

**UNCERTAINTY TREATMENT IN CIVIL
ENGINEERING NUMERICAL MODELS**

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It is the uncertainty of life that make us live...

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The research work presented in this thesis was carried out half in LABEST – Laboratory for the Concrete Research and Structural Behavior, held in the Civil Engineering Department of the Faculty of Engineering, University of Porto (FEUP), and half in MICELab Group – Research Laboratory on Modal Intervals and Control Engineering, held in the University of Girona. Part of this thesis was developed during the SMARTE Project, a research project constituted by the Faculty of Engineering University of Porto (FEUP), the Portuguese Highway Administration (BRISA) and the Opto-Electronic Laboratory (INESC Porto), which was financed by AdI – Innovation Agency through the POSI program. I also would like to express my gratitude to FCT – Portuguese Foundation for Science and Technology due to the received grant that made all this research possible.

ABSTRACT

Uncertainty analysis is a matter of extremely importance in civil engineering field. In fact, uncertainty is not only present during the design process, as the used input variables are not known exactly, but also during the assessment one. Within the assessment procedure, uncertainty is present on the numerical model and also in the sensor obtained data. To characterize such uncertainty the unknown parameters are represented by probability distribution functions. The studied variables are the ones which are used more in numerical systems as the materials, the elements geometry and the applied loads. In the case of an assessment procedure, a distribution type must be attached to the observed data.

To perform an uncertainty analysis of a system it is usually used the Monte Carlo method, as it is well known, easy to use and to control. However, this methodology presents some disadvantages as the associated high computational cost and calculation time spent. In fact, to obtain a liable result, using this technique, the numerical model of the system needs to be run several times. The efficiency grows as the sampling number gets higher. This disadvantage becomes more evident when performing an assessment process. The main objective of an assessment procedure is to obtain a liable and fast decision about any system behaviour. To perform this, an efficient and fast uncertainty analysis methodology needs to be executed, in order to establish a continuous comparison with sensor obtained data.

In order to face this problem, two methodologies for uncertainty analysis were studied and applied in simple civil engineering numerical models. The first, Perturbation technique, was developed within the LABEST, Laboratory for the Concrete Technology and Structural Behavior, Faculty of Engineering, University of Porto (FEUP). This methodology studies the system response due to small perturbations in their input variables. The second, Modal Interval Analysis (MIA), was developed by

MICE Lab Group at Girona University. This technique treats the variables not as point numbers but by intervals with an associated semantic extension.

Two methodologies for structural assessment were also developed and applied with two simple civil engineering systems. The first, consistent comparison technique can be applied with all the three previously presented uncertainty analysis methodologies. It is essentially based in a direct and consistent comparison between numerical results and obtained sensor data, both affected by the associated uncertainty. The second, quantified constraint satisfaction problem (QCSP), is used together with Modal Interval Analysis (MIA). This technique is based in the establishment of a study about the system inconsistency.

RESUMO

A análise de incertezas é um assunto de extrema importância no âmbito da engenharia civil. De facto, a incerteza é não só considerada durante o processo de dimensionamento, uma vez que as variáveis utilizadas não são conhecidas à partida, mas também durante o processo de avaliação. Neste caso, a incerteza encontra-se presente quer no modelo numérico quer na informação proveniente dos sensores. De modo a caracterizar a incerteza as variáveis aleatórias são representadas por distribuições probabilísticas. As variáveis estudadas correspondem aquelas com maior uso em sistemas numéricos tais com as relacionadas com os materiais utilizados, com a geometria dos elementos e com as cargas aplicadas. No caso de um processo de avaliação, um determinado tipo de distribuição probabilística deverá ser considerado para os dados obtidos.

Para realizar uma análise de incertezas de um sistema, geralmente utiliza-se o método de Monte Carlo, uma vez que é extremamente conhecido, fácil de utilizar e de controlar. No entanto, esta metodologia apresenta algumas desvantagens tais como o elevado custo computacional e tempo de cálculo associado. De facto, para obter um resultado fiável, utilizando esta técnica, o modelo numérico do sistema necessita de ser calculado por várias vezes. A eficiência aumenta com o aumento do número de testes realizado. Esta desvantagem torna-se mais evidente nos processos de avaliação. O objectivo principal destes procedimentos corresponde à rápida obtenção de uma decisão fiável sobre o comportamento do sistema. De modo a ir ao encontro desta necessidade, uma metodologia para análise de incertezas rápida e eficiente necessita de ser aplicada, pois, só assim, se conseguirá estabelecer uma comparação de forma contínua com os resultados provenientes dos sensores.

De modo a resolver este problema, duas metodologias para a análise de incertezas foram estudadas e aplicadas em simples modelos numéricos de engenharia civil. A primeira, técnica das Perturbações, foi desenvolvida no LABEST, Laboratório

da Tecnologia do Betão e do Comportamento Estrutural, Faculdade de Engenharia da Universidade do Porto (FEUP). Este método estuda a resposta do sistema devido a pequenas perturbações introduzidas nas variáveis de entrada. A segunda, Análise Intervalar Modal (AIM), foi desenvolvida pelo grupo MICE Lab na Universidade de Girona. Nesta técnica as variáveis aleatórias não são analisadas como pontos mas por intervalos aos quais se associa uma extensão semântica.

Duas metodologias para a avaliação estrutural foram desenvolvidas e aplicadas em dois sistemas simples de engenharia civil. A primeira, consiste num método de comparação consistente e pode ser utilizada com as três metodologias para análise de incertezas anteriormente apresentadas. Baseia-se, essencialmente, numa comparação directa e consistente entre os resultados numéricos e os provenientes da monitorização, ambos afectados pela incerteza associada. A segunda, corresponde a um problema de satisfação de restrições quantificadas (PSRQ), e é utilizado em conjunto com a Análise Intervalar Modal (AIM). Esta técnica baseia-se no estudo da inconsistência do sistema.

KEYWORDS

PALAVRAS CHAVE

Uncertainty Analysis	Análise de Incertezas
Risk Analysis	Análise de Risco
Structural Safety	Segurança Estrutural
Random Variable	Variável Aleatória
Probability Distribution	Distribuição Probabilística
Monte Carlo Method	Método de Monte Carlo
Perturbation Technique	Técnica das Perturbações
Modal Interval Analysis (MIA)	Análise Intervalar Modal (AIM)
Structural Assessment	Avaliação Estrutural
Consistent Comparison Method	Método de Comparação Consistente
Quantified Constraint Satisfaction Problem (QCSP)	Problema de Satisfação de Restrições Quantificadas (PSRQ)
Liability	Fiabilidade
Structural Design	Dimensionamento Estrutural
Sensor Linearity	Linearidade do Sensor

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CHAPTER 1.
INTRODUCTION

1.1. BACKGROUND

Within the civil engineering field, “uncertainty” is an important word that must be always taken into account. Building materials do not present homogeneous properties, applied loads change during the structure life cycle, structural elements connection vary with fatigue, each structural component geometry is usually different from what is on design drawings. Consequently, when analyzing a structure, “uncertainty” must be considered.

The uncertainty analysis becomes more evident when the set of words “structural safety” is introduced. “Structural safety” is the basis for the design or assessment of any structure. On the beginning, codes recommended empiric values, obtained from past experiences, to use in a “structural safety” analysis. Afterwards, the structural numerical model was introduced in order to obtain a better approximation of the real structural behaviour. Firstly, the calculus and the model definition were hand made and so the obtained results were not so exactly as it. With the development of informatics, potent computers started to appear and so robust numerical models and precise calculations become allowable. The finite element framework, used to develop rigorous numerical models, it is essentially a methodology introduced during this era and based in the transformation of a continuous structural system to a discrete one. Regarding safety coefficients, the use of empiric global safety factors, were firstly used. Actually, codes became more strictly with the necessity of performing a rigorous “structural safety” analysis. So the use of partial safety factors, for applied loads and resistance materials, obtained from a strict probabilistic analysis started to appear. Safety analysis using deep probabilistic techniques was only recommended by codes for special occasions, like existing structures which were in possible risk of failure or on the design of special structures.

It is important here to point out the importance of the word “risk” in the civil engineering field. “Risk” is associated to the probability of failure and to the consequences of such failure for the society. There exist several consequences like human injuries, environmental problems or even economy costs. It is important to note that in the last decade several civil engineering structures collapsed (Figure 1.1). The dichotomy failure and cost must always be considered in civil engineering field. To

mitigate such “risk”, rigorous safety analysis of structures, during the design or assessment procedure, must be executed. Such analysis should be executed using deep probabilistic or simulation techniques. However the use of such techniques when an acceptable efficiency is needed presents, as a consequence, a high computational cost. Nowadays, “risk” quantification and interpretation, and safety analysis, are themes with large applicability and importance for our society. There is so a line of civil engineering research that focus both themes. One of the several points of investigation, within such line, is the development of more efficient methodologies that could realize the uncertainty treatment in an easy and fast way (Henriques, 1998; Nowak and Collins, 2000; Singh *et al.*, 2007).



a)



b)



c)

Figure 1.1 – Structural collapses. a) I35W Bridge; b) Hutch Building; c) Teton Dam.

One other point that presents an extreme importance for risk control is the development and implementation of structural monitoring systems and non destructive techniques. Information obtained from such systems is very useful for the updating of the structural numerical model. A large amount of results is usually collected and they could be used to characterize accurately the uncertainties related to structural performance. Techniques to treat uncertainties have also a special role to perform in this field, not only to characterize dispersion but also to control such results (Beck *et al.*, 2001; Casas *et al.*, 2005; Delgado, 2006; Wang *et al.*, 2001). Also, obtained data present an uncertainty due to sensor or to equipment linearity which must be always considered. Due, also, to the development of structural health monitoring systems and computer hardware and software, an increase of structural assessment techniques was verified. Reliable numerical models were needed even more, but also reliable obtained data

should be obtained. In order to perform it, strict uncertainty analysis must be executed both in numerical model and in obtained data.

Within the scope of a research Portuguese project, SMARTE project which was developed with BRISA, Highways Portuguese Administration, INESC Porto, Opto-Electronic Department, and FEUP, Civil Engineering Department, a structural health monitoring system, composed by opto-electronic sensors were developed and implemented in a pre-stressed concrete bridge (Sorraia River Bridge) (Assis *et al.*, 2005; Assis *et al.*, 2006a; Assis *et al.*, 2006b; Figueiras *et al.*, 2004; Matos *et al.*, 2004c; Matos *et al.*, 2007; Perdigão *et al.*, 2004; Perdigão *et al.*, 2006; Sousa *et al.*, 2004a; Sousa *et al.*, 2004b; Sousa *et al.*, 2006). From the obtained experience it was really needed to study and develop a numerical methodology that could treat the obtained data and realize the structural uncertainty analysis in an efficient, fast and reliable way. Once the random obtained sensor data present a distribution similar to a uniform one, Modal Interval Analysis (MIA) technique was a real possibility for this purpose. Then, and with MICELab, Girona University, Department of Applied Mathematics, such technique was studied and implemented in simple examples (Gardenyes *et al.*, 2001; Garcia *et al.*, 2004; Matos *et al.*, 2004a). However, this technique presents some problems when implemented in a real structure like the one of SMARTE project. To face it, another methodology, Perturbation Technique, was studied and implemented with the same simple examples of Modal Interval Analysis (MIA) (Henriques, 2006a; Henriques, 2006b; Veiga *et al.*, 2006a; Veiga *et al.*, 2006b).

At the same time, algorithms for the structural behaviour control, which could realize the structural assessment in an easy and fast way, were developed. It is important to focus such algorithms, the first, based in a direct consistent comparison between numerical and obtained data, affected by the existent uncertainty (Casas *et al.*, 2005; Matos *et al.*, 2004b; Matos *et al.*, 2005a; Matos *et al.*, 2005b), and the second, based in a Quantified Constraint Satisfaction Problem (QCSP) applied together with Modal Interval Analysis (MIA) (Herrero *et al.*, 2004; Herrero *et al.*, 2005; Matos *et al.*, 2006; Garcia *et al.*, 2006), both developed within MICELab group. These techniques are recent and were applied, till now, with very simple structures.

1.2. OBJECTIVES

The research work presented in this thesis is essentially divided in three parts. The first part is based on the study of the distribution type that fits better with random numerical input parameters and sensor obtained data. The second is the study of an efficient methodology for uncertainty analysis, with application in structural design and assessment. The third focus the development of a methodology for structural assessment, by performing damage identification in a reliable and efficient way. Accordingly, the main objectives of the thesis were defined:

- Review distribution types that characterize better typical civil engineering random variables, namely, material, geometry and applied loads, and how actual codes define probability models for them;
- Review the most used numerical methodologies for uncertainty analysis, namely, simulation methods, like Monte Carlo, and probabilistic analysis;
- Perform a research study about Modal Interval Analysis (MIA) within MICELab group, and the way such technique could be applied into structural systems;
- Perform a research study about Perturbation Technique and the way such technique could be applied into civil engineering systems;
- Show the applicability of Modal Interval Analysis (MIA) and Perturbation technique in simple civil engineering examples and compare the results with the ones obtained by already known numerical methodologies;
- Study the structural assessment techniques and propose one consistent algorithm, which uses the developed uncertainty analysis methodologies for the assessment of civil engineering systems;
- Develop an algorithm, mainly based in a Quantified Constraint Satisfaction Problem (QCSP) and in Modal Interval Analysis (MIA), for structural assessment;
- Establish a comparison between both algorithms and apply them to simple laboratory structures and, finally, to a real structure.

The previously presented objectives are ambitious and could not be completely fulfilled within this thesis program. However, it is important to refer that the majority of them were accomplished.

1.3. THESIS OUTLINE

The following thesis is organised in five chapters (Figure 1.2).

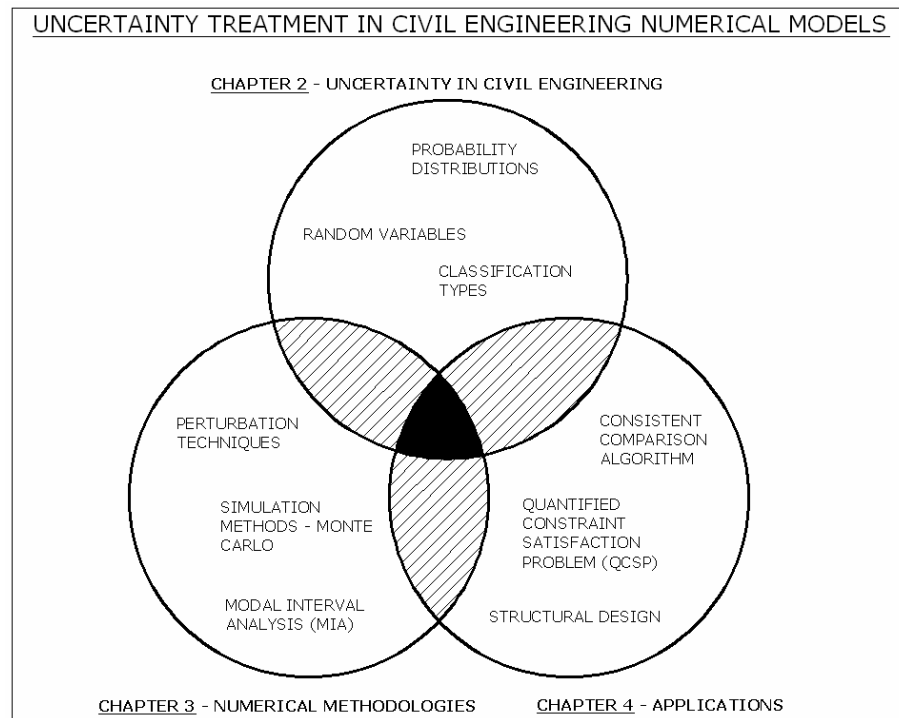


Figure 1.2 – Thesis Outline.

In Chapter 1, introduction, a general background of the problem is provided and the objectives of the current work are stated.

In Chapter 2, Uncertainty in Civil Engineering, two different classification types of uncertainty are presented. The first focus the uncertainty of abstracted, non-abstracted, and unknown aspects of a system and measures to mitigate them. The second one, focus the physical, modelling process, statistic and human errors uncertainty. In this chapter the basis of a probabilistic analysis is introduced. It is presented the different random variables probability distribution types and explained how to determine the distribution that fits better with a sample of observation values. In this chapter it is defined how actual codes treat the problem of uncertainty and what are the suggested models. It is, essentially, focused the uncertainty present in civil engineering materials, in structural elements geometry and in applied loads. The aim of

this chapter is to provide the theoretical background necessary to understand the following parts of the thesis and to present a state of the art of what it is explained in civil engineering codes.

In Chapter 3, Numerical Methodologies, three different techniques for uncertainty treatment are introduced. The first one, Monte Carlo method is one of the most used simulation techniques. A description of such method principles, and how to use it with uniformly and normally distributed random numbers, is realized. It is also presented the Latin Hypercube sampling, a technique that can be executed together with Monte Carlo in order to reduce the number of samplers. Secondly, it is presented the Perturbation method, its basis, when defined in terms of forces or of displacements, and the respective computational approach. Finally it is presented the Modal Interval Analysis (MIA), the arithmetic, the interval extensions for continuous functions, and the respective computational implementation. The aim of this chapter is to present the methodologies, developed to perform a reliable uncertainty analysis of civil infrastructures.

In Chapter 4, Applications, it is studied first the developed uncertainty analysis methodologies (Perturbation Technique and Modal Interval Analysis – MIA). Such study is based in a comparison between them and with the well known Monte Carlo method. Such comparison is focused in two simple examples of structural design processes, namely, a hyperstatic steel beam and a one-floor concrete frame. Then it is presented a developed methodology for structural assessment by a simple consistent comparison between obtained sensor data, from a structural monitoring system, and numerical output of a structural model. At this point, it is defined the uncertainty analysis related to sensor obtained data. The application of this algorithm to a laboratory tested steel and concrete beams is presented. In this chapter it is also defined the quantified constraint satisfaction algorithm (QCSP), applied with the Modal Interval Analysis (MIA), to perform civil engineering structural assessment. An application of it to a simple hyperstatic beam and to the same laboratory tested concrete beam is presented. The aim of this chapter is to make a real comparison between developed methodologies and to define simple and efficient methodologies for structural assessment.

In Chapter 5, Conclusions & Future Developments, the main conclusions of this study are presented and suggestions for future research are stated.

CHAPTER 2.

UNCERTAINTY IN CIVIL ENGINEERING

2.1. INTRODUCTION

2.1.1. GENERAL CONSIDERATIONS

For centuries, scientists had accused religion of being a collection of dogmas and religious people of being dogmatists. However, only with the demonstration of the limitations of classical physics did they realize that they too had become dogmatically attached to their theories. For instance, Newton's law reigned for such a long time and explained so many things that no one believed that it would ever need a correction.

Nowadays the concept of absolute truth is not considered anymore by the scientific community. To analyze their problems uncertainty theories, based in probabilistic and statistical analysis, are so used. The first scientist to define uncertainty was Werner Heisenberg in 1927 (Heisenberg, 1958). For him uncertainty is not due to any fault on the part of the observer; rather, it is part of the nature of reality.

More recently, according to Oberkampf terminology, uncertainty is defined as a deficiency that may or may not occur in a stage of modelling process due to lack of knowledge (Oberkampf et al., 1999). Uncertainties are so within the basic common problems that engineers have to face when modelling structures. They arise due to mismatches and inaccuracies in physical and geometrical parameters. When considering uncertainty, attention must be paid not only to the way the non-deterministic variables are defined but also to the used numerical methodology. Within the structural assessment analysis scope, uncertainties are not only present in model parameters but also in obtained sensor data. In such case, the uncertainty treatment must be executed in a rigorous way.

In this chapter it is described the necessary statistical approaches for the definition of uncertainty variables. It is first made a description of uncertainty types, then, the characterization of typical discrete and continuous statistical distributions present in model parameters is presented and finally it is explained how the used norms define and implement such uncertainty.

2.1.2. IS UNCERTAINTY ALWAYS PRESENT IN CIVIL ENGINEERING?

The engineering community is going through a paradigmatic shift in its treatment of uncertainty in engineering analysis and design, and decision making. In civil engineering, uncertainty is acknowledged in many of its fields such as the development of structural codes, analysis of natural hazards, decision making in infrastructure expenditure and environmental risks. The interest in uncertainty will continue to increase as engineers continue to design complex systems, deal with new technologies, systems and materials, and are increasingly required to make critical decisions with potential high adverse consequences.

Also, political, societal, and financial demands are increasing, thereby adding new dimensions of complexity for engineers. The expectations of society from engineers are becoming larger than ever, and its tolerance to errors is diminishing. The aggregation of these factors produces an environment where engineers must formally consider uncertainty in their decisions at all levels in a framework of system analysis.

Within such scenario, it is necessary to refer the example of structural engineers which are responsible for proportioning the elements of the structure in such a way that satisfy the design criteria related to performance, safety, serviceability or durability under various demands. Handling this responsibility, in everyday practice they have to deal with uncertainties. The sources of uncertainty are various. Most of them are related to the uncertain mechanical parameters of construction materials, uncertain geometry of the structure and uncertain loads. The models describing the behaviour of the structure or of the structural elements are also uncertain. The most rational way to deal with this problem is to treat all the uncertain parameters as random variables and perform reliability analysis.

Uncertainty, associated to risk, is one of the civil engineering major points of interest. All system models cannot preview the real behaviour due to the presence of uncertainty. Once such uncertainty is reduced, using monitoring systems, load tests, and many other techniques, the risk is also reduced as the theoretical system model is achieving the real one. However, and even with a detailed analysis of the real structure, and using the most sophisticated methods, an uncertainty analysis must be always executed.

2.2. CLASSIFICATION OF UNCERTAINTY

Engineers can deal with information for the purpose of system analysis and design. Information in this case are classified, sorted, analyzed, and used to predict system parameters and performances. However, it can be more difficult to classify, sort and analyze the uncertainty in this information, and use it to predict unknown system parameters and performances. As a first step the nature of uncertainty in civil engineering needs to be understood. Then, uncertainty can be classified into types, and different analytical tools can be used for its modelling and analysis.

Uncertainties in civil engineering can be mainly attributed to ambiguity and vagueness in defining the architecture, parameters and governing prediction models for the systems. The ambiguity component is generally due to non-cognitive sources. These sources include (1) physical randomness; (2) statistical uncertainty; (3) lack of knowledge; (4) modelling uncertainty. The vagueness-related uncertainty is due to cognitive sources that include (1) definition of certain parameters; (2) other human factors; (3) defining the inter-relationship among the parameters of the problems. Other sources of uncertainty can include conflict in information, and human and organizational errors.

Analysis of a system commonly starts with a definition of the system that can be viewed as an abstraction of the real system, which is performed at different epistemological levels according to Figure 2.1. During the process of abstraction the engineer needs to make decisions regarding what aspects should or should not be included in the model (Ayyub, 1998). These aspects included the previously identified uncertainty types. In addition to abstracted and non-abstracted aspects, unknown aspects of the system can exist, being more difficult to deal with.

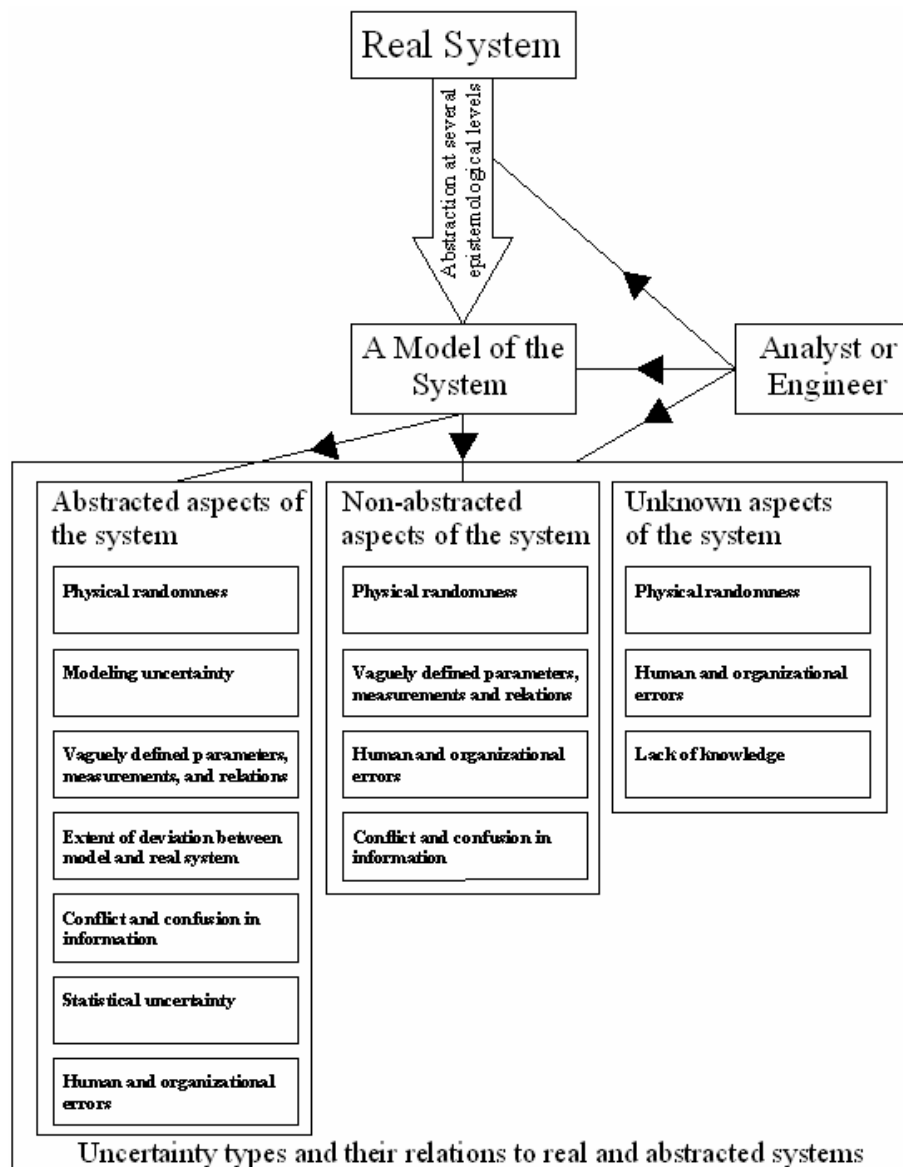


Figure 2.1 – Uncertainty types for engineering systems.

Uncertainty modeling and analysis for the abstracted aspects of the system need to be performed with a proper consideration of the non-abstracted aspects of a system. The division between abstracted and non-abstracted aspects can be a division of convenience that is driven by the objectives of the system modeling, or simplification of the model. However, the unknown aspects of the system are due to ignorance and lack of knowledge.

2.2.1. UNCERTAINTY IN "ABSTRACTED" ASPECTS OF A SYSTEM

2.2.1.1. "NON COGNITIVE" UNCERTAINTY TYPES

Civil Engineers and researchers dealt with the ambiguity types of uncertainty in predicting the behaviour and designing structural systems using the theories of probability and statistics. Probability distributions, as Uniform and Triangular ones were so used to model system parameters that are uncertain. Bayesian techniques are utilized to gain information about such parameters, updating the underlying distributions and probabilities.

2.2.1.2. "COGNITIVE" UNCERTAINTY TYPES

The cognitive types of uncertainty arise from mind-based abstractions of reality. These abstractions are, therefore, subjective and lack crispness. This vagueness is distinct from ambiguity in source and natural properties. The axioms of probability and statistics are limiting for the proper modelling and analysis of this type and are nor not completely relevant nor completely applicable. The vagueness type of uncertainty in civil engineering systems was previously discussed elsewhere along with applications of fuzzy set theory to those systems (Ayyub, 1998). Such theory has been successfully used in (1) strength assessment of existing structures and other structural engineering applications; (2) risk analysis and assessment in structures and other structural engineering applications; (3) analysis of construction failures, scheduling and safety assessment of construction activities, decisions during construction and tender evaluation; (4) impact assessment of engineering projects on the quality of wildlife habitat; (5) planning of river basins; (6) control of engineering systems; (7) computer vision; (8) optimization based on soft constraints.

2.2.2. UNCERTAINTY IN "NON-ABSTRACTED" ASPECTS OF A SYSTEM

In developing a model an analyst or engineer needs to decide at the different levels of modelling a system upon the aspects of the system that need to be abstracted, and the aspects that do not. This division can be done for convenience or to simplify the

model. The resulting division is highly affected by the analyst or engineer, as a result of their knowledge and background, and the general state of knowledge about the system.

The abstracted aspects of a system and their uncertainty models can be developed to account for the non-abstracted aspects of the system to some extent. This process is generally incomplete and therefore a source of uncertainty exists. The uncertainty types in this case include physical randomness, vagueness, human and organizational errors, and conflict and confusion in information. This uncertainty types due to non-abstracted aspects of the system are more difficult to deal with. Such difficult can stem from a lack of knowledge or understanding of the effects of the non-abstracted aspects on the resulting model in terms of its ability to mimic the real system.

2.2.3. UNCERTAINTY DUE TO "UNKNOWN" ASPECTS OF A SYSTEM

Some structural failures occurred in the past because of failure modes that were not accounted for in the design of these structural systems. This can be due to (1) ignorance, negligence, human or organizational errors; (2) general state of knowledge about the system that is incomplete. These unknown aspects depend on the nature of the system under consideration, the knowledge of the analyst, and the state of knowledge about the system in general. The non-accounting of these aspects in the models for the system can result in varying levels of impact on the ability of these models in mimicking the system behaviour. Their effects on these models can range from none to significant. In this case, the uncertainty types can include physical randomness, human and organizational errors, and lack of knowledge.

2.2.4. UNCERTAINTY MEASURES FOR "ABSTRACTED" ASPECTS OF A SYSTEM

Uncertainty measures were developed to quantify the quality of information available about the abstracted aspects of a system. Table 2.1 shows a summary of a selected group of these measures based on their description. The computational aspects for these measures are described elsewhere. They were based on using the $\log_2$ in their mathematical formulation in order to relate them to bits of information. The selection of

a unity for measuring information is essential for establishing a common ground for measuring uncertainty. However, the choice of the unit does not completely satisfy the practical needs of engineers in providing calibrated computational procedures for measuring uncertainty. For example, uncertainty measures can be used to convert the different types of uncertainty to a selected type, therefore enabling them to solve their example problems using the mathematics of a selected type. However, additional research is needed in cross-calibration and validation of such conversions of uncertainty types.

Table 2.1 – Uncertainty measures for available information.

<i>Uncertainty measure (year)</i>	<i>Type of uncertainty</i>	<i>Theoretical basis</i>	<i>Comments</i>	<i>Uncertainty measure range</i>
Hartley (1928)	Ambiguity	Crisp Sets	A basic discrete measure that is proportional to the number of outcomes of a system. The larger the number of outcomes, the larger the uncertainty measure.	$[0, \infty)$
Shannon Entropy (1948)	Ambiguity	Crisp Sets and Probability Theory	The occurrence probability of outcomes is used in this measure. The closer the outcomes to an equally likely state, the larger the measure.	$[0, \infty)$
U-uncertainty (1982)	Ambiguity	Crisp Sets and Possibility Theory	The possibility counterpart to the Shannon entropy, and a generalization of the Hartley measure.	$[0, \infty)$
Fuzziness (1970s)	Vagueness	Fuzzy Sets	It measures the lack of distinction between a fuzzy set and its complement. As the fuzziness of a set approaches the crisp case, it approaches zero.	$[0, \infty)$
Confusion (1983)	Ambiguity and confusion in information	Crisp Sets and Evidence Theory	It measures confusion of evidence using the theory of evidence.	$[0, \infty)$
Dissonance (1983)	Ambiguity and conflict in information	Crisp Sets and Evidence Theory	It measures conflict of evidence using the theory of evidence.	$[0, \infty)$
Nonspecificity (1985)	Ambiguity and confusion in information	Crisp Sets and Evidence Theory	It measures nonspecificity in evidence using the theory of evidence.	$[0, \infty)$

2.2.5. UNCERTAINTY MEASURES FOR "NON-ABSTRACTED" OR UNKNOWN ASPECTS OF A SYSTEM

Structural engineers dealt with non-abstracted and unknown aspects of a system by assessing modelling uncertainty which is defined as the ratio of a predicted system's parameter (based on the model) to the value of the parameter in the real system. This ratio, defined as bias, is commonly treated as a random variable that can consist of objective and subjective components. This approach is based on two assumptions (1) the value of the parameter for the real system is known or can be accurately accessed from historical information or expert judgement; (2) the state of knowledge about the

real system is absolute, complete and reliable. For some systems, the first assumption can be approximately examined for its validity. The second assumption cannot be validated for its absolute strictness.

2.2.6. OTHER CLASSIFICATION TYPES

According to Henriques, 1998, it exist several uncertainty sources that can be divided in: (1) Physical uncertainty; (2) Uncertainty during the modelling process; (3) Statistic uncertainty; (4) Uncertainty due to human errors.

The first group is associated to the inherent uncertainty nature of material properties, elements geometry, variability and simultaneity of loads, and others. This uncertainty kind can be controlled using an extensive data base or via a convenient quality control process. Usually this uncertainty kind is not known “a priori” but it can be estimated by observing the variables or considering previous experiences.

The second group is essentially due to theoretic approximations of the real materials behaviour and to simplifications when considering loads and their effects. This kind of uncertainty can be quantified by one variable which represents the relation between the truth answer and the one that is previewed by the model.

The third group is related to statistical inference, once the estimative of parameters which characterize the probabilistic models is done using a limited quantity of data. Statistical uncertainty can be considered using a probability density function (PDF). It is so possible to use a Bayesian approximation in order to re-define this function in a way that more information, obtained from new data, can be easily inserted.

The last group results from the human involvement during the structure life cycle. This uncertainty kind is not only due to the natural variation during the execution of several tasks, but also to interventions and errors executed during the processes of documentation, design, construction and structure exploitation. The knowledge about this kind of uncertainties is limited, being on it majority of qualitative character. However, it is clear that it effect has, as a consequence, an increase on the uncertainty in structural resistance for a higher value than the one due to mechanic and geometric structural properties.

2.3. PROBABILISTIC VARIABLE DISTRIBUTION

2.3.1. PROBABILISTIC DISTRIBUTIONS

When performing the structural assessment or even during the structural design process, if uncertainty analysis is taken into account, both sensor data and the structural model input parameters should be defined by random variables. Probabilistic analysis is so a way of considering uncertainty both in measured data and in numerical model. At this point it will be presented the most important probabilistic concepts based in Haldar and Mahadevan (2000), Melchers (1999), Nowak and Collins (2000), Reis *et al.* (1999), Schneider (1997) and Singh *et al.* (2007).

If x is a discrete random variable which assumes different values $x_1, x_2 \dots x_n$, the probability density function (PDF) of x , represented by $f(x)$, is defined as:

$$f(x) = \begin{cases} P(X = x) & \text{if } x = X \\ 0 & \text{if } x \neq X \end{cases} \quad (j = 1, 2, \dots, n, \dots) \quad (2.1)$$

which presents the following properties:

$$\begin{aligned} 1) & 0 \leq f(x) \leq 1, \forall x \in R \\ 2) & \text{If "n" is finite, } \sum_{i=1}^n f(x_i) = 1 \text{ and if "n" is infinite } \sum_{i=1}^{\infty} f(x_i) \rightarrow 1 \end{aligned} \quad (2.2)$$

A cumulative distribution function (CDF), represented by $F(x)$, of a random variable is defined as:

$$F(x) = P[X \leq x] \quad (2.3)$$

with domain R and an image set $[0,1]$. The properties of this function are:

- 1) $0 \leq F(x) \leq 1, \forall x \in IR$
 - 2) $F(x_2) \geq F(x_1), \forall x_1, x_2$ with $x_2 > x_1$ ($F(x)$ is a monotonic function not decrescent)
 - 3) $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow +\infty} F(x) = 1$
 - 4) $P[x_1 < x \leq x_2] = F(x_2) - F(x_1), \forall x_1, x_2$ with $x_2 > x_1$
- (2.4)

For continuous random variables the probability density function (PDF), $f(x)$, and the cumulative distribution function (CDF), $F(x)$, can be graphically presented according to Figure 2.2. In this situation $f(x)$ assumes the following properties:

- 1) $f(x) \geq 0, \forall x \in IR$
 - 2) $\int_{-\infty}^{+\infty} f(x) dx = 1$
- (2.5)

and $F(x)$ defined as:

$$F(x) = P[X \leq x] = \int_{-\infty}^x f(x) dx \quad (2.6)$$

Presents the following properties:

- 1) $P[x < a] = P[x \leq a] = F(a) = \int_{-\infty}^a f(x) dx$
 - 2) $P[x > a] = 1 - P[x \leq a] = 1 - F(a) = \int_a^{+\infty} f(x) dx$
 - 3) $P[a < x < b] = P[a \leq x \leq b] = F(b) - F(a) = \int_a^b f(x) dx = \int_{-\infty}^b f(x) dx - \int_{-\infty}^a f(x) dx$
- (2.7)

Figure 2.2 presents the cumulative distribution function (CDF), $F(x)$, and the probability density function (PDF), $f(x)$, of a continuous random variable.

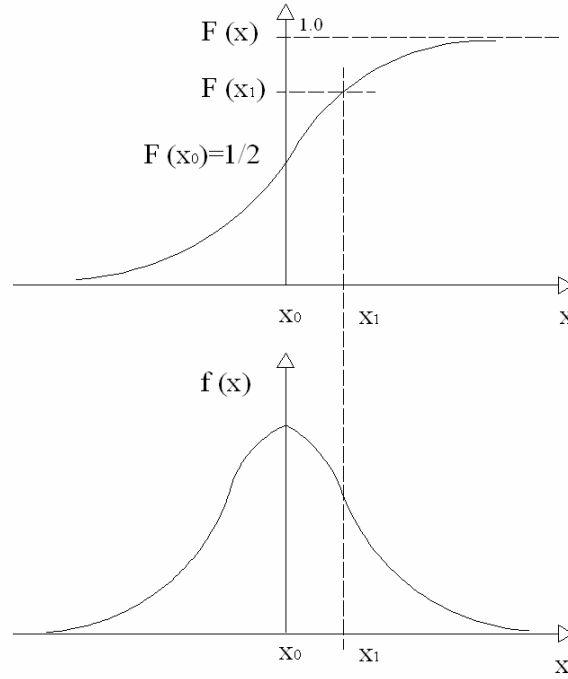


Figure 2.2 – Cumulative distribution function (CDF), $F(x)$, and probability density function (PDF), $f(x)$.

Being x a random variable, the expected value of x , $E[x]$ (μ), when exists, is defined by $E(x) = \sum_i x_i f(x_i)$ if x is a discrete variable, and $E(x) = \int_{-\infty}^{+\infty} x f(x) dx$ if x is a continuous variable. Considering x , y as random variables and k a constant value the following properties can be defined:

- 1) $E[k] = k$
 - 2) $E[kx] = k E[x]$
 - 3) $E[x \pm y] = E[x] \pm E[y]$
 - 4) $E[xy] = E[x] \cdot E[y]$ if they are independent
- (2.8)

Being x a random variable, the variance of x , $VAR[x]$ (σ^2), is represented by $VAR[x] = E[(x - \mu)^2]$ and can be calculated as $VAR[x] = \sum_i (x_i - \mu)^2 f(x_i)$ if x is discrete or

$VAR[x] = \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx$ if x is continuous. The standard deviation, σ , is so defined as

$\sigma = \sqrt{VAR[x]}$. Having this into account the subsequent properties are defined:

- 1) $VAR[k] = 0$
 - 2) $VAR[kx] = k^2 VAR[x]$
 - 3) $VAR[x \pm y] = VAR[x] + VAR[y]$ if x, y are independent
 - 4) $VAR[x] = E[(x - \mu)^2] = E[x^2] - \mu^2 \Rightarrow VAR[x] = E[x^2] - E[x]^2$
 - 5) If x is an aleatory variable in a way that $E[x] = \mu$ and $VAR[x] = \sigma^2$,
the aleatory variable $w = \frac{x - \mu}{\sigma}$ has the following parameters
 $E[w] = 0$ and $VAR[w] = 1$
- (2.9)

Other probability distribution parameter which is used several times to characterize a distribution is the so called coefficient of variation. The coefficient of variation (CV) is a measure of dispersion of a probability distribution and it is defined as the ratio of the standard deviation, σ , to the mean $E[x]$.

$$CV = \frac{\sigma}{E[x]} \quad (2.10)$$

In statistic analysis, the difference between an estimator expected value and the true value of the parameter being estimated is called the bias (λ). Suppose we are trying to estimate the parameter θ using an estimator θ' (some function of observed data). Then the bias of θ' is defined to be:

$$\lambda = E(\theta' - \theta), \quad (2.11)$$

which would read “the expected value of the difference between the estimator and the true value”, or, the so called, bias (λ).

2.3.2. DISCRETE DISTRIBUTIONS

2.3.2.1. UNIFORM DISTRIBUTION

A typical discrete distribution is the uniform distribution characterized by the fact that the probability of x to assume any value being the same. This distribution is

mainly used when there is no pattern, or in other words, the variable can assume any value between a lower and an upper bound.

The discrete uniform distribution associated to the discrete random variable x , and specified as $X \cap U(x; N)$, can be defined by the probability density function (PDF), $f(x)$, by the cumulative distribution function (CDF), $F(x)$, and by the mean, $E[x]$, and variance, $\text{Var}[x]$, values:

$$\begin{aligned}
 1) f(x) = f(x; N) = P(X = x) &= \begin{cases} 1/N & x=1, 2, 3, \dots, N \\ 0 & \text{other values} \end{cases} \\
 2) F(x) = F(x; N) = P[X \leq x] &= \begin{cases} 0 & x < 1 \\ \frac{x_i}{N} & x_i \leq x \leq x_{i+1} \quad x_i = 1, 2, \dots, N-1 \\ 1 & x \geq N \end{cases} \quad (2.12) \\
 3) E[x] = \frac{N+1}{2} \text{ and } \text{VAR}[x] &= \frac{N^2-1}{12}
 \end{aligned}$$

In Figure 2.3 it is represented both the cumulative distribution (CDF) and the probability density (PDF) functions.

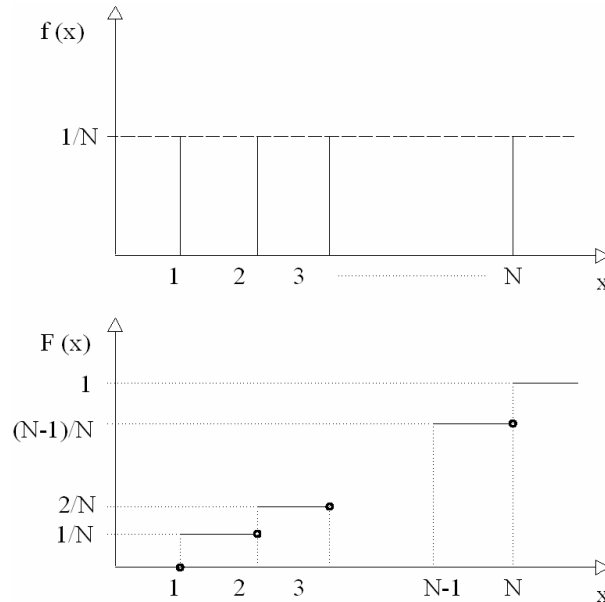


Figure 2.3 – Uniform discrete distribution.

2.3.2.2. OTHER DISCRETE DISTRIBUTIONS

There exist other discrete distributions with a high utility to describe the uncertainty associated to several input model parameters. It is important to refer the Bernoulli distribution and the binomial distribution, both based in Bernoulli Proof Theory.

Consider a random experience with only two possible results, the success A and insuccess $\bar{A}$. The success occurs with probability p and insuccess with probability $q = 1 - p$. A random experience with these characteristics is the so called Bernoulli proof. The stochastic processes are an example of a sequence of Bernoulli proofs.

Consider a random variable, x , which assume only two values, 0, when the Bernoulli proof result is insuccess and 1, when it is a success. To the success is associated the probability p and to insuccess $(1-p) = q$. The discrete random variable x presents a Bernoulli distribution, and it is defined by $X \cap b(x;p)$, if its probability density function (PDF) is given by:

$$f(x) = f(x; p) = P(X = x) = \begin{cases} p^x \cdot (1-p)^{1-x} & x=0,1 \\ 0 & \text{other values} \end{cases} \quad (2.13)$$

with a unique parameter, p , which satisfies the condition $0 \leq p \leq 1$. The cumulative distribution function (CDF) is then defined as:

$$F(x) = F(x; p) = P[X \leq x] = \begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (2.14)$$

If the random variable x presents a Bernoulli distribution, then:

$$\begin{aligned} E[x] &= p \\ Var[x] &= p \cdot (1-p) = p \cdot q \end{aligned} \quad (2.15)$$

The binomial distribution is one of the most used discrete distributions as an appropriate model to a variety of real situations. This distribution also presents an important role in sample theory. In generic way this distribution is considered as a

probabilistic scheme where a finite sample of objects with a determined attribute, associated to probability p , or without it, associated to probability $(1 - p) = q$, exist.

The discrete random variable, x , number of successes in n Bernoulli proofs, presents a binomial distribution $X \sim b(x; n; p)$ if its probability density function (PDF) is given by:

$$f(x) = f(x; n; p) = P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x=0,1,2,\dots,n \\ 0 & \text{other values} \end{cases} \quad (2.16)$$

where n and p are the distribution parameters. The parameter n corresponds to the Bernoulli proofs parameters, being n a positive integer, and p corresponds to the probability associated to the success, with $0 \leq p \leq 1$. The cumulative distribution function (CDF) is so defined as:

$$F(x) = F(x; n; p) = P[X \leq x] = \begin{cases} 0 & x < 0 \\ \sum_{x_i=0}^x \binom{n}{x_i} \cdot p^{x_i} \cdot (1-p)^{n-x_i} & 0 \leq x < n \\ 1 & x \geq n \end{cases} \quad (2.17)$$

The family of binomial distributions is defined by different values of parameters n and p . Tables, present in Appendix A, define different values for the binomial cumulative distribution function (CDF), according to values of n and p . If x is a random variable, then:

$$\begin{aligned} E[x] &= n \cdot p \\ Var[x] &= n \cdot p \cdot q \end{aligned} \quad (2.18)$$

Figure 2.4 presents the probability density function (PDF) of a typical binomial distribution function. The symmetry is achieved when $p = 0.5$, for all possible values of n , or when $n \rightarrow \infty$, for all possible values of p .

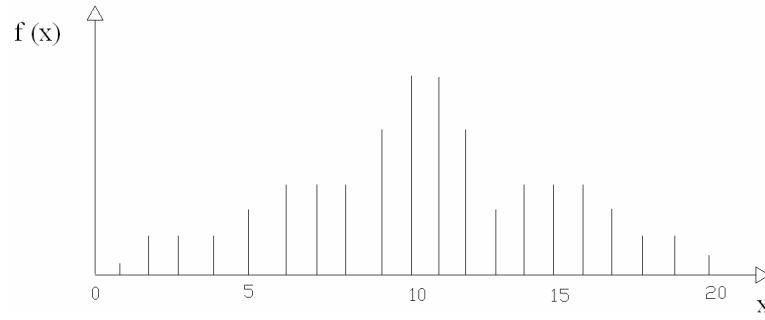


Figure 2.4 – Binomial distribution.

Other discrete distributions like the negative binomial, the multinomial, the Poisson, the geometric or Pascal and the hyper geometric are defined in Reis *et al.* (1999).

2.3.3. CONTINUOUS DISTRIBUTIONS

2.3.3.1. UNIFORM DISTRIBUTION

If the values of one random variable may occur within a limited interval $[a,b]$ and if any subintervals of equal amplitude present an equal probability of occurrence, then we are in the presence of a continuous random variable with a uniform or rectangular distribution. The obtained sensor data follows a typical continuous uniform distribution.

A continuous random variable x presents a uniform distribution within the interval $[a,b]$ and it is written as: $X \cap U(x;a;b)$ if its probability density function (PDF), $f(x)$, is defined as:

$$f(x) = f(x;a;b) = P(X = x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{other values} \end{cases} \quad (2.19)$$

The parameters characterized in this distribution are a , b , that satisfy the condition $-\infty < a < b < +\infty$. The cumulative distribution function (CDF), $F(x)$, is given by:

$$F(x) = F(x; a; b) = P[X \leq x] = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases} \quad (2.20)$$

In Figure 2.5 it is presented the probability density function (PDF) and the cumulative distribution function (CDF) of a typical uniform distribution.

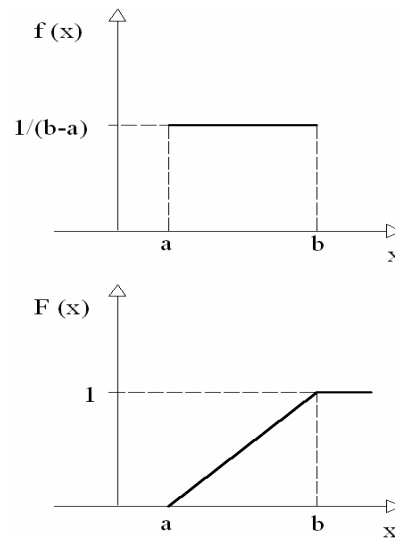


Figure 2.5 – Uniform continuous distribution.

If the continuous random variable x presents a uniform distribution within $[a, b]$ then:

$$\begin{aligned} E[x] &= \frac{a+b}{2} \\ Var[x] &= \frac{(b-a)^2}{12} \end{aligned} \quad (2.21)$$

2.3.3.2. NORMAL DISTRIBUTION

The normal distribution is one of the most used variables distributions in statistically analysis. There are several random variables which describe phenomena's, physical processes or human characteristics and that follow a normal distribution. The

normal distribution presents, also, an important role in statistic inference. In civil engineering almost all model parameters follow a normal distribution.

The continuous random variable x presents a normal probability density function (PDF), written as $X \sim n(x; \mu; \sigma)$ and given by:

$$f(x) = f(x; \mu; \sigma) = P(X = x) = \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2} \text{ with } -\infty < \mu < +\infty \quad (2.22)$$

where μ and σ are the parameters that characterize the distribution and which satisfy $-\infty < \mu < +\infty$ and $\sigma > 0$. Such parameters represent the mean distribution and standard deviation respectively. The normal cumulative distribution function (CDF) is given by:

$$F(x) = F(x; \mu; \sigma) = P[X \leq x] = \int_{-\infty}^x \frac{1}{\sigma \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2} dx \quad (2.23)$$

If x presents a normal distribution, then:

$$\begin{aligned} E[x] &= \mu \\ Var[x] &= \sigma^2 \end{aligned} \quad (2.24)$$

The probability density function (PDF) of a random variable with normal distribution presents a sino shape, is symmetric in relation to the axes $x = \mu$ and presents the inflection points $x = \mu \pm \sigma$.

Each normal distribution is so defined by two parameters: the mean value μ , which localizes the distribution centre and the standard deviation σ , which measure the variability of x in turn of the mean value.

Figure 2.6 present three normal distributions with the same mean μ but with different standard values σ_i ($i = 1, 2, 3$) for which $\sigma_1 > \sigma_2 > \sigma_3$.

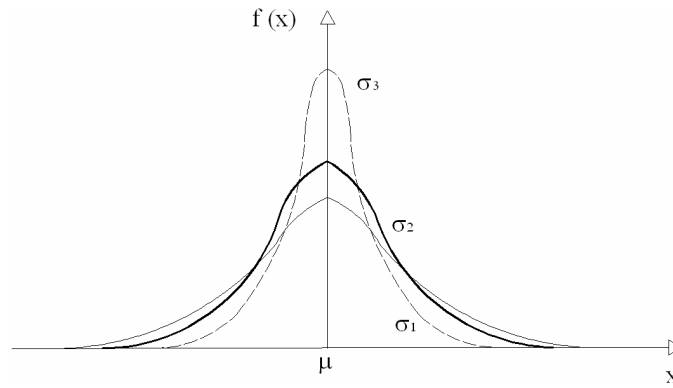


Figure 2.6 – Normal distribution with different standard values (σ).

Figure 2.7 present three normal distributions with the same standard deviation σ but with different mean values μ_i ($i = 1, 2, 3$) with $\mu_1 > \mu_2 > \mu_3$.

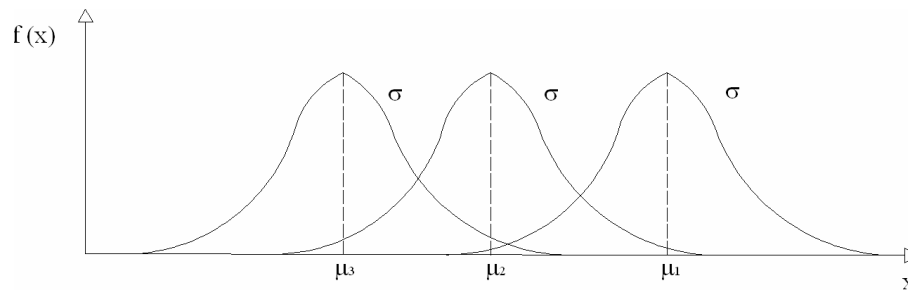


Figure 2.7 - Normal distribution with different mean values (μ).

Once μ and σ may have an infinity of possible values ($-\infty < \mu < +\infty$ and $\sigma > 0$), there is an infinity of normal distributions. So, for the probability calculation, every normal distribution is transformed into a standardized one. The standardization process consists in a change of origin (subtracting by μ) and a scale changing (dividing by σ). So, if the random variable x presents a normal distribution of parameters μ and σ , so $z = (x - \mu) / \sigma$ is defined as the standardized normal.

Having into account that $X \cap n(x; \mu; \sigma)$ and that $E[x] = \mu$ and $\text{Var}[x] = \sigma^2$, in an easy way the standardized normal parameters can be deduced, and so, $E[z] = 0$ and $\text{Var}[z] = 1$. It is so possible to conclude that, $z = (x - \mu) / \sigma \cap n(z; 0; 1)$. The probability density function (PDF) of the standardized normal distribution, z , is defined by:

$$f(z) = P(z = Z) = \frac{1}{\sqrt{2 \cdot \pi}} \cdot e^{-\frac{z^2}{2}} \quad (2.25)$$

with $-\infty < z < +\infty$. The respective cumulative distribution function (CDF), $\phi(z)$, allow calculating the probability within specific intervals:

$$\phi(z) = F(z) = P[Z \leq z] = \int_{-\infty}^z \frac{1}{\sqrt{2 \cdot \pi}} \cdot e^{-\frac{z^2}{2}} dz \quad (2.26)$$

Considering the symmetry of $\phi(z)$ we have:

$$\phi(-z) = 1 - \phi(z) \quad (2.27)$$

As it is possible to deduce from Figure 2.8.

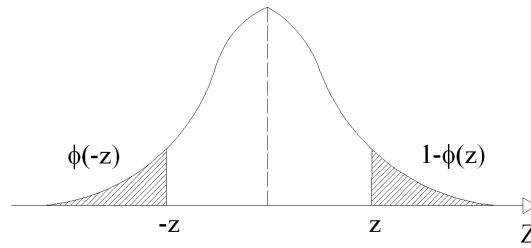


Figure 2.8 – Symmetry within the standardized normal distribution.

As an example, and having the Appendix B table values into consideration, it is possible to determine the following results (Figure 2.9):

$$\begin{aligned} P[\mu - \sigma < X < \mu + \sigma] &= P\left[\frac{\mu - \sigma - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{\mu + \sigma - \mu}{\sigma}\right] = P[-1 < Z < 1] \approx 0.68 \\ P[\mu - 2 \cdot \sigma < X < \mu + 2 \cdot \sigma] &= P[-2 < Z < 2] \approx 0.95 \\ P[\mu - 3 \cdot \sigma < X < \mu + 3 \cdot \sigma] &= P[-3 < Z < 3] \approx 0.99 \end{aligned} \quad (2.28)$$

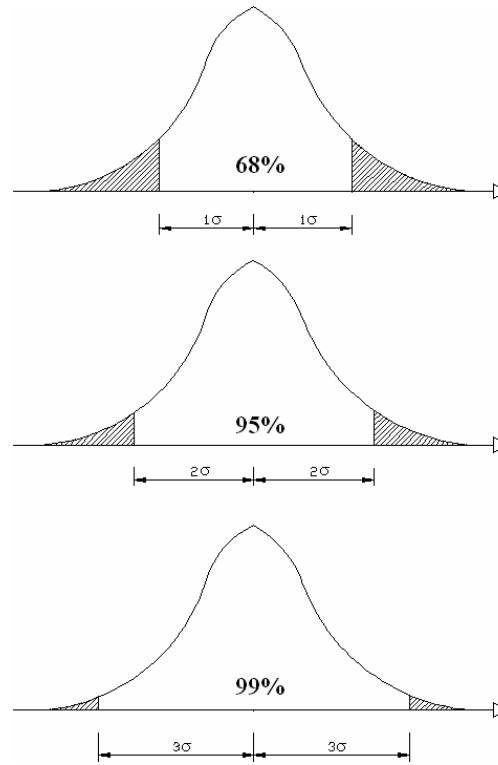


Figure 2.9 – Probability corresponding to different standard deviations (σ).

2.3.3.3. OTHER CONTINUOUS DISTRIBUTIONS

There exist, also, other continuous distributions with application in civil engineering field. One of such distributions is the lognormal one. A random variable x is considered to be lognormal if $y = \ln(x)$ is normally distributed. A lognormal random variable is defined for positive values only ($x \geq 0$). The probability density (PDF) and cumulative distribution (CDF) function can be defined as:

$$f(x) = f(x; \lambda; \zeta) = P(X = x) = \frac{1}{\zeta \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \left(\frac{\ln x - \lambda}{\zeta} \right)^2} \quad (2.29)$$

$$F(x) = F(x; \lambda; \zeta) = P[X \leq x] = \int_0^x \frac{1}{\zeta \cdot \sqrt{2 \cdot \pi}} \cdot e^{-\frac{1}{2} \left(\frac{\ln x - \lambda}{\zeta} \right)^2} dx \quad (2.30)$$

where ζ and λ are the distribution parameters. The lognormal distribution is unsymmetrical and it is applicable for positive values only. Figure 2.10 shows a typical shape of a typical lognormal probability density function (PDF).

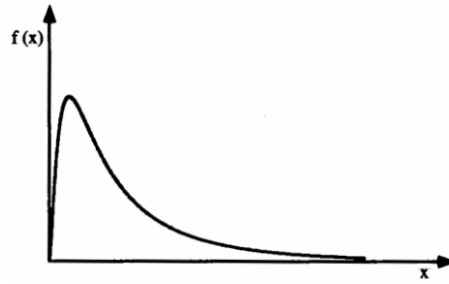


Figure 2.10 – Lognormal distribution (Nowak and Collins, 2000).

The mean value and standard deviation can be defined as:

$$\begin{aligned} E[x] &= \exp\left(\lambda + \frac{\sigma^2}{2}\right) \\ Var[x] &= E[x] \cdot \sqrt{\exp(\sigma^2) - 1} \end{aligned} \quad (2.31)$$

Definitions of other continuous distributions with applicability in civil engineering field, like Gamma, Extreme Type I (Gumbel, Fisher-Tippet Type I), Extreme Type II, Extreme Type III (Weibull) and Poisson distributions may be found in Nowak and Collins (2000) or in Singh *et al.* (2007).

2.3.4. DETERMINATION OF DISTRIBUTION FROM OBSERVATION

In engineering problems very often the distribution type and its parameters describing some random property are unknown and have to be selected based on available experimental data. Furthermore, even when the distribution type can be prescribed to some property arbitrary based on past experience there is necessity to check if the experimental data fit well to the prescribed distribution.

There exist many procedures to determine the type of the probability distribution function for any sample of data. They can be subdivided into groups, visual (histograms, P-P plots, etc) and analytical also known as goodness of fit tests (chi-square and Kolmogorov – Smirnov test).

Histogram is a bar diagram where each bar shows the relative frequency of the data points in predefined interval. By looking at the bar graph it can be observed trends in the data and visually can be determined the theoretical distribution that fits to the data. Figure 2.11 shows a typical histogram for compressive strength of concrete.

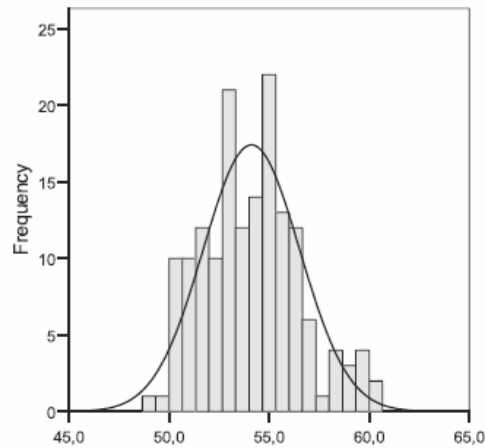


Figure 2.11 – Typical histogram (Wisniewski, 2007).

P-P plot shows the correspondence of the experimental results to the theoretical distribution function. It is basically a plot of the cumulative curve of obtained results against the cumulative theoretical distribution. Figure 2.12 shows a typical P-P plot for concrete compressive strength.

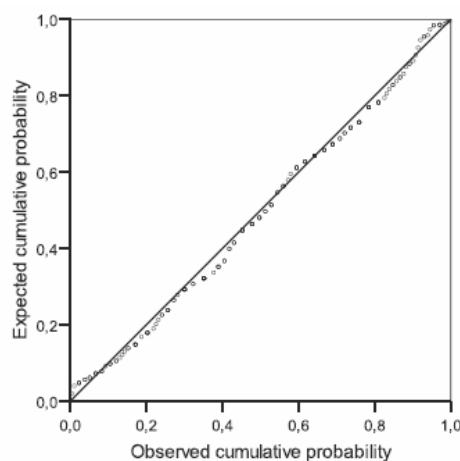


Figure 2.12 – Typical P-P plot (Wisniewski, 2007).

The Kolmogorov – Smirnov (K-S) test compares the observed cumulative frequency with the cumulative distribution function (CDF) of assumed theoretical distribution. It is based on the maximum difference between the two cumulative distributions:

$$D_n = \max |F_x(x_i) - S_n(x_i)| \quad (2.32)$$

Where $F_x(x_i)$ is the theoretical cumulative distribution function (CDF) at the i -th observation of ordered samples x_i and $S_n(x_i)$ is the corresponding observed cumulative distribution function (CDF) of ordered samples. D_n is also a random variable which distribution depends on the sample size n . The cumulative distribution of D_n is related to the significance level α as follows:

$$P(D_n \leq D_n^\alpha) = 1 - \alpha \quad (2.33)$$

D_n^α at various significance levels is tabulated. According to K-S test, if the maximum difference D_n is less than or equal to D_n^α , the assumed distribution is acceptable at the significance level α .

