GENERICITY OF HISTORIC BEHAVIOR
FOR MAPS AND FLOWS

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ABSTRACT. We establish a sufficient condition for a continuous map, acting on a compact metric space, to have a Baire residual set of points exhibiting historic behavior (also known as irregular points). This criterion applies, for instance, to a minimal and non-uniquely ergodic map; to maps preserving two distinct probability measures with full support; to non-trivial homoclinic classes; to some non-uniformly expanding maps; and to partially hyperbolic diffeomorphisms with two periodic points whose stable manifolds are dense, including Mañé and Shub examples of robustly transitive non-hyperbolic diffeomorphisms. This way, our unifying approach recovers a collection of known deep theorems on the genericity of the irregular set, for both additive and sub-additive potentials, and also provides a number of new applications.

1. INTRODUCTION

Let $f : (X, \mathcal{A}) \to (X, \mathcal{A})$ be a measurable transformation and μ be an f-invariant ergodic probability measure on the σ-algebra $\mathcal{A}$. For a measurable function $\varphi : X \to \mathbb{R}$ and $x \in X$, the sequence of Birkhoff averages of φ at x is given by $\left(1/n \sum_{j=0}^{n-1} \varphi(f^j(x))\right)_{n \in \mathbb{N}}$. A point $x \in X$ is said to be φ-regular if the limit of this sequence exists; otherwise x is called a φ-irregular point (and said to have historic behavior [33, 38]). Birkhoff's ergodic theorem asserts that the time averages of φ at μ-almost every point in X converge to the space average $\int_X \varphi \, d\mu$. So the set of φ-regular points carries full μ measure. This result supports Boltzman ergodic hypothesis but fails to describe the behavior and the complexity of the set of points at which the sequence of Birkhoff averages has no limit. Nowadays there is a well established theory to assess how big is the *irregular set* (also called the set of points with historic behavior): contrary to the previous measure theoretical description, the set of these non-typical points may be Baire generic and, moreover, have full topological pressure, full Hausdorff dimension or full metric mean dimension, as attested in [3, 7, 8, 25, 29, 40] and references therein.

Not surprisingly, the research addressing these phenomena started in the realm of uniform hyperbolicity. On the one hand, the existence of many periodic points ensures that there exist continuous observable maps with distinct spaces averages. On the other hand, a hyperbolic structure, even if non-uniform, comes up with a panoply of means to reconstruct true orbits from finite pieces of orbits: specification, gluing orbit property and non-uniform versions of these. Yet, these properties are seldom valid for strong partially hyperbolic transitive diffeomorphisms (see [12, 36] and references therein). Furthermore, according to [37], no minimal and positive entropy homeomorphism has the gluing orbit property, which is one of

the weakest versions on the aforementioned chain of concepts. This may explain the scarcity of results, as far as we know, regarding the irregular set for minimal dynamics.

One may think of minimal and uniquely ergodic dynamics as the natural opposites to hyperbolic dynamics, with much lower level of complexity. This impression is somehow reinforced by the fact that minimal dynamics admit very simple Rohlin towers (see e.g. [5, Lemma 6]), and by the uniform convergence of Birkhoff averages in the case of uniquely ergodic transformations. Notwithstanding, examples of volume preserving analytic diffeomorphisms on $\mathbb{T}^2$ with zero entropy, which are minimal though not uniquely ergodic (due to Furstenberg [19]), or minimal homeomorphisms of the torus with positive entropy (constructed by Herman and Rees [20, 32]) show that the setting is surprisingly rich.

In this paper we present a simple criterion to ensure that the set of points with historic behavior, for maps or flows, is a Baire generic subset of the ambient space, both for additive and sub-additive sequences. The main results are presented in Section 2. Their proofs are a consequence of the existence of an everywhere discontinuous first integral and of a general statement, we will show on Section 4, regarding the accumulation points of sequences of continuous observable functions. This reasoning provides a unified approach to several contexts where this kind of result has already been established, besides bringing forward new applications. Indeed, our assumptions are satisfied by a vast class of discrete and continuous time dynamics, including minimal non-uniquely ergodic homeomorphisms, non-trivial homoclinic classes, continuous maps with the specification property, Viana maps, partially hyperbolic diffeomorphisms and singular-hyperbolic attractors (cf. Corollaries I – X). In the broader context of sub-additive sequences of potentials, we show how to describe more accurately the irregular points for Lyapunov exponents of linear cocycles, as well as those points for which the convergence in the Brin-Katok formula for the metric entropy of weak Gibbs measures fails. Therefore, we establish or improve a wide range of results on the genericity of the irregular set, including [7, 9, 43], as an outcome of an easy criterion.

2. Main results

Given a compact metric space (X, d) and a continuous map $f : X \to X$, the set of irregular points is defined by

$$\mathfrak{I} = \left\{ x \in X \colon \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \quad \text{does not exist in the weak* topology} \right\}. \qquad (2.1)$$

For each continuous function $\varphi : X \to \mathbb{R}$, the set of φ-irregular points is given by

$$\mathfrak{I}_\varphi = \left\{ x \in X \colon \left(\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) \right)_{n \in \mathbb{N}} \quad \text{does not converge} \right\}. \qquad (2.2)$$

Notice that, as φ is bounded, the sequence of Birkhoff averages of φ at x does not converge if and only if it has no limit. Moreover, $\mathfrak{I}_\varphi \subset \mathfrak{I}$. Associated to φ, consider the map

$$x \in X \quad \mapsto \quad L_\varphi(x) = \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)). \qquad (2.3)$$

Observe that L_φ is f-invariant, that is, $L_\varphi \circ f = f$. Thus, L_φ is a so called first integral for f; for more information regarding smooth first integrals for endomorphisms, see [21, 26].

