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MSc

Hecke Algebras and Topological Quantum Field Theories

Bernardo Mariz Ferreira Leite da Cunha

Dissertação de Mestrado apresentada à
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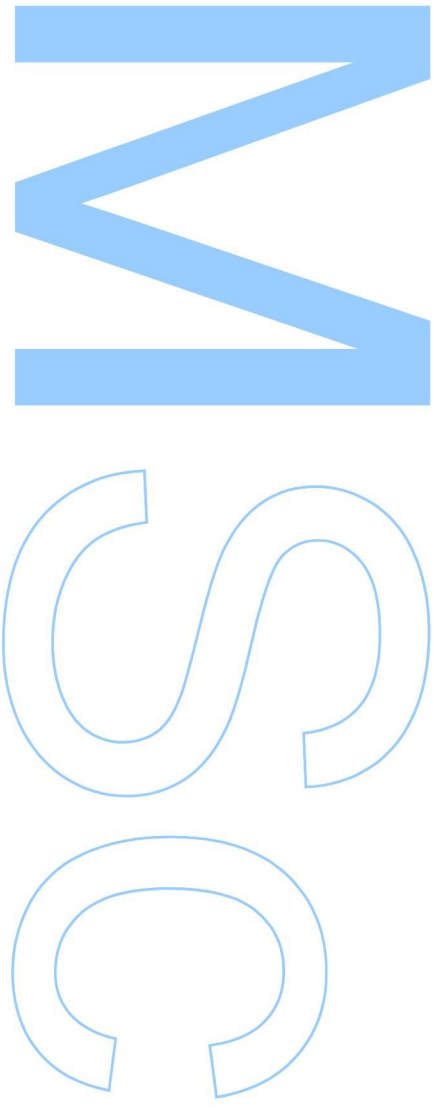
MSc

Hecke Algebras and Topological Quantum Field Theories

Bernardo Mariz Ferreira Leite da Cunha
Mestrado em Matemática
Departamento de Matemática
2022

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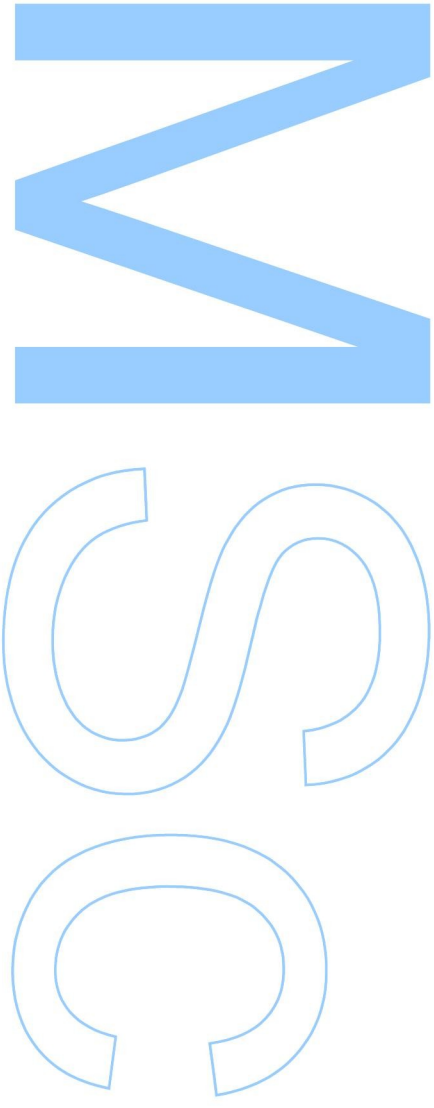
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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, _____/_____/_____



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Abstract

The goal of this work is to summarise and clarify some of the major results of a paper by Fock, Tatitscheff and Thomas titled *Topological quantum field theories from Hecke algebras* [FTT21].

We start by constructing certain topological invariants for a class of smooth marked surfaces named "ciliated surfaces" by use of triangulations (so, for all purposes, we may drop the smoothness requirement and look at the surfaces as combinatorial objects) and with the aid of finite dimensional Hecke algebras over $\mathbb{Z}\left[v^{\pm 1}\right]$, that is, algebras which are deformations of the group algebra $\mathbb{Z}W$ of a finite Coxeter group W.

We generalise these invariants into a more general functor named a "Topological Quantum Field Theory" (TQFT for short), whose target category is the subcategory of $\mathbb{Z}\left[v^{\pm 1}\right]\,\mathsf{Alg}$ whose objects are tensor products of copies of the Hecke algebra and its dual (with any number of factors).

There is one such TQFT on this class of geometric objects for each Hecke algebra. The TQFTs allow us to compute topological invariants for ciliated surfaces if we assign labels to their boundaries and in particular no additional data is required in the case of a closed surface, for which the TQFTs themselves are the original topological invariants defined earlier. We also obtain expressions for these invariants using an arbitrary basis of the Hecke algebra and its dual basis,

Using tools from the representation theory of symmetric algebras (which is a class encompassing Hecke algebras), we obtain some explicit expressions for the invariants using Schur elements and central elements corresponding to irreducible characters of the Hecke algebra, which are easier to compute than the expressions obtained earlier.

Resumo

O objetivo deste trabalho é resumir e clarificar alguns dos principais resultados de um artigo escrito por Fock et al. intitulado *Topological quantum field theories from Hecke algebras* [FTT21].

Começamos por contruir invariantes para uma classe de superfícies diferenciáveis com pontos marcados denominadas "superficies ciliadas" usando triangulações (como tal, para todos os efeitos, podemos ignorar o rquisito de diferenciabilidade para as superfícies e tratá-las como objetos combinatórios) e através do uso de álgebras de Hecke de dimensão finita sobre $\mathbb{Z}\left[v^{\pm 1}\right]$, isto é, álgebras que são deformações da álgebra de grupo $\mathbb{Z}W$ para um grupo de Coxeter finito W.

Generalizamos estes invariantes de modo a obter um functor mais geral denominado Teoria Quântica de Campo Topológica (abreviado para TQFT), cuja categoria alvo é a subcategoria de $\mathbb{Z}\left[v^{\pm 1}\right]-\mathsf{Mod}$ cujos objetos são produtos tensoriais de cópias da álgebra de Hecke e do seu dual (com um número arbitrátio de fatores).

Existe uma TQFT nesta classe de objetos geométricos para cada álgebra de Hecke. A TQFT permite-nos calcular invariantes topológicos para superfícies ciliadas, desde que associemos rótulos às suas componentes de fronteira, e em particular não são necessários mais dados no caso de uma superfície fechada, caso em que recuperamos o invariante construído anteriormente através desta TQFT. Obtemos também expressões para estes invariantes em função dos elementos de uma base arbitrária da álgebra de Hecke e da sua base dual.

Utilizando ferramentas da teoria de representação de álgebras simétricas (que são uma classe que abrange as álgebras de Hecke), obtemos expressões explícitas para os invariantes à custa dos elementos de Schur e dos elementos centrais associados a caracteres irredutíveis da álgebra de Hecke, que são mais fáceis de calcular relativamente às expressões obtidas anteriormente.

Contents

1 Introduction **1**

2 Preliminaries **5**

2.1 Ciliated surfaces 5

2.2 Coxeter groups and Hecke algebras 13

3 The Hecke invariant **21**

3.1 Construction of the invariant 21

3.2 Cutting surfaces 27

3.3 Some examples of invariant contructions 30

4 Hecke topological quantum field theory **37**

4.1 Axioms for a Topological Quantum Field Theory 37

4.2 Generalising the invariant 40

4.3 The tensor invariant as a TQFT 50

5 Explicit expressions for the Hecke TQFT **55**

5.1 Representation theory 55

5.2 Explicit expressions 66

Chapter 1

Introduction

The interplay between algebra and geometry has been a subject of study since antiquity, and in a sense they even were the same subject in the time of Babylon and Ancient Greece. After maturing into different subjects, Algebra and Geometry still maintained a symbiotic relationship, whereby developments in one area were frequently a result of trying to solve problems pertaining to the other, throughout the whole history of mathematics.

In modern mathematics, one of the ways we witness the interaction between Algebra and Geometry is by the study of the algebraic properties of geometric objects, such as the existence of a group law on elliptic curves, or the study of groups of morphisms on a given category of geometric objects; and, more importantly for us, the construction of algebraic invariants for geometric and topological objects, such as the fundamental group of a topological space, as well as homology and cohomology in its various forms.

In this work we aim to summarise and clarify the main results of [FTT21], framing it from a slightly more basic point of view. Whereas [FTT21] focuses on the geometrical origins and interpretation of Iwahori-Hecke algebras, particularly by introducing a graphical calculus to compute the invariants, which is intimately tied with the geometric aspects of the algebraic objects, we forgo that entirely, preferring to leave it in a purely algebraic viewpoint, and focusing more on the use of algebraic tools (notably, representation theory) to aid us.

In Chapter 2 we give some background notions of the geometric and algebraic objects we will be using throughout the work. After we give the definition of a Coxeter group, and in the remainder of the text, all Coxeter groups will be assumed finite without further explanation, and therefore all Hecke algebras will be modules of finite rank.

In Chapter 3 we define invariants for the surfaces under consideration. We first define our invariants for triangles, assigning constants to each triangle, depending on the labelling of the edges, where a label is an element of a Coxeter group W; and then use a triangulation T of our ciliated surface Σ and define the invariant for it by using the invariants for the

triangles and summing them over all labels, as follows:

$$P_{\Sigma,W,\mathsf{T}}(v) = \sum_{\lambda \in \Lambda(\mathsf{T})} \prod_{f \in \mathsf{T}} c_f^{\lambda}(v) \in \mathbb{Z}[v^{\pm 1}],$$

where the product ranges over all faces of the triangulation and the sum is over all labels for the edges of the triangulation. Such a sum is usually called a *state-sum*. Above, we denote the constant associated to the face f when using the labels given by λ by $c_f^{\lambda}(v)$.

It turns out that this state-sum is independent of the chosen triangulation and is, therefore, an invariant of the ciliated surface. Moreover, the invariant is well behaved for the operation of cutting a surface in two along a path through some punctures.

This leads us to the central idea of Chapter 4, where we generalise this invariant with the cutting property in mind, aiming to construct a Topological Quantum Field Theory.

In this generalisation, we work with an arbitrary symmetrising trace of the Hecke algebra, and an arbitrary basis of the Hecke algebra along with its dual basis. We assign tensors to triangles and then to surfaces, rather than scalars. The way we do that allows us to prove that the new tensor invariants are still well defined on surfaces independently of the triangulation, and that they behave well under gluing.

Given a Hecke algebra $\mathcal{H}$ associated to a Coxeter system (W, S), the tensor associated to a ciliated surface is an element of $\mathcal{H}^{\otimes n} \otimes \mathcal{H}^{*\otimes m}$, where the surface has n negatively oriented boundary segments and m positively oriented boundary segments. In particular, a closed surface admits an invariant which is still a scalar, as in the previous chapter.

We also find a way to compute the invariants for closed surfaces:

Theorem (4.2.5). *The invariant for the closed ciliated surface $\Sigma_{g,k}$ is*

$$P_{g,k,W} = \mathsf{tr}\left(\left(\sum_{w \in W} C_w C_w^{\vee}\right)^{k-1} \left(\sum_{x,y \in W} C_x C_y C_x^{\vee} C_y^{\vee}\right)^{g}\right). \tag{1.0.1}$$

Above, $\{C_w\}_{w \in W}$ is an arbitrary basis of $\mathcal{H}$ and $\{C_w^{\vee}\}_{w \in W}$ is the basis of $\mathcal{H}$ such that $\mathsf{tr}(C_w C_v^{\vee}) = \delta_{vw}$.

In Theorem 4.3.1 we also prove that this operation is a functor satisfying the axioms for a TQFT as defined by Atiyah in [Ati88].

Finally, in Chapter 5, we develop some representation theory to aid us, and use it to show we may also express these invariants in terms of the Schur elements $\{s_\lambda\}_{\lambda \in \mathsf{Irr}(\mathcal{H})}$ of the Hecke algebra, and the central elements $\{Z_\lambda\}_{\lambda \in \mathsf{Irr}(\mathcal{H})}$ that correspond to irreducible characters in a natural way. In this way, the computations become much more feasible as the expression is simpler and there are fewer computations that need to be made.

The main theorems in this section are the following, where if we denote an irreducible representation by λ, the associated simple module is V_λ, the associated character is χ_λ and the associated Schur element is s_λ:

Theorem (5.2.3). *The invariant for the closed ciliated surface $\Sigma_{g,k}$ is given by*

$$P_{g,k,W} = \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{2g-2+k}.$$

Theorem (5.2.4). *Consider a ciliated surface $\Sigma = \Sigma_{g,k,\{c_1,\ldots,c_n\}}$ with pre-assigned labels. Denote by h_i a product of the labels of the i-th boundary circle following the orientation induced by Σ. Then the invariant for this surface is*

$$P_{\Sigma,W} = \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{2g-2+k+n} \left(\prod_i \chi_\lambda(h_i) \right).$$

Chapter 2

Preliminaries

In this chapter we introduce the tools we will need to build our invariants. We start by defining our geometric objects of interest and some of the geometric constructions we will need. We bring attention to the fact we define a non standard notion of triangulation.

Following that, we introduce the algebraic objects we will work with, namely Coxeter groups and Iwahori-Hecke algebras, and prove some basic yet important results, with the focus being on proving that a Hecke algebra, defined as an alegbra over the ring $\mathbb{Z}\left[v^{\pm 1}\right]$, is in fact a free module over the same ring.

2.1 Ciliated surfaces

Definition 2.1.1. A *ciliated surface* is a compact and connected topological oriented surface with (possibly empty) boundary consisting of $n \geq 0$ disjoint circles, together with a finite set of marked points on its interior, called *punctures* and on its boundary, called *cilia*. For our purposes we require that all boundary components have at least one cilium.We denote the total number of cilia by c.

It is known that a topological surface always admits a differentiable structure and that two smooth 2-manifolds are diffeomorphic if and only if they are homeomorphic, therefore we may consider ciliated surfaces to be smooth surfaces, which is useful to later make the connection to Atiyah's original definition of a Topological Quantum Field Theory, which we will see on Chapter 4.

An isomorphism of ciliated surfaces is an orientation-preserving homeomorphism of the underlying topological surfaces which maps punctures to punctures and cilia to cilia. With the above in mind, we may also see it as a diffeomorphism between the ciliated surfaces which maps punctures to punctures and cilia to cilia.

With these definitions and remarks we may present a classification of the ciliated surfaces.

Theorem 2.1.2. *The isomorphism class of a ciliated surface is only dependent on its genus, the number of punctures, and the distribution of cilia on the boundary components.*

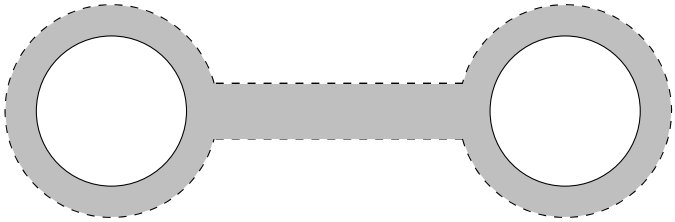
For this reason we abuse notation and talk about the ciliated surface of a particular genus, with a particular number of punctures, and a particular distribution of cilia on the boundary components.

Sketch of Proof. The first part of this proof is to show that a closed, compact and connected topological surface is *strong p-fold transitive*, which informally means that we may take two sets of points $x_1, \ldots, x_p$ and $y_1, \ldots, y_p$ in a topological surface and find a homeomorphism that maps x_i to y_i. The argument is based on [Ban97, Lemma 2.1.10], but whereas this reference proves it in the differentiable manifolds setting, we are working with general topological surfaces, which makes the proof a bit easier.

Fix a closed, compact and connected topological surface Σ. Given just two points x and y on Σ we may prove the existence of a homeomorphism of Σ which swaps the two as follows: consider a path joining x and y inside Σ, which certainly exists by connectedness of Σ. Using the fact that Σ is compact, we "widen" the path to turn it into a strip containing both x and y, and now we can certainly find a homeomorphism of Σ which interchanges x and y and whose support is entirely contained in the strip.

We may do the same for a set of p pairs of points by simply joining each pair by a path and making sure none of the paths intersect. Then, as above, we may create strips around the paths in such a way that they also do not intersect, and find a homeomorphism swapping the first pair of points whose support is contained in the first strip, then a homeomorphism swapping the second pair of points whose support is contained in the second strip, and so on. Given that the supports do not intersect, we may compose all these homeomorphisms, ensuring each one does not interact with the pairs of points "it is not supposed to", and thus we have a homeomorphism as we described.

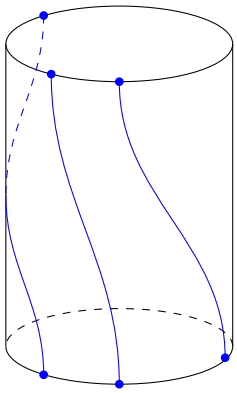
The second thing to prove is that for a topological surface with boundary we may find homeomorphisms that act strong p-fold transitively on the boundary components. The proof is in every respect similar to the above, where instead of creating strips containing a pair of points, we create a strip joining bands around two boundary components, as illustrated below:



As before, we now find a homeomorphism swapping the pair of boundary components and whose support is contained in the shape marked in grey. Now we can swap any number of pairs of boundary components in the same manner as we did above, by composing such homeomorphisms.

The final assertion we need to check is that two boundary components with the same number of cilia are homeomorphic, in order to be able to send cilia to cilia via a homeomorphism of the whole surface.

This is of course quite easy, since "rotating" cilia around their boundary component can be done in a continuous manner. In fact, this can be done via homotopies, and we illustrate a simple example below.



Putting all this together, we can relate two homeomorphic topological surfaces with the same number of punctures and same cilia distribution via an isomorphism of ciliated surfaces.

With this we can justify that the classes of isomorphism are uniquely determined by genus, number of boundary components (as these two together determine the class of homeomorphisms of surfaces if we do not account for marked points), number of punctures, and distribution of cilia by the boundary components. □

Thus we may work with a generic surface for each class of isomorphism and henceforth we denote by $\Sigma_{g,k,\{p_1,\cdots,p_n\}}$ the generic ciliated surface of genus g with k punctures and n

boundary circles, the first of which has c_1 cilia, the second of which has c_2 cilia and so on. In case the surface has no boundary we omit the boundary part and denote it by $\Sigma_{g,k}$.

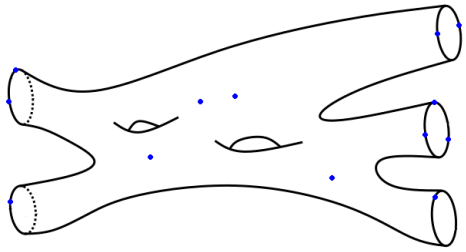


Figure 2.1: The ciliated surface $\Sigma_{2,4,\{2,1,2,3,1\}}$

Remark 2.1.3. When it comes to the distribution of the cilia by the boundary components, the order of the c_i is irrelevant to define the isomorphism class. This means that the surface on Figure 2.1 can also be denoted by $\Sigma_{2,4,\{1,1,2,2,3\}}$, for instance. On the other hand, the repeated numbers are important, as they signify that there is a given number of boundary components with the same number of cilia, therefore the surface represented in the image is not the same as $\Sigma_{2,4,\{1,2,3\}}$. In conclusion, the distribution of cilia across boundary components is denoted by a multiset.

Definition 2.1.4. A *triangulation* T of a ciliated surface Σ is a decomposition of Σ into triangles such that every vertex of a triangle is either a cilium or a puncture. We admit the following degenerate cases:

- two edges of a triangle are identified with each other (and so the two common vertices are also identified);
- two vertices of a triangle are identified with each other (but no edges are);
- all three vertices of a triangle are identified with each other (but no edges are).

We note that because of this, the triangulations mentioned in this work are a generalisation of the usual notion of triangulation of a topological surface. Edges of T that belong to the boundary of Σ are said to be *external* and all others are *internal*.

In this work we will only consider ciliated surfaces for which $k + n \geq 1$, and such that if $g = 0$ then either $k + n \geq 3$ or $k + n = 2$ and $c \geq 1$ or $k + n = 1$ and $c \geq 3$, since the other cases are ciliated surfaces which do not admit triangulations.

Let us see why that is. Firstly we need a powerful Lemma which will allow us to complete this classification.

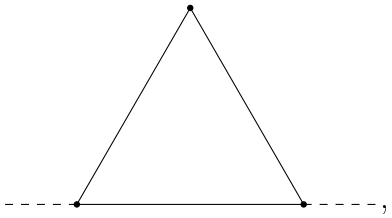
Lemma 2.1.5. *If we take a surface admitting a triangulation and we perform any of the following operations, the resulting surface still admits a triangulation:*

- *Adding a puncture;*
- *Adding a cilium to a boundary component;*
- *Transforming a puncture into a boundary component with one cilium.*

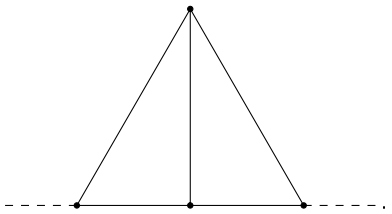
Proof. We will prove this works for each of those operations separately.

Adding a puncture: If we take a surface and a triangulation of it, we may add a puncture to the interior of one of the triangles and connect it to each of the three vertices to form a triangulation of the new surface.

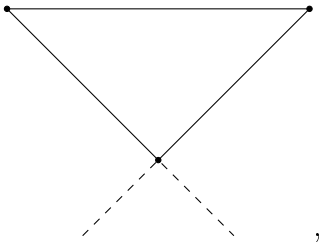
Adding a cilium to a boundary component: Since each boundary component has at least one cilium already, on a triangulation we always have something of the following form:



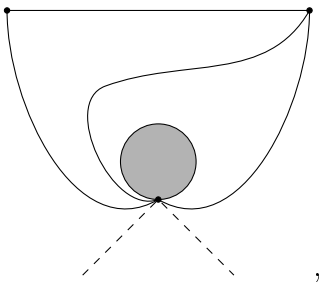
where the line in the bottom represents part of the boundary component (including the case where the points are identified and we have a single cilium) and the top vertex may be another cilium or a puncture. We can then insert a cilium in the boundary component and complete the triangulation as follows:



Transforming a puncture into a boundary component with one cilium: In this case the drawing becomes a little more complicated. Taking a puncture of a triangulated surface, we have the following:



that is, at least a triangle which has the puncture as a vertex, and possibly some other edges coming out of it. If we take the puncture and transform it into a boundary component, we need to add another edge to have a valid triangulation, otherwise we would end up with a quadrilateral. We depict that here



where they greyed out circle means the interior of this circle is not part of the surface (that is, it is the "hole" left by creating a boundary component). Note that there are two triangles in place of the one in the previous figure: the one formed by the top edge, the left edge, and the middle edge coming from the new cilium, and a second one formed by the middle edge, the right edge, and the boundary circle itself, where two of the vertices are the same cilium. □

This is a very powerful Lemma, which lets us verify some combinations of n, k and c are valid and then the rest of the classification is just adding punctures or cilia to these ones, or creating boundary components from punctures.

