U.PORTO
FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

# Metrics on flat bundles

Gabriel da Silva Martinho

Dissertação de Mestrado apresentada à
Faculdade de Ciências da Universidade do Porto em
Matemática

2022

MSc
2.º CICLO
FCUP
2022

U.PORTO

Metrics on flat bundles

Gabriel da Silva Martinho

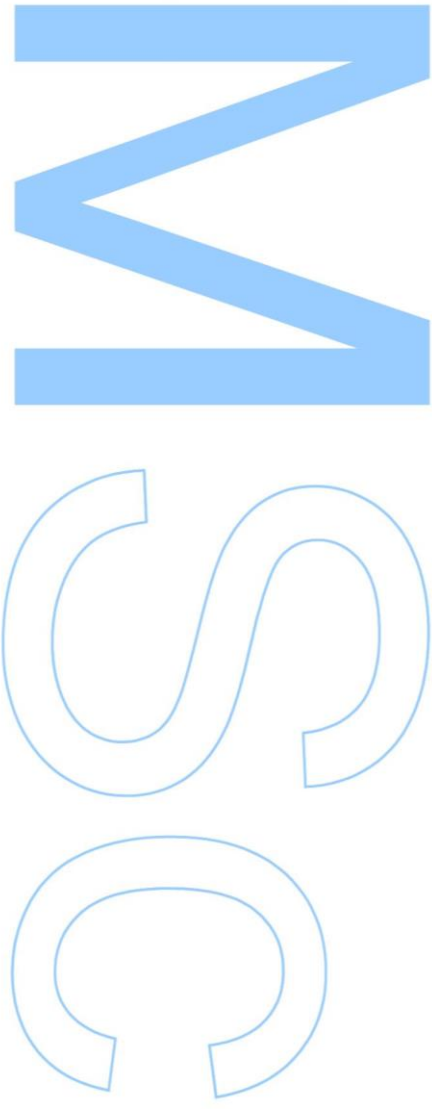
FC

# Metrics on flat bundles

Gabriel da Silva Martinho
Mestrado em Matemática
Departamento de Matemática
2022

**Orientador**
André Gama Oliveira, Doutorado de Nível Inicial,
Faculdade de Ciências da Universidade do Porto

**Coorientador**
Peter Beier Gothen, Professor Associado,
Faculdade de Ciências da Universidade do Porto



U.PORTO

FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, _____/_____/_____

## Acknowledgements

While working on my dissertation, I was supported by the grant with reference UIDB/MAT/00144/2020_GEOM, funded by FCT (Fundação para a Ciência e Tecnologia).

I am very grateful to my advisor, André Gama, and my co-advisor, Peter Gothen, for guiding me throughout this year while working on my dissertation. I hold in a very high esteem their availability to discuss the various topics with me. Evidently, the role of advisor demands willingness to assist, but their care during many sessions of collective head scratching, when compared to what I perceive to be the standard treatment of students like myself, can only be rated as superb. More importantly, their technical prowess has helped me evolve as a student and gain some very valuable experience.

My academic journey has faced many hurdles, moments of fear and indecisiveness, which had to be overcome before deciding to pursue this path. I thank my 12th grade Mathematics teacher for giving me a first taste of the rigor it demands, exposing me to values now embedded within me, which have survived all setbacks. I am grateful to the many teachers that helped me with my struggle to decide what to do after my degree. I thank my dear friends for their unyielding support. I thank my loving family for providing me with the necessary conditions to live, study and evolve as a person, one who aspires to be a good, contributing member of society.

Gabriel Martinho

## Abstract

Differential geometry has a particularly important role in modern science, as much valuable theory is built on the tools it provides. The main goal for this text is to introduce some of its fundamental objects, so as to discuss the energy integral for a bundle metric, motivated by Physics.

We start with vector bundles, which arise as an immediate generalization of tangent/cotangent bundles; the level of abstraction they provide is a fine environment for producing a strong theory with readily visible consequences. An important advantage is the mix of "algebraic" and "differential" points of view. Lie groups are frequently present when studying vector bundles and can be put to good use, via the construction of "frame bundles".

This relation suggests a change of environment, by making Lie groups the protagonists. The main change is the focus on free Lie group actions that are transitive on each fiber, the fundamental ingredients of principal bundles. Although this is a large difference, it is possible to link principal and vector bundles with elegant rewards, by constructing "associated bundles".

Next, we study connections on principal bundles. These differential forms have become central tools in differential geometry, providing complements to certain tangent subspaces (the vertical subspaces) determined by the Lie group action. We start with the essential definitions and examples, after which we highlight a key application: the ability to differentiate sections of associated bundles. We then move on to the study of curvature of connections, focusing on flat connections (with zero curvature). We define the holonomy of a connection and analyze its relation with flatness.

Finally, we discuss the energy integral, adapting it to the context of principal bundles. We relate harmonic maps (those that minimize the energy integral) with solutions to a certain system of equations (called the Hitchin equations). We conclude by stating the Donaldson-Corlette theorem, which relates harmonic maps with certain representations involving fundamental groups and Lie groups.

**Keywords:** vector bundle, principal bundle, reduction, metric, connection, curvature, flat connection, energy, harmonic metric.

## Resumo

A geometria diferencial desempenha um papel particularmente importante na ciência moderna, uma vez que muita teoria valiosa é construída a partir das ferramentas que fornece. O objetivo principal para este texto é introduzir alguns dos seus objetos fundamentais, de modo a discutir o integral de energia com uma métrica de fibrado vetorial, motivado pela Física.

Começamos como fibrados vetoriais, que surgem como uma imediata generalização de fibrados tangentes/cotangentes; o nível de abstração por eles fornecido é um bom ambiente para produzir uma teoria forte e com consequências rapidamente visiveis. Uma importante vantagem é a mistura de pontos de vista "algébricos" e "diferenciais". Os grupos de Lie estão presentes frequentemente no estudo de fibrados vetoriais e podem ser usados com eficácia, através da construção de "fibrados de referenciais".

Esta relação sugere uma mudança de ambiente, fazendo dos grupos de Lie os protagonistas. A maior mudança é a importância dada a ações de grupos de Lie livres e transitivas em cada fibra, os ingredientes fundamentais dos fibrados principais. Ainda que esta seja uma grande diferença, é possível ligar fibrados vetoriais e principais com elegantes recompensas, pela construção de "fibrados associados".

A seguir, passamos aos estudo de conexões em fibrados principais. Estas formas diferenciais tornaram-se ferramentas centrais na geometria diferencial, fornecendo complementos a certos subespaços tangentes (os subespaços verticais) determinados pela ação de grupo de Lie. Começamos com as definições e os exemplos essenciais, após os quais realçamos uma aplicação fulcral: a possibilidade de derivar secções de fibrados associados. A seguir, passamos ao estudo da curvatura de conexões, concentrando-nos nas conexões planas (com curvatura igual a zero). Definimos a holonomia de uma conexão e analisamos a sua relação com a propriedade plana.

Finalmente, discutimos o integral de energia, adaptando-o ao contexto dos fibrados principais. Relacionamos aplicações harmónicas (que minimizam o integral de energia) com soluções de um certo sistema de equações (as equações de Hitchin). Concluímos com o enunciado do teorema de Donaldson-Corlette, que relaciona aplicações harmónicas com certas representações envolvendo grupos fundamentais e grupos de Lie.

**Palavras-chave:** fibrado vetorial, fibrado principal, redução, métrica, conexão, curvatura, conexão plana, energia, métrica harmónica.

# Contents


**1 Introduction** **7**

**2 Vector Bundles** **10**

2.1 Definition and Examples of Vector Bundles . . . . . 10
2.2 Bundle Maps and Sections . . . . . 16
2.3 Differential Forms With Values in a Vector Bundle . . . . . 22
2.4 The Frame Bundle of a Vector Bundle . . . . . 23
2.5 Metrics on Vector Bundles . . . . . 26

**3 Principal Bundles** **29**

3.1 Preliminaries of Lie Group Theory . . . . . 29
3.2 Principal Bundles . . . . . 32
3.3 Sections and Bundle Maps . . . . . 36
3.4 Associated Bundles . . . . . 41
3.5 Reductions of Principal Bundles . . . . . 45
3.6 Sections and Equivariant Maps . . . . . 52

**4 Connections on Principal Bundles** **54**

4.1 Definition and Examples . . . . . 54
4.2 Tensorial Forms on a Principal Bundle . . . . . 59
4.3 Reductions of Connections . . . . . 60
4.4 The Covariant Derivative . . . . . 60

**5 Curvature on Principal Bundles** **63**

5.1 The Curvature Form of a Connection . . . . . 63
5.2 The Holonomy Representation and Parallel Transport . . . . . 66
5.3 Fundamental Group Representations and Flatness . . . . . 72

**6 Metrics on Principal Bundles** **74**

6.1 The Energy Integral . . . . . 74
6.2 Metrics on Principal Bundles . . . . . 75

6.3 The Flatness Equations . . . . . 85
6.4 Statement of the Corlette-Donaldson Theorem . . . . . 86

# 1 Introduction

The foundational role of differential geometry in most of Physics is often the first example of an application of this field of Mathematics which comes to mind; the formalism it provides is the ideal framework for tackling the some of the deepest questions one can ask. Nowadays, Physics is no stranger to abstraction; here, it also natural to switch from Euclidean space to the broader context of manifolds. The concepts analyzed in this text can be regarded as an extra level of abstraction, always inspired by properties of well-known objects. The property that fuels this text is being "locally trivial", which is motivated by charts (local trivializations) of tangent and cotangent bundles of smooth manifolds.

Section 2 introduces some initial vector bundle theory. These objects can be seen as an immediate generalization of tangent/cotangent bundles, the importance of which immediately suggests that these objects appear frequently and have a strong role in producing a robust and wide-ranging theory. However, vector bundles maintain the use of finite-dimensional vector spaces, which preserves the importance and power of finite-dimensional linear algebra. Several examples of vector bundles are introduced, which feature well-known vector spaces. The fundamental notion of section is introduced, noting its natural relation to vector fields and differential forms on manifolds. Transition functions appear as a central tool in analyzing and constructing vector bundles. The notion of differential form is expanded: starting with differential forms on $\mathbb{R}$, we define forms on vector spaces; the abstract notion of section then yields a natural definition of forms with values in vector bundles. Two other bundles are then defined: the frame bundle and the orthogonal frame bundle of a vector bundle; these examples are carried throughout the text, with the second motivating the fundamental notion of bundle metric.

In Section 3, we focus on principal bundles, that are now endowed with a Lie group action. Linear maps are replaced by the notion of equivariance, which uses the group action. The example of the universal covering of a connected manifold will be of particular importance. The notion of section is recovered and a link between sections and trivializations is established. We return to studying transition functions,

now in the context of principal bundles. Next, we use principal bundles to construct a new type of bundle, simply called "associated bundles". These objects link vector bundles and principal $G$-bundles is established, by using of representations of Lie groups on vector spaces. We study reductions of principal bundles, with the goal of being able to consider simpler principal bundles inside pre-existing ones. Notably, if $E$ is a principal $G$-bundle, one can consider reductions of $E$ to a maximal compact subgroup $H$ of $G$; these reductions are known as metrics on $E$. Several equivalent ways of looking at reductions are established, with a more technical approach. Some results on Lie group theory appear at the beginning of the Section.

Section 4 introduces connection forms on principal bundles, which complement certain tangent subspaces (the vertical subspaces) provided by the group action. Some basic properties are established, as well as some fundamental examples. We also focus on horizontal and equivariant forms (known as tensorial forms), relating them to sections of associated bundles. Next, we define the covariant derivative with respect to a connection. This notion, along with the previous result, provides a way to differentiate sections of associated bundles, which is a key application of connections. We also note how connections and reductions of principal bundles are linked, via reduced connections.

In Section 5, we study curvature of connections. Flat connections (with zero curvature) are the main ones studied. Transition functions appear as a useful and fundamental tool to study flatness. The notion of holonomy of a connection is introduced; we study its relation with flatness and note a deep relation between holonomy and representations which involve the fundamental group of the base space. In particular, we define the holonomy representation of a flat principal bundle.

In Section 6, we apply all the theory developed so far, by studying the energy integral and harmonic metrics, those that minimize the energy integral. Eells and Sampson discuss harmonic maps in ([4], Section 11); see also ([9], Section 8). Originally defined for maps $f\colon M \to N$, where $M$ and $N$ are Riemannian manifolds (with $M$ compact), the integral is defined in the context of principal $G$-bundles

$(E, \pi, M)$, when we're provided with a metric (a reduction to a maximal compact subgroup $H$ of $G$), a bundle metric on the adjoint bundle of the reduced subbundle and a flat connection on $E$. We make full use of the tools defined earlier, as well as a very important class of Lie groups, called "reductive" Lie groups, ultimately producing a definition of harmonic metric on a principal bundle. We study the equivalence between having a harmonic metric and solutions to certain equations obtained using properties of curvature. We conclude by stating the Donaldson-Corlette theorem: there is a harmonic metric on a flat principal bundle $E$ if and only if its holonomy representation is reducible.

In this text, all manifolds are assumed to be real and smooth and all submanifolds are considered to be embedded. Unless otherwise specified, all vector spaces are assumed to be finite-dimensional.

# 2 Vector Bundles

Let $M$ be a smooth manifold. If $U$ is an open subset on which there is a chart of $M$, we have a corresponding chart of its tangent bundle $TM$, taking values in $U \times \mathbb{R}^n$. So locally, $TM$ "looks" like $U \times \mathbb{R}^n$, which can be thought of as a family of vector spaces indexed by $U$. We also have the canonical projection $\pi : TM \to M$ given by $\pi(v_p) = p$ if $v_p \in T_pM$. A vector field is a map $X \colon M \to TM$ such that $\pi \circ X(p) = p$ for all $p \in M$. We have the analogue for the cotangent bundle and differential forms.

It is possible to take these properties into the more general context of vector bundles. Here, we summarize some initial theory, focusing on sections (of which vector fields and differential forms are examples) and bundle maps. One can then incorporate Lie groups to acquire a better understanding of vector bundles, via the frame bundle construction. Like one can define a Riemannian metric on a manifold $M$, it is possible to define metrics on vector bundles, of which Riemannian metrics are an example.

## 2.1 Definition and Examples of Vector Bundles

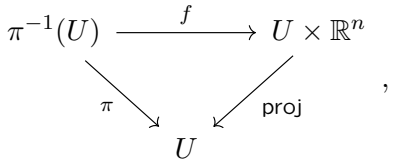
Inspired by tangent and cotangent bundles, we take some of their central properties, studying them at a more abstract level. The key phrase here (and, indeed, throughout the whole text) is "locally trivial".

**2.1.1 Definition.** A *differentiable real vector bundle* of rank $n$ is a triple $(V, \pi, M)$ where $\pi : V \to M$ is a differentiable map between smooth manifolds such that:

1. for every $p \in M$, $V_p = \pi^{-1}(p)$ has the structure of a real vector space of dimension $n$;

2. every point in $M$ has an open neighbourhood $U$ with a diffeomorphism $f : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that

(a) the diagram




,

where proj is the projection onto the first factor, is commutative, that is, $f(V_p) \subseteq p \times \mathbb{R}^n$ for all $p \in U$;

(b) the map $f_p = f|_{V_p} : V_p \to p \times \mathbb{R}^n$ is an isomorphism of vector spaces for each $p \in U$.

We say that $(V, \pi, M)$ is a vector bundle *over* $M$. The manifold $V$ is called the *total space* and $M$ is called the *base space*. If the context is clear, we will write $V$ instead of $(V, \pi, M)$. The vector space $V_p$ is called the *fiber over* $p$. The diffeomorphism $f$ is called a *local trivialization of* $V$ over $U$; we say that $V$ is *locally trivial*. A vector bundle is called *trivial* if it admits a trivialization with domain $V$, called a *global trivialization*.

**2.1.2 Example.** Taking $V = M \times \mathbb{R}^n$, we obtain a vector bundle $(M \times \mathbb{R}^n, \pi, M)$, where $\pi : M \times \mathbb{R}^n \to M$ is the projection, called a *product bundle*. In this case, the identity $M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is a global trivialization.

**2.1.3 Example.** Let $m$ be the dimension of $M$. The tangent bundle of $M$, $TM$, with the projection $\pi : TM \to M$ given by $\pi(v_p) = p$ if $v_p \in T_pM$, is a vector bundle over $M$. Taking local coordinates $x_1, ..., x_m$ over a chart $(U, \phi)$, we have $\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$ and the map $\phi_* : \pi^{-1}(U) \to U \times \mathbb{R}^m$ defined by $\phi_*(\sum_{i=1}^m v_i \frac{\partial}{\partial x^i}\big|_p) = (p, v_1, ..., v_m)$ is a local trivialization over $U$. Analogously, the cotangent bundle $T^*M$ is also a vector bundle over $M$.

**2.1.4 Remark.** Let $V$ and $W$ be vector bundles over the same manifold $M$. Then one can find an open covering $\{U_\alpha\}_{\alpha \in \Sigma}$ of $M$ with local trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$ of $V$ and $\{\psi_\alpha\}_{\alpha \in \Sigma}$ of $W$ (so $V$ and $W$ are trivial over each $U_\alpha$ simultaneously). The same applies if we have a finite family of vector bundles. Note that this fact does not make use of the differentiable structure; it is just topological in nature.

The following result illustrates how it is sufficient to define a vector bundle by constructing the local trivializations, provided that there is a certain compatibility condition on their intersections.

**2.1.5 Proposition.** Let $M$ be a manifold and, for each $p \in M$, let $V_p$ be a vector space of some fixed dimension $n$. Let $V = \bigsqcup_{p \in M} V_p$ and $\pi \colon V \to M$ map $V_p$ to $p$. Furthermore, suppose there is

1. an open cover $\{U_\alpha\}_{\alpha \in \Sigma}$ of $M$;
2. for each $\alpha \in \Sigma$, a bijection $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ which restricts to isomorphisms $V_p \to p \times \mathbb{R}^n \cong \mathbb{R}^n$;
3. for each $\alpha, \beta \in \Sigma$ with $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$ such that the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is of the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(p, v) = (p, g_{\alpha\beta}(p)(v)).$$

Then there is a unique topological and smooth structure on $V$ which makes it a manifold and vector bundle of rank $n$, with $\pi$ being the projection and having the maps $\{\varphi_\alpha\}_{\alpha \in \Sigma}$ as local trivializations.

For a proof, see ([12], Lemma 10.6). On the other hand, if $(V, \pi, M)$ is a vector bundle, let $\{U_\alpha\}_{\alpha \in \Sigma}$ be an open covering of $M$ with trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$. Then the maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are of the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(p, v) = (p, g_{\alpha\beta}(p)(v)).$$

for some differentiable map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})$. The maps $\{g_{\alpha\beta}\}_{\alpha, \beta \in \Sigma}$ are called the *transition functions* for $V$ with respect to the covering $\{U_\alpha\}_{\alpha \in \Sigma}$ and the trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$.

We will analyze this notion more substantially in Section 3, in the context of principal bundles.

**2.1.6 Example.** Let $(V, \pi, M)$ and $(W, \xi, M)$ be vector bundles over the same base space $M$. Let $\mathsf{Hom}(V, W)$ be the set

$$\mathsf{Hom}(V, W) = \bigsqcup_{p \in M} \mathsf{Hom}(V_p, W_p),$$

where on the right side we consider vector space homomorphisms. Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be an open covering of $M$ with trivializations $\{\phi_\alpha\}_{\alpha \in \Sigma}$ of $V$ and $\{\psi_\alpha\}_{\alpha \in \Sigma}$ of $W$. Let $\rho \colon \mathsf{Hom}(V, W) \to M$ be the projection $(p, h_p) \mapsto p$. Consider the maps

$$\begin{aligned} \varphi_\alpha \colon \rho^{-1}(U_\alpha) &\to U_\alpha \times \mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m) \\ (p, f_p) &\mapsto (p, \psi_p \circ f_p \circ \phi_p^{-1}). \end{aligned}$$

The maps $\varphi_\alpha$ are bijective and restrict to vector space isomorphisms. Furthermore, the composite map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is the map $(p, f) \mapsto (p, \psi_{\beta,p} \circ (\psi_{\alpha,p}^{-1} \circ f \circ \phi_{\alpha,p}) \circ \phi_{\beta,p}^{-1})$.

Denote the transition functions of $V$ by $g_{\alpha\beta}$ and by $h_{\alpha\beta}$ the transition functions of $W$. We have a differentiable map

$$\begin{aligned} C \colon \mathsf{GL}(n, \mathbb{R}) \times \mathsf{GL}(m, \mathbb{R}) &\to \mathsf{GL}(\mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \\ (g_n, g_m) &\mapsto (T \mapsto g_m \circ T \circ g_n^{-1}) \end{aligned}$$

which defines

$$\begin{aligned} \gamma_{\alpha\beta} \colon U_{\alpha\beta} &\to \mathsf{GL}(\mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m)) \\ p &\mapsto C(g_{\alpha\beta}(p), h_{\alpha\beta}(p)), \end{aligned}$$

which is also differentiable. It is straightforward to check that

$$\varphi_\beta \circ \varphi_\alpha^{-1}(p, f) = (p, \gamma_{\alpha\beta}(p)(f)).$$

Hence we are in the conditions of Proposition 2.1.5, and so there is a unique vector bundle structure that makes the triple $(\mathsf{Hom}(V, W), \rho, M)$ a vector bundle over $M$ with trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$. Its fiber over $p$ is $\mathsf{Hom}(V_p, W_p) \cong \mathbb{R}^{nm}$.

In particular, if $W = M \times \mathbb{R}$, then $\mathsf{Hom}(V, W) = \bigsqcup_{p \in M} V_p^*$ has fiber $V_p^* \cong (\mathbb{R}^n)^*$, the dual space of $V_p$. The bundle $\mathsf{Hom}(V, M \times \mathbb{R})$ is called the *dual bundle* of $V$. It is denoted by $V^*$. For example, the dual bundle of the tangent bundle $TM$ is the cotangent bundle $T^*M$, as expected.