For a given sample of a random variable the parameters of its distribution can be also estimated using the method of moment or method of maximum likelihood. The basic concept of method of moment is that all the parameters of a distribution can be estimated using the information on its moments. The maximum likelihood method is based in the concept of a likelihood function associated to a range of independent samples of obtained data.

2.3.5. JOINTLY DISTRIBUTED RANDOM VARIABLES

Sometimes it is of interest to observe simultaneously two or more properties each of them being random. The question that can be asked whether there exist any independence between those properties and how important it is. To deal with this problem the concepts of joint distribution and correlation are introduced. The concepts

introduced here are applied to discrete random variables only, as are the ones which are more used.

If some observed property is the result of two (or more) random variables its cumulative distribution functions (CDF) can be described as:

$$F_{x_1, x_2}(x_1, x_2) = P[(X_1 \leq x_1) \cap (X_2 \leq x_2)] \geq 0 \quad (2.34)$$

The corresponding joint probability density function (PDF) can be expressed as follows:

$$f_{x_1, x_2}(x_1, x_2) = \frac{\partial^2 F_{x_1, x_2}(x_1, x_2)}{\partial x_1 \cdot \partial x_2} \quad (2.35)$$

However, a marginal probability density function (PDF) may be obtained from the joint probability density function by integrating over the other variables:

$$f_{x_1}(x_1) = \int_{-\infty}^{+\infty} f_{x_1, x_2}(x_1, x_2) dx_2 \quad (2.36)$$

Bivariate joint probability density function and marginal probability density functions are shown in Figure 2.13.

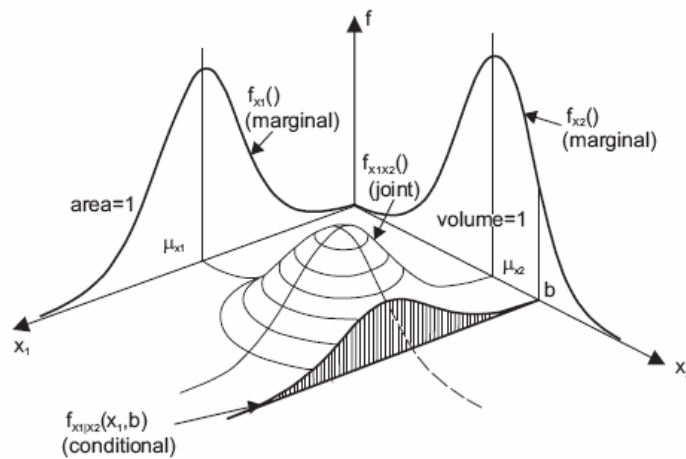


Figure 2.13 - Joint and marginal probability density function (PDF), from Melchers (1999).

The interdependence between any two variables can be checked plotting each pair of variables (x_1, x_2) as points in the corresponding coordinate system. Figure 2.14 shows several possible outputs of such graphical representation. The covariance, denoted as $\text{cov}(x_1, x_2)$, is a measure of the interdependence of two random quantities and, for continuous random variables, it is defined as:

$$\text{cov}(x_1, x_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 - \mu_{x_1}) \cdot (x_2 - \mu_{x_2}) \cdot f_{x_1, x_2}(x_1, x_2) dx_1 dx_2 \quad (2.37)$$

The estimation of the covariance, $\text{cov}(x_1, x_2)$, can be obtained as follows:

$$\text{cov}(X_1, X_2) = \frac{1}{n-1} \cdot \sum_{i=1}^n (x_{1,i} - \mu_{x_1}) \cdot (x_{2,i} - \mu_{x_2}) \quad (2.38)$$

The dimensionless correlation coefficient, $\rho_{x_1 x_2}$, is defined as:

$$\rho_{x_1 x_2} = \frac{\text{cov}(x_1, x_2)}{\sigma_{x_1} \cdot \sigma_{x_2}} \quad (2.39)$$

The correlation coefficient takes values from -1 to 1 and its correspondence to the type of interdependency is illustrated in Figure 2.14.

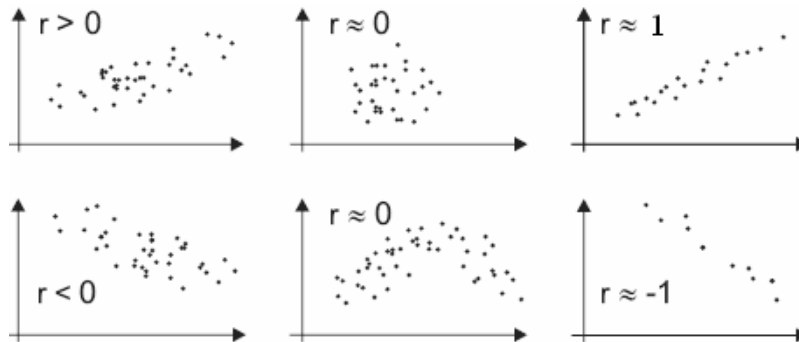


Figure 2.14 - Correlation plots and corresponding correlation coefficients, from Schneider (1997).

2.3.6. FUNCTIONS OF RANDOM VARIABLES

When the response variable is a function of several random variables, its uncertainty analysis is quite complicated. However, for some exceptional cases exists some simple, closed-form solutions. This is the case of sums and differences of independent normal variables and products and quotients of independent lognormal variables (Wisniewski, 2007).

When a random variable y is a sum (or difference) of independent normal random variables x_i , with mean μ_{x_i} and standard deviation σ_{x_i} , expressed as:

$$y = a_1 \cdot x_1 + a_2 \cdot x_2 + \dots + a_i \cdot x_i + \dots + a_n \cdot x_n \quad (2.40)$$

where a_i are constants, it can be shown that y is also a normal random variable with mean and variance defined as follows:

$$\mu_y = \sum_{i=1}^n a_i \cdot \mu_{x_i} \quad (2.41)$$

$$\sigma_y^2 = \sum_{i=1}^n a_i^2 \cdot \mu_{x_i}^2 \quad (2.42)$$

When a random variable y is a product (or quotient) of independent lognormal random variables x_i , with parameters λ_{x_i} and ζ_{x_i} , expressed as:

$$\lambda_y = \sum_{i=1}^n \lambda_{x_i} \quad (2.43)$$

$$\zeta_y^2 = \sum_{i=1}^n \zeta_{x_i}^2 \quad (2.44)$$

According to central limit theorem the sum of large number of variables, where none of them dominates the sum, tends to the normal distribution (regardless to their initial distributions) as the number of variables increases. Similarly for product of a large number of random variables, where none of them dominates the product, tends to the lognormal distribution (regardless to their initial distribution) as the number of variables increases.

2.4. UNCERTAINTY AND CODES

2.4.1. INTRODUCTION

In order to perform the structural design or even the structural assessment, it is necessary a reliable theoretical model which describes the structural behaviour in a rigorous way. Such model needs information about the structure geometry and about materials mechanical properties to run. Therefore, in order to obtain a powerful numerical model, it is necessary to precisely define such parameters. However both of those parameters present a random nature and should be treated as random variables. Consequently, the complete probabilistic models of such variables (probability density function – PDF – and basic statistics) are indispensable. For structural assessment it is also necessary to execute a complete probabilistic model of obtained data. This point will be focus in the following chapters.

At this point, probabilistic models of basic mechanical properties of concretes and steels, of element geometry and of applied loads are analyzed. These models were studied by previous authors from different experimental data and, at this point, it is only performed some references to what is mentioned in codes.

2.4.2. UNCERTAINTY IN USED MATERIALS

2.4.2.1. CONCRETE

The variability of the mechanical properties of concrete depends mostly on the following factors: material components properties; concrete composition; execution; testing procedure; concrete being in the structure rather than in control specimens; maintenance, material degradation, etc. The parameter of concrete which is investigated with higher frequency is the compressive strength f_c as this parameter serves commonly to control quality of concrete during structure execution and is used by legal codes to define the acceptance / rejection criteria. Other mechanical properties like tensile strength, f_{ct} , elasticity modulus, E_c , etc, are sometimes also defined during

experimental tests. However due to the high correlation with the concrete compressive strength they are usually defined via some empirical relations.

According to JCSS (2001) the concrete strength at the particular point I in a given structure j as a function of standard strength f_{c0} is given by:

$$f_{c,ij} = \alpha(t, \tau) \cdot f_{c0,ij}^\lambda \cdot Y_{1j} \quad (2.45)$$

$$f_{c0,ij} = \exp(U_{ij} \sum_j + M_j) \quad (2.46)$$

where $f_{c0,ij}$ is the lognormal variable, independent of Y_{1j} , with distribution parameters M_j and σ_j , M_j is the logarithm mean at job j, σ_j is the logarithm standard deviation at job j, Y_{1j} is a lognormal variable representing additional variations due to the placing, curing and hardening conditions of in-situ concrete at job j, U_{1j} is a standard normal variable representing variability within one structure and λ is a lognormal variable with mean 0.96 and coefficient of variation 0.005.

The remaining concrete properties as tensile strength, elasticity modulus and ultimate strength in compression are given by relationship equations. For elasticity modulus, as an example, we have:

$$Ec, ij = 10.5 \cdot f_{c,ij}^{1/3} \cdot Y_{3j} \cdot (1 + \beta_d \phi(t, \tau))^{-1} \quad (2.47)$$

where the variable Y_{3j} , reflects variations due to factors not well accounted for by concrete compressive strength. The following table give a description of such factors.

Table 2.2 – Data for parameters Y_i (JCSS, 2001).

Related to parameter	Variable	Type of Distribution	Mean	CV (%)
Compression	Y_{1j}	lognormal	1.0	6
Elasticity modulus	Y_{3j}	lognormal	1.0	15

Besides the described JCSS (2001) model of concrete compressive strength in structures, many others works characterizing statistical variations of concrete strength in test cylinders exist.

It is necessary to focus here the work of Mirza *et al.* (1979) who proposed a comprehensive probabilistic model based on observations and analysis of experimental data collected by various authors in United States, Canada and Europe. In their work Mirza *et al.* (1979) concluded that the probability distribution function which describes properly the concrete cylinder compressive strength presents a normal or lognormal shape. Another comprehensive and relatively simple model was proposed by Bartlett and MacGregor (1996) which was developed based on already described work of Mirza *et al.* (1979) and on experimental results performed in Canada. Sobrino (1993) proposed models based on the results collected on the bridge construction sites in Spain. Henriques (1998) studied the variability of strength of concretes used in the constructions of two viaducts in Portugal. Nowak and Szerszen (2003) published the results obtained in the United States and used for the calibration of ACI Design Code for Buildings. A resume of all the results obtained by various authors are presented in Table 2.3 and Table 2.4.

Table 2.3 – Statistical parameters of site-cast concrete.

Origin (Reference)	Nominal f_{ck} [MPa]	Bias λ_c	Standard Deviation. σ_c [MPa]	CV _c (%)
US, Canada and Europe (Mirza <i>et al.</i> , 1979)	< 27	-	-	10 – 20
	≥ 27	-	2.7 – 5.4	-
Spain (Sobrino, 1993)	24 - 40	1.09 – 1.39	2.6 – 4.2	6 – 11
Canada (Bartlett and MacGregor, 1996)	≤ 55	1.25	-	10
Portugal (Henriques, 1998)	20 - 35	1.23 – 1.55	3.9 – 6.6	9 – 17
United States (Nowak and Szerszen, 2003)	21 – 41	1.12 – 1.35	1.5 – 4.9	4 – 15
	48 - 83	1.04 – 1.19	5.4 – 9.0	9 - 12

Table 2.4 – Statistical parameters of plant-cast concrete.

Origin (Reference)	Nominal f_{ck} [MPa]	Bias λ_c	St. Dev. σ_c [MPa]	CV _c (%)
Canada (Bartlett and Macgregor, 1996)	≤ 55	1.19	-	5
United States (Nowak and Szerszen, 2003)	34 - 45	1.14 – 1.38	4.1 - 5.7	8 – 12

As it was already mentioned before the concrete strength in the structure is usually somehow smaller than the strength of the concrete test cylinders. In the literature, there are some models which allowed relating the compressive strength of standard test cylinder with the compressive strength of cores drilled from the structure. At this point it is necessary, besides the works of Mirza *et al.* (1979) and of Bartlett and

MacGregor (1996), to focus the studied performed by Gonçalves (1987). Table 2.5 presents the models proposed by such authors.

Table 2.5 – In-situ concrete compressive strength versus test cylinder strength.

Origin (Reference)	Type of Distribution	Mean	CV (%)
US, Canada and Europe (Mirza <i>et al.</i> , 1979)	Not specified	0.74 – 0.96	10
Portugal (Gonçalves, 1987)	Not specified	0.69 – 1.02	3 – 14
Canada (Bartlett and MacGregor, 1996)	Lognormal	0.95 – 1.03	14

Other concrete properties, like tensile strength (f_{ct}), elasticity modulus (E_c) and ultimate strain in compression (ϵ_{cu}), were probabilistically studied by Mirza *et al.* (1979).

The concrete initial tangent elasticity modulus, according to Mirza *et al.* (1979), can be modelled by the normal distribution with the mean value calculated by:

$$E_{ci} = 5015 \cdot f_c^{1/2} [MPa] \quad (2.48)$$

where f_c is the concrete compressive strength, and a coefficient of variation equal to 8%. The concrete secant elasticity modulus, at 30% of maximum stress, can be modelled by the normal distribution with the mean value defined by equation 2.49 and a coefficient of variation of 12%.

$$E_{cs} = 4600 \cdot f_c^{1/2} [MPa] \quad (2.49)$$

2.4.2.2. STEEL

When characterizing the steel material it is necessary to differentiate reinforcing steel from structural steel. At this point it will be firstly made a probabilistic description of reinforcing steel and then of structural steel.

The variability of mechanical properties of steel is generally lower than variability of parameters describing concrete. This is mostly due to higher industrialization of the production and higher level of the quality control. According to Sobrino (1993) the principal factors that influence the variability of mechanical

behaviour of steel sections (reinforced bars or steel structural shape sections) are: the variability of material strength, the variability of section geometry, the material degradation, the load history and the method of definition of the conventional strength parameters f_{sy} and f_{su} and their experimental evaluation.

The first approach to describe the probabilistic evaluation of reinforced steel bars is the one due to JCSS (2001). Accordingly, the yield stress of the reinforcing steel bar can be defined as the sum of three independent Gaussian variables:

$$X_1(d) = X_{11} + X_{12} + X_{13} \quad (2.50)$$

where $X_{11} = N(\mu_{11}(d), \sigma_{11})$ represents the variations related to different steel producers, $X_{12} = N(0, \sigma_{12})$ represent the batch to batch variation, and $X_{13} = N(0, \sigma_{13})$ represent the variation within the single batch, being d the nominal bar diameter. Table 2.6 presents remaining steel parameters as defined in JCSS (2001). For all the quantities presented in Table 2.6, a normal distribution can be adopted.

Table 2.6 - Statistical parameters of reinforcing steel (JCSS, 2001).

Steel Property	Mean value [MPa] X_{mean}	Standard deviation [MPa] σ_x	CV (%)
Bar area A_s	$A_{s,nom}$	-	2.0
Yield strength f_{sy}	$f_{sy,nom} + 2\sigma$	30	-
Ultimate strength f_{su}	-	40	-
Ultimate strain ϵ_{su}	-	-	9.0

The probabilistic model of steel reinforcement yield strength was already proposed in the end of the 70-ties by Mirza and MacGregor (1979a) when the first generation of partial safety factor design codes were under development. The proposed model bases on the experimental data collected in the United States, Canada and Europe mostly in the fifties and sixties.

In the last decades also some probabilistic models were defined based on the more recent data collected in Europe. Among others, the works of Sobrino (1993) and Pipa (1995) should be mentioned here as the examples of great relevance.

Aiming to verify hypothesis about higher quality of ordinary reinforcing steel produced nowadays in comparison to the steels produced in past decades numerous data of yielding tensile strength were collected recently in the United States.

Based on those data probabilistic models were developed by Nowak and Szerszen (2003) which were used in the calibration of the latest version of the ACI design code for buildings.

In Table 2.7 the resume of the experimental results obtained by various authors is presented. Table 2.7 shows the nominal values, the bias factors and the coefficients of variations of the steel yield strength. It is important to notice that the coefficients of variations presented in Table 2.7 are affected by all the sources of uncertainty as: variation in the strength of material itself, variation in the area of the cross-section, effect of bar diameter on properties of bars and effect of strain at which yield is defined.

Table 2.7 - Experimental results of steel yielding strength.

Origin (Reference)	Normal value [MPa] f_{yk}	Bias factor λ_{sy}	CV (%)
US, Canada and Europe, (Mirza and MacGregor, 1979a)	280	1.20	10.7
	410	1.20	9.3
Spain, (Sobrino, 1993)	500	1.20	8.1
Europe, (Pipa, 1995)	400	1.24	4.7
	500	1.17	5.2
US (Nowak and Szerszen, 2003)	420	1.145	5.0

As can be observed, the bias factor which relates the mean value obtained from the experimental test with the nominal or expected value, regardless to the steel grade, oscillate around value 1.20 excepted in the two last cases when it is close to 1.15. The coefficients of variation obtained for the specimens taken from the same batch or from the same producer are taking values around 4-5%. However the coefficients of variations obtained for the specimens taken from different sources are oscillating around 8-10% except the cases of the most recent results when the coefficients of variation are between 2.5% and 5%. Observed significant difference in the coefficient of variation between data collected in the past decades and that collected recently could be explained by the improvements of the fabrication process and by more strict requirements related to the control of quality.

Regarding the types of the probability distribution functions, various authors suggest different theoretical models. Nowak and Szerszen (2003) use the normal probability distribution function to model the yield strength of reinforcing steel. However, other authors recommend lognormal distribution (Sobrino, 1993) or Beta distribution (Mirza and MacGregor, 1979a) as more appropriate.

Besides the yield strength of steel, some other properties were also observed by several authors. Table 2.8, Table 2.9 and Table 2.10 show the results of experimental test gathered and statistically treated by Sobrino (1993) and Pipa (1995). The results correspond to steel grades S500 and S400 produced in Europe.

Table 2.8 - Experimental results obtained by Sobrino (1993) for S500 steel grade.

Steel Property	Mean value X_{mean}	Minimum value X_{min}	Maximum value X_{max}	CV (%)
Ultimate strength f_{su}	690 MPa	614 MPa	856 MPa	7.8
Yield strength f_{sy}	602 MPa	511 MPa	770 MPa	8.1
Ratio $f_{\text{su}}/f_{\text{sy}}$	1.147	1.055	1.250	4.0
Ultimate strain ϵ_{su}	23.3%	12.6%	30.3%	12.7
Ratio $A_{\text{s, real}}/A_{\text{s, nom}}$	1.005	0.965	1.055	2.1

Table 2.9 - Experimental results obtained by Pipa (1995) for S400 steel grade.

Steel Property	Mean value X_{mean}	Minimum value X_{min}	Maximum value X_{max}	CV (%)
Ultimate strength f_{su}	598 MPa	552 MPa	646 MPa	3.3
Yield strength f_{sy}	496 MPa	431 MPa	544 MPa	4.7
Ultimate strain ϵ_{su}	11.8%	7.5%	16%	14.3
Initiation of hardening ϵ_{sh}	2.2%	1.6%	3.1%	20.0
Modulus of hardening E_{sh}	3.00 GPa	2.17 GPa	5.02 GPa	22.0

Table 2.10 - Experimental results obtained by Pipa (1995) for S500 steel grade.

Steel Property	Mean value X_{mean}	Minimum value X_{min}	Maximum value X_{max}	CV (%)
Ultimate strength f_{su}	680 MPa	613 MPa	752 MPa	4.2
Yield strength f_{sy}	585 MPa	519 MPa	656 MPa	5.2
Ultimate strain ϵ_{su}	9.4%	6.0%	13%	14.9
Initiation of hardening ϵ_{sh}	1.4%	0.7%	2.3%	31.0
Modulus of hardening E_{sh}	3.51 GPa	2.72 GPa	5.33 GPa	15.0

Various authors suggest different types for the probability density functions. However, in general, they recommend being consistent and using the same distribution for yield strength.

In their work, Mirza and MacGregor (1979a) analysed also the ratio of real to notional reinforcing bar area. They concluded that the mean value of this parameter oscillate between 0.96 and 1.20. However the coefficient of variation takes the values between 0.2% and 9%. Sobrino (1993) obtained results of 1.005 for mean value of the ratio and 2.1% for its coefficient of variation. The type of the probability distribution function considered by both authors most appropriate for modelling this parameter is the Gaussian distribution. Mirza and MacGregor (1979a) proposed also the probabilistic models of steel elasticity modulus. The recommended distribution type is the normal distribution with the mean value of 201GPa and coefficient of variation 3.3%.

A significant amount of data were collected and analysed statistically from steel reinforced bars applied in Portuguese construction – site by Wisniewski (2007). The statistical evaluation of data confirmed that a normal and lognormal probability distribution function describes accurately variability of most of the properties of reinforcing steel bars. The basic statistics of all the parameters obtained in the analysis are showed in the Table 2.11 and in the Table 2.12.

Table 2.11 - Experimental results of reinforcing steel strength (Wisniewski, 2007).

Origin (Reference)	Stress	Nominal value [MPa]	Bias factor λ	CV [%]
Portugal	f_{sy}	550	1.21	6.0
	f_{su}	550	1.28	5.9
	f_{sy}	500	1.16	5.5
	f_{su}	575	1.20	4.9

Table 2.12 - Experimental results of other properties of reinforcing steel (Wisniewski, 2007).

Origin (Reference)	Parameter	Nominal value	Bias factor λ	CV [%]
Portugal	f_{us} / f_{ys}	1.10 (1.15)	1.06 (1.04)	3.3 (2.8)
	E_s	200 GPa	1.01 – 1.03	1.0 – 4.9
	ϵ_s	5% (8%)	2.7 (1.64)	24.5 (19.2)
	A_s	10 – 25 mm ²	0.92 – 0.94	4.3 – 4.4

Regarding the structural steel, JCSS (2001), propose a probabilistic model for the random vector $X=(f_y, f_u, E, v, \epsilon_u)$ to be used for any particular steel grade, which may be defined in terms of nominal values verified by standard mill test or in terms of

minimum values given in material specifications. Such vector is based in the structural steel properties: yield strength, f_y [MPa], ultimate tensile strength, f_u [MPa], modulus of elasticity, E [MPa], Poisson's ratio, ν , and ultimate strain, ϵ_u .

Mean values and coefficients of variation for the above vector are given in Table 2.13. A multi-variate log-normal distribution is recommended for all these parameters. The CV values refer to total steel production and are based primarily on European, US, and Canada studies.

Within the same batch, CV can be taken as one fourth of the values given in Table 2.13, but for the modulus of elasticity, E , and Poisson's ratio, ν , this reduction may be neglected. Variations along the length of a rolled section are normally small and may not be considered.

Table 2.13 - Mean and CV values.

Property	Mean value, E [-]	CV, ν
f_y	$f_{ysp} \alpha \exp(-u \nu) - C$	0.07
f_u	$B E [f_u]$	0.04
E	E_{sp}	0.03
ν	ν_{sp}	0.03
ϵ_u	ϵ_{usp}	0.06

being the suffix (sp) used for the code specified or nominal value for the variable considered, α the spatial position factor ($\alpha = 1.05$ for webs of hot rolled sections and $\alpha = 1$ otherwise), u is a factor related to the fractile of distribution used in describing the distance between the code specified or nominal value and the mean value (in the range -1.5 to -2.0 for steel produced in accordance with the relevant EN standards, C is a constant reducing yield strength as obtained from usual mill tests to the static yield strength (it is recommended a value of 20 MPa) and B is constant equal to 1.5 for structural carbon steel, 1.4 for low alloy steel and 1.1 for quenched and tempered steel.

References regarding the probabilistic analysis of structural steel may be found in Baker (1972), Galambos and Ravindra (1978), Kennedy and Baker (1984) and Agostoni *et al.* (1994).

2.4.3. UNCERTAINTY IN GEOMETRY PARAMETERS

Geometric imperfections in concrete elements are caused by deviations from the specified values of the cross-sectional shape and dimensions, the position of active and passive reinforcement, the horizontality and verticality of concrete lines, the alignment of columns and beams and the grades and surfaces of constructed structure, Mirza and MacGregor (1979b).

The following factor affects the basic statistical characteristics of structure geometric imperfections: the structural type, the construction process and applied technology and the execution quality.

According to JCSS (2001), the dimensional deviations of a dimension X can be described by the statistical characteristics of its deviations Y from the nominal value X_{nom} :

$$Y = X - X_{nom} \quad (2.51)$$

The deviations of the external dimensions of pre cast and cast “in situ” concrete components for the nominal dimensions X_{nom} up to 1000 mm, can be modelled by the normal distribution with the mean value and standard deviation defined by the equation 2.52 and equation 2.53 respectively:

$$0 \leq \mu_y = 0.003 \cdot X_{nom} \leq 3 \text{ [mm]} \quad (2.52)$$

$$\sigma_y = 4 + 0.006 \cdot X_{nom} \leq 10 \text{ [mm]} \quad (2.53)$$

The deviations of the concrete cover in the reinforced and pre stressed concrete elements can be generally modelled by the normal distribution with the mean and standard deviation as defined in Table 2.14. However, in some cases other types of distribution, namely one or two side limited distributions (e.g. beta, gamma, shifted lognormal, etc.) can be more appropriate.

Table 2.14 - Statistical parameters of concrete cover (JCSS, 2001).

Element and cover type	Mean value μ_y [mm]	Standard Deviation σ_y [mm]
Column and wall	0 - 5	5 - 10
Slab bottom steel	0 - 10	5 - 10
Beam bottom steel	-10 - 0	5 - 10
Slab and beam top steel	0 - 10	10 - 15

As in the case of probabilistic models of reinforcing steel and concrete mechanical properties, one of the first comprehensive probabilistic models of the geometric deviations in reinforced concrete members were proposed by Mirza and MacGregor (1979b). The models base on the measurement results obtained by other authors on buildings and bridges (precast and cast “in situ”) constructed in Europe and in North America. The values of basic statistical parameters proposed by Mirza and MacGregor (1979b) to model geometric deviations in slabs, beams and columns are presents in Table 2.15, in Table 2.16 and in Table 2.17, respectively.

Table 2.15 - Variability of slab dimensions (Mirza and MacGregor, 1979b).

Dimension description	Technology of execution	Nominal range[mm]	Deviation from Nominal[mm]	Standard deviation [mm]
Thickness	In-situ	100 - 230	0.8	12
	Precast	100 - 230	0.0	5
Effective depth (top reinforcement)	In-situ	100 - 200	± 20	15 - 20
	Precast	100 - 200	± 20	3 - 6
Effective depth (bottom reinforcement)	In-situ	100 - 200	-8 - 9	10 - 15
	Precast	100 - 200	0	3 - 6

Table 2.16 - Variability of beam dimensions (Mirza and MacGregor, 1979b).

Dimension description	Technology of execution	Nominal range[mm]	Deviation from Nominal [mm]	Standard deviation [mm]
Overall depth	In-situ	460 - 690	2.5	4
	Precast	530 - 990	3.2	5
Rib width	In-situ	280 - 305	2.8	4.8
	Precast	480 - 610	4	6.5
Flange width	Precast	280 - 305	2.8	4.8
Concrete cover (bottom reinforcement)	In-situ	12 - 25	-3 - 6	16 - 18
	Precast	50 - 60	3	8 - 9
Concrete cover (top reinforcement)	In-situ	19 - 25	-5 - 2	11 - 13
	Precast	19	3	8 - 9

Table 2.17 - Variability of column dimensions (Mirza and MacGregor, 1979b).

Dimension description	Technology of execution	Nominal range[mm]	Deviation from Nominal [mm]	Standard deviation [mm]
Rectangular (width, thickness)	In-situ	280 - 760	1.6	6.4
	Precast	180 - 410	0.8	3.2
Circular (diameter)	In-situ	280 - 330	0	4.8
	Precast	280 - 330	0	2.4

Regarding the types of distribution functions, Mirza and MacGregor (1979b) recommended Gaussian distribution for all dimensions except concrete cover where normal truncated distribution is suggested in order to avoid negative values.

A study related to probabilistic description of bridge section geometric variations was performed by Sobrino (1993). During the years 1990 – 1993 he collected data on several bridges (precast and cast “in situ”) in Spain. Based on that data he proposed mean values and standard deviation of distribution functions which can be used in modelling characteristic dimensions of the bridge sections (Table 2.18 and Table 2.19). Sobrino (1993) also recommends using normal distribution function for the modelling of geometric variations.

Table 2.18 - Variability of bridge sections dimensions (Sobrino, 1993).

Dimension description	Technology of execution	Nominal range[mm]	Deviation from Nominal [mm]	Standard deviation [mm]
Horizontal	In-situ	≤ 600	-	-
		≥ 600	0.2	3
	Precast	≤ 250	-2	5
		250 – 600 ≥ 600	10 -2 - 1	12 -
Vertical	In-situ	≤ 250	0 - 2	2 - 3
		250 – 600	-2	8 - 10
		≥ 600	40 - 20	15 - 22
	Precast	≤ 250 250 – 600 ≥ 600	5 12 -	5 - 9 5 -

Table 2.19 - Variability of reinforcing position in bridges (Sobrino, 1993).

Dimension description	Nominal range[mm]	Deviation from Nominal [mm]	Standard deviation [mm]
Concrete cover (top reinforcement)	H ≤ 200 H ≥ 200	5 - 10 5 - 30	10 - 12 7 - 10
Effective depth (bottom reinforcement)	D ≤ 200 D ≥ 200	2.5 -7 - 10	6 6
Effective depth (cable)	D ≥ 1000	- 30 - 0	16.3
Transversal spacing	150 - 200	1 - 4	13 - 36

2.4.4. UNCERTAINTY IN APPLIED LOADS

The loads present in the evaluation of any structural static behaviour can be divided in two groups: mechanical loads and non-mechanical loads. On the first group it is inserted loads like self-weight, pre stress, live loads, etc. On the second group it is included the temperature variations and the long term phenomena resulting loads, like shrinkage, creep and reinforcement relaxation. At this point it will be focused, due to their importance, essentially the self-weight and live loads in buildings.

2.4.4.1. SELF-WEIGHT

Self-weight loads are usually characterized by distributed loads in the structure volume (structural elements self-weight) or by the external superficies (non-structural elements that are applied to the structure with a permanent character). This kind of load usually presents a low variability which is implicit in actual norms that identify the characteristic value by the mean one. The self-weight variability is essentially due to geometric imperfections as the material self-weight presents an insignificant variation.

According to (JCSS, 2001), the self-weight concerns the weight of structural and non-structural components. The main characteristics of self-weight are that the probability of occurrence at an arbitrary point in any time is closed to one, the variability with time is normally negligible and the uncertainties in magnitude are normally small in comparison with other kind of loads. Concerning the uncertainties, it can be distinguished the variability within a structural part, between different structural parts of the same structure, and between various structures. The variability within a structural part is normally small and can often be neglected.

The self-weight, G , of a structural part is determined by the relation:

$$G = \int_{Vol} \gamma dV \quad (2.54)$$

where V is the volume described by the boundary of the structural part (volume of V is Vol), and γ the material self-weight. For a part where the material can be assumed to be reasonably homogeneous we may obtain the following equation:

$$G = \gamma_{av} \cdot V \quad (2.55)$$

where γ_{av} is an average weight density for the part.

The weight density and the dimensions of a structural part are assumed to have Gaussian distributions. To simplify the calculations the self-weight, G , may as an approximation be assumed to have a Gaussian distribution too. Mean values, μ_γ , and coefficients of variation, V_γ , for the total variability of the weight density of some building materials are given in Table 2.20.

Table 2.20 – Mean value and coefficient of variation for weight density.

Material	Mean value [kN/m ³]	Coefficient of Variation
Steel	77	< 0.01
Concrete (ordinary concrete)	24	0.04

The other parameter that the structural self-weight depends is the volume of its components. In most cases it may be assumed that the mean values of the dimensions are equal to the nominal values, like the dimensions given on drawings. The volume mean value, V , of the structural part is calculated directly from the dimensions mean values. The standard deviation of the volume, V , is also calculated directly from the values of the standard deviation for the dimensions. Table 2.21 presents, according to JCSS (2001), the mean value and standard deviation for deviations of cross section dimensions related to their nominal values, for common building materials and typical structural elements.

Table 2.21 – Mean value and standard deviation for deviations of cross-section dimensions.

Structure or structural member	Mean value	Standard deviation
Rolled steel		
Steel profiles, area A	0.01 Anom	0.04 Anom
Steel plates, thickness t	0.01 tnom	0.02 tnom
Concrete members		
$\text{anom} \leq 1000 \text{ mm}$	0.03 anom	$4 + 0.006 \text{ anom}$
$\text{anom} \geq 1000 \text{ mm}$	3 mm	10 mm

JCSS(2001) also presents procedures to treat the variability of the self-weight of material or of the structural component volume within a determined component (e.g. variability of the cross sectional area along a beam).

2.4.4.2. LIVE LOADS

Accordingly to actual Eurocodes (EC1-2, 1994), the exploitation loads in buildings are due to: equipments and materials; usual human occupation; exceptional human or equipment occupation; vehicles. Typical loads, due to equipments and materials or to usual human occupation, present a magnitude that may suffer instantaneous oscillations in time due, for instance, to a change in the use. However, within such time steps, variations on load magnitude are so small that can be considered as insignificant. The human occupation loads present a periodic characteristic and act during a very short period. Exceptional loads occur only in very special occasions during a short or even moderate period of time, but with a high frequency within the structural life cycle to be considered as significant. The vehicle load presents a diary fluctuation.

At this point it will be described in a summary way the essential aspects that conduced to the values proposed by Eurocode 1 (EC1-2, 1994) for live loads. The statistic techniques used in the characteristic and design values definitions of live loads had into account the following hypothesis (Sedlacek, 1992; Sedlacek and Gulvanessian, 1996): 1) The spatial live load variation is independent of it time variation; 2) When performing the spatial representation, the discrete live loads are defined by an equivalent uniformly distributed one; 3) The representation of the temporal variations is made through the consideration of two components: the quasi-permanent component (Figure 2.15) which value represents, in an approximate way, the mean time of the real live load fluctuation in between the changes of use and includes the weight of persons that frequently are present; the intermittent component (Figure 2.16) which represents all live loads that are not considered in the permanent component like special use live loads. The combination of both components is represented in Figure 2.17.

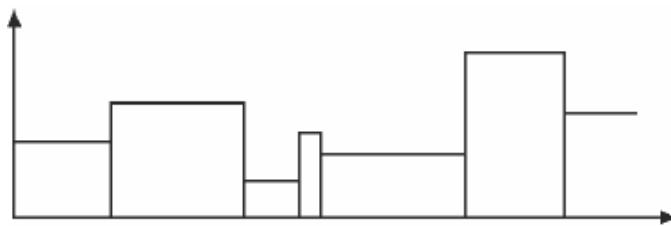


Figure 2.15 – Live load quasi-permanent component in time (Henriques, 1998).



Figure 2.16 - Live load intermittent component in time (Henriques, 1998).

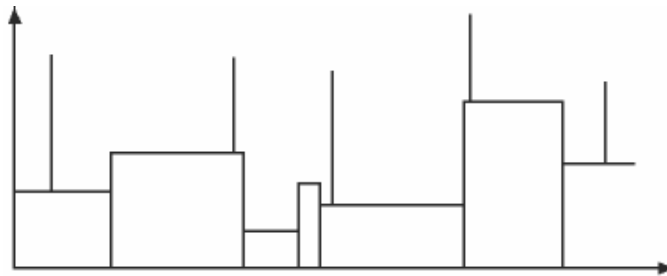


Figure 2.17 – Combination of live load components in time (Henriques, 1998).

The live load design values are determined for a reference period of 50 years. For the design of the structural horizontal elements Eurocode 1 (EC1-2, 1994) considers two possible live loads: uniformly distributed vertical live load, with a characteristic value defined by q_{kv} , affected by a superficial reducing coefficient α_A ; and concentrated applied live load with characteristic value Q_{kv} which is, in general, not accumulated with the uniform distributed live load.

The live load values and the partial combination factors are differentiated accordingly with the function and the aim to which the structure is designed. So, the structures are classified in different categories. Table 2.22 presents Eurocode 1 (EC1-1, 2002) categories for buildings.

Table 2.22 – Eurocode 1 (EC1-1, 2002) categories for buildings.

Category	Specific Use	Example
A	Areas for domestic and residential activities	Rooms in residential buildings and houses; bedrooms and wards in hospitals; bedrooms in hotels and hostels kitchens and toilets.
B	Office areas	
C	Areas where people may congregate (with the exception of areas defined under category A, B, and D ¹⁾)	C1: Areas with tables, etc. e.g. areas in schools, cafés, restaurants, dining halls, reading rooms, receptions. C2: Areas with fixed seats, e.g. areas in churches, theatres or cinemas, conference rooms, lecture halls, assembly halls, waiting rooms, railway waiting rooms. C3: Areas without obstacles for moving people, e.g. areas in museums, exhibition rooms, etc. and access areas in public and administration buildings, hotels, hospitals, railway station forecourts. C4: Areas with possible physical activities, e.g. dance halls, gymnastic rooms, stages. C5: Areas susceptible to large crowds, e.g. in buildings for public events like concert halls, sports halls including stands, terraces and access areas and railway platforms.
D	Shopping areas	D1: Areas in general retail shops D2: Areas in department stores

There exist, also, the category E (areas for storage and industrial activities), F and G (traffic and parking areas for light and medium vehicles, respectively), H, I and K (roofs not accessible, roofs accessible with occupancy according to categories A to D and roofs accessible for special services, respectively). Table 2.23 present the uniformly distributed and point applied live load values for the previously defined categories.

Table 2.23 – Eurocode 1 (EC1-1, 2002) values for live loads.

Categories of loaded areas	q_k [kN/m ²]	Q_k [kN]
Category A		
- Floors	1,5 to <u>2,0</u>	<u>2,0</u> to 3,0
- Stairs	<u>2,0</u> to 4,0	<u>2,0</u> to 4,0
- Balconies	<u>2,5</u> to 4,0	<u>2,0</u> to 3,0
Category B	2,0 to <u>3,0</u>	1,5 to <u>4,5</u>
Category C		
- C1	2,0 to <u>3,0</u>	3,0 to <u>4,0</u>
- C2	3,0 to <u>4,0</u>	2,5 to 7,0 (<u>4,0</u>)
- C3	3,0 to <u>5,0</u>	<u>4,0</u> to 7,0
- C4	4,5 to <u>5,0</u>	3,5 to <u>7,0</u>
- C5	<u>5,0</u> to 7,5	3,5 to <u>4,5</u>
category D		
- D1	<u>4,0</u> to 5,0	3,5 to 7,0 (<u>4,0</u>)
- D2	4,0 to <u>5,0</u>	3,5 to <u>7,0</u>

According to the Joint Committee of Structural Safety – JCSS (2001), the live loads on floors in buildings are caused by the weight of furniture, equipment, stored objects and persons. The live load is distinguished according the intended user category of the building (e.g. domestic buildings, hospitals, etc). At design stage considerations should also be given to eventual changes of use during the life time. Live loads vary in time and in space in a random manner. With respect to the variation in time, live loads are divided in two components, sustained load and intermittent load. A description of such load types is similar to the one made before (EC1-2, 1994; EC1-1, 2002).

A stochastic model for spatial variation is so defined for live load. The load intensity is so represented by a stochastic field $W(x,y)$, whereby the parameters depend on the user category of the building.

$$W(x,y) = m + V + U(x,y) \quad (2.56)$$

where m is the overall mean load intensity for a particular user category [kN/m^2], m_q for the sustained load and m_p for the intermittent load, V is a zero mean normal distributed variable [kN/m^2] and $U(x,y)$ is a zero mean random field [kN/m^2] with a characteristic skewness to the right. The quantities V and U are assumed to be stochastically independent.

The model load effects shall describe the load effects caused by real load with sufficient accuracy. For linear elastic systems, where superposition is possible, the load effect S [kN/m^2] is written as:

$$S = \int_A W(x,y) \cdot i(x,y) dA \quad (2.57)$$

where $W(x,y)$ is the load intensity [kN/m^2] and $i(x,y)$ is the influence function for the load effect over a considered area A . An equivalent uniformly distributed load for the sustained load (q) per unit area [kN/m^2] is that load having the same effect as the original load field, i.e:

$$q = \frac{\int_A W(x, y) \cdot i(x, y) dA}{\int_A i(x, y) dA} \quad (2.58)$$

The statistical parameters of the sustained load are:

$$E[q] = m \quad (2.59)$$

$$Var[q] = \sigma_V^2 + \sigma_U^2 \cdot \frac{A_0}{A} \cdot \kappa \quad (2.60)$$

being A the loaded area [m^2], A_0 a reference loaded area [m^2], σ_V and σ_U the standard deviation of, respectively, the zero mean normal distributed variable [kN/m^2] and of the zero mean random field [kN/m^2] and κ is a factor which depends of the influence line $i(x, y)$ shape (Figure 2.18). Note that $\kappa=1.0$ corresponds to a constant value of $i(x, y)$.

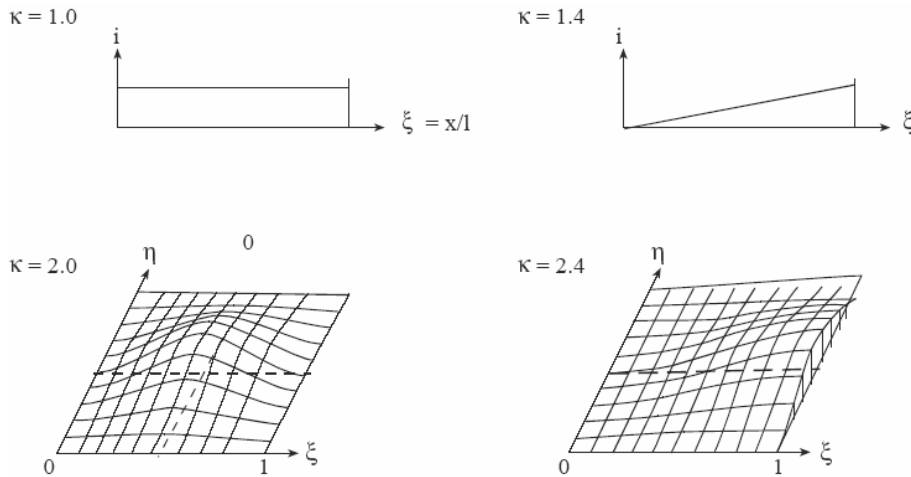


Figure 2.18 – Random fields and corresponding κ values (JCSS, 2001).

The stochastic distribution of variable V is assumed to be normal. The random field $U(x, y)$ has a specific skewness to the right, and, in consequence, also the load effect S and the sustained load q . A Gamma distribution for the sustained load fits best the actual observations, in accordance to JCSS (2001), with parameters defined through the relations $E[q] = k/\mu_U$ and $Var[q] = k/\mu_U^2$.

The load intensity for the intermittent load p [kN/m²] is represented by the same stochastic field as the sustained load, whereby the parameters depend on the user category of the building. The intermittent load can generally be considered as concentrated load. But, for design purposes, the same approach as for the sustained load is used. The intermittent load duration d_p [Years] is considered to be deterministic.

The equivalent uniformly distributed load for intermittent loads p has the statistical properties as the sustained load and can be evaluated in the same manner. Generally, there is a lack of data for this load and so the standard deviation normally gets values in the same magnitude as the mean value $E[p] = \mu_p$. Therefore, the intermittent load is assumed to be exponentially distributed.

For time variations, JCSS (2001) defines also a specific model. The time between load changes is assumed to be exponentially distributed, and then the number of load changes is Poisson distributed. The probability function for the maximum sustained load is given by:

$$F_{q_{\max}}(x) = \exp[-\lambda T(1 - F_q(x))] \quad (2.61)$$

Where $F_q(x)$ is the probability function of the sustained load, T is the reference time [Years], like the anticipated lifetime of the building, and λ is the occurrence rate of sustained load changes [1/Years], being $1/\lambda$ the occurrence time [Years]. Thus λT is the mean value of the structural occupancy change.

The maximum of the intermittent load is defined to occur as a Poisson process in time with the mean occurrence rate ν [1/Years], being $1/\nu$ the occurrence time [Years]. The average duration of the intermittent load depends on the process (e.g. personal, emergency, etc).

The maximum load which will occur in a building is a combination of sustained load and intermittent load. Assuming a stochastic independence between both load types, the maximum obtained load during one occupancy is obtained from the convolution integral. The total maximum load during the reference time T is obtained by employing the extreme value theory. A list of the previously defined parameters, to be used in live load model of JCSS (2001), is presented in Table 2.24.

Table 2.24 – Parameters for live loads depending on the user category.

Type of use	A_0 [m ²]	Sustained Load				Intermittent Load			
		m_q [kN/m ²]	σ_v [kN/m ²]	σ_u [kN/m ²]	$1/\lambda$ [Years]	m_p [kN/m ²]	σ_u [kN/m ²]	$1/v$ [Years]	d_p [Years]
Office	20	0.5	0.3	0.6	5	0.2	0.4	0.3	1 – 3
Lobby	20	0.2	0.15	0.3	10	0.4	0.6	1.0	1 – 3
Residence	20	0.3	0.15	0.3	7	0.3	0.4	1.0	1 – 3
Hotel guest room	20	0.3	0.05	0.1	10	0.2	0.4	0.1	1 – 3
Patient room	20	0.4	0.3	0.6	5 – 10	0.2	0.4	1.0	1 – 3
Laboratory	20	0.7	0.4	0.8	5 – 10				
Libraries	20	1.7	0.5	1.0	> 10				
School classroom	100	0.6	0.15	0.4	> 10	0.5	1.4	0.3	1 – 5
Merchant / retail: first floor upper floor	100	0.9	0.6	1.6	1 – 5	0.4	1.1	1.0	1 – 14
	100	0.9	0.6	1.6	1 – 5	0.4	1.1	1.0	1 – 14
Storage	100	3.5	2.5	6.9	0.1 – 10				
Industrial: Light Heavy	100	1.0	1.0	2.8	5 – 10				
	100	3.0	1.5	4.1	5 – 10				
Concentration of people	20					1.25	2.5	0.02	0.5

2.5. CONCLUSIONS

2.5.1. OVERVIEW

In this chapter it was firstly defined the uncertainty classification, namely uncertainty in “abstracted” (“non-cognitive” and “cognitive”), in “non-abstracted” and in unknown aspects of a system and the necessary measures to diminish it. One other uncertainty classification type, according to Henriques, 1998, is also characterized.

The second point focuses the basis of statistical procedures for uncertainty analysis. It is described the probability density function (PDF), and cumulative distribution function (CDF), of discrete and continuous random variables. The main concepts of typical parameters which describe a probability distribution, like mean value, variance, standard deviation, coefficient of variation and the bias estimator, are then defined. A definition of the most important discrete (uniform, Bernoulli and binomial) and continuous (uniform, normal and lognormal) distributions is then realized. Some techniques to determine the distribution that best fits a set of observation data are presented next. The definitions of the main parameters that characterize jointly distributed random variables, like covariance, are described. The case of functions with more than one random variable, common in most of the civil engineering problems, is also focused.

Subsequently the theme of uncertainty and codes is referred. It is always important to focus how codes describe and define the random parameters which are mostly used in civil engineering field. The applied and suggested models, which include the definition of each parameter probabilistic distribution, are then explained. It is described the mainly numerical models and how codes consider it for materials (concrete and steel), geometry and applied loads (self-weight and live loads) uncertainty analysis.

At this point it is necessary to refer that in this thesis examples, a uniform continuous distribution was adopted for obtained sensor data, in the case of structural assessment methodologies, and normal distributions were considered to define all numerical model uncertainty input parameters.

2.5.2. UNCERTAINTY AND SAFETY

The safety design of any structure, having into account the respective different limit states and the intervenient variables uncertainty, may be considered as a decision process. There exist 4 levels of safety design. Level 0: Pure deterministic analysis. The variables involved in the design procedure present deterministic values, being uncertainties considered via global security coefficients, estimated in an empiric way; Level 1: Semi-probabilistic methods. The variability present in loads and material properties is considered through representative values (nominal or characteristic) related to partial security coefficients (γ). Characteristic values are defined from the average values, the variation coefficients (or standard deviation coefficients) and from distribution function; Level 2: Probabilistic methods. Based on the characterization of all variables that interfere in the design process, through statistical measures that describe the central tendency and their dispersion, and in the calculus of a probability of a specified limit state of being reached. The security probabilistic evaluation is performed by approximate techniques; Level 3: Pure probabilistic methods. Based on techniques that have into account the joined distribution of all variables. The probability of a specified limit state of being reached is analytically calculated or, using simulation methods.

The safety criterion, used in mostly civil engineering codes is based on a Level 1 design, and may be expressed by the expression $S_d \leq R_d$ where S is the stress applied to a structural element and R the strength of such element. Figure 2.19 presents such criterion, being μ_S and μ_R the mean values, R_k and S_k the characteristic values and $S_d = S_k / \gamma_S$ and $R_d = R_k / \gamma_R$ the design values.

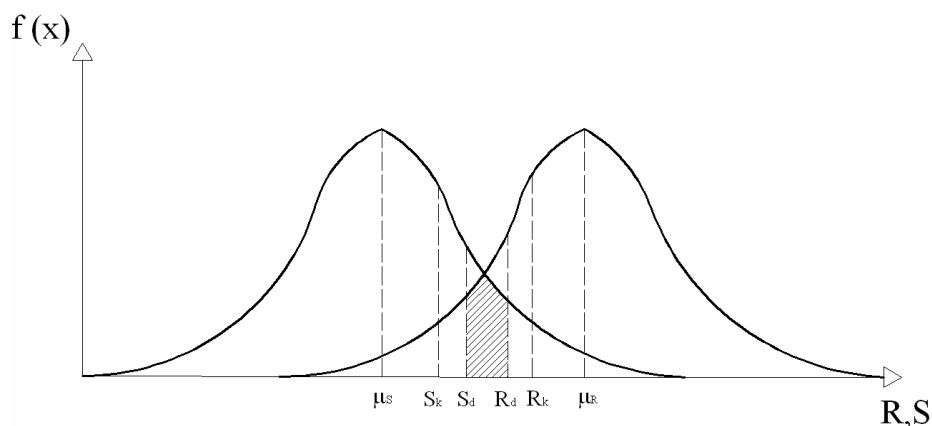


Figure 2.19 – Safety criterion used in most of civil engineering codes.

CHAPTER 3.

NUMERICAL METHODOLOGIES

3.1. INTRODUCTION

3.1.1. GENERAL CONSIDERATIONS

In this chapter it will be presented the formulation of three techniques for the analysis of uncertainty in structural engineering systems, namely, the Monte Carlo analysis, the Perturbation technique and the Modal Interval Analysis (MIA). The introduction of such methodologies into a finite element tool is also described.

Within the field of simulation techniques, the most used one is the Monte Carlo analysis (Rubinstein, 1981). A high amount of samples are needed to evaluate, in an accurate way, the structural response using these techniques. Alternatively, variance reduction techniques, like Latin Hypercube Sampling (Florian, 1992; Olsson *et al.* 2003) are commonly used, to turn the Monte Carlo analysis more efficient.

Perturbation technique consist in the application of the development of first or second order Taylor series of equations, that rule the problem in terms of finite elements (Eibl and Schmidt-Hurtienne, 1995). The structural behaviour is characterized by the mean value and standard deviation of aleatory basic variables. Those techniques are applied into a finite element framework with the name of probabilistic finite element methods (PFEM) (Liu *et al.*, 1988).

The initial idea of interval analysis (Moore, 1966; Neumaier, 1990) is to enclose real numbers in intervals and real vectors in boxes as a method of considering the imprecision of representing real numbers by finite digits in numerical computers. The variables are not deterministic, but taking any value between a lower and an upper limit of an interval. Interval analysis has become a fundamental nonlinear numerical tool for representing uncertainties or errors, proving sets properties, solving sets of equations or inequalities and optimizing globally via interval arithmetic (Hansen, 1992; Jaulin *et al.*, 2001). Modal Interval Analysis (MIA) is a natural extension of Classical Interval Analysis, where the interval concept is widened by the set of predicates that are fulfilled by real numbers (Gardenyes *et al.*, 2001; SIGLA/X group, 1999).

3.1.2. METHODOLOGIES FOR MODELLING UNCERTAINTY

According to Oberkampf *et al.* terminology (1999) and mentioned by Moens and Vandepitte (2004), uncertainty is defined as a deficiency that may or may not occur in a stage of modelling process due to lack of knowledge. Uncertainties are within the basic common problems that engineers have to face when modelling structures. They arise due to mismatches and inaccuracies in physical and geometrical parameters, such as loading, Young modulus, Poisson ratio, length, etc. There are different techniques to deal with uncertainties and randomness (Altus *et al.*, 2005; Ayyub and Gupta, 1997; Biondini *et al.*, 2004; Hurtado, 2002; Klir, 1995; Maglaras *et al.*, 1997; Muhanna and Mullen, 2001; Zimmermann, 2000).

Techniques employing simulation procedures, such as crude Monte Carlo method, have high computational cost in large structural systems even if the computational efficiency is implemented with variance reduction techniques (Mahadevan and Raghothamachar, 2000; Olsson *et al.*, 2003; Rais-Rohani and Singh, 2004; Schueller, 2001).

Reliability techniques, such as FORM (First Order Reliability Methods) and SORM (Second Order Reliability Methods), are essentially used to handle with implicit formulations like the implementation of limit state functions derivatives (Frangopol *et al.*, 1996; Guan and Melchers, 1994; Liu and Der Kiureghian, 1991). However, the problem of efficiency arises due also to the necessity of performing various structural analyses to deal with uncertainty. In order to avoid the efficiency problem, new methodologies start to be applied.

One of such methodologies is the Perturbation Technique, essentially used to evaluate the uncertainty of structural response. The present procedure evaluates the mean response and its standard deviation in only one structural analysis. The obtained results are accurate for linear problems and normal or quasi-normal distributed random variables (Eibl and Schmidt-Hurtienne, 1995).

Lately, several formulations have been proposed to evaluate efficiently the uncertainty of input variables on responses, in different engineering fields, such as, civil, mechanical and aerospace engineering (DeLaurentis and Marvris, 2000; Du and Sudjianto, 2004; Haftka *et al.*, 2006; Huang and Xiaoping, 2006; Rao and Sawyer, 1995; Zou and Mahadevan, 2006). One of them is the Neural Network techniques which have been applied as an alternative methodology (Ayyub and Gupta, 1997; Hurtado,

2002; Papadrakakis and Lagaros, 2002). Another formulation is the Set Theory that embraces the Probability Theory (Bier, 1990), the Fuzzy-Set Theory (Zadeh, 1965), the Possibility Theory (Cayrac *et al.*, 1996), the Rough Sets Theory (Pawlak, 1995; Pawlak, 1996), the Random Sets Theory (Pownuk, 2001) and the Interval Analysis Theory (Moore, 1966).

The Probability Theory is one of the mathematical theories used to characterize uncertainty, due to the impossibility of scientists to access all the issues involved in natural phenomenon. Fuzzy set theory was originally described by Zadeh (1965) as a class with a continuum of grades of membership. Fuzzy set theory was firstly introduced to model uncertainty in subjective information. Fuzzy set has been used in the treatment of uncertainties in structural problems (Abdel-Tawab and Noor, 1999; Mullen and Muhanna, 1999). Possibility Theory is a model that aims at quantifying degrees of possibility and degrees of necessity (Cayrac *et al.*, 1996). Rough Sets Theory is involved in the analysis of imprecise, uncertain or incomplete information or knowledge expressed in terms of data obtained from experience (Pawlak, 1995; Pawlak, 1996). Random Sets Theory has been used for the modeling of uncertain parameters (Pownuk, 2001) in the computation of reliability of structures.

Recently, Interval Analysis Theory has been applied to evaluate uncertainties in structural models (Garcia *et al.*, 2004; Gardenyes *et al.*, 2001; Hansen, 1992; Jaulin *et al.*, 2001; Rao and Berke, 1997). Interval arithmetic is used as a feasible tool for the representation of uncertain, inexact data and inexact computations. The way to represent uncertain values is through intervals, which contains the unknown value of the number in question. Interval arithmetic was first introduced by Moore (1966), recently the use of interval arithmetic has been growing gradually. Kulpa *et al.* (1998) applied interval methods in the analysis of linear mechanical systems with uncertain parameters. Rao and Berke (1997) and Rao and Chen (1998) presented different approaches of interval methods based on the finite element techniques for modelling uncertainty in engineering problems.

Recently a new technique, Modal Interval Analysis (MIA) based in the already developed interval analysis methodology (Moore, 1966; Neumaier, 1990) was developed by SIGLA/X group (1999). In such technique the interval concept is widened by the set of predicates that are fulfilled by the real numbers (Gardenyes *et al.*, 2001).

3.2. SIMULATION AND THE MONTE CARLO METHOD

3.2.1. BASIC CONCEPT

Several techniques may be used to solve structural reliability problems. Simulation techniques are one possible way to solve such problems (Ang and Tang, 1984; Ayyub and McCuen, 1997; Marek *et al.*, 1996; Nowak and Collins, 2000, Ross, 1997; Rubinstein, 1981; Singh *et al.*, 2007). The basic idea behind simulation is, as the name implies, to numerically simulate some phenomenon and then observe the number of times some event of interest occurs. The basic concept behind simulation is relatively straightforward, but the procedure can become computationally intensive.

Assume we have information from N tests, and assume that we put all N test results in a bag as shown in Figure 3.1. Now suppose that we need a sample of n results. Instead of doing n additional tests, we could randomly select n of the N test results from the bag. In Figure 3.1, this sampling technique is referred to as the “special technique”.

The Monte Carlo method is a special technique that we can use to generate some results numerically without actually doing any physical testing. We can use results of previous tests to establish the probability distributions of the important parameters in our problem. Then we use this distribution information to generate samples of data.

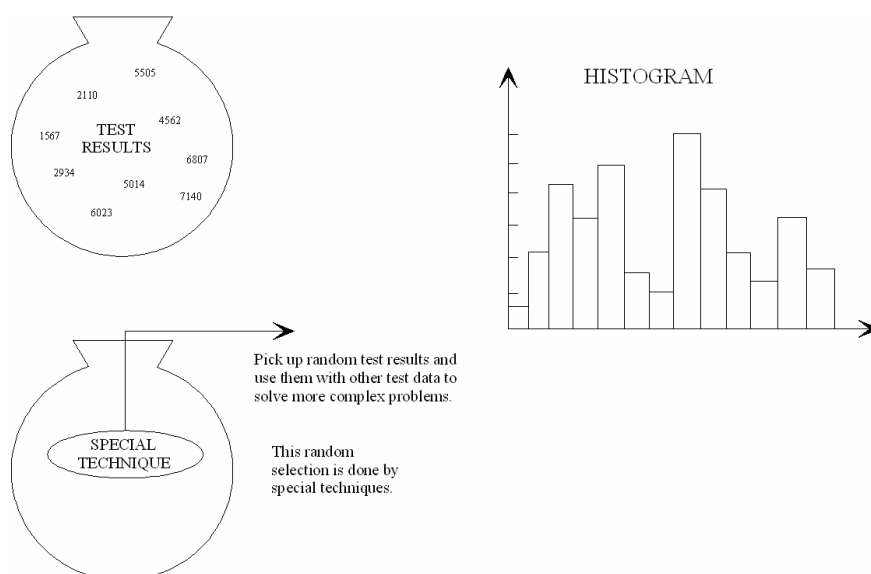


Figure 3.1 – Schematic of the Monte Carlo method.

The Monte Carlo method is often applied in three situations:

- It is used to solve complex problems for which closed-form solutions are either not possible or extremely difficult. For example, probabilistic problems involving complicated nonlinear finite element models can be solved by Monte Carlo simulation provided that necessary computing power is available and the required information is known;
- It is used to solve complex problems that can be solved (at least approximately) in closed form if many simplifying assumptions are made. By using Monte Carlo simulation, the “original” problem can be studied without these assumptions, and more realistic results are obtained;
- It is used to check the results of other solution techniques.

3.2.2. UNIFORMLY DISTRIBUTED RANDOM NUMBERS

The basis of all Monte Carlo simulation procedures is the generation of random numbers that are uniformly distributed between 0 and 1. Tables of randomly generated numbers are available (Rand Corporation, 1955), as are computer subroutines, and many popular mathematical programs have such subroutines built in. Table 3.1 is an example of a table of uniform random variables that was generated using a standard spreadsheet program.

Once we have some realizations u of a uniformly distributed random number U between 0 and 1, we can generate realizations x of a uniformly distributed random number X between any two values a and b ($a \leq x \leq b$), using the following formula:

$$x = a + (b - a) \cdot u \quad (3.1)$$

We can also generate sample values i for a uniformly distributed random integer i between two integer values a and b (including the values a and b) using the following formula:

$$i = a + \text{TRUNC}[(b - a + 1) \cdot u] \quad (3.2)$$

where $\text{TRUNC}[]$ denotes a function which truncates its argument (i.e., removes the fractional part and returns the integer part of a real number).

Before leaving this topic, two comments are in order. First, most (if not all) random number generators require the user to input a “seed” value. This number is an integer that is used by the routine to start the simulation algorithm. By choosing a different seed, you can generate a different set of uniformly distributed numbers. In general, if you use the same seed over and over again, you will generate the same set of uniformly distributed random numbers over and over again. Second, the built-in generators found in many software packages should be used with caution. Some random number generation algorithms work better than others (Press *et al.*, 1992).

Table 3.1 – Simulated values of a uniformly distributed random variable (values between 0 and 1).