As a particular case of a more general statement we prove on Section 4, we will show that the existence of a non-trivial and discontinuous first integral L_φ conveys a topologically large set of irregular points. To the best of our knowledge, this sufficient condition has not appeared before in the literature, so it primarily provides a new criterion for genericity of irregular sets.

Theorem A. *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map and $\varphi : X \to \mathbb{R}$ be a continuous observable. Assume that there exist two dense subsets $\mathcal{A}$ and $\mathcal{B}$ of X such that the restrictions of L_φ to $\mathcal{A}$ and to $\mathcal{B}$ are constant, though the value at $\mathcal{A}$ is different from the one at $\mathcal{B}$. Then $\mathfrak{I}_\varphi$ is a Baire residual subset of X.*

Let us now list a few applications of Theorem A in a variety of settings. Given a measurable map $f : X \to X$ and an f-invariant probability measure μ, the ergodic basin of attraction of μ is the set of points $x \in X$ such that $\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu$ in the weak* topology. A first consequence of Theorem A is the following result, whose main assumption is fulfilled, for instance, whenever f preserves two distinct probability measures with full support.

Corollary I. *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map preserving two distinct Borel probability measures with dense ergodic basins. Then $\mathfrak{I}$ is a Baire residual subset of X.*

Corollary I is suited, for example, to continuous maps with the specification property or the gluing orbit property (we refer the reader to [13] for the definitions), providing a simpler and alternative proof of the genericity of the irregular set in that setting. Indeed, if the map f satisfies the specification property then [15, Propositions 21.12 and 21.14] ensure not only that generic invariant measures are full supported but also that their basins are dense.

Corollary II. *Let X be a compact metric space and $f : X \to X$ be a continuous map satisfying the specification property. There exists a Baire generic subset of points in X with historic behavior.*

Corollary I also applies to endomorphisms with critical or singular behavior (e.g. quadratic maps, Lorenz interval maps or Viana maps) under just a few requirements. Actually, Theorem 5 in [30, p. 928] indicates that any strongly transitive $C^{1+\alpha}$-map on a compact Riemannian manifold with a periodic point which does not lie in the forward orbit of the critical or singular set admits an uncountable number of ergodic and full supported invariant measures. Let us illustrate this assertion by considering the robust class of multidimensional non-uniformly expanding maps with singularities known as Viana maps. These are skew-products of the type

$$f : \begin{array}{ccc} \mathbb{S}^1 \times \mathbb{R} & \to & \mathbb{S}^1 \times \mathbb{R} \\ (x, y) & \mapsto & \big(dx \,(\mathrm{mod}\,1),\, a(x) - y^2\big) \end{array} \qquad (2.4)$$

where $d \geqslant 2$, $a(x) = a_0 + \alpha \sin(2\pi x)$ and $a_0 \in (1, 2)$ is chosen so that 0 is pre-periodic for the quadratic map $h(x) = a_0 - x^2$ (cf. [41] for more details). According to [30, Theorem 10], Viana maps have an uncountable number of ergodic probability measures whose support is the whole set $f(S^1 \times \mathbb{R})$. Thus their basins are dense in the attractor $f(S^1 \times \mathbb{R})$, and so there exists a Baire generic subset $\mathcal{R}$ of points in the attractor $\Lambda = f(\mathbb{S}^1 \times \mathbb{R})$ with historic behavior. Besides, f is a local diffeomorphism except in the critical set $\mathcal{C} = \mathbb{S}^1 \times \{0\}$, which is closed and has empty interior in $\mathbb{S}^1 \times \mathbb{R}$. More precisely, identifying $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$, one can

write $(\mathbb{S}^1 \times \mathbb{R}) \setminus (\mathbb{S}^1 \times \{0\}) = \bigcup_{i=\pm} \bigcup_{j=0}^{d-1} \mathcal{A}_{i,j}$, where

$$\mathcal{A}_{+,j} = \left[\frac{j}{d}, \frac{j+1}{d}\right) \times (0, +\infty) \qquad \text{and} \qquad \mathcal{A}_{-,j} = \left[\frac{j}{d}, \frac{j+1}{d}\right) \times (-\infty, 0)$$

and $f\mid_{\mathcal{A}_{i,j}} \colon \mathcal{A}_{i,j} \to f(\mathcal{A}_{i,j})$ is a diffeomorphism for every $i = \pm$ and $0 \leqslant j \leqslant d-1$. Consequently, the union $\bigcup_{i=\pm} \bigcup_{j=0}^{d-1} (f\mid_{\mathcal{A}_{i,j}})^{-1}(\mathcal{R})$ is a Baire residual subset of $\mathbb{S}^1 \times \mathbb{R}$ whose points exhibit historical behavior.

Corollary III. *Consider a map $f : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ as in* (2.4)*. There exists a Baire generic subset of points in $\mathbb{S}^1 \times \mathbb{R}$ with historic behavior.*

It is known that the irregular set associated to a uniquely ergodic dynamics is empty (cf. [42, Theorem 6.19]). Besides, some of the most interesting known examples of minimal non-uniquely ergodic homeomorphisms have zero topological entropy. Thus, in this setting, it is useful to describe the complexity of the irregular set using Baire category arguments instead of other measurements of chaos. The next consequence of Corollary I benefits precisely from this strategy.