**2.1.7 Remark.** Note that the map $C$ is a Lie group representation, which arises from the Lie group action

$$\begin{aligned} &\mathsf{GL}(n, \mathbb{R}) \times \mathsf{GL}(m, \mathbb{R}) \times \mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to \mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m) \\ &(g_n, g_m, T) \mapsto g_n \circ T \circ g_m^{-1} \end{aligned}$$

of $\mathsf{GL}(n, \mathbb{R}) \times \mathsf{GL}(m, \mathbb{R})$ on $\mathsf{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

**2.1.8 Example.** Let $(V, \pi, M)$ and $(W, \xi, M)$ be two vector bundles over the same base space $M$ of ranks $n$ and $m$, respectively. Denote by $V_p$ (respectively, $W_p$) the fiber over $p \in M$ in $V$ (respectively, $W$). Consider a set $V \otimes W$ defined by

$$V \otimes W = \bigsqcup_{p \in M} V_p \otimes W_p,$$

where $V_p \otimes W_p$ is the tensor product of the vector spaces $V_p$ and $W_p$. We shall construct on $V \otimes W$ a vector bundle structure over $M$. Let $\rho$ be the projection $\rho\colon V \otimes W \to M$ defined by $(p, t) \mapsto p$. Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be an open covering of $M$ with trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$ of $V$ and $\{\psi_\alpha\}_{\alpha \in \Sigma}$ of $W$. Note that

$$\rho^{-1}(U) = \bigsqcup_{p \in U} V_p \otimes W_p$$

for all $U \subseteq M$. Let $p \in U_\alpha$. Then $\varphi_{\alpha,p}$ and $\psi_{\alpha,p}$ are vector space isomorphisms, hence they define a vector space isomorphism

$$\varphi_{\alpha,p} \otimes \psi_{\alpha,p}\colon p \times (V_p \otimes W_p) \to p \times (\mathbb{R}^n \otimes \mathbb{R}^m)$$

which, in turn, induces a bijection

$$\varphi_\alpha \otimes \psi_\alpha\colon \rho^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m)$$

whose restriction to $p \times (V_p \otimes W_p)$ is $\varphi_{\alpha,p} \otimes \psi_{\alpha,p}$. We define the vector bundle structure on $V \otimes W$ by letting the maps $\varphi_\alpha \otimes \psi_\alpha$ be diffeomorphisms. Clearly, we have a commutative diagram

$$\begin{array}{ccc} \rho^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha \otimes \psi_\alpha} & U_\alpha \times (\mathbb{R}^n \otimes \mathbb{R}^m) \\ & {\scriptstyle \rho} \searrow \quad \swarrow {\scriptstyle \text{proj}} & \\ & U_\alpha & \end{array},$$

from which we obtain the smoothness of $\rho$. The vector bundle $(V \otimes W, \rho, M)$ is called the *tensor product* of the vector bundles $(V, \pi, M)$ and $(W, \xi, M)$. It is also a vector bundle of rank $nm$. Note that there is a canonical isomorphism between $V_p \otimes W_p$ and $(V \otimes W)_p$.

## 2.2 Bundle Maps and Sections

Vector bundles form a category, where the morphisms are bundle maps. We require our morphisms to be smooth, preserve fibers and be linear over each fiber. Sections are the generalizations of vector fields and differential forms on the tangent and cotangent bundles, respectively. It turns that these two concepts are linked - the main result here relates bundle maps $V \to W$ to sections of $\mathrm{Hom}(V, W)$.

**2.2.1 Definition.** Let $(V, \pi, M)$ and $(W, \xi, M')$ be two vector bundles. A *bundle map* from $V$ to $W$ is a pair of differentiable mappings $(\overline{\boldsymbol{f}}, f)$ such that

1. the diagram

$$\begin{array}{ccc} V & \xrightarrow{\overline{\boldsymbol{f}}} & W \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \xi} \\ M & \xrightarrow{f} & M' \end{array}$$

   is commutative;

2. $\overline{\boldsymbol{f}}$ restricts to a linear map $\overline{\boldsymbol{f}}_p \colon V_p \to W_p$ for all $p \in M$.

If $V$ and $W$ are vector bundles over the same manifold $M$ and $f = id_M$, then we call $(\overline{\boldsymbol{f}}, f)$ a bundle map *over* $M$. We will often call the map $\overline{\boldsymbol{f}} \colon V \to W$ alone the bundle map. If there is a bundle map $\overline{\boldsymbol{g}} \colon W \to V$ such that $\overline{\boldsymbol{f}} \circ \overline{\boldsymbol{g}} = id_V$ and $\overline{\boldsymbol{g}} \circ \overline{\boldsymbol{f}} = id_W$, then $\overline{\boldsymbol{f}}$ is said to be a *bundle isomorphism over* $M$, and the vector bundles $V$ and $W$ are said to be *isomorphic*.

**2.2.2 Example.** Let $f \colon M \to N$ be a smooth map between manifolds. Then

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{f} & N \end{array}$$

is a bundle map.

**2.2.3 Remark.** It follows from the definition that a bundle map $\overline{\boldsymbol{f}}\colon V \to W$ over $M$ is an isomorphism if and only if $\overline{\boldsymbol{f}}$ is a diffeomorphism of total spaces which restricts to an isomorphism of vector spaces $\overline{\boldsymbol{f}}_p\colon V_p \to W_p$ for all $p \in M$.

We can improve this condition:

**2.2.4 Proposition.** A bundle map $\overline{\boldsymbol{f}}\colon V \to W$ over $M$ is an isomorphism if and only if $\overline{\boldsymbol{f}}_p\colon V_p \to W_p$ is a vector space isomorphism for all $p \in M$.

**Proof.** We only need to show that if $\overline{\boldsymbol{f}}_p\colon V_p \to W_p$ is a vector space isomorphism for all $p \in M$, then $\overline{\boldsymbol{f}}$ is a diffeomorphism. Clearly $\overline{\boldsymbol{f}}$ is smooth and bijective, hence it is sufficient to show that $\overline{\boldsymbol{f}}^{-1}$ is differentiable. This is a local problem, so we can assume that $V = M \times \mathbb{R}^n$ and $W = M \times \mathbb{R}^n$. In this case, $\overline{\boldsymbol{f}}\colon M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is of the form $\overline{\boldsymbol{f}}(p, v) = (p, \overline{\boldsymbol{f}}_p(v))$ where $\overline{\boldsymbol{f}}_p\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. Using local coordinates for $M$, the Jacobian matrix for $\overline{\boldsymbol{f}}$ at every point $(p, v)$ is of the form

$$\begin{pmatrix} I & 0 \\ X & \overline{\boldsymbol{f}}_p \end{pmatrix}$$

where $\overline{\boldsymbol{f}}_p$ is the matrix of $\overline{\boldsymbol{f}}_p\colon \mathbb{R}^n \to \mathbb{R}^n$. Since $\overline{\boldsymbol{f}}_p$ is an isomorphism, the matrix is invertible. The result now follows from the inverse function theorem. □

**2.2.5 Remark.** If $E$ and $F$ are vector spaces, then there is a natural isomorphism between $\mathrm{Hom}(E, F)$ and $E^* \otimes F$. If $V$ and $W$ are vector bundles over the same manifold $M$, one can likewise define a natural vector bundle isomorphism between $\mathrm{Hom}(V, W)$ and $V^* \otimes W$ by considering the isomorphisms $\mathrm{Hom}(V_p, W_p) \to V_p^* \otimes W_p$ fibrewise.

**2.2.6 Example.** Let $(V, \pi, M)$ be a vector bundle of rank $n$ and $f\colon N \to M$ a smooth map. Define a set $f^*V$ by

$$f^*V = \{(n, v) \in N \times V\colon f(n) = \pi(v)\},$$

considered as a subspace of $N \times V$. We have the projection $\eta\colon f^*V \to N$ which maps $(n, v)$ to $n$.

**2.2.7 Proposition.** With this notation, the triple $(f^*V, \eta, N)$ is in a natural way a vector bundle over $N$ of rank $n$.

**Proof.** Let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of $M$ with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$. Define

$$\begin{aligned}\psi_\alpha\colon \eta^{-1}(f^{-1}(U_\alpha)) &\to f^{-1}(U_\alpha) \times \mathbb{R}^n\\ (n, v) &\mapsto (n, p_2 \circ \varphi_\alpha(v)),\end{aligned}$$

where $p_2$ is the obvious projection. For each $\alpha$, the map $\psi_\alpha$ is well defined, continuous and has an inverse $(n, x) \mapsto (n, \varphi_\alpha^{-1}(f(n), x))$, which is also well defined and continuous.

Clearly, the maps $\psi_\alpha$ preserve fibers and are isomorphisms over each fiber. One can then define the vector bundle structure on $f^*V$ by making each $\psi_\alpha$ a diffeomorphism. □

The triple $(f^*V, \eta, N)$ is called the *pullback bundle* of $(V, \pi, M)$ by $f$. Note that there is a bundle map

$$\begin{array}{ccc} f^*V & \xrightarrow{p_2} & V \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}.$$

For more detailed constructions of the previous examples, see ([14], Section 20).

**2.2.8 Definition.** A differentiable (or smooth) *section* of a vector bundle $(V, \pi, M)$ is a differentiable mapping $\sigma : M \to V$ such that $\pi \circ \sigma = \mathrm{id}_M$, that is,

$\sigma(p) \in V_p = \pi^{-1}(p)$ for all $p \in M$. The set of differentiable sections in $V$ is denoted by $\Gamma(V)$. Note that $\Gamma(V)$ is a $C^\infty(M)$-module.

**2.2.9 Example.** A differentiable section of the tangent bundle $TM$ (respectively, the cotagent bundle $T^*M$) of a manifold $M$ is a vector field (respectively, a 1-form) on $M$.

**2.2.10 Remark.** Let $V$ and $W$ be vector bundles over the same manifold $M$ and $f\colon V \to W$ a bundle map. Then $f$ induces a map $F\colon \Gamma(V) \to \Gamma(W)$ between the vector spaces of sections:

$$F(\sigma)(p) = f(\sigma(p)).$$

It is easy to check that $F(\sigma)$ is indeed a (smooth) section of $W$. Since bundle maps restrict to linear maps on fibers, the map $F$ is linear over $C^\infty(M)$. The converse is also true (see [14], Theorem 7.26). Summarizing, we have the following.

**2.2.11 Theorem.** Given two vector bundles $(V, \pi, M)$ and $(W, \xi, M)$ over the same base space $M$, there are bijections between

1. the vector space $\Gamma(\mathrm{Hom}(V, W))$ of sections of the vector bundle $(\mathrm{Hom}(V, W), \rho, M)$;
2. the set of $C^\infty(M)$-linear maps $\Gamma(V) \to \Gamma(W)$;
3. the set $B(V, W)$ of bundle maps $V \to W$.

**Proof.** We prove that there is a bijection between $\Gamma(\mathrm{Hom}(V, W))$ and $B(V, W)$. Consider

$$\begin{aligned} F\colon \Gamma(\mathrm{Hom}(V, W)) &\to B(V, W) \\ \sigma &\mapsto (v \in V_p \mapsto \sigma(p)(v)). \end{aligned}$$

We verify that $F(\sigma)$ is a bundle map; it remains to see that is differentiable. Take an open set $U$ of $M$ with trivializations $\phi$ of $V$ and $\psi$ of $W$. Let $\varphi$ be the corresponding

trivialization of $\mathrm{Hom}(V,W)$ over $U$. The local expression of $\sigma$ over $U$ is

$$\begin{aligned}\varphi\circ\sigma|_U\colon U &\to U\times \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)\\ p &\mapsto (p,\psi_p\circ\sigma(p)\circ\phi_p^{-1})\end{aligned}$$

and the local expression of $F(\sigma)$ over $U$ is

$$\begin{aligned}\psi\circ f\circ\phi^{-1}\colon U\times\mathbb{R}^n &\to U\times\mathbb{R}^m\\ (p,x) &\mapsto (p,(\psi_p\circ F(\sigma)|_{V_p}\circ\phi_p^{-1})\cdot x)\end{aligned}$$

This map factors as

$$\begin{aligned}&U\times\mathbb{R}^n\to U\times\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)\times\mathbb{R}^n\to U\times\mathbb{R}^m\\ &(p,x)\mapsto(\varphi\circ\sigma(p),x)\mapsto(p,\varphi\circ\sigma(p)\cdot x)\end{aligned}$$

and all of these maps are clearly differentiable.

For the inverse, consider

$$\begin{aligned}G\colon B(V,W) &\to \Gamma(\mathrm{Hom}(V,W))\\ b &\mapsto (\sigma_b\colon p\mapsto b|_{V_p}).\end{aligned}$$

We check that, given a bundle map $b\colon V\to W$, the map $\sigma_b$ is a differentiable section. Clearly $\rho\circ\sigma_b=id_M$. Over $U$, the composite $\varphi\circ\sigma_b$ is the map

$$\begin{aligned}&U\to U\times\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^m)\\ &p\mapsto(p,\psi_p\circ b|_{V_p}\circ\phi_p^{-1}).\end{aligned}$$

By hypothesis,

$$\begin{aligned}\psi\circ b\circ\phi^{-1}\colon U\times\mathbb{R}^n &\to U\times\mathbb{R}^m\\ (p,x) &\mapsto (p,(\psi_p\circ b|_{V_p}\circ\psi_p^{-1})\cdot x)\end{aligned}$$

is differentiable. Letting $x$ equal the various vectors of the canonical basis of $\mathbb{R}^n$, we

conclude that the various entries of the matrix $\psi_p \circ b_{|V_p} \circ \phi_p$ are differentiable, hence $\sigma_b$ is differentiable. Furthermore, it is clear that $G$ is the inverse of $F$. $\square$

## 2.3 Differential Forms With Values in a Vector Bundle

Let $\omega \in \Omega^1(M,V)$ be a 1-form on $M$ taking values in a vector space $V$. So, given a point $p \in M$, we have a linear map

$$\omega_p \colon T_pM \to V.$$

More generally, we can allow the vector space $V$ to vary with $p$. Suppose $(V,\pi,M)$ is a vector bundle. One can think about a "1-form $\omega$ on $M$ with values in $V$" as a family of linear maps

$$\omega_p \colon T_pM \to V_p.$$

So $\omega$ would be a map $\omega \colon M \to \mathrm{Hom}(TM,V)$. Furthermore, denoting the projection on the vector bundle $\mathrm{Hom}(TM,V)$ by $\rho$, we have $\rho \circ \omega = id_M$. Naturally, we want $\omega_p$ to vary smoothly with $p$. Our previous discussion makes the desired meaning of "smooth" clear.

**2.3.1 Definition.** A smooth 1-form $\omega$ on $M$ *with values in a vector bundle* $V$ over $M$ is a smooth section of the vector bundle $\mathrm{Hom}(TM,V)$.

**2.3.2 Remark.** There are also natural definitions of $k$-forms on $M$ with values on $V$: for each $p \in M$, we have a $k$-linear and alternating map

$$\omega_p \colon (T_pM)^k \to V_p$$

and a $k$-form $\omega$ on $M$ with values on $V$ can be seen as a section of the tensor product of vector bundles $(\Lambda^k T^*M) \otimes E$. A complete discussion can be found in ([14], Chapter 21).

## 2.4 The Frame Bundle of a Vector Bundle

Let $V$ be a $n$-dimensional real vector space. We can then consider

$$F(V) = \mathsf{Iso}(\mathbb{R}^n, V),$$

the set of linear isomorphisms $\mathbb{R}^n \to V$. These are called *frames* of $V$. There is a bijection between $F(V)$ and the set of basis of $V$ (if $f \in F(V)$, then $\{f(e_i)\}$ is a basis of $V$, where $\{e_i\}$ is the canonical basis of $\mathbb{R}^n$). A choice of basis of $V$ gives an identification of $V$ with $\mathbb{R}^n$, hence it also gives an identification of $F(V)$ with $\mathsf{Iso}(\mathbb{R}^n, \mathbb{R}^n) = \mathsf{GL}(n, \mathbb{R})$, which is an open set of $\mathsf{M}(n, \mathbb{R}) = \mathbb{R}^{n^2}$.

Now if $(V, \pi, M)$ is a vector bundle, one can consider the fibers $V_p = \pi^{-1}(p)$ where $p \in M$. Let $F(V)$ be the disjoint union

$$F(V) = \bigsqcup_{p \in M} F(V_p) = \bigsqcup_{p \in M} \mathsf{Iso}(\mathbb{R}^n, V_p).$$

We shall use the vector bundle structure on $V$ to define on $F(V)$ a topological and smooth structure such that the map $\overline{\boldsymbol{\pi}} : F(V) \to M$ given by $\overline{\boldsymbol{\pi}}(x) = p$ for $x \in F(V_p)$ is differentiable.

The main tools are the local trivializations of $V$. Let $f : \pi^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialization of $V$. Then there is a bijection

$$\begin{aligned} \overline{\boldsymbol{f}} : &\overline{\boldsymbol{\pi}}^{-1}(U) \to U \times \mathsf{GL}(n, \mathbb{R}) \\ &x \mapsto (p, f_p \circ x) \end{aligned}$$

for $x \in F(V_p)$, where $f_p$ is the restriction of the trivialization $f$ to the fiber $V_p$.

**2.4.1 Proposition.** There is a topological and differentiable structure on $F(V)$ which satisfies:

1. $F(V)$ is a differentiable manifold of dimension $m+n^2$, where $m$ is the dimension of $M$;
2. given a local trivialization $f$, the corresponding bijection $\overline{\boldsymbol{f}}$ is a diffeomorphism;
3. the map $\overline{\boldsymbol{\pi}}$ is differentiable and we have commutative diagrams

$$\begin{array}{ccc} \overline{\boldsymbol{\pi}}^{-1}(U) & \xrightarrow{\overline{\boldsymbol{f}}} & U \times \mathsf{GL}(n,\mathbb{R}) \\ & {\scriptstyle \overline{\boldsymbol{\pi}}} \searrow \quad \swarrow {\scriptstyle \text{proj}} & \\ & U & \end{array} \quad ;$$

4. the map $F(V) \times \mathsf{GL}(n,\mathbb{R}) \to F(V)$ given by $x \cdot g = x \circ g$ is a differentiable group action.

Indeed, the differentiable structure on $F(V)$ is constructed by saying that the maps $\overline{\boldsymbol{f}}$ are diffeomorphisms; point 4 can be checked locally, via a diffeomorphism $\overline{\boldsymbol{f}}$. The triple $(F(V), \overline{\boldsymbol{\pi}}, M)$ is called the *frame bundle* of $V$.

**2.4.2 Remark.** The action of $\mathsf{GL}(n,\mathbb{R})$ is free and transitive on each fiber. Each $F(V_p)$ is an orbit of this group action, and we can identify $M$ with the orbit space $F(V)/\mathsf{GL}(n,\mathbb{R})$. If $V$ is trivial, then we have a trivialization $f\colon V \to M \times \mathbb{R}^n$ of $V$ with corresponding diffeomorphism $\overline{\boldsymbol{f}}\colon F(V) \to M \times \mathsf{GL}(n,\mathbb{R})$. So $F(V)$ can also be seen as "trivial", although in the context of principal bundles, which will be studied in Section 3.

One can use the frame bundle construction and sections to establish a property of trivial vector bundles.

**2.4.3 Proposition.** A vector bundle $(V, \pi, M)$ is trivial if and only if its frame bundle $(F(V), \overline{\boldsymbol{\pi}}, M)$ has a section, i.e, a differentiable mapping $\sigma : M \to F(V)$ such that $\overline{\boldsymbol{\pi}} \circ \sigma = \mathsf{id}_M$.

**Proof.** Suppose $V$ is trivial and let $f\colon V \to M \times \mathbb{R}^n$ be a trivialization. Let $\overline{f}\colon F(V) \to M \times \mathrm{GL}(n,\mathbb{R})$ be its corresponding diffeomorphism. Define $\sigma\colon M \to F(V)$ by $p \mapsto f_p^{-1} \in F(V_p)$. The composite map $\overline{f} \circ \sigma$ is the map $p \mapsto (p,1)$, hence $\sigma$ is differentiable.

Now let $\sigma : M \to F(V)$ be a differentiable section. Then $\sigma(p) \in F(V_p)$ is an isomorphism of vector spaces. Consider the map $\varphi\colon M \times \mathbb{R}^n \to V$ defined by $(p,v) \mapsto \sigma(p)(v)$. Taking a local trivialization $f$ of $V$ over an open set $U$ of $M$ and the corresponding trivialization $\overline{f}$ of $F(V)$, one can check that $\varphi$ is a bundle map over $M$. By Proposition 2.2.4, $\varphi$ is a bundle isomorphism, hence its inverse $\varphi^{-1}\colon V \to M \times \mathbb{R}^n$ is a trivialization. □

**2.4.4 Remark.** Let $(V,\pi,M)$ be a vector bundle of rank $n$ and $F(V)$ its frame bundle. Suppose $\sigma\colon M \to F(V)$ is a section (hence $V$ is trivial). For each $i \in \{1,...,n\}$, define $\sigma_i\colon M \to V$ by $\sigma_i(p) = \sigma(p)(e_i)$, where $e_i$ is the $i$-th vector of the canonical basis of $\mathbb{R}^n$. Then $\sigma_i$ is a (differentiable) section of $V$. Note that, for all $p \in M$, the set $\{\sigma_1(p),...,\sigma_n(p)\}$ is linearly independent.

On the other hand, if we are provided with $n$ sections $\sigma_i\colon M \to V$ such that $\{\sigma_1(p),...,\sigma_n(p)\}$ for all $p \in M$, we can define a section $\sigma\colon M \to F(V)$ of $F(V)$ by $\sigma(p)(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i \sigma_i(p)$. Furthermore, these correspondences are inverses of each other.

The set of these $n$ sections $\sigma_i\colon M \to V$ is called a *moving frame* for $V$. So $V$ admits a moving frame if and only if it is trivial. However, since every vector bundle is locally trivial, it always has a local moving frame.

## 2.5 Metrics on Vector Bundles

A Riemannian metric on a manifold is an inner product on each tangent space $T_pM$ that varies smoothly with $p \in M$. This notion can be readily generalized to vector bundles, replacing the tangent spaces $T_pM$ by fibers, which are vector spaces.

**2.5.1 Definition.** A *metric on a vector bundle* $V$ over $M$ is a collection of inner products $g_p$ on $V_p$ such that, given $s_1, s_2 \in \Gamma(V)$, the function $g(s_1, s_2)$ given by $g(s_1, s_2)(p) = g(p)(s_1(p), s_2(p))$ is differentiable.

**2.5.2 Example.** The product bundle $V = M \times \mathbb{R}^n$ has the standard inner product given by the usual inner product in $\mathbb{R}^n$.

Using a partition of unity, one can prove that every vector bundle can be endowed with a metric. Suppose then that $(V, \pi, M)$ is a vector bundle of rank $n$ with a metric $g$.

**2.5.3 Proposition.** Every point in $M$ has an open neighbourhood $U$ with a local trivialization $f : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $f_p$ is a linear isometry for every $p \in M$ (considering the metric in $U \times \mathbb{R}^n$ from the previous example).

