &= \mathbf{1}(p)\omega_p = \omega_p
\end{aligned}$$

[2] In fact, here one could also define the scalar multiplication so that, for $\lambda \in \mathbb{R}$ and $\omega \in \Omega^k(M)$, $(\lambda\omega)_p = \lambda\,\omega_p$, for all $p \in M$. This scalar multiplication with the addition defined above gives $\Omega^k(M)$ the structure of a vector space over $\mathbb{R}$.

where $\mathbf{1} : M \to \mathbb{R}$ is just the function which assigns to any point of the manifold the real number $1 \in \mathbb{R}$. Q.E.D.

Remark 1.1.30. As it is customary, instead of denoting the $C^\infty(M)$-module operation as $\varrho(f)(\omega)$, we denote it simply as $f\omega$.

Remark 1.1.31. As we shall see in Section 1.2, this $C^\infty(M)$-module structure we obtained for the set k-forms is the result of a more general proposition about an equivalent module structure for sections of a vector bundle over a manifold M.

This module structure motivates the introduction of the set of all differential forms $\Omega^\bullet(M) = \bigoplus_{k=0}^{m} \Omega^k(M)$, where, as usual, $m = \dim(M)$, which will be the central object of study in the next subsection.

Exterior Product, Exterior Derivative and Interior Product

In the present subsection, we aim to study with more detail the collection of k-forms for arbitrary $k \in \mathbb{Z}_{\geqslant 0}$. We start by introducing the notion of the *exterior product*:

Definition 1.1.32. Let $p \in M$, $\omega \in \bigwedge^k T_p^*M$ and $\theta \in \bigwedge^\ell T_p^*M$. The *exterior product* or *wedge product* of ω and θ is a $(k+\ell)$-form, denoted by $\omega \wedge \theta \in \bigwedge^{k+\ell} T_p^*M$, such that, for any $v_1, \ldots, v_{k+\ell} \in T_pM$,

$$(\omega \wedge \theta)(v_1, \ldots, v_{k+\ell}) \overset{\text{def}}{=} \sum_{\sigma \in S^{k,\ell}} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \theta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}), \tag{1.14}$$

where $S^{k,\ell}$ represents the set of all (k,ℓ)-shuffles, i.e. permutations σ of $\{1, \ldots, k+\ell\}$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+\ell)$, and $\operatorname{sgn}(\sigma)$ is the sign ± 1 of the permutation σ.

Remark 1.1.33. Since differential k-forms are just maps $M \to \bigwedge^k T^*M$, the previous definition of the exterior product can be extended to a map

$$\begin{aligned} \Omega^k(M) \times \Omega^\ell(M) &\longrightarrow \Omega^{k+\ell}(M) \\ (\omega, \theta) &\longmapsto \omega \wedge \theta \end{aligned}$$

for any $k, \ell \in \mathbb{Z}_{\geqslant 0}$. In this context, we can also define the wedge product when one or both entries are functions (elements of $\Omega^0(M) \overset{\text{def}}{=} C^\infty(M)$). In this context, then, if $f, g \in C^\infty(M)$ and $\omega \in \Omega^k(M)$, $f \wedge g \overset{\text{def}}{=} fg$ and $f \wedge \omega \overset{\text{def}}{=} f\omega$, where, in this last expression, the product is the one coming from the $C^\infty(M)$-module structure of $\Omega^k(M)$.

Proposition 1.1.34. *The exterior product satisfies the following properties:*

(i) it is associative: $(\omega \wedge \theta) \wedge \eta = \omega \wedge (\theta \wedge \eta)$*, for* $\omega \in \Omega^k(M)$*,* $\theta \in \Omega^\ell(M)$ *and* $\eta \in \Omega^h(M)$*;*

(ii) $(\omega + \theta) \wedge \eta = \omega \wedge \eta + \theta \wedge \eta$ *and* $(f\omega) \wedge \theta = f(\omega \wedge \theta)$*, with* $\omega, \theta \in \Omega^k(M)$*,* $\eta \in \Omega^\ell(M)$ *and* $f \in C^\infty(M)$*;*

(iii) it is graded commutative: $\theta \wedge \omega = (-1)^{k\ell}\omega \wedge \theta$*, given* $\omega \in \Omega^k(M)$ *and* $\theta \in \Omega^\ell(M)$*.*

Remark 1.1.35. From the previous proposition we conclude that, together with the exterior product, the collection of differential forms $\Omega^\bullet(M)$ forms a *graded* $C^\infty(M)$*-algebra* and, in particular, $\bigwedge^\bullet T_p^*M$, with this exterior product, forms a *graded* $\mathbb{R}$*-algebra* for all $p \in M$. A more detailed presentation of the notions of graded algebras is given in Section 1.3.

We have previously seen that, using the cotangent lift of a local diffeomorphism, it is possible to define a form in another manifold. However, the more natural way of doing this is by using the pull-back which we now define for arbitrary k-forms:

Definition 1.1.36. If $\varphi : M \to N$ is a smooth map of smooth manifolds M and N and $\omega \in \Omega^k(N)$ is a k-form in N, we define the *pull-back of* ω *by* φ as the k-form $\varphi^*\omega \in \Omega^k(M)$ such that, for $p \in M$ and $v_1, \dots, v_k \in T_pM$,

$$(\varphi^*\omega)_p(v_1, \dots, v_k) \overset{\text{def}}{=} \omega_{\varphi(p)}(\varphi_*(p)(v_1), \dots, \varphi_*(p)(v_k)). \tag{1.15}$$

For 0-forms $f \in C^\infty(N)$, we define $\varphi^* f \overset{\text{def}}{=} f \circ \varphi \in C^\infty(M)$.

Proposition 1.1.37. *Given a smooth map* $\varphi : M \to N$*, the map*

$$\begin{aligned} \varphi^* : \Omega^\bullet(N) &\longrightarrow \Omega^\bullet(M) \\ \omega &\longmapsto \varphi^*\omega \end{aligned} \tag{1.16}$$

is an algebra homomorphism. Additionally, given smooth maps $f : M \to N$ *and* $g : N \to P$ *and* $\omega \in \Omega^\bullet(P)$*, we have that* $(g \circ f)^*\omega = f^*(g^*\omega)$*.*

Proof. By the definition of the pull-back, for all $p \in M$ and $v_1, \dots, v_k \in T_pM$, given $\omega, \omega' \in \Omega^k(N)$ and $a, a' \in \mathbb{R}$, we have that

$$\begin{aligned} (\varphi^*(a\omega + a'\omega'))_p(v_1, \dots, v_k) &= (a\omega + a'\omega')_{\varphi(p)}(\varphi_*(p)(v_1), \dots, \varphi_*(p)(v_k)) \\ &= a\omega_{\varphi(p)}(\varphi_*(p)(v_1), \dots, \varphi_*(p)(v_k)) \\ &\quad + a'\omega'_{\varphi(p)}(\varphi_*(p)(v_1), \dots, \varphi_*(p)(v_k)) \\ &= a(\varphi^*\omega)_p(v_1, \dots, v_k) + a'(\varphi^*\omega')_p(v_1, \dots, v_k) \\ &= (a\varphi^*\omega + a'\varphi^*\omega')_p(v_1, \dots, v_k), \end{aligned}$$

i.e. φ^* is $\mathbb{R}$-linear. Moreover, given $\omega \in \Omega^k(N)$ and $\eta \in \Omega^\ell(N)$ and arbitrary vectors $v_1, \dots, v_{k+\ell} \in T_pM$,

$$\begin{aligned}(\varphi^*(\omega \wedge \eta))_p(v_1, \dots, v_{k+\ell}) &= (\omega \wedge \eta)_{\varphi(p)}(\varphi_*(p)(v_1), \dots, \varphi_*(p)(v_{k+\ell}))\\ &= \sum_{\sigma \in S^{k,\ell}} \operatorname{sgn}(\sigma)\, \omega_{\varphi(p)}(\varphi_*(p)(v_{\sigma(1)}), \dots, \varphi_*(p)(v_{\sigma(k)}))\times\\ &\qquad\qquad \times \eta_{\varphi(p)}(\varphi_*(p)(v_{\sigma(k+1)}), \dots, \varphi_*(p)(v_{\sigma(k+\ell)}))\\ &= \sum_{\sigma \in S^{k,\ell}} \operatorname{sgn}(\sigma)\, (\varphi^*\omega)_p(v_{\sigma(1)}, \dots, v_{\sigma(k)})\times\\ &\qquad\qquad \times (\varphi^*\eta)_p(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})\\ &= (\varphi^*\omega \wedge \varphi^*\eta)_p(v_1, \dots, v_{k+\ell}),\end{aligned}$$

thus, $\varphi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ is an algebra homomorphism. To prove the last portion of the Proposition, let $p \in M$, $\omega \in \Omega^k(P)$ and $v_1, \dots, v_k \in T_pM$. Then,

$$\begin{aligned}((g \circ f)^*\omega)_p(v_1 \dots, v_k) &= \omega_{g\circ f(p)}((g \circ f)_*(p)(v_1), \dots, (g \circ f)_*(p)(v_k))\\ &= \omega_{g(f(p))}(g_*(f(p))(f_*(p)(v_1)), \dots, g_*(f(p))(f_*(p)(v_k)))\\ &= (g^*\omega)_{f(p)}(f_*(p)(v_1), \dots, f_*(p)(v_k)) = (f^*(g^*\omega))_p(v_1, \dots, v_k)\end{aligned}$$

which concludes the proof. Q.E.D.

To finalise this brief study of the operations/structures on the collection of differential forms, we introduce two operators that allow to increase or decrease the degree of a certain k-form.

Definition 1.1.38. Given $X \in \mathfrak{X}(M)$, the *interior product by* X is the map (for all $k \in \mathbb{Z}_{\geqslant 0}$)

$$\begin{aligned} i_X : \Omega^k(M) &\longrightarrow \Omega^{k-1}(M)\\ \omega &\longmapsto i_X\omega \end{aligned} \tag{1.17a}$$

such that, for $k = 0$, $\omega = f \in C^\infty(M)$ and $i_X f \overset{\text{def}}{=} 0$ and, for $k \geq 1$, given $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$,

$$(i_X\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}). \tag{1.17b}$$

Remark 1.1.39. As we shall see in Section 1.3, by introducing the notion of graded algebras and the shift, we may rewrite the definition of the interior product as the map $i_X : \Omega^\bullet(M) \to \Omega^\bullet(M)[-1]$ such that, for any homogeneous element of degree k, $\omega \in \Omega^k(M)$, $i_X\omega \in \Omega^{k-1}(M)$ and $(i_X\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1})$, for any $X_1, \dots, X_{k-1} \in \mathfrak{X}(M)$.

Proposition 1.1.40. *Let $X \in \mathfrak{X}(M)$. Then i_X is a degree -1 graded derivation of $(\Omega^\bullet(M), \wedge)$, i.e., for $\omega \in \Omega^k(M)$ and $\theta \in \Omega^\ell(M)$, $i_X(\omega \wedge \theta) = (i_X\omega) \wedge \theta + (-1)^k \omega \wedge (i_X\theta)$.*

Proof. To obtain the result, let $X_2, \ldots, X_{k+\ell} \in \mathfrak{X}(M)$. Then, by identifying $X_1 = X$, we have that

$$\begin{aligned}
(i_X(\omega \wedge \theta))(X_2, \ldots, X_{k+\ell}) &= (\omega \wedge \theta)(X, X_2, \ldots, X_{k+\ell}) \\
&= \sum_{\sigma \in S^{k,\ell}} \operatorname{sgn}(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\theta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}) \\
&= \sum_{\substack{\sigma \in S^{k,\ell} \\ \sigma(1)=1}} \operatorname{sgn}(\sigma)\omega(X_{\sigma(1)=1}, \ldots, X_{\sigma(k)})\theta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}) \\
&\quad + \sum_{\substack{\sigma \in S^{k,\ell} \\ \sigma(k+1)=1}} \operatorname{sgn}(\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\theta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)})
\end{aligned} \tag{$*$}$$

Noticing that a (k, ℓ)-shuffle with the first entry fixed is just a $(k-1, \ell)$-shuffle and that a (k, ℓ)-shuffle with the $(k+1)$-th entry fixed is just a $(k, \ell-1)$-shuffle (but whose sign is $(-1)^k$ of the original shuffle due to the permutation of that fixed entry), we get that

$$\begin{aligned}
(*) &= \sum_{\sigma \in S^{k-1,\ell}} \operatorname{sgn}(\sigma)\omega(X_1, X_{\sigma(1)+1} \ldots, X_{\sigma(k-1)+1})\theta(X_{\sigma(k)+1}, \ldots, X_{\sigma(k-1+\ell)+1}) \\
&\quad + (-1)^k \sum_{\sigma \in S^{k,\ell-1}} \operatorname{sgn}(\sigma)\omega(X_{\sigma(1)+1} \ldots, X_{\sigma(k)+1})\theta(X_1, X_{\sigma(k+1)+1}, \ldots, X_{\sigma(k+\ell-1)+1}) \\
&= ((i_X\omega) \wedge \theta)(X_2, \ldots, X_{k+\ell}) + (-1)^k(\omega \wedge (i_X\theta))(X_2, \ldots, X_{k+\ell}).
\end{aligned}$$

Thus, we obtain the result. Q.E.D.

Definition 1.1.41. The *exterior derivative on* $\Omega^\bullet(M)$ or the *de Rham differential* is the operator

$$\mathrm{d} : \Omega^k(M) \longrightarrow \Omega^{k+1}(M) \tag{1.18a}$$

such that, for $k = 0$, $\omega = f \in C^\infty(M)$ and $(\mathrm{d}f)_p = f_*(p)$, for any $p \in M$ and, for $k \geq 1$ and $\omega \in \Omega^k(M)$, given $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$:

$$\begin{aligned}
(\mathrm{d}\omega)(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\
&\quad + \sum_{\substack{i,j=1 \\ i<j}}^{k+1}(-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),
\end{aligned} \tag{1.18b}$$

where the notation $\hat{X}_i$ means that we omit the entry X_i from ω.

Remark 1.1.42. The exterior derivative could be seen as a local operator. In fact, given a coordinate chart $(U, (x^1, \ldots, x^m))$ of M, we may write any k-form $\omega \in \Omega^k(M)$ as

$$\omega|_U = \sum_{1 \leq i_1 < \ldots < i_k \leq m} a_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}.$$

Then, we can write $\mathrm{d}\omega$ as

$$(\mathrm{d}\omega)|_U = \sum_{1 \leq i_1 < \ldots < i_k \leq m} \mathrm{d}a_{i_1 \cdots i_k} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} \tag{1.18c}$$

Remark 1.1.43. Similar to the case of the interior product, the exterior derivative can be thought as a map $\mathrm{d} : \Omega^\bullet(M) \to \Omega^\bullet(M)[1]$, which, for every k-form, $\omega \in \Omega^k(M)$, associates a $(k+1)$-form $\mathrm{d}\omega \in \Omega^{k+1}(M)$.

Proposition 1.1.44. *The exterior derivative satisfies the following properties*

(i) d *is* $\mathbb{R}$*-linear;*

(ii) d *is a degree 1 graded derivation on* $(\Omega^\bullet(M), \wedge)$*, i.e. it satisfies the graded Leibniz rule*

$$\mathrm{d}(\omega \wedge \theta) = (\mathrm{d}\omega) \wedge \theta + (-1)^k \omega \wedge (\mathrm{d}\theta)$$

for all $\omega \in \Omega^k(M)$ *and* $\theta \in \Omega^\ell(M)$*;*

(iii) $\mathrm{d}(\mathrm{d}\omega) = 0$ *for any* $\omega \in \Omega^\bullet(M)$*;*

(iv) for any smooth map $\varphi : M \to N$ *and* $\omega \in \Omega^\bullet(N)$, $\mathrm{d}(\varphi^*\omega) = \varphi^*(\mathrm{d}\omega)$.

Definition 1.1.45. A k-form $\omega \in \Omega^k(M)$ is said to be

1) *closed* if $\mathrm{d}\omega = 0$. The collection of closed k-forms is denoted by $\Omega^k_{\mathrm{cl}}(M)$;

2) *exact* if exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = \mathrm{d}\eta$. The collection of exact k-forms is denoted by $\mathrm{d}\Omega^{k-1}(M)$.

Lemma 1.1.46. *If* ω *is an exact* k*-form, then it is closed.*

Proof. For an exact k-form $\omega \in \Omega^k(M)$, let $\eta \in \Omega^{k-1}(M)$ be such that $\omega = \mathrm{d}\eta$. Then, $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\eta) = 0$, therefore $\omega \in \Omega^k(M)$ is closed. Q.E.D.

We could now question when is the opposite relation valid, i.e. when a closed form is exact. This problem leads to the notion of the de Rham cohomology groups, which we briefly introduce.

Definition 1.1.47. Let M be a smooth manifold. The k-th de Rham cohomology group of M is the quotient space

$$H^k_{\mathrm{dR}}(M) \stackrel{\text{def}}{=} \frac{\ker(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M))} = \Omega^k_{\mathrm{cl}}(M)/\mathrm{d}\Omega^{k-1}(M) \tag{1.19}$$

One of the most important results when dealing with the previously posed question comes in the form of the Poincaré lemma, which states that every closed form is *locally* exact.

Lemma 1.1.48 (Poincaré)**.** *Let M be a smooth manifold and $\omega \in \Omega^k_{\mathrm{cl}}(M)$. Then, for any $p \in M$, there exists an open neighbourhood U of p and a $(k-1)$-form $\eta \in \Omega^{k-1}(U)$ such that $\omega|_U = \mathrm{d}\eta$.*

The proof of this concept is beyond the scope of the present document. For the interested reader, we refer to [4, Section 9.10].

More on the Structure of $\mathfrak{X}(M)$

We now return to the study of vector fields and, in particular, the different structures one can define in $\mathfrak{X}(M)$. The first of these is the $C^\infty(M)$-module structure:

Proposition 1.1.49. *The collection of vector fields on a manifold M, $\mathfrak{X}(M)$, has the structure of a $C^\infty(M)$-module, given by $(fX)_p = f(p)X_p$, for all $p \in M$, $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$.*

Proof. The proof of this result is entirely analogous to the homologous result for the module of $\Omega^k(M)$, Proposition 1.1.29. Q.E.D.

Remark 1.1.50. Similarly to how it was done in the case of k-forms, we shall denote the $C^\infty(M)$-module operation as fX.

Remark 1.1.51. Since the ring of smooth functions assumes as operations the pointwise addition and multiplication of functions, if we had only considered the sub-ring of constant functions, we would have obtained a real vector space structure for $\mathfrak{X}(M)$. As we shall see, in some of the structures that we will add to $\mathfrak{X}(M)$ it is not possible to consider the full $C^\infty(M)$-module structure but rather just the real vector space structure.

One other rather important use of vector fields has to do with differential equations. This application comes by considering the vector fields as generators of flows on a manifold:

Definition 1.1.52. Let M be a smooth manifold, $p \in M$ and $X \in \mathfrak{X}(M)$. A curve $\gamma : I \to M$, with $I \subseteq \mathbb{R}$ is an open interval containing 0, is called an *integral curve of X at the point p* if $\gamma(0) = p$ and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = X_{\gamma(t)}$. More generally, the map

$$\begin{aligned} \mathrm{Fl}_X : D \subseteq \mathbb{R} \times M &\longrightarrow M \\ (t,p) &\longmapsto \mathrm{Fl}^t_X(p) \end{aligned} \tag{1.20}$$

such that $t \mapsto \mathrm{Fl}^t_X(p)$ is an integral curve of X starting from p, is called the *flow of the vector field X*.

Proposition 1.1.53. *Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ and $\mathrm{Fl}_X^t : M \to M$ the local flow generated by X. Then,*

(i) $\mathrm{Fl}_X^0 = \mathrm{id}_M$*;*

(ii) $\mathrm{Fl}_X^t \circ \mathrm{Fl}_X^s = \mathrm{Fl}_X^{t+s}$ *for t and s in suitable intervals;*

(iii) Fl_X^t *is a diffeomorphism between open subsets of M, for each $t \in \mathbb{R}$.*

Proof. *(i)* Given $p \in M$, we have that $\mathrm{Fl}_X^0(p) = p$;

(ii) Let $p \in M$, $q = \mathrm{Fl}_X^s(p)$, $\gamma(t)$ the integral curve of X passing through p and $\tilde{\gamma}(t)$ the integral curve of X passing through q. Define the curve δ by $\delta(t) = \gamma(t+s)$. Then $\delta(0) = \gamma(s) = q$ and $\frac{\mathrm{d}}{\mathrm{d}t}\delta(t) = \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma(\tau)|_{\tau=t+s} = X_{\gamma(t+s)} = X_{\delta(t)}$, i.e. δ is the integral curve of X passing through q. By the existence and uniqueness of solutions of differential equations, $\delta = \tilde{\gamma}$, i.e. $\tilde{\gamma}(t) = \gamma(t+s) \iff \mathrm{Fl}_X^t(\mathrm{Fl}_X^s(p)) = \mathrm{Fl}_X^{t+s}$. Thus, since $p \in M$ is arbitrary, we get that $\mathrm{Fl}_X^t \circ \mathrm{Fl}_X^s = \mathrm{Fl}_X^{t+s}$.

(iii) It is clear that Fl_X^t is differentiable and, by *(i)* and *(ii)*, $(\mathrm{Fl}_X^t)^{-1} = \mathrm{Fl}_X^{-t}$. Therefore, $(\mathrm{Fl}_X^t)^{-1}$ is also differentiable and we conclude that Fl_X^t is a local diffeomorphism.

Q.E.D.

Definition 1.1.54. A vector field $X \in \mathfrak{X}(M)$ on a manifold M is said to be *complete* if D, the domain of maximal flow of X, is $D = \mathbb{R} \times M$, i.e. if $\mathrm{dom}(\mathrm{Fl}_X^t) = M$, for all $t \in \mathbb{R}$.

Having now defined the set of vector fields, $\mathfrak{X}(M)$, we are now able to introduce additional structures to this collection. The first of these structures is a Lie bracket:

Definition 1.1.55. The *Lie bracket* of two vector fields $X, Y \in \mathfrak{X}(M)$ is the unique vector field $[X, Y] \in \mathfrak{X}(M)$ defined by

$$[X,Y](f) \overset{\mathrm{def}}{=} X(Y(f)) - Y(X(f)), \quad \forall\, f \in C^\infty(M). \tag{1.21}$$

Lemma 1.1.56. *The Lie bracket has the following properties for $X, Y, Z \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$:*

(i) $[X, Y+Z] = [X,Y] + [X,Z]$ *and* $[X, \lambda Y] = \lambda[X,Y]$*;*

(ii) $[X,Y] = -[Y,X]$*;*

(iii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ *(Jacobi identity).*

Proof. The first two properties are clear from the definition of the Lie bracket. As for the Jacobi identity, let $f \in C^\infty(M)$. Then,

$$\begin{aligned}
[X,[Y,Z]](f) &= X(Y(Z(f)) - X(Z(Y(f))) - Y(Z(X(f)) + Z(Y(X(f))),\\
[Y,[Z,X]](f) &= Y(Z(X(f)) - Y(X(Z(f))) - Z(X(Y(f)) + X(Z(Y(f))),\\
[Z,[X,Y]](f) &= Z(X(Y(f)) - Z(Y(X(f))) - X(Y(Z(f)) + Y(X(Z(f))).
\end{aligned}$$

Thus, by adding the three terms on the LHS, noticing the common tones of black, red, blue, teal, violet and orange, we obtain the result. Q.E.D.

From this lemma, we get that $(\mathfrak{X}(M), [\cdot,\cdot])$ is a real Lie algebra where we consider only the real vector space structure of $\mathfrak{X}(M)$ and not the full $C^\infty(M)$-module structure.

The final concept we shall introduce in this section is the notion of the *Lie derivative*. For that, however, we must first extend the notion of the pull-back of forms to the setting of vector fields:

Definition 1.1.57. Let $\varphi : M \to N$ be a diffeomorphism and $X \in \mathfrak{X}(N)$ a vector field. The *pull-back of the vector field X along φ* is the vector field $\varphi^* X \in \mathfrak{X}(M)$ on M given by:

$$\varphi^* X = \varphi_*^{-1} \circ X \circ \varphi, \tag{1.22a}$$

or, at a point $p \in M$:

$$(\varphi^* X)_p = \varphi_*^{-1}(\varphi(p))(X_{\varphi(p)}). \tag{1.22b}$$

Remark 1.1.58. This definition of the pull-back of vector fields motivates us to introduce the notion of related vector fields. Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ and a smooth map $f : M \to N$, X and Y are said to be *f-related* if $f_*(p)(X_p) = Y_{f(p)}$, for all $p \in M$.

Lemma 1.1.59. *Let $\varphi : M \to N$ be a diffeomorphism, $f \in C^\infty(M)$ and $X \in \mathfrak{X}(N)$. Then $(\varphi^* X)(f) = X(f \circ \varphi^{-1}) \circ \varphi$.*

Proof. Let $p \in M$. Then,

$$\begin{aligned}
(\varphi^* X)(f)(p) &= f_*(p)((\varphi^* X)_p) = f_*(p)(\varphi_*^{-1}(\varphi(p))(X_{\varphi(p)}))\\
&= (f \circ \varphi^{-1})_*(\varphi(p))(X_{\varphi(p)}) = X(f \circ \varphi^{-1})(\varphi(p)),
\end{aligned}$$

which is the intended result. Q.E.D.

We are now ready to introduce the notion of the Lie derivative:

Definition 1.1.60. Let $X \in \mathfrak{X}(M)$ be a vector field and Fl_X^t its flow at time t. We define

1) *the Lie derivative of* $f \in C^\infty(M)$ *along* X to be $\mathcal{L}_X f \in C^\infty(M)$ such that

$$(\mathcal{L}_X f)(p) = \frac{\mathrm{d}}{\mathrm{d}t}((\mathrm{Fl}_X^t)^* f)(p)\Big|_{t=0}; \tag{1.23}$$

2) *the Lie derivative of* $\omega \in \Omega^k(M)$ *along* X to be $\mathcal{L}_X \omega \in \Omega^k(M)$ such that

$$(\mathcal{L}_X \omega)_p(v_1, \ldots, v_k) = \frac{\mathrm{d}}{\mathrm{d}t}((\mathrm{Fl}_X^t)^* \omega)_p(v_1, \ldots, v_k)\Big|_{t=0}; \tag{1.24}$$

3) *the Lie derivative of* $Y \in \mathfrak{X}(M)$ *along* X to be $\mathcal{L}_X Y \in \mathfrak{X}(M)$ such that

$$(\mathcal{L}_X Y)(f)(p) = \frac{\mathrm{d}}{\mathrm{d}t}((\mathrm{Fl}_X^t)^* Y)(f)(p)\Big|_{t=0}. \tag{1.25}$$

In reality, it is much more useful to use the following results:

Proposition 1.1.61. *Let* $X \in \mathfrak{X}(M)$*. Then, the Lie derivative along* X *satisfies the following properties:*

(i) $\mathcal{L}_X f = X(f)$, $\forall\, f \in C^\infty(M)$*;*

(ii) $\mathcal{L}_X Y = [X, Y]$, $\forall\, Y \in \mathfrak{X}(M)$*.*

Proposition 1.1.62. *Given vector fields* $X_1, X_2 \in \mathfrak{X}(M)$ *and a* k*-form* $\omega \in \Omega^k(M)$*, we have the following:*

$$i_{[X_1, X_2]}\omega = \mathcal{L}_{X_1} i_{X_2}\omega - i_{X_2}\mathcal{L}_{X_1}\omega. \tag{1.26}$$

Theorem 1.1.63 (Cartan's magic formula)**.** *Given a vector field* $X \in \mathfrak{X}(M)$ *and a* k*-form,* $\omega \in \Omega^k(M)$*, we have that*

$$\mathcal{L}_X \omega = \mathrm{d}(i_X \omega) + i_X(\mathrm{d}\omega). \tag{1.27}$$

1.2 Vector Bundles and Double Vector Bundles

A fundamental concept that shall be of extreme importance in later chapters is that of fibre bundles and, in particular, vector bundles. In this section, we give a brief overview of the ideas present when one studies such objects.

The study of fibre bundles comes about as a generalisation (and in some sense an abstraction) of the construction made previously for the tangent and cotangent bundles TM and T^*M. These appear to be a product of topological spaces and, in fact, the definition of a fibre bundle embodies this notion (even though only locally):

Definition 1.2.1. A *locally trivial fibration* is a quadruple (E, B, pr, F), where E, B, F are topological spaces and $\mathrm{pr} : E \to B$ is a continuous surjective map such that, for every $p \in B$, exists a neighbourhood $U \subseteq B$ of p and a homeomorphism $\varphi : \mathrm{pr}^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{pr}^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\mathrm{pr}"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd} \tag{1.28}$$

where $\mathrm{proj}_1 : U \times F \to U$ is the projection onto the first factor.

Relative to the quadruple (E, B, pr, F) we call E the *total space*, B the *base space*, pr the *projection map* of the fibration and F the *fibre*. If the neighbourhood U in the definition above is actually the whole base space B, the map φ is called a *(global) trivialisation* of the fibration since it identifies (preserving the fibres) E with $B \times F$.

Remark 1.2.2. It is often easier to denote a locally trivial fibration as $E \xrightarrow{\mathrm{pr}} B$ or, simply, E and we denote the fibre at $p \in B$ by E_p or $\mathrm{pr}^{-1}(p)$. If E, B can be taken as smooth manifolds, pr a smooth map and φ in Definition 1.2.1 a diffeomorphism, then we call $E \xrightarrow{\mathrm{pr}} B$ a *smooth locally trivial fibration*.

Remark 1.2.3. In parts of the literature, the object we here called a locally trivial fibration is also called a *fibre bundle*. However, in others, the name fibre bundle is kept for a locally trivial fibration for which there exists a topological group G (usually called the *structure group*) which acts on the fibres F of the locally trivial fibration (E, B, pr, F).

Example 1.2.4. The simplest example of a locally trivial fibration is that of the *trivial bundle*, which is the case in which we take two topological spaces B, F, and define $E = B \times F$ and take $\mathrm{pr} : B \times F \to B$ the projection onto the first factor. ♮

Example 1.2.5. Both the tangent and cotangent bundles are examples of locally trivial fibrations (more specifically, they correspond to vector bundles) and shall be described in more detail below. ♮

Example 1.2.6. Another less trivial example is that of the Möbius strip, for which we take $B = S^1$, the circle, and $F = \mathbb{R}$. Then, by defining the equivalence relation $\sim$ in $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2) \iff \exists\, n \in \mathbb{Z}\colon (x_2, y_2) = (x + n, (-1)^n y)$, we take $E = \mathbb{R}^2/\sim$ and $\mathrm{pr} : E \to S^1$ defined by $\mathrm{pr}([(x, y)]) = e^{2\pi i x}$ (identifying S^1 with the complex unit circle). ♮

Since we are more interested in dealing with vector bundles, we now turn our attention to these objects and, when necessary, we add a remark to invoke the same underlying fibre bundle structure. A vector bundle comes as a particular interesting case of fibre bundles, since it adds a structure of a vector space over a field $\mathbb{K}$ (which we shall always take as the reals $\mathbb{R}$). As we shall also see, this is precisely the case for the tangent bundle.

Definition 1.2.7. A locally trivial fibration $E \xrightarrow{\mathrm{pr}} B$ is called a *real vector bundle* if, for each $p \in B$, the fibre $E_p \overset{\mathrm{def}}{=} \mathrm{pr}^{-1}(p)$ is endowed with the structure of a real vector space and if each local trivialisation $\varphi : \mathrm{pr}^{-1}(U) \to U \times F$ restricts at each $p \in U$ to a linear isomorphism between E_p and $\{p\} \times F \cong F$.

Remark 1.2.8. In all the cases we shall study, the fibres of a vector bundle $E \xrightarrow{\mathrm{pr}} B$ shall be finite-dimensional. To the dimension of these fibres — which equals the difference of the dimensions of the total space and the base space —, we shall give the name *rank of the vector bundle*, and denote it by $\mathrm{rank}(E) = \dim(E) - \dim(B)$.

The following result proves that the tangent bundle is, indeed, a vector bundle:

Proposition 1.2.9. *Let M be an m-dimensional smooth manifold and TM its tangent bundle. With respect to the natural projection $\tau_M : TM \to M$, TM has the structure of a vector bundle.*

Proof. Let $U \subseteq M$ and $\varphi : U \to V \subseteq \mathbb{R}^m$ be a chart for M. Then, since the map $\Phi : TM|_U \to V \times \mathbb{R}^m$ defined in the proof of Proposition 1.1.12 is a linear isomorphism on the fibres of TM, it is enough to show that the following diagram commutes:

$$\begin{tikzcd}
TM|_U \arrow[r, "\Phi"] \arrow[d, "\tau_M"'] & \varphi(U) \times \mathbb{R}^m \\
U & U \times \mathbb{R}^m \arrow[l, "\mathrm{proj}_1"] \arrow[u, "\varphi \times \mathrm{id}_{\mathbb{R}^m}"']
\end{tikzcd}$$

since, in this case, the φ appearing in Definition 1.2.1 is $\Phi \circ (\varphi \times \mathrm{id}_{\mathbb{R}^m})^{-1}$. However, given $p \in U$ and $[\gamma]_p \in T_pM$,

$$\begin{aligned}
(\mathrm{proj}_1 \circ (\varphi \times \mathrm{id}_{\mathbb{R}^m})^{-1} \circ \Phi)([\gamma]_p) &= (\mathrm{proj}_1 \circ (\varphi \times \mathrm{id}_{\mathbb{R}^m})^{-1})(\varphi(p), \varphi_*(p)([\gamma]_p)) \\
&= \mathrm{proj}_1(p, \varphi_*(p)([\gamma]_p)) = p = \tau_M([\gamma]_p),
\end{aligned}$$

which concludes the proof. Q.E.D.

Definition 1.2.10. A *bundle morphism* between locally trivial fibrations $E \xrightarrow{\mathrm{pr}} B$ and $E' \xrightarrow{\mathrm{pr}'} B'$ is a pair $(\tilde{f}, f)$ of maps $\tilde{f} : E \to E'$ and $f : B \to B'$ such that the induced map on the fibres $\tilde{f}_p : E_p \to E'_{f(p)}$, for $p \in B$, is a homeomorphism and the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\mathrm{pr}"'] & E' \arrow[d, "\mathrm{pr}'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd} \tag{1.29}$$

If $E \xrightarrow{\mathrm{pr}} B$ and $E' \xrightarrow{\mathrm{pr}'} B'$ are vector bundles, we call the pair $(\tilde{f}, f)$ a *vector bundle morphism* if, additionally, for every $p \in B$, the map $E_p \to E'_{f(p)}$ is homomorphism of real vector spaces, i.e. $\tilde{f}$ is fibre-wise linear.

This definition allows us to define the category of vector bundles, denoted as **VB**, whose objects are vector bundles and morphisms are vector bundle morphisms.

Remark 1.2.11. Usually, we will denote a (vector) bundle morphism $(\tilde{f}, f)$ simply as $\tilde{f}$. In this case, we say that $\tilde{f} : E \to E'$ is a (vector) bundle morphism *covering* $f : B \to B'$.

Additionally, given a smooth map $f : B \to B'$, it is always possible to extend it to a bundle map $\tilde{f} : E \to E'$ between the two vector bundles $E \xrightarrow{\mathrm{pr}} B$ and $E' \xrightarrow{\mathrm{pr}'} B'$, by taking, for instance, $\tilde{f} = 0_{E'} \circ f \circ \mathrm{pr}$, where $0_{E'} : B' \to E'$ associates every point $p' \in B'$ to the zero vector of the fibre $E'_{p'}$.

Homogeneity Structures

Having introduced the notions of vector bundles and morphisms, we now give an alternative characterisation of vector bundles using the notion of actions on a manifold:

Definition 1.2.12. Let X be a topological space and G an topological group (i.e. a group $(G, \cdot)$ with a topology structure such that both $\cdot : G \times G \to G$ and $^{-1} : G \to G$ are continuous). An *action of* $(G, \cdot)$ *in* X is a continuous map

$$\begin{aligned} \phi : G \times X &\longrightarrow X \\ (g, x) &\longmapsto \phi_g(x) \end{aligned}$$

satisfying the following conditions

1) $\phi_g : X \to X$ is a homeomorphism for all $g \in G$;

2) for all $g, h \in G$, $\phi_g \circ \phi_h = \phi_{g \cdot h}$;

3) if $e \in G$ is the neutral element of G, then $\phi_e = \mathrm{id}_X$.

To the pair (X, ϕ) we call a *G-set*.

Remark 1.2.13. If X is a smooth manifold, the action is said to be smooth if $G \times X \to X$ is smooth and ϕ_g is a diffeomorphism for all $g \in G$.

Definition 1.2.14. Let X, Y be topological spaces and $(G, \cdot)$ a topological group acting on X and Y. A *G-equivariant map* or a *G-morphism* between G-sets (X, ϕ) and (Y, ψ) is a continuous map $f : X \to Y$ such that, for all $g \in G$ and $x \in X$, $f(\phi_g(x)) = \psi_g(f(x))$ or, equivalently, such that the following diagram commutes:

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\phi"'] & G \times Y \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Remark 1.2.15. With the introduction of G-equivariant maps, it is now possible to introduce the category of G-actions, $\mathbf{Act}_{(G,\cdot)}$, whose objects are G-sets (X, ϕ) and morphisms are G-morphisms.

From now on, we shall always consider the manifold M above to be a vector bundle $E \xrightarrow{\mathrm{pr}} M$ and, given the natural real structure of the manifold, we shall take[3] $G = \mathbb{R}_{\geq 0}$.

Definition 1.2.16. The *vertical lift* associated to an action $\phi : \mathbb{R}_{\geq 0} \times E \to E$ on a vector bundle $E \xrightarrow{\mathrm{pr}} M$ is the map

$$\begin{aligned} \mathrm{vl}_\phi : E &\longrightarrow TE|_{\phi_0(E)} \\ e &\longmapsto [\gamma_e] \in T_{\phi_0(e)}E, \end{aligned}$$

where, for all $t \in \mathbb{R}_{\geq 0}$, $\gamma_e(t) = \phi_t(e)$.

Then, we may define a homogeneity structure on a vector bundle as follows:

Definition 1.2.17. An action $h : \mathbb{R}_{\geq 0} \times E \to E$ on a vector bundle $E \xrightarrow{\mathrm{pr}} M$ is called a *homogeneity structure on E* if given any $e \in E$, $\mathrm{vl}_h(e) = 0$ if and only if $e \in h_0(E)$.

Remark 1.2.18. Notice that, in the definition of homogeneity structure, the implication from right to left (i.e. the implication $e \in h_0(E) \implies \mathrm{vl}_h(e) = 0$) is always satisfied. In fact, we have that, if $e = h_0(\epsilon)$ for some $\epsilon \in E$,

$$h_t(e) = (h_t \circ h_0)(\epsilon) = h_{t \times 0}(\epsilon) = h_0(\epsilon) = e.$$

Thus $\mathrm{vl}_h(e) = [\gamma_e] = [h_0(e)] = 0$, where, in the last equality, we simply note that the element on $T_eE \cong T_{h_0(e)}E$ associated to the equivalence class of constant curves $\gamma_e(t) = h_0(e)$ is the null vector.

[3]Clearly, $(\mathbb{R}_{\geq 0}, \cdot)$ is a monoid and not a group. In this case, the action of $\mathbb{R}_{\geq 0}$ is called a monoid action. The following results are still valid for a monoid action instead of a group one.

For every vector bundle $E \xrightarrow{\mathrm{pr}} M$, there always exists a homogeneity structure on E. In fact, given a chart $(U, (x^1, \ldots, x^m))$ of M ($m = \dim(M)$) and adapted coordinates (x^i, e^a) on E, the action

$$\begin{aligned} h_E : \mathbb{R}_{\geq 0} \times E &\longrightarrow E \\ (t, (x^i, \epsilon^a)) &\longmapsto (x^i, t\,\epsilon^a) \end{aligned} \tag{1.30}$$

is a homogeneity structure on E. This fact is easy to check, since $\mathrm{vl}_{h_E}(x^i, \epsilon^a) = [(x^i, t\,\epsilon^a)]$ and this equivalence class gives, locally, the vector $\epsilon^a \left.\frac{\partial}{\partial e^a}\right|_\epsilon$. Thus, if $\mathrm{vl}_{h_E}(x^i, \epsilon^a) = 0$, then ϵ has coordinates $(x^i, 0) = h_0(x^i, e^a)$, for some $e \in E$ with coordinates (x^i, e^a).

The existence of this homogeneity structure allows us to obtain a functor $\mathcal{H} : \mathbf{VB} \to \mathbf{Act}_{\mathbb{R}_{\geq 0}}$ mapping a vector bundle $E \xrightarrow{\mathrm{pr}} M$ to the $\mathbb{R}_{\geq 0}$-set (E, h_E) (and mapping vector bundle morphisms to themselves). However, given any $\mathbb{R}_{\geq 0}$-set, we can also construct the following functor:

$$\begin{aligned} \mathcal{V} : \mathbf{Act}_{\mathbb{R}_{\geq 0}} &\longrightarrow \mathbf{VB} \\ (M, h) &\overset{\mathrm{Ob}}{\longmapsto} \ker(Th_0) \\ f &\overset{\mathrm{Mor}}{\longmapsto} (\mathcal{V}f, f_0) \end{aligned}$$

where the vector bundle morphism $(\mathcal{V}f, f_0)$ is obtained from $f : (M, h_M) \to (N, h_N)$ by the following commutative diagram:

$$\begin{tikzcd} \mathcal{V}M \arrow[r, hook] \arrow[d, "\mathcal{V}f"'] & TM \arrow[r, "\tau_M"] \arrow[d, "Tf"'] & M \arrow[r, "(h_M)_0"] \arrow[d, "f"] & (h_M)_0(M) \arrow[d, "f_0"] \\ \mathcal{V}N \arrow[r, hook] & TN \arrow[r, "\tau_N"'] & N \arrow[r, "(h_N)_0"'] & (h_N)_0(N) \end{tikzcd}$$

This functor allows us to obtain the following result

Theorem 1.2.19. *Given a homogeneity structure $h : \mathbb{R}_{\geq 0} \times E \to E$ on a manifold E, there exists a unique vector bundle structure on E such that h has the local expression* (1.30).

Remark 1.2.20. We can rewrite the previous discussion using the language of category theory in a more compact way. First, we define a subcategory $\mathbf{rAct}_{\mathbb{R}_{\geq 0}} \subseteq \mathbf{Act}_{\mathbb{R}_{\geq 0}}$ whose objects are pairs (E, h) where h is a homogeneity structure on E. Then, Theorem 1.2.19 and the discussion prior give that the functors $\mathcal{H}$ and $\mathcal{V}|_{\mathbf{rAct}_{\mathbb{R}_{\geq 0}}}$ are inverse, in the sense that $\mathcal{H} \circ \mathcal{V}|_{\mathbf{rAct}_{\mathbb{R}_{\geq 0}}} \cong \mathrm{id}_{\mathbf{rAct}_{\mathbb{R}_{\geq 0}}}$ and $\mathcal{V}|_{\mathbf{rAct}_{\mathbb{R}_{\geq 0}}} \circ \mathcal{H} \cong \mathrm{id}_{\mathbf{VB}}$. Thus, we obtain an equivalence of categories $\mathbf{rAct}_{\mathbb{R}_{\geq 0}} \simeq \mathbf{VB}$.

The proof of Theorem 1.2.19 (or of the claim in the Remark above) is rather long and so, we shall omit it from the current work. The proof for these results can be found, for instance, in [5, 6].

Subbundles and Pull-back Bundle

We now fully focus on vector bundles and on particular construction of vector bundles from a given one. The first of these constructions is of a vector subbundle:

Definition 1.2.21. Let $E \xrightarrow{\mathrm{pr}} B$ be a vector bundle. A *(vector) subbundle of* E is a vector bundle $F \xrightarrow{\mathrm{pr}_F} B$ such that $F \subseteq E$ is a subset of E, $\mathrm{pr}|_F = \mathrm{pr}_F$ and, for every $p \in B$, $\mathrm{pr}_F^{-1}(p) = F_p = F \cap E_p = F \cap \mathrm{pr}^{-1}(p)$ is a vector subspace of E_p and its vector structure is the one inherited from E_p.

Remark 1.2.22. An equivalent way to define a vector subbundle is to consider a submanifold $F \subseteq E$ such that, for all $p \in M$, $F_p \overset{\mathrm{def}}{=} F \cap E_p$ is a vector subspace of E_p of constant dimension with respect to p. With this notion, one clear obtains that $F \xrightarrow{\mathrm{pr}|_F} M$ becomes a vector bundle.

Example 1.2.23. If M is a smooth manifold and $v_1, \ldots, v_k \in \mathfrak{X}(M)$ are point-wise linearly independent vector fields on M, then

$$D = \coprod_{p \in M} \mathrm{span}\big\{(v_1)_p, \ldots, (v_k)_p\big\} \subseteq TM$$

is a vector subbundle of the tangent bundle TM. ♮

Example 1.2.24. Given a smooth manifold M and a submanifold $N \subseteq M$, then the tangent bundle to N, TN, is a vector subbundle of $TM|_N$. ♮

Next, we introduce the notion of an induced bundle, which is a particular important construction which essentially keeps the fibres unchanged but changes the base space:

Definition 1.2.25. Let $E \xrightarrow{\mathrm{pr}} B$ be a vector bundle and $f : B' \to B$ a continuous map. The *induced bundle by* f or *pull-back bundle* is the bundle

$$f^*E \overset{\mathrm{def}}{=} \big\{(b', e) \in B' \times E \,\big|\, f(b') = \mathrm{pr}(e)\big\}, \tag{1.31}$$

equipped with the subspace topology and the projection map $\pi : f^*E \to B'$ given by $\pi(b', e) = b'$. The projection onto the second factor gives a map $\mathrm{proj}_2 : f^*E \to E$ such that the following diagram commutes:

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\pi"'] & E \arrow[d, "\mathrm{pr}"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Example 1.2.26. If, for instance, we take $E = TB$ the tangent bundle of B, we get

$$\varphi^*TB = \big\{(b', v) \in B' \times TB \,\big|\, f(b') = \tau_B(v)\big\} \cong TB|_{f(B')}.$$ ♮

Example 1.2.27. Considering B' to be a submanifold of B and $f = \iota_{B'} : B' \hookrightarrow B$ to be the inclusion map, then

$$\begin{aligned}\iota_{B'}^* E &= \left\{(b', e) \in B' \times E \,\middle|\, \iota_{B'}(b') = \mathrm{pr}(e)\right\}\\ &= \left\{(b', e) \in B' \times E \,\middle|\, \mathrm{pr}(e) = b'\right\}\\ &\cong \mathrm{pr}^{-1}(B') \stackrel{\mathrm{def}}{=} E|_{B'}.\end{aligned} \qquad \natural$$

Example 1.2.28. Let $F \xrightarrow{\mathrm{pr}_F} M$ be a vector bundle and $f = \mathrm{pr}_F$ and denote the $\mathrm{pr}_E : E \to M$ to be the projection of E onto M. Therefore,

$$\mathrm{pr}_F^* E = \left\{(f, e) \in F \times E \,\middle|\, \mathrm{pr}_F(f) = \mathrm{pr}_E(e)\right\} \cong F \oplus E,$$

where we take this fibre-product to be a vector bundle over F with the projection being the projection onto the first factor. In particular, taking the simpler case of $F = M \times N$ with N a smooth manifold, then

$$\mathrm{pr}_F^* E = \left\{((p, q), e) \in (M \times N) \times E \,\middle|\, p = \mathrm{pr}_E(e)\right\} \cong N \times E. \qquad \natural$$

The construction of an induced bundle shall be of significance when we deal with the pull-back of vector fields or, more generally, of smooth sections.

Local and Global Sections on a Bundle

Before concluding this preliminary discussion of vector bundles with the description of associated bundles, we now generalise the notions of vector fields and 1-forms to the general setting of a local trivial fibration:

Definition 1.2.29. A *(smooth) section* on a locally trivial fibration $E \xrightarrow{\mathrm{pr}} B$ is a (smooth) map $\sigma : B \to E$ satisfying $\mathrm{pr} \circ \sigma = \mathrm{id}_B$. The collection of sections in $E \xrightarrow{\mathrm{pr}} B$ is denoted by $\Gamma(E)$.

Example 1.2.30. The most clear examples of a section are precisely the vector fields and k-forms which are the smooth sections on $TM \xrightarrow{\tau_M} M$ and $\bigwedge^k T^*M \xrightarrow{\hat{\tau}_M^{\wedge k}} M$, respectively. $\natural$

Example 1.2.31. One example of a rather important section on a vector bundle $E \xrightarrow{\mathrm{pr}} B$ is the so-called *zero section*, which is the map $0_E : B \to E$ assigning every point in B to the null vector of the fibres of E (this can be effectively thought as the inclusion of the base space B into the total space E). $\natural$

Definition 1.2.32. For a locally trivial fibration $E \xrightarrow{\mathrm{pr}} B$ and an open neighbourhood U of $p \in B$ on B, a *(smooth) local section* is a (smooth) map $\varsigma : U \to E$ such that $\mathrm{pr} \circ \varsigma = \mathrm{id}_U$. We shall denote the collection of all local sections in $E \xrightarrow{\mathrm{pr}} B$ by $\Gamma_{\mathrm{loc}}(E)$.

Remark 1.2.33. When confusion may arise between local sections and sections as in Definition 1.2.29, one usually calls the latter by *global sections.*

As seen in the previous section, both the collection of vector fields and the set of k-forms form a module over the ring of smooth functions. This result was actually a special case of the module structure on the set of global sections of a vector bundle.

Proposition 1.2.34. *Let* $E \xrightarrow{\mathrm{pr}} B$ *be a vector bundle. Then* $\Gamma(E)$ *has the structure of a* $C^\infty(B)$*-module.*

Proof. Defining the addition on the set of sections to be pointwise, so that, for any $p \in B$ and $\sigma, \varsigma \in \Gamma(E)$, $(\sigma + \varsigma)(p) = \sigma(p) + \varsigma(p)$, we may define the following operation

$$\begin{aligned} \varrho : C^\infty(B) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (f, \sigma) &\longmapsto [p \mapsto f(p)\,\sigma(p)] \end{aligned} \tag{$*$}$$

Now, we may check that this operation obeys the necessary properties to endow $\Gamma(E)$ with the structure of a $C^\infty(B)$-module. Let $f, g \in C^\infty(B)$ and $\sigma, \varsigma \in \Gamma(E)$, then

$$\begin{aligned} (\varrho(fg, \sigma))(p) &= (fg)(p)\sigma(p) = f(p)g(p)\sigma(p) = f(p)(\varrho(g, \sigma))(p) \\ &= (\varrho(f, \varrho(g, \sigma)))(p) \\ (\varrho(f + g, \sigma))(p) &= (f + g)(p)\sigma(p) = (f(p) + g(p))\sigma(p) = f(p)\sigma(p) + g(p)\sigma(p) \\ &= (\varrho(f, \sigma))(p) + (\varrho(g, \sigma))(p) \\ (\varrho(f, \sigma + \varsigma))(p) &= f(p)(\sigma + \varsigma)(p) = f(p)(\sigma(p) + \varsigma(p)) = f(p)\sigma(p) + f(p)\varsigma(p) \\ &= (\varrho(f, \sigma))(p) + (\varrho(f, \varsigma))(p) \\ (\varrho(\mathbf{1}, \sigma))(p) &= \mathbf{1}(p)\sigma(p) = \sigma(p) \end{aligned}$$

where $\mathbf{1} : B \to \mathbb{R}$ is just the function assigning the real number $1 \in \mathbb{R}$ to every point of B. Q.E.D.

For local sections, however, we still obtain the following result:

Proposition 1.2.35. *Let* $E \xrightarrow{\mathrm{pr}} B$ *be a vector bundle. Then, if* U_1, U_2 *are neighbourhoods of a point in* B *and* $\varsigma_1, \varsigma_2 \in \Gamma_{\mathrm{loc}}(E)$ *are local sections defined on* U_1, U_2*, respectively, then, for all* $f_1, f_2 \in C^\infty(U_1 \cap U_2)$*,* $f_1\varsigma_1 + f_2\varsigma_2$ *is a local section defined on* $U_1 \cap U_2$*.*

Proof. For the case of local smooth sections, the sum $f_1\varsigma_1 + f_2\varsigma_2$ only makes sense for points in the intersection $U \overset{\mathrm{def}}{=} U_1 \cap U_2$ (thus we may define f_1, f_2 only on $U_1 \cap U_2$). Therefore, it is enough to show that $\mathrm{pr} \circ (f_1\varsigma_1 + f_2\varsigma_2) = \mathrm{id}_U$. Given $p \in U$, since $(f_1\varsigma_1 + f_2\varsigma_2)(p) = f_1(p)\varsigma_1(p) + f_2(p)\varsigma_2(p) \in E_p$,

$$\mathrm{pr} \circ (f_1\varsigma_1 + f_2\varsigma_2)(p) = \mathrm{pr}(f_1(p)\varsigma_1(p) + f_2(p)\varsigma_2(p)) = p = \mathrm{id}_U(p),$$

which concludes the proof. Q.E.D.

For the case of an induced bundle by a map f, for every section $\sigma \in \Gamma(E)$, we also have an induced section $f^*\sigma \in \Gamma(f^*E)$, defined in f^*E as

$$(f^*\sigma)(b') \overset{\text{def}}{=} (b', \sigma(f(b'))), \quad \text{for all } b' \in B'.$$

The set of these induced sections is denoted by $f^*\Gamma(E)$. In fact, these pull-back sections are enough to describe all sections of the induced bundle as the following proposition shows.

Proposition 1.2.36. *Let B, B' be smooth manifolds, $E \xrightarrow{\text{pr}} B$ be a vector bundle and $f : B' \to B$ a smooth map. Then, the space of smooth sections of f^*E is generated as a $C^\infty(B)$-module by the space of induced sections $f^*\Gamma(E)$, i.e.*

$$\Gamma(f^*E) \cong f^*\Gamma(E) \otimes_{f^*C^\infty(B)} C^\infty(B'). \tag{1.32}$$

Proof. Since we can extend any local section on a vector bundle to a global one, it is enough to prove this result locally. Let $U \subseteq M$ be an open subset of M and $\varphi : E|_U = \text{pr}^{-1}(U) \to U \times \mathbb{R}^r$ be a local trivialisation, where $r \in \mathbb{Z}_{>0}$ is the rank of the vector bundle and let $\{e_1, \dots, e_r\}$ be a basis of $\mathbb{R}^r$. Then, consider the following local section

$$\begin{aligned} \sigma_i : U &\longrightarrow E|_U \\ p &\longmapsto \varphi^{-1}(p, e_i) \end{aligned}$$

Notice that, by the trivialisation above, we obtain a local trivialisation $\phi : f^*E|_{f^{-1}(U)} \xrightarrow{\cong} f^{-1}(U) \times \mathbb{R}^r$ constructed as follows

$$\begin{tikzcd} f^*E|_{f^{-1}(U)} \arrow[r, "\phi"] \arrow[d, "\text{proj}_2"'] & f^{-1}(U) \times \mathbb{R}^r \arrow[d, "f \times \text{id}_{\mathbb{R}^r}"] \\ E|_U \arrow[r, "\varphi"'] & U \times \mathbb{R}^r \end{tikzcd}$$

where $\text{proj}_2 : f^*E \to E$ represents the projection onto the second factor of f^*E. Thus, the pull-back section $f^*\sigma_i$ is such that $f^*\sigma_i(q) = \phi^{-1}(q, e_i)$, for all $q \in f^{-1}(U)$. With this pull-back section, it is clear that any other local section $\varsigma \in \Gamma(f^*E)$ can be written as $\varsigma = \sum_{i=1}^{r} \lambda_i \, f^*\sigma_i$ for some $\lambda_i \in C^\infty(f^{-1}(U))$ and $i = 1, \dots, r$. Q.E.D.

Associated Vector Bundles

To finish this preliminary discussion of vector bundles, we now move on to the construction of associated vector bundles. These, in general, require a vector bundle $E \xrightarrow{\mathrm{pr}} B$ and a functor, $\mathcal{F} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$, from the category of (finite) real vector spaces to itself. For our intents and purposes, it is sufficient to talk about the associated bundles coming from tensor products and quotients by ideals of the tensor algebra.

Definition 1.2.37. Let $E \xrightarrow{\mathrm{pr}} B$ and $E' \xrightarrow{\mathrm{pr}'} B$ be vector bundles.

1) The direct sum bundle is the vector bundle $E \oplus E' \longrightarrow B$ defined fibre-wise as $E \oplus E' \stackrel{\mathrm{def}}{=} \coprod_{p\in B}(E_p \oplus E'_p)$;

2) The dual bundle is the vector bundle $E^* \xrightarrow{\hat{\mathrm{pr}}} B$ such that $E^* = \coprod_{p\in B} E^*_p$, where E^*_p is the dual vector space to E_p, for all $p \in B$;

3) The vector bundle of tensors of type $\binom{r}{s}$ is the vector bundle $E^{\otimes r} \otimes (E^*)^{\otimes s}$ (also denoted as $\bigotimes^r E \otimes \bigotimes^s E^*$) defined fibre-wise by
$$E^{\otimes r} \otimes (E^*)^{\otimes s} \stackrel{\mathrm{def}}{=} \coprod_{p\in B} E_p^{\otimes r} \otimes (E_p^*)^{\otimes s},$$
where, by convention, $E^{\otimes 0} = (E^*)^{\otimes 0} = \mathbb{R}$;

4) The tensor bundle is the vector bundle $\bigotimes^\bullet E$ such that $\bigotimes^\bullet E = \coprod_{p\in B} \bigotimes^\bullet E_p$, where $\bigotimes^\bullet E_p = \bigoplus_{k\in\mathbb{Z}_{\geqslant 0}} E_p^{\otimes k}$ is the tensor algebra of E_p for all $p \in B$.

Example 1.2.38. Since the cotangent bundle is defined from the cotangent spaces which are, in effect, the dual to the tangent spaces, $T^*M \xrightarrow{\hat{\tau}_M} M$ is the dual vector bundle to $TM \xrightarrow{\tau_M} M$. ♮

As mentioned above, the fact that all the interesting algebras for a given vector space come from some quotient of the tensor algebra, allows us to define a plenitude of associated vector bundles. For us, the more interesting is, naturally, the one arising from the exterior algebra. To construct this algebra, we consider a vector space V and its tensor algebra $\bigotimes^\bullet V$. Then, by defining the ideal $\mathcal{I}$ generated (as an algebra) by elements of the form $v \otimes v$ for all $v \in V$, i.e.
$$\mathcal{I} = \langle v \otimes v \,|\, v \in V\rangle = \left\{\alpha \otimes (v \otimes v) \otimes \beta \,\middle|\, \alpha, \beta \in \textstyle\bigotimes^\bullet V,\ v \in V\right\} \trianglelefteq \textstyle\bigotimes^\bullet V,$$
we obtain the *exterior algebra* as the quotient of the tensor algebra by $\mathcal{I}$. Since the exterior algebra can be written as the direct sum of tensors with a fixed rank, when doing the quotient, we obtain that the exterior algebra can also be written as the direct sum:
$$\textstyle\bigwedge^\bullet V \stackrel{\mathrm{def}}{=} \bigotimes^\bullet V/\mathcal{I} = \displaystyle\bigoplus_{k\geq 0} \textstyle\bigwedge^k V. \tag{1.33}$$
The elements of each $\bigwedge^k V$ are called *(algebraic) k-forms.*

Definition 1.2.39. Let $E \xrightarrow{\text{pr}} B$ a vector bundle. The *vector bundle of (algebraic) k-forms* is $\bigwedge^k E = \coprod_{p \in B} \bigwedge^k E_p$.

Example 1.2.40. Taking $B = M$ and $E = T^*M$, we get the bundle of (algebraic) k-forms on a manifold, $\bigwedge^k T^*M \to M$. The set of sections is the set of k-forms, $\Gamma(\bigwedge^k T^*M) = \Omega^k(M)$. $\natural$

Example 1.2.41. Similarly, taking $E = TM$, we obtain the bundle of k-vectors $\bigwedge^k TM \to M$, whose sections are the multivector fields $\Gamma(\bigwedge^k TM) = \mathfrak{X}^k(M)$, to be explored in Section 1.4. $\natural$

Double Vector Bundles

As we have seen, the introduction of vector bundles generalises the idea of the tangent and cotangent bundles and allows for a rather large class of examples of manifolds with quite interesting properties. We could now question: 'What additional interesting properties do the tangent (and cotangent) bundle(s) of vector bundles have?' Well, this question motivates the following notion of double vector bundles:

Definition 1.2.42. Let $A \xrightarrow{\text{pr}_A} M$ and $B \xrightarrow{\text{pr}_B} M$ vector bundles. A *double vector bundle* is a quadruple (E, A, B, M) such that $E \xrightarrow{\pi_A} A$ and $E \xrightarrow{\pi_B} B$ are vector bundles and, if $h_A : \mathbb{R} \times E \to E$ and $h_B : \mathbb{R} \times E \to E$ are the homogeneity structures encoding the vector bundle structures of $E \to A$ and $E \to B$, respectively, then $(h_A)_r \circ (h_B)_s = (h_B)_s \circ (h_A)_r$, for all $r, s \in \mathbb{R}$.

$$\begin{tikzcd}
E \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "\text{pr}_A"] \\
B \arrow[r, "\text{pr}_B"'] & M
\end{tikzcd}$$

Remark 1.2.43. Equivalently, the double vector bundle structure can be defined so that the projections, the operations of addition and scalar product and the zero sections of E are vector bundle morphisms.

Definition 1.2.44. A *morphism of double vector bundles* between (E, A, B, M) and (F, C, D, N) is a quadruple $(\Phi, \phi_{CA}, \phi_{DB}, \varphi)$ such that (Φ, ϕ_{CA}), (Φ, ϕ_{DB}), (ϕ_{CA}, φ) and (ϕ_{DB}, φ) are vector bundle morphisms and the following diagram is commutative

$$\begin{tikzcd}
F \arrow[rr] \arrow[dr, "\Phi"] \arrow[dd] & & C \arrow[dr, "\phi_{CA}"] \arrow[dd] & \\
& E \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
D \arrow[rr] \arrow[dr, "\phi_{DB}"'] & & N \arrow[dr, "\varphi"] & \\
& B \arrow[rr] & & M
\end{tikzcd} \tag{1.34}$$

Remark 1.2.45. This definition allows us to define the category of double vector bundles **DVB** whose objects are double vector bundles (E, A, B, M) and whose morphisms are morphisms of double vector bundles $(\Phi, \phi_{CA}, \phi_{DB}, \varphi)$.

Example 1.2.46. For any vector bundle $E \xrightarrow{\mathrm{pr}} M$, the tangent bundle TE has a natural structure of a vector bundle over TM with projection given by the derivative map $\mathrm{pr}_* = T\mathrm{pr}$. Then, we obtain a double vector bundle (TE, TM, E, M) as in the following diagram

$$\begin{tikzcd} TE \arrow[r, "T\mathrm{pr}"] \arrow[d, "\tau_E"'] & TM \arrow[d, "\tau_M"] \\ E \arrow[r, "\mathrm{pr}"'] & M \end{tikzcd}$$

Choosing a chart $(U, \varphi = (x^1, \ldots, x^m))$ in M and adapted coordinates (x^i, e^a) in E, we get that an element in $\epsilon \in TE$ can be written with local coordinates $(x^i, e^a, v^i, \epsilon^a)$ or, by denoting $T\mathrm{pr}(\epsilon) = v = v^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$, $\tau_E(\epsilon) = e = (x^i, e^a) \in E_p$ and $p = \mathrm{pr}(e) = \tau_M(v)$, we may write

$$\epsilon = v^i \left.\frac{\partial}{\partial x^i}\right|_p + \epsilon^a \left.\frac{\partial}{\partial e^a}\right|_e .$$

♮

Example 1.2.47. A particular case of the previous example is to take $E = TM$. Then, we get the so-called *double tangent bundle*, denoted TTM or T^2M which can be represented by the following commutative diagram:

$$\begin{tikzcd} T^2M \arrow[r, "T\tau_M"] \arrow[d, "\tau_{TM}"'] & TM \arrow[d, "\tau_M"] \\ TM \arrow[r, "\tau_M"'] & M \end{tikzcd}$$

Locally, given a chart $(U, \varphi = (x^1, \ldots, x^m))$ of M, the adapted coordinates on TM can be written as (x^i, v^i) (which correspond to the vector $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$, where $p \in U$ has coordinates x^i). Thus, the adapted coordinates on T^2M are (x^i, v^i, u^i, w^i) or, equivalently, writing the local coordinates of $p \in M$ and $v \in T_pM$ as before, we may write $W \in T^2M$ with the previous coordinates as $W = u^i \left.\frac{\partial}{\partial x^i}\right|_p + w^i \left.\frac{\partial}{\partial v^i}\right|_v$. ♮

We now move on to explore in more detail the example of the double tangent bundle (T^2M, TM, TM, M) of a manifold M. Since both the horizontal and vertical bundle $T^2M \xrightarrow{T\tau_M} TM$ and $T^2M \xrightarrow{\tau_{TM}} TM$ are bundles over TM, it is possible to convert between them:

Proposition 1.2.48. *Given any manifold M, there is a unique isomorphism $J : T^2M \to T^2M$ such that, for a coordinate system x^i on M,*

$$J(x^i, v^i, u^i, w^i) = (x^i, u^i, v^i, w^i), \tag{1.35}$$

where (x^i, v^i) denote the local coordinates on TM and (x^i, v^i, u^i, w^i) are the coordinates on T^2M. Additionally, J is an involution, i.e. $J^2 = \mathrm{id}_{T^2M}$.

Remark 1.2.49. To the map introduced in the previous proposition, we call the *canonical flip* and it is a double vector bundle morphism satisfying the diagram:

$$\begin{tikzcd}
T^2M \arrow[rr, "\tau_{TM}"] \arrow[dr, "J"'] \arrow[dd, "T\tau_M"'] & & TM \arrow[dr, equal] \arrow[dd] & \\
& T^2M \arrow[rr, "T\tau_M"] \arrow[dd, "\tau_{TM}"] & & TM \arrow[dd] \\
TM \arrow[rr] \arrow[dr, equal] & & M \arrow[dr, equal] & \\
& TM \arrow[rr] & & M
\end{tikzcd}$$

The introduction of the double tangent bundle, allows us to look again at vector fields on a manifold. Considering vector fields as maps $M \to TM$, it is possible to define the derivative map $TX : TM \to T^2M$. Locally, if a point $p \in M$ has local coordinates $(x^1, \dots, x^m)$, we may write $X_p = X^i(p) \, \frac{\partial}{\partial x^i}\big|_p$, where $X^i \in C^\infty_{\mathrm{loc}}(M)$ are local smooth functions on M, i.e. smooth functions defined on arbitrary open subsets of M, for all $i = 1, \dots, m$. Then, for $v = v^i \, \frac{\partial}{\partial x^i}\big|_p \in T_pM$,

$$TX(v) = v^i \, \frac{\partial}{\partial x^i}\bigg|_p + v^i \frac{\partial X^j}{\partial x^i}(p) \, \frac{\partial}{\partial v^j}\bigg|_{X_p}.$$

Definition 1.2.50. The *tangent lift* of a vector field $X \in \mathfrak{X}(M)$ on a smooth manifold M is the vector field $T^\natural X \in \mathfrak{X}(TM)$, i.e. the smooth map $T^\natural X : TM \to T^2M$ such that $\tau_{TM} \circ T^\natural X = \mathrm{id}_{TM}$, given by $T^\natural X \overset{\text{def}}{=} J \circ TX$. Locally, given adapted coordinates (x^i, v^i) on TM and writing $X = X^i \, \frac{\partial}{\partial x^i}$, we get that

$$T^\natural X\Big(v^i \, \frac{\partial}{\partial x^i}\bigg|_p\Big) = X^i(p) \, \frac{\partial}{\partial x^i}\bigg|_p + v^i \frac{\partial X^j}{\partial x^i}(p) \, \frac{\partial}{\partial v^j}\bigg|_v. \tag{1.36}$$

Proposition 1.2.51. *The flow of the tangent lift $T^\natural X$ of $X \in \mathfrak{X}(M)$ is*

$$\mathrm{Fl}^t_{T^\natural X} = T\mathrm{Fl}^t_X. \tag{1.37}$$

Proof. To check that $T\mathrm{Fl}^t_X$ is the flow of $T^\natural X$, it is enough to see that it obeys the same differential equation with the same initial conditions as $\mathrm{Fl}^t_{T^\natural X}$. For that, let $p \in M$ and $v \in T_pM$. Then, clearly, we have that

$$T_p\mathrm{Fl}^0_X(v) = (\mathrm{Fl}^0_X)_*(p)(v) = \mathrm{id}_{TM}(v) = v = \mathrm{Fl}^0_{T^\natural X}(v).$$

Choosing a chart $(U, \varphi = (x^1, \dots, x^m))$ on M and local adapted coordinates (x^i, v^i) on $TM|_U$, we may write, for each $p \in U \cap \mathrm{dom}(\mathrm{Fl}^t_X)$ and $u = u^i \, \frac{\partial}{\partial x^i}\big|_p \in T_pU$, noting that we may write

$$T_p\mathrm{Fl}^t_X(u) = (\mathrm{Fl}^t_X(p))^i \, \frac{\partial}{\partial x^i}\bigg|_p + u^j \frac{\partial (\mathrm{Fl}^t_X)^i}{\partial x^j}(p) \, \frac{\partial}{\partial v^i}\bigg|_u,$$

where $(\mathrm{Fl}_X^t)^i$ is the i-th entry of the flow of $X = X^i \frac{\partial}{\partial x^i}$. Then, differentiating $T_p\mathrm{Fl}_X^t(u)$ with respect to t, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}T_p\mathrm{Fl}_X^t(u) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^t(p)\right)^i \frac{\partial}{\partial x^i}\bigg|_p + v^i \frac{\partial}{\partial x^i}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^t(x)\right)^j\bigg|_{x=p}\frac{\partial}{\partial v^j}\bigg|_{T_p\mathrm{Fl}_X^t(u)} \\ &= X^i(p)\frac{\partial}{\partial x^i}\bigg|_p + v^i\frac{\partial X^j}{\partial x^i}(p)\frac{\partial}{\partial v^j}\bigg|_{T_p\mathrm{Fl}_X^t(v)} = (T^\natural X)_{T_p\mathrm{Fl}_X^t(u)},\end{aligned}$$

which shows that $T\mathrm{Fl}_X^t$ obeys the same differential equation as $\mathrm{Fl}_{T^\natural X}^t$. By the uniqueness of the solution of differential equations, we conclude the result. Q.E.D.

1.3 Brief Overview of the Theory of Lie Algebras

We now look carefully at the concept of graded algebras and Lie algebras, as they will arise whenever we study the different class of geometries in the following chapters. For that, we start with a quick review of the notion of an algebra and relative concepts but we still assume the reader is familiarised with certain concepts one would study in elementary courses on Algebra.

Definition 1.3.1. Given a ring R, an *R-algebra structure* on an R-module A is an R-bilinear map $\mu : A \times A \to A$ called the *product*. Then, the pair (A, μ) is called an *R-algebra*.

Notice that we do not require $\mu : A \times A \to A$ to satisfy any particular properties. In fact, the addition of certain properties to this map yields special kind of algebras. For us, we shall upgrade the ring R to a field $\mathbb{K}$ and, depending on the type of algebra we will study, assume that the algebra A satisfies (some) of the following properties:

(i) Associative: for any $a, b, c \in A$, $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$;

(ii) Commutative: for any $a, b \in A$, $\mu(a, b) = \mu(b, a)$;

(iii) Anti-symmetry: for any $a, b \in A$, $\mu(a, b) = -\mu(a, b)$;

(iv) Unital: there exists an element $u \in A$ such that $\mu(u, a) = \mu(a, u) = a$, for all $a \in A$;

(v) Jacobi identity: for any $a, b, c \in A$, $\mu(a, \mu(b, c)) + \mu(b, \mu(c, a)) + \mu(c, \mu(a, b)) = 0$, where $0 \in A$ is the zero for the module structure in A.

Remark 1.3.2. From now on, for a $\mathbb{K}$-algebra A, we shall denote the product of two elements $a, b \in A$ simply by $a \cdot b$ or even ab instead of $\mu(a, b)$. Moreover, when the product is understood, we shall say simply that A is a $\mathbb{K}$-algebra

Definition 1.3.3. A map $f : A \to B$ between $\mathbb{K}$-algebras $(A, \cdot_A)$ and $(B, \cdot_B)$ is called a $\mathbb{K}$-*algebra homomorphism* if, for any $a, b \in A$, $f(a \cdot_A b) = f(a) \cdot_B f(b)$. The set of $\mathbb{K}$-algebra homomorphisms between A and B is denoted by $\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{K}}}(A, B)$.

Remark 1.3.4. With this notion of homomorphism we can construct the category of $\mathbb{K}$-algebras, $\mathbf{Alg}_{\mathbb{K}}$, whose objects are $\mathbb{K}$-algebras $(A, \cdot_A)$ and the morphisms are $\mathbb{K}$-algebra homomorphisms.

Definition 1.3.5. Let A be a $\mathbb{K}$-algebra and $S, I \subseteq A$ submodules of A.

1) S is said to be a *subalgebra* of A if, for all $r, s \in S$, $r \cdot s \in S$, i.e. if $S \cdot S \subseteq S$;

2) I is called a *left ideal* (respectively a *right ideal*) of A if, for all $i \in I$ and $a \in A$, $a \cdot i \in I$, i.e. if $A \cdot I \subseteq I$ (resp. $i \cdot a \in I$, i.e. $I \cdot A \subseteq I$).

We shall denote $S \leq A$ if S is a subalgebra of A and $I \trianglelefteq A$ if I is an ideal of A.

Remark 1.3.6. If A is an associative commutative/anti-symmetric algebra, there is no difference between left and right ideals and, thus, they are simply called ideals of A.

Proposition 1.3.7. *If A is a $\mathbb{K}$-algebra, then $I \trianglelefteq A$ is an ideal of A if and only if there is an algebra homomorphism $f : A \to B$ (for some $\mathbb{K}$-algebra B) such that $I = \ker(f)$.*

Definition 1.3.8. A *derivation on a $\mathbb{K}$-algebra A* is a $\mathbb{K}$-linear homomorphism $\mathrm{d} : A \to A$ obeying the Leibniz rule, i.e. for every $a, b \in A$,

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\,\mathrm{d}(b). \tag{1.38}$$

The collection of all derivations on A is denoted $\mathrm{Der}(A)$. Moreover, for a non-zero positive integer $k \in \mathbb{Z}_{>0}$, a k-derivation is a map $\mathrm{d} : A^{\times k} \to A$ which is a derivation in every entry, i.e. it is a $\mathbb{K}$-multilinear map such that for any $a_1, \ldots, b, c, \ldots, a_k \in A$,

$$\mathrm{d}(a_1, \ldots, b\,c, \ldots, a_k) = \mathrm{d}(a_1, \ldots, b, \ldots, a_k)\,c + b\,\mathrm{d}(a_1, \ldots, c, \ldots, a_k).$$

Then we may define the notion of a differential algebra the following way:

Definition 1.3.9. A *differential algebra* is an associative $\mathbb{K}$-algebra A equipped with a derivation $\mathrm{d} \in \mathrm{Der}(A)$.

There are many examples of differential algebras, although we shall delay such examples to the case of differential graded algebras, since those are the ones that are more useful for our study in latter chapters.

Graded Algebras

Definition 1.3.10. Let G be a monoid and A a (associative) $\mathbb{K}$-algebra. Then A is called a *G-graded algebra* if A can be written as a direct sum of subspaces, $A = \bigoplus_{g\in G} A_g$, and, for any $g, h \in G$, $A_g A_h \subseteq A_{gh}$.

If A is a $\mathbb{Z}_{\geqslant 0}$-graded algebra, as is the case, for instance, for $\bigotimes^\bullet V$ for a vector space V, it is common to simply call A a graded algebra.

Definition 1.3.11. Let A be a G-graded algebra. A *homogeneous element of degree g* is an element $a \in A$ such that $a \in A_g$. The degree of a homogeneous element is denoted by $\deg(a)$.[4]

Example 1.3.12. Let V be a vector space over a field $\mathbb{K}$, $v, w \in V$ and consider $\bigotimes^\bullet V = \bigoplus_{k\geq 0} (\bigotimes^k V$. Then

$$\begin{aligned}\deg(1_{\mathbb{K}}) &= 0, \text{ since } 1_{\mathbb{K}} \in \mathbb{K} = \textstyle\bigotimes^0 V \subseteq \bigotimes^\bullet V\\ \deg(v) = \deg(w) &= 1, \text{ because } v, w \in V = \textstyle\bigotimes^1 V \subseteq \bigotimes^\bullet V\\ \deg(v\otimes w) &= 2, \text{ as } v\otimes w \in V\otimes V \subseteq \textstyle\bigotimes^\bullet V.\end{aligned}$$ $\natural$

Remark 1.3.13. Similarly to the case of algebras, G-graded algebras form a category, denoted by ${}_G\mathrm{Alg}_{\mathbb{K}}$, by considering morphisms as algebra homomorphisms $f : A \to B$ between G-graded algebras, such that $f(A_g) \subseteq B_g$ and we shall denote $f \in \mathrm{Hom}_{{}_G\mathbf{Alg}_{\mathbb{K}}}(A, B)$. Additionally, for a given $g \in G$, it is possible to define *homogeneous maps of degree g*, which are those maps $f : A \to B$ such that $f(A_h) \subseteq B_{gh}$, for all $h \in G$. The collection of these maps is denoted by $\mathrm{Hom}^g_{{}_G\mathbf{Alg}_{\mathbb{K}}}(A, B)$.

Since we introduce a grading in our algebra, we also have to extend our notions of commutative and anti-commutative product. This is done by assigning the notion of 'evenness' and 'oddness' to the elements G that grade our algebra. More specifically, that is done by introducing a homomorphism from G to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Definition 1.3.14. A *signed monoid* is a pair (G, ς), where G is a monoid and $\varsigma : G \to \mathbb{Z}/2\mathbb{Z}$ is a monoid homomorphism.

Remark 1.3.15. When we take $G = \mathbb{Z}$, we shall always take $\varsigma : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ to be the canonical projection. This connection between the integers and $\mathbb{Z}/2\mathbb{Z}$ is precisely what we previously meant by the 'evenness' or 'oddness' of the elements of G.

Then, we can extend the notions of commutativity, anti-commutativity and the Jacobi identity to the general setting of graded algebras.

[4] In other words, $\deg\colon \bigcup_{g\in G} A_g \setminus\{0\} \to G$ such that, $\forall\, 0 \neq a \in \bigcup_{g\in G} A_g\colon\ a \in A_{\deg(a)}$, where $\bigcup_{g\in G} A_g \setminus\{0\} \subseteq A$ is the set of homogeneous elements of A.

Definition 1.3.16. Let (G, ς) be a signed monoid and A an (associative) G-graded $\mathbb{K}$-
-algebra.

1) A is said to be a *commutative G-graded algebra* if, for any homogeneous elements $a, b \in A$
$$a\,b = (-1)^{\varsigma(\deg(a))\varsigma(\deg(b))} b\,a; \tag{1.39}$$

2) A is said to be an *anti-commutative G-graded algebra* if
$$a\,b = -(-1)^{\varsigma(\deg(a))\varsigma(\deg(b))} b\,a, \tag{1.40}$$
for all homogeneous elements $a, b \in A$.

3) A satisfies the *graded Jacobi identity* if, for every homogeneous $a, b, c \in A$ we have that
$$(-1)^{\varsigma(\deg(a))\varsigma(\deg(c))} a\,(b\,c) + (-1)^{\varsigma(\deg(a))\varsigma(\deg(b))} b\,(c\,a) + (-1)^{\varsigma(\deg(b))\varsigma(\deg(c))} c\,(a\,b) = 0. \tag{1.41}$$

Now we merge the notions of a differential algebra and of a graded algebra:

Definition 1.3.17. A *differential graded algebra* or *dg-algebra* is a graded algebra A equipped with a homogeneous map of degree ± 1, $\mathrm{d} : A \to A$, such that $\mathrm{d} \circ \mathrm{d} = 0$ and satisfying the graded Leibniz rule, i.e. for every homogeneous $a, b \in A$:
$$\mathrm{d}(a\,b) = \mathrm{d}(a)b + (-1)^{\varsigma(\deg(a))} a\,\mathrm{d}(b). \tag{1.42}$$

Example 1.3.18. The two examples that we have already seen in previous sections are that of the algebra $(\Omega^\bullet(M), \wedge)$ with the exterior derivative d, which is a homogeneous map of degree 1, or with the interior product by a vector field X, i_X, which is a homogeneous map of degree -1. ♮

Example 1.3.19. As we shall see in the subsequent section, there is a bilinear bracket operation in the set of multivector fields, the Schouten-Nijenhuis bracket $[\cdot, \cdot]_{\mathrm{SN}}$, which is such that, for a given bivector $\pi \in \mathfrak{X}^2(M)$, $[\pi, \cdot]_{\mathrm{SN}} : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)$ is a homogeneous map of degree 1, thus making this a dg-algebra whenever $[\pi, \pi]_{\mathrm{SN}} = 0$. ♮

To conclude this discussion about graded algebras, we now define the shift which is valid for ($\mathbb{Z}$-)graded algebras:

Definition 1.3.20. Let V be a graded vector space over $\mathbb{K}$, i.e. $V = \bigoplus_{n \in \mathbb{Z}} V_n$, and $k \in \mathbb{Z}$. The *k-shifted vector space* $V[k]$ is the graded vector space such that, for $n \in \mathbb{Z}$, $(V[k])_n = V_{n+k}$.