Theorem 2.1.6. *All of the following families of surfaces admit triangulations:*

(1) $g > 0$, $k + n \geq 1$;
(2) $g = 0$, $k + n \geq 3$;
(3) $g = 0$, $k + n = 2$, $c \geq 1$;
(4) $g = 0$, $k + n = 1$, $c \geq 3$.

Moreover, this is a complete classification, meaning all other ciliated surfaces (of which only a finite number exists, if we discount surfaces with no marked points) do not admit triangulations.

Proof. We start by showing all of those families admit triangulations.

For family (1) we only need to show it holds for $g > 0$, $n = 0$, $k = 1$, and all other cases are covered by extending this surface using the operations of the previous lemma. We can realise $\Sigma_{g,1}$ as a $4g$-gon identifying the sides as in Figure 2.2, illustrating the case $g = 2$, and then we can apply the triangulation shown also in the same Figure.

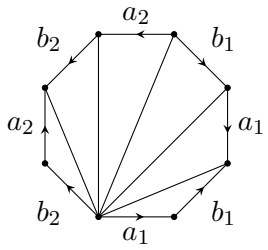
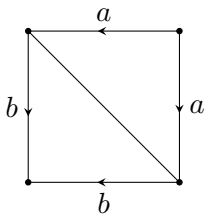


Figure 2.2: Gluing and triangulation of the surface $\Sigma_{2,1}$

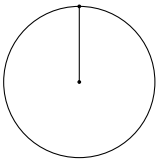
This shows the surface $\Sigma_{g,1}$ admits a triangulation, and as stated before, we use the Lemma to extend the result to the entire family.

For family (2) we do the same procedure: we show it holds for $g = 0$, $k = 3$, $n = 0$, and all other surfaces in the family are extensions of this one by using the operations that we stated before. A valid triangulation is

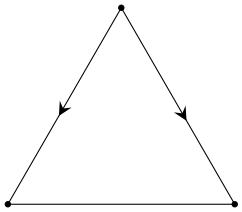


Again, this extends to the whole family and we are done with family (2).

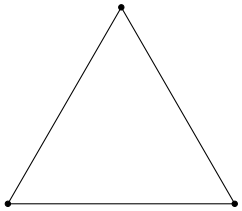
To prove the result holds for family (3) we again only need to prove it for one case: $g = 0$, $k = 1$, $n = 1$, $c = 1$, and all others are obtained from this. Note that a surface of genus 0 with a boundary component is just a disk on the plane. We can use the following triangulation:



where two edges are the same. To make it clearer, we can use a triangle in which we identify two edges:



Finally we move on to the last family. Note that $c \geq 3$ implies there is at least one boundary component, therefore $k + n = 1$ in fact means $k = 0$, $n = 1$, and so this family is actually $g = 0$, $k = 0$, $n = 1$, $c \geq 3$. Again, we prove the result holds for $g = 0$, $k = 0$, $n = 1$, $c = 3$ and to get all others we simply add cilia to the boundary component. A triangulation of the surface in question is just a triangle with no edges or vertices identified:



We thus have proved that all of these families admit valid triangulations, and now we are left to check all other cases yield surfaces which do not.

For any surface, if $k + n = 0$ we can never have a valid triangulation, because we have no vertices.

If $g = 0$, $k + n = 2$ and $c = 0$, we have that $n = 0$ and so we have a sphere with two punctures. It is clear that no matter how many lines we join the punctures with, always have "one sided" or "two sided" shapes, therefore, we cannot have a valid triangulation.

Finally, if $g = 0$, $k + n = 1$ and $c \leq 2$ we have three cases, $c = 0$, $c = 1$ and $c = 2$.

For the first of these, we are left with a sphere with a single puncture, and for the same reason as before, we have no triangulations.

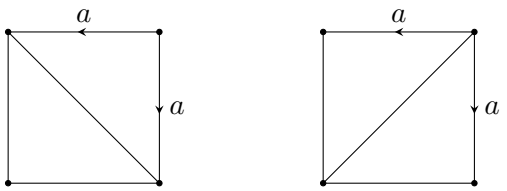
On the other two, we have no punctures and a single boundary component, so we end up with a circle with one or two cilia in its boundary, and in both cases we again only get "two sided" or "one sided" shapes, and no triangulation is possible.

This exhausts all the possibilities for ciliated surfaces and the classification according to the existence or not of triangulations is complete. □

We will need the following proposition throughout this work, which can be found in [Hat91] or in [Mos88].

Proposition 2.1.7. *Given a ciliated surface and two triangulations we may always reach one from the other via a finite number of diagonal flips, where a diagonal flip consists of taking a quadrilateral formed by two adjacent triangles, deleting the common edge of the triangles, and adding the opposite diagonal of the quadrilateral to the triangulation.*

We remark that this result is only valid for the standard notion of triangulation. But a short case by case analysis of the degenerate cases we allow, shows we never introduce new degenerate cases. We illustrate one such case, which is the case where one of the triangles of a square has two edges which are identified. When we swap the diagonal we end up with two non degenerate triangles who share two edges, and we still remain with a valid triangulation.



2.2 Coxeter groups and Hecke algebras

Definition 2.2.1 (Coxeter Group)**.** A *Coxeter group* W is a group presented in the form $W = \langle s \in S \mid (s_i s_j)^{m_{ij}} = e \rangle$ where $S = \{s_1, \ldots, s_n\}$ and $M = (m_{ij}) \in M_n(\mathbb{Z}_{>0} \cup \{\infty\})$ is symmetric and with all diagonal entries equal to 1 and with all other entries strictly larger than 1; such a matrix is called a *Coxeter matrix*. The rank of W is the natural number n. The pair (W, S) is called a *Coxeter system*.

Remark 2.2.2. What is meant by $m_{ij} = \infty$ is that s_i and s_j are not subject to any relation of that kind.

Remark 2.2.3. The entries of the main diagonal all being equal to 1 is equivalent to having $s_i^2 = e$ for all i, and the off-diagonal entries being greater than 1 means that no product of two different generators is the identity. Note also that the relation $(s_i s_j)^{m_{ij}} = e$ is equivalent to $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ where each member has m_{ij} elements. This relation is called a braid relation. In case $m_{ij} = 2$, we obtain that the generators s_i and s_j commute.

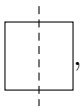
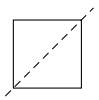
Given $x \in W$ a word/expression for x is a relation $x = s_1 s_2 \cdots s_n$ with $s_1, s_2, \ldots, s_n \in S$. We say that such a word is reduced if the number of elements in the expression, n, is minimal among all expressions for x. That number is said to be the *length* of x, and is denoted by $\ell(x)$.

Example 2.2.4 (Dihedral Groups)**.** The group of symmetries of an equilateral triangle, $\mathrm{Symm}(\Delta)$, is isomorphic to the group with 6 elements with the presentation $\langle s, r \mid s^2 = r^2 = e, srs = rsr \rangle$.

A possible isomorphism is the following:

$s \leftrightarrow$  , $r \leftrightarrow$ 

We can do the same for a square. This time $\mathrm{Symm}(\square) \cong \langle s, r \mid s^2 = r^2 = e, srsr = rsrs \rangle$, with the isomorphism given by

$s \leftrightarrow$  , $r \leftrightarrow$ 

This can be generalised to the symmetries of a regular n-sided polygon. In fact $D_{2n} \cong \langle s, r \mid s^2 = r^2 = (sr)^n = e \rangle$, where the isomorphism is constructed by mapping s to any reflection and r to an adjacent one. In this context the groups is denoted by $I_2(n)$, where 2 is the rank of the Coxeter group.

Example 2.2.5 (More examples)**.** We may also take an "algebraic limit" of the groups from the previous examples, obtaining $U_2 = \langle s, r \mid s^2 = r^2 = e \rangle$. We remark that this is not the same as the symmetry group of a circle. This group is the "universal Coxeter group of rank 2". This name comes from the fact that all Coxeter groups of rank 2 are homomorphic images of U_2.

We can generalise even further, by varying the rank: $U_n = \langle s_1, s_2, \ldots, s_n \mid s_1^2 = s_2^2 = \cdots = s_n^2 = e \rangle$ is the "universal Coxeter group of rank n". As before, this name comes from the fact that all Coxeter groups of rank n are homomorphic images of U_n.

All universal Coxeter groups are infinite, and it turns out that U_2 happens to be the same as the affine Coxeter group $\widetilde{A}_1$.

Example 2.2.6 (Symmetric group on n letters)**.** The group S_n (which can also be seen as the symmetry group of an n-simplex, if we want to focus on the geometric character) is also a Coxeter group, being called the "Weyl group of type A_{n-1}". It admits a Coxeter presentation given by generators $s_1, s_2, \ldots, s_{n-1}$ and relations

- $s_i^2 = e$ for each i;
- $s_i s_j = s_j s_i$ if $|i - j| \geq 2$;
- $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ (equivalently $i = j \pm 1$).

Using the usual permutation notation for the symmetric group, one possible assignment for the generators is $s_i = (i \quad i+1)$.

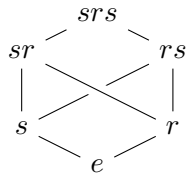
Remark 2.2.7. Although we do not use it explicitly, it is useful to remark that there is a full classification for the finite Coxeter groups, which has been achieved early on by Coxeter

himself in [Cox35]. The irreducible Coxeter groups are classified in this article and, as the name implies, all other Coxeter groups can be constructed from the irreducible ones, where a generating set is the union of the generating sets of the components and the relations are all the previous ones in each component, plus $st = ts$ for all s and t in the generating sets of different components. In terms of the group structure, we simply end up with the direct product of each component group.

Definition 2.2.8 (Bruhat Order)**.** For $x, y \in W$ we say $x \leq y$ if some (not necessarily consecutive) subword of a reduced word for y is a reduced word for x. That is equivalent (see [BB05] Corollary 2.2.3) to some subword of any reduced word for y being a reduced word for x. Note that $x \leq y$ implies $\ell(x) \leq \ell(y)$.

It is clear that this relation is a partial order: it is reflexive, that is, $x \leq x$, because a reduced word is its own subword; it is transitive because a subword of a subword is a subword of the original word; and it is antisymmetric because if $x \leq y$ and $y \leq x$ then in particular, $\ell(x) = \ell(y)$ and given a reduced subword for y its only subword with the same length is itself.

Example 2.2.9 (The Bruhat order lattice in S_3)**.**



Theorem 2.2.10 ([Mat64])**.** *For any $x \in W$ it is possible to obtain a reduced expression for x from any other using only the braid relations. Moreover, for $s \in S$, if $sx < x$ then there is a reduced word for x of the form $x = sr \cdots t$, and if $xs < x$ then there is a reduced word for x of the form $x = r \cdots ts$.*

Definition 2.2.11 (Hecke Algebra)**.** Given a Coxeter system (W, S) of *finite type*, the *Hecke algebra (or Iwahori-Hecke algebra)* $\mathcal{H}_{(W,S)}$ associated to it is the unital and associative $\mathbb{Z}[v^{\pm 1}]$-algebra with generators $\{h_s\}_{s \in S}$ and relations

- $(h_s)^2 = (v^{-1} - v)h_s + 1$;
- $h_s h_r h_s \cdots = h_r h_s h_r \cdots$ for each $r, s \in S$, with m_{ij} elements on each member.

If we take $v = 1$ we obtain the group algebra $\mathbb{Z}W$. So we can see $\mathcal{H}$ as a deformation of the group algebra.

For any reduced expression $\underline{x} = sr \cdots t$ we set $h_{\underline{x}} = h_s h_r \cdots h_t$. By the previous theorem and because the braid relations are also satisfied in $\mathcal{H}$, this expression depends only on x so we can define $h_x = h_s h_r \cdots h_t$.

Lemma 2.2.12. *Let $s \in S$ and $w \in W$. We can rewrite the product in $\mathcal{H}$ as*

$$h_s h_w = \begin{cases} h_{sw} & \text{if } w < sw \\ (v^{-1} - v)h_w + h_{sw} & \text{if } sw > w \end{cases}$$

Proof. In the first case it is clear the equality holds, since $w < sw$ implies that if $r_1 \cdots r_n$ is a reduced word for w then $s \neq r_1$ and $sr_1 \cdots r_n$ is a reduced word for sw, and we can directly use the definition of h_{sw}.

On the other hand if $sw < w$ we can use Theorem 2.2.10 to write a reduced word for w of the form $w = sr_2 \cdots r_n$, and noting that $sw = s^2 r_2 \cdots r_n = r_2 \cdots r_n$ we can see that

$$\begin{aligned} h_s h_w &= h_s h_s h_{r_2} \cdots h_{r_n} \\ &= ((v^{-1} - v)h_s + 1)h_{r_2} \cdots h_{r_n} \\ &= (v^{-1} - v)h_s h_{r_2} \cdots h_{r_n} + h_{r_2} \cdots h_{r_n} \\ &= (v^{-1} - v)h_w + h_{sw} \end{aligned}$$

which shows the formula we stated. □

Theorem 2.2.13 (Iwahori)**.** *$\mathcal{H}$ is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{h_w\}_{w \in W}$. This is called the standard basis of $\mathcal{H}$.*

Remark 2.2.14. In the remainder of this thesis, we will assume all Coxeter groups to be finite, and in light of this theorem, all Hecke algebras will be free modules of finite rank over $\mathbb{Z}[v^{\pm 1}]$.

We will need a short technical Lemma to prove the Theorem.

Lemma 2.2.15. *Let $w \in W$ and $s, t \in S$ be such that $\ell(sw) = \ell(wt)$ and $\ell(swt) = \ell(w)$. Then $sw = wt$.*

Proof. We have two cases to consider, the first being $wt > w$. Since $\ell(swt) = \ell(w) < \ell(wt)$ we can write $wt = sv$ for some $v \in W$ such that $sv > v$, by Theorem 2.2.10. Write $v = s_1 \cdots s_k$ be a reduced expression. We have $w < wt = ss_1 \cdots s_k$ and $\ell(w) = k$, whence we may obtain a reduced expression for w by taking a suitable subexpression of $ss_1 \cdots s_k$ obtained by omitting exactly one factor. Since $sw > w$ that subexpression cannot start with s, therefore that is the factor we must omit and thus $w = s_1 \cdots s_k = v$, whence $wt = sw$ as needed.

Now we consider the case $wt < w$. We set $w' = wt$ which satisfies $\ell(sw't) = \ell(w')$ and $\ell(sw') = \ell(w't) > \ell(w')$, therefore we may use the previous case on w', concluding $w't = sw'$. Inserting $w' = wt$ we obtain $wt^2 = swt$ and cancelling a factor of t we obtain $sw = wt$ as desired. □

Proof of theorem 2.2.13. For simplicity denote $R = \mathbb{Z}[v^{\pm 1}]$.

$\{h_w\}_{w\in W}$ is a generating set: By definition, we have that $h_w = h_{s_1} \cdots h_{s_n}$, where $s_1 \cdots s_n$ is a reduced expression for $w \in W$. Moreover, we have that $(h_s)^2 = (v^{-1} - v)h_s + 1$ so for each expression of the form $\prod h_{s_i}$, possibly with consecutive s_i being the same element, we can reduce it to a linear combination of expressions with no squares of the h_{s_i} via the quadratic relation, and then further reduce it to a linear combination of the h_w since we would be left with expressions corresponding to reduced words. This way we can ultimately reduce any linear combination of products of h_{s_i} to linear combinations of some h_w which is precisely taking an element of $\mathcal{H}$ and writing it as a linear combination of elements in $\{h_w\}_{w\in W}$, showing that this set is a generating set of $\mathcal{H}$ as a $\mathbb{Z}\left[v^{\pm 1}\right]$-module.

$\{h_w\}_{w\in W}$ is linearly independent: (We closely follow the proof of [GP00, Theorem 4.4.6.]) Let V be the free R-module with basis $(e_w \mid w \in W)$. For $s \in S$ we define the R-linear map $\lambda_s : V \to V$ given by

$$\lambda_s(e_w) = \begin{cases} e_{sw} & \text{if } w < sw \\ (v^{-1} - v)e_w + e_{sw} & \text{if } sw > w \end{cases}$$

We claim the map $h_s \mapsto \lambda_s$ extends to an R-algebra homomorphism $\mathcal{H} \to \mathrm{End}_R(V)$. To see this, we have to check the defining relations of the Hecke algebra hold for the λ_s.

Let us check the quadratic relation acting on an element e_w, that is, we will compute $\lambda_s^2(e_w)$. We have to distinguish the case $sw > w$ and $sw < w$.

For the first one we have

$$\begin{aligned} \lambda_s^2(e_w) &= \lambda_s(\lambda_s e_w) \\ &= \lambda_s(e_{sw}) \\ &\overset{(*)}{=} (v^{-1} - v)e_{sw} + e_{s^2w} \\ &= (v^{-1} - v)e_{sw} + e_w, \end{aligned}$$

where the equality marked with $(*)$ is due to the fact that if $sw > w$, then $s(sw) < sw$. We note that this means that λ_s^2 acts as $(v^{-1} - v)\lambda_s + \mathrm{id}_V$ in this case, as we need.

Now if $sw < w$, we perform the computation, obtaining now

$$\begin{aligned} \lambda_s^2(e_w) &= \lambda_s(\lambda_s e_w) \\ &= \lambda_s((v^{-1} - v)e_w + e_{sw}) \\ &= (v^{-1} - v)\lambda_s(e_w) + \lambda_s(e_{sw}) \\ &= (v^{-1} - v)((v^{-1} - v)e_w + e_{sw}) + e_w, \end{aligned}$$

where now we use the fact that if $sw < w$, then $s(sw) > sw$. Now we compare it with

$$\begin{aligned}\left((v^{-1}-v)\lambda_s + \mathrm{id}_V\right)(e_w) &= (v^{-1}-v)\lambda_s(e_w) + e_w \\ &= (v^{-1}-v)((v^{-1}-v)e_w + e_{sw}) + e_w\end{aligned}$$

to see in this case λ_s also acts as desired, proving it satisfies the quadratic relation.

We are left to check that if $w = s_1 \cdots s_n = s_1' \cdots s_n'$ are two reduced expressions, we have $\lambda := \lambda_{s_1 \cdots s_n} - \lambda_{s_1' \cdots s_n'} = 0$, proving that the assignment $w \mapsto \lambda_w = \lambda_{s_1 \cdots s_n}$ is well defined. For this purpose we introduce another map $\rho_s : V \to V$ given by

$$\rho_s(e_w) = \begin{cases} e_{ws} & \text{if } w < sw \\ (v^{-1}-v)e_w + e_{ws} & \text{if } sw > w, \end{cases}$$

which we will shortly see satisfies $\lambda_s \rho_t = \rho_t \lambda_s$ for all $t, s \in S$. Once we prove these maps commute, we may prove that $\lambda(e_w) = 0$, by induction on $\ell(w)$:

If $\ell(w) = 0$, $w = 1$ and it is clear that $\lambda(e_1) = e_{s_1 \cdots s_n} - e_{s_1' \cdots s_n'} = 0$. Now we let $\ell(w) > 0$. We know that there exists a $t \in S$ such that $wt < w$, and so $\lambda(e_{wt}) = 0$ by induction hypothesis. But $\lambda(e_w) = \lambda(e_{(wt)t}) = \lambda(\rho_t(e_{wt})) = \rho_t(\lambda(e_{wt})) = \rho_t(0) = 0$, concluding the proof that $\lambda = 0$.

This shows the map $h_s \mapsto \lambda_s$ extends to an R-algebra homomorphism $\mathcal{H} \to \mathrm{End}_R(V)$ which we denote by φ and use to define a left action of $\mathcal{H}$ on V given by $hv = \varphi(h)(v)$. We use it to finish the proof:

Suppose we have a finite linear combination in $\mathcal{H}$ of the form $\sum_{w \in W} r_w h_w = 0$. If $w = s_1 \cdots s_n$ is a reduced expression, we have $\lambda_w(e_1) = (\lambda_{s_1} \cdots \lambda_{s_n})(e_1) = e_w$, and so $h_w e_1 = e_w$. We apply the above linear combination to e_1 to obtain

$$0 = \left(\sum_{w \in W} r_w h_w\right)(e_1) = \sum_{w \in W} r_w e_w,$$

which means we have $r_w = 0$ for each $w \in W$, since we are now in V, which is a free R-module with basis $(e_w \mid w \in W)$, by construction. This shows that indeed the set $\{h_w\}_{w \in W}$ is linearly independent in $\mathcal{H}$, as needed.

The only missing piece of the proof is showing that $\lambda_s(\rho_t(e_w)) = \rho_t(\lambda_s(e_w))$ for all $t, s \in S$ and $w \in W$, which we will do now. We divide the problem into various cases.

Case 1: $sw > w$ and $swt > sw$. In this case we have also $swt > wt > w$ and so

$$\lambda_s(\rho_t(e_w)) = \lambda_s(e_{wt}) = e_{swt} = \rho_t(e_{sw}) = \rho_t(\lambda_s(e_w)).$$

Case 2: $sw > w$, $swt < sw$, and $wt < w$. In this case we have $swt > sw$ and thus

$$\lambda_s(\rho_t((e_w)) = \lambda_s((v^{-1}-v)e_w + e_{wt}) = (v^{-1}-v)e_{sw} + e_{swt} = \rho_t(e_{sw}) = \rho_t(\lambda_s(e_w)).$$

Case 3: $sw > w$, $swt < sw$, and $wt > w$. In this case we have $\ell(swt) = \ell(w)$ and by Lemma 2.2.15 $sw = wt$ and so we also have $swt < wt$, and we write

$$\rho_t(\lambda_s(e_w)) = \rho_t(e_{sw}) = (v^{-1} - v)e_{sw} + e_{swt}$$

and

$$\lambda_s(\rho_t(e_w)) = \lambda_s(e_{wt}) = (v^{-1} - v)e_{wt} + e_{swt},$$

and because $sw = wt$, we have equality as needed.