0.050203	0.048524	0.046113	0.588458	0.736930	0.184576	0.648091	0.345592
0.619129	0.279794	0.035615	0.509049	0.660878	0.847652	0.868068	0.856594
0.872402	0.176305	0.775964	0.419080	0.468429	0.757653	0.662557	0.759972
0.376568	0.848170	0.609577	0.895810	0.844203	0.610858	0.901730	0.001984
0.139927	0.851497	0.901547	0.678549	0.211646	0.959258	0.858760	0.135502
0.318491	0.975768	0.277444	0.691275	0.454543	0.716910	0.836543	0.507309
0.987671	0.127567	0.003204	0.071810	0.701315	0.955016	0.238868	0.521622
0.033265	0.816553	0.938719	0.669546	0.501053	0.449171	0.877651	0.249245
0.234626	0.661519	0.273812	0.171331	0.582873	0.616138	0.364849	0.775719
0.623157	0.855403	0.903897	0.741417	0.090884	0.842067	0.369976	0.505203
0.964171	0.194403	0.427198	0.423383	0.216071	0.458052	0.422681	0.579150
0.537767	0.594501	0.157109	0.845119	0.663900	0.806421	0.155614	0.503128
0.325022	0.105197	0.946318	0.744987	0.037202	0.439344	0.497818	0.994446
0.323160	0.052919	0.506821	0.346507	0.705832	0.205359	0.705191	0.128910
0.096225	0.431745	0.946715	0.749199	0.017487	0.316660	0.310465	0.206366
0.432936	0.144658	0.045412	0.086886	0.580767	0.043367	0.524674	0.112644
0.079775	0.412336	0.645955	0.434034	0.544053	0.315592	0.154698	0.733207
0.794885	0.026246	0.198859	0.819300	0.751122	0.738884	0.858364	0.087466
0.303201	0.524949	0.843379	0.361492	0.170690	0.457717	0.231391	0.175146
0.455580	0.733726	0.399823	0.206885	0.294961	0.709799	0.365093	0.565569
0.551683	0.314646	0.560930	0.056917	0.756951	0.104068	0.591601	0.619190
0.347545	0.484848	0.185430	0.441237	0.414838	0.840236	0.370464	0.728416
0.976501	0.131230	0.691031	0.705466	0.164586	0.263100	0.134892	0.255898
0.732902	0.533006	0.429029	0.500748	0.680074	0.436018	0.610126	0.611408
0.955382	0.368816	0.774499	0.332865	0.534440	0.086184	0.163640	0.720725

3.2.3. NORMAL DISTRIBUTED RANDOM NUMBERS

Since the normal probability distribution plays such an important role in structural reliability analysis, the capability to simulate normally distributed random variables is important. To begin, consider a standard normal distribution. To generate a set of standard normal random numbers $z_1 \dots z_n$, we first need to generate a

corresponding set of uniformly distributed random variables $u_1 \dots u_n$, between 0 and 1. Then, for each u_i , we can generate a value z_i using:

$$z_i = \Phi^{-1}(u_i) \quad (3.3)$$

where Φ^{-1} is the inverse of the standard normal cumulative distribution function. Figure 3.2 shows this relationship graphically. Many computer programs have a standard normal random number generator built in.

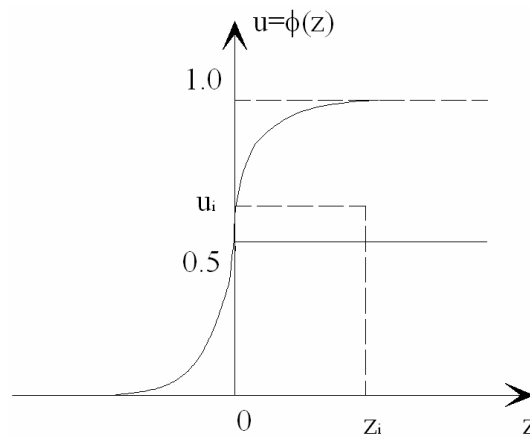


Figure 3.2 – Generation of standard normal random variables.

The anterior procedure is applied for standard normal random numbers but it is possible to generalize it to all normal random numbers. Assuming X as a normally distributed random variable with mean μ_X and standard deviation σ_X . The basic relationship between X and the standard normal variable Z is:

$$X = \mu_X + Z \cdot \sigma_X \quad (3.4)$$

So, given a sample value z_i generated using the approach presented before, the corresponding x_i value can be calculated using:

$$x_i = \mu_X + z_i \cdot \sigma_X \quad (3.5)$$

Till now it was only explained how to generate random numbers for well known distributions. However, there is a general procedure, applicable to any type of distribution function, which can be formulated theoretically.

Consider a random variable X with a cumulative distribution function $F_X(x)$. To generate sample values x_i for the random variable, the following steps can be taken:

- Generate a sample value u_i for a uniformly distributed random variable between 0 and 1;
- Calculate a sample value x_i from the following formula:

$$x_i = \mu_X + z_i \cdot \sigma_X \quad (3.6)$$

where F_X^{-1} is the inverse of $F(x)$. This is a completely general procedure. However it is, in some cases, difficult to determine a closed-form solution for the inverse cumulative distribution function.

3.2.4. CHARACTERISTICS OF MONTE CARLO SIMULATION

The Monte Carlo simulation techniques allow the numerical calculus of integrals with an impracticable analytical resolution using existing methods. In specialized literature, Monte Carlo results are presented in order to verify the rigor of the results obtained using other methodologies.

Monte Carlo method is of general application. It can be applied with all kind of aleatory distributions. The error associated to this kind of techniques is completely controlled through the simulations number. It is verified that, when the number of samples tend to infinite ($n \rightarrow \infty$) the results converges to the exact one. The uncertainty in the analysis decreases as the total number of simulations increase. One generalized critic of Monte Carlo method is the necessary high computational time. Although, the application of variance reduction techniques may turn this method more efficient. The Monte Carlo method does not present any restrictions (Henriques, 1998).

3.2.5. CORRELATED VARIABLES

Till now it were treated the problem of non-correlated variables simulation. At this point, we will analyze correlate variables, and will explain how the simulation

procedure must be able such correlation. A transformation technique for simulating correlated normal random variables is shown here. Although it is strictly valid for normal random variables only, it can be used for other types of random variables as an approximation.

Let $X_1 \dots X_n$ be correlated normal variables. The mean values and covariance matrix are given by:

$$\{\mu_X\} = \{\mu_{X_1} \dots \mu_{X_n}\} \quad (3.7)$$

$$[C_X] = \begin{bmatrix} CoV(X_1, X_1) & CoV(X_1, X_2) & \dots & CoV(X_1, X_n) \\ CoV(X_2, X_1) & CoV(X_2, X_2) & \dots & CoV(X_2, X_n) \\ \dots & \dots & \dots & \dots \\ CoV(X_n, X_1) & CoV(X_n, X_2) & \dots & CoV(X_n, X_n) \end{bmatrix} \quad (3.8)$$

To generate random numbers for $X_1 \dots X_n$ it is necessary to first generate a set of uncorrelated random numbers $Y_1 \dots Y_n$ using the techniques discussed earlier. Then $X_1 \dots X_n$ are calculated using the variable transformation:

$$\{X\} = [T] \cdot \{Y\} \quad (3.9)$$

where $[T]$ is a transformation matrix. To apply this approach, we needed to determine the matrix $[T]$ as well as the mean and the variance values for the uncorrelated Y_i variables. To do this, we must use some concepts from linear algebra, namely, eigenvalues and eigenvectors. Let $[A]$ be a symmetric $n \times n$ matrix. A diagonal matrix $[D]$ and a square matrix $[T]$ can be found such that the following relationship holds:

$$[D] = [T]^T \cdot [A] \cdot [T] \quad (3.10)$$

$$[A] = [T] \cdot [D] \cdot [T]^T \quad (3.11)$$

The superscript T denotes transpose. The matrix $[T]$ contains the orthonormal eigenvectors corresponding to the eigenvalues of the matrix $[A]$. The diagonal matrix $[D]$ contains the eigenvalues of $[A]$.

In the present context of simulating random variables, the matrix $[A]$ is the covariance matrix $[C_X]$ of the original, correlated variables $\{X\}$. The matrix $[T]$ is made up of the orthonormal eigenvectors corresponding to the eigenvalues of the matrix $[C_X]$. Thus, the first column of $[T]$ contains the orthonormal eigenvector corresponding to the first eigenvalues, the second column contains the eigenvector corresponding to the second eigenvalues, and so on. The matrix $[T]$ is an orthogonal matrix, meaning that its inverse is equal to its transpose. The diagonal matrix $[D]$ corresponds to the covariance matrix $[C_Y]$ of the uncorrelated variables $[Y]$. With these changes in notation, equation 3.10 and equation 3.11 become:

$$[C_Y] = [T]^T \cdot [C_X] \cdot [T] = \begin{bmatrix} \sigma_{Y_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{Y_2}^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_{Y_n}^2 \end{bmatrix} \quad (3.12)$$

$$[C_X] = [T] \cdot [C_Y] \cdot [T]^T \quad (3.13)$$

The diagonal elements of $[C_Y]$ contain the variances of the uncorrelated Y variables needed to do the simulation. The mean values of Y_i variables can be obtained using:

$$\{\mu_Y\} = [T]^T \cdot \{\mu_X\} \quad (3.14)$$

Once simulated values of $\{Y\}$ are obtained, equation 3.9 can be used to obtain simulated values of $\{X\}$.

3.2.6. LATIN HYPERCUBE SAMPLING

The technique of random sampling and Monte Carlo simulation discussed before are very powerful and useful techniques. However, in some instances, the problem being analyzed is extremely complex, and the time needed to evaluate the problem for single trial ($N = 1$) may be very long. As a result, the time needed to perform hundreds or thousands of simulations may be unfeasible.

The Latin hypercube is one technique for reducing the number of simulations needed to obtain reasonable results (Florian, 1992; Olsson *et al.*, 2003). In this method, the range of possible values of each random input variable is partitioned into “strata”, and a value from each stratum is randomly selected as a representative value. The representative values for each random variable are then combined so that each representative value is considered once and only once in the simulation process. In this way, all possible values of the random variables are represented in the simulation.

To be specific, let's assume that we need to simulate values of some function Y described by:

$$Y = f(X_1 \dots X_k) \quad (3.15)$$

where $f()$ is some deterministic function (but possibly not known in closed form) and the X_i ($i=1 \dots k$) are the random input variables. The basic steps in Latin hypercube sampling are as follows:

- Partition the range of each X_i into N intervals. The partitioning should be done so that the probability of a value of X_i occurring in each interval is $1/N$ (Iman and Conover 1980);
- For each X_i variable and each of its N intervals, randomly select a representative value for the interval. In practical applications, if the number of intervals is large, the center point of each interval can be used instead of doing random sampling;
- After steps 1 and 2, there will be N representative values for each of the K random variables. There are N^K possible combinations of these representative values. The objective of Latin hypercube sampling is to select N combinations such that each representative value appears once and only once in the N combinations;
- To obtain the first combination, randomly select one of the representative values for each of the K input random variables. To obtain the second combination, randomly select one of the $N-1$ remaining representative values of each random variable. To obtain the third combination, randomly select one of the $N-2$ remaining representative values of each random variable. Continue this selection process until you have N combinations of values of the input variables generated;
- Evaluate equation 3.15 for each of the N combinations of input variables generated above. This will lead to N values of the function. These values will be referred to as y_i ($i=1 \dots N$).

This procedure provides the simulation data. Now we must determine how to use the data to estimate statistical parameters for Y . The most common used formulas include the following:

$$\text{Estimated mean value of } Y = \tilde{Y} = \frac{1}{N} \sum_{i=1}^N y_i \quad (3.16)$$

$$\text{Estimated } m^{\text{th}} \text{ moment of } Y = \frac{1}{N} \sum_{i=1}^N (y_i)^m \quad (3.17)$$

$$\text{Estimated CDF } F_y(y) = \frac{\text{number of times } y_i \leq y}{N} \quad (3.18)$$

3.3. PERTURBATION METHOD

The perturbation method is applied in a finite element formulation level following the steps of a deterministic analysis (Reh *et al.*, 2006; Spanos and Ghanem, 1989; Thomos and Trezos, 2001). The method is based on Taylor series expansion of the governing equations. The uncertainty of structural behaviour is evaluated by taking into account terms around mean values of the basic random variables. Mean response and its variance can be found in terms of mean and variance of the basic random variables and their correlation (Altus *et al.*, 2005; Contreras, 1980; Zhang and Ellingwood, 1996).

Such method consists in the following phases (Henriques, 1998):

- Aleatory variables characterization (material mechanical properties, geometric imperfections and actions) through the mean values, standard deviations and correlation between variables;
- Obtaining the mean structural response through a deterministic analysis, having into account the mean values of the basic aleatory variables;
- Obtaining the covariance matrix of the variables that characterize the structural response.

The well known finite element equation to state the system equilibrium can be defined as:

$$K \cdot U = F \cdot \phi \quad (3.19)$$

where K is the stiffness matrix, U is the nodal displacements vector, $F\phi$ stands for the nodal force vector that represents external actions, being F the load value and $\Phi = [\Phi_1 \dots \Phi_n]$ the load distribution vector along the structure with n degrees of freedom. In a load-test, the load value F increases successively until the studied limit state is achieved, but the load distribution vector Φ remains constant.

When a perturbation δ is applied to the balanced system, the equilibrium can be redefined by the following equation:

$$(K_0 + \delta K) \cdot (u_0 + \delta u) = (F_0 + \delta F) \cdot \phi \quad (3.20)$$

In which variables with indexes 0 represent the central values of their distribution (generally the mean values) and variables with δ -sign stand for perturbations around those central values.

Taking into account that $K_0 \cdot u_0 = F_0 \cdot \Phi$ and neglecting the second order term, the following expression is obtained:

$$K_0 \cdot \delta u + \delta K \cdot u_0 = \delta F \cdot \phi \quad (3.21)$$

The uncertainty of structural response can be defined (Figure 3.3): (a) in terms of forces, for a fixed maximum deformation (or displacement); or, (b) in terms of displacements, for a pre-defined load level. The following sections present the evaluation of uncertainty by perturbation techniques for each case.

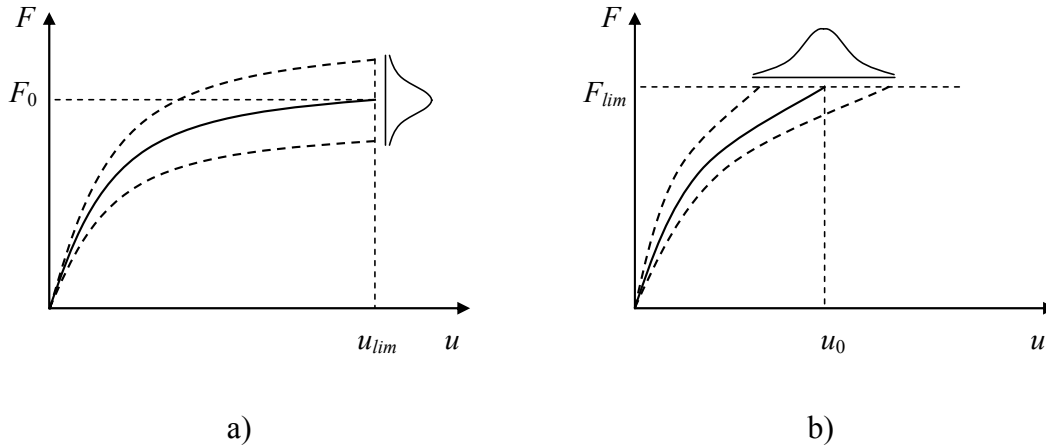


Figure 3.3 – Definition of response uncertainty: (a) in terms of forces; (b) in terms of displacements.

3.3.1. UNCERTAINTY IN TERMS OF FORCES

The uncertainty of structural response in terms of forces should be evaluated at the i -point where the displacement is maximum and, as a consequence, the following condition is set: $\delta u_{i,\max} = 0$. In this way, (equation 3.21) can be defined by:

$$\delta K \cdot u_0 = \delta F_{i,\max} \cdot \phi - K_0 \cdot \delta u \Big|_{\delta u_{i,\max}=0} \quad (3.22)$$

or by the reduced equation:

$$\delta K \cdot u_0 = -K_M \cdot \delta q \quad (3.23)$$

where vector δq and matrix K_M are defined, respectively, by (Eibl and Schmidt-Hurtienne, 1995):

$$\delta q = \left\{ \delta u_1 \quad \dots \quad \delta u_{i-1} \quad \delta F_{i,\max} \quad \delta u_{i+1} \quad \dots \quad \delta u_n \right\}^T \quad (3.24)$$

and,

$$K_M = \begin{bmatrix} k_{11} & \dots & k_{1i-1} & -\Phi_1 & k_{1i+1} & \dots & k_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_{i-11} & \dots & k_{i-1i-1} & -\Phi_{i-1} & k_{i-1i+1} & \dots & k_{i-1n} \\ k_{i1} & \dots & k_{ii-1} & -\Phi_i & k_{ii+1} & \dots & k_{in} \\ k_{i+11} & \dots & k_{i+1i-1} & -\Phi_{i+1} & k_{i+1i+1} & \dots & k_{i+1n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_{n1} & \dots & k_{ni-1} & -\Phi_n & k_{ni+1} & \dots & k_{nn} \end{bmatrix} \quad (3.25)$$

Equating 3.23 in order to δq , it follows:

$$\delta q = -K_M^{-1} \cdot \delta K \cdot u_0 \quad (3.26)$$

The non deterministic nature of the structural design parameters is defined by random variables denoted by x . The covariance matrix of vector q is written as follows:

$$C_q = \frac{\partial q}{\partial x} \cdot C_x \cdot \left(\frac{\partial q}{\partial x} \right)^T \quad (3.27)$$

where C_x is the covariance matrix of random variables x , $\partial q / \partial x$ stands for the partial derivatives of system variables q with respect to the random variables x . The covariance matrix C_x is expressed by the standard deviations δx_i and the correlations ρ_{ij} between the m random variables x_i and x_j (where $i, j = 1 \dots m$):

$$C_x = \delta x \cdot C_p \cdot \delta x^T \quad (3.28)$$

being C_p the correlation matrix of random variables x and δx the vector containing the standard deviations of random variables x . Taking into consideration that the deviation of structural stiffness, δK , defined in equation 3.26, results from the dispersion of random variables x , the deviation of structural response, δq , can be defined by:

$$\delta q = \frac{\partial q}{\partial x} \cdot \delta x = -K_M^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x \quad (3.29)$$

Introducing equation 3.28 and equation 3.29 in equation 3.27, the following expression is defined to compute the covariance matrix of structural response:

$$C_q = \left(K_M^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x \right) \cdot C_p \cdot \left(K_M^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x \right)^T \quad (3.30)$$

In this manner, the uncertainty of structural response, in terms of forces, is defined by the covariance matrix C_q which is evaluated taking into account mean values x_i , standard deviations δx_i and correlations ρ_{ij} ($i, j = 1 \dots m$) between basic random variables.

3.3.2. UNCERTAINTY IN TERMS OF DISPLACEMENTS

The uncertainty of structural response in terms of displacements can be evaluated by the relation:

$$\delta u = -K_0^{-1} \cdot \delta K \cdot u_0 + K_0^{-1} \cdot \delta F \cdot F \quad (3.31)$$

The covariance matrix of displacements, C_u , is calculated by:

$$C_u = \frac{\partial u}{\partial x} \cdot C_x \cdot \left(\frac{\partial u}{\partial x} \right)^T \quad (3.32)$$

where $\partial u / \partial x$ stands for the partial derivatives of displacements u with respect to the random variables x and C_x is the covariance matrix of random variables x defined in equation 3.28 as a function of standard deviations and the correlations between the basic random variables. The deviations of structural stiffness, δK , and forces, δF , result from the dispersion of random variables x . Therefore, the uncertainty of structural response in terms of displacements can be defined by:

$$\delta u = \frac{\partial u}{\partial x} \cdot \delta x = -K_0^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x + K_0^{-1} \cdot \frac{\partial F}{\partial x} \cdot \Phi \cdot \delta x \quad (3.33)$$

When equation 3.32 and equation 3.33 are considered in equation 3.31, the covariance matrix of displacements, C_u , become defined as:

$$C_u = \left(-K_0^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x + K_0^{-1} \cdot \frac{\partial F}{\partial x} \cdot \Phi \cdot \delta x \right) \cdot C_\rho \cdot \left(-K_0^{-1} \cdot \frac{\partial K}{\partial x} \cdot u_0 \cdot \delta x + K_0^{-1} \cdot \frac{\partial F}{\partial x} \cdot \Phi \cdot \delta x \right)^T \quad (3.34)$$

By this way, the uncertainty of structural response in terms of displacements is defined as a function of the mean values, standard deviations of basic random variables and correlations between them.

3.3.3. COMPUTATIONAL APPROACH

The application of present method in a finite element framework should have in concern the following aspects: random field characterized by random variables (mean values, standard deviations and correlations); evaluation of mean structural response by a deterministic analysis, using mean values for the random variables; evaluation of covariance matrix related to structural response variables (displacements or forces) and, as a consequence, their standard deviations.

The implementation to current finite element computational programs requires the addition of new modulus to evaluate the covariance matrix. This implementation is performed by incorporating new instructions to compute the inverse of stiffness matrix, K_0^{-1} (or K_M^{-1}), and the partial derivatives of the stiffness matrix, $\partial K / \partial x$, and nodal forces vector, $\partial F \Phi / \partial x$. Generally, the stiffness matrix K and the nodal forces $F \Phi$ are obtained by integration. Therefore the analytical calculation of the partial derivatives is

not available. In these cases, numerical evaluation is performed, such as finite differentiation.

An algorithm coupled to a finite element program can be divided in the following steps (Figure 3.4):

1. Read data of structural problem: geometry of the finite element mesh, material properties and applied actions. Read data to characterize random variables: mean values, standard deviations and correlation coefficients;
2. Evaluate nodal forces $F_0\Phi$ corresponding to the actions previously defined;
3. Calculate tangent stiffness matrix K_0 ;
4. Compute the displacements vector u_0 by solving the equation: $K_0 \cdot u_0 = F_0\Phi$;
5. Depending on the option chosen to evaluate the response uncertainty, compute the inverse of global stiffness matrix K_0^{-1} or define matrix K_M and compute its inverse K_M^{-1} ;
6. Calculate partial derivatives $\partial K/\partial X$ of stiffness matrix and partial derivatives $\partial F_0\Phi/\partial x$ of forces, with respect to random variables x ;
7. Depending on the previous option, compute covariance matrix C_q , according to equation 3.30, or compute displacements covariance matrix C_u according to equation 3.34;
8. Calculate standard deviation of structural response from the diagonal elements of covariance matrix.

This algorithm can also be applied in a non-linear program of structural analysis that uses finite element techniques. The non-linear structural behaviour is analysed by an incremental and iterative procedure where structural response is a result of actions applied by increments, and, equilibrium is computed iteratively by successive linear steps (tangent stiffness matrices) according to the Newton-Raphson method (Henriques, 1998). The application of this algorithm consists to repeat steps 3 and 4 for each increment of action and for each iteration, until the external actions and internal forces are balanced.

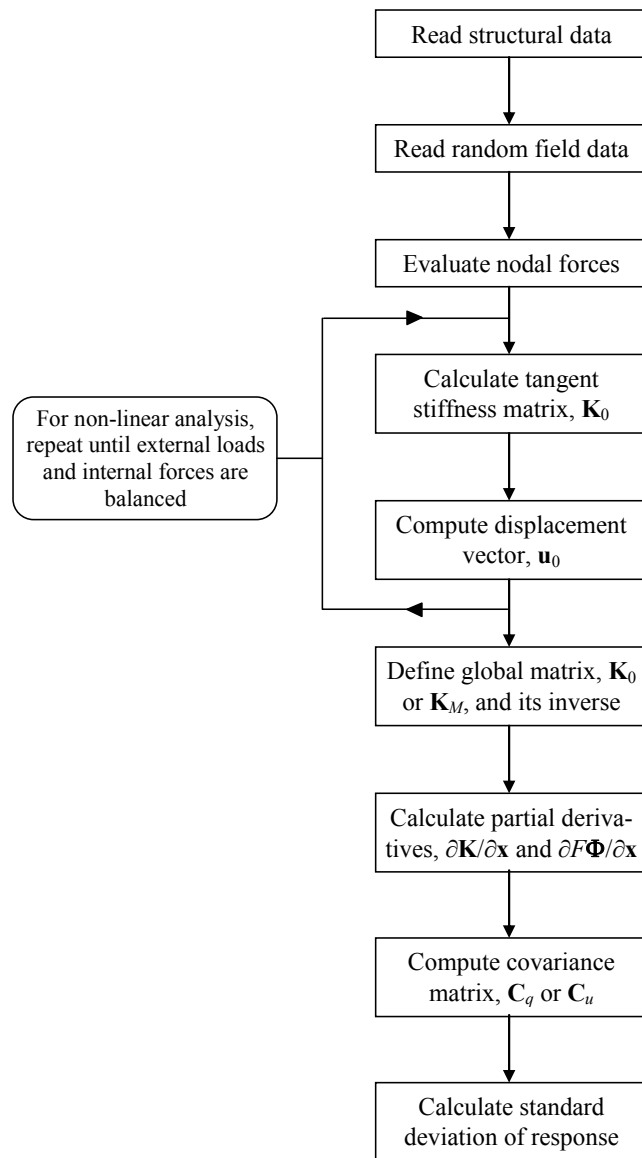


Figure 3.4 – Algorithm for the finite element program.

3.4. INTERVAL METHOD

3.4.1. INTERVAL ARITHMETIC

In the following chapter it will be described the Interval Arithmetic, namely, the Modal Interval Analysis (MIA) methodology. It will be a detailed description, including some illustrative examples, as it is one of the newest methodologies for uncertainty treatment which was specially studied by the author. Such methodology was developed within the SIGLA/X group in Gerona University, Spain (SIGLA/X group, 1996; SIGLA/X group, 1997; SIGLA/X group, 1998a; SIGLA/X group, 1998b; SIGLA/X group, 1998c; SIGLA/X group, 1999).

Numerical results are exact only to a limited extent and they bear a certain amount of numerical error. Numerical results other than those carried by small integer numbers, are only estimated or approximated results of some measurement or computation. Some times a single number seems enough to carry a determinate numerical information, because it is known beforehand what kind of measuring or computing process it comes from and what its corresponding spread of error is but, in general, single numbers are unable to usefully represent numerical information. For example, it is hardly acceptable to guess from the symbol 15 which of the information $15 \pm 0.1 = [14.9, 15.1]$ or $15 \pm 5.0 = [10.0, 20.0]$ is meant. The only way to indicate "the spread of a numerical result" is by pointing to a lower and an upper limit of its possible values, maybe through a notation like 15 ± 0.1 or 15 ± 5.0 , or maybe thorough a more direct interval notation like $[14.9, 15.1]$ or $[10.0, 20.0]$.

Intervals, denoted generally by $[a, b]$ with the condition $a \leq b$, are the actual elementary items of numerical information. But when it comes to operate or doing some mathematics with numerical information, here is the departure point of interval mathematics from the usual way to handle numerical information through numbers. From an interval point of view the common alternative of computing with single numbers is able to provide only an indication of the true result that could be reached using interval calculus: only a single value pointing somewhere inside the complete interval result. Following this way systematically, interval mathematics uses trough the

entire process of a computation, all the range of possible values that correspond to every item of the numerical information.

Real numbers generalise the concept of exact fraction to exact number and set the geometrical model of the real line R , which is the ideal ground -conceptual and intuitive- on which numerical models are conceived. From a theoretical point of view, it is possible to deal with any real number with a finite or infinite amount of digits, but real numbers are not very suitable, as their name could indicate, because in practice it is not possible to deal with numbers with an infinite number of digits. Computer representation of a number contains only a limited amount of digits. We could think that technology allows working with a sufficient number of digits, but this ignores the reality and would apply properties of R to a set of numbers that, in fact, does not possess.

These outline facts are fully recognised and put to work by the Classical Interval Analysis's approach to numerical mathematics, when it decides systematically to keep the two nearest procedurally discernible digital (DI) bounds, a lower bound n_1 (DI), and an upper bound n_2 (DI), to represent any real value x (R) conceptually definite as something compatible enough with a definite measurement or actual computation. Thus every real number is between two consecutive digital numbers n_1 (DI) and n_2 (DI). The identification of the pair of digital numbers (n_1, n_2) bounding a real value to a set-theoretical interval $[n_1, n_2]$ makes the set of classic set-theoretical intervals. In the Classical Interval Analysis approach to numerical computing the digital intervals:

$$[n_1(DI), n_2(DI)] = \{x \in DI \mid n_1(DI) \leq x \leq n_2(DI)\} \quad (3.35)$$

are the computational items. If $I(R)$ denote the set of intervals with real numbers as bounds:

$$I(R) = \{[x_1, x_2] \mid x_1 \in R, x_2 \in R, x_1 \leq x_2\} \quad (3.36)$$

Definitions of classic intervals and operations with them can be found in several references like those of Hansen (1965), Moore (1966) and Neumaier (1990). A bibliography about interval algebra can be found in Bierbaum and Schwietz (1995) and in Sunaga (1958).

An interval number is an approximation of a real number, is a closed set which includes the possible range of an unknown real number. Thus, instead of considering a fixed value a , the following representation is adopted:

$$A = [\underline{a}, \bar{a}] := \{x \in R / \underline{a} \leq x \leq \bar{a}\} \quad (3.37)$$

where $\underline{a}$ is the infimum and $\bar{a}$ is the supremum.

The addition of the intervals $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$ is the set $\{a + b \mid a \in A, b \in B\}$. In a similar way, the different arithmetic operations can be defined. If op denotes an arithmetic operation for real numbers, the corresponding interval arithmetic operation is,

$$C = A \text{ op } B = \{a \text{ op } b / a \in A, b \in B\} \quad (3.38)$$

In interval arithmetic, overestimation is one of the main drawbacks because the range of uncertain is much larger than the range introduced by round off error. Overestimation is due to dependency and failure of some algebraic laws that are valid in real arithmetic. Such overestimation produces extremely and sometimes meaningless results.

What makes intervals non trivial for numerical computation and analysis is that some fundamental regularities of real numbers are lost. This and other shortcomings and defects of Classical Interval Analysis can lead to the idea of giving it up, but actually an effort is necessary to complete its structure through a wider set.

3.4.2. MODAL INTERVAL ANALYSIS (MIA)

Modal Interval Analysis (MIA) is a natural extension of the Classical Interval Analysis, where the concept of interval is widened by the set of predicates that are fulfilled by the real numbers (Gardenyes and Trepate, 1979; Gardenyes *et al.*, 1982; Gardenyes *et al.*, 1985; Gardenyes *et al.*, 2001).

Modal Interval Analysis (MIA) has been applied to simulate the imprecision and uncertainty present in the systems as bounded envelopes. These envelopes have been used for the prediction of fault detection. Different types of simulators that deal with

uncertainty are described. Another interesting application is in the area of control, analysis and design of robust predictive controllers.

Since in a computational operation on digital numerical information, an interval result points to, and bounds, some real number x (or numbers) holding a determinate property $P(x)$, we are forced to consider the context formed by:

- The set of real numbers R ;
- Set of classical intervals:

$$I(R) := \{[a, b]' \mid a, b \in R, a \leq b\} \quad (3.39)$$

- The set of classical predicates on the real line:

$$\text{Pred}(R) := \{P(.) \mid P(.): R \rightarrow \{0, 1\}\} \quad (3.40)$$

- The universal (U) and existential (E) quantifiers.

In order to avoid confusions between classical and modal intervals, a classical interval of bounds a and b will be represented by $[a,b]'$, instead of the standard notation $[a,b]$. Moreover, as the quantifiers are operators which transform real predicates in interval predicates, we will use for them a special notation formalizing their character of change-of-variable and operators; so,

$$E(x, A')P(x) \quad (3.41)$$

$$U(x, A')P(x) \quad (3.42)$$

will replace the apparently more intuitive forms used in the previous examples.

A modal interval X is defined by a pair formed by a classical interval and a quantifier.

$$X := (X', QX) \quad (3.43)$$

This is a similar method to that in which real numbers are associated in pairs having the same absolute value but opposite signs. Modal intervals are associated in pairs too, each member corresponding to the same closed interval of the real line, but

with each having one of the opposite selection modalities, existential or universal. The set of modal intervals will be represented by $I^*(R) := \{(X', \{E, U\}) | X' \in I(R)\}$. The modal coordinates of a modal interval are its point-set domain and its modality. Given $I^*(R) := \{[a, b] | a \in I(R), b \in I(R)\}$, then,

$$\text{Set}(A', QA) := A' \quad (3.44)$$

$$\text{Mod}(A', QA) := QA \quad (3.45)$$

The canonical coordinates of modal intervals are defined by:

$$\text{Inf}(A) := \begin{cases} \text{if Mod}(A) = E \text{ then min}(\text{Set}(A)) \\ \text{if Mod}(A) = U \text{ then min}(\text{Set}(A)) \end{cases} \quad (3.46)$$

$$\text{Inf}(A) := \begin{cases} \text{if Mod}(A) = E \text{ then min}(\text{Set}(A)) \\ \text{if Mod}(A) = U \text{ then min}(\text{Set}(A)) \end{cases} \quad (3.47)$$

The canonical notation of modal intervals is introduced by the definition:

$$[a, b] := \begin{cases} \text{If } a \leq b \text{ then } ([a, b]', E) \\ \text{if } a \geq b \text{ then } ([b, a]', U) \end{cases} \quad (3.48)$$

Canonical notations for the modal and canonical coordinates are:

$$\begin{aligned} \text{Inf}([a, b]) &= a \\ \text{Sup}([a, b]) &= b \\ \text{Set}([a, b]) &= [\min(a, b), \max(a, b)] \end{aligned} \quad (3.49)$$

$$\text{Mod}([a, b]) := \begin{cases} \text{If } a \leq b \text{ then } E \\ \text{if } a \geq b \text{ then } U \end{cases} \quad (3.50)$$

For example, the modal coordinates of the modal interval $([2, 3]', U)$ are:

$$\text{Set}([2, 3]', U) = [2, 3]' \text{ and } \text{Mod}([2, 3]', U) = U \quad (3.51)$$

The canonical coordinates are:

$$\text{Inf}([2, 3]', U) = 3 \text{ and } \text{Sup}([2, 3]', U) = 2 \quad (3.52)$$

The canonical notation is $[3, 2] = ([2, 3]', U)$.

With this canonical notation, “natural” sets of modal intervals are:

$$I^*(R) = \{[a, b] \mid a \in R, b \in R\} \quad (3.53)$$

$$Ie(R) = \{[a, b] \in I^*(R) \mid a \leq b\} \quad (3.54)$$

$$Iu(R) = \{[a, b] \in I^*(R) \mid a \geq b\} \quad (3.55)$$

$$Ip(R) = \{[a, b] \in I^*(R) \mid a = b\} \quad (3.56)$$

An interval $[a, b] \in Ie(R)$ is qualified as a “proper interval”; an interval $[a, b] \in Iu(R)$ as “improper”; an interval $[a, b] \in Ip(R)$ as “pointwise.” A graphical representation is in Figure 3.5.

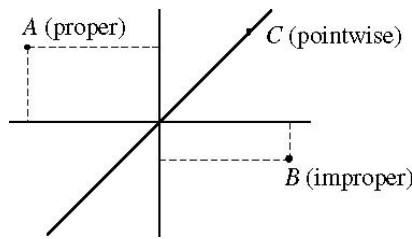


Figure 3.5 - (Inf, Sup) Diagram.

The modal quantifier Q associates to every real predicate $P(.) \in \text{Pred}(R)$ a unique hereditary interval predicate:

$$Q(x, (A', QA)) P(x) := QA(x, A') P(x) \quad (3.57)$$

for a variable x on R and $(A', QA) \in I^*(R)$.

We obtain, for example:

$$Q(x, ([-3, 1]', E)) \ x \geq 0 := E(x, [-3, 1]') \ x \geq 0 \quad (3.58)$$

$$Q(x, ([1, 2]', U)) \ x \geq 0 := U(x, [1, 2]') \ x \geq 0, \quad (3.59)$$

Using the canonical notation, the operation of the Q-quantifier is displayed as follows:

$$Q(x, [a, b]) = \begin{cases} \text{if } a \leq b \text{ then } E(x, [a, b]') \\ \text{if } a \geq b \text{ then } U(x, [b, a]') \end{cases} \quad (3.60)$$

3.4.2.1. INTERVAL RELATIONS

Defining the set of real predicates accepted by a modal interval,

$$\text{Pred}(A', QA) := \{P(\cdot) \in \text{Pred}(R) \mid Qx, (A', QA)P(x)\} \quad (3.61)$$

the parallel relation to the inclusion of two set-theoretical intervals can be introduced into the system of modal intervals.

We can define the set-theoretical modal inclusion, for $A, B \in I^*(R)$, as:

$$A \in B \Leftrightarrow \text{Pred}(A) \subseteq \text{Pred}(B) \quad (3.62)$$

Thus the inclusion among modal intervals, $A \subseteq B$, makes valid the implication $Q(x, A) P(x) \Rightarrow Q(x, B) P(x)$ for any property $P(x)$ on the real numbers. In terms of the canonical notation:

$$[a_1, a_2] \subseteq [b_1, b_2] \Leftrightarrow (a_1 \geq b_1, a_2 \leq b_2) \quad (3.63)$$

formally identical to the $\subseteq$ -relation for $I(R)$. Naming the existential intervals “proper intervals” to the universal ones “improper intervals” comes from the identification of $I_e(R)$ and $I(R)$ suggested by their coinciding inclusion’s programming theorems. Modal

intervals with their inclusion relation $(I^*(R), \subseteq)$ are the structural completion of set-theoretical intervals $(I(R), \subseteq)$.

In a dual way it is possible to define the set of real “copredicates” or predicates rejected by a modal interval, given by its modal coordinates (A', QA) :

$$\text{Copred}(A', QA) := \{P(\cdot) \in \text{Pred}(R) \mid \neg Qx, (A', QA)P(x)\} \quad (3.64)$$

There exists a complementary between Pred and Copred by means of the duality operator:

$$\text{Dual}([a, b]) := [b, a] \quad (3.65)$$

since $A \subseteq B \Leftrightarrow \text{Dual}(A) \supseteq \text{Dual}(B) \Leftrightarrow \text{Copred}(A) \supseteq \text{Copred}(B)$.

The “less or equal” relation, generated by the complementation of the modal inclusion, is defined as follows:

$$[a_1, a_2] \leq [b_1, b_2] := (a_1 \leq b_1, a_2 \leq b_2) \quad (3.66)$$

The same relations are valid for $I(R)$.

In the (Inf, Sup)-diagram, a representation for “inclusion” and “less or equal” relationships is in Figure 3.6.

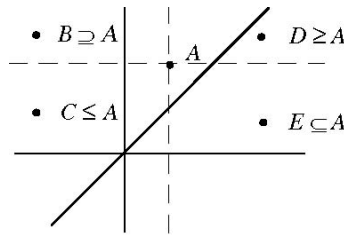


Figure 3.6 - "Inclusion" and "less than" relations.

The system $(I^*(R), \subseteq)$ is a lattice and the infimum and the supremum are called meet and join operators represented by $\wedge$ and $\vee$, respectively: for a family of modal intervals $A(i)$ with $i \in I$ we define,

$$\begin{aligned} \wedge(i, I) A(i) = A \in I^*(R) \text{ is such that } U(i, I) X \subseteq A(i) &\Leftrightarrow X \subseteq A \\ \vee(i, I) A(i) = B \in I^*(R) \text{ is such that } U(i, I) X \supseteq A(i) &\Leftrightarrow X \supseteq B \end{aligned} \quad (3.67)$$

annotated $A \wedge B$ and $A \vee B$ for the corresponding two-operands case. In terms of the canonical notations $A(i)=[a_1(i), a_2(i)]$,

$$\begin{aligned} \wedge(i, I) A(i) &= [\max(i, I) a_1(i), \min(i, I) a_2(i)] \\ \vee(i, I) A(i) &= [\min(i, I) a_1(i), \max(i, I) a_2(i)] \end{aligned} \quad (3.68)$$

The system $(I^*(R), \leq)$ is a lattice and the infimum and the supremum are called min and max operators: For a family of modal intervals $A(i)$ with $i \in I$ we similarly define:

$$\begin{aligned} \min(i, I) A(i) = C \in I^*(R) \text{ is such that } U(i, I) X \leq A(i) &\Leftrightarrow X \leq C \\ \max(i, I) A(i) = D \in I^*(R) \text{ is such that } U(i, I) X \geq A(i) &\Leftrightarrow X \geq D \end{aligned} \quad (3.69)$$

In terms of the canonical notations $A(i)=[a_1(i), a_2(i)]$,

$$\begin{aligned} \min(i, I) A(i) &= [\min(i, I) a_1(i), \min(i, I) a_2(i)] \\ \max(i, I) A(i) &= [\max(i, I) a_1(i), \max(i, I) a_2(i)] \end{aligned} \quad (3.70)$$

The pair $(\subseteq, \leq)$ of relations upon $I^*(R)$ provide the possible algorithms on $I^*(R)$ with a 4-way branching which corresponds to the 2-way branching established in the $\leq$ relation on R .

3.4.2.2. DIGITAL INTERVALS AND ROUNDINGS

The Classical Interval Analysis' approach to numerical mathematics, holds the two nearest procedurally discernible digital bounds, a lower bound n_1 (DI) and an upper bound n_2 (DI), to represent any real value $x(R)$ conceptually compatible with a definite measurement or any actual computation and consistent with the geometrical model directing the interpretation of this measurement or computation.

In the Modal Interval Analysis (MIA), if $DI \subseteq R$ is a digital scale for the real numbers:

$$I^*(DI) := \{[a, b] \in I^*(R) \mid a \in DI, b \in DI\} \quad (3.71)$$

the modal outer and inner roundings of $A \in I^*(R)$ are defined by:

$$\begin{aligned} Inn([a, b]) &= [Right(a), Left(b)] I^*(DI) \\ Out([a, b]) &= [Left(a), Right(b)] I^*(DI) \end{aligned} \quad (3.72)$$

The condition:

$$Inn([a, b]) \subseteq [a, b] \subseteq Out([a, b]) \quad (3.73)$$

is fulfilled and the equality,

$$Inn(A) = Dual \ Out \ Dual(A) \quad (3.74)$$

makes unnecessary the implementation of the inner rounding.

3.4.2.3. N-DIMENSIONAL CASE

The generalization to intervals with n components is direct. The set of n -dimensional intervals:

$$I^*(R^n) := \{([a_1, b_1], \dots, [a_n, b_n]) \mid [a_1, b_1] \in I^*(R), \dots, [a_n, b_n] \in I^*(R)\} \quad (3.75)$$

and the inclusion relation for two n -dimensional intervals $A = (A_1 \dots A_n)$, $B = (B_1 \dots B_n) \in I^*(R^n)$,

$$A \subseteq B \Leftrightarrow (A_1 \subseteq B_1, \dots, A_n \subseteq B_n) \quad (3.76)$$

3.4.3. INTERVAL EXTENSIONS FOR CONTINUOUS FUNCTIONS

In any mathematical model one or several real functions are involved: to get an interpretable result when their variables are intervals is one of the outstanding problems of interval analysis.

3.4.3.1. MODAL SEMANTIC EXTENSIONS

In the context of the Modal Interval Analysis (MIA) it can be expected, as a starting point, that as the R-predicate $P(x)$ leads to the modal interval predicate $Q(x, X)P(x)$, a relation $z = f(x_1 \dots x_n)$ must similarly become some kind of interval relation $z = f(x_1 \dots x_n)$ guaranteeing some sort of $(n+1)$ -dimensional predicate of the form:

$$Q_1(x_1, X_1) \dots Q_n(x_n, X_n) Q_z(z, F(f)(X_1, \dots, X_n)) z = f(x_1, \dots, x_n) \quad (3.77)$$

where an ordering problem obviously arises since the quantifying prefixes are not generally commutative. Given a function f from R^n to R we have to extend it to a function from $I^*(R^n)$ to $I^*(R)$ verifying some conditions. In the Modal Interval Analysis (MIA) we deal with the semantic interval extension of a real continuous function and with two “semantic” functions which play a grounding role in the theory because they are in close relation with the semantic extension and will provide a meaning to the interval calculations: if f is an R^n to R continuous function, $X \in I^*(R^n)$, and if $x=(x_p, x_i)$ is the component-splitting corresponding to $X=(X_p, X_i)$,

$$\begin{aligned} f^*(X) &:= \vee(x_p, X_p') \wedge(x_i, X_i') [f(x_p, x_i), f(x_p, x_i)] \\ &= [\min(x_p, X_p') \max(x_i, X_i') f(x_p, x_i), \max(x_p, X_p') \min(x_i, X_i') f(x_p, x_i)] \end{aligned} \quad (3.78)$$

$$\begin{aligned} f^{**}(X) &:= \wedge(x_i, X_i') \vee(x_p, X_p') [f(x_p, x_i), f(x_p, x_i)] \\ &= [\max(x_i, X_i') \min(x_p, X_p') f(x_p, x_i), \min(x_i, X_i') \max(x_p, X_p') f(x_p, x_i)] \end{aligned} \quad (3.79)$$

In the case $X_i = \emptyset$ it is:

$$\begin{aligned}
f^*(X) &:= f^{**}(X) = \vee(x, X') [f(x), f(x)] \\
&= [\min(x, X') f(x), \max(x, X') f(x)]
\end{aligned} \tag{3.80}$$

which can be identified with the united extension of the classical interval extension.

For example, for the real continuous $f(x_1, x_2) = x_1^2 + x_2^2$, the computation of the *-semantic and the **-semantic functions for $X = [-1, 1], [1, -1]$ yields the following results:

$$\begin{aligned}
f^*([-1, 1], [1, -1]) &= \vee(x_1, [-1, 1])' \wedge (x_2, [-1, 1])' [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
&= \vee(x_1, [-1, 1])' [x_1^2 + 1, x_1^2] = [1, 1]
\end{aligned} \tag{3.81}$$

$$\begin{aligned}
f^{**}([-1, 1], [1, -1]) &= \wedge(x_2, [-1, 1])' \vee (x_1, [-1, 1])' [x_1^2 + x_2^2, x_1^2 + x_2^2] \\
&= \wedge(x_2, [-1, 1])' [x_2^2, 1 + x_2^2] = [1, 1]
\end{aligned} \tag{3.82}$$

For the real continuous function $g(x_1, x_2) = (x_1 + x_2)^2$ the corresponding *-semantic and **-semantic functions for $X = [-1, 1], [1, -1]$ do not have coincident values:

$$\begin{aligned}
g^*([-1, 1], [1, -1]) &= \vee(x_1, [-1, 1])' \wedge (x_2, [-1, 1])' [(x_1 + x_2)^2, (x_1 + x_2)^2] \\
&= \left[\text{if } x_1 < 0 \text{ then } (x_1 - 1)^2 \text{ else } (x_1 + 1)^2, 0 \right] = [1, 0]
\end{aligned} \tag{3.83}$$

$$\begin{aligned}
g^{**}([-1, 1], [1, -1]) &= \wedge(x_2, [-1, 1])' \vee (x_1, [-1, 1])' [(x_1 + x_2)^2, (x_1 + x_2)^2] \\
&= \wedge(x_2, [-1, 1])' \left[0, \text{if } x_2 < 0 \text{ then } (x_2 - 1)^2 \text{ else } (x_2 + 1)^2 \right] \\
&= [0, 1]
\end{aligned} \tag{3.84}$$

The semantic extensions f^* and f^{**} can be equal or not, but both f^* and f are out of reach for any direct computation, except for simple real functions, such as the previous example or arithmetic operations.

In $I^*(\mathbb{R})$ the semantic extensions for the arithmetic operators are the semantic extensions of the continuous functions:

$$w(x, y) = x \ w \ y \quad (3.85)$$

with $w = \{+, -, *, /\}$. According to the Semantic Theorems, presented next, they can be computed from the w^* -extension or the w^{**} -extension, since, in this case, they are equal. If the common value is represented by AwB , for $A=[a_1, a_2]$ and $B=[b_1, b_2]$ the results, depending on the interval bounds, turn to be:

$$\begin{aligned} A + B &= [a_1 + b_1, a_2 + b_2] \\ A + B &= [a_1 + b_2, a_2 + b_1] \end{aligned} \quad (3.86)$$

$$A * B = \begin{cases} \text{if } a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 \geq 0 & \text{then } [a_1 b_1, a_2 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 < 0 & \text{then } [a_1 b_1, a_1 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 \geq 0 & \text{then } [a_2 b_1, a_2 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 b_1, a_1 b_2] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 \geq 0, b_2 \geq 0 & \text{then } [a_1 b_1, a_2 b_1] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 \geq 0, b_2 < 0 & \\ & \text{then } [\max(a_2 b_2, a_1 b_1), \min(a_2 b_1, a_1 b_2)] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 \geq 0 & \text{then } [0, 0] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 b_2, a_1 b_1] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 \geq 0, b_2 \geq 0 & \text{then } [a_1 b_2, a_2 b_2] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 \geq 0, b_2 < 0 & \text{then } [0, 0], \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 \geq 0 & \\ & \text{then } [\min(a_1 b_2, a_2 b_1), \max(a_1 b_1, a_2 b_2)] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 b_1, a_1 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 \geq 0, b_2 \geq 0 & \text{then } [a_1 b_2, a_2 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 \geq 0, b_2 < 0 & \text{then } [a_2 b_2, a_2 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 \geq 0, b_2 \geq 0 & \text{then } [a_1 b_2, a_1 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 b_2, a_1 b_1] \end{cases} \quad (3.87)$$

$$A / B = \begin{cases} \text{if } a_1 \geq 0, a_2 \geq 0, b_1 > 0, b_2 > 0 & \text{then } [a_1 / b_1, a_2 / b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 / b_2, a_1 / b_1] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 > 0, b_2 > 0 & \text{then } [a_1 / b_2, a_2 / b_2] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 / b_1, a_1 / b_1] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 > 0, b_2 > 0 & \text{then } [a_1 / b_1, a_2 / b_1] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 / b_2, a_1 / b_2] \\ \text{if } a_1 < 0, a_2 < 0, b_1 > 0, b_2 > 0 & \text{then } [a_1 / b_1, a_2 / b_2] \\ \text{if } a_1 < 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2 / b_1, a_1 / b_2] \end{cases} \quad (3.88)$$

These results for the arithmetic operators coincide with the definitions which Kaucher (1980) stated for the arithmetic operations in his extended interval space. Appendix C presents the whole Kaucher's Arithmetic operators results.

3.4.3.2. SEMANTIC THEOREMS

In $I^*(R)$ the values of f^* or f^{**} extensions may show, without further thought, no clear meaning concerning the values of the real function f in its domain. Two key theorems reverse this perspective, revealing completely the meaning of the interval results f^* and f^{**} and characterizing them as the key reference for the semantic interval extensions previously defined in logical terms.

THEOREM 3.1 - (Semantic Theorem for f^*). If $A \in I^*(R^n)$, f is continuous on A' and there exists an interval, called $F(A) \in I^*(R)$,

$$f^*(A) \subseteq F(A) \Leftrightarrow U(a_p, A_p') Q(z, F(A)) E(a_i, A_i') z = f(a_p, a_i) \quad (3.89)$$

THEOREM 3.2 - (Semantic Theorem for f^{**}). If $A \in I^*(R^n)$, f is continuous on A' and there exists an interval, called $F(A) \in I^*(R)$,

$$F(A) \subseteq f^{**}(A) \Leftrightarrow U(a_i, A_i') Q(z, Dual(F(A))) E(a_p, A_p') z = f(a_p, a_i) \quad (3.90)$$

The following examples illustrate and show the importance of both theorems. For the real function $f(x,y) = x+y$ we have $[1,3]+[4,8]=[5,11]$ which means:

$$\begin{aligned} U(x, [1,3]') U(y, [4,8]') E(z, [5,11]') z &= x + y \\ U(z, [5,11]') E(x, [1,3]') E(y, [4,8]') z &= x + y \end{aligned} \quad (3.91)$$

If we want the semantics:

$$U(x, [1,3]') U(z, ?) E(y, [4,8]') z = x + y \quad (3.92)$$

it must be $[1,3]+[8,4]=[9,7]$ which means:

$$U(x, [1, 3]') U(z, [7, 9]') E(y, [4, 8]') z = x + y \quad (3.93)$$

We can get other semantics handling the modalities of the operands $[3, 1] + [4, 8] = [7, 9]$ which means:

$$U(y, [4, 8]') U(x, [1, 3]') E(z, [7, 9]') z = x + y \quad (3.94)$$

or, $[3, 1] + [8, 4] = [11, 5]$ which means:

$$U(z, [5, 11]') E(x, [1, 3]') E(y, [4, 8]') z = x + y \quad (3.95)$$

Let us apply the result for the function $f(x, y) = x + y$ to a naturalistic context (Gardenyès *et al.*, 2001).

Suppose we have two cable reels of $a = 10$ and $b = 20$ units of length. Both connected can cover an overall length $c = 30$. This most elementary situation can be expressed for all that computationally matters by the algebraic expression $c = a + b$. Consider the parallel more realistic interval-situation where the first reel of cable has a length known only to lie in a range bounded by the interval $A' = [9, 20]'$. i.e., $a \in [9, 20]'$; about the second reel we know that $b \in B' = [10, 25]'$.

Let us consider the connection between both reels and let us apply the semantic theorem for f^* restricted to $f(a, b) = a + b$.

Case 1: $[9, 20] + [25, 10] = [34, 30]$ means:

$$U(a, [9, 20]') U(c, [30, 34]') E(b, [10, 25]') c = a + b \quad (3.96)$$

i.e., a particular length of the interval $[25, 10]$ can be selected to regulate some, in principle, unknown but specific length c lying within the improper interval $C = [34, 30]$, in spite of the value a belonging to the proper operand $A = [9, 20]$ being understood as $a \in A'$ coming out from some general random selection process.

Case 2: $[9,20]+[20,15]=[29,35]$ means:

$$U(a, [9, 20]')E(c, [29, 35]')E(b, [15, 20]') c = a + b \quad (3.97)$$

i.e., with the same autonomous interval $A = [9,20]$ and a narrower regulating interval $B = [20,15]$, a determined length of $b \in B' = [15,20]'$ should be selected (a regulation operation) just to get some length c lying within the range bounded by the proper interval $C = [29,35]'$.

Case 3: $[9,20]+[10,15]=[19,45]$ means:

$$U(a, [9, 20]')U(b, [10, 25]')E(c, [19, 45]') c = a + b \quad (3.98)$$

so that c will show the joint full indeterminacy coming from a and b .

Case 4: $[20,9]+[25,10]=[45,19]$ will be interpreted by:

$$U(c, [19, 45]')U(a, [9, 20]')E(b, [10, 25]') c = a + b \quad (3.99)$$

As a remark the *-semantic theorem allows the interpretation of universal intervals as “regulating or feedback ranges”, and existential intervals as “fluctuation or autonomous ranges”.

3.4.3.3. PROPERTIES OF SEMANTIC EXTENSIONS

Important inclusion relations between $f^*(X)$ and $f^{**}(X)$ are:

$$a) f^*(X) \subseteq f^{**}(X) \quad (3.100)$$

$$b) X \subseteq Y \Rightarrow f^*(X) \subseteq f^*(Y), f^{**}(X) \subseteq f^{**}(Y) \quad (3.101)$$

The case $f^*(X)=f^{**}(X)$, when X is not uni-modal, is characterized by the following result.

“Let (X_1', X_2') be a component split of $X \in I^*(R^n)$ and f an R^n to R continuous function. If $SDP(f, X_p', X_i') \neq \emptyset$ and $SDP(f, X_i', X_p') \neq \emptyset$ then,

$$f^*(X) = f^{**}(X) = \left[SDV(f, X_p', X_i'), SDV(f, X_i', X_p') \right] \quad (3.102)$$

where SDP means the set of saddle point and SDV is the corresponding saddle value,

$$SDV(f, X_1', X_2') = \min(x_1, X_1') \max(x_2, X_2') f(x_1, x_2) \quad (3.103)$$

In this case we say that f is JM-commutable for $X \in I^*(R^n)$ ”.

The semantic theorems show that $f^*(X)$ and $f^{**}(X)$ are optimal in semantic terms, and which is the right $\subseteq$ -sense to round when $*$ -semantics or $**$ -semantics is to be applied. They provide, therefore, the general norm that computational functions F from $I^*(R^n)$ to $I^*(R)$ must satisfy to be consistent with the f^* - or the f^{**} - semantics (a most suitable norm as long as we compute on a digital line DI), but still not a general procedure by which these functions may be effectively computed. These procedures will be provided by the rational modal extension of rational continuous functions, as far as they obey certain syntactic conditions which qualify them as inner-or outer-rounded computations of the associated f^* or f^{**} functions.

3.4.3.4. MODAL RATIONAL EXTENSIONS

The two orderly applications of the meet-join operators to a continuous function f from R^n to R provide the two semantic extensions f^* and f^{**} . When the continuous real function is a rational function, it can also be operationally extended to a modal rational function fR from $I^*(R^n)$ to $I^*(R)$ by using the computing program implicitly defined by the syntactic tree of the expression defining the function.

The problem with the semantic extensions f^* and f^{**} is that they are not generally computable. The interpretation problem for modal rational functions, which, are the core of numerical computing, consists in relating them to the corresponding semantic functions which have a standard meaning (defined by the semantic theorems) referring to their original real continuous functions.

Modal rational $*$ -extension $fR^*(X)$ is the function from $I^*(R^n)$ to $I^*(R)$ defined by the computational program indicated by the syntax of f when the real operators are transformed into their $*$ -semantic extension.

The outer rounding computation $Out(fR^*(X))$ is the function defined by the computational program of $fR^*(X)$ in which the value of every X -component is replaced by its modal outer rounding and the exact value of every operator is replaced by its actually computed outer rounding.

Modal rational $**$ -extension $fR^{**}(X)$ is the function from $I^*(R^n)$ to $I^*(R)$ defined by the computational program indicated by the syntax of f when the real operators are transformed into their $**$ -semantic extension.

The inner rounding computation $Inn(fR^{**}(X))$ is the function defined by the computational program of $fR^{**}(X)$ in which the value of every X -component is replaced by its modal inner rounding and the exact value of every operator is replaced by its inner rounding.

Modal rational extension $fR(X)$ is the function alike $fR^*(X)$ or $fR^{**}(X)$ when all its operators are JM-commutable (or “rational”).

For example, for the R^2 to R continuous function $f(x_1, x_2) = x_1 x_2 + g(x_1, x_2)$ with the operator $g(x_1, x_2) = (x_1 + x_2)^2$, the computational programs of f , fR^* and fR^{**} are in Figure 3.7.

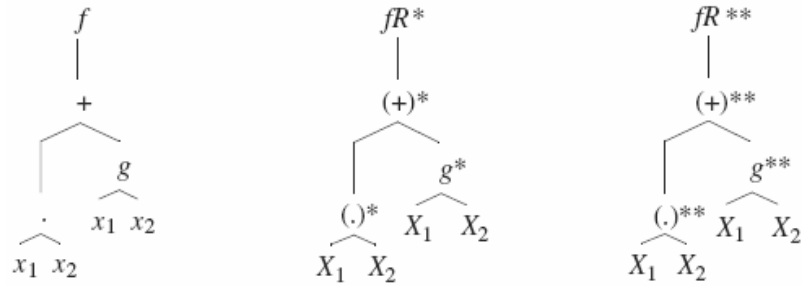


Figure 3.7 - Computational programs of f , fR^* and fR^{**} .

For $X = [-1, 1], [1, -1]$, fR^* and fR^{**} are computed as follows:

Operator $x_1 x_2$

$$\begin{aligned}
*-\text{extension} \quad & \vee(x_1, [-1, 1]') \wedge (x_2, [-1, 1]')[x_1x_2, x_1x_2] = [0, 0] \\
**-\text{extension} \quad & \wedge(x_2, [-1, 1]') \vee (x_1, [-1, 1]')[x_1x_2, x_1x_2] = [0, 0]
\end{aligned} \tag{3.104}$$

Operator $g(x_1, x_2)$

$$\begin{aligned}
*-\text{extension} \quad & \vee(x_1, [-1, 1]') \wedge (x_2, [-1, 1]')[(x_1 + x_2)^2, (x_1 + x_2)^2] = [1, 0] \\
**-\text{extension} \quad & \wedge(x_2, [-1, 1]') \vee (x_1, [-1, 1]')[(x_1 + x_2)^2, (x_1 + x_2)^2] = [0, 1]
\end{aligned} \tag{3.105}$$

Therefore,

$$\begin{aligned}
fR^*([-1, 1], [1, -1]) &= \vee(y_1, [0, 0]') \wedge (y_2, [0, 1]')[y_1 + y_2, y_1 + y_2] = [1, 0] \\
fR^{**}([-1, 1], [1, -1]) &= \vee(y_1, [0, 0]') \vee (y_2, [0, 1]')[y_1 + y_2, y_1 + y_2] = [0, 1]
\end{aligned} \tag{3.106}$$

Note that “rationality” means that the *-or **-semantic theorem ruling the interpretation of these functions will be (for any rational function built up by “rational operators”) determined by the choice of outer or inner truncation along their computing process (as far as the multi-incidences allow the application of both or one of these semantic theorems).

A component x_i is “uni-incident” in a rational function $f(x)$ if it occupies only one leaf of the syntactical tree for this function. Otherwise x_i is “multi-incident” in $f(x)$.

Several results show the relations between the semantic extensions and the rational one.

THEOREM 3.3 - *-Interpretability of a Modal Rational Extension. If the improper (universal) components of X are uni-incident in $fR^*(X)$, and if $\text{Out}(fR^*(X))$ does exist, then:

$$\text{Out}(fR^*(X)) \supseteq f^*(X) \tag{3.107}$$

Note that the existence of $\text{Out}(fR^*(X))$ guarantees the existence of the operands and the operators implied by $\text{Out}(fR^*(X))$.

THEOREM 3.4 - **-Interpretability of a Modal Rational Extension. If the proper (existential) components of X are uni-incident in $fR^{**}(X)$, and if $Out(fR^{**}(X'))$ does exist, then,

$$Inn(fR^{**}(X)) \subseteq f^{**}(X) \quad (3.108)$$

THEOREM 3.5 - Dual Computing Process. If $fR(X)$ is a modal rational function, then,

$$Inn(fR(X)) = Dual \left(Out \left(fR \left(Dual(X) \right) \right) \right) \quad (3.109)$$

This latter theorem allows the implementation of only externally rounded interval arithmetic. Mind the application of the $Out(.)$ operator to $Dual(X)$ in the second term: $Dual(.)$ is not a rational operator and the information about X implied by this expression will be $Inn(X)$.