**Proof.** By Remark 2.4.4, every point in $M$ has a neighbourhood $U$ with a local moving frame, i.e., a set of $n$ sections $s_i \colon U \to \pi^{-1}(U)$ of $(\pi^{-1}(U), \pi, U)$ such that $\{s_1(p), ..., s_n(p)\}$ is linearly independent for all $p \in U$. Apply the Gram-Schmidt process to replace the local frame $\{s_1, ..., s_n\}$ for a new local frame $\{\sigma_1, ..., \sigma_n\}$ (still over $U$) such that $\{\sigma_1(p), ..., \sigma_n(p)\}$ is an orthonormal basis of $V_p$ for all $p \in U$. Note that $\sigma_1, ..., \sigma_n$ are still differentiable. Now we consider $\{\sigma_1, ..., \sigma_n\}$ as a section $\sigma$ of the frame bundle $F(\pi^{-1}(U))$. Using Proposition 2.4.3, we obtain an isomorphism $\varphi \colon U \times \mathbb{R}^n \to \pi^{-1}(U)$ defined by $(p, v) \mapsto \sigma(p)(v)$. Since $\varphi_p(e_i) = \sigma_p(e_i) = \sigma_i(p)$ and $\{\sigma_1(p), ..., \sigma_n(p)\}$ is an orthonormal basis of $V_p$, it follows that $\varphi_p$ is a linear isometry for all $p \in U$. We conclude that $f = \varphi^{-1}$ is the desired trivialization. □

**2.5.4 Example.** This allows us to define the *orthogonal frame bundle* of a vector bundle with a metric. For a single vector space $V$ with an inner product, let

$F_O(V) \subseteq F(V)$ be the set $\mathrm{Isom}(\mathbb{R}^n, V)$ of linear isometries $\mathbb{R}^n \to V$. If we take an orthonormal basis of $V$ we obtain an identification of $F_O(V)$ with the orthogonal group $\mathsf{O}(n)$, which is a submanifold of $\mathsf{GL}(n, \mathbb{R})$.

For a vector bundle with a metric, consider the subset of $F(V)$

$$F_O(V) = \bigsqcup_{p \in M} F_O(V_p) = \bigsqcup_{p \in M} \mathrm{Isom}(\mathbb{R}^n, V_p).$$

**2.5.5 Proposition.**

1. $F_O(V)$ is a submanifold of $F(V)$ and $\overline{\pi}|_{F_O(V)} : F_O(V) \to M$ is differentiable.
2. There is a differentiable right $\mathsf{O}(n)$-action on $F_O(V)$ such that the orbits are the sets $F_O(V_p)$, for $p \in M$.
3. There are local diffeomorphisms

$$\overline{\boldsymbol{f}} : \overline{\pi}^{-1}(U) \cap F_O(V) \to U \times \mathsf{O}(n)$$

such that $\overline{\boldsymbol{f}}_p = \overline{\boldsymbol{f}}|_{F_O(V_p)}$ maps $F_O(V_p)$ to $p \times \mathsf{O}(n)$ for every $p \in U$, and also $\overline{\boldsymbol{f}}(x \cdot g) = \overline{\boldsymbol{f}}(x) \cdot g$ for all $x \in F_O(V_p), g \in \mathsf{O}(n)$.

One can take an open set $U$ with a trivialization $f \colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that $f_p$ is an isometry (see Proposition 2.5.3). The corresponding trivialization of $F(V)$ then maps $\overline{\pi}^{-1}(U) \cap F_O(V)$ to $U \times \mathsf{O}(n)$. Now the result follows from the fact that $\mathsf{O}(n)$ is a submanifold of $\mathsf{GL}(n, \mathbb{R})$ (see ([3], Proposition 2.21)).

**2.5.6 Example.** Let $f \colon M \to N$ be a smooth map of manifolds and let $(V, \pi, N)$ be a vector bundle over $N$ with metric $g$. Consider the pullback bundle $f^*V$, which is a vector bundle over $M$. Then there is a natural metric $f^*g$ on $f^*V$ induced by $g$, defined by

$$(f^*g)(p)((p, v_1), (p, v_2)) = g(f(p))(v_1, v_2).$$

**2.5.7 Example.** Suppose there are metrics $g$ on $V$ and $h$ on $W$. We construct a natural metric on $\mathrm{Hom}(V,W)$ induced by the metrics $g$ and $h$.

First, some linear algebra. Suppose $V$ and $W$ are vector spaces with inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$. Then there are natural isomorphisms (in the sense that they don't depend on a choice of basis) $i_V\colon V \to V^*$ defined by $v \mapsto (u \mapsto \langle v,u\rangle_V)$ and $i_W\colon W \to W^*$ which maps $w$ to the map $u \mapsto \langle w,u\rangle_W$. We define an inner product on the vector space $\mathrm{Hom}(V,W)$ using the inner products on $V$ and $W$. Let $f,g \in \mathrm{Hom}(V,W)$. Let $g^*\colon V \to W$ be the map which makes the diagram

$$\begin{array}{ccc} W & \xrightarrow{g^*} & V \\ \downarrow{\scriptstyle i_W} & & \downarrow{\scriptstyle i_V} \\ W^* & \xrightarrow{g^t} & V^* \end{array}$$

commutative, i.e, $g^* = i_V^{-1} \circ g^t \circ i_W$. Then one can define an inner product on $\mathrm{Hom}(V,W)$ by letting $\langle f,g\rangle = tr(g^* \circ f)$, where the trace of the map $g^* \circ f\colon V \to V$ is obtained by taking a (any) basis of $V$ and calculating the trace of the corresponding matrix.

Now, if $V$ and $W$ are vector bundles over the same base space $M$ with metrics, we can define a metric on the Hom-bundle $\mathrm{Hom}(V,W)$ by letting $\langle f_p,g_p\rangle_p = tr(g_p^* \circ f_p)$ be the inner product on the fiber $\mathrm{Hom}(V_p,W_p)$, as above.

# 3 Principal Bundles

With the case of vector bundles and their frame bundles introduced, we can now focus on principal bundles, of which frame bundles are an example. Much like a vector bundle is locally a family of vector spaces, a principal bundle is locally a family of Lie groups; the fiber-wise linearity of local trivializations being replaced by equivariance. In this section, we study some fundamental properties of principal bundles and relate them to vector bundles, via the "associated bundle" method. We then focus on reductions of principal bundles, paving way to one of the main results, relating sections of associated bundles and certain equivariant maps.

First we summarize some necessary theory on Lie groups. For a reference, see [6] or [10]. Most of the required theory can also be found in [12].

## 3.1 Preliminaries of Lie Group Theory

**3.1.1 Definition.** A *Lie group* $G$ is a topological group $G$ endowed with a differentiable manifold structure such that the product map $G \times G \to G$ and the inverse map $G \to G$ are differentiable.

In particular, the *inversion map* and the *left and right translation maps* $L_g : h \mapsto gh$ and $R_g : h \mapsto gh$ are diffeomorphisms for all $g \in G$.

**3.1.2 Example.** Any group with the discrete topology is a topological group, called a *discrete group*. If in addition the group is countable, then it is a zerodimensional Lie group, called a *discrete Lie group*.

**3.1.3 Definition.** A vector field $X$ in a Lie group $G$ is said to be *left-invariant* if $(L_g)_*(h)(X_h) = X_{gh}$ for all $g, h \in G$.

**3.1.4 Remark.** One can check that

1. every vector field $X$ on $G$ is a linear combination of left-invariant vector fields on $G$ with differentiable coefficients;
2. the set $\mathrm{Lie}(G)$ of left-invariant vector fields on $G$ is a Lie subalgebra of $\mathfrak{X}(G)$

(ie, a vector subspace of $\mathfrak{X}(G)$ closed under Lie brackets). Therefore, it is a Lie algebra, called the Lie algebra of the Lie group $G$.

3. The evaluation map

$$\begin{aligned} \epsilon\colon \mathrm{Lie}(G) &\to T_eG \\ X &\mapsto X_e \end{aligned}$$

is a vector space isomorphism whose inverse is the map $v \mapsto \tilde{v}$, where $\tilde{v} \in \mathrm{Lie}(G)$ is given by $\tilde{v}(g) = L_{g*}(e)(v)$. Using this isomorphism, one can define a natural Lie algebra structure on $T_eG$ which makes $\epsilon$ a Lie algebra isomorphism. We will later refer to $T_eG$ as the Lie algebra of the Lie group $G$. Given $v \in T_eG$, the left-invariant vector field $\tilde{v}$ is called the *left-invariant vector field associated to* $v$.

**3.1.5 Definition.** A *Lie subgroup $H$ of a Lie group $G$* is a subgroup of $G$ which is also a submanifold of $G$.

**3.1.6 Theorem.** If $H$ is a closed subgroup of a Lie group $G$, then $H$ is a Lie subgroup of $G$.

Furthermore, its Lie algebra can be seen in a natural way as a Lie subalgebra of $\mathfrak{g}$, considering the natural inclusion $H \to G$. Its derivative is a Lie algebra homomorphism.

**3.1.7 Theorem.** Let $G$ be a Lie group and $H$ a closed Lie subgroup of $G$. Let $G/H$ be the set of left cosets with the quotient topology. Then there is a unique smooth structure on $G/H$ such that the canonical projection $\pi$ is a smooth submersion.

This smooth structure is constructed using the quotient manifold theorem, which can be found in ([12], Theorem 21.10).

**3.1.8 Definition.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g} = T_1G$. The *exponential map* $\exp\colon \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma(1)$, where $\gamma$ is the integral curve

of $\tilde{X}$ starting at $1 \in G$.

The exponential map and its properties are discussed in Chapter 20 of [12].

**3.1.9 Definition.** Let $G$ be a Lie group. The *adjoint representation* of $G$ on its Lie algebra $\mathfrak{g}$, Ad: $G \to \mathrm{Aut}(\mathfrak{g})$, maps $g \in G$ to the differential at the identity $e$ of the conjugation mapping $x \mapsto gxg^{-1}$.

**3.1.10 Theorem.** With this notation, the differential of Ad at the identity is the map

$$\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),$$

defined by $\mathrm{ad}(X)(Y) = [X, Y]$, where we consider the Lie brackets of the Lie algebra $\mathfrak{g}$.

**3.1.11 Definition.** The *Killing form* on $\mathfrak{g}$ is the symmetric bilinear form $B$ on $\mathfrak{g}$ defined by

$$B(X, Y) = tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y)),$$

calculated with respect to a (any) basis of $\mathfrak{g}$.

**3.1.12 Remark.**

1. The Killing form is *Ad-equivariant*, i.e,

$$B(X, Y) = B(\mathrm{Ad}(g)(X), \mathrm{Ad}(g)(Y))$$

   for all $g \in G, X, Y \in \mathfrak{g}$.

2. The adjoint representation induces a group action of $G$ on its Lie algebra, namely

$$G \times \mathfrak{g} \to \mathfrak{g}$$
$$(g, X) \mapsto \mathrm{Ad}(g)(X).$$

**3.1.13 Theorem.** ([6], Theorem 14.1.3). A Lie group $G$ with finitely many connected components has a *maximal compact subgroup*. The maximal compact subgroups of $G$ are all isomorphic via conjugation, i.e., if $K$ and $L$ are maximal compact subgroups of $G$, there is an element $g \in G$ such that $L = gKg^{-1}$.

## 3.2 Principal Bundles

**3.2.1 Definition.** Let $G$ be a Lie group. A *principal $G$-bundle* is a triple $(E, \pi, M)$ where $\pi : E \to M$ is a differentiable map between smooth manifolds. Furthermore, $E$ is equipped with a differentiable right $G$-action $E \times G \to E$ such that the following conditions hold:

1. the sets $E_p = \pi^{-1}(p)$ are the orbits of the action;
2. every point in $M$ has an open neighbourhood $U$ with a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that
   (a) the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times G \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \text{proj}} & \\ & U & \end{array}$$

   is commutative, that is, the map $\varphi_p = \varphi|_{E_p}$ maps $E_p$ to $p \times G$;
   (b) $\varphi$ is equivariant, that is,

$$\varphi(xg) = \varphi(x)g$$

   for all $x \in \pi^{-1}(U), g \in G$, where $G$ acts on $U \times G$ by $(p, g')g = (p, g'g)$.

$E$ is called the *total space*, $M$ is the *base space* and $E_p$ is the *fiber over* $p$. The diffeomorphism $\varphi$ is called a *local trivialization of* $E$ over $U$. Like in the case of vector bundles, we say that $E$ is *locally trivial*. We also write $E_{|U} = \pi^{-1}(U)$ for an open set $U \subseteq M$ over which $E$ is trivial. A principal $G$-bundle is said to be *trivial* if it admits a

trivialization with domain $E$, called a *global trivialization*.

### 3.2.2 Remark.

1. Given a local trivialization $\varphi$, its inverse $\varphi^{-1}$ is also equivariant, ie, $(\varphi^{-1}(p,g))\cdot h = \varphi^{-1}(p,gh)$ for all $p \in U, g, h \in G$.

2. The $G$-action on $E$ is free (i.e. $x = xg$ implies $g = 1$ for all $g \in G, x \in E$) and transitive on each fiber. The orbit space $E/G$ can be identified with $M$.

3. Any left action of a group $G$ on space $X$

$$\begin{aligned} \phi\colon G \times X &\to X \\ (g,x) &\mapsto g \cdot x \end{aligned}$$

   can be transformed into a right $G$-action $\psi$ by defining $x \cdot g = g^{-1} \cdot x$. In certain situations (see Example 3.2.3.5), this allows us to use left actions to construct principal bundles (which require right actions). Furtermore, if $X$ is a manifold, then $\phi$ is differentiable if and only if $\psi$ is differentiable.

### 3.2.3 Example.

1. The frame bundle $(F(V), \overline{\pi}, M)$ of a vector bundle $(V, \pi, M)$ is a principal $\mathrm{GL}(n,\mathbb{R})$-bundle.

2. If $V$ is equipped with a metric, then $(F_O(V), \overline{\pi}, M)$ is a principal $\mathrm{O}(n)$-bundle.

3. For any manifold $M$ and Lie group $G$, $(M \times G, \pi, M)$, with $\pi$ being the projection onto the first factor, is a principal $G$-bundle, called a *product bundle*, which is a trivial bundle. Like in the vector bundle case, it is important to keep trivial bundles in mind.

4. Let $F(k,n) \subset \mathbb{R}^{n\times k}$ be the set of all $n \times k$ matrices of rank $k$ and let $G(k,n) = F(k,n)/\mathrm{GL}(k,\mathbb{R})$ be the Grassmann manifold of $k$-planes in $\mathbb{R}^n$. Let $\pi : F(k,n) \to G(k,n)$ be the natural projection.

   First, we summarize the differentiable manifold structure on $G(k,n)$ (see [13],

page 83). Let $I$ be a strictly ascending multi-index $1 \leq i_1 < ... < i_k \leq n$. For $F \in F(k,n)$, let $A_I$ be the $k \times k$ submatrix of $A$ obtained by taking the $i_1$th, ..., $i_k$th rows of $A$. Define

$$V_I = \{A \in F(k,n) \colon \det A_I \neq 0\}.$$

Next, define a map $\tilde{\phi}_I \colon V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\phi}_I(A) = (AA_I^{-1})_{I'},$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement $I'$ of the multi-index $I$. We have a smooth right $\mathsf{GL}(k,\mathbb{R})$-action

$$\begin{aligned} \phi \colon F(k,n) \times \mathsf{GL}(k,\mathbb{R}) &\to F(k,n) \\ (F,G) &\mapsto FG, \end{aligned}$$

hence the projection $\pi$ is an open map. In particular, $\pi(V_I)$ is an open subset for all $I$. The map $\tilde{\phi}_I$ induces a homeomorphism $\phi_I \colon \pi(V_I) \to \mathbb{R}^{(n-k)\times k}$, and $\{(\pi(V_I), \phi_I)\}$ is a smooth atlas of $G(k,n)$. Therefore $G(k,n)$ is a smooth manifold of dimension $k(n-k)$. Furthermore, $\pi$ is a smooth submersion.

Now we check that we have a principal bundle. We have the smooth action defined earlier and, given $[F] \in G(k,n)$,

$$\begin{aligned} \pi^{-1}([F]) &= \{F' \in F(k,n) \colon \pi(F') = [F]\} \\ &= \{F' \in F(k,n) \colon F' = FG \text{ for some } G \in \mathsf{GL}(k,\mathbb{R})\} \\ &= F \cdot \mathsf{GL}(k,\mathbb{R}), \end{aligned}$$

which is the orbit of $F$ under $\phi$.
The maps

$$\begin{aligned} \varphi_I \colon V_I &\to \pi(V_I) \times \mathsf{GL}(k,\mathbb{R}) \\ F &\mapsto ([F], F_I) \end{aligned}$$

are local trivializations: $\varphi_I$ is smooth, because of the smoothness of $\pi$ and of the map $F \mapsto F_I$. One can verify that $(FG)_I = F_I G$ for all $F \in F(k,n)$ and $G \in \mathrm{GL}(k,\mathbb{R})$, hence $\varphi_I$ is $\mathrm{GL}(k,\mathbb{R})$-equivariant. Furthermore, $\varphi_I$ is invertible with inverse $([F],G) \mapsto F F_I^{-1} G$, which is also smooth (and well defined).

5. Now we make use of some notions of algebraic topology, namely of covering map theory (see ([5], Chapters 11-13)). Let $G$ be a group acting on the left on a topological space $X$. This action is said to be properly discontinuous if any point in $X$ has a neighbourhood $V$ such that $gV \cap hV = \emptyset$ for all distinct elements $g, h \in G$. A covering map $p\colon Y \to X$ is said to be a $G$-covering if $X \cong Y/G$ for some properly discontinuous left action of $G$ on $Y$. Let $p\colon Y \to X$ and $p'\colon Y' \to X$ be two $G$-coverings of the same base space $X$. An isomorphism of the $G$-coverings $p$ and $p'$ is a homeomorphism $\varphi\colon Y \to Y'$ such that $p' \circ \varphi = p$ and $\varphi(g \cdot y) = g \cdot \varphi(y)$ for all $g$ in $G$ and $y$ in $Y$. The trivial $G$-covering is the projection $X \times G \to X$, where $G$ has the discrete topology.

   Any $G$-covering is locally trivial as a $G$-covering; i.e, if $p\colon Y \to X$ is a $G$-covering, then any point in $X$ has a neighbourhood $U$ such that the $G$-covering $p^{-1}(U) \to U$ is isomorphic to the the trivial $G$-covering $U \times G \to U$.

   Let $M$ be a connected manifold and let $p\colon \tilde{M} \to M$ be its universal covering map. Then $\tilde{M}$ inherits a natural smooth structure from $M$. Now $p$ is a $\pi_1 M$-covering (see [5], Theorem 13.11). Hence we have a homeomorphism $\varphi_U\colon p^{-1}(U) \to U \times \pi_1 M$ (continuing to consider the discrete topology on $\pi_1 M$). Note that $M$, as a manifold, has a countable fundamental group (see [12], Proposition 1.16), hence the discrete topology defines a zerodimensional Lie group structure on $\pi_1 M$, which makes $\varphi_U$ a diffeomorphism. The action of $\pi_1 M$ on $\tilde{M}$ is defined by converting the left action of page 183 of [5] into a (differentiable) right action.
   Clearly, the diagram

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi_U} & U \times \pi_1 M \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}} & \\ & U & \end{array}$$

is commutative, and $\pi_1 M$-equivariance follows from the definition of isomorphism of $G$-coverings above.

This construction then defines a principal $\pi_1 M$-bundle $(\tilde{M}, p, M)$.

6. Let $G$ be a Lie group, $H \subseteq G$ a closed embedded Lie subgroup and $\alpha$ the inclusion. Then the projection $\pi : G \to G/H$ defines a principal $H$-bundle $(G, \pi, G/H)$.

   To see this, let $U \subseteq G$ be a submanifold of $G$ containing the identity element $e$, such that $\pi(U) = W$ is open in $G/H$ and $\pi|_U \colon U \to W$ is a diffeomorphism (such maps are components of a proof of the quotient manifold theorem as written in ([12], Theorem 21.10)). Denote its inverse by $s \colon W \to U$.

   Then the map $f \colon \pi^{-1}(W) \to W \times H$ defined by $f(a) = (\pi(a), s(\pi(a))^{-1}a)$ is a diffeomorphism with inverse $(w, h) \mapsto s(w) \cdot h$. $f$ and $f^{-1}$ are both $H$-equivariant, hence $f$ is a local trivialization. Considering the neighbourhood $gW$, we have the trivialization $f_g \colon \pi^{-1}(gW) \to gW \times H$ defined by $f_g(a) = (\pi(a), s(\pi(g^{-1}a))^{-1}g^{-1}a)$, with inverse $(u, h) \mapsto g \cdot s(g^{-1}u) \cdot h$. This defines a principal $H$-bundle structure on $G$ with projection $\pi$ and base $G/H$. See ([3], Lemma 4.6).

## 3.3 Sections and Bundle Maps

**3.3.1 Definition.** A (differentiable) *section* of a principal $G$-bundle $(E, \pi, M)$ is a differentiable mapping $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$, that is, $\sigma(p) \in E_p = \pi^{-1}(p)$ for all $p \in M$.

**3.3.2 Remark.** Consider a trivial bundle $M \times G$ and let $\sigma \colon M \to M \times G$ be a section. Clearly $\sigma$ is of the form $\sigma(p) = (p, s(p))$ for some differentiable map $s \colon M \to G$. Next, consider

$$\begin{aligned} \phi \colon M \times G &\to M \times G \\ (p, g) &\mapsto (p, s(p)^{-1} \cdot g). \end{aligned}$$

Then $\phi$ is a trivialization of $M \times G$ which satisfies $\phi \circ \sigma(p) = (p, 1)$.

In general, given a principal $G$-bundle $(E, \pi, M)$, suppose there is a local section

$\sigma\colon U \to \pi^{-1}(U) \subseteq E$, where $U$ is an open subset of $M$ over which $E$ has a trivialization $\psi$. We can then consider $\psi \circ \sigma\colon U \to U \times G$, which is a section of the trivial $G$-bundle $U \times G$. Therefore, we can take a trivialization $\phi$ of $U \times G$ such that $\phi \circ \psi \circ \sigma(p) = (p, 1)$. Clearly $\varphi = \phi \circ \psi$ is a trivialization of $E$ over $U$. Again, we have $\varphi \circ \sigma(p) = (p, 1)$.

