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Numerical and Experimental Analysis of Polymeric Foams obtained by 3D Printing (Additive Manufacturing)

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“Everything you want, you can have instantaneously - except job satisfaction and strength of relationships - there is no app for that.”

Simon Sinek

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Numerical and Experimental Analysis of Polymeric Foams obtained by 3D Printing (Additive Manufacturing)

by Nuno Alexandre Mota Ferreira Pereira Nascimento

A thesis submitted in fulfillment of the degree of
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The importance of cellular materials continues to increase in lightweight applications as more technologies realize these materials are becoming more reliable, repeatable and allowing for lower production costs. Among all the common structural applications of cellular architected materials, cores for sandwich panels may perhaps be the most important one, and therefore, will be the focus of this work.

Having widely adjustable architectures, periodic cellular materials naturally lend themselves to optimization. Despite some efforts for the establishment of analytical expressions for the behavior of honeycomb and foam cores, numerical analysis using dedicated software allow for better simulations and predictions. Therefore, a total of 3 methods were applied: the Finite Element Method (FEM) and two meshless methods, the Radial Point Interpolation Method (RPIM) and the Natural Neighbour Radial Point Interpolation Method (NNRPIM).

Two different geometries were studied at the micro-scale, namely the conventional honeycomb and the re-entrant honeycomb. For the linear-elastic domain, they were tested in both x and y directions (in-plane), and the effect of the variation of the relative density was studied, considering a total of 7 relative densities for each. For the elastoplastic domain, the same relative densities were studied, in the y direction, using the Newton-Raphson method for non-linear solution algorithm. Finally, using an homogenization process, multi-scale models were created considering the properties which were obtained from the studied unit cell geometries.

The numerical results should be verified experimentally; hence, prototypes were built and tested, with varying parameters in the cellular cores of sandwich structures. Since additive manufacturing enables the fabrication of these materials with intricate cellular architecture, in this work, 4 different types of specimens were built of a PLA polymer using the FDM 3D printing technique, and tested in compression. The comparison between the obtained results from the numerical analysis, using the FEMAS software, and the experimental analysis is documented.

Keywords: meshless methods, sandwich structures, polymeric foams, additive manufacturing, numerical analysis, elastoplastic analysis, experimental analysis.

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por Nuno Alexandre Mota Ferreira Pereira Nascimento

Tese apresentada à Faculdade de Engenharia da Universidade do Porto, para a obtenção do grau de Mestre em Engenharia Mecânica, sob a orientação de:

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A importância dos materiais celulares sólidos continua a aumentar em aplicações de baixo peso à medida que as indústrias verificam que estes se estão a tornar mais seguros e reprodutíveis com menores custos de produção. De entre várias aplicações estruturais comuns de materiais celulares sólidos, o interior de estruturas em sanduíche será talvez o mais importante, e como tal, será o foco deste trabalho.

Com diversos parâmetros ajustáveis, a utilização de materiais celulares sólidos está naturalmente associada à optimização. Apesar da existência de algumas expressões analíticas para o comportamento de *honeycombs* e espumas, uma análise numérica utilizando software dedicado permite realizar simulações que conduzem a melhores previsões. Assim, foi aplicado um total de 3 métodos: o Método dos Elementos Finitos (FEM) e dois métodos sem malha, o "*Radial Point Interpolation Method*" (RPIM), e o "*Natural Neighbour Radial Point Interpolation Method*" (NNRPIM).

Duas geometrias diferentes foram estudadas a nível da micro-escala, nomeadamente a *honeycomb* convencional e a *honeycomb* re-entrante. No domínio elástico, estas foram testadas nas direções x e y (no plano) e foi estudado o efeito da variação da densidade relativa, considerando 7 densidades relativas para cada. No domínio elastoplástico, as mesmas densidades relativas foram testadas, na direção y , utilizando o método de Newton-Raphson aplicado a algoritmos de solução não linear.

Finalmente, utilizando um processo de homogeneização, foram criados modelos multi-escala com as propriedades obtidas do estudo das células unitárias.

Os resultados numéricos devem ser validados experimentalmente; assim, foram elaborados e testados provetes, variando os parâmetros associados às estruturas em sanduíche. Dado que o fabrico aditivo permite a obtenção de materiais com estruturas celulares de arquitetura selecionada, neste trabalho, 4 tipos diferentes de provetes foram fabricados com um polímero de PLA, e com a técnica de impressão FDM (*fused deposition modeling*), e testados à compressão. A comparação dos resultados obtidos na análise numérica, utilizando o FEMAS, e a análise experimental é aqui documentada.

Palavras-chave: métodos sem malha, estruturas em sanduíche, espumas poliméricas, fabrico aditivo, análise numérica, análise no domínio elastoplástico, análise experimental.

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Chapter 1

Introduction

1.1 The application of polymeric foams

Since the beginning of times, nature has created cellular materials with complex and intricate architectures for many situations in which low density as well as high stiffness and strength were needed. Likewise, in a world of fast progress, such as today's world, the need for lightweight (yet sufficiently strong) structures has significantly increased in various applications, such as engineering applications where the weight is a determinant factor [1].

Therefore, humankind has developed many materials which are clearly mechanically superior to the original biological base materials; however, the cellular architectures developed so far are much less sophisticated than the ones created by nature, and this is an area that may be explored in our favour.

At the simplest level, there are the honeycomb-like materials, which are still the state-of-the-art core for sandwich structures; and in terms of materials, polymeric foams are nowadays used in everything, from common disposable coffee cups to the crash padding of an aircraft cockpit. The use of these foams not only for structural applications but also for thermal and acoustic insulation, cushioning, energy absorption, etc. exploits the unique combination of properties offered by cellular solids, properties which, ultimately, derive from the cellular structure [2].

Therefore, the study of periodic cellular structures is of great interest. The mechanical response of these solids can be modeled by using numerical methods, such as the Finite Element method (FEM) or Meshless methods.

1.2 Meshless methods

In spite of the fact that the FEM is a widely used tool nowadays for the simulation and analysis of structures, there are other more recently developed methods which are capable of meeting the design needs of an engineer, and may as well offer some advantages concerning the reproduction of the studied phenomenon.