THEOREM 3.6 - Interpretability of a Modal Rational Extension. If $fR(X)$ is a rational uni-incident function and all of its operators are JM-commutable,

$$f^*(X) \subseteq fR(X) \subseteq f^{**}(X) \quad (3.110)$$

Important JM-commutable operators will be the arithmetic ones, $w \in \{+, -, *, /\}$. For them we will define, according to the properties of the semantic extensions,

$$A w B = wR(A, B) := w^*(A, B) \quad (3.111)$$

and compute with the results of interval relations.

For example, for the function $g(x_1, x_2, x_3, x_4) = (x_1 + x_2)(x_3 + x_4)$ with $X_1 = [-2, 2]$, $X_2 = [-1, 1]$, $X_3 = [-1, 1]$, $X_4 = [-2, 2]$, we have:

$$\begin{aligned}
g^*(X) &= g^*([-2,2], [-1,1], [-1,1], [-2,2]) = [-9,9] \\
g^{**}(X) &= g^{**}([-2,2], [-1,1], [-1,1], [-2,2]) = [-9,9] \\
gR(X) &= ([-2,2] + [-1,1]) \cdot ([-1,1] + [-2,2]) = [-9,9]
\end{aligned}
\tag{3.112}$$

The case when $fR(X)$ is totally commutable is characterized by the following result.

THEOREM 3.7 - Interpretability of a Totally Commutable Modal Rational Extension. If $fR(X)$ is a rational uni-incident function with all its operators JM-commutable and totally commutable,

$$f^*(X) = fR(X) = f^{**}(X) \tag{3.113}$$

In particular, this occurs if all the X components are uni-incident and with the same modality.

There are rational uni-incident functions f built up by JM-commutable operators, which are not totally JM-commutable. For example, the function $g(x_1, x_2, x_3, x_4) = (x_1 + x_2)(x_3 + x_4)$ has all its operators JM-commutable, but it is not totally JM-commutable since:

$$\begin{aligned}
g^*(X) &= g^*([-2,2], [1,-1], [-1,1], [2,-2]) = [1.5, -1.5] \\
g^{**}(X) &= g^{**}([-2,2], [1,-1], [-1,1], [2,-2]) = [-1.5, 1.5] \\
gR(X) &= ([-2,2] + [1,-1]) \cdot ([-1,1] + [2,-2]) = [0,0]
\end{aligned}
\tag{3.114}$$

Two important results in the case of multi-incident components follow.

THEOREM 3.8 - Coertion to *-Interpretability. If in $fR(X)$ there are multi-incident improper components and if XT^* is obtained from X , transforming, for every multi-incident improper component, all incidences but one into its dual, then,

$$f^*(X) \subseteq fR(XT^*) \tag{3.115}$$

THEOREM 3.9 - Coertion to **-Interpretability. If in $fR(X)$ there are multi-incident proper components and if XT^{**} is obtained from X , transforming, for every multi-incident proper component, all incidences but one into its dual, then,

$$fR(XT^{**}) \subseteq f^{**}(X) \quad (3.116)$$

For example, for the rational function $f(x)=x_1x_2/(x_1+x_2)$ extended for the intervals $X_1=[200,5000]$ and $X_2=[51,49]$ we have:

$$\begin{aligned} fR(X) &= X_1 \cdot X_2 / (X_1 + X_2) \\ &= [200,5000] \cdot [51,49] / ([200,5000] + [51,49]) \subseteq [2.0,976.1] \end{aligned} \quad (3.117)$$

which is not interpretable; but,

$$\begin{aligned} fR_1(XT^*) &= X_1 \cdot X_2 / X_1 + \text{Dual}(X_2) \\ &= [200,5000] \cdot [51,49] / ([200,5000] + [49,51]) \subseteq [2.0,984.0] \end{aligned} \quad (3.118)$$

and,

$$\begin{aligned} fR_2(XT^*) &= X_1 \cdot \text{Dual}(X_2) / (X_1 + X_2) \\ &= [200,5000] \cdot [49,51] / ([200,5000] + [51,49]) \subseteq [1.9,1016.0] \end{aligned} \quad (3.119)$$

do admit interpretations like, for example,

$$U(x_1, [200,5000]') E(z, [1.9,1016.0]') E(x_2, [49,51]') z = x_1x_2 / (x_1 + x_2) \quad (3.120)$$

The computation of a rational interval program $fR(XT^*)$ or $fR(XT^{**})$ may result in a loss of information. This loss can be cancelled or reduced in some cases; for example, if there are multi-incident improper components and $fR(XT_1^*) \dots fR(XT_k^*)$ are k results transforming each of the multi-incident components but one into its dual, then,

$$f^*(X) \subseteq fR(XT_1^*) \wedge \dots \wedge fR(XT_k^*) = B \quad (3.121)$$

and B will possibly be a better result than every $fR(XT_i^*)$. Similarly, for the case of multi-incident proper components,

$$f^{**}(X) \supseteq fR(XT_1^{**}) \vee \dots \vee fR(XT_k^{**}) \quad (3.122)$$

For example, for the rational function $f(x) = x_1 x_2 / (x_1 + x_2)$ of the previous example,

$$\begin{aligned} fR(XT_1^*) &= X_1 \cdot X_2 / X_1 + \text{Dual}(X_2) \\ &= [200, 5000] \cdot [51, 49] / ([200, 5000] + [49, 51]) \subseteq [2.0, 984.0] \end{aligned} \quad (3.123)$$

and,

$$\begin{aligned} fR(XT_2^*) &= X_1 \cdot \text{Dual}(X_2) / (X_1 + X_2) \\ &= [200, 5000] \cdot [49, 51] / ([200, 5000] + [51, 49]) \subseteq [1.9, 1016.0] \end{aligned} \quad (3.124)$$

Then,

$$f^*(X) \subseteq fR(XT_1^*) \wedge fR(XT_2^*) = [2.0, 984.0] \quad (3.125)$$

with the interpretation,

$$U(x_1, [200, 5000])' E(x_2, [49, 51])' E(z, [2.0, 984.0]) z = x_1 x_2 / (x_1 + x_2) \quad (3.126)$$

3.4.4. OPTIMALITY

3.4.4.1. DEFINITION

If, for every $X \in I^*(R^n)$ for which $fR(X')$ is defined, the condition:

$$f^*(X) = fR(X) = f^{**}(X) \quad (3.127)$$

holds, fR is said to be optimal.

In this case both $f^*(X)$ and $f^{**}(X)$ are computable through $fR(X)$ and $fR(X)$ is interpretable through the semantic theorems for $f^*(X)$ and $f^{**}(X)$.

The lack of optimality is a heavy drawback for every computation serving any actual use since the corresponding information loss will usually be far greater than the

one occasioned by common numerical roundings it will be crucial to find optimality conditions characterizing functions for which the rational program fR be optimal.

3.4.4.2. OPTIMAL OPERATORS

a) “Every one-variable continuous function is JM-commutable, and therefore a rational operator”.

The interesting operators are the monotonic operators or other easily programable ones like $\text{abs}(x)$, $\text{power}(x, n)$, $\log(x)$ or $\text{root}(x, n)$. If $X = [x_1, x_2]$, for the logarithmic and exponential operators:

$$\ln X := [\ln x_1, \ln x_2] \text{ whenever } X' > 0 \quad (3.128)$$

$$\exp(X) := [\exp(x_1), \exp(x_2)] \quad (3.129)$$

For the absolute value:

$$|X| := \begin{cases} \text{if } x_1 \geq 0, x_2 \geq 0 \text{ then } [x_1, x_2] \\ \text{if } x_1 < 0, x_2 < 0 \text{ then } [|x_2|, |x_1|] \\ \text{if } x_1 < 0, x_2 \geq 0 \text{ then } [0, \max(|x_1|, |x_2|)] \\ \text{if } x_1 \geq 0, x_2 < 0 \text{ then } [\max(|x_1|, |x_2|), 0] \end{cases} \quad (3.130)$$

For the operator power (x, n) we have:

$$X^n = [x_1, x_2]^n := \begin{cases} \text{if } n \text{ is odd then } [x_1^n, x_2^n] \\ \text{if } n \text{ is even then} \\ \quad \begin{cases} \text{if } x_1 \geq 0, x_2 \geq 0 \text{ then } [x_1^n, x_2^n] \\ \text{if } x_1 < 0, x_2 < 0 \text{ then } [x_2^n, x_1^n] \\ \text{if } x_1 < 0, x_2 \geq 0 \text{ then } [0, \max(|x_1|^n, |x_2|^n)] \\ \text{if } x_1 \geq 0, x_2 < 0 \text{ then } [\max(|x_1|^n, |x_2|^n), 0] \end{cases} \end{cases} \quad (3.131)$$

A n-variable continuous function $f(x,y)$ is x-uniformly monotonic on a domain $(X',Y') \in (R,R^m)$ if it is monotonic for x on X' , and it keeps the same sense of monotony for all the values y on Y' . An n-variable continuous operator $f(x,y)$ is x-partially monotonic on $(X',Y') \in (R,R^m)$ if it may increase with x for some y-values, and may decrease with x for the remaining y-values in the domain Y' .

b) “Every two-variable continuous function $f(x,y)$ which is partially monotonic in a domain (X',Y') is JM-commutable for the corresponding interval argument (X,Y) ”.

Interesting operators are, of course, $x+y$, $x-y$, $x*y$, x/y , x^y , $\max(x,y)$ and $\min(x,y)$.

c) “Every continuous uniformly monotonic function $f(x,y)$ on $(x,y) \in (X',Y)$, if it is x-isotonic and y-antitonic, it is JM-commutable for (X,Y) and,

$$f^*(X,Y) = f^{**}(X,Y) = [f(\text{Inf}(X), \text{Sup}(Y)), f(\text{Sup}(X), \text{Inf}(Y))] \quad (3.132)$$

It is convenient to call “monotonicity split” the partition of the parameter components of a uniformly monotonic function into the two groups of x-isotonic and y-antitonic components. For example, the function $f(x,y,z,t) = (x-y)/(z-t)$ in $X = [-1,1]$, $Y = [2,1]$, $Z = [-1,1]$ and $T = [3,2]$ is:

$$\begin{aligned} x &- \text{antitonic since } f_x < 0 \text{ in } (X',Y',Z',T') \\ y &- \text{isotonic since } f_y > 0 \text{ in } (X',Y',Z',T') \\ z &- \text{isotonic since } f_z > 0 \text{ in } (X',Y',Z',T') \\ t &- \text{antitonic since } f_t < 0 \text{ in } (X',Y',Z',T') \end{aligned} \quad (3.133)$$

then,

$$\begin{aligned} f^*(X,Y,Z,T) &= [f(\text{Sup}(X), \text{Inf}(Y), \text{Inf}(Z), \text{Sup}(T)), f(\text{Inf}(X), \text{Sup}(Y), \text{Sup}(Z), \text{Inf}(T))] \\ &= [(1-2)/(-1-2), (-1-1)/(1-3)] = [1/3, 1] \end{aligned} \quad (3.134)$$

3.4.4.3. ALGEBRAIC PROPERTIES OF ARITHMETIC OPERATORS

In $I^*(R)$ the arithmetic operators can be defined from the w^* -extension or the w^{**} -extension, since w is JM-commutable and computable according to the results of interval relations.

Some fundamental regularities of real numbers were lost in the context of the classical intervals $I(R)$. The addition and multiplication operations lose some of their group properties. If $[a,b]$ is a “non-pointwise interval” ($a=b$), there exists no interval $[x,y]$ such that:

$$[a,b] + [x,y] = [0,0] \quad (3.135)$$

and the equation,

$$[a, b] + [x, y] = [c, d] \quad (3.136)$$

has an interval solution only when $b-a \leq d-c$. Even in this case the $I(R)$ -system fails to obtain the solution from any set-theoretical interval operation between $[a,b]$ and $[c,d]$.

Contrary to the classical intervals, in $I^*(R)$ the equation $A + X = B$ has, as only solution, $X=B\text{-Dual}(A)$ and the equation $A * X = B$ has, as only solution, $X=B/\text{Dual}(A)$, (if $0 \notin \text{Set}(A)$).

Note that to solve the equations $A + X = B$ or $A * X = B$ is a specific modal interval problem, different from the set-theoretical problem of finding an outer or inner interval estimation of the solution set for the equations $a + x = b$ or $a * x = b$ when a and b take values of certain intervals A and B .

For example, the equation $[1,3] + [x,y] = [4,5]$ has the solution:

$$[x, y] = [4, 5] - \text{Dual}([1, 3]) = [4, 5] - [3, 1] = [3, 2] \quad (3.137)$$

The equation $[1, 3] * [x,y] = [4,-5]$ has the solution:

$$[x, y] = [4, -5] / \text{Dual}([1, 3]) = [4, -5] \cdot [1/3, 1] = [4/3, -5/3] \quad (3.138)$$

The set-theoretical problem of finding the inner interval estimation X of the solution set for the equation $a x = b$ when a and b takes values of the intervals $[1,3]$ and $[-5,4]$, respectively, is equivalent to finding an interval X such that:

$$U(x, X')E(a, [1, 3]')E(b, [-5, 4]') ax = b \quad (3.139)$$

Therefore, and according to the theorem *-semantic, X will be the proper interval solution of the equation:

$$[3, 1] \cdot X = [-5, 4] \quad (3.140)$$

which is $X = [-5, 4]$.

As an example (Gardenyes *et al.*, 2001), let us consider a gas container whose volume is to be determined to keep the pressure within certain pre-established bounds (Figure 3.8). Assuming valid the equation:

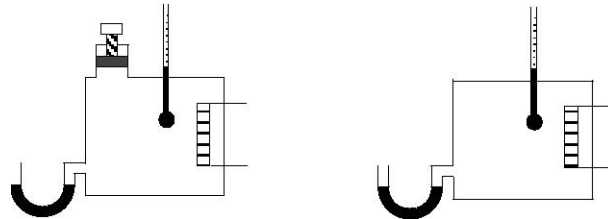


Figure 3.8 - Gas containers.

$$p = kt / v \quad (3.141)$$

with the intervals of variation:

$$k \in K = [0.00366, 0.00367], t \in T = [263, 283] \quad (3.142)$$

If the pressure is in the range of values $P = [0.99, 1.01]$, the solution for the volume is:

$$V = \text{Dual}(K \cdot T) / P = \left[\frac{1.03861}{1.01}, \frac{0.96258}{0.99} \right] \supseteq [1.03, 0.97] \quad (3.143)$$

The resulting semantics is:

$$U(k, [0.00366, 0.00367])' \cup U(t, [263, 283])' \quad (3.144)$$

$$E(v, [0.97, 1.03])' \cap E(p, [0.99, 1.01])' \quad p = kt / v \quad (3.145)$$

The result is an improper “control” interval, because for every value of k and t there exist a volume v between 0.97 and 1.03, depending on the k and t values, which makes the pressure within the desired limits.

The container is to be built with a feedback valve to allow its volume to be regulated within the computed bounds, to keep the stated conditions.

But, allowing the pressure within the range $P = [0.9, 1.1]$ the resulting volume is:

$$V = \text{Dual}(K \cdot T) / P = \left[\frac{1.03861}{1.1}, \frac{0.96258}{0.9} \right] \supseteq [0.95, 1.06] \quad (3.146)$$

and the resulting semantics is:

$$U(k, [0.00366, 0.00367])' \cup U(t, [263, 283])' \cup U(v, [0.95, 1.06])' \\ E(p, [0.9, 1.1])' \quad p = kt / v. \quad (3.147)$$

This result is a proper interval, a “fluctuation” for the would-be fixed volume container since for every k and t and every volume v between 0.95 and 1.06, the pressure falls within the limits.

Externally rounded data are not always enough to compute externally rounded results. An external solution to $A + X = B$ could be obtained from the equation $\text{Inn}(A) + X = B$ but not from $\text{Out}(A) + X = B$.

For example, for the exact equation in R:

$$\left[\frac{131}{99}, \frac{317}{90} \right] + [x, y] = [2, 7] \Rightarrow [x, y] = \left[\frac{67}{99}, \frac{313}{90} \right] \quad (3.148)$$

For Out (A) + X = B:

$$[1.3, 3.6] + [x, y] = [2, 7] \Rightarrow [x, y] = [0.7, 3.4] \quad (3.149)$$

which is not the outer-rounding of $[67/99, 313/90]$.

For Inn(A) + X = B:

$$[1.4, 3.5] + [x, y] = [2, 7] \Rightarrow [x, y] = [0.6, 3.5] \quad (3.150)$$

which is the outer-rounding of $[67/99, 313/90]$.

The + and * operations keep the commutative property and the associativity, when they are not rounded. Thus, the group-property anomaly for the classical interval operations + and * is overcome.

In I(R), the distributive property is weakened into “sub-distributivity”:

$$A \cdot (B + C) \subseteq (A \cdot B) + (A \cdot C) \quad (3.151)$$

In I*(R) the subdistributivity law becomes just more complex and takes the form:

$$A \cdot (B + C) \subseteq A \cdot B + \text{Prop}(A) \cdot C \quad (3.152)$$

with $\text{Prop}(A) = [\min(a_1, a_2), \max(a_1, a_2)]$, and:

$$A \cdot (B + C) \supseteq A \cdot B + \text{Impr}(A) \cdot C \quad (3.153)$$

with $\text{Impr}(A) = [\max(a_1, a_2), \min(a_1, a_2)]$.

3.4.4.4. TREE-OPTIMALITY

We will now construct the fundamental class of uni-incident optimal rational functions. The uni-incident hypothesis is assumed but not explicitly repeated.

The modal rational function $fR(X)$ is tree-optimal if, given any one of its non-uniformly monotonic elementary branches, it is followed down the fR - tree only by one-variable operators.

In this definition the idea of branch follows the intuitive reference to trees when we mean the syntactical form of rational functions, and it develops from the elementary formal connections between any operator and each one of its ordered immediate operands, and of these operands (or the final or root result) with the immediately following operators. The check about the tree-optimality of $fR(X)$ can be restricted to the sub-tree defined by its non-uniformly monotonic variables.

THEOREM 3.10 - Optimality Tree-Optimal Rational Functions. If $fR(X)$ is tree-optimal and X is uni-incident in $fR(X)$, the $fR(X)$ is optimal.

For example, $fR(X)$ is tree-optimal for the function $f(x,y,z,u) = xy + zu$ in any (X,Y,Z,U) , but gR is not tree-optimal for the function $g(x,y,z,u) = (x+y)(z+u)$ in $X = [-2, 2]$, $Y = [1, -1]$, $Z = [-1, 1]$, $U = [2, -2]$ and not optimal since:

$$\begin{aligned} g^*([-2, 2], [1, -1], [-1, 1], [2, -2]) &= [1.5, -1.5] \\ g^{**}([-2, 2], [1, -1], [-1, 1], [2, -2]) &= [-1.5, 1.5] \\ gR([-2, 2], [1, -1], [-1, 1], [2, -2]) &= [0, 0] \end{aligned} \tag{3.154}$$

Nevertheless it is tree-optimal (and thereafter optimal) in other domains, since:

$$\begin{aligned} g^*([1, 3], [0, 3], [4, 2], [3, 1]) &= g^{**}([1, 3], [0, 3], [4, 2], [3, 1]) \\ &= gR([1, 3], [0, 3], [4, 2], [3, 1]) = [7, 18] \end{aligned} \tag{3.155}$$

In the case of $fR(X_1, \dots, X_n)$ being uni-incident and uni-modal, the optimality holds independently of the syntactic structure of the fR tree (note that fR -trees must be dealt with without any operator built from the duality transformation). It is interesting to

note the difference between the fact that a function f is JM commutable in X , i.e. $f^*(X) = f^{**}(X)$, and the existence of an optimal rational computation, as the following example suggests.

As an example (Gardeny *et al.*, 2001), let g be the distance between an object and a thin lens, f the focal length and b the distance between the lens and the image (Figure 3.9).

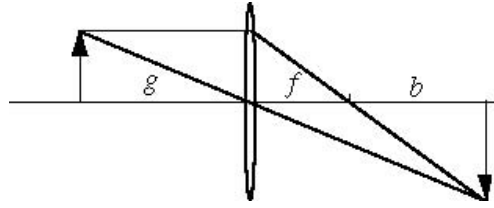


Figure 3.9 - Lens diagram.

They are related through the lens equation:

$$g = 1 / (1/f - 1/b) \quad (3.156)$$

Let us suppose that the focal distance lies between 40 mm and 50 mm. For the domain $B = [51, 60]$ g is JM-commutable, since it is f and b -uniformly monotonic.

The rational computation:

$$gR = 1 / (1/F - 1/B) \quad (3.157)$$

is uni-incident and tree-optimal for $F = [40, 50]$ and $B = [60, 51]$. The result:

$$gR([40, 50], [60, 51]) = 1 / (1/[40, 5000] - 1/[60, 51]) = [185.45, 300] \quad (3.158)$$

is optimal, except for roundings, with the semantical reference:

$$U(f, [40, 50]') E(g, [185.45, 300]') E(b, [51, 60]') g = 1 / (1/f - 1/b) \quad (3.159)$$

The equivalent rational function:

$$g = bf / (b - f) \quad (3.160)$$

is obviously JM-commutable, as well.

But the rational computation:

$$gR([40, 50], [60, 51]) = [60, 51] \cdot [40, 50] / ([60, 51] - [40, 50]) = \left[\frac{2400}{11}, 255 \right] \quad (3.161)$$

is not interpretable, since B is improper and multi-incident.

A real function f is x -totally monotonic for a multi-incident variable $x \in R$ if it is uniformly monotonic for this variable and for each one of its incidences (considering each leaf of the rational function as an independent variable).

THEOREM 3.11 - The *-Partially Optimal Coertion. Let X be an interval vector, and fR defined in the domain X' and totally monotonic for a subset Z of multi-incident components. Let XDT^* be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XDT^* as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the global one of the corresponding Z -component; for the rest, the multi-incident improper components are transformed into their dual in every incidence except one. Then:

$$f^*(X) \subseteq fR(XDT^*) \quad (3.162)$$

Moreover, if $fR(X)$ is tree-optimal,

$$fR(XT^{**}) \subseteq fR(XDT^{**}) \quad (3.163)$$

on the condition that the multi-incident components not belonging to Z suffer in XT^* the same transformation as in XDT^* .

THEOREM 3.12 - The $**$ -Partially Optimal Coertion. Let X be an interval vector, and fR defined in the domain X' and totally monotonic for a subset Z of multi-incident components. Let XDT^{**} be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XDT^{**} as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the global one of the corresponding Z -component; for the rest, the multi-incident proper components are transformed into their dual in every incidence except one. Then:

$$fR(XDT^{**}) \subseteq f^{**}(X) \quad (3.164)$$

Moreover, if $fR(X)$ is tree-optimal,

$$fR(XT^{**}) \subseteq fR(XDT^{**}) \quad (3.165)$$

on the condition that the multi-incident components not belonging to Z suffer in XT^{**} the same transformation as in XDT^{**} .

THEOREM 3.13 - Coertion to Optimality. Let X be an interval vector, and fR defined in the domain X' and totally monotonic for all its multi-incident components. Let XD be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XD as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the overall one of the corresponding X -component. Let fR be tree-optimal in the domain X' ; in which case:

$$f^*(X) = fR(XD) = f^{**}(X) \quad (3.166)$$

For example, for $f(x) = x - x$, it is:

$$fR(XD) = X - \text{Dual}(X) \quad \text{or} \quad fR(XD) = \text{Dual}(X) - X \quad (3.167)$$

for $f(x) = x / x$, it is:

$$f^*(X) = fR(XD) = X / \text{Dual}(X) \text{ or } fR(XD) = \text{Dual}(X) / X, \text{ if } 0 \notin X' \quad (3.168)$$

for $f(x) = 1/(1+x) + 1/(1-x)$ and $X = [1/4, 1/2]$, it is:

$$f^*(X) = fR(XD) = 1/(1 + [1/2, 1/4]) + 1/(1 - [1/4, 1/2]) \quad (3.169)$$

for $f(x,y) = xy + 1/(x+y)$ and $(X,Y) = ([5,10],[2,1])$, it is:

$$\begin{aligned} f^*(X) &= fR(XD) = X \cdot Y + 1/(\text{Dual}(X) + \text{Dual}(Y)) \\ &= [5, 10] \cdot [2, 1] + 1/([10, 5] + [1, 2]) = [71/7, 111/11] \end{aligned} \quad (3.170)$$

and for the rational function $f(x_1, x_2) = x_1 x_2 / (x_1 + x_2)$ continuous in $X_1 = [200, 5000]$ and $X_2 = [51, 49]$ we have the following interpretable computations:

$$fR_1(XT^*) = X_1 \cdot X_2 / (X_1 + \text{Dual}(X_2)) = [2.01, 983.93] \quad (3.171)$$

$$fR_2(XT^*) = X_1 \cdot \text{Dual}(X_2) / (X_1 + X_2) = [1.94, 1015.93] \quad (3.172)$$

$$fR_3(XT^*) = X_1 \cdot X_2 / (\text{Dual}(X_1) + \text{Dual}(X_2)) = [40.63, 50.30] \quad (3.173)$$

where only the last one is optimal.

THEOREM 3.14 - Equivalent Optimality. Any uni-incident rational function $fR(X)$ with an equivalent rational function gR which has an optimal computation for X is optimal for X .

For example, the R^3 to R rational functions:

$$f(a, b, c) = a(b + c) \quad \text{and} \quad g(a, b, c) = ab + ac \quad (3.174)$$

are equivalent and gR is syntactically optimal. For $A = [-1, 1]$, $B = [1, 2]$ and $C = [3, 1]$, g is uniformly monotonic for the variable a and isotonic for the two incidences of a . The computation:

$$gR(A, B, C) = A \cdot B + A \cdot C = [-1, 1] \cdot [1, 2] + [-1, 1] \cdot [3, 1] = [-3, 3] \quad (3.175)$$

is optimal. The theorem of equivalent optimality implies the optimality of the computation:

$$fR(A, B, C) = A \cdot (B + C) = [-1, 1] \cdot ([1, 2] + [3, 1]) = [-3, 3] \quad (3.176)$$

For the intervals $A = [-1, 1]$, $B = [1, 2]$ and $C = [0, -4]$, f is only partially monotonic for the variable a . Then there is no criterion for an acceptable coercion on $gR(.)$ and:

$$fR(A, B, C) = A \cdot (B + C) = [-1, 1] \cdot ([1, 2] + [0, -4]) = [0, 0] \quad (3.177)$$

can be different from f^* , from f^{**} , or in principle from both.

3.4.4.5. CONDITION OPTIMALITY

Other conditions to assess the syntactical optimality of rational functions depend on the modalities of their arguments.

THEOREM 3.15 - The Coercion to Optimality for Uni-Modal Arguments. Let X be an interval uni-modal vector, and fR defined in the domain X' and totally monotonic for all its multi-incident components. Let XD be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XD as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the overall one of the corresponding X -component. In this case:

$$f^*(X) = fR(XD) = f^{**}(X) \quad (3.178)$$

Let us apply this result to simulation for a model of a physical system represented for a general equation:

$$Y = FR(X, P, U) \quad (3.179)$$

where X is the state vector, P the vector of parameters, U the input vector, Y the output vector and FR a rational interval function referring to the interpretable modal extension F^* . The interval semantics of this model could be:

$$U(x, X')U(p, P')U(u, U')E(y, Y')y = f(x, p, u) \quad (3.180)$$

if all the suitable interval modalities were proper, according to the technical operation of the system.

As an example (Gardenyès *et al.*, 2001), suppose the case of a water tank of volume v heated by means a hydraulic system formed by a primary circuit, in which warm water is pumped by the pump B_1 at a flow rate of q_1 , a secondary circuit, in which water is pumped by the pump B_2 at a flow rate of q_2 and a heat exchanger, from which the water of both circuits goes out at the same temperature (Figure 3.10).

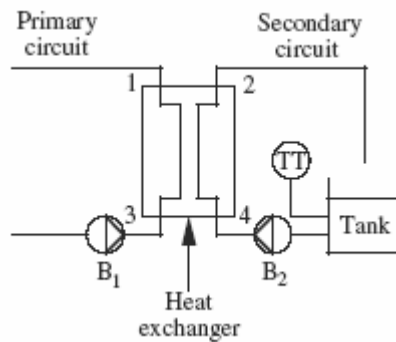


Figure 3.10 - Heat exchanger.

The mathematical model for the energy balance is:

$$\frac{d_{t_4}}{d_t} = \frac{1}{v} \left(\frac{1}{1/q_1 + 1/q_2} (t_3 - t_4) - \frac{k}{dc} (t_3 - t_a) \right) \quad (3.181)$$

where t_4 is the temperature of the water at point 4 and in the tank, measured by TT, t_3 is the temperature of the water in the primary circuit at point 3, t_a is the temperature of the

air surrounding the system, k is the dissipation constant, d is the density of water and c is the heat capacity of the water.

Simulation performed with finite differences scheme:

$$t_4(n+1) = t_4(n) + \frac{\Delta}{v} \left(\frac{1}{1/q_1 + 1/q_2} (t_3 - t_4(n)) - \frac{k}{dc} (t_4(n) - t_a) \right) \quad (3.182)$$

for $K = 7000$ W/K, $d = 1000$ Kg/m³, $c = 4180$ J/K*Kg, $q_1 = 0.005$ m³/s, $q_2 = 0.0025$ m³/s, $v = 1$ m³, $t_a = 300$ K, $t_3 = 340$ K and $t_4(0) = 301$ K gives the results shown in Figure 3.11.

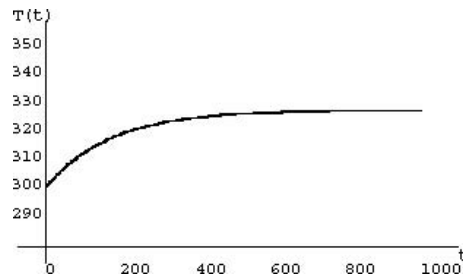


Figure 3.11 - Punctual model.

Let us consider the following intervals of variation: Q_1 for the flow rate q_1 , Q_2 for the flow rate q_2 , T_3 for the temperature t_3 , T_4 for the temperature t_4 , T_a for the temperature t_a and V for the volume v (let us consider k , d and c as constants). The physical system can then be represented by the following rational interval extension:

$$T_4(n+1) = T_4(n) + \frac{\Delta}{V} \left(\frac{1}{1/Q_1 + 1/Q_2} (T_3 - T_4(n)) - \frac{k}{dc} (T_4(n) - T_a) \right) \quad (3.183)$$

A naive simulation performed for $K = 7000$ W/K, $d = 1000$ Kg/m³, $c = 4180$ J/K*Kg, $T_a = [299, 301]$ K, $T_3 = [339, 341]$ K, $Q_1 = [0.005, 0.006]$ m³/s, $Q_2 = [0.024, 0.026]$ m³/s, $V = [0.9, 1.1]$ m³ and $T_4(0) = [301, 303]$ K gives the evolution bounds shown in Figure 3.12.

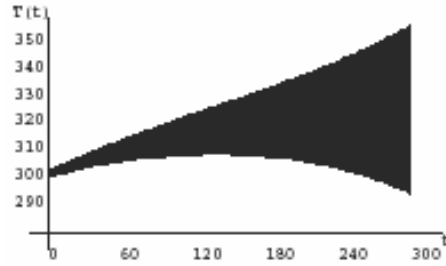


Figure 3.12 - Non optimal model.

However, using the interval model after the application of the coercion to optimality theorem:

$$T_4(n+1) = T_4(n) + \frac{\Delta}{V} \left(\frac{1}{1/Q_1 + 1/Q_2} (T_3 - \text{Dual}(T_4(n))) - \frac{k}{dc} (\text{Dual}(T_4(n)) - T_a) \right) \quad (3.184)$$

the evolution bounds improve, as shown in Figure 3.13.

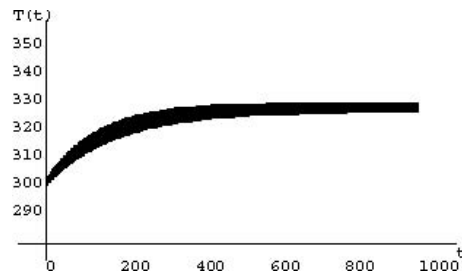


Figure 3.13 - Optimal model.

That means, for example, at the instant $n = 1000$:

$$\begin{aligned} &U(t_a, [299, 301]') U(t_3, [339, 341]') \\ &U(q_1, [0.005, 0.006]') U(q_2, [0.024, 0.026]') \\ &U(v, [0.9, 1.1]') U(t_4(0), [301, 303]') \\ &E(t_4(1000), [325.6, 330.7]') t_4(1000) = f(t_a, t_3, q_1, q_2, v, t_4(0)) \end{aligned} \quad (3.185)$$

Coming back to the main line of discourse, we say that the structure $f(g(X), h(V))$ holds the condition of split modality when f is (g, h) -partially monotonic and X and V are proper and improper uni-modal vectors.

THEOREM 3.16 - The Split Modality Nodes. If the continuous function $f(g(x), h(v))$ verifies the split modality condition, it follows that:

$$f^*(g(X), h(V)) = f_0^*(g^*(X), h^*(V)) = f_0^{**}(g^{**}(X), h^{**}(V)) = f^{**}(g(X), h(V)) \quad (3.186)$$

As a remark, the conventionally $f^*(g_1(X_1), \dots, g_n(X_n))$ designates $f^*(X_1, \dots, X_n)$, where $f(x_1, \dots, x_n) = f_0(g_1(x_1), \dots, g_n(x_n))$ and f_0 is the main operator of the f -syntactic tree.

A modal rational function f is called a modally conditioned optimal (mc-optimal or transitive) rational operator if f has some branches requiring uni-modality restrictions to qualify f as a node of a uni-incident optimal rational function. In a parallel sense, we will speak of conditionally rational operators as those which are not JM-commutable for any modality of their arguments.

If a rational operator has some branches requiring uniracity or uni-modality restrictions to be qualified as a node of a uni-incident optimal rational function, it is called conditionally an optimal operator. Then $fR(X)$ is c-tree-optimal if it contains some conditionally optimal nodes (with branches restricted by uniracity or uni-modality), and the restrictions for optimality upon the developments of these nodes are complied with.

THEOREM 3.17 - Conditional Optimality. If $fR(X)$ is uni-incident and c-tree-optimal, then $fR(X)$ is optimal,

$$f^*(X) = fR(X) = f^{**}(X) \quad (3.187)$$

THEOREM 3.18 - Coertion to Conditional Optimality. Let X be an interval vector, and fR defined in the domain X' and totally monotonic for all its multi-incident components. Let XD be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XD as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the overall one of the corresponding X -component. Let fR be tree-optimal in the domain X' ; in this case:

$$f^*(X) = fR(XD) = f^{**}(X) \quad (3.188)$$

Let X be an interval vector; let $fR(XD)$ be defined and c -tree-optimal on the domain X' and totally monotonic for all its multi-incident components. Let XD be the enlarged vector of X , such that each incidence of every multi-incident component of the subset with total monotonicity is included in XD as an independent component, but transformed into its dual if the corresponding incidence-point has a monotony-sense contrary to the overall one of the corresponding X -component. In this case:

$$f^*(X) = fR(XD) = f^{**}(X) \quad (3.189)$$

THEOREM 3.19 - Lateral Optimality. If the continuous function $f(g(x), h(y, v))$ is h -uniformly monotonic and with h JM-commutable for (Y, V) , and if X is a unimodal vector, Y and V the proper and the improper components of the h -argument, and X , Y and V have no common component, then,

$$\begin{aligned} f^*(g(X), h(Y, V)) &= f_0^*(g^*(X), h^*(Y, V)) \\ &= f_0^{**}(g^{**}(X), h^{**}(Y, V)) = f^{**}(g(X), h(Y, V)) \end{aligned} \quad (3.190)$$

For example, consider the real continuous function $f(u, y, v) = u(y+v)$, expressed by $f_0(g, h) = gh$ with $g(u) = u$ and $h(y, v) = y+v$. For $U = [1, 0]$, $Y = [-4, 0]$, $V = [2, 1]$ this function is not tree-optimal. Nevertheless it is c -tree-optimal and the conditioned optimal coercion theorem gives:

$$f^*(U, Y, V) = f^{**}(U, Y, V) = fR(U, Y, V) = fR([1, 0], [-4, 0], [2, 1]) = [0, 0] \quad (3.191)$$

3.4.4.6. N-DIMENSIONAL EXTENSIONS

THEOREM 3.20 - *-Interpretability of n-Dimensional Computations. Let $X \in I^*(R^m)$ and $f : R^m \rightarrow R^n$ defined by $f(x) = (f_1(x), \dots, f_n(x))$ continuous in X' . Let $(f_1R(A_1), \dots, f_nR(A_n)) \subseteq (B_1, \dots, B_n)$ be a system of interpretable outer-rounded computations of $(f^*_1(X_1), \dots, f^*_n(X_n))$, where $(X_1, \dots, X_n)$ are defined from X so as to have

the same proper components of X , and with each improper component of X transformed into its dual in every X_j except one. In this case:

$$U(x_p, X_p') Q^*(z, B) E(x_i, X_i') z = f(x_p, x_i) \quad (3.192)$$

with $X = (X_p, X_i)$ and $Q^*(z, B)$ being the sequence of n prefixes $Q(z_j, B_j)$ ($j = 1, \dots, n$) with the prefixes for corresponding to the universal delimiters heading the sequence.

THEOREM 3.21 - **-Interpretability of n -Dimensional Computations. Let $X \in I^*(R^m)$ and $f : R^m \rightarrow R^n$ defined by $f(x) = (f_1(x), \dots, f_n(x))$ continuous in X' . Let $(f_1 R(A_1), \dots, f_n R(A_n)) \supseteq (B_1, \dots, B_n)$ be a system of interpretable outer-rounded computations of $(f_1(X_1), \dots, f_n^*(X_n))$, where $(X_1, \dots, X_n)$ are defined from X so to have the same improper components of X , and with each proper component of X transformed into its dual in every X_j except one. In this case:

$$U(x_i, X_i') Q^*(z, Dual(B)) E(x_p, X_p') z = f(x_p, x_i) \quad (3.193)$$

For example, for the function $f: R^3 \rightarrow R^2$ defined by $f(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 x_2 - x_1)$ and continuous in X' with $X = ([6, 1], [6, 2], [-3, 2])$, we have:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\ f_2(x_1, x_2, x_3) &= x_1 x_2 - x_1 \\ A_1 &= ([1, 6], [6, 2], [-3, 2]) \\ A_2 &= ([6, 1], [2, 6]) \end{aligned} \quad (3.194)$$

and,

$$\begin{aligned} fR(A_1) &= [1, 6] + [6, 2] + [-3, 2] = [4, 10] \\ fR(A_2) &= [6, 1] * [2, 6] - [1, 6] = [6, 5] \\ B &= ([4, 10], [6, 5]) \end{aligned} \quad (3.195)$$

The interpretation of this result can be obtained combining those of the corrected components of the system $f_1(X_1)$,

$$U(x_1, [1, 6]') U(x_3, [-3, 2]') E(z_1, [4, 10]') E(x_2, [2, 6]') z_1 = x_1 + x_2 + x_3 \quad (3.196)$$

and,

$$U(x_2, [2, 6]') U(z_2, [5, 6]') E(x_1, [1, 6]') z_2 = x_1 x_2 - x_1 \quad (3.197)$$

giving,

$$U(x_3, [-3, 2]') U(z_2, [5, 6]') E(z_1, [4, 10]') E(x_1, [1, 6]') E(x_2, [2, 6]') \quad (3.198)$$

Reconsidering the problem, the fact is that in a computational program of the form:

$$f_1^*(X) \subseteq B_1, \dots, f_n^*(X) \subseteq B_n \quad (3.199)$$

assuming the *-semantics, no improper component of X can be repeated keeping its own modality in more than one $f_i(X)$, since multiple existential quantifiers over the universal variable cannot be factored out of the n semantic statements corresponding to the equations of the system. Beware the fact that the transformation of all the occurrences but one of each improper variable implies an operational normative for the involved material system. After imposing the previous condition, the functions $f_i^*(X)$ should be individually coerced to optimality (resulting in the computations $f_1R(A_1), \dots, f_nR(A_n)$) to obtain a component-wise optimality (the ceiling of n -dimensional optimality, unless the component computations $f_i(X)$ contain no common variable at all). Appendix D presents a resume, represented by a schematic tree, of all relationships between f^* , f^{**} and some rational computation (except roundings).

3.4.5. COMPUTATIONAL IMPLEMENTATION

In Modal Interval Analysis (MIA), the uncertainty is included in the mathematical model of the structure by replacing the uncertain parameters by intervals. The uncertain parameter can take any value within the limits of the interval. The result of this process is an interval value, obtained once the structural equations have been

analyzed with the previous presented theorems. Such theorems are implemented in specific libraries that can be used “a posteriori” to analyze any system.

The Finite Element Method (FEM) is one of the mainly used algorithms for the analyses of structural systems. Interval implementation can be utilized together with such methodology, Interval Finite Element Method (IFEM), presenting a sharp bound of possible nodal displacements for uncertainty treatment (Garcia *et al* 2004). The static equilibrium system of equations, basis of the structural model, is transformed into a system of interval equations that can be written as:

$$[K]_s \{\Delta\}_s = \{F\}_s \quad (3.200)$$

where K is the global interval matrix of the structure, F the applied interval load vector and Δ the unknown interval displacement vector. The problem introduction can be done through loading it from a local file. The steps used in Modal Interval Analysis (MIA) solver are:

- Definition of the variables (deterministic and intervals);
- Write the system of interval equations;
- Definition of the desired precision between the inner and the outer approximation;
- Execution: Solve the introduced problem. A time limitation of 60 seconds is fixed to avoid extremely long computations. If this time is reached, a partial solution is provided.

3.5. CONCLUSIONS

3.5.1. POINTS AND INTERVALS

In this chapter it were described two methodologies, Monte Carlo and Perturbation Techniques, that uses points and one methodology Modal Interval Analysis (MIA) that uses intervals for the uncertainty analysis of a system. It is now time to make a real comparison between them, points and intervals calculations, providing the scope of applicability of each, considering both what is possible now and what may be possible in the future with better tools and new algorithms.

Advocates of point arithmetic argue that Interval Analysis is useless because the intervals grow too quickly to provide meaningful results. Advocates of Interval Analysis argue that points are useless because the results might be very far from the correct result without warning to the user. The following characterization suggests that there are problems to which only Interval Analysis can be applied, problems to which both can be applied and problems to which only point methods can be applied.

What intervals can do, that points cannot?

1) Guarantee that computational results are correct. Intervals are guaranteed to contain the mathematically correct answer. Points are approximations that have no information regarding their accuracy;

2) Solve global optimization problems. Point methods cannot generally prove that the minimum and maximum they discover is the global minimum or maximum; that would require exhaustively computing the function value for every possible input;

3) Create algebraically closed computer representations. The values of some simple point expressions, like $0/0$ and $1/0$, are not a point but a set. Thus point systems must represent these as errors, just as they would trap the square root of a negative number as an error. Intervals represent numbers, so they easily handle division by zero as well as operations on the sets of numbers that result;

4) Correctly handle re-ordering of computations. In point arithmetic, $(a+b)+c$ is not necessarily equal to $a+(b+c)$ (e.g. $a=10^{20}$, $b=1$ and $c=-10^{20}$ on any system with a standard floating point arithmetic, the first expression evaluates to exactly 0.00 whereas

the second one evaluates top exactly 1.00). In contrast, Interval Analysis produces intervals that may have different endpoints but always contain the correct answer;

5) Reveal loss of accuracy in solutions. Points do not carry any information regarding loss of accuracy, so applications which require this must resort to indirect methods that suggest what the inaccuracy is but cannot bound it. Intervals explicitly express accuracy by how wide or narrow they are, so they rigorously bound the loss of accuracy in the result;

6) Distinguish control flow program faults from numerical program faults. Almost any change to a computer system can change a computed answer with point methods. When the numerical answer differs, there is no way to know whether it is because there is a real flaw in the logic or if it is simply the normal variance inherent in floating-point computations. With intervals, two runs that claim to compute the same thing must always generate overlapping intervals. If they do not overlap, then this proves there is a logic error.

What both intervals and points can do?

1) Obtain solutions to design problems where proof of optimality is not required. The term “optimization” is usually a misnomer, and “improvement” is what is really meant. Engineers often use computers to improve designs, not to prove that no better design is possible. If the workload is defined in terms of a human-meaningful objective, it is often the case both interval and point methods can be applied;

2) Simulate physical behaviour where the method is known to be numerically stable. For some problems, it is easy to show that numerical errors are attenuated instead of amplified as the computation progresses (e.g. if x is inexact and positive and you compute $(x+10)/10$, the inexactness of the result will be less than that in x). Both interval and point methods benefit from this stability;

3) Explore space of answers resulting from inexact inputs. The inputs to programs are often treated as exact numbers, but a more realistic approach is to recognize that the initial data and physical constants are only known to a few decimal places. The point method can sample the space of possible answers that result from this inexactness simply by repeatedly running the program with slightly different point inputs chosen from their possible range, like in Monte Carlo analysis. The interval method explicitly expresses inexactness as an interval of nonzero width and propagates the error with a single run. In general, the point method generates a subset of the solution space and the interval method generates a superset. The ideal bound on the set

of possible solutions lies between. Either methods can, more closely, approach the ideal, through more computation;

4) Compute with precisely expressive numbers. It is possible to restrict floating-point computation to numbers and operations that involving no rounding or overflow. For example, whole numbers within half the range of the mantissa can be added and subtracted perfectly, and fractions involving only powers of two can be divided and multiplied within the dynamic range. “Reverse error analysis” sometimes shows that a rounded point calculation is the exact solution to a problem that differs only slightly from the one specified, and a user may be satisfied with this argument. A rigorous error bound on the value of an integral, for instance, can be computed entirely with floating-point numbers and no rounding control. The same rigorous bound could also be computed with intervals;

5) Solve problems involving ranges that are large compared to rounding error. If the natural data type in a problem is a range of numbers, then intervals are an efficient way to express them. If the range is large compared to the rounding error, then it is possible to simply use point values to represent the top and bottom of the range and do all the computations for each case. The point values will not round up and down correctly to guarantee containment, but for some applications that does not matter.

What points can do, that intervals cannot?

1) Run Monte Carlo (simulate) methods. Point methods excel at guessing. When guessing is done at very high speed with a statistically sound pseudo-random number generator, computes can generate useful guidance via the Law of Large Numbers. The range of the solution is not rigorous as it is with intervals, but the Monte Carlo approach can automatically show how probability densities in input propagate to the probability density of the outputs. The random numbers can be used to simulate events that are actually random in nature. Because the accuracy of Monte Carlo methods tends to increase as the square root of the number of trials, independent of the number of degrees of freedom in the problem, it can render tractable some problems that have many degrees of freedom. Interval methods cannot match any of these features of the Monte Carlo approach;

2) Solve large sets of linear equations. Physical processes are often linear. Point methods require time, in the worst case, that increases as the cube of the number of unknown variables. Interval methods do not work well, because a tight bound cannot be obtained. It is important to note that linear systems that arise in much of scientific

computing are the results of a discretization of a continuous system domain, so an error is committed in using a single value to represent a set of varying values. A reformulation of such problems might allow interval methods to work and even exceed the utility of the point method. In addition, interval methods exist for iterative solution of certain important types of linear equations, having the general case, no good solution;

3) Handle the “single use expression” problem where no expression folding is possible or tractable. If x is a point, then $x-x$ always evaluates to zero. If x is an interval of width w , then $x-x$ is an interval about zero with a width $2w$. Similarly, x divides by x is not identically to 1.0 but instead an interval that contains 1.0. The general problem is that interval arithmetic has no knowledge of whether it is representing an entire set or a single point within a set. So whereas points lose accuracy as a computation progresses and have no way to record that loss, intervals lose the ability to distinguish point-of-a-set from entire-set computations and have no way to record that loss. This is a primary reason why the best interval algorithm is seldom the same as the best point algorithm. One way of describing this category is “Algorithms for which practical point methods are known but all interval methods known create interval results too wide to be useful”.

4) Perform sensitivity analysis on existing programs. A point algorithm can be run multiple times with a perturbed input to check how much the answer is affected. This property allows identifying instabilities in the algorithm that an interval method cannot. If interval arithmetic is applied to an unstable algorithm, it will produce a wide result but the converse is not always true.

5) Estimate time-evolved behaviour where no rigorous model exists. Scientific programs start as mathematical models of some aspects of the world that cannot be made rigorous. The use of interval arithmetic will not restore rigor and will often produce wide intervals of little use as the system is evolved through time. Many point methods are empirically successful at estimating time-evolved behaviour.

6) Partial differential equations where no interval approach has been formulated. Physical problems that are typically expressed as partial differential equations remain a challenge for interval methods. Innovative approaches could change this, but even a complete set of libraries and tools in product form will not solve it. Some problems need not to be expressed by partial differential equations but as integral equations or as the limiting behaviour of physical systems with discrete instead of continuous elements. However these application areas cannot be solved by interval analysis, also.

3.5.2. OVERVIEW

The simulation techniques, as Monte Carlo, allow the integral numerical calculus with an impossible analytical resolution. The Monte Carlo method is applied in a general way wherever the variables distribution type. The error associated to these kinds of techniques is perfectly controlled through the sample's number. It is verified that to a sample with a size tending to infinite ($n \rightarrow \infty$), the result converges to the right one. A generalized critic to Monte Carlo method is the high computational time as it is necessary to run several times the structural model to obtain a high resolution output result. However the application of variance reduction techniques may turn this method more efficient. There are no restrictions in Monte Carlo application (Henriques, 1998).

Perturbation Method presents the following characteristics: it is a very efficient method in terms of time spent in computational calculation; to take into account the uncertainties in a finite element framework, it requires additionally the calculation of an inverse of a matrix with equal dimensions of a stiffness matrix, the evaluation of partial derivatives and some product of matrices to calculate matrix of covariance; the dispersion of structural response is available with a single structural analysis; correlation between variables could be easily taken into account; information about variables uncertainty is defined only by two parameters (mean value and standard deviation); the implementation of this method in a finite element framework requires addition of new modulus in the computer code; the uncertainties are defined by statistical parameters and not by intervals.

Modal Interval Analysis (MIA) is a methodology which considers the whole uncertainty, warranting that the real answer is within the range of output values. The computational cost during the analysis was considerable, because the tolerance has to be incremented so the response could be the most certain one. It was also verified that the computational cost is directly influenced by the dimension of the stiffness matrix. Higher dimension corresponds to a higher cost maintaining the output accuracy.

CHAPTER 4.

APPLICATIONS

4.1. INTRODUCTION

4.1.1. GENERAL CONSIDERATIONS

The behavior assessment or the design of structural systems involves uncertainty evaluation. This is mainly due to error estimation of theoretical models used in analysis, to geometrical imperfections, and to inherent variability of materials and actions (Ayyub, 1998; Gayton *et al.*, 2004). To analyze such uncertainty, probabilistic and reliability techniques have been applied increasingly in the last years, but, till now, the generalized application of these techniques has been delayed by the inefficiency to solve complex or large problems (Kharmanda *et al.*, 2002; Schueremans and Gemert, 2003). In fact, their application is rather simple when an explicit formulation of the structural problem exists. However, when there are not explicit relations between variables, such as in finite element method (FEM), usually several analysis of the same problem should be performed to evaluate the uncertainty of structural response (Ghanem and Spanos, 2003; Schenk and Schueller, 2005). To solve it, two techniques were introduced in Chapter 3, namely, the Modal Interval Analysis (MIA) (Gardenyes *et al.*, 2001; Hansen, 1992; Jaulin *et al.*, 2001; Rao and Berke, 1997, SIGLA/X group, 1999) and the Perturbation Technique (Altus *et al.*, 2005; Eibl and Schmidt-Hurtienne, 1995). In this chapter, it will be presented their application into civil engineer problems.

The behavior assessment of real structures is an important issue in civil engineer field. The main objective of such assessment is to identify any structural abnormal behavior. To perform it, more efficient structural monitoring systems have been recently developed (Frangopol *et al.*, 2001; Matos *et al.*, 2005b). In this chapter it is presented a methodology which is based in a direct and consistent comparison between collected data and results obtained from numerical system (Matos *et al.*, 2005b). The existent uncertainty is easily treated by each of the previously presented techniques. Another methodology for structural assessment, based on Modal Interval Analysis (MIA) applied to a numerical Quantified Constraint Satisfaction Problem (QCSP), is also described (Herrero *et al.*, 2004, Herrero *et al.*, 2005). This is a recent developed technique which uses the finite element formulation and the respective system of equations.

4.1.2. CIVIL ENGINEERING APPLICATIONS

It is necessary, at this point, to distinguish assessment from design of civil engineer structures. The standard procedure in the design process of any structure is as follows. At first, the geometry is defined. Afterwards the structure typology is chosen, namely, support types and span lengths. The static system is so defined and the cross-sections dimensions are assumed. The applied loads are so considered, according to information from design codes. In the next step, the load effects in the structural elements are calculated and the capacity of the structural members is determined using values of the strength properties of materials and the design formulas provided in the codes. When the capacity of all members is greater than the calculated load effects the process may stop.

When performing the structural assessment, at first, since the structure exists, its geometry is already determined and can be measured. Furthermore, the material properties can be quantified using one of the available non-destructive methods. Alternatively they can be assumed based on design specifications and data obtained within quality control procedures. The acting loads can also be obtained due to measurements. For example, the self weighting can be determined by weighing the bridge with hydraulic jacks and the temperature obtained by temperature sensors. Additionally, the load effects in any structure element can be determined using, for example, displacement transducers. The numerical model of the structure can so be calibrated in order to predict distribution of internal forces between the structural members with greater accuracy. Sometimes, load tests are realized in order to calibrate a model that can be so used to predict the structure long term behavior (Sousa *et al.*, 2005). The main objective of any structural assessment is to evaluate its performance in real time face to real loads.

Due to all above mentioned facts it is evident that the amount of information available in the process of assessment of existent structures is significantly higher than in the structure design. Therefore, the uncertainty related to the assessment is generally lower than uncertainty characteristic for the design. Nevertheless, it has to be stated that in practical situations not all information about an existing structure will be available from measurements and some of the available measurement data may be of low quality. Thus, the uncertainty related to our limited knowledge of the actual structure state may be still significant (Wisniewski, 2007).

4.2. COMPARISON OF DIFFERENT NUMERICAL METHODOLOGIES

4.2.1. FINITE ELEMENT ANALYSIS

Systems in civil engineering are complex. To face this problem a methodology, Finite Element Method (FEM), is adopted. This methodology consists in dividing a continuous system in small parts (finite elements). The continuous analysis is converted in a discrete one and, from the mathematical point of view, a complex equation transforms in a system of equations. The equilibrium static equation, used with FEM for the analysis of most civil engineering systems, can be defined as:

$$[K]_s \{\Delta\}_s = \{F\}_s \quad (4.1)$$

where the sub index s means ‘static’, $[K]$ is the stiffness matrix, $\{\Delta\}$ the generalized displacement vector and $\{F\}$ the load vector. The Non-linear FEM is used when exists material or geometrical non linearity. On such analysis the system of equations is continuously used in an iterative way, till the final solution is achieved (Bathe, 1996).

It is necessary, at this point, to introduce the concept of Interval Finite Element Method (IFEM), which is the direct application of Modal Interval Analysis (MIA) with the FEM to analyze civil engineering structures. In Garcia *et al.* (2004) and Matos *et al.* (2004a), the uncertainty is included in the mathematical model of the structure by replacing the uncertain parameters by intervals. The result of this process is an interval value, which is obtained once the structural equations have been analyzed with interval theorems. In IFEM the equation 4.1., can be expressed as follows:

$$\begin{bmatrix} [\underline{K}_{11}, \bar{K}_{11}] & [\underline{K}_{12}, \bar{K}_{12}] & \dots & [\underline{K}_{1n}, \bar{K}_{1n}] \\ [\underline{K}_{21}, \bar{K}_{21}] & [\underline{K}_{22}, \bar{K}_{22}] & \dots & [\underline{K}_{2n}, \bar{K}_{2n}] \\ \vdots & \vdots & \ddots & \vdots \\ [\underline{K}_{n1}, \bar{K}_{n1}] & [\underline{K}_{n2}, \bar{K}_{n2}] & \dots & [\underline{K}_{nn}, \bar{K}_{nn}] \end{bmatrix} \begin{Bmatrix} [\underline{\Delta}_1, \bar{\Delta}_1] \\ [\underline{\Delta}_2, \bar{\Delta}_2] \\ \vdots \\ [\underline{\Delta}_n, \bar{\Delta}_n] \end{Bmatrix} = \begin{Bmatrix} [\underline{f}_1, \bar{f}_1] \\ [\underline{f}_2, \bar{f}_2] \\ \vdots \\ [\underline{f}_n, \bar{f}_n] \end{Bmatrix} \quad (4.2)$$

being $[K]$ the interval stiffness matrix, $\{\Delta\}$ the interval displacement vector and $\{F\}$ the interval load vector.

4.2.2. HYPERSTATIC BEAM

The random response of a hyperstatic steel beam due to random fluctuation in material properties and applied loads is studied here (Henriques, 2006a). To show their advantages and disadvantages, both Modal Interval Analysis (MIA) and Perturbation Techniques were used. The obtained results are then compared with the ones due to Monte Carlo Analysis and with deterministic values.

It is a beam, clamped at the left side and simply supported at the right one, with a total span of 4m. This structure is subjected to a uniform distributed load (p). Due to the low level load, this beam response follows a linear elastic regime. The steel modulus of elasticity, E , and the uniform load level, p , are considered as uncertain parameters of the model and are defined by random variables which are not correlated between them. Other mechanical parameters are considered to be deterministic.

To analyze such structure it is used a Finite Element Methodology (FEM). The beam is divided in two 1D Euler-Bernoulli beam finite elements (Figure 4.1). Each element is composed by two nodes with two degrees of freedom per node. According to the equilibrium static equation 4.1 it is necessary to define the beam stiffness matrix $[K]$ and its load vector $\{F\}$. To do so, and in a first step, the element stiffness matrix $[K_e]$ and its load vector $\{F_e\}$ must be determined.

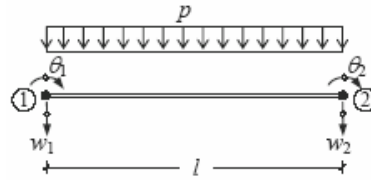


Figure 4.1 – 1D Euler-Bernoulli beam element.

$$K_e = \begin{bmatrix} \frac{4EI}{l} & \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} \\ \frac{2EI}{l} & \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} \\ -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} \end{bmatrix} \quad (4.3)$$

$$F_e = \left\{ \frac{pl^2}{12} \quad \frac{pl}{2} \quad -\frac{pl^2}{12} \quad \frac{pl}{2} \right\}^T \quad (4.4)$$

The general stiffness matrix $[K]$ and load vector $\{F\}$ of the structure (Figure 4.2) is then obtained by regrouping the stiffness matrix and the load vectors of both elements. It is, also, defined the generalized displacement vector $\{\Delta\}$.

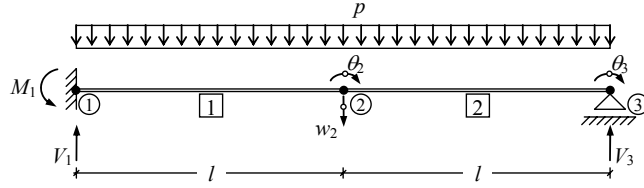


Figure 4.2 – Finite element model.

$$K = \frac{EI}{l} \begin{bmatrix} 4 & \frac{6}{l} & 2 & -\frac{6}{l} & 0 & 0 \\ \frac{6}{l} & \frac{12}{l^2} & \frac{6}{l} & -\frac{12}{l^2} & 0 & 0 \\ 2 & \frac{6}{l} & 8 & 0 & 2 & -\frac{6}{l} \\ -\frac{6}{l} & -\frac{12}{l^2} & 0 & \frac{24}{l^2} & \frac{6}{l} & -\frac{12}{l^2} \\ 0 & 0 & 2 & \frac{6}{l} & 4 & -\frac{6}{l} \\ 0 & 0 & -\frac{6}{l} & -\frac{12}{l^2} & -\frac{6}{l} & \frac{12}{l^2} \end{bmatrix} \quad (4.5)$$

$$F = \frac{pl}{2} \left\{ \frac{l}{6} \quad 1 \quad 0 \quad 2 \quad -\frac{l}{6} \quad 1 \right\}^T \quad (4.6)$$

$$\Delta = \{\theta_1 \quad w_1 \quad \theta_2 \quad w_2 \quad \theta_3 \quad w_3\}^T \quad (4.7)$$

Considering the restrictions imposed by the supports (clamped in node 1 and simply supported at node 3) the displacement components θ_1 , w_1 and w_3 are null. Accordingly, the incognita vector $\{U\}$ is composed by three displacement components (θ_2 , w_2 and θ_3) and by the support reactions (M_1 – moment reaction at node 1, V_1 and V_3 – vertical reaction at node 1 and 3).

$$U = \{M_1 \quad V_1 \quad \theta_2 \quad w_2 \quad \theta_3 \quad V_3\}^T \quad (4.8)$$

The system of equations that controls the static equilibrium (equation 4.1) of this structure is then defined by:

$$\underbrace{\begin{bmatrix} 1 & 0 & \frac{2EI}{l} & -\frac{6EI}{l^2} & 0 & 0 \\ 0 & 1 & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & 0 & 0 \\ 0 & 0 & \frac{8EI}{l} & 0 & \frac{2EI}{l} & 0 \\ 0 & 0 & 0 & \frac{24EI}{l^3} & \frac{6EI}{l^2} & 0 \\ 0 & 0 & \frac{2EI}{l} & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 \\ 0 & 0 & -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 1 \end{bmatrix}}_{[K]} \cdot \underbrace{\begin{Bmatrix} M_1 \\ V_1 \\ \theta_2 \\ w_2 \\ \theta_3 \\ V_3 \end{Bmatrix}}_{\{U\}} = \underbrace{\begin{Bmatrix} \frac{pl^2}{12} \\ \frac{pl}{2} \\ 0 \\ pl \\ -\frac{pl^2}{12} \\ \frac{pl}{2} \end{Bmatrix}}_{\{F\}} \quad (4.9)$$

All the applied methodologies are based on the equilibrium static equation of the structure, defined in equation 4.9. The idea is to obtain the range of values for the displacements (θ_2 , w_2 and θ_3) of node 2 and node 3, using the specified techniques.