Here are some applications:

1. Consider

$$\begin{aligned} f\colon U \times G &\to \pi^{-1}(U) \\ (p, g) &\mapsto \sigma(p) \cdot g. \end{aligned}$$

   We can take a trivialization $\varphi$ of $E$ over $U$ such that $\varphi \circ \sigma(p) = (p, 1)$. Therefore, the map $f \circ \varphi$ is the identity over $U \times G$ and so $f$ is a diffeomorphism. Noting that it is also equivariant, we conclude that $f^{-1}$ is a trivialization of $E$ over $U$.

2. For each $x \in \pi^{-1}(U)$, let $p = \pi(x)$. Denote by $a(x)$ the element of $G$ such that $x = \sigma(p) \cdot a(x)$. This defines a map $a\colon \pi^{-1}(U) \to G$. By definition, the composite $a \circ f\colon U \times G \to G$ is the map $(p, g) \mapsto g$, which is differentiable. Since $f$ is a diffeomorphism, we conclude that $a$ is differentiable.

In particular, we obtain

**3.3.3 Proposition.** A principal $G$-bundle $E$ over a manifold $M$ is trivial if and only if it admits a global section $\sigma\colon M \to E$.

**3.3.4 Definition.** Let $(E, \pi, M)$ and $(F, \xi, M')$ be two principal $G$-bundles. A *bundle map* from $E$ to $F$ is a pair of differentiable mappings $(\overline{\boldsymbol{f}}, f)$ such that

1. the diagram

$$\begin{array}{ccc} E & \xrightarrow{\overline{\boldsymbol{f}}} & F \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \xi} \\ M & \xrightarrow{f} & M' \end{array}$$

is commutative;

2. $\overline{\boldsymbol{f}}$ is $G$-equivariant, that is,

$$\overline{\boldsymbol{f}}(x \cdot g) = \overline{\boldsymbol{f}}(x)g$$

for all $x \in E, g \in G$.

If $M = M'$, $\overline{\boldsymbol{f}}$ is a $G$-equivariant diffeomorphism and $f = id$, then $\overline{\boldsymbol{f}}$ is called an isomorphism between $E$ and $F$.

Note the similiarities with the definition for vector bundles, linearity being replaced by equivariance.

**3.3.5 Example.** Let $(E, \pi, M)$ be a principal $G$-bundle and $f\colon N \to M$ a smooth map of manifolds. Like in the vector bundle case, we can define a set $f^*E$ by

$$f^*E = \{(n, e) \in N \times E\colon f(n) = \pi(e)\},$$

considered as a subspace of $N \times E$, and take the natural projection $\eta\colon f^*E \to N, (n, e) \mapsto n$. Then the triple $(f^*E, \eta, N)$ is a principal $G$-bundle over $N$: $G$ acts naturally on $f^*E$ on the right and, given an open covering $\{U_\alpha\}_{\alpha\in\Sigma}$ of $M$ with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$, we define

$$\begin{aligned}\psi_\alpha\colon \eta^{-1}(f^{-1}(U_\alpha)) &\to f^{-1}(U_\alpha) \times G\\ (n, e) &\mapsto (n, p_2 \circ \varphi_\alpha(e)).\end{aligned}$$

For each $\alpha$, the map $\psi_\alpha$ is well defined, continuous and has an inverse $(n, g) \mapsto (n, \varphi_\alpha^{-1}(f(n), g))$, which is also well defined and continuous.

Clearly, the maps $\psi_\alpha$ preserve fibers and equivariance follows from the equivariance of the respective maps $\varphi_\alpha$. One can then define the principal $G$-bundle structure on $f^*E$ by making each $\psi_\alpha$ a diffeomorphism. The bundle $(f^*E, \eta, N)$ is called the pullback of $E$ by $f$.

Furthermore, denote by $p_2$ the map $f^*E \to E, (n, e) \mapsto e$. Then we have a bundle map

$$\begin{array}{ccc} f^*E & \xrightarrow{p_2} & E \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{f} & M \end{array}.$$

Much like with vector bundles, it is important to know how to compare the various local trivializations in their intersections. For this, we return to the study of transition functions, starting with the trivial case.

**3.3.6 Lemma.** Every isomorphism $\varphi : M \times G \to M \times G$ is of the form

$$\varphi(p, a) = (p, g(p) \cdot a) \qquad (*)$$

for some differentiable mapping $g\colon M \to G$.

**Proof.** It is easy to verify that $\varphi$ defined by $(*)$ is an isomorphism with inverse given by $(p, b) \mapsto (p, g(p)^{-1} \cdot b)$. On the other hand, let $\varphi : M \times G \to M \times G$ be an arbitrary isomorphism. It is straightforward to verify that there is a differentiable mapping $g\colon M \to G$ such that $\varphi(p, 1) = (p, g(p))$. The equivariance of $\varphi$ now implies

$$\varphi(p, a) = \varphi((p, 1)a) = (\varphi(p, 1))a = (p, g(p) \cdot a),$$

hence $(*)$ holds. □

For an arbitrary principal $G$-bundle $(E, \pi, M)$, take now an open covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$. If $U_\alpha \cap U_\beta \neq \emptyset$, we can consider the isomorphism $\varphi_\beta \circ \varphi_\alpha^{-1}\colon U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G$. The previous Lemma implies that

$$\varphi_\beta \circ \varphi_\alpha^{-1}(p, a) = (p, g_{\alpha\beta}(p) \cdot a)$$

for some differentiable map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$.

**3.3.7 Definition.** The differentiable maps $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ are called the *transition functions* for $E$ with respect to the covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and the trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$.

**3.3.8 Remark.**

1. Note that $g_{\alpha\beta}(p) = (p_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1})(p, 1)$, where $p_2$ is the projection onto the second factor.

2. For $\alpha, \beta, \gamma \in \Sigma$ (such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$) the following *cocycle condition* is verified:

$$g_{\gamma\beta}(p) \cdot g_{\beta\alpha}(p) = g_{\gamma\alpha}(p)$$
$$g_{\alpha\alpha}(p) = 1.$$

3. Conversely, let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of a manifold $M$ and suppose $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ is a family of differentiable mappings $U_{\alpha\beta} \to G$ satisfying the cocycle condition. Let

$$E = \big( \bigsqcup_{\alpha\in\Sigma} U_\alpha \times G \big) / \sim,$$

where $(\alpha, p, a) \sim (\beta, q, b)$ if and only if $p = q$ and $b = g_{\beta\alpha}(p)a$. The cocycle condition implies that $\sim$ is an equivalence relation. Consider the map

$$\pi \colon E \to M$$
$$[(\alpha, p, a)] \mapsto p$$

and the bijections

$$\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$$
$$[(\alpha, p, a)] \mapsto (p, a)$$

Then one can define a differentiable structure on $E$ such that $(E, \pi, M)$ is a principal $G$-bundle with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ with $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ being the associated family of transition functions (see ([3], Proposition 3.6)).

**3.3.9 Example.** (continuation of Example 3.2.3.4) The transition functions for the principal $\mathsf{GL}(k,\mathbb{R})$-bundle $(F,(k,n),\pi,G(k,n))$ are

$$g_{IJ}\colon \pi(V_I)\cap\pi(V_J)\to \mathsf{GL}(k,\mathbb{R})$$
$$[F]\mapsto F_I F_J^{-1}.$$

## 3.4 Associated Bundles

In Section 2.4, we associated to a vector bundle $(V,\pi,M)$ its frame bundle $(F(V),\overline{\boldsymbol{\pi}},M)$, which is a principal $\mathsf{GL}(n,\mathbb{R})$-bundle. We can think about inverting this correspondence, i.e., associating to a principal $\mathsf{GL}(n,\mathbb{R})$-bundle $(E,\pi,M)$ a vector bundle over the same base space $M$. This will be a particular case of the following construction.

Let $G$ be a Lie group and $(E,\pi,M)$ a principal $G$-bundle. Consider a manifold $N$ with a left $G$-action $G\times N\to N$. Then there is a right $G$-action on $E\times N$, defined by

$$(x,u)\cdot g=(xg,g^{-1}u),$$

where $x\in E, u\in N, g\in G$. Let $E_N=E\times_G N=(E\times N)/G$ be the orbit space for this action. One can check that $E_N$ is the quotient space for the equivalence relation $\sim$, where $(x,u)\sim(y,v)$ if and only if $y=xg$ and $u=gv$ for some $g\in G$. Furthermore, consider the (well-defined) map $\pi_N: E_N\to M$ which sends a class $[(e,n)]$ to $\pi(e)$.

### 3.4.1 Proposition.

1. $E_N$ is in a natural way a differentiable manifold and $\pi_N: E_N\to M$ is differentiable.
2. There are local trivializations: every point in $M$ has a neighbourhood $U$ and a diffeomorphism $f:\pi_N^{-1}(U)\to U\times N$ such that the diagram

$$\begin{array}{ccc} \pi_N^{-1}(U) & \xrightarrow{\ f\ } & U\times N \\ & {\scriptstyle\pi_N}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & U & \end{array}$$

commutes.

**Proof.** Let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of $M$ with local trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ of $E$. Define $f_\alpha\colon \pi_N^{-1}(U) \to U \times N$ by $[(e,n)] \mapsto (\pi(e), p_2 \circ \varphi_\alpha(e)\cdot n)$, where $p_2\colon U\times G \to G$ is the projection onto the second factor. Its inverse is defined by $(p,n)\mapsto [(\varphi_\alpha^{-1}(p,1),n)]$; the differentiable structure on $E_N$ is defined by making the maps $f_\alpha$ diffeomorphisms. Indeed, the coordinate transformations are

$$\begin{aligned} f_\beta \circ f_\alpha^{-1}\colon U_{\alpha\beta}\times N &\to U_{\alpha\beta}\times N \\ (p,n) &\mapsto (p, g_{\alpha\beta}(p)\cdot n), \end{aligned}$$

where $g_{\alpha\beta}$ is the transition function (of $E$) from $\varphi_\alpha$ to $\varphi_\beta$. Furthermore, we have the required commutative diagrams. □

The triple $(E_N, \pi_N, M)$ is called the bundle with fiber $N$ *associated* to the principal $G$-bundle $(E,\pi,M)$.

**3.4.2 Example.** Let $M$ be a connected manifold and consider the principal $\pi_1 M$-bundle $(\tilde{M}, p, M)$ defined in Example 3.2.3.5. Let $G$ be a Lie group such that there is a representation (a group homomorphism) $\rho\colon \pi_1 M \to G$. Then $\rho$ defines a left action of $\pi_1 M$ on $G$

$$\begin{aligned} \pi_1 M \times G &\to G \\ ([\gamma], g) &\mapsto \rho([\gamma])g. \end{aligned}$$

Note that this action is smooth, as we are considering $\pi_1 M$ as a discrete Lie group. Hence we can form the associated bundle with fibre $G$, using the action of $\pi_1 M$ on $G$ provided by $\rho$, which we denote by $(\tilde{M}\times_\rho G, p_G, M)$. In this case, we have a commutative diagram

$$\begin{array}{ccc} p_G^{-1}(U) & \xrightarrow{\quad f \quad} & U\times G \\ & {\scriptstyle p_G}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & U & \end{array},$$

Furthermore, one can also define a differentiable right $G$-action on $\tilde{M} \times_\rho G$ by $[(y,g)] \cdot h = [(y,gh)]$; $f$ then becomes $G$-equivariant. This defines a principal $G$-bundle structure on $\tilde{M} \times_\rho G$.

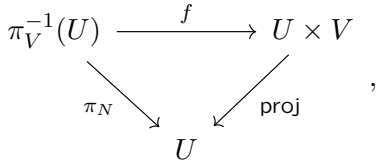
### 3.4.3 Proposition.

1. Let $(E, \pi, M)$ be a principal $G$-bundle and $V$ a vector space such that there is a representation $\rho\colon G \to \mathsf{GL}(V)$. Then $E \times_\rho V$, the bundle with fibre $V$ associated to $E$ (defined using the natural action of $G$ on $V$ using $\rho$) is in natural way a vector bundle.

2. If $G = \mathsf{GL}(n, \mathbb{R})$ and $V = \mathbb{R}^n$, then the frame bundle of $E_{\mathbb{R}^n}$ is isomorphic to $E$.

**Proof.**

1. By definition, we have commutative diagrams




   and the maps $f$ restrict to bijections $f_p\colon \pi_V^{-1}(p) \to p \times V$. Define on $\pi_V^{-1}(p)$ the vector space structure such that $f_p$ is a vector space isomorphism for each $p \in U$. This construction satisfies conditions 1. and 2. of Definition 2.1.1.

2. We want a fibre-preserving equivariant diffeomorphism $\varphi\colon E \to F(E_{\mathbb{R}^n})$. Let $\varphi(e) = \tilde{e}$, where $\tilde{e} \in \mathsf{Iso}(\mathbb{R}^n, \pi_{\mathbb{R}^n}^{-1}(\pi(e))$ is defined by $x \mapsto [(e,x)]$. Clearly $\varphi$ preserves fibers. Since $[(eg,x)] = [(e,gx)]$ for all $e \in E$, $g \in \mathsf{GL}(n,\mathbb{R})$ and $x \in \mathbb{R}^n$, $\varphi$ is equivariant.

   Let $U \subseteq M$ be an open subset with trivializations $f\colon \pi^{-1}(U) \to U \times \mathsf{GL}(n,\mathbb{R})$ of $E$ and $\overline{\boldsymbol{f}}\colon \overline{\boldsymbol{\pi}}_{\mathbb{R}^n}^{-1}(U) \to U \times \mathsf{GL}(n,\mathbb{R})$ of $F(E_{\mathbb{R}^n})$. One can verify that $\overline{\boldsymbol{f}} \circ \varphi|_U \circ f^{-1}$ is the identity map, hence $\varphi$ is differentiable. In particular, it is a local diffeomorphism. Since $\varphi|_U$ maps $\pi^{-1}(U)$ bijectively to $\overline{\boldsymbol{\pi}}^{-1}(U)$ and the sets $\overline{\boldsymbol{\pi}}^{-1}(U)$ cover $F(E_{\mathbb{R}^n})$, it follows that $\varphi$ is bijective. So $\varphi$ is bijective and a local diffeomorphism, hence it is a diffeomorphism.

□

**3.4.4 Remark.** If $E$ is a principal $G$-bundle, then we can take the associated bundle $E_G$, where $G$ acts on itself via left translations. In this case, we have a natural isomorphism $E \to E_G$ which sends $e \in E$ to the class $[(e, 1)]$.

**3.4.5 Example.** Let $\rho\colon G \to \mathsf{GL}(V)$ be a representation of a Lie group $G$ on a vector space $V$. We can consider the *dual representation* of $\rho$ on the dual vector space $V^*$:

$$\begin{aligned} \rho^*\colon G &\to \mathsf{GL}(V^*) \\ g &\mapsto \rho(g^{-1})^t, \end{aligned}$$

where $\rho(g^{-1})^t\colon V^* \to V^*$ is the transpose map which sends $f \in V^*$ to $f \circ \rho(g^{-1})$.

Now let $(V, \pi, M)$ be a vector bundle of rank $n$ and consider the identity representation $\rho = id\colon \mathsf{GL}(n, \mathbb{R}) \to \mathsf{GL}(n, \mathbb{R})$. Then we can form the vector bundle $F(V)_{\mathbb{R}^n}$, which is naturally isomorphic to $V$.

If we instead consider the dual representation $\rho^*\colon \mathsf{GL}(n, \mathbb{R}) \to \mathsf{GL}((\mathbb{R}^n)^*)$ defined by $\rho^*(A)(f) = f \circ \rho(A^{-1}) = f \circ A^{-1}$ we can form a new vector bundle $F(V)_{(\mathbb{R}^n)^*}$ using the action provided by $\rho^*$, which has fibre $(\mathbb{R}^n)^*$.

Next, let

$$V^* = \bigsqcup_{p \in M} V_p^*,$$

be the dual bundle of $V$. Consider the map

$$\begin{aligned} \varphi\colon F(V)_{(\mathbb{R}^n)^*} &\to V^* \\ [(e, f)] &\mapsto (\pi(e), f \circ e^{-1}). \end{aligned}$$

It's easy to see that $\varphi$ is well defined and preserves fibers. Furthermore, noting that, given $f \in V_p^*$, $[(e_1, f \circ e_1)] = [(e_2, f \circ e_2)]$ for all $e_1, e_2 \in F(V_p)$, we have a well-defined map $(p, f_p) \mapsto [(e_p, f_p \circ e_p)]$ (where $e_p \in F(V_p)$), which is the inverse of $\varphi$. Over an open set $U$

with trivializations $\phi$ of $F(V)_{(\mathbb{R}^n)^*}$ and $\psi$ of $V^*$, the composite map $\psi \circ \varphi \circ \phi^{-1}$ is the identity map over $U \times (\mathbb{R}^n)^*$, hence $\varphi$ is an isomorphism.

So the "associated bundle method" provides an alternative construction of the dual bundle $V^*$. It can also be applied to construct some vector bundles mentioned in Section 2, such as $V^* \otimes W \cong \mathrm{Hom}(V, W)$.

## 3.5 Reductions of Principal Bundles

We now study ways to relate principal bundles over the same base space with different structure groups. Let $G$ and $H$ be two Lie groups and $\alpha : H \to G$ a Lie group homomorphism. For example, one can consider the case where $H$ is a closed Lie subgroup (so $H$ is a submanifold of $G$) and $\alpha$ is the inclusion.

**3.5.1 Definition.** Let $(F, \xi, M)$ be a principal $H$-bundle and $(E, \pi, M)$ a principal $G$-bundle. A *reduction of $E$ to $H$ relative to $\alpha$* (or an extension of $F$ to $G$ relative to $\alpha$) is a differentiable mapping $\varphi : F \to E$ which satisfies the following conditions:

1. (Fibre preservation) the diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & E \\ & {\scriptstyle \xi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

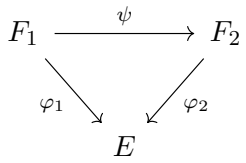
   is commutative, that is, the map $\varphi_p = \varphi|_{F_p}$ maps $F_p$ to $E_p$ for all $p \in M$;

2. $\varphi$ is $\alpha$-equivariant, that is,

$$\varphi(xh) = \varphi(x) \cdot \alpha(h)$$

   for all $x \in F, h \in H$.

Two reductions $\varphi_1 \colon F_1 \to E$, $\varphi_2 \colon F_2 \to E$ of $E$ to $H$ relative to $\alpha$ are said to be *equivalent* if there is an isomorphism $\psi \colon F_1 \to F_2$ such that the diagram




commutes. Note that this is an equivalence relation.

### 3.5.2 Example.

1. Let $(V, \pi, M)$ be a vector bundle with a metric. Then the inclusion $F_O(V) \subseteq F(V)$ defines a reduction of the principal $\mathrm{GL}(n, \mathbb{R})$-bundle $F(V)$ to $\mathrm{O}(n)$ (which is the maximal compact subgroup of $\mathrm{GL}(n, \mathbb{R})$).

2. (continuation of Example 3.2.3.4) Let $S(k, n) = \{A \in F(k, n) \colon A^T A = id\} \subseteq F(k, n)$ be the Stiefel manifold. Then the inclusion $S(k, n) \subseteq F(k, n)$ defines a reduction of the principal $\mathrm{GL}(k, \mathbb{R})$-bundle $(F, (k, n), \pi, G(k, n))$ to the group $\mathrm{O}(k) \subseteq \mathrm{GL}(k, \mathbb{R})$.

Let $(F, \xi, M)$ be a principal $H$-bundle, $(E, \pi, M)$ a principal $G$-bundle, $\alpha\colon H \to G$ a Lie group homomorphism and $\varphi\colon F \to E$ a reduction of $E$ to $H$ relative to $\alpha$. Let $\{U_i\}_{i\in I}$ be an open covering of $M$ with trivializations $\psi_i\colon \xi^{-1}(U_i) \to U_i \times H$ and $\phi_i\colon \pi^{-1}(U_i) \to U_i \times G$ of $F$ and $E$, respectively. By definition, $\varphi$ restricts to maps $\xi^{-1}(U_i) \to \pi^{-1}(U_i)$ (which we also denote by $\varphi$), hence we can consider the composite maps

$$U_i \times H \xrightarrow{\psi_i^{-1}} \xi^{-1}(U_i) \xrightarrow{\varphi} \pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times G.$$

Given $p \in U_i$, we have $\phi_i \circ \varphi \circ \psi_i^{-1}(p, 1_H) = (p, g(p))$ for some differentiable map $g\colon U_i \to G$, which implies

$$\begin{aligned}\phi_i \circ \varphi \circ \psi_i^{-1}(p,h) &= \phi_i \circ \varphi \circ \psi_i^{-1}((p,1_H)\cdot h)\\ &= \phi_i \circ \varphi(\psi_i^{-1}(p,1_H)\cdot h)\\ &= (\phi_i \circ \varphi \circ \psi_i^{-1}(p,1_H))\cdot \alpha(h)\\ &= (p,g(p))\cdot\alpha(h) = (p, g(p)\alpha(h)).\end{aligned}$$

Define $\tilde{\phi}_i\colon \pi^{-1}(U_i) \to U_i \times G$ by $\tilde{\phi}_i(e) = (\pi(e), g(\pi(e))^{-1} \cdot (p_2 \circ \phi)(e))$. Then $\tilde{\phi}_i$ is also a trivialization of $E$, with inverse $(p, a) \mapsto \phi_i^{-1}(p, g(p) \cdot a)$. It is easy to see that $\tilde{\phi}_i \circ \varphi \circ \psi_i^{-1}(p, 1_H) = (p, 1_G)$, hence $\tilde{\phi}_i \circ \varphi \circ \psi_i^{-1}(p, h) = (p, \alpha(h))$. Now we have commutative diagrams

$$\begin{array}{ccc}\xi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times H\\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id\times\alpha}\\ \pi^{-1}(U_i) & \xrightarrow{\tilde{\phi}_i} & U_i \times G\end{array}.$$

Let $\{h_{ij}\}_{i,j\in I}$ be the transition functions of $F$ associated to the trivializations $\{\psi_i\}_{i\in I}$ and $\{\tilde{g}_{ij}\}_{i,j\in I}$ the transition functions of $E$ associated to the trivializations $\{\tilde{\phi}_i\}_{i\in I}$. In particular, we have commutative diagrams

$$\begin{array}{ccc}\xi^{-1}(U_{ij}) & \xrightarrow{\psi_i} & U_{ij} \times H\\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id\times\alpha}\\ \pi^{-1}(U_{ij}) & \xrightarrow{\tilde{\phi}_i} & U_{ij} \times G\end{array},$$

therefore $\varphi \circ \psi_i^{-1} = \tilde{\phi_i}^{-1} \circ (id \times \alpha)$. From $(id \times \alpha) \circ \psi_j = \tilde{\phi_j} \circ \varphi$, we obtain $(id \times \alpha) \circ \psi_j \circ \psi_i^{-1} = \tilde{\phi_j} \circ \tilde{\phi_i}^{-1} \circ (id \times \alpha)$. Applying both sides to an element $(p, h) \in U_{ij} \times H$, we obtain $(p, (\alpha \circ h_{ij})(p) \cdot \alpha(h)) = (p, \tilde{g_{ij}}(p) \cdot \alpha(h))$, which implies $\tilde{g_{ij}} = \alpha \circ h_{ij}$.