Case 4: $swt < sw < w$. In this case we have

$$\rho_t(\lambda_s(e_w)) = \rho_t((v^{-1} - v)e_w + e_{sw}) = (v^{-1} - v)((v^{-1} - v)e_w + e_{wt}) + (v^{-1} - v)e_{sw} + e_{swt},$$

and

$$\lambda_s(\rho_t(e_w)) = \lambda_s((v^{-1} - v)e_w + e_{wt}) = (v^{-1} - v)(e_w + (v^{-1} - v)e_{sw}) + (v^{-1} - v)e_{wt} + e_{swt}$$

and we can check the expressions are in fact equal.

Case 5: $sw < w$, $swt > sw$ and $wt > w$. Then we have $swt < wt$ and thus

$$\rho_t(\lambda_s(e_w)) = \rho_t((v^{-1} - v)e_w + e_{sw}) = (v^{-1} - v)e_{wt} + e_{swt} = \lambda_s(e_{wt}) = \lambda_s(\rho_t(e_w)).$$

Case 6: $sw < w$, $swt > sw$ and $wt < w$. As in case 3 this implies $sw = wt$ and we have

$$\rho_t(\lambda_s(e_w)) = \rho_t((v^{-1} - v)e_w + e_{sw}) = (v^{-1} - v)(e_w + (v^{-1} - v)e_{wt}) + e_{swt},$$

and

$$\lambda_s(\rho_t(e_w)) = \lambda_s((v^{-1} - v)e_w + e_{wt}) = (v^{-1} - v)(e_w + (v^{-1} - v)e_{sw}) + e_{swt},$$

and again these expressions are the same.

This concludes the proof. □

Chapter 3

The Hecke invariant

In this chapter we define our first invariant for ciliated surfaces. The invariant is defined by resorting to triangulations of the surface, but we prove that it is independent of the chosen triangulation, and is therefore an invariant in a geometrical sense. We bring to attention the fact that this is only a "true" geometric invariant when the surfaces are closed, since when the surfaces have boundary components, we need to assign labels to them, in order to compute the invariant.

We also prove equation (3.2.1), which is very helpful for computing the invariant in various contexts, by cutting the surfaces into smaller "pieces".

We end the chapter by computing a few examples in some special cases.

3.1 Construction of the invariant

Definition 3.1.1. Given a Coxeter system (W, S) let $\mathcal{H}$ be the associated Hecke algebra. Let $\{h_w\}_{w\in W}$ be its standard basis and let $\mathcal{H}^*$ be the dual $\mathbb{Z}[v^{\pm 1}]$-module with dual basis $\{h^w\}_{w\in W}$.

A *trace* in $\mathcal{H}$ is a $\mathbb{Z}[v^{\pm 1}]$-linear map $\mathsf{tr} : \mathcal{H} \to \mathbb{Z}[v^{\pm 1}]$ such that $\mathsf{tr}(hh') = \mathsf{tr}(h'h)$. A trace is said to be *symmetrising* if the map $h' \mapsto \mathsf{tr}(hh')$ is non degenerate for all $h \neq 0$.

Proposition 3.1.2. *A Hecke algebra $\mathcal{H}$ admits a symmetrising trace given by the map $\mathsf{tr}(\sum c_w h_w) = h^e(\sum c_w h_w) = c_e$, which we call the standard trace. We have that, explicitly, $\mathsf{tr}(h_x h_y) = \delta_{x,y^{-1}}$, for all $x, y \in W$.*

Proof. To show the map in question is a symmetrising trace, we need to show it is a trace and that the form $\mathsf{tr}(h_x \cdot -)$ is non degenerate, for $h_x \neq 0$. If we prove the given expression,

we prove both these assertions, since of course $\delta_{x,y^{-1}} = \delta_{y,x^{-1}}$, and $h_y \mapsto \delta_{x,y^{-1}}$ is non-zero since the element $h_{x^{-1}}$ is mapped to 1.

Let us then prove the expression. We will use induction on $\ell(x)$, as well as Lemma 2.2.12 and Theorem 2.2.10.

Base case: For $\ell(x) = 1$ we have $x \in S$ and we may use the Lemma directly, obtaining two cases.

Case $x < xy$: In this case, we forcibly have $y \neq x$ and so we should have that $\mathsf{tr}(h_x h_y) = 0$. In fact, using the lemma, we have $h_x h_y = h_{xy}$ and of course the coefficient of h_e in this case is 0, as needed.

Case $x > xy$: Here, by Theorem 2.2.10, we have a reduced expression for y of the form $y = xy_0$. If $y_0 = e$ we have $x = y = y^{-1}$ and therefore we should have $\mathsf{tr}(h_x h_y) = 1$. Indeed, $h_x h_y = (h_x)^2 = (v^{-1} - v)h_x + h_{x^2} = (v^{-1} - v)h_x + h_e$. The coefficient of h_e on this element is 1 and the result is proved for the case $x = y$.

Now, if we have $y_0 \neq e$, we once again use the Lemma to write $h_x h_y = (v^{-1} - v)h_y + h_{xy} = (v^{-1} - v)h_y + h_{y_0}$, and note that there is no term in h_e and therefore the result of applying the map is zero, as needed. This proves the base case.

Induction step: Fix x with $\ell(x) > 1$ and assume the result is true for all x' such that $\ell(x') < \ell(x)$. Write $x = x_0 x_1$ as a reduced expression, where $x_1 \in S$. This implies $\ell(x_0) = \ell(x) - 1$ and we can apply the induction hypothesis to x_0. Note that since the expression for x is reduced, it is true that $h_x = h_{x_0} h_{x_1}$.

We again distinguish two cases:

Case $x_1 < x_1 y$: In this case we have that $h_x h_y = h_{x_0} h_{x_1} h_y = h_{x_0} h_{x_1 y}$, where in the last equality we used the Lemma once again. As we know that we can apply the induction hypothesis to x_0, we can write

$$\begin{aligned} \mathsf{tr}(h_x h_y) &= \mathsf{tr}\left(h_{x_0} h_{x_1 y}\right) \\ &= \delta_{x_0,(x_1 y)^{-1}} \\ &= \delta_{x,y^{-1}} \end{aligned}$$

where this last equality comes from

$$\begin{aligned} x_0 = (x_1 y)^{-1} &\Leftrightarrow x_0 = y^{-1}(x_1)^{-1} \\ &\Leftrightarrow x_0 x_1 = y^{-1} \\ &\Leftrightarrow x = y^{-1}. \end{aligned}$$

This proves the case $x_1 < x_1 y$.

Case $x_1 > x_1 y$: Here, again by Theorem 2.2.10, we can write a reduced expression for y in the form $y = x_1 y_1$, for some $y_1 \in W$. We thus have

$$h_x h_y = h_{x_0} h_{x_1} h_y = h_{x_0} \left((v^{-1} - v) h_y + h_{y_1} \right),$$

where, again, the last equality is due to the Theorem. By linearity of the function tr and the induction hypothesis, we have that

$$\begin{aligned} \mathsf{tr}(h_x h_y) &= \mathsf{tr}\left(h_{x_0} \left((v^{-1} - v) h_y + h_{y_1} \right) \right) \\ &= (v^{-1} - v)\, \mathsf{tr}\left(h_{x_0} h_y \right) + \mathsf{tr}\left(h_{x_0} h_{y_1} \right) \\ &= (v^{-1} - v) \delta_{x_0, y^{-1}} + \delta_{x_0, (y_1)^{-1}}. \end{aligned}$$

Now we are left with showing that $\delta_{x_0, y^{-1}} = 0$ and that $\delta_{x_0,(y_1)^{-1}} = \delta_{x, y^{-1}}$, which would conclude the induction step.

Firstly, if $x_0 = y^{-1}$ we would have $x_0 x_1 y_1 = e$, that is, $y_1 = (x_1)^{-1}(x_0)^{-1} = x_1 (x_0)^{-1}$. But this would mean the expression $y = x_1 y_1 = x_1 x_1 (x_0)^{-1}$ would not be in reduced form, contradicting the hypothesis, therefore, $\delta_{x_0, y^{-1}}$ can only be zero.

On the other hand the following shows that $\delta_{x_0,(y_1)^{-1}} = \delta_{x, y^{-1}}$:

$$\begin{aligned} x_0 = (y_1)^{-1} &\Leftrightarrow x_0 x_1 = (y_1)^{-1} (x_1)^{-1} \\ &\Leftrightarrow x_0 x_1 = (x_1 y_1)^{-1} \\ &\Leftrightarrow x = y^{-1}, \end{aligned}$$

thereby proving the induction step fully, and hence the result. □

We recall that, for a commutative ring R and free and finitely generated R-modules A and B, we have $A^* \otimes_R B \cong \mathrm{Hom}_R(A, B)$, where the map is determined by the rule

$$\begin{aligned} A^* \otimes_R B &\to \mathrm{Hom}_R(A, B) \\ \alpha \otimes b &\mapsto [a \mapsto \alpha(a) b], \end{aligned} \tag{3.1.1}$$

and $A^* = \mathrm{Hom}_R(A, R)$ is an R-module by $(r\alpha)(a) = r\alpha(a)$ for all $a \in A$ and $\alpha \in A^*$. and the inverse is given by the formula

$$\begin{aligned} \mathrm{Hom}(A, B) &\to A^* \otimes_R B \\ f &\mapsto \sum_i a^i \otimes f(a_i), \end{aligned} \tag{3.1.2}$$

where $\{a_i\}_{i \in I}$ is a basis for A and $\{a^i\}_{i \in I}$ is the dual basis (recall that I is finite).

With this equivalence in mind, the product in $\mathcal{H}$ corresponds to a tensor $c \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$ whose components in the standard basis are $c_{xy}{}^z = h^z(h_x h_y)$. These constants are called *structure constants*.

We define $c_{xyz} = c_{xy}{}^{z^{-1}} = h^{z^{-1}}(h_x h_y)$.

Proposition 3.1.3. *For $x, y, z \in W$ we have that $c_{xyz} = \mathsf{tr}(h_x h_y h_z)$. In particular c_{xyz} is cyclic: $c_{xyz} = c_{zxy} = c_{yzx}$.*

Proof. By the definition of the structure constants we have that

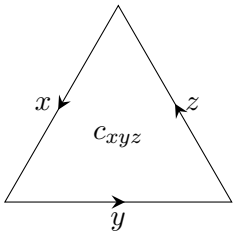
$$h_x h_y = \sum_{z'} c_{xyz'} h_{z'^{-1}}$$

so we conclude that

$$\begin{aligned} h_x h_y h_z = \sum_{z'} c_{xyz'} h_{z'^{-1}} h_z \Leftrightarrow \\ \Leftrightarrow \mathsf{tr}(h_x h_y h_z) = \sum_{z'} c_{xyz'} \mathsf{tr}(h_{z'^{-1}} h_z) = c_{xyz} \end{aligned}$$

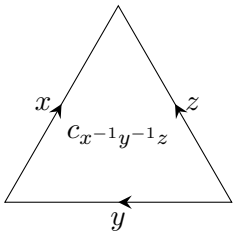
□

Let us then begin constructing our invariant, one piece at a time. To make use of the fact that we triangulate the surfaces, we define it first for triangles. Firstly, we fix a Coxeter system and the associated Hecke algebra. Given an oriented triangle, we define the orientation of an edge as being positive or negative relative to the orientation of the triangle. Given a triangle whose edges are positively oriented and labelled by $x, y, z \in W$, we associate to it the constant $c_{xyz} \in \mathbb{Z}[v^{\pm 1}]$ as in the following image:



By the previous proposition, the constant associated to a given triangle is independent from the place where we start reading the labels since $c_{xyz} = c_{zxy} = c_{yzx}$.

More generally, to the triangle labelled by the elements x, y, z we associate the constant $c_{x^a y^b z^c} \in \mathbb{Z}[v^{\pm 1}]$, where $a = 1$ if the edge labelled by x is positively oriented and $a = -1$ if it is negatively oriented, and so on. We present another example:



Proposition 3.1.4. *The structure constants satisfy $c_{xyz} = c_{z^{-1}y^{-1}x^{-1}}$. Therefore the structure constant associated to each triangle depends only on the relative orientation of the edges and not on the orientation of the triangle.*

Proof. Consider $\sigma : \mathcal{H} \to \mathcal{H}$ defined by $\sigma(h_w) = h_{w^{-1}}$ and extended by linearity. We claim that σ is an anti-isomorphism of $\mathbb{Z}[v^{\pm 1}]$-modules, that is, $\sigma(h_a h_b) = \sigma(h_b)\sigma(h_a)$. It is only necessary to show this equality holds on the basic relations of the Hecke algebra, that is, the quadratic relation in the definition and the one involving a product $w = sw'$ where $s \in S$ and sw' is a reduced expression for w, since every product of two elements in the Hecke algebra can be described with these. The quadratic relation satisfies

$$\sigma((h_s)^2) = \sigma((v^{-1} + v)h_s + 1) = (v^{-1} + v)\sigma(h_s) + 1 = (v^{-1} + v)h_{s^{-1}} + 1 = \sigma(h_s)^2$$

For $w = sw'$ where $s \in S$ and w' is a reduced word, we have

$$\sigma(h_s h_{w'}) = \sigma(h_w) = h_{w^{-1}} = h_{w'^{-1}s} = \sigma(h_{w'})\sigma(h_s)$$

which proves our claim that σ is an anti-isomorphism.

We complete the proof using σ: by definition

$$h_x h_y = \sum_z c_{xyz} h_{z^{-1}}$$

and applying σ on both sides we obtain

$$h_{y^{-1}} h_{x^{-1}} = \sum_z c_{xyz} h_z$$

which implies $c_{y^{-1}x^{-1}z^{-1}} = c_{xyz}$, completing the proof. □

Having proved this we may now start constructing the invariants for surfaces.

First let us fix some notation. Let T be a triangulation of a ciliated surface (whose vertices are either punctures or cilia). We denote by $F(\mathsf{T})$ the set of faces of T and by $E(\mathsf{T})$ the set of edges of T. A *labelling* of T with values on a Coxeter group W is a function $\lambda : E(\mathsf{T}) \to W$, and the set of all such labellings is denoted by $\Lambda(\mathsf{T})$. For a labelling λ and a face $f \in F(\mathsf{T})$ such that its edges are labelled with x, y, z we denote by $c_f^\lambda(v)$ the constant c_{xyz} associated to this labelling of f.

We will now define the invariant for ciliated surfaces. Let W be a Coxeter group, let Σ be a ciliated surface and let T be a triangulation of Σ. We will construct the invariant via the triangles of the triangulation, therefore we have to choose an orientation for the edges of the triangulation, so that each $c_f^\lambda(v)$ is well defined. As before, each edge can be positively or negatively oriented relative to a triangle of which it is a side of. The orientation of the faces is induced by the orientation of the surface.

Definition 3.1.5. For a ciliated surface Σ we choose boundary data consisting of an element of W associated to each boundary component. The invariant associated to Σ given such boundary data via the triangulation T is the Laurent polynomial

$$P_{\Sigma,W,\mathsf{T}}(v) = \sum_{\lambda\in\Lambda(\mathsf{T})}\prod_{f\in\mathsf{T}} c_f^{\lambda}(v) \in \mathbb{Z}[v^{\pm 1}]$$

where the sum runs over the labellings of the inner edges of the triangulation, and when a face of the triangulation has a boundary component as an edge, we compute the triangle invariant using the given boundary data.

In particular when the surface is closed we need not choose any boundary data. For the closed surface of genus g and with k punctures, we denote the invariant by $P_{g,k,W,\mathsf{T}}$.

Remark 3.1.6. Since we are summing over all possible labellings, the polynomial $P_{g,k,W,\mathsf{T}}$ does not depend on the chosen orientation of the edges, since changing the orientation of an edge corresponds to switching its label w for w^{-1}. This means that the choice of orientation for each edge of the triangulation ends up playing no role in the invariant and so we can speak of the invariant constructed via the triangulation T with no mention of the orientation of each individual edge.

Besides that, we also have the following result:

Theorem 3.1.7. *The polynomial invariant that we have constructed does not depend on the triangulation T, therefore it is a topological invariant for the ciliated surface. Moreover it is invariant with respect to changing the orientation of the surface.*

Proof. The second part of the Theorem is a direct consequence of the definition of the polynomial and of Proposition 3.1.4. The first part follows from the associativity of the product in $\mathcal{H}$. Indeed, if $x, y, z \in \mathcal{H}$, then by definition

$$h_x h_y = \sum_w c_{xyw} h_{w^{-1}}$$

and with this we can check that

$$\begin{aligned}
\sum_{v,w} c_{xyw}c_{w^{-1}zv}h_{v^{-1}} &= \sum_w c_{xyw}\left(\sum_v c_{w^{-1}zv}h_{v^{-1}}\right)\\
&= \sum_w c_{xyw}h_{w^{-1}}h_z\\
&= (h_x h_y)h_z\\
&= h_x(h_y h_z)\\
&= \sum_{t,v} c_{yzt}c_{xt^{-1}v}h_{v^{-1}}
\end{aligned}$$

Comparing the coefficients we have that, for all $x, y, z, v \in W$,

$$\sum_w c_{xyw} c_{w^{-1}zv} = \sum_t c_{yzt} c_{xt^{-1}v}$$

which in turn means that

$$\sum_{w \in W} \; y \boxed{\begin{matrix} & x & \\ & w & \\ z & & v \end{matrix}} \; = \sum_{t \in W} \; y \boxed{\begin{matrix} & x & \\ & t & \\ z & & v \end{matrix}}$$

showing that the invariant remains the same up to a diagonal flip on the triangulation. Since we can reach any triangulation from any other with a finite number of these diagonal flips (Theorem 2.1.7), the Theorem is proven in general. □

3.2 Cutting surfaces

We now show a useful property to calculate these polynomial invariants. Given a ciliated surface, we may cut it into two smaller ones along some punctures and the punctures then become cilia of each of the separate components. We present an expression for the invariant associated to the original surface that makes use of this "cutting" operation. Let us make that notion precise.

Definition 3.2.1. Fix a Coxeter system, and let Σ be a ciliated surface. Fix some punctures $p_1, p_2, \ldots, p_n$ and join them with a loop not containing any other punctures. If the loop is removed we get two connected components Σ_1 and Σ_2, which are to be seen as two ciliated surfaces when we see the loop as part of each, which we now name the "common boundary" and denote by ∂. It is clear that the cilia on the common boundary are the original chosen punctures on Σ. We also denote $\Sigma = \Sigma_1 \cup_\partial \Sigma_2$ on the expression to emphasise the "cutting".

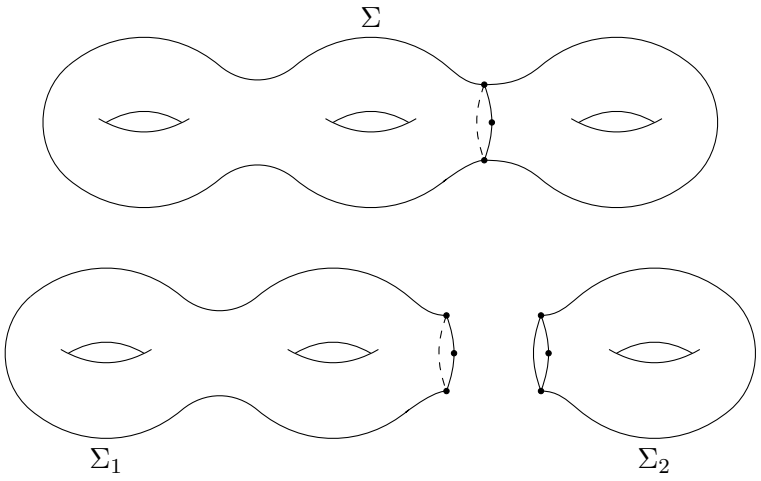


Figure 3.1: Cutting a surface along a common boundary with three punctures

Theorem 3.2.2. *Let Σ be a ciliated surface and perform a cut as defined earlier to obtain $\Sigma = \Sigma_1 \cup_\partial \Sigma_2$. Let $\Lambda(\partial)$ be the set of all the possible labellings on ∂. Then for a given label $\lambda \in \Lambda(\partial)$ we can take $(\partial, \lambda(\partial))$ as boundary data to compute $P_{(\partial,\lambda(\partial))}(\Sigma_i)$ for $i = 1, 2$. Here, the subscript indicates that on the boundary ∂ we fix the labels given by $\lambda(\partial)$. Then the following holds:*

$$P(\Sigma_1 \cup_\partial \Sigma_2) = \sum_{\lambda \in \Lambda(\partial)} P_{(\partial,\lambda(\partial))}(\Sigma_1) P_{(\partial,\lambda(\partial))}(\Sigma_2). \tag{3.2.1}$$

Proof. We can fix triangulations T_1 in Σ_1 and T_2 in Σ_2, where $\mathsf{T} = \mathsf{T}_1 \cup_\partial \mathsf{T}_2$ is a triangulation of $\Sigma_1 \cup_\partial \Sigma_2$. Expanding the right hand side of equation (3.2.1) we obtain the following:

$$\begin{aligned}
\sum_{\lambda \in \Lambda(\partial)} P_{(\partial,\lambda(\partial))}(\Sigma_1) P_{(\partial,\lambda(\partial))}(\Sigma_2) &= \sum_{\lambda \in \Lambda(\partial)} \left(\sum_{\lambda \in \Lambda(\mathsf{T}_1)} \prod_{f_1 \in \mathsf{T}_1} c^\lambda_{f_1}(v) \right) \left(\sum_{\lambda \in \Lambda(\mathsf{T}_2)} \prod_{f_2 \in \mathsf{T}_2} c^\lambda_{f_2}(v) \right) \\
&= \sum_{\lambda \in \Lambda(\mathsf{T})} \left(\prod_{f_1 \in \mathsf{T}_1} c^\lambda_{f_1}(v) \right) \left(\prod_{f_2 \in \mathsf{T}_2} c^\lambda_{f_2}(v) \right) \\
&= \sum_{\lambda \in \Lambda(\mathsf{T})} \left(\prod_{f \in \mathsf{T}} c^\lambda_f(v) \right) \\
&= P(\Sigma_1 \cup_\partial \Sigma_2)
\end{aligned}$$

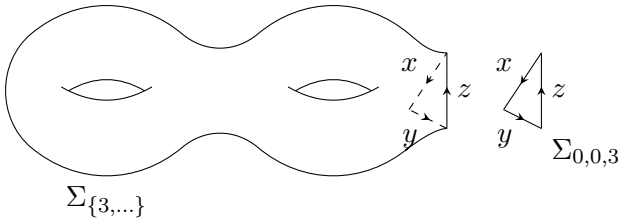
where passing from the first line to the second line is simply swapping the order of the sum and product since we are dealing with finite sets, and noting that then the sum ranges through all labellings of all the edges. This proves the Theorem. □

Notice that in the limit where we keep cutting the surface until we are left only with triangles, we get the original expression we used to define the invariant.

