Cosmic Strings in Modified Theories of Gravity

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MSc

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Masters Thesis

Cosmic Strings in Modified Theories of Gravity

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“ We came out of the cave and we looked over the hill and we saw fire. And we crossed the ocean and we pioneered the West and we took to the sky. The history of man is hung on a timeline of exploration, and this is what’s next. ”

Rob Lowe as Sam Seaborn in The West Wing

Acknowledgements

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This thesis is also dedicated to my sister, Diana, my oldest friend. With her, I shared most of my life, and annoyed all of hers. She is truly what a sister should be: a friend and a companion; I am sorry for those who were not lucky enough to have someone like her in their childhood and throughout their lives... Waxend crew forever!

To my grandmother Adelaide, twice a mother to me, to whom my small victories were cause for celebration, who always supported me unconditionally, who loves me every second. To my grandfather Manuel, who I miss so much, and who left at the beginning of this journey; I so regret not sharing this moment with you, but I hope you would be proud! I will keep you both forever in my heart. To my grandparents Emília e Tomás,

that shaped who my father is, the best father I could have asked for, and to whom I'll be forever grateful for that.

My parents were the first ones to believe in me. They never questioned what I wanted to do when I grew up as long as it was honest and that I fought for it. Mom, Dad, this is it. I'm here. And I'm here because of you; because you took the trouble to take an insisting 10 years old boy to watch seminars at CAUP that he was too young to understand, because you took a 11 years old boy to watch a comet, in an experience that changed my life. This is the mission of a parent: to love their child, to sparkle and fuel their passions, so they can find what makes them truly happy. I hope I can be to my son the kind of parents you are to me. Thank you forever.

About 10 years ago I had given up. I convinced myself that I would never amount to anything in Astrophysics, and that it was too late to fulfill my dream of becoming a Cosmologist. I was taking the easy way out. Fortunately I had someone in my life with more sense than I, someone that told me: "If you give up because you want to do something else, I'll support you. If you give up because it's too late, I can't accept that. If you do that, sooner or later, you'll regret it for the rest of your life." That person was, and still is, my best friend. The person I share all my dreams and fears, my doubts and concerns. The one who puts up with all my rants, mood swings and frustrations. Someone that takes great personal sacrifice to help me achieve my goals and dreams. I'm writing this thesis because she reminded me, a long time ago, of the happiness I draw from this field, this work. I was fortunate enough to find her, I was fortunate enough to love her and I was fortunate enough to marry her. Claudete, I didn't build my dreams around you, I build them with you. No one more than you.

Last, but not least, to my son Tomás. The reason for everything, the motivation and the final goal. He may not be able to understand any of these pages, but in time he will. And when that happens I want him to know that, if he wants it enough, he can bring his dreams to reality, because I did. Tomás, you are the best thing that ever happened in my life and I want you to be happy in the path you choose. Nothing else is anywhere near as important to me. Never settle, never compromise who you want to be for what is easier or more comfortable; growth is born from discomfort. Sometimes I may not have the time to play with you as much as you (and me) would like me to, but I hope someday you will understand. And when that day comes, I hope you can be proud of your father, like I am of you, every single day. I love you, Dunfas.

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Abstract

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

MSc. Astronomy and Astrophysics

Cosmic Strings in Modified Theories of Gravity

by Hilberto Manuel Rocha da Silva

The theory of General Relativity, the standard model for gravitation, has been a pillar of Physics for almost a century now. It's arguably one of the most astounding feats of the human intellect and has attracted and inspired generations of physicists not only for its accuracy in the description of physical phenomena, but also for its predictive capability, elegance, and (apparent) simplicity. However, despite being well studied, tested and understood in a vast array of environments, frailties in the theory have become too severe to be ignored: the need to (re)introduce the cosmological constant to account for the late-time accelerated expansion, the seemingly inevitable existence of dark matter (both rendering more than 95% of the Universe non-interactive with electromagnetic radiation) allied to the fact that GR continuously resists quantization have put Einstein theory into question, paving the way to more general theories of gravity that hope to address at least some of these issues. In this work we made use of one of the possible extensions to General Relativity, the Hybrid metric-Palatini Gravity theory (both in its original and generalised versions) to study the existence and properties of local $U(1)$ (Abelian-Higgs) cosmic strings. Cosmic strings are topological defects that may have been formed after phase transitions earlier in the Universe history through spontaneous symmetry breaking. This type of defects is a common byproduct of several Grand Unified scenarios and thus can be used to constrain, or even exclude, some of the proposed GUT models. Although their existence is yet to be observed, their properties are object of several future missions, like LISA, that may shed more light over these relics of a distant past.

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Resumo

Faculdade de Ciências da Universidade do Porto
Departamento de Física e Astronomia

Mestrado em Astronomia e Astrofísica

Cordas cósmicas em teorias modificadas da gravidade

por Hilberto Manuel Rocha da Silva

A Teoria da Relatividade Geral, o modelo padrão da gravitação tem sido, nos últimos cem anos, um dos pilares da Física. É indiscutivelmente um dos mais impressionantes feitos do intelecto humano e atraiu e inspirou gerações de físicos não só pela sua precisão na descrição de fenómenos físicos, mas também pela sua capacidade preditiva, elegância e (aparente) simplicidade. Contudo, apesar de ser bem estudada, testada e compreendida numa vasta gama de ambientes, as fragilidades da teoria tornaram-se demasiado severas para serem ignoradas: a necessidade da (re-)introdução da constante cosmológica para descrever a recente expansão acelerada do universo, a aparente inevitável existência de matéria escura (ambas tornando mais de 95 % do Universo não interactivo com radiação electromagnética), aliadas ao facto de a Relatividade Geral continuar a resistir à quantização, colocaram a teoria de Einstein em questão, abrindo caminho a teorias mais gerais da gravitação que esperam resolver, no mínimo, alguns destes problemas. Neste trabalho fizemos uso de uma das possíveis extensões à Relatividade geral, a teoria híbrida métrica-Palatini (tanto na sua formulação original como na sua generalização) para estudar a existência e propriedades físicas de cordas cósmicas locais $U(1)$ (Abelian-Higgs). Cordas cósmicas são defeitos topológicos que poderão ter sido formados após transições de fase no Universo primitivo através de quebras espontâneas de simetria. Este tipo de defeitos é um subproduto comum a várias teorias da Grande Unificação e podem, assim, ser usados para colocar constrangimentos, ou até excluir, alguns modelos propostos para teorias unificadas. Apesar de a sua presença ainda não ter sido confirmada cosmologicamente, as suas propriedades são objecto de várias missões futuras, como a missão LISA, que poderão lançar mais luz sobre estas relíquias de um passado distante.

Contents

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Glossary

BH	Black Hole
CDM	Cold Dark Matter
CMB	Cosmic Microwave Background
COBE	Cosmic Background Explorer
DGP	Dvali, Gabadadze and Porrati
EFE	Einstein Field Equations
EFT	Effective Field Theory
ESA	European Space Agency
GR	General Relativity
GUT	Grand Unified Theory
HMPG	Hybrid Metric-Palatini Gravity
IA	Instituto de Astrofísica e Ciências do Espaço
ISW	Integrated Sachs-Wolfe
LIGO	Laser Interferometer Gravitational-Wave Observatory
LIP	Laboratório de Instrumentação e Física Experimental de Partículas
LISA	Laser Interferometer Space Antenna
NS	Neutron Star
SDSS	Sloan Digital Sky Survey
VEV	Vacuum Expectation Value
ΛCDM	Cold Dark matter plus cosmological constant

Preface

The research presented in this work was conducted at the Instituto de Astrofísica e Ciências do Espaço, and was partially supported by FCT in the context of the project DarkRipple – PTDC/FIS-OUT/29048/2017.

I declare that this thesis is not substantially the same as any that I have submitted for a degree, diploma or other qualification at any other university and that no part of it has already been or is concurrently submitted for any such degree, diploma or other qualification.

This work was done in collaboration with my supervisors, Professor Francisco Lobo and Professor Pedro Avelino and also in collaboration with Professor Tibeiru Harko and Doctor João Luís and has been published, or submitted for publication, as follows:

- T. Harko, F. S. N. Lobo and H. M. R. da Silva, 'Cosmic stringlike objects in hybrid metric-Palatini gravity," Phys. Rev. D 101 (2020) no.12, 124050 [arXiv:2003.09751 [gr-qc]]. (Chapter 5).
- H. M. R. da Silva, T. Harko, F. S. N. Lobo and J. L. Rosa, 'Cosmic strings in generalized hybrid metric-Palatini gravity," [arXiv:2104.12126 [gr-qc]], submitted to Phys. Rev. D. (Chapter 6)

Additionally, this work has also been presented by the author in the following conferences/meetings:

- XXX ENAA - poster presentation
- IA-On 2020 - poster presentation
- XV Iberian Cosmology meeting (IberiCos 2020)
- XVI Marcel Grossmann meeting - Cosmic Strings parallel session
- XVI Marcel Grossmann meeting - Extended Theories of Gravity and Quantum Cosmology parallel session

In parallel to this work, the author concluded two internships, one at the LIP-CMS group under supervision of Jonathan Hollar in 2020, and another one at IA under supervision of Noemi Frusciante in 2021.

To my family, even those who did not live to see this moment, for always being there, even in the darkest of times.

Chapter 1

Introduction

> "The more important fundamental laws and facts of physical science have all been discovered, our future discoveries must be looked for in the sixth place of decimals."
>
> Albert Abraham Michelson,1894

It is a truth, universally acknowledged, that news of the completeness of our understanding of Nature are, invariably, grossly exaggerated.

It is difficult to find, if not impossible, a more defining moment to our present understanding of the Universe than the beginning of the XX century. In the 20 years that followed Michelson's statement that "the more important fundamental laws and facts of physical science have all been discovered", our understanding of physical processes was revolutionised, emphasizing the idea that, no matter how well a theory may be established, understood, and studied, it is always subject to intense scrutiny and revision, either because it fails to address all observations, or by the simple curious nature of the human spirit. And that is exactly where the beauty of Physics resides.

One of those theories was born out of the mind of a young physicist at the turn of the century. Moved by an acute curiosity and physical intuition, Albert Einstein shattered centuries of well-established physical theories in an attempt to simply understand the world [1–4].

General Relativity was conceived in an attempt to generalize Special Relativity to include non-inertial reference frames. In this quest, Einstein was guided by a set of fundamental principles that are the basis of the theory: Mach's principle, the Equivalence principle, and the principles of Relativity, general covariance and correspondence.

The General theory of gravity is now a well established theory, with an impressive curriculum of successes and confirmed predictions. However, it is widely regarded now as an incomplete (if not wrong) theory, and the reasons will become apparent later in this work.

1.1 General Relativity

With an article written in 1905, published under the underwhelming title of "Electrodynamics of moving bodies" [5], Einstein responded to the Michelson-Morley experiment [6] by incorporating the Lorentz transformations in a more vast concept: spacetime; and with this, centuries of well established physics stood in check.

The concept of spacetime is now, of course, of central importance to Cosmology and the framework for gravity as a relativistic theory. The main paradigm shift came from the realization that gravity is not the action at a distance of the Newtonian school of thought, but a rather more subtle consequence of matter-related deformations of said spacetime [4]. Einstein further realized that spacetime could be modeled as a curved pseudo-Riemannian manifold, a concept born of one of the most prolific mathematical minds in the context of differential geometry. The fundamental element of this pseudo-Riemannian manifold (pseudo due to the relaxation of the positive-definedess of its line element) is the metric tensor, $g_{\mu\nu}$, a non-degenerate, smooth, symmetric, bi-linear map that assigns a real number to pairs of tangent vectors at each tangent space of the manifold.

In such a pseudo-Riemannian manifold, the line element is defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

which is not, as mentioned, positive defined.

The intimate relation between the geometry of spacetime and the matter-energy content responsible for its curved nature can be cast, quite elegantly, as:

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{1.2}$$

where the left hand side, the Einstein tensor, describes the geometry of spacetime and the right hand side is the energy-momentum tensor, where κ^2 is a constant. This equation highlights the deep connection between the contents of the Universe (whose distribution will determine the shape of spacetime) and the geometry that will define the dynamics of its content.

From the static weak-field approximation produced by a non-relativistic mass density ρ, and in order to recover the Newtonian dynamics, the time-time component of the metric tensor must be of the form

$$g_{00} \simeq -(1+2\Phi) \tag{1.3}$$

where Φ is the Newtonian potential arising from the Poisson's equation ($\nabla^2\Phi = 4\pi G\rho$). The energy density for such a matter configuration is simply $T_{00} \simeq \rho$. This means that the constant κ^2 must be equal to $8\pi G$, where G is the Newton constant.

Regarding the form of $G_{\mu\nu}$, there are some important consequences of equation (1.2) that may help in the construction of the Einstein tensor. The first one is that, in an empty spacetime, the Einstein tensor should be $G_{\mu\nu} = 0$, since the curvature is null. The second characteristic of $G_{\mu\nu}$ is that it should not be constructed with derivatives higher than second order on the metric tensor, in order to reproduce the (second order) Euler Lagrange equations, and, finally, like the energy-momentum tensor, should be divergence-free (as the conservation equations dictates $\nabla^\mu T_{\mu\nu} = 0$). Additionally, the Einstein tensor must be symmetric on its components, of second order and, in the weak field limit, reproduce the Newtonian equations for gravity:

$$\vec{a} = -\nabla\Phi\,, \tag{1.4}$$

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{1.5}$$

A natural candidate for the form of the Einstein tensor is a contraction of the Riemann tensor. The Riemann curvature tensor contains information about the intrinsic curvature of the Riemannian manifold, as it expresses its non-holonomy (the fact that vectors transported along a closed loop do not necessarily end up “pointing in the same direction”). The Riemann tensor can be defined as:

$$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}\,, \tag{1.6}$$

where Γ are the Christoffel symbols, the affine connection for the metric tensor.

The Riemann tensor, through the Bianchi identities, is divergence-free and encompasses the information about the geometry of the manifold; however it is not second order, since it is not an element of the tensor product of two vector spaces. So we will look at the contraction of this tensor, the Ricci tensor:

$$R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}\,, \tag{1.7}$$

which is second order and symmetric on its components.

Riemannian manifolds have a particularly simple form for the contracted Bianchi identities,

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \tag{1.8}$$

and so, one can construct a second order symmetric tensor that is divergence-free:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \tag{1.9}$$

where $R \equiv g_{\mu\nu} R^{\mu\nu}$ is the Ricci Scalar.

All the components of these equations are constructed entirely in terms of the metric tensor, $g_{\mu\nu}$. So in this sense, General Relativity is a metric gravity theory. Einstein and Grossmann arrived to this equation in November 1915, although they had previously, mistakenly, rejected it.

However, the lack of static stable solutions when eq.(1.9) was applied to the Cosmological scale led to a later extension of the field equations. In Einstein's own words:

> "In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation, which is not justified by our actual knowledge of gravitation. It has to be emphasized, however, that a positive curvature of space is given by our results, even if the supplementary term is not introduced. That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars."

And so, in 1917, the Cosmological Constant, Λ, was born. Not out of some religious views (as it is often pointed), but rather from a very sound (at the time) scientific rationale: in 1917 there were no observations that could indicate a non-static Universe.

Thus, in a little more than a decade, not only the notion of space and time as separated and background entities was abandoned, but we also let go of an "action-at-a-distance" view of gravity for a more profound geometric phenomenon, translated by Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{1.10}$$

In the words of Hermann Minkowski:

> "Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Equation (1.10) is the most notable face of a different way of thinking about gravity, and is a very powerful, and fruitful, tool to study and understand our Universe.

As all fundamental field theories, the Einstein field equations, postulated from symmetry considerations, can also be obtained from variational methods. In this case, a "least action principle" may be used to attain eq.(1.10) by assuming the following action, known as the Einstein-Hilbert action (for a detailed derivation see Appendix A):

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, (R - 2\Lambda)\, d^4x. \tag{1.11}$$

The story of General Relativity, and of our understanding of the Universe, knew another chapter in the 1920's. At Mount Wilson observatory, Edwin Hubble measured the recession velocity of galaxies against their luminosity-distance (Fig. 1.1) and determined that the Universe must be expanding [7].

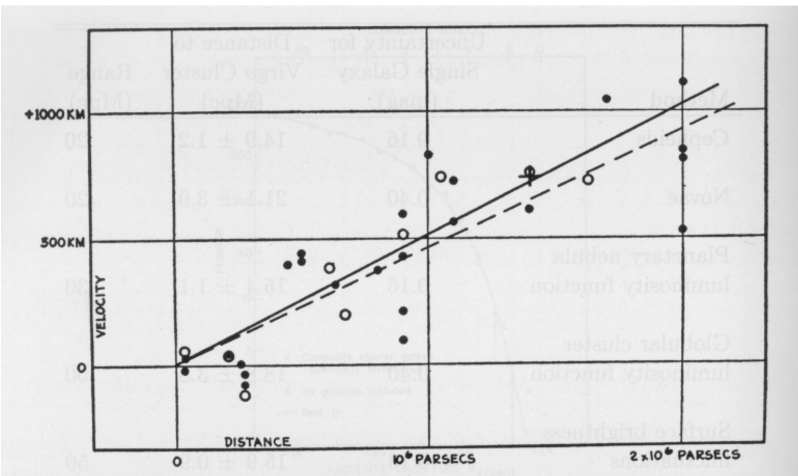

Figure 1.1: Hubble diagram of recession velocities vs. distance. From [7]

In light of this new information, the stable static solutions, the motivation behind the cosmological constant, were not in agreement with observations anymore. The cosmological constant was left to fade into oblivion. However, a more interesting fate was reserved for Λ...

1.1.1 Experimental test of General Relativity - classical tests.

General Relativity is unquestionably one of the most successful theories in physics. It has been tested, and confirmed, in a wide variety of scenarios.

The firsts tests, the so-called classical tests, were put forward by Einstein himself in 1915 [4].

1.1.1.1 Perihelion precession of Mercury.

The perihelion of the planets in our Solar System is known to precess around the Sun due to numerous effects, the main one being the gravitational influence of other solar system bodies.

The special case of Mercury was “uncovered” by Le Verrier in 1859 [8]. According to his calculations, the rate of the perihelion precession of Mercury (observed to be $\sim 574''$ per Julian century) was in disagreement with the Newtonian prediction by $38''$ (later re-estimated at $43''$ by Simon Newcomb in 1882). A number of explanations for this discrepancy was proposed (even the existence of an interior planet - Vulcano) but none of them successfully accounted for the disparity, or if they did, they also introduced other problems.

When relativistic effects are considered, the correcting factor to the perihelion precession of Mercury is 42.98” [9], remarkably close to the 43” estimated by Newcomb. So, by replacing the Newtonian dynamics by GR, a (then) century-old problem was solved.

1.1.1.2 Deflection of light by the Sun

Even though Newton’s theory predicted that light paths would be bent when passing in the vicinity of a massive object (Cavendish, 1784 and Soldner, 1801) [10], the value that it predicted was half of the expected one when GR was used ($1.75''$).

In 1919 two expeditions (to Ceará, Brazil and São Tomé e Príncipe) were organized by Arthur Eddington to observe and measure the effect during the total solar eclipse of May 29th. The objective was to measure the apparent position of stars near the solar disk at the time of the eclipse (in the constellation of Taurus) and compare it to their position without the presence of the solar disk (Fig.1.2). Although riddled with difficulties (technical and meteorological), the observations confirmed the GR prediction of 1.75” [11] and marked a decisive shift in the paradigm for gravitation.

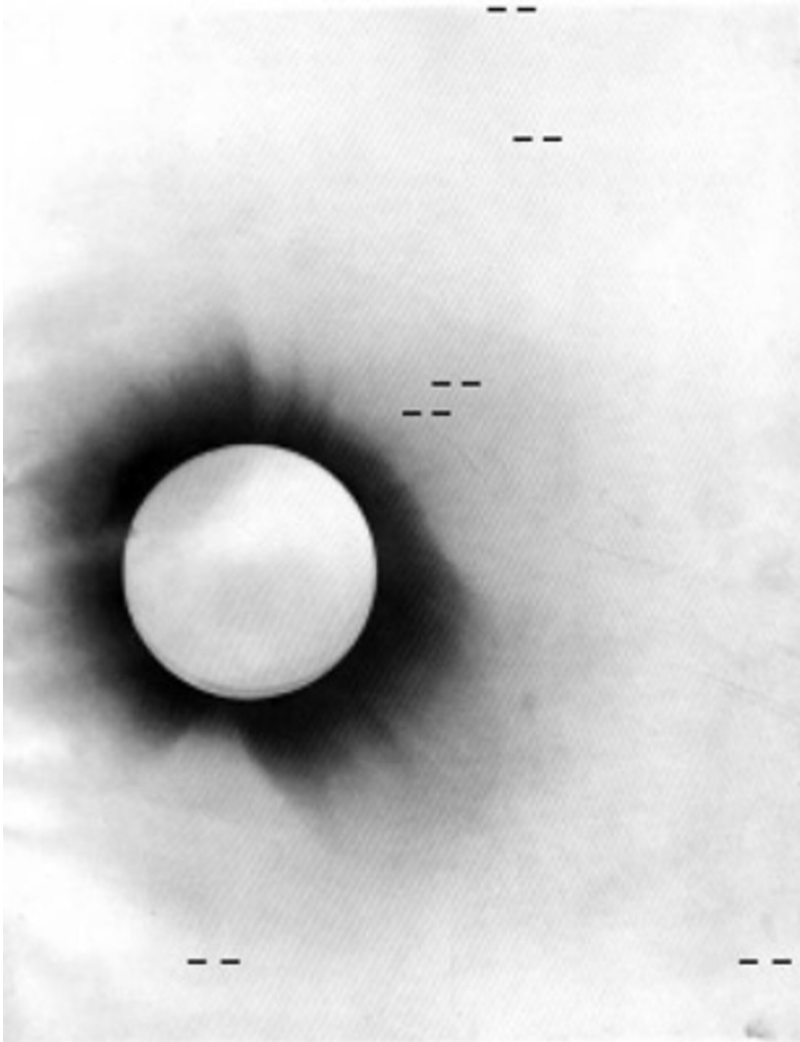

Figure 1.2: Negative photo of the 1919 solar eclipse at Principe Island. From Memoirs of the Royal Astronomical Society LXII, Appendix Plate 1

Several teams repeated the procedure (in 1922, 1953, 1973, etc.) and all of them validated the initial Eddington results.

1.1.1.3 Gravitational redshift

This experimental test, proposed by Einstein by means of the equivalence principle [12], was actually the first one to be put forward in 1907 and is the only completely relativistic effect of the classical tests.

The first attempt to measure the effect, by Walter Adams in 1925, was based on the analysis of spectral lines of stars with a strong gravitational field (white dwarfs, in the case). Even though his experiment using Sirius B was considered flawed [13], due to the light contamination coming from the companion star Sirius A, his work led to similar attempts (by Popper in 1954 [14] and Greenstein in 1974) that confirmed the existence of a gravitational redshift.

This relativistic effect was also observed on Earth, through the Pound-Rebka experiment [15], in 1959 (Fig.1.3).

Figure 1.3: Glen Rebka at the lower end of the Pound-Rebka experimental apparatus at the Jefferson Towers, Harvard University

1.1.2 Experimental test of General Relativity - modern tests.

The experimental tests considered in the last section constitute the classical tests, proposed by Einstein in the early days of the theory, but they are far from being the only tests and confirmations of the General theory of Relativity.

The modern tests of GR will not be reviewed in full, but one in particular deserves a reference.

1.1.2.1 Gravitational Waves

One of the most striking predictions of GR was stated by Einstein in 1916: perturbed massive bodies emit gravitational energy in the form of gravitational waves, ripples in the spacetime itself, propagating away from the source [16].

This physical result was seen as nothing more than a mathematical feature, without the possibility of confirmation due to the lack of technology at the time. The first indirect detection of such gravitational waves was achieved almost 60 years later: the Hulse-Taylor binary pulsar [17], whose orbital period decayed in perfect concordance with the gravitational waves emission predicted from GR (Fig.1.4).

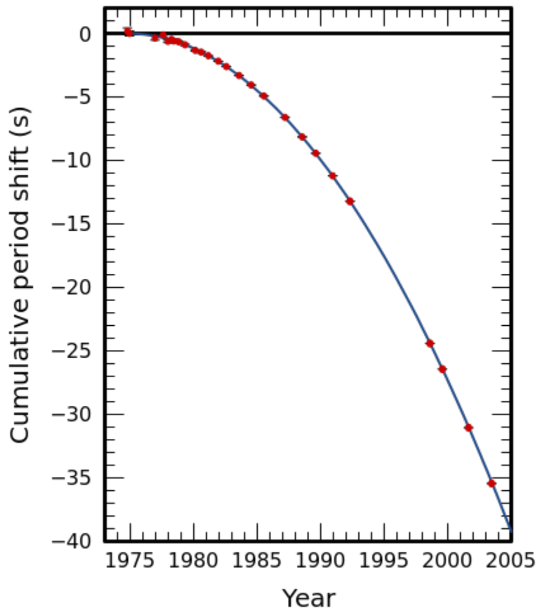

Figure 1.4: Orbit decay of Hulse-Taylor binary pulsar. The continuous line is the GR prediction. From [18]

For a direct detection we had to wait until 2015, one hundred years after Einstein's prediction. Using laser interferometry, the LIGO collaboration was able to detect the direct passage of gravitational waves, coming from a black holes merger event, through the experimental apparatus (Fig. 1.5). This detection is considered the crown of glory of the theory [19].

Presently, the direct detection of gravitational waves was possible in more than 20 astrophysical events, mainly BH-BH and NS-NS merger events.

An ESA mission is planned (for within the next couple of decades) to detect gravitational waves from a space-based interferometer (LISA), which will lead to the improvement of the capability of detection of gravitational waves of lower frequency (Fig. 1.6), allowing us to detect gravitational radiation coming from, among others, binary pulsars [20].

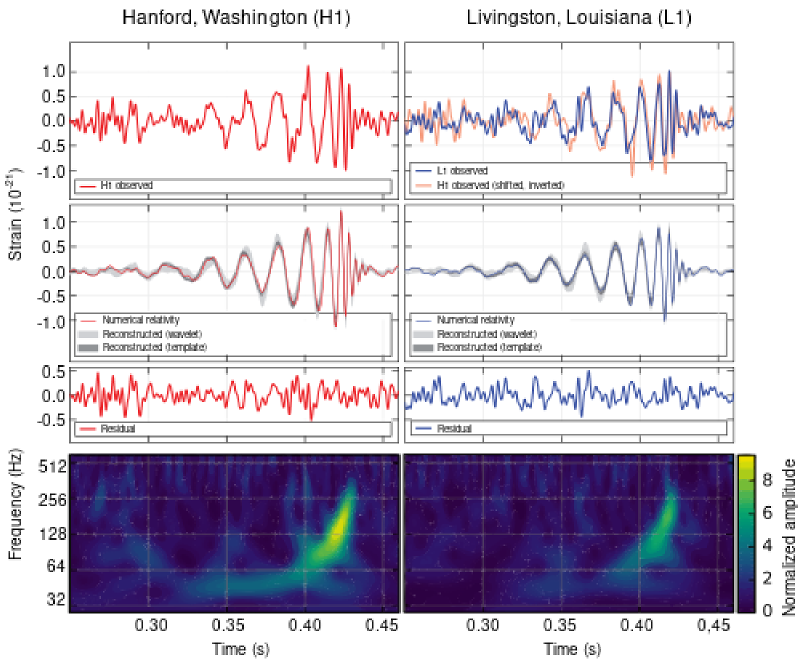

Figure 1.5: First gravitational waves direct detection. From [19]

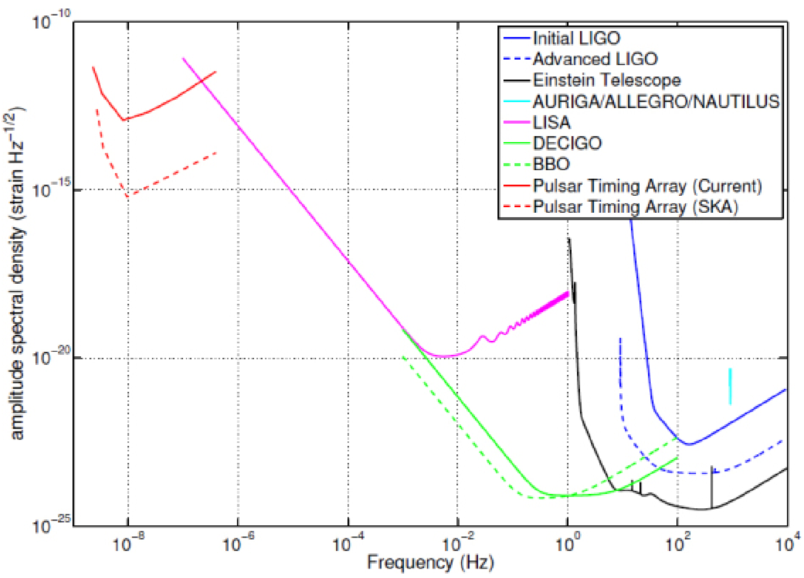

Figure 1.6: The sensitivity of various gravitational-wave detection techniques across 13 orders of magnitude in frequency. From [21]

1.2 The contents of the Universe and the late-time acceleration.

The cosmological solutions to the EFE started as soon as GR was presented. Both de Sitter (1917) [22, 23] and Lemaitre (1927) [24] proposed non-static models for the Universe. The Lemaitre-Eddington solution was particularly interesting, since it not only required an expanding Universe (like the de Sitter solution) but also required that the Universe must have had a beginning, a state of high density and energy which started to expand [25]. Although Fred Hoyle, mockingly, dubbed this model “The Big Bang Model”, George Gamow took it seriously enough to calculate the relative amounts of H and He formed from this initial state (which were in accordance with the composition of stars) [26] and also to predict (along with Alpher and Herman) the existence of some kind of relic-radiation of this initial state flooding the Universe [27].