Hence the transitions functions $\tilde{g_{ij}}$ take values in $\alpha(H)$. In particular, if $H$ is a closed embedded Lie subgroup of $G$ and $\alpha$ is the inclusion, they take values in $H$.

On the other hand, if $E$ has transition functions of the form $\{\alpha \circ h_{ij}\}_{i,j \in J}$, where $h_{ij} \colon U_{ij} \to H$ are differentiable functions satisfying the cocycle condition, consider the principal $H$-bundle $(F, \xi, M)$ built using Remark 3.3.8.3 and let

$$\begin{aligned} \varphi \colon F &\to E \\ [(i, p, h)] &\mapsto \phi_i^{-1}(p, \alpha(h)). \end{aligned}$$

We check that $\varphi$ is a reduction of $E$ to $H$. First, we see it's well defined. If $[(i, p, h)] = [(j, q, k)]$, then $p = q$ and $k = h_{ij}(p)h$. We want $\phi_i^{-1}(p, \alpha(h)) = \phi_j^{-1}(p, \alpha(h_{ij}(p)h)$. Since

$$\begin{aligned} &\phi_i^{-1}(p, \alpha(h)) = \phi_j^{-1}(p, \alpha(h_{ij}(p)h) \\ &\Leftrightarrow \phi_j \circ \phi_i^{-1}(p, \alpha(h)) = (p, \alpha(h_{ij}(p)h) \\ &\Leftrightarrow (p, (\alpha \circ h_{ij})(p) \cdot \alpha(h)) = (p, \alpha(h_{ij}(p)h)), \end{aligned}$$

$\varphi$ is well defined. Clearly $\varphi$ preserves fibers and is $\alpha$-equivariant. Denote the trivializations of $F$ by $\psi_i$. It is easy to see that there are commutative diagrams

$$\begin{array}{ccc} \xi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times H \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id \times \alpha} \\ \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times G \end{array}$$

and so $\varphi$ is differentiable. Hence $\varphi$ is a reduction of $E$ to $H$.

Summarizing, we have

**3.5.3 Proposition.** Let $E$ be a principal $G$-bundle and $\alpha\colon H \to G$ a Lie group homomorphism. Then the following conditions are equivalent:

1. $E$ has a reduction to $H$ relative to $\alpha$;
2. there is a covering $\{U_i\}_{i\in I}$ of $M$ and transition functions for $E$ of the form $\{\alpha \circ h_{ij}\}_{i,j\in I}$, where $h_{ij}\colon U_{ij} \to H$ are differentiable functions satisfying the cocycle condition.

Now, we focus on the case where $G$ is a Lie group, $H \subseteq G$ is a closed embedded Lie subgroup and $\alpha$ is the inclusion. It follows from the previous Proposition that $E$ has a reduction to $H$ if and only if there is a covering $\{U_i\}_{i\in I}$ of $M$ and transition functions for $E$ mapping into $H$.

**3.5.4 Remark.** It also follows from the discussion preceding Proposition 3.5.3 that if $\alpha\colon H \to G$ is the inclusion, then the reduction $\varphi\colon F \to E$ is an embedding. Hence we can see $F$ as a submanifold of $E$, called a *reduced subbundle* of $E$. In this case, the local trivializations of $F$ are the restrictions of the local trivializations of $E$, which now take values in $H$.

Recall from Section 3.4 that, given a principal $G$-bundle $(E, \pi, M)$, we can form associated bundles $(E_N, \pi_N, M)$. In particular, we can let $N = G/H$, noting that $G$ acts naturally on $G/H$ on the left. We now have one of the main results of the Section, relating reductions of $E$ to $H$ with sections of the bundle $E_{G/H} = E \times_G (G/H)$.

**3.5.5 Theorem.** Let $H \subseteq G$ be a closed embedded Lie subgroup, $\alpha\colon H \to G$ the inclusion and $(E, \pi, M)$ a principal $G$-bundle.

1. Let $\overline{\pi}\colon E \to E_{G/H}$ be defined by $\overline{\pi}(e) = [(e, H)]$. Then $(E, \overline{\pi}, E_{G/H})$ is a principal $H$-bundle.
2. There is a 1-1 correspondence between the set of sections of the bundle $(E \times_G (G/H), \pi_{G/H}, M)$ and the set of equivalence classes of reductions of $E$ to $H$ (relative to $\alpha$).

**Proof.** 1) Note that $\overline{\pi}^{-1}(\pi_{G/H}^{-1}(U)) = \pi^{-1}(U)$. By definition of the differentiable structure of $E_{G/H}$, over suitable neighbourhoods $U \subseteq M$, we have commutative diagrams

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{f_G} & U \times G \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle id\times p} \\ \pi_{G/H}^{-1}(U) & \xrightarrow[f_{G/H}]{} & U \times G/H \end{array},$$

where the maps $f_G$ and $f_{G/H}$ are local trivializations of $E$ and $E_{G/H}$ over $U$, respectively, and $p$ is the projection of the principal $H$-bundle $(G, p, G/H)$ of Example 3.2.3.6. Consider a trivialization $\phi\colon p^{-1}(W) \to W \times H$ associated to a diffeomorphism $s$ as in the Example. Let $\psi$ be the diffeomorphism which factors as follows:

$$f_G^{-1}(U \times p^{-1}(W)) \xrightarrow{f_G} U \times p^{-1}(W) \xrightarrow{id\times\phi} U \times W \times H \xrightarrow{f_{G/H}^{-1}\times id} f_{G/H}^{-1}(U \times W) \times H,$$

$$f_G^{-1}(p, g) \mapsto (p, g) \mapsto (p, gH, s(gH)^{-1} \cdot g) \mapsto ([(f_G^{-1}(p, 1), gH)], s(gH)^{-1} \cdot g).$$

Clearly $\psi$ is $H$-equivariant. It is also straightforward to check that we have a commutative diagram

$$\begin{array}{ccc} f_G^{-1}(U \times p^{-1}(W)) & \xrightarrow{\quad\psi\quad} & f_{G/H}^{-1}(U \times W) \times H \\ & {\scriptstyle \overline{\pi}}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & f_{G/H}^{-1}(U \times W) & \end{array}$$

This shows that the maps $\psi$ are local trivializations which define a principal $H$-bundle $(E, \overline{\pi}, E_{G/H})$.

2) Suppose we have a reduction $\varphi\colon F \to E$, where $F$ is a principal $H$-bundle over $M$. Recall that $M$ is diffeomorphic to the orbit space $F/H$ (see Remark 3.2.2.2). The

reduction $\varphi$ defines a map

$$\begin{aligned} s_\varphi \colon M = F/H &\to E/H \\ [f] &\mapsto [\varphi(f)]. \end{aligned}$$

Note that the map $\beta \colon E/H \to E \times_G (G/H)$ defined by $\beta([e]) = [(e, H)]$ is a homeomorphism (see ([3], Theorem 4.7)), which defines a differentiable structure on $E/H$; now $\beta$ becomes a diffeomorphism. So $s_\varphi$ becomes

$$\begin{aligned} s_\varphi \colon M = F/H &\to E \times_G (G/H) \\ [f] &\mapsto [(\varphi(f), H)]. \end{aligned}$$

Let $\{U_i\}_{i \in I}$ be an open covering of $M$ with trivializations $\{\psi_i\}_{i \in I}$ of $F$ and $\{\phi_i\}_{i \in I}$ of $E$ such that we have commutative diagrams

$$\begin{array}{ccc} \xi^{-1}(U_i) & \xrightarrow{\psi_i} & U_i \times H \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id \times \alpha} \\ \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times G \end{array}$$

as in the discussion preceding Proposition 3.5.3 (see also Remark 3.5.4). Let $\{\overline{\phi}_i\}_{i \in I}$ be the corresponding trivializations of $E \times_G (G/H)$. Given $p \in U_i$ and $f \in \xi^{-1}(p)$, we have $(p_2 \circ (id \times \alpha) \circ \psi_i)(f) \in H$, hence the composite map $\overline{\phi}_i \circ s_\varphi \colon U_i \to U_i \times G/H$ is the map $p \mapsto (p, H)$, which is differentiable. Hence $s_\varphi$ is differentiable. It is also clear that if $\varphi'$ is equivalent to $\varphi$, then $s_\varphi = s_{\varphi'}$.

On the other hand, let $\sigma \colon M \to E \times_G (G/H)$ be a section. We can then consider the pullback bundle $\sigma^* E$ of the principal $H$-bundle $(E, \overline{\pi}, E \times_G (G/H))$. We have a bundle

map

$$\begin{array}{ccc} \sigma^*E & \xrightarrow{p_2} & E \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{\sigma} & E/H \end{array}$$

and it follows that $p_2 \colon \sigma^*E \to E$ is a reduction of $E$ to $H$. □

## 3.6 Sections and Equivariant Maps

**3.6.1 Theorem.** Let $(E, \pi, M)$ be a principal $G$-bundle and $N$ be a manifold such that $G$ acts on $N$ on the left. Then there is a 1-1 correspondence between the set of sections of $E_N$ and the set of differentiable maps $s \colon E \to N$ such that

$$s(e \cdot g) = g^{-1} \cdot s(e)$$

for all $e \in E, g \in G$.

**Proof.** Suppose $s \colon E \to N$ is $G$-equivariant. For each $p \in M$ let $e_p \in \pi^{-1}(p)$. Define $\sigma \colon M \to E_N$ by $\sigma(p) = [(e_p, s(e_p))]$. We verify that $\sigma$ is a section of $E_N$.

1. $\sigma$ is well-defined, i.e, it does not depend on the choice of $e_p$. Indeed, let $e'_p \in \pi^{-1}(p)$ and $g \in G$ be such that $e'_p = e_p g$. We have

   $$s(e'_p) = s(e_p g) = g^{-1} s(e_p),$$

   hence $[(e_p, s(e_p))] = [(e'_p, s(e'_p))]$.

2. $\sigma$ is differentiable. Fix an open subset $U$ of $M$ over which there is a trivialization $\varphi$ of $E$. Let $\tilde{\sigma} \colon U \to \pi^{-1}(U)$ be the associated local section, which sends $p$ to $\varphi^{-1}(p, 1)$. Let $\phi$ be the corresponding trivialization of $E_N$. Clearly, we have $\sigma(p) = [(\tilde{\sigma}(p), s(\tilde{\sigma}(p))]$. The composite map $\phi \circ \sigma$ is the map $p \mapsto (p, s(\tilde{\sigma}(p)))$, which is differentiable. Hence $\sigma$ is differentiable.

It is now clear that $\sigma$ is a section of $E_N$.

On the other hand, suppose $\sigma\colon M \to E_N$ is a section. Let $x \in E$ and $p = \pi(x)$. Suppose $\sigma(p) = [x', n]$, with $x' = x \cdot h$. Set $f(x) = h^{-1} \cdot n$. It is easy to see that $f$ is well defined and is $G$-equivariant. To check that it is differentiable, take a local section $s$ of $E$ over $U$, denote by $\varphi$ the trivialization of $E$ associated to $s$ and by $\phi$ be the corresponding trivialization of $E_N$. Then, again over $U$,

$$\sigma(p) = [(s(p), \lambda(p))]$$

for some map $\lambda\colon U \to N$. This implies $f(s(p)) = \lambda(p)$ and so

$$f(x) = (p_2 \circ \varphi)(x)^{-1} \cdot \lambda(\pi(x))$$

where $x \in U$. It is sufficient to prove that $\lambda$ is differentiable. But $\lambda$ is just the composite map $p_2 \circ \phi \circ \sigma$, so it is differentiable.

Finally, it is easy to check that these two correspondences are inverses of each other. □

With Theorem 3.5.5, we obtain another of our main results.

**3.6.2 Corollary.** Let $(E, \pi, M)$ be a principal $G$-bundle and $H$ be a closed embedded subgroup of $G$. Then there is a 1-1 correspondence between the set of (equivalence classes of) reductions of $E$ to $H$ and the set of differentiable maps $s\colon E \to G/H$ such that

$$s(e \cdot g) = g^{-1} \cdot s(e)$$

for all $e \in E, g \in G$.

# 4 Connections on Principal Bundles

## 4.1 Definition and Examples

Let $G$ be a Lie group and let $\mathfrak{g} = T_eG$ be its Lie algebra. Let Ad: $G \to \text{Aut}(\mathfrak{g})$, be the adjoint representation (see Definition 3.1.9). Let $(E, \pi, M)$ be a principal $G$-bundle. For a fixed $x \in E$, the mapping $G \to E$ which sends $g$ to $xg$ is an embedding, whose derivative $v_x : \mathfrak{g} \to T_xE$ is injective. Furthermore, the quotient space by the image of $v_x$ is mapped isomorphically onto $T_{\pi(x)}M$ by $\pi_*$, hence we have an exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_x} T_xE \xrightarrow{\pi_*(x)} T_{\pi(x)}M \longrightarrow 0.$$

**4.1.1 Definition.** Let $x \in E$. A tangent vector $X \in v_x(\mathfrak{g}) = \ker(\pi_*(x))$ is said to be *vertical* (at $x$).

**4.1.2 Remark.** Note that, given $A \in \mathfrak{g}$, we can define a vector field $A^*$ on $E$ by $A^*_x = v_x(A)$, called the *vertical vector field associated to* $A$. Furthemore, we have $[A, B]^* = [A^*, B^*]$ for all $A, B \in \mathfrak{g}$. This follows from the definition of the Lie bracket on $\mathfrak{g}$ as the Lie bracket of the associated left-invariant vector fields on $G$ (see Remark 3.1.4.3 and ([3], page 56)).

We want to choose a complement of the vertical subspace $v_x(\mathfrak{g}) \subseteq T_xE$, i.e., a subspace $H_x \subseteq T_xE$ such that $H_x$ is mapped isomorphically onto $T_{\pi(x)}M$ by $\pi_*$. It is a standard fact of linear algebra that this choice is equivalent to choosing a linear mapping $\omega_x : T_xE \to \mathfrak{g}$ such that $\omega_x \circ v_x = id_{\mathfrak{g}}$ and $\text{Ker}(\omega_x) = H_x$. Furthermore, we want the map $\omega_x$ to vary smoothly, that is, we want the family $\{\omega_x : x \in E\}$ to define a differential form $\omega \in \Omega^1(E, \mathfrak{g})$.

**4.1.3 Definition.** Let $(E, \pi, M)$ be a principal $G$-bundle. For $g \in G$, consider the map $R_g : E \to E, x \mapsto xg$. A *connection* on $E$ is a differential 1-form $\omega \in \Omega^1(E, \mathfrak{g})$ such that

1. for all $x \in E$, $\omega_x \circ v_x = id$, where $v_x : \mathfrak{g} \to T_xE$ is the differential of mapping $g \mapsto xg$;
2. $\omega$ is *Ad-equivariant*, i.e, $R_g^*\omega = \text{Ad}(g^{-1}) \circ \omega$ for all $g \in G$.

Clearly $H_x = \text{Ker}(\omega_x)$ is mapped isomorphically onto $T_{\pi(x)}M$ by $\pi_*$. The subspace $H_x$ is called the *horizontal subspace at $x$ given by $\omega$*, and a vector $X \in H_x$ is said to be a *horizontal tangent vector at $x$* in $E$. Also, condition 2 is equivalent to $(R_g)_*(H_x) = H_{xg}$ for all $x \in E, g \in G$. That is, the horizontal subspaces are permuted by the natural right action of $G$ on $TE$ (induced by the $G$-action on $E$).

**4.1.4 Definition.** Let $(E, \pi, M)$ be a principal $G$-bundle.

1. A differential $k$-form $\omega \in \Omega^k(E, V)$ is called *horizontal* if $\omega_x(X_1, ..., X_k) = 0$ for all $k$-tuples of tangent vectors $X_1, ..., X_k \in T_xE$ such that at least one $X_i$ is vertical.

2. Let $\rho : G \to \text{GL}(V)$ be a Lie group representation (i.e., a smooth group homomorphism) of $G$ on $V$. Then $\omega \in \Omega^k(E, V)$ is said to be *$\rho$-equivariant* if $R_g^*\omega = \rho(g^{-1}) \circ \omega$ for all $g \in G$.

3. If $\omega$ is horizontal and $\rho$-equivariant, then it is called a *tensorial form of type $\rho$*.

**4.1.5 Remark.** Note that the notion of vertical vector does not depend on the connection; all connections have the same behaviour on vertical vectors. Therefore, given two connections $A, B$ on $E$, their difference $A - B$ is a 1-form $A - B \in \Omega^1(E, \mathfrak{g})$ which is horizontal (so it is no longer a connection). Note that it is also Ad-equivariant.

**4.1.6 Example.**

1. Consider a product bundle $(M \times G, \pi, M)$ and denote an element of $M \times G$ by $(p, g)$, where $p \in M$ and $g \in G$. Let $\pi_2 : M \times G \to G$ be the projection onto the second factor. We verify that $\omega_{MC}$, defined by

$$(\omega_{MC})_{(p,g)}(X, Y) = (L_{g^{-1}} \circ \pi_2)_*(p, g)(X, Y)$$

   is a connection on $(M \times G, \pi, M)$. First, we show that is a differential form $\omega_{MC} \in \Omega^1(E, \mathfrak{g})$. Note that $\omega_{MC} = \pi_2^*\omega_0$, where $\omega_0$ is the 1-form on $G$ defined by $(\omega_0)_g = (L_{g^{-1}})_*(g)$, hence it is sufficient to show that $\omega_0$ is differentiable. This is equivalent to showing that for all vector fields $X$ on $G$, the function $\omega_0(X)$ is differentiable (see [13], Proposition 17.9). First, if $X$ is a left-invariant vector field on $G$, then $X = \tilde{v}$ for some $v \in T_eG$, hence $\omega_0(X) = v$ is constant, hence

differentiable. For a generic vector field $Y$, the result follows from Remark 3.1.4.1.

Next, we verify that $\omega_{MC}$ is a connection. Writing $x = (p, g)$, the map $v_x$ is the differential at the identity of the map $G \to M \times G, a \mapsto (p, ga)$, ie, it is the map

$$\begin{aligned} v_x \colon \mathfrak{g} &\to T_x(M \times G) \cong T_pM \times T_gG \\ X &\mapsto (0, (L_g)_*(e)(X)). \end{aligned}$$

We verify conditions 1 and 2 of Definition 4.1.3. First, we compute

$$\begin{aligned} (\omega_{MC})_x \circ v_x(X) &= (\omega_{MC})_x(0, (L_g)_*(e)(X)) \\ &= (L_{g^{-1}} \circ \pi_2)_*(x)(0, (L_g)_*(e)(X)) \\ &= (L_{g^{-1}})_*(\pi_2(x))((\pi_2)_*(x)(0, (L_g)_*(e)(X))) \\ &= (L_{g^{-1}})_*(g)((L_g)_*(e)(X)) = id_*(X) = X \end{aligned}$$

and so condition 1 is satisfied. For condition 2, since $\omega_{MC} = \pi_2^*\omega_0$, it is sufficient to verify condition 2 for $\omega = \omega_0$. Here, we have, for $a \in G$,

$$\begin{aligned} (R_g^*\omega)_a &= \omega_{ag} \circ (R_g)_* \\ &= (L_{ag^{-1}})_* \circ (R_g)_* \\ &= (L_{g^{-1}})_* \circ (L_{a^{-1}})_* \circ (R_g)_* \\ &= \mathsf{Ad}(g^{-1}) \circ (L_{a^{-1}})_* \\ &= \mathsf{Ad}(g^{-1}) \circ \omega_a, \end{aligned}$$

as we wanted. The form $\omega_{MC}$ is called the *Maurer-Cartan connection*. Note that $\omega_0$ is the Maurer-Cartan connection when $M$ is just a point. Furthermore, denoting an element of $T_x(M \times G)$ by $(X, Y)$, we have

$$\omega_{MC}(x)(X, Y) = 0 \Leftrightarrow (L_{g^{-1}})_*(g)(Y) = 0 \Leftrightarrow Y = 0,$$

since $(L_{g^{-1}})_*(g)$ is an isomorphism. So the horizontal tangent vectors at $x$ are the elements of $T_pM \times 0$.

2. Let $\omega$ be any connection on a trivial principal $G$-bundle $E = M \times G$. We already have the maps $v_x$ from the previous example:

$$v_x \colon \mathfrak{g} \to T_x(M \times G) \cong T_pM \times T_gG$$
$$X \mapsto (0, (L_g)_*(e)(X)),$$

where we write $x = (p, g)$. Therefore, the vertical vectors tangent to $x$ are the elements of $0 \times T_gG$, regardless of the connection. There are many complements of $0 \times T_gG$, but, since we are considering a product bundle, we have the canonical choice of $T_pM \times 0$, which is precisely the horizontal subspace at $x$ provided by the Maurer-Cartan connection $\omega_{MC}$. For this reason, it is also called the trivial connection. As a particular case of Remark 4.1.5, if we take another connection $\omega$ on $M \times G$, the difference $\omega - \omega_{MC}$ is a horizontal $\mathfrak{g}$-valued 1-form on $E$. Since we are considering a trivial bundle, this form can be identified with a $\mathfrak{g}$-valued 1-form on $M$. Putting it differently, one can write $\omega = \omega_{MC} + A$ for a unique horizontal $\mathfrak{g}$-valued 1-form $A$ on $E$. Note that $A$ is also Ad-equivariant. See ([3], Proposition 6.8) for a more detailed exposition.

3. Consider the case $G = \mathrm{GL}(k, \mathbb{R})$. The Lie algebra is $M(n, \mathbb{R})$, the set of $n \times n$ real matrices with Lie bracket $[A, B] = AB - BA$. Here, the adjoint representation is $\mathrm{Ad}(g)(A) = gAg^{-1}$. Given a principal $G$-bundle $(E, \pi, M)$, a connection on $E$ is then a form $\omega \in \Omega^1(E, M(n, \mathbb{R}))$ satisfying Definition 4.1.3.