We may use this Theorem to recursively compute the invariant for some surfaces. Some examples follow:

Example 3.2.3 (Removing a triangle)**.** Let $\Sigma = \Sigma_{g,k,\{...\}}$ be a surface of genus g with $k \geq 3$ punctures and possibly some boundary components. We remove a triangle $\Sigma_1 = \Sigma_{0,0,\{3\}}$ formed by joining three punctures and we are left with $\Sigma_2 = \Sigma_{g,k-3,\{3,...\}}$, which is a surface with the same genus as Σ and 3 less punctures, and an extra boundary component containing 3 cilia. Denoting by $(\Delta, \{x, y, z\})$ a triangle with labels $\{x, y, z\}$, equation (3.2.1) becomes

$$P(\Sigma_{g,k,\{...\}}) = \sum_{x,y,z \in W} c_{xyz} P_{(\Delta,\{x,y,z\})}(\Sigma_{g,k-3,\{3,...\}})$$



Example 3.2.4. Suppose we have a punctured surface with triangle boundary labelled by x, y and z. We can reduce the number of punctures, using a quadrilateral with boundary z, x, x'^{-1}, z'^{-1} as Σ_2 in the formula for cutting surfaces. We depict the construction on Figure 3.2:

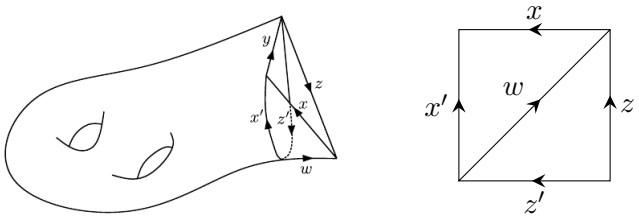


Figure 3.2: Original surface with the square shown and "unfolding" of the square.

The polynomial invariant for the quadrilateral is easily shown to be $\sum_w c_{xx'^{-1}w} c_{zw^{-1}z'^{-1}}$. Hence, from the cutting formula, we get that

$$P_{(\Delta,\{x,y,z\})}(\Sigma_{g,k+1}) = \sum_{x',z',w} c_{xx'^{-1}w} c_{zw^{-1}z'^{-1}} P_{(\Delta,\{x',y,z'\})}(\Sigma_{g,k}) \tag{3.2.2}$$

3.3 Some examples of invariant contructions

Invariants for punctured surfaces at $q = 1$

Let us set $v^{-2} = q$. It will turn out that, for closed surfaces, our invariants will be polynomial in q and symmetrical under the transformation $q \mapsto q^{-1}$.

At $q = 1$ the Hecke algebra is reduced to the group algebra $\mathbb{Z}W$. We have the following important result:

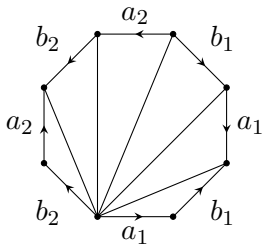
Proposition 3.3.1. *For a punctured surface $\Sigma_{g,k}$ the value at $q = 1$ of the polynomial invariant is*

$$P_{g,k,W}(1) = (\#W)^{k-1} \times \# \left\{ \textit{Solutions in } W \textit{ to } \prod_{i=1}^{g} [a_i, b_i] = 1 \right\}, \tag{3.3.1}$$

where $[a_i, b_i]$ is the commutator $a_i b_i a_i^{-1} b_i^{-1}$. In particular $P_{0,k,W}(1) = (\#W)^{k-1}$.

Remark 3.3.2. It is well known that the equation $\prod_{i=1}^{g} [a_i, b_i] = 1$ is the relation which appears in the presentation of the fundamental group of a surface of genus g. The polynomial $P_{g,k,W}(1)/(\#W)^{k-1}$ thus counts the number of group homomorphisms from $\pi_1(\Sigma_g)$ to W, where Σ_g is the topological surface with no marked points from which we obtain $\Sigma_{g,k}$.

Proof. Starting by the case $g > 0$, $k = 1$, we can realise $\Sigma_{g,k}$ as a $4g$-gon, gluing the border in a specific way, triangulating, and then counting the contributions of each triangle. We illustrate the case $g = 2$ in the following figure, already with a chosen triangulation. Note that all the vertices of the polygon are the same point in $\Sigma_{g,1}$, corresponding to the puncture.



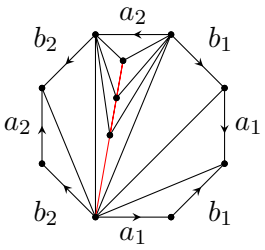
The structure constants in $\mathcal{H}$ at $q = 1$ satisfy $c_{xyz} = 1$ if $xyz = 1$ in W and $c_{xyz} = 0$ in all other instances, since $h^{z^{-1}}(h_x h_y)$ reduces to $\delta_{z^{-1},xy}$ in the group algebra, and $xy = z^{-1}$ if and only if $xyz = 1$. Therefore, to have non zero contributions coming from the faces of a triangle, each label on a side of a triangle can be determined from the labels on the other two. Let $a_1, b_1, \ldots, a_n, b_n$ be the labels on the edges of the polygon and let T be a triangulation whose triangles are all incident on the same vertex (as in the previous figure). When the

contribution of a labelling is non zero the labels of the inner edges of the triangulation are completely determined by $a_1, b_1, \ldots, a_n, b_n$ as we just noted. Moreover we have following constraint:

$$\prod_{i=1}^{g} [a_i, b_i] = 1, \tag{3.3.2}$$

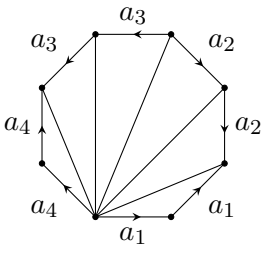
which can be seen as the consequence of the fact that the products of labels assigned to each triangle whose boundary is oriented counterclockwise has to be 1. Therefore each solution of equation (3.3.2) has contribution 1 to the polynomial whilst all other labellings do not contribute. This proves the proposition in the case $g > 0$ and $k = 1$.

Now for $k > 1$ we add $k - 1$ punctures in the interior of the polygon and complete the triangulation as seen here:



Now, if we choose labels for the edges in red, all other labels are determined in order for the whole labelling to have contribution 1, exactly as before. That is to say, fixing labels in the outer edges of the polygons (still with the same constraint as before), we have $(\#W)^{k-1}$ possible choices for labelling the red edges, which have no constraints. Therefore equation (3.3.1) is satisfied in this case as well.

Finally, we deal with the case $g = 0$, which imposes $k > 2$ since only in this case triangulating the surface is possible. We represent $\Sigma_{0,k}$ as a $(2k-2)$-gon, with labels $a_1, a_1^{-1}, \ldots, a_{k-1}, a_{k-1}^{-1}$ on the outer edges, and triangulating as before. We illustrate the case $k = 5$:



Again, we need only notice that the inner edges are still completely determined by the outer ones for us to have a non zero contribution from a particular labelling. But now

no further restrictions are needed on the a_i, leaving us the total number of options to be $P_{0,k,W} = \#W^{k-1}$, finalising the proof. □

The case $(W, S) = (S_2, \{s\})$

Another very important example is when we consider our Coxeter group to be the symmetric group on 2 symbols. This is the only abelian Coxeter group, and the fact it is abelian has consequences for our invariant.

The Hecke algebra $\mathcal{H}_{(S_2,\{s\})}$ is a free $\mathbb{Z}\left[v^{\pm 1}\right]$-module of rank 2 with basis $\{h_e = 1, h_s\}$. The only non trivial product is the quadratic relation ${h_s}^2 = (v^1 - v)h_s + 1$ so the structure constants are

$$\begin{cases} c_{eee} = 1 \\ c_{ees} = c_{ese} = c_{see} = 0 \\ c_{ess} = c_{ses} = c_{ess} = 1 \\ c_{sss} = v^{-1} - v = q^{\frac{1}{2}} - q^{-\frac{1}{2}} \end{cases}$$

We know from an earlier result that the invariant does not depend on the triangulation, so let us compute a couple of examples explicitly. For a sphere with 3 punctures and a torus with a single puncture we use the triangulations in Figure 3.3:

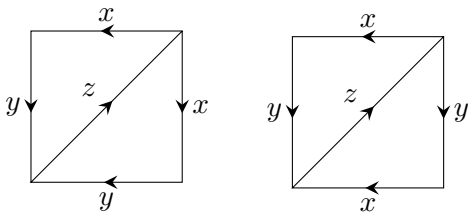


Figure 3.3: Triangulations of $\Sigma_{0,3}$ and of $\Sigma_{1,1}$.

Using the definition and the fact that $x^{-1} = x$ for every element in W we have that

$$\begin{aligned} P_{0,3,S_2} &= \sum_{(x,y,z)\in S_2\times S_2\times S_2} c_{xyz}c_{z^{-1}y^{-1}x^{-1}} \\ &= \sum_{(x,y,z)\in S_2\times S_2\times S_2} (c_{xyz})^2 \\ &= (c_{eee})^2 + 3(c_{see})^2 + 3(c_{ess})^2 + (c_{sss})^2 \\ &= 1 + 0 + 3 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \\ &= q + 2 + q^{-1} \end{aligned}$$

where those factors of 3 come from the structure constants being cyclic so we only need to have one of each "kind". Essentially the same computation shows that $P_{1,1,S_2} = q + 2 + q^{-1}$ but we take one extra step, noting that S_2 is abelian:

$$\begin{aligned} P_{1,1,S_2} &= \sum_{(x,y,z)\in S_2\times S_2\times S_2} c_{xyz}c_{z^{-1}x^{-1}y^{-1}} \\ &= \sum_{(x,y,z)\in S_2\times S_2\times S_2} c_{xyz}c_{yxz} \\ &= \sum_{(x,y,z)\in S_2\times S_2\times S_2} (c_{xyz})^2 \\ &= q + 2 + q^{-1} \end{aligned}$$

From these two examples we can gain the intuition for the proof of the following Proposition:

Proposition 3.3.3. *The polynomial P_{g,k,S_2} does not strictly depend on (g,k), but only on the absolute value of the Euler characteristic (of the triangulation[1]) $\chi = 2g - 2 + k$.*

To prove this we will need an additional lemma:

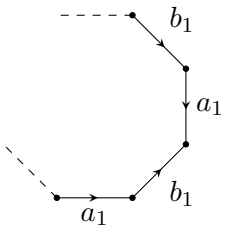
Lemma 3.3.4. *Given a surface $\Sigma_{g,k}$ with Euler characteristic (of the triangulation) $\chi = 2g-2+k$ it is possible to realise it as the quotient of an $4n$-sided polygon where $n = (\chi+1)/2$ if χ is odd or $n = \chi/2$ if it is even. Moreover the labels on the edges are a permutation of $(a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_n, a_n^{-1}, b_n, b_n^{-1})$.*

Proof. We will prove that for any surface of odd characteristic, we can realise it as stated in the proposition, for $n = (\chi + 1)/2$. So for a given χ we will proceed in a "downwards induction" on g.

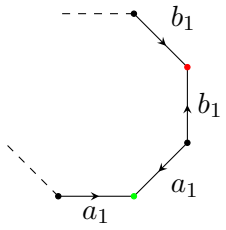
Let $\Sigma_{g,k} = \Sigma_{G,1}$ for our base case, where $G = (\chi + 1)/2$. As seen before we can realise this surface as a $4G$-sided polygon whose labels are $(a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_G, b_G, a_G^{-1}, b_G^{-1})$, without permuting anything.

Now assume the proposition is true for a given pair (g,k) with $g > 0$ and with the same fixed χ. Since its genus is positive, we can realise it as stated, where the labelling is $(a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots)$, where the part after the dots is a permutation of the labels $a_2, a_2^{-1}, \ldots$, as seen in the figure below. We remark that all marked points are identified and therefore represent a single puncture.

[1] Which is completely different from the Euler characteristic of the original unpunctured surface.



Now taking this realisation of the surface as a polygon, we can realise $\Sigma_{g-1,k+2}$ as

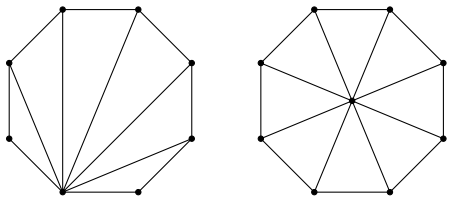


noting that the labels of the polygon are still a permutation of the original ones. We also note that the genus is reduced by one, and that the points marked in green and in red are not identified with the ones in black or with each other, hence increasing the number of punctures by two, preserving χ. Moreover we can proceed in this way and we cover all the pairs (g, k) with the given characteristic χ, thereby proving the assertion.

For a surface of even characteristic the base case is $(G, 2)$ where $G = \chi/2$ and the steps are exactly the same, except we add an extra puncture to the interior of the polygon.

This proves the Lemma in its full generality. □

Proof of Proposition 3.3.3. We use the previous lemma and one of the following triangulations, depending on whether χ is odd or even:



Now, remembering the calculations we did in the previous example, we note that the way in which we glue the sides becomes irrelevant due to the fact that S_2 is abelian, since it will only depend on the set of labels on the edges. This means that the final result will

not depend on the surface but only on χ, since the labels will always be permutations of $(a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_n, a_n^{-1}, b_n, b^{-1})$. □

With the previous proposition in hand we may prove a more general one:

Proposition 3.3.5. *We have* $P_{g,k,S_2} = (1+q)^{2g-2+k} + (1+q^{-1})^{2g-2+k}$.

By Prop. 3.3.3 we only need to compute the polynomial for the punctured spheres $\Sigma_{0,k}$. We will prove a more general formula for punctured spheres with triangle boundary using induction and the cutting formula 3.2.2.

Lemma 3.3.6. *For all $k \geq 3$ we have:*

$$P_{(\Delta,\{e,e,e\})}(\Sigma_{0,k}) = (1+q)^{k-4} + (1+q^{-1})^{k-4} \tag{3.3.3}$$

$$P_{(\Delta,\{s,e,e\})}(\Sigma_{0,k}) = q^{1/2}(1+q)^{k-4} - q^{-1/2}(1+q^{-1})^{k-4} \tag{3.3.4}$$

$$P_{(\Delta,\{s,s,e\})}(\Sigma_{0,k}) = q(1+q)^{k-4} + q^{-1}(1+q^{-1})^{k-4} \tag{3.3.5}$$

$$P_{(\Delta,\{s,s,s\})}(\Sigma_{0,k}) = q^{3/2}(1+q)^{k-4} - q^{-3/2}(1+q^{-1})^{k-4} \tag{3.3.6}$$

Proof. The proof is by induction on k. It's straightforward to verify that for $k = 3$ we get back the constants c_{xyz} as seen in the beginning of this section. For the induction step we use the formula (3.2.2). We will prove the induction step for the first equation; the other three are essentially the same.

For the boundary data (e, e, e) the formula gives

$$P_{(\Delta,\{e,e,e\})}(\Sigma_{0,k+1}) = P_{(\Delta,\{e,e,e\})}(\Sigma_{0,k}) + P_{(\Delta,\{s,s,e\})}(\Sigma_{0,k})$$

(because $c_{ees} = 0$ these are the only terms that survive). By the induction hypothesis we get

$$\begin{aligned} P_{(\Delta,\{e,e,e\})}(\Sigma_{0,k+1}) &= (1+q)^{k-4} + (1+q^{-1})^{k-4} + q(1+q)^{k-4} + q^{-1}(1+q^{-1})^{k-4} \\ &= (1+q)^{k-3} + (1+q^{-1})^{k-3} \end{aligned}$$

□

Proof of Proposition 3.3.5. Using the cutting formula again and the previous proposition, we now quickly check that

$$\begin{aligned} P(\Sigma_{0,k}) &= P_{(\Delta,\{e,e,e\})}(\Sigma_{0,k}) + 3P_{(\Delta,\{s,s,e\})}(\Sigma_{0,k}) + (q^{1/2} - q^{-1/2})P_{(\Delta,\{s,s,s\})}(\Sigma_{0,k}) \\ &= (1+q)^{k-4} + (1+q^{-1})^{k-4} + 3\left(q(1+q)^{k-4} + q^{-1}(1+q^{-1})^{k-4}\right) + \\ &\qquad + (q^{1/2} - q^{-1/2})\left(q^{3/2}(1+q)^{k-4} - q^{-3/2}(1+q^{-1})^{k-4}\right) \\ &= (1+q)^{k-4}(1+3q+q^2-q) + (1+q^{-1})^{k-4}(1+3q^{-1}+q^{-2}-q^{-1}) \\ &= (1+q)^{k-2} + (1+q^{-1})^{k-2} \end{aligned}$$

□

Chapter 4

Hecke topological quantum field theory

In this chapter we define a generalisation for the invariant from the previous section. Motivated by the appearance of a "cutting formula", we make the connection to Topological Quantum Field Theories (TQFT), starting by giving its definition.

We follow up by developing some algebraic machinery to help us reach our goal of generalising our invariant, which we will show to be a TQFT on the class of ciliated surfaces. The construction is somewhat similar to the first invariant, in that we define it first for triangles, then we glue triangles together while defining what the invariant is for gluings of triangles, and we prove the independence from the triangulation.

We also obtain expressions for the invariants of polygons and consequently for all closed surfaces, by obtaining them as quotients of polygons.

We again end the chapter by providing some examples, to make sense of the construction and to show its nature as a TQFT.

4.1 Axioms for a Topological Quantum Field Theory

We will now present the axioms for a Topological Quantum Field Theory (TQFT) as described by Atiyah in [Ati88], slightly adapted to our case.

In the following section all manifolds are compact, smooth and oriented, and all diffeomorphisms are orientation preserving unless otherwise stated.

We also keep in mind the equivalence between $A^* \otimes_R B$ and $\mathrm{Hom}_R(A, B)$, for a commutative

ring R and free and finitely generated R-modules A and B, given in Chapter 3.

Definition 4.1.1. A TQFT in dimension d defined over an integral domain R consists of the following data:

1. A finitely generated R-module $F(\Sigma)$ associated to each closed d-dimensional manifold Σ;
2. An element $F(M) \in F(\partial M)$ associated to each $(d+1)$-dimensional manifold (with boundary) M.

For a given manifold T (of dimension d or $d+1$) $F(T)$ being a module or an element of a module is never ambiguous, since we may check the dimension of T to clear up any doubt. The only particular case is $\emptyset$, as we may see it as a manifold of any dimension, but in this case we will always specify the dimension.

These data are subject to the following axioms:

i. F is functorial with respect to orientation preserving diffeomorphisms of Σ and M;
ii. F is involutory, i.e. $F(\Sigma^*) = F(\Sigma)^*$ where Σ^* denotes the manifold Σ with the opposite orientation and $F(\Sigma)^*$ is the dual module to $F(\Sigma)$;
iii. $F(\emptyset)$ is non-trivial, that is, $F(\emptyset) = R$ when we see the empty set as a d-dimensional manifold and $F(\emptyset) = 1 \in R$ when we see the empty set as a $(d+1)$-dimensional manifold (the reason we call this "non-triviality" will be clear in a moment);
iv. F is multiplicative in the following sense:
We impose first and foremost the most obvious form of multiplicativity,

$$F(\Sigma_1 \sqcup \Sigma_2) = F(\Sigma_1) \otimes F(\Sigma_2). \tag{iv.a}$$

Moreover if $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup \Sigma_3^*$ and $M = M_1 \cup_{\Sigma_3} M_2$ is the manifold obtained by gluing M_1 and M_2 along the common Σ_3 boundary component (for $\Sigma_1, \Sigma_2, \Sigma_3$ of dimension d and M, M_1, M_2 of dimension $d+1$) then we have the following:

$$F(M) = \langle F(M_1), F(M_2) \rangle, \tag{iv.b}$$

where $\langle -, - \rangle$ denotes the natural pairing

$$F(\Sigma_1) \otimes F(\Sigma_3) \otimes F(\Sigma_3)^* \otimes F(\Sigma_2) \to F(\Sigma_1) \otimes F(\Sigma_2)$$

Remark 4.1.2. In the above case, when $\Sigma_3 = \emptyset$, we have a disjoint union and the axiom becomes the obvious multiplicative axiom with respect to the $(d+1)$-dimensional manifolds:

$$F(M) = F(M_1) \otimes F(M_2) \tag{iv.c}$$

Remark 4.1.3. We will explain how axiom (iii.) is natural and why we call it "non-triviality". From (iv.a) we deduce immediately that, if $\Sigma = \emptyset$ as a d-dimensional manifold, then $F(\Sigma)$

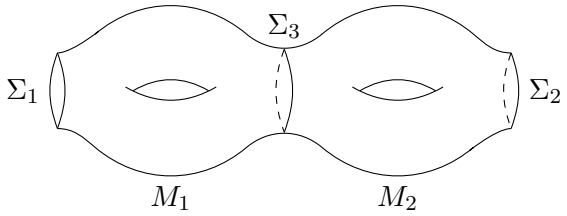


Figure 4.1: Gluing of two manifolds

is an idempotent module (in the sense "A is an idempotent module if $A \otimes A = A$"), therefore it's either the zero module or canonically isomorphic to our base ring R. To exclude the trivial case we then impose

$$F(\emptyset) = R \text{ (for the empty } d\text{-dimensional manifold).} \tag{iii.a}$$

If we don't impose this, then the whole theory becomes trivial: for any Σ we have $F(\Sigma) = F(\Sigma \sqcup \emptyset) = F(\Sigma) \otimes F(\emptyset)$ and if $F(\emptyset)$ were the zero module then all $F(\Sigma)$ would be zero as well.