Less than two decades later, when mapping the microwave noise in the sky, Penzias and Wilson stumbled across a persistent noise coming from all directions in the sky [28]. This radiation, named Cosmic Microwave Background, was studied by Dicke in 1965 and was associated to a blackbody spectrum with $T \sim 3K$ and was interpreted as being the relic-radiation predicted in 1948. This discovery represents one of the most remarkable findings in Modern Astronomy and a decisive victory for the Big Bang model.

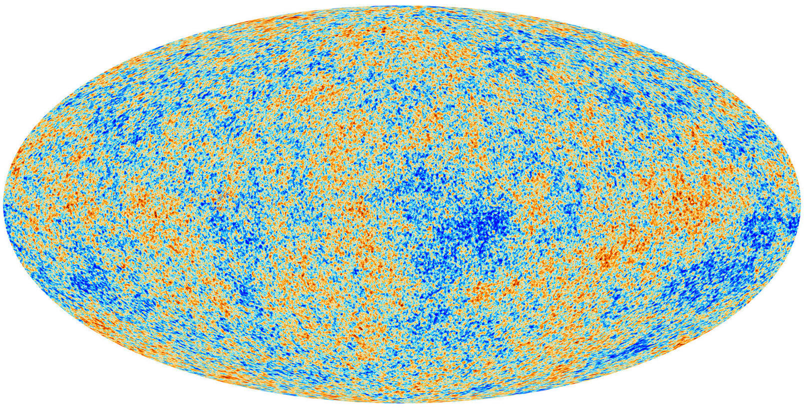

Figure 1.7: The anisotropies of the Cosmic microwave background (CMB) as observed by Planck. From [29]

In the following decades, cosmologists struggled to find models that could account for all observations. The earlier models, which considered all matter to be baryonic, consistently failed to explain large scale structures (the formation and distribution of galaxies). The death blow to baryonic-only models came in 1965 (Fig. 1.8): Rubin and Ford showed that, in globular clusters, the rotational curves revealed that stars farther away from the gravitational center would not remain gravitationally bound if only visible matter was considered [30, 31].

A new form of matter, Dark Matter, that only interacted gravitationally with baryonic matter, was needed to fit not only the theoretical Big Bang nucleosynthesis model, but also the observations of galaxies light curves, lensing observations and large scale structures.

These Cold Dark Matter (CDM) models [33], in which Dark Matter represented 95% of all matter-energy contents in the Universe, were partially successful. In fact, they

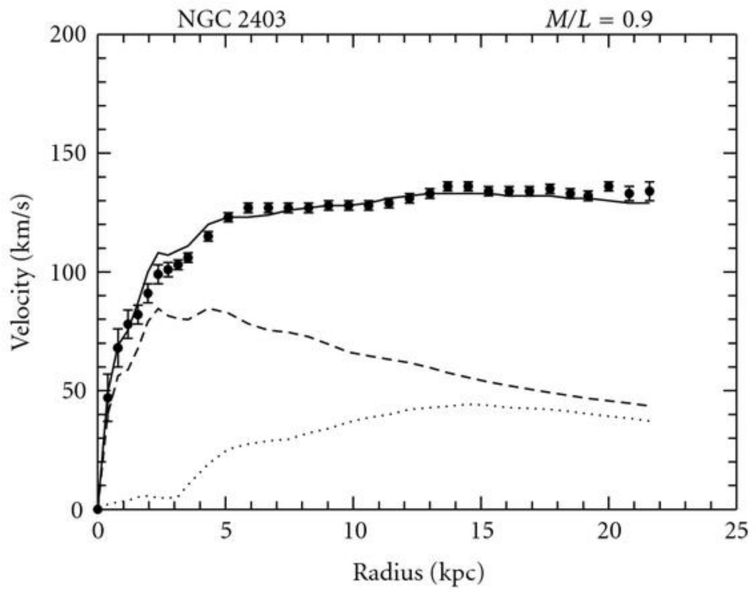

Figure 1.8: The 21 cm line rotation curve of the spiral galaxy NGC 2403 (points with error bars). The dashed curve shows the Newtonian rotation curve of the visible disk, assuming a constant mass-to-light ratio of 0.9 in solar units. The dotted curve is Newtonian rotation curve due to the gaseous component (hydrogen plus primordial helium). From [32]

were accurate in predicting the galaxies' rotation curves and galaxy clustering; however they required a slower expansion rate than the observed one, and also anticipated a lower galaxy clustering at larger scales. In the early 90's, the satellite COBE further confirmed these issues with the CDM models [34].

In addition to these questions about the matter-energy content of the Universe, there were some more fundamental concerns with the observation of the CMB radiation, which showed that the energy budget of the Universe was very close to the critical density needed to render the geometry flat, which seemed to implicate a large degree of fine-tuning early on in the history of the Universe (the flatness problem); and also implying that regions of the Universe not causally connected exhibit the same physical properties (the horizon problem).

Alan Guth in 1981 proposed a solution for both these problems: a period of exponential acceleration early on in the history of the Universe [35]. This inflationary phase became a key component of the standard model of the Universe.

The Cosmological Constant introduced by Einstein in 1917 as an ad-hoc resource to achieve stable static solutions to the field equations seemed to have lost its pertinence in 1920, by the discovery of the expansion of the Universe by Hubble. However, it was kept in many works due to its usefulness in accounting for an apparent inconsistency of the observations: an initial estimation of the Hubble constant, about one order of magnitude higher than the current estimation, rendered the age of the Universe shorter than the Earth formation timescale. And so, the repulsive nature of the Cosmological constant was used to correct that factor. Even so, as before, a solid theoretical reasoning behind it was still missing.

In the late 90's, one of the most puzzling discoveries in the history of Cosmology was done by the observations of supernovae type Ia (Fig.1.9): not only was the Universe expanding, but it was expanding at an accelerated rate [36]. This late-time acceleration led to the conclusion that the matter-energy content of the Universe needed a form of vacuum energy, of repulsive nature, to account for the observations [37]. It was dubbed as Dark Energy, and its most obvious candidate was already there: the cosmological constant, Λ.

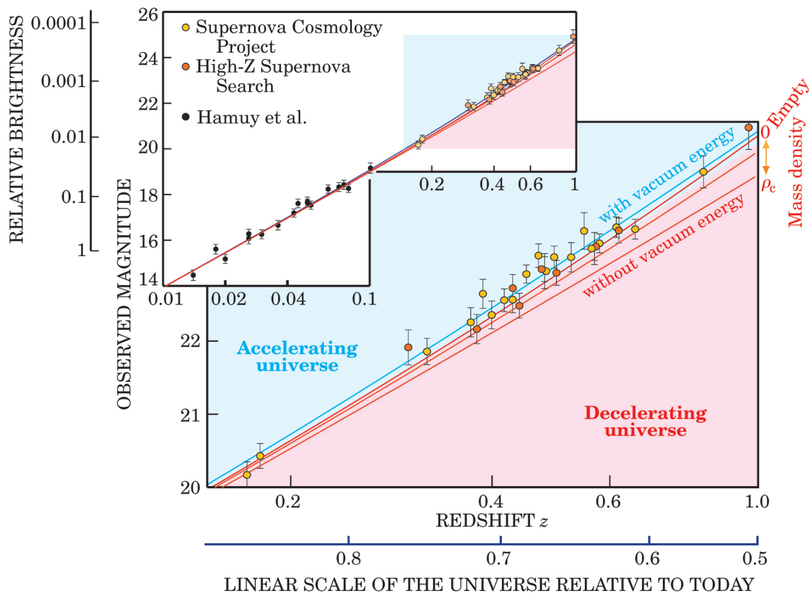

Figure 1.9: Observed magnitude versus redshift is plotted for well-measured distant and nearby type Ia supernovae. At redshifts beyond z= 0.1 (distances greater than about 109 light-years), the cosmological predictions (indicated by the curves) begin to diverge, depending on the assumed cosmic densities of mass and vacuum energy. The red curves represent models with zero vacuum energy and mass densities ranging from the critical density ρ_c down to zero (an empty cosmos). The best fit (blue line) assumes a mass density of about $\rho_c/3$ plus a vacuum energy density twice that large—implying an accelerating cosmic expansion. From [38]

And with this, we have a full picture of the Standard Model for Cosmology, the ΛCDM model: the Universe is around 13.8×10^9 years old, it started in a state of extreme density and temperature; soon after the "Big Bang", that led to the production of the CMB radiation, the Universe underwent a phase of exponential expansion (inflation) and is now dominated by the Dark Energy component, which accounts for its present state of accelerated expansion. The energy budget of the Universe is: $\sim 5\%$ Baryonic Matter, $\sim 25\%$ Cold Dark Matter and $\sim 70\%$ Dark Energy [39, 40].

The evidence for the ΛCDM model is strong: the existence of dark matter is supported by observations of the rotation curves of galaxies, that tend to flatten at large radii [41], clusters of galaxies that appear to have deeper potential wells than the ones inferred by baryonic matter [42], small scale structures surviving the diffusion damping during recombination [43], etc. While the measurements of the luminosity distance of type Ia supernovae [44], of the CMB anisotropies [45] combined with galaxy clustering from the SDSS [46], cross correlation between ISW effect and large-scale structures [47, 48], etc. strongly support the presence of an energy density with equation of state parameter $w \simeq -1$, the cosmological constant.

1.3 Thesis outline

This thesis is organized as follows: in chapter 2 we introduce the motivation behind modified gravity theories, and also Lovelock's theorem that guides this search. In chapter 3, we introduce the hybrid metric-Palatini theory, in its original form, $R + f(\mathcal{R})$ and also in its generalized version $f(R, \mathcal{R})$ and will deduce the equations of motion of the theory in the scalar-tensor representation. In chapter 4, we will make an overview of the process of formation of topological defects, in particular cosmic strings, as a result of phase transitions of the Universe and as a common byproduct of Grand Unified theories. In chapter 5, we study local $U(1)$ cosmic strings configurations in the hybrid metric-Palatini framework for a variety of potentials. A similar analysis is performed in chapter 6, for the generalisation of the hybrid theory. In chapter 7 we present the final remarks, conclusions and future prospects.

The convention followed for the metric signature was (-+++) and Planck natural units ($c = G = 1$) were used, except when stated otherwise.

Chapter 2

Modified Gravity

> "Physics is about questioning, studying, probing nature. You probe, and, if you're lucky, you get strange clues."
>
> Lene Hau, 2007

Albeit its tremendous success in explaining some poorly understood natural phenomena and also its impressive predictive power (as we've seen in the previous chapter), General Relativity presents some interesting challenges that pave the way to modifications of the standard theory of gravity.

The history of modified gravity theories is almost as old as General Relativity itself, with the first ones being proposed just a few years after General Relativity became complete.

2.1 Motivation for Modified Gravity theories

Paradoxically, the ΛCDM model discussed in the previous chapter is one of GR's "Achilles's heels".

In the previous chapter we've noticed that the construction of the ΛCDM model was achieved by addressing some inconsistencies between the observations and the theoretical predictions. Historically, physics addresses such imbalances in the known theories either by adding a "missing" component that was previously unaccounted for, or by changing the governing equations of the theory itself.

The Concordance model, as we've seen, has almost always chosen the first route: adding inflation, adding Dark Matter and adding Dark Energy.

One can argue that perhaps we should look into the other route [49]: that, in fact, the late-time acceleration of the Universe, the galaxies' rotation curves and the dynamics of large gravitationally bond structures are all telling us that we are simply outside the validity of GR and we need another, more general, gravity theory instead of additional fields or particles of which we know very little about. Furthermore, the fact that even more precise, grand (and expensive) experiments to detect Dark Matter particles remain devoid of practical results, reinforces this argument.

In addition, the nature of Dark Energy still eludes us. As noted, the appearance of the cosmological constant as a mathematical artifact lacked a solid theoretical motivation and that hasn't changed. The most common theoretical interpretation is that Λ represents the vacuum energy density. However, when looking into quantum field theories, the majority of those predict a value of about 120 (!!) orders of magnitude higher than the observed value for the cosmological constant [50]. Apart from this fine-tuning problem, another issue arises from the cosmological observations pertaining the Cosmological Constant: it has an energy density of the same order of magnitude as the average matter density of the Universe today ($\rho_{\Lambda(a=1)} \sim \rho_{m(a=1)}$), as ρ_Λ and ρ_m scale differently with the size of the Universe. This poses the "coincidence problem" [51].

The Cosmological constant is not, however, the only troublesome part of the ΛCDM model. Cold Dark Matter is also not problem-free: CDM particles are thought to be produced thermally, since weak interaction cross-sections naturally give rise to the correct dark matter abundance via thermal production; while baryonic matter, on the other hand, is produced non-thermally [52]. So, the question is: how do these two different mechanism produce energy-densities so close to each other? Two other issues can be pointed out regarding dark matter, the "cusp problem" on the center of galaxies [53] and the "missing satellites problem" [54].

Besides the possible inadequacy in explaining cosmological dynamics, another common referred issue with General Relativity is that the theory is not quantizable [55, 56], meaning that a full quantum field theory of gravity either is impossible, and gravity cannot be unified with the other fundamental interactions [57], or that GR is not a final gravity theory.

These arguments, along with the existence of solutions with mathematical singularities in GR, make Modified Gravity theories a fertile field of study.

2.2 Lovelocks theorem and the array of modified gravity theories.

In the early 1970's the British physicist David Lovelock constructed the most general action that can be formed from a Lagrangian that depends only on the metric tensor [58]. If we assume that such lagrangian contains up to second derivatives of the metric, extremising the action with respect to the metric will give rise to the Euler Lagrange equation:

$$E^{\mu\nu}[\mathcal{L}] = \frac{d}{dx^{\rho}}\left[\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho}} - \frac{d}{dx^{\lambda}}\left(\frac{\partial\mathcal{L}}{\partial g_{\mu\nu,\rho\lambda}}\right)\right] - \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = 0 \tag{2.1}$$

Lovelock found that, in 4 dimensions, the only possible solution is

$$E^{\mu\nu} = \alpha\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right] + \lambda\sqrt{-g}g^{\mu\nu} \tag{2.2}$$

which is a general form of the EFE with a cosmological constant (if we make the relation $\alpha = 1$ and $\lambda = \Lambda$). This result has profound implications not only to GR, but also to Modified Gravity theories.

Lovelock's theorem can be seen as a triumph for GR, as it sets that the EFE are the only possible solution for an action that respects simultaneously the following conditions:

- The metric is the only dynamical variable.
- It's constructed with the metric and up to it's second derivatives.
- Has 4 dimensions.
- Linear in the second derivatives of the metric.
- It's divergence free.

However, Lovelock's theorem can in fact be used as a "roadmap" to Modified Gravity theories, since it shows us exactly what assumptions we need to break in order to modify the theory:

- Include more fields, other (or rather) than the metric.
- Allow for higher order derivatives of the metric in the field equations.
- Allow for more than 4 dimensions.
- Abandon rank (2,0) field equations, symmetry on the exchange of indexes, or diverge-free field equations.

- Give up locality.

The roadmap provided by Lovelock allowed for the flourishing of a wide variety of Modified Gravity theories (Fig.2.1), each of them breaking one (or more) of the assumptions in the theorem.

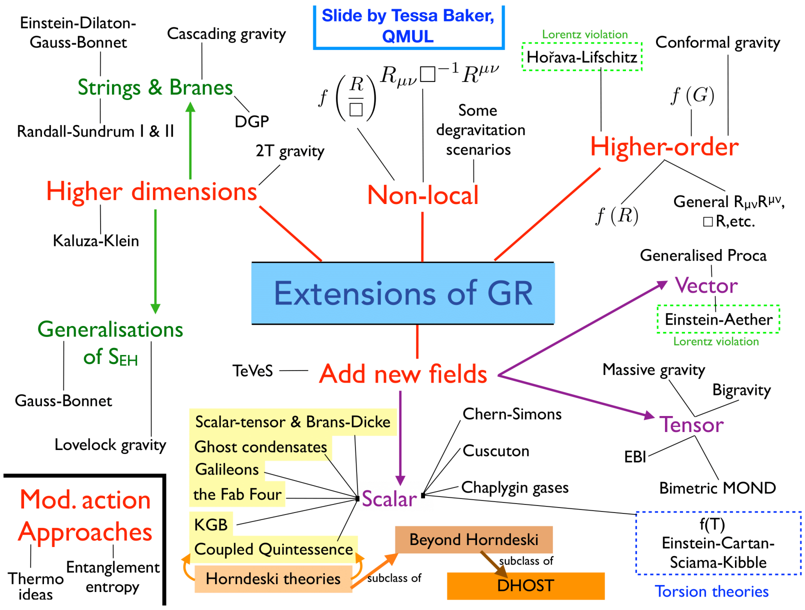

Figure 2.1: The Modified Gravity family tree.

2.2.1 $f(R)$ theories

Although fascinating theories can be found in the modified-gravity picture (Fig. 2.1), for the purpose of this work the key modified gravity theories are the so called $f(R)$ theories, proposed by Hans Buchdahl in 1970 [59].

The underlying concept is to modify the Einstein-Hilbert action, eq.(1.11), from where we can obtain the EFE, to feature not the Ricci scalar per se, but a more general function of it:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, f(R)\, d^4x. \tag{2.3}$$

In order to study this kind of theories, one is presented with two options: the metric approach and the Palatini approach.

In the metric approach, the connection is assumed to be the Levi-Civita (metric compatible and torsion-free) of the metric tensor $g_{\mu\nu}$, and hence the metric is the only dynamical variable to be considered, since the connection can be constructed from the metric. In the Palatini approach the connection is considered an independent structure of the manifold, and thus independent of the metric tensor. This results in two distinct dynamical variables, the metric and the connection.

In GR, both these formalism give rise to the same field equations, but that will not be true, in general, for more complex forms of the action, like the ones considered in $f(R)$ gravity (see appendix A).

Even though $f(R)$ theories can accurately reproduce the expansion rate of the Universe [60, 61], as well as account for inflation [62], on both metric and Palatini formalism, this type of theories have severe drawbacks. The metric $f(R)$, save from the GR-limit, was shown to be inconsistent with solar-system constraints [63] (one of GR's strongest foothold) unless chameleon mechanisms, in which GR is restored at Solar System scales, are considered. While the Palatini-$f(\mathcal{R})$ induces microscopic instabilities, surface singularities in polytropic star models [64], and is unable to describe the evolution of cosmological perturbations.

And so, in 2011 a new approach was proposed: the hybrid metric-Palatini gravity theory [65].

Chapter 3

Hybrid Metric-Palatini Gravity

"Nature uses human imagination to lift her work of creation to even higher levels."

Luigi Pirandello, 1921

The hybrid metric-Palatini theory was proposed by Capozzielo, Harko, Koivisto, Lobo and Olmo in 2011 [65] in order to overcome the difficulties faced by $f(R)$ theories of gravity and successfully unifies the late-time cosmic acceleration period with the weak-field solar system dynamics without the need for chameleon mechanisms [65]. For a more in-depth review of the theory, we refer the reader to the review article [66] and to the full-fledged book on the hybrid theory [67].

3.1 $R + f(\mathcal{R})$ Hybrid metric-Palatini gravity

In this hybrid regime, the metric and Palatini approaches are combined, by adding a new term, $f(\mathcal{R})$ to the Einstein-Hilbert action (1.11):

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, d^4x \ \left[R + f(\mathcal{R}) + \mathcal{L}_m\right]. \tag{3.1}$$

where $\mathcal{R}$ is the Ricci-Palatini scalar, the geometrical analogous to the Ricci scalar, constructed in terms of an independent connection, $\hat{\Gamma}$:

$$\mathcal{R}_{\mu\nu} = \partial_\alpha \hat{\Gamma}^\alpha_{\mu\nu} - \partial_\nu \hat{\Gamma}^\alpha_{\mu\alpha} + \hat{\Gamma}^\alpha_{\alpha\beta}\hat{\Gamma}^\beta_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\beta}\hat{\Gamma}^\beta_{\alpha\nu}. \tag{3.2}$$

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{3.3}$$

In this theory we have two dynamical variables, the metric and the (independent) connection. And so, we apply variation calculations of the action with respect to both.

Varying (3.1) with respect to the metric, $g_{\mu\nu}$ we obtain

$$G_{\mu\nu} + \frac{df}{d\mathcal{R}}\mathcal{R}_{\mu\nu} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{3.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor defined, as usual, as:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta(g^{\mu\nu})} \tag{3.5}$$

When we perform the variation of the action (3.1) with respect to the independent connection, $\hat{\Gamma}$, the resulting equation is:

$$\hat{\nabla}_\alpha = \left(\sqrt{-g}\frac{df}{d\mathcal{R}}g^{\mu\nu}\right) \tag{3.6}$$

which has an interesting feature: if we consider a metric conformally related to $g_{\mu\nu}$ by a $\frac{df}{d\mathcal{R}}$ factor, $h_{\mu\nu} = \frac{df}{d\mathcal{R}}g_{\mu\nu}$, equation (3.6) implies that the independent connection $\hat{\Gamma}$ is the Levi-Civita connection of such metric.

3.1.1 Scalar-tensor representation

One of the useful features of the hybrid metric-Palatini theory is that it admits a scalar-tensor representation, which simplifies the analysis of the dynamics of the theory.

Let's introduce an auxiliary field, A, so that the action (3.1) becomes

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\ d^4x \left[\Omega_A R + f(A) + \frac{df}{dA}(\mathcal{R} - A) + \mathcal{L}_m\right]. \tag{3.7}$$

where the coupling constant Ω_A is introduced for generality. If we further define $\phi \equiv \frac{df}{dA}$ and $V(\phi) \equiv A\frac{df}{dA} - f(A)$, the action becomes:

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\ d^4x \left[\Omega_A R + \phi\mathcal{R} - V(\phi) + \mathcal{L}_m\right] . \tag{3.8}$$

Since we now have three dynamical variables, the metric $g_{\mu\nu}$, the independent connection, $\hat{\Gamma}$ and the scalar field ϕ, we perform the variation of the action (3.8) with respect to each of the dynamical variables, resulting in the following equations of motion (for a detailed derivation refer to appendix C):

$$\Omega_A R_{\mu\nu} + \phi\mathcal{R}_{\mu\nu} - \frac{1}{2}(\Omega_A R + \phi\mathcal{R} - V(\phi))g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{3.9}$$

$$\mathcal{R} - V_\phi = 0 \tag{3.10}$$

$$\hat{\nabla}_\alpha = \left(\sqrt{-g}\phi g_{\mu\nu}\right) \tag{3.11}$$

As we've seen before, the last equation implies that the independent connection is the Levi-Civita connection to a conformal related metric to $g_{\mu\nu}$: $h_{\mu\nu} = \phi g_{\mu\nu}$, allowing us to write the Ricci-Palatini tensor from the Ricci tensor and the conformal factor ϕ (more details in appendix C):

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{\phi}\left(\nabla_\mu \nabla_\nu \phi + \frac{1}{2} g_{\mu\nu} \Box \phi\right) + \frac{3}{2\phi^2} \partial_\mu \phi \partial_\nu \phi \tag{3.12}$$

$$R = \mathcal{R} + \frac{3}{\phi} \Box \phi - \frac{3}{2\phi^2} \partial_\mu \phi \partial^\mu \phi \tag{3.13}$$

where $\Box \equiv \nabla_\mu \nabla^\mu$ in the D'Alembert operator. Eq.(3.13) can be used to recast the action (3.8) into the following form:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, d^4x \left[(\Omega_A + \phi) R + \frac{3}{2\phi} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \mathcal{L}_m \right] . \tag{3.14}$$

We can note now three different cases for the value of the coupling constant Ω_A, for $\Omega_A = 1$ we have the original hybrid theory, for $\Omega_A = 0$ we have the Palatini-$f(\mathcal{R})$ gravity and for $\Omega_A \to \infty$ we recover the metric-$f(R)$ theory.

It is interesting to note that the theory offers a bridge between two different Brans-Dicke type theories, Palatini-$f(\mathcal{R})$, with $w_{BD} = -3/2$ and metric-$f(R)$, with $w_{BD} = 0$, by generalizing the Brans-Dicke coupling

$$w_{BD} = -\frac{3\phi}{2\phi - 2\Omega_A} \tag{3.15}$$

Using eqs.(3.2) and (3.10), we can rewrite eq.(3.9) in the scalar-tensor representation for the original hybrid theory as:

$$(1+\phi) R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \frac{1}{2} g_{\mu\nu} (V + \Box \phi) + \nabla_\mu \nabla_\nu \phi - \frac{3}{2\phi} \partial_\mu \phi \partial_\nu \phi \tag{3.16}$$

In this equation we can conclude that the curvature of spacetime is due not only to the matter distribution, but also to the presence of the scalar field.

Taking the trace of equation (3.9) with $g^{\mu\nu}$, and using equation (3.10), we get

$$2V - \phi V_\phi = \kappa^2 T + \Omega_A R \tag{3.17}$$

There is a striking difference between the hybrid theory and the purely Palatini-$f(\mathcal{R})$. As presented in appendix C, in the purely Palatini case, the field ϕ is a function of T alone, while in the hybrid theory the field is a function of both T and R.

If we now rewrite eq. (3.17) using eqs. (3.13) and (3.10), we get an effective Klein-Gordon equation for the scalar field:

$$-\Box\phi + \frac{1}{2\phi}\partial_\mu\phi\partial^\mu\phi + \frac{\phi\left[2V-(1+\phi)V_\phi\right]}{3} = \frac{\phi\kappa^2}{3}T \tag{3.18}$$

Which demonstrates that, unlike the purely Palatini case, the scalar field is dynamic and the theory is therefore not affected by the instabilities found in the Palatini gravity [61].

3.2 $f(R,\mathcal{R})$ Generalized Hybrid metric-Palatini gravity

In the year of 2013, a generalization of the hybrid metric-Palatini was proposed by Tamanini and Boehmer [68]. In this generalization, the action is no longer constructed by the Ricci scalar plus a general function of a Ricci-Palatini scalar, but rather by general functions of both, resulting in

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\, d^4x\left[f(R,\mathcal{R}) + \mathcal{L}_m\right] . \tag{3.19}$$

As before, we maintain two dynamical variables, the metric and the independent connection, and so, we should perform the variation of the action with respect to both. This yields

$$\frac{\partial f}{\partial R}R_{\mu\nu} + \frac{\partial f}{\partial \mathcal{R}}\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f\left(R,\mathcal{R}\right) - \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\frac{\partial f}{\partial R} = \kappa^2 T_{\mu\nu}, \tag{3.20}$$

$$\hat{\nabla}_\alpha\left(\sqrt{-g}\frac{\partial f}{\partial\mathcal{R}}g^{\mu\nu}\right) = 0, \tag{3.21}$$

We should now note three limits to the above equations:

- $f(R,\mathcal{R}) = f(R)$, in which eq.(3.20) becomes eq.(B.2) and eq.(3.21) becomes trivial.
- $f(R,\mathcal{R}) = f(\mathcal{R})$, in which eq.(3.20) becomes eq.(B.8) and eq.(3.21) remains unchanged.
- $f(R,\mathcal{R}) = \{R,\mathcal{R}\}$, where we retrieve the Einstein Field equations from eq.(3.20) and from eq.(3.21) we obtain the condition for the connection to be Levi-Civita for the metric $g_{\mu\nu}$.

