
Vibration Control of Adaptive Structures

*Modeling, Simulation and Implementation of
Viscoelastic and Piezoelectric Damping Technologies*

by

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(MSc, Mechanical Engineering)

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Abstract

In the last decade the interest demonstrated, mainly by public and private companies from the aerospace and space engineering sectors, in the development and use of high performance structures, in general, and adaptive structures, in particular, comprising piezo-visco-elastic damping treatments as technological solutions to tackle vibration- and noise-related problems has grown. In the meantime, a relative maturity in the adequate modeling and design of piezo-visco-elastic damping treatments has also been achieved. However, with the ever increasing strong demands of lighter, stiffer, lower cost and more efficient and reliable structures, engineers and scientists are faced nowadays with the requirement of having at their disposal refined and more accurate multiphysics models of those complex multilayer structures and damping technologies. That problem pushes the underlying complexity of more representative models to higher levels and poses a sustained and application-motivated challenge to the adaptive structures research community.

In that context, this dissertation addresses issues on the vibration control of adaptive structures, tentatively covering the design, analysis and application of the emerging piezo-visco-elastic damping technologies focusing, in general, main five research fields/topics. (1) MODELING: new refined finite element (FE) models of multilayer piezo-visco-elastic beam, plate and shell structures with a high degree of accuracy and a suitable trade-off between accuracy and complexity are developed; (2) DAMPING: viscoelastic (frequency-dependent) damping modeling and solution approaches are developed, assessed and implemented into FE modeling; (3) CONTROL: passive, active and hybrid damping treatments and control strategies considering both feedback and adaptive feedforward algorithms used separately or blended are developed, simulated, assessed and implemented in real-time control; (4) DESIGN: the efficiency of the damping treatments/technologies in terms of material properties, location, geometric configuration, thicknesses of the layers and control law is analyzed and discussed, and a general design methodology to assess their design is proposed and demonstrated; (5) APPLICATION: FE routines devoted to hybrid damping design and simulation, real-time test and analysis, validation and application of hybrid (with multi-algorithms and multi-damping materials) damping technologies are developed and implemented into a FE modeling and programming environment.

Keywords: beam, plate, shell, finite element, adaptive structure, vibration, damping, control, piezoelectric, viscoelastic, feedback, feedforward.

Résumé

Au cours de la dernière décennie, l'intérêt démontré, principalement par les compagnies publiques et privées dans les secteurs de l'ingénierie aéronautique et aérospatiale, dans le développement et l'emploi de structures à haute performance, en général, et structures adaptatives, en particulier, incorporant des traitements d'amortissement piezo-visco-élastique pour régler des problèmes de vibrations et bruit, a augmenté. Cependant, avec les réquisits fonctionnels de plus en plus exigeants qui amènent à des structures plus légères, rigides et à bas coût et qui, simultanément, soient aussi efficaces et fiables lorsqu'elles sont soumises à des charges dynamiques, les ingénieurs et les chercheurs se rencontrent actuellement confrontés avec la nécessité d'avoir à leur disposition des modèles multi-physiques plus raffinés et précis qui soient représentatives de ces structures complexes multicouches et des technologies d'amortissement. Ce problème déplace la complexité intrinsèque de ces modèles plus raffinés et plus représentatives pour un plus haut niveau et il défie, de une forme soutenue et motivée par l'application, la communauté des chercheurs de structures adaptatives à donner la réponse à ce problème.

Dans ce contexte, cette dissertation adresse des questions sur le contrôle des vibrations des structures adaptatives, en essayant de couvrir le projet, l'analyse et l'application des technologies émergentes d'amortissement piezo-visco-élastique exploitant, en général, cinq aires/thèmes principales de recherche. (1) MODÉLISATION: nouveaux modèles d'éléments finis raffinés sont développés pour les structures piezo-visco-élastique multicouche du type poutre, plaque et coque avec un haut degré d'exactitude et un compromis adéquat entre exactitude et complexité; (2) AMORTISSEMENT: différents modèles d'amortissement viscoélastique dépendant de la fréquence (hystérétique au sens général) et des méthodes de solution sont développés, évalués et implémentés dans des modèles d'éléments finis; (3) CONTRÔLE: traitements d'amortissement passif, actif et hybride ainsi que des stratégies de contrôle en considérant les approches par feedback et feedforward adaptatif, utilisés séparément ou ensemble, sont développés, simulés, évalués et implémentés en temps-réel; (4) PROJET: l'efficacité des traitements/technologies d'amortissement, par rapport à leurs propriétés matérielles, emplacement, configuration géométrique, épaisseur des couches et loi du contrôle, est analysée et discutée et une méthodologie générale pour aider le projet des traitements est proposée et démontrée; (5) APPLICATION: des routines d'éléments finis dévoués au projet et simulation des traitements d'amortissement hybride, test en temps-réel et analyse, validation et application des technologies d'amortissement hybride (avec plusieurs algorithmes et matériaux amortisseurs) sont développées et implémentées dans un environnement de modélisation et programmation d'éléments finis.

Mots-clés: poutre, plaque, coque, élément fini, structure adaptative, vibration, amortissement, contrôle, piézoélectrique, viscoélastique, feedback, feedforward.

Resumo

Na última década o interesse demonstrado, principalmente por empresas públicas e privadas nos sectores de engenharia aeronáutica e aeroespacial, no desenvolvimento e utilização de estruturas de elevado desempenho, em geral, e estruturas adaptativas, em particular, incorporando tratamentos de amortecimento piezo-visco-elástico para tratar problemas relacionados com vibrações e ruído, tem aumentado. Entretanto, com os cada vez mais exigentes requisitos funcionais que obrigam a dispor de estruturas mais leves, rígidas e de reduzido custo que, simultaneamente, sejam também eficientes e fiáveis quando sujeitas a solicitações dinâmicas, engenheiros e cientistas encontram-se nos dias de hoje confrontados com a necessidade de terem à sua disposição modelos multi-disciplinares mais refinados e precisos representativos dessas complexas estruturas multi-camada e tecnologias de amortecimento. Esse problema desloca a subjacente complexidade de modelos mais refinados e mais representativos para um nível mais elevado e desafia de uma forma fundamentada e motivada pela aplicação a comunidade científica investigadora de estruturas adaptativas a dar resposta a esse problema.

Nesse contexto, esta dissertação investiga questões relacionadas com o controlo de vibração de estruturas adaptativas, tentando cobrir o projecto, análise e a aplicação de tecnologias emergentes de amortecimento piezo-visco-elástico focando, em geral, cinco áreas/tópicos principais. (1) MODELIZAÇÃO: são desenvolvidos novos modelos de elementos finitos de estruturas piezo-visco-elásticas multi-camada do tipo viga, placa e casca, com um elevado grau de precisão e com uma admissível relação de compromisso entre a precisão e complexidade computacional; (2) AMORTECIMENTO: são desenvolvidas, avaliadas e implementadas em modelos de elementos finitos diferentes abordagens de modelização do amortecimento viscoelástico (dependente da frequência) e dos respectivos métodos de solução; (3) CONTROLO: são desenvolvidos, simulados, avaliados e implementados em tempo-real tratamentos de amortecimento passivo, activo e híbrido e estratégias de controlo considerando abordagens por feedback e feedforward adaptativo utilizados individualmente ou combinados; (4) PROJECTO: é analisada e discutida a eficiência dos tratamentos/tecnologias de amortecimento em termos das suas propriedades materiais, localização, configuração geométrica, espessuras das camadas e tipo de estratégia de controlo e é proposta e demonstrada uma metodologia geral de auxílio ao projecto desses mesmos tratamentos; (5) APLICAÇÃO: são desenvolvidas e implementadas num ambiente de modelização e programação de elementos finitos rotinas dedicadas ao projecto e simulação, teste em tempo-real e análise, e validação e aplicação de tecnologias de amortecimento híbrido considerando vários algoritmos de controlo e materiais de amortecimento.

Palavras-chave: viga, placa, casca, elemento finito, estrutura adaptativa, vibração, amortecimento, controlo, piezoelectrico, viscoelástico, feedback, feedforward.

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Glossary

Abbreviations

1-D	one-dimensional
2-D	two-dimensional
3-D	three-dimensional
ACLD	active constrained layer damping
AD	active damping
ADF	anelastic displacement fields
CAVF	constant amplitude velocity feedback
CGVF	constant gain velocity feedback
CL	closed-loop
CLD	constrained layer damping
CLT	classical laminate theory
CMA	complex modulus approach
CT	classical theory
DFR	direct frequency response
DLT	discrete-layer theory
DoF	degree of freedom
ESLT	equivalent single-layer theory
FE	finite element
FRF	frequency response function
FSDT	first-order shear deformation theory
GHM	Gholla-Hughes-McTavish
HSDT	higher-order shear deformation theory
ICL	intelligent constrained layer
LWT	layerwise theory
MEMS	micro-electromechanical systems
MIMO	multiple-input multiple-output
OL	open-loop
PCLD	passive constrained layer damping
PLD	passive layer damping
SDT	Structural Dynamics Toolbox
SHM	structural health monitoring
SISO	single-input single-output

SOL	stand-off layer
TSDT	third-order shear deformation theory
ULD	unconstrained layer damping
YAPD	hybrid active-passive damping

Part I

INTRODUCTION

Part I

INTRODUCTION

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CHAPTER 1

Introduction

1.1 Motivations

1.1.1 High Performance Structures

In the last two decades so-called *adaptive structure technologies* have been vastly employed in the development of high performance structural systems for the aerospace, aeronautic, automotive and military industries, to name just a few. A lot of evidence has been shown demonstrating that *adaptive structures* have the potential to outperform conventional structures in a number of different ways and applications.

At this point, and before discussing the motivations for their use, it seems appropriate to discuss the concept from a philosophical viewpoint. Looking at the etymology of *adaptive*, it comes from the Latin “ad” (toward) + “aptus” (suitable, appropriate, fitting) and dates to the beginning of the 17th century. In biology, “adaptation” means the *adjustment of living matter to environmental conditions and to other living things*¹. The adaptation process can be achieved in different time-scales according to the reason behind the need of the organism to adapt, i.e. the rate at which the environmental conditions change. Well known examples of adaptation processes in two different time-scales are the remarkable adaptive coloring capability of chameleons (relatively short-time adaptation) and the evolution of the living organisms (long-term adaptation), which in biology is referred to as the *evolution of species*, named after the English naturalist Charles Darwin in 1859 with his seminal book “On the Origin of Species”, which, among other things, may concern the emergence of humans as a distinct species from other primates.

The ability to adapt is a fundamental property of life and constitutes a basic difference between living and nonliving matter. However, under the auspices of the technological revolution ongoing in the 21st century, biologically inspired technologies are becoming a reality and the concept of *adaptation* is being extended to the so-called *adaptive structures* making the difference between living and nonliving matter not so evident. As in living matter, the time-scale of the adaptation process is a crucial parameter which can even compromise the success of it and the “survival” of the structural system. As obvious, for structural engineering applications,

¹Adaptation. (n.d.). Columbia Electronic Encyclopedia. Retrieved January 11, 2008, from Reference.com website: <http://www.reference.com/browse/columbia/adaptati>.

short-term adaptation, with a time-scale of the same order of magnitude of the rate of change of the environmental conditions, is more important and the design purpose to be retained.

Some say that the biological sciences, as an area of application and motivation for technological development, will play in the 21st century for engineering a similar role that physics and chemistry played in the 20th century. In fact, following the reflections of the 17th century French philosopher, mathematician and scientist René Descartes on mind and mechanisms, which began the strain of western thought, living matter, in general, and human beings, in particular, can be seen as more “perfect machines”, encompassing all the “rudimentary” technologies and devices (and more) usually utilized by engineering in “real” machines. Therefore, bioengineering seems to be a natural field of engineering concepts application which puts a serious challenge to science. It definitively is a multidisciplinary field which has motivated scientists in both areas to deepen and widen their knowledge. Therefore, in the pursuit of those goals, engineers can learn a lot from biology, an vice versa, seeing living matter as a source of inspiration for improving their designs. The ultimate challenge of engineers would of course be to reproduce, or being able to build the “perfect machine”, or synthesize an intelligent form of life, or part of it, which is a much more difficult task though. This new science involving the study and imitation of nature’s methods, designs and processes has been coined *Biomimetics* by Otto Schmitt in 1969. An excellent monograph on the subject has been recently edited by Bar-Cohen (2005).

From an engineering viewpoint, and as attested by the recently created Adaptive Structures Technical Committees of two world recognized scientific organizations whose members represent all sectors of the field including academia, government agencies and industries, namely the AIAA² and ASME³, created, respectively, in 1997 and 1993, and two well know international journals created in the early 1990s, namely, the journal of *Smart Materials and Structures* and *Journal of Intelligent Material Systems and Structures*, adaptive structures are nowadays more popular as aerospace technological solutions and related technologies are transitioning into the marketplace. Adaptive structures became also nowadays a discipline with particular scientific characteristics and has its own identity and design signature.

Through time other names have been used interchangeably, though, to refer to similar concepts, namely *smart*, *intelligent* and *active structures*, which are the most currently used ones, demonstrating the ever-increasing volume of vocabulary to denote some type of structural system merging sensors, actuators and controls, and the structure itself into a single system (Figure 1.1). Therefore, that terminology usually defines some type of system integrating all or some of these three electronic ingredients, with the main differences between them being the simultaneous (or not) combination of all of them and the degree of structural and electronic integration into the structure (Figure 1.2). Though conceptually similar, the alternative definitions should be used with care and depend pretty much upon the author’s preference, which might produce some different misleading interpretations among different people. Furthermore, when analyzed with care, there are slight differences between them that make them more suitable for describing some applications than others. That issue has been the subject of discussion for some time [see the review paper by Wada et al. (1990) for more information] and nowadays, however not fully understood by the scientific community in general, clear and distinct definitions and interpretations of these have been established for a few people.

In what follows six different definitions are given to denote structures integrating all or

²American Institute of Aeronautics and Astronautics; see <http://www.aiaa.org/tc/as/index.html>.

³American Society of Mechanical Engineers; see <http://divisions.asme.org/aerospace/committees/asms.html>.

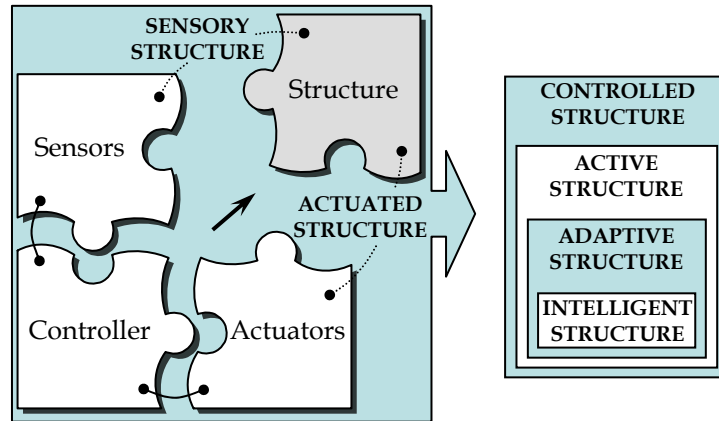


Figure 1.1: Conceptual definition of high performance structures and their key constituents.

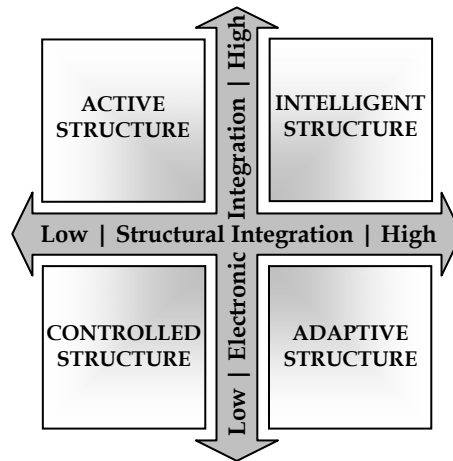


Figure 1.2: Schematic definition of various types of structures whose properties or states can be influenced by the presence of a closed-loop control system according to the degree of structural and electronic integration of the sensors, actuator and controls.

some of the key structural control ingredients, namely actuators, sensors and controls, with different levels of mechanical and electronic integration. These different definitions represent, in some extent, the personal interpretation and preference of the author and are given taking into account the early discussion and general framework for categorizing the field given by [Wada et al. \(1990\)](#) and the nomenclature developments and adjustment to a more realistic and up-to-date categorization with some minor alterations to the original definitions which were later reported by experts in the field [[Crawley \(1994\)](#), [Clark et al. \(1998\)](#), [Preumont \(2002\)](#)]. Therefore, the general framework embracing a broad context of structural approaches has the following six categories:

- *Sensory structures* are structural systems that possess sensors that enable monitoring some system states and other characteristics which can be used to determine the actual state of the structure, its integrity, damage, etc.; the system may possess different types of sensors but no actuators; this type of configuration might be seen as a passive category, i.e. where no direct (or significant) interference with the structural system is performed, and is often used, for example, for structural health monitoring purposes;

- Conversely, *actuated structures*, originally referred to as *adaptive structures*, are structural systems (sometimes articulated) that possess actuators, but possess no sensors, which enable the alteration of the structural states or characteristics in a desired (or “controlled”) manner; examples of this type of structures are deployable truss structures, manipulator arms, space antennas, conventional aircraft wings with articulated leading- and trailing-edge control surfaces, etc.; the motivation for Wada et al. (1990) to use the *adaptive structures* designation for this category has been given in part by the concept proposed by Miura and Furuya (1988) that used it in the context of geometric adaptation of truss structures for future space applications; none of them gave much importance to the degree of structural integration which, for these structures, is established in a discrete “truss-member” sense; as discussed in the next lines, in the author’s opinion, *adaptive structures* should be seen as more sophisticated structures with a higher degree of integration; the adoption of designation of actuated instead of adaptive structures has also been reported by Crawley (1994) and Clark et al. (1998);
- *Controlled structures* are the result of combining sensory and actuated structures into a single one; the sensing and actuating behaviors are coupled by a controller with the purpose of actively controlling the system states and characteristics; the degree of structural and electrical integration is weak and the structure and control system are distinct and well separated; the sensors and actuators have small structural functionality; common applications are the shape control of antennas, active adaptation of structures to optimal fluid flow, isolators, vehicle suspensions, etc.;
- *Active structures* are a subset of controlled structures and contain sensors and/or actuators, usually of concentrated nature (“force”- and “displacement”-based ones), with highly integrated electronics with some structural functionality; the degree of structural integration is weak; typical applications of this designation are active tendon control of cable stayed bridges, truss structures with active members, structures with inertial actuators (“sky-hook” dampers), hexapod isolators (Stewart platform), etc.;
- A subset of active structures designated as *adaptive structures* are autonomous structural systems which incorporate sensors, controls and actuators to enable adaptation to changing environmental conditions, thereby enhancing, among other things, safety, stability, vibration damping, structural acoustic suppression, aerodynamic performance, accuracy, load redistribution, damage response, structural integrity, etc; the actuators and sensors are often of distributed nature (“strain” actuators) and consist of so-called “smart” materials having geometry or characteristics that can be remotely or automatically changed to respond to internal and external stimulation; adaptive control theory should be used; examples of so-called *smart materials* are summarized in Table 1.2 along with their main features; they have a high degree of structural integration but the controls are not so well electronically integrated; examples are smart panels for noise and vibration reduction, and some of the others pointed out before as long as the sensors and actuators are of distributed nature and structural functionality;
- Taken to the extreme, *intelligent structures* are a subset of adaptive structures and the most exclusive subset of controlled structures; they are those which incorporate actuators and sensors that are highly integrated into the structure and have structural functionality (structural integration), as well as highly integrated control logic, signal conditioning and power amplification electronics (electronic integration); such actuating, sensing and

signal processing elements are incorporated into a structure for the purpose of influencing its states or characteristics, be they mechanical, thermal, optical, chemical, electrical, magnetic and so forth, by means of distributed or hierarchic control architectures; for example, a mechanically intelligent structure is capable of altering both its mechanical states (its position or velocity) or its mechanical characteristics (its stiffness or damping); they have self-learning, self-adaptive and decision capabilities; perfect examples fulfilling all the forementioned features are difficult to give since to date such design challenge has not been filed, but applications following that line of thought include “smart skins” and morphing structures for aeronautic and aerospace end-products, optically intelligent structure which could, for example, change color to match its background, etc.

1.1.2 Smart (Multi-Functional) Materials

In recent years the term *smart materials* has been vastly used in the literature in the context of analysis and design of engineering systems incorporating high performance structures such as the ones mentioned before. Smart materials are defined here as those that convert energy between multiple physical domains. A *physical domain*, as presented in Figure 1.3, is composed by a set of two energy conjugated state variables which can be used to define in more precise way the term *coupling*. Multi-physics coupling occurs when a change in a state variable in one physical domain causes a change in the state variable of a separate physical domain and is usually denoted by a blend of the names associated with the involved physical domains (e.g. *electromechanical* coupling). Some materials exhibit a *one-way coupling* and others a *two-way coupling* functionality between two different physical domains. These couplings are expressed in terms of the different state variables of the different physical domains and have specific names depending upon the type of domain in which a stimulus (action) in one of its states variables is performed and the type of physical domain and its associated state variable which is changed (reaction). Some examples are the electric output produced by a thermal stimulus, which is termed *pyroelectric effect*, or the mechanical output produced by an electric stimulus, which is termed as *converse piezoelectric effect*. In respect to each individual physical domain, *internal coupling* also occurs between the associated state variables as is the case, for example, of *elasticity* which couples the *stress* and *strain* state variables of the *mechanical* domain.

Mechanical	Electrical	Thermal	Magnetic	Chemical
Stress	Electric field	Temperature	Magnetic field	Concentration
Strain	Electric displacement	Entropy	Magnetic flux	Volumetric flux

Figure 1.3: Examples of physical domains and associated energy conjugated state variables.

Structures incorporating smart materials and taking advantage of their multi-physics coupling properties to provide functionality are denoted as *smart structures* or *smart material systems*. Other names used to denote these types of materials are *intelligent* or *adaptive materials* or even more sophisticated ones such as *adaptronic* and *structronic materials*. Again, these alternative definitions allows us to characterize these materials in terms of their most general and common features, such as their multi-physics coupling capabilities. Other designation for this area of activity merging mechanical engineering, electronic engineering and software engineering is the field of *mechatronics* which is very often used also. This field has been originally devoted with the smart materials science to large systems where smart materials are included

in the bulk form but is increasingly learning towards miniaturization with the popularization of the micro-electromechanical systems (MEMS).

Looking at the etymology of the different designations, they do not necessarily represent exactly the same features for all the materials and they can be used intuitively to add some extra information creating specific subsets of smart materials classes. One example is readily understood by making the following question and reflection: *What should an intelligent material be?* A definite answer is not available; they may be reasonably defined, however, as materials that besides the same multi-physics coupling capabilities as smart materials have, they have some degree of intelligence close to – and, if possible, exceeding – those found in living matter. For early discussions on the concept and developments of these types of materials the reader is referred to Takagi (1990), Takahashi (1990) and Rogers (1993); more recent ones can be found in Tani et al. (1998), Takagi (1999) and the monograph of Varadan et al. (2006, Chapter 1).

As mentioned before, the transduction (“structural”) key constituents of high performance structures are the sensors and actuators. A large number of different transduction devices are available for sensing and actuation purposes; these smart material devices, which might have different levels of electronic integration, along with some of their main features, are summarized in Table 1.2 [see Srinivasan and McFarland (2001), Schwartz (2002), Chopra (2002), Tzou et al. (2004) and Janocha (2007) for further information]. The most common application (actuation and/or sensing) of each device and the designation of the best known materials are also pointed out.

Investigation of the potential of piezoelectric materials began during the Cold War era; among all different smart materials, piezoelectric materials are nowadays the most popular and widely used as distributed actuators and sensors in structural control applications. As presented in Table 1.2 there are two classes of piezoelectric materials: ceramics and polymers. The best known *piezoceramic* is the *Lead Zirconate Titanate* (PZT), discovered by Jaffe et al. (1954); it has a recoverable strain which might go up to 0.1% and is widely used for high precision actuators and sensors for a wide range of frequencies, including ultrasonic applications. The second class of piezoelectric materials are the *piezopolymers*. They differ from PZTs since they are, in general terms, 20 times more compliant, 4 times less dense and have an actuation capability 300 times lower. The latter performance indicator demonstrates their little authority over the dynamics of stiff structures, constructed from steel or aluminum, making their use more appropriate as sensors. However, motivated by the documented benefits of modal sensors, their use as spatially shaped sensors is facilitated due to the high compliance which facilitates tailoring. The best known is the *Polyvinylidene Fluoride* (PVDF) which was first studied by Kawai (1969) and, along with other types of polarized film, was made commercially available in the early 80’s.

1.1.3 Applications of High Performance Structures

It is well known that the general architecture of high performance structures, i.e. the type of transducing devices and control system strategy, are inherently dependent upon the motivations for their application, the established performance criteria and the design constraints, such as the type of disturbance and the environmental conditions over which the system should operate. Once the appropriate transducers have been appropriately selected, they must be considered at the remaining design stages and “linked” with the “logical” key constituents of high performance structures, i.e. the controls. As obvious, they represent a step forward in the problem-solving motivated research and impose constraints which determine the most adequate control strategy to be used. Thus, before going any further in the discussion of the last

Table 1.2: Transduction devices for high performance structures.

Device/Material	S	A	Designation/Features
Accelerometer	👍	×	Measures point acceleration, requires wiring, low structural integration, various sizes.
Load cell	👍	×	Measures point force, expensive, requires wiring, low structural integration.
Laser vibrometer	👍	×	Measures velocity, no contact, expensive, difficult to integrate into the structure, good adaptability to different measurement locations, allows structural scanning.
Electrodynamic shaker	×	👍	Applies point force, limited operating temperature and stroke, needs a reaction support.
Strain gauge	👍	×	Measures local strain, suitable for static and dynamic applications, sensitive to temperature variations and aging, relatively inexpensive, low mass, good sensitivity, good accuracy.
Piezoelectric ceramics	👍	👍	Lead Zirconate Titanate (PZT) / Distributed strain transducer, brittle, heavy, good recoverable strain (up to 0.1%), high precision in the nanometer range, fast response.
Piezoelectric polymers	👍	👎	Polyvinylidene Fluoride (PVDF) / Distributed strain transducer, flexible, lightweight, require high voltages, limited control authority.
Electrostrictive	👎	👍	Lead Magnesium Niobate (PMN) / Low hysteresis, no electric aging, very sensitive to temperature, non-linear constitutive behavior.
Magnetostrictive	👎	👍	TerFeNoL-D / Long life-time, good recoverable strain (up to 0.15%), work under compressive pre-stress, aging resistant, larger and bulkier than piezoelectrics, adequate for large forces and strains, low efficiency, appropriate for load-carrying.
Shape memory alloy	👍	👍	NiTiNOL / Little use in vibration control, best suited for one-way applications, relatively large actuation force and strain, large hysteresis, slow response, simple in design, bad aging resistance, excellent recovery strain (up to 8%), low voltage, low efficiency, allows complex movements and shapes.
Optical fibers	👍	👎	Fiber Bragg gratings (FBGs) / High sensitivity, small, lightweight, chemically inert, highly distributed, allow an high and easier structural integration in composites, electrically passive, cheap, fragile.
Electrorheological (ER) fluid	×	×	Indirect coupling used in semi-active devices obtained by changing the rheological behavior in a reversible manner – “semi-smart” behavior (no reciprocal effect), fast response, high temperature sensitivity, requires high voltage.
Magnetorheological (MR) fluid	×	×	“Semi-smart” behavior, has better performance than ER fluids, maximum yield stress is an order of magnitude or more greater than the best ER fluids, requires low voltage, relative insensitivity to temperature changes and contamination.

Notes: ‘S’, sensing; ‘A’, actuation; ‘👍’, not applicable; ‘👎’, interesting feature; ‘👍’, uninteresting feature.

“logical” key constituents of high performance structures, i.e. the controls, it is appropriate to discuss at this point the typical motivations for their application.

The most common applications of high performance structures depend pretty much upon the different branches of pure (e.g. physical, earth and life) and applied (engineering) sciences and specific disciplines where the problem motivating the use of high performance structures occurs. Typically the concerns behind these motivations involve the following four areas of activity:

- *Structural vibration control*: structural damping augmentation by means of passive, semi-active, active or hybrid damping means with respect to improving different vibration related efficiency criteria such as aeroelastic and flutter performance, passengers comfort, weight reduction, chatter reduction, earthquake/seismic resistance of buildings and bridges, etc.; typically of interest in aeronautics, astronautics and automotive, machine tool and manufacturing plant industries;
- *Structural acoustics control*: vibroacoustic damping augmentation reducing the radiated acoustic energy by means of passive, semi-active, active or hybrid damping means with respect to improving different acoustics related efficiency criteria such as the reduction of noise generated through aerodynamics and/or engines, passengers acoustic comfort, etc.; typically of interest in aeronautics and automotive industries;
- *Shape, position and fluid flow control*: static deformation control of structures to improve for example communication performance of antennas or adapt structures to optimum aerodynamic fluid flow with a reduced aerodynamic drag; typically of interest in aeronautics, astronautics, astrophysics and the automobile industries;
- *Structural health monitoring (SHM) and damage prognosis*: the integration of sensing and possibly even actuation devices for either monitoring the operational and/or damaging condition of structures and predicting the remaining structural life; typically of interest in aeronautics, astronautics, civil engineering structures, etc.

Other examples of areas of activity can be found in life sciences, such as biology and medicine, where smart structures technologies comprising biologic actuation (artificial muscles), monitoring of biological system with different sensor modalities (physical, electrical, chemical), implants of artificial adaptive organs, adaptive prostheses, controlled drug delivery, to name only a few, have been under great development lately by the bioengineering research community. A further extension of these technologies is the self-repair and self-healing bio-inspired capability, highly desirable in high performance structures, which has recently started to give its first steps towards a feasible reality.

1.1.4 Passive versus Active Design of Damping Technologies

Structure-borne noise and vibration have long been known for their capacity for disturbance, discomfort, damage and destruction. These disastrous and sometimes inevitable side effects of structures, manifesting themselves at different levels affecting the health and structural integrity of human beings and machines, motivate the ever increasing existence and establishment of multi-million-dollar industries devoted to structural vibration and acoustics control, continuously developing, producing and improving devices and materials to control these outcomes. These two areas are inherently related and implicitly coupled since in the majority of the

cases fluid-structure interactions are always present and as the origin of these phenomenon we can have fluid-flow inducing structural vibrations or noise being radiated by vibrating structures. In order to reduce the overall sound radiation of the structural source there are mainly two approaches: (i) completely mitigating vibrations, which represents the outcomes of the first area of activity when the control efficiency is taken to maximum extent or, (ii) since the various modes of vibration have different radiation efficiencies and some are better coupled with the acoustic field than others, changing the vibration distribution patterns in a convenient manner. The latter approach, when compared with the former “brute-force” approach, is by far the most interesting one for structural acoustic control since it requires in general less control power; it has the drawback though of requiring some physical knowledge of the process and acoustic engineering abilities at the design and simulation stages.

Directing our attention towards situations where unacceptable vibration-related problems need to be remedied, it is desirable and often necessary, first of all, to understand its whole nature. This includes understanding its source, its spatial nature, direction and energy propagation path and the frequency content of the vibration problem. The next step would be trying to reduce the disturbance power acting at the source and/or isolating the system from the source as much as possible. If isolation or dealing with the source does not sufficiently alleviate the problem one must consider other alternatives.

The foregoing discussion has concerned an *active design* approach to the development of high performance structures which employs additional structural and logical active constituents, i.e. transducers and controls, to enhance the functionality and structural integrity of structures. However, with the purpose of solving similar problems, *passive design* techniques have been used successfully for a long time to solve vibration-related phenomena and were, in fact, the source of inspiration to active design developments, which also involve the modification of structural stiffness, mass and damping to make the system less responsive to its vibratory environment. These modifications when introduced by passive vibration control techniques may take the form of basic structural changes (structural design) or the addition of “passive” elements which are connected with the structural system, such as masses (which can be chunks of the same or other material), springs (such as vibration isolators), spring-mass systems (vibration absorbers), fluid dampers or damped rubbers (added damping). In comparison with active elements, these passive elements do not require any external assistance (e.g. power, command, “intelligence”) to do their job; they simply react passively in opposition to the stimuli imposed upon them by the vibration, and therefore have less failure probability. In opposition, active control methods are usually more costly than passive methods but allow an higher degree of adaptability and controllability. Therefore, the expensive measures taken with active control can only be justified in applications where all passive design techniques have been exhausted without success. A milestone book on passive vibration control is the one by [Mead \(1998\)](#) where a comprehensive description of passive design techniques for vibration control by structural design, localized additions, added damping and resilient isolation is presented.

Although active systems can be manufactured with a high degree of reliability, the possibility of occurring a failure somehow must be accounted for at the design stage. A secondary passive system should therefore be considered as a back-up to prevent complete disaster, insuring continued admissible functionality and performance under operation even if the primary system fails; quite often, passive systems are also used to enhance and assure control stability. An obvious conclusion is that passive design should always be considered at the design stage whether by an adequate structural design, which may reduce the control power needed by active design, or by the addition of passive elements, in order to increase the efficiency and reliability of the global system. As obvious, this approach is denoted as *hybrid design*, blend-

ing the fail-safe capabilities of passive means with the adaptability, controllability and high performance of active ones.

Despite serving as source of inspiration to the development of active techniques, from a technological viewpoint, passive design is less demanding. However, from a physical and applied engineering viewpoint, it is more stringent and requires more technical and empirical skills, usually obtained as the outcome of experience and a large number of years working in the field. For these reasons, nowadays the active design approach is much more appellative to young engineers and scientists due to the level of high technology involved and the degree of automation involved in the design process. It should not be forgotten, though, that in order to have the best solution one should take advantage from the best of the two “worlds”, using intelligence both to develop the best passive solution and to incorporate some in engineering products as part of the solution. This line of thought constitutes the engineering design paradigm of this new millennium implicitly accounting all the lessons learned with more conventional strategies and designs.

Among these passive strategies, the modification of the structural system damping by the insertion of layers of high damping materials within or on a structure, acting essentially as distributed damping systems, as a means of increasing damping in structures, has been used for over 50 years. The materials commonly used are viscoelastic polymers (synthetic rubbers) which have been suitably developed to yield high damping capacities in the frequency and temperature ranges of interest. These materials are relatively inexpensive and may constitute a very efficient lightweight vibration reduction solution when carefully selected and appropriately inserted and positioned into the structure. As obvious, they have some limitations and are often better suited to attenuate high frequency vibrations. For that reason, *hybrid damping* solutions considering both active piezoelectric distributed actuators, more efficient in a low frequency range, and passive viscoelastic absorbers, appropriate for higher frequency ranges, have been used lately.

1.1.5 Control System Architectures

Up to this point we have discussed active, passive and hybrid design technologies to improve the functionality and performance of structures. These design strategies employ combinations of distributed passive damping elements and active ones; the latter usually have some degree of intelligence which is given, in general, by the directions given by a controller capable of modifying the static and dynamic response of the structure in the presence of time-varying environmental and operational conditions. As previously referred, structures with a high degree of structural integration of passive and active damping elements incorporating some degree of intelligence are denoted as *adaptive structures*. Once that both its passive and active structural key constituents have been described it is time to make some considerations regarding the definition of the appropriate control system architectures to be used with adaptive structures.

As previously discussed, the design process of high performance structures, in general, and adaptive structures, in particular, typically begins with the definition of some performance objective. This objective is then used to establish the adequate control strategy which should implicitly consider, at least, a basic understanding of the dynamics of the structural system under study and the type of disturbances involved. To do that, a structural model with more or less degree of accuracy and representativeness is often used to gain insight of its dynamic behavior and assessing the design scenario in terms of feasibility and performance criteria. However, all structural models are inaccurate in their representation of the structural entities and systems we wish them to represent. That situation even gets worst when we are considering adaptive,

or even intelligent, structures which have a high degree of electronic and structural integration of transducers and controls. This model dependence is a fundamental characteristic of any controlled system and is a result of implementing real-time controllers on real structures whose infinite-dimensional dynamics can not be entirely described, or even accurately measured, over the entire operational bandwidth. Therefore, its implications must be considered at the design stage and the goal should be to design controllers with sufficient insensitivity to either changing or misrepresented structural model dynamics, i.e. with increased *robustness*.

As far as the controller is concerned, and according to the degree of adaptiveness of it, there are two fundamental forms of adaptive structures that directly derive from the a priori adaptiveness design constraints that may be thought of [cf. [Clark et al. \(1998\)](#)]: so-called *constrained adaptive* or *purely adaptive* structures. A constrained adaptive structure is designed to operate under the direction of a fixed-gain control law and therefore is constrained to performance limits within a pre-determined range which is determined ideally when the structural characteristics and disturbing environment do not change at all or change in a time-scale that do not compromise the controller efficiency. This latter assumption is in fact very restrictive since, as previously referred, the underlying mathematical models are never accurate enough, making the simulation and control system design limited to theoretical performances, and does not apply to the generality of the situations. Therefore, the alleviation of this restrictions gives a lot more flexibility to the designer and increases the robustness of the controller to adverse unpredicted phenomena. This leads us to the concept of purely adaptive structures which operate under the direction of a control law whose parameters are adaptively modified in a convenient way, usually as a function of some measured system output parameters, or a fixed control law that operates on real-time estimates of the plant parameters (model-based control). Both concepts inevitably have some degree of model dependence; however, as obvious, the latter concept is less sensitive to the model accuracy, meaning that the control response will be more insensible to a change in the system parameters and therefore less deviations in performance between the expected (predicted) and achieved outputs is expected.

In the choice of the adequate control architecture meeting the desired performance requirements there are two quite different and somewhat complementary approaches: *feedback* and *feedforward* control. The principle behind feedback control is based upon a process whereby some proportion of a measured output signal of a system is passed (fed back) to the input in order to cancel the effects of external disturbances. In this case it is assumed that no prior knowledge or correlated information with the external disturbances affecting the control plant is available and, in the context of lightly damped structures, it is usually utilized for two distinct purposes: *active damping* and *model-based* feedback. The former has the objective of reducing the effects of the high magnitudes of the system near the resonances (resonant peaks) while the latter has more ambitious objectives comprising also disturbance rejection over the bandwidth of the disturbance and regulating (making a certain control variable to track a desired value) capabilities. Regarding the feedforward approach, it is based upon digital filtering approaches and characterizes a controlled system that reacts to a measurement or correlated information of the disturbance in a pre-determined way where some proportion of the input disturbance is passed (feed forward) as another input. Therefore, it must have some information of the disturbance, the effect of the disturbance to the output of the system must be known and the time it takes for the disturbance to affect the output must be longer than the time it takes the feedforward controller to affect the output. If these conditions are met, feedforward can be tuned to be extremely effective. Furthermore, it can respond more quickly to known and measurable kinds of disturbances but, when used with fixed-filtering coefficients, it cannot do much with novel disturbances. For that reason, due to the relative robustness and ease of implementation

of adaptive filtering, it has been preferred as a way of conferring to feedforward control a self-adjusting signal processing capability in response to real-time system response measurements.

A decisive parameter influencing the adequate choice of the controller and its design is the statistics and spectral content of the inputs disturbing the structural system. Quite often, the input disturbances are some combination of persistent and impulsive disturbances as presented in Table 1.3. That is also the case of the impulse disturbances caused by the adaptive feedforward controller which during the adaptation process, i.e. the discrete change in the filter coefficients, produces a discrete time-varying control input. In such cases, a combination of both adaptive feedforward and feedback control in a single architecture would be appropriated; this latter combination is defined as *hybrid* control. The use of hybrid control architectures represents a more complex adaptive structure concept, but in turn it affords also significant performance and flexibility advantages.

Table 1.3: Recommended control architectures for different types of single and multiple input disturbances.

Disturbance	Adaptive feedforward	Feedback	Hybrid
Persistent			
Harmonic	👍		👍
Multi-harmonic	👍		👍
Broadband	👍	👍	👍
Impulsive		👍	👍

Feedback control is usually utilized with a fixed-gain design philosophy, which simplifies pretty much its design and implementation; when coupled with a fully instrumented structure it constitutes an example of a *constrained adaptive structure* as referred early. Instrumented structures using adaptive feedforward control, however, are a good example of *fully adaptive structures* since the controller relies on signal processing methods which inherently have a higher degree of adaptation capabilities associated since the control input to the structure evolves with time as a result of the continuous adjustment of the compensator filter coefficients. As obvious, the use of a hybrid, fixed-gain feedback combined with adaptive feedforward, controller will yield a blend of constrained and fully adaptive structures. However, many times this degree of adaptiveness is not enough to deal with insufficient degree of a priori knowledge of the plant dynamics which will make closed-loop instability a likely result. A solution to solve these real world obstacles can be obtained through a further increase in adaptiveness of the control algorithms by using some of the large number of enhanced adaptive control architectures proposed by control theorists [see for example the textbook by [Aström and Wittenmark \(1995\)](#) for further information] both for feedback and feedforward strategies, such as different adaptive signal processing control methods, deterministic and stochastic self-tuning regulators, model reference adaptive controllers or gain scheduling, to name just a few. It is worthy to mention that adaptive control is different from robust control in the sense that it does not need a priori information about the bounds on these uncertain or time-varying parameters; robust control guarantees that if the changes are within given bounds the control law need not be changed, while adaptive control is precisely concerned with control law changes. This concept of *hybrid fully adaptive structures* represents the forementioned capabilities taken to its maximum extent; in some way they represent an *automated version of self-learning* capabilities and resemble a certain degree of *intelligence* being conferred to the structure. Other ways of obtaining it would probably be to enter in the realm of artificial intelligence science.

1.1.6 Outlook and Future Perspectives

Controlled structures, in general, and adaptive (or even intelligent, in some extent) structures, in particular, available in the market or on production lines of many industries, in the air, in the field, in the water, in combat or in our pocket, are currently revolutionizing several corners of technology. From the previous reflection, it is obvious that this growing field is highly multi-disciplinary involving several, if not all, branches of sciences. New capabilities and improved performance have been concretely demonstrated in so many different things as the human body, cars and aircraft from the size of an ant (or even less) to the size of the biggest mammal on earth. Therefore, one way or the other, they have already started to affect our lives. In the near future they are expected to make part of our everyday life providing us, among other things, a more comfortable life and capabilities to do things that not too long ago had space only in our imagination. But the most lasting impact will be that the philosophy of engineering design will begin to change. Engineers of the future will not have to add mass, stiffness or damping to a structure to assure safety in structures that are used outside their initially intended envelope. Engineers will not have to learn from structural failures, but will be able to learn from the life experiences of the structure. They will allow the silent systems we create to inform us, to enlighten us, to educate us of the physics, science, and interaction of the environment on our designs, as well as to self-learn and self-repair in order to “survive”. However, today, the area still faces a profound challenge in that too few practicing technologists have been trained to properly exploit the capabilities of these unique features of smart material systems and structures.

1.2 Background and History of Piezo-Visco-Elastic Damping Treatments

In this section a brief look at the background, concepts and terminology of damping treatments, significant historic events and some recent developments of adaptive structures comprising viscoelastic and piezoelectric damping strategies will be performed. The focus will be on the cases where the established performance criteria concern structural vibration and acoustics control by means of hybrid design techniques using piezo-viscoelastic damping treatments. Since these subjects will be further dissected and reviewed in the following chapters, it is not intended here to exhaust these subjects but to point out the most important aspects to gain some background and historic insight into the field.

The first experimental observation of *material* (or *internal*) damping in metals was performed by Coulomb in the 1780s, as reported in his *mémoire* "Sur la Force de Torsion & Sur l'Elasticité des Fils de Metal" [Coulomb (1784)]. One century later, the Austrian physicist Ludwig Boltzmann developed the first formulation of a three-dimensional theory of material damping regarding linear isotropic viscoelastic stress-strain relations [Boltzmann (1874)]. The assumptions and developments presented by Boltzmann were very important and constitute nowadays the pillars of linear viscoelasticity theory. Since then, various mathematical models of damping have been postulated and validated by experimental observation of damping amazing phenomena [cf. Bishop (1955), Demer (1956), Lazan (1959) and Peters (2005)] and the great importance of damping on vibration related phenomena has been recognized [Crandall (1970)].

Viscoelastic materials are known by their damping capabilities due to the phase lag between stress and strain under steady-state harmonic vibrations; this latter effect promotes an

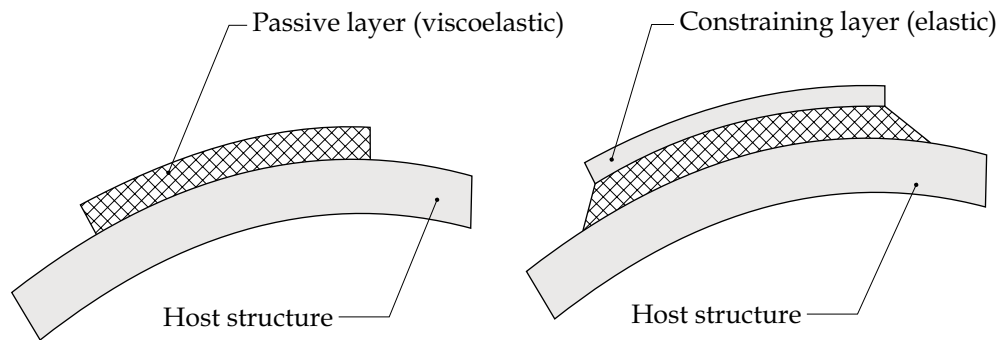


Figure 1.4: Viscoelastic unconstrained (ULD) and passive constrained layer damping (PCLD) treatments.

increased internal molecular friction which yields energy dissipation due to the heating effect. Therefore, passive damping treatments comprising different arrangements of viscoelastic damping layers were first proposed as an effective means to artificially increase structural damping by “adding” passive damping to the systems. These damping treatments have been extensively used in engineering practice to reduce vibration and noise radiation [Nashif et al. (1985), Johnson (1995), Mead (1998), Jones (2001)].

The primary methods of increasing the passive damping in a distributed structure are the so-called viscoelastic *free layer* (or *unconstrained*) and *constrained layer* damping treatments (see Figure 1.4) which both rely on the strain of a viscoelastic material to dissipate energy. The former method, denoted as *unconstrained layer damping* (ULD), was alleged first proposed and studied by Liénard (1951) and Oberst (1952) in the early 1950s, which provided methods for predicting and measuring the damping of bars with homogeneous viscoelastic damping layers. Looking at the physics of the treatment, when the structure vibrates, energy is dissipated in the viscoelastic layers mainly due to the cyclic extensional deformation. Increasing the thickness and length of the viscoelastic treatment would increase the energy dissipation and consequently the damping. However, if the weight of the treatment is of critical importance, the thickness and length of the treatment have to be limited. The solution is to find new alternatives where the efficiency of the treatment is improved. These were in the late 1950s the motivations behind the development of the latter method, denoted as *passive constrained layer damping* (PCLD), which has been proposed for the first time by Kerwin (1959) that demonstrated its superior damping performance with reduced weight increase. Collaborating directly with the viscoelastic material manufacturer, which provided detailed information regarding the frequency and temperature constitutive dependence of the utilized viscoelastic material, with this article Kerwin presented a very detailed study, originally postulating (however as suggestion of J. Baruch) that damping by a constrained damping layer is attributable to the shear deformation of the damping layer and to the resultant underlying dissipation of energy, and later demonstrating, both experimentally and theoretically, that indeed the shear mechanism is the predominant damping mechanism. Some examples of its subsequent use and further study can be found in [Ross et al. (1959), Mead and Markus (1969), Macé (1994)].

However, while passive damping treatments can greatly improve the damping of the system, there are again other limitations. Viscoelastic materials have frequency and temperature dependent mechanical properties which can make the damping change, bringing limitations

to the effective temperature and frequency range of the treatment. In order to provide adequate damping over a broad frequency band, different viscoelastic materials must be chosen which often complicates the analysis and design of the system. Therefore, while viscoelastic treatments are easy to apply, the damping is often of limited bandwidth. These latter reasons motivated the use of piezoelectric materials as a means to increase structural damping, denoted as *active damping* (AD), by “adding” active damping to structural systems by means of the active design philosophy of adaptive structures or hybrid damping using blends of piezoelectrics and viscoelastics (see Figure 1.5).

As recently reported by Greg Reich from the Wright-Patterson AFRL in a historical note in the recently published first ASME & AIAA Adaptive Structures & Material Systems Newsletter⁴, perhaps THE seminal article in adaptive structures, entitled “Use of piezoelectric actuators as elements of intelligent structures”, was published in the AIAA Journal in October 1987 by Edward Crawley and his student Javier de Luis. The paper addressed at that time many of the issues that would become central to the field: it defined (although not for the first time) intelligent structures as a highly distributed network of sensors, actuators and processing networks; it developed an analytic model for the behavior of distributed piezoelectric actuators coupled with the structural system, and verified that model experimentally for static and dynamic cases; it created a scaling parameter to help determine the effectiveness of an actuator in relation to the substructure; it addressed actuator placement in terms of the expected behavior of the system; and it addressed manufacturing issues for implementation of adaptive structures. The impact that this paper has had on the field can’t be overstated. According to the Scopus Citation Tracker, this paper has been cited an amazing 961 times since its publication (as of 01/31/08) and no other paper in the field of adaptive structures comes close. (It seems important also to refer that one and a half year ago it had been cited 802 times, which demonstrates the ever increasing interest in the field from year to year.)

Another landmark article in the field, however receiving less interest by the scientific community, with 473 citations (as of 01/31/08), is the article by Bailey and Hubbard (1985), published two years early. It is also cited by Crawley and de Luis (1987) and is alleged the first article experimentally using distributed piezoelectric actuators using different distributed-parameter control theories. It considers, though, an uncoupled model of a piezopolymer distributed actuator and assesses mainly the feasibility and controllability of the active damping approach with different control strategies using an accelerometer as sensor. A summary of the developments made in generalizing the control design to beams with arbitrary boundary conditions and prescribing a set of design guidelines for the application of film controllers for these configurations, and considering also (alleged for the first time) the design and implementation of non-uniformly linearly varying spatially distributed film actuators for the control, was presented two years later by James Hubbard and his student Shawn Burke [Burke and Hubbard (1987)].

In fact, in 1986 James Hubbard was granted the U. S. patent on the “Method and Apparatus using a Piezoelectric Film for Active Control of Vibrations”⁵ which is worthy to look into; several new ideas which would reveal key elements in forthcoming developments in adaptive structures were already outlined in his patent; some limitation would be though that only piezopolymer (PVF₂) transducers were being considered. Other pioneer works in the adaptive structures field, somehow related and inspiring Bailey and Hubbard (1985) and Crawley and

⁴Published in the Summer of 2006; Retrieved January 31, 2008, from the AIAA Adaptive Structures Technical Committee website: <http://www.aiaa.org/tc/as/reports.html>.

⁵U. S. patent number 4565940; James E. Hubbard Jr., Massachusetts Institute of Technology (Cambridge, MA); Filed: August 14, 1984; Issued: January 21, 1986.

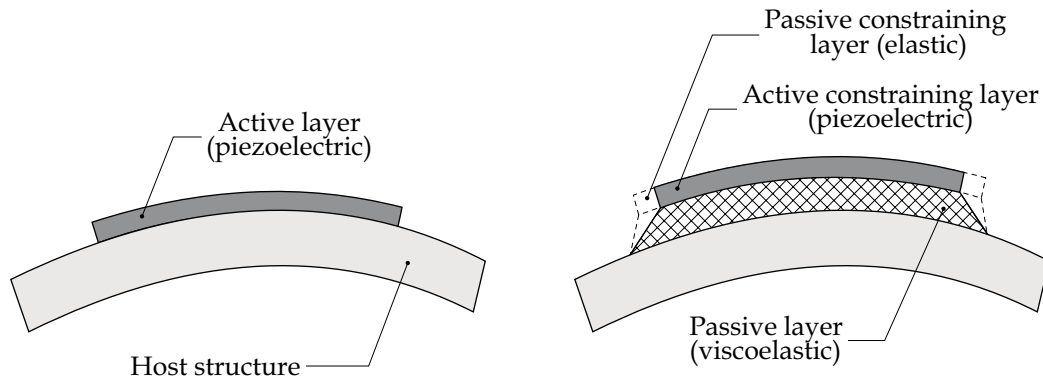


Figure 1.5: Active damping (AD) and active constrained layer damping (ACLD) treatments.

de Luis (1987), applied to actuate or damping simple dynamic structures, which should also receive some credit as the prosecutors of adaptive structures developments, are the ones by Olson (1956), Forward (1979, 1981) and Hanagud et al. (1985).

Motivated by the low actuation authority of polyvinylidene fluoride (PVF₂), which was the piezoelectric actuating material being used by Hubbard and co-workers in the 1980s, Hubbard and his student John Plump directed their seminal efforts towards an alternate design philosophy consisting in increasing the passive damping by means of actively constrained viscoelastic passive damping treatments (an “active damper”, as originally denoted); this original and pioneer idea reported in [Plump and Hubbard (1986), Plump (1986)] gave rise to the designation *active constrained layer damping* (ACLD) treatments which became nowadays an interesting damping strategy which blends the advantages of both passive and active design. In his master thesis, Plump (1986) addressed the most important issues that also would be the central and vastly used in the design of ACLD treatments: an analytical model was developed based on good physical assumptions for the viscoelastic layer damping effects, the three interrelated energy dissipation mechanisms (denoted as passive dissipation, active dissipation and boundary work) were identified and discussed, the implications of the active stiffness value of the constraining layer in terms of the viscoelastic material thickness necessary to obtain equal loss factor with a passive constrained layer was examined, the modulated active nature of passive energy dissipation was pointed out and the “boundary work”, related with the active efforts transmitted to the host structure due to the actuation of the piezoelectric film, was quantified. From a design viewpoint these are also nowadays the most important issues which were early discussed by Plump. In summary, the use of ACLD as an effective means to increase damping was envisaged and recommendations for future work regarding the development of appropriate control laws and the use of piezoelectric films as sensors were given. The ACLD treatments combine the high capacity of passive viscoelastic materials to dissipate vibratory energy at high frequencies with the active capacity of piezoelectric materials at low frequencies. Therefore, in the same damping treatment, a broader band control is achieved benefiting from the advantages of both passive (simplicity, stability, fail-safe, low-cost) and active (adaptability, high-performance) systems.

These issues and others, related with modeling, control strategies, robustness, thicknesses and location optimizations, to name a few, have later been exploited by several researchers, being the most important of these the works by Baz and co-workers. His first article on ACLD was presented in *Damping'93* conference [Baz (1993)] and since then he has published more than 40 articles in the subject with the words ACLD in the title and is nowadays recognized

as the ACLD mentor; he also holds the U. S. patent for ACLD entitled “Method and Device for Active Constrained Layer Damping for Vibration and Sound Control”⁶ (however the previous Hubbard’s 1986 patent on active control of vibrations already refers the ACLD design approach). Curiously, the most highly cited article in ACLD, denoted with a different name though, namely *intelligent constrained layer* (ICL), is the one by Shen (1994b) which has been cited 104 times (as of 01/31/08); although not the first Shen’s article on the subject it makes already an interesting review of the field describing the most interesting and pioneer contributions to the study and development of ACLD without making any reference to Hubbard and Plump’s original idea. Some further examples of ACLD design can be found for example in [Agnes and Napolitano (1993), Ray and Baz (1997), Park and Baz (1999a), Baz (2001)].

Nowadays, these damping treatments are becoming more popular in aerospace, aeronautical and automotive industries, and research is being done towards the study and optimization of the various configurations of active and passive layers treatments. These configurations have been proposed in an attempt to improve performance and are denoted as general hybrid active-passive damping (or arbitrary ACLD) treatments, which involve arbitrary arrangements of active, passive and constraining (active or passive) layers [Azvine et al. (1995), Ro et al. (1997), Inman and Lam (1997), Baz (1998), Park and Baz (2001), Trindade and Benjeddou (2002)]. Furthermore, piezoelectric sensing layers are also usually considered to establish a control law for the hybrid treatments. A survey of advances in hybrid active-passive vibrations and noise control via piezoelectric and viscoelastic constrained layer treatments can be found in references [Park and Baz (1999b), Benjeddou (2001), Stanway et al. (2003)].

In Figure 1.6 a history timeline graphic pointing out the most relevant historic happenings and discoveries which led to what the passive, active and hybrid design noise and vibration reductions technologies for high performance structures, employing viscoelastics, piezoelectrics and controls, are today, are summarized. The names of the authors, the year and a brief description of their achievements are also briefly depicted in the graphic.

Among these, some works worthy to mention are the early developments in the 1960s on variational techniques for piezoelectric media using whether energy (Hamilton’s principle) or equilibrium approaches, performed by Tiersten (1967) and Eer Nisse (1967), respectively, and the seminal book by Tiersten (1969) on “Linear Piezoelectric Plate Vibrations”. Another interesting book pioneering the use of variational principles in electromechanical systems in general, is the one by Crandall et al. (1968) entitled “Dynamics of mechanical and electromechanical systems”. These variational principles have triggered the subsequent development of approximate methods and the first piezoelectric finite element was developed by Allik and Hughes (1970).

1.3 Objectives of the Work

The interest in high performance structures, in general, and adaptive structures, in particular, has been, since the beginning, mainly demonstrated by public and private companies from the aerospace and space engineering sectors. Since their end-products are usually very expensive and sophisticated systems demanding an high degree of performance and reliability, the underlying research, which requires a lot of money investment, could only be afforded by these companies which end-up triggering the developments in the field. So in this case, the ends DID

⁶U. S. patent number 5485053; Amr M. Baz (Rockville, MD); Filed: October 15, 1993; Issued: January 16, 1996.

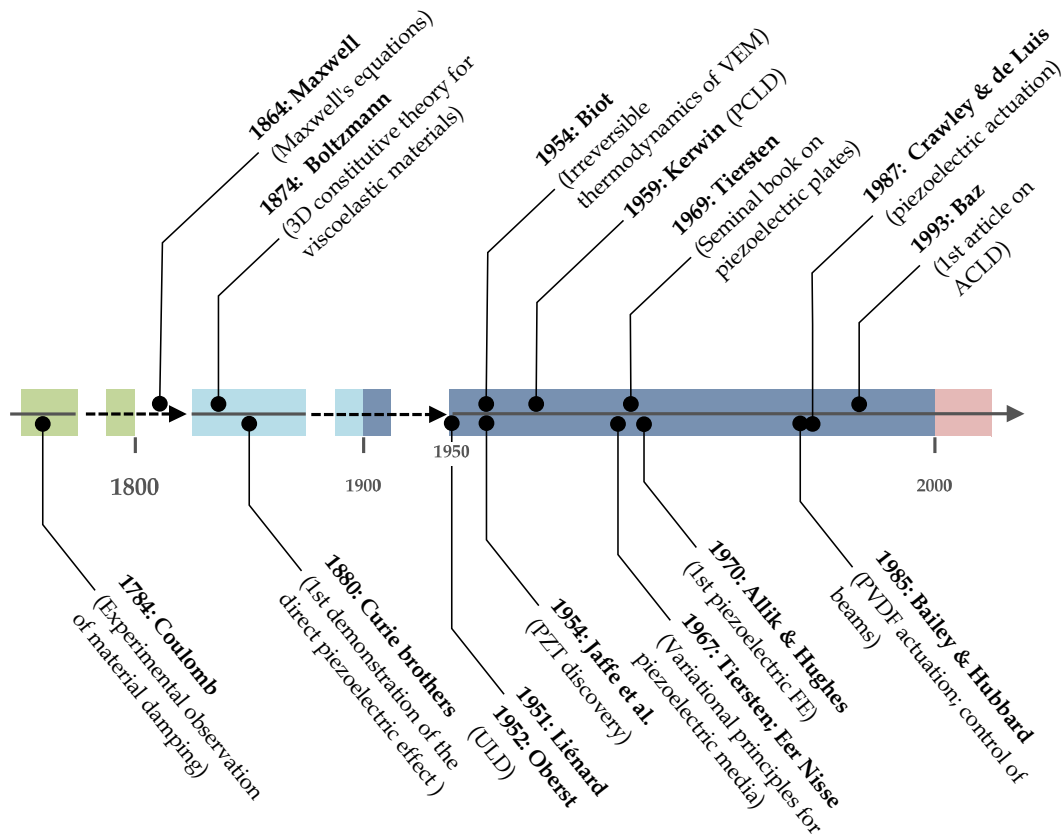


Figure 1.6: Summary of relevant historic happenings and discoveries in piezo-viscoelastic damping treatments technologies from the 18th to the 21st centuries.

justify the means! However, so far, a lot of research has been performed in aerospace-related smart structures technologies but only very little has gone into true application. Arguments to justify it could be the fact that in general every technology takes some time (some say it takes half a century!) to achieve its breakthrough from the very first beginning when the idea was born (which is in fact very young with “only” 20 years old) or the the fact that this technology may have more intellectual than commercial rewards. In summary, when looking at the past, we can conclude that the outcome of high performance structures technologies deserves our respect and when analyzing the potential of these technologies for the future, we might say that they have still not been adequately explored. Likewise, at the present, we should recognize that we are still nearly halfway from its breakthrough and that, as usual in science, it still will take some time to see what will happen. However, the field seems very promising...

Looking at the field as it is today, it has brought together communities from different disciplines of science which used to work more in isolation during the past. The physical outcome is still not evident but from an intellectual viewpoint it definitely generates an unbelievable scientific and technical enrichment of the people involved towards a broader and thus holistic understanding of physics and multiphysics problem-solving engineering, forecasting the technical skills needed by engineers in the future and envisioning the bright future that the involved people will have.

In the last decade, these technologies started to receive an increasing amount of attention from sound and vibration related companies around the world and a relative maturity in the adequate modeling and design of piezo-viscoelastic damping treatments has been achieved.

However, with the ever increasing strong demands of lighter, stiffer, lower cost and more efficient and reliable structures, engineers and scientists are faced nowadays with the requirement of having at their disposal refined and more accurate multiphysics models of those complex layered structures. That problem pushes the underlying complexity of representative models to higher levels and poses a sustained and application-motivated challenge to the adaptive structures research community. Therefore, allied with the ever increasing processing capabilities of modern computers, the 21st century has emerged with a "new" challenge for science, demanding the development of new, refined, more accurate and fully representative modeling technologies which may help to solve the "classical" structural noise- and vibration-related problems that have been tackled by scientists and engineers for a long time, and the education and training of engineers (which historically have shown to have the role in science of applying the knowledge into real constructions) qualified not only with theoretical knowledge but also with technical, experimental, empirical and practical know-how which is acquired and developed only by applied engineers when these technologies are tried to and effectively put into practice.

Therefore, the objectives of this work are to address structural vibrations control, tentatively covering the design, analysis and application of emerging piezo-visco-elastic damping technologies focusing, in general, the following main five research fields/topics:

1. **MODELING:** Development and application of new refined finite element (FE) models of multilayer piezo-visco-elastic beam, plate and shell structures with a high degree of accuracy and a suitable trade-off between accuracy and complexity;
2. **DAMPING:** Development, assessment and FE implementation of general temperature- and frequency-dependent viscoelastic (or general hysteretic) damping models;
3. **CONTROL:** Development, simulation, assessment and real-time implementation of passive, active and hybrid control strategies considering both feedback and adaptive feed-forward algorithms used separately or blended;
4. **DESIGN:** Analysis, simulation and development of optimal design strategies to improve the efficiency of the damping treatments in terms of material properties, location, geometric configuration, thicknesses of the layers and control law;
5. **APPLICATION:** Development and implementation of FE routines devoted to hybrid damping design and simulation into the MATLAB/SDT⁷ programming environment, real-time test and analysis, validation and application of hybrid (with multi-algorithms and multi-damping materials) damping technologies.

⁷The Structural Dynamics Toolbox (SDT) for MATLAB enhances MATLAB core capabilities in controls and signal processing through extensions linked to general finite element modeling, experimental modal analysis and test/analysis correlation capabilities. It provides a general open finite element architecture and a range of specialized solvers to study static and dynamic problems. Among other things, it includes also interface capabilities with popular FEM codes, Graphical User Interface (GUI) visualization tools to pre-/post-processing SDT or other codes computations and a multiphysics finite element library (OpenFEM). The open specification of data structures and element function services allows integration of user developments in the pre-/post-processors and solution methods. SDT is a software product commercialized by SDTools which is an independent and privately owned software development company that provides off the shelf scientific libraries called from the MATLAB environment and consulting services to implement custom applications using these libraries (<http://www.sdtools.com>).

Due to the multidisciplinary nature of the present work, and for easiness of presentation of the different subjects, the different forementioned research fields/topics (mainly the first three) will be presented somewhat separately in this dissertation. In fact, this dissertation results in part from a compilation of several research articles from the author already published in different journals and conferences which, along with some unpublished new material, where tentatively given a unified and sequential character of a self-contained dissertation. However, this research field is very vast and embraces different disciplines in science. Therefore, it is the author's intention to limit the tutorial nature of this dissertation to the minimum extent, pointing out whenever possible appropriate references where detailed explanations of some classical subjects are given, allowing him to give more emphasis to the real innovative and developed work. However, the integration of all the subjects, including the design and application, is implicitly considered when presented and discussing the design, simulation and real-time application of the case studies considered.

To sum up, the ultimate and general objective of this dissertation is therefore to *embrace multiphysics noise- and vibration-related problem-solving engineering science through the study of piezo-viscoelastic damping treatments and controls* applied to structures in order to become acquainted with not only theoretical know-how but also qualified with technical, experimental, empirical and practical skills in classical and emerging sound and vibration related technologies.

1.4 Organization of the Dissertation

Considering the aforementioned objectives, this dissertation is organized in five parts which are described as follows:

- An introduction to the dissertation is performed in Part **I** – *Introduction* where the motivations behind this work, along with some background concepts/terminology and relevant historic events interconnected with of piezo-viscoelastic damping treatments, and the general objectives and organization of the dissertation are presented and discussed; the reader will notice that the state-of-the-art review in this part was intentionally presented from a philosophical/conceptual and historical viewpoint living the deepness and detailed review with more technical details for the remaining parts as the different subjects are subsequently introduced;
- Part **II** – *Structural Mathematical Models* addresses the analytical modeling and approximated FE solution of arbitrary hybrid active-passive damping treatments applied to multilayer beams, plates and shells. With that purpose four different chapters are considered where the former two are devoted to one-dimensional coupled-kinematics layerwise models of structures (beams) and the latter two to two-dimensional fully and partially refined discrete-kinematics layerwise models of structures (plates and shells) which are originally proposed in this work taking into consideration a somewhat new “regeneration” principle. In order to efficiently model these treatments, the piezo-visco-elastic couplings between the elastic host structure and viscoelastic and piezoelectric damping layers must be accurately considered. It is well known that modeling pure mechanical laminated structures is already a very complicated subject involving a lot of thinking concerning, among other things, kinematic assumptions, displacement-strain-stress relationships, different variational principles, constitutive relations, consistency of the resultant governing equations, with the extra complication of in the case of curved structures having the problem formulated in curvilinear coordinates, which is the case for

the chapter on shells. In addition, the viscoelastic and piezoelectric layers behaviors are also incorporated in the model which increases its complexity. That involves viscoelastic damping modeling issues in the one hand, utilized to describe the damping behavior of the viscoelastic counterpart of the structural system, and electro-mechanical coupling modeling assumptions for the piezoelectric layers counterpart in the other hand. With that in mind, these are the specific subjects addressed in this part to develop multilayer piezo-visco-elastic beam, plate and shell FEs with a high degree of refinement/accuracy and simultaneously with a good trade-off between accuracy and FEs complexity (computational cost). This part is complemented with two appendixes for the one- and two-dimensional formulations;

- Vibration damping, in general, and viscoelastic damping, in particular, is addressed in Part **III** – *Vibration Damping*. This part comprises two chapters: the former addresses the modeling issues of viscoelastic damping, presenting and discussing different frequency- and time-domain modeling approaches and FE implementations, while the latter is devoted to the experimental identification of viscoelastic damping materials and to the assessment of the different damping models;
- The contents in Part **IV** – *Vibration Control* establish the bridge between modeling/simulation and effective real-time control implementation in structural vibrations and noise control applications of adaptive structural systems. The part is divided in two chapters where the first addresses active and hybrid control of vibrations and the second the assessment and application of the previous vibration control strategies. Both passive, active and hybrid damping treatments and control configurations are considered and analyzed. The implemented experiments and real-time control of vibrations are also described and the lessons learned regarding the advantages and disadvantages of the practical implementation, performance, robustness and stability of classical and optimal feedback and adaptive feedforward control strategies are discussed.
- Lastly, the dissertation is concluded in Part **V** – *Conclusion*, with a single chapter presenting a summary of the contributions made, discussing the new and most significant aspects, and giving suggestions for further work in the field.

As these technologies enter the marketplace, relevant research issues must be addressed for successful transition. An adequate design and applications of these technologies is addressed with this dissertation which is disseminated following the organization previously referred.

Part II

STRUCTURAL MATHEMATICAL MODELS

Part II

STRUCTURAL MATHEMATICAL MODELS

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CHAPTER 2

Analytical Model of Piezo-Visco-Elastic Beams

2.1 Introduction

In an attempt to furnish designers of multilayer damped composite structures, vastly used in the aerospace, aeronautic, automotive and military companies, to name just a few, with analytical and FE models with better representativeness than the actual alternatives available in home-made or commercial FE codes, this part is dedicated to the modeling of damping treatments in structures taking into account the piezo-visco-elastic couplings between the elastic host structures and the viscoelastic and piezoelectric damping layers. With that purpose, four different chapters are considered where the former two are devoted to one-dimensional coupled-kinematics layerwise models of structures (beams) and the latter two to two-dimensional fully and partially refined discrete-kinematics layerwise models of structures (plates and shells), which are originally proposed in this work taking into consideration a somewhat new “regeneration” principle.

At first sight it might seem that the current research subject has already been satisfactorily developed in the last two decades by the scientific community, and reached a relative stage of maturity. However, as later reviewed in the subsequent section and in Chapter 4, that does not correspond to reality. Accurate adaptive piezo-visco-elastic multilayer FEs with laminated composite individual layers or not developed to date rely on theories with an undesirable complexity, sometimes with some important inaccurate electrical and damping model restrictions and simplifications being introduced. Furthermore, they require a high computational burden (e.g. mixed and/or displacement based high-order layerwise theories), making their implementation and use in well-known FE commercial codes not appealing. In addition, they are also unable (at least in a straightforward and generalized manner) to model structural systems with segmented (or discontinuous) layers, which is a very sensitive and key aspect when modeling host structures with damping treatments.

A strong inconvenience and drawback of using these outdated models and “simplified” FE codes to model adaptive structures is the inaccuracy of the used models and the overestimated adaptive structure project and control system design, increasing the gap and discrepancy between the real working conditions and the ones obtained by model simulation, making the simulation less appealing to structural designers and “dynamicists”, contributing to the use of high safety design coefficients of the project variables.

In this first chapter an analytical model for the free and forced vibration of beams with hybrid (active/passive), or arbitrary ACLD, damping treatments, incorporating an arbitrary number of elastic, viscoelastic and piezoelectric layers, is developed. At this point, only the elastic and piezo-elastic constitutive behaviors will be considered to the formulation of a generic piezo-elastic structure with an arbitrary number of layers. The inclusion of viscoelastic effects can be performed afterward in several ways, as discussed in detail in Section 6. For now, only a frequency-dependent viscoelastic damping model will be considered by means of a complex modulus approach (CMA) where the “elastic” moduli are complex. Using the CMA, the resulting piezo-visco-elastic model is formally identical to the piezo-elastic one, making the extension and application of the piezo-elastic model to structures with viscoelastic layers more straightforward.

The key steps to the development of the weak and strong forms, governing the motion and electric charge equilibrium, are presented here. For the sake of brevity, when convenient, the reader will be referred to Appendix A where more details about their formulation can be found. The resultant set of partial differential equations, namely, the equations of motion and electric charge equilibrium and electro-mechanical boundary conditions, are obtained from the strong forms. Regarding the weak forms, they will be used in the next chapter as the basis for the development of a one-dimensional FE solution model.

2.2 Review of Mathematical Models of Beams

Among all typical structural elements commonly used by structural designers, *beams* are a particular form of three-dimensional solids whose cross-sectional dimensions, i.e. the thickness and width, are small as compared with the length. Probably they are used in every engineering structure and usually they are flat slender structures with a constant cross-section which are primarily subjected to bending and transverse shear stresses. However, in general, they may be curved with a varying cross-section and quite often they support also membrane stresses (typically produced in *bars*) and shear stresses due to its torsion (typically produced in *shafts*).

The long history of the development of man’s understanding of the structural behavior of beams is an interesting subject well described in Timoshenko’s book, *History of Strength of Materials* [Timoshenko (1983)]. Modern investigation of beams began with Galileo Galilei (1564-1642) in the end of the 16th century who developed a theory for the bending strength of cantilever beams and the ultimate breaking loads; the latter theory was then shown to have errors and a more accurate theory was later developed by Mariote (1620-1684). Starting near the end of the 17th century, the Bernoulli family produced for more than a hundred years several outstanding mathematicians that, among other things, were responsible for the initial developments on the understanding of the deflection of beams. Considering Hooke’s law, after Robert Hooke (1635-1703), Jacob Bernoulli (1654-1705) and his younger brother Johann (1667-1748), developed, respectively, the equation of elastic curves (deflections) of beams and the formulation of the principle of virtual displacements. In fact, whereas Galileo and Mariote were concerned with the strength of beams, the Bernoullis were much more interested in the determination of their deflections and in the utilized calculus approach. It was Johann Bernoulli’s son, Daniel Bernoulli (1700-1782), after recognizing his pupil, Leonhard Euler (1707-1783), as the master of the calculus of variations, that gave Euler information on the expression of the strain energy and that suggested to Euler that he should apply the variational calculus in deriving the equations of the elastic curves of beams. This suggestion triggered Euler to obtain Jacob Bernoulli’s differential equation for elastic curves of beams by means of the “method of final

causes" (variational calculus) putting side by side the efficiency of the "direct method" (equilibrium approach), used by Jacob, with the "method of final causes" (variational approach), used by Euler, to arrive at the same equation governing the deflection of beams. Furthermore, Daniel was also the first to derive the differential equation governing lateral vibrations of prismatic beams and to study particular modes of its motion whereas the integration of this equation was done by Euler that developed solutions for various support conditions. The latter developments gave rise to what is nowadays known as the *Euler-Bernoulli beam theory* (or *classical beam theory*). It is worthy to mention also that it is to the credit of Saint-Venant (1797-1886) and his semi-inverse method that the accuracy of the forementioned beam theory and the distortion of the shape of the cross section (warping) were first examined and that the beam theory was extended to non-uniform bending, large deflections and to non-Hookean materials in beams with various shapes and cross-sections.

Inquiries on the distribution of stresses in beams and the position of the neutral axis were raised by Parent (1666-1716); his work was not known to Coulomb (1736-1806) that analyzed also the distribution of stresses and was the first to realize that the effects of shearing forces (and stresses) on the strength of beams can be neglected as long as the beam is thin, i.e. its thickness to length ratio is small. That was something that originated critics to the Euler-Bernoulli beam theory on the ground that it does not consider shear stresses and therefore is not accurate for "short" (thick) beams subjected to transverse loads imposing bending moments varying along the beam; furthermore, the equilibrium and compatibility equations of elasticity are also not satisfied for transverse loads. The rigorous elasticity solution of the problem of shearing stresses in beams was given by Saint-Venant for simple shapes of cross sections and by Jourawski (1821-1891) for more complicated ones. However, despite the significant efforts of Saint-Venant's pupils [e.g. Boussinesq (1842-1929), Levy (1838-1910)] and others to the development of the *theory of elasticity*, the exact solutions are burdensome to be obtained and can only be obtained for certain load and geometric conditions. Nevertheless, experimental evidence and comparison with some exact solutions from the theory of elasticity have shown that the *engineering theory* of the stresses in beams [see for example [Crandall et al. \(1978\)](#)], as it is frequently referred to nowadays, which assumes that the Euler-Bernoulli beam theory and the underlying bending-stress distribution are valid even when the bending moment varies along the beam, i.e. when a shear force is present, and estimates the stress distribution *a posteriori* by the use of the equilibrium equations, yields displacement and stress responses which are usually accurate enough to be considered nowadays in the design of most structural systems.

Following the previous line of thought, and according to Coulomb's observations on the importance of transverse shear stresses, various *shear deformation theories* (SDTs) for beams have been proposed since, in an attempt to improve Euler-Bernoulli's theory. The most famous and pioneering one, was given by [Timoshenko \(1921\)](#) where not only the shear effects are approximated by assuming that the cross section of the beam after deformation remains plane but not necessarily normal to the neutral axis, but the rotatory inertia is taken into consideration, as suggested by Lord [Rayleigh \(1945\)](#) in his seminal books *The Theory of Sound*, with volume one on the mechanics of a vibrating medium which produces sound being first published in 1877 and the second one on acoustic wave propagation one year later. The latter theory is known as *Timoshenko beam theory* [or *first-order shear deformation theory* (FSDT)] and represents an improvement to the Euler-Bernoulli theory since it includes the effects of transverse shear stresses; however, these shear stresses are through-the-thickness constant which does not represent the exact solution in all cases since the theory can not represent stress states with null transverse shear stresses at the top and bottom surfaces. The conclusions obtained in Timoshenko's seminal article that the correction for shear, in a representative example, was four times as important

as the correction for rotatory inertia when the wave-length of transverse vibration is of the same order of the thickness, are confirmed one year later by [Timoshenko \(1922\)](#).

As obvious, the wide use of plates and shells in modern structures has brought about advances in the theory of bending of plates and shells which somehow can be seen as natural extension of the aforementioned considerations on the mathematical modeling of beams. In fact, it was Euler's student Jacob Bernoulli II (Daniel Bernoulli's nephew) who attempted in 1788 to derive an equation for the vibration of a rectangular plate as a gridwork of beams. Subsequent extensions can be attested by the historical evidences left by several researchers which contributed to the pioneering development of these theories [e.g. Gustave Kirchhoff (1824-1887) for the theory of bending of plates, after the first erroneous attempts of its development by Sophie Germain (1776-1831), or Augustus Love (1863-1940) for the theory of thin shells]. Credit to creation of the first deformation plate theories considering the shear effects are given to [Reissner \(1945\)](#), in the static case, and to [Mindlin \(1951\)](#), in the dynamic case, that, similarly to what Timoshenko did, also included the rotatory inertia as suggested by Rayleigh. These theories are known as *Kirchhoff plate theory* and *Reissner-Mindlin plate theory* for thin and thick plates, respectively. As far as shells are concerned, Love derived in 1888 the equations for vibration of thin shells based on the assumptions introduced by Kirchhoff for plates [see [Love \(1888\)](#)] and gave a significant contribution in unifying the theory of dynamic deformation of continuous structural elements whose thickness is small compared with the other dimensions, as can be seen in his book *A Treatise on the Mathematical Theory of Elasticity* [[Love \(1944\)](#)]. After Love's basic development, several investigators tried to improve on his theory. By the end of the 19th century, [Basset \(1890a\)](#) pointed out the importance of the effects of the out-of-plane stresses, until then neglected by Love and others, and considered that the strains and displacement in a shell could be expanded in a power series of the thickness coordinate. The incorporation of the transverse shear and normal stress effects on shell theory was later substantiated by [Hildebrand et al. \(1949\)](#), [Naghdi and Cooper \(1956\)](#) and [Sanders \(1959\)](#). Since then, researchers realized that for thick beams, plates or shells, both rotary inertia and shear deformation have to be included in any reliable theory of such components. It is worthy to stress that Basset was a pioneer in the area of *higher-order shear deformation theories* (HSDT), which many authors adopted in the following decades, with the particularity of considering the displacement field expressed in terms of the generalized displacements and their derivatives in a Taylor series expansion. Basset's work did not receive enough attention during the subsequent decades but the recognition of his work was given by [Hildebrand et al. \(1949\)](#) that also considered a quadratic expansion of the three components of the displacement field. Issues on the modeling of plates and shells will be further discussed and reviewed in Section 4.2 and the emphasis given on this section in what follows will be given to beam structures; it is worthy to stress, though, that beams are particular cases of plate and shell structures and some of the works are cited assuming that.

In the preceding paragraphs the discussion addressed homogeneous structural systems with isotropic material properties. However, as discussed in Chapter 1, the need for lightweight high performance structures with the well known advantages in terms of strength, weight, stiffness, fatigue, thermal resistance and damping, to name just a few, motivated the development and use of composite multilayer structures composed of different materials, be they passive elastic and viscoelastic or active piezoelectric materials. The forementioned qualities of composite high performance structures have generated considerable interest in the development of reliable mathematical models, capable of efficiently describing their behavior. As a result of that, shear deformable theories are intrinsically related to the advent of multilayer high performance structures. The importance of including these effects comes from the fact that, on the

one hand, and as far as the host structure is concerned, composite laminated materials have very high ratios of in-plane moduli to transverse shear moduli which makes theories neglecting shear and normal out-of-plane deformation prohibitively inaccurate both for the global and local static and dynamic responses, and, on the other hand, as far as piezo-visco-elastic damping treatments mounted in the host structure are concerned, the estimation of hybrid (active/passive) damping strongly depends of the accuracy and representativeness of localized stress states of the damping layers which in turn strongly influences the global damped response. Additionally, since these damping treatments are often of segmented and of limited dimensions, the edge effects on the damping layers are also important to be retained.

The classic and shear (first and higher orders) deformation theories discussed so far when applied to composite multilayer structures can be classified as *equivalent single layer theories* (ESLTs) which are derived from the 3-D elasticity theory by making suitable assumptions concerning the kinematics of deformation or the stress state through the thickness of the multilayer composite structure. These assumptions, in turn, allow the reduction of a 3-D problem to a 1-D problem for the case of beams. Another alternative relying on a 3-D representation of the continua are the *discrete-layer theories* (DLTs), also known as *layerwise theories* (LWTs), where each layer has individual kinematic assumptions. Although these HSDTs and LWTs are capable of accurately predicting the global behavior of thick beams, when used with displacement-based formulations they fail to fulfill the continuity conditions for the out-of-plane stresses (transverse shear and normal stresses). Another alternative, with a reduced computational effort, in the framework of ESL theories, is the use of the so-called zig-zag theories (ZZTs), which have their origins and most significant contributions coming from the Russian school. The fundamental ideas in developing and using ZZTs consists to assume a certain displacement and/or stress model in each layer and then to use compatibility and equilibrium conditions at the interface to reduce the number of the unknown variables and keep the number of variables independent of the number of layers. Further details on these theories will be given latter in Section 4.2 in the context of plate and shell structures.

The use of Euler-Bernoulli's beam theory in the analysis of laminates is often referred to as the *classical laminate theory* (CLT). Its use in composite laminates can be found for example in [Ashton and Whitney (1970)] and its application to piezo-laminates in [Baz and Poh (1988), Lee (1990), Hwang and Park (1993)] and piezo-visco-elastic laminates in [Shieh (1994)]. Regarding viscoelastic damping treatments, since the CLT does not allow to consider shear deformations, its use should be restricted to applications of unconstrained layer damping (ULD). A laminated version of Timoshenko's theory has also been applied to piezo-laminated structures by Di Sciuva and Icardi (1995) and Tzou (1993).

In attempt to improve even further Timoshenko's theory and to adapt it to multilayer structures, various laminated HSDT have been proposed [see, for example, Reddy et al. (1997), Zhen and Wanji (2008)]. Their use to model piezo-elastic multilayer structures can be found in [Mitchell and Reddy (1995), Chattopadhyay and Seeley (1997), Reddy (1999)] and piezo-visco-elastic damped structures in [Plagianakos and Saravanos (2003), Ray and Reddy (2004a,b)].

Regarding the ZZTs, they were conceptually first proposed by Lekhnitskii (1968) to the analytical modeling of multilayer cantilever beams and further used by many others [see, for example, Icardi (2003) and Kapuria et al. (2004)]. Application to piezo-elastic structures was done by Icardi and DiSciuva (1996), Kim et al. (2002), Ganapathi et al. (2004), Ballhause et al. (2005) and Oh et al. (2007), and to piezo-visco-elastic damped structures was done by Chattopadhyay et al. (2001).

In the context of the LWTs, the credit of first allegedly adopting piecewise linear continuous displacement definitions should be given to Yu in the late 1950s that, as mentioned in his book

[Yu (1996)], first used it to model sandwich structures. Thick laminated composites were also modeled using the LWT in [Sun and Whitney (1973), Reddy et al. (1989), Robbins and Reddy (1993)] and multilayer piezo-elastic beams by Robbins and Reddy (1991) and Saravanan and Heyliger (1995). The extension to plates and shells was performed by Heyliger et al. (1994) and Heyliger et al. (1996), respectively. The damping effects in laminates was analyzed by Saravanan (1993, 1994). So-called *variable kinematic models* were also considered by Robbins and Reddy (1996) to model piezo-elastic plates. Higher-order displacement-based and mixed LWTs were considered in [Suzuki et al. (2003), Garção et al. (2004), Plagianakos and Saravanan (2005), Carrera and Boscolo (2007)].

Works in the framework of considering different displacement order approximations and stress assumptions for the different layers and considering the shear only in the core were presented for sandwich beams [Ross et al. (1959), Kao and Ross (1968), Khatua and Cheung (1973)] and for multilayer piezo-visco-elastic beams [Baz (1993), Lesieutre and Lee (1996), Baz (1997), Trindade et al. (2001a,b)], plates [Park and Baz (2001)] and shells [Ray and Balaji (2007)], usually in the latter case reducing the multilayer problem to a three-layer one as typically encountered in sandwich structures. The alleviation of the layer reducing simplification assuming individual approximations for each layer following the LWT was performed for visco-elastic multilayer damped beams by Zapfe and Lesieutre (1999) and to piezo-visco-elastic ones by Vasques, Mace, Gardonio and Rodrigues (2006).

Segmented damping treatments make solutions with ESL theories more difficult to be obtained, if not impossible, and make LWTs (or DLTs) more appropriate. A general theory and an excellent solution to the segmented nature of damping treatments and the underlying modeling problems and correspondent finite element solution for thick beams using a sublaminar approximation concept was remarkably presented by Averill and Yip (1996a,b). The latter theory has an excellent trade-off between computational cost and accuracy and was developed considering a hybrid theory, combining the general layerwise theories with independent layerwise kinematic approximations and the efficient zig-zag theories in which the layerwise DoFs are eliminated by enforcing transverse shear stress continuity conditions. To the author's knowledge an extension of the theory to piezo-visco-elastic damped structures has not been performed yet but seems a very promising modeling strategy.

Mechanics and computational models for laminated and multilayer piezo-visco-elastic beams, plates and shells are discussed, for example, by Reddy (1989, 1990b), Ghugal and Shimpi (2001, 2002), Benjeddou (2000, 2001), Chee et al. (1998) and Saravanan and Heyliger (1999). Lastly, variations on modeling of constrained-layer damping treatments are reviewed by Austin (1999) and the effect of laminate kinematic assumptions on the global response of piezo-elastic plates is assessed by Robbins and Chopra (2006). The reader is referred to the previous references and to Section 4.2 for complementary details on high performance structures modeling issues.

2.3 Displacements and Strains

Consider the layered beam illustrated in Figure 2.1. The composite beam consists of a host beam, layer 0, of thickness $2h_0$, to which other viscoelastic or piezoelectric layers (treatments) are attached. In order to be able to model several configurations of the treatments, a composite discrete-layer (layerwise) beam theory with coupled-kinematics allows an arbitrary number of layers of elastic, piezoelectric and viscoelastic materials, in arbitrary positions. There are $\bar{n}$ layers on the top surface and $-\bar{m}$ layers on the bottom surface. The displacement field is

defined according to a partial layerwise theory where the axial and transverse displacements, $\tilde{u}_k(x, z_k, t)$ and $\tilde{w}_k(x, t)$, of the top ($n = 1, \dots, \bar{n}$), core ($c = 0$) and bottom ($m = \bar{m}, \dots, -1$) layers are given by

$$\tilde{u}_n(x, z_n, t) = u_0(x, t) + h_0\theta_0(x, t) + \sum_{i=1}^{n-1} 2h_i\theta_i(x, t) + (z_n + h_n)\theta_n(x, t), \quad (2.1a)$$

$$\tilde{u}_c(x, z_c, t) = u_0(x, t) + z_0\theta_0(x, t), \quad (2.1b)$$

$$\tilde{u}_m(x, z_m, t) = u_0(x, t) - h_0\theta_0(x, t) - \sum_{i=m+1}^{-1} 2h_i\theta_i(x, t) + (z_m - h_m)\theta_m(x, t), \quad (2.1c)$$

$$\tilde{w}_k(x, z_k, t) \equiv \tilde{w}_n(x, z_n, t) \equiv \tilde{w}_c(x, z_c, t) \equiv \tilde{w}_m(x, z_m, t) = w_0(x, t), \quad (2.1d)$$

where $2h_k$ is the thickness of layers k , with $k = \bar{m}, \dots, -1, 0, 1, \dots, \bar{n}$, $u_0(x, t)$, $w_0(x, t)$ and $\theta_0(x, t)$ are, respectively, the generalized axial and transverse displacements and the rotation of the beam's mid-plane, and $\theta_n(x, t)$ and $\theta_m(x, t)$ are the rotation of each n -th top and m -th bottom layer. It is worth noting that positive indices are used to denote the top layers and negative indices are used to the bottom ones, i.e., $n > 0$ and $m < 0$. The z -coordinates in Equations (2.1a) and (2.1c) are measured from the interface between layers n and $n - 1$ and m and $m + 1$ through the terms $z_n + h_n$ and $z_m - h_m$. They represent a translation of the rotation axis of each top and bottom layer from the layer mid-plane to the interface of the adjacent layer. Furthermore, note that axial displacement continuity at the interfaces of the layers is assured, leading to coupling terms in the axial displacements of the layers, and that a constant through-the-thickness transverse displacement $w_0(x, t)$ is assumed.

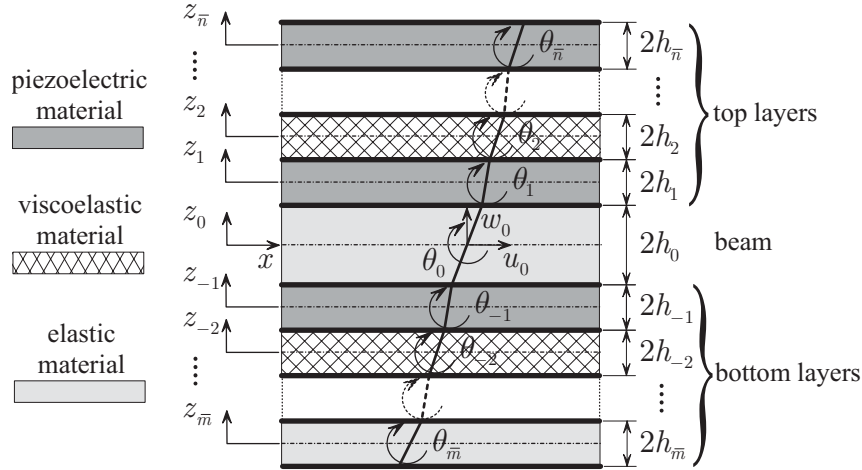


Figure 2.1: Coupled-kinematics discrete-layer (layerwise) displacement field of the beam with arbitrary ACLD treatments.

According to the displacement field (2.1) the extensional and shear strains of the layers are determined by the usual linear strain-displacement relations, to be

$$S_{xx}^n = \frac{\partial \tilde{u}_n}{\partial x} = u'_0 + h_0\theta'_0 + \sum_{i=1}^{n-1} 2h_i\theta'_i + (z_n + h_n)\theta'_n, \quad (2.2a)$$

$$S_{xx}^c = \frac{\partial \tilde{u}_c}{\partial x} = u'_0 + z_0\theta'_0, \quad (2.2b)$$

$$S_{xx}^m = \frac{\partial \tilde{u}_m}{\partial x} = u'_0 - h_0 \theta'_0 - \sum_{i=m+1}^{-1} 2h_i \theta'_i + (z_m - h_m) \theta'_m, \quad (2.2c)$$

$$S_{zx}^k = \frac{\partial \tilde{w}_k}{\partial x} + \frac{\partial \tilde{u}_k}{\partial z_k} = w'_0 + \theta_k, \quad (2.2d)$$

where the notation $(\cdot)'$ is used to denote the spatial derivative in the x -direction. The kinematic hypotheses considered previously lead to null transverse strains and a first-order shear deformation theory (FSDT) for each layer.

2.4 Constitutive Equations

The linear piezoelectric constitutive equations in compact matrix notation [IEEE (1988)] are given by

$$\mathbf{T} = \mathbf{c}^E \mathbf{S} - \mathbf{e}^T \mathbf{E}, \quad (2.3a)$$

$$\mathbf{D} = \mathbf{e} \mathbf{S} + \boldsymbol{\varepsilon}^S \mathbf{E}, \quad (2.3b)$$

where $\mathbf{T}$, $\mathbf{S}$, $\mathbf{E}$ and $\mathbf{D}$ are, respectively, the stress, strain, electric field and electric displacement vectors, and $\mathbf{c}^E$, $\mathbf{e}^T$ and $\boldsymbol{\varepsilon}^S$ are, respectively, the elasticity (at constant electric field), transpose piezoelectric and dielectric (at constant strain) matrices appropriate for the material.

The material of the piezoelectric layers is assumed to be orthotropic, with the axes of orthotropy parallel to the axes of the beam, x , y , z_k , and polarized in the transverse direction z_p . They have the behavior of normal piezoelectric materials, with the symmetry properties of an orthorhombic crystal of the class $mm2$ [Nye (1985), IEEE (1988)]. Representing Equations (2.3), with their full matrix and vector terms, for a generic piezoelectric layer p , yields

$$\begin{Bmatrix} T_{xx}^p \\ T_{yy}^p \\ T_{zz}^p \\ T_{yz}^p \\ T_{zx}^p \\ T_{xy}^p \end{Bmatrix} = \begin{bmatrix} c_{11}^p & c_{12}^p & c_{13}^p & 0 & 0 & 0 \\ c_{12}^p & c_{22}^p & c_{23}^p & 0 & 0 & 0 \\ c_{13}^p & c_{23}^p & c_{33}^p & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44}^p & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55}^p & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}^p \end{bmatrix} \begin{Bmatrix} S_{xx}^p \\ S_{yy}^p \\ S_{zz}^p \\ S_{yz}^p \\ S_{zx}^p \\ S_{xy}^p \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31}^p \\ 0 & 0 & e_{32}^p \\ 0 & 0 & e_{33}^p \\ 0 & e_{24}^p & 0 \\ e_{15}^p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_x^p \\ E_y^p \\ E_z^p \end{Bmatrix}, \quad (2.4a)$$

$$\begin{Bmatrix} D_x^p \\ D_y^p \\ D_z^p \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15}^p & 0 \\ 0 & 0 & 0 & e_{24}^p & 0 & 0 \\ e_{31}^p & e_{32}^p & e_{33}^p & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} S_{xx}^p \\ S_{yy}^p \\ S_{zz}^p \\ S_{yz}^p \\ S_{zx}^p \\ S_{xy}^p \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11}^p & 0 & 0 \\ 0 & \varepsilon_{22}^p & 0 \\ 0 & 0 & \varepsilon_{33}^p \end{bmatrix} \begin{Bmatrix} E_x^p \\ E_y^p \\ E_z^p \end{Bmatrix}. \quad (2.4b)$$

The displacement field considered in (2.1) is independent of the y -axis and null along that direction. As consequence,

$$S_{yy}^p = S_{yz}^p = 0. \quad (2.5)$$

Moreover, assuming $T_{zz}^p \approx 0$ as a priority choice to the one of $S_{zz}^p = 0$, which results from the kinematic assumptions, yields

$$S_{zz}^p = \frac{1}{c_{33}^p} (e_{33}^p E_z^p - c_{13}^p S_{xx}^p). \quad (2.6)$$

Thus, considering (2.5) and substituting (2.6) in (2.4), the constitutive equations of a generic top or bottom piezoelectric layer p can be written in the following reduced form,

$$\begin{Bmatrix} T_{xx}^p \\ T_{zx}^p \end{Bmatrix} = \begin{bmatrix} c_{11}^{*p} & 0 \\ 0 & c_{55}^p \end{bmatrix} \begin{Bmatrix} S_{xx}^p \\ S_{zx}^p \end{Bmatrix} - \begin{bmatrix} 0 & e_{31}^{*p} \\ e_{15}^p & 0 \end{bmatrix} \begin{Bmatrix} E_x^p \\ E_z^p \end{Bmatrix}, \quad (2.7a)$$

$$\begin{Bmatrix} D_x^p \\ D_z^p \end{Bmatrix} = \begin{bmatrix} 0 & e_{15}^p \\ e_{31}^{*p} & 0 \end{bmatrix} \begin{Bmatrix} S_{xx}^p \\ S_{zx}^p \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11}^p & 0 \\ 0 & \varepsilon_{33}^{*p} \end{bmatrix} \begin{Bmatrix} E_x^p \\ E_z^p \end{Bmatrix}, \quad (2.7b)$$

where

$$c_{11}^{*p} = c_{11}^p - \frac{c_{13}^p{}^2}{c_{33}^p}, \quad e_{31}^{*p} = e_{31}^p - e_{33}^p \frac{c_{13}^p}{c_{33}^p}, \quad \varepsilon_{33}^{*p} = \varepsilon_{33}^p + \frac{e_{33}^p{}^2}{c_{33}^p}. \quad (2.8a-c)$$

The modification of constants c_{11}^p , e_{31}^p and ε_{33}^p is due to the transverse stress assumption, $T_{zz}^p \approx 0$, and to the fact that the components T_{yz}^p , T_{xy}^p , S_{yz}^p and S_{xy}^p , and the y -component of the electric field E_y^p and electric displacement D_y^p , are considered negligible to the electro-mechanical energy.

Equations (2.7) are the reduced constitutive equations of a generic piezoelectric layer, typically utilized when considering FSDT. As can be seen, the electro-mechanical coupling takes place between the extensional strain and the transverse electric field, and the shear strain and axial electric field components. Furthermore, for the characterization of the core layer or any other top or bottom viscoelastic or elastic layers, Equation (2.7a) can also be used by setting the piezoelectric constants e_{31}^{*p} and e_{15}^p to zero and by using the appropriate values of the Young's and shear modulus c_{11}^{*p} and c_{55}^p in the elasticity matrix.

2.5 Electric Field and Potential Assumptions

In the present work, a fully coupled electro-mechanical theory which takes into account the direct piezoelectric effect with a non-linear distribution of the electric potential is utilized. The electric displacement vector in Equation (2.7b) can be written as

$$\begin{Bmatrix} D_x^p \\ D_z^p \end{Bmatrix} = \begin{bmatrix} \varepsilon_{11}^p & 0 \\ 0 & \varepsilon_{33}^{*p} \end{bmatrix} \begin{Bmatrix} E_x^p - \bar{E}_x^p \\ E_z^p - \bar{E}_z^p \end{Bmatrix}, \quad (2.9)$$

with

$$\bar{E}_x^p = -\frac{e_{15}^p}{\varepsilon_{11}^p} S_{zx}^p, \quad \bar{E}_z^p = -\frac{e_{31}^{*p}}{\varepsilon_{33}^{*p}} S_{xx}^p, \quad (2.10a,b)$$

where $\bar{E}_x^p$ and $\bar{E}_z^p$ are the electric fields induced by the mechanical strains.

According to [Krommer and Irschik \(1999\)](#), for electroded layers with the electric potential being prescribed and with the assumption of zero electric displacement in the x -direction, the axial and transverse electric fields, E_x^p and E_z^p , and the electric potential φ_p , are given by

$$E_x^p = \bar{E}_x^p, \quad E_z^p = -\frac{\phi_p}{2h_p} + \bar{E}_z^p - \frac{1}{2h_p} \int_{-h_p}^{h_p} \bar{E}_z^p dz_p, \quad (2.11a,b)$$

$$\varphi_p = \frac{\phi_p}{2h_p} (z_p + h_p) - \int_{-h_p}^{z_p} \bar{E}_z^p dz_p + \frac{(z_p + h_p)}{2h_p} \int_{-h_p}^{h_p} \bar{E}_z^p dz_p, \quad (2.12)$$

where ϕ_p denotes the electrical potential difference between the electrodes of each piezoelectric layer. Substituting into Equations (2.11) and (2.12) the induced electric fields in (2.10) and

considering the strain definitions in (2.2), the electric field and potential become [Vasques and Rodrigues (2005)]

$$E_x^p = -\frac{e_{15}^p}{\varepsilon_{11}^p} (w'_0 + \theta_p), \quad E_z^p = -\frac{\phi_p}{2h_p} - \frac{e_{31}^{*p}}{\varepsilon_{33}^{*p}} z_p \theta'_p, \quad (2.13a,b)$$

$$\varphi_p = \frac{\phi_p}{2h_p} (z_p + h_p) + \frac{1}{2} \frac{e_{31}^{*p}}{\varepsilon_{33}^{*p}} (z_p^2 - h_p^2) \theta'_p. \quad (2.14)$$

It is worth noting that the first part of Equation (2.14) is a linear through-the-thickness electric potential term concerning the applied electric potential difference and that the second part is a parabolic term concerning the electric potential induced by the mechanical strains.

2.6 Variational Formulation

In order to derive the weak and strong forms, governing the motion and electric charge equilibrium of the composite beam with ACLD treatments, Hamilton's principle is used. The lagrangian and the work of the applied forces are adapted for the electrical and mechanical contributions [Tiersten (1967, 1969)], so that

$$\delta \int_{t_0}^{t_1} (T - H + W) dt = 0, \quad (2.15)$$

where t_0 and t_1 define the time interval, δ denotes the variation, T is the kinetic energy, H is the electro-mechanical enthalpy (energy stored in the piezoelectric and non-piezoelectric layers) and W denotes the work done by the applied mechanical forces and electrical charges. In the following, it will be assumed that all the top and bottom layers are piezoelectric. However, the formulation still holds for non piezoelectric layers by considering only the mechanical virtual work terms for those layers.

2.6.1 Virtual Work of the Internal Electro-Mechanical Forces

The work of the internal electro-mechanical forces is given by the sum of the virtual work contributions of all layers. Considering a generic piezoelectric layer p , with $p = (n, m) = \bar{m}, \dots, -1, 1, \dots, \bar{n}$, and separating the total virtual work δH^p into mechanical δH_{uu}^p , piezoelectric $\delta H_{u\phi}^p$ and $\delta H_{\phi u}^p$, and dielectric $\delta H_{\phi\phi}^p$ terms, for the piezoelectric layer of volume V_p , yields

$$\delta H^p = \delta H_{uu}^p - \delta H_{u\phi}^p - \delta H_{\phi u}^p - \delta H_{\phi\phi}^p, \quad (2.16)$$

where

$$\delta H_{uu}^p = \int_{V_p} (\delta S_{xx}^p c_{11}^{*p} S_{xx}^p + \delta S_{zx}^p c_{55}^p S_{zx}^p) dV_p, \quad (2.17)$$

$$\delta H_{u\phi}^p = \int_{V_p} (\delta S_{xx}^p e_{31}^{*p} E_z^p + \delta S_{zx}^p e_{15}^p E_x^p) dV_p, \quad (2.18)$$

$$\delta H_{\phi u}^p = \int_{V_p} (\delta E_z^p e_{31}^{*p} S_{xx}^p + \delta E_x^p e_{15}^p S_{zx}^p) dV_p, \quad (2.19)$$

$$\delta H_{\phi\phi}^p = \int_{V_p} (\delta E_x^p \varepsilon_{11}^p E_x^p + \delta E_z^p \varepsilon_{33}^{*p} E_z^p) dV_p. \quad (2.20)$$

Furthermore, for the definition of the core's internal forces virtual work, $\delta H^c = \delta H_{uu}^c$, Equation (2.17) can also be used by considering the appropriate elastic coefficients and strain components.

Substituting the strain definitions (2.2) and the electric field components (2.13) into Equations (2.17)-(2.20) and integrating with respect to the cross section, the mechanical terms for the top, core and bottom layers, are given by

$$\begin{aligned} \delta H_{uu}^n = \int_L & \left[\delta u'_0 \left(c_{11}^{*n} A_n u'_0 + c_{11}^{*n} h_0 A_n \theta'_0 + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i A_n \theta'_i + c_{11}^{*n} \bar{I}_n \theta'_n \right) + \delta w'_0 \left(c_{55}^n A_n w'_0 \right. \right. \\ & + c_{55}^n A_n \theta_n \left. \right) + \delta \theta'_0 \left(c_{11}^{*n} h_0 A_n u'_0 + c_{11}^{*n} h_0^2 A_n \theta'_0 + c_{11}^{*n} h_0 \sum_{i=1}^{n-1} 2h_i A_n \theta'_i \right. \\ & + c_{11}^{*n} h_0 \bar{I}_n \theta'_n \left. \right) + \sum_{i=1}^{n-1} \delta \theta'_i \left(c_{11}^{*n} 2h_i A_n u'_0 + c_{11}^{*n} 2h_i h_0 A_n \theta'_0 \right. \\ & + c_{11}^{*n} 2h_i \sum_{j=1}^{n-1} 2h_j A_n \theta'_j + c_{11}^{*n} 2h_i \bar{I}_n \theta'_n \left. \right) + \delta \theta'_n \left(c_{11}^{*n} \bar{I}_n u'_0 + c_{11}^{*n} h_0 \bar{I}_n \theta'_0 \right. \\ & \left. \left. + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i \bar{I}_n \theta'_i + c_{11}^{*n} I_n \theta'_n \right) + \delta \theta_n \left(c_{55}^n A_n w'_0 + c_{55}^n A_n \theta'_n \right) \right] dL, \quad (2.21) \end{aligned}$$

$$\delta H_{uu}^c = \int_L \left[\delta u'_0 c_{11}^{*c} A_c u'_0 + \delta w'_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) + \delta \theta'_0 c_{11}^{*c} I_c \theta'_0 + \delta \theta_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) \right] dL, \quad (2.22)$$

$$\begin{aligned} \delta H_{uu}^m = \int_L & \left[\delta u'_0 \left(c_{11}^{*m} A_m u'_0 - c_{11}^{*m} h_0 A_m \theta'_0 - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i + c_{11}^{*m} \bar{I}_m \theta'_m \right) \right. \\ & + \delta w'_0 \left(c_{55}^m A_m w'_0 + c_{55}^m A_m \theta_m \right) + \delta \theta'_0 \left(-c_{11}^{*m} h_0 A_m u'_0 + c_{11}^{*m} h_0^2 A_m \theta'_0 \right. \\ & + c_{11}^{*m} h_0 \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i - c_{11}^{*m} h_0 \bar{I}_m \theta'_m \left. \right) + \sum_{i=m+1}^{-1} \delta \theta'_i \left(-c_{11}^{*m} 2h_i A_m u'_0 \right. \\ & + c_{11}^{*m} 2h_i h_0 A_m \theta'_0 + c_{11}^{*m} 2h_i \sum_{j=m+1}^{-1} 2h_j A_m \theta'_j - c_{11}^{*m} 2h_i \bar{I}_m \theta'_m \left. \right) + \delta \theta'_m \left(c_{11}^{*m} \bar{I}_m u'_0 \right. \\ & \left. \left. - c_{11}^{*m} h_0 \bar{I}_m \theta'_0 - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i \bar{I}_m \theta'_i + c_{11}^{*m} I_m \theta'_m \right) + \delta \theta_m \left(c_{55}^m A_m w'_0 + c_{55}^m A_m \theta'_m \right) \right] dL, \quad (2.23) \end{aligned}$$

where L is the length of the beam and A_k , $\bar{I}_p$ and I_k represent the zero-, first- and second-order moments of area, respectively. They are given by

$$A_k = 2h_k b, \quad \bar{I}_n = \frac{(2h_n)^2 b}{2}, \quad \bar{I}_m = -\frac{(2h_m)^2 b}{2}, \quad I_p = \frac{(2h_p)^3 b}{3}, \quad I_c = \frac{(2h_c)^3 b}{12}, \quad (2.24a-e)$$

where b is the width of the beam. Similarly, the electrical terms become

$$\delta H_{u\phi}^p = -\delta H_{u\phi}^{p(\phi)} - \delta \bar{H}_{u\phi}^{p(\phi)}, \quad (2.25)$$

$$\delta H_{\phi u}^p = -\delta H_{\phi u}^{p(\phi)} - \delta \bar{H}_{\phi u}^{p(\phi)}, \quad (2.26)$$

$$\delta H_{\phi\phi}^p = \delta H_{\phi\phi}^{p(\phi)} + \delta \bar{H}_{\phi\phi}^{p(\phi)}, \quad (2.27)$$

where

$$\delta H_{u\phi}^{n(\phi)} = \int_L \frac{e_{31}^{*n}}{2h_n} \phi_n \left(A_n \delta u'_0 + h_0 A_n \delta \theta'_0 + \sum_{i=1}^{n-1} 2h_i A_n \delta \theta'_i + \bar{I}_n \delta \theta'_n \right) dL, \quad (2.28)$$

$$\delta H_{u\phi}^{m(\phi)} = \int_L \frac{e_{31}^{*m}}{2h_m} \phi_m \left(A_m \delta u'_0 - h_0 A_m \delta \theta'_0 - \sum_{i=m+1}^{-1} 2h_i A_m \delta \theta'_i + \bar{I}_m \delta \theta'_m \right) dL, \quad (2.29)$$

$$\delta H_{\phi u}^{n(\phi)} = \int_L \frac{e_{31}^{*n}}{2h_n} \delta \phi_n \left(A_n u'_0 + h_0 A_n \theta'_0 + \sum_{i=1}^{n-1} 2h_i A_n \theta'_i + \bar{I}_n \theta'_n \right) dL, \quad (2.30)$$

$$\delta H_{\phi u}^{m(\phi)} = \int_L \frac{e_{31}^{*m}}{2h_m} \delta \phi_m \left(A_m u'_0 - h_0 A_m \theta'_0 - \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i + \bar{I}_m \theta'_m \right) dL, \quad (2.31)$$

$$\delta H_{\phi\phi}^{p(\phi)} = \int_L \frac{\varepsilon_{33}^{*p} A_p}{(2h_p)^2} \delta \phi_p \phi_p dL, \quad (2.32)$$

$$\begin{aligned} \delta \bar{H}_{u\phi}^{p(\phi)} &= \delta \bar{H}_{\phi u}^{p(\phi)} = \delta \bar{H}_{\phi\phi}^{p(\phi)} \\ &= \int_L \left\{ \frac{e_{31}^{*p}}{\varepsilon_{33}^{*p}} I_p \delta \theta'_p \theta'_p + \frac{e_{15}^p}{\varepsilon_{11}^p} A_p \left[\delta w'_0 (w'_0 + \theta_p) + \delta \theta_p (w'_0 + \theta_p) \right] \right\} dL. \end{aligned} \quad (2.33)$$

The induced terms $\delta \bar{H}_{u\phi}^{p(\phi)}$, $\delta \bar{H}_{\phi u}^{p(\phi)}$ and $\delta \bar{H}_{\phi\phi}^{p(\phi)}$ in (2.33), are expressed only in terms of the mechanical strains. Therefore, those terms can be summed with the purely mechanical ones in (2.21) and (2.23),

$$\delta H_{uu}^{p(\phi)} = \delta H_{uu}^p + \delta \bar{H}_{u\phi}^{p(\phi)} + \delta \bar{H}_{\phi u}^{p(\phi)} - \delta \bar{H}_{\phi\phi}^{p(\phi)}, \quad (2.34)$$

which after some algebra yields

$$\begin{aligned} \delta H_{uu}^{n(\phi)} &= \int_L \left[\delta u'_0 \left(c_{11}^{*n} A_n u'_0 + c_{11}^{*n} h_0 A_n \theta'_0 + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i A_n \theta'_i + c_{11}^{*n} \bar{I}_n \theta'_n \right) + \delta w'_0 \left(\bar{c}_{55}^{n(\phi)} A_n w'_0 \right. \right. \\ &\quad \left. \left. + \bar{c}_{55}^{n(\phi)} A_n \theta_n \right) + \delta \theta'_0 \left(c_{11}^{*n} h_0 A_n u'_0 + c_{11}^{*n} h_0^2 A_n \theta'_0 + c_{11}^{*n} h_0 \sum_{i=1}^{n-1} 2h_i A_n \theta'_i + c_{11}^{*n} h_0 \bar{I}_n \theta'_n \right) \right. \\ &\quad \left. + \sum_{i=1}^{n-1} \delta \theta'_i \left(c_{11}^{*n} 2h_i A_n u'_0 + c_{11}^{*n} 2h_i h_0 A_n \theta'_0 + c_{11}^{*n} 2h_i \sum_{j=1}^{n-1} 2h_j A_n \theta'_j + c_{11}^{*n} 2h_i \bar{I}_n \theta'_n \right) \right. \\ &\quad \left. + \delta \theta'_n \left(c_{11}^{*n} \bar{I}_n u'_0 + c_{11}^{*n} h_0 \bar{I}_n \theta'_0 + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i \bar{I}_n \theta'_i + \bar{c}_{11}^{n(\phi)} I_n \theta'_n \right) \right. \\ &\quad \left. + \delta \theta_n \left(\bar{c}_{55}^{n(\phi)} A_n w'_0 + \bar{c}_{55}^{n(\phi)} A_n \theta'_n \right) \right] dL, \quad (2.35) \end{aligned}$$

$$\begin{aligned} \delta H_{uu}^{m(\phi)} &= \int_L \left[\delta u'_0 \left(c_{11}^{*m} A_m u'_0 - c_{11}^{*m} h_0 A_m \theta'_0 - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i + c_{11}^{*m} \bar{I}_m \theta'_m \right) \right. \\ &\quad \left. + \delta w'_0 \left(\bar{c}_{55}^{m(\phi)} A_m w'_0 + \bar{c}_{55}^{m(\phi)} A_m \theta_m \right) + \delta \theta'_0 \left(-c_{11}^{*m} h_0 A_m u'_0 + c_{11}^{*m} h_0^2 A_m \theta'_0 \right. \right. \\ &\quad \left. \left. + c_{11}^{*m} h_0 \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i - c_{11}^{*m} h_0 \bar{I}_m \theta'_m \right) + \sum_{i=m+1}^{-1} \delta \theta'_i \left(-c_{11}^{*m} 2h_i A_m u'_0 + c_{11}^{*m} 2h_i h_0 A_m \theta'_0 \right. \right. \\ &\quad \left. \left. + c_{11}^{*m} 2h_i \sum_{j=m+1}^{-1} 2h_j A_m \theta'_j - c_{11}^{*m} 2h_i \bar{I}_m \theta'_m \right) + \delta \theta'_m \left(c_{11}^{*m} \bar{I}_m u'_0 - c_{11}^{*m} h_0 \bar{I}_m \theta'_0 \right. \right. \\ &\quad \left. \left. - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i \bar{I}_m \theta'_i + \bar{c}_{11}^{m(\phi)} I_m \theta'_m \right) + \delta \theta_m \left(\bar{c}_{55}^{m(\phi)} A_m w'_0 + \bar{c}_{55}^{m(\phi)} A_m \theta'_m \right) \right] dL, \quad (2.36) \end{aligned}$$

where

$$\bar{c}_{11}^{p(\phi)} = c_{11}^{*p} + \frac{e_{31}^{*p2}}{4\varepsilon_{33}^{*p}}, \quad \bar{c}_{55}^{p(\phi)} = c_{55}^p + \frac{e_{15}^p}{\varepsilon_{11}^p}. \quad (2.37a,b)$$

Expressions (2.37) are the so-called *effective stiffness parameters* (Vasques and Rodrigues, 2005) and they represent the stiffness increase due to the direct piezoelectric effect. As can be seen in (2.35) and (2.36), the parameter $\bar{c}_{11}^{p(\phi)}$ only affects the bending stiffness and it represents the effects of the induced parabolic electric potential. Regarding $\bar{c}_{55}^{p(\phi)}$, it represents the stiffness increase due to the assumption of zero electric displacement in the axial direction and it affects only the shear stiffness.