Remark 1.3.21. Given a graded $\mathbb{K}$-algebra A, considering the vector space structure of A, it is possible to define the k-shifted graded vector space $A[k]$. Then, we may define a bilinear map $\cdot : A[k] \times A[k] \to A[k]$ such that $A_{n+k} \cdot A_{m+k} \subseteq A_{n+m+k}$ for all $n, m \in \mathbb{Z}$. Then, $A[k]$ with this operation is the k-shifted graded algebra.

Remark 1.3.22. A differential graded algebra can be defined, using the shift, by a graded algebra equipped with a homogeneous map (of degree 0) $\mathrm{d} : A \to A[1]$ satisfying relation (1.42).

Lie Algebras

Now we shift our attention to the study to a particular kind of non-associative algebras: the Lie algebras. These algebras are of extreme importance in a diverse range of topics in Mathematics as well as in physics. Naturally, we start with the definition of such algebras:

Definition 1.3.23. An R-algebra (A, μ) is called a *Lie algebra* if μ is anti-symmetric and satisfies the Jacobi identity, i.e. for $a, b, c \in A$

$$\mu(a, \mu(b, c)) + \mu(b, \mu(c, a)) + \mu(c, \mu(a, b)) = 0. \tag{1.43}$$

For a Lie algebra, it is usual to denote the product as a bracket $[\cdot, \cdot]$ which is called the *Lie bracket.*

Example 1.3.24. An important example comes from the linear maps of a vector space V. Denoting, $\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, W)$ to be the set of linear maps between vector spaces V, W over a field $\mathbb{K}$ (thus, the set of morphism in the category of vector spaces over $\mathbb{K}$, $\mathbf{Vect}_{\mathbb{K}}$), we shall call $\mathrm{GL}(V) = \mathrm{Aut}_{\mathbf{Vect}_{\mathbb{K}}}(V)$ the group of linear automorphisms of V. Defining

$$[f, g] \overset{\text{def}}{=} f \circ g - g \circ f, \quad \forall f, g \in \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V, V),$$

we obtain the Lie algebra of endomorphisms of V, denoted by $\mathfrak{gl}(V)$.

More generally, given an associative $\mathbb{K}$-algebra $(A, \cdot)$, we can define the commutator

$$\begin{aligned} [\cdot, \cdot] : A \times A &\longrightarrow A \\ (x, y) &\longmapsto x \cdot y - y \cdot x \end{aligned}$$

Then, it is easy to verify that $(A, [\cdot, \cdot])$ has the structure of a Lie algebra. In fact, the $\mathbb{K}$--bilinearity and anti-symmetry are obvious, whilst the Jacobi identity comes directly from the associativity of the product $\cdot$. $\natural$

Example 1.3.25. The most usual cases of Lie algebras come from groups of matrices with the Lie bracket being the commutator of matrices

Matrix Group	Associated Lie Algebra
$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$	$\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$
$\mathrm{SL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$	$\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$
$\mathrm{O}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^\intercal A = I_n\}$	$\mathfrak{o}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^\intercal = -A\}$
$\mathrm{SO}_n(\mathbb{R}) = \{A \in \mathrm{O}_n(\mathbb{R}) \mid \det(A) = 1\}$	$\mathfrak{so}_n(\mathbb{R}) = \mathfrak{o}_n(\mathbb{R})$
$\mathrm{Sp}_n(\mathbb{R}) = \{A \in M_{2n}(\mathbb{R}) \mid A^\intercal J A = J\}$	$\mathfrak{sp}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^\intercal J + JA = 0\}$
$\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid A^\dagger A = I_n\}$	$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A^\dagger = -A\}$
$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}$	$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) \mid \mathrm{tr}(A) = 0\}$
$H = \left\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) \in M_3(\mathbb{R}) \,\middle|\, a,b,c \in \mathbb{R} \right\}$	$\mathfrak{h} = \left\{ \left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{smallmatrix}\right) \in M_3(\mathbb{R}) \,\middle|\, a,b,c \in \mathbb{R} \right\}$

Table 1.1: Examples of matrix groups and their associated Lie algebras

In Table 1.1 we have denoted I_n to be the $n \times n$ identity matrix, $J = \left(\begin{smallmatrix} 0_n & -I_n \\ I_n & 0_n \end{smallmatrix}\right)$ to be the symplectic matrix and $A^\dagger$ to be the conjugate transpose matrix of A. $\natural$

Remark 1.3.26. The previous notion of algebra homomorphisms, subalgebras and ideals have a natural application to Lie algebras but now with the Lie bracket instead of the general product introduced earlier:

– Lie algebra homomorphisms: $f : \mathfrak{g} \to L$ such that $f([\ell_1, \ell_2]_\mathfrak{g}) = [f(\ell_1), f(\ell_2)]_L$, for any $\ell_1, \ell_2 \in \mathfrak{g}$;

– Lie subalgebra: $L \leq \mathfrak{g} \iff [L, L]_\mathfrak{g} \subseteq L$;

– Lie ideal: $I \trianglelefteq \mathfrak{g} \iff [I, \mathfrak{g}]_\mathfrak{g} \subseteq I$.

As previously, the introduction of Lie algebra homomorphisms allows us to define the category of Lie algebras over a field $\mathbb{K}$, $\mathbf{LieAlg}_\mathbb{K}$.

One other important characteristic feature of Lie algebras is their structure constants. Since each Lie algebra $\mathfrak{g}$ is also a (possibly infinite-dimensional) vector space — or, more generally, a free module —, we may consider a basis $\{e_a \in \mathfrak{g} \mid a = 1, \ldots, n = \dim(\mathfrak{g})\}$ of $\mathfrak{g}$. Then, we define the *structure constants of the Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ as the coefficients f_{ab}^c in

$$[e_a, e_b] = f_{ab}^c\, e_c, \tag{1.44}$$

where we also employed Einstein's summation convention[5]. Since any two elements $\ell_1, \ell_2 \in \mathfrak{g}$ of the Lie algebra can be written from the given basis as $\ell_i = \ell_i^a\, e_a$ ($\ell_i^a \in \mathbb{K}$ and $i = 1, 2$), we may simply write the Lie bracket of these elements as

$$[\ell_1, \ell_2] = [\ell_1^a\, e_a, \ell_1^b\, e_b] = \ell_1^a \ell_2^b\, [e_a, e_b] = \ell_1^a \ell_2^b f_{ab}^c\, e_c.$$

Thus, we may write the properties of the Lie algebra solely from this basis. Given that the Lie algebra is anti-symmetric and satisfies the Jacoby identity, by the former, $f_{ba}^c = -f_{ab}^c$ and, moreover,

$$[e_a, [e_b, e_c]] = [e_a, f_{bc}^d\, e_d] = f_{bc}^d f_{ad}^e\, e_e.$$

Therefore, the Jacobi identity boils down to

$$f_{bc}^d f_{ad}^e + f_{ca}^d f_{bd}^e + f_{ab}^d f_{cd}^e = 0, \tag{1.45}$$

for all $a, b, c = 1, \ldots, \dim(\mathfrak{g})$.

Lie Group of a Lie algebra: Even though Lie algebras can be treated as we have been doing so far, they are much more interesting when thought of as the tangent spaces to some specific manifolds: the Lie groups. We now give a quick look to these manifolds:

Definition 1.3.27. A *Lie group* G is a smooth manifold which is also a group such that the operations (which are the group structure maps)

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto g\, h \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are smooth.

Example 1.3.28. The matrix groups given in Example 1.3.25 are all examples of Lie groups. ♮

Theorem 1.3.29. *If G is a connected topological group and $U \subseteq G$ is a neighbourhood of the neutral element $e \in G$, then G is generated by U.*

Proof. We may assume, without loss of generality, that U is open and $U = U^{-1} = \{g^{-1} \in G \,|\, g \in U\}$ (if that is not the case, we restrict U to be $U \cap U^{-1} \supseteq \{e\} \neq \emptyset$). Let $H = \bigcup_{n \geq 1} U^n$ which is clearly open. Given $g \in G \backslash H$, gU is an open neighbourhood of g such that $gU \cap H = \emptyset$, otherwise it would exist $u \in U$ such that $gu \in H$, i.e. $g \in H$. Thus $G \backslash H$ is open. Since G is connected and H is both open and closed, $H = G$. Q.E.D.

[5] This convention essentially states that, in a product, whenever there are two repeating indices, one in subscript and another in superscript, then there is an implied summation over all the possible values for that index.

Since, by the previous theorem, if a Lie Group G is connected (which we shall assume from now on), it is generated by a neighbourhood of the neutral element, the *left translation by an element* $g \in G$, defined by

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\longmapsto g\,h \end{aligned} \tag{1.46}$$

is a diffeomorphism. Therefore, its differential for an arbitrary element $h \in G$,

$$(L_g)_*(h) : T_hG \xrightarrow{\cong} T_{gh}G,$$

is an isomorphism. This motivates the definition of the Lie algebra of the Lie group G to be T_eG. However, we still have no definition of the Lie bracket. To obtain such a bracket, firstly notice that the set of vector fields of G forms a Lie algebra with the Lie bracket of said vector fields. Thus, the natural way to realise the notion of a Lie bracket for T_eG is by introducing the notion of *left-invariant* vector fields:

Definition 1.3.30. Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is said to be *left-invariant* if, for every $g \in G$, $(L_g)_*X = X$ or, equivalently, for every $g, h \in G$, $(L_g)_*(h)(X_h) = X_{gh}$. The set of left-invariant vector fields of G is denoted $\mathrm{Lie}(G)$.

Clearly, a left-invariant vector field is entirely defined by its value on $e \in G$, i.e., as sets, $\mathrm{Lie}(G) \cong T_eG$. Hence, given the inherited Lie algebra structure on $\mathrm{Lie}(G)$, we may now define a Lie bracket on T_eG by:

$$[\![u, v]\!] \overset{\text{def}}{=} [X, Y]_e, \tag{1.47}$$

where $X, Y \in \mathrm{Lie}(G)$ are left-invariant vector fields such that $X_e = u$ and $Y_e = v$. To see that this Lie bracket is well-defined, it is enough to show that the following map is bijective

$$\begin{aligned} \mathrm{Lie}(G) &\longrightarrow T_eG \\ X &\longmapsto X_e \end{aligned}$$

Clearly, this assignment is surjective, hence, we need only to show that it is injective. For that, take $X, \hat{X} \in \mathrm{Lie}(G)$ such that $\hat{X}_e = X_e$. Then, by the defining property of left-invariant vector fields, for any $g \in G$,

$$\hat{X}_g = (L_g)_*(e)(\hat{X}_e) = (L_g)_*(e)(X_e) = X_g,$$

thus $\hat{X} = X$, which proves the injectivity of the map above. In reality, given this bijection, we get that, for every $u \in T_eG$, we may write $X \in \mathrm{Lie}(G)$ used in Eq. (1.47) to be such that, for every $g \in G$, $X_g = (L_g)_*(e)(u)$.

Therefore, we may now define a finite-dimensional (real) Lie algebra associated to any Lie group by:

Definition 1.3.31. The Lie algebra associated to a Lie group G is $\mathfrak{g} = T_eG$, with the bracket $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined in Eq. (1.47)

Remark 1.3.32. From the way we have defined the Lie bracket on T_eG, it is clear that the Lie algebra of G could have been taken as $\mathrm{Lie}(G)$ instead of T_eG since these are isomorphic as Lie algebras. In the sequel, we shall treat $\mathrm{Lie}(G)$ and T_eG interchangeably.

Now that we have seen that every Lie group has an associated Lie algebra, one could ask if the opposite is also true. It turns out it is as long as the Lie algebra is finite-dimensional (and real):

Theorem 1.3.33. *Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Then there exists a connected Lie group G, such that* $\mathrm{Lie}(G) = \mathfrak{g}$.

Representations of a Lie Group/Algebra: An important application of the theory of Lie groups and algebras comes in the study of symmetries in certain spaces. To properly solve these problems, it is necessary to understand the notion of a representation of these Lie groups and algebras. This is the purpose of this small paragraph. We start by defining what a representation is in this context:

Definition 1.3.34. Let G be a Lie group, $\mathfrak{g}$ a Lie algebra over $\mathbb{K}$ and V a vector space over $\mathbb{K}$.

1) A *representation* of G on V is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$;

2) A *representation of* $\mathfrak{g}$ *on* V is Lie algebra homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Remark 1.3.35. For the reader who is more familiarised with the language of modules, saying that (V, ρ) is a representation of the Lie group G is nothing more than saying that V is a $\mathbb{K}[G]$-module, where $\mathbb{K}[G]$ is the group ring of G over $\mathbb{K}$. As for the representation of the Lie algebra, if we restate the definition of the map ϱ to be as a map

$$\begin{aligned} \tilde{\varrho} : \mathfrak{g} \times V &\longrightarrow V, \\ (\ell, v) &\longmapsto \varrho(\ell)(v) \overset{\text{def}}{=} \ell \cdot v \end{aligned}$$

we can show that a representation of a Lie algebra satisfies the same properties as a module over an algebra and, additionally,

$$[\![\ell_1, \ell_2]\!] \cdot v = \ell_1 \cdot (\ell_2 \cdot v) - \ell_2 \cdot (\ell_1 \cdot v).$$

In these circumstances we say, in agreement with the notion of a module, that V is a $\mathfrak{g}$--module.

There are innumerous different examples of representations of both Lie groups and Lie algebras. For us, though, the important ones are the adjoint and coadjoint representations of the Lie group and Lie algebras. These are representations of either a Lie group or Lie algebra on the Lie algebra or its dual:

– **Adjoint representation of G:** Let, for every $g \in G$, $C_g : G \to G$ be the conjugation map, so that $C_g(h) = ghg^{-1}$, for all $h \in G$. Then, the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is defined by:

$$\begin{aligned} \mathrm{Ad}_g = (C_g)_*(e) : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ \ell &\longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\big(C_g(\exp(t\ell))\big)\Big|_{t=0} \end{aligned} \tag{1.48}$$

where $\exp : \mathfrak{g} \to G$ is the *exponential map* which is such that $\exp(t\ell) = \mathrm{Fl}^t_X(e)$, where $X \in \mathrm{Lie}(G)$ is such that $X_e = \ell$.

– **Adjoint representation of $\mathfrak{g}$:** Given $\ell_1, \ell_2 \in \mathfrak{g}$, there are two equivalent ways to define the adjoint representation of $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$: either as the map

$$\begin{aligned} \mathrm{ad}_{\ell_1} : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ \ell_2 &\longmapsto [\![\ell_1, \ell_2]\!] \end{aligned} \tag{1.49}$$

or, simply as $\mathrm{ad} = \mathrm{Ad}_*(e) : T_eG \cong \mathfrak{g} \to T_{\mathrm{id}_\mathfrak{g}}(\mathrm{GL}(\mathfrak{g})) \cong \mathfrak{gl}(\mathfrak{g})$. To show that these definitions are equivalent, let $\ell_1, \ell_2 \in \mathfrak{g}$ and $X_1, X_2 \in \mathrm{Lie}(G)$ such that $(X_i)_e = \ell_i$ $(i = 1, 2)$. Then,

$$\begin{aligned} \mathrm{Ad}_*(\ell_1)(\ell_2) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\ell_1)}(\ell_2)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}C_{\exp(t\ell_1)}(\exp(s\ell_2))\Big|_{s,t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\exp(t\ell_1)\exp(s\ell_2)\exp(-t\ell_1)\Big|_{s,t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Fl}^{-t}_{X_1} \circ \mathrm{Fl}^s_{X_2}(\exp(t\ell_1))\Big|_{s,t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}^{-t}_{X_1})_*(\exp(t\ell_1))\big((X_2)_{\exp(t\ell_1)}\big)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big((\mathrm{Fl}^t_{X_1})^*X_2\big)_e\Big|_{t=0} = \big(\mathcal{L}_{X_1}X_2\big)_e = [\![\ell_1, \ell_2]\!] = \mathrm{ad}_{\ell_1}(\ell_2), \end{aligned}$$

where we used the fact that $(\exp(t\ell_i))^{-1} = \exp(-t\ell_i)$ and $\mathrm{Fl}^t_{X_i}(g) = g\exp(t\ell_i)$ for all $g \in G$ and $i = 1, 2$.

– **Coadjoint representation of G:** Let $\mathfrak{g}^*$ be the dual of the Lie algebra $\mathfrak{g}$. Then, we define the coadjoint representation of G to be the map $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ such that, for $g \in G$, $\ell \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$,

$$(\mathrm{Ad}^*_g(\lambda))(\ell) = \lambda\big(\mathrm{Ad}_{g^{-1}}(\ell)\big). \tag{1.50}$$

– **Coadjoint representation of $\mathfrak{g}$:** In this case, given $\ell_1, \ell_2 \in \mathfrak{g}$ and $\lambda \in \mathfrak{g}^*$, the coadjoint representation of the Lie algebra is the map $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ such that

$$(\mathrm{ad}^*_{\ell_1}(\lambda))(\ell_2) = -\lambda\big(\mathrm{ad}_{\ell_1}(\ell_2)\big) = -\lambda\big([\![\ell_1, \ell_2]\!]\big). \tag{1.51}$$

These last two representations of the Lie group and Lie algebra will prove to be of particular interest once we study the symplectic foliation of certain Poisson manifolds.

Graded Lie algebras

Definition 1.3.36. A *graded Lie algebra* is a graded algebra which is also a Lie algebra, i.e. for a signed monoid (G, ς), it is a pair $(L, [\cdot,\cdot])$ such that $L = \bigoplus_{g\in G} L_g$ is an algebra with respect to $[\cdot,\cdot] : L \times L \to L$, satisfying:

– $[L_g, L_h] \subseteq L_{gh}$;

– Graded anti-commutativity: $[\ell_1, \ell_2] = -(-1)^{\varsigma(\deg(\ell_1))\varsigma(\deg(\ell_2))}[\ell_2, \ell_1]$ for any $\ell_1, \ell_2 \in L$;

– Graded Jacobi identity: for $\ell_1, \ell_2, \ell_3 \in L$ and $i_j = \varsigma(\deg(\ell_j))$ $(j = 1, 2, 3)$

$$(-1)^{i_1 i_3}[\ell_1, [\ell_2, \ell_3]] + (-1)^{i_1 i_2}[\ell_2, [\ell_3, \ell_1]] + (-1)^{i_2 i_3}[\ell_3, [\ell_1, \ell_2]] = 0.$$

A *differential graded Lie algebra* is a graded Lie algebra L with a graded derivation $\mathrm{d} : L \to L$, i.e. a homogeneous degree 0 graded map $\mathrm{d} : L \to L[1]$ such that $\mathrm{d}\circ\mathrm{d} = 0$ and , for $\ell_1, \ell_2 \in L$,

$$\mathrm{d}([\ell_1, \ell_2]) = [\mathrm{d}\ell_1, \ell_2] + (-1)^{\varsigma(\deg(\ell_1))}[\ell_1, \mathrm{d}\ell_2].$$

Example 1.3.37. A graded Lie algebra that is not often thought as one is $\mathfrak{sl}_2(\mathbb{R})$. In fact, calling $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and defining the Lie bracket as the commutator of matrices, we get that

$$[E, F] = H,\ [H, E] = 2E,\ [H, F] = -2F.$$

Thus, if we call $L_0 = \mathrm{span}\{H\}$, $L_1 = \mathrm{span}\{E\}$ and $L_2 = \mathrm{span}\{F\}$, we get that $\mathfrak{sl}_2(\mathbb{R}) = L_0 \oplus L_1 \oplus L_2$ is a $\mathbb{Z}/3\mathbb{Z}$-graded Lie algebra. ♮

Example 1.3.38. Let V be a dg-algebra, equipped with derivation d. Then, it is easy to check that the algebra $\mathrm{Hom}^\bullet_{\mathbb{Z}\mathbf{Alg}_{\mathbb{R}}}(V, V) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^g_{\mathbb{Z}\mathbf{Alg}_{\mathbb{R}}}(V, V)$ equipped with the bracket and derivation

$$\begin{aligned} [f, g] &= f \circ g - (-1)^{\deg(f)\deg(g)}\, g \circ f, \\ \partial f &= [\mathrm{d}, f], \end{aligned}$$

for all $f, g \in \mathrm{Hom}^\bullet_{\mathbb{Z}\mathbf{Alg}_{\mathbb{R}}}(V, V)$, is a differential graded Lie algebra. ♮

1.4 Multivectors and their Lie Algebra Structure

In Section 1.1, we have introduced the essential definitions of vector fields, k-forms and the wedge product of these forms. But a question arises: is it possible (or even useful) to define alternating k-vectors? As we have seen in Section 1.2, the answer turns out to be positive and makes use of the exterior algebra of a vector space to be able to define such k-vectors. These are rather useful, for example, in the study of Poisson manifolds since, as we shall see in Chapter 5, the Poisson structure can be defined using these alternating multivector fields[6].

Let M be a (smooth) manifold and $k \in \mathbb{Z}_{>0}$ a non-zero positive integer. We have previously studied what happens when we take elements from $\bigwedge^k T^*M$ or even (differential) k-forms, $\Omega^k(M) \overset{\text{def}}{=} \Gamma(\bigwedge^k T^*M)$, however, we now give the same treatment to the vector bundle $\bigwedge^k TM \to M$ and its set of sections $\mathfrak{X}^k(M) \overset{\text{def}}{=} \Gamma(\bigwedge^k TM)$. Firstly, recall that a k-form $\omega \in \Omega^k(M)$ can be identified with a $C^\infty(M)$-multilinear alternating map of degree k on the space of vector fields $\mathfrak{X}(M)$ of M. Hence, we may consider a k-vector field to be a $C^\infty(M)$-multilinear alternating map of degree k on the space of 1-forms $\Omega^1(M)$ of M, i.e. if $\theta \in \mathfrak{X}^k(M)$, then we may consider this multivector field to be a map $\theta : \Omega^1(M) \times \cdots \times \Omega^1(M) \to C^\infty(M)$. Locally, for a chart $(U, \varphi = (x^1, \ldots, x^m))$ centred at a point $p \in M$, this essentially means that $\left\{ \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \;\middle|\; 1 \leq i_1 < \ldots < i_k \leq m \right\}$ is a local frame of $\bigwedge^k TM \to M$ and, thus, any k-vector field θ can be expressed as

$$\theta|_U = \sum_{i_1 < \ldots < i_k} \theta^{i_1 \cdots i_k} \; \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \; , \tag{1.52}$$

where $\theta^{i_1 \cdots i_k} \in C^\infty(U)$ are smooth maps and the wedge product $\wedge$ is thought as the anti--symmetric operations arising from the quotient of the tensor algebra $\bigotimes^k TM$ with the ideal generated by the elements $\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^i}$ for all $i = 1, \ldots, m$. As it turns out, this wedge product satisfies the same properties as the exterior product defined in Section 1.1 and, thus, we shall treat them as indistinguishable. From the graded commutativity of this exterior product, the coefficients $\theta^{i_1 \cdots i_k}$ are also anti-symmetric in the exchange of any two indices: $\theta^{i_1 i_2 \cdots i_k} = -\theta^{i_2 i_1 \cdots i_k}$. The natural identification of k-vector fields and alternating derivations comes by considering the following map, associated with each a k-vector field $\theta \in \mathfrak{X}^k(M)$:

$$\begin{aligned} \mathfrak{D}_\theta : C^\infty(M)^{\times k} &\longrightarrow C^\infty(M) \\ (f_1, \ldots, f_k) &\longmapsto \theta(\mathrm{d}f_1, \ldots, \mathrm{d}f_k). \end{aligned} \tag{1.53}$$

[6] The standard way of thinking about multivector fields on a manifold and of multiderivations of $C^\infty(M)$ is by assuming that they are always alternating. Thus, we shall omit the word 'alternating' whenever dealing with alternating multivectors and/or multiderivations.

It is quite clear that this map is a multi-derivation since

$$\begin{aligned}
\mathfrak{D}_\theta(f_1,\ldots,gh,\ldots,f_k) &= \theta(\mathrm{d}f_1,\ldots,\mathrm{d}(gh),\ldots,\mathrm{d}f_k)\\
&= \theta(\mathrm{d}f_1,\ldots,(\mathrm{d}g)h + g\,\mathrm{d}h,\ldots,\mathrm{d}f_k)\\
&= \theta(\mathrm{d}f_1,\ldots,(\mathrm{d}g)h,\ldots,\mathrm{d}f_k) + \theta(\mathrm{d}f_1,\ldots,g\,\mathrm{d}h,\ldots,\mathrm{d}f_k)\\
&= \theta(\mathrm{d}f_1,\ldots,\mathrm{d}g,\ldots,\mathrm{d}f_k)\,h + g\,\theta(\mathrm{d}f_1,\ldots,\mathrm{d}h,\ldots,\mathrm{d}f_k)\\
&= \mathfrak{D}_\theta(f_1,\ldots,g,\ldots,f_k)\,h + g\,\mathfrak{D}_\theta(f_1,\ldots,h,\ldots,f_k),
\end{aligned}$$

where, in the third and fourth equalities we used the fact that $\theta \in \mathfrak{X}^k(M)$ is $C^\infty(M)$--multilinear map from $\Omega^1(M)^{\times k}$ to $C^\infty(M)$. Furthermore, $\mathfrak{D}_\theta$ is alternating because θ is a k-vector field. The reciprocal of this result is also valid, thus leading us to the following result:

Proposition 1.4.1. *The map $\theta \mapsto \mathfrak{D}_\theta$ generates a one-to-one correspondence:*

$$\left\{\begin{array}{c}\textit{Multivector fields}\\ \theta \in \mathfrak{X}^k(M)\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\textit{Alternating multi-derivations}\\ \mathfrak{D}\colon C^\infty(M)^{\times k} \to C^\infty(M)\end{array}\right\}, \tag{1.54}$$

Proof. We have already seen that every multivector field has an associated alternating multiderivation from the map $\theta \mapsto \mathfrak{D}_\theta$. Conversely, let $\mathfrak{D} : C^\infty(M)^{\times k} \to C^\infty(M)$ be an alternating k-derivation. Assuming that there exists a map $\theta : M \to \bigwedge^k TM$ such that

$$\mathfrak{D}(f_1,\ldots,f_k)(p) = \theta_p((\mathrm{d}f_1)_p,\ldots,(\mathrm{d}f_k)_p), \tag{$*$}$$

it is clear that such θ must be a section of $\bigwedge^k TM \to M$, since by the definition of θ, $\theta_p \in \bigwedge^k T_pM$. As for the existence, following a similar procedure as the one used for the case of vector fields, taking a chart $(U,(x_1,\ldots,x_m))$ of M centred at $p \in M$, for functions $f_1,\ldots,f_k \in C^\infty(M)$, we have that

$$\mathfrak{D}(f_1,\ldots,f_k)(q) = \frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{m} a_{i_1\cdots i_k}(q)\,\left.\frac{\partial f_1}{\partial x_{i_1}}\right|_q \cdots \left.\frac{\partial f_k}{\partial x_{i_k}}\right|_q,$$

where $q \in U$ and $a_{i_1\cdots i_k} \in C^\infty(U)$ is alternating for all $(i_1,\ldots,i_k) \in \{1,\ldots,m\}^{\times k}$. Then, we may define $\theta : U \to \bigwedge^k TU$ by

$$\theta_q = \sum_{1\le i_1<\ldots<i_k\le m} a_{i_1\cdots i_k}(q)\,\left.\frac{\partial}{\partial x_{i_1}}\right|_q \wedge\cdots\wedge \left.\frac{\partial}{\partial x_{i_k}}\right|_q,$$

and extend this definition to a map $\theta : M \to \bigwedge^k TM$ satisfying property $(*)$, by using coordinate charts. Q.E.D.

As expressed in Eq. (1.52), a multivector field can be expressed locally using the exterior product. Naturally, one can also use this product with multivector fields of different degree to obtain another multivector field. Therefore, given $\theta \in \mathfrak{X}^k(M)$ and $\eta \in \mathfrak{X}^\ell(M)$, $\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^\ell(M) \to \mathfrak{X}^{k+\ell}(M)$, and, for $\alpha_1, \ldots, \alpha_{k+\ell} \in \Omega^1(M)$, we have the following:

$$(\theta \wedge \eta)(\alpha_1, \ldots, \alpha_{k+\ell}) = \sum_{\sigma \in S^{k,\ell}} \operatorname{sgn}(\sigma)\theta(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)})\eta(\alpha_{\sigma(k+1)}, \ldots, \alpha_{\sigma(k+\ell)}),$$

where the sum is taken over all the (k, ℓ)-shuffles. In particular, for $k = 0$, we have $\mathfrak{X}^0(M) = C^\infty(M)$ and $f \wedge \theta = f\theta$, for any $f \in C^\infty(M)$. Also, from the way we defined the bundle of k-forms of a vector bundle $E \xrightarrow{\mathrm{pr}} M$ on Section 1.2 (here, taking $E = TM$), the exterior product for multivectors satisfies the same properties as the exterior product for forms (cf. Proposition 1.1.34). Hence, we may define the space of multivectors as

$$\mathfrak{X}^\bullet(M) \overset{\text{def}}{=} \bigoplus_{k=0}^{m} \mathfrak{X}^k(M) = \Gamma(\textstyle\bigwedge^\bullet TM), \tag{1.55}$$

which, together with the exterior product, the addition of k-vectors and the scalar multiplication, forms a graded commutative algebra.

An additional structure $\mathfrak{X}(M)$ has that $\Omega^1(M)$ does not is that of the Lie bracket of vector fields. To obtain a similar structure now in $\mathfrak{X}^\bullet(M)$ we introduce the following properties that uniquely extend the Lie bracket structure in $\mathfrak{X}(M)$:

Definition 1.4.2. The *Schouten-Nijenhuis bracket* (also just Schouten bracket) on $\mathfrak{X}^\bullet(M)$, denoted $[\cdot,\cdot]_{\mathrm{SN}}$, is uniquely determined by

1) $[\cdot,\cdot]_{\mathrm{SN}}$ is a graded Lie bracket on $\mathfrak{X}^\bullet(M)[1]$;
2) $[\cdot,\cdot]_{\mathrm{SN}}$ is a graded derivation of the graded algebra $(\mathfrak{X}^\bullet(M), \wedge)$ in each entry separately, i.e., $\forall\, \theta \in \mathfrak{X}^{k+1}(M),\ \lambda \in \mathfrak{X}^{\ell+1}(M),\ \eta \in \mathfrak{X}^{h+1}(M)$:
$$[\theta, \lambda \wedge \eta]_{\mathrm{SN}} = [\theta, \lambda]_{\mathrm{SN}} \wedge \eta + (-1)^{k(\ell+1)}\, \lambda \wedge [\theta, \eta]_{\mathrm{SN}};$$
3) action on degree 0 and degree 1 elements:
$$[X, Y]_{\mathrm{SN}} = [X, Y],\ \ [X, f]_{\mathrm{SN}} = X(f).$$

Remark 1.4.3. The previous definition implies that the Schouten bracket is such that $[\mathfrak{X}^k(M), \mathfrak{X}^\ell(M)]_{\mathrm{SN}} \subseteq \mathfrak{X}^{k+\ell-1}(M)$, for all $k, \ell \in \mathbb{Z}_{\geqslant 0}$ and, thus, since $\mathfrak{X}^{-1}(M) = 0$, for any $f, g \in \mathfrak{X}(M) = C^\infty(M)$, $[f, g]_{\mathrm{SN}} = 0$.

In the following text, since the Schouten bracket is the unique bracket that obeys the above properties (which are the natural properties to expect from an extension of the Lie bracket of vector fields) we shall drop the subscript 'SN' in the notation of said bracket, whenever no confusion with other objects arise. With these properties, the Schouten bracket of the multivector fields $\theta \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^\ell(M)$, $[\theta, \zeta] \in \mathfrak{X}^{k+\ell-1}(M)$ is the unique multivector field satisfying:

$$\mathfrak{D}_{[\theta,\zeta]} = \mathfrak{D}_\theta \mathfrak{D}_\zeta - (-1)^{(k-1)(\ell-1)} \mathfrak{D}_\zeta \mathfrak{D}_\theta, \tag{1.56}$$

where $\mathfrak{D}_\theta$ represents the (alternating) multi-derivation of $C^\infty(M)$ associated with the multivector field θ as in Eq. (1.53) and the map $\mathfrak{D}_\theta \mathfrak{D}_\zeta : C^\infty(M)^{\times(k+\ell-1)} \to C^\infty(M)$ is defined by

$$\mathfrak{D}_\theta \mathfrak{D}_\zeta(f_1, \ldots, f_{k+\ell-1}) = \sum_{\sigma \in S^{k-1,\ell}} \operatorname{sgn}(\sigma) \mathfrak{D}_\theta(f_{\sigma(1)}, \ldots, f_{\sigma(k-1)}, \mathfrak{D}_\zeta(f_{\sigma(k)}, \ldots, f_{\sigma(k+\ell-1)})), \tag{1.57}$$

for all $f_1, \ldots, f_{k+\ell-1} \in C^\infty(M)$. We also define the map $\mathfrak{D}_\zeta \mathfrak{D}_\theta : C^\infty(M)^{\times(k+\ell-1)} \to C^\infty(M)$ in a similar manner.

With this Schouten bracket, the algebra $(\mathfrak{X}^\bullet(M), \wedge, [\cdot, \cdot]_{\mathrm{SN}})$ is, in fact, what is called a Gerstenhaber algebra (cf. for instance, [7]). In the following chapters, we shall deal more with these multivectors and analyse the importance they have in several types of geometries.

Chapter 2

Tangent Distributions and Foliations

Tangent distributions arise as a natural generalisation of the integral curves of a certain vector field to some higher-dimensional structure. It is in the study of the integrability of these spaces that we obtain one of the most important normal form results of differential geometry: Frobenius' Theorem. In general, normal form theorems give a local description of a geometric structure that makes the study of such a geometry to be, simpler. In the case of Frobenius, the statement gives an equivalence between an algebraic condition (involutivity of a distribution) and a geometrical one (existence of a foliation), that simplifies the structure of a manifold into submanifolds whose tangent spaces inherit this simpler structure of the distribution. The statement and proof of this theorem in their multiple forms will be the fundamental problem to be developed throughout the present chapter.

The chapter is organised the following way: a first section where we describe the basic notions of a distribution as well as involutivity and integrability. In the second section, we develop the theory in order to prove the local version of Frobenius theorem for regular distributions. The third section delves into the notion of regular foliations and gives an equivalence between involutive distributions and foliations which is the global version of Frobenius theorem. In the fourth and final section, we consider singular distributions to obtain a generalised version of Frobenius theorem in this context. This latter result shall be of extreme importance as it is allows us to properly study Poisson manifolds in Chapters 5 and 6.

2.1 Smoothness, Integrability and Involutivity of Distributions

We start our discussion of tangent distributions with the basic definitions that will accompany us for the rest of the chapter. First and foremost, we give the precise definition of a tangent distribution:

Definition 2.1.1. Let M be a smooth manifold.

1) A *(tangent) distribution* $\mathcal{D}$ on M is a subset of TM such that $\mathcal{D} = \coprod_{p\in M} \mathcal{D}_p$, where $\mathcal{D}_p \subseteq T_pM$ is a linear subspace of T_pM, for all $p \in M$;

2) A tangent distribution $\mathcal{D}$ is called *k-dimensional* or said to be of *constant rank k* if, $\dim \mathcal{D}_p = k$, for all $p \in M$.

For a tangent distribution $\mathcal{D} \subseteq TM$, it is usual to call the linear subspace $\mathcal{D}_p$ $(p \in M)$ a *fibre* of $\mathcal{D}$.

Remark 2.1.2. In the rest of the present thesis, we will omit the word *tangent* for tangent distributions.

Definition 2.1.3. For a smooth manifold M, a distribution $\mathcal{D} \subseteq TM$ is said to be *smooth* if and only if there exists a family of vector fields $\big\{X_\lambda \,\big|\, \lambda \in \Lambda\big\} \subseteq \mathfrak{X}(M)$ such that, $\mathcal{D}_p = \operatorname{span}\big\{(X_\lambda)_p \,\big|\, \lambda \in \Lambda\big\}$, for all $p \in M$.

In particular, a k-dimensional distribution $\mathcal{D}$ is smooth if, for any $p \in M$, there is a neighbourhood U of p in M and $X_1, \dots, X_k \in \mathfrak{X}(U)$ such that $\{(X_1)_q, \dots, (X_k)_q\}$ is a basis of $\mathcal{D}_q$, for all $q \in U$.

Lemma 2.1.4. *Let M be a smooth manifold and $\mathcal{D} \subseteq TM$ k-dimensional distribution. Then $\mathcal{D}$ is smooth if and only if it has the structure of a vector subbundle.*

Proof. $\Longrightarrow$) Since $\mathcal{D}$ is smooth, we may take a neighbourhood U of $p \in M$ in M for which there exist $X_1, \dots, X_k \in \mathfrak{X}(U)$ such that $\{(X_1)_q, \dots, (X_k)_q\}$ is a basis of $\mathcal{D}_q$, for all $q \in U$, in particular for p itself. Thus, $\mathcal{D}_p = \operatorname{span}\big\{(X_1)_p, \dots, (X_k)_p\big\} \subseteq T_pM$ is a vector subspace of T_pM. Then, by recalling Definition 1.2.21, we get that $\mathcal{D} \xrightarrow{\tau_M|_{\mathcal{D}}} M$ is a vector subbundle of $TM \xrightarrow{\tau_M} M$.

$\Longleftarrow$) If $\mathcal{D} \subseteq TM$ is a vector subbundle, in particular, $\mathcal{D}$ is a vector bundle so, for each point $p \in M$ we may consider an open neighbourhood $U_p \subseteq M$ of p in M and a smooth local trivialisation $\varphi_p : \mathcal{D}|_{U_p} \xrightarrow{\cong} U_p \times \mathbb{R}^k$. Let $\{e_1, \dots, e_k\}$ be a basis of $\mathbb{R}^k$ and construct the smooth sections $\tilde{e}_i : U_p \to U_p \times \mathbb{R}^k$ such that $\tilde{e}_i(q) = (q, e_i)$, for all $q \in U_p$ and

$i = 1, \dots, k$. Then, since φ is a diffeomorphism, we may construct vector fields on U_p, $X_{(p,i)}$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{D}|_{U_p} \arrow[rr, "\varphi_p"] \arrow[dr, "\tau_M"] & & U_p \times \mathbb{R}^k \arrow[dl, "\mathrm{proj}_1"'] \\
& U_p \arrow[ul, bend left, "X_{(p,i)}"] \arrow[ur, bend right, "\tilde{e}_i"'] &
\end{tikzcd}$$

where proj_1 is the projection onto U_p. Let $\lambda_p : M \to M$ be a bump function supported on U_p. Then, $\lambda_p X_{(p,i)}$ is a vector field on M and it is easy to see that the family $\{\lambda_p X_{(p,i)} \in \mathfrak{X}(M) \,|\, (p,i) \in M \times \{1, \dots, k\}\}$ is such that, for all $p \in M$, $\mathcal{D}_p = \mathrm{span}\{(\lambda_q X_{(q,i)})_p \,|\, (q,i) \in M \times \{1, \dots, k\}\}$. Q.E.D.

Definition 2.1.5. A smooth distribution that has constant rank is called *regular*. Otherwise, the distribution is called *singular*, *generalised* or *Stefan-Sussmann*

Remark 2.1.6. In the following, we shall omit the word singular when dealing with singular distributions, whenever no confusion arises to the regularity of such a distribution.

Definition 2.1.7. An immersed submanifold $N \xrightarrow{i} M$ is called an *integral (sub)manifold* of the distribution $\mathcal{D}$ if $i_*(p)(T_pN) \subseteq \mathcal{D}_{i(p)}$, for all $p \in N$. If the equality takes place at any point $p \in N$ (respectively, at every point of N), then N is called an integral (sub)manifold *of maximal dimension at* p (resp. *everywhere*).

Definition 2.1.8. A smooth distribution $\mathcal{D}$ is called *integrable* if any point of M is in some integral submanifold of maximal dimension everywhere of $\mathcal{D}$.

Remark 2.1.9. In the rest of this document, whenever we consider integral submanifolds, we shall always consider them to be of maximal dimension everywhere. Thus, we shall omit this final part and always call 'integral submanifolds of maximal dimension everywhere' simply by 'integral submanifolds'.

Example 2.1.10. If $V \in \mathfrak{X}(M)$ is a vector field in M, we may define $\mathcal{D}_p = \mathrm{span}\{V_p\}$, $\forall\, p \in M$ and $\mathcal{D} = \coprod_{p \in M} \mathcal{D}_p$ is a smooth distribution. With this distribution, the image of an integral curve of V is an integral manifold of $\mathcal{D}$. Hence, this distribution is integrable. Moreover, if the vector field V is non-vanishing, then the distribution is one-dimensional (and so, regular). ♮

Example 2.1.11. For $M = \mathbb{R}^m$, $\mathcal{D}_p = \mathrm{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p \right\}$ are the fibres of a k--dimensional distribution. The integral submanifolds (of maximal dimension everywhere) of this distribution are of the form

$$N_c = \{(x^1, \dots, x^m) \in M \,|\, x^{k+1} = c^{k+1}, \dots, x^m = c^m\},$$

where $c^{k+1}, \dots, c^m \in \mathbb{R}$ are arbitrary real numbers. Hence, this distribution is integrable. ♮

Example 2.1.12. Let $\mathcal{D}$ be the smooth regular 2-dimensional distribution in $\mathbb{R}^3$ spanned by $X = \dfrac{\partial}{\partial x} + y\,\dfrac{\partial}{\partial z}$ and $Y = \dfrac{\partial}{\partial y}$. We shall now see that $\mathcal{D}$ is not integrable. Suppose N is an integral manifold (of maximal dimension everywhere) of $\mathcal{D}$ passing through the origin. Since $X,\ Y$ are tangents to N, any integral curve of these vector fields starting in N must remain in N at least for small enough times. Thus, N contains an open subset of the x-axis and it must also contain an open subset of the line parallel to the y-axis passing through $(x, 0, 0)$. So, N contains a subset of the (x, y)-plane. However, the tangent plane to the (x, y)-plane at any point p outside the x-axis is not equal to $\mathcal{D}_p$. This shows that there exists no integral submanifold (of maximal dimension everywhere) of $\mathcal{D}$ passing through 0 and, so, $\mathcal{D}$ is not integrable. ♮

Example 2.1.13. For $M = \mathbb{R}^4$, the Engel distribution is the distribution $\mathcal{D}$ spanned by the vector fields $\dfrac{\partial}{\partial x} + z\,\dfrac{\partial}{\partial y} + w\,\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial w}$. Following the same argument as of the previous example, it is easy to check that this distribution is not integrable. ♮

Following the definition and notations given for vector bundles, consider local and global sections of a distribution $\mathcal{D}$ to be the following sets:

$$\Gamma(\mathcal{D}) \overset{\text{def}}{=} \big\{X \in \mathfrak{X}(M) \,\big|\, X_p \in \mathcal{D}_p,\ \forall\, p \in M\big\}$$
$$\Gamma_{\text{loc}}(\mathcal{D}) \overset{\text{def}}{=} \bigcup_{U \subseteq M \text{ open}} \big\{X \in \mathfrak{X}(U) \,\big|\, X_p \in \mathcal{D}_p,\ \forall\, p \in U\big\}$$

Then, certain properties of these sections yield rather important features for distributions:

Definition 2.1.14. A smooth distribution $\mathcal{D}$ on a smooth manifold M is said to be *involutive* if, given any two local sections of $\mathcal{D}$, their Lie bracket is also a local section of $\mathcal{D}$, i.e. $[\Gamma_{\text{loc}}(\mathcal{D}), \Gamma_{\text{loc}}(\mathcal{D})] \subseteq \Gamma_{\text{loc}}(\mathcal{D})$.

A direct consequence of the notion of involutivity comes is that it is actually not a local property, as the following proposition shows:

Proposition 2.1.15. *Let $\mathcal{D}$ be a smooth distribution in a manifold M. Then $\mathcal{D}$ is involutive if and only if $\Gamma(\mathcal{D})$ is a Lie subalgebra of $\mathfrak{X}(M)$, i.e. , $[\Gamma(\mathcal{D}), \Gamma(\mathcal{D})] \subseteq \Gamma(\mathcal{D})$.*

Proof. $\Longleftarrow$) Let $X \in \mathfrak{X}(U)$ and $Y \in \mathfrak{X}(V)$ be local sections of $\mathcal{D}$, we want to show that $[X, Y] \in \mathfrak{X}(U \cap V)$ is also a local section of $\mathcal{D}$. Fix $p \in U \cap V$ arbitrary and let λ be a bump function on M taking the value 1 on an open neighbourhood W of p with $\operatorname{supp}(\lambda) \subseteq U \cap V$. Then $\lambda X,\ \lambda Y \in \Gamma(\mathcal{D})$ and, by hypothesis, $[\lambda X, \lambda Y] \in \Gamma(\mathcal{D})$. However,

$$\begin{aligned}
[\lambda X, \lambda Y]|_W &= \big(\lambda X(\lambda Y) - \lambda Y(\lambda X)\big)\big|_W = \big(\lambda^2[X, Y] + \lambda\, X(\lambda)\, Y - \lambda\, Y(\lambda)\, X\big)\big|_W \\
&= (\lambda|_W)^2[X, Y]|_W + \lambda|_W(X(\lambda))|_W Y|_W - \lambda|_W(Y(\lambda))|_W X|_W \\
&= [X, Y]|_W.
\end{aligned}$$

Thus, locally we have that $[X, Y]_p = [\lambda X, \lambda Y]_p \in \mathcal{D}_p$.

$\Longrightarrow$) Let $X, Y \in \Gamma(\mathcal{D})$. Since every global section of $\mathcal{D}$ is, in particular, a local section (defined on the whole space M), i.e. $\Gamma(\mathcal{D}) \subseteq \Gamma_{\text{loc}}(\mathcal{D})$, we have that $X, Y \subseteq \Gamma_{\text{loc}}(\mathcal{D})$. Additionally, because $\mathcal{D}$ is involutive (so $[\Gamma_{\text{loc}}(\mathcal{D}), \Gamma_{\text{loc}}(\mathcal{D})] \subseteq \Gamma_{\text{loc}}(\mathcal{D})$), $[X, Y] \subseteq \Gamma_{\text{loc}}(\mathcal{D})$. Since $X, Y \in \Gamma(\mathcal{D})$, $[X, Y]$ is defined on the entire M. Because $\Gamma(\mathcal{D})$ consists exactly of those elements in $\Gamma_{\text{loc}}(\mathcal{D})$ whose domain is the entire manifold M, we conclude that $[X, Y] \in \Gamma(\mathcal{D})$. Q.E.D.

Theorem 2.1.16. *Any integrable distribution is involutive.*

Proof. Let X, Y be smooth local sections of an integrable distribution $\mathcal{D}$ defined on an open subset $U \subseteq M$ and let $p \in U$ be an arbitrary point. Then, by hypothesis, there exists an immersed submanifold $N \xrightarrow{i} M$ which is an integral submanifold (of maximal dimension everywhere) of $\mathcal{D}$ with $p \in i(N) \subseteq U$. Since, $X_{i(q)}, Y_{i(q)} \in i_*(q)(T_qN) = \mathcal{D}_{i(q)}$, for any $q \in N$, there exist vector fields $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$ which are i-related to X, Y, respectively. Then, because $i_*(q)([\tilde{X}, \tilde{Y}]_q) = [X, Y]_{i(q)}$, for any $q \in N$ (i.e. $[\tilde{X}, \tilde{Y}]$ is i-related to $[X, Y]$) and $[\tilde{X}, \tilde{Y}]_q \in T_qN$, we get that $[X, Y]_{i(q)} \in i_*(q)(T_qN) = \mathcal{D}_{i(q)}$, for any $q \in N$, and, particularly since $p \in i(N)$, $[X, Y]_p \in \mathcal{D}_p$. Q.E.D.

2.1.1 Regularity of Distributions using Differential forms

So far, we have been dealing with distributions from the point of view of vector fields. However, it is also possible to have a similar discussion of the theory of distributions from the perspective of forms, as long as are comfortable with more algebraic terminology. Unfortunately, the price to pay for switching the treatment to differential forms is that we now can only work with regular distributions and not singular ones. Regardless, we shall still see that this way of dealing with distributions has its benefits and, so, before we discuss the Frobenius' theorem, let us give a quick look at this alternative description.

Definition 2.1.17. A k-form $\omega \in \Omega^k(M)$, $0 \leq k \leq \dim M$, *annihilates* a distribution $\mathcal{D}$ in an open subset $U \subseteq M$ if $(\omega(X_1, \ldots, X_k))|_U = 0$ for any local sections $X_1, \ldots, X_k$ of $\mathcal{D}$ supported on U.

In the case of 1-forms, if ω annihilates $\mathcal{D}$, we denote $\ker(\omega_p) \supseteq \mathcal{D}_p$, for any $p \in U$.

Given this notion of annihilating forms, we can now introduce smoothness and involutivity of a distribution for this equivalent description.

Proposition 2.1.18. *Let M be a smooth manifold and $\mathcal{D} \subseteq TM$ a k-dimensional distribution and denote*

$$\mathcal{D}^\circ = \left\{\eta \in \Omega^1(M) \,\middle|\, \ker(\eta|_U) \supseteq \mathcal{D}|_U \text{ for some } U \subseteq M\right\}.$$

Then $\mathcal{D}$ is smooth if and only if, for all $p \in M$,

$$\mathcal{D}_p = \bigcap_{\eta \in \mathcal{D}^\circ} \ker(\eta_p).$$

Proof. $\Longrightarrow$) Firstly, it is clear that $\mathcal{D}_p \subseteq \bigcap_{\eta\in\mathcal{D}^\circ} \ker(\eta_p)$, for all $p \in M$. Since $\mathcal{D}$ is smooth, there exist $X_1, \ldots, X_k \in \mathfrak{X}(M)$, such that $\mathcal{D}_p = \operatorname{span}\{(X_1)_p, \ldots, (X_k)_p\}$, for all $p \in M$. Considering vector fields $X_{k+1}, \ldots, X_m \in \mathfrak{X}(M)$ such that $\{(X_1)_p, \ldots, (X_m)_p\}$ form a basis of T_pM and the corresponding dual basis $\{(\omega_1)_p, \ldots, (\omega_m)_p\}$ of T_p^*M (which is such that, for $i, j = 1, \ldots, m$, $\omega_i(X_j) = \delta_{ij}\,\mathrm{id}_M$, where δ_{ij} is the Kroenecker delta), we conclude that $\bigcap_{\eta\in\mathcal{D}^\circ} \ker(\eta_p) = \ker((\omega_{k+1})_p) \cap \ldots \cap \ker((\omega_m)_p)$ and, thus $\dim\left(\bigcap_{\eta\in\mathcal{D}^\circ} \ker(\eta_p)\right) = k = \dim(\mathcal{D}_p)$, giving the equality of the statement.

$\Longleftarrow$) Since $\dim(\mathcal{D}_p) = k$, by a similar argument done for the previous implication, we may find $\omega_1, \ldots, \omega_{m-k} \in \Omega^1(M)$ such that $\mathcal{D}_p = \ker((\omega_1)_p) \cap \ldots \cap \ker((\omega_{m-k})_p)$. Then. completing the basis of T_p^*M with forms $\omega_{m-k+1}, \ldots, \omega_m \in \Omega^1(M)$, we can obtain the dual basis of T_pM with vector fields $X_1, \ldots, X_m \in \mathfrak{X}(M)$ such that $\omega_i(X_j) = \delta_{ij}\mathrm{id}_M$, for all $i, j = 1, \ldots, m$. To conclude the proof, simply note that $\{(X_{m-k+1})_p, \ldots, (X_m)_p\}$ form a basis of $\mathcal{D}_p$, which gives the definition of smoothness of distributions. Q.E.D.

Remark 2.1.19. Since we are dealing with k-dimensional distributions, the previous proposition is equivalently stated the following way: $\mathcal{D}$ is smooth if and only if there are 1-forms $\omega_1, \ldots, \omega_{m-k} \in \Omega^1(M)$ defined in $U \subseteq M$ such that, for every $p \in U$,

$$\mathcal{D}_p = \ker((\omega_1)_p) \cap \ldots \cap \ker((\omega_{m-k})_p).$$

Proposition 2.1.20. *Let $\mathcal{D} \subseteq TM$ be a regular distribution. Then $\mathcal{D}$ is involutive if and only if, given any 1-form η that annihilates $\mathcal{D}$ in an open subset $U \subseteq M$, $\mathrm{d}\eta$ also annihilates $\mathcal{D}$ in U.*

Proof. $\Longrightarrow$) Let $\eta \in \Omega^1(M)$ be a 1-form that annihilates $\mathcal{D}$ on open set $U \subseteq M$ and let X_1, X_2 be local sections of $\mathcal{D}$ defined on U. Then, since $\mathcal{D}$ is involutive, $[X_1, X_2] \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ defined on U. Thus, by using the definition of the exterior derivative and the fact that η annihilates $\mathcal{D}$:

$$\mathrm{d}\eta(X_1, X_2) = X_1(\eta(X_2)) - X_2(\eta(X_1)) - \eta([X_1, X_2]) = 0,$$

which shows that $\mathrm{d}\eta$ also annihilates $\mathcal{D}$ on U.

$\Longleftarrow$) Take $\eta \in \Omega^1(M)$ a 1-form such that both η and $\mathrm{d}\eta$ annihilate $\mathcal{D}$ on open set $U \subseteq M$. Let $X_1, X_2 \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ be local sections defined on U. Then

$$0 = \mathrm{d}\eta(X_1, X_2) = X_1(\eta(X_2)) - X_2(\eta(X_1)) - \eta([X_1, X_2]) = -\eta([X_1, X_2]),$$

where, in the third equality, we use the fact that η annihilates $\mathcal{D}$ in U. Therefore, we conclude that $[X_1, X_2]_p \in \ker(\eta_p) \supseteq \mathcal{D}_p$ and, since $\eta \in \Omega^1(M)$ is an arbitrary 1-form annihilating $\mathcal{D}$, we get that $[X, Y]_p \in \bigcap\limits_{\eta\in\mathcal{D}^\circ} \ker(\eta_p)$, where

$$\mathcal{D}^\circ = \left\{\eta \in \Omega^1(M) \,\middle|\, \ker(\eta|_U) \supseteq \mathcal{D}|_U \text{ for a neighbourhood } U \text{ of } p\right\}.$$

By the previous proposition, $\mathcal{D}_p = \bigcap_{\eta\in\mathcal{D}^\circ} \ker(\eta_p)$, thus we conclude that $[X, Y]_p \in \mathcal{D}_p$, as required. Q.E.D.

Introducing more algebraic terminology, we can express, elegantly, the involutivity in terms of differential forms. Firstly, recalling the notions given in Section 1.3, the algebra of differential forms in a smooth manifold M (with $\dim(M) = m$) is

$$\Omega^\bullet(M) = \bigoplus_{k=0}^{m} \Omega^k(M).$$

Then, we can define the notions of graded ideal for this algebra:

Definition 2.1.21. A *graded ideal* of $\Omega^\bullet(M)$ is a graded linear subspace $\mathcal{I} \subseteq \Omega^\bullet(M)$ such that, for any $\omega \in \mathcal{I}$ and $\eta \in \Omega^\bullet(M)$, $\omega \wedge \eta \in \mathcal{I}$ or, equivalently, such that $\mathcal{I} \wedge \Omega^\bullet(M) \subseteq \mathcal{I}$. As usual, we denote $\mathcal{I} \trianglelefteq \Omega^\bullet(M)$ for a graded ideal $\mathcal{I}$ of $\Omega^\bullet(M)$.

Since we have an additional structure on the set of forms (the exterior derivative), it is important to also distinguish graded ideals which are closed with respect to this additional operation on $\Omega^\bullet(M)$:

Definition 2.1.22. A graded ideal $\mathcal{I} \trianglelefteq \Omega^\bullet(M)$ is called a *differential ideal* if, for any $\omega \in \mathcal{I}$, $\mathrm{d}\omega \in \mathcal{I}$, or, equivalently, if $\mathrm{d}\mathcal{I} \subseteq \mathcal{I}$.

We now return to dealing with distributions:

Proposition 2.1.23. *Let $\mathcal{D}$ be a regular distribution on a manifold M, $\mathcal{I}^k(\mathcal{D}) \subseteq \Omega^k(M)$ the space of k-forms that annihilate $\mathcal{D}$ (with $\mathcal{I}^0(\mathcal{D}) = 0$) and*

$$\mathcal{I}^\bullet(\mathcal{D}) = \bigoplus_{k=0}^{m} \mathcal{I}^k(\mathcal{D}) \subseteq \Omega^\bullet(M).$$

Then $\mathcal{I}^\bullet(\mathcal{D}) \trianglelefteq \Omega^\bullet(M)$ is a graded ideal of $\Omega^\bullet(M)$.

Proof. Since $\mathcal{I}^\bullet(\mathcal{D}) \subseteq \Omega^\bullet(M)$ is a graded vector space, it is enough to show that $\mathcal{I}^\bullet(\mathcal{D})$ is closed under the exterior product. Let $\omega \in \Omega^k(\mathcal{D})$ be an arbitrary k-form, $\eta \in \mathcal{I}^\ell(\mathcal{D})$ be an ℓ-form that annihilates the distribution $\mathcal{D}$ and let $X_1, \dots, X_{k+\ell} \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ be local sections of $\mathcal{D}$ supported on an open subset $U \subseteq M$. Then, by the definition of the wedge product,

$$(\omega \wedge \eta)(X_1, \dots, X_{k+\ell}) = \sum_{\sigma \in S^{k,\ell}} \mathrm{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)}) = 0$$

since η annihilates $\mathcal{D}$. Therefore, $\omega \wedge \eta$ annihilates the distribution $\mathcal{D}$, i.e. $\omega \wedge \eta \in \mathcal{I}^{k+\ell}(\mathcal{D})$.

Q.E.D.

Definition 2.1.24. The graded ideal $\mathcal{I}^\bullet(\mathcal{D}) \trianglelefteq \Omega^\bullet(M)$ of forms annihilating the distribution $\mathcal{D}$ is called a *Pfaffian system of* $\mathcal{D}$.

Remark 2.1.25. For a regular k-dimensional distribution $\mathcal{D}$ on an m-dimensional smooth manifold M, since, at every point $p \in M$, there is a local frame $X_1, \dots, X_k \in \mathfrak{X}(M)$ of $\mathcal{D}$, the Pfaffian system of $\mathcal{D}$ is such that $\mathcal{I}^n(\mathcal{D}) = \Omega^n(M)$ for all $n \in \{k+1, \dots, m\}$.

Theorem 2.1.26. *Given a regular distribution $\mathcal{D}$ on M, $\mathcal{D}$ is involutive if and only if the Pfaffian system for $\mathcal{D}$, $\mathcal{I}^\bullet(\mathcal{D})$, is a differential ideal of $\Omega^\bullet(M)$.*

Proof. $\Longrightarrow$) Let $\omega \in \mathcal{I}^n(\mathcal{D})$ be an n-form that annihilates $\mathcal{D}$ and $X_0, X_1, \ldots, X_n \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ local sections of $\mathcal{D}$ supported on an open subset $U \subseteq M$. Then, using Eq. (1.18b) for the exterior derivative,

$$\begin{aligned} \mathrm{d}\omega(X_0, \ldots, X_n) = &\sum_k (-1)^k X_k(\omega(X_0, \ldots, \widehat{X}_k, \ldots, X_n)) \\ &+ \sum_{k<\ell} (-1)^{k+\ell} \omega([X_k, X_\ell], X_0, \ldots, \widehat{X}_k, \ldots, \widehat{X}_\ell, \ldots, X_n) \end{aligned}$$

and noting that ω annihilates $\mathcal{D}$ and, because $\mathcal{D}$ is involutive, $[X_k, X_\ell] \in \Gamma_{\mathrm{loc}}(\mathcal{D})$, we conclude that $\mathrm{d}\omega(X_0, \ldots, X_n) = 0$, i.e. $\mathrm{d}\omega$ annihilates $\mathcal{D}$ and so, $\mathrm{d}\omega \in \mathcal{I}^{n+1}(\mathcal{D})$.

$\Longleftarrow$) If $\mathcal{I}^\bullet(\mathcal{D})$ is a differential ideal, in particular, $\mathrm{d}\mathcal{I}^1(\mathcal{D}) \subseteq \mathcal{I}^2(\mathcal{D})$, thus, by Proposition 2.1.20, $\mathcal{D}$ is involutive. Q.E.D.

2.2 Frobenius Theorem for Regular Distributions

In this section and the following one, we shall always consider that a tangent distribution $\mathcal{D}$ is regular, unless stated otherwise.

Definition 2.2.1. Let M be a smooth m-dimensional manifold and $\mathcal{D} \subseteq TM$ a smooth k-dimensional distribution.

1) A chart $(U, \varphi = (x_1, \ldots, x_m))$ in M is said to be *flat* for $\mathcal{D}$ if $\varphi(U)$ is a 'cube' in $\mathbb{R}^m$ and,
$$\mathcal{D}\big|_U = \operatorname{span}\left\{ \frac{\partial}{\partial x_1}\bigg|_U, \frac{\partial}{\partial x_2}\bigg|_U, \ldots, \frac{\partial}{\partial x_k}\bigg|_U \right\}. \tag{2.1}$$

2) A distribution $\mathcal{D}$ is called *completely integrable* if there is a flat chart for $\mathcal{D}$ in a neighbourhood of any point $p \in M$.

Remark 2.2.2. In the definition, the notion of a 'cube' is the usual notion that $\varphi(U)$ can be written as the $\{(x_1, \ldots, x_m) \in U \,|\, m_i \leq x_i \leq M_i,\ i = 1, \ldots, m\}$ for $m_i, M_i \in \mathbb{R}$ for all $i = 1, \ldots, m$. Then, if $(U, \varphi = (x_1, \ldots, x_m))$ is a flat chart for $\mathcal{D}$, we can represent integral manifolds of $\mathcal{D}$, locally, as slices of the form

$$\{(x_1, \ldots, x_m) \in U \,|\, x_{k+1} = c_{k+1}, \ldots, x_m = c_m\},$$

for some $c_{k+1}, \ldots, c_m \in \mathbb{R}$. Therefore, any completely integrable distribution is integrable.

Hence, we have the following:

$$\text{completely integrable} \implies \text{integrable} \implies \text{involutive} \tag{2.2}$$

It is clear that the definition given for complete integrability is only valid for regular distributions. In fact, this definition is only useful in the study of regular distributions, as it makes it simpler to close the cycle of implications in Eq. (2.2).

Before we introduce the actual result that gives name to the present section, we present the following lemma which will make the proof of Frobenius theorem much simpler:

Lemma 2.2.3. *Let M be a smooth m-dimensional manifold and $X_1, \dots, X_k$ be linearly independent vector fields defined on a neighbourhood V of $p \in M$ such that*

$$[X_i, X_j] = 0, \quad 1 \leq i, j \leq k.$$

Then there exists a chart $(U \subseteq V; (x_1, \dots, x_m))$ on M centred at p, such that

$$X_i\big|_U = \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq k.$$

Proof. Let Fl_i^t be the flow of X_i and $(U \subseteq V; (x_1, \dots, x_m))$ be a chart (containing p) such that $\left\{(X_1)_p, \dots, (X_k)_p, \frac{\partial}{\partial x_{k+1}}\Big|_p, \dots, \frac{\partial}{\partial x_m}\Big|_p\right\}$ spans T_pV. Since the lemma is purely local, we can assume, without loss of generality, that $U \subseteq \mathbb{R}^m$ is a neighbourhood of $p = 0$. Let $\varepsilon > 0$ and let W be a neighbourhood of 0 in U such that $\mathrm{Fl}_1^{t_1} \circ \cdots \circ \mathrm{Fl}_k^{t_k} : W \to U$ if $|t_i| < \varepsilon$, for all $i = 1, \dots, k$. Then, define

$$\Omega \stackrel{\text{def}}{=} \left\{(x_{k+1}, \dots, x_m) \in \mathbb{R}^{m-k} \,\middle|\, (0, \dots, 0, x_{k+1}, \dots, x_m) \in W\right\}$$

and $\Phi : \,]-\varepsilon, \varepsilon[\,^{\times k} \times \Omega \to U$ such that

$$\Phi(x_1, \dots, x_m) = (\mathrm{Fl}_1^{x_1} \circ \cdots \circ \mathrm{Fl}_k^{x_k})(0, \dots, 0, x_{k+1}, \dots, x_m).$$

Thus, denoting $x = (x_1, \dots, x_m)$ and taking $f \in C^\infty(U)$, we get that, for all $i = 1, \dots, k$,

$$\begin{aligned}
\Phi_*(x)\left(\frac{\partial}{\partial x_i}\bigg|_x\right)(f) &= \frac{\partial}{\partial x_i}\Big(f \circ \Phi\Big)\bigg|_x \\
&= \frac{\partial}{\partial x_i}\Big(f \circ \mathrm{Fl}_1^{x_1} \circ \cdots \circ \mathrm{Fl}_k^{x_k}(0, \dots, 0, x_{k+1}, \dots, x_m)\Big)\bigg|_x \\
&= \frac{\partial}{\partial x_i}\Big(f \circ \mathrm{Fl}_i^{x_i} \circ \mathrm{Fl}_1^{x_1} \circ \cdots \circ \widehat{\mathrm{Fl}_i^{x_i}} \circ \cdots \circ \mathrm{Fl}_k^{x_k})(0, \dots, 0, x_{k+1}, \dots, x_m)\Big)\bigg|_x \\
&= (X_i(f))(\mathrm{Fl}_i^{x_i} \circ \mathrm{Fl}_1^{x_1} \circ \cdots \circ \widehat{\mathrm{Fl}_i^{x_i}} \circ \cdots \circ \mathrm{Fl}_k^{x_k})(0, \dots, 0, x_{k+1}, \dots, x_m)) \\
&= (X_i(f))(\Phi(x)),
\end{aligned}$$

where, in the third equality we used the fact that, since the vector fields X_i, X_j commute, the flows of these fields also commute, i.e. $\mathrm{Fl}_i^t \circ \mathrm{Fl}_j^s = \mathrm{Fl}_j^s \circ \mathrm{Fl}_i^t$.

Therefore,

$$\Phi_*(0)\left(\frac{\partial}{\partial x_i}\Big|_0\right)=\begin{cases}(X_i)_p, & i=1,\dots,k\\ \dfrac{\partial}{\partial x_i}\Big|_p, & i=k+1,\dots,m\end{cases},$$

and, by the Inverse Function Theorem Φ is a diffeomorphism at 0 with $\Phi(0)=p$ and $\Phi_*^{-1}(\frac{\partial}{\partial x_i})=X_i$. Q.E.D.

We now arrive at the central theorem, whose different versions are the central focus of this chapter. In simple terms, the point of Frobenius theorem is to take into account the implications presented in (2.2) and close the cycle, making all the definitions equivalent:

Theorem 2.2.4 (Local Frobenius)**.** *Any involutive distribution is completely integrable.*

Proof. Since Lemma 2.2.3 guarantees that any distribution locally spanned by independent smooth commuting vector fields is completely integrable (the chart given by the Lemma is flat), it suffices to show that if a distribution $\mathcal{D}$ is involutive, then $\mathcal{D}$ is the local span of independent commuting vector fields.

Therefore, let $(U;(x_1,\dots,x_m))$ be a chart on M centred at a point $p\in M$ and, up to permuting $x_1,\dots,x_m$ if necessary, we can always assume that the following set

$$\operatorname{span}\left\{\frac{\partial}{\partial x_{k+1}}\Big|_p,\dots,\frac{\partial}{\partial x_m}\Big|_p\right\}\subseteq T_pM$$

is complementary to $\mathcal{D}_p$. Then, there is an open neighbourhood W of 0 in U such that, for all $q\in W$,

$$T_qM=\mathcal{D}_q\oplus\operatorname{span}\left\{\frac{\partial}{\partial x_{k+1}}\Big|_q,\dots,\frac{\partial}{\partial x_m}\Big|_q\right\}.$$

If we take $\pi:\mathbb{R}^m\to\mathbb{R}^k$ to be projection onto the first k coordinates, it is clear that $\pi_*|_{\mathcal{D}}:\mathcal{D}|_W\to T\mathbb{R}^k$, defined as the composition $\pi_*\circ i$ of the inclusion $\mathcal{D}|_W\overset{i}{\hookrightarrow}TW$ and

$$\begin{aligned}\pi_*:TW=W\times\mathbb{R}^m&\longrightarrow W\times\mathbb{R}^k\\ (x,(v_1,\dots,v_m))&\longmapsto(x,(v_1,\dots,v_k)),\end{aligned}$$

is a vector bundle isomorphism. Thus, considering a neighbourhood W of p and a point $q\in W$, we define

$$(X_i)_q=(\pi_*|_{\mathcal{D}_q})^{-1}(\pi(q))\left(\frac{\partial}{\partial x_i}\Big|_{\pi(q)}\right)\in\mathcal{D}_q,$$

which are local sections of $\mathcal{D}\xrightarrow{\tau_M|_{\mathcal{D}}}M$. The proof is concluded if $[X_i,X_j]=0$, for all $i,j=1,\dots,k$. However, because $\mathcal{D}$ is involutive and $X_1,\dots,X_k$ are local sections of $\mathcal{D}$,

$[X_i, X_j] \in \Gamma_{\text{loc}}(\mathcal{D})$ and

$$\begin{aligned}
\pi_*(q)([X_i, X_j]_q) &= \pi_*(q)\left(\left[(\pi_*|_{\mathcal{D}_q})^{-1}(\pi(q))\left(\left.\frac{\partial}{\partial x_i}\right|_{\pi(q)}\right), (\pi_*|_{\mathcal{D}_q})^{-1}(\pi(q))\left(\left.\frac{\partial}{\partial x_j}\right|_{\pi(q)}\right)\right]_q\right)\\
&= \pi_*(q)\left((\pi_*|_{\mathcal{D}_q})^{-1}(\pi(q))\left(\left[\left.\frac{\partial}{\partial x_i}\right|_{\pi(q)}, \left.\frac{\partial}{\partial x_j}\right|_{\pi(q)}\right]_{\pi(q)}\right)\right)\\
&= (\pi_* \circ (\pi_*|_{\mathcal{D}_q})^{-1})(\pi(q))\left(\left[\left.\frac{\partial}{\partial x_i}\right|_{\pi(q)}, \left.\frac{\partial}{\partial x_j}\right|_{\pi(q)}\right]_{\pi(q)}\right)\\
&= \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]_{\pi(q)} = 0.
\end{aligned}$$

Since $\pi_*(q)|_{\mathcal{D}_q} : \mathcal{D}_q \hookrightarrow T_{\pi(q)}\mathbb{R}^k$ is injective, we conclude that $[X_i, X_j]_q = 0$, for all $q \in W$, concluding the proof. Q.E.D.

Notice that the proof of Frobenius theorem gives a technique to find integral submanifolds. The idea of such a method is to use the projection to find commuting vector fields spanning the same distribution. The following example illustrates this idea.

Example 2.2.5. Let $\mathcal{D} \subseteq T\mathbb{R}^3$ be the distribution spanned by the vector fields $X = x\,\frac{\partial}{\partial x} + xy\,\frac{\partial}{\partial z}$ and $Y = xe^{-y^2}\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{x^2}{2}(1+2ye^{-y^2})\,\frac{\partial}{\partial z}$. Then, by an easy computation, we obtain that $[X, Y] = e^{-y^2}X$ and, so, $\mathcal{D}$ is an involutive distribution. Since, for all $(x, y, z) \in \mathbb{R}^3$, $T_{(x,y,z)}\mathbb{R}^3 = \mathcal{D}_{(x,y,z)} \oplus \mathbb{R}\,\frac{\partial}{\partial z}$, we obtain that the projection $\text{proj} : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\text{proj}(x, y, z) = (x, y)$ gives an isomorphism $\text{proj}_*|_{\mathcal{D}_{(x,y,z)}} : \mathcal{D}_{(x,y,z)} \to T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2$. By the proof of the Frobenius theorem, there exist commuting sections $V, W \in \Gamma(\mathcal{D})$ that generate $\mathcal{D}$ such that

$$\text{proj}_*(x, y, z)(V_{(x,y,z)}) = \left.\frac{\partial}{\partial x}\right|_{(x,y)} \quad \text{and} \quad \text{proj}_*(x, y, z)(W_{(x,y,z)}) = \left.\frac{\partial}{\partial y}\right|_{(x,y)}.$$

It is straightforward that $V = \frac{\partial}{\partial x} + v(x, y)\,\frac{\partial}{\partial z}$ and $W = \frac{\partial}{\partial y} + w(x, y)\,\frac{\partial}{\partial z}$ for some smooth functions $v, w \in C^\infty(\mathbb{R}^3)$. By analysing combinations of X and Y, we may take $V = X$ and $W = Y - xe^{-y^2}X = \frac{\partial}{\partial y} + \frac{x^2}{2}\,\frac{\partial}{\partial z}$. By (the proof of) Lemma 2.2.3, a flat chart of $\mathcal{D}$ is obtained by considering the inverse of the map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ (in this particular case, we can extend the domain and codomain to all of $\mathbb{R}^3$) given by

$$\Phi(t_1, t_2, \tau) = (\text{Fl}_V^{t_1} \circ \text{Fl}_W^{t_2})(0, 0, \tau).$$

The flows of V and W are quite simply to compute and give

$$\begin{aligned}
\text{Fl}_V^t(x, y, z) &= \left(x + t, y, z + yt\left(x + \frac{t}{2}\right)\right)\\
\text{Fl}_W^t(x, y, z) &= \left(x, y + t, z + \frac{x^2}{2}t\right),
\end{aligned}$$

Thus, $\Phi(t_1,t_2,\tau) = \big(t_1,t_2,\tau+\frac{1}{2}t_1t_2^2\big)$ and its inverse gives

$$\Phi^{-1}(x,y,z) = \left(x,y,z-\frac{1}{2}yx^2\right).$$

The integral manifolds of $\mathcal{D}$ are, then, the level sets $\phi(x,y,z) = z - \frac{1}{2}yx^2$. ♮

To conclude the present section, we simply include the following proposition which gives a description of the local structure of integral submanifolds of a certain regular distribution $\mathcal{D}$ that is involutive (or, equivalently, integrable).

Proposition 2.2.6 (Local structure of Integral Manifolds)**.** *Let $\mathcal{D}$ be an involutive k--dimensional distribution in M and $(U,\varphi=(x_1,\dots,x_m))$ a flat chart for $\mathcal{D}$. If N is an integral manifold of $\mathcal{D}$, then $N\cap U$ is a disjoint union of k-dimensional parallel open sets of U of the form $\{x_{k+1}=c_{k+1},\dots,x_m=c_m\}$ with $c_{k+1},\dots,c_m\in\mathbb{R}$, each of them being an open set in N embedded in M.*

Proof. Let N be an integral manifold of $\mathcal{D}$ with inclusion map $\iota : N \hookrightarrow M$. Since ι is continuous, $N\cap U = \iota^{-1}(U)$ is open in N, thus is a (countable) disjoint union of connected components, each of these open in N.

Moreover, since N is an integral manifold and $(U,\varphi=(x_1,\dots,x_m))$ is a flat chart for $\mathcal{D}$, $N\cap U$ consists of slices S_i of the form

$$S_i = \big\{(x_1,\dots,x_m)\in N\cap U \,\big|\, x_{k+1}=c^i_{k+1},\dots,x_m=c^i_m\big\},$$

for some $c^i_{k+1},\dots,c^i_m\in\mathbb{R}$. Additionally, considering V to be one of the disjoint components of $N\cap U$, since the 1-forms $\mathrm{d}x_{k+1},\dots,\mathrm{d}x_m$ annihilate the distribution $\mathcal{D}$, they vanish in V and, so, V must lie in a single slice of $N\cap U$.

To finish the proof, we need only prove that V is embedded in N. However, since each slice S_i of $N\cap U$ is embedded in M and the inclusion $V\hookrightarrow S_i$ is a smooth injective immersion of manifolds with the same dimension, it is a local diffeomorphism (thus a homeomorphism onto its image), therefore it is an embedding. Composing the inclusions $V\hookrightarrow S_i$ and $S_i\hookrightarrow M$, we conclude that V is a smooth embedding. Q.E.D.

2.3 Foliations

We have seen how the Frobenius theorem is a powerful tool to describe the properties of a manifold endowed with an involutive distribution locally. However, it would also be of interest to be able to describe a manifold (endowed with an involutive distribution) in a more global way. This can be done by understanding a bit more about the nature of the integral manifolds arising from a given involutive regular distribution $\mathcal{D}$. To arrive at this non-local description of Frobenius theorem, we need to introduce the notion of a foliation and, in particular, of a flat chart for a partition $\mathcal{F}$ of a manifold M:

Definition 2.3.1. Let $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a partition of an m-dimensional manifold M in smooth immersed submanifolds. Then, a *flat chart for* $\mathcal{F}$ is a chart (U, φ) such that $\varphi(U)$ is a 'cube' and, for each $\lambda \in \Lambda$, either $\mathcal{F}_\lambda \cap U = \emptyset$, or

$$\mathcal{F}_\lambda \cap U = \bigcup_{\eta\in H} \varphi^{-1}\big(\mathbb{R}^k \times \{\eta\}\big),$$

where H is a countable subset of $\mathbb{R}^{m-k}$.

The previous definition of a flat chart allows us now to define a foliation:

Definition 2.3.2. A *foliation of dimension* k in M is a collection $\mathcal{F}$ of immersed non-empty connected disjoint k-dimensional submanifolds of M, called the *leaves of the foliation* $\mathcal{F}$, whose union is M such that, around each $p \in M$, there is a flat chart for $\mathcal{F}$.

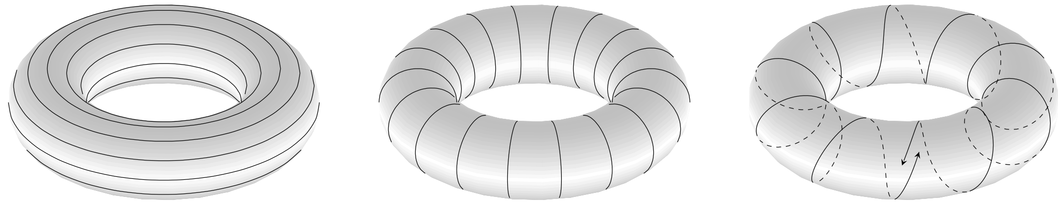
Remark 2.3.3. An equivalent way of expressing Definition 2.3.2 would be the following: A foliation is a partition $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of M in (immersed) k-dimensional submanifolds that locally 'looks like' $\{\mathbb{R}^k \times \{\alpha\}\}_{\alpha\in\mathbb{R}^{m-k}}$.

Example 2.3.4. From the previous observation, it is clear that $\{\mathbb{R}^k \times \{\alpha\}\}_{\alpha\in\mathbb{R}^{m-k}}$ is a k-dimensional foliation of $\mathbb{R}^k$. ♮

Example 2.3.5. $\big\{\lambda x \,\big|\, \lambda > 0\big\}_{x\in\mathbb{R}^m\setminus\{0\}}$ is a one-dimensional foliation of $\mathbb{R}^m\setminus\{0\}$. Similarly, the collection of the spheres centred at 0 forms an $(m-1)$-dimensional foliation of $\mathbb{R}^m\setminus\{0\}$. ♮

Example 2.3.6. Let M, N be connected manifolds, then the collections $\{M \times \{q\}\}_{q\in N}$ and $\{\{p\} \times N\}_{p\in M}$ are foliations of $M \times N$ whose leaves are diffeomorphic to M and N, respectively. If $M = N = S^1$, then $\{S^1 \times \{q\}\}_{q\in S^1}$ and $\{\{p\} \times S^1\}_{p\in S^1}$ are the foliations (a) and (b) of the torus $T^2 = S^1 \times S^1$, respectively, represented in Fig 2.1. ♮

Example 2.3.7. Still considering the torus T^2, if $\alpha \in \mathbb{R}$, the image of the curves $\gamma_\theta(t) = (e^{it}, e^{i(\alpha t+\theta)})$ $(\theta \in \mathbb{R})$ generates a one-dimensional foliation of the torus ((c) in Fig 2.1). If $\alpha \in \mathbb{Q}$, each leaf is submersed on (and diffeomorphic to) S^1, whilst if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$, each leaf is diffeomorphic to $\mathbb{R}$ and is dense in T^2. ♮



(a) (b) (c)

Figure 2.1: Representation of different foliations of the torus.

Proposition 2.3.8. *Let $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ a foliation of a manifold M. The collection of tangent spaces to the leaves of $\mathcal{F}$ forms an involutive regular distribution $\mathcal{D} \subseteq TM$ (also denoted by $T\mathcal{F}$) on M given by $\mathcal{D}_p \overset{\text{def}}{=} T_p\mathcal{F}_\lambda$, for all $p \in M$ and $\lambda \in \Lambda$ such that $p \in \mathcal{F}_\lambda$.*

Proof. The fact that the collection of the tangent leaves to the foliation $\mathcal{F}$ is a distribution comes directly from the definition of a foliation (use flat charts). Therefore, we only need to show that $T\mathcal{F}$ is involutive. Let $p \in M$ and $\lambda \in \Lambda$ such that $p \in \mathcal{F}_\lambda$ and take $X, Y \in \mathfrak{X}(M)$ local sections of $T\mathcal{F}$. Since $\mathcal{F}$ is a foliation, $\mathcal{F}_\lambda$ is an immersed submanifold and, once again, since X and Y are tangent to $\mathcal{F}_\lambda$, $[X, Y]$ is tangent to $\mathcal{F}_\lambda$ and so, in particular, $[X, Y]_p \in T_p\mathcal{F}_\lambda = T_p\mathcal{F}$. Hence, $T\mathcal{F}$ is an involutive regular distribution. Q.E.D.