Corollary IV. *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous minimal map. Then either there exists an f-invariant ergodic Borel probability measure μ such that, in the weak* topology,*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu \qquad \forall\, x \in X$$

or the set $\mathfrak{I}$ is Baire residual in X.

Corollary IV applies, for instance, to the minimal and non-uniquely ergodic homeomorphisms constructed in [19] and to the minimal and non-uniquely ergodic interval exchange transformation provided by [23]. Its statement appears, though with a distinct formulation, in [20, Proposition 6.3] and [18, Lemma 3].

A fifth consequence of Theorem A concerns the irregular set for partially hyperbolic diffeomorphisms. Given a smooth compact manifold X with dimension $\dim(X) \geqslant 2$, one says that a diffeomorphism $f \in \mathrm{Diff}^1(X)$ is *partially hyperbolic* if there exists a Df-invariant splitting $TX = E^s \oplus F$ and constants $C > 0$ and $\lambda \in (0, 1)$ such that, for every $x \in X$ and every $n \in \mathbb{N}$, one has:

(a) $\|Df^n(x)(v)\| \leqslant C\lambda^n \|v\|$ for every $v \in E_x^s$;

(b) for every pair of unitary vectors $v \in E_x^s$ and $w \in F_x$,

$$\frac{\|Df^n(x)v\|}{\|Df^n(x)w\|} \leqslant C\lambda^n.$$

Property (a) means that the sub-bundle E^s is uniformly contracting, while a splitting satisfying property (b) is called *dominated*. It is well known that, under these assumptions, for every point $x \in X$ there exists a C^1-submanifold called stable manifold $W^s(x)$ passing at x and tangent to E_x^s, and the collection of stable manifolds defines a stable foliation on X. In several relevant examples of partially hyperbolic diffeomorphisms it is known that the stable foliation is minimal, meaning that all the stable manifolds are dense in X. We will establish the following variant of Theorem A within the setting of those partially hyperbolic diffeomorphisms.

Corollary V. *Let X be a compact Riemannian manifold and $f : X \to X$ be a partially hyperbolic diffeomorphism with two distinct periodic points whose stable manifolds are dense in X. Then $\mathfrak{I}$ is Baire residual in X.*

This corollary applies, for instance, to the open set of robustly transitive diffeomorphisms on $\mathbb{T}^4$ with minimal stable foliations constructed by Shub in [35] and to Mañé's examples [27]. A dual statement for partially hyperbolic diffeomorphisms having a dominated splitting into center and unstable sub-bundles with a minimal unstable foliation can be proved similarly.

It is worth noticing that our results also allow us to deal with important classes of proper subsets that are invariant by the dynamics, as is the case of non-trivial homoclinic classes. If X is a compact Riemannian manifold, the *homoclinic class $H(p, f)$* associated to a hyperbolic saddle periodic point p of a diffeomorphism $f \in \text{Diff}^1(X)$ is defined as the closure of transverse intersections of the unstable and the stable manifolds of the orbit of p. It is known that there exists a Baire residual subset of irregular points on each non-hyperbolic homoclinic class of C^1-generic diffeomorphisms of X (see [9]). The next modification of Corollary V improves [9] since it actually applies to arbitrary non-trivial homoclinic classes of any C^1-diffeomorphism.

Corollary VI. *Let X be a compact Riemannian manifold. Given $f \in \text{Diff}^1(X)$ and a hyperbolic saddle periodic point p by f, then either $H(p, f) = \{p\}$ or the set $\mathfrak{I} \cap H(p, f)$ is a Baire residual subset of $H(p, f)$.*

The previous result complements [1, Proposition 9.1], which asserts that the set of points with historic behavior is Baire generic in the closure $\overline{W^s(\mathcal{O}(p))}$ of the basin of attraction of any non-trivial homoclinic class $H(p)$. This proposition does not impart the same information of the previous corollary since it is not directly applicable to the homoclinic class itself and $H(p)$ is a meager subset of $\overline{W^s(\mathcal{O}(p))}$ even in the hyperbolic context. However, the proof of [1, Proposition 9.1] might be adjusted to convey the statement of Corollary VI.

At this moment, it is natural to ask whether Theorem A and its corollaries can be adapted to deal with the broader context of non-additive sequences, which are relevant to the computation of several dynamical quantities, such as entropy and Lyapunov exponents. The case of almost additive or asymptotically additive sequences of continuous functions carries no extra difficulties (cf. Remark 7.1); on the contrary, the case of continuous sub-additive sequences requires further explanation.

A sequence $\Phi = (\varphi_n)_{n \in \mathbb{N}}$ of continuous maps $\varphi_n : X \to \mathbb{R}$ is *sub-additive* if

$$\varphi_{m+n} \leqslant \varphi_m \circ f^n + \varphi_n \qquad \forall\, m, n \in \mathbb{N}.$$

Accordingly, the set of Φ-irregular points is defined by

$$\mathfrak{I}_\Phi = \left\{ x \in X \colon \left(\frac{1}{n}\, \varphi_n(x) \right)_{n \in \mathbb{N}} \quad \text{does not converge} \right\}. \tag{2.5}$$

Kingman's sub-additive ergodic theorem guarantees that $\mathfrak{I}_\Phi$ has zero measure with respect to every f-invariant Borel probability measure on X. In this context, the map M_Φ defined by

$$x \in X \quad \mapsto \quad M_\Phi(x) = \inf_{n \in \mathbb{N}} \frac{1}{n}\, \varphi_n(x) \tag{2.6}$$

is a Lyapunov function associated to the discrete dynamical system f. More precisely, the inequality

$$\frac{1}{n+1}\, \varphi_{n+1}(x) \leqslant \frac{1}{n+1} \left[\, \varphi_n(f(x)) + \sup_{z \in X}\, \varphi_1(z) \,\right]$$

ensures that $M_\Phi(x) \leqslant M_\Phi(f(x))$ at every $x \in X$.