Finally, the total virtual work of the internal electro-mechanical forces in all the layers (elastic, piezoelectric or viscoelastic layers) is given by

$$\begin{aligned} \delta H = \delta H_{uu}^c + \sum_{n=1}^{\bar{n}} \left(\delta H_{uu}^{n(\phi)} + \delta H_{u\phi}^{n(\phi)} + \delta H_{\phi u}^{n(\phi)} - \delta H_{\phi\phi}^{n(\phi)} \right) \\ + \sum_{m=\bar{m}}^{-1} \left(\delta H_{uu}^{m(\phi)} + \delta H_{u\phi}^{m(\phi)} + \delta H_{\phi u}^{m(\phi)} - \delta H_{\phi\phi}^{m(\phi)} \right) \end{aligned} \quad (2.38)$$

2.6.2 Virtual Work of the Inertial Forces

The virtual work of the inertial forces in a generic layer k is given by

$$\delta T^k = - \int_{V_k} \rho_k \left(\delta \tilde{u}_k \ddot{u}_k + \delta \tilde{w}_k \ddot{w}_k \right) dV_k, \quad (2.39)$$

where ρ_k is the density of the material of the layer. Substituting the displacement field defined in equation (2.1) into (2.39) gives, after integration, for the top, bottom and core layers,

$$\begin{aligned} \delta T^n = - \int_L \left[\delta u_0 \left(\rho_n A_n \ddot{u}_0 + \rho_n h_0 A_n \ddot{\theta}_0 + \rho_n \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i + \rho_n \bar{I}_n \ddot{\theta}_n \right) + \delta w_0 \rho_n A_n \ddot{w}_0 \right. \\ + \delta \theta_0 \left(\rho_n h_0 A_n \ddot{u}_0 + \rho_n h_0^2 A_n \ddot{\theta}_0 + \rho_n h_0 \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i + \rho_n h_0 \bar{I}_n \ddot{\theta}_n \right) \\ + \sum_{i=1}^{n-1} \delta \theta_i \left(\rho_n 2h_i A_n \ddot{u}_0 + \rho_n 2h_i h_0 A_n \ddot{\theta}_0 + \rho_n 4h_i \sum_{j=1}^{n-1} h_j A_n \ddot{\theta}_j + \rho_n 2h_i \bar{I}_n \ddot{\theta}_n \right) \\ \left. + \delta \theta_n \left(\rho_n \bar{I}_n \ddot{u}_0 + \rho_n h_0 \bar{I}_n \ddot{\theta}_0 + \rho_n \sum_{i=1}^{n-1} 2h_i \bar{I}_n \ddot{\theta}_i + \rho_n I_n \ddot{\theta}_n \right) \right] dL, \end{aligned} \quad (2.40)$$

$$\delta T^c = - \int_L \left(\delta u_0 \rho_c A_c \ddot{u}_0 + \delta \theta_0 \rho_c I_c \ddot{\theta}_0 + \delta w_0 \rho_c A_c \ddot{w}_0 \right) dL, \quad (2.41)$$

$$\begin{aligned} \delta T^m = - \int_L \left[\delta u_0 \left(\rho_m A_m \ddot{u}_0 - \rho_m h_0 A_m \ddot{\theta}_0 - \rho_m \sum_{i=m+1}^{-1} 2h_i A_m \ddot{\theta}_i + \rho_m \bar{I}_m \ddot{\theta}_m \right) + \delta w_0 \rho_m A_m \ddot{w}_0 \right. \\ + \delta \theta_0 \left(-\rho_m h_0 A_m \ddot{u}_0 + \rho_m h_0^2 A_m \ddot{\theta}_0 + \rho_m h_0 \sum_{i=m+1}^{-1} 2h_i A_m \ddot{\theta}_i - \rho_m h_0 \bar{I}_m \ddot{\theta}_m \right) \\ + \sum_{i=m+1}^{-1} \delta \theta_i \left(-\rho_m 2h_i A_m \ddot{u}_0 + \rho_m 2h_i h_0 A_m \ddot{\theta}_0 + \rho_m 4h_i \sum_{j=m+1}^{-1} h_j A_m \ddot{\theta}_j - \rho_m 2h_i \bar{I}_m \ddot{\theta}_m \right) \\ \left. + \delta \theta_m \left(\rho_m \bar{I}_m \ddot{u}_0 + \rho_m h_0 \bar{I}_m \ddot{\theta}_0 - \rho_m \sum_{i=m+1}^{-1} 2h_i \bar{I}_m \ddot{\theta}_i + \rho_m I_m \ddot{\theta}_m \right) \right] dL, \end{aligned} \quad (2.42)$$

The total virtual work of the inertial forces is then given by the sum of the virtual work of all the layers,

$$\delta T = \delta T^c + \sum_{n=1}^{\bar{n}} \delta T^n + \sum_{m=\bar{n}}^{-1} \delta T^m. \quad (2.43)$$

2.6.3 Virtual Work of the External Forces

In the determination of the virtual work of the mechanical external forces, two types of prescribed mechanical forces are considered, namely, axial and transverse volume forces F_x^k and F_z^k . The virtual work of those forces for the generic layer k is given by

$$\delta W_u^k = \int_{V_k} \left(F_x^k \delta \tilde{u}_k + F_z^k \delta \tilde{w}_k \right) dV_k. \quad (2.44)$$

Substituting the displacement field (2.1) into (2.44) and integrating, yields

$$\delta W_u^n = \int_L \left[\left(\delta u_0 + h_0 \delta \theta_0 + \sum_{i=1}^{n-1} 2h_i \delta \theta_i + h_n \delta \theta_n \right) X_n + \delta \theta_n M_n + \delta w_0 Z_n \right] dL, \quad (2.45)$$

$$\delta W_u^c = \int_L \left(\delta u_0 X_c + \delta \theta_0 M_c + \delta w_0 Z_c \right) dL, \quad (2.46)$$

$$\delta W_u^m = \int_L \left[\left(\delta u_0 - h_0 \delta \theta_0 - \sum_{i=m+1}^{-1} 2h_i \delta \theta_i - h_m \delta \theta_m \right) X_m + \delta \theta_m M_m + \delta w_0 Z_m \right] dL, \quad (2.47)$$

where

$$\left(X_k, M_k, Z_k \right) = \int_{A_k} \left(F_x^k, z_k F_x^k, F_z^k \right) dA_k. \quad (2.48)$$

Thus, the total virtual work of the applied mechanical forces is given by

$$\delta W_u = \delta W_u^c + \sum_{n=1}^{\bar{n}} \delta W_u^n + \sum_{m=\bar{n}}^{-1} \delta W_u^m. \quad (2.49)$$

The virtual work of the electric charge density in each piezoelectric layer is defined by

$$\delta W_\phi^p = - \int_{A_p^e} \delta \varphi_p \tau_p dA_p^e = - \int_L \delta \phi_p b \tau_p dL, \quad (2.50)$$

where A_p^e is the electrode area and τ_p is the applied electric charge density at the electrode. Note that from the definition of the electric potential (2.14), and considering only the applied potential term, one finds that $\varphi_p(z_p = -h_p) = 0$ and $\varphi_p(z_p = h_p) = \phi_p$. The total virtual work of the applied electric charge density is given by

$$\delta W_\phi = \sum_{n=1}^{\bar{n}} \delta W_\phi^n + \sum_{m=\bar{n}}^{-1} \delta W_\phi^m. \quad (2.51)$$

Finally, the total virtual work δW of the applied external forces, comprising the virtual work of the applied mechanical forces δW_u and that of the applied electric charge density at the electrodes δW_ϕ , is given by

$$\delta W = \delta W_u + \delta W_\phi = \sum_{k=\bar{n}}^{\bar{n}} \delta W_u^k + \sum_{n=1}^{\bar{n}} \delta W_\phi^n + \sum_{m=\bar{n}}^{-1} \delta W_\phi^m. \quad (2.52)$$

2.7 Strong Forms

As derived in Appendix A, the equations of motion are obtained from the Hamilton's principle by substituting expressions (2.38), (2.43) and (2.52) into (4.52), integrating by parts and collecting the terms involving the variations δu_0 , δw_0 , $\delta \theta_0$, $\delta \theta_{\hat{n}}$ ($\hat{n} = 1, \dots, \bar{n} - 1$), $\delta \theta_{\bar{n}}$, $\delta \theta_{\bar{m}}$ and $\delta \theta_{\hat{m}}$ ($\hat{m} = \bar{m} + 1, \dots, -1$), independent and arbitrary in the interval $[0, L]$. They are given by

$\delta u_0 :$

$$\begin{aligned} & \sum_{k=\bar{m}}^{\bar{n}} c_{11}^{*k} A_k u_0'' + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) \theta_0'' + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \theta_{\hat{n}}'' \\ & + c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} \theta_{\bar{n}}'' + c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} \theta_{\bar{m}}'' + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' \\ & + \sum_{n=1}^{\bar{n}} A_n \frac{e_{31}^{*n}}{2h_n} \phi_n' + \sum_{m=\bar{m}}^{-1} A_m \frac{e_{31}^{*m}}{2h_m} \phi_m' + \sum_{k=\bar{m}}^{\bar{n}} X_k = \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k \ddot{u}_0 + \left(\sum_{n=1}^{\bar{n}} \rho_n h_0 A_n \right. \\ & - \sum_{m=\bar{m}}^{-1} \rho_m h_0 A_m \Big) \ddot{\theta}_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} + \rho_{\bar{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}} \\ & + \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}}, \quad (2.53) \end{aligned}$$

$\delta w_0 :$

$$\begin{aligned} & \left(\sum_{n=1}^{\bar{n}} \bar{c}_{55}^{n(\phi)} A_n + c_{55}^0 A_0 + \sum_{m=\bar{m}}^{-1} \bar{c}_{55}^{m(\phi)} A_m \right) w_0'' + c_{55}^e A_c \theta_0' + \sum_{\hat{n}=1}^{\bar{n}-1} \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}}' + \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}}' \\ & + \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}}' + \sum_{\hat{m}=\bar{m}+1}^{-1} \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}}' + \sum_{k=\bar{m}}^{\bar{n}} Z_k = \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k \ddot{w}_0, \quad (2.54) \end{aligned}$$

$\delta \theta_0 :$

$$\begin{aligned} & \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) u_0'' + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0^2 A_n + c_{11}^{*0} I_0 + \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0^2 A_m \right) \theta_0'' \\ & + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta_{\hat{n}}'' + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta_{\bar{n}}'' - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \theta_{\bar{m}}'' \\ & + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\hat{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' - c_{55}^0 A_0 w_0' - c_{55}^0 A_0 \theta_0 + \sum_{n=1}^{\bar{n}} \frac{e_{31}^{*n}}{2h_n} h_0 A_n \phi_n' \\ & - \sum_{m=\bar{m}}^{-1} \frac{e_{31}^{*m}}{2h_m} h_0 A_m \phi_m' + \sum_{n=1}^{\bar{n}} h_0 X_n + M_c - \sum_{m=\bar{m}}^{-1} h_0 X_m = \left(\sum_{n=1}^{\bar{n}} \rho_n h_0 A_n - \sum_{m=\bar{m}}^{-1} \rho_m h_0 A_m \right) \ddot{u}_0 \\ & + \ddot{\theta}_0 \left(\sum_{n=1}^{\bar{n}} \rho_n h_0^2 A_n + \rho_0 I_0 + \sum_{m=\bar{m}}^{-1} \rho_m h_0^2 A_m \right) + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i A_i + \rho_{\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} \\ & + \rho_{\bar{n}} h_0 \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}} - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\hat{m}}^{\hat{m}-1} \rho_i A_i - \rho_{\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}} \quad (2.55) \end{aligned}$$

$\delta\theta_{\hat{n}} :$

$$\begin{aligned}
& \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) u_0'' + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta_0'' + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j \right. \\
& \quad \left. + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \theta_i'' + \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}}^2 A_i + \bar{c}_{11}^{\hat{n}(\phi)} \bar{I}_{\hat{n}} \right) \theta_{\hat{n}}'' + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}} h_j A_i + c_{11}^{*j} 2h_{\hat{n}} \bar{I}_j \right) \theta_j'' \\
& \quad + c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \theta_{\bar{n}}'' - \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} w_0' - \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}} + \frac{e_{31}^{*\hat{n}}}{2h_{\hat{n}}} \bar{I}_{\hat{n}} \phi_{\hat{n}}' + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i' + h_{\hat{n}} X_{\hat{n}} \\
& \quad + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} X_i + M_{\hat{n}} = \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \ddot{u}_0 + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i A_i + \rho_{\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \ddot{\theta}_0 \\
& \quad + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \ddot{\theta}_i + \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 4h_{\hat{n}}^2 A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} \\
& \quad + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} \rho_i 4h_{\hat{n}} h_j A_i + \rho_j 2h_{\hat{n}} \bar{I}_j \right) \ddot{\theta}_j + \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}}, \quad (2.56)
\end{aligned}$$

$\delta\theta_{\bar{n}} :$

$$\begin{aligned}
& c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} u_0'' + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta_0'' + \sum_{\hat{n}=1}^{\bar{n}-1} c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \theta_{\hat{n}}'' + \bar{c}_{11}^{\bar{n}(\phi)} \bar{I}_{\bar{n}} \theta_{\bar{n}}'' - \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} w_0' - \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}} \\
& \quad + \frac{e_{31}^{*\bar{n}}}{2h_{\bar{n}}} \bar{I}_{\bar{n}} \phi_{\bar{n}}' + h_{\bar{n}} X_{\bar{n}} + M_{\bar{n}} = \rho_{\bar{n}} \bar{I}_{\bar{n}} \ddot{u}_0 + \rho_{\bar{n}} h_0 \bar{I}_{\bar{n}} \ddot{\theta}_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\hat{n}} + \rho_{\bar{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}}, \quad (2.57)
\end{aligned}$$

$\delta\theta_{\bar{m}} :$

$$\begin{aligned}
& c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} u_0'' - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \theta_0'' + \bar{c}_{11}^{\bar{m}(\phi)} \bar{I}_{\bar{m}} \theta_{\bar{m}}'' - \sum_{\hat{m}=\bar{m}+1}^{-1} c_{11}^{*\bar{m}} 2h_{\hat{m}} \bar{I}_{\bar{m}} \theta_{\hat{m}}'' - \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} w_0' - \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}} \\
& \quad + \frac{e_{31}^{*\bar{m}}}{2h_{\bar{m}}} \bar{I}_{\bar{m}} \phi_{\bar{m}}' - h_{\bar{m}} X_{\bar{m}} + M_{\bar{m}} = \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{u}_0 - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \ddot{\theta}_0 - \sum_{\hat{m}=\bar{m}+1}^{-1} \rho_{\bar{m}} 2h_{\hat{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\hat{m}} + \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}}, \quad (2.58)
\end{aligned}$$

$\delta\theta_{\hat{m}} :$

$$\begin{aligned}
& \left(- \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \right) u_0'' + \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta_0'' + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} c_{11}^{*i} 4h_{\hat{m}} h_j A_i \right. \\
& \quad \left. - c_{11}^{*j} 2h_{\hat{m}} \bar{I}_j \right) \theta_j'' + \left(\sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 4h_{\hat{m}}^2 A_i + \bar{c}_{11}^{\hat{m}(\phi)} \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' + \sum_{i=\hat{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \theta_i'' \\
& \quad - c_{11}^{*\bar{m}} 2h_{\hat{m}} \bar{I}_{\bar{m}} \theta_{\bar{m}}'' - \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} w_0' - \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}} + \frac{e_{31}^{*\hat{m}}}{2h_{\hat{m}}} \bar{I}_{\hat{m}} \phi_{\hat{m}}' - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i' - h_{\hat{m}} X_{\hat{m}} \\
& \quad - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} X_i + M_{\hat{m}} = \left(- \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{u}_0 + \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i A_i - \rho_{\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \ddot{\theta}_0 \\
& \quad + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} \rho_i 4h_{\hat{m}} h_j A_i - \rho_j 2h_{\hat{m}} \bar{I}_j \right) \ddot{\theta}_j + \left(\sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 4h_{\hat{m}}^2 A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}} \\
& \quad + \sum_{i=\hat{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \ddot{\theta}_i + \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}}, \quad (2.59)
\end{aligned}$$

In a similar way (see Appendix A), the electric charge equilibrium equations are obtained by collecting the terms related to the variations $\delta\phi_n$ ($n = 1, \dots, \bar{n}$) and $\delta\phi_m$ ($m = -1, \dots, \bar{m}$), yielding

$$\delta\phi_n : \quad \frac{e_{31}^{*n}}{2h_n} A_n u'_0 + \frac{e_{31}^{*n}}{2h_n} h_0 A_n \theta'_0 + \sum_{i=1}^{n-1} \frac{e_{31}^{*n}}{2h_n} 2h_i A_n \theta'_i + \frac{e_{31}^{*n}}{2h_n} \bar{I}_n \theta'_n - \frac{\varepsilon_{33}^{*n} A_n}{4h_n^2} \phi_n = -b\tau_n, \quad (2.60)$$

$$\delta\phi_m : \quad \frac{e_{31}^{*m}}{2h_m} A_m u'_0 - \frac{e_{31}^{*m}}{2h_m} h_0 A_m \theta'_0 + \frac{e_{31}^{*m}}{2h_m} \bar{I}_m \theta'_m - \sum_{i=m+1}^{-1} \frac{e_{31}^{*m}}{2h_m} 2h_i A_m \theta'_i + \frac{\varepsilon_{33}^{*m} A_m}{4h_m^2} \phi_m = -b\tau_m. \quad (2.61)$$

Finally, as derived in Appendix A, the electro-mechanical boundary conditions at $x = 0, L$ are expressed as

$$\begin{aligned} \delta u_0 \left[\sum_{k=\bar{m}}^{\bar{n}} c_{11}^{*k} A_k u'_0 + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) \theta'_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \theta'_{\hat{n}} + c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} \theta'_{\bar{n}} \right. \\ \left. + c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} \theta'_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \theta'_{\hat{m}} + \sum_{n=1}^{\bar{n}} A_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \sum_{m=\bar{m}}^{-1} A_m \frac{e_{31}^{*m}}{2h_m} \phi_m \right] = 0, \end{aligned} \quad (2.62)$$

$$\begin{aligned} \delta w_0 \left[\left(\sum_{n=1}^{\bar{n}} \bar{c}_{55}^{n(\phi)} A_n + c_{55}^0 A_0 + \sum_{m=\bar{m}}^{-1} \bar{c}_{55}^{m(\phi)} A_m \right) w'_0 + c_{55}^c A_c \theta_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}} + \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}} \right. \\ \left. + \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}} \right] = 0, \end{aligned} \quad (2.63)$$

$$\begin{aligned} \delta\theta_0 \left[\left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) u'_0 + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0^2 A_n + c_{11}^{*0} I_0 + \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0^2 A_m \right) \theta'_0 \right. \\ \left. + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta'_{\hat{n}} + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta'_{\bar{n}} - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \theta'_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\hat{m}}^{\hat{m}-1} c_{11}^{*i} A_i \right. \right. \\ \left. \left. - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta'_{\hat{m}} + \sum_{n=1}^{\bar{n}} \frac{e_{31}^{*n}}{2h_n} h_0 A_n \phi_n - \sum_{m=\bar{m}}^{-1} \frac{e_{31}^{*m}}{2h_m} h_0 A_m \phi_m \right] = 0, \end{aligned} \quad (2.64)$$

$$\begin{aligned} \delta\theta_{\hat{n}} \left[\left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) u'_0 + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta'_0 \right. \\ \left. + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \theta'_i + \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}}^2 A_i + \bar{c}_{11}^{\hat{n}(\phi)} I_{\hat{n}} \right) \theta'_{\hat{n}} \right. \\ \left. + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}} h_j A_i + c_{11}^{*j} 2h_{\hat{n}} \bar{I}_j \right) \theta'_j + c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \theta'_{\bar{n}} + \frac{e_{31}^{*\hat{n}}}{2h_{\hat{n}}} \bar{I}_{\hat{n}} \phi_{\hat{n}} + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \right] = 0, \end{aligned} \quad (2.65)$$

$$\delta\theta_{\bar{n}} \left(c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} u'_0 + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta'_0 + \sum_{\hat{n}=1}^{\bar{n}-1} c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\hat{n}} \theta'_{\hat{n}} + \bar{c}_{11}^{\bar{n}(\phi)} I_{\bar{n}} \theta'_{\bar{n}} + \frac{e_{31}^{*\bar{n}}}{2h_{\bar{n}}} \bar{I}_{\bar{n}} \phi_{\bar{n}} \right) = 0, \quad (2.66)$$

$$\delta\theta_{\bar{m}} \left(c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} u'_0 - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \theta'_0 + \bar{c}_{11}^{\bar{m}(\phi)} I_{\bar{m}} \theta'_{\bar{m}} - \sum_{\hat{m}=\bar{m}+1}^{-1} c_{11}^{*\bar{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \theta'_{\hat{m}} + \frac{e_{31}^{*\bar{m}}}{2h_{\bar{m}}} \bar{I}_{\bar{m}} \phi_{\bar{m}} \right) = 0, \quad (2.67)$$

$$\begin{aligned}
\delta\theta_{\hat{m}} \Big(& - \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \Big) u'_0 + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta'_0 - c_{11}^{*\hat{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \theta'_{\hat{m}} \\
& + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} c_{11}^{*i} 4h_{\hat{m}} h_j A_i - c_{11}^{*j} 2h_{\hat{m}} \bar{I}_j \right) \theta'_j + \left(\sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 4h_{\hat{m}}^2 A_i + \bar{c}_{11}^{\hat{m}(\phi)} \bar{I}_{\hat{m}} \right) \theta'_{\hat{m}} \\
& + \sum_{i=\bar{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \theta'_i + \frac{e_{31}^{*\hat{m}}}{2h_{\hat{m}}} \bar{I}_{\hat{m}} \phi_{\hat{m}} - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \Big] = 0. \quad (2.68)
\end{aligned}$$

Equations (2.53)-(2.68) represent the analytical electro-mechanical model of the layered beam with arbitrary ACLD treatments, where the electric potential differences and the generalized mechanical displacements are the unknown independent variables. It is worthy to mention that no surface prescribed forces were prescribed in the boundary of the beam.

2.8 Summary

In this chapter a generic analytical weak and strong formulation for the study of beams with arbitrary hybrid damping treatments with also an arbitrary number of layers was presented. A coupled-kinematics partial layerwise theory, in the sense that a generic coupled global formulation of the displacement field was explicitly considered, was utilized leading to a FSDT for each layer; however, the global kinematics, depending upon the number of physical or non-physical layers considered, as usual in layerwise theories, may yield an higher-order approximation. For thick laminates this allows a more accurate representation of the intra- and inter-laminar effects and the strain and stress distributions become better approximations to the three-dimensional elasticity theory. Moreover, the rotary inertia is considered in a more accurate way, which might be of importance at high frequencies. For modeling the electric field and potential of the piezoelectric layers a fully coupled electro-mechanical theory was utilized and the viscoelastic damping behavior is considered through a CMA. The use of effective stiffness parameters allowed a representation of the system similar to a pure mechanical one to be obtained merely by modifying the elastic constitutive parameters. The hybrid behavior of the elastic, viscoelastic and piezoelectric materials was thus considered in an accurate and consistent way.

The main differences from classical layerwise theories are that the present one does not need an *a posteriori* assemblage of the contributions of each discrete layer, ensuring displacement continuity at the interfaces, to obtain the global model. Therefore, this generic analytical model is passible of being used with different approximate methods, using whether the weak (as usual in FE modeling) or strong (e.g. meshless and spectral methods) forms, or utilized to derive closed-form analytical solutions. The developed weak forms, governing the motion and electrical charge equilibrium, will be used in the next chapter for the development of a FE model.

CHAPTER 3

Finite Element Model of Piezo-Visco-Elastic Beams

3.1 Introduction

In this chapter a FE model of beams with hybrid (active/passive), or arbitrary ACLD, damping treatments, incorporating an arbitrary number of elastic, viscoelastic and piezoelectric layers is developed. To this end, the weak forms of the analytical model presented in Chapter 2 are utilized and a one-dimensional FE model with two nodes is developed; the nodal mechanical degrees of freedom are the axial displacement, transverse displacement and the rotation of the mid-plane of the host beam and the rotations of the individual layers, and the electrical elemental degrees of freedom are the electrical potential difference of each piezoelectric layer. In order to validate the FE model, frequency response functions were measured experimentally and evaluated numerically for a freely suspended aluminum beam with an ACLD patch, and the results are presented and discussed.

3.2 Spatial Model

The weak forms of the analytical model presented in Chapter 2, governing the motion and electric charge equilibrium of the layered beam with piezoelectric and viscoelastic layers, are derived in the Appendix A. For convenience, the generalized mechanical displacements and electric potential differences are grouped in the generalized vectors of displacements and potential differences,

$$\mathbf{u}(x, t) = \{u_0(x, t), w_0(x, t), \theta_0(x, t), \theta_1(x, t), \dots, \theta_{\bar{n}}(x, t), \theta_{\bar{m}}(x, t), \dots, \theta_{-1}(x, t)\}^T, \quad (3.1)$$

$$\boldsymbol{\phi}(x, t) = \{\phi_1(x, t), \dots, \phi_{\bar{n}}(x, t), \phi_{\bar{m}}(x, t), \dots, \phi_{-1}(x, t)\}^T, \quad (3.2)$$

and the weak forms, given in equations (A.26) and (A.27), are given by

$$\int_L \left[\delta \mathbf{u}^T \mathbf{J} \ddot{\mathbf{u}} + \delta \mathbf{u}^T (\mathbf{L}_{xx}^T \mathbf{Y} \mathbf{L}_{xx} + \mathbf{L}_{zx}^T \mathbf{G} \mathbf{L}_{zx}) \mathbf{u} + \delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{P}^T \boldsymbol{\phi} \right] dL = \int_L \delta \mathbf{u}^T \mathbf{f} dL, \quad (3.3)$$

$$\int_L (\delta \boldsymbol{\phi}^T \mathbf{P} \mathbf{L}_{xx} \mathbf{u} + \delta \boldsymbol{\phi}^T \mathbf{C} \boldsymbol{\phi}) dL = \int_L \delta \boldsymbol{\phi}^T \boldsymbol{\tau} dL, \quad (3.4)$$

where the non-zero terms of the symmetric positive definite matrix $\mathbf{J}$, which is related with the inertia, and of the symmetric semi-positive definite matrices $\mathbf{Y}$ and $\mathbf{G}$, on which the extensional and shear strain energies depend, of size $[(\bar{k} - 1) \times (\bar{k} - 1)]$, are defined in (A.19), (A.20) and (A.21), respectively. The matrices $\mathbf{P}$ and $\mathbf{C}$, related with the piezoelectric coupling effects and the capacitance of the piezoelectric layers, of size $[\bar{p} \times (\bar{k} - 1)]$ and $(\bar{p} \times \bar{p})$, respectively, have non-zero terms defined by (A.22) and (A.23a,b). The applied mechanical forces and electric charge density vectors $\mathbf{f}$ and $\boldsymbol{\tau}$, of size $[(\bar{k} - 1) \times 1]$ and $(\bar{p} \times 1)$, have elements defined by (A.24a-g) and (A.25a,b). Here, $\bar{p} = \bar{n} - \bar{m}$ is the number of piezoelectric layers and $\bar{k} - 1$, with $\bar{k} = \bar{p} + 4$, is the number of generalized displacements. Furthermore, the differential operators

$$\mathbf{L}_{xx} = \text{diag} \left(\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial x}, \dots, \frac{\partial}{\partial x} \right), \quad \mathbf{L}_{zx} = \text{diag} \left(0, \frac{\partial}{\partial x}, 1, \dots, 1 \right), \quad (3.5a,b)$$

of size $[(\bar{k} - 1) \times (\bar{k} - 1)]$, are used for the definition of the generalized extensional and shear strains.

The FE method is utilized here with the purpose of obtaining an approximated solution of Equations (3.3) and (3.4). That involves a transformation of the global integral forms to a representation composed of the sum of the elemental integral forms, leading to the definition of the local (elemental) matrices and vectors. Thus, the beam's total length L is divided and the domain of integration is discretized into q finite elements of length L_e . With the sum taken over all q elements, the global weak forms become

$$\sum_{e=1}^q \int_{L_e} \delta \mathbf{u}^e \mathbf{T} \left[\mathbf{J} \ddot{\mathbf{u}}^e + \left(\mathbf{L}_{xx}^T \mathbf{Y} \mathbf{L}_{xx} + \mathbf{L}_{zx}^T \mathbf{G} \mathbf{L}_{zx} \right) \mathbf{u}^e + \mathbf{L}_{xx}^T \mathbf{P}^T \boldsymbol{\phi}^e - \mathbf{f} \right] dL_e = 0, \quad (3.6)$$

$$\sum_{e=1}^q \int_{L_e} \delta \boldsymbol{\phi}^e \mathbf{T} (\mathbf{P} \mathbf{L}_{xx} \mathbf{u}^e + \mathbf{C} \boldsymbol{\phi}^e - \boldsymbol{\tau}^e) dL_e = 0. \quad (3.7)$$

The mechanical (and its second time derivative) and electrical variables of (3.6) and (3.7), $\mathbf{u}^e(x, t)$ and $\boldsymbol{\phi}^e(x, t)$, are restricted to the domain of integration and an approximation is considered at a local level in each sub-domain, where only the nodal values of the variables at the element boundaries contribute to the approximation. The FE mesh is then composed of $q + 1$ nodal points and the global mechanical degree of freedom (DoF) vectors, $\bar{\mathbf{u}}(t)$ and $\bar{\boldsymbol{\phi}}(t)$, of size $[(q + 1)(\bar{p} + 4) \times 1]$ and $(q\bar{p} \times 1)$, are defined as

$$\bar{\mathbf{u}}(t) = \{ \dots, \bar{u}_0^r(t), \bar{w}_0^r(t), \bar{\theta}_0^r(t), \bar{\theta}_1^r(t), \dots, \bar{\theta}_n^r(t), \bar{\theta}_m^r(t), \dots, \bar{\theta}_{-1}^r(t), \dots \}^T, \quad (3.8)$$

$$\bar{\boldsymbol{\phi}}(t) = \{ \dots, \phi_1^s(t), \dots, \phi_n^s(t), \phi_m^s(t), \dots, \phi_{-1}^s(t), \dots \}^T, \quad (3.9)$$

where $r = 1, \dots, q + 1$ and $s = 1, \dots, q$.

For the definition of the local approximation of the generalized mechanical and electrical DoFs in each sub-domain, a generic element with two nodes (1 and 2) is isolated from the FE mesh (Figure 3.1). The correspondence between each node of the element and the global node enumeration is established through the mechanical and electrical connectivity matrices $\mathbf{R}_u^e$ and $\mathbf{R}_\phi^e$, of size $[\bar{k}(q + 1) \times \bar{k}(q + 1)]$ and $(\bar{p}q \times \bar{p}q)$, expressed as

$$\bar{\mathbf{u}}^e = \mathbf{R}_u^e \bar{\mathbf{u}}, \quad \bar{\boldsymbol{\phi}}^e = \mathbf{R}_\phi^e \bar{\boldsymbol{\phi}}, \quad (3.10a,b)$$

where the mechanical and electrical elemental DoFs vectors $\bar{\mathbf{u}}^e(t)$ and $\bar{\boldsymbol{\phi}}^e(t)$ are given by

$$\bar{\mathbf{u}}^e(t) = \{ \bar{u}_0^1(t), \bar{w}_0^1(t), \bar{\theta}_0^1(t), \bar{\theta}_1^1(t), \dots, \bar{\theta}_n^1(t), \bar{\theta}_m^1(t), \dots, \bar{\theta}_{-1}^1(t), \bar{u}_0^2(t), \bar{w}_0^2(t), \bar{\theta}_0^2(t), \bar{\theta}_1^2(t), \dots, \bar{\theta}_n^2(t), \bar{\theta}_m^2(t), \dots, \bar{\theta}_{-1}^2(t) \}^T, \quad (3.11)$$

$$\bar{\phi}^e(t) = \{\bar{\phi}_1(t), \dots, \bar{\phi}_{\bar{n}}(t), \bar{\phi}_{\bar{m}}(t), \dots, \bar{\phi}_{-1}(t)\}^T. \quad (3.12)$$

The superscripts 1 and 2 denote the node at which the DoF is defined, while the subscript identifies the layer to which the DoF refers. As usual, a variable from the mechanical DoFs global vector appears on two elemental DoFs vectors, except for the boundary nodes of the global domain. However, for the electrical DoFs global vector that won't be the case because continuity of the electric potential difference between adjacent FEs won't be assured.

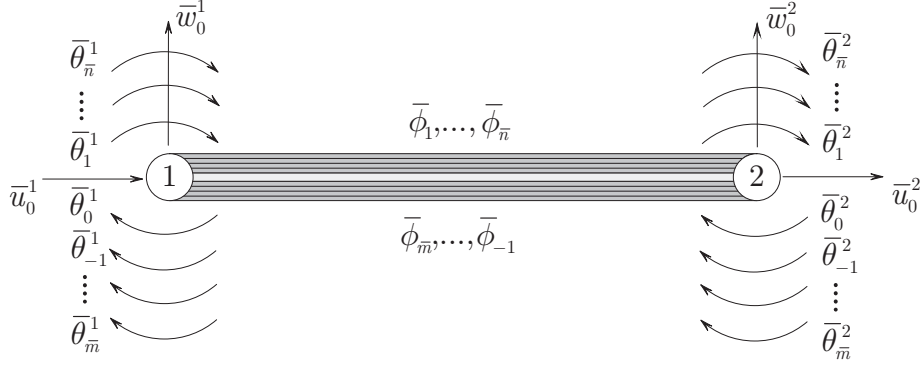


Figure 3.1: One-dimensional FE of the beam with arbitrary ACLD treatments.

From the differential operators (3.5) it is possible to see that the variational Equations (3.3) and (3.4) contain at most first order derivatives of the generalized displacements and zero order derivatives of the electric potential differences. Thus, the displacement variables within each element domain must be, at least, approximated by linear interpolation functions. For the electric potential differences, constant functions are utilized and the electric potential difference becomes constant in each element. The interpolation functions are grouped in the mechanical and electrical interpolation matrices $\mathbf{N}_u(x)$ and $\mathbf{N}_\phi$. Therefore, the generalized displacement and electric potential vectors in each sub-domain are approximated by

$$\mathbf{u}^e(x, t) = \mathbf{N}_u(x) \bar{\mathbf{u}}^e(t), \quad \phi^e(t) = \mathbf{N}_\phi \bar{\phi}^e(t). \quad (3.13a,b)$$

Substituting Equations (3.13) into (3.6) and (3.7) yields

$$\sum_{e=1}^q \delta \bar{\mathbf{u}}^{eT} (\mathbf{M}_{uu}^e \ddot{\bar{\mathbf{u}}}^e + \mathbf{K}_{uu}^e \bar{\mathbf{u}}^e + \mathbf{K}_{u\phi}^e \bar{\phi}^e - \mathbf{f}_u^e) = 0, \quad (3.14)$$

$$\sum_{e=1}^q \delta \bar{\phi}^{eT} (\mathbf{K}_{\phi u}^e \bar{\mathbf{u}}^e + \mathbf{K}_{\phi\phi}^e \bar{\phi}^e - \mathbf{q}^e) = 0, \quad (3.15)$$

where the elemental matrices and vectors are defined as

$$\mathbf{M}_{uu}^e = \int_{L_e} \mathbf{N}_u^T \mathbf{J} \mathbf{N}_u \, dL_e, \quad \mathbf{K}_{uu}^e = \int_{L_e} (\mathbf{B}_{xx}^T \mathbf{Y} \mathbf{B}_{xx} + k_s \mathbf{B}_{zx}^T \mathbf{G} \mathbf{B}_{zx}) \, dL_e, \quad (3.16a,b)$$

$$\mathbf{K}_{u\phi}^e = \mathbf{K}_{\phi u}^{eT} = \int_{L_e} \mathbf{B}_{xx}^T \mathbf{P}^T \mathbf{N}_\phi \, dL_e, \quad \mathbf{K}_{\phi\phi}^e = \int_{L_e} \mathbf{N}_\phi^T \mathbf{C} \mathbf{N}_\phi \, dL_e, \quad (3.17a,b)$$

$$\mathbf{f}_u^e = \int_{L_e} \mathbf{N}_u^T \mathbf{f} \, dL_e, \quad \mathbf{q}^e = \int_{L_e} \mathbf{N}_\phi^T \boldsymbol{\tau} \, dL_e, \quad (3.18a,b)$$

and the extensional and shear deformation matrices utilized are given by

$$\mathbf{B}_{xx} = \mathbf{L}_{xx} \mathbf{N}_u, \quad \mathbf{B}_{zx} = \mathbf{L}_{zx} \mathbf{N}_u. \quad (3.19a,b)$$

Furthermore, a shear correction factor k_s was introduced in Equation (3.16b) in order to approximate the effects of the non-linear shear deformation distribution (Birman and Bert, 2002). Moreover, reduced integration of the higher order terms of the shear stiffness matrix should be used in order to overcome the over-stiffening of the element at low thickness (shear locking).

Considering Equations (3.10) and substituting them into (3.14) and (3.15), yields

$$\forall \delta \bar{\mathbf{u}} : \quad \delta \bar{\mathbf{u}}^T (\mathbf{M}_{uu} \ddot{\bar{\mathbf{u}}} + \mathbf{K}_{uu} \bar{\mathbf{u}} + \mathbf{K}_{u\phi} \bar{\phi} - \mathbf{f}_u) = 0, \quad (3.20)$$

$$\forall \delta \bar{\phi} : \quad \delta \bar{\phi}^T (\mathbf{K}_{\phi u} \bar{\mathbf{u}} + \mathbf{K}_{\phi\phi} \bar{\phi} - \mathbf{q}) = 0, \quad (3.21)$$

where the global matrices and vectors are given by

$$\mathbf{M}_{uu} = \sum_{e=1}^q \mathbf{R}_u^{eT} \mathbf{M}_{uu}^e \mathbf{R}_u^e, \quad \mathbf{K}_{uu} = \sum_{e=1}^q \mathbf{R}_u^{eT} \mathbf{K}_{uu}^e \mathbf{R}_u^e, \quad (3.22a,b)$$

$$\mathbf{K}_{u\phi} = \mathbf{K}_{\phi u}^T = \sum_{e=1}^q \mathbf{R}_u^{eT} \mathbf{K}_{u\phi}^e \mathbf{R}_\phi^e, \quad \mathbf{K}_{\phi\phi} = \sum_{e=1}^q \mathbf{R}_\phi^{eT} \mathbf{K}_{\phi\phi}^e \mathbf{R}_\phi^e, \quad (3.23a,b)$$

$$\mathbf{f}_u = \sum_{e=1}^q \mathbf{R}_u^{eT} \mathbf{F}^e, \quad \mathbf{q} = \sum_{e=1}^q \mathbf{R}_\phi^{eT} \mathbf{Q}^e. \quad (3.24a,b)$$

Finally, the global equations of motion and charge equilibrium of the discrete system are given by the non-trivial solution of Equations (3.20) and (3.21), yielding

$$\mathbf{M}_{uu} \ddot{\bar{\mathbf{u}}}(t) + \mathbf{K}_{uu} \bar{\mathbf{u}}(t) + \mathbf{K}_{u\phi} \bar{\phi}(t) = \mathbf{f}_u(t), \quad (3.25)$$

$$\mathbf{K}_{\phi u} \bar{\mathbf{u}}(t) + \mathbf{K}_{\phi\phi} \bar{\phi}(t) = \mathbf{q}(t). \quad (3.26)$$

Therefore, the FE model of the beam with arbitrary ACLD treatments is described by the elemental matrices and vectors in Equations (3.16)-(3.18) and by the global FE equations of motion and charge equilibrium in Equations (3.25)-(3.26).

3.3 Sensing and Actuating Governing Equations

The electrical DoFs vector in Equations (3.25) and (3.26) can be partitioned into the actuating and sensing DoFs, $\bar{\phi}(t) = \{\bar{\phi}_a(t), \bar{\phi}_s(t)\}^T$, where the subscripts 'a' and 's' denote the actuating and sensing capabilities. Furthermore, the stiffness matrix can be written as the sum of the stiffness matrices of the elastic and piezoelectric layers, $\mathbf{K}_{uu}^E$ and $\mathbf{K}_{uu}^P$. Hence, considering the electrodes in open-circuit, and in that case $\mathbf{Q}(t) = 0$, the non specified potential differences in (3.26) can be statically condensed in (3.25), and the equations of motion and charge equilibrium become

$$\mathbf{M}_{uu} \ddot{\bar{\mathbf{u}}}(t) + (\mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{P*}) \bar{\mathbf{u}}(t) = -\mathbf{K}_{u\phi a} \bar{\phi}_a(t) + \mathbf{f}_u(t), \quad (3.27)$$

$$\bar{\phi}_s(t) = -\mathbf{K}_{\phi\phi s}^{-1} \mathbf{K}_{\phi\phi s} \bar{\mathbf{u}}(t), \quad (3.28)$$

where

$$\mathbf{K}_{uu}^{P*} = \mathbf{K}_{uu}^P - \mathbf{K}_{u\phi s} \mathbf{K}_{\phi\phi s}^{-1} \mathbf{K}_{\phi\phi s}^T. \quad (3.29)$$

It is worthy to note that a non-null through-the-thickness parabolic distribution of the electric potential within the piezoelectric layers was already considered in the variational formulation through the use of effective stiffness parameters. Moreover, the static condensation in Equation (3.29) only considers the linear term of the electrical potential distribution, which is the one that in fact contributes to the sensor output voltage [Vasques and Rodrigues (2005)].

3.4 Experimental Validation

In order to validate the FE model some measurements were taken on an aluminum beam with a partial ACLD treatment (viscoelastic layer sandwiched between the piezoelectric patch and the base beam). The aims were the validation of both the dynamics of the system and the actuating and sensing capabilities of the piezoelectric patches. Thus, three FRFs were measured experimentally and evaluated numerically: acceleration per unit force (accelerance), acceleration per unit voltage into the piezoelectric patch and induced voltage per unit force. The FE model implementation was realized in MATLAB.

The test rig comprised a beam that was 400 mm long, 2.92 mm thick and 30 mm wide, and a partial ACLD treatment that was 50 mm long and 30 mm wide, with viscoelastic and piezoelectric layers of thickness 0.127 mm and 0.5 mm, respectively. Taking one end of the beam to be $x = 0$, the ACLD treatment was positioned slightly off-center on the beam, starting at $x = 202$ mm, and was realized with a passive viscoelastic layer of a material manufactured by 3M (ISD112) and a piezoelectric constraining patch manufactured by PI (PIC 255).

The mechanical and electrical material properties of the aluminum beam, viscoelastic layer and piezoelectric patch are presented in Table 9.1. The shear storage modulus and loss factor of the viscoelastic material at the ambient temperature (22 °C) were extracted from the manufacturer's nomogram [3M (1993)] and are presented in Figure 3.2. The Young's storage modulus was obtained assuming a frequency independent Poisson's ratio equal to 0.45 and the viscoelastic damping capabilities were considered in the FE model through a CMA (see Chapter 6 and Section 6.6 for further details).

Table 3.1: Properties of the aluminium, viscoelastic layer and piezoelectric patch.

	Aluminium	3M ISD112	PIC 255		PIC 255
c_{11}^*	69.0×10^9 Pa	($\nu = 0.45$)	62.1×10^9 Pa	e_{31}^*	-11.18 C m ⁻²
c_{55}	26.5×10^9 Pa	3M (1993)	23.9×10^9 Pa	e_{15}	11.94 C m ⁻²
ρ	2700 kg m ⁻³	1130 kg m ⁻³	7800 kg m ⁻³	ε_{11}	12.6×10^{-9} F m ⁻¹
				ε_{33}^*	9.96×10^{-9} F m ⁻¹

Regarding the numerical analysis parameters, the beam was discretized into 500 equally spaced FEs, a shear correction factor equal to one was utilized for all the layers and an hysteretic damping model with a loss factor equal to 2×10^{-3} was considered.

The analysis concerned free-free boundary conditions. The beam was suspended on two wires attached to the beam approximately at 22.4% and 77.6% of the total length of the beam as schematically presented in Figure 3.3. The placement of the wires was chosen at the locations of the nodes of the first flexural mode which minimizes the interference of the attached wires in the system. The beam was excited by an impact hammer from PCB Piezotronics (model 086C03) with a hard tip (model 084B03) mounted on it. The impulsive force was applied at $x = 275$ mm and measured by the hammer's force sensor. The response (acceleration) was measured with a PCB Piezotronics accelerometer (model 352C22), with a frequency range up to 10 kHz and a weight equal to 0.5 g, placed on the other side of the beam at the same location of the excitation force.

In the measurements both the impact hammer and accelerometer were used in ICP mode and an 8-channel dynamic signal analyzer HP 35650 with a signal generator module was utilized for the data acquisition, signal generation and computations of the FRFs. Furthermore,

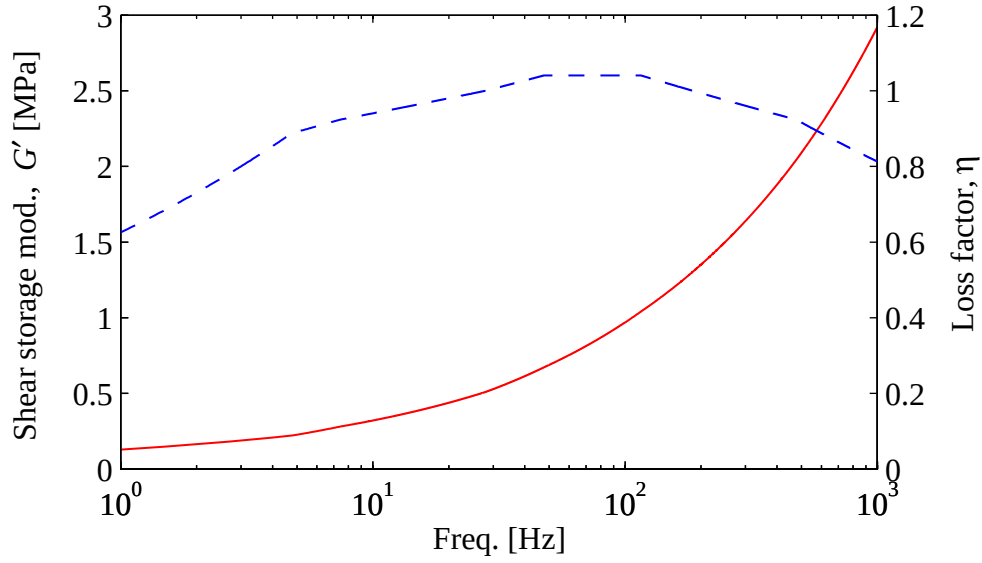


Figure 3.2: Frequency dependent properties of 3M ISD112 at 22 °C: loss factor (dashed line) and shear storage modulus (solid line).

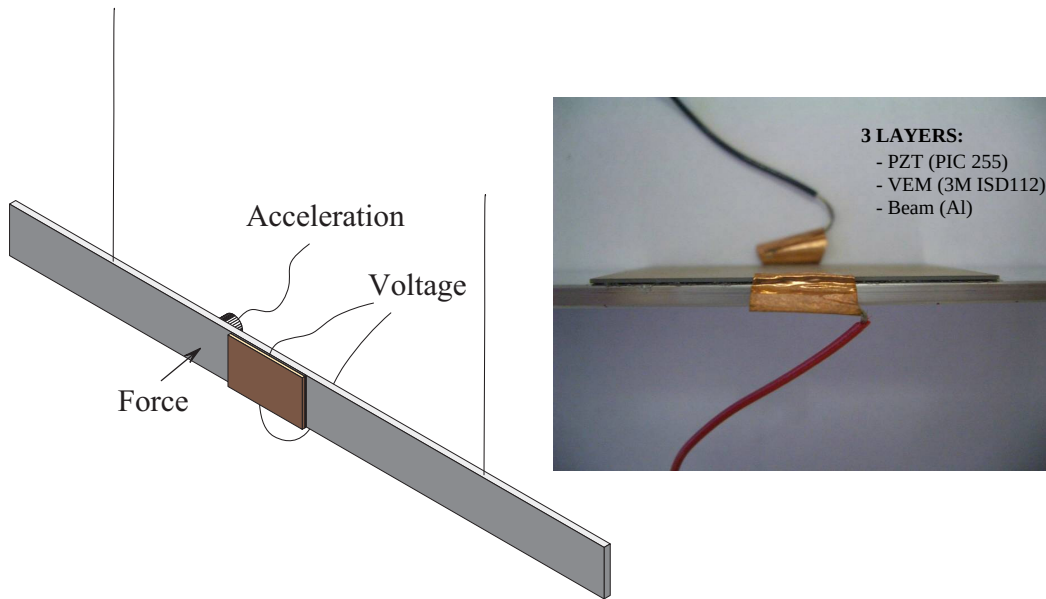


Figure 3.3: Test rig and ACLD patch.

in the analysis the frequency span was set to 1.6 kHz and a frequency resolution of 0.5 Hz was defined. The impact excitation was utilized for the measurement of the acceleration and induced voltage per unit force FRFs and 10 averages were considered in that procedure. For the acquisition of the piezoelectric patch (sensor) induced voltage a Brüel & Kjaer charge amplifier (type 2635) was utilized. Concerning the measurement of the acceleration per unit voltage into the actuator, the beam was excited with a stepped sine voltage into the piezoelectric actuator which was previously amplified by a PCB Piezotronics single channel power amplifier (AVC

instrumentation, series 790) with an output voltage range equal to 200 V. The acceleration was measured by the accelerometer which was kept at the same location as that referred to above.

As described in Section 6.6 the damping behavior of the viscoelastic layers is modeled by a CMA (complex modulus approach) and isothermal conditions are considered. Therefore, considering simple harmonic mechanical and electrical excitations, the frequency response model is generated from the solution of discrete frequency calculations of the equations of motion and charge equilibrium, where the complex stiffness matrix of the viscoelastic layers is re-calculated at each discrete frequency value. That constitutes one of the disadvantages of the CMA: the re-calculation and assemblage of the stiffness matrix anew at each frequency which, for relatively large problems, can become very time consuming.

What makes the present approach so useful is that the effects of the inherent damping of viscoelastic materials can easily be incorporated simply by using a frequency varying complex stiffness. Thus, the present methodology is particularly interesting when the characterization of the system under study in the frequency domain is necessary and enough. However, in certain situations, time domain analysis might be needed and in that case the use of inverse Fourier transforms of the FRFs to determine for example transient responses might be cumbersome or even impossible to obtain. In that case, as discussed in Part III, time domain models (e.g. GHM, ADF), with the drawback of including additional (dissipative) DoFs, which for relatively large problems might prohibitively increase the dimension of the problem, can alternatively be used.

The three measured and predicted FRFs, namely, acceleration per unit force (accelerance), acceleration per unit voltage into the piezoelectric patch and induced voltage per unit force, are shown in Figures 3.4 to 3.6. Furthermore, Figure 3.4 also presents the FE predicted accelerance of the bare beam without the ACLD treatment and a comparison of the natural frequencies and loss factors of the treated and untreated beam is presented in Table 3.2. As can be seen, there is a very good agreement between the predicted and measured FRFs of the treated beam. Small differences are found around the fourth natural frequency in all the FRFs and at some anti-resonances. The reason for that discrepancy might be related to uncertainties concerning the viscoelastic material properties and with the fact that the wavelength at the fourth natural frequency is closer to the ACLD patch length than at the lower natural frequencies. Furthermore, the ACLD patch is located between consecutive nodes of the fourth mode, where the modal strains are higher, and as a consequence the influence of the uncertainties of the viscoelastic material properties becomes more pronounced. Moreover, among other things, the accelerometer adds a little mass, the wires have some effect and the FE involves assumptions and approximations.

Table 3.2: Natural frequencies [Hz] (and loss factors [%]) of the first four modes.

Beam without treatment				
FE Predicted	94.94 (0.20)	261.2 (0.20)	512.0 (0.20)	845.8 (0.20)
Beam with the ACLD treatment				
Measured	92.00 (1.54)	258.5 (0.56)	502.0 (1.35)	835.5 (1.12)
FE Predicted	91.94 (1.12)	257.8 (0.50)	499.7 (0.86)	826.1 (0.98)
Deviation [%]	0.06 (27.3)	0.27 (10.7)	0.46 (36.3)	1.13 (12.5)

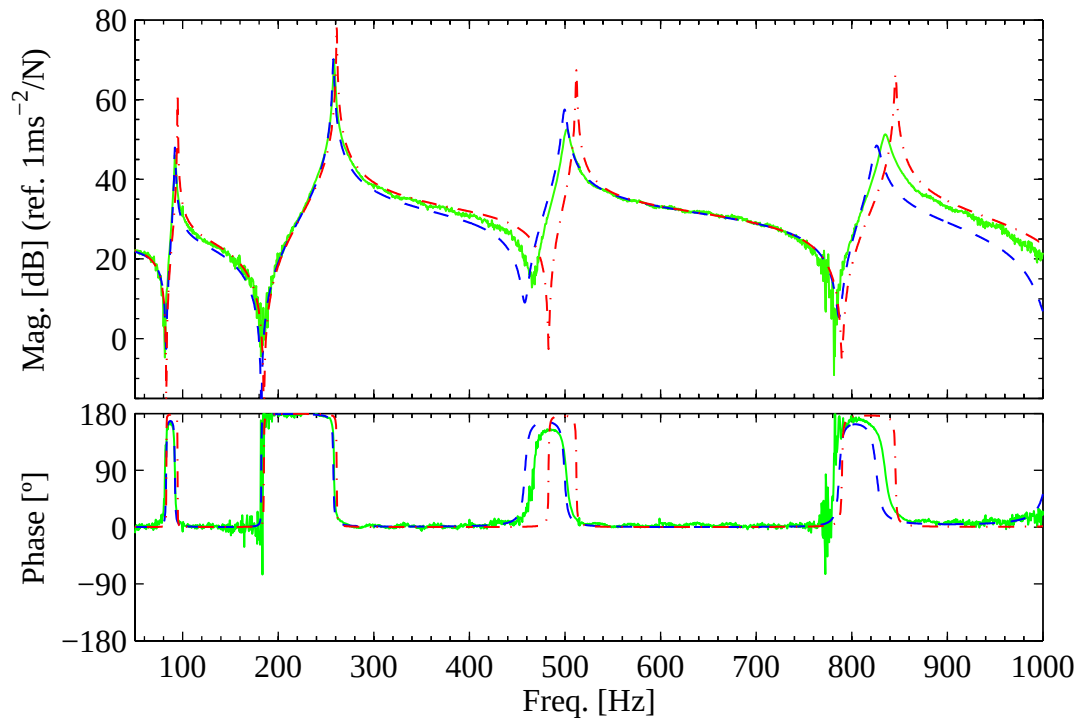


Figure 3.4: Frequency response function (acceleration per unit force) of the beam with and without ACLD treatment: FE predicted without treatment (dash-dot line), FE predicted with treatment (dashed line) and measured with treatment (solid line).

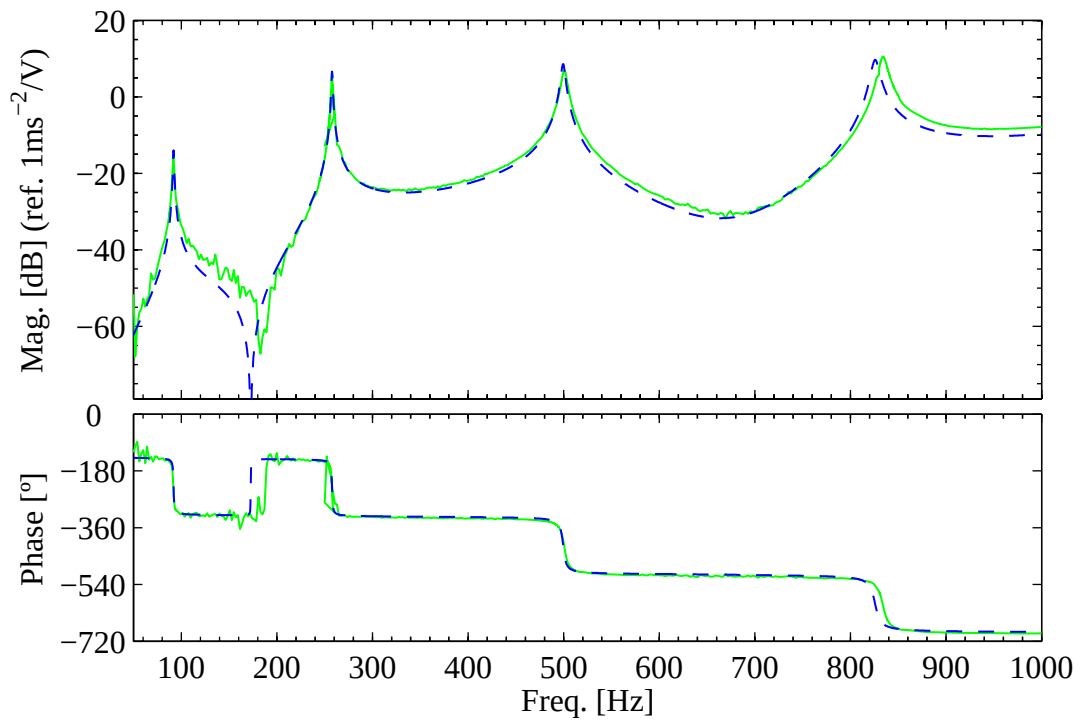


Figure 3.5: Frequency response function (acceleration per unit voltage) of the beam with ACLD treatment: FE predicted (dashed line) and measured (solid line).

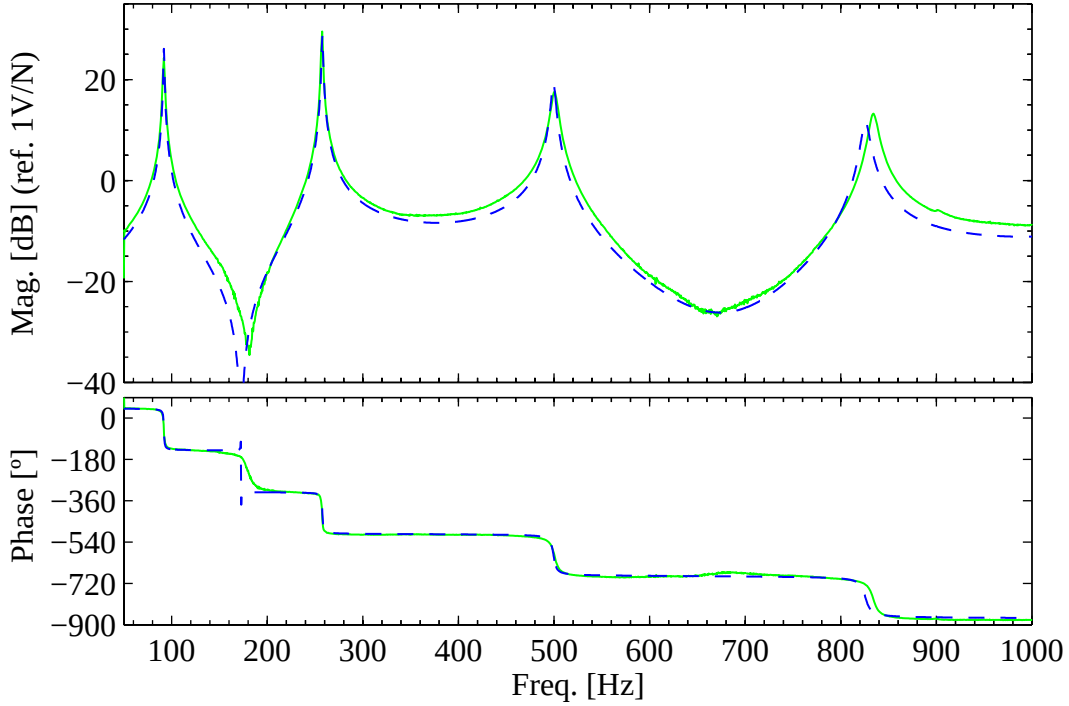


Figure 3.6: Frequency response function (voltage per unit force) of the beam with ACLD treatment: FE predicted (dashed line) and measured (solid line).

It is worth noting that a relatively small percentage of the beam (12.5%) is covered with the ACLD patch. When the beam is excited by the force the patch acts as a passive constraining layer. As shown in Figure 3.4 and Table 3.2, the layer adds significant damping: the measured and predicted damping loss factors, estimated by circle fitting [Ewins (2000)] and using characteristic response function procedures [Maia et al. (1994)], are in good agreement, especially for the second and fourth modes. Furthermore the patch adds mass (and, to a lesser extent stiffness) to the beam, hence reducing the natural frequencies somewhat.

The accelerance of the beam in Figure 3.4 is dominated by the properties of the host beam, except for damping which is dominated by the properties and geometry of the viscoelastic and covering piezoelectric patches. However the FRFs in Figures 3.5 and 3.6, which describe the behavior of the piezoelectric patch as an actuator and as a sensor respectively, also depend on the transmissibility between the beam and piezoelectric patch. The latter FRFs allow to infer about the modeling of the dynamics of the sandwiched viscoelastic layer. Therefore, since the agreement between the measured and predicted results is in general very good, both the dynamics of the system, beam and ACLD patch, and the actuating and sensing capabilities of the piezoelectric patch are satisfactorily validated.

3.5 Summary

The generic analytical formulation for the study of beams with arbitrary ACLD treatments proposed in the previous chapter was utilized here to the development of the underlying FE. A partial layerwise theory with explicit coupled-kinematics was considered for the definition of the displacement field leading to a first-order shear deformation theory (FSDT) for each layer; for

thick laminates this allows a more accurate representation of the intra- and inter-laminar effects and the strain and stress distributions become better approximations to the three-dimensional elasticity theory. For modeling the electric field and potential of the piezoelectric layers, a fully coupled electromechanical theory was utilized; the use of effective stiffness parameters allowed a representation of the system similar to a pure mechanical one to be obtained merely by modifying the elastic constitutive parameters. Regarding the complex frequency-dependent viscoelastic constitutive behavior, a simple CMA was utilized to model the damping behavior of the viscoelastic materials. The resultant weak forms, governing the motion and electrical charge equilibrium, were therefore used for the development of the FE model.

Thus, to sum up, the hybrid behavior of the elastic, viscoelastic and piezoelectric materials was considered in an accurate and consistent way and the presented FE model solution, with the well known advantages of the FE approximated method, allows the study of partial ACLD treatments of beams with more complex geometries. The FE model was validated against experimental results and a very good agreement was obtained validating both the dynamics of the system and the actuator and sensor capabilities of the piezoelectric layers. The results demonstrate the capacity and robustness of the proposed theory. Thus, the presented analytical formulation and FE model solution can be utilized for the study of arbitrary ACLD treatments on beams with various configurations and arrangements of the layers (more sophisticated hybrid active/passive damping treatments) and with different control strategies. However, while the material properties and damping are modeled in a precise way, the solution requires the re-calculation of the stiffness matrix of the viscoelastic layers at each frequency considered, which, for a relatively large number of degrees of freedom, can be very time consuming, requiring a high computational effort. For those reasons other alternatives to model the complex damping behavior of viscoelastic materials will be considered in the forthcoming chapters.

CHAPTER 4

Analytical Model of Piezo-Visco-Elastic Shells

4.1 Introduction

As reported by Felippa (2006), “The word *shell* is an old one and is commonly used to describe the hard covering of eggs, crustacea, tortoises, etc. The dictionary says that the word shell is derived from the Latin *scalus*, as in fish scale. But to us now there is a clear difference between the tough but flexible scaly covering of a fish and the tough but rigid shell of, say, a turtle.” In fact, a shell is a structural element that appears to us in our everyday life in many machines and structural systems, such as the sheet metal in automobile and the fuselage of aircraft, the housings of cellphones or washing machines, or the roof of an elegant building, to name just a few. Here, one will refer to man-made shell structures as used in the various branches of engineering and damping treatments applied to those shells will be considered.

Among the many interesting aspects of the use of shells in engineering, the *structural* aspect alone stands out as being of paramount importance and has “entertained”, and still is, the scientific community for a long time. In general, the structural behavior of shells is characterized by two principally different states of deformation, namely, membrane and bending action. Taking advantage of the high membrane stiffness characteristics, if properly designed as load-carrying structures, the structural behavior should be dominated by membrane action, where bending and shear effects may be present only due to unavoidable circumstances (e.g. concentrated forces or boundary effects), and typically exhibit an optimal stiffness to weight ratio; on the other hand, shells, especially thin ones, can be extremely sensitive to imperfections. The mixture of the latter characteristics makes shells being denoted as the *primadonna of structures* due to the difficulties and challenges encountered when modeling and designing them.

The *theory of structures* tends to deal with abstract mathematical models stripped of many of the features that make them recognizable as representative models of structural elements in engineering practice. Thus a *beam* is often idealized as a line endowed with certain mechanical and geometrical properties, irrespective of whether it is a large bridge or an aircraft wing, which is modeled as a 1-D system. In a similar way, the theory of shell structures tends to deal also with idealized 2-D mathematical models of surface entities endowed with certain mechanical and geometrical properties which are representative of the 3-D shell structure (e.g. conical, cylindrical and spherical shells which are idealized as the correspondent surfaces endowed with certain mechanical and geometrical properties). This makes the treatment the same irre-

spective of the type of shell structure being considered.

Mathematical models of shells are constructed in principle following the same general idea used for flat plates. The actual shell, which is a 3-D object, is reduced to a surface endowed with certain mechanical and geometrical properties where the transfer of such properties onto the surface is done by making suitable kinematic assumptions similar to those made in the theory of plates. As a result of these assumptions one obtains a set of mechanical properties expressed, in the case of an elastic material, in the form of a generalized Hooke's law relating the deformation of a small material element to the stresses applied to it (*constitutive equations*). Subsequent steps in the theory of shells are similar to those followed for beams and plates: obtain *equilibrium equations* relating the stress resultants in the structure to the applied external forces, and kinematic relations, also called *compatibility equations*, that connect strains and displacements. For the study of the dynamic effects, as in vibration problems, the equilibrium equations include inertial and possibly damping terms. Those equations can be obtained directly from the balance of forces and momentum or, alternatively, from variational methods (e.g. Hamilton's principle and Principle of Virtual Works) which may take a slightly different form from the classical mechanical formulation when considering piezoelectric continua.

In linear shell theory the equations of equilibrium and compatibility are written in terms of the initial geometry of the structure. This assumption necessarily restricts the displacements to be very small and the so-called *infinitesimal strains* are used (or *engineering strains*).

The three sets of field equations: kinematic, constitutive and equilibrium, along with appropriate boundary conditions, comprise the governing equations of the mathematical model. What makes the shell problem more complex is the fact that the equations have to be set up with respect to a generally curved surface in a 3-D space which is usually done resorting to tri-orthogonal curvilinear coordinate systems. By comparison, in the theory of flat plates, the governing equations are set up merely over a planar surface, which of course might be obtained from the shell equations by imposing certain geometrical simplifications, i.e., infinite radii of curvature.

Loads applied to a shell structure are carried in general through a combination of bending and stretching actions, which generally vary from point to point of application. Its resistance to deformation in terms of separable stretching and bending effects are described by the mechanical and geometrical properties of the shell structure. One of the major difficulties in the theory of shells is to find a relatively simple way of describing the interaction between the two effects (stretching and bending). This aspect of the theory has been troublesome from the beginning. Rayleigh (1881) argued that the deformation of a thin hemispherical bowl would be primarily inextensional, and accordingly he developed a special method of analysis which took into account only the bending energy of the shell. On the other hand, Love (1888) argued that for thin shells stretching was the dominant effect. At that time Love had not grasped the strong contrast between open and closed shells. The controversy was resolved by Lamb (1890) and Basset (1890b), who solve Love's general equations for a cylindrical shell and demonstrated the possibility of a narrow boundary layer in which there was a rapid transition between bending and stretching effects. The width of the layer was determined by the interaction between those effects.

Shell mathematical models, as in the case of plates, might be classified in terms of the ratio of the thickness to a characteristic dimension (e.g. radii of curvature for shells or side length for plates and shells):

- Very thick (3-D effects dominate);
- Thick (stretching, bending and higher order transverse shear effects are the most rele-

vant);

- Moderately thick (stretching, bending and first order transverse shear are the most relevant);
- Thin shells (stretching and bending energy considered but transverse shear neglected);
- Very thin shells (dominated by stretching effects; also called *membranes*).

Depending of the characteristic dimension value, different deformation theories have been proposed to model these structures with enough accuracy to capture the global (macroscopic) behavior. Therefore, if one is interested in the local behavior and more complex state of strain and stress are under analysis, 3-D theories are more appropriate and are usually employed. However, as far as computational efficiency and cost are concerned, they imply a burden (which usually is prohibitive for practical problems) and instead alternative simplified refined theories might be envisaged, when appropriate, without a significant loss of accuracy. It is up to the designer and analyst to define which theory and simplifying assumptions should be considered to the problem to be analyzed.

Obviously, the first approach for the analysis of shells can be implemented by solving, in a strong or weak sense, the fundamental differential equations of 3-D elasticity, which are summarized by the equilibrium and compatibility equations, and by the physical constitutive relations. If the problem is grasped by the direct solution of the 3-D differential equations, with its inherent cumbersome complexity, "exact" information about the 3-D strain and stress state is provided. However, in general, these solutions are difficult, if not impossible to obtain, and they cannot be given in strong form for the most general case of geometry, laminate layout, boundary and loading conditions [Carrera (2002)]. Some well known and vastly used 3-D solutions were obtained, among others, for example by Pagano and his co-workers [Pagano (1969, 1970), Pagano and Hatfield (1972), Noor (1973), Ren (1987), Ye and Soldatos (1994), Reddy (1994)].

It is well known that physical 3-D shells are usually modeled recurring to approximated 2-D mathematical representations. They are obtained by imposing some chosen kinematic and mechanical assumptions to the 3-D continuum and by explicitly assuming a through-the-thickness axiomatic displacement field definition. When compared to 3-D solid brick FEs, 2-D shell FEs allow a significant reduction of the computational cost without losing much accuracy. However, besides the FEM, other analytical or numerical solution methods might be used (e.g. state space methods, assumed modes method and closed-form solutions). For the modeling of multiphysics structures see the review performed by Benjeddou (2004).

Following the same line of thought of the previous chapters on damped multi-layered beams, this chapter is concerned with the mathematical modeling of multi-layered shells with hybrid (active/passive), or arbitrary ACLD, damping treatments, incorporating an arbitrary number of elastic, viscoelastic and piezoelectric layers. Different deformation theories are reviewed and discussed in their effectiveness to conveniently represent the mechanical, electrical and damping behavior of complex multi-layered shell structures. An analytical formulation is presented for the electro-mechanical behavior and both weak and strong forms are derived. The weak forms are used in Chapter 5 to obtain the solution by means of the FEM using a "regeneration" approach and a discrete-layer model with an a priori uncoupled-kinematics. Since plates are particular cases of shells, and beams particular cases of plates, the theories will be presented here for the shell case and assumed extensible to plates and beams. Regarding the viscoelastic damping behavior, in a similar way to what has been considered in Chapter 2 for

beams, the frequency-dependent viscoelastic damping model will be considered by means of a CMA where the "elastic" moduli are complex. Using the CMA, the resulting piezo-visco-elastic model is formally identical to the piezo-elastic one, making the extension and application of the piezo-elastic model to structures with viscoelastic layers more straightforward.

4.2 Review of Modeling Approaches

The finite element (FE) method is usually the preferable way of obtaining solutions for structures with more complicated geometries, boundary conditions and applied loads types. "Perfect" FEs of general shell structures without any numerical pathologies are still an issue to be solved by the scientific community and a challenge to deal with. However, models have been developed to simulate the structure and damping treatments and to assess their performance. The ultimate aim of this simulation is to define the optimal configuration (e.g., thickness, locations, number and type of sensors and actuators, control law, etc.) and put them in practice, making them work in order to solve some of the problems involving noise and vibration issues that our society nowadays has to deal with. Hence, one should also be familiarized with control theory aspects both for simulation or real-time implementation. These reasons attest the interesting multidisciplinary and attracting challenge of modeling this type of coupled physical system, already reasonably studied over the last two decades, however with many aspects that still need to be solved. Further refinement of the models, fully coupling the physical ingredients, and coming up with some new approaches should be the main tendencies. There is still much work to be done and this research issue is far from being fully understood.

The derivation of shell theories has been one of the most prominent challenges in solid mechanics for many years. The idea is to develop appropriate models that can accurately simulate the effects of shear deformations and transverse normal strains in laminated shells with good trade-off between accuracy and complexity, which is a big mathematical difficulty. Physical 3-D shells are usually modeled recurring to approximated mathematical 2-D models. They are obtained by imposing some chosen kinematic and mechanical assumptions to the 3-D continuum, e.g. by explicitly assuming a through-the-thickness axiomatic displacement field definition and assuming a plane-stress state. When compared to 3-D solid FEs, 2-D shell FEs allow a significant reduction of the computational cost without losing much accuracy. However, to make matters worse, this sort of approximations lead to so-called *locking* effects (e.g. shear and membrane locking), which produces an overstiffening of the FE model which in turn produces erroneous results. Furthermore, for shell-type structures, with a more complex shear-membrane-bending coupling behavior, the locking effects are not yet fully understood, making these numerical pathologies difficult to remedy. Fully modeling the viscoelastic damping behavior is another important difficulty since their constitutive behavior is temperature and frequency dependent. Usually, for simplicity, only the frequency dependent material properties are considered and isothermal conditions are assumed.

When modeling this kind of structural system, various aspects have to be taken into account: the advances and alternative formulations of representative mathematical models of the physical behavior of shells with damping treatments; the developments (e.g., improved alternative configurations, control approaches, etc.) that the damping treatments suffered in the last decades; the developments and alternative approaches to viscoelastic damping modeling; considerations concerning the electro-mechanical coupling of piezoelectric layers; and the implementation and development of simple and reliable, locking-free, easily implemented FE solution methods. Over the years many authors have developed models for this type of

problem. It's very difficult to comprehensively review their contributions because the multidisciplinary nature of the subject makes the literature vast and the technical developments and improvements to appear dispersed in technical journals of different areas. However, the major disciplines, considered relevant for this work, are shortly reviewed here, namely: (1) elastic deformation theories, (2) piezoelectric materials and electro-mechanical coupling assumptions, (3) viscoelastic damping, (4) coupled piezo-visco-elastic FE formulations and (5) control strategies. This will allow to introduce the subject of this chapter and to justify the options and assumptions taken for developing the model. Furthermore, it will allow to identify the aspects of the work that are new and significant in a more founded way.

In the development of structural mathematical models, different theories have been considered to axiomatically define the kinematics of laminated structural systems, where the planar dimensions are one to two orders of magnitude larger than their thickness (e.g., beams, plates or shells). Usually, these structural systems are formed by stacking layers of different isotropic or orthotropic composite materials with arbitrary fiber orientation or then, in this specific case, by arbitrary stacking sequences of active piezoelectric or passive viscoelastic damping layers.

These theories, following Kraus (1967), were originally developed for single layer "monocoque" thin structures made of traditional isotropic materials. Generally speaking, they can be grouped into two classes of alternate theories: one in which all of the Love's original assumptions (Love, 1944) are preserved, and other, following higher-order linear theories in which one or another of Love's assumptions are suspended. Many additional theories of thin and thick elastic shells have been proposed and the chronicle of these efforts are presented for example by Naghdi (1956) and Leissa (1993).

Two different approaches are often used. The first one, the so-called *equivalent single layer theories* (ESLTs), where the number of independent generalized variables doesn't depend of the number of layers, are derived from 3-D elasticity by making suitable assumptions concerning the kinematics of deformation or the stress state through the thickness of the laminate, allowing the reduction of a 3-D problem to a 2-D one. The second one, the so-called *layerwise theories* (LWTs) or *discrete-layer theories* (DLTs), where the number of generalized variables depends on the number of physical (or nonphysical) layers, rely on the basis that the kinematic assumptions are established for each individual layer, which might be modeled (or not) as a 3-D solid. The problem is then reduced to a 2-D problem, however, retaining the 3-D intralaminar and interlaminar effects.

As reported by Yang et al. (2000), plate and shell structures made of laminated composite materials have often been modeled as an ESL using the *classical laminate theory* (CLT) [see for example the textbook of Reddy (2004)], in which the out-of-plane stress components are ignored. The CLT is a direct extension of the well-known Kirchhoff-Love kinematic hypothesis, i.e., plane sections before deformation remain plane and normal to the mid-plane after deformation and that normals to the middle surface suffer no extension (Kirchhoff contribution) and others [cf. Leissa (1993)], however applied to laminate composite structures. This theory is adequate when the ratio of the thickness to length (or other similar dimension) is small, the dynamic excitations are within the low-frequency range and the material anisotropy is not severe. The application of such theories to layered anisotropic composite shells could lead to 30% or more errors in deflections, stresses and frequencies (Reddy, 2004). In order to overcome the deficiencies in the CLT, new refined laminate theories have been proposed relaxing some of the Love's postulates according to Koiter's recommendations (Koiter, 1960), where it is stated that "... a refinement of Love's approximation theory is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time." However, as stated by Carrera (1999), for 2-D modelings of multi-layered structures (such as laminated constructions,

sandwich panels, layered structures used as thermal protection, intelligent structural systems embedding piezoelectric and/or viscoelastic layers) amendments to Koiter's recommendation are required. Among these, the inclusion of continuity of displacements, *zig-zag* effects, and of transverse shear and normal stresses inter-laminar continuity at the interface between two adjacent layers, are some of the amendments necessary. The role played by *zig-zag* effects and inter-laminar continuity has been confirmed by many 3-D analysis of layered plates and shells [Srinivas et al. (1970), Srinivas (1974), Ren (1987), Varadan and Bhaskar (1991), Bhaskar and Varadan (1994), Soldatos (1994), Reddy (1994)]. These amendments become more significant when complicating effects such as high in-plane and/or out-of-plane transverse anisotropy are present. Hence, as referred by Carrera (1999), Koiter's recommendation concerning isotropic shells could be re-written for the case of multi-layered shells as "*... a refinement of ... unless the effects of interlaminar continuous transverse shear and normal stresses are taken into account at the same time.*" This enforces the need of also assuring interlaminar continuity of the out-of-plane stresses, i.e. the C_z^0 requirements defined by Carrera (1997).

A refinement of the CLT, in which the transverse shear stresses are taken into account, was achieved with the extension to laminates of the so-called *Reissner-Mindlin theory*, or *first-order shear deformation theory* (FSDT). It provides improved global response estimates for deflections, vibration frequencies and buckling loads of moderately thick composites when compared to the CLT [see Reddy (2004)]. Both approaches (CLT and FSDT) consider all layers as one anisotropic ESL and, as a consequence, they cannot model the warping effect of cross-sections. Furthermore, the assumption of a non-deformable normal results in incompatible shearing stresses between adjacent layers. The latter approach, because it assumes constant transverse shear stress, also requires the introduction of an arbitrary shear correction factor which depends on the lamination parameters for obtaining accurate results. As reported by Reddy and Arciniega (2004) in a review of SDTs for plates and shells, the first ESL theory for laminated isotropic plates including shear deformation effects is due to Stavsky in 1959 and is known as the YNS theory (which stands for Yang, Norris, and Stavsky). The pioneer YNS theory represents an extension of Reissner-Mindlin plate theory for homogeneous isotropic plates to arbitrarily laminated anisotropic plates and includes shear deformation and rotatory inertia effects. A comparison of different SDT for laminated plates was also performed by Sun and Whitney (1973).

Such a theory is adequate to predict only the gross behavior of laminates. *Higher-order theories* (HOTs), overcoming some of these limitations, were presented for example by Reddy and Liu (1985) and Reddy (1990a) for laminated plates and shells. However, because of the material mismatch at the intersection of the layers, the HOTs also lead to transverse shear and normal stress mismatch at the intersection. In conclusion, ESL theories are found to be inadequate for detailed, accurate, local stress analysis of laminated structures.

If detailed response of individual layers is required, as is the case for piezoelectric layers, and if significant variations in displacements gradients between layers exist, as is the case of local phenomena usually in viscoelastic layers, LWT (discrete layer) become more suitable to model the intralaminar and interlaminar effects and the warping of the cross section. The LWT corresponds to the implementation of CLT, FSDT or HOT at a layer level. That is, each layer is seen as an individual plate or shell and compatibility (continuity) of displacement (and eventually out-of-plane stress) components with correspondence to each interface is imposed as a constraint. As reported by Garção et al. (2004) and Lage et al. (2004), high-order displacement-based or mixed LWT have been successfully used to accurately model the behavior of laminates taking into account the interlaminar and intralaminar effects.

Another alternative, with a reduced computational effort, in the framework of ESL theories,

is the use of the so-called *zig-zag theories* (ZZTs), which have their origins and most significant contributions coming from the Russian school. Refined ZZTs have therefore been motivated to fulfill a priori (in a complete or partial form) the C_z^0 requirements. The fundamental ideas in developing ZZT consists to assume a certain displacement and/or stress model in each layer and then to use compatibility and equilibrium conditions at the interface to reduce the number of the unknown variables and keep the number of variables independent of the number of layers.

As stated by [Carrera \(2003a\)](#), the first contribution to the ZZTs was supposedly given by Lekhnitskii in the 1930s (see the brief treatment concerning a layered beam in the English translation of his book [[Lekhnitskii \(1968, Section 18\)](#)]). Apart from the method proposed by Lekhnitskii, which was almost ignored, two other independent contributions, which received much more attention from the scientific community, have been proposed in the literature in the second half of last century. The first of these was originally given by Ambartsumian in the 1950s, motivated by the attempt to refine the CLT to include partially or completely the C_z^0 requirements, and was applied to anisotropic single and multilayer plates and shells (see his textbooks [[Ambartsumyan \(1991\)](#), [Ambartsumian \(1991\)](#)]). Several variations have been presented which consisted in direct or particular applications of the original Ambartsumian's idea. [Ashton and Whitney \(1970, Chapter 7\)](#) introduced the theory in the Western community and applied it to non-symmetrical plates whereas the extension to multilayer shells and dynamic problems was made by [Rath and Das \(1973\)](#). However, several useless works concerning particular applications of Ambartsumian's original theory were developed presenting progressive refinements towards the original idea. It was only in the 1990s that the original theory was re-obtained as can be found, among others, in the works of [Cho and Parmerter \(1993\)](#), [Beakou and Touratier \(1993\)](#) and [Soldatos and Timarci \(1993\)](#). Regarding the second independent contribution, it was given by Reissner in 1980s [[Reissner \(1984\)](#)], who proposed a mixed variational theorem that allows both displacements and stress assumptions to be made. Significant contributions to the theory proposed by Reissner were made by [Murakami \(1986\)](#) that introduced a zig-zag form of displacement field and [Carrera \(2004\)](#) that presented a systematic generalized manner of using the Reissner mixed variational principle to develop FE applications of ESL theories and LWT of plates and shells. For further details, an historical review of ZZT was performed by [Carrera \(2003a,b\)](#). A discussion on the theories and FE for multilayered structures, with numerical assessment and a benchmarks for plate and shell structures can be found in the literature [[Noor and Burton \(1989\)](#), [Noor et al. \(1996\)](#), [Ghugal and Shimpi \(2001, 2002\)](#), [Carrera \(2002, 2003b\)](#), [Reddy and Arciniega \(2004\)](#)].

When modeling piezoelectric sensors and actuators, different electrical assumptions can be taken into account in the theoretical model when considering the electro-mechanical (or piezo-elastic) coupling. These assumptions regard mainly the use (or not) of electric degrees of freedom (DoFs) and the approximations of the in-plane and out-of-plane components of the electric displacement vector and/or through-the-thickness variation of the electric potential. Therefore, they might lead to *decoupled*, *partial* and *fully coupled* electro-mechanical theories, which in turn can lead to different modifications of the structure's stiffness and different approximations of the physics of the system. Those electro-mechanical coupling theories can be considered by the use of *effective stiffness parameters* (ESP), defined according to the electric boundary condition considered, as shown in [[Krommer and Irschik \(1999, 2000\)](#), [Vasques and Rodrigues \(2005\)](#)] for a smart piezoelectric beam and plate. This ESP formulation, however, can also be extended to other types of structures such as shells. Interesting textbooks concerning the piezoelectric modeling of beams, plates and shells can be found in [[Tiersten \(1969\)](#), [Tzou \(1993\)](#), [Rogacheva \(1994\)](#)].

This Chapter concerns the mathematical modeling and FE solution of general anisotropic shells with hybrid active-passive damping treatments. A fully-coupled piezo-visco-elastic mathematical model of the shell (host structure) and segmented arbitrarily stacked layers of damping treatments is considered in the framework of a discrete layer approach. Thus, the weak form of the governing equations is derived for a single generic layer of the multilayer shell using Hamilton's principle and a mixed (displacement/stresses) definition of the displacement field.

First, a *fully refined* deformation theory of the generic layer, based on postulated out-of-plane shear stress components and the in-plane stresses obtained with a Reissner-Mindlin type shell theory, is outlined. A *semi-inverse* iterative procedure is used to derive the layer *mixed* non-linear displacement field, in terms of a blend of the generalized displacements of the Love-Kirchhoff and Reissner-Mindlin theories and stress components at the generic layer interfaces, without any simplifying assumptions regarding the thinness of the shell being considered. The electrical potential is defined assuming negligible in-plane electric displacement field components and a constant transverse electric displacement. Regarding the electro-mechanical coupling, a fully coupled theory is considered where the *direct piezoelectric effect* is adequately taken into account and the *converse piezoelectric effect* is considered by the action of a prescribed electric potential difference in the generic piezoelectric layer surface electrodes.

Then, the weak forms of a *partially refined* theory, where only the zero-order term of the non-linear *fully refined* transverse displacement is retained, are derived for an orthotropic doubly-curved piezo-elastic generic layer. Based on the weak forms a FE solution is developed for the generic shell layer. The degrees of freedom (DoFs) of the resultant four-noded generic piezo-elastic single layer FE are then "regenerated" into an equivalent eight-node 3-D formulation in order to allow through-the-thickness assemblage of displacements and stresses and the generation of a *partially refined* multilayer piezo-elastic FE. A dynamic condensation technique is employed to eliminate the stress DoFs and to cast the problem in an equivalent displacement-based FE model form.

Regarding the deformation theory developed in this work, it is inspired in Ambartsumian's contributions for the deformation theory of single layer anisotropic plates and shells [Ambartsumian (1958), Ambartsumyan (1991), Ambartsumian (1991)]. Ambartsumian basically used a *semi-inverse* method to develop refined shear deformation theories. They are based on assuming a refined distribution of the transverse shear stresses and in the use of the equilibrium and constitutive equations to derive expressions for the in-plane displacements, which in turn become non-linear in the thickness coordinate. Improvements including the effect of transverse normal strain were also considered for plates and shells. In some of these refined theories transverse shear stress distributions are assumed to follow a parabolic law and to satisfy zero shear stress conditions at the top and bottom surfaces of the plate or shell. However, if out-of-plane stresses continuity is required, a mixed formulation should consider also the stresses on the interfaces of the single layer, which, for the sake of simplicity, were, in general, assumed to be nil in the single layer theories developed by Ambartsumian. That puts some limitations in the generalization of the theory to multilayer structures since neither the displacement nor the normal stresses are available in the interfaces and interlayer continuity can not be imposed. Furthermore, his multilayer approach doesn't allow to consider segmented layers, which is something required in the study of segmented hybrid damping treatments. When that is the case, individual refined theories must be considered for each individual discrete-layer.

When compared with Ambartsumian's *first* and *second improved theories* of anisotropic shells [see Ambartsumian (1991)], the proposed *fully* and *partially refined* theories, similar to Ambartsumian's *first* and *second improved theories*, respectively, assume as a first approximation of the

in-plane stresses the ones obtained with the FSDT instead of the CLT. Additionally, all the surface shear stresses are retained in the formulation since they will be used afterward with the "regenerated" 3-D element to generalize the theory to segmented multilayered shells. Furthermore, the theory is extended to coupled piezo-visco-elastic multilayered shells and a FE solution is developed. Thus, however strongly inspired in Ambartsumian's work, the underlying deformation theory of the present work has some important differences and represents a further step towards the increasing refinement of this type of multiphysics problem. It is worthy to mention that it would be very complicated and cumbersome to fulfill a priori all the C_z^0 requirements for a multilayer anisotropic shell. Instead, a discrete layer approach is used, which allows interlayer displacement and out-of-plane stresses continuity to be imposed a posteriori in a more straightforward manner, by means of a through-the-thickness assemblage of the "regenerated" single layer FE.

Regarding the variational approach used to get the governing equations, similar analytical works using "partial" mixed theories where the displacement field is defined in terms of generalized displacements and generalized surface and transverse stresses can be found for beams and plates in the works of Rao and colleagues [Rao et al. (2001), Rao and Desai (2004), Rao et al. (2004)]. In contrast to other "fully" mixed methods, as the ZZT based on Reissner's contribution, where mixed variational principles are used, Hamilton's principle has also been employed to derive the governing equations (i.e., no mixed-enhanced variational principles are considered). The present work extends a similar concept to multilayered shells and a FE solution is developed. Concerning piezo-visco-elastic FEs, a similar work about a three-layered coupled piezo-visco-elastic plate FE was developed by Chattopadhyay et al. (2001), where a HOT was used for the definition of the displacement field of each individual layer and the displacement and shear stress continuity were assured. However, the formulation is limited to the study of active constrained layer damping (ACLD) treatments on plates and doesn't allow the study of arbitrary damping treatments (multilayer structures). As far as the "regeneration" concept is concerned, the concept has been employed also by Cho and Averill (2000) in the framework of the ZZTs, where a plate FE, avoiding the shortcomings of requiring C^1 continuity of the transverse displacement, has been developed using a first-order zig-zag sublaminate theory for laminated composite and sandwich panels. However, the formulation is based on the decomposition of the whole structure in sublaminate, a linear piecewise function is assumed for the displacement of each sublaminate and the interlaminar displacement and normal shear stress continuity is imposed between the layers of the same sublaminate but not between the sublaminate. In comparison, the present work uses a similar "regeneration" concept to shell-type structures with a more refined deformation theory being employed which allows displacement and interlayer continuity between all layers to be imposed.

To sum up, this chapter will reveal some new developments to the study of shells with hybrid damping treatments. Only a few articles concerning hybrid active-passive damping treatments in shells are found in the open literature. They were mainly devoted to simple shell geometries and a few to arbitrary shells. Therefore further studies on shells are necessary. Here one intends to develop an accurate mechanical structural model of shells, considering also an accurate model of the electro-mechanical coupling and an effective time domain damping model of the viscoelastic layers. Furthermore, works in the open literature are usually devoted to three-layered sandwich shells and usually concern only modeling aspects.

4.3 Fully Refined Mathematical Model of General Shells

4.3.1 Physical Problem Description

The physical problem to be mathematically modeled here concerns a general anisotropic shell to which damping treatments of piezoelectric and viscoelastic materials are attached on both top and bottom surfaces of the host structure (Figure 4.1). All the damping layers and the host structure have constant thickness. The damping treatments are required to be able to be discontinuous (or segmented) and arbitrary stacking sequences of hybrid damping treatments comprising piezoelectric sensing or actuating, viscoelastic or elastic layers are considered. Isotropic viscoelastic materials are used and the piezoelectric materials are general orthotropic materials of the class $mm2$ [Nye (1985), IEEE (1988)]. The system is supposed to operate under moderate changing thermal environments, well below the Curie temperature of the piezoelectric material.

Thus, the physical problem involves visco-elastic (or mechanical) and piezo-elastic (or electro-mechanical) interactions mainly devoted to increase the energy dissipation of the structural system.

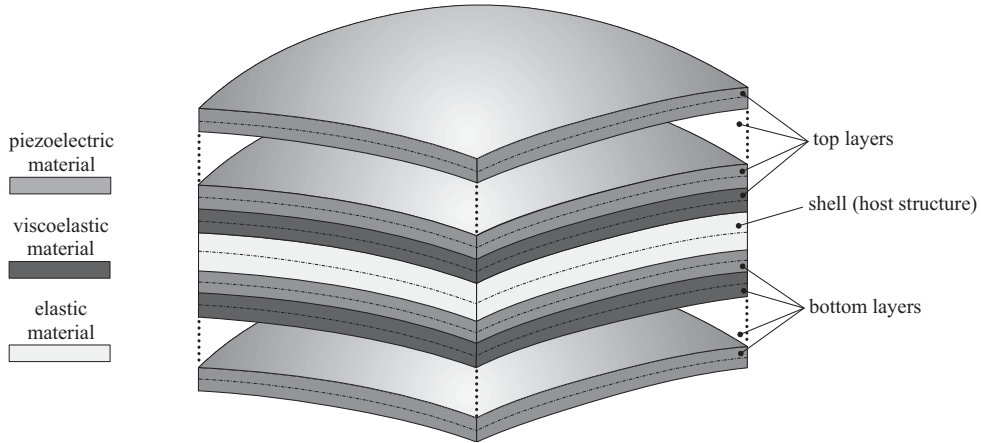


Figure 4.1: Multilayer shell with arbitrary hybrid damping treatments.

4.3.2 Shell Differential Geometry

A theory is established by treating the shell with hybrid damping treatments as a multilayer shell of arbitrary materials. Then, a generic layer (elastic, viscoelastic or piezoelectric) is taken from the global laminate, and the weak form of the governing equations is derived for the individual layer. Interlaminar (or interlayer) continuity conditions of displacements and stresses (surface tractions) and homogeneous (or not) traction boundary conditions at inner and outer boundary surfaces of the global laminate (or damped shell) are imposed later, at the multilayer FE level, by assembling all the individual layers contributions.

Consider a generic piezo-visco-elastic layer (Figure 4.2) extracted from the multilayer shell in Figure 4.1. The general orthotropic (anisotropic) layer has a constant thickness of $2h$ and a plane of elastic symmetry parallel to the middle surface Ω_0 . The latter surface is used as a reference surface referred to the curvilinear orthogonal coordinates α and β , which coincide with the lines of principal curvature of the middle surface. Let z denote the distance comprised

in the interval $[-h, h]$ and measured along the normal of a point (α, β) of the middle surface Ω_0 and a point (α, β, z) of the shell layer and Ω a surface at a distance z and parallel to Ω_0 . The square of an arbitrary differential element of arc length ds , the infinitesimal area of a rectangle in Ω and an infinitesimal volume dV are given by

$$ds^2 = H_\alpha^2 d\alpha^2 + H_\beta^2 d\beta^2 + H_z^2 dz^2, \quad (4.1a)$$

$$d\Omega = H_\alpha H_\beta d\alpha d\beta, \quad (4.1b)$$

$$dV = H_\alpha H_\beta H_z d\alpha d\beta dz, \quad (4.1c)$$

where $H_\alpha = H_\alpha(\alpha, \beta, z)$, $H_\beta = H_\beta(\alpha, \beta, z)$ and H_z are the so-called *Lamé parameters* given by

$$H_\alpha = A_\alpha \left(1 + \frac{z}{R_\alpha} \right), \quad (4.2a)$$

$$H_\beta = A_\beta \left(1 + \frac{z}{R_\beta} \right), \quad (4.2b)$$

$$H_z = 1. \quad (4.2c)$$

$A_\alpha = A_\alpha(\alpha, \beta)$ and $A_\beta = A_\beta(\alpha, \beta)$ are given by the square root of the coefficients of the *first fundamental form* and $R_\alpha = R_\alpha(\alpha, \beta)$ and $R_\beta = R_\beta(\alpha, \beta)$ the principal radii of curvature of the middle surface Ω_0 [see for example [Kraus \(1967\)](#) for further details].

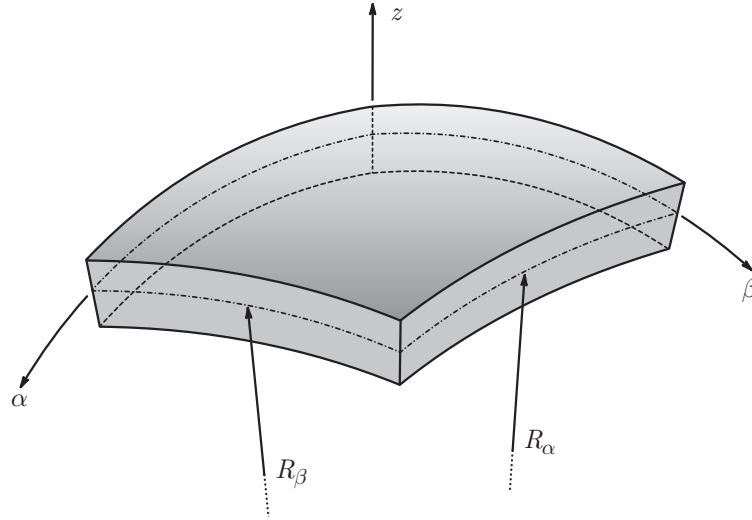


Figure 4.2: Generic piezo-visco-elastic shell layer.

4.3.3 General Assumptions and Relationships

Different assumptions are made regarding the mechanical, electrical and electro-mechanical behavior. By now, one will be only concerned with the mechanical assumptions which are as follows:

- (a) Strains and displacements are sufficiently small so that the quantities of the second- and higher-order magnitude in the strain-displacement relationships may be neglected in comparison with the first-order terms (infinitesimal strains) and linear elasticity is considered;

- (b) The shear stresses $\sigma_{z\alpha}(\alpha, \beta, z)$ and $\sigma_{\beta z}(\alpha, \beta, z)$, or the corresponding strains, namely $\varepsilon_{z\alpha}(\alpha, \beta, z)$ and $\varepsilon_{\beta z}(\alpha, \beta, z)$, vary in the shell layer thickness according to a specified law, defined by a "correction" even function $f(z)$, symmetric relative to the middle surface (e.g. parabolic, trigonometric), and the shear angle rotations of a normal to the reference mid-surface about the α and β axes, $\psi_\alpha(\alpha, \beta)$ and $\psi_\beta(\alpha, \beta)$, which are obtained from the FSDT; additionally non-homogeneous conditions are assumed at the top and bottom surfaces;
- (c) The normal stresses $\sigma_{zz}(\alpha, \beta, z)$ at areas parallel to the middle surface are not negligible and are obtained through the out-of-plane equilibrium equation in orthogonal curvilinear coordinates, and non-homogeneous conditions are assumed at the top and bottom surfaces;
- (d) The strain $\varepsilon_{zz}(\alpha, \beta, z)$ is determined assuming as first approximations of the in-plane stresses the ones obtained with the FSDT for shells defined in Appendix B.2, $\sigma_{\alpha\alpha}^*(\alpha, \beta, z)$, $\sigma_{\beta\beta}^*(\alpha, \beta, z)$ and $\sigma_{\alpha\beta}^*(\alpha, \beta, z)$, without any simplification regarding the thinness of the shell being made (i.e., the terms $z/R_\alpha(\alpha, \beta)$ and $z/R_\beta(\alpha, \beta)$ are fully retained);

According to assumption (b) the out-of-plane shear stress components are postulated as

$$\sigma_{\beta z}(\alpha, \beta, z) = \frac{1}{H_\beta} \left[\bar{\tau}_{\beta z}(\alpha, \beta) + \frac{z}{2h} \tilde{\tau}_{\beta z}(\alpha, \beta) + f(z) \bar{c}_{44}^* \psi_\beta(\alpha, \beta) \right], \quad (4.3a)$$

$$\sigma_{z\alpha}(\alpha, \beta, z) = \frac{1}{H_\alpha} \left[\bar{\tau}_{z\alpha}(\alpha, \beta) + \frac{z}{2h} \tilde{\tau}_{z\alpha}(\alpha, \beta) + f(z) \bar{c}_{55}^* \psi_\alpha(\alpha, \beta) \right], \quad (4.3b)$$

where $\bar{c}_{44}^*$ and $\bar{c}_{55}^*$ are stress-strain shear constitutive parameters [cf. Equation B.20 in Appendix], $\psi_\alpha = \psi_\alpha(\alpha, \beta)$ and $\psi_\beta = \psi_\beta(\alpha, \beta)$ are the shear angles obtained with the FSDT (defined in Appendix B.2) and the *bar* and *tilde* above $\tau_{\beta z}$ and $\tau_{z\alpha}$ are used to denote *mean* imposed surface shear tractions, given by

$$\bar{\tau}_{\beta z}(\alpha, \beta) = \frac{1}{2} \left[\sigma_{\beta z}^t(\alpha, \beta) - \sigma_{\beta z}^b(\alpha, \beta) \right], \quad (4.4a)$$

$$\bar{\tau}_{z\alpha}(\alpha, \beta) = \frac{1}{2} \left[\sigma_{z\alpha}^t(\alpha, \beta) - \sigma_{z\alpha}^b(\alpha, \beta) \right], \quad (4.4b)$$

and *relative* ones, given by

$$\tilde{\tau}_{\beta z}(\alpha, \beta) = \sigma_{\beta z}^t(\alpha, \beta) + \sigma_{\beta z}^b(\alpha, \beta), \quad (4.5a)$$

$$\tilde{\tau}_{z\alpha}(\alpha, \beta) = \sigma_{z\alpha}^t(\alpha, \beta) + \sigma_{z\alpha}^b(\alpha, \beta), \quad (4.5b)$$

where $(\cdot)^t$ and $(\cdot)^b$ denote prescribed shear tractions at the top and bottom surfaces (i.e., at $z = \pm h$), respectively, of the shell layer (see Figure 4.3).

By substituting the values of $\sigma_{z\alpha}(\alpha, \beta, z)$ and $\sigma_{\beta z}(\alpha, \beta, z)$ in Equations (4.3) into the static transverse equilibrium equation in orthogonal curvilinear coordinates in Equation (B.2c) of the Appendix, in the absence of body forces, taking into account the definitions in (4.4) and (4.5) and integrating with respect to z , yields the following relation for the normal stress component $\sigma_{zz}(\alpha, \beta, z)$ defined in terms of coefficients of powers of z (since they are not important here, for the sake of simplicity, the necessary definitions will be given latter),

$$\sigma_{zz}(\alpha, \beta, z) = \sigma_{zz}^{(0)}(\alpha, \beta) + \sigma_{zz}^{(1)}(\alpha, \beta, z) + \sigma_{zz}^{(2)}(\alpha, \beta, z) + \sigma_{zz}^{(f)}(\alpha, \beta, z), \quad (4.6)$$

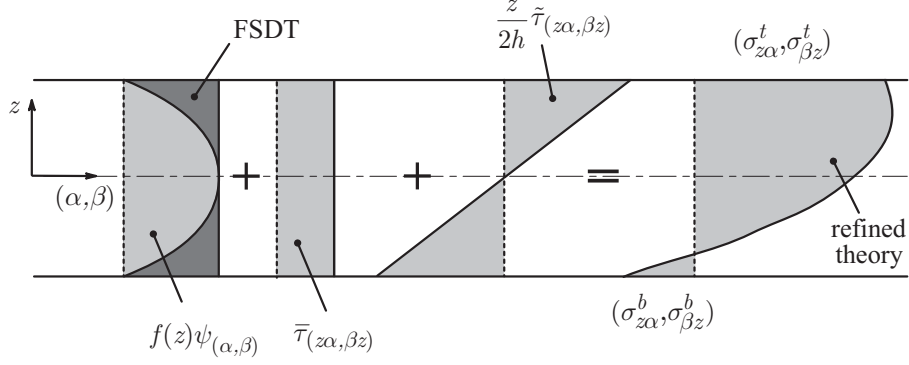


Figure 4.3: Postulated shear strains distributions of the refined theory.

where $\sigma_{zz}^{(0)}(\alpha, \beta)$ is an integration function independent of z that is determined from the boundary conditions on the top and bottom surfaces of the shell layer, $\sigma_{zz}^t(\alpha, \beta)$ and $\sigma_{zz}^b(\alpha, \beta)$, respectively. Regarding the dependencies of the different order terms, the first-order one, $\sigma_{zz}^{(1)}(\alpha, \beta, z)$, depends of the mean shear tractions $\bar{\tau}_{z\alpha}(\alpha, \beta)$ and $\bar{\tau}_{\beta z}(\alpha, \beta)$, the second-order term, $\sigma_{zz}^{(2)}(\alpha, \beta, z)$, depends of the relative shear tractions $\tilde{\tau}_{z\alpha}(\alpha, \beta)$ and $\tilde{\tau}_{\beta z}(\alpha, \beta)$, and the term depending on $f(z)$, $\sigma_{zz}^{(f)}(\alpha, \beta, z)$, is related with the shear angles $\psi_\alpha(\alpha, \beta)$ and $\psi_\beta(\alpha, \beta)$.

By satisfying the prescribed stress conditions $\sigma_{zz}^t(\alpha, \beta)$ and $\sigma_{zz}^b(\alpha, \beta)$ at the top and bottom surfaces, after some algebra, the function dependent of (α, β) (integration constant in z) that results from the integration is given as

$$\begin{aligned} \sigma_{zz}^{(0)}(\alpha, \beta) &= \bar{\tau}_{zz}(\alpha, \beta) - h^2 \sigma_{zz}^{(2)}(\alpha, \beta) - \frac{[F(h) + F(-h)]}{2F(z)} \sigma_{zz}^{(f)}(\alpha, \beta, z) \\ &= \bar{\tau}_{zz}(\alpha, \beta) - h^2 \sigma_{zz}^{(2)}(\alpha, \beta), \end{aligned} \quad (4.7)$$

where

$$F(z) = \int f(z) dz. \quad (4.8)$$

Equation (4.7) is simplified since $\sigma_{zz}^{(0)}(\alpha, \beta)$ can't depend on z , which is confirmed since $F(z)$ is an odd function, i.e., $F(h) = -F(-h)$. Additionally, an extra equation is obtained by imposing the top and bottom conditions, which defines the term $\sigma_{zz}^{(f)}(\alpha, \beta, z)$ as

$$\sigma_{zz}^{(f)}(\alpha, \beta, z) = \frac{F(z)}{F(h) - F(-h)} \left[\tilde{\tau}_{zz}(\alpha, \beta) - 2h \sigma_{zz}^{(1)}(\alpha, \beta, z) \right], \quad (4.9)$$

where similarly to Equations (4.4) and (4.5), the *bar* and *tilde* have been used to denote *mean* and *relative* prescribed transverse stresses,

$$\bar{\tau}_{zz}(\alpha, \beta) = \frac{1}{2} \left[\sigma_{zz}^t(\alpha, \beta) - \sigma_{zz}^b(\alpha, \beta) \right], \quad (4.10a)$$

$$\tilde{\tau}_{zz}(\alpha, \beta) = \sigma_{zz}^t(\alpha, \beta) + \sigma_{zz}^b(\alpha, \beta). \quad (4.10b)$$

After some algebra, the remaining undefined terms of Equation (4.6) are given by

$$\begin{aligned} \sigma_{zz}^{(1)}(\alpha, \beta, z) &= \frac{z}{H_\alpha H_\beta} \left[H_\beta \frac{A_\alpha}{R_\alpha} \sigma_{\alpha\alpha}^{*(0)}(\alpha, \beta, z) + H_\alpha \frac{A_\beta}{R_\beta} \sigma_{\beta\beta}^{*(0)}(\alpha, \beta, z) \right. \\ &\quad \left. - \frac{\partial \bar{\tau}_{z\alpha}(\alpha, \beta)}{\partial \alpha} - \frac{\partial \bar{\tau}_{\beta z}(\alpha, \beta)}{\partial \beta} \right], \end{aligned} \quad (4.11)$$

$$\sigma_{zz}^{(2)}(\alpha, \beta, z) = \frac{z^2}{H_\alpha H_\beta} \left[H_\beta \frac{A_\alpha}{2R_\alpha} \sigma_{\alpha\alpha}^{*(1)}(\alpha, \beta, z) + H_\alpha \frac{A_\beta}{2R_\beta} \sigma_{\beta\beta}^{*(1)}(\alpha, \beta, z) - \frac{1}{4h} \frac{\partial \tilde{\tau}_{z\alpha}(\alpha, \beta)}{\partial \alpha} - \frac{1}{4h} \frac{\partial \tilde{\tau}_{\beta z}(\alpha, \beta)}{\partial \beta} \right]. \quad (4.12)$$

At this point, it's worthy to mention that in order to keep the formulation of the transverse stress $\sigma_{zz}(\alpha, \beta, z)$ in Equation (4.6) general, its last term depends of the integral of the shear "correction" function, $F(z)$. Depending of the type and/or order in z of the shear function $f(z)$, (e.g. a quadratic polynomial, trigonometric function), different expansions in z can be obtained.

Considering the strain-stress constitutive behavior of the out-of-plane shear strains $\varepsilon_{z\alpha}$ and $\varepsilon_{\beta z}$ expressed in Equation (B.24) in Appendix, the shear strains are expressed as

$$\varepsilon_{\beta z}(\alpha, \beta, z) = \frac{1}{H_\beta} \left[\bar{\Sigma}_{\beta z}(\alpha, \beta) + \frac{z}{2h} \tilde{\Sigma}_{\beta z}(\alpha, \beta) + f(z) \Psi_\beta(\alpha, \beta) \right], \quad (4.13a)$$

$$\varepsilon_{z\alpha}(\alpha, \beta, z) = \frac{1}{H_\alpha} \left[\bar{\Sigma}_{z\alpha}(\alpha, \beta) + \frac{z}{2h} \tilde{\Sigma}_{z\alpha}(\alpha, \beta) + f(z) \Psi_\alpha(\alpha, \beta) \right], \quad (4.13b)$$

where the following notations regarding the mean surface tractions,

$$\bar{\Sigma}_{\beta z}(\alpha, \beta) = \bar{s}_{44}^* \bar{\tau}_{\beta z}(\alpha, \beta) + \bar{s}_{45}^* \bar{\tau}_{z\alpha}(\alpha, \beta), \quad (4.14a)$$

$$\bar{\Sigma}_{z\alpha}(\alpha, \beta) = \bar{s}_{45}^* \bar{\tau}_{\beta z}(\alpha, \beta) + \bar{s}_{55}^* \bar{\tau}_{z\alpha}(\alpha, \beta), \quad (4.14b)$$

relative surface tractions,

$$\tilde{\Sigma}_{\beta z}(\alpha, \beta) = \bar{s}_{44}^* \tilde{\tau}_{\beta z}(\alpha, \beta) + \bar{s}_{45}^* \tilde{\tau}_{z\alpha}(\alpha, \beta), \quad (4.15a)$$

$$\tilde{\Sigma}_{z\alpha}(\alpha, \beta) = \bar{s}_{45}^* \tilde{\tau}_{\beta z}(\alpha, \beta) + \bar{s}_{55}^* \tilde{\tau}_{z\alpha}(\alpha, \beta), \quad (4.15b)$$

and shear angle rotations,

$$\Psi_\beta(\alpha, \beta) = \bar{s}_{44}^* \bar{c}_{44}^* \psi_\beta(\alpha, \beta) + \bar{s}_{45}^* \bar{c}_{55}^* \psi_\alpha(\alpha, \beta), \quad (4.16a)$$

$$\Psi_\alpha(\alpha, \beta) = \bar{s}_{45}^* \bar{c}_{44}^* \psi_\beta(\alpha, \beta) + \bar{s}_{55}^* \bar{c}_{55}^* \psi_\alpha(\alpha, \beta), \quad (4.16b)$$

are used in terms of *effective* shear compliance coefficients (see Appendix B.3). Alternatively, for simplicity, the shear strains in Equations (4.13) can be expressed in terms of coefficients of increasing powers and functions of z as

$$\varepsilon_{\beta z}(\alpha, \beta, z) = \frac{1}{H_\beta} \left[\varepsilon_{\beta z}^{(0)}(\alpha, \beta) + z \varepsilon_{\beta z}^{(1)}(\alpha, \beta) + \varepsilon_{\beta z}^{(f)}(\alpha, \beta, z) \right], \quad (4.17a)$$

$$\varepsilon_{z\alpha}(\alpha, \beta, z) = \frac{1}{H_\alpha} \left[\varepsilon_{z\alpha}^{(0)}(\alpha, \beta) + z \varepsilon_{z\alpha}^{(1)}(\alpha, \beta) + \varepsilon_{z\alpha}^{(f)}(\alpha, \beta, z) \right], \quad (4.17b)$$

where the definitions of the terms in the previous equations are obvious from the analysis of Equations (4.13).

In a similar way, considering the transverse strain-stress constitutive behavior of $\varepsilon_{zz}(\alpha, \beta, z)$ expressed in Equation (B.24) of Appendix, neglecting the converse piezoelectric effects, taking as approximations of the in-plane stress components the ones obtained with the FSDT for shells described in Equations (B.8) of Appendix, i.e., $\sigma_{\alpha\alpha}^*(\alpha, \beta, z)$, $\sigma_{\beta\beta}^*(\alpha, \beta, z)$ and $\sigma_{\alpha\beta}^*(\alpha, \beta, z)$, and considering $\sigma_{zz}(\alpha, \beta, z)$ as defined in (4.6), the transverse strain component is given by

$$\varepsilon_{zz}(\alpha, \beta, z) \approx \bar{s}_{13} \sigma_{\alpha\alpha}^*(\alpha, \beta, z) + \bar{s}_{23} \sigma_{\beta\beta}^*(\alpha, \beta, z) + \bar{s}_{33} \sigma_{zz}(\alpha, \beta, z) + \bar{s}_{36} \sigma_{\alpha\beta}^*(\alpha, \beta, z). \quad (4.18)$$

4.3.4 Mixed Displacement Field and Strains

In this section the displacement field of the shell layer is derived by considering the out-of-plane strains $\varepsilon_{\beta z}$, $\varepsilon_{z\alpha}$ and ε_{zz} presented in the previous section in Equations (4.17) and (4.18), respectively, and the out-of-plane strain-displacement relations in the third to fifth equations of (B.1) in Appendix. By virtue of the assumptions previously considered,

$$\begin{aligned}\varepsilon_{zz} &= \frac{\partial w}{\partial z} \\ &\approx \bar{s}_{13} \left[\sigma_{\alpha\alpha}^{*(0)}(\alpha, \beta, z) + \sigma_{\alpha\alpha}^{*(1)}(\alpha, \beta, z) \right] + \bar{s}_{23} \left[\sigma_{\beta\beta}^{*(0)}(\alpha, \beta, z) + \sigma_{\beta\beta}^{*(1)}(\alpha, \beta, z) \right] \\ &\quad + \bar{s}_{33} \left[\sigma_{zz}^{(0)}(\alpha, \beta, z) + \sigma_{zz}^{(1)}(\alpha, \beta, z) + \sigma_{zz}^{(2)}(\alpha, \beta, z) + \sigma_{zz}^{(f)}(\alpha, \beta, z) \right] \\ &\quad + \bar{s}_{36} \left[\sigma_{\alpha\beta}^{*(0)}(\alpha, \beta, z) + \sigma_{\alpha\beta}^{*(1)}(\alpha, \beta, z) \right].\end{aligned}\quad (4.19)$$

Thus, integrating the previous equation with respect to z over the limits from 0 to z , considering that when $z = 0$ we have $w(\alpha, \beta, z) = w_0(\alpha, \beta)$, and taking into account the time dependence of the strains and stresses definitions, the transverse displacement $w = w(\alpha, \beta, z, t)$ is given by

$$\begin{aligned}w(\alpha, \beta, z, t) &= w^{(0)}(\alpha, \beta, t) + w^{(1)}(\alpha, \beta, z, t) \\ &\quad + w^{(2)}(\alpha, \beta, z, t) + w^{(3)}(\alpha, \beta, z, t) + w^{(f)}(\alpha, \beta, z, t),\end{aligned}\quad (4.20)$$

where

$$w^{(0)}(\alpha, \beta, t) = w_0(\alpha, \beta, t), \quad (4.21a)$$

$$\begin{aligned}w^{(1)}(\alpha, \beta, z, t) &= \int_0^z \left[\bar{s}_{13} \sigma_{\alpha\alpha}^{*(0)}(\alpha, \beta, z, t) + \bar{s}_{23} \sigma_{\beta\beta}^{*(0)}(\alpha, \beta, z, t) \right. \\ &\quad \left. + \bar{s}_{33} \sigma_{zz}^{(0)}(\alpha, \beta, z, t) + \bar{s}_{36} \sigma_{\alpha\beta}^{*(0)}(\alpha, \beta, z, t) \right] dz, \quad (4.21b)\end{aligned}$$

$$\begin{aligned}w^{(2)}(\alpha, \beta, z, t) &= \int_0^z \left[\bar{s}_{13} \sigma_{\alpha\alpha}^{*(1)}(\alpha, \beta, z, t) + \bar{s}_{23} \sigma_{\beta\beta}^{*(1)}(\alpha, \beta, z, t) \right. \\ &\quad \left. + \bar{s}_{33} \sigma_{zz}^{(1)}(\alpha, \beta, z, t) + \bar{s}_{36} \sigma_{\alpha\beta}^{*(1)}(\alpha, \beta, z, t) \right] dz, \quad (4.21c)\end{aligned}$$

$$w^{(3)}(\alpha, \beta, z, t) = \int_0^z \bar{s}_{33} \sigma_{zz}^{(2)}(\alpha, \beta, z, t) dz, \quad (4.21d)$$

$$w^{(f)}(\alpha, \beta, z, t) = \int_0^z \bar{s}_{33} \sigma_{zz}^{(f)}(\alpha, \beta, z, t) dz. \quad (4.21e)$$

In a similar way, using the relations

$$\begin{aligned}\varepsilon_{\beta z}(\alpha, \beta, z) &= H_\beta \frac{\partial}{\partial z} \left[\frac{v(\alpha, \beta, z)}{H_\beta} \right] + \frac{1}{H_\beta} \frac{\partial w(\alpha, \beta, z)}{\partial \beta} \\ &\approx \frac{1}{H_\beta} \left[\varepsilon_{\beta z}^{(0)}(\alpha, \beta) + z \varepsilon_{\beta z}^{(1)}(\alpha, \beta) + \varepsilon_{\beta z}^{(f)}(\alpha, \beta, z) \right],\end{aligned}\quad (4.22a)$$

$$\begin{aligned}\varepsilon_{z\alpha}(\alpha, \beta, z) &= H_\alpha \frac{\partial}{\partial z} \left[\frac{u(\alpha, \beta, z)}{H_\alpha} \right] + \frac{1}{H_\alpha} \frac{\partial w(\alpha, \beta, z)}{\partial \alpha} \\ &\approx \frac{1}{H_\alpha} \left[\varepsilon_{z\alpha}^{(0)}(\alpha, \beta) + z \varepsilon_{z\alpha}^{(1)}(\alpha, \beta) + \varepsilon_{z\alpha}^{(f)}(\alpha, \beta, z) \right],\end{aligned}\quad (4.22b)$$

integrating in order to z over the limits from 0 to z and considering that for $z = 0$, the displacements on the middle surface are given by $u(\alpha, \beta, 0) = u_0(\alpha, \beta)$ and $v(\alpha, \beta, 0) = v_0(\alpha, \beta)$, the time dependent tangential displacements of any point of the shell are given by

$$\begin{aligned}
 u(\alpha, \beta, z, t) &= -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \frac{\partial w(\alpha, \beta, z, t)}{\partial \alpha} dz \\
 &\quad + H_\alpha \int_0^z \frac{1}{H_\alpha^2} \left[\varepsilon_{z\alpha}^{(0)}(\alpha, \beta, t) + z\varepsilon_{z\alpha}^{(1)}(\alpha, \beta, t) + \varepsilon_{z\alpha}^{(f)}(\alpha, \beta, z, t) \right] dz \\
 &= u^{(0)}(\alpha, \beta, z, t) + u^{(1)}(\alpha, \beta, z, t) + u^{(2)}(\alpha, \beta, z, t) + u^{(3)}(\alpha, \beta, z, t) \\
 &\quad + u^{(4)}(\alpha, \beta, z, t) + u^{(f)}(\alpha, \beta, z, t),
 \end{aligned} \tag{4.23}$$

$$\begin{aligned}
 v(\alpha, \beta, z, t) &= -H_\beta \int_0^z \frac{1}{H_\beta^2} \frac{\partial w(\alpha, \beta, z, t)}{\partial \beta} dz \\
 &\quad + H_\beta \int_0^z \frac{1}{H_\beta^2} \left[\varepsilon_{\beta z}^{(0)}(\alpha, \beta, t) + z\varepsilon_{\beta z}^{(1)}(\alpha, \beta, t) + \varepsilon_{\beta z}^{(f)}(\alpha, \beta, z, t) \right] dz \\
 &= v^{(0)}(\alpha, \beta, z, t) + v^{(1)}(\alpha, \beta, z, t) + v^{(2)}(\alpha, \beta, z, t) + v^{(3)}(\alpha, \beta, z, t) \\
 &\quad + v^{(4)}(\alpha, \beta, z, t) + v^{(f)}(\alpha, \beta, z, t),
 \end{aligned} \tag{4.24}$$

where

$$u^{(0)}(\alpha, \beta, z, t) = \frac{H_\alpha}{A_\alpha} u_0(\alpha, \beta, t), \tag{4.25a}$$

$$u^{(1)}(\alpha, \beta, z, t) = -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \left[\frac{\partial w^{(0)}(\alpha, \beta, t)}{\partial \alpha} - \varepsilon_{z\alpha}^{(0)}(\alpha, \beta, t) \right] dz, \tag{4.25b}$$

$$u^{(2)}(\alpha, \beta, z, t) = -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \left[\frac{\partial w^{(1)}(\alpha, \beta, z, t)}{\partial \alpha} - z\varepsilon_{z\alpha}^{(1)}(\alpha, \beta, t) \right] dz, \tag{4.25c}$$

$$u^{(3)}(\alpha, \beta, z, t) = -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \frac{\partial w^{(2)}(\alpha, \beta, z, t)}{\partial \alpha} dz, \tag{4.25d}$$

$$u^{(4)}(\alpha, \beta, z, t) = -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \frac{\partial w^{(3)}(\alpha, \beta, z, t)}{\partial \alpha} dz, \tag{4.25e}$$

$$u^{(f)}(\alpha, \beta, z, t) = -H_\alpha \int_0^z \frac{1}{H_\alpha^2} \left[\frac{\partial w^{(f)}(\alpha, \beta, z, t)}{\partial \alpha} - \varepsilon_{z\alpha}^{(f)}(\alpha, \beta, z, t) \right] dz, \tag{4.25f}$$

and

$$v^{(0)}(\alpha, \beta, z, t) = \frac{H_\beta}{A_\beta} v_0(\alpha, \beta, t), \tag{4.26a}$$

$$v^{(1)}(\alpha, \beta, z, t) = -H_\beta \int_0^z \frac{1}{H_\beta^2} \left[\frac{\partial w^{(0)}(\alpha, \beta, t)}{\partial \beta} - \varepsilon_{\beta z}^{(0)}(\alpha, \beta, t) \right] dz, \tag{4.26b}$$

$$v^{(2)}(\alpha, \beta, z, t) = -H_\beta \int_0^z \frac{1}{H_\beta^2} \left[\frac{\partial w^{(1)}(\alpha, \beta, z, t)}{\partial \beta} - z\varepsilon_{\beta z}^{(1)}(\alpha, \beta, t) \right] dz, \tag{4.26c}$$

$$v^{(3)}(\alpha, \beta, z, t) = -H_\beta \int_0^z \frac{1}{H_\beta^2} \frac{\partial w^{(2)}(\alpha, \beta, z, t)}{\partial \beta} dz, \tag{4.26d}$$

$$v^{(4)}(\alpha, \beta, z, t) = -H_\beta \int_0^z \frac{1}{H_\beta^2} \frac{\partial w^{(3)}(\alpha, \beta, z, t)}{\partial \beta} dz, \quad (4.26e)$$

$$v^{(f)}(\alpha, \beta, z, t) = -H_\beta \int_0^z \frac{1}{H_\beta^2} \left[\frac{\partial w^{(f)}(\alpha, \beta, z, t)}{\partial \beta} - \varepsilon_{\beta z}^{(f)}(\alpha, \beta, z, t) \right] dz, \quad (4.26f)$$

Equations (4.20), (4.23) and (4.24) show that in comparison with the CLT of shells, following Love's first approximation, and the FSDT of shells discussed in Appendix B.2, the in-plane and transverse displacements of any point of the shell are nonlinearly dependent on z . Additionally, the same five time dependent generalized displacements of the FSDT (see Appendix B.2), $u_0 = u_0(\alpha, \beta, t)$, $v_0 = v_0(\alpha, \beta, t)$ and $w_0 = w_0(\alpha, \beta, t)$, which are the tangential and transverse displacements referred to a point on the middle surface, respectively, and the shear angle rotations of a normal to the reference middle surface,

$$\psi_\alpha(\alpha, \beta, t) = \frac{\partial w_0(\alpha, \beta, t)}{\partial \alpha} + A_\alpha(\alpha, \beta) \theta_\alpha(\alpha, \beta, t) - \frac{A_\alpha(\alpha, \beta)}{R_\alpha(\alpha, \beta)} u_0(\alpha, \beta, t), \quad (4.27a)$$

$$\psi_\beta(\alpha, \beta, t) = \frac{\partial w_0(\alpha, \beta, t)}{\partial \beta} + A_\beta(\alpha, \beta) \theta_\beta(\alpha, \beta, t) - \frac{A_\beta(\alpha, \beta)}{R_\beta(\alpha, \beta)} v_0(\alpha, \beta, t), \quad (4.27b)$$

are used to define the "generalized" displacements of the proposed refined theory which represent non-linear functions in z of the generalized displacements of FSDT. Thus, based on the assumptions (b)-(d) the 3-D problem of the theory of elasticity has been fully brought to a 2-D problem of the theory of the shell, with Equations (4.20), (4.23) and (4.24) establishing the geometrical model of the deformed state of the *fully refined* theory of the generic shell layer.