Now, using (iv.b), we can check also that for $M = \emptyset$ as a $(d+1)$-dimensional manifold we have $F(M) \in F(\partial M) = F(\emptyset) = R$ is an idempotent element. In our case where the base ring is an integral domain we have that the only options are 0 or 1, and so we impose

$$F(\emptyset) = 1 \in R \text{ (for the empty } (d+1)\text{-dimensional manifold).} \tag{iii.b}$$

Cobordisms

If we now want to look at our d-dimensional manifolds as objects in a category and have our $(d+1)$-dimensional manifolds act as cobordisms (morphisms in such a category) we can check that a TQFT is also functorial in this setting.

Let M be a $(d+1)$-dimensional manifold whose boundary is $\partial M = \Sigma_0^* \sqcup \Sigma_1$, that is, M is a cobordism between Σ_0 and Σ_1. Then

$$F(M) \in F(\Sigma_1 \sqcup \Sigma_0^*) \overset{(*)}{=} F(\Sigma_0)^* \otimes F(\Sigma_1) \overset{(3.1.2)}{\cong} \mathrm{Hom}(F(\Sigma_0), F(\Sigma_1)),$$

where the equality marked with $(*)$ comes from axioms (ii.) and (iv.a).

Therefore a cobordism M induces a morphism $F(M) : F(\Sigma_0) \to F(\Sigma_1)$, and moreover, axiom (iv.b) asserts that this respects composition of cobordisms, hence F is functorial in this sense.

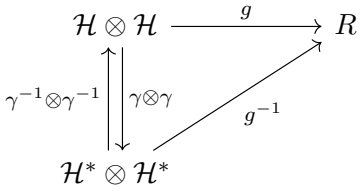
4.2 Generalising the invariant

Algebraic interlude

In the following part we fix a Coxeter system (W, S) and the associated Hecke algebra $\mathcal{H}$.

We apply the previous construction in (3.1.2) to our case, obtaining $\mathrm{Hom}(\mathcal{H}, \mathcal{H}^*) \cong \mathcal{H}^* \otimes \mathcal{H}^* \cong (\mathcal{H} \otimes \mathcal{H})^*$ where the homomorphisms and tensor products are taken over $\mathbb{Z}[v^{\pm 1}]$ (which we will now denote by R for simplicity). So choosing an isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$ is the same as choosing a non degenerate element of $(\mathcal{H} \otimes \mathcal{H})^*$. In what follows we will consider an arbitrary symmetrising trace on $\mathcal{H}$ and denote it by tr.

We can define $g : \mathcal{H} \otimes \mathcal{H} \to R$ by $g(h \otimes h') = \mathsf{tr}(hh')$ and the induced isomorphism $\gamma : \mathcal{H} \to \mathcal{H}^*$ by $\gamma(h) = g(h \otimes -)$. We have the following commutative diagram:



Note that by definition of g and γ we have that for $h, h' \in \mathcal{H}$, $\langle \gamma(h), h' \rangle = g(h \otimes h') = \mathsf{tr}(hh')$, and so for any $\alpha, \beta \in \mathcal{H}^*$ we have

$$\begin{aligned} g^{-1}(\alpha \otimes \beta) &= g(\gamma^{-1}(\alpha) \otimes \gamma^{-1}(\beta)) \\ &= \left\langle \gamma(\gamma^{-1}(\alpha)), \gamma^{-1}(\beta) \right\rangle \\ &= \left\langle \alpha, \gamma^{-1}(\beta) \right\rangle, \end{aligned}$$

giving us

$$g^{-1}(\alpha \otimes \beta) = \left\langle \alpha, \gamma^{-1}(\beta) \right\rangle. \tag{4.2.1}$$

We also present some results related to doing computations using an arbitrary basis, as this will be useful in later sections.

We fix a basis $\{C_i\}_{i \in I}$ of $\mathcal{H}$ and let $\{C^i\}_{i \in I}$ be its dual basis in $\mathcal{H}^*$ so that $C^i(C_j) = \delta_{i,j}$, and also define a basis $\{C_i^\vee\}_{i \in I}$ of $\mathcal{H}$ such that $\mathsf{tr}(C_i C_j^\vee) = \delta_{ij}$. We call the latter the *trace-dual basis of* $\{C_i\}_{i \in I}$. We also introduce the constants g_{ij} and g^{ij} defined by $g = \sum_{i,j} g_{ij} C^i \otimes C^j$ and $g^{-1} = \sum_{i,j} g^{ij} C_i \otimes C_j$.

In the remainder of the text we will use that notation with no further explanation, keeping in mind it makes reference to an arbitrary Hecke algebra and an arbitrary basis, unless stated otherwise.

We can now prove the following:

Proposition 4.2.1. *For any tensor of the form $\sum_i c^i C_i \in \mathcal{H}$ we have that*

$$\gamma\left(\sum_i c^i C_i\right) = \sum_i \left(\sum_j c^j g_{ji}\right) C^i, \tag{4.2.2}$$

and for any tensor of the form $\sum_i c_i C^i \in \mathcal{H}^*$ we have

$$\gamma^{-1}\left(\sum_i c_i C^i\right) = \sum_i \left(\sum_j c_j g^{ji}\right) C_i\,. \tag{4.2.3}$$

Proof. For the first equation we let $\gamma\left(\sum_i c^i C_i\right) = \sum_i c_i C^i$ and we will prove that $c_i = \sum_j c^j g_{ji}$. To do that we apply $\langle C_j, -\rangle$ to both sides and obtain

$$\begin{aligned} c_j &= \left\langle C_j, \gamma\left(\sum_i c^i C_i\right)\right\rangle \\ &= g\left(C_j \otimes \sum_i c^i C_i\right) \\ &= \sum_i c^i g_{ji}, \end{aligned}$$

and to prove the other equation we proceed in a similar fashion, letting $\gamma^{-1}\left(\sum_i c_i C^i\right) = \sum_i c^i C_i$ applying $\left\langle C^k, -\right\rangle$ to both sides, yielding

$$\begin{aligned} c^k &= \left\langle C^k, \gamma^{-1}\left(\sum_i c_i C^i\right)\right\rangle \\ &\overset{(4.2.1)}{=} g^{-1}\left(C^k \otimes \sum_i c_i C^i\right) \\ &= \sum_i c_i g^{ki}. \end{aligned}$$

□

Remark 4.2.2. It is trivial to check, from the definition of γ, that for $a, b \in \mathcal{H}$ we have $\langle \gamma(a), b\rangle = \langle a, \gamma(b)\rangle$. In the same way, for $\alpha \in \mathcal{H}^*$ and $b \in \mathcal{H}$ we have $\langle \alpha, b\rangle = \left\langle \gamma^{-1}(\alpha), \gamma(b)\right\rangle$.

Remark 4.2.3. It may also be very useful to remark that $\gamma^{-1}(C^i) = C_i^\vee$. To see this, let us make the following computation:

$$\begin{aligned} \mathsf{tr}(\gamma^{-1}(C^i)C_j) &= \left\langle \gamma(C_j), \gamma^{-1}(C^i)\right\rangle \\ &= \left\langle C_j, C^i\right\rangle \\ &= \delta_{i,j} \end{aligned}$$

where in the second line we used the previous remark. This implies that indeed $\gamma^{-1}(C^i) = C_i^\vee$.

Now we present a couple of results which involve the trace-dual basis, the latter of which is central to a later result in our construction.

Lemma 4.2.4. *We can express any element $P \in \mathcal{H}$ in the basis of $\mathcal{H}$ or in its trace-dual basis with the aid of the trace as follows:*

$$\sum_m \mathsf{tr}(PC_m^\vee)C_m = P = \sum_m \mathsf{tr}(PC_m)C_m^\vee.$$

Proof. The proof is quite straightforward and is just a couple of computations. To check the first equality we simply multiply both sides by $C_m^\vee$ and take the trace (this works because the trace was required to be symmetrising), and for the second one we do the same, using C_m. □

Proposition 4.2.5. *For generic elements $A, B \in \mathcal{H}$ the following holds:*

$$\mathsf{tr}(AB) = \sum_i \mathsf{tr}(AC_i)\,\mathsf{tr}(C_i^\vee B)$$

Proof. The proof is again just a simple computation. Using the previous lemma we can write

$$AB = \sum_i \mathsf{tr}(AC_i)C_i^\vee B$$

and now we simply take the trace on both sides and obtain the formula stated on the proposition. □

Associating tensors to triangles

Given a triangle as on the left of Figure 4.2 (which we say to have signature $++-$) we assign to it the structure tensor of the Hecke algebra $\mathcal{H}$. This tensor $c \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$ arises from the product on $\mathcal{H}$ and is defined by $c(h \otimes h' \otimes \alpha) = \alpha(hh')$, for all $h, h' \in \mathcal{H}$ and $\alpha \in \mathcal{H}^*$. Now if we label the triangle with suitable elements (elements in $\mathcal{H}$ on the positively oriented sides and elements in $\mathcal{H}^*$ on the negatively oriented sides) we assign to it the scalar obtained by applying the tensor to the labels, as seen on the right side of the figure.

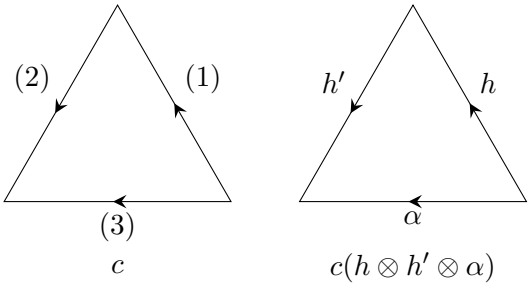


Figure 4.2: Assigning tensors to triangles

If we now have a triangle with a different signature we use γ and γ^{-1} applied to c in such a way as to make the components "match" the side. For instance since $(\mathrm{id} \otimes \gamma^{-1} \otimes \mathrm{id})(c) \in \mathcal{H}^* \otimes \mathcal{H} \otimes \mathcal{H}$ we assign it to a triangle with signature $+--$. We illustrate a couple of examples in Figure 4.3:

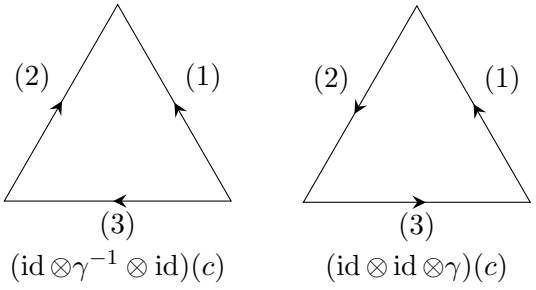


Figure 4.3: Signatures $+--$ and $+++$

When labelling edges we may always use elements of $\mathcal{H}$ to simplify things, where if an element $x \in \mathcal{H}$ is associated to a negatively oriented edge we mean that edge is actually labelled with $\gamma(x) \in \mathcal{H}^*$ and use that in the computations. So the scalar associated to a "standard" triangle (with signature $++-$) whose labels are x, y, z is $c(x, y, \gamma(z)) = \langle \gamma(z), xy \rangle = \mathsf{tr}(xyz)$.[1] In the same way, when we assign the labels x, y and z to a triangle whose signature is $-+-$ we are actually using the elements $\gamma(x) \in \mathcal{H}^*$, y and $\gamma(z) \in \mathcal{H}^*$, and so on.

We now present an easy yet remarkable result which fully justifies this abuse of notation.

Theorem 4.2.6. *For a labelled triangle, the computation of the invariant is independent of the orientation of the edges. This means that after labelling we may always use c for the computations.*

Proof. The statement means that for a given labelling x, y, z we may compute the invariant

[1] Note that, if we use the standard trace and restrict the elements we can use as labels to just the elements of the standard basis, we get the original invariant for the labelled triangles.

as $c(x \otimes y \otimes \gamma(z))$ or as $(\mathrm{id} \otimes \mathrm{id} \otimes \gamma)(c)(x \otimes y \otimes z)$, or as any other combination of id, γ and γ^{-1} before and after c that still yields a valid computation. For that we only need to show that if ϕ is a function mapping from $\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}$ consisting only of id, γ or γ^{-1} on the components, then $\phi \circ c = c \circ \phi$.

The proof stems from remark 4.2.2. We illustrate for the particular example we just gave, and it is clear how it generalises. Writing $c = \sum_i \theta_i \otimes \sigma_i \otimes h_i$ we have

$$\begin{aligned} c(x \otimes y \otimes \gamma(z)) &= \sum_i \langle \theta_i, x \rangle \otimes \langle \sigma_i, y \rangle \otimes \langle h_i, \gamma(z) \rangle \\ &\overset{(*)}{=} \sum_i \langle \theta_i, x \rangle \otimes \langle \sigma_i, y \rangle \otimes \langle \gamma(h_i), z \rangle \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \gamma)(c)(x \otimes y \otimes z) \end{aligned}$$

where we use the note in the equality marked with $(*)$. □

Remark 4.2.7. While it might not be entirely obvious how that algebraic proof actually implies the proposition, we illustrate one more example to convince ourselves.

Consider a triangle with signature $+-+$. If we assign the labels x, y, z we compute the invariant as $(\mathrm{id} \otimes \gamma^{-1} \otimes \gamma)(c)(x \otimes \gamma(y) \otimes z)$. If we now look at this as $(\mathrm{id} \otimes \gamma^{-1} \otimes \gamma) \circ c \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id})(x \otimes y \otimes z)$ we may use the algebraic fact to write it as $c \circ (\mathrm{id} \otimes \gamma^{-1} \otimes \gamma) \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id})(x \otimes y \otimes z)$ and now simply composing the middle factors we are left with $c \circ (\mathrm{id} \otimes \mathrm{id} \otimes \gamma)(x \otimes y \otimes z)$ which is precisely the same invariant as if we were dealing with the standard $++-$ triangle.

A particular case of this is that the constants associated to triangles are invariant under reversing the orientation. This turns out to be a key property that allows us to define constants associated to any ciliated surface in a way that is also invariant under changing the orientation of the surface.

Another important question is that even if we ignore the orientations of the edges when computing the constant for a triangle with a given set of labels x, y, z we still have to be careful to check that the computation yields the same scalar for a cyclic permutation of the same labels. But that is easily resolved by remembering we computed $c(x \otimes y \otimes \gamma(z)) = \mathsf{tr}(xyz)$. For any cyclic permutation of the labels we may use c again, by using the previous Theorem, and get for instance $c(y \otimes z \otimes \gamma(x)) = \mathsf{tr}(yzx)$, and now using the fact that the trace is invariant under cyclic permutations we get the desired equality.

We may also describe these tensors and scalars associated to triangles with the aid of coordinates. We firstly remark that if we describe the tensor c in coordinates as $c = \sum_{i,j,k} {c_{ij}}^k C^i \otimes C^j \otimes C_k$ it is easily proven that in fact ${c_{ij}}^k = \mathsf{tr}\,(C_i C_j C_k^\vee)$ as follows.

Keeping in mind the cyclicity of the trace and the fact that c is the multiplication tensor

and therefore $C_iC_j = \sum_\ell {c_{ij}}^\ell C_\ell$ we have

$$\begin{aligned}\mathsf{tr}\left(C_iC_jC_k^\vee\right) &= \mathsf{tr}\left(C_k^\vee\left(\sum_\ell {c_{ij}}^\ell C_\ell\right)\right)\\ &= \sum_\ell {c_{ij}}^\ell \,\mathsf{tr}\left(C_k^\vee C_\ell\right)\\ &= \sum_\ell {c_{ij}}^\ell \delta_{k,\ell}\\ &= {c_{ij}}^k\end{aligned}$$

With this in hand, we have that the tensor associated to a triangle with signature $++-$, which is the structure tensor of the Hecke algebra, can be written as

$$c = \sum_{i,j,k} \mathsf{tr}\left(C_iC_jC_k^\vee\right) C^i \otimes C^j \otimes C_k. \tag{4.2.4}$$

It can easily be shown that the same is true for triangles of other signatures, for instance

$$(\gamma^{-1}\otimes 1\otimes 1)(c) = \sum_{i,j,k} \mathsf{tr}\left(C_i^\vee C_jC_k^\vee\right) C_i \otimes C^j \otimes C_k$$

and

$$(1\otimes 1\otimes \gamma)(c) = \sum_{i,j,k} \mathsf{tr}\left(C_iC_jC_k\right) C^i \otimes C^j \otimes C^k.$$

Gluing triangles

The natural pairing $\mathcal{H}^*\otimes\mathcal{H}\to R$ allows us to glue triangles along an edge. Consider two triangles drawn as on the left of Figure 4.4, which we intend to glue to make the square on the right.

To keep track of which copy of $\mathcal{H}$ or $\mathcal{H}^*$ we are associating to each side, and to which triangle we are assigning each tensor, we will indicate that with subscripts. So $c_d \in \mathcal{H}_1^*\otimes\mathcal{H}_2^*\otimes\mathcal{H}_3$ means that the tensor associated to the upper triangle on the figure "lives" on a tensor product $\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{H}$ where we use the labels to imply that the first component corresponds to the edge marked with (1) and so on. In the same way, $c_u \in \mathcal{H}_4^*\otimes\mathcal{H}_5^*\otimes\mathcal{H}_6^*$ and $c_q \in \mathcal{H}_1^*\otimes\mathcal{H}_2^*\otimes\mathcal{H}_5^*\otimes\mathcal{H}_6^*$. It is of course clear that c_d is c itself and that c_u is $(\mathrm{id}\otimes\mathrm{id}\otimes\gamma)(c)$, and again we just keep the labels to aid us in remembering which triangle the tensors correspond to.

When gluing the triangles, the obvious thing to do on the tensors is to apply the map $\mathcal{H}_1^*\otimes\mathcal{H}_2^*\otimes\mathcal{H}_3\otimes\mathcal{H}_4^*\otimes\mathcal{H}_5^*\otimes\mathcal{H}_6^* \to \mathcal{H}_1^*\otimes\mathcal{H}_2^*\otimes\mathcal{H}_5^*\otimes\mathcal{H}_6^*$ which simply consists of pairing

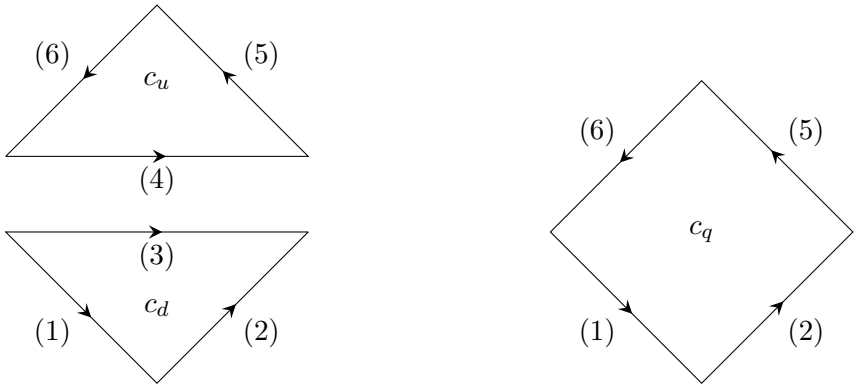


Figure 4.4: Gluing triangles

the elements in the middle, that is, for pure tensors we have $\alpha \otimes \beta \otimes a \otimes \theta \otimes \sigma \otimes \tau \mapsto \langle a, \theta \rangle \, \alpha \otimes \beta \otimes \sigma \otimes \tau$ and extended by linearity to any sum of pure tensors. Of course the same principle applies if we do the same for any other types of triangles as long as we glue sides with opposite orientations, and as we will see later we may also glue any kind of polygon, as soon as we define the tensors for arbitrary polygons.

As we did before with the original invariants, we will now show that the tensor associated to a square is invariant with respect to the orientation of the edge where the gluing occurs.

Proposition 4.2.8. *For a square as on the right of the above figure, the constant obtained is the same regardless of the orientation of the edges (3) and (4) on the triangles on the left. Moreover this is true for any gluing of this kind regardless of the signature of the triangles involved.*[2] *In algebraic terms this means that for any elements $a \in \mathcal{H}$ and $\alpha \in \mathcal{H}^*$ we have $\langle \alpha, a \rangle = \langle \gamma^{-1}(\alpha), \gamma(a) \rangle$.*

Proof. The proof is a direct consequence of Remark 4.2.2. □

Proposition 4.2.9. *Given a quadrilateral, the associated tensor does not depend on the diagonal along which we glue the triangle components. That is, gluing both pairs of triangles of Figure 4.5 yelds the same tensor.*

Proof. To prove this we will simply compute both invariants using generic coordinates to see they are equal. For any other combination of orientations of the outer edges of the quadrilateral, the proof carries over exactly the same.

Let us start with the gluing on the left. We pair up the tensors $\sum_{i,j,k} \mathsf{tr}\,(C_i C_j C_k)\, C^i \otimes C^j \otimes C^k$

[2] As long as we glue edges with opposite orientations.

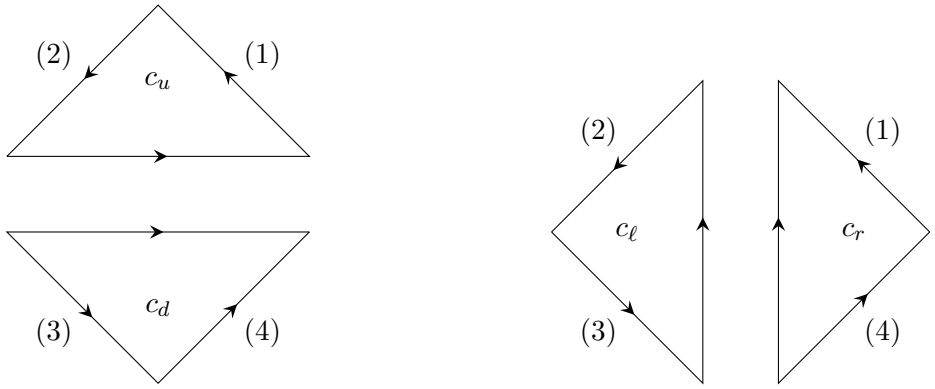


Figure 4.5: Invariance with respect to the diagonal.

and $\sum_{\ell,m,n} \mathsf{tr}\left(C_\ell^\vee C_m C_n\right) C_\ell \otimes C^m \otimes C^n$ to obtain

$$\sum_{i,j,k,\ell,m,n} \mathsf{tr}\left(C_i C_j C_k\right) \mathsf{tr}\left(C_\ell^\vee C_m C_n\right) \left\langle C^k, C_\ell \right\rangle C^i \otimes C^j \otimes C^m \otimes C^n.$$

Substituting $\left\langle C^k, C_\ell \right\rangle = \delta_{k,\ell}$ we get

$$\sum_{i,j,m,n} \left(\sum_k \mathsf{tr}\left(C_i C_j C_k\right) \mathsf{tr}\left(C_k^\vee C_m C_n\right) \right) C^i \otimes C^j \otimes C^m \otimes C^n,$$

and using proposition 4.2.5 we finally obtain that the tensor associated to the gluing on the left is

$$\sum_{i,j,m,n} \mathsf{tr}\left(C_i C_j C_m C_n\right) C^i \otimes C^j \otimes C^m \otimes C^n.$$

Doing the exact same kind of computations, and taking care to "preserve" the place of each copy of $\mathcal{H}$ on the tensor product, we obtain that the tensor on the right is given by

$$\sum_{i,j,m,n} \mathsf{tr}\left(C_n C_i C_j C_m\right) C^i \otimes C^j \otimes C^m \otimes C^n.$$

By cyclicity of the trace we confirm that both the tensors are the same and complete the proof. □

We are now ready to prove the central result of this Thesis.