4. (continuation of Example 3.2.3.4) Denote the principal $\mathrm{GL}(k, \mathbb{R})$-bundle $(F(k, n), \pi, G(k, n))$ by $E$ and the set of $n \times k$ real matrices by $\mathbb{R}^{n \times k}$. We check that the 1-form

$$\omega = (X^T X)^{-1} X^T dX$$

is a connection on $E$. Here, the form $dX \in \Omega^1(E, M(n, \mathbb{R}))$ is the matrix of standard 1-forms $dX = (dx_{ij})_{i,j}$. In this case, $dx_{ij} \in \Omega^1(E, \mathbb{R})$ is defined by $dx_{ij,A}(B) = B_{ij}$ for all $A \in F(k, n), B \in \mathbb{R}^{n \times k}$. Clearly $\omega$ is a smooth 1-form, so we only need to check conditions 1 and 2 of Definition 4.1.3.

1. Let $A \in F(k,n)$. We consider the map $v_A$ defined using the notation at the beginning of the section, ie, $v_A$ is the differential at the identity matrix of the map $L_A\colon \mathsf{GL}(k,\mathbb{R}) \to F(k,n)$ which maps $g$ to $Ag$. But $L_A$ is linear, hence $v_A$ is the map $v_A\colon M(k,\mathbb{R}) \to \mathbb{R}^{n\times k}, g \mapsto Ag$. Let $B \in \mathbb{R}^{n\times k}$. We have

$$\omega_A \circ v_A(B) = (A^TA)^{-1}A^TAB = B,$$

hence condition 1 is verified.

2. Given $g \in F(k,n)$, the map $R_g$ is the linear map $F(k,n) \to F(k,n), A \mapsto Ag$, hence its derivative at any point is the map $\mathbb{R}^{n\times k} \to \mathbb{R}^{n\times k}, A \mapsto Ag$. Given $A \in F(k,n), B \in \mathbb{R}^{n\times k}$, we have

$$\begin{aligned}
(R_g^*\omega)_A(B) &= \omega_{Ag}(R_{g*,A}(B)) = \omega_{Ag}(Bg)\\
&= ((Ag)^TAg)^{-1}(Ag)^T dX_{Ag}(Bg)\\
&= ((Ag)^TAg)^{-1}(Ag)^TBg\\
&= (g^TA^TAg)^{-1}(Ag)^TBg\\
&= g^{-1}(A^TA)^{-1}(g^T)^{-1}(Ag)^TBg\\
&= g^{-1}(A^TA)^{-1}(g^{-1})^T(Ag)^TBg\\
&= g^{-1}(A^TA)^{-1}(Agg^{-1})^TBg\\
&= g^{-1}(A^TA)^{-1}A^TBg\\
&= \mathsf{Ad}(g^{-1})(\omega_A(B)),
\end{aligned}$$

as we wanted. Hence $\omega$ is a connection on $E$.

The next result requires just a direct computation.

### 4.1.7 Proposition.

1. Let $(\overline{\boldsymbol{f}}, f)\colon (E,\pi,M) \to (F,\xi,M')$ be a bundle map of principal $G$-bundles and let $\omega \in \Omega^1(F,\mathfrak{g})$ be a connection on $F$. Then $\overline{\boldsymbol{f}}^*\omega$ is a connection on $E$.
2. Suppose $\omega_1,...,\omega_k$ are connections on $E$ and $\lambda_1,...,\lambda_k \in C^\infty(M)$ satisfy $\sum_i \lambda_i = 1$. Then $\omega = \sum_i \lambda_i\omega_i$ is a connection on $E$.

**4.1.8 Corollary.** Every principal $G$-bundle can be endowed with a connection.

**Proof.** Let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of $M$ with trivializations $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$. By Proposition 4.1.7, the Maurer-Cartan connections on each $U_\alpha \times G$ can be pulled back to connections $\omega_\alpha$ on $\pi^{-1}(U_\alpha)$ by $\varphi_\alpha$. Choose a partition of unity $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ adapted to the covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and define $\omega = \sum_\alpha \lambda_\alpha\omega_\alpha$. Then $\omega$ is a 1-form on $E$ which, by the same Proposition, satisfies the requirements for a connection. □

## 4.2 Tensorial Forms on a Principal Bundle

The next result is the $k = 1$ version of Theorem 3.6.1. It holds when $k$ is any non-negative integer; it is proved in all its generality in [14].

**4.2.1 Theorem.** Let $V$ be a vector space and $\rho\colon G \to \mathrm{Aut}(V)$ be a smooth representation. There is a 1-1 correspondence between

1. the set of $\rho$-equivariant and horizontal 1-forms $\omega \in \Omega^1(E, V)$;
2. the set of $\Omega^1(M, E \times_\rho V)$ of 1-forms on $M$ taking values in the associated bundle $E \times_\rho V$.

**Proof.** (from [14], Section 31.2). Let $p \in M$ and $x \in E_p$. It is straightforward to check that

$$\begin{aligned} f_x\colon V &\to (E \times_\rho V)_p \\ v &\mapsto [x, v] \end{aligned}$$

is a vector space isomorphism which satisfies $f_{xg} = f_x \circ \rho(g)$. We now define our correspondence. Consider a $\rho$-equivariant and horizontal 1-form $\omega \in \Omega^1(E, V)$ and define a 1-form $\omega^\flat \in \Omega^1(M, E \times_\rho V)$ by

$$\omega^\flat_p(v) = f_x(\omega_x(u)) \in (E \times_\rho V)_p$$

where $x \in E_p$ and $\pi_*(u) = v$. On the other hand, given $\theta \in \Omega^1(M, E \times_\rho V)$, define a 1-form $\omega \in \Omega^1(E, V)$ by

$$\theta^\sharp_x(u) = f_x^{-1}(\theta_{\pi(x)}(\pi_*(u))) \in V.$$

It is easy to check that $\omega^\flat$ and $\theta^\sharp$ are well-defined and that these correspondences are inverses of each other. □

### 4.2.2 Remark.

1. Note that a smooth local section of $E$ provides a smooth choice of $x$ and $u$ for the definition of $\omega^\flat$.

2. A tensorial form of type Ad induces a form on the bundle $E \times_{\mathrm{Ad}} \mathfrak{g}$, called the *adjoint bundle* of $E$. In particular, this applies to the difference of two connections (see Remark 4.1.5).

## 4.3 Reductions of Connections

Here we make some auxiliary notes regarding the relation between reductions and connections, based on ([3], page 53).

**4.3.1 Definition.** Let $(E, \pi, M)$ be a principal $G$-bundle, $(F, \xi, M)$ a principal $H$-bundle, $\alpha\colon H \to G$ be a Lie group homomorphism and $\varphi\colon F \to E$ a reduction of $E$ to $H$. Let $\omega_F$ be a connection on $F$ and $\omega_E$ a connection on $E$. Then $\omega_F$ is said to be a *reduction of* $\omega_E$ *relative to* $\alpha$ if $\alpha_* \circ \omega_E = \omega_F$.

**4.3.2 Proposition.** With the notation of Definition 4.3.1, the horizontal subspaces in $F$ are mapped isomorphically into the horizontal subspaces of $E$ by $\varphi_*$.

**4.3.3 Corollary.** Suppose $H$ is a closed embedded subgroup of $G$ and $\alpha$ is the inclusion. Then the horizontal subspaces defined by the connection $\omega_F$ are exactly the horizontal subspaces defined by $\omega_E$.

## 4.4 The Covariant Derivative

Finally, we note a key application of connections. We follow ([14], Sections 31.4 and 31.5). Let $E$ be a principal $G$-bundle over $M$ and $\alpha$ be a $k$-form on $E$ with values on a vector space $V$. A connection $\omega$ on $E$ provides a decomposition of tangent vectors:

if $x \in E$ and $X \in T_x E$ one can write

$$X = X^h + X^v,$$

where $X^h$ and $X^v$ are the horizontal and vertical components of $X$, respectively (the horizontal component $x^h$ being determined by the connection $\omega$).

In general, $d\alpha$, the exterior derivative of $\alpha$, is not horizontal, even if $\alpha$ is horizontal. The connection $\omega$ provides a way to obtain a horizontal form from $\alpha$.

**4.4.1 Definition.** Let $E$ be a principal $G$-bundle over $M$ with connection $\omega$ and $\alpha$ be a $k$-form on $E$ with values on a vector space $V$.

1. The *horizontal component* of $\alpha$ is the $k$-form $\alpha^h \in \Omega^k(E, V)$ defined by

$$\alpha^h(x)(v_1, ..., v_k) = \alpha(x)(v_1^h, ..., v_k^h),$$

   where the horizontal components of the vectors $v_i$ are determined by the connection $\omega$.

2. The *covariant derivative* of $\alpha$ with respect to $\omega$ is the $(k+1)$-form $d_\omega \alpha \in \Omega^{k+1}(E, V)$ defined by $d_\omega \alpha = (d\alpha)^h$.

**4.4.2 Proposition.** Let $\rho$ be a representation of $G$ on $V$ and $\alpha$ be a $k$-form on $E$ with values on a vector space $V$. If $\alpha$ is $\rho$-equivariant, then so are $\alpha_h$ and $d\alpha$. Moreover, the covariant derivative $d_\omega \alpha$ is horizontal and $\rho$-equivariant, hence it defines a k-form on $M$ with values in $E \times_\rho V$.

Denoting the set of horizontal and $\rho$-equivariant $k$-forms on $E$ with values in $V$ by $\Omega^k(E, V)^\rho_h$, the covariant derivative then defines a linear map

$$d_\omega \colon \Omega^k(E, V)^h_\rho \to \Omega^{k+1}(E, V)^h_\rho,$$

which, using Theorem 4.2.1, induces another linear map

$$d_\omega \colon \Omega^k(M, E \times_\rho V) \to \Omega^{k+1}(M, E \times_\rho V).$$

That is, we define the covariant derivative of $\theta \in \Omega^k(M, E \times_\rho V)$ in order to make the following diagram commutative:

$$\begin{array}{ccc} \Omega^k(E,V)^h_\rho & \xrightarrow{d_\omega} & \Omega^{k+1}(E,V)^h_\rho \\ \sharp \uparrow & & \downarrow \flat \\ \Omega^k(M, E \times_\rho V) & \xrightarrow{d_\omega} & \Omega^{k+1}(M, E \times_\rho V) \end{array},$$

where the vertical arrows are determined by Theorem 4.2.1. In particular, it is now possible to *differentiate sections* $\sigma \in \Omega^0(E,V)^h_\rho$, this possibility being one the main motivations for connections.

# 5 Curvature on Principal Bundles

## 5.1 The Curvature Form of a Connection

Let $G$ be a Lie group and $\mathfrak{g} = T_eG$ be its Lie algebra. Let $(E, \pi, M)$ be a principal $G$-bundle with connection $\omega \in \Omega^1(E, \mathfrak{g})$. Define a 2-form $[\omega, \omega] \in \Omega^2(E, \mathfrak{g})$ by

$$[\omega, \omega]_x(X, Y) = [\omega_x(X), \omega_x(Y)] - [\omega_x(Y), \omega_x(X)] = 2[\omega_x(X), \omega_x(Y)],$$

where we are taking the Lie brackets of the Lie algebra $\mathfrak{g}$ of $G$.

**5.1.1 Definition.** The *curvature form* $F_\omega \in \Omega^2(E, \mathfrak{g})$ of the connection $\omega$ is defined by the equation

$$d\omega = -\frac{1}{2}[\omega, \omega] + F_\omega.$$

The connection $\omega$ is called *flat* if $F_\omega \equiv 0$. In this case, the principal bundle $E$ is said to be *flat*.

It turns out that flatness is strongly related to transition functions.

**5.1.2 Theorem.** Let $(E, \pi, M)$ be a principal $G$-bundle with connection $\omega$. Then the following conditions are equivalent:

1. $\omega$ is flat;
2. there is an open covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ such that $\omega|_{E_{|U_\alpha}} = \varphi_\alpha^*(\omega_{MC})$.

Moreover, one can choose the covering $\{U_\alpha\}_{\alpha\in\Sigma}$ in a way such that all intersections $U_{\alpha\beta}$ are connected, and the transition functions associated to the trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ are constant.

For a proof, see ([3], Theorem 7.5 and Corollary 7.6).

**5.1.3 Theorem.** For a connection $\omega$ on a principal $G$-bundle $E$, the curvature form $F_\omega$ is tensorial of type Ad, hence it defines a 2-form (also denoted by $F_\omega$) on $E$ with values in the adjoint bundle $E \times_{\mathrm{Ad}} \mathfrak{g}$.

For a proof, see ([3], Theorem 7.2 (2)) or ([14], Theorem 30.4).

**5.1.4 Example.**

1. Let $\omega$ be the Maurer-Cartan connection defined on Example 4.1.6.1. We verify that $\omega$ is a flat connection. Using the notation of the Example, we have $\omega = \pi_2^*\omega_0$, where $\omega_0$ is the Maurer-Cartan connection on the principal $G$-bundle $(G, \pi, pt)$. Therefore, $d\omega = d(\pi_2^*\omega_0) = \pi_2^*(d\omega_0)$. Furthermore, given $x \in E$ and $X, Y \in T_xE$,

$$\begin{aligned}
[\omega, \omega]_x(X, Y) &= 2[\omega_x(X), \omega_x(Y)] \\
&= 2[\pi_2^*\omega_{0,x}(X), \pi_2^*\omega_{0,x}(Y)] \\
&= 2[\omega_{0,\pi_2(x)}(\pi_{2,*}(x)(X), \omega_{0,\pi_2(x)}(\pi_{2,*}(x)(Y))] \\
&= [\omega_0, \omega_0]_{\pi_2(x)}(\pi_{2,*}(x)(X), \pi_{2,*}(x)(Y)) \\
&= (\pi_2^*[\omega_0, \omega_0])_x(X, Y),
\end{aligned}$$

   hence $F_\omega = \pi_2^* F_{\omega_0}$. In order to prove that $\omega$ is flat, it is then sufficient to prove that $F_{\omega_0} = 0$, ie, that $\omega_0$ is a flat connection on the bundle $(G, \pi, pt)$. In this case, given $g \in G$, the map $v_g$ is an isomorphism $\mathfrak{g} \to T_gG$, hence all tangent vectors are vertical. Let $g \in G$ and let $X, Y \in T_gG$. Then $X = A_g^*$ and $Y = B_g^*$ for some $A, B \in \mathfrak{g}$. We have $d\omega_0(X, Y) = (d\omega_0(A^*, B^*))(g)$. Now

$$\begin{aligned}
d\omega_0(A^*, B^*) &= A^*(\omega_0(B^*)) - B^*(\omega_0(A^*)) - \omega_0([A^*, B^*]) \\
&= A^*(B) - B^*(A) - \omega_0([A^*, B^*]),
\end{aligned}$$

   noting that, since $\omega_0$ is a connection, we have $\omega_0(A^*) = A$ and $\omega_0(B^*) = B$. Using Remark 4.1.2, we obtain

$$\begin{aligned}
d\omega_0(A^*, B^*) &= A^*(B) - B^*(A) - \omega_0([A, B]^*) \\
&= -[A, B]
\end{aligned}$$

and so $d\omega_0(X,Y) = -[A,B]$. Next, we have

$$\begin{aligned}[\omega_0,\omega_0](X,Y) &= [\omega_0,\omega_0](A_g^*, B_g^*) \\ &= 2[\omega_{0,g}(A_g^*), \omega_{0,g}(B_g^*)] \\ &= 2[A,B].\end{aligned}$$

Hence $d\omega_0 = -\frac{1}{2}[\omega_0,\omega_0]$ and we conclude that $\omega_0$ is flat.

2. (continuation of Example 3.2.3.4) We return to the principal $\mathrm{GL}(k,\mathbb{R})$-bundle $(F(k,n),\pi,G(k,n))$, with the connection

$$\omega = (X^TX)^{-1}X^TdX.$$

The curvature form is given by

$$F_\omega = (X^TX)^{-1}dX^T \wedge dX - (X^TX)^{-1}dX^T \wedge X(X^TX)^{-1}X^TdX.$$

In general, $\omega$ is not a flat connection. However, if $k=1$ (and so we have the $\mathbb{R}^*$-bundle $(\mathbb{R}^n \setminus \{0\}, \pi, \mathbb{R}P^{n-1})$), $\omega$ is flat. In this case, $\omega$ is given by

$$\omega = \frac{x^T}{|x|^2}dx.$$

and

$$F_\omega = \frac{1}{|x|^2}dx \wedge dx - \frac{1}{|x|^2}dx \wedge \frac{|x|^2}{|x|^2}dx = 0,$$

hence the connection $\omega$ is flat for $k=1$. However, the transition functions (see Example 3.3.9) are not constant. However, this is not a contradiction of Theorem 5.1.2; we just learn that there is an open covering with trivializations whose transition functions are indeed constant.

3. Consider the principal $\pi_1 M$-bundle $(\tilde{M}, p, M)$ defined in Example 3.2.3.5. Let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of $M$ with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$. Denote its transition functions by $g_{\alpha\beta}\colon U_{\alpha\beta} \to \pi_1 M$. These are continuous functions that take values in a discrete space, hence they are locally constant. One can then

restrict the trivializations in such a way that the corresponding transition functions are constant. Hence $(\tilde{M}, p, M)$ is a flat $\pi_1 M$-bundle.

Assume now that the transition functions $g_{\alpha\beta}$ are constant, and consider the bundle associated to $(\tilde{M}, p, M)$ with fibre $G$, $(\tilde{M}_G, p_G, M) = \tilde{M} \times_\rho G$. We compute its transition functions:

$$U_{\alpha\beta} \times G \xrightarrow{f_\alpha^{-1}} p_g^{-1}(U_{\alpha\beta}) \xrightarrow{f_\beta} U_{\alpha\beta} \times G$$

$$\begin{aligned}(x, g) \mapsto [(\varphi_\alpha^{-1}(x, 1), g)] &\mapsto (p(\varphi_\alpha^{-1}(x, 1)), (p_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1})(x, 1) \cdot g)) \\ &= (x, g_{\alpha\beta}(x) \cdot g) \\ &= (x, (\rho \circ g_{\alpha\beta})(x) g).\end{aligned}$$

The transition functions of $\tilde{M} \times_\rho G$ are then $\rho \circ g_{\alpha\beta} \colon U_{\alpha\beta} \to G$, which are also constant. Hence $\tilde{M} \times_\rho G$ is also a flat bundle.

**5.1.5 Remark.** Let $\alpha \in \Omega^1(E, \mathfrak{g})$ be tensorial of type Ad. Suppose $E$ has a connection $\omega$ (not necessarily flat). Recall Definition 4.4.1. Then one can establish an equation similar to that of Definition 5.1.1 (see [14], Theorem 31.19):

$$d_\omega \alpha = d\alpha + [\omega, \alpha].$$

In particular, the form $d\alpha + [\omega, \alpha]$ is also tensorial of type Ad.

## 5.2 The Holonomy Representation and Parallel Transport

In the previous section, we established a correspondence from representations $\rho \colon \pi_1 M \to G$ (with $M$ connected) to flat principal $G$-bundles over $M$: given a representation $\rho$, we have the flat $G$-bundle $\tilde{M} \times_\rho G$. Now we establish a correspondence in the opposite direction. We follow ([8], Section 4). Let $(E, \pi, M)$ be a principal $G$-bundle over a connected manifold $M$ with connection $\omega$.

**5.2.1 Defintion.** A piecewise differentiable path $\alpha\colon [0,1] \to E$ is said to be *horizontal with respect to* $\omega$ if, for all $t \in [0,1]$, the tangent vectors $\alpha'(t) \in T_{\gamma(t)}E$ are horizontal.

**5.2.2 Lemma.** Let $\gamma\colon [0,1] \to M$ be a piecewise differentiable path and let $x \in E$ be such that $\pi(x) = \gamma(0)$. Then there is a unique horizontal path $\tilde{\gamma}_x\colon [0,1] \to E$ with respect to $\omega$ such that $\pi \circ \tilde{\gamma}_x = \gamma$. The path $\tilde{\gamma}_x$ is called the *horizontal lift* of $\gamma$ starting at $x$.

**Proof.** Since we can cover the image of $\gamma$ by open sets $U_\alpha$ over which $E$ is trivial, we can, without loss of generality, assume that the image of $\gamma$ is contained in an open set $U$ with a trivialization $\varphi\colon \pi^{-1}(U) \to U \times G$.

Every lift of $\gamma$ to $\pi^{-1}(U)$ is of the form

$$\begin{aligned} \tilde{\gamma}_x\colon [0,1] &\to \pi^{-1}(U) \\ t &\mapsto \varphi^{-1}(\gamma(t), g(t)), \end{aligned}$$

for some differentiable map $g\colon [0,1] \to G$. Since we want the starting point to be $x$, $g(0)$ is determined by the equation $\tilde{\gamma}_x(0) = x$.

We want $\tilde{\gamma}_x$ to be horizontal, that is, $\omega_{\tilde{\gamma}_x(t)}(\tilde{\gamma}_x'(t)) = 0$. We can consider the pullback connection $A = (\varphi^{-1})^*\omega$ on $U \times G$, making the horizontality condition equivalent to $A_{(\gamma(t),g(t))}((\gamma'(t), g'(t)) = 0$. Now $A$ can be written uniquely as $A = \omega_{MC} + \tilde{A}$, where $\omega_{MC}$ is the Maurer-Cartan connection on $U \times G$ and $A$ is a $\mathfrak{g}$-valued 1-form on $U \times G$ which is zero on vertical vectors (see Example 4.1.6.2). Manipulating this equation, we obtain

$$A_{(\gamma(t),g(t))}(\gamma'(t), g'(t)) = 0 \Leftrightarrow g'(t) = -[(L_{g(t)^{-1}})_*(g(t))]^{-1}(\tilde{A}_{(\gamma(t),g(t))}(\gamma'(t), 0)).$$

This is a first order differential equation for $g$ (of the form $g'(t) = f(t)$ where $f$ is smooth). Since we have the initial condition already noted, there is a unique solution. □

### 5.2.3 Definition.

1. The *parallel transport* of a piecewise differentiable path $\gamma\colon [0,1] \to M$ with respect to $\omega$ is the map

$$\begin{aligned} E^{\omega}_{\gamma}\colon E_{\gamma(0)} &\to E_{\gamma(1)} \\ x &\mapsto \tilde{\gamma}_x(1), \end{aligned}$$

   where $\tilde{\gamma}_x\colon [0,1] \to E$ is the unique horizontal lift of $\gamma$ starting at $x$.

2. Let $\gamma\colon [0,1] \to M$ be a loop at $p \in M$ and $x \in \pi^{-1}(p)$. The holonomy of $\gamma$ at $x$ with respect to $\omega$ is the element $\mathsf{Hol}_x(\omega,\gamma) \in G$ such that

$$x \cdot \mathsf{Hol}_x = E^{\omega}_{\gamma}(x).$$

3. The image of the induced map

$$\begin{aligned} \mathsf{Hol}_x(\omega,-)\colon \{\gamma\colon [0,1] \to M\colon \gamma \text{ is a loop at } x\} &\to G \\ \gamma &\mapsto \mathsf{Hol}_x(\omega,\gamma). \end{aligned}$$

   is called the *holonomy group* of $\omega$ at $x$. We'll denote it by $H_x(\omega)$.