On the basis of the refined displacement field defined by Equations (4.20), (4.23) and (4.24) and the general strain-displacement relations in Equations (B.1) in Appendix, the not yet defined in-plane strain components of the proposed theory may be determined. For the sake of brevity their definitions will not be given here since they are quite long equations, in terms of high-order derivatives of the generalized displacements of the FSDT, which can be derived from the previous definitions.

4.3.5 Electric Field and Potential

Resuming the basic hypotheses defined in Section 4.3.3, and regarding the electrical and electro-mechanical behavior, the following considerations and assumptions regarding the physical problem are made:

- (e) The piezoelectric layers are polarized in the thickness direction and electroded only in the top and bottom surfaces;
- (f) Only a transverse electric field $\hat{E}_z$ is externally applied;
- (g) The in-plane electric displacement components D_x and D_y are nil at the non-electroded lateral edges and are considered negligible inside the piezoelectric volume;
- (h) The out-of-plane component of the electric displacement field D_z is constant with respect to the thickness;

The behavior of the electromagnetic field is described by Maxwell's equations. However, as presented by Tiersten (1969), considering that the electromagnetic waves essentially uncouple

from the elastic waves and that the elastic wavelengths are much shorter than the electromagnetic ones at the same frequency, Maxwell's equations can be replaced by electrostatic equations in vector form,

$$\operatorname{div} \mathbf{D} = 0, \quad (4.28)$$

$$-\operatorname{grad} \varphi = \mathbf{E}, \quad (4.29)$$

where $\varphi = \varphi(\alpha, \beta, z)$ is the electric potential, and $\mathbf{D} = \mathbf{D}(\alpha, \beta, z)$ and $\mathbf{E} = \mathbf{E}(\alpha, \beta, z)$ are the electric displacement and field vectors. In scalar form, according to the definitions of the *divergence* and *gradient* operators in arbitrary orthogonal curvilinear coordinates [see [Weisstein \(2003\)](#)] and Equations (4.2), the electrostatic equations are given as

$$\frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial \alpha} (H_\beta D_\alpha) + \frac{\partial}{\partial \beta} (H_\alpha D_\beta) + \frac{\partial}{\partial z} (H_\alpha H_\beta D_z) \right] = 0, \quad (4.30)$$

$$E_\alpha = -\frac{1}{H_\alpha} \frac{\partial \varphi}{\partial \alpha}, \quad (4.31a)$$

$$E_\beta = -\frac{1}{H_\beta} \frac{\partial \varphi}{\partial \beta}, \quad (4.31b)$$

$$E_z = -\frac{\partial \varphi}{\partial z}. \quad (4.31c)$$

According to what has been discussed in Appendix B.3, for the present problem the electrodes are located on the top and bottom surfaces and only a transverse electric field $\hat{E}_z$ is externally applied. The conditions $D_\alpha = D_\beta = 0$ hold in the shell surfaces not covered with electrodes and in contact with a medium with low permittivity, e.g., vacuum or air [see [Rogacheva \(1994\)](#), [Krommer and Irschik \(1999\)](#) and [Vasques and Rodrigues \(2005\)](#) for further details] and the induced electric fields $\bar{E}_\alpha$ and $\bar{E}_\beta$ (which are defined as a function of the strains) where condensed in the constitutive Equation (B.20). Thus, Equation (4.30) can be written as

$$\frac{1}{H_\alpha H_\beta} \left[\frac{\partial}{\partial z} (H_\alpha H_\beta D_z) \right] = 0. \quad (4.32)$$

In view of the Gauss and Mainardi-Codazzi formulas [cf. [Leissa \(1973\)](#) and [Rogacheva \(1994\)](#)],

$$\frac{\partial H_\alpha}{\partial \beta} = \left(1 + \frac{z}{R_\beta} \right) \frac{\partial A_\alpha}{\partial \beta}, \quad (4.33a)$$

$$\frac{\partial H_\beta}{\partial \alpha} = \left(1 + \frac{z}{R_\alpha} \right) \frac{\partial A_\beta}{\partial \alpha}, \quad (4.33b)$$

$$\frac{\partial H_\alpha}{\partial z} = \frac{A_\alpha}{R_\alpha}, \quad (4.33c)$$

$$\frac{\partial H_\beta}{\partial z} = \frac{A_\beta}{R_\beta}, \quad (4.33d)$$

after some algebra, and taking into account Equations (4.2), Equation (4.32) is expressed as

$$\left(\frac{1}{R_\alpha + z} + \frac{1}{R_\beta + z} \right) D_z + \frac{\partial D_z}{\partial z} = 0. \quad (4.34)$$

Assuming a constant value of α and β , the previous partial differential equation can be solved as an ordinary differential equation in z with solution

$$D_z = C(\alpha, \beta) \frac{(-R_\alpha + h)(-R_\beta + h)}{(R_\alpha + h)(R_\beta + h)} = C(\alpha, \beta) \lambda(\alpha, \beta). \quad (4.35)$$

From Equation (4.35) it can be seen that the transverse component of the electric displacement is constant with respect to the thickness direction where the constant $\lambda(\alpha, \beta)$ is a correction factor that takes into account the effects of the radii of curvature (and half thickness of the layer) on the electric displacement. Thus, taking into account (B.21) of Appendix, it can be reformulated [Krommer and Irschik (1999, 2000), Vasques and Rodrigues (2005)] taking into account the curvature effects via $\lambda(\alpha, \beta)$ as

$$D_z = \frac{\lambda}{2h} \int_{-h}^{+h} \bar{\epsilon}_{33} (\hat{E}_z - \bar{E}_z) dz = -\lambda \bar{\epsilon}_{33} \frac{\phi}{2h} - \frac{\lambda}{2h} \bar{\epsilon}_{33} \int_{-h}^{+h} \bar{E}_z dz, \quad (4.36)$$

In the previous equation the prescribed transverse electric field component has been defined as $\hat{E}_z = -\phi/(2h)$ where $\phi = \phi(\alpha, \beta)$ denotes the electric potential difference between the top and bottom electrodes. Considering the previous equation and the general electrostatic constitutive behavior in (B.11), the net transverse electric field E_z is given by the sum of the effects of the externally applied electric potential difference ϕ and net induced term which depends of the strains $\tilde{E}_z$, and is expressed as

$$E_z = \lambda (\hat{E}_z + \tilde{E}_z) = -\lambda \frac{\phi}{2h} + \left(\lambda \bar{E}_z - \frac{\lambda}{2h} \int_{-h}^{+h} \bar{E}_z dz \right). \quad (4.37)$$

It is worthy to mention that the terms inside parenthesis in the previous equation are related with the mechanical strains and represent the net effects of the induced electric fields due to the *direct piezoelectric effect*. Finally, from the definition of the electric potential in the third equation of (4.31), substituting (4.37) and integrating with respect to the thickness of the layer, the through-the-thickness distribution of the electric potential is given by

$$\varphi = \lambda \frac{\phi}{2h} (z + h) + \left(\frac{\lambda}{2h} \int_{-h}^{+h} \bar{E}_z dz \right) (z + h) - \lambda \int_{-h}^z \bar{E}_z dz. \quad (4.38)$$

It can be seen that the net transverse electric field $E_z = E_z(\alpha, \beta, z)$ and electric potential distribution $\varphi = \varphi(\alpha, \beta, z)$ for the curvilinear coordinate system presented in Equations (4.37) and (4.38) are composed by terms externally applied and by induced terms due to the mechanical deformations. Furthermore, the formulation is kept general, since the induced terms are not explicitly defined at this point, and can be applied for any piezoelectric curved layer (and obviously for 1-D and 2-D planar structures where the term λ is equal to unity).

4.4 Partially Refined Mathematical Model of Doubly-Curved Shells

4.4.1 Restrictions and Simplifications

The definitions of the displacement field presented in Equations (4.20), (4.23) and (4.24) are quite general and applicable to anisotropic shells of arbitrary curvature. They result from a

fully refined interactive shell theory based, as a first approximation, on the in-plane stresses of the FSDT. Furthermore, all the terms regarding the thickness coordinate to radii or curvature ratios were retained and no simplifications were made regarding thin shell assumptions. Additionally, transverse shear strains and stresses weren't considered negligible and as a result a non-linearly dependent on z transverse displacement was obtained by the iterative procedure. That theory was denoted as *fully refined* since all the strain and stress components of the 3-D elasticity are obtained directly from the *mixed* (in terms of surface stresses and generalized displacements) displacement field by using the strain-displacement relations and an anisotropic constitutive law. It can also be seen from the definitions of the terms of the in-plane displacements in Equations (4.25) and (4.26) that they involve high-order derivatives (which become even higher for the strain and stress components) of the generalized displacements of the FSDT, which complicates the formulation and FE solution dramatically. Additionally, the *fully refined* mathematical model allows full out-of-plane interlayer (or interlaminar) stress continuity (and, as obvious, displacements too) to be imposed when assembling all the layers contributions at the "regenerated" FE level (further details will be discussed later). This renders a 2-D theory representative of the *full* 3-D behavior of a shell with arbitrary geometry (curvature).

However, the *fully refined* mixed displacement definitions are quite tedious and, for the sake of making the calculations less cumbersome, the general mixed displacement field definitions will be restricted to orthotropic shells with constant curvatures, i.e., doubly-curved shells (cylindrical, spherical, toroidal geometries) for which

$$A_\alpha(\alpha, \beta) = A_\alpha, \quad (4.39a)$$

$$A_\beta(\alpha, \beta) = A_\beta, \quad (4.39b)$$

$$R_\alpha(\alpha, \beta) = R_\alpha, \quad (4.39c)$$

$$R_\beta(\alpha, \beta) = R_\beta, \quad (4.39d)$$

Additionally, the non-linear transverse displacement field definition will be discarded and only the zero-order term, i.e., $w(\alpha, \beta, z, t) = w^{(0)}(\alpha, \beta, t)$, will be retained. This simplification makes the theory less complicated and more suitable to be implemented, since it avoids the higher-order derivatives of the generalized variables, and is denoted as *partial refined* theory. It is well known, however, that in the major part of the problems, the transverse stress is small when compared with the other stress components. An exception is, for example, in thermo-mechanical analysis where the transverse stress σ_{zz} plays an important role [see Carrera (1999, 2005)], which is not the case here. This simplification implies also the need to make the usual plane-stress assumption, which doesn't allow to impose interlayer transverse normal stress continuity. Regarding the shear stress correction function $f(z)$, several functions can be used. However, as stated by Ambartsumyan (1991, p. 37), some arbitrariness in the reasonable selection of $f(z)$ will not introduce inadmissible errors into the refined theory, which in this work will be assumed to follow the law of a quadratic parabola as

$$f(z) = 1 - \frac{z^2}{h^2}. \quad (4.40)$$

4.4.2 Displacements and Strains

For the sake of simplicity and brevity of writing the mathematical definitions, from this point henceforth, the spatial (α, β, z) and time t dependencies will be omitted from the equations when convenient and only written when necessary for the comprehension of the equations.

Thus, under the doubly-curved shell restrictions to the general problem, and following the previously defined *partial refined* theory, the displacement definitions in Equations (4.20), (4.23) and (4.24), for an orthotropic shell layer, are given by

$$u(\alpha, \beta, z, t) = \frac{H_\alpha}{A_\alpha} \left[u_0 + \frac{z_\alpha^{*(f)}}{A_\alpha} \psi_\alpha - \frac{z_\alpha^{*(0)}}{A_\alpha} \frac{\partial w_0}{\partial \alpha} + \frac{z_\alpha^{*(0)}}{A_\alpha} \bar{s}_{55}^* \bar{\tau}_{z\alpha} + \frac{z_\alpha^{*(1)}}{A_\alpha} \frac{\bar{s}_{55}^*}{2h} \tilde{\tau}_{z\alpha} \right], \quad (4.41a)$$

$$v(\alpha, \beta, z, t) = \frac{H_\beta}{A_\beta} \left[v_0 + \frac{z_\beta^{*(f)}}{A_\beta} \psi_\beta - \frac{z_\beta^{*(0)}}{A_\beta} \frac{\partial w_0}{\partial \beta} + \frac{z_\beta^{*(0)}}{A_\beta} \bar{s}_{44}^* \bar{\tau}_{\beta z} + \frac{z_\beta^{*(1)}}{A_\beta} \frac{\bar{s}_{44}^*}{2h} \tilde{\tau}_{\beta z} \right], \quad (4.41b)$$

$$w(\alpha, \beta, z, t) = w_0, \quad (4.41c)$$

where

$$z_\alpha^{*(i)} = \int_0^z \frac{z^i}{(1 + z/R_\alpha)^2} dz, \quad z_\beta^{*(i)} = \int_0^z \frac{z^i}{(1 + z/R_\beta)^2} dz, \quad (4.42a,b)$$

$$z_\alpha^{*(f)} = \int_0^z \frac{f(z)}{(1 + z/R_\alpha)^2} dz, \quad z_\beta^{*(f)} = \int_0^z \frac{f(z)}{(1 + z/R_\beta)^2} dz, \quad (4.43a,b)$$

with $i = 0, 1$. Taking into account the shear angles definition in Equations (4.27), the displacement field can still be expressed as

$$u(\alpha, \beta, z, t) = \frac{H_\alpha}{A_\alpha} u_0 + \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(f)} \left(\frac{\partial w_0}{\partial \alpha} + A_\alpha \theta_\alpha - \frac{A_\alpha}{R_\alpha} u_0 \right) - \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(0)} \frac{\partial w_0}{\partial \alpha} + \frac{H_\alpha}{A_\alpha^2} \bar{s}_{55}^* z_\alpha^{*(0)} \bar{\tau}_{z\alpha} + \frac{H_\alpha}{A_\alpha^2} \frac{\bar{s}_{55}^*}{2h} z_\alpha^{*(1)} \tilde{\tau}_{z\alpha}, \quad (4.44a)$$

$$v(\alpha, \beta, z, t) = \frac{H_\beta}{A_\beta} v_0 + \frac{H_\beta}{A_\beta^2} z_\beta^{*(f)} \left(\frac{\partial w_0}{\partial \beta} + A_\beta \theta_\beta - \frac{A_\beta}{R_\beta} v_0 \right) - \frac{H_\beta}{A_\beta^2} z_\beta^{*(0)} \frac{\partial w_0}{\partial \beta} + \frac{H_\beta}{A_\beta^2} \bar{s}_{44}^* z_\beta^{*(0)} \bar{\tau}_{\beta z} + \frac{H_\beta}{A_\beta^2} \frac{\bar{s}_{44}^*}{2h} z_\beta^{*(1)} \tilde{\tau}_{\beta z}, \quad (4.44b)$$

$$w(\alpha, \beta, z, t) = w_0. \quad (4.44c)$$

As can be seen in the *mixed* displacement field definition in Equations (4.41), the displacement field is defined in terms of the generalized displacements of the FSDT and mean and relative surface shear stresses. Additionally, as would be expected, if $f(z)$ is assumed equal to one, which corresponds to case where no correction to the FSDT constant through-the-thickness shear stresses/strains is made, the displacement is the same as the one obtained with the FSDT, with the extra surface shear stress terms. In the limit case where $R_\alpha = R_\beta = \infty$, which corresponds to the case of planar structures such as plates, and considering the parabolic definition of $f(z)$, the displacements are consistent and are expanded in a power series of z up to z^3 . In the present case, since the terms z/R_α and z/R_β were fully retained, the displacements are defined with more complex coefficients in terms of powers of z and $\ln(R_\alpha + z)$ and $\ln(R_\beta + z)$. For convenience, the *mixed* displacement field in Equations (4.41) can be expressed in matrix form as

$$\mathbf{u}(\alpha, \beta, z, t) = \mathbf{z}^u(z) \mathbf{u}_0(\alpha, \beta, t) + \mathbf{z}^\tau(z) \boldsymbol{\tau}(\alpha, \beta, t), \quad (4.45)$$

or, alternatively,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} z_{11}^u & 0 & z_{13}^u \partial_\alpha & z_{14}^u & 0 \\ 0 & z_{22}^u & z_{23}^u \partial_\beta & 0 & z_{25}^u \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} z_{11}^\tau & z_{12}^\tau & 0 & 0 \\ 0 & 0 & z_{23}^\tau & z_{24}^\tau \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix}, \quad (4.46)$$

where the coefficients of matrices $\mathbf{z}^u(z)$ and $\mathbf{z}^\tau(z)$ are derived from the displacements as explained in Appendix B.4. In the specific case of $z_{13}^u(z)$ and $z_{23}^u(z)$, which are related with $\partial w_0 / \partial \alpha$ and $\partial w_0 / \partial \beta$, partial derivative operators ∂_α and ∂_β in order to α and β , respectively, were also included.

From the definition of the displacement field in Equation (4.45), the out-of-plane shear strains are given by Equations (4.17), taking into account the restrictions in (4.39) and the transverse strain $\varepsilon_{zz} = 0$ [due to the fact that $w(\alpha, \beta, z, t)$ was assumed independent of z]. The in-plane strain components are obtained according to the displacement-strain relations in the first two and last equations of Equations (B.1) by taking into account (4.39). Thus, the strains vector without the null component ε_{zz} may be expressed in matrix form as

$$\begin{aligned} \varepsilon(\alpha, \beta, z, t) &= \partial_\varepsilon(z) \mathbf{z}^u(z) \mathbf{u}_0(\alpha, \beta, t) + \partial_\varepsilon(z) \mathbf{z}^\tau(z) \boldsymbol{\tau}(\alpha, \beta, t) \\ &= \mathbf{z}^{\varepsilon u}(z) \mathbf{u}_0(\alpha, \beta, t) + \mathbf{z}^{\varepsilon \tau}(z) \boldsymbol{\tau}(\alpha, \beta, t), \end{aligned} \quad (4.47)$$

where $\partial_\varepsilon(z)$ is a matrix differential operator given by

$$\partial_\varepsilon(z) = \begin{bmatrix} \frac{1}{H_\alpha} \partial_\alpha & 0 & \frac{1}{H_\alpha} \frac{A_\alpha}{R_\alpha} \\ 0 & \frac{1}{H_\beta} \partial_\beta & \frac{1}{H_\beta} \frac{A_\beta}{R_\beta} \\ 0 & \partial_z - \frac{1}{H_\beta} \frac{A_\beta}{R_\beta} & \frac{1}{H_\beta} \partial_\beta \\ \partial_z - \frac{1}{H_\alpha} \frac{A_\alpha}{R_\alpha} & 0 & \frac{1}{H_\alpha} \partial_\alpha \\ \frac{1}{H_\beta} \partial_\beta & \frac{1}{H_\alpha} \partial_\alpha & 0 \end{bmatrix}, \quad (4.48)$$

and ∂_z is an other partial differential operator, this time in order to z . Considering the previous operator matrix in Equation (4.47), the strains vector is defined in terms of the matrices $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon \tau}(z)$ as

$$\begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} z_{11}^{\varepsilon u} \partial_\alpha & 0 & z_{13}^{\varepsilon u(1)} + z_{13}^{\varepsilon u(2)} \partial_\alpha^2 & z_{14}^{\varepsilon u} \partial_\alpha & 0 \\ 0 & z_{22}^{\varepsilon u} \partial_\beta & z_{23}^{\varepsilon u(1)} + z_{23}^{\varepsilon u(2)} \partial_\beta^2 & 0 & z_{25}^{\varepsilon u} \partial_\beta \\ 0 & z_{32}^{\varepsilon u} & z_{33}^{\varepsilon u} \partial_\beta & 0 & z_{35}^{\varepsilon u} \\ z_{41}^{\varepsilon u} & 0 & z_{43}^{\varepsilon u} \partial_\alpha & z_{44}^{\varepsilon u} & 0 \\ z_{51}^{\varepsilon u} \partial_\beta & z_{52}^{\varepsilon u} \partial_\alpha & z_{53}^{\varepsilon u} \partial_\alpha \partial_\beta & z_{54}^{\varepsilon u} \partial_\beta & z_{55}^{\varepsilon u} \partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} z_{11}^{\varepsilon \tau} \partial_\alpha & z_{12}^{\varepsilon \tau} \partial_\alpha & 0 & 0 \\ 0 & 0 & z_{23}^{\varepsilon \tau} \partial_\beta & z_{24}^{\varepsilon \tau} \partial_\beta \\ 0 & 0 & z_{33}^{\varepsilon \tau} & z_{34}^{\varepsilon \tau} \\ z_{41}^{\varepsilon \tau} & z_{42}^{\varepsilon \tau} & 0 & 0 \\ z_{51}^{\varepsilon \tau} \partial_\beta & z_{52}^{\varepsilon \tau} \partial_\beta & z_{53}^{\varepsilon \tau} \partial_\alpha & z_{54}^{\varepsilon \tau} \partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix} \quad (4.49)$$

where ∂_{α}^2 , ∂_{β}^2 and $\partial_{\alpha\beta}$ are double and crossed partial differential operators. The coefficients of $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon \tau}(z)$ are given in Appendix B.4. It is worthy to mention that the strain field is defined in terms of zero- and/or first-order derivatives of the generalized in-plane displacements, rotations and surface stresses, and zero-, second-order and cross derivatives of the generalized transverse displacement w_0 .

4.4.3 Induced Electric Field and Net Strain Field

The net electric field E_z in Equation (4.37) is expressed in terms of external prescribed and induced effects. Considering the electric eigenfield definition in the third equation of Equations (B.18), and since $\varepsilon_{zz} = 0$, the net induced term $\tilde{E}_z$ for the orthotropic shell layer is given by

$$\tilde{E}_z = \frac{\lambda}{\bar{\epsilon}_{33}^*} \left[-\bar{e}_{31}^* \varepsilon_{\alpha\alpha} - \bar{e}_{32}^* \varepsilon_{\beta\beta} + \frac{1}{2h} \int_{-h}^{+h} (\bar{e}_{31}^* \varepsilon_{\alpha\alpha} + \bar{e}_{32}^* \varepsilon_{\beta\beta}) dz \right], \quad (4.50)$$

where $\bar{e}_{31}^*$ and $\bar{e}_{32}^*$ are the reduced (modified) piezoelectric plane-stress constants.

Considering the definitions of the in-plane strain components $\varepsilon_{\alpha\alpha}$ and $\varepsilon_{\beta\beta}$ given in Equation (4.49), the previous equation can be expressed in terms of the generalized displacements, and after some algebra, the net strain field accounting for the *direct piezoelectric effects* taking into account the plane-stress version of the electrical relation of Equation (B.24), is given by

$$\begin{aligned} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} &= \begin{bmatrix} z_{11}^{\varepsilon u} \partial_{\alpha} & z_{12}^{\varepsilon u} \partial_{\beta} & z_{13}^{\varepsilon u(1)} + z_{13}^{\varepsilon u(2)} \partial_{\alpha}^2 & z_{14}^{\varepsilon u} \partial_{\alpha} & z_{15}^{\varepsilon u} \partial_{\beta} \\ z_{21}^{\varepsilon u} \partial_{\alpha} & z_{22}^{\varepsilon u} \partial_{\beta} & z_{23}^{\varepsilon u(1)} + z_{23}^{\varepsilon u(2)} \partial_{\beta}^2 & z_{24}^{\varepsilon u} \partial_{\alpha} & z_{25}^{\varepsilon u} \partial_{\beta} \\ 0 & z_{32}^{\varepsilon u} & z_{33}^{\varepsilon u} \partial_{\beta} & 0 & z_{35}^{\varepsilon u} \\ z_{41}^{\varepsilon u} & 0 & z_{43}^{\varepsilon u} \partial_{\alpha} & z_{44}^{\varepsilon u} & 0 \\ z_{51}^{\varepsilon u} \partial_{\beta} & z_{52}^{\varepsilon u} \partial_{\alpha} & z_{53}^{\varepsilon u} \partial_{\alpha\beta} & z_{54}^{\varepsilon u} \partial_{\beta} & z_{55}^{\varepsilon u} \partial_{\alpha} \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_{\alpha} \\ \theta_{\beta} \end{Bmatrix} \\ &+ \begin{bmatrix} z_{11}^{\varepsilon \tau} \partial_{\alpha} & z_{12}^{\varepsilon \tau} \partial_{\alpha} & z_{13}^{\varepsilon \tau} \partial_{\beta} & z_{14}^{\varepsilon \tau} \partial_{\beta} \\ z_{21}^{\varepsilon \tau} \partial_{\alpha} & z_{22}^{\varepsilon \tau} \partial_{\alpha} & z_{23}^{\varepsilon \tau} \partial_{\beta} & z_{24}^{\varepsilon \tau} \partial_{\beta} \\ 0 & 0 & z_{33}^{\varepsilon \tau} & z_{34}^{\varepsilon \tau} \\ z_{41}^{\varepsilon \tau} & z_{42}^{\varepsilon \tau} & 0 & 0 \\ z_{51}^{\varepsilon \tau} \partial_{\beta} & z_{52}^{\varepsilon \tau} \partial_{\beta} & z_{53}^{\varepsilon \tau} \partial_{\alpha} & z_{54}^{\varepsilon \tau} \partial_{\alpha} \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix}, \quad (4.51) \end{aligned}$$

where the modified (taking into account the electrical induced strains) terms of $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon \tau}(z)$ are given in Appendix B.5.

4.4.4 Variational Formulation

In order to derive the weak form of the equations governing the motion and electric charge equilibrium of the shell layer, Hamilton's principle is used. The Lagrangian and the work of the applied forces are adapted for the electrical and mechanical contributions [Tiersten (1969)], so that

$$\delta \int_{t_0}^{t_1} (T - H + W) dt = 0, \quad (4.52)$$

where t_0 and t_1 define the time interval, δ denotes the variation, T is the *kinetic energy*, H is the *electro-mechanical enthalpy* and W denotes the *work of the external non-conservative actions* done by the applied mechanical forces and electrical charges.

Since the stresses have been replaced and considered by means of internal forces and moments due to the thickness integration, it is appropriate to alter the definition of the fundamental element of the shell. Accordingly, it will be assumed, henceforth, that the element which was formerly defined to be dz thick, is replaced, on account of the integrations with respect to z , with an element of thickness h . Such an element is acted upon by the internal forces (stress resultants) and moments per unit arc length and by external effects which include mechanical forces and electrical charge loads. The internal forces act upon the edges of the element while the mechanical forces and electrical charges act upon the inner and outer surfaces/electrodes.

According to Equation (4.45), the kinetic energy is given by

$$T = \frac{1}{2} \int_V \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV, \quad (4.53)$$

where $\dot{\mathbf{u}} = \dot{\mathbf{u}}(\alpha, \beta, z, t)$ is the vector of generalized velocities taking into account the time differentiation of the three components of the displacement field expressed in the tri-orthogonal curvilinear coordinate system. The first variation of the kinetic energy yields the virtual work of the inertial forces, given by

$$\delta T = - \int_{\Omega_0} \left[\int_{-h}^{+h} \rho \delta \mathbf{u}^T \ddot{\mathbf{u}} H_\alpha H_\beta H_z dz \right] d\alpha d\beta, \quad (4.54)$$

which is expanded with more detail in terms of the variations of the generalized displacements and stresses in Appendix B.6.1.

The electro-mechanical enthalpy of the piezoelectric medium is expressed in terms of mechanical and electrical quantities by

$$H = \frac{1}{2} \int_V (\boldsymbol{\sigma}^T \boldsymbol{\varepsilon} - \mathbf{D}^T \mathbf{E}) dV. \quad (4.55)$$

Taking into account the plane-stress piezoelectric constitutive Equation (B.25) in the Appendix, the first variation of the previous equation yields

$$\begin{aligned} \delta H &= \delta H_\varepsilon - \delta H_E \\ &= \int_{\Omega_0} \left[\int_{-h}^{+h} \boldsymbol{\sigma}^T \delta \boldsymbol{\varepsilon} H_\alpha H_\beta H_z dz - \int_{-h}^{+h} D_z \delta E_z H_\alpha H_\beta H_z dz \right] d\alpha d\beta, \end{aligned} \quad (4.56)$$

which is expanded with more detail in terms of the variations of the generalized displacements and internal forces and moments in Appendix B.6.2.

The last term of Equation (4.52) involves the work done by the applied mechanical forces W_u , applied on the inner and outer surfaces and lateral edges of the shell, and the work done by the applied electrical charges density W_ϕ on the top and bottom electroded surfaces. To write the expressions for the net external loads work recall that Ω denotes a surface at a distance z and parallel to the middle-surface, where Ω^t and Ω^b denote the top and bottom surfaces for which $z = \pm h$, and that Γ denotes the boundary of the shell element, with Γ_α and Γ_β being the boundary edges of constant β and α coordinates, respectively (with the circle on the integral implying that it includes the total boundary of the shell). Thus, the work of the non-conservative loads is given by

$$W = W_u - W_\phi. \quad (4.57)$$

The mechanical counterpart is given by

$$W_u = \int_{\Omega^t} F_z^t w d\Omega^t + \int_{\Omega^b} F_z^b w d\Omega^b + \oint_{\Gamma_\alpha} \left[\int_{-h}^{+h} (\hat{\sigma}_{\beta\beta} v + \hat{\sigma}_{\beta\alpha} u + \hat{\sigma}_{\beta z} w) H_\alpha dz \right] d\alpha \\ + \oint_{\Gamma_\beta} \left[\int_{-h}^{+h} (\hat{\sigma}_{\alpha\alpha} u + \hat{\sigma}_{\alpha\beta} v + \hat{\sigma}_{z\alpha} w) H_\beta dz \right] d\beta, \quad (4.58)$$

where $F_z^t = F_z^t(\alpha, \beta, t)$ and $F_z^b = F_z^b(\alpha, \beta, t)$ are transverse normal forces applied on the top and bottom surfaces, and the *hat* over the stresses, $\hat{\sigma}_{\alpha\alpha} = \hat{\sigma}_{\alpha\alpha}(\beta, z, t)$, $\hat{\sigma}_{\alpha\beta} = \hat{\sigma}_{\alpha\beta}(\beta, z, t)$ and $\hat{\sigma}_{z\alpha} = \hat{\sigma}_{z\alpha}(\beta, z, t)$, for the edges normal to α , and $\hat{\sigma}_{\beta\beta} = \hat{\sigma}_{\beta\beta}(\alpha, z, t)$, $\hat{\sigma}_{\beta\alpha} = \hat{\sigma}_{\beta\alpha}(\alpha, z, t)$ and $\hat{\sigma}_{\beta z} = \hat{\sigma}_{\beta z}(\alpha, z, t)$, for the edges normal to β , denotes prescribed stresses on the boundary edges. Regarding the electrical counterpart, it is given by

$$W_\phi = \int_{\Omega^t} \tau^t \varphi d\Omega^t + \int_{\Omega^b} \tau^b \varphi d\Omega^t, \quad (4.59)$$

where $\tau^t = \tau^t(\alpha, \beta, t)$ and $\tau^b = \tau^b(\alpha, \beta, t)$ are prescribed electric charge densities on the top and bottom electroded surfaces.

Considering in Equation (4.38) that the induced electric field has no contribution for the electric potential at the surface electrodes, assuming the bottom electrode is grounded, i.e. that $\varphi(\alpha, \beta, z = -h, t) = 0$, and retaining only the normal mechanical load on the top surface F_z^t , the first variation of Equation (4.56) yields the net virtual work of the non-conservative external effects

$$\delta W = \delta W_u - \delta W_\phi, \quad (4.60)$$

with its terms given by

$$\delta W_u = \int_{\Omega_0} Z \delta w_0 d\alpha d\beta + \oint_{\Gamma_\alpha} \left[\int_{-h}^{+h} (\hat{\sigma}_{\beta\beta} \delta v + \hat{\sigma}_{\beta\alpha} \delta u + \hat{\sigma}_{\beta z} \delta w) H_\alpha dz \right] d\alpha \\ + \oint_{\Gamma_\beta} \left[\int_{-h}^{+h} (\hat{\sigma}_{\alpha\alpha} \delta u + \hat{\sigma}_{\alpha\beta} \delta v + \hat{\sigma}_{z\alpha} \delta w) H_\beta dz \right] d\beta, \quad (4.61)$$

$$\delta W_\phi = \int_{\Omega_0} \tau \delta \phi d\alpha d\beta, \quad (4.62)$$

where $Z = F_z^t A_\alpha A_\beta (1 + h/R_\alpha) (1 + h/R_\beta)$ and $\tau = \tau^t \lambda A_\alpha A_\beta (1 + h/R_\alpha) (1 + h/R_\beta)$. The virtual work of the non-conservative forces, δW_u , is expressed with more detail in terms of the variations of the generalized displacements and prescribed forces and moments in Appendix B.7.

4.4.5 Weak Forms of the Governing Equations in Terms of Internal Forces and Moments

The weak form of the governing electro-mechanical equations in terms of internal forces and moments are obtained by substituting the variational terms in Equations (4.54), (4.56) and (4.60) into the Hamilton's principle in Equation (4.52). The virtual generalized displacements, surface stresses and electric potential difference are zero where the corresponding variables are specified. For time-dependent problems, the admissible virtual generalized variables must also vanish at time $t = t_0$ and $t = t_1$. Thus, using the fundamental lema of variational calculus

and collecting the coefficients of each variation of the different generalized variables (displacements, surface stresses and electric potential) into independent equations yields for the generalized displacements δu_0 , δv_0 , δw_0 , $\delta \theta_\alpha$ and $\delta \theta_\beta$:

$$\int_{\Omega_0} \left[\delta u_0 (I_{11}^{uu} \ddot{u}_0 + I_{13}^{uu} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{14}^{uu} \ddot{\theta}_\alpha + I_{11}^{\tau u} \ddot{\tau}_{z\alpha} + I_{21}^{\tau u} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial u_0}{\partial \alpha} (N_{\alpha\alpha}^{*11} + N_{\beta\beta}^{*21}) \right. \\ \left. + \delta \frac{\partial u_0}{\partial \beta} (N_{\alpha\beta}^{51}) + \delta u_0 (Q_{z\alpha}^{41}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta u_0 (\hat{N}_{\alpha\beta}^{11}) d\alpha - \oint_{\Gamma_\beta} \delta u_0 (\hat{N}_{\alpha\alpha}^{11}) d\beta = 0, \quad (4.63)$$

$$\int_{\Omega_0} \left[\delta v_0 (I_{22}^{uu} \ddot{v}_0 + I_{23}^{uu} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{25}^{uu} \ddot{\theta}_\beta + I_{32}^{\tau u} \ddot{\tau}_{\beta z} + I_{42}^{\tau u} \ddot{\tau}_{\beta z}) + \delta \frac{\partial v_0}{\partial \beta} (N_{\alpha\alpha}^{*12} + N_{\beta\beta}^{*22}) \right. \\ \left. + \delta \frac{\partial v_0}{\partial \alpha} (N_{\alpha\beta}^{52}) + \delta v_0 (Q_{\beta z}^{32}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta v_0 (\hat{N}_{\beta\beta}^{22}) d\alpha - \oint_{\Gamma_\beta} \delta v_0 (\hat{N}_{\alpha\beta}^{22}) d\beta = 0, \quad (4.64)$$

$$\int_{\Omega_0} \left[\delta w_0 (I_{33}^{uu} \ddot{w}_0 + \delta \frac{\partial w_0}{\partial \alpha} (I_{13}^{uu} \ddot{u}_0 + I_{33}^{u\alpha} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{34}^{uu} \ddot{\theta}_\alpha + I_{13}^{\tau u} \ddot{\tau}_{z\alpha} + I_{23}^{\tau u} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial w_0}{\partial \beta} (I_{23}^{uu} \ddot{v}_0 + I_{33}^{u\beta} \frac{\partial \ddot{w}_0}{\partial \beta} \right. \\ \left. + I_{35}^{uu} \ddot{\theta}_\beta + I_{33}^{\tau u} \ddot{\tau}_{\beta z} + I_{43}^{\tau u} \ddot{\tau}_{\beta z}) + \delta w_0 (M_{\alpha\alpha}^{*13(1)} + M_{\beta\beta}^{*23(1)}) + \delta \frac{\partial^2 w_0}{\partial \alpha^2} (M_{\alpha\alpha}^{*13(2)}) + \delta \frac{\partial^2 w_0}{\partial \beta^2} (M_{\beta\beta}^{*23(2)}) \right. \\ \left. + \delta \frac{\partial^2 w_0}{\partial \alpha \partial \beta} (M_{\alpha\beta}^{53}) + \delta \frac{\partial w_0}{\partial \alpha} (Q_{z\alpha}^{43}) + \delta \frac{\partial w_0}{\partial \beta} (Q_{\beta z}^{33}) - \delta w_0 Z \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \left[\delta \frac{\partial w_0}{\partial \beta} (\hat{M}_{\beta\beta}^{23}) \right. \\ \left. + \delta \frac{\partial w_0}{\partial \alpha} (\hat{M}_{\alpha\beta}^{13}) + \delta w_0 (\hat{Q}_{\beta z}^{33}) \right] d\alpha - \oint_{\Gamma_\beta} \left[\delta \frac{\partial w_0}{\partial \alpha} (\hat{M}_{\alpha\alpha}^{13}) + \delta \frac{\partial w_0}{\partial \beta} (\hat{M}_{\alpha\alpha}^{23}) + \delta w_0 (\hat{Q}_{z\alpha}^{33}) \right] d\beta = 0, \quad (4.65)$$

$$\int_{\Omega_0} \left[\delta \theta_\alpha (I_{14}^{uu} \ddot{u}_0 + I_{43}^{uu} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{44}^{uu} \ddot{\theta}_\alpha + I_{14}^{\tau u} \ddot{\tau}_{z\alpha} + I_{24}^{\tau u} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial \theta_\alpha}{\partial \alpha} (M_{\alpha\alpha}^{*14} + M_{\beta\beta}^{*24}) \right. \\ \left. + \delta \frac{\partial \theta_\alpha}{\partial \beta} (M_{\alpha\beta}^{54}) + \delta \theta_\alpha (Q_{z\alpha}^{44}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \theta_\alpha (\hat{M}_{\alpha\beta}^{14}) d\alpha - \oint_{\Gamma_\beta} \delta \theta_\alpha (\hat{M}_{\alpha\alpha}^{14}) d\beta = 0, \quad (4.66)$$

$$\int_{\Omega_0} \left[\delta \theta_\beta (I_{25}^{uu} \ddot{v}_0 + I_{35}^{uu} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{55}^{uu} \ddot{\theta}_\beta + I_{35}^{\tau u} \ddot{\tau}_{\beta z} + I_{45}^{\tau u} \ddot{\tau}_{\beta z}) + \delta \frac{\partial \theta_\beta}{\partial \beta} (M_{\alpha\alpha}^{*15} + M_{\beta\beta}^{*25}) \right. \\ \left. + \delta \frac{\partial \theta_\beta}{\partial \alpha} (M_{\alpha\beta}^{55}) + \delta \theta_\beta (Q_{\beta z}^{35}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \theta_\beta (\hat{M}_{\beta\beta}^{25}) d\alpha - \oint_{\Gamma_\beta} \delta \theta_\beta (\hat{M}_{\alpha\beta}^{25}) d\beta = 0. \quad (4.67)$$

In a similar way, for the generalized surface stress variables $\delta \bar{\tau}_{z\alpha}$, $\delta \bar{\tau}_{z\alpha}$, $\delta \bar{\tau}_{\beta z}$ and $\delta \bar{\tau}_{\beta z}$ one gets

$$\int_{\Omega_0} \left[\delta \bar{\tau}_{z\alpha} (I_{11}^{\tau u} \ddot{u}_0 + I_{13}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{14}^{\tau u} \ddot{\theta}_\alpha + I_{11}^{\tau\tau} \ddot{\tau}_{z\alpha} + I_{12}^{\tau\tau} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} (T_{\alpha\alpha}^{*11} + T_{\beta\beta}^{*21}) \right. \\ \left. + \delta \bar{\tau}_{z\alpha} (T_{z\alpha}^{41}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \beta} (T_{\alpha\beta}^{51}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \bar{\tau}_{z\alpha} (\hat{T}_{\beta\alpha}^{11}) d\alpha - \oint_{\Gamma_\beta} \delta \bar{\tau}_{z\alpha} (\hat{T}_{\alpha\alpha}^{11}) d\beta = 0, \quad (4.68)$$

$$\int_{\Omega_0} \left[\delta \bar{\tau}_{z\alpha} (I_{21}^{\tau u} \ddot{v}_0 + I_{23}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{24}^{\tau u} \ddot{\theta}_\alpha + I_{12}^{\tau\tau} \ddot{\tau}_{z\alpha} + I_{22}^{\tau\tau} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} (T_{\alpha\alpha}^{*12} + T_{\beta\beta}^{*22}) \right. \\ \left. + \delta \bar{\tau}_{z\alpha} (T_{z\alpha}^{42}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \beta} (T_{\alpha\beta}^{52}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \bar{\tau}_{z\alpha} (\hat{T}_{\beta\alpha}^{12}) d\alpha - \oint_{\Gamma_\beta} \delta \bar{\tau}_{z\alpha} (\hat{T}_{\alpha\alpha}^{12}) d\beta = 0, \quad (4.69)$$

$$\int_{\Omega_0} \left[\delta \bar{\tau}_{\beta z} (I_{32}^{\tau u} \ddot{v}_0 + I_{33}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{35}^{\tau u} \ddot{\theta}_\beta + I_{33}^{\tau \tau} \ddot{\tau}_{\beta z} + I_{34}^{\tau \tau} \ddot{\tau}_{\beta z}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} (T_{\alpha\alpha}^{*13} + T_{\beta\beta}^{*23}) \right. \\ \left. + \delta \bar{\tau}_{\beta z} (T_{\beta z}^{33}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \alpha} (T_{\alpha\beta}^{53}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \bar{\tau}_{\beta z} (\hat{T}_{\beta\beta}^{23}) d\alpha - \oint_{\Gamma_\beta} \delta \bar{\tau}_{\beta z} (\hat{T}_{\alpha\beta}^{23}) d\beta = 0, \quad (4.70)$$

$$\int_{\Omega_0} \left[\delta \bar{\tau}_{\beta z} (I_{42}^{\tau u} \ddot{v}_0 + I_{43}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{45}^{\tau u} \ddot{\theta}_\beta + I_{34}^{\tau \tau} \ddot{\tau}_{\beta z} + I_{44}^{\tau \tau} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} (T_{\alpha\alpha}^{*14} + T_{\beta\beta}^{*24}) \right. \\ \left. + \delta \bar{\tau}_{\beta z} (T_{\beta z}^{34}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \alpha} (T_{\alpha\beta}^{54}) \right] d\alpha d\beta - \oint_{\Gamma_\alpha} \delta \bar{\tau}_{\beta z} (\hat{T}_{\beta\beta}^{24}) d\alpha - \oint_{\Gamma_\beta} \delta \bar{\tau}_{\beta z} (\hat{T}_{\alpha\beta}^{24}) d\beta = 0. \quad (4.71)$$

The last equation regards the electrostatic equilibrium and is obtained by collecting the coefficients of the variation of the generalized electric potential difference $\delta\phi$ as

$$\int_{\Omega_0} \left[\delta\phi \left[(P_{\alpha\alpha}^{*11} + P_{\beta\beta}^{*21}) \frac{\partial u_0}{\partial \alpha} + (P_{\alpha\alpha}^{*12} + P_{\beta\beta}^{*22}) \frac{\partial v_0}{\partial \beta} + (P_{\alpha\alpha}^{*13} + P_{\alpha\alpha}^{*23}) w_0 + (P_{\alpha\alpha}^{*13}) \frac{\partial^2 w_0}{\partial \alpha^2} \right. \right. \\ \left. \left. + (P_{\beta\beta}^{*23}) \frac{\partial^2 w_0}{\partial \beta^2} + (P_{\alpha\alpha}^{*14} + P_{\beta\beta}^{*24}) \frac{\partial \theta_\alpha}{\partial \alpha} + (P_{\alpha\alpha}^{*15} + P_{\beta\beta}^{*25}) \frac{\partial \theta_\beta}{\partial \beta} + (R_{\alpha\alpha}^{*11} + R_{\beta\beta}^{*21}) \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} \right. \right. \\ \left. \left. + (R_{\alpha\alpha}^{*12} + R_{\beta\beta}^{*22}) \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} + (R_{\alpha\alpha}^{*13} + R_{\beta\beta}^{*23}) \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} + (R_{\alpha\alpha}^{*14} + R_{\beta\beta}^{*24}) \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} + (S_{\phi\phi}) \phi - \tau \right] d\alpha d\beta = 0. \quad (4.72)$$

Equations (4.63)-(4.72) are the weak forms of the governing equations of the doubly-curved orthotropic generic piezoelectric shell layer. As can be seen, the 3-D problem has been brought to a 2-D form in function of the reference surface curvilinear coordinates α and β . Hence, the FE solution of the shell problem can be derived in a manner similar to that of plates with some additional terms regarding the curvatures. It is worthy to mention that in the present refined shell theory no assumptions regarding the thinness of the shell were considered and as a consequence the formulation fully accounts for the effects of the z/R_α and z/R_β terms. Additionally, the "mixed" *partially refined* theory also considers additional generalized variables concerning the shear stresses on the top and bottom surfaces of the shell layer which will be used at the elemental FE level to impose transverse interlaminar (interlayer) continuity of the shear stresses and homogeneous shear stress conditions on the top and bottom global surfaces of the multilayer shell.

4.4.6 Constitutive Equations of the Internal Forces and Moments

The strains, and thereby the stresses, of the proposed theory were shown to be non-linearly dependent across the thickness of a thick anisotropic elastic shell. Thus, as far as the mathematical model is concerned, it is convenient to integrate the stress distributions through the thickness of the shell and to replace the usual consideration of stress by statically equivalent internal forces and moments. By performing such integration, the variations with respect to the thickness coordinate z are completely eliminated to yield a 2-D abstract mathematical model of the 3-D physical problem. These integrations were carried out in Appendix B.6, and the required virtual work quantities in the Hamilton's principle enunciated in Equation (4.52) were expressed in terms of internal forces and moments.

Contrarily to what is often presented in the literature, and since the thickness terms z/R_α and z/R_β were fully retained in the formulation, in the definitions of force and moment resultants given in Equations (B.41a-j) and (B.45a-j) of Appendix B.6 one may notice that the symmetry of the stress tensor (that is, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$) doesn't necessarily implies that the correspondent

force resultants or the moment resultants are equal, even if we consider the restriction of dealing with a doubly-curved shell as stated in Equations (4.39). That relation holds only for a spherical shell, flat plate or a thin shell of any type where the assumptions $1 + z/R_\alpha \approx 1$ and $1 + z/R_\beta \approx 1$ are taken into account. Vanishing of the moments about the normal to the differential element yields an additional relation between the twisting moments and twisting shear forces [cf. Reddy (2004, Sec. 8.2.4)]. In order to avoid inconsistency associated with rigid body rotations (i.e., rigid body rotation gives a non vanishing torsion except for flat plates or spherical shells) the additional relation must be accounted for in the formulation (see the treatment of Sanders which is described for example by Kraus (1967, Sec. 3.2), Leissa (1993, Sec. 1.4.5) or Reddy (2004, Sec. 8.2.4)). However, if the rotation is of the same order of magnitude as the strain components, which is actually the case in most problems, then, as noted by Koiter (1960), the torsion is negligible. Thus, for general engineering purposes the foregoing inconsistencies can generally be overlooked which will be the case in this work.

Next the constitutive equations that relate the internal forces and moments in Equations (B.41a-j) of Appendix with the strains of the layer and/or generalized displacements are derived. To this end it is recalled that an orthotropic piezoelectric material is considered for the generic shell layer and that it obeys Hooke's law under plane-stress assumption as defined in Equation (B.25) of Appendix. Thus, the internal in-plane forces are collected as

$$\begin{Bmatrix} (N_{\alpha\alpha}^{*11}, N_{\alpha\alpha}^{*12}) \\ (N_{\beta\beta}^{*21}, N_{\beta\beta}^{*22}) \\ (N_{\alpha\beta}^{*51}, N_{\alpha\beta}^{*52}) \end{Bmatrix} = \left\langle \begin{Bmatrix} \sigma_{\alpha\alpha}(z_{11}^{*\varepsilon u}, z_{12}^{*\varepsilon u}) \\ \sigma_{\beta\beta}(z_{21}^{*\varepsilon u}, z_{22}^{*\varepsilon u}) \\ \sigma_{\alpha\beta}(z_{51}^{\varepsilon u}, z_{52}^{\varepsilon u}) \end{Bmatrix}^u \right\rangle - \left\langle \begin{Bmatrix} \sigma_{\alpha\alpha}(z_{11}^{*\varepsilon u}, z_{12}^{*\varepsilon u}) \\ \sigma_{\beta\beta}(z_{21}^{*\varepsilon u}, z_{22}^{*\varepsilon u}) \\ \sigma_{\alpha\beta}(z_{51}^{\varepsilon u}, z_{52}^{\varepsilon u}) \end{Bmatrix}^\phi \right\rangle, \quad (4.73)$$

where the superscripts $(\cdot)^u$ and $(\cdot)^\phi$ denote quantities referred to the mechanical displacements (or strains) and electrical potential, and for convenience $\langle \dots \rangle$ denotes thickness integration defined as

$$\langle \dots \rangle = \int_{-h}^{+h} (\dots) H_\alpha H_\beta dz. \quad (4.74)$$

Therefore, considering the converse piezoelectric constitutive behavior in Equation (B.25) and considering only the prescribed electric field component $\hat{E}_z$ yields

$$\begin{Bmatrix} N_{\alpha\alpha}^{*1i} \\ N_{\beta\beta}^{*2i} \\ N_{\alpha\beta}^{*5i} \end{Bmatrix} = \left\langle \begin{bmatrix} z_{1i}^{*\varepsilon u} & 0 & 0 \\ 0 & z_{2i}^{*\varepsilon u} & 0 \\ 0 & 0 & z_{5i}^{\varepsilon u} \end{bmatrix} \begin{bmatrix} \bar{c}_{11}^* & \bar{c}_{12}^* & 0 \\ \bar{c}_{12}^* & \bar{c}_{22}^* & 0 \\ 0 & 0 & \bar{c}_{66}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} \right\rangle - \left\langle \begin{Bmatrix} \bar{e}_{31} z_{1i}^{*\varepsilon u} \\ \bar{e}_{32} z_{2i}^{*\varepsilon u} \\ 0 \end{Bmatrix} \hat{E}_z \right\rangle. \quad (4.75)$$

($i = 1, 2$)

Next, taking the appropriate definitions of the in-plane strains, i.e. $\varepsilon_{\alpha\alpha}$, $\varepsilon_{\beta\beta}$ and $\varepsilon_{\alpha\beta}$, defined in Equation (4.51) and substituting them into the previous equation, the internal in-plane forces are expressed in terms of the generalized variables as

$$\begin{Bmatrix} N_{\alpha\alpha}^{*1i} \\ N_{\beta\beta}^{*2i} \\ N_{\alpha\beta}^{*5i} \end{Bmatrix} = \begin{bmatrix} A_{11}^{1i} \partial_\alpha & A_{12}^{1i} \partial_\beta & A_{13}^{1i(1)} + A_{13}^{1i(2)} \partial_\alpha^2 + A_{13}^{1i(3)} \partial_\beta^2 & A_{14}^{1i} \partial_\alpha & A_{15}^{1i} \partial_\beta \\ A_{21}^{2i} \partial_\alpha & A_{22}^{2i} \partial_\beta & A_{23}^{2i(1)} + A_{23}^{2i(2)} \partial_\alpha^2 + A_{23}^{2i(3)} \partial_\beta^2 & A_{24}^{2i} \partial_\alpha & A_{25}^{2i} \partial_\beta \\ A_{51}^{5i} \partial_\beta & A_{52}^{5i} \partial_\alpha & A_{53}^{5i} \partial_{\alpha\beta} & A_{54}^{5i} \partial_\beta & A_{55}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} \\ + \begin{bmatrix} A_{16}^{1i} \partial_\alpha & A_{17}^{1i} \partial_\alpha & A_{18}^{1i} \partial_\beta & A_{19}^{1i} \partial_\beta \\ A_{26}^{2i} \partial_\alpha & A_{27}^{2i} \partial_\alpha & A_{28}^{2i} \partial_\beta & A_{29}^{2i} \partial_\beta \\ A_{56}^{5i} \partial_\beta & A_{57}^{5i} \partial_\beta & A_{58}^{5i} \partial_\alpha & A_{59}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix} + \begin{Bmatrix} P_{\alpha\alpha}^{*1i} \\ P_{\beta\beta}^{*2i} \\ 0 \end{Bmatrix} \phi, \quad (4.76)$$

$$(i = 1, 2)$$

where the non-zero coefficients A_{rs}^{ri} , $P_{\alpha\alpha}^{\star 1i}$ and $P_{\alpha\alpha}^{\star 2i}$ (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 1, 2$) are presented in Appendixes B.8 and B.6.2, respectively.

Similarly, the out-of-plane forces are collected as

$$\left\{ \begin{pmatrix} Q_{\beta z}^{32}, Q_{\beta z}^{33}, Q_{\beta z}^{35} \\ Q_{z\alpha}^{41}, Q_{z\alpha}^{43}, Q_{z\alpha}^{44} \end{pmatrix} \right\} = \left\langle \begin{pmatrix} \sigma_{\beta z}(z_{32}^{\varepsilon u}, z_{33}^{\varepsilon u}, z_{35}^{\varepsilon u}) \\ \sigma_{z\alpha}(z_{41}^{\varepsilon u}, z_{43}^{\varepsilon u}, z_{44}^{\varepsilon u}) \end{pmatrix} \right\rangle, \quad (4.77)$$

which, since they are not coupled with the transverse electric field but only with the induced in-plane components which effects were already considered through the use of *effective shear stiffness parameters*, are expressed in terms of the shear strains as

$$\begin{pmatrix} Q_{\beta z}^{3i} \\ Q_{z\alpha}^{4j} \end{pmatrix} = \left\langle \begin{bmatrix} z_{3i}^{\varepsilon u} & 0 \\ 0 & z_{4j}^{\varepsilon u} \end{bmatrix} \begin{bmatrix} \bar{c}_{44}^{\star} & 0 \\ 0 & \bar{c}_{55}^{\star} \end{bmatrix} \begin{pmatrix} \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \end{pmatrix} \right\rangle. \quad (4.78)$$

$$(i = 2, 3, 5; j = 1, 3, 4)$$

Again, taking the definitions of the out-of-plane strains in Equation (4.51) and substituting them into the previous equation, we get

$$\begin{pmatrix} Q_{\beta z}^{3i} \\ Q_{z\alpha}^{4j} \end{pmatrix} = \begin{bmatrix} 0 & A_{32}^{3i} & A_{33}^{3i} \partial_{\beta} & 0 & A_{35}^{3i} \\ A_{41}^{4j} & 0 & A_{43}^{4j} \partial_{\alpha} & A_{44}^{4j} & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_{\alpha} \\ \theta_{\beta} \end{pmatrix} + \begin{bmatrix} 0 & 0 & A_{38}^{3i} & A_{39}^{3i} \\ A_{46}^{4j} & A_{47}^{4j} & 0 & 0 \end{bmatrix} \begin{pmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{pmatrix}, \quad (4.79)$$

$$(i = 2, 3, 5; j = 1, 3, 4)$$

where the non-zero coefficients A_{ps}^{pi} (with $p = 3, 4$, $s = 1, \dots, 9$ and $i = 1, \dots, 5$) are presented in Appendix B.8.

The moment resultants of the in-plane stresses are also collected and expressed by

$$\begin{pmatrix} (M_{\alpha\alpha}^{\star 13(1)}, M_{\alpha\alpha}^{\star 13(2)}, M_{\alpha\alpha}^{\star 14}, M_{\alpha\alpha}^{\star 15}) \\ (M_{\beta\beta}^{\star 23(1)}, M_{\beta\beta}^{\star 23(2)}, M_{\beta\beta}^{\star 24}, M_{\beta\beta}^{\star 25}) \\ (M_{\alpha\beta}^{53}, M_{\alpha\beta}^{54}, M_{\alpha\beta}^{55}) \end{pmatrix} = \left\langle \begin{pmatrix} \sigma_{\alpha\alpha}(z_{13}^{\star \varepsilon u(1)}, z_{13}^{\star \varepsilon u(2)}, z_{14}^{\star \varepsilon u}, z_{15}^{\star \varepsilon u}) \\ \sigma_{\beta\beta}(z_{23}^{\star \varepsilon u(1)}, z_{23}^{\star \varepsilon u(2)}, z_{24}^{\star \varepsilon u}, z_{25}^{\star \varepsilon u}) \\ \sigma_{\alpha\beta}(z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u}) \end{pmatrix}^u \right\rangle$$

$$- \left\langle \begin{pmatrix} \sigma_{\alpha\alpha}(z_{13}^{\star \varepsilon u(1)}, z_{13}^{\star \varepsilon u(2)}, z_{14}^{\star \varepsilon u}, z_{15}^{\star \varepsilon u}) \\ \sigma_{\beta\beta}(z_{23}^{\star \varepsilon u(1)}, z_{23}^{\star \varepsilon u(2)}, z_{24}^{\star \varepsilon u}, z_{25}^{\star \varepsilon u}) \\ \sigma_{\alpha\beta}(z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u}) \end{pmatrix}^{\phi} \right\rangle. \quad (4.80)$$

Similarly to what has been considered for the in-plane forces in Equation (4.75), the internal moments are re-written as

$$\begin{pmatrix} M_{\alpha\alpha}^{\star 1i(j)} \\ M_{\beta\beta}^{\star 2i(j)} \\ M_{\alpha\beta}^{5i} \end{pmatrix} = \left\langle \begin{bmatrix} z_{1i}^{\star \varepsilon u(j)} & 0 & 0 \\ 0 & z_{2i}^{\star \varepsilon u(j)} & 0 \\ 0 & 0 & z_{5i}^{\varepsilon u} \end{bmatrix} \begin{bmatrix} \bar{c}_{11}^{\star} & \bar{c}_{12}^{\star} & 0 \\ \bar{c}_{12}^{\star} & \bar{c}_{22}^{\star} & 0 \\ 0 & 0 & \bar{c}_{66}^{\star} \end{bmatrix} \begin{pmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\alpha\beta} \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} \bar{e}_{31} z_{1i}^{\star \varepsilon u(j)} \\ \bar{e}_{32} z_{2i}^{\star \varepsilon u(j)} \\ 0 \end{pmatrix} \hat{E}_z \right\rangle. \quad (4.81)$$

$$(i = 3, 4, 5; j = 0, 1, 2)$$

For the sake of simplifying the equations, it is worthy to note that in the previous equation it was tacitly assumed that for $j = 0$ we have $M_{\alpha\alpha}^{\star 1i(0)}$ denoting $M_{\alpha\alpha}^{\star 1i}$ with $(z_{1i}^{\star \varepsilon u(0)}, z_{2i}^{\star \varepsilon u(0)})$

being the same as $(z_{1i}^{\star\epsilon u}, z_{2i}^{\star\epsilon u})$. Thus, substituting the definitions of the out-of-plane strains in Equation (4.51) into the previous equation yields

$$\begin{aligned} \begin{Bmatrix} M_{\alpha\alpha}^{\star 1i(j)} \\ M_{\beta\beta}^{\star 2i(j)} \\ M_{\alpha\beta}^{5i} \end{Bmatrix} &= \begin{bmatrix} B_{11}^{1i(j)} \partial_\alpha & B_{12}^{1i(j)} \partial_\beta & B_{13}^{1i(j1)} + B_{13}^{1i(j2)} \partial_\alpha^2 + B_{13}^{1i(j3)} \partial_\beta^2 & B_{14}^{1i(j)} \partial_\alpha & B_{15}^{1i(j)} \partial_\beta \\ B_{21}^{2i(j)} \partial_\alpha & B_{22}^{2i(j)} \partial_\beta & B_{23}^{2i(j1)} + B_{23}^{2i(j2)} \partial_\alpha^2 + B_{23}^{2i(j3)} \partial_\beta^2 & B_{24}^{2i(j)} \partial_\alpha & B_{25}^{2i(j)} \partial_\beta \\ B_{51}^{5i} \partial_\beta & B_{52}^{5i} \partial_\alpha & B_{53}^{5i} \partial_{\alpha\beta} & B_{54}^{5i} \partial_\beta & B_{55}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} \\ &+ \begin{bmatrix} B_{16}^{1i(j)} \partial_\alpha & B_{17}^{1i(j)} \partial_\alpha & B_{18}^{1i(j)} \partial_\beta & B_{19}^{1i(j)} \partial_\beta \\ B_{26}^{2i(j)} \partial_\alpha & B_{27}^{2i(j)} \partial_\alpha & B_{28}^{2i(j)} \partial_\beta & B_{29}^{2i(j)} \partial_\beta \\ B_{56}^{5i} \partial_\beta & B_{57}^{5i} \partial_\beta & B_{58}^{5i} \partial_\alpha & B_{59}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix} + \begin{Bmatrix} P_{\alpha\alpha}^{\star 1i(j)} \\ P_{\beta\beta}^{\star 2i(j)} \\ 0 \end{Bmatrix} \phi, \quad (4.82) \\ &(i = 3, 4, 5; j = 0, 1, 2) \end{aligned}$$

where the non-zero coefficients $B_{rs}^{ri(jk)}$, $P_{\alpha\alpha}^{\star 1i(j)}$ and $P_{\alpha\alpha}^{\star 2i(j)}$ (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 3, 4, 5$, $j = 0, 1, 2$, $k = 0, \dots, 3$) are presented in Appendixes B.8 and B.6.2, respectively.

Lastly, in a similar way to what has been done in Equations (4.75) and (4.78), the resultants of the stresses related with the interlayer surface stresses are given by

$$\begin{aligned} \begin{Bmatrix} T_{\alpha\alpha}^{\star 1i} \\ T_{\beta\beta}^{\star 2i} \\ T_{\alpha\beta}^{5i} \end{Bmatrix} &= \left\langle \begin{bmatrix} z_{1i}^{\star\epsilon\tau} & 0 & 0 \\ 0 & z_{2i}^{\star\epsilon\tau} & 0 \\ 0 & 0 & z_{5i}^{\star\epsilon\tau} \end{bmatrix} \begin{bmatrix} \bar{c}_{11}^* & \bar{c}_{12}^* & 0 \\ \bar{c}_{12}^* & \bar{c}_{22}^* & 0 \\ 0 & 0 & \bar{c}_{66}^* \end{bmatrix} \begin{Bmatrix} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \epsilon_{\alpha\beta} \end{Bmatrix} \right\rangle - \left\langle \begin{bmatrix} \bar{e}_{31} z_{1i}^{\star\epsilon\tau} \\ \bar{e}_{32} z_{2i}^{\star\epsilon\tau} \\ 0 \end{bmatrix} \hat{E}_z \right\rangle, \quad (4.83) \\ &(i = 1, \dots, 4) \end{aligned}$$

and

$$\begin{aligned} \begin{Bmatrix} T_{\beta z}^{3i} \\ T_{z\alpha}^{4j} \end{Bmatrix} &= \left\langle \begin{bmatrix} z_{3i}^{\epsilon\tau} & 0 \\ 0 & z_{4j}^{\epsilon\tau} \end{bmatrix} \begin{bmatrix} \bar{c}_{44}^* & 0 \\ 0 & \bar{c}_{55}^* \end{bmatrix} \begin{Bmatrix} \epsilon_{\beta z} \\ \epsilon_{z\alpha} \end{Bmatrix} \right\rangle, \quad (4.84) \\ &(i = 3, 4; j = 1, 2) \end{aligned}$$

which after substituting the appropriate definitions of the in-plane strains defined in Equation (4.51) are expressed as

$$\begin{aligned} \begin{Bmatrix} T_{\alpha\alpha}^{\star 1i} \\ T_{\beta\beta}^{\star 2i} \\ T_{\alpha\beta}^{5i} \end{Bmatrix} &= \begin{bmatrix} C_{11}^{1i} \partial_\alpha & C_{12}^{1i} \partial_\beta & C_{13}^{1i(1)} + C_{13}^{1i(2)} \partial_\alpha^2 + C_{13}^{1i(3)} \partial_\beta^2 & C_{14}^{1i} \partial_\alpha & C_{15}^{1i} \partial_\beta \\ C_{21}^{2i} \partial_\alpha & C_{22}^{2i} \partial_\beta & C_{23}^{2i(1)} + C_{23}^{2i(2)} \partial_\alpha^2 + C_{23}^{2i(3)} \partial_\beta^2 & C_{24}^{2i} \partial_\alpha & C_{25}^{2i} \partial_\beta \\ C_{51}^{5i} \partial_\beta & C_{52}^{5i} \partial_\alpha & C_{53}^{5i} \partial_{\alpha\beta} & C_{54}^{5i} \partial_\beta & C_{55}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} \\ &+ \begin{bmatrix} C_{16}^{1i} \partial_\alpha & C_{17}^{1i} \partial_\alpha & C_{18}^{1i} \partial_\beta & C_{19}^{1i} \partial_\beta \\ C_{26}^{2i} \partial_\alpha & C_{27}^{2i} \partial_\alpha & C_{28}^{2i} \partial_\beta & C_{29}^{2i} \partial_\beta \\ C_{56}^{5i} \partial_\beta & C_{57}^{5i} \partial_\beta & C_{58}^{5i} \partial_\alpha & C_{59}^{5i} \partial_\alpha \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix} + \begin{Bmatrix} R_{\alpha\alpha}^{\star 1i} \\ R_{\alpha\alpha}^{\star 2i} \\ 0 \end{Bmatrix} \phi, \quad (4.85) \\ &(i = 1, \dots, 4) \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} T_{\beta z}^{3i} \\ T_{z\alpha}^{4j} \end{Bmatrix} &= \begin{bmatrix} 0 & C_{32}^{3i} & C_{33}^{3i} \partial_\beta & 0 & C_{35}^{3i} \\ C_{41}^{4j} & 0 & C_{43}^{4j} \partial_\alpha & C_{44}^{4j} & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_\alpha \\ \theta_\beta \end{Bmatrix} + \begin{bmatrix} 0 & 0 & C_{38}^{3i} & C_{39}^{3i} \\ C_{46}^{4j} & C_{47}^{4j} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix}. \quad (4.86) \\ &(i = 3, 4; j = 1, 2) \end{aligned}$$

The non-zero coefficients C_{rs}^{ri} , $R_{\alpha\alpha}^{\star 1i}$ and $R_{\alpha\alpha}^{\star 2i}$ (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 1, \dots, 4$) and C_{ps}^{pi} (with $p = 3, 4$, $s = 1, \dots, 9$ and $i = 1, \dots, 4$) are presented in Appendixes B.8 and B.6.2.

4.5 Summary

Based on the proposed *fully refined* mathematical model of general anisotropic shells a fully coupled piezo-visco-elastic FE model has been conceptually proposed in this chapter for multilayered shells. However, for practical reasons, related with the complexity of the formulation, simplifications regarding the through-the-thickness distribution of the transverse displacement were considered and a *partially refined* theory was derived with additional restrictions inherent to doubly-curved orthotropic shells physics. No simplifications regarding the thinness of the shell were considered and a plane-stress state was considered for the partially refined theory.

It was shown that the refined assumptions and relaxation of some of Love's classical assumptions lead to a "mixed" definition of the displacement field in terms of the same generalized displacements of the FSDT and CLT, and shear stresses on the top and bottom surfaces.

The electrical potential was defined assuming negligible in-plane electric displacement field which allowed the use of *effective shear stiffness parameters* which account for the effects of the induced in-plane electric field components. Regarding the transverse (out-of-plane) electrical behavior, the induced transverse electric field was considered by means of a modification of the mechanical in-plane strains, which in an analogous way were modified into *effective strains*. Thus, the direct piezoelectric effect was condensed into the model through *effective* stiffness and strains definitions, and the converse counterpart was considered by the action of prescribed electric potential differences in each piezoelectric layer.

The weak forms of the governing equations of a generic single layer of the multilayered shell were derived with Hamilton's principle in conjunction with the "mixed" displacement field and the electric potential difference definitions. The resultant weak forms will be used to the development of a FE solution in the next chapter.

CHAPTER 5

Finite Element Model of Piezo-Visco-Elastic Shells

5.1 Introduction

In this chapter the weak form of the governing equations of a generic piezo-visco-elastic layer of a multilayered shell derived following the *partially refined* shell model proposed in the previous chapter is utilized to develop a multilayered shell FE solution method. Based on the weak forms a FE solution is initially developed for the single layer. The degrees of freedom (DoFs) of the resultant four-noded generic piezo-elastic single layer FE are then “regenerated” into an equivalent eight-node 3-D formulation in order to allow through-the-thickness assemblage of displacements and stresses, yielding a partially refined multilayer FE assuring displacement and shear stress interlayer continuity and homogeneous shear stress conditions at the outer surfaces. Afterward, the deformation theory and implemented FE model are preliminarily assessed and validated through the FE free vibration analysis of a single layer plate and cylindrical shell.

5.2 Comments on the FE Solution of the Refined Models

In this section the FE solution of the weak form of the governing electro-mechanical coupled equations of the partially refined mathematical model of doubly curved shells in Equations (4.63)-(4.72) is developed. Regarding the FE solution of the fully refined model, it will not be derived here for reasons related with the complexity of the formulation. As can be seen from the fully refined definitions of the displacement field presented in Equations (4.20), (4.23) and (4.24), the fully refined weak forms would involve high-order derivatives of the generalized displacements which would complicate the formulation and FE solution dramatically. That would require higher order continuity of the variables, which would be cumbersome for FE solutions, with the outcome of considering an equivalent 2-D theory representative of the full 3-D elasticity problem. It is well known, however, that in the major part of the problems, the transverse stress is small when compared with the other stress components. An exception is, for example, in thermo-mechanical analysis where the transverse stress σ_{zz} plays an important role [see Carrera (1999, 2005)], which is not the case here. That refinement is not pursued here since the trade-off between accuracy and complexity is not appellative for the physical problem

to be treated in this work.

5.3 Spatial Approximation

For the sake of brevity the weak forms of the partially refined model in Equations (4.63)-(4.72) are expressed in terms of the internal forces and moments (stress resultants). However, if the constitutive equations of the internal forces and moments, presented in Sec. 4.4.6 and detailed in Appendix B.8, are taken into account, the weak forms can still be expressed in terms of the generalized variables. That will not be made here explicitly, but those relations will be implicitly taken into account to derive the elemental matrices and vectors.

Thus, from the analysis of the weak forms in (4.63)-(4.72) and/or the internal forces and moments in Appendix B.8, it can be seen that they contain at the most first-order derivatives of the generalized displacements u_0 , v_0 , θ_α and θ_β and surface stresses $\bar{\tau}_{z\alpha}$, $\tilde{\tau}_{z\alpha}$, $\bar{\tau}_{\beta z}$ and $\tilde{\tau}_{\beta z}$, requiring C^0 continuity, and in contrast to what is traditionally obtained with the FSDT, the present partial refined model contain also at the most second-order derivatives of the transverse displacement w_0 , requiring C^1 continuity, which is something that is obtained with the CLT. Thus, the partially refined model, at first sight, yields something that resembles a blend of the CLT and FSDT. Additionally, the electric potential difference requires zero-order derivatives, and is assumed constant at the elemental electrodes, which is convenient since it imposes at least at the elemental level the physical equipotential area condition of the electrodes, which should also be assumed between adjacent FEs. Thus, the displacement variables, u_0 , v_0 , θ_α , θ_β , w_0 , $\partial w_0/\partial\alpha$, $\partial w_0/\partial\beta$ and (or not) $\partial^2 w_0/\partial\alpha\partial\beta$ (*nonconforming* or *conforming* elemental approaches), and shear stress variables, $\bar{\tau}_{z\alpha}$, $\tilde{\tau}_{z\alpha}$, $\bar{\tau}_{\beta z}$ and $\tilde{\tau}_{\beta z}$, must be carried as nodal variables in order to enforce their interelement continuity. Regarding the electric potential difference variable, ϕ , is carried out as an elemental variable and interelement continuity is not enforced.

For the FE solution, linear Lagrange C^0 continuity rectangular interpolation functions might be used to approximate all the displacement and stress variables whereas the generalized transverse displacement w_0 should be approximated using Hermite C^1 continuity rectangular interpolation functions over a four-noded element Ω_0^e . The combined conforming or nonconforming elements have a total of 12 or 11 degrees of freedom (DoFs) per node, respectively, and 1 electrical DoF per element. Therefore, consider the following approximations for the displacements,

$$u_0(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{u}_0^j(t) L_j^e(\alpha, \beta), \quad (5.1a)$$

$$v_0(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{v}_0^j(t) L_j^e(\alpha, \beta), \quad (5.1b)$$

$$\theta_\alpha(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{\theta}_\alpha^j(t) L_j^e(\alpha, \beta), \quad (5.1c)$$

$$\theta_\beta(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{\theta}_\beta^j(t) L_j^e(\alpha, \beta), \quad (5.1d)$$

$$w_0(\alpha, \beta, t) \approx \sum_{r=1}^m \bar{w}_0^r(t) H_r^e(\alpha, \beta), \quad (5.1e)$$

surface shear stresses,

$$\bar{\tau}_{z\alpha}(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{\tau}_{z\alpha}^j(t) L_j^e(\alpha, \beta), \quad (5.1f)$$

$$\tilde{\tau}_{z\alpha}(\alpha, \beta, t) \approx \sum_{j=1}^n \tilde{\tau}_{z\alpha}^j(t) L_j^e(\alpha, \beta), \quad (5.1g)$$

$$\bar{\tau}_{\beta z}(\alpha, \beta, t) \approx \sum_{j=1}^n \bar{\tau}_{\beta z}^j(t) L_j^e(\alpha, \beta), \quad (5.1h)$$

$$\tilde{\tau}_{\beta z}(\alpha, \beta, t) \approx \sum_{j=1}^n \tilde{\tau}_{\beta z}^j(t) L_j^e(\alpha, \beta), \quad (5.1i)$$

and electrical potential difference,

$$\phi(\alpha, \beta, t) \approx \bar{\phi}(t), \quad (5.1j)$$

where $\bar{u}_0^j, \bar{v}_0^j, \bar{\theta}_\alpha^j, \bar{\theta}_\beta^j$ and $\bar{\tau}_{z\alpha}^j, \bar{\tau}_{z\alpha}^j, \bar{\tau}_{\beta z}^j, \bar{\tau}_{\beta z}^j$ (with $j = 1, \dots, n$) denote the values of the generalized in-plane displacements, rotations and surface shear stresses at the j th node of the Lagrange elements, $\bar{w}_0^r$ denote the values of w_0 and its derivatives with respect to α and β at the nodes of the Hermite elements (with $r = 1, \dots, m$), which for a conforming element with the same number of nodes as the Lagrange elements are

$$\bar{w}_0^j = \bar{w}_0^j, \quad \bar{w}_0^{j+n} = \frac{\partial \bar{w}_0^j}{\partial \alpha}, \quad \bar{w}_0^{j+2n} = \frac{\partial \bar{w}_0^j}{\partial \beta}, \quad \bar{w}_0^{j+3n} = \frac{\partial^2 \bar{w}_0^j}{\partial \alpha \partial \beta}, \quad (5.2)$$

and L_j^e and H_r^e are the Lagrange and Hermite elemental interpolation functions, respectively. For the conforming four-noded rectangular element ($n = 4$ and $m = 12$) the total number of DoFs per element is 49 and for the nonconforming is 45.

5.4 Discrete Finite Element Equations of the Shell Layer

Substituting the spatial approximations of the generalized displacements, surface stresses and electric potential difference in Equations (5.1) into the weak forms in Equations (4.63)-(4.72), the i th equation associated with each weak form is given as

$$\begin{aligned} \sum_{j=1}^n (M_{ij}^{11} \ddot{u}_0^j + M_{ij}^{14} \ddot{\theta}_\alpha^j + M_{ij}^{16} \ddot{\tau}_{z\alpha}^j + M_{ij}^{17} \ddot{\tau}_{z\alpha}^j + K_{ij}^{11} \bar{u}_0^j + K_{ij}^{12} \bar{v}_0^j + K_{ij}^{14} \bar{\theta}_\alpha^j + K_{ij}^{15} \bar{\theta}_\beta^j \\ + K_{ij}^{16} \bar{\tau}_{z\alpha}^j + K_{ij}^{17} \bar{\tau}_{z\alpha}^j + K_{ij}^{18} \bar{\tau}_{\beta z}^j + K_{ij}^{19} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{13} \ddot{w}_0^r + K_{ir}^{13} \bar{w}_0^r) + K_{i\phi}^1 \bar{\phi} - F_i^1 = 0, \end{aligned} \quad (5.3a)$$

$$\begin{aligned} \sum_{j=1}^n (M_{ij}^{22} \ddot{v}_0^j + M_{ij}^{25} \ddot{\theta}_\beta^j + M_{ij}^{28} \ddot{\tau}_{\beta z}^j + M_{ij}^{29} \ddot{\tau}_{\beta z}^j + K_{ij}^{21} \bar{u}_0^j + K_{ij}^{22} \bar{v}_0^j + K_{ij}^{24} \bar{\theta}_\alpha^j + K_{ij}^{25} \bar{\theta}_\beta^j \\ + K_{ij}^{26} \bar{\tau}_{z\alpha}^j + K_{ij}^{27} \bar{\tau}_{z\alpha}^j + K_{ij}^{28} \bar{\tau}_{\beta z}^j + K_{ij}^{29} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{23} \ddot{w}_0^r + K_{ir}^{23} \bar{w}_0^r) + K_{i\phi}^2 \bar{\phi} - F_i^2 = 0, \end{aligned} \quad (5.3b)$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{rj}^{31} \ddot{u}_0^j + M_{rj}^{32} \ddot{v}_0^j + M_{rj}^{34} \ddot{\theta}_\alpha^j + M_{rj}^{35} \ddot{\theta}_\beta^j + M_{rj}^{36} \ddot{\tau}_{z\alpha}^j + M_{rj}^{37} \ddot{\tau}_{z\alpha}^j + M_{rj}^{38} \ddot{\tau}_{\beta z}^j + M_{rj}^{39} \ddot{\tau}_{\beta z}^j \\
& + K_{rj}^{31} \bar{u}_0^j + K_{rj}^{32} \bar{v}_0^j + K_{rj}^{34} \bar{\theta}_\alpha^j + K_{rj}^{35} \bar{\theta}_\beta^j + K_{rj}^{36} \bar{\tau}_{z\alpha}^j + K_{rj}^{37} \bar{\tau}_{z\alpha}^j + K_{rj}^{38} \bar{\tau}_{\beta z}^j + K_{rj}^{39} \bar{\tau}_{\beta z}^j) \\
& + \sum_{s=1}^m (M_{rs}^{33} \ddot{w}_0^s + K_{rs}^{33} \bar{w}_0^s) + K_{r\phi}^3 \bar{\phi} - F_r^3 = 0, \quad (5.3c)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{41} \ddot{u}_0^j + M_{ij}^{44} \ddot{\theta}_\alpha^j + M_{ij}^{46} \ddot{\tau}_{z\alpha}^j + M_{ij}^{47} \ddot{\tau}_{z\alpha}^j + K_{ij}^{41} \bar{u}_0^j + K_{ij}^{42} \bar{v}_0^j + K_{ij}^{44} \bar{\theta}_\alpha^j + K_{ij}^{45} \bar{\theta}_\beta^j \\
& + K_{ij}^{46} \bar{\tau}_{z\alpha}^j + K_{ij}^{47} \bar{\tau}_{z\alpha}^j + K_{ij}^{48} \bar{\tau}_{\beta z}^j + K_{ij}^{49} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{43} \ddot{w}_0^r + K_{ir}^{43} \bar{w}_0^r) + K_{i\phi}^4 \bar{\phi} - F_i^4 = 0, \quad (5.3d)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{52} \ddot{v}_0^j + M_{ij}^{55} \ddot{\theta}_\beta^j + M_{ij}^{58} \ddot{\tau}_{\beta z}^j + M_{ij}^{59} \ddot{\tau}_{\beta z}^j + K_{ij}^{51} \bar{u}_0^j + K_{ij}^{52} \bar{v}_0^j + K_{ij}^{54} \bar{\theta}_\alpha^j + K_{ij}^{55} \bar{\theta}_\beta^j \\
& + K_{ij}^{56} \bar{\tau}_{z\alpha}^j + K_{ij}^{57} \bar{\tau}_{z\alpha}^j + K_{ij}^{58} \bar{\tau}_{\beta z}^j + K_{ij}^{59} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{53} \ddot{w}_0^r + K_{ir}^{53} \bar{w}_0^r) + K_{i\phi}^5 \bar{\phi} - F_i^5 = 0, \quad (5.3e)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{61} \ddot{u}_0^j + M_{ij}^{64} \ddot{\theta}_\alpha^j + M_{ij}^{66} \ddot{\tau}_{z\alpha}^j + M_{ij}^{67} \ddot{\tau}_{z\alpha}^j + K_{ij}^{61} \bar{u}_0^j + K_{ij}^{62} \bar{v}_0^j + K_{ij}^{64} \bar{\theta}_\alpha^j + K_{ij}^{65} \bar{\theta}_\beta^j \\
& + K_{ij}^{66} \bar{\tau}_{z\alpha}^j + K_{ij}^{67} \bar{\tau}_{z\alpha}^j + K_{ij}^{68} \bar{\tau}_{\beta z}^j + K_{ij}^{69} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{63} \ddot{w}_0^r + K_{ir}^{63} \bar{w}_0^r) + K_{i\phi}^6 \bar{\phi} - F_i^6 = 0, \quad (5.3f)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{71} \ddot{u}_0^j + M_{ij}^{74} \ddot{\theta}_\alpha^j + M_{ij}^{76} \ddot{\tau}_{z\alpha}^j + M_{ij}^{77} \ddot{\tau}_{z\alpha}^j + K_{ij}^{71} \bar{u}_0^j + K_{ij}^{72} \bar{v}_0^j + K_{ij}^{74} \bar{\theta}_\alpha^j + K_{ij}^{75} \bar{\theta}_\beta^j \\
& + K_{ij}^{76} \bar{\tau}_{z\alpha}^j + K_{ij}^{77} \bar{\tau}_{z\alpha}^j + K_{ij}^{78} \bar{\tau}_{\beta z}^j + K_{ij}^{79} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{73} \ddot{w}_0^r + K_{ir}^{73} \bar{w}_0^r) + K_{i\phi}^7 \bar{\phi} - F_i^7 = 0, \quad (5.3g)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{82} \ddot{v}_0^j + M_{ij}^{85} \ddot{\theta}_\beta^j + M_{ij}^{88} \ddot{\tau}_{\beta z}^j + M_{ij}^{89} \ddot{\tau}_{\beta z}^j + K_{ij}^{81} \bar{u}_0^j + K_{ij}^{82} \bar{v}_0^j + K_{ij}^{84} \bar{\theta}_\alpha^j + K_{ij}^{85} \bar{\theta}_\beta^j \\
& + K_{ij}^{86} \bar{\tau}_{z\alpha}^j + K_{ij}^{87} \bar{\tau}_{z\alpha}^j + K_{ij}^{88} \bar{\tau}_{\beta z}^j + K_{ij}^{89} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{83} \ddot{w}_0^r + K_{ir}^{83} \bar{w}_0^r) + K_{i\phi}^8 \bar{\phi} - F_i^8 = 0, \quad (5.3h)
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^n (M_{ij}^{92} \ddot{v}_0^j + M_{ij}^{95} \ddot{\theta}_\beta^j + M_{ij}^{98} \ddot{\tau}_{\beta z}^j + M_{ij}^{99} \ddot{\tau}_{\beta z}^j + K_{ij}^{91} \bar{u}_0^j + K_{ij}^{92} \bar{v}_0^j + K_{ij}^{94} \bar{\theta}_\alpha^j + K_{ij}^{95} \bar{\theta}_\beta^j \\
& + K_{ij}^{96} \bar{\tau}_{z\alpha}^j + K_{ij}^{97} \bar{\tau}_{z\alpha}^j + K_{ij}^{98} \bar{\tau}_{\beta z}^j + K_{ij}^{99} \bar{\tau}_{\beta z}^j) + \sum_{r=1}^m (M_{ir}^{93} \ddot{w}_0^r + K_{ir}^{93} \bar{w}_0^r) + K_{i\phi}^9 \bar{\phi} - F_i^9 = 0, \quad (5.3i)
\end{aligned}$$

where $i = 1, \dots, n$, and $r = 1, \dots, m$. Regarding the weak form of the equation governing the electrostatic equilibrium in Equation (4.72) yields

$$K_{\phi i}^1 \bar{u}_0^i + K_{\phi i}^2 \bar{v}_0^i + K_{\phi r}^3 \bar{w}_0^r + K_{\phi i}^4 \bar{\theta}_\alpha^i + K_{\phi i}^5 \bar{\theta}_\beta^i + K_{\phi i}^6 \bar{\tau}_{z\alpha}^i + K_{\phi i}^7 \bar{\tau}_{z\alpha}^i + K_{\phi i}^8 \bar{\tau}_{\beta z}^i + K_{\phi i}^9 \bar{\tau}_{\beta z}^j + K_{\phi\phi} \bar{\phi} - Q_\phi = 0. \quad (5.3j)$$

The coefficients of the mass matrix $M_{ij}^{xy} = M_{ji}^{yx}$, stiffness matrix $K_{ij}^{xy} = K_{ji}^{yx}$, piezoelectric coupling matrix $K_{\phi i}^x = K_{\phi i}^x$, capacitance matrix $K_{\phi\phi}$, force vectors F_i^x and electric charge Q_ϕ are defined in Appendix B.9.

In matrix notation Equations (5.3a)-(5.3j) can be expressed in terms of the elemental matrices and vectors of the generic layer l as

$$\begin{bmatrix} \mathbf{M}_{uu}^l & \mathbf{M}_{u\tau}^l \\ \mathbf{M}_{\tau u}^l & \mathbf{M}_{\tau\tau}^l \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}^l(t) \\ \ddot{\boldsymbol{\tau}}^l(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu}^l & \mathbf{K}_{u\tau}^l \\ \mathbf{K}_{\tau u}^l & \mathbf{K}_{\tau\tau}^l \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^l(t) \\ \bar{\boldsymbol{\tau}}^l(t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{K}_{u\phi}^l \\ \mathbf{K}_{\tau\phi}^l \end{Bmatrix} \bar{\phi}(t) = \begin{Bmatrix} \mathbf{F}_u^l(t) \\ \mathbf{F}_\tau^l(t) \end{Bmatrix}, \quad (5.4a)$$

$$\begin{Bmatrix} \mathbf{K}_{\phi u}^l & \mathbf{K}_{\phi\tau}^l \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^l(t) \\ \bar{\boldsymbol{\tau}}^l(t) \end{Bmatrix} + K_{\phi\phi} \bar{\phi}(t) = Q_\phi(t), \quad (5.4b)$$

where, since $\mathbf{K}^{yx} = (\mathbf{K}^{xy})^T$ and $\mathbf{M}^{yx} = (\mathbf{M}^{xy})^T$, one gets $\mathbf{M}_{\tau u}^l = (\mathbf{M}_{u\tau}^l)^T$, $\mathbf{K}_{\tau u}^l = (\mathbf{K}_{u\tau}^l)^T$, $\mathbf{K}_{\phi u}^l = (\mathbf{K}_{u\phi}^l)^T$ and $\mathbf{K}_{\tau\phi}^l = (\mathbf{K}_{\phi\tau}^l)^T$, and the matrices and vectors are

$$\mathbf{M}_{uu}^l = \begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{M}^{13} & \mathbf{M}^{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{0} & \mathbf{M}^{25} \\ \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} & \mathbf{M}^{35} \\ \mathbf{M}^{41} & \mathbf{0} & \mathbf{M}^{43} & \mathbf{M}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{52} & \mathbf{M}^{53} & \mathbf{0} & \mathbf{M}^{55} \end{bmatrix}, \quad \mathbf{K}_{uu}^l = \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix}, \quad (5.5a,b)$$

$$\mathbf{M}_{u\tau}^l = \begin{bmatrix} \mathbf{M}^{16} & \mathbf{M}^{17} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{28} & \mathbf{M}^{29} \\ \mathbf{M}^{36} & \mathbf{M}^{37} & \mathbf{M}^{38} & \mathbf{M}^{39} \\ \mathbf{M}^{46} & \mathbf{M}^{47} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{58} & \mathbf{M}^{59} \end{bmatrix}, \quad \mathbf{M}_{\tau\tau}^l = \begin{bmatrix} \mathbf{M}^{16} & \mathbf{M}^{17} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{28} & \mathbf{M}^{29} \\ \mathbf{M}^{36} & \mathbf{M}^{37} & \mathbf{M}^{38} & \mathbf{M}^{39} \\ \mathbf{M}^{46} & \mathbf{M}^{47} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{58} & \mathbf{M}^{59} \end{bmatrix}, \quad (5.6a,b)$$

$$\mathbf{M}_{\tau\tau}^l = \begin{bmatrix} \mathbf{M}^{66} & \mathbf{M}^{67} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{76} & \mathbf{M}^{77} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{88} & \mathbf{M}^{89} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{98} & \mathbf{M}^{99} \end{bmatrix}, \quad \mathbf{K}_{\tau\tau}^l = \begin{bmatrix} \mathbf{K}^{66} & \mathbf{K}^{67} & \mathbf{K}^{68} & \mathbf{K}^{69} \\ \mathbf{K}^{76} & \mathbf{K}^{77} & \mathbf{K}^{78} & \mathbf{K}^{79} \\ \mathbf{K}^{86} & \mathbf{K}^{87} & \mathbf{K}^{88} & \mathbf{K}^{89} \\ \mathbf{K}^{96} & \mathbf{K}^{97} & \mathbf{K}^{98} & \mathbf{K}^{99} \end{bmatrix}, \quad (5.7a,b)$$

$$\mathbf{K}_{u\phi}^l = \begin{Bmatrix} \mathbf{K}_\phi^1 \\ \mathbf{K}_\phi^2 \\ \mathbf{K}_\phi^3 \\ \mathbf{K}_\phi^4 \\ \mathbf{K}_\phi^5 \end{Bmatrix}, \quad \mathbf{K}_{\tau\phi}^l = \begin{Bmatrix} \mathbf{K}_\phi^6 \\ \mathbf{K}_\phi^7 \\ \mathbf{K}_\phi^8 \\ \mathbf{K}_\phi^9 \end{Bmatrix}, \quad (5.8a,b)$$

$$\bar{\mathbf{u}}^l(t) = \begin{Bmatrix} \bar{\mathbf{u}}_0 \\ \bar{\mathbf{v}}_0 \\ \bar{\mathbf{w}}_0 \\ \bar{\boldsymbol{\theta}}_\alpha \\ \bar{\boldsymbol{\theta}}_\beta \end{Bmatrix}, \quad \bar{\boldsymbol{\tau}}^l(t) = \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix}, \quad \mathbf{F}_u^l(t) = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \mathbf{F}^4 \\ \mathbf{F}^5 \end{Bmatrix}, \quad \mathbf{F}_\tau^l(t) = \begin{Bmatrix} \mathbf{F}^6 \\ \mathbf{F}^7 \\ \mathbf{F}^8 \\ \mathbf{F}^9 \end{Bmatrix}. \quad (5.9a-d)$$

The coupled "mixed" electro-mechanical FE model in Equations (5.4a) and (5.4b) is based on the weak forms of the equations of motion and electrostatic equilibrium, which were obtained through Hamilton's principle, and is expressed in terms of displacement (and the derivatives of the transverse displacement), surface shear stresses and electric potential difference DoFs, which are all coupled. It should be noted that the contributions of the internal forces defined in vectors $\mathbf{F}_u^l$ and $\mathbf{F}_\tau^l$ to the force vector will cancel when element equations are assembled. They will remain in the force vector only when the element boundary coincides with the boundary of the domain being modeled. However, as is well known, the contributions of the distributed applied loads $Z(\alpha, \beta)$ to a node will add up from elements connected at the node and remain as a part of the force vector [see Reddy (1993, pp. 313-318)]. Regarding the electric potential difference it is assumed constant in each element and, as can be seen in Equation (5.4a), it produces an equivalent mechanical load through the $\mathbf{K}_{u\phi}^l$ and $\mathbf{K}_{\tau\phi}^l$ vectors. Regarding the Equation (5.4b), associated with the sensing capabilities of the piezoelectric shell layer, it establishes the electrostatic relation at the shell layer FE level between the strains (related with the "mixed" displacements), induced electric potential difference and surface electric charge.

5.5 Assemblage of Matrices from Layer to Multilayer Level

In this section the elemental equations derived for the generic single shell layer are adapted in order to allow the generalization of the present theory to a multilayer, or discrete layer, type formulation. To that end, since the displacements DoFs of the elemental equations of the shell layer are defined in terms of in-plane generalized displacements in the middle surface and rotations of the normals to the middle surface, first the DoFs are transformed to equivalent in-plane displacements on the top and bottom surfaces of the generic shell layer element. Additionally, since the effects of the surface top and bottom shear stresses have been represented in terms of *mean* and *relative* quantities, another transformation is required to the stress DoFs to dispose of top and bottom shear stresses DoFs. The transverse displacement is assumed constant in the multilayer shell (i.e., is constant, and the same, for all layers). These transformations allow the displacement and stress DoFs of different layers to be assembled imposing not only displacement continuity but also shear stress continuity across the interfaces of the multilayer shell FE. Thus, the FE is "regenerated" (in opposition to the well-known "degeneration" approach) in the form of an equivalent eight-noded 3-D element with 2 in-plane displacement and 2 shear stress DoFs per node, and one transverse displacement (and its derivatives) and one electric potential difference per element. Therefore, the "regenerated" formulation is suitable for assemblage of elemental matrices from single layer to multilayer level.

The effects of the pairs of generalized variables (u_0, θ_α) and (v_0, θ_β) in the global displacement field are taken into account through new equivalent pairs of generalized variables (u_t, u_b) and (v_t, v_b) , with each pair containing the in-plane translations at the top and bottom surfaces, respectively. Thus, rather than describing the in-plane displacement field by a translation and a rotation at one point, it can more conveniently be described here by the translation at two points on the top and bottom surfaces.

According to Equation (4.46), and using the adequate coefficients of matrices $\mathbf{z}^u(z = h)$ and $\mathbf{z}^\tau(z = h)$, the displacement field $u(\alpha, \beta, z)$ on the top surface is given as

$$u_t = z_{11}^u(h)u_0 + z_{13}^u(h)\frac{\partial w_0}{\partial \alpha} + z_{14}^u(h)\theta_\alpha + z_{11}^\tau(h)\bar{\tau}_{z\alpha} + z_{12}^\tau(h)\bar{\tau}_{z\alpha}. \quad (5.10)$$

Then, from the previous equation, $u_0 = u_0(\alpha, \beta)$ is written as

$$u_0 = \frac{1}{z_{11}^u(h)} u_t - \frac{z_{13}^u(h)}{z_{11}^u(h)} \frac{\partial w_0}{\partial \alpha} - \frac{z_{14}^u(h)}{z_{11}^u(h)} \theta_\alpha - \frac{z_{11}^\tau(h)}{z_{11}^u(h)} \bar{\tau}_{z\alpha} - \frac{z_{12}^\tau(h)}{z_{11}^u(h)} \bar{\tau}_{z\alpha}. \quad (5.11)$$

Substituting the definition of $u_0 = u_0(\alpha, \beta, z = 0)$ in terms of $u_t = u_t(\alpha, \beta, z = h)$ yields

$$u = \frac{z_{11}^u}{z_{11}^u(h)} u_t + \left[z_{13}^u - z_{11}^u \frac{z_{13}^u(h)}{z_{11}^u(h)} \right] \frac{\partial w_0}{\partial \alpha} + \left[z_{14}^u - z_{11}^u \frac{z_{14}^u(h)}{z_{11}^u(h)} \right] \theta_\alpha + \left[z_{11}^\tau - z_{11}^u \frac{z_{11}^\tau(h)}{z_{11}^u(h)} \right] \bar{\tau}_{z\alpha} + \left[z_{12}^\tau - z_{11}^u \frac{z_{12}^\tau(h)}{z_{11}^u(h)} \right] \bar{\tau}_{z\alpha}. \quad (5.12)$$

From the previous equation it can be seen that some transformations to the first line of matrices $\mathbf{z}^u(z)$ and $\mathbf{z}^\tau(z)$ was performed in order to make a transformation of the generalized in-plane displacement u_0 on the middle surface to the translation on the top surface u_t . Performing a similar process to eliminate the rotation θ_α of Equation (5.12) and express the displacement u also in terms of the translation on the bottom surface u_b , another transformation is performed considering also the terms of the first line of $\mathbf{z}^u(z = -h)$ and $\mathbf{z}^\tau(z = -h)$. Similar relations hold for the second pair of variables (v_0, θ_β) . For the sake of brevity the algebra of these relations will not be presented here but can easily be derived from the previous explanation.

Other required transformation to "regenerate" the 2-D element is performed according to Equations (4.4) and (4.5), where relationships between the *mean* and *relative* shear stresses and the shear stresses on the interfaces of the generic shell layer $\sigma_{z\alpha}^t, \sigma_{z\alpha}^b, \sigma_{\beta z}^t$ and $\sigma_{\beta z}^b$ can be easily established.

According to the previous discussion, the relationship between the original and "regenerated" set of generalized variables used to defined the in-plane displacement field can be established by means of a transformation matrices $\mathbf{T}_u$ and $\mathbf{T}_\tau$ as

$$\begin{Bmatrix} \bar{\mathbf{u}}_0 \\ \bar{\mathbf{v}}_0 \\ \bar{\mathbf{w}}_0 \\ \bar{\theta}_\alpha \\ \bar{\theta}_\beta \end{Bmatrix} = \mathbf{T}_u \begin{Bmatrix} \bar{\mathbf{u}}_t \\ \bar{\mathbf{u}}_b \\ \bar{\mathbf{v}}_t \\ \bar{\mathbf{v}}_b \\ \bar{\mathbf{w}} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{\tau}_{z\alpha} \\ \bar{\tau}_{z\alpha} \\ \bar{\tau}_{\beta z} \\ \bar{\tau}_{\beta z} \end{Bmatrix} = \mathbf{T}_\tau \begin{Bmatrix} \bar{\sigma}_{z\alpha}^t \\ \bar{\sigma}_{z\alpha}^b \\ \bar{\sigma}_{\beta z}^t \\ \bar{\sigma}_{\beta z}^b \end{Bmatrix}. \quad (5.13a,b)$$

Performing the previous transformations into the FE elemental matrices in Equations (5.4a) and (5.4b), where the elemental matrices and vectors are transformed according to (similar relations hold for the stiffness matrices)

$$\mathbf{M}_{uu}^{*l} = \mathbf{T}_u^T \mathbf{M}_{uu}^l \mathbf{T}_u, \quad \mathbf{M}_{\tau\tau}^{*l} = \mathbf{T}_\tau^T \mathbf{M}_{\tau\tau}^l \mathbf{T}_\tau, \quad \mathbf{M}_{u\tau}^{*l} = \mathbf{T}_u^T \mathbf{M}_{u\tau}^l \mathbf{T}_\tau, \quad (5.14a-c)$$

$$\mathbf{K}_{u\phi}^{*l} = \mathbf{T}_u^T \mathbf{K}_{u\phi}^l, \quad \mathbf{K}_{\tau\phi}^{*l} = \mathbf{T}_\tau^T \mathbf{K}_{\tau\phi}^l, \quad \mathbf{F}_u^{*l} = \mathbf{T}_u^T \mathbf{F}_u^l, \quad (5.15a-c)$$

yields

$$\begin{bmatrix} \mathbf{M}_{uu}^{*l} & \mathbf{M}_{u\tau}^{*l} \\ \mathbf{M}_{\tau u}^{*l} & \mathbf{M}_{\tau\tau}^{*l} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}^{*l}(t) \\ \ddot{\bar{\tau}}^{*l}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu}^{*l} & \mathbf{K}_{u\tau}^{*l} \\ \mathbf{K}_{\tau u}^{*l} & \mathbf{K}_{\tau\tau}^{*l} \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^{*l}(t) \\ \bar{\bar{\tau}}^{*l}(t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{K}_{u\phi}^{*l} \\ \mathbf{K}_{\tau\phi}^{*l} \end{Bmatrix} \bar{\phi}(t) = \begin{Bmatrix} \mathbf{F}_u^{*l}(t) \\ \mathbf{F}_\tau^{*l}(t) \end{Bmatrix}, \quad (5.16)$$

$$\begin{Bmatrix} \mathbf{K}_{\phi u}^{*l} & \mathbf{K}_{\phi\tau}^{*l} \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^{*l}(t) \\ \bar{\bar{\tau}}^{*l}(t) \end{Bmatrix} + K_{\phi\phi} \bar{\phi}(t) = Q_\phi(t), \quad (5.17)$$

From this point forward, the generic layer elemental matrices can be assembled in the thickness direction in order to create the desired multilayer FE according to the representative multilayer shell model to be generated. Displacement and shear stress continuity at the through-the-thickness interfaces of adjacent elements (discrete layers) is imposed in the assemblage process, as is usually done with the displacement DoF of 3-D elements, and it is assumed that no slip-page occurs in the interfaces between adjacent layers. It is worthy to mention that the resultant multilayer elemental matrices are needed, for example, when there is the need to consider segmented layers, as is the case when dealing with arbitrary damping treatments (piezoelectric or viscoelastic patches) mounted on a host shell structure. After the through-the-thickness assemblage, the multilayer elemental matrices are written as

$$\begin{bmatrix} \mathbf{M}_{uu}^e & \mathbf{M}_{u\tau}^e \\ \mathbf{M}_{\tau u}^e & \mathbf{M}_{\tau\tau}^e \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}^e(t) \\ \ddot{\boldsymbol{\tau}}^e(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{uu}^e & \mathbf{K}_{u\tau}^e \\ \mathbf{K}_{\tau u}^e & \mathbf{K}_{\tau\tau}^e \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^e(t) \\ \bar{\boldsymbol{\tau}}^e(t) \end{Bmatrix} + \begin{Bmatrix} \mathbf{K}_{u\phi}^e \\ \mathbf{K}_{\tau\phi}^e \end{Bmatrix} \bar{\boldsymbol{\phi}}^e(t) = \begin{Bmatrix} \mathbf{F}_u^e(t) \\ \mathbf{F}_\tau^e(t) \end{Bmatrix}, \quad (5.18)$$

$$\begin{Bmatrix} \mathbf{K}_{\phi u}^e & \mathbf{K}_{\phi\tau}^e \end{Bmatrix} \begin{Bmatrix} \bar{\mathbf{u}}^e(t) \\ \bar{\boldsymbol{\tau}}^e(t) \end{Bmatrix} + \mathbf{K}_{\phi\phi}^e \bar{\boldsymbol{\phi}}^e(t) = \mathbf{Q}_\phi^e(t), \quad (5.19)$$

where the superscript $(\cdot)^e$ is used to denote multilayer elemental matrices and vectors. Regarding the electrical part of the FE equations, it should be noted that since the multilayer shell can contain several piezoelectric layers, there is more than one electrical DoF and therefore instead of a single electrical potential we have a vector $\bar{\boldsymbol{\phi}}^e$ of electrical potential differences and the correspondent capacitance matrices $\mathbf{K}_{\phi\phi}^e$ and electric charge vector $\mathbf{Q}_\phi^e$.

Assuming homogeneous shear stress conditions on the free top and bottom surfaces of the multilayer shell element and performing a dynamic condensation to the shear stress DoFs, as suggested for a generic system, for example, by [Kidder \(1973\)](#) or [O'Callahan \(1989\)](#) and [Gordis \(1994\)](#), yields the multilayer FE elemental equations in terms of elemental reduced matrices and vectors in terms of only displacement variables as

$$\hat{\mathbf{M}}_{uu}^e \ddot{\mathbf{u}}^e(t) + \hat{\mathbf{K}}_{uu}^e \bar{\mathbf{u}}^e(t) + \hat{\mathbf{K}}_{u\phi}^e \bar{\boldsymbol{\phi}}^e(t) = \hat{\mathbf{F}}_u^e(t), \quad (5.20)$$

$$\hat{\mathbf{K}}_{\phi u}^e \bar{\mathbf{u}}^e(t) + \mathbf{K}_{\phi\phi}^e \bar{\boldsymbol{\phi}}^e(t) = \mathbf{Q}_\phi^e(t). \quad (5.21)$$

A generic fully discretized global electro-mechanical system is obtained by "in-plane" assembling the elemental multilayer FE matrices and vectors yielding

$$\mathbf{M}_{uu} \ddot{\mathbf{u}}(t) + \mathbf{K}_{uu} \bar{\mathbf{u}}(t) + \mathbf{K}_{\phi u}^T \bar{\boldsymbol{\phi}}(t) = \mathbf{F}_u(t), \quad (5.22)$$

$$\mathbf{K}_{\phi u} \bar{\mathbf{u}}(t) + \mathbf{K}_{\phi\phi} \bar{\boldsymbol{\phi}}(t) = \mathbf{Q}_\phi(t), \quad (5.23)$$

where the superscript in $(\cdot)^e$ and the *hat* above the elemental matrices and vectors, i.e. in $(\hat{\cdot})$, have been dropped to denote global matrices and vectors of the fully discretized FE model.

The electrical DoFs vector in Equations (5.22) and (5.23) can be partitioned into the actuating and sensing DoFs,

$$\bar{\boldsymbol{\phi}}(t) = \text{col}[\bar{\boldsymbol{\phi}}_a(t), \bar{\boldsymbol{\phi}}_s(t)], \quad (5.24)$$

where the subscripts in $(\cdot)_a$ and $(\cdot)_s$ denote the actuating and sensing capabilities. Furthermore, the stiffness matrix can be written as the sum of the elastic and piezoelectric layers stiffness matrices $\mathbf{K}_{uu}^E$ and $\mathbf{K}_{uu}^P$, respectively. Hence, considering open-circuit electrodes, and in that case $\mathbf{Q}_\phi(t) = 0$ [see [Vasques and Rodrigues \(2005\)](#)], the non specified potential differences in (5.23) can be statically condensed in (5.22) and the equations of motion and charge equilibrium become

$$\mathbf{M}_{uu} \ddot{\mathbf{u}}(t) + (\mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{P*}) \bar{\mathbf{u}}(t) = -\mathbf{K}_{\phi ua}^T \bar{\boldsymbol{\phi}}_a(t) + \mathbf{F}_u(t), \quad (5.25)$$

$$\bar{\boldsymbol{\phi}}_s(t) = -\mathbf{K}_{\phi\phi s}^{-1} \mathbf{K}_{\phi\phi s} \bar{\mathbf{u}}(t), \quad (5.26)$$

where

$$\mathbf{K}_{uu}^{P*} = \mathbf{K}_{uu}^P - \mathbf{K}_{\phi us}^T \mathbf{K}_{\phi\phi s}^{-1} \mathbf{K}_{\phi us}. \quad (5.27)$$

It is worthy to mention that a through-the-thickness distribution of the induced electric potential within the piezoelectric layers was already considered in the formulation through the modification of the strain definition, as described in Section 4.4.3, and the use of *effective shear stiffness parameters* in Equations (B.3) of Appendix. Thus, the static condensation in Equation (5.25) only considers the linear counterpart of the electrical potential distribution, which is the one that in fact contributes to the sensor voltage. Moreover, a second alternative where the equipotential area condition is satisfied by means of a modified static condensation of the non-specified potentials might be utilized, which corresponds to a more realistic approach, that becomes more significant for bare piezoelectric structures or as the length of the piezoelectric layers approaches the length of the host structure [see Vasques (2003) and Vasques and Rodrigues (2005) for further details].

5.6 Preliminary Assessment and Validation of the Shell Finite Element

In this section the previously established FE solution is implemented into the MATLAB/SDT programming FE environment. A preliminary assessment of the mechanical counterpart of the partially refined theory and FE solution is performed here for isotropic structures. In a first analysis, the nucleus of the regenerated multilayer concept, i.e. a single layer, is considered and the theory is applied to model an isotropic homogeneous elastic rectangular plate and a half-cylinder shell. The aims of this preliminary analysis are mainly to assess and validate the proposed deformation theory for thick (or moderately thick) structures and for that purpose a free vibration analysis (natural frequencies and mode shapes) is performed. Both stiffness and mass FE matrices should be accurate only if the theory is representative of the kinematics and kinetics of the structural system and therefore the natural frequencies values are used as global assessment parameters.

The emphasis of this study is put in thick structures but, however, thin cases are also considered in some extent. It is well known that thin FEs with unsuitable thickness to length aspect ratios often lead to shear and membrane locking resulting from the inability of plate and shell FEs to represent deformation states in which the transverse shear should vanish (pure bending) and the inability of shell FEs to represent inextensional modes of deformation (bending without stretching), respectively. These issues are strongly dependent upon the degree of interpolation used to approximate the different generalized displacements and, among other solutions, underintegrating (selective integration) the appropriate terms of the stiffness matrix often solves the problem in some extent. These finite element approximation issues will not be addressed or circumvented at this stage. Full integration of the system matrices is utilized and numerical (or approximation) instabilities typical in thin limit cases will not be tackled.

5.6.1 Rectangular Plate

Let us consider a rectangular aluminum plate with geometric and material characteristics as depicted in Figure 5.1. For the case of a plate, the derived shell FE model is transformed to the particular case of a plate geometry where $A_\alpha = A_\beta = 1$ and $1/R_\alpha = 1/R_\beta = 0$. For the present

case study, the plate is considered to be clamped at two edges ($\alpha = \pm a/2$) and free at the other two ($\beta = \pm b/2$).

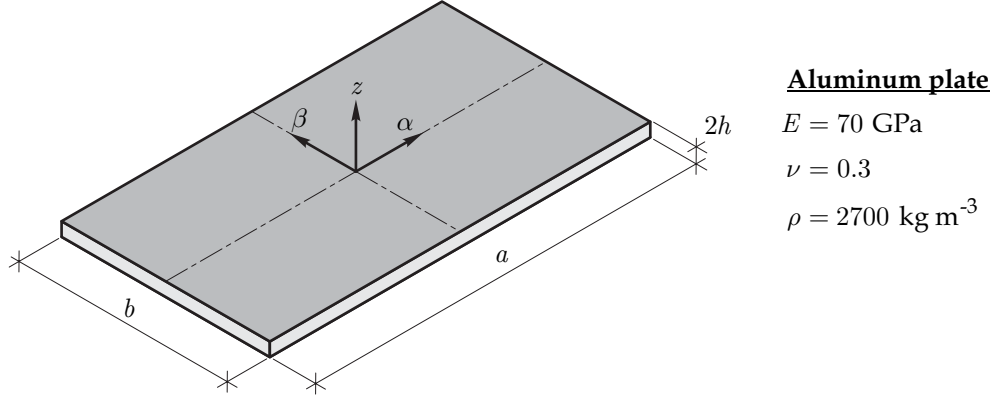


Figure 5.1: Geometric and material description of the rectangular plate.

The free vibration analysis of the clamped-free-clamped-free plate was performed considering three different deformation theories which directly come from the present one by defining the shear stress correction function $f(z)$ in Equation (4.40) in three different ways:

TSDT: Since in the present case only the mechanical counterpart of the FE formulation is used and since when considering single-layer plate structures the top and bottom transverse shear stresses of the layer should vanish, for single-layer plate structures, the theory essentially becomes a third-order shear deformation theory (TSDT) with a quadratic through-the-thickness distribution of the transverse shear stresses as depicted in Figure 4.3.

FSDT: By considering $f(z) = 1$ the first-order shear deformation theory (FSDT) is enforced since no correction is performed to the constant through-the-thickness distribution of the transverse shear stresses; the utilized FSDT FE is then obtained from the TSDT one only by modifying the shear correction function and no shear correction factor is used. It should be mentioned, though, that the FSDT FE, contrarily to what would be necessary, uses Hermite interpolation for the transverse displacement which somehow presents some differences with respect to the “typical” FSDT bi-linear FE formulations.

CT: This last case is obtained by setting $f(z) = 0$ and corresponds exactly to the classical theory (CT) used for plates where transverse shear stresses are neglected. Regarding the FE model, it uses a conforming interpolation scheme using Lagrange interpolation polynomials for the in-plane generalized displacements and Hermite interpolation of the transverse displacement.

The natural frequencies of a clamped-free-clamped-free rectangular plate for different thicknesses, mesh sizes, deformation theories and FE solutions are presented in Table 5.1 and the first ten mode shapes of the correspondent thick rectangular plate are presented in Figure 5.2. The results are compared with the ones obtained with the FE software COSMOSWorks 2008, denoted as 3-D in the table, by using a very refined mesh of parabolic tetrahedral solid elements. The results demonstrate that the plate FE formulation is robust and valid for thin structures with high length to thickness aspect ratios for all the two shear deformation theories, i.e. the FSDT and TSDT, and that for the thin case, as expected, the CT is the most interesting one

Table 5.1: Natural frequencies [Hz] of a clamped-clamped ($\alpha = \pm a/2$) free-free ($\beta = \pm b/2$) rectangular plate for different thicknesses, mesh sizes, deformation theories and FE solutions.