For the analysis of this structure, the following values are considered: total span length $2l = 4\text{m}$, moment of inertia $I = 2 \times 10^{-5} \text{m}^4$, modulus of elasticity (E) defined by a normal distribution with a mean of 200GPa and a coefficient of variation of 5% (Figure 4.3) and an applied load (p) defined by a normal distribution with a mean of 10kN/m and a coefficient of variation of 15% (Figure 4.4).

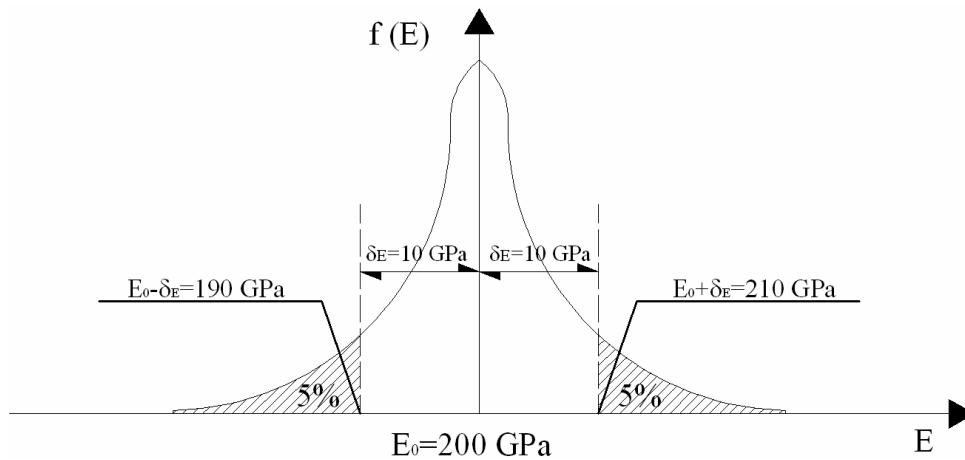


Figure 4.3 – Normal Distribution of Elasticity Modulus (E).

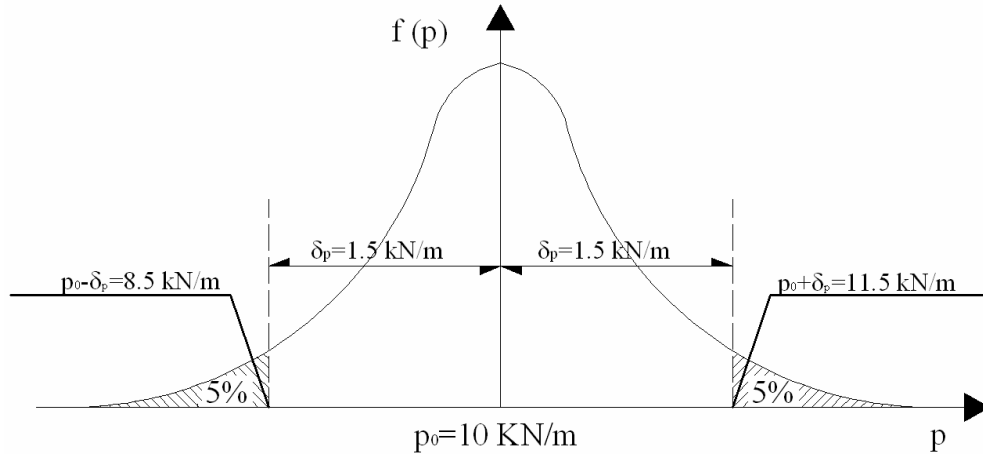


Figure 4.4 – Normal Distribution of Applied Load (p).

Initially, the structure will be analyzed by the Perturbation Technique. Firstly, it will be determined the partial derivatives of $[K]$ and $\{F\}$ in order to the aleatory variables E and p .

$$\frac{\partial K}{\partial p} = \frac{\partial F}{\partial E} = 0 \quad (4.10)$$

$$\frac{\partial K}{\partial E} = \frac{I}{l} \begin{bmatrix} 0 & 0 & 2 & -\frac{6}{l} & 0 & 0 \\ 0 & 0 & \frac{6}{l} & -\frac{12}{l^2} & 0 & 0 \\ 0 & 0 & 8 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{24}{l^2} & \frac{6}{l} & 0 \\ 0 & 0 & 2 & \frac{6}{l} & 4 & 0 \\ 0 & 0 & -\frac{6}{l} & -\frac{12}{l^2} & -\frac{6}{l} & 0 \end{bmatrix} \quad (4.11)$$

$$\frac{\partial F}{\partial p} = \frac{l}{2} \left\{ \frac{l}{6} \quad 1 \quad 0 \quad 2 \quad -\frac{l}{6} \quad 1 \right\}^T \quad (4.12)$$

Once it is defined the stiffness matrix $[K]$, the load vector $\{F\}$, and their partial derivatives, we proceed to the calculus of the unknowns mean values $\{U_0\}$, by solving the system of equations 4.9 and taking the mean values of the parameters (E_0 and p_0):

$$U_0 = \begin{Bmatrix} M_1 \\ V_1 \\ \theta_2 \\ w_2 \\ \theta_3 \\ V_3 \end{Bmatrix}_0 = \begin{Bmatrix} 20 \text{ kN.m} \\ 25 \text{ kN} \\ 0.00083(3) \text{ rad} \\ 0.0033(3) \text{ m} \\ -0.0033(3) \text{ rad} \\ 15 \text{ kN} \end{Bmatrix} \quad (4.13)$$

Matrix $[K_0]^{-1}$ is then obtained substituting the parameter mean values in the stiffness matrix $[K]$ (equation 4.9) and then processing it inversion. The partial derivate matrix (equation 4.11) is determined substituting it parameters by their mean values. Considering δX as the standard deviation of the aleatory variables, E and p, respectively, $\delta E = 200 \cdot 5\% = 10 \text{ GPa}$ and $\delta p = 10 \cdot 15\% = 1.5 \text{ kN/m}$, we may calculate the column, corresponding to the contribution of the aleatory variable E.

$$\begin{aligned} & \left(K_0^{-1} \cdot \frac{\partial K}{\partial E} \right) \cdot U_0 \cdot \delta E = \\ & = 0.5 \cdot 10^{-8} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{Bmatrix} 20 \\ 25 \\ 0.00083(3) \\ 0.0033(3) \\ -0.0033(3) \\ 15 \end{Bmatrix} \cdot 10^7 = \\ & = \{0 \quad 0 \quad 0.000041(6) \quad 0.00016(6) \quad -0.00016(6) \quad 0\}^T \end{aligned} \quad (4.14)$$

Then, it is possible to obtain the column that respects to the contribution of the aleatory variable p.

$$\left(-K_0^{-1} \cdot \frac{\partial F}{\partial p} \right) \cdot \delta p = \begin{Bmatrix} 2 \\ 2.5 \\ 0.00083(3) \\ 0.0033(3) \\ -0.0033(3) \\ 1.5 \end{Bmatrix} \cdot 1.5 = \begin{Bmatrix} 3 \\ 3.75 \\ 0.000125 \\ 0.0005 \\ -0.0005 \\ 2.25 \end{Bmatrix} \quad (4.15)$$

Having into account that $[C_p]$ is an identity matrix (the variables are not correlated between each other) it is possible to calculate the following covariance matrix of structural response $[C_u]$:

$$C_u = \begin{bmatrix} 9 & 11.25 & 3.75 \times 10^{-4} & 1.5 \times 10^{-3} & -1.5 \times 10^{-3} & 6.75 \\ 11.25 & 14.06 & 4.69 \times 10^{-4} & 1.88 \times 10^{-3} & -1.88 \times 10^{-3} & 8.44 \\ 3.75 \times 10^{-4} & 4.69 \times 10^{-4} & 1.736 \times 10^{-8} & 6.94 \times 10^{-8} & -6.94 \times 10^{-8} & 2.81 \times 10^{-4} \\ 1.5 \times 10^{-3} & 1.88 \times 10^{-3} & 6.94 \times 10^{-8} & 2.778 \times 10^{-7} & -2.78 \times 10^{-7} & 1.12 \times 10^{-3} \\ -1.5 \times 10^{-3} & -1.88 \times 10^{-3} & -6.94 \times 10^{-8} & -2.78 \times 10^{-7} & 2.778 \times 10^{-7} & -1.12 \times 10^{-3} \\ 6.75 & 8.44 & 2.81 \times 10^{-4} & 1.12 \times 10^{-3} & -1.12 \times 10^{-3} & 5.06 \end{bmatrix} \quad (4.16)$$

The variances of displacement components, θ_2 , w_2 and θ_3 , correspond to the values obtained at the diagonal of matrix $[C_u]$, respectively, at lines (or columns) 3, 4 and 5. Hence, standard deviation of θ_2 , w_2 and θ_3 are:

$$\delta\theta_2 = \sqrt{1.736 \times 10^{-8}} = 0.000132 \text{ rad}$$

$$\delta w_2 = \sqrt{2.778 \times 10^{-7}} = 0.000527 \text{ m}$$

$$\delta\theta_3 = \sqrt{2.778 \times 10^{-7}} = 0.000527 \text{ rad}$$

Using the mean value (deterministic value) from equation 4.13, and the calculated standard variation, it is possible to determine the output range for each displacement. The obtained range is, for θ_2 , between $\theta_2 - \delta\theta_2 = 0.00083(3) - 0.000132 = 0.0007013$ and $\theta_2 + \delta\theta_2 = 0.00083(3) + 0.000132 = 0.0009653$, for w_2 , within $w_2 - \delta w_2 = 0.00333(3) - 0.000527 = 0.002806$ and $w_2 + \delta w_2 = 0.00333(3) + 0.000527 = 0.003860$, and for θ_3 , among $\theta_3 - \delta\theta_3 = -0.00333(3) - 0.000527 = -0.003860$ and $\theta_3 + \delta\theta_3 = -0.00333(3) + 0.000527 = -0.002806$.

In Modal Interval Analysis (MIA), only uniform distributions can be considered. In order to obtain a good agreement with the results obtained by the application of the other techniques, the following approximation is executed (Figure 4.5 and Figure 4.6). The minimum ($\underline{X}$) and the maximum ($\bar{X}$) of each interval variable, characterized by warranting a probability of occurrence in uniform distribution of 100%, are defined by a probability of occurrence in a normal distribution of 90%, and so, the previously presented values for coefficients of variation are used (Figure 4.3 and Figure 4.4). The interval limits are so obtained by adding and subtracting the standard deviation of each random variable to the respective mean value.

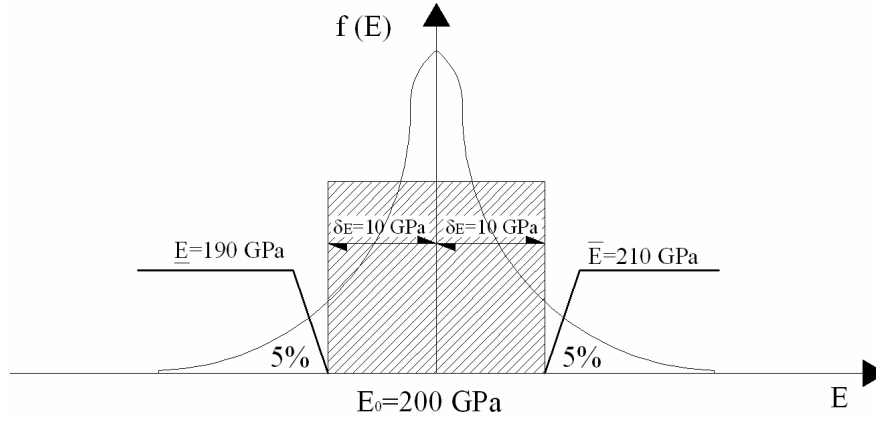


Figure 4.5 – Approximation for the aleatory variable E.

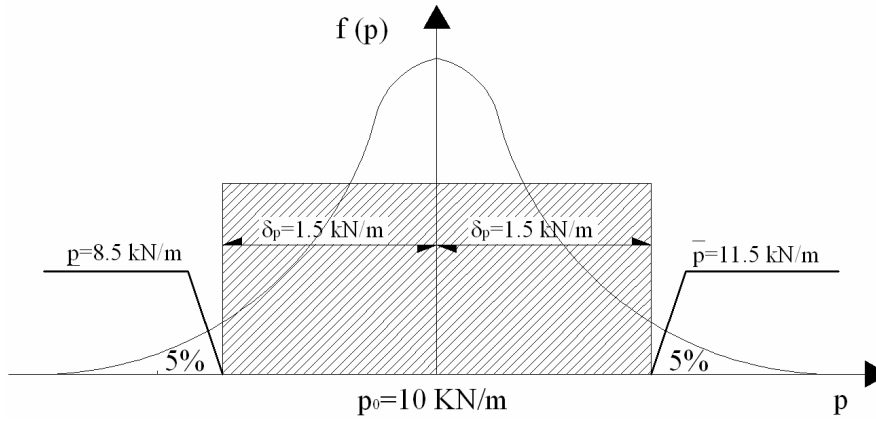


Figure 4.6 – Approximation for the aleatory variable p.

For both aleatory variables, the normal distribution is converted into a uniform one with equal mean value (X_0), and a lower and an upper limit value of $[X_0 - \delta_X, X_0 + \delta_X]$. While the modulus of elasticity (E) presents a mean value of 200 GPa and is represented by an interval value, $E = [190, 210]$ GPa, the applied load (p) presents a mean value of 10 kN/m and is defined by the interval value $P = [8.5, 11.5]$ kN/m.

In such analysis the stiffness matrix $[K]$ and the load vector $\{F\}$ is converted into an interval stiffness matrix $[\underline{K}, \overline{K}]$ and in an interval load vector $\{\underline{F}, \overline{F}\}$, replacing the parameters E and p by their interval values. It is then applied the Interval Finite Element Methodology (IFEM), in order to obtain the interval incognita vector $\{\underline{U}, \overline{U}\}$. Using equation 4.9 as the system of interval equations node 2 and node 3 interval displacements (θ_2 , w_2 and θ_3) are obtained. The calculus process is the following:

$$[\underline{K}, \overline{K}] \cdot \{\underline{U}, \overline{U}\} = \{\underline{F}, \overline{F}\} \Leftrightarrow \{\underline{U}, \overline{U}\} = [\underline{K}, \overline{K}]^{-1} \cdot \{\underline{F}, \overline{F}\} \quad (4.17)$$

$$\begin{aligned}
 & \underbrace{\begin{Bmatrix} \overline{M_1}, \overline{M_1} \\ \overline{V_1}, \overline{V_1} \\ \overline{\theta_2}, \overline{\theta_2} \\ \overline{w_2}, \overline{w_2} \\ \overline{\theta_3}, \overline{\theta_3} \\ \overline{V_3}, \overline{V_3} \\ \overline{U}, \overline{U} \end{Bmatrix}}_{\{U, \overline{U}\}} = \frac{[8.5, 11.5]}{[190, 210] \cdot 10^6 \cdot 2 \cdot 10^{-5}} \underbrace{\begin{bmatrix} 1 & 0 & -0.125 & 0.75 & -0.5 & 0 \\ 0 & 1 & -0.28125 & 0.6875 & -0.375 & 0 \\ 0 & 0 & 0.3125 & 0.125 & -0.25 & 0 \\ 0 & 0 & 0.125 & 0.583(3) & -0.5 & 0 \\ 0 & 0 & -0.25 & -0.5 & 1 & 0 \\ 0 & 0 & 0.28125 & 0.3125 & 0.375 & 1 \end{bmatrix}}_{[K, \overline{K}]} \underbrace{\begin{Bmatrix} 0.33(3) \\ 1 \\ 0 \\ 2 \\ -0.33(3) \\ 1 \end{Bmatrix}}_{\{F, \overline{F}\}} = \\
 & \frac{[8.5, 11.5]}{[190, 210] \cdot 10^6 \cdot 2 \cdot 10^{-5}} \begin{Bmatrix} 2 \\ 2.5 \\ 0.33(3) \\ 1.33(3) \\ -1.33(3) \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 17.8947, 21.9048 \\ 22.3684, 27.3810 \\ 0.0007456, 0.0009127 \\ 0.002982, 0.003650 \\ -0.003650, -0.002982 \\ 13.4211, 16.4286 \end{Bmatrix}
 \end{aligned} \tag{4.18}$$

In order to validate the results obtained with the two previous methods, it was applied the Monte Carlo analysis. It were considered 10000 samplers of the problem, assigning to the variables E and p aleatory generated values accordingly to normal distribution laws, defined by the same parameters as the ones used in Perturbation Technique. Figure 4.7 presents the oscillation of the average value and Figure 4.8 of standard deviation of displacement w_2 in function of the number of samplers. It is verified that the average stabilizes for values placed in between 3.340 mm and 3.350 mm and the standard deviation for values situated between 0.530 and 0.535 mm.

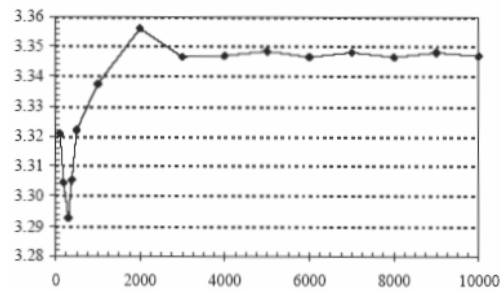


Figure 4.7 – Oscillation of average value of w_2 with the sampler's number.

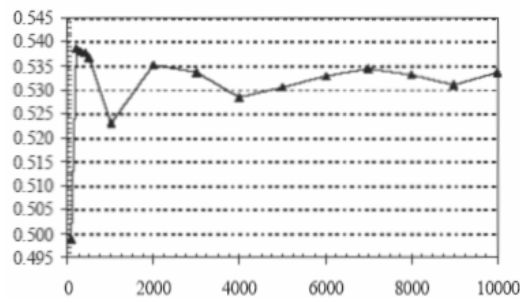


Figure 4.8 – Oscillation of standard deviation of w_2 with the number of sampler's number.

Table 4.1 presents the results obtained with the application of Monte Carlo analysis (average value and standard deviation for each variable).

Table 4.1 – Monte Carlo analysis. Results obtained with 10000 samplers simulation.

E (GPa)		p (kN/m)			
Average value	Standard deviation	Average value	Standard deviation		
200.1	10.08	10.02	1.514		
θ_2 (10^{-3} rad)		w_2 (mm)		θ_3 (10^{-3} rad)	
Average value	Standard deviation	Average value	Standard deviation	Average value	Standard deviation
0.837	0.133	3.347	0.533	-3.347	0.533

Table 4.2 presents the main results obtained with all the applied methods and the deterministic ones. The nomenclature used here is the same of Figure 4.2 being the under scored variables the minimum value and the upper scored the maximum one. For deterministic analysis, just one value is presented. Figure 4.9 presents the beam deformed shape obtained for all methodologies.

Table 4.2 – Obtained results - Comparison of methods.

Solution	$\underline{w}_2$ (m)	$\overline{w}_2$ (m)	$\underline{\theta}_2$ (rad)	$\overline{\theta}_2$ (rad)	$\underline{\theta}_3$ (rad)	$\overline{\theta}_3$ (rad)
Perturbation Technique	$2,806 \times 10^{-3}$	$3,860 \times 10^{-3}$	$7,013 \times 10^{-4}$	$9,653 \times 10^{-4}$	$-3,860 \times 10^{-3}$	$-2,806 \times 10^{-3}$
Modal Interval Analysis	2.982×10^{-3}	3.651×10^{-3}	$7,456 \times 10^{-4}$	$9,127 \times 10^{-4}$	$-3,651 \times 10^{-3}$	$-2,982 \times 10^{-3}$
Monte Carlo Analysis	2.810×10^{-3}	3.880×10^{-3}	7.040×10^{-4}	9.700×10^{-4}	-3.880×10^{-3}	-2.810×10^{-3}
Deterministic Values	$3.33(3) \times 10^{-3}$		$8.33(3) \times 10^{-4}$		$-3.33(3) \times 10^{-3}$	

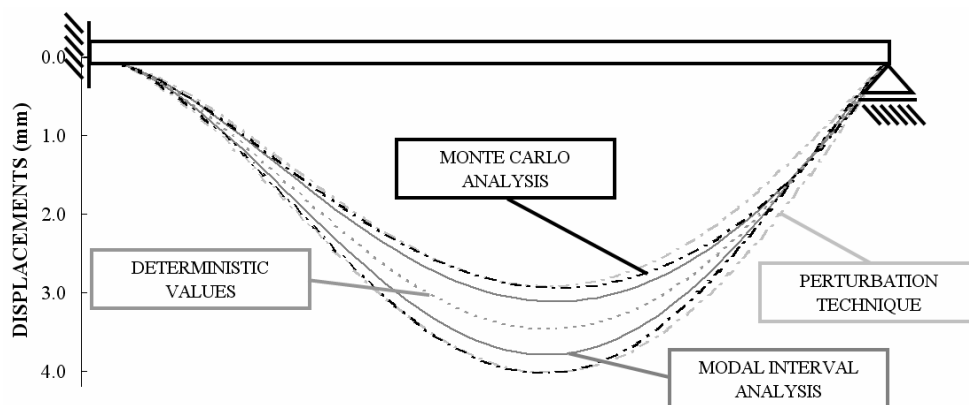


Figure 4.9 – Obtained results – Comparison of methods.

4.2.3. ONE-FLOOR FRAME

To evaluate the accuracy of the proposed methods (Perturbation Techniques and Modal Interval Analysis - MIA), a simple frame, subjected to different load cases, is analyzed and the obtained results are compared with values from a deterministic analysis and from the application of Monte Carlo Methodology (Veiga *et al*, 2006 a; Veiga *et al*, 2006 b).

The studied frame is composed by two piles and one beam (Figure 4.10). A uniform distributed load along the beam, p , and a vertical force, F , at the beam end are applied. To analyze such structure it is used a Finite Element Methodology (FEM). The structure is divided in five 1D Euler-Bernoulli beam finite elements. Each element is composed by two nodes with three degrees of freedom per node (one rotation and two displacements). Firstly, it is specified the element stiffness matrix $[K_e]$ and load vector $\{F_e\}$. Then the general stiffness matrix $[K]$ and load vector $\{F\}$ of the structure is obtained by regrouping the stiffness matrix and the load vector of all elements. At the same time, the generalized displacement vector $\{\Delta\}$ is defined. Considering the support restrictions (clamped at node 1 and 4) it is determined the incognita vector $\{U\}$. Finally the structural system of equations, basis of all applied methodologies, is obtained.

The system presents three basic non correlated random variables, p , F and Young modulus, E , with normal distributions. The geometrical values are defined by deterministic parameters (Figure 4.10).

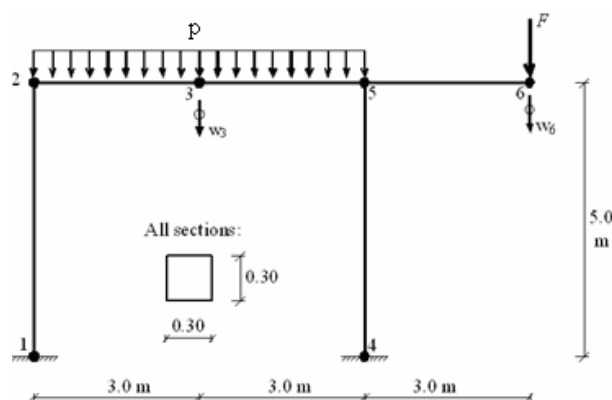


Figure 4.10 – Frame subjected to vertical loads.

Different load cases are studied in this example, considering different mean values and dispersions for load F . Statistical parameters for all considered load cases are

presented in Table 4.3. Cases 1 and 2 have the same properties, being the dispersion of force F higher for the second case (20 kN instead of 7.5 kN). Cases 3 and 4 are similar to cases 1 and 2, being force F pointed to the opposite side.

Table 4.3 – Statistical parameters for different cases.

Cases	E (GPa)		p (kN/m)		F (kN/m)	
	mean	st. dev.	mean	st. dev.	Mean	st. dev.
1	29	2	20	2	50	7.5
2	29	2	20	2	50	20
3	29	2	20	2	-50	7.5
4	29	2	20	2	-50	20

The study focus on the uncertainty evaluation of vertical displacement at beam mid-span, w_3 , and at beam end, w_6 . Using Perturbation Technique and Monte Carlo Analysis it is determined their average value and standard deviation. The maximum and minimum value of such displacements is obtained by, respectively, adding and subtracting the standard deviation to the average value. For Monte Carlo Analysis, it was considered 15000 samples of the problem, assigning to variables p , F , and E aleatory generated values. In Modal Interval Analysis (MIA), only uniform distributions can be considered, and so the same approximation of the previous example is executed (Figure 4.5 and Figure 4.6). In such analysis it is obtained an interval value for each displacement. Results are summarized in Table 4.4.

Table 4.4 – Summary of results for the different cases (mm).

Cases	Perturbation Technique				Modal Interval Analysis			
	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$
1	1.513	3.687	27.998	41.842	1.608	3.456	29.754	39.218
2	0.441	4.759	17.562	52.278	0.469	4.461	18.664	48.999
3	11.267	14.033	-58.319	-43.561	11.974	13.153	-54.661	-46.293
4	10.329	14.971	-68.486	-33.394	10.977	14.032	-64.190	-35.489

Cases	Monte Carlo Analysis				Deterministic Values	
	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$	w_3	w_6
1	1.539	3.701	28.029	41.931	2.600	34.920
2	0.477	4.823	17.425	52.115	2.600	34.920
3	11.278	14.082	-58.409	-43.591	12.650	-50.940
4	10.322	14.978	-68.475	-33.365	12.650	-50.940

4.3. A METHODOLOGY FOR THE DETECTION OF AN ABNORMAL BEHAVIOUR

4.3.1. THE PROPOSED METHODOLOGY

Every structure experiences damage during its lifetime. Damage can appear in the form of corrosion, cracking, loss of section or material deterioration. The process of structural assessment, by visual inspection, is a laborious and at the same time subjective process. In the last years, researchers have investigated ways of automating the procedure of identifying and assessing damage in civil structures.

Nowadays numerous structural assessment techniques are available. Some of them are based on measuring the changes in the dynamic properties of the structure. However there are a lot of difficulties when these techniques are applied to large massive civil structures. It is difficult to excite dynamically a large civil structure to an extent that it will elicit the changes caused by local damage in the measurements of the structural response (Farrar and Doebling, 1997).

Different static methods have also been studied. The first ones to appear require multiple load cases involving the application of concentrated loads at different structure locations. Then, it appeared another type of structural assessment technique that uses the static dead load strain measurements (Hu and Shenton, 2002, Zhao and Shenton, 2002). Although these methodologies are promising, there are still some problems to be solved like the consideration of uncertainty in the model and in the obtained sensor data.

The proposed methodology, which main objective is to detect any structural abnormal behavior, is based on a direct comparison between the sensor obtained data ($X_p \pm \delta X_p$) and the results given by the numerical model ($X_t \pm \delta X_t$) (Figure 4.11). The uncertainty in numerical model is represented by δX_t . Such uncertainty is an output of the model and is due to the fact that some input variables are not known exactly ($Y_{inp} \pm \delta Y_{inp}$). Such output is determined using the previously presented methodologies (Perturbation Technique, Modal Interval Analysis - MIA and Monte Carlo Method). The uncertainty presented in sensor data, symbolized by δX_p , can be due to its linearity, specified by each fabricator, and to the presence of noise, which must be minimized by the introduction of appropriate filters (Matos *et al.*, 2004b; Matos *et al.*, 2005b).

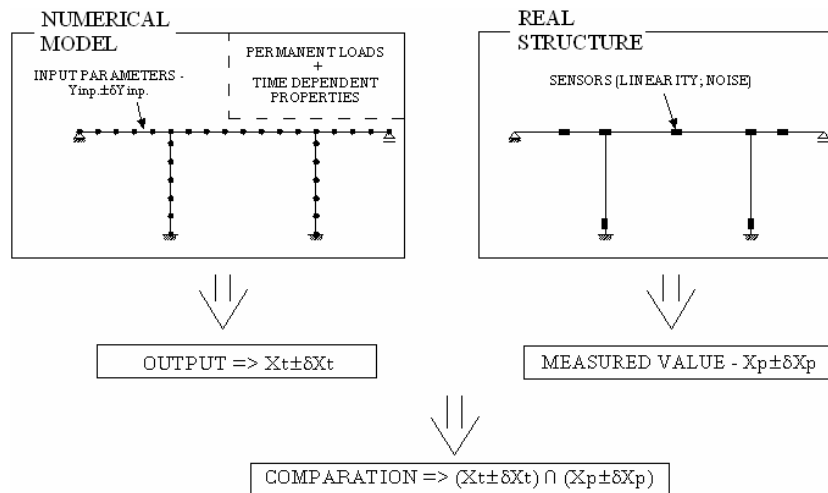


Figure 4.11 – Algorithm for the detection of a structural abnormal behaviour.

If the obtained experimental data is within the range of admissible values given by the numerical analysis, then the structure is behaving well. If not, something is wrong. It could be sensor damage, and then the sensor has to be replaced or structural damage.

The discrepancy between the results obtained by numerical model and the measured data can also be due to incorrect modeling. To prevent that, the numerical model should be calibrated with a load test and other necessary field measurements. For each structure there will be always one specific model capable of assessing its behavior. The specified algorithm is intended to be applied for short and long term structural assessment, and so, the updated numerical model must have into account the permanent loads and the time dependent properties of the structure. An abnormal structural behavior, usually due to the presence of damage, is identified when the structure presents a behavior which differentiates from the one specified by the previous calibrated model.

In this methodology a direct comparison between obtained numerical and sensor data is performed, having into account the presence of uncertainty. It is so a consistent way of detecting any damage in the structure. Once the identification of any structural abnormal behavior must be fast and efficient, the used technique for the consideration of uncertainty in numerical model must be the most appropriate one.

4.3.2. UNCERTAINTY IN MEASURED DATA

4.3.2.1. SENSOR LINEARITY

The measure executed by any sensor is dotted of uncertainty. The used parameter that defines such uncertainty is the linearity, previously determined by calibration tests and provided by the sensor fabricator. If a sensor presents an uncertainty of δX_p , being X_p the measured parameter, and the obtained value X_p , it is possible to affirm that the real value is within an interval defined by $[X_p - \delta X_p; X_p + \delta X_p]$.

Linearity expresses the interval between two straight lines, defined in relation to the measured value or to the end scale value (Figure 4.12a and Figure 4.12b), in which the measured parameter is located. It corresponds to the maximum deviation from the linear relation, expressed in percentage, by the expression 4.19.

$$Linearity = \left(1 - \frac{a}{b}\right) \cdot 100 [\%] \quad (4.19)$$

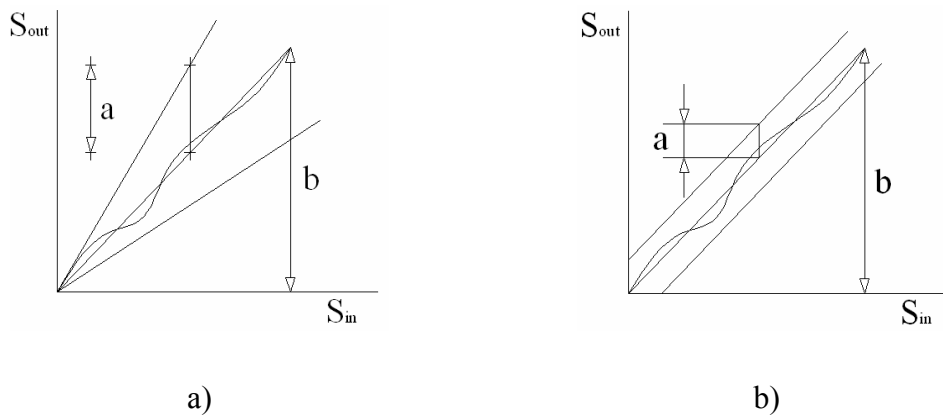


Figure 4.12 – Linearity. a) Percentage of measured value; b) Percentage of end scale value.

If a sensor measures a determined parameter X_p , its input signal is a deterministic value (S_{in}), but, its output signal must be represented as an interval value (S_{out}). Such uncertainty is so due to its linearity. When performing a study to choose the most appropriate sensor to be applied it must be a comparison between its linearity and the specified cost. A low linearity corresponds to a high cost and vice versa.

4.3.2.2. TECHNICAL PARAMETERS

In Table 4.5 the principal technical parameters of metal resistance strain gages, sensors, which measure the structural strain, are described. Such characteristics are well identified by the fabricator in almost catalogues. Recently, a new kind of sensors (Fiber Optic Bragg Grating sensors), which measure also the structural strain, is being applied. From several advantages, those sensors present a lower linearity. However they are still more expensive in the market and that is the main reason why the electrical sensors are the most applied ones.

Table 4.5 – Metal resistance strain gages technical parameters.

Parameter	Design Value
Gage Length	0.5 ... 150 mm
Measurable Strain Range	≤ 50 mm/m (normally) / up to 200 mm/m (specially)
Linearity	≥ 1 μ m/m (normally) / up to 0.1 μ m/m (specially)
Temperature Range	- 200 ... + 200 °C (non-zero referenced measurements; constant grid) / - 20 ... + 70 °C (high accuracy)
Gage Factor (k)	2.05 ... 2.2 (usually), 4 (platinum-tungsten)
Transverse Sensitivity	- 0.9 % ... + 2 % (ratio between transverse and longitudinal gage factor)
Gage Resistance	120, 350, 500, 700 and 1000 Ohms

In Table 4.6 it is presented the principal technical parameters of typical displacement transducer sensors, which are, LVDC (Linear Variable Differential Choke Coil), LVDT (Linear Variable Differential Transformer), PS (Conventional Potentiometer Sensor), PSI (Inductive-Potentiometer Sensor) and MSS (Magnetostrictive Sensor). In this case the sensor linearity (or uncertainty parameter) is referred, in a percentage way, of the equipment full scale (FS) (Bergmeister, 2003). The displacement sensor which presents a best relation, in the market, of cost / linearity is the LVDT (Linear Variable Differential Transformer) and that is why it is the most applied one.

Table 4.6 – Displacement sensors technical parameters.

Sensor Type (abbreviation)	Technical Parameters			
	Measuring Ranges	Linearity	Resolution	Temperature Range
LVDC	± 1 mm ... ± 50 mm	$\geq \pm 0.1$ % FS	Quasi infinite	- 20 °C ... + 120 °C
LVDT	± 1 mm ... ± 50 mm	$\geq \pm 0.1$ % FS	Quasi infinite	- 20 °C ... + 120 °C
PS	± 5 mm ... ± 200 mm	$\geq \pm 0.5$ % FS	0.5 % FS	- 25 °C ... + 125 °C
PSI	50 mm ... 200 mm	$\geq \pm 0.1$ % FS	0.05 % FS	- 40 °C ... + 60 °C
MSS	100 mm ... 1000 mm	$\geq \pm 0.05$ % FS	0.01 % FS	- 20 °C ... + 80 °C

4.3.3. STEEL BEAM ANALYSIS

The following example illustrates the application of the previously presented methodology to a simply supported four point load steel beam which has been tested in laboratory. Such beam was tested with an actuator applied at mid span – $\frac{1}{2} l$ (Test 1), being l the beam length span. Figure 4.13a presents one picture and Figure 4.13b the scheme of the following test. The actuator was previously programmed to comply a specified time – load graphic. It is a time step graphic, composed by growing up and stabilized periods of 5 minutes each. The load step is of 5 kN, between a pre-load of 5 kN and a maximum one of 40 kN (Casas *et al.*, 2005; Garcia, 2005; Matos *et al.*, 2004a; Matos *et al.*, 2004b; Matos *et al.*, 2005b).

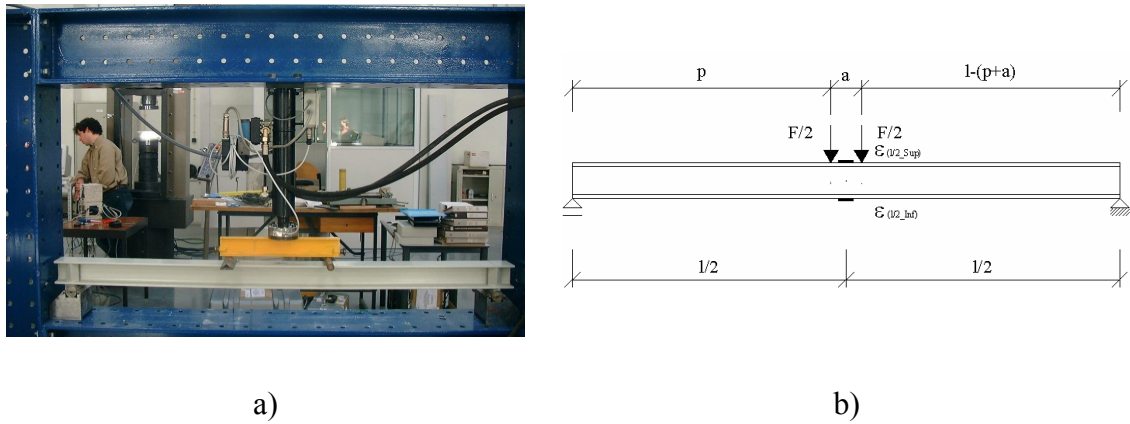


Figure 4.13 – Tested steel beam. a) Photo; b) Scheme.

There are two sensors (strain gages) at mid span, being one at the bottom and other at the top flange of the beam. The obtained data is designated by ϵ_p . These values are interval values $[\epsilon_p - \delta\epsilon_p; \epsilon_p + \delta\epsilon_p]$, where $\delta\epsilon_p$ is the sensor linearity ($\pm 1 \mu\text{m/m}$ of the measured value). Such data is then compared with the results obtained from the analytical model (ϵ_t), which can be presented as a range of values $[\epsilon_t - \delta\epsilon_t; \epsilon_t + \delta\epsilon_t]$.

Due to the elastic behavior of the steel beam during the whole test, an analytical model, based on Hooke Law, is considered:

$$\sigma = \epsilon_t E, \quad (4.20)$$

where σ is the stress, ϵ_t the strain and E the Elasticity Modulus of the structure. From the elastic equation it is possible to determine a direct relation between applied loads and installed stresses:

$$\sigma = \frac{M(F)}{w}, \quad (4.21)$$

where M is the moment, F the applied load and w the flexural modulus. From equation 4.20 and equation 4.21 it is obtained the numerical strain:

$$\varepsilon_t = \frac{M(F)}{w \cdot E} \quad (4.22)$$

The presented theoretical model is developed assuming specific hypothesis, and so it can be only applied within particular situations. In the analytical model, the applied load (F), the elasticity modulus (E) with a mean of 200 GPa and standard deviation $\delta E = 10$ GPa, the flexural modulus (w) with a mean of $9.00 \cdot 10^{-5} \text{ m}^3$ and a standard deviation $\delta w = 0.05 \cdot 10^{-5} \text{ m}^3$ and the geometric parameters, ($l_0 = 1.80 \text{ m}$ / $\delta l = 0.02 \text{ m}$; $a_0 = 0.40 \text{ m}$ / $\delta a = 0.02 \text{ m}$) are uncertainty parameters, defined by normal distributions. In order to obtain a range of values for ε_t , the Perturbation Technique, Modal Interval Analysis (MIA) and Monte Carlo Analysis were used.

Firstly, it is presented the application of Perturbation Technique. The first step is to define the structural analytical model, which is:

$$\varepsilon_t^{1/2} = \frac{M^{1/2}(F; l; p)}{w E} = \frac{F/2 \cdot (l/2 - a/2)}{w \cdot E} \Leftrightarrow \underbrace{\frac{w \cdot E}{(l/2 - a/2)}}_{[K]} \cdot \underbrace{\varepsilon_t^{1/2}}_{\{U\}} = \underbrace{F/2}_{\{F\}} \quad (4.23)$$

By substituting the mean values of each input parameter in the expression of the stiffness matrix [K], it is obtained the stiffness matrix $[K_0] = 25714.30$, being $[K_0]^{-1} = 3.888 \cdot 10^{-5}$. Then, it is defined the partial derivatives of the stiffness matrix and load vector in order to their uncertainty variables. Substituting each variable by their respective mean value, the following is obtained:

$$\partial K / \partial E = \frac{w}{(l/2 - a/2)} = 1.285 \cdot 10^{-4} \quad (4.24)$$

$$\partial K / \partial w = \frac{E}{(l/2 - a/2)} = 2.857 \cdot 10^8 \quad (4.25)$$

$$\partial K / \partial l = -\frac{2 \cdot E \cdot w}{(l - a)^2} = 18367.30 \quad (4.26)$$

$$\frac{\partial K}{\partial a} = \frac{2 \cdot E \cdot w}{(a-l)^2} = 18367.30 \quad (4.27)$$

$$\frac{\partial F}{\partial F} = 1/2 \quad (4.28)$$

Afterwards, it is necessary to define the applied load variable (F) as a normal distribution with a mean value (10.009 kN; 19.996 kN; 29.974 kN; 39.551 kN) and a standard deviation (δF), represented by the actuator linearity, which is 5% of the mean value (0.50 kN; 1.00 kN; 1.50 kN; 2.00 kN). For each load step, and using a typical deterministic analysis, it is obtained the mean displacements (U_0) ($F = 10.009 \text{ kN} \rightarrow \epsilon_t = 1.946 \cdot 10^{-4}$; $F = 19.996 \text{ kN} \rightarrow \epsilon_t = 3.888 \cdot 10^{-4}$; $F = 29.974 \text{ kN} \rightarrow \epsilon_t = 5.828 \cdot 10^{-4}$; $F = 39.551 \text{ kN} \rightarrow \epsilon_t = 7.689 \cdot 10^{-4}$).

Considering the first load step ($F = 10.009 \text{ kN}$), the following values were determined:

$$K_0^{-1} \cdot \frac{\partial K}{\partial E} \cdot U_0 \cdot \delta E = 9.624 \cdot 10^{-6} \quad (4.29)$$

$$K_0^{-1} \cdot \frac{\partial K}{\partial w} \cdot U_0 \cdot \delta w = 1.081 \cdot 10^{-6} \quad (4.30)$$

$$K_0^{-1} \cdot \frac{\partial K}{\partial L} \cdot U_0 \cdot \delta L = 2.779 \cdot 10^{-6} \quad (4.31)$$

$$K_0^{-1} \cdot \frac{\partial K}{\partial a} \cdot U_0 \cdot \delta a = 2.779 \cdot 10^{-6} \quad (4.32)$$

$$-K_0^{-1} \cdot \frac{\partial F}{\partial F} \cdot \delta F = -9.526 \cdot 10^{-6} \quad (4.33)$$

Then, it is possible to calculate the covariance matrix of the structural response (C_u), which is:

$$C_u = [9.624 \quad 1.081 \quad 2.779 \quad 2.779 \quad -9.526] \cdot 10^{-6} \cdot \begin{bmatrix} 9.624 \\ 1.081 \\ 2.779 \\ 2.779 \\ -9.526 \end{bmatrix} \cdot 10^{-6} \Leftrightarrow C_u = [1.999 \cdot 10^{-10}] \quad (4.34)$$

And, finally, it is obtained the standard deviation of the midspan strain:

$$\delta \varepsilon_t^{1/2} = \sqrt{1.999 \cdot 10^{-10}} = 1.414 \cdot 10^{-5} \quad (4.35)$$

The obtained range value for the midspan strain using Perturbation Technique and for step load 9.909 kN is within $[\varepsilon_t - \delta \varepsilon_t; \varepsilon_t + \delta \varepsilon_t] = [1.926 \cdot 10^{-4} - 1.414 \cdot 10^{-5}; 1.926 \cdot 10^{-4} + 1.414 \cdot 10^{-5}] = [1.785 \cdot 10^{-4}; 2.067 \cdot 10^{-4}]$. The same procedures are executed for the other load steps (Table 4.7).

Secondly, it is presented the application of Modal Interval Analysis (MIA). In this case, the input variables are assumed to present a uniform distribution. Consequently, it is necessary to execute an approximation, for each variable, of a normal distribution into a uniform one (Figure 4.5 and Figure 4.6). The following values are then obtained:

$$E = [190; 210] \text{ GPa};$$

$$W = [8.95; 9.05] \cdot 10^{-5} \text{ m}^3;$$

$$L = [1.78; 1.82] \text{ m};$$

$$A = [0.38; 0.42] \text{ m}.$$

As an example, the application of Modal Interval Analysis (MIA) for load step $F = [9.508; 10.509] \text{ kN}$, is detailed next:

$$\begin{aligned} \varepsilon_t^{1/2} &= \frac{M^{1/2}(F; l; p)}{w E} = \frac{F/2 \cdot (l/2 - a/2)}{w \cdot E} \Leftrightarrow \underbrace{\frac{w \cdot E}{(l/2 - a/2)}}_{[K]} \cdot \underbrace{\varepsilon_t^{1/2}}_{\{U\}} = \underbrace{F/2}_{\{F\}} \Leftrightarrow \\ &\Leftrightarrow \frac{[8.95; 9.05] \cdot 10^{-5} \cdot [190; 210] \cdot 10^6}{([1.78; 1.82]/2 - [0.38; 0.42]/2)} \cdot \varepsilon_t^{1/2} = [9.508; 10.509] / 2 \Leftrightarrow \varepsilon_t^{1/2} = [1.880; 2.010] \cdot 10^{-4} \end{aligned} \quad (4.36)$$

The same procedures can be executed for all other load steps. Accordingly, a program based on Modal Interval Analysis (MIA), was specifically developed in C++, to analyze this example (Matos *et al.*, 2005b). Such program executes the previous presented algorithm for the detection of any structural abnormal behavior. In order to perform it, a study for each load step is elaborated and a consistent comparison realized. Figure 4.14 presents the output of such program. The first graphic presents the results from MIA (red), the measured values from the sensors placed at the bottom (green) and at the top flange of the beam (blue). The second graphic shows the applied load (F).

If the data obtained from the sensors, affected by its linearity, is within the range of admissible values given by the analytical model $([\varepsilon_p - \delta \varepsilon_p; \varepsilon_p + \delta \varepsilon_p] \cap [\varepsilon_t - \delta \varepsilon_t; \varepsilon_t + \delta \varepsilon_t] \neq [0; 0])$ then the structure is behaving well and nothing appears on the two bottom fail detection indicators. If not, a signal appears detecting a possible damage. The upper indicator refers to the sensor placed at the bottom flange (ε_p Bot) while the bottom refers

to the sensor placed at the top flange (ε_p Top) of the beam. In this specific case the algorithm has identified damage for lower applied loads. This damage is understandable as it happens for very low measured values. This is essentially due to the sensor resolution, which is low, when comparable with measured values.

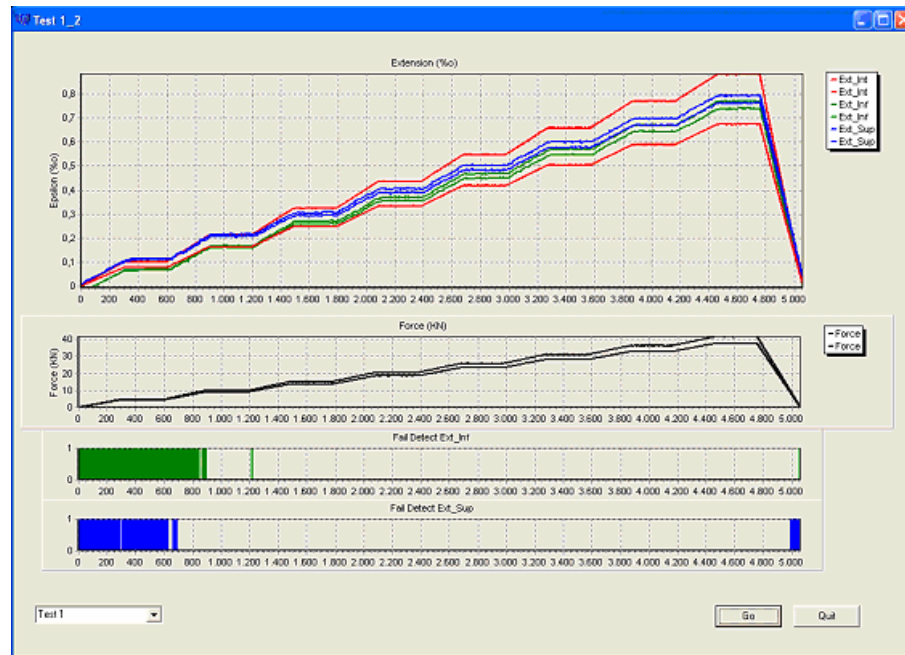


Figure 4.14 – Output of software for the detection of a structural abnormal behaviour.

This example was also analyzed with Monte Carlo Methodology in order to compare with the previous presented methodologies. Monte Carlo Analysis was applied by running 1,000 possible combinations of all input variables assigning a normal distribution to each one, defined by the same parameters as the ones calculated for Perturbation Technique.

In Table 4.7 it is possible to observe, for a specific load step (F), the Perturbation Technique - ε_t PT, the Modal Interval Analysis (MIA) - ε_t MIA, the Monte Carlo Analysis - ε_t MC and the deterministic results - ε_t Det. Such values were all obtained from the same numerical model (equation 4.21). The data obtained from the sensors placed at the bottom flange - ε_p Bot - and at the top flange - ε_p Top - of the beam, for the same load steps, is also presented.

The comparison between numerical results and obtained data showed that the structure behaves well for the applied loads. In fact the data from the sensors is frequently within the range of admissible values. This comparison is only possible with

uncertainty analysis numerical methodologies. In Table 4.8 it is presented the main results of such comparison.

Table 4.7 – Numerical results and obtained data of strains for different load steps ($\cdot 10^{-3}$).

F (kN)		ε_t PT		ε_t MIA		ε_t MC		ε_t Det	ε_p Bot		ε_p Top	
Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.		Min.	Max.	Min.	Max.
9.508	10.509	0.178	0.206	0.188	0.201	0.184	0.203	0.194	0.211	0.216	0.171	0.174
18.996	20.996	0.360	0.417	0.376	0.402	0.318	0.452	0.388	0.393	0.400	0.362	0.369
28.475	31.472	0.539	0.625	0.563	0.603	0.415	0.692	0.582	0.588	0.599	0.555	0.566
37.573	41.528	0.710	0.824	0.743	0.795	0.650	0.908	0.768	0.773	0.788	0.745	0.760

Table 4.8 – Comparison between Numerical results and obtained data for different load steps.

F (kN)		ε_t PT $\cap$ ε_p Bot	ε_t PT $\cap$ ε_p Top	ε_t MIA $\cap$ ε_p Bot	ε_t MIA $\cap$ ε_p Top	ε_t MC $\cap$ ε_p Bot	ε_t MC $\cap$ ε_p Top
Min.	Max.						
9.508	10.509	NO	NO	NO	NO	NO	NO
18.996	20.996	YES	YES	YES	YES	YES	YES
28.475	31.472	YES	YES	YES	YES	YES	YES
37.573	41.528	YES	YES	YES	YES	YES	YES

4.3.4. CONCRETE BEAM ANALYSIS

The following example illustrates the application of the previously presented methodology to a reinforced concrete beam which has been tested in laboratory up to failure. The structure presents a square section of 0.15 (b) x 0.15 (h) m² and a total length of 2.10 m (Figure 4.15). The beam is loaded by two punctual forces F applied at third parts of the span. Such forces are applied by an actuator. The error in the measurement of applied load can be modeled as a random variable with a mean equivalent to the load value and a standard deviation equal to 5% of it. Compressive tests were executed in some samples made during the pouring of the beam to determine the concrete elasticity modulus (E_c). From the samples it is estimated a mean value of 32.6 GPa and standard deviation of $\delta E_c = 1.94$ GPa.



Figure 4.15 - Laboratory test of concrete beam.

A finite element model of this beam was considered in order to analyze its behavior (Figure 4.16). The model is constituted by 1D Euler Bernoulli beam elements, with two unknown displacements by each node. In both ends the beam is simply supported, being the rotational displacements also partially restrained due to the test set-up at the supports (Figure 4.15). To take into account such restraint, a rotational spring was considered in the model. It was also considered a normal distribution for the spring stiffness with mean $k = 6000 \text{ kN.m/rad}$ and standard deviation $\delta k = 60 \text{ kN.m/rad}$. The static structural equilibrium equation, basis of all methodologies, was consequently defined.

The uncertainty is considered in the elasticity modulus (E_c), applied load (F) and in the spring stiffness (k). Normal distributions were considered for all uncertain variables, except in Modal Interval Analysis (MIA) where a transformation into Uniform Distributions must be executed before (Figure 4.5 and Figure 4.6). Other variables are considered to be deterministic (Length $l = 0.35 \text{ m}$; Inertia $I = 4.21875 \times 10^{-5} \text{ m}^4$). During the test, displacement transducers (LVDT - Linear Variable Differential Transformer), with a linearity equal to $\pm 0.1 \%$ of the transducer Full Scale (FS), were placed to control the beam deflection, namely w_3, w_5, w_7, w_9 and w_{11} .

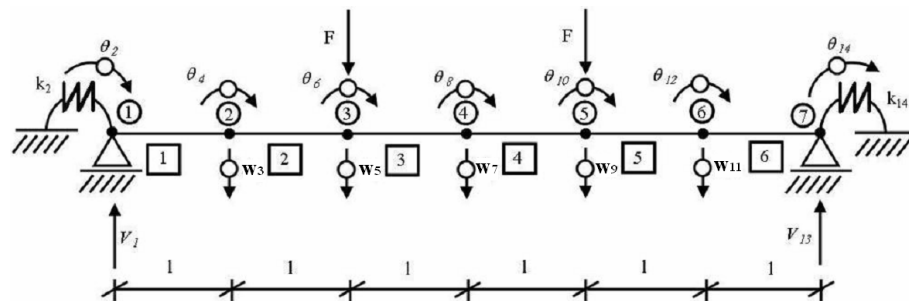


Figure 4.16 – Finite element model.

The developed algorithm consists in a direct comparison between the obtained data, affected by the transducers linearity $[\varepsilon_p - \delta\varepsilon_p; \varepsilon_p + \delta\varepsilon_p]$ with the numerical results

obtained from the application of the uncertainty analysis methodologies $[\varepsilon_t - \delta\varepsilon_t; \varepsilon_t + \delta\varepsilon_t]$ (Perturbation Technique; Modal Interval Analysis - MIA; Monte Carlo Analysis). If there is any intersection between both sets, then the structure presents a behavior within the expected one. If not, something is wrong and it might be an error in obtained data or even a structural abnormal behavior. On the last hypothesis it might be an error in numerical model, and so it needs to be updated, or a structural damage.

The algorithm was applied here for each level of load used during the test. As the obtained data and the numerical model were rigorously controlled, when no intersection between numerical data and experimental results was identified, it was possible to state that the beam presents a different behavior from the elastic one, and so, a potential damage was recognized. In this case, the damage is the beam cracking and therefore the prediction of the moment when cracking appears (“Cracking Load”) is the main point of interest of the algorithm.

The first methodology to be applied was the Perturbation Technique. The procedures were the same of the previous analysis (Appendix E). It was used the previous defined static equilibrium equation. In such technique, the input uncertainty variables were considered as presenting normal distributions. Figure 4.17 presents the results for the control variable w_7 . From the application of the algorithm it was obtained a “Cracking Load” of 16.90 kN for a 1.55 mm vertical displacement.

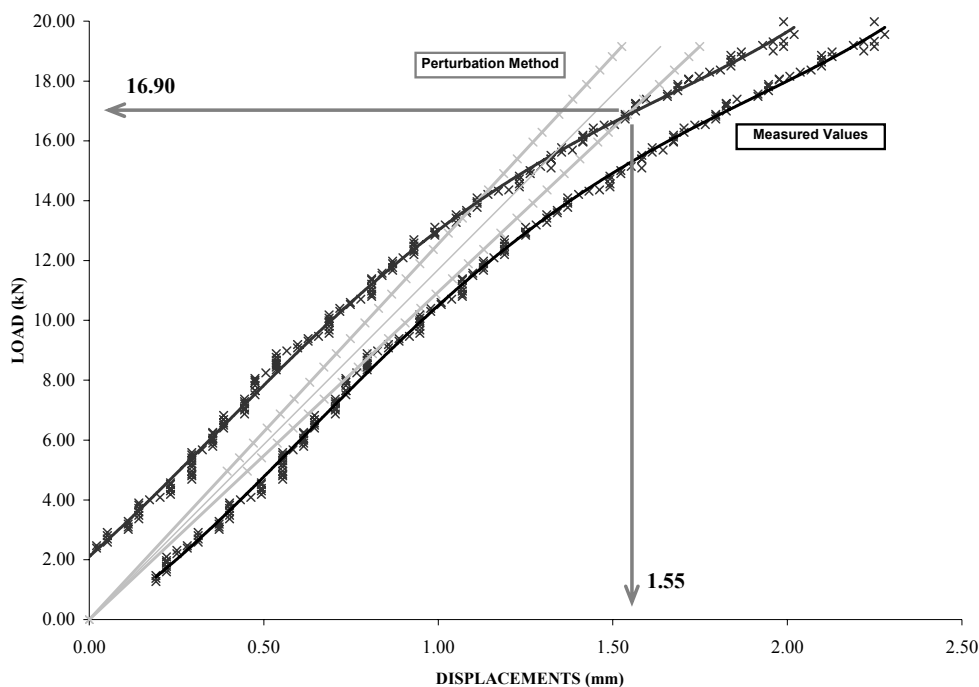


Figure 4.17 – Obtained results – Perturbation Method.

Secondly it was used the Modal Interval Analysis (MIA). The interval calculation was realized on the basis of the symbolic static equilibrium equation, defined before from the Finite Element Methodology (FEM). It was applied, here, the IFEM. In Figure 4.18 it is present the results obtained with Modal Interval Analysis (MIA) for the control variable w_7 . From the application of the algorithm it was determined a “Cracking Load” of 16.40 kN for a 1.45 mm vertical displacement.

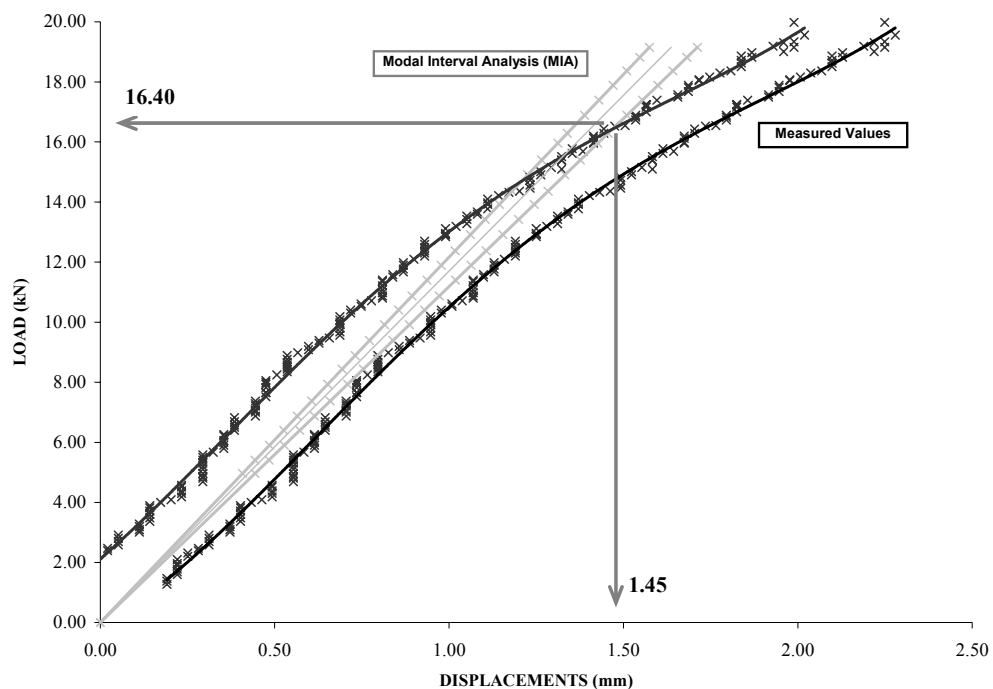


Figure 4.18 – Obtained results – Modal Interval Analysis (MIA).

Finally, it was applied the Monte Carlo Analysis by running 500 possible combinations of all input variables assigning a normal distribution to each one, defined by the same parameters as the ones calculated for Perturbation Technique. In Figure 4.19 it is present the results obtained with Modal Interval Analysis (MIA) for the control variable w_7 . In order to determine the “Cracking Load” a comparison was made between numerical and obtained data for each load step. From the range of applied load values, no intersection was determined and so it was not possible to define the “Cracking Load”. Eventually, more samples were needed to obtain a result similar to the ones obtained with the previous methodologies.

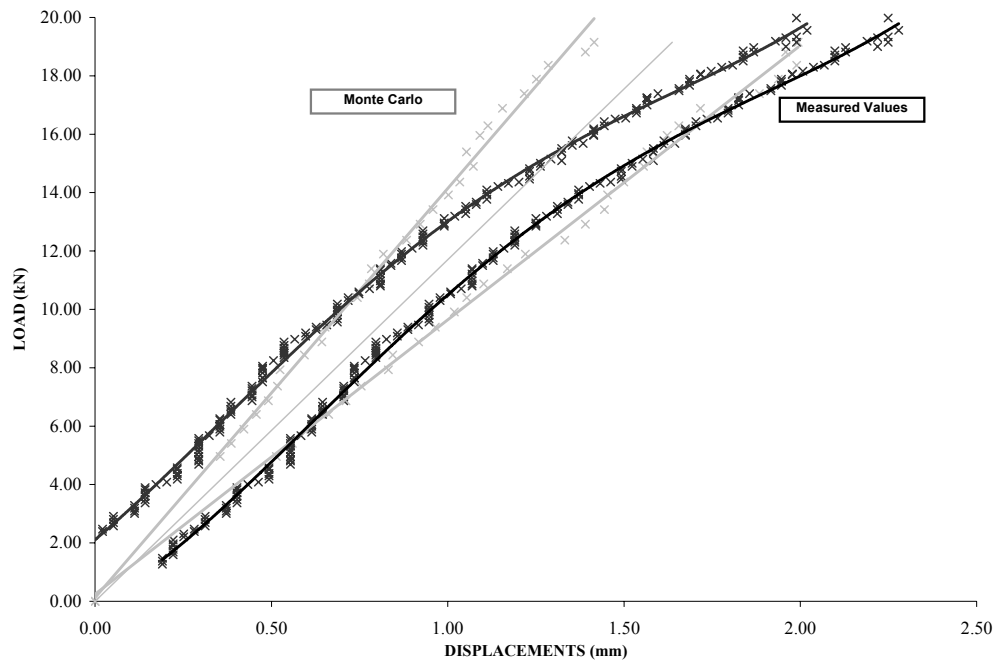


Figure 4.19 – Obtained results – Monte Carlo Analysis.

