From Non-minimally Coupled Curvature-Matter Gravity to the Newton-Schrödinger System

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João Pedro de Araújo Novo

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"O Binómio de Newton é tão belo como a Vénus de Milo.
O que há é pouca gente para dar por isso."

— Álvaro de Campos

"If you can't say it simply and clearly, keep quiet, and keep working on it till you can."

— Karl Popper

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Resumo

Nesta tese são estudadas teorias não minimamente acoplados da gravitação. Tomando o limite Newtoniano da teoria é possível deduzir as equações hidrodinâmicas para um fluído ideal. Estas diferem das obtidas a partir da Relatividade Geral num contexto específico pelo aparecimento de um potencial de Yukawa e uma força proporcional ao gradiente do da densidade. Efectuando uma transformação de Madelung é possível obter um sistema de Newton-Schrödinger a partir das equações hidrodinâmicas resultantes.

Utilizando métodos numéricos desenvolvidos para sistemas ópticos não lineares e não locais, descritos pelo mesmo tipo de sistemas de equações, foi possível obter previsões sobre a teoria alternativa proposta. Desta forma descobriu-se que a teoria considerada admite soluções estacionárias auto semelhantes. De destacar que estas soluções estacionárias são possíveis mesmo desprezando a pressão do fluído, ou seja, a força extra proporcional ao gradiente do quadrado da densidade é suficiente para contrariar a atração gravítica.

Palavras-chave: Teorias Alternativas da Gravitação, Equação de Schrödinger não linear, Transformação de Madelung, Sistemas de N-corpos, Soluções Estacionárias.

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Abstract

In this work Non-minimally Coupled Curvature-Matter theories of gravity are studied. In the Newtonian limit of this theory, in a particular context, two extra terms arise when compared with General Relativity, a Yukawa force density and an extra force density proportional to the gradient of the mass density. Performing a Madelung transformation the resulting hydrodynamical equations yield a Newton-Schrödinger system.

Using numerical methods developed for the study of non-linear non-local optical systems, described by the same system of equations, it was possible to make predictions about the gravitational model. Namely the theory considered admits self-similar stationary solutions. This stationary solutions hold even when the pressure of the fluid is neglected, this means the extra force proportional to the gradient of the squared matter density is enough to oppose the gravitational attractive forces.

Keywords: Alternative Theories of Gravity, Non-linear Schrödinger Equation, Madelung Transformation, N-body Systems, Stationary Solutions.

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Abbreviations

- GR - General Relativity;
- NMCCM - Non-minimally Coupled Curvature-Matter;
- EH - Einstein-Hilbert;
- NLS - Non-linear Schrödinger;
- FFT - Fast Fourier transforms;
- NS - Newton-Schrödinger;
- SSFM - Split-step Fourier Method;
- i.e. - *id est*;
- GPGPU - General Purpose Graphics Processing Unit.

Chapter 1

Introduction

Proposed by Einstein in 1915 General Relativity (GR) is currently the best description of gravitation and consequently of the Universe at large scales. In GR matter responds to the curvature of the space-time, described by the metric, and the other fields can interact with matter by their contribution to the metric tensor.

GR was immediately very successful explaining the anomalous perihelion advance of Mercury in 1915, that could not be adequately explained using Newtonian Gravity. Since then the theory has passed a series of experimental tests, like the deflection of light by gravity and the prediction of gravitational waves. Despite its immediate success, it took only four years to be questioned by Weyl in 1919 [1] and even Eddington (the scientist responsible for the first experimental verification of GR by measuring the bending of light from a distant star during a solar eclipse in 1919) proposed a modification of the theory in 1923 [2].

Those first attempts of modifying GR were manly driven by scientific curiosity and were viewed as a way to better understand the original theory. However, in the recent decades the original formulation of GR has been challenged in the light of recent astrophysical observations. To explain the flattening of rotation curves of spiral galaxies, dark matter was introduced. The accelerated expansion of the Universe has lead to the introduction of dark energy. There is also the problem of compatibility with a quantum theory, as to describe gravity in the same way as the other three fundamental forces.

The currently accepted model that fits the cosmological observations is the ΛCDM model (Λ-Cold Dark Matter), supplemented by a inflationary scenario normally based on a scalar field, the inflaton. Besides being unable to explain the origin of the inflaton this model is also burdened by the cosmological constant problem, as the value of the cosmological constant based on the vacuum energy of matter fields is huge when compared to the observed value [3].

These problems have motivated various attempts to modify General Relativity. One of these attempts is to generalise the usual Einstein-Hilbert (EH) action by replacing the linear term R in the EH action by an arbitrary function of the curvature scalar, $f(R)$, the so called $f(R)$ theories. Choosing a suitable function of the Ricci scalar to EH action it is possible to explain the evolution of the Universe without the need for dark matter or dark energy [4]. These alternative theories are particularly relevant among higher order gravitation theories as they seem to be the only ones that can avoid Ostrogradski instabilities [5].

Another class of alternative theories of gravitation are the Non-minimally Coupled Curvature-Matter (NMCCM) theories of gravitation [6]. These theories introduce a coupling between matter and curvature by means of a term $f_2(R)\mathcal{L}_m$ in the action, where f_2 is an arbitrary function of R. One of the most striking features of these theories is that the energy-momentum tensor is no longer covariantly conserved due to this coupling. This model has been applied to several problems in modern astrophysics: it can mimic galactic [7] and cluster [8] dark matter; it can mimic a Cosmological Constant at astrophysical scales [9]; it admits post inflationary reheating [10] and the current accelerated expansion of the Universe [11]. This class of theories will be the main subject of this thesis. The non-minimal coupling leads to very convoluted equations of motion such that a complete analytical study is impossible, so numerical tools must be employed if one wants to make predictions of the theory. The aim of this work is to construct such numerical tools to evolve density distributions into the mildly non-linear regime, beyond the point where linear perturbation theory breaks down. This is done by obtaining a Newton-Schrödinger system for the model, starting from the weak-field non-relativistic limit of the field equations for a perfect fluid to obtain the hydrodynamical equations and then performing a Madelung transformation. This approach was first developed by Widrow and Keiser in 1993 [12].

This kind of systems are very common in the field of non-linear non-local optics and optical analogue models, namely, for instance, for boson stars which have been studied experimentally and numerically [13]. So using numerical solvers developed for optical systems it is possible to obtain predictions for these non-minimally coupled theories of gravity. This kind of simulations allows to test NMCCM gravity at large scales in contrast with what was done in Ref. [14].

This thesis is structured as follows: in Chapter 2 formal aspects of GR and alternative theories, namely $f(R)$ and NMCCM, are discussed. In Chapter 3 the weak field non-relativistic limit of a particular model of NMCCM gravity is considered, giving rise to corrections to the hydrodynamical equations for a perfect fluid that are characteristic of the theory. In Chapter 4 Madelung transformations are presented allowing to transform the hydrodynamical equations into a Newton-Schrödinger system, that often appears in non-linear non-local optics, and then are used to study a particular NMCCM gravity model.

In this thesis the metric signature chosen will be $(-, +, +, +)$ with the usual conventions for the indices: Greek indices range from 0 to 3, while Latin ones range from 1 to 3.

Chapter 2

Alternative Theories of Gravity

Alternative theories of gravity have been the object of great interest for solving the problems of GR. The need to explain some astrophysical observations, like the flattening of rotation curves of galaxies and the accelerated expansion of the Universe, has lead physicists in the quest for alternative theories of gravity. In order to explain these phenomena in the light of GR, it is necessary to introduce of dark matter and energy, whose contribution to the matter-energy content of the Universe are 27% and 69%, respectively, in the ΛCDM model [15]. The hope is to find a theory that does not need these *ad-hoc* forms of matter and energy. One of the alternatives is to include higher order curvatures in the EH action. These are called $f(R)$-theories of gravity.

There is an even more general alternative theory of gravity, the non-minimally coupled gravity where a second function, $f_2(R)$, of the curvature scalar is introduced in the action multiplying the matter Lagrangian. This gives rise to a non-minimal coupling between matter and curvature.

In this chapter a brief review of GR, $f(R)$ and NMCCM theories is presented.

2.1 General Relativity

Einstein envisioned gravity as a geometric phenomenon. Thus, in his theory space-time is modelled as a pseudo-Riemannian manifold, a differentiable manifold equipped with a metric tensor that is everywhere non-degenerate (its determinant is non vanishing $g \equiv \det(g_{\mu\nu}) \neq 0$).

The space-time curvature manifests itself by the derivatives of a connection. In standard GR a particular connection is considered, the Levi-Civita connection, which corresponds to the Christoffel symbols:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu} \right) . \tag{2.1}$$

This connection has two useful properties:

- torsion free: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{(\mu\nu)}$;
- metric compatibility: $\nabla_{\rho} g_{\mu\nu} = 0$.

The field equations of GR, can be obtained from the least action principle by choosing an appropriate Lagrangian density. The action that leads to the field equations is known as Einstein-Hilbert action:

$$S = \int \mathrm{d}^4 x \sqrt{|g|} \left[\frac{1}{2\kappa} R + \mathcal{L}_m \right] , \tag{2.2}$$

where R is the curvature scalar, $\kappa = 8\pi G/c^4$, $\mathcal{L}_m = \mathcal{L}_m \left(\Psi_m, \nabla_{\mu} \Psi_m, g_{\mu\nu} \right)$ is the matter Lagrangian that can depend on the matter fields, Ψ_m, and their derivatives as well as the metric $g_{\mu\nu}$, and g is the metric determinant.

Considering a variation of the metric tensor, $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, and imposing $\delta S/\delta g^{\mu\nu} = 0$ one obtains the field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu} , \tag{2.3}$$

with the energy-momentum tensor defined as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|}\mathcal{L})}{\delta g^{\mu\nu}} . \tag{2.4}$$

A relation between the trace of the energy-momentum tensor and the curvature scalar can be obtained by taking the trace of Eq. (2.3):

$$R + \kappa T = 0 . \tag{2.5}$$

This implies that in the vacuum, $T = 0$, the Universe is Ricci flat.

2.2 $f(R)$ theories of gravity

The $f(R)$ theories of gravity consist in introducing a function of the curvature scalar in the action that replaces the linear term of the EH action. If one considers a series of terms in the scalar curvature GR can be regarded as the low curvature limit of $f(R)$ theories. The new action is then:

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{2\kappa}f(R) + \mathcal{L}_m\right]. \tag{2.6}$$

The field equations that follow from this action using the metric formalism are:

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]F(R) = \kappa T_{\mu\nu}\,, \tag{2.7}$$

where $F(R) \equiv \mathrm{d}f/\mathrm{d}R$. Taking the covariant derivative of this equation it is clear that the energy-momentum tensor is still covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$.

The trace of Eq. (2.7) is:

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T\,. \tag{2.8}$$

With the last two equations it is possible to define an effective energy-momentum tensor that encapsulates the effect of the function $f(R)$:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa}{F(R)}\left(T_{\mu\nu} + T^{(eff)}_{\mu\nu}\right)\,, \tag{2.9}$$

where,

$$T^{(eff)}_{\mu\nu} \equiv \left[\frac{f(R) - RF(R)}{2}g_{\mu\nu} + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)F(R)\right]. \tag{2.10}$$

Contrary to GR where R and T are related algebraically through Eq. (2.5), $f(R)$ theories give rise to a differential relation, Eq. (2.8). From this is clear that $f(R)$ theories have a larger set of solutions than GR. Considering the two theories for $T_{\mu\nu} = 0$, in GR the solution is $R = 0$; however, in $f(R)$ theories R needs not to be zero or even constant. Searching for the solutions with constant scalar curvature, that represent the maximally symmetric solutions, it follows from Eq. (2.8):

$$F(R)R - 2f(R) = 0\,. \tag{2.11}$$

This equation has two kinds of solutions with constant curvature. Either $f(R = 0) = 0$ and therefore $R = 0$ is a solution, inserting this result in Eq. (2.7) yields $R_{\mu\nu} = 0$. Or $R = 0$ is not a root of f, and solving the differential equation one finds that $f(R) \propto R^2$, giving rise to two roots $R = \pm C$, then the Ricci tensor, from Eq. (2.7) will be:

$$R_{\mu\nu} = \pm \frac{C}{4} g_{\mu\nu} \, . \tag{2.12}$$

With the two maximally symmetric solutions being the de Sitter $(+C)$ and anti-de Sitter $(-C)$

For a full review of $f(R)$ theories the reader is referred to Refs. [16, 17].