THIN RECTANGULAR PLATE ($b/2h = 200$; $a/b = 3/2$; $2h = 1$ mm)											
Elements		103396	12×8			24×16			36×24		
DoFs		601701	594	792	792	2346	3128	3128	4650	6200	6200
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,0)	60.27	60.22	61.48	61.89	60.20	60.51	60.48	60.19	60.33	60.31
2	(1,2)	83.90	83.35	107.8	98.45	83.98	85.65	84.96	83.87	84.27	84.13
3	(2,0)	166.4	166.2	172.9	172.7	166.1	167.6	167.6	166.1	166.8	166.7
4	(1,3)	193.1	193.1	251.7	228.5	193.1	197.8	195.9	193.1	194.1	193.7
5	(2,2)	201.5	201.4	277.4	248.5	201.3	207.2	205.1	201.3	202.7	202.3
6	(2,3)	321.8	321.9	522.6	445.1	321.7	337.4	331.0	321.7	325.1	323.8
7	(3,0)	326.6	326.5	347.4	347.4	326.3	331.4	331.2	326.3	328.4	328.3
8	(3,2)	366.8	366.7	536.1	476.5	366.4	380.3	375.9	366.4	370.0	369.1
9	(1,4)	424.4	424.9	593.1	499.2	424.6	434.7	431.5	424.6	427.3	426.6
10	(3,3)	499.4	499.4	549.7	783.7	499.1	533.6	520.0	499.0	506.6	503.8

MODERATLY THICK RECTANGULAR PLATE ($b/2h = 20$; $a/b = 3/2$; $2h = 10$ mm)											
Elements		498514	12×8			24×16			36×24		
DoFs		2179923	594	792	792	2346	3128	3128	4650	6200	6200
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,0)	597.2	601.8	606.7	604.4	601.6	599.5	597.8	601.6	598.2	596.8
2	(1,2)	822.5	837.6	841.3	836.0	837.3	828.8	825.7	837.2	826.2	823.6
3	(2,0)	1631	1658	1698	1675	1658	1647	1637	1657	1639	1631
4	(1,3)	1881	1921	1924	1911	1920	1897	1890	1920	1892	1885
5	(2,2)	1959	2007	2035	2014	2007	1982	1970	2006	1973	1962
6	(2,3)	3089	3197	3214	3171	3195	3131	3190	3195	3116	3098
7	(3,0)	3161	3251	3355	3309	3248	3214	3182	3248	3189	3164
8	(3,2)	3525	3646	3745	3684	3643	3587	3550	3643	3560	3531
9	(1,4)	4096	4200	4245	4205	4198	4144	4118	4197	4126	4104
10	IPB 1	4577	4620	4620	4620	4567	4567	4567	4558	4558	4558

THICK RECTANGULAR PLATE ($b/2h = 5$; $a/b = 3/2$; $2h = 40$ mm)											
Elements		882916	12×8			24×16			36×24		
DoFs		3647685	594	792	792	2346	3128	3128	4650	6200	6200
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,0)	2151	2387	2207	2154	2385	2181	2145	2386	2177	2144
2	(1,2)	2799	3264	2895	2824	3263	2858	2805	3262	2851	2801
3	IPB 1	4555	4620	4620	4620	4567	4567	4567	4557	4557	4557
4	(2,0)	5288	6428	5561	5321	6425	5440	5289	6425	5420	5283
5	(2,2)	6146	7658	6467	6217	7654	6337	6166	7654	6314	6159
6	(1,3)	6158	7114	6329	6200	7111	6268	6170	7111	6255	6159
7	IPM 1	8576	8586	8586	8586	8551	8551	8551	8545	8545	8545
8	IPB 2	8673	8880	8880	8880	8729	8729	8729	8703	8703	8703
9	(2,3)	9149	11640	9572	9268	11635	9421	9190	11634	9390	9170
10	(3,0)	9242	12189	9930	9347	12181	9619	9275	12180	9568	9260

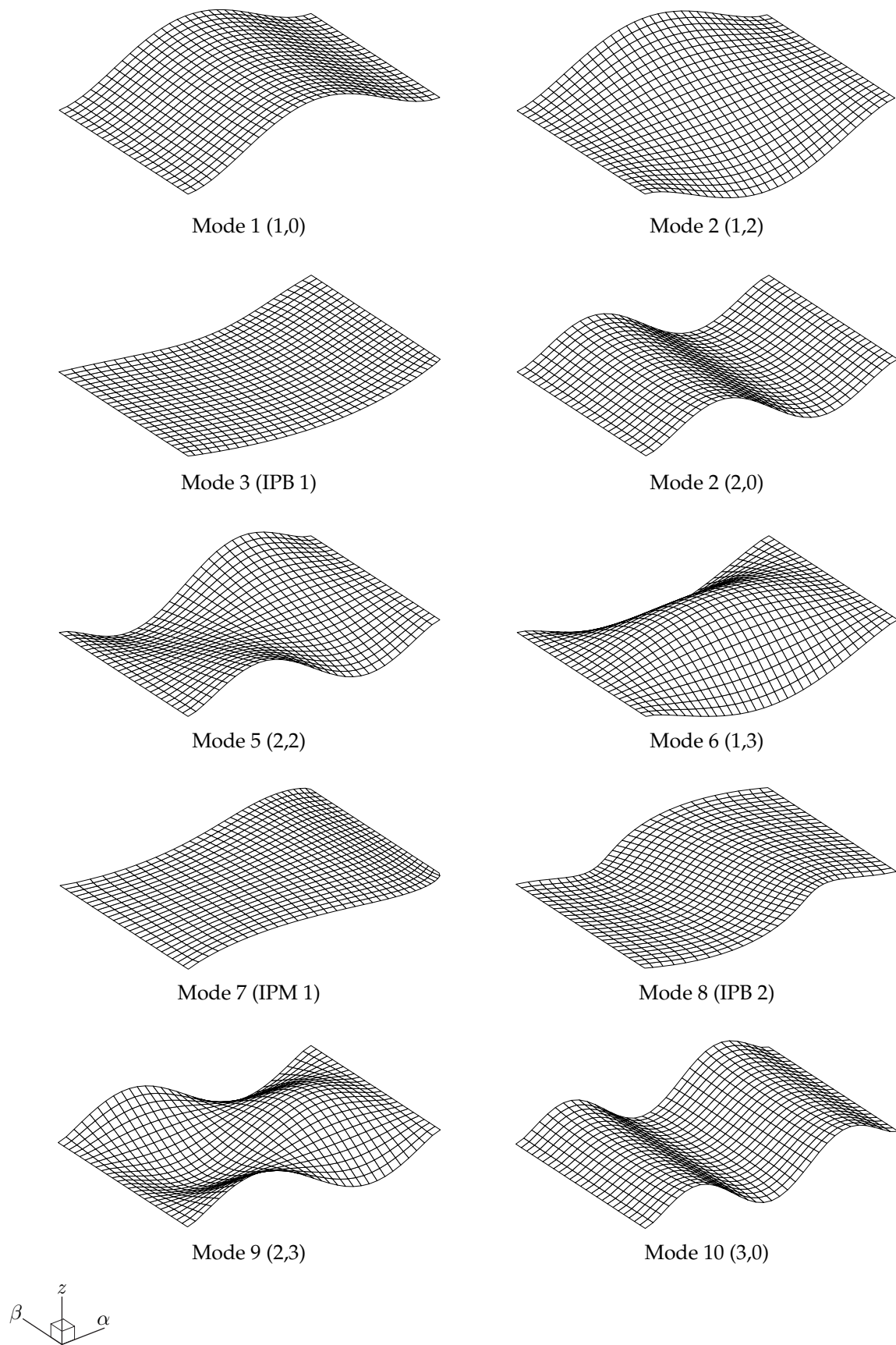


Figure 5.2: First ten mode shapes of a clamped-clamped ($\alpha = \pm a/2$) free-free ($\beta = \pm b/2$) thick rectangular plate ($b/2h = 5$; $a/b = 3/2$; $2h = 40$ mm).

followed by the TSDT and FSDT which have a similar behavior. As the thickness increases, the shear deformation theories become more representative than the CT, with the difference between the FSDT and TSDT being more pronounced for the thick plate where the CT presents significant errors.

From this case study, it may be concluded that the single-layer plate FE formulation proposed here is valid and representative in a wide range of thicknesses. It should be stressed, though, that the differences between the different deformation theories are not very significant for single layer isotropic plate structures. However, for multilayer and composite structures, the inter- and intra-layer stress states become more complex making the use of HSDT more appropriate and accurate whether using ESLTs or DLTs; the extension of the theory to regenerated multilayered plates would allow to investigate these issues. Lastly, the rigid body mode performance of the plate FE and elemental distortion effects need to be investigated in order to yield accurate and robust plate FEs.

5.6.2 Half-Cylinder Shell

In this second case let us consider a half-cylinder aluminum shell with the material and geometric properties as depicted in Figure 5.3. In the case of a cylindrical shell, the shell surface is defined by curvilinear (cylindrical) coordinates where $R_\beta = R$, $\beta = R\theta$, $A_\alpha = 1$, $A_\beta = 1$, and $1/R_\alpha = 0$. Again, the considered boundary conditions impose a clamped condition at two edges ($\alpha = \pm a/2$) and free conditions at the other two ($\beta = \pm R\pi/2$).

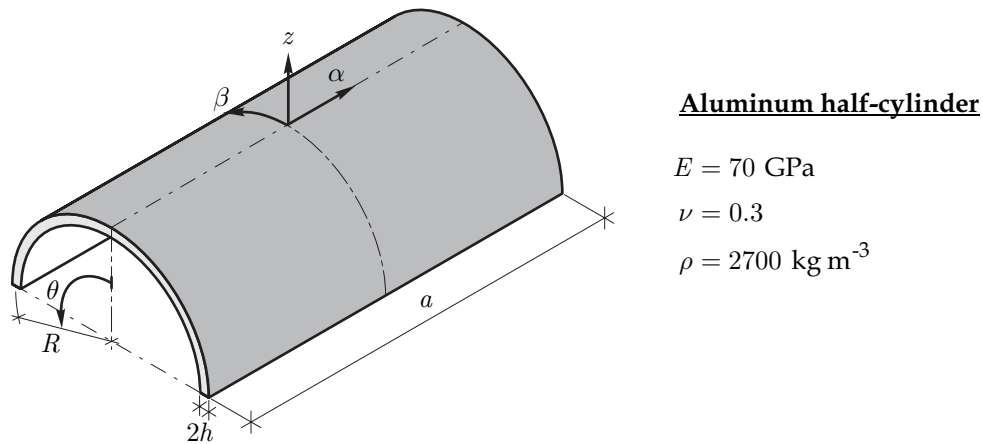


Figure 5.3: Geometric and material description of the half-cylinder shell.

In a similar way to what was considered for the rectangular plate case, the free vibration analysis of the clamped-free-clamped-free half-cylinder was performed considering the same three different deformation theories which directly come by defining the shear stress correction function $f(z)$ in Equation (4.40) in three different ways. Furthermore, a second variant of the theories where the z/R_α and z/R_β coming from the definition of the Lamé parameters of the *second fundamental form* given in Equations 4.2 are neglected was also considered. However, for the shell case there are some additional considerations that need to be done in order to recover the appropriate shell theory, which are as follows :

FSDT: The definition $f(z) = 1$ still holds but in the definition of the shear angle rotations in Equations 4.27 the terms involving the in-plane generalized displacements should be discarded.

CT: This last case for shells is obtained by setting $f(z) = 1$ and by considering the shear angle rotations in Equations 4.27 defined only by positive values of the terms involving the in-plane generalized displacements (the other terms are discarded).

The natural frequencies of a clamped-free-clamped-free half-cylinder shell for different thicknesses, mesh sizes, deformation theories, thickness assumptions and FE solutions are presented in Table 5.2 and the first ten mode shapes of the correspondent thick half-cylinder shell are presented in Figure 5.4. Again, the results are compared with the ones obtained with the FE software COSMOSWorks 2008, denoted as 3-D in the table, by using a very refined mesh of parabolic tetrahedral solid elements, which is taken as the reference solution.

The results demonstrate that the shell FE formulation is robust and valid for thick shell structures but that the FE solution as it is presents some problems when trying to model thinner structures with high length to thickness aspect ratios. These difficulties are evidenced even for the CT since it also suffers from an inefficient approximation order of the membrane stresses which results in a deficiency of representing inextensional deformations in pure bending states which in turn introduces spurious membrane stiffness terms that increase the stiffness and consequently yield higher natural frequencies. That effect can be circumvented by developing and implementing suitable methods of reducing locking, which were not introduced here in this preliminary analysis, and are pretty much mode dependent. This latter characteristic is justified by the fact that the clamped-clamped free-free boundary conditions impose a mixture of bending and membrane dominated states which according to the mode being considered might be more or less pronounced. As obvious, since for thin structures transverse shear effects are negligible, the CT for the case of thin shell presents some advantages in terms of not having shear locking but, despite of that, the differences in performance for the shear deformation theories in thin cases are not evident. As the thickness increases, the shear deformation theories become more representative than the CT, with the difference between the FSDT and TSDT being pronounced for the thick cylindrical shell where the CT presents significant errors. When neglecting the z/R_α and z/R_β terms coming from the definition of the Lamé parameters in thick shell analysis the representativeness of the theories become significantly worse with the FSDT somehow giving better results.

It may be concluded therefore that the single-layer shell FE formulation as it is is valid and representative only for moderately thick and thick structures. Again, it should be stressed, though, that the differences between the different deformation theories are not very significant for single layer isotropic shell structures. However, for multilayer and composite structures, the inter- and intra-layer stress states become more complex making the use of HSDT more appropriate and accurate whether using ESLTs or DLTs; the extension of the theory to regenerated multilayered shells would allow to investigate these issues. Lastly, it is evident that some procedures to circumvent membrane locking effects need to be implemented in order to yield a more robust formulation for thin shells and the rigid body mode performance of the shell FE and elemental distortion effects need to be investigated in order to yield accurate and robust shell FEs for thin and thick shells.

5.7 Summary

Based on a *fully refined* mathematical model of general anisotropic shells, a fully coupled piezo-visco-elastic FE model has been proposed for multilayer shells. However, for practical reasons, related with the complexity of the formulation, simplifications regarding the through-the-thick-

Table 5.2: Natural frequencies [Hz] of a clamped-clamped ($\alpha = \pm a/2$) free-free ($\beta = \pm R\pi/2$) half-cylinder shell for different thicknesses, mesh sizes, deformation theories, thickness assumptions and FE solutions.

MODERATLY THICK CYLINDRICAL SHELL ($R/2h = 20$; $a/2h = 64$; $2h = 12.5$ mm)											
Elements		512861	24×24			48×48			72×72		
DoFs		2403936	3450	4600	4600	13818	18424	18424	31098	41464	41464
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,3)	305.6	399.4	398.9	398.7	338.8	333.9	333.7	325.0	318.8	318.6
2	(1,4)	312.2	422.7	421.2	421.0	350.3	347.1	346.9	332.6	328.7	328.4
3	(2,4)	599.4	708.2	708.8	707.7	635.1	630.6	629.7	619.8	613.9	613.1
4	(2,5)	607.8	715.3	714.0	712.9	644.9	638.6	637.7	629.8	622.1	621.3
5	(1,2)	689.8	750.0	764.85	764.4	695.9	708.8	708.5	685.8	698.3	698.0
6	(1,5)	701.1	854.3	883.5	882.6	721.7	751.9	751.1	694.0	724.6	723.8
7	(3,4)	965.6	1076	1078	1076	1006	998.5	996.4	990.0	981.2	979.4
8	(3,5)	967.3	1079	1081	1079	1008	1000	997.7	992.9	982.8	981.0
9	(1,6)	1020	1194	1132	1130	1045	1078	1076	1015	1046	1045
10	(1,0)	1109	1127	1129	1128	1113	1113	1112	1110	1110	1110

THICK CYLINDRICAL SHELL ($R/2h = 5$; $a/2h = 16$; $2h = 50$ mm; $z/R \neq 0$)											
Elements		891481	24×24			48×48			72×72		
DoFs		3767988	3450	4600	4600	13818	18424	18424	31098	41464	41464
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,2)	666.0	716.0	692.2	684.8	703.1	673.4	669.7	700.6	669.4	666.7
2	(1,3)	717.1	773.5	756.8	751.8	748.7	729.4	725.1	744.0	723.7	719.9
3	(1,0)	1172	1181	1186	1186	1171	1176	1175	1169	1174	1173
4	(1,4)	1355	1371	1406	1417	1327	1375	1371	1319	1369	1362
5	(2,2)	1475	1579	1535	1502	1563	1496	1480	1560	1488	1475
6	(2,3)	1479	1588	1543	1510	1568	1501	1484	1564	1492	1479
7	IPB 1	1810	1818	1815	1814	1814	1810	1811	1813	1810	1810
8	(2,4)	2202	2293	2269	2248	2264	2235	2216	2259	2228	2209
9	(1,5)	2280	2361	2347	2347	2311	2311	2293	2302	2304	2284
10	(2,0)	2288	2319	2317	2305	2307	2300	2294	2305	2296	2291

THICK CYLINDRICAL SHELL ($R/2h = 5$; $a/2h = 16$; $2h = 50$ mm; $z/R = 0$)											
Elements		891481	24×24			48×48			72×72		
DoFs		3767988	3450	4600	4600	13818	18424	18424	31098	41464	41464
Mode (m,n)		3-D	CT	FSDT	TSDT	CT	FSDT	TSDT	CT	FSDT	TSDT
1	(1,2)	666.0	692.2	693.7	756.8	677.9	678.3	742.9	675.1	675.3	740.1
2	(1,3)	717.1	766.3	772.8	808.5	742.1	747.0	782.9	737.5	742.0	778.0
3	(1,0)	1172	1198	1191.7	1302	1187	1179	1293	1185	1177	1291
4	(1,4)	1355	1477	1459	1472	1435	1412	1429	1427	1403	1420
5	(2,2)	1475	1556	1514	1523	1539	1489	1501	1535	1484	1496
6	(2,3)	1479	1581	1529	1529	1561	1500	1504	1556	1494	1499
7	IPB 1	1810	1815	1817	1988	1811	1813	1985	1810	1812	1984
8	(2,4)	2202	2406	2300	2291	2380	2262	2259	2374	2255	2253
9	(1,5)	2280	2482	2401	2380	2435	2342	2326	2426	2330	2316
10	(2,0)	2288	2351	2322	2353	2340	2306	2341	2337	2303	2339

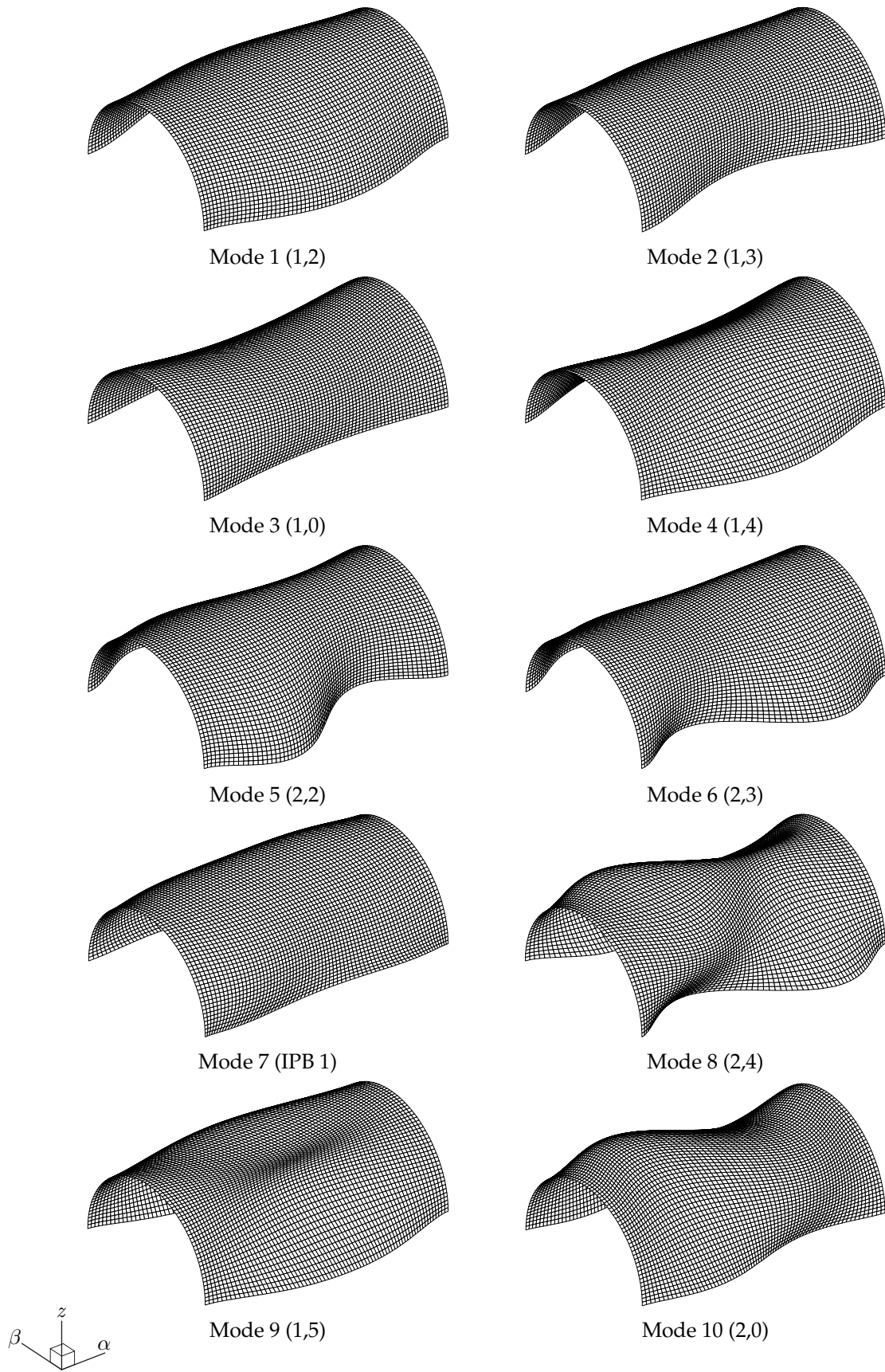


Figure 5.4: First ten mode shapes of a clamped-clamped ($\alpha = \pm a/2$) free-free ($\beta = \pm R\pi/2$) half-cylinder shell ($R/2h = 5$; $a/2h = 16$; $2h = 50$ mm; $z/R \neq 0$).

ness distribution of the transverse displacement were considered and a *partially refined* theory was derived with additional restrictions inherent to doubly-curved orthotropic shells physics. No simplifications regarding the thinness of the shell were considered and a plane stress state was considered for the partially refined theory.

It was shown that the refined assumptions and relaxation of some of Love's classical assumptions lead to a "mixed" definition of the displacement field in terms of the same generalized displacements of the FSDT and CLT, and shear stresses on the top and bottom surfaces.

The electrical potential was defined assuming negligible in-plane electric displacement field which allowed the use of *effective shear stiffness parameters* which account for the effects of the induced in-plane electric field components. Regarding the transverse (out-of-plane) electrical behavior, the induced transverse electric field was considered by means of a modification of the mechanical in-plane strains, which in an analogous way were modified into *effective strains*. Thus, the direct piezoelectric effect was condensed into the model through *effective* stiffness and strains definitions, and the converse counterpart was considered by the action of prescribed electric potential differences in each piezoelectric layer.

The governing equations of a generic single layer of the multilayered shell were derived with Hamilton's principle in conjunction with the "mixed" displacement field and the electric potential difference definitions. The DoFs of the resultant four-noded generic piezo-elastic single layer FE model were then "regenerated" into an equivalent eight-node 3-D formulation in terms of top and bottom translations and shear stresses, and a transverse displacement (and its derivatives) constant in the elemental volume. The through-the-thickness assemblage of the "regenerated" FE model of the single layer allowed the generation of a refined multilayer FE assuring displacement and shear stress interlayer continuity. The dynamic condensation of the stress DoFs allowed the reduction of the refined multilayer piezo-elastic FE to an equivalent representation similar in structure to the one obtained with a first-order partial layerwise theory, but considering nonlinear in-plane displacement and quadratic shear stresses definitions and also interlayer continuity and homogeneous conditions, at the top and bottom surfaces of the refined multilayer FE, of the shear stresses.

The resultant partially refined piezo-visco-elastic coupled FE model was assessed and validated in a mechanical (elastic and inertial) sense and its extension to damped shells is expected to be able to be used to model shells with segmented hybrid (active-passive) damping treatments and to be used to design damping treatments for vibration and/or sound radiation suppression in an accurate way.

Part III

VIBRATION DAMPING

Part III

VIBRATION DAMPING

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CHAPTER 6

Viscoelastic Damping: Mathematical Modeling

6.1 Introduction

Although viscoelastic treatments are very effective solutions as damping mechanisms for light structures, its design and analysis (numerical and/or analytical) are quite difficult and cumbersome. One of these difficulties is related with the development, application and implementation of efficient constitutive mathematical models able to represent the frequency- and temperature-dependent constitutive properties of general viscoelastic materials. Likewise, another difficulty is related with the reduced efforts that have been made in the publication and dissemination of databases with material data of general viscoelastic materials making it difficult sometimes to find accurate data for some materials. Furthermore, since a generalized viscoelastic constitutive model is not available (or has not yet been recognized as such), material information in the form of complex moduli data is usually identified from low resolution and outdated nomograms published by the material manufacturers, which has been the most current and “universal” means of providing material data used by the manufacturers, or eventually obtained by adverse published data related with a series of fortuitous heuristic or secondhand theoretical models that have been proposed or even by non-standardized experimental characterization procedures. In fact, quite often, when dealing with “new” or non-standard viscoelastic materials (e.g. cork compounds), experimental identification tests for these materials need to be carried out in order to furnish the designer with the most important information at the start of the design stage: the frequency- and temperature-dependent constitutive law of the material. These issues are discussed and analyzed in this part of the dissertation which comprises two chapters; the first addresses mathematical modeling and implementation issues and the latter experimental identification ones.

Based on a phenomenological approach to represent the linear viscoelastic constitutive behavior, this chapter is devoted to mathematical descriptions of viscoelastic material damping. To that end, the main difficulties of the different approaches to modeling, FE implementation and solution methods proposed in the last decades are shortly reviewed and the new trends in both time and frequency domain viscoelastic (or in a more general sense, “arbitrary frequency dependent hysteretic”) damping modeling and their FE implementations are hereby discussed and presented. Additionally, the spatial modeling of the damping treatments is also addressed in this work and the most common applied modeling approaches, their advantages, limita-

tions and new trends are highlighted. Both time and frequency domain damping models, typically necessary in FE analysis of structural systems with embedded or surface mounted hybrid or purely passive damping treatments, are presented. Internal variables models, such as the *Golla-Hughes-McTavish* (GHM) and *anelastic displacement fields* (ADF) models, *direct frequency response* (DFR), based on the *complex modulus approach* (CMA), *iterative modal strain energy* (IMSE) approach and a new variant based on an *iterative complex eigensolution* (ICE) approach are described and implemented at the "global" FE level. Additionally, a state space representation of the integrated damped FE models is presented, model reduction procedures are discussed and the use of damping modeling solutions in commercial FE codes is investigated. Afterwards, the experimental identification of the viscoelastic material 3M ISD112 and the assessment and validation of the forementioned integrated FE damping modeling approaches are presented in Chapter 7.

6.2 Review of Damping Modeling Approaches

Nowadays, in certain technological areas (e.g. automotive, aeronautics, aerospace, acoustics) there is a growing need of lighter and stiffer structural components. However, the weight reduction, maintaining the rigidity unaffected, might lead to a significant reduction of the inherent damping capacity of those flexible structures. Viscoelastic materials can be used as an effective means of controlling the dynamics of structures, reducing and controlling the structural vibrations and/or noise radiation from structures. They can be used as surface mounted or embedded damping treatments, utilizing passive viscoelastic materials alone, the so-called *passive treatments*, or in an unified way, with active materials such as piezoelectric polymers or ceramics, the so-called *hybrid treatments*. When adequately designed, the use of these materials in hybrid damping treatments might provide high damping capability over wide temperature and frequency ranges.

Applications with passive treatments alone have been widely used since the end of the 1950s, as can be found in reference textbooks [e.g. [Nashif et al. \(1985\)](#), [Sun and Lu \(1995\)](#), [Mead \(1998\)](#), [Jones \(2001\)](#), [Harris and Piersol \(2002\)](#) and [Harris and Piersol \(2002\)](#)]. However, hybrid treatments are a class of treatments usually more expensive and difficult to implement, but with the advantage of leading to a higher damping control performance and adaptability. Therefore, a considerable research effort has been undertaken by the scientific community towards their effective use in real life applications. A review of hybrid damping treatments is addressed in [[Park and Baz \(1999b\)](#), [Benjeddou \(2001\)](#), [Trindade and Benjeddou \(2002\)](#), [Stanway et al. \(2003\)](#)] and some analytical and experimental studies concerning beams, plates and shells can be found, for example, in [[Azvine et al. \(1995\)](#), [Ro et al. \(1997\)](#), [Ray and Baz \(1997\)](#), [Park and Baz \(1999a\)](#), [Liu et al. \(2000\)](#), [Baz \(2001\)](#), [Chantalakhana and Stanway \(2001\)](#), [Ko et al. \(2004\)](#), [Vasques and Rodrigues \(2006\)](#), [Vasques, Mace, Gardonio and Rodrigues \(2006\)](#)]. Three landmark papers reviewing investigations dealing with vibration control using viscoelastic materials, addressing specific topics such as the analysis of configurations, properties of the materials used and applications were written by [Nakra \(1976, 1981, 1984\)](#).

In the last decade the technologies associated with a more refined design and adequate implementation of viscoelastic-based damping treatments has achieved a relative maturity and is starting to be widely applied in structures with an high level of integrated technology, such as components for the automobile or aerospace industries, household appliances, and other simple structures where damping is becoming an essential design parameter. However, in order to adequately design these viscoelastic damping treatments, the viscoelastic material properties

need to be accurately measured and characterized and their constitutive and phenomenological damping behaviors must be properly taken into account in the underlying mathematical models and simulation tools.

It is well known that the FE solution method is a valuable approximation method widely applied in the simulation of multiphysics structural systems. Due to the inherent complexity when modeling the damping mechanisms of the structural systems, typically developed by embedded viscoelastic layers in surface treatments or in sandwich configurations, reliable simulation tools based upon the FE method are frequently necessary. Therefore, a continuous effort to develop accurate and trustworthy numerical tools is currently one of the major investigation interests of the structural damping research community. Nevertheless, this mission has some difficulties, which are related to the specific shear deformation patterns typically developed inside the viscoelastic layers, which are the basis of the damping mechanism, requiring the application of specific spatial models and FE discretizations. Furthermore, the significant temperature and frequency dependency of the viscoelastic materials demand a special constitutive modeling approach and analysis method, which must provide the requested solution using the selected constitutive model.

When applying the FE method to simulate viscoelastically damped structures, the solutions to the aforementioned difficulties are not available or cannot be straightforwardly accessed when using commercially available FE packages. In fact, even specific FE codes devoted to the structural dynamic simulation do not provide a direct modeling capability to represent the viscoelastic behavior for time or frequency domain analysis. The alternative is the use of some artifices to account for the frequency dependency of the damping exerted by the viscoelastic materials. This feature has motivated a considerable research effort on the subject, which covers the development of reliable and low-cost constitutive models compatible with current structural FE solutions and analysis techniques.

In general, the elastic and dissipative properties of viscoelastic materials depend on the frequency, operating temperature, amplitude and type of excitation, to name just a few [Ferry (1980), Riande et al. (2000), Jones (2001)]. These dependencies of the viscoelastic material properties make a mathematical description of the viscoelastic constitutive behavior and material damping more difficult and complicated to obtain, and might turn the underlying FE implementations and solution methods more troublesome and difficult endeavors. Thus, for simplicity, the amplitude and type of excitation effects are often overlooked, isothermal conditions are usually assumed and only the frequency dependency of the viscoelastic constitutive behavior is usually taken into account.

The *direct frequency response* (DFR) method, which is based upon the *complex modulus approach* (CMA), early presented by Myklestad in the 1950s [Myklestad (1952)] and further discussed and utilized by Snowdon (1968) and Bert (1973), is a frequency domain method that utilizes a time-domain based model which is limited to steady state vibrations and single-frequency harmonic excitations. That corresponds to a simple way of modeling viscoelastic damping effects where the material properties are defined for each discrete frequency value [Moreira and Rodrigues (2004)]. However, the structural model can alternatively be formulated in the frequency domain, yielding the so-called *wave models*, which are based on the definition of a dynamic stiffness matrix. These wave methods can also be extended to discontinuous structures and damping treatments, for example by the use of the so-called *spectral finite element method* (SFEM) proposed by Doyle (1997) which utilizes dynamic shape functions. These wave-based methods, which also allow viscoelastic damping effects to be accurately considered by defining the material constitutive behavior for the current waves specific frequency, have been used in [Wang and Wereley (1998), Baz (2000), Wang and Wereley (2002)]. That so-

lution method is denoted as *wave propagation method* (WPM) and, according to Wang and Wereley (2002), Douglas was the first to explore wave solutions in order to implicitly account for the frequency dependent complex modulus of viscoelastic components in the solution method [Douglas (1977), Douglas and Yang (1978)].

The CMA is also the basis of the so-called *modal strain energy* (MSE) method, first derived and utilized by Mead (1960) and popularized later by Johnson et al. (1980), where the loss factor of each individual mode is determined from the ratio between the dissipated modal strain energy of the viscoelastic counterpart and the storage modal strain energy of the global structural system. Iterative versions of the MSE have been successfully used for moderate damping values [Trindade et al. (2000b)]. The MSE based methods are known to lead to poor viscoelastic damping estimation of highly damped structural systems.

Time domain models, such as the *Golla-Hughes-McTavish* (GHM), after Hughes and his colleagues [Golla and Hughes (1985), McTavish and Hughes (1993)], *anelastic displacement fields* (ADF), after Lesieutre [Lesieutre and Bianchini (1995), Lesieutre et al. (1996)], and others presented by Yiu (1993) and more recently by Silva (2003), utilizing additional *internal* (or *dissipation*) variables [see Johnson (1999)], have been successfully utilized and were shown to yield good damping estimates.

Alternatively, the use of *fractional calculus* (FC) models, initially proposed by Bagley and Torvik (1983, 1985) and recently revisited, among others, by Enelund and Lesieutre (1999), Jones (2001), Schmidt and Gaul (2002) and Galucio et al. (2004, 2005, 2006), provides a simpler and more economic descriptor of the complex constitutive behavior of viscoelastic materials, being able to represent the storage and loss modulus of frequency dependent viscoelastic materials using a low order parameter set formed by one or two model parameter series and based upon the use of fractional derivatives. It has the drawback, though, of generating a "non-standard" FE time domain formulation, with a more burdensome characteristic solution procedure making use of some special artifices that allow to obtain good damping estimates of the system.

Other relevant contributions for damping, in general, and viscoelastic damping modeling, in particular, were given, among others, by Adhikari and Woodhouse (2003) and the references therein on non-viscous damping in discrete linear systems, Balmès and his co-workers [Balmès (1997), Plouin and Balmès (1998, 1999)], who have used different variations of modal subspace models specially defined and assumed to be representative of the damped system response, and Kelly and Stevens (1989) and Lin and Lim (1996), that have used perturbation method approaches. Since they fall out of the scope of the damping strategies utilized in this work, it is not intended to review those works here and the interested readers are referred to the aforementioned references for further details on the methods used.

Regarding the temperature effects, studies taking into account the temperature dependence of the properties and the self-heating of viscoelastic materials have been performed by Lesieutre and his co-workers [Brackbill et al. (1996), Lesieutre and Govindswamy (1996)], that extended the ADF model for these cases, leading, however, to nonlinear differential equations. The effect of the operating temperature upon the performance of hybrid damping treatments and viscoelastic damping efficiency were analyzed, for example, by Baz (1998), Friswell and Inman (1998), Trindade et al. (2000a), Silva et al. (2005) and Pradeep and Ganesan (2006).

6.3 Configurations of Viscoelastic Damping Treatments

Viscoelastic materials are rubber-like polymers possessing stiffness and damping characteristics which, among other factors, vary strongly with temperature and frequency. These polymers are materials composed of long intertwined and cross-linked molecular chains, each containing thousands or even millions of atoms. The internal molecular interactions that occur during deformation in general, and vibration in particular, give rise to macroscopic properties such as stiffness and energy dissipation during cyclic deformation, which in turn characterizes the mechanism of viscoelastic damping [Ferry (1980), Riande et al. (2000)].

Over the last decades, this particular, and in some way unique, property of viscoelastic materials has been used to develop surface mounted or embedded damping treatments, which take advantage of the damping capabilities of the viscoelastic materials. The different and most significant damping configurations that have been proposed, which were already introduced in part in Section 1.2, along with the different alternatives to model them, both in terms of their phenomenological behavior and spatial distribution, are shortly reviewed in what follows.

Passive viscoelastic damping treatments have been extensively used in engineering to reduce vibration and/or noise radiation [Nashif et al. (1985), Sun and Lu (1995), Mead (1998), Jones (2001), Harris and Piersol (2002), Mead (2002)]. From a phenomenological point of view, when a damping layer is applied to a structural system, it dissipates energy by virtue of the heat generated by the strain field that it undergoes. The strain field varies over the surface and depth of the structure and is proportional to the fluctuating extensional strains, due to the membrane and bending loads, and shear strains, due to the shear loads. These damping treatments are usually designed and incorporated so that the energy is dissipated primarily by one type of strain or the other, yielding different configurations and material properties for each type in order to achieve the greatest damping efficiency.

The simplest form of employing passive viscoelastic damping treatments is the one where a layer of viscoelastic material is attached to the outer surfaces of the structure. When the layer is attached in this way, the outer surface of the damping layer is quite free and *unconstrained* by any shear stress. This configuration is therefore known as *unconstrained layer damping* (ULD) and was originally proposed by Liénard (1951) and Oberst (1952) in the early 1950s. When the structure vibrates, energy is dissipated by the viscoelastic layer, mainly due to the heat generated by the extensional strains which, as the distance of the damping layer to the neutral surface increases, become greater. Increasing the thickness and/or the distance from the neutral surface and length of the viscoelastic treatment in an appropriate manner would increase the energy dissipation and consequently the damping. However, in applications where the weight and size of the structure and damping treatments are of critical importance, these constraints demand a more efficient treatment, and other alternatives to increase damping must be found.

Alternatively, a damping treatment designed to dissipate energy mainly by virtue of shear strain can be achieved by the inclusion of an elastic constraining layer sandwiching the viscoelastic layer between itself and the host structure. Since the maximum shear strain usually occurs close to the neutral surface and is zero at the outer surfaces, it is logical finding configurations where the viscoelastic layer is incorporated close to the neutral surface. The resultant high shear stress and strain values developed in the damping layer, justify the use of the damping layer in a sandwich configuration (Figure 6.1). However, it is not essential for the damping layer to be close to the neutral surface to achieve a good damping performance. Provided that it is sandwiched between an host structure and a stiff constraining layer, reasonable levels of shear stress and strain are developed which allow a significant damping to be achieved.

This configuration of damping treatment is known as *passive constrained layer damping* (PCLD). Some examples of their use can be found in [Macé (1994), Bhimaraddi (1995), Cupial and Nizioł (1995)].

While the ULD treatment performance mainly depends upon the distance from the damping layer to the neutral surface and the value of the so-called *loss* extensional modulus, PCLD treatments are more complex to design since their best damping performance is achieved by a trade-off between different parameters. Therefore, optimal values for the so-called *storage* shear modulus of the viscoelastic material and elastic properties and thicknesses of the remaining layers in the configuration must be carefully chosen in order to have optimal damping performance [Mead (1998)]. When comparing the ULD and PCLD damping layers thicknesses, the latter usually requires smaller thicknesses and therefore the influence of the damping treatment in the dynamic properties of the structural systems where it is applied is mainly devoted to the damping increase and smaller mass and stiffness modifications are introduced.

However, while passive damping treatments, constrained or not, can greatly improve the damping of the system, there are some limitations. The frequency and temperature dependent mechanical properties of the viscoelastic materials can make the damping change, bringing limitations to the effective temperature and frequency range of the treatment over which it is efficient. In order to provide adequate damping over a broad frequency band and different temperature ranges, different viscoelastic materials must be simultaneously applied in the same damping treatment, which often complicates the analysis and design of the system. Therefore, while viscoelastic treatments are easy to apply, the damping is often of limited frequency bandwidth and temperature range.

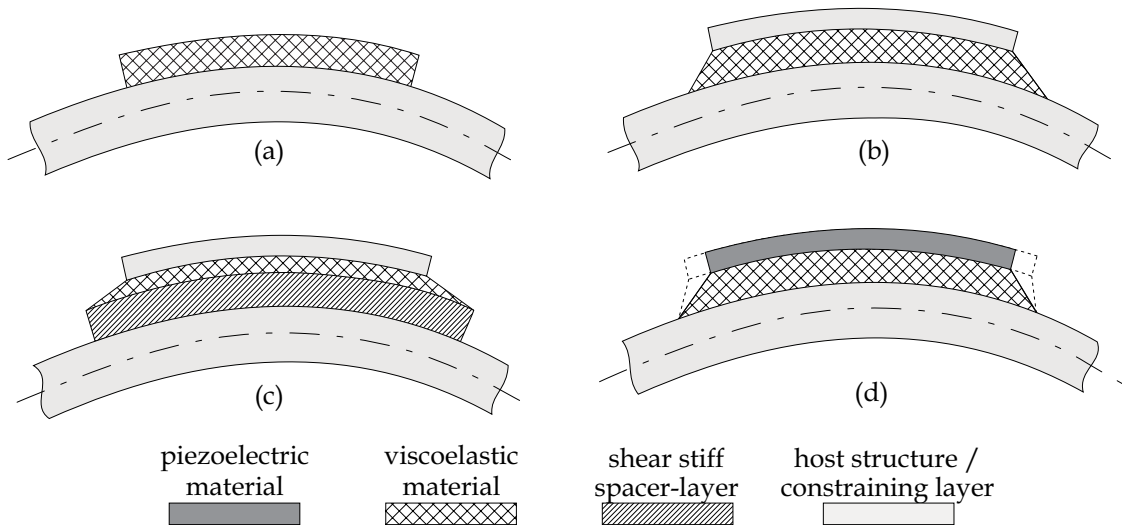


Figure 6.1: Viscoelastic damping treatments configurations: (a) unconstrained layer damping (ULD); (b) passive constrained layer damping (PCLD); (c) PCLD with the strain magnifying effect of a spacer-layer; (d) active constrained layer damping (ACLD).

From the early 1990s, so-called *active constrained layer damping* (ACLD) treatments have been analyzed and applied to structures [Agnes and Napolitano (1993), Baz (1993), Shen (1994a)]. These are hybrid treatments with active constraining layers made of piezoelectric materials (Figure 6.1d) which have the unique feature of having the ability to be used both as sensors and/or actuators [Cady (1964)]. If utilized as actuators, and according to an appropriate control law, an active constraining layer can actively increase, in a convenient way, the shear deformation of the viscoelastic layer and therefore overcome some of the PCLD limitations by adapting

the treatment to the temperature and spectral content of the undergoing operating conditions. Thus, ACLD treatments combine the high capacity of passive viscoelastic materials to dissipate vibrational energy at medium to high frequencies with the active capacity of piezoelectric materials at low frequencies. Therefore, in the same damping treatment, a broader control over the temperature and frequency ranges of interest is achieved benefiting from the advantages of both passive (simplicity, stability, fail-safe and low-cost) and active (adaptability, observability, controllability and high-performance) systems. Some examples can be found in the works of Baz and his colleagues [Ray and Baz (1997), Park and Baz (1999a), Baz (2001)].

Various configurations of active and passive layers surface mounted or embedded in an host structure have been proposed in an attempt to improve damping performance even further. In general, *hybrid active-passive damping* (YAPD), or arbitrary ACLD, treatments, involving arbitrary arrangements of constraining and passive layers, integrating piezoelectric sensors and actuators, might be utilized [Azvine et al. (1995), Ro et al. (1997), Baz (1998), Park and Baz (2001), Trindade et al. (2001a,b), Trindade and Benjeddou (2002), Suzuki et al. (2003)]. Surveys on the advances in hybrid active-passive vibrations and/or noise control via piezoelectric and viscoelastic constrained layer treatments can be found in [Park and Baz (1999b), Benjeddou (2001), Trindade and Benjeddou (2002), Stanway et al. (2003)].

Damping treatments with *stand-off layer* (SOL) have been widely accepted as an attractive alternative to conventional constrained layer damping (CLD) treatments. Such an acceptance stems from the fact that the SOL, which is simply a slotted spacer layer sandwiched between the viscoelastic layer and the base structure, acts as a strain magnifier that considerably amplifies the shear strain and hence the energy dissipation characteristics of the viscoelastic layer. Accordingly, more effective vibration suppression can be achieved by using SOL as compared to employing CLD [Yellin et al. (2000, 2005), Chaudry and Baz (2006)] however with some mass and stiffness modifications in exchange.

6.4 Finite Element Models of Viscoelastic Structural Systems

As reported in the previous section, according with the operating thermal conditions and spectral content of the excitation of the host structure, the optimal performance of viscoelastic damping treatments demands alternative configurations to be tested and designed. In the limit case, these configurations might comprise arbitrary arrangements and stacking sequences of a single or multiple kinds of viscoelastic materials with different thicknesses and different locations.

For shear-type damping mechanisms, an accurate representation of the shear strain and stress fields inside the viscoelastic layer(s) is a key issue in the accurate physical representation which must be properly taken into account into the mathematical modeling and numerical dynamic simulation of structural systems with viscoelastic damping treatments. In the development of the mathematical model the spatial distribution of the damping treatments is a special requirement which demands specific modeling strategies to be developed and utilized. These modeling strategies are special in the sense that the extensional and/or shear strains and stresses developed in the viscoelastic layers must be accurately modeled in order to obtain good damping estimates and in the sense that the treatments are very often discontinuous which makes analytical models difficult to use.

It is well known that a general approach able to circumvent these complications corre-

sponds to the use of the FE method, which can be used as a reliable and useful tool to characterize the structure response and analyzing the effects of any damping treatment configuration and assessing the influence of the design parameters upon the structural dynamic performance. However, despite the obvious and well known capabilities of the FE method, there are three main issues that still need to be carefully considered in its application:

1. The first is related with the specific deformation patterns developed in viscoelastic laminated structures which demands a special and dedicated deformation theory able to capture the high shear deformation pattern developed in the viscoelastic layer during the structure vibration;
2. The second is related with the meshing and spatial modeling of the damped structural system which, again, demands specific and dedicated discretization procedures, employing 2-D or 3-D meshes, according to the spatial model approach utilized involving "composite" assemblies of standard FEs available in commercial FE codes or the development of specific and dedicated new discrete-layer FEs;
3. The last issue is related with the specific description of the viscoelastic material constitutive behavior considered in the FE model and its implementation.

The former two issues are discussed in what follows. The latter issue is thoroughly presented and discussed in Section 6.6.

6.4.1 Some Comments on Deformation Theories

The deformation theory typically applied to structural laminates [see for example the textbook of Reddy (2004)], i.e., the *classical laminate theory* (CLT), cannot be used for this purpose, since it is not able to represent any shear strain distribution within the layers. Similarly, *equivalent single-layer* (ESL) theories and even *high-order theories* (HOT), do not take into account the inter-laminar and intralaminar effects and therefore suffer from lack of accuracy and generalization. These issues were in fact analyzed by Kosmatka and Liguore (1993) that reported a situation where plate/solid/plate "composite" (assembled) approaches accurately predict both the natural frequencies and the loss factors of a viscoelastic sandwich plate, while ESL approaches based either in layered, or composite, single solid or plate FEs approaches generally overpredict and underpredict the natural frequencies, respectively.

Despite the assumed continuity of the displacement field between the viscoelastic and the stiff constraining layers, the continuity of the out-of-plane stress components is not usually considered and the localized effects in the highly sensitive shear strained viscoelastic layers are not accurately captured. However, as described in Section 4.2, refined deformation theories, taking into account the so-called C_z^0 -requirements [Carrera (1997)], can be used to accurately model layered structures with arbitrary configurations of damping layers. A comprehensive overview and assessment of some refined deformation theories of plates and shells, e.g. first- or higher-order *discrete-layer* (or *layerwise*) or *zig-zag theories* (ZZT), in some cases using mixed variational principles, in static and dynamic situations, can be found for example in the works of Carrera (1999, 2002, 2003b).

6.4.2 Spatial Modeling and Meshing

Regarding the solution method utilized, as previously stated, the FE method is the most interesting approach which, as far as the spatial modeling is concerned, is often utilized considering

two main strategies: (i) by connecting multiple FEs (i.e., beam, plate, shell and/or solid FEs) into a single "composite" FE; (ii) or developing new dedicated higher-order discrete-layer (or zig-zag) models. Therefore, in what follows, the underlying spatial mesh model approaches, related with the aforementioned deformation theories, are discussed from a FE method implementation standpoint.

6.4.2.1 "Composite" Elemental Models

In the past, researchers developed various FE approaches for structures with constrained layer damping treatments. The easiest way of doing that was utilizing the modeling capabilities of existing FE codes, such as NASTRAN, ANSYS, ABAQUS, etc., to analyze constrained layer beams, plates and shells. Since these structural viscoelastic damped models involve different physical layers with different material properties, the aforementioned first approach, in an attempt to represent the real physical distribution of the layers, was therefore often obtained by combining the existing FE modeling capabilities available at that time into a "composite" FE model involving various FE types.

Consider the sandwiched or viscoelastically constrained generic plate depicted in Figure 6.2. As previously discussed, the CLT is not adequate to accurately describe the shear deformation of the viscoelastic layer and therefore it is necessary to use a different model. (Similar conclusions might also be applied to more complex shell-type structures if adequate care is taken with numerical FE pathologies such as shear and/or membrane locking.) In order to overcome the aforementioned issues, special layered schemes of beam, plate and brick FEs, used also in conjunction with rigid-links, were utilized in conjunction with commercial FE codes and tools to model the host structure, viscoelastic and constraining layer [cf. [Nashif et al. \(1985\)](#), [Sun and Lu \(1995\)](#), [Jones \(2001\)](#)]. Several distinct approaches available in the open literature are depicted in Figure 6.3. For simplicity, only three-layered sandwiched plate structures have been considered to illustrate the "composite" modeling approach. However, the underlying modeling idea and concept could also be employed and extended to multilayered plate and shell structures using also higher-order solid FEs. The "composite" and mesh models main characteristics including the most relevant works where these approaches are first developed and utilized are summarized in Table 6.1.

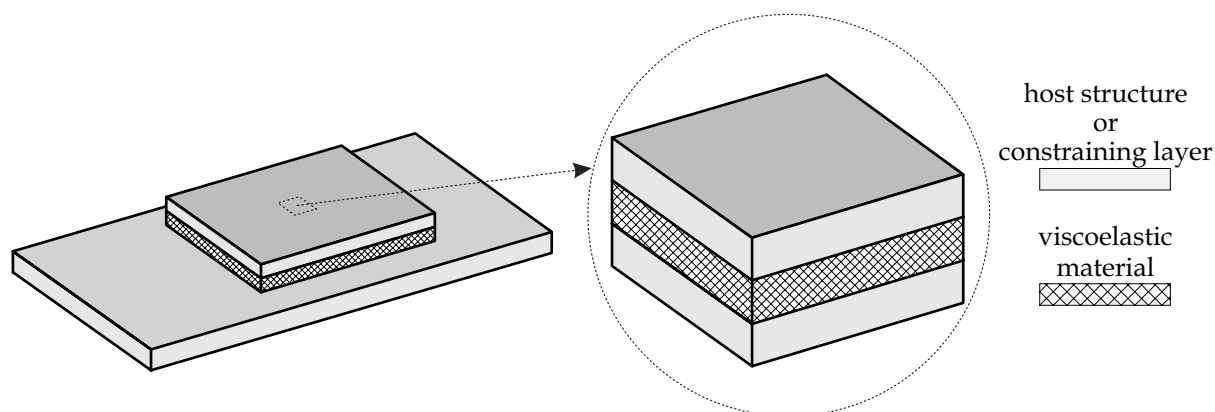


Figure 6.2: Generic viscoelastically damped sandwich or constrained plate and elemental volume.

The first model, model 1, due to its simplicity and easiness of implementation with commercial FE codes (e.g. NASTRAN, ANSYS, ABAQUS), benefiting from the usual standard

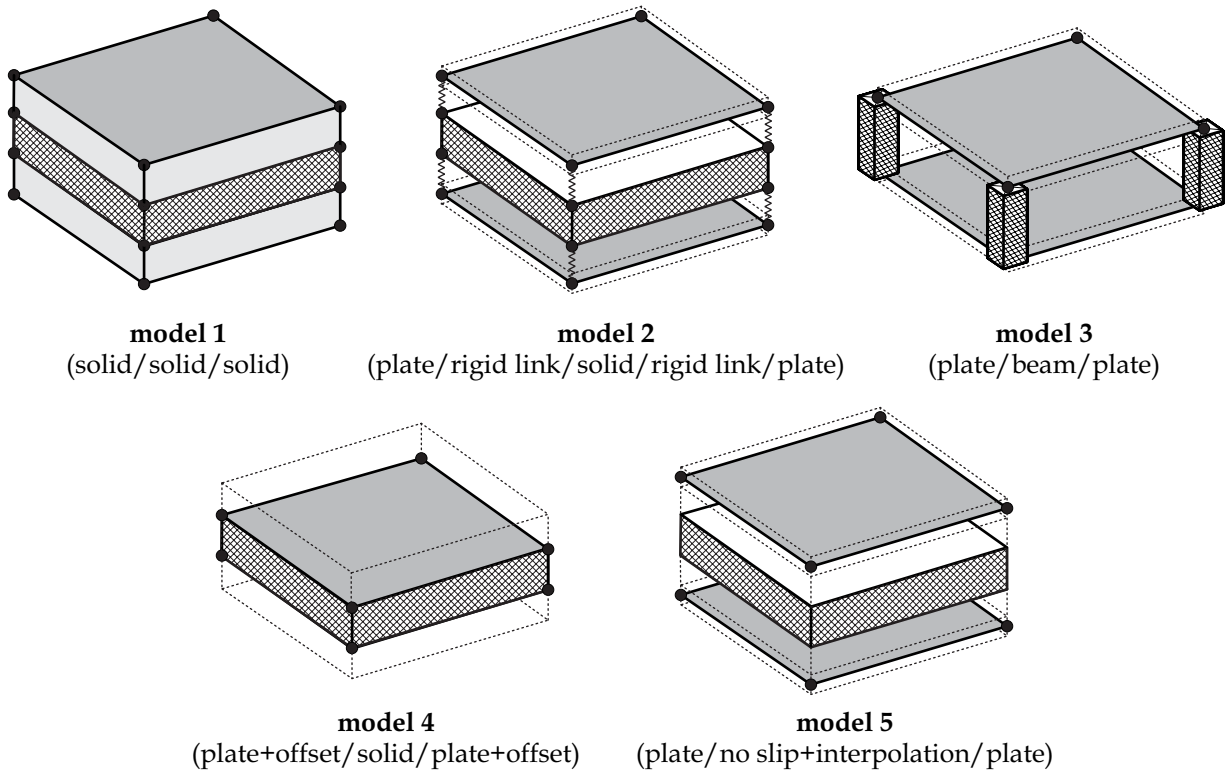


Figure 6.3: "Composite" FE plate (or shell) models of viscoelastic damping treatments using existing or in-house FE codes/elements.

discretization procedures available in these codes, was probably the first approach to be utilized. It uses standard linear (or higher-order) solid brick elements to model all the layers and, for a three-layered configuration, yields four planes (layers) of nodal points. In this model, since a 3-D formulation is used, a 3-D stress-strain state is considered and all the components, i.e. in-plane and out-of-plane extensional and shear components, are considered and a full 3-D stress-strain state is captured. However, from a computational cost point of view, this model yields a "brute-force" formulation with too many degrees of freedom (DoFs), making it uninteresting for large-scale problems. Since this is a very common approach, it is difficult to point out the first work using it. However, the reader is referred to some of the more relevant works in the last decade [Moreira and Rodrigues (2004), Zhang and Chen (2006), Hambric et al. (2007)] fully representative of the results obtained with model 1. As observed by Moreira and Rodrigues (2004), with modeling approaches considering solid brick FEs to model the viscoelastic layer, special care must be taken in order to avoid numerical pathologies such as shear (or in the case of shells also membrane) locking [MacNeal (1994)].

A comparative and assessment study, using conventional FEs from NASTRAN, between models 1, 2 and 4, looking at their limitations, accuracy and applicability, was performed by Moreira and Rodrigues (2004). All the three models used in that study share a common FE representation of the viscoelastic layer which is performed using solid brick FEs (HEXA8). The base plate and the constraining layer of the surface treatments, or the skin plates of the sandwiched layered configuration, were both modeled by either plate (QUAD4) or brick (HEXA8) FEs.

It was shown that the relatively low storage modulus of the viscoelastic layers, when compared with the stiff materials used in the host structure and constraining layer, attenuates the

locking effects. However, the locking effects can also be a problem if the host structure and constraining layer are also modeled with solid brick elements, as is the case in model 1. According to [Johnson and Kienholz \(1982\)](#), [Liguore and Kosmatka \(1988\)](#) and [Kosmatka and Liguore \(1993\)](#), numerical pathologies in viscoelastically damped plates occur mainly due to the shear locking effects within the viscoelastic core for aspect ratios of the layer (length to thickness ratio of the layer) higher than 5000. However, for the constraining layers the limit aspect ratio where shear locking effects become more pronounced is 100 when using selective reduced integration or other shear correction mechanisms (e.g. MITC, EAS, etc.) and its value is limited to 5 when normal integration schemes are used. Obviously, the previous conclusions hold only for planar structures. Nevertheless, they illustrate well the difficulties encountered when using FE solution methods for planar structures and it is expected to have a somewhat different behavior when considering more complex curved structures such as shells or curved beams, due to the more complex stress state where membrane locking may occur [[MacNeal \(1994\)](#)].

As shown by [Moreira and Rodrigues \(2004\)](#), in general, when using FEs from NASTRAN, the convergence rate of model 1, considering selective reduced integration only for the viscoelastic layer, is better than the one obtained with model 4. Furthermore, elements of the external layers should also use adequate shear locking reduction strategies when they become too thin in order to make the response less dependent of the mesh refinement.

Another more recent alternative to model 1 is the use of models 2 and 4; the latter two models both have in common the use of solid brick elements to model the viscoelastic core. However, regarding the outer stiffer layers modeling, there are two distinct approaches.

In the first one, model 2, the translational degrees of freedom of the plate are connected to the brick ones by means of rigid links [[Lu et al. \(1979\)](#), [Balmès and Germès \(2002\)](#)]. Using this model, the most complex one, it is possible to simulate bonding failure between the viscoelastic layer and the adjacent plates by simply removing those links in specific nodes of the FE method mesh. However, as in model 1, there are still four planes of nodal points but the locking effects of the stiff thin external layers might be more easily circumvented. Other structures such as a continuously constrained damped beam and ring (curved beam) were also modeled with similar and analogous procedures by [[Lu and Everstine \(1980\)](#)].

The second approach, model 4, as proposed by [Johnson and Kienholz \(1982\)](#) and further employed by [Kosmatka and Liguore \(1993\)](#), [Plouin and Balmès \(1999\)](#) and [Plouin and Balmès \(2000\)](#), is not as straightforward since while attempting to attach plate elements to a solid element surface there are two problems that must be overcome: (i) the first problem involves offsetting the nodes from the plate mid-plane to the plate surface adjacent to the solid element; (ii) the second problem involves developing and incorporating a set of constraint equations used to correct continuity discrepancies between the plate and solid elements. However, since there are coincident nodes and translational degrees of freedom for the plate and the adjacent face of the solid element, it allows the reduction of the total number of nodal points requiring only two nodal planes. A similar approach to model 4 was also performed by [Soni and Bogner \(1982\)](#) that used a dedicated code called MAGNA to analyze constrained layer treatments using eight-node thin shell elements or penalty function shell elements to model the constraining layers and the base structures, respectively.

Although providing accurate results two decades ago, FE solutions using solid brick elements to model the thin viscoelastic layer used to require a great deal of computational time. Therefore, when trying to study more complex structural problems, that approach was limited by the current computing capabilities at that time. These limitations motivated [Killian and Lu \(1984\)](#) to develop and propose the approach depicted in model 3. They had in mind that it was needed to introduce an approximation which is simple and yet provides accurate predictions.

Their objective was to present a FE approximation with simple elements to replace solid elements which are not really necessary for linear vibratory systems. The modeling approach is the same as that given in model 2 for the host structure and the constraining layer. However, the brick elements to model the thin viscoelastic damping core are not used and, instead, the thin viscoelastic layer is approximated by viscoelastic beams connecting the base plate and the constraining plate. The structural properties and damping for these beams were then specified in order to simulate the shear and extensional damping provided by the thin viscoelastic layer.

With this approach, the number of nodal points for the composite system is reduced from four rows to two, and the rigid link elements used in model 2 to assure the geometric compatibility are also eliminated. When compared with model 2, model 3 has been shown [Killian and Lu (1984)] to provide better results than those obtained by using 3-D solid elements, not only predicting the gross features of the system behavior, but also the resonances and antiresonances. However, a very interesting and creative approach at the time of its creation, in the forthcoming years, and with the fast development of modern computing capabilities, since its use in commercial FE codes is not straightforward, requiring special procedures to define the beam element properties and assembling process, to the knowledge of the author the approach in model 3 has become less attractive.

The last model, model 5, so-called *degenerate constrained layer element*, has been proposed by Jeung and Shen (2001) for plate and shell structures. They adopted the degenerated shell element proposed by Ahmad et al. (1970) for the constraining layer and the base shell structure. The displacement field of the viscoelastic layer is linearly interpolated through no-slip boundary conditions at the layer interfaces, therefore creating pseudo-nodes in the viscoelastic layer. In fact, this model is very similar in principle to model 2; the main difference lies in the fact that the rigid link effect is imposed a priori at the formulation level by no-slip conditions, yielding a composite FE that does not need to be assembled at the elemental level and which requires a 2-D global assemblage. This type of composite approach is slightly different from the others in the sense that it requires special efforts to the user to formulate a new element and implementing it in commercial or in-house FE codes. For thin plate structures, numerical results have shown that the present approach can predict natural frequencies, loss factors, and mechanical impedances that are as accurate as the ones obtained with NASTRAN, however requiring substantially fewer elements. For thin shell structures, applications of model 5 are possible, if appropriate numerical pathologies are implemented to control the tragic spurious modes effects.

Other models also based in the use of existing FEs of commercial codes, applied to damped beams, plates and shells, based on ESL approaches [Kosmatka and Liguore (1993)] or on the previous models with slight modifications [Lu and Everstine (1980), Lumsdaine and Scott (1998), Nallainathan et al. (2003)], can still be found in the open literature.

6.4.2.2 Discrete-Layer (Layerwise) Elemental Models

Some general drawbacks of the composite spatial models previously described are the higher intervention of the designer during the spatial modeling task, which has to decide which combination of FEs is more appropriate and to develop special elemental assembling procedures for the elements to be combined, and the need to completely redefine the 3-D FE mesh when changing the geometric parameters of the damping treatments such as the thickness of the layers or the location(s) of the damping treatment(s). Therefore the latter approach is not well suited for design optimization procedures of single and multi-layer viscoelastic damping treatments. Furthermore, when using this discretization approach to obtain the spatial model of

Table 6.1: Main features of the "composite" and discrete-layer elemental models.

MODELS	STRENGTHS	WEAKNESSES
"Composite" elemental models		
Model 1 (s/s/s)	• Simplicity and easiness of implementation • Standard and readily available FEs • Standard discretization procedures • 3D stress-strain state may be captured	• "Brute-force" modeling • High computational cost (many DoFs) • Uninteresting for large-scale problems • Numerical pathologies may occur for thin layers
Model 2 (p/r/s/r/p)	• Moderate computational cost • Less sensible to numerical pathologies for thin external layers • Ideal for sandwich structures • Possibility to simulate bonding failure	• Rigid links impose more complexity • Non-standard discretization and assemblage procedures • Less-standard FEs
Model 3 (p/b/p)	• It exempts the use of brick FEs • Better accuracy with less DoFs • Insensitive to numerical pathologies due to issues related with the core thickness	• Requires special procedures to define the beam element properties • Non-standard discretization and assemblage procedures
Model 4 (p+o/s/p+o)	• Less computational cost (allows the reduction of DoFs) • Less sensible to numerical pathologies for thin external layers	• Requires off-setting of plate DoFs • Requires external constraint equations • Non-standard discretization and assemblage procedures
Model 5 (p/n+i/p)	• Lower computational cost with only plate DoFs • Less sensible to numerical pathologies for thin external layers • More standard discretization and assemblage procedures	• Requires an a priori coupled-kinematics formulation • Non-standard composite FE • Not readily available
Discrete-layer elemental models		
Model 6 (single-sided)	• High degree of automation in spatial modeling and mesh generation • Best suited for optimization procedures • Requires a standard 2D mesh	• Not readily available • Segmented double-sided damping configurations can not be considered • Non-standard coupled-kinematics formulation
Model 7 (double-sided)	• The same as model 6 • Segmented symmetric and asymmetric double-sided configurations can be considered • Highest spatial modeling generalization capabilities	• Not readily available • Non-standard coupled-kinematics formulation • More cumbersome formulation
Model 8 (regenerated)	• Same as 6 and 7 • More straightforward to formulate and implement • Accurate with excellent trade-off with complexity • Higher degree of versatility	• Same as 7 • High degree of technical expertise required to the formulation

Notes: 's', solid; 'p', plate; 'r', rigid link; 'b', beam; 'o', offset; 'n', no slip; 'i', interpolation.

structures, a cumbersome and time consuming 3-D FE mesh generation process is required. The forementioned reasons motivated the use of alternative methods circumventing these limitations.

A more efficient approach using new dedicated FEs, which are not readily available in commercial FE codes, which inherently yields an higher degree of automation in the spatial modeling and mesh generation processes, can alternatively be used to model viscoelastically damped structures. These new FEs can vary substantially in terms of nodal topology because for their formulation different assumptions regarding deformation theories, kinematics, constitutive behavior, number of layers, interpolation degree, shape, etc., may be considered. Following that line of thought, special FEs based on first- or higher-order *discrete-layer* (or *layerwise*) or *zig-zag theories* (Figure 6.4), have been proposed and successfully applied to beam, plate and shell-type structures with active and/or passive damping treatments [Macé (1994), Saravanos (1994), Cupial and Niziol (1995), Bhimaraddi (1995), Heyliger et al. (1996), Suzuki et al. (2003), Carrera (2003a), Vasques, Mace, Gardonio and Rodrigues (2006), Moreira et al. (2006), Vasques and Dias Rodrigues (2006b)]. As far as the deformation theory and kinematics are concerned, with a discrete-layer approach each physical layer (or sub-laminate) is treated individually according with the desired accuracy. Different deformation theories of different orders might be postulated for each layer, and by the use of “simple” or “mixed” variational principles and *a priori* assemblages at the elemental level or layered formulations with coupled kinematics, the elemental equations of motion and the correspondent layerwise FE formulation are established in a 2-D sense. Due to the specificity of these formulations, the introduction of these new and dedicated FEs into general commercial FE codes for the composite, sandwich materials and damping treatments end-users is progressing well but far from being fully accomplished.

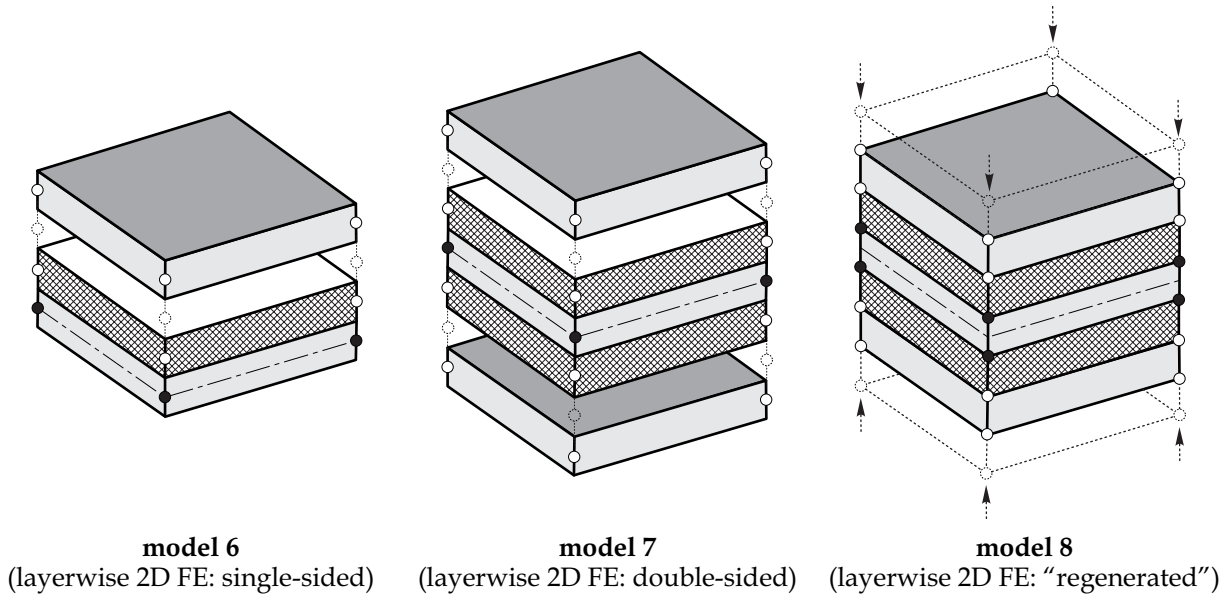


Figure 6.4: Discrete-layer (layerwise) 2-D FE (or shell) models of viscoelastic damping treatments using new dedicated FE codes/elements.

These layerwise FEs allow a significant simplification on the spatial model generation, since the out-of-plane description, which depends on the morphology of the locally applied damping treatment (coverage, numbers of layers, layering scheme and layer materials), is defined during the FE formulation, due to the coupled-layer kinematics, or during the *a priori* assemblage at

the elemental level, going from a single- to a multi-layer FE. Thus, this modeling technique enables the external definition of the damping treatment parameters through a user defined data structure containing the properties which can be dynamically modified or updated in a numerical optimization process analysis, which is an impractical feature when using the previously presented layered composite models.

As it was mentioned before, the out-of-plane description of the damping treatment configuration is performed either by coupled-layer kinematics or *a priori* assemblage at the elemental FE level so that the 3-D continuum assemblage, at the global FE level, is reduced to a 2-D discretization. In general, the damping treatments might be sandwiched or surface mounted in a single or both sides of the host structure. For those cases, and since the continuity of the host structure must be assured, dedicated single- or double-sided layerwise FEs using coupled-layer kinematics must be employed to model the equivalent 2-D continuum. These modeling approaches are represented by models 6 and 7, respectively.

As proposed in Chapter 5 for the piezo-visco-elastic shell FE, an alternative approach, and a more general one, instead of using formulations based on coupled-layer kinematics for an arbitrary number of layers [Vasques, Mace, Gardonio and Rodrigues (2006)], as used in Chapter 2 for piezo-visco-elastic beams, uses an *a priori* assemblage of a single-layer FE before performing the global assemblage of the equivalent 2-D continuum. That approach was first proposed by Vasques and Dias Rodrigues (2006b) and is depicted by the model 8, that represents a "regenerated" layerwise 2-D approach which, in opposition to the well known Ahmad's "degenerated" approach [Ahmad et al. (1970)], some transformations of the elemental DoFs are performed allowing the displacement and stress DoFs of different layers to be assembled imposing not only displacement continuity but also shear stress continuity across the interfaces of multi-layer FEs, making it suitable for assemblage of elemental matrices from single- to multi-layer level.

6.5 Viscoelastic Constitutive Modeling

6.5.1 Comments on the Historical Developments of Viscoelasticity

In general, the constitutive behavior of viscoelastic materials might be said to depend upon the frequency, working temperature, amplitude and type of excitation [Nashif et al. (1985), Jones (2001)]. A mathematical model considering all these effects simultaneously has not yet been developed and is very difficult to conceive. Thus, for simplicity, since the amplitude and type of excitation effects have been reported to be of reduced importance, they are often overlooked. However, the temperature and frequency dependent physical properties of the viscoelastic materials still introduce serious difficulties in the definition of an accurate mathematical model able to simulate properly the dynamic behavior of the damped structure. Therefore, for practical reasons, isothermal conditions are usually assumed in the simulation conditions and merely the frequency dependent constitutive behavior is directly taken into account upon the constitutive mathematical model. Following this assumption, the design of passive viscoelastic damping treatments, for broad temperature range applications, is usually conducted at several constant temperature levels, selected within the temperature range, considering isothermal conditions.

A few exceptions, though, where both the temperature and frequency effects are captured in a coupled model, were reported by Schapery (1964), who extended Biot's theory [Biot (1954, 1955)] in the application of the *thermodynamics of irreversible processes* to viscoelastic materials, yielding a unified theory of the thermomechanical behavior of viscoelastic materials with

an explicit temperature dependence and a consistent thermodynamically inclusion of the time-temperature superposition principle, and by Lesieutre and his co-workers [Lesieutre and Govindswamy (1996), Brackbill et al. (1996)], that extended the ADF method to include thermal effects by means of the use of the linear *time-temperature superposition principle*, valid for *thermorheologically simple materials*, yielding a nonlinear amplitude-dependent FE which requires specific nonlinear solution strategies. Other studies where the temperature effects are also considered can be found in [Taylor et al. (1970), Zocher et al. (1997), Hammerand and Kapania (1999), Demirdžić et al. (2005), Clifford et al. (2006)].

Literature covering the three primary mechanisms of damping which are important in the study of mechanical systems, namely *internal damping* (or *material*), *structural damping* (at joints and interfaces) and *fluid damping* (through fluid-structure interaction), is very vast and it is not the purpose here to survey it. The interested readers are referred to some textbooks and articles on general damping theory [Bishop (1955), Crandall (1970), Ewins (2000), Lesieutre (2001), Inman (2001), Cremer et al. (2005), de Silva (2005)] in order to become acquainted with the developments on damping theories. Worth to mention is a recent historical and contemporary critical discussion on damping theory, also pointing out the perspective on how the treatment of damping is likely to evolve in the future, performed by Peters (2005).

Directing now our attention to material damping, since the first experimental observation of *material* (or *internal*) damping performed by Coulomb in the 1780s with his *mémoire* "Sur la Force de Torsion & Sur l'Elasticité des Fils de Metal" [Coulomb (1784)], where he not only hypothesized regarding the microstructural mechanisms of damping but also undertook experiments which proved that the damping of torsional oscillations is not caused by air friction but by internal losses in the material, over the following centuries different methods to characterize frequency dependent damping, in general, and the viscoelastic material's constitutive behavior (viscoelastic damping), in particular, were proposed.

As discussed by Snowdon (1968), the mechanical properties of rubber-like materials may be considered at two different damping levels: (i) *low-damping materials*, which have the dynamic modulus and damping factor (typically with a low value of the order 0.1) varying slowly with frequency, which enables them to be considered as constants through the range of frequencies normally of concern in vibration problems; and (ii) *high-damping materials*, where both the dynamic modulus and damping factor vary significantly with frequency. For the latter case, during the transition zone, where the material undergoes a change from rubber to glassy dominated behavior, the dynamic modulus increases very rapidly with frequency and the damping factor is large (typically with values of the order 1) and might vary slowly or more strongly with frequency. Under the foregoing assumptions, Snowdon has considered four different models of damping, which he called damping of the *solid type* I and II, of the *viscous type* and of a *three-element spring and dashpot combination*. The first damping type is nowadays referred to in the literature as *hysteretic damping* [Banks and Pinter (2001)], where both the storage modulus and loss factor do not depend on frequency and is typically applied to model the small energy dissipation effects due to the internal friction which occurs in metals in general. The second and third damping types alleviate, respectively, the assumption that the storage modulus or the loss factor do not depend on frequency. The interest and applicability of the second and third models is somewhat reduced since usually models of rubber-like materials need to have explicit frequency dependent mechanical properties in order to capture the real physical behavior. Regarding the third damping type, it was considered by means of a parallel Hookean spring and Newtonian dashpot equivalent constitutive model, which justifies the term *viscous* used for its designation, where the loss factor is frequency-dependent (the well know *viscous damping*). It should be mentioned, though, that the *viscous* equivalent constitutive model, also

known as Kelvin-Voigt model, poorly represents the dynamic mechanical properties of rubber-like materials justifying the need for other type of models. Lastly, the fourth model, known as *standard linear solid*, also known as Kelvin or Zener model [Zener (1948)], considering a three-element combination of a spring and dashpot in series (Maxwell model) placed in parallel with another spring, allows both the dynamic modulus and damping factor to depend on frequency. In fact, all the first three damping types can be seen as particular cases of this more general last model.

The aforementioned four different types of damping illustrate well the real physical damping behavior observed in the different kinds of materials that we have at our disposal and it is up to the designer to choose the model that best suits the real physical behavior of the material under consideration. Usually, in the author's opinion, the designation of "viscoelastic" damping refers to the fourth case mentioned above where both the storage modulus and loss factor are physically observed to depend significantly with frequency. However, depending on the degree of frequency dependence, some viscoelastic models might be preferred in place of others depending on how well the observed creep and relaxation, in the time domain, and broadband, in the frequency domain, behaviors are captured by the viscoelastic approximate damping model in hands.

The four models mentioned before can be seen as "classical" viscoelastic damping models. These models were gradually developed since the 18th century in an attempt to mathematically represent the real physically observed behavior of a class of materials with a somewhat material behavior composed by a blend of pure viscous fluids and elastic solid materials constitutive behaviors. These mechanical model analogies of viscoelastic materials consider arbitrary combinations of elastic and viscous damping elements, ranging from basic discrete systems such as the Maxwell (established by Maxwell in 1868) or Kelvin-Voigt (first established by Kelvin in 1875 and later by Voigt in 1892, sometimes referred to in the literature as only Kelvin or Voigt) models, more complex models combining these basic elements in parallel or series, namely the *linear standard solid* (also known as Kelvin model) and Burgers (1935) models (which represent more realistic viscoelastic models), generalized versions combining these elements (e.g. generalized Maxwell and Kelvin models) which yield frequency- and time-domain series representations, the latter containing exponentials, using mathematical representations in terms of the so-called Dirichlet or, alternatively, Prony's series [Prony (1795), Williams (1964)] [see some works using Prony's series in Park and Schapery (1999) and Slanik et al. (2000)], or even more complex arbitrary distributions of the basic spring and dashpot elements. Lastly, all these models, though, can also be seen as particular cases of a more general formulation, namely, the so-called *differential equation form* of the linear viscoelastic constitutive behavior under isothermal conditions. The interested reader can find further details on classical models and differential equation forms using mechanical analogies in [Alfrey and Doty (1945), Williams (1964), Ferry (1980), Findley et al. (1989), Johnson (1999), Fung and Tong (2001), Ottosen and Ristinmaa (2005)] and references therein.

Certainly, a very comprehensive and accurate model may be constructed by combining generalized models in an arbitrary way. Irrespective of how we combine springs and dashpots, it comes as no surprise that it is always possible to write the constitutive relation in a differential equation form. An advantage of the differential equation approach is that each model is easy to physically understand and interpret and models can be constructed in an intuitive fashion. The drawback, however, is that more advanced models soon become cumbersome since the initial conditions become difficult to deal with when using higher-order differential equation in time [cf. Ottosen and Ristinmaa (2005)].

Since usually to have good accuracy representing complex rate-dependent materials, such

as viscoelastic materials, more complex networks of springs and dashpots are required, an alternative approach is to change the differential equation for the evolution of the dissipative stresses by means of the tools of *fractional calculus* [see, for example, [Gemant \(1936\)](#), [Stiassnie \(1979\)](#), [Bagley and Torvik \(1983\)](#), [Koeller \(1984\)](#), [Bagley and Torvik \(1985\)](#), [Koeller \(1986\)](#), [Padovan \(1987\)](#), [Bagley \(1989\)](#), [Pritz \(1996\)](#) and [Pritz \(1996\)](#)]. In that case, the viscoelastic constitutive behavior is defined by a fractional differential equation form where the time derivatives order are not integer numbers. As suggested by [Koeller \(1984\)](#), the dashpots used in the classical rheological models might be substituted by the so-called *spring-pot* element, yielding a fractional differential model where the spring-pots are elements that, depending of the non-integer order (or memory parameter) of the derivative, allow a continuous transition of the constitutive behavior from the solid state (full memory representing a spring with complete elastic recovery) to the fluid state (no memory representing a dashpot with unrecoverable behavior) when the memory parameter varies from zero to one.

By using the fractional derivative model instead of the classical differential approach, one advantage is the easiness how the viscoelastic material properties are modeled in the frequency domain with a reduced number of parameters; typically three [[Bagley and Torvik \(1985\)](#)], four [[Pritz \(1996\)](#), [Galucio et al. \(2004\)](#)] or five [[Bagley and Torvik \(1983\)](#)]. However, its representation and use in the time domain is far more complicated than in the frequency domain requiring specific solution procedures to integrate the governing equations [see, for example, [Enelund and Josefson \(1997\)](#)].

As discussed previously, there were several early contributions to the theory of linear viscoelasticity stated in terms of differential equations, such as the ones given by Maxwell, Kelvin and Voigt. However, integral equations could be used as well, and in general terms, the integral approach might be seen as the general approach and starting point from which the viscoelastic theory should start from. The Austrian physicist Ludwig Boltzmann apparently developed in 1874 the first formulation of a three-dimensional theory of linear isotropic viscoelastic stress-strain relations [[Boltzmann \(1874\)](#)] while [Volterra \(1909, 1913\)](#) obtained comparable forms for anisotropic solids in 1909. Boltzmann initiated a linear hereditary theory of material damping by formulating an integral representation of the stress-strain relation which allows greater freedom when constructing models than the differential approach that relies on the concepts of certain combinations of springs and dashpots. Integrals are summing operations, and this view of viscoelasticity takes the response of the material at a certain instant of time to be the sum of the responses to excitations imposed at all previous instants of time. The ability to sum these individual responses requires the material to obey a more general statement of linearity than we have invoked previously, specifically that the response to a number of individual excitations be the sum of the responses that would have been generated by each excitation acting alone.

Thus far we have devoted our attention to time-dependent formulations by means of differential or integral equations forms where the typical creep and relaxation behavior of viscoelastic materials were more directly observable and identified from experimental measurements. However, since one of the simplest ways of experimentally determine the viscoelastic properties is to subject the material to periodic dynamic oscillations, the dynamic representation of the material properties by means of a complex modulus became a current practice and is perhaps the most commonly used approach when modeling the material in the frequency domain.

As far as material damping is concerned, and in particular the viscoelastic damping, the interested readers in further details are referred to a report with a unique comprehensive bibliography survey until the early-mid of the 20th Century of the material damping field compiled by Demer in the 1950s [[Demer \(1956\)](#)] and to some articles [[Alfrey and Doty \(1945\)](#), [Lazan](#)

(1959), Ungar and Kerwin (1962), Williams (1964), Bert (1973), Tschoegl (1997), Johnson (1999), Mead (2002)] and textbooks [Zener (1948), Snowdon (1968), Christensen (1982), Nashif et al. (1985), Tschoegl (1989), Fung (1993), Sun and Lu (1995), Mead (1998), Riande et al. (2000), Fung and Tong (2001), Jones (2001), Goodman (2002), Ottosen and Ristinmaa (2005)] which illustrate well the later developments in the field.

Within the scope of linear theory, time and frequency domain constitutive models are presented in what follows.

6.5.2 Time Domain Representation: Hereditary Approach

Viscoelastic materials are sometimes called materials with infinite “memory”, in the sense that their actual mechanical response is modulated by the past. Therefore, the viscoelastic constitutive behavior relies on the assumption that the current value of the stress tensor depends upon the complete past history of the strain tensor components. This latter assumption therefore implies that the behavior of any viscoelastic material may be represented by a hereditary approach. Furthermore, if the net strain response of the material to an arbitrary sequence of stimulus can be determined through the sum of the responses which would have been obtained if all stimuli had act independently, *time dependence linearity*, and if an increase in the strain stimulus by an arbitrary factor increases the stress response by the same factor, *stress-strain linearity*, the material is said to have a *linear viscoelastic* behavior. These are the type of viscoelastic materials discussed in what follows.

Considering an isotropic viscoelastic material under isothermal conditions and under small (infinitesimal) deformation conditions, the theory of *linear viscoelasticity* [Christensen (1982), Fung and Tong (2001)] states that the constitutive relationship for a generic one-dimensional isotropic stress-strain system (e.g. in shear or extension) can be given by a Riemann convolution integral,

$$\sigma(t) = \int_{-\infty}^t G(t - \tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau, \quad (6.1)$$

where $\sigma(t)$ and $\varepsilon(t)$ are the time dependent stress and arbitrary strain component histories and $G(t)$ is called the constitutive time varying *characteristic relaxation function* (utilized in the literature under many different names, such as *damping kernel*, *retardation*, *hereditary* or *after-effect* function) of the material. Equation (6.1) expresses an essential feature of linear behavior of viscoelastic materials known as *Boltzmann's superposition principle* [Boltzmann (1874)]. Since the lower limit of integration is taken as $-\infty$, it is to mean that the integration is to be taken before the very beginning of the motion. Thus, if the motion starts at time $t = 0$, meaning that the stress and strain are equal to zero up until time zero where the loading begins, i.e., $\sigma(t) = \varepsilon(t) = 0$ for $t < 0$, and discontinuous strain histories with a step discontinuity at $t = 0$ are to be considered, Equation (6.1) reduces to

$$\sigma(t) = G(t)\varepsilon(0) + \int_0^t G(t - \tau) \frac{\partial \varepsilon(\tau)}{\partial \tau} d\tau, \quad (6.2)$$

where $\varepsilon(0)$ is the limiting value of $\varepsilon(t)$ when $t \rightarrow 0$ from the positive side. The first term in Equation (6.2) gives the effect of the initial disturbance and it arises from allowing a jump of $\varepsilon(t)$ at $t = 0$. Furthermore, when Equation (6.1) was written it was tacitly assumed that $\varepsilon(t)$ is continuous and differentiable [see Fung and Tong (2001) for further details].

Considering initial conditions equal to zero, i.e., $\varepsilon(0) = 0$, the Laplace transform of Equation (6.2) yields

$$\tilde{\sigma}(s) = \tilde{Q}(s)\tilde{\varepsilon}(s) \equiv s\tilde{G}(s)\tilde{\varepsilon}(s), \quad (6.3)$$

where $\tilde{Q}(s) \equiv s\tilde{G}(s)$ is a *characteristic material function*, which should be experimentally identified somehow. Among all the possible identification tests [see [Tschoegl \(1989\)](#) for further details], the *relaxation test*, where a unit relaxation strain stimulus with $\tilde{\varepsilon}(s) = 1/s$ is applied, is considered here. Substituting it into Equation 6.3, the *relaxation stress* is given by

$$\tilde{\sigma}_{\text{rel}}(s) = \tilde{G}(s). \quad (6.4)$$

Considering the equality $\tilde{G}(s) = G_{\infty}[1 + \tilde{h}(s)]$, and substituting it into (6.4), yields

$$\tilde{\sigma}_{\text{rel}}(s) = G_{\infty} [1 + \tilde{h}(s)], \quad (6.5)$$

where G_{∞} is the so-called *relaxed* (also known as *static*) modulus, which for a isotropic material might be the shear or extension modulus, obtained after the material relaxation, i.e., G_{∞} is the limiting value of $G(t)$ when $t \rightarrow \infty$. The resultant relaxation stress has two components; since G_{∞} is constant, the first component, G_{∞} , represents the recoverable (elastic) counterpart of the material stress, while the second component, $G_{\infty}\tilde{h}(s)$, represents the non-recoverable (dissipated) one. For that reason, $\tilde{h}(s)$ is usually termed as *dissipation* and *relaxation* function. These two parameters, G_{∞} and $\tilde{h}(s)$, are unique characteristic properties of the viscoelastic material which are obtained by curve fitting experimental data, obtained from a relaxation identification test where a step function deformation stimulus is utilized, according to the mathematical viscoelastic damping representation chosen by the analyst. In fact, it is worthy to emphasize here that in order to characterize the viscoelastic material behavior to any type of strain stimulus, Equation 6.3 should be used; the necessary characteristic material function, $\tilde{Q}(s) \equiv s\tilde{G}(s)$, can readily be obtained after the relaxation stress has been determined in a relaxation test.

Many authors have proposed different mathematical representations of the relaxation behavior $\tilde{h}(s)$ in the last decades, which in turn leads to different mathematical models of the viscoelastic damping behavior and different solution methods of the governing equations in the time and frequency domains. Some different relaxation functions are summarized in [[Golla and Hughes \(1985\)](#), [Park et al. \(1999\)](#), [Trindade and Benjeddou \(2002\)](#), [Adhikari and Woodhouse \(2003\)](#)] as a result of a compilation from several works, where the different models and solution methods of the resultant mathematical models are thoroughly discussed. Some of the most well known relaxation functions, frequently referred in the open literature, are the ones used by the internal variables GHM and ADF models, which will be considered in this work and described in the following sections. The correspondent relaxation functions along with some other mathematical representations that have been used and presented in the literature are summarized in Table 6.2.

Following the thermodynamics of irreversible phenomena and applying it to viscoelasticity, the engineer, physicist and applied mathematician Maurice Biot was a pioneer first giving the subject in the 1950s an original treatment based upon relaxation modes and hidden variables and establishing a linear viscoelastic constitutive theory [[Biot \(1954\)](#)]¹. It was shown that the proposed model was very general in nature and that it could represent the behavior of a wide variety of phenomena in the process of their response to external actions which may involve, for example, the application of external stresses, chemical reactions, heat transfer, etc. In fact,

¹To the author's opinion, the first treatment given by Biot is brilliant and a cornerstone in science that over the following decades has been shown to include other theories and relaxation models, as the ones presented in Table 6.2 which, with the exception of the fractional calculus approach proposed by [Bagley and Torvik \(1983\)](#) and presented in the third line of Table 6.2, correspond to particular cases of Biot's treatment.

Table 6.2: Mathematical representations of the relaxation function in the Laplace and time domains.

Function $\tilde{h}(s)$	Function $h(t)$, $t \geq 0$	Author (year)
$\sum_{s=1}^n \frac{D_s}{p + \sigma_s}^*$	$\sum_{s=1}^n D_s e^{-\sigma_s t}$	Biot (1954)
$\int_0^\infty \frac{D(\sigma)\gamma(\sigma)}{p + \sigma} d\sigma$	$\int_0^\infty D(\sigma)\gamma(\sigma) e^{-\sigma t} d\sigma$	Biot (1954) Buhariwala and Hansen (1988)
$\frac{1 + as^\alpha}{1 + bs^\beta}^\dagger$	$--^\ddagger$	Bagley and Torvik (1983)
$\sum_{i=1}^n \alpha_i \frac{s + 2\hat{\zeta}_i \hat{\omega}_i}{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s + \hat{\omega}_i^2}$	$\sum_{i=1}^n \alpha_i \frac{b_{2i} e^{-b_{1i} t} - b_{1i} e^{-b_{2i} t}}{b_{2i} - b_{1i}}^\S$	Golla and Hughes (1985) McTavish and Hughes (1993)
$\sum_{i=1}^n \frac{\Delta_i}{s + \Omega_i}$	$\sum_{i=1}^n \Delta_i e^{-\Omega_i t}$	Lesieutre (1992)
$\sum_{i=1}^n \frac{c_i}{\tau_i s + 1}$	$\sum_{i=1}^n (c_i / \tau_i) e^{-t/\tau_i}$	Yiu (1993, 1994)

*In this expression a one-dimensional constitutive behavior was considered and the original notation used by Biot (1954) was retained, where $p = d/dt$ is a time operator which in the Laplace domain corresponds to considering $p = s$.

[†]A non-integer order of the exponents based upon the fractional calculus assumptions is considered so that $0 < \alpha < 1$ and $0 < \beta < 1$; for simplicity, the constant a was considered equal to $a = G_1/G_0$.

[‡]The relaxation function in the time domain can not be deduced and therefore is not presented; however, the full constitutive behavior considering the fractional derivative operator D^α , with the property in the Laplace domain $\mathcal{L}\{D^\alpha x(t)\} = s^\alpha \mathcal{L}\{x(t)\}$, is given by $\sigma(t) + bD^\beta \sigma(t) = G_0 \varepsilon(t) + G_1 D^\alpha \varepsilon(t)$.

[§]The constants b_{1i} and b_{2i} are given by $b_{1i}, b_{2i} = \hat{\omega}_i \hat{\zeta}_i \mp \hat{\omega}_i (\hat{\zeta}_i^2 - 1)^{1/2}$.

Biot's work comprised an unusually broad range of scientific and technological areas including applied mechanics, acoustics, heat transfer, thermodynamics, aeronautics, geophysics and electromagnetism, to which his contributions on the thermodynamics of irreversible phenomena were extremely important. His approach is presented in the first two lines of Table 6.2.

As reported by Biot (1954), the summation presented in the first line of Table 6.2 is extended to all internal relaxation constants, σ_s . In order to fully represent the effects of all the internal variables of a solid, since it has too many degrees of freedom, there may be in some cases an almost continuous distribution of relaxation modes. That will correspond to a spectrum or spectral density distribution of the relaxation constant $D(\sigma)$ with a density distribution function $\gamma(\sigma)$, as presented in the second line of Table 6.2. That constitutive model was later utilized to model viscoelastic beams and plates as reported by Buhariwala and Hansen (1988) and Biot (1972), respectively.

The exponential function introduced by Biot is probably the simplest physically realistic non-viscous damping model [Adhikari and Woodhouse (2001)] and has been used extensively in the context of viscoelastic systems. In fact, Biot has shown that, not only in the context of

mechanical models but also involving other coupled phenomena such as chemical, thermodynamic, etc., any relaxation phenomena may be represented by a spring, a dashpot and a sum of a great many elements made up of a Maxwell type material. As particular cases of Biot's model we have the double exponential relaxation function, known as the GHM model, or the single exponential relaxation function of the ADF model. Regarding Yiu's model, it was proposed in the context of finite element solutions of structures with viscoelastic treatments and is equal to the model proposed by Biot, however with a slight change in the constants of the coefficients.

6.5.3 Frequency Domain Representation: Complex Modulus

There are practical situations in which structures with viscoelastic materials may be subjected to steady-state oscillatory forcing conditions. Under these conditions, assuming s as pure imaginary variable, $s = j\omega$, the term $\tilde{Q}(s)$ in the previously derived stress-strain relationship in Equation (6.3), referring to the shear constitutive term, might be expressed as a function of the frequency ω in the complex form

$$G(j\omega) = G'(\omega) + jG''(\omega), \quad (6.6)$$

where $G'(\omega)$ is the shear *storage* modulus, which accounts for the recoverable energy, and $G''(\omega)$ is the shear *loss* modulus, which represents the energy dissipation effects [Christensen (1982), Nashif et al. (1985), Jones (2001)]. The *loss* factor of the viscoelastic materials is defined as

$$\eta(\omega) = \frac{G''(\omega)}{G'(\omega)}, \quad (6.7)$$

which alternatively allows writing Equation (6.6) as

$$G(j\omega) = G'(\omega) [1 + j\eta(\omega)]. \quad (6.8)$$

For a linear, homogeneous and isotropic viscoelastic material, equivalent representations of the previous equations hold for the extensional modulus $E(j\omega)$ and the relationship

$$G(j\omega) = \frac{E(j\omega)}{2[1 + \nu(j\omega)]}, \quad (6.9)$$

where $\nu(j\omega)$ is the Poisson's ratio, establishes a relationship between the two. However, for simplicity, a real frequency independent Poisson's ratio $\nu(j\omega) = \nu$ is assumed, leading to identical loss factors of the shear and extensional complex moduli, i.e., $\eta_E(\omega) = \eta_G(\omega) = \eta(\omega)$. Further details on the dynamic properties of the Poisson's ratio in linearly viscoelastic solids can be found in [Chen and Lakes (1993), Pritz (2000), Tschoegl et al. (2002), Lakes and Wineman (2006)].

6.6 Finite Element Implementation of Viscoelastic Damping

In the analysis of viscoelastically damped structures the choice of the most adequate solution method is strongly dictated by the applied temperature- and time-dependent viscoelastic constitutive model and by the desired type and accuracy of the system responses. This relation

is strikingly noticeable and both issues, viscoelastic model and solution method, are often incorrectly associated and no distinction between them is usually reported in the open literature. However, a clear distinction between the two must be made keeping still in mind that a close relation between the two exists. The solution and analysis methods can be divided in frequency and time domain based solution strategies, which clearly can admit distinct specific constitutive models. Comparative analysis and surveys of modern methods for modeling frequency dependent damping in FE models can be found in [Slater et al. (1993), Trindade et al. (2000b), Benjeddou (2001), Trindade and Benjeddou (2002)].

The next sections address the two alternative solution methods and describe different alternatives as far as the FE implementation is concerned.

6.6.1 Frequency Domain Based Approaches

6.6.1.1 Direct Frequency Response (DFR) Analysis Model

The general time dependent FE equations of motion of a general structural system with piezo-electric and viscoelastic damping treatments, are obtained by considering the isothermal frequency dependent physical properties of the viscoelastic materials presented in the previous sections into the piezo-elastic FE beam and shell models presented in Equations 3.27 and 5.25, yielding

$$\mathbf{M}_{uu}\ddot{\mathbf{u}}(t) + \mathbf{D}_{uu}\dot{\mathbf{u}}(t) + [\mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{P*} + \mathbf{K}_{uu}^V(j\omega)]\mathbf{u}(t) = -\mathbf{K}_{u\phi a}\bar{\phi}_a(t) + \mathbf{f}_u(t), \quad (6.10)$$

where $\mathbf{D}_{uu}$ is a global viscous damping matrix and $\mathbf{K}_{uu}^V(j\omega)$ is a complex global stiffness matrix of the viscoelastic layers, whose terms are frequency dependent. It is worthy to mention that two types of damping models are considered in the previous equation: (i) an arbitrarily frequency dependent hysteretic (or viscoelastic) damping type, represented by $\text{Im}[\mathbf{K}_{uu}^V(j\omega)]$, whose terms are frequency dependent, representing the viscoelastic relaxation behavior; and (ii) a viscous damping type, which is described by $\mathbf{D}_{uu}$, representing other general sources of damping (e.g. the material damping of the elastic part of the structural system, air-based damping, energy dissipation in the supports, etc.), which are assumed to be proportional to the velocity.

Using Equation (6.10) as it is, the frequency dependent matrix definition implies that its use and analysis can only be performed in the frequency domain, based in the CMA, where the material properties of the stiffness matrix of the viscoelastic layers are defined for each discrete frequency value [Vasques et al. (2004), Moreira and Rodrigues (2004)]. Thus, the DFR is a frequency domain method where the *frequency response model* [Ewins (2000)] can be generated in a straightforward manner from the results of many discrete frequency calculations of the equations of motion, in which the complex stiffness matrix of the viscoelastic layers is re-calculated for each frequency value of the desired discrete frequency range.

Considering simple harmonic mechanical or electric excitation, with

$$\mathbf{f}_u(t) = \bar{\mathbf{F}}e^{j\omega t}, \quad \bar{\phi}_a(t) = \bar{\Phi}_a e^{j\omega t}, \quad (6.11a,b)$$

where $\bar{\mathbf{F}}$ and $\bar{\Phi}_a$ are the amplitude vectors of the applied mechanical forces and electrical potential differences, the steady state mechanical and electrical harmonic responses of the system can be written as

$$\mathbf{u}(t) = \bar{\mathbf{U}}(j\omega)e^{j\omega t}, \quad \bar{\phi}_s(t) = \bar{\Phi}_s(j\omega)e^{j\omega t}, \quad (6.12a,b)$$

where $\bar{\mathbf{U}}(j\omega)$ and $\bar{\Phi}_s(j\omega)$ are the mechanical and electrical complex response vectors (displacements and sensing voltages phasors). Substituting Equations (6.11) and (6.12) into (6.10) and into the sensing voltage expression given in (3.28) and (5.26) yields

$$[\mathbf{K}_{uu}(j\omega) + j\omega\mathbf{D}_{uu} - \omega^2\mathbf{M}_{uu}]\bar{\mathbf{U}}(j\omega) = -\mathbf{K}_{u\phi a}\bar{\Phi}_a + \bar{\mathbf{F}}, \quad (6.13)$$

$$\bar{\Phi}_s(j\omega) = -\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\bar{\mathbf{U}}(j\omega), \quad (6.14)$$

where $\mathbf{K}_{uu}(j\omega) = \mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{P*} + \mathbf{K}_{uu}^V(j\omega)$, from which the complex solution vectors $\bar{\mathbf{U}}(j\omega)$ and $\bar{\Phi}_s(j\omega)$ can be obtained.

The FRF (frequency response function) for a mechanical force applied in the i th DoF and a displacement measured in the o th DoF, can be obtained by solving (6.13) for different values of frequency,

$$[\mathbf{K}_{uu}(j\omega_l) + j\omega_l\mathbf{D}_{uu} - \omega_l^2\mathbf{M}_{uu}]\bar{\mathbf{U}}(j\omega_l) = \bar{\mathbf{F}}_i, \quad (6.15)$$

where $\bar{\mathbf{F}}_i$ denotes a force vector with a non-zero component in the i th DoF and all other elements equal to zero, and $\bar{\mathbf{U}}(j\omega_l)$ is the resulting complex response vector (displacements) solution at frequency ω_l . Moreover, the resultant sensing voltage vector $\bar{\Phi}_s(j\omega_l)$ is obtained by substituting the displacement response vector $\bar{\mathbf{U}}(j\omega_l)$ into (6.14), yielding

$$\bar{\Phi}_s(j\omega_l) = -\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\bar{\mathbf{U}}(j\omega_l). \quad (6.16)$$

Thus, the displacement and induced voltage per unit force frequency response at frequency ω_l , $H_{oi}^{uu}(j\omega_l)$ and $H_{oi}^{\phi u}(j\omega_l)$, are given by

$$H_{oi}^{uu}(j\omega_l) = \frac{\bar{U}_o(j\omega_l)}{\bar{F}_i}, \quad (6.17)$$

$$H_{oi}^{\phi u}(j\omega_l) = \frac{\bar{\phi}_s^o(j\omega_l)}{\bar{F}_i} = \frac{\text{avg}[-\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\bar{\mathbf{U}}(j\omega_l)]}{\bar{F}_i}, \quad (6.18)$$

where $\bar{F}_i$ is the amplitude of the mechanical force input, $\bar{U}_o(j\omega_l)$ is the displacement response amplitude extracted from the o th DoF of the vector $\bar{\mathbf{U}}(j\omega_l)$, and $\bar{\phi}_s^o(j\omega_l)$ is the o th sensor averaged voltage output determined with the electrical DoFs of the o th sensor. The notation $\text{avg}[\cdot]$ is utilized here to denote that the voltage of the sensor is calculated from an average of the electrical DoFs (electric potential differences) where an electrical FE separation of the electrode was performed.

Similarly, for a piezoelectric i th actuator voltage input vector $\bar{\Phi}_a^i$, where a non-zero voltage is considered only in the electrical DoFs which correspond to the i th actuator, the mechanical displacements amplitude vector is given by solving

$$[\mathbf{K}_{uu}(j\omega_l) + j\omega_l\mathbf{D}_{uu} - \omega_l^2\mathbf{M}_{uu}]\bar{\mathbf{U}}(j\omega_l) = -\mathbf{K}_{u\phi a}\bar{\Phi}_a^i, \quad (6.19)$$

and the o th sensor output voltage vector is given by substituting $\bar{\mathbf{U}}(j\omega_l)$ into (6.16). Thus, the displacement and induced voltage per unit voltage applied into the actuator frequency response at frequency ω_l , $H_{oi}^{u\phi}(j\omega_l)$ and $H_{oi}^{\phi\phi}(j\omega_l)$, are given by

$$H_{oi}^{u\phi}(j\omega_l) = \frac{\bar{U}_o(j\omega_l)}{\bar{\phi}_a^i}, \quad (6.20)$$

$$H_{oi}^{\phi\phi}(j\omega_l) = \frac{\bar{\phi}_s^o(j\omega_l)}{\bar{\phi}_a^i} = \frac{\text{avg}[-\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\bar{\mathbf{U}}(j\omega_l)]}{\bar{\phi}_a^i}, \quad (6.21)$$

where $\bar{\phi}_a^i$ is the amplitude of the voltage applied to the i th actuator.

Finally, the frequency response model (FRFs) can be generated from the results of many discrete frequency calculations of Equations (6.15), (6.16) and (6.19), in which the complex stiffness matrix of the viscoelastic layers is re-calculated at each frequency value comprised in the discrete frequency range $\omega = \omega_0, \dots, \omega_l, \dots, \omega_f$, as shown in Figure 6.5.

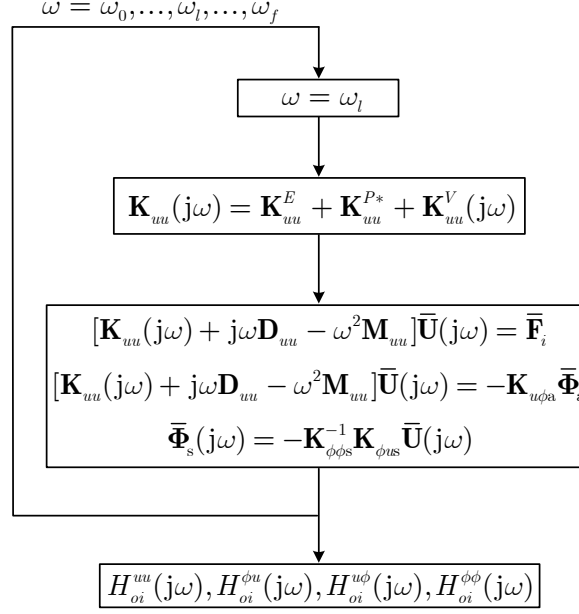


Figure 6.5: Direct frequency response analysis model generation diagram.

6.6.1.2 Modal Strain Energy (MSE) Approach

Assuming that the normal modes obtained from the undamped system are representative of the damped system, which, obviously, is merely true for lightly damped structures, an approximation to the modal loss factors can therefore be obtained from the ratio between the loss and the storage parts of the modal strain energy. Based on this assumption, [Johnson and Kienholz \(1982\)](#) proposed a direct method for the FE prediction of the modal damping ratio in structures with viscoelastic damping layers. While the storage energy is calculated from the modal strain energy of the entire structure for each mode shape, the dissipation energy is calculated multiplying the modal strain energy corresponding to the viscoelastic material FEs by the respective material loss factor at the current natural frequency value. Thus, establishing a relationship between the loss and storage modal strain energies of a generic mode r , namely $\phi_r^T \text{Im}(\mathbf{K}) \phi_r$ and $\phi_r^T \text{Re}(\mathbf{K}_{uu}) \phi_r$, where $\mathbf{K}_{uu} = \mathbf{K}_{uu}(j\omega)$, the modal loss factor might be expressed as

$$\eta_r = \frac{\phi_r^T \text{Im}[\mathbf{K}_{uu}(j\omega)] \phi_r}{\phi_r^T \text{Re}[\mathbf{K}_{uu}(j\omega)] \phi_r}, \quad (6.22)$$

which physically represents the ratio between the dissipated energy, proportional to the imaginary component of the complex stiffness matrix, and the stored energy, proportional to the real counterpart.

Conceptually, the MSE approach was initially introduced by Mead (1960) and Ungar and Kerwin (1962) in the early 1960s and later popularized and applied to FE analysis by Johnson and Kienholz (1982) and Soni and Bogner (1982). This approximated method is simple to implement in a FE simulation and is computationally cost-effective, providing a good damping estimator in comparative design analysis or optimization procedures [Hwang and Gibson (1992), Moreira and Rodrigues (2006)]. However, its usage is restricted to lightly damped structures wherein the undamped real modal model can effectively represent the damped structure. For highly damped structures, where the modal coupling is noteworthy, and especially for those where the viscoelastic layer induces a considerable mass and stiffness modification, the originally proposed MSE approach usage may lead to considerable approximation errors. With that in mind, modified versions have been recently proposed in an attempt to find more accurate approaches. Among others, the one proposed by Hu et al. (1995) is one of these modified MSE versions and is based on the application of a mode shape matrix derived from a modified eigenvalue problem which considers also the imaginary counterpart of the stiffness matrix.

6.6.1.3 Iterative Modal Strain Energy (IMSE) Approach

The MSE approach is based in the principle that the undamped natural modes of vibration of the structure are representative of the damped modal model. However, and despite the fact that this approximation might be valid for low to moderate damping treatments, the application of single or multi-layer viscoelastic sandwiched damping treatments to structures might modify the stiffness substantially due to the decoupling effects derived from the viscoelastic core/layers. Alternatively, the mechanical properties might be assumed constant in the neighborhood of each mode and the complex eigensolution of each individual mode might be determined independently, which corresponds to a more realistic approach. This methodology can, however, be applied under an iterative approach where the complex stiffness matrix is updated iteratively with the new corrected complex stiffness of the calculated mode. In general, this procedure allows to obtain more realistic values of the modal loss factors and natural frequencies with the drawback of requiring more processing time.

Thus, a modification to the original MSE algorithm is proposed in order to introduce the stiffness changes of the structural damped system through an iterative procedure to consider the variation of the storage and loss moduli with frequency. As a consequence, an iterative calculation of the real (undamped) eigensolution is performed using the continuously iteratively updated, by means of the adequate viscoelastic material properties at the frequency of the natural mode determined in the previous iteration, real and imaginary parts of the stiffness matrices. Based on the real eigensolution performed using FE analysis, which is the starting solution, once the convergence of the undamped natural frequency of the natural mode under analysis is verified, the correspondent modal loss factor is determined by means of the same methodology used in the original MSE method. When necessary, this method has to be repeated according to the modal density of the bandwidth of interest and employed to determine the modal loss factors and natural frequencies of each of the individual modes. Some examples of the use of the IMSE in the context of FE models can be found for example in [Trindade et al. (2000b), Zhang and Chen (2006)]. A schematic procedure describing the IMSE approach is presented in Figure 6.6 where it is assumed that the eigensolution algorithm calculates the smallest eigenvalues.

IMSE algorithm**Step 1.** Eigensolution with $\omega = \omega^0$

$$\text{Re}[\mathbf{K}_{uu}(j\omega^0)]\phi_r = (\omega_r^0)^2 \mathbf{M}_{uu}\phi_r$$

Step 2. Loop for each eigenpair (ω_r^0, ϕ_r) with $r = 1, \dots, p$

i. Initial value

$$\omega_r^i = \omega_r^0$$

ii. Iterative loop for each natural frequency and mode shape

- Eigensolution (ω_s^0, ϕ_s) with $s = 1, \dots, r$

$$\text{Re}[\mathbf{K}_{uu}(j\omega_r^i)]\phi_s = (\omega_s^{i+1})^2 \mathbf{M}_{uu}\phi_s$$

- Iterated natural frequency

$$\begin{cases} \omega_s^{i+1} : \text{rejected,} & \text{for } s < r \\ \omega_r^{i+1} = \sqrt{(\omega_s^{i+1})^2}, & \text{for } s = r \end{cases}$$

- Convergence condition test

$$\Delta_\omega = \frac{|\omega_r^{i+1} - \omega_r^i|}{\omega_r^{i+1}} \leq \Delta_{\max}$$

iii. Modal loss factor

$$\eta_r = \frac{\phi_r^T \text{Im}[\mathbf{K}_{uu}(j\omega_r^{i+1})]\phi_r}{\phi_r^T \text{Re}[\mathbf{K}_{uu}(j\omega_r^{i+1})]\phi_r}$$

Figure 6.6: Iterative modal strain energy (IMSE) algorithm.**6.6.1.4 Iterative Complex Eigensolution (ICE) Approach**

Similar to the IMSE, an alternative *iterative complex eigensolution* (ICE) algorithm is proposed in this section. This algorithm is exact in the sense that it does not estimate the modal loss factor in the same way as the MSE but, instead, it uses the determined complex eigenvalue to calculate the exact modal loss factor value. However, it has the drawback of requiring a higher computational effort since, for each mode, repeated complex eigensolutions are sought before the convergence is achieved.

The outcome of the effort of seeking a more accurate and generally applicable method, is the new ICE method presented in this dissertation. Based on the real eigensolution performed using FE analysis, the proposed method successively updates the complex eigenvalues and eigenvectors until the convergence for the necessary accuracy condition is achieved. As compared with conventional methods, the ICE method improves considerably the numerical

accuracy while it maintains the computational efficiency and simplicity required for practical applications. A schematic procedure describing the ICE approach is presented in Figure 6.7 where, in a similar way, it is assumed that the eigensolution algorithm calculates the smallest magnitude eigenvalues.

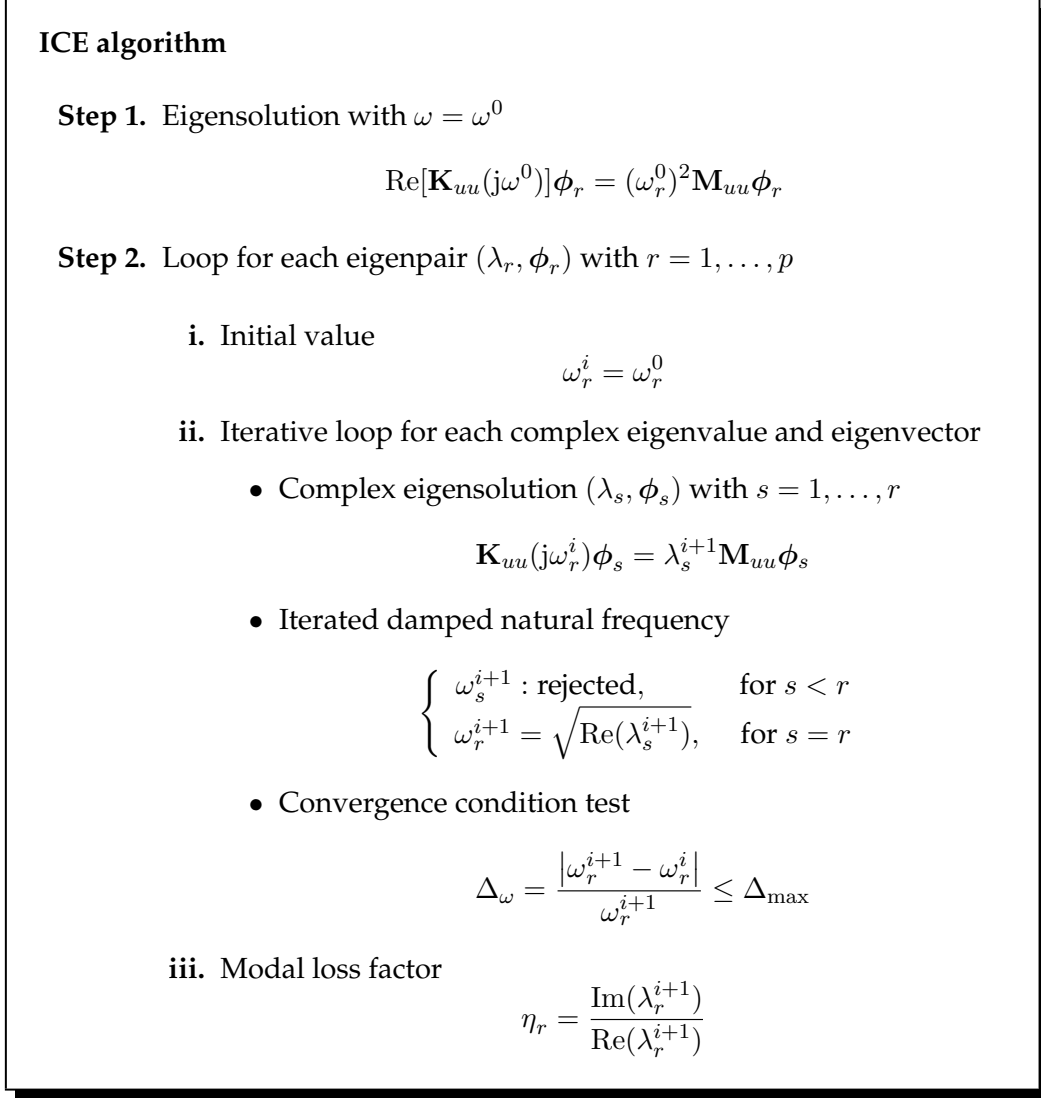


Figure 6.7: Iterative complex eigensolution (ICE) algorithm.

Regarding the two iterative approaches, namely the IMSE and ICE, as the iteration continues, the estimated ω_r^{i+1} , η_r^{i+1} and ϕ_r^{i+1} will converge to their exact solutions, ω_r , η_r and ϕ_r . However, the computation effort involved might increase with the modal density and/or the bandwidth of interest, since, for each mode, several real or complex eigensolutions are required before the convergence is achieved.

The MSE approach is an approximate method and its accuracy depends on the level of damping and the degree of isolation of the mode to be analyzed. The proposed ICE method makes use of classical complex eigensolution algorithms and represents an extension and improvement to IMSE method, where only real eigensolution algorithms are utilized. In fact,

to the author's best knowledge, this method is for the first time proposed here and there has not been any discussion in the open literature regarding its application to the characterization of viscoelastically damped structures. When compared with the DFR, where the modal parameters are approximated by modal identification procedures [Maia and Silva (1997), Ewins (2000)], the IMSE and ICE allow the modal parameters ϕ_r , η_r , and ω_r of the r th mode of a generic viscoelastically damped structural system to be obtained directly. However, with the aforementioned advantages and disadvantages and modeling strategies, all the three models can be used to build a modal model of the damped structural system, which can be used to estimate both frequency and time domain responses.

6.6.2 Time Domain Based Approaches

6.6.2.1 Golla-Hughes-McTavish (GHM) Damping Model

When general transient responses are required, time domain models are more suitable and versatile, and they might represent better alternatives to the CMA-based frequency domain approaches, since they allow the reduction of the computational burden due to the re-calculation of the stiffness matrix for each discrete frequency value (DFR) and the use of iterative eigenproblem calculations (IMSE and ICE). One alternative is the GHM (*Golla-Hughes-McTavish*) model [Golla and Hughes (1985), McTavish and Hughes (1993)], which assumes that the *characteristic material function* $\tilde{Q}(s) \equiv s\tilde{G}(s)$ in Equation (6.3) might be represented in terms of a series of *mini-oscillator* terms (see Table 6.2),

$$s\tilde{G}(s) = G_\infty \left[1 + \tilde{h}(s) \right] = G_\infty \left(1 + \sum_{i=1}^n \alpha_i \frac{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s}{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s + \hat{\omega}_i^2} \right), \quad (6.23)$$

where G_∞ is the relaxed modulus and each i th mini-oscillator term is a second order rational function involving three positive constant parameters, α_i , $\hat{\omega}_i$ and $\hat{\zeta}_i$. These parameters govern the shape of the modulus function over the complex plane (or over the frequency domain, assuming only the imaginary part of the complex plane), and depending upon the nature of the viscoelastic characteristic material function (usually the shear modulus for isotropic materials) and the range of s (or frequency) over which it is to be modeled, the number of mini-oscillator terms is defined according to the required accuracy of the representation.