Although the function M_Φ is measurable, because it is the infimum of a sequence of continuous functions, in general one can not rely on higher regularity. This is an evidence that the previous concept of Lyapunov function scarcely describes the irregular set $\mathfrak{I}_\Phi$, though it suggests the following version of Theorem A for sub-additive sequences.

Theorem B. *Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map and $\Phi = (\varphi_n)_{n \in \mathbb{N}}$ be a sub-additive sequence of continuous maps on X. Assume that there exist two dense subsets $\mathcal{A}$ and $\mathcal{B}$ of X such that the restrictions of M_Φ to $\mathcal{A}$ and to $\mathcal{B}$ are constant, though the value at $\mathcal{A}$ is different from the one at $\mathcal{B}$. Then $\mathfrak{I}_\Phi$ is Baire residual in X.*

In the aftermath of the previous result, we secure a more precise description of the Lyapunov irregular points, that is, points that are non-typical for Oseledets' theorem. Given a compact metric space (X, d), a continuous $f : X \to X$ and a continuous *linear cocycle* $A : X \to GL(k, \mathbb{R})$, $k \geqslant 2$, we assign to the cocycle the skew-product given by

$$F_A: \quad \begin{array}{ccc} X \times \mathbb{R}^k & \to & M \times \mathbb{R}^k \\ (x, v) & \mapsto & (f(x), A(x)v). \end{array}$$

Set $A^n(x) := A(f^{n-1}(x)) \dots A(f(x)) A(x)$ for each $x \in X$ and $n \in \mathbb{N}$. By Oseledets' theorem the *largest Lyapunov exponent* (respectively the smallest Lyapunov exponent) associated to an f-invariant ergodic Borel probability measure μ is given by

$$\lambda^+(A, \mu) = \inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x)\| \qquad \text{for } \mu\text{-a.e. } x$$

(respectively $\lambda^-(A, \mu) = \sup_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1}$ for μ-a.e. x). Using Theorem B, whenever the dynamics f is minimal we get additional information on the set of points whose Lyapunov exponent fails to be well defined.

Corollary VII. *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous minimal map. Given $k \geqslant 2$ and a continuous linear cocycle $A \in C^0(X, GL(k, \mathbb{R}))$, either*

$$\inf_{x \in X} \inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x)\| = \sup_{x \in X} \inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x)\|$$

and the previous value is the unique possible largest Lyapunov exponent associated to any f-invariant ergodic probability measure, or there exists a Baire residual subset $\mathcal{R} \subset X$ such that

$$\forall x \in \mathcal{R} \qquad \liminf_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\| < \limsup_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$

A dual statement, concerning the smallest Lyapunov exponent, holds as well. We note that Corollary VII improves [18, Theorem 4], where the author considers $GL(2, \mathbb{R})$-valued cocycles over minimal and uniquely ergodic maps, and whose argument is exclusive to 2-dimensional linear cocycles. We also observe that, in a complementary direction, the non-existence of Lyapunov exponents for Hölder continuous matrix cocycles over maps satisfying exponential specification has been considered in [39]. As far as we know, no other references address this problem of identification of topologically big sets of irregular points for sub-additive sequences of potentials.

As another illustration of the scope of Theorem B's applications, let us consider a continuous map $f : X \to X$ on a compact metric space, a sub-additive sequence of continuous

functions $\Phi = (\varphi_n)_{n \in \mathbb{N}} \in C(X)^{\mathbb{N}}$ and a probability measure μ which is *weak Gibbs* with respect to Φ. For such a measure μ and each $x \in X$ we may find a sequence of positive constants $(K_n(x))_{n \in \mathbb{N}}$ satisfying $\lim_{n \to +\infty} \frac{1}{n} \log K_n(x) = 0$ and

$$K_n^{-1}(x) \leqslant \frac{\mu(B_n(x, \varepsilon))}{e^{-n\,P(\Phi) + \varphi_n(x)}} \leqslant K_n(x) \qquad \forall\, x \in X \quad \forall\, n \in \mathbb{N} \tag{2.7}$$

where $B_n(x, \varepsilon) := \{y \in X : d_n(x, y) < \varepsilon\}$ stands for the dynamical ball centered at x with radius ε and length n. These measures appear naturally in the context of equilibrium states for matrix cocycles $A : \{1, \ldots, d\}^{\mathbb{N}} \to GL(d, \mathbb{C})$ over the shift map σ (cf. [17]), with the weak-Gibbs condition then rewritten as

$$K^{-1} e^{-n\,P(q)} \|A^{(n)}(x)\|^q \leqslant \mu(B_n(x, \varepsilon)) \leqslant K e^{-n\,P(q)} \|A^{(n)}(x)\|^q \tag{2.8}$$

where $A^{(n)}(x) = A(\sigma^{n-1}(x)) \ldots A(\sigma(x)) A(x)$, $P(q)$ is the pressure function of the family $\Phi_q = (q \log \|A^n(\cdot)\|)_{n \in \mathbb{N}}$ and $K > 0$ is a constant such that $K_n(x) = K$ for every $n \in \mathbb{N}$ and any $x \in X$.