2.2.1 $f(R)$ gravity as a scalar field theory

The field equations in the metric formalism of $f(R)$ gravity, Eq. (2.7), can be rewritten by performing a conformal transformation on the metric. In this conformally transformed frame the curvature scalar decouples from the scalar fields such that the field equations have the same form as the usual Einstein's equation with the energy-momentum tensor coupled to a new scalar field. Hence the new frame is referred to as the Einstein frame while the original is called the Jordan frame. The quantities with a tilde will be referring to the Einstein frame. A brief review on conformal transformations is given in Appendix A. Conformal transformations correspond to a transformation of the metric with the form

$$\tilde{g}_{\mu\nu} = \Omega^2(x) \, g_{\mu\nu} \tag{2.13}$$

here $\Omega(x)$ is referred to as the conformal factor.

Recalling the action for the $f(R)$ theories of gravity:

$$S = \int \mathrm{d}^4x \sqrt{|g|} \left\{ \frac{1}{2\kappa} f(R) + \mathcal{L}_m (g_{\mu\nu}, \Psi_m) \right\} \, , \tag{2.14}$$

where the matter Lagrangian depends not only on the matter fields Ψ_m, but also on the metric. This action is presented in the Jordan frame since the matter Lagrangian is minimally coupled to the curvature.

Using the inverse relations presented on Appendix A it is possible to write the action in terms of quantities in the Einstein frame as follows:

$$
\begin{aligned}
S = \int \mathrm{d}^4x\sqrt{|\tilde{g}|}\Bigg[\frac{1}{2\kappa}\left(\tilde{R} + 6\frac{\tilde{\Box}\Omega}{\Omega} - 6\tilde{g}^{\mu\nu}\frac{\tilde{\nabla}_\mu\Omega\tilde{\nabla}_\nu\Omega}{\Omega^2}\right) - \Omega^{-4}V \\
+\Omega^{-4}\mathcal{L}_m\left(\Omega^{-2}\tilde{g}_{\mu\nu}, \Psi_m\right)\Bigg] ,
\end{aligned}
\tag{2.15}
$$

where

$$
V = \frac{\Omega^2 R(\Omega) - f\left(R(\Omega)\right)}{2\kappa} , \tag{2.16}
$$

here R takes the role of a scalar field. Now it is possible to define a new scalar field ϕ as:

$$
\phi = \sqrt{\frac{6}{\kappa}}\ln\Omega . \tag{2.17}
$$

The term $\tilde{\Box}\ln\Omega$ on the action is a boundary term and can therefore be set to zero, on account of the divergence theorem, resulting in

$$
\begin{aligned}
S = \int d^4x\sqrt{|\tilde{g}|}\Bigg[\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi - U(\phi) \\
+\mathrm{e}^{-4\sqrt{\frac{2\kappa}{3}}\phi}\mathcal{L}_m\left(\mathrm{e}^{-2\sqrt{\frac{2\kappa}{3}}\phi}\tilde{g}_{\mu\nu}, \Psi_m\right)\Bigg] ,
\end{aligned}
\tag{2.18}
$$

where

$$
U\left(\phi\right) = \frac{V(\phi)}{\mathrm{e}^{\sqrt{8\kappa/3}\phi}} = \frac{\mathrm{e}^{\sqrt{2\kappa/3}\phi}\left(R(\phi)\right) - f\left(R(\phi)\right)}{2\kappa\mathrm{e}^{\sqrt{8\kappa/3}\phi}} , \tag{2.19}
$$

and the matter Lagrangian is affected by the conformal transformation of the metric $\tilde{g}_{\mu\nu} = \exp\left[2\sqrt{\frac{2\kappa}{3}}\phi\right] g_{\mu\nu}$.

So, in the Einstein frame, the field equation for the metric $\tilde{g}_{\mu\nu}$ takes the form of the Einstein equation, but the energy-momentum tensor is now coupled to the field ϕ.

In the Jordan frame, the geometrical part of the field equations is coupled to the field ϕ, while the matter field does not show this dependence.

For general discussions see Refs. [18, 19].

2.3 Non-minimally Coupled Curvature-Matter gravity

In NMCCM theories a second function of the curvature scalar, $f_2(R)$, is introduced that couples to the matter Lagrangian [6]:

$$S = \int \mathrm{d}^4x \sqrt{|g|} \left[\frac{1}{2} f_1(R) + (1 + f_2(R)) \mathcal{L}_m \right] , \tag{2.20}$$

where $f_i(R)\,(i = 1, 2)$ are functions of the scalar curvature, $\mathcal{L}_m$ is the matter Lagrangian density, that can depend on the metric and on the matter fields and their derivatives. GR is recovered when:

$$f_1(R) = \frac{R}{\kappa} , \quad f_2(R) = 0 . \tag{2.21}$$

The field equations for this model, obtained in the metric formalism are:

$$(F_1 + 2F_2\mathcal{L}_m) R_{\mu\nu} - \frac{f_1}{2} g_{\mu\nu} = (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)(F_1 + 2F_2\mathcal{L}_m) + (1 + f_2) T_{\mu\nu} , \tag{2.22}$$

thus in vacuum $T_{\mu\nu} = 0$, $\mathcal{L}_m = 0$ these equations reduce to the ones for $f(R)$ theories in vacuum. From these equations it is possible to deduce one of the most distinctive features of the NMCCM gravity, the energy-momentum tensor is not covariantly conserved. In fact taking the covariant derivative of Eq. (2.22) one gets:

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{1 + f_2} (g^{\mu\nu}\mathcal{L}_m - T^{\mu\nu}) \nabla_\mu R , \tag{2.23}$$

where $F_i \equiv \partial f_i / \partial R$, and $\Box$ is the D'Alembertian operator defined as $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. To reach this equation metric compatibility was assumed and the identity

$$(\Box\nabla_\nu - \nabla_\nu\Box) F_i = R_{\mu\nu}\nabla^\mu F_i \tag{2.24}$$

was also used[1].

Another feature of NMCCM gravity that will be useful later on is the trace of the field equations:

$$(F_1 + 2F_2\mathcal{L}_m) R - 2f_1 + 3\Box F_1 + 6\Box (F_2\mathcal{L}_m) = (1 + f_2) T . \tag{2.25}$$

[1]This equality follows from the commutator of two covariant derivatives on a vector field

$$[\nabla_\alpha, \nabla_\beta] V_\sigma = R^\lambda_{\sigma\beta\alpha} V_\lambda .$$

This non-minimal coupling between matter and curvature when considering a perfect fluid, with pressure p and energy density ρ, gives rise to an additional acceleration, orthogonal to the four-velocity [6]:

$$f^\alpha = \frac{1}{\rho + p}\left[\frac{F_2}{1+f_2}(\mathcal{L}_m - p)\nabla_\nu R + \nabla_\nu p\right] h^{\alpha\nu}\,, \tag{2.26}$$

where $h^{\alpha\nu} = g^{\alpha\nu} - u^\alpha u^\nu$ is the projection operator. This extra acceleration leads to a non-geodesic motion for test particles. In fact the action that yields the equations of motion for NMCCM gravity is [20]

$$S = -mc \int \mathrm{d}\tau\ (1 + f_2(R)) \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\,, \tag{2.27}$$

where τ is the affine parameter that parameterises the trajectory. Varying this action with respect to each component of the coordinates yields a modified geodesic equation:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\sigma\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{F_2(R)}{1+f_2(R)} g^{\mu\alpha}\frac{\partial R}{\partial x^\alpha}\,. \tag{2.28}$$

This non-geodesic motion of test particles provides a possible probe for testing NMCCM gravity.

Chapter 3

The Newtonian limit of NMCCM gravity

In this chapter the Newtonian limit of GR is presented to introduce the formalism and to contrast with the same limit of NMCCM gravity. Then, considering NMCCM gravity, a perturbative expansion on powers of $1/c$ of several quantities, Ricci tensor and scalar, energy-momentum tensor, will be made in order to arrive at the hydrodynamic equations on this regime.

3.1 The Newtonian limit of GR

The non-relativistic limit of GR yields Newtonian gravity. This limit corresponds to slow velocities and weak fields, where test particles acceleration is given by:

$$\mathbf{a} = -\nabla U \,, \tag{3.1}$$

with U being the Newtonian gravitational potential.

In GR, free particles move in geodesics described by:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0 \,. \tag{3.2}$$

Where λ is the affine parameter that parametrises the motion.

The Newtonian limit of GR requires that:

- the test particles move slowly (here the affine parameter is the proper time)

$$\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau}\,; \tag{3.3}$$

- the gravitational field is weak, so the metric can be perturbed around the Minkowski metric;

- the field is static.

From the first of these conditions the geodesic equation can be rewritten as:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0\,, \tag{3.4}$$

as the spatial derivatives are negligible.

Since the field (metric) is static the relevant Christoffel symbol can be calculated:

$$\begin{aligned}\Gamma^\mu_{00} &= \frac{1}{2}g^{\mu\rho}\left(g_{\rho 0,0} + g_{\rho 0,0} - g_{00,\rho}\right)\\ &= -\frac{1}{2}g^{\mu\rho}g_{00,\rho}\,.\end{aligned} \tag{3.5}$$

As the fields considered are assumed to be small the metric can be thought as a small perturbation on the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad |h_{\mu\nu}| \ll 1\,. \tag{3.6}$$

So the relevant Christoffel symbol, Eq. (3.5), takes the form (keeping only linear terms in the perturbation the indices of $h_{\mu\nu}$ can be raised and lowered using the flat space metric):

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}h_{00,\nu}\,, \tag{3.7}$$

and the geodesic equation, Eq. (3.4) becomes:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\eta^{\mu\nu}h_{00,\nu}\,. \tag{3.8}$$

As the metric is assumed to be static in this regime the 0-th component of this equation is simply:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0\,, \tag{3.9}$$

That is $\mathrm{d}t/\mathrm{d}\tau$ is a constant, thus $t \propto \tau$.

The spatial components of the Minkowski metric are the identity matrix so upper and lower indices are equivalent. Thus the space components of the geodesic equation are:

$$\begin{aligned} \frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} &= \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 h_{00,i} \\ \Leftrightarrow \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} &= \frac{c^2}{2} h_{00,i}\,. \end{aligned} \tag{3.10}$$

This equation is very similar to the Newtonian one, in fact comparing this equation with Eq. (3.1) the relationship between the metric and the Newtonian potential is clear:

$$h_{00} = -\frac{2U}{c^2} \implies g_{00} = -\left(1 + \frac{2U}{c^2}\right)\,. \tag{3.11}$$

Thus, the Newtonian potential manifests itself through the 00 component of the metric in the Newtonian limit.

3.2 Metric and curvature

Now a perturbative analysis of the metric and the subsequent curvature tensors will be presented to achieve a non-relativistic limit of NMCCM theories. In order to do so a small perturbation is considered on the flat space metric, of the form (3.6). So that the metric can be expanded around $h_{\mu\nu} = 0$ in powers of $1/c$ and keeping only the terms relevant for the non-relativistic regime [21]:

$$\begin{aligned} g_{00} &= -1 + h_{00}^{(2)} + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \\ g_{ij} &= \delta_{ij} + h_{ij}^{(2)} + \mathcal{O}\left(\frac{1}{c^4}\right)\,, \\ g_{0i} &= \mathcal{O}\left(\frac{1}{c^3}\right)\,, \end{aligned} \tag{3.12}$$

where:

$$h_{\mu\nu}^{(n)} = \mathcal{O}\left(\frac{1}{c^n}\right)\,. \tag{3.13}$$

Now the gauge freedom of the metric is used to impose:

$$h_{ij,i}^{(3)} = \frac{1}{2c} h_{ii,0}^{(2)} + \mathcal{O}\left(\frac{1}{c^5}\right) ,$$
$$h_{ij,j}^{(2)} = \frac{1}{2} h_{jj,i}^{(2)} - \frac{1}{2} h_{00,i}^{(2)} + \mathcal{O}\left(\frac{1}{c^4}\right) . \tag{3.14}$$

With this choice the Ricci tensor, $R_{\mu\nu}$ can be expanded as:

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00}^{(2)} - \frac{1}{2}\nabla^2 h_{00}^{(4)} - \frac{1}{2}\left|\nabla h_{00}^{(2)}\right|^2 + \frac{1}{2} h_{ij}^{(2)} h_{00,ij}^{(2)} + \mathcal{O}\left(\frac{1}{c^6}\right) , \tag{3.15}$$

$$R_{0i} = -\frac{1}{2}\nabla^2 h_{0i}^{(3)} - \frac{1}{4c} h_{00,i0}^{(2)} + \mathcal{O}\left(\frac{1}{c^5}\right) , \tag{3.16}$$

$$R_{ij} = -\frac{1}{2}\nabla^2 h_{ij}^{(2)} + \mathcal{O}\left(\frac{1}{c^4}\right) , \tag{3.17}$$

where ∇^2 is the usual Laplacian operator in 3-dimensional Euclidean space.