Remark 2.3.9. An alternative way to prove the previous result comes from the fact that, if $\mathcal{D} = T\mathcal{F}$ and $\mathcal{F}$ is a foliation, then we can take a flat chart (U, φ) which is a 'cube' and $U \cap \mathcal{F}_\lambda$ is either empty or a countable union of sets of the form $\big\{x_i = c_i \,\big|\, i = k+1, \ldots, m\big\}$, for $(c_{k+1}, \ldots, c_m) \in \mathbb{R}^{m-k}$. Thus,

$$\begin{aligned}\mathcal{D}|_U = T\mathcal{F}|_U &= \bigcup_{\lambda\in\Lambda} T(\mathcal{F}_\lambda \cap U) = \bigcup_{\lambda\in\hat\Lambda}\bigcup_{\eta\in H_\lambda} T\big\{x_i = c^i_\eta \,\big|\, c^i_\eta \in \mathbb{R},\ i = k+1, \ldots, m\big\} \\ &= \bigcup_{\lambda\in\hat\Lambda}\bigcup_{\eta\in H_\lambda} \operatorname{span}\left\{\left.\frac{\partial}{\partial x_{k+1}}\right|_U, \ldots, \left.\frac{\partial}{\partial x_m}\right|_U\right\} = \operatorname{span}\left\{\left.\frac{\partial}{\partial x_{k+1}}\right|_U, \ldots, \left.\frac{\partial}{\partial x_m}\right|_U\right\},\end{aligned}$$

where $\hat\Lambda \subseteq \Lambda$ represents the set of indices for which $\mathcal{F}_\lambda \cap U \neq \emptyset$. So, we conclude that the distribution is completely integrable and, by Equation (2.2), that is, in fact, involutive.

To arrive at the global version of Frobenius theorem, we need to introduce the following lemma:

Lemma 2.3.10. *Let $\mathcal{D} \subseteq TM$ be a distribution and $\{N_\lambda\}_{\lambda\in\Lambda}$ a collection of connected integral manifolds of $\mathcal{D}$ such that $\bigcap_{\lambda\in\Lambda} N_\lambda \neq \emptyset$. Then $N = \bigcup_{\lambda\in\Lambda} N_\lambda$ has a unique manifold structure that makes it a connected integral manifold of maximal dimension of $\mathcal{D}$.*

The proof of this result essentially entails the construction of a topological structure and a manifold structure on N making N an integral manifold of $\mathcal{D}$. Since this proof is rather long and intricate, we leave it out of the current text. For the interested reader, a complete proof can be found in [3, Lemma 19.22].

Remark 2.3.11. Notice that, given a distribution $\mathcal{D}$ on M, if we take $\mathcal{N}_p$ to be the collection of all connected integral manifolds containing a certain point $p \in M$ then, assuming that $\mathcal{N}_p$ is non-empty, by the previous lemma, there is a maximal element $S \in \mathcal{N}_p$ in the sense that any other $N \in \mathcal{N}_p$ is contained in S. Again using the previous lemma, it is easy to discover what is this integral manifold: it must be the union of all connected integral manifolds of $\mathcal{D}$ containing p, $\bigcup_{N \in \mathcal{N}_p} N$.

Theorem 2.3.12 (Global Frobenius)**.** *Let $\mathcal{D}$ be an involutive distribution on a manifold M. Then, the collection of all connected integral manifold of $\mathcal{D}$ constitutes a foliation of M.*

Proof. Let $p \in M$ and N_p be the union of all connected integral manifolds of $\mathcal{D}$ containing p. Then, by the previous lemma, N_p is a (maximal) integral manifold of $\mathcal{D}$ containing p. To conclude the proof, it is enough to show that $\{N_p\}_{p \in M}$ is a foliation of M.

Let (U, φ) be a flat chart for $\mathcal{D}$. Then, by Proposition 2.2.6, for each $p \in M$, $N_p \cap U$ is a countable disjoint union of k-dimensional open sets (parallel in U). Taking S to be such a slice of $N_p \cap U$, then $N_p \cap S = S$ since, otherwise, we could take $N_p \cup S$ to be an integral manifold of $\mathcal{D}$, contradicting the fact that N_p is maximal. Therefore, $N_p \cap U$ is a countable union of slices and, hence, $\{N_p\}_{p \in M}$ is a foliation of M. Q.E.D.

2.4 The case of Singular Distributions

We have previously seen the case of regular distributions and how important Frobenius Theorem is for these distributions. However, the theorem is not valid for the more general singular distributions as the following example shows:

Example 2.4.1. Let $M = \mathbb{R}^2$ and $\mathcal{D}$ be a smooth distribution defined by setting, for all $(x, y) \in \mathbb{R}^2$, $\mathcal{D}_{(x,y)} = \operatorname{span}\left\{ \dfrac{\partial}{\partial x}, \phi(x) \dfrac{\partial}{\partial y} \right\}$, where

$$\phi(y) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0 \end{cases} .$$

This distribution is involutive since

$$\left[\frac{\partial}{\partial x}, \phi(x) \frac{\partial}{\partial y} \right]_{(x,y)} = \begin{cases} 0 & x \leq 0 \\ D\phi(x)\, \phi(x) \left. \frac{\partial}{\partial y} \right|_{(x,y)} & x > 0 \end{cases} ,$$

where $D\phi(x)$ denotes the derivative of ϕ. Thus, we cannot construct a partition of $\mathbb{R}^2$ by integral manifolds of $\mathcal{D}$ because no integral manifold of $\mathcal{D}$ contains points on the y-axis. Additionally, it is easy to verify that, for $y \in \mathbb{R}$ and U a neighbourhood of $(0, y) \in \mathbb{R}^2$, there are points $(-\varepsilon, y)$ and (ε, y) such that $\dim \mathcal{D}_{(-\varepsilon,y)} = 1 \neq 2 = \dim \mathcal{D}_{(\varepsilon,y)}$, which shows that it is not possible to define a flat chart for every neighbourhood of each point in $\mathbb{R}^2$. ♮

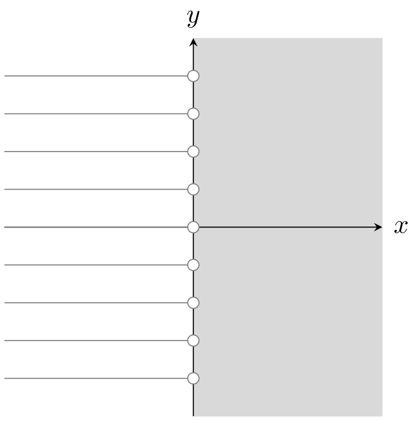


Figure 2.2: Integral manifolds of the distribution from Example 2.4.1

Thus, for a 'Frobenius-like' theorem to be valid, we need to add some conditions that should be trivially satisfied by regular k-dimensional distributions.

With this in mind, we introduce the following definition:

Definition 2.4.2. Let $\mathfrak{D} \subseteq \mathfrak{X}(M)$. A (singular) distribution $\mathcal{D} \subseteq TM$ is said to be $\mathfrak{D}$-*-invariant* if both conditions apply:

1) $\mathcal{D}_p = \operatorname{span}\{X_p \mid X \in \mathfrak{D}\}$, for all $p \in M$;

2) $(\mathrm{Fl}^t_X)_*(p)(\mathcal{D}_p) = \mathcal{D}_{\mathrm{Fl}^t_X(p)}$, for all $X \in \mathfrak{D}$, $t \in \mathbb{R}$ and $p \in \operatorname{dom}(\mathrm{Fl}^t_X)$.

As explained previously, even though the definitions of involutivity and integrability are valid for any distribution, the definition of complete integrability is not. Thus, in our study of general distributions, we only deal with the notion of an integrable distribution. In this case, we can introduce the following theorem that can already be considered to be the generalisation of Frobenius theorem for singular distributions:

Theorem 2.4.3 (Stefan-Sussmann)**.** *A smooth distribution is integrable if and only if there exists $\mathfrak{D} \subseteq \mathfrak{X}(M)$ such that $\mathcal{D}$ is a $\mathfrak{D}$-invariant distribution.*

Proof. $\Longrightarrow$) If $\mathcal{D}$ is an integrable distribution, then we can check that $\mathcal{D}$ is a $\Gamma(\mathcal{D})$-invariant distribution. In fact, taking

$$\mathfrak{D} \overset{\text{def}}{=} \Gamma(\mathcal{D}) = \{X \in \mathfrak{X}(M) \mid \operatorname{im}(X) \subseteq \mathcal{D}\} \tag{$*$}$$

in Definition 2.4.2, condition 1) is clearly satisfied, because $\mathcal{D}$ is smooth. For condition 2), since an arbitrary element of $\mathcal{D}_p$ can be written as $\sum_{\lambda\in\Lambda} a_\lambda\,(X_\lambda)_p$, with Λ a set of indices, $a_\lambda \in \mathbb{R}$, $\forall\,\lambda \in \Lambda$ and $X_\lambda \in \Gamma(\mathcal{D})$ a set of generators of $\mathcal{D}$, using the linearity of the derivative $(\mathrm{Fl}^t_X)_*(p)$, it is enough to show that

$$\{(\mathrm{Fl}^t_X)_*(p)(Y_p) \mid Y \in \Gamma(\mathcal{D})\} = \{Y_{\mathrm{Fl}^t_X(p)} \mid Y \in \Gamma(\mathcal{D})\}. \tag{$\dagger$}$$

Let $i : N \to M$ be an integral manifold of $\mathcal{D}$, let $p \in i(N)$ and $X \in \Gamma(\mathcal{D})$. Since $T_p i(N) = \operatorname{span}\{Y_p \mid Y \in \Gamma(\mathcal{D})\}$, $X_p \in T_p i(N)$ and $\mathrm{Fl}^t_X(p) \in i(N)$ for all t. Thus, because Fl^t_X is a local diffeomorphism, $(\mathrm{Fl}^t_X)_*(p) : T_p i(N) \to T_{\mathrm{Fl}^t_X(p)} i(N)$ is a local isomorphism and, so given $Y \in \Gamma(\mathcal{D})$ exists $Z \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ supported in a neighbourhood of p such that $(\mathrm{Fl}^t_X)_*(p)(Y_p) = Z_{\mathrm{Fl}^t_X(p)}$ (and, similarly, exists $Z' \in \Gamma_{\mathrm{loc}}(\mathcal{D})$ such that $Y_{\mathrm{Fl}^t_X(p)} = (\mathrm{Fl}^t_X)_*(p)(Z'_p)$). Finally, using bump functions, we can guarantee that booth Z, Z' are defined globally, showing the equality ($\dagger$).

$\Longleftarrow$) Given $p \in M$ and $k = \dim \mathcal{D}_p$, take $X_1, \ldots, X_k \in \mathfrak{D}$ such that $(X_1)_p, \ldots, (X_k)_p$ form a basis of $\mathcal{D}_p$. Let W be an open neighbourhood of $0 \in \mathbb{R}^k$ such that

$$\begin{aligned} \eta : W &\longrightarrow M \\ (t_1, \ldots, t_k) &\longmapsto (\mathrm{Fl}^{t_1}_{X_1} \circ \cdots \circ \mathrm{Fl}^{t_k}_{X_k})(p) \end{aligned}$$

is well-defined. Furthermore, because η is the composition of (local) integral curves, η is differentiable and

$$\begin{aligned}\eta_*(0)\left(\left.\frac{\partial}{\partial t_i}\right|_0\right) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big(\eta(0,\dots,0,s,0,\dots,0)\Big)\Big|_{s=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Big((\mathrm{Fl}^0_{X_1}\circ\dots\circ\mathrm{Fl}^0_{X_{i-1}}\circ\mathrm{Fl}^s_{X_i}\circ\mathrm{Fl}^0_{X_{i+1}}\circ\dots\circ\mathrm{Fl}^0_{X_k})(p)\Big)\Big|_{s=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Fl}^s_{X_i}(p)\big)\Big|_{s=0} = (X_i)_{\mathrm{Fl}^s_{X_i}(p)}\Big|_{s=0} = (X_i)_p\end{aligned}$$

therefore, $\eta_*(0)$ is injective, so, up to shrinking W around 0, we get that η is an immersion and, thus $\mathrm{im}(\eta)=\eta(W)$ is an immersed submanifold of M. Moreover, we get that

$$\begin{aligned}T_{\eta(t_1,\dots,t_k)}\big(\mathrm{im}(\eta)\big) &= \eta_*(t_1,\dots,t_k)(T_{(t_1,\dots,t_k)}W)\\ &= \eta_*(t_1,\dots,t_k)\left(\mathrm{span}\left\{\left.\frac{\partial}{\partial t_1}\right|_{(t_1,\dots,t_k)},\dots,\left.\frac{\partial}{\partial t_k}\right|_{(t_1,\dots,t_k)}\right\}\right)\\ &= \mathrm{span}\left\{\eta_*(t_1,\dots,t_k)\left(\left.\frac{\partial}{\partial t_i}\right|_{(t_1,\dots,t_k)}\right)\,\middle|\, i=1,\dots,k\right\}\\ &= \mathrm{span}\Big\{\big((\mathrm{Fl}^{t_1}_{X_1})_*\circ\dots\circ(\mathrm{Fl}^{t_i}_{X_i})_*\circ X_i\big)(\mathrm{Fl}^{t_{i+1}}_{X_{i+1}}\circ\dots\circ\mathrm{Fl}^{t_k}_{X_k}(p))\,\Big|\, i=1,\dots,k\Big\}\\ &\subseteq \mathcal{D}_{\eta(t_1,\dots,t_k)},\end{aligned}$$

where, to conclude the last inclusion, we used the hypothesis and condition 2) of Definition 2.4.2. Also by these conditions, we get that $\dim(\mathcal{D}_{\eta(t_1,\dots,t_k)})=\dim(\mathcal{D}_p)=k$, for all $t_1,\dots,t_k$. Thus, we conclude that $T_{\eta(t_1,\dots,_k)}(\mathrm{im}(\eta))=\mathcal{D}_{\eta(t_1,\dots,t_k)}$ and, therefore, $\mathrm{im}(\eta)$ is an integral manifold of $\mathcal{D}$, i.e. $\mathcal{D}$ is integrable. Q.E.D.

We may now rephrase the previous theorem to have a statement that more closely resembles that of the Frobenius theorem for regular distributions:

Theorem 2.4.4 (Generalised Frobenius). *A distribution $\mathcal{D}$ is integrable if and only if there exists a Lie subalgebra $\mathfrak{L}$ of $\mathfrak{X}(M)$ such that, for all $p\in M$:*

(i) $\mathcal{D}_p=\mathrm{span}\{X_p\mid X\in\mathfrak{L}\}$*;*

(ii) for all $X\in\mathfrak{L}$, we have that $\dim\mathcal{D}_{\mathrm{Fl}^t_X(p)}=\dim\mathcal{D}_p$ $(t\in\mathbb{R})$.

Proof. $\Longrightarrow$) Firstly, we have seen in Section 2.1 that any integrable distribution is involutive, so we know that $\Gamma(\mathcal{D})$ is a Lie subalgebra (Proposition 2.1.16). Therefore, we may take $\mathfrak{L}=\Gamma(\mathcal{D})$ and *(i)* is trivially satisfied.

To prove the second condition, similarly to the proof of Stefan-Sussmann theorem, we note that $\mathcal{D}$ is a $\Gamma(\mathcal{D})$-invariant distribution. Thus, if $\dim\mathcal{D}_p=k$, for any $X\in\Gamma(\mathcal{D})$, $p\in M$ and $t\in\mathbb{R}$ such that $p\in\mathrm{dom}(\mathrm{Fl}^t_X)$, we get that $(\mathrm{Fl}^t_X)_*(p):T_pM\to$

$T_{\mathrm{Fl}_X^t(p)}M$ is a linear isomorphism and $(\mathrm{Fl}_X^t)_*(p)(\mathcal{D}_p) = \mathcal{D}_{\mathrm{Fl}_X^t(p)}$. Therefore, $\dim(\mathcal{D}_p) = \dim(\mathcal{D}_{\mathrm{Fl}_X^t(p)})$, for all $X \in \Gamma(\mathcal{D})$, $p \in M$ and $t \in \mathbb{R}$ such that $p \in \mathrm{dom}(\mathrm{Fl}_X^t)$, proving condition *(ii)*.

$\Longleftarrow$) Let $p \in M$, $X \in \mathfrak{L}$ and Fl_X^t be the flow of X at time $t \in \mathbb{R}$. Then, once again, since $(\mathrm{Fl}_X^t)_*(p) : T_pM \to T_{\mathrm{Fl}_X^t(p)}M$ is a linear isomorphism, we get that $\dim \mathcal{D}_p = \dim(\mathrm{Fl}_X^t)_*\mathcal{D}_p$ and, by hypothesis *(ii)*, $\dim \mathcal{D}_{\mathrm{Fl}_X^t(p)} = \dim(\mathrm{Fl}_X^t)_*\mathcal{D}_p$. Since the fibres of $\mathcal{D}$ are vector spaces, to show that $(\mathrm{Fl}_X^t)_*(p)(\mathcal{D}_p) = \mathcal{D}_{\mathrm{Fl}_X^t(p)}$ it is enough to show one of the inclusions. Let us then show that

$$(\mathrm{Fl}_X^t)_*(p)(\mathcal{D}_p) \supseteq \mathcal{D}_{\mathrm{Fl}_X^t(p)},$$

or equivalently, since $(\mathrm{Fl}_X^t)_*(p) : T_pM \to T_{\mathrm{Fl}_X^t(p)}M$ is a linear isomorphism,

$$\mathcal{D}_p \supseteq (\mathrm{Fl}_X^{-t})_*(\mathrm{Fl}_X^t(p))(\mathcal{D}_{\mathrm{Fl}_X^t(p)}).$$

Take $Y_1, \ldots, Y_k \in \mathfrak{L}$ such that $\{(Y_1)_p, \ldots, (Y_k)_p\}$ is a basis of $\mathcal{D}_p$. By hypothesis *(ii)*, $\dim \mathcal{D}_{\mathrm{Fl}_X^t(p)} = k$, thus $\{(Y_1)_{\mathrm{Fl}_X^t(p)}, \ldots, (Y_k)_{\mathrm{Fl}_X^t(p)}\}$ is also a basis of $\mathcal{D}_{\mathrm{Fl}_X^t(p)}$ for $t \in\,]-\epsilon, \epsilon[$. Because $\mathfrak{L}$ is a Lie subalgebra, $\big([X, Y_i]\big)_{\mathrm{Fl}_X^t(p)} \in \mathcal{D}_{\mathrm{Fl}_X^t(p)}$, for all $t \in\,]-\epsilon, \epsilon[$ and, therefore, there exist $f_{ij} \in C^\infty(\ ]-\epsilon, \epsilon[\ )$ $(i, j = 1, \ldots, k)$ such that

$$\big([X, Y_i]\big)_{\mathrm{Fl}_X^t(p)} = \sum_{j=1}^{k} f_{ij}(t)(Y_j)_{\mathrm{Fl}_X^t(p)}.$$

Well, $\Big(\big((\mathrm{Fl}_X^t)^*Y_i\big)_p\Big)_{i=1,\ldots,k}$ is a smooth curve in $(T_pM)^{\times k}$ satisfying the linear first order ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((\mathrm{Fl}_X^t)^*Y_i\big)_p = (\mathrm{Fl}_X^t)^*\big([X, Y_i]\big)_p = \sum_{j=1}^{k} f_{ij}(t)\big((\mathrm{Fl}_X^t)^*Y_j\big)_p,$$

with initial value $\Big(\big((\mathrm{Fl}_X^0)^*Y_i\big)_p\Big)_{i=1,\ldots,k} = \big((Y_1)_p, \ldots, (Y_k)_p\big) \in \mathcal{D}_p$. Thus, by existence and uniqueness of solutions of ODE's, the integral curve $\Big(\big((\mathrm{Fl}_X^t)^*Y_i\big)_p\Big)_{i=1,\ldots,k}$ is also in $\mathcal{D}_p^{\times k}$, therefore $\big((\mathrm{Fl}_X^t)^*Y_i\big)_p \in \mathcal{D}_p$. Hence, we conclude that $\mathcal{D}_{\mathrm{Fl}_X^t(p)} \subseteq (\mathrm{Fl}_X^t)_*(p)(\mathcal{D}_p)$, finishing the proof. Q.E.D.

Remark 2.4.5. Following this version of the generalised Frobenius theorem, we may extend the notion of involutivity to include condition *(ii)* of Theorem 2.4.4. We, however. will stick with the notion of involutivity from Definition 2.1.14 and, when necessary, prove that a distribution has local constant rank, i.e. it satisfies condition *(ii)* of Theorem 2.4.4.

Remark 2.4.6. If the distribution $\mathcal{D}$ is regular, then condition *(ii)* of Theorem 2.4.4 is trivially satisfied. Therefore, we may take $\mathfrak{L} = \Gamma(\mathcal{D})$ and, therefore, the general Frobenius theorem reduces to its local version, Theorem 2.2.4.

From the procedure we described in Section 2.3, we could now obtain a global version of the generalised Frobenius theorem. However, as seen in Example 2.4.1, the integral submanifolds of a singular distribution need not be of the same dimension. Regardless, we still obtain a partition of the manifold by integral manifolds of maximal dimension of a singular distribution $\mathcal{D}$. To the collection of all of these submanifolds we shall still call the *integral foliation of* $\mathcal{D}$. This foliation is not in the same sense as in Definition 2.3.2 but in the sense of a singular foliation, which is defined the following way:

Definition 2.4.7. A *singular foliation* on a manifold M is a $C^\infty(M)$-submodule $\mathcal{F} \subseteq \mathfrak{X}(M)$ which is

1) closed under the Lie bracket, i.e. $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$;

2) locally finitely generated as a $C^\infty(M)$-module, i.e. there exists an open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of M and, for any $\lambda \in \Lambda$, vector fields $X_1, \dots, X_k \in \Gamma(U_\lambda, \mathcal{F})$, for some $k \in \mathbb{Z}_{\geqslant 0}$, such that
$$\Gamma(U_\lambda, \mathcal{F}) = C^\infty(U_\lambda)X_1 + \cdots + C^\infty(U_\lambda)X_k,$$
where $\Gamma(U_\lambda, \mathcal{F})$ represents the local sections of $\mathcal{F}$ supported by U_λ.

One can show that regular foliations, in the sense of Definition 2.3.2 are also singular foliations. To a singular distribution $\mathcal{F}$, we can always associate a tangent distribution, denoted by $T\mathcal{F}$:
$$T\mathcal{F} = \coprod_{p \in M} T_p\mathcal{F} = \coprod_{p \in M} \{X_p \mid X \in \mathcal{F}\} \subseteq TM.$$
This distribution is, in general, singular, and we obtain the following result:

Proposition 2.4.8. *For a manifold M, a singular distribution $\mathcal{D}$ in M is integrable if and only if there exists a singular foliation $\mathcal{F}$ such that $\mathcal{D} = T\mathcal{F}$.*

The study of singular foliations is still quite in its infancy, and many results that are valid for regular foliations are still unknown for the singular case. For the interested reader, we refer to [8, 9] and references therein.

Chapter 3

Lie Algebroids

As we have seen, a vector bundle can be thought of an extension of the idea of a vector space to a sort of family of vector spaces (one for each point of a manifold) ‘glued’ in a some smooth way. Similarly, we could think if there is an equivalent construction for the case of Lie algebras. The answer to this problem is a Lie algebroid. However, the construction of such a geometrical structure is more complicated that the one done for vector bundles and require a more careful treatment. These Lie algebroids came to be as the infinitesimal versions of Lie groupoids, similarly to how Lie algebras are the infinitesimal version of a Lie group and, also similarly to Lie algebras, can be entirely described independently of their global versions and, so, we shall focus on these Lie algebroids the present chapter. Beyond their usage in mathematics, Lie algebroids are also extremely important in several different areas of physics since the integrability problem for Lie algebroids can be relevant in the quantisation procedures of pre-symplectic manifolds, as sections of Lie algebroids can be seen as the Yang-Mills fields appearing in field theories.

The present chapter starts by discussing anchored vector bundles, i.e. vector bundles for which we may define a vector bundle morphism with codomain the tangent bundle of the base manifold. The present of this anchor adds additional structure and so certain concepts such as infinitesimal automorphisms, or the pull-back bundle have to be redefined to include this new vector bundle morphism. In the second section, we further introduce structure in anchored vector bundles, giving Lie algebroids and also study the morphisms for these algebroids as well as generalise the notion of the Schouten-Nijenhuis bracket to the setting of Lie algebroids.

3.1 Anchored Vector Bundles

In Section 1.2, we gave a brief overview of the theory of vector bundles as a way to extend the ideas of vector spaces to a manifold/bundle setting. However, the true interest of the study of vector bundles come from the innumerous constructions that arise from this type of fibre bundles. One such construction is of Anchored Vector bundles (which later serve as the basis for the definition of Lie algebroids) that we shall now introduce.

Definition 3.1.1. Let $E \xrightarrow{\mathrm{pr}} M$ be a vector bundle and $\mathsf{a} : E \to TM$ be a vector bundle morphism covering the identity map id_M. Then the pair $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ or, more usually, (E, a) is called an *anchored vector bundle* (AVB) over M and a is called the *anchor*.

Example 3.1.2. The simplest example of an anchored vector bundle is the tangent bundle TM to a manifold M. Clearly, $TM \xrightarrow{\tau_M} M$ is a vector bundle and we simply take the anchor as the identity $\mathsf{a} = \mathrm{id}_{TM}$. ♮

Example 3.1.3. If the anchor a is injective, then we may consider E to be a vector subbundle of TM. This implies that E is a regular tangent distribution. ♮

Example 3.1.4. For a Poisson manifold (M, π), the cotangent bundle T^*M is an anchored vector bundle with anchor $\mathsf{a} = \pi^\sharp : T^*M \to TM$. ♮

Example 3.1.5. Let V be a $\mathbb{K}$-vector space, A an associative $\mathbb{K}$-algebra and $\rho : A \to \mathrm{End}_{\mathbb{K}}(V)$ a representation of A on V. Then $(V \times A \xrightarrow{\mathrm{pr}_1} V, \mathsf{a})$ (with pr_1 the projection onto the first factor) is an anchored vector bundle with anchor $\mathsf{a}(v, a) = (v, \rho(a)(v)) \in TV$, where, for all $v \in V$, we identify $T_vV \cong V$. ♮

As usual in mathematical constructions, it is important to introduce the notion of maps between anchored vector bundles that preserve this structure:

Definition 3.1.6. A *morphism of anchored vector bundles* $(E \xrightarrow{\mathrm{pr}_E} M, \mathsf{a}_E)$ and $(F \xrightarrow{\mathrm{pr}_F} N, \mathsf{a}_F)$ is a vector bundle morphism (Φ, φ) from E to F such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\mathsf{a}_E"'] & F \arrow[d, "\mathsf{a}_F"] \\ TM \arrow[r, "T\varphi"'] & TN \end{tikzcd} \tag{3.1}$$

With this definition of a morphism of anchored vector bundles, we can conclude that the anchored vector bundles with their morphisms form a category, denoted **AVB**.

3.1.1 Anchored Vector Bundle (Infinitesimal) Automorphisms

The introduction of the anchor on a vector bundle allows us to not only link vector fields on a vector bundle to a vector field on the base manifold but also with its tangent lift. Firstly, we introduce the notion of an infinitesimal AVB automorphism:

Definition 3.1.7. Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be and anchored vector bundle. An infinitesimal anchored vector bundle automorphisms is vector field $X \in \mathfrak{X}(E)$ on E such that its flow is an automorphism of anchored vector bundles, $\mathrm{Fl}^t_X \in \mathrm{Aut}_{\mathbf{AVB}}(E)$, for all t. The collection of infinitesimal anchored vector bundle automorphisms is denoted by $\mathfrak{aut}_{\mathbf{AVB}}(E)$.

The next result is valid even in the case where we do not have an anchor (i.e. for a vector bundle) and links any vector bundle morphism of E to a diffeomorphism of M:

Lemma 3.1.8. *Let $\Phi \in \mathrm{Aut}_{\mathbf{VB}}(E)$ be an automorphism of a vector bundle $E \xrightarrow{\mathrm{pr}} M$. Then Φ restricts to a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ satisfying*

$$\mathrm{pr} \circ \Phi = \varphi \circ \mathrm{pr}.$$

Likewise, any infinitesimal vector bundle morphism $\hat{X} \in \mathfrak{aut}_{\mathbf{VB}}(E)$ restricts to a vector field $X \in \mathfrak{X}(M)$ such that $T\mathrm{pr} \circ \hat{X} = X \circ \mathrm{pr}$.

Proof. By the definition of a vector bundle morphism, $\Phi : E \to E$ (which is smooth) restricts to a smooth map $\varphi : M \to M$ satisfying $\mathrm{pr} \circ \Phi = \varphi \circ \mathrm{pr}$. Since pr is surjective $\Phi : E_p \to E_{\varphi(p)}$ is linear, φ is bijective if and only if Φ is bijective. Then, there exists the inverse φ^{-1} of φ which is smooth by the smoothness of Φ^{-1}.

For the case of $\hat{X} \in \mathfrak{aut}_{\mathbf{VB}}(E)$, $\mathrm{Fl}^t_{\hat{X}} \in \mathrm{Aut}_{\mathbf{VB}}(E)$ so, by the previous argument, for all t, there are $\varphi^t \in \mathrm{Diff}(M)$ such that $\mathrm{pr} \circ \mathrm{Fl}^t_{\hat{X}} \circ \mathrm{pr}$. Firstly, notice that φ^t define the flow of a vector field since, for any $e \in E$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\varphi^t \circ \mathrm{pr})(e) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{pr} \circ \mathrm{Fl}^t_{\hat{X}})(e) = \mathrm{pr}_*(\mathrm{Fl}^t_{\hat{X}}(e)) \left(\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Fl}^t_{\hat{X}}(e) \right) \\
&= \mathrm{pr}_*(\mathrm{Fl}^t_{\hat{X}}(e)) \Big(\hat{X}_{\mathrm{Fl}^t_{\hat{X}}(e)} \Big) = \big(\mathrm{pr}_* \hat{X}\big)_{(\mathrm{pr} \circ \mathrm{Fl}^t_{\hat{X}})(e)} \\
&= \big(\mathrm{pr}_* \hat{X}\big)_{(\varphi^t \circ \mathrm{pr})(e)}.
\end{aligned}$$

Calling $X = \mathrm{pr}_* \hat{X}$, the previous equation gives $\mathrm{pr}_* \circ \hat{X} = X \circ \mathrm{pr}$. By the uniqueness of solution of ODE's, since $\varphi^0 = \mathrm{id}_M = \mathrm{Fl}^0_X$ and both φ^t and Fl^t_X satisfy the same differential equation, we obtain that $\varphi^t = \mathrm{Fl}^t_X$. Q.E.D.

Moreover, to every infinitesimal vector bundle automorphism, we may introduce a derivation on E:

Definition 3.1.9. Let $\hat{X} \in \mathfrak{aut}_{\mathbf{VB}}(E)$ be an infinitesimal vector bundle automorphism restricting to a vector field $X \in \mathfrak{X}(M)$ on M. Then, the *derivation of* $\Gamma(E)$ *associated to* $\hat{X}$ is

$$\begin{aligned} D_{\hat{X}} : \Gamma(E) &\longrightarrow \Gamma(E) \\ \sigma &\longmapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{Fl}_{\hat{X}}^{-t} \circ \sigma \circ \mathrm{Fl}_X^t\Big)\Big|_{t=0}. \end{aligned}$$

By introducing the anchor, we obtain another link, now to the derivative of the base diffeomorphism:

Proposition 3.1.10. *Let* $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ *be an anchored vector bundle. If* $\hat{X} \in \mathfrak{aut}_{\mathbf{AVB}}(E)$ *restricts to* $X \in \mathfrak{X}(M)$*, the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "X"] \arrow[d, "\mathsf{a}"'] & TE \arrow[d, "T\mathsf{a}"] \\ TM \arrow[r, "T^{\natural}X"'] & T^2M \end{tikzcd}$$

Proof. By the same type of argument of Lemma 3.1.8, since the flow $\mathrm{Fl}_{\hat{X}}^t$ of $\hat{X}$ restricts to the flow Fl_X^t of X and $\mathrm{Fl}_{\hat{X}}^t$ is an anchored vector bundle automorphism, we get that $\mathsf{a} \circ \mathrm{Fl}_{\hat{X}}^t = T\mathrm{Fl}_X^t \circ \mathsf{a}$, by the definition of morphism of anchored vector bundles. Thus, noting that, by Proposition 1.2.51, $T\mathrm{Fl}_X^t = \mathrm{Fl}_{T^{\natural}X}^t$, differentiating both sides with respect to t at $t = 0$, we get

$$T\mathsf{a} \circ \hat{X} = T^{\natural}X \circ \mathsf{a}.$$

Q.E.D.

3.1.2 Pull-back of Anchored Vector Bundles

An important construction when studying differential geometry is that of the pullback by a map $\varphi : M \to N$ between manifolds M and N. However, the introduction of the anchor makes the previous condition of an induced bundle not enough. To extend the notion of a induced anchored bundle we recall the definition of transverse maps:

Definition 1.1.23. *One says that two maps* $f_1 : N_1 \to M$ *and* $f_2 : N_2 \to M$ *are* transverse to each other*, and denotes it by* $f_1 \pitchfork f_2$*, if, for all* $p_i \in N_i$ $(i = 1, 2)$*, with* $p = f_1(p_1) = f_2(p_2)$*, one gets that*

$$T_pM = (f_1)_*(p_1)(T_{p_1}N_1) + (f_2)_*(p_2)(T_{p_2}N_2). \tag{1.8}$$

Then, we can adapt this definition for the case in which one of the maps is the anchor of an AVB $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ as follows

Definition 3.1.11. Let (E, a) be an anchored vector bundle over M.

1) A smooth map $f : N \to M$ is *transverse to the anchor* a if for all $p \in N$:

$$f_*(p)(T_pN) + \mathsf{a}(E_{f(p)}) = T_{f(p)}M;$$

2) A submanifold N of M is *transverse to the anchor* a if the inclusion map $i : N \hookrightarrow M$ is transverse to the anchor, i.e. if for all $p \in N$,

$$T_pN + \mathsf{a}(E_p) = T_pM.$$

A submanifold transverse to the anchor is also called a *transversal for the anchored vector bundle* $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$.

Proposition 3.1.12. *Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be an anchored vector bundle and $\varphi : N \to M$ be a smooth map. Define a vector bundle morphism $\Phi : TN \oplus \varphi^*E \to \varphi^*TM$ covering the identity $\mathrm{id}_N : N \to N$ by setting (for all $p \in N$)*

$$\Phi\big(v_p, (p, e_{\varphi(p)})\big) = \big(p, \varphi_*(p)(v_p) - \mathsf{a}(e_{\varphi(p)})\big).$$

Then the map φ is transverse to the anchor a if and only if Φ is surjective in each fibre.

Proof. $\Longrightarrow$) Since Φ is a vector bundle morphism, it is fibre-wise linear, thus, to show that Φ is fibre-wise surjective it is enough to show that $\dim(\mathrm{im}(\Phi|_{T_pN\oplus E_{\varphi(p)}})) = \dim(T_{\varphi(p)}M)$, for all $p \in N$. Given the definition of the map Φ, we get that

$$\dim\big(\mathrm{im}(\Phi|_{T_pN\oplus E_{\varphi(p)}})\big) = \dim\big(\varphi_*(p)(T_pN) + \mathsf{a}(E_{\varphi(p)})\big) = \dim(T_{\varphi(p)}M),$$

where, in the second equality, we used the fact that φ is transverse to the anchor.

$\Longleftarrow$) Because Φ is fibre-wise surjective, we have that

$$\dim(T_{\varphi(p)}M) = \dim\big(\mathrm{im}(\Phi|_{T_pN\oplus E_{\varphi(p)}})\big) = \dim\big(\varphi_*(p)(T_pN) + \mathsf{a}(E_{\varphi(p)})\big).$$

Since $\varphi_*(p)(T_pN) + \mathsf{a}(E_{\varphi(p)}) \subseteq T_{\varphi(p)}M$, we conclude the result. Q.E.D.

Thus, we finally arrive at the definition of the induced anchored bundle:

Definition 3.1.13. Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be an anchored vector bundle and $\varphi : N \to M$ a smooth map transverse to the anchor a. The *anchored vector bundle induced by* φ or, simply the pull-back bundle, is

$$\begin{aligned} \varphi^!E &\overset{\mathrm{def}}{=} \big\{(v, (p, e)) \in TN \oplus \varphi^*E \,\big|\, \tau_N(v) = p \text{ and } \varphi_*(p)(v) = \mathsf{a}(e)\big\} \\ &= \ker\big(\Phi : TN \oplus \varphi^*E \to \varphi^*TM\big), \end{aligned} \tag{3.2}$$

with anchor $\varphi^!\mathsf{a} : \varphi^!E \to TN$ given by the projection onto the first factor.

Remark 3.1.14. The requirement of $\varphi : N \to M$ to be transverse to the anchor is to guarantee that $\varphi^! E$ is, in fact, a vector subbundle of $TN \oplus \varphi^* E$. Had we not imposed this condition on φ, analysing Definition 1.2.1, it would be possible not to have a unique topological space F such that, for any open set $U \subseteq N$, $(\varphi^! \mathrm{pr})^{-1}(U)$ is homeomorphic to $U \times F$ (essentially, for each U we could have a different topological space F_U satisfying this property, so $\varphi^! E$ is not a locally trivial fibration). The transversality then implies that $(\varphi^! \mathrm{pr})^{-1}(U) \cong TM|_{\varphi(U)} \cong \varphi(U) \times \mathbb{R}^m$, which solves this issue.

Example 3.1.15. Taking $E = TM$ we get

$$\begin{aligned}\varphi^! TM &= \big\{(v,(p,u)) \in TN \oplus \varphi^* TM \,\big|\, \tau_N(v) = p \text{ and } \varphi_*(p)(v) = u\big\}\\ &= \big\{(v,(p,\varphi_*(p)(v))) \in TN \oplus \varphi^* TM \,\big|\, \tau_N(v) = p\big\}\\ &\cong \{(v,\varphi_*(v)) \in TN \oplus TM|_{\varphi(N)}\} \cong TN,\end{aligned}$$

where, in the third equality we used the known relation that $\varphi^* TM \cong TM|_{\varphi(N)}$. ♮

Example 3.1.16. Taking $N \subseteq M$ to be a submanifold of M and $\varphi = \iota_N : N \hookrightarrow M$ to be the inclusion map, then

$$\begin{aligned}\iota_N^! E &= \big\{(v,(p,e)) \in TN \oplus \iota_N^* E \,\big|\, \tau_N(v) = p \text{ and } (\iota_N)_*(p)(v) = \mathsf{a}(e)\big\}\\ &= \{(\mathsf{a}(e),(p,e)) \in TN \oplus \iota_N^* E\} = \{(\mathsf{a}(e),(p,e)) \in TN \oplus E|_N\} \cong \mathsf{a}^{-1}(TN),\end{aligned}$$

where in the second equality we used that, for all $p \in N$, $(\iota_N)_*(p) = \iota_{TN}$, where $\iota_{TN} : TN \hookrightarrow TM$ is the inclusion of TN on TM and, in the third equality, that $\iota_N^* E = E|_N = \mathrm{pr}^{-1}(N)$. ♮

Example 3.1.17. Let $F \xrightarrow{\mathrm{pr}_F} M$ be a vector bundle and $\varphi = \mathrm{pr}_F$. Then, recalling that $\mathrm{pr}_F^* E = F \oplus E$ as a bundle over F, we obtain

$$\begin{aligned}\mathrm{pr}_F^! E &= \big\{(\phi,(f,e)) \in TF \oplus \mathrm{pr}_F^* E \,\big|\, \tau_F(\phi) = f,\ (\mathrm{pr}_F)_*(f)(\phi) = \mathsf{a}(e)\big\}\\ &= \big\{(\phi,e) \in TF \times E \,\big|\, \mathrm{pr}_F \circ \tau_F(\phi) = \mathrm{pr}_E(e),\ (\mathrm{pr}_F)_*\phi = \mathsf{a}(e)\big\}\end{aligned}$$

In particular, for $F = M \times N$ with N a smooth manifold, we get that

$$\begin{aligned}\mathrm{pr}_F^! E &= \big\{((u,v),e) \in (TM \times TN) \times E \,\big|\, \tau_M(u) = \mathrm{pr}_E(e),\ u = \mathsf{a}(e)\big\}\\ &= \{((\mathsf{a}(e),v),e) \in (TM \times TN) \times E\} \cong TN \times E.\end{aligned}$$ ♮

Definition 3.1.18. An anchored vector bundle $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ is said to be *involutive* if $\mathsf{a}(\Gamma(E)) \leq \mathfrak{X}(M)$ is a Lie subalgebra, i.e.

$$[\mathsf{a}(\Gamma(E)), \mathsf{a}(\Gamma(E))] \subseteq \mathsf{a}(\Gamma(E))$$

Proposition 3.1.19. *Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be an anchored vector bundle and $\varphi : N \to M$ a smooth map transverse to the anchor. If E is involutive, the anchored vector bundle $(\varphi^! E \to N, \varphi^! \mathsf{a})$ is involutive.*

Proof. As we have seen previously, the anchored vector bundle $\varphi^! E$ can be identified with $\ker(\Phi : TN \oplus \varphi^* E \to \varphi^* TM)$. So, to show that $\varphi^! E$ is involutive, let $\sigma_1, \sigma_2 \in \Gamma(\varphi^! E)$ and choose $\varsigma_1, \varsigma_2 \in \Gamma(TN \oplus \varphi^* E)$ to be extensions of σ_1, σ_2 to $TN \oplus \varphi^* E$, respectively.

Notice that $\Gamma(TN \oplus \varphi^* E) = \mathfrak{X}(N) \oplus \Gamma(\varphi^* E)$ and the anchor map of $TN \oplus \varphi^* E$ is just the projection onto the first factor, $\mathrm{proj}_1 : TN \oplus \varphi^* E \to TN$. Thus, given any section $s \in \Gamma(\varphi^* E)$, the section $[\mathrm{proj}_1(\varsigma_1), \mathrm{proj}_1(\varsigma_2)] + s \in \Gamma(TN \oplus \varphi^* E)$ is such that

$$\mathrm{proj}_1\big([\mathrm{proj}_1(\varsigma_1), \mathrm{proj}_1(\varsigma_2)] + s\big) = [\mathrm{proj}_1(\varsigma_1), \mathrm{proj}_1(\varsigma_2)],$$

i.e. $TN \oplus \varphi^* E$ is involutive. Restricting $[\mathrm{proj}_1(\varsigma_1), \mathrm{proj}_1(\varsigma_2)] + s$ to $\varphi^! E$ gives a section $[\varphi^! \mathsf{a}(\sigma_1), \varphi^! \mathsf{a}(\sigma_2)] + s'$ that gives satisfying the necessary properties. Q.E.D.

3.2 Lie Algebroids, morphisms and Isotropy Lie algebra

Now we shall properly introduce Lie algebroids.

Definition 3.2.1. A *Lie algebroid* over a manifold M is a triplet $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$, where $(A \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ is an anchored vector bundle and $[\cdot,\cdot]_A$ is a Lie algebra structure in the space of sections $\Gamma(A)$, subject to the following condition:

$$[\sigma, f\varsigma]_A = f[\sigma, \varsigma]_A + \mathsf{a}(\sigma)(f)\varsigma, \text{ for all } \sigma, \varsigma \in \Gamma(A) \text{ and } f \in C^\infty(M). \tag{3.3}$$

Proposition 3.2.2. *Given a Lie algebroid $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$, the map induced by the anchor $\mathsf{a} : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.*

Proof. Let us consider the following $\mathbb{R}$-bilinear anti-symmetric bracket:

$$\begin{aligned} \langle\cdot,\cdot\rangle : \Gamma(A) \times \Gamma(A) &\longrightarrow \Gamma(A) \\ (\sigma_1, \sigma_2) &\longmapsto [\mathsf{a}(\sigma_1), \mathsf{a}(\sigma_2)] - \mathsf{a}\big([\sigma_1, \sigma_2]_A\big) \end{aligned}$$

Then, using Eq. (3.3), for sections $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(A)$ and $f \in C^\infty(M)$, we get:

$$\begin{aligned} [\sigma_1, [\sigma_2, f\sigma_3]_A]_A &= [\sigma_1, f[\sigma_2, \sigma_3]_A]_A + [\sigma_1, (\mathsf{a}(\sigma_2)(f))\sigma_3]_A \\ &= f[\sigma_1, [\sigma_2, \sigma_3]_A]_A + \mathsf{a}(\sigma_1)(f)[\sigma_2, \sigma_3]_A + \mathsf{a}(\sigma_2)(f)[\sigma_1, \sigma_3]_A \\ &\qquad + \mathsf{a}(\sigma_1)(\mathsf{a}(\sigma_2)(f))\sigma_3, \\ [\sigma_2, [f\sigma_3, \sigma_1]_A]_A &= f[\sigma_2, [\sigma_3, \sigma_1]_A]_A + \mathsf{a}(\sigma_2)(f)[\sigma_3, \sigma_1]_A - \mathsf{a}(\sigma_1)(f)[\sigma_2, \sigma_3]_A \\ &\qquad - \mathsf{a}(\sigma_2)(\mathsf{a}(\sigma_1)(f))\sigma_3, \\ [f\sigma_3, [\sigma_1, \sigma_2]_A]_A &= f[\sigma_3, [\sigma_1, \sigma_2]_A]_A - \mathsf{a}([\sigma_1, \sigma_2]_A).(f)\sigma_3 \end{aligned}$$

So, adding the three expressions above, we get

$$
\begin{aligned}
&[\sigma_1,[\sigma_2,f\sigma_3]_A]_A+[\sigma_2,[f\sigma_3,\sigma_1]_A]_A+[f\sigma_3,[\sigma_1,\sigma_2]_A]_A\\
&\qquad\qquad=f\big([\sigma_1,[\sigma_2,\sigma_3]_A]_A+[\sigma_2,[\sigma_3,\sigma_1]_A]_A+[\sigma_3,[\sigma_1,\sigma_2]_A]_A\big)+\langle\sigma_1,\sigma_2\rangle(f)\sigma_3.
\end{aligned}
$$

By the Jacobi property of $[\cdot,\cdot]_A$ we conclude that $\langle\sigma_1,\sigma_2\rangle(f)\sigma_3=0$ and, since $\sigma_3\in\Gamma(A)$ and $f\in C^\infty(M)$ are arbitrary, we conclude that $\langle\sigma_1,\sigma_2\rangle=0$, for all $\sigma_1,\sigma_2\in\Gamma(A)$, i.e. $\mathsf{a}:\Gamma(A)\to\mathfrak{X}(M)$ is a Lie algebra homomorphism. Q.E.D.

The simplest examples of these structures follow from the similar cases for anchored vector bundles.

Example 3.2.3. The tangent bundle to a manifold M, TM, is clearly the case of a Lie algebroid taking the anchor to be the identity and the bracket structure to be the usual Lie bracket of vector fields. ♮

Example 3.2.4. As we shall see in Chapter 5, given a bivector $\pi\in\mathfrak{X}^2(M)$ satisfying $[\pi,\pi]_{\mathrm{SN}}=0$, the pair $(T^*M,\pi^\sharp)$, where $\pi^\sharp:T^*M\to TM$ is the induced map from π, is an anchored vector bundle. Moreover, the fact that $[\pi,\pi]_{\mathrm{SN}}=0$ allows us to define a bracket structure satisfying Eq. (3.3) in the space of 1-forms. Thus, for Poisson manifolds (which can be described by pairs (M,π) with $[\pi,\pi]_{\mathrm{SN}}=0$), the cotangent bundle is an example of a Lie algebroid. ♮

Example 3.2.5. One interesting example is that of a *Lie algebra bundle.* This object is a vector bundle $A\xrightarrow{\mathrm{pr}}M$ together with a bracket $[\cdot,\cdot]:\Gamma(A)\times\Gamma(A)\to\Gamma(A)$ such that, for every $p\in M$, there is an open neighbourhood U of p in M, a Lie algebra $\mathfrak{g}$ and a trivialisation $\varphi:\mathrm{pr}^{-1}(U)\to U\times\mathfrak{g}$ where $\varphi|_{A_p}:A_p\to\{p\}\times\mathfrak{g}$ is a Lie algebra isomorphism.

Similar to a Lie algebra bundle is a *field of Lie algebras.* The latter correspond to the case of a vector bundle $A\xrightarrow{\mathrm{pr}}M$ together with Lie algebra structures $[\cdot,\cdot]_p$ on each fibre A_p varying smoothly on M and such that, for every $\sigma_1,\sigma_2\in\Gamma(A)$, $[\sigma_1,\sigma_2]\in\Gamma(A)$ and $[\sigma_1,\sigma_2](p)=[\sigma_1(p),\sigma_2(p)]_p$. In the case where the manifold M is just a singleton $\{*\}$, then this structure 'collapses' and we get that a Lie algebra is also an example of a Lie algebroid.

Both examples above are cases of *intransitive Lie algebroids*, i.e. Lie algebroids with a vanishing anchor map. ♮

Lie algebroid morphisms

We now introduce the notion of morphisms in a Lie algebroid.

Definition 3.2.6. Let $(A, [\cdot,\cdot]_A, \mathsf{a}_A)$ and $(L, [\cdot,\cdot]_L, \mathsf{a}_L)$ be Lie algebroids over the same base manifold M. A *morphism of Lie algebroids over* M is an anchored vector bundle morphism $\Phi : A \to L$ covering the identity id_M that is also a Lie algebra homomorphism from $(\Gamma(A), [\cdot,\cdot]_A)$ to $(\Gamma(L), [\cdot,\cdot]_L)$. In more detail, $\Phi : A \to L$ is a *Lie algebroid morphism* if, for all $\sigma, \varsigma \in \Gamma(A)$, we have that $\Phi([\sigma,\varsigma]_A) = [\Phi(\sigma), \Phi(\varsigma)]_L$ and the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\mathsf{a}_A"'] & L \arrow[d, "\mathsf{a}_L"] \\ TM \arrow[r, "\mathrm{id}_{TM}"'] & TM \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\mathsf{a}_A"'] & L \arrow[d, "\mathsf{a}_L"] \\ M \arrow[r, "\mathrm{id}_{M}"'] & M \end{tikzcd}$$

For the case of a morphism between Lie algebroids over different base manifolds, the definition is not as straightforward, since it is not always possible for sections of A to be pushed forward to sections of L. Thus, the definition of these morphisms are usually expressed in two ways:

- Given a Lie algebroid A, by defining A-forms to be elements of $\Gamma(\bigwedge^\bullet A^*)$, we may introduce the A-differential, $\mathrm{d}_A : \Gamma(\bigwedge^\bullet A^*) \to \Gamma(\bigwedge^\bullet A^*)$ to be a graded 1 derivation of $\Gamma(\bigwedge^\bullet A^*)$, satisfying a similar expression to that of the forms on a manifold (1.18b). Then, a Lie algebroid morphism from $(A \xrightarrow{\mathrm{pr}_A} M, [\cdot,\cdot]_A, \mathsf{a}_A)$ to $(L \xrightarrow{\mathrm{pr}_L} N, [\cdot,\cdot]_L, \mathsf{a}_L)$ is an anchored vector bundle morphism $\Phi : A \to L$ covering a smooth map $\varphi : M \to N$ such that
$$\Phi^* \mathrm{d}_B = \mathrm{d}_A \Phi^*;$$
- Given $A \xrightarrow{\mathrm{pr}_A} M$ and $L \xrightarrow{\mathrm{pr}_L} N$ to be Lie algebroids, one defines the direct product algebroid $A \times L$ to be the anchored vector bundle $A \times L \to M \times N$ with anchor given by $\mathsf{a}_A \oplus \mathsf{a}_L : A \times L \to TM \oplus TN$, and such that
$$\begin{aligned} \Gamma(A) \oplus \Gamma(L) &\longrightarrow \Gamma(A \times L) \\ (\sigma, \varsigma) &\longmapsto \mathrm{pr}_A^* \sigma + \mathrm{pr}_L^* \varsigma. \end{aligned}$$
  Then, $\Phi : A \to L$ is a Lie algebroid morphism if $\mathrm{gr}(\Phi)$ is a Lie subalgebroid of $A \times L$, i.e. it is a vector subbundle $\mathrm{gr}(\Phi) \to P \subseteq M \times N$, such that $(\mathsf{a}_A \oplus \mathsf{a}_L)|_{\mathrm{gr}(\Phi)}$ takes values on TP and, given any sections $\sigma, \varsigma \in \Gamma(A \times L)$ such that $\sigma|_P, \varsigma|_P \in \mathrm{gr}(\Phi)$, then $[\sigma, \varsigma]_{A \times L}|_P \in \mathrm{gr}(\Phi)$.

By introducing this notion of morphisms of Lie algebroids, together with Lie algebroids, we now get a category **LA**. We shall not explore these concepts further so, the interested reader can find more about these results in, for instance, [10, 11, 12].

Isotropy Lie algebra and Characteristic Distribution of Lie algebroids

The introduction of a Lie algebra structure on the space of sections give the anchor special properties. In particular, the kernel and images of the anchor gives rise to interesting features, we now introduce.

Definition 3.2.7. Given a Lie algebroid $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$ and a point $p \in M$, the *isotropy Lie algebra of A at p* is $\mathfrak{g}_p(A) \overset{\mathrm{def}}{=} \ker(\mathsf{a}_p : A_p \to T_pM)$.

Proposition 3.2.8. *The isotropy Lie algebra at a point $p \in M$ of a Lie algebroid A has an induced Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_p(A)}$ such that, for any $\sigma, \varsigma \in \Gamma(A)$ with $\sigma(p), \varsigma(p) \in \mathfrak{g}_p(A)$,*

$$[\sigma, \varsigma]_A(p) = [\sigma(p), \varsigma(p)]_{\mathfrak{g}_p(A)}. \tag{3.4}$$

Proof. Given $\sigma, \varsigma \in \Gamma(A)$ with $\sigma(p), \varsigma(p) \in \mathfrak{g}_p(A)$ for $p \in M$, the Leibniz property of the Lie algebroid implies that, for $f \in C^\infty(M)$,

$$[\sigma, f\varsigma]_A(p) = f(p)[\sigma, \varsigma]_A(p). \tag{$*$}$$

Thus, we may define a bracket $[\cdot,\cdot]_{\mathfrak{g}_p(A)}$ in $\mathfrak{g}_p(A)$ such that

$$[\sigma(p), \varsigma(p)]_{\mathfrak{g}_p(A)} = [\sigma, \varsigma]_A(p).$$

By $(*)$, $[\cdot,\cdot]_{\mathfrak{g}_p(A)}$ is $\mathbb{R}$-linear and, by the properties of $[\cdot,\cdot]_A$, it is anti-symmetric and satisfies the Jacobi identity. Hence, $[\cdot,\cdot]_{\mathfrak{g}_p(A)}$ is a Lie bracket on $\mathfrak{g}_p(A)$. Q.E.D.

Remark 3.2.9. With the result of the previous proposition, we conclude that the isotropy Lie algebra $\ker(\mathsf{a}) = \mathfrak{g}(A)$ is a Lie algebra bundle, usually called the *isotropy Lie algebra bundle.*

As for the image of the anchor, we have the following:

Definition 3.2.10. Let $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$ be a Lie algebroid and $p \in M$ a point. The *characteristic distribution of A at p* is $\mathrm{im}(\mathsf{a}_p) \subseteq T_pM$.

Proposition 3.2.11. *If $(A, [\cdot,\cdot], \mathsf{a})$ is a Lie algebroid, then $\mathrm{im}(\mathsf{a})$ is an involutive distribution.*

Proof. It is clear that $\mathrm{im}(\mathsf{a})$ is a distribution, since a is a vector bundle morphism, which is linear on the fibres. To check that this distribution is involutive, we need only show that $\Gamma(\mathsf{a}(A))$ is a Lie subalgebra of $\mathfrak{X}(M)$.

Firstly, note that, for any vector field $X \in \Gamma(\mathsf{a}(A))$, we can always find a section $\sigma \in \Gamma(A)$ such that $X = \mathsf{a}(\sigma)$ (this section is not unique as it can differ by a section $\varsigma \in \Gamma(A)$ such that $\mathrm{im}(\varsigma) \subseteq \ker(\mathsf{a})$). Thus, $\mathsf{a}(\Gamma(A))$ is a generator of $\Gamma(\mathsf{a}(A))$ and it is enough to check that the former space is a Lie subalgebra of $\mathfrak{X}(M)$.

To check $\mathsf{a}(\Gamma(A)) \leq \mathfrak{X}(M)$, we take two sections $\sigma, \varsigma \in \Gamma(A)$. Then, since a is a Lie algebra homomorphism of Lie algebras, $[\mathsf{a}(\sigma), \mathsf{a}(\varsigma)] = \mathsf{a}([\sigma, \varsigma]_A)$. Thus $\mathsf{a}(\Gamma(A))$ is a Lie subalgebra of $\mathfrak{X}(M)$. Q.E.D.

Proposition 3.2.12. *If* $(A, [\cdot,\cdot], \mathsf{a})$ *is a* regular *Lie algebroid (i.e. if* a *has constant rank), then* $\operatorname{im}(\mathsf{a})$ *is integrable.*

Proof. By the previous proposition, we know that $\Gamma(\mathsf{a}(E))$ is a Lie subalgebra and, clearly, for all $p \in M$,

$$\mathsf{a}(A)_p = \operatorname{span}\{\mathsf{a}(\sigma)_p \in T_pM\}\sigma \in \Gamma(A) = \operatorname{span}\{X_p\}X \in \Gamma(\mathsf{a}(A)).$$

Moreover, since A is a vector bundle, the dimensions of the fibres are the same, thus, for every $\sigma \in \Gamma(A)$,

$$\dim(\mathsf{a}(A)_{\mathrm{Fl}^t_{\mathsf{a}(\sigma)}(p)}) = \dim(\mathsf{a}(A_{\mathrm{Fl}^t_{\mathsf{a}(\sigma)}(p)})) = \dim(\mathsf{a}(A_p)) = \dim(\mathsf{a}(A)_p),$$

where, in the second equality we used the fact that a has constant rank. Therefore, by the generalised Frobenius theorem 2.4.4, $\mathsf{a}(A)$ is integrable. Q.E.D.

Remark 3.2.13. The previous Proposition guarantees that, if $(A, [\cdot,\cdot], \mathsf{a})$ is a regular Lie algebroid, then M is foliated by immersed submanifolds $\mathcal{F}_A = \{\mathcal{F}_{A,\lambda} \mid \lambda \in \Lambda\}$ (Λ a set of indices) such that $T_p\mathcal{F}_{A,\lambda} = \operatorname{im} \mathsf{a}_p$, for all $p \in \mathcal{F}_{A,\lambda}$ and $\lambda \in \Lambda$. As we shall see in Chapter 6, this foliation gives us an equivalent way to describe Poisson manifolds.

Pull-Backs of Lie algebroids

Following the construction of the induced anchored bundle, it is possible to define the pull-back Lie algebroid by defining the Lie bracket structure on the induced anchored bundle.

Definition 3.2.14. Let $\varphi : N \to M$ be a smooth map and $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$ be a Lie algebroid. Then the Lie bracket structure in $\varphi^! A$ is defined, for $\sigma_1, \sigma_2 \in \Gamma(A)$ and $X_1, X_2 \in \mathfrak{X}(N)$, as

$$[(X_1, \sigma_1), (X_2, \sigma_2)]_{\varphi^! A} = ([X_1, X_2], [\sigma_1, \sigma_2]_A). \tag{3.5}$$

Remember that the elements of the pull-back anchored bundle are of the form $(v, \alpha) \in TN \times \varphi^* A$ such that $\mathsf{a}(\alpha) = \varphi_* v$ so that the sections are generated by elements of the form $(X, \sigma) \in \mathfrak{X}(N) \times \Gamma(A)$ with X φ-related to $\mathsf{a}(\sigma)$.

Schouten Brackets of Lie algebroids

Similarly to what was done in Section 1.4, given a Lie algebroid $(A \xrightarrow{\text{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$, because of the Lie algebra structure introduced in $\Gamma(A)$, it is possible to define a Schouten bracket (and thus a Gerstenhaber algebra) on the set of sections of the exterior powers of A.

Definition 3.2.15. The *Schouten-Nijenhuis bracket on a Lie algebroid* $(A \xrightarrow{\text{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$ is uniquely determined by

1) $[\cdot,\cdot]_{\text{SN}}$ endows $\Gamma(\bigwedge^\bullet A)[1]$ with a graded Lie algebra structure;

2) $[\cdot,\cdot]_{\text{SN}}$ is a graded derivation of $(\Gamma(\bigwedge^\bullet A), \wedge)$;

3) its action on degree 0 and degree 1 elements:

$$[\sigma, \varsigma]_{\text{SN}} = [\sigma, \varsigma]_A,\ [\sigma, f]_{\text{SN}} = \mathsf{a}(\sigma)(f),\ [f, g]_{\text{SN}} = 0,$$

for $\sigma, \varsigma \in \Gamma(A)$ and $f, g \in C^\infty(M)$.

Similar to the case of the multivector fields, we could now find a particular formal expression for the Schouten bracket of a Lie algebroid using derivations. However, unlike the case for multivector fields, such a expression does not allows us to recuperate the Schouten bracket of two sections of the exterior algebra of A. Thus, we introduce the following proposition:

Proposition 3.2.16. *Let $\bigwedge^\bullet \mathsf{a} : \bigwedge^\bullet A \to \bigwedge^\bullet TM$ be the map such that, for $\alpha \in \bigwedge^k A$ and $\eta_1, \ldots, \eta_k \in T^*M$,*

$$\big(\textstyle\bigwedge^\bullet \mathsf{a}(\alpha)\big)(\eta_1, \ldots, \eta_k) = \alpha(\eta_1 \circ \mathsf{a}, \ldots, \eta_k \circ \mathsf{a}).$$

Then, $\bigwedge^\bullet \mathsf{a}$ is a vector bundle morphism covering the identity id_M that preserves the exterior product and brackets in $\bigwedge^\bullet A$ and $\bigwedge^\bullet TM$.

Proof. The fact that $\bigwedge^\bullet \mathsf{a}$ is a vector bundle morphism covering the identity comes directly from the definition of the anchor a and of the construction of associated vector bundles.

As for the preservation of the exterior product, we note that, for $\sigma_1, \sigma_2 \in \Gamma(A)$ and $\eta_1, \eta_2 \in \Omega^1(M)$, by using the definition of $\bigwedge^\bullet \mathsf{a}$ and the fact that a is a vector bundle morphism, we get that

$$\begin{aligned}
(\textstyle\bigwedge^\bullet \mathsf{a})(\sigma_1 \wedge \sigma_2)(\eta_1, \eta_2) &= (\sigma_1 \wedge \sigma_2)(\eta_1 \circ \mathsf{a}, \eta_2 \circ \mathsf{a}) \\
&= \sum_{\tau \in S_2} (-1)^\tau (\eta_{\tau(1)} \circ \mathsf{a})(\sigma_1)((\eta_{\tau(2)} \circ \mathsf{a})(\sigma_2) \\
&= \sum_{\tau \in S_2} (-1)^\tau \eta_{\tau(1)}(\mathsf{a}(\sigma_1))\eta_{\tau(2)}(\mathsf{a}(\sigma_2)) \\
&= (\mathsf{a}(\sigma_1) \wedge \mathsf{a}(\sigma_2))(\eta_1, \eta_2).
\end{aligned}$$

For a general element of $\gamma(\bigwedge^\bullet A)$, let $p \in M$ such that $\dim A_p = k$ and $\{e_1, \ldots, e_k\}$ be a basis of A_p. Then, any $\sigma \in \Gamma(A)$ and $\vartheta \in \Gamma(\bigwedge^\ell A)$ can be written, at p, as

$$\sigma_p = \sum_{a=1}^{k} \sigma^a(p)\, e_a, \qquad \vartheta_p = \sum_{1 \le a_1 < \ldots < a_\ell \le k} \vartheta^{a_1 \cdots a_\ell}(p)\, e_{a_1} \wedge \ldots \wedge e_{a_\ell}.$$

Then, we simply get that

$$\begin{aligned}\big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(\vartheta_p) &= \sum_{1 \le a_1 < \ldots < a_\ell \le k} \vartheta^{a_1 \cdots a_\ell}(p)\, \big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(e_{a_1} \wedge \ldots \wedge e_{a_\ell}) \\ &= \sum_{1 \le a_1 < \ldots < a_\ell \le k} \vartheta^{a_1 \cdots a_\ell}(p)\big(\mathsf{a}_p(e_{a_1}) \wedge \ldots \wedge \mathsf{a}_p(e_{a_\ell})\big),\end{aligned}$$

so, given $\vartheta \in \Gamma(\bigwedge^\ell A)$ and $\varsigma \in \Gamma(\bigwedge^h A)$,

$$\begin{aligned}\big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(\vartheta_p \wedge \varsigma_p) &= \sum_{\substack{1 \le a_1 < \ldots < a_\ell \le k \\ 1 \le b_1 < \ldots < b_h \le k}} \vartheta^{a_1 \cdots a_\ell}(p) \varsigma^{b_1 \cdots b_h}(p)\, \big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(e_{a_1} \wedge \ldots \wedge e_{a_\ell} \wedge e_{b_1} \wedge \ldots \wedge e_{b_h}) \\ &= \sum_{\substack{1 \le a_1 < \ldots < a_\ell \le k \\ 1 \le b_1 < \ldots < b_h \le k}} \vartheta^{a_1 \cdots a_\ell}(p) \varsigma^{b_1 \cdots b_h}(p)\, (\mathsf{a}_p(e_{a_1}) \wedge \ldots \wedge \mathsf{a}_p(e_{a_\ell}) \wedge \mathsf{a}_p(e_{b_1}) \wedge \ldots \wedge \mathsf{a}_p(e_{b_h})) \\ &= \big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(\vartheta_p) \wedge \big(\textstyle\bigwedge^\bullet \mathsf{a}\big)_p(\varsigma_p).\end{aligned}$$

Finally, for the fact that $\bigwedge^\bullet \mathsf{a}$ preserves the bracket structures, since the Schouten bracket is a graded derivation of $(\Gamma(\bigwedge^\bullet A), \wedge)$ and its action of degree 1 elements is just the bracket $[\cdot,\cdot]_A$ of the Lie algebroid, we can use the local expression of elements of $\Gamma(\bigwedge^\bullet A)$ to conclude that

$$(\textstyle\bigwedge^\bullet a)\big([\vartheta, \varsigma]_{\mathrm{SN}}\big) = [(\textstyle\bigwedge^\bullet a)(\vartheta), (\textstyle\bigwedge^\bullet a)(\varsigma)]_{\mathrm{SN}}. \qquad \text{Q.E.D.}$$

Remark 3.2.17. As explained at the end of Section 1.4, $(\mathfrak{X}^\bullet(M), \wedge, [\cdot,\cdot]_{\mathrm{SN}})$ is the example of an Gerstenhaber algebra. With the introduction of this Schouten bracket to $\Gamma(\bigwedge^\bullet A)$, we obtain another class of examples of these algebras. Then, Proposition 3.2.16 says that $\bigwedge^\bullet \mathsf{a}$ is what is called an Gerstenhaber algebra homomorphism.

Chapter 4

Symplectic Manifolds

Since its conception in the XVI and XVII centuries with Newton, Galileo, Huygens and others, Mechanics has been deeply interconnected with geometric structures. This connection only rose with the inception of Analytical Mechanics with Euler, Lagrange, Hamilton and Jacobi one century later and gave birth to the way physics is done today. Naturally, it became essential to understand the mathematics involved in these physical theories. Symplectic geometry came as the solution to this search and has shown to be one of the most important branches of differential geometry and topology. Symplectic manifolds can be defined simply by adding a new structure to the tangent bundle of a manifold, a 2-form, which gave a simple way to understand all of Hamiltonian mechanics as the solution of the flow of a vector field associated with the Hamiltonian of the system (which is identified with the energy of that system).

We start this chapter by giving the physical motivation for the introduction of symplectic geometry. In the second section, we introduce the notions of (pre-)symplectic manifolds as well as we study the maps that preserve this structure and the most important types of submanifolds we may define in this geometry. The third and last section deals with the normal form theorems arising once we introduce a symplectic structure on a manifold, in particular, Darboux's theorem shows that every symplectic manifold is locally isomorphic to a canonical symplectic structure. To understand this theorem, one needs to use Moser's method which also involves the concept of time-dependent vector fields. All of these nuances are explored in this last section of the chapter.

4.1 Introduction and Motivation from Physics

As it is well-known, symplectic geometry arises from classical mechanics as a generalisation of the geometrical models employed to obtain the classical motion of particles. In this section, we intend to give a brief overview of how these concepts came to be and how they are applied to the physics of classical mechanics.

Let us consider a system of N particles moving in the so-called *configuration space* $Q \subseteq \mathbb{R}^{3N}$, which is a submanifold of the Euclidean space $\mathbb{R}^{3N}$. Then, the motion of the particles is obtained from a variational problem, i.e. they are the solutions of a system of ordinary differential equations (the *Euler-Lagrange equations*) which are obtained from the minimisation of the *action functional*:

$$S[q(t)] = \int_{[t_0,t_1]} L(q(t), \dot{q}(t), t)\, \mathrm{d}t, \tag{4.1}$$

where $\dot{q}(t)$ represents the time-derivative of the map $q(t) \in C^2(\mathbb{R}, Q)$ and $L(q(t), \dot{q}(t), t)$ is an at-least C^2 function, called the *Lagrangian* of the system.

Remark 4.1.1. Notice that the Lagrangian of the system $L(q, \dot{q}, t)$ can be thought as a map $L : TQ \times \mathbb{R} \to \mathbb{R}$ from velocity space (+ time) to the real numbers.

From the action functional, the equations of motion for the system are as follows:

Proposition 4.1.2. *Let $q(t) = (q^1(t), \dots, q^n(t)) \in Q$ be the position of a particle at a time $t \in \mathbb{R}$ with respect to some local chart on Q. The equations of motion for the particle are the Euler-Lagrange equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n \tag{4.2}$$

When the system satisfies the Legendre condition:

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0, \tag{4.3}$$

then, the Euler-Lagrange equations form a regular system of n 2^{nd}-order ordinary differential equations and it is possible to invert the equations

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \tag{4.4}$$

which define the *canonical momenta* of the system. For the case of symplectic and Poisson geometry, we will always consider that the Legendre condition is satisfied. Whenever such a condition is not satisfied, we are in the realm of Hamiltonian mechanics with constraints which can be solved using Dirac's theory of constraints. More abstractly, the Dirac theory of constraints gives rise to Dirac geometry, which we shall study in Chapter 7.

Performing a Legendre transformation, we obtain the *Hamiltonian of the system*,

$$H(q,p,t) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q,\dot{q},t), \tag{4.5}$$

and the Euler Lagrange equations give way to the *Hamilton canonical equations*,

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{4.6}$$

which are a set of $2n$ 1^{st}-order differential equations.

Remark 4.1.3. Since the Hamiltonian of the system is obtained from the Lagrangian of the system by a Legendre transformation, we may consider the pair (q,p) to be an element of the cotangent bundle T^*Q of Q, which is called the *phase space* of the system. This identification is what allows us to define symplectic manifolds in their generality and, as we shall see in more detail later, the symplectic manifolds built from the cotangent bundle are a special class of symplectic manifolds.

Once again, these equations arise from a variational problem, for which the action can be written as

$$S[q(t),p(t)] = \int_{[t_0,t_1]} \left(\sum_{i=1}^{n} p_i(t)\, \mathrm{d}q^i(t) - H(q(t),p(t),t)\, \mathrm{d}t \right). \tag{4.7}$$

Moreover, when considering $\xi = (q,p) \in T^*Q$ to be adapted coordinates on T^*Q associated to a chart of coordinates $q \in Q$, Eq. (4.6) can be written in a special way:

$$\dot{\xi} = -J\,\frac{\partial H}{\partial \xi}, \tag{4.8}$$

where

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}, \qquad \frac{\partial H}{\partial \xi} = \begin{pmatrix} \frac{\partial H}{\partial q^1} & \cdots & \frac{\partial H}{\partial q^n} & \frac{\partial H}{\partial p^1} & \cdots & \frac{\partial H}{\partial p^n} \end{pmatrix}^{\mathsf{T}}$$

and 0_n and I_n represent, respectively, the zero and identity $n \times n$ matrices. Recalling the study made in Section 1.1 of vector fields, defining a vector field associated to the Hamiltonian H by

$$X_H = -J\frac{\partial H}{\partial \xi}, \tag{4.9}$$

the Hamilton canonical equations essentially become the equations for the flow of this vector field, $\mathrm{Fl}^t_{X_H}(q_0,p_0)$ for initial conditions (q_0,p_0). This is, in essence, the motivation that lead to the introduction of symplectic manifolds. When shifting our attention to the mathematics side of this question, we are interested in solving this (and other) problem(s) for an arbitrary manifold M (in our previous discussion we would have taken $M = T^*Q$). Thus, we need to, somehow, transform Eq. (4.9) into an expression valid on M. That is done by converting

$\frac{\partial H}{\partial \xi}$ into $\mathrm{d}H$ and by changing $-J$ into some map $M \to TM \otimes TM \cong \mathrm{Hom}(T^*M, TM)$. It turns out that it is more natural to multiply both sides of (4.9) by J and deal with a map $\omega : M \to T^*M \otimes T^*M$ such that, for every $p \in M$, the induced map $\omega^\flat(p) : T_pM \to T_p^*M$ must be an isomorphism of vector spaces and $\omega_p \in T_p^*M \otimes T_p^*M$ must be anti-symmetric, by the properties of J. Thus, $\omega \in \Omega^2(M)$ and Eq. (4.9) becomes

$$i_{X_H}\omega = \mathrm{d}H.$$

Finally, the fact that the J does not change along the motion of the particle implies that ω does not change along flow lines, i.e. $\mathcal{L}_{X_H}\omega = 0$. Applying Cartan's formula, we conclude that $\mathrm{d}\omega = 0$, i.e. ω is a closed 2-form. Therefore, as we shall introduce in the next section, the natural way to introduce symplectic manifolds is by considering a pair (M, ω), where $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

Returning to the physics, in many mechanical problems, it is more important to know conserved quantities, i.e. quantities whose time-derivative is zero, rather than the solutions to the equations of motion. Thus, we may take a map $f : T^*Q \times \mathbb{R} \to A$ (where A can be, for instance, $\mathbb{R}$, Q — in general, any tensor of type $\binom{r}{s}$ of TQ, i.e. elements of $(TQ)^{\otimes r} \otimes (T^*Q)^{\otimes s}$) and check if its time-derivative evaluated at the motion vanishes. Sticking with the case when $A = \mathbb{R}$, we want to check if, given the Hamiltonian H and initial conditions (q_0, p_0),

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ (\mathrm{Fl}^t_{X_H} \times \mathrm{id}_{\mathbb{R}}))(q_0, p_0, t) = 0. \tag{4.10}$$

Expanding this expression, using chain rule and (4.6)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(f(\mathrm{Fl}^t_{X_H} \times \mathrm{id}_{\mathbb{R}}))(q_0, p_0, t) = \frac{\mathrm{d}}{\mathrm{d}t} f(q(t), p(t), t) &= \frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\mathrm{d}q_i}{\mathrm{d}t}\frac{\partial f}{\partial q^i} + \frac{\mathrm{d}p_i}{\mathrm{d}t}\frac{\partial f}{\partial p_i} \right) \\
&= \frac{\partial f}{\partial t} + \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} \right) \\
&= \frac{\partial f}{\partial t} + \{H, f\},
\end{aligned}$$

where we define the *Poisson bracket* of the maps H and f by

$$\{H, f\} \stackrel{\text{def}}{=} \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial f}{\partial p_i} \right). \tag{4.11}$$

Remark 4.1.4. In physics textbooks, it is more common for the definition of the Poisson bracket to have the opposite sign to Eq. (4.11). As we shall see, particularly when we study Poisson manifolds, it is more useful to define the Poisson brackets in this way as it makes simpler the discussion of these brackets in Chapters 5 and 6.

Thus, a map $f : T^*Q \to \mathbb{R}$ that does not depend (explicitly) on time is a conserved quantity if and only if its Poisson bracket with the Hamiltonian vanishes on $\mathrm{Fl}^t_{X_H}(q_0, p_0) =$

$(q(t), p(t)) \in T^*Q$, for all t. Moreover, the Poisson bracket, turns out to have the following properties:

(i) Anti-symmetry:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) = -\sum_{i=1}^{n} \left(\frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) = -\{g, f\}$$

(ii) Bilinearity: It is enough to show linearity on the first argument. Hence,

$$\begin{aligned} \{af_1 + bf_2, g\} &= \sum_{i=1}^{n} \left(\left(a\frac{\partial f_1}{\partial p_i} + b\frac{\partial f_2}{\partial p_1} \right) \frac{\partial g}{\partial q^i} - \left(a\frac{\partial f_1}{\partial q^i} + b\frac{\partial f_2}{\partial q^i} \right) \frac{\partial g}{\partial p_i} \right) \\ &= a\sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f_1}{\partial q^i} \frac{\partial g}{\partial p_i} \right) + b\sum_{i=1}^{n} \left(\frac{\partial f_2}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f_2}{\partial q^i} \frac{\partial g}{\partial p_i} \right) \\ &= a\,\{f_1, g\} + b\,\{f_2, g\} \end{aligned}$$

(iii) Jacobi Identity:

$$\begin{aligned} \{f, \{g, h\}\} &= \sum_{j=1}^{n} \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q^j} \left(\sum_{i=1}^{n} \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial h}{\partial p_i} \right) - \frac{\partial f}{\partial q^j} \frac{\partial}{\partial p_j} \left(\sum_{i=1}^{n} \frac{\partial g}{\partial p_i} \frac{\partial h}{\partial q^i} - \frac{\partial g}{\partial q^i} \frac{\partial h}{\partial p_i} \right) \\ &= \sum_{i,j=1}^{n} \frac{\partial f}{\partial p_j} \frac{\partial^2 g}{\partial q^j \partial p_i} \frac{\partial h}{\partial q^i} + \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial q^j \partial q^i} - \frac{\partial f}{\partial p_j} \frac{\partial^2 g}{\partial q^j \partial q^i} \frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^i} \frac{\partial^2 h}{\partial q^j \partial p_i} \\ &\quad - \frac{\partial f}{\partial q^j} \frac{\partial^2 g}{\partial p_j \partial p_i} \frac{\partial h}{\partial q^i} - \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_i} \frac{\partial^2 h}{\partial p_j \partial q^i} + \frac{\partial f}{\partial q^j} \frac{\partial^2 g}{\partial p_j \partial q^i} \frac{\partial h}{\partial p_i} + \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial q^i} \frac{\partial^2 h}{\partial p_j \partial p_i} \end{aligned}$$

$$\begin{aligned} \{g, \{h, f\}\} &= \sum_{i,j=1}^{n} \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial q^j \partial p_i} \frac{\partial f}{\partial q^i} + \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial p_i} \frac{\partial^2 f}{\partial q^j \partial q^i} - \frac{\partial g}{\partial p_j} \frac{\partial^2 h}{\partial q^j \partial q^i} \frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_j} \frac{\partial h}{\partial q^i} \frac{\partial^2 f}{\partial q^j \partial p_i} \\ &\quad - \frac{\partial g}{\partial q^j} \frac{\partial^2 h}{\partial p_j \partial p_i} \frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^j} \frac{\partial h}{\partial p_i} \frac{\partial^2 f}{\partial p_j \partial q^i} + \frac{\partial g}{\partial q^j} \frac{\partial^2 h}{\partial p_j \partial q^i} \frac{\partial f}{\partial p_i} + \frac{\partial g}{\partial q^j} \frac{\partial h}{\partial q^i} \frac{\partial^2 f}{\partial p_j \partial p_i} \end{aligned}$$

$$\begin{aligned} \{h, \{f, g\}\} &= \sum_{i,j=1}^{n} \frac{\partial h}{\partial p_j} \frac{\partial^2 f}{\partial q^j \partial p_i} \frac{\partial g}{\partial q^i} + \frac{\partial h}{\partial p_j} \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial q^j \partial q^i} - \frac{\partial h}{\partial p_j} \frac{\partial^2 f}{\partial q^j \partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial h}{\partial p_j} \frac{\partial f}{\partial q^i} \frac{\partial^2 g}{\partial q^j \partial p_i} \\ &\quad - \frac{\partial h}{\partial q^j} \frac{\partial^2 f}{\partial p_j \partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial h}{\partial q^j} \frac{\partial f}{\partial p_i} \frac{\partial^2 g}{\partial p_j \partial q^i} + \frac{\partial h}{\partial q^j} \frac{\partial^2 f}{\partial p_j \partial q^i} \frac{\partial g}{\partial p_i} + \frac{\partial h}{\partial q^j} \frac{\partial f}{\partial q^i} \frac{\partial^2 g}{\partial p_j \partial p_i} \end{aligned}$$

Adding the three terms on the left-hand side of the previous expressions and using the fact that f, g, h are at least C^2 functions, the terms in black, red, blue, violet, orange, lime, brown, light blue, pink, dark red and grey cancel out and we obtain that

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

(iv) Leibniz rule:

$$\{f, gh\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial gh}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial gh}{\partial p_i} \right) = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial p_i} \left(\frac{\partial g}{\partial q^i} h + g \frac{\partial h}{\partial q^i} \right) - \frac{\partial f}{\partial q^i} \left(\frac{\partial g}{\partial p_1 i} h + g \frac{\partial h}{\partial p_i} \right) \right]$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right) h + g \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial h}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial h}{\partial p_i} \right) = \{f, g\} h + g \{f, h\} .$$

Given a general manifold, the study of $\mathbb{R}$-bilinear maps $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying properties (i)–(iv) above on such a manifold yields Poisson geometry and is the subject of Chapters 5 and 6.