Now, when a probability measure μ is f-invariant, the entropy $h_\mu(f)$ can be estimated using dynamical balls and the Brin-Katok formula [6] by

$$h_\mu(f) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) \qquad \text{at } \mu \text{ a.e. } x. \tag{2.9}$$

If, in addition, f is expansive, the previous $\limsup$ does not depend on the value of ε if this is kept small enough. When μ is also weak Gibbs with respect to a sub-additive sequence $\Phi = (\varphi_n)_{n \in \mathbb{N}} \in C(X)^{\mathbb{N}}$, then there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ one has

$$\limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = P(\Phi) + \limsup_{n \to +\infty} \frac{1}{n} \varphi_n(x) \qquad \forall\, x \in X$$

and

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = P(\Phi) + \liminf_{n \to +\infty} \frac{1}{n} \varphi_n(x) \qquad \forall\, x \in X.$$

For instance, f may be the one-sided full shift on a finite alphabet (whose pre-orbit of every point is dense) and μ be the measure of maximal entropy of f (which is weak Gibbs with respect to $(\varphi_n \equiv 0)_{n \in \mathbb{N}}$). In this setting, Theorem B implies that either

$$h_{\text{top}}(f) = \lim_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon))$$

for every $x \in X$ and any $0 < \varepsilon \leqslant \varepsilon_0$, or the set of points for which the limit in Brin-Katok's formula does not exist is a Baire residual subset of X. More generally:

Corollary VIII. *Let (X, d) be a compact metric space and $f : X \to X$ be a continuous expansive map. Given a Borel f-invariant probability measure μ which is weak Gibbs with respect to a sub-additive sequence $\Phi = (\varphi_n)_{n \in \mathbb{N}} \in C(X)^{\mathbb{N}}$, there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ one has either*

$$h_\mu(f) = \lim_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = P(\Phi) + \lim_{n \to +\infty} \frac{1}{n} \varphi_n(x) \qquad \forall\, x \in X$$

or the set of points for which

$$\liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) < \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon))$$

is a Baire residual subset of X.

3. The case of flows

The proofs of Theorems A and B apply *verbatim* to continuous $\mathbb{R}$-actions if one replaces Birkhoff averages by the suitable means obtained by integration along the orbits of the flow. Nevertheless, in some particular cases one can reduce the analysis of the continuous-time to the discrete-time setting. In order to illustrate such an application, given a continuous flow $(Y_t)_{t \in \mathbb{R}}$ on a compact metric space X define the irregular set by

$$\mathfrak{I} = \left\{ x \in X \colon \lim_{t \to +\infty} \frac{1}{t} \int_0^t \delta_{Y_s(x)} \, ds \quad \text{does not exist in the weak* topology} \right\}. \tag{3.1}$$

If there exists a pair of distinct Borel probability measures with dense ergodic basins, the situation can be reduced to the discrete-time setting in the sense that the following result is a consequence of Corollary I.

Corollary IX. *Let (X, d) be a compact metric space and $(Y_t)_{t \in \mathbb{R}}$ be a continuous flow on X preserving two distinct Borel probability measures with dense ergodic basins. Then $\mathfrak{I}$ is a Baire residual subset of X.*

Proof. Let μ_1 and μ_2 be two distinct $(Y_t)_{t \in \mathbb{R}}$-invariant Borel probability measures whose ergodic basins $\mathbb{B}(\mu_1)$ and $\mathbb{B}(\mu_2)$ are dense in X. Pick $\varphi \in C^0(X, \mathbb{R})$ such that $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$. As a consequence of [31], for each $i = 1, 2$ there exists a Baire residual subset $\mathcal{R}_i \subset \mathbb{R}$ of times such that μ_i is ergodic for the time-t map Y_t associated to $t \in \mathcal{R}_i$. Now, fix an arbitrary $T \in \mathcal{R}_1 \cap \mathcal{R}_2$, and consider the homeomorphism $f = Y_T$ and the potential $\varphi_T := \frac{1}{T} \int_0^T \varphi(Y_s(x)) \, ds$.

The sets $\mathcal{A} = \mathbb{B}(\mu_1)$ and $\mathcal{B} = \mathbb{B}(\mu_2)$ are f-invariant and, by assumption, dense in X. Besides, by the ergodicity of μ_1 and μ_2 with respect to f, the map L_{φ_T} is constant in $\mathcal{A}$ and $\mathcal{B}$, and equal to $\int \varphi \, d\mu_1$ and $\int \varphi \, d\mu_2$, respectively. Thus, Corollary I implies that $\mathfrak{I}$ is Baire residual in X. $\qquad \square$

Corollary IX implies, along the same lines used in the proof of Corollary VI, that every non-trivial homoclinic class of a vector field has a Baire generic subset of points with historic behavior. Since every singular-hyperbolic attractor in dimension three is a homoclinic class (cf. [4] for the definition and proofs), we conclude the following:

Corollary X. *Let M be a three-dimensional compact Riemannian manifold, $(X_t)_t$ be a C^1 flow and Λ be a singular-hyperbolic attractor. The set of points with historic behavior is Baire residual in Λ.*

We note that, as a singular-hyperbolic attractor of a C^2 vector field supports a non-atomic ergodic hyperbolic measure (cf. [2, 24]), the conclusion of Corollary X improves item (7) of [4, Theorem A] since the corollary's statement comprises C^1 vector fields as well.

Observe that Corollary X also holds for multidimensional singular-hyperbolic attractors of typical C^1 vector fields, extending the recent result of D. Yang in [43]. Indeed, by [14, Theorem B], for a C^1 open and dense set of vector fields their singular-hyperbolic Lyapunov stable chain-recurrence class is a homoclinic class.