The scalar curvature can also be expanded as:

$$R = R^{(2)} + \mathcal{O}\left(\frac{1}{c^4}\right) , \tag{3.18}$$

where $R^{(n)} = \mathcal{O}(1/c^n)$.

3.3 Energy-momentum tensor

Considering a perfect fluid, the relevant components of its energy-momentum tensor are [21]:

$$T_{00} = \rho c^2 \left(1 + \frac{v^2}{c^2} + \frac{\Pi}{c^2} - h_{00}^{(2)}\right) + \mathcal{O}\left(\frac{1}{c^2}\right) ,$$
$$T_{0i} = -\rho c v_i + \mathcal{O}\left(\frac{1}{c}\right) ,$$
$$T_{ij} = \rho v_i v_j + p\delta_{ij} + \mathcal{O}\left(\frac{1}{c^2}\right) . \tag{3.19}$$

As can be seen this is the energy-momentum tensor for a perfect fluid with matter density ρ, velocity field $\mathbf{v}$, pressure p and specific energy density ratio Π (the ratio of energy density to rest mass density). Therefore the trace of the energy-momentum tensor is:

$$T = -\rho c^2 \left(1 + \frac{\Pi}{c^2}\right) + 3p + \mathcal{O}(\frac{1}{c^2}) \tag{3.20}$$

The definition of the matter Lagrangian for a perfect fluid is not unique so a choice has to be made. Two distinct possibilities are $\mathcal{L}_m = p$ or $\mathcal{L}_m = -\rho c^2$, for this thesis the chosen Lagrangian is the latter, because the first one leads to the vanishing of the extra force discussed in Section 2.3. For a thorough discussion on this matter see Refs. [22, 23].

3.4 Newtonian limit of NMCCM gravity

The goal of this section is to compute the perturbation to the metric relevant at order $\mathcal{O}(1/c^2)$, that is $h_{00}^{(2)}$ which can be interpreted as the gravitational potential in a Newtonian perspective as seen in the previous section [20]. To do so the functions f_i are assumed to be analytical around $R = 0$ and so admit the following Taylor series expansion around $R = 0$:

$$\begin{aligned} f_1(R) &= \frac{1}{\kappa}(a_1 R + a_2 R^2) + \mathcal{O}(R^3)\,, \\ f_2(R) &= q_1 R + \mathcal{O}(R^2)\,. \end{aligned} \tag{3.21}$$

So this model reduces to the usual General Relativity when $a_1 = 1, a_2 = 0$ and $f_2(R) \equiv 0$. The effect of this choice of parameters will be the study subject for this thesis.

First the trace of the field equations, Eq. (2.25) is used to compute $R^{(2)}$. At the relevant order the equation reduces to:

$$\nabla^2 R^{(2)} - \frac{R^{(2)}}{6a_2} = -\frac{4\pi G}{3c^2 a_2}\left(\rho - 6q_1 \nabla^2 \rho\right)\,. \tag{3.22}$$

It is assumed that $a_2 > 0$ (this corresponds to the condition needed to avoid Dolgov-Kawasaki instabilities which will be discussed later on) such that it is possible to define $\lambda = \sqrt{6a_2}$ so that the equation can be expressed as:

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right) R^{(2)} = -\frac{8\pi G}{c^2 \lambda^2}\left(\rho - 6q_1 \nabla^2 \rho\right)\,. \tag{3.23}$$

This equation is easily solved introducing the Green function:

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right) G\left(\mathbf{x}-\mathbf{y}\right) = \delta\left(\mathbf{x}-\mathbf{y}\right) \,. \tag{3.24}$$

This is the Klein-Gordon equation and its Green function is the one corresponding to the Yukawa potential:

$$G(\mathbf{x}-\mathbf{y}) = -\frac{1}{4\pi}\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|} \,. \tag{3.25}$$

The solution for this inhomogeneous equation is:

$$R^{(2)} = \frac{G}{3c^2 a_2}\int \mathrm{d}^3 y \frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\left[\rho(t,\mathbf{y}) - 6q_1\nabla^2\rho(t,\mathbf{y})\right] \,. \tag{3.26}$$

Integrating by parts the Laplacian term yields:

$$\begin{aligned}\int \nabla^2\rho(t,\mathbf{y})\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3 y &= \int \rho(t,\mathbf{x})\nabla^2\left(\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\right)\mathrm{d}^3 y \\ &= -4\pi\rho + m^2\int \rho(t,\mathbf{x})\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3 y \,,\end{aligned} \tag{3.27}$$

where surface terms were neglected due to the boundary conditions:

$$\rho(t,x) = 0 \,, \quad \nabla\rho(t,x) = 0 \,, \quad x \in \partial\Omega \,, \tag{3.28}$$

with Ω denoting a region in three-dimensional space with mass density ρ. The above conditions follow from the continuity of both the density function $\rho(t,x)$ and its derivative on the boundary $\partial\Omega$.

Thus at order $\mathcal{O}(1/c^2)$ the Ricci scalar is:

$$R^{(2)} = \frac{8\pi G}{c^2}\frac{q_1}{a_2}\rho + \frac{G}{3c^2 a_2}\left(1 - \frac{q_1}{a_2}\right)\int \rho(t,\mathbf{y})\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3 y \,. \tag{3.29}$$

It is now clear that if a_2 were to be negative this solution would be oscillatory, due to λ being purely imaginary, resulting in a nonphysical behaviour at large distances. This condition, $a_2 > 0$, is needed to avoid Dolgov-Kawasaki instabilities. The general formulation of this condition, that is for arbitrary functions $f_i(R)$, is [24, 25]:

$$f_1''(R) + 2\mathcal{L}_m f_2''(R) \geq 0\,, \tag{3.30}$$

And for the choice of Eq. (3.21) this yields precisely $a_2 > 0$. The Dolgov-Kawasaki instability does not occur in GR since the Ricci scalar is fixed by an algebraic relation with the trace of the energy-momentum tensor, however in $f(R)$ and NMCCM theories the trace of the field equations gives rise to a differential equation for R. So in order for the field R to be stable its effective mass, $m = 1/\lambda$, must be positive. This agrees with the interpretation of the mass of the field in the Klein-Gordon equation

This solution for the Ricci scalar can now be inserted into the 00 component of the field equations in order to extract $h_{00}^{(2)}$. Using Eqs. (3.15) and (3.19) the 00 component of the field equations, Eq. (2.22), takes the form:

$$\nabla^2\left(h_{00}^{(2)} + 4a_2 R^{(2)} - \frac{32\pi G q_1}{c^2}\rho\right) = R^{(2)} - \frac{16\pi G}{c^2}\rho c^2\,. \tag{3.31}$$

From Eq. (3.22) it is possible to write the right hand side as:

$$6a_2\nabla^2 R - \frac{8\pi G}{c^2}\left(\rho + 6q_1\nabla^2\rho\right)\,. \tag{3.32}$$

So introducing a function U' such that:

$$\nabla^2 U' = -\frac{8\pi G}{c^2}\rho\,, \tag{3.33}$$

it is possible to write Eq. (3.31) as:

$$\nabla^2\left(h_{00}^{(2)} - 2a_2 R^{(2)} + \frac{16\pi G q_1}{c^2}\rho - U'\right) = 0\,. \tag{3.34}$$

Imposing that the argument of the Laplacian is identically zero[1] the solution for h_{00} is:

$$h_{00}^{(2)} = 2\left(\frac{U}{c^2} + a_2 R^{(2)} - \frac{8\pi G}{c^2}q_1\rho\right)\,, \tag{3.35}$$

where $U = c^2 U'/2$. The potential U' can be determined from Eq. (3.33), which is a Poisson's

[1] It could also be a constant, but in that case the constant could be absorbed in the definition of the potentials.

equation whose solution is the Newtonian potential:

$$U' = \frac{2G}{c^2}\int \frac{\rho(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3y\,. \tag{3.36}$$

Using the expression for $R^{(2)}$:

$$h_{00}^{(2)} = \frac{2}{c^2}\left[U + \frac{1}{3}\left(1-\frac{q_1}{a_2}\right)\mathcal{Y}\right], \tag{3.37}$$

where $\mathcal{Y}$ is the Yukawa potential:

$$\mathcal{Y} = G\int \rho(t,\mathbf{y})\frac{e^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3y\,. \tag{3.38}$$

This shows that in this regime NMCCM gravity consists of a Newtonian potential and a Yukawa potential, in the same sense as Eq. (3.11), with range controlled by the parameter a_2 by $\lambda = \sqrt{6a_2}$. Other alternative theories also give rise to a Yukawa term in the potential, namely massive dilaton gravity (Brans-Dicke theory with $\omega = 0$ [2])[26].

3.5 Hydrodynamic equations

The hydrodynamic equations for a perfect fluid, with energy-momentum tensor given in Eq. (3.19), are obtained from the covariant divergence of the energy-momentum tensor introduced in Eq. (2.23), but repeated here for the sake of convenience:

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{1+f_2}\left(g^{\mu\nu}\mathcal{L}_m - T^{\mu\nu}\right)\nabla_\mu R\,. \tag{3.39}$$

The continuity equation follows from the 0-th component of this equation. Considering the left hand side first:

$$\nabla_\mu T^{\mu 0} = \frac{\partial T^{\mu 0}}{\partial x^\mu} + \Gamma^\mu_{\mu\alpha}T^{\alpha 0} + \Gamma^0_{\mu\alpha}T^{\mu\alpha}\,. \tag{3.40}$$

[2] Brans-Dicke theory is and example of a scalar-tensor theory of gravitation where the gravitational interaction is mediated by a scalar field, φ. Its equations of motion follow from the action:

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left(\varphi R - \frac{\omega}{\varphi}\varphi_{,\mu}\varphi^{,\mu} + \mathcal{L}_m\right)$$

where ω is a parameter of the theory.

Neglecting terms of $\mathcal{O}(1/c)$ and taking into account that terms involving Christoffel symbols, give a contribution of $\mathcal{O}(1/c)$ to the 0-th component of the covariant divergence of the energy-momentum tensor [14]:

$$\begin{aligned}\nabla_\mu T^{\mu 0} &= \frac{\partial T^\mu}{\partial x^\mu} + \mathcal{O}(1/c) \\ &= c\frac{\partial \rho}{\partial t} + c\frac{\partial \rho v^i}{\partial x^i} + \mathcal{O}(1/c)\,.\end{aligned} \tag{3.41}$$

Considering now the right-hand side of Eq. (3.39):

$$\frac{F_2}{1+f_2}\left(g^{\mu 0}\mathcal{L}_m - T^{\mu 0}\right)\nabla_\mu = \mathcal{O}(1/c)\,. \tag{3.42}$$

Hence the full equation is:

$$c\frac{\partial \rho}{\partial t} + c\frac{\partial\left(\rho v^i\right)}{\partial x^i} = \mathcal{O}(1/c) \implies \frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho v^i\right)}{\partial x^i} = 0\,, \tag{3.43}$$

neglecting terms $\mathcal{O}(1/c^2)$. So at this order the continuity equation remains unchanged by the non-minimal coupling between matter and curvature.

Euler's equation follows from the spatial components of Eq. (3.39). Starting with the right-hand side at order $\mathcal{O}(1/c^2)$:

$$\nabla_\mu T^{\mu i} = \frac{\partial T^{\mu i}}{\partial x^\mu} + \Gamma^i_{00}T^{00} + \mathcal{O}(1/c^2)\,. \tag{3.44}$$

From the metric considered, Eq. (3.12), and the result for h_{00}, Eq. (3.37), the relevant Christoffel symbol can be computed:

$$\Gamma^i_{00} = -\frac{1}{c^2}\frac{\partial}{\partial x^i}\left(U + \frac{1-\theta}{3}\mathcal{Y}\right) + \mathcal{O}(1/c^4)\,, \tag{3.45}$$

with $\theta = q_1/a_2$. Inserting this expression into Eq. (3.44) and recalling the components of the energy-momentum tensor, Eq. (3.19):

$$\nabla_\mu T^{\mu i} = \frac{\partial \rho v^i}{\partial t} + \frac{\partial(\rho v^i v^j)}{\partial x^j} - \rho\frac{\partial}{\partial x^i}\left(U + \frac{1-\theta}{3}\mathcal{Y}\right) + \frac{\partial p}{\partial x^i} + \mathcal{O}(1/c^2)\,. \tag{3.46}$$

Using the continuity equation this can be rearranged as:

$$\nabla_\mu T^{\mu i} = \rho\frac{\mathrm{d}v^i}{\mathrm{d}t} - \rho\frac{\partial U}{\partial x^i} - \rho\frac{1-\theta}{3}\frac{\partial \mathcal{Y}}{\partial x^i} + \frac{\partial p}{\partial x^i}\,, \tag{3.47}$$

where $\mathrm{d}/\mathrm{dt} \equiv \partial/\partial t + v^i\partial/\partial x^i$ is the convective derivative.