4.2 Symplectic forms, manifolds

Having introduced the physical motivation for the introduction of an additional anti- -symmetric structure on a manifold, we now look more closely at the actual mathematical construction of such a theory so that, whenever possible, we come back to connect with the physical world where we introduce a symplectic form in the phase space, i.e. the cotangent bundle of a particular configuration space.

Naturally, we start by introducing the relevant notion of a symplectic form and a symplectic manifold:

Definition 4.2.1. Let M be a smooth manifold and $\omega \in \Omega^2(M)$ a 2-form.

1) ω is called a *symplectic form on* M if it is closed, i.e. $\mathrm{d}\omega = 0$ and non-degenerate, i.e. for any $p \in M$, given $v \in T_pM$, if $\omega_p(v, u) = 0$, for all $u \in T_pM$, then $v = 0$;

2) a *symplectic manifold* is a pair (M, ω), where ω is a symplectic form.

Remark 4.2.2. If we relax the condition of non-degeneracy, i.e. if $\omega \in \Omega^2(M)$ is closed, the pair (M, ω) is called a *pre-symplectic manifold.*

Lemma 4.2.3. *If* (M, ω) *is a symplectic manifold, then the dimension of* M *is even.*

Proof. To prove that the dimension of M is even, we shall prove that the dimension of the tangent spaces T_pM must be even for every $p \in M$. Since T_pM is a vector space, let $(e_1, \ldots, e_n)$ be a basis of T_pM (and, let $n = \dim(T_pM)$). Then, we may write $\omega_p(u, v) = \mathbf{v}^\intercal \mathbf{J} \mathbf{u}$, where $\mathbf{J} \in M_n(\mathbb{R})$ is an $n \times n$ matrix with real entries and we identify $u = u_1 e_1 + \ldots + u_n e_n$ with the column vector $\mathbf{u} = (u_1 \ \cdots \ u_n)^\intercal$, with $\mathbf{v}^\intercal$ denoting the transpose of $\mathbf{v}$.

Since ω_p is anti-symmetric, it is clear that $\mathbf{J}$ is an anti-symmetric matrix. Thus,

$$\det(\mathbf{J}) = \det(\mathbf{J}^\intercal) = \det(-\mathbf{J}) = (-1)^n \det(\mathbf{J}),$$

i.e. either $\det(\mathbf{J}) = 0$ or n is even. However, the former cannot occur otherwise we would be able to find $u \in T_pM$ such that $\mathbf{Ju} = 0$, i.e. $\omega(u, v) = \mathbf{v}^\intercal \mathbf{Ju} = 0$, for all $v \in T_pM$, contradicting the non-degeneracy of ω_p. Hence, we conclude that n is even. Q.E.D.

Example 4.2.4. The simplest case of a symplectic manifold occurs by taking $M = \mathbb{R}^{2s}$, consider the coordinates $\{x_1, \dots, x_s, y_1, \dots, y_s\}$ and define the canonical symplectic form:

$$\omega_{\text{can}} = \sum_{i=1}^{s} \mathrm{d}x_i \wedge \mathrm{d}y_i. \tag{4.12}$$

Then, one can quickly show that ω_{can} is closed and non-degenerate. ♮

Example 4.2.5. Given any two-dimensional oriented submanifold $M \subseteq \mathbb{R}^3$, considering $\nu(p)$ to be the normal vector to M at p, we may define a symplectic form on M by

$$\omega_p(u, v) = \nu(p) \cdot (u \times v), \text{ for all } u, v \in T_pM$$

where $\cdot$ and $\times$ denote the dot and vector products, respectively. The fact that ω is closed comes directly from the fact that $\mathrm{d}\omega \in \Omega^3(M)$ and $\dim(M) = 2$. As for the non-degeneracy, if $u \in T_pM$ is such that $\omega_p(u, v) = 0$, for all $v \in T_pM$, then, by the properties of the dot product, this means that $\nu(p)$ and $u \times v$ are perpendicular, i.e. $u \times v \in T_pM$, for all $v \in T_pM$. However, $\nu(p)$ is parallel to $u \times v$ by the properties of the vector product. Thus $u \times v = 0$ for all $v \in T_pM$ and, therefore, u is parallel to all $v \in T_pM$. So, we conclude that $u = 0$. ♮

Example 4.2.6. The example that is most important for a physicist is the case where the symplectic manifold is the cotangent bundle T^*Q of the configuration space Q. In this bundle, we define the *Liouville 1-form* $\lambda \in \Omega^1(T^*Q)$ as

$$\lambda_{p_q}(v) = p_q\big((\hat{\tau}_Q)_*(p_q)(v)\big), \tag{4.13}$$

where $q \in Q$, $p_q \in T_q^*Q$ and $v \in T_{p_q}(T^*Q)$.

Theorem 4.2.7. *The pair $(T^*Q, -\mathrm{d}\lambda)$, where Q is a smooth manifold and $\lambda \in \Omega^1(T^*Q)$ its Liouville 1-form, is a symplectic manifold.*

Proof. Clearly, $-\mathrm{d}\lambda$ is closed, thus we need only show that $-\mathrm{d}\lambda$ is non-degenerate. Let $(U, \Phi = (x^1, \dots, x^m, y_1, \dots, y_m))$ be a chart of T_q^*Q containing p_q. Then,

$$\left\{ \frac{\partial}{\partial x^1}, \dots \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_m} \right\} \quad \text{and} \quad \left\{\mathrm{d}x^1, \dots, \mathrm{d}x^m, \mathrm{d}y_1, \dots, \mathrm{d}y_m\right\}$$

are local frames of $T(T^*Q)|_U$ and $T^*(T^*Q)|_U$, respectively. Without loss of generality, we may assume there is a chart (V, φ) of Q such that the following diagram commutes:

$$\begin{tikzcd} T^*Q \supseteq U \arrow[r, "\Phi"] \arrow[d, "\hat{\tau}_Q"'] & \mathbb{R}^{2m} \arrow[d] & (x,y) \arrow[d, mapsto] \\ Q \supseteq V \arrow[r, "\varphi"'] & \mathbb{R}^m & x \end{tikzcd}$$

Moreover, there exist functions a_i, $b^i \in C^\infty(U)$ such that

$$\lambda|_U = \sum_{i=1}^m a_i \, \mathrm{d}x^i + b^i \, \mathrm{d}y_i.$$

Thus, for a local vector field $V = \sum_{i=1}^m v^i \frac{\partial}{\partial x^i} + \upsilon_i \frac{\partial}{\partial y_i} \in \mathfrak{X}(U)$ and $\Phi(p_q) = (q^1, \ldots, q^m, p_1, \ldots, p_m)$ or, equivalently, $p_q = \sum_{i=1}^m p_i (\mathrm{d}x^i)_{x^j = q^j} \in T_q^*Q$,

$$\begin{aligned}\lambda_{p_q}(V_{p_q}) &= \sum_{i=1}^m a_i(p_q) v^i(p_q) + b^i(p_q)\upsilon_i(p_q) \\ &= p_q\big((\hat{\tau}_Q)_*(p_q)(V_{p_q})\big) = p_q\left(\sum_{i=1}^m v^i(p_q) \left.\frac{\partial}{\partial x^i}\right|_{p_q}\right) = \sum_{i=1}^m p_i \, v^i(p_q).\end{aligned}$$

Thus, by successively choosing only one of v^i or υ_i to be non-zero, we conclude that $\lambda|_U = \sum_{i=1}^m y_i \, \mathrm{d}x^i$. Therefore, $\mathrm{d}\lambda|_U = \sum_{i=1}^m \mathrm{d}y_i \wedge \mathrm{d}x^i$, i.e. in this chart, $-\mathrm{d}\lambda|_U$ is the canonical symplectic form on $\Phi(U)$, which is non-degenerate. Q.E.D.

Moreover, the Liouville 1-form satisfies a characteristic property:

Proposition 4.2.8. *Let Q be a smooth manifold and $\lambda \in \Omega^1(T^*Q)$ a 1-form on its cotangent bundle. Then, λ is the Liouville 1-form of Q if and only if, for all $\eta \in \Omega^1(Q)$, $\eta^*\lambda = \eta$.*

Proof. Let $q \in Q$ and $v \in T_qQ$. Then, by taking a 1-form $\eta \in \Omega^1(Q)$ and treating it as a map $\eta : Q \to M = T^*Q$ (with the additional property $\hat{\tau}_Q \circ \eta = \mathrm{id}_Q$), it makes sense to apply the tangent, cotangent and section functors:

$$\begin{tikzcd}
TQ \arrow[d, "T\eta = \eta_*"'] & Q \arrow[l, "T"'] \arrow[r, "\Gamma \circ T^*"] \arrow[d, "\eta"] & \Omega^1(Q) \\
T(T^*Q) & T^*Q \arrow[l, "T"] \arrow[r, "\Gamma \circ T^*"'] & \Omega^1(T^*Q) \arrow[u, "\Gamma(T^*\eta) = \eta^*"']
\end{tikzcd}$$

Thus, we obtain,

$$\begin{aligned}(\eta^*\lambda)_q(v) &= \lambda_{\eta_q}(\eta_*(q)(v)) = \eta_q\big((\hat{\tau}_Q)_*(\eta_q)(\eta_*(q)(v))\big) = \eta_q\big((\hat{\tau}_Q \circ \eta)_*(q)(v))\big) \\ &= \eta_q\big((\mathrm{id}_Q)_*(q)(v))\big) = \eta_q(v),\end{aligned}$$

which proves the statement. Q.E.D.

In the rest of the chapter, due to its importance, we shall return to this example whenever possible. ♮

Given two symplectic manifolds it is also possible to define a symplectic manifold from the product of the original two manifolds:

Proposition 4.2.9. *Let* (M_1, ω_1) *and* (M_2, ω_2) *be symplectic manifolds and* $M = M_1 \times M_2$ *be the product manifold. Then the 2-form* $\Omega \in \Omega^2(M)$ *defined by*

$$\Omega = \lambda_1\, \pi_1^*\omega_1 + \lambda_2\, \pi_2^*\omega_2 \tag{4.14}$$

where $\pi_i : M \twoheadrightarrow M_i$ $(i = 1, 2)$ *is the canonical projection, is a symplectic form in* M*, for any* $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$.

Proof. The fact that Ω is a 2-form on M is clear. Also, given that both ω_1, ω_2 are symplectic forms, we have that

$$\mathrm{d}\Omega = \mathrm{d}(\lambda_1\, \pi_1^*\omega_1 + \lambda_2\, \pi_2^*\omega_2) = \lambda_1\, \pi_1^*\mathrm{d}\omega_1 + \lambda_2\, \pi_2^*\mathrm{d}\omega_2 = 0.$$

As for the non-degeneracy, let $p = (p_1, p_2) \in M$ and $v \in T_pM$ such that $\Omega_p(v, u) = 0$, for all $u \in T_pM$. Since $T_pM \cong T_{p_1}M_1 \oplus T_{p_2}M_2$, we may write $v = v_1 + v_2$ and $u = u_1 + u_2$ for $v_i, u_i \in T_{p_i}M_i$ $(i = 1, 2)$ and obtain

$$\Omega_p(v, u) = \lambda_1(\pi_1^*\omega_1)_p(v, u) + \lambda_2(\pi_2^*\omega_2)_p(v, u) = \lambda_1(\omega_1)_{p_1}(v_1, u_1) + \lambda_2(\omega_2)_{p_2}(v_2, u_2).$$

Given that this expression is valid for all $u \in T_pM$, in particular it is valid for the cases when $u_1 = 0$ and $u_2 = 0$. Thus, by the non-degeneracy of ω_1 and ω_2,

$$\begin{aligned} u_1 = 0 &\implies \lambda_2(\omega_2)_{p_2}(v_2, u_2) = 0,\ \forall\, u_2 \in T_{p_2}M_2 \implies v_2 = 0 \text{ or } \lambda_2 = 0 \\ u_2 = 0 &\implies \lambda_1(\omega_1)_{p_1}(v_1, u_1) = 0,\ \forall\, u_1 \in T_{p_1}M_1 \implies v_1 = 0 \text{ or } \lambda_1 = 0 \end{aligned}$$

Therefore, because $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$, we get that $v = v_1 + v_2 = 0 \in T_pM$. Q.E.D.

Proposition 4.2.10. *Any symplectic manifold* (M, ω) *determines a vector bundle isomorphism between* TM *and* T^*M *by*

$$\begin{aligned} \omega^\flat : TM &\xrightarrow{\cong} T^*M \\ v &\longmapsto i_v\omega \end{aligned} \tag{4.15}$$

where i_v *is the interior product defined in Definition 1.1.38.*

Proof. Given the definition of $\omega^\flat$, it is clear that it is a vector bundle morphism covering the identity id_M. To check that it is in fact an isomorphism, it is enough to show that $\omega_p^\flat : T_pM \to T_p^*M$ is a vector isomorphism, for all $p \in M$. However,

$$\ker(\omega_p^\flat) = \big\{v \in T_pM \,\big|\, \omega_p^\flat(v) = 0\big\} = \big\{v \in T_pM \,\big|\, \omega_p(v, u) = 0,\ \forall\, u \in T_pM\big\} = \{0\},$$

where, in the last equality, we used the fact that ω is a symplectic form, thus, is non--degenerate, so $\omega_p^\flat$ is injective. To conclude the surjectivity of $\omega_p^\flat$, we simply note that $\mathrm{im}(\omega_p^\flat) \subseteq T_p^*M$ and

$$\dim(T_p^*M) = \dim(\mathrm{im}(\omega_p^\flat)) + \dim(\ker(\omega_p^\flat)) = \dim(\mathrm{im}(\omega_p^\flat)).$$

Hence, we conclude that $\mathrm{im}(\omega_p^\flat) = T_p^*M$ and, therefore, that $\omega^\flat$ is a vector bundle isomorphism. Q.E.D.

Definition 4.2.11. A map $\varphi : M_1 \to M_2$ between symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is called a *symplectic map* or *symplectic morphism* if

$$\varphi^*\omega_2 = \omega_1.$$

If, additionally, φ is a diffeomorphism, we call $\varphi : M_1 \to M_2$ a *symplectomorphism* or *symplectic isomorphism.*

Remark 4.2.12. By introducing the notion of a symplectic morphism, we are able to define the category of symplectic manifolds, **SympMan** whose objects are symplectic manifolds (M, ω) and morphisms are, precisely, symplectic morphisms $\varphi : M_1 \to M_2$.

An interesting result pertaining to the symplectic structure of the tangent bundle is the following:

Proposition 4.2.13. *Let Q_1, Q_2 be smooth manifolds and let $\lambda_1 \in \Omega^1(T^*Q_1)$ and $\lambda_2 \in \Omega^1(T^*Q_2)$ be the Liouville 1-forms on T^*Q_1 and T^*Q_2, respectively. Then, if $\varphi : Q_1 \to Q_2$ is a (local) diffeomorphism, its cotangent lift $T^*\varphi : T^*Q_1 \to T^*Q_2$ is a symplectic morphism between $(T^*Q_1, -\mathrm{d}\lambda_1)$ and $(T^*Q_2, -\mathrm{d}\lambda_2)$.*

Proof. Recall that, given $q \in Q_1$ and $p_q \in T_q^*Q_1$, $T_q^*\varphi(p_q) = p_q \circ \varphi_*^{-1}(\varphi(q)) \in T_{\varphi(q)}Q_2$, i.e. $T^*\varphi$ is a vector bundle morphism (between the cotangent bundles of Q_1 and Q_2) covering $\varphi : Q_1 \to Q_2$. Therefore, for $v \in T_{p_q}(T_q^*Q_1)$, we obtain

$$\begin{aligned}
((T^*\varphi)^*\lambda_2)_{p_q}(v) &= (\lambda_2)_{T^*\varphi(p_q)}\big((T^*\varphi)_*(p_q)(v)\big) \\
&= (T^*\varphi(p_q))\Big((\hat{\tau}_{Q_2})_*(T^*\varphi(p_q))\big((T^*\varphi)_*(p_q)(v)\big)\Big) \\
&= \big(p_q \circ \varphi_*^{-1}(\varphi(q))\big)\Big((\hat{\tau}_{Q_2} \circ T^*\varphi)_*(p_q)(v)\Big) \\
&= p_q\Big(\varphi_*^{-1}(\varphi(q))\big((\varphi \circ \hat{\tau}_{Q_1})_*(p_q)(v)\big)\Big) \\
&= p_q\Big(\varphi_*^{-1}(\varphi(q)) \circ \varphi_*(q) \circ (\hat{\tau}_{Q_1})_*(p_q)(v)\Big) \\
&= p_q\big((\hat{\tau}_{Q_1})_*(p_q)(v)\big) = (\lambda_1)_{p_q}(v).
\end{aligned}$$

Thus, $(T^*\varphi)^*\lambda_2 = \lambda_1$ and so $(T^*\varphi)^*(-\mathrm{d}\lambda_2) = -\mathrm{d}\lambda_1$, i.e. $T^*\varphi$ is a symplectic morphism. Q.E.D.

4.2.1 Submanifolds of a Symplectic Manifold

As noted in Section 1.1, beyond understanding the structures of certain manifolds, it is also quite important to understand the submanifolds and the induced structures obtained in those submanifolds. In the case of symplectic manifolds, unfortunately, a general submanifold does not inherit a symplectic structure but a pre-symplectic one.

Proposition 4.2.14. *Let (M, ω) be a symplectic manifold and $i : N \to M$ and immersed submanifold. Then $(N, i^*\omega)$ is a pre-symplectic manifold.*

Proof. To show the result, it is enough to show that $i^*\omega \in \Omega^2(M)$ is closed. By the properties of the exterior derivative and pull-back, it is easy to show that, since ω is symplectic, it is closed, hence,

$$\mathrm{d}(i^*\omega) = i^*\mathrm{d}\omega = 0.$$

Therefore, $i^*\omega \in \Omega^2_{\mathrm{cl}}(N)$ and $(N, i^*\omega)$ is a pre-symplectic manifold. Q.E.D.

Remark 4.2.15. Using the same notations as above, we can clearly see that, in general, $i^*\omega$ is not non-degenerate. Let $q \in N$ and $v \in T_qN$ such that $(i^*\omega)_p(v, w) = 0$, for all $w \in T_qN$. Then, by the definition of the pull-back, we get that

$$\omega_{i(q)}\big(i_*(q)(v), i_*(q)(w)\big) = 0, \quad \text{for all } w \in T_qN.$$

Since $i : N \to M$ is an immersion, $\dim(\mathrm{im}(i_*(q))) = \dim(N)$ which, in general, is different from $m = \dim(M)$. Thus, there could be $u \in T_{i(q)}M$ such that $\omega_{i(q)}\big(i_*(q)(v), u\big) \neq 0$, meaning we cannot use the non-degeneracy of ω to guarantee the non-degeneracy of $i^*\omega$. One such way to guarantee the non-degeneracy would be if $i_*(q) : T_qN \to T_{i(q)}M$ was surjective, for all $q \in N$, i.e. if we guarantee that $i : N \to M$ is a local diffeomorphism.

To understand the special kinds of embedded manifolds, we must first study the special types of vector subspaces that arise in linear symplectic structures. These will then be used to define certain embedded submanifolds on a symplectic manifold (M, ω).

Definition 4.2.16. Let (M, ω) be a symplectic manifold and $V \leq T_pM$ a vector subspace of the tangent space at a point p. The *symplectic orthogonal to V* is the vector subspace $V^{\perp_{\omega_p}}$ defined as

$$V^{\perp_{\omega_p}} \stackrel{\text{def}}{=} \big\{u \in T_pM \,\big|\, \omega_p(u, v) = 0, \ \forall\, v \in V\big\}. \tag{4.16}$$

Remark 4.2.17. For the previous definition, instead of choosing a symplectic manifold (M, ω), we could have simply taken a vector space W equipped with a non-degenerate 2-form $\omega \in \bigwedge^2 W^*$. The pair (W, ω) is then called a *symplectic vector space*. Thus, to define the symplectic orthogonal, one just considers a vector subspace $V \leq W$ and define $V^{\perp_\omega} = \{u \in W \,|\, \omega(u, v) = 0, \ \forall\, v \in V\}$. This approach is a simplification of our definition by taking $M = W$ and $\omega \in \Omega^2(W)$ to be constant, i.e. such that

$$\begin{aligned} \omega : W &\longrightarrow \textstyle\bigwedge^2 T^*W \cong W \times \bigwedge^2 W^* \\ p &\longmapsto (p, \varpi), \end{aligned}$$

with $\varpi \in \bigwedge^2 W^*$ non-degenerate.

This definition is essential to define the four distinct types of vector subspaces that will be essential for the different types of submanifolds we shall define in a symplectic manifold. Before that, though, we consider the following result:

Proposition 4.2.18. *Let (M, ω) be a symplectic manifold and $V \leq T_pM$ be a vector subspace for a given $p \in M$. Then,*

(i) $\dim(V) + \dim(V^{\perp_{\omega_p}}) = \dim(M)$*;*

(ii) $(V^{\perp_{\omega_p}})^{\perp_{\omega_p}} = V$*.*

Proof. *(i)* Firstly, note that

$$\omega_p^\flat(V^{\perp_{\omega_p}}) = \{\omega_p^\flat(u) \in T_p^*M \,|\, \omega_p(u, v) = 0, \ \forall\, v \in V\} \subseteq V^\circ,$$

where $V^\circ \subseteq T_p^*M$ is the annihilator of V in T_pM. Since $\omega_p^\flat$ is an isomorphism by Proposition 4.2.10, for every $\alpha \in V^\circ$, there is $u \in T_pM$ such that $\alpha = \omega_p^\flat(u)$ and, moreover, because $\alpha \in V^\circ$, for every $v \in V$, $\omega_p(u, v) = \omega_p^\flat(u)(v) = \alpha(v) = 0$, i.e. $u \in V^{\perp_{\omega_p}}$. Therefore, $V^\circ = \omega_p^\flat(V^{\perp_{\omega_p}})$. It is common knowledge that $\dim(V^\circ) = \dim(T_pM) - \dim(V)$, thus, we get the result.

(ii) By *(i)*, we have that

$$\dim(V^{\perp_{\omega_p}}) + \dim((V^{\perp_{\omega_p}})^{\perp_{\omega_p}}) = \dim(M) = \dim(V) + \dim(V^{\perp_{\omega_p}}),$$

so, $\dim((V^{\perp_{\omega_p}})^{\perp_{\omega_p}}) = \dim(V)$. Moreover, given $u \in V$ and $v \in V^{\perp_{\omega_p}}$, $\omega_p(u, v) = 0$ since, v is in the symplectic orthogonal of V. But, because this expression is valid for any $v \in V^{\perp_{\omega_p}}$, u must be in the symplectic orthogonal of $(V^{\perp_{\omega_p}})$, i.e. $V \subseteq (V^{\perp_{\omega_p}})^{\perp_{\omega_p}}$. By the dimension argument, we obtain the equality $V = (V^{\perp_{\omega_p}})^{\perp_{\omega_p}}$. Q.E.D.

Definition 4.2.19. Let (M, ω) be a symplectic manifold and $V \leq T_pM$ a vector subspace of the tangent space at a point p. Then V is called

1) *isotropic*, if $V \subseteq V^{\perp_{\omega_p}}$;
2) *co-isotropic*, if $V \supseteq V^{\perp_{\omega_p}}$;
3) *Lagrangian*, if $V = V^{\perp_{\omega_p}}$;
4) *symplectic*, if $V \cap V^{\perp_{\omega_p}} = 0$.

Proposition 4.2.20. *For a symplectic manifold (M, ω), given a point $p \in M$, $V \leq T_pM$ is a symplectic vector subspace if and only if*

$$T_pM = V \oplus V^{\perp_{\omega_p}}. \tag{4.17}$$

Proof. Since, by Proposition 4.2.18*(i)*, the dimensions of V and $V^{\perp_{\omega_p}}$ sum to the dimension of T_pM and, furthermore, V is symplectic (i.e. $V \cap V^{\perp_{\omega_p}} = 0$), we obtain the result. Q.E.D.

We now turn our attention to the actual submanifolds of a symplectic manifold, instead of the vector subspaces of a particular tangent space.

Definition 4.2.21. Let (M, ω) be a symplectic manifold. Then, an embedded submanifold N of M is called:

1) *isotropic*, if $T_pN \leq T_pM$ is an isotropic vector subspace for all $p \in N$, i.e. if
$$T_pN \subseteq (T_pN)^{\perp_{\omega_p}}, \text{ for all } p \in N; \tag{4.18a}$$
2) *co-isotropic*, if $T_pN \leq T_pM$ is a co-isotropic vector subspace for all $p \in N$, i.e. if
$$T_pN \supseteq (T_pN)^{\perp_{\omega_p}}, \text{ for all } p \in N; \tag{4.18b}$$
3) *Lagrangian*, if $T_pN \leq T_pM$ is a Lagrangian vector subspace for all $p \in N$, i.e. if
$$T_pN = (T_pN)^{\perp_{\omega_p}}, \text{ for all } p \in N; \tag{4.18c}$$
4) *symplectic*, if $T_pN \leq T_pM$ is a symplectic vector subspace for all $p \in N$, i.e. if
$$T_pN \cap (T_pN)^{\perp_{\omega_p}} = 0 \text{ or } T_pM = T_pN \oplus (T_pN)^{\perp_{\omega_p}}, \text{ for all } p \in N. \tag{4.18d}$$

Proposition 4.2.22. *Let (M, ω) be a symplectic manifold and N be an (embedded) submanifold of M. Then,*

(i) N is isotropic if and only if $\omega|_N = i^*\omega = 0$;

(ii) N is Lagrangian if and only if $\omega|_N = 0$ and $\dim(N) = \frac{1}{2}\dim(M)$;

(iii) N is symplectic if and only if $(N, \omega|_N)$ is a symplectic manifold.

Proof. *(i)* Let $p \in N$. Assuming first that N is isotropic, since $T_pN \subseteq (T_pN)^{\perp_{\omega_p}}$, given $v \in T_pN$, $\omega_p(v,u) = 0$, for all $u \in T_pN$, thus $\omega_p|_{T_pN} = 0$. Thus, since p is arbitrary, we get that $\omega|_N = 0$.

Reciprocally, if, for every $p \in N$, $\omega_p(u,v) = 0$, for all $u, v \in T_pN$, in particular, for a fixed $v \in T_pN$, $\omega_p(u,v) = 0$, for all $u \in T_pN$. Thus, $v \in (T_pN)^{\perp_{\omega_p}}$ and, therefore, $T_pN \subseteq (T_pN)^{\perp_{\omega_p}}$, i.e. N is isotropic.

(ii) If N is Lagrangian, by *(i)*, $\omega|_N = 0$. Moreover, by Proposition 4.2.18*(i)*,

$$\begin{aligned}\dim(M) = \dim(T_pM) &= \dim(T_pN) + \dim((T_pN)^{\perp_{\omega_p}}) \\ &= \dim(T_pN) + \dim(T_pN) = 2\dim(N).\end{aligned}$$

On the other hand, since $\omega|_N = 0$, by *(i)*, N is isotropic. To show that, for all $p \in N$, $(T_pN)^{\perp_{\omega_p}} = T_pN$, because $\dim(N) = \frac{1}{2}\dim(M)$,

$$\dim((T_pN)^{\perp_{\omega_p}}) = \dim(M) - \dim(N) = \dim(N) = \dim(T_pN).$$

Thus, N is also co-isotropic, so N is Lagrangian.

(iii) The fact that $\omega|_N$ is closed comes directly from the fact that ω is symplectic. As for the non-degeneracy, let $p \in N$ and let $v \in T_pN$ such that $\omega_p(v,u) = 0$, for all $u \in T_pN$. Then, $v \in (T_pN)^{\perp_{\omega_p}}$ and, because N is symplectic, $T_pN \cap (T_pN)^{\perp_{\omega_p}} = 0$, thus $v = 0$. Q.E.D.

Example 4.2.23. Let us begin by giving a few examples of isotropic, co-isotropic, Lagrangian and symplectic vector subspaces of T_pM for a given $p \in M$. Taking $M = \mathbb{R}^{2s}$ with coordinates $(x^1, \ldots, x^s, y_1, \ldots, y_s)$ and $\omega_{\text{can}} = \sum\limits_{i=1}^{s} \mathrm{d}x^i \wedge \mathrm{d}y_i$, it is easy to show that, for all $k \leq s$:

- $V = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^k}\big|_p \right\}$ is an isotropic vector subspace, since

$$V^{\perp_{(\omega_{\text{can}})_p}} = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^s}\right|_p, \left.\frac{\partial}{\partial y_{k+1}}\right|_p, \left.\frac{\partial}{\partial y_s}\right|_p \right\} \supseteq V;$$

- $V = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^s}\big|_p \right\}$ is a Lagrangian vector subspace, because

$$V^{\perp_{(\omega_{\text{can}})_p}} = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^s}\right|_p \right\} = V;$$

- $V = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^s}\big|_p, \frac{\partial}{\partial y_1}\big|_p, \ldots, \frac{\partial}{\partial y_k}\big|_p \right\}$ is a co-isotropic vector subspace, as

$$V^{\perp_{(\omega_{\text{can}})_p}} = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x^{k+1}}\right|_p, \ldots, \left.\frac{\partial}{\partial x^s}\right|_p \right\} \subseteq V;$$

– $V = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^k}\big|_p, \frac{\partial}{\partial y_1}\Big|_p \ldots, \frac{\partial}{\partial y_k}\Big|_p \right\}$ is a symplectic vector subspace, due to

$$V^{\perp_{(\omega_{\mathrm{can}})_p}} = \operatorname{span}\left\{ \frac{\partial}{\partial x^{k+1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^s}\bigg|_p, \frac{\partial}{\partial y_{k+1}}\bigg|_p, \ldots, \frac{\partial}{\partial y_s}\bigg|_p \right\},$$

so that $V \cap V^{\perp_{(\omega_{\mathrm{can}})_p}} = \{0\}$); ♮

Example 4.2.24. Considering the same symplectic manifold as in the previous example, we can describe the following submanifolds $N \subseteq \mathbb{R}^{2s}$:

– $N = \left\{(x^1, \ldots, x^k, 0, \ldots, 0) \in \mathbb{R}^{2s} \,\middle|\, x^1, \ldots, x^k \in \mathbb{R}\right\}$ is an isotropic submanifold since, for any $p \in N$, $T_pN = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^k}\big|_p \right\}$ which is an isotropic vector subspace of $T_p\mathbb{R}^{2s} \cong \mathbb{R}^{2s}$;

– $N = \left\{(x^1, \ldots, x^s, y_1, \ldots, y_s) \in \mathbb{R}^{2s} \,\middle|\, y_1^2 + \ldots + y_s^2 = 1\right\}$ is a co-isotropic submanifold because, for any $p = (q^1, \ldots, q^s, p_1, \ldots, p_s) \in N$,

$$T_pN = \operatorname{span}\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^s}\bigg|_p, \frac{\partial}{\partial y_1}\bigg|_p - p_1 \frac{\partial}{\partial y_s}\bigg|_p, \ldots, \frac{\partial}{\partial y_{s-1}}\bigg|_p - p_{s-1} \frac{\partial}{\partial y_s}\bigg|_p \right\}$$

and $(T_pN)^{\perp_{(\omega_{\mathrm{can}})_p}} = \operatorname{span}\left\{ p_1 \frac{\partial}{\partial x^1}\big|_p + \ldots + p_s \frac{\partial}{\partial x^s}\big|_p \right\} \subseteq T_pN$;

– $N = \left\{(x^1, \ldots, x^s, y_1, \ldots, y_s) \in \mathbb{R}^{2s} \,\middle|\, (x^i)^2 + y_i^2 = 1, \text{ for all } i = 1, \ldots, s\right\}$ is a Lagrangian submanifold as, for any $p = (q^1, \ldots, q^s, p_1, \ldots, p_s) \in N$,

$$T_pN = \operatorname{span}\left\{ p_i \frac{\partial}{\partial x^i}\bigg|_p - q^i \frac{\partial}{\partial y_i}\bigg|_p \,\middle|\, i = 1, \ldots, s \right\} = (T_pN)^{\perp_{(\omega_{\mathrm{can}})_p}}$$

– $N = \left\{(x^1, \ldots, x^k, 0, \ldots, 0, y_1, \ldots, y_k, 0, \ldots, 0) \in \mathbb{R}^{2s} \,\middle|\, x^1, \ldots, x^k, y_1, \ldots, y_k \in \mathbb{R}\right\}$ is a symplectic submanifold given that $(N, \omega_{\mathrm{can}}|_N) \cong (\mathbb{R}^{2k}, \omega'_{\mathrm{can}})$, where, clearly $\omega'_{\mathrm{can}} = \sum_{i=1}^{k} \mathrm{d}x^i \wedge \mathrm{d}y_i$ is a symplectic form. ♮

Example 4.2.25. Returning to the example of a symplectic manifold on the cotangent bundle of Q, we have the following result:

Proposition 4.2.26. *Let $\eta \in \Omega^1(Q)$ be a 1-form. Then, η is closed if and only if, when considering η to be a map from Q to T^*Q, $\operatorname{im}(\eta)$ is a Lagrangian submanifold of $(T^*Q, -\mathrm{d}\lambda)$.*

Proof. Firstly, note that, by treating $\eta \in \Omega^1(Q)$ as a map from Q to T^*Q, η is an immersion ($T\eta = \eta_*$ is injective) and, thus $\dim(\operatorname{im}(\eta)) = \dim(Q) = \frac{1}{2}\dim(T^*Q)$. Hence, by Proposition 4.2.22, it is sufficient to show that η is closed if and only if $-\mathrm{d}\lambda|_{\operatorname{im}(\eta)} = 0$. This latter expression is equivalent to say that $\eta^*(-\mathrm{d}\lambda) = 0$, which, by Proposition 4.2.8, is equivalent to $\mathrm{d}\eta = 0$. Q.E.D.

This proposition implies, in particular, that the image of the zero section θ_{T^*Q} is a Lagrangian submanifold of $(T^*Q, -\mathrm{d}\lambda)$. One other example of a Lagrangian submanifold that is not necessarily included in the previous Proposition is that of the fibres at a certain point, i.e. the cotangent spaces T_q^*Q for every $q \in Q$. Clearly, $\dim(T_q^*Q) = \frac{1}{2}\dim(T^*Q)$ and, considering the inclusion $\iota : T_q^*Q \hookrightarrow T^*Q$ (which is an embedding),

$$-\mathrm{d}\lambda|_{\iota(T_q^*Q)} = 0 \iff \iota^*(-\mathrm{d}\lambda) = 0 \iff \mathrm{d}(\iota^*\lambda) = 0.$$

However, given $p_q \in T_q^*Q$ and $v \in T_{p_q}(T_q^*Q)$, we get that

$$(\iota^*\lambda)_{p_q}(v) = \lambda_{p_q}(\iota_*(p_q)(v)) = p_q\big((\hat{\tau}_Q)_*(p_q)(\iota_*(p_q)(v))\big) = p_q\big((\hat{\tau}_\circ\iota)_*(p_q)(v)\big) = 0,$$

since in our case, $\hat{\tau}_Q \circ \iota = q$ (the constant function assigning the point q). Hence, T_q^*Q is a Lagrangian submanifold of $(T^*Q, -\mathrm{d}\lambda)$ for all $q \in Q$. ♮

Proposition 4.2.27. *Let (M, ω) be a symplectic manifold and $\varphi : M \to M$ be a diffeomorphism. Then φ is a symplectic diffeomorphism if and only if its graph*

$$\mathrm{gr}(\varphi) = \big\{(p, \varphi(p)) \,\big|\, p \in M\big\} \subseteq M \times M \tag{4.19}$$

is a Lagrangian submanifold of $(M \times M, (-\omega) \oplus \omega)$.

Proof. Firstly, notice that $\mathrm{gr}(\varphi)$ is an embedded submanifold of $M \times M$ (it is the image of $f = (\mathrm{id}_M \times \varphi) \circ \delta$, where $\delta(p) = (p, p) \in \Delta \subseteq M \times M$). It is clear that $\dim(\mathrm{gr}(\varphi)) = \dim(M) = \frac{1}{2}\dim(M \times M)$, thus it is sufficient to prove that φ is a symplectic diffeomorphism if and only if $(-\omega) \oplus \omega|_{\mathrm{gr}(\varphi)} = 0$. Let $p \in M$ and $v_1, v_2 \in T_{(p,\varphi(p))}\mathrm{gr}(\varphi)$. Then, since $\mathrm{gr}(\varphi) = \mathrm{im}(f)$,

$$T_{(p,\varphi(p))}\mathrm{gr}(\varphi) = T_{f(p)}\,\mathrm{im}(f) = \mathrm{im}(T_p f)$$

and, so we may simply take $v_i = f_*(p)(u_i)$ for some $u_i \in T_pM$ for all $i = 1, 2$. Thus,

$$\begin{aligned}((-\omega) \oplus \omega)_{(p.\varphi(p)}(v_1, v_2) &= ((-\omega) \oplus \omega)_{f(p)}(f_*(p)(u_1), f_*(p)(u_2)) \\ &= (f^*((-\omega) \oplus \omega))_p(u_1, u_2) = (-f^*\pi_1^*\omega + f^*\pi_2^*\omega)_p(u_1, u_2) \\ &= (-(\pi_1 \circ f)^*\omega + (\pi_2 \circ f)^*\omega)_p(u_1, u_2) \\ &= (-\mathrm{id}_M^*\omega + \varphi^*\omega)_p(u_1, u_2) = (-\omega + \varphi^*\omega)_p(u_1, u_2),\end{aligned}$$

which vanishes for all $u_1, u_2 \in T_pM$ if and only if $\varphi^*\omega = \omega$, i.e. if φ is a symplectic isomorphism. Q.E.D.

4.2.2 Symplectic and Hamiltonian (Infinitesimal) Automorphisms

We now return to the study of certain morphisms in the category of symplectic manifolds and, in particular, their infinitesimal counterparts. In many applications, it is rather important to understand the symplectomorphisms between the same manifold. Borrowing the nomenclature employed in category theory, we shall call these such maps by *symplectic automorphisms* and denote

$$\mathrm{Aut}(M,\omega) = \left\{\Phi \in \mathrm{Diff}(M) \,\middle|\, \Phi^*\omega = \omega\right\}. \tag{4.20}$$

Proposition 4.2.28. *The pair* $(\mathrm{Aut}(M,\omega), \circ)$ *forms a group and it is a subgroup of* $(\mathrm{Diff}(M), \circ)$, *the group of diffeomorphisms of a manifold* M.

Proof. It is clear that $\mathrm{id}_M \in \mathrm{Aut}(M,\omega)$. To check that $(\mathrm{Aut}(M,\omega), \circ)$ forms a (sub)group (of $(\mathrm{Diff}(M), \circ)$) it is enough to show that it is closed under the composition and the inversion map $\mathrm{Diff}(M) \to \mathrm{Diff}(M)$, $\Phi \mapsto \Phi^{-1}$. Let $\Phi, \Psi \in \mathrm{Aut}(M,\omega)$. Then,

$$(\Phi \circ \Psi)^*\omega = \Psi^*\Phi^*\omega = \Psi^*\omega = \omega.$$

Thus $\Phi \circ \Psi \in \mathrm{Aut}(M,\omega)$. Q.E.D.

Similar to how the vector fields are the infinitesimal version of the diffeomorphisms of a manifold, we can now consider the infinitesimal version of these symplectic automorphisms:

Definition 4.2.29. A vector field $X \in \mathfrak{X}(M)$ on a symplectic manifold (M,ω) is said to be *locally Hamiltonian* or called an *infinitesimal symplectic automorphism* if its flow Fl_X^t is a symplectic automorphism, i.e. for all $t \in \mathbb{R}$:

$$(\mathrm{Fl}_X^t)^*\omega = \omega, \text{ on } \mathrm{dom}(\mathrm{Fl}_X^t).$$

Remark 4.2.30. Notice that, since symplectic automorphisms form a Lie group, there is no problem in calling the elements of its Lie algebra by *infinitesimal symplectic automorphisms.* In the sequel we shall adopt this nomenclature for a locally Hamiltonian vector field $X \in \mathfrak{X}(M)$ and denote

$$\mathfrak{aut}(M,\omega) = \left\{X \in \mathfrak{X}(M) \,\middle|\, X \text{ is a locally Hamiltonian vector field of } (M,\omega)\right\}. \tag{4.21}$$

Definition 4.2.31. Let (M,ω) be a symplectic manifold. Then, a vector field $X \in \mathfrak{X}(M)$ is called a

1) *symplectic vector field* if $\mathrm{d}(i_X\omega) = 0$;

2) *Hamiltonian vector field* if there exists $H \in C^\infty(M)$ such that $i_X\omega = \mathrm{d}H$.

Remark 4.2.32. From the definition of a Hamiltonian vector field, it is clear that, if a vector field X is Hamiltonian, then the function H is not unique (we could add any constant function to H). However, the opposite is not true. In fact, since $\omega^\flat$ is a vector bundle isomorphism covering the identity id_M, for any $\alpha \in \Omega^1(M)$, there is a unique $X \in \mathfrak{X}(M)$ such that $\alpha = \omega^\flat(X)$. In particular, this is valid for exact 1-forms.

The previous remark motivates the following definition:

Definition 4.2.33. For a symplectic manifold (M, ω), the *Hamiltonian vector field associated to $H \in C^\infty(M)$* is the unique vector field $X_H \in \mathfrak{X}(M)$ satisfying

$$i_{X_H}\omega = \mathrm{d}H. \tag{4.22}$$

Therefore we shall denote this collection of Hamiltonian vector fields of a symplectic manifold by:

$$\begin{aligned} \mathfrak{ham}(M,\omega) &= \left\{X \in \mathfrak{X}(M) \,\middle|\, X \text{ is a Hamiltonian vector field of } (M,\omega)\right\} \\ &= (\omega^\flat)^{-1}(\mathrm{d}C^\infty(M)). \end{aligned} \tag{4.23}$$

This notation will be justified later.

As for symplectic vector fields, we have the following:

Lemma 4.2.34. *Let (M, ω) be a symplectic manifold and $X \in \mathfrak{X}(M)$. Then X is a symplectic vector field if an only if X is an infinitesimal symplectic automorphism.*

Proof. $\Longrightarrow$) For a symplectic vector field X, given its flow Fl^t_X, we obtain,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}^t_X)^*\omega &= \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{Fl}^s_X)^*\omega\Big|_{s=t} = \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{Fl}^{s+t}_X)^*\omega\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{Fl}^t_X)^*(\mathrm{Fl}^s_X)^*\omega\Big|_{s=0} = (\mathrm{Fl}^t_X)^*\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{Fl}^s_X)^*\omega\Big|_{s=0} \\ &= (\mathrm{Fl}^t_X)^*\mathcal{L}_X\omega = (\mathrm{Fl}^t_X)^*(\mathrm{d}(i_X\omega) + i_X\mathrm{d}\omega) = 0. \end{aligned}$$

Thus, $(\mathrm{Fl}^t_X)^*\omega = (\mathrm{Fl}^0_X)^*\omega = \omega$, i.e. X is an infinitesimal symplectic automorphism.

$\Longleftarrow$) If X is an infinitesimal symplectic automorphism, since $(\mathrm{Fl}^t_X)^*\omega = \omega$, on $\mathrm{dom}(\mathrm{Fl}^t_X)$, we have that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}^t_X)^*\omega = 0$. Thus, in particular, we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}^t_X)^*\omega\Big|_{t=0} = \mathcal{L}_X\omega = \mathrm{d}(i_X\omega) + i_X\mathrm{d}\omega = \mathrm{d}(i_X\omega).$$

Hence, X is a symplectic vector field.

Q.E.D.

Theorem 4.2.35. *Let (M, ω) be a symplectic manifold. Then, given $X, Y \in \mathfrak{aut}(M, \omega)$, we have that*

$$i_{[X,Y]}\omega = -\mathrm{d}(\omega(X, Y)).$$

Proof. Let $Z \in \mathfrak{X}(M)$. By the definition of the exterior derivative, we have that

$$\begin{aligned}\mathrm{d}(i_X\omega)(Y,Z) &= Y(\omega(X,Z)) - Z(\omega(X,Y)) - \omega(X,[Y,Z]),\\ \mathrm{d}\omega(X,Y,Z) &= X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) - \omega([X,Y],Z)\\ &\quad + \omega([X,Z],Y) - \omega([Y,Z],X),\end{aligned}$$

and similarly for $\mathrm{d}(i_Y\omega)(X,Z)$. Then, since ω is a symplectic form and X, Y are infinitesimal symplectic automorphisms, ω, $i_X\omega$ and $i_Y\omega$ are closed and, therefore,

$$\begin{aligned}0 &= X(\omega(Y,Z)) - Y(\omega(X.Z)) + Z(\omega(X,Y)) - \omega([X,Y],Z) + \mathrm{d}(i_Y\omega)(X,Z)\\ &\quad - X(\omega(Y,Z)) + Z(\omega(Y,X)) - \mathrm{d}(i_X\omega)(Y,Z) + Y(\omega(X,Z)) - Z(\omega(X,Y))\\ &= -\omega([X,Y],Z) - Z(\omega(X,Y)) = -\big(i_{[X,Y]}\omega + \mathrm{d}(\omega(X,Y))\big)(Z)\end{aligned}$$

where the terms in red, blue and teal cancel out. Thus, since $Z \in \mathfrak{X}(M)$ is an arbitrary vector field, we conclude that $i_{[X,Y]}\omega + \mathrm{d}(\omega(X,Y)) = 0$. Q.E.D.

From Theorem 4.2.35, we can organise the different Lie algebras of vector fields the following way: both Lie algebras of Hamiltonian and symplectic vector fields are Lie subalgebras of the algebra of vector fields and, furthermore, the algebra of Hamiltonian vector fields is a Lie ideal of the algebra of symplectic vector fields. More compactly, we write:

$$\mathfrak{ham}(M,\omega) \trianglelefteq \mathfrak{aut}(M,\omega) \leq \mathfrak{X}(M) \quad \text{as Lie algebras.} \tag{4.24}$$

Additionally, Theorem 4.2.35 motivates the introduction of the Poisson bracket:

Definition 4.2.36. The *Poisson bracket* of two functions $f, g \in C^\infty(M)$ on a symplectic manifold (M,ω) is the bracket

$$\{f,g\} = -\omega(X_f, X_g), \tag{4.25}$$

where $X_f, X_g \in \mathfrak{ham}(M,\omega)$ are the Hamiltonian vector fields associated to the functions f, g.

Remark 4.2.37. As mentioned previously, the addition of the sign on Eq. (4.25) follows the convention used in Poisson geometry as it turns the discussion of said manifolds much simpler. Alternatively, in other references such as [13], it is often preferred to use a minus sign in the definition of the Lie bracket, so that $[X,Y] = YX - XY$. For a more detailed discussion of sign convention in symplectic geometry, we redirect the reader to Remark 3.1.6 of [13].

Remark 4.2.38. With this definition of the Poisson bracket, Theorem 4.2.35 shows that, given two functions $f, g \in C^\infty(M)$,

$$[X_f, X_g] = X_{\{f,g\}}. \tag{4.26}$$

Proposition 4.2.39. *The Poisson bracket defined by*

$$\begin{aligned} \{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) &\longrightarrow C^\infty(M) \\ (f,g) &\longmapsto \{f,g\} \end{aligned} \tag{4.27}$$

is anti-symmetric, $\mathbb{R}$-bilinear and satisfies, for all $f, g, h \in C^\infty(M)$:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \qquad \textit{(Jacobi identity)}$$
$$\{f,gh\} = \{f,g\}\ h + g\ \{f,h\} \qquad \textit{(Leibniz rule)}$$

Proof. The anti-symmetry and bilinearity of the Poisson bracket come directly from the fact that ω is a 2-form and that d is $\mathbb{R}$-linear. As for the Leibniz rule, we have:

$$\begin{aligned} \{f,gh\} &= -\omega(X_f, X_{gh}) = (i_{X_{gh}}\omega)(X_f) = \mathrm{d}(gh)(X_f) = \mathrm{d}g(X_f)h + g\mathrm{d}h(X_f) \\ &= -\omega(X_f,X_g)h - g\omega(X_f,X_h) = \{f,g\}\, h + g\,\{f,h\}\,. \end{aligned}$$

Finally, to prove the Jacobi identity, since ω is closed,

$$\begin{aligned} 0 = \mathrm{d}\omega(X_f,X_g,X_h) &= X_f(\omega(X_g,X_h)) - X_g(\omega(X_f,X_h)) + X_h(\omega(X_f,X_g)) \\ &\qquad - \omega([X_f,X_g],X_h) + \omega([X_f,X_h],X_g) - \omega([X_g,X_h],X_f) \\ &= -\mathrm{d}\,\{g,h\}\,(X_f) + \mathrm{d}\,\{f,h\}\,(X_g) - \mathrm{d}\,\{f,g\}\,(X_h) \\ &\qquad - \omega(X_{\{f,g\}},X_h) + \omega(X_{\{f,h\}},X_g) - \omega(X_{\{g,h\}},X_f) \\ &= -\omega(X_{\{g,h\}},X_f) + \omega(X_{\{f,h\}},X_g) - \omega(X_{\{f,g\}},X_h) \\ &\qquad - \omega(X_{\{f,g\}},X_h) + \omega(X_{\{f,h\}},X_g) - \omega(X_{\{g,h\}},X_f) \\ &= -2(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}) \end{aligned}$$

which concludes the proof. Q.E.D.

Remark 4.2.40. From the proposition we get that the pair $(C^\infty(M), \{\cdot,\cdot\})$ is a (real infinite-dimensional) Lie algebra.

Proposition 4.2.41. *Let (M,ω) be a symplectic manifold.*

(i) The following map is a Lie algebra homomorphism:

$$\begin{aligned} (C^\infty(M), \{\cdot,\cdot\}) &\longrightarrow (\mathfrak{ham}(M,\omega), [\cdot,\cdot]) \\ f &\longmapsto X_f \end{aligned}$$

(ii) If $\varphi \in \mathrm{Aut}(M,\omega)$ and $f \in C^\infty(M)$, then $\varphi^* X_f = X_{\varphi^* f}$.

Proof. *(i)* By the properties of the exterior derivative, for $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$, we simply get that

$$\omega^\flat(X_{af+bg}) = \mathrm{d}(af + bg) = a\mathrm{d}f + b\mathrm{d}g = a\omega^\flat(X_f) + b\omega^\flat(X_g),$$

thus, since $\omega^\flat$ is an isomorphism, $X_{af+bg} = aX_f + bX_g$. Additionally, by Eq. (4.26), we get that the assignment $f \mapsto X_f$ is a Lie algebra homomorphism.

(ii) In the conditions of the statement,

$$\begin{aligned}\omega^\flat(X_{\varphi^* f}) &= i_{X_{\varphi^* f}}\omega = \mathrm{d}(\varphi^* f) = \varphi^*\mathrm{d}f = \varphi^*(i_{X_f}\omega) \\ &= i_{\varphi^* X_f}(\varphi^*\omega) = i_{\varphi^* X_f}\omega = \omega^\flat(\varphi^* X_f).\end{aligned}$$

Since $\omega^\flat$ is an isomorphism, we then conclude that $X_{\varphi^* f} = \varphi^* X_f$. Q.E.D.

Even though we started by defining the infinitesimal symplectic automorphisms from their non-infinitesimal versions, such was not so evident to do for Hamiltonian vector fields. We shall now remedy this fact by introducing the notion of Hamiltonian isotopies:

Definition 4.2.42. Let (M, ω) be a symplectic manifold.

1) A *Hamiltonian isotopy* is a smooth family $\{\Phi^t \mid t \in [0,1]\}$ of symplectic automorphisms such that $\Phi^0 = \mathrm{id}_M$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p) = (X_{H_t})_{\Phi^t(p)}, \ \forall\, (p,t) \in M \times [0,1],$$

where $\{H_t \in C^\infty(M) \mid t \in [0,1]\}$ is a smooth family of smooth functions.

2) A symplectic automorphism $\phi : M \to M$ is called a *Hamiltonian automorphism* if there exists a Hamiltonian isotopy $\{\Phi^t \mid t \in [0,1]\}$ such that $\Phi^1 = \phi$. The collection of all Hamiltonian automorphisms is denoted by $\mathrm{Ham}(M, \omega)$.

Proposition 4.2.43. *The pair* $(\mathrm{Ham}(M,\omega), \circ)$ *forms a group, which is a normal subgroup of* $\mathrm{Aut}(M,\omega)$*, which we shall denote as* $\mathrm{Ham}(M,\omega) \trianglelefteq \mathrm{Aut}(M,\omega)$.

Proof. The fact that $(\mathrm{Ham}(M,\omega), \circ)$ forms a group boils down to show that the composition of two Hamiltonian automorphisms is still a Hamiltonian automorphism (the other necessary group properties are evident). Therefore, let $\phi, \psi \in \mathrm{Ham}(M,\omega)$ with Hamiltonian isotopies $\{\Phi^t \mid t \in [0,1]\}$ and $\{\Psi^t \mid t \in [0,1]\}$, respectively, such that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p) = (X_{H_t})_{\Phi^t(p)}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Psi^t(p) = (X_{K_t})_{\Psi^t(p)}$. We want to show that $\{\Phi^t \circ \Psi^t \mid t \in [0,1]\}$ is a Hamiltonian isotopy of $\phi \circ \psi$. For that, notice that, given a symplectic automorphism $\varphi \in \mathrm{Aut}(M,\omega)$, by the properties of the interior product,

$$\begin{aligned}i_{(X_{H_t} + \varphi^* X_{K_t})}\omega &= i_{X_{H_t}}\omega + i_{\varphi^* X_{K_t}}\omega = \mathrm{d}H_t + \varphi^* i_{X_{K_t}}(\varphi^{-1})^*\omega \\ &= \mathrm{d}H_t + \varphi^* i_{X_{K_t}}\omega = \mathrm{d}H_t + \varphi^*\mathrm{d}K_t = \mathrm{d}(H_t + \varphi^* K_t).\end{aligned} \tag{$*$}$$

Thus, we get,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t\circ\Psi^t(p) &= \frac{\mathrm{d}}{\mathrm{d}s}\Phi^s\circ\Psi^t(p)\bigg|_{s=t} + (\Phi^t)_*(\Psi^t(p))\left(\frac{\mathrm{d}}{\mathrm{d}t}\Psi^t(p)\right)\\ &= (X_{H_t})_{\Phi^t\circ\Psi^t(p)} + (\Phi^t_* X_{K_t})_{(\Phi^t\circ\Psi^t)(p)} = (X_{H_t}+\Phi^t_* X_{K_t})_{(\Phi^t\circ\Psi^t)(p)}\\ &= (X_{H_t}+(\Phi^{-t})^* X_{K_t})_{(\Phi^t\circ\Psi^t)(p)}.\end{aligned}$$

Then, by Eq. $(*)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t\circ\Psi^t(p) = (X_{(H_t+(\Phi^{-t})^*K_t)})_{(\Phi^t\circ\Psi^t)(p)},$$

hence $\left\{\Phi^t\circ\Psi^t \,\middle|\, t\in[0,1]\right\}$ is a Hamiltonian isotopy and, therefore, $\mathrm{Ham}(M,\omega)$ is a subgroup of $\mathrm{Aut}(M,\omega)$. To prove that it is normal, let $\psi\in\mathrm{Aut}(M,\omega)$ and $\phi\in\mathrm{Ham}(M,\omega)$ such that $\left\{\Phi^t \,\middle|\, t\in[0,1]\right\}$ is its Hamiltonian isotopy with functions H_t. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi^{-1}\circ\Phi^t\circ\psi(p) &= \psi^{-1}_*(\Phi^t\circ\psi(p))\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t\circ\psi(p)\right) = \psi^{-1}_*(\Phi^t\circ\psi(p))((X_{H_t})_{\Phi^t\circ\psi(p)})\\ &= (\psi^{-1}_* X_{H_t})_{\Phi^t\circ\psi(p)} = (\psi^* X_{H_t})_{\psi^{-1}\circ\Phi^t\circ\psi(p)} = (X_{\psi^* H_t})_{\psi^{-1}\circ\Phi^t\circ\psi(p)},\end{aligned}$$

thus, $\psi^{-1}\circ\phi\circ\psi\in\mathrm{Ham}(M,\omega)$, i.e. $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Aut}(M,\omega)$. Q.E.D.

Therefore, similarly to what we have previously obtained for their infinitesimal counterpart, we get

$$\mathrm{Ham}(M,\omega)\trianglelefteq\mathrm{Aut}(M,\omega)\leq\mathrm{Diff}(M)\quad\text{as groups.}\tag{4.28}$$

4.3 Darboux and Moser Theorems

The next step in our journey through symplectic manifolds is to obtain one of the most important theorems of this geometry, Darboux's Theorem:

Theorem 4.3.13. *Let (M, ω) be a $2s$-dimensional symplectic manifold. Then, given $p \in M$, there exist an open neighbourhood $U \subseteq M$ of p and a diffeomorphism $\varphi : U \to V \subseteq \mathbb{R}^{2s}$ such that $\varphi^*\big(\omega_{\mathrm{can}}|_V\big) = \omega|_U$.*

To prove such a result it is necessary to introduce the notion of Moser isotopy which firstly requires to fully understand the concept of time-dependent vector fields, as well as a similar result for symplectic vector spaces. Let us, then, start by the latter:

Theorem 4.3.1. *Let (M, ω) be a $2s$-dimensional symplectic manifold. Then, given $p \in M$, there is a basis $\{\chi_1, \dots, \chi_s, \upsilon_1, \dots, \upsilon_s\}$ of T_pM such that, for all $i, j = 1, \dots, s$,*

$$\omega_p(\chi_i, \chi_j) = \omega_p(\upsilon_i, \upsilon_j) = 0 \quad \text{and} \quad \omega_p(\chi_i, \upsilon_j) = \delta_{ij},$$

where δ_{ij} is the Kroenecker delta. Moreover, there exists a (vector space) isomorphism $\varphi : T_pM \xrightarrow{\cong} \mathbb{R}^{2s}$ such that

$$(\omega_{\mathrm{can}})_0 \circ \textstyle\bigwedge^2 \varphi = \omega_p,$$

or, with more detail, for $u, v \in T_pM$, $\omega_p(u, v) = (\omega_{\mathrm{can}})_0(\varphi(u), \varphi(v))$.

Proof. Induction over s: for $s = 1$, let $\{\chi, \upsilon\}$ be an arbitrary basis of T_pM. Since ω is a symplectic form, it is non-degenerate and, so, $\omega_p(\chi, \upsilon) \neq 0$. Thus, redefining the basis elements to be χ, $\frac{\upsilon}{\omega_p(\chi,\upsilon)}$, we obtain the conditions of the theorem.

For the induction step, we simply note that, since ω is non-degenerate, we can always find two linearly independent $\chi_1, \upsilon_1 \in T_pM$ such that $\omega_p(\chi_1, \upsilon_1) \neq 0$ and, so, the subspace spanned by $\big\{\chi_1, \frac{\upsilon_1}{\omega_p(\chi_1,\upsilon_1)}\big\}$ is a symplectic vector subspace of (T_pM, ω_p). Then, its symplectic orthogonal is also symplectic and has dimension $2(s-1)$, thus, by the induction hypothesis, there is a basis $\{\bar\chi_1, \dots, \bar\chi_{s-1}, \bar\upsilon_1, \dots, \bar\upsilon_{s-1}\}$ such that

$$\omega_p(\bar\chi_i, \bar\chi_j) = \omega_p(\bar\upsilon_i, \bar\upsilon_j) = 0 \quad \text{and} \quad \omega_p(\bar\chi_i, \bar\upsilon_j) = \delta_{ij},$$

for $i, j = 1, \dots, s-1$. Therefore, $\{\chi_1, \bar\chi_1, \dots, \bar\chi_{s-1}, \frac{\upsilon_1}{\omega_p(\chi_1,\upsilon_1)}, \bar\upsilon_1, \dots, \bar\upsilon_{s-1}\}$ forms a basis of T_pM satisfying the conditions of the theorem.

Finally, we construct the linear map $\varphi : T_pM \to \mathbb{R}^{2s} \cong T_0\mathbb{R}^{2s}$ such that

$$\varphi\left(\sum_{i=1}^{s} \big(v_i^\chi \chi_i + v_i^\upsilon \upsilon_i\big)\right) = \sum_{i=1}^{s} \Big(v_i^\chi \left.\frac{\partial}{\partial x_i}\right|_0 + v_i^\upsilon \left.\frac{\partial}{\partial y_i}\right|_0\Big),$$

where $(x_1, \dots, x_s, y_1, \dots, y_s)$ are the standard coordinates on $\mathbb{R}^{2s}$ such that the canonical symplectic form takes the form $\omega_{\mathrm{can}} = \sum_{i=1}^{s} \mathrm{d}x_i \wedge \mathrm{d}y_i$. Then it is clear that φ is a vector isomorphism it satisfies $(\omega_{\mathrm{can}})_0 \circ \bigwedge^2 \varphi = \omega_p$. Q.E.D.

This theorem is almost precisely what we require for Darboux theorem, however it is not necessary that the bases for the tangent spaces glue well together. The response to this problem is precisely what we shall discuss in the rest of this section.

4.3.1 Time-dependent vector fields

When we first dealt with vector fields, we learnt that they can be used in the setting of differential equations for their flow/integral curves. However, in many scenarios, the ODE's describing certain processes are not autonomous, i.e. the process itself may depend on time. This also motivates the extension of the notion of vector fields to allow them to depend on time:

Definition 4.3.2. A *time-dependent or non-autonomous vector field* on a manifold M is a smooth map $X : \Omega \to TM$, where $\Omega \subseteq \mathbb{R} \times M$ is an open subset, such that $X(t,p) \overset{\text{def}}{=} X_t(p) \in T_pM$, for all $(t,p) \in \Omega$.

Remark 4.3.3. Notice that, in essence, a time-dependent vector field is nothing more than a family of vector fields $\{X_t\}_{t\in I}$ (for $I \subseteq \mathbb{R}$ an open interval and $\Omega = I \times M$) that glues sufficiently well as we change the time $t \in I$.

Remark 4.3.4. Following the same notation introduced in Section 1.1, we shall denote the time-dependent vector field at a time t, X_t, evaluated at a point $p \in M$ by $(X_t)_p$.

Example 4.3.5. The simplest cases of time-dependent vector fields are, in fact, time--independent vector fields $X \in \mathfrak{X}(M)$. Clearly, if we consider $\Omega = \mathbb{R} \times M$ and $X_t = X$, for all $t \in \mathbb{R}$, then X_t is a time-dependent vector field. ♮

Example 4.3.6. Let $M = \mathbb{R}^m$. Then, for all $t \in \mathbb{R}_{>0}$ we can consider

$$X_t(x_1, \ldots, x_m) = \sum_{i=1}^{m} \frac{1 - x_i^2}{t} \frac{\partial}{\partial x_i} ,$$

which is a time-dependent vector field we shall discuss in more detail below. ♮

The discussion of time-dependent vector fields cannot be concluded without the notion of their flows and integral curves.

Definition 4.3.7. Given a time-dependent vector field X on a manifold M, an *integral curve of* X is a smooth curve $\gamma : I \to M$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = X(t, \gamma(t))$, for all $t \in I$ with $(t, \gamma(t)) \in \Omega$.

We could define the flow of such a vector field as well, however to do so it is more useful to think about time-dependent vector fields on M as time-independent vector fields on $\mathbb{R} \times M$. In fact, for every non-autonomous vector field X, we may define an autonomous vector field on an open subset of $\mathbb{R} \times M$ (with domain $\Omega \subseteq \mathbb{R} \times M$) by $\tilde{X} = \frac{\partial}{\partial t} + X$, where $\frac{\partial}{\partial t}$ is the vector field on $\mathbb{R} \times M$ obtained by the pullback by the projection $\mathbb{R} \times M \twoheadrightarrow \mathbb{R}$. Then, we may define the flow of $\tilde{X}$ by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Fl}^s_{\tilde{X}}(t,p) = \tilde{X}_{\mathrm{Fl}^s_{\tilde{X}}(t,p)} \quad \text{and} \quad \mathrm{Fl}^0_{\tilde{X}}(t,p) = (t,p) \in \mathbb{R} \times M.$$

Expanding the equation on the left by rewriting $\mathrm{Fl}^s_{\tilde{X}}(t,p) = (\tau^s(t,p), \chi^s(t,p))$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\tau^s(t,p) = \left.\frac{\partial}{\partial t}\right|_{\tau^s(t,p)} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}\chi^s(t,p) = (X_{\tau^s(t,p)})_{\chi^s(t,p)}.$$

The first equation yields $\tau^s(t,p) = s+t$ (where we already use the initial condition $\tau^0(t,p) = t$) and, thus, the second equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}\chi^s(t,p) = (X_{s+t})_{\chi^s(t,p)},$$

with the initial condition $\chi^0(t,p) = p$. Thus, to obtain an expression similar to an integral curve for X, we perform a change in coordinates $s \to s-t$ and call $\mathrm{Fl}^{s,t}_X(p) = \chi^{s-t}(t,p)$ the *flow of the non-autonomous vector field* X (we shall prove the uniqueness of this map in a subsequent proposition). This flow must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Fl}^{s,t}_X(p) = (X_s)_{\mathrm{Fl}^{s,t}_X(p)} \quad \text{and} \quad \mathrm{Fl}^{t,t}_X(p) = p. \tag{4.29}$$

Proposition 4.3.8. *Let M be a smooth manifold and X a time-dependent vector field.*

(i) The map $\mathrm{Fl}^{s,t}_X$ defined previously is unique and it is a diffeomorphism between two open subsets of M, with inverse $\mathrm{Fl}^{t,s}_X$, for all $s,t \in \mathbb{R}$;

(ii) If $p \in \mathrm{dom}(\mathrm{Fl}^{u,t}_X)$ and $\mathrm{Fl}^{u,t}_X(p) \in \mathrm{dom}(\mathrm{Fl}^{s,u}_X)$, then $\mathrm{Fl}^{s,u}_X \circ \mathrm{Fl}^{u,t}_X(p) = \mathrm{Fl}^{s,t}_X(p)$.

Proof. *(i)* By construction, $\mathrm{Fl}^{s,t}_X(p) = \chi^{s-t}(t,p)$ is a local diffeomorphism since $\chi^s(t,p)$ is the second component of the flow of $\frac{\partial}{\partial t} + X$. Moreover, the inverse of the flow $\mathrm{Fl}^s_{\tilde{X}}$ is $(\mathrm{Fl}^s_{\tilde{X}})^{-1} = \mathrm{Fl}^{-s}_{\tilde{X}}$, so, in order to compute the inverse of $\mathrm{Fl}^{s,t}_X$, take for $p,q \in M$ and $s,t \in \mathbb{R}$ such that $p = \mathrm{Fl}^{s,t}_X(q) = \chi^{s-t}(t,q)$. Then, we have that

$$(s,p) = \mathrm{Fl}^{s-t}_{\tilde{X}}(t,q) \implies (t,q) = \mathrm{Fl}^{t-s}_{\tilde{X}}(s,p) \implies q = \chi^{t-s}(s,p) = \mathrm{Fl}^{t,s}_{\tilde{X}}(p).$$

Therefore, the inverse of the flow $\mathrm{Fl}^{s,t}_{\tilde{X}}$ is $(\mathrm{Fl}^{s,t}_{\tilde{X}})^{-1} = \mathrm{Fl}^{t,s}_{\tilde{X}}$.

As for the uniqueness of the flow, assuming that $\phi^{s,t}$ also gives the flow of X, it is quite straightforward to prove that, for any $p \in M$, $(s+t, \phi^{s+t,t}(p))$ gives the flow of $\frac{\partial}{\partial t} + X$ at the point p. Thus, by the uniqueness of the flow of (time-independent) vector fields, we conclude that $\phi^{s,t} = \mathrm{Fl}^{s,t}_X$.

(ii) In the conditions of the statement, we have that

$$\begin{aligned}\mathrm{Fl}_X^{s,u}\circ\mathrm{Fl}_X^{u,t}(p) &= \chi^{s-u}(u,\chi^{u-t}(u,p)) = \chi^{s-u}(u-t+t,\chi^{u-t}(t,p))\\ &= \chi^{s-u}(\tau^{u-t}(t,p),\chi^{u-t}(t,p)) = \chi^{s-u}\circ\mathrm{Fl}_{\tilde{X}}^{u-t}(t,p).\end{aligned}$$

Since, from the properties of the flow of autonomous vector fields, $\mathrm{Fl}_{\tilde{X}}^{t_1}\circ\mathrm{Fl}_{\tilde{X}}^{t_2} = \mathrm{Fl}_{\tilde{X}}^{t_1+t_2}$ and χ^s is the 'spatial' component of the flow $\mathrm{Fl}_{\tilde{X}}^s$, we have that

$$\chi^{s-u}\circ\mathrm{Fl}_{\tilde{X}}^{u-t}(t,p) = \chi^{s-u+u-t}(t,p) = \chi^{s-t}(t,p) = \mathrm{Fl}_X^{s,t}(p).$$ Q.E.D.

Example 4.3.9. For a time-independent vector field $X\in\mathfrak{X}(M)$, constructing the associated non-autonomous vector field $\hat{X}$ as $\hat{X}_t = X$, for all $t\in\mathbb{R}$, the flow of $\hat{X}$ satisfies, for $p\in M$ and $t,s\in\mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Fl}_{\hat{X}}^{s,t}(p) = (\hat{X}_s)_{\mathrm{Fl}_{\hat{X}}^{s,t}(p)} = X_{\mathrm{Fl}_{\hat{X}}^{s,t}(p)} \quad\text{and}\quad \mathrm{Fl}_{\hat{X}}^{t,t} = \mathrm{id}_M$$

Thus $\mathrm{Fl}_{\hat{X}}^{s,t}(p)$ satisfies the same ODE as the flow of X, $\mathrm{Fl}_X^{s-t}(p)$. By the uniqueness of solutions of ODE's, these two maps must be the same in a neighbourhood of p. In particular, $\mathrm{Fl}_{\hat{X}}^{s,t} = \mathrm{Fl}_X^{s-t}$ for all triples $(s,t,p)\in\big\{(s,t,p)\in\mathbb{R}^2\times M\,\big|\,(s-t,p)\in\mathrm{dom}(\mathrm{Fl}_X)\big\}$. $\natural$

Example 4.3.10. For the case of example 4.3.6, given the vector field $X_t(x_1,\ldots,x_m) = \sum\limits_{i=1}^{m}\frac{1-x_i^2}{t}\frac{\partial}{\partial x_i}$ and writing the flow of this vector field as

$$\mathrm{Fl}_X^{s,t}(p) = (\phi_1(s,t,p),\ldots,\phi_m(s,t,p))$$

and $p=(x_1,\ldots,x_m)$, the conditions for the flow give, for all $i=1,\ldots,m$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_i(s,t,p) = \frac{1-(\phi_i(s,t,p))^2}{s} \quad\text{and}\quad \phi_i(t,t,p) = x_i.$$

Therefore, by solving the differential equation on the left, we get

$$\begin{aligned}\int_{x_i}^{\phi_i(s,t,p)}\frac{\mathrm{d}\phi_i}{1-\phi_i^2} = \int_t^s\frac{\mathrm{d}s'}{s'} &\implies \mathrm{arctanh}(\phi_i(s,t,p)) - \mathrm{arctanh}(x_i) = \ln\left|\frac{s}{t}\right|\\ &\implies \phi_i(s,t,p) = \tanh\left(\ln\left|\frac{s}{t}\right| + \mathrm{arctanh}(x_i)\right),\end{aligned}$$

where $\mathrm{arctanh}:\,]-1,1[\,\to\mathbb{R}$ is the hyperbolic arctangent function. Thus, the solution for the flow of X is

$$\mathrm{Fl}_X^{s,t}(x_1,\ldots,x_m) = \left(\frac{x_1(s^2+t^2)+s^2-t^2}{x_1(s^2-t^2)+s^2+t^2},\ldots,\frac{x_m(s^2+t^2)+s^2-t^2}{x_m(s^2-t^2)+s^2+t^2}\right).$$ $\natural$

4.3.2 Moser Method

To more easily prove Darboux's theorem, we shall utilise a method, first introduced by Moser in Ref. [14], which essentially follows the proof of the following theorem:

Theorem 4.3.11 (Moser)**.** *Let M be a compact m-dimensional manifold, let $\{\omega_t\}_{t\in[0,1]}$ be a family of symplectic forms such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega_t \in \mathrm{d}\big(\Omega^1(M)\big).$$

Then, there is a family of diffeomorphisms $\{\phi_t\}_{t\in[0,1]}$ of M such that

$$\phi_t^*\omega_t = \omega_0 \quad \textit{and} \quad \phi_0 = \mathrm{id}_M. \tag{4.30}$$

Proof. To obtain the intended result, let us instead obtain a time-dependent vector field X such that $\phi_t = \mathrm{Fl}_X^{t,0}$. Since ϕ_t must satisfy Eq. (4.30), differentiating both sides of this equation yields:

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\omega_t &= \phi_t^*\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}s}\phi_s^*\omega_t\bigg|_{s=t} = \phi_t^*\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Fl}_X^{s,0}\big)^*\omega_t\bigg|_{s=t} \\ &= \phi_t^*\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{Fl}_X^{s+t,0}\big)^*\omega_t\bigg|_{s=0} = \phi_t^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\omega_t + \mathcal{L}_{X_t}\omega_t\right) \end{aligned}$$

Taking $\eta_t \in \Omega^1(M)$ such that $\mathrm{d}\eta_t = \frac{\mathrm{d}}{\mathrm{d}t}\omega_t$ and using Cartan's magic formula, we get that

$$\mathrm{d}(\eta_t + i_{X_t}\omega_t) = 0.$$

So, in particular, given some $f_t \in C^\infty(M)$, if $\omega_t^\flat(X_t) = -\eta_t + \mathrm{d}f_t$, the equation above is satisfied. Since ω_t is a symplectic form for all t, there is a unique X_t satisfying this property. Having now defined a vector field, to obtain the diffeomorphism ϕ_t required by the theorem is just the process of solving the ODE determining the flow of X. The compactness of M guarantees that X is complete and, so $\phi_t = \mathrm{Fl}_X^{t,0}$ is a symplectomorphism from (M,ω_0) to (M,ω_t) for all t. Q.E.D.

Remark 4.3.12. The proof of this theorem gives a particular useful method to solve many problems in differential geometry. This method, called *Moser's trick*, even though only entirely rigorous when we consider M to be compact (otherwise we would obtain that $\phi_t^*\omega_t = \omega_0$ only on $\mathrm{dom}(\mathrm{Fl}_X^{t,0})$), it can still be applied in general since we are only interested in finding local diffeomorphisms.

4.3.3 Darboux and Foliated Darboux Theorems

We are now ready to state and prove Darboux's theorem:

Theorem 4.3.13 (Darboux)**.** *Let (M, ω) be a $2s$-dimensional symplectic manifold. Then, given $p \in M$, there exist an open neighbourhood $U \subseteq M$ of p and a diffeomorphism $\varphi : U \to V \subseteq \mathbb{R}^{2s}$ such that $\varphi^*\omega_{\mathrm{can}}|_V = \omega|_U$.*

Proof. Let $U \subseteq M$ be an open neighbourhood of $p \in M$ and $\psi : U \to V \subseteq \mathbb{R}^{2s}$ a chart. Calling $\omega_0 = \psi^*\omega_{\mathrm{can}}|_V$, we define, for all $t \in [0,1]$

$$\omega_t = \omega_0 + t(\omega|_U - \omega_0).$$

Clearly, ω_t defines a family of symplectic forms on U and, moreover, $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \omega|_U - \omega_0$ which is closed. Consequently, by Poincaré's lemma, up to shrinking U around p, we can assume that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t$ is exact, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\eta_t$ for some $\eta_t \in \Omega^1(U)$ for $t \in [0,1]$. Thus, we may apply Moser's trick and obtain a family of local diffeomorphisms $\{\phi_t\}_{t\in[0,1]}$ such that $\phi_t^*\omega_t = \omega_0$ with $\phi_0 = \mathrm{id}_M$. In particular, for $t = 1$,

$$\psi^*\omega_{\mathrm{can}}|_V = \omega_0 = \phi_1^*\omega_1 = \phi_1^*\omega|_U,$$

or, reorganising the previous expression and defining $\varphi = \psi \circ \phi_1^{-1} : U \to V$ (which is a local diffeomorphism),

$$\varphi^*\omega_{\mathrm{can}}|_V = \omega|_U. \qquad \text{Q.E.D.}$$

Corollary 4.3.14. *Let (M, ω) be a $2s$-dimensional symplectic manifold and $p \in M$. Then, there is local chart $(U, \varphi = (Q^1, \ldots, Q^s, P_1, \ldots, P_s))$ on M around p such that*

$$\omega|_U = \sum_{i=1}^{s} \mathrm{d}Q^i \wedge \mathrm{d}P_i. \tag{4.31}$$

Remark 4.3.15. A chart satisfying this corollary is called a *Darboux chart.*

Having now obtained one of the most fundamental theorems in symplectic geometry, we state certain important consequences of such a powerful theorem. One such consequence is, in fact, an extension of Darboux's theorem due to Weinstein:

Theorem 4.3.16 (Darboux-Weinstein)**.** *Let M be $2s$-dimensional manifold, $N \subseteq M$ a compact submanifold and ω_1, ω_2 two symplectic forms on M such that $(\omega_1)_p = (\omega_2)_p \in \bigwedge^2 T_p^*M$, for all $p \in N$. Then, there exist open neighbourhoods U_1 and U_2 of N in M and a diffeomorphism $\phi : U_1 \to U_2$ such that $\phi : (U_1, \omega_1|_{U_1}) \to (U_2, \omega_2|_{U_2})$ is a symplectomorphism.*

The proof of this theorem uses a similar procedure as the one from Theorem 4.3.13 but also uses the concepts of integration of fibres as well as of Riemannian metrics, both of which are out of the scope of the current thesis. The interested reader can check the proof of this result, for instance, in [13, Lemma 3.2.1].

We now look at specific types of submanifolds, as seen in the previous section.

Proposition 4.3.17. *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and let $N_i \subseteq M_i$ $(i = 1, 2)$ be compact symplectic submanifolds. Suppose there is a diffeomorphism $\varphi : N_1 \to N_2$ covered by a symplectic vector bundle isomorphism $\hat{\varphi} : TM_1|_{N_1} \to TM_2|_{N_2}$ restricting to $\varphi_* : TN_1 \to TN_2$. Then φ extends to a symplectic isomorphism $\psi : U_1 \to U_2$ where U_i is a neighbourhood of N_i in M_i for $i = 1, 2$.*

This result is essentially a consequence of the Darboux-Weinstein theorem, as the proof of this statement boils down to said theorem once we reduce to the case where $M_1 = M_2$ and $N_1 = N_2$. For more details, cf. [13, Theorem 3.4.10], for instance.

Proposition 4.3.18. *Let (M, ω) be a symplectic manifold and $L \subseteq M$ a Lagrangian submanifold. Then, there exist open neighbourhoods U and V of $L \cong \mathrm{im}(\theta_{T^*L})$ in M and T^*L, respectively, and a diffeomorphism $\phi : U \to V$ such that*

$$\phi|_L = \theta_{T^*L} \quad \textit{and} \quad \phi^*(-\mathrm{d}\lambda_L) = \omega \quad \textit{on} \quad U, \tag{4.32}$$

where $\lambda_L \in \Omega^1(T^*L)$ is the Liouville form on T^*L.

The proof of this result requires the usage of almost complex structures on a manifold, which is out of the scope for the current text. The interested reader can check the proof of this result, for instance, in [13, Theorem 3.4.13]. If we also add the requirement that the Lagrangian submanifold is compact, the proof of this weaker statement uses the notion of tubular neighbourhoods, that we shall introduce in Chapter 8. For the proof of this altered statement, we redirect the interested reader to [15, Theorem 9.3]

Foliated Darboux theorem

To conclude this section, we shall give a look at foliated manifolds and foliated forms.

Definition 4.3.19. A *k-dimensional foliated manifold* is a pair $(M, \mathcal{F})$, where M is a manifold and $\mathcal{F} = \{\mathcal{F}_\lambda \mid \lambda \in \Lambda\}$ is a k-dimensional foliation of M.

Given this additional structure of a foliation on the manifold, it is more natural to study the forms relative to this foliation.

Definition 4.3.20. Let $(M,\mathcal{F})$ be a foliated manifold. Then, if we denote by $\mathfrak{X}(\mathcal{F})=\Gamma(T\mathcal{F})$ the space of vector fields tangent to the foliation $\mathcal{F}$, we define the space of foliated k-forms by $\Omega^k(\mathcal{F}) \overset{\text{def}}{=} \Gamma(\bigwedge^k T^*\mathcal{F})$.