**5.2.4 Remark.** Let $p,q,r$ be points in $M$, $\gamma$ be a path in $M$ from $p$ to $q$, $\gamma'$ a path in $M$ from $q$ to $r$ and $x \in \pi^{-1}(p)$. Then

$$\widetilde{(\gamma \cdot \gamma')}_x = \tilde{\gamma}_x \cdot \widetilde{\gamma'}_{E^{\omega}_{\gamma}(x)}.$$

Indeed, $\tilde{\gamma}_x \cdot \widetilde{\gamma'}_{E^{\omega}_{\gamma}(x)}$ is a horizontal path and is a lift of $\gamma \cdot \gamma'$ starting at $x$. The equality then follows from uniqueness. We also obtain a property of parallel transport:

$$E^{\omega}_{\gamma\cdot\gamma'}(x) = E^{\omega}_{\gamma'}(E^{\omega}_{\gamma}(x)).$$

If $\gamma' = \gamma^{-1}$, it follows from the same argument that

$$\widetilde{(\gamma \cdot \gamma')}_x = \tilde{\gamma}_x \cdot (\tilde{\gamma}_x)^{-1},$$

hence $(\tilde{\gamma}_x)^{-1} = \widetilde{\gamma^{-1}}_{E^\omega_\gamma(x)}$ and $E^\omega_{\gamma\cdot\gamma^{-1}}(x) = x$. We now note some more properties of the holonomy map and parallel transport.

**5.2.5 Lemma.** Let $\gamma, \gamma' \colon [0,1] \to M$ be loops at $p \in M$ and $\delta \colon [0,1] \to M$ a path from $p$ to $q \in M$. Let $\alpha \colon [0,1] \to E$ be horizontal. Then, for all $x \in \pi^{-1}(p)$ and $g \in G$, the following statements hold:

1. $R_g \circ \alpha \colon [0,1] \to E$ is also horizontal;
2. $(E^\omega_\gamma \circ R_g)(x) = (R_g \circ E^\omega_\gamma)(x)$;
3. $\mathsf{Hol}_{xg}(\omega, \gamma) = g^{-1}\mathsf{Hol}_x(\omega, \gamma)g$;
4. $\mathsf{Hol}_x(\omega, \gamma \cdot \gamma') = \mathsf{Hol}_x(\omega, \gamma) \cdot \mathsf{Hol}_x(\omega, \gamma')$;
5. $\mathsf{Hol}_x(\omega, \gamma^{-1}) = \mathsf{Hol}_x(\omega, \gamma)^{-1}$;
6. $\mathsf{Hol}_x(\omega, \gamma) = \mathsf{Hol}_{E^\omega_\delta(x)}(\omega, \delta \cdot \gamma \cdot \delta^{-1})$.

(Putting it differently, item 3. says that if $x$ and $y$ are in the same fiber, then the holonomy maps associated to $x$ and $y$ are conjugate; we will return to this notion at the end of the section, after defining holonomy representations).

**Proof.**

1. We have

$$\begin{aligned}\omega((R_g \circ \alpha)(t))((R_g \circ \alpha)'(t)) &= R_g^*\omega(\alpha(t))(\alpha'(t)) \\ &= \mathsf{Ad}(g^{-1})(\omega(\alpha(t))(\alpha'(t)) \\ &= 0\end{aligned}$$

   since $\alpha$ is horizontal. Hence $R_g \circ \alpha$ is also horizontal.

2. We want to prove that $\tilde{\gamma}_{xg} = \tilde{\gamma}_x \cdot g$. Clearly $(\tilde{\gamma}_x \cdot g)(0) = xg$ and the curve $\tilde{\gamma}_x \cdot g$ is horizontal by item 1. Since $x$ and $xg$ are in the same fiber, given $t \in [0,1]$, $\tilde{\gamma}_{xg}(t)$ and $(\tilde{\gamma}_x \cdot g)(t)$ are in the same fiber, hence $\tilde{\gamma}_x \cdot g$ is a horizontal lift of $\gamma$ starting at $xg$. By uniqueness, $\tilde{\gamma}_{xg} = \tilde{\gamma}_x \cdot g$.

3. By definition, $xg \cdot \mathsf{Hol}_{xg}(\omega, \gamma) = E^{\omega}_{\gamma}(xg) = E^{\omega}_{\gamma}(x)g$ by item 2 and $E^{\omega}_{\gamma}(x) = x \cdot \mathsf{Hol}_x(\omega, \gamma)$. This item now follows from a simple computation, recalling that the $G$-action on $E$ is free.

4. We have

$$\begin{aligned} E^{\omega}_{\gamma\cdot\gamma'}(x) = E^{\omega}_{\gamma'}(E^{\omega}_{\gamma}(x)) &\Leftrightarrow x \cdot \mathsf{Hol}_x(\omega, \gamma\cdot\gamma') = E^{\omega}_{\gamma'}(x \cdot \mathsf{Hol}_x(\omega, \gamma)) \\ &\Leftrightarrow x \cdot \mathsf{Hol}_x(\omega, \gamma\cdot\gamma') = E^{\omega}_{\gamma'}(x) \cdot \mathsf{Hol}_x(\omega, \gamma) \\ &\Leftrightarrow x \cdot \mathsf{Hol}_x(\omega, \gamma\cdot\gamma') = x \cdot \mathsf{Hol}_x(\omega, \gamma') \cdot \mathsf{Hol}_x(\omega, \gamma). \end{aligned}$$

   Since the $G$-action on $E$ is free, we obtain $\mathsf{Hol}_x(\omega, \gamma\cdot\gamma') = \mathsf{Hol}_x(\omega, \gamma') \cdot \mathsf{Hol}_x(\omega, \gamma)$, as desired.

5. This follows from $E^{\omega}_{\gamma\cdot\gamma^{-1}}(x) = x$ and from item 4.

6. Applying the previous items, we obtain

$$\begin{aligned} \mathsf{Hol}_{E^{\omega}_{\delta}(x)}(\omega, \delta\cdot\gamma\cdot\delta^{-1}) &= \mathsf{Hol}_{x\cdot\mathsf{Hol}_x(\omega,\delta)}(\omega, \delta\cdot\gamma\cdot\delta^{-1}) \\ &= \mathsf{Hol}_x(\omega, \delta)^{-1} \cdot \mathsf{Hol}_x(\omega, \delta) \cdot \mathsf{Hol}_x(\omega, \gamma) \cdot \mathsf{Hol}_x(\omega, \delta^{-1}) \cdot \mathsf{Hol}_x(\omega, \delta) \\ &= \mathsf{Hol}_x(\omega, \gamma) \cdot \mathsf{Hol}_x(\omega, \delta)^{-1} \cdot \mathsf{Hol}_x(\omega, \delta) \\ &= \mathsf{Hol}_x(\omega, \gamma), \end{aligned}$$

   as we wanted.

□

**5.2.6 Theorem.** If $\omega$ is flat and $\gamma$ is homotopic to $\beta$, then they have the same holonomy, i.e., $\mathsf{Hol}_x(\omega, \gamma) = \mathsf{Hol}_x(\omega, \beta)$.

**Proof.** Let $H$ be a homotopy from $\gamma$ to $\beta$. For a fixed $s \in ]0, 1[$, let $\alpha_s(t)$ be the path $H(s, t)$. By compactness, it is sufficient to show that there is $\epsilon > 0$ such that $\mathsf{Hol}_x(\omega, \gamma) = \mathsf{Hol}_x(\omega, \alpha_s)$ for all $s < \epsilon$. Use Theorem 5.1.2 to obtain open sets $U_0, U_1, ..., U_n$ covering the trace of $\gamma$ such that

1. the intersections $U_{ij} = U_i \cap U_j$ are non-empty and connected;

2. for each $i$, there is a trivialization $\phi_i$ of $E$ over $U_i$ such that $(\phi_i^{-1})^*\omega = \omega_{MC}$ and the transition functions associated to the trivializations $\phi_i$ are constant.

We assume $n = 2$. Let $\delta > 0$ and $a, b \in ]0, 1[$ be such that $U_0 \cap \gamma = \gamma([0, a + \delta])$, $U_1 \cap \gamma = \gamma([a - \delta, b + \delta])$ and $U_2 \cap \gamma = \gamma([b - \delta, 1])$. For each $i$, let $\gamma_i$ be the obvious restriction. Let $\tilde{\gamma_i}$ be the (unique) horizontal lift of $\gamma_i$ to $U_i \times G$ with respect to $A_i = (\varphi_i^{-1})^*\omega$. In particular, for $\tilde{\gamma_0}$, we have

$$\begin{aligned} \tilde{\gamma_0} \colon [0, a + \delta] &\to U_0 \times G \\ t &\mapsto (\gamma(t), g(t)) \end{aligned}$$

for a unique differentiable map $g \colon [0, a + \delta] \to G$. Following the same notation as in Lemma 5.2.2, we now have $A_i = \omega_{MC}$, therefore $\tilde{A}_i = 0$ and so $g'(t) = 0$. So $g$ is determined by the initial condition; in this case, we want $g = g(0) = \phi_0(p)$. We then have $\tilde{\gamma_0}(a) = (\gamma(a), \phi_0(p))$. Next, write

$$\begin{aligned} \tilde{\gamma_1} \colon [a - \delta, b + \delta] &\to U_1 \times G \\ t &\mapsto (\gamma(t), h(t)) \end{aligned}$$

for a unique differentiable map $h \colon [0, a + \delta] \to G$. We again have $h'(t) = 0$, so $h$ is constant. By uniqueness of horizontal lifting, we must have $h = g_{10}(\gamma_0(a)) \cdot \phi_0(p)$, where $g_{10}$ is the constant transition function from $\phi_0$ to $\phi_1$. We can repeat this process for $\tilde{\gamma_2}$.

Next, let $\epsilon > 0$ be such that $\alpha_s([0, a + \delta]) \subset U_0, \alpha_s([a - \delta, b + \delta]) \subset U_1$ and $\alpha_s([b - \delta, 1]) \subset U_2$ for all $0 < s < \epsilon$. For each $i$, let $\alpha_s^i$ be the obvious restriction. Again, by flatness, the horizontal lift of $\alpha_s^0$ is $(\gamma(t), \phi_0(p))$. We can then repeat the process applied to $\gamma$ to conclude that $\gamma$ and $\alpha_s$ have the same holonomy.

An analogous argument now works for any natural number $n$. □

Since the holonomy of a constant path is clearly trivial, we obtain

**5.2.7 Corollary.** If $\gamma$ is contractible, then it has trivial holonomy.

Combining Lemma 5.2.5, Theorem 5.2.6 and Corollary 5.2.7, there is a well-defined homomorphism

$$\begin{aligned} \rho_{\omega,x} \colon \pi_1 M &\to G \\ [\gamma] &\mapsto \mathsf{Hol}_x(\omega, \gamma). \end{aligned}$$

Considering $\pi_1(M)$ as a discrete Lie group, the map $\rho_{\omega,x}$ is a Lie group representation. Note that item 3 of Lemma 5.2.5 implies that if $x$ and $y$ are in the same fiber, the holonomy representations associated to $x$ and $y$ are conjugate.

## 5.3 Fundamental Group Representations and Flatness

We continue assuming that $M$ is connected. We now have two correspondences:

$$\begin{gathered} \{\text{flat principal } G\text{-bundles over } M\} \leftrightarrow \{\text{representations } \pi_1 M \to G\} \\ (E, \omega) \mapsto \rho_{\omega,x} \\ \tilde{M} \times_\rho G \hookleftarrow \rho \end{gathered}$$

for $x \in E$. These correspondences are not inverses of each other (indeed, the first one requires a choice of $x \in E$), but, if we consider *gauge equivalences* of connections (two connections $\omega_1$ and $\omega_2$ are said to be gauge equivalent if there is an automorphism $\varphi$ of $E$ - a gauge equivalence - such that $\varphi^* \omega_1 = \omega_2$) and *conjugacy classes* of representations (see Corollary 5.2.7), we arrive at a very important result.

**5.3.1 Theorem.** ([8], Theorem 4.7). There is a bijection between the set of conjugacy classes of representations $\rho \colon \pi_1 M \to G$ and the set of gauge equivalence classes of flat connections on principal $G$-bundles over $M$.

**5.3.2 Remark.** The method used in [8] is different from the one used in Theorem 5.2.6, and more powerful and general results are mentioned there. Indeed, it is proved in ([11], Theorem 4.2, p.73) that the holonomy group $H_x(\omega)$ is a Lie subgroup of $G$; this result is a key component in the Reduction Theorem (see [11], Theorem 7.1,

p.83). Finally, a theorem by Ambrose and Singer (see [11], Theorem 8.1, page 89) relates holonomy and curvature: it produces generators of the Lie algebra of $H_x(\omega)$ using the curvature form. As a particular case, it follows that the Lie algebra of the holonomy group is trivial when $\omega$ is flat. Hence, in this case, $H_x(\omega)$ is discrete. This relation is also present in our proof, where flatness plays a key role.

# 6 Metrics on Principal Bundles

In this section, we apply the theory we've developed so far to introduce and study the energy integral. The goal is to be in condition to state the Donaldson-Corlette theorem. For this, we highlight the importance of harmonic maps (those that minimize the energy integral). Harmonic maps are discussed by Eells and Sampson in ([4], Section 11) and in ([9], Section 8). In the preface, it is noted the focus on "representations involving fundamental groups and Lie groups". We now know that these are the holonomy representations.

## 6.1 The Energy Integral

Let $f\colon (M,g) \to (N,h)$ be a differentiable map between Riemannian manifolds, with $\dim(M) = m$ and $\dim(N) = n$. Assume $M$ is oriented and compact. In local coordinates, the metric on $M$ can be written as a matrix $(g_{ij})_{i,j=1,\dots,m}$, and it defines a volume form $\mathrm{vol}_M$ on $M$, given locally by $\mathrm{vol}_M = \sqrt{\det((g_{ij}))}dx_1 \wedge \dots \wedge dx_m$.

We consider the derivative $df$ as the bundle map

$$\begin{aligned} df\colon TM &\to f^*TN \\ (p, v_p) &\mapsto (p, df(p)(v_p)). \end{aligned}$$

Since both bundles have the same base space $M$, Theorem 2.2.11 allows us to consider $df$ as a section of the vector bundle $\mathrm{Hom}(TM, f^*TN)$ over $M$. Recall that $f^*TN$ inherits a natural metric from the one on $TN$, given $(p,v_1),(p,v_2) \in f^*TN_{|p}$ (so for $v_1, v_2 \in T_{f(p)}N$) we define $\langle (p,v_1),(p,v_2)\rangle = h_p(v_1,v_2)$. Like in Example 2.5.7, we have commutative diagrams

$$\begin{array}{ccc} f^*TN_{|p} & \xrightarrow{df(p)^*} & T_pM \\ \downarrow{\scriptstyle j_p} & & \downarrow{\scriptstyle i_p} \\ (f^*TN_{|p})^* & \xrightarrow{df(p)^t} & T_p^*M \end{array}$$

with the natural isomorphisms $i_p$ and $j_p$ provided by the metrics on $TM$ and $f^*TN$, respectively. This construction defines a metric on $\mathrm{Hom}(TM, f^*TN)$. Considering $df$ as a section of this bundle, we have a new notion.

### 6.1.1 Definition.

1. With this notation, the *energy density* of $f$ is
$$e(f)(p) = \frac{1}{2}||df(p)||^2 = \frac{1}{2}tr(df(p)^* \circ df(p)).$$
2. The *energy* of $f$ is
$$E(f) = \int_M e(f)\mathrm{vol}_M.$$
3. The map $f$ is called *harmonic* if it minimizes the energy function (among all differentiable maps $M \to N$).

The existence of harmonic maps is proved in ([4], Section 11). Jost discusses harmonic maps and the energy integral in ([9], Section 8).

## 6.2 Metrics on Principal Bundles

Harmonic maps can also be studied in the context of principal bundles. Applying Definition 6.1.1 to a principal $G$-bundle $(E, \pi, M)$ and to some map defined on $E$, we would have to take an integral over $E$, which, in general, does not converge. It is then necessary to find an alternate definition. Just like in the previous section, we will later assume that $M$ is compact and oriented; we will then pursue a sensible way to obtain an integral over $M$. A fundamental property will also assumed be for the Lie group $G$ - that it is *reductive*. There are many definitions of a reductive Lie group; we follow the definition used by Knapp in [10].

Here are the properties of $G$ we will use: $G$ has a maximal compact subgroup $H$ (this is true if, for example, $G$ has finitely many connected components; see Theorem 3.1.13) with Lie algebra $\mathfrak{h}$ (which is a Lie subalgebra of $\mathfrak{g}$). There is an

Ad-invariant bilinear form $B$ on $\mathfrak{g}$ which extends the Killing form on $[\mathfrak{g},\mathfrak{g}]$ (the semisimple part of $\mathfrak{g}$). This is negative definite on $\mathfrak{h}$. We take the orthogonal complement $\mathfrak{m}$ of $\mathfrak{h}$ under $B$; here it actually defines an inner product. This subspace is not a Lie subalgebra of $\mathfrak{g}$ in general, but there is a diffeomorphism

$$\begin{aligned} f\colon \mathfrak{m} &\to G/H \\ X &\mapsto \pi_G(\exp(X)), \end{aligned}$$

which defines a $G$-invariant Riemannian metric on $G/H$ such that $f_*(0)$ is an isometry. Furthermore, we have

1. if $X, Y \in \mathfrak{h}$, then $[X,Y] \in \mathfrak{h}$;
2. if $X, Y \in \mathfrak{m}$, then $[X,Y] \in \mathfrak{h}$;
3. if $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$, then $[X,Y] \in \mathfrak{h}$;
4. $H$ acts on $\mathfrak{m}$ via the adjoint representation, i.e, $\mathrm{Ad}(h)$ restricts to an automorphism of $\mathfrak{m}$ for all $h \in H$.

We also make use of the diffeomorphism

$$\begin{aligned} m\colon H \times \mathfrak{m} &\to G \\ (h, X) &\mapsto h\exp(X). \end{aligned}$$

**6.2.1 Definition.** Let $(E,\pi,M)$ be a principal $G$-bundle. A *metric* on $E$ is a reduction of $E$ to a maximal compact subgroup $H$ of $G$ (relative to the inclusion $H \subseteq G$).

Note that Definition 6.2.1 does not require $G$ to be reductive or $M$ to be compact, however, from now on, we assume these properties. Like in Section 6.1, we assume $M$ is endowed with a Riemannian metric, with $\mathrm{vol}_M$ being the associated volume form.

So a metric on $E$ amounts to a section $\sigma\colon M \to E_{G/H}$, using Theorem 3.5.5, which in turn defines a map $s\colon E \to G/H$ such that $s(xg) = g^{-1}s(x)$ for all $g \in G, x \in E$ as in Theorem 4.2.1. $E$ is not compact (unless $G$ is compact, but in this case $s$ is constant), so we can't call $s$ "harmonic" using Definition 6.1.1. However, with our hypothesis that $G$ is reductive and with the further assumption that $E$ has a connection $\omega$, it is possible to look at $s$ in a similar context. The idea is to use all of the tools provided by principal bundles and the theorems to treat $\sigma$ in a way similar to Section 6.1. In particular, we make use of the connection $\omega$ to differentiate the section $\sigma$, which allows us to define the energy like before. Since harmonic maps are usually studied in the context of flat connections, we also assume that $\omega$ is flat.

We consider $ds$ as the bundle map

$$\begin{aligned} ds\colon TE &\to s^*T(G/H) \\ (x, X) &\mapsto (x, s_*(x)(X)). \end{aligned}$$

Consider the following left $H$-actions on $H \times \mathfrak{m}$ and $G$, respectively:

$$\begin{aligned} k \cdot (h, X) &= (khk^{-1}, \mathrm{Ad}(h)(X)) \\ k \cdot g &= kgk^{-1}. \end{aligned}$$

For all $k \in H$, the exponential map satisfies (see [12], Proposition 20.8g)

$$\exp \circ \mathrm{Ad}(k) = C_k \circ \exp \tag{1}$$

hence $m$ is $H$-equivariant with respect to the actions just defined. We have a commutative diagram

$$\begin{array}{ccc} G & \xleftarrow{m} & H \times \mathfrak{m} \\ \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle p_2} \\ G/H & \xleftarrow[f]{} & \mathfrak{m} \end{array} . \tag{2}$$

Recall that

$$l_k \circ \pi_G = \pi_G \circ L_k \tag{3}$$

for all $k \in H$, where $l_k$ the left translation by $k$ in $G/H$. Using (1) and (3), we obtain

$$f \circ \mathsf{Ad}(k) = l_k \circ f, \tag{4}$$

from which it follows that $f^{-1} \circ \pi_G \circ L_k = \mathsf{Ad}(k) \circ f^{-1} \circ \pi_G$. Using (2), this is equivalent to

$$\mathsf{Ad}(k) \circ p_2 \circ m^{-1} = p_2 \circ m^{-1} \circ L_k. \tag{5}$$

Define $\alpha \colon G/H \to G$ by $\alpha(c) = m(1, f^{-1}(c))$. In particular, $\alpha(1H) = 1$. Since $c = \pi_G(\mathsf{exp}(X))$ for a unique $X \in \mathfrak{m}$, we have $\alpha(c) = \mathsf{exp}(X)$. If $h \in H$, then

$$\alpha(\pi_G(h\mathsf{exp}(X)) = \alpha(\pi_G(\mathsf{exp}(\mathsf{Ad}(h)(X)))) = \mathsf{exp}(\mathsf{Ad}(h)(X))) = h\mathsf{exp}(X)h^{-1},$$

therefore $\alpha$ is $H$-equivariant when considering left translations in $G/H$ and conjugation in $G$. Next, let $\tilde{s} = \alpha \circ s \colon E \to G$. The equivariances of $\alpha$ and $s$ yield

$$\tilde{s}(xh) = \alpha(h^{-1}s(x)) = h^{-1}\tilde{s}(x)h \tag{6}$$

for all $h \in H, x \in E$. Clearly $s(x\tilde{s}(x)) = 1$ for all $x \in E$.