The basic concept behind the Finite Element Method is the discretization of the analysed problem into smaller parts, called elements [3]. It is the association of these elements that originates a full mesh, where the nodal connectivity can be found. However, this method presents some limitations, namely in problems involving large deformations, where it becomes hard to ensure the nodal connectivity without highly distorting the mesh (which might end up leading to inaccurate results) or crack propagation problems, just to name a few. While re-meshing can be a solution to this problem, the main drawbacks are the associated time consumption and the high computational costs.

In order to solve some of the aforementioned issues, meshless methods appeared around 1977 with the introduction of the Smooth Particle Hydrodynamics Method [4], and then extended in terms of application to solid mechanics problems a few years later, with the first global weak form based meshless method being presented in 1994 as the Element Free Galerkin Method [5].

In opposition to the Finite Element Method, in the meshless methods the concepts of mesh or elements are non-existent. Instead, in these methods, the nodes can be arbitrarily distributed and the field functions are approximated within an "influence domain", rather than an element [6]. This can be a influence-domain consisting of an area or volume, which can be concentric with an interest point or an "influence-cell", which is constructed in the problem's domain, resulting in a node dependent integration background mesh [7]. In addition to that, while in the FEM there is a no-overlap rule between the elements, in the formulation of meshless methods, the nodal connectivity is imposed by the actual overlap of these influence-domains [6].

Therefore, a meshless method can be classified, regarding the used numerical integration, as a truly meshless method or a not truly meshless method. The former makes use of an integration mesh which is constructed in the problem's domain, considering the nodal discretization, while the latter uses a background integration mesh (with Gauss-Legendre integration points) to perform the numerical integration, which, in fact, eliminates the literal mesh-free characteristic of the method [6]. An example of a not truly meshless method is the Radial Point Interpolation Method (RPIM), and an example of a truly meshless method is the Natural Neighbour Radial Point Interpolation Method (NNRPIM).

1.3 Additive Manufacturing

The recent advances in computational power and computational methods now enable the optimization of an architected material to meet all desired objectives. In parallel, there is a notable progress in additive manufacturing technologies, which, in turn, enable the creation of complex materials with location-specific architectures [2].

When referring to polymers in particular, this technology has demonstrated the ability to fabricate architectures that are impossible to fabricate through conventional processing techniques [8], [9].

Therefore, the same engineering and architectural principles that have been thoroughly used to increase the mechanical efficiency of many structures can now be applied at the material level.

As the complexity of lightweight structures continues to grow with ever-increasing structural, aerodynamic, and packaging requirements, performance and weight benefits may be achieved using complex-shaped sandwich cores with spatially graded properties.

Thus, using additive manufacturing techniques to produce the cores of sandwich structures offers several advantages over the conventional foam manufacturing, including the ability to form net shape cores, to create cores with complex curvature, and the possibility of designing cores with graded cellular architectures, in such a way that the target application is maximally enabled.

With the recent advances in additive manufacturing and its continuous, rapid evolution, the production limits are progressively vanishing.

1.4 Thesis objectives

The main purpose of this thesis is to perform, for the first time, a numerical structural analysis of polymeric honeycombs (used as the core of sandwich structures) using a total of three methods: the Finite Element Method (FEM), and two recently developed meshless methods: the Radial Point Interpolation Method (RPIM) and the Natural Neighbour Radial Point Interpolation Method (NNRPIM). The numerical results shall be then validated through an experimental analysis.

Therefore, the objectives of this work include:

- Defining a state of the art concerning cellular solids, namely honeycombs and foams (which constitute the core of sandwich structures) applied in lightweight structural applications;
- Understanding the principles and foundations of the meshless methods used in the numerical analysis;
- Understanding the behaviour of sandwich structures and the effects of varying the geometry and relative density of their cores;
- Developing 2D honeycombs cells using a CAD software and obtaining the stress/strain curves by isolating one inner honeycomb at a micro scale, as well as correlations of the mechanical properties with the honeycomb density in the form of phenomenological curves;
- Obtaining the mechanical properties of the honeycomb designs both for the linear-elastic domain and the elastoplastic domain, considering a bi-linear model.
- Repeating the whole process for the re-entrant honeycomb design, as an alternative for the conventional honeycomb design.
- Creating multi-scale models using an homogenization technique;
- Producing 3D specimens containing honeycomb structures in the core with Additive Manufacturing, for different relative densities;
- Validating the multi-scale model and the numerical analysis with the experimental testing of the corresponding 3D printed specimens.

1.5 Thesis outline

This thesis is divided in 8 chapters: Introduction, Cellular Solid Materials, Meshless Methods, Continuum Mechanics of Plasticity, Mechanics of honeycombs and foams, Additive Manufacturing, Numerical and Experimental Analysis, and Conclusions and Future works.

In the first chapter, **Introduction**, the purposes of the thesis are presented. A brief overview on cellular materials, meshless methods and additive manufacturing is given.

In chapter 2, **Cellular Solid Materials**, a description about the constitution and potentialities of this type of solids, with focus on the cores of sandwich structures, is given.

In chapter 3, **Meshless Methods**, the two used methods - RPIM and NNRPIM - are described and their formulation is presented. The necessary Solid Mechanics fundamentals are presented and explained.

In chapter 4, **Plasticity**, the foundation concepts for the elastoplastic numerical analysis are explained.

In chapter 5, **Mechanics of honeycombs and foams**, the analytical expressions obtained throughout the years are reviewed. These allow for a better understanding and prediction of the numerically and experimentally analyzed specimens.

In chapter 6, **Additive Manufacturing**, a short overview on this process is given, focusing on the FDM technique and the manufacturing of sandwich structures with polymeric foam cores.

In chapter 7, **Numerical and Experimental Analysis**, the numerical simulations are described thoroughly and the obtained results are presented, as well as validated by the experimental analysis.

In chapter 8, **Conclusions and Future work**, the conclusions retained from this work are summarized, and suggestions for future works in this context are given.