It is known that geometric Lorenz attractors on three or higher dimensional compact Riemannian manifold are homoclinic classes [10, 11]. Thus, Corollary X improves [22], where the authors show that the set of points with historic behavior for the geometric Lorenz attractor is residual in a trapping region of the attractor.

We refer the reader to [28], where the authors introduce the concept of length averages for singular foliations and study their existence for codimension one C^1 foliations on compact surfaces.

4. PRELIMINARY RESULT

Let (X, d) be a compact metric space and consider a sequence $\psi = (\psi_n)_{n \in \mathbb{N}}$ of continuous functions $\psi_n \colon X \to \mathbb{R}$. Define the map

$$x \in X \quad \mapsto \quad U_\psi(x) = \limsup_{n \in \mathbb{N}} \psi_n(x)$$

and consider the set

$$C_\psi = \left\{ x \in X \colon \left(\psi_n(x) \right)_{n \in \mathbb{N}} \quad \text{does not converge} \right\}.$$

Theorem 4.1. *Assume that there exist two dense subsets $\mathcal{A}$ and $\mathcal{B}$ of X such that the restrictions of the map U_ψ to $\mathcal{A}$ and to $\mathcal{B}$ are constant, though the value at $\mathcal{A}$ is different from the one at $\mathcal{B}$. Then C_ψ is a Baire residual subset of X.*

Proof. Suppose that the constant value of U_ψ at the dense sets $\mathcal{A}$ and $\mathcal{B}$ are α and β, respectively, with $\alpha \neq \beta$. Fix $0 < \varepsilon < \frac{1}{6} |\alpha - \beta|$. Since the map ψ_n is continuous for every $n \in \mathbb{N}$, given a positive integer N the set

$$\Lambda_N = \left\{ x \in X \colon \left| \psi_n(x) - \psi_m(x) \right| \leqslant \varepsilon \qquad \forall\, m, n \geqslant N \right\} \tag{4.1}$$

is closed in X.

Proposition 4.2. *Λ_N has empty interior for every $N \in \mathbb{N}$.*

Proof. Assume that there exists $N \in \mathbb{N}$ such that the interior of Λ_N, we denote by $\mathrm{int}(\Lambda_N)$, is non-empty, and take $\lambda \in \mathrm{int}(\Lambda_N)$. As $\mathcal{A}$ and $\mathcal{B}$ are dense in X, there exist sequences $(p_n)_{n \in \mathbb{N}} \in \mathcal{A}$ and $(q_n)_{n \in \mathbb{N}} \in \mathcal{B}$ such that

$$\forall\, n \in \mathbb{N} \quad p_n, q_n \in \mathrm{int}(\Lambda_N) \qquad \text{and} \qquad \lim_{n \to +\infty} p_n = \lambda = \lim_{n \to +\infty} q_n. \tag{4.2}$$

Lemma 4.3. *For every $N \in \mathbb{N}$, if a sequence $(x_k)_{k \in \mathbb{N}}$ of elements of Λ_N converges, then*

$$\left| \limsup_{k \to +\infty} U_\psi(x_k) - U_\psi\left(\lim_{k \to +\infty} x_k \right) \right| \leqslant 3\,\varepsilon.$$

Proof. Given $N \in \mathbb{N}$, take a convergent sequence $(x_k)_{k \in \mathbb{N}}$ contained in Λ_N and consider $\ell = \lim_{k \to \infty} x_k$, which is in Λ_N. By the definition of Λ_N, one has

$$\left| \psi_n(x_k) - \psi_m(x_k) \right| \leqslant \varepsilon \qquad \forall\, m, n \geqslant N \quad \forall\, k \in \mathbb{N}$$

$$\left| \psi_n(\ell) - \psi_m(\ell) \right| \leqslant \varepsilon \qquad \forall\, m, n \geqslant N.$$

Fixing $m = N$ and taking the limit as n goes to $+\infty$ in the first inequality along subsequences that attain the lim sup, we conclude that

$$\forall\, k \in \mathbb{N} \quad \left|U_\psi(x_k) - \psi_N(x_k))\right| \leqslant \varepsilon \qquad \text{and} \qquad \left|U_\psi(\ell) - \psi_N(\ell)\right| \leqslant \varepsilon.$$

By the compactness of (X, d) and the uniform continuity of ψ_N, we may choose $\delta_N > 0$ such that

$$d(z, w) < \delta_N \quad \Rightarrow \quad \left|\psi_N(z) - \psi_N(w)\right| < \varepsilon.$$

Altogether, this proves that, for $k \in \mathbb{N}$ large enough so that $d(x_k, \ell) < \delta_N$, one has

$$\left|U_\psi(x_k) - U_\psi(\ell)\right| \leqslant 3\,\varepsilon.$$

In particular,

$$\left|\limsup_{k \to +\infty} U_\psi(x_k) - U_\psi(\ell)\right| \leqslant 3\,\varepsilon$$

as claimed. □

Let us resume the proof of the Proposition 4.2. As $U_\psi(p_n) = \alpha$ and $U_\psi(q_n) = \beta$ for every $n \in \mathbb{N}$, the conditions (4.2) and Lemma 4.3 imply that

$$|\alpha - U_\psi(\lambda)| \leqslant 3\,\varepsilon \quad \text{and} \quad |\beta - U_\psi(\lambda)| \leqslant 3\,\varepsilon.$$

So $|\alpha - \beta| \leqslant 6\,\varepsilon$, contradicting the choice of ε. Thus, Λ_N must have empty interior. This completes the proof of the proposition. □