Considering the FE equations of motion of a viscoelastically damped general structural system in Equation (6.10), with the shear modulus of the viscoelastic counterpart factored out of the viscoelastic stiffness matrix [note that both shear and extensional stiffness terms may be considered in the viscoelastic stiffness matrix by assuming a frequency independent Poisson's coefficient in the extensional and shear modulus relationship in (6.9)], yielding $\bar{\mathbf{K}}_{uu}^V$, and assuming that the structural system possesses only one type of viscoelastic material, following the hereditary stress-strain law in (6.2) yields

$$\mathbf{M}_{uu} \ddot{\mathbf{u}}(t) + \mathbf{D}_{uu} \dot{\mathbf{u}}(t) + \mathbf{K}_{uu}^{\text{EP}*} \bar{\mathbf{u}}(t) + G(t) \bar{\mathbf{K}}_{uu}^V \bar{\mathbf{u}}(0) + \int_0^t G(t-\tau) \bar{\mathbf{K}}_{uu}^V \frac{\partial \bar{\mathbf{u}}(\tau)}{\partial \tau} d\tau = \mathbf{f}_\phi(t) + \mathbf{f}_u(t), \quad (6.24)$$

where $\mathbf{K}_{uu}^{\text{EP}*} = \mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{\text{P}*}$ and $\mathbf{f}_\phi(t) = -\mathbf{K}_{u\phi a} \bar{\phi}_a(t)$. Transforming Equation (6.24) to the Laplace domain yields

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{EP}*}) \tilde{\mathbf{u}}(s) + s\tilde{G}(s) \bar{\mathbf{K}}_{uu}^V \tilde{\mathbf{u}}(s) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s). \quad (6.25)$$

Substituting the characteristic material function representation in Equation (6.23) into (6.25) yields

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{EP}*}) \tilde{\mathbf{u}}(s) + G_\infty \bar{\mathbf{K}}_{uu}^V \left(1 + \sum_{i=1}^n \alpha_i \frac{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s}{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s + \hat{\omega}_i^2} \right) \tilde{\mathbf{u}}(s) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s). \quad (6.26)$$

Then, introducing a set of n series of *dissipation* (or *internal*) variables $\tilde{\mathbf{u}}_i^D(s)$ ($i = 1, \dots, n$), for each series, one can assume the relationship

$$\tilde{\mathbf{u}}(s) - \tilde{\mathbf{u}}_i^D(s) = \frac{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s}{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s + \hat{\omega}_i^2} \tilde{\mathbf{u}}(s). \quad (6.27)$$

Substituting Equation (6.27) into (6.26) yields

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{V0}) \tilde{\mathbf{u}}(s) + \mathbf{K}_{uu}^{V\infty} \sum_{i=1}^n \alpha_i (\tilde{\mathbf{u}}(s) - \tilde{\mathbf{u}}_i^D(s)) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s), \quad (6.28)$$

where $\mathbf{K}_{uu}^{V\infty} = G_\infty \bar{\mathbf{K}}_{uu}^V$ is the *relaxed* (or *static*) stiffness matrix of the viscoelastic components. Next, in order to describe the dissipative behavior of the internal DoFs, after some algebra, Equation (6.27) can be written as

$$\tilde{\mathbf{u}}_i^D(s) = \frac{\hat{\omega}_i^2}{s^2 + 2\hat{\zeta}_i \hat{\omega}_i s + \hat{\omega}_i^2} \tilde{\mathbf{u}}(s). \quad (6.29)$$

Furthermore, considering Equations (6.28) and (6.29), after some algebra, we get the following augmented coupled system,

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{V\infty}) \tilde{\mathbf{u}}(s) - \mathbf{K}_{uu}^{V\infty} \sum_{i=1}^n \hat{\alpha}_i \tilde{\mathbf{u}}_i^D(s) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s), \quad (6.30a)$$

$$\left(s^2 \frac{1}{\hat{\omega}_i^2} + s \frac{2\hat{\zeta}_i}{\hat{\omega}_i} + 1 \right) \tilde{\mathbf{u}}_i^D(s) - \tilde{\mathbf{u}}(s) = \mathbf{0}, \quad (6.30b)$$

where

$$\mathbf{K}_{uu}^{V0} = \left(1 + \sum_{i=1}^n \alpha_i \right) \mathbf{K}_{uu}^{V\infty}. \quad (6.31)$$

In order to obtain the time-dependent behavior of the augmented system, multiplying Equation (6.30b) by $\alpha_i \mathbf{K}_{uu}^{V\infty}$, and since all matrices are independent of s , a linear time domain model is easily recovered by the inverse Laplace transform of Equations (6.30), yielding

$$\mathbf{M}_{uu} \ddot{\mathbf{u}}(t) + \mathbf{D}_{uu} \dot{\mathbf{u}}(t) + (\mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{V0}) \mathbf{u}(t) - \mathbf{K}_{uu}^{V\infty} \sum_{i=1}^n \alpha_i \mathbf{u}_i^D(t) = \mathbf{f}_\phi(t) + \mathbf{f}_u(t), \quad (6.32a)$$

$$\frac{\alpha_i}{\hat{\omega}_i^2} \mathbf{K}_{uu}^{V\infty} \ddot{\mathbf{u}}_i^D(t) + \frac{2\alpha_i \hat{\zeta}_i}{\hat{\omega}_i} \mathbf{K}_{uu}^{V\infty} \dot{\mathbf{u}}_i^D(t) + \alpha_i \mathbf{K}_{uu}^{V\infty} \mathbf{u}_i^D(t) - \alpha_i \mathbf{K}_{uu}^{V\infty} \mathbf{u}(t) = \mathbf{0}. \quad (6.32b)$$

The augmented coupled system in Equations (6.32) may still be expressed in compact matrix form as

$$\mathbf{M} \ddot{\mathbf{z}}(t) + \mathbf{D} \dot{\mathbf{z}}(t) + \mathbf{K} \mathbf{z}(t) = \mathbf{r}(t), \quad (6.33)$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{DD} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{DD} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{EE} & \mathbf{K}_{ED} \\ \mathbf{K}_{DE} & \mathbf{K}_{DD} \end{bmatrix}, \quad (6.34a-c)$$

$$\mathbf{z}(t) = \text{col} [\bar{\mathbf{u}}(t), \mathbf{u}_1^D(t), \dots, \mathbf{u}_n^D(t)], \quad \mathbf{r}(t) = \text{col} [\mathbf{f}_\phi(t) + \mathbf{f}_u(t), \mathbf{0}, \dots, \mathbf{0}], \quad (6.35a,b)$$

and

$$\mathbf{M}_{DD} = \text{diag} \left(\frac{\alpha_1}{\hat{\omega}_1^2} \mathbf{K}_{uu}^{V\infty}, \dots, \frac{\alpha_n}{\hat{\omega}_n^2} \mathbf{K}_{uu}^{V\infty} \right), \quad \mathbf{D}_{DD} = \text{diag} \left(\frac{2\alpha_1 \hat{\zeta}_1}{\hat{\omega}_1} \mathbf{K}_{uu}^{V\infty}, \dots, \frac{2\alpha_n \hat{\zeta}_n}{\hat{\omega}_n} \mathbf{K}_{uu}^{V\infty} \right), \quad (6.36a,b)$$

$$\mathbf{K}_{DD} = \text{diag} (\alpha_1 \mathbf{K}_{uu}^{V\infty}, \dots, \alpha_n \mathbf{K}_{uu}^{V\infty}), \quad \mathbf{K}_{ED} = [-\alpha_1 \mathbf{K}_{uu}^{V\infty}, \dots, -\alpha_n \mathbf{K}_{uu}^{V\infty}], \quad (6.37a,b)$$

$$\mathbf{K}_{EE} = \mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{V0}, \quad \mathbf{K}_{DE} = \mathbf{K}_{ED}^T. \quad (6.38a,b)$$

Equation (6.33) represents a time domain mathematical realization of the damped FE multi-layer structural system comprising viscoelastic, piezoelectric and elastic layers. For now the model is presented in second-order form. However, first-order state space formulations and model reduction techniques, adapted to the problem and matrix topology in hands, are often more convenient to solve the system. These techniques will be presented in Chapter 8 in the scope of the control of vibrations using hybrid damping treatments and hybrid control strategies.

6.6.2.2 Anelastic Displacement Fields (ADF) Damping Model

A time domain model based on a variation, or Laplace transformed, formulation of the ADF (*anelastic displacement fields*) model, originally proposed by Lesieutre and his co-workers [Lesieutre and Bianchini (1995), Lesieutre et al. (1996)], is presented in this work. It takes a definition of the complex modulus in the frequency (or Laplace) domain (see Table 6.2) and utilizes the so-called *internal*, *dissipation* or *anelastic* (after Lesieutre) variables to simplify the equations, although with the drawback of increasing the size of the problem. Afterward, through an inverse Laplace transform one obtains an amenable and computationally tractable augmented system of linear ordinary differential equations that can be solved by standard numerical methods applicable to first-order linear systems. With this procedure, the FE model implementation of the ADF model is more straightforward when compared with the Lesieutre's original direct time domain formulation, based upon the method of irreversible thermodynamics and a decomposition of the total displacement field in an *elastic* and *anelastic* counterpart. Thus, the process of deriving the augmented coupled elastic-anelastic system (using the original designation of Lesieutre) is similar to what has been presented for the GHM method. In this case, a different definition of the *characteristic material function* is utilized. It also worthy to mention that Lesieutre's definition of the relaxation function is in fact a particular case of the relaxation function derived by Biot and presented in Table 6.2, where the viscous term proportional to velocity is not considered in the relaxation function but instead the velocity proportional damping is considered by means of a viscous damping model through the viscous damping matrix $\mathbf{D}_{uu}$. That is in fact expected since both are derived from thermodynamic principles. Furthermore, when compared with the GHM model, which can also be seen as a particular case of Biot's model assuming a time-domain relaxation function as a series of double exponential terms, the ADF approach has been shown to yield the same damping capabilities with less parameters per each series.

As reported by Lesieutre, the frequency dependent viscoelastic shear modulus described by the ADF model, represented by a series of functions in the Laplace domain, has the form [Lesieutre (1992), Lesieutre and Bianchini (1995)] (cf. Table 6.2)

$$s\tilde{G}(s) = G_\infty [1 + \tilde{h}(s)] = G_\infty \left(1 + \sum_{i=1}^n \frac{\Delta_i s}{s + \Omega_i} \right), \quad (6.39)$$

where, again, G_∞ is the *relaxed* (or *static, low-frequency*) shear modulus, and Ω_i is the inverse of the characteristic *relaxation time* at constant strain and Δ_i the correspondent *relaxation resistance*. To take into consideration the relaxation behavior, the entire ADF model itself may be comprised of several individual fields, where n series of ADF are used to describe the material behavior. Given measured values of the shear modulus in the form a of frequency dependent complex modulus $G(j\omega)$, the relaxed shear modulus G_∞ and the series of material parameters Δ_i and Ω_i can be determined using curve fitting or optimization techniques. The number of series of ADF parameters determines the accuracy of the matching of the measured data over the frequency range of interest.

Substituting Equation (6.39) into (6.25) yields

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{EP}*}) \tilde{\mathbf{u}}(s) + G_\infty \bar{\mathbf{K}}_{uu}^V \left(1 + \sum_{i=1}^n \frac{\Delta_i s}{s + \Omega_i} \right) \tilde{\mathbf{u}}(s) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s). \quad (6.40)$$

Then, in a similar way to the GHM, introducing a set of n series of *anelastic* (or *internal, dissipation*) $\tilde{\mathbf{u}}_i^A(s)$ variables, with $i = 1, \dots, n$, for each series, one can assume the relationship

$$\tilde{\mathbf{u}}(s) - \tilde{\mathbf{u}}_i^A(s) = \frac{s}{s + \Omega_i} \tilde{\mathbf{u}}(s). \quad (6.41)$$

Substituting Equation (6.41) into (6.40) and considering the dissipative behavior of the *anelastic* DoFs given from Equation (6.41) as

$$\tilde{\mathbf{u}}_i^A(s) = \frac{\Omega_i}{s + \Omega_i} \tilde{\mathbf{u}}(s), \quad (6.42)$$

after some algebra, we get the following augmented *elastic-anelastic* coupled system,

$$(s^2 \mathbf{M}_{uu} + s \mathbf{D}_{uu} + \mathbf{K}_{uu}^{\text{PE}*} + \mathbf{K}_{uu}^{\text{V0}}) \tilde{\mathbf{u}}(s) - \mathbf{K}_{uu}^{\text{V}\infty} \sum_{i=1}^n \Delta_i \tilde{\mathbf{u}}_i^A(s) = \tilde{\mathbf{f}}_\phi(s) + \tilde{\mathbf{f}}_u(s), \quad (6.43a)$$

$$\left(\frac{s}{\Omega_i} + 1 \right) \tilde{\mathbf{u}}_i^A(s) - \tilde{\mathbf{u}}(s) = \mathbf{0}, \quad (6.43b)$$

where, in this case,

$$\mathbf{K}_{uu}^{\text{V0}} = \left(1 + \sum_{i=1}^n \Delta_i \right) \mathbf{K}_{uu}^{\text{V}\infty}. \quad (6.44)$$

The time-dependent behavior of the augmented system, multiplying Equation (6.43b) by $\Delta_i \mathbf{K}_{uu}^{\text{V}\infty}$, is recovered by the inverse Laplace transform of Equations (6.43), yielding

$$\mathbf{M}_{uu} \ddot{\mathbf{u}}(t) + \mathbf{D}_{uu} \dot{\mathbf{u}}(t) + (\mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{\text{V0}}) \bar{\mathbf{u}}(t) - \mathbf{K}_{uu}^{\text{V}\infty} \sum_{i=1}^n \Delta_i \mathbf{u}_i^A(t) = \mathbf{f}_\phi(t) + \mathbf{f}_u(t), \quad (6.45a)$$

$$\frac{\Delta_i}{\Omega_i} \mathbf{K}_{uu}^{\text{V}\infty} \dot{\mathbf{u}}_i^A(t) + \Delta_i \mathbf{K}_{uu}^{\text{V}\infty} \mathbf{u}_i^A(t) - \Delta_i \mathbf{K}_{uu}^{\text{V}\infty} \bar{\mathbf{u}}(t) = \mathbf{0}. \quad (6.45b)$$

The augmented coupled system in Equations (6.45) can also be expressed in compact matrix form, as in Equation (6.33), where for the ADF model one gets

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{AA} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{EE} & \mathbf{K}_{EA} \\ \mathbf{K}_{AE} & \mathbf{K}_{AA} \end{bmatrix}, \quad (6.46a-c)$$

$$\mathbf{z}(t) = \text{col} [\bar{\mathbf{u}}(t), \mathbf{u}_1^A(t), \dots, \mathbf{u}_n^A(t)], \quad (6.47)$$

where

$$\mathbf{D}_{AA} = \text{diag}\left(\frac{\Delta_1}{\Omega_1} \mathbf{K}_{uu}^{V\infty}, \dots, \frac{\Delta_n}{\Omega_n} \mathbf{K}_{uu}^{V\infty}\right), \quad \mathbf{K}_{AA} = \text{diag}\left(\Delta_1 \mathbf{K}_{uu}^{V\infty}, \dots, \Delta_n \mathbf{K}_{uu}^{V\infty}\right), \quad (6.48a,b)$$

$$\mathbf{K}_{EE} = \mathbf{K}_{uu}^{\text{EP}*} + \mathbf{K}_{uu}^{V0}, \quad \mathbf{K}_{EA} = [-\Delta_1 \mathbf{K}_{uu}^{V\infty}, \dots, -\Delta_n \mathbf{K}_{uu}^{V\infty}], \quad \mathbf{K}_{AE} = \mathbf{K}_{EA}^T. \quad (6.49a-c)$$

Regarding the augmented damped system in Equation (6.33), one may notice from $\mathbf{M}$ that, when compared with the GHM model, the anelastic DoFs of the ADF model have no inertia and therefore the global mass matrix $\mathbf{M}$ is singular and hence is not positive-definite. However, the singularity of the mass matrix can be overcome if, instead of solving the second-order system (6.33) of the ADF, a state-space representation with an adequate design of the state variables is considered. The number of flexible modes is kept the same and the dissipative modes, which correspond to the internal relaxations of the viscoelastic material, are overdamped with low observability. The use of state space realizations will be addressed in Chapter 8.

6.6.3 Spatial Model Reduction

As can be seen in Equations (6.32) and (6.45), the main disadvantage of GHM and ADF models is that, associated with a FE discretization and in order to account for the frequency dependence of the viscoelastic material, they lead to larger systems, since they add auxiliary DoFs, which for each GHM or ADF series must be equal to the number of elastic DoFs. However, model reduction techniques might be utilized in order to reduce the size of the problem.

As suggested by Trindade et al. (2000b), which follows a similar treatment to the one given by Biot (1954) to the unobservable variables of a generic irreversible physical system, the matrices corresponding to the internal DoFs might be reduced and diagonalized through a projection in a suitable reduced modal base, to reduce the computational cost. Thus, defining $\mathbf{I} = (\mathbf{D}, \mathbf{A})$ and using it here to denote *internal* variables in general, which may be both the *dissipative* or *anelastic* variables definitions of the GHM and ADF approaches, respectively, and considering a linear coordinate transformation $\mathbf{u}_i^I(t) = \Psi_I \hat{\mathbf{u}}_i^I(t)$, where $\Lambda_I = \Psi_I^T \mathbf{K}_{uu}^{V\infty} \Psi_I$ is a diagonal matrix composed by the non-zero eigenvalues of $\mathbf{K}_{uu}^{V\infty}$ and Ψ_I is the correspondent matrix of normalized eigenvectors, such that $\Psi_I^T \Psi_I = \mathbf{I}$, the vector $\mathbf{z}(t)$ is alternatively modified to

$$\mathbf{z}(t) = \text{col}[\bar{\mathbf{u}}(t), \hat{\mathbf{u}}_1^I(t), \dots, \hat{\mathbf{u}}_n^I(t)] \quad (6.50)$$

and matrices $\mathbf{M}_{DD}$, $\mathbf{D}_{II}$, $\mathbf{K}_{II}$ and $\mathbf{K}_{IE}$ are modified to

$$\mathbf{M}_{DD} = \text{diag}\left(\frac{\alpha_1}{\hat{\omega}_1^2} \Lambda_D, \dots, \frac{\alpha_n}{\hat{\omega}_n^2} \Lambda_D\right), \quad \mathbf{D}_{DD} = \text{diag}\left(\frac{2\alpha_1 \hat{\zeta}_1}{\hat{\omega}_1} \Lambda_D, \dots, \frac{2\alpha_n \hat{\zeta}_n}{\hat{\omega}_n} \Lambda_D\right), \quad (6.51a,b)$$

$$\mathbf{K}_{DD} = \text{diag}(\alpha_1 \Lambda_D, \dots, \alpha_n \Lambda_D), \quad \mathbf{K}_{ED} = [-\alpha_1 \mathbf{K}_{uu}^{V\infty} \Psi_D, \dots, -\alpha_n \mathbf{K}_{uu}^{V\infty} \Psi_D], \quad (6.52a,b)$$

for the GHM model, and to

$$\mathbf{D}_{AA} = \text{diag}\left(\frac{\Delta_1}{\Omega_1} \Lambda_A, \dots, \frac{\Delta_n}{\Omega_n} \Lambda_A\right), \quad \mathbf{K}_{AA} = \text{diag}(\Delta_1 \Lambda_A, \dots, \Delta_n \Lambda_A), \quad (6.53a,b)$$

$$\mathbf{K}_{EA} = [-\Delta_1 \mathbf{K}_{uu}^{V\infty} \Psi_A, \dots, -\Delta_n \mathbf{K}_{uu}^{V\infty} \Psi_A], \quad (6.54)$$

for the ADF model.

The advantages of this alternative (transformed) representation are that in the case where only some part of the structure has surface mounted or embedded viscoelastic layers, only

some FEs have viscoelastic components and $\mathbf{K}_{uu}^{V\infty}$ can have several rows and columns of zeros, which in turn leads to some nil eigenvalues. Thus, the size of $\hat{\mathbf{u}}_i^I(t)$ can be substantially smaller than that of $\mathbf{u}_i^I(t)$. Therefore, through an adequate coordinate transformation based on the eigenvalues and eigenvectors of $\mathbf{K}_{uu}^{V\infty}$ and elimination of the nil (spurious) eigenvalues, the size of the problem can be substantially reduced. Furthermore, truncated state space modal models might also be used in order to reduce the system size even further. A study addressing reduced-order FE models of viscoelastically damped beams, where the selective significance of the "non-physical" modes to the net mechanical response is performed through internal variables projection in order to significantly reduce the computational cost, was recently presented by Trindade [Trindade \(2006\)](#). The reader is also referred to [[Friswell and Inman \(1999\)](#), [Park et al. \(1999\)](#), [Trindade et al. \(2000b\)](#), [Vasques and Rodrigues \(2007b\)](#)] for further details about state space design and model reduction techniques which will be addressed in Chapter 8.

6.7 Commercial Finite Element Codes Solutions

During the last decades commercial FE codes have been used to study and design viscoelastic damping treatments. The use of commercial FE codes to validate and compare other FE and analytical models and experimental tests using NASTRAN can be found, for example, in [[Lu et al. \(1979\)](#), [Lu and Everstine \(1980\)](#), [Johnson and Kienholz \(1982\)](#), [Killian and Lu \(1984\)](#), [Macé \(1994\)](#), [Jeung and Shen \(2001\)](#), [Chattopadhyay et al. \(2001\)](#), [Moreira and Rodrigues \(2004\)](#), [Hambric et al. \(2007\)](#)], with ANSYS in [[Allen et al. \(2001\)](#), [Yellin \(2004\)](#), [Yellin et al. \(2005\)](#), [Zhang and Chen \(2006\)](#), [Chaudry and Baz \(2006\)](#)] and with ABAQUS in [[Lumsdaine and Scott \(1998\)](#), [Slanik et al. \(2000\)](#), [Allen et al. \(2001\)](#)]. Attempts to use the GHM and ADF internal variables models with commercial FE codes such as NASTRAN have been performed in [[Gibson et al. \(1995\)](#), [Bianchini et al. \(1995\)](#)]. A general discussion regarding the basic procedures and major capabilities of commercial FE analysis software packages, such as ABAQUS, ADINA, ALGOR, ANSYS, COSMOSWorks, NASTRAN, ABAQUS/Explicit, DYNA3D and LS-DYNA, for vibration analysis was presented by [Song et al. \(2007\)](#).

Apart from the accuracy and representativeness of the structural model, the damping modeling capability is a key ingredient in the design of damping treatments. In fact, an adequate damping design is strongly influenced by the degree of accuracy of how the damping quantification factors are determined and how they are utilized to optimally define the damping treatment location, geometry and thicknesses of the layers. It is well known that the structural model has received substantial attention by the scientific community in order to refine the structural models and that these developments have reached a relative stage of maturity, with some of these refined models being adopted and implemented by the major part of the FE codes (an exception in the field of structural mechanics is for example the use of HOT and layerwise elements to the study of composite laminated structures which is not a common tool in commercial FE codes). However, damping modeling is a more uncertain and specific area of interest (mainly to dynamicists) which has not reached the maturity stage yet and the transition of contemporary damping modeling approaches to commercial softwares is far from being done. As a result of that, when intending to design viscoelastic damping treatments using FE codes, the available damping models of the commercial FE codes are an important aspect to take into account. With that in mind, a compilation of the damping modeling capabilities and general features available in some well known softwares is presented in Table 6.3.

Table 6.3: Damping modeling capabilities and general features available in some commercial FE softwares.

	ANSYS v11	ABAQUS v6.7	MD NASTRAN R2	ALGOR v20	LS-DYNA v971	ADINA	COSMOSWorks 2008	LUSAS v14	ACTRAN/VA 2007
GENERAL DAMPING									
Viscous	✓	✓	✓	✓	✓	✓	✓	✓	✓
Hysteretic	✓	✓	✓	N	✓	N	N	✓	✓
Composite modal damping	✓	✓	N	N	✓	N	✓	N	N
VISCOELASTIC CONSTITUTIVE MODELS									
Small deformation									
Small strain (linear viscoelasticity)	✓	✓	✓	✓	✓	✓	✓	✓	✓
Large deformation									
Small strain (visco-hypoelasticity)	✓	N	N	✓	N	✓	N	✓	N
Large strain (visco-hyperelasticity)	✓	✓	✓	✓	✓	✓	✓	✓	N
Time-domain									
Prony's series	✓	✓	✓	✓	✓	✓	✓	✓	N
Frequency-domain									
Complex modulus	✓	✓	✓	N	N	N	N	N	✓
Curve-fitting to Prony's series									
Creep test data	✓	✓	✓	N	N	N	N	N	N
Relaxation test data	✓	✓	✓	N	✓	✓	✓	N	N
Frequency-dependent test data	N	✓	N	N	N	N	N	N	N
Temperature effects/Shift-functions (TRM)									
Williams-Landel-Ferry (WLF)	✓	✓	✓	✓	✓	✓	✓	N	N
Tool-Narayananaswamy (TN)	✓	N	✓	N	N	N	N	N	N
SOLUTION METHODS									
Eigenanalysis									
Undamped (real)	✓	✓	✓	✓	✓	✓	✓	✓	✓
Damped (complex)	X	X	X	X	X	X	N	X	N
Direct frequency response	✓	✓	✓	N	N	✓	N	N	✓
Direct integration method									
Transient response	✓	✓	✓	✓	✓	✓	✓	✓	N
Mode superposition method									
Transient response	X	X	X	X	X	X	X	X	N
Frequency response	X	X	X	X	X	X	X	X	N
Response spectrum analysis	X	X	N	N	N	X	X	X	✓
Random response analysis	X	X	N	N	N	X	X	N	✓
Other features									
Reduced solution methods	✓	✓	✓	N	N	N	N	✓	✓
Frequency spanning tools	✓	✓	✓	N	N	N	N	✓	N
Substructuring analysis	✓	✓	✓	N	N	✓	N	✓	✓
Subspace solution method	✓	✓	N	N	N	N	N	✓	✓
VISCOELASTIC FE TYPES									
2-D/3-D	✓	✓	✓	✓	N/✓	✓	N/✓	✓	✓
Beam/Truss	✓	✓	✓	N	✓/N	N	N	N/✓	N
Plate/Shell	✓	✓	✓	✓	✓	✓	✓	N	N/✓
Layered (composite)	✓	✓	N	N	✓	✓	N	N	✓
Constitutive behavior									
Isotropy	✓	✓	✓	✓	✓	✓	✓	✓	✓
Orthotropy	N	N	✓	✓	✓	✓	N	✓	✓
Anisotropy	N	N	✓	N	N	N	N	N	N

Notes: '✓', procedure fully available; 'X', approximated procedure not admissible/accurate for frequency dependent materials (however, feasible using internal variables models); 'N', not available or not known.

When using more complex high-order or layerwise FEs with segmented viscoelastic patches in composite laminated structures, the pre- and post-processing capabilities of these softwares should furnish the designer with the adequate tools to discretize and generate the mesh of the structure along with special analysis options (e.g. modal strain energy distribution) which might be useful to the design analyst.

Since most FE softwares cannot extract complex modes from models with frequency-varying material properties, the analysis procedure might be subdivided into frequency ranges, as was the case used by [Hambric et al. \(2007\)](#), and an incremental approach considering interpolation of estimated modal parameters might be used to estimate the modal parameters at other frequencies. This procedure, when compared with approaches that allow for frequency-dependent material properties to be used in a FE complex modal analysis, such as the GHM and ADF methods, might be considered inefficient, since it requires multiple calculations of a complex eigenvalue problem while the internal variables models require only one, however with a higher order problem, and it follows a similar approach to ICE method proposed earlier.

6.8 Summary

This chapter presents FE-based mathematical strategies to model the constitutive and damping behavior of viscoelastic materials which might be used as surface mounted, constrained or embedded damping treatments in structures, in order to reduce vibrations and/or noise radiation. As far as the spatial modeling is concerned, different FE spatial modeling approaches have been presented and their main features outlined. Regarding the constitutive behavior, both time and frequency domain based techniques and their FE implementation and use of commercial FE codes were presented and discussed. Time domain techniques regard the use of the GHM and ADF internal variables models. As will be shown later in Chapter 8, the ADF model is known to lead to an augmented model of the damped structural system with a lower size than the GHM model. In the author's opinion the ADF model represents the best internal variables alternative to accurately model the damping behavior since it yields good trade-off between accuracy and complexity. One major disadvantage in using internal variables models, such as the GHM or ADF, is the creation of additional dissipation variables increasing the size of the coupled damped FE model. Frequency domain techniques comprises the use of CMA-based approaches, namely the DFR, IMSE and ICE (which is proposed as a new alternative in this work), where the FE spatial model is used by re-calculating the complex viscoelastic stiffness matrix for each discrete frequency value, in the case of the DFR, or during the iterative process, in the case of the IMSE and ICE. These approaches are more straightforward to use and implement at the global FE level. That is the reason why the CMA-based approach, DFR, is the most common approach implemented in commercial FE codes incorporating viscoelastic damping modeling capabilities.

The ultimate aim of these models is to be able to simulate the time and frequency response of viscoelastically damped structural systems. With that purpose, the extraction of the model parameters can be performed directly or by identification procedures, as is the case for the DFR approach. However, with the aforementioned advantages and disadvantages, all the approaches can be used to build a truncated modal model of the damped structural system, which can be used to estimate both frequency and time domain responses.

The experimental identification of the 3M ISD112 and the aforementioned FE integrated damped modeling approaches are performed, implemented, assessed and validated in the following chapter [[Vasques et al. \(2007a\)](#)].

CHAPTER 7

Viscoelastic Damping: Experimental Identification and Validation of the Models

7.1 Introduction

Integrated approaches to the mathematical modeling and experimental characterization of viscoelastic surface mounted or embedded damping treatments, typically used to reduce structural vibration and/or noise radiation from structures, are presented in this part of the dissertation, which comprises two chapters. To that end, viscoelastic damping has been tackled from a mathematical point of view and the implementation, at the FE model level, of time and frequency domain methods, namely the internal variables models, GHM (*Golla-Hughes-McTavish*) and ADF (*anelastic displacement fields*), and the CMA-based ones, DFR (*direct frequency response*), IMSE (*iterative modal strain energy*) and ICE (*iterative complex eigensolution*), were described and formulated in Chapter 6.

Therefore, this chapter is a complement to Chapter 6 where the GHM and ADF model parameters, which are used for modeling the viscoelastic damping behavior, are identified via an experimental procedure. The viscoelastic material considered here is the 3M ISD112 [see the manufacturer product information 3M (1993)], which was chosen because of its commercial availability. However, the methodology described below might in general be applied to any viscoelastic material. The chapter starts by making a brief review of available experimental identification procedures. Next, a generic methodology to identify the complex shear modulus of viscoelastic materials and the obtained nomogram of the 3M ISD112 viscoelastic material are presented. The identified complex shear modulus of the 3M ISD112 is then curve-fitted in order to obtain the modeling parameters of the GHM and ADF models. Lastly, a viscoelastic sandwich plate (three-layered) specimen and the correspondent FE model are considered numerically and experimentally. Measured and predicted FRFs are compared with the purpose of assessing the performance of the damping models presented in Chapter 6. The analysis allows to assess the validity of the methodology to determine the frequency dependent complex modulus, the GHM and ADF parameters identification and the outcomes and drawbacks of the DFR, IMSE, ICE, GHM and ADF viscoelastic damping modeling strategies and their FE implementations.

An analytical empirical expression for the complex modulus and the shift factor obtained

by curve fitting master curves of 3M ISD112 is given by [Drake and Soovere \(1984\)](#). However, the accuracy of these expressions is not clear and, with time, manufacturers have changed the production process, with implications in the properties of the viscoelastic materials, making old data inaccurate. All these aspects strongly justify the development and usage of an experimental methodology to measure *in loco* the viscoelastic material complex modulus, which is hereby proposed and validated. The frequency dependent material properties of cork compounds were also identified with the same experimental procedure and apparatus presented here [[Lopes et al. \(2006\)](#)].

It should be mentioned, though, that in this section, the credits of developing the experimental procedure to measure the complex modulus of the utilized viscoelastic material, developing the numerical procedure to processing the measured data in order to obtain the nomogram of the material, the utilized 2-dimensional layered plate FE and the test setup and measuring of the FRFs, should be given to a colleague and co-worker of the author, which undertook those tasks during his PhD [[Moreira \(2005\)](#)], with whom the author also published some works related with the subject [[Vasques, Moreira and Dias Rodrigues \(2006\)](#), [Vasques et al. \(2007b,a\)](#)]. However, for completeness, these contents will be included in this dissertation.

7.2 Brief Review of Experimental Identification Procedures

The extensive use of passive or hybrid damping treatments using viscoelastic materials to reduce vibration and/or noise radiation from structures has motivated the integration of the aforementioned viscoelastic damping models into commercial or home-made FE codes, which are very interesting approaches used by the major part of the structural designers for studying complex damped structures. However, in order to choose the most appropriate material for a specific application, the designer needs some information regarding the damping capabilities of these materials. The latter is usually obtained through the analysis of the complex (frequency dependent) constitutive behavior of viscoelastic materials at different temperatures. To that end, normalized and proprietary experimental procedures to the measurement and analysis of the dynamic constitutive behavior (usually isotropic) and numerical procedures for the identification and graphical representation of the complex moduli of viscoelastic materials have been proposed.

There are numerous methods for evaluating the performance of damping materials. These methods can be roughly divided into two categories: those whose purpose is to rank the performance of damping materials on a defined structure [e.g. the [SAE J1637](#) test of the Society of Automotive Engineers which is based on the so-called *Oberst Bar Test Method* and is used to rank order materials used in passenger vehicle applications] and those whose purpose is to measure the properties of the damping material alone so that mathematical models can be used to predict its damping performance when applied to many different types of structures.

As far as measuring the complex moduli of viscoelastic damping materials is concerned, there are various testing methodologies that have been devised through time and they can be divided in two distinct and somewhat complementary categories: (i) *direct methods*, which are based on dynamic measurements of pure material samples, and (ii) *indirect methods*, where the properties are inferred from dynamic measurements made on bars with surface mounted or sandwiched viscoelastic damping layers and from assumed underlying mathematical models which relate the composite bar damping behavior to raw viscoelastic material one. Both meth-

ods, though, rely on the *time-temperature superposition principle* to construct master curves of the storage modulus and loss factor from data sets measured at several distinct temperatures, inherently assuming that the material is *thermorheologically simple* [see [Ferry \(1980\)](#) for further details on these concepts].

Regarding the direct methods, they usually consider raw material samples in a single DoF (SDoF) system configuration, which is conceptually the simplest means for obtaining complex moduli data, where the damping material sample is deformed in a specific mode, such as shear or extension. These tests can be dynamic or pseudo-static and involve the measurement and acquisition of force and displacement time histories, which are later processed to obtain the complex modulus data. In the former case, the material is dynamically (usually harmonically) deformed and the frequency dependent complex modulus is directly obtained through transmissibility- or impedance-based approaches. Regarding the latter approach, usually a step deformation (relaxation test) or force (creep test), or other types of stimuli, is used to deform the material and it yields the complex moduli indirectly through time-frequency equivalences, based upon the identification of a representative time domain model by fitting the measured response to the model.

On the other hand, indirect methods consider test specimens where the damping material is surface mounted or embedded in beam structures, with the most popular configuration of the test specimen being the *Oberst beam* where the damping material is surface mounted only in one side of the beam and where the material suffer extensional deformations. Other configurations have also been used considering a symmetric *Oberst beam* with damping material on both sides (so-called *Van Hoort beam*) or sandwiched beams [cf. [Nashif et al. \(1985\)](#), [Jones \(2001\)](#), [ASTM E756-05 \(2005\)](#)], or even beams with constrained damping layers [e.g. [Hambric et al. \(2007\)](#)]. These test specimens are used to infer complex elastic moduli by resonance- or impedance-based methods which use analytic expressions or approximated numerical models of distributed parameters systems which relate the complex modulus of the viscoelastic material to the resonances and loss factors or FRFs, respectively, of the test specimen (beam plus the damping material).

The different experimental apparatus for measurement of the viscoelastic constitutive behavior by the aforementioned approaches (resonant beam tests, SDoF resonant and dynamic modulus testing apparatus, creep and relaxation tests, etc.), along with measurement and determination procedures and representation of viscoelastic materials properties are described and discussed, for example, in monographs and books [[Drake and Soovere \(1984\)](#), [Nashif et al. \(1985\)](#), [Tschoegl \(1989\)](#), [Findley et al. \(1989\)](#), [Sun and Lu \(1995\)](#), [Jones \(2001\)](#), [Cremer et al. \(2005\)](#)], articles [[Paxson \(1975\)](#), [Ferguson \(1984\)](#), [Allen and Pinson \(1991\)](#), [Allen \(1996\)](#), [Willis et al. \(2001\)](#), [Etchessahar et al. \(2005\)](#), [Sorvari and Malinen \(2007\)](#)] and standards [ISO 10112:1991, ASTM D5026-95a, ANSI S2.21-1998 (R2007), ANSI S2.22-1998 (R2007), ANSI S2.23-1998 (R2007), ANSI S2.24-2001 (R2006), ASTM D5023-01, ASTM E756-05, ASTM D5418-07, ASTM D5024-07] and the references therein. Furthermore, some information of proprietary apparatus such as the Dynamic Mechanical Analyzer (DMA), Dynamic Mechanical Thermal Analyzer (DMTA), Rheovibron, Autovibron, Viscoanalyzer, RSA II, to name a few, from companies and laboratories such as CSA Engineering (Mountain View, CA, US), Polymer Laboratories (Amherst, MA, US), Roush Technologies (Livonia, MI, US), etc., can be found in [[Allen \(1996\)](#), [Jones \(2001\)](#), [ASTM E756-05 \(2005\)](#), [Melo and Radford \(2005\)](#)] and the references therein.

In the previous references viscoelastic material master curves, tables and empirical and fractional derivative analytical expressions of the complex modulus and shift factor are presented for several types of viscoelastic materials. However, as referred by [Jones \(2001\)](#), that information is provided from various sources and is fairly accurate, in general, but it is by no

means the best data which can be obtained. It is, however, the best data available in the public domain, and better data ("real" data) may be obtained only by additional testing or purchase of data from in-house and proprietary data bases [e.g. Drake and Terborg (1980), Drake (1988)], often at considerable cost.

7.3 Experimental Identification of Viscoelastic Materials

7.3.1 Underlying Analytical Model

The proposed experimental methodology for the dynamic characterization of the complex modulus of viscoelastic materials is based on the direct identification of the complex equivalent stiffness $K(j\omega)$ of a discrete SDoF system [Allen (1996)]. The complex stiffness is physically materialized by a thin viscoelastic layer which is subjected to cyclic shear deformation imposed by a dynamic exciter. The dynamic response of the considered SDoF system is then used to evaluate the shear *storage modulus* and *loss factor* variation with frequency and temperature.

The *receptance* FRF of the a SDoF system, assuming steady state harmonic motion, is given by

$$H(j\omega) = \frac{\bar{X}(j\omega)}{\bar{F}} = \frac{1}{K(j\omega) - \omega^2 M}, \quad (7.1)$$

where $\bar{F}$ is the amplitude of the dynamic force, $\bar{X}(j\omega)$ is the displacement response phasor and M is the mass of the system. Thus, the complex value stiffness $K(j\omega)$ of the viscoelastic sample can be directly determined through the inverse of the receptance FRF, the *dynamic stiffness* function $Z(j\omega) = H(j\omega)^{-1}$, as

$$K(j\omega) = \omega^2 M + Z(j\omega). \quad (7.2)$$

The viscoelastic material storage modulus, $G'(\omega)$, and loss factor, $\eta(\omega)$, functions can then be evaluated from Equation (7.2) yielding

$$G'(\omega) = \frac{h}{A_S} (\omega^2 M + \text{Re} [Z(j\omega)]), \quad (7.3a)$$

$$\eta(\omega) = \frac{\text{Im} [Z(j\omega)]}{\omega^2 M + \text{Re} [Z(j\omega)]}, \quad (7.3b)$$

where h is the thickness of the viscoelastic sample and A_S its shear area.

7.3.2 Experimental Apparatus

The test rig representing the dynamic SDoF system used to identify the complex shear modulus of viscoelastic materials is illustrated in Figure 7.1. The underlying analytical model of the SDoF system previously described comprises a moving mass M , represented by the upper bar (2), and a complex stiffness $K(j\omega)$, represented by the thin viscoelastic layer sample (4) introduced between two rigid blocks, as shown in Figure 7.2. The upper bar is guided by two thin lamina springs (3), which provide the restraint of the spurious DoFs, allowing the viscoelastic specimen to deform mainly in shear due to the relative motion between the moving bar (2) and the fixed bar (1).

The dynamic excitation of the moving bar is provided by an electrodynamic shaker (Ling Dynamic Systems - model 201), using a thin stinger to minimize the rotation and lateral excitation, driven by a random signal generated by the signal generator of a dynamic signal

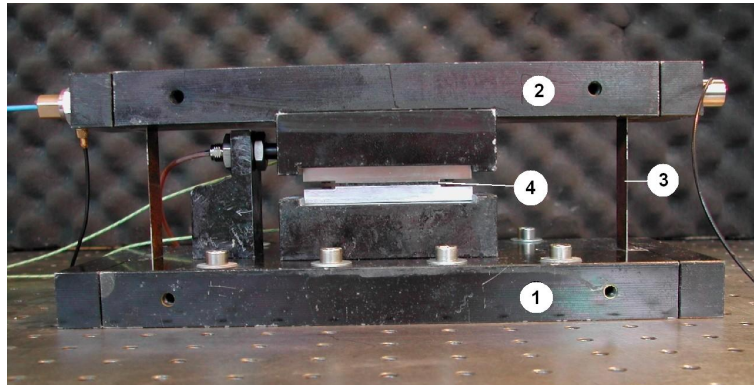


Figure 7.1: Test rig used to identify the complex shear modulus [Moreira (2005)].

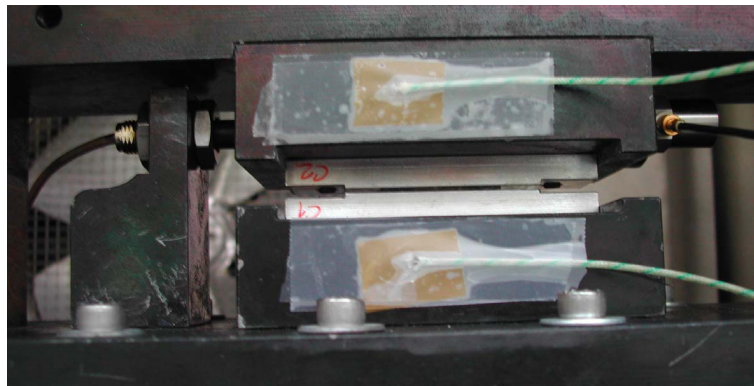


Figure 7.2: Detailed view of the viscoelastic specimen [Moreira (2005)].

analyzer (Brüel & Kjær - model 2035) and amplified at a power amplifier (Ling Dynamic Systems - PA25E). The applied excitation force is measured using a piezoelectric force transducer (Brüel & Kjær - model 8200). The acceleration response of the moving mass is measured with a piezoelectric accelerometer (Brüel & Kjær - model 4371) and the displacement response with a proximity probe (Philips PR6423). The signal conditioning and the FRFs are determined with the aforementioned dynamic signal analyzer.

The test rig and the electrodynamic shaker are rigidly assembled onto an inertial block (Figure 7.3), which is supported by rubber pads in order to minimize the influence of the rigid body modes of the assembly. Furthermore, the experimental apparatus was introduced in a thermal chamber, providing near isothermal conditions between 0 and 35 °C, with the temperature range limited by the shaker admissible range of operating temperature. The analysis at various temperatures allows the use of the time-temperature (or frequency-temperature) correspondence principle [Jones (2001)] extending the frequency range of the characterization and allowing data correlation. The temperature of the viscoelastic material is evaluated using two thermocouple probes located near the specimen, as depicted in Figure 7.2.

7.3.3 Numerical Analysis of Measured Complex Modulus Data

The *receptance* and *accelerance* FRFs of the SDoF dynamic system described in the previous sections are measured using the transducer response signals (proximity probe, accelerometer and force transducer) for different temperatures. Since two different FRFs were measured using

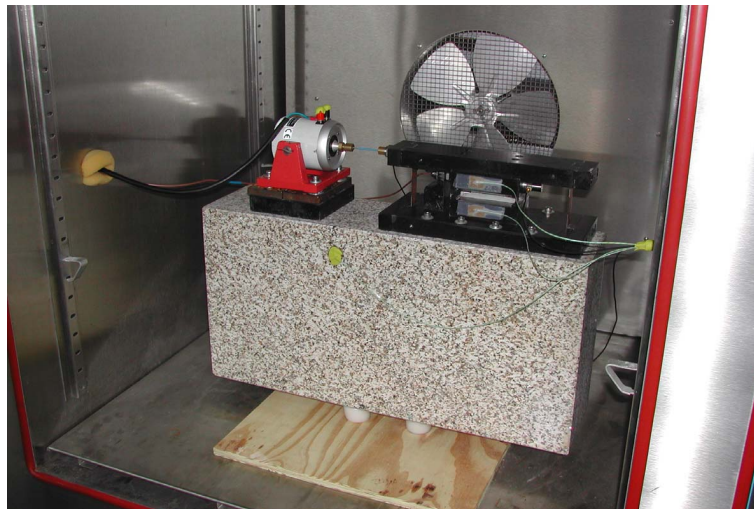


Figure 7.3: Experimental apparatus inside the thermal chamber [Moreira (2005)].

two different transducer responses (displacement and acceleration), the corresponding FRFs can be correlated in order to identify high frequency noise generated by the thin springs, the stinger and shaker trunnion (highly evidenced by the receptance results), and the rigid body modes effects at low frequencies (evidenced in the accelerance results). Therefore, using both transducers, the useful frequency range is enlarged.

Viscoelastic damping is a property exhibited by a wide variety of polymeric materials, ranging from natural or synthetic rubber to industrial plastics, or even cork [Lopes et al. (2006)]. The class of polymer materials is extremely wide and many polymeric compounds, displaying somewhat different complex modulus properties, available from commercial manufacturers and other sources [see Nashif et al. (1985) and Jones (2001) for lists of manufacturers] can be found. Some commonly available commercially viscoelastic damping polymers are the ISD110, 112 and 113 series, from 3M™, DYAD601, 606 and 609 series, from Soundcoat®, to name just a few. Here attention is given to the 3M ISD112, which is the viscoelastic material identified and utilized in the damping treatment of the analyzed case study. However, as obvious, the present experimental procedure might be used to any other type of viscoelastic material.

The receptance and accelerance FRFs of a $0.57 \text{ mm} \times 480 \text{ mm}^2$ 3M ISD112 viscoelastic material specimen, measured in a bandwidth of $[0 - 400] \text{ Hz}$ at nine different temperatures between 2.7 and 33.5°C are represented in Figure 7.4. The obtained FRFs, whether using directly the receptance or the accelerance, are the inputs for the complex modulus description calculated using Equations (7.3).

In order to identify and filter possible random errors due to the high frequency noise, rigid body modes or other error sources, the identified complex modulus functions are plotted point by point in a *wicket plot* which is a useful graphical representation to highlight the errors [see Jones (2001) for further details]. In this graphical representation the storage modulus calculated from the data at different temperatures are plotted against the corresponding loss modulus values. Figure 7.5a represents the wicket plot of the identified complex modulus data of the 3M ISD112 specimen. It stands out the master curve of the complex modulus, being possible to identify erroneous data sets, and the data lying outside the main sequence is eliminated. After filtering the data, removing all scattered points, a reliable data set of complex modulus is obtained, as illustrated in Figure 7.5b, which can be used to characterize the viscoelastic material under analysis.

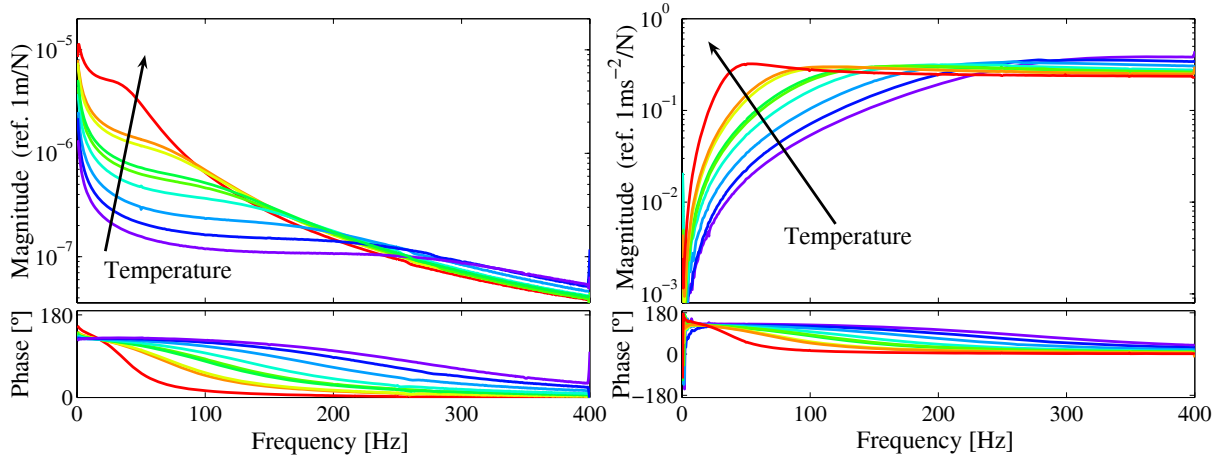


Figure 7.4: Measured receptance and accelerance FRFs at 2.7, 5.8, 7.1, 11.2, 14.4, 14.5, 18.8, 21.5 and 33.5°C [Moreira (2005)].

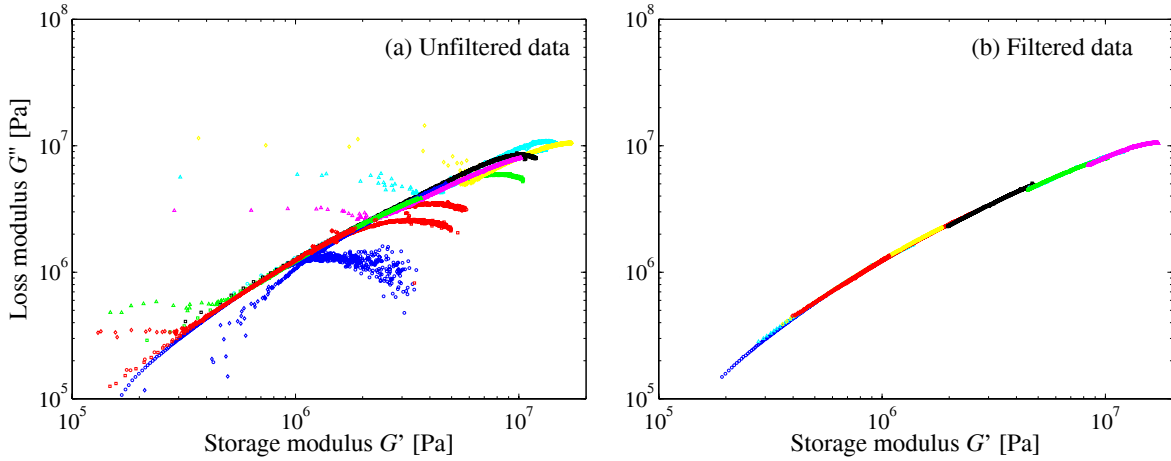


Figure 7.5: Wicket plot of the identified complex modulus functions (spectral filtering) [Moreira (2005)].

The set of complex modulus functions, identified at different temperatures, should finally be correlated to identify the shift factor $\alpha_T(T)$ distribution of the material by using the frequency-temperature equivalence principle. The shift factor relationship might be described by the Arrhenius equation as

$$\log [\alpha_T(T)] = T_A \left(\frac{1}{T} - \frac{1}{T_0} \right), \quad (7.4)$$

where T is the absolute temperature of each data set, T_0 is the reference temperature and T_A is identified by fitting the experimental data to the model. Using the identified shift factor distribution, the identified complex modulus functions can be represented in the unified and universal representation, the so-called reduced-frequency *nomogram*. This representation is very useful and broadly applied since it represents simultaneously the frequency and the temperature dependence of the complex modulus of a viscoelastic material by a single pair of curves: one for the storage modulus and the other representing the loss factor distribution. Figure 7.6 overlaps the original nomogram of the 3M ISD112 viscoelastic material provided by the manufacturer [3M (1993)] with the identified nomogram of the 3M ISD112 specimen under analysis.

The identified nomogram agrees well with the one published by the viscoelastic material manufacturer and is consistent with other nomograms obtained with other specimens with different dimensions using the same methodology. The reader is referred to the well-known textbooks of [Nashif et al. \(1985\)](#) and [Jones \(2001\)](#) for further details about the frequency-temperature equivalence principle and construction of the nomogram.

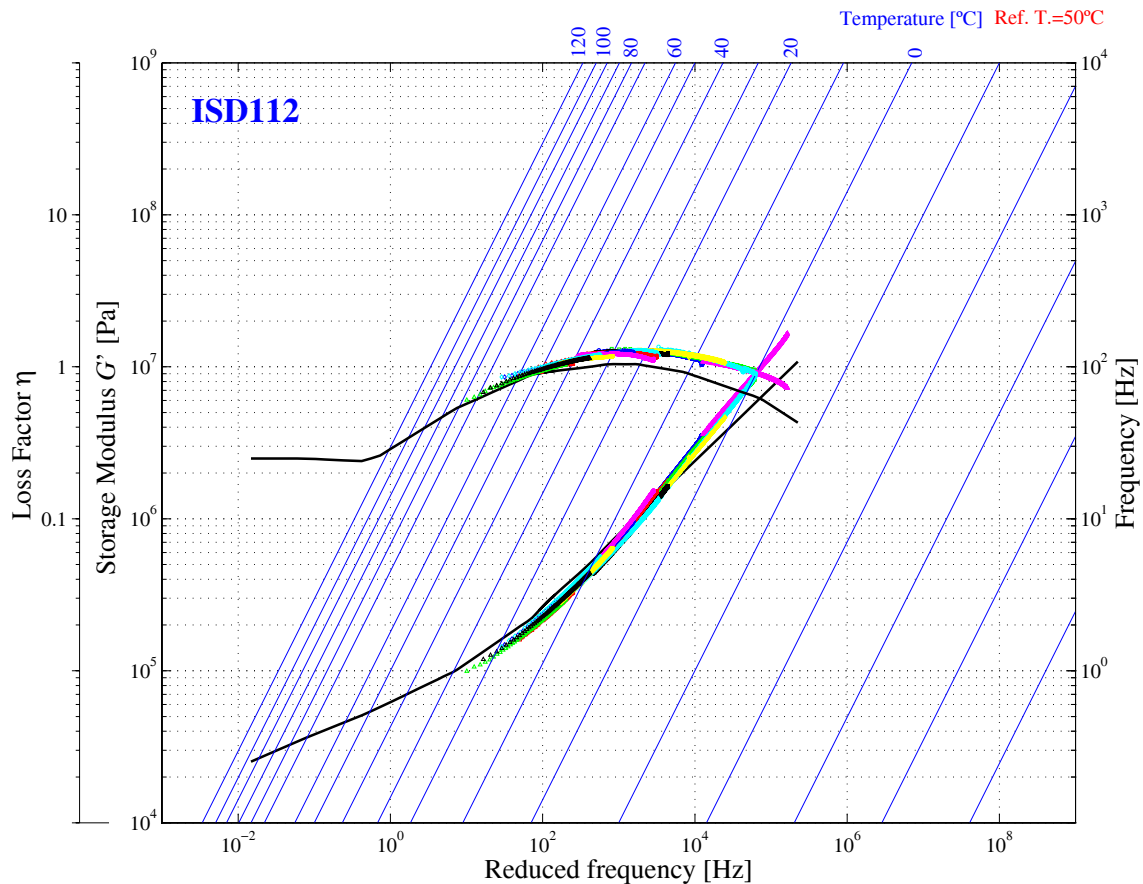


Figure 7.6: Nomogram of 3M ISD112: manufacturer's (black) and identified (color) [[Moreira \(2005\)](#)].

7.4 Parameters Curve Fitting

Time domain models represent better alternatives to CMA-based models, which do not require any curve fitting procedure, allowing the reduction of the computational burden and the study of the transient response in a more straightforward manner, even for highly damped structural systems. Among the time domain models, internal variables models are more interesting from the computational point of view and easiness of implementation into FE codes. Thus, the GHM and ADF models are alternative methods used to model the damping behavior of viscoelastic materials in FE analysis, which yield a standard FE formulation (however with the addition of some "non-physical" dissipative variables). In order to use them, one needs the GHM and ADF characteristic parameters which allow characterizing the complex (frequency dependent) constitutive behavior of the viscoelastic material being used. To do that, experimental procedures

to measure the isotropic constitutive behavior (usually the shear modulus) and numerical identification procedures of the measured data need to be developed.

As reported by [Trindade and Benjeddou \(2002\)](#), the parameters of internal variables models, in general, are adjusted by curve-fitting the viscoelastic material master curves provided by the manufacturer, in order to minimize the difference between the measured and estimated data. In fact, such procedure was carried out in the 1990s by [Lesieutre and Bianchini \(1995\)](#) which presented the results of the curve-fitting of the 3M ISD112 material data at 27 °C in the frequency range [8 – 8000] Hz. They concluded that five ADF series (with two ADF parameters per each series) would represent accurately the behavior of the material shear modulus and loss factor. [Friswell et al. \(1997\)](#) presented the same analysis for the GHM model, with three or four parameters per series. They used the 3M ISD112 material at 20 °C in the frequency-range [10 – 4800] Hz and DYAD601 material at 24 °C in the frequency range [2 – 4800] Hz. Their results indicated that the model with four parameters represented the material data better than that with three parameters. In general, GHM and ADF models fit well the master curves of materials whose properties have strong frequency dependence. Nevertheless, the number of parameters needed is somehow related with the degree of frequency dependence of the material properties and usually increases with that dependence. These reasons motivated [Enelund and Lesieutre \(1999\)](#) to propose a combination of the ADF model with a fractional derivatives model, where the relaxation equation for the anelastic strain is taken as an in time differential equation of fractional-order, combining the best features of the ADF model and the FC (*fractional calculus*). It was shown that this fractional-order ADF model can predict the instantaneous transient response of the material over a wide frequency range using a single anelastic strain field (only one ADF serie) governed by a fractional-order equation with the drawback of yielding a "non-standard" FE formulation requiring a more complex characteristic solution procedure typical of FC.

From the measured material properties (identified nomogram) of the 3M ISD112, and using the shift factor relationship previously determined, one can obtain the frequency dependent complex modulus for different temperature values. In order to use the internal variables viscoelastic damping models GHM and ADF defined in Chapter 6, the characteristic model parameters (three and two for each GHM and ADF series, respectively) were determined using a curve-fitting technique based upon the minimization (optimization procedure) of the least mean square error between the measured and predicted data, using the GHM and ADF definitions of the complex shear modulus, for a specified frequency range of interest. The number of series used in the GHM and ADF models determines how well the models are capable of matching the data over the frequency range of interest.

For each viscoelastic model, one and three series of parameters were used to curve fit the identified complex modulus of the 3M ISD112 material at 27 °C over the frequency range [10 – 3000] Hz. The fitted curves of both models are presented in Figures 7.7 and 7.8 and compared with the measured data.

As can be verified, the quality of the fit depends on the number of terms (series) retained in the models. Using more series improves the accuracy of the models material description, however increasing the size of the problem. The values of the parameters determined in the fitting process for both GHM and ADF models are presented in Table 7.1. It is worthy to mention that the values of the parameters determined are guaranteed to define the material properties with accuracy only over the frequency range specified in the fitting process which in this case is [10 – 3000] Hz. From the analysis of Figures 7.7 and 7.8 one can conclude that both models fit the measured data with a satisfactory accuracy with 3 series of parameters (one serie wouldn't be enough).

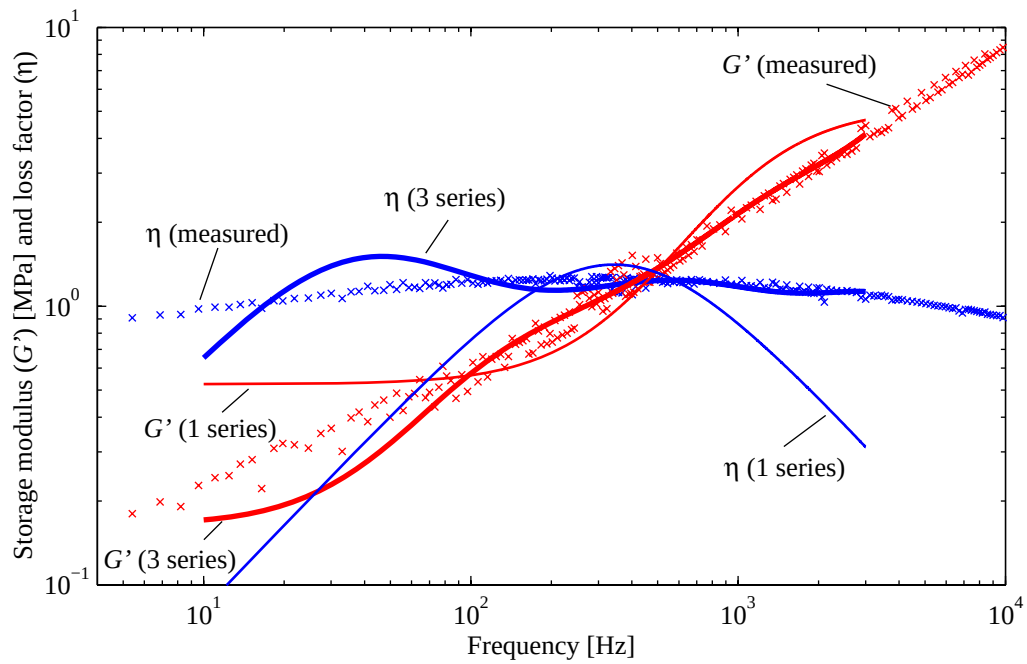


Figure 7.7: Curve fitted GHM curves with 1 and 3 series of parameters at 27°C.

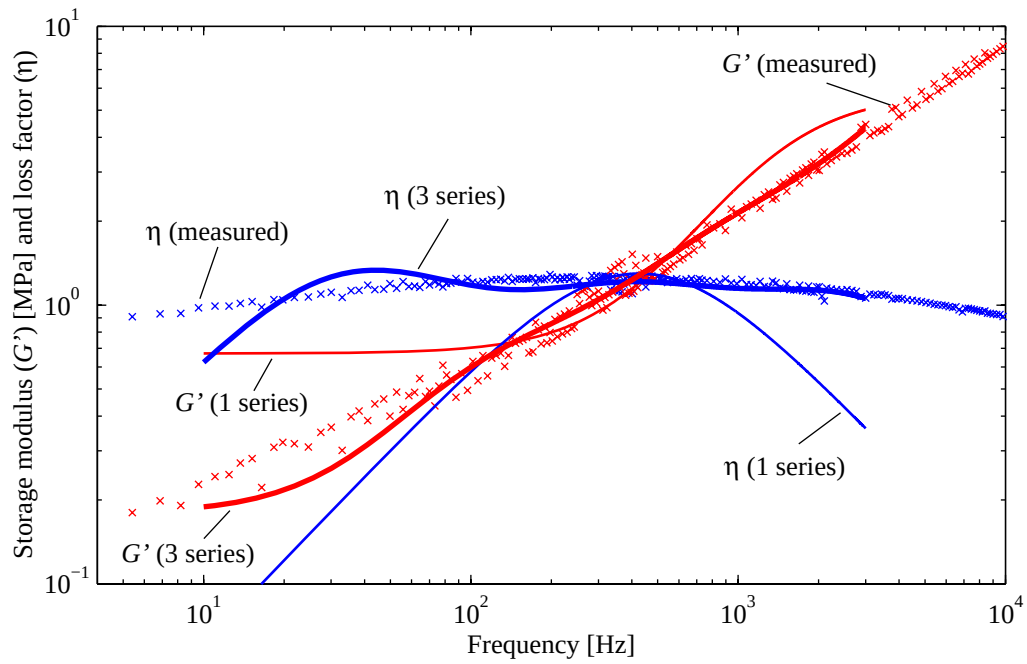


Figure 7.8: Curve fitted ADF curves with 1 and 3 series of parameters at 27°C.

Table 7.1: Identified GHM and ADF parameters for material 3M ISD112 at 27°C using three series ($n = 3$).

Model	GHM				ADF		
	G_0 [MPa]	$\hat{\alpha}_i$	$\hat{\omega}_i$	$\hat{\zeta}_i$	G_0 [MPa]	Δ_i	Ω_i
1	0.1633	4.8278	28045	22.013	0.1789	3.5286	504.20
2		14.548	41494	3.1275		8.7533	4286.5
3		40.043	41601	0.6165		60.324	39313

7.5 Assessment and Validation of the Damping Models

7.5.1 Experimental specimen

The aim of this section is to assess and validate the time domain based GHM and ADF viscoelastic damping models, and their correspondent FE implementation and solution method, and the frequency domain based ones, IMSE and ICE, presented and described in Chapter 6. Thus, a comparison between the FRFs of a viscoelastic sandwich plate, predicted with the aforementioned models and by a DFR approach, and measured results, is performed.

To perform the experimental analysis, a sandwich plate specimen with an embedded viscoelastic damping treatment was produced. The viscoelastic material considered is the 3M ISD112 [3M (1993)] which was used to produce the viscoelastic layer applied in the core of the sandwich plate. The specimen was manufactured using two aluminum plates (aluminum alloy 1050A H24) with dimensions $200 \times 100 \times 1$ mm, to produce the sandwich skins, and a viscoelastic core with thickness 0.127 mm (Figure 7.9). The specimen was produced following the manufacturer instructions and it was found that the 3M ISD112 can be easily bonded to the metallic substrate at room temperature. The properties of the viscoelastic material and aluminum applied in the specimen are presented in Table 7.2.

Table 7.2: Material properties of the viscoelastic sandwich plate specimen.

Material	G [G Pa]	ν	ρ [kg m ⁻³]
AA 1050A H24	26.5	0.32	2708
3M ISD112	Figure 7.6	0.49	1140

7.5.2 Experimental setup

The aim of the experimental study was the determination of a representative set of FRFs providing a reliable basis for comparison and validation of the numerical models. The experimental specimen was suitably suspended in the air through lightweight nylon wires attached to a rigid frame to get approximate free boundary conditions, which minimizes the boundary error effects. A grid with 15 points, as depicted in Figure 7.9, was defined for measurement. An electrodynamic shaker (Ling Dynamic Systems - model 201), suspended from an independent rigid frame, was utilized to generate a random $[0 - 800]$ Hz excitation applied in point 5 of the specimen. A thin and flexible stinger was used to link the shaker to the miniature force transducer

(Brüel & Kjær - model 8203) attached to the plate surface, which provided the measurement of the applied dynamic force (Figure 7.10). The specimen response was evaluated by using a laser vibrometer (Polytec - model OFV303) to measure the velocity of each point of the measuring mesh (Figure 7.11). The FRFs determination was performed by a dynamic signal analyzer (Brüel & Kjær - model 2035). The temperature was evaluated by a thermocouple located near the specimen. Fifteen FRFs (mobility functions) were determined for the specimen. However, here, for comparison of the damping models, only the driving point mobility FRF measured in point 5 is considered.

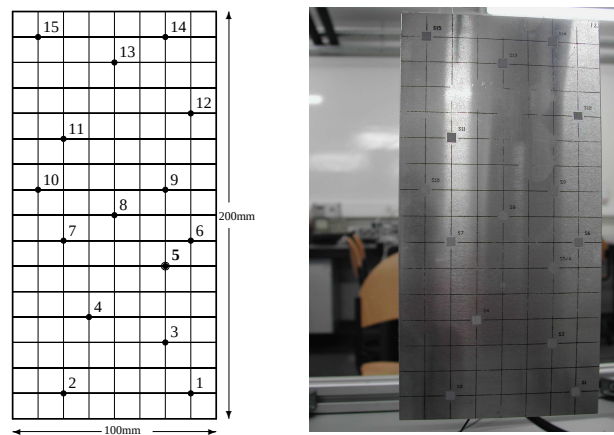


Figure 7.9: Measuring mesh and viscoelastic sandwich plate specimen [Moreira (2005)].

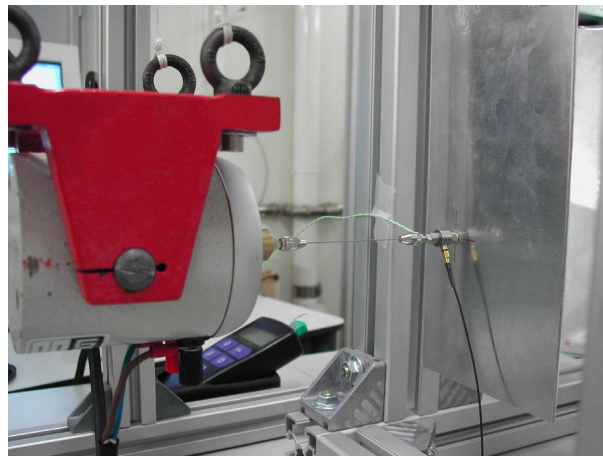


Figure 7.10: Experimental setup (specimen excitation) [Moreira (2005)].

7.5.3 Assessment and comparison of the results

For comparison of results, the driving point mobility FRF measured in point 5 of the mesh depicted in Figure 7.9 is utilized. The predicted FRFs were determined using a 4-node facet type quadrangular shell FE, based on a layerwise theory, developed for the dynamic modeling of laminated structures [the reader is referred to Moreira et al. (2006) for further details about the

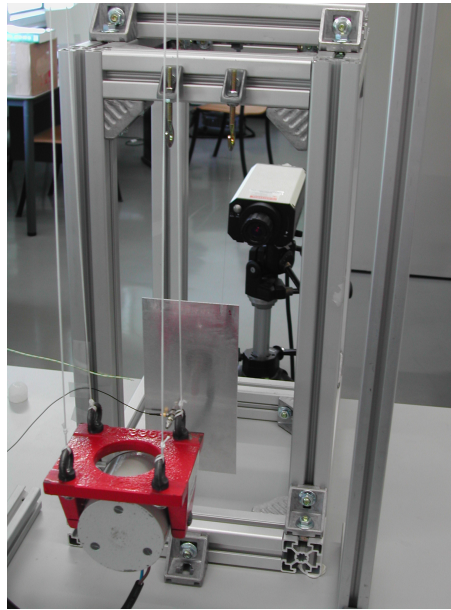


Figure 7.11: Experimental setup (response measurement) [Moreira (2005)].

FE model utilized]. The sandwich plate was modeled with a FE mesh discretization of 32×16 elements. The damping behavior of the viscoelastic sandwiched layer was incorporated by the five methods described in Chapter 6: (i) DFR, where the stiffness matrix is re-calculated at each frequency value; (ii) IMSE, where an iterative procedure is used to estimate the modal parameters with the modal loss factor determined through a MSE approach; (iii) ICE, where, similarly to the IMSE method, the modal parameters are determined through an iterative solution where, in this case, a complex eigenvalue problem is considered instead of a real one, which allows a direct estimation of the loss factor and damped natural frequency from the complex eigenvalue; (iv and v) and the GHM and ADF models, based in the use of additional internal (or dissipative) variables, utilizing the three series of parameters previously identified (Table 7.1). The measured and predicted FRFs are compared in Figure 7.12. Results of the natural frequencies and damping ratios for the first 4 modes are presented in Table 7.3. In order to quantify the correlation level between the measured and predicted FRFs, the *Global Amplitude Criterion* (GAC), proposed by Zang et al. (2001) is utilized and the frequency distribution of the GAC indicators are presented in Figure 7.13. It worthy to mention that this correlation criterion will only be unitary if the compared FRFs have identical phase and magnitude, and since it is correlating mainly the amplitude, it allows more clearly to identify the damping deviation between the measured and each of the five predicted FRFs. The interested readers looking for further details on FRF correlation indicators are also referred to [Moreira and Rodrigues (2002)].

From the results it can be seen that the predicted FRFs are well correlated with the measured one. Both natural frequencies and damping ratios are well estimated with the internal variables models, GHM and ADF, and with the DFR and ICE approaches. The worst results, both in terms of natural frequency and damping estimation, are obtained with the IMSE, which is shown to be less accurate (at least for highly damped systems, as demonstrated in the present case study). It is worthy to mention that the DFR approach is very time-consuming and that it does not provide directly the modal parameters of the viscoelastically damped structure. In opposition, both the internal variables models and the two iterative approaches allow the definition of a mathematical model of the structure allowing, as in this specific case, to determine

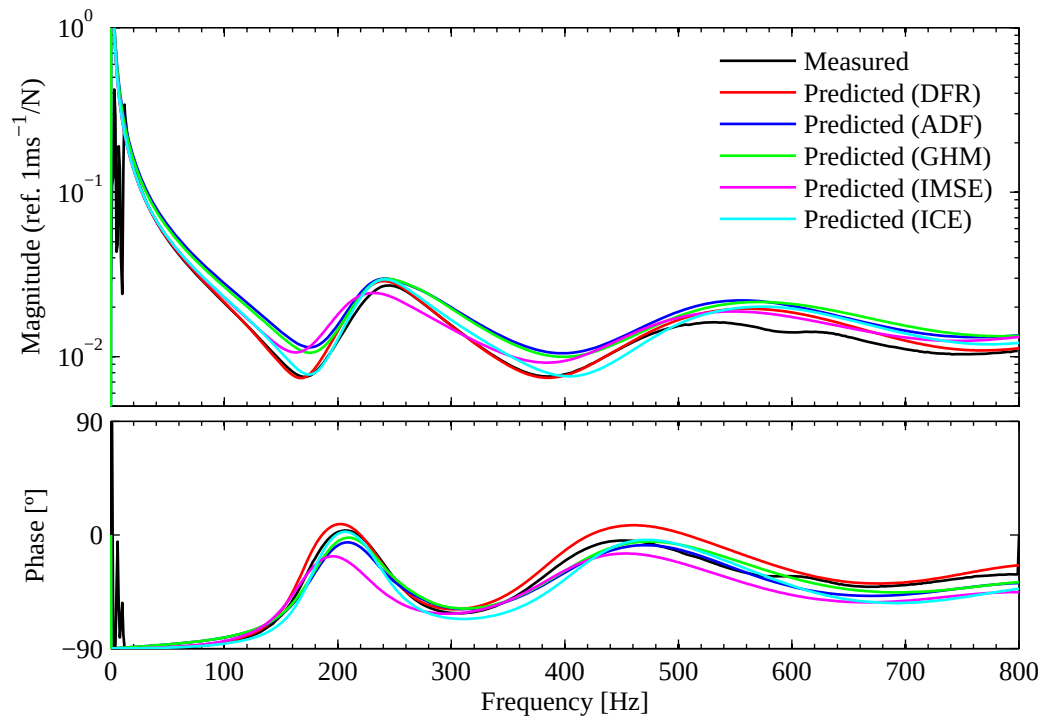


Figure 7.12: Direct mobility FRF of the viscoelastic sandwich plate at 27°C.

Table 7.3: Measured and predicted natural frequencies [Hz] and modal damping ratios [%] of the first 4 modes of the viscoelastic sandwich plate specimen.

	1st mode	2nd mode	3rd mode	4th mode
Measured	235.1 (14.1)	—	521.0 (12.1)	—
Predicted (DFR)	233.9 (13.0)	—	518.6 (15.1)	—
Predicted (IMSE)	212.7 (19.5)	223.1 (19.0)	476.7 (19.1)	526.6 (20.7)
Predicted (ICE)	230.3 (13.2)	237.1 (14.9)	504.6 (15.5)	558.9 (17.0)
Predicted (GHM)	235.7 (13.7)	268.2 (12.7)	524.3 (16.8)	556.8 (19.3)
Predicted (ADF)	233.8 (14.1)	269.4 (13.3)	524.2 (16.5)	554.5 (18.7)

directly and iteratively the natural frequencies and damping ratios from the extracted complex eigenvalues, identifying all the complex modes, which sometimes might be difficult to identify from the post-processing of highly damped FRFs. Furthermore, these mathematical modal models can be used both in time and frequency domain analysis. Moreover, it is also shown that the GHM and ADF models conduct to similar results, with the size of the ADF model smaller than the GHM one, and it was concluded that the fitting process is easier to perform with the ADF model. Last but not the least, the new proposed ICE approach is shown to be very accurate in terms of damping estimation and a little worst in the natural frequency estimation, with the drawback of requiring a complex eigenvalue solution but without the need to increase the size of the problem, as is the case in the internal variables models.

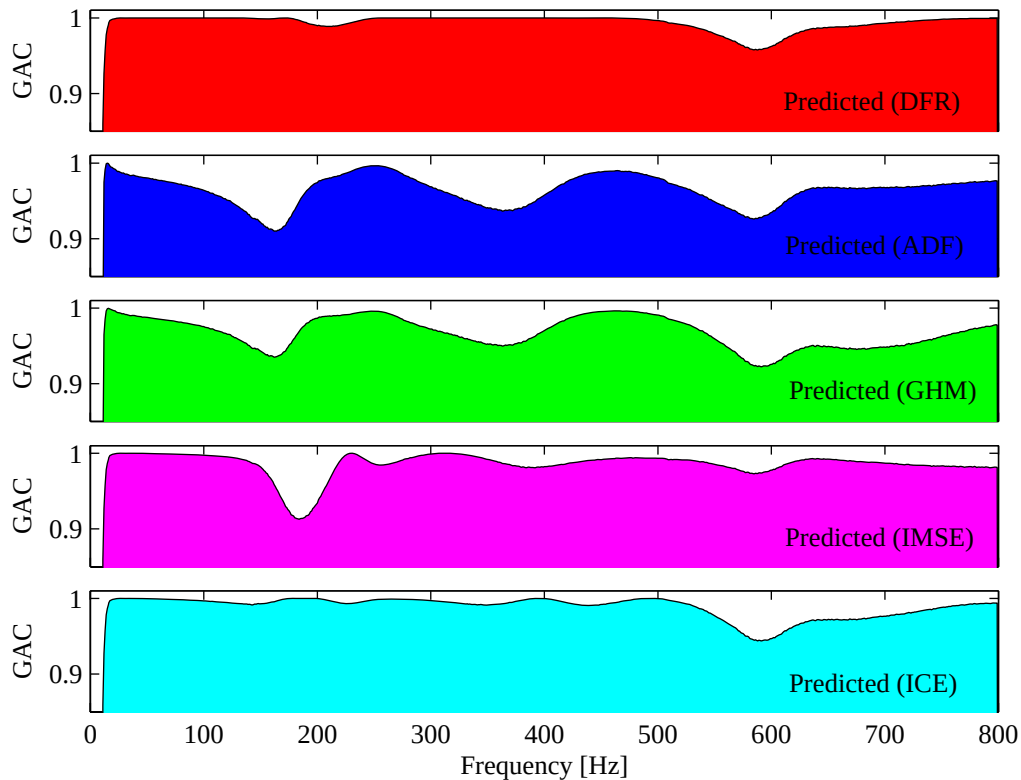


Figure 7.13: GAC correlation indicators for the five damping models.

7.6 Summary

This Chapter has assessed FE-based analytical strategies to model the damping behavior of viscoelastic materials, which might be used (surface mounted, constrained or embedded) as damping treatments in structures to control noise and vibration levels. An experimental procedure to identify the complex shear modulus of viscoelastic materials was presented, and the obtained data was used to identify the characteristic parameters of the internal variables models, GHM and ADF, for the 3M ISD112 viscoelastic material. Measured and FE-based predicted FRFs based on a DFR, internal variables models (GHM and ADF), and two iterative approaches (IMSE and ICE), were compared in order to assess the damping models and validate the experimental procedure for the material properties identification and the curve fitting process.

The application of a discrete dynamic system, described by a SDoF analytical model, has been shown to provide a reliable identification methodology, since it is based on the direct (in opposition to indirect measuring approaches like the resonant beam technique) characterization of the complex stiffness of a viscoelastic material sample in shear deformation. The obtained material results were shown to be similar to those published by the material manufacturer.

Regarding the internal variables models under analysis here, which were implemented at the FE model level, the ADF model is known to lead to an augmented model of the damped structural system with a lower size than the GHM model. To the opinion of the author, the ADF model represents the best alternative to accurately model the general damping behavior, since it yields good trade-off between accuracy and complexity. One major disadvantage

in using internal variables models such as the GHM or ADF is the creation of additional internal non-observable dissipation (or anelastic) variables increasing the size of the coupled damped FE model. However, an alternative based on the DFR, using the FE spatial model and re-calculating the complex viscoelastic stiffness matrix for each discrete frequency value, might be used with the outcome of being simpler to implement and the drawback of being time-consuming and not providing directly the modal parameters of the damped structural system. Nevertheless, and despite its huge computational cost, the DFR approach should be used in validation tasks when experimental data is a set of frequency response functions. Alternatively, the ICE approach was successfully utilized and it was shown that this strategy has some advantages over the ADF model: it does not need to increase the size of the problem and the eigensolution can consider only a small number of modes necessary to build the required truncated modal model. However, it has the drawback of requiring several complex eigenproblem (of a smaller size than the ADF approach) calculations during the iterative process until convergence is achieved.

In general, it can be stated that all the models but the IMSE one have similar accuracy and yield representative results of viscoelastically damped structural systems.

Part IV

VIBRATION CONTROL

Part IV

VIBRATION CONTROL

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CHAPTER 8

Active and Hybrid Control of Vibrations

8.1 Introduction

This part of the dissertation is dedicated to the active and hybrid control of vibrations. With that in mind, feedback and feedforward control strategies are considered and used in closed-loop controlled hybrid structural systems by means of viscoelastic and piezoelectric damping treatments in order to increase the stability and damping performance of the host structure.

The term *hybrid* is used here in a two-fold sense: on the one hand, it is used to denote that two types of materials, viscoelastic and piezoelectric materials, are used and blended as a unified damping device/technology, which acts upon the structure simultaneously in a passive and active (*hybrid damping*) fashion; on the other hand, it is used also to denote that the two distinct and somewhat complementary feedback and feedforward control strategies may also be used in a simultaneous fashion and blended into a unified *hybrid control* system taking advantage of the individual outcomes of each strategy.

Therefore, this part of the dissertation is organized in two chapters. The first addresses the theoretical aspects of feedback and adaptive feedforward control systems used in modeling and simulation of structural systems with hybrid damping devices. The latter intends to establish the bridge between modeling/simulation and effective real-time control implementation in structural vibrations and noise control applications of adaptive structural systems. Both passive, active and hybrid damping treatments and control configurations will be considered and analyzed from both numerical simulation and experimental implementation viewpoints.

Regarding this chapter, it starts with a brief review of hybrid damping and control technologies. Next, in order to optimize the design of arbitrary hybrid damping treatments, a discussion concerning their phenomenological behavior and the identification and quantification of the efficiency of the distinct underlying damping mechanisms is performed. After that, the state space representation of the FE governing equations of active and hybrid structural systems and a modal reduction procedure are presented. Lastly, both feedback and adaptive feedforward control strategies, used separately or combined, and the underlying theory are described.

8.2 Brief Review of Hybrid Damping and Control Technologies

The aim of active vibration control is to reduce the vibration of a mechanical system by the automatic modification of the system's structural response. An adaptive structure, as previously discussed in Chapter 1, consists of a structure provided with a set of highly integrated sensors (to detect the vibration) and actuators (which influence the structural response of the system) coupled by a controller (to suitably manipulate the signal from the sensor and change the system's response in the required manner), which can be used to automatically modify the structure's response. Additionally, these structures may still incorporate in a hybrid or separated fashion some passive damping devices such as viscoelastic materials and may use adaptive hybrid control strategies. These high performance structures, which have the distinctive feature of having sensors and actuators that are often distributed and have a high degree of structural and/or electronic integration into the structure are often called *smart*, *adaptive* or *intelligent* structures [see Section 1.1.1 for a discussion on this terminology]. Reviews describing the state of the art of smart material systems and related technologies can be found in [Chee et al. (1998), Sunar and Rao (1999), Giurgiutiu (2000), Chopra (2002), Tzou et al. (2004)] and related books in [Tzou and Anderson (1992), Clark et al. (1998), Suleman (2001), Srinivasan and McFarland (2001), Schwartz (2002), Preumont (2002), Tokhi and Veres (2002), Varadan et al. (2006), Preumont (2006), Janocha (2007), Leo (2007), Wagg et al. (2007)].

It was already referred that some of the most widely used distributed sensors and actuators are made of piezoelectric materials. The piezoelectric effect consists of the ability of certain crystalline materials to generate an electrical charge in proportion of an externally applied force (*direct piezoelectric effect*) and to induce an expansion on the piezoelectric material in proportion to an electric field parallel to the direction of polarization (*converse piezoelectric effect*) [Cady (1964)]. Among the piezoelectric materials there are two broad classes used in vibration control: polymers and ceramics. The piezopolymers are used mostly as sensors because they require extremely high actuating voltages; the piezoceramics are extensively used both as sensors and actuators, since they require lower actuating voltages, and are used for a wide range of frequency. Significant advances in smart material actuators have taken place in the past decade and the most current trends in piezoelectric actuation architectures are presented in a review paper by Niezrecki et al. (2001).

A brief look at the background, concepts and terminology of damping treatments, significant historic events and some recent developments of adaptive structures comprising viscoelastic and piezoelectric damping strategies, was already performed in Section 1.2. However, for completeness, and to give the reader a quick insight into the terminology and main features of damping treatments, some contents are repeated here in the following paragraphs. Therefore, damping treatments comprising piezoelectric and viscoelastic materials can be divided as follows:

- *Active damping* (AD) treatments (see Figure 1.5) comprise piezoelectric material patches (or layers) alone. Usually they are surface mounted (or embedded) on structures and produce distributed dynamic forces and/or moments upon the structure which originate "negative" power flows to balance the effects of the primary disturbance(s). Some of the pioneer works concerning active damping are the ones presented by Bailey and Hubbard (1985) and Crawley and de Luis (1987) in the 1980s, and over the following decades several works appeared in the open literature with further developments [see, for exam-

ple, Lee and Moon (1990), Collins et al. (1994), Smith et al. (1996), Chandrashekhara and Varadarajan (1997)];

- *Passive damping* treatments (see Figure 1.4) usually comprise viscoelastic layers either sandwiched between a host structure and an elastic constraining layer or left unconstrained. They add damping to the system due to heat dissipation mainly caused by extensional or shear deformation of the viscoelastic layer. If an elastic constraining layer is considered, which makes the shear deformation become dominant to the detriment of the extensional one, the configuration is usually known as *passive constrained layer damping* (PCLD);
- *Hybrid damping* treatments are characterized by a blend of *active* piezoelectric and *passive* viscoelastic damping capabilities. In order to further increase the shearing and energy dissipation of the viscoelastic layer in a PCLD configuration, an active constraining piezoelectric layer might be used to actively increase the shearing in a convenient way. That was originally proposed by Plump and Hubbard (1986) and afterward further developed and widely studied by Baz and his co-workers [Baz (1993), Ray and Baz (1997), Park and Baz (1999a)] and many others [e.g. Kapadia and Kawiecki (1997), Lam et al. (1997), Liu and Chattopadhyay (2000), Liu and Wang (2002)]. The latter treatment is the so-called *active constrained layer damping* (ACLD) treatment (see Figure 1.5). However, when used in a generalized sense, where several layers of viscoelastic, piezoelectric and elastic materials are used in arbitrary arrangements, they should be denoted as *hybrid damping* or *arbitrary ACLD* treatments. This types of treatments combine the passive capabilities (fail-safe, robustness and simplicity) of viscoelastic materials to dissipate vibratory energy at high frequencies with the active capabilities (high performance and adaptability) of piezoelectric materials at low frequencies. Reviews, surveys and assessments on hybrid active-passive damping treatments and related technologies were performed by Park and Baz (1999b), Benjeddou (2001), Trindade and Benjeddou (2002) and Stanway et al. (2003).

In the last decades the advances in digital signal processing and sensors and actuators technology have prompted interest in active control and a considerable effort has been put in the development and implementation of *active noise control* (ANC) and *active vibration control* (AVC) theories as testified by the related textbooks [Inman (1989), Meirovitch (1990), Tzou (1993), Fuller et al. (1996), Hansen and Snyder (1997), Clark et al. (1998), Elliott (2001), Preumont (2002, 2006), Inman (2006), Moheimani and Fleming (2006), Leo (2007)]. In the past two decades various strategies to actively control the vibrations of structures with piezoelectric layers acting as sensors or actuators have been applied. Some examples of analytical and experimental studies concerning the actuation and vibration control of smart piezoelectric beams can be found in [Burke and Hubbard (1987), Crawley and de Luis (1987), Baz and Poh (1988), Lee and Elliott (2001)]. An analytical and experimental study utilizing classic and optimal feedback control techniques for the vibration control of beams and plates was accomplished by Han et al. (1997) and FE models for plates and shells are presented in [Balamurugan and Narayanan (2001), Wang et al. (2004)]. Studies about modal sensors and actuators concerning independent modal space control applied to beams are presented in [Lee and Moon (1990), Lee et al. (1991), Chen and Shen (1997)].

The different algorithms utilized in active vibration control can be classified under two general categories: *feedforward* and *feedback* control. The former control strategy has been shown to be most suitable in applications where the structure is under impulsive or stochastic unknown

disturbances and the latter to the case where deterministic or correlated information about the disturbance is known. Variations of the two general methods exist, each with advantages, disadvantages and limitations. A review about the active structural vibration control is presented by [Alkhatib and Golnaraghi \(2003\)](#).

In the open literature similar studies concerning different active control of vibration strategies (e.g. feedforward wave suppression, proportional and velocity feedback, optimal control) and with different types of actuation (e.g. point forces, pair of moments, piezoelectric patch actuation) in different structural systems (beams with piezoelectric actuators or hybrid treatments also with passive viscoelastic layers) can be found in [[Vipperman et al. \(1993\)](#), [Brennan et al. \(1995\)](#), [Baz and Chen \(2000\)](#), [Trindade et al. \(2001c\)](#), [Baz \(2001\)](#), [Gandhi and Munsky \(2002\)](#), [Ray and Reddy \(2004b\)](#)].

From the analysis of the literature we conclude that feedback theories have been vastly applied in the vibration control of structures with AD and ACLD treatments. Usually it is shown that ACLD treatments have better performances than AD ones, with higher damping values being obtained with lower control voltages. A relevant work in the field is given by [Trindade et al. \(2001c\)](#) which presents a study concerning an FE model of sandwich beams with piezoelectric laminated surface layers and viscoelastic cores where the control design and performance are evaluated for a cantilever sandwich beam model using three control algorithms, namely, LQR, LQG and derivative feedback. One of the well known advantages of having treatments with viscoelastic layers is that they increase the passive damping and consequently the stability margin of the control system; that makes it substantially different from a pure active system with only piezoelectric layers. However, in that work, time domain results for an initial velocity field, for the three algorithms, and frequency response functions, determined by analytical definitions, only for the LQR and derivative feedback algorithms, have been reported and the effects of the statistical content of noise have only been analyzed to assess the state estimator convergence in the time domain.

Comparisons of different classical and/or optimal feedback control strategies were also performed, for example, by [Gandhi and Munsky \(2002\)](#) and [Vasques and Dias Rodrigues \(2006a\)](#). Works concerning adaptive feedforward control theory in active damping applications were performed, for example, by [Vipperman et al. \(1993\)](#) and [Vipperman and Burdisso \(1995\)](#). However, only a few works utilizing ACLD treatments, concerning harmonic *active structural acoustic control* (ASAC) [[Poh et al. \(1996\)](#)] or broadband [[Illaire \(2004\)](#)] and harmonic vibration reduction [[Vasques and Rodrigues \(2007b\)](#)], might be found. The most typical application of feedforward control presented in textbooks concerns the noise attenuation in ducts and applications concerning vibration reduction are mainly devoted to ASAC.

Hybrid (combined feedback/feedforward) control strategies with active damping treatments were successfully used in the past for noise and vibration suppression [see, for example, [Saunders et al. \(1993\)](#), [Clark \(1995\)](#), [de Man and Preumont \(1996\)](#), [Preumont et al. \(1996\)](#) and [Saunders et al. \(1996\)](#)]. However, only one study concerning hybrid control with ACLD treatments applied to a beam was performed by [Vasques and Rodrigues \(2007b\)](#), which is also described in this dissertation. A comprehensive survey on advances in piezoelectric finite element modeling of adaptive structural elements and hybrid active-passive vibration and noise control was performed by [Benjeddou \(2000, 2001\)](#). From the analysis of the state of the art it can be concluded that over the time the efforts of the scientific community have mainly been devoted to analytical and finite element studies, mainly concerning adaptive beams and plates and only a few to shells, and that there is a lack of experimental studies and real-time implementations of AVC and ASAC in adaptive structures.

To sum up, now that finally adaptive structural systems and technologies are transitioning

into the marketplace, it is clear that instead of modeling and simulation the research efforts should be directed towards the real-time implementation and experimental assessment of active control strategies. Besides active control techniques, passive techniques have been successfully used in the past to reduce vibration and noise radiated from structures. It is well known that only after all passive damping possibilities have been applied the designer should start using active damping strategies. The main characteristics and advantages of passive techniques are their stability and robustness. However, adaptability is something that is only achieved with active techniques. In order to take advantages of both strategies, hybrid configurations are good alternatives to blend all the advantages obtained with each of the damping treatments used individually. Among others, passive damping techniques comprise the use of passive viscoelastic materials alone, in hybrid configurations with active piezoelectric or passive elastic constraining layers, or the use of active piezoelectric materials in a passive way where the electrical charge (energy) generated due to the direct piezoelectric effect is dissipated through an electrical network creating a desired impedance in the electrode terminals which might comprise arbitrary mountings of electronic components such as resistors, capacitors and inductors, which is known in the literature as “shunt” control. The latter techniques have been successfully utilized as reported for example in [Hagood and von Flotow (1991), Lesieutre (1998), Moheimani and Fleming (2006), Benjeddou and Ranger (2006)].

8.3 Design and Efficiency of Hybrid Damping Mechanisms

Hybrid damping treatments attenuate vibrations and sound radiation through different damping mechanisms. In order to optimize the design of arbitrary hybrid active-passive damping treatments, where the damping layers are arbitrarily stacked and mounted on the host structure, it is essential to understand their phenomenological behavior and to be able to identify and quantify the efficiency of the different damping mechanisms. These treatments comprise, on the one hand, the damping effects due to the internal molecular interactions that occur during deformation, in general, and vibration, in particular, of the viscoelastic materials, which give rise to macroscopic properties such as stiffness and energy dissipation during cyclic deformation, and, on the other hand, the effects due to the piezoelectric actuation, which exerts forces and moments upon the structure and viscoelastic layers, which depending upon the control law might also give rise to stiffness modifications and damping.

In order to assess and analyze their independent contributions to the net “controlled” structural behavior, two approaches are proposed here. The first approach is based upon the analysis of the contributions of the different damping mechanisms along the frequency, making it more suitable for situations where the untreated structure or system is non-resonant or highly damped (small peaks at the resonances with a flatted frequency response), has an high modal density without clearly uncoupled modes or when feedforward control, aiming at reducing the response more evenly over a specific bandwidth, is being used; structural parameters which vary along the frequency, such as energetic parameters (e.g. the global kinetic or strain energies) or the FRFs (e.g. receptance or mobility) are used to derive *reduction factors* which are used as design and assessment parameters of the damping mechanisms. The second approach is based upon the analysis of the damping contributions of the different damping mechanisms to the peak responses of each individual mode and a method based on the use of *damping ratios* of each mechanism is devised; it is more suitable for situations where the untreated structure

is poorly damped and when feedback control is being used since the damping effects are pronounced only close to the resonant peaks. These two approaches are described in what follows.

Assuming a general closed-loop control system, considering an arbitrary control strategy applied in conjunction with arbitrary hybrid damping treatments to reduce the vibration and/or noise radiation of a general structural system, the resultant *hybrid* damping mechanism basically represents the net interaction and contribution of the following, somewhat different but complementary, three damping mechanisms (as an example consider Figure 8.1 for an illustration in the case of the ACLD configuration):

- A *passive mechanism*, where the system is assumed in open-loop and where only the damping mainly due to the shearing of the viscoelastic layers (constrained or not) alone is considered;
- An *active mechanism*, which decreases the total input power into the structure due to the forces applied by the active piezoelectric layers, which might be in constraining configurations, and in that case these actions are exerted through the viscoelastic layers which reduces the transmissibility of efforts, or directly mounted on the structure, and in that case the transmissibility is not affected;
- A *coupling mechanism*, which represents the energy dissipation effects of the increase of shearing in the viscoelastic layers due to the convenient motion (actuation) of the active (constraining or not) piezoelectric layers.

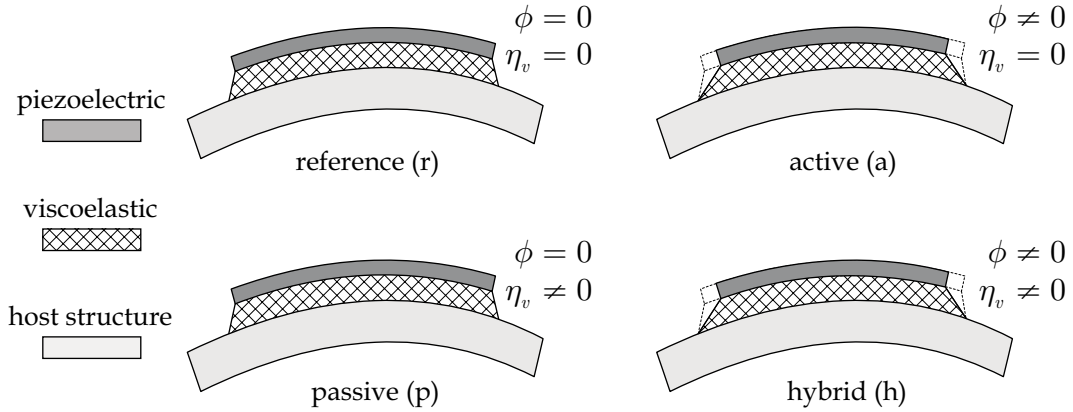


Figure 8.1: Schematic of the methodology to individually quantify the reduction factors of the damping mechanisms for the ACLD configuration.

Thus, based on the previous definitions, and neglecting other general sources of damping (e.g. the material damping of the elastic part of the structural system, air-based damping, energy dissipation in the supports, etc.), which were assumed to be proportional to the velocity and were considered through the viscous proportional damping model in the global FE model, a normalized *reduction factor* correlated with the net damping achieved with the *hybrid mechanism* along the frequency bandwidth, $\eta_h(\omega)$, might be defined as

$$\eta_h(\omega) = \eta_p(\omega) + \eta_a(\omega) + \eta_c(\omega), \quad (8.1)$$

where $\eta_p(\omega)$, $\eta_a(\omega)$ and $\eta_c(\omega)$ are normalized reduction factors correlated with the energy dissipated by the *passive*, *active* and *coupling mechanisms*, given by

$$\eta_h(\omega) = \frac{E_r(\omega) - E_h(\omega)}{E_r(\omega)}, \quad \eta_p(\omega) = \frac{E_r(\omega) - E_p(\omega)}{E_r(\omega)}, \quad \eta_a(\omega) = \frac{E_r(\omega) - E_a(\omega)}{E_r(\omega)}, \quad (8.2a-c)$$

where $E_r(\omega)$ is the *reference* energetic (e.g. kinetic or strain) or FRF quantity of the structural system and $E_h(\omega)$, $E_p(\omega)$ and $E_a(\omega)$ are the correspondent hybrid, passive and active energetic or FRF quantities. These quantities are determined by setting the electric potential difference ϕ on or off (open- or closed-loop system) and by assuming or not the loss factor of the viscoelastic material $\eta_v(\omega)$ equal to zero, as depicted in Fig. 8.1 for the ACLD configuration. Since the reduction factors in Equations (8.2) are in fact the ones that we can determine, at the simulation level, from the outputs of the system, the reduction factor of the coupling mechanism can be determined from the others as

$$\eta_c(\omega) = \eta_h(\omega) - \eta_p(\omega) - \eta_a(\omega). \quad (8.3)$$

As previously mentioned, the analysis and assessment of the different damping mechanisms for resonant structures can be quantified following the same conceptual approach previously presented but adopting as performance indices “fixed” frequency parameters, i.e. the modal damping ratios, in opposition to the previously presented frequency dependent reduction factors. Therefore, in a similar way, the modal damping ratios obtained with active, hybrid and coupling damping mechanisms, can be used to evaluate the coupling damping ratio as before.

Assuming a general structural system with hybrid damping treatments represented by a modal model, as far the modal parameters are concerned, the effects of hybrid damping treatments are twofold. On the one hand, considering a general control law with an arbitrary complex gain composed by real and imaginary counterparts, comprehending real proportional (in-phase or opposite-phase) and/or imaginary derivative (quadrature) terms, the real component of the forces and moments generated by the piezoelectric actuators and exerted upon the structure generate an *active stiffness modification* which produces negative or positive shifts of the resonant frequencies, de-tuning the structure from its original (or passive) dynamic behavior. On the other hand, the imaginary component of the active forces and moments exerted upon the structure produces *active modal damping* which modifies the initial (or passive) damping ratio; depending of the control law the latter forces and moments can still increase or decrease the *passive modal damping* within the viscoelastic, which is denoted here as *coupling modal damping*.

Assuming that the effects of the damping mechanisms are independent, the net *hybrid modal damping* of a generic i th mode, ξ_i^h , may be expressed by the sum of the active, passive and coupling damping ratios,

$$\xi_i^h = \xi_i^a + \xi_i^p + \xi_i^c, \quad (8.4)$$

where $-1 < \xi_i^a < \xi_i^h$, $\xi_i^p > 0$ and $-\xi_i^p < \xi_i^c < 1$. It is worthy to mention that the system was considered always underdamped, i.e. with $0 < \xi_i^h < 1$, and that “negative dampings” can be introduced by the active mechanism, which corresponds to destabilizing the system, adding energy instead of removing it, and by the coupling mechanism which can at most eliminate the damping effects of the passive mechanism with the control switched off.

In a similar way to the reduction factors, the damping ratios that we can in fact determine, at the simulation level, from the outputs of the system, are the active, passive and hybrid mechanisms ones, which can be used to estimate the damping ratio of the coupling mechanism, determined from the others as

$$\xi_i^c = \xi_i^h - \xi_i^a - \xi_i^p. \quad (8.5)$$

It is important also to mention that the previous equation yields an estimation of the coupling damping since it was tacitly assumed that the passive damping ratio ξ_i^p , i.e. the damping ratio due to the dissipation of energy within the viscoelastic layer without considering any effect of the piezoelectric actuation, was the same in both actuated and non-actuated configurations.

However, that may not be always true since a collateral effect of the piezoelectric actuation is the frequency shift, which makes the viscoelastic frequency dependent damping effect to work at a different frequency; if the frequency shift is small and/or if the viscoelastic material constitutive behavior does not change significantly within the frequency bandwidth where the shift occurs, the latter assumption seems to be very reasonable and practical for design application.

The latter coupling reduction factor and damping ratio are of great importance since they allow the designer to infer about the advantages of hybrid damping configurations when compared with pure active or passive ones. Furthermore, when the hybrid ACLD treatments are incorrectly designed, the active mechanism might have a reduced importance since the viscoelastic layer usually reduces the transmissibility of efforts to the host structure. In fact, usually the transmissibility (stiffness) increases with frequency. Thus, in order to dissipate energy also at low frequencies, the aim would be to actively increase the shearing in the viscoelastic layer. However, due to the loss factor behavior of viscoelastic materials (usually smaller at low frequencies), the treatment is usually more efficient for frequencies with higher loss factors, if we have also significant shearing strains, because the energy released is related with the loss factor times the shearing strain. Regarding the "pure" active treatments, the transmissibility of efforts to the host structure is higher, since the patch is bonded to the structure, and the aim is actively reducing the input power coming into the structure. Hence, this set of frequency dependent or modal design indices provide an efficient and straightforward means to design hybrid damping treatments since the contributions and trade-off between the different damping "ingredients" are fully known and quantified.

It is still worthy to mention that, for simplicity, the present approach to quantify the damping mechanisms has been presented and discussed for a structural system with an ACLD treatment. However, as obvious, the methodology is general and can be applied to structural systems with arbitrarily stacked configurations of hybrid damping treatments.

8.4 Structural Modal Model of Piezo-Elastic Structures

In the process of designing an active control system one can utilize a full model of the system, such as the spatial beam and shell structural models presented in Part II, where, as a consequence, a higher computational effort is needed, or a reduced model of the system, which requires a lower computational effort. However, the structural mathematical model and controller design are not independent aspects of vibration control. Flexible structures are distributed parameter systems that have an infinite number of DoFs, and a controller based on a finite reduced modal model can destabilize the residual modes (unmodeled dynamics) leading to observation and control spillover problems. They both degrade the system's performance, and the former can even cause the system to become unstable. Methods to reduce the effects of spillover are discussed by Balas (1978) and Meirovitch (1990).

When excited a structure presents preferable modes of vibration which depend of the spectral content of the excitation. Assuming that the lower order modes, which have lower energy associated and consequently are the most easily excitable ones, are the most significant to the global response of the system, a truncated modal matrix $\hat{\Psi}$ can be utilized as a transformation matrix between the generalized coordinates $\bar{\mathbf{u}}(t)$ and the modal coordinates $\boldsymbol{\eta}(t)$. Thus, considering a FE model of general piezo-elastic structural systems, such as the ones presented in Equations (3.27) and (3.28), for beams, and in Equations (5.25) and (5.26), for shell structures,

which is recalled here as

$$\mathbf{M}_{uu}\ddot{\mathbf{u}}(t) + (\mathbf{K}_{uu}^E + \mathbf{K}_{uu}^{P*})\dot{\mathbf{u}}(t) = -\mathbf{K}_{u\phi a}\bar{\phi}_a(t) + \mathbf{f}_u(t), \quad (8.6)$$

$$\bar{\phi}_s(t) = -\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\mathbf{u}(t), \quad (8.7)$$

the displacement vector $\mathbf{u}(t)$ can be approximated by the modal superposition of the first r modes,

$$\mathbf{u}(t) \approx \sum_{i=1}^r \psi_i \eta_i(t) = \hat{\Psi}\boldsymbol{\eta}(t), \quad (8.8)$$

where $\hat{\Psi} = [\psi_1, \dots, \psi_r]$ is the truncated modal matrix and $\boldsymbol{\eta}(t) = \{\eta_1(t), \dots, \eta_r(t)\}^T$ the correspondent modal coordinates vector. Hence, the system's size does not need anymore to be the total number of DoFs of the FE model and instead it can be represented by a reduced subset of the total number of modes chosen to model it.

The spatial stiffness and mass matrices obtained with the FE method typically present a band structure, which represents the coupling between the several DoFs, and the spatial damping matrix is usually difficult to be obtained. In this study a proportional viscous damping model is utilized and the spatial damping matrix $\mathbf{D}_{uu}$, according to the orthogonality properties of the modal vectors with respect to the mass and stiffness matrices, is diagonalized by the truncated modal matrix $\hat{\Psi}$,

$$\hat{\Psi}^T \mathbf{D}_{uu} \hat{\Psi} = \mathbf{\Lambda}, \quad (8.9)$$

and a diagonal modal damping matrix $\mathbf{\Lambda}$ with the generic term $2\xi_i\omega_i$, where ξ_i is the modal damping ratio and ω_i the undamped natural frequency of the i th mode, is obtained. Therefore, the inclusion of the inherent damping effects in the model is considerably simplified. Thus, considering (8.8) and a normalization scheme for unitary modal masses, Equations (8.6) and (8.7) with the damping effects included become

$$\mathbf{I}\ddot{\boldsymbol{\eta}}(t) + \mathbf{\Lambda}\dot{\boldsymbol{\eta}}(t) + \mathbf{\Omega}\boldsymbol{\eta}(t) = -\hat{\Psi}^T\mathbf{K}_{u\phi a}\bar{\phi}_a(t) + \hat{\Psi}^T\mathbf{f}_u(t), \quad (8.10)$$

$$\bar{\phi}_s(t) = -\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}\hat{\Psi}\boldsymbol{\eta}(t), \quad (8.11)$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{\Omega}$ is a diagonal matrix with the generic term ω_i^2 . Equations (8.11) represent the reduced (truncated) modal FE model of a generic piezo-elastic structure.

8.5 State Space Design

The state space approach is the basis of the modern control theories and is strongly recommended in the design and analysis of control systems with a great amount of inputs and outputs. In this method, dynamic systems are described by a set of first-order differential equations in variables called the *state*. See related textbooks in references [Kwakernaak and Sivan (1972), Burl (1999)] for further details.

The state space design application to adaptive structures modeling is twofold. If the adaptive structure considers only active piezoelectric damping treatments, the piezo-elastic structural model can be directly recast in a state space form in terms of the generalized modal coordinates through a modal projection of the spatial coordinates. However, if the structural damping model is hybrid, i.e., contains both piezoelectric and viscoelastic materials, the frequency (or time) dependent viscoelastic behavior imposes some restrictions and the piezo-visco-elastic structural spatial model is used first in order to obtain a spatial state space model which is then reduced through modal projection yielding in a less straightforward manner (indirect way) the desired complex modal model. In what follows these two procedures are described.

8.5.1 Active Damping Treatments

According to Equations (8.10) and (8.11) the state variables for a general piezo-elastic structural system are chosen as

$$\mathbf{x}(t) = \begin{Bmatrix} \boldsymbol{\eta}(t) \\ \dot{\boldsymbol{\eta}}(t) \end{Bmatrix}, \quad (8.12)$$

and the open-loop system is represented by two first-order matrix differential equations expressed in terms of the state variables vector $\mathbf{x}(t)$,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_\phi \mathbf{u}_\phi(t) + \mathbf{B}_u \mathbf{u}_u(t), \quad (8.13a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad (8.13b)$$

where $\mathbf{A}$ is the system matrix, $\mathbf{B}_u$ and $\mathbf{B}_\phi$ are the mechanical and electrical input matrices, $\mathbf{C}$ is the output matrix, $\mathbf{u}_u(t)$ and $\mathbf{u}_\phi(t)$ are the mechanical and electrical input vectors and $\mathbf{y}(t)$ is the output vector, given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{\Omega} & -\boldsymbol{\Lambda} \end{bmatrix}, \quad \mathbf{B}_u = \begin{bmatrix} \mathbf{0} \\ \hat{\boldsymbol{\Psi}}^T \end{bmatrix}, \quad \mathbf{B}_\phi = \begin{bmatrix} \mathbf{0} \\ -\hat{\boldsymbol{\Psi}}^T \mathbf{K}_{u\phi a} \end{bmatrix}, \quad (8.14a-c)$$

$$\mathbf{C} = \text{avg}(-\mathbf{K}_{\phi\phi s}^{-1} \mathbf{K}_{\phi u s} \hat{\boldsymbol{\Psi}}), \quad \mathbf{y}(t) = \bar{\boldsymbol{\phi}}_s(t), \quad (8.15a,b)$$

$$\mathbf{u}_u(t) = \mathbf{f}_u(t), \quad \mathbf{u}_\phi(t) = \bar{\boldsymbol{\phi}}_a(t). \quad (8.16a,b)$$

It is worthy to mention that in the output matrix $\mathbf{C}$ the notation $\text{avg}(\cdot)$ is utilized to denote that the signal induced by the piezoelectric sensors should be calculated from an average of the electrical DoFs where an electrical FE separation of the electrodes was performed. That is an approximation procedure utilized to estimate the induced voltage if an equipotential area condition was considered, which corresponds to a more realistic case for electroded layers in open-circuit [Vasques and Rodrigues (2005)].

The open-loop system in Equations (8.13) considers two different input vectors, a mechanical disturbance force $\mathbf{u}_u(t)$ (primary excitation) and a control voltage $\mathbf{u}_\phi(t)$ (secondary excitation), and one output vector $\mathbf{y}(t)$ (with the output voltage of the sensors). However, in general, one can consider that multiple inputs are applied to the structure (multiple forces and piezoelectric actuators) and that multiple outputs are obtained from multiple piezoelectric and position-based localized sensors (e.g. vibrometer, accelerometer).

8.5.2 Hybrid Damping Treatments

In order to apply the general augmented damped FE model in control design, the system in Equation (6.33) is transformed into a state space form. However, depending if we are considering its definition following the GHM or ADF approaches, different definitions of the state vector are chosen. Therefore, for the GHM approach, the state space vector $\mathbf{x}(t)$ is chosen as

$$\mathbf{x}_D(t) = \begin{Bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{Bmatrix}, \quad (8.17)$$

where the chosen state variables are the reduced augmented vector $\mathbf{z}(t)$ given in Equation (6.50), which is composed by the mechanical *elastic* DoFs vector $\bar{\mathbf{u}}(t)$ and a set of reduced *dissipative* (internal) variables vectors $\hat{\mathbf{u}}_i^D(t)$, and its time derivative.

However, for the ADF approach, in order to overcome the singularity of the mass matrix, the state space vector $\mathbf{x}(t)$ is chosen as

$$\mathbf{x}_A(t) = \begin{Bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{u}}(t) \end{Bmatrix}, \quad (8.18)$$

where the chosen state variables are the reduced augmented vector $\mathbf{z}(t)$, composed by the mechanical *elastic* DoFs vector $\bar{\mathbf{u}}(t)$ and a set of reduced *anelastic* (internal) variables vectors $\hat{\mathbf{u}}_i^A(t)$, and the time derivative of the mechanical DoFs vector, $\dot{\mathbf{u}}(t)$. It is worthy to stress that the time derivatives of $\hat{\mathbf{u}}_i^A(t)$ are not considered here since these variables are massless, i.e. the matrix $\mathbf{M}$ is singular.

Thus, taking the forementioned state space vector definitions, the augmented piezo-visco-elastic coupled system given in Equations (6.32) and (6.45) for the GHM and ADF models, and the sensing equation generically given in (8.11) for any system incorporating piezoelectric sensors, the governing equations of piezo-visco-elastic structural systems can be expressed in a first-order state space form in terms of the state variables vector $\mathbf{x}_I(t)$, with $\mathbf{I} = (\mathbf{D}, \mathbf{A})$, yielding

$$\dot{\mathbf{x}}_I(t) = \mathbf{A}\mathbf{x}_I(t) + \mathbf{B}_\phi \mathbf{u}_\phi(t) + \mathbf{B}_u \mathbf{u}_u(t), \quad (8.19a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}_I(t), \quad (8.19b)$$

where the state space system, input and output matrices given in Equations (8.14) and (8.15a) for the GHM model are given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -\mathbf{M}_{uu}^{-1}\mathbf{K}_{EE} & -\mathbf{M}_{uu}^{-1}\mathbf{K}_{ED} & -\mathbf{M}_{uu}^{-1}\mathbf{D}_{uu} & \mathbf{0} \\ -\mathbf{M}_{DD}^{-1}\mathbf{K}_{DE} & -\mathbf{M}_{DD}^{-1}\mathbf{K}_{DD} & \mathbf{0} & -\mathbf{M}_{DD}^{-1}\mathbf{D}_{DD} \end{bmatrix}, \quad (8.20)$$

$$\mathbf{B}_u = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_{uu}^{-1} \end{bmatrix}, \quad \mathbf{B}_\phi = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\mathbf{M}_{uu}^{-1}\mathbf{K}_{u\phi a} \end{bmatrix}, \quad (8.21a,b)$$

$$\mathbf{C} = \text{avg}(-\mathbf{K}_{\phi\phi s}^{-1}\mathbf{K}_{\phi us}), \quad (8.22)$$

and for the ADF model the system matrix $\mathbf{A}$ is changed to

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -\mathbf{D}_{AA}^{-1}\mathbf{K}_{AE} & -\mathbf{D}_{AA}^{-1}\mathbf{K}_{AA} & \mathbf{0} \\ -\mathbf{M}_{uu}^{-1}\mathbf{K}_{EE} & -\mathbf{M}_{uu}^{-1}\mathbf{K}_{EA} & -\mathbf{M}_{uu}^{-1}\mathbf{D}_{uu} \end{bmatrix}, \quad (8.23)$$

The foregoing matrices define the modifications that need to be performed to state space matrices in order to make the model representative of piezo-visco-elastic structures utilizing the GHM and ADF internal variables models. It is worth to stress that the alternative transformed (reduced) definitions of $\mathbf{D}_{AA}$, $\mathbf{K}_{AA}$ and $\mathbf{K}_{EA}$ presented in Section 6.6.3 should be used in order to reduce the size of the system, leading to a lower computational burden, and that the system matrix $\mathbf{A}$ for the ADF model has a smaller dimension than the one obtained with the GHM, with the difference between the two matrices being the total number of internal DoFs times the number of series used by the model to fit the master curves of the viscoelastic material modulus.

8.5.3 Modal Reduction of the State Space Models

One major disadvantage in using internal variables models such as the GHM or ADF to model the damping introduced by the viscoelastic materials is the creation of additional dissipation coordinates. Even with a modal reduction of the DoFs of the non-viscoelastic elements the order of the system quickly increases as the number of series of dissipation GHM or ADF parameters used in the summation is increased. This size is determined by the number of series of parameters necessary for an accurate curve fitting of the frequency-dependent complex shear modulus. Larger order makes control design more difficult, especially when these states are non-physical and can not be directly sensed [Lam et al. (2000)]. It is therefore advantageous to look at model reduction techniques to reduce the system's size.

Model reduction in structural dynamics, where the original state space system is approximated by an equivalent system with a lower dimension, can be achieved by a complex modal projection of the original system and a subsequent truncation of the number of modes considered. Thus, the complex eigensolution of the system matrix $\mathbf{A}$ is derived such that

$$\mathbf{A}\Psi_R = \Lambda\Psi_R, \quad (8.24a)$$

$$\mathbf{A}^T\Psi_L = \Lambda\Psi_L, \quad (8.24b)$$

where Ψ_R and Ψ_L are the right and left complex eigenvectors matrices normalized by $\Psi_L^T\Psi_R = \mathbf{I}$ and Λ the correspondent eigenvalues matrix. However, if one looks at the complex eigenvalues of $\mathbf{A}$ one finds that some of them are underdamped, usually associated with the elastic DoFs (flexible modes), and others overdamped, which are associated with the dissipative DoFs (internal relaxations of the viscoelastic materials). Thus, considering a modal decomposition of the eigensolution in the underdamped and overdamped modes, yields

$$\Lambda = \begin{bmatrix} \Lambda_U & \mathbf{0} \\ \mathbf{0} & \Lambda_O \end{bmatrix}, \quad (8.25)$$

and

$$\Psi_R = \begin{bmatrix} \Psi_{RU} & \Psi_{RO} \end{bmatrix}, \quad (8.26a)$$

$$\Psi_L = \begin{bmatrix} \Psi_{LU} & \Psi_{LO} \end{bmatrix}, \quad (8.26b)$$

where the subscripts $(\cdot)_U$ and $(\cdot)_O$ are used to denote the underdamped and overdamped counterparts of the complex eigensolution. Since the contribution of the low observable overdamped modes to the dynamic response of the system is negligible, the state space vector can be approximated only by the contribution of the elastic (underdamped) modes, such that

$$\mathbf{x}(t) \approx \Psi_{RU}\mathbf{x}_U(t), \quad (8.27)$$

and the state space system in Equations (8.19) is reduced to a state space modal system with the same outputs and state space vector $\mathbf{x}_U(t)$, given by

$$\dot{\mathbf{x}}_U(t) = \Lambda_U\mathbf{x}_U(t) + \Psi_{LU}^T\mathbf{B}_\phi\mathbf{u}_\phi(t) + \Psi_{LU}^T\mathbf{B}_u\mathbf{u}_u(t), \quad (8.28a)$$

$$\mathbf{y}(t) = \mathbf{C}\Psi_{RU}\mathbf{x}_U(t). \quad (8.28b)$$

However, as previously referred, when excited a structure presents preferable modes of vibration which depend of the spectral content of the excitation. Therefore, in a similar way to what has been considered in Section 8.4, assuming that the lower order modes, which have

lower energy associated and consequently are the more easily excitable ones, are the more significant to the global response of the system in the frequency range of interest, a truncated complex modal matrix $\hat{\Psi}_{RU}$ can be utilized instead where only the first r modes are considered,

$$\mathbf{x}(t) \approx \sum_{i=1}^r \psi_{RU}^i x_U^i(t) = \hat{\Psi}_{RU} \hat{\mathbf{x}}_U(t), \quad (8.29)$$

where $\hat{\Psi}_{RU} = [\psi_{RU}^1, \dots, \psi_{RU}^r]$ is the truncated modal matrix and the correspondent modal coordinates vector is given by $\hat{\mathbf{x}}_U(t) = \{x_U^1(t), \dots, x_U^r(t)\}^T$. Hence, the system's size is not anymore the total number of DoFs of the augmented FE model but twice the number of modes chosen to model it.

Finally, considering the correspondent truncated matrix of underdamped eigenvalues, $\hat{\Lambda}_U$, the state space system in Equations (8.19) is reduced to a truncated state space modal system with the same outputs and truncated state space vector $\hat{\mathbf{x}}_U(t)$, given by

$$\dot{\hat{\mathbf{x}}}_U(t) = \hat{\Lambda}_U \hat{\mathbf{x}}_U(t) + \hat{\Psi}_{LU}^T \mathbf{B}_\phi \mathbf{u}_\phi(t) + \hat{\Psi}_{LU}^T \mathbf{B}_u \mathbf{u}_u(t), \quad (8.30a)$$

$$\mathbf{y}(t) = \mathbf{C} \hat{\Psi}_{RU} \hat{\mathbf{x}}_U(t). \quad (8.30b)$$

8.6 Control Strategies

8.6.1 Feedback Control

With the purpose of reducing vibration and noise radiation from structures with surface mounted or embedded distributed piezoelectric sensors and actuators, a feedback control loop, somehow amplifying the signals produced by piezoelectric sensor(s) and feeding it(them) back to the actuator(s), may be considered. In essence, it means that a controlled secondary excitation is applied to the structure in order to cancel the effects of the primary one according to a pre-determined control law which uses the information of some exact or correlated information of some sort of structural quantity which is measured (e.g. displacement, velocity, acceleration, strain).