Chapter 2

Cellular Solid Materials

2.1 An introduction to cellular solid materials

A cellular solid material is one made up of an interconnected network of solid struts or plates which form the edges and faces of the cells [10]. Nature evolved architected cellular materials for many situations in which low density as well as high stiffness and strength are needed. Examples are beaks and bones of birds that consist of thin, solid skins attached to a highly porous, cellular core (Figure 2.1). The architecture of the core is complex, with intricately shaped ligaments and gradients in density [2].

Likewise, the need for lightweight structures has significantly increased in many engineering applications: for instance, in any transportation machine, the lower the weight of a machine, the higher the payload that can be delivered at the same expense: the lighter the airplane structure, the lower is the fuel spending per passenger, etc. In the case of a product to be delivered from the point of its fabrication to a store point (and further on, to a consumer), the lighter an equipment producing a given quantity of a useful performance, the cheaper is its delivery - and consequently, the higher is the profit [1].

Although humankind has developed synthetic materials which are mechanically superior to the biological base materials, the cellular architectures developed by humans are much less sophisticated than the ones created by nature. At the simplest level, there are the honeycomb-like materials, made up of parallel, prismatic cells, used for lightweight structural components. Also familiar are the polymeric foams used in everything, from disposable coffee cups to the crash padding of an aircraft cockpit. The use of these foams for thermal and acoustic insulation, cushioning, energy absorption, etc. exploits the unique combination of properties offered by cellular solids, properties which, ultimately, derive from the cellular structure [2]. Nevertheless, this work will be focused entirely on the structural aspects of these cellular materials.

2.2 Architected cellular materials - mechanics and potentialities

Architecture has always played an important role in most large and common structures around us, such as buildings or wheels. For example, in modern bicycle wheels, which have a superior architecture (in which more than 95% of the material has been replaced with air), excellent strength and stiffness are achieved despite the light weight, because the spokes are loaded in tension, where their performance is stretch dominated, and are not loaded in compression, where they would buckle and bend easily. This same concept is used to reduce the mass of materials in many

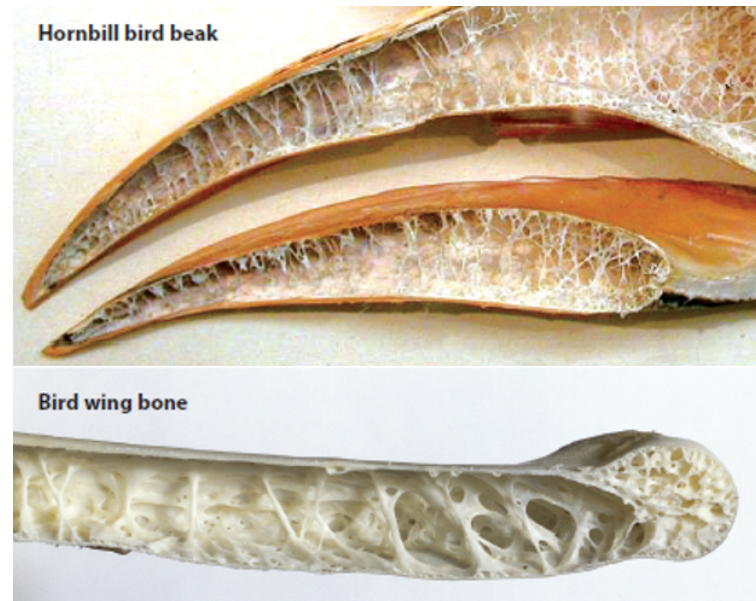


FIGURE 2.1: Architected cellular materials in the core of bird beaks and bones provide optimum strength and stiffness at low density [2].

weight-critical applications; for example, the cores of surfboards and rotor blades are not solid but are made from foam and honeycomb, respectively - Figure 2.3 [11], [12]. The same engineering and architectural principles that have been used to increase the mechanical efficiency of these structures can be applied at the material scale [2]. Figure 2.2 shows general scenarios for adding architecture to materials.

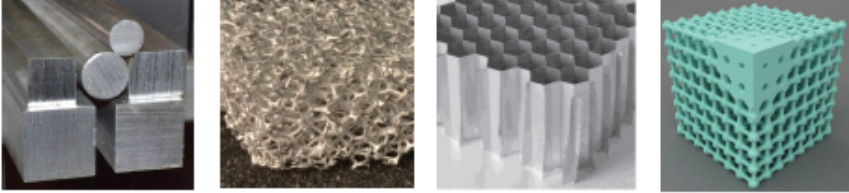
Cellular architecture				
	None	Random	Ordered	Ordered and location specific
Properties	Continuous and homogeneous	Homogeneous at scales > cell	Homogeneous and highly anisotropic	Inhomogeneous and highly anisotropic
Design degrees of freedom	Solid constituent	Solid constituent, cell size	Solid constituent, cell size, and orientation	Cell size/shape, node topology, ligament shape, material...

FIGURE 2.2: Adding architecture to materials [2].

The single most important feature of a cellular solid is its relative density, ρ^*/ρ_s ; that is, the density of the cellular material, ρ^* , divided by that of the solid from which the cell walls are made, ρ_s [13]. Thus, the properties of cellular materials are determined not only by the solid constituents but also by the spatial configuration of voids and solids—that is, the cellular architecture [2]. This is easily observed in engineering honeycombs and foams. Honeycombs, with their prismatic cells, are two-dimensional cellular solids, while foams, with their polyhedral cells, are three-dimensional cellular solids – Figure 2.4. Periodic honeycombs are straightforward