Now the right hand side of the spatial components of Eq. (3.39), at order $\mathcal{O}(1)$, using the Taylor series expansion

$$\begin{aligned}\frac{f_R^2}{1+f^2}\left(g^{\mu i}\mathcal{L}_m - T^{\mu i}\right)\frac{\partial R}{\partial x^\mu} &= \frac{f_R^2}{1+f^2}g^{ji}\mathcal{L}_m\frac{\partial R}{\partial x^j}\\ &= -c^2 q_1\rho\frac{\partial R}{\partial x^i}\\ &= -\frac{1}{3}\theta(1-\theta)\rho\frac{\partial \mathcal{Y}}{\partial x^i} - \frac{2}{3}\pi G\lambda^2\theta^2\frac{\partial \rho^2}{\partial x^i}\,.\end{aligned} \tag{3.48}$$

Combining this result with Eq. (3.47) one obtains the hydrodynamical equations for a perfect fluid in the Newtonian limit of NMCCM gravity:

$$\rho\frac{dv^i}{dt} = \rho\frac{\partial U}{\partial x^i} - \frac{\partial p}{\partial x^i} + \frac{1}{3}(1-\theta)^2\rho\frac{\partial \mathcal{Y}}{\partial x^i} - \frac{2}{3}\pi G\lambda^2\theta^2\frac{\partial \rho^2}{\partial x^i}\,. \tag{3.49}$$

Compared with the Newtonian hydrodynamics, the non-minimal coupling gives rise to two new terms: (i) a Yukawa force density, (ii) a force density proportional to the gradient of the mass density. The force density proportional to the mass density has opposite signs to the ones of the gravitational Newton's potential and the Yukawa term.

3.5.1 Oceanic experiment

The theory must respect solar system constraints, given the observations of planetary precessions and Lunar Laser Ranging measurements [27], hence the only way that the Yukawa range, λ, can reach astronomical scales is if the strength of the Yukawa force is very small. This imposes:

$$(1-\theta)^2 \ll 1 \implies q_1 \simeq a_2\,. \tag{3.50}$$

However, if this condition is met then the extra force proportional to the mass density

$$-\frac{2}{3}\pi G\lambda^2\theta^2\frac{\partial \rho^2}{\partial x^i}, \tag{3.51}$$

can become non-negligible and lead to relevant corrections to the hydrostatic equilibrium of a compressible fluid, $\partial\rho/\partial p \neq 0$, on Earth. Therefore it is possible to devise an experiment to test the presence of a Yukawa force on a compressible fluid and establish an upper bound on λ with Earth based experiments. This analysis was performed in Ref. [14] considering an oceanic experiment as described in Ref. [28]. In this work it is shown that the NMCCM gravity leads to a correction in the pressure with depth. It was assumed that the range of the Yukawa force was much smaller than the radius of the Earth, $\lambda \ll R_\oplus$, this lead to restrictions on the parameters.

Nevertheless this analysis neglects the possibility of a chameleon mechanism. This mechanism first proposed by Khoury and Weltman [29, 30] gives rise to a scalar field whose mass depends on the local matter density. In regions with large densities, for example Earth, the effective mass is very large (resulting in a small range) while in regions of low density, like the interstellar medium, the mass can be very small. The effect of such field in $f(R)$ theories of gravity, which are equivalent to a scalar field theory as seen in Ch. 2, is discussed in Ref. [31]. The effective potential for this scalar field, ϕ, is $V_{\text{eff}}(\phi) = V(\phi) + \mathrm{e}^{a\phi}\rho$, with a an appropriate constant, such that the position of its minimum depends on the local density ρ, and the effective mass of the field grows with the density. This makes this case particularly relevant in a cosmological context where the light background density yields a light, long range, field.

Indeed in Ref. [31] it is speculated that accounting for this effect the constraints of the mass scale of the Starobinsky model[3][4] would change. Thus, the inclusion of a coupling between curvature, corresponding to the scalar field, and matter, as is the case of NMCCM, can be relevant.

Therefore, the constraints derived in Ref. [14] can hold at the local Earth's density, but might be irrelevant when considering cosmological scales, where the local density is significantly smaller, and, in particular, the range of the Yukawa interaction can be far greater.

[3]The Starobinsky model is a form of $f(R)$ theories of gravity with action:

$$S = \int \mathrm{d}^4x \sqrt{|g|} \left[\frac{1}{2\kappa} \left(R + \frac{R^2}{6M^2} \right) + \mathcal{L}_m \right] . \tag{3.52}$$

where M has the dimensions of a mass. This corresponds to the case of $q_1 = 0$ and $a_2 = \frac{1}{6M^2}$ in the proposed theory with non-minimal coupling between matter and geometry.

Chapter 4

Newton-Schrödinger Systems for Non-minimally Coupled Curvature-Matter Gravity

In this chapter the Madelung transformation will be introduced. This transformation together with the hydrodynamical equations from the previous chapter results in a Newton-Schrödinger system describing the evolution of matter subject to the proposed NMCCMCM model. Numerical methods developed for non-linear non-local optical systems, also described by Newton-Schrödinger systems, are employed to obtain predictions for our NMCCMCM model of gravity.

4.1 Newton-Schrödinger systems

The Newton-Schrödinger system corresponds to a non-linear non-local modification of the Schrödinger equation for non-local potentials $\mathcal{V}_i$ as well as non-linear terms [32]. In its most general form it can be written as:

$$\mathrm{i}\frac{\partial \psi}{\partial t} = A\nabla^2\psi + \sum_i B_i \mathcal{V}_i \psi + G\left(|\psi|^2\right)\psi \tag{4.1}$$

$$\left(\nabla^2 - m_i^2\right)\mathcal{V}_i = D_i|\psi|^2\,. \tag{4.2}$$

Where the field $\psi(\mathbf{x}, t)$ represents the wave function of the system. The function $G(|\psi|^2)$ accounts for the possibility of the system having different local non-linearities. The potentials, $\mathcal{V}_i$, each governed by an equation of the form of Eq. 4.2, describe the non-local character of the system. A, B, D_i and m_i are constants to be adjusted to each physical system.

The non-linear nature of the wave equation leads to some conceptual difficulties. The theory of non-linear quantum dynamics is used to describe the macroscopic quantum effects and quasi-particles such as solitons, which have a Hamiltonian which depends non-linearly on their wave function. This differs from linear quantum mechanics in many important ways. For example, the square of the wave function can no longer be interpreted as a probability density, but rather as the mass or number density of its constituent microscopic particles at that point. Operators are no longer linear, and hence wavefunctions no longer obey the principle of superposition.

Newton-Schrödinger systems are used to describe the behaviour of N body systems and are frequently used to describe light propagating in a non-local and non-linear system, under the paraxial approximation. These are described by a non-linear Schrödinger (NLS) equation for the propagation of the beam through the medium, and a Newton-like potential describing the distribution of the refractive indices in the medium. Therefore it is possible to use numerical tools specifically developed for optical systems to study the behaviour of matter in gravitational fields. This analogy has permitted the study of several systems: dark energy [33], dark matter [34], structure formation [35] and alternative theories of gravity [32]. But the analogy is deeper and also allows for analogue optical experiments on a laboratory that reproduce the behaviour of astrophysical objects like boson stars [13].

This approach to study matter distributions has several advantages when compared to some analytical methods like Eulerian fluid-bases perturbation theory or Lagrangian approaches such as the Zel'dovich approximation. The three most interesting advantages are [35]

- perturbation theory: standard perturbation methods do not guarantee that the matter density is everywhere positive;
- shell-crossing: methods such as the Zel'dovich approximation break down at shell-crossing where the density becomes infinite;
- gas pressure: analytical techniques for modelling the effects of gas pressure are scarce.

This emphasises the relevance of the method developed in this thesis. This approach is able to describe perturbations of scale up to the Hubble radius where beyond the horizon perturbations effects appear and the non-relativistic weak-field approximation breaks down.

Some expect that the Newton-Schrödinger system applies at a fundamental level to a single particle wave function and not just to some semiclassical mean field. It is proposed that this equation could explain the localisation of macroscopic particles by suppressing the quantum dispersion due to the particles own gravitational field [36]. Later Penrose proposed that the inclusion of gravity in the Schrödinger equation could lead to a dynamical collapse of the wavefunction [37], although it was later shown that the gravitational potential is not able to describe the wave function collapse [38]. These works show how this line of research can entangle gravity and quantum mechanics.

4.2 Newtonian potential

At large scales matter can be modelled as a perfect fluid, hence it obeys the following set of hydrodynamics equations [39]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{4.3}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} + \frac{\nabla p}{\rho} + \nabla \phi = 0\,. \tag{4.4}$$

Here ϕ represents the Newtonian potential, that obeys the Poisson's equation $\nabla^2 \phi = 4\pi G \rho$. It will be useful to use the vector calculus identity $\nabla(\mathbf{v}^2) = 2(\mathbf{v} \cdot \nabla)\mathbf{v} + 2\mathbf{v} \times (\nabla \times \mathbf{v})$ to rewrite Euler's equation as:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla\left(\mathbf{v}^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v}) - \frac{\nabla p}{\rho} + \nabla \phi = 0\,. \tag{4.5}$$

A polytropic relation of the form $p = \omega \rho^n$ will be the choice for the equation of state, with ω and n being coefficients to be chosen for each physical situation. Assuming an irrotational flow the velocity can be expressed as a gradient of some scalar function ξ, therefore $\mathbf{v} = \nabla \xi$. With this assumption Euler's equation can be written as:

$$\nabla\left[\left(\frac{\partial \xi}{\partial t}\right) + \frac{1}{2}\left(\nabla \xi\right)^2 + \phi\right] + \frac{\nabla p}{\rho} = 0\,. \tag{4.6}$$

One would like to turn the pressure gradient term into a full gradient, for that the aforementioned polytropic relation will be used. Therefore the pressure potential V_p is defined:

$$V_p = \begin{cases} \omega \ln \rho \quad , & n = 1 \\ \frac{n\omega}{n-1}\rho^{n-1} \, , & n < 0 \vee n > 1 \end{cases} \, , \tag{4.7}$$

such that $\nabla V_p = \nabla(p)/\rho$. Now it is clear to see that Euler's equation reduces to:

$$\left(\frac{\partial \xi}{\partial t}\right) + \frac{1}{2}\left(\nabla \xi\right)^2 + \phi + V_p = 0 \, . \tag{4.8}$$

This equations governs the fluid's velocity field.

Now the Madelung transformation is introduced by writing [40]:

$$\psi = \sqrt{\rho} \mathrm{e}^{\mathrm{i}\xi/\nu} \, . \tag{4.9}$$

In this way the fluid density $\rho = |\psi|^2$ will always be non-negative. Here the constant ν acts as a de Broglie wavelength giving the spatial resolution of the solution [35], it can also be related to the Planck's constant by $\nu = \hbar/m$, such that $m^2 s^{-1} = [\nu]$, this definition may suggest the introduction of a momentum field rather than a velocity field. Now the time derivative of ψ can be computed:

$$\frac{\partial \psi}{\partial t} = \frac{\partial \sqrt{\rho}}{\partial t} \mathrm{e}^{\mathrm{i}\xi/\nu} + \frac{\mathrm{i}}{\nu} \psi \frac{\partial \xi}{\partial t} \, . \tag{4.10}$$

The second term on the right hand side has already been computed so the focus will be the term proportional to the time derivative of ρ:

$$\frac{\partial \sqrt{\rho}}{\partial t} \mathrm{e}^{\mathrm{i}\xi/\nu} = \frac{\mathrm{e}^{\mathrm{i}\xi/\nu}}{2\sqrt{\rho}} \frac{\partial \rho}{\partial t} \, . \tag{4.11}$$

Using the continuity equation it is possible to write:

$$\frac{\partial \rho}{\partial t} = -\rho \nabla^2 \xi - \nabla \xi \cdot \nabla \rho \, , \tag{4.12}$$

inserting this result into Eq. 4.11 one gets:

$$\frac{\psi}{2|\psi|^2} \frac{\partial \rho}{\partial t} = -\frac{\psi}{2|\psi|} \left(|\psi| \nabla^2 \xi + 2 \nabla \xi \cdot \nabla |\psi| \right) \, . \tag{4.13}$$