Theorem 4.2.10. *We can associate a tensor to any ciliated surface, provided we set an orientation for the boundary segments.*

Proof. It is obvious that by triangulating a ciliated surface and successively applying the gluing construction we obtain a tensor. This tensor is independent of the triangulation because of Proposition 4.2.9 and the fact that any two triangulations of a ciliated surface are related by a finite number of diagonal flips of quadrilaterals (Theorem 2.1.7). □

Since our construction behaves well under gluing, we may create invariants for ciliated surfaces from a triangulation in any way we find more convenient. In particular, we may construct ciliated surfaces from polygons by identifying some edges in a suitable manner, and compute the invariants that way. Firstly, we present and prove an elementary lemma that is needed for this construction.

Lemma 4.2.11. *Consider a polygon with n edges. The tensor associated to it is*

$$c_n = \sum_{w_i \in W} \mathsf{tr}\left(C'_{w_1} \cdots C'_{w_n}\right) C^{w_1}_{w_1} \otimes \cdots \otimes C^{w_n}_{w_n}$$

where $C'_{w_i} = C^{\vee}_{w_i}$ and $C^{w_i}_{w_i} = C_{w_i}$ if the i-th edge is positively oriented; or $C'_{w_i} = C_{w_i}$ and $C^{w_i}_{w_i} = C^{w_i}$ in case the i-th edge is negatively oriented.

Proof. We will prove the lemma by induction on the number of edges. For the case $n = 3$ we already know it is true from the previous discussion on triangles. Now assume it is true for all polygons with n or less sides and we will check it holds for a polygon with $n+1$ sides. We can decompose such a polygon into a triangle and an n-gon, as in Figure 4.6.

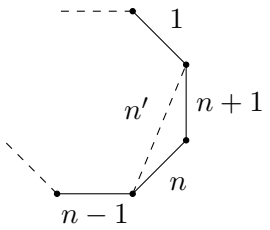


Figure 4.6: Induction step

By the induction hypothesis we know that

$$c_n = \sum_{\substack{w_i \in W \\ w_{n'} \in W}} \mathsf{tr}\left(C'_{w_1} \cdots C'_{w_{n-1}} C'_{w_{n'}}\right) C^{w_1}_{w_1} \otimes \cdots \otimes C^{w_{n-1}}_{w_{n-1}} \otimes C^{w_{n'}}_{w_{n'}}$$

and

$$c_3 = \sum_{w_i \in W} \mathsf{tr}\left(C'_{w_{n'}} C'_{w_n} C'_{w_{n+1}}\right) C^{w_{n'}}_{w_{n'}} \otimes C^{w_n}_{w_n} \otimes C^{w_{n+1}}_{w_{n+1}},$$

where we note that the edge marked with n' has the opposite orientation in the triangle as it does in the n-gon.

When we glue these polygons along this same edge, we obtain

$$\sum_{w_i\in W}\left(\sum_{w_{n'}\in W}\mathsf{tr}\left(C'_{w_1}\cdots C'_{w_{n-1}}C'_{w_{n'}}\right)\mathsf{tr}\left(C'_{w_{n'}}C'_{w_n}C'_{w_{n+1}}\right)\right)C^{w_1}_{w_1}\otimes\cdots\otimes C^{w_{n-1}}_{w_{n-1}}\otimes C^{w_n}_{w_n}\otimes C^{w_{n+1}}_{w_{n+1}}.$$

Now, using Proposition 4.2.5, we finally obtain

$$\sum_{w_i\in W}\mathsf{tr}\left(C'_{w_1}\cdots C'_{w_{n-1}}C'_{w_n}C'_{w_{n+1}}\right)C^{w_1}_{w_1}\otimes\cdots\otimes C^{w_{n-1}}_{w_{n-1}}\otimes C^{w_n}_{w_n}\otimes C^{w_{n+1}}_{w_{n+1}},$$

which completes the induction step. □

Remark 4.2.12. For the tensor invariants, the point at which we start reading the edges of a polygon (including a triangle) is relevant, in the sense that starting at different points may yield different tensors. But the tensors we obtain differ only up to a permutation of the components, and when we apply such tensors to labels, this distinction is lost. The same is true if we glue all edges together and end up with a tensor which is in fact a scalar., as the order of the computations we made is not relevant. An example of this will be seen in the next theorem.

With this, we may now compute the invariants for closed surfaces by realising them as gluings of polygons. We remark that for closed surfaces all edges of such a polygon will be paired with another one, meaning the invarian will be a scalar in $\mathbb{Z}\left[v^{\pm 1}\right]$.

We may realise the surface $\Sigma_{g,k}$ as a polygon with $4g+2k-2$ sides with the labels $x_1, y_1, x'_1, y'_1, \ldots, x_g, y_g, x'_g, y'_g, w_1, w'_1, \ldots, w_{k-1}, w'_{k-1}$, where a regular label stands for an edge that's directly oriented and the same label with a prime means the label is the same but the orientation is reversed. To realise the surface we glue the edges pairwise, according to the labels.

Theorem 4.2.13. *The invariant for the closed ciliated surface $\Sigma_{g,k}$ is*

$$P_{g,k,W}=\mathsf{tr}\left(\left(\sum_{w\in W}C_wC_w^\vee\right)^{k-1}\left(\sum_{x,y\in W}C_xC_yC_x^\vee C_y^\vee\right)^g\right) \tag{4.2.5}$$

Proof. The proof is a simple computation, taking into account the construction of $\Sigma_{g,k}$ specified above, and using 4.2.11. For the polygon, befoe gluing, the invariant is

$$\sum_{x_i,y_i,w_i\in W}\mathsf{tr}\left(C_{x_1}C_{y_1}C_{x'_1}^\vee C_{y'_1}^\vee\cdots C_{w_1}C_{w'_1}^\vee\cdots\right)C^{x_1}\otimes C^{y_1}\otimes C_{x'_1}\otimes C_{y'_1}\otimes\cdots\otimes C^{w_1}\otimes C_{w'_1}\cdots.$$

Now, gluing the polygon to form $\Sigma_{g,k}$ is done by pairing up each t_i with the corresponding t'_i, where t is any of x, y or w. This gives us

$$P_{g,k,W} = \sum_{x_i,y_i,w_i \in W} \mathsf{tr}\left(C_{x_1}C_{y_1}C^{\vee}_{x_1}C^{\vee}_{y_1}C_{x_2}C_{y_2}C^{\vee}_{x_2}C^{\vee}_{y_2}\cdots C_{w_1}C^{\vee}_{w_1}C_{w_2}C^{\vee}_{w_2}\cdots\right).$$

Now we use the linearity of the trace to express this as

$$P_{g,k,W} = \mathsf{tr}\left(\sum_{x_i,y_i,w_i \in W} C_{x_1}C_{y_1}C^{\vee}_{x_1}C^{\vee}_{y_1}C_{x_2}C_{y_2}C^{\vee}_{x_2}C^{\vee}_{y_2}\cdots C_{w_1}C^{\vee}_{w_1}C_{w_2}C^{\vee}_{w_2}\cdots\right),$$

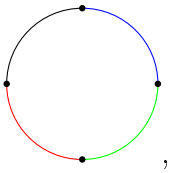
and using the fact we are summing over a finite set of indices, we may further simplify the expression and write

$$P_{g,k,W} = \mathsf{tr}\left(\left(\sum_{x_1,y_1 \in W} C_{x_1}C_{y_1}C^{\vee}_{x_1}C^{\vee}_{y_1}\right)\left(\sum_{x_2,y_2 \in W} C_{x_2}C_{y_2}C^{\vee}_{x_2}C^{\vee}_{y_2}\right)\cdots \left(\sum_{w_1 \in W} C_{w_1}C^{\vee}_{w_1}\right)\left(\sum_{w_2 \in W} C_{w_2}C^{\vee}_{w_2}\right)\cdots\right),$$

which is of course the same as 4.2.5, completing the proof. □

4.3 The tensor invariant as a TQFT

Let $\mathcal{C}$ be the category whose objects are disjoint unions of oriented segments and whose morphisms between two sets of disjoint segments A and B are ciliated surfaces whose boundary is $A^* \sqcup B$, that is, the morphisms are the cobordisms. We note that we see a circle boundary as a disjoint union of segments, whose number is the same as the number of cilia on it, as if the segments are the "pieces" in between the cilia. By this, we mean that a boundary circle, such as the one that follows



,

is to be seen as the disjoint union of the segments between the cillia,



.

Given a Hecke algebra $\mathcal{H}$ let us now consider the functor $F : \mathcal{C} \to \mathbb{Z}\left[v^{\pm 1}\right]-\mathsf{Mod}$ which associates a copy of $\mathcal{H}$ to a positively oriented segment, a copy of $\mathcal{H}^*$ to a negatively oriented segment, and associates the tensor product of the appropriate number of $\mathcal{H}$ and $\mathcal{H}^*$ to a disjoint union of segments.

For a cobordism $\Sigma \in \mathrm{Hom}(A, B)$, where A and B are disjoint unions of segments, the homomorphism of modules $F(\Sigma) \in F(A)^* \otimes F(B)$ is the tensor invariant which we just constructed, and which is well defined by the previous Theorem.

Theorem 4.3.1. *This functor satisfies the axioms of a TQFT presented in Definition 4.1.1.*

Proof. The proof is straightforward, we just need to check the axioms are satisfied one by one.

Axiom i. is trivially satisfied for the segments, and since orientation preserving diffeomorphisms also map a triangulation of the first surface to a triangulation of the second and our invariant only depends on the triangulation, the axiom is also satisfied for the ciliated surfaces.

Axiom ii. is also immediate from the construction: if we reverse the orientation of all boundary segments then the associated module is the tensor product of the duals of the first ones, therefore the dual of the tensor product.

Axiom iii. is immediate in the case of the segments since we do have a non-trivial module, $\mathbb{Z}\left[v^{\pm 1}\right]$ itself, associated to an empty set of segments. As for the empty ciliated surface, this choice is truly free so we simply add that its invariant is $1 \in \mathbb{Z}\left[v^{\pm 1}\right]$.

Finally, axiom iv. follows from the gluing construction, which was defined precisely to match the axiom. □

Some examples

We will now see a couple of examples to understand what it means for this tensor invariant to be a TQFT.

Consider a cylinder whose boundary components are oriented in different directions and have a cilium each. We also label the boundary components, to index them for the computations which will follow.

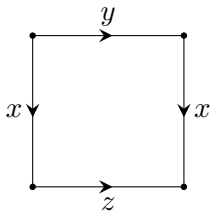


Figure 4.7: Cylinder

Using Lemma 4.2.11 and gluing the appropriate edges, we know that the invariant for this surface is

$$\sum_{x,y,z} \mathsf{tr}(C_x C_z C_x^\vee C_y^\vee) C^z \otimes C_y \quad \in \mathcal{H}^* \otimes \mathcal{H}. \tag{4.3.1}$$

We may see this cylinder as a cobordism from one of its boundary components to the other one, say from the one labelled with z to the one labelled with y in the figure. Taking advantage of the new description of this invariant as a TQFT we may also take the expression (4.3.1) and interpret it as a map from $\mathcal{H}$ to itself, using (3.1.1). This would then be the map

$$\begin{aligned} &\mathcal{H} \to \mathcal{H} \\ &h \mapsto \sum_{x,y,z} \mathsf{tr}(C_x C_z C_x^\vee C_y^\vee) C^z(h) \otimes C_y. \end{aligned}$$

We may also compose cobordisms, which will be the same as composing the maps, in terms of the TQFT. To illustrate this, consider now the next figure, which is a disk with a cilium and a puncture.

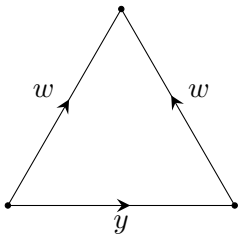


Figure 4.8: Punctured disk with a single cilium

The invariant for this figure is

$$\sum_{w,y} \mathsf{tr}(C_y C_w C_w^\vee) C^y \quad \in \mathcal{H}^*, \tag{4.3.2}$$

and by the same considerations as before, this can be seen as a cobordism from a circle with a cilium to a single point. In terms of the TQFT this is a map from $\mathcal{H}$ to R.

We may now compose these cobordisms, giving us another cobordism from a circle with a cilium to a single point, which in algebraic terms is a map from $\mathcal{H}$ to R, whose expression is

$$h \mapsto \sum_{w,x,z} \mathsf{tr}(C_x C_z C_x^\vee C_w C_w^\vee) C^z(h),$$

which is obtained by pairing both expressions on the components labelled by y and contracting the product of the traces by using Proposition 4.2.5.

Chapter 5

Explicit expressions for the Hecke TQFT

In this section we will present some explicit expressions for the more general tensor invariant. When we have a closed surface, the invariant is itself a polynomial already, which we will compute using the Schur elements of the Hecke algebra we are working with. We will also compute the invariant for a ciliated surface with a single boundary component, and in this case we will need both the Schur elements as well as the characters of the Hecke algebra. Due to this, we start by developing some representation theory.

5.1 Representation theory

Generalities

We will need to present some general module and ring theoretical results to be able to develop the representation theory we need. In this section, unless otherwise mentioned, all algebras are unital and finitely generated, and all modules are right-modules, and finitely generated.

A *semisimple algebra* $\mathcal{A}$ over a commutative ring R is an algebra which, when seen as a right module over itself, decomposes into a direct sum of simple modules. Equivalently, no element in $\mathcal{A}$ other than 0 annihilates all simple modules and $\mathcal{A}$ is artinian (see [Lam01, Theorem 4.14]).

We may write that as

$$\mathcal{A}_{\mathcal{A}} \cong \bigoplus_i V_i^{n_i}, \tag{5.1.1}$$

where the direct sum has a finite number of summands and each $V_i \ncong V_j$ if $i \neq j$. Moreover, the V_i are all the simple $\mathcal{A}$-modules (up to isomorphism).

By the theorems of Wedderburn and Schur, we have that

$$\mathcal{A} \cong \prod_i M_{n_i}(D_i), \tag{5.1.2}$$

as R-algebras, where $D_i = \mathrm{End}_{\mathcal{A}}(V_i)$ is a division ring over R, for each i.

Later we will be working in the case where $R = \mathbb{K}$ is a field of characteristic 0, and in that case we may have that an $\mathcal{A}$-module V is such that $\mathrm{End}_A(V) \cong \mathbb{K}$, and in that case we say V is a *split simple* module. If an algebra $\mathcal{A}$ is such that all its simple modules are split simple, we say it is a *split* algebra, and in that case the Wedderburn decomposition is

$$\mathcal{A} \cong \prod_i M_{n_i}(\mathbb{K}), \tag{5.1.3}$$

and it can be easily shown that each n_i is in fact the dimension of V_i, whence the decomposition can be further expressed as

$$\mathcal{A} \cong \prod_i \mathrm{End}_{\mathbb{K}}(V_i), \tag{5.1.4}$$

We remark that any $\varphi \in \mathrm{End}_{\mathbb{K}}(V_i)$ (when seen as an element of $\mathcal{A}$) acts as 0 on V_j, when $i \neq j$.

We may also use the language of representations, which will be useful.

Definition 5.1.1. Let V be an $\mathcal{A}$-module which is free and finitely generated over R. We write the action of $\mathcal{A}$ on V as a right multiplication, that is, if $a \in \mathcal{A}$ and $v \in V$, we write a acting on v as va. We have a homomorphism of algebras

$$\rho_V : \mathcal{A} \to \mathrm{End}_R^{\mathrm{op}}(V),$$

where $v\rho_V(a) = va$ for $a \in \mathcal{A}$ and $v \in V$. We call ρ_V the *representation afforded by* V. The *character afforded by* V is the R-linear map $\chi_V : \mathcal{A} \to R$ defined by $\chi_V(a) = \mathsf{Tr}(\rho_V(a))$, where Tr denotes the usual matrix trace. Since ρ_V is a homomorphism of algebras and Tr satisfies the property $\mathsf{Tr}(AB) = \mathsf{Tr}(BA)$, we have that indeed χ_V is a trace function on $\mathcal{A}$.

In this language, simple modules correspond to *irreducible representations* and *irreducible characters*. We denote by $\mathsf{Irr}(\mathcal{A})$ the set of all irreducible representations of $\mathcal{A}$.

In this framework, when we are working over a field and $\mathcal{A}$ is split, we write the Wedderburn decomposition as

$$\mathcal{A} = \prod_{\lambda \in \mathsf{Irr}(\mathcal{A})} \mathrm{End}_{\mathbb{K}}(V_\lambda), \tag{5.1.5}$$

where the V_λ are all the simple $\mathcal{A}$-modules and λ is the irreducible representation afforded by V_λ.

Moreover, we can deduce that the irreducible characters form a basis of the space of trace functions on $\mathcal{A}$. This is due to the fact that on a matrix algebra the only trace functions are scalar multiples of the standard matrix trace, and on a direct sum of matrix algebras the traces are the sums of traces on each summand, therefore the irreducible characters span the space of trace functions. We are left to check the linear independence of the irreducible characters, but that can easily be seen as follows:

Suppose that $\sum_{\lambda_i \in \mathsf{Irr}(\mathcal{A})} a_i \chi_{\lambda_i} = 0$, with $a_i \in \mathbb{K}$, for all i, and consider the element id_i of $\mathcal{A}$ which acts as the identity on V_{λ_i} and as 0 on the other V_{λ_j} (cf. (5.1.5)). Applying the linear combination to this element yields simply $a_i \dim_{\mathbb{K}}(V_{\lambda_1}) = 0$, which forces $a_i = 0$, since the characteristic of the field is 0. Since this works for all i, we obtain the linear independence of the irreducible characters.

Using this fact, we can prove the following:

Proposition 5.1.2. *Let $\mathcal{A}$ be a split algebra over a field $\mathbb{K}$ and let $\rho_V : \mathcal{A} \to \mathrm{End}_R(V)$ and $\rho'_V : \mathcal{A} \to \mathrm{End}_R(V')$ be representations. We have that $\chi_V = \chi'_V$ if and only if V and V' are isomorphic as $\mathcal{A}$-modules.*

Proof. ($\Leftarrow$): Let $\varphi : V \mapsto V'$ be an isomorphism such that $\rho'_V(a) = \varphi \circ \rho_V(a) \circ \varphi^{-1}$, for all $a \in \mathcal{A}$. Then

$$\begin{aligned} \chi'_V &= \mathsf{Tr}(\rho'_V) \\ &= \mathsf{Tr}(\varphi \circ \rho_V \circ \varphi^{-1}) \\ &= \mathsf{Tr}(\rho_V) \\ &= \chi_V, \end{aligned}$$

where in the third equality we use a well known property of the matrix trace.

($\Rightarrow$): Using the representations point of view, write $V \cong \bigoplus_{\lambda \in \mathsf{Irr}(\mathcal{A})} V_\lambda^{m_\lambda}$ and $V' \cong \bigoplus_{\lambda \in \mathsf{Irr}(\mathcal{A})} V_\lambda^{n_\lambda}$, where m_λ and n_λ are the multiplicities of the factor V_λ on the decompositions, which may be 0. Since clearly $\chi_{W \oplus W'} = \chi_W + \chi_{W'}$, for all $\mathcal{A}$-modules W and W', we get

$$\chi_V = \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} m_\lambda \chi_\lambda,$$

and similarly,

$$\chi'_V = \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} n_\lambda \chi_\lambda.$$

From the equality of these two expressions and the fact the irreducible characters form a basis of the space of trace functions, we conclude that $m_\lambda = n_\lambda$ for each λ, whence the modules are isomorphic, as stated. □

Some representation theory of symmetric algebras

As we know by now, a Hecke algebra admits a symmetrising trace (see Definition 3.1.1), which means it is what we call a *symmetric algebra*. There is a more general theory for the representations of symmetric algebras which we will develop here, to later apply to our particular case.

We fix similar notation as before: R is a commutative unital ring, $\mathcal{A}$ is an unital, free and finitely generated associative algebra over R. The set of all trace functions in $\mathcal{A}$ is an R-module with operations defined pointwise.

If τ is a symmetrising trace on $\mathcal{A}$ and $\{B_i\}_{i\in I}$ is a basis of $\mathcal{A}$, we may define its dual basis $\{B_i^\vee\}_{i\in I}$ in relation to τ to be the basis such that $\tau\left(B_iB_j^\vee\right) = \delta_{i,j}$, as before. From now on we fix a symmetric algebra $\mathcal{A}$ with symmetrising trace τ and we use the notation B_i and $B_i^\vee$ without further explanation when we use an arbitrary basis and its τ-dual basis.

We will now present a way to associate an element of $\mathcal{A}$ to each element of $\mathcal{A}^* := \mathrm{Hom}_R(\mathcal{A}, R)$, in a similar fashion as we did before with γ^{-1}. The purpose of this will become clear after the proposition that follows.

Given $\alpha \in \mathcal{A}^*$ we can uniquely define an element $\alpha^* \in \mathcal{A}$ by the condition $\tau(\alpha^* a) = \alpha(a)$ for all $a \in \mathcal{A}$.

This is well defined since $\mathcal{A} \to \mathcal{A}^*; \alpha^* \mapsto \tau(\alpha^* -)$ is injective, and since $\mathcal{A} \cong \mathcal{A}^*$ surjectivity is also immediate. This means that for a given α there is indeed an unique corresponding α^* satisfying the definition. Moreover this also shows this operation to be an isomorphism.