Similarly to the Palatini procedure followed in appendix B, in which we take advantage of (3.21) to build a relation between $R_{\mu\nu}$ and $\mathcal{R}_{\mu\nu}$, we can write:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{3}{2f_{\mathcal{R}}^2}\partial_\mu f_{\mathcal{R}}\partial_\nu f_{\mathcal{R}} - \frac{1}{f_{\mathcal{R}}}\left(\nabla_\mu\nabla_\nu + \frac{1}{2}g_{\mu\nu}\Box\right)f_{\mathcal{R}}, \tag{3.22}$$

$$\mathcal{R} = R + \frac{3}{2f_{\mathcal{R}}^2}\partial_\mu f_{\mathcal{R}}\partial^\mu f_{\mathcal{R}} - \frac{3}{f_{\mathcal{R}}}\Box f_{\mathcal{R}}. \tag{3.23}$$

3.2.1 Scalar-tensor representation

Alike the HMP case, the generalized theory admits a scalar-tensor representation. In order to obtain it, let's start with two auxiliary fields, A and B, and rewrite the action (3.19) as

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\left[f(A,B) + \frac{\partial f}{\partial A}(R-A) + \frac{\partial f}{\partial B}(\mathcal{R}-B)\right]d^4x. \tag{3.24}$$

If one makes the substitutions $A = R$ and $B = \mathcal{R}$, we go back to (3.19). We will also define the potential as

$$V(R,\mathcal{R}) = -f(R,\mathcal{R}) + R\frac{\partial f(R,\mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}}. \tag{3.25}$$

And defining two scalar fields as $\varphi = f_R$ and $\psi = -f_{\mathcal{R}}$ (the minus sign of the ψ definition guarantees that the kinetic energy of the field remains positive), the previous equation becomes

$$V(\varphi,\psi) = -f(A,B) + \varphi A - \psi B \tag{3.26}$$

With these definitions, the action (3.19) becomes

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\,d^4x\left[\varphi R - \psi\mathcal{R} - V(\varphi,\psi)\right], \tag{3.27}$$

that can be further simplified by making use of eq. (3.23) to eliminate the explicit dependence in $\mathcal{R}$:

$$S = \frac{1}{2\kappa^2}\int\sqrt{-g}\,d^4x\left[(\varphi-\psi)R - \frac{3}{2\psi}\partial^\mu\psi\partial_\mu\psi - V(\varphi,\psi)\right]. \tag{3.28}$$

Performing the variations of this action with respect to the three dynamical variables (the metric and both scalar fields), we get the dynamical equations for the theory in the scalar-tensor representation:

$$\begin{aligned}(\varphi-\psi)\,G_{\mu\nu} =& \kappa^2 T_{\mu\nu}-\left[\Box\left(\varphi-\psi\right)+\frac{1}{2}V+\frac{3}{4\psi}\partial^\alpha\psi\partial_\alpha\psi\right]g_{\mu\nu}+\\ &+\frac{3}{2\psi}\partial_\mu\psi\partial_\nu\psi+\nabla_\mu\nabla_\nu\left(\varphi-\psi\right)\end{aligned} \tag{3.29}$$

$$R = V_\varphi \tag{3.30}$$

$$-R-\frac{3}{2\psi^2}\partial_\mu\psi\partial^\mu\psi+\frac{3}{\psi}\Box\psi-V_\psi=0 \tag{3.31}$$

The (modified) Klein Gordon equations for the fields can be obtained by taking the trace of eq.(3.29) and use the result in eqs.(3.30) and (3.31) to eliminate the explicit dependence in R. This procedure yields two dynamical equations for the scalar fields:

$$\Box\varphi+\frac{1}{3}\left(2V-\psi V_\psi-\varphi V_\varphi\right)=\frac{\kappa^2 T}{3}, \tag{3.32}$$

$$\Box\psi-\frac{1}{2\psi}\partial^\mu\psi\partial_\mu\psi-\frac{\psi}{3}\left(V_\varphi+V_\psi\right)=0. \tag{3.33}$$

These last equations reveal an interesting feature of the theory, the fact that only one of the scalar fields, in this case φ, couples to matter. This fact will have important consequences in the search for cosmological solutions in the presence of matter fields as can be seen in [67].

Chapter 4

Cosmic Strings

> "Nothing novel or interesting happens unless it is on the border between order and chaos."
>
> R. A. Delmonico

Inspired by the success of electroweak theory [69–71], which unifies the weak and electromagnetic interaction under the gauge group $SU(2) \times U(1)$ at a scale of around $10^2 GeV$, Grand Unified Theories (GUT) propose the unification of electroweak and strong interactions under a more general symmetry that takes place at higher energy scales, around $10^{16} GeV$, the Grand Unification Scale. GUT theories are supported by the observation that the coupling "constants" of the Standard Model for Particle Physics seem to slowly vary with the energy scale, converging to a common value at the Grand Unification Scale [72].

These symmetries presented at higher energies are spontaneously broken as the system lowers it's energy state. In several GUT scenarios proposed, a universal covering group G, which would be effective above the GUT scale, would spontaneously break into the Standard Model $SU(3) \times SU(2) \times U(1)$, where $SU(3)$ is the symmetry group of quantum chromodynamics, describing the strong interaction, and $SU(2) \times U(1)$ is the aforementioned electroweak group.

These phase transitions may have left behind some relics that can help shed some light into earlier times of our Universe [73]. These relics are known as topological defects and are a well known, and studied, phenomena in physics, particularly in the context of condensed matter (namely metal crystallization [74], liquid crystals [75, 76], superfluid helium-3 [77] and helium-4 [78], and superconductivity [79]).

The underlying idea behind topological defect formation is the one of Spontaneous symmetry breaking, which is the principle behind the Higgs-Englert mechanism [80].

Spontaneous Symmetry Breaking occurs when a system that is governed by laws which are invariant under a set of transformations evolve to a state that does not respect the same invariant properties. To illustrate such a transition, we will make use of a simple Goldstone model [81] for a complex field ϕ with a Lagrangian density of the form:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \bar{\phi} - \frac{\lambda}{4}\left(|\phi|^2 - \eta^2\right)^2 \tag{4.1}$$

where λ and η are positive constants. This lagrangian possesses a global $U(1)$ symmetry, remaining invariant under global transformations of the $U(1)$ group $\phi(x) \longrightarrow e^{i\alpha}\phi(x)$ (global since α is a constant).

The potential term in eq.(4.1) corresponds to the "mexican hat" potential (Fig. 4.1). This potential has a circle of minima ($|\phi| = \eta$) and therefore the vacuum state of the field corresponds to a value of $\phi = \eta e^{i\theta}$, where θ is an arbitrary phase. Once the field reaches this VEV (Vacuum Expectation Value), transformations of the $U(1)$ group no longer leave the ground state invariant, but instead result in a change in phase $\theta \to \theta + \alpha$. The symmetry is said to be spontaneously broken.

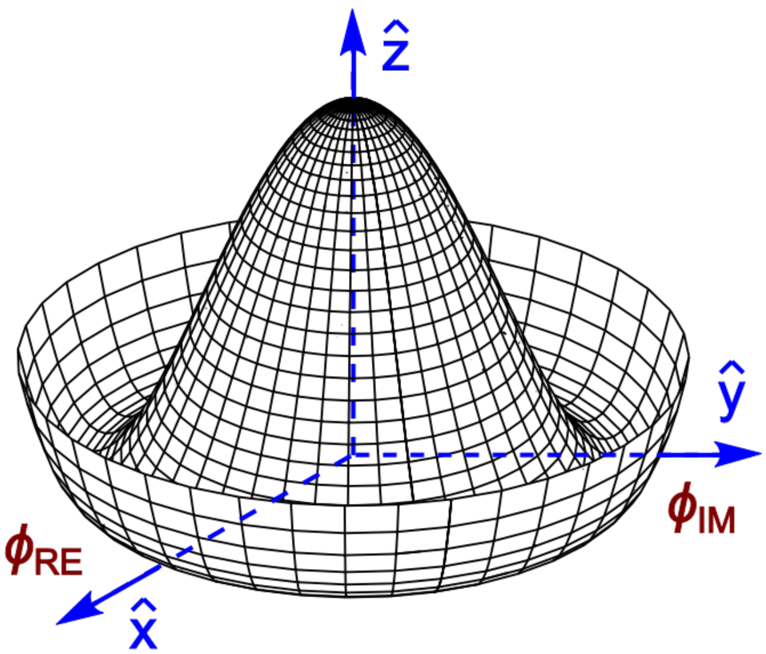

Figure 4.1: Mexican Hat potential

To understand how this phase transition can happen, we will use the temperature-dependent effective potential for the Goldstone model [82]:

$$V_{eff}(\phi, T) = \frac{\lambda}{12}\left(T^2 - 6\eta^2\right)|\phi|^2 + \frac{\lambda}{4}|\phi|^4 . \tag{4.2}$$

At sufficiently high temperatures, the first term of the effective potential is positive and the expectation value of the field is equal to 0. The field is thus in its symmetric state. However, if the temperature decreases below the critical value of $T_c = \sqrt{6}\eta$, the first term becomes negative and the value of the field that minimizes the potential is no longer zero, but instead

$$|\phi| = \frac{1}{\sqrt{6}}\left(T_c^2 - T^2\right)^{1/2} . \tag{4.3}$$

Hence, when the system cools and reaches a temperature below the critical temperature, a phase transition happens and, as a consequence, the field no longer has a zero VEV and is no longer in its symmetric state. A spontaneous symmetry breaking occurred.

We observe that in the new VEV of the field, in which the modulus of ϕ is defined, the phase θ is left undefined. The choice of this phase will depend on random fluctuations at the time of the phase transition. Correlations cannot be established at a speed greater than the speed of light. Therefore, regions that are causally disconnected at the time of the phase transition are "free" to choose different phases of the scalar field ϕ.

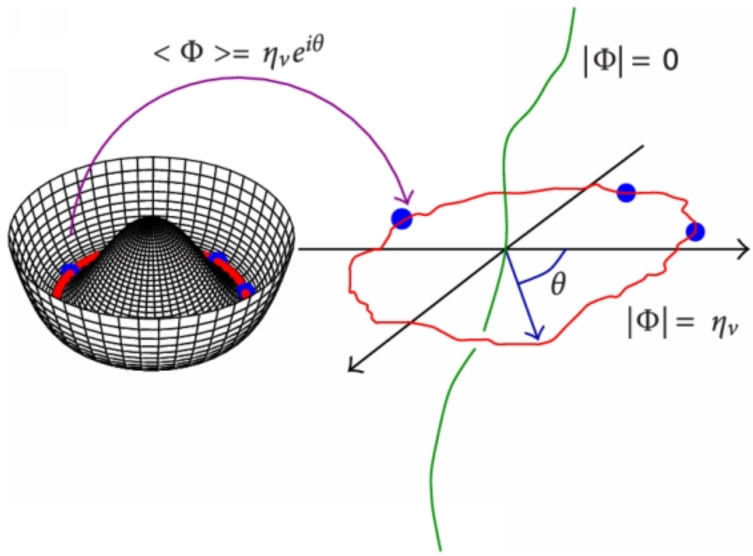

Figure 4.2: The formation of a cosmic string (green).

This mechanism, known as Kibble mechanism [83], is responsible for the surge of topological defects in cosmological phase transitions. The requirement that ϕ is continuous in space, will lead to patches of the manifold that are not in the ground state. In fact, in 2D inside a closed path where the phase changes by 2π, there is a region of trapped energy where the phase of the field is not defined, as can be seen schematically in Figure 4.2. This is a topological defect. The generalization for a spatial 3D manifold is a line-like defect known as cosmic string. But these are not the only defects that can be formed in the wake of a cosmological phase transition. The type of defects that can be formed are linked to the topology of the minima manifold formed after a $G \longrightarrow K$ symmetry breaking, in particular the non-triviality of the n-th homotopy group of the quotient space G/K: for $n = 0$ we have domain walls, for $n = 1$ cosmic strings, for $n = 2$ monopoles, for $n = 3$ textures are formed (for a detailed review see [84]).

The type of strings to be considered in this work are local $U(1)$ cosmic strings, which are an extension of the global $U(1)$ strings to include gauge fields. Local strings differ from the global cosmic strings in what concerns the symmetry that is effective above the spontaneous breaking scale; in the case of local strings, the lagrangian remains invariant under local transformations of the type $\phi(x) \longrightarrow e^{i\alpha(x)}\phi(x)$, where α in no longer a constant, but a more general function. This Abelian-Higgs model can be cast as a lagrangian of the form

$$\mathcal{L}_m = (D_\mu\phi)(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\sigma}{2}\left(\phi\phi^* - \eta^2\right)^2 \tag{4.4}$$

where $D_\mu\phi = \nabla_\mu\phi - ieA_\mu\phi$ is the covariant derivative associated with the complex scalar field. The field strength tensor is $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ of the $U(1)$ gauge potential A_μ with the coupling constant e. The parameter σ denotes the self-coupling of the scalar field, whereas η is its vacuum expectation value.

It is at this point worth mentioning that the production of networks of topological defects as a consequence of cosmological phase transitions is ubiquitous in grand unified scenarios [73]. Although all types of topological defects are interesting and worthy of study, in this work we decided to focus on cosmic strings for several reasons: unlike other types of defects that are inherently unstable [84], stable cosmic strings solutions have been found; and unlike local monopoles or domain walls [85], cosmic strings do not tend to dominate energetically the evolution of the Universe, making them more benign from a cosmological point of view; on the other hand, the presence of cosmic strings can have important cosmological consequences including potentially relevant contributions to Cosmic Microwave Background (CMB) anisotropies [86], small scale structure formation [87], the reionization history of the Universe [88], gravitational lensing observations [89], Gamma ray bursts [90], as well as the formation of super-massive black holes in the early universe [91]. Cosmic strings are also expected to contribute to the stochastic gravitational waves background [92],as a consequence of cosmic strings interactions, which produce closed loops that oscillate under their tension and decay emitting gravitational radiation [93, 94].

Chapter 5

Cosmic Strings in hybrid metric-Palatini gravity

The study of cosmic strings in the context of modified gravity theories is not new, as several works have been conducted for other modified gravity theories. String-type solutions have been studied in the framework of Brans-Dicke theory [95], general scalar-tensor theories [96–99] and $f(R, L_m)$ gravity [100, 101], just to cite a few examples. In this first part, we will analyze local $U(1)$ strings in the context of the hybrid metric-Palatini gravity.

5.1 Metric and equations of motion

For the study of local $U(1)$ string in the context of the hybrid metric-Palatini gravity, we will start by defining the energy-momentum tensor of a straight, infinite cosmic string along the z-direction. Following Vilenkin's prescription [102], one possible energy-momentum tensor is

$$T^r_r = T^t_t = -\sigma(r) \tag{5.1}$$

where $\sigma(r)$ is the linear energy density of the string. The studies done on the dynamics of cosmic strings show us that, although an infinite string is possible, a straight string is a rather more unrealistic assumption, since the interaction of the string with itself and with other strings might produce "kinks" and loops. Additionally, the energy-momentum tensor defined in eq.(5.1) is not completely general, as it ignores the pressure components perpendicular to the direction of the string that will not be, in general, null [103]. Nevertheless, we chose this energy-momentum tensor because it allow us some simplifications and will be a good starting point for other, more complex, string configurations.

We consider a general cylindrically symmetric static metric

$$ds^2 = -e^{(K-U)}dt^2 + e^{2(K-U)}dr^2 + e^{-2U}W^2d\theta^2 + e^{2U}dz^2 \tag{5.2}$$

Where t, r, θ and z denote the time, radial, angular and axial cylindrical coordinates respectively, and K, U and W are functions of r alone.

Inserting the metric (5.2) into the gravitational field equation (3.16) provides the following components

$$\begin{aligned}(1+\phi)\left(-U'^2 + K'\frac{W'}{W} - \frac{W''}{W}\right) &= \phi'' - \frac{3}{4\phi}\phi'^2 \\ -\left(K' - U' - \frac{W'}{W}\right)\phi' &+ \left(\kappa^2\sigma + \frac{1}{2}V\right)e^{2(K-U)}\,,\end{aligned} \tag{5.3}$$

$$(1+\phi)\left(-U'^2 + K'\frac{W'}{W}\right) = -\frac{3}{4\phi}\phi'^2 - \left(K' - U' + \frac{W'}{W}\right)\phi' - \frac{1}{2}Ve^{2(K-U)}\,, \tag{5.4}$$

$$(1+\phi)\left(U'^2 + K''\right) = -\phi'' + \frac{3}{4\phi}\phi'^2 - U'\phi' - \frac{1}{2}Ve^{2(K-U)}\,, \tag{5.5}$$

$$\begin{aligned}(1+\phi)\left(U'^2 + K'' - 2U'' - 2U'\frac{W'}{W} + \frac{W''}{W}\right) &= -\phi'' \\ +\frac{3}{4\phi}\phi'^2 + \left(U' - \frac{W'}{W}\right)\phi' &- \left(\kappa^2\sigma + \frac{1}{2}V\right)e^{2(K-U)}.\end{aligned} \tag{5.6}$$

Additionally, we can use Eq. (3.18) to determine the effective Klein-Gordon equation for the scalar field ϕ:

$$e^{-2(K-U)}\left(-\phi'' - \frac{W'}{W}\phi' + \frac{\phi'^2}{2\phi}\right) + \frac{\phi}{3}\left[2V - (\phi+1)V_{,\phi}\right]\frac{2\phi\kappa^2\sigma}{3} = 0\,. \tag{5.7}$$

Since in this model the matter field couples minimally with curvature, it's possible to show that the energy conservation equation still holds, i.e.,

$$\nabla_\mu T^\mu{}_\nu = 0 \tag{5.8}$$

which yields $K'\sigma = 0$, and apart from the trivial vacuum solution, $\sigma = 0$, this implies that $K' = 0$. Thus, we consider from now on that $e^K = 1$, so that Eqs. (5.3)–(5.6) simplify to the following relations

$$
\begin{aligned}
(1+\phi)\left(-U'^2 - \frac{W''}{W}\right) &= \phi'' - \frac{3}{4\phi}\phi'^2 + \left(U' + \frac{W'}{W}\right)\phi' \\
&+ \left(\kappa^2\sigma + \frac{1}{2}V\right)e^{-2U}\,,
\end{aligned}
\tag{5.9}
$$

$$
(1+\phi)U'^2 = \frac{3}{4\phi}\phi'^2 + \left(-U' + \frac{W'}{W}\right)\phi' + \frac{1}{2}Ve^{-2U}\,, \tag{5.10}
$$

$$
(1+\phi)U'^2 = -\phi'' + \frac{3}{4\phi}\phi'^2 - U'\phi' - \frac{1}{2}Ve^{-2U}\,, \tag{5.11}
$$

$$
\begin{aligned}
(1+\phi)\left(U'^2 - 2U'' - 2U'\frac{W'}{W} + \frac{W''}{W}\right) &= -\phi'' + \frac{3}{4\phi}\phi'^2 \\
+ \left(U' - \frac{W'}{W}\right)\phi' - \left(\kappa^2\sigma + \frac{1}{2}V\right)e^{-2U}\,,
\end{aligned}
\tag{5.12}
$$

With the effective Klein-Gordon equation for the scalar field ϕ reducing to

$$
e^{2U}\left(-\phi'' - \frac{W'}{W}\phi' + \frac{\phi'^2}{2\phi}\right) + \frac{\phi}{3}\left[2V - (\phi+1)V_{,\phi}\right] + \frac{2\phi\kappa^2\sigma}{3} = 0\,. \tag{5.13}
$$

If we further consider that local gauge strings preserve boost invariance along the t and z directions [102], meaning, in this case, that $U = 0$, the gravitational field equations simplify considerably:

$$
(1+\phi)\left(-\frac{W''}{W}\right) = \phi'' - \frac{3}{4\phi}\phi'^2 + \frac{W'}{W}\phi' + \kappa^2\sigma + \frac{1}{2}V\,, \tag{5.14}
$$

$$
0 = \frac{3}{4\phi}\phi'^2 + \frac{W'}{W}\phi' + \frac{1}{2}V\,, \tag{5.15}
$$

$$
0 = -\phi'' + \frac{3}{4\phi}\phi'^2 - \frac{1}{2}V\,, \tag{5.16}
$$

$$
(1+\phi)\frac{W''}{W} = -\phi'' + \frac{3}{4\phi}\phi'^2 - \frac{W'}{W}\phi' - \kappa^2\sigma - \frac{1}{2}V\,. \tag{5.17}
$$

where we can see now that Eqs. (5.14) and (5.17) become redundant. Combining Eqs. (5.15) and (5.16) yields the following relation for the potential V:

$$
V = -\phi'' - \frac{W'}{W}\phi'\,, \tag{5.18}
$$

substituting into the Klein-Gordon equation (5.13), the latter reduces to:

$$V\left(3+2\phi\right)-V_{\phi}\phi\left(\phi+1\right)+2\kappa^{2}\sigma\phi+\frac{3\phi'^{2}}{2\phi}=0\,. \tag{5.19}$$

Additionally, we can further deduce:

$$\kappa^{2}\sigma=\frac{1}{W}\left[(1+\phi)W'\right]'\,, \tag{5.20}$$

and

$$\frac{\left[(1+\phi)W\right]''}{W}=-\left(V+\kappa^{2}\sigma\right). \tag{5.21}$$

An important physical parameter characterizing the cosmic string properties is the mass per unit length of the string, which is defined as

$$\begin{aligned} m(r) &= \int_{0}^{2\pi}d\theta\int_{0}^{R_s}\sigma(r)W(r)dr \\ &= 2\pi\int_{0}^{R_s}\sigma(r)W(r)dr, \end{aligned} \tag{5.22}$$

where R_s is the string radius.

The set of the previous equations allows us to write the gravitational equations of a cosmic string in hybrid metric-Palatini gravity in an exact(closed) form, with all the geometric and physical quantities expressed in a parametric form, in ϕ taken as the parameter.

By taking into account Eq. (5.16), the field equations (5.14) and (5.17) reduce to the form

$$\left(1+\phi\right)\frac{W''}{W}=-\frac{W'}{W}\phi'-\kappa^{2}\sigma. \tag{5.23}$$

5.1.1 Parametric form of the dynamical equations

From a mathematical point of view Equation (5.16) is independent of the metric tensor coefficient W and it represents a second order nonlinear differential equation. In order to solve it we first rescale the radial coordinate r according to the transformation $r=\beta\xi$. Hence Eq. (5.16) takes the form

$$\frac{d^{2}\phi}{d\xi^{2}}-\frac{3}{4\phi}\left(\frac{d\phi}{d\xi}\right)^{2}+\frac{1}{2}\beta^{2}V(\phi)=0. \tag{5.24}$$

In order to solve Eq. (5.24) we introduce the transformations

$$\frac{d\phi}{d\xi}=u,\quad \frac{d^{2}\phi}{d\xi^{2}}=\frac{du}{d\xi}=\frac{du}{d\phi}\frac{d\phi}{d\xi}=u\frac{du}{d\phi}=\frac{1}{2}\frac{d}{d\phi}u^{2}, \tag{5.25}$$

and

$$u^2 = v, \tag{5.26}$$

respectively. Then Eq. (5.24) becomes a first order linear differential equation of the form

$$\frac{dv}{d\phi} - \frac{3}{2\phi}v + \beta^2 V(\phi) = 0, \tag{5.27}$$

with the general solution given by

$$v(\phi) = \phi^{3/2}\left[C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi\right], \tag{5.28}$$

where C is an arbitrary constant of integration. We immediately obtain

$$u(\phi) = \phi^{3/4}\sqrt{\left[C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi\right]}, \tag{5.29}$$

and

$$\xi + C_0 = \int \frac{\phi^{-3/4} d\phi}{\sqrt{\left[C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi\right]}}, \tag{5.30}$$

respectively, where C_0 is an arbitrary constant of integration.

Equation (5.15) can be successively transformed as

$$\frac{1}{W}\frac{dW}{d\xi}\frac{d\phi}{d\xi} = -\frac{3}{4\phi}\left(\frac{d\phi}{d\xi}\right)^2 - \frac{\beta^2}{2}V(\phi), \tag{5.31}$$

and

$$\begin{aligned}\frac{1}{W}\frac{dW}{d\phi} &= -\frac{3}{4\phi} - \frac{\beta^2}{2}\frac{\phi^{-3/2}V(\phi)}{\left[C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi\right]} \\ &= -\frac{3}{4\phi} + \frac{1}{2}\frac{d}{d\phi}\ln\left[C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi\right],\end{aligned} \tag{5.32}$$

yielding

$$W(\phi) = W_0 \phi^{-3/4}\sqrt{C - \beta^2 \int \phi^{-3/2} V(\phi) d\phi}, \tag{5.33}$$

where W_0 is an arbitrary constant of integration.

As a last step we need to obtain the expression of σ. With the use of Eq. (5.15), then Eq. (5.23) can be rewritten as

$$(1+\phi)\frac{1}{W}\frac{d^2W}{d\xi^2} = \frac{3}{4\phi}\left(\frac{d\phi}{d\xi}\right)^2 + \frac{1}{2}\beta^2 V(\phi) - \beta^2\kappa^2\sigma. \tag{5.34}$$

With the use of the mathematical identities

$$\frac{dW}{d\xi} = \frac{dW}{d\phi}\frac{d\phi}{d\xi} = \frac{dW}{d\phi}u, \tag{5.35}$$

$$\frac{d^2W}{d\xi^2} = \frac{d^2W}{d\phi^2}v + \frac{1}{2}\frac{dW}{d\phi}\frac{dv}{d\phi}, \tag{5.36}$$

Eq. (5.34) takes the form

$$(1+\phi)\left(\frac{1}{W}\frac{d^2W}{d\phi^2}v + \frac{1}{2}\frac{1}{W}\frac{dW}{d\phi}\frac{dv}{d\phi}\right) = \frac{3}{4\phi}v + \frac{1}{2}\beta^2V(\phi) - \beta^2\kappa^2\sigma. \tag{5.37}$$

Finally, after some simple calculations we obtain

$$\begin{aligned}\kappa^2\sigma(\phi) =& \frac{1}{4\phi}\left\{\left[2\phi(\phi+1)V'(\phi) + 3\sqrt{\phi}\int\frac{V(\phi)}{\phi^{3/2}}\,d\phi\right.\right.\\ &\left.\left. -2(2\phi+3)V(\phi)\right] - 3\left(C/\beta^2\right)\sqrt{\phi}\right\}.\end{aligned} \tag{5.38}$$

Equation (5.30), (5.33) and (5.38) give the complete solution of the field equations describing the geometry of a cosmic string in hybrid metric-Palatini gravity. The solution is obtained in a parametric form, with ϕ taken as a parameter. It also contains three arbitrary integration constants ξ_0, C, and W_0 which must be obtained from the initial or boundary conditions imposed on the cosmic string configuration.

As for the mass of the string, in the dimensionless variable ξ it can be obtained as

$$m(\xi) = 2\pi\beta\int_0^{\xi_s}\sigma(\xi)W(\xi)d\xi, \tag{5.39}$$

where $\xi_s = R_s/\beta$.

In order to study specific cosmic string solutions in the hybrid metric-Palatini framework, we need now to specify the form of the potential $V(\phi)$.

5.2 String solutions for specific scalar potentials

In this section we will investigate the application of the set of parametric equations deduced on the previous section to different potential configurations.

5.2.1 Constant potential V=V$_0$

The first type of potential to be studied is the constant potential. Given a constant potential, we can separate two diferent cases, the one where the potential is identically zero, and the one where it is a constant different from zero.

5.2.1.1 Constant potential V=V$_0$=0

In the particular case $V = 0$ the field equations describing the cosmic string configuration can be solved exactly. From Eq. (5.30) we immediately obtain

$$\phi(\xi) = C\left(\xi - 4C_0\right)^4. \tag{5.40}$$

The initial conditions $\phi(0) = \phi_0$ and $\phi'(0) = \phi'_0$ fix the constants C_0 and C as

$$C_0 = -\frac{\phi_0}{\phi'_0}, \tag{5.41}$$

$$C = \frac{{\phi'_0}^4}{256{\phi_0}^3}. \tag{5.42}$$

For the metric tensor component W we find

$$W^2(\xi) = \frac{{w_0}^2}{\left(\xi - 4C_0\right)^6}. \tag{5.43}$$

This metric tensor component does not satisfy the condition $W(0) = 0$.On the string axis the metric takes the finite value $W^2(0) = {W_0}^2 = {w_0}^2/4096{C_0}^6 = {w_0}^2{\phi'_0}^6/4096{\phi_0}^6$, a condition that fixes the integration constant ${w_0}^2$ as ${w_0}^2 = 4096{W_0}^2{\phi_0}^6/{\phi'_0}^6$. As for the energy density σ of the string, it is given by

$$\kappa^2\sigma(\xi) = -\frac{12}{\left(\xi - 4C_0\right)^2}. \tag{5.44}$$

Both the metric and the energy density are singular at $\xi = 4C_0$. However, if $C_0 = -\phi_0/\phi'_0 < 0$, implying that both ϕ_0 and ϕ'_0 are positive, there is no infinite type singularity in the metric or energy density. The metric tensor and the energy density are monotonically increasing functions of the distance, while the scalar field is an increasing function of the radial coordinate, becoming infinite for $\xi \to \infty$.