From a direct analysis of the structural behaviour, given by the displacement transducer, positioned (w_7) at the beam midspan it was possible to evaluate the “Cracking Load”. In this case, the load was defined by a changing in the structural behaviour. In a first step, the structure presented an elastic behaviour. When the cracking process begun it changed to a non elastic one. From a non rigorous procedure, and having into account only experimental data, the “Cracking Load” is 13.00 kN for a displacement of 1.05 mm (Figure 4.20).

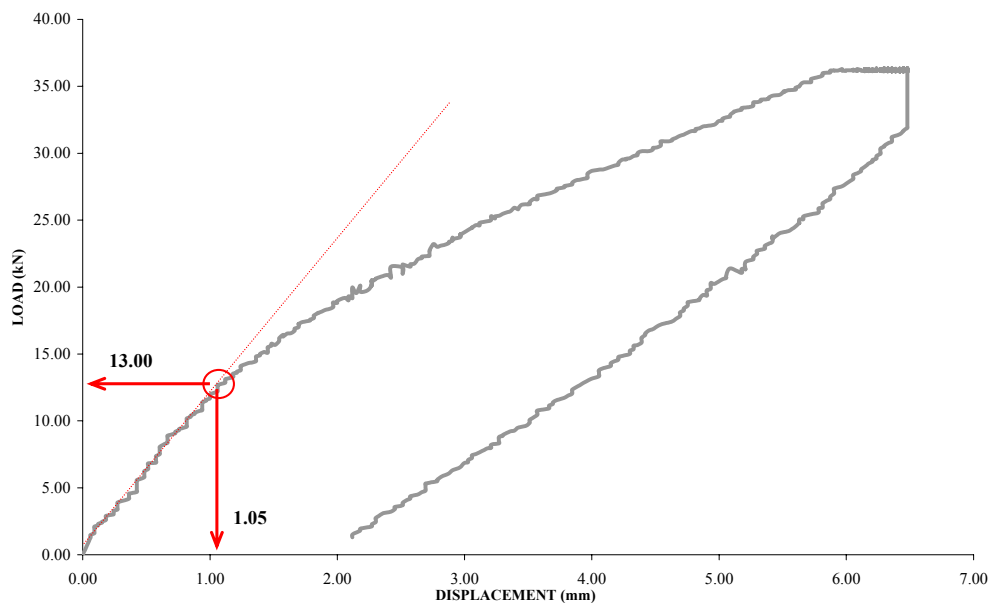


Figure 4.20 – Experimental data from displacement transducer (w_7).

Table 4.9 presents a comparison between measured values and the results obtained with Perturbation Technique, Modal Interval Analysis (MIA) and Monte Carlo Analysis. It is necessary to note that all this results were determined using the obtained data of the displacement transducer positioned at midspan. The values obtained from the application of all methodologies and also the measured values are presented in Appendix F.

Table 4.9 – Beam results - Comparison of methods.

Solution	“Cracking Load” (kN)	Vertical Displacement (mm)
Perturbation Method	16.40	1.45
Modal Interval Analysis	16.90	1.55
Monte Carlo Analysis	-	-
Measured Value	13.00	1.05

Figure 4.21 presents the beam deformed shape determined using the numerical data from the application of Perturbation Technique, Modal Interval Analysis (MIA) and Monte Carlo Analysis for two different load cases. To calculate the respective deformation, an approximation was executed. It was assumed that the beam deformation was similar to a polynomial function of sixth degree. Having into consideration the data obtained from all sensors (w_3 , w_5 , w_7 , w_9 and w_{11}) and the support restraints it was possible to determine the unknown parameters of such function. Afterwards, it was defined the polynomial function, which represents the structural deformed shape for each methodology and load step (Appendix G).

The measured results for each displacement transducer affected by its linearity (w_3 , w_5 , w_7 , w_9 and w_{11}) are also displayed. For a load level $F = 8.4390 \text{ kN} < \text{“Cracking Load”}$ (Figure 4.21 - Top) all the measured data is within the deformed shape for all methodologies, while for $F = 19.1510 \text{ kN} > \text{“Cracking Load”}$ (Figure 4.21 - Bottom) all transducers present results outside the respective range for the case of Perturbation Technique and Modal Interval Analysis (MIA). When cracks start to appear the structure presents a non-linear behavior and so the elastic model can not be applied anymore. The damage in the beam due to cracking is clearly identified by the two proposed techniques, using the previously proposed algorithm.

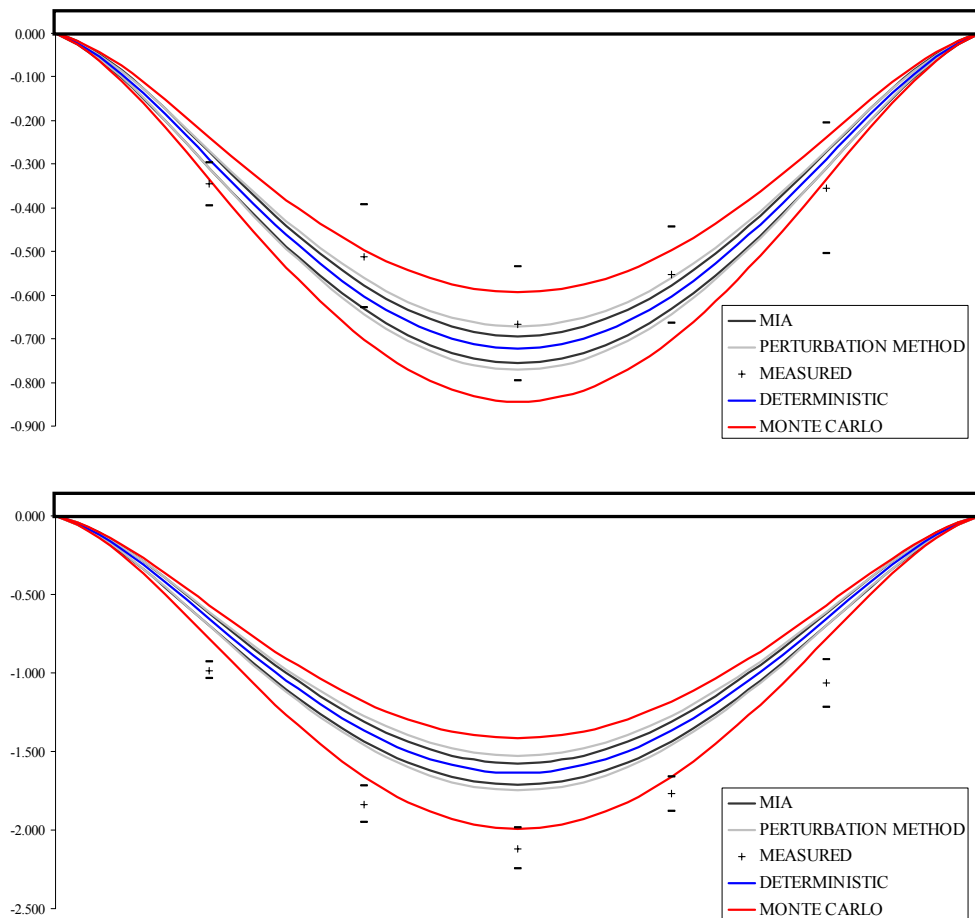


Figure 4.21 – Deformed shape determined with different methods and measured data (Top – $F = 8.4390$ kN; Bottom – $F = 19.1510$ kN).

4.3.5. QUANTIFIED CONSTRAINT SATISFACTION PROBLEMS (QCSP)

The use of Modal Interval Analysis (MIA) with large scale numerical models still presents some barriers that need to be over passed. In order to avoid such difficulties, another algorithm for the detection of a structural abnormal behavior under the presence of uncertain parameters, with Modal Interval Analysis (MIA), is being studied. Such algorithm is the so called Quantified Constraint Satisfaction Problem (QCSP).

A Quantified Constraint (QC) is an algebraic expression over the reals which contains quantifiers ($\exists, \forall$), predicate symbols (e.g. $=, <, \leq$), function symbols (e.g. $+, -$,

*), rational constants and variables $x = \{x_1 \dots x_n\}$ ranging over reals domains $D = \{D_1 \dots D_n\}$. An example of a Quantified Constraint (QC) is the following one:

$$\forall x \in \mathbb{R} \quad x^4 + px^2 + qx + r \geq 0, \quad (4.37)$$

where x is a universally ($\forall$) quantified variable and p and r are free variable.

A numerical Constrain Satisfaction Problem (CSP) (Shary 2002) is a triple $CSP = (x; D; C(x))$ defined by:

- A set of numeric variables $x = \{x_1 \dots x_n\}$;
- A set of domains $D = \{D_1 \dots D_n\}$ where D_i , a set of numerical values, is the domain associated with the variable x_i ;
- A set of constraints $C(x) = \{C_1(x) \dots C_m(x)\}$ where a constraint $C_i(x)$ is determined by any numeric relation (equation, inequality, inclusion, etc.) linking a set of variables under consideration.

A solution to a numeric constraint satisfaction problem $CSP = (x; D; C(x))$ is an instantiation of the variables of x for which both inclusion in the associated domains and all the constraints of $C(x)$ are satisfied. All the solutions of a constraint satisfaction problem thus constitute the set:

$$\Sigma = \{x \in D \mid C(x) \text{ is satisfied}\} \quad (4.38)$$

Now suppose that the constraints $C(x)$ depend on some parameters $p_1 \dots p_n$ about which we only know that they belong to some intervals $P_1 \dots P_n$. Moreover these parameters have an associated quantifier $Q \in \{\exists, \forall\}$. Taking into account the dual character of Modal Interval Analysis (MIA), the most general definition of the set of solutions to such Quantified Constraint Satisfaction Problem (QCSP) should have the form:

$$\Sigma = \{x \in D \mid Q_1(p_1, P_1) \dots Q_n(p_n, P_n) C(x)\} \quad (4.39)$$

where:

- Q_i is a logical quantifier ($\exists, \forall$);
- $\{p_1 \dots p_n\}$ is the set of parameters of the constraints system considered;

- $\{P_1 \dots P_n\}$ is the set of intervals containing the possible values of these parameters.

The sets of the form (equation 4.39) will be referred to as quantified solution sets to the numerical quantified constraints satisfaction problem $QCSP = (x, D, C(x))$.

The inconsistency of a Quantified Constraint Satisfaction Problem (QCSP) is identified when one or more of its constraints are inconsistent, what is expressed by the following expression:

$$QCSP \text{ is inconsistent if } (\forall x \in X) \{ \neg(C_1(x)) \vee, \dots, \vee \neg(C_m(x)) \} \quad (4.40)$$

Let us consider the case when the constraints are under the form $C(x) = f(x) = 0$, with f a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. The logic formulation needed for proving the inconsistency of a constraint $C_i(x)$ is as follows:

$$(\forall x \in X) \neg(f(x) = 0) \quad (4.41)$$

In order to prove this logic formulation, Modal Interval Analysis (MIA) is used. Then, the evaluation of formula 4.41 is equivalent to proving the following interval exclusion:

$$0 \notin Out(f^*(X)) \quad (4.42)$$

where $Out(f^*(X))$ is an outer approximation of the range of the continuous function f . Proving the inconsistency of a Quantified Constraint Satisfaction Problem (QCSP) has been reduced to proving the exclusion of zero from the range of a set of continuous functions. However, computing the range of a continuous function f by means of rational extensions given by MIA provokes an overestimation of the interval evaluation, when the rational computation are not optimal, due to the possible multi occurrences of some variable. An algorithm, based on the results of Modal Interval Analysis (MIA) and branch-and-bound techniques, which allow to efficiently compute an inner and an outer approximation of f^* , has been built. This algorithm uses the optimality theorems of MIA according with (Herrero *et al.*, 2004; Herrero *et al.*, 2005).

The structural assessment is made based on a numerical Quantified Constraint Satisfaction Problem (QCSP), for which it is needed to know all the system parameters.

Modal Interval Analysis (MIA) is used to simulate the unknown variables by solving the static equilibrium equation of the system. In a first step the unknown structural parameters are determined using the Finite Element Methodology (FEM). Secondly, the input parameters are substituted by uncertain interval parameters $[x_t - \delta x_t; x_t + \delta x_t]$, the unknown non-controlled parameters by deterministic value and, finally, the unknown controlled parameters (defined as the parameters which are measured by transducer devices) by the measured values affected by their linearity $[x_p - \delta x_p; x_p + \delta x_p]$. Then, all data is put in the algorithm based on Modal Interval Analysis (MIA) and branch-and-bound techniques to prove the inconsistency of the defined constraints. If any inconsistency is detected, then it could be sensor damage, a problem with the chosen numerical model, or even a structural abnormal behaviour.

4.3.5.1. HYPERSTATIC BEAM

This example illustrates the application of Modal Interval Analysis (MIA) in a Quantified Constraint Problem (QCSP) with the objective of performing the structural assessment having into account the uncertainty present in some variables (Matos *et al.* 2006, Casas *et al.* 2005). The structure is a reinforced concrete beam with a rectangular section $0.10 \text{ (b)} * 0.20 \text{ (h)} \text{ m}^2$ in its extremities and $0.10 \text{ (b)} * 0.30 \text{ (h)} \text{ m}^2$ in the middle. It is a C25/30 concrete ($E=30.5\text{GPa}$) and A500 steel type ($E_s=200\text{GPa}$) (Figure 4.22 a).

To analyze the structure, a finite element model, with 1D Euler-Bernoulli beam elements (three degrees of freedom per node), is used. The structure is supported in four points, namely on its extremities and where the section suffers a height variation. It is considered that on its extremities the beam is fixed on both directions while in mid points only the vertical displacement is blocked. To simulate supports, springs (k) were used in all nodes and in all directions (horizontal, vertical and rotation) (Figure 4.22 b).

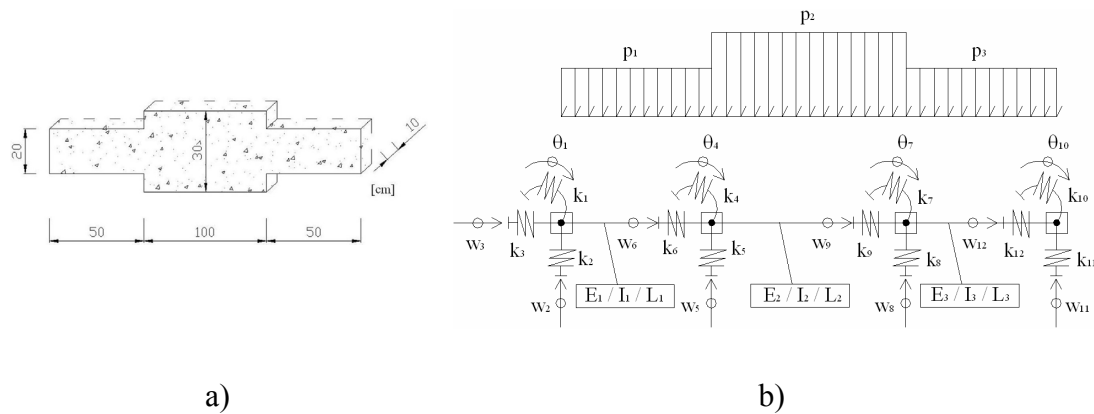


Figure 4.22 - Reinforced concrete beam. a) Scheme; b) Simplified model.

Three different elements with proper Young modulus (E), inertia (I) and length (L) are used to model the beam. The structure is only loaded by its self weight (p). First it is determined the stiffness matrix (K) and the load vector (F) of the system. Then the static equilibrium equation ($KU=F$) is formulated. This matrix equation can be represented by a system of 12 parametric equations being each, one restriction. Those equations depend on uncertain variables such as the applied load (p), the Young modulus (E), the inertia (I) and the obtained displacement (θ and w).

Limit values for each interval variable are defined. For the stiffness modulus (EI) are $[1647, 2460.33]$ kN.m² for (EI_1), $[3362.625, 11597.625]$ kN.m² for (EI_2) and $[1647, 2460.33]$ kN.m² for (EI_3), and for the loads (p) are $[0.45, 0.55]$ kN/m for p_1 , $[0.675, 0.825]$ kN/m for p_2 and $[0.45, 0.55]$ kN/m for p_3 . The length ($L_1 = 0.5$ m; $L_2 = 1.0$ m; $L_3 = 0.5$ m) of each element have deterministic values. The values for each spring constant (k), also deterministic, were properly defined according to each support type.

Combinations of different values for each variable are defined. Some of them associated with damage in some element, namely, loss of stiffness (EI), and some with no damage. For each combination, all displacements are obtained. In this situation the unknown displacements are all rotations (θ). Those rotations, supposed to be obtained by real inclinometers, are also uncertainty intervals. In this situation it is admitted that the model unknown displacements are experimentally measured. For such transducers it is considered an uncertainty equal to 2% of measured value.

Having into account the support types it is possible to reduce the system from 12 to 4 parametric equations. In order to perform this structure assessment a consistency analysis is necessary to verify all restrictions. The idea is to determine if the values of the uncertain parameters satisfy the restrictions, warranting the normal behaviour of the

beam. To warrantee the consistency of the restrictions it is necessary to work with constraints (C_i). The inconsistency test is made based on the following constraints:

$$C_1 : \left(\frac{4EI_1}{L_1} + k_1 \right) \cdot \theta_1 + \left(\frac{2EI_1}{L_1} \right) \cdot \theta_4 - \left(\frac{p_1 L_1^2}{12} \right) = 0 \quad (4.43)$$

$$C_2 : \left(\frac{2EI_1}{L_1} \right) \cdot \theta_1 + \left(\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} + K_4 \right) \cdot \theta_4 + \left(\frac{2EI_2}{L_2} \right) \cdot \theta_7 - \left(\frac{p_2 L_2^2}{12} + \frac{p_1 L_1^2}{12} \right) = 0 \quad (4.44)$$

$$C_3 : \left(\frac{2EI_2}{L_2} \right) \cdot \theta_4 + \left(\frac{4EI_2}{L_2} + \frac{4EI_3}{L_3} + K_7 \right) \cdot \theta_7 + \left(\frac{2EI_3}{L_3} \right) \cdot \theta_{10} - \left(\frac{p_2 L_2^2}{12} + \frac{p_3 L_3^2}{12} \right) = 0 \quad (4.45)$$

$$C_4 : \left(\frac{2EI_3}{L_3} \right) \cdot \theta_7 + \left(\frac{4EI_3}{L_3} + k_{10} \right) \cdot \theta_{10} + \left(\frac{p_3 L_3^2}{12} \right) = 0 \quad (4.46)$$

The goal is to perform the structural assessment, identifying any possible structural abnormal behavior with the monitored data and taking into account the uncertainty presented in the variables. The methodology also allows determining the damage approximate location. So, if C_1 is inconsistent a damage is found in element 1 (EI_1), if C_4 is inconsistent a damage exists in element 3 (EI_3) and, finally, damage in EI_2 needs to be checked with C_2 and C_3 . Some results are presented in Table 4.10. In Table 4.11 a sensitive analysis of the method is carried out. To perform that, all variables have a fixed value, within the acceptable interval, except the bar 3 stiffness (EI_3).

Table 4.10. Simulations for structural assessment.

θ_1^*	θ_4^*	θ_7^*	θ_{10}^*	Constraints			
rad.	rad.	rad.	rad.	C_1	C_2	C_3	C_4
-2.2505	17.1815	-17.1815	2.2505	True	True	True	True
-3.4295	19.6663	-22.6608	3.8487	True	True	True	False

Displacements in rad. (* 10^{-3}).

Table 4.11. Sensitivity analysis.

Case	EI_3	Constraint				Damage
	KN.m ²	C_1	C_2	C_3	C_4	
1	2053.667	True	True	True	True	No
2	1700.000	True	True	True	True	No
3	1650.000	True	True	True	True	No
4	1600.000	True	True	True	True	No
5	1550.000	True	True	True	False	Yes
6	1500.000	True	True	False	False	Yes

According to Table 4.10 it is possible to confirm that the methodology can detect any structural damage. The displacements for the upper case were determined using the midpoints of the interval of each variable and so no damage was detected. The bottom case simulates a stiffness degradation in element 3 ($EI_3 = 857.970 \text{ kN.m}^2$) and so a damage is found on the structure, namely the constraint C_4 is not consistent what means that damage is localized in EI_3 .

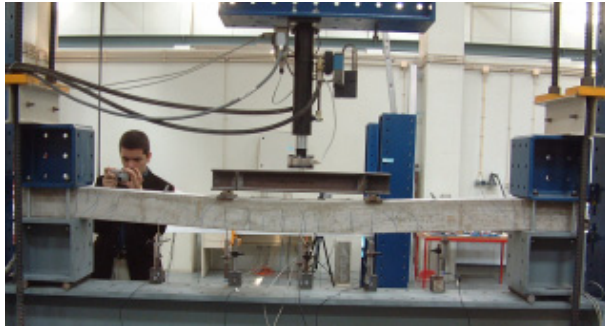
From Table 4.11 it is possible to observe that when the stiffness modulus is close to the admissible interval ($EI_3 = 1600 \text{ kN.m}^2$) the methodology presents some difficulties in detecting the damage (Case 4). When the stiffness modulus presents values far from it interval limit, then the algorithm identifies a structural abnormal behaviour by localizing an inconsistency in C_4 (Case 5). The constraint C_4 is, in fact, the most sensitive constraint due to changes in stiffness modulus EI_3 .

4.3.5.2. CONCRETE BEAM ANALYSIS

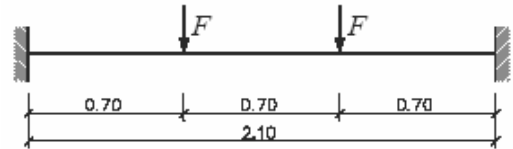
This example illustrates the application of Modal Interval Analysis (MIA) in a Quantified Constraint Problem (QCSP) with the objective of performing the structural assessment having into account the uncertainty present in some variables (Matos *et al.* 2006, Garcia *et al.* 2006, Garcia *et al.*, 2007). The structure analyzed here is the same that was studied by the previous presented methodology for damage assessment, at point 4.3.4. The main objective is the same, which is to determine the “Cracking Load”.

The analyzed structure is a reinforced concrete beam with a square section of $0.15\text{m(b)} \times 0.15\text{m(h)}$ and a total length of 2.10m was tested in the laboratory up to failure (Figure 4.23 a). The beam, clamped on both extremities, is loaded by two concentrated forces F applied at third of the span (Figure 4.23 b). Such forces are applied by actuator. The error in the measurement of applied load can be modeled as a random variable with a mean equivalent to the applied load and a standard deviation equal to 5% of it. Compressive tests were executed in some samples casted during the pouring of the beam to determine the concrete elasticity modulus (E_c). From the samples it is estimated

a mean value of 32.6 GPa and standard deviation of $\delta E_c = 1.94$ GPa. To apply the proposed methodology, as Modal Interval Analysis (MIA) is used, it is necessary to transform such normal distributions into uniform ones (Figure 4.5 and Figure 4.6).



a)



b)

Figure 4.23 - Tested laboratory concrete beam. a) Photo; b) Scheme.

In Figure 4.24 it is presented the reinforced concrete beam scheme. The structure is composed by a longitudinal reinforcement of $2\phi 10 + 2\phi 10$ for negative moments and $2\phi 10$ for positive moments. At midspan it was placed a constructive longitudinal reinforcement of $2\phi 6$. The implemented stirrups are $\phi 6$ with 0.10m space in the support region and 0.25m space at midspan. The nominal cover is of 9mm (Figure 4.25).

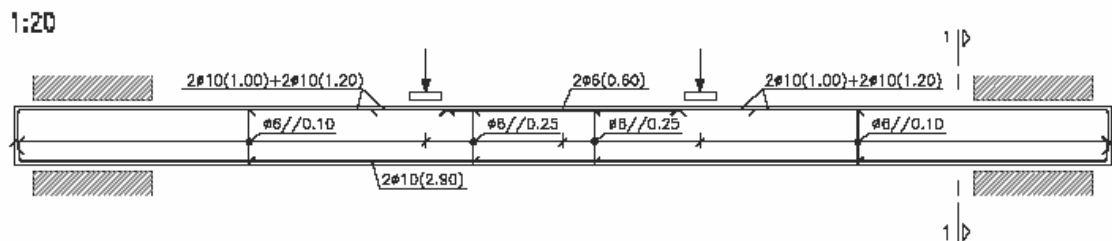


Figure 4.24 – Longitudinal elevation of reinforced concrete beam.

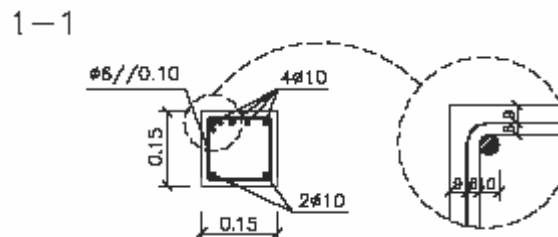


Figure 4.25 – Cross section (cut 1-1 - Figure 4.24) of reinforced concrete beam.

By analyzing the load-strain diagram of this test (Figure 4.26), using a non-rigorous procedure, it is possible to obtain an approximate value for the “Cracking Load”, which is approximately 9.00 kN for a strain of $0.15 \cdot 10^{-3}$ (Matos *et al.* 2006, Garcia *et al.* 2006, Garcia *et al.*, 2007). In this case, strains are measured at midspan section (bottom fiber) by a strain gage. Such sensor is implemented in a clip gauge which is applied to the concrete surface. The obtained result is lower than the one obtained using the load-displacement diagram of Figure 4.20, considering the measured midspan displacement values, which indicates the more sensitiveness of strain gauge.

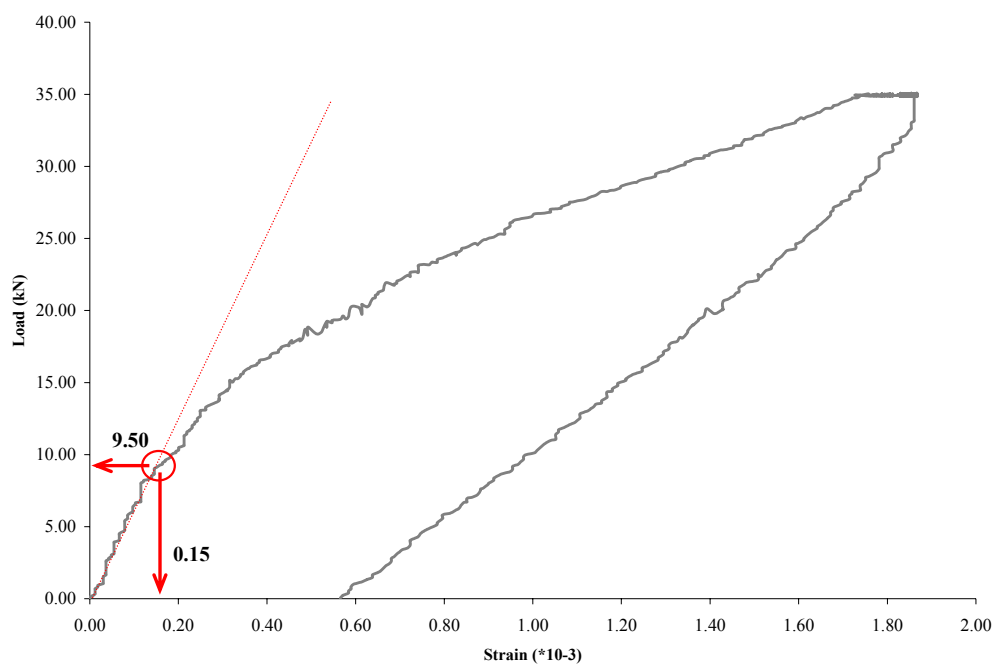


Figure 4.26 - Obtained load – strain diagram (Matos *et al.* 2006, Garcia *et al.* 2006, Garcia *et al.*, 2007).

From an analytical point of view it is also possible to determine the “Cracking Load”. Once the involved variables present an uncertainty which cannot be considered in the deterministic calculus the obtained value is only an approximate one. According to bending diagram (Figure 4.27) the first crack will appear in the support region where stresses are higher and, secondly, at midspan. From analytical analysis the “Cracking Load” is:

$$\begin{aligned}
m &= \frac{E_s}{E_c} = \frac{200}{32.6} = 6.135 \\
y_G &= \frac{(4 \cdot 0.785 \cdot 10^{-4} \cdot (150 - (9+8+5)) \cdot 10^{-3} + 2 \cdot 0.785 \cdot 10^{-4} \cdot (9+8+5) \cdot 10^{-3}) \cdot 6.135 + 0.150 \cdot 0.150 \cdot \frac{0.150}{2}}{(4 \cdot 0.785 + 2 \cdot 0.785) \cdot 10^{-4} \cdot 6.135 + 0.150 \cdot 0.150} = 0.0770m \\
I_G &= \frac{0.150 \cdot 0.150^3}{12} + 0.150 \cdot 0.150 \cdot \left(\frac{0.150}{2} - y_G\right)^2 + \left(4 \cdot 0.785 \cdot 10^{-4} \cdot ((150 - (9+8+5)) \cdot 10^{-3} - y_G)^2 + 2 \cdot 0.785 \cdot 10^{-4} \cdot (y_G - (9+8+5) \cdot 10^{-3})^2\right) \cdot 6.135 = 4.229 \cdot 10^{-5} m^4 \quad (4.47) \\
w_G &= \frac{I_G}{(150 - y_G)} = \frac{4.229 \cdot 10^{-5}}{(150 - y_G)} = 5.793 \cdot 10^{-4} m^3 \\
f_{ctm} &= \frac{M(F_{cr})}{w} \Rightarrow 2.9 \cdot 10^3 \cdot w_G = 0.4667 \cdot F_{cr} \Rightarrow F_{cr} \approx 3.60 kN
\end{aligned}$$

where m is the homogenization coefficient, y_G the mass centre, I_G the inertia, w_G is the flexural modulus of the reinforced beam section and f_{ctm} the mean value of concrete axial tensile strength. The determined analytical “Cracking Load” is equal to 3.60 kN, much lower than the experimental one.

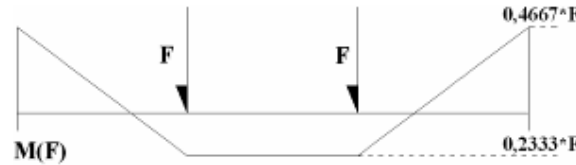


Figure 4.27 – Bending Diagram.

As we can see on Figure 4.26, the structural behaviour is divided in two parts, elastic behaviour at beginning and non-linear behaviour when the structural damage starts to appear. Application of Modal Interval Analysis (MIA) in a Quantified Constraint Satisfaction Problem (QCSP) was used to determine the load where the structure changes from elastic to non-linear behaviour (“Cracking Load”). To formulate the Quantified Constraint Satisfaction Problem (QCSP), numerical models, constituted by 1D Euler-Bernoulli beam elements, two displacements per node, were used. From the established static equilibrium equation $KU=F$, the respective constraints were obtained. Those equations depend on uncertain variables such as the applied load (F), the Young’s modulus (E), and the obtained vertical displacements from LVDT – Linear Variable Differential Transformer (w). Uniform distributions were considered for all uncertain variables. Other variables are considered to be deterministic (Length $L = 0.35$ m; Inertia $I = 4.21875 \times 10^{-5} m^4$).

Having into account the uncertainty due to material properties, the inconsistency test is made with $EI = [1293.4688, 1457.1562]$ kN.m². The load (F) is variable within the test and its value is affected by the actuator linearity. During the Quantified

Constraint Satisfaction Problem (QCSP), the analysis rotations (θ) were considered to be deterministic. The used vertical displacements are measured values affected by the transducer linearity, which is equal to ± 0.1 % of the transducer Full Scale (FS). Two cases, as presented in Figure 4.28 and Figure 4.29, have been studied to develop the numerical model of the beam for this structural assessment (Matos *et al.* 2006, Garcia *et al.* 2006, Garcia *et al.*, 2007). One last situation was also analyzed, considering an analytic redundancy reduction (ARR) of the structural model (Case 1).

4.3.5.2.1. CASE 1

The simplified model has been developed having into account the beam symmetry (Figure 4.28).

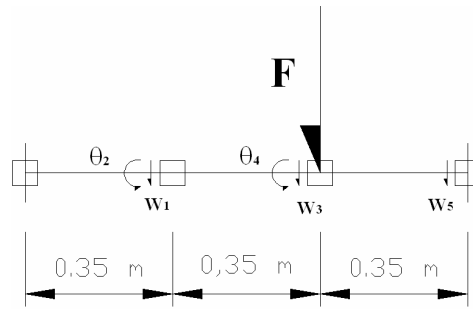


Figure 4.28 - Simplified numerical model (5 Constraints).

Values obtained from the transducers (vertical displacements) are represented by the variables w_1 , w_3 and w_5 . Rotations are first determined by solving the structural static equilibrium equation. F , E , L and I are simulated with the previously mentioned values. The following constraints are defined for the structural assessment:

$$C_1: \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_1 + \left(\frac{6 \cdot EI}{L^2} - \frac{6 \cdot EI}{L^2} \right) \cdot \theta_2 - \left(\frac{12 \cdot EI}{L^3} \right) \cdot w_3 - \left(\frac{6 \cdot EI}{L^2} \right) \cdot \theta_4 \leq 0 \quad (4.48)$$

$$C_2: \left(\frac{6 \cdot EI}{L^2} - \frac{6 \cdot EI}{L^2} \right) \cdot w_1 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_2 + \left(\frac{6 \cdot EI}{L^2} \right) \cdot w_3 + \left(\frac{2 \cdot EI}{L} \right) \cdot \theta_4 \leq 0 \quad (4.49)$$

$$C_3 : -\left(\frac{12 \cdot EI}{L^3}\right) \cdot w_1 + \left(\frac{6 \cdot EI}{L^2}\right) \cdot \theta_2 + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3}\right) \cdot w_3 + \left(\frac{6 \cdot EI}{L^2} - \frac{6 \cdot EI}{L^2}\right) \cdot \theta_4 - \left(\frac{12 \cdot EI}{L^3}\right) \cdot w_5 - F \leq 0 \quad (4.50)$$

$$C_4 : -\left(\frac{6 \cdot EI}{L^2}\right) \cdot w_1 + \left(\frac{2 \cdot EI}{L}\right) \cdot \theta_2 + \left(\frac{6 \cdot EI}{L^2} - \frac{6 \cdot EI}{L^2}\right) \cdot w_3 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L}\right) \cdot \theta_4 + \left(\frac{6 \cdot EI}{L^2}\right) \cdot w_5 \leq 0 \quad (4.51)$$

$$C_5 : -\left(\frac{12 \cdot EI}{L^3}\right) \cdot w_3 + \left(\frac{6 \cdot EI}{L^2}\right) \cdot \theta_4 + \left(\frac{12 \cdot EI}{L^3}\right) \cdot w_5 \leq 0 \quad (4.52)$$

From the Quantified Constraint Satisfaction Problem (QCSP) analysis, first inconsistency appeared in C2 for the value of $F = 8.439$ kN. It seems to be the load where the structural behaviour changes of elastic to non-linear (“Cracking Load”). Secondly, for $F = 9.195$ kN the constraint C1 becomes inconsistent too, confirming the appearance of a possible damage in support region (Table 4.12). Quantified Constraint Satisfaction Problem (QCSP) detects the appearance of this change nearby the experimental results. This is due to the fact that the used rotations, deterministic values, were obtained by a previous elastic analysis of the structure.

Table 4.12 – Results for structural assessment with 5 constraints.

No.	F (kN)	C ₁	C ₂	C ₃	C ₄	C ₅
1	8.439	True	False	True	True	True
2	9.195	False	False	True	True	True

4.3.5.2.2. CASE 2

To simulate the clamped-clamped boundary conditions, the following model has been developed having into account the existence of rotational springs ($k=k_2=k_{14}=6000$ kN.m/rad; $\delta k=60$ kN.m/rad). During the test it was implemented displacement transducers to control the beam deflection (w), namely w_3 , w_5 , w_7 , w_9 and w_{11} (Figure 4.29). Rotations are assumed by solving the structural static equilibrium equation. F , E , L , and I are simulated with the previously mentioned values.

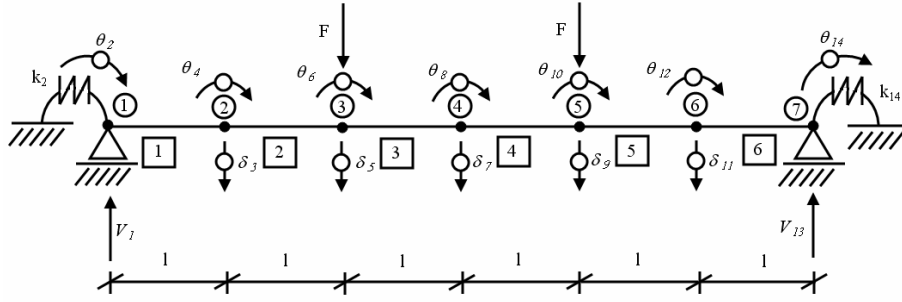


Figure 4.29 – Numerical model (14 Constraints).

The following constraints are defined for the structural assessment:

$$C_1 : F + \frac{6 \cdot EI}{L^2} \cdot \theta_2 - \frac{12 \cdot EI}{L^3} \cdot w_3 + \frac{6 \cdot EI}{L^2} \cdot \theta_4 \leq 0 \quad (4.53)$$

$$C_2 : \left(\frac{4 \cdot EI}{L} + k_2 \right) \cdot \theta_2 - \frac{6 \cdot EI}{L^2} \cdot w_3 \leq 0 \quad (4.54)$$

$$C_3 : -\frac{6 \cdot EI}{L^2} \cdot \theta_2 + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_3 - \frac{12 \cdot EI}{L^3} \cdot w_5 + \frac{6 \cdot EI}{L^2} \cdot \theta_6 \leq 0 \quad (4.55)$$

$$C_4 : \frac{2 \cdot EI}{L} \cdot \theta_2 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_4 - \frac{6 \cdot EI}{L^2} \cdot w_5 + \frac{2 \cdot EI}{L} \cdot \theta_6 \leq 0 \quad (4.56)$$

$$C_5 : -\frac{12 \cdot EI}{L^3} \cdot w_3 - \frac{6 \cdot EI}{L^2} \cdot \theta_4 + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_5 - \frac{12 \cdot EI}{L^3} \cdot w_7 + \frac{6 \cdot EI}{L^2} \cdot \theta_8 - F \leq 0 \quad (4.57)$$

$$C_6 : \frac{6 \cdot EI}{L^2} \cdot w_3 + \frac{2 \cdot EI}{L} \cdot \theta_4 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_6 - \frac{6 \cdot EI}{L^2} \cdot w_7 + \frac{2 \cdot EI}{L} \cdot \theta_8 \leq 0 \quad (4.58)$$

$$C_7 : -\frac{12 \cdot EI}{L^3} \cdot w_5 - \frac{6 \cdot EI}{L^2} \cdot \theta_6 + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_7 - \frac{12 \cdot EI}{L^3} \cdot w_9 + \frac{6 \cdot EI}{L^2} \cdot \theta_{10} \leq 0 \quad (4.59)$$

$$C_8 : \frac{6 \cdot EI}{L^2} \cdot w_5 + \frac{2 \cdot EI}{L} \cdot \theta_6 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_8 - \frac{6 \cdot EI}{L^2} \cdot w_9 + \frac{2 \cdot EI}{L} \cdot \theta_{10} \leq 0 \quad (4.60)$$

$$C_9 : -\frac{12 \cdot EI}{L^3} \cdot w_7 + \frac{6 \cdot EI}{L^2} \cdot \theta_8 + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_9 - \frac{12 \cdot EI}{L^3} \cdot w_{11} + \frac{6 \cdot EI}{L^2} \cdot \theta_{12} - F \leq 0 \quad (4.61)$$

$$C_{10} : \frac{6 \cdot EI}{L^2} \cdot w_7 + \frac{2 \cdot EI}{L} \cdot \theta_8 + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_{10} - \frac{6 \cdot EI}{L^2} \cdot w_{11} + \frac{2 \cdot EI}{L} \cdot \theta_{12} \subseteq 0 \quad (4.62)$$

$$C_{11} : -\frac{12 \cdot EI}{L^3} \cdot w_9 - \frac{6 \cdot EI}{L^2} \cdot \theta_{10} + \left(\frac{12 \cdot EI}{L^3} + \frac{12 \cdot EI}{L^3} \right) \cdot w_{11} + \frac{6 \cdot EI}{L^2} \cdot \theta_{14} \subseteq 0 \quad (4.63)$$

$$C_{12} : \frac{6 \cdot EI}{L^2} \cdot w_9 + \frac{2 \cdot EI}{L} \cdot \theta_{10} + \left(\frac{4 \cdot EI}{L} + \frac{4 \cdot EI}{L} \right) \cdot \theta_{12} + \frac{2 \cdot EI}{L} \cdot \theta_{14} \subseteq 0 \quad (4.64)$$

$$C_{13} : -\frac{12 \cdot EI}{L^3} \cdot w_{11} - \frac{6 \cdot EI}{L^2} \cdot \theta_{12} + -\frac{6 \cdot EI}{L^2} \cdot \theta_{14} + F \subseteq 0 \quad (4.65)$$

$$C_{14} : \frac{6 \cdot EI}{L^2} \cdot w_{11} + \frac{2 \cdot EI}{L} \cdot \theta_{12} + \left(\frac{4 \cdot EI}{L} + k_{14} \right) \cdot \theta_{14} \subseteq 0 \quad (4.66)$$

For this numerical model, the Quantified Constraint Satisfaction Problem (QCSP) analysis first results were not the expected, because of the huge number of variables involved. Two events were considered, first with the original experimental results and then with a signal treatment, because of the noise and delay presented on them. Structural behavior for Case 2 is improved after the signal treatment as seen in Table 4.13. Inconsistence appeared in C_{13} and C_{14} for the value $F = 4.805$ kN. Simulation of a huge number of constraints do not warranty a better solution as the number of deterministic rotations also increases.

Table 4.13 – Results for structural assessment with 14 constraints and signal treatment.

No.	F (kN)	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆	C ₇	C ₈	C ₉	C ₁₀	C ₁₁	C ₁₂	C ₁₃	C ₁₄
1	1.271	T	T	T	T	T	T	T	T	T	T	T	T	T	T
2	4.085	T	T	T	T	T	T	T	T	T	T	T	T	F	F
3	4.184	T	T	T	T	T	T	T	T	T	T	F	T	F	F
4	4.688	T	T	T	T	T	T	T	T	T	F	F	T	F	F
5	5.794	T	T	T	T	T	T	T	F	T	F	F	T	F	F
6	7.989	T	T	T	F	T	T	T	F	T	F	F	T	F	F
7	11.082	T	T	T	F	T	T	T	F	T	F	F	T	F	F
8	11.212	T	T	F	F	F	T	T	F	T	F	F	T	F	F
9	13.674	F	T	F	F	F	T	T	F	T	F	F	T	F	F
10	14.902	F	F	F	F	F	T	T	F	T	F	F	T	F	F
11	16.192	F	F	F	F	F	T	T	F	T	F	F	F	F	F
12	18.365	F	F	F	F	F	T	T	F	F	F	F	F	F	F
13	20.997	F	F	F	F	F	F	T	F	F	F	F	F	F	F
14	25.927	F	F	F	F	F	F	F	F	F	F	F	F	F	F

T = True, F = False

4.3.5.2.3. CASE 3: ANALYTICAL REDUNDANCY REDUCTION (ARR)

In order to minimize uncertainty, a variable reduction, based on Analytical Redundancy Reduction (ARR), was executed. There exists analytical redundancy if there are two or more different ways to determine one variable. A constraint that applies to only known variables and parameters constitutes an Analytical Redundancy Reduction (ARR) and it can be evaluated from only observed variables in order to be used in structural assessment. Based on the simplified numerical model of 5 constraints (Case 1) the following Analytical Redundancy Reductions (ARR) are defined:

$$ARR_1: \left(\frac{24 \cdot EI}{L^3} \right) \cdot w_1 - \left(\frac{24 \cdot EI}{L^3} \right) \cdot w_3 + \left(\frac{12 \cdot EI}{L^3} \right) \cdot w_5 \subseteq 0 \quad (4.67)$$

$$ARR_2: 3 \cdot EI \cdot (12 \cdot w_1 - 15 \cdot w_3 + 8 \cdot \theta_5) + 2 \cdot F \cdot L^3 \subseteq 0 \quad (4.68)$$

$$ARR_3: 6 \cdot EI \cdot (15 \cdot w_1 - 12 \cdot w_3 + 5 \cdot w_5) + FL^3/4L^3 \subseteq 0 \quad (4.69)$$

Results (Table 4.14) based only in measured values (w_1 , w_3 and w_5) give a quite approximation of experimental results.

Table 4.14 – Results for structural assessment with 3 Analytical Redundancy Reductions (ARR).

No.	F (kN)	C ₁	C ₂	C ₃
1	10.603	True	True	False
2	11.501	False	True	False
3	10.603	False	False	False

4.4. CONCLUSIONS

4.4.1. NUMERICAL METHODOLOGIES

In chapter 4, it was described the application of the previously presented methodologies (Perturbation Technique; Modal Interval Analysis (MIA)) to civil engineering structures. The obtained values were then compared with those due to the application of Monte Carlo Analysis, a well known methodology. It is so possible to make a comparison between those methodologies.

Perturbation Method presents the following advantages: it is a very efficient method in terms of time spent in computational calculation; to take into account the uncertainties in a finite element framework, it requires additionally the calculation of an inverse of a matrix with equal dimensions of a stiffness matrix, the evaluation of partial derivatives and some product of matrices to calculate matrix of covariance; the dispersion of structural response is available with a single structural analysis. As disadvantages it presents: the implementation of this method in a Finite Element framework requires addition of new modulus in the computer code, therefore it is almost “impossible” to apply this technique to a black-box type computer programs; the uncertainties are defined by statistical parameters and not by intervals.

From Interval Analysis it is possible to conclude that is a methodology which considers the whole uncertainty, warranting that the real answer is within the range of output values. The computational cost during the analysis was considerable, because the tolerance has to be incremented so the response could be the most certain one. It was also verified that the computational cost is directly influenced by the dimension of the stiffness matrix. Higher dimension corresponds to higher cost maintaining the output accuracy and this can be a serious problem in the application to large structures with an important number of degrees of freedom.

To obtain the same output value as the previous methodologies, by using Monte Carlo Analysis, it is necessary to run several times the structural numerical model. If n is the number of samples, the Monte Carlo Analysis gives better results when n tend to infinite. Although, for each sample the structure must be calculated and so, the computational cost is very high when compared to the other methodologies that need

only one calculation to obtain the same output. Anyway it is a well known methodology, with a large amount of applications in civil engineering field. It is, also, a methodology that uses both normal and uniform distributions.

Table 4.15 and Table 4.16 shows the output results for the examples presented at point 4.2.2 and point 4.2.3, respectively. The results are given in an absolute percentage (%) of the deterministic value, also referred in the table. Comparing those results, it can be observed that all methodologies give similar values, however, Monte Carlo Analysis gives wider tolerance intervals, Modal Interval Analysis (MIA) method gives narrower intervals and Perturbation Method gives intermediate values. The Perturbation Technique values are very similar to the ones obtained by Monte Carlo Analysis and that is, probably, due to the fact that both calculus are based in the same distributions of input parameters. A sharper bound is obtained with Modal Interval Analysis (MIA) which allows to concluding about its accuracy. In Table 4.16 it is possible to verify that the results for Case 1 and Case 3 are narrower than the ones obtained in Case 2 and Case 4. That is due to the fact that the applied load (F) in last situations presents a higher standard deviation than in the first ones.

Table 4.15 – Obtained results - Comparison of methods (point 4.2.2).

Solution	$\underline{w}_2$ (%)	$\overline{w}_2$ (%)	$\underline{\theta}_2$ (%)	$\overline{\theta}_2$ (%)	$\underline{\theta}_3$ (%)	$\overline{\theta}_3$ (%)
Perturbation Technique	15.81	15.81	15.84	15.84	15.81	15.81
Modal Interval Analysis	10.53	9.54	10.52	9.53	9.54	10.53
Monte Carlo Analysis	15.69	16.41	15.52	16.40	16.41	15.69
Deterministic Values	$3.33(3) \times 10^{-3} \text{ m}$		$8.33(3) \times 10^{-4} \text{ rad}$		$-3.33(3) \times 10^{-3} \text{ rad}$	

Table 4.16 – Summary of results for the different cases (%) (point 4.2.3).

Cases	Perturbation Technique				Modal Interval Analysis			
	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$
1	41.81	41.81	19.82	19.82	38.15	32.92	14.79	12.31
2	83.04	83.04	49.71	49.71	81.96	71.57	46.55	40.32
3	10.93	10.93	14.49	14.49	5.34	3.98	7.30	9.12
4	18.35	18.35	34.44	34.44	13.23	10.92	26.01	30.33

Cases	Monte Carlo Analysis				Deterministic Values (mm)	
	$\underline{w}_3$	$\overline{w}_3$	$\underline{w}_6$	$\overline{w}_6$	w_3	w_6
1	40.81	42.34	19.73	20.08	2.600	34.920
2	81.65	85.50	50.10	49.24	2.600	34.920
3	10.85	11.32	14.66	14.43	12.650	-50.940
4	18.40	18.40	34.42	34.50	12.650	-50.940

4.4.2. STRUCTURAL ASSESSMENT METHODOLOGY

In this chapter, it was also introduced two methodologies for the detection of any structural abnormal behaviour. The first one was based in a direct comparison between obtained data, affected by the transducer linearity, and numerical results, those, calculated using a finite element formulation (FEM) and the previously presented methodologies (Perturbation Technique; Modal Interval Analysis (MIA)) for the consideration of uncertainty. Once the application of Modal Interval Analysis (MIA) to large numerical models is still very difficult to realize, another methodology, based on a Quantified Constraint Satisfaction Problem (QCSP) and in Modal Interval Analysis (MIA) was presented here.

The first methodology was tested with two examples, at point 4.3.3 and point 4.3.4. The first example showed that the steel beam presented a normal behaviour during all test. It is considered as normal behaviour, an elastic one, due to the fact that the numerical model was made based in elastic assumptions. Only for small applied loads the algorithm identified a possible damage. This was due to the fact that, for small loads, the measured values are small, also, and the sensor resolution is of the same order of magnitude of such value. In fact, the structure and the utilized sensors present good behaviour for small and higher loads.

In the second example the methodology was applied to a reinforced concrete beam that was tested in laboratory up to failure. The idea is to determine the “Cracking Load”, the load for which the structural elastic behaviour changes to a non-linear one, and so, the initial numerical model cannot be applied anymore. This algorithm considers the uncertainty in input parameters and in measured variables (noise and sensor linearity). The results were approximate from the experimental ones, when analyzing the graph load-displacement for the displacement transducer positioned at beam midspan. In fact, the comparison is made between the numerical output and experimental data for the midspan displacement. Once the structure is clamped in its extremities, the first zone to crack it will be the supporting region and so the real “Cracking Load” may be lower than the one determined here. For this methodology it is possible to conclude that the transducer position is extremely important and so it is extremely important to determine the critical structural regions before applying this algorithm.

The second methodology was tested with two examples at point 4.3.5.1 and point 4.3.5.2. The first example, a theoretical one, showed the algorithm sensitivity. In fact, small changes in the rigidity modulus of the structure could not be detected by the developed methodology. While in the previous methodology it was possible to analyse the whole structure or any instrumented section, this algorithm only analyses, at the same time, all numerical constraints, and so, the overall structure. This is the main reason why the other methodology is more sensitive than this one. However the robustness of this algorithm is higher than the previous one.

The second example shows the application of this algorithm to the same reinforced concrete structure, which was analyzed by the previous methodology. First it was determined, having into account the measured midspan strain, the experimental and analytical “Cracking Load”. It was verified that such load was less than the previously determined one, that the first cracks start to appear in the support region, that the analytical result gives a lower value as it is considered the mean value of concrete axial tensile strength (f_{ctm}), and that the strain device is more sensitive than the displacement transducer to changes in structural behaviour. Three cases were studied using the developed algorithm. The first one was analysed by applying a simplified numerical model, the second by using a complete numerical model and the third performed with the numerical model applied in the first situation but with an Analytical Redundancy Reduction (ARR). It was verified that simplified models give better results as less deterministic variables are considered in numerical model.

CHAPTER 5.

CONCLUSIONS & FUTURE DEVELOPMENTS

5.1. GENERAL CONCLUSIONS

This thesis deals with different topics related to the uncertainty treatment for civil engineering numerical models. At first, the uncertainty classification, the necessary basis for a probabilistic analysis and the probabilistic models used in the design codes of civil engineering structures are presented. Secondly, three methodologies for uncertainty analysis are introduced, namely, Monte Carlo method, Perturbation technique and Modal Interval Analysis (MIA). Thirdly it is presented some applications of these techniques in order to execute a simple comparison between them, and two algorithms which were developed for structural assessment.

A review of distribution types for the characterization of civil engineering variables is presented in this thesis. It is also explained the Monte Carlo methodology, the Perturbation technique (Henriques, 2006a; Henriques, 2006b; Veiga *et al.*, 2006a; Veiga *et al.*, 2006b) and the Modal Interval Analysis (MIA) (Gardenyes *et al.*, 2001; Garcia *et al.*, 2004; Matos *et al.*, 2004a). A study of each methodology, namely, a presentation of its basis and how to use such techniques in the uncertainty analysis of civil engineering structures is also defined. It is important to refer that it is also referred the scope of applicability of each technique.

From the analysis of the examples used to compare the developed numerical methodologies, namely the hyperstatic steel beam and one-floor frame, it is possible to conclude that:

- All techniques give a bound for the structural response, accordingly to the variability in numerical parameters;
- In general, the used techniques give a superset of the numeric solution space;
- Modal Interval Analysis (MIA) presents the structural response narrowest bound, while Perturbation technique gives approximated values from Monte Carlo ones;
- Modal Interval Analysis (MIA) can only be used with uniform distributions. Several times it is needed an approximation to other distribution types, introducing a source of error;
- In general, normal distributions are attributed to all numerical input parameters, like material (steel or concrete), geometry and applied load;

- One other important point to introduce is the methodology efficiency. Monte Carlo needs, in order to give an acceptable result, to be applied several times (elevated number of samplers) being necessary a high computational cost. Modal Interval Analysis (MIA) and Perturbation technique to give the same efficient result need only to be ran one unique time, being the associated computational cost lower;

- A disadvantage of Modal Interval Analysis (MIA), which led the author to recommend the Perturbation technique as a methodology for an efficient structural uncertainty analysis, is the fact it runs very badly in a finite element framework. Once the finite element model of the structure gets more precise, more variables are introduced and the system of equations becomes higher. The problem of overestimation grows as the possibility of multi occurrence of variables in one equation gets higher, and so, as the system of equations gets bigger. Modal Interval Analysis (MIA) should so be only used with simple numerical civil engineering problems of uncertainty analysis and problems of variable control.

These methodologies were developed in order to obtain a fast and efficient way of uncertainty treatment. The final conclusion is that the most appropriate method is the Perturbation technique, while Modal Interval Analysis (MIA) presents a high potential for civil engineering field once some steps are over passed. Monte Carlo is a well known methodology, with a high applicability in civil engineering field, that will continue to have it applications, but, it is inefficient when performing a structural assessment. Table 5.1 presents a resume of the main disadvantages and advantages of uncertainty analysis techniques.

Table 5.1 – Comparison between uncertainty analysis methodologies.

Methodology	Advantages	Disadvantages
Monte Carlo	1) Allow integral numerical calculus with impossible analytical resolution; 2) Applied in a general way wherever the variables distribution type; 3) The error associated to these kinds of techniques is perfectly controlled through the sample's number. It is verified that to a sample with a size tending to infinite ($n \rightarrow \infty$), the result converges to the right one; 4) There are no restrictions in Monte Carlo application.	1) High computational time as it is necessary to run several times the structural model to obtain a high resolution output result. However the application of variance reduction techniques may turn this method more efficient.
Perturbation Technique	1) Very efficient method in terms of time spent in computational calculation; 2) To take into account the uncertainties in a finite element framework, it requires additionally the calculation of an inverse of a matrix with equal dimensions of a stiffness matrix, the evaluation of partial derivatives and some product of matrices to calculate matrix of covariance; 3) The dispersion of structural response is available with a single structural analysis; 4) Correlation between variables could be easily taken into account; 5) Information about variables uncertainty is defined only by two parameters (mean value and standard deviation).	1) The implementation of this method in a finite element framework requires addition of new modulus in the computer code, therefore it is almost "impossible" to apply this technique to a black-box type computer programs; 2) The uncertainties are defined by statistical parameters and not by intervals.
Modal Interval Analysis (MIA)	1) Methodology which considers the whole uncertainty, warranting that the real answer is within the range of output values.	1) The computational cost during the analysis was considerable, because the tolerance has to be incremented so the response could be the most certain one; 2) The computational cost is directly influenced by the dimension of the stiffness matrix. As a conclusion, the computational effort is relatively low and the calculus velocity adequate for the simple examples presented here. However, this can be a serious problem in the application to large structures with an important number of degrees of freedom.

A study of different structural assessment techniques was executed. Two methodologies for structural assessment were developed. The first consists in a direct and consistent comparison between numerical values and obtained data (Casas *et al.*, 2005; Matos *et al.*, 2004b; Matos *et al.*, 2005a; Matos *et al.*, 2005b). This methodology was applied with a steel and a concrete beams tested in laboratory.

The steel beam, which was tested in its elastic regime presents damage for low applicable loads. This happened due to the fact of the sensor linearity being of the same order of magnitude of numerical output for such load intensity. Low applicable loads, being the uncertainty in applied loads directly dependent of the load magnitude, means narrower bound for possible structural response. As a result, comparison intervals become also narrower and probability of error gets higher. To mitigate such error, uncertainty in applied load should be a percentage of actuator full scale. This algorithm, although very simple and efficient, presents disadvantages, like the problem of how to differentiate a structural abnormal behaviour from sensor damage or non appropriate numerical model. These disadvantages can be over passed using updated models and using logic algorithms that could help to distinguish the cause of identification.

The concrete beam was tested in laboratory till rupture. The aim of this application was to obtain the “Cracking Load”. The obtained results are similar to the laboratory ones. Those results are compared with the ones obtained by a displacement transducer positioned at the beam mid span which is not a sensitive sensor for the “Cracking Load” target. This is one other point as this algorithm is directly dependent of the used sensors. Once they are not the more appropriate ones, the error gets higher. One possibility to face it is to use more than one sensor at the same time, and to establish a structural model, based on the results obtained for one set of sensor data.

One other methodology, based in a Quantified Constraint Satisfaction Problem (QCSP) (Herrero *et al.*, 2004; Herrero *et al.*, 2005; Matos *et al.*, 2006; Garcia *et al.*, 2006), for structural assessment was developed. Such algorithm was tested, initially, with a theoretical example where structural damage was obtained and then detected by introducing a reduction in the stiffness factor of one beam element. This algorithm was then tested with the previously presented concrete laboratory beam. The results were obtained using three different possible numerical models of the beam, all of them based in the finite element method (FEM). It was verified that simplified models give better results as less deterministic variables are considered in numerical model.

5.2. FUTURE DEVELOPMENTS

Whilst presenting and explaining the numerical methodologies for uncertainty analysis and the structural assessment techniques, several areas that required future research have been identified, namely:

- Definition of model uncertainties. What is the distribution type that fits better with numerical input parameters, how to pass such distributions to uniform ones and vice-versa, and, for structural assessment techniques, how to define the distribution type that best fits the sensor linearity;

- Developed numerical methodologies. An algorithm, for Perturbation technique (Henriques, 2006a; Henriques, 2006b; Veiga *et al.*, 2006a; Veiga *et al.*, 2006b), that leads with all uncertainty input parameters of a model should be developed in a programming language and introduced in an open source finite element program. Modal Interval Analysis (MIA) (Gardenyes *et al.*, 2001; Garcia *et al.*, 2004; Matos *et al.*, 2004a) should be studied in a detailed way. Applications of it to control of variables should be executed in civil engineering field. However, there are obstacles that need to be over passed when applying such methodology with a finite element framework;

- Structural assessment problem using static measured data and a consistent comparison algorithm (Casas *et al.*, 2005; Matos *et al.*, 2004b; Matos *et al.*, 2005a; Matos *et al.*, 2005b). This methodology should be tested in laboratory with other structural types and measured variables. An algorithm to differentiate a structural abnormal behavior from sensor damage must be developed. These algorithms should be implemented into an open source finite element program. Finally, these algorithms should be executed in a real structure;

- Structural assessment problem using static measured data and a Quantified Constraint Satisfaction Problem (QCSP) algorithm (Herrero *et al.*, 2004; Herrero *et al.*, 2005; Matos *et al.*, 2006; Garcia *et al.*, 2006). A detailed studied and more applications regarding this methodology should be executed as it is very recent and so the results are not controllable as with the previous methodology. Afterwards, this methodology could be implemented in an open source finite element program and then applied into a real structure.

CHAPTER 6.

BIBLIOGRAPHY



Abdel-Tawab, K, Noor, AK (1999) A fuzzy-set analysis for a dynamic thermo-elasto-viscoplastic damage response. *Computer and Structures*, Vol. 70: 91-107.

Agostoni, N, Ballio, G, Poggi, C (1994) Statistical analysis of the mechanical properties of structural steel. *Costruzioni metalliche*, No. 2: pp. 31 – 39.

Altus, E, Totry, E, Givli, S (2005) Optimized functional perturbation method and morphology based effective properties of randomly heterogeneous beams. *International Journal of Solids and Structures* 42 (8): 2345 – 2359.