It is easy to see that, given an R-basis $\{B_i\}_{i\in I}$ of $\mathcal{A}$ and its τ-dual basis, we have

$$\alpha^* = \sum_i \alpha(B_i)B_i^\vee.$$

To see this, simply compute $\tau\left(\left(\sum_i \alpha(B_i)B_i^\vee\right)B_j\right) = \sum_i \alpha(B_i)\delta_{ij} = \alpha(B_j) = \tau(\alpha^* B_j)$.

We define the commutator of two elements a, a' of $\mathcal{A}$ as $[a, a'] := aa' - a'a$, and $[\mathcal{A}, \mathcal{A}]$ as the R-submodule of $\mathcal{A}$ spanned by all the commutators of elements of $\mathcal{A}$. By definition we have that $\alpha \in \mathcal{A}^*$ is a trace function if and only if $[\mathcal{A}, \mathcal{A}] \subseteq \ker(\alpha)$.

Lemma 5.1.3 ([GP00] Lemma 7.1.7)**.** *Assume $\mathcal{A}$ admits a symmetrising trace τ. Let α be an element of $\mathcal{A}^*$ and let α^* be its corresponding element in $\mathcal{A}$. Then α is a trace function if and only if α^* is in the centre of $\mathcal{A}$.*

Proof. We know that α is a trace function if and only if $\alpha(aa') = \alpha(a'a)$ for all $a, a' \in \mathcal{A}$, or equivalently, using the definition of α^*, $\tau(\alpha^*(aa' - a'a)) = 0$.

On the other hand we have that $\alpha^* \in Z(\mathcal{A})$ if and only if $\alpha^* a - a\alpha^* = 0$, for all $a \in \mathcal{A}$. This last statement is also equivalent to $\tau((\alpha^* a - a\alpha^*)a') = 0$ for all $a, a' \in \mathcal{A}$, due to τ being non-degenerate.

We also have that $\tau(\alpha^*(aa' - a'a)) = 0$ is equivalent to $\tau(\alpha^* aa') - \tau(\alpha^* a'a) = 0$. Furthermore, $\tau((\alpha^* a - a\alpha^*)a') = 0$ is equivalent to $\tau(\alpha^* aa') - \tau(a\alpha^* a') = 0$. But now it is immediate that the two simplified expressions are equivalent due to τ being cyclic, which establishes the original equivalence, proving the result. □

This means that the space of trace functions $T(\mathcal{A})$ is isomorphic to $Z(\mathcal{A})$. Now we also take the time to give a new definition which makes use of this.

Definition 5.1.4. Assume $\mathcal{A}$ admits a symmetrising trace. If $\lambda : \mathcal{A} \to \mathrm{End}_R(V)$ is a representation of $\mathcal{A}$, we define Z_λ to be the element of $\mathcal{A}$ corresponding to the character χ_λ in the way stated above. By the previous lemma, we have that $Z_\lambda \in Z(\mathcal{A})$, since characters are trace functions.

Remark 5.1.5. From the fact that the irreducible characters form a basis of the space of trace functions, we immediately get that the central elements $\{Z_\lambda\}_{\lambda \in \mathsf{Irr}(\mathcal{A})}$ form a basis of $Z(\mathcal{A})$.

Remark 5.1.6. We remark that for any $a \in \mathcal{A}$, by definition, $\tau(Z_\lambda a) = \chi_\lambda(a)$.

Proposition 5.1.7. *We can compute the central element Z_λ associated to a representation λ using the formula*

$$Z_\lambda = \sum_i \chi_\lambda(B_i) B_i^\vee,$$

where $\{B_i\}_{i \in I}$ is a basis of $\mathcal{A}$.

Proof. Using the fact that $h = \sum_i \tau(B_i h) B_i^\vee$ we compute

$$Z_\lambda = \sum_i \tau(B_i Z_\lambda) B_i^\vee = \sum_i \chi_\lambda(B_i) B_i^\vee$$

using the definition of Z_λ in the second equality. □

Definition 5.1.8. Let V and V' be right $\mathcal{A}$-modules. For any $\varphi \in \mathrm{Hom}_R(V, V')$ we define $I(\varphi) \in \mathrm{Hom}_R(V, V')$ as

$$I(\varphi)(v) := \sum_i \varphi(vB_i) B_i^\vee \tag{5.1.6}$$

for any $v \in V$.

Remark 5.1.9. This definition is unambiguous: it does not depend on the basis we use to define it. This will be shown in the next result.

Lemma 5.1.10. *For any $\varphi \in \mathrm{Hom}_R(V, V')$, the homomorphism $I(\varphi) \in \mathrm{Hom}_R(V, V')$ defined above is in $\mathrm{Hom}_{\mathcal{A}}(V, V')$ and it does not depend on choice of the basis $\{B_i\}$.*

Proof. Let us check the independence of the basis. Let $\{C_i\}_{i\in I}$ be another basis of $\mathcal{A}$ and denote by $\{C_i^\vee\}_{i\in I}$ its τ-dual basis, as usual. If we write $C_j = \sum_i \mu_{ij} B_i$, where $(\mu_{ij})_{ij}$ is an invertible matrix with entries in R, we can easily check that $B_j^\vee = \sum_i \mu_{ji} C_i^\vee$. In fact $\tau(B_j^\vee C_i) = \sum_k \mu_{ki}\tau(B_j^\vee B_k) = \mu_{ji}$. We can now compute

$$\begin{aligned}\sum_j \varphi(vC_j)C_j^\vee &= \sum_{i,j} \mu_{ij}\varphi(vB_i)C_j^\vee \\ &= \sum_i \varphi(vB_i)\left(\sum_j \mu_{ij} C_j^\vee\right) \\ &= \sum_i \varphi(vB_i)B_i^\vee,\end{aligned}$$

for all $v \in V$, as we wished to show.

We now check that the action of $\mathcal{A}$ commutes with $I(\varphi)$, establishing that $I(\varphi)$ is indeed a homomorphism of $\mathcal{A}$-modules. For $a \in \mathcal{A}$ write $aB_j = \sum_i \rho(a)_{ij} B_i$, for $\rho(a)_{ij} \in R$. Then we have that

$$\tau\left(B_i^\vee a B_j\right) = \sum_k \rho(a)_{kj}\tau(B_i^\vee B_k) = \rho(a)_{ij},$$

and thus, $B_i^\vee a = \sum_j \rho(a)_{ij} B_j^\vee$. With this we compute

$$\begin{aligned}I(\varphi)(va) &= \sum_j \varphi(vaB_j)B_j^\vee \\ &= \sum_{i,j} \rho(a)_{ij}\varphi(vB_i)B_j^\vee \\ &= \sum_i \varphi(vB_i)\left(\sum_j \rho(a)_{ij} B_j^\vee\right) \\ &= \sum_i \varphi(vB_i)B_i^\vee a \\ &= (I(\varphi)(v))a,\end{aligned}$$

proving the result. □

Remark 5.1.11. Let V, V', and V'' be $\mathcal{A}$-modules and let $\varphi \in \mathrm{Hom}_{\mathbb{K}}(V, V')$. If $\psi : V' \to V''$ is an $\mathcal{A}$-module homomorphism, then $I(\psi \circ \varphi) = \psi \circ I(\varphi)$. By the definitions, each term, when applied to an element $v \in V$, is respectively equal to

$$\sum_i \psi\left(\varphi(vB_i)\right) B_i^\vee,$$

$$\psi\left(\sum_i \varphi(vB_i) B_i^\vee\right).$$

It is easy to see that these are equal if ψ commutes with the action of $\mathcal{A}$.

Similarly, if $\pi : V'' \to V$ is an $\mathcal{A}$-module homomorphism, then $I(\varphi \circ \pi) = I(\varphi) \circ \pi$.

Schur elements

Let us now fix a finite dimensional symmetric algebra $\mathcal{A}$ over a field $\mathbb{K}$ with symmetrising trace τ. Let V be a simple $\mathcal{A}$-module. Recall the Wedderburn decomposition and the definition of a split simple module and of a split algebra given at the beginning of this section.

Theorem 5.1.12. *Let V be a split simple $\mathcal{A}$-module. Then there is a unique element $c_V \in \mathbb{K}$ such that*

$$I(\varphi) = c_V \,\mathsf{Tr}(\varphi)\,\mathrm{id}_V, \tag{5.1.7}$$

for all $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$. Moreover, the element c_V only depends on the isomorphism class of V.

Proof. Firstly we note that given $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$, $I(\varphi)$ is actually in $\mathrm{End}_{\mathcal{A}}(V)$ by Lemma 5.1.10, and since V is split simple, $I(\varphi) = c_{V,\varphi}\,\mathrm{id}_V$ always holds for some $c_{V,\varphi}$ in $\mathbb{K}$.

To prove this expression is the one presented in the statement of the Theorem, we will assume the dimension of V over $\mathbb{K}$ is n and we take a basis $\{v_1, \ldots, v_n\}$ of V. For $1 \leq i, j \leq n$ we take $\varphi_{ij} \in \mathrm{End}_{\mathbb{K}}(V)$ to be defined by $\varphi_{ij}(v_\ell) = \delta_{i\ell}(v_j)$. In terms of matrices, these are the elementary matrices, whose entries are all 0 except for the entry i, j, which is 1; it is therefore clear that these endomorphisms form a basis of $\mathrm{End}_{\mathbb{K}}(V)$. To simplify notation, write c_{ij} instead of $c_{V,\varphi_{ij}}$.

Let us remind ourselves of the matrix representation afforded by V (Definition 5.1.1), where in this case, $\mathrm{End}_{\mathbb{K}}(V) \cong M_n(\mathbb{K})$, and we can write $\rho : A \to M_n(\mathbb{K})$ (dropping the V subscript as to not overload the notation). Note that, for $a \in \mathcal{A}$, we can write $v_\ell a = \sum_k \rho(a)_{k\ell} v_k$.

Keeping in mind the defining expression for $I(-)$ we can compute

$$\begin{aligned} I(\varphi_{ij})(v_\ell) &= \sum_t \varphi_{ij}(v_\ell B_t) B_t^\vee \\ &= \sum_t \varphi_{ij}\left(\sum_k \rho(B_t)_{k\ell} v_k\right) B_t^\vee \\ &= \sum_t \rho(B_t)_{i\ell} v_j B_t^\vee \\ &= \sum_{t,k} \rho(B_t)_{i\ell}\rho(B_t^\vee)_{kj} v_k. \end{aligned}$$

Now, using that $I(\varphi_{ij}) = c_{ij}\,\mathrm{id}_V$ we conclude

$$\sum_t \rho(B_t)_{i\ell}\rho(B_t^\vee)_{kj} = \delta_{k\ell} c_{ij} \quad \text{for all } i,j,k,\ell.$$

We can interchange the roles of B_t and $B_t^\vee$ and perform the same computation, obtaining instead

$$\sum_t \rho(B_t^\vee)_{i'\ell'}\rho(B_t)_{k'j'} = \delta_{k'\ell'} c_{i'j'} \quad \text{for all } i',j',k',\ell',$$

and now setting $i' = k$, $\ell' = j$, $k' = i$ and $j' = \ell$, we obtain equality on the left hand side of both expressions, and hence the right hand side is also equal. This leaves us with

$$\delta_{k\ell} c_{ij} = \delta_{ij} c_{k\ell}.$$

Set $c_V := c_{11}$ and set $k = 1, \ell = 1$, thus obtaining $c_{ij} = \delta_{ij} c_V$, that is, $I(\varphi_{ij}) = \delta_{ij} c_V\,\mathrm{id}_V$. Noting that $\mathsf{Tr}(\varphi_{ij}) = \delta_{ij}$, it holds true that $I(\varphi_{ij}) = c_V\,\mathsf{Tr}(\varphi_{ij})\,\mathrm{id}_V$. From the defining expression of $I(\varphi)$, we see that $I(-)$ is linear in the arguments, therefore so is the assignment $\varphi \mapsto c_{V,\varphi}$. As remarked before, the φ_{ij} form a basis of $\mathrm{End}_{\mathbb{K}}(V)$, which together with the linearity of the assignment and the fact that $I(\varphi_{ij}) = c_V\,\mathsf{Tr}(\varphi_{ij})\,\mathrm{id}_V$, finally shows that the expression holds for all $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$.

It remains to show that $c_V = c_{V'}$ if V and V' are isomorphic modules. Let $\sigma : V' \to V$ be an isomorphism of modules and let $\varphi' := \sigma^{-1} \circ \varphi \circ \sigma$. We compute

$$c_{V'}\,\mathsf{Tr}(\varphi')\,\mathrm{id}_{V'} = I(\varphi') = I(\sigma^{-1} \circ \varphi \circ \sigma) = \sigma^{-1} \circ I(\varphi) \circ \sigma = c_V\,\mathsf{Tr}(\varphi)\sigma^{-1} \circ \sigma \circ \mathrm{id}_V,$$

where in the middle we use Note 5.1.11, because σ and its inverse are clearly module homomorphisms. Once we cancel out σ^{-1} and σ and remark that $\mathsf{Tr}(\varphi) = \mathsf{Tr}(\varphi')$, we choose some $\varphi \in \mathrm{End}_{\mathbb{K}}(V)$ whose trace is nonzero and conclude that $c_V = c_{V'}$. □

Definition 5.1.13. Given V a split simple $\mathcal{A}$-module we say c_V is the *Schur element* associated to V.

Remark 5.1.14. When we are using the language of representations, we denote by s_λ the Schur element associated to the irreducible representation λ, where we use the split simple $\mathcal{A}$-module V_λ in the defining expression.

Corollary 5.1.15 (Schur relations)**.** *Let V and V' be split simple $\mathcal{A}$-modules affording matrix representations $\rho : \mathcal{A} \to M_m(\mathbb{K})$ and $\rho' : \mathcal{A} \to M_n(\mathbb{K})$, respectively. Then we have*

$$\sum_{t\in I} \rho(B_t)_{i\ell}\rho'(B_t^\vee)_{jk} = \begin{cases} \delta_{ij}\delta_{k\ell}c_V \text{ if } V = V' \text{ and } \rho = \rho', \\ 0 \text{ if } V \ncong V', \end{cases} \tag{5.1.8}$$

for all $0 \le i, \ell \le m$ and $0 \le j, k \le n$.

Proof. Choose a basis $\{v_1, \dots, v_n\}$ of V and a basis $\{v'_1, \dots, v'_m\}$ of V', and let $\varphi_{ij} \in \mathrm{Hom}_{\mathbb{K}}(V, V')$ be defined by $\varphi_{ij}(v_\ell) = \delta_{i\ell}v'_j$, for $1 \le i \le n$ and $1 \le j \le m$. This is, in terms of matrices, the elementary $m \times n$ matrix whose entries are all 0 except for the entry i, j, which is 1. As in the previous proof, we note that we can write $v_\ell a = \sum_{k=1}^m \rho(a)_{k\ell}v_k$ and $v'_\ell a = \sum_{k=1}^n \rho'(a)_{k\ell}v_k$ using the matrix representations. Again, we compute

$$\begin{aligned} I(\varphi_{ij})(v_\ell) &= \sum_t \varphi_{ij}(v_\ell B_t)B_t^\vee \\ &= \sum_t \varphi_{ij}\left(\sum_{k=1}^m \rho(B_t)_{k\ell}v_k\right) B_t^\vee \\ &= \sum_t \rho(B_t)_{i\ell}v'_j B_t^\vee \\ &= \sum_t \rho(B_t)_{i\ell} \sum_{k=1}^n \rho'(B_t^\vee)_{kj}v'_k. \end{aligned}$$

If $V = V'$ then by the previous theorem we know that $I(\varphi_{ij})(v_\ell) = \delta_{ij}c_V v_\ell$, and therefore we conclude that $\sum_{t\in I} \rho(B_t)_{i\ell}\rho'(B_t^\vee)_{jk} = \delta_{ij}\delta_{k\ell}c_V$

By Schur's lemma, if $V \ncong V'$, $I(\varphi_{ij}) = 0$, whence we get the second line of the relations. □

Theorem 5.1.16 (Orthogonality relations)**.** *Let V and V' be split simple $\mathcal{A}$-modules affording characters χ_V and $\chi_{V'}$ respectively. Then*

$$\sum_{t\in I} \chi_V(B_t)\chi_{V'}(B_t^\vee) = \begin{cases} c_V \dim_{\mathbb{K}} V & \text{if } \chi_{V'} = \chi_V, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1.9}$$

Proof. Using the notation of the previous proof, we may write $\chi_V(a) = \sum_{i=1}^m \rho_{ii}(a)$ and $\chi'_V(a) = \sum_{j=1}^n \rho'_{jj}(a)$ (noting each character is the trace of the corresponding matrix repre-

sentation). Thus, the left hand side of the equation becomes

$$\sum_{t\in I}\chi_V(B_t)\chi_{V'}(B_t^\vee) = \sum_{t\in I}\left(\sum_{i=1}^{m}\rho_{ii}(B_t)\right)\left(\sum_{j=1}^{n}\rho'_{jj}(B_t^\vee)\right)$$
$$= \sum_{i,j=1}^{m,n}\left(\sum_{t}\rho_{ii}(B_t)\rho'_{jj}(B_t^\vee)\right).$$

We now evaluate the terms inside the sum using the Schur relations. Using the fact that $\chi_V = \chi'_V$ if and only if $V \cong V'$ (Prop. 5.1.2) we get that the terms evaluate to 0 if $\chi_V \neq \chi'_V$. When $\chi_V = \chi'_V$, we have $V \cong V'$ (and so $n = m$), and by the Schur relations, each summand is equal to $\delta_{ij}c_V$, whence the expression equals $\sum_{i=1}^{m} c_V = \dim_V c_V$. □

Proposition 5.1.17. *Given irreducible representations λ and μ we have that Z_μ acts as 0 on V_λ if $\lambda \neq \mu$ and Z_λ acts on V_λ as multiplication by s_λ. Moreover, the Schur elements are all nonzero.*

Proof. We rewrite the orthogonality relations in terms of irreducible characters as such:

$$\sum_{t\in I}\chi_\lambda(B_t)\chi_\mu(B_t^\vee) = \begin{cases} s_\lambda \dim_{\mathbb{K}} V_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1.10}$$

and we take the left hand side, rewriting it as

$$\sum_{t\in I}\chi_\lambda(B_t)\chi_\mu(B_t^\vee) = \chi_\lambda\left(\sum_{t\in I} B_t\chi_\mu(B_t^\vee)\right)$$
$$= \chi_\lambda(Z_\mu),$$

where in the second equality we used the Proposition 5.1.7.

Now, since V_λ is a split simple module, we know that all $\mathcal{A}$-module endomorphisms are given by a scalar multiplication. We note what if z is a central element of $\mathcal{A}$, then for any $\mathcal{A}$-module M, the action of z on M is an endomorphism of $\mathcal{A}$-modules, since $(ma)z = m(az) = m(za)$, for any $m \in M$. This means that in particular we may write the endomorphism given by the action of Z_μ on V_λ as $vZ_\mu = \xi_\mu v$, for all $v \in V_\lambda$ and for some $\xi_\mu \in \mathbb{K}$. With this in mind, we have on one hand

$$\chi_\lambda(Z_\mu) = \mathsf{Tr}(\rho(Z_\mu)) = \xi_\mu \dim_{\mathbb{K}} V_\lambda,$$

and on the other, by the orthogonality relations,

$$\chi_\lambda(Z_\mu) = \begin{cases} s_\lambda \dim_{\mathbb{K}} V_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

leading us to conclude that $\xi_\mu = 0$ if $\lambda \neq \mu$ and $\xi_\lambda = s_\lambda$. Finally, if $s_\lambda = 0$ for any irreducible representation λ, we would have Z_λ acting as 0 on all simple modules V_λ, contradicting the fact that $\mathcal{A}$ is semisimple. This concludes the proof. □

Proposition 5.1.18. *We may write*

$$1 = \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} Z_\lambda, \tag{5.1.11}$$

or, equivalently,

$$\tau = \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} \chi_\lambda. \tag{5.1.12}$$

Proof. Taking into account the previous proposition, it is easy to see that $\sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} Z_\lambda$ acts as 1 on all simple modules: for a given simple module V_μ, all terms of the sum act as 0 except for the one corresponding to μ, which acts as $\frac{1}{s_\mu} s_\mu = 1$. This shows the expression to be equal to $1 \in \mathcal{A}$.

With this expression we now obtain the second one, applying $\tau(-a)$ to both sides for some $a \in \mathcal{A}$. We thus obtain

$$\begin{aligned} \tau(a) &= \tau\left(\sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} Z_\lambda a \right) \\ &= \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} \tau(Z_\lambda a) \\ &= \sum_{\lambda \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\lambda} \chi_\lambda(a). \end{aligned}$$

Since this is true for all $a \in \mathcal{A}$ we obtain the desired expression. □

Proposition 5.1.19. *The set* $\{Z_\lambda\}_{\lambda \in \mathsf{Irr}(\mathcal{A})}$ *forms a basis of* $Z(\mathcal{A})$ *such that*

$$Z_\lambda Z_\mu = \delta_{\lambda\mu} s_\lambda Z_\lambda \tag{5.1.13}$$

for all $\lambda, \mu \in \mathsf{Irr}(\mathcal{A})$

Proof. The fact the elements $\{Z_\lambda\}_{\lambda \in \mathsf{Irr}(\mathcal{A})}$ form a basis of the centre of the algebra has been established in Remark 5.1.5. To prove the expression, let us first assume $\mu \neq \lambda$. In this case, it is clear that $Z_\lambda Z_\mu$ acts as 0 on all simple $\mathcal{A}$-modules: if $\gamma \neq \lambda$, taking $v \in V_\gamma$ we see that $v(Z_\lambda Z_\mu) = (vZ_\lambda)Z_\mu = 0$, and for $v \in V_\lambda$ we have $v(Z_\lambda Z_\mu) = v(Z_\mu Z_\lambda) = (vZ_\mu)Z_\lambda = 0$, since both Z_λ and Z_μ are central. We conclude that $Z_\lambda Z_\mu = 0$ by the semisimplicity of $\mathcal{A}$.