To simplify this result it is possible to use the identity

$$\nabla^2(\phi\psi) = \psi\nabla^2\phi + 2\nabla\psi\cdot\nabla\phi + \phi\nabla^2\psi \tag{4.14}$$

to write:

$$\nabla^2\psi = \nabla^2\left(|\psi|e^{i\xi/\nu}\right) = |\psi|\nabla^2 e^{i\xi/\nu} + 2\nabla|\psi|\cdot\nabla e^{i\xi/\nu} + e^{i\xi/\nu}\nabla^2|\psi| \tag{4.15}$$

$$= \frac{\psi}{|\psi|}\nabla^2|\psi| + \frac{2i}{\nu}\frac{\psi}{|\psi|}\nabla|\psi|\cdot\nabla\xi + \frac{i}{\nu}\psi\nabla^2\xi - \frac{1}{\nu^2}\left(\nabla\xi\right)^2 . \tag{4.16}$$

Now it is easy to see that

$$\frac{\psi}{2|\psi|^2}\frac{\partial\rho}{\partial t} = \frac{i\nu}{2}\nabla^2\psi - \frac{i\nu}{2}\frac{\psi}{|\psi|}\nabla^2|\psi| + \frac{i}{2\nu}\psi\left(\nabla\xi\right)^2 . \tag{4.17}$$

Finally it is possible to obtain a Schrödinger like equation for ψ:

$$i\nu\frac{\partial\psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2\psi + \left(\nu^2 V_B + V'\right)\psi , \tag{4.18}$$

where $V' = \phi + V_p$ and $V_B = \nabla^2\left(|\psi|\right)/(2|\psi|)$ is Bohm's quantum potential. This quantum potential can also be expressed as:

$$V_B = \frac{\psi}{2|\psi|}\nabla^2|\psi| = \frac{1}{2}\left[\frac{\nabla^2|\psi|^2}{|\psi|^2} - \frac{1}{2}\frac{\left(\nabla|\psi|^2\right)^2}{|\psi|^4}\right] . \tag{4.19}$$

This form is relevant because it only depends on integer powers of the matter density $\rho = |\psi|^2$.

4.3 Newton and Yukawa

In the non-relativistic limit of a NMCCM gravity of the form (3.21) the solution of the field equations contains both a Newtonian and a Yukawa potentials [14]. In this regime the hydrodynamic equations are, as seen in Subsection 3.5:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left[U + \alpha\mathcal{Y} - V_p - \frac{4\pi}{3}G\lambda^2\theta^2\rho\right] , \tag{4.20}$$

where $\alpha = (1-\theta)^2/3$, the same polytropic relation between p and ρ was assumed giving rise to the same pressure potential defined in Eq. (4.7). In Eq. (4.20) U is the Newtonian potential while $\mathcal{Y}$ is the Yukawa potential, defined as:

$$\mathcal{Y} = G\int \rho(t,\mathbf{y})\frac{\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|/\lambda}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3y\,, \tag{4.21}$$

$$U = G\int \frac{\rho(t,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}^3y\,. \tag{4.22}$$

Or in the differential form, and making the identification $\rho = |\psi|^2$:

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\mathcal{Y} = -4\pi G|\psi|^2\,, \tag{4.23}$$

$$\nabla^2 U = -4\pi G|\psi|^2\,. \tag{4.24}$$

Following the same methods as before the NLS equation for this system is:

$$\mathrm{i}\nu\frac{\partial\psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2\psi + \left(\frac{\nu^2}{2}\frac{1}{|\psi|}\nabla^2|\psi| - U - \alpha\mathcal{Y} + V_p + \frac{4\pi}{3}G\lambda^2\theta^2|\psi|^2\right)\psi\,. \tag{4.25}$$

It is important to note that the presence of a Gross-Pitaevskii like term in this equation may allow for stationary solutions, even when neglecting the pressure. To find these stationary solutions we impose that the time derivative vanishes for all times

$$\frac{\partial\psi}{\partial t} \equiv 0\,. \tag{4.26}$$

then, assuming the solutions have spherical symmetry, using the freedom to choose the phase of the velocity field (which for a static solution is a constant), and choosing the phase to be zero such that $\psi = |\psi|$:

$$-\frac{\nu^2}{2}\nabla^2\psi + \frac{\nu^2}{2}\frac{\psi}{|\psi|}\nabla^2|\psi| = 0\,. \tag{4.27}$$

Therefore the static solutions are those that obey the condition:

$$U + \alpha\mathcal{Y} = V_p + \frac{4\pi}{3}G\lambda^2\theta^2|\psi|^2\,. \tag{4.28}$$

For numerical calculations it is well advised to work with a dimensionless wave function.

So ψ needs to be re-scaled to obtain a dimensionless wave function ψ' defined as:

$$\psi' = \sqrt{\frac{\rho}{\rho_0}} \mathrm{e}^{\mathrm{i}\xi/\nu} = \frac{\psi}{\sqrt{\rho_0}} \,. \tag{4.29}$$

where ρ_0 is a parameter with the same dimensions as ρ, this is particularly important for gravitational simulations as when considering astrophysical/cosmological structures the densities can be very small. Thus, with this redefinition the wave function can still be of the order of the unity which is important for numerical computations of low density systems.

The dimensionless wave function obeys the following NLS equation:

$$\begin{aligned} \mathrm{i}\nu\frac{\partial\psi'}{\partial t} &= -\frac{\nu^2}{2}\nabla^2\psi' + \\ &+ \left[\nu^2\frac{\nabla^2|\psi'|}{2|\psi'|} - 4\pi G\rho_0\left(U' + \alpha\mathcal{Y}' - \frac{\lambda^2\theta^2}{3}|\psi'|^2\right) + V_p\right]\psi' \,, \end{aligned} \tag{4.30}$$

where the primed potentials obey the following equations:

$$\nabla^2 U' = -|\psi'|^2 \,, \tag{4.31}$$

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\mathcal{Y}' = -|\psi'|^2 \,. \tag{4.32}$$

The pressure potential must also be redefined as:

$$V_p = \begin{cases} \omega \ln\left(\rho_0|\psi'|^2\right) & n = 1 \\ \frac{n\omega}{n-1}\left(\rho_0|\psi'|^2\right)^{n-1} & n < 0 \vee n > 1 \end{cases} \,, \tag{4.33}$$

From this point on the primes will be dropped for simplicity of notation.

4.4 Normalisation

Assuming that $n > 1$ the NLS equation of interest is in the polytropic relation:

$$\mathrm{i}\nu\frac{\partial\psi}{\partial t} = -\frac{\nu^2}{2}\nabla^2\psi + \left[\nu^2\frac{\nabla^2|\psi|}{2|\psi|} - 4\pi G\rho_0\left(U + \alpha\mathcal{Y} - \frac{\lambda^2\theta^2}{3}|\psi|^2\right) + \frac{n\omega\rho_0^{n-1}}{n-1}|\psi|^{2n-2}\right]\psi \,, \tag{4.34}$$

with the non-local terms being controlled by Eqs. (4.31) and (4.32). The Bohm potential will not be considered since it transforms the same way as the kinetic term and it will eventually be neglected in the numerical calculations, mainly because it leads to numerical instabilities on account of the dependence on $1/|\psi|$ and it is negligible when compared with the other terms when the solution varies slowly in the scales of interest there are other reasons that will be discussed later on this chapter. Considering a general system

$$\mathrm{i}a_1\frac{\partial\psi}{\partial t} = -\frac{a_2}{2}\nabla^2\psi - a_3U\psi - a_3\alpha\mathcal{Y}\psi + a_3\frac{\lambda^2\theta^2}{3}|\psi|^2\psi + a_4\frac{n}{n-1}|\psi|^{2n-2}\psi\,, \tag{4.35}$$

$$\nabla^2 U = -a_5|\psi|^2\,, \tag{4.36}$$

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\mathcal{Y} = -a_5|\psi|^2\,. \tag{4.37}$$

Lets now consider the following normalisation:

$$t = \beta t'\,, \tag{4.38}$$

$$x_i = \Delta x_i'\,, \tag{4.39}$$

$$U = TU'\,, \tag{4.40}$$

$$\mathcal{Y} = T\mathcal{Y}'\,. \tag{4.41}$$

In terms of these new primed variables the previous system becomes

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{a_2\beta}{2a_1\Delta^2}\nabla^2\psi - \frac{a_3T\beta}{a_1}\left(U' + \alpha\mathcal{Y}' - \frac{\lambda^2\theta^2}{3T}|\psi|^2\right)\psi + \frac{a_4\beta}{a_1}\frac{n}{n-1}|\psi|^{2n-2}\psi\,, \tag{4.42}$$

$$\nabla'^2U' = -\frac{a_5\Delta^2}{T}|\psi|^2\,, \tag{4.43}$$

$$\left(\nabla'^2 - \frac{\Delta^2}{\lambda^2}\right)\mathcal{Y}' = -\frac{a_5\Delta^2}{T}|\psi|^2\,. \tag{4.44}$$

Now the following conditions are imposed for the normalisation parameters:

$$\frac{a_5\Delta^2}{T} = 1\,, \tag{4.45}$$
$$\frac{a_2\beta}{a_1\Delta^2} = 1\,, \tag{4.46}$$
$$\frac{a_3T\beta}{a_1} = 1\,, \tag{4.47}$$
$$\frac{\lambda}{\Delta} = \Gamma\,. \tag{4.48}$$

Here Γ is the normalised range of the Yukawa interaction. Comparing with Eq. (4.34) the a_i parameters are

$$a_1 = \nu\,, \tag{4.49}$$
$$a_2 = \nu^2\,, \tag{4.50}$$
$$a_3 = 4\pi G\rho_0\,, \tag{4.51}$$
$$a_4 = \omega\rho_0^{n-1}\,, \tag{4.52}$$
$$a_5 = 1\,, \tag{4.53}$$

meaning that the normalisation parameters are given by:

$$\beta = \frac{1}{\sqrt{4\pi G\rho_0}}\,, \tag{4.54}$$
$$\Delta = \left(\frac{\nu^2}{4\pi G\rho_0}\right)^{1/4}\,, \tag{4.55}$$
$$T = \sqrt{\frac{\nu^2}{4\pi G\rho_0}}\,, \tag{4.56}$$
$$\frac{a_4\beta}{a_1} = \frac{\omega\rho_0^{n-3/2}}{\nu\sqrt{4\pi G}} \equiv \gamma_0\,. \tag{4.57}$$

With these transformations the normalised Newton-Schrödinger system, with the inclusion of the Bohm potential that transforms in the same way as the kinetic term, becomes (the primes were dropped for simplicity of notation):

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\nabla^2\psi}{2} - \left(U + \alpha\mathcal{Y} - \frac{\Gamma^2\theta^2}{3}|\psi|^2 - \frac{\nabla^2|\psi|}{2|\psi|}\right)\psi + \gamma_0\frac{n}{n-1}\,|\psi|^{2n-2}\,\psi\,, \tag{4.58}$$

$$\nabla^2 U = -\,|\psi|^2\;, \tag{4.59}$$

$$\left(\nabla^2 - \frac{1}{\Gamma^2}\right)\mathcal{Y} = -\,|\psi|^2\;. \tag{4.60}$$

4.5 Lagrangian and Hamiltonian densities

The field equations can be derived from the Euler-Lagrange equation with the choice of a proper Lagrangian density. It will be assumed that the fields are fixed by Eqs. (4.59) and (4.60), therefore the focus will be the Lagrangian density that yields Eq. (4.58). A first choice for the Lagrangian density would be

$$\mathcal{L} = \psi^*\left(\mathrm{i}\partial_t - H\right)\psi\,, \tag{4.61}$$

with

$$H = \left(-\frac{1}{2}\nabla^2 - U - \alpha\mathcal{Y} + \frac{\Gamma^2\theta^2}{3}|\psi|^2 + \gamma_0\frac{n}{n-1}\,|\psi|^{2n-2} + \frac{\nabla^2|\psi|}{2|\psi|}\right)\,. \tag{4.62}$$

In fact this choice does not give the correct field equations due to the non-linear terms, that depend on ψ^*. The Lagrangian density is not unique, in fact as the Euler-Lagrange equation is obtained from the least action principle, all choices that lead to the same action functional give the same field equations. With this in mind it is possible to use integration by parts for the spatial derivatives (the boundary terms are assumed to be zero, corresponding to the matter distribution being zero at spatial infinity) to obtain the more symmetric Lagrangian density:

$$\mathcal{L} = \mathrm{i}\psi^*\partial_t\psi - \frac{1}{2}\,|\nabla\psi|^2 + U\,|\psi|^2 + \alpha\mathcal{Y}\,|\psi|^2 - \frac{\Gamma^2\theta^2}{3\kappa_1}\,|\psi|^4 - \frac{\gamma_0}{\kappa_2}\frac{n}{n-1}\,|\psi|^{2n} + \frac{1}{4}\frac{\left(\nabla\,|\psi|^2\right)^2}{|\psi|^2}\;, \tag{4.63}$$