The lift $\tilde{s}$ of $s$ to $G$ induces a well-defined bundle map

$$\begin{aligned} s^{-1}ds \colon TE &\to E \times T_{1H}(G/H) \\ (x, X) &\mapsto (x, (l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X)). \end{aligned}$$

Using $f$, we can instead write

$$\begin{aligned} s^{-1}ds \colon TE &\to E \times \mathfrak{m} \\ (x, X) &\mapsto (x, (f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X)), \end{aligned}$$

which in turn defines a 1-form $s^{-1}ds \in \Omega^1(E, \mathfrak{m})$ by

$$s^{-1}ds(x)(X) = (f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X).$$

To apply Theorem 4.2.1, we need this form to be horizontal. We can apply the covariant derivative with respect to $\omega$, taking $s^{-1}d_\omega s$, defined by

$$s^{-1}d_\omega s(x)(X) = (f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X^h),$$

where $X^h$ is the horizontal component of $X$ determined by $\omega$. We also require $s^{-1}d_\omega s$ to be equivariant with respect to some representation $G \to \mathrm{Aut}(\mathfrak{m})$, which, in general, is not possible. To overcome this, we take the reduced subbundle $E_H := \sigma^*(E)$ defined by $\sigma$, which satisfies the commutative diagram

$$\begin{array}{ccc} E_H & \xrightarrow{\ i\ } & E \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi_{G/H}} \\ M & \xrightarrow{\ \sigma\ } & E_{G/H} \cong E/H \end{array}$$

as in Theorem 3.5.5. The inclusion $i$ allows us to take the pullback form $i^*(s^{-1}d\omega s) \in \Omega^1(E_H, \mathfrak{m})$, which we will also denote by $s^{-1}d_\omega s$ from now on. Note that $s^{-1}d_\omega s$ is still horizontal. It is now possible to ask whether $s^{-1}d_\omega s$ is Ad-equivariant. Taking $x \in E_H$,

$X \in T_xE_H$ and $h \in H$, we have, using the formulas deduced above,

$$\begin{aligned}
R_h^*(s^{-1}d_\omega s)(x)(X) &= s^{-1}d_\omega s(xh)((R_h)_*(x)(X)) \\
&= (f^{-1} \circ l_{\tilde{s}(xh)^{-1}} \circ s)_*(x)((R_h)_*(x)(X)^h) \\
&= (f^{-1} \circ l_{\tilde{s}(xh)^{-1}} \circ s)_*(x)((R_h)_*(x)(X^h)) \\
&= (f^{-1} \circ l_{\tilde{s}(xh)^{-1}} \circ s \circ R_h)_*(x)(X^h) && \text{(since } \omega \text{ is a connection)} \\
&= (f^{-1} \circ l_{\tilde{s}(xh)^{-1}} \circ l_h^{-1} \circ s)_*(x)(X^h) && \text{(equivariance of } \rho) \\
&= (f^{-1} \circ l_{h^{-1}} \circ l_{\tilde{s}(x)^{-1}} \circ l_h \circ l_h^{-1} \circ s)_*(x)(X^h) && \text{(from (6))} \\
&= (f^{-1} \circ l_{h^{-1}} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X^h) \\
&= (f^{-1} \circ f \circ \mathsf{Ad}(h)^{-1} \circ f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X^h) && \text{(from (4))} \\
&= \mathsf{Ad}(h)^{-1}(f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(X^h)) \\
&= \mathsf{Ad}(h)^{-1}(s^{-1}d_\omega s(x)(X)),
\end{aligned}$$

so $s^{-1}d_\omega s \in \Omega^1(E_H, \mathfrak{m})$ is horizontal and Ad-equivariant. Using Theorem 4.2.1, $s^{-1}d_\omega s$ descends to a 1-form $s^{-1}d_\omega s \in \Omega^1(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$, defined by

$$s^{-1}d_\omega s(p)(v) = [x, (f^{-1} \circ l_{\tilde{s}(x)^{-1}} \circ s)_*(x)(u^h)],$$

where $x \in \pi^{-1}(p)$ and $\pi_*(x)(u) = v$. Take the bundle metric $g$ on $E_H \times_{\mathsf{Ad}} \mathfrak{m}$ defined by

$$g(p)([x, X], [x, Y]) = B(X, Y), \tag{7}$$

where $\pi(x) = p \in M$. Note that it is well defined, as $B$ is Ad-invariant. We can now define the notion of harmonic metric on a principal bundle. Again, we assume flatness in the definition, however, it should be noted that during this discussion, we did not use flatness; only the choice of the connection was necessary.

**6.2.2 Definition.** A metric $s\colon E \to G/H$ on $E$ is said to be *harmonic* if it minimizes the energy integral

$$E(f) = \int_M ||f^{-1}d_\omega f|| \mathrm{vol}_M,$$

among all metrics $f\colon E \to G/H$ on $E$.

Note the choice of the flat connection $\omega$. Now we focus on the section $\sigma\colon M \to E_{G/H}$. Again, we have a commutative diagram

$$\begin{array}{ccc} E_H & \xrightarrow{i} & E \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi_{G/H}} \\ M & \xrightarrow{\sigma} & E_{G/H} \end{array}.$$

Consider the pullback form $i^*\omega \in \Omega^1(E_H, \mathfrak{g})$. It is not a connection, but it is easy to check that the 1-form $A^s \in \Omega^1(E_H, \mathfrak{h})$ obtained by composition with the projection of $\mathfrak{g}$ onto $\mathfrak{h}$ is indeed a connection. Essentially, the one property that $i^*\omega$ is missing is taking values in $\mathfrak{h}$ (note that $E_H$ is a principal $H$-bundle, not a $G$-bundle). The difference $\theta^s = i^*\omega - A^s$, which is a 1-form in $\Omega^1(E_H, \mathfrak{m})$, is readily seen to be horizontal and Ad-equivariant, hence it defines a 1-form $\theta^s \in \Omega^1(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$. Note that $\theta^s$ is obtained by taking the projection $p_{\mathfrak{m}}$ onto $\mathfrak{m}$. Furthermore, this decomposition exists even if $\omega$ is not flat. We also have the following:

**6.2.3 Lemma.** The connection $A^s$ on $E_H$ is a reduction of $\omega$ relative to the inclusion $i$. Moreover, $\omega$ is flat if and only if $i^*F_\omega = 0$.

**Proof.** It is straightforward to check that $A^s$ is a reduction of $\omega$. The second statement follows from Corollary 4.3.3, noting that $F_\omega$ is horizontal. □

Finally, we check that $\theta^s = s^{-1}d_\omega s$. This is the final component of one of our main results.

**6.2.4 Theorem.** Let $E$ be a flat principal $G$-bundle with flat connection $\omega$ over a compact and oriented manifold $M$, equipped with a Riemannian metric. Let $\mathrm{vol}_M$ be the associated volume form. Assume that $G$ is reductive. Suppose $E$ admits a reduction to a maximal compact subgroup $H$ of $G$, considered as a map $s\colon E \to G/H$ such that $s(xg) = g^{-1}s(x)$ for all $g \in G, x \in E$. Let $E_H$ be the associated reduced subbundle. Then we have the following:

1. The connection $\omega$ induces a 1-form $s^{-1}d_\omega s \in \Omega^1(E_H, \mathfrak{m})$ which is horizontal and Ad-equivariant, hence it defines a 1-form $s^{-1}d_\omega s \in \Omega^1(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$;

2. The projection $\theta^s$ onto $\mathfrak{m}$ of the pullback form $i^*\omega \in \Omega^1(E_H, \mathfrak{g})$ is horizontal and Ad-equivariant, hence it defines a 1-form $\theta^s \in \Omega^1(M, E_H \times_{\mathsf{Ad}_\mathfrak{m}} \mathfrak{m})$.

3. We have $\theta^s = s^{-1}d_\omega s$, which implies

$$E(s) = \int_M ||\theta^s|| \mathsf{vol}_M.$$

**Proof.** We first calculate $i^*\omega$. By taking local trivializations, we may assume that $E$ is trivial and that $\omega$ is the Maurer-Cartan connection. In this case, $\sigma$ takes values in $M \times G/H$, so it is determined by its projection onto the second factor, which we denote by $\sigma_2 \colon M \to G/H$. Using $\alpha$, we can lift $\sigma_2$ to a map $\tilde{s}_2 \colon M \to G$. First, by definition,

$$\begin{aligned} E_H &= \{(p, (q, g)) \in M \times (M \times G) \colon \sigma(p) = (q, gH)\} \\ &= \{(p, (p, g)) \in M \times (M \times G) \colon \sigma_2(p) = gH\}, \end{aligned}$$

which is clearly diffeomorphic to $\{(p, g) \in M \times G \colon \sigma_2(p) = gH\}$, hence we can now consider $E_H$ as this set. So if $(p, g) \in E_H$, then $g^{-1}\tilde{s}_2(p) \in H$, hence we can also consider

$$E_H = \{(p, \tilde{s}_2(p)h) \in M \times G \colon h \in H\}$$

via the diffeomorphism

$$\begin{aligned} &E_H \to \{(p, \tilde{s}_2(p)h) \in M \times G \colon h \in H\} \\ &(p, g) \mapsto (p, \tilde{s}_2(p)g^{-1}\tilde{s}_2(p)). \end{aligned}$$

Finally, we have a principal $H$-bundle isomorphism

$$\begin{aligned} t \colon M \times H &\to E_H \\ (p, h) &\mapsto (p, \tilde{s}_2(p)h), \end{aligned}$$

whose inverse is a global trivialization of $E_H$.

We focus on the case $h = 1$. Let $(p, \tilde{s}_2(p)) \in E_H$ and let $(v, w) \in T_{(p,\tilde{s}_2(p))}E_H \subseteq M \times G$. Note that

$$(v, w) = t_*(p, 1)(v', w')$$

for some tangent vector $(v', w') \in T_{(p,1)}(M \times H) = T_pM \times \mathfrak{h}$. Clearly $v' = v$. Next, we have

$$w = (p_2 \circ t)_*(p, 1)(v, w'),$$

where $p_2$ is the obvious projection. So we have

$$(p_2 \circ t)(p, h) = \tilde{s}_2(p)h = L_{\tilde{s}_2(p)}(h) = R_h \circ \tilde{s}_2(p).$$

The derivative of $p_2 \circ t$ at $(p, h)$ is obtained by differentiating with respect to $p$ and $h$ separately:

$$(p_2 \circ t)_*(p, h)(X, Y) = L_{\tilde{s}_2(p)*}(h)(Y) + (R_h)_*(\tilde{s}_2(p))((\tilde{s}_2)_*(p)(X)).$$

In particular, if $h = 1$,

$$w = (p_2 \circ t)_*(p, 1)(v, w') = L_{\tilde{s}_2(p)*}(1)(w') + (\tilde{s}_{2*})(p)(v).$$

Now

$$\begin{aligned} i^*\omega(p, \tilde{s}_2(p))(v, w) &= \omega(p, \tilde{s}_2(p))(v, w) \\ &= L_{\tilde{s}_2(p)^{-1}*}(\tilde{s}_2(p))(L_{\tilde{s}_2(p)*}(1)(w')) + L_{\tilde{s}_2(p)^{-1}*}(\tilde{s}_2(p))((\tilde{s}_{2*})(p)(v)). \\ &= w' + (L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2)_*(p)(v). \end{aligned}$$

So

$$\theta^s(p, \tilde{s}_2(p))(v, w) = p_{\mathfrak{m}}(w' + (L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2)_*(p)(v)) = p_{\mathfrak{m}}((L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2)_*(p)(v)),$$

since $w' \in \mathfrak{h}$. As a form on $E_H$, $s^{-1}d_\omega s$ is defined by

$$s^{-1}d_\omega s(p, \tilde{s}_2(p))(v, w) = (f^{-1} \circ l_{\tilde{s}(p,\tilde{s}_2(p))^{-1}} \circ s)_*(p, \tilde{s}_2(p))(v, 0).$$

Since $\tilde{s}(p, \tilde{s}_2(p)) = \alpha(s((p, 1) \cdot \tilde{s}(p, 1))) = \alpha(1H) = 1$, we just have

$$s^{-1}d_\omega s(p, \tilde{s}_2(p))(v, w) = (f^{-1} \circ s)_*(p, \tilde{s}_2(p))(v, 0).$$

We check that $\theta^s = s^{-1}d_\omega s$. It is sufficient to verify equality after applying the isomorphism $f_*(0)$ to both sides, that is, we can check that

$$f_*(0) \circ p_{\mathfrak{m}}((L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2)_*(p)(v)) = s_*(p, \tilde{s}_2(p))(v, 0),$$

Since $f_*(0) \circ p_{\mathfrak{m}} = (\pi_G)_*(1)$, this is equivalent to

$$(\pi_G \circ L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2)_*(p)(v)) = s_*(p, \tilde{s}_2(p))(v, 0).$$

Note that $(v, 0) = (R_{\tilde{s}_2(p)})_*(v, 0)$. Denoting the inclusion $M \to M \times 1$ by $j$, this implies $s_*(p, \tilde{s}_2(p))(v, 0) = (s \circ R_{\tilde{s}_2(p)} \circ j)_*(p)(v)$. The equality now follows from $s \circ R_{\tilde{s}_2(p)} \circ j = \pi \circ L_{\tilde{s}_2(p)^{-1}} \circ \tilde{s}_2$. □

### 6.2.5 Remark.

1. Note that we proved that a reduction of $M \times G$ to the maximal compact subgroup $H$ admits a global trivialization (though it is not a product bundle).

2. We verified that $\theta^s = s^{-1}d_\omega s$ on $E_H$; however, since the correspondence of Theorem 4.2.1 is bijective, this implies equality after descending to $M$.

3. If $E = M \times G$, then the energy of $s$ is the energy of $\sigma_2$ in the sense of Definition 6.1.1 (because $l_{\tilde{s}_2(p)} \colon G/H \to G/H$ is an isometry for all $p \in M$).

## 6.3 The Flatness Equations

Using Lemma 6.2.3, we have

$$\begin{aligned} F_\omega = 0 &\Leftrightarrow i^*(F_\omega) = 0 \\ &\Leftrightarrow d(i^*\omega) + \frac{1}{2}[i^*\omega, i^*\omega] = 0 \\ &\Leftrightarrow dA^s + d\theta^s + \frac{1}{2}[A^s, A^s] + \frac{1}{2}[A^s, \theta^s] + \frac{1}{2}[\theta^s, A^s] + \frac{1}{2}[\theta^s, \theta^s] = 0 \\ &\Leftrightarrow F_{A^s} + \frac{1}{2}[\theta^s, \theta^s] + d\theta^s + [A^s, \theta^s] = 0, \end{aligned}$$

where the forms $F_{A^s} + \frac{1}{2}[\theta^s, \theta^s]$ and $d\theta^s + [A^s, \theta^s]$ take values in $\mathfrak{h}$ and $\mathfrak{m}$, respectively. Here we also use the straightforward fact that $[A^s, \theta^s] = [\theta^s, A^s]$. Recalling from Remark 5.1.4 that

$$d\theta^s + [A^s, \theta^s] = d^s_A \theta^s,$$

we obtain

**6.3.1 Theorem.** Let $E$ be a principal $G$-bundle over a compact and oriented manifold $M$, with $G$ reductive. Assume there is a reduction $s$ of $G$ to a maximal compact subgroup $H$. Let $\omega$ be a connection on $E$. Let $i$ be inclusion of the reduced subbundle $E_H$. Write $i^*\omega = A^s + \theta^s$, as before. Then $\omega$ is flat if and only if $A^s$ and $\theta^s$ satisfy the equations

$$\begin{cases} F_{A^s} + \frac{1}{2}[\theta^s, \theta^s] = 0 \\ d_{A^s}\theta^s = 0 \end{cases}.$$

Next, recall that $F_{A^s}$ is tensorial of type Ad (see Theorem 5.1.5). One can also verify that $[\theta^s, \theta^s]$ is Ad-equivariant (see [12], Corollary 8.31, noting that the definition of the Lie bracket on $\mathfrak{g}$ is based on the isomorphism of Remark 3.1.4.3) and horizontal, hence their sum $F_{A^s} + \frac{1}{2}[\theta^s, \theta^s]$ is also tensorial of type Ad, which then descends to a 1-form in $\Omega^1(M, E_H \times_{\mathrm{Ad}} \mathfrak{h})$, which we denote by $F_{A^s} + \frac{1}{2}[\theta^s, \theta^s]$. Note that $F_{A^s}$, as a form on $M$, is the one corresponding to $F_{A^s}$ as the curvature form of $A$ on $E_H$. Since $d_{A^s}\theta^s$ is tensorial of type Ad (see Proposition 4.4.2), we also have the 1-form $d_{A^s}\theta^s \in \Omega^1(M, E_H \times_{\mathrm{Ad}} \mathfrak{m})$.

Note that this does not rely on $\omega$ being flat. Since these correspondences are bijective, we have a result just like Theorem 6.3.1 after descending to $M$.

## 6.4 Statement of the Corlette-Donaldson Theorem

Like at the end of Section 4, the covariant derivative with respect to the connection $A^s$ induces linear maps

$$d_{A^s} \colon \Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m}) \to \Omega^{k+1}(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$$

for all $k \geq 0$. If $\nu, \tau \in \Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$, we can define

$$\begin{aligned} \langle \nu, \tau \rangle \colon M &\to \mathbb{R} \\ p &\mapsto g(p)(\nu(p), \tau(p)), \end{aligned}$$

where $g$ is the metric on $E_H \times_{\mathsf{Ad}} \mathfrak{m}$ defined in (7). This induces a $L^2$- inner product

$$\begin{aligned} &\Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m}) \times \Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m}) \to \mathbb{R} \\ &(\nu, \tau) \mapsto \int_M \langle \nu, \tau \rangle \mathsf{vol}_M. \end{aligned}$$

There is a unique *formal adjoint operator* to $d_{A^s}$, denoted by

$$d^*_{A^s} \colon \Omega^{k+1}(M, E_H \times_{\mathsf{Ad}} \mathfrak{m}) \to \Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m}),$$

defined for all $k \geq 0$, satisfying

$$\langle d_A \nu, \tau \rangle = \langle \nu, d^*_{A^s} \tau \rangle.$$

for all $\nu, \tau \in \Omega^k(M, E_H \times_{\mathsf{Ad}} \mathfrak{m})$. Jost constructs the formal adjoint operator in ([9], Section 3.3); it is also used in Section 4.2.

The formal adjoint operator provides a way of testing for harmonic metrics:

**6.4.1 Theorem.** The metric $s\colon E \to G/H$ on $E$ is harmonic if and only if $d^*_{A^s}\theta^s = 0$.

This is proved in [2] (see Equation (9iii)) and in ([1], Proposition 2.2).

Now, assume that $M$ is connected. Using Theorem 5.3.1, we can write $E \cong \tilde{M} \times_\rho G$, where $\rho\colon \pi_1 M \to G$ is the holonomy representation of $E$. This defines an action of $\pi_1 M$ on $\mathfrak{g}$ via the representation $\mathrm{Ad} \circ \rho : \pi_1 M \to \mathrm{GL}(\mathfrak{g})$. Note that if $\rho$ is trivial, then $E = M \times G$, recovering Remark 6.2.5.3. To state the Donaldson-Corlette theorem, we require a final definition.

**6.4.2 Definition.** A representation $\rho\colon \pi_1 M \to G$ is said to be *reductive* (or *semisimple*) if every subspace of $\mathfrak{g}$ invariant under $\mathrm{Ad} \circ \rho$ has an invariant complement.

With the same hypothesis used throughout this section, we can state the theorem.

**6.4.3 Theorem.** (Donaldson-Corlette) The holonomy representation $\rho$ of $E$ is reductive if and only if $E$ admits a harmonic metric.

The theorem is proved by Donaldson in [2] in the $\mathrm{PSL}(2,\mathbb{C})$ case (for the case where $M$ is an oriented compact Riemann surface with no boundary) and proved by Corlette in [1] for a general reductive Lie group $G$ and general oriented compact $M$.

Compiling the results of the section, we have

**6.4.4 Theorem.** Let $E = \tilde{M} \times_\rho G$ be a principal $G$-bundle as in Theorem 6.2.4 (with $M$ connected). Then the following conditions are equivalent:

1. $\rho$ is reductive;
2. there is a harmonic metric on $E$;
3. there is a metric $s$ on $E$ satisfying

$$\begin{cases} F_{A^s} + \frac{1}{2}[\theta^s, \theta^s] = 0 \\ d_{A^s}\theta^s = 0 \\ d^*_{A^s}\theta^s = 0 \end{cases} . \tag{8}$$

**6.4.5 Remark.** The equations (8) are a version of *Hitchin's self-duality equations*. Recall Theorem 5.3.1. We then have a bijection between the set of conjugacy classes of reductive representations $\pi_1 M \to G$ and the set of gauge equivalence classes of solutions to Hitchin's equations. This correspondence is a substantial step towards the *non-abelian Hodge correspondence*, between representations $\pi_1 M \to G$ and isomorphism classes of certain holomorphic objects over a Riemann surface, with $M$ being the underlying surface. These objects are called *polystable Higgs $G$-bundles*. This is discussed by Hitchin in [7].

# References


[1] Kevin Corlette. Flat $G$-bundles with canonical metrics. *Journal of Differential Geometry*, 28(3):361 – 382, 1988.

[2] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proceedings of the London Mathematical Society*, s3-55(1):127–131, 1987.

[3] Johan Dupont. *Fibre Bundles and Chern-Weil Theory*. University of Aarhus, August 2003.

[4] James Eells and J. H. Sampson. Harmonic mappings of riemannian manifolds. *American Journal of Mathematics*, 86(1):109–160, 1964.

[5] William Fulton. *Algebraic Topology*, volume 153 of *Graduate Texts in Mathematics*. Springer New York, September 1997.

[6] Joachim Hilgert and Karl-Hermann Neeb. *Structure and Geometry of Lie Groups*, volume 159 of *Springer Monographs in Mathematics*. Springer New York Dordrecht Heidelberg London, 01 2012.

[7] Nigel Hitchin. The self-duality equations on a riemann surface. *Proceedings of The London Mathematical Society*, 55:59–126, 1987.

[8] Victoria Hoskins. On algebraic aspects of the moduli space of flat connections. 2013.

[9] Jürgen Jost. *Riemannian Geometry and Geometric Analysis*. Universitext. Springer-Verlag Berlin Heidelberg, 2011.

[10] Anthony W. Knapp. *Lie Groups Beyond an Introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, MA, 1996.

[11] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry - Volume 1*. Wiley Classics Library. Wiley, New York, 1996.

[12] John Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer New York, August 2012.

[13] Loring W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York, October 2010.

[14] Loring W. Tu. *Differential Geometry - Connections, Curvature, and Characteristic Classes*, volume 275 of *Graduate Texts in Mathematics*. Springer-Verlag GmbH, July 2017.