As for the mass of the string, it is obtained as

$$m = 24\pi\beta W_0\left[\frac{1}{1024{C_0}^4} - \frac{1}{4(\xi_s - 4C_0)^4}\right], \tag{5.45}$$

where ξ_s is the string radius. If $\xi_s = 4C_0$, the total mass of the string is (negative) infinite. On the other hand, for $C_0 < 0$, the string extends to infinity, but its mass is finite, taking the value

$$m = \frac{3\pi\beta W_0}{128{C_0}^4} = \frac{3\pi\beta W_0{\phi'_0}^4}{128{\phi_0}^4}. \tag{5.46}$$

5.2.1.2 Constant potential $V = V_0 \neq 0$

We will proceed now to the general case of a constant potential, $V = V_0 \neq 0$. Moreover, we will choose the scaling parameter of the radial coordinate r so that $\beta^2 V_0 = 1$, giving $\beta = 1/\sqrt{V_0}$, and $\xi = \sqrt{V_0} r$. Then the variation of the scalar field as a function of the radial coordinate is obtained from Eq. (5.30) as

$$\xi + C_0 = \int \frac{\phi^{-3/4} d\phi}{\sqrt{C + 2\phi^{-1/2}}}, \tag{5.47}$$

giving

$$C\left(\xi + C_0\right) = 4\sqrt[4]{\phi}\sqrt{C + \frac{2}{\sqrt{\phi}}}, \tag{5.48}$$

and

$$\phi(\xi) = \frac{\left[C^2\left(\xi + C_0\right)^2 - 32\right]^2}{256 C^2}, \tag{5.49}$$

The integration constants C_0 and C must be determined from the initial conditions $\phi\left(\xi_0\right) = \phi_0$ and $\phi'\left(\xi_0\right) = \phi_0'$, respectively, and they are given by

$$C_0 = \pm \frac{2\phi_0 - {\phi_0}'^2}{{\phi_0}^{3/2}}, \qquad C = \frac{4\,\phi_0\,\phi_0'}{{\phi_0}'^2 - 2\phi_0} - \xi_0. \tag{5.50}$$

The variation of the scalar field is represented in Fig. 5.1. For large values of the radial coordinate $r = \xi/\sqrt{V_0}$ the scalar field is a monotonically decreasing function and, at large distances from the string, it reaches the value zero. The variation of ϕ is strongly dependent, from a quantitative point of view, on the initial conditions for the field on the string axis. For large values of ϕ_0' and near the axis of the string, the scalar field is an increasing function and, after reaching a maximal value at a finite r, ϕ begins to decrease tending towards zero for very large values of r.

For the metric tensor coefficient W we obtain

$$W(\phi) = W_0 \phi^{-3/4}\sqrt{C + 2\phi^{-1/2}}, \tag{5.51}$$

or

$$W(\xi) = \frac{64 C^3 W_0\left(\xi + C_0\right)}{\left[C^2\left(\xi + C_0\right)^2 - 32\right]^2}. \tag{5.52}$$

For $\xi = 0$ (on the string axis) we have

$$W(0) = \frac{64 C^3 W_0 C_0}{\left(C^2 {C_0}^2 - 32\right)^2} \tag{5.53}$$

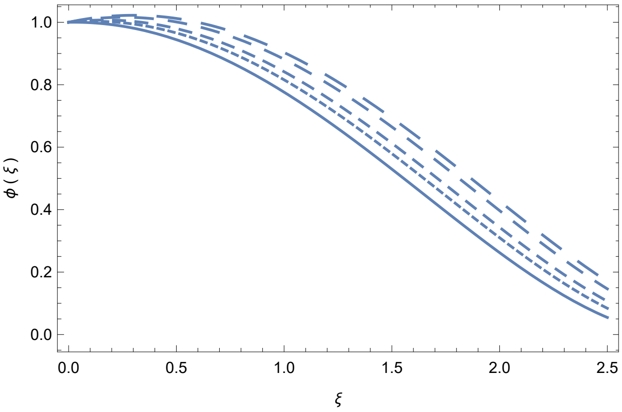

Figure 5.1: Variation of the scalar field of the cosmic string configuration in hybrid metric-Palatini gravity in the presence of a constant potential for $\phi(0) = \phi_0 = 1$, and for different values of ϕ_0': $\phi_0' = 0.012$ (solid curve), $\phi_0' = 0.056$ (dotted curve), $\phi_0' = 0.084$ (short dashed curve), $\phi_0' = 0.126$ (dashed curve), and $\phi_0' = 0.148$ (long dashed curve), respectively.

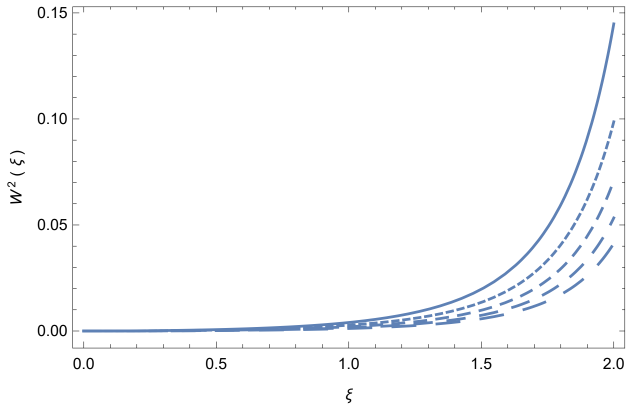

Figure 5.2: Variation of the metric tensor component $W^2(\xi)$ of the cosmic string configuration in the hybrid metric-Palatini gravity in the presence of a constant potential for $\phi\left(10^{-7}\right) = 1$, $W\left(10^{-7}\right) = 10^{-3}$, and for different values of ϕ_0': $\phi_0' = 0.010$ (solid curve), $\phi_0' = 0.012$ (dotted curve), $\phi_0' = 0.014$ (short dashed curve), $\phi_0' = 0.016$ (dashed curve), and $\phi_0' = 0.018$ (long dashed curve), respectively.

The condition $W(0) = 0$ would require to take $C_0 = 0$, which imposes the relation $2\phi_0 = {\phi_0}'^2$ between the initial values of the field and of its derivative on the string axis. But if this relation is satisfied, as can be seen immediately from the second of the Eqs. (5.50), the constant C is undefined, and diverges for $\xi = 0$. Therefore, the metric tensor is not defined on the string axis $r = 0$. The variation of the metric tensor coefficient $W^2(\xi)$ is represented, for $\phi\left(10^{-7}\right) = 1$, $W\left(10^{-7}\right) = 10^{-3}$, and for different values of ϕ_0', in Fig. 5.2.

The metric tensor component is divergent for $C^2\left(\xi + C_0\right)^2 - 32 = 0$, which gives for the value of the singular point ξ_∞ the expression

$$\xi_\infty = \left({\phi_0}'^2 - 2\phi_0\right)\left[\frac{4\sqrt{2}}{4\phi_0\phi_0' - \xi_0\left({\phi_0}'^2 - 2\phi_0\right)} \pm \frac{1}{{\phi_0}^{3/2}}\right]. \tag{5.54}$$

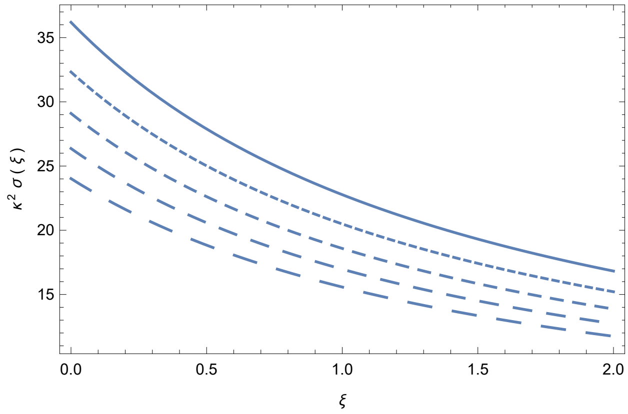

Figure 5.3: Variation of the energy density $\kappa^2\sigma(\xi)$ of the cosmic string configuration in hybrid metric-Palatini gravity in the presence of a constant potential for $\phi(0) = 1$, and for different values of ϕ_0': $\phi_0' = 1.272$ (solid curve), $\phi_0' = 1.258$ (dotted curve), $\phi_0' = 1.244$ (short dashed curve), $\phi_0' = 1.230$ (dashed curve), and $\phi_0' = 1.216$ (long dashed curve), respectively.

The position of the singular point is essentially determined by the initial values of the scalar field and of its derivative near the string axis. At the metric singularity the scalar field vanishes, as one can see immediately from Eq. (5.49). However, a different physical behavior is also possible, if near the origin the integration constants C_0 and C satisfy the condition $C^2C_0^2 \gg 32$, or, equivalently, $\phi_0' \gg 2\phi_0$. In this case the metric tensor can be approximated as

$$W^2(\xi) \approx \frac{4096{W_0}^2}{C^2\left(\xi + C_0\right)^6}. \tag{5.55}$$

For $\xi \to \infty$, $W^2(\xi) \to 0$, and there are no infinity type singularities in the metric. However, a zero type singularity in the metric cannot be avoided even for this choice of the initial conditions.

As for σ, we easily find the expression

$$\kappa^2\sigma(\phi) = V_0\left(-\frac{3C}{4\sqrt{\phi}} - \frac{(\phi+3)}{\phi}\right), \tag{5.56}$$

or

$$\begin{aligned}\kappa^2\sigma(\xi) \quad = \quad & \frac{V_0}{\left[C^2\left(\xi+C_0\right)^2 - 32\right]^2}\Bigg\{C^2\Big\{\left(\xi+C_0\right)^2 \times \\ & \left[-\left(C^2\left(\left(\xi+C_0\right)^2+12\right)-64\right)\right] - 384\Big\} - 1024\Bigg\}.\end{aligned} \tag{5.57}$$

The variation of $\sigma(\xi)$ is represented, for $\phi(0) = 1$ and different values of $\phi'(0)$ in Fig. 5.3.

In order to have positive values of σ the integration constant C must be negative, $C < 0$, a condition that imposes some strong constraints on the initial values of the scalar field, and its derivative. The energy density of the string is a monotonically decreasing function of the distance and, at least for the present choice of the initial conditions, it does not have any singularities.

For the total mass of the string we obtain the expression

$$\begin{aligned} m &= \frac{64\pi}{\kappa^2}\beta V_0 C W_0 \left\{ -\frac{{C_0}^2\left({C_0}^2+24\right)C^4-64\left({C_0}^2+8\right)C^2+1024}{\left({C_0}^2C^2-32\right)^3} \right. \\ &+ \left. \frac{C^4(C_0+\xi_s)^2\left[(C_0+\xi_s)^2+24\right]-64C^2\left[(C_0+\xi_s)^2+8\right]+1024}{\left[C^2(C_0+\xi_s)^2-32\right]^3} \right\}. \end{aligned} \tag{5.58}$$

The mass is divergent for ${C_0}^2C^2 = 32$, and for $C^2(C_0+\xi_s)^2 \to 32$. If the string radius tends to infinity, the mass of the string is finite, and it is given by

$$\lim_{\xi_s\to\infty} m = \frac{64\pi\beta C W_0}{\left(32-{C_0}^2C^2\right)^3} \times \left[{C_0}^2\left({C_0}^2+24\right)C^4-64\left({C_0}^2+8\right)C^2+1024\right], \tag{5.59}$$

where we have assumed that ${C_0}^2C^2 < 32$.

5.2.2 Power law potential $V = V_0\phi^{3/4}$

The gravitational field equations describing a cosmic string in hybrid metric-Palatini gravity also admit another exact solution, corresponding to the power law type scalar field potential $V(\phi) = V_0\phi^{3/4}$. We rescale the radial coordinate r by imposing the condition $\beta^2 V_0 = 1$, which gives $r = \xi/\sqrt{V_0}$. With these choices from Eq. (5.30) we obtain explicitly the scalar field as a function of ξ, given by

$$\phi(\xi) = \frac{\left(\xi^2{\phi_0}^{3/4} \pm 2\xi\phi_0' \pm 8\phi_0\right)^4}{4096{\phi_0}^3}, \tag{5.60}$$

where we have used the usual initial conditions $\phi(0) = \phi_0$ and $\phi'(0) = \phi_0'$, respectively. For the metric tensor component W we obtain

$$W(\xi) = \frac{W_0}{\left(\xi^2{\phi_0}^{3/4} \pm 2\xi\phi_0' \pm 8\phi_0\right)^3\sqrt{2\xi \pm 2\phi_0'/{\phi_0}^{3/4}}}. \tag{5.61}$$

where W_0 is an arbitrary constant of integration. On the string axis, i.e., $\xi = 0$, we obtain $W^2(0) = \pm{W_0}^2/524288\,{\phi_0}^{21/4}\phi_0'$. Since the metric tensor component W^2 must be positive for all $\xi \geq 0$, it follows that the physical solution for the string configuration is the one with the positive sign. Hence in the case of the $V(\phi) = V_0\phi^{3/4}$ potential, the solutions of the field equations describing a cosmic string in hybrid metric-Palatini gravity are

$$\begin{aligned}
\phi(\xi) &= \frac{\left(\xi^2{\phi_0}^{3/4} + 2\xi\phi_0' + 8\phi_0\right)^4}{4096{\phi_0}^3}, \\
W^2(\xi) &= \frac{{W_0}^2}{\left(\xi^2{\phi_0}^{3/4} + 2\xi\phi_0' + 8\phi_0\right)^6\left(2\xi + 2\phi_0'/{\phi_0}^{3/4}\right)},
\end{aligned} \tag{5.62}$$

respectively, with ${W_0}^2 = 524288\, W^2(0){\phi_0}^{21/4}\phi_0'$, a condition that implies $\phi_0 > 0$ and $\phi_0' > 0$. For the string tension as a function of ϕ and ξ we obtain the expressions

$$\kappa^2\sigma(\phi) = V_0\frac{-6C - 5(\phi - 3)\sqrt[4]{\phi}}{8\sqrt{\phi}}, \tag{5.63}$$

and

$$\begin{aligned}
\kappa^2\sigma(\xi) = \frac{V_0\phi_0^{3/4}}{\left(\xi^2\phi_0^{3/4}+2\xi\phi_0'+8\phi_0\right)^2}\Bigg\{ &-48C\phi_0^{3/4} - 5\left(\xi^2\phi_0^{3/4} + 2\xi\phi_0' + 8\phi_0\right) \times \\
&\left[\frac{(\xi^2\phi_0^{3/4}+2\xi\phi_0'+8\phi_0)^4}{4096\phi_0^3} - 3\right]\Bigg\},
\end{aligned} \tag{5.64}$$

respectively.

In this case, the scalar field is a monotonically increasing function of the radial distance from the string axis and tends to infinity for $\xi \to \infty$. On the other hand, the metric tensor component decreases monotonically from a finite value on the string axis to zero at infinity. For $\xi = 0$, the string tension takes the finite value

$$\sigma(0) = V_0\left[-48C\phi_0^{3/4} - 40\left(\phi_0 - 3\right)\phi_0\right]/64\phi_0^{5/4} \tag{5.65}$$

while $\lim_{\xi\to\infty}\sigma(\xi) = -\infty$, indicating that σ is a monotonically decreasing function of the radial coordinate. In the first order of approximation we obtain for the mass of the string of radius ξ_s the expression

$$\begin{aligned}
m &= \frac{\pi\beta W_0\xi_s}{8192\phi_0^{33/8}\phi_0'^{3/2}}\Bigg\{6C\xi_s\phi_0^{7/4} + 15\xi_s{\phi_0'}^2\left(C - 2\sqrt[4]{\phi_0}\right) \\
&- 4\phi_0\phi_0'\left[6C + 5\left(\phi_0 - 3\right)\sqrt[4]{\phi_0}\right] + 5\xi_s\left(\phi_0 - 3\right)\phi_0^2\Bigg\}.
\end{aligned} \tag{5.66}$$

In this approximation the mass monotonically increases with the string radius.

5.2.3 Exponential potential $V = V_0 e^{-\lambda\phi}$

As another example of a string type configuration, we will consider the configuration generated by an exponential type potential, with $V(\phi) = V_0 e^{-\lambda\phi}$, where V_0 and $\lambda > 0$ are constants. The solutions of the gravitational field equations for different scalar field models with exponential potentials have been intensively investigated in the recent physical literature [104–108], including the cases of both homogeneous and inhomogeneous scalar fields. In four-dimensional effective Kaluza-Klein or string type theories an exponential potential is generated from the compactification of the higher dimensions [109]. Due to the curvature of the internal spaces or to the interaction with form fields on the internal spaces, the moduli fields may acquire exponential type potentials.

In the case of the exponential potential Eq. (5.30) giving the scalar field-radial coordinate dependence becomes

$$\xi + C_0 = \int \frac{\phi^{-3/4} d\phi}{\sqrt{C + 2\sqrt{\pi}\beta^2 V_0 \sqrt{\lambda}\,\mathrm{erf}\left(\sqrt{\lambda\phi}\right) + 2\beta^2 V_0 e^{-\lambda\phi}/\sqrt{\phi}}}, \tag{5.67}$$

where $\mathrm{erf}(x)$ is the error function, and cannot be represented in a closed form, therefore we will use a numerical approach to solve the field equations. We rescale first the scalar field so that $\phi = \Phi/\lambda$, and we choose the scaling parameter β of the radial coordinate as $\beta = \sqrt{2/V_0\lambda}$. Then Eq. (5.24), which gives the variation of the scalar field, takes the form

$$\frac{d^2\Phi}{d\xi^2} - \frac{3}{4\Phi}\left(\frac{d\Phi}{d\xi}\right)^2 + e^{-\Phi} = 0. \tag{5.68}$$

The variation of the metric tensor component W^2 can be obtained from the equation

$$\frac{1}{W}\frac{dW}{d\xi}\frac{d\Phi}{d\xi} = -\frac{3}{4\Phi}\left(\frac{d\Phi}{d\xi}\right)^2 - e^{-\Phi}. \tag{5.69}$$

The behavior of the scalar field with exponential type potential is represented in Fig. 5.4. For the sake of comparison we have chosen the same initial values for the field Φ and for its derivative as in the case of the constant potential.

The variation of the metric tensor component $W^2(\xi)$ is represented in Fig. 5.5.

For the adopted set of initial values the behavior of the scalar field and of the metric is very similar to the constant potential case. The scalar field is a monotonically decreasing function of the distance, and it reaches the value zero at a greater distance from the string axis than in the case of the constant potential. The behavior of the field is strongly

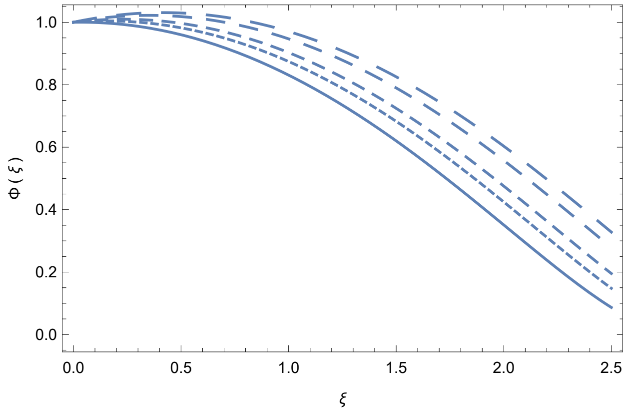

Figure 5.4: Variation of the scalar field of the cosmic string configuration in the presence of an exponential potential $V(\Phi) = e^{-\Phi}$ for $\Phi(0) = \Phi_0 = 1$, and for different values of Φ_0': $\Phi_0' = 0.012$ (solid curve), $\Phi_0' = 0.056$ (dotted curve), $\Phi_0' = 0.084$ (short dashed curve), $\Phi_0' = 0.126$ (dashed curve), and $\Phi_0' = 0.148$ (long dashed curve), respectively.

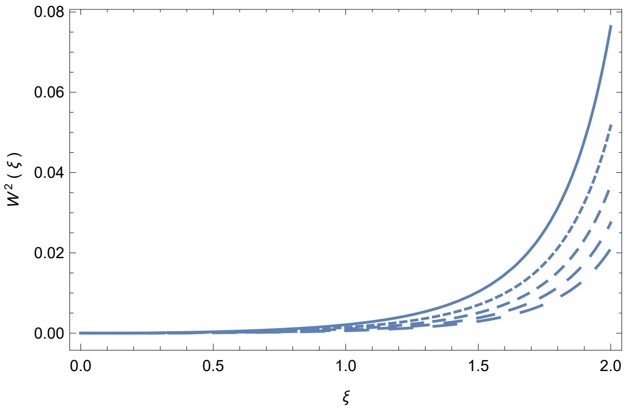

Figure 5.5: Variation of the metric tensor component $W^2(\xi)$ of the cosmic string configuration in the presence of an exponential potential $V(\Phi) = e^{-\Phi}$ for $\Phi(0) = 1$, $W(0) = 10^{-3}$, and for different values of Φ_0': $\Phi_0' = 0.010$ (solid curve), $\Phi_0' = 0.012$ (dotted curve), $\Phi_0' = 0.014$ (short dashed curve), $\Phi_0' = 0.016$ (dashed curve), and $\Phi_0' = 0.018$ (long dashed curve), respectively.

dependent on the initial conditions. The metric tensor is a monotonically increasing function of ξ, and it is defined properly on the string axis. However, it becomes singular at a finite distance from the axis of the string, tending to infinity at a finite ξ. For distances in the range $\xi \in (0,1)$, or $r \in (0, \sqrt{2/\lambda V_0})$, the metric tensor is practically a constant, and its behavior is basically independent on the initial conditions of the scalar field. For the exponential potential the energy density of the string can be obtained generally as a function of the scalar field in the form

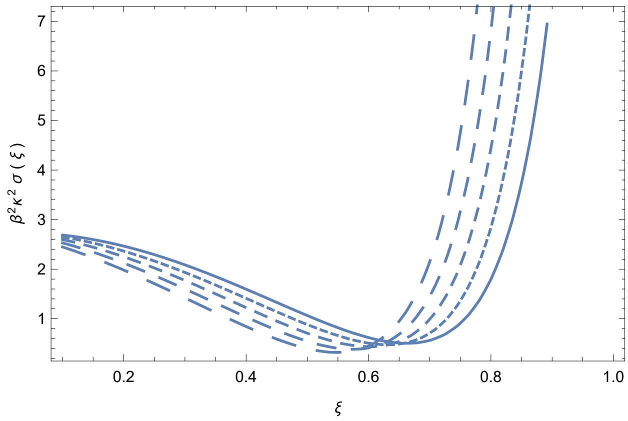

Figure 5.6: Variation of the energy density $\kappa^2\sigma(\xi)$ of the cosmic string configuration in the Hybrid Metric-Palatini Gravity in the presence of an exponential potential $V(\Phi) = e^{-\Phi}$ for $\Phi(0) = 1$, and for different values of Φ_0': $\Phi_0' = -0.20$ (solid curve), $\Phi_0' = -0.30$ (dotted curve), $\phi_0' = -0.40$ (short dashed curve), $\Phi_0' = -0.50$ (dashed curve), and $\Phi_0' = -0.60$ (long dashed curve), respectively.

$$\begin{aligned}\beta^2\sigma(\phi) = & -\frac{3}{4\sqrt{\phi}}\left[C + 2\sqrt{\pi}\beta^2\sqrt{\lambda}V_0\mathrm{erf}\left(\sqrt{\phi\lambda}\right)\right] \\ & -\frac{2\beta^2V_0e^{-\lambda\phi}}{4\phi}\left[\phi(\lambda\phi+\lambda+2)+6\right]. \end{aligned} \tag{5.70}$$

However, since the numerical solutions for $\phi(\xi)$ and $W(\xi)$ are known, it is more convenient to obtain $\sigma(\xi)$ from the equation

$$\beta^2\kappa^2\sigma(x) = -\frac{1+\phi}{W}\left(\frac{d^2W}{d\xi^2} + \frac{1}{1+\phi}\frac{dW}{d\xi}\frac{d\phi}{d\xi}\right). \tag{5.71}$$

The variation of σ as a function of the dimensionless radial coordinate distance is represented in Fig. 5.6.

In order to obtain positive energy densities the initial values of Φ_0' must be negative. There is a significant difference between the behavior of the energy density σ as compared to the constant potential case. The energy density initially decreases for increasing values of the radial coordinate, but for $\xi > \xi_{cr}$, the energy density begins to increase, and tends to infinity, thus experiencing a singularity at large distances from the string axis.

In the first order of approximation, and after rescaling the variable ϕ, the integrand in Eq. (5.67) can be approximated as

$$
\begin{aligned}
&\frac{\Phi^{-3/4}d\Phi}{\sqrt{C+2\sqrt{\pi}\mathrm{erf}\left(\sqrt{\Phi}\right)+2e^{-\Phi}/\sqrt{\Phi}}}\approx\sqrt{2\Phi}-\frac{C\Phi}{4\sqrt{2}}\\
&+\frac{\left(3C^2-16\right)\Phi^{3/2}}{48\sqrt{2}}+\frac{C\left(48-5C^2\right)\Phi^2}{256\sqrt{2}}+O\left(\Phi^{9/4}\right),
\end{aligned}
\tag{5.72}
$$

yielding

$$
\begin{aligned}
\xi+C_0\approx\lambda^{-1/4}\Bigg[&\sqrt{2\Phi}-\frac{C\Phi}{4\sqrt{2}}+\frac{\left(3C^2-16\right)\Phi^{3/2}}{48\sqrt{2}}\\
&+\frac{C\left(48-5C^2\right)\Phi^2}{256\sqrt{2}}+O\left(\Phi^{9/4}\right)\Bigg].
\end{aligned}
\tag{5.73}
$$

However, this equation is not particularly useful in the study of the behavior of the string models with exponential potential. On the other hand at infinity, we obtain

$$
\frac{\Phi^{-3/4}d\Phi}{\sqrt{C+2\sqrt{\pi}\mathrm{erf}\left(\sqrt{\Phi}\right)+2e^{-\Phi}/\sqrt{\Phi}}}\approx\frac{1}{\sqrt{C+2\sqrt{\pi}}\Phi^{3/4}},
\tag{5.74}
$$

which provides

$$
\xi+C_0\approx\lambda^{-1/4}\frac{4\sqrt[4]{\Phi}}{\sqrt{C+2\sqrt{\pi}}},
\tag{5.75}
$$

and

$$
\Phi(\xi)\approx\frac{\lambda}{256}\left(C+2\sqrt{\pi}\right)^2\left(C_0+\xi\right)^4.
\tag{5.76}
$$

Within the framework of this approximation the metric tensor component W is given by the differential equation

$$
\frac{1}{W}\frac{dW}{d\xi}=-\frac{64e^{-\frac{1}{256}\left(C+2\sqrt{\pi}\right)^2\lambda\left(C_0+\xi\right)^4}}{\left(C+2\sqrt{\pi}\right)^2\lambda\left(C_0+\xi\right)^3}-\frac{3}{C_0+\xi},
\tag{5.77}
$$

with the general solution given by

$$
\begin{aligned}
W(\xi) &= \frac{1}{\left(C_0+\xi\right)^3}\exp\Bigg\{2\Bigg[\frac{16e^{-\frac{1}{256}\left(C+2\sqrt{\pi}\right)^2\lambda\left(C_0+\xi\right)^4}}{\left(C_0+\xi\right)^2\left(C+2\sqrt{\pi}\right)^2\lambda}\\
&+\frac{\sqrt{\pi}}{\left(C+2\sqrt{\pi}\right)\sqrt{\lambda}}\mathrm{erf}\left(\frac{1}{16}\left(C+2\sqrt{\pi}\right)\sqrt{\lambda}\left(C_0+\xi\right)^2\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{5.78}
$$

Even in this approximation the full analysis of the behaviour of the cosmic string configuration in hybrid metric-Palatini gravity in the presence of an exponential type potential can only be achieved using numerical methods.

5.2.4 The Higgs-type potential

Next we consider the case where the scalar field potential is of the Higgs type, given by

$$V(\phi) = \pm\frac{\bar{\mu}^2}{2}\phi^2 + \frac{\nu}{4}\phi^4, \tag{5.79}$$

where $\bar{\mu}^2$ and ν are constants. In the following we will investigate only the case with $\bar{\mu}^2 < 0$, meaning we will adopt the minus sign in the definition of the potential. By following the standard approach in elementary particle physics, we assume that the constant $\bar{\mu}^2$ is related to the mass of the scalar field particle as $m_\phi^2 = 2\xi v^2 = 2\bar{\mu}^2$, where $v^2 = \bar{\mu}^2/\xi$ gives the minimum value of the potential. The Higgs self-coupling constant ν can be obtained, in the case of strong interactions, from the determination of the mass of the Higgs boson in laboratory experiments, and its numerical value is of the order of $\nu \approx 1/8$ [110]. By rescaling the radial coordinate and the scalar field according to

$$r = \sqrt{2}\bar{\mu}\xi, \qquad \phi = \frac{\Phi}{(\nu\bar{\mu})^{1/3}}, \tag{5.80}$$

then Eq. (5.30) provides the profile of the scalar field in the following form

$$\frac{d^2\Phi}{d\xi^2} - \frac{3}{4\phi}\left(\frac{d\Phi}{d\xi}\right)^2 - \Phi^2 + \Phi^4 = 0. \tag{5.81}$$

The general solution for this equation is given in a closed form by

$$\xi + C_0 = \int \frac{1}{\Phi^{3/4}\sqrt{C + \frac{2}{21}\left(7 - 3\Phi^2\right)\Phi^{3/2}}}d\Phi. \tag{5.82}$$

However, this solution cannot be expressed in an analytical form in terms of known functions. In the first order approximation, we obtain

$$\xi + C_0 \approx \frac{4\sqrt[4]{\Phi}}{\sqrt{C}} - \frac{4\Phi^{7/4}}{21C^{3/2}} + O\left(\Phi^{9/4}\right), \tag{5.83}$$

but this representation is not particularly useful from the point of view of concrete calculations.