Moreover, since, by Frobenius theorem, $T\mathcal{F}$ is involutive, the space $\mathfrak{X}(\mathcal{F})$ is equipped with a Lie algebra structure, so we may also define a foliated de Rham differential as

Definition 4.3.21. For a foliated manifold $(M,\mathcal{F})$, the *foliated de Rham differential* $\mathrm{d}_{\mathcal{F}}$ is a derivation in the algebra $(\Omega^\bullet(\mathcal{F}),\wedge)$ which is also a homogeneous map of degree 1, i.e. $\mathrm{d}_{\mathcal{F}}:\Omega^k(\mathcal{F})\to\Omega^{k+1}(\mathcal{F})$ such that, given $\omega\in\Omega^k(\mathcal{F})$ and $X_1,\dots,X_{k+1}\in\mathfrak{X}(\mathcal{F})$,

$$\begin{aligned}\mathrm{d}_{\mathcal{F}}\omega(X_1,\dots,X_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i-1}X_i(\omega(X_1,\dots,\hat{X}_i,\dots,X_{k+1})) \\ &\quad+\sum_{\substack{i,j=1\\ i<j}}^{k+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{k+1}),\end{aligned} \tag{4.33}$$

where, once again, the notation $\hat{X}_i$ means that we omit the entry X_i from ω.

The notions introduced in Section 1.1 can be generalised to the setting of foliated manifolds. In what concerns us, we are interested in the foliated Darboux theorem. To get to that result, we must firstly generalise the Poincaré lemma in this context:

Lemma 4.3.22 (Foliated Poincaré)**.** *Let $(M,\mathcal{F})$ be a k-dimensional foliated manifold and let $\omega\in\Omega^\ell(\mathcal{F})$ such that $\mathrm{d}_{\mathcal{F}}\omega=0$. Then, for every $p\in M$, there exists a neighbourhood $U\subseteq M$ of p and $\eta\in\Omega^{\ell-1}(\mathcal{F}|_U)$ such that $\omega|_U=\mathrm{d}_{\mathcal{F}|_U}\eta$.*

Proof. Let $(U,\varphi=(x_1,\dots,x_k,z_1,\dots,z_{m-k}))$ be a chart centred at $p\in M$ such that $T_p\mathcal{F}=\operatorname{span}\left\{\frac{\partial}{\partial x_i}\;\middle|\; i=1,\dots,k\right\}$ (it is possible to choose such a chart by the global Frobenius theorem 2.3.12). Considering the map $\psi:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^m$ such that $\psi(t,(x,z))=\psi_t(x,z)=(tx,z)$, notice that $\psi_{e^t}=\mathrm{Fl}_X^t$ for $X=\sum\limits_{i=1}^k x_i\frac{\partial}{\partial x_i}\in\mathfrak{X}(\varphi(U))$. Then, for $t>0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\mathrm{Fl}_X^{\ln(t)}\big)^*\omega=\frac{1}{t}\big(\mathrm{Fl}_X^{\ln(t)}\big)^*\mathcal{L}_X\omega=\frac{1}{t}\big(\mathrm{Fl}_X^{\ln(t)}\big)^*(\mathrm{d}_{\mathcal{F}}i_X\omega+i_X\mathrm{d}_{\mathcal{F}}\omega)\\ &=\frac{1}{t}\mathrm{d}_{\mathcal{F}}\big(\mathrm{Fl}_X^{\ln(t)}\big)^*i_X\omega=\frac{1}{t}\mathrm{d}_{\mathcal{F}}\psi_t^*i_X\omega\end{aligned}$$

and, since $(\psi_t)_*(x,z)=(t\,\mathrm{id}_{\mathbb{R}^k})\times\mathrm{id}_{\mathbb{R}^{m-k}}$, given $v_2,\dots,v_\ell\in T_{(x,z)}\mathcal{F}$,

$$\begin{aligned}\left(\frac{1}{t}\psi_t^*i_X\omega\right)_{(x,z)}(v_2,\dots,v_\ell) &= \frac{1}{t}(i_X\omega)_{(tx,z)}(tv_2,\dots,tv_\ell)\\ &=\frac{1}{t}\omega_{(tx,z)}(X_{(tx,z)},tv_2,\dots,tv_\ell)\\ &=t^{k-1}\omega_{(tx,z)}(X_{(x,z)},v_2,\dots,v_\ell)\end{aligned}$$

so we get that, for all $\ell \geq 1$, $\frac{1}{t}\psi_t^* i_X \omega$ is a unique foliated ℓ-form of $(M, \mathcal{F})$. Since $\omega \in \Omega^\ell(\mathcal{F})$, it is clear that $\psi_1^*\omega = \omega$ and $\psi_0^*\omega = 0$. Thus,

$$\begin{aligned}\omega = \psi_1^*\omega - \psi_0^*\omega &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\omega \,\mathrm{d}t = \int_0^1 \frac{1}{t}\mathrm{d}_{\mathcal{F}}\psi_t^* i_X\omega \,\mathrm{d}t \\ &= \int_0^1 \mathrm{d}_{\mathcal{F}}\left(\frac{1}{t}\psi_t^* i_X \omega\right)\mathrm{d}t = \mathrm{d}_{\mathcal{F}}\left(\int_0^1 \frac{1}{t}\psi_t^* i_X\omega \,\mathrm{d}t\right),\end{aligned}$$

so, we define $\eta = \int_0^1 \frac{1}{t}\psi_t^* i_X \omega \,\mathrm{d}t$ to obtain the result. Q.E.D.

Similarly to what is done in 'regular' symplectic geometry, we may now introduce the notion of foliated symplectic manifold:

Definition 4.3.23. A *(regular) symplectic foliation* is a triple $(M, \mathcal{F}, \omega_{\mathcal{F}})$, where $(M, \mathcal{F})$ is a foliated manifold and $\omega_{\mathcal{F}} \in \Omega^2(\mathcal{F})$ is a foliated 2-form which is leaf-wise symplectic, i.e. such that for a every leaf $\mathcal{F}_\lambda$ of $\mathcal{F}$, $(\mathcal{F}_\lambda, \omega_{\mathcal{F}}|_{\mathcal{F}_\lambda})$ is a symplectic manifold.

This definition allows us to still define the Hamiltonian vector field of a map $H \in C^\infty(M)$ the same way as before by defining as the foliated vector field $X_H \in \mathfrak{X}(\mathcal{F})$ satisfying

$$i_{X_H}\omega_{\mathcal{F}} = \mathrm{d}_{\mathcal{F}} H$$

and we also obtain an equivalent normal form theorem for these symplectic foliations:

Proposition 4.3.24 (Foliated Darboux). *Let $(M, \mathcal{F}, \omega_{\mathcal{F}})$ be a regular symplectic foliation of dimension $2s$ and codimension $n = m - 2s$. Then, for every point p, there is a chart $(U, \varphi = (x^1, \dots, x^s, y_1, \dots, y_s, z^1, \dots, z^n))$ such that:*

(i) $T\mathcal{F} = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_s}\right\}$ *or, equivalently, the leafs of the foliation are given by the level sets for which $(z^1, \dots, z^n)$ are constant;*

(ii) the leaf-wise symplectic form is given by $\omega_{\mathcal{F}}|_U = \sum_{i=1}^{s} \mathrm{d}_{\mathcal{F}} x^i \wedge \mathrm{d}_{\mathcal{F}} y_i$.

Proof. Let $(U, \varphi = (x^1, \dots, x^s, y_1, \dots, y_s, z^1, \dots, z^n))$ be a chart centred at a point $p \in M$ such that $\varphi_*(q)(T_q\mathcal{F}) = \mathbb{R}^{2s} \subseteq \mathbb{R}^m \cong T_{\varphi(q)}\mathbb{R}^m$, for all $q \in U$, which can be always done by the global Frobenius theorem 2.3.12. Then, by Darboux theorem 4.3.13 applied to $T_p^*\varphi(\omega_p) \in \bigwedge^2 T_0^*\mathbb{R}^{2s} \subseteq \bigwedge^2 T_0^*\mathbb{R}^m$, where $T_p^*\varphi$ denotes the cotangent lift of φ at p, we get a diffeomorphism $\psi : V \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^m$ such that

$$T_p^*(\psi \circ \varphi)(\omega_p) = \sum_{i=1}^{s} (\mathrm{d}x^i)_0 \wedge (\mathrm{d}y_i)_0.$$

Calling $\omega_0 = \omega|_U$ and $\omega_1 = (\psi \circ \varphi)^*\left(\sum_{i=1}^s \mathrm{d}x^i \wedge \mathrm{d}y_i\right)$, it is clear that we have that $(\omega_0)_p = (\omega_1)_p \in \bigwedge^2 T_p^*\mathcal{F}$. By the foliated Poincaré lemma 4.3.22, up to shrinking U, there exists a foliated 1-form $\eta \in \Omega^1(\mathcal{F})$ such that $\omega_1 - \omega_0 = \mathrm{d}_{\mathcal{F}}\eta$ and $\eta_p = 0 \in T_p^*\mathcal{F}$. Now, we follow a similar procedure as the proof of Darboux theorem, using a method that resembles Moser's trick. Defining a time-dependent 2-form

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega|_U + t\left(\omega|_U - (\psi \circ \varphi)^*\left(\sum_{i=1}^{s} \mathrm{d}x^i \wedge \mathrm{d}y_i\right)\right),$$

it is clear that $\mathrm{d}_{\mathcal{F}}\omega_t = 0$ for all $t \in \mathbb{R}$ and, up to shrinking U around p, $(\omega_t)_p \in \bigwedge^2 T_p^*\mathcal{F}$ is non-degenerate for all $t \in [0,1]$. Then, there exists a unique time-dependent vector field $X_t \in \mathfrak{X}(\mathcal{F})$, depending smoothly on $t \in [0,1]$ such that $i_{X_t}\omega_t + \eta = 0$, for all $t \in [0,1]$, by the same argument as used in the proof of Moser's theorem 4.3.11. Taking $\phi_t : U_t \to \phi_t(U_t)$ to be the foliated local diffeomorphism of $(M, \mathcal{F})$ describing the flow of X_t,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\omega_t = \phi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right) &= \phi_t^*\left(\mathrm{d}_{\mathcal{F}} i_{X_t}\omega_t + i_{X_t}\omega_t \mathrm{d}_{\mathcal{F}}\omega_t + (\omega_1 - \omega_0)\right)\\ &= \phi_t^*\mathrm{d}_{\mathcal{F}}(i_{X_t}\omega_t + \eta) = 0,\end{aligned}$$

therefore, $(\phi_t^*\omega_t)_q = (\phi_0^*\omega_0)_q$ for all $t \in [0,1]$ and $q \in U_t$. In particular, $\phi_1^*\omega_1 = \omega_0$ on U_1, which gives a local diffeomorphism such that $(\phi_1)_*(q)(T_q\mathcal{F}) = T_{\phi_1(q)}\mathcal{F}$ for all $q \in U_1$ and $\phi_1^*\omega_1 = \omega_0$ in U_1. To conclude the result, notice that $\chi = \psi \circ \varphi \circ \phi_1 : U_1 \to \mathbb{R}^m$ is a coordinate chart on M centred at p such that

$$\chi_*(T\mathcal{F}) = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_s}\right\} \text{ and } \chi^*\left(\sum_{i=1}^{s} \mathrm{d}x^i \wedge \mathrm{d}y_i\right) = \omega|_{U_1}.$$

Q.E.D.

Poisson Manifolds

Chapter 5

Introduction to Poisson Structures

We now turn our sights to the study of Poisson geometry. This type of geometry arises naturally when one studies classical mechanics, similarly to Symplectic geometry, as well as Lie's theory for solving partial differential equations (also very prominent in physics problems). In fact, as we shall see in these and the next chapter, Poisson geometry is, in a certain sense, like an organism with its core components (these being symplectic geometry, Lie theory and foliation theory) working together to form a beautiful and cohesive geometry that, in the last decades, has spewed an incredible amount of new research topics. In this chapter, however, we lay the groundwork by exploring the direct consequences of the definitions and by viewing Poisson manifolds in two distinct yet equivalent ways, using the Poisson bracket, which appears more notably when studying physics, and using bivector fields which turns out to be the more natural way of understanding Poisson geometry.

The chapter is divided into the following: we start in the first section by introducing the Poisson manifolds using a bilinear anti-symmetric bracket as well as give the standard examples of this class of manifolds. Section 5.2 deals with the morphism one can define on a Poisson manifold as well as the special types of vector fields and their Lie algebra structures. On Section 5.3, we introduce another way of looking at Poisson manifolds making use of multivectors and consider the Lie algebroid structure arising on the cotangent bundle of a Poisson manifold. The fourth and final section discusses the Poisson structure one can define on the dual of a Lie algebroid as an extension of a classic example of the linear Poisson structure on the dual of Lie algebras.

5.1 Poisson brackets

Let us start by defining the essential operation that is at the centre of the discussion of Poisson geometry:

Definition 5.1.1. Let M be a smooth manifold.

1) A bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that is an $\mathbb{R}$-bilinear map, anti--symmetric, and obeys the following properties, $\forall\, f, g, h \in C^\infty(M)$:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \qquad \text{(Jacobi identity)} \tag{5.1}$$

$$\{f, gh\} = \{f, g\}\ h + g\ \{f, h\} \qquad \text{(Leibniz rule)} \tag{5.2}$$

is called a *Poisson bracket* on the set $C^\infty(M)$ of smooth maps on M;

2) A *Poisson manifold* is a pair $(M, \{\cdot,\cdot\})$ where $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a Poisson bracket.

Notice that, in the first definition, a Poisson bracket is nothing more than a Lie bracket on the set $C^\infty(M)$ with the additional property of it being a derivation in each entry. This allows us to say that $C^\infty(M)$ is a Poisson algebra which we define as follows:

Definition 5.1.2. A *Poisson algebra* is an R-algebra (where R is a commutative ring or more usually a field) A together with a Lie bracket $\{\cdot,\cdot\} : A \times A \to A$ such that, for every $a \in A$, $\{a, \cdot\} : A \to A$ is a derivation of A, i.e. $\{\cdot,\cdot\}$ is a Poisson bracket.

As it is usual in constructions in Mathematics, it is useful, once introduced a space with certain properties, to also introduce a type of maps that preserve the new structure we just added to our spaces. For the case of Poisson manifolds, we introduce the notion of a Poisson map/morphism:

Definition 5.1.3. A *Poisson map* or *Poisson morphism* between Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ is a smooth map $\Phi : M_1 \to M_2$ inducing a Lie algebra homomorphism:

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi, \quad \forall\, f, g \in C^\infty(M_2). \tag{5.3}$$

Remark 5.1.4. With the introduction of Poisson morphisms, we can define the category of Poisson manifolds, **PoissMan**, whose objects are Poisson manifolds $(M, \{\cdot,\cdot\})$ and morphisms are Poisson maps.

We could now question whether it is possible to express the Poisson bracket locally in an unique way. In fact that is possible by noticing that the Poisson bracket is local:

Proposition 5.1.5. *A Poisson bracket $\{\cdot,\cdot\}$ on a manifold M is local, i.e. for all $f, g \in C^\infty(M)$,*

$$\operatorname{supp}\big(\{f,g\}\big) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g),$$

where the support *of the map f is defined as* $\operatorname{supp}(f) \stackrel{\text{def}}{=} \overline{\{p \in M \mid f(p) \neq 0\}}$. *In equivalent terms, a Poisson structure is such that, for any open subset $U \subseteq M$ and any $f, g \in C^\infty(M)$, if $f|_U = 0$, then $\{f,g\}|_U = 0$.*

Proof. Given any $p \in U$, we may always find a function $\varphi \in C^\infty(M)$ with compact support such that $\operatorname{supp}(\varphi) \subseteq U$ and $\varphi(p) = 1$. Then, if $f|_U = 0$ for a function $f \in C^\infty(M)$, $\varphi f = 0$ and, for any $g \in C^\infty(M)$,

$$0 = \{0, g\}(p) = \{\varphi f, g\}(p) = \{\varphi, g\}(p) f(p) + \varphi(p)\{f, g\}(p) = \{f, g\}(p),$$

where, in the third equality we used the fact that $\varphi(p) = 1$ and $f(p) = 0$. Since $p \in U$ is arbitrary, we conclude that $\{f,g\}|_U = 0$. Q.E.D.

Remark 5.1.6. This local characteristic of the Poisson structure is nothing particular to a Poisson structure. In fact, given any anchored vector bundle $(E \xrightarrow{\text{pr}} M, \mathsf{a})$, an $\mathbb{R}$-bilinear, anti-symmetric bracket $\langle\cdot,\cdot\rangle : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying the Leibniz rule (with respect to the anchor), i.e. such that, for $\sigma_1, \sigma_2 \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$\langle \sigma_1, f\sigma_2 \rangle = f\langle \sigma_1, \sigma_2 \rangle + \mathsf{a}(\sigma_1)(f)\,\sigma_2,$$

satisfies this local property (the proof is entirely analogous to the one given for the Poisson structure). The triple $(E \xrightarrow{\text{pr}} M, \mathsf{a}, \langle\cdot,\cdot\rangle)$ as above is called a *Leibniz algebroid.*

Remark 5.1.7. Given an open subset $U \subseteq M$ of a manifold M and a point $p \in U$, for any function $f \in C^\infty(U)$, we may always find a function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_V = f|_V$ for some $V \subseteq U$ and open neighbourhood of a point $p \in U$ in U. In fact, by considering a function $\varphi \in C^\infty(M)$ with compact support such that $\operatorname{supp}(\varphi) \subseteq U$, and $\varphi|_V = 1$ for some open neighbourhood V of p in U, with define $\tilde{f}$ at any $q \in M$ as

$$\tilde{f}(q) = \begin{cases} 0, & \text{if } q \notin \operatorname{supp}(\varphi) \\ \varphi(q) f(q), & \text{if } q \in U \end{cases}.$$

This construction allows us to finally define show that any Poisson structure can always be expressed locally.

Proposition 5.1.8. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. For any open subset U of M, there exists a unique Poisson structure $\{\cdot,\cdot\}_U$ on U such that the inclusion map $\iota : (U, \{\cdot,\cdot\}_U) \hookrightarrow (M, \{\cdot,\cdot\})$ is a Poisson map or, equivalently, given any $f, g \in C^\infty(M)$,*

$$\{f|_U, g|_U\}_U = \{f, g\}\,|_U.$$

Proof. The previous Remark and the local character of the Poisson structure guarantee that there exists a unique function $\{f,g\}_U \in C^\infty(U)$ such that, for all $p \in U$ $\{f,g\}_U(p) = \{\tilde{f}, \tilde{g}\}(p)$ for some functions $\tilde{f}, \tilde{g} \in C^\infty(M)$ such that $\tilde{f}|_V = f|_V$ and $\tilde{g}|_V = g|_V$ for some open neighbourhood V of p in U. Then, this property allows us to show that $\{\cdot,\cdot\}_U$ is, in fact, a Poisson structure on U. Q.E.D.

Thus, Proposition 5.1.8 essentially allows any Poisson bracket to be expressed (locally for a chart $(U, \varphi = (x^1, \ldots, x^m))$), for any two maps $f, g \in C^\infty(U)$, as

$$\{f, g\}_U = \sum_{i,j=1}^{m} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \tag{5.4}$$

for some smooth functions $\pi^{ij} \in C^\infty(U)$ with $i, j = 1, \ldots, m$, where, the anti-symmetry of the Poisson bracket reflects unto the anti-symmetry of the coefficients π^{ij}, i.e. for all $i, j = 1, \ldots, m$, $\pi^{ij} = -\pi^{ji}$. Moreover, we can show that, in order for the Poisson bracket (5.4) to obey the Jacobi identity, one must have the following, for $f, g, h \in C^\infty(U)$:

$$\begin{aligned}
\{f, \{g,h\}_U\}_U &= \left\{ f, \sum_{j,k=1}^{m} \pi^{jk}\, \partial_j g\, \partial_k h \right\}_U = \sum_{i,\ell=1}^{m} \pi^{i\ell}\, \partial_i f\, \partial_\ell \left(\sum_{j,k=1}^{m} \pi^{jk}\, \partial_j g\, \partial_k h \right) \\
&= \sum_{i,j,k,\ell=1}^{m} \pi^{i\ell}\, \partial_i f \big(\partial_\ell \pi^{jk}\, \partial_j g\, \partial_k h + \pi^{jk}\, (\partial_\ell \partial_j g)\, \partial_k h + \pi^{jk}\, \partial_j g\, (\partial_\ell \partial_k h) \big),
\end{aligned}$$

where we used the simplified notation $\partial_i \overset{\text{def}}{=} \frac{\partial}{\partial x^i}$. Adding the corresponding terms cycling the entries in the Poisson brackets and changing the indices when necessary, we get

$$\begin{aligned}
0 &= \{f, \{g,h\}_U\}_U + \{g, \{h,f\}_U\}_U + \{h, \{f,g\}_U\}_U \\
&= \sum_{i,j,k,\ell=1}^{m} \big(\pi^{i\ell} \partial_\ell \pi^{jk} + \pi^{k\ell} \partial_\ell \pi^{ij} + \pi^{j\ell} \partial_\ell \pi^{ki} \big) \partial_i f\, \partial_j g\, \partial_k h \\
&\quad + \sum_{i,j,k,\ell=1}^{m} \Big((\partial_\ell \partial_i f)\, \partial_j g\, \partial_k h (\pi^{k\ell}\pi^{ij} + \pi^{j\ell}\pi^{ki}) + \partial_i f\, (\partial_\ell \partial_j g)\, \partial_k h (\pi^{i\ell}\pi^{jk} + \pi^{k\ell}\pi^{ij}) \\
&\qquad\qquad + \partial_i f\, \partial_j g\, (\partial_\ell \partial_k h)(\pi^{i\ell}\pi^{jk} + \pi^{jl}\pi^{ki}) \Big)
\end{aligned}$$

where the terms in the second sum vanish since $\partial_\ell \partial_i f = \partial_i \partial_\ell f$ ($f \in C^\infty(U) \subseteq C^2(U)$) but $\pi^{k\ell}\pi^{ij} + \pi^{j\ell}\pi^{ki} = -\pi^{ki}\pi^{\ell j} - \pi^{ji}\pi^{k\ell}$, so each term contains the product of a symmetric matrix with an anti-symmetric one, which always vanishes. Thus, the Jacobi identity implies that the matrices of smooth maps $\pi^{ij} \in C^\infty(U)$ obey the following condition, for all $i, j, k = 1, \ldots, m$,

$$\sum_{\ell=1}^{m} (\pi^{i\ell}\partial_\ell \pi^{jk} + \pi^{k\ell}\partial_\ell \pi^{ij} + \pi^{j\ell}\partial_\ell \pi^{ki}) = 0. \tag{5.5}$$

Let us now look at some important examples of Poisson manifolds that will follow us as we expand our knowledge of this geometric theory:

Example 5.1.9. The simplest possible case of a Poisson manifold is one in which the Poisson brackets are identically 0, i.e.

$$\{f, g\} = 0, \ \forall\, f, g \in C^\infty(M).$$

Naturally, such a bracket obeys all the necessary properties for it to be a Poisson bracket but, naturally, it is not an interesting case of a Poisson manifold. ♮

Example 5.1.10. A slightly more interesting example of a Poisson bracket is that of a constant one, in which $M = \mathbb{R}^m$ and $\pi^{ij}(p) = c^{ij} \in \mathbb{R}$, for all $p \in \mathbb{R}^m$. Thus, we may write the *constant Poisson bracket* of the maps $f, g \in C^\infty(M)$ as

$$\{f, g\} = \sum_{i,j=1}^{m} c^{ij} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j},$$

or, without mentioning any specific coordinate basis,

$$\{f, g\} = (\nabla f)^t\, \mathbf{C}\, \nabla g,$$

where $\mathbf{C} \in M_m(\mathbb{R})$ is an arbitrary anti-symmetric $m \times m$ matrix with real entries c^{ij}. As one can easily see, the bracket defined previously obeys all the properties necessary of a Poisson bracket including Jacobi identity, since it trivially satisfies Eq. (5.5). ♮

Example 5.1.11. One other class of important examples of Poisson manifolds are precisely symplectic manifolds. As seen in Chapter 4, it is possible to define Hamiltonian vector fields by $i_{X_H}\omega = \mathrm{d}H$ for each $H \in C^\infty(M)$ by this definition, the Lie bracket of two Hamiltonian vector fields X_f and X_g is also a Hamiltonian vector field $X_{\omega(X_f, X_g)}$. This fact prompts us to define the canonical Poisson bracket:

$$\{f, g\}_{\text{can}} = -\omega(X_f, X_g). \tag{5.6}$$

In particular, for $\mathbb{R}^{2s}$ with coordinates $(q^1, \ldots, q^s, p_1, \ldots, p_s)$, the canonical symplectic structure takes the form $\omega = \sum\limits_{i=1}^{s} \mathrm{d}q^i \wedge \mathrm{d}p_i$. For this structure, we can show that the Poisson bracket takes the canonical form:

$$\{f, g\} = \sum_{i=1}^{s} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} \right).$$

As we shall see in the future, symplectic structures are essential to the study of Poisson structures as they are, in some sense, the building blocks of all the possible Poisson structures we can introduce on a manifold. ♮

Example 5.1.12. The final class of examples is that of *linear Poisson brackets.* This type of Poisson brackets arises by considering a vector space V and a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(V)$ that produces a linear function if both its entries are linear, i.e. since the linear functions on V are just the elements of the dual of V ($C^\infty_{\mathrm{lin}}(V) = V^*$), we have that $\{V^*, V^*\} \subseteq V^*$. Using the local definition of the brackets for a (global) coordinate chart $(U, (x^1, \ldots, x^m))$ on M such that, any element $v \in V$ can be written as $v = \sum_{i=1}^{m} x^i\, e_i$ for some basis $\{e_1, \ldots, e_m\}$ of V, we obtain that the coefficients appearing in Eq. (5.4) are of the form $\pi^{ij} = \sum_{k=1}^{m} c_k^{ij}\, x^k$, so the bracket for linear maps $f, g \in V^*$ is just

$$\{f, g\}\,|_U = \sum_{i,j,k=1}^{m} c_k^{ij}\, x^k \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}.$$

Furthermore, if we look at the Jacobi identity that the bracket $\{\cdot,\cdot\}$ must obey or, equivalently, the PDE in Eq. (5.5) that the given π^{ij}'s must satisfy, we obtain a relation for the constants c_k^{ij}:

$$\sum_{\ell=1}^{m} \left(c_p^{i\ell}\, c_\ell^{jk} + c_p^{k\ell}\, c_\ell^{ij} + c_p^{j\ell}\, c_\ell^{ki}\right) = 0,$$

which is the same relation obeyed by the structure constants of a Lie algebra on V^*.

Conversely, and more generally, if $(\mathfrak{g}, [\cdot,\cdot])$ is a real finite-dimensional Lie algebra, we may associate to each element $\ell \in \mathfrak{g}$ to the evaluation map $\mathrm{ev}_\ell : \mathfrak{g}^* \to \mathbb{R}$, defined by

$$\begin{aligned} \mathrm{ev}_\ell : \mathfrak{g}^* &\longrightarrow \mathbb{R} \\ \lambda &\longmapsto \lambda(\ell) \end{aligned}$$

Then, for these evaluation maps, we may define a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ as

$$\{\mathrm{ev}_{\ell_1}, \mathrm{ev}_{\ell_2}\} \stackrel{\mathrm{def}}{=} \mathrm{ev}_{[\ell_1,\ell_2]}, \quad \forall\, \ell_1, \ell_2 \in \mathfrak{g}. \tag{5.7}$$

For the general case of a map $f \in C^\infty(\mathfrak{g}^*)$, we must consider the differential structure present in $\mathfrak{g}$. Let $\lambda \in \mathfrak{g}^*$ and $f_*(\lambda) : T_\lambda\mathfrak{g}^* \to \mathbb{R}$ be the derivative map of f. We may identify $T_\lambda\mathfrak{g}^* \cong \mathfrak{g}^*$ and, so, $f_*(\lambda) \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$ which allows us to rewrite the Poisson bracket in (5.7) as

$$\{f, g\}\,(\lambda) = \lambda([f_*(\lambda), g_*(\lambda)]), \quad \forall\, \lambda \in \mathfrak{g}^*. \tag{5.8}$$

If we give a basis $\{e_1, \ldots, e_n\}$ to the Lie algebra $\mathfrak{g}$, with associate coordinates in $\mathfrak{g}^*$ given by $\{x^1, \ldots, x^n\}$, then it is possible to rewrite Eq. (5.8) in a local form since, now,

$$\lambda = \sum_{a=1}^{n} \lambda_a\, x^a, \quad f_*(\lambda) = \sum_{a=1}^{n} \frac{\partial f}{\partial x^a}(\lambda)\, e_a \quad \text{(analogous for } g\text{)}.$$

Then, by defining the structure constants k_{ab}^c of the Lie algebra $\mathfrak{g}$ by $[e_a, e_b] = \sum_{c=1}^{n} k_{ab}^c\, e_c$,

$$\begin{aligned}\{f, g\}(\lambda) &= \lambda\left(\left[\sum_{a=1}^{n} \frac{\partial f}{\partial x^a}(\lambda)\, e_a, \sum_{b=1}^{n} \frac{\partial g}{\partial x^b}(\lambda)\, e_b\right]\right) = \lambda\left(\sum_{a,b=1}^{n} \frac{\partial f}{\partial x^a}(\lambda)\frac{\partial g}{\partial x^b}(\lambda)[e_a, e_b]\right) \\ &= \sum_{a,b=1}^{n} \frac{\partial f}{\partial x^a}(\lambda)\frac{\partial g}{\partial x^b}(\lambda)\lambda\left(\sum_{c=1}^{n} k_{ab}^c\, e_c\right) = \sum_{a,b,c=1}^{n} \frac{\partial f}{\partial x^a}(\lambda)\frac{\partial g}{\partial x^b}(\lambda) k_{ab}^c\, \lambda(e_c) \\ &= \sum_{a,b,c=1}^{n} k_{ab}^c\, \lambda_c \frac{\partial f}{\partial x^a}(\lambda)\frac{\partial g}{\partial x^b}(\lambda).\end{aligned}$$

Thus, we obtain a linear Poisson bracket on $\mathfrak{g}^*$. These prove the following statement:

Proposition 5.1.13. *For any finite-dimensional vector space $\mathfrak{g}$, there is a natural one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Linear Poisson brackets} \\ \{\cdot,\cdot\}\ \textit{on}\ \mathfrak{g}^*\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\textit{Lie algebra structures} \\ [\cdot,\cdot]\ \textit{on}\ \mathfrak{g}\end{array}\right\},$$

established by relation (5.7).

For a Lie algebroid A, given its space of sections has the structure of a Lie algebra, it is also possible to define a 'linear' Poisson structure on the dual A^* of A. Since such a discussion requires introduction of more concepts that would break the natural progression of the chapter, we shall delay such a discussion for the section at the end of the present chapter. ♮

5.2 Poisson and Hamiltonian Automorphisms and Vector fields

Following the notion of a Poisson morphism and the category of Poisson manifolds, it is also important to explore particular structures obtained by studying the isomorphisms (or, in particular, the automorphisms) of this category. Thus, given a Poisson manifold $(M, \{\cdot,\cdot\})$, if a Poisson morphism $\Phi : M \to M$ is also a diffeomorphism, we shall call Φ a *Poisson automorphism* or *Poisson diffeomorphism* and denote:

$$\mathrm{Aut}(M, \{\cdot,\cdot\}) = \big\{\Phi \in \mathrm{Diff}(M) \,\big|\, \{f\circ\Phi, g\circ\Phi\} = \{f, g\}\circ\Phi\ \forall\, f, g \in C^\infty(M)\big\}. \tag{5.9}$$

Proposition 5.2.1. *The pair* $\big(\mathrm{Aut}(M, \{\cdot,\cdot\}), \circ\big)$ *forms a group which is a subgroup of* $(\mathrm{Diff}(M), \circ)$*, the group of diffeomorphism in a manifold M, i.e.* $\mathrm{Aut}(M, \{\cdot,\cdot\}) \leq \mathrm{Diff}(M)$ *as groups.*

Proof. It is clear that $\mathrm{id}_M \in \mathrm{Aut}(M, \{\cdot,\cdot\})$ and that, if $\Phi \in \mathrm{Aut}(M, \{\cdot,\cdot\})$ then also $\Phi^{-1} \in \mathrm{Aut}(M, \{\cdot,\cdot\})$. To conclude that $\big(\mathrm{Aut}(M, \{\cdot,\cdot\}), \circ\big)$ forms a subgroup of $(\mathrm{Diff}(M), \circ)$, let $\Phi, \Psi \in \mathrm{Aut}(M, \{\cdot,\cdot\})$ and $f, g \in C^\infty(M)$. Then:

$$\begin{aligned}\{f \circ (\Phi \circ \Psi), g \circ (\Phi \circ \Psi)\} &= \{(f \circ \Phi) \circ \Psi, (g \circ \Phi) \circ \Psi\} = \{f \circ \Phi, g \circ \Phi\} \circ \Psi \\ &= (\{f, g\} \circ \Phi) \circ \Psi = \{f, g\} \circ (\Phi \circ \Psi),\end{aligned}$$

so $\Phi \circ \Psi \in \mathrm{Aut}(M, \{\cdot,\cdot\})$ and we obtain the result. Q.E.D.

Infinitesimally, we may also define Poisson vector fields the following way:

Definition 5.2.2. In a Poisson manifold $(M, \{\cdot,\cdot\})$, a *Poisson vector field* X is a vector field such that its flow Fl_X^t is a (local) Poisson diffeomorphism, i.e. for all $t \in \mathbb{R}$ and $f, g \in C^\infty(M)$, one gets that

$$(\mathrm{Fl}_X^t)^* \{f, g\} = \left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\}, \text{ on } \mathrm{dom}(\mathrm{Fl}_X^t).$$

Remark 5.2.3. Following the fact that Poisson diffeomorphisms are also considered to be automorphisms in the category of Poisson manifolds, we can conclude that a Poisson vector field is, in fact, an *infinitesimal Poisson automorphism*. Hence, in the sequel, we may treat these vector fields as such.

These Poisson vector fields on $(M, \{\cdot,\cdot\})$ admit the following characterisation:

Proposition 5.2.4. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and $X \in \mathfrak{X}(M)$ a vector field on M. Then, X is a Poisson vector field on $(M, \{\cdot,\cdot\})$ if and only if X is a derivation of the Poisson bracket $\{\cdot,\cdot\}$, i.e. if, for all $f, g \in C^\infty(M)$,*

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}\,. \tag{5.10}$$

Proof. $\Longrightarrow$) Let Fl_X^t be the flow of X. Since X is a Poisson vector field, Fl_X^t is a Poisson diffeomorphism and, thus, by differentiating the defining expression of such Poisson diffeomorphisms at $t = 0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\{f, g\} \circ \mathrm{Fl}_X^t\big)\Big|_{t=0} = X(\{f, g\}), \tag{$*$}$$

by the definition of the flow of a vector field, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\{f \circ \mathrm{Fl}_X^t, g \circ \mathrm{Fl}_X^t\}\big)\Big|_{t=0} &= \left\{\frac{\mathrm{d}}{\mathrm{d}t} f \circ \mathrm{Fl}_X^t, g \circ \mathrm{Fl}_X^t\right\}\Bigg|_{t=0} + \left\{f \circ \mathrm{Fl}_X^t, \frac{\mathrm{d}}{\mathrm{d}t} g \circ \mathrm{Fl}_X^t\right\}\Bigg|_{t=0} \\ &= \left\{\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}_X^t)^* f\Big|_{t=0}, g \circ \mathrm{Fl}_X^0\right\} + \left\{f \circ \mathrm{Fl}_X^0, \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}_X^t)^* g\Big|_{t=0}\right\} \\ &= \{X(f), g\} + \{f, X(g)\}\,.\end{aligned} \tag{$\dagger$}$$

This implication is obtained by comparing Eqs. ($*$) and ($\dagger$).

$\Longleftarrow$) Let us now assume that $X \in \mathfrak{X}(M)$ is a vector field satisfying Eq. (5.10) for any $f, g \in C^\infty(M)$. Let $p \in M$ and consider a smooth function

$$\begin{aligned}\gamma : I \subseteq \mathbb{R} &\longrightarrow \mathbb{R}\\ t &\longmapsto ((\mathrm{Fl}_X^{-t})^* \left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\})(p),\end{aligned}$$

where $I \subseteq \mathbb{R}$ is a real interval containing 0. Then, by the properties of the flow of a vector field, $\gamma(0) = \{f, g\}(p)$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) &= \frac{\mathrm{d}}{\mathrm{d}t}((\mathrm{Fl}_X^{-t})^* \left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\})(p)\\ &= \left(\left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\} \right)_*(\mathrm{Fl}_X^{-t}(p)) \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}_X^{-t}(p))\right)\\ &\qquad + \left((\mathrm{Fl}_X^{-t})^* \left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\}\right)\right)(p)\\ &= \left(\left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\} \right)_*(\mathrm{Fl}_X^{-t}(p))(-X_{\mathrm{Fl}_X^{-t}(p)})\\ &\qquad + \left((\mathrm{Fl}_X^{-t})^* \left(\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\} + \left\{(\mathrm{Fl}_X^t)^* f, \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Fl}_X^t)^* g\right\}\right)\right)(p)\\ &= -X(\left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\})(\mathrm{Fl}_X^{-t}(p)) + \Big((\mathrm{Fl}_X^{-t})^* \big(\left\{X((\mathrm{Fl}_X^t)^* f), (\mathrm{Fl}_X^t)^* g\right\}\\ &\qquad\qquad + \left\{(\mathrm{Fl}_X^t)^* f, X((\mathrm{Fl}_X^t)^* g)\right\} \big)\Big)(p)\\ &= \Big((\mathrm{Fl}_X^{-t})^* \big(- X(\left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\}) + \left\{X((\mathrm{Fl}_X^t)^* f), (\mathrm{Fl}_X^t)^* g\right\}\\ &\qquad\qquad + \left\{(\mathrm{Fl}_X^t)^* f, X((\mathrm{Fl}_X^t)^* g)\right\} \big)\Big)(p)\\ &= 0\end{aligned}$$

where, in the last equality we used the fact that X satisfies Eq. (5.10). Therefore, since $\gamma(0) = \{f, g\}(p)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = 0$ for all $t \in I$, we conclude that $\gamma(t) = \{f, g\}(p)$, for all $t \in I$. In conclusion, through an obvious transformation, we get the result, for all $t \in \mathbb{R}$:

$$(\mathrm{Fl}_X^t)^* \{f, g\} = \left\{(\mathrm{Fl}_X^t)^* f, (\mathrm{Fl}_X^t)^* g\right\} \quad \text{on} \quad \mathrm{dom}(\mathrm{Fl}_X^t).$$

Q.E.D.

Hence, we may define the real vector space of all Poisson vector fields as the set of vector fields obeying a Leibniz rule relative to the Poisson bracket. We shall denote such a vector space as

$$\mathfrak{aut}(M, \{\cdot,\cdot\}) = \left\{X \in \mathfrak{X}(M) \,\middle|\, X \text{ is a Poisson vector field of } (M, \{\cdot,\cdot\})\right\}. \tag{5.11}$$

Proposition 5.2.5. *For a Poisson manifold $(M, \{\cdot,\cdot\})$, the Lie algebra of infinitesimal Poisson automorphisms $\big(\mathfrak{aut}(M, \{\cdot,\cdot\}), [\cdot,\cdot]\big)$ is a Lie subalgebra of $(\mathfrak{X}(M), [\cdot,\cdot])$, i.e.*

$$\big[\mathfrak{aut}(M, \{\cdot,\cdot\}), \mathfrak{aut}(M, \{\cdot,\cdot\})\big] \subseteq \mathfrak{aut}(M, \{\cdot,\cdot\})$$

and we shall denote $\mathfrak{aut}(M, \{\cdot,\cdot\}) \leq \mathfrak{X}(M)$ as Lie algebras.

Proof. Let $X, Y \in \mathfrak{aut}(M, \{\cdot,\cdot\})$ be Poisson vector fields and $f, g \in C^\infty(M)$. Then,

$$\begin{aligned}
[X,Y](\{f,g\}) &= X(Y(\{f,g\})) - Y(X(\{f,g\})) \\
&= X(\{Y(f),g\} + \{f,Y(g)\}) - Y(\{X(f),g\} + \{f,X(g)\}) \\
&= \{X(Y(f)),g\} + \{Y(f),X(g)\} + \{X(f),Y(g)\} + \{f,X(Y(g))\} \\
&\qquad - \{Y(X(f)),g\} - \{X(f),Y(g)\} - \{Y(f),X(g)\} - \{f,Y(X(g))\} \\
&= \{[X,Y](f),g\} + \{f,[X,Y](g)\},
\end{aligned}$$

where, in the second and third equalities, we used the fact that X, Y are Poisson vector fields, in the second equality, the $\mathbb{R}$-linearity of the vector fields and we note that the terms in red and teal cancel out. Thus, we obtain that $[X,Y] \in \mathfrak{aut}(M, \{\cdot,\cdot\})$ and conclude the result. Q.E.D.

One other very important type of vector fields comes directly from the fact that a Poisson bracket must obey the Leibniz rule. These shall be called the Hamiltonian vector fields and they can be defined so we have an agreement with the same notion in symplectic geometry:

Definition 5.2.6. The *Hamiltonian vector field* associated to a smooth map $H \in C^\infty(M)$ is the unique vector field X_H such that, for any $f \in C^\infty(M)$:

$$X_H(f) \overset{\text{def}}{=} \{H, f\}. \tag{5.12}$$

We shall denote the real vector space of all Hamiltonian vector fields by

$$\mathfrak{ham}(M, \{\cdot,\cdot\}) \overset{\text{def}}{=} \big\{X_f \in \mathfrak{X}(M) \,\big|\, f \in C^\infty(M)\big\}, \tag{5.13}$$

and we obtain the following result:

Proposition 5.2.7. *For a Poisson manifold $(M, \{\cdot,\cdot\})$, the set of Hamiltonian vector fields, $\mathfrak{ham}(M, \{\cdot,\cdot\})$, forms a Lie subalgebra of $\mathfrak{X}(M)$, i.e.*

$$\big[\mathfrak{ham}(M, \{\cdot,\cdot\}), \mathfrak{ham}(M, \{\cdot,\cdot\})\big] \subseteq \mathfrak{ham}(M, \{\cdot,\cdot\}).$$

More precisely, for any $f, g \in C^\infty(M)$, one gets that

$$[X_f, X_g] = X_{\{f,g\}}. \tag{5.14}$$

Proof. Expression is equivalent to the Jacobi identity, since, by applying the LHS of Eq. (5.14) to $h \in C^\infty(M)$, we get

$$\begin{aligned}
[X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) = X_f(\{g,h\}) - X_g(\{f,h\}) \\
&= \{f,\{g,h\}\} + \{g,\{h,f\}\} = -\{h,\{f,g\}\} \\
&= X_{\{f,h\}}(h),
\end{aligned}$$

where we used the anti-symmetry of the Poisson brackets as well as the definition of Hamiltonian vector fields to obtain the result. Q.E.D.

In particular, we can relate the Hamiltonian vector fields with the infinitesimal Poisson automorphisms the following way:

Proposition 5.2.8. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Then the Lie algebra of Hamiltonian vector fields forms a Lie ideal of $\mathfrak{aut}(M, \{\cdot,\cdot\})$, i.e.*

$$\big[\mathfrak{aut}(M, \{\cdot,\cdot\}), \mathfrak{ham}(M, \{\cdot,\cdot\})\big] \subseteq \mathfrak{ham}(M, \{\cdot,\cdot\})$$

and we shall denote $\mathfrak{ham}(M, \{\cdot,\cdot\}) \trianglelefteq \mathfrak{aut}(M, \{\cdot,\cdot\})$ as Lie algebras.

Proof. Let $f, g \in C^\infty(M)$ and $Y \in \mathfrak{aut}(M, \{\cdot,\cdot\})$. Then,

$$\begin{aligned}[][Y, X_f](g) &= Y(X_f(g)) - X_f(Y(g)) = Y(\{f, g\}) - \{f, Y(g)\} \\ &= \{Y(f), g\} + \{f, Y(g)\} - \{f, Y(g)\} = \{Y(f), g\} = X_{Y(f)}(g),\end{aligned}$$

where, in the second and fifth equalities we used the definition of Hamiltonian vector fields and, in the third equality, that Y is an infinitesimal Poisson automorphism. Thus, we conclude that $[Y, X_f] = X_{Y(f)} \in \mathfrak{ham}(M, \{\cdot,\cdot\})$ and, therefore, that $\mathfrak{ham}(M, \{\cdot,\cdot\}) \trianglelefteq \mathfrak{aut}(M, \{\cdot,\cdot\})$ as Lie algebras. Q.E.D.

From Propositions 5.2.7 and 5.2.8, we can reorganise the Lie algebras of vector fields, Hamiltonian vector fields and infinitesimal Poisson automorphisms as:

$$\mathfrak{ham}(M, \{\cdot,\cdot\}) \trianglelefteq \mathfrak{aut}(M, \{\cdot,\cdot\}) \leq \mathfrak{X}(M) \quad \text{as Lie algebras.} \tag{5.15}$$

One could now question if we find an equivalent relation between the different collections of diffeomorphisms but, unfortunately, that is not so evident because we do not necessarily know whether there is a collection of Poisson automorphisms which, infinitesimally, correspond to the Lie algebra of Hamiltonian vector fields. In an attempt to remedy the lack of such a set, we introduce the following notion of a Hamiltonian diffeomorphism:

Definition 5.2.9. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold.

1) A diffeomorphism $\phi : M \to M$ is called a *Hamiltonian automorphism* or *Hamiltonian diffeomorphism* if there is a smooth family of diffeomorphisms $\Phi^t : M \to M$ $(t \in [0,1])$ with $\Phi^0 = \mathrm{id}_M$ and $\Phi^1 = \phi$ such that there exists a smooth family of functions $\big\{H_t \in C^\infty(M) \,\big|\, t \in [0,1]\big\}$ on M with

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p) = (X_{H_t})_{\Phi^t(p)}, \ \forall\, (p,t) \in M \times [0,1].$$

2) The collection of Hamiltonian automorphisms is denoted $\mathrm{Ham}(M, \{\cdot,\cdot\})$ and it is called the *Hamiltonian Group* of $(M, \{\cdot,\cdot\})$.

A family $\left\{\Phi^t \,\middle|\, t \in [0,1]\right\}$ as in the previous definition is called a *Hamiltonian isotopy* of $(M, \{\cdot,\cdot\})$. Note that Φ^t is the flow of the time-dependent vector field X_{H_t}. With this definition, we can show that an analogue of the relation in equation (5.15) is still valid. Firstly, though, we shall show that the algebra of Hamiltonian vector fields is, in fact the infinitesimal version of this Lie group:

Proposition 5.2.10. *For a Poisson manifold* $(M, \{\cdot,\cdot\})$*, the group of Hamiltonian automorphisms,* $\mathrm{Ham}(M, \{\cdot,\cdot\})$*, forms a normal subgroup of* $\mathrm{Aut}(M, \{\cdot,\cdot\})$*, and we shall denote* $\mathrm{Ham}(M, \{\cdot,\cdot\}) \trianglelefteq \mathrm{Aut}(M, \{\cdot,\cdot\})$ *as groups.*

Proof. Firstly, it is clear that $\mathrm{Ham}(M, \{\cdot,\cdot\}) \subseteq \mathrm{Aut}(M, \{\cdot,\cdot\})$. In fact, given $\phi \in \mathrm{Ham}(M, \{\cdot,\cdot\})$, if $\left\{\Phi^t \,\middle|\, t \in [0,1]\right\}$ is the Hamiltonian isotopy associated with ϕ, then, since X_{H_t} is a Hamiltonian vector field,

$$\begin{aligned}
X_{H_t}(\{f \circ \Phi^t, g \circ \Phi^t\}) &= \left\{X_{H_t}(f \circ \Phi^t), g \circ \Phi^t\right\} + \left\{f \circ \Phi^t, X_{H_t}(g \circ \Phi^t)\right\} \\
&= \left\{\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \Phi^t), g \circ \Phi^t\right\} + \left\{f \circ \Phi^t, \frac{\mathrm{d}}{\mathrm{d}t}(g \circ \Phi^t)\right\} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left\{f \circ \Phi^t, g \circ \Phi^t\right\}.
\end{aligned}$$

Moreover, by the properties of the pull-back, we get that

$$X_{H_t}(\{f, g\} \circ \Phi^t) = \frac{\mathrm{d}}{\mathrm{d}t}(\{f, g\} \circ \Phi^t).$$

Thus, for any $p \in M$, both functions of $t \in \mathbb{R}$ given by $\{f \circ \Phi^t, g \circ \Phi^t\}$ and $\{f, g\} \circ \Phi^t$ are solutions of the same initial-value problem of the same linear ODE. Hence, they must coincide, i.e. $\{f \circ \Phi^t, g \circ \Phi^t\} = \{f, g\} \circ \Phi^t$ on M for all $t \in \mathbb{R}$ and, so $\Phi^t \in \mathrm{Aut}(M, \{\cdot,\cdot\})$, for all $t \in [0,1]$. In particular, for $t = 1$, we get that $\phi \in \mathrm{Aut}(M, \{\cdot,\cdot\})$.

Next, we show that $(\mathrm{Ham}(M, \{\cdot,\cdot\}), \circ)$ forms a group and, in particular, a subgroup of $\mathrm{Aut}(M, \{\cdot,\cdot\})$. Naturally, the Hamiltonian isotopy related to the identity map is just the $\Phi^t = \mathrm{id}_M$, $\forall\, t \in [0,1]$ (which corresponds to $H_t \equiv 0$, $\forall\, t \in [0,1]$) and for the inverse map ϕ^{-1} of ϕ, the Hamiltonian isotopy is just $\left\{(\Phi^t)^{-1} \,\middle|\, t \in [0,1]\right\}$. Then one can quickly note that, given $\phi, \psi \in \mathrm{Ham}(M, \{\cdot,\cdot\})$ with Hamiltonian isotopies $\left\{\Phi^t \,\middle|\, t \in [0,1]\right\}$ and $\left\{\Psi^t \,\middle|\, t \in [0,1]\right\}$, respectively, such that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p) = (X_{H_t})_{\Phi^t(p)}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Psi^t(p) = (X_{K_t})_{\Psi^t(p)}$ we can construct the Hamiltonian isotopy for the composition $\psi \circ \phi$ as $\left\{\Psi^t \circ \Phi^t \,\middle|\, t \in [0,1]\right\}$. Therefore, for any $p \in M$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Psi^t \circ \Phi^t)(p) &= (\Psi^t)_*(\Phi^t(p))\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(p)\right) + \frac{\mathrm{d}}{\mathrm{d}s}\Big(\Psi^s \circ \Phi^t(p)\Big)\Big|_{s=t} \\
&= (\Psi^t)_*(\Phi^t(p))((X_{H_t})_{\Phi^t(p)}) + (X_{K_t})_{(\Psi^t \circ \Phi^t)(p)} \\
&= ((\Psi^t)_* X_{H_t})_{(\Psi^t \circ \Phi^t)(p)} + (X_{K_t})_{(\Psi^t \circ \Phi^t)(p)} \\
&= ((\Psi^{-t})^* X_{H_t})_{(\Psi^t \circ \Phi^t)(p)} + (X_{K_t})_{(\Psi^t \circ \Phi^t)(p)} \\
&= \big((\Psi^{-t})^* X_{H_t} + X_{K_t}\big)_{(\Psi^t \circ \Phi^t)(p)}.
\end{aligned}$$

Since X_{H_t} is a Hamiltonian vector field of $(M, \{\cdot,\cdot\})$ and Ψ^t is a Poisson automorphism of $(M, \{\cdot,\cdot\})$ for all t, given an arbitrary function $f \in C^\infty(M)$, by Lemma 1.1.59,

$$\begin{aligned}\big((\Psi^t)^* X_{H_t}\big)(f) &= X_{H_t}(f \circ \Psi^{-t}) \circ \Psi^t = \big\{H_t, f \circ \Psi^{-t}\big\} \circ \Psi^t \\ &= \big\{H_t \circ \Psi^t, f\big\} = X_{(\Psi^t)^* H_t}(f).\end{aligned}$$

Therefore, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi^t \circ \Phi^t)(p) = \big(X_{(\Psi^{-t})^* H_t} + X_{K_t}\big)_{\Psi^t \circ \Phi^t(p)} = \big(X_{(\Psi^{-t})^* H_t + K_t}\big)_{\Psi^t \circ \Phi^t(p)},$$

where, in the second equality we used the fact that, given two Hamiltonian vector fields $X_f, X_g \in \mathfrak{ham}(M, \{\cdot,\cdot\})$ of two functions $f, g \in C^\infty(M)$, for any $h \in C^\infty(M)$,

$$(X_f + X_g)(h) = X_f(h) + X_g(h) = \{f, h\} + \{g, h\} = \{f + g, h\} = X_{f+g}(h).$$

Thus, we conclude that $\psi \circ \phi$ is Hamiltonian diffeomorphism and, consequently, that $\mathrm{Ham}(M, \{\cdot,\cdot\})$ is a subgroup of $\mathrm{Aut}(M, \{\cdot,\cdot\})$.

Finally, to show that $\mathrm{Ham}(M, \{\cdot,\cdot\})$ is a normal subgroup, let $\psi \in \mathrm{Aut}(M, \{\cdot,\cdot\})$ and $\phi \in \mathrm{Ham}(M, \{\cdot,\cdot\})$. Then, we claim that $\big\{\alpha^t = \psi^{-1} \circ \Phi^t \circ \psi \,\big|\, t \in [0,1]\big\}$ is a Hamiltonian isotopy of $\psi^{-1} \circ \phi \circ \psi$. Clearly $\alpha^0 = \mathrm{id}_M$ and $\alpha^1 = \psi^{-1} \circ \phi \circ \psi$. Furthermore, given $p \in M$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\alpha^t(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\psi^{-1} \circ \Phi^t \circ \psi(p)\Big) &= (\psi^{-1})_*(\Phi^t \circ \psi(p))\left(\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(\psi(p))\right) \\ &= (\psi^{-1})_*(\Phi^t \circ \psi(p))((X_{H_t})_{\Phi^t \circ \psi(p)}) = (\psi_*^{-1} X_{H_t})_{(\psi^{-1} \circ \Phi^t \circ \psi)(p)} \\ &= (\psi^* X_{H_t})_{\psi^{-1} \circ \Phi^t \circ \psi(p)} = (X_{\psi^* H_t})_{\psi^{-1} \circ \Phi^t \circ \psi(p)},\end{aligned}$$

where, in the second-to-last equality we used the definition of the pull-back of vector fields and, in the last equality, that ψ is a Poisson automorphism. Hence, $\psi^{-1} \circ \phi \circ \psi \in \mathrm{Ham}(M, \{\cdot,\cdot\})$ is a Hamiltonian automorphism and $\mathrm{Ham}(M, \{\cdot,\cdot\})$ is a normal subgroup of $\mathrm{Aut}(M, \{\cdot,\cdot\})$. Q.E.D.

Thus, as we previously obtained for the vector fields, we conclude that

$$\mathrm{Ham}(M, \{\cdot,\cdot\}) \trianglelefteq \mathrm{Aut}(M, \{\cdot,\cdot\}) \leq \mathrm{Diff}(M) \quad \text{as groups.} \tag{5.16}$$

5.3 Poisson Bivectors

An alternative way to analyse Poisson structures is through the introduction of multi-vector fields. Motivated by expression (5.4) and repeated here for convenience

$$\{f,g\}_U = \sum_{i,j=1}^{m} \pi^{ij} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}, \quad \pi^{ij} \in C^\infty(U), \tag{5.4}$$

we shall explore the case of biderivations, i.e. the case of elements in $\mathfrak{X}^2(M)$. This case is a special one since it corresponds to anti-symmetric $\mathbb{R}$-bilinear maps from $C^\infty(M)^{\times 2}$ to $C^\infty(M)$ satisfying the Leibniz rule, i.e., all the properties of a Poisson bracket apart from the Jacobi identity. To introduce this identity, it is necessary to make use of the Schouten bracket. The following proposition succinctly expresses the connection between Poisson brackets and bivector fields:

Theorem 5.3.1. *Let M be a smooth manifold. There is a natural one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Poisson brackets} \\ \{\cdot,\cdot\} \ \textit{on } M \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{Bivector fields } \pi \in \mathfrak{X}^2(M) \\ \textit{such that } [\pi,\pi]=0 \end{array} \right\}. \tag{5.17}$$

established by the relation

$$\pi(\mathrm{d}f,\mathrm{d}g) = \mathfrak{D}_\pi(f,g) = \{f,g\}, \quad \textit{for all } f,g \in C^\infty(M). \tag{5.18}$$

Proof. To conclude the result, we need only to compute, for any $\pi \in \mathfrak{X}^2(M)$ under the identification (5.18), the Schouten bracket $[\pi,\pi] \in \mathfrak{X}^3(M)$ of any functions $f,g,h \in C^\infty(M)$. For that purpose, we use Eqs. (1.56) and (1.57) and identify $f_1 = f,\ f_2 = g,\ f_3 = h$:

$$\begin{aligned}
[\pi,\pi](\mathrm{d}f,\mathrm{d}g,\mathrm{d}h) &= \mathfrak{D}_{[\pi,\pi]}(f,g,h) \\
&= \sum_{\sigma\in S^{1,2}} \mathrm{sgn}(\sigma)\Big(\mathfrak{D}_\pi(f_{\sigma(1)},\mathfrak{D}_\pi(f_{\sigma(2)},f_{\sigma(3)})) \\
&\qquad\qquad - (-1)^{1\times 1}\mathfrak{D}_\pi(f_{\sigma(1)},\mathfrak{D}_\pi(f_{\sigma(2)},f_{\sigma(3)}))\Big) \\
&= 2\sum_{\sigma\in S^{1,2}} \mathrm{sgn}(\sigma)\mathfrak{D}_\pi\big(f_{\sigma(1)},\mathfrak{D}_\pi(f_{\sigma(2)},f_{\sigma(3)})\big) \\
&= 2\sum_{\sigma\in S^{1,2}} \mathrm{sgn}(\sigma)\pi\big(\mathrm{d}f_{\sigma(1)},\mathrm{d}(\pi(\mathrm{d}f_{\sigma(2)},\mathrm{d}f_{\sigma(3)}))\big) \\
&= 2\Big(\pi\big(\mathrm{d}f,\mathrm{d}(\pi(\mathrm{d}g,\mathrm{d}h))\big) - \pi\big(\mathrm{d}g,\mathrm{d}(\pi(\mathrm{d}f,\mathrm{d}h))\big) + \pi\big(\mathrm{d}h,\mathrm{d}(\pi(\mathrm{d}f,\mathrm{d}g))\big)\Big) \\
&= 2\big(\{f,\pi(\mathrm{d}g,\mathrm{d}h)\} - \{g,\pi(\mathrm{d}f,\mathrm{d}h)\} + \{h,\pi(\mathrm{d}f,\mathrm{d}g)\}\big) \\
&= 2\big(\{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\}\big) \\
&= 2\big(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\big)
\end{aligned}$$

Thus, we get that

$$\frac{1}{2}[\pi,\pi](\mathrm{d}f,\mathrm{d}g,\mathrm{d}h) = \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\,, \tag{5.19}$$

and, thus, the bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity if and only if its associated biderivation $\pi \in \mathfrak{X}^2(M)$ satisfies $[\pi,\pi] = 0$. Q.E.D.

Locally, if $\{\cdot,\cdot\}$ is expressed as in (5.4), its associated bivector field $\pi \in \mathfrak{X}^2(M)$ can locally be written as

$$\pi|_U = \frac{1}{2}\sum_{i,j=1}^{m} \pi^{ij}\,\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}\;.$$

Furthermore, this identification allows us to define

Definition 5.3.2. A bivector field $\pi \in \mathfrak{X}^2(M)$ satisfying $[\pi,\pi] = 0$ is called a *Poisson structure* or *Poisson bivector field* in M. Following this, we also call (M,π) a *Poisson manifold.*

Proposition 5.3.3. *Let $X \in \mathfrak{X}(M)$ be a vector field in a Poisson manifold (M,π). The following conditions are equivalent:*

(i) X is a Poisson vector field, i.e. $X \in \mathfrak{aut}(M,\pi)$;

(ii) π is invariant over the flow of X, i.e. $(\mathrm{Fl}^t_X)^*\pi = \pi$ on $\mathrm{dom}(\mathrm{Fl}^t_X)$, for all $t \in \mathbb{R}$;

(iii) $\mathcal{L}_X\pi = [X,\pi] = 0$.

Proof. (i) $\Longleftrightarrow$ (ii): Let X be a Poisson vector field and $\mathrm{Fl}^t_X : M \to M$ its flow. Since X is (also) an infinitesimal Poisson automorphism, Fl^t_X is a Poisson automorphism. Thus, given $p \in \mathrm{dom}(\mathrm{Fl}^t_X)$ and $f,g \in C^\infty(M)$,

$$\begin{aligned}\big(\pi(\mathrm{d}f,\mathrm{d}g)\circ \mathrm{Fl}^t_X\big)_p &= \big(\{f,g\}\circ \mathrm{Fl}^t_X\big)_p = \big(\{f\circ \mathrm{Fl}^t_X, g\circ \mathrm{Fl}^t_X\}\big)_p\\ &= \big(\pi(\mathrm{d}(f\circ \mathrm{Fl}^t_X),\mathrm{d}(g\circ \mathrm{Fl}^t_X))\big)_p = \big(\pi((\mathrm{Fl}^t_X)^*\mathrm{d}f,(\mathrm{Fl}^t_X)^*\mathrm{d}g)\big)_p\\ &= \big(((\mathrm{Fl}^t_X)_*\pi)(\mathrm{d}f,\mathrm{d}g)\big)_{\mathrm{Fl}^t_X(p)} = \big(((\mathrm{Fl}^t_X)_*\pi)(\mathrm{d}f,\mathrm{d}g)\circ \mathrm{Fl}^t_X\big)_p,\end{aligned}$$

where, in the fifth equality, we used the fact that, for any multivector field $\theta \in \mathfrak{X}^k(M)$ and any map $\phi : M \to M$:

$$(\phi_*\theta)(\alpha_1,\dots,\alpha_k) = \phi_*\big(\theta(\phi^*\alpha_1,\dots,\phi^*\alpha_k)\big), \quad \forall\,\alpha_1,\dots,\alpha_k \in \Omega^1(M).$$

Thus, for points on $\mathrm{im}(\mathrm{Fl}^t_X)$, we get that $(\mathrm{Fl}^t_X)_*\pi = \pi$ or, equivalently, for points on $\mathrm{dom}(\mathrm{Fl}^t_X)$, we get $(\mathrm{Fl}^t_X)^*\pi = \pi$.

(i) $\Longleftrightarrow$ *(iii)*: Let $f, g \in C^\infty(M)$. Using Eqs. (1.56) and (1.57), for any vector field $X \in \mathfrak{X}(M)$ we can write the Schouten bracket $[X, \pi]$ as

$$\begin{aligned}[X,\pi](\mathrm{d}f,\mathrm{d}g) &= (-1)^0\, X(\pi(\mathrm{d}f,\mathrm{d}g)) - (-1)^{1\times 0}(-1)^0\pi(\mathrm{d}f,\mathrm{d}(X(g))) \\ &\quad - (-1)^{0\times 1}(-1)^1\pi(\mathrm{d}g,\mathrm{d}(X(f))) \\ &= X(\pi(\mathrm{d}f,\mathrm{d}g)) - \pi(\mathrm{d}f,\mathrm{d}(X(g))) - \pi(\mathrm{d}(X(f)),\mathrm{d}g).\end{aligned}$$

Now, noting that $\pi(\mathrm{d}f,\mathrm{d}g) = \{f,g\}$ we obtain

$$[X,\pi](\mathrm{d}f,\mathrm{d}g) = X(\{f,g\}) - \{f,X(g)\} - \{X(f),g\}\,.$$

Thus, since $f, g \in C^\infty(M)$ are arbitrary, we get that X is a Poisson vector field if and only if $[X,\pi] = 0$. Q.E.D.

5.3.1 Induced Poisson Map: $\pi^\sharp : T^*M \to TM$

Given a bivector field $\pi \in \mathfrak{X}^2(M)$, it is possible to obtain an induced map

$$\begin{aligned}\pi^\sharp : T^*M &\longrightarrow TM \\ \alpha &\longmapsto i_\alpha \pi\end{aligned}$$

where $\alpha \in T_p^*M$ for some $p \in M$ and

$$\begin{aligned}i_\alpha : \textstyle\bigwedge^{k+1} T_pM &\longrightarrow \textstyle\bigwedge^{k} T_pM \\ \theta &\longmapsto i_\alpha\theta\end{aligned}$$

is the interior product (of multivectors) defined by $(i_\alpha\theta)(\beta_1,\dots,\beta_k) = \theta(\alpha,\beta_1,\dots,\beta_k)$, for all $\beta_1,\dots,\beta_k \in T_p^*M$. Naturally, we shall denote the induced map between sections of the tangent and cotangent bundles by the same symbol $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$. Thus, it is possible to express the condition $[\pi,\pi] = 0$ in terms of $\pi^\sharp$ using the following operation between 1-forms:

$$[\alpha,\beta]_\pi \overset{\text{def}}{=} \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - \mathrm{d}(\pi(\alpha,\beta)). \tag{5.20}$$

Using Cartan's magical formula, the previous expression takes the form

$$\begin{aligned}[\alpha,\beta]_\pi &= i_{\pi^\sharp\alpha}\mathrm{d}\beta + \mathrm{d}(\beta(\pi^\sharp\alpha)) - i_{\pi^\sharp\beta}\mathrm{d}\alpha - \mathrm{d}(\alpha(\pi^\sharp\beta)) - \mathrm{d}(\pi(\alpha,\beta)) \\ &= i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(\beta(\pi^\sharp\alpha)) - \mathrm{d}(\alpha(\pi^\sharp\beta)) - \mathrm{d}(\pi(\alpha,\beta)) \\ &= i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,\beta)) - \mathrm{d}(\pi(\beta,\alpha)) - \mathrm{d}(\pi(\alpha,\beta))\end{aligned}$$

Thus, cancelling out the terms in red and using the anti-symmetry of π, it is also possible to describe the bracket as

$$[\alpha,\beta]_\pi = i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,\beta)). \tag{5.21}$$

We now introduce some of the properties of this bracket.

Proposition 5.3.4. *Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field associated with the biderivation $\{\cdot,\cdot\}$. The bracket $[\cdot,\cdot]_\pi$ defined on $\Omega^1(M)$ is the unique $\mathbb{R}$-bilinear, anti-symmetric operation in $\Omega^1(M)$ such that*

(i) $[\mathrm{d}f, \mathrm{d}g]_\pi = \mathrm{d}\,\{f,g\}$, *for all $f,g \in C^\infty(M)$ (Action of $[\cdot,\cdot]_\pi$ on exact 1-forms);*

(ii) $[\alpha, f\beta]_\pi = f\,[\alpha,\beta]_\pi + (\pi^\sharp\alpha(f))\,\beta$, *for all $f \in C^\infty(M)$ and $\alpha,\beta \in \Omega^1(M)$ (Leibniz rule).*

Proof. Firstly, it is clear that the bracket $[\cdot,\cdot]_\pi$ defined in Eq. (5.21) is $\mathbb{R}$-bilinear and anti--symmetric. Furthermore, we can quickly check that it obeys properties (i) and (ii) given:

(i) Let $f,g \in C^\infty(M)$. Then we can easily get

$$[\mathrm{d}f,\mathrm{d}g]_\pi = i_{\pi^\sharp \mathrm{d}f}\mathrm{d}^2 g - i_{\pi^\sharp \mathrm{d}g}\mathrm{d}^2 f + \mathrm{d}(\pi(\mathrm{d}f,\mathrm{d}g)) = \mathrm{d}\,\{f,g\}\,;$$

(ii) Given $f \in C^\infty(M)$ and $\alpha,\beta \in \Omega^1(M)$, we have:

$$\begin{aligned}
[\alpha, f\beta]_\pi &= i_{\pi^\sharp\alpha}\mathrm{d}(f\beta) - i_{\pi^\sharp(f\beta)}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,f\beta))\\
&= i_{\pi^\sharp\alpha}(\mathrm{d}f\wedge\beta + f\mathrm{d}\beta) - i_{\pi^\sharp(f\beta)}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,f\beta))\\
&= (i_{\pi^\sharp\alpha}\mathrm{d}f)\beta - (i_{\pi^\sharp\alpha}\beta)\mathrm{d}f + f\,i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{f\,\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(f\pi(\alpha,\beta))\\
&= (\pi^\sharp\alpha(f))\beta - (i_{\pi^\sharp\alpha}\beta)\mathrm{d}f + f(i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,\beta))) + \pi(\alpha,\beta)\mathrm{d}f\\
&= (\pi^\sharp\alpha(f))\beta + f\,[\alpha,\beta]_\pi,
\end{aligned}$$

where the terms in red cancel out.

As for the uniqueness, suppose there is an $\mathbb{R}$-bilinear, anti-symmetric bracket $\langle\cdot,\cdot\rangle$ obeying properties (i) and (ii). Then, given a chart $(U,\varphi = (x^1,\dots,x^m))$ on M, we may write $\alpha,\beta \in \Omega^1(M)$ and $\pi \in \mathfrak{X}^2(M)$ locally as

$$\alpha|_U = \sum_{i=1}^m \alpha_i\,\mathrm{d}x^i, \quad \beta|_U = \sum_{i=1}^m \beta_i\,\mathrm{d}x^i \quad \text{and} \quad \pi|_U = \frac{1}{2}\sum_{i,j=1}^m \pi^{ij}\,\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}\,.$$

Therefore,

$$\pi(\alpha,\beta)|_U = \sum_{i,j=1}^m \pi^{ij}\alpha_i\beta_j,$$

so we get

$$\mathrm{d}(\pi(\alpha,\beta)) = \sum_{i,j,k=1}^m \left(\frac{\partial \pi^{ij}}{\partial x^k}\alpha_i\beta_j + \pi^{ij}\frac{\partial\alpha_i}{\partial x^k}\beta_j + \pi^{ij}\alpha_i\frac{\partial\beta_j}{\partial x^k}\right)\mathrm{d}x^k$$

and also

$$\pi^\sharp\alpha = i_\alpha\pi = \sum_{i,j=1}^m \pi^{ij}\alpha_i\,\frac{\partial}{\partial x^j}\,,$$

with a similar expression for $\pi^\sharp\beta$. Moreover,

$$\begin{aligned}
i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha &= \sum_{i,j=1}^m \frac{\partial\beta_j}{\partial x^i}\mathrm{d}x^i\wedge\mathrm{d}x^j\left(\sum_{k,\ell=1}^m \pi^{k\ell}\alpha_k\,\frac{\partial}{\partial x^\ell}\right)\\
&\qquad - \sum_{i,j=1}^m \frac{\partial\alpha_j}{\partial x^i}\mathrm{d}x^i\wedge\mathrm{d}x^j\left(\sum_{k,\ell=1}^m \pi^{k\ell}\beta_k\,\frac{\partial}{\partial x^\ell}\right)\\
&= \sum_{i,j,k=1}^m \pi^{ij}\left(\frac{\partial\alpha_k}{\partial x^i}\beta_j + \frac{\partial\alpha_j}{\partial x^k}\beta_i - \alpha_j\frac{\partial\beta_k}{\partial x^i} - \alpha_i\frac{\partial\beta_j}{\partial x^k}\right)\mathrm{d}x^k.
\end{aligned}$$

Joining the previous expressions, we get that

$$\begin{aligned}
[\alpha,\beta]_\pi &= i_{\pi^\sharp\alpha}\mathrm{d}\beta - i_{\pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}(\pi(\alpha,\beta))\\
&= \sum_{i,j,k=1}^m\left(\pi^{ij}\frac{\partial\alpha_k}{\partial x^i}\beta_j - \alpha_j\pi^{ij}\frac{\partial\beta_k}{\partial x^i} + \alpha_i\beta_j\frac{\partial\pi^{ij}}{\partial x^k}\right)\mathrm{d}x^k.
\end{aligned}$$

Using the fact that $\langle\cdot,\cdot\rangle$ obeys properties *(i)* and *(ii)*, we obtain

$$\begin{aligned}
\langle\alpha,\beta\rangle|_U &= \sum_{i,j=1}^m\left(\beta_j\alpha_i\,\mathrm{d}\left\{x^i,x^j\right\} - \beta_j(\pi^\sharp(\mathrm{d}x^j)(\alpha_i))\mathrm{d}x^i + \alpha_i\pi^\sharp(\mathrm{d}x^i)(\beta_j)\mathrm{d}x^j\right)\\
&= \sum_{i,j=1}^m\Bigg(\beta_j\alpha_i\sum_{n=1}^m\sum_{k,\ell=1}^m\frac{\partial\pi^{k\ell}}{\partial x^n}\delta^i_k\delta^j_\ell\,\mathrm{d}x^n - \beta_j\frac{1}{2}\sum_{k,\ell=1}^m\pi^{k\ell}\Big(\delta^j_k\frac{\partial\alpha_i}{\partial x^\ell} - \delta^j_\ell\frac{\partial\alpha_i}{\partial x^k}\Big)\mathrm{d}x^i\\
&\qquad + \alpha_i\frac{1}{2}\sum_{k,\ell=1}^m\pi^{k\ell}\Big(\delta^i_k\frac{\partial\beta_j}{\partial x^\ell} - \delta^i_\ell\frac{\partial\beta_j}{\partial x^k}\Big)\mathrm{d}x^j\Bigg)\\
&= \sum_{i,j,k=1}^m\left(\alpha_i\beta_j\frac{\partial\pi^{ij}}{\partial x^k} - \beta_j\pi^{ji}\frac{\partial\alpha_k}{\partial x^i} + \alpha_j\pi^{ji}\frac{\partial\beta_k}{\partial x^i}\right)\mathrm{d}x^k
\end{aligned}$$

So, we conclude that, locally, $\langle\alpha,\beta\rangle = [\alpha,\beta]_\pi$ and, since, by the Leibniz property, the brackets $[\cdot,\cdot]_\pi$ and $\langle\cdot,\cdot\rangle$ are local, we conclude they coincide for all $\alpha,\beta\in\Omega^1(M)$. Q.E.D.

Proposition 5.3.5. *If $\pi\in\mathfrak{X}^2(M)$ is a bivector field with associated biderivation $\{\cdot,\cdot\}$, and bracket $[\cdot,\cdot]_\pi$ in $\Omega^1(M)$, the following conditions are equivalent:*

(i) $[\pi,\pi]_{\mathrm{SN}} = 0$*;*

(ii) $\pi^\sharp : (\Omega^1(M),[\cdot,\cdot]_\pi)\to(\mathfrak{X}(M),[\cdot,\cdot])$ *is a bracket preserving map;*

(iii) $[\cdot,\cdot]_\pi$ *obeys the Jacobi identity.*

Proof. (i) $\Longleftrightarrow$ *(ii):* Consider the following $\mathbb{R}$-bilinear anti-symmetric bracket:

$$\begin{aligned}
\langle\cdot,\cdot\rangle : \Omega^1(M)\times\Omega^1(M) &\longrightarrow \Omega^1(M)\\
(\alpha,\beta) &\longmapsto [\pi^\sharp\alpha,\pi^\sharp\beta] - \pi^\sharp[\alpha,\beta]_\pi
\end{aligned}$$

Then, by the Leibniz rule obeyed both by the Lie bracket $[\cdot,\cdot]$ and by $[\cdot,\cdot]_\pi$, we get that $\langle\cdot,\cdot\rangle$ is a $C^\infty(M)$-bilinear map. Indeed, for $f\in C^\infty(M)$ and $\alpha,\beta\in\mathfrak{X}(M)$,

$$\begin{aligned}\langle\alpha,f\beta\rangle &= [\pi^\sharp\alpha,\pi^\sharp(f\beta)]-\pi^\sharp[\alpha,f\beta]_\pi=[\pi^\sharp\alpha,f\pi^\sharp\beta]-\pi^\sharp(f[\alpha,\beta]_\pi+(\pi^\sharp\alpha(f))\beta)\\ &= (\pi^\sharp\alpha(f))\pi^\sharp\beta+f[\pi^\sharp\alpha,\pi^\sharp\beta]-f\pi^\sharp[\alpha,\beta]_\pi-(\pi^\sharp\alpha(f))\pi^\sharp\beta\\ &= f\langle\alpha,\beta\rangle,\end{aligned}$$

where the terms in red cancel out. Notice that condition (ii) is equivalent to $\langle\alpha,\beta\rangle=0$, for all $\alpha,\beta\in\Omega^1(M)$. Moreover, since, locally, $\Omega^1(M)$ is generated as a $C^\infty(M)$-module by $\mathrm{d}C^\infty(M)$, condition (ii) is, then, equivalent to $\langle\mathrm{d}f,\mathrm{d}g\rangle=0$, for all $f,g\in C^\infty(M)$. But, because the Hamiltonian vector field associated to a function $f\in C^\infty(M)$ can be written as $X_f=\pi^\sharp\mathrm{d}f$, we conclude that

$$\langle\mathrm{d}f,\mathrm{d}g\rangle=[X_f,X_g]-\pi^\sharp[\mathrm{d}f,\mathrm{d}g]_\pi=[X_f,X_g]-\pi^\sharp\mathrm{d}\{f,g\}=[X_f,X_g]-X_{\{f,g\}}.$$

As explained in the proof of Proposition 5.2.7, $X_{\{f,g\}}=[X_f,X_g]$ is equivalent to the Jacobi identity, which, in turn, is equivalent to $[\pi,\pi]_{\mathrm{SN}}=0$, proving the equivalence $(i)\iff(ii)$.

(ii) $\iff$ *(iii):* Using the same notation as in the previous equivalence for $\langle\alpha,\beta\rangle=[\pi^\sharp\alpha,\pi^\sharp\beta]-\pi^\sharp[\alpha,\beta]_\pi$, given $\alpha,\beta,\gamma\in\Omega^1(M)$ and $f\in C^\infty(M)$, by the properties of $[\cdot,\cdot]_\pi$,

$$\begin{aligned}[][\alpha,[\beta,f\gamma]_\pi]_\pi &= [\alpha,f[\beta,\gamma]_\pi]_\pi+[\alpha,(\pi^\sharp\beta(f))\gamma]_\pi\\ &= f[\alpha,[\beta,\gamma]_\pi]_\pi+\pi^\sharp\alpha(f)[\beta,\gamma]_\pi+\pi^\sharp\beta(f)[\alpha,\gamma]_\pi+\pi^\sharp\alpha(\pi^\sharp\beta(f))\gamma,\\ [\beta,[f\gamma,\alpha]_\pi]_\pi &= f[\beta,[\gamma,\alpha]_\pi]_\pi+\pi^\sharp\beta(f)[\gamma,\alpha]_\pi-\pi^\sharp\alpha(f)[\beta,\gamma]_\pi-\pi^\sharp\beta(\pi^\sharp\alpha(f))\gamma\\ [f\gamma,[\alpha,\beta]_\pi]_\pi &= f[\gamma,[\alpha,\beta]_\pi]_\pi-\pi^\sharp[\alpha,\beta]_\pi(f)\gamma\end{aligned}$$

so,

$$\begin{aligned}&[\alpha,[\beta,f\gamma]_\pi]_\pi+[\beta,[f\gamma,\alpha]_\pi]_\pi+[f\gamma,[\alpha,\beta]_\pi]_\pi\\ &\qquad=f\big([\alpha,[\beta,\gamma]_\pi]_\pi+[\beta,[\gamma,\alpha]_\pi]_\pi+[\gamma,[\alpha,\beta]_\pi]_\pi\big)+\langle\alpha,\beta\rangle(f)\gamma.\end{aligned}$$

From this expression, we directly obtain that, if *(iii)* holds, then $\langle\alpha,\beta\rangle(f)=0$ for all $f\in C^\infty(M)$, i.e. $\langle\alpha,\beta\rangle=0$ and *(ii)* holds. Reciprocally, if *(ii)* holds, then, again, it is equivalent to say that $\langle\mathrm{d}f,\mathrm{d}g\rangle=0$ for all $f,g\in C^\infty(M)$. Using the properties of the bracket $[\cdot,\cdot]_\pi$, we get that

$$\begin{aligned}&[\mathrm{d}f,[\mathrm{d}g,\mathrm{d}h]_\pi]_\pi+[\mathrm{d}g,[\mathrm{d}h,\mathrm{d}f]_\pi]_\pi+[\mathrm{d}h,[\mathrm{d}f,\mathrm{d}h]_\pi]_\pi\\ &\qquad=\mathrm{d}\big(\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}\big)=0,\end{aligned}$$

and, thus, condition *(iii)* holds. Q.E.D.

The bracket $[\cdot,\cdot]_\pi$ together with the map $\pi^\sharp : T^*M \to TM$ endows the vector bundle $T^*M \xrightarrow{\hat{\tau}_M} M$ with a structure of a Lie algebroid.

Returning to the description of the map $\pi^\sharp : T^*M \to TM$, we remember that the definition of a Hamiltonian vector field X_H of a map $H \in C^\infty(M)$ implies that

$$X_H(f) \stackrel{\text{def}}{=} \{H,f\} = \pi(\mathrm{d}H,\mathrm{d}f) = i_{\mathrm{d}f} i_{\mathrm{d}H}\pi = i_{\mathrm{d}f}(\pi^\sharp \mathrm{d}H),\ \forall f \in C^\infty(M), \tag{5.22}$$

thus, an alternative way to describe an Hamiltonian vector field is

$$X_H \stackrel{\text{def}}{=} \pi^\sharp \mathrm{d}H. \tag{5.23}$$

Therefore, since, for $p \in M$, we have that[1] $C^\infty(M) \xrightarrow{(\mathrm{d}\,\cdot)_p} T_p^*M$, we conclude that for a given point $p \in M$ the image of $\pi_p^\sharp : T_p^*M \to T_pM$ is equal to the set of all Hamiltonian vector fields evaluated at p:

$$\mathrm{im}(\pi_p^\sharp) = \left\{ (X_H)_p \in T_pM \,\middle|\, H \in C^\infty(M) \right\} \subseteq T_pM, \tag{5.24}$$

which is a linear subspace of T_pM. The union of all these subspaces,

$$\mathrm{im}(\pi^\sharp) = \coprod_{p\in M} \mathrm{im}(\pi_p^\sharp) \subseteq TM$$

is a tangent distribution which, in general, is a singular smooth distribution.