Therefore, following a feedback control philosophy and considering that only distributed piezoelectric actuation is available, the actuating control voltage is generically defined as

$$\mathbf{u}_\phi(t) = -\mathbf{G}\mathbf{y}(t), \quad (8.31)$$

where $\mathbf{G}$ is a generic feedback gain matrix defined according to the control law of interest. Substituting Equation (8.31) into any of the system equations governing the behavior of structures with active or hybrid damping treatments, which are given by Equations (8.13) and (8.30), respectively, and considering, for simplicity, that the latter system can also be equivalently represented by a state space system following the notation of the former in (8.13), the closed-loop system state equation of a generic damped system is given by

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_\phi \mathbf{G} \mathbf{C}) \mathbf{x}(t) + \mathbf{B}_u \mathbf{u}_u(t), \quad (8.32)$$

where the control vector $\mathbf{u}_\phi(t)$ is condensed in the state equations. From Equation (8.32) one can see that the gain matrix $\mathbf{G}$ controls the system through the modification of the closed-loop system poles. Therefore, if an adequate gain matrix and control law are established, the system vibration modes are attenuated through an increase in the modal damping ratio (the initial open-loop poles of matrix $\mathbf{A}$ are transformed into the highly damped ones of $\mathbf{A} - \mathbf{B}_\phi \mathbf{G} \mathbf{C}$).

The stability is guaranteed provided the actuator and sensor are collocated [Preumont (2002)]. However, perfect collocation with an asymmetric piezoelectric sensor/actuator pair is unfeasible due to the in-plane and out-of plane coupling and can lead to instabilities of the control system. Therefore, some care must be taken in the design of the controller and, as pointed out by Yang and Huang (1998), in principle, a symmetric collocated sensor/actuator technique would make the controller absolutely stable.

The gain selection and control system design of such feedback controllers can be done using either a pole-zero representation of the system (e.g. root-locus method, pole placement) or a frequency response representation (e.g. Nyquist method) or even automated processes. The way the control gain matrix is defined, involving the designer in less or greater extent, dictates the type of feedback control strategy that is used. These will be discussed in what follows.

8.6.1.1 Classical Control

Control systems where indirect methods such as pole-zero or frequency representations of the system are used for the definition of the feedback gains are often referred to as *classical control* [Franklin et al. (1991)]. These methods require an high degree of knowledge and intervention of the control system designer and may not be appropriated for control systems with a high number of inputs and outputs. This latter issue can however be circumvented by using decentralized control approaches [see for example Elliott (2007) and Fahy and Gardonio (2007) for further information].

Assuming that a velocity feedback scheme is utilized, where the sensing outputs are differentiated, amplified, and then fed back into the actuators, the signals used in the velocity feedback are representative of the strain rate of the structure. Hence, the conventional term, *velocity feedback*, is used here to denote that approach.

Two velocity feedback control algorithms are considered here. In the first one, which is termed *constant amplitude* velocity feedback (CAVF), the individual gain of the i th piezoelectric actuator is defined according to the polarity of the i th sensor voltage and, to denote that, the function $\text{sign}(\cdot)$ is utilized. Hence, the feedback control voltage amplitude is constant, non-linear and discontinuous, and the gain matrix is defined by

$$\mathbf{G} = \text{diag} (A_1, \dots, A_i, \dots, A_n), \quad (8.33)$$

where $\text{diag}(\cdot)$ denotes a diagonal matrix with the individual amplitudes A_i of the i th control voltage, with $i = 1, \dots, n$, and the control voltage is given by

$$\mathbf{u}_\phi(t) = -\mathbf{G}\text{sign}[\dot{\mathbf{y}}(t)]. \quad (8.34)$$

Alternatively, a *constant gain* velocity feedback (CGVF) can be used, with the gain matrix defined by

$$\mathbf{G} = \text{diag} (G_1, \dots, G_i, \dots, G_n), \quad (8.35)$$

where G_i is the individual gain of the i th actuator and the control voltage is given by

$$\mathbf{u}_\phi(t) = -\mathbf{G}\dot{\mathbf{y}}(t). \quad (8.36)$$

In this case the feedback control voltage is linear, continuous and decreases as the vibration velocity decays. It is worthy to mention that in Equation (8.32) the output matrix $\mathbf{C}$ assumes that all the sensor voltages are known for every sensor and that the velocity feedback control laws in Equations (8.34) and (8.36) indicate that the voltage needs to be differentiated and that it is the sensing voltage velocity that is being considered in the feedback control loop.

8.6.1.2 Optimal Control

In the previous section the values of the feedback gains were chosen to achieve some prescribed change in the dynamic response of the system. However, the ultimate aim of the feedback control is often to reduce the motion of the mechanical system to the greatest possible extent and, in that case, the control system is said to act as a *regulator* (in opposition to the *track following* problem). Systems where direct methods of designing feedback control systems which achieve the greatest possible reduction in the dynamic response are used, are known as *optimal control* systems [Kwakernaak and Sivan (1972), Burl (1999)].

In optimal control the feedback control system is designed to minimize a cost function, or performance index, which is proportional to the required measure of the system's response and to the control inputs required to attenuate the response. The cost function can be chosen to be quadratically dependent on the output response and control input,

$$J = \int_0^{t_f} \left[\mathbf{y}^T(t) \mathbf{Q}_y \mathbf{y}(t) + \mathbf{u}_\phi^T(t) \mathbf{R} \mathbf{u}_\phi(t) \right] dt + \mathbf{y}^T(t_f) \mathbf{S}_y \mathbf{y}(t_f), \quad (8.37)$$

where $\mathbf{Q}_y$, $\mathbf{R}$ and $\mathbf{S}_y$ are the output, control input and terminal output condition positive definite weighting matrices, respectively. Alternatively, the cost function J can also be written in another form,

$$J = \int_0^{t_f} \left[\mathbf{x}^T(t) \mathbf{Q}_x \mathbf{x}(t) + \mathbf{u}_\phi^T(t) \mathbf{R} \mathbf{u}_\phi(t) \right] dt + \mathbf{x}^T(t_f) \mathbf{S}_x \mathbf{x}(t_f), \quad (8.38)$$

where $\mathbf{Q}_x$ and $\mathbf{S}_x$ are the state variable and terminal state condition positive semi-definite weighting matrices. Equation (8.38) is the form of cost function generally considered in optimal control. However, that all depends on the way the designer chooses to model the system, i.e., the way he suitably chooses the state variables and the system outputs.

As noted by Inman (1989), modal control can be cast either in "state space" form or "physical space" form, i.e., in terms of the physical modes of the mechanical system. The present dissertation considers an approach to control system design where the dynamics of the mechanical system is considered in terms of its modal response. Hence, one can choose as a performance index the cost function minimizing the outputs of the system (e.g. the sensing voltages and/or their derivatives), Equation (8.37), or, alternatively, one can choose to minimize the state variables (modal amplitudes), Equation (8.38). By a convenient definition of the state weighting matrix $\mathbf{Q}_x$, the modal gain matrix can be "tuned" to give different design objectives. Assuming that all the modes (state variables) are observable and controllable, the cost function in Equation (8.38) provides independent control over the natural frequencies and damping ratios of each mode. That strategy is called *independent modal-space control*, or IMSC [Meirovitch (1990)]. Some convenient choices for the definition of the weighting matrices can be, for example,

$$\mathbf{R} = \text{diag}(\mathbf{I}, \mathbf{I}), \quad (8.39a)$$

$$\mathbf{Q}_y = \text{diag}(\mathbf{I}, \mathbf{I}), \quad (8.39b)$$

$$\mathbf{Q}_x = \text{diag}(\mathbf{\Omega}, \mathbf{0}), \quad (8.39c)$$

where an identity matrix $\mathbf{I}$ and diagonal matrix $\mathbf{\Omega}$ with the generic term ω_i^2 (squared natural frequency of the i th mode) of size $(r \times r)$ are utilized.

It can be seen in [Kwakernaak and Sivan (1972), Burl (1999)] that the feedback control system which minimizes the cost function in Equation (8.38) for the linear time-invariant system

defined in Equations (8.13) or (8.30), uses state feedback with a time-varying feedback gain matrix $\mathbf{K}_g^*(t)$, so that

$$\mathbf{u}_\phi(t) = -\mathbf{K}_g^*(t)\mathbf{x}(t). \quad (8.40)$$

The optimal time-varying feedback gain is given by $\mathbf{K}_g^* = \mathbf{R}^{-1}\mathbf{B}_\phi^T\mathbf{P}(t)$, where $\mathbf{P}(t)$ is the solution of the *matrix Riccati equation*,

$$\dot{\mathbf{P}}(t) = -\mathbf{P}(t)\mathbf{A} - \mathbf{A}^T\mathbf{P}(t) - \mathbf{Q}_x + \mathbf{P}(t)\mathbf{B}_\phi\mathbf{R}^{-1}\mathbf{B}_\phi^T\mathbf{P}(t). \quad (8.41)$$

This control philosophy is called *linear quadratic regulator* (LQR). That regulator requires the knowledge of all the optimal gain values $\mathbf{K}_g^*(t)$ in the time interval $[0, t_f]$. However, the feedback gains of the LQR usually approach steady-state values far from the final time which justifies the use of a steady-state LQR controller, considerably simplifying the controller design and the analog and digital implementation [Burl (1999)]. The steady-state feedback gain matrix is then given by

$$\mathbf{K}_g = \mathbf{R}^{-1}\mathbf{B}_\phi^T\mathbf{P}^\infty, \quad (8.42)$$

where $\mathbf{P}^\infty$ is the steady-state solution, i.e., the limiting value of $\mathbf{P}(t)$ when $t \rightarrow \infty$, of the matrix Riccati equation. Therefore, considering a steady-state feedback gain $\mathbf{K}_g$ in Equation (8.40), the closed-loop state equation is given by

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}_\phi\mathbf{K}_g)\mathbf{x}(t) + \mathbf{B}_u\mathbf{u}_u(t). \quad (8.43)$$

In the previous equation it was assumed that all the states were completely observable and therefore could be directly related to the outputs and used by the control system. However, that is not always the case and a more realistic approach would consider that only the outputs $\mathbf{y}(t)$ can be known and measured. In order to be able to use the states information in the control system, it will be necessary to estimate the states from a model of the system and a limited number of observations of the outputs. That estimation is made usually by a *state estimator* or *observer*. It is fed by the same known input signals as the mechanical system, $\mathbf{u}_\phi(t)$, and has its output $\hat{\mathbf{y}}(t)$ constantly compared with the output of the mechanical system, $\mathbf{y}(t)$.

However, the state variable estimates are very sensitive to any uncorrelated noise in the system, particularly measurement noise from the observed outputs of the mechanical system. Knowing the statistical properties of the various sources of noise in the mechanical system, and assuming white uncorrelated noise uniformly distributed in the bandwidth, an “optimal” state observer which minimizes the effects of plant and measurement noise is known as *Kalman-Bucy filter*.

Considering the generic form of an open-loop system as given in Equations (8.13) adjoined with plant and measurement noise, $\mathbf{w}(t)$ and $\mathbf{v}(t)$, yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}_\phi\mathbf{u}_\phi(t) + \mathbf{B}_u\mathbf{u}_u(t) + \mathbf{B}_w\mathbf{w}(t), \quad (8.44a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{v}(t). \quad (8.44b)$$

where $\mathbf{B}_w$ is a plant noise input matrix. The plant and measurement noise are both assumed to be white (stationary), have a Gaussian probability density function and are assumed uncorrelated with the inputs. The correlation properties of the plant and measurement noise vectors are given by the correlation matrices

$$E[\mathbf{B}_w\mathbf{w}(t)\mathbf{w}^T(t)\mathbf{B}_w^T] = \mathbf{W}, \quad (8.45a)$$

$$E[\mathbf{v}(t)\mathbf{v}^T(t)] = \mathbf{V}, \quad (8.45b)$$

where E denotes the expectation operator. Considering a state estimator with the same dynamics as the system under control, which is assumed to be known, and the same known inputs, the estimator states $\hat{\mathbf{x}}$ would thus be governed by the equation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}_\phi \mathbf{u}_\phi(t) + \mathbf{K}_e^*(t) [\mathbf{C}\mathbf{x} + \mathbf{v}(t) - \mathbf{C}\hat{\mathbf{x}}(t)], \quad (8.46)$$

where $\hat{\mathbf{x}}(0) = 0$ and $\mathbf{K}_e^*(t)$ is known as the *Kalman-Bucy gain matrix*. Defining the error between the true and estimated states as $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, the dynamic behavior of the Kalman-Bucy filter can now be expressed as a coupled set of first-order differential equations through the difference of Equations (8.44a) and (8.46),

$$\dot{\mathbf{e}}(t) = [\mathbf{A} - \mathbf{K}_e^*(t)\mathbf{C}] \mathbf{e}(t) + \mathbf{B}_u \mathbf{u}_u(t) + \mathbf{B}_w \mathbf{w}(t) - \mathbf{K}_e^*(t)\mathbf{v}(t). \quad (8.47)$$

The Kalman-Bucy gain matrix which leads to the optimal feedback controller is shown by [Burl \(1999\)](#) to be of the form $\mathbf{K}_e^*(t) = \mathbf{M}(t)\mathbf{C}^T\mathbf{V}^{-1}$, where $\mathbf{M}(t)$ is the correlation matrix of the estimation error which is the solution of another Riccati equation,

$$\dot{\mathbf{M}}(t) = \mathbf{M}(t)\mathbf{A}^T + \mathbf{A}\mathbf{M}(t) + \mathbf{W} - \mathbf{M}(t)\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{M}(t), \quad (8.48)$$

which is usually solved with the initial condition $\mathbf{M}(0) = E[\mathbf{x}(0)\mathbf{x}^T(0)]$. In a similar way to the steady-state LQR, the “optimal” Kalman-Bucy gain matrix $\mathbf{K}_e^*(t)$ in Equation (8.47) typically experiences a transient and then approaches steady-state as time increases from the initial time. In applications where the estimator is designed to operate for time periods that are long compared to the transient times of the Kalman-Bucy gains, it is reasonable to ignore the transient and exclusively use the steady-state gains $\mathbf{K}_e$ [[Burl \(1999\)](#)].

The equations for computing the Kalman-Bucy gain have a striking resemblance to the equations for computing the LQR gain. As pointed out by [Burl \(1999\)](#) the steady-state Kalman-Bucy filter problem is shown to be mathematically equivalent to the steady-state LQR problem when appropriate substitutions are made.

Combining the steady-state Kalman-Bucy filter with the steady state LQR, the inter-related dynamic system will take the form

$$\begin{Bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{Bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}_\phi \mathbf{K}_g & \mathbf{B}_\phi \mathbf{K}_g \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_e \mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{B}_u \\ \mathbf{B}_u \end{bmatrix} \mathbf{u}_u(t) + \begin{bmatrix} \mathbf{B}_w & \mathbf{0} \\ \mathbf{B}_w & -\mathbf{K}_e \end{bmatrix} \begin{Bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{Bmatrix}. \quad (8.49)$$

Because we must assume that the random perturbations (force disturbance or measurement noise) are Gaussian, this control philosophy is called *linear quadratic gaussian* (LQG) control. The dynamics of the coupled controller and observer system is determined by the eigenvalues of the square system matrix in Equation (8.49) and the closed-loop coupled system stability depends upon the two sub-systems. The error will asymptotically be stable provided the observer poles have negative real components collocated, in the complex plan, as far as possible from the system poles so that the observer error reduces more rapidly than the system response.

8.6.2 Feedforward Control

Feedforward control system design and implementation rely upon digital filtering fundamentals and adaptive filter theory, which can be found in reference textbooks such as [[Widrow and Stearns \(1985\)](#), [Elliott \(2001\)](#), [Haykin \(2001\)](#)]. Contrarily to *feedback* systems, *feedforward* control assumes that one has information about the original (primary) excitation of the system and

that a reference signal correlated with the excitation is available. The aims are then to produce a secondary excitation that will cancel the effects of the primary excitation at the chosen location(s).

The *filtered-reference LMS* algorithm is the most widely accepted feedforward control algorithm because of its ease of implementation and remarkable performance. It is assumed that a detection system, which produces a signal correlated with the primary disturbance, exists (or not, and in that case the true excitation signal is utilized), and produces a reference signal $r(k)$ correlated with the excitation. This signal is then adaptively filtered to generate the necessary control action, u_ϕ , to cancel the effect of the *primary excitation* (disturbance). A schematic of the discrete-time (digital) generalized plant for a feedforward controller is presented in Figure 8.2.

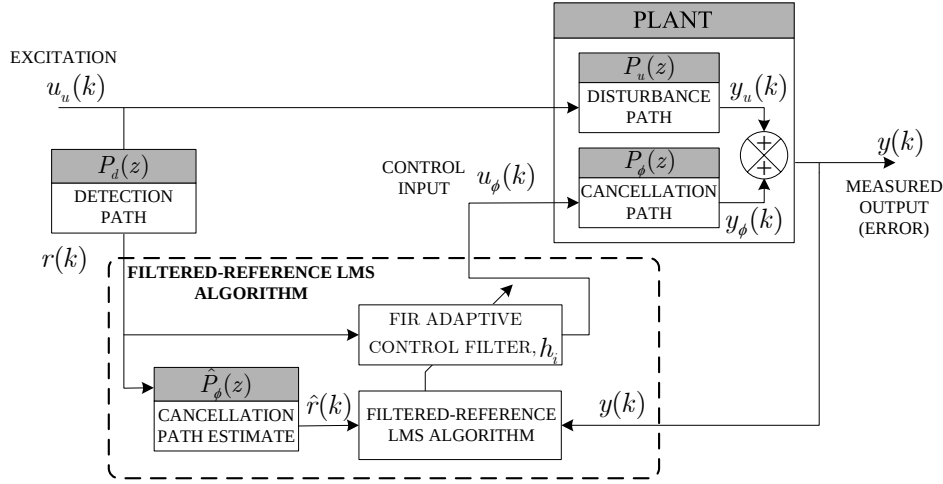


Figure 8.2: Schematic of the discrete-time (digital) generalized plant for a feedforward SISO controller.

The control filtering process utilizes an adaptive *finite impulse response* (FIR) filter whose i th coefficient at the k th sample time is $h_i(k)$. The filter output $u_\phi(k)$ is given by

$$u_\phi(k) = \sum_{i=0}^{N-1} h_i(k)r(k-i), \quad (8.50)$$

where N is the number of filter coefficients. Then, the control signal has to pass through a part of the physical system before the error sensor measures the output. This physical path, $P_\phi(z)$, is called the *cancellation path* (or *error path*). Therefore, the output of the plant due to the feedforward control input only, $y_\phi(k)$, is given by

$$y_\phi(k) = \sum_{j=0}^{M-1} g_j \sum_{i=0}^{N-1} h_i(k)r(k-i-j), \quad (8.51)$$

where g_j is the discrete impulse response of the control input-to-output path $P_\phi(z)$, which is assumed to be of order M . Thus, the net output of the system, $y(k)$, can be written as

$$y(k) = y_u(k) + \sum_{j=0}^{M-1} g_j \sum_{i=0}^{N-1} h_i(k)r(k-i-j), \quad (8.52)$$

where $y_u(k)$ is the response due to the effects of the primary disturbance alone. However, the order of convolution can be interchanged without changing the result, yielding

$$y(k) = y_u(k) + \sum_{i=0}^{N-1} h_i(k) \bar{r}(k-i), \quad (8.53)$$

where

$$\bar{r}(k-i) = \sum_{j=0}^{M-1} g_j r(k-i-j). \quad (8.54)$$

By rearranging the convolution, a signal $\bar{r}(k-i)$ is created, which is to be estimated by the filtered-reference operation. If the true impulse response of the plant $P_\phi(z)$ is estimated by a FIR or *infinite impulse response* (IIR) filter, then an estimate of $\bar{r}(k)$ is given as $\hat{r}(k)$.

The problem of how best to adapt the filter coefficients $h_i(k)$ can now be addressed. They can be adapted in order to minimize a *cost function* quadratically dependent of the output, $J = E[y^2(k)]$. However, as for the LMS algorithm [cf. [Widrow and Stearns \(1985\)](#)], the instantaneous value of $y^2(k)$ is used as an estimate of the expected value of J . A simple gradient descent algorithm is thus guaranteed to converge to the globally optimal solution of the problem of minimizing the cost function. Such an adaptive algorithm can be written as

$$h_i(k+1) = h_i(k) - \mu \frac{\partial J}{\partial h_i(k)}, \quad (8.55)$$

where μ is a convergence coefficient. Thus, from the definition of the cost function the derivative in Equation (8.55) is written as

$$\frac{\partial J}{\partial h_i(k)} = 2y(k) \frac{\partial y(k)}{\partial h_i(k)}. \quad (8.56)$$

From Equation (8.53) the derivative of $y(k)$ with respect to $h_i(k)$ is simply $\bar{r}(k-i)$. If one uses the estimated filtered reference signal $\hat{r}(k-i)$, the steepest descent algorithm required to adapt the coefficients of the digital controller, given by Equation (8.55), can thus be written as

$$h_i(k+1) = h_i(k) - \alpha y(k) \hat{r}(k-i), \quad (8.57)$$

where $\alpha = 2\mu$ is another convergence coefficient parameter that determines the speed and stability of adaptation. The convergence properties of the *filtered-reference LMS* algorithm are similar to those of the normal *LMS* algorithm, whose properties are described in detail for example by [Widrow and Stearns \(1985\)](#).

8.6.3 Hybrid (Feedback/Feedforward) Control

Hybrid (combined feedback/feedforward) control is defined as the combination of both *feedback* and *adaptive feedforward* control, as outlined by [Clark et al. \(1998\)](#). Therefore, the net control input u_ϕ is defined by the sum of both feedback, u_ϕ^{fb} , and feedforward, u_ϕ^{ff} , control inputs contributions. A schematic diagram of the discrete-time (digital) generalized plant for a SISO hybrid (combined feedback/ feedforward) control system is depicted in Figure 8.3.

The primary purpose of hybrid control systems is to design the disturbance control problem with the architecture best suited for overall performance. If a reverberant plant is exposed to exogenous inputs consisting of both stochastic and harmonic disturbances, for which a reference correlated with the harmonic disturbance is available, it is likely that feedback control

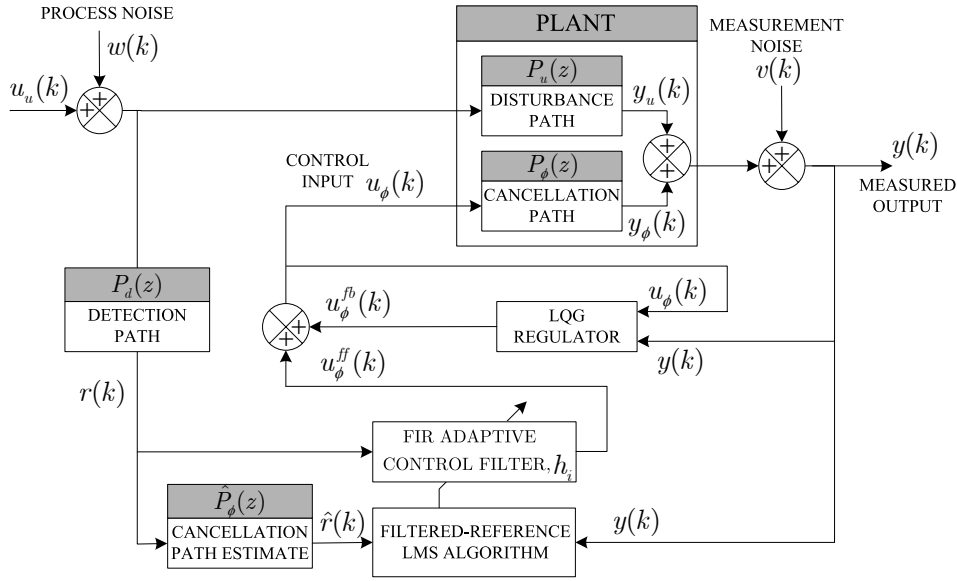


Figure 8.3: Schematic diagram of the discrete-time generalized plant for the hybrid (combined feedback/feedforward) SISO controller.

strategies are best suited for the stochastic disturbances, while adaptive feedforward control strategies are better suited for the harmonic disturbances. Hybrid control utilizes both forms of control to maximize the performance of the adaptive structure to mixed-signal disturbances. As discussed in Section 1.1.5, the essential features of hybrid control for adaptive structures are: robust disturbance rejection of narrowband, periodic signals; simultaneous transient suppression of impulsive and stochastic disturbances; improved performance and convergence of the adaptive feedforward controller.

8.7 Summary

This chapter has been devoted to mathematical modeling and theoretical control issues regarding the active and hybrid control of vibrations comprising both active and passive damping materials (hybrid damping) and a combination of feedback and adaptive feedforward control algorithms (hybrid control).

The chapter started with a brief review of hybrid damping and control technologies. Next, in order to optimize the design of arbitrary hybrid damping treatments, a discussion concerning their phenomenological behavior and the identification and quantification of the efficiency of the distinct underlying damping mechanisms was performed. After that, the state space representation of the FE governing equations of active and hybrid structural systems and a modal reduction procedure were presented. By last, both feedback and adaptive feedforward control strategies, used separately or combined, and the underlying theory was described.

The next chapter intends to establish the bridge between modeling/simulation and effective real-time control implementation in structural vibrations control applications of adaptive structural systems. Both passive, active and hybrid damping treatments and control configurations will be considered and analyzed from both numerical simulation and experimental implementation viewpoints.

CHAPTER 9

Application and Assessment of Vibration Control Strategies

9.1 Introduction

This chapter intends to establish the bridge between modeling/simulation and effective real-time control implementation in structural vibrations control applications of adaptive structures with damping treatments (damped structures). Both passive, active and hybrid damping treatments, comprising the use of active piezoelectric patches alone or passive viscoelastic materials in hybrid configurations with active piezoelectric or passive elastic constraining layers, will be considered and analyzed from both numerical simulation and experimental implementation viewpoints. As far as control strategies are concerned, the simulation and real-time implementation of feedback, adaptive feedforward and hybrid (combined feedback/feedforward) control strategies on the control of vibrations of adaptive damped beams with sensing and actuating elements will be performed and their damping performance, feasibility, robustness, limitations and capabilities will be assessed. The FE implementation, numerical calculations and real-time simulation were performed in MATLAB and SIMULINK working and control environments.

With that in mind the chapter is organized in two sections: the first addresses the use of active damping treatments, which consider only piezoelectric patches as distributed sensing and actuating devices; the second one considers hybrid damping treatments where passive viscoelastic damping is also incorporated into the treatment. The analyzed case studies address the application of these damping treatments to beams and for that purpose the mathematical models developed in the previous chapters for piezo-visco-elastic beams are utilized. Several stacking arrangements and configurations of piezoelectric, viscoelastic and elastic layers, and feedback control algorithms, namely, classical (velocity feedback) and optimal (LQR and LQG) control strategies, and an adaptive feedforward technique, the so-called filtered-reference LMS algorithm, will be considered. The different configurations and algorithms will be compared and their main features discussed.

9.2 Active Damping Treatments

9.2.1 Control Through Piezoelectric Actuation and Laser Vibrometer Sensing

9.2.1.1 Introduction

This case study concerns the use of a SISO control system used to attenuate vibrations in a beam with a pair of symmetrically collocated piezoelectric patches being used as actuators and a laser vibrometer being used to measure the velocity of a point of the beam. With that purpose, the piezo-elastic beam FE model presented in Chapter 3 is utilized here to design and simulate the system and the model and design procedure are also assessed via comparison of the simulation and experimental results. First, the simultaneous quasi static action of the piezoelectric actuators is considered. Then, the free and forced dynamic responses of the beam with the vibration control system being used are presented. The results presented here can also be found in a publication of the author in article format in [Vasques and Rodrigues (2007a)].

9.2.1.2 Experimental Setup

The test rig comprises a cantilever aluminum beam, 400 mm long, 15 mm wide and 2 mm thick, with two piezoelectric patches (Philips Components - PLT 30/15/1-PX5-N)[Waanders (1991)], 30 mm long, 15 mm wide and 1 mm thick, which are bonded 5 mm away from the clamped edge (Figure 9.1).

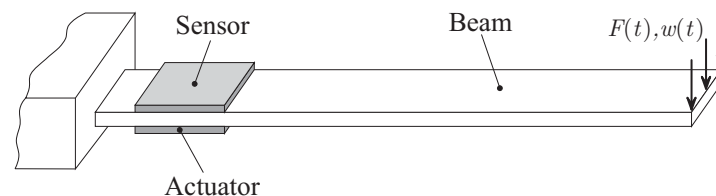


Figure 9.1: Cantilever smart piezoelectric beam.

The mechanical and electrical material properties of the aluminum beam and piezoelectric patches (PXE-5) are defined in Table 9.1 [see the standard IEEE (1988) for details about notation]. The modal damping ratios of the smart beam presented in Table 9.2 were identified experimentally from the open-loop FRFs, and are used in the numeric model as input parameters into the adopted proportional viscous damping model.

The clamped smart beam was mounted in a Newport vibration isolation table (Figure 9.2) and the boundary conditions were materialized by means of a rigid support that inhibits the translation and rotation of the clamped extremity of the beam. Regarding the piezoelectric ceramic patches (Figure 9.3), they were mounted on the beam by means of a nonconductive cyanoacrylate adhesive (Loctite - 496) with high rigidity.

Table 9.1: Material properties of the aluminium and PXE-5.

Aluminium			PXE-5		
E	$70 \times 10^9 \text{ Pa}$	c_{11}^E	$131.1 \times 10^9 \text{ Pa}$	d_{31}	$-215 \times 10^{-12} \text{ m V}^{-1}$
ν	0.3	c_{12}^E	$7.984 \times 10^9 \text{ Pa}$	d_{33}	$500 \times 10^{-12} \text{ m V}^{-1}$
ρ	2710 kg m^{-3}	c_{13}^E	$8.439 \times 10^9 \text{ Pa}$	d_{15}	$515 \times 10^{-12} \text{ m V}^{-1}$
		c_{33}^E	$12.31 \times 10^9 \text{ Pa}$	$\varepsilon_{11}^T/\varepsilon_0$	1800
		c_{44}^E	$2.564 \times 10^9 \text{ Pa}$	$\varepsilon_{33}^T/\varepsilon_0$	2100
		c_{66}^E	$2.564 \times 10^9 \text{ Pa}$	ρ	7800 kg m^{-3}

Table 9.2: Experimental modal damping ratios of the open-loop system [%].

Mode	1st	2nd	3rd	4th
ξ	1.71	0.72	0.42	0.41

9.2.1.3 Quasi-static Analysis

For the application of the electric voltage to the piezoelectric actuators two power sources were used with a maximum voltage of 300 V (DC). The tip displacement was measured by means of a Phillips - PR6423 capacitive displacement transducer with a measurement range equal to 1 mm and a resolution of $1 \mu\text{m}$ (Figure 9.4).

In order to determine the sensibility of the capacitive transducer for the aluminum, an experimental calibration of the transducer was previously performed, and the sensibility was found to be equal to 13.62 V/mm. For the acquisition of the values supplied by the displacement transducer, proportional to the measured displacement, a digital multimeter (APPA - 69) with a resolution of 1 mV was used.

Results: Quasi-Static Actuation The tip displacement of the clamped smart beam under the simultaneous quasi-static actuation of the two piezoelectric actuators is presented in Figure 9.5. The displacement was measured by the capacitive transducer 5 mm away from the free end for different values of voltage applied simultaneously into the two piezoelectric actuators. In Figure 9.5 it is shown that a good match between the measured and predicted results is obtained, and the static actuation of the proposed FE model is experimentally validated. The maximum deviation between the measured and predicted displacements is 4.1 % for 250 V. Furthermore, it is worthy to note that the measured results underestimate the predicted ones. This fact is due to the bonding of the piezoelectric actuators, where a perfect transmissibility of the efforts is usually no completely assured. Thus, the procedure of bonding the piezoelectric ceramic patches is a very important step when dealing with this type of sensors and actuators.

9.2.1.4 Vibration Analysis

In order to experimentally implement the proposed velocity feedback control scheme a laser Doppler vibrometer Polytec - OFV 303 was utilized to measure the velocity at one point in the beam. For the data acquisition a dynamic signal analyzer Brüel & Kjær - 2035 with two channels and a generator module was utilized. The experimental setup and utilized equipment are presented in Figures 9.6 and 9.7, respectively. The free-vibration excitation is considered

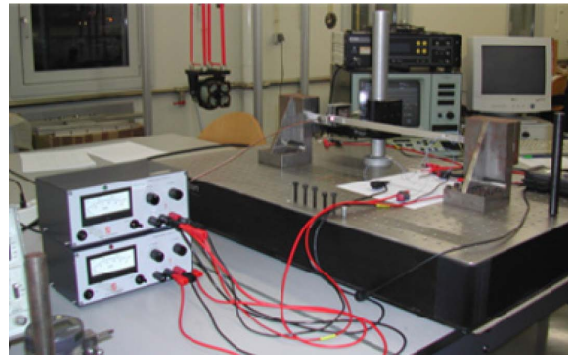


Figure 9.2: Experimental setup of the smart beam when utilized in the analysis of the static action of the piezoelectric patches.

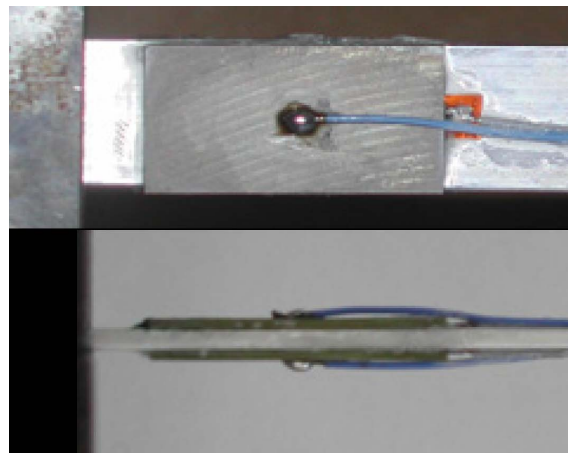


Figure 9.3: Mounting of the piezoelectric ceramic patches: top and side views.

through an initial displacement applied in the free end of the beam. Regarding the forced harmonic vibration, it is considered through a voltage that is applied to one of the two piezoelectric ceramic patches (exciter). The excitation signal is generated by the signal generator of the dynamic analyzer, amplified and applied to the actuator. A schematic of the velocity feedback control system is depicted in Figure 9.8. The tip displacement was measured by the laser vibrometer at 15 mm from the free end. To implement the velocity feedback control system, the signal (voltage) produced by the laser transducer (proportional to the velocity) was used as feedback control signal and applied to the second piezoelectric ceramic patch (actuator). To that end, the signal produced by the laser transducer is conditioned by a Krohn Hite - 3550 filter and by an operational amplifier with variable gain, with a maximum gain of 101 and a maximum output voltage of ± 150 V. A band-pass filter regulated to the first natural mode of vibration revealed to be necessary to guarantee the system stability (further details will be given in the next sections).

Root-Locus Analysis The developed FE model is utilized here to predict the root-locus of the closed-loop system with velocity feedback for values of gain comprised in the intervals $[-\infty, 0]$ and $[0, +\infty]$, as presented in Figures 9.9 and 9.10, respectively. From the analysis of the root locus one can observe that an increase in the control gain displaces the closed loop poles ($\times$) towards the zeros ($\circ$). That displacement, in case it moves the poles away from the

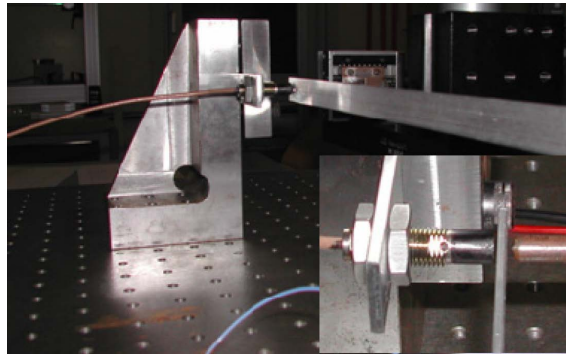


Figure 9.4: Displacement measurement with a Phillips - PR6423 capacitive transducer.

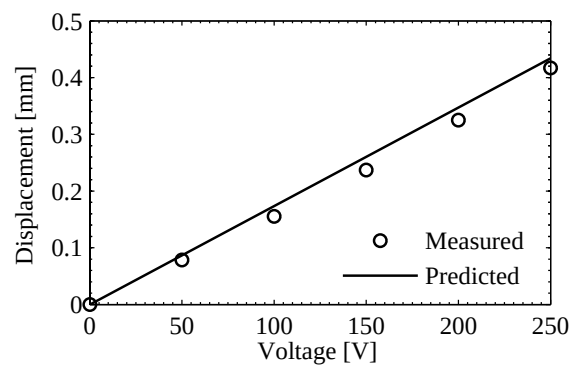


Figure 9.5: Tip displacement versus applied voltage results.

imaginary axis, $\text{Re}(s) = 0$, and provided they are located in the left half-plane, represents an increase of the modal damping ratio ξ , which for the case where $\xi \ll 1$ is approximately given by $\xi = -\text{Re}(s) / |\text{Im}(s)|$.

Furthermore, from the analysis of Figures 9.9 and 9.10 one verifies that depending on how the polarity of the gain is chosen, namely positive or negative gains, some kind of opposite (or symmetric) behavior in the locus of the closed-loop poles is obtained. Thus, for positive values of gain, the even numbered modes are easily excited even when the gain is too low. On the contrary, for negative values of gain, an opposite behavior is somehow obtained and the odd numbered modes become the easily excited ones. Therefore, in order to damp out the first mode (fundamental mode), the value of the gain, that affects all modes, would have to be limited to the maximum possible and acceptable value without the second and fourth modes (even modes) being excited. However, in these conditions, the control system would be very inefficient. An alternative would be to consider a feedback control voltage with only the contribution of the first mode velocity. That can be achieved by the use of a band pass filter tuned to the first natural mode of vibration. That would allow the gain to be chosen freely without the risk of exciting and destabilizing the even modes, just limited by the saturation voltage of the piezoelectric actuators and the physical limitations of the utilized equipment. The correspondent root locus diagram of the closed-loop band-pass filtered system is presented in Figure 9.11.

Results: Forced Harmonic Vibration The open loop FRF, velocity (measured 15 mm from the free end) per unit voltage applied to one of the piezoelectric patches, is presented in Figure

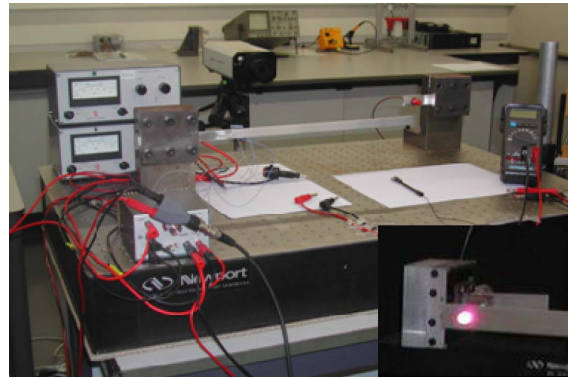


Figure 9.6: Experimental setup of the smart beam utilized in the active vibration control scheme and measurement point of the laser vibrometer.



Figure 9.7: Equipment utilized: (a) operational amplifier; (b) dynamic signal analyzer Brüel & Kjær - 2035 and laser vibrometer controller Polytec - OFV 3001; (c) Digital oscilloscope LeCroy - 9450A and filter Krohn-Hite - 3550; (d) laser vibrometer measurement unit Polytec - OFV 303.

9.12. A good agreement between the measured and predicted results is found. The resonance frequencies of the first two modes present reduced deviations and a maximum deviation of 4.75 % is found in the fourth mode (see Table 9.3). It is worthy to note that the predicted results overestimate the effective values of the resonance frequencies. That fact is due, once again, to the bonding of the piezoelectric ceramic patches. Given the difficulty in guaranteeing the complete transmissibility of the efforts on the interfaces with the beam, it creates a decrease in the stiffness of the structure, maintaining, however, the mass unaltered. Therefore, the measured resonance frequencies take lower values than the predicted ones. The coherence of the open-loop FRF is presented in Figure 9.13. The coherence evidences a good efficiency of the piezoelectric actuator in exciting of the structure, demonstrating the causality relationship between the response and the excitation, and reveals a high signal to noise ratio.

The measured and predicted FRFs of the band-pass filtered closed-loop system obtained for different values of gain are presented in Figure 9.14. Both results, namely measured and predicted, demonstrate that for increasing values of gain only the first mode is affected with its amplitude being decreased and the damping being increased. The agreement between results is quite good and the amplitude of the first mode in open-loop (Figure 9.12), with an aver-

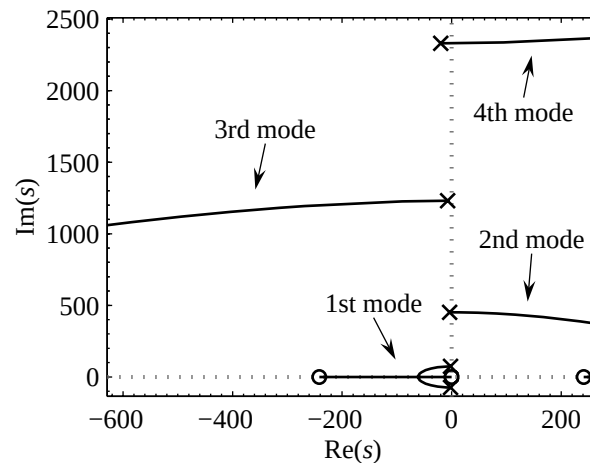


Figure 9.10: Root-locus of the closed-loop system for feedback gains comprised in $[0, +\infty]$.

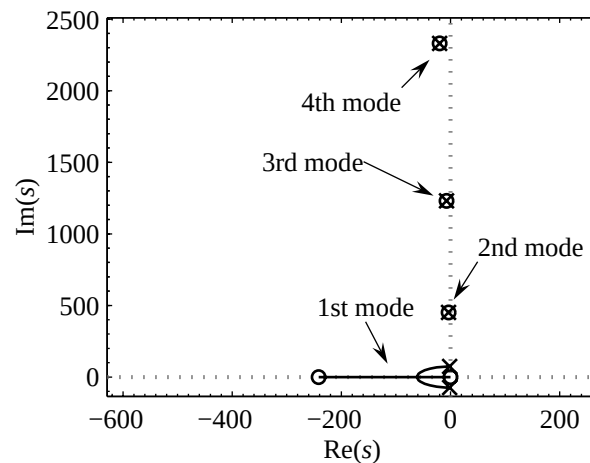


Figure 9.11: Root-locus of the closed-loop band-pass filtered system for feedback gains comprised in $[0, +\infty]$.

which is mandatory for a good damping estimation with the half-power bandwidth method.

As previously referred, in order to optimize the attenuation (or modal damping ratio) of the first mode without exciting and destabilizing the others, a band-pass filter is applied to the measured velocity signal utilized in the feedback loop. In Figures 9.16 and 9.17 the FRFs of the closed-loop system, for different values of gain, without recurring to the band pass filter are presented.

The measured and predicted unfiltered FRFs demonstrate that for increasing positive values of gain the even modes (2nd and 4th) are excited and the odd ones (1st and 3rd) damped out (see the graphics on the left of Figures 9.16 and 9.17). However, when the gains are negative the opposite behavior is observed (see the graphics on the right of Figures 9.16 and 9.17), confirming the indications given by the root locus of the unfiltered closed loop system (Figures 9.9 and 9.10).

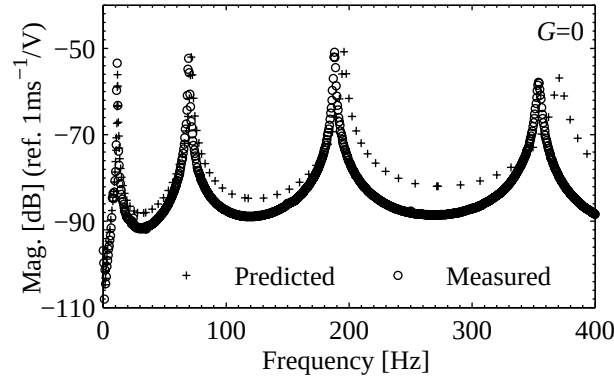


Figure 9.12: Open-loop FRF.

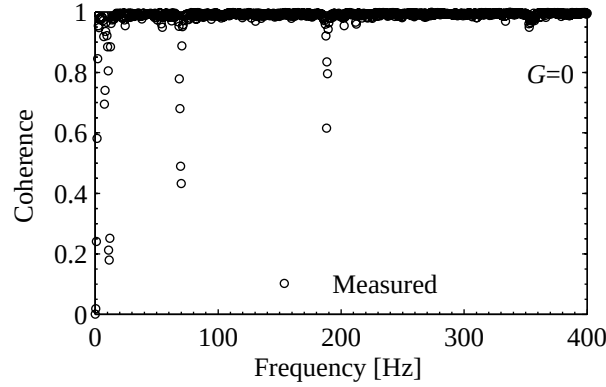
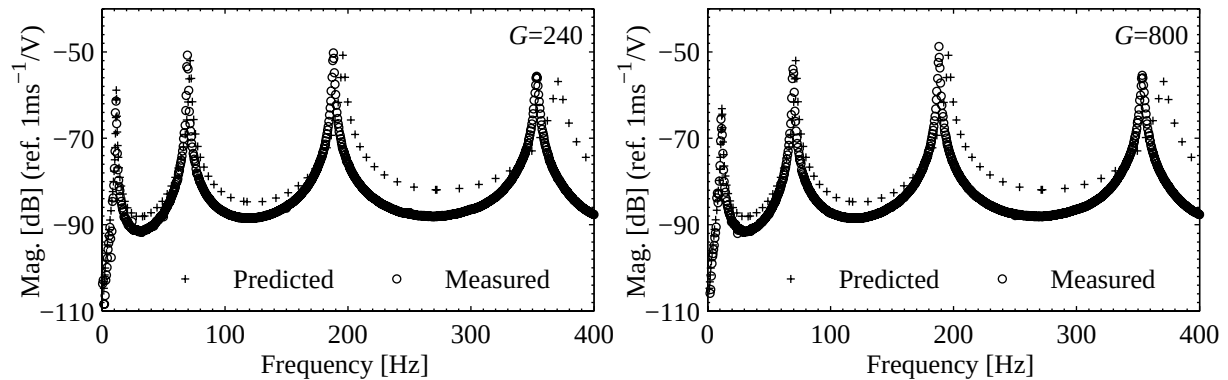


Figure 9.13: Coherence of the open-loop measured FRF.

Results: Free Vibration The free vibration response (velocity) of the smart beam subjected to an initial displacement field of 5 mm at the measurement point of the laser, which is located 15 mm away from the free end, are presented in Figures 9.18 and 9.19. The measured and predicted open-loop responses are presented in Figure 9.18 and a good agreement between the results is verified. For the same reasons referred in the previous section, a band pass filter tuned to the first mode of vibration (10 – 12 Hz) is considered. The measured and predicted closed-loop responses, obtained for different values of gain, and the predicted control voltages utilized are presented in Figure 9.19. For the different values of gain one can observe that a small difference between the measured and predicted results is obtained. That small difference is shown to be independent of the control gain and is probably due to the band pass filter implementation. In the experimental model the efficiency of the filter is not perfect and the harmonic components outside the bandwidth of interest (10 – 12 Hz) are not completely eliminated. However, the numerical model considers an ideal filter. Thus, the numerical model considers optimal operational conditions which are difficult to reproduce experimentally. Despite all of that, for the same values of control gain, both results show similar behaviors. From the results one can see that the control voltage, proportional to the measured velocity, increases for higher values of gain. In the experimental setup, the maximum control voltage, limited to the maximum admissible voltage of the amplifier (± 150 V), was regulated for ± 120 V. However, for values of gain lower than 300, the control voltage does not go above the regulated value. However, for higher values of gain, the numerical model, without limitation of control

Table 9.3: Resonance frequencies of the open-loop system [Hz].

Mode	1st	2nd	3rd	4th
Measured	11.50	69.50	188.0	354.0
Predicted	11.71	71.75	195.9	370.8
Deviation [%]	1.83	3.24	4.20	4.75

**Figure 9.14:** Closed-loop FRFs of the filtered system.

voltage, overestimates the attenuation of the response.

The damping ratios of the measured and predicted free responses obtained by the *logarithmic decrement* method [Meirovitch (1986)], assuming that the response is mainly governed by the response of the first mode, which is the most excited one for the initial displacement field considered, are presented on the left-hand side of Figure 9.20 for different values of control gain. For the same reasons previously referred, on the one hand the limitation of the control voltage and on the other hand the implementation of the filter, in free vibration the numeric simulation overestimates the measured values of damping ratio. In this case, the time resolution (2048 lines) is not as important as it was before in the FRFs and the results are more influenced by the experimental implementation of the band pass filter. In fact, a bandwidth of 100 Hz was considered in experimental measurements, which is assumed to have enough resolution for the correct measurement of the time responses. Furthermore, the linear trend of the predicted response was approximately obtained with the measured ones (see Figure 9.20). The 2% settling time of the measured and predicted results for different values of control gain is presented in the right-hand side of Figure 9.20. For the same reasons that influenced the value of the damping ratios, the predicted settling time results underestimate the measured ones. However, once again, the same type of behavior is obtained.

In order to demonstrate the capability of the FE model to predict instabilities of the control system, the closed loop free vibration responses without filtering the velocity were also measured and predicted for different values of control gain (Figure 9.21). As can be seen, for $G = 200$, the response is lightly damped in the beginning and after a while is destabilized. Since the applied initial displacement field excites mainly the first mode, the response in the beginning represents mainly the contribution of the first mode. Thus, at the same time that the first mode is damped, the second mode, with a lower gain margin (see Figure 9.10), starts to be excited and later on is destabilized until it starts to resonate. In a similar manner, for higher values of gain, the first mode is attenuated more quickly, but, in compensation, the second

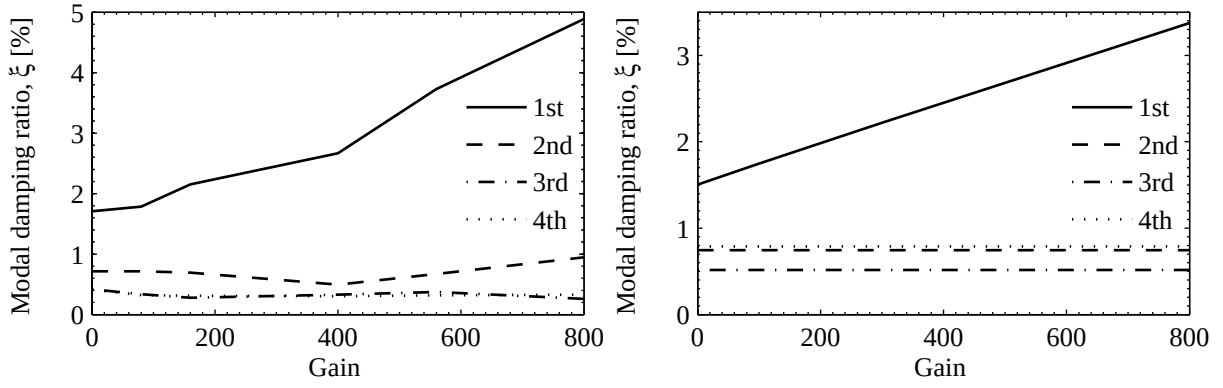


Figure 9.15: Modal damping ratios of the measured (left) and predicted (right) results.

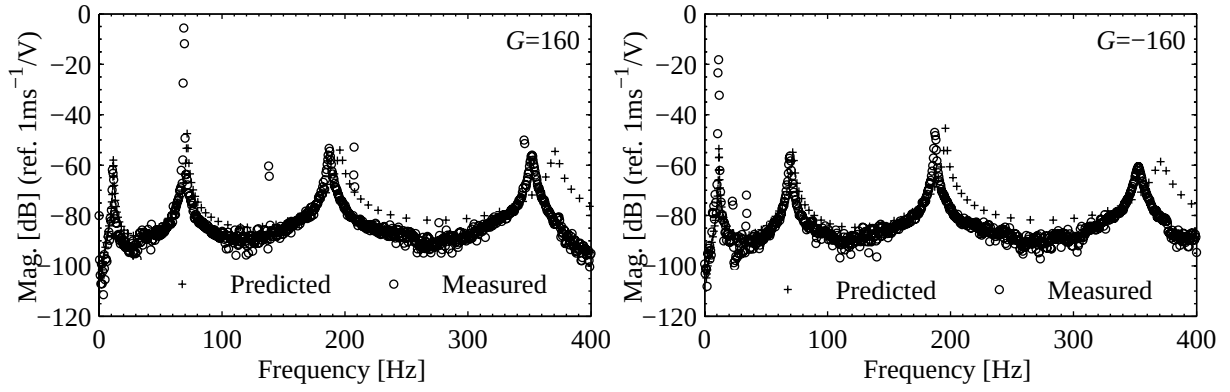


Figure 9.16: Closed-loop FRF of the unfiltered system with positive (left) and negative (right) gain.

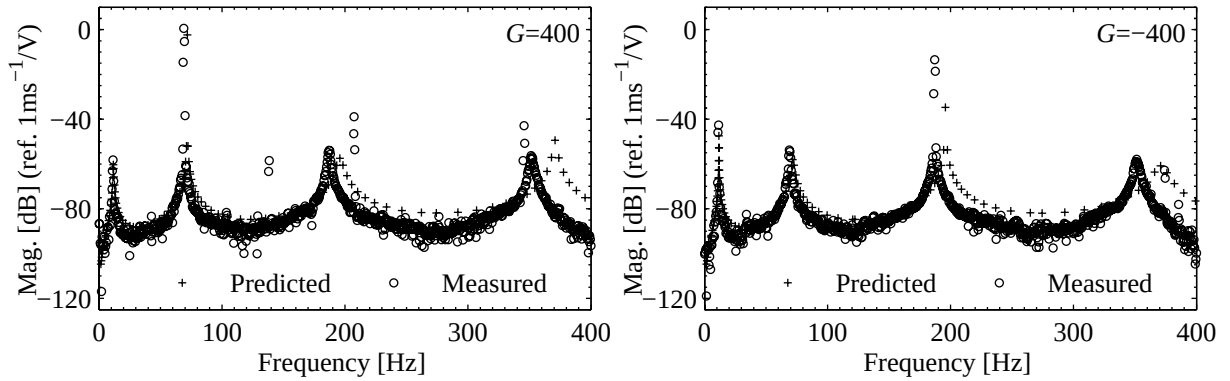


Figure 9.17: Closed-loop FRF of the unfiltered system with positive (left) and negative (right) gain.

mode (or the even modes) is also destabilized more quickly. Furthermore, due to the fact that the control voltage cannot be increased indefinitely (is limited to ± 120 V), note in Figure 9.21 that the measured response for $G = 240$, contrarily to the predicted response, stabilizes with an amplitude of about 0.75 ms^{-1} .

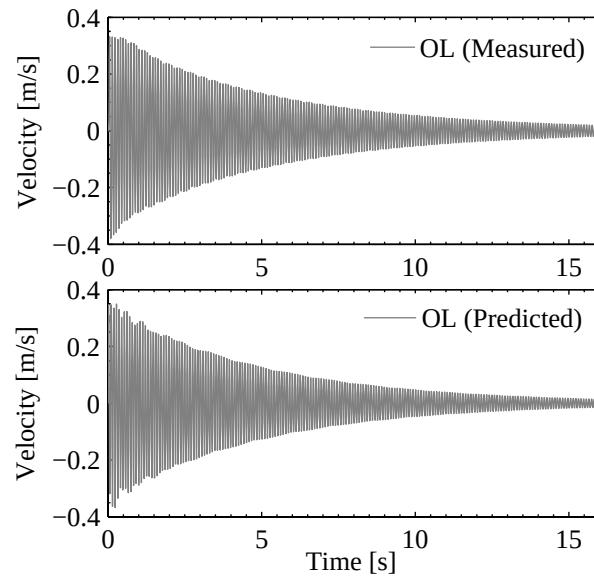


Figure 9.18: Free vibration response (velocity) of the open-loop (OL) system.

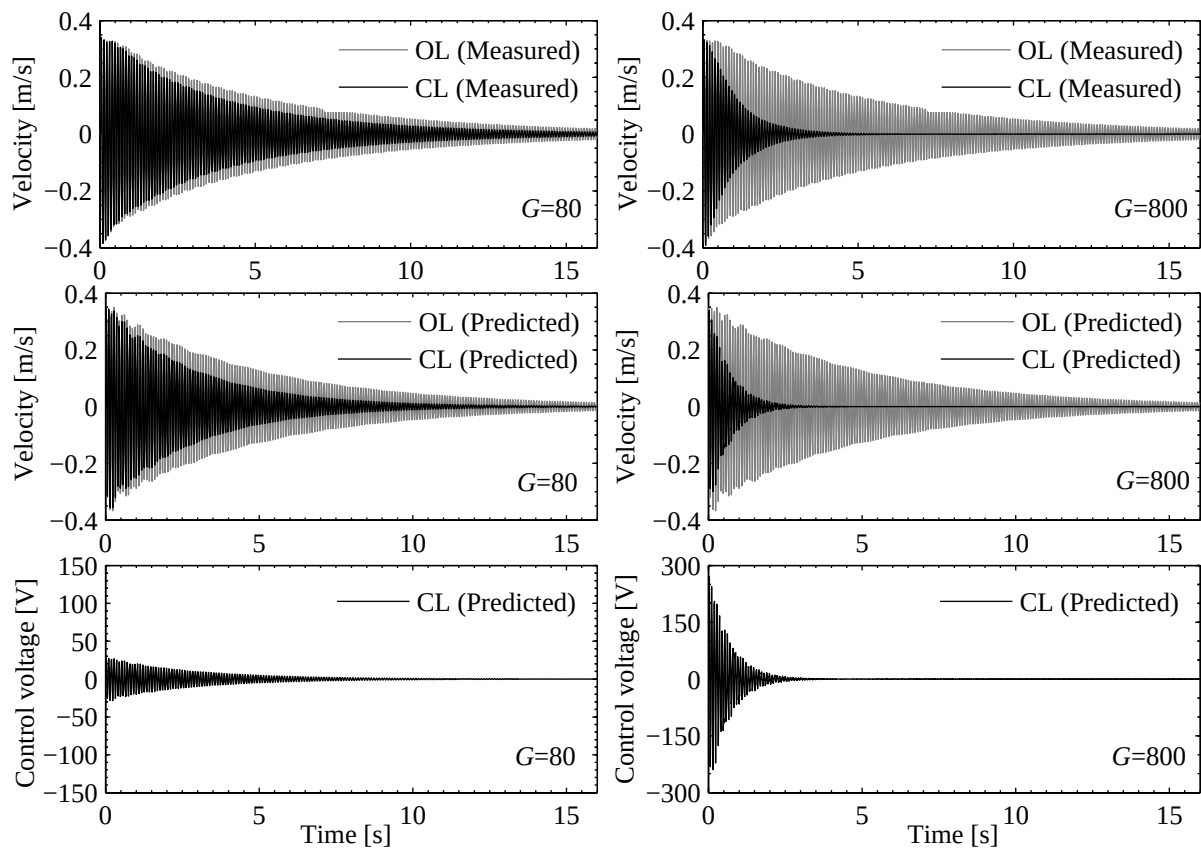


Figure 9.19: Free vibration response of the open- (OL) and closed-loop (CL) systems and control voltage with filtered velocity feedback control.

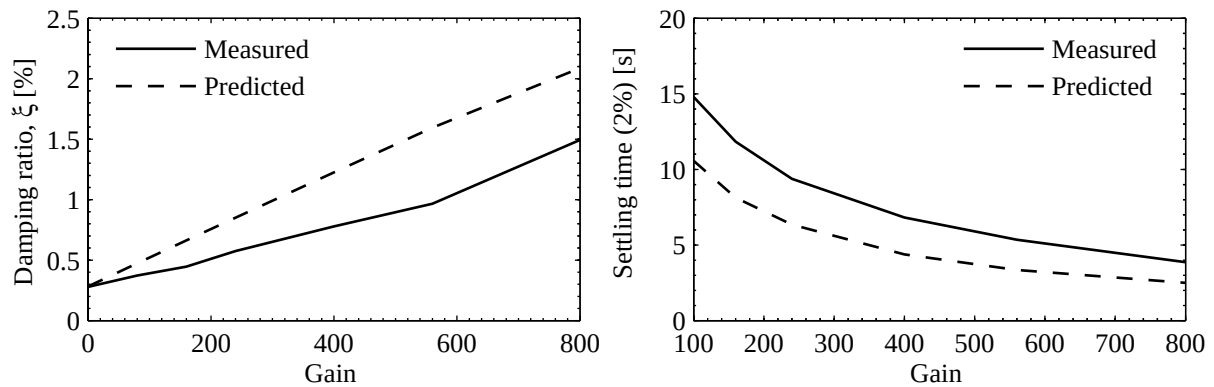


Figure 9.20: Damping ratio and 2 % settling time of the free vibration response.

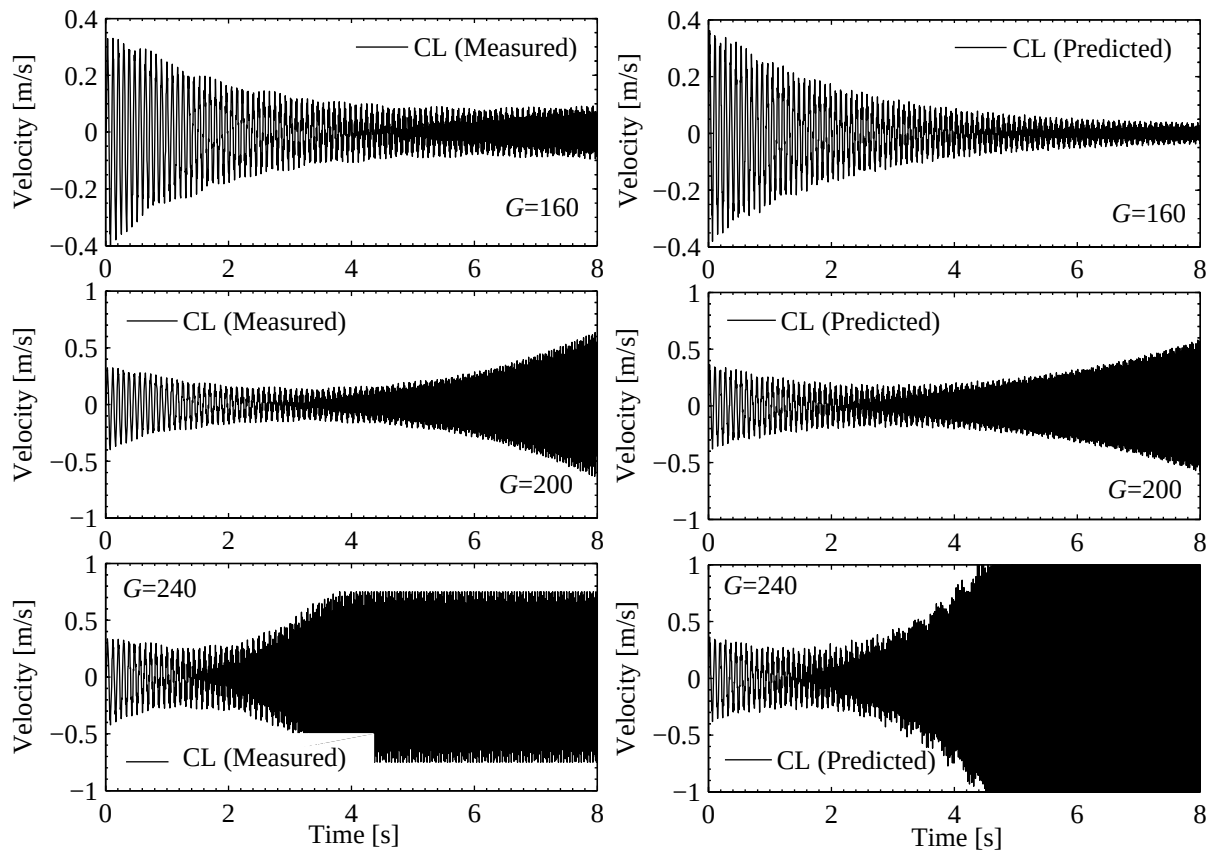


Figure 9.21: Measured (left) and predicted (right) free vibration response of the unfiltered closed-loop (CL) system.

9.2.1.5 Overview and Lessons Learned

The simulation and experimental implementation of an active vibration control system based on the feedback of the velocity measured by a laser Doppler vibrometer was described in this section. The system was applied to a smart beam with piezoelectric ceramic patches and was utilized to attenuate both the free and forced harmonic vibrations. For the simulation a previously developed FE was utilized for an adequate design of the derivative control gains and to

predict and avoid instabilities of the control system.

From the experimental implementation it was demonstrated that the mounting (bonding) of the piezoelectric ceramic patches may reduce the actuating efficiency since a perfect transmissibility of the efforts is usually difficult to obtain. Therefore, some care must be taken with that task. The quasi-static analysis allowed demonstrating the efficiency of the bonding and a good agreement between the measured and predicted results was obtained. The utilized velocity feedback control scheme, using a laser Doppler vibrometer as sensor to measure the velocity, was shown to be efficient in the vibration control of the first (fundamental) mode. A considerable increase of the damping ratio was obtained both in free and forced vibration conditions. However, the predicted and measured results showed that high control gains destabilize some of the modes while damping out others. To avoid this phenomenon a band-pass filter tuned to the first natural mode was utilized. Furthermore, a digital or analog bank of band pass filters could also be used with independent gains for each mode, which would allow to independently damp out each individual mode, however with the drawback of increasing the controller complexity.

The developed FE model demonstrated, through the comparison of measured and predicted results, a high representativeness of the structural system and was shown to be a valuable tool to simulate, design and predict eventual instabilities of vibration control systems. As a result, the experimental implementation is facilitated and the performance of the controller is optimized.

9.2.2 Comparison of Classical and Optimal Feedback Control Strategies

9.2.2.1 Introduction

In this section the piezo-elastic beam FE model is utilized in the evaluation of the classical and optimal active vibration control of a cantilever aluminum beam with an asymmetric collocated piezoelectric sensor/actuator pair mounted in the surface (Figure 9.1). A comparison between the classical control strategies, constant gain and amplitude velocity feedback (CGVF and CAVF), and optimal control strategies, LQR and LQG controller, is performed in order to investigate their effectiveness to suppress vibrations in beams with piezoelectric patches acting as sensors or actuators. With that in mind, the transverse displacement time history, for an initial displacement field and white noise force disturbance, and point receptance at the free end are evaluated with the open- and closed-loop classical and optimal control systems. The case studies allow the comparison of their performances demonstrating some of their advantages and disadvantages. It is also worthy to mention that the results presented here can also be found in a publication of the author in article format in [Vasques and Dias Rodrigues (2006a)].

The cantilever beam is the same utilized in Section 9.2.1 and is 400 mm long, 2 mm thick and 15 mm wide, and the two piezoelectric patches, Philips Components - PLT 30/15/1-PX5-N [Waanders (1991)], are 30 mm long, 1 mm thick and 15 mm wide, and were mounted at 5 mm from the clamped edge. The mechanical and electrical material properties of the aluminum beam and piezoelectric patches, PXE-5, are presented in Table 9.1. In the analysis a truncated modal model with the first four flexural modes was considered. The correspondent modal damping ratios, as presented in Table 9.2, were determined experimentally and their values are as follows: 1.71%, 0.72%, 0.42% and 0.41%. Furthermore, the beam with the sensor/actuator pair was discretized into 160 elements of equal length and a shear correction factor equal to

5/6 was utilized. All the numerical calculations and FE implementation were performed in the MATLAB environment.

9.2.2.2 Initial Displacement Field

Consider an initial displacement field applied to the beam which is obtained by a mechanical force applied at the free tip that induces a tip displacement equal to 1.5 mm. The tip displacement time history and control voltage for the CGVF and CAVF control systems are presented in Figure 9.22. The control gain for the CGVF, $G = 0.4$, was chosen from a root locus analysis in order to increase the damping of the first mode as much as possible and, simultaneously, by a trial-and-error method in order to use control voltages within the desired range. According to the limit electric field strength of the piezoelectric patches (300 V mm^{-1}), the control voltage should be lower than 300 V. Exceeding that voltage may result in depolarization of the material so that the piezoelectric properties become less pronounced or disappear completely. For the CAVF, a constant amplitude with $A = 250 \text{ V}$ was chosen and the control voltage was turned on at $t = 0 \text{ s}$ and turned off at $t = 0.2 \text{ s}$. Furthermore, in the CAVF only the velocity polarity of the first-mode was considered in the feedback loop (perfect band-pass filter tuned to the first mode).

As can be seen in Figure 9.22, the CGVF control manages to significantly attenuate the free tip displacement with an admissible control voltage. The closed-loop 5% settling time is equal to 0.5 s, which reveals a great improvement on the response attenuation when compared with the open-loop one (2.3 s). Moreover, it presents a better performance than the CAVF (0.58 s). However, if we look at the 10% settling time, the CAVF control shows a faster attenuation capacity (0.24 s) than the CGVF (0.38 s).

In Figure 9.23 the LQR and LQG controllers with output weighting (sensor voltage and its derivative) and state weighting (modal amplitudes and modal velocities) are utilized. It is assumed in the state estimation that only the sensor voltage, and not its derivative, is measured. That assumption corresponds to a more realistic approach and avoids the necessity of differentiating the sensor signal. For the output weighting case, the weighting matrices $\mathbf{Q}_y = \text{diag}(1, 1)$ and $R = 15$ were utilized, and for the state weighting case, the matrix $\mathbf{Q}_x = 1 \times 10^{10} \text{diag}(\omega_1^2, 0, \dots, 0)$ was defined in order to mainly damp the first mode, and $R = 30$ was chosen. For the LQG controller design, a white noise mechanical disturbance applied in the free end of the beam is modeled as plant noise. Therefore, in order to define the correlation matrix in Equation (8.45b), the equality $\mathbf{B}_w = \mathbf{B}_u$ should be considered. A plant noise vector (force disturbance) with $E[\mathbf{w}(t)\mathbf{w}^T(t)] = 4 \times 10^{-4} \text{ N}^2$ and a sensor white noise disturbance with $E[\mathbf{v}(t)\mathbf{v}^T(t)] = 1 \text{ V}^2$ are considered for the definition of the noise correlation matrices and Kalman-Bucy gain design. However, the effects of the stochastic mechanical disturbance in the displacement time history will be considered only in the following sections. The results in Figures 9.23a and 9.23b show that the state estimator of the LQG controller with state weighting (only the first mode) has a better performance than the one with output weighting. Furthermore, in Figure 9.23a the displacements with the LQR and LQG have a significant difference in attenuation, with 5% settling times equal to 0.49 s and 0.89 s, and the control voltage of the LQG is too high with the risk of depolarizing the actuator. Moreover, the LQR and LQG with state weighting in Figure 9.23b have the same 5% settling time (0.34 s), with the LQG requiring a lower control voltage and the LQR requiring an inadmissible control voltage. The LQG with state weighting, which corresponds to the realistic approach, when compared with the velocity feedback results presents a better performance.

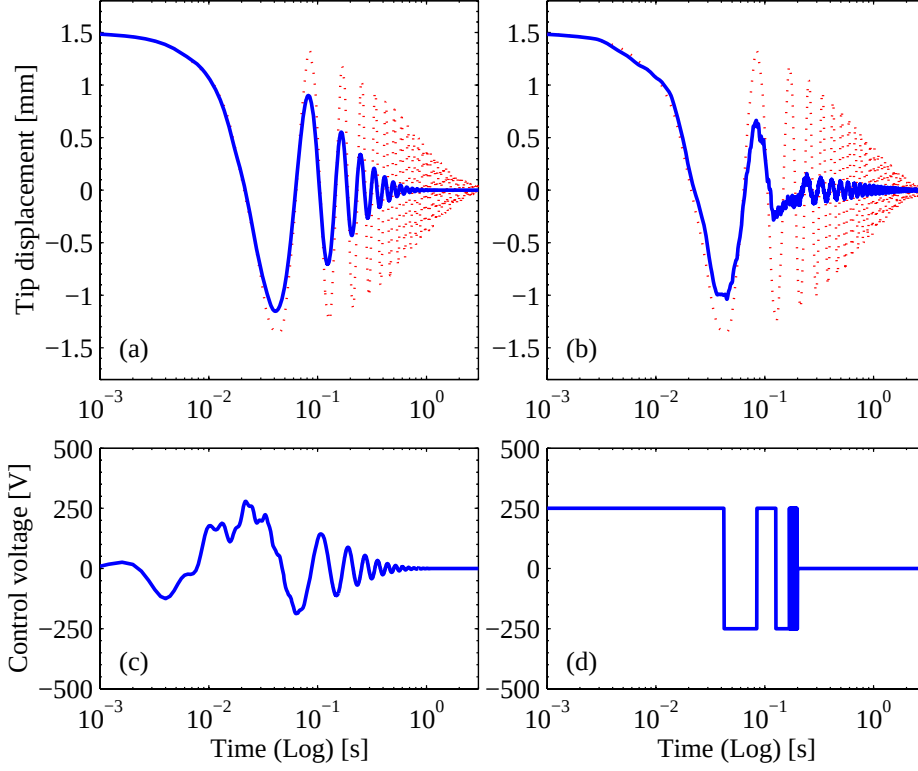


Figure 9.22: Tip displacement and control voltage for an initial displacement field with CGVF (a, c) and CAVF (b, d): open-loop (dotted line); closed-loop (solid line).

9.2.2.3 White Noise Disturbance

As referred in the previous section, a white noise point force disturbance with variance equal to $4 \times 10^{-4} \text{ N}^2$ is applied in the free end of the beam. The gains, amplitude and control parameters for all the control strategies are the same that were utilized in the previous section and only the output and state weighting parameters are changed to $R = 1$ and $\mathbf{Q}_x = 1 \times 10^{10} \text{diag}(\omega_1^2, \omega_2^2, \omega_3^2, \omega_4^2, 0, \dots, 0)$. This choice for the state weighting matrix is made in order to distribute the vibration control effort in bandwidth. However, the analysis is limited to the modal model bandwidth (only the first four flexural modes are considered).

The open- and closed-loop tip displacements and control voltages for the white noise force disturbance evaluated with the CGVF and CAVF are presented in Figure 9.24. In Figure 9.25 the displacement and control voltages are evaluated with the LQR and LQG with output and state weighting. It can be seen that all the control systems manage to reduce the tip displacement significantly. When compared to the open-loop response standard deviation, which is equal to 0.57 mm, the closed-loop response standard deviations (with the respective control voltage standard deviations) for the CGVF and CAVF, 0.26 mm (79.6 V) and 0.20 mm (250 V), demonstrate the vibration control efficiency. Moreover, the LQR and LQG with output weighting present responses with standard deviations equal to 0.26 mm (97.4 V) and 0.28 mm (118 V), and the LQR and LQG with state weighting have their values equal to 0.16 mm (80 V) and 0.20 mm (72 V). The most effective control system is the LQR with state weighting. However, a state estimator is necessary, and the results achieved with the LQG (0.20 mm) are the realistic ones. Furthermore, they are equal to the ones obtained with the CAVF. However, if we look at the control voltages utilized, the most interesting one is the LQG with state weighting, with only

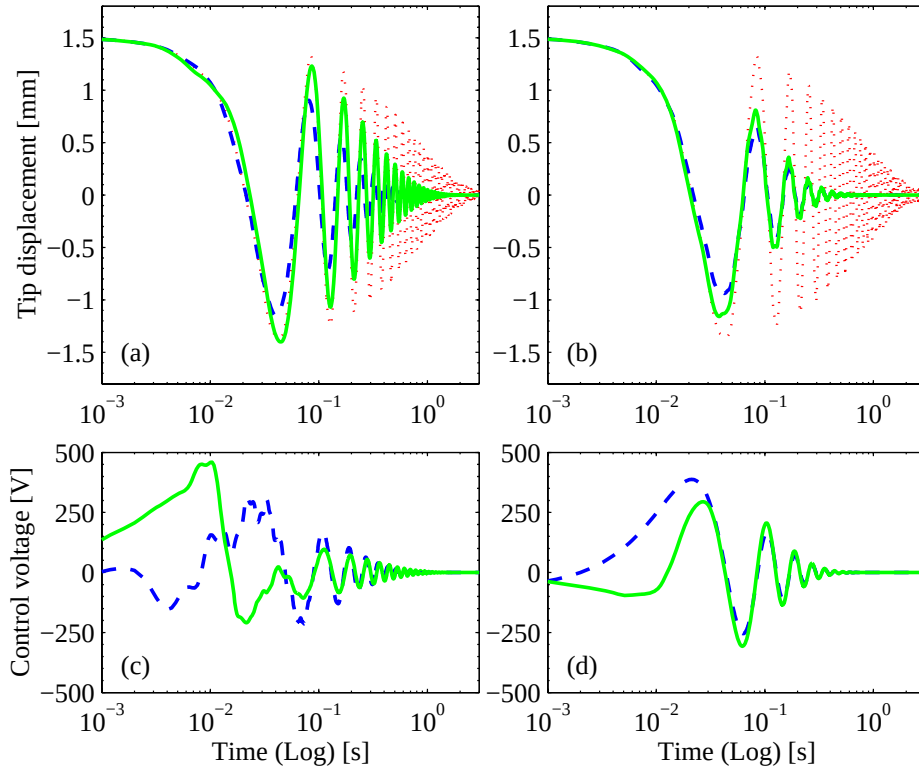


Figure 9.23: Tip displacement and control voltage for an initial displacement field with output y (a, c) and state x (b, d) weighting: open-loop (dotted line); closed-loop LQR (dashed line); closed-loop LQG (solid line).

72 V, and the most inefficient, with the higher control voltage (250 V), is the CAVF.

9.2.2.4 Frequency Response Function

In order to analyze the capacity of the various control strategies in the frequency domain, the point receptances at the free end of the beam evaluated with the open- and closed-loop control systems are presented. With that purpose, the displacement time histories and white noise mechanical disturbance of the previous section are utilized to estimate the frequency response functions by a signal processing analysis procedure through a fast Fourier transform (FFT) algorithm. The open- and closed-loop point receptances for the CGVF and CAVF control are presented in Figure 9.26.

The CAVF control significantly reduces the magnitude of the first resonance in approximately 27 dB, and increases the magnitudes of the other modes. In comparison, the CGVF does not manage to have such a good performance in the attenuation of the first mode, with a 8.6 dB reduction, but, in compensation, attenuation in all modes is achieved; the fourth mode is completely eliminated and the natural frequencies are shifted. Furthermore, a severe spillover effect is visible in Figure 9.26b above the first natural frequency. That is related with the fact that for the CAVF algorithm a band pass filter was considered and only the polarity of the first mode of the sensor voltage was considered in the closed-loop response to the white noise force disturbance. Therefore, since the control voltage is non-linear and discontinuous and a filter is being used for the sensor signal (the controller doesn't consider the dynamics of the higher modes), the effects of the control voltage originate a control spillover problem which leads to

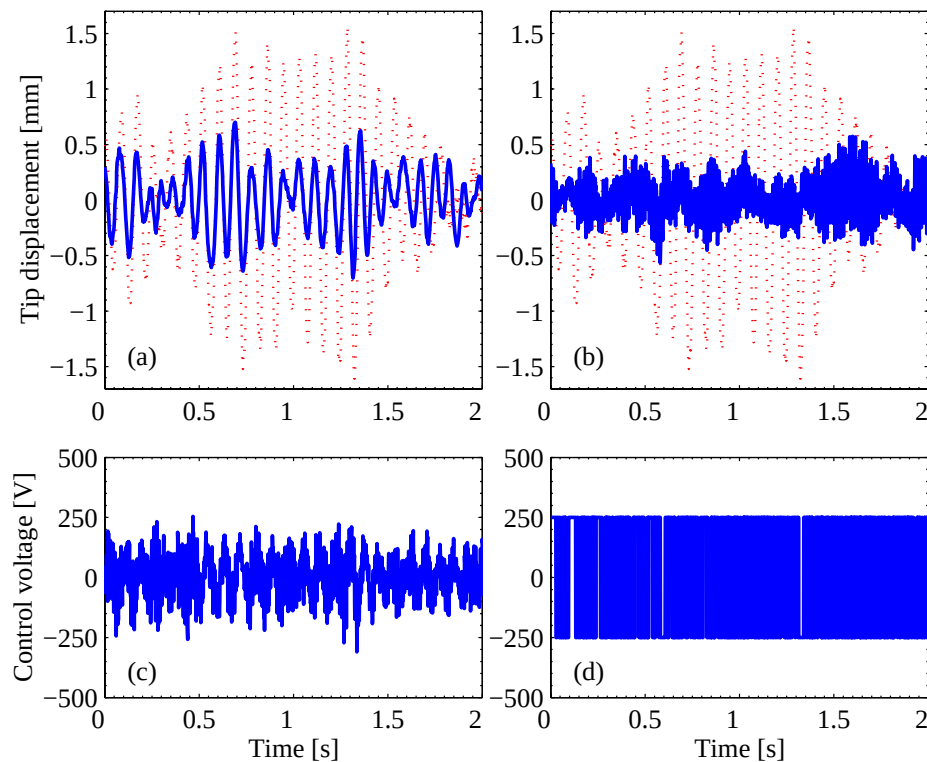


Figure 9.24: Tip displacement and control voltage for white noise point force disturbance with CGVF (a, c) and CAVF (b, d): open-loop (dotted line); closed-loop (solid line).

instabilities outside the bandwidth under control (first mode). Since a truncated modal model has been used, which can for instance be implemented by an adequate filter design, the problem of observation spillover might be prevented and is not at issue here.

In Figures 9.27 and 9.28 the point receptance for output y and state x weighting with the LQR and LQG controller are presented. The results show that the effects of the state estimation don't compromise the stability of the system, and that the output weighting LQG controller can have even a better performance than the LQR. A 8.8 dB and 9.4 dB attenuation of the first mode is achieved with the LQR and LQG controller with output weighting, respectively, and the resonant frequencies are shifted. If one now looks at the results with state weighting (Figure 9.28), one can see that only the peak amplitudes are affected by the state estimator, and the natural frequencies remain the same. Furthermore, all the modes are damped, and a 12.2 dB and 10.7 dB reductions of the first mode are achieved with the LQR and LQG controller with state weighting, respectively.

9.2.2.5 Overview and Lessons Learned

In this section an analysis and comparison of the classical control strategies, *constant amplitude* and *constant gain* velocity feedback (CAVF and CGVF), and optimal control strategies, *linear quadratic regulator* (LQR) and *linear quadratic Gaussian* (LQG) controller, concerning the active vibration control of smart piezoelectric beams, was presented. The control models assumed that one of the piezoelectric layers acts as a distributed sensor and the other one as a distributed actuator, and the sensor signal was used as a feedback reference in the closed-loop control systems.

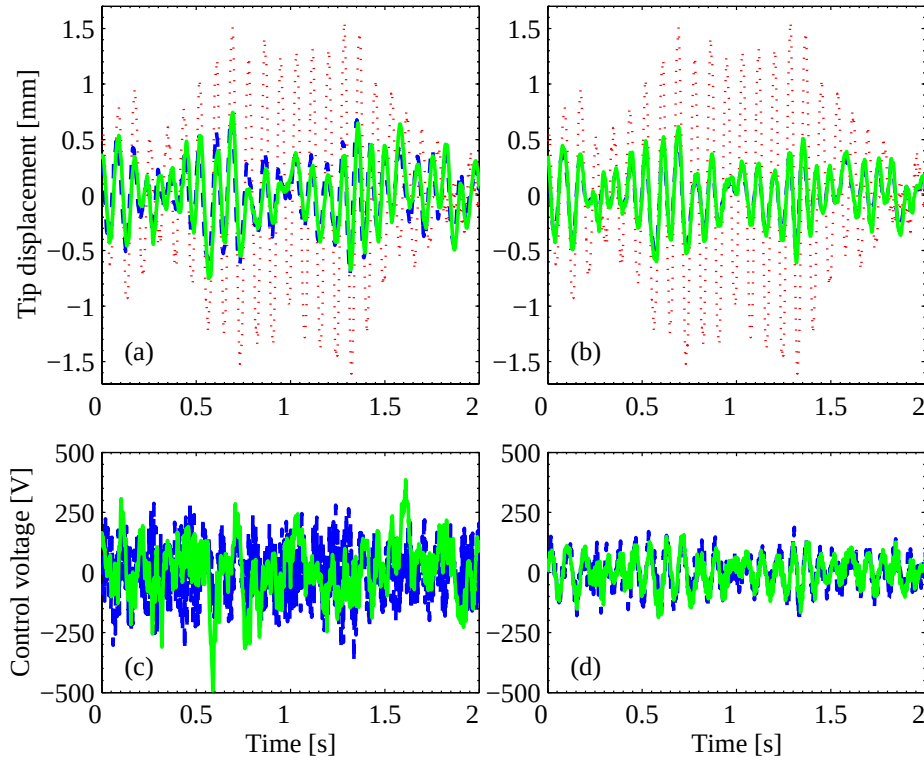


Figure 9.25: Tip displacement and control voltage for white noise point force disturbance with output y (a, c) and state x (b, d) weighting: open-loop (dotted line); closed-loop LQR (dashed line); closed-loop LQG (solid line).

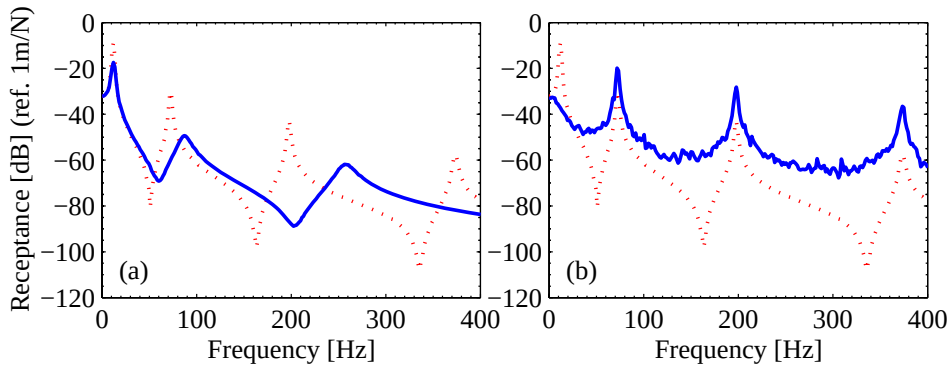


Figure 9.26: Point receptance with CGVF (a) and CAVF (b): open-loop (dotted line); closed-loop (solid line).

A case study concerning the active vibration control of a cantilever aluminum beam with an asymmetric pair of collocated piezoelectric patches mounted on the surface was analyzed. The transverse displacement time history, for an initial displacement field and white noise force disturbance, and point receptance, both measured at the free end of the beam, allowed to compare the performances of the classical and optimal strategies in the time and frequency domains. It was shown that for an initial displacement field the CAVF and LQG with the first mode weighted are the most interesting solutions. However, for a white noise disturbance, the CAVF only manages to reduce the mode under control and the others are destabilized. Fur-

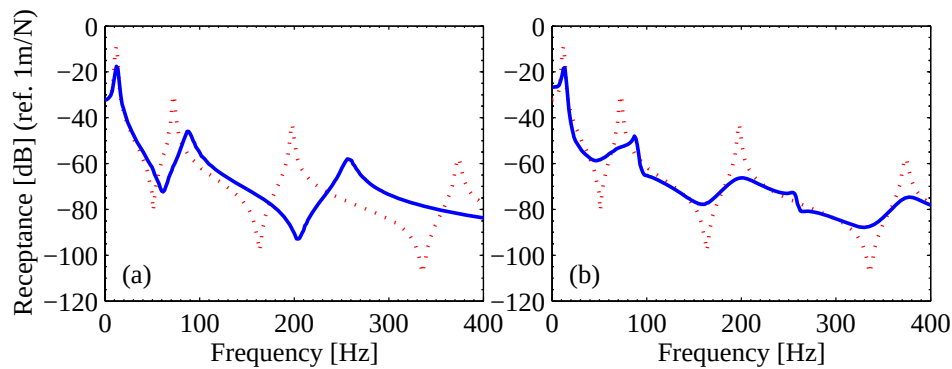


Figure 9.27: Point receptance with the LQR (a) and LQG (b) controller with output y weighting: open-loop (dotted line); closed-loop (solid line).

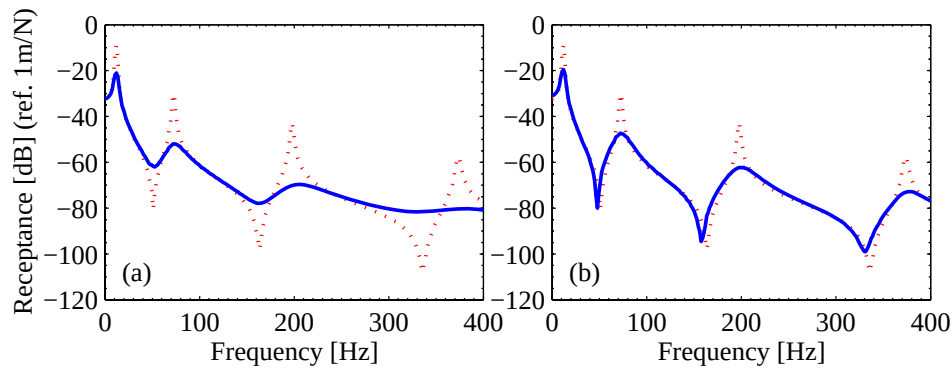


Figure 9.28: Point receptance with the LQR (a) and LQG (b) controller with state x weighting: open-loop (dotted line); closed-loop (solid line).

thermore, the LQG with all the modes weighted presents a better control in bandwidth with a lower control voltage. Moreover, the output weighting LQG can also be an interesting strategy in situations where attenuation of the outputs is of interest and a resemblance in performance with the CGVF was shown in the present analysis.

One of the major advantages of the classical techniques is that they can avoid the necessity of digital control, reducing the time delays in the control system, and that stability is guaranteed provided the actuator/sensor pair is perfectly collocated. However, the CGVF and CAVF strategies require the differentiation of the sensor voltage, which for noisy measurements can become troublesome. If one looks for adaptability in design, the optimal techniques, with the disadvantage of requiring a model of the system, can be more interesting. The quantification of the control system's performance by a quadratic cost function provides the designer with lots of flexibility to perform trade-offs among various performance criteria. The relationship between cost function weights and performance criteria hold even for high-order and multiple input-output systems, where classical control becomes cumbersome. A major limitation of the LQR is that all the states must be measured when generating the control. The LQG controller overcomes that by estimating the states using a Kalman-Bucy filter, which utilizes noisy partial output information, and some modifications in the LQR performance, which sometimes can improve the efficiency, often occur. However, observation and control spillover problems related with the reduced (truncated) or unobserved model dynamics may compromise stability.

9.3 Hybrid (Active/Passive) Damping Treatments

9.3.1 Simulation of Hybrid (Feedback/Feedforward) Control of Vibrations

9.3.1.1 Introduction

The primary purpose of this case study is to investigate the design of feedback, adaptive feedforward and hybrid (combined feedback/feedforward) controllers for the vibration suppression of a beam with partial symmetrically collocated ACLD treatments (viscoelastic layers sandwiched between the piezoelectric patches and the base beam). With this purpose, the piezo-visco-elastic beam FE and ADF model presented in the previous chapters were implemented in MATLAB and the control algorithms were simulated with SIMULINK.

The structural response is assumed to derive from the presence of either periodic or stochastic point-force disturbances. First, a feedback (LQG) controller is designed to mitigate the effects of stochastic disturbances. Then, an adaptive feedforward controller (filtered-reference LMS) is implemented for a periodic disturbance. In a final example both the feedback and adaptive feedforward elements are combined to realize a *hybrid* controller.

9.3.1.2 Problem Description and Results

The cantilever aluminum beam with the ACLD patches is depicted in Figure 9.29. The beam is 300 mm long and 2 mm thick, the partial ACLD treatment is 70 mm long, with viscoelastic and piezoelectric layers of thickness 0.03 mm and 1 mm, respectively, and the symmetrically collocated ACLD treatment is positioned 4 mm away from the clamped edge. The mechanical and electrical material properties of the passive viscoelastic layers, 3M ISD112 [3M (1993)], and of the piezoelectric constraining patches, PXE-5, are presented in Table 9.4 [see the standard IEEE (1988) for details about notation]. The shear storage modulus and loss factor of the viscoelastic material at 27 °C, presented in Figure 9.30, were defined by means of a three-series ADF model with parameters $G_0 = 0.5$ MPa, $\Delta = [0.743, 3.265, 43.284]$, $\Omega = [468.7, 4742.4, 71532.5]$ rad s⁻¹, which were determined by curve fitting experimental data in the frequency range 20 – 5000 Hz [Trindade et al. (2000b)]. Furthermore, the extensional storage modulus was obtained assuming a frequency independent Poisson's ratio equal to 0.45.

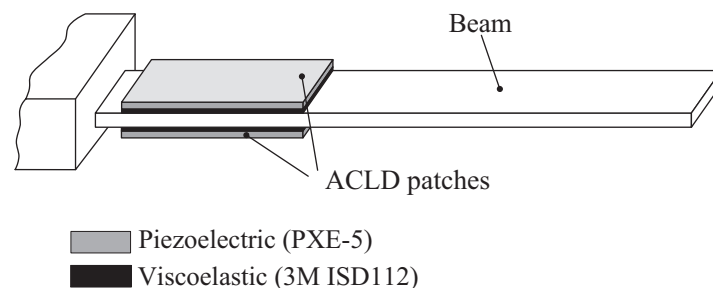
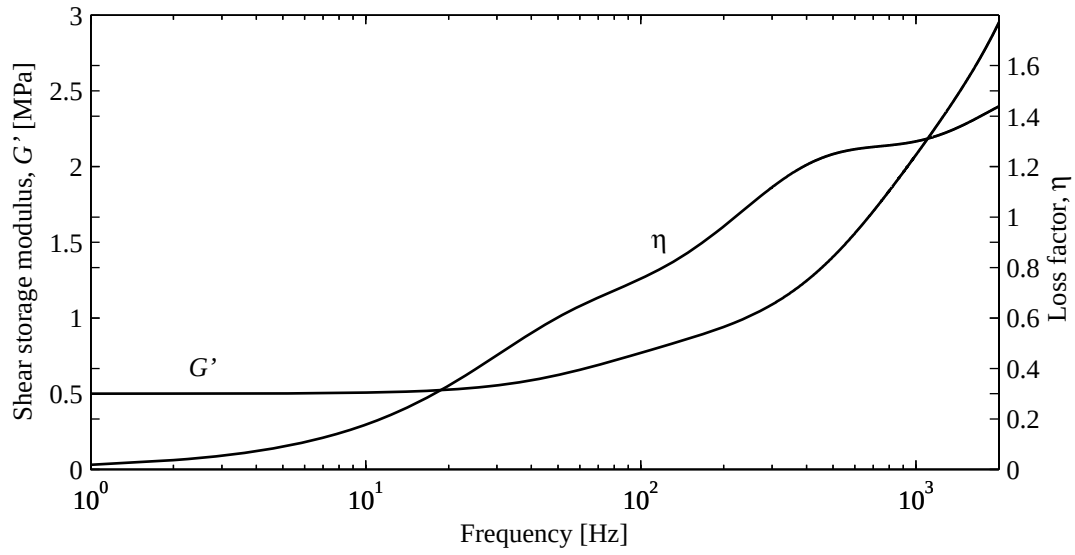


Figure 9.29: Clamped beam with a pair of ACLD patches.

In order to infer the amount of damping introduced by the PCLD treatment, where in that case the constraining piezoelectric layers are not actuated, the length of the piezoelectric constraining layer was defined according to the maximum length of commercially available piezo-

Table 9.4: Material properties of the aluminium, PXE-5 and 3M ISD112.

Aluminium		3M ISD112		PXE-5			
E	70 GPa	G	Figure 9.30	c_{11}^E	131.1 GPa	d_{31}	$-215 \times 10^{-12} \text{ m V}^{-1}$
ν	0.3	ν	0.45	c_{12}^E	7.984 GPa	d_{33}	$500 \times 10^{-12} \text{ m V}^{-1}$
ρ	2710 kg m^{-3}	ρ	1600 kg m^{-3}	c_{13}^E	8.439 GPa	d_{15}	$515 \times 10^{-12} \text{ m V}^{-1}$
				c_{33}^E	12.31 GPa	$\varepsilon_{11}^T/\varepsilon_0$	1800
				c_{44}^E	2.564 GPa	$\varepsilon_{33}^T/\varepsilon_0$	2100
				c_{66}^E	2.564 GPa	ρ	7800 kg m^{-3}

**Figure 9.30:** Frequency dependent properties of 3M ISD112 at 27 °C defined through a three-series ADF model.

electric actuators. Then, a geometric optimization study, through a parametric study of the influence of only varying the thickness of the viscoelastic layers in the overall damping performance of the PCLD treatment, was performed. The beam was discretized into 150 equal length FEs, with a layerwise (discrete-layer) configuration of 1/1/3/1/1 for the five physical layers, a state space model with 20 modes with a viscous modal damping ratio equal to 0.2 % for all the modes was considered, and the operating temperature was supposed to be constant and equal to 27 °C. The damping ratios of the first five modes for viscoelastic thickness in the range 0.001 – 1 mm are presented in Figure 9.31.

From the analysis of Figure 9.31, one can see that the optimal thickness depends on the frequency range (modes) to attenuate, with the maximum values of damping ratio occurring at different thickness. Furthermore, it can be seen that the optimal thickness for the first three modes is comprised in the range 0.01 – 0.02 mm, and for the higher frequency modes in the range 0.05 – 0.08 mm, with damping ratios more or less comprised between 2 – 4% (note that a 0.2% damping ratio was considered for the untreated beam). Furthermore, it should be noted that the values of the natural frequencies remain practically the same for the different thicknesses, with values for the thickness 0.03 mm equal to 22.57 Hz, 119.9 Hz, 281.7 Hz, 578.9 Hz and 1040 Hz. Since the aims of the treatment are to attenuate significantly the first 5 modes, and

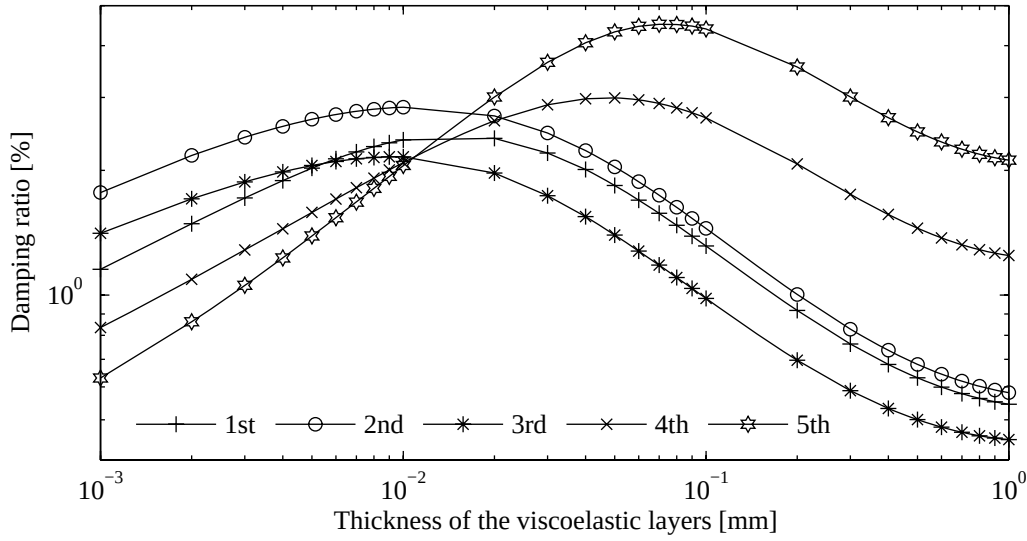


Figure 9.31: Influence of the viscoelastic layers thickness on the modal damping ratios of the beam with the symmetric PCLD treatment.

since a trade-off in modal damping performance occurs for different thicknesses, the optimal thickness value was set to 0.03 mm, revealing the effects of the passive piezoelectric (without control) treatment. That is the thickness value optimized for the PCLD treatment utilized herein.

With the optimal thickness chosen, the next step is to actively utilize the piezoelectric layers, through connection with the LQG controller, to enhance the PCLD treatment performance. Here, a SISO control system is considered with the input being the control voltage applied into the piezoelectric layers, which since the treatment is symmetrically collocated is the same for both active layers but with opposite polarity, and the output being the velocity of the free tip of the beam, which is assumed to be measured. In order to damp the first five modes, and since they occur in complex conjugated pairs, the LQR state weighting matrix for the real equivalent system, as pointed out in Section 8.6.1.2, is set to $\mathbf{Q}_x = 1 \times 10^6 \text{diag}(\omega_1^2, \omega_1^2, \dots, \omega_5^2, \omega_5^2, 0, \dots, 0)$ and the control input weight to $R = 1$. For the LQG controller design, a white noise mechanical disturbance applied on the free end of the beam is modeled as plant noise. Therefore, the equality $\mathbf{B}_w = \mathbf{B}_u$ in Equation (8.45b) should be considered, and a plant noise vector (force disturbance) with $E[\mathbf{w}(t)\mathbf{w}^T(t)] = 0.25 \text{ N}^2$ and a sensor white noise disturbance with $E[\mathbf{v}(t)\mathbf{v}^T(t)] = 1 \times 10^{-5} \text{ V}^2$ are considered for the definition of the noise correlation matrices and Kalman-Bucy gain design. It should be mentioned that the only output measured considered in the Kalman filter estimation is the velocity at the free tip. Furthermore, in order to avoid the depolarization of the piezoelectric actuators, the maximum control voltage applied is set to 250 V, leading to a maximum electric field of 250 V mm^{-1} .

The control voltage and time displacement at the free end, for a white noise mechanical force disturbance applied at the free end too, for the bare beam and PCLD and ACLD treated beam configurations are obtained with SIMULINK, with the sampling frequency set to 10000 Hz, and presented in Figure 9.32. It can be seen that both the PCLD and ACLD treatments manage to reduce the tip displacement significantly. When compared to the bare beam displacement standard deviation, which is equal to 5.9 mm, the standard deviations of the PCLD and ACLD treatment cases, which is equal to 1.59 mm and 0.76 mm, respectively, demonstrate the vibration control efficiency. Moreover, it can be seen that when compared with the fail safe PCLD treat-

ment, the ACLD treatment improves even further (around 50%) the damping performance, with the actuating control voltage not exceeding 250 V.

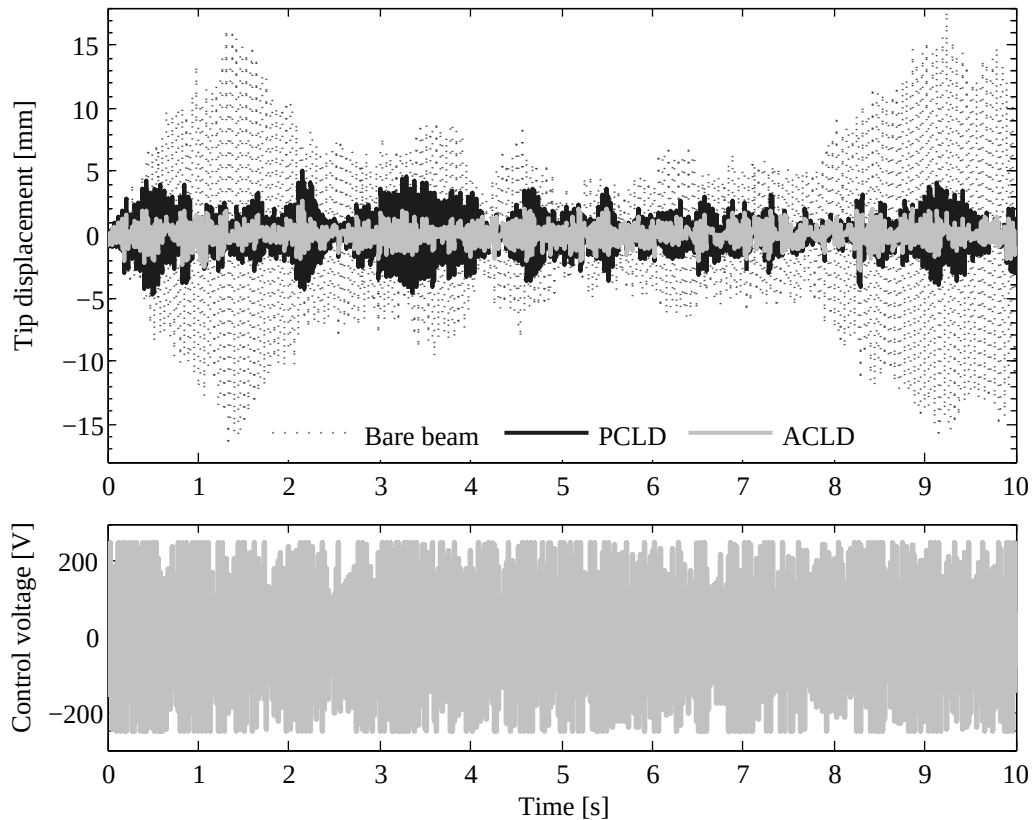


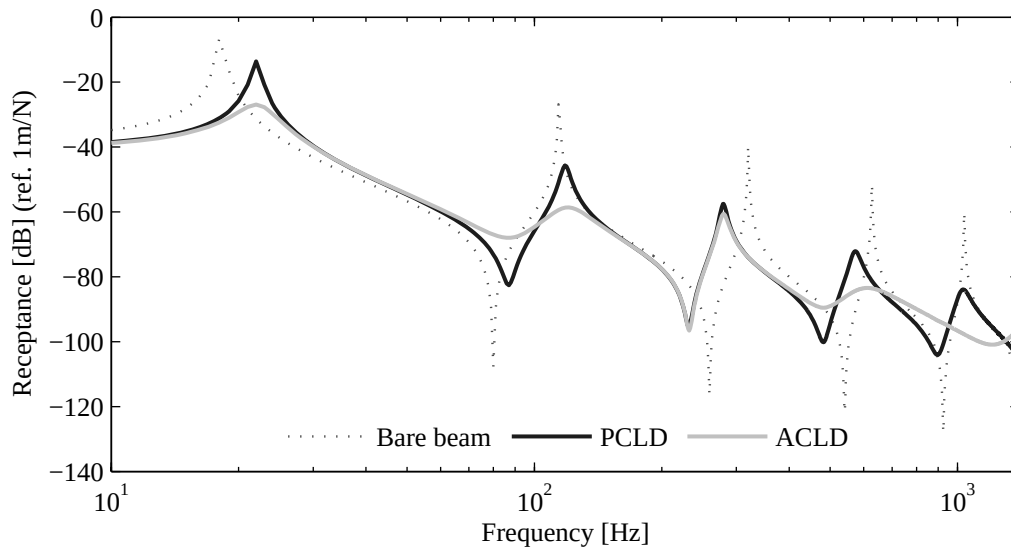
Figure 9.32: Control voltage and tip displacement histories at the free tip for a white noise mechanical force disturbance applied at the free end of the beam for the bare beam, PCLD and ACLD treatment configurations.

In order to analyze the capacity of the PCLD and ACLD treatments in the frequency domain, the point receptances at the free end of the beam are presented in Figure 9.33. The PCLD treatment significantly reduces the magnitude of the first resonance in approximately 7 dB and the ACLD treatment in approximately 20 dB. Furthermore, the PCLD manages to significantly attenuate all modes. In comparison, the ACLD attenuates them even more, except the third mode which rests almost unaltered, and the fifth is completely eliminated. The damping ratios of the first five modes obtained with PCLD and ACLD are compared with the 0.2% for all modes obtained for the bare beam in Table 9.5. Both Figures 9.32 and 9.33 show the adaptability and improvement in the damping capabilities obtained with the ACLD treatment and the robustness and highly satisfactory performance of the PCLD (in case the active system controller fails) when the beam is under stochastic mechanical disturbances.

In order to attenuate the effects of an harmonic point-force disturbance oscillating at a frequency of 600 Hz and in the vicinity of the fourth mode, the adaptive filtered-reference LMS algorithm is utilized. First, the LMS algorithm is used to identify the plant *cancellation path* (control voltage to measured velocity path) at the disturbance frequency, which for a single frequency only requires two filter weights, with the convergence being achieved in nearly 0.5 s. After identifying the plant along the cancellation path, the filtered-reference LMS algorithm was implemented. Again, only two filter weights were considered in the adaptive feedforward

Table 9.5: Modal damping ratios of the beam with and without the damping treatments [%].

Mode	1st	2nd	3rd	4th	5th
Bare beam	0.20	0.20	0.20	0.20	0.20
PCLD	2.20	2.46	1.74	2.88	3.65
ACLD	10.0	12.0	2.46	15.3	100

**Figure 9.33:** Point receptance at the free tip for the bare beam, PCLD and ACLD configurations.

controller and the results are presented in Figure 9.34. As can be seen, the control voltage is limited to 250 V in absolute value, and the ACLD feedforward controller manages to reduce the free tip displacement significantly when compared with the PCLD one. Furthermore, the filter weights did not converge yet at the time of 5 s.

While the filtered-reference LMS algorithm can be used to successfully control a periodic disturbance, as shown in Figure 9.34, LQG control is often more appropriate for control of stochastic disturbances. This is particularly true when an uncontrollable reference signal can not be identified and the system is subject to both periodic and stochastic disturbances. To demonstrate the complementary performance of the two control approaches, the *hybrid* control approach described in Section 8.6.3 is implemented for the same harmonic disturbance. The same SISO system in the previous example is used here. However, the LQG dynamic compensator designed earlier for the stochastic disturbances is also incorporated in the control path, which modifies the plant that is acted on by the feedforward controller. Therefore, the modified *cancellation path* is identified again. The results of the hybrid controller are presented in Figure 9.35.

As can be seen from the analysis of Figures 9.34 and 9.35, when the hybrid control system is engaged, linearly combining both feedback and adaptive feedforward, and for the same convergence parameter, the displacement converges faster. The displacement results are plotted on the same scale to display the improved convergence characteristics and transient response performance. Furthermore, at the end of the period of time considered in the simulation ($t = 5$ s), the displacement with the adaptive feedforward controller is comprised in an interval of amplitude 0.3 mm while with the hybrid controller a 0.05 mm amplitude was obtained. These values

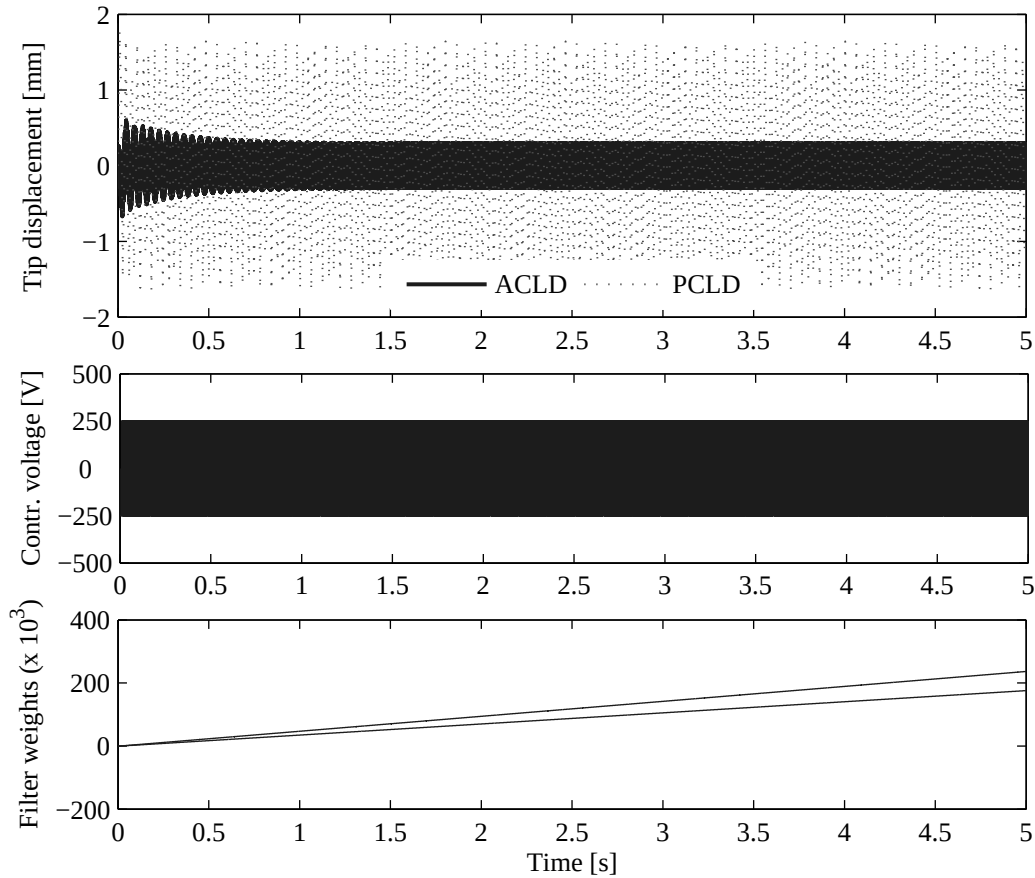


Figure 9.34: Tip displacement, control voltage and filter weights, for a sinusoidal disturbance applied at the free end of the beam, obtained with the adaptive feedforward controller for the PCLD and ACLD treatment configurations.

are obtained for the ACLD case and when compared with the results of the PCLD system, 2 mm amplitude, demonstrate the effectiveness of the ACLD treatment. Moreover, note that the transient response of the structure when the harmonic disturbance is introduced at $t = 0$ s decays much faster with the hybrid controller and that the stationary response converges to a lower displacement amplitude.

9.3.1.3 Overview and Lessons Learned

In this section the generic formulation for the study of beams with arbitrary ACLD treatments presented in the preceding chapters was utilized. The FE model has considered a partial layer-wise displacement formulation and a fully coupled electro-mechanical theory and the damping behavior of the viscoelastic layers was considered by a Laplace transformed ADF method. Furthermore, the optimal feedback (LQG) and adaptive feedforward (filtered-reference LMS) controllers described in the preceding chapter were discussed when used individually or combined in a unified hybrid controller.

The analyzed case study regards the disturbance rejection of an aluminum beam with a pair of symmetrically collocated surface mounted ACLD patches. For the design and simulation of the control system, MATLAB and SIMULINK softwares were utilized, and a SISO configuration, with the output being the velocity at the free end of the beam, and the input being the control

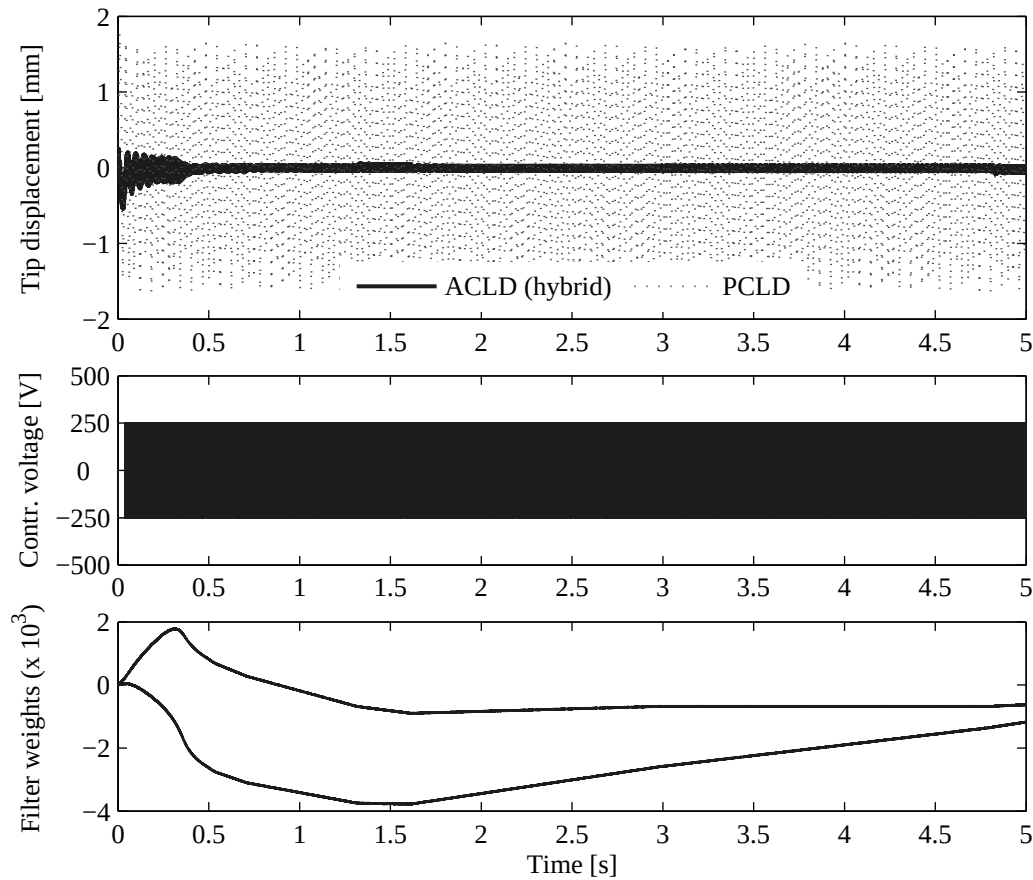


Figure 9.35: Tip displacement, control voltage and filter weights, for a sinusoidal disturbance applied at the free end of the beam, obtained with the adaptive hybrid controller for the PCLD and ACLD treatment configurations.

voltage applied into the piezoelectric constraining layers, was considered. First, a broadband stochastic disturbance was considered and a feedback controller was utilized to suppress vibration. Then, considering that the designer has access to a periodic reference signal that is correlated with the disturbance, an adaptive feedforward strategy was employed to cancel the effects of the disturbance on the plant. Finally, both controllers were combined into a unified hybrid controller.

The case study allowed to assess and discuss the outcomes and drawbacks of the feedback and feedforward controllers when used individually and the advantages of the hybrid controller. The hybrid controller improved the feedback performance and led to an improved performance and convergence of the adaptive feedforward controller. The transient response of the adaptive hybrid control system was observed to converge on a much faster timescale due to the modification of the system dynamics resulting from the feedback control. The rate of convergence for the adaptive controller is thus increased in hybrid configurations.

9.3.2 Experimental Implementation of Feedforward Control of Vibrations

9.3.2.1 Introduction

This case study concerns the adaptive feedforward control of vibration of a freely supported beam with two distinct surface mounted hybrid (active/passive) damping treatments. The first configuration concerns the use of an ACLD patch alone, where the piezoelectric constraining layer is actively utilized to increase the shear deformation of the sandwiched passive viscoelastic layer and at the same time to apply forces and moments upon the structure, which will balance the power flows coming into the structure, and is denoted by ACLD configuration. The second configuration regards the use, as an active element in the control, of the piezoelectric patch alone, denoted by AD (*active damping*), and since the constraining layer of the ACLD treatment also bonded on the other side of the beam is not actively utilized, a PCLD treatment is utilized in combination with an AD one, yielding an AD+PCLD configuration. A FE model of the beam with the damping treatments is used for the simulation of the adaptive feedforward controller which is also implemented and tested in real-time. The aims are to compare the predicted and measured damping performances of the two treatments in terms of vibration reduction, control effort, stability and robustness, when a filtered-reference LMS algorithm is used to cancel the effects of a broadband voltage disturbance applied into a third surface mounted piezoelectric patch which is used to excite the beam.