The variation of the scalar field with Higgs potential supporting a string configuration in hybrid metric-Palatini gravity is represented in Fig. 5.7. There is a significant qualitative

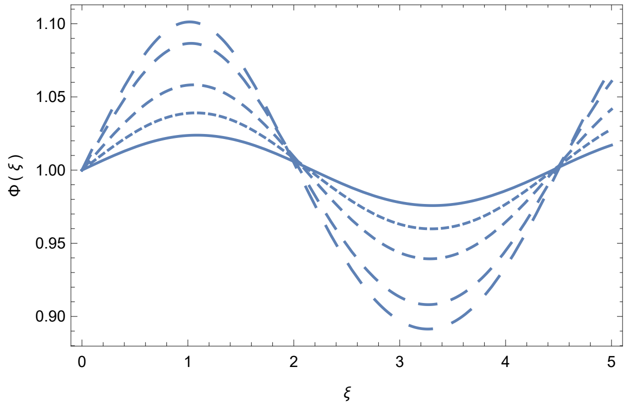

Figure 5.7: Variation of the scalar field for the cosmic string configuration in the presence of a Higgs-type potential $V(\Phi) = -\Phi^2 + \Phi^4$ for $\Phi(0) = \Phi_0 = 1$, and for different values of Φ_0': $\Phi_0' = 0.034$ (solid curve), $\Phi_0' = 0.056$ (dotted curve), $\Phi_0' = 0.084$ (short dashed curve), $\Phi_0' = 0.126$ (dashed curve), and $\Phi_0' = 0.148$ (long dashed curve), respectively.

difference between this string model and the constant, simple power law or exponential potentials. Note that the scalar field for the Higgs-type potential shows a basically periodic structure, changing between successive maxima and minima. There are singularities in the field. Its behavior is strongly affected by the initial conditions on the string axis, and the field extends to infinity.

The variation of the metric tensor component $W^2(\xi)$ in the presence of a Higgs potential is represented in Fig. 5.8. The same oscillatory pattern can also be observed in the case of the metric tensor component W^2. However, there is a difference in the phase of these to quantities. When the field reaches its maximum at $\xi \approx 1$, the metric tends to zero, $W^2(1) \approx 0$. Then, while the scalar field decreases, the metric tensor increases, reaching its maximum at the minimum of the field, corresponding to $\xi \approx 2$. This pattern is repeated up to infinity.

The variation of the string tension with respect to the radial coordinate in the presence of the Higgs potential is depicted in Fig. 5.9. The variation of σ is in phase with that of the scalar field, and both quantities reach their maxima and minima at the same position. The string tension also has an oscillatory behaviour, which is a general property of all physical and geometrical parameters of the string configurations supported by scalar fields with a Higgs-type potential.

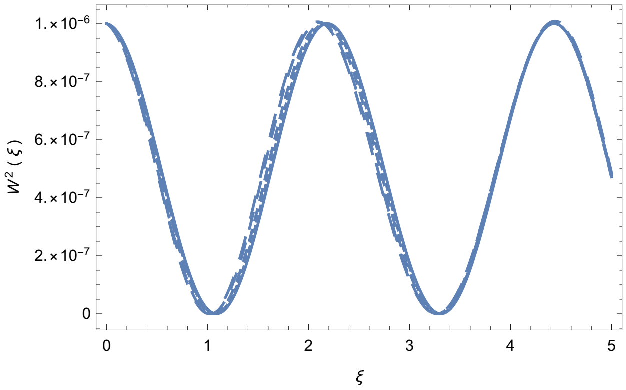

Figure 5.8: Variation of the metric tensor component $W^2(\xi)$ of the cosmic string configuration in the presence of a Higgs type potential $V(\Phi) = -\Phi^2 + \Phi^4$ for $\Phi(0) = 1$, $W(0) = 10^{-3}$, and for different values of Φ_0': $\Phi_0' = 0.034$ (solid curve), $\Phi_0' = 0.056$ (dotted curve), $\Phi_0' = 0.084$ (short dashed curve), $\Phi_0' = 0.126$ (dashed curve), and $\Phi_0' = 0.148$ (long dashed curve), respectively.

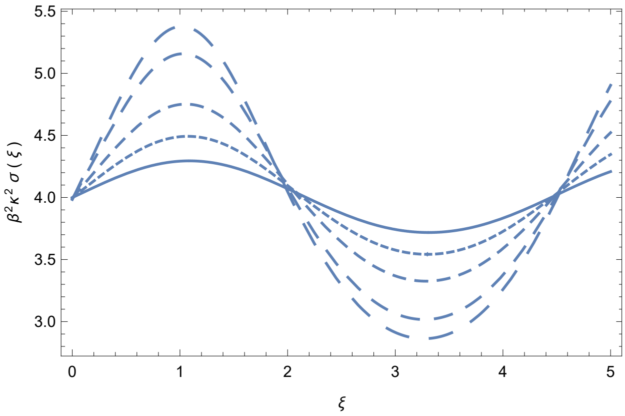

Figure 5.9: Variation of the string tension $\kappa^2\sigma(\xi)$ of the cosmic string configuration in the presence of a Higgs type potential $V(\Phi) = -\Phi^2 + \Phi^4$ for $\Phi(0) = 1$, $W(0) = 10^{-3}$, and for different values of Φ_0': $\Phi_0' = 0.034$ (solid curve), $\Phi_0' = 0.056$ (dotted curve), $\Phi_0' = 0.084$ (short dashed curve), $\Phi_0' = 0.126$ (dashed curve), and $\Phi_0' = 0.148$ (long dashed curve), respectively.

Chapter 6

Cosmic Strings in Generalized hybrid metric-Palatini gravity

We will now turn our attention to the study of local $U(1)$ cosmic strings in the framework of the generalized version of the hybrid metric-Palatini theory, by extending our analysis of the previous section. We will apply the same energy-momentum tensor for the string as before (eq. (5.1)) and we will also use the same general cylindrically symmetric static metric, eq. (5.2).

For this analysis, we will use the dynamical equations for the generalized theory as stated in eq. (3.29), in addition to the modified Klein Gordon equations for both scalar fields, eqs. (3.30) and (3.31).

We can notice from the action of the theory, Eq.(3.28), that the coupling between the scalar fields and the Ricci scalar is the combination $\varphi - \psi$. Since φ and ψ are arbitrary functions, it is not guaranteed that this combination preserves the positivity of the coupling. We thus introduce a redefinition of the scalar field φ as $\xi^2 = \varphi - \psi$. With this redefinition, any solution obtained for which ξ is a real function preserves the positivity of the coupling $(\varphi - \psi)\, R$.

With this redefinition, the equations (3.29), (3.30) and (3.31) become:

$$\xi^2 G_{\mu\nu} = \kappa^2 T_{\mu\nu} + \nabla_\mu \nabla_\nu \xi^2 + \frac{3}{2\psi}\partial_\mu\psi\partial_\nu\psi - \left(\Box\xi^2 + \frac{1}{2}\bar{V} + \frac{3}{4\psi}\partial^\gamma\psi\partial_\gamma\psi\right) g_{\mu\nu}, \tag{6.1}$$

$$\Box\xi^2 + \frac{1}{2\psi}\partial^\mu\psi\partial_\mu\psi + \frac{1}{6}\left(4\bar{V} - \xi\bar{V}_\xi\right) = \frac{\kappa^2 T}{3}, \tag{6.2}$$

$$\Box\psi - \frac{1}{2\psi}\partial^\mu\psi\partial_\mu\psi - \frac{\psi}{3}\left(\frac{1}{2\xi}\bar{V}_\xi + \bar{V}_\psi\right) = 0, \tag{6.3}$$

where $\bar{V}\left(\xi,\psi\right)$ is the potential written in terms of the scalar fields ξ and ψ and the subscript ξ denotes a partial derivative with respect to this scalar field. Finally, one can also obtain a relationship between the potential $\bar{V}$ and the function $f\left(R,\mathcal{R}\right)$ from Eq.(3.26) as

$$\bar{V}\left(\xi,\psi\right)=-f\left(R,\mathcal{R}\right)+\xi^2R+\psi\left(R-\mathcal{R}\right), \tag{6.4}$$

where we have used the fact that the scalar-tensor representation is only defined if $A=R$ and $B=\mathcal{R}$. This equation becomes a PDE for $f\left(R,\mathcal{R}\right)$ by replacing $\psi=f_{\mathcal{R}}$ and $\xi^2=f_R+f_{\mathcal{R}}$.

6.1 Metric and equations of motion

Applying the same requirements as before (conservation of energy and boost invariance along the t and z directions), equation (6.1) provides three independent field equations, which are

$$\xi^2\frac{W''}{W}+2\xi\xi'\frac{W'}{W}+\frac{3\psi'^2}{4\psi}+2\left(\xi'^2+\xi\xi''\right)+\frac{\bar{V}}{2}=-\kappa^2\sigma, \tag{6.5}$$

$$2\xi\xi'\frac{W'}{W}-\frac{3\psi'^2}{4\psi}+\frac{\bar{V}}{2}=0, \tag{6.6}$$

$$2\left(\xi'^2+\xi\xi''\right)+\frac{3\psi'^2}{4\psi}+\frac{\bar{V}}{2}=\frac{d^2}{dr^2}\xi^2+\frac{3\psi'^2}{4\psi}+\frac{\bar{V}}{2}=0\,, \tag{6.7}$$

whereas the scalar field equations for ξ and ψ, provided by Eqs. (6.2) and (6.3), give

$$2\left(\xi'^2+\xi\xi''\right)+2\xi\xi'\frac{W'}{W}+\frac{\psi'^2}{2\psi}+\frac{1}{6}\left(\bar{V}-\xi\bar{V}_\xi\right)=-\frac{2\kappa^2}{3}\sigma, \tag{6.8}$$

$$\psi''+\frac{W'}{W}\psi'-\frac{\psi'^2}{2\psi}-\frac{\psi}{3}\left(\bar{V}_\psi+\frac{1}{2\xi}\bar{V}_\xi\right)=0\,. \tag{6.9}$$

The system of Eqs.(6.5)-(6.9) is a system of five equations from which only four are linearly independent. This statement can be proven by taking a radial derivative of Eq.(6.6), using Eq.(6.5) to cancel the factors W'', using Eq.(6.8) to cancel σ, and finally using Eqs.(6.7) and (6.9) to cancel the second-order derivatives of the scalar fields ξ'' and ψ'', respectively. As a result, one recovers Eq.(6.6), thus proving that the system of equations is linearly dependent. Thus, one only needs to consider four of these equations to completely determine the solution in the sections that follow. Given its complexity, we chose to discard Eq.(6.5) from the analysis.

Furthermore, an equation for the potential $\bar{V}$ can be obtained by summing the field equations in Eqs.(6.6) and (6.7), yielding

$$\bar{V} = -2\left(\xi'^2 + \xi\xi''\right) - 2\xi\xi'\frac{W'}{W}. \tag{6.10}$$

This equation is particularly useful to obtain an equation for W' in terms of the scalar fields ξ and ψ and their derivatives after setting an explicit form of the potential $\bar{V}$, which will be explored later.

The system of basic equations describing the structure of a cosmic string can thus be reformulated in the form of a first-order dynamical system. By defining $\alpha = \xi^2$, and by introducing two extra dynamical variables as $u = \alpha'$ and $v = \psi'$, the dynamical system takes the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.11}$$

$$\frac{dW}{dr} = \frac{1}{u}\left(\frac{3v^2}{4\psi} - \frac{\overline{V}}{2}\right)W, \tag{6.12}$$

$$\frac{du}{dr} = -\frac{3v^2}{4\psi} - \frac{\overline{V}}{2}, \tag{6.13}$$

$$\frac{dv}{dr} = -\frac{v}{u}\left(\frac{3v^2}{4\psi} - \frac{\overline{V}}{2}\right) + \frac{v^2}{2\psi} + \frac{\psi}{3}\left(\overline{V}_\psi + \frac{1}{2\sqrt{\alpha}}\overline{V}_{\sqrt{\alpha}}\right), \tag{6.14}$$

where Eq.(6.11) is the explicit definition of u and v, and Eqs. (6.12)–(6.14) are reformulations of Eqs.(6.6), (6.7), and (6.9), respectively. Once the functional form of the potential $\overline{V}(\xi,\psi)$ is specified, the system of Eqs. (6.11)-(6.14) represents a system of ordinary, strongly nonlinear, differential equations for the variables $\left(\alpha = \xi^2, \psi, W, u, v\right)$. To solve this system, one has to impose a set of boundary conditions at some radius $r = r_0$, i.e., $\alpha\left(r_0\right) = \alpha_0$, $\psi\left(r_0\right) = \psi_0$, $W\left(r_0\right) = W_0$, $u\left(r_0\right) = u_0$, and $v\left(r_0\right) = v_0$, respectively, which specify the boundary values of the variables on, or nearby the string axis. Moreover, we will also impose the condition $u\left(r_0\right) \neq v\left(r_0\right)$. Once the system is solved, the string tension can be obtained from Eq. (6.8), and it is given by

$$\frac{2}{3}\kappa^2\sigma = \overline{V} - \frac{v^2}{2\psi} - \frac{1}{6}\left(\overline{V} - \sqrt{\alpha}\overline{V}_{\sqrt{\alpha}}\right). \tag{6.15}$$

Note that, in general, the solutions obtained for σ do not satisfy the property $\sigma(r) \equiv 0, \forall r \geq R_s$, and this condition must be imposed manually by performing a matching between the string spacetime and an exterior cosmological spacetime. This matching must be performed via the use of the junction conditions of the theory.

6.2 Specific string solutions.

The solutions of the gravitational field equations for a cosmic string-like configuration in generalized HMPG essentially depend on the functional form of the potential $\overline{V} = \overline{V}(\xi, \psi)$. Once the potential and the boundary conditions are given, solutions of the gravitational field equations can be obtained that uniquely fix the metric tensor component W, as well as the two scalar fields ξ and ψ.

6.2.1 Exact solutions

6.2.1.1 Constant null potential $\overline{V} = 0$

As before, we begin our investigations of the string-like structures in generalized HMPG by considering the simple case in which the potential $\overline{V}$ vanishes, $\overline{V} = 0$. In this case Eq. (3.25) takes the form

$$-f(R,\mathcal{R}) + R\frac{\partial f(R,\mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} = 0, \tag{6.16}$$

and it has the general solution

$$f(R,\mathcal{R}) = Rg\left(\frac{\mathcal{R}}{R}\right) + \mathcal{R}h\left(\frac{R}{\mathcal{R}}\right), \tag{6.17}$$

where g and h are arbitrary functions.

From Eq. (6.10) we obtain

$$\frac{W'}{W} = -\frac{\left(\xi^2\right)''}{\left(\xi^2\right)'}, \tag{6.18}$$

giving immediately

$$W = \frac{W_0}{\left(\xi^2\right)'} = \frac{W_0}{2\xi\xi'}, \tag{6.19}$$

where W_0 is an integration constant. On the other hand, from Eq. (6.9) we obtain

$$\frac{\psi''}{\psi'} + \frac{W'}{W} - \frac{1}{2}\frac{\psi'}{\psi} = 0, \tag{6.20}$$

and

$$W = \frac{C\sqrt{\psi}}{\psi'}, \tag{6.21}$$

respectively, where C is an arbitrary integration constant. Eqs (6.18) and (6.21) can be combined to obtain

$$\frac{d}{dr}\xi^2 = \frac{w_0\psi'}{\sqrt{\psi}} = 2w_0\frac{d}{dr}\sqrt{\psi}, \tag{6.22}$$

where we have defined $w_0 = W_0/C$. Taking the radial derivative of Eq. (6.22) and inserting the result into Eq. (6.7) yields an ODE for ψ in the form

$$\psi'' - \frac{1}{2}\frac{\psi'^2}{\psi} + \frac{3}{4w_0}\frac{\psi'^2}{\sqrt{\psi}} = 0. \tag{6.23}$$

Dividing through by ψ' allows one to integrate Eq. (6.23) directly twice and obtain the solution for $\psi(r)$ in the form

$$\psi(r) = \frac{4}{9}w_0^2 \ln^2\left[\frac{3c_1(r+c_2)}{4w_0}\right], \tag{6.24}$$

where c_1 and c_2 are arbitrary integration constants. From Eq. (6.22) we obtain

$$\xi^2 = \xi_0^2 + \frac{4}{3}w_0^2 \ln\frac{3c_1(r+c_2)}{4w_0}. \tag{6.25}$$

Thus we find for the metric tensor component W the expression

$$W(r) = \frac{3C^2(r+c_2)}{4W_0}. \tag{6.26}$$

Finally, the string tension can be computed directly from Eq. (6.8) and it is given by

$$-\kappa^2\sigma = \frac{4W_0^2}{3C^2(c_2+r)^2}. \tag{6.27}$$

We should note that the above solutions have been obtained under the assumption $\ln\left[3c_1(c_2+r)/4w_0\right] > 0$. By assuming the integration constant $c_2 \neq 0$, it follows that W takes on the string axis the boundary value $W(0) = 3c_1c_2/4w_0$. However, for the choice $c_2 = 0$, W vanishes on the string axis. But, on the other hand, this choice would imply a divergent string tension and scalar fields at $r = 0$. The circular radius $W(r)$ monotonically increases with the radial distance from the center, and for $r \to \infty$, $\lim_{r\to\infty} W^2(r) = \infty$. In the same limit, the string tension tends to zero, indicating a vanishing string tension at infinity.

An interesting property of the present zero potential solution for cosmic strings in generalized HMPG theory is that the string tension is negative. In [111], wormhole configurations representing an alternative of the cosmic string solutions of standard GR have been obtained under the assumption of a negative string tension and mass. The properties of such configurations have been further investigated in [112], where it was pointed out that a wormhole mouth embedded in high background matter density, and which accretes mass, can give the other mouth a net negative mass. The lensing of such gravitationally negative anomalous objects will have observable lensing properties. However, for the present vanishing potential generalized HMPG theory cosmic string solution, the mass of the string is given by

$$m_s\left(R_s\right) = \frac{2\pi W_0}{\kappa^2}\ln\frac{c_2}{c_2+R_s}. \tag{6.28}$$

For $R_s \to \infty$, the mass of the cosmic string diverges logarithmically to minus infinity.

For $\sigma = 0$, that is, in the vacuum, the field equations (6.5)-(6.9) do admit the simple solution

$$\psi = \psi_0 = \text{constant} \tag{6.29}$$

$$W = \mathcal{W}_0 = \text{constant} \tag{6.30}$$

$$\xi^2 = c_3 r + c_4 \tag{6.31}$$

where c_1 and c_2 are integration constants. The corresponding vacuum metric is given by

$$ds^2 = -dt^2 + dr^2 + \mathcal{W}_0^2 d\theta^2 + dz^2. \tag{6.32}$$

A matching of this vacuum metric with the string metric tensor component $W(r)$ at a finite radius R_s determines the string radius as

$$R_s = \frac{4\mathcal{W}_0 W_0}{3C^2} - c_2. \tag{6.33}$$

The string tension takes then the surface value

$$-\kappa^2\sigma = \frac{3}{4}\frac{C^2}{\mathcal{W}_0^2}. \tag{6.34}$$

Hence there is a sudden transition from a finite (negative) value of the string tension to its zero vacuum value.

6.2.1.2 Constant potential $\overline{V} = \Lambda$

Next we consider the case where the potential V is a constant, so that $V = \Lambda = \text{constant}$. In this case Eq. (3.25) takes the form

$$-f\left(R,\mathcal{R}\right) + R\frac{\partial f\left(R,\mathcal{R}\right)}{\partial R} + \mathcal{R}\frac{\partial f\left(R,\mathcal{R}\right)}{\partial \mathcal{R}} = \Lambda, \tag{6.35}$$

and it has the general solution

$$f\left(R,\mathcal{R}\right) = Rg\left(\frac{\mathcal{R}}{R}\right) + \mathcal{R}h\left(\frac{R}{\mathcal{R}}\right) - \Lambda, \tag{6.36}$$

where g and h are arbitrary functions.

For a constant potential Eq. (6.9) simplifies to

$$\frac{W'}{W} = -\frac{\psi''}{\psi'} + \frac{1}{2}\frac{\psi'}{\psi}, \tag{6.37}$$

which allows us to write Eq. (6.6) in the form

$$\frac{d}{dr}\xi^2 = \frac{3\psi'^2/4\psi - \Lambda/2}{W'/W} = \frac{3\psi'^2/4\psi - \Lambda/2}{-\psi''/\psi' + (1/2)\psi'/\psi}. \tag{6.38}$$

To facilitate the analysis, we introduce now a new function $h = \psi'^2/\psi$. The radial derivative of this function can be written in terms of ψ and its derivatives as

$$h' = 2h\left(\frac{\psi''}{\psi'} - \frac{1}{2}\frac{\psi'}{\psi}\right). \tag{6.39}$$

This definition allows us to rewrite Eq. (6.38) in terms of h as

$$\frac{d}{dr}\xi^2 = -\frac{(3/4)h - \Lambda/2}{h'/2h}, \tag{6.40}$$

which can then be differentiated with respect to r and inserted into Eq. (6.7) to cancel the dependency in $d^2\xi^2/dr^2$ and ψ''. As a result, we obtain an equation depending solely in h of the form

$$\frac{(2\Lambda - 3h)\left(3h'^2 - 2hh''\right)}{4h'^2} = 0, \tag{6.41}$$

This equation is undefined for $h' = 0$, as the denominator vanishes in this case. Thus, in the following we ignore the solution corresponding to $h = 2\Lambda/3 = \text{constant}$, giving $h' = 0$, and $W = \text{constant}$. The general solution of Eq. (6.41) is given by

$$h(r) = \frac{c_2}{(r + 2c_1)^2}, \tag{6.42}$$

where c_1 and c_2 are arbitrary integration constants. Recalling that $h' = \psi'^2/\psi$, Eq. (6.42) becomes a separable ODE for ψ which can be directly integrated and provides the general solution

$$\psi(r) = \left[c_3 \pm \frac{\sqrt{c_2}}{2}\ln(r + 2c_1)\right]^2, \tag{6.43}$$

where c_3 is an arbitrary integration constant. For $c_3 = 0$, we recover the expression (6.24) of $\psi(r)$, corresponding to the case $V = 0$. Inserting Eq. (6.43) into Eq. (6.40) and integrating gives for ξ^2 the expression

$$\xi^2(r) = \xi_0^2 + \frac{1}{4}\left[3c_2\ln(r + 2c_1) - \Lambda r(r + 4c_1)\right], \tag{6.44}$$

where ζ_0^2 is an integration constant. Inserting Eq. (6.43) into Eq. (6.37) we obtain the solution for W

$$W(r) = W_0 \left(r + 2c_1\right), \tag{6.45}$$

where W_0 is a constant of integration. Hence the cosmic string metric tensor component $W(r)$ is the same in both $V = 0$ and $V = \Lambda$ cases. Finally, the string tension can be computed via Eq. (6.8), leading to

$$\kappa^2\sigma = \frac{\Lambda}{2} - \frac{3c_2}{4\left(2c_1 + r\right)^2}. \tag{6.46}$$

On the string axis $r = 0$ we obtain for the string tension the value

$$\kappa^2\sigma_0 = \kappa^2\sigma(0) = \frac{\Lambda}{2} - \frac{3c_2}{16c_1^2}. \tag{6.47}$$

The condition of the positivity of the string tension imposes $c_2/c_1^2 < 8\Lambda/3$ on the integration constants.

For $\Lambda = 0$ we re-obtain the expression corresponding to the case $V = 0$. However, by an appropriate choice of the integration constants, and by assuming $\Lambda > 0$, the string tension can be made positive in this model for all $r > 0$. Moreover, $\lim_{r\to\infty}\sigma(r) = \Lambda/2\kappa^2$, and hence at infinity the string tension becomes equal to the cosmological constant. However, in this case one can obtain a finite radius string configuration, with the radius R_s determined by the condition $\sigma\left(R_s\right) = 0$, and given by

$$R_s = \sqrt{\frac{3}{2}\frac{c_2}{\Lambda}} - 2c_1. \tag{6.48}$$

In order to obtain a positive radius for the string, the string tension at the origin $r = 0$ must be negative.

As for the mass of the string we obtain

$$\begin{aligned} m_s\left(R_s\right) = \frac{2\pi W_0}{\kappa^2} & \left\{ \frac{1}{2}\Lambda\left[R_s + c_1(\Lambda W_0 + 2) - \frac{3c_2W_0}{8c_1}\right] \right. \\ & \left. - \frac{6c_1c_2}{8c_1\left[R_s + c_1(\Lambda W_0 + 2)\right] - 3c_2W_0}\right\}. \end{aligned} \tag{6.49}$$

By an appropriate choice of the integration constants, giving the boundary conditions of the fields φ and ψ for $r = 0$, one can always satisfy the condition $m_s\left(R_s\right) > 0, \forall R_s$. In the limit $R_s \to \infty$, we obtain $m_s\left(R_s\right) \approx \left(\pi W_0\Lambda/\kappa^2\right) R_s$, that is, for large distances the mass of the string linearly increases with its radius.

6.2.2 Numerical solutions

Unlike the simple cases considered in the previous section, for potentials having more complicated functional forms one must resort to numerical methods in order to construct cosmic string models in the generalized HMPG theory. In the present Section, we obtain a number of such numerical solutions describing cosmic string configurations, by fixing first the form of the potential. We will adopt several forms for $V(\xi, \psi)$, by assuming that it has an additive, and a multiplicative structure, we will also consider the cases where V depends on only one variable, ξ or ψ, so that $V = V(\xi)$ or $V = V(\psi)$, respectively. Given the dependence of the scalar field ξ in both the derivatives of $f(R, \mathcal{R})$ with respect to R and $\mathcal{R}$, the symmetric structure of Eq. (3.25) when the potential V depends solely on ξ allows for the analytic derivation of families of solutions for the function $f(R, \mathcal{R})$. In the following sections, when one considers potentials that depend also on ψ, this symmetry is broken and one can only obtain particular solutions of Eq. (3.25). The numerical solutions strongly depend on the boundary values of the geometrical and physical parameters used to integrate the system of equations (6.11)–(6.14), which, at least in the present approach, can be chosen arbitrarily. Moreover, the potential also depends on some numerical parameters. Hence, in the following, we will also study two types of different effects on the numerical cosmic string configurations, related to the variation of the potential parameters, and of the boundary conditions.

6.2.2.1 Polynomial potential $\overline{V}(\xi, \psi) = \overline{V}(\psi^2)$

We begin our numerical investigations by first considering the case in which the potential $\overline{V}(\xi, \psi)$ is independent on the field ξ, $\overline{V} = \overline{V}(\psi)$. We assume that the potential $\overline{V}$ has the form $\overline{V} = \overline{V}_0\psi^2$, with $\overline{V}_0$ a constant. For this choice of the potential, Eq. (3.25) becomes

$$-f(R, \mathcal{R}) + R\frac{\partial f(R, \mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R, \mathcal{R})}{\partial \mathcal{R}} = \bar{V}_0\frac{\partial f(R, \mathcal{R})}{\partial \mathcal{R}} \tag{6.50}$$

and a particular solution for the function $f(R, \mathcal{R})$ is

$$f(R, \mathcal{R}) = \frac{(\mathcal{R} + c_1 R)^2}{4\bar{V}_0(1 + c_2\mathcal{R})} + c_3 R, \tag{6.51}$$

where the c_i are constants.

The system of field equations describing the cosmic string behavior in the Generalized Hybrid Metric Palatini gravity take the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.52}$$

$$\frac{dW}{dr} = \frac{1}{2u}\left(\frac{3v^2}{2\psi} - \overline{V}_0\psi^2\right)W, \tag{6.53}$$

$$\frac{du}{dr} = -\frac{3v^2}{4\psi} - \frac{\overline{V}_0\psi^2}{2}, \tag{6.54}$$

$$\frac{dv}{dr} = -\frac{v}{u}\left(\frac{3v^2}{4\psi} - \frac{\overline{V}_0\alpha}{2}\right) + \frac{v^2}{2\psi} + \frac{2V_0}{3}\psi^2. \tag{6.55}$$

The string tension can be obtained as

$$\frac{2\kappa^2}{3}\sigma = \frac{5}{6}\overline{V}_0\psi^2 - \frac{v^2}{2\psi}. \tag{6.56}$$

The variations of the metric function W^2 and of the string tension are represented in Fig. 6.1. Both W^2 and σ are monotonically decreasing functions, with $\sigma(r)$ vanishing for a finite value of $r = R_s$, which represents the radius of the string. To obtain the plots we have varied the initial condition for $\psi'(0) = v_0$. There is a significant impact on the numerical value of v_0 on the string properties. However, even that for small r the effect on the string tension is rather large, the string radius is less affected by the variation of this initial condition.

The variations of the potential $\overline{V}$ and of the function ψ are represented in Fig. 6.2. Both quantities monotonically decrease radially, and their behavior depends on the adopted initial conditions.

The variation of $\xi^2(r)$ is represented in Fig. 6.3. ξ^2 is positive in the range $0 \leq r \leq R_s$, thus ensuring the physical nature of the gravitational coupling. ξ^2 is a monotonically increasing function of r, and at large values of the radial coordinate, near the string boundary, its variation depends on the initial condition for $\psi'(0)$.