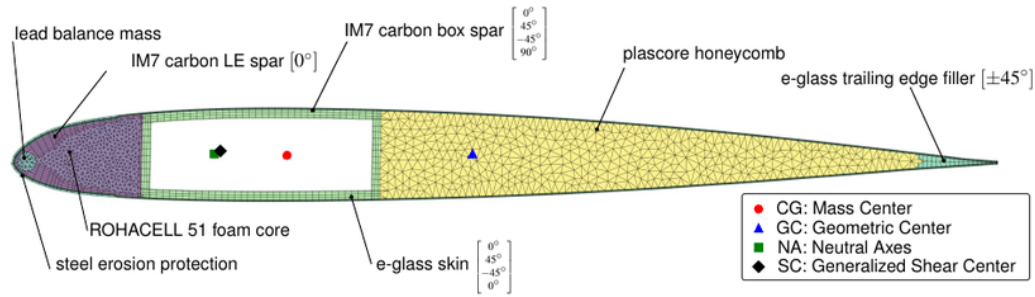


FIGURE 2.3: Cross section view of a composite rotor blade [11].

to analyze; insight gained from them can be applied to the analysis of foams, as both generally deform and fail by the same mechanisms. Open-cell foams are solid only at the edges of the polyhedral, while closed-cell foams have solid membranes over the faces of the polyhedral. The properties of the foam depend on those of the solid from which the honeycomb or foam is made [13], [10].

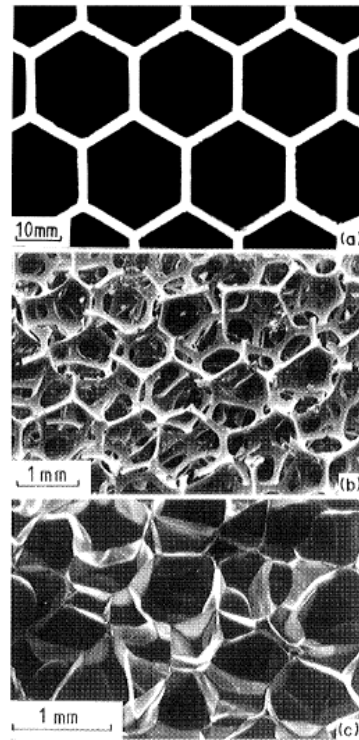


FIGURE 2.4: Examples of engineering cellular solids: (a) two-dimensional honeycomb; (b) three-dimensional foam with open-cells; (c) three-dimensional foam with closed-cells [10].

The stress-strain curve for a cellular solid in compression is characterized by three regimes (see Figure 2.5(a)): a linear elastic regime, corresponding to cell edge bending or face stretching; a stress *plateau*, corresponding to the progressive cell collapse by elastic buckling, plastic yielding or brittle crushing, depending on the nature of the solid from which the honeycomb or foam is made; and a final regime of densification, corresponding to the collapse of the cells throughout the material and subsequent loading of the cell edges and faces against one another. Cellular solids typically have low relative densities (10-20%), usually allowing large strains (70-80%) before densification occurs. In tension, at small strains, the linear elastic

response is the same as in compression (see Figure 2.5(b)). As the strain increases, the cells become more oriented with the loading direction, increasing the stiffness until tensile failure occurs [13].

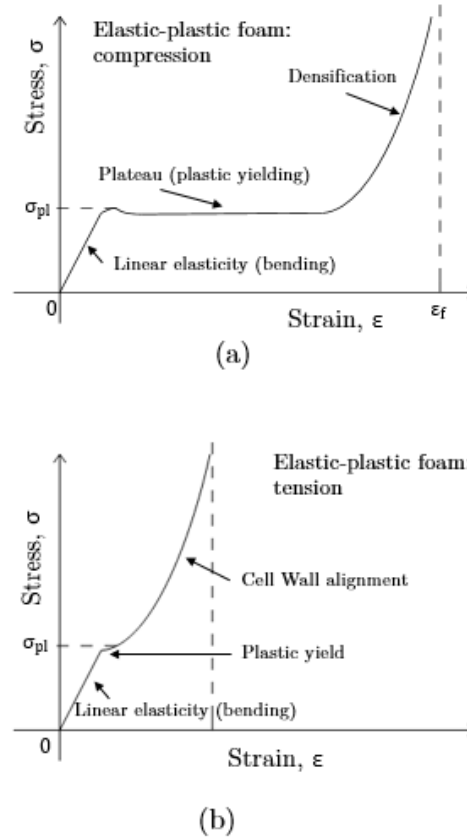


FIGURE 2.5: Schematic uniaxial stress-strain curves for (a) elastic-plastic foam in compression and (b) elastic-plastic foam in tension. Adapted from [13].

The mechanical response of cellular solids has been modeled by representing the cellular structure in several ways. Initial models analyzed a unit cell such as a hexagon in two dimensions, or a dodecahedron (a 12-faced polyhedron) in three dimensions (Figure 2.6). The geometry of the unit cell makes the analysis tractable but may not give an exact representation of the real material (e.g. foams). An even simpler approach is to use dimensional analysis to model the mechanics of deformation and failure observed in the cellular material without specifying the exact cell geometry. This approach assumes that the cell geometry is similar in foams of different relative densities. It gives the dependence of the foam properties on the relative density and the solid properties, but requires experiments to determine the constants related to the cell geometry [10]; finally, a third approach is to use numerical methods, such as the finite element method or meshless methods - both used in this work - in order to analyze either regular or random cellular structures [13].

In three dimensional foams, the random cellular architecture of these foams results in bending-dominated deformation of the ligaments, thus having in a rapid decrease in strength and stiffness as porosity is increased. [10], [2]. On the other hand, certain ordered lattice-type cellular architectures can have nearly optimal stretching-dominated properties, resulting in materials in which the strength and stiffness scale

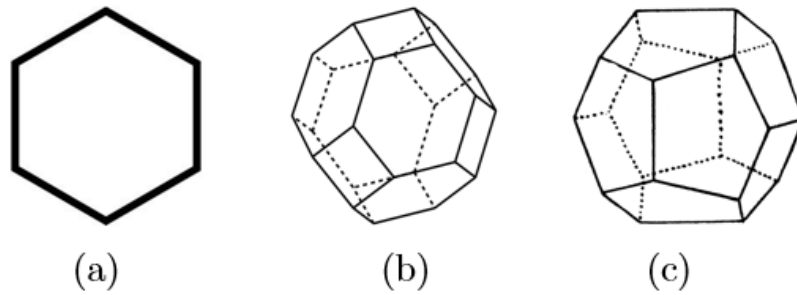


FIGURE 2.6: Different unit cells: (a) hexagon; (b) pentagonal dodecahedron; and (c) tetrakaidekahedron.

proportional with the solid volume fraction of the material. The first comprehensive treatise of mechanics of cellular materials was provided in 1997 by Gibson and Ashby [10], who describe the mechanics of foams and honeycombs in detail [2]. A reference to this work will be made in Chapter 5.