Ang, AH-S, Tang, WH (1984) Probability concepts in engineering planning and design: Volume 2. Decision Risk and Reliability. New York. Wiley.

Assis, WS, Matos, JC, Sousa, H, Figueiras, JA, Bittencourt, TN (2005) Sistema de Visualização do comportamento estrutural da Ponte sobre o Rio Sorraia. 47º Congresso Brasileiro do Concreto (IBRACON), Olinda – PE - Brasil, 2 - 7 Setembro 2005.

Assis, WS, Sá, J, Sousa, H, Matos, JC, Pereira, D, Bittencourt, T, Dias, I, Figueiras, JA (2006a) Avaliação do sistema de monitorização da Ponte sobre Rio Sorraia. VI Simpósio EPUSP sobre Estruturas de Concreto, São Paulo – Brasil, 8 – 11 Abril 2006.

Assis, W, Bittencourt, TN, Matos, JC, Sousa, H, Figueiras, JA (2006b) An Integrated System for Structural Behaviour Visualization. The Second fib Congress, Naples – Italy, 5 – 8 June 2006.

Ayyub, BM, McCuen, RH (1997) Probability, Statistics and Reliability for Engineers. Boca Raton, FL. CRC Press.

Ayyub, BM, Gupta, MM (1997) Uncertainty Analysis in Engineering and Sciences: Fuzzy Logic, Statistics, and Neural Network approach. Kluwer Academic Publishers, Boston.

Ayyub, BM (1998) Uncertainty modelling and analysis in civil engineering. CRC Press LLC.



Baker, MJ (1972) Variability in the strength of structural steels – a study in material variability; Part 1: Material variability. CIRIA Technical Note 44.

Bartlett, FM, MacGregor, JG (1996) Statistical analysis of compressive strength of concrete in structures. *ACI materials journal*, V. 93: pp. 158 - 168.

Bathe, K-J (1996) *Finite Element Procedures*. Prentice Hall, Upper Saddle River, NJ.

Beck, JL, Au, S-K, Vanik, MW (2001) *Monitoring Structural Health Using a Probabilistic Measure*. *Computer Aided Civil and Infrastructure Engineering*, Vol. 16 (1): pp. 1-11.

Bergmeister, K (2003) *Monitoring and safety evaluation of existing concrete structures*. State-of-Art Report. International Federation for Structural Concrete (fib), N. 22.

Bier, VM (1990) Is Fuzzy more general than probability theory? First international symposium on Uncertainty Modelling and Analysis, pp. 297-301.

Bierbaum, F, Schwartz, KP (1995) *A Bibliography on Interval-Mathematics*. Appendix A. *Methods and Applications of Interval Analysis*, SIAM Studies in Applied Mathematics, Philadelphia.

Biondini, F, Bontempi, F, Malerba, PG (2004) Fuzzy reliability analysis of concrete structures. *Computational Structures*, Vol. 82(13-14): 1033-1052.

T

Casas, JR, Matos, JC, Figueiras, JA, Vehi, J, Garcia, O, Herrero, P. (2005) Bridge monitoring and assessment under uncertainty via Interval Analysis. In: Augusti, G, Schueller, GI, Ciampoli (ed), *Proceedings of the 9th International Conference on Structural Safety and Reliability - ICOSSAR'05*, Rome, June 19-23, Millpress, Rotterdam, pp 487-494.

Cayrac, D, Dubois, D, Prade, H (1996) Handling uncertainty with possibility theory and fuzzy sets in satellite fault diagnosis application. *Fuzzy Systems*, Vol. 4(3): 251-269.

Contreras, H (1980) The stochastic finite element. *Computational Structures* 12 (3): 341-348.

D

DeLaurentis, D, Marvris, D (2000) Uncertainty modelling and management in multidisciplinary analysis and synthesis. *Proceedings of the 38th AIAA Aerospace Sciences Meeting*, paper No. AIAA 2000-0422, Reno, NV, January 10-13.

Delgado, LEM (2006) *A hybrid approach of knowledge-based reasoning for structural assessment*. PhD Thesis, University of Girona, Spain, 142 pp.

Du, X, Sudjianto, A (2004) A saddle point approximation method for uncertainty analysis. *Proceedings of the DETC'04 – ASME 2004 International Design*

Engineering Technical Conferences and the Computers and Information in Engineering Conference, Salt Lake City, Utah, USA, September 28 – October 2.



EC1-2 (1994) Eurocode 1 – Part 2.4: Imposed loads on buildings. Comité Européen de Normalisation (CEN). European Pre standard ENV 1991 2-4.

EC1-1 (2002) Eurocode 1 - Part 1.1: General actions - Densities, self-weight, imposed loads for buildings. Comité Européen de Normalisation (CEN). European Pre standard ENV 1991 2-1.

Eibl, J, Schmidt-Hurtienne, B (1995) General outline of a new safety format. CEB . Bulletin d'Information 229: 33-48.



Farrar, CR, Doebling, SW (1997) Lessons learned from applications of Vibration – Based damage identification methods to a large bridge structure. International workshop on structural health monitoring, Stanford, USA, September, pp 18-20.

Figueiras, J, Felix, C, Matos, JC, Sousa, H (2004) An integrated system for structural health monitoring – application to the Sorraia River Bridge. IABMAS'04, 18-22 October 2004, Kyoto – Japan.

Florian, A (1992) An efficient sampling scheme: Updated Latin Hypercube Sampling. Probabilistic Engineering Mechanics 7 (2): 123-130.

Frangopol, DM, Lee, Y-H, William, KJ (1996) Nonlinear finite element reliability analysis of concrete. Journal of Engineering Mechanics ASCE, Vol. 122(12): 1174-1182.

Frangopol DM, Dong JS, Gharaibeh ES (2001) Reliability based life cycle management of highway bridges. Journal Computational Civil Engineering ASCE 15(1): 27-34.



Galambos, TV, Ravindra, MK (1978) Properties of steel for use in LFRD. Journal of structural division, ASCE, Vol. 104, ST9.

Garcia, O, Vehi, J, Sainz, MA, Matos, JC, Casas, JR, Rodellar, J, Mujica, LE (2004). Application of Interval Analysis to solve engineering problems – Some examples. 2nd European Conference on Structural Health Monitoring, Munich, Germany, 7-9 July, pp 93-100.

Garcia, OD (2005). Structural Damage Identification in presence of uncertain parameters through Modal Interval Analysis. Research Report. Departament d'Electrònica, Informàtica I Automàtica, Universidad de Girona, Girona, Spain.

Garcia, OD, Vehí, J, Matos, JC, Henriques, AA, Figueiras, JA, Casas, JR (2006) Bridge Assessment under Uncertain Parameters via Interval Analysis. Third International Conference on Bridge Maintenance, Safety and Management, Porto – Portugal, 16 – 19 July 2006.

Garcia, O, Vehí, J, Matos, JC, Henriques, AA, Casas, JR (2007) Structural Assessment under Uncertain Parameters via Interval Analysis. Special Issue of Journal of Computational and Applied Mathematics. Editor: S. Brenner, MJ, Goovaerts, T, Mitsui, M, Ng, B, Sommeijer, L, Wuytack (doi: 10.1016/j.cam.2007.04.047).

Gardenyes, E, Trepát, A (1979) The Interval Computing System. Sigla-PL/I(0), Freiburger Intervall-Berichte 79 (8).

Gardenyes, E, Trepát, A, Mielgo, H (1982) Present Perspective of the SIGLA Interval System, Freiburger Intervall-Berichte 82 (9).

Gardenyes, E, Mielgo, H, Trepát, A (1985) Modal Intervals: Reason and Ground Semantics. Interval Mathematics, Springer, Heidelberg, pp. 27–35.

Gardenyes E, Sainz MA, Jorba L, Calm R, Estela R, Mielgo H, Trepát A (2001) Modal Intervals. Reliable Computing 7(2): 77-111.

Gayton N, Mohamed A, Sorensen JD, Pendola M, Lemaire M (2004) Calibration methods for reliability-based design codes. Structural Safety 26(1): 91-121.

Ghanem RG, Spanos PD (2003) Stochastic Finite Elements: A Spectral Approach. Courier Dover Publications, New York.

Gonçalves, AF (1987) Resistência do betão nas estruturas. PhD Thesis, Civil Engineering National Laboratory (LNEC), Lisbon.

Guan, XL, Melchers, RE (1994) An efficient formulation for limit state functions gradient calculation. Computational and Structures, Vol. 53(4): 929-935.



Haftka, RT, Rosca, RI, Nikolaidis, E (2006) An approach for testing methods for modelling uncertainty. Journal of Mechanical Design, Vol. 128(5): 1038-1049.

Haldar, A, Mahadevan, S (2000) Probability, reliability and statistical methods in engineering design. John Wiley & Sons, New York.

Hansen, E (1965) Interval arithmetic in matrix computation. Journal of Numerical Analysis, SIAM, Part I (2): 308-320.

Hansen, E (1992) Global Optimization Using Interval Analysis. Marcel Dekker, New York.

Heisenberg, W. (1958) Physics and Philosophy. New York: Harper.

Henriques, AA (1998) Application of new safety concepts in the design of structural concrete. Ph. D. Thesis. Faculty of Engineering of the University of Porto, 502 pp (In Portuguese).

Henriques, AA (2006a) Avaliação eficiente da incerteza estrutural aplicando técnicas de perturbação. RPEE - Revista Portuguesa de Engenharia de Estruturas, LNEC, N. 55, pp. 5-13.

Henriques, AA (2006b) Uncertainty analysis and assessment of structural systems, 6th European Solid Mechanics Conference, Budapest (Ed. CD-ROM: 2 pp.).

Herrero, P, Sainz, MA, Vehí, J, Jaulin, L (2004) Quantified Set Inversion With Applications To Control. Proceedings of the IEEE CCA/ISIC/CACSD Conference, Taipei, Taiwan, September 1 (CD-ROM Edition).

Herrero, P, Vehí, J, Sainz, MA, Jaulin, L (2005) Quantified Set Inversion Algorithm. Reliable Computing 11(5): 369-382.

Hu, X, Shenton, HW (2002) Structural damage identification using static dead load strain measurements. 15th ASCE Engineering mechanic conference, New York, USA, June 2-5.

Hurtado, JE (2002) Analysis of one-dimensional stochastic finite elements using neural networks. Probabilistic Engineering Mechanics, Vol. 17(1): 35-44.

Huang, B, Xiaoping, D (2006) Uncertainty analysis by dimension reduction integration and saddle point approximations. Journal of Mechanic Design, Vol. 128(1): 26-33.

J

Iman, RL, Conover, WJ (1980) Small sample sensitivity analysis techniques for computer models, with an application to risk assessment. Communications in statistics: theory and methods A9, no. 17, pp. 1749-1842.

J

Jaulin L, Kieffer M, Didrit O, Walter E (2001) Applied Interval Analysis. Springer-Verlag, London.

JCSS (2001) Probabilistic Model Code. Joint Committee on Structural Safety, <http://www.jcss.ethz.ch>, 12 -th draft.

K

Kaucher, E (1980) Interval Analysis in the Extended Interval Space IR, Computing Supplementum 2, Springer, Heidelberg,, pp. 33–49.

Kennedy, DJL, Baker, KA (1984) Resistance factors for steel highway bridges. Canadian Journal of Civil Engineering, Vol. 11: pp. 324 – 334.

Kharmanda G, Mohamed A, Lemaire M (2002) Efficient reliability-based design optimization using a hybrid space with application to finite element analysis. Structure Multidisciplinary Optimization 24(3): 233-245.

Klir, GJ (1995) Principles of uncertainty: What are they? Why do we need them? Fuzzy Sets and Systems, Vol. 74 (1): 15-31.

Kulpa, Z, Pownuk, A, Skalna, I (1998) Analysis of linear mechanical structures with uncertainties by means of methods. Computer Assisted Mechanics and Engineering Sciences, Vol. 5: 443-477.

L

Liu, WK, Besterfield, GH, Belytschko, T (1988) Variational approach to probabilistic finite elements. ASCE Journal of Engineering Mechanical Division, Vol. 114 (12): 2115-2133.

Liu, P, Der Kiureghian, A (1991) Finite element reliability of geometrically nonlinear uncertain structures. Journal of Engineering Mechanics ASCE, Vol. 117(8): 1806-1825.

M

Maglaras, G, Nikolaidis, E, Haftka, RT, Cudney, HH (1997) Analytical-experimental comparison of probabilistic methods and fuzzy set based methods for designing under uncertainty. Structural Multidisciplinary Optimization, Vol. 13 (2-3): 69-80.

Mahadevan, S, Raghathamachar, P (2000) Adaptive simulation for system reliability analysis of large structures. Computational Structures, Vol. 77(6): 725-734.

Marek, P, Gustar, M, Anagnos, T (1996) Simulation-Based Reliability Assessment for Structural Engineers. Boca Raton, FL. CRC Press.

Matos, JC, Garcia, O, Vehi, J, Sainz, M (2004a) Aplicación del análisis intervalar a algunos problemas de ingeniería civil. Métodos Computacionais em Engenharia. 2004. Lisboa - Portugal - pp. 396.

Matos, JC, Figueiras, J, Casas, JR (2004b) Efficient health monitoring of structures. 5th International PhD Symposium in Civil Engineering, 16-19 June 2004, Delft – The Netherlands.

Matos, JC, Pedro, JO, Sousa, H, Félix, C, Reis, A, Figueiras, JA (2004c) Análise do comportamento da ponte sobre o Rio Sorraia durante a fase construtiva. Encontro Nacional do Betão Estrutural 2004, 17-19 Novembro 2004, Porto – Portugal.

Matos, JC, Sousa, H, Assis, W, Figueiras, JA (2005a) Structure assessment by continuous monitoring - Application to Sorraia River Bridge. fib Symposium Keep Concrete Attractive 2005, Budapest – Hungary.

Matos, JC, Casas, JR, Figueiras, JA (2005b) A new methodology for damage assessment of bridges through instrumentation: Application to the Sorraia River Bridge. Structure and Infrastructure Engineering Maintenance, Management, Life-Cycle Design and Performance Journal (IABMAS). Editor-in-Chief: Prof. Dan M. Frangopol, University of Colorado, Boulder, CO, USA. Volume 1, Issue 4, September 2005.

Matos, JC, Henriques, AA, Figueiras, JA, Casas, JR, Garcia, OD, Vehi, J (2006) Long Term Assessment of Concrete Structures. The Second fib Congress, Naples – Italy, 5 – 8 June 2006.

Matos, JC, Sousa, H, Figueiras, JA, Casas, JR (2007) Structural Health Monitoring (SHM) system implemented in Sorraia River Bridge. ICASP 10, 10th International Conference on Applications of Statistics and Probability in Civil Engineering, Tokyo – Japan, 31 July – 3 August 2007.

Melchers, RE (1999) Structural reliability analysis and prediction. John Wiley & Sons, Chichester, 2nd edn.

Mirza, SA, MacGregor, JG (1979a) Variability of mechanical properties of reinforcing bars. Journal of the structural division, ASCE, Vol. 105: pp. 921 – 937.

Mirza, SA, MacGregor, JG (1979b) Variation in dimensions of reinforced concrete members. Journal of the structural division, ASCE, Vol. 105: pp. 921 – 937.

Mirza, SA, Hatzinikolas, M, MacGregor, JG (1979) Statistical descriptions of Strength of Concrete. Journal of the Structural Division, ASCE, V. 105: pp. 751 - 766.

Moens, D and Vandepitte, D (2004) Non-probabilistic approaches for non-deterministic dynamic FE analysis of imprecisely defined structures. Proceedings of International Conference on Noise and Vibration Engineering, ISMA2004, pp. 3095-3120.

Moore, RE (1966) Interval Analysis. Prentice-Hall, Englewood Cliffs.

Muhanna, RL, Mullen, RL (2001) Uncertainty in mechanics problems - Interval-based approach. Journal Engineering Mechanics ASCE, Vol. 127 (6): 557-566.

Mullen, RL and Muhanna, RL (1999) Bounds of structural response for all possible loading combinations. Journal of Structural Engineering, ASCE, Vol. 125(1): 98-106.

N

Neumaier, A (1990) Interval methods for system of equations. Cambridge University Press.

Nowak, AS, Collins, KR (2000) Reliability of structures. McGraw-Hill Higher Education, New York.

Nowak, AS, Szerszen, MM (2003) Calibration of design code for buildings (ACI 318): Part 1 – Statistical models for resistance. ACI Structural Journal, V. 100: pp. 377 – 382.

O

Oberkampf, WL, Deland, S, Rutherford, BM, Diegert, KV, Alvin, KF (1999) A new methodology for the estimation of total uncertainty in computational simulation. 40th AIAA / ASME / ASCE / AHS / ASC structures, structural dynamics and materials conference. AIAA. V. 99 (1612): pp. 3061 – 3083.

Olsson, A, Sandberg, G, Dahlblom, O (2003) On Latin Hypercube Sampling for structural reliability analysis. Structural Safety, 25 (1): 47-68.

P

Papadrakakis, M, Lagaros, ND (2002) Reliability-based structural optimization using neural networks and Monte Carlo simulation. Computational Methods Applied to Mechanical Engineering, Vol. 191: 3491–3507.

Pawlak, Z (1995) Rough set approach to knowledge-based decision support. 14th European Conference on Operational Research.

Pawlak, Z (1996) Rough sets, rough relations and rough functions. Fundamenta Informaticae, Vol. 27: 103-108.

Perdigão, V, Barros, P, Dias, I, Pereira, D, Alves, J, Sousa, H, Matos, JC, Figueiras, JA (2004) Sistema de monitorização de obras de arte aplicado à ponte sobre o Rio Sorraia. Encontro Nacional do Betão Estrutural 2004, 17-19 Novembro 2004, Porto – Portugal.

Perdigão, V, Barros, P, Matos, JC, Henriques, AA, Figueiras, JA, Dias, I, Pereira, D (2006) SMARTE – Development and Implementation of a Long-Term Structural Health Monitoring. Third International Conference on Bridge Maintenance, Safety and Management, Porto – Portugal, 16 – 19 July 2006.

Pipa, MJ (1995) Ductilidade de elementos de betão armado sujeitos a acções cíclicas. Influência das características mecânicas das armaduras. PhD Thesis, School of Engineering of the Technical University of Lisbon, Department of Civil Engineering.

Pownuk, A (2001) Calculation of reliability of structures using random sets. AI-MECH 2001, Symposium on Methods of Artificial Intelligence in Mechanics and Mechanical Engineering, Gliwice, 14-16 November, pp. 217-220.

Press, WH, Teukolsky, SA, Vetterling, WT, Flannery, BP (1992) Numerical recipes in Fortran, 2nd Edition. New York. Cambridge University Press. Pp. 266-276, 305-306.



Rais-Rohani, M, Singh, MN (2004) Comparison of global and local response surface techniques in reliability-based optimization of composite structures. Structures and Multidisciplinary Optimization, Vol. 26(5): 333-345.

Rand Corporation (1955) A Million Random Digits with 1,000,000 Normal Deviates. New York. Free Press.

Rao, SS, Sawyer, P (1995) Fuzzy finite element approach for analysis of imprecisely defined systems. AIAA Journal, Vol. 33(2): 2364-2370.

Rao, SS, Berke, L (1997) Analysis of uncertain structural systems using interval analysis. AIAA Journal, Vol. 35(4): 727-735.

Rao, SS, Chen, L (1998) Numerical solution of fuzzy linear equations in engineering analysis, International Journal of Numerical Methods in Engineering, Vol. 43: 391-408.

Reh, S, Beley, J-D, Mukherjee, S, Khor, EK (2006) Probabilistic finite element analysis using ANSYS. Structural Safety 28 (1-2): 17-43.

Reis, E, Melo, P, Andrade, R, Calapez, T (1999) Estatística aplicada - Volume 1. Edições Sílabo, Lda.

Ross, S (1997) Simulation, 2nd Edition. San Diego. Academic Press.

Rubinstein, RY (1981) Simulation and the Monte Carlo Method. New York. Wiley.



Schenk CA, Schueller GI (2005) Uncertainty Assessment of Large Finite Element Systems. Springer, Berlin Heidelberg New York.

Schneider, J (1997) Introduction to safety and reliability of structures. IABSE, Zurich.

Schueller, GI (2001) Computational stochastic mechanics - recent advances. *Computational Structures*, Vol. 79(22-25): 2225-2234.

Schueremans L, Gemert DV (2003) System reliability methods using advanced sampling techniques. In: Bedford T, Van Gelder P (ed) *Safety and Reliability, Proceedings of the Esrel 2003 Conference*, Maastricht, The Netherlands, June 15-18, Routledge, pp. 1425-1432.

Sedlacek, G (1992) Imposed loads on buildings. *IABSE Reports*, Vol. 65, IABSE Conference Structural Eurocodes, Davos, pp. 51 – 57.

Sedlacek, G, Gulvanessian, H (1996) Basis of design and actions on structures. Part 2.1: densities, self-weight, imposed loads, *IABSE Reports*, Vol. 74, IABSE Conference Basis of design and actions on structures, Background of application of Eurocode 1, Delft, pp. 61 – 70.

Shary, SP (2002) A new technique in system analysis under interval uncertainty and ambiguity. *Reliable Computing* 8: 321-418.

SIGLA/X group (1996) Construcción de los intervalos modales, Report IMA 96-07-RR, Departamento de Informática Aplicada, Universidad de Girona.

SIGLA/X group (1997) Aritmética intervalar en el entorno C++, Report IMA 97-06-RR, Departamento de Informática Aplicada, Universidad de Girona.

SIGLA/X group (1998a) Extensiones de las funciones continuas, Report IMA 98-04-RR, Departamento de Informática Aplicada, Universidad de Girona.

SIGLA/X group (1998b) Optimalidad condicionada de las funciones racionales modales, Report IMA 98-02-RR, Departamento de Informática Aplicada, Universidad de Girona.

SIGLA/X group (1998c) Optimalidad de las funciones racionales modales, Report IMA 98-05-RR, Departamento de Informática Aplicada, Universidad de Girona.

SIGLA/X group (1999) Modal Intervals. Basic Tutorial. Applications of Interval Analysis to Systems and Control. Proceedings of the Workshop MISC'99. University of Girona, Girona, Spain, pp. 157-227.

Singh, V. P., Jain, S. K., Tyagi, A (2007) Risk and Reliability Analysis. ASCE Press.

Sobrino, JA (1993) Evaluación del comportamiento funcional y de la seguridad estructural de puentes existentes de hormigón armado y pretensado. PhD Thesis, Technical University of Catalonia, Department of Construction Engineering.

Sousa, H, Matos, JC, Félix, C, Figueiras, J (2004a) A monitorização estrutural como parte integrante de obras de arte de engenharia civil – Aplicação à ponte sobre o rio Sorraia. 46º Congresso Brasileiro do Concreto, 14-18 Agosto 2004, Florianópolis – Brasil.

Sousa, H, Matos, JC, Silva, H, Esteves, JL, Vieira, P, Marques, AT, Figueiras, JA (2004b) Desenvolvimento e caracterização de novas cabeças sensoras para embeber no betão. Encontro Nacional do Betão Estrutural 2004, 17-19 Novembro 2004, Porto – Portugal.

Sousa, H, Matos, JC, Assis, W, Felix, C, Figueiras, JA 2005. Load Test of Sorraia River Bridge in Highway A13 (Almeirim-Marateca). Technical Report. June 2005.

Sousa, H, Matos, JC, Figueiras, JA (2006) Development of an Embedded Sensor Holder for Concrete Structures Monitoring. The Second fib Congress, Naples – Italy, 5 – 8 June 2006.

Spanos, PD, Ghanem, R (1989) Stochastic finite element expansion for random media. Journal of Engineering Mechanics ASCE, 115 (5): 1035-1053.

Sunaga, T (1958) Theory of Interval Algebra and Its Applications to Numerical Analysis, RAAG Memoirs 2, pp. 547–564.

T

Thomos, GC, Trezos, CG (2001) Examination of the probabilistic response of reinforced concrete structures under static non-linear analysis. Engineering Structures 28 (1): 120-133.

Tp

Veiga, JM, Henriques, AA, Delgado, JM (2006 a) An efficient evaluation of structural safety applying perturbation techniques. III European Conference on Computational Mechanics, Solids, Structures and Coupled Problems in Engineering, Lisbon, Portugal, June 5 – 9, pp. 126 (CD-ROM Edition: pp. 9).

Veiga, JM, Henriques, AA, Delgado, JM (2006 b) Aplicação de técnicas de perturbação na análise da incerteza estrutural. 4^{as} Jornadas Portuguesas de Engenharia de Estruturas, Lisboa, Portugal, Dezembro 13 – 16, pp. 92-93 (CD-ROM Edition: pp. 13).

Wp

Wang, CS, Wu, F, Chang, FK (2001) Structural health monitoring from fibre-reinforced composites to steel-reinforced concrete. Smart Materials and Structures, Vol. 10: pp. 548-552.

Wisniewski, DF (2007) Safety Formats for the assessment of Concrete Bridges with special focus on precast concrete. Doctoral Thesis. Department of Civil Engineering. School of Engineering – University of Minho. Guimarães, Portugal.

Z

Zadeh, LA (1965) Fuzzy Sets. Information and Control, Vol. 8: 338-353.

Zhang, J, Ellingwood, B (1996) SFEM for reliability with material nonlinearities. Journal Structural Engineering ASCE 122 (6): 701-704.

Zhao, L, Shenton, HW (2002) Direct identification of damage in beam structures using dead load measurements. 15th ASCE Engineering mechanic conference, New York, USA, June 2-5.

Zimmermann, H-J (2000) An application-oriented view of modelling uncertainty. European Journal of Operational Research, Vol. 122: 190-198.

Zou, T, Mahadevan, S (2006) Versatile formulation for multiobjective reliability-based design optimization. Journal of Mechanic Design, Vol. 128(6): 1217-1226.

APPENDIX A

BINOMIAL DISTRIBUTION

BINOMIAL DISTRIBUTION

The binomial discrete distribution presents a cumulative distribution function (CDF), $F(x)$, characterized by the following expression:

$$F(x) = F(x; n; p) = P[X \leq x] = \begin{cases} 0 & x < 0 \\ \sum_{x_i=0}^x \binom{n}{x_i} \cdot p^{x_i} \cdot (1-p)^{n-x_i} & 0 \leq x < n \\ 1 & x \geq n \end{cases} \quad (\text{A.1})$$

being the parameters n and p enough to specify any binomial distribution and so to calculate the correspondent probability of X being lesser or equal to x . The use of this formula gives rise to laborious and monotonous calculus. The Tables presented in this Appendix, allow realizing the probability calculus in a fast and much easier way. Such tables are defined by the parameters n and p , which characterize any distribution type, and for all possible values of x . In this case, x can assume any value between 0 to n , in intervals of 0.05, being the limit value of n equal to 20.

TABLE

n	x	p									
		0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
1	0	0,9500	0,9000	0,8500	0,8000	0,7500	0,7000	0,6500	0,6000	0,5500	0,5000
	1	0,0500	0,1000	0,1500	0,2000	0,2500	0,3000	0,3500	0,4000	0,4500	0,5000
2	0	0,9025	0,8100	0,7225	0,6400	0,5625	0,4900	0,4225	0,3600	0,3025	0,2500
	1	0,0950	0,1800	0,2550	0,3200	0,3750	0,4200	0,4550	0,4800	0,4950	0,5000
	2	0,0025	0,0100	0,0225	0,0400	0,0625	0,0900	0,1225	0,1600	0,2025	0,2500
3	0	0,8574	0,7290	0,6141	0,5120	0,4219	0,3430	0,2746	0,2160	0,1664	0,1250
	1	0,1354	0,2430	0,3251	0,3840	0,4219	0,4410	0,4436	0,4320	0,4084	0,3750
	2	0,0071	0,0270	0,0574	0,0960	0,1406	0,1890	0,2389	0,2880	0,3341	0,3750
	3	0,0001	0,0010	0,0034	0,0080	0,0156	0,0270	0,0429	0,0640	0,0911	0,1250
4	0	0,8145	0,6561	0,5220	0,4096	0,3164	0,2401	0,1785	0,1296	0,0915	0,0625
	1	0,1715	0,2916	0,3685	0,4096	0,4219	0,4116	0,3845	0,3456	0,2995	0,2500
	2	0,0135	0,0480	0,0975	0,1536	0,2109	0,2646	0,3105	0,3456	0,3675	0,3750
	3	0,0005	0,0036	0,0115	0,0256	0,0469	0,0756	0,1115	0,1536	0,2005	0,2500
	4	0,0000	0,0001	0,0005	0,0016	0,0039	0,0081	0,0150	0,0256	0,0410	0,0625
5	0	0,7738	0,5905	0,4437	0,3277	0,2373	0,1681	0,1160	0,0778	0,0503	0,0312
	1	0,2036	0,3281	0,3915	0,4096	0,3955	0,3601	0,3124	0,2592	0,2059	0,1563
	2	0,0214	0,0729	0,1382	0,2048	0,2637	0,3087	0,3364	0,3456	0,3369	0,3125
	3	0,0011	0,0081	0,0244	0,0512	0,0879	0,1323	0,1811	0,2304	0,2757	0,3125
	4	0,0000	0,0005	0,0022	0,0064	0,0146	0,0283	0,0488	0,0768	0,1128	0,1563
	5	0,0000	0,0000	0,0001	0,0003	0,0010	0,0024	0,0053	0,0102	0,0185	0,0312
6	0	0,7351	0,5314	0,3771	0,2621	0,1780	0,1176	0,0754	0,0467	0,0277	0,0156
	1	0,2321	0,3543	0,3993	0,3932	0,3560	0,3025	0,2437	0,1866	0,1359	0,0938
	2	0,0305	0,0984	0,1762	0,2458	0,2966	0,3241	0,3280	0,3110	0,2780	0,2344
	3	0,0021	0,0146	0,0415	0,0819	0,1318	0,1852	0,2355	0,2765	0,3032	0,3125
	4	0,0001	0,0012	0,0055	0,0154	0,0330	0,0595	0,0951	0,1382	0,1861	0,2344
	5	0,0000	0,0001	0,0004	0,0015	0,0044	0,0102	0,0205	0,0369	0,0609	0,0938
	6	0,0000	0,0000	0,0000	0,0001	0,0002	0,0007	0,0018	0,0041	0,0083	0,0156
7	0	0,6983	0,4783	0,3206	0,2097	0,1335	0,0824	0,0490	0,0280	0,0152	0,0078
	1	0,2573	0,3720	0,3960	0,3670	0,3115	0,2471	0,1848	0,1306	0,0872	0,0547
	2	0,0406	0,1240	0,2097	0,2753	0,3115	0,3177	0,2985	0,2613	0,2140	0,1641
	3	0,0036	0,0230	0,0617	0,1147	0,1730	0,2269	0,2679	0,2903	0,2918	0,2734
	4	0,0002	0,0026	0,0109	0,0287	0,0577	0,0972	0,1442	0,1935	0,2388	0,2734
	5	0,0000	0,0002	0,0012	0,0043	0,0115	0,0250	0,0466	0,0774	0,1172	0,1641
	6	0,0000	0,0000	0,0001	0,0004	0,0013	0,0036	0,0084	0,0172	0,0320	0,0547
	7	0,0000	0,0000	0,0000	0,0000	0,0001	0,0002	0,0006	0,0016	0,0037	0,0078
8	0	0,6634	0,4305	0,2725	0,1678	0,1001	0,0576	0,0319	0,0168	0,0084	0,0039
	1	0,2793	0,3826	0,3847	0,3355	0,2570	0,1977	0,1373	0,0896	0,0548	0,0313
	2	0,0515	0,1488	0,2376	0,2936	0,3115	0,2965	0,2587	0,2090	0,1569	0,1094
	3	0,0054	0,0331	0,0839	0,1468	0,2076	0,2541	0,2786	0,2787	0,2568	0,2188
	4	0,0004	0,0046	0,0185	0,0459	0,0865	0,1361	0,1875	0,2322	0,2627	0,2734
	5	0,0000	0,0004	0,0026	0,0092	0,0231	0,0467	0,0808	0,1239	0,1719	0,2188
	6	0,0000	0,0000	0,0002	0,0011	0,0038	0,0100	0,0217	0,0413	0,0703	0,1094
	7	0,0000	0,0000	0,0000	0,0001	0,0004	0,0012	0,0033	0,0079	0,0164	0,0313
	8	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0002	0,0007	0,0017	0,0039

n	x	p									
		0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
9	0	0,6302	0,3874	0,2316	0,1342	0,0751	0,0404	0,0207	0,0101	0,0046	0,0020
	1	0,2985	0,3874	0,3679	0,3020	0,2253	0,1556	0,1004	0,0605	0,0339	0,0176
	2	0,0629	0,1722	0,2597	0,3020	0,3003	0,2668	0,2162	0,1612	0,1110	0,0703
	3	0,0077	0,0446	0,1069	0,1762	0,2336	0,2668	0,2716	0,2508	0,2119	0,1641
	4	0,0006	0,0074	0,0283	0,0661	0,1168	0,1715	0,2194	0,2508	0,2600	0,2461
	5	0,0000	0,0008	0,0050	0,0165	0,0389	0,0735	0,1181	0,1672	0,2128	0,2461
	6	0,0000	0,0001	0,0006	0,0028	0,0087	0,0210	0,0424	0,0743	0,1160	0,1641
	7	0,0000	0,0000	0,0000	0,0003	0,0012	0,0039	0,0098	0,0212	0,0407	0,0703
	8	0,0000	0,0000	0,0000	0,0000	0,0001	0,0004	0,0013	0,0035	0,0083	0,0176
	9	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0008	0,0020
10	0	0,5987	0,3487	0,1969	0,1074	0,0563	0,0282	0,0135	0,0060	0,0025	0,0010
	1	0,3151	0,3874	0,3474	0,2684	0,1877	0,1211	0,0725	0,0403	0,0207	0,0098
	2	0,0746	0,1937	0,2759	0,3020	0,2816	0,2335	0,1757	0,1209	0,0763	0,0439
	3	0,0105	0,0574	0,1298	0,2013	0,2503	0,2668	0,2522	0,2150	0,1665	0,1172
	4	0,0010	0,0112	0,0401	0,0881	0,1460	0,2001	0,2377	0,2508	0,2384	0,2051
	5	0,0001	0,0015	0,0085	0,0264	0,0584	0,1029	0,1536	0,2007	0,2340	0,2461
	6	0,0000	0,0001	0,0012	0,0055	0,0162	0,0368	0,0689	0,1115	0,1596	0,2051
	7	0,0000	0,0000	0,0001	0,0008	0,0031	0,0090	0,0212	0,0425	0,0746	0,1172
	8	0,0000	0,0000	0,0000	0,0001	0,0004	0,0014	0,0043	0,0106	0,0229	0,0439
	9	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0016	0,0042	0,0098
	10	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0010
11	0	0,5688	0,3138	0,1673	0,0859	0,0422	0,0198	0,0088	0,0036	0,0014	0,0005
	1	0,3293	0,3835	0,3248	0,2362	0,1549	0,0932	0,0518	0,0286	0,0125	0,0054
	2	0,0867	0,2131	0,2866	0,2953	0,2581	0,1998	0,1395	0,0887	0,0513	0,0269
	3	0,0137	0,0710	0,1517	0,2215	0,2581	0,2568	0,2254	0,1774	0,1259	0,0806
	4	0,0014	0,0158	0,0536	0,1107	0,1721	0,2201	0,2428	0,2365	0,2060	0,1611
	5	0,0001	0,0025	0,0132	0,0388	0,0803	0,1321	0,1830	0,2207	0,2360	0,2256
	6	0,0000	0,0003	0,0023	0,0097	0,0268	0,0566	0,0985	0,1471	0,1931	0,2256
	7	0,0000	0,0000	0,0003	0,0017	0,0064	0,0173	0,0379	0,0701	0,1128	0,1611
	8	0,0000	0,0000	0,0000	0,0002	0,0011	0,0037	0,0102	0,0234	0,0462	0,0806
	9	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0018	0,0052	0,0126	0,0269
	10	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0007	0,0021	0,0054
	11	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0005
12	0	0,5404	0,2824	0,1422	0,0687	0,0317	0,0138	0,0057	0,0022	0,0008	0,0002
	1	0,3413	0,3766	0,3012	0,2062	0,1267	0,0712	0,0368	0,0174	0,0075	0,0029
	2	0,0988	0,2301	0,2924	0,2835	0,2323	0,1678	0,1088	0,0639	0,0339	0,0161
	3	0,0173	0,0852	0,1720	0,2362	0,2581	0,2397	0,1954	0,1419	0,0923	0,0537
	4	0,0021	0,0213	0,0683	0,1329	0,1936	0,2311	0,2367	0,2128	0,1700	0,1208
	5	0,0002	0,0038	0,0193	0,0532	0,1032	0,1585	0,2039	0,2270	0,2225	0,1934
	6	0,0000	0,0005	0,0040	0,0155	0,0401	0,0792	0,1281	0,1766	0,2124	0,2256
	7	0,0000	0,0000	0,0006	0,0033	0,0115	0,0291	0,0591	0,1009	0,1489	0,1934
	8	0,0000	0,0000	0,0001	0,0005	0,0024	0,0078	0,0199	0,0420	0,0762	0,1208
	9	0,0000	0,0000	0,0000	0,0001	0,0004	0,0015	0,0048	0,0125	0,0277	0,0537
	10	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0008	0,0025	0,0068	0,0161
	11	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0010	0,0029
	12	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0002
13	0	0,5133	0,2542	0,1209	0,0550	0,0238	0,0097	0,0037	0,0013	0,0004	0,0001
	1	0,3512	0,3672	0,2774	0,1787	0,1029	0,0540	0,0259	0,0113	0,0045	0,0016
	2	0,1109	0,2448	0,2937	0,2680	0,2059	0,1388	0,0836	0,0453	0,0220	0,0095
	3	0,0214	0,0987	0,1900	0,2457	0,2517	0,2181	0,1651	0,1107	0,0660	0,0349
	4	0,0028	0,0277	0,0838	0,1535	0,2097	0,2337	0,2222	0,1845	0,1350	0,0873

		p									
n	x	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
14	5	0,0003	0,0055	0,0266	0,0691	0,1258	0,1803	0,2154	0,2214	0,1989	0,1571
	6	0,0000	0,0008	0,0063	0,0230	0,0559	0,1030	0,1546	0,1968	0,2169	0,2095
	7	0,0000	0,0001	0,0011	0,0058	0,0186	0,0442	0,0833	0,1312	0,1775	0,2095
	8	0,0000	0,0000	0,0001	0,0011	0,0047	0,0142	0,0336	0,0656	0,1089	0,1571
	9	0,0000	0,0000	0,0000	0,0001	0,0009	0,0034	0,0101	0,0243	0,0495	0,0873
	10	0,0000	0,0000	0,0000	0,0000	0,0001	0,0006	0,0022	0,0065	0,0162	0,0349
	11	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0012	0,0036	0,0095
	12	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0016
	13	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001
	0	0,4877	0,2288	0,1028	0,0440	0,0178	0,0068	0,0024	0,0008	0,0002	0,0001
	1	0,3593	0,3559	0,2539	0,1539	0,0832	0,0407	0,0181	0,0073	0,0027	0,0009
	2	0,1229	0,2570	0,2912	0,2501	0,1802	0,1134	0,0634	0,0317	0,0141	0,0058
	3	0,0259	0,1142	0,2056	0,2501	0,2402	0,1943	0,1366	0,0845	0,0462	0,0222
	4	0,0037	0,0349	0,0998	0,1720	0,2202	0,2290	0,2022	0,1549	0,1040	0,0611
15	5	0,0004	0,0078	0,0352	0,0860	0,1468	0,1963	0,2178	0,2066	0,1701	0,1222
	6	0,0000	0,0013	0,0093	0,0322	0,0734	0,1262	0,1759	0,2066	0,2088	0,1833
	7	0,0000	0,0002	0,0019	0,0092	0,0280	0,0618	0,1082	0,1574	0,1952	0,2095
	8	0,0000	0,0000	0,0003	0,0020	0,0082	0,0232	0,0510	0,0918	0,1398	0,1833
	9	0,0000	0,0000	0,0000	0,0003	0,0018	0,0066	0,0183	0,0408	0,0762	0,1222
	10	0,0000	0,0000	0,0000	0,0000	0,0003	0,0014	0,0049	0,0136	0,0312	0,0611
	11	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0010	0,0033	0,0093	0,0222
	12	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0019	0,0058
	13	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0002	0,0009
	14	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001
	0	0,4633	0,2059	0,0874	0,0352	0,0134	0,0047	0,0016	0,0005	0,0001	0,0000
	1	0,3658	0,3432	0,2312	0,1319	0,0668	0,0305	0,0126	0,0047	0,0016	0,0005
	2	0,1348	0,2669	0,2856	0,2309	0,1559	0,0916	0,0476	0,0219	0,0090	0,0032
	3	0,0307	0,1285	0,2184	0,2501	0,2252	0,1700	0,1110	0,0634	0,0318	0,0139
16	4	0,0049	0,0428	0,1156	0,1876	0,2252	0,2186	0,1792	0,1268	0,0780	0,0417
	5	0,0006	0,0105	0,0449	0,1032	0,1651	0,2061	0,2123	0,1859	0,1404	0,0916
	6	0,0000	0,0019	0,0132	0,0430	0,0917	0,1472	0,1906	0,2066	0,1914	0,1527
	7	0,0000	0,0003	0,0030	0,0138	0,0393	0,0811	0,1319	0,1771	0,2013	0,1964
	8	0,0000	0,0000	0,0005	0,0035	0,0131	0,0348	0,0710	0,1181	0,1647	0,1964
	9	0,0000	0,0000	0,0001	0,0007	0,0034	0,0116	0,0298	0,0612	0,1048	0,1527
	10	0,0000	0,0000	0,0000	0,0001	0,0007	0,0030	0,0096	0,0245	0,0515	0,0916
	11	0,0000	0,0000	0,0000	0,0000	0,0001	0,0006	0,0024	0,0074	0,0191	0,0417
	12	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0004	0,0016	0,0052	0,0139
	13	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0010	0,0032
	14	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005
	15	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	0	0,4401	0,1853	0,0743	0,0281	0,0100	0,0033	0,0010	0,0003	0,0001	0,0000
	1	0,3706	0,3294	0,2097	0,1126	0,0535	0,0228	0,0087	0,0030	0,0009	0,0002
	2	0,1463	0,2745	0,2775	0,2111	0,1336	0,0732	0,0353	0,0150	0,0056	0,0018
	3	0,0359	0,1423	0,2285	0,2463	0,2079	0,1465	0,0888	0,0468	0,0215	0,0085
4	0,0061	0,0514	0,1311	0,2001	0,2252	0,2040	0,1553	0,1014	0,0572	0,0278	
5	0,0008	0,0137	0,0555	0,1201	0,1802	0,2099	0,2008	0,1623	0,1123	0,0667	
6	0,0001	0,0028	0,0180	0,0550	0,1101	0,1649	0,1982	0,1983	0,1684	0,1222	
7	0,0000	0,0004	0,0045	0,0197	0,0524	0,1010	0,1524	0,1889	0,1969	0,1746	
8	0,0000	0,0001	0,0009	0,0055	0,0197	0,0487	0,0923	0,1417	0,1812	0,1964	
9	0,0000	0,0000	0,0001	0,0012	0,0058	0,0185	0,0442	0,0840	0,1318	0,1746	

n	x	p									
		0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
10	0	0,0000	0,0000	0,0000	0,0002	0,0014	0,0056	0,0167	0,0392	0,0755	0,1222
	1	0,0000	0,0000	0,0000	0,0000	0,0002	0,0013	0,0049	0,0142	0,0337	0,0667
	2	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0011	0,0040	0,0115	0,0278
	3	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0008	0,0029	0,0085
	4	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0018
	5	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0002
	6	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	7	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	8	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	9	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
17	0	0,4181	0,1668	0,0631	0,0225	0,0075	0,0023	0,0007	0,0002	0,0000	0,0000
	1	0,3741	0,3150	0,1893	0,0957	0,0426	0,0169	0,0060	0,0019	0,0005	0,0001
	2	0,1575	0,2800	0,2673	0,1914	0,1136	0,0581	0,0260	0,0102	0,0035	0,0010
	3	0,0415	0,1556	0,2359	0,2393	0,1893	0,1245	0,0701	0,0341	0,0144	0,0052
	4	0,0076	0,0605	0,1457	0,2093	0,2209	0,1868	0,1320	0,0796	0,0411	0,0182
	5	0,0010	0,0175	0,0668	0,1361	0,1914	0,2081	0,1849	0,1379	0,0875	0,0472
	6	0,0001	0,0039	0,0236	0,0680	0,1276	0,1784	0,1991	0,1839	0,1432	0,0944
	7	0,0000	0,0007	0,0065	0,0267	0,0668	0,1201	0,1685	0,1927	0,1841	0,1484
	8	0,0000	0,0001	0,0014	0,0084	0,0279	0,0644	0,1134	0,1606	0,1883	0,1855
	9	0,0000	0,0000	0,0003	0,0021	0,0093	0,0276	0,0611	0,1070	0,1540	0,1855
18	0	0,0000	0,0000	0,0000	0,0004	0,0025	0,0095	0,0263	0,0571	0,1008	0,1484
	1	0,0000	0,0000	0,0000	0,0001	0,0005	0,0026	0,0090	0,0242	0,0525	0,0944
	2	0,0000	0,0000	0,0000	0,0000	0,0001	0,0006	0,0024	0,0081	0,0215	0,0472
	3	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0005	0,0021	0,0068	0,0182
	4	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0004	0,0016	0,0052
	5	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0003	0,0010
	6	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001
	7	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	8	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
	9	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
18	0	0,3972	0,1501	0,0536	0,0180	0,0056	0,0016	0,0004	0,0001	0,0000	0,0000
	1	0,3763	0,3002	0,1704	0,0811	0,0338	0,0126	0,0042	0,0012	0,0003	0,0001
	2	0,1683	0,2835	0,2556	0,1723	0,0958	0,0458	0,0190	0,0069	0,0022	0,0006
	3	0,0473	0,1680	0,2406	0,2297	0,1704	0,1046	0,0547	0,0246	0,0095	0,0031
	4	0,0093	0,0700	0,1592	0,2153	0,2130	0,1681	0,1104	0,0614	0,0291	0,0117
	5	0,0014	0,0218	0,0787	0,1507	0,1988	0,2017	0,1664	0,1146	0,0666	0,0327
	6	0,0002	0,0052	0,0301	0,0816	0,1436	0,1873	0,1941	0,1655	0,1181	0,0708
	7	0,0000	0,0010	0,0091	0,0350	0,0820	0,1376	0,1792	0,1892	0,1657	0,1214
	8	0,0000	0,0002	0,0022	0,0120	0,0376	0,0811	0,1327	0,1734	0,1864	0,1659
	9	0,0000	0,0000	0,0004	0,0033	0,0139	0,0386	0,0794	0,1284	0,1694	0,1855
18	10	0,0000	0,0000	0,0001	0,0008	0,0042	0,0149	0,0385	0,0771	0,1248	0,1669
	11	0,0000	0,0000	0,0000	0,0001	0,0010	0,0046	0,0151	0,0374	0,0742	0,1214
	12	0,0000	0,0000	0,0000	0,0000	0,0002	0,0012	0,0047	0,0145	0,0354	0,0708
	13	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0012	0,0045	0,0134	0,0327
	14	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0011	0,0039	0,0117
	15	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0002	0,0009	0,0031
	16	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001	0,0006
	17	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0001
	18	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000	0,0000
19	0	0,3774	0,1351	0,0456	0,0144	0,0042	0,0011	0,0003	0,0001	0,0000	0,0000
	1	0,3774	0,2852	0,1529	0,0685	0,0268	0,0093	0,0029	0,0008	0,0002	0,0000
	2	0,1787	0,2852	0,2428	0,1540	0,0803	0,0358	0,0138	0,0046	0,0013	0,0003
	3	0,0533	0,1796	0,2428	0,2182	0,1517	0,0869	0,0422	0,0175	0,0062	0,0018
	4	0,0112	0,0798	0,1714	0,2182	0,2023	0,1491	0,0909	0,0467	0,0203	0,0074

APPENDIX B

STANDARDIZED NORMAL DISTRIBUTION

STANDARDIZED NORMAL DISTRIBUTION

The cumulative distribution function (CDF) of a standardized normal shape continuous distribution is defined by the expression:

$$F(z) = P[Z \leq z] = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{t^2}{2}} dt \quad (\text{B.1})$$

being z the random variable in analysis. The analytical calculation of each cumulative probability is laborious and monotonous. To face this problem it is often used Tables which gives the cumulative probability for any value of z between 0.00 and 3.49 in fractions of 0.01.

TABLE

z	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998

z	1,282	1,645	1,960	2,326	2,576	3,090	3,291	3,891	4,417
$F(z)$	0,90	0,95	0,975	0,99	0,995	0,999	0,9995	0,99995	0,999995
$2[1 - F(z)]$	0,20	0,10	0,05	0,02	0,01	0,002	0,001	0,0001	0,00001

APPENDIX C
KAUCHER'S ARITHMETIC

KAUCHER'S ARITHMETIC

The basis of Modal Interval Analysis (MIA) arithmetic is Kaucher's arithmetic. This arithmetic and the generalized interval concept appeared in order to solve some algebraic deficiencies of Classic Interval Analysis. It is now possible to execute operations that maintain the isotonic property, to complete the interval algebraic structure with the sum and product operations allowing solving equations and to solve the problem of inner rounding. It is defined the set of generalized intervals as:

$$R = \{[a, b] \mid a, b \in R\} \quad (C.1)$$

The representation of this set is realized by all points of Moore plan. In the first quarter diagonal it is represented the intervals with both extremes with equal values, punctual intervals. In superior semi plan it is represented the intervals with the inferior extreme lesser or equal to the superior one, proper intervals, and in inferior semi plan, the intervals with inferior extreme greater or equal to the superior one, improper intervals. Dual operator establishes a relationship between both kind of intervals and it is defined as:

$$Dual[a, b] := [b, a] \quad (C.2)$$

The equality and inclusion relations are a generalisation of the interval classic correspondent relations and so $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are defined as:

$$\begin{aligned} A = B &\leftrightarrow a_1 = b_1, a_2 = b_2 \\ A \subseteq B &\leftrightarrow a_1 \geq b_1, a_2 \leq b_2 \end{aligned} \quad (C.3)$$

In this Appendix it is presented the arithmetic operations between generalized intervals and their respective properties. These are the basis for Modal Interval Analysis (MIA) calculations. It is important to note that generalized intervals do not solve the semantic deficiencies of classic intervals due to it lack of significance. To solve this problem it was developed and implemented the Modal Interval Analysis (MIA).

ARITHMETIC OPERATIONS

Sum

$$A + B := [a_1 + b_1, a_2 + b_2] \quad (C.4)$$

Subtraction

$$A - B := [a_1 - b_2, a_2 - b_1] \quad (C.5)$$

Product

$$A * B := \begin{cases} \text{if } a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 \geq 0 \text{ then } [a_1 b_1, a_2 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 \geq 0, b_2 < 0 \text{ then } [a_1 b_1, a_1 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 \geq 0 \text{ then } [a_2 b_1, a_2 b_2] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 < 0 \text{ then } [a_2 b_1, a_1 b_2] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 \geq 0, b_2 \geq 0 \text{ then } [a_1 b_1, a_2 b_1] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 \geq 0, b_2 < 0 \\ \quad \text{then } [\max(a_1 b_1, a_2 b_2), \min(a_2 b_1, a_1 b_2)] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 \geq 0 \text{ then } [0, 0] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 < 0 \text{ then } [a_2 b_2, a_1 b_2] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 \geq 0, b_2 \geq 0 \text{ then } [a_1 b_2, a_2 b_2] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 \geq 0, b_2 < 0 \text{ then } [0, 0], \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 \geq 0 \\ \quad \text{then } [\min(a_1 b_2, a_2 b_1), \max(a_1 b_1, a_2 b_2)] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 < 0 \text{ then } [a_2 b_1, a_1 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 \geq 0, b_2 \geq 0 \text{ then } [a_1 b_2, a_2 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 \geq 0, b_2 < 0 \text{ then } [a_2 b_2, a_2 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 < 0, b_2 \geq 0 \text{ then } [a_1 b_2, a_1 b_1] \\ \text{if } a_1 < 0, a_2 < 0, b_1 < 0, b_2 < 0 \text{ then } [a_2 b_2, a_1 b_1] \end{cases} \quad (C.6)$$

Product by a scalar

$$r \cdot A := [r, r][a_1, a_2] = \text{if } r \geq 0 \text{ then } [ra_1, ra_2] \text{ else } [ra_2, ra_1] \quad (C.7)$$

Division

$$A/B = \begin{cases} \text{if } a_1 \geq 0, a_2 \geq 0, b_1 > 0, b_2 > 0 & \text{then } [a_1/b_2, a_2/b_1] \\ \text{if } a_1 \geq 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2/b_2, a_1/b_1] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 > 0, b_2 > 0 & \text{then } [a_1/b_2, a_2/b_2] \\ \text{if } a_1 \geq 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2/b_1, a_1/b_1] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 > 0, b_2 > 0 & \text{then } [a_1/b_1, a_2/b_1] \\ \text{if } a_1 < 0, a_2 \geq 0, b_1 < 0, b_2 < 0 & \text{then } [a_2/b_2, a_1/b_2] \\ \text{if } a_1 < 0, a_2 < 0, b_1 > 0, b_2 > 0 & \text{then } [a_1/b_1, a_2/b_2] \\ \text{if } a_1 < 0, a_2 < 0, b_1 < 0, b_2 < 0 & \text{then } [a_2/b_1, a_1/b_2] \end{cases} \quad (\text{C.8})$$

PROPERTIES

Sum

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + \vee(i, I)B(i) = \vee(i, I)(A + B(i))$
 $A + \wedge(i, I)B(i) = \wedge(i, I)(A + B(i))$
4. $A \subseteq B, C \subseteq D$ imply $A + C \subseteq B + D$
5. $A \leq B, C \leq D$ imply $A + C \leq B + D$
6. $\text{Dual}(A + B) = \text{Dual}(A) + \text{Dual}(B)$
7. $A + [0, 0] = A$
8. *exists the opposite of* $A = [a_1, a_2]$ *and* $\text{Opp}(A) = [-a_1, -a_2]$

(C.9)

Subtraction

1. $A \subseteq B, C \subseteq D$ imply $A - C \subseteq B - D$
2. $A - \text{Dual}(B) = A + \text{Opp}(B)$
3. *The equation* $A + X = B$ *has a unique solution*
 $X = B - \text{Dual}(A) = B + \text{Opp}(A)$

(C.10)

Product

1. $A \cdot B = B \cdot A$
2. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
3. $r(A \cdot B) = (rA) \cdot B = A \cdot (rB)$
4. $(-A) \cdot (-B) = A \cdot B$
5. $\text{Opp}(A) \cdot \text{Opp}(B) = \text{Dual}(A \cdot B)$
6. $\text{Dual}(A) \cdot \text{Dual}(B) = \text{Dual}(A \cdot B)$
7. $rA = [r, r] \cdot A$
8. $rs \geq 0$ *imply* $\begin{cases} rA \vee sA = [\min\{r, s\}, \max\{r, s\}] \cdot A \\ rA \wedge sA = [\max\{r, s\}, \min\{r, s\}] \cdot A \end{cases}$
9. $[1, 1] \cdot A = A$
10. $(A \subseteq B, C \subseteq D)$ *imply* $A \cdot C \subseteq B \cdot D$
11. $([0, 0] \leq A \leq B, [0, 0] \leq C \leq D)$ *imply* $A \cdot C \leq B \cdot D$
12. *A proper imply* $A \cdot (B + C) \subseteq A \cdot B + A \cdot C$
A improper imply $A \cdot (B + C) \supseteq A \cdot B + A \cdot C$
13. *There exist distributivity conditions, Moore plan zones, where*
 $A \cdot (B_1 + \dots + B_n) = A \cdot B_1 + \dots + A \cdot B_n$
14. *If* $0 \notin A$ *exist it inverse and is* $\frac{1}{A} = \left[\frac{1}{a_2}, \frac{1}{a_1} \right]$
15. *If there exist inverses, so*

(C.11)

$$A \subseteq B \text{ imply } \frac{1}{A} \subseteq \frac{1}{B}$$

$$A \leq B \text{ imply } \frac{1}{A} \geq \frac{1}{B}$$

$$\text{Dual}\left(\frac{1}{A}\right) = \frac{1}{\text{Dual}(A)}$$

Product by a scalar

1. If $\phi \in \{Dual, -, Opp, Prop\}$, then $\phi(rA) = r\phi(A)$
2. $r \vee (i, I)A(i) = \vee (i, I)rA(i)$ and $r \wedge (i, I)A(i) = \wedge (i, I)rA(i)$
3. $A \subseteq B$ imply $rA \subseteq rB$
4. $A \leq B$ imply $\begin{cases} rA \leq rB & \text{if } r \geq 0 \\ rA \geq rB & \text{if } r < 0 \end{cases}$ (C.12)
5. $r(A + B) = rA + rB$
6. $r(sA) = (rs)A$
7. If $rs \geq 0$ so $(r + s)A = rA + sA$

Division

1. $\frac{A}{B} = A \cdot \frac{1}{B}$
2. $(A \subseteq B, C \subseteq D)$ imply $A/C \subseteq B/D$
3. $Dual\left(\frac{A}{B}\right) = \frac{Dual(A)}{Dual(B)}$ (C.13)
4. The equation $A \cdot X = B$ has a unique solution if $0 \notin Prop(A)$ and is

$$X = \frac{B}{Dual(A)}$$

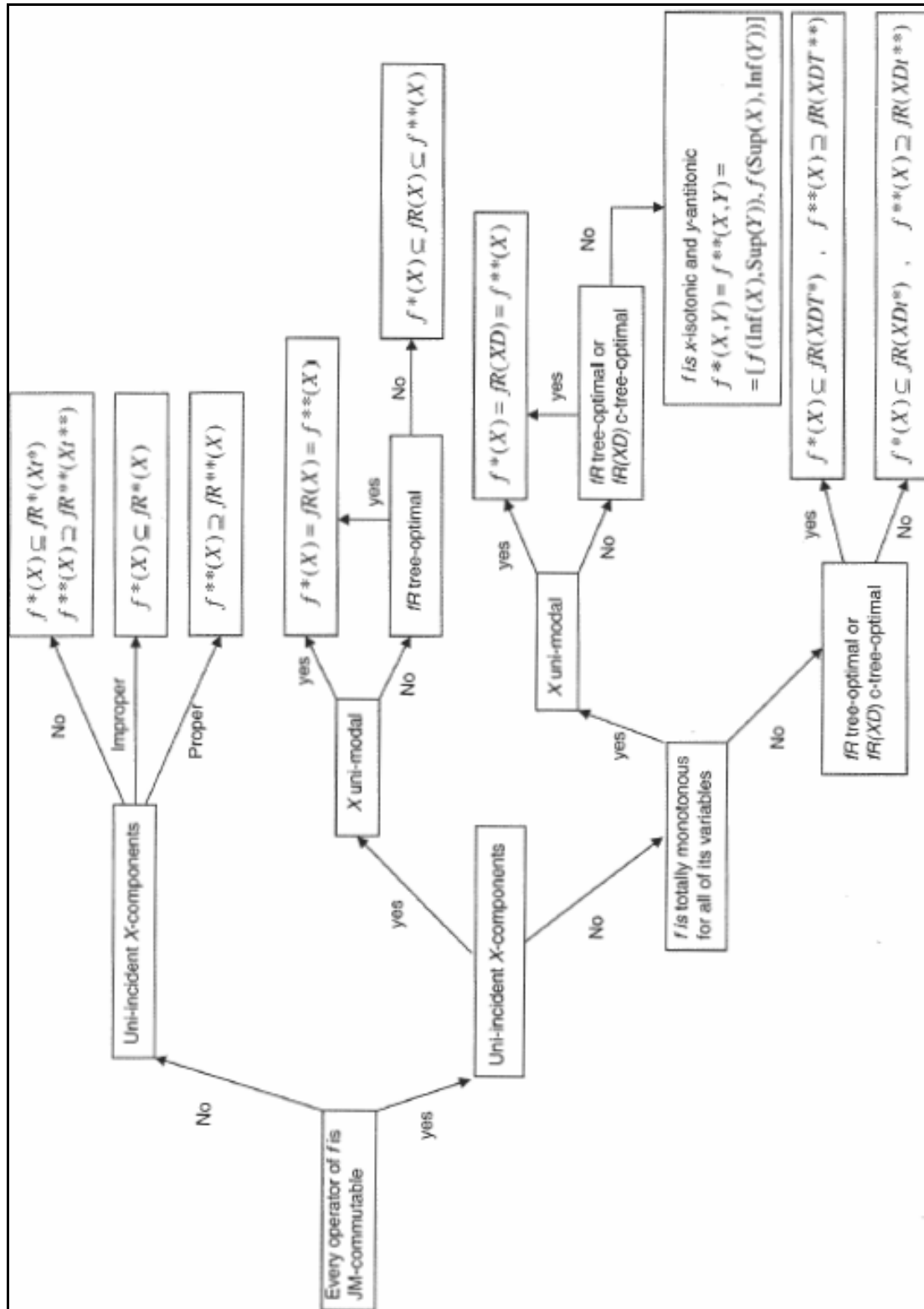
APPENDIX D

RATIONAL COMPUTATION

RATIONAL COMPUTATION

In this Appendix it is presented a Tree Diagram of Rational Computations, very useful to perform an uncertainty analysis, by using Modal Interval. At first is made a division whether the operator is JM-commutable or not. If not, a second division is made for uni-incident (proper or improper) or not X-component. If yes, then, it is realized a second division whether it is uni-incident or multi-incident X-component. If uni-incident, a third division is made whether X it is uni-modal or not. If multi-incident, a division is realized next, whether f is totally monotonous for all variables or not. At this point, and if yes, a fourth division is realized whether X is uni-modal or not. If not, a division is made whether fR is tree-optimal or fR (XD) c-tree-optimal, or not both of them. For each situation it is possible to apply a specific theorem, from the ones presented before, and to make a comparison between the calculated fR with f^* and f^{**} . This Tree Diagram is very important as all necessary and important semantic theorems are presented here.