Now using this fact and the expression for 1 given by the previous proposition, we have that

$$\begin{aligned} Z_\lambda &= 1 \cdot Z_\lambda \\ &= \left(\sum_{\mu \in \mathsf{Irr}(\mathcal{A})} \frac{1}{s_\mu} Z_\mu \right) Z_\lambda \\ &= \frac{1}{s_\lambda} Z_\lambda^2, \end{aligned}$$

concluding the proof. □

Hecke algebras

Now with all the machinery we developed above, we are ready to apply it to the case where we have a Hecke algebra $\mathcal{H}$ with trace tr, defined over $R = \mathbb{Z}\left[v^{\pm 1}\right]$. To make use of the above section, we would somehow need to find a way to make our Hecke algebras be defined over a field and split semisimple, in order to use the "better" version of the Wedderburn decomposition and to be able to define Schur elements for $\mathcal{H}$.

The first of these requirements is easy to deal with. We simply consider the field $\mathbb{K}' = \mathbb{C}(v)$ and tensor our Hecke algebra with it, obtaining the $\mathbb{K}'$-algebra $\mathcal{H}' = \mathcal{H} \otimes_R \mathbb{K}'$.

This still leaves us with an obstacle to overcome, namely the fact that we need to consider a Hecke algebra which is split semisimple. Fortunately there is a very powerful (albeit not easy to prove) theorem which deals with this.

Theorem 5.1.20 ([GP00] 8.1.7)**.** *Let $\mathcal{H}$ be a Hecke algebra. Then there exists a finite Galois extension $\mathbb{K} \supseteq \mathbb{C}(v)$ such that the $\mathbb{K}$-algebra $\mathbb{K}\mathcal{H} = \mathcal{H} \otimes_R \mathbb{K}$ is split semisimple and isomorphic to the group algebra $\mathbb{K}W$.*

With this in hand we might now work over $\mathbb{K}\mathcal{H}$ to make use of all the definitions and results we saw thus far.

5.2 Explicit expressions

In this section we replace our Hecke algebra $\mathcal{H}$ by $\mathbb{K}\mathcal{H}$ where $\mathbb{K}$ is a Galois extension $\mathbb{K} \supseteq \mathbb{C}(v)$ such that $\mathbb{K}\mathcal{H}$ is split semisimple, as in the previous Theorem.

Obtaining the expressions

For a closed surface we know, by 4.2.5, that expressions of the form

$$P_{g,k,W} = \mathsf{tr}\left(\left(\sum_{w\in W} C_w C_w^\vee\right)^{k-1}\left(\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee\right)^g\right)$$

arise on our computations. We will show that elements of the form

$$\left(\sum_{w\in W} C_w C_w^\vee\right)^{k-1}\left(\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee\right)^g$$

are in the centre of the Hecke algebra, and we will find expressions for them in terms of the characters and Schur elements of the Hecke algebra. In the process we will do the the same for elements of the form $\sum_{w\in W} C_w h C_w^\vee$ (for $h \in \mathcal{H}$), which arise when computing invariants for surfaces with a single boundary component.

Theorem 5.2.1. *The elements of the form*

$$\left(\sum_{w\in W} C_w C_w^\vee\right)^{k-1}\left(\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee\right)^g$$

and the elements of the form

$$\sum_{w\in W} C_w h C_w^\vee\ , \quad \text{with } h \in \mathcal{H}$$

are in the centre of the Hecke algebra.

Proof. This argument is geometric and using the fact that these expressions are precisely the ones that arise in our invariants, and is therefore not dependent on us working on $\mathbb{K}\mathcal{H}$ instead of $\mathcal{H}$.

Let $s = \left(\sum_{w\in W} C_w C_w^\vee\right)^{k-1}\left(\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee\right)^g$. To show that s is in the centre of the Hecke algebra, we may show that $\mathsf{tr}(sh_1h_2) = \mathsf{tr}(h_1sh_2)$, due to the trace being non-degenerate. The second member is also $\mathsf{tr}(sh_2h_1)$ since the trace is cyclically invariant.

But note that $\mathsf{tr}(sh_1h_2)$ is the invariant for the surface of genus g and k punctures with a single boundary disk with two cilia, $\Sigma_{g,k,\{2\}}$, where we assign the labels h_1 and h_2 to the boundary segments. To see this simply take the expression in Lemma 4.2.11 and adapt the proof of Theorem 4.2.13 to the case where we glue all but two edges labelled h_1 and h_2. On the other hand, $\mathsf{tr}(sh_2h_1)$ is the invariant for the same surface, where instead we assign the labels in the order h_2, h_1.

It is clear that we can obtain one of these surfaces from the other by simply rotating it in space, as in Figure 5.1.

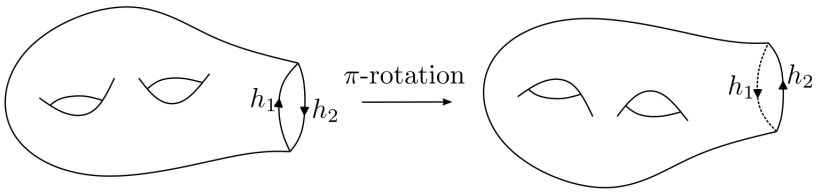


Figure 5.1: Exchanging labels on a surface with a single boundary with two cilia

We thus show the expressions to be equal, since they are both invariants for the same surface, thereby concluding that s is in the centre of the Hecke algebra, as needed.

Fixing some $h \in \mathcal{H}$, we show $t = \sum_{w\in W} C_w h C_w^\vee$ is in the centre of $\mathcal{H}$ in a similar fashion. We show $\mathsf{tr}(th_1h_2) = \mathsf{tr}(th_2h_1)$ by realising both as the same surface, apart from a rotation. This time it is a cylinder with two boundary components, the first with a single cilium and labelled by h and the second with two cilia and labelled by h_1 and h_2. Again, a rotation of the surface exchanges h_1 and h_2. □

We are now going to deduce explicit expressions for the "elementary building blocks" of our invariants, $\sum_{w\in W} C_w h C_w^\vee$ and $\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee$ in terms of the Schur elements and the central elements of Prop. 5.1.19. With these, we are able to obtain explicit and more easily computed expressions for our invariants.

Theorem 5.2.2. *For any $h \in \mathcal{H}$ we have that*

$$\sum_{w\in W} C_w h C_w^\vee = \sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} \chi_\lambda(h) Z_\lambda, \tag{5.2.1}$$

and

$$\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee = \sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} s_\lambda Z_\lambda. \tag{5.2.2}$$

In particular, setting $h = 1$ in equation (5.2.1),

$$\sum_{w\in W} C_w C_w^\vee = \sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} \dim(V_\lambda) Z_\lambda. \tag{5.2.3}$$

Proof. For the first assertion, we fix a simple module V_λ and we let $\varphi_h \in \mathrm{End}_{\mathbb{K}}(V_\lambda)$ be the action induced by right multiplication by h, which is the same as $\rho_{V_\lambda}(h)$ (c.f. Def. 5.1.1). Let $h' \in V_\lambda$ and write $\sum_w C_w h C_w^\vee = \sum_\mu e_\mu Z_\mu$, which we can do since the elements Z_λ form

a basis of the centre and we proved that the left hand side of the equation is in the centre as well. We have

$$\begin{aligned}\sum_w h'C_whC_w^\vee &= \sum_w \varphi_h(h'C_w)C_w^\vee \\ &\overset{(5.1.6)}{=} I(\varphi_h)(h') \\ &\overset{(5.1.7)}{=} s_\lambda \,\mathsf{Tr}(\varphi_h)h' \\ &= s_\lambda \,\mathsf{Tr}(\rho_{V_\lambda}(h))h' \\ &= s_\lambda\chi_\lambda(h)h'.\end{aligned}$$

On the other hand,

$$\sum_w h'C_whC_w^\vee = \sum_\mu e_\mu Z_\mu h' = e_\lambda s_\lambda h',$$

since Z_μ acts as 0 on V_λ when $\mu \neq \lambda$ and as the scalar s_λ when $\mu = \lambda$. We therefore conclude that $e_\lambda = \chi_\lambda(h)$.

For the second assertion, we consider $\varphi_{\lambda,x} : V_\lambda \to V_\lambda$ which maps $v \in V_\lambda$ to $\varphi_{\lambda,x}(v) = vC_x^\vee$.

On one hand, using the defining expression for $I(-)$ we have

$$I(\varphi_{\lambda,x})(v) = \sum_y \varphi_{\lambda,x}(vC_y)C_y^\vee = \sum_y vC_yC_x^\vee C_y^\vee,$$

and on the other, using the definition of Schur elements,

$$I(\varphi_{\lambda,x})(v) = s_\lambda \,\mathsf{Tr}(\varphi_{\lambda,x})v = s_\lambda\chi_\lambda(C_x^\vee)v.$$

We thus conclude that

$$\sum_y vC_yC_x^\vee C_y^\vee = s_\lambda\chi_\lambda(C_x^\vee)v \quad \text{for all } v \in V_\lambda. \tag{5.2.4}$$

Now we recall the Wedderburn decomposition

$$\mathcal{H} = \bigoplus_{\lambda\in\mathsf{Irr}(\mathcal{H})} End_{\mathbb{K}}(V_\lambda).$$

Let $(B_i)_{i\in I}$ be an adapted basis of $\mathcal{H}$ for this matrix decomposition, that is, each element is an elementary matrix on one of the summands. Of course $|I| = |W|$ but to avoid confusion we will not index the basis by elements of the group. Let $\lambda(i)$ be the unique irreducible representation such that B_i is in $\mathrm{End}_{\mathbb{K}}(V_{\lambda(i)})$. As usual, denote by $(B_i^\vee)_{i\in I}$ its dual basis.

If we write $B_i = \sum_j b_{ij}B_j^\vee$ we have $b_{ij} = \mathsf{tr}(B_iB_j) = 0$, if $\lambda(i) \neq \lambda(j)$, that is, $B_i \in \mathsf{span}\left\{B_j^\vee \mid \lambda(i) = \lambda(j)\right\}$.

Thus, fixing some λ, that $\mathsf{span}\{B_i \mid \lambda(i) = \lambda\} \subseteq \mathsf{span}\{B_i^\vee \mid \lambda(i) = \lambda\}$ and by a dimensionality argument, the two vector spaces are the same.

This implies, again by Schur's lemma, that $B_i^\vee B_j = 0$ if $\lambda(i) \neq \lambda(j)$ and so $\chi_\lambda(B_i^\vee) = 0$ for $\lambda \neq \lambda(i)$.

From equation (5.2.4), using $v = B_i \in V_{\lambda(i)}$ we obtain

$$\sum_j B_i B_j B_i^\vee B_j^\vee = s_{\lambda(i)} \chi_{\lambda(i)}(B_i^\vee) B_i, \tag{5.2.5}$$

which yields

$$\begin{aligned}
\sum_{i,j} B_i B_j B_i^\vee B_j^\vee &= \sum_i s_{\lambda(i)} \chi_{\lambda(i)}(B_i^\vee) B_i \\
&= \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} s_\lambda \sum_{i \mid \lambda(i) = \lambda} \chi_\lambda(B_i^\vee) B_i \\
&\overset{(*)}{=} \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} s_\lambda \sum_i \chi_\lambda(B_i^\vee) B_i \\
&= \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} s_\lambda Z_\lambda,
\end{aligned}$$

where in the last equality we used Proposition 5.1.7 and in the equality marked with $(*)$ we used the fact that $\chi_\lambda(B_i^\vee) = 0$ for $\lambda \neq \lambda(i)$. □

With this, we are finally ready to prove the explicit expressions we sought after.

Theorem 5.2.3. *The invariant for the closed ciliated surface $\Sigma_{g,k}$ is given by*

$$P_{g,k,W} = \sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{2g-2+k}. \tag{5.2.6}$$

Proof. We already know that the expression for the polynomial is given by

$$P_{g,k,W} = \mathsf{tr}\left(\left(\sum_{w \in W} C_w C_w^\vee \right)^{k-1} \left(\sum_{x,y \in W} C_x C_y C_x^\vee C_y^\vee \right)^g \right).$$

Using the previous Theorem, this is equal to

$$\mathsf{tr}\left(\left(\sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} \dim(V_\lambda) Z_\lambda \right)^{k-1} \left(\sum_{\lambda \in \mathsf{Irr}(\mathcal{H})} s_\lambda Z_\lambda \right)^g \right),$$

and using 5.1.13, we may write this as

$$\mathsf{tr}\left(\sum_{\lambda\in\mathsf{Irr}(\mathcal{H})}(\dim V_\lambda)^{k-1}{Z_\lambda}^{k-1}{s_\lambda}^g{Z_\lambda}^g\right),$$

since all other terms involve products of the form $Z_\mu Z_\lambda$ with $\mu\neq\lambda$, which are zero. Using again the same expression we may reduce this further to

$$\mathsf{tr}\left(\sum_{\lambda\in\mathsf{Irr}(\mathcal{H})}(\dim V_\lambda)^{k-1}{s_\lambda}^{2g+k-2}Z_\lambda\right).$$

Finally, we note that $\mathsf{tr}\, Z_\lambda=\chi_\lambda(1)=\dim(V_\lambda)$, which finally yields the expression in the statement. □

We may also now compute the invariant for any ciliated surface as long as we label the boundary components beforehand. Consider then a ciliated surface $\Sigma=\Sigma_{g,k,\{c_1,\dots,c_n\}}$ with labels pre-assigned. Denote by h_i a product of the labels of the i-th boundary circle following the orientation induced by Σ. We say *a product* because, of course, this is only defined up to a cyclic permutation depending on the cilia we start our reading from, but the formulas will be constant on the conjugacy classes, therefore there is no issue with this.

Theorem 5.2.4. *Let Σ and h_i, with i ranging from 1 to n be as above. Then the invariant is*

$$P_{\Sigma,W}=\sum_{\lambda\in\mathsf{Irr}(\mathcal{H})}(\dim V_\lambda)^k{s_\lambda}^{2g-2+k+n}\left(\prod_i\chi_\lambda(h_i)\right)\tag{5.2.7}$$

Proof. We can assume $n\geq 1$ since we have considered the case of a surface with no boundary in the previous Theorem. We may obtain the ciliated surface Σ as follows:

1. start with a disk with a single cilium;
2. add g handles, obtaining a term $\left(\sum_{x,y\in W}C_xC_yC_x^\vee C_y^\vee\right)^g$;
3. add k punctures, obtaining a term $\left(\sum_{w\in W}C_wC_w^\vee\right)^{k-1}$;
4. add $n-1$ boundary components labelled $h_2,\dots,h_n$, obtaining a term $\prod_{i=2}^n\left(\sum_{w\in W}C_wh_iC_w^\vee\right)$;
5. add the label h_1 to the initial boundary component.

We thus obtain

$$\begin{aligned}
P_{\Sigma,W} &= \mathsf{tr}\left(\left(\sum_{x,y\in W} C_x C_y C_x^\vee C_y^\vee\right)^g \left(\sum_{w\in W} C_w C_w^\vee\right)^{k-1} \prod_{i=2}^{n}\left(\sum_{w\in W} C_w h_i C_w^\vee\right) h_1\right) \\
&= \mathsf{tr}\left(\left(\sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{g+k-1} Z_\lambda\right) \prod_{i=2}^{n}\left(\sum_{\lambda_i\in \mathsf{Irr}(\mathcal{H})} \chi_{\lambda_i}(h_1) Z_{\lambda_i}\right)\right) \\
&\overset{(*)}{=} \sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{g+k-2+1} \left(\prod_{i=2}^{n} \chi_\lambda(h_i)\right) \mathsf{tr}\left(Z_\lambda h_1\right) \\
&= \sum_{\lambda\in \mathsf{Irr}(\mathcal{H})} (\dim V_\lambda)^k {s_\lambda}^{2g-2+k+n} \left(\prod_i \chi_\lambda(h_i)\right),
\end{aligned}$$

as we wanted to show. Note that in the equality marked with $(*)$ we used the expression (5.1.13) again, as well as $\mathsf{tr}(Z_\lambda h_1) = \chi_\lambda(h_1)$ for the last equality.

□

Remark 5.2.5. From the expression we just computed we can now affirm that a puncture is equivalent to a boundary component labelled by 1: by substituting one of the h_i by 1 we obtain an extra factor of $\dim V_\lambda = \chi_\lambda(1)$.

Remark 5.2.6. We can see this formula remains consistent with the cutting property: when summing expressions of this form over all the possible boundary components we may use formula (5.1.9) to note that when we have a product of the characters on the formula, the only ones that count towards it are the ones where we are computing the same character, and that counts with a factor of $s_\lambda \dim V_\lambda$, reflecting the fact that punctures along which we cut the surface correspond to cilia of the component surfaces.

Some explicit computations

In this final part we will use the expressions we have just obtained to present some examples of the computation of invariants.

One needs only the Schur elements and the dimensions of the irreducible representations of the particular Hecke algebra we choose to work with. In [GP00] a way is provided for computing character tables for the classical types of Hecke algebras, as well as for the exceptional ones (where the dimensions of the irreducible representations feature), and in [Chl16, Example 2.5], there are references for finding the Schur elements for all known families of Hecke algebras.

Example 5.2.7. Let us compute by hand a small example. We fix the Coxeter group $W = I_2(3)$ (c.f. Example 2.2.4) and work over its Hecke algebra, $\mathcal{H} = \mathcal{H}_W$. We will

compute the polynomial invariant for a sphere with 3 punctures, using 5.2.3. In this case, $g = 0$ and $k = 3$.

We use [GP00, Theorem 8.3.4] to obtain the Schur elements:

- $s_1 = 1 + 2q + 2q^2 + q^3$, whose corresponding representation has degree 1;
- $s_2 = q^{-3} + 2q^{-2} + 2q^{-1} + 1$, whose corresponding representation has degree 1;
- $s_3 = q^{-1} + 1 + q$, whose corresponding representation has degree 2.

We are now ready to compute the invariant. Using the formula and the fact that $2g-2+k = 1$ we obtain

$$P_{0,3,I_2(3)} = 1^3 s_1 + 1^3 s_2 + 2^3 s_3 = q^{-3} + 2q^{-2} + 10q^{-1} + 10 + 10q + 2q^2 + q^3$$

Example 5.2.8. We will now compute a lager example, for which we will be using GAP3 (see [S+97]) with the module CHEVIE (see [GHL+96]) to obtain the needed Schur elements and degrees of representations, as well as to compute the invariant itself, for which we will perform all the computations using CHEVIE, following the procedure seen in [FTT21].

Let us use $W = S_4$, which in the world of Coxeter groups, is called "Weyl group of type A_3" (c.f. 2.2.6). For computing, for instance, the invariant over its Hecke algebra on the surface of genus 2 with 2 punctures, $\Sigma_{2,2}$, we run the following code:

```
g:=2;;
k:=2;;
W:= CoxeterGroup("A",3);;
v:=X(Cyclotomics);; v.name:="v";;
H:=Hecke(W,[[v,-v^-1]]);;
T:=Basis(H,"T");;
schur:=SchurElements(H);;
list:=[1..Length(schur)];;
dim:=HeckeCharValues(T());;
Sum(list,i->dim[i]^k*schur[i]^(2*g-2+k));
```

obtaining the output

```
v^48 + 12*v^46 + 74*v^44 + 312*v^42 + 1007*v^40 + 2640*v^38 + 5826*v^36
+ 11076*v^34 + 18427*v^32 + 27120*v^30 + 35572*v^28 + 41784*v^26
+ 44083*v^24 + 41856*v^22 + 35860*v^20 + 27912*v^18 + 20123*v^16
+ 14076*v^14 + 10426*v^12 + 9096*v^10 + 9714*v^8 + 11832*v^6 + 14754*v^4
+ 17340*v^2 + 18378 + 17340*v^(-2) + 14754*v^(-4) + 11832*v^(-6)
+ 9714*v^(-8) + 9096*v^(-10) + 10426*v^(-12) + 14076*v^(-14)
+ 20123*v^(-16) + 27912*v^(-18) + 35860*v^(-20) + 41856*v^(-22)
+ 44083*v^(-24) + 41784*v^(-26) + 35572*v^(-28) + 27120*v^(-30)
+ 18427*v^(-32) + 11076*v^(-34) + 5826*v^(-36) + 2640*v^(-38)
+ 1007*v^(-40) + 312*v^(-42) + 74*v^(-44) + 12*v^(-46) + v^(-48)
```

On the program, the first two lines are setting the parameters, the following four are defining the Coxeter group and corresponding Hecke algebra, the next three are computing the Schur elements and dimensions of the corresponding characters, and finally, the last line is computing the invariant using Theorem 5.2.3.

As stated before, we see this is still a function of $q = v^{-2}$ and moreover, we observe it is symmetric under the transformation $q \mapsto q^{-1}$. In fact, it can be proven that all of these invariants have this symmetry, since it is well known that the Schur elements are symmetric in this way (see [Chl16, Formula (2.2)] or [FTT21, Prop. B2]). We also remark that the fact that the invariants for punctured surfaces are functions of $q = v^{-2}$ is proved in [FTT21, Theorem 4.21], but in this work we lack the means to prove it ourselves.

Looking at these examples, we may also be led to wonder if the coefficients are always positive. This is not the case, as can be seen if we perform the computation of the invariant for the surface $\Sigma_{1,3}$ using the Coxeter group $W = I_2(5)$, whose output is reproduced below, where we see, among others, that the constant term has a negative coefficient:

```
v^30 + 6*v^28 + 18*v^26 + 38*v^24 + 66*v^22 + 99*v^20 + 128*v^18 + 144*v^16
+ 144*v^14 + 128*v^12 + 99*v^10 + 66*v^8 + 438*v^6 - 582*v^4 + 1806*v^2
- 1198 + 1806*v^(-2) - 582*v^(-4) + 438*v^(-6) + 66*v^(-8) + 99*v^(-10)
+ 128*v^(-12) + 144*v^(-14) + 144*v^(-16) + 128*v^(-18) + 99*v^(-20)
+ 66*v^(-22) + 38*v^(-24) + 18*v^(-26) + 6*v^(-28) + v^(-30)
```

Fock, Tatitscheff and Thomas tackle the question, obtaining a result which classifies the families of Coxeter systems which always yield invariants whose coefficients are all positive ([FTT21, Theorem 6.10]).

Theorem 5.2.9. *The folowing positivity properties hold for punctured surfaces:*

(i) The polynomial invariant $P_{g,k,W}(q)$ has positive coefficients for all classical W and for the exceptional types H_3, E_6 and E_7;

(ii) For all other types (that is, $I_2(m)$ for $m \geq 5$, H_4, F_4 and E_8), the invariants may have negative coefficients.

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