The variation of the potential parameter $\overline{V}_0$ does not change the qualitative behavior of the solution, despite having an important effect on the numerical characteristics of the string.

6.2.2.2 Polynomial potential $\overline{V}(\xi, \psi) = \overline{V}(\xi^n)$

We will study only potentials $V(\xi)$ that have a simple polynomial dependence on ξ, $\overline{V} = \overline{V}_0\xi^n$, more specifically we will restrict our analysis to the cases $n = 2$ and $n = 4$,

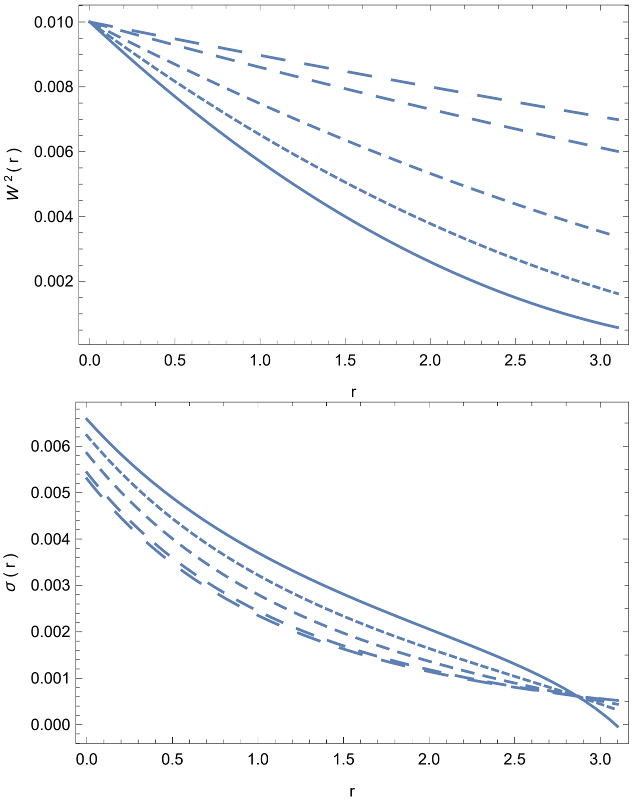

Figure 6.1: Variation of the metric function $W^2(r)$ (upper panel), and of the string tension $\sigma(r)$ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = \overline{V}_0\psi^2$ potential, for $v_0 = -0.01$ (solid curve), $v_0 = -0.011$ (dotted curve), $v_0 = -0.012$ (short dashed curve), $v_0 = -0.013$ (dashed curve), and $v_0 = -0.0133$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 11$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.001$, $\alpha_0 = 0.10$, $W(0) = 0.10$, and $\psi_0 = 0.03$, respectively.

respectively. These forms of the potential are chosen because the associated forms of the function $f(R,\mathcal{R})$ have been proven to provide interesting cosmological behaviors in other works [113], e.g., for $n = 2$ one can obtain the de-Sitter solution as well as cosmological bounces, and for $n = 4$ one can reproduce the matter-dominated era.

We assume now that $\bar{V}$ is a simple quadratic power law function, so that $\bar{V} = \bar{V}_0\xi^2$. In this case, Eq. (3.25) becomes

$$-f(R,\mathcal{R}) + R\frac{\partial f(R,\mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} = \bar{V}_0\left(\frac{\partial f(R,\mathcal{R})}{\partial R} + \frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}}\right), \tag{6.57}$$

which can be solved with respect to $f(R,\mathcal{R})$ to obtain

$$f(R,\mathcal{R}) = \left(R - \bar{V}_0\right) g\left(\frac{\mathcal{R}-\bar{V}_0}{R-\bar{V}_0}\right) + \left(\mathcal{R}-\bar{V}_0\right) h\left(\frac{R-\bar{V}_0}{\mathcal{R}-\bar{V}_0}\right), \tag{6.58}$$

where g and h are arbitrary functions.

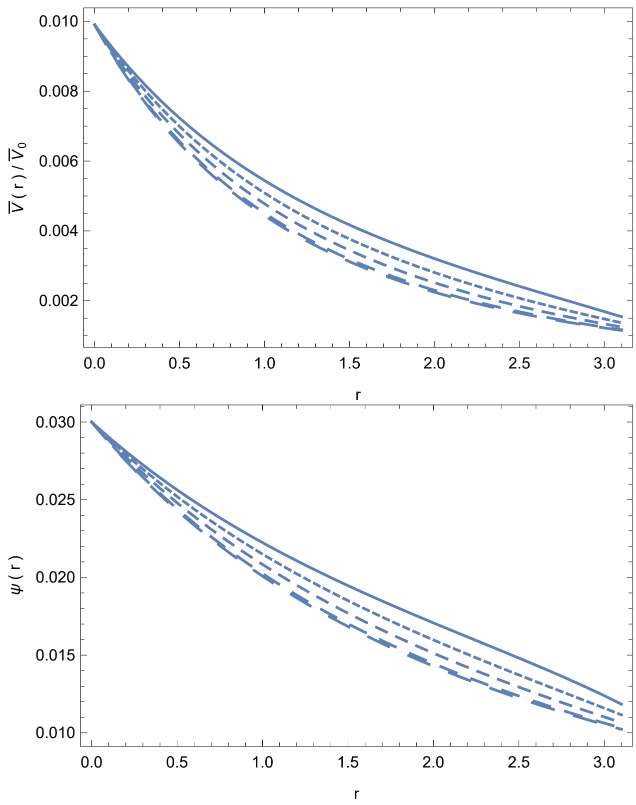

Figure 6.2: Variation of the potential $\overline{V}(r)$ (upper panel), and of the function $\psi(r)$ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi)=\overline{V}_0\psi^2$ potential, for $v_0=-0.01$ (solid curve), $v_0=-0.011$ (dotted curve), $v_0=-0.012$ (short dashed curve), $v_0=-0.013$ (dashed curve), and $v_0=-0.0133$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0=11$, while the boundary conditions used to numerically integrate the field equations are $u_0=-0.001$, $\alpha_0=0.10$, $W(0)=0.10$, and $\psi_0=0.03$, respectively.

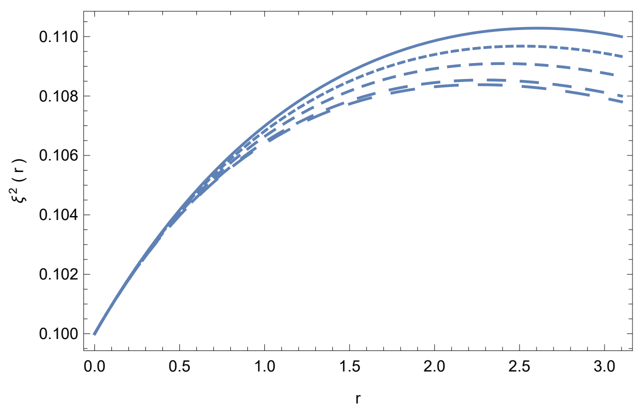

Figure 6.3: Variation of ξ^2 as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi)=\overline{V}_0\psi^2$ potential, for $v_0=-0.01$ (solid curve), $v_0=-0.011$ (dotted curve), $v_0=-0.012$ (short dashed curve), $v_0=-0.013$ (dashed curve), and $v_0=-0.0133$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0=11$, while the boundary conditions used to numerically integrate the field equations are $u_0=-0.001$, $\alpha_0=0.10$, $W(0)=0.10$, and $\psi_0=0.03$, respectively.

The system of field equations to be solved takes the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.59}$$

$$\frac{dW}{dr} = \frac{1}{2u}\left(\frac{3v^2}{2\psi} - \bar{V}_0\alpha\right)W, \tag{6.60}$$

$$\frac{du}{dr} = -\frac{3v^2}{4\psi} - \frac{\bar{V}_0\alpha}{2}, \tag{6.61}$$

$$\frac{dv}{dr} = -\frac{v}{u}\left(\frac{3v^2}{4\psi} - \frac{\bar{V}_0\alpha}{2}\right) + \frac{v^2}{2\psi} + \frac{V_0}{3}\psi. \tag{6.62}$$

The string tension is given by

$$\frac{2\kappa^2}{3} = \frac{7}{6}\bar{V}_0\alpha - \frac{v^2}{2\psi}. \tag{6.63}$$

The system of Eqs. (6.59)-(6.62) must be integrated with the initial conditions $\alpha(0) = \alpha_0$, $\psi(0) = \psi_0$, $W(0) = W_0$, $u(0) = u_0$, and $v(0) = v_0$, respectively. The variations of the metric tensor component $W^2(r)$ and of the string tension $\sigma(r)$ are represented, for different values of α_0, and for fixed initial conditions of the other parameters, in Fig. 6.4, respectively,

The metric tensor component W^2 inside the string takes a finite value for $r = 0$, and initially it increases with increasing r, reaching a maximum value at $r = r_{max}$. For $r > r_{max}$, $W^2(r)$ becomes a monotonically decreasing function. The behavior of W^2 is strongly influenced by the initial conditions, as one can see from its dependence on α_0. The string tension σ is a monotonically decreasing function of distance, and it identically vanishes at $r = R_s$, with $\sigma(R_s) = 0$. This condition allows us to uniquely define R_s as the string radius. The behavior of the string tension depends strongly on the initial conditions, and this dependence also induces a significant variation of the string radius on the initial values of the string parameters. The variations of the potential $\bar{V}$ and of the function ψ are represented in Fig. 6.5.

The potential $\bar{V}(\xi) = \bar{V}_0\xi^2$ is a decreasing function of r, having an approximate parabolic dependence on r. Inside the string $\xi^2 > 0$ for $0 \leq r \leq R_s$, which guaranties that the gravitational coupling has the correct sign. The function ψ monotonically increases inside the string, and takes only positive values. The behavior of both functions V and ψ depends strongly on the initial conditions. The string geometric and physical characteristics also depend on the potential parameter V_0, but its variation does not change the qualitative behavior of the numerical solution.

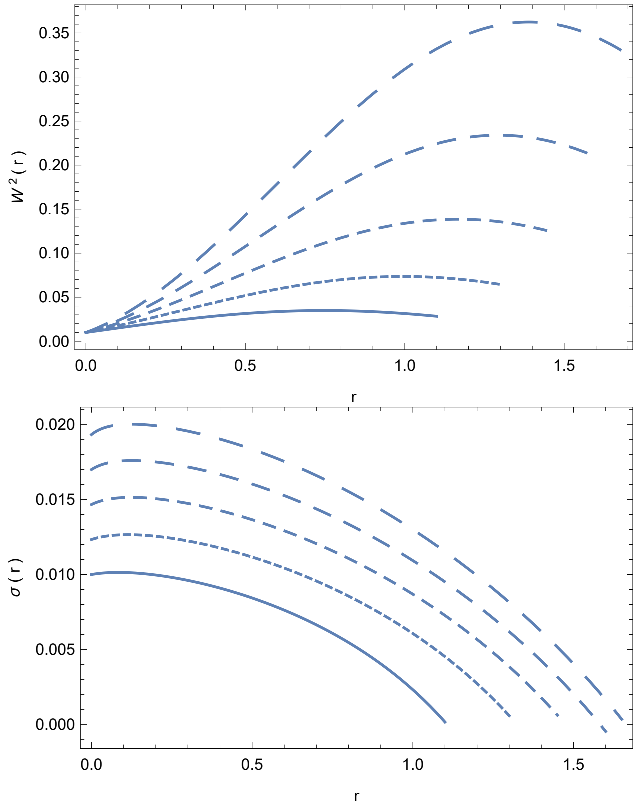

Figure 6.4: Variations of the metric function $W^2(r)$ (upper panel), and of the string tension σ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\bar{V}(\xi,\psi) = \bar{V}_0\xi^2$ potential, for $\alpha_0 = 0.010$ (solid curve), $\alpha_0 = 0.012$ (dotted curve), $\alpha_0 = 0.014$ (short dashed curve), $\alpha_0 = 0.016$ (dashed curve), and $\alpha_0 = 0.018$ (long dashed curve), respectively. For $\bar{V}_0$ we have adopted the value $\bar{V}_0 = 1$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.001$, $v_0 = 0.01$, $W(0) = 0.10$, and $\psi_0 = 0.03$, respectively.

Regarding the other polynomial solution to be studied, the $\overline{V}(\xi,\psi) = \overline{V}_0\xi^4$ potential, Eq. (3.25) takes the form

$$-f(R,\mathcal{R}) + R\frac{\partial f(R,\mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} = \bar{V}_0\left(\frac{\partial f(R,\mathcal{R})}{\partial R} + \frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}}\right)^2, \tag{6.64}$$

which yields the following solution to $f(R,\mathcal{R})$

$$f(R,\mathcal{R}) = \frac{(R+\mathcal{R})^2}{16\bar{V}_0} + g(R-\mathcal{R}), \tag{6.65}$$

where g is an arbitrary function.

The structure equations describing the cosmic string configuration in generalized Hybrid Metric-Palatini Gravity take the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.66}$$

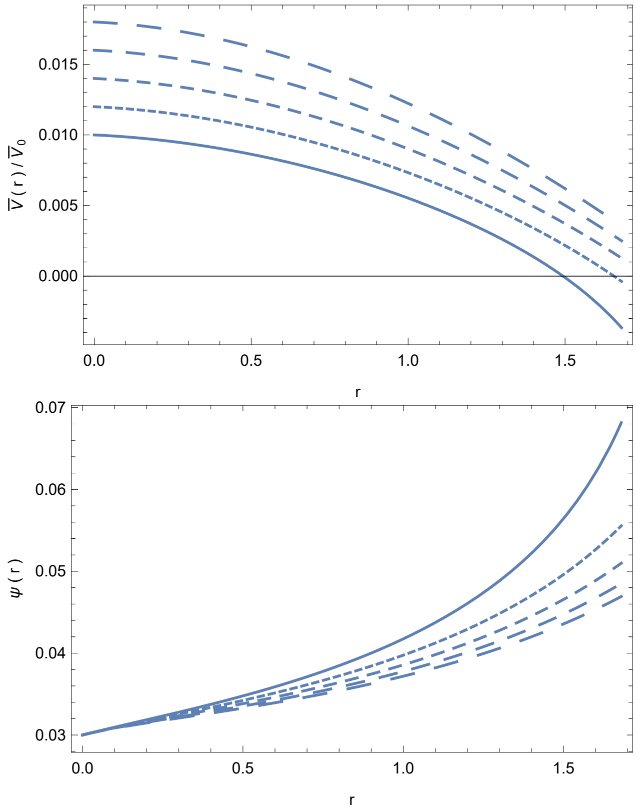

Figure 6.5: Variations of the potential $\bar{V}(\xi,\psi)=\bar{V}_0\xi^2$ (upper panel), and of the function ψ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\bar{V}(\xi)=\bar{V}_0\xi^2$ potential, for $\alpha_0=0.010$ (solid curve), $\alpha_0=0.012$ (dotted curve), $\alpha_0=0.014$ (short dashed curve), $\alpha_0=0.016$ (dashed curve), and $\alpha_0=0.018$ (long dashed curve), respectively. For $\bar{V}_0$ we have adopted the value $\bar{V}_0=1$, while the boundary conditions used to numerically integrate the field equations are $u_0=-0.001$, $v_0=0.01$, $W(0)=0.10$, and $\psi_0=0.03$, respectively.

$$\frac{dW}{dr}=\frac{1}{2u}\left(\frac{3v^2}{2\psi}-\overline{V}_0\alpha^2\right)W, \tag{6.67}$$

$$\frac{du}{dr}=-\frac{3v^2}{4\psi}-\frac{\overline{V}_0\alpha^2}{2}, \tag{6.68}$$

$$\frac{dv}{dr}=-\frac{v}{u}\left(\frac{3v^2}{4\psi}-\frac{\overline{V}_0\alpha^2}{2}\right)+\frac{v^2}{2\psi}+\frac{2V_0}{3}\psi\alpha. \tag{6.69}$$

For the string tension we obtain

$$\frac{2\kappa^2}{3}=\frac{3}{2}\overline{V}_0\alpha^2-\frac{v^2}{2\psi}. \tag{6.70}$$

The solutions of the system of Eqs. (6.66)-(6.69) can be obtained only numerically, and for its integration the initial conditions $\alpha(0)=\alpha_0$, $\psi(0)=\psi_0$, $W(0)=W_0$, $u(0)=u_0$, and $v(0)=v_0$ must be specified. The metric tensor component $W^2(r)$ and the string tension $\sigma(r)$ are represented, for different values of v_0, and for fixed initial conditions of the other parameters, in Fig. 6.6, respectively,

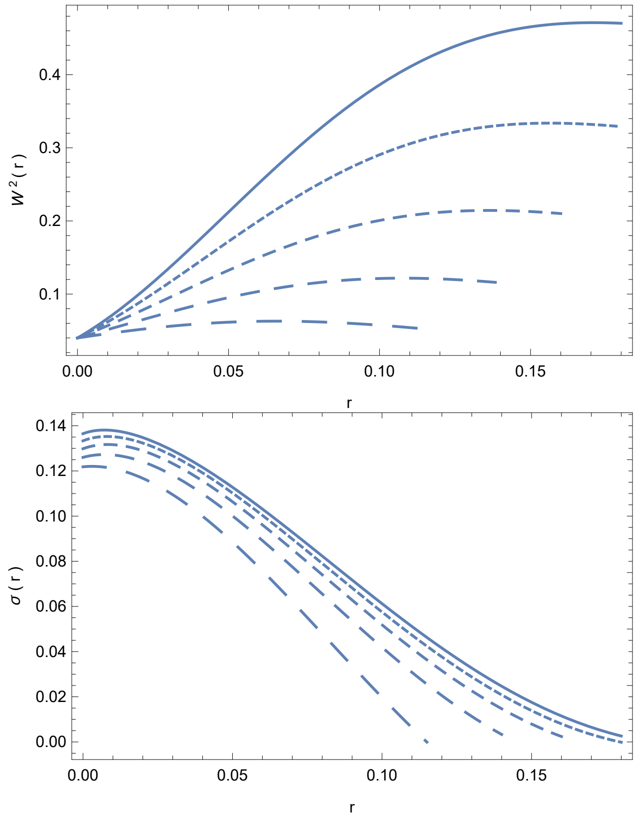

Figure 6.6: Variation of the metric function $W^2(r)$ (upper panel), and of the string tension σ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = \overline{V}_0\xi^4$ potential, for $v_0 = 0.00009$ (solid curve), $v_0 = 0.0001$ (dotted curve), $v_0 = 0.00011$ (short dashed curve), $v_0 = 0.00012$ (dashed curve), and $v_0 = 0.00013$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 10^5$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.001$, $\alpha_0 = 0.001$, $W(0) = 0.20$, and $\psi_0 = 0.0000003$, respectively.

The qualitative behavior of the metric tensor component W^2 inside the string shows some differences as compared to the $\overline{V}(\xi) = \overline{V}_0\xi^2$ case. While in the quadratic case W^2 reaches a maximum inside the string, for the quartic potential the maximal value of the metric tensor is attained on the string surface. The behavior of W^2 is strongly dependent on the initial conditions, and significant variations may occur once v_0 is modified. The string tension σ monotonically decreases with increasing r, and it becomes zero at $r = R_s$, with $\sigma(R_s) = 0$. Hence, we can uniquely define R_s as the string radius. The string tension, as well as the string radius depend significantly on the initial condition. The variations of the potential $\overline{V}$ and of the function ψ are represented in Fig. 6.7.

The potential $\overline{V}(\xi) = \overline{V}_0\xi^4$ monotonically decreases inside the string, and it becomes zero on the string surface. Similarly to the previous case, $\xi^2 > 0$ for $0 \leq r \leq R_s$, and thus the gravitational coupling has the correct sign. The function ψ monotonically increases,

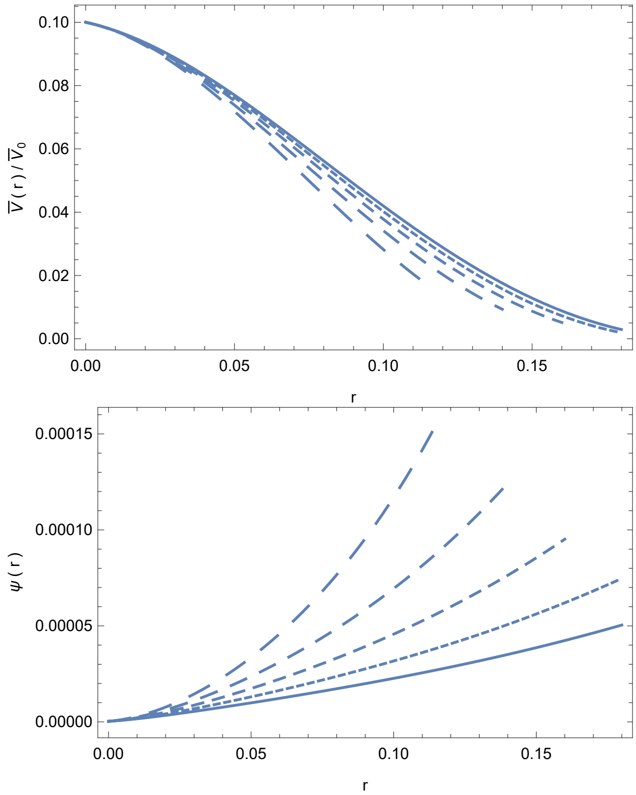

Figure 6.7: Variation of the potential $\overline{V}(\xi,\psi) = \overline{V}_0\xi^4$ (upper panel), and of the function ψ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi) = \overline{V}_0\xi^4$ potential, for $v_0 = 0.00009$ (solid curve), $v_0 = 0.0001$ (dotted curve), $v_0 = 0.00011$ (short dashed curve), $v_0 = 0.00012$ (dashed curve), and $v_0 = 0.00013$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 10^5$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.001$, $\alpha_0 = 0.001$, $W(0) = 0.20$, and $\psi_0 = 0.0000003$, respectively.

and takes only positive values. The behavior of both functions V and ψ is significantly dependent on the initial conditions, and on the potential parameter V_0. However, major changes in the numerical value of V_0 do not affect significantly the qualitative behavior of the numerical string solution for the quartic potential of the generalized Hybrid Metric-Palatini Gravity theory.

6.2.2.3 Multiplicative structure potential$\overline{V}\left(\xi,\psi\right) = \overline{V}_0\xi^2\psi^2$

Next we will consider string type solutions in the Generalized Hybrid Metric Palatini Gravity under the assumption that the potential $\overline{V}$ is given by $\overline{V} = \overline{V}_0\xi^2\psi^2 = \bar{V}_0\alpha\psi^2$, with $\overline{V}_0$ constant. Equation (3.25) then becomes

$$-f\left(R,\mathcal{R}\right) + R\frac{\partial f\left(R,\mathcal{R}\right)}{\partial R} + \mathcal{R}\frac{\partial f\left(R,\mathcal{R}\right)}{\partial \mathcal{R}} = \bar{V}_0\frac{\partial f\left(R,\mathcal{R}\right)}{\partial \mathcal{R}}\left(\frac{\partial f\left(R,\mathcal{R}\right)}{\partial \mathcal{R}} + \frac{\partial f\left(R,\mathcal{R}\right)}{\partial R}\right) \tag{6.71}$$

and a particular solution for the function $f(R,\mathcal{R})$ is

$$f(R,\mathcal{R}) = \sqrt{\frac{R}{\overline{V}_0}}\,(R-\mathcal{R})\,. \tag{6.72}$$

For this potential the field equations describing the string-like structure take the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.73}$$

$$\frac{dW}{dr} = \frac{1}{2u}\left(\frac{3v^2}{2\psi} - \overline{V}_0\alpha\psi^2\right)W, \tag{6.74}$$

$$\frac{du}{dr} = -\frac{3v^2}{4\psi} - \frac{\overline{V}_0\alpha\psi^2}{2}, \tag{6.75}$$

$$\frac{dv}{dr} = -\frac{v}{u}\left(\frac{3v^2}{4\psi} - \frac{\overline{V}_0\alpha}{2}\right) + \frac{v^2}{2\psi} + \frac{2V_0}{3}\psi^2\left(\alpha + \frac{\psi}{2}\right). \tag{6.76}$$

For this model the string tension is given by

$$\frac{2\kappa^2}{3}\sigma = \frac{7}{6}\overline{V}_0\alpha\psi^2 - \frac{v^2}{2\psi}. \tag{6.77}$$

The metric function W^2 and the string tension σ are depicted in Fig. 6.8, for a varying initial condition $\psi'(0) = \psi_0$, while all the other initial conditions are fixed. In this case the radial metric function is an increasing function of the radial coordinate r, and its rate of increase is strongly dependent on the variations in the numerical values of ψ_0. Similarly to the previous cases, the string tension is a monotonically decreasing function of r, and it vanishes at a finite value of r, $r = R_s$, which uniquely defines the string radius. The string radius is weakly dependent on the variation of ψ_0, however, significant variations in σ do appear for small values of r.

The variations of the potential and of the function ψ are represented in Fig. 6.9. $\overline{V}$ is a slowly decreasing positive function of r, strongly dependent on the initial condition for ψ'. The function ψ takes negative values, and show a strong dependence on ψ_0.

The behavior of the function $\xi^2(r)$ is depicted in Fig. 6.10. ξ^2 is positive for $r \in [0, R_s]$, and thus the physical nature of the gravitational coupling in the present model is guaranteed. ξ^2 is a monotonically decreasing function of r, and its variation depends significantly on the numerical values of the initial conditions for $\psi'(0)$.

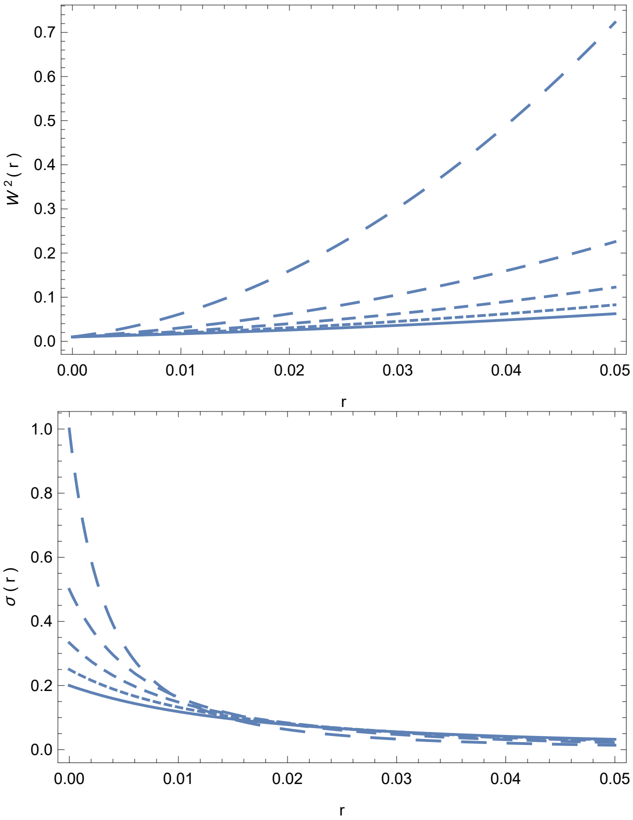

Figure 6.8: Variation of the metric function $W^2(r)$ (upper panel), and of the string tension $\sigma(r)$ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = \overline{V}_0\xi^2\psi^2$ potential, for $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.020$ (dotted curve), $\psi_0 = -0.015$ (short dashed curve), $\psi_0 = -0.01$ (dashed curve), and $\psi_0 = -0.005$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 10$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.025$, $W(0) = 0.10$, and $v_0 = 0.10$, respectively.

6.2.2.4 Additive structure potential $\overline{V}(\xi,\psi) = a\xi^2 + b\psi^2$

Finally, we considered string type solutions in the Generalized Hybrid Metric Palatini Gravity by assuming that the potential $\overline{V}$ is given by the quadratic expression $\overline{V} = a\xi^2 + b\psi^2$, with a, b constants. In this case, Eq. (3.25) takes the form

$$-f(R,\mathcal{R}) + R\frac{\partial f(R,\mathcal{R})}{\partial R} + \mathcal{R}\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} = a\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} + b\left(\frac{\partial f(R,\mathcal{R})}{\partial \mathcal{R}} + \frac{\partial f(R,\mathcal{R})}{\partial R}\right) \tag{6.78}$$

and a particular solution for the function $f(R,\mathcal{R})$ is

$$f(R,\mathcal{R}) = \frac{1}{4b}\left[a(a-R) + (R-\mathcal{R})^2\right] + c(R-a), \tag{6.79}$$

where c is a constant.