Various ordered cellular architectures with improving mechanical properties were studied shortly after, such as the octet truss lattice; Figure 2.7 illustrates the influence of cellular architecture on the scaling of mechanical properties. At very low densities, even small differences in scaling can have a large effect on strength and modulus. Foams and honeycombs, the conventional cellular materials, scale far from ideal: foams exhibit poor scaling due to their bending-dominated architecture, whereas honeycombs exhibit a linear scaling until the failure mode shifts from yielding to cell wall buckling - which will be explained as well in more detail throughout Chapter 5. Honeycombs are highly anisotropic and exhibit very little strength and stiffness in the lateral directions. The octet truss can outperform both foam and honeycomb. However, difficulties in manufacturing have limited the applications of such octet truss materials. Additive manufacturing offers new avenues to realize complex cellular architectures, including octet truss lattices [14], [15], [16], [17].

Furthermore, architected materials not only have the advantage of allowing higher strength and stiffness despite their light weight (thus increasing the mechanical efficiency of structures), but they also offer unprecedented opportunities for multifunctionality [18]. By tailoring the architecture, the properties can be varied to match location-specific requirements. Figure 2.8 shows a simple example: The relative density of the core of a wing can be varied by adjusting pore and ligament dimensions to increase strength where necessary (e.g., in areas exposed to higher air loads or impact), whereas the density is minimized in other locations in order to save mass; and additional functions can be addressed as well - for instance, the core architecture can be tailored to ensure that the center of gravity is aligned with design targets, or the open cellular architecture can be optimized for thermal management. Another function that can be fulfilled by architected cellular materials is energy storage, specifically liquid fuel storage in open-celled porosity and chemical energy storage in structural batteries [2].

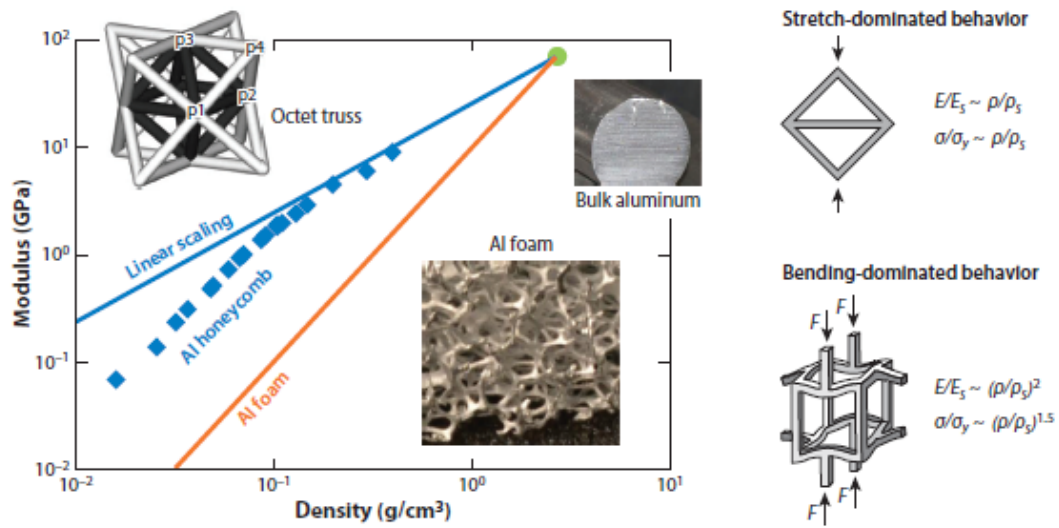


FIGURE 2.7: Influence of cellular architecture on the scaling of mechanical properties with density exemplified by aluminum foam, honeycomb, and octet truss [2].

2.3 Sandwich structures

Among all the possible structural applications of architected materials, cores for sandwich panels could be the most important one. Sandwich structures are widely used in lightweight design, especially in aerospace, but also increasingly in sports and automotive applications [12]. Covering the surface of the architected material with a thin and stiff face sheet has implications for the material's mechanical properties. Sandwich panels are typically loaded in bending, which subjects the face sheets to compression and tension, whereas the core is loaded in shear (Figure 2.9). The different architectures can be optimized for specific loads by identifying the failure modes and tailoring the architecture to suppress the performance-limiting mechanisms.

When the yield strength of all commercially available materials is plotted versus their density, an illustration of the material property space that can be readily analyzed is obtained - Figure 2.10 [19].

Note that the white space in this material property plot means that no materials are available with this property combination. Especially at low densities (less than 1 g/cm^3) there are very few options available — only foams, honeycombs, and some natural materials like balsa wood and cork [20].

Nevertheless, through the design of lattice structures, the "property space" of a known material can be extended. For instance, fabricating nickel microlattices extends the property space that can be accessed with nickel across a wide range of density, modulus, and strength. The intrinsic property of the constitutive material defines the upper bound because linear scaling with density is the best possible scenario. The architecture will, then, define the extent to which the properties deviate from the ideal scaling.[2]

In most cases, in which architected cellular materials are bearing a load, face sheets are attached to the outward-facing surfaces. By attaching thin, stiff face sheets to the top and bottom surfaces of a relatively thick, lightweight core, mechanically