Theorem 5.3.6. *The distribution* $\mathrm{im}(\pi^\sharp)$ *is involutive.*

Proof. It is enough to show that, for a Poisson manifold (M,π), given $\alpha,\beta \in \Omega^1(M)$, $[\pi^\sharp\alpha, \pi^\sharp\beta] \in \mathrm{im}(\pi^\sharp)$. However, by Proposition 5.3.5, $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism, thus $[\pi^\sharp\alpha,\pi^\sharp\beta] = \pi^\sharp[\alpha,\beta]_\pi \in \mathrm{im}(\pi^\sharp)$. Q.E.D.

As we shall see, this distribution plays a central role in the local description of Poisson manifolds.

Returning to our previous examples, we can see that:

Example 5.3.7. For the Zero Poisson bracket, it is clear that the Poisson bivector is identically 0, $\pi \equiv 0$. Similarly, for the case of the constant Poisson brackets on $\mathbb{R}^m$, considering $(x^1,\dots,x^m)$ to be the coordinates on $\mathbb{R}^m$ associated to the canonical basis of $\mathbb{R}^m$, $\{f,g\} = \sum\limits_{i,j=1}^m c^{ij} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} = \pi(\mathrm{d}f,\mathrm{d}g)$, so the Poisson bivector is a constant section of $\bigwedge^2 T\mathbb{R}^m \to \mathbb{R}^m$, since $\pi = \frac{1}{2}\sum\limits_{i,j=1}^m c^{ij}\, \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. ♮

[1] To obtain the surjectivity of this map, simply consider, for a chart $(U \subseteq M, (x^1,\dots,x^m))$ and for any covector $\eta = \sum_i \eta_i(\mathrm{d}x^i)_p \in T_p^*M$ ($\eta_i \in \mathbb{R}$, $\forall i = 1,\dots,m$), the linear functions $H = \sum_i \eta_i x^i$, whose derivative at p is $H_*(p) = (\mathrm{d}H)_p = \sum_i \eta_i(\mathrm{d}x^i)_p = \eta$.

Example 5.3.8. Following the Symplectic structures introduced in Example 5.1.11, the Poisson bivector field associated to the canonical Poisson bracket on $\mathbb{R}^{2s}$ is

$$\pi_{\text{can}} = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} \, . \tag{5.25}$$

More generally, as seen in Chapter 4, for a symplectic manifold (M, ω), it is possible to define a vector bundle isomorphism $\omega^\flat : TM \to T^*M$. Calling $\pi^\sharp = (\omega^\flat)^{-1} : T^*M \to TM$, which is an isomorphism, we may define a bivector field $\pi \in \mathfrak{X}^2(M)$ such that,

$$\pi(\alpha, \beta) = -\omega\big((\omega^\flat)^{-1}\alpha, (\omega^\flat)^{-1}\beta\big), \quad \forall\, \alpha, \beta \in \Omega^1(M), \tag{5.26}$$

which, by the non-degeneracy of ω is also non-degenerate. In fact, if, for every $p \in M$, $\alpha \in T_p^*M$ is such $\pi_p(\alpha, \beta) = 0$, for all $\beta \in T_p^*M$, then $-\omega_p(\pi_p^\sharp\alpha, \pi_p^\sharp\beta) = 0$, for all $\beta \in T_p^*M$, so, since $\pi^\sharp$ is an isomorphism (for every $p \in M$), $\omega_p(\pi_p^\sharp\alpha, v) = 0$, for all $v \in T_pM$. Thus, $\pi_p^\sharp\alpha = 0$ by the non-degeneracy of ω and, therefore, $\alpha = 0$ because $\pi^\sharp$ is an isomorphism.

Expression (5.26) extends Eq. (5.6) to non-exact 1-forms. Moreover, we can prove the converse result and arrive at the following proposition:

Proposition 5.3.9. *There is a canonical one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Non-degenerate bivector} \\ \textit{fields } \pi \in \mathfrak{X}^2(M) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{Non-degenerate} \\ \textit{2-forms } \omega \in \Omega^2(M) \end{array} \right\}, \tag{5.27}$$

established by the condition

$$\pi^\sharp = (\omega^\flat)^{-1} : T^*M \longrightarrow TM \tag{5.28}$$

Particularly, one gets that

$$\left\{ \begin{array}{c} \textit{Non-degnenerate Poisson} \\ \textit{bivectors } \pi \in \mathfrak{X}^2(M) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{Symplectic} \\ \textit{2-forms } \omega \in \Omega^2(M) \end{array} \right\}.$$

Proof. One of the implications was already seen above. For the other implication, let $\pi \in \mathfrak{X}^2(M)$ be a non-degenerate bivector field and define a 2-form:

$$\omega(X, Y) = -\pi\big((\pi^\sharp)^{-1}X, (\pi^\sharp)^{-1}Y\big), \quad \forall\, X, Y \in \mathfrak{X}(M). \tag{5.29}$$

Following the same procedures as previously, let $p \in M$ and $v \in T_pM$ such that $\omega_p(v, u) = 0$, for all $u \in T_pM$. Since $\omega^\flat = (\pi^\sharp)^{-1}$ is an isomorphism, we get that $\pi_p(\omega_p^\flat v, \alpha) = 0$, for all $\alpha \in T_p^*M$. Thus, by the non-degeneracy of π, $\omega_p^\flat v = 0$ and, because $\omega^\flat$ is an isomorphism, $v = 0$. Hence, we conclude that ω is non-degenerate.

To show that the correspondence extends to Poisson bivectors and symplectic forms, it is enough to relate $\mathrm{d}\omega$ with $[\pi, \pi]$. For that, it suffices to compute $\mathrm{d}\omega$ on Hamiltonian vector fields $\pi^\sharp \mathrm{d}f$ ($f \in C^\infty(M)$). Taking $f, g, h \in C^\infty(M)$ and using the definition of the exterior derivative of forms (1.18b):

$$\begin{aligned}
\mathrm{d}\omega(\pi^\sharp\mathrm{d}f, \pi^\sharp\mathrm{d}g, \pi^\sharp\mathrm{d}h) &= \pi^\sharp\mathrm{d}f\big(\omega(\pi^\sharp\mathrm{d}g, \pi^\sharp\mathrm{d}h)\big) - \pi^\sharp\mathrm{d}g\big(\omega(\pi^\sharp\mathrm{d}f, \pi^\sharp\mathrm{d}h)\big) + \pi^\sharp\mathrm{d}h\big(\omega(\pi^\sharp\mathrm{d}f, \pi^\sharp\mathrm{d}g)\big)\\
&\quad - \omega([\pi^\sharp\mathrm{d}f, \pi^\sharp\mathrm{d}g], \pi^\sharp\mathrm{d}h) - \omega([\pi^\sharp\mathrm{d}g, \pi^\sharp\mathrm{d}h], \pi^\sharp\mathrm{d}f) - \omega([\pi^\sharp\mathrm{d}h, \pi^\sharp\mathrm{d}f], \pi^\sharp\mathrm{d}g)\\
&= -\pi^\sharp\mathrm{d}f\big(\pi(\mathrm{d}g, \mathrm{d}h)\big) + \pi^\sharp\mathrm{d}g\big(\pi(\mathrm{d}f, \mathrm{d}h)\big) - \pi^\sharp\mathrm{d}h\big(\pi(\mathrm{d}f, \mathrm{d}g)\big)\\
&\quad - \omega(\pi^\sharp[\mathrm{d}f, \mathrm{d}g]_\pi, \pi^\sharp\mathrm{d}h) - \omega(\pi^\sharp[\mathrm{d}g, \mathrm{d}h]_\pi, \pi^\sharp\mathrm{d}f) - \omega(\pi^\sharp[\mathrm{d}h, \mathrm{d}f]_\pi, \pi^\sharp\mathrm{d}g)\\
&= -\{f, \{g, h\}\} + \{g, \{f, h\}\} - \{h, \{f, g\}\} + \pi(\mathrm{d}\{f, g\}, \mathrm{d}h)\\
&\quad + \pi(\mathrm{d}\{g, h\}, \mathrm{d}f) + \pi(\mathrm{d}\{h, f\}, \mathrm{d}g)\\
&= -2\big(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}\big)\\
&= -[\pi, \pi](\mathrm{d}f, \mathrm{d}g, \mathrm{d}h).
\end{aligned}$$

Hence, $[\pi, \pi] = 0$ if and only if $\mathrm{d}\omega = 0$, finishing the proof. Q.E.D.

Remark 5.3.10. Even though expression (5.26) linking a bivector field $\pi \in \mathfrak{X}^2(M)$ with a closed 2-form $\omega \in \Omega^2(M)$ contains a minus sign, we shall still denote this relation by $\pi = \omega^{-1}$. ♮

Example 5.3.11. For the case of linear Poisson structures, considering a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ we shall denote the Poisson bivector field by $\pi_\mathfrak{g} \in \mathfrak{X}^2(\mathfrak{g}^*)$ so that, for all $\lambda \in \mathfrak{g}^*$,

$$(\pi_\mathfrak{g})_\lambda((\mathrm{d}f)_\lambda, (\mathrm{d}g)_\lambda) = \lambda([f_*(\lambda), g_*(\lambda)]) \tag{5.30}$$

If we consider a basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$ and the associated coordinates $\{x^1, \dots, x^n\}$ on $\mathfrak{g}^*$, then we may write $\pi_\mathfrak{g}$ as

$$\pi_\mathfrak{g} = \frac{1}{2} \sum_{a,b,c=1}^{n} k_{ab}^c x^c \, \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b} \ ,$$

where $[e_a, e_b] = \sum_{c=1}^{n} k_{ab}^c \, e_c$ and the factor $\frac{1}{2}$ comes from the fact that the wedge product is anti-symmetric. ♮

Example 5.3.12. One particular class of examples we could not introduce until now are the log-symplectic manifolds. These occur when a Poisson manifold (M, π), $\dim(M) = 2n$, is such that the n^{th} exterior power of π is transverse to the zero section of $\bigwedge^{2n} TM$, i.e. $\pi^{\wedge n} \pitchfork 0_{\bigwedge^{2n} TM}$. Since $\bigwedge^{2n} TM$ is a rank 1 bundle over M, we may write $\pi^{\wedge n}$ in a local chart $(U, (x^1, \dots, x^{2n}))$ as

$$\pi^{\wedge n}\big|_U = f(x^1, \dots, x^{2n}) \, \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^{2n}} \ ,$$

where $f \in C^\infty(U)$. Then, the transversality condition for $\pi^{\wedge n}$ implies that $f_*(x) \neq 0$ for any $x \in U$ such that $f(x) = 0$. Also, the set $Z \stackrel{\text{def}}{=} (\pi^{\wedge n})^{-1}(0)$, which we call the *singular locus* of (M, π), is a $(2n-1)$-dimensional submanifold of M. The importance of this class of examples comes from the following proposition, whose proof we shall delay until Section 6.2:

Proposition 5.3.13. *Let (M, π) be a $2n$-dimensional log-symplectic manifold with singular locus Z. Then, for any $x \in Z$, there is a local coordinate chart $(U, (u, v, q^1, \ldots, q^{n-1}, p_1, \ldots, p_{n-1}))$ on M centred at x such that*

$$U \cap Z = \big\{(0, v, q^1, \ldots, q^{n-1}, p_1, \ldots, p_{n-1}) \,\big|\, v, q^1, \ldots, q^{n-1}, p_1, \ldots, p_{n-1} \in \mathbb{R}\big\}$$

and

$$\pi|_U = u \, \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} + \sum_{i=1}^{n-1} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} \, .$$

Thus, the Poisson structure near the singular locus is 'close' to be symplectic. Moreover, we can write this 'close'-symplectic 2-form as

$$(\pi|_U)^{-1} = \frac{1}{u} \mathrm{d}v \wedge \mathrm{d}u + \sum_{i=1}^{n-1} \mathrm{d}q^i \wedge \mathrm{d}p_i = \mathrm{d}v \wedge \mathrm{d}\ln|u| + \sum_{i=1}^{n-1} \mathrm{d}q^i \wedge \mathrm{d}p_i,$$

which justifies the term 'log-symplectic' used for these type of Poisson structures. ♮

5.4 The Poisson Structure on the dual of Lie Algebroids

As we have seen at the end of Section 5.1, every Lie algebra has a linear Poisson structure on its dual. Since Lie algebroids are, in some sense, an extension of the concept of a Lie algebra to the bundle setting, is it possible to still have some kind of Poisson structure on the dual of a Lie algebroid? The answer turns out to be yes and, in fact, this Poisson structure is linear, although only fibre-wise.

Proposition 5.4.9. *For any vector bundle $A \to M$, there is a natural one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Fibre-wise linear Poisson bivectors} \\ \pi \in \mathfrak{X}^2(A^*) \textit{ on } A^* \to M \end{array} \right\} \xleftrightarrow{\ 1:1\ } \left\{ \begin{array}{c} \textit{Lie Algebroid Structures} \\ ([\cdot,\cdot], \mathsf{a}) \textit{ on } A \to M \end{array} \right\},$$

established by the relations

$$\begin{aligned} \{\mathrm{ev}_\sigma, \mathrm{ev}_\varsigma\}_{A^*} &= \mathrm{ev}_{[\sigma,\varsigma]_A}, \ \forall\, \sigma, \varsigma \in \Gamma(A); \\ \{\mathrm{ev}_\sigma, f \circ \mathrm{pr}\}_{A^*} &= \mathsf{a}(\sigma)(f) \circ \mathrm{pr}, \ \forall\, \sigma \in \Gamma(A), \ f \in C^\infty(M). \end{aligned}$$

Let us start, then, by introducing the notion of a fibre-wise linear multivector field.

Definition 5.4.1. For a vector bundle $E \xrightarrow{\mathrm{pr}} M$, a multivector field $\theta \in \mathfrak{X}^k(E)$ is said to be *fibre-wise linear* if $(h_t)_*\theta = t^{k-1}\theta, \ \forall\, t > 0$, where h_t is the homogeneity structure on E.

Since we are only interested in Poisson structures being fibre-wise linear, one also finds an equivalent definition for (Poisson) bivectors which we shall later prove (Proposition 5.4.7). Firstly, we require two additional definitions:

Definition 5.4.2. Let $E \xrightarrow{\mathrm{pr}} M$ be a vector bundle. A *basic* or *fibre-wise constant function* on E is a map $f \circ \mathrm{pr} : E \to \mathbb{R}$, where $f : M \to \mathbb{R}$ is a function on M. The collection of basic functions on E is denoted by $\mathrm{pr}^*(C^\infty(M))$.

Definition 5.4.3. For a vector bundle $E \xrightarrow{\mathrm{pr}} M$, a *fibre-wise linear function* on E is a function $f : E \to \mathbb{R}$ such that, for any $p \in M$, $f|_{E_p}$ is a linear function. The collection of fibre-wise linear functions on E is denoted by $C^\infty_{\mathrm{lin}}(E)$.

Proposition 5.4.4. *Let $E \xrightarrow{\mathrm{pr}} M$ be a vector bundle and $h_t : E \times E \to E$ the homogeneity structure associated to vector bundle structure of E. Then,*

(i) for any function $f \in C^\infty(E)$, f is a fibre-wise constant function on E if and only if $(h_t)^* f = f$, for all $t \in \mathbb{R}$;

(ii) for any function $g \in C^\infty(E)$, g is a fibre-wise linear function on E if and only if $(h_t)^* g = t\, g$, for all $t \in \mathbb{R}$;

Proof. *(i)* The implication from left to right is trivial since $\mathrm{pr} \circ h_t = \mathrm{pr}$. Reciprocally, given local adapted coordinates (x^i, e^a) on E, since $(h_t)^* f = f$, we have that

$$f(x^i, e^a) = (h_t)^* f(x^i, e^a) = f(x^i, t\, e^a), \quad \text{for all } t \in \mathbb{R},$$

hence f does not depend on the coordinates e^a in the fibres of E. Since that is the case, there is a function $\phi : M \to \mathbb{R}$ such that $f = \phi \circ \mathrm{pr}$, i.e. f is fibre-wise constant.

(ii) Once again, the implication from left to right is clear because, given adapted coordinates (x^i, e^a) on E, any fibre-wise linear function g can be written as $g(x^i, e^a) = \sum_a g_a(x)\, e^a$ for some functions $g_a(x^i)$ of the x^i's. Thus,

$$((h_t)^* g)(x^i, e^a) = g(x^i, t\, e^a) = \sum_a g_a(x^i)\, te^a = t \sum_a g_a(x^i)\, e^a = tg(x^i, e^a).$$

As for the opposite implication, we want to check that if $g \in C^\infty(E)$ satisfies $(h_t)^* g = tg$, for all $t \in \mathbb{R}$, then $g|_{E_p}$ is a linear function. Once again, let (x^i, e^a) be local adapted coordinates of E. Then we get that

$$\begin{aligned} g(x^i, e^a) &= \frac{\mathrm{d}}{\mathrm{d}t}\big(t\, g(x^i, e^a)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(((h_t)^* g)(x^i, e^a)\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(x^i, t\, e^a) = \sum_a e^a \frac{\partial g}{\partial e^a}. \end{aligned}$$

The solution to this partial differential equation is simply that $g(x^i, e^a) = \sum_a g_a(x^i)\, e^a$ for some functions $g_a(x^i)$ of the x^i's. Thus, we conclude that $g \in C^\infty(E)$ is a fibre-wise linear function on E. Q.E.D.

Proposition 5.4.5. *Let $E \xrightarrow{\mathrm{pr}} M$ be a vector bundle and $\mathcal{D} : C^\infty(E) \to C^\infty(E)$ a derivation of the algebra of smooth functions on E. Then, $\mathcal{D}$ is completely determined by its action on basic functions and fibre-wise linear functions.*

Proof. Note that, given that E is a vector bundle, we may prove this result only locally and then use the different local trivialisation to obtain the result globally. Let $p \in M$ and $(U, \varphi = (x^1, \ldots, x^m))$ be a chart of M centred at p. Also, choose adapted coordinates (x^i, e^a) $(a = 1, \ldots, k = \mathrm{rank}(E))$ in $E|_U$ and fix arbitrary $\epsilon \in E_p$ so that ϵ has coordinates $(0, \epsilon^a)$. Then, any function $f \in C^\infty(E)$ restricts to $f|_{E|_U} : E|_U \cong U \times \mathbb{R}^k \to \mathbb{R}$ so, by Hadamard lemma,

$$f(x^i, e^a) = f(0, \epsilon^a) + \sum_{j=1}^{m} x^j\, g_j(x^i, e^a) + \sum_{b=1}^{k} (e^b - \epsilon^b)\, h_b(x^i, e^a),$$

where $g_i, h_a \in C^\infty(U \times \mathbb{R}^k)$ are functions such that

$$g_i(0, \epsilon^a) = \frac{\partial f}{\partial x^i}(0, \epsilon^a) \quad \text{and} \quad h_b(0, \epsilon^a) = \frac{\partial f}{\partial e^b}(0, \epsilon^a),$$

for all $i = 1, \ldots, m$ and $a = 1, \ldots, k$. Then, by the definition of the derivation $\mathcal{D}$,

$$\begin{aligned}\mathcal{D}(f)(x^i, e^a) = &\sum_{j=1}^{m} \mathcal{D}(x^j)\, g_j(x^i, e^a) + \sum_{j=1}^{m} x^j\, \mathcal{D}(g_j)(x^i, e^a) \\ &+ \sum_{b=1}^{k} \mathcal{D}(e^b - \epsilon^b)\, h_b(x^i, e^a) + \sum_{b=1}^{k} (e^b - \epsilon^b)\, \mathcal{D}(h_b)(x^i, e^a).\end{aligned}$$

Evaluating this expression at ϵ, we get

$$\begin{aligned}\mathcal{D}(f)(0, \epsilon^a) &= \sum_{j=1}^{m} \mathcal{D}(x^j)\big|_{x=0}\, g_j(0, \epsilon^a) + \sum_{b=1}^{k} \mathcal{D}(e^b - \epsilon^b)\big|_{e=\epsilon}\, h_b(0, \epsilon^a) \\ &= \sum_{j=1}^{m} \mathcal{D}(x^j)\big|_{x=0}\, \frac{\partial f}{\partial x^j}(0, \epsilon^a) + \sum_{b=1}^{k} \mathcal{D}(e^b - \epsilon^b)\big|_{e=\epsilon}\, \frac{\partial f}{\partial e_b}(0, \epsilon^a).\end{aligned}$$

So, at ϵ, which is an arbitrary element of E, $\mathcal{D}$ is completely determined by the coefficients $\mathcal{D}(x^j)\big|_{x=0}$ and $\mathcal{D}(e^b - \epsilon^b)\big|_{e=\epsilon}$. Therefore, taking f to be fibre-wise constant, $f(x^i, e^a) = f(x^i)$ and we get that

$$\mathcal{D}(f)(0, \epsilon^a) = \sum_{j=1}^{m} \mathcal{D}(x^j)\big|_{x=0} \frac{\partial f}{\partial x^j}(0),$$

so, knowing the action of $\mathcal{D}$ on fibre-wise constant functions we may find the value of the coefficients $\mathcal{D}(x^j)\big|_{x=0}$. Afterwards, we may choose f to be fibre-wise linear, so $f(x^i, e^a) = \sum_{a=1}^{k} f_a(x^i) e^a$ for some $f_a \in C^\infty(U)$ and

$$\mathcal{D}(f)(0, \epsilon^a) = \sum_{b=1}^{k} \left(\sum_{j=1}^{m} \mathcal{D}(x^j)\big|_{x=0}\, \epsilon^b \frac{\partial f_b}{\partial x^j}(0) + \mathcal{D}(e^b - \epsilon^b)\big|_{e=\epsilon} f_b(0) \right),$$

allowing us to obtain the coefficients $\mathcal{D}(e^b - \epsilon^b)\big|_{e=\epsilon}$. Thus, we conclude that $\mathcal{D}$ is determined at any point in E by its action on fibre-wise constant and fibre-wise linear functions, as required. Q.E.D.

Remark 5.4.6. Since a multi-derivation is also a derivation when all but one of its entries is fixed, Proposition 5.4.5 can be extended to include these multi-derivations of E.

Proposition 5.4.7. *A bivector field $\pi \in \mathfrak{X}^2(E)$ is fibre-wise linear if and only if the associated bilinear form $\{\cdot,\cdot\}_\pi : C^\infty(E) \times C^\infty(E) \to C^\infty(E)$ satisfies the following conditions:*

(i) if $f, g \in C^\infty(M)$, then $\{f \circ \mathrm{pr}, g \circ \mathrm{pr}\}_\pi = 0$;

(ii) given $f \in C^\infty(M)$ and $g \in C^\infty_{\mathrm{lin}}(E)$, then $\{f \circ \mathrm{pr}, g\}_\pi \in \mathrm{pr}^*(C^\infty(M))$;

(iii) for $f, g \in C^\infty_{\mathrm{lin}}(E)$, then $\{f, g\}_\pi \in C^\infty_{\mathrm{lin}}(E)$.

Proof. Let $h_t : E \times E$ be the homogeneity structure associated to the vector bundle structure of $E \xrightarrow{\mathrm{pr}} M$, let $p \in M$ and let $(U, \varphi = (x^1, \dots, x^m))$ be a chart centred p in M. Then, by taking adapted coordinates (x^i, e^a) on E arising from the given chart in M, for any $e = (x^i, e^a) \in E_p$, locally, $h_t(e) = (x^i, te^a)$. Additionally, for any functions $f, g \in C^\infty(E)$, we get that

$$\begin{aligned}((h_t)_*\pi)_e((\mathrm{d}f)_e, (\mathrm{d}g)_e) &= \pi_{h_t^{-1}(e)}\big(((h_t)^*\mathrm{d}f)_{h_t^{-1}(e)}, ((h_t)^*\mathrm{d}g)_{h_t^{-1}(e)}\big)\\ &= \pi_{h_{t^{-1}}(e)}(\mathrm{d}(f \circ h_t)_{h_{t^{-1}}(e)}, \mathrm{d}(g \circ h_t)_{h_{t^{-1}}(e)}),\end{aligned}$$

therefore, if $\{f, g\}_\pi(e) = \pi_e((\mathrm{d}f)_e, (\mathrm{d}g)_e)$,

$$(h_t)_*\pi(\mathrm{d}f, \mathrm{d}g) = \{f \circ h_t, g \circ h_t\}_\pi \circ h_{t^{-1}}. \tag{$*$}$$

$\Longrightarrow$) From $(*)$, and the definition of a fibre-wise multivector field, we get that, for $f, g \in C^\infty(M)$ and $\kappa, \ell \in C^\infty_{\mathrm{lin}}(E)$,

$$\begin{aligned}t\{f \circ \mathrm{pr}, g \circ \mathrm{pr}\}_\pi(e) &= \{f \circ \mathrm{pr} \circ h_t, g \circ \mathrm{pr} \circ h_t\}_\pi(h_{t^{-1}}(e))\\ &= \{f \circ \mathrm{pr}, g \circ \mathrm{pr}\}_\pi(h_{t^{-1}}(e));\end{aligned}$$

$$\begin{aligned}t\{\kappa, \ell\}_\pi(e) &= \{\kappa \circ h_t, \ell \circ h_t\}_\pi(h_{t^{-1}}(e)) = \{t\,\kappa, t\,\ell\}_\pi(h_{t^{-1}}(e))\\ &= t^2\{\kappa, \ell\}_\pi(h_{t^{-1}}(e));\end{aligned}$$

$$\begin{aligned}t\{f \circ \mathrm{pr}, \kappa\}_\pi(e) &= \{f \circ \mathrm{pr} \circ h_t, \kappa \circ h_t\}_\pi(h_{t^{-1}}(e))\\ &= \{f \circ \mathrm{pr}, t\,\kappa\}_\pi(h_{t^{-1}}(e)) = t\{f \circ \mathrm{pr}, \kappa\}_\pi(h_{t^{-1}}(e)).\end{aligned}$$

Since these expressions are valid for all $t > 0$, the first of these expressions implies that, $\{f \circ \mathrm{pr}, g \circ \mathrm{pr}\}_\pi = 0$. As for the second expression, we obtain that $\{\kappa, \ell\}_\pi(h_{t^{-1}}(e)) = t^{-1}\{\kappa, \ell\}_\pi(e)$, therefore, by Proposition 5.4.4, $\{\kappa, \ell\}_\pi \in C^\infty_{\mathrm{lin}}(E)$. Finally, for the third expression, we conclude that $\{f \circ \mathrm{pr}, \kappa\}_\pi(e) = \{f \circ \mathrm{pr}, \kappa\}_\pi(h_{t^{-1}}(e))$, which, by Proposition 5.4.4, implies that $\{f \circ \mathrm{pr}, \kappa\}_\pi \in \mathrm{pr}^*(C^\infty(M))$.

$\Longleftarrow$) If the bilinear form associated to the bivector field $\pi \in \mathfrak{X}^2(E)$ satisfies conditions *(i)–(iii)*, then, by $(*)$, we get that

$$\begin{aligned}(h_t)_*\pi(\mathrm{d}(f\circ \mathrm{pr}),\mathrm{d}(g\circ \mathrm{pr})) &= \{f\circ \mathrm{pr}\circ h_t, g\circ \mathrm{pr}\circ h_t\}_\pi \circ h_{t^{-1}}\\ &= \{f\circ \mathrm{pr}, g\circ \mathrm{pr}\}_\pi \circ h_{t^{-1}} \overset{(i)}{=} 0 = t\,\{f\circ \mathrm{pr}, g\circ \mathrm{pr}\}_\pi\end{aligned}$$

$$\begin{aligned}(h_t)_*\pi(\mathrm{d}\kappa,\mathrm{d}\ell) &= \{\kappa\circ h_t, \ell\circ h_t\}_\pi \circ h_{t^{-1}} = \{t\,\kappa, t\,\ell\}_\pi \circ h_{t^{-1}}\\ &= t^2\,\{\kappa,\ell\}_\pi \circ h_{t^{-1}} \overset{(iii)}{=} t^{2-1}\,\{\kappa,\ell\}_\pi = t\,\{\kappa,\ell\}_\pi\end{aligned}$$

$$\begin{aligned}(h_t)_*\pi(\mathrm{d}(f\circ \mathrm{pr}),\mathrm{d}\kappa) &= \{f\circ \mathrm{pr}\circ h_t, \kappa\circ h_t\}_\pi \circ h_{t^{-1}} = \{f\circ \mathrm{pr}, t\,\kappa\}_\pi \circ h_{t^{-1}}\\ &= t\,\{f\circ \mathrm{pr},\kappa\}_\pi \circ h_{t^{-1}} \overset{(ii)}{=} t\,\{f\circ \mathrm{pr},\kappa\}_\pi\end{aligned}$$

By the previous proposition, since the algebra of functions $C^\infty(E)$ is generated by the set of fibre-wise constant and linear functions, and, because for every point p, T_e^*E is generated by $(\mathrm{d}C^\infty(E))_p$, we conclude that $(h_t)_*\pi = t\pi$, i.e. $\pi \in \mathfrak{X}^2(M)$ is a fibre-wise linear bivector. Q.E.D.

Back to the study of Lie algebroids, we now want to check if, given a Lie algebroid $(A \xrightarrow{\mathrm{pr}} M, [\cdot,\cdot]_A, \mathsf{a})$, we can define a Poisson structure on the dual A^*. Similarly to what was done in the case of Lie algebras, we start by defining the Poisson bracket in A^* for the evaluation $\mathrm{ev}_\sigma \in C^\infty_{\mathrm{lin}}(A^*)$ for $\sigma \in \Gamma(A)$ as

$$\{\mathrm{ev}_\sigma, \mathrm{ev}_\varsigma\}_{A^*} = \mathrm{ev}_{[\sigma,\varsigma]_A}. \tag{5.31}$$

In fact, it can be shown that all linear functions on A^* come from sections of A:

Proposition 5.4.8. *The following map is a $C^\infty(M)$-module isomorphism:*

$$\begin{aligned}\Gamma(A) &\xrightarrow{\cong} C^\infty_{\mathrm{lin}}(A^*)\\ \sigma &\longmapsto \mathrm{ev}_\sigma\end{aligned} \tag{5.32}$$

where $\mathrm{ev}_\sigma : A^* \to \mathbb{R}$ is the evaluation map such that, for $p \in M$ and $\alpha \in A_p^*$, $\mathrm{ev}_\sigma(\alpha) = \alpha(\sigma(p))$.

Proof. Let x^i be coordinates in M, and e_a a local frame in A such that (x^i, v_a) are local coordinates on A^*. Given a section $\sigma \in \Gamma(A)$, we can represent it locally as $\sigma(x^i) = \sum_a \sigma^a(x^i)\, e_a$, for $\sigma^a \in C^\infty(M)$ for all indices a. Then, the evaluation map is, locally, $\mathrm{ev}_\sigma(x^i, v_a) = \sum_a v_a \sigma^a(x^i)$ or, without mention of the point in A^*, $\mathrm{ev}_\sigma = \sum_a \sigma^a e_a^*$, which shows that the map given in (5.32) is a (injective) $C^\infty(M)$-module homomorphism.

To finish the proof, it is enough to show that any element of $C^\infty_{\text{lin}}(A^*)$ comes from some section of A. Since a fibre-wise linear function $f \in C^\infty_{\text{lin}}(A^*)$ on A^* must have that $(h_t)^* f = tf$, for all $t \in \mathbb{R}$, where $h_t : A^* \times A^* \to A^*$, we have that $f(x^i, tv_a) = tf(x^i, v_a)$, meaning that we can locally write f as $f = \sum_a f^a\, e_a^*$, with $f^a \in C^\infty(M)$, i.e. $f = \mathrm{ev}_\sigma$ for $\sigma = \sum_a f^a\, e_a$, proving the statement of the proposition. Q.E.D.

Then, by Propositions 5.4.5 and 5.4.7, to fully define a fibre-wise linear bivector on A, we need an additional relation between a fibre-wise linear function and a fibre-wise constant function. Given the Lie algebroid structure on A, we define, for a section $\sigma \in \Gamma(A)$ and a function $f \in C^\infty(M)$,

$$\{\mathrm{ev}_\sigma, f \circ \mathrm{pr}'\}_{A^*} = \mathsf{a}(\sigma)(f) \circ \mathrm{pr}', \tag{5.33}$$

where $\mathrm{pr}' = \mathrm{Hom}(\mathrm{pr}, \mathbb{R}) : A^* \to M$ is the projection obtained from the one in $A \to M$ by dualisation. To check that the associated bivector field $\pi \in \mathfrak{X}^2(A^*)$ is Poisson, we need only to check the Jacobi identity, which, by Proposition 5.4.5 is sufficient if we consider only all possible combinations of three maps of fibre-wise constant and fibre-wise linear maps. Let $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(A)$ and $f_1, f_2, f_3 \in C^\infty(M)$. Then,

$$\begin{aligned}
\{f_1 \circ \mathrm{pr}, \{f_2 \circ \mathrm{pr}, f_3 \circ \mathrm{pr}\}_{A^*}\}_{A^*} + \{f_2 \circ \mathrm{pr}, \{f_3 \circ \mathrm{pr}, f_1 \circ \mathrm{pr}\}_{A^*}\}_{A^*}& \\
+ \{f_3 \circ \mathrm{pr}, \{f_1 \circ \mathrm{pr}, f_2 \circ \mathrm{pr}\}_{A^*}\}_{A^*}& \\
= \{f_1 \circ \mathrm{pr}, 0\}_{A^*} + \{f_2 \circ \mathrm{pr}, 0\}_{A^*} + \{f_3 \circ \mathrm{pr}, 0\}_{A^*} = 0&
\end{aligned}$$

$$\begin{aligned}
\{f_1 \circ \mathrm{pr}, \{f_2 \circ \mathrm{pr}, \mathrm{ev}_{\sigma_1}\}_{A^*}\}_{A^*} + \{f_2 \circ \mathrm{pr}, \{\mathrm{ev}_{\sigma_1}, f_1 \circ \mathrm{pr}\}_{A^*}\}_{A^*}& \\
+ \{\mathrm{ev}_{\sigma_1}, \{f_1 \circ \mathrm{pr}, f_2 \circ \mathrm{pr}\}_{A^*}\}_{A^*}& \\
= \{f_1 \circ \mathrm{pr}, -\mathsf{a}(\sigma_1)(f_2) \circ \mathrm{pr}\}_{A^*} + \{f_2 \circ \mathrm{pr}, \mathsf{a}(\sigma_1)(f_1) \circ \mathrm{pr}\}_{A^*} + \{\mathrm{ev}_{\sigma_1}, 0\}_{A^*}& \\
= 0 + 0 + 0 = 0&
\end{aligned}$$

$$\begin{aligned}
\{f_1 \circ \mathrm{pr}, \{\mathrm{ev}_{\sigma_1}, \mathrm{ev}_{\sigma_2}\}_{A^*}\}_{A^*} &+ \{\mathrm{ev}_{\sigma_1}, \{\mathrm{ev}_{\sigma_2}, f_1 \circ \mathrm{pr}\}_{A^*}\}_{A^*} + \{\mathrm{ev}_{\sigma_2}, \{f_1 \circ \mathrm{pr}, \mathrm{ev}_{\sigma_1}\}_{A^*}\}_{A^*} \\
&= \left\{f_1 \circ \mathrm{pr}, \mathrm{ev}_{[\sigma_1,\sigma_2]_A}\right\}_{A^*} + \{\mathrm{ev}_{\sigma_1}, \mathsf{a}(\sigma_2)(f_1) \circ \mathrm{pr}\}_{A^*} + \{\mathrm{ev}_{\sigma_2}, -\mathsf{a}(\sigma_1)(f_1) \circ \mathrm{pr}\}_{A^*} \\
&= -\mathsf{a}([\sigma_1, \sigma_2]_A)(f_1) \circ \mathrm{pr} + \mathsf{a}(\sigma_1)(\mathsf{a}(\sigma_2)(f_1)) \circ \mathrm{pr} - \mathsf{a}(\sigma_2)(\mathsf{a}(\sigma_1)(f_1) \circ \mathrm{pr} \\
&= -\big(\mathsf{a}([\sigma_1, \sigma_2]_A) - \mathsf{a}(\sigma_1) \circ \mathsf{a}(\sigma_2) + \mathsf{a}(\sigma_2) \circ \mathsf{a}(\sigma_1)\big)(f_1) \circ \mathrm{pr} \\
&= -\big([\mathsf{a}(\sigma_1), \mathsf{a}(\sigma_2)] - [\mathsf{a}(\sigma_1), \mathsf{a}(\sigma_2)]\big)(f_1) \circ \mathrm{pr} = 0
\end{aligned}$$

$$\begin{aligned}
\{\mathrm{ev}_{\sigma_1}, \{\mathrm{ev}_{\sigma_2}, \mathrm{ev}_{\sigma_3}\}_{A^*}\}_{A^*} &+ \{\mathrm{ev}_{\sigma_2}, \{\mathrm{ev}_{\sigma_3}, \mathrm{ev}_{\sigma_1}\}_{A^*}\}_{A^*} + \{\mathrm{ev}_{\sigma_3}, \{\mathrm{ev}_{\sigma_1}, \mathrm{ev}_{\sigma_2}\}_{A^*}\}_{A^*} \\
&= \left\{\mathrm{ev}_{\sigma_1}, \mathrm{ev}_{[\sigma_2,\sigma_3]_A}\right\}_{A^*} + \left\{\mathrm{ev}_{\sigma_2}, \mathrm{ev}_{[\sigma_3,\sigma_1]_A}\right\}_{A^*} + \left\{\mathrm{ev}_{\sigma_3}, \mathrm{ev}_{[\sigma_1,\sigma_2]_A}\right\}_{A^*} \\
&= \mathrm{ev}_{[\sigma_1,[\sigma_2,\sigma_3]_A]_A} + \mathrm{ev}_{[\sigma_2,[\sigma_3,\sigma_1]_A]_A} + \mathrm{ev}_{[\sigma_3,[\sigma_1,\sigma_2]_A]_A} \\
&= \mathrm{ev}_{[\sigma_1,[\sigma_2,\sigma_3]_A]_A + [\sigma_2,[\sigma_3,\sigma_1]_A]_A + [\sigma_3,[\sigma_1,\sigma_2]_A]_A} = \mathrm{ev}_0 = 0
\end{aligned}$$

Thus, $\{\cdot,\cdot\}_{A^*}$ defines a fibre-wise linear Poisson bracket on A^*. In general, we have the following result:

Proposition 5.4.9. *For any vector bundle $A \to M$, there is a natural one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Fibre-wise linear Poisson bivectors}\\ \pi\in\mathfrak{X}^2(A^*)\ \textit{on}\ A^*\to M\end{array}\right\}\xleftrightarrow{1:1}\left\{\begin{array}{c}\textit{Lie Algebroid Structures}\\ ([\cdot,\cdot],\mathsf{a})\ \textit{on}\ A\to M\end{array}\right\},$$

established by relations

$$\begin{aligned}\{\mathrm{ev}_\sigma,\mathrm{ev}_\varsigma\}_{A^*}&=\mathrm{ev}_{[\sigma,\varsigma]_A},\ \forall\,\sigma,\varsigma\in\Gamma(A);\\ \{\mathrm{ev}_\sigma,f\circ\mathrm{pr}\}_{A^*}&=\mathsf{a}(\sigma)(f)\circ\mathrm{pr},\ \forall\,\sigma\in\Gamma(A),\ f\in C^\infty(M).\end{aligned}$$

Proof. We have already seen that every Lie algebroid structure has an inherited Poisson structure on its dual. Now, we check the converse result.

Let $\pi\in\mathfrak{X}^2(A^*)$ be a fibre-wise linear Poisson structure on a vector bundle $A^*\to M$. We want to see that the dual bundle $A\to M$ inherits a Lie algebroid structure. Since π is a fibre-wise linear bivector field, by Propositions 5.4.7 and 5.4.8, we get that, if $\{\cdot,\cdot\}_\pi : C^\infty(A^*)\times C^\infty(A^*)\to C^\infty(A^*)$ is the bilinear form associated to π,

$$\{f\circ\mathrm{pr},g\circ\mathrm{pr}\}_\pi=0,\quad \{f\circ\mathrm{pr},\mathrm{ev}_\varsigma\}_\pi=g\circ\mathrm{pr}\quad\text{and}\quad\{\mathrm{ev}_{\sigma_1},\mathrm{ev}_{\sigma_2}\}_\pi=\mathrm{ev}_\varsigma \qquad (*)$$

for some $f,g\in C^\infty(M)$ and $\sigma_1,\sigma_2,\varsigma\in\Gamma(A)$. Then, define $[\cdot,\cdot]_A$ to be bilinear bracket structure on $\Gamma(A)$ such that $\{\mathrm{ev}_{\sigma_1},\mathrm{ev}_{\sigma_2}\}=\mathrm{ev}_{[\sigma_1,\sigma_2]_A}$. This expression automatically gives that $[\cdot,\cdot]_A$ satisfies the Jacobi identity.

The second expression of $(*)$, allows us to define a map $\alpha:\Gamma(A)\times C^\infty(M)\to C^\infty(M)$ such that, for any $f\in C^\infty(M)$ and $\sigma\in\Gamma(A)$, $\{f\circ\mathrm{pr},\mathrm{ev}_\sigma\}=\alpha(\sigma,f)\circ\mathrm{pr}$. Using the Leibniz rule of the Poisson bracket, we get that

$$\begin{aligned}\alpha(\sigma,fg)\circ\mathrm{pr}&=\{(fg)\circ\mathrm{pr},\mathrm{ev}_\sigma\}_\pi=\{(f\circ\mathrm{pr})(g\circ\mathrm{pr}),\mathrm{ev}_\sigma\}_\pi\\&=\{f\circ\mathrm{pr},\mathrm{ev}_\sigma\}_\pi\,(g\circ\mathrm{pr})+(f\circ\mathrm{pr})\,\{g\circ\mathrm{pr},\mathrm{ev}_\sigma\}_\pi\\&=(\alpha(\sigma,f)\circ\mathrm{pr})(g\circ\mathrm{pr})+(f\circ\mathrm{pr})(\alpha(\sigma,g)\circ\mathrm{pr})\\&=(\alpha(\sigma,f)\,g)\circ\mathrm{pr}+(f\,\alpha(\sigma,g))\circ\mathrm{pr}=(\alpha(\sigma,f)\,g+f\,\alpha(\sigma,g))\circ\mathrm{pr}\end{aligned}$$

hence, α is a derivation in the second entry. Moreover, noting that $\mathrm{ev}_{f\sigma}=(f\circ\mathrm{pr})\,\mathrm{ev}_\sigma$, by the first expression of $(*)$ and again by the Leibniz rule for the Poisson bracket,

$$\begin{aligned}\alpha(f\sigma,g)\circ\mathrm{pr}&=\{g\circ\mathrm{pr},\mathrm{ev}_{f\sigma}\}_\pi=\{g\circ\mathrm{pr},(f\circ\mathrm{pr})\mathrm{ev}_\sigma\}_\pi\\&=\{g\circ\mathrm{pr},f\circ\mathrm{pr}\}_\pi\,(\mathrm{ev}_\sigma)+(f\circ\mathrm{pr})\,\{g\circ\mathrm{pr},\mathrm{ev}_\sigma\}_\pi\\&=(f\circ\mathrm{pr})(\alpha(\sigma,g)\circ\mathrm{pr})=(f\,\alpha(\sigma,g))\circ\mathrm{pr}\end{aligned}$$

i.e. α is $C^\infty(M)$-linear in the first entry. Thus, there exists a vector bundle morphism $\mathsf{a} : A \to TM$ covering the identity such that[2] $\mathsf{a}(\sigma)(f) = -\alpha(\sigma, f)$, for all $\sigma \in \Gamma(A)$ and $f \in C^\infty(M)$ and,

$$\begin{aligned}
\mathrm{ev}_{[\sigma_1, f\sigma_2]_A} &= \{\mathrm{ev}_{\sigma_1}, \mathrm{ev}_{f\sigma_2}\}_\pi = \{\mathrm{ev}_{\sigma_1}, (f \circ \mathrm{pr})\, \mathrm{ev}_{\sigma_2}\}_\pi \\
&= \{\mathrm{ev}_{\sigma_1}, f \circ \mathrm{pr}\}_\pi\, \mathrm{ev}_{\sigma_2} + (f \circ \mathrm{pr})\, \{\mathrm{ev}_{\sigma_1}, \mathrm{ev}_{\sigma_2}\}_\pi \\
&= (\mathsf{a}(\sigma_1)(f) \circ \mathrm{pr})\mathrm{ev}_{\sigma_2} + (f \circ \mathrm{pr})\mathrm{ev}_{[\sigma_1,\sigma_2]_A} \\
&= \mathrm{ev}_{\mathsf{a}(\sigma_1)(f)\,\sigma_2} + \mathrm{ev}_{f\,[\sigma_1,\sigma_2]_A} = \mathrm{ev}_{+\mathsf{a}(\sigma_1)(f)\,\sigma_2 + f\,[\sigma_1,\sigma_2]_A},
\end{aligned}$$

hence, we get the Leibniz rule $[\sigma_1, f\sigma_2]_A = f\,[\sigma_1, \sigma_2]_A + \mathsf{a}(\sigma_1)(f)\,\sigma_2$ and we conclude that the triple $(A, [\cdot,\cdot]_A, \mathsf{a})$ is a Lie algebroid. Q.E.D.

To finish this section, we give a few examples of this fibre-wise linear Poisson structure on the dual of Lie algebroids:

Example 5.4.10. Naturally, linear Poisson structures in the dual of a Lie algebra are special cases of the previous construction. ♮

Example 5.4.11. As we have seen previously, TM is a Lie algebroid with the anchor being the identity and bracket the usual Lie bracket of vector fields. This implies that T^*M always has a fibre-wise linear Poisson structure. The evaluations in this case are maps $\mathrm{ev}_X : T^*M \to \mathbb{R}$ such that $\mathrm{ev}_X(\alpha) = \alpha(X_{\hat{\tau}_M(\alpha)})$. Locally, choosing adapted coordinates (x^i, α_i) on T^*M, the evaluation of a vector field X (locally expressed as $X = X^i\,\frac{\partial}{\partial x^i}$), then the evaluation takes the form

$$\mathrm{ev}_X(x^i, \alpha_i) = \sum_{j=1}^m X^j(x^i)\alpha_j.$$

Then, we get that

$$\begin{aligned}
\{\mathrm{ev}_X, \mathrm{ev}_Y\}_{T^*M}\,(x^i, \alpha^i) &= \mathrm{ev}_{[X,Y]}(x^i, \alpha^i) = \sum_{j=1}^m [X,Y]^j(x^i)\,\alpha_j \\
&= \sum_{j,k=1}^m \left(X^k(x^i)\frac{\partial Y^j}{\partial x^k} - Y^k(x^i)\frac{\partial X^j}{\partial x^k} \right) \alpha_j \\
&= \sum_{k=1}^m \left(\frac{\partial}{\partial \alpha^k}\,(X^\ell \alpha_\ell)\,\frac{\partial}{\partial x^k}\,(Y^j\alpha_j) - \frac{\partial}{\partial \alpha^k}\,(Y^\ell \alpha_\ell)\,\frac{\partial}{\partial x^k}\,(X^j\alpha_j) \right) \\
&= \sum_{k=1}^m \left(\frac{\partial\,\mathrm{ev}_X}{\partial \alpha_k}\frac{\partial\,\mathrm{ev}_Y}{\partial x^k} - \frac{\partial\,\mathrm{ev}_Y}{\partial \alpha_k}\frac{\partial\,\mathrm{ev}_X}{\partial x^k} \right),
\end{aligned}$$

where, in the third line, the sums in j and ℓ are implicit. Thus, the fibre-wise linear Poisson structure on the cotangent bundle (as the dual of the tangent bundle as a Lie algebroid) is the canonical Poisson bracket on T^*M. ♮

[2] In this expression, the negative sign serves only so that the Leibniz rule expression for a Lie algebroid (3.3) gets the familiar positive sign.

Chapter 6

Local Structure of Poisson Manifolds

In this chapter, we continue our study of Poisson manifolds by delving deeper into the local structure one can find in this class of geometries. In the first section, we reflect on the symplectic structures arising from the foliation induced by the Poisson bivector as well as the orbits obtained by Hamiltonian vector fields. Section 6.2 deals with one of the most important results in Poisson geometry, Weinstein's splitting theorem, which allows us to be able to always rewrite the Poisson structure into a symplectic part and a remainder and we also discuss the notion of the isotropy Lie algebra arising from the Poisson bivector. The final section finalises the chapter by studying the remainder part obtained from Weinstein's theorem and show that, even though this portion of the Poisson structure could be different at each point and each symplectic leaf, they are, in fact isomorphic to each other.

6.1 Symplectic Foliation

As seen in the Section 5.3, the introduction of the induced map $\pi^\sharp : T^*M \to TM$ allowed us to define a singular smooth distribution from its image, $\operatorname{im}\pi^\sharp = \pi^\sharp(T^*M) \subseteq TM$. This image is a rather important object when studying Poisson structures on a manifold and, hence, it deserves a special name.

Definition 6.1.1. The *characteristic distribution* of a Poisson manifold (M, π) is the singular involutive distribution $\operatorname{im}(\pi^\sharp) = \pi^\sharp(T^*M)$ whose fibres are given by the Hamiltonian vector fields at each point:

$$\operatorname{im}(\pi_p^\sharp) = \pi_p^\sharp(T_p^*M) = \big\{(X_f)_p \,\big|\, f \in C^\infty(M)\big\}. \tag{6.1}$$

Since the characteristic distribution is singular, the dimension of its integral leaves are not the same. Thus, it is useful to introduce the notion of the individual dimensions of the characteristic distribution at each point of the manifold

Definition 6.1.2. Given a Poisson manifold (M, π) and $p \in M$, the *rank* of the Poisson structure at a point $p \in M$, denoted by $\operatorname{rank}(\pi_p)$, is the integer $\dim(\operatorname{im}(\pi_p^\sharp)) \in \mathbb{Z}_{\geqslant 0}$.

Proposition 6.1.3. *Let (M, π) be a Poisson manifold and $\operatorname{im}(\pi^\sharp)$ its characteristic distribution.*

(i) The rank of (M, π) is even at every point;

(ii) The map $\operatorname{rank}(\pi) : M \to \mathbb{R}$ is bounded and lower semi-continuous;

Proof. Let $(U, \varphi = (x_1, \ldots, x_m))$ be a chart and define, for each $p \in U$, the matrix $\Pi_p \in M_n(\mathbb{R})$ whose entries are $(\Pi_p)_{ij} = \{x_i, x_j\}(p)$. Then, it is rather straightforward that Π_p is an anti-symmetric matrix, for all $p \in U$, thus its rank is even and, as a function of the point $p \in U$, is bounded and lower semi-continuous. Q.E.D.

Moreover, given this (involutive) distribution, we have the following results:

Theorem 6.1.4. *For a Poisson manifold (M, π), the characteristic distribution $\operatorname{im}(\pi^\sharp)$ is integrable.*

Proof. Since, by Theorem 5.3.6, the characteristic distribution is involutive, to apply the generalised Frobenius theorem 2.4.4 it is enough to show that the rank of π is constant along the flow of any Hamiltonian vector field. Let X be a Hamiltonian vector field and

$p \in M$. Since, by Proposition 5.3.3, $(\mathrm{Fl}_X^t)_*\pi = \pi$, the following diagram commutes

$$\begin{tikzcd} T_p^*M \arrow[r, "\pi_p^\sharp"] \arrow[d, "T_p^*\mathrm{Fl}_X^t"'] & T_pM \arrow[d, "T_p\mathrm{Fl}_X^t"] \\ T_{\mathrm{Fl}_X^t(p)}^*M \arrow[r, "\pi_{\mathrm{Fl}_X^t(p)}^\sharp"'] & T_{\mathrm{Fl}_X^t(p)}M \end{tikzcd}$$

Moreover, locally, Fl_X^t is a diffeomorphism, thus both $T_p^*\mathrm{Fl}_X^t$ and $T_p\mathrm{Fl}_X^t$ are isomorphisms (of vector spaces) and, therefore, $\dim(\mathrm{im}(\pi_p^\sharp)) = \dim(\mathrm{im}(\pi_{\mathrm{Fl}_X^t(p)}^\sharp))$. Q.E.D.

Theorem 6.1.5. *Let (M, π) be a Poisson manifold and N be an integral submanifold of $\mathrm{im}(\pi^\sharp)$. Then, the Poisson structure π induces a symplectic structure on N given, for each $p \in N$, by:*

$$(\omega_N)_p(u, v) = -\pi_p(\alpha, \beta), \tag{6.2}$$

for all $u, v \in T_pN$ and $\alpha, \beta \in T_p^*M$ such that $\pi_p^\sharp\alpha = u$ and $\pi_p^\sharp\beta = v$. Moreover, this symplectic structure is such that the inclusion $\iota : (N, \omega_N^{-1}) \hookrightarrow (M, \pi)$ is a Poisson morphism.

Proof. Let $N \subseteq M$ be an integral manifold of $\mathrm{im}(\pi^\sharp)$ and $\iota : N \hookrightarrow M$ be the inclusion map. To conclude the result, we need only to show that $\pi|_N = \iota^*\pi : T^*N \to TN$ is non-degenerate. Take $p \in N$ and $\alpha \in T_p^*N$ such that $(\pi|_N)_p(\alpha, \beta) = 0$, for all $\beta \in T_p^*N$. In this condition, $\beta\big((\pi|_N)_p^\sharp\alpha\big) = 0$, for all $\beta \in T_p^*N$, thus $(\pi|_N)_p^\sharp\alpha = 0$, i.e. $\alpha \in \ker((\pi|_N)_p^\sharp)$. Because the induced map is fibre-wise linear we get that $\dim\big(\mathrm{codom}((\pi|_N)_p^\sharp)\big) = \dim\big(\mathrm{im}((\pi|_N)_p^\sharp)\big) + \dim\big(\ker((\pi|_N)_p^\sharp)\big)$, thus

$$\begin{aligned} \dim\big(\ker((\pi|_N)_p^\sharp)\big) &= \dim\big(\mathrm{codom}((\pi|_N)_p^\sharp)\big) - \dim\big(\mathrm{im}((\pi|_N)_p^\sharp)\big) \\ &= \dim T_pN - \dim\mathrm{im}(\pi_p^\sharp) = 0 \end{aligned}$$

So, we conclude that $\alpha = 0$ and $\pi|_N$ is non-degenerate. Hence, by Proposition 5.3.9, we may find a symplectic form $\omega_N \in \Omega^2(N)$ satisfying Eq. (6.2).

Finally, we note that $\iota : N \hookrightarrow M$ is a Poisson morphism. For that, let $f, g \in C^\infty(M)$ and we, then, get

$$\begin{aligned} \omega_N^{-1}(\mathrm{d}(f \circ \iota), \mathrm{d}(g \circ \iota)) = \pi|_N(\iota^*\mathrm{d}f, \iota^*\mathrm{d}g) &= \iota_*^{-1}\big((\iota_*\pi|_N)(\mathrm{d}f, \mathrm{d}g)\big) \\ &= \iota^*\big((\iota_*\iota^*\pi)(\mathrm{d}f, \mathrm{d}g)\big) = \pi(\mathrm{d}f, \mathrm{d}g) \circ \iota, \end{aligned}$$

which concludes the proof. Q.E.D.

Definition 6.1.6. The integral leaves of $\mathrm{im}(\pi^\sharp)$ are called the *symplectic leaves* of the Poisson manifold (M, π) and the integral foliation of $\mathrm{im}(\pi^\sharp)$ is also called the *symplectic foliation* of M.

Theorem 6.1.7. *Let M be a manifold and $\mathcal{D} \subseteq TM$ a smooth involutive distribution with integral foliation $\mathcal{F}$ so that $\mathcal{D} = T\mathcal{F}$. If $\mathcal{D}$ is such that*

(i) for every point $p \in M$, the leaf $\mathcal{F}_p$ of $\mathcal{F}$ at p has a symplectic structure $\omega_{\mathcal{F}_p}$;

(ii) for any $f \in C^\infty(M)$, the rough vector field X_f given, for all $p \in M$, by $(X_f)_p = (\omega^\flat_{\mathcal{F}_p})^{-1}_p(\mathrm{d}(f|_{\mathcal{F}_p}))_p$ is differentiable in M;

then M has a unique Poisson structure π such that $\mathrm{im}(\pi^\sharp) = T\mathcal{F} = \mathcal{D}$ and $\omega^{-1}_{\mathcal{F}_p} = \pi|_{\mathcal{F}_p}$, for all $p \in M$.

Proof. Since each leaf $\mathcal{F}_p$ has a symplectic structure such that $\omega_{\mathcal{F}_p} = \omega_{\mathcal{F}_q}$ for every $p, q \in \mathcal{F}_p = \mathcal{F}_q$, it also has a Poisson structure such that $\pi_{\mathcal{F}_p} = \omega^{-1}_{\mathcal{F}_p}$ and, by *(ii)*, we can write, for any $f \in C^\infty(M)$, a vector field $(X_f)_p = \left(\pi^\sharp_{\mathcal{F}_p}\mathrm{d}(f|_{\mathcal{F}_p})\right)_p$. Thus, any two Poisson structures $\pi, \varpi \in \mathfrak{X}^2(M)$ satisfying $\mathrm{im}(\pi^\sharp) = T\mathcal{F} = \mathrm{im}(\varpi^\sharp)$ and $\pi|_{\mathcal{F}_p} = \omega^{-1}_{\mathcal{F}_p} = \varpi|_{\mathcal{F}_p}$ give rise to the same Hamiltonian vector fields in M and, hence, are equal in M.

Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field such that $\pi|_{\mathcal{F}_p} = \pi_{\mathcal{F}_p}$. We want to show that π is a Poisson bivector field for M, i.e. that $[\pi, \pi] = 0$. For this, since we have seen that $[\pi, \pi] = 0$ is equivalent to the Jacobi identity and $\{f, g\}(p) = \{f, g\}|_{\mathcal{F}_p}(p)$, it is enough to show that $[\pi, \pi]|_{\mathcal{F}_p} = 0$ for all $p \in M$. But, $[\pi, \pi]|_{\mathcal{F}_p} = [\pi|_{\mathcal{F}_p}, \pi|_{\mathcal{F}_p}] = [\pi_{\mathcal{F}_p}, \pi_{\mathcal{F}_p}] = 0$ because $\pi_{\mathcal{F}_p} = \omega^{-1}_{\mathcal{F}_p}$, which is a Poisson structure. Moreover, for this Poisson structure, the characteristic distribution at each point $p \in M$ is given by

$$\mathrm{im}(\pi^\sharp_p) = \left\{(X_f)_p \,\middle|\, f \in C^\infty(M)\right\} = \left\{\left(\pi^\sharp_{\mathcal{F}_p}\mathrm{d}(f|_{\mathcal{F}_p})\right)_p \,\middle|\, f \in C^\infty(M)\right\} = \mathrm{im}\left((\omega^\flat_{\mathcal{F}_p})^{-1}_p\right).$$

Since, by *(i)*, $\omega_{\mathcal{F}_p}$ is a symplectic structure on the leaf $\mathcal{F}_p$, it is non-degenerate and $\omega^\flat_{\mathcal{F}_p}$ is an isomorphism. Therefore, $\mathrm{im}((\omega^\flat_{\mathcal{F}_p})^{-1}) = T\mathcal{F}_p$, for all $p \in M$ and we conclude that $\mathrm{im}(\pi^\sharp) = T\mathcal{F} = \mathcal{D}$. Q.E.D.

In other words, the previous theorem guarantees that a Poisson structure can be defined by its symplectic foliation instead of its Poisson bivector field.

Having introduced these results, one could now ask: 'How does one obtain the symplectic foliation of a Poisson manifold (M, π)?' Since the characteristic distribution is generated by the Lie algebra of Hamiltonian vector fields, given N an integral submanifold of $\mathrm{im}(\pi^\sharp)$,

$$T_pN = \mathrm{im}(\pi^\sharp_p) = \left\{(X_H)_p \,\middle|\, H \in C^\infty(M)\right\}$$

hence, since the flow of a vector field tangent to a submanifold stays in said submanifold, choosing any point $p \in N$,

$$\left\{\mathrm{Fl}^t_{X_H}(p) \,\middle|\, H \in C^\infty(M) \text{ and for all } t \text{ defined}\right\} \subseteq N$$

Repeating this procedure for other points in these flows we may define the following equivalence relation:

$$p \sim q \iff \exists\, X_1, \ldots, X_k \in \mathfrak{ham}(M,\pi)\colon\; q = \mathrm{Fl}^1_{X_1} \circ \cdots \circ \mathrm{Fl}^1_{X_k}(p). \tag{6.3}$$

Then any equivalence class $[p]_\sim$ for a point $p \in M$ is open and entirely contained in a single symplectic leaf.

Moreover, we can easily check that $N \subseteq [p]_\sim$, otherwise, we would be able to take two distinct points $p, q \in N$ such that $[q]_\sim \neq [p]_\sim$ and, therefore, $N = [q]_\sim \cup [p]_\sim$. However, these equivalence classes are disjoint, thus, N is a disconnected integral submanifold of $\mathrm{im}(\pi^\sharp)$, which contradicts the fact that the collection of all integral manifolds of $\mathrm{im}(\pi^\sharp)$ forms a singular foliation. Therefore, we conclude the following:

Theorem 6.1.8. *The symplectic leaves of a Poisson manifold* (M,π) *are the elements of the quotient space* $M/\sim$ *for the equivalence relation* (6.3)*.*

This theorem then allows us to define the notion of a *leaf space* of a Poisson manifold:

Definition 6.1.9. The *Leaf Space* of a Poisson manifold (M,π) is the quotient space $M/\sim$ equipped with the quotient topology. To avoid confusion what equivalence relation we are assuming, we shall denote the leaf space of (M,π) by $M/\mathcal{F}_\pi$, where $\mathcal{F}_\pi$ is the symplectic foliation of (M,π).

Notice that Theorem 6.1.8 shows that the computation of the symplectic foliation of a Poisson manifold is rather complicated. In an attempt to partially solve this problem, we introduce the notion of a Casimir:

Definition 6.1.10. A *Casimir function* on a Poisson manifold (M,π) is a function $C \in C^\infty(M)$ such that $\pi(\mathrm{d}C, \mathrm{d}f) = \{C, f\} = 0$, for all $f \in C^\infty(M)$.

Proposition 6.1.11. *A Casimir function* $C \in C^\infty(M)$ *on a Poisson manifold* (M,π) *is constant along a symplectic leaf of* (M,π)*.*

Proof. Let $(M.\pi)$ be a Poisson manifold, $C \in C^\infty(M)$ be a Casimir function, $H \in C^\infty(M)$ an arbitrary function and $p \in M$. Then, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}(C \circ \mathrm{Fl}^t_{X_H}(p)) = (X_H(C))(\mathrm{Fl}^t_{X_H}(p)) = \{H, C\}\,(\mathrm{Fl}^t_{X_H}(p)) = 0,$$

i.e. $C(\mathrm{Fl}^t_{X_H}(p)) = C(p)$ for any Hamiltonian vector field $X_H \in \mathfrak{ham}(M,\pi)$. Thus, since, given any symplectic leaf S and two points $p, q \in S$, by Theorem 6.1.8, $q = \mathrm{Fl}^1_{X_1} \circ \cdots \circ \mathrm{Fl}^1_{X_k}(p)$ for some $X_1, \ldots, X_k \in \mathfrak{ham}(M,\pi)$, we get that

$$C(q) = C(\mathrm{Fl}^1_{X_1} \circ \cdots \circ \mathrm{Fl}^1_{X_k}(p)) = C(\mathrm{Fl}^1_{X_2} \circ \cdots \circ \mathrm{Fl}^1_{X_k}(p)) = \ldots = C(p)$$

Therefore, we conclude that a Casimir function is constant on an integral leaf of $\mathrm{im}(\pi^\sharp)$.

Q.E.D.

Thus, even though we might not be able to accurately describe the symplectic leaves of a Poisson manifold, the level sets of Casimir functions will give us a finer (or complete) description of the integral manifolds of $\mathrm{im}(\pi^\sharp)$.

Example 6.1.12. Clearly, for symplectic manifolds, $\mathrm{im}(\pi_p^\sharp) = T_p^*M$ and, therefore, the symplectic foliation is degenerate (i.e. the leaves are the connected components of M). ♮

Example 6.1.13. Recalling now the example of Log-Symplectic structures, there are two types of symplectic leaves in (M,π): the connected components of $(M\backslash Z,\eta)$, where $\eta^\flat = (\pi^\sharp|_{M\backslash Z})^{-1}$ and the leaves inside Z. The leaves in Z are of the form $(S,\omega|_S)$, where S is a leaf of $\mathcal{F}$, $T\mathcal{F} = \ker\theta$ and $(\theta,\omega) \in \Omega^1(M)\times\Omega^2(M)$ is the co-symplectic structure of Z. ♮

Example 6.1.14. Coming back to the linear Poisson structures, the symplectic leaves of $(\mathfrak{g}^*,\pi_{\mathfrak{g}})$ correspond to the coadjoint orbits of $\mathfrak{g}^*$. Given $\lambda\in\mathfrak{g}^*$, the coadjoint orbit is

$$\mathcal{O}_\lambda = \left\{\mathrm{Ad}_g^*\lambda \,\middle|\, g\in G\right\},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ is the Lie algebra of a connected Lie group G and, for all $\ell\in\mathfrak{g}$,

$$(\mathrm{Ad}_g^*\lambda)(\ell) = \lambda(\mathrm{Ad}_{g^{-1}}(\ell)) = \lambda\left(\frac{\mathrm{d}}{\mathrm{d}t} g e^{t\ell} g^{-1}\Big|_{t=0}\right).$$

Additionally, given the expression for the Poisson structure, the Hamiltonian vector field for a function $f\in C^\infty(\mathfrak{g}^*)$ at a point $\lambda\in\mathfrak{g}^*$ is

$$(\pi_{\mathfrak{g}}^\sharp \mathrm{d}f)_\lambda = i_{(\mathrm{d}f)_\lambda}(\pi_{\mathfrak{g}})_\lambda = \lambda([f_*(\lambda),\cdot]) = \mathrm{ad}_{f_*(\lambda)}^*\lambda,$$

where $\mathrm{ad}^* : \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ given, for all $\ell_1,\ell_2\in\mathfrak{g}$ and $\lambda\in\mathfrak{g}^*$, by $(\mathrm{ad}_{\ell_1}^*\lambda)(\ell_2) = \lambda([\ell_1,\ell_2])$ is the coadjoint representation of the Lie algebra on its dual. Thus, we get that $\mathrm{im}\left((\pi_{\mathfrak{g}}^\sharp)_\lambda\right) = \left\{\mathrm{ad}_\ell^*\lambda \,\middle|\, \ell\in\mathfrak{g}\right\} = T_\lambda\mathcal{O}_\lambda$, as expected for the integral submanifolds of the characteristic distribution. Then, by the proof of Theorem 6.1.5,

$$\omega_{\mathcal{O}_\lambda}(\mathrm{ad}_{\ell_1}^*\lambda, \mathrm{ad}_{\ell_2}^*\lambda) = -\lambda([\ell_1,\ell_2]),\ \ \lambda\in\mathfrak{g}^*,\ \ \ell_1,\ell_2\in\mathfrak{g}. \tag{6.4}$$

This symplectic form is called the *Kirillov-Konstant-Souriau (KKS) symplectic form.*

Some more specific examples of this construction is the following:

– $(\mathfrak{so}_3(\mathbb{R}))^*$: Consider the following generators of $\mathfrak{so}_3(\mathbb{R})$:

$$L_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\quad L_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad L_3 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Then, the Lie bracket relations for these generators are $[L_i,L_j] = \sum_{k=1}^{3}\varepsilon_{ijk}L_k$, where ε_{ijk} is the Levi-Civita symbol. Furthermore, it is easy to see that $C = (L_1)^2+(L_2)^2+(L_3)^2$

(or, more accurately, ev_C) is a Casimir function since, for $i = 1, 2, 3$,

$$\pi_{\mathfrak{so}_3(\mathbb{R})}(\mathrm{d}(\mathrm{ev}_C), \mathrm{d}(\mathrm{ev}_{L_i})) = \mathrm{ev}_{[C,L_i]} = \mathrm{ev}_{\sum_{j,k=1}^{3} \varepsilon_{jik} 2L_j L_k} = 0$$

and the differential of any other function $f \in C^\infty((\mathfrak{so}_3(\mathbb{R}))^*)$ can be written from the generators as a linear combination of the three generators L_1, L_2, L_3. Since the level sets of this Casimir function are the spheres $(L_1)^2 + (L_2)^2 + (L_3)^2 = r^2$ for $r^2 \neq 0$ and the point $(0, 0, 0)$ (interpreted as $0\,(L_1)^* + 0\,(L_2)^* + 0\,(L_3)^*$), we know that the symplectic leaves for this Poisson structure are contained on these spheres. The rank of $\pi_{\mathfrak{so}_3(\mathbb{R})}$ is 2 for all $\lambda \in \mathfrak{g}^* \backslash \{(0, 0, 0)\}$ since, if we write $\lambda = \lambda_1 L_1^* + \lambda_2 L_2^* + \lambda_3 L_3^*$, we get that

$$\mathrm{im}((\pi_{\mathfrak{so}_3(\mathbb{R})})_\lambda) = \left\{ \mathrm{ad}_\lambda^* \ell \,\middle|\, \ell \in \mathfrak{so}_3(\mathbb{R}) \right\} = \left\{ \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \lambda_i \ell_j L_k^* \,\middle|\, (\ell_1, \ell_2, \ell_3) \in \mathbb{R}^3 \right\}$$

which has dimension[1] 2. Thus, the rank of $\pi_{\mathfrak{so}_3(\mathbb{R})}$ is either 0 or 2 and, therefore, the symplectic leaves are, precisely, the connected components of the level sets of the Casimir C (or, more correctly, of the evaluation $\mathrm{ev}_C = (L_1^*)^2 + (L_2^*)^2 + (L_3^*)^2$).

– $(\mathfrak{sl}_2(\mathbb{R}))^*$: Let $\{E, F, H\}$ be the generators of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

such that their Lie bracket relations are

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

Following the same procedure done for the case of $\mathfrak{so}_3(\mathbb{R})$, showing that $C = 2(EF + FE) + H^2$ (or more accurately, ev_C) is a Casimir function boils down to proving that $[C, E] = [C, F] = [C, H] = 0$:

$$\begin{aligned}
[C, E] &= 2[E, E]F + 2E[F, E] + 2[F, E]E + 2F[E, E] + H[H, E] + [H, E]H \\
&= -2EH - 2HE + 2HE + 2EH = 0 \\
[C, F] &= 2[E, F]F + 2E[F, F] + 2[F, F]E + 2F[E, F] + [H, F]H + H[H, F] \\
&= 2HF + 2FH - 2FH - 2HF = 0 \\
[C, H] &= 2[E, H]F + 2E[F, H] + 2[F, H]E + 2F[E, H] + [H^2, H] \\
&= -4EF + 4EF + 4FE - 4FE = 0
\end{aligned}$$

[1] The entries of each L_i^* correspond to the coordinates of the vector product $a \times b$ where $a = (\lambda_1, \lambda_2, \lambda_3)$ and $b = (\ell_1, \ell_2, \ell_3)$ with a fixed, so, as we vary b we get the vectors tangent to the plane with normal a

To compute the rank of the Poisson structure, notice that, choosing $\lambda \in (\mathfrak{sl}_2(\mathbb{R}))^*$, written as $\lambda = \lambda_e E^* + \lambda_f F^* + \lambda_h H^*$, for any $\ell, l \in \mathfrak{sl}_2(\mathbb{R})$ with $\ell = \ell_e E + \ell_f F + \ell_h H$ (and similar for l with the different font),

$$\begin{aligned}(\mathrm{ad}^*_\lambda \ell)(l) &= \lambda([\ell, l]) = \lambda\left(2(-\ell_e l_f + \ell_h l_e)E + 2(\ell_f l_h - \ell_h l_f)F + (\ell_e l_f - \ell_f l_e)H\right)\\ &= 2\lambda_e(-\ell_e l_f + \ell_h l_e) + 2\lambda_F(\ell_f l_h - \ell_h l_f) + \lambda_h(\ell_e l_f - \ell_f l_e)\\ &= (2\lambda_e \ell_h - \lambda_h \ell_f)E^*(l) + (\lambda_h \ell_e - 2\lambda_f \ell_h)F^*(l) + (2\lambda_f \ell_f - 2\lambda_e \ell_e)H^*(l).\end{aligned}$$

Hence, we get that

$$\mathrm{im}((\pi_{\mathfrak{sl}_2(\mathbb{R})})_\lambda) = \left\{(2\lambda_e\ell_h - \lambda_h\ell_f)E^* - (2\lambda_f\ell_h - \lambda_h\ell_e)F^* - 2(\lambda_e\ell_e - \lambda_f\ell_f)H^* \,\middle|\, (\ell_e, \ell_f, \ell_h) \in \mathbb{R}^3\right\},$$

which, for $\lambda = 0 \in (\mathfrak{sl}_2(\mathbb{R}))^*$ has dimension 0 and, for any $\lambda \in (\mathfrak{sl}_2(\mathbb{R}))^* \backslash \{0\}$, has dimension 2.

Then, once again, the rank of the Poisson structure $\pi_{\mathfrak{sl}_2(\mathbb{R})}$ is either 0 or 2, thus the symplectic leaves are the connected components of the level sets of the Casimir function C. Therefore, there are 4 types of symplectic leaves:

1) $C > 0$: the hyperboloid of one sheet;
2) $C < 0$: the two components of the hyperboloid of two sheets;
3) $C = 0$: $EF + FE = -\frac{1}{2}H^2$ has two components of the conical surface;
4) $(0, 0, 0)$.

– Heisenberg algebra: $\mathfrak{h} = \left\{aX + bY + cZ \,\middle|\, a, b, c \in \mathbb{R}\right\}$, where

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the Lie bracket is the usual commutator of matrices, which means that $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. It is immediate that any Casimir must be independent of X and Y, since, otherwise, considering any Casimir function $C(X, Y, Z)$, we get that $[C(X, Y, Z), Z] = 0$ but $[C(X, Y, Z), X] = C(X, -Z, Z)$ and $[C(X, Y, Z), Y] = C(Z, Y, Z)$ which are not both necessarily equal to zero. Since the Poisson structure must have even rank, it must be either 0 or 2. Taking points in $\mathfrak{h}^*$ of the form $\alpha X^* + \beta Y^* + \gamma Z^*$, if $\gamma = 0$,

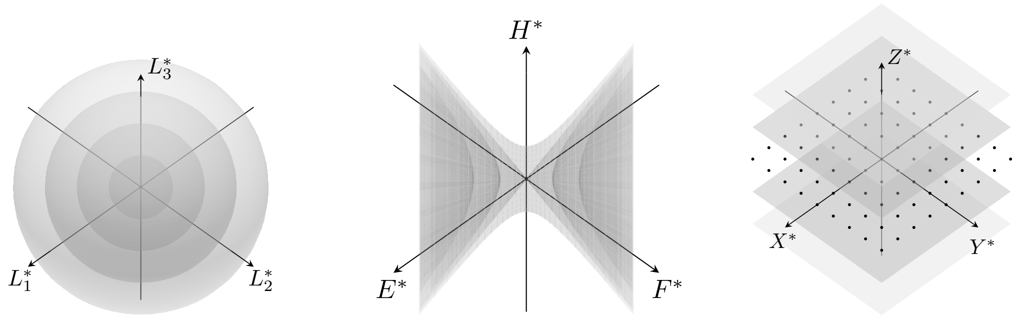
$$\left(\mathrm{ad}^*_{aX+bY+cZ}(\alpha X^* + \beta Y^*)\right)(a'X + b'Y + c'Z) = (\alpha X^* + \beta Y^*)((ab' - ba')Z) = 0,$$

so the rank of the Poisson structure for these points is 0. Otherwise, the rank of $\pi_\mathfrak{h}$ is 2, so we have two types of symplectic leaves:

1) the points $\{\alpha\, X^* + \beta\, Y^*\}$ for fixed $\alpha, \beta \in \mathbb{R}$;

2) the planes $\{\alpha\, X^* + \beta\, Y^* + \gamma\, Z^* \mid \alpha, \beta \in \mathbb{R}\}$ with fixed $\gamma \in \mathbb{R}$.

The following figure shows the examples of symplectic foliations for the dual of the Lie algebras mentioned above:



(a) $(\mathfrak{so}_3(\mathbb{R}))^*$ (b) $(\mathfrak{sl}_2(\mathbb{R}))^*$ (c) $(\mathfrak{h})^*$

Figure 6.1: Representation of the symplectic leaves for certain linear Poisson structures.

♮

6.2 Weinstein Splitting Theorem

Given the study done previously, we can now discuss one of the most fundamental theorems in Poisson geometry. For the proof of such theorem we require to use Lemma 2.2.3 which we repeat here for convenience:

Lemma 2.2.3. *Let M be a smooth m-dimensional manifold and $X_1, \ldots, X_k$ be linearly independent vector fields defined on a neighbourhood V of $p \in M$ such that*

$$[X_i, X_j] = 0, \quad 1 \leq i, j \leq k.$$

Then there is a chart on M centred at p, $(U \subseteq V; (x_1, \ldots, x_m))$ such that

$$X_i\big|_U = \frac{\partial}{\partial x_i}\,, \quad 1 \leq i \leq k.$$

Then we introduce the splitting theorem, which is one of the most fundamental theorems of Poisson geometry and is one of the central results of our study.

Theorem 6.2.1 (Weinstein Splitting). *Let (M,π) be a Poisson manifold and $x \in M$ with* $\operatorname{rank}(\pi_x) = 2s$. *Then there are local coordinates $(U; q^1, \dots, q^s, p_1, \dots, p_s, y^1, \dots, y^n)$ on M centred at x such that*

$$\pi|_U = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \sum_{1 \le a < b \le n} \theta^{ab} \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b} \tag{6.5}$$

where $\theta^{ab}(y)$ are smooth functions of $(y^1, \dots, y^n)$ with $\theta^{ab}(0) = 0$.

Proof. Induction on $\operatorname{rank}(\pi_x) = 2s$: In the case $\operatorname{rank}(\pi_x) = 0$, there is nothing to prove and, indeed, π looks like Eq. (6.5) for any choice of a local coordinate chart $(U, \varphi = (y^1, \dots, y^n))$ on M centred at x. Assume $\operatorname{rank}(\pi_x) = 2s$ with $s \ge 1$. Since $\pi_x \ne 0$, there is a local function p on M, defined around x, such that $p(x) = 0$ and $(X_p)_x = \pi_x^\sharp((\mathrm{d}p)_x) \ne 0$. Using Lemma 2.2.3, there exists a coordinate chart $(W \subseteq M; x^1, \dots, x^m)$ on M centred at x such that $X_p|_W = \frac{\partial}{\partial x^1}$. Call $q \overset{\text{def}}{=} x^1$ and note that

$$X_q(q) = 0, \quad X_p(p) = 0, \quad X_q(p) = \{q,p\} = -1, \quad X_p(q) = \{p,q\} = 1.$$

This means that X_q and X_p are linearly independent commuting vector fields (since $[X_q, X_p] = X_{\{q,p\}} = 0$) and, once again, we may apply Lemma 2.2.3. Then there exists a coordinate chart $(U \subseteq M; \tilde{x}^1, \dots, \tilde{x}^m)$ on W centred at x such that, $X_q|_U = \frac{\partial}{\partial \tilde{x}^1}$ and $X_p|_U = \frac{\partial}{\partial \tilde{x}^2}$. Up to restricting U around x if necessary, we can take the chart to be $(U; q, p, \tilde{x}^3, \dots, \tilde{x}^m)$ satisfying

$$\{q,p\} = -1, \quad \left\{p, \tilde{x}^i\right\} = \left\{q, \tilde{x}^i\right\} = 0 \quad (i = 3, \dots, m).$$

In these conditions, the Poisson bivector field becomes

$$\pi|_U = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \sum_{3 \le a < b \le m} \theta^{ab}(q, p, \tilde{x}^3, \dots, \tilde{x}^m) \frac{\partial}{\partial \tilde{x}^a} \wedge \frac{\partial}{\partial \tilde{x}^b}. \tag{6.6}$$

To conclude the result, we need only to show that the functions θ^{ab} do not depend on the coordinates q, p. For this, since the Jacobi identity applies to π,

$$\frac{\partial}{\partial q}\left(\theta^{ab}\right) = X_p(\theta^{ab}) = \left\{p, \left\{\tilde{x}^a, \tilde{x}^b\right\}\right\} = \left\{\left\{p, \tilde{x}^a\right\}, \tilde{x}^b\right\} + \left\{\tilde{x}^a, \left\{p, \tilde{x}^b\right\}\right\} = 0$$

and analogously $\frac{\partial}{\partial p}\left(\theta^{ab}\right) = 0$, hence θ^{ab} does not depend on q, p. Again by the Jacobi identity, the second factor in Eq. (6.6) defines a Poisson structure around $(\tilde{x}^3 = 0, \dots, \tilde{x}^m = 0)$ with rank equal to $\operatorname{rank}(\pi_x) - 2$. Thus, by the induction hypothesis, the result follows.