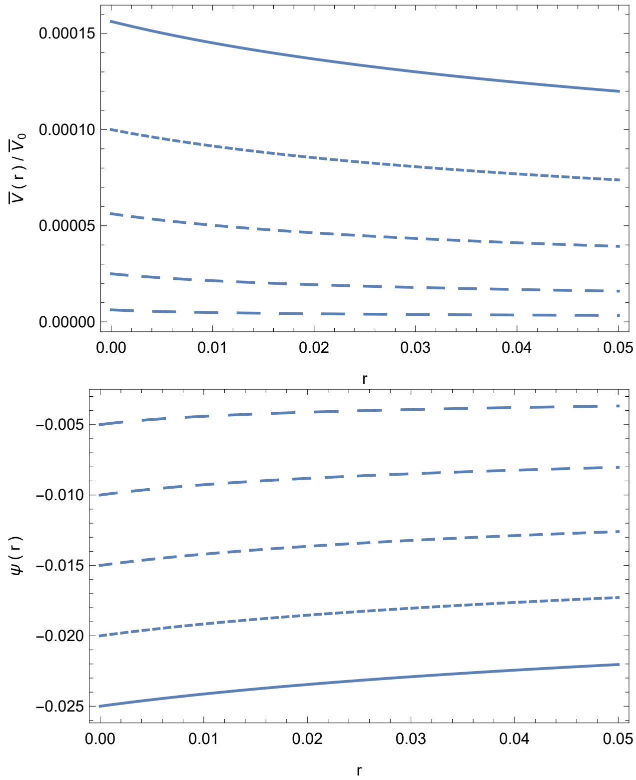

Figure 6.9: Variation of the potential $\overline{V}\left(\xi,\psi\right) = \overline{V}_0\xi^2\psi^2$ (upper panel), and of the function ψ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi) = \overline{V}_0\xi^2\psi^2$ potential, for $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.020$ (dotted curve), $\psi_0 = -0.015$ (short dashed curve), $\psi_0 = -0.01$ (dashed curve), and $\psi_0 = -0.005$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 10$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.025$, $W(0) = 0.10$, and $v_0 = 0.10$, respectively.

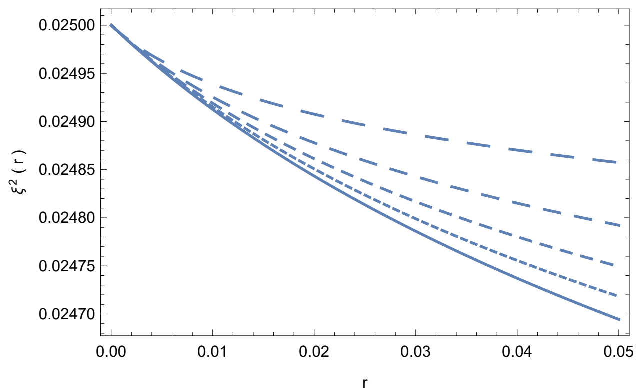

Figure 6.10: Variation of ξ^2 as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = \overline{V}_0\xi^2\psi^2$ potential, for $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.020$ (dotted curve), $\psi_0 = -0.015$ (short dashed curve), $\psi_0 = -0.01$ (dashed curve), and $\psi_0 = -0.005$ (long dashed curve), respectively. For $\overline{V}_0$ we have adopted the value $\overline{V}_0 = 10$, while the boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.025$, $W(0) = 0.10$, and $v_0 = 0.10$, respectively.

The field equations describing the string-like structure take the form

$$\frac{d\alpha}{dr} = u, \qquad \frac{d\psi}{dr} = v, \tag{6.80}$$

$$\frac{dW}{dr} = \frac{1}{2u}\left(\frac{3v^2}{2\psi} - a\alpha - b\psi^2\right)W, \tag{6.81}$$

$$\frac{du}{dr} = -\frac{3v^2}{4\psi} - \frac{1}{2}\left(a\alpha + b\psi^2\right), \tag{6.82}$$

$$\frac{dv}{dr} = -\frac{v}{u}\left(\frac{3v^2}{4\psi} - \frac{\overline{V}_0\alpha}{2}\right) + \frac{v^2}{2\psi} + \frac{\psi}{3}\left(a + 2b\psi\right). \tag{6.83}$$

For this model the string tension is given by

$$\frac{2\kappa^2}{3}\sigma = \frac{7}{6}a\alpha + \frac{5}{6}b\psi^2 - \frac{v^2}{2\psi}. \tag{6.84}$$

In the following we will consider two class of models described by this potential, obtained by varying the potential parameters (a, b) for fixed initial conditions, and models in which the potential parameters are fixed, while the initial conditions for $r = 0$ are slightly modified.

As a first example of string solutions with the quadratic potential in ξ and ψ we consider a configuration with fixed initial conditions but for different values of a and b. For the sake of concreteness we fix a and we vary b. The variation of the metric function W^2 and of the string tension are represented in Fig. 6.11 for different values of b. $W^2(r)$ is a monotonically increasing function of r, reaching its maximum value on the string vacuum boundary. The string tension monotonically decreases from its value at $r = 0$ to zero, with the corresponding value of r uniquely determining the string boundary. Both the variations of W^2 and σ are basically independent of the variations of the potential parameters.

The variations of the potential $\overline{V}$ and of the function ψ are depicted in Fig. 6.12. $\overline{V}$ is a decreasing function of r, which near the vacuum boundary of the string takes negative values. The function ψ is negative inside the string. Both $\overline{V}$ and ψ are basically independent on the variation of the potential parameter b.

The behavior of the function $\xi^2(r)$ is shown in Fig. 6.13. ξ^2 is a monotonically decreasing positive function of r for $r \in [0, R_s]$, which vanishes on the vacuum boundary of the string $\overline{V}(R_s) = 0$. Inside the string the gravitational coupling is positive, but in the vacuum outside the string it changes sign. The variation of ξ^2 is also basically independent on the numerical values of the potential parameter b.

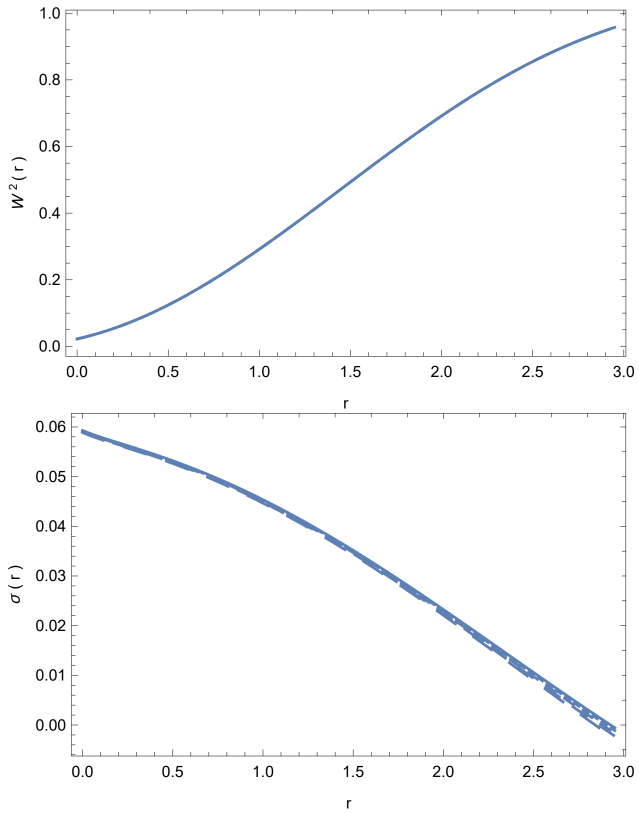

Figure 6.11: Variation of the metric function $W^2(r)$ (upper panel), and of the string tension $\sigma(r)$ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = a\xi^2 + b\psi^2$ potential, for $a = 0.5$ and $b = -2$ (solid curve), $b = -2.2$ (dotted curve), $b = -2.4$ (short dashed curve), $b = -2.6$ (dashed curve), and $b = -2.8$ (long dashed curve), respectively. The boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

We consider now the effects of the variation of the initial conditions on the string-like structures in the presence of the quadratic potential $\overline{V}\left(\xi,\psi\right) = a\xi^2 + b\psi^2$. The variation of the metric function W^2 and of the string tension are represented in Fig. 6.14, for fixed a and b, and varying $\psi'(0)$. The metric function is an increasing function of r, and its variation is strongly influenced by the variation of the initial conditions. The string tension is a monotonically decreasing function of r that monotonically decreases, and identically vanishes on the vacuum boundary $r = R_s$ of the string. Thus the string radius is uniquely determined, but its numerical value depends on the adopted initial conditions.

The behaviors of the potential $\overline{V}$ and of the function ψ are shown in Fig. 6.15. $\overline{V}$ is a decreasing function of r, which becomes negative near the string boundary. The function ψ is negative inside the string. Both $\overline{V}$ and ψ are strongly dependent on the numerical value of ψ_0.

The variation of the function $\xi^2(r)$ with respect to r is represented in Fig. 6.16. For all

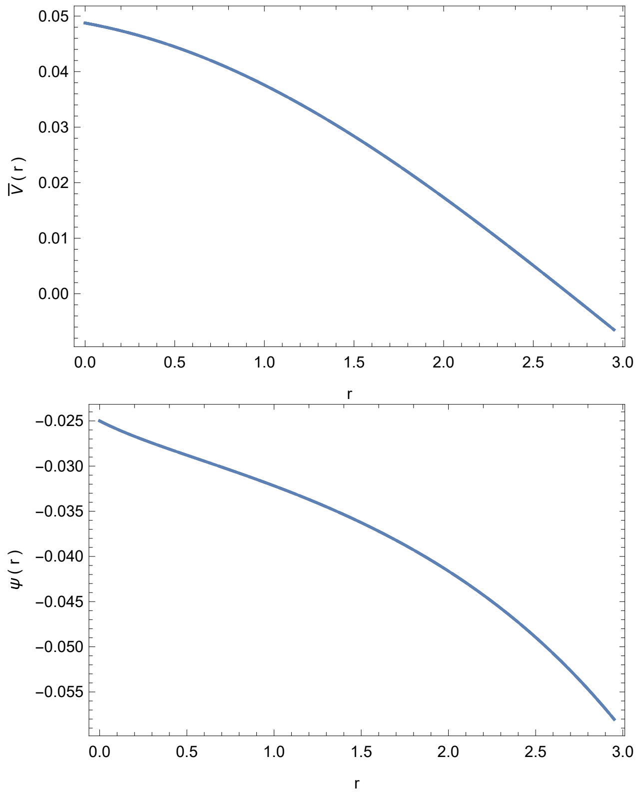

Figure 6.12: Variation of the potential $\overline{V}\left(\xi, \psi\right) = a\xi^2 + b\psi^2$ (upper panel), and of the function ψ (lower panel) as a function of r (with all quantities in arbitrary units) for $a = 0.5$ and $b = -2$ (solid curve), $b = -2.2$ (dotted curve), $b = -2.4$ (short dashed curve), $b = -2.6$ (dashed curve), and $b = -2.8$ (long dashed curve), respectively. The boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

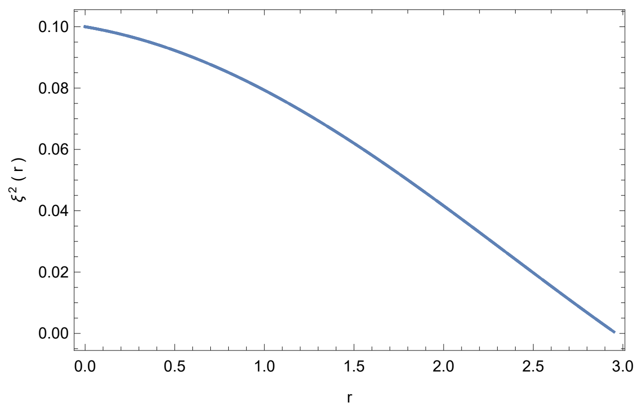

Figure 6.13: Variation of ξ^2 as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi, \psi) = a\xi^2 + b\psi^2$ potential, for $a = 0.5$ and $b = -2$ (solid curve), $b = -2.2$ (dotted curve), $b = -2.4$ (short dashed curve), $b = -2.6$ (dashed curve), and $b = -2.8$ (long dashed curve), respectively. The boundary conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

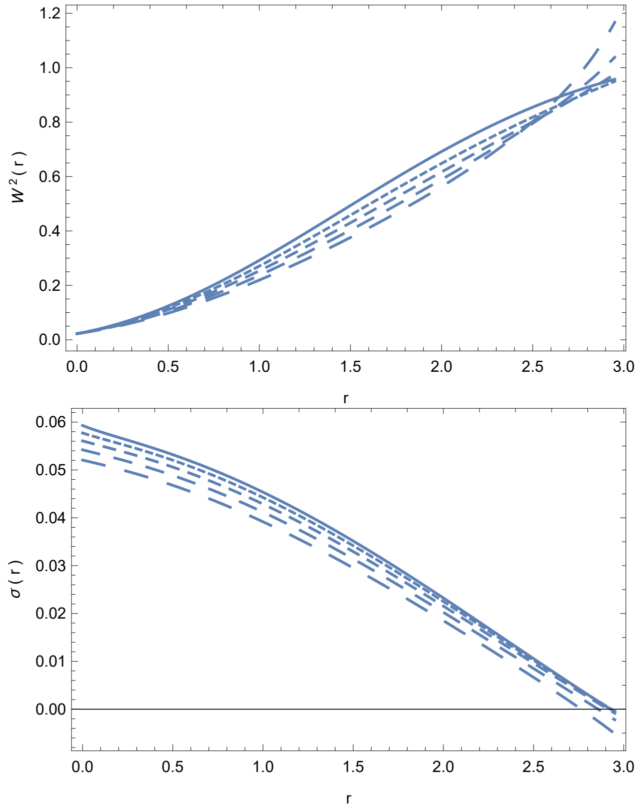

Figure 6.14: Variation of the metric function $W^2(r)$ (upper panel), and of the string tension $\sigma(r)$ (lower panel) as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi,\psi) = a\xi^2 + b\psi^2$ potential, for $a = 0.5$ and $b = -2$, and for different values of ψ_0: $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.035$ (dotted curve), $\psi_0 = -0.045$ (short dashed curve), $\psi_0 = -0.055$ (dashed curve), and $\psi_0 = -0.065$ (long dashed curve), respectively. The initial conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

r in the range $r \in [0, R_s]$, ξ^2 is a monotonically decreasing positive function, which vanishes on the vacuum boundary of the string $\xi^2(R_s) = 0$. Similarly to the previously considered case, inside the string the gravitational coupling is positive. However, for $r > R_s$, ξ^2 changes sign, indicating the presence of the unphysical coupling between the scalar field and the Ricci scalar. The variation of ξ^2 is also strongly dependent on the initial conditions.

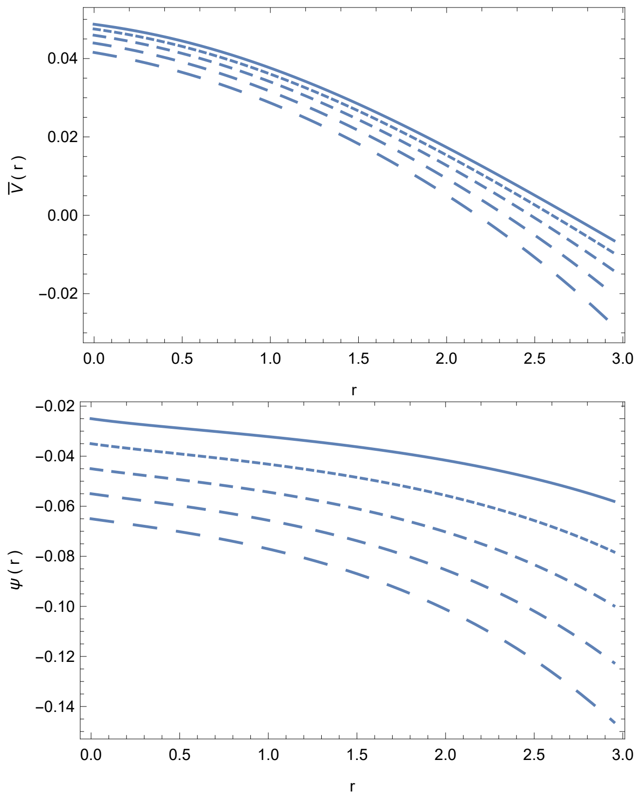

Figure 6.15: Variation of the potential $\overline{V}\left(\xi, \psi\right) = a\xi^2 + b\psi^2$ (upper panel), and of the function ψ (lower panel) as a function of r (with all quantities in arbitrary units) for $a = 0.5$ and $b = -2$, and for different values of ψ_0: $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.035$ (dotted curve), $\psi_0 = -0.045$ (short dashed curve), $\psi_0 = -0.055$ (dashed curve), and $\psi_0 = -0.065$ (long dashed curve), respectively. The initial conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

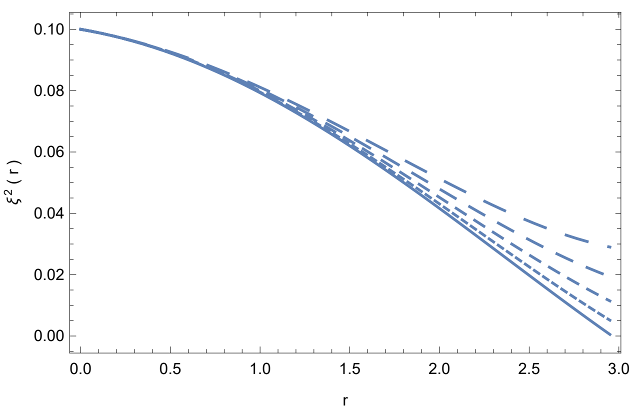

Figure 6.16: Variation of ξ^2 as a function of r (with all quantities in arbitrary units) for the $\overline{V}(\xi, \psi) = a\xi^2 + b\psi^2$ potential, for $a = 0.5$ and $b = -2$, and for different values of ψ_0: $\psi_0 = -0.025$ (solid curve), $\psi_0 = -0.035$ (dotted curve), $\psi_0 = -0.045$ (short dashed curve), $\psi_0 = -0.055$ (dashed curve), and $\psi_0 = -0.065$ (long dashed curve), respectively. The initial conditions used to numerically integrate the field equations are $u_0 = -0.01$, $\alpha_0 = 0.10$, $W(0) = 0.15$, and $v_0 = -0.01$, respectively.

Chapter 7

Final Remarks

In this work we studied the existence and physical properties of cosmic strings in the context of the hybrid metric-Palatini gravity, both in its original formulation and its subsequent generalization. The theory is an extension to General Relativity, combining both metric and Palatini formalism. A main success of the theory is the possibility to generate long-range forces that pass the classical local tests of gravity at the Solar System level, thus avoiding some problematic features of the standard $f(R)$ theories. Another interesting advantage of the theory is that it admits an equivalent scalar-tensor representation, simplifying greatly the dynamical equations. The type of strings studied in this work are local gauge strings, using an approximation to the Vilenkin-prescribed energy-momentum tensor. The most general form of the metric for a straight cosmic string oriented along the z-axis can be written as

$$ds^2 = -dt^2 + dr^2 + \left[1 - \frac{\mu(r)}{2\pi}\right]^2 r^2 d\theta^2 + dz^2 \tag{7.1}$$

where the Riemann curvature is zero everywhere except for the t-z hyperplane. In the linear approximation of the GR, the non-trivial element of the previous metric takes the form $(1 - 8\pi G\mu)r^2$, in this case the geometry of the string can be described as a conical singularity with a deficit angle proportional to the linear density of the string (μ). In terms of the modified-azimuthal coordinate $\theta' = (1 - 4G\mu)\theta$, ranging from 0 to $2\pi(1 - 4G\mu)$, the geometry is Minkowski, which implies that the gravitational acceleration of massive objects towards the string is null. The metric adopted in this work, after several simplifications from symmetry considerations, is of the form similar to eq. (7.1), with $W^2(r) = [1 - \mu(r)/2\pi]^2 r^2$. We showed that, by adopting the scalar-tensor representation of the hybrid theory, the gravitational field equations can be formulated in terms of the circular radius

W, the scalar field (or fields), their derivatives and the string tension. A crucial ingredient for the model is the form of the scalar field potential, V.

7.1 Remarks on the solutions obtained in the hybrid metric-Palatini theory

In the case of the the original hybrid theory, $R + f(\mathcal{R})$, the general solution of the field equations, for an arbitrary potential, can be obtained in a closed form, and for at least three particular choices of the potential they can be expressed in an exact analytical form. We have showed that the behaviour of the solutions is strongly dependent on the initial conditions for the scalar field ϕ and its derivative at $r = 0$. Depending on the initial conditions given, rather arbitrary due to the large number of field configurations that can be constructed, we can distinguish two classes of solutions.

The first class are the solutions that become singular for a finite value of the rescaled radial coordinate, ξ. One of such solutions arises in the null potential case, if the initial value of the field is positive and its derivative is negative, then $C_0 > 0$ and both the metric and the string tension become infinite at $\xi_\infty = 4\phi_0\phi_0'$, and the scalar field vanishes at that point. On the other hand, if both ϕ_0 and ϕ_0' are positive, both the string tension and the metric tend to zero at infinity, and the scalar field becomes singular when ξ tends to infinity.

The second class of solutions can be found in the case of the potential of the form $V = V_0\phi^{3/4}$. The requirement of the positivity of the metric tensor for $r = 0$ imposes the condition that the metric tensor and the string tension are decreasing functions of r, since both ϕ_0 and ϕ_0' are required to be positive. In this case it is the scalar field that becomes singular at infinity. However, it is possible to construct finite string configurations: defining a string radius by introducing an effective cuttoff length, ξ_{co} for both the metric and the scalar field. This radius would allow us to get finite values for the scalar field, the metric tensor components and the string tension. For this type of potential the choice of ξ_{co} should be made based on empirical considerations, such as consistency with observational data.

For the exponential scalar field potentials, one can impose the condition of the vanishing of the tension $\sigma(r)$ to define the radius of the string. Despite being strongly dependent of the initial conditions of the scalar field and its derivatives at $r = 0$, the string tension vanishes at $\xi_{cr} \approx 0.6$, which would allows us to define the string radius of the order of magnitude $R_s \approx 0.6\sqrt{2/V_0\lambda}\xi_{cr}$.

The case of the Higgs-type potential is quite different, since the string tension does not vanish for any value of the radial coordinate but it does reaches its minimum value at $\xi \approx 3.5$, where the scalar field is also at its minimum and $W^2(\xi)$ is singular and tends to zero. Another possible choice for the string radius could be $\xi \approx 1$, the first zero of the metric tensor component; in this case both the string tension and the scalar field are at their maxima. Or we could, as in the exponential potential, introduce a cutoff radius to be determined by confrontation with observations, matching the solution with the cosmological background or another GR string solution, thus achieving a well behaved structure throughout the radius of the string.

Following [114], the standard regularity and asymptotic conditions required for string-like solutions of the gravitational field equations (regularity on the axis, at infinity and finite mass per unit length) are not, in general, satisfied by our solutions. Either the metric tensor components, the scalar field or the total mass become singular at infinity. From a physical point of view, this may be due to two factors, the first one being the use of the scalar tensor representation of the hybrid theory. As pointed out in [114], string solutions in scalar-tensor field models do not satisfy the asymptotic conditions, which may also be the case for the scalar-tensor formulation of the hybrid theory. The second one is a natural consequence of the theory: the hybrid metric-Palatini gravity naturally explains the recent accelerations of the Universe, and represents a viable alternative to dark energy models. Thus, an effective cosmological constant appears as a consequence of the theory. Far from gravitating sources, and using spherical symmetry, the scalar field behaves as:

$$\phi(r) \approx \phi_0^{(\mathrm{vac})} + \frac{2G\phi_0^{(b)}M}{3r}e^{-m_\phi r}, \tag{7.2}$$

with the effective mass m_ϕ given by

$$m_\phi = \frac{1}{3}\left[2V - V_{,\phi} - \phi(1+\phi)V_{,\phi\phi}\right]\Big|_{\phi=\phi_0^{(b)}},$$

where $\phi_0^{(b)}$ is the background value of the scalar field. For the metric perturbations we obtain [65, 115],

$$h_{00}^{(2)} = \frac{2G_{\mathrm{eff}}M}{r} + \frac{V_0^{(\mathrm{vac})}r^2}{6\left(1+\phi_0^{(b)}\right)}, \tag{7.3}$$

where $V_0^{(b)} = V\left(\phi_0^{(b)}\right)$ is the background value of the scalar field potential, and G_{eff} is the effective gravitational constant. It is obvious that $h_{00}^{(2)}$ diverges as $r \to \infty$. This implies that the vacuum at infinity is of the de Sitter type, and that generally the solutions of the field equations in hybrid metric-Palatini gravity are not asymptotically flat. Hence the divergence at infinity of our solutions is an indicator of the presence of a vacuum energy (an effective cosmological constant) at infinity. This also indicates that our string solutions are of cosmological type, with the singular behaviour happening at a radius of the order of the Hubble radius. Hence, on can construct regular string solutions in the range of astrophysically relevant distances.

Since the hybrid metric-Palatini gravity theory is intrinsically "cosmological", with the vacuum at infinity being of de Sitter type, in order to construct physically realistic solutions a cutoff procedure must be introduced, in order to avoid any pathological behavior of the geometrical and physical quantities. If we assume that the cosmic strings extend indefinitely with the increasing distance from the central circular line, the resulting spacetime is not asymptotically flat, but of de Sitter type. This is due to the presence of the (effective) cosmological constant in the Universe. There are two possibilities to estimate the upper bound for the cutoff of the strings. The first one, which we call cosmological cutoff, is based on the definition of the string radius as the distance from the circular line at which the decreasing density profile of the string becomes smaller than the average energy density of the Universe. More specifically, we define the cosmological cutoff radius ξ_s of the string as the finite distance from the $r = 0$ circular line that satisfies the following two conditions: a) the scalar field takes its background value $\phi(\xi_s) = \phi_0$, and b) the string tension satisfies the condition $\sigma(\xi_s) \leq \rho_{\text{univ}}$, where ρ_{univ} is the mean energy density of the Universe. The mean density of the Universe as well as the numerical value of the effective cosmological constant can be also expressed by using the density parameters $\Omega = \rho_{\text{univ}}/\rho_{\text{crit}}$ and $\Omega_\Lambda = \Lambda c^2/3H_0^2$, where H_0 is the Hubble constant, and $\rho_{\text{crit}} = 3H_0^2/8\pi G$ is the critical density of the Universe.

We can obtain a simple qualitative estimate of the extension of a cosmic string from the following simple computation: by assuming that the (effective) cosmological constant has a numerical value of the order of $\Lambda = 3 \times 10^{-56}$ cm^{-2} [29], at a distance from the circular line of the order of $r = 100$ kpc, for a supermassive cosmic string with mass $M = 10^{10} M_\odot$, the quantities $GM/r = 4.94 \times 10^{-9}$ and $\Lambda r^2 = 2.7 \times 10^{-9}$, appearing in Eq. (7.3), are roughly of the same order of magnitude. For radial distances greater than 100 kpc the effects of the cosmological constant become dominant. This example shows that even in the case of an extremely heavy cosmic string the infinities of the theory are of astrophysical type, of the order of hundreds of kiloparsecs, much smaller than the "cosmological infinities" of the order of the Hubble radius of the Universe, which is around 3000 megaparsecs. On the other hand, for the case of a small mass cosmic string with $M = 10^{-20} M_\odot$, the gravitational potential energy and cosmological expansion energy are of the same order of magnitude for $r = 0.11$ kpc only.

As an example of the application of the cosmological cutoff procedure we will consider the case of the simplest string solution, corresponding to $V = 0$, and given by Eqs. (5.40), (5.43), and (5.44), respectively. Here, in order to define a proper radius of the string, we require that at some finite radius ξ_s the scalar field reaches its background value, $\phi\left(\xi_s\right) = \phi_0^{(b)}$, which gives for ξ_s the expression

$$\xi_s = 4C_0 + \left(\frac{\phi_0^{(b)}}{C}\right)^{1/4}. \tag{7.4}$$

The condition of the equality of the string tension with the critical density of the Universe, $\left|\sigma\left(\xi_s\right)\right| = \rho_{\text{crit}} = 3H_0^2/8\pi G$, gives for the integration constant C the expression $C = \left(H_0^2/32\pi G\right)^2 \phi_0^{(b)}$, while for $W^2\left(\xi_s\right)$ we obtain $W^2\left(\xi_s\right) = w_0^2 C^{3/2}\left(\phi_0^{(b)}\right)^{-3/2}$. The metric is not asymptotically flat, and for $\xi > \xi_s$ the cosmological background dominates the string dynamics.

A second possibility for defining a cutoff radius for the string is to match the hybrid metric-Palatini gravity string-like solutions with another, exterior string solution. If at radii exceeding some radius specific r_s the effects of the scalar field may become negligibly small, and hybrid metric-Palatini gravity reduces to standard general relativity, then the present string solutions may go smoothly into a general relativistic cylindrically symmetric solution of the Einstein field equations. For example, by assuming that in a certain limit the string-like metrics obtained in the present study reduce to the metric (7.1), we can again define a finite string radius by imposing the conditions $W^2\left(\xi_s\right) = \left(1 - 8\pi G\mu\right)\beta^2\xi_s^2$, and $\sigma\left(\xi_s\right) = \mu$, where the mass per unit length of the string must satisfy the condition $\mu = \text{constant}$ for all $\xi \geq \xi_s$.