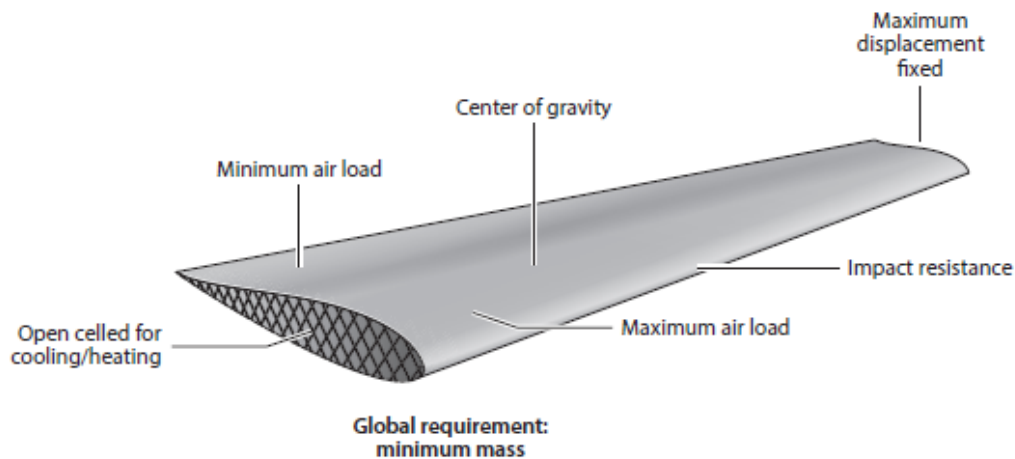


FIGURE 2.8: Multiple requirements can be fulfilled with a continuous architected material by tailoring the cellular architecture to local requirements [2].

efficient, lightweight sandwich panels that are widely used in the aerospace industry can be created. The thickness of the core separates the face sheets, thus providing the structure with a high area moment of inertia. In a sandwich structure, the face sheets carry in-plane and bending loads, whereas the core carries transverse shear and transverse compression loads. The honeycomb cellular architecture is the current state-of-the-art sandwich core; aluminum alloys, aramid fiber and glass fiber-reinforced polymers represent the most common constituent materials. Closed-cell polyurethane or PVC foam is an alternative with generally lower performance [21]. More recently, alternative core materials have been studied, among which the most compelling are tetrahedral and pyramidal truss structures [22]. Hollow truss structures, when designed to suppress buckling, offer an opportunity for improved compressive and shear strengths of honeycomb foams [2].

There are various design factors that need to be taken into account during manufacturing of sandwich structures in order to obtain the desired mechanical properties. For instance, aluminum foam cores can be used for energy absorption purposes, which makes them useful specifically in impact-related applications against dynamic loads [23], [24], whereas polymer foams with fiber skins can be used to obtain a composite material having extremely high flexural resistance to weight ratio [1].

When compared to metal foams, polymer foam core materials represent the best candidates whenever designing the weight critical components in various industrial applications, and provide high flexural or bending strength as well. Other factors can be considered as the type of matrix, fiber/matrix ratio, fiber orientation, number of fiber plies on each layer, core density or core thickness. [25], [26], [27], [28].

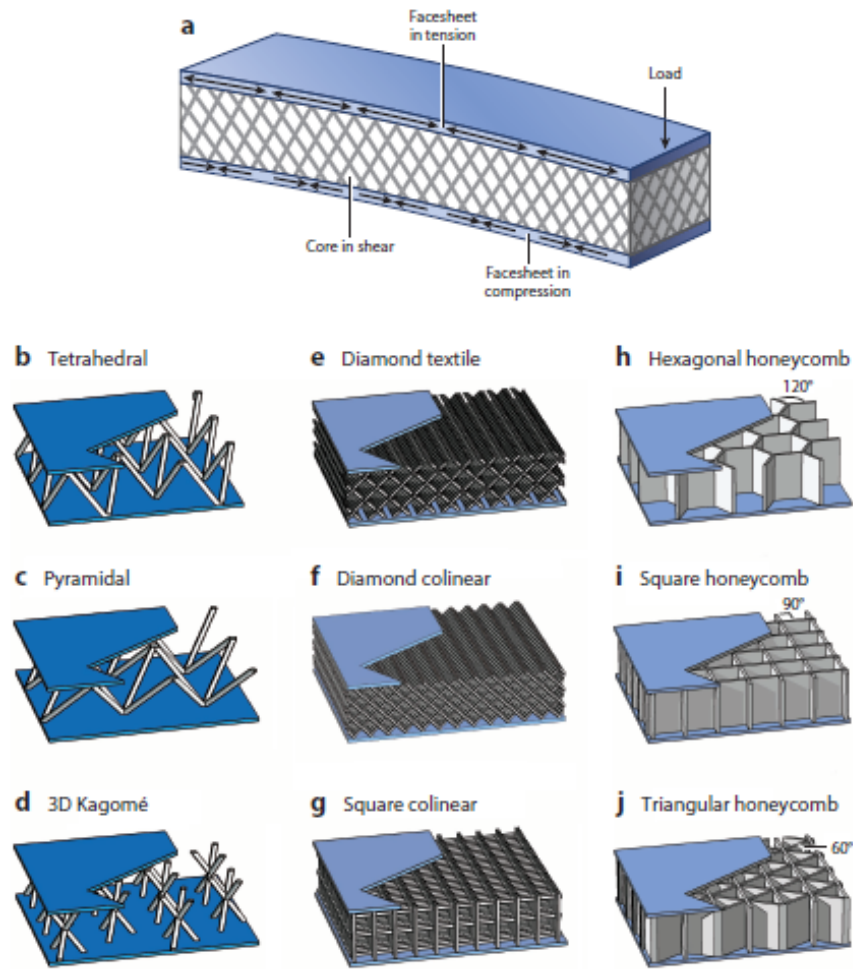


FIGURE 2.9: (a) Illustration of a sandwich structure commonly used in lightweight design under loading. (b–j) Examples of sandwich panels with periodic cellular metal cores [2].

2.4 Optimization of architected cellular materials

Ultimately, to take full advantage of the unique freedom in designing with architected materials, the architecture itself should be a variable in the optimization process. Various topology optimization algorithms have been developed to define the ideal arrangement of two or more materials to achieve optimum macroscopic properties. Although this work will not be mainly focused on the algorithms for topology optimization, some insight on the whole optimization process will be given here, specifically on the homogenization process.