TREE DIAGRAM



APPENDIX E
PERTURBATION TECHNIQUE PROCEDURES

PERTURBATION TECHNIQUE PROCEDURES

In this Appendix it is represented a detailed description of all steps for numerical analysis, using Perturbation Technique, of concrete beam tested in laboratory for a load equal to 19.151 KN. At first it is made a description of Stiffness Matrix $[K]$ and Load Vector $\{F\}$ and of deterministic analysis obtained results, then it is realized the calculation of the structural partial derivatives in order to the random variables. Consequently it is calculated the product of matrixes in order to obtain the contribution of each random variable in the whole system response. Finally it is determined the covariance matrix which allow to obtain the standard deviation of each unknown displacement (the average value it is determined by the previously deterministic analysis). The interval of variation is then defined by the mean value affected (sum or subtraction) by the standard deviation.

Stiffness Matrix [K] and Load Vector {F}

1. Input Data

Data:

	mean value	standard deviation
E =	32600000	1940000
b =	0.15	
h =	0.15	
L =	0.35	
F =	19.151	0.95755
K ₀ =	6000	600
K ₁₄ =	6000	600

I = 4.21875E-05 m⁴

2. Stiffness Matrix [K]

Stiffness Matrix [K]:														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	67362.2449	-384927.11	67362.2449	0	0	0	0	0	0	0	0	0	0
2	0	21717.85714	-67362.245	7858.928571	0	0	0	0	0	0	0	0	0	0
3	0	-67362.2449	769854.227	0	-384927.114	67362.2449	0	0	0	0	0	0	0	0
4	0	7858.928571	0	31435.71429	-67362.2449	7858.928571	0	0	0	0	0	0	0	0
5	0	0	-384927.11	-67362.2449	769854.2274	0	-384927.1137	67362.2449	0	0	0	0	0	0
6	0	0	67362.2449	7858.928571	0	31435.71429	-67362.2449	7858.928571	0	0	0	0	0	0
7	0	0	0	0	-384927.114	-67362.2449	769854.2274	0	-384927.1137	67362.2449	0	0	0	0
8	0	0	0	0	67362.2449	7858.928571	0	31435.71429	-67362.2449	7858.92857	0	0	0	0
9	0	0	0	0	0	0	-384927.1137	-67362.2449	769854.2274	0	-384927.114	67362.2449	0	0
10	0	0	0	0	0	0	67362.2449	7858.928571	0	31435.7143	-67362.2449	7858.92857	0	0
11	0	0	0	0	0	0	0	0	-384927.1137	-67362.2449	769854.227	0	0	67362.2449
12	0	0	0	0	0	0	0	0	67362.2449	7858.92857	0	31435.7143	0	7858.92857
13	0	0	0	0	0	0	0	0	0	0	-384927.114	-67362.2449	1	-67362.2449
14	0	0	0	0	0	0	0	0	0	0	67362.2449	7858.92857	0	21717.8571

3. Load Vector [F]

Vector {F}:

0
0
0
0
19.151
0
0
0
19.151
0
0
0
0
0
0
0

4. Inverse of Stiffness Matrix [K]⁻¹

Stiffness Matrix Inverse [K] ⁻¹ :														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	-0.18844657	0.88928354	-0.428233158	0.71142683	-0.572105112	0.5	-0.62006243	0.28857317	-0.57210511	0.11071646	-0.42823316	0	-0.18844657
2	0	0.000118756	2.9742E-05	5.39933E-05	3.97536E-05	6.01531E-06	3.59106E-05	-2.5178E-05	2.40875E-05	-3.9585E-05	1.0159E-05	-3.7208E-05	0	-1.8046E-05
3	0	2.97416E-05	1.3736E-05	3.55503E-05	2.08399E-05	6.68526E-06	1.95666E-05	-1.2318E-05	1.33677E-05	-2.146E-05	5.695E-06	-2.074E-05	0	-1.0159E-05
4	0	5.39933E-05	3.555E-05	0.000155509	6.87706E-05	4.06791E-05	6.84757E-05	-3.6007E-05	4.80156E-05	-7.4551E-05	2.074E-05	-7.4951E-05	0	-3.7208E-05
5	0	3.97536E-05	2.084E-05	6.87706E-05	4.07466E-05	3.44205E-05	4.27372E-05	-1.8762E-05	3.06122E-05	-4.624E-05	1.3368E-05	-4.8016E-05	0	-2.4087E-05
6	0	6.01531E-06	6.6853E-06	4.06791E-05	3.44205E-05	0.000126301	5.65055E-05	8.39253E-06	4.62403E-05	-5.8558E-05	2.146E-05	-7.4551E-05	0	-3.9585E-05
7	0	3.59106E-05	1.9567E-05	6.84757E-05	4.27372E-05	5.65055E-05	5.39246E-05	8.30662E-20	4.27372E-05	-5.6506E-05	1.9567E-05	-6.8476E-05	0	-3.5911E-05
8	0	-2.5178E-05	-1.232E-05	-3.60072E-05	-1.8762E-05	8.39253E-06	1.16407E-19	0.000108022	1.87616E-05	8.3925E-06	1.2318E-05	-3.6007E-05	0	-2.5178E-05
9	0	2.40875E-05	1.3368E-05	4.80156E-05	3.06122E-05	4.62403E-05	4.27372E-05	1.87616E-05	4.07466E-05	-3.442E-05	2.084E-05	-6.8771E-05	0	-3.9754E-05
10	0	-3.9585E-05	-2.146E-05	-7.45506E-05	-4.624E-05	-5.85579E-05	-5.65055E-05	8.39253E-06	-3.44205E-05	0.0001263	-6.6853E-06	4.0679E-05	0	6.0153E-06
11	0	1.0159E-05	5.695E-06	2.07403E-05	1.33677E-05	2.14601E-05	1.95666E-05	1.23182E-05	2.08399E-05	-6.6853E-06	1.3736E-05	-3.555E-05	0	-2.9742E-05
12	0	-3.7208E-05	-2.074E-05	-7.49509E-05	-4.8016E-05	-7.45506E-05	-6.84757E-05	-3.6007E-05	-6.87706E-05	4.0679E-05	-3.555E-05	0.00015551	0	5.3993E-05
13	0	0.188446568	0.11071646	0.428233158	0.28857317	0.572105112	0.5	0.62006243	0.71142683	0.57210511	0.88928354	0.42823316	1	0.18844657
14	0	-1.8046E-05	-1.016E-05	-3.72082E-05	-2.4087E-05	-3.95854E-05	-3.59106E-05	-2.5178E-05	-3.97536E-05	6.0153E-06	-2.9742E-05	5.3993E-05	0	0.00011876

5. Results of System of Equations (Deterministic Value)

System of Equations Result:

R_1		19.151
Δ_2		1.223E-03
Δ_3		6.551E-04
Δ_4		2.237E-03
Δ_5		1.367E-03
Δ_6		1.545E-03
Δ_7		1.637E-03
Δ_8		3.361E-18
Δ_9		1.367E-03
Δ_{10}		-1.545E-03
Δ_{11}		6.551E-04
Δ_{12}		-2.237E-03
R_{13}		19.151
Δ_{14}		-1.223E-03

6. Results of System of Equations (Mean Value and Standard Deviation)

Result in terms of mean values and standard deviations:

Variable	mean value	standard deviation	mean - st. dev.	mean + st. dev.
R_1	19.151	0.977303489	18.17369651	20.12830349
Δ_2	1.223E-03	0.000108035	0.001114585	0.001330654
Δ_3	6.551E-04	4.54929E-05	0.000609618	0.000700604
Δ_4	2.237E-03	0.000153366	0.002083207	0.00238994
Δ_5	1.367E-03	9.32152E-05	0.001273377	0.001459808
Δ_6	1.545E-03	0.000109613	0.001435122	0.001654348
Δ_7	1.637E-03	0.000111582	0.001525339	0.001748503
Δ_8	3.361E-18	2.61199E-05	-2.61199E-05	2.61199E-05
Δ_9	1.367E-03	9.32152E-05	0.001273377	0.001459808
Δ_{10}	-1.545E-03	0.000109613	-0.001654348	-0.001435122
Δ_{11}	6.551E-04	4.54929E-05	0.000609618	0.000700604
Δ_{12}	-2.237E-03	0.000153366	-0.00238994	-0.002083207
R_{13}	19.151	0.977303489	18.17369651	20.12830349
Δ_{14}	-1.223E-03	0.000108035	-0.001330654	-0.001114585

1. Input Data

Data:

E =	32600000	kPa
b =	0.15	m
h =	0.15	m
L =	0.35	m
F =	19.151	kN
K ₂ =	6000	kN.m/rad
K ₁₄ =	6000	kN.m/rad
I =	4.21875E-05	m ⁴

2. Partial Derivate of Stiffness Matrix [K] in order to Elasticity Modulus (E)

Matrix $[\partial K/\partial E]$:														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0.002066	-0.011808	0.002066	0	0	0	0	0	0	0	0	0	0
2	0	0.000482	-0.002066	0.000241	0	0	0	0	0	0	0	0	0	0
3	0	-0.002066	0.023615	0	-0.011808	0.002066	0	0	0	0	0	0	0	0
4	0	0.000241	0	0.000964	-0.002066	0.000241	0	0	0	0	0	0	0	0
5	0	0	-0.011808	-0.002066	0.023615	0	-0.011808	0.002066	0	0	0	0	0	0
6	0	0	0.002066	0.000241	0	0.000964	-0.002066	0.000241	0	0	0	0	0	0
7	0	0	0	0	-0.011808	-0.002066	0.023615	0	-0.011808	0.002066	0	0	0	0
8	0	0	0	0	0.002066	0.000241	0	0.000964	-0.002066	0.000241	0	0	0	0
9	0	0	0	0	0	0	-0.011808	-0.002066	0.023615	0	-0.011808	0.002066	0	0
10	0	0	0	0	0	0	0.002066	0.000241	0	0.000964	-0.002066	0.000241	0	0
11	0	0	0	0	0	0	0	0	-0.011808	-0.002066	0.023615	0	0	0.002066
12	0	0	0	0	0	0	0	0	0.002066	0.000241	0	0.000964	0	0.000241
13	0	0	0	0	0	0	0	0	0	0	-0.011808	-0.002066	0	-0.002066
14	0	0	0	0	0	0	0	0	0	0	0.002066	0.000241	0	0.000482

3. Partial Derivate of Stiffness Matrix [K] in order to Stiffness Parameter (K_2)

[illegible]

4. Partial Derivate of Stiffness Matrix [K] in order to Stiffness Parameter (K_{14})

[illegible]

5. Partial Derivate of Stiffness Matrix [K] in order to Applied Load (F)

Vector $\{\partial F / \partial F\}$:

0
0
0
0
1
0
0
0
1
0
0
0
0
0
0

1. Contribution of Modulus of Elasticity (E) variable

$$- [K]^{-1} * [\partial K / \partial E]$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	-3.47E-05	5.64E-18	-2.06E-18	3.88E-17	1.141E-17	-3.9E-17	7.54E-18	-7.45E-17	-1E-17	8.57E-17	-8.27E-18	0	-3.47E-05
2	0	-8.82E-09	2.12E-21	1.44E-22	-3.32E-21	-1.14E-22	2.329E-21	-2.6E-22	4.103E-22	1.6E-22	-8.8E-22	1.43E-22	0	-3.32E-09
3	0	5.474E-09	-3.1E-08	-6.66E-23	-2.68E-22	-1.93E-22	6.353E-22	-1.1E-22	4.368E-22	7.4E-23	-3.84E-22	9.97E-23	0	-1.87E-09
4	0	9.937E-09	-5E-21	-3.07E-08	1.59E-21	-1.18E-21	1.747E-21	-4.5E-22	3.203E-21	4.1E-22	-2.58E-21	5.58E-22	0	-6.85E-09
5	0	7.317E-09	-1.1E-21	-1.34E-22	-3.07E-08	-3.17E-22	6.617E-22	-1.7E-22	1.019E-21	1.1E-22	-7.21E-22	1.82E-22	0	-4.43E-09
6	0	1.107E-09	1.43E-21	1.85E-22	-1.16E-21	-3.07E-08	-5.29E-22	-5.6E-23	5.294E-22	-2E-23	-6.09E-22	3.47E-23	0	-7.29E-09
7	0	6.609E-09	-4.4E-22	-5.29E-23	-2.78E-22	-1.66E-22	-3.07E-08	-1.1E-22	4.235E-22	5E-23	-3.97E-22	1.16E-22	0	-6.61E-09
8	0	-4.63E-09	1.88E-21	2.42E-22	-7.94E-23	4.897E-22	-6.32E-22	-3.1E-08	-2.49E-21	-2E-22	1.57E-21	-3.16E-22	0	-4.63E-09
9	0	4.433E-09	1.32E-23	-4.96E-24	-1.59E-22	-9.1E-24	5.294E-23	-3.6E-23	-3.07E-08	0	-1.06E-22	3.31E-23	0	-7.32E-09
10	0	-7.29E-09	2.65E-23	-2.98E-23	2.38E-22	2.771E-23	0	4.63E-23	-1.32E-22	-3E-08	1.67E-22	-4.53E-23	0	1.107E-09
11	0	1.87E-09	4.63E-23	-1.65E-24	-6.95E-23	3.309E-24	-4.96E-24	-2E-23	6.617E-23	-8E-24	-3.07E-08	1.82E-23	0	-5.47E-09
12	0	-6.85E-09	-1.9E-22	-1.99E-23	4.5E-22	2.482E-23	-9.26E-23	7.61E-23	-5.29E-23	-7E-24	1.72E-22	-3.07E-08	0	9.937E-09
13	0	3.468E-05	-2.2E-19	2.71E-20	0	-2.71E-20	8.674E-19	-3.3E-19	-1.41E-18	1.1E-19	-3.79E-19	-1.83E-19	0	3.468E-05
14	0	-3.32E-09	-5.3E-23	-3.31E-24	2.85E-22	-2.16E-23	-1.7E-22	4.98E-23	2.647E-23	-2E-23	1.06E-22	-3.97E-23	0	-8.82E-09

$$- [K]^{-1} * [\partial K / \partial E] * \{U_0\} * \delta E$$

1	-5.84E-15
2	-1.3E-05
3	-2.16E-05
4	-9.33E-05
5	-5.35E-05
6	-7.2E-05
7	-6.61E-05
8	-3.49E-19
9	-5.35E-05
10	7.202E-05
11	-2.16E-05
12	9.328E-05
13	1.926E-16
14	1.304E-05

2. Contribution of Stiffness Parameter (K_2) variable

Variable K_2 contribution:

$$- [K]^{-1} * [\partial K / \partial K_2]$$
[illegible]

$$- [K]^{-1} * [\partial K / \partial K_2] * \{U_0\} * \delta K_2$$

1	0.1382391
2	-8.71E-05
3	-2.18E-05
4	-3.96E-05
5	-2.92E-05
6	-4.41E-06
7	-2.63E-05
8	1.847E-05
9	-1.77E-05
10	2.904E-05
11	-7.45E-06
12	2.729E-05
13	-0.138239
14	1.324E-05

3. Contribution of Stiffness Parameter (K_{14}) variable

Variable K_{14} contribution:

$$- [K]^{-1} * [\partial K / \partial K_{14}]$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1884466
2	0	0	0	0	0	0	0	0	0	0	0	0	0	1.805E-05
3	0	0	0	0	0	0	0	0	0	0	0	0	0	1.016E-05
4	0	0	0	0	0	0	0	0	0	0	0	0	0	3.721E-05
5	0	0	0	0	0	0	0	0	0	0	0	0	0	2.409E-05
6	0	0	0	0	0	0	0	0	0	0	0	0	0	3.959E-05
7	0	0	0	0	0	0	0	0	0	0	0	0	0	3.591E-05
8	0	0	0	0	0	0	0	0	0	0	0	0	0	2.518E-05
9	0	0	0	0	0	0	0	0	0	0	0	0	0	3.975E-05
10	0	0	0	0	0	0	0	0	0	0	0	0	0	-6.02E-06
11	0	0	0	0	0	0	0	0	0	0	0	0	0	2.974E-05
12	0	0	0	0	0	0	0	0	0	0	0	0	0	-5.4E-05
13	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.188447
14	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.000119

$$- [K]^{-1} * [\partial K / \partial K_{14}] * \{U_0\} * \delta K_{14}$$

1	-0.138239
2	-1.32E-05
3	-7.45E-06
4	-2.73E-05
5	-1.77E-05
6	-2.9E-05
7	-2.63E-05
8	-1.85E-05
9	-2.92E-05
10	4.413E-06
11	-2.18E-05
12	3.961E-05
13	0.1382391
14	8.712E-05

4. Contribution of Applied Load (F) variable

Variable F contribution:

	$[K]^{-1} * [\partial F / \partial F] * \delta F$
1	0.95755
2	6.113E-05
3	3.276E-05
4	0.0001118
5	6.833E-05
6	7.724E-05
7	8.185E-05
8	1.687E-19
9	6.833E-05
10	-7.72E-05
11	3.276E-05
12	-0.000112
13	0.95755
14	-6.11E-05

Covariance Matrix

1. Matrix of variable contributions [A]

Matrix A:

	variable E	variable K ₂	variable K ₁₄	variable F
1	-5.84122E-15	0.138239118	-0.138239118	0.95755
2	-1.30371E-05	-8.71163E-05	-1.3238E-05	6.1131E-05
3	-2.15668E-05	-2.18176E-05	-7.45239E-06	3.27556E-05
4	-9.32833E-05	-3.9608E-05	-2.72949E-05	0.000111829
5	-5.34555E-05	-2.91621E-05	-1.76699E-05	6.83296E-05
6	-7.20193E-05	-4.41267E-06	-2.90388E-05	7.72367E-05
7	-6.60588E-05	-2.6343E-05	-2.6343E-05	8.1846E-05
8	-3.49468E-19	1.84696E-05	-1.84696E-05	1.68704E-19
9	-5.34555E-05	-1.76699E-05	-2.91621E-05	6.83296E-05
10	7.20193E-05	2.90388E-05	4.41267E-06	-7.72367E-05
11	-2.15668E-05	-7.45239E-06	-2.18176E-05	3.27556E-05
12	9.32833E-05	2.72949E-05	3.9608E-05	-0.000111829
13	1.92568E-16	-0.138239118	0.138239118	0.95755
14	1.30371E-05	1.3238E-05	8.71163E-05	-6.1131E-05

2. Transpose of Matrix [A]

Matrix A Transpose:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
variable E	-5.84E-15	-1.304E-05	-2.2E-05	-9.3E-05	-5.3E-05	-7.2E-05	-6.6E-05	-3.49E-19	-5.35E-05	7.2E-05	-2E-05	9.3E-05	1.9E-16	1.3E-05
variable K ₂	0.138239	-8.712E-05	-2.2E-05	-4E-05	-2.9E-05	-4.4E-06	-2.6E-05	1.847E-05	-1.77E-05	2.9E-05	-7E-06	2.7E-05	-0.1382	1.32E-05
variable K ₁₄	-0.138239	-1.324E-05	-7.5E-06	-2.7E-05	-1.8E-05	-2.9E-05	-2.6E-05	-1.85E-05	-2.92E-05	4.41E-06	-2E-05	4E-05	0.13824	8.71E-05
variable F	0.95755	6.113E-05	3.28E-05	0.000112	6.83E-05	7.72E-05	8.18E-05	1.687E-19	6.83E-05	-7.72E-05	3.3E-05	-0.00011	0.95755	-6.1E-05

3. Covariance Matrix

Covariance Matrix:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.955122	4.832E-05	2.94E-05	0.000105	6.38E-05	7.74E-05	7.84E-05	5.106E-06	6.7E-05	-7.06E-05	3.3E-05	-0.00011	0.87868	-6.9E-05
2	4.83E-05	1.167E-08	4.28E-09	1.19E-08	7.65E-09	6.43E-09	8.51E-09	-1.36E-09	6.8E-09	-8.25E-09	3.2E-09	-1.1E-08	6.9E-05	-6.2E-09
3	2.94E-05	4.283E-09	2.07E-09	6.74E-09	4.16E-09	4.4E-09	4.88E-09	-2.65E-10	3.99E-09	-4.75E-09	1.9E-09	-6.6E-09	3.3E-05	-3.2E-09
4	0.000105	1.186E-08	6.74E-09	2.35E-08	1.43E-08	1.63E-08	1.71E-08	-2.27E-10	1.41E-08	-1.66E-08	6.6E-09	-2.3E-08	0.00011	-1.1E-08
5	6.38E-05	7.648E-09	4.16E-09	1.43E-08	8.69E-09	9.77E-09	1.04E-08	-2.12E-10	8.56E-09	-1.01E-08	4E-09	-1.4E-08	6.7E-05	-6.8E-09
6	7.74E-05	6.429E-09	4.4E-09	1.63E-08	9.77E-09	1.2E-08	1.2E-08	4.548E-10	1.01E-08	-1.14E-08	4.7E-09	-1.7E-08	7.1E-05	-8.2E-09
7	7.84E-05	8.508E-09	4.88E-09	1.71E-08	1.04E-08	1.2E-08	1.25E-08	3.79E-23	1.04E-08	-1.2E-08	4.9E-09	-1.7E-08	7.8E-05	-8.5E-09
8	5.11E-06	-1.365E-09	-2.7E-10	-2.3E-10	-2.1E-10	4.55E-10	3.79E-23	6.823E-10	2.12E-10	4.55E-10	2.7E-10	-2.3E-10	-5E-06	-1.4E-09
9	6.7E-05	6.799E-09	3.99E-09	1.41E-08	8.56E-09	1.01E-08	1.04E-08	2.123E-10	8.69E-09	-9.77E-09	4.2E-09	-1.4E-08	6.4E-05	-7.6E-09
10	-7.06E-05	-8.249E-09	-4.7E-09	-1.7E-08	-1E-08	-1.1E-08	-1.2E-08	4.548E-10	-9.77E-09	1.2E-08	-4E-09	1.6E-08	-8E-05	6.43E-09
11	3.34E-05	3.222E-09	1.86E-09	6.57E-09	3.99E-09	4.75E-09	4.88E-09	2.653E-10	4.16E-09	-4.4E-09	2.1E-09	-6.7E-09	2.9E-05	-4.3E-09
12	-0.000109	-1.095E-08	-6.6E-09	-2.3E-08	-1.4E-08	-1.7E-08	-1.7E-08	-2.27E-10	-1.43E-08	1.63E-08	-7E-09	2.4E-08	-0.0001	1.19E-08
13	0.878682	6.875E-05	3.34E-05	0.000109	6.7E-05	7.06E-05	7.84E-05	-5.11E-06	6.38E-05	-7.74E-05	2.9E-05	-0.00011	0.95512	-4.8E-05
14	-6.87E-05	-6.213E-09	-3.2E-09	-1.1E-08	-6.8E-09	-8.2E-09	-8.5E-09	-1.36E-09	-7.65E-09	6.43E-09	-4E-09	1.2E-08	-5E-05	1.17E-08

APPENDIX F
RESULTS FROM CONCRETE BEAM ANALYSIS

RESULTS FROM CONCRETE BEAM ANALYSIS

In this Appendix it is presented all values obtained for all load steps of the concrete laboratory beam. It is represented the exactly load step, measured by the actuator (RefF Act 100kN), and the values obtained by each displacement transducer (LVDT – Linear Variable Differential Transformer), which are Lvdt 1452, Lvdt 437, Lvdt 443, Lvdt 553 and Lvdt 550. It is presented those values affected by the equipment linearity. It is also showed the deterministic, the Monte Carlo, the Perturbation Technique and the Modal Interval Analysis (MIA) results, obtained using the respective system of equations of the structure and the algorithms associated to each methodology.

Values obtained from different methodologies

1. Measured Values from Displacement Transducers and Actuator

Measured Values					
RefFAct100kN (F)	Lvdt1452 (w3)	Lvdt437 (w5)	Lvdt443 (w7)	Lvdt553 (w9)	Lvdt550 (w11)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.9640	0.2350	0.3010	0.4240	0.3040	0.2190
5.4060	0.2490	0.3010	0.4240	0.3320	0.2460
5.9000	0.2630	0.3310	0.4840	0.3870	0.2460
6.4000	0.2910	0.3910	0.5150	0.4150	0.2730
6.8730	0.3050	0.4510	0.5750	0.4700	0.2730
7.3730	0.3320	0.4510	0.5750	0.4700	0.3010
7.9330	0.3460	0.4810	0.6050	0.4980	0.3280
8.4390	0.3460	0.5110	0.6660	0.5530	0.3550
8.8870	0.3600	0.5720	0.6660	0.6080	0.3830
9.3910	0.4020	0.6020	0.7570	0.6360	0.4100
9.9130	0.4160	0.6620	0.8170	0.6910	0.4100
10.4040	0.4290	0.7220	0.8480	0.7190	0.4650
10.8760	0.4570	0.7520	0.9390	0.8020	0.5190
11.3960	0.4710	0.8120	0.9390	0.8850	0.5470
11.8930	0.4990	0.8420	0.9990	0.8850	0.5740
12.3750	0.5400	0.8730	1.0600	0.9400	0.6010
12.9160	0.5680	0.9630	1.1200	0.9960	0.6010
13.4190	0.5960	0.9630	1.1810	1.0230	0.6010
13.9200	0.6230	1.0530	1.2410	1.1060	0.6560
14.3610	0.6510	1.0830	1.3320	1.1060	0.6830
14.9020	0.6650	1.1430	1.3930	1.1890	0.7110
15.3960	0.6930	1.2030	1.4530	1.2170	0.7650
15.9590	0.7340	1.2940	1.5440	1.2720	0.7930
16.2880	0.7760	1.3540	1.5740	1.3550	0.8200
16.8910	0.8170	1.3840	1.6650	1.4110	0.8750
17.3920	0.8310	1.4440	1.7260	1.4660	0.9290
17.8860	0.8590	1.5650	1.8160	1.5490	0.9570
18.3650	0.9280	1.6550	1.9380	1.6320	0.9570
18.8150	0.9420	1.7450	1.9980	1.7150	1.0110
19.1510	0.9830	1.8350	2.1190	1.7700	1.0660

2. Measured Values from Displacement Transducers and Actuator (affected by the respective Linearity)

Measured Values (Linearity)											
F (kN)		w3 (mm)		w5 (mm)		w7 (mm)		w9 (mm)		w11 (mm)	
RefFAct100kN		Lvdt1452		Lvdt437		Lvdt443		Lvdt553		Lvdt550	
a	b	a	b	a	b	a	b	a	b	a	b
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.7158	5.2122	0.1850	0.2850	0.1839	0.4181	0.2940	0.5540	0.1940	0.4140	0.0690	0.3690
5.1357	5.6763	0.1990	0.2990	0.1839	0.4181	0.2940	0.5540	0.2220	0.4420	0.0960	0.3960
5.6050	6.1950	0.2130	0.3130	0.2139	0.4481	0.3540	0.6140	0.2770	0.4970	0.0960	0.3960
6.0800	6.7200	0.2410	0.3410	0.2739	0.5081	0.3850	0.6450	0.3050	0.5250	0.1230	0.4230
6.5294	7.2167	0.2550	0.3550	0.3339	0.5681	0.4450	0.7050	0.3600	0.5800	0.1230	0.4230
7.0044	7.7417	0.2820	0.3820	0.3339	0.5681	0.4450	0.7050	0.3600	0.5800	0.1510	0.4510
7.5364	8.3297	0.2960	0.3960	0.3639	0.5981	0.4750	0.7350	0.3880	0.6080	0.1780	0.4780
8.0171	8.8610	0.2960	0.3960	0.3939	0.6281	0.5360	0.7960	0.4430	0.6630	0.2050	0.5050
8.4427	9.3314	0.3100	0.4100	0.4549	0.6891	0.5360	0.7960	0.4980	0.7180	0.2330	0.5330
8.9215	9.8606	0.3520	0.4520	0.4849	0.7191	0.6270	0.8870	0.5260	0.7460	0.2600	0.5600
9.4174	10.4087	0.3660	0.4660	0.5449	0.7791	0.6870	0.9470	0.5810	0.8010	0.2600	0.5600
9.8838	10.9242	0.3790	0.4790	0.6049	0.8391	0.7180	0.9780	0.6090	0.8290	0.3150	0.6150
10.3322	11.4198	0.4070	0.5070	0.6349	0.8691	0.8090	1.0690	0.6920	0.9120	0.3690	0.6690
10.8262	11.9658	0.4210	0.5210	0.6949	0.9291	0.8090	1.0690	0.7750	0.9950	0.3970	0.6970
11.2984	12.4877	0.4490	0.5490	0.7249	0.9591	0.8690	1.1290	0.7750	0.9950	0.4240	0.7240
11.7563	12.9938	0.4900	0.5900	0.7559	0.9901	0.9300	1.1900	0.8300	1.0500	0.4510	0.7510
12.2702	13.5618	0.5180	0.6180	0.8459	1.0801	0.9900	1.2500	0.8860	1.1060	0.4510	0.7510
12.7481	14.0900	0.5460	0.6460	0.8459	1.0801	1.0510	1.3110	0.9130	1.1330	0.4510	0.7510
13.2240	14.6160	0.5730	0.6730	0.9359	1.1701	1.1110	1.3710	0.9960	1.2160	0.5060	0.8060
13.6430	15.0791	0.6010	0.7010	0.9659	1.2001	1.2020	1.4620	0.9960	1.2160	0.5330	0.8330
14.1569	15.6471	0.6150	0.7150	1.0259	1.2601	1.2630	1.5230	1.0790	1.2990	0.5610	0.8610
14.6262	16.1658	0.6430	0.7430	1.0859	1.3201	1.3230	1.5830	1.1070	1.3270	0.6150	0.9150
15.1611	16.7570	0.6840	0.7840	1.1769	1.4111	1.4140	1.6740	1.1620	1.3820	0.6430	0.9430
15.4736	17.1024	0.7260	0.8260	1.2369	1.4711	1.4440	1.7040	1.2450	1.4650	0.6700	0.9700
16.0465	17.7356	0.7670	0.8670	1.2669	1.5011	1.5350	1.7950	1.3010	1.5210	0.7250	1.0250
16.5224	18.2616	0.7810	0.8810	1.3269	1.5611	1.5960	1.8560	1.3560	1.5760	0.7790	1.0790
16.9917	18.7803	0.8090	0.9090	1.4479	1.6821	1.6860	1.9460	1.4390	1.6590	0.8070	1.1070
17.4468	19.2833	0.8780	0.9780	1.5379	1.7721	1.8080	2.0680	1.5220	1.7420	0.8070	1.1070
17.8743	19.7558	0.8920	0.9920	1.6279	1.8621	1.8680	2.1280	1.6050	1.8250	0.8610	1.1610
18.1935	20.1086	0.9330	1.0330	1.7179	1.9521	1.9890	2.2490	1.6600	1.8800	0.9160	1.2160

3. Deterministic Values from Numerical Analysis

Deterministic Values					
F	w3	w5	w7	w9	w11
kN	mm	mm	mm	mm	mm
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.9640	0.1698	0.3542	0.4243	0.3542	0.1698
5.4060	0.1849	0.3858	0.4620	0.3858	0.1849
5.9000	0.2018	0.4210	0.5043	0.4210	0.2018
6.4000	0.2189	0.4567	0.5470	0.4567	0.2189
6.8730	0.2351	0.4904	0.5875	0.4904	0.2351
7.3730	0.2522	0.5261	0.6302	0.5261	0.2522
7.9330	0.2714	0.5661	0.6781	0.5661	0.2714
8.4390	0.2887	0.6022	0.7213	0.6022	0.2887
8.8870	0.3040	0.6342	0.7596	0.6342	0.3040
9.3910	0.3212	0.6701	0.8027	0.6701	0.3212
9.9130	0.3391	0.7074	0.8473	0.7074	0.3391
10.4040	0.3559	0.7424	0.8893	0.7424	0.3559
10.8760	0.3720	0.7761	0.9296	0.7761	0.3720
11.3960	0.3898	0.8132	0.9740	0.8132	0.3898
11.8930	0.4068	0.8487	1.0165	0.8487	0.4068
12.3750	0.4233	0.8830	1.0577	0.8830	0.4233
12.9160	0.4418	0.9217	1.1040	0.9217	0.4418
13.4190	0.4590	0.9576	1.1470	0.9576	0.4590
13.9200	0.4762	0.9933	1.1898	0.9933	0.4762
14.3610	0.4913	1.0248	1.2275	1.0248	0.4913
14.9020	0.5098	1.0634	1.2737	1.0634	0.5098
15.3960	0.5267	1.0986	1.3160	1.0986	0.5267
15.9590	0.5459	1.1388	1.3641	1.1388	0.5459
16.2880	0.5572	1.1623	1.3922	1.1623	0.5572
16.8910	0.5778	1.2053	1.4437	1.2053	0.5778
17.3920	0.5949	1.2410	1.4866	1.2410	0.5949
17.8860	0.6118	1.2763	1.5288	1.2763	0.6118
18.3650	0.6282	1.3105	1.5697	1.3105	0.6282
18.8150	0.6436	1.3426	1.6082	1.3426	0.6436
19.1510	0.6551	1.3666	1.6369	1.3666	0.6551

4. Monte Carlo Numerical Analysis

Monte Carlo											
F (kN)		w3 (mm)		w5 (mm)		w7 (mm)		w9 (mm)		w11 (mm)	
RefFact100kN		Lvd1452		Lvd1437		Lvd1443		Lvd1553		Lvd1550	
a	b	a	b	a	b	a	b	a	b	a	b
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.7158	5.2122	0.1429	0.2026	0.2968	0.4281	0.3549	0.5139	0.2969	0.4279	0.1433	0.2026
5.1357	5.6763	0.1556	0.2206	0.3232	0.4662	0.3865	0.5596	0.3233	0.4660	0.1561	0.2206
5.6050	6.1950	0.1699	0.2408	0.3527	0.5088	0.4218	0.6108	0.3529	0.5086	0.1704	0.2408
6.0800	6.7200	0.1842	0.2612	0.3826	0.5519	0.4575	0.6625	0.3828	0.5517	0.1848	0.2612
6.5294	7.2167	0.1979	0.2805	0.4109	0.5927	0.4913	0.7115	0.4111	0.5925	0.1985	0.2805
7.0044	7.7417	0.2095	0.2961	0.4326	0.6278	0.5171	0.7545	0.4328	0.6280	0.2097	0.2964
7.5364	8.3297	0.2144	0.3273	0.4396	0.6921	0.5246	0.8315	0.4393	0.6928	0.2140	0.3282
8.0171	8.8610	0.2375	0.3346	0.4966	0.7016	0.5937	0.8441	0.4969	0.7025	0.2381	0.3349
8.4427	9.3314	0.2606	0.3582	0.5382	0.7623	0.6433	0.9172	0.5379	0.7633	0.2602	0.3595
8.9215	9.8606	0.2633	0.3823	0.5447	0.8045	0.6517	0.9650	0.5454	0.8036	0.2642	0.3811
9.4174	10.4087	0.2779	0.4035	0.5750	0.8492	0.6879	1.0186	0.5758	0.8482	0.2789	0.4023
9.8838	10.9242	0.2998	0.4209	0.6219	0.8799	0.7437	1.0542	0.6211	0.8793	0.2988	0.4202
10.3322	11.4198	0.3134	0.4400	0.6501	0.9198	0.7775	1.1020	0.6492	0.9192	0.3123	0.4392
10.8262	11.9658	0.3201	0.4643	0.6563	0.9749	0.7831	1.1692	0.6558	0.9747	0.3195	0.4641
11.2984	12.4877	0.3340	0.4845	0.6849	1.0174	0.8173	1.2202	0.6844	1.0172	0.3334	0.4843
11.7563	12.9938	0.3569	0.5269	0.7388	1.1098	0.8835	1.3321	0.7382	1.1103	0.3562	0.5275
12.2702	13.5618	0.3725	0.5499	0.7711	1.1583	0.9221	1.3903	0.7704	1.1588	0.3717	0.5505
12.7481	14.0900	0.3870	0.5713	0.8012	1.2034	0.9580	1.4445	0.8004	1.2040	0.3862	0.5720
13.2240	14.6160	0.4060	0.5748	0.8385	1.2112	1.0020	1.4530	0.8377	1.2094	0.4050	0.5726
13.6430	15.0791	0.4188	0.5930	0.8650	1.2495	1.0338	1.4990	0.8642	1.2477	0.4178	0.5907
14.1569	15.6471	0.4346	0.6153	0.8976	1.2966	1.0727	1.5555	0.8968	1.2947	0.4336	0.6130
14.6262	16.1658	0.4303	0.6185	0.8823	1.3007	1.0529	1.5648	0.8817	1.3024	0.4295	0.6196
15.1611	16.7570	0.4461	0.6411	0.9146	1.3483	1.0914	1.6221	0.9139	1.3500	0.4452	0.6423
15.4736	17.1024	0.4553	0.6543	0.9334	1.3761	1.1139	1.6555	0.9328	1.3779	0.4544	0.6555
16.0465	17.7356	0.4721	0.6786	0.9680	1.4270	1.1551	1.7168	0.9673	1.4289	0.4712	0.6798
16.5224	18.2616	0.4939	0.7359	1.0190	1.5660	1.2168	1.8840	1.0164	1.5680	0.4906	0.7384
16.9917	18.7803	0.5079	0.7568	1.0479	1.6105	1.2514	1.9375	1.0452	1.6126	0.5046	0.7594
17.4468	19.2833	0.5215	0.7770	1.0760	1.6536	1.2849	1.9894	1.0732	1.6558	0.5181	0.7798
17.8743	19.7558	0.5577	0.7687	1.1617	1.6298	1.3901	1.9587	1.1610	1.6304	0.5583	0.7695
18.1935	20.1086	0.5677	0.7825	1.1825	1.6589	1.4149	1.9937	1.1817	1.6595	0.5683	0.7832

5. Perturbation Technique Numerical Analysis

Perturbation Method											
F (kN)		w3 (mm)		w5 (mm)		w7 (mm)		w9 (mm)		w11 (mm)	
RefFact100kN		Lvdt1452		Lvdt437		Lvdt443		Lvdt553		Lvdt550	
a	b	a	b	a	b	a	b	a	b	a	b
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.7158	5.2122	0.1580	0.1816	0.3301	0.3784	0.3954	0.4532	0.3301	0.3784	0.1580	0.1816
5.1357	5.6763	0.1721	0.1978	0.3595	0.4121	0.4306	0.4936	0.3595	0.4121	0.1721	0.1978
5.6050	6.1950	0.1878	0.2158	0.3923	0.4497	0.4699	0.5387	0.3923	0.4497	0.1878	0.2158
6.0800	6.7200	0.2037	0.2341	0.4255	0.4878	0.5097	0.5843	0.4255	0.4878	0.2037	0.2341
6.5294	7.2167	0.2188	0.2514	0.4570	0.5239	0.5474	0.6275	0.4570	0.5239	0.2188	0.2514
7.0044	7.7417	0.2347	0.2697	0.4902	0.5620	0.5872	0.6732	0.4902	0.5620	0.2347	0.2697
7.5364	8.3297	0.2525	0.2902	0.5275	0.6047	0.6318	0.7243	0.5275	0.6047	0.2525	0.2902
8.0171	8.8610	0.2686	0.3087	0.5611	0.6433	0.6721	0.7705	0.5611	0.6433	0.2686	0.3087
8.4427	9.3314	0.2829	0.3251	0.5909	0.6774	0.7078	0.8114	0.5909	0.6774	0.2829	0.3251
8.9215	9.8606	0.2989	0.3436	0.6244	0.7158	0.7480	0.8574	0.6244	0.7158	0.2989	0.3436
9.4174	10.4087	0.3156	0.3626	0.6591	0.7556	0.7896	0.9051	0.6591	0.7556	0.3156	0.3626
9.8838	10.9242	0.3312	0.3806	0.6918	0.7931	0.8287	0.9499	0.6918	0.7931	0.3312	0.3806
10.3322	11.4198	0.3462	0.3979	0.7232	0.8290	0.8663	0.9930	0.7232	0.8290	0.3462	0.3979
10.8262	11.9658	0.3628	0.4169	0.7577	0.8687	0.9077	1.0405	0.7577	0.8687	0.3628	0.4169
11.2984	12.4877	0.3786	0.4351	0.7908	0.9066	0.9473	1.0858	0.7908	0.9066	0.3786	0.4351
11.7563	12.9938	0.3939	0.4527	0.8228	0.9433	0.9856	1.1298	0.8228	0.9433	0.3939	0.4527
12.2702	13.5618	0.4111	0.4725	0.8588	0.9845	1.0287	1.1792	0.8588	0.9845	0.4111	0.4725
12.7481	14.0900	0.4272	0.4909	0.8922	1.0229	1.0688	1.2252	0.8922	1.0229	0.4272	0.4909
13.2240	14.6160	0.4431	0.5092	0.9256	1.0611	1.1087	1.2709	0.9256	1.0611	0.4431	0.5092
13.6430	15.0791	0.4571	0.5254	0.9549	1.0947	1.1438	1.3112	0.9549	1.0947	0.4571	0.5254
14.1569	15.6471	0.4744	0.5452	0.9909	1.1359	1.1869	1.3606	0.9909	1.1359	0.4744	0.5452
14.6262	16.1658	0.4901	0.5632	1.0237	1.1736	1.2263	1.4057	1.0237	1.1736	0.4901	0.5632
15.1611	16.7570	0.5080	0.5838	1.0611	1.2165	1.2711	1.4571	1.0611	1.2165	0.5080	0.5838
15.4736	17.1024	0.5185	0.5959	1.0830	1.2416	1.2973	1.4871	1.0830	1.2416	0.5185	0.5959
16.0465	17.7356	0.5377	0.6179	1.1231	1.2875	1.3453	1.5422	1.1231	1.2875	0.5377	0.6179
16.5224	18.2616	0.5536	0.6363	1.1564	1.3257	1.3852	1.5879	1.1564	1.3257	0.5536	0.6363
16.9917	18.7803	0.5694	0.6543	1.1893	1.3634	1.4246	1.6330	1.1893	1.3634	0.5694	0.6543
17.4468	19.2833	0.5846	0.6718	1.2211	1.3999	1.4627	1.6767	1.2211	1.3999	0.5846	0.6718
17.8743	19.7558	0.5989	0.6883	1.2510	1.4342	1.4986	1.7178	1.2510	1.4342	0.5989	0.6883
18.1935	20.1086	0.6096	0.7006	1.2734	1.4598	1.5253	1.7485	1.2734	1.4598	0.6096	0.7006

6. Modal Interval Analysis (MIA)

Modal Interval Analysis											
F (kN)		w3 (mm)		w5 (mm)		w7 (mm)		w9 (mm)		w11 (mm)	
RefFact100kN		Lvdt1452		Lvdt437		Lvdt443		Lvdt553		Lvdt550	
a	b	a	b	a	b	a	b	a	b	a	b
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4.7158	5.2122	0.1601	0.1812	0.3396	0.3717	0.4080	0.4437	0.3396	0.3717	0.1601	0.1812
5.1357	5.6763	0.1737	0.1981	0.3689	0.4052	0.4435	0.4841	0.3698	0.4048	0.1744	0.1973
5.6050	6.1950	0.1903	0.2154	0.4036	0.4418	0.4850	0.5274	0.4036	0.4418	0.1903	0.2154
6.0800	6.7200	0.2064	0.2336	0.4378	0.4792	0.5261	0.5721	0.4378	0.4792	0.2064	0.2336
6.5294	7.2167	0.2217	0.2509	0.4701	0.5147	0.5650	0.6144	0.4701	0.5147	0.2217	0.2509
7.0044	7.7417	0.2378	0.2691	0.5043	0.5521	0.6061	0.6591	0.5043	0.5521	0.2378	0.2691
7.5364	8.3297	0.2559	0.2896	0.5426	0.5940	0.6521	0.7091	0.5426	0.5940	0.2559	0.2896
8.0171	8.8610	0.2722	0.3080	0.5773	0.6319	0.6937	0.7544	0.5773	0.6319	0.2722	0.3080
8.4427	9.3314	0.2867	0.3244	0.6079	0.6655	0.7305	0.7944	0.6079	0.6655	0.2867	0.3244
8.9215	9.8606	0.3029	0.3428	0.6424	0.7032	0.7720	0.8395	0.6424	0.7032	0.3029	0.3428
9.4174	10.4087	0.3197	0.3618	0.6781	0.7423	0.8149	0.8861	0.6781	0.7423	0.3197	0.3618
9.8838	10.9242	0.3356	0.3798	0.7117	0.7791	0.8552	0.9300	0.7117	0.7791	0.3356	0.3798
10.3322	11.4198	0.3508	0.3970	0.7439	0.8144	0.8940	0.9722	0.7439	0.8144	0.3508	0.3970
10.8262	11.9658	0.3676	0.4160	0.7795	0.8533	0.9368	1.0187	0.7795	0.8533	0.3676	0.4160
11.2984	12.4877	0.3836	0.4341	0.8135	0.8906	0.9776	1.0631	0.8135	0.8906	0.3836	0.4341
11.7563	12.9938	0.3992	0.4517	0.8465	0.9267	1.0172	1.1062	0.8465	0.9267	0.3992	0.4517
12.2702	13.5618	0.4166	0.4715	0.8835	0.9672	1.0617	1.1546	0.8835	0.9672	0.4166	0.4715
12.7481	14.0900	0.4328	0.4898	0.9179	1.0048	1.1031	1.1995	0.9179	1.0048	0.4328	0.4898
13.2240	14.6160	0.4490	0.5081	0.9522	1.0423	1.1442	1.2443	0.9522	1.0423	0.4490	0.5081
13.6430	15.0791	0.4632	0.5242	0.9823	1.0754	1.1805	1.2837	0.9823	1.0754	0.4632	0.5242
14.1569	15.6471	0.4807	0.5440	1.0193	1.1159	1.2250	1.3321	1.0193	1.1159	0.4807	0.5440
14.6262	16.1658	0.4966	0.5620	1.0531	1.1529	1.2656	1.3763	1.0531	1.1529	0.4966	0.5620
15.1611	16.7570	0.5148	0.5825	1.0916	1.1950	1.3119	1.4266	1.0916	1.1950	0.5148	0.5825
15.4736	17.1024	0.5254	0.5946	1.1141	1.2197	1.3389	1.4560	1.1141	1.2197	0.5254	0.5946
16.0465	17.7356	0.5448	0.6166	1.1554	1.2648	1.3885	1.5099	1.1554	1.2648	0.5448	0.6166
16.5224	18.2616	0.5610	0.6349	1.1897	1.3023	1.4296	1.5547	1.1897	1.3023	0.5610	0.6349
16.9917	18.7803	0.5769	0.6529	1.2235	1.3393	1.4703	1.5988	1.2235	1.3393	0.5769	0.6529
17.4468	19.2833	0.5924	0.6704	1.2562	1.3752	1.5096	1.6417	1.2562	1.3752	0.5924	0.6704
17.8743	19.7558	0.6069	0.6868	1.2870	1.4089	1.5466	1.6819	1.2870	1.4089	0.6069	0.6868
18.1935	20.1086	0.6177	0.6991	1.3100	1.4341	1.5742	1.7119	1.3100	1.4341	0.6177	0.6991

APPENDIX G
STRUCTURAL DEFORMED SHAPE

STRUCTURAL DEFORMED SHAPE

In this Appendix it is presented the main procedures necessary to obtain the structural deformed shape of the concrete beam analyzed in laboratory, once the numerical output of the system is determined using one of the previously uncertainty analysis methodologies (Monte Carlo; Perturbation Technique; Modal Interval Analysis – MIA) or even using a deterministic analysis. For each vector of output results, and considering the necessary constraints, it is obtained the parameter values (A; B; C; D; E; F; G). In this situation a sixth degree polynomial is considered to represent the structural deformed shape. It is also presented the structural deformed shape, using the previous obtained unknowns for each vector of output results, for all load steps that were applied to the beam during the laboratory test (0.000 kN to 19.1510 kN). It is interesting, at this point, to compare the sensor obtained data with the possible range of admissible values given by the numerical model, and to observe the load at which crack starts to appear or, in a formal way, cracking phenomena is detected.

Calculation procedures of structural deformed shape

1. Constraints for Deformed Shape calculation

$$Y = Ax^6 + Bx^5 + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

CONSTRAINTS

$Y(x=0)=0$
$Y(x=0.35m)=w3$
$Y(x=0.70m)=w5$
$Y(x=1.05m)=w7$
$Y(x=1.40m)=w9$
$Y(x=1.75m)=w11$
$Y(x=2.10m)=0$

2. Matrix [A], and it inverse $[A]^{-1}$, of considered constraints

Matrix A

85.766	40.841	19.448	9.261	4.410	2.100
0.002	0.005	0.015	0.043	0.123	0.350
0.118	0.168	0.240	0.343	0.490	0.700
1.340	1.276	1.216	1.158	1.103	1.050
7.530	5.378	3.842	2.744	1.960	1.400
28.723	16.413	9.379	5.359	3.063	1.750

Matrix A(-1)

0.756	-4.533	11.333	-15.111	11.333	-4.533
-3.967	31.733	-75.365	95.198	-67.432	25.386
7.867	-86.075	190.199	-223.981	148.549	-52.756
-7.269	112.731	-224.004	241.011	-149.174	50.534
3.107	-71.020	119.358	-115.193	67.347	-22.041
-0.476	17.143	-21.429	19.048	-10.714	3.429

3. Values of parameters (A;B;C;D;E;F;G=0) for the correspondent constrains

Vector B

0.000
0.272
0.577
0.694
0.577
0.272

Vector A

-0.13 (A)
0.84 (B)
-2.38 (C)
3.79 (D)
-2.55 (E)
-0.26 (F)

0.000
0.308
0.632
0.754
0.632
0.308

-0.13 (A)
0.83 (B)
-2.33 (C)
3.72 (D)
2.44 (E)
-0.39 (F)

0.000
0.269
0.561
0.672
0.561
0.269

-0.13 (A)
0.80 (B)
-2.24 (C)
3.54 (D)
-2.35 (E)
-0.29 (F)

0.000
0.309
0.643
0.770
0.643
0.309

-0.14 (A)
0.87 (B)
-2.47 (C)
3.96 (D)
-2.64 (E)
-0.35 (F)

0.000
0.289
0.602
0.721
0.602
0.289

-0.13 (A)
0.84 (B)
-2.36 (C)
3.75 (D)
-2.50 (E)
-0.32 (F)

0.000
0.238
0.497
0.594
0.497
0.238

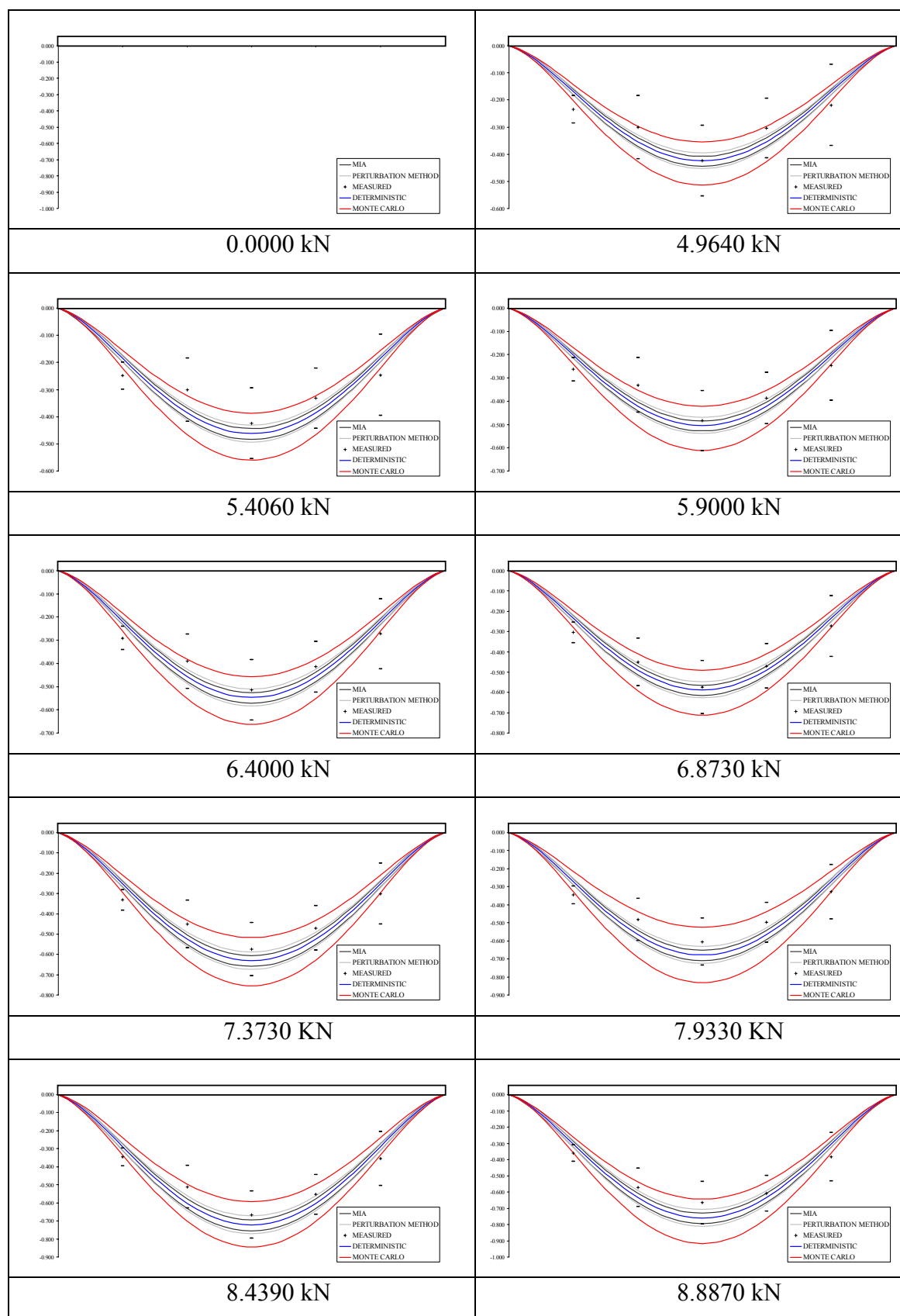
-0.13 (A)
0.83 (B)
-2.27 (C)
3.46 (D)
-2.24 (E)
-0.23 (F)

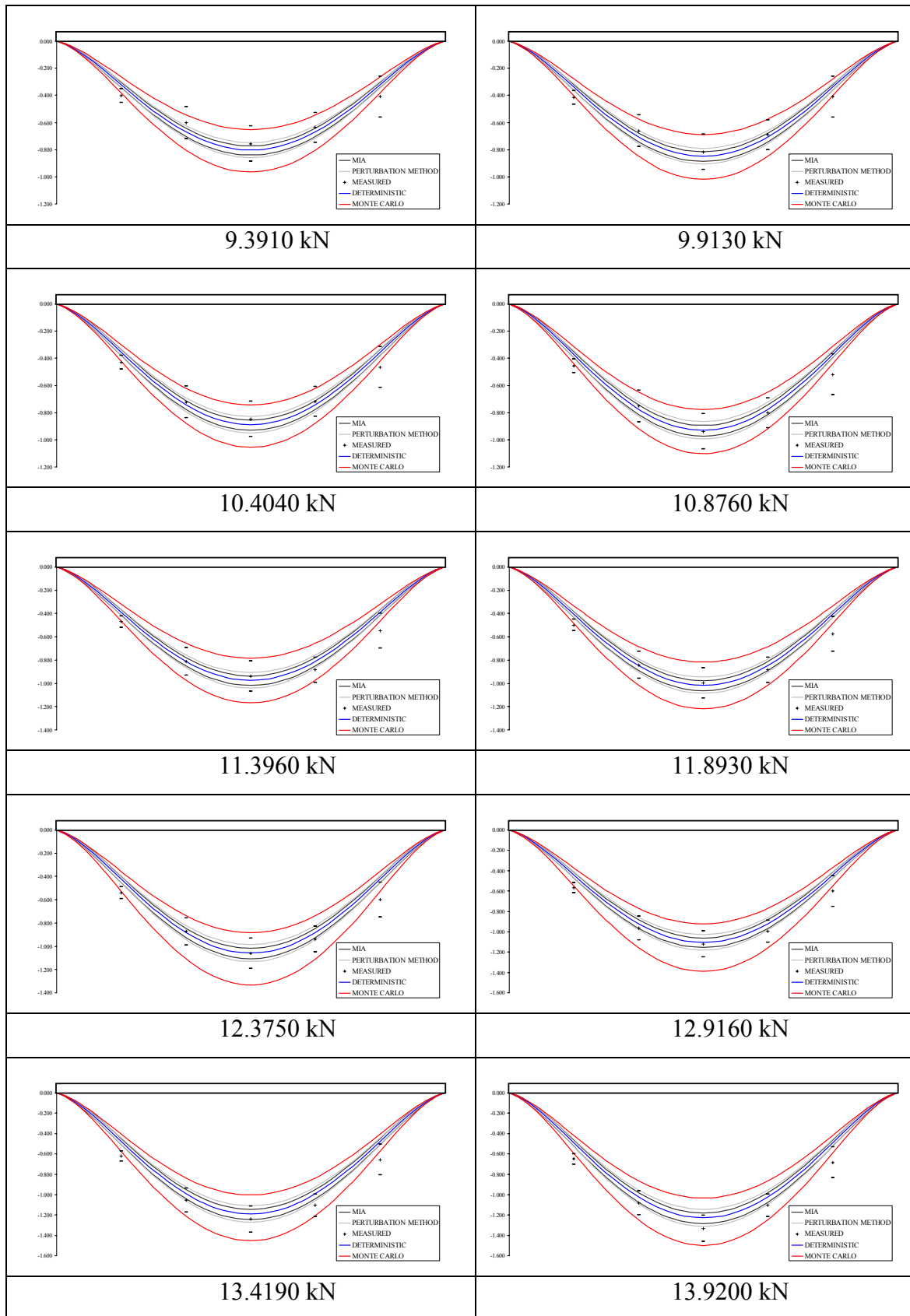
0.000
0.335
0.702
0.844
0.702
0.335

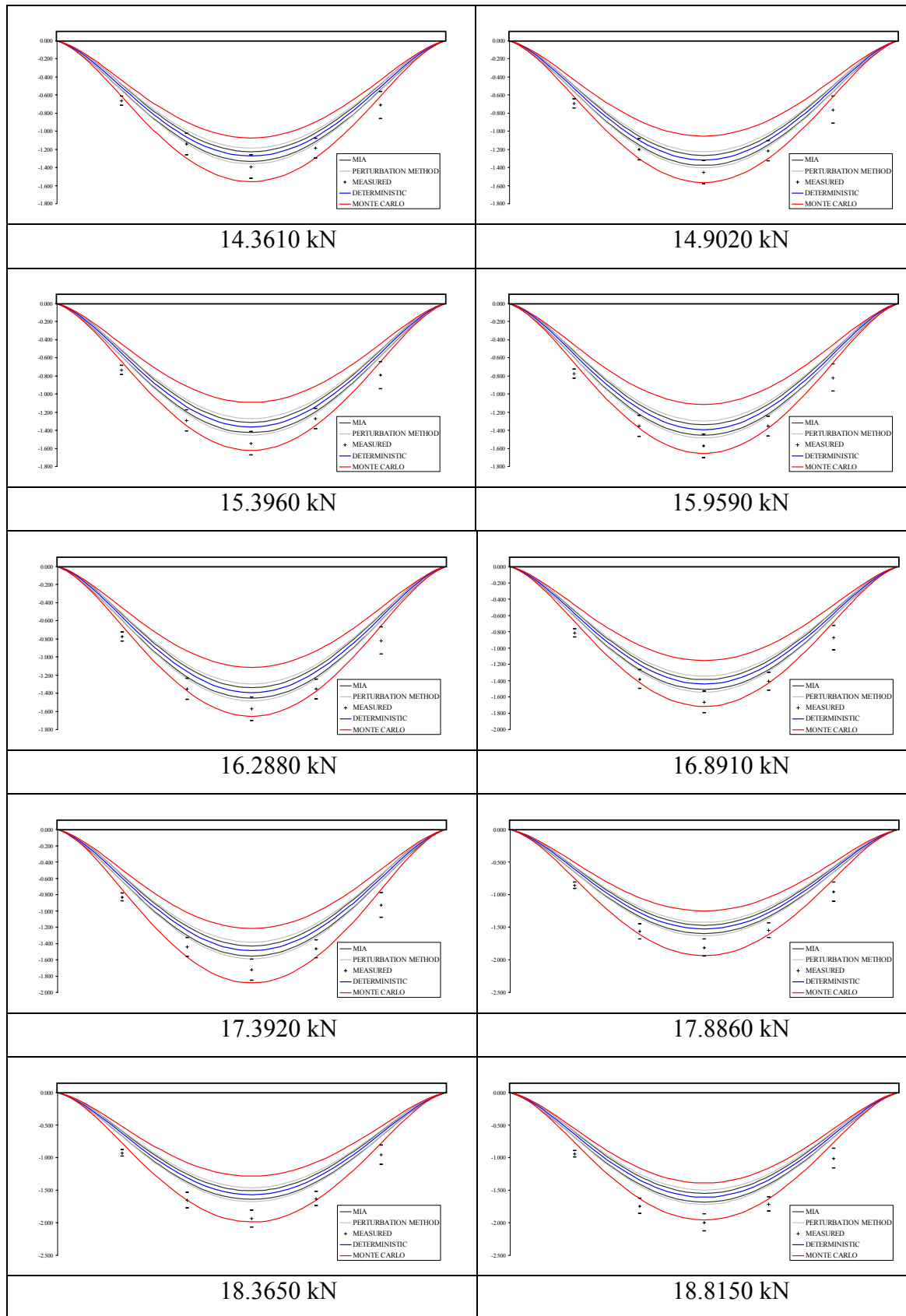
-0.12 (A)
0.77 (B)
-2.27 (C)
3.88 (D)
-2.70 (E)
-0.40 (F)

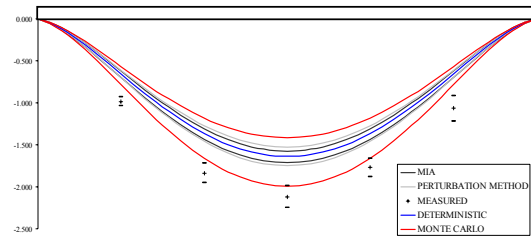
[illegible]

Structural deformed shape









19.1510 kN