Q.E.D.

Alternatively, we may as well describe this local structure of Poisson manifolds without making any mention to specific charts.

Theorem 6.2.2. *Let (M, π) an m-dimensional Poisson manifold, $p \in M$ and* $\operatorname{rank}(\pi_p) = 2h$*. Then, for an open neighbourhood $U \subseteq M$ of p, there is a Poisson isomorphism $\phi = (\phi_S, \phi_N) : (U, \pi|_U) \xrightarrow{\cong} (S, \omega_S^{-1}) \times (N, \pi_N)$, where (S, ω_S) is a $2h$-dimensional symplectic manifold and N is a $(m-2h)$-dimensional manifold such that* $\operatorname{rank}\big((\pi_N)\phi_N(p)\big) = 0$*. Additionally, S and N are unique up to local isomorphism.*

This previous theorem defines the so-called *Weinstein splitting charts*.

Weinstein's splitting theorem is the analogous in Poisson geometry to Frobenius' theorem or Darboux's theorem for tangent distributions and symplectic structures, respectively. In the following section, we shall further generalise Weinstein's theorem to include the notion of *Poisson transversals*.

Regular Points

As with the case of distributions, it is possible to distinguish points of a Poisson manifold according to the rank of the Poisson structure:

Definition 6.2.3. Let (M, π) be a Poisson manifold. A point $p \in M$ is called a *regular point* if the rank of π is constant on a neighbourhood of p. If said rank is not constant, then p is called a *singular point*.

Lemma 6.2.4. *The set of regular points of a Poisson manifold (M, π) is open and dense in M.*

Proof. Let (M, π) be a Poisson manifold and $\mathcal{R}$ be its set of regular points.

Open: To see that $\mathcal{R}$ is open in M, let us prove that the set of singular points $\mathcal{S} = M \backslash \mathcal{R}$ is closed. Let $p \in \overline{\mathcal{S}}$ be a point in the closure of $\mathcal{S}$. Then, by the definition of closure, any open neighbourhood $U \subseteq M$ of p in M intersects $\mathcal{S}$ and, therefore, there exists a singular point $s \in U$. Since U is open and contains s, it is a neighbourhood of s in M. Thus, by definition of a singular point, there exists at least one point $q \in U$ such that $\operatorname{rank}(\pi_q) \neq \operatorname{rank}(\pi_s)$ and, so, the rank of π is not constant on U (which is a neighbourhood of p). Hence, p is a singular point and $\overline{\mathcal{S}} = \mathcal{S}$.

Dense: Given a point $p \in \mathcal{S}$, since $\operatorname{rank}(\pi) : M \to \mathbb{Z}_{\geqslant 0}$ is bounded (and is integer-valued), it must have a maximum in any neighbourhood U of p and this point of maximum for $\operatorname{rank}(\pi|_U)$ turns out to be a regular point for π. Thus, $p \in \overline{\mathcal{R}}$. Q.E.D.

For a regular point $p \in M$, since the rank of π must be constant, given a splitting chart $(U, \varphi = (q^1, \ldots, q^s, p_1, \ldots, p_s, y^1, \ldots, y^n))$ centred at p, the Poisson structure must be constant up to shrinking U around p if necessary, i.e. $\pi|_U = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$, and the characteristic distribution is a linear space:

$$\mathrm{im}(\pi^\sharp|_U) = \mathrm{span}\left\{ \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^s}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_s} \right\}.$$

Thus, near a regular point, $\mathrm{im}(\pi^\sharp)$ is an involutive regular distribution. So, by Theorem 2.3.12 there is a foliation associated to this distribution whose leaves precisely correspond to the level curves

$$\big\{(q^1, \ldots, q^s, p_1, \ldots, p_s, y^1, \ldots, y^n) \in M \,\big|\, (y^1, \ldots, y^n) = (y_0^1, \ldots, y_0^n)\big\},$$

as $(y_0^1, \ldots, y_0^n)$ varies in $\mathbb{R}^n$. These leaves are symplectic in nature and, for each leaf, the corresponding symplectic 2-form is $\omega_{y_0} = \sum_{i=1}^{s} \mathrm{d}q^i \wedge \mathrm{d}p_i$. Hence, in the case of regular points, Weinstein splitting theorem becomes just Darboux theorem for regular foliations of symplectic manifolds, Proposition 4.3.24. We can also make this result independent of coordinate charts:

Proposition 6.2.5. *Let M be a smooth manifold. There is a natural one-to-one correspondence:*

$$\left\{ \begin{array}{c} \textit{Regular Poisson} \\ \textit{structures on } M \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \textit{Regular Symplectic} \\ \textit{Foliations } (\mathcal{F}, \omega_{\mathcal{F}}) \textit{ on } M \end{array} \right\},$$

established by the relation $\pi = \omega_{\mathcal{F}}^{-1}$ and $\mathrm{im}(\pi^\sharp) = T\mathcal{F}$.

Proof. Given a regular Poisson structure $\pi \in \mathfrak{X}^2(M)$, by Theorem 6.1.5, $\mathrm{im}(\pi^\sharp)$ gives a singular foliation $\{N_\lambda\}_{\lambda \in \Lambda}$ such that each leaf N_λ has a symplectic structure ω_{N_λ}. Since this Poisson structure is regular, for any $p \in N_\lambda$ and $q \in N_{\lambda'}$, $\dim(N_\lambda) = \dim(\mathrm{im}(\pi_p^\sharp)) = \dim(\mathrm{im}(\pi_q^\sharp)) = \dim(N_{\lambda'})$, so the foliation $\{N_\lambda\}_{\lambda \in \Lambda}$ is a regular symplectic foliation.

Conversely, if $(\mathcal{F}, \omega_{\mathcal{F}})$ is a regular symplectic foliation, by Theorem 6.1.7, there is a unique Poisson structure π on M satisfying the properties of the Proposition. The regularity of the foliation guarantees that, given two points $p, q \in M$ in distinct leafs of the foliation $\mathcal{F}$, $\dim(\mathrm{im}(\pi_p^\sharp)) = \dim(\mathcal{F}_p) = \dim(\mathcal{F}_q) = \dim(\mathrm{im}(\pi_q^\sharp))$, i.e. π is a regular Poisson structure. Q.E.D.

Singular Points

For singular points, the Poisson structures may have rather complicated behaviours. In $\mathbb{R}^2$, take $\pi = f(x,y)\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ which is a Poisson structure. For a singular point (x_0, y_0), $f(x_0, y_0) = 0$ but $f \neq 0$ in a neighbourhood of (x_0, y_0). Although f could be any smooth map with these properties, the local classification of these bivectors is not simple. The case of generic singular points is covered by the following result:

Proposition 6.2.6. *Let $\pi = f(x,y)\,\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \in \mathfrak{X}^2(\mathbb{R}^2)$ with $f(0) = 0$ and $f_*(0) \neq 0$. Then, there exists a chart $(U,(u,v))$ centred at 0 such that $\pi|_U = u\,\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$.*

Proof. Since $f_*(0) = 0$, we may define a vector field $X = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}$ and apply Lemma 2.2.3. Hence, there is a chart $(U,(v,u))$ around $(0,0) \in \mathbb{R}^2$ such that $X|_U = \frac{\partial}{\partial v}\big|_U$ to which we may choose $u = f$. Thus, $\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x} = X(v) = 1$ and

$$\{u,v\} = u\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) = u\,X(v) = u.$$

Therefore, we conclude that $\pi|_U = u\,\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$. Q.E.D.

We are now in condition to prove Proposition 5.3.13 detailing the local structure of log-symplectic manifolds in a neighbourhood of a point in its singular locus.

Proposition 5.3.13. *Let (M,π) be a $2n$-dimensional log-symplectic manifold with singular locus Z. Then, for any $x \in Z$, there is a local coordinate chart $(U,(u,v,q^1,\dots,q^{n-1},p_1,\dots,p_{n-1}))$ on M centred at x such that*

$$U \cap Z = \left\{(0,v,q^1,\dots,q^{n-1},p_1,\dots,p_{n-1}) \,\middle|\, v,q^1,\dots,q^{n-1},p_1,\dots,p_{n-1} \in \mathbb{R}\right\}$$

and

$$\pi|_U = u\,\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} + \sum_{i=1}^{n-1} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}\,.$$

Proof. Let $(U,(q^1,\dots,q^s,p_1,\dots,p_s,y^1,\dots,y^{2r}))$ be a Weinstein splitting chart centred at $x \in Z$ with $n = s + r$, where $\mathrm{rank}(\pi_x) = 2s$. We want to show that, since (M,π) is log-symplectic and $x \in Z$, $\mathrm{rank}(\pi_x) = 2(n-1)$, i.e. $s = n-1$ and $r = 1$. Then the Poisson structure on M can be written as

$$\pi|_U = \pi_{\mathrm{can}} + \theta = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \sum_{1 \le a < b \le 2r} \theta^{ab}(y)\,\frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b}\,,$$

for some local functions $\theta^{ab}(y)$ vanishing at $y = 0$. Computing the n^{th} exterior power of π

we get

$$
\begin{aligned}
(\pi|_U)^{\wedge n} = (\pi_{\text{can}} + \theta)^{\wedge n} &= \sum \left\{ \begin{array}{c} \text{All possible combinations containing} \\ s \text{ terms of } \pi_{\text{can}} \text{ and } r \text{ terms of } \theta \end{array} \right\} \\
&= \frac{n!}{s!\,r!}\theta^{\wedge r} \wedge (\pi_{\text{can}})^{\wedge s} = \frac{n!}{s!\,r!}\theta^{\wedge r} \wedge \left(s!\,\frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} \wedge \ldots \wedge \frac{\partial}{\partial p_s} \wedge \frac{\partial}{\partial q^s} \right) \\
&= \frac{n!}{r!}\theta^{\wedge r} \wedge \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} \wedge \ldots \wedge \frac{\partial}{\partial p_s} \wedge \frac{\partial}{\partial q^s}\ ,
\end{aligned}
$$

where

$$
\theta^{\wedge r} = \bigwedge_{i=1}^{r} \sum_{1 \le a_i < b_i \le 2r} \theta^{a_i b_i}(y)\, \frac{\partial}{\partial y^{a_i}} \wedge \frac{\partial}{\partial y^{b_i}}
$$

contains a product of r coefficients $\theta^{a_i b_i}(y)$. Thus, since we can write

$$
\pi^{\wedge n} = f(y)\, \frac{\partial}{\partial q^1} \wedge \ldots \wedge \frac{\partial}{\partial q^s} \wedge \frac{\partial}{\partial p_1} \wedge \cdots \wedge \frac{\partial}{\partial p_s} \wedge \frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^{2r}}\ ,
$$

we get that $f(y)$ has a product of r coefficients $\theta^{a_i b_i}(y)$ and, so, $f_*(x)$ has the product of $r-1$ coefficients $\theta^{a_i b_i}(0) = 0$. This clearly contradicts the hypothesis that (M, π) is a log-symplectic manifold unless $r = 1$. Hence,

$$
\pi|_U = \pi_{\text{can}} + \theta = \sum_{i=1}^{n-1} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \theta^{12}(y)\, \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2}
$$

and, applying the previous proposition to θ, we obtain the result. Q.E.D.

It is important to understand that the quick study made here for singular points is, in fact, very extensive, since the number of Poisson structures near a singular point whose singularity is of larger order becomes bigger very quickly.

6.3 Poisson Transversals

In Section 6.1, we have introduced the notion of the symplectic foliation of a Poisson manifold. However, the leaves of this foliation do not need to have the same dimension of the manifold itself as expressed by Weinstein splitting theorem[2]. Hence, it is important to be able to describe the 'remainder' of the dimension of the manifold at each point $p \in M$. This is precisely what the concepts of Poisson transversals and cosymplectic submanifolds intend to do.

Definition 6.3.1. Let (M, π) a Poisson manifold, $p \in M$ and let S be the symplectic leaf passing through p. A *Poisson transversal* of (M, π) relative to S at p is any embedded submanifold $X \subseteq M$ containing p satisfying

$$T_pM = T_pS \oplus T_pX, \tag{6.7a}$$

or, equivalently,

$$T_pM = \operatorname{im}(\pi_p^\sharp) \oplus T_pX. \tag{6.7b}$$

Remark 6.3.2. Given any symplectic leaf S on a Poisson manifold (M, π) and a point $p \in S$, there always exists a Poisson transversal of (M, π) relative to S at p.

Recall that, since $\operatorname{im}(\pi^\sharp) = \pi^\sharp(T^*M)$ is a generalised distribution, the dimension of the characteristic leaf S' through a point $q \in X$ near p may be strictly larger than the dimension of the leaf S passing through p; hence Eq. (6.7b) may fail at certain points in X because we might have $\operatorname{im}(\pi_q^\sharp) \cap T_qX \neq 0$. Regardless, given the continuity of $\pi^\sharp$, for any point $q \in X$ sufficiently close to p, X is still transverse to the leaf S' passing through q, i.e. $T_qM = T_qS' + T_qX$ or, equivalently, $T_qM = \operatorname{im}(\pi_q^\sharp) + T_qX$.

Moreover, these Poisson transversals are well-behaved with respect to the flow of infinitesimal Poisson automorphisms:

Proposition 6.3.3. *Let (M, π) be a Poisson manifold, S a symplectic leaf of (M, π) passing through a point $p \in M$ and X a Poisson transversal of (M, π) relative to S at p. Then, for any Hamiltonian vector field $Y \in \mathfrak{ham}(M, \pi)$ such that $p \in \operatorname{dom}(\mathrm{Fl}_Y^t)$, $\mathrm{Fl}_Y^t(S) \subseteq S$ and $\mathrm{Fl}_Y^t(X)$ is a Poisson transversal of (M, π) relative to S at $\mathrm{Fl}_Y^t(p)$.*

Proof. The fact that $\mathrm{Fl}_Y^t(S) \subseteq S$ is clear by the definition of the characteristic distribution and the fact that S is an integral submanifold of $\operatorname{im}(\pi^\sharp)$. Thus, to conclude the result, we need only prove that $\mathrm{Fl}_Y^t(X)$ is a Poisson transversal on (M, π) relative to S at $\mathrm{Fl}_Y^t(p)$. For that, simply note that, given any $v \in T_pM$, $(\mathrm{Fl}_Y^t)_*(p)(v) \in T_{\mathrm{Fl}_Y^t(p)}M$. Since X is a Poisson transversal relative to S at p, $v = u + w$, with $u \in T_pS$ and $w \in T_pX$. Thus,

$$(\mathrm{Fl}_Y^t)_*(p)(v) = (\mathrm{Fl}_Y^t)_*(p)(u) + (\mathrm{Fl}_Y^t)_*(p)(w) = u' + w',$$

[2]In fact, these leaves of the symplectic foliation have the same dimension as the manifold only when the Poisson structure is non-degenerate, which means that it must be precisely a symplectic structure.

where $u' \in T_{\mathrm{Fl}_Y^t(p)}S$ and $w' \in T_{\mathrm{Fl}_Y^t(p)}(\mathrm{Fl}_Y^t(X))$. To conclude the result, we need only to show that $T_{\mathrm{Fl}_Y^t(p)}S \cap T_{\mathrm{Fl}_Y^t(p)}(\mathrm{Fl}_Y^t(X)) = 0$. For that, suppose that $v \in T_{\mathrm{Fl}_Y^t(p)}S \cap T_{\mathrm{Fl}_Y^t(p)}(\mathrm{Fl}_Y^t(X))$. Then, by calling ω_S the symplectic form on S, taking any $u \in T_{\mathrm{Fl}_Y^t(p)}S$ and by noticing that Fl_Y^t is a local diffeomorphism, we get that

$$\begin{aligned}(\omega_S)_{\mathrm{Fl}_Y^t(p)}(v, u) &= ((\mathrm{Fl}_Y^t)^*\omega_S)_p((\mathrm{Fl}_Y^t)_*^{-1}(\mathrm{Fl}_Y^t(p))(v), (\mathrm{Fl}_Y^t)_*^{-1}(\mathrm{Fl}_Y^t(p))(u)) \\ &= (\omega_S)_p(v', u') = 0\end{aligned}$$

where in the second equality we used the fact that Y is an Hamiltonian vector field (so, in particular, $Y \in \mathfrak{aut}(M, \pi)$) to conclude that, when restricted to S, it is an infinitesimal symplectomorphism and, in the third equality, we notice that $v' = (\mathrm{Fl}_Y^t)_*^{-1}(\mathrm{Fl}_Y^t(p))(v) \in T_pS \cap T_pX = 0$, since X is a Poisson transversal to S at p. Thus, by the non-degeneracy of the symplectic form ω_S, we conclude that $v = 0$, hence, $\mathrm{Fl}_Y^t(X)$ is a Poisson transversal of (M, π) relative to S at p. Q.E.D.

Definition 6.3.4. Let (M, π) be a Poisson manifold. A *cosymplectic submanifold* of (M, π) is any embedded submanifold $X \subseteq M$ such that

$$T_pM = T_pX + \pi_p^\sharp((T_pX)^\circ), \text{ for all } p \in X, \tag{6.8}$$

where $(T_pX)^\circ = \{\alpha \in T_p^*M \mid \alpha|_{T_pX} = 0\}$ is the annihilator of T_pX.

Remark 6.3.5. If (S, ω_S) is the symplectic leaf of a Poisson manifold (M, π) passing through a point $p \in M$ and X is any submanifold of M containing p, then it is clear that $\pi_p^\sharp((T_pX)^\circ) = \{\pi_p^\sharp\alpha \in T_pM \mid \ker(\alpha) \supseteq T_pX\} \subseteq \mathrm{im}(\pi_x^\sharp) = T_pS$. Additionally, we have that

$$\begin{aligned}\pi_p^\sharp((T_pX)^\circ) &\subseteq \{\pi_p^\sharp\alpha \in T_pM \mid \alpha|_{T_pX\cap T_pS} = 0\} = \pi_p^\sharp((T_pX \cap T_pS)^\circ) \\ &= (\omega_S^\flat)_p^{-1}((T_pX \cap T_pS)^\circ) = (T_pX \cap T_pS)^{\perp_{(\omega_S)_p}}.\end{aligned}$$

To prove the reciprocal inclusion, consider an arbitrary element $v \in (T_pX \cap T_pS)^{\perp_{(\omega_S)_p}}$. Since ω_S is a symplectic form, $\omega_S^\flat$ is an isomorphism, thus there exists a unique $\alpha \in T_p^*S$ such that $\alpha = (\omega_S^\flat)_p(v)$. Extending α to $\tilde{\alpha} \in T_p^*M$ so that $v = \pi_p^\sharp\tilde{\alpha}$, we conclude that, for any $u \in T_pX$, $\tilde{\alpha}(u) = 0$ (either because $\tilde{\alpha}|_{T_p^*M \setminus T_p^*S} = 0$ or because $\tilde{\alpha}|_{T_pX\cap T_pS} = (\omega_S^\flat)_p(v)|_{T_pX\cap T_pS} = 0$). Thus, $v \in \pi_p^\sharp((T_pX)^\circ)$ and

$$\pi_p^\sharp((T_pX)^\circ) = (T_pX \cap T_pS)^{\perp_{(\omega_S)_p}}.$$

Finally, since $\dim(\pi_p^\sharp((T_pX)^\circ) \leq \dim((T_pX)^\circ) = \dim(M) - \dim(X)$, we conclude that the defining condition of a cosymplectic submanifold X is equivalent to

$$T_pM = T_pX \oplus \pi_p^\sharp((T_pX)^\circ), \text{ for all } p \in X. \tag{6.9}$$

In the case of symplectic vector spaces, one can show the following results:

Lemma 6.3.6. *Let (V,ω) be a symplectic vector space with an associated Poisson structure $\pi : V^* \times V^* \to \mathbb{R}$ such that, for all $\alpha, \beta \in V^*$,*

$$\pi(\alpha,\beta) = -\omega((\omega^\flat)^{-1}\alpha, (\omega^\flat)^{-1}\beta).$$

Then, if $W \leq V$ is a vector subspace, we have that

$$\pi^\sharp(W^\circ) = W^{\perp_\omega}.$$

Proof. Firstly, note that, since W° is the annihilator of W, by the definition of π, we have that $\pi^\sharp = (\omega^\flat)^{-1} : V^* \to V$ and, hence, $\pi^\sharp(W^\circ) = \{\pi^\sharp\alpha \in V \,\big|\, \ker(\alpha) \supseteq W\}$. Thus,

$$\begin{aligned} W^{\perp_\omega} &= \{v \in V \,|\, \omega(v,w) = 0,\ \forall\, w \in W\} = \{v \in V \,|\, (\omega^\flat v)(w) = 0,\ \forall\, w \in W\} \\ &= \{\pi^\sharp_p\alpha \in V \,|\, \alpha(w) = 0,\ \forall\, w \in W\} = \pi^\sharp(W^\circ), \end{aligned}$$

where, in the second line, we used the fact that $\pi^\sharp = (\omega^\flat)^{-1}$ is an isomorphism. Q.E.D.

For symplectic manifolds, then, we get that the notion of a cosymplectic manifold is the same of a symplectic submanifold:

Proposition 6.3.7. *Let (M,ω) a symplectic manifold and $X \subseteq M$ an embedded submanifold. Then X is a cosymplectic submanifold of (M,ω^{-1}) if and only if $(X, \omega|_X)$ is a symplectic submanifold of (M,ω).*

Proof. $\Longrightarrow$) Let X a cosymplectic submanifold. We want to show that $\omega|_X$ is a symplectic form on X. Clearly, since ω is a symplectic form on M, it is closed M and, because $X \subseteq M$ is an (embedded) submanifold, it is also closed on X. As for the non--degeneracy on X, let $p \in X$ and $v \in T_pX$ be such that $\omega_p(v,u) = 0$ for all $u \in T_pX$. Then, since

$$\pi^\sharp_p((T_pX)^\circ) = (\omega^\flat)^{-1}_p((T_pX)^\circ) = \{w \in T_pM \,|\, \omega_p(w,u) = 0,\ \forall\, u \in T_pX\},$$

and X is a cosymplectic manifold, any $w \in T_pM$ can be written as $w = w_x + w_c$ with $w_x \in T_pX$ and $w_c \in \pi^\sharp_p((T_pX)^\circ)$ and

$$\omega_p(v,w) = \omega_p(v,w_x) + \omega_p(v,w_c) = 0.$$

Thus, $\omega_p(v,u) = 0$ for all $u \in T_pM$ and, because ω is a symplectic form on M, we conclude that $v = 0$. Therefore, $(X, \omega|_X)$ is a symplectic submanifold.

$\Longleftarrow$) If X is a symplectic submanifold with symplectic form $\omega|_X$, taking $p \in X$, we know that $T_pM = T_pX \oplus (T_pX)^{\perp_\omega}$, where

$$(T_pX)^{\perp_\omega} = \{w \in T_pM \,|\, \omega_p(w,u) = 0,\ \forall\, u \in T_pX\} = \pi^\sharp_p((T_pX)^\circ).$$

Therefore, in particular, $T_pM = T_pX + \pi^\sharp_p((T_pX)^\circ)$ and X is a cosymplectic submanifold. Q.E.D.

For the more general case of a Poisson manifold, there is still a link between the Definitions 6.3.1 and 6.3.4 above, although only locally:

Proposition 6.3.8. *Let (M, π) be a Poisson manifold and $S \subseteq M$ be a symplectic leaf of M passing through a point $p \in M$. If X is a Poisson transversal to S at $p \in M$, then a sufficiently small neighbourhood $U \subseteq X$ of p is a cosymplectic submanifold.*

Proof. If X is a Poisson transversal to S at p, then $T_pX \cap T_pS = 0$ and, by Remark 6.3.5 and by Lemma 6.3.6, we have that $\pi_p^\sharp((T_pX)^\circ) = (T_pX \cap T_pS)^{\perp_{(\omega_S)_p}} = T_pS$. Therefore, $T_pM = T_pX \oplus T_pS = T_pX \oplus \pi_p^\sharp((T_pX)^\circ)$. Q.E.D.

Furthermore, one rather important result when dealing with Poisson transversals and cosymplectic manifolds is that they can inherit a Poisson structure from the original manifold. A first step in that direction comes from identifying cosymplectic manifolds with submanifolds that intersect the symplectic leaves transversally:

Proposition 6.3.9. *Given a Poisson manifold (M, π), an embedded submanifold $X \subseteq M$ is a cosymplectic submanifold if and only if the following conditions apply for every symplectic leaf (S, ω_S):*

(i) $T_pM = T_pX + T_pS$, *for all* $p \in X \cap S$;

(ii) $(\omega_S)_p|_{T_pX \cap T_pS}$ *is non-degenerate, for all* $p \in X \cap S$.

In particular, $(X \cap S, \omega_S|_{X \cap S})$ *is a smooth symplectic manifold.*

Proof. Let $p \in X$ and (S, ω_S) the symplectic leaf passing through p.

$\Longrightarrow$) If X co-symplectic, then $T_pM = T_pX + \pi_p^\sharp((T_pX)^\circ)$ and, by Remark 6.3.5, $\pi_p^\sharp((T_pX)^\circ) = (T_pX \cap T_pS)^{\perp_{(\omega_S)_p}} \subseteq T_pS$. Thus, we conclude condition *(i)*, i.e. $T_pM = T_pX + T_pS$. Moreover, notice that condition *(ii)* is equivalent to

$$(T_pX \cap T_pS)^{\perp_{(\omega_S)_p}} \cap (T_pX \cap T_pS) = 0, \tag{$*$}$$

which, in turn, is equivalent to $T_pX \cap \pi_p^\sharp((T_pX)^\circ) = 0$. This is precisely what we obtain in Remark 6.3.5.

$\Longleftarrow$) If both *(i)* and *(ii)* are satisfied, since $\dim(\pi_p^\sharp((T_pX)^\circ) = \dim(T_pS) - \dim(T_pX \cap T_pS)$ and, by *(i)*, $\dim(T_pM) = \dim(T_pX) + \dim(T_pS) - \dim(T_pX \cap T_pS)$, we get that $\dim(\pi_p^\sharp((T_pX)^\circ) = \dim(T_pM) - \dim(T_pX)$. By relation $(*)$, equivalent to relation *(ii)*, we conclude that $T_pM = T_pX \oplus \pi_p^\sharp((T_pX)^\circ)$, thus X is a cosymplectic manifold.

Q.E.D.

Now we arrive at the result that allows cosymplectic submanifolds (and in the following corollary for Poisson transversals) to inherit Poisson structures from their parent Poisson manifolds.

Theorem 6.3.10. *Let X be a cosymplectic submanifold in a Poisson manifold (M, π). Then, X inherits a unique Poisson structure π_X such that the symplectic leaves of (X, π_X) are the connected components of $(X \cap S, \omega_S|_{X\cap S})$, for $S \in \mathcal{F}_\pi$, where $\mathcal{F}_\pi$ is the symplectic foliation of (M, π).*

Proof. For a cosymplectic manifold, by Remark 6.3.5, we have, for every $p \in X$ that $T_pM = T_pX \oplus \pi_p^\sharp((T_pX)^\circ)$. Thus, we can obtain the dual description as

$$T_p^*M = (T_pX)^\circ \oplus (\pi_p^\sharp((T_pX)^\circ))^\circ \tag{$*$}$$

and, by applying $\pi_p^\sharp$ on both sides of the equation, we get

$$T_pM \supseteq \operatorname{im}(\pi_p^\sharp) = \pi_p^\sharp((T_pX)^\circ) \oplus \pi_p^\sharp((\pi_p^\sharp((T_pX)^\circ))^\circ),$$

where the direct sum is obtained because,

$$\begin{aligned}
\pi_p^\sharp((\pi_p^\sharp((T_pX)^\circ))^\circ) &= \left\{\pi_p^\sharp\alpha \in T_pM \,\middle|\, \alpha \in (\pi_p^\sharp((T_pX)^\circ))^\circ\right\} \\
&= \left\{\pi_p^\sharp\alpha \in T_pM \,\middle|\, \pi_p(\alpha, \beta) = 0,\ \forall\, \beta \in (T_pX)^\circ)\right\} \\
&= \left\{\pi_p^\sharp\alpha \in T_pM \,\middle|\, \pi_p^\sharp\alpha \in T_pX\right\} \subseteq T_pX.
\end{aligned}$$

Moreover, the dual description $(*)$ also implies that we have an isomorphism $T_p^*X \cong (\pi_p^\sharp((T_pX)^\circ))^\circ$, allowing us to define the following bivector field on X:

$$\pi_X(\alpha, \beta) \overset{\text{def}}{=} \pi(\overline{\alpha}, \overline{\beta}), \tag{6.10}$$

where $\overline{\alpha}, \overline{\beta} \in T_p^*M$ are the unique extensions of $\alpha, \beta \in T_p^*X$, respectively, vanishing on $\pi_p^\sharp((T_pX)^\circ)$. To show that this bivector field is, in fact, a Poisson structure, we need to check that $[\pi_X, \pi_X] = 0$. Since $(\mathrm{d}C^\infty(X))_p = T_p^*X$, it is enough to show that the Schouten bracket vanishes when evaluated on (the differential of) the functions $f_1, f_2, f_3 \in C^\infty(X)$. Thus,

$$\begin{aligned}
[\pi_X, \pi_X](\mathrm{d}f_1, \mathrm{d}f_2, \mathrm{d}f_3) &= 2 \sum_{\sigma \in S^{1,2}} \pi_X\big(\mathrm{d}f_{\sigma(1)}, \mathrm{d}(\pi_X(\mathrm{d}f_{\sigma(2)}, \mathrm{d}f_{\sigma(3)}))\big) \\
&= 2 \sum_{\sigma \in S^{1,2}} \pi_X\big(\mathrm{d}f_{\sigma(1)}, \mathrm{d}(\pi(\overline{\mathrm{d}f_{\sigma(2)}}, \overline{\mathrm{d}f_{\sigma(3)}}))\big) \\
&= 2 \sum_{\sigma \in S^{1,2}} \pi\big(\overline{\mathrm{d}f_{\sigma(1)}}, \overline{\mathrm{d}(\pi(\overline{\mathrm{d}f_{\sigma(2)}}, \overline{\mathrm{d}f_{\sigma(3)}}))}\big) \\
&= 2 \sum_{\sigma \in S^{1,2}} \pi\big(\overline{\mathrm{d}f_{\sigma(1)}}, \mathrm{d}(\pi(\overline{\mathrm{d}f_{\sigma(2)}}, \overline{\mathrm{d}f_{\sigma(3)}}))\big) \\
&= [\pi, \pi](\overline{\mathrm{d}f_1}, \overline{\mathrm{d}f_2}, \overline{\mathrm{d}f_3}) = 0.
\end{aligned}$$

To conclude the proof, it remains to be proved that the symplectic leaves of (X, π_X) are the connected components of $(X \cap S, \omega_S|_{X\cap S})$, where S are the symplectic leaves of the symplectic foliation of (M, π). For that, let $p \in X \cap S$. Then, by the results from Remark 6.3.5, we get

$$\operatorname{im}((\pi_X^\sharp)_p) = (\pi_X^\sharp)_p(T_p^* X) \cong (\pi_X^\sharp)_p((\pi_p^\sharp((T_pX)^\circ))^\circ) = T_pX \cap T_pS = T_p(X \cap S),$$

where in the last equality, we used the fact that X and S intersect transversely. Moreover, from the fact that $\omega_S|_{X\cap S} = \pi^{-1}|_{X\cap S} = \pi_X^{-1}|_{X\cap S}$, we conclude that (the connected components of) $(X \cap S, \omega_S|_{X\cap S})$ are the symplectic leaves of (X, π_X). Q.E.D.

Corollary 6.3.11. *Given a Poisson manifold (M, π), with a symplectic leaf S and Poisson transversal X of S at a point $p \in S$, there exists a small enough neighbourhood $U \subseteq X$ of p with an induced Poisson structure, such that its symplectic leaves are the connected components of $(X \cap S, \omega_S|_{X\cap S})$.*

Proof. This result comes directly from the previous theorem and Proposition 6.3.8. Q.E.D.

Since any cosymplectic submanifold has a Poisson structure in a small enough neighbourhood, it is possible to talk about the symplectic foliation for this submanifold, or, in particular, its leaf space. Even though, as we have seen, it is, in general, rather difficult to understand what this leaf space is, Theorem 6.3.10 shows that, given a Poisson manifold (M, π) and a cosymplectic manifold X with induced Poisson structure π_X, we have an induced map $X/\mathcal{F}_{\pi_X} \to M/\mathcal{F}_\pi$ which may be used to study the structure of the leaf space. In fact, $X/\mathcal{F}_{\pi_X}$ may be much simpler and the previous map has the following properties:

Proposition 6.3.12. *Let (M, π) be a Poisson manifold, $X \subseteq M$ a cosymplectic submanifold with induced Poisson structure π_X, and let $\mathcal{F}_\pi$, $\mathcal{F}_{\pi_X}$ be the symplectic foliations of M and X, respectively. The map $X/\mathcal{F}_{\pi_X} \to M/\mathcal{F}_\pi$ is continuous and open. In particular, if X intersects each leaf at most a single time, $X/\mathcal{F}_{\pi_X}$ is homeomorphic to an open subspace of $M/\mathcal{F}_\pi$.*

To prove this result, we must first introduce the following Weinstein theorem near cosymplectic submanifolds:

Theorem 6.3.13. *Let X be a cosymplectic submanifold of a Poisson manifold (M, π) with co-dimension $2s$. Then, for every $x \in X$, there exists a Poisson isomorphism $\chi : (U, \pi|_U) \xrightarrow{\cong} (S, \omega_S^{-1}) \times (W, \pi_X)$, where $U \subseteq M$ is a neighbourhood of x, (S, ω_S^{-1}) is a symplectic submanifold of the symplectic leaf passing through x in M and $W \subseteq X$ is a neighbourhood of x in X.*

Proof of Proposition 6.3.12. Let $x \in X$ and U be an open neighbourhood of x in M. Then, by Theorem 6.3.13, there exists a Poisson isomorphism

$$\chi = (\chi_S, \chi_W) : (U, \pi|_U) \to (S, \omega_S^{-1}) \times (W, \pi_W),$$

where S can be taken as a symplectic submanifold of the symplectic leaf containing x and W is an open neighbourhood of x in X. If $\mathcal{F}_\pi$ and $\mathcal{F}_{\pi_X}$ are the symplectic foliations of M and X, respectively, then we have the quotient maps $q : M \to M/\mathcal{F}_\pi$ and $q_X : X \to X/\mathcal{F}_{\pi_X}$, which are continuous, surjective and restrict to maps $q|_U : U \to U/\mathcal{F}_{\pi|_U}$ and $q_X|_W : W \to W/\mathcal{F}_{\pi_X|_W}$. Let $\psi : U/\mathcal{F}_{\pi|_U} \to W/\mathcal{F}_{\pi_X|_W}$ be the map assigning the symplectic leaf of (M, π) containing q to the symplectic leaf of (X, π_X) containing $\chi_W(q)$, i.e. making the following diagram commutative:

$$\begin{tikzcd}
(U, \pi|_U) \arrow[r, "\chi_W"] \arrow[d, "q|_U"'] & (W, \pi_X|_W) \arrow[d, "q_X|_W"] \\
U/\mathcal{F}_{\pi|_U} \arrow[r, "\psi"'] & W/\mathcal{F}_{\pi_X|_W}
\end{tikzcd}$$

Then, this map is clearly continuous as $q|_U$, $q_X|_W$ and χ_W are continuous. As for the openness of ψ, notice that, by the commutative diagram above, it is enough to prove that $q_X|_W$ is an open map. Since $q_X|_W$ is a quotient map, it is open if and only if, given any open subset $\mathcal{W} \subseteq W$ of W, $(q_X|_W)^{-1}\big(q_X|_W(\mathcal{W})\big)$ is an open subset of W. However,

$$(q_X|_W)^{-1}\big(q_X|_W(\mathcal{W})\big) = \bigcup \left\{ W \cap \mathcal{S} \;\middle|\; \begin{array}{c} \mathcal{S} \text{ is a symplectic leaf} \\ \text{containing some point of } \mathcal{W} \end{array} \right\}.$$

Because W and the symplectic leaves of (X, π_X) are open subsets of X, we conclude that $(q_X|_W)^{-1}\big(q_X|_W(\mathcal{W})\big)$ is open.

Finally, if X intersects each leaf at most one time, χ can be extended to a Poisson isomorphism $\chi : (U, \pi|_U) \to (S, \omega_S^{-1}) \times (X, \pi_X)$ and, by a similar argument as before, $q|_U : U \to U/\mathcal{F}_{\pi|_U}$ is an open quotient map. Thus, we conclude that $X/\mathcal{F}_{\pi_X}$ is homeomorphic to a subspace of $M/\mathcal{F}_\pi$. Q.E.D.

For the generalised version of Weinstein Splitting theorem, we firstly introduce the following lemma, which is also an extension of Lemma 2.2.3:

Lemma 6.3.14. *Let M be a smooth manifold and $X_1, \ldots, X_k$ linearly independent commuting vector fields in a neighbourhood of a point $p \in M$. Then, for any submanifold $N \subseteq M$ with $p \in N$ and $T_pM = T_pN \oplus \operatorname{span}\{(X_1)_p, \ldots, (X_k)_p\}$, there is a coordinate chart $(U, \varphi = (x_1, \ldots, x_m))$ on M centred at p such that:*

(i) $U \cap N = \big\{(x_1, \ldots, x_m) \in M \,\big|\, x_1 = \ldots = x_k = 0\big\}$;

(ii) *for* $i = 1, \ldots, k$, $X_i|_U = \dfrac{\partial}{\partial x_i}$.

Proof. The proof of this result follows a procedure similar to that of Lemma 2.2.3. Let V and W be open neighbourhoods of 0 in $\mathbb{R}^k$ and $\mathbb{R}^{m-k}$, respectively, and consider U to be an open neighbourhood of p in M. Moreover, we may always construct the following local diffeomorphism $\phi : U \longrightarrow V \times W$ such that $\phi(p) = 0$ and $\phi(U \cap N) = \{0\} \times W$. Then, for a neighbourhood $\mathcal{W}$ of 0 in $\mathbb{R}^k \times W$, consider the map $\Phi : \mathcal{W} \to M$, such that

$$\Phi(x_1, \ldots, x_m) = (\mathrm{Fl}^{x_1}_{X_1} \circ \ldots \circ \mathrm{Fl}^{x_k}_{X_k})\big(\phi^{-1}(0, \ldots, 0, x_{k+1}, \ldots, x_m)\big).$$

Clearly, we have that $\Phi(0) = p$ and $\Phi_*(0)\big(\frac{\partial}{\partial x^i}\big|_0\big) = (X_i)_p$, for all $i = 1, \ldots, k$. Thus, we have that

$$\mathrm{im}(\Phi_*(0)) = \mathrm{span}\big\{(X_1)_p, \ldots, (X_k)_p\big\} + T_pN = T_pM.$$

Shrinking $\mathcal{W}$ around 0, if necessary, $\Phi : \mathcal{W} \to \Phi(\mathcal{W})$ is a diffeomorphism from an open neighbourhood of 0 in $\mathbb{R}^m$ to an open neighbourhood of p in M, mapping 0 into p. Finally, by the definition of Φ, we can also quickly obtain that

$$\Phi(\mathcal{W} \cap (\{0\} \times W)) = \Phi(\mathcal{W}) \cap N.$$

Then, the coordinate chart of the lemma is just $(\Phi(\mathcal{W}), \Phi^{-1})$. Q.E.D.

Therefore, by using this result we obtain

Theorem 6.3.15 (Weinstein)**.** *Let (M, π) a Poisson manifold and S a symplectic leaf. Given a Poisson transversal X to S at a point $x \in S$, there exists a coordinate chart $(U, \varphi = (q^1, \ldots, q^s, p_1, \ldots, p_s, y^1, \ldots, y^r))$ centred at x such that:*

(i) $X \cap U = \big\{(q^1, \ldots, q^s, p_1, \ldots, p_s, y^1, \ldots, y^r) \in M \,\big|\, q^1 = \ldots = q^s = p_1 = \ldots = p_s = 0\big\}$;

(ii) $\displaystyle \pi|_U = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \sum_{1 \le a < b \le r} \theta^{ab}(y)\, \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b}$, *where $\theta^{ab}(y)$ are local functions solely depending on y^a's vanishing at $x = (0, \ldots, 0)$.*

Proof. The proof of this statement is entirely similar to that of Theorem 6.2.1 now using the improved Lemma 6.3.14.

Induction on $\mathrm{rank}(\pi_x) = 2s$: for $s = 0$, $S = \{0\}$ and $M = X$, thus there is nothing to prove. For $s \ge 1$, we assume the statement is valid for $\mathrm{rank}(\pi_x) \le 2s - 2$. Then there is $p_1 \in C^\infty(M)$ vanishing on X with $(p_1)_*(x) \ne 0$ (because $X \ne M$). Since X is transverse to S, there exists $g \in C^\infty(M)$ such that $(p_1)_*(x)((X_g)_x) \ne 0$, and $(X_{p_1})_x \ne 0$ is tangent to S. By Lemma 6.3.14, there exists a chart $(U, (x^1, \ldots, x^m))$ such that $X_{p_1}|_U = \frac{\partial}{\partial x_1}$ and $X \cap U = \{x^2 = \ldots = x^m = 0\}$. Calling $q^1 = x^1$, then $\pi(\mathrm{d}q^1, \mathrm{d}p_1) = -X_{p_1}(q^1) = -1$ and, furthermore, X_{p_1}, X_{q^1} are commuting linearly independent vector fields. Once again, by the lemma, up to restricting U around x if necessary, there exists a chart $(U, (\tilde{x}^1, \dots, \tilde{x}^m))$ with $X_{p_1}|_U = \frac{\partial}{\partial \tilde{x}^2}$, $X_{q^1}|_U = \frac{\partial}{\partial \tilde{x}^1}$, $X \cap U \subseteq \{\tilde{x}^1 = \tilde{x}^2 = 0\}$ and

$$\pi|_U = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q^1} + \sum_{3 \le a < b \le m} \theta^{ab}(q^1, p_1, \tilde{x}^3, \dots, \tilde{x}^m) \frac{\partial}{\partial \tilde{x}^a} \wedge \frac{\partial}{\partial \tilde{x}^b} .$$

Once again, it is quite simple to observe that θ^{ab} does not depend on the coordinates q^1, p_1, thus the result is obtained by induction. Q.E.D.

Another fundamental result about Poisson transversals is the fact that the transverse structure in a Poisson manifold is unique and determined from any choice of a symplectic leaf at any point. So, every Poisson transversal found in each point in the same symplectic leaf should be isomorphic. This next result is a first step into showing that result:

Proposition 6.3.16. *Let X be a cosymplectic manifold in a Poisson manifold (M, π). Then, for any $x \in X$, the isotropy Lie algebras of (X, π_X) and (M, π), $\ker(\pi_X^\sharp)$ and $\ker(\pi^\sharp)$, respectively, are isomorphic.*

Proof. Let $p \in X$, let S be the symplectic leaf of (M, π) containing p and Y be a transversal to $X \cap S$ at p. Then, since we know that $\operatorname{im}((\pi_X^\sharp)_p) \cong T_p^* X / \operatorname{im}((\pi_X^\sharp)_p)$ and $T_p(X \cap S) = \operatorname{im}((\pi_X^\sharp)_p)$, the defining property of a transversal gives an isomorphism $T_p Y \cong \ker((\pi_X^\sharp)_p)$. Moreover, by Remark 6.3.5, we get

$$\begin{aligned} T_p M &= T_p X \oplus \pi_p^\sharp((T_p X)^\circ) = T_p Y \oplus T_p(X \cap S) \oplus \pi_p^\sharp((T_p X)^\circ) \\ &= T_p Y \oplus ((T_p X \cap T_p S) \oplus \pi_p^\sharp((T_p X)^\circ)), \end{aligned}$$

where, in the last equality, we used the fact that X and S intersect transversely. Because $\pi_p^\sharp((T_p X)^\circ) \subseteq T_p S$, and $T_p X \subseteq \pi_p^\sharp((T_p X)^\circ) = \{0\}$, we conclude that

$$(T_p X \cap T_p S) \oplus \pi_p^\sharp((T_p X)^\circ) = T_p S = \operatorname{im}(\pi_p^\sharp)$$

and, therefore, we obtain an isomorphism $\ker(\pi_p^\sharp) \cong T_p Y \cong \ker((\pi_X^\sharp)_p)$. Q.E.D.

Thus, as a final result, we get the previously mentioned isomorphism between Poisson transversals:

Theorem 6.3.17. *Let (M, π) be a Poisson manifold and let S be a symplectic leaf of (M, π). Also assume that N_0 and N_1 are Poisson transversals of (M, π) relative to S at $x_0 \in S$ and $x_1 \in S$, respectively. Then, there exist open neighbourhoods U_0, U_1 of x_0, x_1 in N_0, N_1, respectively, and a Poisson diffeomorphism $\phi : (U_0, \pi_{U_0}) \to (U_1, \pi_{U_1})$, with $\phi(x_0) = x_1$.*

Proof. Since $x_0,\ x_1 \in S$, by Proposition 6.3.3, there exists a Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}(M,\pi)$ with $\psi(x_0) = x_1$. Thus, without loss of generality, we may take N_1 to be $\psi(N_0)$ and $x_0 = x_1 \overset{\text{def}}{=} x$. Take a splitting chart $(V \subseteq \mathbb{R}^{2s}, \pi_{\text{can}}) \times (W, \pi_W)$ centred at x, adapted to N_0, i.e. $N_0 \overset{\text{def}}{=} \{0\} \times W$ and $S \overset{\text{def}}{=} V \times \{0\}$. Since N_1 is transverse to $S = V \times \{0\}$ at x, there exists a smooth map $F : W \to \mathbb{R}^{2s}$, with $F(0) = 0$ such that $N_1 = \big\{(F(w), w) \,\big|\, w \in W\big\}$.

Define a smooth family of Poisson transversals in (M,π) relative to S at x that interpolates between N_0 and N_1, $N_t \overset{\text{def}}{=} \big\{(tF(w), w) \,\big|\, w \in W\big\}$, $t \in [0,1]$. For the proof to be concluded, we just need to find a family of functions $(H_t)_{t\in[0,1]}$ with $H_t \in C^\infty(M)\ \forall\, t \in [0,1]$, such that the flow of their associated Hamiltonian vectors, $X_{H_t} \in \mathfrak{ham}(M,\pi)$, satisfies

$$\mathrm{Fl}^t_{X_{H_t}}(x) = x \quad \& \quad \mathrm{Fl}^t_{X_{H_t}}(U_0) \subseteq N_t, \qquad \forall\, t \in [0,1], \tag{$*$}$$

for a small (enough) neighbourhood $U_0 \subseteq N_0$ of x. Notice that the second expression in Eq. $(*)$ is valid if $X_{H_t} + \frac{\partial}{\partial t}$ is tangent to $\tilde{N} = \big\{(u,t) \,\big|\, u \in N_t\big\} \subseteq M \times [0,1]$. This is equivalent to $F(w) - X_{H_t}|_{i_t(w)} \in T_{i_t(w)}N_t,\ \forall\, (w,t) \in W \times [0,1]$, where $i_t(w) = (tF(w), w)$.

To find such a family, notice that, since N_t is a Poisson transversal relative to S at x, shrinking $V \times W$ around x if necessary, N_t is a cosymplectic manifold. Thus, we may write $F(w) = T_t(w) + n_t(w)$, with $T_t(w) \in T_{i_t(w)}N_t$ and $n_t(w) \in \pi^\sharp_{i_t(w)}((T_{i_t(w)}N_t)^\circ) \cong (T_{i_t(w)}N_t)^\circ$, which implies that

$$n_t(w) = \pi^\sharp(\beta_t(w)), \quad \beta_t(w) \in (T_{i_t(w)}N_t)^\circ.$$

Because the elements of $(T_{i_t(w)}N_t)^\circ$ can be written as $(\alpha, (tF)^*\alpha)$ for some $\alpha \in T^*_{tF(w)}\mathbb{R}^{2s} \cong \mathbb{R}^{2s}$, we may write $\beta_t(w)$ as

$$\beta_t(w) = (\alpha_t(w), (tF)^*(\alpha_t(w))), \quad \alpha_t(w) \in T^*_{tF(w)}\mathbb{R}^{2s}.$$

Define $H_t(u,w) \overset{\text{def}}{=} \alpha_t(w)(u - tF(w))$, where $(u,w) \in V \times W$ and we are considering $\alpha_t(w) \in T^*_{tF(w)}\mathbb{R}^{2s}$ and $u - tF(w) \in \mathbb{R}^{2s} \cong T_{tF(w)}\mathbb{R}^{2s}$. Then,

$$(H_t)_*(i_t(w)) = \beta_t(w) \implies (X_{H_t})_{i_t(w)} = n_t(w) = \pi^\sharp(\beta_t(w)).$$

Hence, $F(w) - (X_{H_t})_{i_t(w)} \in T_{i_t(w)}N_t$ and $i_t(0) = (0,0)$, which means that $n_t(0) = 0$, thus $(X_{H_t})_x = 0$, i.e. $\mathrm{Fl}^t_{X_{H_t}}(x) = x$. Q.E.D.

This concludes our study of Poisson geometry.

Chapter 7

Dirac Structures

Firstly introduced by Weinstein and Courant [20], Dirac structures came to be from the physics of Hamiltonian mechanics of constrained systems, a theory first proposed by Dirac [21]. In this theory, we study a physical system constrained to a submanifold of the cotangent bundle and, so, we need to generalise the notion of the Poisson bracket into the notion of the Dirac bracket in order to obtain the correct equations of motion of the system. Mathematically, Dirac structures correspond to the general case of a submanifold in a Poisson manifold as those structures define a pre-symplectic foliation. So, Dirac geometry generalise several geometries including Poisson, pre-symplectic, as well as foliations but are also an example of a Lie algebroid on a subbundle of the generalised tangent bundle. Clearly, Dirac structures form a rather important class of geometries and also one of the most recent ones which makes them quite interesting structures to be studied.

The chapter starts with a discussion on the physical origin of Dirac structures, namely the Dirac theory of Hamiltonian constrained systems. These systems are obtained when the Lagrangian of a physical system does not satisfy the Legendre condition, i.e. when it is not possible to invert the definitions of the canonical momenta with respect to the velocities of the system. In Section 7.2, we introduce the notions of the generalised tangent bundle and the Courant algebroid structure on this bundle as well as the definition of Dirac structures and show important class of examples such as foliations, pre-symplectic manifolds and Poisson manifolds. Finally, in the third and last section, we explore the properties of Dirac manifolds, from the pre-symplectic foliation of such structures to the discussion of the different types of morphisms we may introduce in Dirac geometry.

7.1 Introduction and Motivation for Dirac Structures from Physics

As we have seen in Section 4.1, symplectic and Poisson geometry arise from classical mechanics. However, in that section we always assumed the Legendre condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0, \quad i,j = 1,\ldots,n, \tag{4.3}$$

for the Lagrangian $L(q,\dot{q},t)$ and $q = (q^1,\ldots,q^n) \in Q$ a point in configuration space. Now, we shall disregard this condition which is precisely the case for a constrained physical system. We start our study with an example:

Example 7.1.1. Let us consider a two dimensional configuration space, with coordinates (q^1,q^2) and Lagrangian given by $L = \frac{1}{2}(\dot{q}^1 - \dot{q}^2)^2$. Then, by (4.4), the canonical momenta are given by

$$p_1 = \dot{q}^1 - \dot{q}^2 \quad \text{and} \quad p_2 = \dot{q}^2 - \dot{q}^1.$$

By analysing the expressions of the canonical momenta, it is clear that these expressions are not invertible. Moreover, we can clearly obtain a constraint for this system:

$$\phi \overset{\text{def}}{=} p_1 + p_2 = 0,$$

thus, the solution for the motion of the system occurs in a submanifold of T^*Q. ♮

Now, from this example, we may expand the general theory presented by Dirac in [21], following the treatment done in [22]. Since the Legendre condition (4.3) is not satisfied, the defining equation for the canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1,\ldots,n \tag{4.4}$$

is not invertible. Thus, there are L maps $\phi_\ell : T^*Q \to \mathbb{R}$ such that the motion of the particles (in phase space) are constrained to the submanifold

$$M_1 \overset{\text{def}}{=} \left\{(q,p) \in T^*Q \,\middle|\, \phi_\ell(q,p) = 0, \text{ for all } \ell = 1,\ldots,\mathrm{L}\right\} \subseteq T^*Q. \tag{7.1}$$

These constraints ϕ_ℓ obtained directly from the definition of the canonical momenta (4.4) are called *primary constraints*. These primary constraints need not be independent of each other and, in general, there are multiple ways to define M_1 using different constraints ϕ_ℓ (for instance, the line defined by $p_1 = 0$ is the same as for $p_1^2 = 0$). However, to describe the Hamiltonian formalism of the system, we need to impose some conditions to these ϕ_ℓ. These conditions, called the *regularity conditions* of the primary constraints, are as follows:

Proposition 7.1.2 (Regularity conditions of primary constraints)**.** *Let M_1 be the primary constraint surface, locally defined by primary constraints ϕ_ℓ, $\ell = 1, \dots, \text{L}$. Then, we may always choose $\phi_1, \dots, \phi_{\dim(M_1)}$ to be independent constraints, i.e. such that*

$$\mathrm{d}\phi_1 \wedge \cdots \wedge \mathrm{d}\phi_{\dim(M_1)} \neq 0.$$

Moreover, the constraints $\phi_{\ell'} = 0$, for $\ell' = \dim(M_1) + 1, \dots, \text{L}$, hold as a consequence of $\phi_\ell = 0$ for $\ell = 1, \dots, \dim(M_1)$.

The next step is to introduce the Hamiltonian of the system. However, since Eq. (4.4) is not invertible, the usual definition of the Hamiltonian as the Legendre transformation of the Lagrangian is not possible. So, we instead define the Hamiltonian as a function of the velocities:

$$H(q, \dot{q}, t) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q})\dot{q}^i - L(q, \dot{q}, t). \tag{7.2}$$

In reality, by doing the variation of this Hamiltonian, we can see that this function is not dependent in the velocities apart from their dependency on the canonical momenta:

$$\begin{aligned} \delta H &= \sum_{i=1}^{n} \left(p_i\, \delta\dot{q}^i + \dot{q}^i\, \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i \right) - \frac{\partial L}{\partial t}\delta t \\ &= \sum_{i=1}^{n} \left(\dot{q}^i\, \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \right) - \frac{\partial L}{\partial t}\delta t, \end{aligned}$$

where the terms in red cancel out because of the definition of the canonical momenta (4.4). Thus, we actually can describe the Hamiltonian as a function of the momenta and not of the velocity, $H(q, p, t)$.

To introduce these constraints in the action, we simply introduce Lagrange multipliers. Then, the total action becomes

$$S_T[q(t), p(t), u(t)] = \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} p_i\dot{q}^i - H(q, p, t) - \sum_{\ell=1}^{\text{L}} u^\ell \phi_\ell(q, p) \right) \mathrm{d}t. \tag{7.3}$$

The equations of motion are obtained by variational calculus on q, p, u, obtaining:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \sum_{\ell=1}^{\text{L}} u^\ell \frac{\partial \phi_\ell}{\partial p_i} \tag{7.4a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \sum_{\ell=1}^{\text{L}} u^\ell \frac{\partial \phi_\ell}{\partial q^i} \tag{7.4b}$$

$$\phi_\ell(q, p) = 0 \tag{7.4c}$$

or, by introducing an arbitrary function of the variable in phase space, $F(q, p, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q(t), p(t), t) = \frac{\partial F}{\partial t} + \{H, F\} + \sum_{\ell=1}^{\text{L}} u^\ell \{\phi_\ell, F\}, \tag{7.4d}$$

where $\{\cdot,\cdot\}$ are the Poisson brackets defined in (4.11). The primary constraints, beyond being subject to regularity conditions, should also satisfy *consistency conditions* that guarantee that the primary constraint surface M_1 does not change with time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\ell(q(t),p(t)) = 0, \quad \ell = 1,\dots,\mathrm{L}. \tag{7.5}$$

Solving these equations yield four possibilities:

1) They result in an impossible equation (meaning that the Lagrangian does not give consistent equations of motion);

2) They lead to a relation involving only the primary constraints $\phi_\ell(q,p)$, which reduces to an identically true statement after imposing the constraints $\phi_\ell(q,p) = 0$;

3) They impose a condition on the Lagrange multipliers u^ℓ;

4) They lead to a relation involving q,p independent of the primary constraints.

The third of these cases will give an expression $u^\ell = u^\ell(q,p)$ for $\ell \in \Lambda \subseteq \{1,\dots,\mathrm{L}\}$, which allows us to define the so-called *first-class Hamiltonian*:

$$H_1(q,p,t) \stackrel{\text{def}}{=} H(q,p,t) + \sum_{\ell\in\Lambda} u^\ell(q,p)\phi_\ell(q,p), \tag{7.6}$$

whose relevance will come up later. On the other hand, the last case 4. introduces additional constraints $\phi_l(q,p) = 0$, with $l = \mathrm{L}+1,\dots,\mathrm{L}+\mathrm{L}'$, called the *secondary constraints*. Joining the primary and secondary constraints $\phi_\ell(q,p) = 0$, with $\ell = 1,\dots,\mathrm{L}+\mathrm{L}'$, the motion of the particles of the system now should occur in the submanifold:

$$M_2 \stackrel{\text{def}}{=} \left\{(q,p) \in T^*Q \,\middle|\, \phi_\ell(q,p) = 0, \text{ for all } \ell = 1,\dots,\mathrm{L}+\mathrm{L}'\right\} \subseteq T^*Q.$$

Remark 7.1.3. When dealing with constrained system, one usually introduces the notion of an *weak equality*, $\approx$. This equality is defined so that, given two phase functions $F, G : T^*Q \times \mathbb{R} \to \mathbb{R}$, one says $F \approx G$ if and only if $F|_{M_2} = G|_{M_2}$.

Example 7.1.4. Returning to the example discussed above, we can compute the Hamiltonian the following way:

$$\begin{aligned} H(q^1,q^2,p_1,p_2) &= p_1\dot{q}^1 + p_2\dot{q}^2 - L(q^1,q^2,\dot{q}^1,\dot{q}^2) \\ &= (\dot{q}^1-\dot{q}^2)\dot{q}^1 - (\dot{q}^1-\dot{q}^2)\dot{q}^2 - \frac{1}{2}(\dot{q}^1-\dot{q}^2)^2 \\ &= \frac{1}{2}(\dot{q}^1-\dot{q}^2)^2 = \frac{1}{2}p_1^2 = \frac{1}{2}p_2^2 \end{aligned}$$

Then the equations of motion are

$$\dot{q}^1 = \frac{\partial H}{\partial p_1} + u\frac{\partial \phi}{\partial p_1} = p_1 + u, \qquad \dot{p}_1 = -\frac{\partial H}{\partial q^1} - u\frac{\partial \phi}{\partial q^1} = 0,$$

$$\dot{p}^2 = \frac{\partial H}{\partial p_2} + u\frac{\partial \phi}{\partial p_2} = u, \qquad \dot{p}_2 = -\frac{\partial H}{\partial q^2} - u\frac{\partial \phi}{\partial q^2} = 0,$$

$$0 = \phi(q^1, q^2, p_1, p_2) = p_1 + p_2.$$

Moreover, the consistency condition (7.5) for the constraint ϕ gives:

$$\dot{\phi} = \{H, \phi\} + u\,\{\phi, \phi\} = 2p_1\,\{p_1, p_1 + p_2\} = 0$$

hence, we are in case 2. described above and so ϕ is the only constraint for the system. ♮

It turns out that when dealing with Hamiltonian theory with constraints, the notions of primary and secondary constraints is not very useful. Instead, we introduce the notion of first- and second-class constraints that result from the equivalent notion for arbitrary phase functions:

Definition 7.1.5. Given a Hamiltonian system with constraints $\phi_\ell(q,p) = 0$, $\ell = 1, \ldots, \mathrm{L} + \mathrm{L}'$, a phase space function $F : T^*Q \times \mathbb{R} \to \mathbb{R}$ (with possible explicit dependency on time) is called a *first-class function* if, for all $\ell = 1, \ldots, \mathrm{L} + \mathrm{L}'$,

$$\{F, \phi_\ell\} \approx 0, \text{ or, equivalently, } \{F, \phi_\ell\}\,|_{M_2} = 0. \tag{7.7}$$

If at least one of these Poisson brackets is non-zero on M_2, the phase function F is said to be a *second-class function*.

Remark 7.1.6. In more mathematical terms, taking $\lambda \in \Omega^1(T^*Q)$ to be the Liouville 1--form on T^*Q, the pair $(T^*Q, -\mathrm{d}\lambda)$ is a symplectic manifold and, so, for any phase space function $F : T^*Q \times \mathbb{R} \to \mathbb{R}$, we may still define the associated Hamiltonian vector field $X_F = (-\mathrm{d}\lambda)^\flat(\mathrm{d}F)$. Then, we can express condition (7.7) as $\{F, \phi_\ell\}\,|_{M_2} = (X_F(\phi_\ell)) = 0$ for all $\ell = 1, \ldots, \mathrm{L} + \mathrm{L}'$, which is equivalent to say that $X_F|_{M_2} \in \Gamma\big(\bigcap_{\ell=1}^{\mathrm{L}+\mathrm{L}'} \ker(\mathrm{d}\phi_\ell)|_{M_2}\big) = TM_2$. Thus, we conclude that a phase function is first-class if its Hamiltonian vector field $X_F \in \mathfrak{X}(T^*Q)$ is tangent to M_2.

Note that the first-class Hamiltonian introduced previously is an example of a first-class function, since, by definition, the $u^\ell(q,p)$, with $\ell \in \Lambda$ are the solutions to (7.5). Applying the previous definition to constraints, we now can reorganise our constraints into *first-class constraints*, which we denote by γ_a, with $a = 1, \ldots, A$, and *second-class constraints*, denoted by χ_α, with $\alpha = 1, \ldots, A'$ and $A + A' = \mathrm{L} + \mathrm{L}'$.

When studying the effects of the first-class constraints, one obtains that these generate gauge transformations of the system (for the interested reader, we refer to [22] for a proof of this statement) and, so, since any possible physical motion should allow for an arbitrary gauge transformation, we may introduce these first-class constraints into the Hamiltonian. The resulting first-class function is called the *extended Hamiltonian*:

$$H_E(q,p,t) = H_1(q,p,t) + \sum_{a=1}^{A} u^a \gamma_a, \tag{7.8}$$

where, once again, the u^a's are unknown Lagrange multipliers.

In relation to the second-class constraints, since these do not correspond to gauge generators, the natural way to treat them in the physical context is by changing the canonical Poisson bracket in order to 'eliminate' these constraints. To do so, one considers the matrix of Poisson brackets $(C_{\alpha\beta})_{1\leq\alpha,\beta\leq A'}$ whose entry α,β correspond to the Poisson bracket $\{\chi_\alpha,\chi_\beta\}$. As Dirac proved in [21], this matrix is invertible, with inverse denoted by $(C^{\alpha\beta})_{1\leq\alpha,\beta\leq A'}$. Then, we define the Dirac bracket of two phase functions F,G by

$$\{F,G\}^* \overset{\text{def}}{=} \{F,G\} - \{F,\chi_\alpha\}\, C^{\alpha\beta}\, \{\chi_\beta, G\}\,. \tag{7.9}$$

Thanks to the properties of the canonical Poisson bracket, the Dirac bracket is also a Poisson bracket. With this Dirac bracket, the equations of motion turn out to return to the familiar version of the system without constraints by substituting the strong equality with a weak equality, i.e. the dynamics of a phase function becomes

$$\left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{M_2} = \left.\frac{\partial F}{\partial t}\right|_{M_2} + \{H_E, F\}^*\,|_{M_2}. \tag{7.10}$$

To conclude this study, let us look at two final examples.

Example 7.1.7. Let us consider a system with coordinates (q,ξ_1,ξ_2,ξ_3) and conjugate momenta (p,π_1,π_2,π_3) with primary constraints $\phi_1 \overset{\text{def}}{=} \pi_1 = 0$ and $\phi_2 \overset{\text{def}}{=} \pi_2 = 0$ and with a Hamiltonian given by

$$H = e^{\xi_1} q^2 + e^{-\xi_2} p^2 + \pi_3(\pi_3 + q).$$

Imposing the consistency conditions (7.5), we obtain,

$$\begin{aligned} 0 = \dot{\phi}_1 &= \{H,\phi_1\} + u^1\{\phi_1,\phi_1\} + u^2\{\phi_2,\phi_1\} = q^2\left\{e^{\xi_1},\pi_1\right\} = -e^{\xi_1}q^2;\\ 0 = \dot{\phi}_2 &= \{H,\phi_2\} + u^1\{\phi_1,\phi_2\} + u^2\{\phi_2,\phi_2\} = p^2\left\{e^{-\xi_2},\pi_2\right\} = e^{-\xi_2}p^2. \end{aligned}$$

Thus we obtain two secondary constraints $\phi_3 \overset{\text{def}}{=} q = 0$ and $\phi_4 \overset{\text{def}}{=} p = 0$, to which, once again, we impose condition (7.5):

$$\begin{aligned} 0 = \dot{\phi}_3 &= \{H,\phi_3\} + u^1\{\phi_1,\phi_3\} + u^2\{\phi_2,\phi_3\} = e^{-\xi_2}\left\{p^2,q\right\} = 2pe^{-\xi_2};\\ 0 = \dot{\phi}_4 &= \{H,\phi_4\} + u^1\{\phi_1,\phi_4\} + u^2\{\phi_2,\phi_4\} = e^{\xi_1}\left\{q^2,p\right\} + \pi_3\{q,p\} = -2qe^{\xi_1} - \pi_3, \end{aligned}$$

resulting in another constraint $\phi_5 \overset{\text{def}}{=} \pi_3 = 0$, whose consistency condition implies

$$0 = \dot{\phi}_5 = \{H, \phi_5\} + u^1 \{\phi_1, \phi_5\} + u^2 \{\phi_2, \phi_5\} = 0.$$

Thus, we have 5 total constraints and, with a quick computation of the Poisson bracket,

$$\begin{aligned} \{\pi_1, \pi_2\} = \{\pi_1, q\} = \{\pi_1, p\} = \{\pi_1, \pi_3\} = \{\pi_2, q\} = 0 \\ \{\pi_2, p\} = \{\pi_2, \pi_3\} = \{q, \pi_3\} = \{p, \pi_3\} = 0, \quad \{p, q\} = 1 \end{aligned}$$

we conclude that constraints ϕ_1, ϕ_2, ϕ_5 are first-class, whilst ϕ_3, ϕ_4 are second-class. Thus, the extended Hamiltonian is (note that, in this case the original Hamiltonian and the first--class Hamiltonian are equal)

$$H_E = e^{\xi_1} q^2 + e^{-\xi_2} p^2 + \pi_3(\pi_3 + q) + u^1\pi_1 + u^2\pi_2 + u^3\pi_3,$$

and the matrix $(C_{\alpha\beta})_{\alpha\beta=2,3}$ of Poisson brackets of second-class constraints is $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore, the Dirac bracket for two phase functions F, G is given by:

$$\begin{aligned} \{F, G\}^* &= \{F, G\} - \{F, q\} \{p, G\} + \{F, p\} \{q, G\} \\ &= \{F, G\} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial q} + \left(-\frac{\partial F}{\partial q}\right) \left(-\frac{\partial G}{\partial p}\right) = \{F, G\} |_{(\xi, \pi)}, \end{aligned}$$

where the final Poisson bracket is the canonical Poisson bracket restricted to the coordinates $(\xi_1, \xi_2, \xi_3, \pi_1, \pi_2, \pi_3)$. $\natural$

Example 7.1.8. A more physically interesting example is that of a particle of mass m in $\mathbb{R}^3$ restricted to a sphere of radius R with a central force (as, for instance, gravity), whose Hamiltonian is given by:

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - \frac{k}{\sqrt{x^2 + y^2 + z^2}},$$

with $k \in \mathbb{R}$. The constraint for this system is $\phi \overset{\text{def}}{=} x^2 + y^2 + z^2 - R^2 = 0$ and the consistency condition implies:

$$0 = \dot{\phi} = \{H, \phi\} + u_1 \{\phi, \phi\} = \frac{1}{2m}((2p_x)(2x) + (2p_y)(2y) + (2p_z)(2z)) = \frac{2}{m}(p_x x + p_y y + p_z z).$$

Considering the usual basis $\{e_x, e_y, e_z\}$ of $\mathbb{R}^3$ and the position and momentum vectors, $\vec{r} = x\, e_x + y\, e_y + z\, e_z$ and $\vec{p} = p_x\, e_x + p_y\, e_y + p_z\, e_z$, respectively, the previous condition gives a secondary constraint $\phi' \overset{\text{def}}{=} \vec{p} \cdot \vec{r} = 0$, where $\cdot$ represents the usual inner product in $\mathbb{R}^3$. Then, the consistency condition of ϕ' gives

$$\begin{aligned} 0 = \dot{\phi}' &= \{H, \phi'\} + u_1 \{\phi, \phi'\} \\ &= \frac{1}{2m}(2p_x^2 + 2p_y^2 + 2p_z^2) + \frac{k}{(x^2 + y^2 + z^2)^{3/2}}(-x^2 - y^2 - z^2) + u_1(-2x^2 - 2y^2 - 2z^2) \\ &= \frac{1}{m}\|\vec{p}\|^2 - \frac{k}{\|\vec{r}\|^2} - 2u_1\|\vec{r}\|^2 \implies u_1 = \frac{1}{2m} \frac{\|\vec{p}\|^2}{\|\vec{r}\|^2} - \frac{k}{2\|\vec{r}\|^3}, \end{aligned}$$

where $\|\cdot\|$ represents the norm associated to the inner product $\cdot$. Thus, we obtain the first-class Hamiltonian:

$$\begin{aligned} H_1 = H + u_1\phi &= \frac{1}{2m}\|\vec{p}\|^2\left(1+\frac{\|\vec{r}\|^2-R^2}{\|\vec{r}\|^2}\right) - \frac{k}{\|\vec{r}\|}\left(1+\frac{\|\vec{r}\|^2-R^2}{\|\vec{r}\|^2}\right) \\ &= \left(\frac{1}{2m}\|\vec{p}\|^2 - \frac{k}{\|\vec{r}\|}\right)\left(1+\frac{\|\vec{r}\|^2-R^2}{\|\vec{r}\|^2}\right). \end{aligned}$$

It is clear that both ϕ and ϕ' are second-class constraints, since $\left\{\|\vec{r}\|^2 - R^2, \vec{p}\cdot\vec{r}\right\} = -2\|\vec{r}\|^2 \neq 0$ and, so, the extended Hamiltonian is equal to the first-class Hamiltonian. Moreover, the matrix of Poisson brackets of the second-class constraints is $2\|\vec{r}\|^2\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, thus the Dirac bracket of two phase functions F, G is

$$\{F,G\}^* = \{F,G\} - \frac{1}{2\|\vec{r}\|^2}\left(\left\{F, \|\vec{r}\|^2 - R^2\right\}\{\vec{p}\cdot\vec{r}, G\} - \{F, \vec{p}\cdot\vec{r}\}\left\{\|\vec{r}\|^2 - R^2, G\right\}\right). \quad \natural$$

Coming back to the side of Mathematics, if we consider a general Poisson manifold (M, π), the equivalent to consider the constrained system is to take a general submanifold and understand their structure. As we have seen, a submanifold of a symplectic manifold is, in general, a pre-symplectic manifold, i.e. a manifold equipped with a closed 2-form (not necessarily non-degenerate). For the case of a Poisson manifold, we have seen that the Poisson structure can be equivalently defined by the Poisson bivector or by the symplectic foliation on this manifold. Thus, assuming that any submanifold of a Poisson manifold intersects each symplectic leaf into a submanifold, the result is a a pre-symplectic (sub)manifold. This structure of a foliation by pre-symplectic leaves is what we call a Dirac structure on a manifold. In the next section, we shall give a more abstract way to introduce Dirac structures and later prove that this abstract characterisation is equivalent to a manifold having a pre-symplectic foliation.

7.2 Dirac Structures

As explained previously, we shall now approach a more algebraic and also more abstract definition of Dirac structures on a manifold. We start by introducing the notion of the generalised tangent bundle.

Definition 7.2.1. Let M be a smooth manifold.

1) The *generalised tangent bundle* is the fibre product of the tangent and cotagent bundles or, equivalently,
$$\mathbb{T}M \overset{\text{def}}{=} TM \oplus T^*M = \coprod_{p\in M} (T_pM \oplus T_p^*M). \tag{7.11}$$

2) The canonical pairing in the generalised tangent bundle is $\langle\cdot,\cdot\rangle : \mathbb{T}M \oplus \mathbb{T}M \to \mathbb{R}$ such that, for $p \in M$ and $v+\alpha, w+\beta \in \mathbb{T}_pM$,
$$\langle v+\alpha, w+\beta\rangle = \beta(v) + \alpha(w). \tag{7.12}$$

Remark 7.2.2. The above definition could be perfectly applied to an arbitrary vector space V. In that case, we obtain the generalised vector space $\mathbb{V} = V \oplus V^*$ and the canonical pairing $\langle\cdot,\cdot\rangle : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ with a similar definition from the one above.

Remark 7.2.3. With the definition of the generalised tangent bundle, we may consider the two projections
$$\begin{aligned} \mathrm{pr}_{TM} : \mathbb{T}M &\longrightarrow TM \\ v+\alpha &\longmapsto v \end{aligned} \quad\text{and}\quad \begin{aligned} \mathrm{pr}_{T^*M} : \mathbb{T}M &\longrightarrow T^*M \\ v+\alpha &\longmapsto \alpha \end{aligned}$$
Then, one can quickly check that $(\mathbb{T}M, \mathrm{pr}_{TM})$ forms an anchored vector bundle.

In addition to this pairing, it is possible to define a bracket on the set of sections of $\mathbb{T}M$ which was first introduced by T. Courant in [20] and later modified by I. Dorfman:

Definition 7.2.4. Let M be a smooth manifold. The *Dorfman bracket* is a bilinear bracket $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ such that, for $X+\omega, Y+\eta \in \mathfrak{X}(M)\oplus\Omega^1(M) = \Gamma(\mathbb{T}M)$,
$$[\![X+\omega, Y+\eta]\!] \overset{\text{def}}{=} [X,Y] + \mathcal{L}_X\eta - i_Y\mathrm{d}\omega. \tag{7.13a}$$

Remark 7.2.5. The Dorfman bracket can be alternatively written as
$$[\![X+\omega, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\omega + \mathrm{d}(\omega(Y)) \tag{7.13b}$$
$$= [X,Y] + i_X\mathrm{d}\eta - i_Y\mathrm{d}\omega + \mathrm{d}(\eta(X)) \tag{7.13c}$$

Remark 7.2.6. In the literature, it is sometimes convenient to define the Dorfman bracket so to include a 3-form $\Xi \in \Omega^3(M)$. This defines the *twisted Dorfman bracket* the following way for $X+\omega, Y+\eta \in \Gamma(\mathbb{T}M)$:
$$[\![X+\omega, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - i_Y\mathrm{d}\omega + i_Xi_Y\Xi.$$

Proposition 7.2.7. *The Dorfman bracket satisfies the following properties:*

(i) Anti-symmetric up to an exact form, i.e. for $\sigma_1, \sigma_2 \in \Gamma(\mathbb{T}M)$,

$$[\![\sigma_2, \sigma_1]\!] = -[\![\sigma_1, \sigma_2]\!] + \mathrm{d}\langle \sigma_1, \sigma_2 \rangle;$$

(ii) Leibniz-like property, i.e. for $\sigma_1, \sigma_2 \in \Gamma(\mathbb{T}M)$ *and* $f \in C^\infty(M)$,

$$[\![\sigma_1, f\sigma_2]\!] = f[\![\sigma_1, \sigma_2]\!] + \mathrm{pr}_{TM}(\sigma_1)(f)\sigma_2;$$

(iii) For all $\sigma \in \Gamma(\mathbb{T}M)$, $[\![\sigma, \cdot]\!]$ *is a derivation of the algebra* $(\Gamma(\mathbb{T}M), [\![\cdot, \cdot]\!])$ *or, equivalently, for* $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(\mathbb{T}M)$,

$$[\![\sigma_1, [\![\sigma_2, \sigma_3]\!]]\!] = [\![[\![\sigma_1, \sigma_2]\!], \sigma_3]\!] + [\![\sigma_2, [\![\sigma_1, \sigma_3]\!]]\!];$$

(iv) Given $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(\mathbb{T}M)$,

$$\mathrm{pr}_{TM}(\sigma_1)(\langle \sigma_2, \sigma_3 \rangle) = \langle [\![\sigma_1, \sigma_2]\!], \sigma_3 \rangle + \langle \sigma_2, [\![\sigma_1, \sigma_3]\!] \rangle.$$

Proof. The proof of the previous proposition comes from direct application of the definition of the Dorfman bracket. Q.E.D.

Remark 7.2.8. By this proposition, because the Dorfman bracket $[\![\cdot, \cdot]\!]$ is not anti-symmetric, it is clear that $(\Gamma(\mathbb{T}M), [\![\cdot, \cdot]\!])$ does not form a Lie algebra. In fact, the properties introduced in this proposition are the defining properties of a so-called *Courant algebroid.* The discussion of this type of algebroids is out of the scope of the present thesis.

The introduction of the Dorfman bracket allows us to define a Dirac structure the following way:

Definition 7.2.9. A Dirac structure on a manifold M is a vector subbundle $\mathbb{L} \subseteq \mathbb{T}M$ of rank $m = \dim(M)$ which is isotropic with respect to the canonical pairing $\langle \cdot, \cdot \rangle$ and whose sections are closed under the Dorfman bracket $[\![\cdot, \cdot]\!]$, i.e. is a rank-m subbundle such that

$$\langle \mathbb{L}, \mathbb{L} \rangle = 0 \quad \text{and} \quad [\![\Gamma(\mathbb{L}), \Gamma(\mathbb{L})]\!] \subseteq \Gamma(\mathbb{L}). \tag{7.14}$$

Then, the pair $(M, \mathbb{L})$ is called a *Dirac manifold.*

Remark 7.2.10. Given a Dirac structure $\mathbb{L}$ on M, the condition of isotropy is equivalent to say that $\mathbb{L} \subseteq \mathbb{L}^\perp$, where

$$\mathbb{L}^\perp \stackrel{\text{def}}{=} \left\{ u \in \mathbb{T}_p M \,\middle|\, p \in M \text{ and } \langle u, l \rangle = 0, \text{ for all } l \in \mathbb{L}_p \right\}$$

denotes the orthogonal complement of $\mathbb{L}$ in $\mathbb{T}M$ with respect to the canonical pairing $\langle \cdot, \cdot \rangle$. However, the condition that $\mathrm{rank}(\mathbb{L}) = m$ means that $\mathbb{L} = \mathbb{L}^\perp$. In the literature, a vector subbundle $\mathbb{L}$ that is such that $\mathbb{L} = \mathbb{L}^\perp$ is called *Lagrangian* or *maximally isotropic.*

Relaxing the condition of $(\Gamma(\mathbb{L}), [\![\cdot,\cdot]\!])$ being a subalgebra of $(\Gamma(\mathbb{T}M), [\![\cdot,\cdot]\!])$ we say that $\mathbb{L}$ is an *almost-Dirac structure on M* or, since these are just $\mathbb{L} \subseteq \mathbb{T}M$ such that $\mathbb{L} = \mathbb{L}^{\perp}$, we say that $\mathbb{L}$ is a Lagrangian subbundle of $(\mathbb{T}M, \langle\cdot,\cdot\rangle)$.

Remark 7.2.11. For a Dirac structure $\mathbb{L}$ on M, since $\mathbb{L}$ is Lagrangian with respect to $\langle\cdot,\cdot\rangle$ and $\Gamma(\mathbb{L})$ is closed under $[\![\cdot,\cdot]\!]$, $\mathbb{L}$ is endowed with a Lie algebroid structure with Lie bracket $[\cdot,\cdot]$ and anchor a given by $[\sigma_1, \sigma_2] = [\![\sigma_1, \sigma_2]\!]$ and $\mathsf{a}(\sigma_1) = \mathrm{pr}_{TM}(\sigma_1)$, for all $\sigma_1, \sigma_2 \in \Gamma(\mathbb{L})$.

Example 7.2.12. The simplest examples of a Dirac structure are the tangent and cotangent bundles. In fact, for the former, since the Dorfman bracket extends the Lie bracket of vector fields, we can take $\mathbb{L} = TM \oplus \{0 \in T^*M\}$, so that $\mathrm{rank}(\mathbb{L}) = \dim(M)$, $\langle\mathbb{L},\mathbb{L}\rangle = 0$ and $[\![\Gamma(\mathbb{L}), \Gamma(\mathbb{L})]\!] \cong [\mathfrak{X}(M), \mathfrak{X}(M)] \subseteq \mathfrak{X}(M) \cong \Gamma(\mathbb{L})$.