A good example of a topological optimization process is presented in [8]. Homogenization is used to derive properties representative of these structures through direct numerical simulation of unit cell models. The resulting homogenized properties are then used assuming uniform distribution of the cellular structure to compute the macroscale structure. Specifically, periodic cellular structures can be characterized by Representative Volume Elements - RVEs - that capture the evident repeated structure in the material. When the RVE is of a size much smaller than the part they compose, homogenization theory can be employed to provide representative material properties of the RVE, which may be used to accurately describe the macroscale

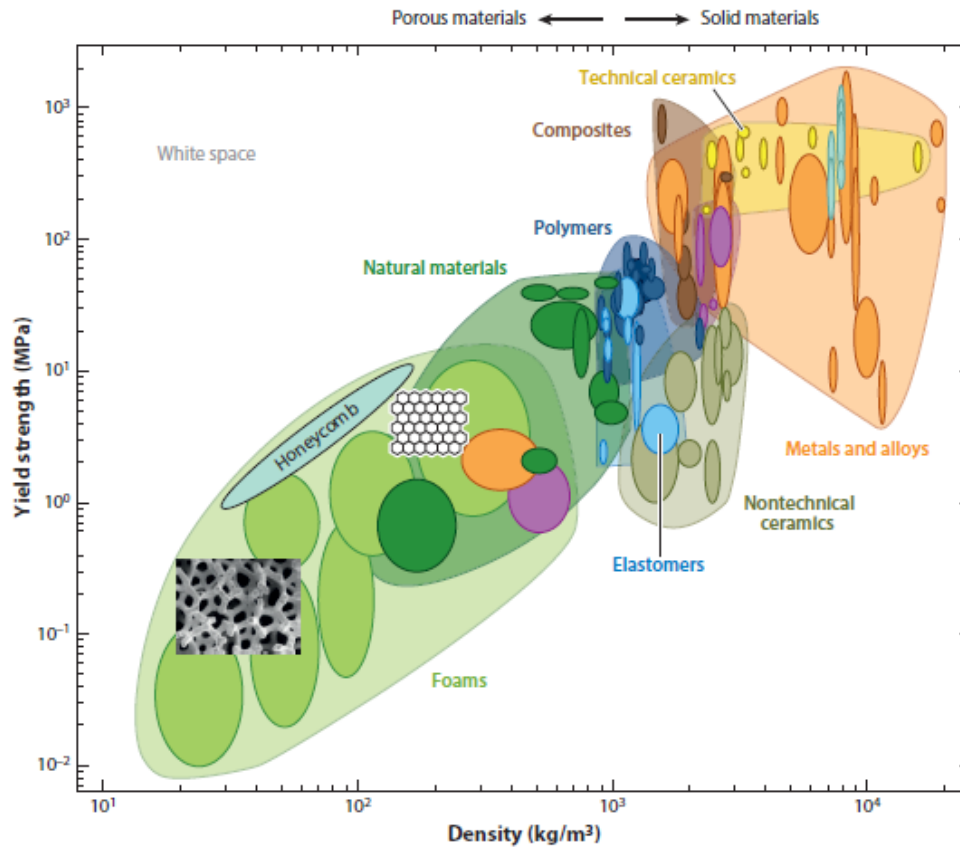


FIGURE 2.10: Material property space obtained by plotting yield strength versus density of all commercially available materials with the CES Selector database [19].

response of the structure. [8], [29], [30], [31].

Following the taxonomy of Wadley [8], cellular structures fall into one of two broad categories: stochastic and periodic. Stochastic structures are those where the cellular structure is randomly distributed. While this cellular structure is among the easiest to manufacture using traditional techniques, such as foaming and sintering, its random layout and varying density hinders the control of mechanical properties and, consequently, it does not allow to maximize its performance [13]. It becomes clear that better control of material properties are possible with periodic structures but their fabrication with traditional approaches is limited [8], [32], [17].

Thus, although conventional manufacturing approaches are still available and dominate the fabrication of these cellular materials, their use becomes problematic for more complex cell geometries such as lattice structures [10], [17]. These manufacturing limitations may be overcome by using Additive Manufacturing techniques, which can build parts that are nearly impossible to realize with traditional subtractive processes due to (for example) geometric and material complexities accessible by these methods. [14], [15], [16].

With widely adjustable architectures, these periodic cellular materials naturally lend themselves to optimization. In addition to the selection of the ideal constitutive (base) material (or combination of materials in the case of a hybrid), multiple parameters of the cellular architecture (such as pore size, ligament diameter, ligament orientation) can be optimized for a specific objective or multiple objectives, subject to a few constraints.

Therefore, a general procedure for optimizing architected materials can be summarized as follows [33]:

1. The fundamental properties of the base material(s) are chosen.
2. The variables of interest (e.g., strength, stiffness, heat transfer) are analytically modeled as a function of the architecture and the applied loads (e.g., mechanical, thermal);
3. The constraints for the application under consideration—including manufacturing constraints, geometric constraints (e.g., no buckling), and environmental constraints (e.g., no melting)—are formulated;
4. Numerical analyses are performed (using, for example, the finite element method or meshless methods) to verify the validity of the assumptions made in steps 2 and 3;
5. An objective function is established and then optimized with an optimization routine subject to the constraints established in step 3.

Examples of works featuring optimizations of architected sandwich cores optimized for mechanical properties include [8], [34], [35], [29], [30], [31] and [36]. A common conclusion is that the integration of finite element analyses within the optimization loop usually becomes computationally intense, and therefore, it remains as a challenge for the optimization process. Thus, in this work, numerical analyses will be, in fact, integrated, but meshless methods will be applied as well, as they generally allow for faster yet precise simulations.

Advantages of meshless methods in general include: [6]

- These methods can provide more accurate approximations for structures with complex geometries when compared with the FEM;
- The test functions are constructed in terms of higher-order continuous weight functions and possess compact support;
- Possessing significant advantages in handling problems with large deformations, moving discontinuities and phase transformation in the development of advanced materials.
- Nodes can be easily added (h-adaptivity) on portions where refinement is needed for a higher solution accuracy.
- The coupling possibility with FEM or boundary element methods (BEM) to avoid their inherent drawbacks.