Here the constants κ_i were introduced in the non linear terms and are to be fixed in order to obtain the correct field equations. The Euler-Lagrange equation for ψ^* yields

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \left(\nabla \psi^* \right)} \right) - \partial_t \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t \psi^* \right)} \right) = 0$$

$$\implies \mathrm{i}\partial_t \psi = -\frac{1}{2}\nabla^2 \psi - \left(U + \alpha \mathcal{Y} \right) \psi + 2\frac{\Gamma^2 \theta^2}{3\kappa_1} \left| \psi \right|^2 \psi + \frac{\nabla^2 |\psi|}{2|\psi|}\psi + \frac{\gamma_0}{\kappa_2}\frac{n^2}{n-1} \left| \psi \right|^{2n-2} \psi \, . \tag{4.64}$$

Comparing with Eq. (4.58) follows that $\kappa_1 = 2$ and $\kappa_2 = n$, and the correct Lagrangian density is:

$$\mathcal{L} = \mathrm{i}\psi^* \partial_t \psi - \frac{1}{2} \left| \nabla \psi \right|^2 + U \left| \psi \right|^2 + \alpha \mathcal{Y} \left| \psi \right|^2 - \frac{\Gamma^2 \theta^2}{6} \left| \psi \right|^4 - \gamma_0 \frac{1}{n-1} \left| \psi \right|^{2n} + \frac{1}{4} \frac{\left(\nabla \left| \psi \right|^2 \right)^2}{\left| \psi \right|^2} \, . \tag{4.65}$$

Note that this Lagrangian is not Hermitian, but it does not need to be. One could be tempted to cast it into an Hermitian form by performing integration by parts in the temporal derivatives. However, this cannot be done as that would imply the field would be zero at temporal infinity, which is inconsistent with the conservation of matter. This happens because the NLS equation is of first order in time. This would have the subsequent problem of not yielding the correct Hamilton equations of motion due to the change in the definition of the conjugate momenta. This happens because the conjugate momenta are not independent of the fields, i.e. the field velocities cannot be expressed in terms of the conjugate momenta. In fact the Legendre transformation requires that the fields are independent of the conjugate momenta[41], which clearly is not the case.

The Hamiltonian density associated to the Lagrangian density, Eq. (4.65), is given by

$$\mathcal{H} = \pi \partial_t \psi + \pi^* \partial_t \psi^* - \mathcal{L} \, , \tag{4.66}$$

where $\pi = \partial \mathcal{L} / \partial \left(\partial_t \psi \right) = \mathrm{i}\psi^*$ and $\pi^* = 0$ are the fields' conjugate momenta. Therefore the Hamiltonian density is:

$$\mathcal{H} = \frac{1}{2} \left| \nabla \psi \right|^2 - U \left| \psi \right|^2 - \alpha \mathcal{Y} \left| \psi \right|^2 + \frac{\Gamma^2 \theta^2}{6} \left| \psi \right|^4 + \gamma_0 \frac{1}{n-1} \left| \psi \right|^{2n} - \frac{1}{4} \frac{\left(\nabla \left| \psi \right|^2 \right)^2}{\left| \psi \right|^2} \, . \tag{4.67}$$

To obtain the field equation for ψ^* the following Hamilton's equation is used:

$$\partial_t \pi = -\frac{\partial \mathcal{H}}{\partial \psi} + \nabla \cdot \left(\frac{\partial \mathcal{H}}{\partial \left(\nabla \psi \right)} \right) + \partial_t \left(\frac{\partial \mathcal{H}}{\partial \left(\partial_t \psi \right)} \right) . \tag{4.68}$$

This yields

$$\mathrm{i}\partial_t \psi^* = \frac{1}{2}\nabla^2 \psi^* + \left(U + \alpha \mathcal{Y} \right) \psi^* - \frac{\Gamma^2 \theta^2}{3} \left| \psi \right|^2 \psi^* - \gamma_0 \frac{n}{n-1} \left| \psi \right|^{2n-2} \psi^* - \frac{\nabla^2 |\psi|}{2|\psi|} \psi^* . \tag{4.69}$$

Taking the complex conjugate yields Eq. (4.58) as expected, thus justifying the Hamiltonian density choice Eq. (4.67).

4.6 Numerical simulations

The Newton-Schrödinger systems have non-linear and non-local terms, and normally are non-integrable, so numerical methods are necessary in order to obtain information about the systems they model. Although these methods do not give a complete set of solutions they allow for exploring different scenarios that would be impossible to study analytically.

The Newton-Schrödinger system is comprised by two different kind of equations, NLS and Poisson, that require different approaches in order to obtain solutions.

4.6.1 Non-linear Schrödinger equation

The method chosen to obtain solutions for this equation is the Split-step Fourier method (SSFM), since it is the most efficient one [42]. This method is well established for numerically solving the time-dependent Schrödinger equation, written here in 1D

$$\mathrm{i}\hbar \frac{\partial \psi \left(x, t \right)}{\partial t} = \hat{H} \psi \left(x, t \right) . \tag{4.70}$$

The Hamiltonian can be divided into two terms, kinetic energy and potential energy, $\hat{H} = \hat{T} + \hat{V}$. Integration the Schrödinger equation from t to $t + \Delta t$ yields the wave function

$$\psi \left(x, t + \Delta t \right) = \mathrm{e}^{-\mathrm{i} \Delta t H / \hbar} \psi \left(x, t \right) . \tag{4.71}$$

Since the operators $\mathrm{e}^{-\mathrm{i}\Delta t T/\hbar}$ and $\mathrm{e}^{-\mathrm{i}\Delta t V/\hbar}$ do not commute and are diagonal in the momentum and position spaces, respectively, the splitting technique consists in arranging these two operators with appropriate coefficients to reproduce the action of the full operator $\mathrm{e}^{-\mathrm{i}\Delta t H/\hbar}$. This back and forth between position and momentum space is computationally inexpensive, especially when compared to the cost of solving the kinetic energy operator in position space, and is achieved by Fast Fourier transforms (FFT). One way to compose the two operators is the symmetric two step (Strang) splitting

$$\mathrm{e}^{-\mathrm{i}\Delta t H/\hbar}\psi \simeq \mathrm{e}^{-\mathrm{i}\Delta t T/2\hbar}\mathrm{e}^{-\mathrm{i}\Delta t V/\hbar}\mathrm{e}^{-\mathrm{i}\Delta t T/2\hbar}\psi\,, \tag{4.72}$$

that holds with an error $\mathcal{O}\left(\Delta t^3\right)$. This means Eq. (4.71) can be written as

$$\psi\left(x, t+\Delta t\right) \simeq \mathcal{F}^{-1}\left[\mathrm{e}^{\mathrm{i}\hbar\Delta t k^2/4m}\mathcal{F}\left[\mathrm{e}^{-\mathrm{i}\Delta t V(x)/2\hbar}\mathcal{F}^{-1}\left[\mathrm{e}^{\mathrm{i}\hbar\Delta t k^2/4m}\mathcal{F}\left[\psi\left(x,t\right)\right]\right]\right]\right]\,. \tag{4.73}$$

Here $\mathcal{F}\left[\cdot\right]$ denotes the Fourier transform. The computational cost of performing forward and backward Fourier transforms to evaluate the time evolution is small, particularly when compared to the expense of evaluating the kinetic energy operator in position space. It should be noted that the split-step method naturally incorporates periodic boundary conditions. These time splitting methods are appropriate for the numerical solution of NLS equations as the time step Δt is independent of the mesh size, this means that a finer mesh does not require a smaller time step. So the extra computational power needed for a smaller mesh comes only from the FFT.

Numerically this method only involves two easy to parallelise operations: FFT for the calculation of the Fourier transforms and matrix multiplication. This is particularly relevant as these operations are well optimised in GPGPUs.

Considering the Schrödinger equation, Eq. (4.70), the fact that the Hamiltonian is a Hermitian operator means that its eigenvalues, E_m, are real and their corresponding eigenstates, ϕ_i, form an orthogonal basis in the Hilbert space. A widely used method to determine the ground state solution of time-dependent Schrödinger equations is the imaginary time propagation method. This method corresponds to a transformation of the time parameter, also called Wick rotation:

$$t \to -\mathrm{i}\tau\,. \tag{4.74}$$

This transforms the NLS equation to a diffusion type equation:

$$\frac{\partial\psi}{\partial\tau} = \frac{1}{2}\nabla^2\psi + \psi U + \alpha Y\psi - \frac{\theta^2\Gamma^2}{3}\left|\psi\right|^2\psi - \gamma_0\frac{n}{n-1}\left|\psi\right|^{2n-2}\psi\,. \tag{4.75}$$

Any initial condition ψ_0 can be decomposed on the basis of eigenfunctions ϕ_m:

$$\psi_0(x) = \sum_m c_m \phi_m(x) \, , \quad c_m = \langle \phi_m | \psi_0 \rangle \, . \tag{4.76}$$

Therefore its time evolution is given by

$$\psi_0(x, \tau) = \sum_m c_m \mathrm{e}^{-\mathrm{i}(-\mathrm{i}\tau)E_m} \phi_m(x) \, . \tag{4.77}$$

Asymptotically, for $\tau \to \infty$ only the ground state survives and $\psi_0(x,\tau) \to \mathrm{e}^{-\tau E_0} c_0 \phi_0$, since excited states with higher energy decay faster. Any initial state is propagated according to this equation. To arrive at a physical solution the conservation of the norm $\int \mathrm{d}^3x \, |\psi|^2$, which corresponds to conservation of mass, must be enforced after every step. It was assumed that the ground state is non-degenerate; if this were not the case, the system would evolve to a linear combination of eigenfunctions, and repeating the process with different initial conditions would yield a complete set of independent eigenvectors of the subspace which could be orthogonalised.

The normalisation of the asymptotic value gives the eigenfunction ϕ_0, whose energy can be obtained integrating the Hamiltonian density, Eq. (4.67). Excited states may be obtained by simultaneously propagating a number of different initial wave function and imposing that they remain orthogonal after each step [43].

This method is very useful because codes developed to solve the Schrödinger equation, like the SSFM discussed above, can be adapted to find stationary solutions.

4.6.2 Poisson type equation

Solving the Poisson equations is a well known problem in computational physics, as such there are many ways to tackle it. The methods can be divided into two categories: (i) direct methods, which use a finite number of steps to reach a solution; (ii) iterative methods which try to obtain the final result given an initial configuration through successive approximations.

Since the purpose is the analysis of the dynamics of the system, the Poisson type equations must to be solved after each step, since the non-local fields U and $\mathcal{Y}$ depend implicitly on time through the field ψ. Thus, direct methods are the most suited and among these, spectral based methods are the ones that converge faster, since they only rely on FFT to solve the equations.

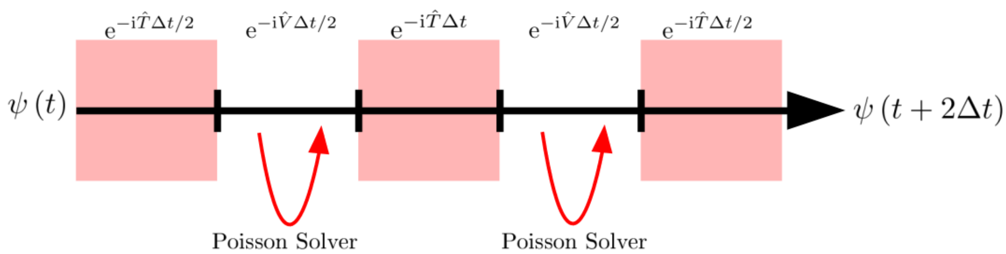

Figure 4.1: Graphical representation of the Split-step Fourier method coupled with the Poisson Solver. The black arrow indicates the time evolution over two time steps, with the corresponding evolution operators. The steps in the red shaded regions are performed in Fourier space, while the ones in the white regions are performed in real space.

Considering a general Poisson type equation:

$$\left(\nabla^2 - m^2\right)\varphi = \alpha\left|\psi\right|^2 \tag{4.78}$$

in Fourier space the solution is simply given by

$$\mathcal{F}\left[\varphi\right] = -\frac{\alpha\mathcal{F}\left[\left|\psi\right|^2\right]}{k^2+m^2}\,, \tag{4.79}$$

applying the inverse Fourier transform to this equation yields the non-local field $\varphi = U, \mathcal{Y}$. In the case of the Newtonian potential U where $m = 0$, an extra condition must be imposed in order to obtain $\mathcal{F}\left[\varphi\right]|_{k=0}$, namely $U\left(\mathbf{x}\to\infty, t\right) = 0$.