For the case of the cotangent bundle, taking $\mathbb{L} = \{0 \in TM\} \oplus T^*M$, once again $\mathrm{rank}(\mathbb{L}) = \dim(M)$, $\langle\mathbb{L},\mathbb{L}\rangle = 0$ and the Dorfman bracket restricts to the zero bracket. Thus, $[\![\Gamma(\mathbb{L}), \Gamma(\mathbb{L})]\!] = 0 \in \Omega^1(M) \cong \Gamma(\mathbb{L})$. ♮

Example 7.2.13. A more interesting example occurs whenever we have a subbundle $E \subseteq TM$ of the tangent space (or, in fact, one of the cotangent bundle). Then, if we denote

$$E^{\circ} = \left\{\alpha \in T^*M \,\middle|\, \alpha(v) = 0,\ \forall\, v \in E\right\}$$

the annihilator of E, we obtain that $\mathbb{L}_E = E \oplus E^{\circ}$ is an almost-Dirac structure on M. For $\mathbb{L}_E$ to be a Dirac structure on M, we must have that $[\![\Gamma(\mathbb{L}_E), \Gamma(\mathbb{L}_E)]\!] \subseteq \Gamma(\mathbb{L}_E)$. In more detail, taking $X_1 + \eta_1, X_2 + \eta_2 \in \Gamma(\mathbb{L}_E)$, we have that

$$[\![X_1 + \eta_1, X_2 + \eta_2]\!] = [X_1, X_2] + i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(X_1)) = [X_1, X_2] + i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1,$$

where we used the fact that $\eta_i \in \Gamma(E^{\circ})$ to conclude that $\eta_i(X_j) = 0$. To conclude that $[\![X_1 + \eta_1, X_2 + \eta_2]\!] \in \Gamma(E \oplus E^{\circ})$ we need only to show that $i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1 \in \Gamma(E^{\circ})$. For that, let $Y \in \Gamma(E)$. Then, using the definition of the exterior derivative (1.18b), we obtain,

$$\begin{aligned}
(i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1)(Y) =& (\mathrm{d}\eta_2)(X_1, Y) - (\mathrm{d}\eta_1)(X_2, Y) \\
=& X_1(\eta_2(Y)) - Y(\eta_2(X_1)) - \eta_2([X_1, Y]) \\
&- X_2(\eta_1(Y)) + Y(\eta_1(X_2)) + \eta_1([X_2, Y]) \\
=& \eta_2([Y, X_1]) + \eta_1([X_2, Y]),
\end{aligned}$$

which only vanishes if and only if $[Y, X_1], [X_2, Y] \in \Gamma(E)$. Hence, we conclude that $\Gamma(\mathbb{L}_E)$ is a subalgebra if and only if $\Gamma(E)$ is a Lie subalgebra or, equivalently, if and only if E is involutive.

In particular, for a foliation $\mathcal{F}$, by Frobenius theorem, there is an involutive distribution $\mathcal{D}_{\mathcal{F}} \subseteq TM$, such that $T\mathcal{F} = \mathcal{D}_{\mathcal{F}}$ and, therefore, $\mathbb{L}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}} \oplus \mathcal{D}_{\mathcal{F}}^{\circ}$ is a Dirac structure on M. ♮

Example 7.2.14. Consider a closed 2-form $\omega \in \Omega^2_{\mathrm{cl}}(M)$ on a manifold M (or equivalently, let (M, ω) be a pre-symplectic manifold) and define

$$\mathbb{L}_\omega = \left\{ v + i_v \omega_{\tau_M(v)} \in \mathbb{T}M \,\middle|\, v \in TM \right\} = \mathrm{gr}(\omega^\flat). \tag{7.15}$$

We shall now prove that $\mathbb{L}_\omega$ is a Dirac structure (in fact, more in general, we will prove that a $\mathbb{L}_\omega$ defined in this way for arbitrary $\omega \in \Omega^2(M)$ is Dirac if and only if ω is closed). Firstly, it is clear that $\mathrm{rank}(\mathbb{L}_\omega) = \dim(T_pM) = \dim(M)$. The fact that $\langle \mathbb{L}_\omega, \mathbb{L}_\omega \rangle = 0$ comes naturally since, given $p \in M$ and $u, v \in T_pM$,

$$\langle v + i_v\omega_p, u + i_u\omega_p \rangle = \omega_p^\flat(v)(u) + \omega_p^\flat(u)(v) = \omega_p(v,u) + \omega_p(u,v) = 0.$$

As for the other condition, let $X_1 + \eta_1, X_2 + \eta_2 \in \Gamma(\mathbb{L}_\omega)$. Then, by the definition of $\mathbb{L}_\omega$, $\eta_i = i_{X_i}\omega$ for $i = 1, 2$ and, furthermore,

$$\begin{aligned}
[\![X_1 + \eta_1, X_2 + \eta_2]\!] &= [X_1, X_2] + i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(X_1)) \\
&= [X_1, X_2] + i_{X_1}\mathrm{d}i_{X_2}\omega - i_{X_2}\mathrm{d}i_{X_1}\omega + \mathrm{d}i_{X_1}i_{X_2}\omega \\
&= [X_1, X_2] + i_{X_1}\mathcal{L}_{X_2}\omega - i_{X_1}i_{X_2}\mathrm{d}\omega - \mathcal{L}_{X_2}i_{X_1}\omega + \mathrm{d}i_{X_2}i_{X_1}\omega + \mathrm{d}i_{X_1}i_{X_2}\omega \\
&= [X_1, X_2] + (i_{X_1}\mathcal{L}_{X_2} - \mathcal{L}_{X_2}i_{X_1})\omega - i_{X_1}i_{X_2}\mathrm{d}\omega \\
&= [X_1, X_2] + i_{[X_1,X_2]}\omega - i_{X_1}i_{X_2}\mathrm{d}\omega,
\end{aligned}$$

where, in the last equality, we used Eq. (1.26). Thus, for $\Gamma(\mathbb{L}_\omega)$ to be closed under the Dorfman bracket, we must have that $i_{X_1}i_{X_2}\mathrm{d}\omega = 0$, for all $X_1, X_2 \in \mathfrak{X}(M)$. This happens if and only if $\mathrm{d}\omega = 0$ or, in other words, if and only if ω is a closed 2-form. Hence, we conclude that any pre-symplectic manifold gives rise to a Dirac structure. $\natural$

Example 7.2.15. Following the same idea as of the previous example, we could now do the same procedure but with bivector fields instead. Let $\pi \in \mathfrak{X}^2(M)$ be a bivector field on a manifold M and define

$$\mathbb{L}_\pi = \left\{ i_\alpha\pi + \alpha \,\middle|\, \alpha \in T^*M \right\} = \mathrm{gr}(\pi^\sharp), \tag{7.16}$$

Then, $\mathrm{rank}(\mathbb{L}_\pi) = \dim(T_p^*M) = \dim(M)$ and, given $\alpha, \beta \in T_p^*M$, we get that

$$\langle i_\alpha\pi_p + \alpha, i_\beta\pi + \beta \rangle = \alpha(\pi_p^\sharp\beta) + \beta(\pi_p^\sharp\alpha) = \pi_p(\beta,\alpha) + \pi_p(\alpha,\beta) = 0.$$

For the Dorfman bracket, we take $X_1 + \eta_1, X_2 + \eta_2 \in \Gamma(\mathbb{L}_\pi)$ and, noting that $X_i = \pi^\sharp\eta_i$ for $i = 1, 2$, we get that

$$\begin{aligned}
[\![X_1+\eta_1, X_2+\eta_2]\!] &= [\pi^\sharp\eta_1, \pi^\sharp\eta_2] + \mathcal{L}_{\pi^\sharp\eta_1}\eta_2 - \mathcal{L}_{\pi^\sharp\eta_2}\eta_1 + \mathrm{d}(\eta_1(\pi^\sharp\eta_2)) \\
&= [\pi^\sharp\eta_1, \pi^\sharp\eta_2] + \mathcal{L}_{\pi^\sharp\eta_1}\eta_2 - \mathcal{L}_{\pi^\sharp\eta_2}\eta_1 - \mathrm{d}(\pi(\eta_1, \eta_2)) \\
&= [\pi^\sharp\eta_1, \pi^\sharp\eta_2] + [\eta_1, \eta_2]_\pi,
\end{aligned}$$

hence $\Gamma(\mathbb{L}_\pi)$ is closed under the Dorfman bracket if and only if $[\pi^\sharp\eta_1, \pi^\sharp\eta_2] = \pi^\sharp[\eta_1, \eta_2]_\pi$, i.e., if $\pi^\sharp : (\Omega^1(M), [\cdot,\cdot]_\pi) \to (\mathfrak{X}(M), [\cdot,\cdot])$ preserves the bracket structure. By Proposition 5.3.5, this is only the case whenever $[\pi,\pi]_{\mathrm{SN}} = 0$, i.e. when π is a Poisson bivector. Therefore, we conclude that all Poisson manifolds are also examples of Dirac manifolds. ♮

7.3 Properties of Dirac Structures

As we have remarked in the previous section, the definition of a Dirac structure $\mathbb{L}$ on a manifold M implies that $\mathbb{L}$ becomes a Lie algebroid. This fact is what allows, as we shall see, to prove the connection between the algebraic definition of a Dirac structure given in the previous section and the noted link to pre-symplectic foliations, which we now prove:

Theorem 7.3.1. *Any Dirac structure $\mathbb{L}$ on a smooth manifold M determines a singular foliation $\mathcal{F}$ endowed with a leaf-wise pre-symplectic structure $\omega_{\mathcal{F}}$. Additionally, the Dirac structure $\mathbb{L}$ is fully determined by its pre-symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$.*

Proof. From the discussion of Lie algebroids, we know that $\mathrm{pr}_{TM}(\mathbb{L})$ is an integrable distribution (so, in particular, it is involutive) and so, by Frobenius theorem, there is a (singular) foliation $\mathcal{F}$ of M such that $T\mathcal{F} = \mathrm{pr}_{TM}(\mathbb{L})$. Then, each leaf $\mathcal{F}_\lambda$ of $\mathcal{F}$ naturally inherits a 2-form $\omega_{\mathbb{L}} \in \Omega^2(\mathcal{F}_\lambda)$ such that

$$(\omega_{\mathbb{L}})_p(v,u) = \alpha(u), \quad \text{for all } v,u \in T_p\mathcal{F}_\lambda \text{ such that } v+\alpha \in \mathbb{L}. \tag{7.17}$$

To see that this 2-form is well-defined, let us first see that it is anti-symmetric and independent of α chosen. If $v+\alpha_1, v+\alpha_2, u+\beta \in \mathbb{L}|_p \subseteq \mathbb{T}_pM$, then we get that

$$0 = \langle v+\alpha_1, u+\beta\rangle = \beta(v) + \alpha_1(u), \quad \text{and} \quad 0 = \langle v+\alpha_2, u+\beta\rangle = \beta(v) + \alpha_2(u)$$

thus concluding that $\alpha_1(u) = \alpha_2(u) = -\beta(v)$. So, the 2-form $\omega_{\mathbb{L}}$ above is well-defined. To show that $\omega_{\mathbb{L}}$ is closed, let $X_1, X_2, X_3 \in \Gamma(T\mathcal{F})$ such that $X_i + \eta_i \in \Gamma(\mathbb{L})$ for $i = 1,2,3$.

Then,

$$\begin{aligned}
\omega_{\mathbb{L}}([X_1,X_2],X_3) &= (i_{X_1}\mathrm{d}\eta_2 - i_{X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(X_1)))(X_3)\\
&= X_1(\eta_2(X_3)) - X_3(\eta_2(X_1)) - \eta_2([X_1,X_3]) - X_2(\eta_1(X_3))\\
&\qquad + X_3(\eta_1(X_2)) + \eta_1([X_2,X_3]) + X_3(\eta_2(X_1))\\
&= X_1(\eta_2(X_3)) - X_2(\eta_1(X_3)) + X_3(\eta_1(X_2)) - \eta_2([X_1,X_3]) + \eta_1([X_2,X_3])\\
&= X_1(\omega_{\mathbb{L}}(X_2,X_3)) - X_2(\omega_{\mathbb{L}}(X_1,X_3)) + X_3(\omega_{\mathbb{L}}(X_1,X_2))\\
&\qquad - \omega_{\mathbb{L}}(X_2,[X_1,X_3]) + \omega_{\mathbb{L}}(X_1,[X_2,X_3])\\
&= (\mathrm{d}\omega_{\mathbb{L}})(X_1,X_2,X_3) + \omega_{\mathbb{L}}([X_1,X_2],X_3),
\end{aligned}$$

where the terms in red cancel out. Thus, we conclude that $(\mathrm{d}\omega_{\mathbb{L}})(X_1,X_2,X_3) = 0$ and, since X_1, X_2, X_3 are arbitrary in $\Gamma(T\mathcal{F})$, $\omega_{\mathbb{L}}$ is a closed foliated 2-form and $(\mathcal{F},\omega_{\mathbb{L}})$ is a pre-symplectic foliation of M.

As for the last part of the theorem, let $(\mathcal{F},\omega_{\mathcal{F}})$ be the pre-symplectic foliation of M determined by a Dirac structure $\mathbb{L}$. Then, writing $\mathcal{F} = \{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, we consider a vector subbundle $\mathbb{F} \subseteq \mathbb{T}M$ such that, for a point $p \in \mathcal{F}_\lambda \subseteq M$ in the leaf $\mathcal{F}_\lambda$,

$$\mathbb{F}_p = \left\{ v + \alpha \in \mathbb{T}_pM \,\middle|\, v \in T_p\mathcal{F}_\lambda \;\text{ and }\; \alpha|_{T_p\mathcal{F}_\lambda} = (\omega_{\mathcal{F}}^\flat)_p(v) \right\}.$$

The proof is concluded if we show that $\mathbb{F}_p = \mathbb{L}_p$, for all $p \in \mathcal{F}_\lambda \subseteq M$ and for all $\lambda \in \Lambda$. For the inclusion $\mathbb{L}_p \subseteq \mathbb{F}_p$, let $p \in M$, $\lambda \in \Lambda$ and $v + \alpha \in \mathbb{L}_p$ such that $v \in T_p\mathcal{F}_\lambda$. Then, for any $u \in T_p\mathcal{F}_\lambda$, we get

$$(\omega_{\mathcal{F}})_p(v,u) = (\omega_{\mathbb{L}})_p(v,u) = \alpha(u),$$

where we used the fact that the pre-symplectic form defined in the leaves $\mathcal{F}_\lambda$ of $\mathcal{F}$ is $\omega_{\mathbb{L}} \in \Omega^2(\mathcal{F}_\lambda)$ defined in (7.17). Since $u \in T_p\mathcal{F}_\lambda$ is arbitrary, we conclude that $\alpha|_{T_p\mathcal{F}_\lambda} = (\omega_{\mathcal{F}}^\flat)_p(v)$, and, so, $v + \alpha \in \mathbb{F}_p$. Conversely, if $v + \alpha \in \mathbb{F}_p$, given $u \in T_p\mathcal{F}_\lambda$,

$$\alpha(u) = \big((\omega_{\mathcal{F}}^\flat)_p(v)\big)(u) = (\omega_{\mathcal{F}})_p(v,u) = (\omega_{\mathbb{L}})_p(v,u).$$

Thus, $v + \alpha \in \mathbb{L}_p$ and we obtain that $\mathbb{F}_p = \mathbb{L}_p$ for all p, concluding the proof. Q.E.D.

Since Dirac manifolds include both symplectic and Poisson manifolds, we could now question whether some of the important characteristics of these manifolds also extend to the setting of Dirac structures. One such case is that of Hamiltonian vector fields. Indeed, it is possible to introduce this notion for Dirac manifolds, however, more is required to do so.

Definition 7.3.2. Let $(M,\mathbb{L})$ be a Dirac manifold. A function $f \in C^\infty(M)$ is called $\mathbb{L}$-*-admissible* if there exists a vector field $X \in \mathfrak{X}(M)$ such that $X + \mathrm{d}f \in \Gamma(\mathbb{L})$. We shall denote the collection of $\mathbb{L}$-admissible functions by $C^\infty_{\mathbb{L}}(M)$.

Remark 7.3.3. It is important to stress that the vector field in the previous definition needs not to be unique. Regardless, we still call the vector field X in the definition a *Hamiltonian vector field related to* f.

Remark 7.3.4. In the case of symplectic and Poisson manifolds, the collection of admissible functions is the entire space of functions $C^\infty(M)$.

Proposition 7.3.5. *For a Dirac manifold* $(M, \mathbb{L})$*, the collection of admissible functions* $C^\infty_{\mathbb{L}}(M)$ *is a Poisson algebra.*

Proof. Firstly, notice that $C^\infty_{\mathbb{L}}(M) \subseteq C^\infty(M)$ is an $\mathbb{R}$-subalgebra. In fact, it is clear that $\mathbb{R} \subseteq C^\infty_{\mathbb{L}}(M)$ and, if $f_1, f_2 \in C^\infty_{\mathbb{L}}(M)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $X_1 \mathrm{d}f_1, X_2 + \mathrm{d}f_2 \in \Gamma(\mathbb{L})$, then $\lambda_1 X_1 + \lambda_2 X_2 + \mathrm{d}(\lambda_1 f_1 + \lambda_2 f_2) \in \Gamma(\mathbb{L})$ and $f_1 X_2 + f_2 X_1 + \mathrm{d}(f_1 f_2) \in \Gamma(\mathbb{L})$. Then, introduce the following anti-symmetric $\mathbb{R}$-bilinear bracket $\{\cdot, \cdot\} : C^\infty_{\mathbb{L}}(M) \times C^\infty_{\mathbb{L}}(M) \to C^\infty_{\mathbb{L}}(M)$ which, for any $f, g \in C^\infty_{\mathbb{L}}(M)$, is given by

$$\{f, g\} = X(g), \tag{7.18}$$

where $X \in \mathfrak{X}(M)$ is a Hamiltonian vector field related to f. Let us prove that this bracket is well-defined. Given a function $f \in C^\infty_{\mathbb{L}}(M)$, if X_1, X_2 are both Hamiltonian vector fields relative to f, then if $Y + \mathrm{d}g \in \Gamma(\mathbb{L})$, we have that

$$X_1(g) + Y(f) = \langle X_1 + \mathrm{d}f, Y + \mathrm{d}g \rangle = 0 = \langle X_2 + \mathrm{d}f, Y + \mathrm{d}g \rangle = X_2(g) + Y(f).$$

Thus, $Y(f) = -X_1(g) = -X_2(g)$ showing that the right-hand side of Eq. (7.18) only depends of f and g and not on the choice of X and, moreover, that $\{f, g\} = -\{g, f\}$. Furthermore, we note that, if $f, g \in C^\infty_{\mathbb{L}}(M)$ with $X + \mathrm{d}f, Y + \mathrm{d}g \in \Gamma(\mathbb{L})$, then

$$[X, Y] + \mathrm{d}\{f, g\} = [X, Y] + \mathrm{d}(X(g)) = [\![X + \mathrm{d}f, Y + \mathrm{d}g]\!] \in \Gamma(\mathbb{L}). \tag{7.19}$$

This shows that $\{f, g\} \in C^\infty_{\mathbb{L}}(M)$, with Hamiltonian vector field given by $[X, Y]$. So, $\{\cdot, \cdot\}$ is a well-defined $\mathbb{R}$-bilinear anti-symmetric bracket on $C^\infty_{\mathbb{L}}(M)$.

To finish the proof, we need to check that the bracket defined above satisfies the Leibniz rule and the Jacobi identity. The former comes directly from the fact that X act as a derivation on functions. Hence, given $f, g, h \in C^\infty_{\mathbb{L}}(M)$ and $X \in \mathfrak{X}(M)$ such that $X + \mathrm{d}f \in \Gamma(\mathbb{L})$,

$$\{f, gh\} = X(gh) = X(g)h + gX(h) = \{f, g\} h + g \{f, h\}.$$

For the Jacobi identity, taking $f, g, h \in C^\infty_{\mathbb{L}}(M)$ and $X, Y, Z \in \mathfrak{X}(M)$ such that $X + \mathrm{d}f, Y + \mathrm{d}g, Z + \mathrm{d}h \in \Gamma(\mathbb{L})$, using Eq. (7.19), we get

$$\begin{aligned} \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= \{f, Y(h)\} + \{g, Z(f)\} - \{h, Y(f)\} \\ &= -[Y, Z](f) + Y(Z(f)) - Z(Y(f)) = 0. \end{aligned}$$

Hence, we conclude that $\{\cdot, \cdot\}$ defined in Eq. (7.18) is a Poisson bracket and, thus $(C^\infty_{\mathbb{L}}(M), \{\cdot, \cdot\})$ forms a Poisson algebra. Q.E.D.

Morphisms of Dirac Manifolds

Throughout the thesis, whenever we define some kind of new geometric structure, we have always defined the respective morphisms that respect said structure. In the case of Dirac manifolds, however, the introduction of such morphisms turns out to be much more complicated, since there is not only one possible notion of a morphism. This comes from the fact that, in symplectic geometry, there are two ways to describe the symplectic structure: either as

(1) a pair (M, ω), where $\omega \in \Omega^2_{\mathrm{cl}}(M)$ is a closed, non-degenerate 2-form;

(2) a pair (M, π), where $\pi \in \mathfrak{X}^2(M)$ is a non-degenerate bivector with $[\pi, \pi]_{\mathrm{SN}} = 0$.

These two descriptions arise from the fact that symplectic manifolds are specific cases of pre-symplectic and Poisson structures, respectively. Hence, there are two ways of seeing symplectic morphisms $\varphi : M_1 \to M_2$, according to the natural operations acting on forms or multivectors:

(1) As a map $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$, such that $\varphi^*\omega_2 = \omega_1$;

(2) As a map $\varphi : (M_1, \pi_1) \to (M_2, \pi_2)$, with $\varphi_*\pi_1 = \pi_2$.

These notions above, as it can be easily seen, coincide in the case of symplectic manifolds, however, in general, they do not need to. Since both pre-symplectic and Poisson structures are examples of Dirac structures, we conclude that there are two alternative notions of morphisms for a Dirac manifold.

Definition 7.3.6. Let $(M_1, \mathbb{L}_1)$ and $(M_2, \mathbb{L}_2)$ be Dirac manifolds and $\varphi : M_1 \to M_2$ a smooth map.

1) The *forward image of* $\mathbb{L}_1$ is the set

$$\mathfrak{F}_\varphi(\mathbb{L}_1) \overset{\text{def}}{=} \coprod_{p \in M_1} \left\{ \varphi_*(p)(v) + \alpha \in \mathbb{T}_{\varphi(p)}M_2 \,\middle|\, v + \alpha \circ \varphi_*(p) \in (\mathbb{L}_1)_p \right\} \subseteq \varphi^*\mathbb{T}M_2. \tag{7.20a}$$

The map $\varphi : M_1 \to M_2$ is called a *forward Dirac map* if $\varphi^*\mathbb{L}_2 = \mathfrak{F}_\varphi(\mathbb{L}_1) \subseteq \varphi^*\mathbb{T}M_2$, i.e. for all $p \in M_1$,

$$(\mathbb{L}_2)_{\varphi(p)} = (\mathfrak{F}_\varphi(\mathbb{L}_1))_p. \tag{7.20b}$$

2) The *backward image of* $\mathbb{L}_2$ is the collection

$$\mathfrak{B}_\varphi(\mathbb{L}_2) \overset{\text{def}}{=} \coprod_{p \in M_1} \left\{ v + \alpha \circ \varphi_*(p) \in \mathbb{T}_pM_1 \,\middle|\, \varphi_*(p)(v) + \alpha \in (\mathbb{L}_2)_{\varphi(p)} \right\} \subseteq \mathbb{T}M_1. \tag{7.21a}$$

The map $\varphi : M_1 \to M_2$ is called a *backward Dirac map* if $\mathbb{L}_1 = \mathfrak{B}_\varphi(\mathbb{L}_2)$, i.e. for all $p \in M_1$,

$$(\mathbb{L}_1)_p = (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)}. \tag{7.21b}$$

Proposition 7.3.7. *Let N be a smooth manifold, $(M, \mathbb{L})$ a Dirac manifold and $\varphi : M \to N$ and $\phi : N \to M$ smooth maps. We have the following:*

(i) If $\ker(T\varphi) \cap \mathbb{L}$ *has constant rank, then* $\mathfrak{F}_\varphi(\mathbb{L})$ *is a Dirac structure on* N*;*

(ii) If $\ker((T\phi)^*) \cap \phi^*\mathbb{L}$ *has constant rank, then* $\mathfrak{B}_\phi(\mathbb{L})$ *is a Dirac structure on* N*.*

For a proof of this statement, one can check [23, Propositions 5.6 and 5.9].

Example 7.3.8. For a Dirac manifold $(M, \mathbb{L})$, given a submanifold $i : N \hookrightarrow M$, if $\mathbb{L} \cap (TN)^\circ$ has constant rank, then $\mathbb{L}|_N \overset{\text{def}}{=} \mathfrak{B}_i(\mathbb{L})$ is the only Dirac structure on N such that $i : (N, \mathbb{L}|_N) \hookrightarrow (M, \mathbb{L})$ is a backward Dirac map. ♮

Example 7.3.9. In a Poisson manifold (M, π), a Poisson-Dirac submanifold is an immersed submanifold $i : N \to M$ equipped with a Poisson structure π_N such that $i : (N, \mathbb{L}_{\pi_N}) \to (M, \mathbb{L}_\pi)$ is a backward Dirac map. ♮

We now proceed to introduce some important results that give sufficient condition for when the two notions of forward and backward Dirac maps coincide:

Proposition 7.3.10. *Let $(M_1, \mathbb{L}_1)$ and $(M_2, \mathbb{L}_2)$ be Dirac manifolds. If a smooth immersion $\varphi : M_1 \to M_2$ is a forward Dirac map, then φ is a backward Dirac map.*

Proof. Let $p \in M$ and suppose that φ is a forward Dirac map. Taking $v + \alpha \circ \varphi_*(p) \in (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)}$. Then, by the definition of the backward image of $\mathbb{L}_2$, $\varphi_*(p)(v) + \alpha \in (\mathbb{L}_2)_{\varphi(p)} = (\mathfrak{F}_\varphi(\mathbb{L}_1))_p$. Thus, since $\varphi : M_1 \to M_2$ is an immersion, i.e. $\ker(\varphi_*(p)) = 0$, by the definition of the forward image of $\mathbb{L}_1$, $v + \alpha \circ \varphi_*(p) \in (\mathbb{L}_1)_p$, proving that $(\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)} \subseteq (\mathbb{L}_1)_p$.

Reciprocally, if $u + \beta \in (\mathbb{L}_1)_p$, since φ is an immersion, for any $w \in \operatorname{im}(\varphi_*(p)) \subseteq T_{\varphi(p)}M_2$ there is a unique $v \in T_pM_1$ such that $w = \varphi_*(p)(v)$ and, thus, there exists $\alpha \in T^*_{\varphi(p)}M_2$ such that $\beta = \alpha \circ \varphi_*(p)$. Therefore, $u + \alpha \circ \varphi_*(p) \in (\mathbb{L}_1)_p$ and, by the definition of the forward image of $\mathbb{L}_1$, $\varphi_*(p)(u) + \alpha \in (\mathfrak{F}_\varphi(\mathbb{L}_1))_p$. Once again, since φ is a forward Dirac map, $\varphi_*(p)(u) + \alpha \in (\mathbb{L}_2)_{\varphi(p)}$ and, by the definition of the backward image of $\mathbb{L}_2$, $u + \beta = u + \alpha \circ \varphi_*(p) \in (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)}$. This show that $(\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)} \supseteq (\mathbb{L}_1)_p$, completing the proof. Q.E.D.

Proposition 7.3.11. *Let M be a smooth manifold, $(N, \mathbb{L})$ a Dirac manifold and $\varphi : M \to N$ a smooth submersion. Then $\mathfrak{B}_\varphi(\mathbb{L})$ is a Lie algebroid, isomorphic to the pullback Lie algebroid $\varphi^! \mathbb{L}$.*

Proof. Consider the following vector bundle morphism covering the identity, id_M:

$$\begin{aligned}\psi : \mathfrak{B}_\varphi(\mathbb{L}) &\longrightarrow \varphi^!\mathbb{L}\\ v+\alpha\circ\varphi_*(p) &\longmapsto (v, \varphi_*(p)(v)+\alpha)\end{aligned}$$

This vector bundle morphism is well-defined ($\varphi_*(p) = \mathrm{pr}_{TM}(\varphi_*(p)(v)+\alpha)$) and, since φ is a submersion, it is a vector bundle isomorphism. To conclude the proof, we need only check that ψ is a Lie algebroid morphism. Firstly, since the anchors of $\mathfrak{B}_\varphi(\mathbb{L})$ and $\varphi^!\mathbb{L}$ are, respectively, pr_{TM} and the projection onto the first factor, it is clear that, given $v+\alpha\circ\varphi_*(p) \in \mathfrak{B}_\varphi(\mathbb{L})_p$,

$$\varphi^!\mathrm{pr}_{TN}\circ\psi(v+\alpha\circ\varphi_*(p)) = v = \mathrm{id}_{TM}\circ\mathrm{pr}_{TM}(v+\alpha\circ\varphi_*(p))$$

hence, ψ preserves the anchors. As for the ψ being a Lie algebra morphism between $\Gamma(\mathfrak{B}_\varphi(\mathbb{L}))$ and $\Gamma(\varphi^!\mathbb{L})$, let $X_1, X_2 \in \mathfrak{X}(M)$ and $\eta_1,\eta_2 \in \Omega^1(N)$ such that $X_i+\varphi^*\eta_i \in \Gamma(\mathfrak{B}_\varphi(\mathbb{L}))$, for $i=1,2$. Then, considering the bracket structure on $\Gamma(\mathfrak{B}_\varphi(\mathbb{L}))$ to be the Dorfman bracket on $\mathbb{T}M$ restricted to the relevant space, we get

$$\begin{aligned}[\![X_1+\varphi^*\eta_1, X_2+\varphi^*\eta_2]\!] &= [X_1,X_2] + i_{X_1}\mathrm{d}\varphi^*\eta_2 - i_{X_2}\mathrm{d}\varphi^*\eta_1 + \mathrm{d}((\varphi^*\eta_2)(X_1))\\ &= [X_1,X_2] + i_{X_1}\varphi^*\mathrm{d}\eta_2 - i_{X_2}\varphi^*\mathrm{d}\eta_1 + \mathrm{d}(\varphi^*(\eta_2(\varphi^*X_1)))\\ &= [X_1,X_2] + \varphi^*(i_{\varphi^*X_1}\mathrm{d}\eta_2) - \varphi^*(i_{\varphi^*X_2}\mathrm{d}\eta_1) + \varphi^*\mathrm{d}(\eta_2(\varphi^*X_1))\\ &= [X_1,X_2] + \varphi^*\big(i_{\varphi^*X_1}\mathrm{d}\eta_2 - i_{\varphi^*X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(\varphi^*X_1))\big)\end{aligned}$$

thus, applying ψ to this expression yields

$$\begin{aligned}\psi\big([\![X_1+\varphi^*\eta_1, X_2+\varphi^*\eta_2]\!]\big) &= \big([X_1,X_2], \varphi^*[X_1,X_2] + i_{\varphi^*X_1}\mathrm{d}\eta_2 - i_{\varphi^*X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(\varphi^*X_1))\big)\\ &= \big([X_1,X_2], [\varphi^*X_1,\varphi^*X_2] + i_{\varphi^*X_1}\mathrm{d}\eta_2 - i_{\varphi^*X_2}\mathrm{d}\eta_1 + \mathrm{d}(\eta_2(\varphi^*X_1))\big)\\ &= \big([X_1,X_2], [\![\varphi^*X_1+\eta_1, \varphi^*X_2+\eta_2]\!]\big)\\ &= [(X_1,\varphi^*X_1+\eta_1),(X_2,\varphi^*X_2+\eta_2)]_{\varphi^!\mathbb{L}}\\ &= [\psi(X_1+\varphi^*\eta_1),\psi(X_2+\varphi^*\eta_2)]_{\varphi^!\mathbb{L}}\end{aligned}$$

which shows that ψ is a Lie algebroid isomorphism. Q.E.D.

Corollary 7.3.12. *For Dirac manifolds $(M_1,\mathbb{L}_1)$ and $(M_2,\mathbb{L}_2)$ and a submersion $\varphi : M_1 \to M_2$, if φ is a backward Dirac map, then it is a forward Dirac map and $\mathbb{L}_1$ is isomorphic as a Lie algebroid to $\varphi^!\,\mathbb{L}_2$.*

Proof. Since φ is a backward Dirac map, by the previous proposition, $\mathbb{L}_1 = \mathfrak{B}_\varphi(\mathbb{L}_2)$ is isomorphic as a Lie algebroid to $\varphi^!\mathbb{L}_2$. To show that φ is also a forward Dirac map, let $p \in M$ and take $\varphi_*(p)(v)+\alpha \in (\mathfrak{F}_\varphi(\mathbb{L}_1))_p$. Then, by the definition of the forward image of $\mathbb{L}_1$, $v+\alpha\circ\varphi_*(p) \in (\mathbb{L}_1)_p$ and, since φ is a backward Dirac map, $(\mathbb{L}_1)_p = (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)}$, thus $v+\alpha\circ\varphi_*(p) \in (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)}$. By the definition of the backward image of $\mathbb{L}_2$, since

$\varphi : M_1 \to M_2$ is a submersion, i.e. the map $T_{\varphi(p)}M_2 \to T_pM_1$, $\eta \mapsto \eta \circ \varphi_*(p)$ is injective for all $p \in M_1$, we conclude that $\varphi_*(p)(v)+\alpha \in (\mathbb{L}_2)_{\varphi(p)}$. This shows that $(\mathfrak{F}_\varphi(\mathbb{L}_1))_p \subseteq (\mathbb{L}_2)_{\varphi(p)}$, for all $p \in M_1$.

Conversely, if $u+\beta \in (\mathbb{L}_2)_{\varphi(p)}$, since φ is a submersion, there exists $v \in T_pM_1$ such that $u = \varphi_*(p)(v)$. Hence, since φ is a backward Dirac map, by the definition of the backward image of $\mathbb{L}_2$, $v + \beta \circ \varphi_*(p) \in (\mathfrak{B}_\varphi(\mathbb{L}_2))_{\varphi(p)} = (\mathbb{L}_1)_p$. By the definition of the forward image of $\mathbb{L}_1$ we, then conclude that $u + \beta = \varphi_*(p)(v) + \beta \in (\mathfrak{F}_\varphi(\mathbb{L}_1))_p$. This proves that $(\mathfrak{F}_\varphi(\mathbb{L}_1))_p \supseteq (\mathbb{L}_2)_{\varphi(p)}$, for all $p \in M_1$, and so concludes the proof. Q.E.D.

The previous results lead us to the following proposition giving an equivalence of the definitions of forward and backward Dirac map:

Proposition 7.3.13. *Let $(M_1, \mathbb{L}_1)$ and $(M_2, \mathbb{L}_2)$ be Dirac manifolds and $\varphi : M_1 \to M_2$ a diffeomorphism. The following conditions are equivalent:*

(i) φ is a forward Dirac map;

(ii) φ is a backward Dirac map;

(iii) the lift of φ to $\mathbb{T}M$, given by

$$\begin{aligned} \mathbb{T}\varphi : \mathbb{T}M_1 &\longrightarrow \mathbb{T}M_2 \\ v+\alpha &\longmapsto T\varphi(v) + T^*\varphi(\alpha) \end{aligned} \tag{7.22}$$

is such that $\mathbb{T}\varphi(\mathbb{L}_1) = \mathbb{L}_2$.

Proof. (i) $\Longleftrightarrow$ (ii): Since φ is a diffeomorphism, it is, in particular, an immersion and a submersion, therefore, this equivalence comes directly from Proposition 7.3.10 and Corollary 7.3.12.

(i) $\Longleftrightarrow$ (iii): Let $p \in M$. Then, we can obtain the image of $\mathbb{T}_p\varphi$ as follows:

$$\begin{aligned} \mathbb{T}_p\varphi((\mathbb{L}_1)_p) &= \left\{ T_p\varphi(v) + T_p^*\varphi(\alpha) \in \mathbb{T}_{\varphi(p)}M_2 \,\middle|\, v+\alpha \in (\mathbb{L}_1)_p \right\} \\ &= \left\{ T_p\varphi(v) + T_p^*\varphi(\beta \circ T_p\varphi) \in \mathbb{T}_{\varphi(p)}M_2 \,\middle|\, v + \beta \circ T_p\varphi \in (\mathbb{L}_1)_p \right\} \\ &= \left\{ T_p\varphi(v) + \beta \in \mathbb{T}_{\varphi(p)}M_2 \,\middle|\, v + \beta \circ T_p\varphi \in (\mathbb{L}_1)_p \right\} = (\mathfrak{F}_\varphi(\mathbb{L}_1))_p, \end{aligned}$$

where, in the second equality, we used that φ is a diffeomorphism. Then, we can clearly conclude that $\mathbb{T}_p\varphi((\mathbb{L}_1)_p) = (\mathbb{L}_2)_{\varphi(p)}$ for all $p \in M$ if and only if φ is a forward Dirac map. Q.E.D.

Remark 7.3.14. A diffeomorphism $\varphi : M_1 \to M_2$ satisfying the equivalent conditions from the previous Proposition is called a *Dirac diffeomorphism*.

Chapter 8

Splitting Theorem for Anchored Vector Bundles

As mentioned in the Introduction, normal form theorems have been rather important to differential geometry, as these results allow for a better understanding of the local geometry. Throughout the thesis, we have explored some of the most notable (and well-known) normal form results, from Frobenius theorem, studied in Chapter 2, and Darboux's theorem, discussed in Chapter 4, to Weinstein's splitting theorem for Poisson geometry, explored in Chapter 6. In this chapter, following the treatment of [1], we explore another such example of a normal form theorem, this time for anchored vector bundles, using Euler-like vector fields to induce a splitting around transversals for these anchored vector bundles.

We start the present chapter by revisiting some notions introduced throughout the previous chapters and expand them focusing on the necessary results in order to understand the statement and proofs of the most relevant theorems from [1]. For that, we introduce the normal bundle to a submanifold as well as the notions and results pertaining to the Euler vector field of a vector bundle and tubular neighbourhoods. To conclude this section, we explore the notion of transversals and involutivity of anchored vector bundles. The last section delves into the splitting theorem for anchored vector bundles proposed in [1], as well as the uniqueness statements for the transversals of these anchored vector bundles. As this chapter is based on an article recently published, we shall omit most of the proofs that are presented therein opting instead to give proofs for results that are only stated or omitted in [1].

8.1 Prelude

We start the present chapter by introducing some additional concepts to previously introduced contents that are required to a better understanding of the splitting theorem proposed in [1].

Homogeneity Structures and Euler vector field

Following the discussion of homogeneity structures introduced in Section 1.2, we now introduce the Euler vector field on a vector bundle. As we shall see, this vector field is deeply connected to the homogeneity structure defining the structure of the vector bundle and will be essential to understand its properties when discussing the splitting theorem in Section 8.2.

Definition 8.1.1. Given a vector bundle $E \xrightarrow{\mathrm{pr}} M$ with homogeneity structure $h : \mathbb{R} \times E \to E$ and a point $p \in M$, the *Euler vector field on* E, denoted $\mathcal{E} \in \mathfrak{X}(E)$, is the vector field such that, for $e \in E_p$,

$$\mathcal{E}_e = \mathrm{vl}_{(p,e)}(e), \tag{8.1}$$

where vl is the vertical lift of the vector bundle, defined as,

$$\begin{aligned} \mathrm{vl}_{(p,e)} : E_p &\longrightarrow T_e E_p \\ \xi &\longmapsto [e + h_t(\xi)]_e \end{aligned}$$

Remark 8.1.2. Locally, for a chart $(U, \varphi = (x^1, \ldots, x^m))$ on M and adapted coordinates e^a on E_p, given $\xi \in E_p$, $\mathcal{E}_\xi = \epsilon^a(\xi) \left.\frac{\partial}{\partial e^a}\right|_\xi$ and $\epsilon^a(\xi) = [\xi^a + t\xi^a]_\xi = \xi^a$, i.e.

$$\mathcal{E} = e^a \, \frac{\partial}{\partial e^a} \ . \tag{8.2}$$

Proposition 8.1.3. *Let* $E \xrightarrow{\mathrm{pr}} M$ *be a vector bundle with homogeneity structure* $h : \mathbb{R} \times E \to E$ *and* $\mathcal{E} \in \mathfrak{X}(E)$ *the Euler vector field. Then the flow of* $\mathcal{E}$ *is* $\mathrm{Fl}^t_{\mathcal{E}} = h_{e^t}$*.*

Proof. Let $(U, \varphi = (x^1, \ldots, x^m))$ be a chart of M and let (x^i, ϵ^a) be adapted coordinates in $E|_U = \mathrm{pr}^{-1}(U)$. Then, clearly, we have that

$$h_{e^0}(p, \epsilon) = h_1(p, \epsilon) = (p, \epsilon) = \mathrm{Fl}^0_{\mathcal{E}}(p, \epsilon)$$

As for the time derivative of h_{e^t}, writing

$$h_{e^t}(p, \epsilon) = (\chi^i(e^t, p, \epsilon), \varepsilon^a(e^t, p, \epsilon)) = (p^i, e^t \, \epsilon^a),$$

we have the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}h_{e^t}(p,\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}t}\chi^i(e^t,p,\epsilon)\left.\frac{\partial}{\partial x^i}\right|_{\chi^i(e^t,p,\epsilon)} + \frac{\mathrm{d}}{\mathrm{d}t}\epsilon^a(e^t,p,\epsilon)\left.\frac{\partial}{\partial e^a}\right|_{\epsilon^a(e^t,p,\epsilon)}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}p^i\left.\frac{\partial}{\partial x^i}\right|_{p^i} + \frac{\mathrm{d}}{\mathrm{d}t}e^t\,\epsilon^a\left.\frac{\partial}{\partial e^a}\right|_{e^t\,\epsilon^a} = e^t\,\epsilon^a\left.\frac{\partial}{\partial e^a}\right|_{e^t\,\epsilon^a}\\
&= \varepsilon^a(e^t,p,\epsilon)\left.\frac{\partial}{\partial e^a}\right|_{\varepsilon^a(e^t,p,\epsilon)} = \mathcal{E}_{h_{e^t}(p,\epsilon)}.\end{aligned}$$

Since both $\mathrm{Fl}^t_{\mathcal{E}}$ and h_{e^t} obey the same Cauchy problem, by the uniqueness of the solution of differential equations, we conclude that $\mathrm{Fl}^t_{\mathcal{E}} = h_{e^t}$. Q.E.D.

8.1.1 Normal Bundle, Linear Approximation and Tubular Neighbourhoods

One other important concept related to vector bundles is the concept of a normal bundle to a submanifold $N \subseteq M$. This bundle essentially gives an orthogonal decomposition of the tangent space to a manifold at points of a submanifold into the tangent space to the submanifold and this normal bundle. Moreover, as we shall see, the concept of tubular neighbourhoods is deeply connected to that of normal bundles and, so, it is important to understand this bundle.

Definition 8.1.4. Let M be a smooth manifold and $N \subseteq M$ a submanifold of M. The *normal bundle of* N is the vector bundle with base N defined by

$$\nu(M,N) \overset{\text{def}}{=} TM|_N/TN, \tag{8.3a}$$

with the projection given by

$$\begin{aligned}\mathrm{npr}_N : \nu(M,N) &\longrightarrow N\\ v_p + T_pN &\longmapsto p\end{aligned} \tag{8.3b}$$

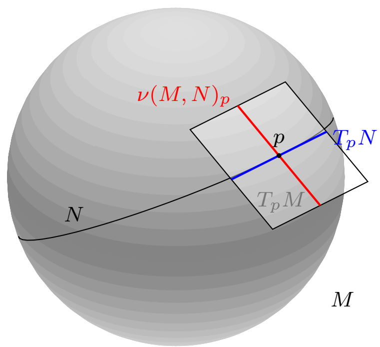


Figure 8.1: Intuitive view of the Normal bundle at a point

Remark 8.1.5. Considering $\mathbf{Man}^2$ to be the category of pairs of manifolds whose objects are pairs (M, N) where M is a smooth manifold and $N \subseteq M$ is a submanifold and whose morphisms $f : (M, N) \to (M', N')$ are maps $f : M \to M'$ such that $f(N) \subseteq N'$, the construction of the normal bundle as in Definition 8.1.4 introduces a functor:

$$\nu : \mathbf{Man}^2 \longrightarrow \mathbf{VB},$$
$$(M, N) \xmapsto{\text{Ob}} \nu(M, N),$$
$$f \xmapsto{\text{Mor}} \nu(f),$$

where the map $\nu(f) : \nu(M, N) \to \nu(M', N')$ is a vector bundle morphism covering $f|_N$ such that, for $p \in N$ and $v \in T_pM$,

$$\nu(f)(v + T_pN) = f_*(p)(v) + T_{f(p)}N'.$$

Proposition 8.1.6. *Let M be a smooth manifold and $N \subseteq M$ a submanifold. Then, $\nu(TM, TN) \cong T\nu(M, N)$.*

Proof. Both $\nu(TM, TN)$ and $T\nu(M, N)$ have the following structure of double vector bundle:

$$\begin{tikzcd}
T\nu(M,N) \arrow[r, "T\mathrm{npr}_N"] \arrow[d, "\tau_{\nu(M,N)}"'] & TN \arrow[d, "\tau_N"] \\
\nu(M,N) \arrow[r, "\mathrm{npr}_N"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
\nu(TM,TN) \arrow[r, "\mathrm{npr}_{TN}"] \arrow[d, "\nu(\tau_M)"'] & TN \arrow[d, "\tau_N"] \\
\nu(M,N) \arrow[r, "\mathrm{npr}_N"'] & N
\end{tikzcd}$$

We want to find a DVB isomorphism between these two double vector bundles. Notice that, $T(TM|_N)$ and $T^2M|_{TN}$ are subbundles of T^2M and, by the definition of the canonical flip $J : T^2M \to T^2M$ of the double tangent bundle T^2M of M, J restricts to an isomorphism between $T(TM|_N)$ and $T^2M|_{TN}$. Then, we may construct short exact sequences of double vector bundles

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dd] & & TN \arrow[rr] \arrow[dd] & & TM|_N \arrow[rr] \arrow[dd] & & \nu(M,N) \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
0 \arrow[rr] \arrow[ur] \arrow[dd] & & T^2N \arrow[rr] \arrow[ur] \arrow[dd] & & T^2M|_{TN} \arrow[rr] \arrow[ur] \arrow[dd] & & \nu(TM,TN) \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
& N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \\
TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dd] & & TN \arrow[rr] \arrow[dd] & & TM|_N \arrow[rr] \arrow[dd] & & \nu(M,N) \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
0 \arrow[rr] \arrow[ur] \arrow[dd] & & T^2N \arrow[rr] \arrow[ur] \arrow[dd] & & T(TM|_N) \arrow[rr] \arrow[ur] \arrow[dd] & & T\nu(M,N) \arrow[rr] \arrow[ur] \arrow[dd] & & 0 \arrow[ur] \arrow[dd] & \\
& N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \arrow[rr, equal] & & N \\
TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[rr, equal] \arrow[ur] & & TN \arrow[ur] &
\end{tikzcd}$$

and, moreover, we can construct

$$\begin{tikzcd} 0 \arrow[r] & T^2N \arrow[r] \arrow[d, "J|_{T^2N}"'] & T^2M|_{TN} \arrow[r] \arrow[d, "J"', "\cong"] & \nu(TM,TN) \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & T^2N \arrow[r] & T(TM|_N) \arrow[r] & T\nu(M,N) \arrow[r] & 0 \end{tikzcd}$$

Therefore, we obtain a double vector bundle isomorphism between $T\nu(M,N)$ and $\nu(TM,TN)$, as we required. Q.E.D.

By the definition and functoriality of ν we also get the following result:

Proposition 8.1.7. *Let M be a smooth manifold, $N \subseteq M$ be a submanifold and let $f : (M,N) \to (M,N)$ be a diffeomorphism of pairs. Then $\nu(f) : \nu(M,N) \to \nu(M,N)$ is a vector bundle automorphism of $\nu(M,N)$.*

Corollary 8.1.8. *Given a vector field $X \in \mathfrak{X}(M)$ tangent to N, then $\nu(X) \in \mathfrak{aut}_{\mathbf{VB}}(\nu(M,N))$.*

The previous results motivate the following definition:

Definition 8.1.9. The *linear approximation of a diffeomorphism* $f : (M,N) \to (M,N)$ preserving N is $\nu(f) \in \mathrm{Aut}_{\mathbf{VB}}(\nu(M,N))$. Similarly, the *linear approximation of a vector field $X \in \mathfrak{X}(M)$ tangent to N* is $\nu(X) \in \mathfrak{aut}_{\mathbf{VB}}(\nu(M,N))$.

Another important concept we need to introduce for the statement and proof of the splitting theorem is that of tubular neighbourhoods, These link the notions of a vector bundle with that of the topology of the manifold, or, in more detail, associate to each submanifold a vector bundle whose structure is closely related to the one of the normal bundle, thus codifying the topological structure of a neighbourhood of the submanifold.

Definition 8.1.10. Let M be a smooth manifold and $i : N \hookrightarrow M$ a submanifold. A *tubular neighbourhood* of N in M is a pair (E,ψ) where $E \xrightarrow{\mathrm{pr}} N$ is a vector bundle and $\psi : E \to M$ is a map such that

1) if $\theta_E : N \to E$ is the zero section, then $\psi \circ \theta_E = i$, i.e. the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "\psi"] & M \\ & N \arrow[ul, "\theta_E"] \arrow[u, hook, "i"'] \end{tikzcd}$$

2) there exist open sets $U \subseteq E$ and $V \subseteq M$ such that $\theta_E(N) \subseteq U$, $N \subseteq V$ and $\psi|_U : U \to V$ is a diffeomorphism.

Theorem 8.1.11. *Every submanifold N of a smooth manifold M has a tubular neighbourhood.*

The proof of this theorem requires the use of Riemannian geometry or the theory of sprays, which are out of the scope of this thesis. In said proof, it is shown that we may take the tubular neighbourhood to be $(\nu(M,N),\psi)$ such that $\psi : \nu(M,N) \to M$ is an embedding and the vector bundle morphism $\nu(\psi) : \nu(\nu(M,N),N) \cong \nu(M,N) \to \nu(M,N)$ induced by the map of pairs $\psi : (\nu(M,N),N) \to (M,N)$ coincides with the map $\mathrm{id}_{\nu(M,N)}$. Thus, in the sequel, we may always consider this to be the case and call ψ a *tubular neighbourhood embedding.*

Definition 8.1.12. A vector field $X \in \mathfrak{X}(M)$ tangent to N is called *linearisable* if there exists a tubular neighbourhood embedding $\psi : \nu(M,N) \to M$ such that $\nu(X) = \psi^* X$ on a neighbourhood of N.

Lemma 8.1.13. *Let $X \in \mathfrak{X}(M)$ be a vector field such that $X|_N = 0$ and $\nu(X) = \mathcal{E}$, where $\mathcal{E} \in \mathfrak{X}(\nu(M,N))$ is the Euler-vector field on $\nu(M,N)$. Then, X is linearisable.*

The proof of the previous statement presented in [1, Lemma 2.4] is rather straightforward, so we shall leave it out of the present review.

Definition 8.1.14. Let $N \subseteq M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is said to be *Euler-like along N* if it is complete, $X|_N = 0$ and $\nu(X)$ is the Euler vector field on $\nu(M,N)$.

One of the first results we can introduce relating to the notion of Euler-like vector fields is the following:

Proposition 8.1.15. *Let $X \in \mathfrak{X}(M)$ be an Euler-like vector field along N. Then, the tangent lift $T^\natural X \in \mathfrak{X}(TM)$ is Euler-like along TN.*

Proof. To prove that $T^\natural X$ is Euler-like, it is enough to show that $T^\natural X|_{TN} = 0$ and that $\nu(T^\natural X)$ is the Euler vector field in $\nu(TM,TN)$. For the proof of the first of these conditions, we note that, given the local expression of $T^\natural X$, if $p \in N$ and $v \in T_pN$,

$$T^\natural X(v) = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p + v^i \frac{\partial X^j}{\partial x^i}(p) \left.\frac{\partial}{\partial v^j}\right|_v ,$$

which vanishes because the first term is just $X_p = 0$ and the second term contains $v(X_j) = 0$, since $X|_N = 0$.

As for the second condition, by Lemma 8.1.13 and Definition 8.1.14, X is linearisable, i.e. $\psi^*X = \mathcal{E}$, for some tubular neighbourhood embedding ψ of N in M, where $\mathcal{E} \in \mathfrak{X}(\nu(M,N))$ is the Euler vector field on $\nu(M,N)$. Further

$$\nu(T^\natural X) = \nu(J \circ TX) = \nu(J) \circ \nu(TX) = \nu(J) \circ T(\nu(X)) = \nu(J) \circ T\mathcal{E} = J' \circ T\mathcal{E} = T^\natural \mathcal{E},$$

where we used the fact that ν is functorial, that $\nu \circ T \cong T \circ \nu$ and that $\nu(J) = J' : T^2\nu(M,N) \to T^2\nu(M,N)$ is the canonical flip of the double tangent bundle of $\nu(M,N)$. Thus, to conclude the result, we need only to prove that $T^\natural\mathcal{E}$ is the Euler vector field on $\nu(TM,TN) \cong T\nu(M,N)$. Locally, if $(U, \varphi = (x^1, \ldots, x^m))$ is a chart on N and (x^i, n^a) are adapted coordinates on $\nu(M,N)$, the Euler vector field on $\nu(M,N)$ is given by $\mathcal{E}|_{\mathrm{npr}_N^{-1}(N)} = n^a \frac{\partial}{\partial n^a}$. On $\nu(TM,TN)$, the adapted coordinates take the form (x^i, v^i, n^a, m^a) and, thus, the Euler vector field on this bundle is given by $\tilde{\mathcal{E}} = n^a \frac{\partial}{\partial n^a} + m^a \frac{\partial}{\partial m^a}$ and the tangent lift of an element $\upsilon = u^i \frac{\partial}{\partial x^i}\big|_p + \eta^a \frac{\partial}{\partial n^a}\big|_n$ is

$$\begin{aligned} T^\natural\mathcal{E}(\upsilon) &= \mathcal{E}^i(p) \left.\frac{\partial}{\partial x^i}\right|_p + \mathcal{E}^a(n) \left.\frac{\partial}{\partial n^a}\right|_n + u^i \frac{\partial \mathcal{E}^j}{\partial x^i}(p) \left.\frac{\partial}{\partial v^j}\right|_u + u^i \frac{\partial \mathcal{E}^a}{\partial x^i}(n) \left.\frac{\partial}{\partial m^a}\right|_\eta \\ &\quad + \eta^a \frac{\partial \mathcal{E}^i}{\partial n^a}(p) \left.\frac{\partial}{\partial v^i}\right|_u + \eta^a \frac{\partial \mathcal{E}^b}{\partial n^a}(n) \left.\frac{\partial}{\partial m^b}\right|_\eta \\ &= 0 + n^a \left.\frac{\partial}{\partial n^a}\right|_n + 0 + 0 + 0 + \eta^a \delta_a^b \left.\frac{\partial}{\partial m^b}\right|_\eta = n^a \left.\frac{\partial}{\partial n^a}\right|_n + \eta^a \left.\frac{\partial}{\partial m^a}\right|_\eta, \end{aligned}$$

which is precisely the local expression for the Euler vector field $\tilde{\mathcal{E}}$. Q.E.D.

An important result regarding Euler-like vector fields is that these determine tubular neighbourhood embeddings.

Proposition 8.1.16. *If $X \in \mathfrak{X}(M)$ is an Euler-like vector field along N, there exists an unique tubular neighbourhood $\psi : \nu(M,N) \to M$ such that $\psi^*X = \mathcal{E}$, for the Euler vector field $\mathcal{E} \in \mathfrak{X}(\nu(M,N))$ on $\nu(M,N)$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd} \nu(M,N) \arrow[r, "\mathcal{E}"] \arrow[d, "\psi"'] & \nu(TM,TN) \arrow[d, "T\psi"] \\ M \arrow[r, "X"'] & TM \end{tikzcd}$$

Idea of the Proof. The existence of a tubular neighbourhood embedding satisfying the conditions of the statement comes directly from Lemma 8.1.13. As for the uniqueness, taking $h : \mathbb{R} \times \nu(M,N) \to \nu(M,N)$ to be the homogeneity structure encoding the bundle structure of $\nu(M,N)$, one considers the flows of X and the Euler vector field $\mathcal{E}$, Fl_X^t and h_{e^t}, respectively and note that one can extend $\mathrm{Fl}_X^{\ln(t)}$ to $t = 0$, since $\nu(\mathrm{Fl}_X^t) = h_{e^t}$. Thus, one can define,

$$U = \left\{p \in M \;\middle|\; \lim_{t \to 0} \mathrm{Fl}_X^{\ln(t)}(p) \text{ exists and lies in } N \subseteq M\right\}. \tag{$*$}$$

Since $\nu(\psi) = \mathrm{id}_{\nu(M,N)}$ and, by hypothesis, $\psi \circ h_t = \mathrm{Fl}_X^{\ln(t)} \circ \psi$, differentiating this expression at $t = 0$ and applying the functor ν, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^{\ln(t)}(p)\bigg|_{t=0} + T_xN = \frac{\mathrm{d}}{\mathrm{d}t}h_t(n)\bigg|_{t=0} + T_xN,$$

where $n \in \nu(M,N)$ is such that $\psi(n) = p$ and $x = \lim_{t\to 0} \mathrm{Fl}_X^{\ln(t)}(p)$. Because $\frac{\mathrm{d}}{\mathrm{d}t}h_t = \mathrm{id}_{\nu(M,N)}$, we finally obtain that the map $\psi^{-1} : U \to \nu(M,N)$ can be written as

$$\psi^{-1}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^{\ln(t)}(p)\bigg|_{t=0} + T_xN. \tag{†}$$

Equations (∗) and (†) depend only on the flow of X, thus showing the uniqueness of the tubular neighbourhood embedding $\psi : \nu(M,N) \to U$. Q.E.D.

Following the previous result, we can now obtain the tubular neighbourhood determined by the tangent lift of an Euler-like vector field. For the proof of said result, however, we require the following lemma:

Lemma 8.1.17. *Let $\varphi : (M,N) \to (M',N')$ be a smooth map of pairs, X and X' be Euler-like vector fields along N and N', respectively, such that $T\varphi \circ X = X' \circ \varphi$. Then, if $\psi : \nu(M,N) \to M$ and $\psi' : \nu(M',N') \to M'$ are tubular neighbourhood embeddings determined by X and X', respectively, the following diagram commutes:*

$$\begin{tikzcd} \nu(M,N) \arrow[r, "\psi"] \arrow[d, "\nu(\varphi)"'] & M \arrow[d, "\varphi"] \\ \nu(M',N') \arrow[r, "\psi'"'] & M' \end{tikzcd}$$

Proof. To check that the diagram above is commutative, we may, instead, show that $\nu(\varphi) \circ \psi^{-1} = \psi'^{-1} \circ \varphi$. Proving the commutativity of the diagram this way allows us to use the definition of the inverse of the tubular neighbourhood associated with X from (the proof of) Proposition 8.1.16. Let $p \in U = \big\{ q \in M \;\big|\; \lim_{t\to 0} \mathrm{Fl}_X^{\ln(t)}(q) \text{ exists and lies in } N \subseteq M \big\}$ and write $x = \lim_{t\to 0} \mathrm{Fl}_X^{\ln(t)}(p)$. Then,

$$\begin{aligned} \big(\nu(\varphi) \circ \psi^{-1}\big)(p) &= \nu(\varphi)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^{\ln(t)}(p)\bigg|_{t=0} + T_xN \right) \\ &= \varphi_*(x)\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_X^{\ln(t)}(p)\bigg|_{t=0} \right) + T_{\varphi(x)}N' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi \circ \mathrm{Fl}_X^{\ln(t)}\big)(p)\bigg|_{t=0} + T_{\varphi(x)}N'. \end{aligned}$$

Because $T\varphi \circ X = X' \circ \varphi$, we get, for all t, $\varphi \circ \mathrm{Fl}_X^t = \mathrm{Fl}_{X'}^t \circ \varphi$. Since φ is smooth, it is, in particular, continuous, thus the previous expression can be extended for $t \to -\infty$. So, given that $\varphi(N) \subseteq N'$, we conclude that

$$\varphi(p) \in U' = \left\{q' \in M' \,\middle|\, \lim_{t\to 0} \mathrm{Fl}_{X'}^{\ln(t)}(q') \text{ exists and lies in } N' \subseteq M'\right\}.$$

Therefore, we obtain that

$$\varphi(x) = \varphi\left(\lim_{t\to 0} \mathrm{Fl}_X^{\ln(t)}(p)\right) = \lim_{t\to 0}\left(\varphi \circ \mathrm{Fl}_X^{\ln(t)}\right)(p) = \lim_{t\to 0}\left(\mathrm{Fl}_{X'}^{\ln(t)} \circ \varphi\right)(p),$$

and, hence,

$$\begin{aligned}\left(\psi'^{-1} \circ \varphi\right)(p) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}_{X'}^{\ln(t)}(\varphi(p))\bigg|_{t=0} + T_{\varphi(x)}N' = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi \circ \mathrm{Fl}_X^{\ln(t)}\right)(p)\bigg|_{t=0} + T_{\varphi(x)}N' \\ &= \left(\nu(\varphi) \circ \psi^{-1}\right)(p),\end{aligned}$$

which gives the intended result. Q.E.D.

Proposition 8.1.18. *Let $X \in \mathfrak{X}(M)$ be an Euler-like vector field along N and $\psi : \nu(M,N) \to M$ be the tubular neighbourhood embedding determined by X. Then, $T\psi : \nu(TM,TN) \to TM$ is the tubular neighbourhood embedding defined by $T^\natural X$.*

Proof. Notice that $\tau_M^* X = T^\natural X$ by the following diagram:

$$\begin{tikzcd} TM \arrow[r, "TX"] \arrow[d, "\tau_M"'] & T^2M \arrow[r, "J"] \arrow[d, "\tau_{TM}"'] & T^2M \arrow[dl, "T\tau_M"] \\ M \arrow[r, "X"'] & TM & \end{tikzcd}$$

Thus, if $\Psi : \nu(TM,TN) \to TM$ and $\psi : \nu(M,N) \to M$ are the tubular neighbourhood embeddings of $T^\natural\mathcal{E}$ and $\mathcal{E}$, respectively, we have the commutative diagram below, which is precisely the same commutative diagram for the derivative map $T\psi$ of ψ.

$$\begin{tikzcd} T\nu(M,N) \cong \nu(TM,TN) \arrow[r, "\Psi"] \arrow[d, "\tau_{\nu(M,N)} = \nu(\tau_M)"'] & TM \arrow[d, "\tau_M"] \\ \nu(M,N) \arrow[r, "\psi"'] & M \end{tikzcd}$$

Hence, we conclude $\Psi = T\psi$. Q.E.D.

8.1.2 Transversals and Involutivity of AVB

We recall that, given an anchored vector bundle $(E \xrightarrow{\text{pr}} M, \mathsf{a})$ and a submanifold $i : N \hookrightarrow M$, N is called a transversal of (E, a) if, for all $p \in N$, $T_pM = T_pN + \mathsf{a}(E_p)$ and the pull-back bundle by i is $i^!\,E = \mathsf{a}^{-1}(TN)$. Then we can introduce the following lemma:

Lemma 8.1.19. *Given a transversal $i : N \hookrightarrow M$ of an anchored vector bundle (E, a), we have the following isomorphism of double vector bundles*

$$\nu(E, i^!\,E) \cong \mathrm{npr}_N^!\, i^! E. \tag{8.4}$$

Idea of the Proof. As seen previously, $i^! E = \mathsf{a}^{-1}(TN)$ which is a subbundle of E. Therefore, we may consider the anchor to be a map of pairs $\mathsf{a} : (E, i^! E) \to (TM, TN)$ and obtain the following double vector bundle:

$$\begin{tikzcd}
\nu(E, i^! E) \arrow[r, "\nu(\mathsf{a})"] \arrow[d, "\mathrm{npr}_{i^! E}"'] & \nu(TM, TN) \arrow[d, "\mathrm{npr}_N"] \\
i^! E \arrow[r, "\mathsf{a}"'] & TN
\end{tikzcd}$$

To conclude the result, then, we construct a morphism of double vector bundles

$$\begin{tikzcd}
\nu(E, i^! E) \arrow[rr, "\nu(\mathsf{a})"] \arrow[dr, "f"'] \arrow[dd, "\mathrm{npr}_{i^! E}"'] & & \nu(TM, TN) \arrow[dr, "\cong"] \arrow[dd, "\mathrm{npr}_N"] & \\
& \mathrm{npr}_N^! i^! E \arrow[rr, "\mathrm{npr}_N^! \mathsf{a}"] \arrow[dd] & & T\nu(M, N) \arrow[dd, "T\mathrm{npr}_N"] \\
i^! E \arrow[rr, "\mathsf{a}"'] \arrow[dr, equal] & & TN \arrow[dr, equal] & \\
& i^! E \arrow[rr, "\mathsf{a}"'] & & TN
\end{tikzcd}$$

where, by identifying $\mathrm{npr}_N^! i^! E$ as the bundle $T\nu(M, N) \oplus i^! E$ as bundles over TN, we define

$$\begin{aligned} f : \nu(E, i^! E) &\longrightarrow T\nu(M, N) \oplus i^! E \\ \varepsilon &\longmapsto \nu(\mathsf{a})(\varepsilon) + \mathrm{npr}_{i^! E}(\varepsilon) \end{aligned}$$

which can be proven to be an isomorphism. Q.E.D.

Lemma 8.1.20. *Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be an anchored vector bundle and $i : N \hookrightarrow M$ transversal for (E, a). Then, there exists a section $\sigma \in \Gamma(E)$ vanishing on N such that $X = \mathsf{a}(\sigma)$ is an Euler-like vector field on M along N.*

Idea of the Proof. From the definition of $\nu(M, N)$ we get the following commutative diagram of vector bundle morphisms, over id_M, whose rows are exact (and so split):

$$\begin{tikzcd}
0 \arrow[r] & TN \arrow[r] & TM|_N \arrow[r, "q"] & \nu(M,N) \arrow[r] & 0 \\
0 \arrow[r] & i^! E \arrow[r] \arrow[u, "\mathsf{a}"] & E|_N \arrow[r, "q\circ\mathsf{a}"] \arrow[u, "\mathsf{a}"] & \nu(M,N) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Then, choosing $\sigma \in \Gamma(E)$ with $\sigma|_N = 0$ such that $\nu(\sigma) : \nu(M, N) \to \nu(E, M) \cong E|_N$ is a right inverse of the composition $q \circ \mathsf{a}$ of the quotient map $q : TM|_N \to \nu(M, N)$ and the anchor, i.e. it is such that $q \circ \mathsf{a} \circ \nu(\sigma) = \mathrm{id}_{\nu(M,N)}$. To conclude the proof, one needs only check that $X = \mathsf{a}(\sigma)$ is Euler-like along N. Q.E.D.

Proposition 8.1.21. *An anchored vector bundle (E, a) is involutive if and only if the anchor $\mathsf{a} : \Gamma(E) \to \mathfrak{X}(M)$ admits a lift $\hat{\mathsf{a}} : \Gamma(E) \to \mathfrak{aut}_{\mathbf{AVB}}(E)$ such that the following diagram commutes:*

$$\begin{tikzcd}
& \mathfrak{aut}_{\mathbf{AVB}}(E) \arrow[d] \\
\Gamma(E) \arrow[ur, "\hat{\mathsf{a}}"] \arrow[r, "\mathsf{a}"'] & \mathfrak{X}(M)
\end{tikzcd}$$

The proof of this proposition requires the usage of connections which are out of scope of the present work. For the interested reader, we refer back to Ref. [1, Proposition 3.17] for the complete proof of this statement.

8.2 Splitting Theorem

Now that we have introduced the necessary results, we proceed to state the splitting theorem for anchored vector bundles:

Theorem 8.2.1. *Let $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ be an anchored vector bundle and $i : N \to M$ be a transversal.*

(i) If (E, a) is involutive, then there exists an infinitesimal AVB automorphism $\hat{X} \in \mathfrak{aut}_{\mathbf{AVB}}(E)$ vanishing on $i^! E = \mathsf{a}^{-1}(TN)$ such that its projection $X \in \mathfrak{X}(M)$ is Euler--like along N;

(ii) Any $\hat{X} \in \mathfrak{aut}_{\mathbf{AVB}}(E)$ vanishing on $\mathsf{a}^{-1}(TN)$ whose projection $X \in \mathfrak{X}(M)$ is Euler-like along N determines an AVB isomorphism $\hat{\psi} : \mathrm{npr}_N^! i^! E \to E|_U$ with base map a tubular neighbourhood embedding $\psi : \nu(M, N) \to U \subseteq M$.

The proof of this statement requires the use of most of the results introduced in the precious section. However, instead of proving the theorem outright, we want to discuss why we are using the notions of tubular neighbourhoods to induce a transverse structure on an anchored vector bundle and, later, a local splitting for these geometric structures.

The idea for the introduction of the tubular neighbourhoods and Euler-like vector fields is that, as in the case of Frobenius, Darboux and Weinstein theorems, we want to find a local model for the neighbourhood of a submanifold N in M in which the local structure is simplified. In the case of the well-known theorems mentioned, what is done is a substitution of a neighbourhood of the manifold near a point with a vector space with an adequate choice of chart/local coordinates (which is equivalent to find special local coordinates around that point):

- in the case of Frobenius, we foliate the manifold by integral submanifolds which locally have the vector structure of the distribution;
- for Darboux's theorem, near a point, we approximate the structure of the manifold with that of the canonical symplectic structure on $\mathbb{R}^m$;
- in Weinstein's splitting theorem, the local model correspond to the sum of a symplectic portion which, by the previous item, is the canonical symplectic structure, and a 'remainder' portion, such that the structures are disjoint.

Thus, when trying to find a splitting in the more general context of anchored vector bundles, we want to find a normal form of the geometric structure near $N \subseteq M$ and, similar to before, we try to substitute a neighbourhood of N in M by a vector bundle over N by an adequate diffeomorphism. This notion corresponds precisely to the notion of a tubular neighbourhood of N in M.

As a corollary of Theorem 8.2.1, we obtain the aforementioned splitting theorem for anchored vector bundles:

Corollary 8.2.2. *Let* $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ *be an involutive anchored vector bundle,* $p \in M$ *and* $N \subseteq M$ *a submanifold containing* p *such that* $T_pM = T_pN \oplus \mathsf{a}(E_p)$*. Then,* E *is isomorphic, around* p*, to the direct product* $i^!E \times T(\mathsf{a}(E_p))$*.*

Proof. Firstly, note that, for a sufficiently small open set $U \subseteq N \subseteq M$ containing p, $TM|_U = U \times T_pM$, therefore

$$\nu(M,N)|_U = TM|_U/TN|_U \cong (U \times (T_pN \oplus \mathsf{a}(E_p)))/(U \times T_pN) \cong U \times \mathsf{a}(E_p).$$

Therefore, the projection $\mathrm{npr}_N : \nu(M,N) \to N$ is, around $p \in U$, the projection onto the first factor and, by the properties of the pull-back, we have that $\mathrm{npr}_N^! i^! E = i^! E \times T\mathsf{a}(E_p)$. Applying the previous theorem, we obtain an isomorphism of anchored vector bundles between $E|_N$ and $i^!E \times T\mathsf{a}(E_p)$. Q.E.D.

As it was done in the case of Poisson manifolds, we can show that the transverse structures arising from the splitting theorem are actually unique (up to isomorphism). To obtain those results, we introduce the following notions:

Definition 8.2.3. Let M, Q be smooth manifolds, $\chi : M \to Q$ a submersion and denote $M_q = \chi^{-1}(q)$, for $q \in Q$. A *family of anchored vector bundles* $E_q \to M_q$ is an anchored vector bundle $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ such that $E|_{M_q} = E_q$ and $\mathsf{a} : E \to TM$ is tangent to $\ker(\chi_*)$, i.e. $\mathrm{im}(\mathsf{a}) \subseteq \ker(T\chi)$.

Definition 8.2.4. A family of anchored vector bundles is said to be *infinitesimally trivial* if every vector field $X \in \mathfrak{X}(Q)$ admits a lift $Y \in \mathfrak{X}(M)$ (such that $T\chi \circ Y = X \circ \chi$) which is the base vector field of some $\hat{Y} \in \mathfrak{aut}_{\mathbf{AVB}}(E)$.

Proposition 8.2.5. *Let* $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ *be an involutive anchored vector bundle,* $\chi : N \to Q$ *a submersion and* $\phi : N \to M$ *a family of transversals, i.e. a smooth map such that* $i_q = \phi|_{N_q} : N_q = \chi^{-1}(q) \hookrightarrow M$ *is a family of transversals, for all* $q \in Q$*. Then, the family of anchored vector bundles* $i_q^!E \to N_q$ *is infinitesimally trivial.*

Idea of the Proof. Using the fact that $(\phi^!E, \phi^!\mathsf{a})$ is involutive, by Proposition 3.1.19, we can show that the fibres N_q are transversals for $\phi^!E$. Then, this transversality implies that $\psi_* \circ \phi^!\mathsf{a} : \phi^!E \to TQ$ is fibre-wise surjective. Thus, given any $X \in \mathfrak{X}(Q)$, we may find a section $\sigma \in \Gamma(\phi^!E)$ such that $\psi_* \circ \phi^!\mathsf{a}(\sigma) = X \circ \psi$. Since $\phi^!E$ is involutive, $\phi^!\mathsf{a}$ admits a lift $\widehat{\phi^!\mathsf{a}} : \Gamma(\phi^!E) \to \mathfrak{aut}_{\mathbf{AVB}}(\phi^!E)$, by Proposition 8.1.21. To conclude the result, one needs only to show that $\widehat{\phi^!\mathsf{a}}(\sigma)$ restricts to an element of $\mathfrak{aut}_{\mathbf{AVB}}\Big(\coprod_{q \in Q} i_q^! E\Big)$. Q.E.D.

Now, we get to the more interesting result, the following corollary which mirrors the uniqueness of transversals for Poisson structures:

Corollary 8.2.6. *Let* $(E \xrightarrow{\mathrm{pr}} M, \mathsf{a})$ *be an involutive anchored vector bundle,* $S \subseteq M$ *a leaf (i.e. an integral submanifold of* $\mathsf{a}(E)$*) and* $p_0, p_1 \in S$*. Moreover, let* $N \xhookrightarrow{i_n} M$ *be transversals for* (E, a) *with* $\dim(N_n) = \dim(M) - \dim(S)$ *and* $i_n(N) \cap S = \{p_n\}$*, for* $n = 0, 1$*. Then, up to shrinking* N_n *around* p_n*, there is an isomorphism of (induced) anchored vector bundles* $i_0^! E \to i_1^! E$ *taking* p_0 *to* p_1*.*

Idea of the Proof. Start by constructing a path $\gamma : \mathbb{R} \to S$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$ and a family of transversals $i_x : N_x \hookrightarrow M$ with $N_x \cap S = \{\gamma(x)\}$. Then, the union of these transversals gives a smooth map

$$\begin{aligned} N \times \mathbb{R} &\longrightarrow \mathbb{R} \times M \\ N_y \ni q &\longmapsto (y, i_y(q)). \end{aligned}$$

Call $\psi : N \times \mathbb{R} \to \mathbb{R}$ and $\phi : N \times \mathbb{R} \to M$ the projections onto the first and second factors of the displayed map above, respectively. Notice that ψ is a submersion, $\psi \circ \gamma = \mathrm{id}_{\mathbb{R}}$ and, additionally, $\mathrm{im}(\gamma)$ is integral submanifold of $\phi^! \mathsf{a}(\phi^! E)$.

By (the proof of) Proposition 8.2.5, there exists $\sigma \in \Gamma(\phi^! E)$ such that $\hat{Y} = \widehat{\phi^! \mathsf{a}}(\sigma) \in \mathfrak{aut}_{\mathbf{AVB}}(\phi^! E)$ restricts to an element of $\mathfrak{aut}_{\mathbf{AVB}}\left(\coprod_{x \in \mathbb{R}} i_x^! E\right)$ and its projection to $\mathfrak{X}(N)$, $Y = \phi^! \mathsf{a}(\sigma)$, satisfies $T\psi \circ Y = \frac{\partial}{\partial x} \circ \psi$ and is tangent to the integral manifolds of $\phi^! \mathsf{a}(\phi^! E)$ (Y is one of the generators of $\phi^! \mathsf{a}(\phi^! E)$). Thus, for every $x \in \mathbb{R}$ there exists an open subset $U_x \subseteq N_x$, which is a neighbourhood of $\gamma(x)$, where the flow Fl_Y^t is defined for $t = 1$. Since $\coprod_{x \in \mathbb{R}} i_x^! E \xrightarrow{\phi^! \mathrm{pr}} N$ is such that $i_x^! E \xrightarrow{(\phi^! \mathrm{pr})|_{i_x^! E}} N_x$ and $\mathrm{Fl}_Y^1(i_x^! E) \subseteq i_{x+1}^! E$, the flow of $\hat{Y}$, $\mathrm{Fl}_{\hat{Y}}^t : \coprod_{x \in \mathbb{R}} i_x^! E \to \coprod_{x \in \mathbb{R}} i_x^! E$, which restricts to $\mathrm{Fl}_{\hat{Y}}^t|_{i_x^! E} : i_x^! E \to i_{x+t}^! E$, can also be defined for $t = 1$ on a small enough neighbourhood of $\gamma(x)$ in N_x. Therefore, $\mathrm{Fl}_{\hat{Y}}^1|_{i_0^! E} : i_0^! E \to i_1^! E$ produces the desired isomorphism. Q.E.D.

Chapter 9

Conclusion

In this work, we have studied the several different geometrical structures, from tangent distributions and symplectic geometry to Poisson and Dirac manifolds and Lie algebroids, in order to better understand the local structure of these different geometries.

We started by the simplest of the above mentioned structures: tangent distributions. These are defined to be subsets of the tangent bundle whose fibres at points of the manifolds are vector subspaces of the tangent spaces at those points. We then studied in detail the case where these distributions are both smooth and with constant dimensions throughout the fibres, the so-called regular distributions, including the notions of involutivity and integrability, which reflect the possible Lie subalgebra structure of the sections of the distribution and the possible existence of submanifolds whose tangent spaces are the fibres of the distribution, respectively. The objective of the chapter was to prove Frobenius' theorem, which shows that the notions of involutivity and integrability of distributions are equivalent, for regular distribution as well as for singular distributions (for the latter, so long as we add that the rank of the distribution stays constant along the flow of sections generating the distribution).

Following the discussion of distributions, we studied anchored vector bundles and Lie algebroids which are the natural extension of vector bundles into bundles that have some type of anchor to the tangent bundle of the base manifold. In the case of Lie algebroids, the additional structure of a Lie algebra on the set of sections gives it a additional importance, not only as the geometrical structure present in Yang-Mills theories in physics but, in the mathematical side, as the infinitesimal version of Lie groupoids, a nice way to link back to a structure (these Lie groupoids) which often gives a better geometrical picture of certain phenomena. Both anchored vector bundles and Lie algebroids are known to have normal form results and, in the case of the former, we give an overview of a splitting theorem around transversals in the last chapter of the thesis, following the treatment done in [1].

The fourth chapter deals with the first geometric structure to be highly influenced by physical theories: Symplectic geometry. As it is well-known, symplectic manifolds are obtained by introducing a closed, non-degenerate 2-form on a manifold, mirroring the case of Riemannian manifolds where one introduces a symmetric element of $\Gamma(\bigotimes^2 T^*M)$ which is fibre-wise positive definite. For these manifolds, one can define many types of important submanifolds and obtain many different results, however none larger that Darboux's theorem, which describes how, locally, all symplectic forms have a similar canonical form.

In the following two chapters, we explored Poisson manifolds and their local structure. Also a geometry based on physics, a Poisson manifold is obtained by equipping a manifold M with a bilinear, anti-symmetric bracket on $C^\infty(M)$ satisfying the Leibniz rule and the Jacobi identity, or, equivalently, by introducing a bivector field on M whose Schouten bracket with itself vanishes. The introduction of this structure turns the cotangent bundle T^*M into a Lie algebroid and we also showed that Lie algebroid structures on a vector bundle $A \to M$ are in one-to-one correspondence with fibre-wise linear Poisson structures on A^*. This geometry can be described as a foliated manifold whose leaves of the foliation are symplectic manifolds and, additionally, the transverse structure to the leaves of the so--called symplectic foliation of the manifold is unique up to isomorphism. Regarding the local structure of Poisson manifolds, Weinstein's splitting theorem shows that, around each point of a Poisson manifold, the Poisson bivector field can be written as the sum of a non--degenerate Poisson structures (which is equivalent to a symplectic one) and a Poisson structure vanishing at a point, thus greatly simplifying the local structure of such a manifold.

The final geometry we discussed was that of Dirac manifolds. These also arise from physical theories but now of Hamiltonian systems with constraints. Mathematically, this means that Dirac manifolds are the usual structure arising in submanifolds of Poisson manifolds and, since submanifolds of symplectic manifolds are usually pre-symplectic, a Dirac manifold can be defined as a foliated manifold whose leaves are pre-symplectic. A different (and in some sense easier to understand) description of Dirac manifolds arises by considering it to be a subset of the generalised tangent bundle $\mathbb{T}M = TM \oplus T^*M$ which is closed under the Dorfman bracket and Lagrangian with respect to the natural pairing on $\mathbb{T}M$. This more algebraic description then allowed us to explore several properties of these structures, such as the two distinct descriptions of morphisms of Dirac structures and the aforementioned pre-symplectic foliation of a Dirac manifold.

Summarising, knowing the local structure of different types of geometries gives us a better understanding of the geometry we are studying and, even when not dealing with these kinds of problems, it is still worth understanding the methods employed in proving normal form results. Even though this thesis showcases some of these results, many more can still be uncovered around different types of submanifolds of these and other more generalised geometric structures.

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