9.3.2.2 Experimental and Simulation Setup

The filtered-reference LMS algorithm was tested on a freely supported aluminum beam measuring $380 \times 15 \times 2$ mm, with two piezoelectric patches of material PXE-5 (Phillips Components - PLT 30/15/1-PX5-N) measuring $30 \times 15 \times 1$ mm mounted each one 52 mm from the free ends, and an ACLD patch, with a constrained layer made of the same piezoelectric material and a viscoelastic layer 3M ISD112, measuring $30 \times 15 \times 0.127$ mm mounted on the same location but opposite side (Figure 9.36). The piezoelectric patches were bonded on the beam by means of a nonconductive cyanoacrylate adhesive (Loctite - 496) with high rigidity and the ACLD patch was bonded through the use of the self-adhesive of the viscoelastic layer.

The mechanical and electrical material properties of the aluminum, 3M ISD112 [3M (1993)] and PXE-5 are the same utilized in the previous section and are presented in Table 9.4. The shear complex modulus of the viscoelastic material at 27°C was defined by means of a three-series ADF model with its parameters defined as $G_0 = 0.5 \text{ MPa}$, $\Delta = [0.743, 3.265, 43.284]$, $\Omega = [468.7, 4742.4, 71532.5] \text{ rad s}^{-1}$, which were determined by curve fitting experimental data in the frequency range $20 - 5000 \text{ Hz}$ [Trindade et al. (2000b)]. Furthermore, the extensional storage modulus was obtained assuming a frequency independent Poisson's ratio equal to 0.45. For the simulation, the beam was discretized into 190 FEs and a coupled piezo-visco-elastic structural FE modal model with 20 elastic modes plus 3 rigid body modes was considered.

A schematic of the experimental setup of the real time feedforward controller is depicted in Figure 9.37. The velocity at one point of the beam was measured with a Doppler vibrometer Polytec OFV 303 with a controller unit Polytec OFV 3001. A Krohn-Hite dual channel (model 3362) and single channel (model 3550) low-pass filters with four-pole Butterworth characteristics with the cut-off frequency set to 1200 Hz were utilized as anti-aliasing and reconstruction filters. Furthermore, two power amplifiers were used, one LDS PA100E with a maximum output voltage of 20 V rms for the excitation, and another constructed in the laboratory with a

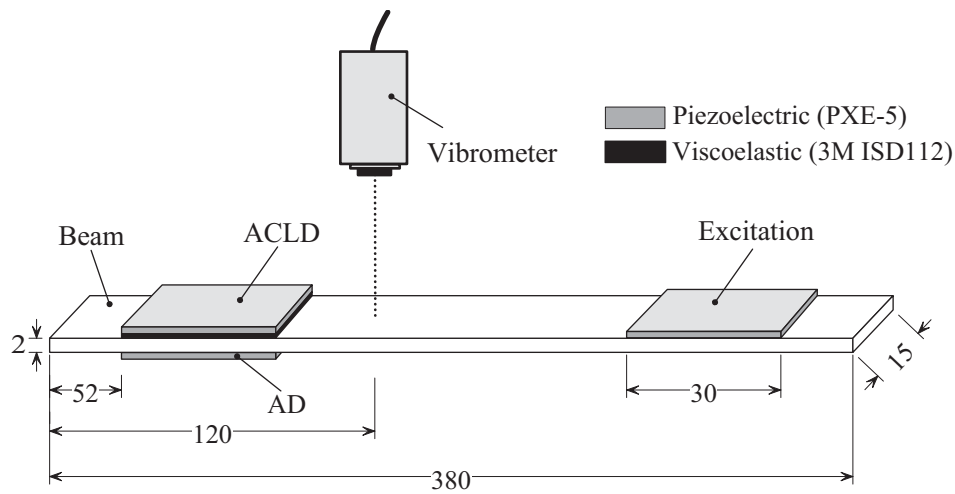


Figure 9.36: Freely supported test beam configuration with the AD, ACLD and excitation patches (dimensions in mm).

maximum output voltage of 150 V for the control voltage. All real time processing was carried out with a National Instruments Lab-PC+ DAQ board with 12-bit analog resolution, and the sampling rate was set to 3600 Hz both for simulation and real time.

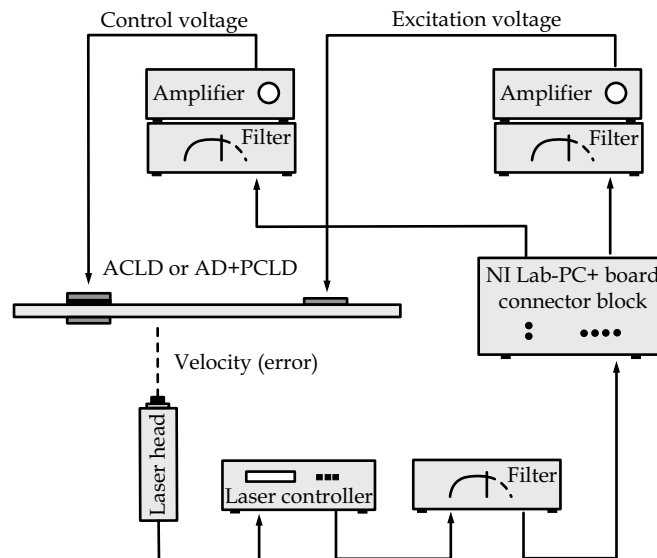


Figure 9.37: Schematic of the experimental setup of the real time feedforward controller.

In the simulation and real-time implementation of the control system a SISO configuration, with the input being the velocity at one point of the beam and the output being the voltage applied into the piezoelectric constraining layer, for the ACLD configuration, or the piezoelectric patch, for the AD+PCLD configuration, is considered here. Assuming that the designer has access to a reference signal that is correlated with the disturbance, the filtered-reference LMS algorithm is employed to cancel the effects of the disturbance on the plant at the chosen output location. The simulation and all real-time processing were performed using MATLAB and SIMULINK software incorporating the REAL-TIME WORKSHOP and the REAL-TIME WINDOWS TARGET.

9.3.2.3 System Identification

The first step in operating the control is to perform the off-line system identification. Thus, a band limited random noise from 10 to 1200 Hz, which was generated by SIMULINK, was fed into the two piezoelectric patches of the AD and ACLD treatments and the FRFs between the voltage applied and the measured velocity were estimated in MATLAB. An IIR filter was used to identify the measured and simulated FRFs using the MATLAB `invfreqz.m` routine and the identified coefficients were utilized as coefficients of the IIR discrete filters of the cancellation paths. The results are presented in Figure 9.38. Note that the identified FRFs of the discrete filters match the predicted and measured FRFs in magnitude and phase, with the exception of some parts of the phase of the measured FRFs, however presenting the same trend. The phase has been “unwrapped” so that it can be seen that the phase does not remain within a $[-90^\circ, 90^\circ]$ boundary and therefore the control loops are indeed non minimum phase in the bandwidth of interest. The resulting nearly linear decreasing trend in the phase verified in the measured FRFs is due to a combination of the delays from the DSP system, analog filters, amplifiers, etc., and demonstrates the “non causality” of the controller. Furthermore, it is also worthy to mention that the predicted FRFs match the measured ones for both AD and ACLD treatments, and that there is almost two orders of magnitude of difference between the actuating capacity of the AD and ACLD patches, where the viscoelastic layer decreases the transmissibility of efforts, mainly in the low stiffness range (low frequencies). That is evident in the measured FRF of the ACLD patch, where in the range from 0 to more or less 150 Hz (first mode) the measured results show a very low transmissibility.

9.3.2.4 Results

All controlled measurements have been obtained with the adequate adaptation rates for the different damping treatments. A filtered-reference LMS algorithm with 50 adaptive coefficients was considered both in the simulation and real time control. The time responses after convergence of the adaptive feedforward algorithm were saved and post processed to obtain the measured and predicted FRFs of the open- and closed-loop control systems for the two damping configurations, namely AD+PCLD and ACLD, as presented in Figure 9.39. The open-loop FRFs show again a good match between the predicted and measured FRFs at all modes. It is shown by both measured and predicted FRFs that the closed-loop response with the AD+PCLD treatment is more effective than with the ACLD treatment. Furthermore, a similar trend in the measured and predicted AD+PCLD results is achieved. However, the predicted results for the ACLD are too optimistic when compared with the measured ones, where the control was inefficient. That is justified by the fact that the control voltage of the simulation does not have any constraint and can increase indefinitely. However, in practice, the real-time controller was limited to a 150 V range output, and since ACLD requires higher voltages to have a similar actuating capacity to AD (recall Figure 9.38 where a two order magnitude difference between the actuating capabilities of AD and ACLD is shown), the real-time controller produces low damping. Of course, in order to be able to increase the excitation to actuating voltage ratio one can decrease the magnitude of the excitation. However, if the values are too low, the signal to noise ratio increases and the controller may become unstable. Other alternative to increase ACLD performance is trying to implement a controller that can increase specifically the shearing of the viscoelastic layer, instead of trying to reduce the input power into the structure.

The measured and predicted ACLD to AD control voltage spectral density ratios are presented in Figure 9.40. It is shown that the predicted ratio is well above the measured one,

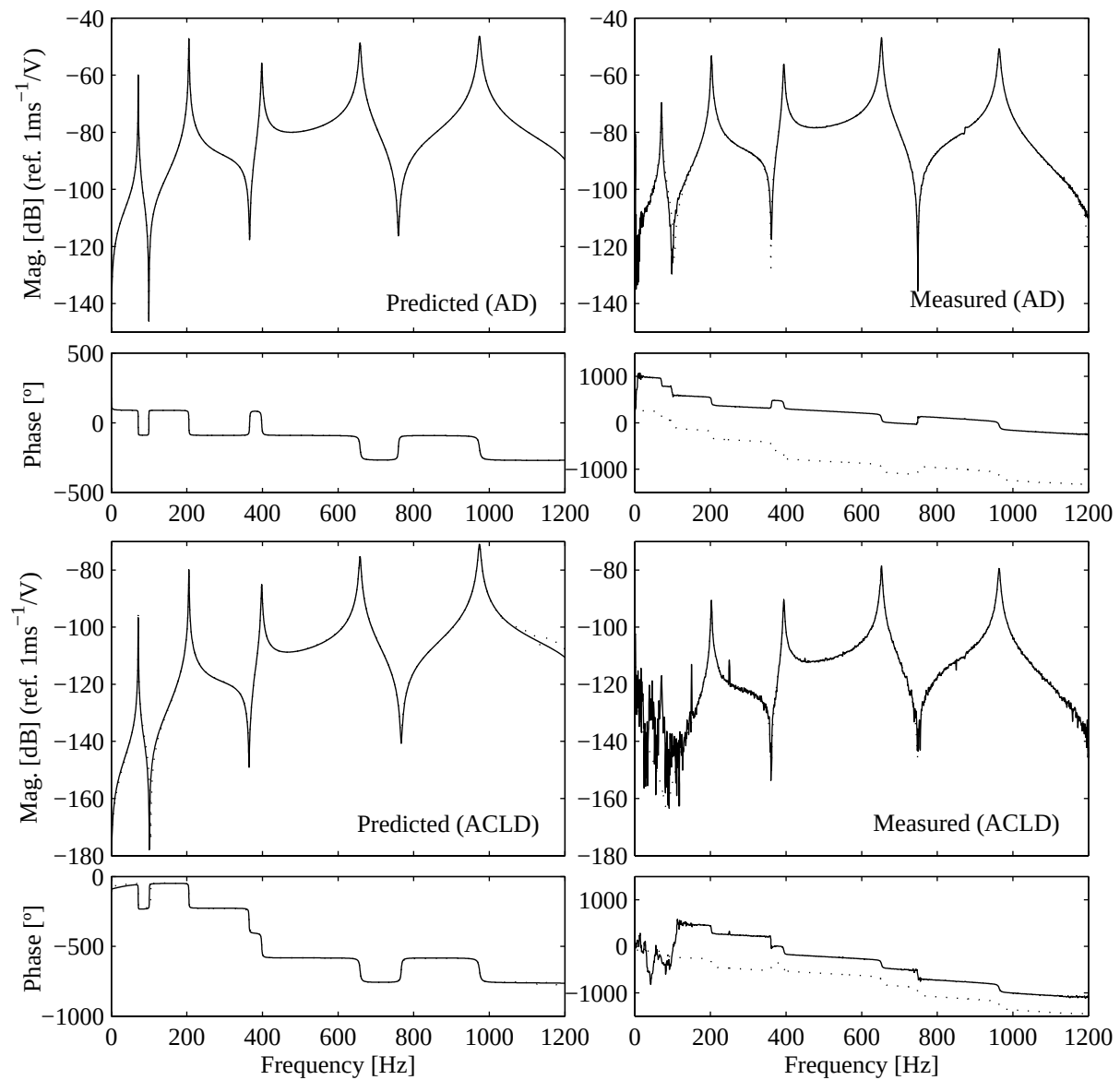


Figure 9.38: Comparison of the predicted and measured (solid lines) AD and ACLD cancellation path FRFs with the identified (dotted lines) ones.

showing that the real-time controller has not been able to feed the ACLD patch with the required control voltage to achieve a similar behavior to the simulated controller.

9.3.2.5 Overview and Lessons Learned

In this section an adaptive feedforward controller using the filtered-reference LMS algorithm was utilized in the active control of vibrations of a beam excited by a piezoelectric patch and damped by an AD or ACLD treatment. The two damping treatments were simulated and experimentally implemented in real-time with the purpose of comparing their damping performance. It was shown that the FE with the viscoelastic damping model utilized here presents a good accuracy and it is a reliable and useful tool for the controller and treatment design. In

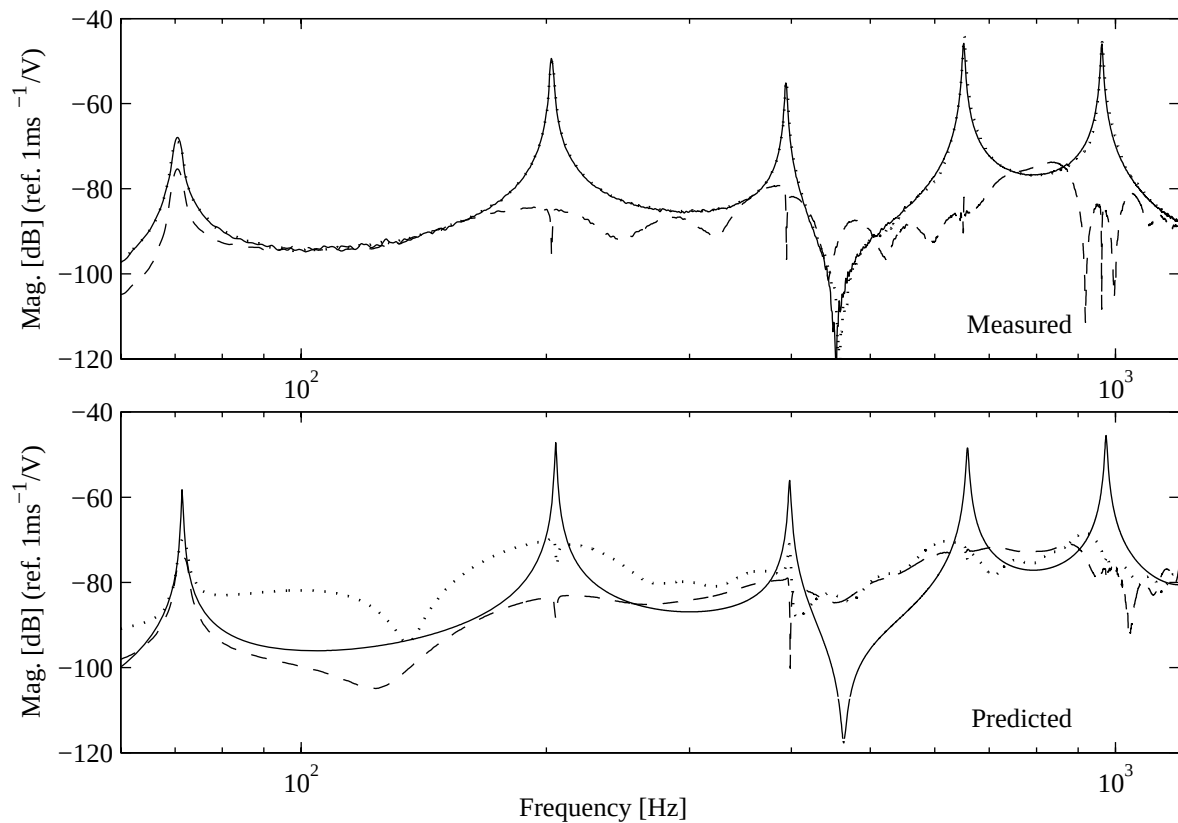


Figure 9.39: Measured and predicted open-loop and closed-loop FRFs of the SISO control system: OL (solid line); AD/PCLD (dashed line); ACLD (dotted line).

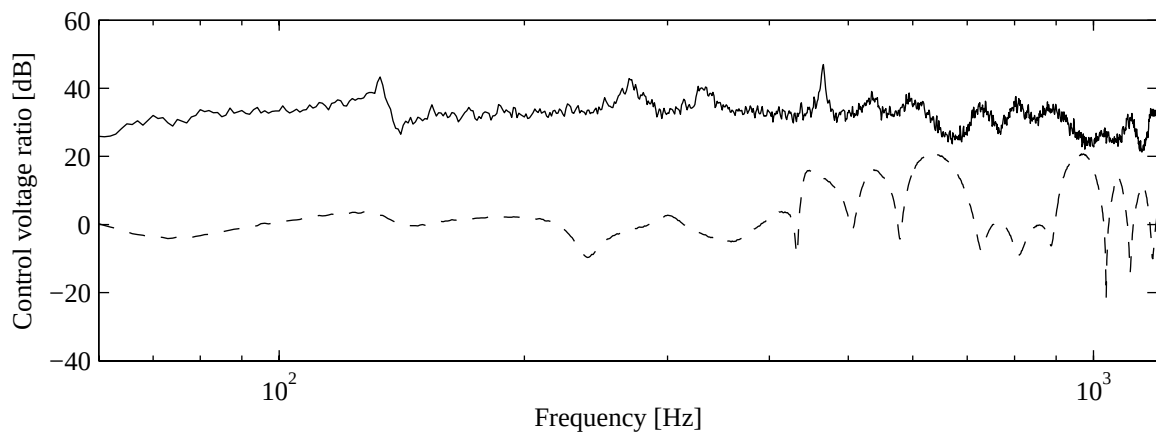


Figure 9.40: Measured (dashed line) and predicted (solid line) ACLD to AD/PCLD control voltage spectral density ratios.

this specific application the AD+PCLD treatment is more efficient than the ACLD one requiring lower control voltages. The need of higher voltages of the ACLD treatment may constitute a problem in real-time applications due to the constraint of the saturating voltage of the piezoelectric patches or the limited output voltage of the amplifiers. However, one has to be cautious

on generalizing the conclusions taken from this work, which may not hold in other cases such as when using feedback control, or when other performance indexes, or cost functions minimizing other parameters, are utilized. To conclude, it should be mentioned that since the transmissibility of efforts from the viscoelastic layer to the host structure is usually reduced, mainly at low frequencies, the performance of the ACLD treatment would only be improved if one designs a controller trying to optimize the damping due to the shearing of the viscoelastic layer, which in turn may require lower control voltages.

9.3.3 Analysis and Assessment of Hybrid Damping Configurations

9.3.3.1 Introduction

Hybrid damping treatments attenuate vibrations and sound radiation through different damping mechanisms. In order to optimize the design of arbitrary hybrid (active/passive) damping treatments, where the damping layers are arbitrarily stacked and mounted on the host structure, it is essential to understand their phenomenological behavior and to be able to identify and quantify the efficiency of the different damping mechanisms. These treatments comprise, on the one hand, the damping effects due to the internal molecular interactions that occur during deformation, in general, and vibration, in particular, of the viscoelastic materials, which give rise to macroscopic properties such as stiffness and energy dissipation during cyclic deformation, and, on the other hand, the effects due to the piezoelectric actuation which applies forces and moments on the structure.

As discussed in Section 8.3, the damping achieved with hybrid damping treatments involves mainly three damping mechanisms:

1. The shearing (or general deformation) of the viscoelastic layer(s) in open-loop;
2. The increase of deformation in the viscoelastic layer(s) due to the convenient motion of the active piezoelectric layer(s);
3. The decrease of total input power into the structure due to the forces/moments applied by the active piezoelectric layer(s).

As obvious, the latter mechanism (iii) has a reduced importance for ACLD treatments since the viscoelastic layer usually reduces the transmissibility of efforts to the host structure. In fact, usually the transmissibility (stiffness) increases with frequency. Thus, in order to dissipate energy also at low frequencies, the aim would be to actively increase the low frequency shearing in the viscoelastic layer. However, due to the loss factor behavior of viscoelastic materials (usually smaller at low frequencies), the treatment is usually more efficient for frequencies with higher loss factors, if we have also significant shearing strains. (The energy released is related with the loss factor times the shearing strain.) Regarding AD treatments, the transmissibility of efforts to the host structure is higher, since the piezoelectric patches are directly bonded to the structure, and the aim would be actively reducing the input power coming into the structure.

9.3.3.2 Damping Configurations and Results

Let us consider a cantilever aluminum beam with three distinct hybrid damping configurations as depicted in Figure 9.41. The considered aluminum beam has dimensions $300 \times 15 \times 1.5$ mm and the hybrid damping treatments of length 70 mm were mounted 4 mm away from

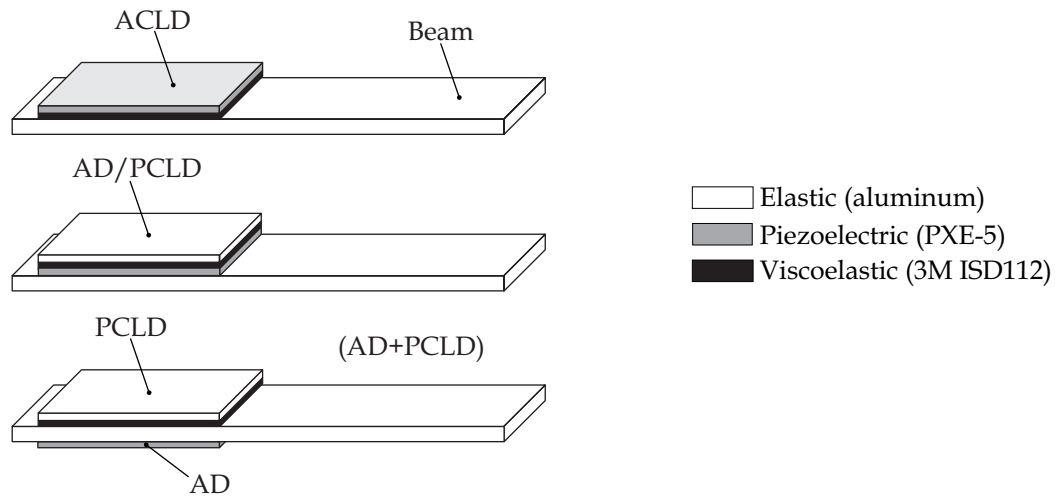


Figure 9.41: Hybrid (active/passive) damping configurations.

the clamped edge. The piezoelectric and viscoelastic materials are the same considered in the previous sections, and the mechanical and electrical material properties of the aluminum, 3M ISD112 [3M (1993)] and PXE-5 are presented in Table 9.4. The shear complex modulus of the viscoelastic material at 27 °C was defined by means of a three-series ADF model with its parameters defined as $G_0 = 0.5 \text{ MPa}$, $\Delta = [0.743, 3.265, 43.284]$, $\Omega = [468.7, 4742.4, 71532.5] \text{ rad s}^{-1}$, which were determined by curve fitting experimental data in the frequency range 20 – 5000 Hz [Trindade et al. (2000b)], and the extensional storage modulus was obtained assuming a frequency independent Poisson's ratio equal to 0.45. For the simulation, the beam with the damping treatments was discretized into 160 FEs and a coupled piezo-visco-elastic structural FE modal model with 20 elastic modes with a viscous modal damping ratio equal to 0.2 % for each mode to model the structural (clamped edge) and fluid (air) damping of the beam structural system were considered.

In the first configuration the viscoelastic layer is sandwiched between the beam and an active piezoelectric layer, yielding the well-known ACLD configuration. In the other two configurations, an ACLD patch with an “elastic” constraining layer, i.e. a PCLD patch, is considered in conjunction with an active piezoelectric layer which is placed either below the PCLD patch or on the other side of the beam. These two latter configurations are denoted as AD/PCLD and AD+PCLD, respectively, and somehow combine active damping (AD) features, where the piezoelectric patch has a perfect transmissibility of efforts to the structure, with ACLD features, i.e. a direct or indirect influence upon the deformation of the viscoelastic layer.

Regarding the thickness of the passive and active layers, for comparison purposes, all the configurations consider the same thicknesses, i.e. 0.03 mm for the viscoelastic layer and 1 mm for the piezoelectric (and “elastic” constraining) layers. Regarding the constraining layers, and again for comparison purposes, the passive “elastic” constraining layers of the AD/PCLD and AD+PCLD configurations have the same mechanical behavior as the piezoelectric constraining layer of the ACLD configuration, i.e. the same material properties.

It is well known that the efficiency of different hybrid damping treatments configurations strongly depends upon geometric parameters, such as the location and thicknesses of the layers, the properties of the materials involved or even the control law considered, which turns a rigorous coupled interrelated analysis and comparison of the optimal maximum efficiency attainable by each configuration burdensome. Instead of considering all these variables at the

same time in a “brute force” optimization process, an alternative would be to establish multi-level optimization procedures addressing configuration, geometric, material and control law issues independently, which would allow the designer to gain some insight into the influence some of these parameters have in the net damping behavior. Therefore, the emphasis of this analysis will be given to the comparison of the configurations, assuming with some confidence that the influence of the location and material properties is the same for all the configurations and with less confidence that the layer’s thicknesses do not influence their behavior significantly. This later assumption is questionable but is introduced here to simplify the analysis. In order to compare the damping efficiency of the three configurations according to the previous assumptions, an optimal LQR controller with state weighting with the same design targets is considered for all the configurations in order to minimize the influence of the control law in the damping performance.

The LQR controller design was performed assuming that a disturbance force is applied to the beam at the free tip and that the control voltage applied to the piezoelectric patch is the input to the system which will cancel somehow the effects of the disturbance. In order to define the gain, two design scenarios were considered for a controlled bandwidth comprising the first five modes: (i) the state weighting matrix was defined in order to minimize the sum of the modal coordinates, i.e. $\mathbf{Q}_x = 1 \times 10^{10} \text{diag}(1, 1, \dots, 1)$, and (ii) an output weight $Q_y = 1 \times 10^{10}$ to be applied to minimize the free tip transverse velocity of the beam with the damping treatments. As far as the control voltage input weighting is concerned, the value $R = 1$ was chosen for both design scenarios. It is worthy to mention that the first scenario addresses the reduction of the “global” spatial response of the beam, which might be important for example for sound radiation and/or fatigue control purposes, while the latter addresses the reduction of the “local” free tip velocity of the beam without being concerned (“weighting”) with the global spatial response, which might be more important for applications requiring high point wise precision and vibration stability.

As a first assessment of the efficiency of the three configurations of damping treatments proposed, the kinetic energy and point mobility obtained at the free end of the beam for the open- and closed loop systems with the two design scenarios are presented in Figure 9.42. As can be seen in the figure, the effect of having the tip velocity as a design parameter in the definition of the LQR cost function conducts to greater kinetic energy and mobility attenuations for all the damping configurations than the cost function of the first design scenario. These results in fact make some sense since, on the one hand, for the particular case of a clamped-free beam, the maximum displacement for all modes occurs at the free end of the beam making the attenuation of the free tip displacement better correlated with the global kinetic energy and, on the other hand, the cost function of the first scenario does not implies a direct kinetic energy reduction since it somewhat accounts a spatial average of the displacement and velocity DoFs of the augmented damped structural system with the internal DoFs.

Looking at the efficiency of the damping treatments, it is evident in Figure 9.42f that the AD/PCLD configuration is the most efficient one over the frequency bandwidth considered. It should be noted though, that the 3rd mode, probably due to the location and length of the actuator which yields a lower controllability for this mode, is not so well damped. However, the attenuation in the response alone can not be seen as a sufficient argument to conclude that this configuration is the best one, since real systems usually have physical control effort constraints related with the maximum electric field strength that the piezoelectric patches may undergo before depoling (which for the PXE-5 is 300 V/mm). Therefore, in order to infer about the required control effort with each different configuration, the open- to closed-loop reduction in the kinetic energy and mobility per unit control volt was analyzed and is presented in Figure 9.43.

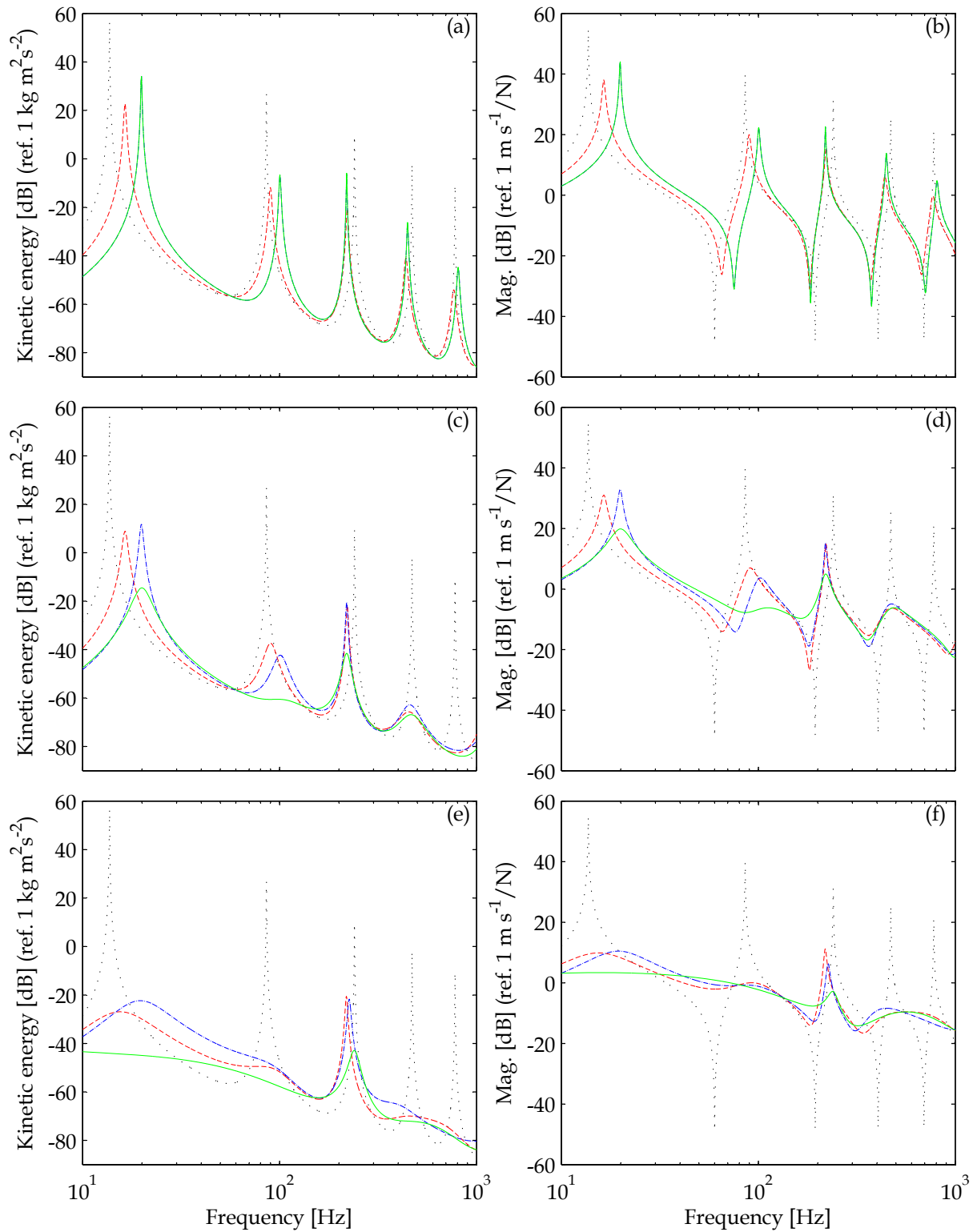


Figure 9.42: Kinetic energies and point mobilities at the free end of the beam for the open-loop system [(a) and (b)] and closed-loop system with the different design scenarios [(c) and (d): state weighting; (e) and (f): tip velocity weighting] with the different configurations of damping treatments: bare beam (dotted line), ACLD (dashed line), AD+PCLD (dash-dot line) and AD/PCLD (solid line).

As can be seen in the figure, despite being the configuration that yields better reduction of kinetic energy and mobility, the AD/PCLD configuration is also the one that requires lower control voltages, mainly for the lower modes, with the difference being less pronounced for the higher modes.

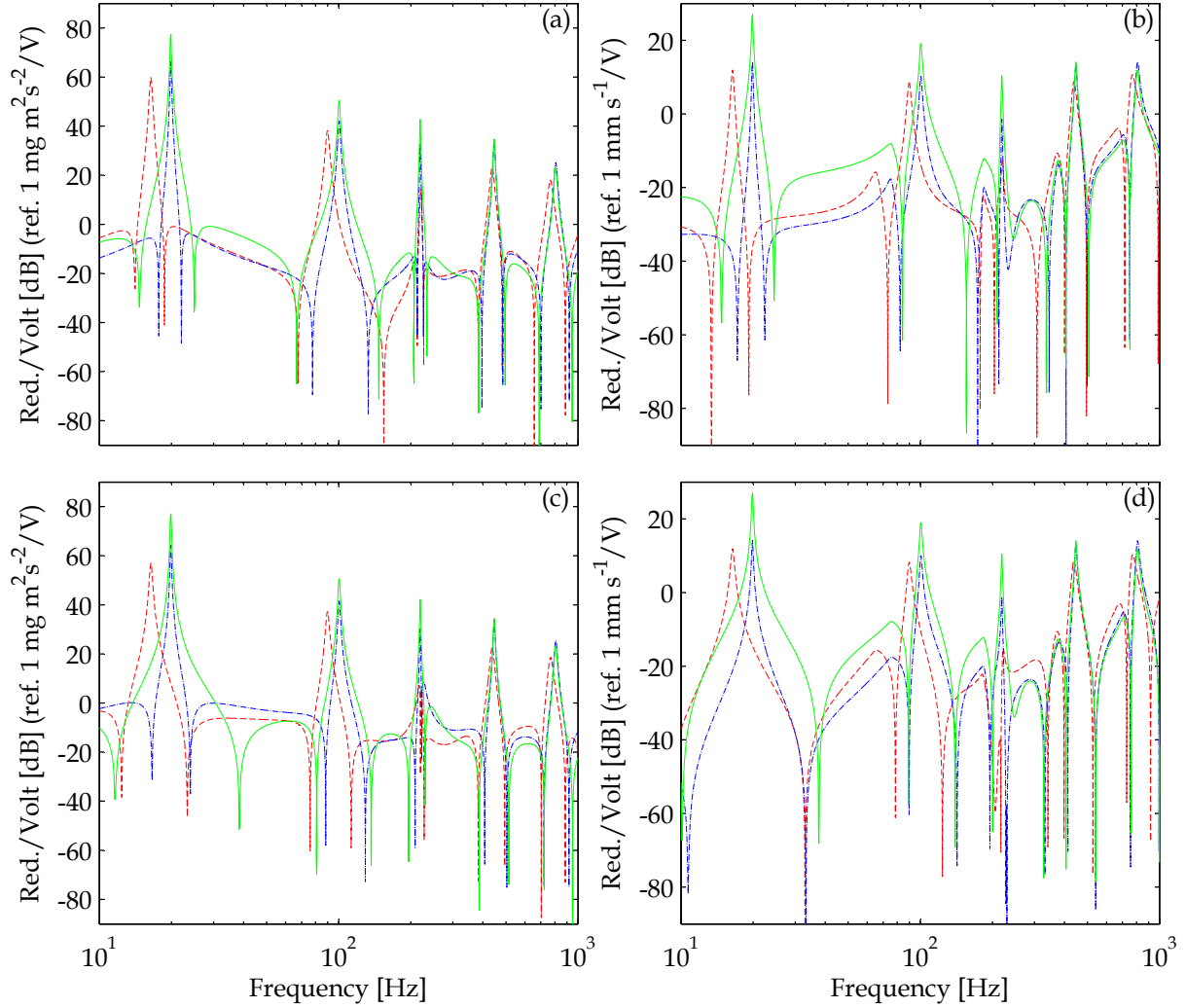


Figure 9.43: Open- to closed-loop kinetic energy and free tip point mobility reduction per unit control voltage with the different design scenarios [(a) and (b): state weighting; (c) and (d): tip velocity weighting] for the different configurations of damping treatments: ACLD (dashed line), AD+PCLD (dash-dot line) and AD/PCLD (solid line).

As mentioned before, the effects of the hybrid damping treatments are twofold: (i) frequency shifting and (ii) increased damping. These two effects upon the 5 controlled modes are presented for the three damping configurations in Figures 9.44 and 9.45. It can be seen that the frequency shifts on the resonance frequencies of the beam due to the damping treatments is significant for all the configurations and since the treatments are placed close to the clamped edge of the beam the stiffness change is more significant. However, frequency shifting is not the ultimate aim of the treatments and for structures subjected to impulsive or broadband disturbances the frequency shift has a reduced importance; its interest may be found only for

situations where single or multi-harmonic disturbances are present since the structure may be de-tuned. Therefore, looking at the modal damping introduced by the damping treatments when the control is switched off or if for some reason the control system fails, it can be seen in Figure 9.45a that the ACLD treatment is the most efficient in all modes, with the damping ratio increasing as the mode number (frequency) increases, and that the other two configurations

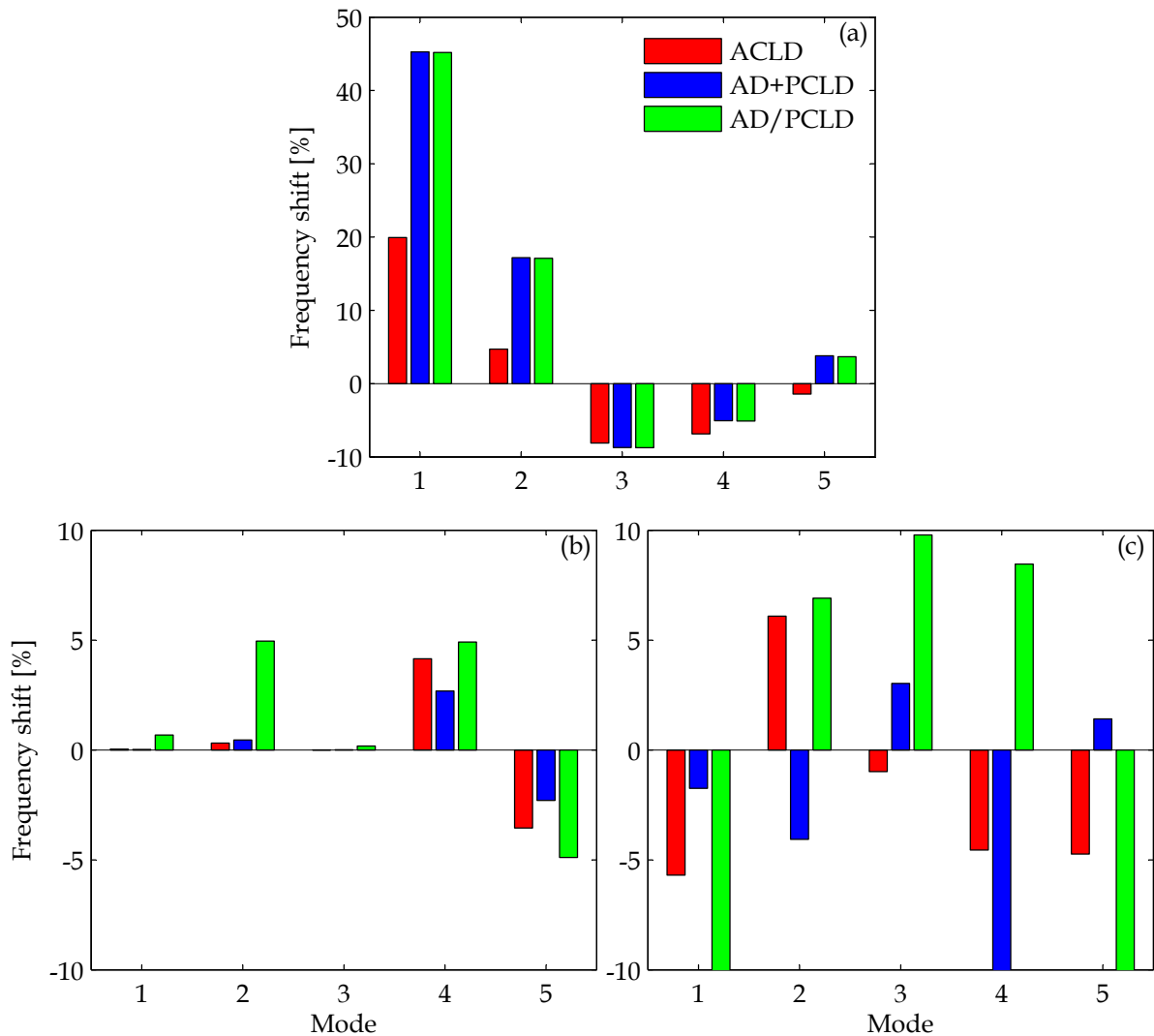


Figure 9.44: Frequency shift due to the application of the hybrid damping treatments to the bare beam (a) and due to the effects of the actuated hybrid damping treatments with the two design scenarios [(b): state weighting; (c): tip velocity weighting].

yield grossly the same half ACLD damping ratio. Therefore, the ACLD configuration is the most interesting non-actuated configuration. However, when the damping treatments are in closed-loop (actuated) the AD/PCLD configuration is again shown to have superior damping augmentation capabilities, with the advantage of requiring lower control voltages, with the superiority being more pronounced for the case where the cost function minimizes the tip velocity. Furthermore, the AD/PCLD configuration is also shown to have an excellent low frequency damping performance and as the frequency increases it approximates the damp-

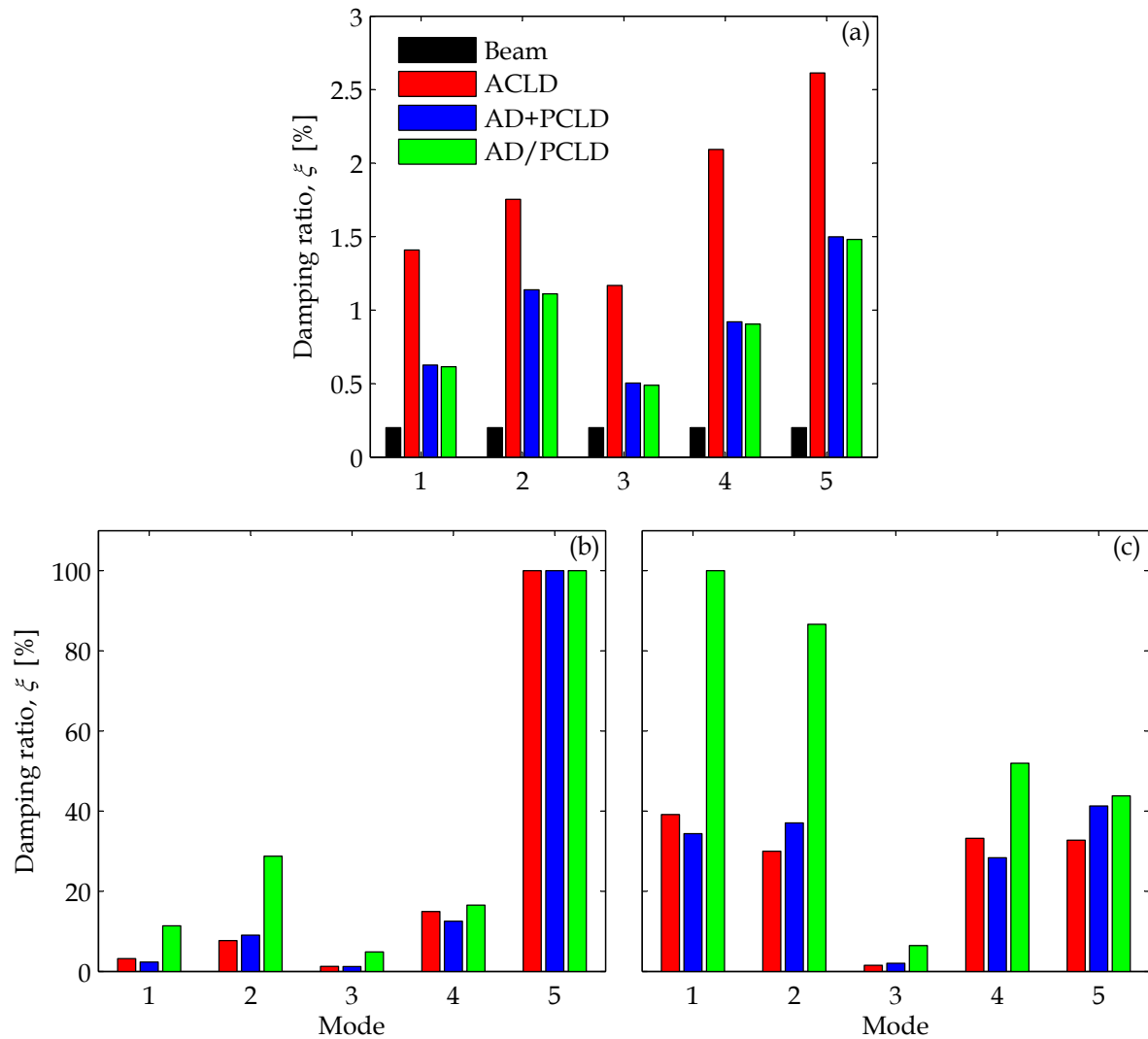


Figure 9.45: Damping ratios of the beam with the hybrid damping treatments in open-loop (a) and closed-loop with the two design scenarios [(b): state weighting; (c): tip velocity weighting].

ing performances of the other two ACLD and AD+PCLD configurations which have similar damping behavior over the 5 modes, however with the latter requiring a slightly lower control voltage. Lastly, it should be noted also in Figure 9.42b that for some reason the 5th mode has become overdamped.

9.3.3.3 Overview and Lessons Learned

In this section the configuration design of hybrid damping treatments was addressed by the comparison of three distinct hybrid damping configurations. The first is the well known ACLD treatment which has been thoroughly studied in the last 15 years or so. It was shown that its efficiency at low frequencies requires a high control voltage since the damping performance is mainly achieved by the forces and moments exerted by the piezoelectric constraining layer upon the structure through the soft (at least at low frequencies) and with low transmissibil-

ity viscoelastic layer. However, since the passive viscoelastic layer is more close to the maximum shear strain zone, which depending of the stiffness and thickness of the constraining layer is moved away from the beam's neutral axis towards the middle of the viscoelastic layer, when compared with the other configurations, the ACLD treatment has a better "fail-safe" (or passive) performance. The second configuration studied, namely the AD+PCLD, was shown to be uninteresting since when actuated it has more or less the same behavior as the ACLD with the disadvantage of not having a "fail-safe" behavior such as good as the ACLD since the placement of the actuator on the opposite side moves the neutral axis (and the zone of maximum shear strain) to the vicinity of the central axis of the beam. On the other hand, the AD/PCLD configuration require lower control voltages, nearly ten times less at the frequency of the first mode, and yields superior closed-loop damping performance. The main advantage of the AD/PCLD configuration is that the active element of the damping treatment, i.e. the piezoelectric patch, actuates directly and simultaneously in both the structure and viscoelastic layer conferring an high degree of adaptability to both the dissipation of energy in the viscoelastic layer (passive plus the coupling mechanism) and to the inhibition of the input power flowing into the structural system (active mechanism).

It should be emphasized, though, that the conclusions taken previously may not hold for all the cases, since the optimization of other design parameters, such as, for example, the thickness of the layers and/or the stiffness of the constraining layers, may lead to other conclusions. However, the latter optimization may be performed to all configurations and better damping performances may still be obtained to all configurations. Therefore, the latter argument, creates, at least, a reasonable certainty that the AD/PCLD configuration is the most interesting and efficient with better trade-off between the different damping mechanisms, simultaneously conferring good levels of controllability to both the dissipation of energy within the viscoelastic layer and the reduction of the input power coming into the structure.

9.4 Summary

This chapter has been devoted to the assessment and application of vibration control strategies using hybrid damping treatments comprising both active piezoelectric and passive viscoelastic materials. Modeling, simulation and effective real-time control implementation of hybrid damping treatments in beams were performed and reported.

Both passive, active and hybrid damping treatments, comprising the use of active piezoelectric patches alone, denoted here as *active damping treatments*, or passive viscoelastic materials in hybrid configurations with active piezoelectric or passive elastic constraining layers, denoted as *hybrid (active/passive) damping treatments*, were considered and analyzed from both numerical simulation and experimental implementation viewpoints. As far as control strategies are concerned, the simulation and real-time implementation of feedback, adaptive feedforward and hybrid (combined feedback/feedforward) control strategies on the control of vibrations of adaptive damped beams with sensing and actuating elements was performed and their damping performance, feasibility, robustness, limitations and capabilities were assessed and discussed. The FE implementation, numerical calculations and real-time simulation were performed in MATLAB and SIMULINK mathematical and control working environments.

Part V

CONCLUSION

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CONCLUSION

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CHAPTER 10

General Discussion and Conclusion

10.1 Introduction

This last chapter presents a general discussion of this dissertation on the use of hybrid damping treatments to reduce vibration levels of adaptive structures, and the main features and conclusions to be taken from it are presented. It is divided in four sections. In the first one, an overview of the dissertation is performed in order to give the reader a quick insight into what has been done by fast scanning it and commenting its different parts and chapters. Next, the main conclusions and lessons learned in the development, physical understanding, modeling, design and application of hybrid damping treatments comprising piezoelectric and viscoelastic damping layers to adaptive structures technologies, in particular, and high performance structures, in general, are presented. The new and most significant aspects and contributions of this dissertation are also highlighted and stand out from the dissertation in a subsequent section. Lastly, since no research work is ever finished or complete, founded suggestions for further research and work in the field of hybrid damping treatments and vibration control modeling and design are given.

10.2 Overview of the Dissertation

In general, the development of bi-hybrid (hybrid active/passive damping plus hybrid feedback/feedforward) vibration control systems/technologies, might be said to comprise four distinct, complementary and intertwined tasks:

1. Analysis of the vibration performance requirements and of the physical system design constraints to ensure that a feasible control strategy is devised;
2. Evaluation of the maximum vibration reduction that can be achieved assuming an exact mathematical model with perfect dynamics and an ideal controller;
3. Simulation of various control strategies and algorithms possibly accounting also with some known and possible to be modeled real physical constraints;

4. Application and real-time implementation of the designed bi-hybrid control system/technology.

With that in mind, this dissertation, organized in six distinct and complementary parts, addressed somewhat these latter four tasks. The description and main features of each of the parts are summarized in the following itemized paragraphs.

- An introduction to the dissertation was performed in Part **I** – *Introduction* – where the problems motivating this work, along with the description of some background concepts/terminology, technical issues, state of the art and relevant historic events interconnected with piezo-visco-elastic damping treatments, and the general objectives and organization of the dissertation are presented and discussed; the reader may notice that the state of the art review was intentionally presented from a philosophical/conceptual and historical viewpoint leaving the deepness and detailed review with more technical details for the remainder parts as the different subjects were subsequently introduced;
- Part **II** – *Structural Mathematical Models* – addressed the analytical modeling and approximated FE solution of arbitrary hybrid active-passive damping treatments applied to multilayer beams, plates and shells. For that purpose, four different chapters were considered: the former two were devoted to one-dimensional coupled-kinematics layerwise models of structures (beams) and the latter two to two-dimensional fully and partially refined discrete-kinematics layerwise models of structures (plates and shells). In order to efficiently model these treatments, the piezo-visco-elastic couplings between the elastic host structure and viscoelastic and piezoelectric damping layers were rigorously considered. This part was still complemented with two appendixes for the one- and two-dimensional formulations which can be found in Part **VI**;
- Vibration damping, in general, and viscoelastic damping, in particular, were addressed in Part **III** – *Vibration Damping*. This part comprises two chapters: the former addresses the modeling issues of viscoelastic damping, presenting and discussing different frequency- and time-domain modeling approaches and FE implementations, while the latter is devoted to the experimental identification of viscoelastic damping materials and to the assessment and validation of the different viscoelastic damping modeling approaches proposed;
- The contents in Part **IV** – *Vibration Control* – establish the bridge between modeling/simulation and effective real-time control implementation in structural vibrations control applications of adaptive structural systems. The part is divided in two chapters where the first addresses active and hybrid control of vibrations strategies and algorithms while the second addresses the assessment and application of the previous vibration control strategies. Both passive, active and hybrid damping treatments and hybrid control configurations (bi-hybrid damping control) are considered and analyzed. The implemented experiments and real-time control of vibrations were also described and the lessons learned regarding the main features of the practical implementation, performance, robustness and stability of classical and optimal feedback and adaptive feedforward control strategies were discussed;
- The main achievements, conclusions and outcomes of the dissertation are discussed in the present Part **V** – *Conclusion*;

- As previously referred, the dissertation still has a last part, Part **VI** – *Appendix*, comprising two appendices, which give some further details and sustained foundations of some of the theoretical developments and assumptions utilized in the chapters contained in Part **II** – *Structural Mathematical Models* – which allowed a simplification of the reading of the text by avoiding the need to include laborious mathematical calculations and definitions within these chapters.

10.3 Conclusions

The conclusions to be taken from this dissertation may be collected into four main groups, which somehow reflect the contents placed in Parts **I** to **IV** of this dissertation.

As far as the analysis of the literature and of the state of the art are concerned we may conclude the following:

- The “proof of concept” that high performance structures incorporating piezoelectric and viscoelastic damping technologies have a wide potential to solve structural vibration related problems has already been done.
- These damping technologies are gradually receiving more and more attention, acceptance and interest as practical engineering solutions for commercial products in different branches of science and related products are transitioning into the marketplace.
- This growing research field is highly multi-disciplinary and due to that the area faces a profound challenge in that too few practicing technologists and engineers have been trained to properly exploit the capabilities of these damping technologies enforcing the idea that efforts in that direction should be done.

Regarding the developed structural mathematical models of piezo-visco-elastic structures the following issues should be stressed:

- The efforts done to derive an explicit coupled-kinematics theory for piezo-visco-elastic beams with an arbitrary number of layers in arbitrary positions were successful and the general analytical model is passible of being used with different approximate methods, using whether the weak (as usual in FE modeling) or strong (e.g. meshless and spectral methods) forms, or utilized to derive closed-form analytical solutions.
- The coupled-kinematics partial layerwise analytical theory for piezo-visco-elastic beams and correspondent FE are accurate and representative of both the dynamics of piezo-visco-elastic beams and the actuator and sensing capabilities of the piezoelectric patches/layers; the FE presents a good trade-off between accuracy and FEs complexity (computational cost).
- The segmented nature of surface mounted or embedded piezo-visco-elastic damping technologies applied to host structures demand special models that must be able to consider cross-section discontinuities of the global structural system; the use of reduced abstract 1-D or 2-D FE models of these structures simplifies the mesh generation and automates the modeling process.

- The representativeness of the model in terms of localized effects such as the intra- and inter-layer strain and stress effects of the piezoelectric and viscoelastic damping layers strongly influences the global response and therefore also the predictability of the amount of damping introduced with an adequate design of the piezo-visco-elastic damping technologies.
- A quadratic definition of the electric potential was shown to be accurate and representative of both the direct and converse piezoelectric effects in the 1-D beam model.
- The use of effective stiffness parameters simplified the electro-mechanical coupled analysis since it renders modified mechanical stiffness parameters which take into account both the shear and extensional stiffness changes due to the direct piezoelectric effects not directly measurable from the electrodes.
- The 1-D piezo-visco-elastic FE is robust and can be utilized for the study of arbitrary ACLD treatments on beams with various configurations and arrangements of the layers (more sophisticated hybrid active/passive damping treatments) and with different control strategies; it may also be used to assess the control effectiveness and to implement model-based controllers.
- Semi-inverse refined deformation theories may be used to further refine “traditional” ones, opening new sustained modeling opportunities towards the development of more refined general FEs to be used to model piezo-visco-elastic damping technologies; the complexity, though, may dramatically increase if the refinement is considered also for the transverse displacement (fully refined theory) without evident gains in accuracy and representativeness.
- The relatively new concept of “regeneration”, in opposition to the Ahmad’s “degeneration” concept, allows the mathematical treatment of a single layer following “conventional” abstract 2-D formulations which are later transformed (“regenerated”) to allow the extension of a single-layer FE to a multilayer one without complicating the formulation and still yielding a multilayer abstract FE through elemental assemblage.
- Refined stress assumptions lead to a “mixed” definition of the displacement field in terms of the same generalized displacements of the FSDT and CLT, and shear stresses on the top and bottom surfaces which allow to impose also transverse shear stress continuity across the layers; the price to pay, in the dynamic case, is an increase in the number of DoFs which, for the static case, can be reduced (condensed) without any loss of accuracy.
- As far as the FE solution of the partial refined theory is concerned, C^1 continuity is required which implies also an increase in the number of elemental DoFs using conforming or non-conforming elements; it is not clear, yet, how the element would behave in terms of locking and rigid body motion when compared with equivalent displacement formulations which use C^0 continuity approximation functions through the use of mathematical artifices to circumvent the C^1 continuity requirement.
- The relaxation of Love’s classical assumptions regarding the thinness of the structure as compared with the radii of curvature is possible to be done; it increases, however, the complexity of the formulation but appears to have a reduced influence on the computational weight of the FE solution with the advantage, though, of yielding a much better representativeness in thick shell cases.

- However conceptually proposed and not validated and assessed yet, special non-standard considerations regarding the electric potential mathematical representation in curved shell structures were shown to be necessary.
- Since both the sensing/actuating and damping accuracy of piezo-visco-elastic structures are strongly dependent upon the accuracy of the “pure” structural mechanical (elastic) model counterpart, the developed fully coupled piezo-visco-elastic shell models consider a very refined mechanical (elastic) formulation, which is the first step towards the development of a very accurate fully coupled global model with excellent mechanical, electrical and damping effects representativeness.

Directing next our attention to issues on vibration damping, in general, and viscoelastic damping, in particular, the following conclusions hold:

- There are different ways of considering viscoelastic damping that have been proposed through time with all of them explicitly considering the frequency (or time) dependency of the damping behavior (viscoelastic damping), whether by differential or integral representations. Some methods are better suited for time-domain analysis (ADF) and others to frequency-domain analysis (DFR) but it is difficult to ascertain which one is best since their performance is strongly dependent of analysis parameters such as the problem size, frequency bandwidth or the amount of damping and the trade-offs are not always evident.
- The temperature and other effects are usually not coupled with the frequency dependency and its effects are considered mainly by assuming non-varying conditions of these effects (e.g. isothermal conditions).
- Frequency domain (CMA based) viscoelastic damping modeling approaches, namely the DFR, IMSE and ICE, are more straightforward to use and implement at the global FE level.
- The DFR is the most common approach implemented in commercial FE codes incorporating viscoelastic damping modeling capabilities with the outcome of being simpler to implement and the drawback of being time-consuming and not providing directly the modal parameters of the damped structural system; it should be used only in validation tasks when experimental data is a set of frequency response functions.
- The ICE approach was shown to be robust and it was shown that this strategy has some advantages over the internal variables (GHM and ADF) models: it does not need to increase the size of the problem and the eigensolution can consider only a small number of modes necessary to build the required truncated modal model. However, it has the drawback of requiring several complex eigenproblem (of a smaller size than the GHM and ADF approaches) calculations during the iterative process until convergence is achieved.
- When compared with the ICE approach, the IMSE method requires iterative calculations of a real eigenvalue problem at the cost of, at least for highly damped systems, yielding bad estimates of the modal damping ratios.
- Internal variables models have a mandatory requirement that is not needed with the other analyzed viscoelastic damping models/approaches: they require a curve fitting of the material master curves. This task is not an automated process since it requires some

experience and intervention of the user to define the required number of series and to assess the quality of the fitting and identified model parameters.

- Only internal variables models allow a direct incorporation of the damping effects in time-domain analysis performed by the mode superposition method; the increase in the systems dimension strongly depends on the number of series of model parameters to accurately represent the material. While the GHM may be solved by standard second-order numerical solution methods the ADF can only be solved using first-order ones.
- A time domain model based on a variation, or Laplace transformed, formulation of the ADF model, taking the definition of the complex modulus in the frequency (or Laplace) domain and utilizing internal, or anelastic (after Lesieutre), variables to simplify the equations, in a similar way to what is done with the GHM method but utilizing a different definition of the characteristic material function, makes the FE model implementation of the ADF model more straightforward when compared with the Lesieutre's original direct time domain formulation, based upon the method of irreversible thermodynamics and a decomposition of the total displacement field in an elastic and anelastic counterpart. It should also be remarked that Lesieutre's definition of the relaxation function is in fact a particular case of the relaxation function derived by Biot.
- The use of fractional calculus seems useful to explicitly defining the material properties through a simple mathematical equation but seems not interesting as a solution method since it imposes a "non-standard" engineering solution method.
- The viscoelastic damping modeling capabilities of commercial FE softwares mainly allow to perform time-domain analysis of viscoelastically damped structures by direct integration methods and frequency domain analysis by a direct frequency response approach. None considers iterative or internal variables approaches where truncated modal models and modal superposition methods may be used.
- Some FE software use Prony's series and have dedicated curve-fitting tools to identify Prony's parameters from time-domain creep and relaxation data and frequency dependent test data. For temperature effects, thermorheological simplicity is considered and the Williams-Landel-Ferry (WLF) method is the model available for simulation of mechanical response under prescribed temperature histories.
- The spatial modeling of multilayer damping treatments in commercial FE software is usually obtained by combining the existent FEs into a "composite" elemental FE that may require non-standard discretization (mesh generation) and assemblage procedures and a high computational cost.

Lastly, from the assessment and application of vibration control strategies, encompassing active and hybrid (active/passive) damping treatments, the following major conclusions may be drawn:

- From the experimental implementation it was verified that the mounting (bonding) of the piezoelectric ceramic and viscoelastic patches is a matter of considerable difficulty. Piezoelectric ceramics are brittle and it is very easy to fracture and damage them while soldering the wires and bonding them. Care must be taken during this process because it may seriously compromise the actuating and damping efficiency of the damping patches. A perfect transmissibility of the efforts is usually difficult to obtain.

- Active damping through piezoelectric actuation on beams may be designed and accurately predicted by using the developed FE model for simulation. Additionally, the model allows the determination of the optimal gains and stability design.
- Active damping is efficient in reducing low-frequency vibrations but from an application viewpoint there are still aspects that may make the strategy unfeasible for some cases: it usually requires considerable processing and power heavy electronics that may be unpractical to use in some real life design scenarios. The degree of electronic integration is in the author's opinion one of the key issues making this damping approaches not so appealing. On the other hand, sensing, for example for structural health monitoring, only requires low voltage electronics which may turn this application more appealing.
- It is well known that observation and control spillover problems related with the reduced (truncated) or unobserved model dynamics may compromise stability. The adequate use of analog or digital filtering can remedy that situation with an increase in the required electronics though. Spatially distributed piezoelectric sensors (filters) may alleviate this issue.
- One great advantage of the classical control techniques is that they can avoid the necessity of digital control, reducing the time delays in the control system, and that stability is guaranteed provided the actuator/sensor pair is perfectly collocated.
- Velocity feedback strategies (e.g. CGVF or CAVF) require the differentiation of the sensor voltage, which for noisy measurements can become troublesome making it necessary special analog or digital filtering "countermeasures".
- Optimal control strategies provide the designer with lots of flexibility to perform trade-offs among various performance criteria conferring to the control design an higher degree of adaptability; they have the disadvantage of requiring a model of the system but the advantage of holding even for high-order and MIMO systems, where classical control becomes cumbersome.
- Realistic implementations of the LQR in distributed parameters systems can only be achieved by appending a state observer; LQG control remedies the influence on noise by estimating the states using a Kalman-Bucy filter, which utilizes noisy partial output information.
- When some information correlated with the excitation is available, hybrid (feedback/adaptive feedforward) control may improve the feedback authority and the performance and convergence of the adaptive feedforward controller.
- In this specific feedforward control application considered, the AD+PCLD treatment was shown to be more efficient than the ACLD one requiring lower control voltages. However, one has to be cautious on generalizing the conclusions taken from this case which may not hold in other cases such as when other performance indexes, or cost functions minimizing other parameters, are utilized. However, a reasonable certainty that in feedforward control applications AD+PCLD when compared with ACLD is more efficient in the bandwidth up to roughly 1 kHz can be stated.
- When considering viscoelastic layers in the way of the feedforward "actuation path" to the structure, the transmissibility of efforts from the viscoelastic layer to the host structure is usually reduced, mainly at low frequencies, and the performance of the ACLD

treatment, or similar ones, would only be improved if one designs a feedforward controller trying to optimize the damping due to the shearing of the viscoelastic layer, which in turn may require lower control voltages, and if the dissipation of energy in the viscoelastic layer produces damping effects more significant than the reduction of the input disturbance power coming into the structure.

- The efficiency of ACLD at low frequencies requires a high control voltage since the damping performance is mainly achieved by the forces and moments exerted by the piezoelectric constraining layer upon the structure through the soft (at least at low frequencies) and with low transmissibility viscoelastic layer. However, since the passive viscoelastic layer is more close to the maximum shear strain zone, which depending of the stiffness and thickness of the constraining layer is moved away from the beam's neutral axis towards the middle of the viscoelastic layer, the ACLD treatment when compared with the other configurations has a better "fail-safe" (or passive) performance.
- The AD+PCLD configuration was shown to be uninteresting since when actuated it has more or less the same behavior as the ACLD with the disadvantage of not having a "fail-safe" behavior such as good as the ACLD since the placement of the actuator on the opposite side moves the neutral axis (and the zone of maximum shear strain) to the vicinity of the central axis of the beam.
- On the other hand, the AD/PCLD configuration requires lower control voltages, nearly ten times less at the frequency of the first mode, and yields superior closed-loop damping performance. The main advantage of the AD/PCLD configuration is that the active element of the damping treatment, i.e. the piezoelectric patch, actuates directly and simultaneously in both the structure and viscoelastic layer conferring an high degree of adaptability to both the dissipation of energy in the viscoelastic layer (passive plus the coupling mechanism) and to the inhibition of the input power flowing into the structural system (active mechanism).
- It should be emphasized, though, that the conclusions taken previously regarding the damping treatments configurations may not hold for all the cases, since the optimization of other design parameters, such as, for example, the thickness of the layers and/or the stiffness of the constraining layers, may lead to other conclusions. However, the latter optimization may be performed to all configurations and better damping performances may still be obtained to all configurations. Therefore, the latter argument, creates, at least, a reasonable certainty that the AD/PCLD configuration is the most interesting and efficient with better trade-off between the different damping mechanisms, simultaneously conferring good levels of controllability to both the dissipation of energy within the viscoelastic layer and the reduction of the input power coming into the structure.
- At this point it seems that the developed models are very useful for understanding the physics of the damping technologies and for control strategies simulation at the design stage. It is the author's opinion, though, that on-line system identification, adaptive hybrid control strategies and developments in order to further integrate the electronics into the structure are realistic issues that need to be seriously taken into account to make this technologies off our bookshelves.

10.4 New and Most Significant Contributions

From this dissertation there are a few aspects that by its relative novelty character and significance deserve to be stood out. These are the following:

- A coupled-kinematics partial layerwise analytical theory for piezo-visco-elastic beams was developed and a FE was developed and implemented. The main differences from classical layerwise theories are that the present one does not need an a posteriori assemblage of the contributions of each discrete layer, ensuring displacement continuity at the interfaces, to obtain the global model. Therefore, this generic analytical model is possible of being used with different approximate methods, using whether the weak (as usual in FE modeling) or strong (e.g. meshless and spectral methods) forms, or utilized to derive closed-form analytical solutions.
- As far as the developed piezo-visco-elastic shell FE is concerned, this dissertation proposes some advances to the classical approach to “traditional” FEs, opening new sustained modeling opportunities towards the development of more refined FE solutions. That refinement is made taking into account semi-inverse deformation theories and analytical solutions developed and established in the middle of the 20th century by the Eastern Europe and Russian school. In this dissertation they were combined with some of the more recent developments and trends of what computational solid mechanics is nowadays. The method is based in a semi-inverse theory where the postulates are done in the opposite direction to what is usually done. One starts by postulating first a stress field in a single layer of the multilayer structure and through the use of the equilibrium equations, constitutive laws and strain-stress relationships, three-dimensional perfectly consistent displacement fields (obtained from the equilibrium equations) are determined.
- A relatively new concept of “regeneration”, in opposition to the Ahmad’s “degeneration” concept, is used to go from a single layer to a multilayer structural model following the same discrete layer kinematic assumptions and allowing through-the-thickness displacement and transverse stresses continuity.
- Regarding the modeling of the viscoelastic damping behavior, both measured and predicted results were utilized to validate the methodologies to include viscoelastic damping in FE models. Additionally, the characteristic material parameters of the GHM and ADF models of the 3M ISD112 were obtained by curve-fitting the measured shear storage modulus and loss factor of “actual” 3M ISD112 material. Furthermore, the experimental data is usually taken by a “manual” procedure from the plot of the nomogram given by the manufacturer.
- A Laplace transformed formulation of the ADF model is proposed and a variant of the IMSE method, based on an iterative complex eigensolution, i.e. the ICE approach, is described and implemented at the “global” FE level. The revisited (and in some extent new) ICE approach is assessed and compared with other damping modeling strategies. To the author’s best knowledge, the ICE method is for the first time proposed here and there has not been any discussion in the open literature regarding its application to the characterization of viscoelastically damped structures.
- The effectiveness of hybrid control strategies was accessed numerically and the use of adaptive feedforward control of vibrations with hybrid damping strategies was investigated both numerically and experimentally.

- A contribution to understanding how hybrid damping treatments behave and to know which one is best and easily implemented in practice was done. Furthermore, different damping treatments configurations were compared and their damping performances assessed through a new hybrid damping design assessment methodology able to quantify the independent damping contributions of the so-called active, passive and hybrid damping mechanisms.

10.5 Suggestions for Further Research

As previously mentioned, no study is ever complete or finished. Therefore, the research efforts conducted in this dissertation throws open several sub-topics and areas of research somewhat addressing the design, analysis and application of new emerging piezoelectric and viscoelastic damping technologies, which immediately emanate from this study and are worth to pursue in the future. These can be said to comprise, in general, the following main five research fields/topics:

Structural Modeling: Motivated by the necessity of designers of high performance structures to have at their disposal FE models with a better representativeness than the actual alternatives available in home-made or commercial FE codes, the further development and application of the fully and partially refined modeling approaches introduced in this dissertation to model structural systems (beams, plates and shells) comprising piezo-visco-elastic damping treatments in passive, active and hybrid damping configurations, is suggested.

Viscoelastic Damping: Though viscoelastic treatments are very effective solutions as damping mechanisms for light structures, its numerical analysis is quite difficult and cumbersome. One of these difficulties is related with the development and application of efficient constitutive models able to represent the highly frequency and temperature dependent properties of the viscoelastic materials. The currently available constitutive models are merely able to represent the frequency dependence of the viscoelastic complex moduli at isothermal conditions, making it necessary to apply the frequency-temperature correspondence principle and to run several analysis to take different temperature conditions into account in a numerical analysis of viscoelastic treatments. Since this correspondence principle is also applied during the identification process of the reduced-frequency nomograms determined from isothermal experimental data obtained from experimental viscoelastic materials characterization devices, it would be a better and more accurate solution to use a generalized constitutive model, simultaneously considering the frequency and temperature dependence, which would be able to directly use in the fitting process not only frequency dependent experimental data but also the temperature dependence. Therefore, the development and FE implementation of general temperature- and frequency-dependent viscoelastic damping models is suggested.

Vibration and Structural Acoustics Control: The extension to, further development, simulation and real-time implementation of passive viscoelastic and shunt damping (through the use of passive electrical networks) and active/hybrid control strategies considering both adaptive feedback and feedforward algorithms with hybrid damping treatments (bi-hybrid control) and their use to control both structural vibrations and acoustic radiation is strongly encouraged. The use of adaptive control techniques and the assessment

of the real-time implementation feasibility, considering both centralized and decentralized SISO and MIMO control configurations, are, without any doubt, a mandatory step towards the use of robust and adaptive damping technologies as engineering solutions.

Design and Optimization: Analysis, simulation and development of optimal design strategies and technical solutions to improve the efficiency of the damping treatments in terms of location, geometric configuration (number, type and stacking sequence of the damping layers), control law, material properties, thicknesses, etc.

Application: As these technologies start entering the marketplace and capturing the attention and interest of their potential end-users, relevant research issues must be addressed for successful transition. Among them, one is the integration and miniaturization of hybrid damping technologies, consolidating mechanical systems, smart materials and electronics, which should be integrated into a unique compact device. Therefore, despite the real-time test and analysis and validation of the models, further research efforts addressing the development, application and adaptation of bi-hybrid damping technologies to industrial needs should be pursued.

Part VI

APPENDIX

Part VI

APPENDIX

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APPENDIX A

Complements to the Beam Formulation

A.1 Hamilton's Principle

From Hamilton's principle in Equation (4.52) we have

$$\delta \int_{t_0}^{t_1} (T - H + W) dt = 0. \quad (\text{A.1})$$

Substituting the virtual works of the total internal electro-mechanical, inertial and external forces and electric charge density, presented in Chapter 2 in Equations (2.38), (2.43), (2.49) and (2.51), into (A.1), and considering the contributions of all the layers, yields

$$\int_{t_0}^{t_1} \left\{ \sum_{n=1}^{\bar{n}} \left[-\delta T^n + \delta H_{uu}^{n(\phi)} + \delta H_{u\phi}^{n(\phi)} + \delta H_{\phi u}^{n(\phi)} - \delta H_{\phi\phi}^{n(\phi)} - \delta W_u^n - \delta W_\phi^n \right] - \delta T^c + \delta H_{uu}^c - \delta W_u^c + \sum_{m=\bar{m}}^{-1} \left[-\delta T^m + \delta H_{uu}^{m(\phi)} + \delta H_{u\phi}^{m(\phi)} + \delta H_{\phi u}^{m(\phi)} - \delta H_{\phi\phi}^{m(\phi)} - \delta W_u^m - \delta W_\phi^m \right] \right\} dL dt. \quad (\text{A.2})$$

For convenience the signs in the previous equation were reversed and it will be used in the following for the definition of the weak and strong forms of the analytical model.

A.2 Weak Forms

Substituting the virtual work terms defined in Chapter 2 into Equation (A.2) and collecting the terms relative to the variations of the generalized displacements and electric potential differ-

ences, yields

$$\begin{aligned}
& \int_{t_0}^{t_1} \int_L \left\{ \sum_{n=1}^{\bar{n}} \left[\delta u_0 \left(\rho_n A_n \ddot{u}_0 + \rho_n h_0 A_n \ddot{\theta}_0 + \rho_n \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i + \rho_n \bar{I}_n \ddot{\theta}_n \right) + \delta w_0 \rho_n A_n \ddot{w}_0 \right. \right. \\
& + \delta \theta_0 \left(\rho_n h_0 A_n \ddot{u}_0 + \rho_n h_0^2 A_n \ddot{\theta}_0 + \rho_n h_0 \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i + \rho_n h_0 \bar{I}_n \ddot{\theta}_n \right) + \sum_{i=1}^{n-1} \delta \theta_i \left(\rho_n 2h_i A_n \ddot{u}_0 \right. \\
& + \rho_n 2h_i h_0 A_n \ddot{\theta}_0 + \rho_n 4h_i \sum_{j=1}^{n-1} h_j A_n \ddot{\theta}_j + \rho_n 2h_i \bar{I}_n \ddot{\theta}_n \left. \right) + \delta \theta_n \left(\rho_n \bar{I}_n \ddot{u}_0 + \rho_n h_0 \bar{I}_n \ddot{\theta}_0 \right. \\
& + \rho_n \sum_{i=1}^{n-1} 2h_i \bar{I}_n \ddot{\theta}_i + \rho_n \bar{I}_n \ddot{\theta}_n \left. \right) + \delta u'_0 \left(c_{11}^{*n} A_n u'_0 + c_{11}^{*n} h_0 A_n \theta'_0 + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i A_n \theta'_i + c_{11}^{*n} \bar{I}_n \theta'_n \right) \\
& + \delta w'_0 \left(\bar{c}_{55}^{n(\phi)} A_n w'_0 + \bar{c}_{55}^{n(\phi)} A_n \theta_n \right) + \delta \theta'_0 \left(c_{11}^{*n} h_0 A_n u'_0 + c_{11}^{*n} h_0^2 A_n \theta'_0 + c_{11}^{*n} h_0 \sum_{j=1}^{n-1} 2h_j A_n \theta'_j \right. \\
& + c_{11}^{*n} h_0 \bar{I}_n \theta'_n \left. \right) + \sum_{i=1}^{n-1} \delta \theta'_i \left(c_{11}^{*n} 2h_i A_n u'_0 + c_{11}^{*n} 2h_i h_0 A_n \theta'_0 + c_{11}^{*n} 2h_i \sum_{j=1}^{n-1} 2h_j A_n \theta'_j + c_{11}^{*n} 2h_i \bar{I}_n \theta'_n \right) \\
& + \delta \theta'_n \left(c_{11}^{*n} \bar{I}_n u'_0 + c_{11}^{*n} h_0 \bar{I}_n \theta'_0 + c_{11}^{*n} \sum_{i=1}^{n-1} 2h_i \bar{I}_n \theta'_i + \bar{c}_{11}^{n(\phi)} \bar{I}_n \theta'_n \right) + \delta \theta_n \left(c_{55}^n A_n w'_0 + c_{55}^n A_n \theta'_n \right) \\
& + \delta u'_0 A_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \delta \theta'_0 h_0 A_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \sum_{i=1}^{n-1} \delta \theta'_i 2h_i A_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \delta \theta'_n \bar{I}_n \frac{e_{31}^{*n}}{2h_n} \phi_n - \delta u_0 X_n - \delta \theta_0 h_0 X_n \\
& - \sum_{i=1}^{n-1} \delta \theta_i 2h_i X_n - \delta \theta_n h_n X_n - \delta \theta_n M_n - \delta w_0 Z_n + \delta \phi_n \frac{e_{31}^{*n}}{2h_n} \left(A_n u'_0 + h_0 A_n \theta'_0 + \sum_{i=1}^{n-1} 2h_i A_n \theta'_i \right. \\
& + \bar{I}_n \theta'_n \left. \right) - \delta \phi_n \frac{\varepsilon_{33}^{*n} A_n}{4h_n^2} \phi_n + \delta \phi_n b \tau_n \left. \right] + \delta u_0 \rho_c A_c \ddot{u}_0 + \delta w_0 \rho_c A_c \ddot{w}_0 + \delta \theta_0 \rho_c I_c \ddot{\theta}_0 \\
& + \delta u'_0 c_{11}^{*c} A_c u'_0 + \delta w'_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) + \delta \theta'_0 c_{11}^{*c} I_c \theta'_0 + \delta \theta_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) \\
& - \delta u_0 X_c - \delta \theta_0 M_c - \delta w_0 Z_c + \sum_{m=\bar{m}}^{-1} \left[\delta u_0 \left(\rho_m A_m \ddot{u}_0 - \rho_m h_0 A_m \ddot{\theta}_0 - \rho_m \sum_{i=m+1}^{-1} 2h_i A_m \ddot{\theta}_i \right. \right. \\
& + \rho_m \bar{I}_m \ddot{\theta}_m \left. \right) + \delta w_0 \rho_m A_m \ddot{w}_0 + \delta \theta_0 \left(-\rho_m h_0 A_m \ddot{u}_0 + \rho_m h_0^2 A_m \ddot{\theta}_0 + \rho_m h_0 \sum_{i=m+1}^{-1} 2h_i A_m \ddot{\theta}_i \right. \\
& - \rho_m h_0 \bar{I}_m \ddot{\theta}_m \left. \right) + \sum_{j=m+1}^{-1} \delta \theta_j \left(-\rho_m 2h_j A_m \ddot{u}_0 + \rho_m 2h_j h_0 A_m \ddot{\theta}_0 + \rho_m 4h_j \sum_{i=m+1}^{-1} h_i A_m \ddot{\theta}_i \right. \\
& - \rho_m 2h_j \bar{I}_m \ddot{\theta}_m \left. \right) + \delta \theta_m \left(\rho_m \bar{I}_m \ddot{u}_0 + \rho_m h_0 \bar{I}_m \ddot{\theta}_0 - \rho_m \sum_{i=m+1}^{-1} 2h_i \bar{I}_m \ddot{\theta}_i + \rho_m \bar{I}_m \ddot{\theta}_m \right) \\
& + \delta u'_0 \left(c_{11}^{*m} A_m u'_0 - c_{11}^{*m} h_0 A_m \theta'_0 - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i + c_{11}^{*m} \bar{I}_m \theta'_m \right) + \delta w'_0 \left(\bar{c}_{55}^{m(\phi)} A_m w'_0 \right. \\
& + \bar{c}_{55}^{m(\phi)} A_m \theta_m \left. \right) + \delta \theta'_0 \left(-c_{11}^{*m} h_0 A_m u'_0 + c_{11}^{*m} h_0^2 A_m \theta'_0 + c_{11}^{*m} h_0 \sum_{j=m+1}^{-1} 2h_j A_m \theta'_j \right. \\
& - c_{11}^{*m} h_0 \bar{I}_m \theta'_m \left. \right) + \sum_{i=m+1}^{-1} \delta \theta'_i \left(-c_{11}^{*m} 2h_i A_m u'_0 + c_{11}^{*m} 2h_i h_0 A_m \theta'_0 + c_{11}^{*m} 2h_i \sum_{j=m+1}^{-1} 2h_j A_m \theta'_j \right. \\
& - c_{11}^{*m} 2h_i \bar{I}_m \theta'_m \left. \right) + \delta \theta'_m \left(c_{11}^{*m} \bar{I}_m u'_0 - c_{11}^{*m} h_0 \bar{I}_m \theta'_0 - c_{11}^{*m} \sum_{i=m+1}^{-1} 2h_i \bar{I}_m \theta'_i + \bar{c}_{11}^{m(\phi)} \bar{I}_m \theta'_m \right) \\
& + \delta \theta_m \left(\bar{c}_{55}^{m(\phi)} A_m w'_0 + \bar{c}_{55}^{m(\phi)} A_m \theta'_m \right) + \delta u'_0 A_m \frac{e_{31}^{*m}}{2h_m} \phi_m - \delta \theta'_0 h_0 A_m \frac{e_{31}^{*m}}{2h_m} \phi_m
\end{aligned}$$

$$\begin{aligned}
& - \sum_{i=m+1}^{-1} \delta\theta'_i 2h_i A_m \frac{e_{31}^{*m}}{2h_m} \phi_m + \delta\theta'_m \bar{I}_m \frac{e_{31}^{*m}}{2h_m} \phi_m - \delta u_0 X_m + \delta\theta_0 h_0 X_m + \sum_{i=m+1}^{-1} \delta\theta_i 2h_i X_m \\
& + h_m \delta\theta_m X_m + \delta\theta_m M_m + \delta w_0 Z_m + \delta\phi_m \frac{e_{31}^{*m}}{2h_m} \left(A_m u'_0 - h_0 A_m \theta'_0 - \sum_{i=m+1}^{-1} 2h_i A_m \theta'_i \right. \\
& \quad \left. + \bar{I}_m \theta'_m \right) - \delta\phi_m \frac{\varepsilon_{33}^{*m} A_m}{4h_m^2} \phi_m + \delta\phi_m b\tau_m \Big] dLdt = 0. \quad (A.3)
\end{aligned}$$

Taking the following properties of sums into consideration in the previous equation,

$$\sum_{n=1}^{\bar{n}} \sum_{i=1}^{n-1} f_n f_i \delta\theta_i = \sum_{i=1}^{\bar{n}} \left(\sum_{n=i+1}^{\bar{n}} f_n f_i \right) \delta\theta_i, \quad (A.4)$$

$$\sum_{m=\bar{m}}^{-1} \sum_{i=m+1}^{-1} f_m f_i \delta\theta_i = \sum_{i=\bar{m}}^{-1} \left(\sum_{m=i+1}^{-1} f_m f_i \right) \delta\theta_i, \quad (A.5)$$

$$\sum_{n=i+1}^{\bar{n}} \sum_{j=1}^{n-1} f_n f_i f_j \theta_j = \sum_{j=1}^{i-1} \left(\sum_{n=i+1}^{\bar{n}} f_n f_i f_j \right) \theta_j + \sum_{j=i}^{\bar{n}-1} \left(\sum_{n=j+1}^{\bar{n}} f_n f_i f_j \right) \theta_j, \quad (A.6)$$

$$\sum_{m=\bar{m}}^{i-1} \sum_{j=m+1}^{-1} f_m f_i f_j \theta_j = \sum_{j=i+1}^{-1} \left(\sum_{m=\bar{m}}^{i-1} f_m f_i f_j \right) \theta_j + \sum_{j=\bar{m}+1}^i \left(\sum_{m=\bar{m}}^{j-1} f_m f_i f_j \right) \theta_j, \quad (A.7)$$

where f_n , f_m , f_i and f_j are generic functions of the displacements and electric potential differences with the n , m , i and j subscripts, the variational terms are transformed to a more convenient representation where the variations become decoupled from the sums. For the sake of brevity only some changes in the terms are shown in the following equations. However, similar transformations are obtained for the stiffness terms and for the bottom layers.

$$\sum_{n=1}^{\bar{n}} \rho_n \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i = \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \rho_n 2h_i A_n \ddot{\theta}_i = \sum_{n=1}^{\bar{n}} \sum_{i=n+1}^{\bar{n}} \rho_i 2h_n A_i \ddot{\theta}_n, \quad (A.8)$$

$$\sum_{n=1}^{\bar{n}} \rho_n h_0 \sum_{i=1}^{n-1} 2h_i A_n \ddot{\theta}_i = \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \rho_n h_0 2h_i A_n \ddot{\theta}_i = \sum_{n=1}^{\bar{n}} \sum_{i=n+1}^{\bar{n}} \rho_i h_0 2h_n A_i \ddot{\theta}_n, \quad (A.9)$$

$$\begin{aligned}
& \sum_{n=1}^{\bar{n}} \sum_{i=1}^{n-1} \delta\theta_i \left(\rho_n 2h_i A_n \ddot{u}_0 + \rho_n 2h_i h_0 A_n \ddot{\theta}_0 + \rho_n 4h_i \sum_{j=1}^{n-1} h_j A_n \ddot{\theta}_j + \rho_n 2h_i \bar{I}_n \ddot{\theta}_n \right) \\
& = \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \delta\theta_i \left(\rho_n 2h_i A_n \ddot{u}_0 + \rho_n 2h_i h_0 A_n \ddot{\theta}_0 + \rho_n 2h_i \bar{I}_n \ddot{\theta}_n \right) + \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \delta\theta_i \rho_n 4h_i \sum_{j=1}^{n-1} h_j A_n \ddot{\theta}_j \\
& = \sum_{n=1}^{\bar{n}} \sum_{i=n+1}^{\bar{n}} \delta\theta_n \left(\rho_i 2h_n A_i \ddot{u}_0 + \rho_i 2h_n h_0 A_i \ddot{\theta}_0 + \rho_i 2h_n \bar{I}_i \ddot{\theta}_i \right) + \sum_{i=1}^{\bar{n}} \delta\theta_i \left(\sum_{n=i+1}^{\bar{n}} \rho_n 4h_i \sum_{j=1}^{n-1} h_j A_n \ddot{\theta}_j \right), \quad (A.10)
\end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^{\bar{n}} \delta\theta_i \left(\sum_{n=i+1}^{\bar{n}} \sum_{j=1}^{n-1} \rho_n 4h_i h_j A_n \ddot{\theta}_j \right) = \sum_{i=1}^{\bar{n}} \delta\theta_i \left(\sum_{j=1}^{i-1} \sum_{n=i+1}^{\bar{n}} \rho_n 4h_i h_j A_n \ddot{\theta}_j \right) \\
& \quad + \sum_{i=1}^{\bar{n}} \delta\theta_i \left(\sum_{j=i}^{\bar{n}-1} \sum_{n=j+1}^{\bar{n}} \rho_n 4h_i h_j A_n \ddot{\theta}_j \right) = \sum_{n=1}^{\bar{n}} \delta\theta_n \left(\sum_{i=1}^{n-1} \sum_{j=n+1}^{\bar{n}} \rho_j 4h_n h_i A_j \ddot{\theta}_i \right) \\
& \quad + \sum_{n=1}^{\bar{n}} \delta\theta_n \left(\sum_{j=n}^{\bar{n}-1} \sum_{i=j+1}^{\bar{n}} \rho_i 4h_n h_j A_i \ddot{\theta}_j \right), \quad (A.11)
\end{aligned}$$

$$\sum_{n=1}^{\bar{n}} \sum_{i=1}^{n-1} \delta\theta_i 2h_i X_n = \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \delta\theta_i 2h_i X_n = \sum_{n=1}^{\bar{n}} \delta\theta_n \sum_{i=n+1}^{\bar{n}} 2h_n X_i, \quad (\text{A.12})$$

$$\sum_{n=1}^{\bar{n}} \sum_{i=1}^{n-1} \delta\theta'_i 2h_i A_n \frac{e_{31}^{*n}}{2h_n} \phi_n = \sum_{i=1}^{\bar{n}} \sum_{n=i+1}^{\bar{n}} \delta\theta'_i 2h_i A_n \frac{e_{31}^{*n}}{2h_n} \phi_n = \sum_{n=1}^{\bar{n}} \sum_{i=n+1}^{\bar{n}} \delta\theta'_n 2h_n A_i \frac{e_{31}^{*i}}{2h_i} \phi_i. \quad (\text{A.13})$$

Thus, considering the transformations, after some algebra Equation (A.3) becomes

$$\begin{aligned} & \int_{t_0}^{t_1} \int_L \left\{ \sum_{n=1}^{\bar{n}} \left[\delta u_0 \left(\rho_n A_n \ddot{u}_0 + \rho_n h_0 A_n \ddot{\theta}_0 + \left\{ \sum_{i=n+1}^{\bar{n}} \rho_i 2h_n A_i + \rho_n \bar{I}_n \right\} \ddot{\theta}_n \right) + \delta w_0 \rho_n A_n \ddot{w}_0 \right. \right. \\ & + \delta \theta_0 \left(\rho_n h_0 A_n \ddot{u}_0 + \rho_n h_0^2 A_n \ddot{\theta}_0 + \left\{ \sum_{i=n+1}^{\bar{n}} \rho_i h_0 2h_n A_i + \rho_n h_0 \bar{I}_n \right\} \ddot{\theta}_n \right) + \delta \theta_n \left(\left\{ \sum_{i=n+1}^{\bar{n}} \rho_i 2h_n A_i \right. \right. \\ & + \left. \left. \rho_n \bar{I}_n \right\} \ddot{u}_0 + \left\{ \sum_{i=n+1}^{\bar{n}} \rho_i 2h_n h_0 A_i + \rho_n h_0 \bar{I}_n \right\} \ddot{\theta}_0 + \sum_{i=1}^{n-1} \left\{ \sum_{j=n+1}^{\bar{n}} \rho_j 4h_n h_i A_j + \rho_n 2h_i \bar{I}_n \right\} \ddot{\theta}_i \right. \\ & + \left. \left\{ \sum_{i=n+1}^{\bar{n}} \rho_i 4h_n h_n A_i + \rho_n \bar{I}_n \right\} \ddot{\theta}_n + \sum_{j=n+1}^{\bar{n}} \left\{ \sum_{i=j+1}^{\bar{n}} \rho_i 4h_n h_j A_i + \rho_j 2h_n \bar{I}_j \right\} \ddot{\theta}_j \right) \\ & + \delta u'_0 \left(c_{11}^{*n} A_n u'_0 + c_{11}^{*n} h_0 A_n \theta'_0 + \left\{ \sum_{i=n+1}^{\bar{n}} c_{11}^{*i} 2h_n A_i + c_{11}^{*n} \bar{I}_n \right\} \theta'_n + A_n \frac{e_{31}^{*n}}{2h_n} \phi_n \right) \\ & + \delta w'_0 \left(\bar{c}_{55}^{n(\phi)} A_n w'_0 + \bar{c}_{55}^{n(\phi)} A_n \theta_n \right) + \delta \theta'_0 \left(c_{11}^{*n} h_0 A_n u'_0 + c_{11}^{*n} h_0^2 A_n \theta'_0 + \left\{ \sum_{i=n+1}^{\bar{n}} c_{11}^{*i} h_0 2h_n A_i \right. \right. \\ & + \left. \left. c_{11}^{*n} h_0 \bar{I}_n \right\} \theta'_n + h_0 A_n \frac{e_{31}^{*n}}{2h_n} \phi_n \right) + \delta \theta'_n \left(\left\{ \sum_{i=n+1}^{\bar{n}} c_{11}^{*i} 2h_n A_i + c_{11}^{*n} \bar{I}_n \right\} u'_0 + \left\{ \sum_{i=n+1}^{\bar{n}} c_{11}^{*i} 2h_n h_0 A_i \right. \right. \\ & + \left. \left. c_{11}^{*n} h_0 \bar{I}_n \right\} \theta'_0 + \sum_{i=1}^{n-1} \left\{ \sum_{j=n+1}^{\bar{n}} c_{11}^{*j} 4h_n h_i A_j + c_{11}^{*n} 2h_i \bar{I}_n \right\} \theta'_i + \left\{ \sum_{i=n+1}^{\bar{n}} c_{11}^{*i} 4h_n h_n A_i + \bar{c}_{11}^{n(\phi)} \bar{I}_n \right\} \theta'_n \right. \\ & + \left. \sum_{j=n+1}^{\bar{n}-1} \left\{ \sum_{i=j+1}^{\bar{n}} c_{11}^{*i} 4h_n h_j A_i + c_{11}^{*j} 2h_n \bar{I}_j \right\} \theta'_j + \bar{I}_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \sum_{i=n+1}^{\bar{n}} 2h_n A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \right) \\ & + \delta \theta_n \left(c_{55}^n A_n w'_0 + c_{55}^n A_n \theta_n \right) - \delta u_0 X_n - \delta \theta_0 h_0 X_n - \delta \theta_n \left\{ \sum_{i=n+1}^{\bar{n}} 2h_n X_i + h_n X_n + M_n \right\} \\ & - \delta w_0 Z_n \left. \right] + \delta \phi_n \left(\frac{e_{31}^{*n}}{2h_n} A_n u'_0 + \frac{e_{31}^{*n}}{2h_n} h_0 A_n \theta'_0 + \sum_{i=1}^{n-1} \frac{e_{31}^{*i}}{2h_n} 2h_i A_n \theta'_i + \frac{e_{31}^{*n}}{2h_n} \bar{I}_n \theta'_n \right) - \delta \phi_n \frac{\varepsilon_{33}^{*n} A_n}{4h_n^2} \phi_n \\ & + \delta \phi_n b \tau_n + \delta u_0 \rho_c A_c \ddot{u}_0 + \delta w_0 \rho_c A_c \ddot{w}_0 + \delta \theta_0 \rho_c I_c \ddot{\theta}_0 + \delta u'_0 c_{11}^{*c} A_c u'_0 + \delta w'_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) \\ & + \delta \theta'_0 c_{11}^{*c} I_c \theta'_0 + \delta \theta_0 \left(c_{55}^c A_c w'_0 + c_{55}^c A_c \theta_0 \right) - \delta u_0 X_c - \delta \theta_0 M_c - \delta w_0 Z_c + \sum_{m=\bar{m}}^{-1} \left[\delta u_0 \left(\rho_m A_m \ddot{u}_0 \right. \right. \\ & - \left. \left. \rho_m h_0 A_m \ddot{\theta}_0 + \left\{ - \sum_{i=\bar{m}}^{m-1} \rho_i 2h_m A_i + \rho_m \bar{I}_m \right\} \ddot{\theta}_m \right) + \delta w_0 \rho_m A_m \ddot{w}_0 + \delta \theta_0 \left(-\rho_m h_0 A_m \ddot{u}_0 \right. \right. \\ & + \left. \left. \rho_m h_0^2 A_m \ddot{\theta}_0 + \left\{ \sum_{i=\bar{m}}^{m-1} \rho_i h_0 2h_m A_i - \rho_m h_0 \bar{I}_m \right\} \ddot{\theta}_m \right) + \delta \theta_m \left(\left\{ - \sum_{i=\bar{m}}^{m-1} \rho_i 2h_m A_i + \rho_m \bar{I}_m \right\} \ddot{u}_0 \right. \right. \\ & + \left. \left\{ \sum_{i=\bar{m}}^{m-1} \rho_i 2h_m h_0 A_i - \rho_m h_0 \bar{I}_m \right\} \ddot{\theta}_0 + \sum_{i=m+1}^{-1} \left\{ \sum_{j=\bar{m}}^{m-1} \rho_j 4h_m h_i A_j - \rho_m 2h_i \bar{I}_m \right\} \ddot{\theta}_i \right. \\ & + \left. \left\{ \sum_{i=\bar{m}}^{m-1} \rho_i 4h_m h_m A_i + \rho_m \bar{I}_m \right\} \ddot{\theta}_m + \sum_{j=\bar{m}+1}^{m-1} \left\{ \sum_{i=\bar{m}}^{j-1} \rho_i 4h_m h_j A_i - \rho_j 2h_m \bar{I}_j \right\} \ddot{\theta}_j - \rho_{\bar{m}} 2h_m \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}} \right) \\ & + \left. \delta u'_0 \left(c_{11}^{*m} A_m u'_0 - c_{11}^{*m} h_0 A_m \theta'_0 + \left\{ \sum_{i=\bar{m}}^{m-1} -c_{11}^{*i} 2h_m A_i + c_{11}^{*m} \bar{I}_m \right\} \theta'_m + A_m \frac{e_{31}^{*m}}{2h_m} \phi_m \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \delta w'_0 \left(\bar{c}_{55}^{m(\phi)} A_m w'_0 + \bar{c}_{55}^{m(\phi)} A_m \theta_m \right) + \delta \theta'_0 \left(-c_{11}^{*m} h_0 A_m u'_0 + c_{11}^{*m} h_0^2 A_m \theta'_0 + \left\{ \sum_{i=\bar{m}}^{m-1} c_{11}^{*i} h_0 2h_m A_i \right. \right. \\
& - c_{11}^{*m} h_0 \bar{I}_m \left. \right\} \theta'_m - h_0 A_m \frac{e_{31}^{*m}}{2h_m} \phi_m \left. \right) + \delta \theta'_m \left(\left\{ \sum_{i=\bar{m}}^{m-1} -c_{11}^{*i} 2h_m A_i + c_{11}^{*m} \bar{I}_m \right\} u'_0 + \left\{ \sum_{i=\bar{m}}^{m-1} c_{11}^{*i} 2h_m h_0 A_i \right. \right. \\
& - c_{11}^{*m} h_0 \bar{I}_m \left. \right\} \theta'_0 + \sum_{i=m+1}^{-1} \left\{ \sum_{j=\bar{m}}^{m-1} c_{11}^{*j} 4h_m h_i A_j - c_{11}^{*m} 2h_i \bar{I}_m \right\} \theta'_i + \left\{ \sum_{i=\bar{m}}^{m-1} c_{11}^{*i} 4h_m h_m A_i + \bar{c}_{11}^{m(\phi)} I_m \right\} \theta'_m \\
& + \sum_{j=\bar{m}}^{m-1} \left\{ \sum_{i=\bar{m}}^{j-1} c_{11}^{*i} 4h_m h_j A_i - c_{11}^{*j} 2h_m \bar{I}_j \right\} \theta'_j + \bar{I}_m \frac{e_{31}^{*m}}{2h_m} \phi_m - \sum_{i=\bar{m}}^{m-1} 2h_m A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \left. \right) \\
& + \delta \theta_m \left(c_{55}^m A_m w'_0 + c_{55}^m A_m \theta_m \right) - \delta u_0 X_m + \delta \theta_0 h_0 X_m - \delta \theta_m \left\{ \sum_{i=\bar{m}}^{m-1} -2h_m X_i - h_m X_m + M_m \right\} \\
& - \delta w_0 Z_m + \delta \phi_m \left(\frac{e_{31}^{*m}}{2h_m} A_m u'_0 - \frac{e_{31}^{*m}}{2h_m} h_0 A_m \theta'_0 - \sum_{i=m+1}^{-1} \frac{e_{31}^{*m}}{2h_m} 2h_i A_m \theta'_i + \frac{e_{31}^{*m}}{2h_m} \bar{I}_m \theta'_m \right) \\
& \left. - \delta \phi_m \frac{\epsilon_{33}^{*m} A_m}{4h_m^2} \phi_m + \delta \phi_m b \tau_m \right\} dL dt = 0. \quad (\text{A.14})
\end{aligned}$$

Considering the generalized mechanical displacements and electric potential differences grouped in the generalized vectors of displacement and potential difference

$$\mathbf{u}(x, t) = \{u_0(x, t), w_0(x, t), \theta_0(x, t), \theta_1(x, t), \dots, \theta_{\bar{n}}(x, t), \theta_{\bar{m}}(x, t), \dots, \theta_{-1}(x, t)\}^T, \quad (\text{A.15})$$

$$\phi(x, t) = \{\phi_1(x, t), \dots, \phi_{\bar{n}}(x, t), \phi_{\bar{m}}(x, t), \dots, \phi_{-1}(x, t)\}^T, \quad (\text{A.16})$$

and since Equation (A.14) can only be satisfied if the integral term relative to L is zero, the weak form of the analytical model can be expressed in matrix form as

$$\begin{aligned}
& \int_L \left[\delta \mathbf{u}^T \mathbf{J} \ddot{\mathbf{u}} + (\mathbf{L}_{xx} \delta \mathbf{u})^T \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} + (\mathbf{L}_{zx} \delta \mathbf{u})^T \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + (\mathbf{L}_{xx} \delta \mathbf{u})^T \mathbf{P}^T \phi - \delta \mathbf{u}^T \mathbf{f} \right. \\
& \left. + \delta \phi^T \mathbf{P} \mathbf{L}_{xx} \mathbf{u} + \delta \phi^T \mathbf{C} \phi - \delta \phi^T \boldsymbol{\tau} \right] dL = 0. \quad (\text{A.17})
\end{aligned}$$

The differential operators

$$\mathbf{L}_{xx} = \text{diag} \left(\frac{\partial}{\partial x}, 0, \frac{\partial}{\partial x}, \dots, \frac{\partial}{\partial x} \right), \quad \mathbf{L}_{zx} = \text{diag} \left(0, \frac{\partial}{\partial x}, 1, \dots, 1 \right), \quad (\text{A.18a,b})$$

are used for the definition of the generalized extensional and shear strains and the non-zero terms of the symmetric positive definite inertia matrix $\mathbf{J}$ and of the symmetric semi-positive definite extensional and shear stiffness related matrices $\mathbf{Y}$ and $\mathbf{G}$, of size $[(\bar{k} - 1) \times (\bar{k} - 1)]$, piezoelectric coupling related and capacitance matrices $\mathbf{P}$ and $\mathbf{C}$, of size $[\bar{p} \times (\bar{k} - 1)]$ and $(\bar{p} \times \bar{p})$, and applied mechanical forces and electric charge density vectors $\mathbf{f}$ and $\boldsymbol{\tau}$, of size $[(\bar{k} - 1) \times 1]$ and $(\bar{p} \times 1)$, have elements defined in (A.14). Here, $\bar{p} = \bar{n} - \bar{m}$ is the total number of piezoelectric layers and $\bar{k} - 1$, with $\bar{k} = \bar{p} + 4$, is the total number of generalized displacements. The non-zero symmetric inertial terms $J_{(i,j)} = J_{(j,i)}$, extensional stiffness related terms $Y_{(i,j)} = Y_{(j,i)}$ and shear stiffness related terms $G_{(i,j)} = G_{(j,i)}$, and the piezoelectric coupling terms $P_{(l,i)}$ and capacitance terms $C_{(l,l)}$, with $i, j = 1, \dots, \bar{k} - 1$ and $l = 1, \dots, \bar{p}$, of the matrices utilized in (A.17), are given by

$$\begin{aligned}
J_{(1,1)} &= \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k, \quad J_{(1,3)} = \sum_{n=1}^{\bar{n}} \rho_n h_0 A_n - \sum_{m=\bar{m}}^{-1} \rho_m h_0 A_m, \quad J_{(1,\hat{n}+3)} = \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}}, \\
J_{(1,\bar{n}+3)} &= \rho_{\bar{n}} \bar{I}_{\bar{n}}, \quad J_{(1,\bar{n}+4)} = \rho_{\bar{m}} \bar{I}_{\bar{m}}, \quad J_{(1,\bar{k}+\hat{m})} = - \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}}, \\
J_{(2,2)} &= \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k, \quad J_{(3,3)} = \sum_{n=1}^{\bar{n}} \rho_n h_0^2 A_n + \sum_{m=\bar{m}}^{-1} \rho_m h_0^2 A_m + \rho_0 I_0, \\
J_{(3,\hat{n}+3)} &= 2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i A_i + \rho_{\hat{n}} h_0 \bar{I}_{\hat{n}}, \quad J_{(3,\bar{n}+3)} = \rho_{\bar{n}} h_0 \bar{I}_{\bar{n}}, \quad J_{(3,\bar{n}+4)} = -\rho_{\bar{m}} h_0 \bar{I}_{\bar{m}}, \\
J_{(3,\bar{k}+\hat{m})} &= 2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i A_i - \rho_{\hat{m}} h_0 \bar{I}_{\hat{m}}, \quad J_{(\hat{n}+3,\hat{n}+3)} = \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 4h_{\hat{n}}^2 A_i + \rho_{\hat{n}} I_{\hat{n}}, \\
J_{(\hat{n}+3,r+3)} &= \sum_{i=r+1}^{\bar{n}} \rho_i 4h_{\hat{n}} h_r A_i + \rho_r 2h_{\hat{n}} \bar{I}_r, \quad J_{(\hat{n}+3,\bar{n}+3)} = \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}}, \quad J_{(\bar{n}+3,\bar{n}+3)} = \rho_{\bar{n}} I_{\bar{n}}, \\
J_{(\bar{n}+4,\bar{n}+4)} &= \rho_{\bar{m}} I_{\bar{m}}, \quad J_{(\bar{n}+4,\bar{k}+\hat{m})} = -\rho_{\bar{m}} 2h_{\hat{m}} \bar{I}_{\bar{m}}, \quad J_{(\bar{k}+\hat{m},\bar{k}+\hat{m})} = \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 4h_{\hat{m}}^2 A_i + \rho_{\hat{m}} I_{\hat{m}}, \\
J_{(\bar{k}+\hat{m},\bar{k}+s)} &= \sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_s A_j - \rho_{\hat{m}} 2h_s \bar{I}_{\hat{m}}, \tag{A.19a-t}
\end{aligned}$$

$$\begin{aligned}
Y_{(1,1)} &= \sum_{k=\bar{m}}^{\bar{n}} c_{11}^{*k} A_k, \quad Y_{(1,3)} = \sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m, \quad Y_{(1,\hat{n}+3)} = \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}}, \\
Y_{(1,\bar{n}+3)} &= c_{11}^{*\bar{n}} \bar{I}_{\bar{n}}, \quad Y_{(1,\bar{n}+4)} = c_{11}^{*\bar{m}} \bar{I}_{\bar{m}}, \quad Y_{(1,\bar{k}+\hat{m})} = - \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}}, \\
Y_{(3,3)} &= \sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0^2 A_n + \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0^2 A_m + c_{11}^{*0} I_0, \quad Y_{(3,\hat{n}+3)} = 2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}}, \\
Y_{(3,\bar{n}+3)} &= c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}}, \quad Y_{(3,\bar{n}+4)} = -c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}}, \quad Y_{(3,\bar{k}+\hat{m})} = 2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}}, \\
Y_{(\hat{n}+3,\hat{n}+3)} &= \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}}^2 A_i + \bar{c}_{11}^{\hat{n}(\phi)} I_{\hat{n}}, \quad Y_{(\hat{n}+3,r+3)} = \sum_{i=r+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}} h_r A_i + c_{11}^{*r} 2h_{\hat{n}} \bar{I}_r, \\
Y_{(\hat{n}+3,\bar{n}+3)} &= c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}}, \quad Y_{(\bar{n}+3,\bar{n}+3)} = \bar{c}_{11}^{\bar{n}(\phi)} I_{\bar{n}}, \quad Y_{(\bar{n}+4,\bar{n}+4)} = \bar{c}_{11}^{\bar{m}(\phi)} I_{\bar{m}}, \\
Y_{(\bar{n}+4,\bar{k}+\hat{m})} &= -c_{11}^{*\bar{m}} 2h_{\hat{m}} \bar{I}_{\bar{m}}, \quad Y_{(\bar{k}+\hat{m},\bar{k}+\hat{m})} = \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 4h_{\hat{m}}^2 A_i + \bar{c}_{11}^{\hat{m}(\phi)} I_{\hat{m}}, \\
Y_{(\bar{k}+\hat{m},\bar{k}+s)} &= \sum_{j=\bar{m}}^{\hat{m}-1} c_{11}^{*j} 4h_{\hat{m}} h_s A_j - c_{11}^{*\hat{m}} 2h_s \bar{I}_{\hat{m}}, \tag{A.20a-s}
\end{aligned}$$

$$\begin{aligned}
G_{(2,2)} &= \sum_{n=1}^{\bar{n}} \bar{c}_{55}^{n(\phi)} A_n + c_{55}^0 A_0 + \sum_{m=\bar{m}}^{-1} \bar{c}_{55}^{m(\phi)} A_m, \quad G_{(2,3)} = c_{55}^0 A_0, \quad G_{(2,\hat{n}+3)} = \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}}, \\
G_{(2,\bar{n}+3)} &= \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}}, \quad G_{(2,\bar{n}+4)} = \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}}, \quad G_{(2,\bar{k}+\hat{m})} = \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}}, \quad G_{(3,3)} = c_{55}^0 A_0, \\
G_{(\hat{n}+3,\hat{n}+3)} &= \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}}, \quad G_{(\bar{n}+3,\bar{n}+3)} = \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}}, \quad G_{(\bar{n}+4,\bar{n}+4)} = \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}}, \\
G_{(\bar{k}+\hat{m},\bar{k}+\hat{m})} &= \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}}, \tag{A.21a-k}
\end{aligned}$$

$$\begin{aligned}
P_{(n,1)} &= \frac{e_{31}^{*n}}{2h_n} A_n, & P_{(n,3)} &= \frac{e_{31}^{*n}}{2h_n} h_0 A_n, & P_{(n,\hat{n}+3)} &= \frac{e_{31}^{*n}}{2h_n} 2h_{\hat{n}} A_n, & P_{(n,n+3)} &= \frac{e_{31}^{*n}}{2h_n} \bar{I}_n, \\
P_{(\bar{p}+1+m,1)} &= \frac{e_{31}^{*m}}{2h_m} A_m, & P_{(\bar{p}+1+m,3)} &= -\frac{e_{31}^{*m}}{2h_m} h_0 A_m, & P_{(\bar{p}+1+m,\bar{k}+m)} &= \frac{e_{31}^{*m}}{2h_m} \bar{I}_m, \\
P_{(\bar{p}+1+m,\bar{k}+\hat{m})} &= -\frac{e_{31}^{*m}}{2h_m} 2h_{\hat{m}} A_m,
\end{aligned} \tag{A.22a-h}$$

$$C_{(n,n)} = -\frac{\varepsilon_{33}^{*n} A_n}{4h_n^2}, \quad C_{(\bar{p}+1+m,\bar{p}+1+m)} = -\frac{\varepsilon_{33}^{*m} A_m}{4h_m^2}. \tag{A.23a,b}$$

In the equations above the indices $\hat{n}$ and $\hat{m}$ of the top and bottom intermediate layers comprise the values $\hat{n} = 1, \dots, \bar{n} - 1$ and $\hat{m} = \bar{m} + 1, \dots, -1$, the indices r and s used in definitions (A.19) and (A.20) comprise the values $r = n + 1, \dots, \bar{n}$ and $s = m + 1, \dots, -1$, and the indices $\tilde{n}$ and $\tilde{m}$ used in definitions (A.22) comprise the values $\tilde{n} = 1, \dots, n - 1$ and $\tilde{m} = m + 1, \dots, -1$. Furthermore, the applied mechanical forces and electric charge densities terms $f_{(i)}$ and $\tau_{(l)}$ are defined as

$$\begin{aligned}
f_{(1)} &= \sum_{k=\bar{m}}^{\bar{n}} X_k, & f_{(2)} &= \sum_{k=\bar{m}}^{\bar{n}} Z_k, & f_{(3)} &= \sum_{n=1}^{\bar{n}} h_0 X_n + M_c - \sum_{m=\bar{m}}^{-1} h_0 X_m, \\
f_{(\hat{n}+3)} &= h_{\hat{n}} X_{\hat{n}} + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} X_i + M_{\hat{n}}, & f_{(\bar{n}+3)} &= h_{\bar{n}} X_{\bar{n}} + M_{\bar{n}},
\end{aligned}$$

$$f_{(\bar{n}+4)} = -h_{\bar{m}} X_{\bar{m}} + M_{\bar{m}}, \quad f_{(\bar{k}+\hat{m})} = -h_{\hat{m}} X_{\hat{m}} - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} X_i + M_{\hat{m}}, \tag{A.24a-g}$$

$$\tau_{(n)} = -b\tau_n, \quad \tau_{(\bar{p}+1+m)} = -b\tau_m. \tag{A.25a,b}$$

Thus, splitting Equation (A.17) into two equations in terms of the variations of the generalized displacements and electric potential differences, $\delta \mathbf{u}$ and $\delta \phi$, yields

$$\int_L \left[\delta \mathbf{u}^T \mathbf{J} \ddot{\mathbf{u}} + \delta \mathbf{u}^T (\mathbf{L}_{xx}^T \mathbf{Y} \mathbf{L}_{xx} + \mathbf{L}_{zx}^T \mathbf{G} \mathbf{L}_{zx}) \mathbf{u} + \delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{P}^T \phi \right] dL = \int_L \delta \mathbf{u}^T \mathbf{f} dL, \tag{A.26}$$

$$\int_L (\delta \phi^T \mathbf{P} \mathbf{L}_{xx} \mathbf{u} + \delta \phi^T \mathbf{C} \phi) dL = \int_L \delta \phi^T \boldsymbol{\tau} dL. \tag{A.27}$$

Equations (A.26) and (A.27) are the weak forms, governing the motion and electric charge equilibrium, of the layered beam with arbitrary ACLD treatments, and the generalized displacements and electrical potential differences vectors are the unknown independent variables.

A.3 Strong Forms

In order to get rid of the differentiation in the $\mathbf{L}_{xx} \delta \mathbf{u}$ and $\mathbf{L}_{zx} \delta \mathbf{u}$ terms, Equation (A.17) is integrated by parts, yielding

$$\begin{aligned}
& \left[\delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} + (\mathbf{L}_{zx}^T \delta \mathbf{u})^T \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{P}^T \phi \right]_0^L \\
& + \int_L \left[\delta \mathbf{u}^T \mathbf{J} \ddot{\mathbf{u}} - \delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{L}_{xx} \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} - \delta \mathbf{u}^T \mathbf{L}_{zx}^T \mathbf{L}_{zx}^2 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \delta \mathbf{u}^T \mathbf{L}_{zx}^T \mathbf{G} \mathbf{L}_{zx} \mathbf{u} \right. \\
& \left. - \delta \mathbf{u}^T \mathbf{L}_{xx}^T \mathbf{L}_{xx} \mathbf{P}^T \phi - \delta \mathbf{u}^T \mathbf{f} + \delta \phi^T \mathbf{P} \mathbf{L}_{xx} \mathbf{u} + \delta \phi^T \mathbf{C} \phi - \delta \phi^T \boldsymbol{\tau} \right] dL = 0, \tag{A.28}
\end{aligned}$$

where

$$\begin{aligned}\mathbf{L}_{zx}^1 &= \text{diag}(0, 1, 0, \dots, 0), & \mathbf{L}_{zx}^2 &= \text{diag}\left(0, \frac{\partial}{\partial x}, 0, \dots, 0\right), \\ \mathbf{L}_{zx}^3 &= \text{diag}(0, 0, 1, \dots, 1), & \mathbf{L}_{xx}^1 &= \text{diag}(1, 0, 1, \dots, 1).\end{aligned}\quad (\text{A.29a-d})$$

Rearranging Equation (A.28) and collecting the terms involving the variations yields

$$\begin{aligned}\left[\delta \mathbf{u}^T (\mathbf{L}_{xx}^1 \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} + \mathbf{L}_{zx}^1 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \mathbf{L}_{xx}^1 \mathbf{P}^T \phi)\right]_0^L + \int_L \delta \mathbf{u}^T (\mathbf{J} \ddot{\mathbf{u}} - \mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} - \mathbf{L}_{zx}^1 \mathbf{L}_{zx}^2 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} \\ + \mathbf{L}_{zx}^3 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} - \mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{P}^T \phi - \mathbf{f}) dL + \int_L \delta \phi^T (\mathbf{P} \mathbf{L}_{xx} \mathbf{u} + \mathbf{C} \phi - \boldsymbol{\tau}) dL = 0.\end{aligned}\quad (\text{A.30})$$

Since the variational generalized displacements and electric potential differences in Equation (A.30) are independent and arbitrary in the interval $[0, L]$, the equation can only be satisfied if the coefficients of the variational displacements and electric potential differences are zero. Thus, the electro-mechanical equations of motion, electric charge equilibrium equations and electro-mechanical boundary conditions can be derived accordingly.