A graphical depiction of how the method works in conjunction with the SSFM to evolve the wave function is presented in Fig. (4.1), where the evolution after two time steps is considered.

4.7 Stationary solutions

To make use of these methods an initial state $\psi_0 = A + \varepsilon\left(r\right)$ is propagated, where A is a homogeneous distribution and ε is a white noise signal. The evolution is halted when the variation of the Hamiltonian after each step is less than 10^{-16} (which corresponds to the numerical precision of the computer since the simulations are made using double precision),

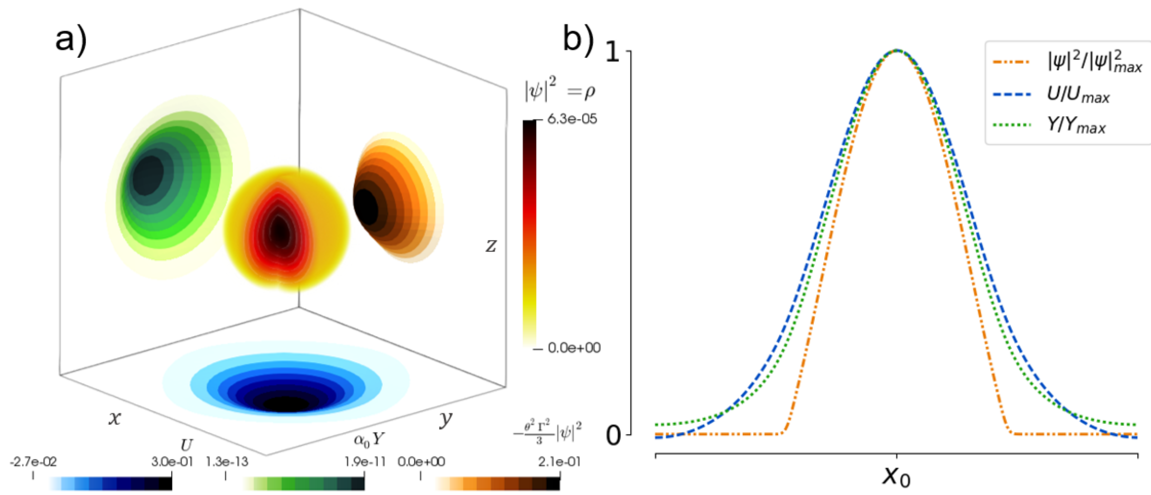

Figure 4.2: Stationary solutions of the Newton-Schrödinger system, a) density plot of the solution with the non-local potentials, in blue the Newtonian potential and in green the Yukawa potential, this results were obtained considering $\Gamma = 100$ and $\alpha = 10^{-10}$, b) the density profile, Newtonian potential and Yukawa potential along an axial direction normalised to their respective maximum value.

as it was assumed that the system has reached the ground state by this point, and this energy is assumed to be the ground state one. The Hamiltonian, $H(t)$ corresponds to the integral over the space of the Hamiltonian density $\mathcal{H}(\mathbf{x}, t)$, given by Eq. (4.67). The energy of the system corresponds to the Hamiltonian only when the field is normalised, as in the present case the norm of the field corresponds to the total mass and this is not necessarily one the energy is given by:

$$E(t) = \frac{H(t)}{\int \mathrm{d}^3x \, |\psi(\mathbf{x}, t)|^2} . \tag{4.80}$$

The matter density profile for the solutions obtained is best adjusted by the product of two super-Gaussian functions:

$$\rho(r) \equiv |\psi(r)|^2 = A\mathrm{e}^{-(r/\sigma_1)^{2m_1} - (r/\sigma_2)^{2m_2}} . \tag{4.81}$$

A graphical three dimensional representation of the solution as well as the density profile along a radial direction are depicted in Fig. (4.2).

It was also studied how these solutions depend on the pressure and the range of the potential. The results are presented in Fig. (4.3). Where the width of the solutions are plotted against the range of the Yukawa potential, this width corresponds the second order momentum, defined as

$$\sigma(t) = \sqrt{\frac{2}{M} \int_{-\infty}^{+\infty} \mathrm{d}x \, [x - x_c(t)] |\psi(\mathbf{x}, t)|^2} \,, \tag{4.82}$$

where $x_c(t)$ is the centre of the mass distribution, and M is the total mass.

Analysing the graph it is possible to observe two different regimes depending on the polytropic exponent: (i) for large n, corresponding to the case when the pressure term is neglected the solutions are sustained only by the extra force and lie on a straight line on the plot; (ii) when n is small the pressure term dominates and for small ranges of the Yukawa potential, λ, the width of the solution remains approximately constant. However for large enough values of λ the extra force dominates and the solutions tend to the straight line obtained when neglecting the pressure. This indicates that stationary solutions seem to exist even when the pressure term is very small when compared to the other potentials, corresponding to $n \to 2$ on the plot.

This means that in the NMCCM model pressure is not necessary to prevent the complete gravitational collapse. The extra force proportional to the gradient of the mass density is enough to counteract the attractive gravitational fields.

The Bohm potential leads to numerical instabilities in the tail regions of the matter distributions since it is proportional to $|\psi|^{-1}$, so, since it also occurs in the usual formulation of GR and the purpose here is to find effects associated with the non-minimal coupling of matter and curvature, the Bohm potential was neglected in the simulations. This term is also negligible when the matter distribution varies slowly over the scales of interest. It is important to note that, as discussed in Sec. 4.3, for stationary spherically symmetric solutions the Bohm potential cancels with the kinetic term. Thus assuming the solutions are wide enough so that the kinetic term is negligible, the solution obtained considering the Bohm potential should match the one obtained as both potentials involving derivatives are suppressed by the other terms and cancel each other. It also follows from the same discussion that the solution obtained do not depend on the parameter ν introduced in the Madelung transformation.

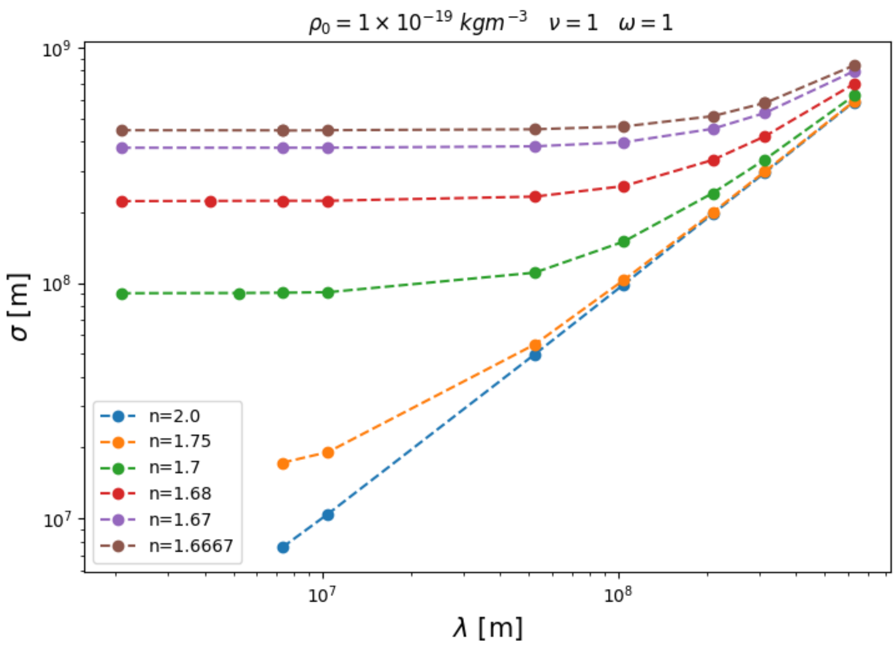

Figure 4.3: Width of the solution as a function of the range of the Yukawa potential for several polytropic exponents.

4.8 Scale invariance and Noether's theorem

It is possible to search for self-similar solutions of the above NLS equation. To do so a scale transformation is considered

$$\mathbf{x} \to \mathbf{x}' = \lambda_S^{-1}\mathbf{x} : \tag{4.83}$$

$$t \to t' = \lambda_S^{-2} t ; \tag{4.84}$$

$$\psi(\mathbf{x}, t) \to \psi'(\mathbf{x}', t') = \lambda_S^{\Delta}\psi\left(\lambda_S \mathbf{x}', \lambda_S^2 t'\right) . \tag{4.85}$$

Lets consider each term separately, first the ones involving derivatives:

$$\frac{\partial \psi'(\mathbf{x}', t')}{\partial t'} = \lambda_S^{\Delta}\frac{\partial t}{\partial t'}\frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \lambda_S^{\Delta+2}\frac{\partial \psi(\mathbf{x}, t)}{\partial t} ; \tag{4.86}$$

$$\nabla^2_{\mathbf{x}'}\psi'(\mathbf{x}', t') = \left(\frac{\partial \mathbf{x}}{\partial \mathbf{x}'}\right)^2 \lambda_S^{\Delta}\nabla^2_{\mathbf{x}}\psi(\mathbf{x}, t) = \lambda_S^{\Delta+2}\nabla^2_{\mathbf{x}}\psi(\mathbf{x}, t) . \tag{4.87}$$

The non-local potentials transform as

$$\nabla^2_{\mathbf{x}'}U(\mathbf{x}', t) = -\left|\psi(\mathbf{x}', t')\right|^2 \to \lambda_S^{2-2\Delta}\nabla^2_{\mathbf{x}}U(\mathbf{x}, t) = -\left|\psi(\mathbf{x}, t)\right|^2 , \tag{4.88}$$

$$\left(\nabla^2_{\mathbf{x}'} - \frac{1}{\Gamma^2}\right)\mathcal{Y}(\mathbf{x}, t) = -\left|\psi(\mathbf{x}', t')\right|^2 \to \lambda_S^{2-2\Delta}\left(\nabla^2_{\mathbf{x}'} - \frac{1}{\lambda_S^2\Gamma^2}\right)\mathcal{Y}(\mathbf{x}, t) = -\left|\psi(\mathbf{x}, t)\right|^2 . \tag{4.89}$$

Now the potentials are redefined to absorb the constants $\lambda_S^{2-2\Delta}U(\mathbf{x}, t) \to U(\mathbf{x}, t)$, $\lambda_S^{2-2\Delta}\mathcal{Y}(\mathbf{x}, t) \to \mathcal{Y}(\mathbf{x}, t)$ and $\lambda_S^2\Gamma^2 \to \Gamma^2$, such that Eqs. (4.59) and (4.60) remain unchanged. Then the NLS equation becomes:

$$\begin{aligned} \mathrm{i}\lambda_S^{\Delta+2}\frac{\partial \psi}{\partial t} = & -\lambda_S^{\Delta+2}\frac{\nabla^2\psi}{2} - \lambda_S^{3\Delta-2}\left(U + \alpha\mathcal{Y}\right)\psi + \lambda_S^{3\Delta-2}\frac{\Gamma^2\theta^2}{3}|\psi|^2\psi \\ & + \gamma_0\frac{n}{n-1}\lambda_S^{\Delta(2n-1)}\left|\psi\right|^{2n-2}\psi . \end{aligned} \tag{4.90}$$

If the pressure term is neglected then the NLS equation is scale invariant if $\Delta = 2$. If the polytropic relation has exponent $n = 3/2$ then this symmetry holds even with the pressure term, however in this case the pressure becomes very large when compared to the other potentials due to the definition of γ_0, Eq. (4.57), and the simulations break down.

This property was verified numerically. To do so an initial homogeneous distribution A_0 is evolved until it reaches a stationary solution ψ_0, then this solution is compared to the one resulting from an initial distribution A_0/λ_S^2, the simulation grid is scaled accordingly.

It is important to note that the total mass of the solution also scales, as:

$$\int \mathrm{d}^3x' \left|\psi'\left(\mathbf{x}', t'\right)\right|^2 = \int \mathrm{d}^3x \lambda_S \left|\psi\left(\mathbf{x}, t\right)\right|^2 . \tag{4.91}$$

Noether's theorem [44, 45] states that if there is a continuous transformation group that leaves the action invariant then there is an associated conserved local current, which when integrated yields a conserved charge. For the present case the conserved charge associated with the scaling transformation is [46]:

$$\frac{\mathrm{i}}{2} \int \mathrm{d}^3x\, \mathbf{x} \cdot \left(\psi \nabla \psi^* - \psi^* \nabla \psi\right) - 4Ht\,. \tag{4.92}$$

Chapter 5

Conclusions

In this thesis a special class of Non-minimally Coupled Curvature-Matter theories of gravity were studied. If both functions $f_{1,2}$ are analytical around $R = 0$ then they admit a Taylor series expansion. The terms in the series were kept until the first order that does not appear in General Relativity, this means terms of order $\mathcal{O}\left(R^3\right)$ for f_1 and $\mathcal{O}\left(R^2\right)$ for f_2 were neglected. The complexity introduced by these two functions make a complete analytical study of the theory impossible and thus numerical methods are fundamental to obtain predictions of the theory to later restrict the parameters.