Finally, observe that the set of points $x \in X$ whose sequence $\big(\psi_n(x)\big)_{n \in \mathbb{N}}$ converges is contained in the countable union $\bigcup_{N=1}^{+\infty} \Lambda_N$ of closed sets with empty interior. This ends the proof of Theorem 4.1. □

5. Proof of Theorem A

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous map and $\varphi : X \to \mathbb{R}$ be a continuous observable such that there exist two dense subsets $\mathcal{A}$ and $\mathcal{B}$ of X such that the restrictions of the map L_φ to $\mathcal{A}$ and to $\mathcal{B}$ are constant, equal to α and β, respectively, and $\alpha \neq \beta$. To prove Theorem A we just run the argument used to show Theorem 4.1 with the following adaptations:

(1) The sequence $(\psi_n)_{n \in \mathbb{N}}$ is made of the Birkhoff averages of f with respect to φ, that is, for every $x \in X$ and every $n \in \mathbb{N}$,

$$\psi_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)).$$

(2) The map U_ψ is precisely L_φ (cf. definition (2.3)).

(3) For every $N \in \mathbb{N}$, the set Λ_N is now defined by

$$\Lambda_N = \left\{ x \in X \colon \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \frac{1}{m} \sum_{j=0}^{m-1} \varphi(f^j(x)) \right| \leqslant \varepsilon \qquad \forall\, m, n \geqslant N \right\}.$$

(4) The set C_ψ becomes $\mathfrak{I}_\varphi$ (cf. definition (2.2)).

6. Proof of Corollary I

Let (X, d) be a compact metric space, $f : X \to X$ be a continuous map and μ_1 and μ_2 two distinct f-invariant Borel probability measures whose ergodic basins $\mathbb{B}(\mu_1)$ and $\mathbb{B}(\mu_2)$ are dense in X. Choose $\varphi \in C^0(X, \mathbb{R})$ such that $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$, and consider $\mathcal{A} = \mathbb{B}(\mu_1)$ and $\mathcal{B} = \mathbb{B}(\mu_2)$. In $\mathcal{A}$ and $\mathcal{B}$ the map L_φ is constant, and equal to $\int \varphi \, d\mu_1$ and $\int \varphi \, d\mu_2$, respectively. Moreover, by assumption, these sets are dense in X. Thus, Theorem A guarantees that $\mathfrak{I}_\varphi$ is Baire residual in X, and so $\mathfrak{I}$ is generic as well.

Remark 6.1. Another proof of this corollary could be obtained as follows. For each f-invariant Borel probability measure μ take $\mathcal{E}_N(\mu)$ as the set of points $x \in X$ for which there exists $n > N$ such that $\left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi \, d\mu \right| < \frac{1}{N}$. If there are $\mu_1 \neq \mu_2$ with dense basins, then the set $\mathcal{E}_N(\mu_1) \cap \mathcal{E}_N(\mu_2)$ is open and dense in X for every $N \geqslant 1$. Moreover, the set of φ-irregular points contains the Baire residual subset $\bigcap_{N \geqslant 1} \mathcal{E}_N(\mu_1) \cap \mathcal{E}_N(\mu_2)$. This argument is similar to the one in [34, Lemma 3.3].

7. Proof of Corollary IV

Let (X, d) be a compact metric space and $f : X \to X$ be a continuous minimal map. If f is uniquely ergodic and μ denotes the unique f-invariant probability measure, then the sequences of Birkhoff averages of every continuous observable map $\varphi \in C^0(X, \mathbb{R})$ are uniformly convergent in X to the constant $\int_X \varphi \, d\mu$. Thus, in the weak* topology, one has

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu \qquad \forall \, x \in X.$$

Assume now that there exist two distinct f-invariant Borel ergodic probability measures μ_1 and μ_2. Choose $\varphi \in C^0(X, \mathbb{R})$ such that $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$ and take two points $p_1 \in \mathbb{B}(\mu_1)$ and $p_2 \in \mathbb{B}(\mu_2)$, where $\mathbb{B}(\mu_i)$ denotes the ergodic basin of attraction of μ_i. These are dense subsets of X, since each contains a dense orbit, within which L_φ is constant, equal to $\int \varphi \, d\mu_1$ in $\mathbb{B}(\mu_1)$ and to $\int \varphi \, d\mu_2$ in $\mathbb{B}(\mu_2)$. Therefore, we may apply Corollary I, concluding that the set $\mathfrak{I}$ is Baire residual in X, as claimed.

Remark 7.1. One may ask whether results analogous to Corollary IV hold for non-additive sequences (e.g. almost additive, asymptotically additive or sub-additive). In view of the very recent work [16], the limits of almost additive or asymptotically additive sequence of continuous maps coincide with Birkhoff averages of a suitable continuous observable. In particular, the conclusion of Corollary IV is valid for this more general class of limits and sequences. However, if the sequences are just sub-additive, then this is no longer true. Indeed, for any uniquely ergodic system there exists sub-additive sequences of continuous maps such that the set of non-typical points is Baire generic (cf. [18]).