In the case of the zero potential, the matching of the two metrics gives the string radius as

$$\xi_s = 4C_0 + \sqrt{\frac{12}{\mu}}, \tag{7.5}$$

with μ obtained as a solution of the algebraic equation

$$w_0^2\left(\frac{\mu}{12}\right) = \left(1-8\pi G\mu\right)\beta^2\left(4C_0+\sqrt{\frac{12}{\mu}}\right)^2. \tag{7.6}$$

For the scalar field at the string boundary we obtain the expression

$$\phi\left(\xi_s\right) = \frac{144C}{\mu^2}. \tag{7.7}$$

On the other hand, for some scalar field potential types one could define the boundary of the string as the point where the energy density of the string vanishes. But even in this case one should match the string solution either with the cosmological background, or with an exterior string solution.

An important geometrical quantity, the angular deficit $\Delta\theta$ in the cylindrical symmetry, due to the presence of the string, is given by [116]

$$\Delta\theta = 2\pi\left[1-\lim_{r\to\infty} W'\left(r\right)\right], \tag{7.8}$$

In the first order of approximation, and for strings with finite extension, we may replace $W'\left(\infty\right)$ with $W'\left(R_s\right)$ in Eq. (7.8), where R_s is the string radius, thus obtaining

$$\Delta\theta \approx 2\pi\left[1-W'\left(R_s\right)\right]. \tag{7.9}$$

In the variable ξ, the angular deficit can be represented as

$$\Delta\theta \approx 2\pi\left[1-\frac{1}{\beta}W'\left(\xi_s\right)\right]. \tag{7.10}$$

Since the variation of W depends on the initial conditions of the scalar field on the $r=0$ circular line, the string-like geometries obtained in the present study allow for a very large range of deficit angles, which significantly impact the geometry of the spacetime near the string. For the solutions with $\lim_{\xi\to\infty} W(\xi)\to 0$, we generally also find $\lim_{\xi\to\infty} W'(\xi)\to 0$, like, for example, in the zero potential case with ϕ_0 and ϕ_0' positive. In this case we obtain $\Delta\theta \approx 2\pi$. In the opposite limit of $\lim_{\xi\to\infty} W'(\xi)\to\infty$, the deficit angle is formally infinite. We can still define a finite deficit angle by introducing a cut-off radius R_s that formally defines the radius of the string. For the exponential and Higgs potentials one can define explicitly the radius of the string-like objects, which also allows the explicit estimation of the deficit angle.

7.2 Remarks on the solutions obtained in the generalised theory

In the study of cosmic strings in the generalized version of the hybrid theory, the gravitational field equations can be obtained in terms of the metric tensor component $W^2(r)$, and of the two scalar fields φ and ψ. However, in the Lagrangian of the scalar-tensor formulation of the theory, the coupling between the scalar fields and the Ricci tensor is of the form $(\phi - \psi)R$. In order to obtain realistic gravitational theories the coupling term $\phi - \psi$ must be positive. This important physical aspect can be implemented by performing a transformation of the fields so that $\phi, \psi) \rightarrow \left(\xi^2, \psi\right)$, where $\xi^2 = \phi - \psi$. The positivity of the gravitational coupling, giving the physical consistency of the solutions, can then be formulated as $\xi^2(r) > 0, \forall r$. Regions of the space-time where $\xi^2 < 0$ can be excluded.

As we've seen in section 6.2.1.1, for the case of zero potential, the metric of the string can be written as

$$ds^2 = -dt^2 + dr^2 + C^2\left(1 + \frac{c_2}{r}\right)^2 r^2 d\theta^2 + dz^2, \tag{7.11}$$

where c_2 and C are constants. For $c_2 = 0$ the metric of the string reduces to the standard general relativistic form, with $C^2 = 1 - \pi G\mu$. However, there is an important distinction between the cosmic string solutions in generalized HMPG theory, and standard GR, namely, that the string tension is negative for the solution with $V = 0$, and, for $c_2 = 0$, $-\kappa^2\sigma(r) \propto 1/r^2$. In this situation the tension diverges for $r = 0$. On the other hand, for $c_2 \neq 0$, the string tension $\sigma(0)$ is finite (negative), but one needs to impose a boundary value for $W^2(0)$. It has been shown in [111] that the energy-momentum tensor of a specific type of wormhole is identical to the energy-momentum tensor of a negative tension classical string. On the other hand, standard field theoretical models of strings predict positive string tensions. Thus, generalized HMPG theories provide a natural mechanism for generating negative string tension, and hence these models may open some new avenues of research in the field of traversable wormholes.

In the case of the constant, nonzero potential, the field equations can again be solved exactly, and some simple expressions for the geometrical and physical parameters can be obtained. The metric has a similar functional form as for the $V = 0$ case, but the string tension can be made positive by an appropriate choice of the potential. A solution with a constant string tension $\kappa^2\sigma = \Lambda/2$ can also be constructed, as well as a solution having $W(r) = W_0 r$, which can describe the standard general relativistic string if $W_0^2 = 1 - 8\pi G\mu$. The string radius R_s can be uniquely defined, and it is given in terms of the constant potential, as well as of two integration constants. Assuming that the string tension is positive in $r = 0$, the string radius is also positive.

For other, more complicated forms of the potential it seems very difficult to obtain exact analytical solutions of the field equations, therefore they must be solved numerically. This way, it is possible to construct large classes of numerical cosmic string models in generalized HMPG theory. In our study we have considered two distinct types of potentials, having specific algebraic structures in the two scalar fields, and obtained either in an additive, or a multiplicative way. We have considered four such forms of the potential. In the first two cases we have considered that the potential depends on only one of the two scalar fields, so that $\bar{V}(\xi,\psi) = \bar{V}_0\xi^2$, $\bar{V}(\xi,\psi) = \bar{V}_0\xi^4$, and $\bar{V}(\xi,\psi) = \bar{V}_0\psi^2$, respectively. For the other two potentials, we have adopted a multiplicative algebraic structure, $\bar{V}(\xi,\psi) = \bar{V}_0\xi^2\psi^2$, and an additive structure, $\bar{V}(\xi,\psi) = a\xi^2 + b\psi^2$, respectively.

Since the potentials depend on at least one extra constant parameter, together with the five boundary conditions we obtain a very large boundary parameter space, containing from six to nine arbitrary parameters. The large number of parameters permits the construction of a large number of different numerical cosmic string models. However, we have restricted the set of parameters, as well as the physical nature of the solutions, by imposing three physical constraints, namely, that the string tension should be positive inside the string, and vanish at the vacuum boundary, that the string must have a well defined and unique radius R_s, obtained from the condition $\sigma(R_s) = 0$, and that $\xi^2 > 0$, $\forall r \in [0, R_s]$. Even after imposing this set of restrictions, a large variety of string models in generalized HMPG theory can be obtained.

For the four considered potentials we have obtained two types of behavior of the metric tensor component $W^2(r)$. It is a monotonically increasing function of r for the potential $V(\xi,\psi) - \bar{V}_0\psi^2$, and a monotonically decreasing function for the other three potentials. For the numerical integration of the field equations we have chosen a nonzero value for $W(0)$, $W(0) \neq 0$, since, at least for the considered cases, no numerical solution satisfying the condition $W(0) = 0$ exists. From the point of view of the sign of $\sigma(r)$, we have considered only solutions, with $\sigma(r)$ taking positive values, monotonically decreasing inside the string, and vanishing at the vacuum boundary, respectively. The string radius is obtained from the condition $\sigma(R_s) = 0$, which uniquely determines its numerical value.

We have also investigated in detail the effects of the variation of the boundary conditions on the cosmic string configuration. Generally, the string models in generalized HMPG theory are very sensitive to any variations of the boundary conditions for all potentials. The functions $W^2(r)$, $\xi^2(r)$ and $\psi(r)$ are strongly affected by any small modification of the five boundary values that describe the string configuration. Since the parameter space of the boundary conditions is very large, the results presented in this investigation did not cover all the possible cosmic string structures that could be generated from the variations of the values in the set $\left(\alpha_0, \psi_0, W(0), u_0, v_0\right)$

7.3 Future prospects

Cosmic strings have a number of very intriguing properties. For example, as suggested by Witten [117], strings behave like superconducting wires, hence they can interact with external cosmic electromagnetic fields, and as they move through cosmic magnetic fields they can develop electric currents. Therefore, short electromagnetic and highly beamed bursts can be emitted from some peculiar points (cusps), located on small string segments, where the velocity approaches the speed of light [118–122]. So, the cusp is a powerful source of electromagnetic radiation that may produce a jet of accelerated particles, which, in turn, may assume an important part in many astrophysical phenomena, such as Gamma Ray Bursts and afterglow emissions. It would be interesting to consider superconducting strings in the framework of modified theories of gravity, and in particular in hybrid metric-Palatini gravity. Such a study would offer some possibilities regarding discriminating standard cosmic strings from string-like structures that appear in modified theories of gravity.

Another important physical effect that could, at least in principle, discern standard general relativistic cosmic strings, cosmic strings in modified gravity, and other filamentary matter distributions is Gravitational Lensing. According to the standard general relativistic string scenario, the curvature of the spacetime is not changed by a vacuum string. However, the topology of the spacetime is modified [123]. Hence photon beams are not bent by a cosmic string, but if two light rays travel on different sides of the string, the presence of the specific conical structure of the spacetime geometry determines their later convergence at the same point of observation [124]. Therefore, for a cosmic string located between a terrestrial observer and a distant cosmological source, the observer will detect two images of the light emitting source, separated by an angle $\delta\theta = 8\mu_s \sin\alpha D_{\rm LS}/D_{\rm OS}$, where by μ_s we have denoted the linear mass density of the string, α represents the angle between the observer-source direction and the string, while $D_{\rm OS}$ and $D_{\rm LS}$ represent the distance between the observer and the source, and the lens and the source, respectively. It follows that, for the case of the standard general relativistic conical string, due to the string presence, the two images formed are identical to the original source, without any distortion or amplification [124]. This effect is very different from the gravitational lensing by gas filaments, which show a very different image structure, formed from one or three elongated images [124].The lensing properties of the string solutions in hybrid metric-Palatini gravity obtained in the present study could help in discriminating between these string solutions and the corresponding solutions obtained in standard string theory, or other modified gravity models.

The array of modified gravity models at our disposal is tremendous, and very successful theories have emerged in recent years, the detailed study of which is crucial for our understanding of the Universe. $f(R,T)$[125], DGP models [126] and other brane-world approaches, as well as lagrangian generalizations, by including non-linear curvature invariants, show promising results, but need a more profound investigation.

The study of topological defects, as well as black hole solutions and the impact on cosmological observables, using Einstein-Boltzmann codes like EFTCAMB [127] (in which the EFT formalism is used in a perturbative approach where an extra degree of freedom appears only at the perturbations level [128]) in the context of different types of modified gravity theories is fundamental to accompany the effort put in state-of-the-art facilities to come to fruition in the upcoming decades.

Appendix A

General Relativity from variational principles

In this section we will deduce the Einstein Field Equations, eq (1.10), by applying variational methods to the action (1.11). We will start by using the metric procedure and than the Palatini procedure, to conclude that, in general, they don't genrate the same equations of motion, which happens only after assumptions are made regarding the connection of the pseudo-Riemannian manifold.

A.1 Metric procedure

In the metric approach, the only dynamical variable of the manifold is the metric tensor, $g_\mu \nu$. So, we will vary the Einstein-Hilbert action with respect to the inverse metric tensor, $g^{\mu\nu}$ to recover the Einstein Field Equations.

$$\delta S = \int \left[\frac{\sqrt{-g}}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} + \sqrt{-g} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} + \mathcal{L}_m \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu} \mathrm{d}^4 x \tag{A.1}$$

After taking $\sqrt{-g}$ outside the square brackets, and applying the action principle $\delta S = 0$, we get the following equation:

$$\frac{1}{2\kappa} \frac{\delta R}{\delta g^{\mu\nu}} + \frac{R}{2\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} - \frac{\Lambda}{\kappa} \frac{1}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} + \frac{\mathcal{L}_m}{\sqrt{-g}} \frac{\delta \sqrt{-g}}{\delta g^{\mu\nu}} = 0 \tag{A.2}$$

Using the results

$$\frac{\delta R}{\delta g^{\mu\nu}} = R_{\mu\nu} \tag{A.3}$$

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\sqrt{-g}\frac{g_{\mu\nu}}{2} \tag{A.4}$$

$$T_{\mu\nu} = \mathcal{L}_m g_{\mu\nu} - 2\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} \tag{A.5}$$

We obtain the Einstein field equations, eq. (1.10).

A.2 Palatini Procedure

In the Palatini procedure the metric and the connection are independent variables, meaning that the connection is no longer assumed to be metric compatible a priori.

The action to be varied in this case is

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\ \mathcal{R} - 2\Lambda + \mathcal{L}_m\ d^4x. \tag{A.6}$$

where the Palatini-Ricci scalar is constructed by the contraction of the Palatini-Ricci tensor, obtained from the independent connection

$$\mathcal{R}_{\mu\nu} = \partial_\alpha\hat{\Gamma}^\alpha_{\mu\nu} - \partial_\nu\hat{\Gamma}^\alpha_{\mu\alpha} + \hat{\Gamma}^\alpha_{\alpha\beta}\hat{\Gamma}^\beta_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\beta}\hat{\Gamma}^\beta_{\alpha\nu}. \tag{A.7}$$

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{A.8}$$

In general, the Palatini-Ricci tensor, and the Einstein tensor derived from it, are asymmetric. In fact, it can be shown that the general form of the connection is

$$\hat{\Gamma}^\mu_{\alpha\beta} = \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} + K^\mu_{\alpha\beta} + L^\mu_{\alpha\beta} \tag{A.9}$$

where the term in brackets represents the Levi-Civita connection, the tensor field $K^\mu_{\alpha\beta}$ is the contortion tensor, defined in terms of the torsion $S^\mu_{\alpha\beta} = \Gamma^\mu_{[\alpha\beta]}$

$$K^\mu_{\alpha\beta} = \frac{1}{2}\left(S^\mu{}_{\alpha\beta} - S_{\alpha\beta}{}^\mu - S_{\beta\alpha}{}^\mu\right) \tag{A.10}$$

and the tensor field $L^\mu_{\alpha\beta}$ is defined in terms of the non-metricity tensor $Q_{\mu\alpha\beta} = \nabla_\mu g_{\alpha\beta}$ as

$$L^\mu_{\alpha\beta} = \frac{1}{2}\left(Q^\mu{}_{\alpha\beta} - Q_{\alpha\beta}{}^\mu - Q_{\beta\alpha}{}^\mu\right) \tag{A.11}$$

If we now procede to perform variations of the action (A.6) with respect to the metric tensor, we arrive to the analogous of the Eisntein field equations,

$$G_{(\mu\nu)} + \Lambda g_{\mu\nu} = \kappa^2 \hat{T}_{\mu\nu} \tag{A.12}$$

where only the symmetric part of Einstein tensor appears and the analogous of the energy-momentum, defined at constant $\Gamma^{\mu}_{\alpha\beta}$ in the variation, is present. Since the matter fields do not couple with the connection, the Palatini energy-momentum tensor will be the same as the "original" energy-momentum tensor, but that will not be true in general.

Performing now the variation of the action with respect to the connection results in the vanishing of the so-called Palatini tensor:

$$P^{\alpha\beta}_{\mu} = \frac{\kappa^2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}R)}{\delta\Gamma^{\mu}_{\alpha\beta}} = 0 \tag{A.13}$$

This equation reveals that the Palatini approach does not give rise to a unique set of field equations, since the connection has 64 independent components and the previous equation only provides 60 constrains, thus leaving 4 degrees of freedom. However, the Palatini approach uniquely yields the EFE if one makes further assumptions regarding the connection, for instance if the connection is assumed torsionless or metric-compatible, both of which should be imposed through Lagrange multipliers on the action.

Appendix B

Variational Approach to f(R) theories

In this appendix, we will deduce the dynamical equations from the variation of the action for the $f(R)$ theories. We will use both the metric and the Palatini formalisms and show that, in the most general case, they don't yield the same equations, which happens for the GR-limit (where some assumptions are made regarding the connection).

B.1 Dynamic Equations from the metric procedure.

We shall start with the action for the $f(R)$ theories:

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\, f(R) + \mathcal{L}_m \, d^4x. \tag{B.1}$$

In the metric formalism, the only dynamical entity is the metric $g_{\mu\nu}$, so we'll perform variations on the action with respect to the metric and make use of the action principle ($\delta S = 0$), which will result in the following equation :

$$f'(R)\, R_{\mu\nu} - \frac{1}{2} f(R)\, g_{\mu\nu} - \left[\nabla_\mu \nabla_\nu - \Box g_{\mu\nu}\right] f'(R) = \kappa^2 T_{\mu\nu}, \tag{B.2}$$

where f' denotes the derivative of $f(R)$ with respect to the argument. And the matter-energy tensor is defined in the usual way as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \tag{B.3}$$

General Relativity theory can be seen as a particular case of the $f(R)$ theories, in which $f(R) = R$. In this case the third and fourth terms on the l.h.s of B.2 vanish and the equation reduces to the Einstein Field Equations 1.9.

In this approach we can "force" the appearance of the Einstein tensor $G_{\mu\nu} = \kappa^2 T^{(eff)}_{\mu\nu}$ by moving the extra terms to the r.h.s. and encapsulate them in an effective energy-momentum tensor of the form

$$T^{(eff)}_{\mu\nu} = \frac{1}{f'(R)} \left\{ T_{\mu\nu} + \frac{1}{\kappa^2} \left[\frac{1}{2} g_{\mu\nu} \left(f(R) - R f'(R) \right) + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box \right) f'(R) \right] \right\}. \quad \text{(B.4)}$$

As expected, in the GR limit, where $f(R) = R$, the effective energy-momentum tensor reduces to the original $T_{\mu\nu}$.

B.2 Dynamic Equations from the Palatini procedure.

We will now consider the Palatini approach to $f(R)$ theories.

In the Palatini case, the action for the theory is

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g}\; f(\mathcal{R}) + \mathcal{L}_m \; d^4x. \quad \text{(B.5)}$$

where $\mathcal{R}$ is the Ricci-Palatini scalar, obtained through the contraction of the Ricci-Palatini tensor that, in turn, is obtained as a function of an independent connection $\hat{\Gamma}$:

$$\mathcal{R}_{\mu\nu} = \partial_\alpha \hat{\Gamma}^\alpha_{\mu\nu} - \partial_\nu \hat{\Gamma}^\alpha_{\mu\alpha} + \hat{\Gamma}^\alpha_{\alpha\beta} \hat{\Gamma}^\beta_{\mu\nu} - \hat{\Gamma}^\alpha_{\mu\beta} \hat{\Gamma}^\beta_{\alpha\nu}. \quad \text{(B.6)}$$

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} \quad \text{(B.7)}$$

Since we have now two independent dynamic variables, the metric and the connection, we make variations of B.5 with respect to both, yielding

$$f'(\mathcal{R}) \mathcal{R}_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R}) g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad \text{(B.8)}$$

$$\hat{\nabla}_\alpha \left[\sqrt{-g} f'(\mathcal{R}) g^{\mu\nu} \right] = 0, \quad \text{(B.9)}$$

where $\mathcal{R}_{(\mu\nu)}$ is the symmetric part of the Ricci-Palatini tensor. We can see now two important features of the Palatini approach regarding the limit $f(\mathcal{R}) = \mathcal{R}$; from the first equation we get the Einstein Field Equations (1.9), while from the second equation we see that the independent connection is the Levi-Civita connection for the metric. Thus, we conclude that in the GR limit, both formalism give rise to the same equations of motion but, unlike in the metric approach, in the Palatini formalism the fact that the connection is Levi-Civita is not an assumption, but rather a dynamical consequence.

In order to illustrate the difference between the purely Palatini gravity and the hybrid theory, we will now obtain the scalar-tensor representation of the $f(\mathcal{R})$ theory. In order to do so, we will define an auxiliary field B and rewrite the action as

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[f(B) + f'(B)(\mathcal{R} - B)\right] d^4x + S_m. \tag{B.10}$$

Defining the scalar field as $\phi = f'(\mathcal{R})$, and the potential as $V(\phi) = B\phi - f(B)$ eq. (B.10) becomes

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[\phi\mathcal{R} - V(\phi)\right] d^4x + S_m \tag{B.11}$$

The action is now dependent on two dynamical variables, the metric and the scalar field. Performing the variation with respect to both yields

$$\phi G_{\mu\nu} = \kappa^2 T_{\mu\nu} - \frac{3}{2\phi}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi\right) + \left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\phi - \frac{V}{2}g_{\mu\nu}, \tag{B.12}$$

$$\Box\phi = \frac{\phi}{3}\left[R - V'(\phi)\right] + \frac{1}{2\phi}\nabla_\alpha\phi\nabla^\alpha\phi. \tag{B.13}$$

As a last step towards the dynamics of the scalar field, we take the trace of eq. (B.12) and with it cancel the R-term in eq. (B.13), obtaining

$$\phi V'(\phi) - 2V(\phi) = -\kappa^2 T. \tag{B.14}$$

which is a Klein Gordon equation for the scalar field. As we can see, the only dependence of ϕ is with the matter distribution, unlike the hybrid theory, where the dependence is not only with T, but also with R (eq. (3.18)).

Appendix C

Equations of motions for the generalized hybrid metric-Palatini theory

In this appendix, we will show the full derivation of the equations of motion for the Generalized hybrid metric-Palatini gravity, both in the geometric representation (eqs. (3.20) and (3.21)) and the scalar-tensor representation (eqs. (3.29) to (3.33)).

C.1 Equations of motion in the geometric representation

We shall start by recalling the action for the theory, eq. (3.19):

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\, f(R, \mathcal{R}) + \mathcal{L}_m \, d^4x. \tag{C.1}$$

In order to obtain the equations of motion, we will have to make the variation with respect to each independent variable, in this case the metric and the independent connection.

C.2 Equations of motion in the scalar-tensor representation

The action for the scalar-tensor representation of the theory reads

$$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[\left(\varphi - \psi\right) R - \frac{3}{2\psi}\partial^\mu \psi \partial_\mu \psi - V\left(\varphi, \psi\right)\right] d^4x. \tag{C.2}$$

Considering the three independent variables, the metric and both scalar fields, we will start with varying the action with respect to the metric.

$$\begin{aligned}\delta\mathcal{L}_g =&\frac{1}{2\kappa^2}\left\{\delta\sqrt{-g}\left[(\varphi-\psi)\,R-\frac{3}{2\psi}\partial^\alpha\psi\partial_\alpha\psi-V\right]\right.\\&\left.+\sqrt{-g}\left[(\varphi-\psi)\,\delta R-\frac{3}{2\psi}\partial_\mu\psi\partial_\nu\psi\delta g^{\mu\nu}\right]\right\}\end{aligned}\tag{C.3}$$

The variation of the volume element, $\delta\sqrt{-g}$, is given by

$$\delta\sqrt{-g}=-\frac{1}{2\sqrt{-g}}\delta g=-\frac{1}{2\sqrt{-g}}gg^{\mu\nu}\delta g_{\mu\nu}=\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}.\tag{C.4}$$

Writing the variation of the Ricci scalar as $\delta R=R_{\mu\nu}\delta g^{\mu\nu}+g^{\mu\nu}\delta R_{\mu\nu}$ and using

$$\delta g^{\mu\nu}=-g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}.\tag{C.5}$$

$$g^{\mu\nu}\left(\varphi-\psi\right)\delta R_{\mu\nu}=\left(\nabla^\mu\nabla^\nu\left(\varphi-\psi\right)-g^{\mu\nu}\Box\left(\varphi-\psi\right)\right)\delta g_{\mu\nu}\tag{C.6}$$

we obtain the following

$$\begin{aligned}\delta\mathcal{L}_g \;=\;& \frac{1}{2\kappa^2}\sqrt{-g}\left\{\frac{1}{2}g^{\mu\nu}\left[(\varphi-\psi)\,R-\frac{3}{2\psi}\partial^\alpha\psi\partial_\alpha\psi-V\right]+\frac{3}{2\psi}\partial^\mu\psi\partial^\nu\psi+\right.\\&\left.+\nabla^\mu\nabla^\nu\left(\varphi-\psi\right)-g^{\mu\nu}\Box\left(\varphi-\psi\right)-\left(\varphi-\psi\right)R^{\mu\nu}\right\}\delta g_{\mu\nu}\end{aligned}\tag{C.7}$$

If we apply the definition of the energy-momentum tensor given in eq. (3.5) we get

$$\begin{aligned}\kappa^2T_{\mu\nu}=-\frac{1}{2}&\left[(\varphi-\psi)\,R-\frac{3}{2\psi}\partial^\alpha\psi\partial_\alpha\psi-V\right]g_{\mu\nu}-\frac{3}{2\psi}\partial_\mu\psi\partial_\nu\psi\\&-\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box-R_{\mu\nu}\right)(\varphi-\psi).\end{aligned}\tag{C.8}$$

finally arriving to eq. (3.29) by making use of the definition of the Einstein tensor, $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$ to obtain a more convenient form of the previous equation:

$$\begin{aligned}\left(\varphi-\psi\right)G_{\mu\nu}=&\kappa^2T_{\mu\nu}-\left(\Box\left(\varphi-\psi\right)+\frac{1}{2}V+\frac{3}{4\psi}\partial^\alpha\psi\partial_\alpha\psi\right)g_{\mu\nu}+\\&+\frac{3}{2\psi}\partial_\mu\psi\partial_\nu\psi+\nabla_\mu\nabla_\nu\left(\varphi-\psi\right).\end{aligned}\tag{C.9}$$

As for the variation of the action with respect to the scalar fields, we will start by varying with respect to ψ:

$$\delta\mathcal{L}_\psi = \frac{1}{2\kappa^2}\sqrt{-g}\left(\frac{3}{2\psi^2}\partial^c\psi\partial_\alpha\psi\delta\psi - \frac{3}{\psi}\partial^\alpha\psi\partial_\alpha\delta\psi - V_\psi\delta\psi - R\delta\psi\right) \tag{C.10}$$

We will now integrate the second term on the r.h.s. by part, which yields

$$\int\sqrt{-g}\frac{3}{\psi}\partial^\mu\psi\partial_\mu\delta\psi d^4x = \int\left[\partial_\mu\left(\sqrt{-g}\frac{3}{\psi}\delta\psi\partial^\mu\psi\right) - \partial_\mu\left(\sqrt{-g}\frac{3}{\psi}\partial^\mu\psi\right)\delta\psi\right]d^4x. \tag{C.11}$$

The first term is a boundary term, that vanishes by definition. The second term can be expanded as

$$\partial_\mu\left(\sqrt{-g}\frac{3}{\psi}\partial^\mu\psi\right) = -\frac{3}{\psi^2}\sqrt{-g}\partial_\mu\psi\partial^\mu\psi + \frac{3}{\psi}\partial_\mu\left(\sqrt{-g}\partial^\mu\psi\right). \tag{C.12}$$

The second term of the previous equation can be written as a D'Alembert operator, the general expression of which can be put as $\Box\psi = \nabla^\mu\nabla_\mu\psi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right)$. As so, by inserting these results in equation (C.10), we get

$$\delta\mathcal{L}_\psi = \frac{1}{2\kappa^2}\sqrt{-g}\left(-R - \frac{3}{2\psi^2}\partial_\mu\psi\partial^\mu\psi + \frac{3}{\psi}\Box\psi - V_\psi\right)\delta\psi, \tag{C.13}$$

which leads to the modified Klein-Gordon equation for the scalar field ψ

$$-R - \frac{3}{2\psi^2}\partial_\mu\psi\partial^\mu\psi + \frac{3}{\psi}\Box\psi - V_\psi = 0. \tag{C.14}$$

The variation with respect to the field φ is more straightforward:

$$\delta\mathcal{L}_\varphi = \frac{1}{2\kappa^2}\sqrt{-g}\left(R - V_\varphi\right), \tag{C.15}$$

which leads directly to

$$R - V_\varphi = 0. \tag{C.16}$$

In order to obtain eqs. (3.32) and (3.33), we must take the trace of equation (C.9), to remove the Ricci scalar from (C.14) and (C.16). This will lead to the following equations, which give the dynamics of the combination of both fields:

$$\Box\psi - \frac{\psi}{\varphi}\Box\varphi - \frac{1}{2\psi}\partial^\mu\psi\partial_\mu\psi - \frac{\psi}{3\varphi}\left[2V + (\varphi-\psi)V_\psi\right] = -\frac{\kappa^2\psi}{3\varphi}T \tag{C.17}$$

$$\Box\varphi - \Box\psi + \frac{1}{2\psi}\partial^{\mu}\psi\partial_{\mu}\psi + \frac{1}{3}\left[2V - \left(\varphi - \psi\right)V_{\varphi}\right] = \frac{\kappa^2}{3}T \tag{C.18}$$

But after some algebraic manipulation, we finally arrive to the effective Klein Gordon equations for the fields, in the form of eqs. (3.32) and (3.33):

$$\Box\psi - \frac{1}{2\psi}\partial^{\mu}\psi\partial_{\mu}\psi - \frac{\psi}{3}\left(V_{\varphi} + V_{\psi}\right) = 0, \tag{C.19}$$

$$\Box\varphi + \frac{1}{3}\left(2V - \psi V_{\psi} - \varphi V_{\varphi}\right) = \frac{\kappa^2 T}{3}. \tag{C.20}$$

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