Considering the Newtonian limit of the theory, two extra terms appear in the hydrodynamical equations for a perfect fluid: (i) a Yukawa force density, which is expected as the non-minimal coupling introduces a new massive degree of freedom, given by $m = 1/\sqrt{6a_2}$, and (ii) an extra force density proportional to the gradient of the squared matter density. To study the effects of these two terms these hydrodynamical equations were transformed into a Newton-Schrödinger system by means of a Madelung transformation. These Newton-Schrödinger systems consist of a Non-linear Schrödinger equation controlling the wave function, containing information about the mass density and velocity field, and Poisson type equations describing the non-local fields. Optical systems are also well described by such systems, so it was possible to use numerical solvers developed for the study of non-linear non-local optical systems to study the proposed alternative theory of gravity.

The Split-step Fourier method combined with a Wick rotation, and considering spherical symmetry, made it possible to determine the 'ground state' of the system, by minimising the Hamiltonian. The matter density profile corresponding to the 'ground state' obtained is best fitted by a product of two super-Gaussians. A particular feature of the stationary solutions is that the solutions do not depend on the chosen velocity scale, controlled by the constant ν introduced in the Madelung transformation. These solutions also exhibit self similarity. This theory leads to an interesting result, the extra force density proportional to the squared matter density is enough to prevent the gravitational collapse, even for a pressureless fluid.

The work developed here is the starting point of a more general analysis and can be further developed namely by relaxing the spherical symmetry and searching for solutions with, for instance, axial symmetry, or by searching higher order stationary solutions. These further explorations of the theory could be then compared to observed matter density profiles of cosmological objects and could be used to constrain the parameters of the theory.

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Appendix A

Conformal transformations

This appendix gives a brief introduction to the conformal transformations following Refs. [47, 48]. A conformal transformation of the metric, from the old metric $g_{\mu\nu}$ to the new one $\tilde{g}_{\mu\nu}$, is defined as:

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}\,, \tag{A.1}$$

where Ω^2 (x denotes all space-time coordinates x^μ) is the conformal factor, a non-vanishing, regular function. Since the metric defines the space-time interval the above transformation is equivalent to:

$$\mathrm{d}\tilde{s}^2 = \Omega^2(x)\mathrm{d}s^2\,. \tag{A.2}$$

So it is clear that a conformal transformation represents a local change of scale. As Ω is nonzero everywhere the inverse transformation $g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}$ is well defined.

One important property of the conformal transformations is that null curves are fixed points of these transformations. This is easily seen by considering a null curve $x^\mu(\lambda)$, that is a curve whose tangent vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ is null,

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = 0\,. \tag{A.3}$$

Then in the transformed frame:

$$\tilde{g}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \Omega^2(x) g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = 0\,. \tag{A.4}$$

This means that null curves defined to a given metric will also be null in any conformally related metric. It is said that conformal transformations leave light cones invariant. In fact conformal transformations leave all angles between 4-vectors unchanged.

The conformal transformation is not a change in the coordinates, but actually a change in the geometry, for example time-like geodesics in the original frame need not to be time-like geodesics in the transformed one. But is important to note that conformal transformations preserve the causal structure of the space-time [47]. However conformal transformations can affect the dynamical variables of the system, as any function of $g_{\mu\nu}$ can be thought as a function of $\tilde{g}_{\mu\nu}$ and $\Omega(x)$. The tilde frame is usually called conformal frame. Lets consider the effect of this transformation.

First the Christoffel symbols. As the connection is linear in the derivatives of the metric and on the inverse metric the transformed symbols should be of the form:

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + C^{\rho}_{\mu\nu}\,, \tag{A.5}$$

where $C^{\rho}_{\mu\nu}$ is a tensor (as it is the difference between two connections). This tensor as the explicit form[1]:

$$C^{\rho}_{\mu\nu} = \Omega^{-1}\left(2\delta^{\rho}_{(\mu}\nabla_{\nu)} + -g_{\mu\nu}g^{\rho\lambda}\nabla_{\lambda}\Omega\right)\,. \tag{A.6}$$

From this transformation law it is possible to show that geodesics are not conformally invariant as said before, that is, geodesics with respect to ∇_{μ} are not necessarily the same as with respect to $\tilde{\nabla}_{\mu}$. The tangent vector, v^{μ} to an affinely parametrised geodesic γ with respect to ∇_{μ} obeys the geodesic equation:

$$v^{\mu}\nabla_{\mu}v^{\nu} = 0\,. \tag{A.7}$$

Hence:

$$v^{\mu}\tilde{\nabla}_{\mu}v^{\nu} = v^{\mu}\nabla_{\mu}v^{\nu} + v^{\mu}C^{\nu}_{\mu\rho}v^{\rho} = 2v^{\nu}v^{\rho}\nabla_{\rho}\ln\Omega - \left(g_{\mu\rho}v^{\mu}v^{\rho}\right)g^{\nu\sigma}\nabla_{\sigma}\ln\Omega\,. \tag{A.8}$$

Thus, in general, γ is not a geodesic with respect to $\tilde{\nabla}_{\mu}$. However null geodesics are conformally invariant. This can be seen through an affine parametrisation, for null geodesics the above transformation law reads:

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\sigma\rho}\frac{\mathrm{d}x^{\sigma}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\lambda} = 2\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\lambda} \tag{A.9}$$

[1]The brackets represent the symmetric part of a tensor:

$$A_{(\mu\nu)} = \frac{A_{\mu\nu} + A_{\nu\mu}}{2}\,.$$

so transforming λ to the affine parameter $\tilde{\lambda}$ yields the geodesic equation, as long as the two parameters are related by:

$$\frac{\partial \tilde{\lambda}}{\partial \lambda} = \Omega^2 \, . \tag{A.10}$$

The transformation of the Chrystoffel connection gives a simple way to determine how the Riemann tensor transforms. In fact for any kind of transformation of the form of Eq. (A.5) the Riemann tensor transforms as:

$$\tilde{R}^{\rho}_{\sigma\mu\nu} = R^{\rho}_{\sigma\mu\nu} + \nabla_\mu C^{\rho}_{\nu\sigma} - \nabla_\nu C^{\rho}_{\mu\sigma} + C^{\rho}_{\mu\lambda} C^{\lambda}_{\nu\sigma} - C^{\rho}_{\nu\lambda} C^{\lambda}_{\mu\sigma} \, . \tag{A.11}$$

Thus it is now a matter of plugging the explicit form of $C^{\rho}_{\mu\nu}$ and after some manipulations the result is[2]:

$$\begin{aligned} \tilde{R}^{\rho}_{\sigma\mu\nu} =& R^{\rho}_{\sigma\mu\nu} - 2\left(\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - g_{\sigma[\mu}\delta^{\alpha}_{\nu]}g^{\rho\beta}\right)\Omega^{-1}\left(\nabla_\alpha\nabla_\beta\Omega\right) \\ &+ 2\left(2\delta^{\rho}_{[\mu}\delta^{\alpha}_{\nu]}\delta^{\beta}_{\sigma} - 2g_{\sigma[\mu}\delta^{\alpha}_{\nu]}g^{\rho\beta} + g_{\sigma[\mu}\delta^{\rho}_{\nu]}g^{\alpha\beta}\right)\Omega^{-2}\left(\nabla_\alpha\Omega\right)\left(\nabla_\beta\Omega\right) \, . \end{aligned} \tag{A.12}$$

Contracting the first and third indices yields the Ricci tensor:

$$\begin{aligned} \tilde{R}_{\sigma v} =& R_{\sigma v} - \left[(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{v} + g_{\sigma v}g^{\alpha\beta}\right]\Omega^{-1}\left(\nabla_\alpha\nabla_\beta\Omega\right) \\ &+ \left[2(n-2)\delta^{\alpha}_{\sigma}\delta^{\beta}_{v} - (n-3)g_{\sigma v}g^{\alpha\beta}\right]\Omega^{-2}\left(\nabla_\alpha\Omega\right)\left(\nabla_\beta\Omega\right) \, , \end{aligned} \tag{A.13}$$

where n is the number of dimensions. Raising one index, with the transformed metric gives the curvature scalar:

$$\tilde{R} = \Omega^{-2}R - 2(n-1)g^{\alpha\beta}\Omega^{-3}\left(\nabla_\alpha\nabla_\beta\Omega\right) - (n-1)(n-4)g^{\alpha\beta}\Omega^{-4}\left(\nabla_\alpha\Omega\right)\left(\nabla_\beta\Omega\right) \, . \tag{A.14}$$

The new Levi-Civita connection defines a new covariant derivative, that allows for studying how the covariant derivative is transformed in successive applications to a scalar field ϕ. The first covariant derivative is unchanged as it corresponds to the partial derivative in both frames:

$$\tilde{\nabla}_\mu\phi = \nabla_\mu\phi = \partial_\mu\phi \, . \tag{A.15}$$

[2]The square brackets represent the symmetric part of a tensor:

$$A_{[\mu\nu]} = \frac{A_{\mu\nu} - A_{\nu\mu}}{2} \, .$$

Applying a second covariant derivative makes a Christoffel symbol appear, and therefore the second derivative has a nontrivial transformation:

$$\tilde{\nabla}_\mu \tilde{\nabla}_v \phi = \nabla_\mu \nabla_v \phi - \left(\delta^\alpha_\mu \delta^\beta_v + \delta^\beta_\mu \delta^\alpha_v - g_{\mu\nu} g^{\alpha\beta}\right) \Omega^{-1} \left(\nabla_\alpha \Omega\right) \left(\nabla_\beta \phi\right) . \tag{A.16}$$

Contracting the indices gives the D'Alembertian:

$$\tilde{\Box} = \Omega^{-2} \Box \phi + (n-2) g^{\alpha\beta} \Omega^{-3} (\nabla_\alpha \Omega)(\nabla_\beta \phi) . \tag{A.17}$$

For the sake of completion, the inverse transformations (from the tilde frame to the original frame) are presented:

$$\begin{aligned} R^\rho_{\sigma\mu\nu} =& \widetilde{R}^\rho_{\sigma\mu\nu} + 2\left(\delta^\rho_{[\mu} \delta^\alpha_{\nu]} \delta^\beta_\sigma - \tilde{g}_{\sigma[\mu} \delta^\alpha_{\nu]} \tilde{\delta}^{\rho\beta}\right) \Omega^{-1} \left(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega\right) \\ &+ 2\tilde{g}_{\sigma[\mu} \delta^\rho_{\nu]} \tilde{g}^{\alpha\beta} \Omega^{-2} \left(\tilde{\nabla}_\alpha \Omega\right) \left(\tilde{\nabla}_\beta \Omega\right) ; \end{aligned} \tag{A.18}$$

$$\begin{aligned} R_{\sigma\nu} =& \widetilde{R}_{\sigma\nu} + \left[(n-2) \delta^\alpha_\sigma \delta^\beta_\nu + \tilde{g}_\sigma \tilde{g}^{\alpha\beta}\right] \Omega^{-1} \left(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega\right) \\ &- (n-1) \tilde{g}_\sigma \tilde{g}^{\alpha\beta} \end{aligned} \tag{A.19}$$

$$R = \Omega^2 \tilde{R} + 2(n-1) \tilde{g}^{\alpha\beta} \Omega \left(\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \Omega\right) - n(n-1) \tilde{g}^{\alpha\beta} \left(\tilde{\nabla}_\alpha \Omega\right) \left(\tilde{\nabla}_\beta \Omega\right) ; \tag{A.20}$$

$$\nabla_\mu \nabla_v \phi = \tilde{\nabla}_\mu \tilde{\nabla}_v \phi + \left(\delta^\alpha_\mu \delta^\beta_v + \delta^\beta_\mu \delta^\alpha_v - \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta}\right) \Omega^{-1} \left(\tilde{\nabla}_\alpha \Omega\right) \left(\tilde{\nabla}_\beta \phi\right) ; \tag{A.21}$$

$$\Box \phi = \Omega^2 \tilde{\Box} \phi - (n-2) \tilde{g}^{\alpha\beta} \Omega \left(\tilde{\nabla}_\alpha \Omega\right) \left(\tilde{\nabla}_\beta \phi\right) . \tag{A.22}$$