8. Proof of Corollary V

Let (X, d) be a compact Riemannian manifold and $f : X \to X$ be a partially hyperbolic diffeomorphism. Suppose that f has two distinct periodic points p_1 and p_2 whose stable manifolds, we denote by $W_f^s(p_1)$ and $W_f^s(p_2)$, respectively, are dense in X. We may assume that p_1 and p_2 are fixed by f, taking an appropriate power of f otherwise. Choose $\varphi \in C^0(X, \mathbb{R})$ such that $\varphi(p_1) \neq \varphi(p_2)$ and define $\mathcal{A} = W_f^s(p_1)$ and $\mathcal{B} = W_f^s(p_2)$. These are f-invariant, dense subsets of X, and the map L_φ is constant in each of them, equal to $\varphi(p_1)$ and

$\varphi(p_2)$, respectively. Indeed, as p_1 and p_2 are fixed points by f and φ is continuous, then one has $L_\varphi(x) = L_\varphi(p_1) = \varphi(p_1)$ for every $x \in W_f^s(p_1)$ (and, analogously, $L_\varphi(y) = L_\varphi(p_2) = \varphi(p_2)$ for every $y \in W_f^s(p_2)$) due to the following immediate chain of deductions:

$$x \in W_f^s(p_1) \quad \Leftrightarrow \quad \lim_{n \to +\infty} f^n(x) = p_1 \quad \Rightarrow \quad \lim_{n \to +\infty} \varphi(f^n(x)) = \varphi(p_1)$$

$$\Rightarrow \quad \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \varphi(p_1).$$

Consequently, Theorem A ensures that $\mathfrak{I}$ is a Baire residual subset of X.

9. Proof of Corollary VI

Let X be a compact Riemannian manifold and $f \in \mathrm{Diff}^1(X)$. Assume that $H(p,f) \neq \{p\}$ is a homoclinic class for f associated to a hyperbolic saddle periodic point p. By Birkhoff's theorem, every transversal homoclinic point in $H(p,f)$ is accumulated by hyperbolic periodic orbits with the same index (that is, the dimension of the stable subbundle) as p. In particular, there exists a hyperbolic saddle $q \in H(p,f)$ which does not belong to the orbit $\mathcal{O}(p)$ of p and is homoclinically related to p. This ensures that $W^s(\mathcal{O}(p)) \pitchfork W^u(\mathcal{O}(q)) \neq \emptyset$ and $W^s(\mathcal{O}(q)) \pitchfork W^u(\mathcal{O}(p)) \neq \emptyset$. Now, the λ-lemma guarantees that $W^s(\mathcal{O}(q))$ is dense in $H(p,f)$. This property together with Theorem A imply that $\mathfrak{I}_\varphi \cap H(p,f)$ is a Baire residual subset of $H(p,f)$ for every continuous observable φ whose averages along the orbits of the periodic points p and q differ. This proves the corollary.

10. Proof of Theorem B

Let (X,d) be a compact metric space and $f : X \to X$ be a continuous map and $\Phi := (\varphi_n)_{n \in \mathbb{N}}$ be a sub-additive sequence of continuous functions. Assume that $\mathcal{A}$ and $\mathcal{B}$ are dense subsets of X such that the restrictions of M_Φ to $\mathcal{A}$ and to $\mathcal{B}$ are constant, equal to α and β, respectively, and $\alpha \neq \beta$. Again, to show Theorem B we repeat the proof of Theorem 4.1 after some adjustments:

(1) The sequence $(\psi_n)_{n \in \mathbb{N}}$ is now $(\frac{1}{n}\varphi_n)_{n \in \mathbb{N}}$.

(2) As the sequence $(\psi_n)_{n \in \mathbb{N}}$ is sub-additive, the map U_ψ coincides with M_φ (cf. definition (2.6)) whenever the limit of the sequence exists.

(3) For every $N \in \mathbb{N}$, the set Λ_N is given by

$$\Lambda_N = \left\{ x \in X : \left| \frac{1}{n}\varphi_n(x) - \frac{1}{m}\varphi_m(x) \right| \leqslant \varepsilon \quad \forall\, m, n \geqslant N \right\}.$$

(4) The set C_ψ is precisely what we denoted by $\mathfrak{I}_\Phi$ (cf. definition (2.5)).

11. Proof of Corollary VII

Let (X,d) be a compact metric space, $f : X \to X$ be a continuous minimal map and $A \in C^0(X, GL(k, \mathbb{R}))$, for $k \geqslant 2$. Consider the sequence $\Phi_A = (\varphi_n)_{n \in \mathbb{N}}$, where

$$\varphi_n(x) = \log \|A^n(x)\| \qquad \forall\, x \in X.$$

On the one hand, by the sub-additivity of the sequence $\left(\log \|A^n(x)\| \right)_{n \in \mathbb{N}}$ we conclude that the map

$$M_{\Phi_A}(\cdot) = \inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(\cdot)\|$$

satisfies $M_{\Phi_A}(x) \leqslant M_{\Phi_A}(f(x))$ for every $x \in X$. On the other hand, for each $x \in X$, the reverse inequality $M_{\Phi_A}(x) \geqslant M_{\Phi_A}(f(x))$ follows from the estimate

$$\|A^n(x)\| = \|A^{n-1}(f(x))A(x)\| \geqslant \min_{z \in X} \|A(z)^{-1}\|^{-1} \cdot \|A^{n-1}(f(x))\| \qquad \forall \, n \in \mathbb{N}.$$

Therefore, M_{Φ_A} is a first integral with respect to f. In particular, by the minimality of f, if there are $x_1 \neq x_2 \in X$ such that

$$\inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x_1)\| < \inf_{n \in \mathbb{N}} \frac{1}{n} \log \|A^n(x_2)\|$$

then there exist dense sets $\mathcal{A}$ and $\mathcal{B}$ (namely, the orbits of x_1 and x_2 by f) satisfying the requirements of Theorem B. Thus $\mathfrak{I}_\Phi$ is Baire residual in X.

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