A.3.1 Electro-Mechanical Equations of Motion

The electro-mechanical equations of motion are derived from Equation (A.30) by considering the integral term involving $\delta \mathbf{u}$,

$$\int_L \delta \mathbf{u}^T (\mathbf{J} \ddot{\mathbf{u}} - \mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} - \mathbf{L}_{zx}^1 \mathbf{L}_{zx}^2 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \mathbf{L}_{zx}^3 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} - \mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{P}^T \phi - \mathbf{f}) dL = 0, \quad (\text{A.31})$$

where the non-trivial solution is given by

$$\mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} + \mathbf{L}_{zx}^1 \mathbf{L}_{zx}^2 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} - \mathbf{L}_{zx}^3 \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \mathbf{L}_{xx}^1 \mathbf{L}_{xx} \mathbf{P}^T \phi + \mathbf{f} = \mathbf{J} \ddot{\mathbf{u}}. \quad (\text{A.32})$$

Thus, rewriting the previous equation in non-matrix form, the system of partial differential equations involving the variations δu_0 , δw_0 , $\delta \theta_0$, $\delta \theta_{\hat{n}}$ ($\hat{n} = 1, \dots, \bar{n} - 1$), $\delta \theta_{\bar{n}}$, $\delta \theta_{\bar{m}}$ and $\delta \theta_{\hat{m}}$ ($\hat{m} = \bar{m} + 1, \dots, -1$), is given by

δu_0 :

$$\begin{aligned}\sum_{k=\bar{m}}^{\bar{n}} c_{11}^{*k} A_k u_0'' + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) \theta_0'' + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \theta_{\hat{n}}'' \\ + c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} \theta_{\bar{n}}'' + c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} \theta_{\bar{m}}'' + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' \\ + \sum_{n=1}^{\bar{n}} A_n \frac{e_{31}^{*n}}{2h_n} \phi_n' + \sum_{m=\bar{m}}^{-1} A_m \frac{e_{31}^{*m}}{2h_m} \phi_m' + \sum_{k=\bar{m}}^{\bar{n}} X_k = \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k \ddot{u}_0 + \left(\sum_{n=1}^{\bar{n}} \rho_n h_0 A_n \right. \\ \left. - \sum_{m=\bar{m}}^{-1} \rho_m h_0 A_m \right) \ddot{\theta}_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} + \rho_{\bar{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}} \\ + \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}},\end{aligned}\quad (\text{A.33})$$

$\delta w_0 :$

$$\begin{aligned} & \left(\sum_{n=1}^{\bar{n}} \bar{c}_{55}^{n(\phi)} A_n + c_{55}^0 A_0 + \sum_{m=\bar{m}}^{-1} \bar{c}_{55}^{m(\phi)} A_m \right) w_0'' + c_{55}^c A_c \theta_0' + \sum_{\hat{n}=1}^{\bar{n}-1} \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}}' + \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}}' \\ & + \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}}' + \sum_{\hat{m}=\bar{m}+1}^{-1} \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}}' + \sum_{k=\bar{m}}^{\bar{n}} Z_k = \sum_{k=\bar{m}}^{\bar{n}} \rho_k A_k \ddot{w}_0, \quad (\text{A.34}) \end{aligned}$$

 $\delta \theta_0 :$

$$\begin{aligned} & \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) u_0'' + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0^2 A_n + c_{11}^{*0} I_0 + \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0^2 A_m \right) \theta_0'' \\ & + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta_{\hat{n}}'' + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta_{\bar{n}}'' - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \theta_{\bar{m}}'' \\ & + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' - c_{55}^0 A_0 w_0' - c_{55}^0 A_0 \theta_0 + \sum_{n=1}^{\bar{n}} \frac{e_{31}^{*n}}{2h_n} h_0 A_n \phi_n' \\ & - \sum_{m=\bar{m}}^{-1} \frac{e_{31}^{*m}}{2h_m} h_0 A_m \phi_m' + \sum_{n=1}^{\bar{n}} h_0 X_n + M_c - \sum_{m=\bar{m}}^{-1} h_0 X_m = \left(\sum_{n=1}^{\bar{n}} \rho_n h_0 A_n - \sum_{m=\bar{m}}^{-1} \rho_m h_0 A_m \right) \ddot{u}_0 \\ & + \ddot{\theta}_0 \left(\sum_{n=1}^{\bar{n}} \rho_n h_0^2 A_n + \rho_0 I_0 + \sum_{m=\bar{m}}^{-1} \rho_m h_0^2 A_m \right) + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i A_i + \rho_{\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} \\ & + \rho_{\bar{n}} h_0 \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}} - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i A_i - \rho_{\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}} \quad (\text{A.35}) \end{aligned}$$

 $\delta \theta_{\hat{n}} :$

$$\begin{aligned} & \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) u_0'' + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta_0'' + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j \right. \\ & \left. + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \theta_i'' + \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}}^2 A_i + \bar{c}_{11}^{\hat{n}(\phi)} I_{\hat{n}} \right) \theta_{\hat{n}}'' + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}} h_j A_i + c_{11}^{*j} 2h_{\hat{n}} \bar{I}_j \right) \theta_j'' \\ & + c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \theta_{\bar{n}}'' - \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} w_0' - \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}} + \frac{e_{31}^{*\hat{n}}}{2h_{\hat{n}}} \bar{I}_{\hat{n}} \phi_{\hat{n}}' + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i' + h_{\hat{n}} X_{\hat{n}} \\ & + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} X_i + M_{\hat{n}} = \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \ddot{u}_0 + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} \rho_i A_i \right. \\ & \left. + \rho_{\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \ddot{\theta}_0 + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \ddot{\theta}_i + \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 4h_{\hat{n}}^2 A_i + \rho_{\hat{n}} I_{\hat{n}} \right) \ddot{\theta}_{\hat{n}} \\ & + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} \rho_i 4h_{\hat{n}} h_j A_i + \rho_j 2h_{\hat{n}} \bar{I}_j \right) \ddot{\theta}_j + \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \ddot{\theta}_{\bar{n}}, \quad (\text{A.36}) \end{aligned}$$

 $\delta \theta_{\bar{n}} :$

$$\begin{aligned} & c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} u_0'' + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta_0'' + \sum_{\hat{n}=1}^{\bar{n}-1} c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\hat{n}} \theta_{\hat{n}}'' + \bar{c}_{11}^{\bar{n}(\phi)} I_{\bar{n}} \theta_{\bar{n}}'' - \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} w_0' - \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}} \\ & + \frac{e_{31}^{*\bar{n}}}{2h_{\bar{n}}} \bar{I}_{\bar{n}} \phi_{\bar{n}}' + h_{\bar{n}} X_{\bar{n}} + M_{\bar{n}} = \rho_{\bar{n}} \bar{I}_{\bar{n}} \ddot{u}_0 + \rho_{\bar{n}} h_0 \bar{I}_{\bar{n}} \ddot{\theta}_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \rho_{\bar{n}} 2h_{\hat{n}} \bar{I}_{\hat{n}} \ddot{\theta}_{\hat{n}} + \rho_{\bar{n}} I_{\bar{n}} \ddot{\theta}_{\bar{n}}, \quad (\text{A.37}) \end{aligned}$$

$\delta\theta_{\bar{m}} :$

$$\begin{aligned} & c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} u_0'' - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \theta_0'' + \bar{c}_{11}^{\bar{m}(\phi)} \bar{I}_{\bar{m}} \theta_{\bar{m}}'' - \sum_{\hat{m}=\bar{m}+1}^{-1} c_{11}^{*\bar{m}} 2h_{\bar{m}} \bar{I}_{\bar{m}} \theta_{\hat{m}}'' - \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} w_0' - \bar{c}_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}} \\ & + \frac{e_{31}^{*\bar{m}}}{2h_{\bar{m}}} \bar{I}_{\bar{m}} \phi_{\bar{m}}' - h_{\bar{m}} X_{\bar{m}} + M_{\bar{m}} = \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{u}_0 - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \ddot{\theta}_0 - \sum_{\hat{m}=\bar{m}+1}^{-1} \rho_{\bar{m}} 2h_{\bar{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\hat{m}} + \rho_{\bar{m}} \bar{I}_{\bar{m}} \ddot{\theta}_{\bar{m}}, \end{aligned} \quad (\text{A.38})$$

$\delta\theta_{\hat{m}} :$

$$\begin{aligned} & \left(- \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \right) u_0'' + \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta_0'' + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} c_{11}^{*i} 4h_{\hat{m}} h_j A_i \right. \\ & \left. - c_{11}^{*j} 2h_{\hat{m}} \bar{I}_j \right) \theta_j'' + \left(\sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 4h_{\hat{m}}^2 A_i + \bar{c}_{11}^{\hat{m}(\phi)} \bar{I}_{\hat{m}} \right) \theta_{\hat{m}}'' + \sum_{i=\hat{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \theta_i'' \\ & - c_{11}^{*\hat{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \theta_{\hat{m}}'' - \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} w_0' - \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}} + \frac{e_{31}^{*\hat{m}}}{2h_{\hat{m}}} \bar{I}_{\hat{m}} \phi_{\hat{m}}' - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i' - h_{\hat{m}} X_{\hat{m}} \\ & - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} X_i + M_{\hat{m}} = \left(- \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{u}_0 + \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} \rho_i A_i - \rho_{\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \ddot{\theta}_0 \\ & + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} \rho_i 4h_{\hat{m}} h_j A_i - \rho_j 2h_{\hat{m}} \bar{I}_j \right) \ddot{\theta}_j + \left(\sum_{i=\bar{m}}^{\hat{m}-1} \rho_i 4h_{\hat{m}}^2 A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \ddot{\theta}_{\hat{m}} \\ & + \sum_{i=\hat{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \ddot{\theta}_i + \rho_{\hat{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \ddot{\theta}_{\bar{n}}, \end{aligned} \quad (\text{A.39})$$

A.3.2 Electric Charge Equilibrium Equations

Similarly, the electric charge equilibrium equations are derived from Equation (A.30) by considering the integral term involving $\delta\phi$,

$$\int_L \delta\phi^T (\mathbf{P}\mathbf{L}_{xx}\mathbf{u} + \mathbf{C}\phi - \boldsymbol{\tau}) \, dL = 0, \quad (\text{A.40})$$

where the non-trivial solution is given by

$$\mathbf{P}\mathbf{L}_{xx}\mathbf{u} + \mathbf{C}\phi = \boldsymbol{\tau}. \quad (\text{A.41})$$

Thus, rewriting the previous equation in non-matrix form, the system of partial differential equations involving the variations of the electric potential differences of the generic top and bottom piezoelectric layers, $\delta\phi_n$ ($n = 1, \dots, \bar{n}$) and $\delta\phi_m$ ($m = \bar{m}, \dots, -1$), is given by

$$\delta\phi_n : \quad \frac{e_{31}^{*n}}{2h_n} A_n u_0' + \frac{e_{31}^{*n}}{2h_n} h_0 A_n \theta_0' + \sum_{i=1}^{n-1} \frac{e_{31}^{*n}}{2h_n} 2h_i A_n \theta_i' + \frac{e_{31}^{*n}}{2h_n} \bar{I}_n \theta_n' - \frac{\varepsilon_{33}^{*n} A_n}{4h_n^2} \phi_n = -b\tau_n, \quad (\text{A.42})$$

$$\delta\phi_m : \quad \frac{e_{31}^{*m}}{2h_m} A_m u_0' - \frac{e_{31}^{*m}}{2h_m} h_0 A_m \theta_0' + \frac{e_{31}^{*m}}{2h_m} \bar{I}_m \theta_m' - \sum_{i=m+1}^{-1} \frac{e_{31}^{*m}}{2h_m} 2h_i A_m \theta_i' + \frac{\varepsilon_{33}^{*m} A_m}{4h_m^2} \phi_m = -b\tau_m. \quad (\text{A.43})$$

A.3.3 Electro-Mechanical Boundary Conditions

The electro-mechanical boundary conditions are obtained from the boundary terms at $x = 0, L$ in Equation (A.30),

$$\delta \mathbf{u}^T \left[\mathbf{L}_{xx}^{1T} \mathbf{Y} \mathbf{L}_{xx} \mathbf{u} + \mathbf{L}_{zx}^{1T} \mathbf{G} \mathbf{L}_{zx} \mathbf{u} + \mathbf{L}_{xx}^{1T} \mathbf{P}^T \phi \right]_0^L = 0, \quad (\text{A.44})$$

which can be expressed in non matrix form as

$$\begin{aligned} \delta u_0 \left[\sum_{k=\bar{m}}^{\bar{n}} c_{11}^{*k} A_k u'_0 + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) \theta'_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \left(\sum_{i=\hat{n}+1}^{\bar{n}} \rho_i 2h_{\hat{n}} A_i + \rho_{\hat{n}} \bar{I}_{\hat{n}} \right) \theta'_{\hat{n}} + c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} \theta'_{\bar{n}} \right. \\ \left. + c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} \theta'_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(- \sum_{i=\hat{m}}^{\hat{m}-1} \rho_i 2h_{\hat{m}} A_i + \rho_{\hat{m}} \bar{I}_{\hat{m}} \right) \theta'_{\hat{m}} + \sum_{n=1}^{\bar{n}} A_n \frac{e_{31}^{*n}}{2h_n} \phi_n + \sum_{m=\bar{m}}^{-1} A_m \frac{e_{31}^{*m}}{2h_m} \phi_m \right] = 0, \end{aligned} \quad (\text{A.45})$$

$$\begin{aligned} \delta w_0 \left[\left(\sum_{n=1}^{\bar{n}} \bar{c}_{55}^{n(\phi)} A_n + c_{55}^0 A_0 + \sum_{m=\bar{m}}^{-1} \bar{c}_{55}^{m(\phi)} A_m \right) w'_0 + c_{55}^c A_c \theta_0 + \sum_{\hat{n}=1}^{\bar{n}-1} \bar{c}_{55}^{\hat{n}(\phi)} A_{\hat{n}} \theta_{\hat{n}} + \bar{c}_{55}^{\bar{n}(\phi)} A_{\bar{n}} \theta_{\bar{n}} \right. \\ \left. + c_{55}^{\bar{m}(\phi)} A_{\bar{m}} \theta_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \bar{c}_{55}^{\hat{m}(\phi)} A_{\hat{m}} \theta_{\hat{m}} \right] = 0, \end{aligned} \quad (\text{A.46})$$

$$\begin{aligned} \delta \theta_0 \left[\left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0 A_n - \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0 A_m \right) u'_0 + \left(\sum_{n=1}^{\bar{n}} c_{11}^{*n} h_0^2 A_n + c_{11}^{*0} I_0 + \sum_{m=\bar{m}}^{-1} c_{11}^{*m} h_0^2 A_m \right) \theta'_0 \right. \\ \left. + \sum_{\hat{n}=1}^{\bar{n}-1} \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta'_{\hat{n}} + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta'_{\bar{n}} - \rho_{\bar{m}} h_0 \bar{I}_{\bar{m}} \theta'_{\bar{m}} + \sum_{\hat{m}=\bar{m}+1}^{-1} \left(2h_0 h_{\hat{m}} \sum_{i=\hat{m}}^{\hat{m}-1} c_{11}^{*i} A_i \right. \right. \\ \left. \left. - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta'_{\hat{m}} + \sum_{n=1}^{\bar{n}} \frac{e_{31}^{*n}}{2h_n} h_0 A_n \phi_n - \sum_{m=\bar{m}}^{-1} \frac{e_{31}^{*m}}{2h_m} h_0 A_m \phi_m \right] = 0, \end{aligned} \quad (\text{A.47})$$

$$\begin{aligned} \delta \theta_{\hat{n}} \left[\left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 2h_{\hat{n}} A_i + c_{11}^{*\hat{n}} \bar{I}_{\hat{n}} \right) u'_0 + \left(2h_0 h_{\hat{n}} \sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} A_i + c_{11}^{*\hat{n}} h_0 \bar{I}_{\hat{n}} \right) \theta'_0 \right. \\ \left. + \sum_{i=1}^{\hat{n}-1} \left(\sum_{j=\hat{n}+1}^{\bar{n}} \rho_j 4h_{\hat{n}} h_i A_j + \rho_{\hat{n}} 2h_i \bar{I}_{\hat{n}} \right) \theta'_i + \left(\sum_{i=\hat{n}+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}}^2 A_i + \bar{c}_{11}^{\hat{n}(\phi)} I_{\hat{n}} \right) \theta'_{\hat{n}} \right. \\ \left. + \sum_{j=\hat{n}+1}^{\bar{n}-1} \left(\sum_{i=j+1}^{\bar{n}} c_{11}^{*i} 4h_{\hat{n}} h_j A_i + c_{11}^{*j} 2h_{\hat{n}} \bar{I}_j \right) \theta'_j + c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\bar{n}} \theta'_{\bar{n}} + \frac{e_{31}^{*\hat{n}}}{2h_{\hat{n}}} \bar{I}_{\hat{n}} \phi_{\hat{n}} + \sum_{i=\hat{n}+1}^{\bar{n}} 2h_{\hat{n}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \right] = 0, \end{aligned} \quad (\text{A.48})$$

$$\delta \theta_{\bar{n}} \left(c_{11}^{*\bar{n}} \bar{I}_{\bar{n}} u'_0 + c_{11}^{*\bar{n}} h_0 \bar{I}_{\bar{n}} \theta'_0 + \sum_{\hat{n}=1}^{\bar{n}-1} c_{11}^{*\bar{n}} 2h_{\hat{n}} \bar{I}_{\hat{n}} \theta'_{\hat{n}} + \bar{c}_{11}^{\bar{n}(\phi)} I_{\bar{n}} \theta'_{\bar{n}} + \frac{e_{31}^{*\bar{n}}}{2h_{\bar{n}}} \bar{I}_{\bar{n}} \phi_{\bar{n}} \right) = 0, \quad (\text{A.49})$$

$$\delta \theta_{\bar{m}} \left(c_{11}^{*\bar{m}} \bar{I}_{\bar{m}} u'_0 - c_{11}^{*\bar{m}} h_0 \bar{I}_{\bar{m}} \theta'_0 + \bar{c}_{11}^{\bar{m}(\phi)} I_{\bar{m}} \theta'_{\bar{m}} - \sum_{\hat{m}=\bar{m}+1}^{-1} c_{11}^{*\bar{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \theta'_{\hat{m}} + \frac{e_{31}^{*\bar{m}}}{2h_{\bar{m}}} \bar{I}_{\bar{m}} \phi_{\bar{m}} \right) = 0, \quad (\text{A.50})$$

$$\begin{aligned}
& \delta\theta_{\hat{m}} \left(- \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 2h_{\hat{m}} A_i + c_{11}^{*\hat{m}} \bar{I}_{\hat{m}} \right) u'_0 + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(2h_0 h_{\hat{m}} \sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} A_i - c_{11}^{*\hat{m}} h_0 \bar{I}_{\hat{m}} \right) \theta'_0 - c_{11}^{*\hat{m}} 2h_{\hat{m}} \bar{I}_{\hat{m}} \theta'_{\bar{m}} \\
& + \sum_{j=\bar{m}+1}^{\hat{m}-1} \left(\sum_{i=\bar{m}}^{j-1} c_{11}^{*i} 4h_{\hat{m}} h_j A_i - c_{11}^{*j} 2h_{\hat{m}} \bar{I}_j \right) \theta'_j + \left(\sum_{i=\bar{m}}^{\hat{m}-1} c_{11}^{*i} 4h_{\hat{m}}^2 A_i + \bar{c}_{11}^{\hat{m}(\phi)} I_{\hat{m}} \right) \theta'_{\hat{m}} \\
& + \sum_{i=\hat{m}+1}^{-1} \left(\sum_{j=\bar{m}}^{\hat{m}-1} \rho_j 4h_{\hat{m}} h_i A_j - \rho_{\hat{m}} 2h_i \bar{I}_{\hat{m}} \right) \theta'_i + \frac{e_{31}^{*\hat{m}}}{2h_{\hat{m}}} \bar{I}_{\hat{m}} \phi_{\hat{m}} - \sum_{i=\bar{m}}^{\hat{m}-1} 2h_{\hat{m}} A_i \frac{e_{31}^{*i}}{2h_i} \phi_i \Big] = 0. \quad (\text{A.51})
\end{aligned}$$

APPENDIX B

Complements to the Shell Formulation

B.1 Strain-Displacement and Equilibrium Equations in Orthogonal Curvilinear Coordinates

Taking into account that $H_z = 1$, from the equations of 3-D theory of elasticity, the strain components of the shell layer are defined as a function of displacements by [Sokolnikoff \(1956, pp. 177-184\)](#) as

$$\begin{aligned} \varepsilon_{\alpha\alpha} &= \frac{1}{H_\alpha} \frac{\partial u}{\partial \alpha} + \frac{1}{H_\alpha H_\beta} \frac{\partial H_\alpha}{\partial \beta} v + \frac{1}{H_\alpha} \frac{A_\alpha}{R_\alpha} w, & \varepsilon_{\beta\beta} &= \frac{1}{H_\alpha H_\beta} \frac{\partial H_\beta}{\partial \alpha} u + \frac{1}{H_\beta} \frac{\partial v}{\partial \beta} + \frac{1}{H_\beta} \frac{A_\beta}{R_\beta} w, \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z}, & \varepsilon_{\beta z} &= H_\beta \frac{\partial}{\partial z} \left(\frac{v}{H_\beta} \right) + \frac{1}{H_\beta} \frac{\partial w}{\partial \beta}, \\ \varepsilon_{z\alpha} &= H_\alpha \frac{\partial}{\partial z} \left(\frac{u}{H_\alpha} \right) + \frac{1}{H_\alpha} \frac{\partial w}{\partial \alpha}, & \varepsilon_{\alpha\beta} &= \frac{H_\alpha}{H_\beta} \frac{\partial}{\partial \beta} \left(\frac{u}{H_\alpha} \right) + \frac{H_\beta}{H_\alpha} \frac{\partial}{\partial \alpha} \left(\frac{v}{H_\beta} \right), \end{aligned} \quad (\text{B.1a-f})$$

where $u = u(\alpha, \beta, z)$, $v = v(\alpha, \beta, z)$ and $w = w(\alpha, \beta, z)$ are the displacement components of an arbitrary point of the shell in the directions of the tangents to the coordinate lines (α, β, z) , respectively. The equilibrium equations of a differential element of the body of the shell layer in the tri-orthogonal system of curvilinear coordinates [[Sokolnikoff \(1956\)](#)] are represented by the partial differential equations

$$\begin{aligned} \frac{\partial}{\partial \alpha} (H_\beta \sigma_{\alpha\alpha}) + \frac{\partial}{\partial \beta} (H_\alpha \sigma_{\alpha\beta}) + \frac{\partial}{\partial z} (H_\alpha H_\beta \sigma_{z\alpha}) \\ - \frac{\partial H_\beta}{\partial \alpha} \sigma_{\beta\beta} - \frac{\partial H_\alpha}{\partial \beta} \sigma_{\alpha\beta} + H_\beta \frac{\partial H_\alpha}{\partial z} \sigma_{z\alpha} + H_\alpha H_\beta P_\alpha = 0, \end{aligned} \quad (\text{B.2a})$$

$$\begin{aligned} \frac{\partial}{\partial \beta} (H_\alpha \sigma_{\beta\beta}) + \frac{\partial}{\partial \alpha} (H_\beta \sigma_{\alpha\beta}) + \frac{\partial}{\partial z} (H_\alpha H_\beta \sigma_{\beta z}) \\ - \frac{\partial H_\alpha}{\partial \beta} \sigma_{\alpha\alpha} - \frac{\partial H_\beta}{\partial \alpha} \sigma_{\alpha\beta} + H_\alpha \frac{\partial H_\beta}{\partial z} \sigma_{z\beta} + H_\alpha H_\beta P_\beta = 0, \end{aligned} \quad (\text{B.2b})$$

$$\begin{aligned} \frac{\partial}{\partial z} (H_\alpha H_\beta \sigma_{zz}) + \frac{\partial}{\partial \alpha} (H_\beta \sigma_{z\alpha}) + \frac{\partial}{\partial \beta} (H_\alpha \sigma_{\beta z}) \\ - H_\beta \frac{\partial H_\alpha}{\partial z} \sigma_{\alpha\alpha} - H_\alpha \frac{\partial H_\beta}{\partial z} \sigma_{\beta\beta} + H_\alpha H_\beta P_z = 0, \end{aligned} \quad (\text{B.2c})$$

where $P_\alpha = P_\alpha(\alpha, \beta, z)$, $P_\beta = P_\beta(\alpha, \beta, z)$ and $P_z = P_z(\alpha, \beta, z)$ are the corresponding projections of the volumetric force in the direction of the tangents to the shell curvilinear coordinate system.

B.2 First-Order Shear Deformation Theory (FSDT) of Anisotropic Shells

According to the *Love's first approximation* assumptions for thin shells (the so-called classical *Kirchhoff-Love theory* of shells), the following strain and stress definitions are derived for anisotropic shells, by relaxing in part the so-called Kirchhoff's hypothesis that *normals to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension* [see [Leissa \(1993, Sec. 1.3\)](#)]. Here it is considered that *normals before deformation remain straight but not necessarily normal after deformation*, which basically relaxes the condition of nil out-of-plane shear strains. That theory is known as FSDT or *Reissner-Mindlin theory* applied to shells.

Consistent with the assumptions of a moderately thick shell theory, the displacement components are

$$u^*(\alpha, \beta, z) = u_0(\alpha, \beta) + z\theta_\alpha(\alpha, \beta), \quad (\text{B.3a})$$

$$v^*(\alpha, \beta, z) = v_0(\alpha, \beta) + z\theta_\beta(\alpha, \beta), \quad (\text{B.3b})$$

$$w^*(\alpha, \beta, z) = w_0(\alpha, \beta), \quad (\text{B.3c})$$

where $u_0 = u_0(\alpha, \beta)$, $v_0 = v_0(\alpha, \beta)$ and $w_0 = w_0(\alpha, \beta)$ are the tangential and transverse displacements referred to a point on the middle surface, respectively, and $\theta_\alpha = \theta_\alpha(\alpha, \beta)$ and $\theta_\beta = \theta_\beta(\alpha, \beta)$ are the rotations of a normal to the reference middle surface.

Thus, taken into account the strain-displacement equations of the 3-D elasticity in orthogonal curvilinear coordinates in Equations (B.1), as proposed by Byrne, Flügge, Goldenveizer, Lur'ye and Novozhilov between the 1940s and 1960s [cf. [Leissa \(1993, Sec. 1.4\)](#)], and in a similar form to what has been presented, for example, by [Reddy \(2004, Sec. 8.2.3\)](#) or [Leissa \(1993, Sec. 1.4.1\)](#), the in-plane strain definitions are given as,

$$\varepsilon_{\alpha\alpha}^*(\alpha, \beta, z) = z_\alpha^{(0)} \varepsilon_{\alpha\alpha}^{*(0)}(\alpha, \beta) + z_\alpha^{(1)} \varepsilon_{\alpha\alpha}^{*(1)}(\alpha, \beta), \quad (\text{B.4a})$$

$$\varepsilon_{\beta\beta}^*(\alpha, \beta, z) = z_\beta^{(0)} \varepsilon_{\beta\beta}^{*(0)}(\alpha, \beta) + z_\beta^{(1)} \varepsilon_{\beta\beta}^{*(1)}(\alpha, \beta), \quad (\text{B.4b})$$

$$\varepsilon_{\alpha\beta}^*(\alpha, \beta, z) = z_{\alpha\beta}^{(0)} \varepsilon_{\alpha\beta}^{*(0)}(\alpha, \beta) + z_{\alpha\beta}^{(1)} \varepsilon_{\alpha\beta}^{*(1)}(\alpha, \beta), \quad (\text{B.4c})$$

where [cf. [Leissa \(1993, Sec. 1.4.1\)](#)]

$$z_\alpha^{(i)} = \frac{z^i}{(1 + z/R_\alpha)}, \quad z_\beta^{(i)} = \frac{z^i}{(1 + z/R_\beta)}, \quad (\text{B.5a,b})$$

$$z_{\alpha\beta}^{(0)} = z_\alpha^{(0)} z_\beta^{(0)} \left(1 - \frac{z^2}{R_\alpha R_\beta}\right), \quad z_{\alpha\beta}^{(1)} = z_\alpha^{(0)} z_\beta^{(0)} z \left(1 + \frac{z}{2R_\alpha} + \frac{z}{2R_\beta}\right), \quad (\text{B.6a,b})$$

with $i = 0, 1$ and

$$\begin{aligned}
\varepsilon_{\alpha\alpha}^{*(0)}(\alpha, \beta) &= \frac{1}{A_\alpha} \frac{\partial u_0}{\partial \alpha} + \frac{v_0}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} + \frac{w_0}{R_\alpha}, \\
\varepsilon_{\beta\beta}^{*(0)}(\alpha, \beta) &= \frac{1}{A_\beta} \frac{\partial v_0}{\partial \beta} + \frac{u_0}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} + \frac{w_0}{R_\beta}, \\
\varepsilon_{\alpha\beta}^{*(0)}(\alpha, \beta) &= \frac{A_\alpha}{A_\beta} \frac{\partial}{\partial \beta} \left(\frac{u_0}{A_\alpha} \right) + \frac{A_\beta}{A_\alpha} \frac{\partial}{\partial \alpha} \left(\frac{v_0}{A_\beta} \right), \\
\varepsilon_{\alpha\alpha}^{*(1)}(\alpha, \beta) &= \frac{1}{A_\alpha} \frac{\partial \theta_\alpha}{\partial \alpha} + \frac{1}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} \theta_\beta, \\
\varepsilon_{\beta\beta}^{*(1)}(\alpha, \beta) &= \frac{1}{A_\beta} \frac{\partial \theta_\beta}{\partial \beta} + \frac{1}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} \theta_\alpha, \\
\varepsilon_{\alpha\beta}^{*(1)}(\alpha, \beta) &= \frac{A_\alpha}{A_\beta} \frac{\partial}{\partial \beta} \left(\frac{\theta_\alpha}{A_\alpha} \right) + \frac{A_\beta}{A_\alpha} \frac{\partial}{\partial \alpha} \left(\frac{\theta_\beta}{A_\beta} \right) \\
&\quad + \frac{1}{R_\alpha} \left(\frac{1}{A_\beta} \frac{\partial u_0}{\partial \beta} - \frac{1}{A_\alpha A_\beta} \frac{\partial A_\beta}{\partial \alpha} v_0 \right) + \frac{1}{R_\beta} \left(\frac{1}{A_\alpha} \frac{\partial v_0}{\partial \alpha} - \frac{1}{A_\alpha A_\beta} \frac{\partial A_\alpha}{\partial \beta} u_0 \right). \quad (\text{B.7a-f})
\end{aligned}$$

Regarding the previous strain definitions it is worthy to mention that the zero-order terms $\varepsilon_{\alpha\alpha}^{*(0)}$, $\varepsilon_{\beta\beta}^{*(0)}$ and $\varepsilon_{\alpha\beta}^{*(0)}$ represent the normal (membrane) and shearing strains of the reference surface, respectively, and the first order terms $\varepsilon_{\alpha\alpha}^{*(1)}$, $\varepsilon_{\beta\beta}^{*(1)}$ and $\varepsilon_{\alpha\beta}^{*(1)}$ represent the linearly distributed bending components of strain and the torsion of the reference surface during deformation.

Considering the mechanical counterpart of the anisotropic (general orthotropic) plane-stress constitutive behavior presented in Equation (B.25), the in-plane stresses are given as

$$\begin{aligned}
\sigma_{\alpha\alpha}^*(\alpha, \beta, z) &= \bar{c}_{11}^* \varepsilon_{\alpha\alpha}^*(\alpha, \beta, z) + \bar{c}_{12}^* \varepsilon_{\beta\beta}^*(\alpha, \beta, z) + \bar{c}_{16}^* \varepsilon_{\alpha\beta}^*(\alpha, \beta, z) \\
&= \sigma_{\alpha\alpha}^{*(0)}(\alpha, \beta, z) + \sigma_{\alpha\alpha}^{*(1)}(\alpha, \beta, z), \quad (\text{B.8a})
\end{aligned}$$

$$\begin{aligned}
\sigma_{\beta\beta}^*(\alpha, \beta, z) &= \bar{c}_{12}^* \varepsilon_{\alpha\alpha}^*(\alpha, \beta, z) + \bar{c}_{22}^* \varepsilon_{\beta\beta}^*(\alpha, \beta, z) + \bar{c}_{26}^* \varepsilon_{\alpha\beta}^*(\alpha, \beta, z) \\
&= \sigma_{\beta\beta}^{*(0)}(\alpha, \beta, z) + \sigma_{\beta\beta}^{*(1)}(\alpha, \beta, z), \quad (\text{B.8b})
\end{aligned}$$

$$\begin{aligned}
\sigma_{\alpha\beta}^*(\alpha, \beta, z) &= \bar{c}_{16}^* \varepsilon_{\alpha\alpha}^*(\alpha, \beta, z) + \bar{c}_{26}^* \varepsilon_{\beta\beta}^*(\alpha, \beta, z) + \bar{c}_{66}^* \varepsilon_{\alpha\beta}^*(\alpha, \beta, z) \\
&= \sigma_{\alpha\beta}^{*(0)}(\alpha, \beta, z) + \sigma_{\alpha\beta}^{*(1)}(\alpha, \beta, z), \quad (\text{B.8c})
\end{aligned}$$

where in a similar way to the zero-order and first-order strain definitions, Equations (B.8) are splitted into the zero-order and first-order stress components, where, for example, for the first Equation of (B.8)

$$\sigma_{\alpha\alpha}^{*(0)}(\alpha, \beta, z) = z_\alpha^{(0)} \bar{c}_{11}^* \varepsilon_{\alpha\alpha}^{*(0)}(\alpha, \beta) + z_\beta^{(0)} \bar{c}_{12}^* \varepsilon_{\beta\beta}^{*(0)}(\alpha, \beta) + z_{\alpha\beta}^{(0)} \bar{c}_{16}^* \varepsilon_{\alpha\beta}^{*(0)}(\alpha, \beta), \quad (\text{B.9a})$$

$$\sigma_{\alpha\alpha}^{*(1)}(\alpha, \beta, z) = z_\alpha^{(1)} \bar{c}_{11}^* \varepsilon_{\alpha\alpha}^{*(1)}(\alpha, \beta) + z_\beta^{(1)} \bar{c}_{12}^* \varepsilon_{\beta\beta}^{*(1)}(\alpha, \beta) + z_{\alpha\beta}^{(1)} \bar{c}_{16}^* \varepsilon_{\alpha\beta}^{*(1)}(\alpha, \beta). \quad (\text{B.9b})$$

Similar relations hold for the second and third Equations of (B.8) which for the sake of brevity are not presented here.

B.3 Piezoelectric Constitutive Behavior

The linear piezoelectric constitutive equations in compact matrix notation [IEEE (1988)] are given by

$$\boldsymbol{\sigma} = \mathbf{c}^E \boldsymbol{\varepsilon} - \mathbf{e}^T \mathbf{E}, \quad (\text{B.10})$$

$$\mathbf{D} = \mathbf{e} \boldsymbol{\varepsilon} + \boldsymbol{\epsilon}^S \mathbf{E}, \quad (\text{B.11})$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{E}$ and $\mathbf{D}$ are, respectively, the stress, strain, electric field and electric displacement vectors, and $\mathbf{c}^E$, $\mathbf{e}^T$ and $\boldsymbol{\epsilon}^S$ are, respectively, the elasticity (at constant electric field), transpose piezoelectric and dielectric (at constant strain) matrices appropriate for the material.

The material of piezoelectric layers is assumed to be general orthotropic, with the axes of orthotropy not necessarily parallel (arbitrary orientation of the piezoelectric shell layer) to the axes of principal curvature of the shell layer (α, β, z), and polarized in the transverse direction z . Furthermore, it has the symmetry properties of an orthorhombic material of the class $mm2$ [Nye (1985), IEEE (1988)]. Representing Equations (B.10) and (B.11), with their full matrix and vector terms in *engineering notation* yields

$$\begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{zz} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & \bar{c}_{16} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} & 0 & 0 & \bar{c}_{26} \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} & 0 & 0 & \bar{c}_{36} \\ 0 & 0 & 0 & \bar{c}_{44} & \bar{c}_{45} & 0 \\ 0 & 0 & 0 & \bar{c}_{45} & \bar{c}_{55} & 0 \\ \bar{c}_{16} & \bar{c}_{26} & \bar{c}_{36} & 0 & 0 & \bar{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{33} \\ \bar{e}_{14} & \bar{e}_{24} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \\ 0 & 0 & \bar{e}_{36} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_z \end{Bmatrix}, \quad (\text{B.12})$$

$$\begin{Bmatrix} D_\alpha \\ D_\beta \\ D_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{e}_{14} & \bar{e}_{15} & 0 \\ 0 & 0 & 0 & \bar{e}_{24} & \bar{e}_{25} & 0 \\ \bar{e}_{31} & \bar{e}_{32} & \bar{e}_{33} & 0 & 0 & \bar{e}_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} + \begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_z \end{Bmatrix}. \quad (\text{B.13})$$

For simplicity the superscripts $(\cdot)^E$ and $(\cdot)^S$ of Equations (B.10) and (B.11) have been dropped. The relationships between the problem quantities $\bar{c}_{ij}$, $\bar{e}_{rj}$ and $\bar{\epsilon}_{rs}$ ($i, j = 1, \dots, 6$; $r, s = 1, 2, 3$) and the original material c_{ij} , e_{rj} and ϵ_{rs} when the material system $\alpha\beta$ -plane is rotated an angle $+\theta$ (rotating from α to β) around the z -axis are given by

$$\begin{aligned} \bar{c}_{11} &= c_{11} \cos^4 \theta + 2(c_{12} + 2c_{66}) \sin^2 \theta \cos^2 \theta + c_{22} \sin^4 \theta, \\ \bar{c}_{12} &= c_{12} (\sin^4 \theta + \cos^4 \theta) + (c_{11} + c_{22} - 4c_{66}) \sin^2 \theta \cos^2 \theta, \\ \bar{c}_{13} &= c_{13} \cos^2 \theta + c_{23} \sin^2 \theta, \\ \bar{c}_{16} &= (c_{11} - c_{12} - 2c_{66}) \sin \theta \cos^3 \theta + (c_{12} - c_{22} + 2c_{66}) \sin^3 \theta \cos \theta, \\ \bar{c}_{22} &= c_{11} \sin^4 \theta + 2(c_{12} + 2c_{66}) \sin^2 \theta \cos^2 \theta + c_{22} \cos^4 \theta, \\ \bar{c}_{23} &= c_{13} \sin^2 \theta + c_{23} \cos^2 \theta, \\ \bar{c}_{26} &= (c_{11} - c_{12} - 2c_{66}) \sin^3 \theta \cos \theta + (c_{12} - c_{22} + 2c_{66}) \sin \theta \cos^3 \theta, \\ \bar{c}_{33} &= c_{33}, \\ \bar{c}_{36} &= (c_{13} - c_{23}) \sin \theta \cos \theta, \\ \bar{c}_{44} &= c_{44} \cos^2 \theta + c_{55} \sin^2 \theta, \end{aligned}$$

$$\begin{aligned}
\bar{c}_{45} &= (c_{55} - c_{44}) \sin \theta \cos \theta, \\
\bar{c}_{55} &= c_{55} \cos^2 \theta + c_{44} \sin^2 \theta, \\
\bar{c}_{66} &= 2(c_{11} + c_{22} - 2c_{12}) \sin^2 \theta \cos^2 \theta + c_{66}(\sin^2 \theta - \cos^2 \theta)^2,
\end{aligned} \tag{B.14a-m}$$

and

$$\begin{aligned}
\bar{e}_{14} &= (e_{15} - e_{24}) \sin \theta \cos \theta, & \bar{e}_{31} &= e_{31} \cos^2 \theta + e_{32} \sin^2 \theta, \\
\bar{e}_{15} &= e_{15} \cos^2 \theta + e_{24} \sin^2 \theta, & \bar{e}_{32} &= e_{31} \sin^2 \theta + e_{32} \cos^2 \theta, \\
\bar{e}_{24} &= e_{15} \sin^2 \theta + e_{24} \cos^2 \theta, & \bar{e}_{33} &= e_{33}, \\
\bar{e}_{25} &= (e_{15} - e_{24}) \sin \theta \cos \theta, & \bar{e}_{36} &= (e_{31} - e_{32}) \sin \theta \cos \theta,
\end{aligned} \tag{B.15a-h}$$

$$\begin{aligned}
\bar{\epsilon}_{11} &= \epsilon_{11} \cos^2 \theta + \epsilon_{22} \sin^2 \theta, & \bar{\epsilon}_{22} &= \epsilon_{11} \sin^2 \theta + \epsilon_{22} \cos^2 \theta, \\
\bar{\epsilon}_{12} &= (\epsilon_{11} - \epsilon_{22}) \sin \theta \cos \theta, & \bar{\epsilon}_{33} &= \epsilon_{33}.
\end{aligned} \tag{B.16a-d}$$

The electrostatic equilibrium of a piezoelectric media expressed in the constitutive Equation (B.12) can be re-written has

$$\begin{Bmatrix} D_\alpha \\ D_\beta \\ D_z \end{Bmatrix} = \begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} & 0 \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} & 0 \\ 0 & 0 & \bar{\epsilon}_{33} \end{bmatrix} \begin{Bmatrix} \hat{E}_\alpha - \bar{E}_\alpha \\ \hat{E}_\beta - \bar{E}_\beta \\ \hat{E}_z - \bar{E}_z \end{Bmatrix}, \tag{B.17}$$

where $\hat{E}_\alpha$, $\hat{E}_\beta$ and $\hat{E}_z$ are the electric field components due to the converse prescribed (externally applied) piezoelectric effects and $\bar{E}_\alpha$, $\bar{E}_\beta$ and $\bar{E}_z$ are the electric field components due to the direct (induced) piezoelectric effect given by

$$\bar{E}_\alpha = -\frac{\bar{\epsilon}_{22}}{\bar{\epsilon}_{12}^*} (\bar{e}_{14}\varepsilon_{\beta z} + \bar{e}_{15}\varepsilon_{z\alpha}) + \frac{\bar{\epsilon}_{12}}{\bar{\epsilon}_{12}^*} (\bar{e}_{24}\varepsilon_{\beta z} + \bar{e}_{25}\varepsilon_{z\alpha}), \tag{B.18a}$$

$$\bar{E}_\beta = +\frac{\bar{\epsilon}_{12}}{\bar{\epsilon}_{12}^*} (\bar{e}_{14}\varepsilon_{\beta z} + \bar{e}_{15}\varepsilon_{z\alpha}) - \frac{\bar{\epsilon}_{11}}{\bar{\epsilon}_{12}^*} (\bar{e}_{24}\varepsilon_{\beta z} + \bar{e}_{25}\varepsilon_{z\alpha}), \tag{B.18b}$$

$$\bar{E}_z = -\frac{1}{\bar{\epsilon}_{33}} (\bar{e}_{31}\varepsilon_{\alpha\alpha} + \bar{e}_{32}\varepsilon_{\beta\beta} + \bar{e}_{33}\varepsilon_{zz} + \bar{e}_{36}\varepsilon_{\alpha\beta}), \tag{B.18c}$$

where $\bar{\epsilon}_{12}^* = \bar{\epsilon}_{11}\bar{\epsilon}_{22} - (\bar{\epsilon}_{12})^2$.

Since in the present problem the electrodes are located on the top and bottom surfaces and only a transverse electric field $\hat{E}_z$ is externally applied, the conditions $D_\alpha = D_\beta = 0$ hold in the shell surfaces not covered with electrodes and in contact with a medium with low permittivity, e.g., vacuum or air [see Rogacheva (1994), Krommer and Irschik (1999), Vasques and Rodrigues (2005) and Vasques and Rodrigues (2005) for further details]. Thus, considering the relationships in (B.17) the prescribed in-plane electric field components $\hat{E}_\alpha$ and $\hat{E}_\beta$ are equal to the induced ones, and in matrix form they are determined from Equation (B.13),

$$\begin{Bmatrix} E_\alpha \\ E_\beta \end{Bmatrix} = -\begin{bmatrix} \bar{\epsilon}_{11} & \bar{\epsilon}_{12} \\ \bar{\epsilon}_{12} & \bar{\epsilon}_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & \bar{e}_{14} & \bar{e}_{15} & 0 \\ 0 & 0 & 0 & \bar{e}_{24} & \bar{e}_{25} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix}. \tag{B.19}$$

Condensing the in-plane induced electric fields in Equation (B.12) yields

$$\begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{zz} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} & 0 & 0 & \bar{c}_{16} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} & 0 & 0 & \bar{c}_{26} \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} & 0 & 0 & \bar{c}_{36} \\ 0 & 0 & 0 & \bar{c}_{44}^* & \bar{c}_{45}^* & 0 \\ 0 & 0 & 0 & \bar{c}_{45}^* & \bar{c}_{55}^* & 0 \\ \bar{c}_{16} & \bar{c}_{26} & \bar{c}_{36} & 0 & 0 & \bar{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} \bar{e}_{31} \\ \bar{e}_{32} \\ \bar{e}_{33} \\ 0 \\ 0 \\ \bar{e}_{36} \end{bmatrix} E_z, \quad (\text{B.20})$$

$$D_z = \begin{bmatrix} \bar{e}_{31} & \bar{e}_{32} & \bar{e}_{33} & 0 & 0 & \bar{e}_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} + \bar{e}_{33} E_z, \quad (\text{B.21})$$

where the modified constants (or *effective shear stiffness parameters*) $\bar{c}_{44}^*$, $\bar{c}_{45}^*$ and $\bar{c}_{55}^*$ are given by

$$\bar{c}_{44}^* = \bar{c}_{44} - \frac{\bar{e}_{11} (\bar{e}_{24})^2 - 2\bar{e}_{14}\bar{e}_{12}\bar{e}_{24} + \bar{e}_{22} (\bar{e}_{14})^2}{\bar{e}_{11}\bar{e}_{22} - (\bar{e}_{12})^2}, \quad (\text{B.22a})$$

$$\bar{c}_{55}^* = \bar{c}_{55} - \frac{\bar{e}_{11} (\bar{e}_{25})^2 - 2\bar{e}_{15}\bar{e}_{12}\bar{e}_{25} + \bar{e}_{22} (\bar{e}_{15})^2}{\bar{e}_{11}\bar{e}_{22} - (\bar{e}_{12})^2}, \quad (\text{B.22b})$$

$$\bar{c}_{45}^* = \bar{c}_{45} - \frac{\bar{e}_{15}\bar{e}_{22}\bar{e}_{14} - \bar{e}_{15}\bar{e}_{12}\bar{e}_{24} - \bar{e}_{25}\bar{e}_{12}\bar{e}_{14} + \bar{e}_{25}\bar{e}_{11}\bar{e}_{24}}{\bar{e}_{11}\bar{e}_{22} - (\bar{e}_{12})^2}. \quad (\text{B.22c})$$

Similar relations to Equation (B.10) can be expressed for the full strain-stress relationship,

$$\boldsymbol{\varepsilon} = \mathbf{s}^E \boldsymbol{\sigma} - \mathbf{d}^T \mathbf{E}, \quad (\text{B.23})$$

where $\mathbf{s}^E$ is the compliance matrix (at constant electric field) and $\mathbf{d}^T$ the transpose piezoelectric strain constant matrix. From Equation (B.20), the reduced form in full matrix and vector terms reads

$$\begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{zz} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{s}_{11} & \bar{s}_{12} & \bar{s}_{13} & 0 & 0 & \bar{s}_{16} \\ \bar{s}_{12} & \bar{s}_{22} & \bar{s}_{23} & 0 & 0 & \bar{s}_{26} \\ \bar{s}_{13} & \bar{s}_{23} & \bar{s}_{33} & 0 & 0 & \bar{s}_{36} \\ 0 & 0 & 0 & \bar{s}_{44}^* & \bar{s}_{45}^* & 0 \\ 0 & 0 & 0 & \bar{s}_{45}^* & \bar{s}_{55}^* & 0 \\ \bar{s}_{16} & \bar{s}_{26} & \bar{s}_{36} & 0 & 0 & \bar{s}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{zz} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} \bar{d}_{31} \\ \bar{d}_{32} \\ \bar{d}_{33} \\ 0 \\ 0 \\ \bar{d}_{36} \end{bmatrix} E_z, \quad (\text{B.24})$$

where the problem quantities $\bar{s}_{ij}$, with the shear effective (modified) compliance parameters, and $\bar{d}_{rj}$ might be obtained from the relationships $\mathbf{s}^E = (\mathbf{c}^E)^{-1}$, taking the matrix $\mathbf{c}^E$ defined in Equation (B.20), and $\mathbf{d} = \mathbf{e}\mathbf{s}^E$.

If one now reduces Equations (B.20) and (B.21) to the plane-stress constitutive behavior,

where $\sigma_{zz} \approx 0$, yields

$$\begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\beta z} \\ \sigma_{z\alpha} \\ \sigma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \bar{c}_{11}^* & \bar{c}_{12}^* & 0 & 0 & \bar{c}_{16}^* \\ \bar{c}_{12}^* & \bar{c}_{22}^* & 0 & 0 & \bar{c}_{26}^* \\ 0 & 0 & \bar{c}_{44}^* & \bar{c}_{45}^* & 0 \\ 0 & 0 & \bar{c}_{45}^* & \bar{c}_{55}^* & 0 \\ \bar{c}_{16}^* & \bar{c}_{26}^* & 0 & 0 & \bar{c}_{66}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} \bar{e}_{31}^* \\ \bar{e}_{32}^* \\ 0 \\ 0 \\ \bar{e}_{36}^* \end{bmatrix} E_z, \quad (\text{B.25})$$

$$D_z = [\bar{e}_{31}^* \quad \bar{e}_{32}^* \quad 0 \quad 0 \quad \bar{e}_{36}^*] \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\beta z} \\ \varepsilon_{z\alpha} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} + \bar{\epsilon}_{33}^* E_z = \bar{\epsilon}_{33}^* (\hat{E}_z - \bar{E}_z), \quad (\text{B.26})$$

where the elastic stiffness constants have been modified to take the plane-stress assumption into account and are defined as

$$\begin{aligned} \bar{c}_{11}^* &= \bar{c}_{11} - \frac{(\bar{c}_{13})^2}{\bar{c}_{33}}, & \bar{c}_{22}^* &= \bar{c}_{22} - \frac{(\bar{c}_{23})^2}{\bar{c}_{33}}, & \bar{c}_{66}^* &= \bar{c}_{66} - \frac{(\bar{c}_{36})^2}{\bar{c}_{33}}, \\ \bar{c}_{12}^* &= \bar{c}_{12} - \frac{\bar{c}_{13}\bar{c}_{23}}{\bar{c}_{33}}, & \bar{c}_{16}^* &= \bar{c}_{16} - \frac{\bar{c}_{13}\bar{c}_{36}}{\bar{c}_{33}}, & \bar{c}_{26}^* &= \bar{c}_{26} - \frac{\bar{c}_{23}\bar{c}_{36}}{\bar{c}_{33}}, \end{aligned} \quad (\text{B.27a-f})$$

and $\bar{e}_{31}^*$, $\bar{e}_{32}^*$ and $\bar{e}_{36}^*$ are reduced (modified) piezoelectric plane-stress constants and $\bar{\epsilon}_{33}^*$ the dielectric one, given by

$$\begin{aligned} \bar{e}_{31}^* &= \bar{e}_{31} - \bar{e}_{33} \frac{\bar{c}_{13}}{\bar{c}_{33}}, & \bar{e}_{32}^* &= \bar{e}_{32} - \bar{e}_{33} \frac{\bar{c}_{23}}{\bar{c}_{33}}, \\ \bar{e}_{36}^* &= \bar{e}_{36} - \bar{e}_{33} \frac{\bar{c}_{36}}{\bar{c}_{33}}, & \bar{\epsilon}_{33}^* &= \bar{\epsilon}_{33} + \frac{(\bar{e}_{33})^2}{\bar{c}_{33}}. \end{aligned} \quad (\text{B.28a-d})$$

Similar relations to the previous equations hold for the reduced compliances and piezoelectric strain constants when the plane-stress is considered.

B.4 Mixed Displacement Field and Strains

Considering the shear angles definition in Equations (4.27) and substituting them into Equations (4.41), the in-plane *mixed* displacements, defined in Equations (4.44), can be expressed as

$$\begin{aligned} u(\alpha, \beta, z, t) &= \frac{H_\alpha}{A_\alpha} u_0 + \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(f)} \left(\frac{\partial w_0}{\partial \alpha} + A_\alpha \theta_\alpha - \frac{A_\alpha}{R_\alpha} u_0 \right) - \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(0)} \frac{\partial w_0}{\partial \alpha} \\ &\quad + \frac{H_\alpha}{A_\alpha^2} \bar{s}_{55}^* z_\alpha^{*(0)} \bar{\tau}_{z\alpha} + \frac{H_\alpha}{A_\alpha^2} \frac{\bar{s}_{55}^*}{2h} z_\alpha^{*(1)} \bar{\tau}_{z\alpha} \\ &= z_{11}^u u_0 + z_{13}^u \frac{\partial w_0}{\partial \alpha} + z_{14}^u \theta_\alpha + z_{11}^\tau \bar{\tau}_{z\alpha} + z_{12}^\tau \bar{\tau}_{z\alpha}, \end{aligned} \quad (\text{B.29a})$$

$$\begin{aligned} v(\alpha, \beta, z, t) &= \frac{H_\beta}{A_\beta} v_0 + \frac{H_\beta}{A_\beta^2} z_\beta^{*(f)} \left(\frac{\partial w_0}{\partial \beta} + A_\beta \theta_\beta - \frac{A_\beta}{R_\beta} v_0 \right) - \frac{H_\beta}{A_\beta^2} z_\beta^{*(0)} \frac{\partial w_0}{\partial \beta} \\ &\quad + \frac{H_\beta}{A_\beta^2} \bar{s}_{44}^* z_\beta^{*(0)} \bar{\tau}_{\beta z} + \frac{H_\beta}{A_\beta^2} \frac{\bar{s}_{44}^*}{2h} z_\beta^{*(1)} \bar{\tau}_{\beta z} \\ &= z_{22}^u v_0 + z_{23}^u \frac{\partial w_0}{\partial \beta} + z_{25}^u \theta_\beta + z_{23}^\tau \bar{\tau}_{\beta z} + z_{24}^\tau \bar{\tau}_{\beta z}, \end{aligned} \quad (\text{B.29b})$$

where the $z_{ij}^u = z_{ij}^u(z)$ and $z_{ir}^\tau = z_{ir}^\tau(z)$ coefficients are given by

$$\begin{aligned} z_{11}^u &= \frac{H_\alpha}{A_\alpha} - \frac{H_\alpha}{A_\alpha R_\alpha} z_\alpha^{*(f)}, & z_{13}^u &= \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(f)} - \frac{H_\alpha}{A_\alpha^2} z_\alpha^{*(0)}, & z_{14}^u &= \frac{H_\alpha}{A_\alpha} z_\alpha^{*(f)}, \\ z_{22}^u &= \frac{H_\beta}{A_\beta} - \frac{H_\beta}{A_\beta R_\beta} z_\beta^{*(f)}, & z_{23}^u &= \frac{H_\beta}{A_\beta^2} z_\beta^{*(f)} - \frac{H_\beta}{A_\beta^2} z_\beta^{*(0)}, & z_{25}^u &= \frac{H_\beta}{A_\beta} z_\beta^{*(f)}, \end{aligned} \quad (\text{B.30a-f})$$

$$\begin{aligned} z_{11}^\tau &= \frac{H_\alpha}{A_\alpha^2} \bar{s}_{55}^* z_\alpha^{*(0)}, & z_{12}^\tau &= \frac{H_\alpha}{A_\alpha^2} \frac{\bar{s}_{55}^*}{2h} z_\alpha^{*(1)}, \\ z_{23}^\tau &= \frac{H_\beta}{A_\beta^2} \bar{s}_{44}^* z_\beta^{*(0)}, & z_{24}^\tau &= \frac{H_\beta}{A_\beta^2} \frac{\bar{s}_{44}^*}{2h} z_\beta^{*(1)}. \end{aligned} \quad (\text{B.31a-d})$$

It is worthy to mention that these terms are functions of z and incorporate elastic constants of the material and geometric variables of the shell layer.

The non-zero coefficients of matrices $\mathbf{z}^{\varepsilon u}(z)$ and $\mathbf{z}^{\varepsilon \tau}(z)$ used to define the strain field in Equation (4.49) are given by

$$\begin{aligned} z_{11}^{\varepsilon u} &= \frac{z_{11}^u}{H_\alpha}, & z_{13}^{\varepsilon u(1)} &= \frac{A_\alpha}{H_\alpha R_\alpha}, & z_{13}^{\varepsilon u(2)} &= \frac{z_{13}^u}{H_\alpha}, \\ z_{14}^{\varepsilon u} &= \frac{z_{14}^u}{H_\alpha}, & z_{22}^{\varepsilon u} &= \frac{z_{22}^u}{H_\beta}, & z_{23}^{\varepsilon u(1)} &= \frac{A_\beta}{H_\beta R_\beta}, \\ z_{23}^{\varepsilon u(2)} &= \frac{z_{23}^u}{H_\beta}, & z_{25}^{\varepsilon u} &= \frac{z_{25}^u}{H_\beta}, & z_{32}^{\varepsilon u} &= \frac{dz_{22}^u}{dz} - \frac{A_\beta}{H_\beta R_\beta} z_{22}^u, \\ z_{33}^{\varepsilon u} &= \frac{1}{H_\beta} + \frac{dz_{23}^u}{dz} - \frac{A_\beta}{H_\beta R_\beta} z_{23}^u, & z_{35}^{\varepsilon u} &= \frac{dz_{25}^u}{dz} - \frac{A_\beta}{H_\beta R_\beta} z_{25}^u, & z_{41}^{\varepsilon u} &= \frac{dz_{11}^u}{dz} - \frac{A_\alpha}{H_\alpha R_\alpha} z_{11}^u, \\ z_{43}^{\varepsilon u} &= \frac{1}{H_\alpha} + \frac{dz_{13}^u}{dz} - \frac{A_\alpha}{H_\alpha R_\alpha} z_{13}^u, & z_{44}^{\varepsilon u} &= \frac{dz_{14}^u}{dz} - \frac{A_\alpha}{H_\alpha R_\alpha} z_{14}^u, & z_{51}^{\varepsilon u} &= \frac{z_{11}^u}{H_\beta}, \\ z_{52}^{\varepsilon u} &= \frac{z_{22}^u}{H_\alpha}, & z_{53}^{\varepsilon u} &= \frac{z_{13}^u}{H_\beta} + \frac{z_{23}^u}{H_\alpha}, & z_{54}^{\varepsilon u} &= \frac{z_{14}^u}{H_\beta}, \\ z_{55}^{\varepsilon u} &= \frac{z_{25}^u}{H_\alpha}, \end{aligned} \quad (\text{B.32a-s})$$

and

$$\begin{aligned} z_{11}^{\varepsilon \tau} &= \frac{z_{11}^\tau}{H_\alpha}, & z_{12}^{\varepsilon \tau} &= \frac{z_{12}^\tau}{H_\alpha}, & z_{23}^{\varepsilon \tau} &= \frac{z_{23}^\tau}{H_\beta}, \\ z_{24}^{\varepsilon \tau} &= \frac{z_{24}^\tau}{H_\beta}, & z_{33}^{\varepsilon \tau} &= \frac{dz_{23}^\tau}{dz} - \frac{A_\beta}{H_\beta R_\beta} z_{23}^\tau, & z_{34}^{\varepsilon \tau} &= \frac{dz_{24}^\tau}{dz} - \frac{A_\beta}{H_\beta R_\beta} z_{24}^\tau, \\ z_{41}^{\varepsilon \tau} &= \frac{dz_{11}^\tau}{dz} - \frac{A_\alpha}{H_\alpha R_\alpha} z_{11}^\tau, & z_{42}^{\varepsilon \tau} &= \frac{dz_{12}^\tau}{dz} - \frac{A_\alpha}{H_\alpha R_\alpha} z_{12}^\tau, & z_{51}^{\varepsilon \tau} &= \frac{z_{11}^\tau}{H_\beta}, \\ z_{52}^{\varepsilon \tau} &= \frac{z_{12}^\tau}{H_\beta}, & z_{53}^{\varepsilon \tau} &= \frac{z_{23}^\tau}{H_\alpha}, & z_{54}^{\varepsilon \tau} &= \frac{z_{24}^\tau}{H_\alpha}. \end{aligned} \quad (\text{B.33a-l})$$

B.5 Effective Strains Coefficients

The modified (taking into account the electrical induced strains) terms of $\mathbf{z}^{\varepsilon u}(z)$ are

$$\begin{aligned}
z_{11}^{*\varepsilon u} &= z_{11}^{u\varepsilon} - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{31}^* (z_{11}^{*u\varepsilon} - z_{11}^{u\varepsilon}), \\
z_{12}^{*\varepsilon u} &= -\bar{d}_{31}^* \bar{\lambda} \bar{e}_{32}^* (z_{22}^{*\varepsilon u} - z_{22}^{u\varepsilon}), \\
z_{13}^{*\varepsilon u(1)} &= z_{13}^{u\varepsilon 1} - \bar{d}_{31}^* \bar{\lambda} [\bar{e}_{31}^* (z_{13}^{*u\varepsilon 1} - z_{13}^{u\varepsilon 1}) - \bar{e}_{32}^* (z_{23}^{*\varepsilon u 1} - z_{23}^{u\varepsilon 1})], \\
z_{13}^{*\varepsilon u(2)} &= z_{13}^{u\varepsilon 2} - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{31}^* (z_{13}^{*u\varepsilon 2} - z_{13}^{u\varepsilon 2}) - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{32}^* (z_{23}^{*\varepsilon u 2} - z_{23}^{u\varepsilon 2}), \\
z_{14}^{*\varepsilon u} &= z_{14}^{u\varepsilon} - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{31}^* (z_{14}^{*u\varepsilon} - z_{14}^{u\varepsilon}), \\
z_{15}^{*\varepsilon u} &= -\bar{d}_{31}^* \bar{\lambda} \bar{e}_{32}^* (z_{25}^{*\varepsilon u} - z_{25}^{u\varepsilon}), \\
z_{21}^{*\varepsilon u} &= -\bar{d}_{32}^* \bar{\lambda} \bar{e}_{31}^* (z_{11}^{*u\varepsilon} - z_{11}^{u\varepsilon}), \\
z_{22}^{*\varepsilon u} &= z_{22}^{u\varepsilon} - \bar{d}_{32}^* \bar{\lambda} \bar{e}_{32}^* (z_{22}^{*\varepsilon u} - z_{22}^{u\varepsilon}), \\
z_{23}^{*\varepsilon u(1)} &= z_{23}^{u\varepsilon 1} - \bar{d}_{32}^* \bar{\lambda} [\bar{e}_{31}^* (z_{13}^{*u\varepsilon 1} - z_{13}^{u\varepsilon 1}) - \bar{e}_{32}^* (z_{23}^{*\varepsilon u 1} - z_{23}^{u\varepsilon 1})], \\
z_{23}^{*\varepsilon u(2)} &= z_{23}^{u\varepsilon 2} - \bar{d}_{32}^* \bar{\lambda} [\bar{e}_{31}^* (z_{13}^{*u\varepsilon 1} - z_{13}^{u\varepsilon 1}) - \bar{e}_{32}^* (z_{23}^{*\varepsilon u 1} - z_{23}^{u\varepsilon 1})], \\
z_{24}^{*\varepsilon u} &= -\bar{d}_{32}^* \bar{\lambda} \bar{e}_{31}^* (z_{14}^{*u\varepsilon} - z_{14}^{u\varepsilon}), \\
z_{25}^{*\varepsilon u} &= z_{25}^{u\varepsilon} - \bar{d}_{32}^* \bar{\lambda} \bar{e}_{32}^* (z_{25}^{*\varepsilon u} - z_{25}^{u\varepsilon}),
\end{aligned} \tag{B.34a-l}$$

and of $\mathbf{z}^{\varepsilon \tau}(z)$ are

$$\begin{aligned}
z_{11}^{*\varepsilon \tau} &= z_{11}^{\varepsilon \tau} - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{31}^* (z_{11}^{*\varepsilon \tau} - z_{11}^{\varepsilon \tau}), & z_{12}^{*\varepsilon \tau} &= z_{12}^{\varepsilon \tau} - \bar{d}_{31}^* \bar{\lambda} \bar{e}_{31}^* (z_{12}^{*\varepsilon \tau} - z_{12}^{\varepsilon \tau}), \\
z_{13}^{*\varepsilon \tau} &= -\bar{d}_{31}^* \bar{\lambda} \bar{e}_{32}^* (z_{23}^{*\varepsilon \tau} - z_{23}^{\varepsilon \tau}), & z_{14}^{*\varepsilon \tau} &= -\bar{d}_{31}^* \bar{\lambda} \bar{e}_{32}^* (z_{24}^{*\varepsilon \tau} - z_{24}^{\varepsilon \tau}), \\
z_{21}^{*\varepsilon \tau} &= -\bar{d}_{32}^* \bar{\lambda} \bar{e}_{31}^* (z_{11}^{*\varepsilon \tau} - z_{11}^{\varepsilon \tau}), & z_{23}^{*\varepsilon \tau} &= -\bar{d}_{32}^* \bar{\lambda} \bar{e}_{31}^* (z_{12}^{*\varepsilon \tau} - z_{12}^{\varepsilon \tau}), \\
z_{23}^{*\varepsilon \tau} &= z_{23}^{\varepsilon \tau} - \bar{d}_{32}^* \bar{\lambda} \bar{e}_{32}^* (z_{23}^{*\varepsilon \tau} - z_{23}^{\varepsilon \tau}), & z_{24}^{*\varepsilon \tau} &= z_{24}^{\varepsilon \tau} - \bar{d}_{32}^* \bar{\lambda} \bar{e}_{32}^* (z_{24}^{*\varepsilon \tau} - z_{24}^{\varepsilon \tau}),
\end{aligned} \tag{B.35a-h}$$

where $\bar{\lambda} = \lambda/\bar{e}_{33}^*$ and the $z_{ij}^{* \dots}$ denotes integration in order to z as

$$z_{ij}^{* \dots} = \frac{1}{2h} \int_{-h}^{+h} z_{ij}^{* \dots} dz. \tag{B.36}$$

B.6 Virtual Work Terms

B.6.1 Virtual Work of the Inertial Forces

Taking into account Equations (4.2) and the doubly-curved shells restrictions in Equations (4.39) into Equation (4.54), yields

$$\begin{aligned}
\delta T &= - \int_{\Omega_0} \left[\int_{-h}^{+h} \rho (\delta u \ddot{u} + \delta v \ddot{v} + \delta w \ddot{w}) H_\alpha H_\beta dz \right] d\alpha d\beta \\
&= - \int_{\Omega_0} \left[\delta u_0 (I_{11}^{uu} \ddot{u}_0 + I_{13}^{uu} \frac{\partial \ddot{u}_0}{\partial \alpha} + I_{14}^{uu} \ddot{\theta}_\alpha + I_{11}^{\tau u} \ddot{\tau}_{z\alpha} + I_{21}^{\tau u} \ddot{\tau}_{z\alpha}) + \delta v_0 (I_{22}^{uu} \ddot{v}_0 + I_{23}^{uu} \frac{\partial \ddot{v}_0}{\partial \beta} \right. \\
&\quad + I_{25}^{uu} \ddot{\theta}_\beta + I_{32}^{\tau u} \ddot{\tau}_{\beta z} + I_{42}^{\tau u} \ddot{\tau}_{\beta z}) + \delta w_0 (I_{33}^{uu} \ddot{w}_0) + \delta \frac{\partial w_0}{\partial \alpha} (I_{13}^{uu} \ddot{u}_0 + I_{33}^{u\alpha} \frac{\partial \ddot{u}_0}{\partial \alpha} + I_{34}^{uu} \ddot{\theta}_\alpha + I_{13}^{\tau u} \ddot{\tau}_{z\alpha} \\
&\quad \left. + I_{23}^{\tau u} \ddot{\tau}_{z\alpha}) + \delta \frac{\partial w_0}{\partial \beta} (I_{23}^{uu} \ddot{v}_0 + I_{33}^{u\beta} \frac{\partial \ddot{v}_0}{\partial \beta} + I_{35}^{uu} \ddot{\theta}_\beta + I_{33}^{\tau u} \ddot{\tau}_{\beta z} + I_{43}^{\tau u} \ddot{\tau}_{\beta z}) + \delta \theta_\alpha (I_{14}^{uu} \ddot{u}_0 \right.
\end{aligned}$$

$$\begin{aligned}
& + I_{34}^{uu} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{44}^{uu} \ddot{\theta}_\alpha + I_{14}^{\tau u} \ddot{\tau}_{z\alpha} + I_{24}^{\tau u} \ddot{\tau}_{z\alpha} + \delta \theta_\beta (I_{25}^{uu} \ddot{w}_0 + I_{35}^{uu} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{55}^{uu} \ddot{\theta}_\beta + I_{35}^{\tau u} \ddot{\tau}_{\beta z} \\
& + I_{45}^{\tau u} \ddot{\tau}_{\beta z}) + \delta \bar{\tau}_{z\alpha} (I_{11}^{\tau u} \ddot{w}_0 + I_{13}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{14}^{\tau u} \ddot{\theta}_\alpha + I_{11}^{\tau \tau} \ddot{\tau}_{z\alpha} + I_{12}^{\tau \tau} \ddot{\tau}_{z\alpha}) + \delta \bar{\tau}_{z\alpha} (I_{21}^{\tau u} \ddot{w}_0 \\
& + I_{23}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \alpha} + I_{24}^{\tau u} \ddot{\theta}_\alpha + I_{12}^{\tau \tau} \ddot{\tau}_{z\alpha} + I_{22}^{\tau \tau} \ddot{\tau}_{z\alpha}) + \delta \bar{\tau}_{\beta z} (I_{32}^{\tau u} \ddot{w}_0 + I_{33}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{35}^{\tau u} \ddot{\theta}_\beta + I_{33}^{\tau \tau} \ddot{\tau}_{\beta z} \\
& + I_{34}^{\tau \tau} \ddot{\tau}_{\beta z}) + \delta \bar{\tau}_{\beta z} (I_{42}^{\tau u} \ddot{w}_0 + I_{43}^{\tau u} \frac{\partial \ddot{w}_0}{\partial \beta} + I_{45}^{\tau u} \ddot{\theta}_\beta + I_{34}^{\tau \tau} \ddot{\tau}_{\beta z} + I_{44}^{\tau \tau} \ddot{\tau}_{z\alpha}) \Big] d\alpha d\beta, \tag{B.37}
\end{aligned}$$

where

$$\begin{aligned}
(I_{11}^{uu}, I_{13}^{uu}, I_{14}^{uu}) &= \rho \langle z_{11}^u(z_{11}^u, z_{13}^u, z_{14}^u) \rangle, \\
(I_{22}^{uu}, I_{23}^{uu}, I_{25}^{uu}) &= \rho \langle z_{22}^u(z_{22}^u, z_{23}^u, z_{25}^u) \rangle, \\
(I_{33}^{uu}) &= \rho \langle 1 \rangle, \\
(I_{33}^{u\alpha}, I_{34}^{uu}) &= \rho \langle z_{13}^u(z_{13}^u, z_{14}^u) \rangle, \\
(I_{33}^{u\beta}, I_{35}^{uu}) &= \rho \langle z_{23}^u(z_{23}^u, z_{25}^u) \rangle, \\
(I_{44}^{uu}, I_{55}^{uu}) &= \rho \langle (z_{14}^u z_{14}^u, z_{25}^u z_{25}^u) \rangle, \\
(I_{11}^{\tau u}, I_{13}^{\tau u}, I_{14}^{\tau u}, I_{11}^{\tau \tau}, I_{12}^{\tau \tau}) &= \rho \langle z_{11}^\tau(z_{11}^u, z_{13}^u, z_{14}^u, z_{11}^\tau, z_{12}^\tau) \rangle, \\
(I_{21}^{\tau u}, I_{23}^{\tau u}, I_{24}^{\tau u}, I_{22}^{\tau \tau}) &= \rho \langle z_{12}^\tau(z_{11}^u, z_{13}^u, z_{14}^u, z_{12}^\tau) \rangle, \\
(I_{32}^{\tau u}, I_{33}^{\tau u}, I_{35}^{\tau u}, I_{33}^{\tau \tau}, I_{34}^{\tau \tau}) &= \rho \langle z_{23}^\tau(z_{22}^u, z_{23}^u, z_{25}^u, z_{23}^\tau, z_{34}^\tau) \rangle, \\
(I_{42}^{\tau u}, I_{43}^{\tau u}, I_{45}^{\tau u}, I_{44}^{\tau \tau}) &= \rho \langle z_{24}^\tau(z_{22}^u, z_{23}^u, z_{25}^u, z_{24}^\tau) \rangle. \tag{B.38a-j}
\end{aligned}$$

For convenience $\langle \dots \rangle$ denotes thickness integration and it is defined by

$$\langle \dots \rangle = \int_{-h}^{+h} (\dots) H_\alpha H_\beta dz. \tag{B.39}$$

B.6.2 Virtual Work of the Internal Electro-Mechanical Forces

Considering the strain definitions in Equation (4.47), the term δH_ε of Equation (4.56) is given by

$$\begin{aligned}
\delta H_\varepsilon &= \int_{\Omega_0} \left[\int_{-h}^h (\sigma_{\alpha\alpha} \delta \varepsilon_{\alpha\alpha} + \sigma_{\beta\beta} \delta \varepsilon_{\beta\beta} + \sigma_{\beta z} \delta \varepsilon_{\beta z} + \sigma_{z\alpha} \delta \varepsilon_{z\alpha} + \sigma_{\alpha\beta} \delta \varepsilon_{\alpha\beta}) H_\alpha H_\beta dz \right] d\alpha d\beta \\
&= \int_{\Omega_0} \left[\delta \frac{\partial u_0}{\partial \alpha} (N_{\alpha\alpha}^{*11} + N_{\beta\beta}^{*21}) + \delta \frac{\partial u_0}{\partial \beta} (N_{\alpha\beta}^{51}) + \delta u_0 (Q_{z\alpha}^{41}) + \delta \frac{\partial v_0}{\partial \beta} (N_{\alpha\alpha}^{*12} + N_{\beta\beta}^{*22}) \right. \\
&+ \delta \frac{\partial v_0}{\partial \alpha} (N_{\alpha\beta}^{52}) + \delta v_0 (Q_{\beta z}^{32}) + \delta w_0 (M_{\alpha\alpha}^{*13(1)} + M_{\beta\beta}^{*23(1)}) + \delta \frac{\partial^2 w_0}{\partial \alpha^2} (M_{\alpha\alpha}^{*13(2)}) + \delta \frac{\partial^2 w_0}{\partial \beta^2} (M_{\beta\beta}^{*23(2)}) \\
&+ \delta \frac{\partial^2 w_0}{\partial \alpha \partial \beta} (M_{\alpha\beta}^{53}) + \delta \frac{\partial w_0}{\partial \alpha} (Q_{z\alpha}^{43}) + \delta \frac{\partial w_0}{\partial \beta} (Q_{\beta z}^{33}) + \delta \frac{\partial \theta_\alpha}{\partial \alpha} (M_{\alpha\alpha}^{*14} + M_{\beta\beta}^{*24}) + \delta \frac{\partial \theta_\alpha}{\partial \beta} (M_{\alpha\beta}^{54}) \\
&+ \delta \theta_\alpha (Q_{z\alpha}^{44}) + \delta \frac{\partial \theta_\beta}{\partial \beta} (M_{\alpha\alpha}^{*15} + M_{\beta\beta}^{*25}) + \delta \frac{\partial \theta_\beta}{\partial \alpha} (M_{\alpha\beta}^{55}) + \delta \theta_\beta (Q_{\beta z}^{35}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} (T_{\alpha\alpha}^{*11} + T_{\beta\beta}^{*21}) \\
&+ \delta \bar{\tau}_{z\alpha} (T_{z\alpha}^{41}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \beta} (T_{\alpha\beta}^{51}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} (T_{\alpha\alpha}^{*12} + T_{\beta\beta}^{*22}) + \delta \bar{\tau}_{z\alpha} (T_{z\alpha}^{42}) + \delta \frac{\partial \bar{\tau}_{z\alpha}}{\partial \beta} (T_{\alpha\beta}^{52}) \\
&+ \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} (T_{\alpha\alpha}^{*13} + T_{\beta\beta}^{*23}) + \delta \bar{\tau}_{\beta z} (T_{\beta z}^{33}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \alpha} (T_{\alpha\beta}^{53}) + \delta \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} (T_{\alpha\alpha}^{*14} + T_{\beta\beta}^{*24}) \Big]
\end{aligned}$$

$$+ \delta \tilde{\tau}_{\beta z} (T_{\beta z}^{34}) + \delta \frac{\partial \tilde{\tau}_{\beta z}}{\partial \alpha} (T_{\alpha \beta}^{54}) \Big] d\alpha d\beta, \quad (\text{B.40})$$

where

$$\begin{aligned} (N_{\alpha\alpha}^{*11}, N_{\alpha\alpha}^{*12}, M_{\alpha\alpha}^{*13(1)}, M_{\alpha\alpha}^{*13(2)}, M_{\alpha\alpha}^{*14}, M_{\alpha\alpha}^{*15}) &= \langle \sigma_{\alpha\alpha}(z_{11}^{*\varepsilon u}, z_{12}^{*\varepsilon u}, z_{13}^{*\varepsilon u(1)}, z_{13}^{*\varepsilon u(2)}, z_{14}^{*\varepsilon u}, z_{15}^{*\varepsilon u}) \rangle, \\ (N_{\beta\beta}^{*21}, N_{\beta\beta}^{*22}, M_{\beta\beta}^{*23(1)}, M_{\beta\beta}^{*23(2)}, M_{\beta\beta}^{*24}, M_{\beta\beta}^{*25}) &= \langle \sigma_{\beta\beta}(z_{21}^{*\varepsilon u}, z_{22}^{*\varepsilon u}, z_{23}^{*\varepsilon u(1)}, z_{23}^{*\varepsilon u(2)}, z_{24}^{*\varepsilon u}, z_{25}^{*\varepsilon u}) \rangle, \\ (Q_{\beta z}^{32}, Q_{\beta z}^{33}, Q_{\beta z}^{35}) &= \langle \sigma_{\beta z}(z_{32}^{\varepsilon u}, z_{33}^{\varepsilon u}, z_{35}^{\varepsilon u}) \rangle, \\ (Q_{z\alpha}^{41}, Q_{z\alpha}^{43}, Q_{z\alpha}^{44}) &= \langle \sigma_{z\alpha}(z_{41}^{\varepsilon u}, z_{43}^{\varepsilon u}, z_{44}^{\varepsilon u}) \rangle, \\ (N_{\alpha\beta}^{51}, N_{\alpha\beta}^{52}, M_{\alpha\beta}^{53}, M_{\alpha\beta}^{54}, M_{\alpha\beta}^{55}) &= \langle \sigma_{\alpha\beta}(z_{51}^{\varepsilon u}, z_{52}^{\varepsilon u}, z_{53}^{\varepsilon u}, z_{54}^{\varepsilon u}, z_{55}^{\varepsilon u}) \rangle, \\ (T_{\alpha\alpha}^{*11}, T_{\alpha\alpha}^{*12}, T_{\alpha\alpha}^{*13}, T_{\alpha\alpha}^{*14}) &= \langle \sigma_{\alpha\alpha}(z_{11}^{*\varepsilon\tau}, z_{12}^{*\varepsilon\tau}, z_{13}^{*\varepsilon\tau}, z_{14}^{*\varepsilon\tau}) \rangle, \\ (T_{\beta\beta}^{*21}, T_{\beta\beta}^{*22}, T_{\beta\beta}^{*23}, T_{\beta\beta}^{*24}) &= \langle \sigma_{\beta\beta}(z_{21}^{*\varepsilon\tau}, z_{22}^{*\varepsilon\tau}, z_{23}^{*\varepsilon\tau}, z_{24}^{*\varepsilon\tau}) \rangle, \\ (T_{\beta z}^{33}, T_{\beta z}^{34}) &= \langle \sigma_{\beta z}(z_{33}^{\varepsilon\tau}, z_{34}^{\varepsilon\tau}) \rangle, \\ (T_{z\alpha}^{41}, T_{z\alpha}^{42}) &= \langle \sigma_{z\alpha}(z_{41}^{\varepsilon\tau}, z_{42}^{\varepsilon\tau}) \rangle, \\ (T_{\alpha\beta}^{51}, T_{\alpha\beta}^{52}, T_{\alpha\beta}^{53}, T_{\alpha\beta}^{54}) &= \langle \sigma_{\alpha\beta}(z_{51}^{\varepsilon\tau}, z_{52}^{\varepsilon\tau}, z_{53}^{\varepsilon\tau}, z_{54}^{\varepsilon\tau}) \rangle. \end{aligned} \quad (\text{B.41a-j})$$

Since the direct piezoelectric effect (the stiffness increase due to the induced electric field) was already considered in Section 4.4.3, the electric enthalpy term δH_E will only consider the applied electric potential counterpart of the transverse electric displacement D_z and electric field E_z defined in Equations (4.36) and (4.37). Thus, considering (B.26) and an orthotropic layer, the second term of Equation (4.56), δH_E , is given by

$$\begin{aligned} \delta H_E &= \int_{\Omega_0} \left[\int_{-h}^{+h} \delta \phi \left(-\frac{\lambda \bar{e}_{31}^*}{2h} \varepsilon_{\alpha\alpha} - \frac{\lambda \bar{e}_{32}^*}{2h} \varepsilon_{\beta\beta} + \frac{\lambda^2 \bar{e}_{33}^*}{4h^2} \phi \right) H_\alpha H_\beta dz \right] d\alpha d\beta \\ &= \int_{\Omega_0} \delta \phi \left[(P_{\alpha\alpha}^{*11} + P_{\beta\beta}^{*21}) \frac{\partial u_0}{\partial \alpha} + (P_{\alpha\alpha}^{*12} + P_{\beta\beta}^{*22}) \frac{\partial v_0}{\partial \beta} + (P_{\alpha\alpha}^{*13} + P_{\beta\beta}^{*23}) w_0 + (P_{\alpha\alpha}^{*13}) \frac{\partial^2 w_0}{\partial \alpha^2} \right. \\ &\quad + (P_{\beta\beta}^{*23}) \frac{\partial^2 w_0}{\partial \beta^2} + (P_{\alpha\alpha}^{*14} + P_{\beta\beta}^{*24}) \frac{\partial \theta_\alpha}{\partial \alpha} + (P_{\alpha\alpha}^{*15} + P_{\beta\beta}^{*25}) \frac{\partial \theta_\beta}{\partial \beta} + (R_{\alpha\alpha}^{*11} + R_{\beta\beta}^{*21}) \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} \\ &\quad \left. + (R_{\alpha\alpha}^{*12} + R_{\beta\beta}^{*22}) \frac{\partial \bar{\tau}_{z\alpha}}{\partial \alpha} + (R_{\alpha\alpha}^{*13} + R_{\beta\beta}^{*23}) \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} + (R_{\alpha\alpha}^{*14} + R_{\beta\beta}^{*24}) \frac{\partial \bar{\tau}_{\beta z}}{\partial \beta} + (S_{\phi\phi}) \phi \right] d\alpha d\beta, \end{aligned} \quad (\text{B.42})$$

where

$$\begin{aligned} (P_{\alpha\alpha}^{*11}, P_{\alpha\alpha}^{*12}, P_{\alpha\alpha}^{*13}, P_{\alpha\alpha}^{*13}, P_{\alpha\alpha}^{*14}, P_{\alpha\alpha}^{*15}) &= \left\langle \frac{\lambda \bar{e}_{31}^*}{2h} (z_{11}^{*\varepsilon u}, z_{12}^{*\varepsilon u}, z_{13}^{*\varepsilon u(1)}, z_{13}^{*\varepsilon u(2)}, z_{14}^{*\varepsilon u}, z_{15}^{*\varepsilon u}) \right\rangle, \\ (P_{\beta\beta}^{*21}, P_{\beta\beta}^{*22}, P_{\beta\beta}^{*23}, P_{\beta\beta}^{*23}, P_{\beta\beta}^{*24}, P_{\beta\beta}^{*25}) &= \left\langle \frac{\lambda \bar{e}_{32}^*}{2h} (z_{21}^{*\varepsilon u}, z_{22}^{*\varepsilon u}, z_{23}^{*\varepsilon u(1)}, z_{23}^{*\varepsilon u(2)}, z_{24}^{*\varepsilon u}, z_{25}^{*\varepsilon u}) \right\rangle, \\ (R_{\alpha\alpha}^{*11}, R_{\alpha\alpha}^{*12}, R_{\alpha\alpha}^{*13}, R_{\alpha\alpha}^{*14}) &= \left\langle \frac{\lambda \bar{e}_{31}^*}{2h} (z_{11}^{*\varepsilon\tau}, z_{12}^{*\varepsilon\tau}, z_{13}^{*\varepsilon\tau}, z_{14}^{*\varepsilon\tau}) \right\rangle, \\ (R_{\beta\beta}^{*21}, R_{\beta\beta}^{*22}, R_{\beta\beta}^{*23}, R_{\beta\beta}^{*24}) &= \left\langle \frac{\lambda \bar{e}_{32}^*}{2h} (z_{21}^{*\varepsilon\tau}, z_{22}^{*\varepsilon\tau}, z_{23}^{*\varepsilon\tau}, z_{24}^{*\varepsilon\tau}) \right\rangle, \\ S_{\phi\phi} &= \left\langle \frac{\lambda^2 \bar{e}_{33}^*}{4h^2} \right\rangle. \end{aligned} \quad (\text{B.43a-e})$$

B.7 Virtual Work of the Non-Conservative Forces

In a similar way to what has been done before, the integration with respect to z is conveniently carrying out, and the virtual work of the non-conservative forces δW_u can be expressed in terms of prescribed forces and moments as

$$\begin{aligned} \delta W_u = & \int_{\Omega_0} \delta w_0(Z) d\alpha d\beta + \oint_{\Gamma_\alpha} \left[\delta u_0(\hat{N}_{\beta\alpha}^{11}) + \delta v_0(\hat{N}_{\beta\beta}^{22}) + \delta \frac{\partial w_0}{\partial \beta}(\hat{M}_{\beta\beta}^{23}) + \delta \frac{\partial w_0}{\partial \alpha}(\hat{M}_{\beta\alpha}^{13}) + \delta w_0(\hat{Q}_{\beta z}^{33}) \right. \\ & + \delta \theta_\alpha(\hat{M}_{\beta\alpha}^{14}) + \delta \theta_\beta(\hat{M}_{\beta\beta}^{25}) + \delta \bar{\tau}_{z\alpha}(\hat{T}_{\beta\alpha}^{11}) + \delta \bar{\tau}_{z\alpha}(\hat{T}_{\beta\alpha}^{12}) + \delta \bar{\tau}_{\beta z}(\hat{T}_{\beta\beta}^{23}) + \delta \bar{\tau}_{\beta z}(\hat{T}_{\beta\beta}^{24}) \Big] d\alpha \\ & + \oint_{\Gamma_\beta} \left[\delta u_0(\hat{N}_{\alpha\alpha}^{11}) + \delta v_0(\hat{N}_{\alpha\beta}^{22}) + \delta \frac{\partial w_0}{\partial \alpha}(\hat{M}_{\alpha\alpha}^{13}) + \delta \frac{\partial w_0}{\partial \beta}(\hat{M}_{\alpha\alpha}^{23}) + \delta w_0(\hat{Q}_{z\alpha}^{33}) + \delta \theta_\alpha(\hat{M}_{\alpha\alpha}^{14}) \right. \\ & + \delta \theta_\beta(\hat{M}_{\alpha\beta}^{25}) + \delta \bar{\tau}_{z\alpha}(\hat{T}_{\alpha\alpha}^{11}) + \delta \bar{\tau}_{z\alpha}(\hat{T}_{\alpha\alpha}^{12}) + \delta \bar{\tau}_{\beta z}(\hat{T}_{\alpha\beta}^{23}) + \delta \bar{\tau}_{\beta z}(\hat{T}_{\alpha\beta}^{24}) \Big] d\beta, \end{aligned} \quad (\text{B.44})$$

where

$$\begin{aligned} (\hat{N}_{\alpha\alpha}^{11}, \hat{M}_{\alpha\alpha}^{13}, \hat{M}_{\alpha\alpha}^{14}) &= \langle \hat{\sigma}_{\alpha\alpha} (z_{11}^u, z_{13}^u, z_{14}^u) \rangle_\beta, \\ (\hat{N}_{\alpha\beta}^{22}, \hat{M}_{\alpha\beta}^{23}, \hat{M}_{\alpha\beta}^{25}) &= \langle \hat{\sigma}_{\alpha\beta} (z_{22}^u, z_{23}^u, z_{25}^u) \rangle_\beta, \\ (\hat{Q}_{z\alpha}^{33}) &= \langle \hat{\sigma}_{z\alpha} \rangle_\beta, \\ (\hat{T}_{\alpha\alpha}^{11}, \hat{T}_{\alpha\alpha}^{12}) &= \langle \hat{\sigma}_{\alpha\alpha} (z_{11}^\tau, z_{12}^\tau) \rangle_\beta, \\ (\hat{T}_{\alpha\beta}^{23}, \hat{T}_{\alpha\beta}^{24}) &= \langle \hat{\sigma}_{\alpha\beta} (z_{23}^\tau, z_{24}^\tau) \rangle_\beta, \\ (\hat{N}_{\beta\beta}^{22}, \hat{M}_{\beta\beta}^{23}, \hat{M}_{\beta\beta}^{25}) &= \langle \hat{\sigma}_{\beta\beta} (z_{22}^u, z_{23}^u, z_{25}^u) \rangle_\alpha, \\ (\hat{N}_{\beta\alpha}^{11}, \hat{M}_{\beta\alpha}^{13}, \hat{M}_{\beta\alpha}^{14}) &= \langle \hat{\sigma}_{\beta\alpha} (z_{11}^u, z_{13}^u, z_{14}^u) \rangle_\alpha, \\ (\hat{Q}_{\beta z}^{33}) &= \langle \hat{\sigma}_{\beta z} \rangle_\alpha, \\ (\hat{T}_{\beta\beta}^{23}, \hat{T}_{\beta\beta}^{24}) &= \langle \hat{\sigma}_{\beta\beta} (z_{23}^\tau, z_{24}^\tau) \rangle_\alpha, \\ (\hat{T}_{\beta\alpha}^{11}, \hat{T}_{\beta\alpha}^{12}) &= \langle \hat{\sigma}_{\beta\alpha} (z_{11}^\tau, z_{12}^\tau) \rangle_\alpha. \end{aligned} \quad (\text{B.45a-j})$$

Once again, for convenience, $\langle \dots \rangle_\alpha$ and $\langle \dots \rangle_\beta$ denote thickness integration and are defined as

$$\langle \dots \rangle_\alpha = \int_{-h}^{+h} (\dots) H_\alpha dz, \quad \langle \dots \rangle_\beta = \int_{-h}^{+h} (\dots) H_\beta dz. \quad (\text{B.46a,b})$$

B.8 Internal Forces and Moments Coefficients

Considering the internal force or moment terms in Equations (4.75), (4.81) and (4.83) defined by the generic matrix product

$$\left\langle \begin{bmatrix} g_{1i}^{(j)} & 0 & 0 \\ 0 & g_{2i}^{(j)} & 0 \\ 0 & 0 & g_{5i} \end{bmatrix} \begin{bmatrix} \bar{c}_{11}^* & \bar{c}_{12}^* & 0 \\ \bar{c}_{12}^* & \bar{c}_{22}^* & 0 \\ 0 & 0 & \bar{c}_{66}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\alpha\beta} \end{Bmatrix} \right\rangle, \quad (\text{B.47a,b})$$

where $g_{ri}^{(j)}$ (with $r = 1, 2, 5$, $i = 1, \dots, 5$ and $j = 0, 1, 2$) is a generic term used to denote the coefficients $z_{ri}^{\star \varepsilon u}$, $z_{ri}^{\star \varepsilon u(j)}$ and $z_{ri}^{\star \varepsilon \tau}$ (note that for $j = 0$ the superscript is omitted and $g_{ri}^{(0)}$ means

the same as g_{ri}), the non-zero coefficients A_{rs}^{ri} (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 1, 2$), $B_{rs}^{ri(jk)}$ (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 3, 4, 5$, $j = 0, 1, 2$, $k = 0, \dots, 3$) and C_{rs}^{ri} (with $r = 1, 2, 5$, $s = 1, \dots, 9$ and $i = 1, \dots, 4$), used in Equations (4.76), (4.82) and (4.85), respectively, are defined by the generic terms $G_{rs}^{ri(jk)}$ (with $G = A, B, C$), given by

$$\begin{aligned}
G_{11}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{11}^{\star \varepsilon u} + \bar{c}_{12}^* z_{21}^{\star \varepsilon u}] \rangle, & G_{12}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{12}^{\star \varepsilon u} + \bar{c}_{12}^* z_{22}^{\star \varepsilon u}] \rangle, \\
G_{13}^{1i(j1)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{13}^{\star \varepsilon u(1)} + \bar{c}_{12}^* z_{23}^{\star \varepsilon u(1)}] \rangle, & G_{13}^{1i(j2)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{13}^{\star \varepsilon u(2)}] \rangle, \\
G_{13}^{1i(j3)} &= \langle g_{1i}^{(j)} [\bar{c}_{12}^* z_{23}^{\star \varepsilon u(2)}] \rangle, & G_{14}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{14}^{\star \varepsilon u} + \bar{c}_{12}^* z_{24}^{\star \varepsilon u}] \rangle, \\
G_{15}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{15}^{\star \varepsilon u} + \bar{c}_{12}^* z_{25}^{\star \varepsilon u}] \rangle, & G_{21}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{11}^{\star \varepsilon u} + \bar{c}_{22}^* z_{21}^{\star \varepsilon u}] \rangle, \\
G_{22}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{12}^{\star \varepsilon u} + \bar{c}_{22}^* z_{22}^{\star \varepsilon u}] \rangle, & G_{23}^{2i(j1)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{13}^{\star \varepsilon u(1)} + \bar{c}_{22}^* z_{23}^{\star \varepsilon u(1)}] \rangle, \\
G_{23}^{2i(j2)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{13}^{\star \varepsilon u(2)}] \rangle, & G_{23}^{2i(j3)} &= \langle g_{2i}^{(j)} [\bar{c}_{22}^* z_{23}^{\star \varepsilon u(2)}] \rangle, \\
G_{24}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{14}^{\star \varepsilon u} + \bar{c}_{22}^* z_{24}^{\star \varepsilon u}] \rangle, & G_{25}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{15}^{\star \varepsilon u} + \bar{c}_{22}^* z_{25}^{\star \varepsilon u}] \rangle, \\
G_{51}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{51}^{\varepsilon u}] \rangle, & G_{52}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{52}^{\varepsilon u}] \rangle, \\
G_{53}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{53}^{\varepsilon u}] \rangle, & G_{54}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{54}^{\varepsilon u}] \rangle, \\
G_{55}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{55}^{\varepsilon u}] \rangle, & &
\end{aligned} \tag{B.48a-s}$$

$$\begin{aligned}
G_{16}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{11}^{\star \varepsilon \tau} + \bar{c}_{12}^* z_{21}^{\star \varepsilon \tau}] \rangle, & G_{17}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{12}^{\star \varepsilon \tau} + \bar{c}_{12}^* z_{22}^{\star \varepsilon \tau}] \rangle, \\
G_{18}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{13}^{\star \varepsilon \tau} + \bar{c}_{12}^* z_{23}^{\star \varepsilon \tau}] \rangle, & G_{19}^{1i(j)} &= \langle g_{1i}^{(j)} [\bar{c}_{11}^* z_{14}^{\star \varepsilon \tau} + \bar{c}_{12}^* z_{24}^{\star \varepsilon \tau}] \rangle, \\
G_{26}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{11}^{\star \varepsilon \tau} + \bar{c}_{22}^* z_{21}^{\star \varepsilon \tau}] \rangle, & G_{27}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{12}^{\star \varepsilon \tau} + \bar{c}_{22}^* z_{22}^{\star \varepsilon \tau}] \rangle, \\
G_{28}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{13}^{\star \varepsilon \tau} + \bar{c}_{22}^* z_{23}^{\star \varepsilon \tau}] \rangle, & G_{29}^{2i(j)} &= \langle g_{2i}^{(j)} [\bar{c}_{12}^* z_{14}^{\star \varepsilon \tau} + \bar{c}_{22}^* z_{24}^{\star \varepsilon \tau}] \rangle, \\
G_{56}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{51}^{\varepsilon \tau}] \rangle, & G_{57}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{52}^{\varepsilon \tau}] \rangle, \\
G_{58}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{53}^{\varepsilon \tau}] \rangle, & G_{59}^{5i} &= \langle g_{5i} [\bar{c}_{66}^* z_{54}^{\varepsilon \tau}] \rangle.
\end{aligned} \tag{B.49a-l}$$

$$\begin{aligned}
(A_{rs}^{ri(j)} &\equiv A_{rs}^{ri} \rightarrow i = 1, 2; j = 0) \\
(B_{rs}^{ri(jk)} &\rightarrow i = 3, 4, 5; j = 0, 1, 2; k = 0, \dots, 3) \\
(C_{rs}^{ri(j)} &\equiv C_{rs}^{ri} \rightarrow i = 1, \dots, 4; j = 0)
\end{aligned}$$

It is worth to stress that, again, for generalization, in a similar way to what has been referenced before, for $j = 0$ and $k = 0$ the null superscripts are omitted and $G_{rs}^{ri(00)}$ means the same as G_{rs}^{ri} . Furthermore, the generic coefficients $G_{rs}^{ri(jk)} = G_{rs}^{ri(jk)}(g_{ri}^{(j)})$ is used here to denote $A_{rs}^{ri} = A_{rs}^{ri}(z_{ri}^{\star \varepsilon u})$, $B_{rs}^{ri(jk)} = B_{rs}^{ri(jk)}(z_{ri}^{\star \varepsilon u(j)})$ and $C_{rs}^{ri} = C_{rs}^{ri}(z_{ri}^{\star \varepsilon \tau})$ where the appropriate dependencies of the zs coefficients are explicitly declared.

Regarding the non-zero shear coefficients A_{ps}^{pi} (with $p = 3, 4$, $s = 1, \dots, 9$ and $i = 1, \dots, 5$) in Equation (4.79) they are given by

$$\begin{aligned}
A_{32}^{3i} &= \langle z_{3i}^{\varepsilon u} [\bar{c}_{44}^* z_{32}^{\varepsilon u}] \rangle, & A_{33}^{3i} &= \langle z_{3i}^{\varepsilon u} [\bar{c}_{44}^* z_{33}^{\varepsilon u}] \rangle, \\
A_{35}^{3i} &= \langle z_{3i}^{\varepsilon u} [\bar{c}_{44}^* z_{35}^{\varepsilon u}] \rangle, & A_{41}^{4j} &= \langle z_{4j}^{\varepsilon u} [\bar{c}_{55}^* z_{41}^{\varepsilon u}] \rangle, \\
A_{43}^{4j} &= \langle z_{4j}^{\varepsilon u} [\bar{c}_{55}^* z_{43}^{\varepsilon u}] \rangle, & A_{44}^{4j} &= \langle z_{4j}^{\varepsilon u} [\bar{c}_{55}^* z_{44}^{\varepsilon u}] \rangle,
\end{aligned} \tag{B.50a-f}$$

$$\begin{aligned}
A_{38}^{3i} &= \langle z_{3i}^{\varepsilon u} [\bar{c}_{44}^* z_{33}^{\varepsilon \tau}] \rangle, & A_{39}^{3i} &= \langle z_{3i}^{\varepsilon u} [\bar{c}_{44}^* z_{34}^{\varepsilon \tau}] \rangle, \\
A_{46}^{4j} &= \langle z_{4j}^{\varepsilon u} [\bar{c}_{55}^* z_{41}^{\varepsilon \tau}] \rangle, & A_{47}^{4j} &= \langle z_{4j}^{\varepsilon u} [\bar{c}_{55}^* z_{42}^{\varepsilon \tau}] \rangle,
\end{aligned} \tag{B.51a-d}$$

$$(A_{ps}^{pi} \rightarrow i = 2, 3, 5; j = 1, 3, 4)$$

B.9 Coefficients of the Finite Element Matrices

$$\begin{aligned}
(M_{ij}^{11}, M_{ij}^{14}, M_{ij}^{16}, M_{ij}^{17}) &= (I_{11}^{uu}, I_{14}^{uu}, I_{11}^{\tau u}, I_{21}^{\tau u}) L_{ij}^{00}, \\
M_{ir}^{13} &= I_{13}^{uu} R_{ir}^{01}, \\
(M_{ij}^{22}, M_{ij}^{25}, M_{ij}^{28}, M_{ij}^{29}) &= (I_{22}^{uu}, I_{25}^{uu}, I_{32}^{\tau u}, I_{42}^{\tau u}) L_{ij}^{00}, \\
M_{ir}^{23} &= I_{23}^{uu} R_{ir}^{02}, \\
M_{rs}^{33} &= I_{33}^{uu} H_{rs}^{00} + I_{33}^{u\alpha} H_{rs}^{11} + I_{33}^{u\beta} H_{rs}^{22}, \\
(M_{rj}^{34}, M_{rj}^{36}, M_{rj}^{37}) &= (I_{34}^{uu}, I_{13}^{\tau u}, I_{23}^{\tau u}) S_{rj}^{10}, \\
(M_{rj}^{35}, M_{rj}^{38}, M_{rj}^{39}) &= (I_{35}^{uu}, I_{33}^{\tau u}, I_{43}^{\tau u}) S_{rj}^{20}, \\
(M_{ij}^{44}, M_{ij}^{46}, M_{ij}^{47}) &= (I_{44}^{uu}, I_{14}^{\tau u}, I_{24}^{\tau u}) L_{ij}^{00}, \\
(M_{ij}^{55}, M_{ij}^{58}, M_{ij}^{59}) &= (I_{55}^{uu}, I_{35}^{\tau u}, I_{45}^{\tau u}) L_{ij}^{00}, \\
(M_{ij}^{66}, M_{ij}^{67}) &= (I_{11}^{\tau\tau}, I_{12}^{\tau\tau}) L_{ij}^{00}, \\
M_{ij}^{77} &= I_{22}^{\tau\tau} L_{ij}^{00}, \\
(M_{ij}^{88}, M_{ij}^{89}) &= (I_{33}^{\tau\tau}, I_{34}^{\tau\tau}) L_{ij}^{00}, \\
M_{ij}^{99} &= I_{44}^{\tau\tau} L_{ij}^{00}.
\end{aligned} \tag{B.52a-m}$$

$$\begin{aligned}
K_{ij}^{11} &= [A_{11}^{11} + A_{21}^{21}] L_{ij}^{11} + [A_{51}^{51}] L_{ij}^{22} + [A_{41}^{41}] L_{ij}^{00}, \\
K_{ij}^{12} &= [A_{12}^{11} + A_{22}^{21}] L_{ij}^{12} + [A_{52}^{51}] L_{ij}^{21}, \\
K_{ir}^{13} &= [A_{13}^{11(1)} + A_{23}^{21(1)}] R_{ir}^{10} + [A_{13}^{11(2)} + A_{23}^{21(2)}] R_{ir}^{14} + [A_{13}^{11(3)} + A_{23}^{21(3)}] R_{ir}^{15} \\
&\quad + [A_{53}^{51}] R_{ir}^{23} + [A_{43}^{41}] R_{ir}^{01}, \\
K_{ij}^{14} &= [A_{14}^{11} + A_{24}^{21}] L_{ij}^{11} + [A_{54}^{51}] L_{ij}^{22} + [A_{44}^{41}] L_{ij}^{00}, \\
K_{ij}^{15} &= [A_{15}^{11} + A_{25}^{21}] L_{ij}^{12} + [A_{55}^{51}] L_{ij}^{21}, \\
K_{ij}^{16} &= [A_{16}^{11} + A_{26}^{21}] L_{ij}^{11} + [A_{56}^{51}] L_{ij}^{22} + [A_{46}^{41}] L_{ij}^{00}, \\
K_{ij}^{17} &= [A_{17}^{11} + A_{27}^{21}] L_{ij}^{11} + [A_{57}^{51}] L_{ij}^{22} + [A_{47}^{41}] L_{ij}^{00}, \\
K_{ij}^{18} &= [A_{18}^{11} + A_{28}^{21}] L_{ij}^{12} + [A_{58}^{51}] L_{ij}^{21}, \\
K_{ij}^{19} &= [A_{19}^{11} + A_{29}^{21}] L_{ij}^{12} + [A_{59}^{51}] L_{ij}^{21}, \\
K_{ij}^{22} &= [A_{12}^{12} + A_{22}^{22}] L_{ij}^{22} + [A_{52}^{52}] L_{ij}^{11} + [A_{32}^{32}] L_{ij}^{00}, \\
K_{ir}^{23} &= [A_{13}^{12(1)} + A_{23}^{22(1)}] R_{ir}^{20} + [A_{13}^{12(2)} + A_{23}^{22(2)}] R_{ir}^{24} + [A_{13}^{12(3)} + A_{23}^{22(3)}] R_{ir}^{25} \\
&\quad + [A_{53}^{52}] R_{ir}^{13} + [A_{33}^{32}] R_{ir}^{02},
\end{aligned}$$

$$\begin{aligned}
K_{ij}^{24} &= [A_{14}^{12} + A_{24}^{22}] L_{ij}^{21} + [A_{54}^{52}] L_{ij}^{12}, \\
K_{ij}^{25} &= [A_{15}^{12} + A_{25}^{22}] L_{ij}^{22} + [A_{55}^{52}] L_{ij}^{11} + [A_{35}^{32}] L_{ij}^{00}, \\
K_{ij}^{26} &= [A_{16}^{12} + A_{26}^{22}] L_{ij}^{21} + [A_{56}^{52}] L_{ij}^{12}, \\
K_{ij}^{27} &= [A_{17}^{12} + A_{27}^{22}] L_{ij}^{21} + [A_{57}^{52}] L_{ij}^{12}, \\
K_{ij}^{28} &= [A_{18}^{12} + A_{28}^{22}] L_{ij}^{22} + [A_{58}^{52}] L_{ij}^{11} + [A_{38}^{32}] L_{ij}^{00}, \\
K_{ij}^{29} &= [A_{19}^{12} + A_{29}^{22}] L_{ij}^{22} + [A_{59}^{52}] L_{ij}^{11} + [A_{39}^{32}] L_{ij}^{00}, \\
K_{rs}^{33} &= [B_{13}^{13(11)} + B_{23}^{23(11)}] H_{rs}^{00} + [B_{13}^{13(12)} + B_{23}^{23(12)}] H_{rs}^{04} + [B_{13}^{13(13)} + B_{23}^{23(13)}] H_{rs}^{05} \\
&\quad + [B_{13}^{13(21)}] H_{rs}^{40} + [B_{13}^{13(22)}] H_{rs}^{44} + [B_{13}^{13(23)}] H_{rs}^{45} + [B_{23}^{23(21)}] H_{rs}^{50} + [B_{23}^{23(22)}] H_{rs}^{54} \\
&\quad + [B_{23}^{23(23)}] H_{rs}^{55} + [B_{53}^{53}] H_{rs}^{33} + [A_{43}^{43}] H_{rs}^{11} + [A_{33}^{33}] H_{rs}^{22}, \\
K_{rj}^{34} &= [B_{14}^{13(1)} + B_{24}^{23(1)}] S_{rj}^{01} + [B_{14}^{13(2)}] S_{rj}^{41} + [B_{24}^{23(2)}] S_{rj}^{51} + [B_{54}^{53}] S_{rj}^{32} + [A_{44}^{43}] S_{rj}^{10}, \\
K_{rj}^{35} &= [B_{15}^{13(1)} + B_{25}^{23(1)}] S_{rj}^{02} + [B_{15}^{13(2)}] S_{rj}^{42} + [B_{25}^{23(2)}] S_{rj}^{52} + [B_{55}^{53}] S_{rj}^{31} + [A_{35}^{33}] S_{rj}^{20}, \\
K_{rj}^{36} &= [B_{16}^{13(1)} + B_{26}^{23(1)}] S_{rj}^{01} + [B_{16}^{13(2)}] S_{rj}^{41} + [B_{26}^{23(2)}] S_{rj}^{51} + [B_{56}^{53}] S_{rj}^{32} + [A_{46}^{43}] S_{rj}^{10}, \\
K_{rj}^{37} &= [B_{17}^{13(1)} + B_{27}^{23(1)}] S_{rj}^{01} + [B_{17}^{13(2)}] S_{rj}^{41} + [B_{27}^{23(2)}] S_{rj}^{51} + [B_{57}^{53}] S_{rj}^{32} + [A_{47}^{43}] S_{rj}^{10}, \\
K_{rj}^{38} &= [B_{18}^{13(1)} + B_{28}^{23(1)}] S_{rj}^{02} + [B_{18}^{13(2)}] S_{rj}^{42} + [B_{28}^{23(2)}] S_{rj}^{52} + [B_{58}^{53}] S_{rj}^{31} + [A_{38}^{33}] S_{rj}^{20}, \\
K_{rj}^{39} &= [B_{19}^{13(1)} + B_{29}^{23(1)}] S_{rj}^{02} + [B_{19}^{13(2)}] S_{rj}^{42} + [B_{29}^{23(2)}] S_{rj}^{52} + [B_{59}^{53}] S_{rj}^{31} + [A_{39}^{33}] S_{rj}^{20}, \\
K_{ij}^{44} &= [B_{14}^{14} + B_{24}^{24}] L_{ij}^{11} + [B_{54}^{54}] L_{ij}^{22} + [A_{44}^{44}] L_{ij}^{00}, \\
K_{ij}^{45} &= [B_{15}^{14} + B_{25}^{24}] L_{ij}^{12} + [B_{55}^{54}] L_{ij}^{21}, \\
K_{ij}^{46} &= [B_{16}^{14} + B_{26}^{24}] L_{ij}^{11} + [B_{56}^{54}] L_{ij}^{22} + [A_{46}^{44}] L_{ij}^{00}, \\
K_{ij}^{47} &= [B_{17}^{14} + B_{27}^{24}] L_{ij}^{11} + [B_{57}^{54}] L_{ij}^{22} + [A_{47}^{44}] L_{ij}^{00}, \\
K_{ij}^{48} &= [B_{18}^{14} + B_{28}^{24}] L_{ij}^{12} + [B_{58}^{54}] L_{ij}^{21}, \\
K_{ij}^{49} &= [B_{19}^{14} + B_{29}^{24}] L_{ij}^{12} + [B_{59}^{54}] L_{ij}^{21}, \\
K_{ij}^{55} &= [B_{15}^{15} + B_{25}^{25}] L_{ij}^{22} + [B_{55}^{55}] L_{ij}^{11} + [A_{35}^{35}] L_{ij}^{00}, \\
K_{ij}^{56} &= [B_{16}^{15} + B_{26}^{25}] L_{ij}^{21} + [B_{56}^{55}] L_{ij}^{12}, \\
K_{ij}^{57} &= [B_{17}^{15} + B_{27}^{25}] L_{ij}^{21} + [B_{57}^{55}] L_{ij}^{12}, \\
K_{ij}^{58} &= [B_{18}^{15} + B_{28}^{25}] L_{ij}^{22} + [B_{58}^{55}] L_{ij}^{11} + [A_{38}^{35}] L_{ij}^{00}, \\
K_{ij}^{59} &= [B_{19}^{15} + B_{29}^{25}] L_{ij}^{22} + [B_{59}^{55}] L_{ij}^{11} + [A_{39}^{35}] L_{ij}^{00}, \\
K_{ij}^{66} &= [C_{16}^{11} + C_{26}^{21}] L_{ij}^{11} + [C_{56}^{51}] L_{ij}^{22} + [C_{46}^{41}] L_{ij}^{00}, \\
K_{ij}^{67} &= [C_{17}^{11} + C_{27}^{21}] L_{ij}^{11} + [C_{57}^{51}] L_{ij}^{22} + [C_{47}^{41}] L_{ij}^{00}, \\
K_{ij}^{68} &= [C_{18}^{11} + C_{28}^{21}] L_{ij}^{12} + [C_{58}^{51}] L_{ij}^{21}, \\
K_{ij}^{69} &= [C_{19}^{11} + C_{29}^{21}] L_{ij}^{12} + [C_{59}^{51}] L_{ij}^{21}, \\
K_{ij}^{77} &= [C_{17}^{12} + C_{27}^{22}] L_{ij}^{11} + [C_{57}^{52}] L_{ij}^{22} + [C_{47}^{42}] L_{ij}^{00}, \\
K_{ij}^{78} &= [C_{18}^{12} + C_{28}^{22}] L_{ij}^{12} + [C_{58}^{52}] L_{ij}^{21}, \\
K_{ij}^{79} &= [C_{19}^{12} + C_{29}^{22}] L_{ij}^{12} + [C_{59}^{52}] L_{ij}^{21}, \\
K_{ij}^{88} &= [C_{18}^{13} + C_{28}^{23}] L_{ij}^{22} + [C_{58}^{53}] L_{ij}^{11} + [C_{38}^{33}] L_{ij}^{00}, \\
K_{ij}^{89} &= [C_{19}^{13} + C_{29}^{23}] L_{ij}^{22} + [C_{59}^{53}] L_{ij}^{11} + [C_{39}^{33}] L_{ij}^{00},
\end{aligned}$$

$$K_{ij}^{99} = [C_{19}^{14} + C_{29}^{24}] L_{ij}^{22} + [C_{59}^{54}] L_{ij}^{11} + [C_{39}^{34}] L_{ij}^{00}, \quad (\text{B.53a-as})$$

$$\begin{aligned} F_i^1 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{N}_{\alpha\beta}^{11} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{N}_{\alpha\alpha}^{11} d\beta, \\ F_i^2 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{N}_{\beta\beta}^{22} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{N}_{\alpha\beta}^{22} d\beta, \\ F_r^3 &= \int_{\Omega_0^e} H_r^e Z d\alpha d\beta + \oint_{\Gamma_\alpha^e} \left(\frac{\partial H_r^e}{\partial \beta} \hat{M}_{\beta\beta}^{23} + \frac{\partial H_r^e}{\partial \alpha} \hat{M}_{\alpha\beta}^{13} + H_r^e \hat{Q}_{\beta z}^{33} \right) d\alpha \\ &\quad + \oint_{\Gamma_\beta^e} \left(\frac{\partial H_r^e}{\partial \alpha} \hat{M}_{\alpha\alpha}^{13} + \frac{\partial H_r^e}{\partial \beta} \hat{M}_{\alpha\alpha}^{23} + H_r^e \hat{Q}_{z\alpha}^{33} \right) d\beta, \\ F_i^4 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{M}_{\alpha\beta}^{14} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{M}_{\alpha\alpha}^{14} d\beta, \\ F_i^5 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{M}_{\beta\beta}^{25} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{M}_{\alpha\beta}^{25} d\beta, \\ F_i^6 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\alpha}^{11} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\alpha}^{11} d\beta, \\ F_i^7 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\alpha}^{12} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\alpha}^{12} d\beta, \\ F_i^8 &= \oint_{\Gamma_\alpha^e} L_i^e \hat{T}_{\beta\beta}^{23} d\alpha + \oint_{\Gamma_\beta^e} L_i^e \hat{T}_{\alpha\beta}^{23} d\beta, \end{aligned} \quad (\text{B.54a-h})$$

$$\begin{aligned} K_{i\phi}^1 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \alpha} (P_{\alpha\alpha}^{*11} + P_{\beta\beta}^{*21}) d\alpha d\beta, \\ K_{i\phi}^2 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} (P_{\alpha\alpha}^{*12} + P_{\beta\beta}^{*22}) d\alpha d\beta, \\ K_{r\phi}^3 &= \int_{\Omega_0^e} \left[H_r^e (P_{\alpha\alpha}^{*13(1)} + P_{\beta\beta}^{*23(1)}) + \frac{\partial^2 H_r^e}{\partial \alpha^2} P_{\alpha\alpha}^{*13(2)} + \frac{\partial^2 H_r^e}{\partial \beta^2} P_{\beta\beta}^{*23(2)} \right] d\alpha d\beta, \\ K_{i\phi}^4 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \alpha} (P_{\alpha\alpha}^{*14} + P_{\beta\beta}^{*24}) d\alpha d\beta, \\ K_{i\phi}^5 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} (P_{\alpha\alpha}^{*15} + P_{\beta\beta}^{*25}) d\alpha d\beta, \\ K_{i\phi}^6 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \alpha} (R_{\alpha\alpha}^{*11} + R_{\beta\beta}^{*21}) d\alpha d\beta, \\ K_{i\phi}^7 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \alpha} (R_{\alpha\alpha}^{*12} + R_{\beta\beta}^{*22}) d\alpha d\beta, \\ K_{i\phi}^8 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} (R_{\alpha\alpha}^{*13} + R_{\beta\beta}^{*23}) d\alpha d\beta, \\ K_{i\phi}^9 &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \beta} (R_{\alpha\alpha}^{*14} + R_{\beta\beta}^{*24}) d\alpha d\beta, \end{aligned} \quad (\text{B.55a-i})$$

$$K_{\phi\phi} = \int_{\Omega_0^e} S_{\phi\phi} d\alpha d\beta, \quad (\text{B.56})$$

$$Q_\phi = \int_{\Omega_0^e} \tau d\alpha d\beta. \quad (\text{B.57})$$

For illustration purposes, as an example, the following relations of the interpolation functions should be considered

$$\begin{aligned} L_{ij}^{00} &= \int_{\Omega_0^e} L_i^e L_j^e d\alpha d\beta, & L_{ij}^{01} &= \int_{\Omega_0^e} L_i^e \frac{\partial L_j^e}{\partial \alpha} d\alpha d\beta, \\ H_{rs}^{03} &= \int_{\Omega_0^e} H_r^e \frac{\partial^2 H_s^e}{\partial \alpha \partial \beta} d\alpha d\beta, & H_{rs}^{45} &= \int_{\Omega_0^e} \frac{\partial^2 H_r^e}{\partial \alpha^2} \frac{\partial^2 H_s^e}{\partial \beta^2} d\alpha d\beta, \\ R_{ir}^{02} &= \int_{\Omega_0^e} L_i^e \frac{\partial H_r^e}{\partial \beta} d\alpha d\beta, & R_{ir}^{14} &= \int_{\Omega_0^e} \frac{\partial L_i^e}{\partial \alpha} \frac{\partial^2 H_r^e}{\partial \alpha^2} d\alpha d\beta, \\ S_{rj}^{10} &= \int_{\Omega_0^e} \frac{\partial H_r^e}{\partial \alpha} L_j^e d\alpha d\beta, & R_{rj}^{32} &= \int_{\Omega_0^e} \frac{\partial^2 H_r^e}{\partial \alpha \partial \beta} \frac{\partial L_j^e}{\partial \beta} d\alpha d\beta, \end{aligned} \quad (\text{B.58a-e})$$

where the superscript 0 denotes zero-order derivative of the Lagrange (L) and/or Hermite (H) interpolation functions, and the superscripts from 1 to 5 denote the first- and second-order derivatives, with the following correspondence:

$$1 \rightarrow \frac{\partial}{\partial \alpha}, \quad 2 \rightarrow \frac{\partial}{\partial \beta}, \quad 3 \rightarrow \frac{\partial^2}{\partial \alpha \partial \beta}, \quad 4 \rightarrow \frac{\partial^2}{\partial \alpha^2}, \quad 5 \rightarrow \frac{\partial^2}{\partial \beta^2}.$$

For the sake of brevity only a few terms are presented since the rest are obvious from the previous relations. It is also worth to note that the $L_{ij}^{\dots}$ and $H_{rs}^{\dots}$ integrals involve only the Lagrange or Hermite interpolation functions alone while the $R_{ir}^{\dots}$ and $S_{rj}^{\dots}$ integrals involve crossed relations of the interpolation functions with $S_{rj}^{\dots} = (R_{ir}^{\dots})^T$.

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