Chapter 3

Meshless Methods

Throughout this chapter, two meshless methods are highlighted: the Radial Point Interpolation Method (RPIM) and the Natural Neighbour Radial Point Interpolation Method (NNRPIM). First, the generic procedure of a meshless method is outlined. Then, the RPIM and the NNRPIM are introduced along with the concepts of influence-domain and nodal connectivity, as well as the integration scheme and the interpolations functions that are used in both methods. For the NNRPIM, the notion and construction of influence-cell through Voronoi diagrams is presented.

3.1 Generic procedure of a meshless method

Most meshless methods follow a standard procedure, which begins by defining the problem's geometry, the solid's domain and contour, and then the essential and natural boundary conditions. At this point, the problem's domain and boundary are discretized in a nodal mesh, which can be regular or irregular - Figure 3.1.

Similarly with mesh dependent discretization numerical methods, in meshless methods the nodal density of the discretization and the nodal spatial distribution have an influence on the method's performance. On one hand, a fine nodal distribution generally leads to more accurate results; however, the computational cost grows with the increase of the total number of nodes. On the other hand, the unbalanced distribution of the nodes discretizing the problem domain can lead to a lower accuracy, but it is useful for locations with predictable stress concentrations, such as crack tips [6].

Subsequently, a background integration mesh is constructed. The integration mesh can be nodal dependent (NNRPIM) or nodal independent (RPIM), and it can have the size of the problem domain or even a larger one, without affecting too much the final results.

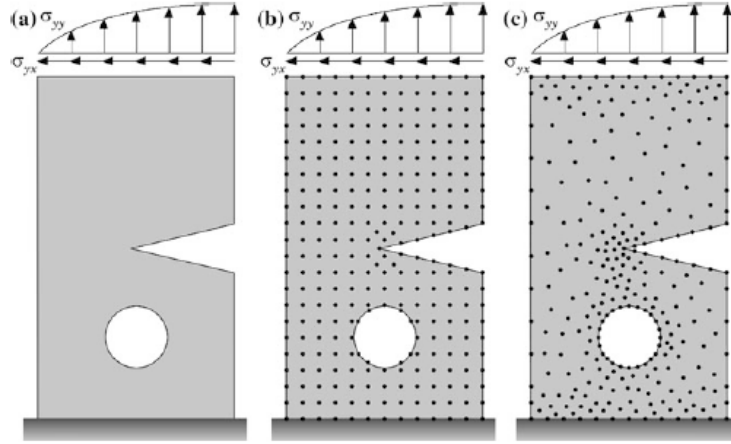


FIGURE 3.1: (a) Solid domain; (b) Regular nodal discretization; (c) Irregular nodal discretization [6].

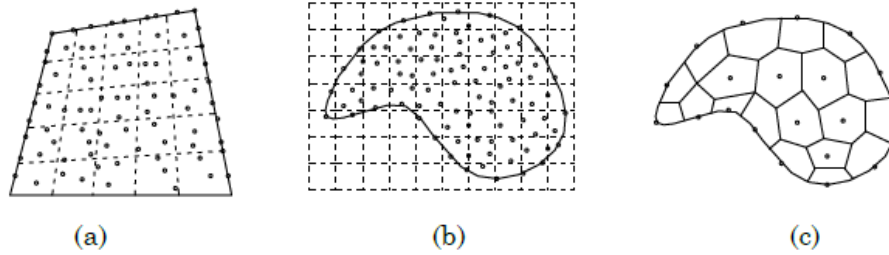


FIGURE 3.2: (a) Fitted Gaussian integration mesh; (b) General Gaussian integration mesh; (c) Voronoï diagram for nodal integration [6].

While it is common to use Gaussian integration meshes fitted to the problem's domain (as in the FEM), other approaches in meshless methods are also valid (see Figure 3.2). An alternative way to integrate the weak form equations is by using a nodal integration scheme, which makes use of Voronoï diagrams in order to obtain the integration weight on each node - Figure 3.2 (c).

After the definition of the nodal distribution and integration mesh, the nodal connectivity can be imposed. Although the nodal connectivity for the FEM is obtained by the interaction of nodes belonging to the same element with each other and also with the boundary nodes of neighbour finite elements, for meshless methods, the nodal connectivity is imposed by the overlap of influence domains. Thus, for each interest point, x_I , of the problem domain, concentric areas (for 2D problems) or volumes (for 3D) are generated, and then, the nodes inside these areas or volumes belong to the 'influence-domain' of point x_I . The shape and size of this influence-domain (which depends on the relative position of the interest point) also affects the quality of the results. It is recommend that all influence-domains possess approximately the same number of nodes - therefore, the size of the influence-domain should depend on the nodal density around the interest point.

For the majority of meshless methods, the interest points are the same as the integration points from the background integration mesh; however, there are meshless methods using the collocation point methodology or the nodal integration scheme,

in which the interest points are the nodes of the nodal discretization. For the NNR-PIM, the areas or volumes defined are Voronoï cells that are geometric constructions based on the nodal distribution. This concept will be presented in later sections.

The next step is to obtain the field variables, using interpolation functions - note that both RPIM and NNRPIM use the same interpolation functions.

Consider the displacement field $\mathbf{u}$, whose components $\mathbf{u}(x_I) = (u, v, w)$ are obtained at an interest point, x_I , within the problem domain and are interpolated using the nodal displacement of the n nodes contained in the influence-domain of this point, $u(x_i)$:

$$\mathbf{u}(\mathbf{x}_I) = \sum_i^n \phi_i(\mathbf{x}_I) \mathbf{u}(\mathbf{x}_i) \quad (3.1)$$

where $\phi_i(\mathbf{x}_I)$ is the approximation or interpolation function value of the i th node, obtained using only the n nodes inside the influence-domain. Both RPIM and NNR-PIM use interpolation functions based on the combination of multiquadric (MQ) radial basis functions (RBF) with polynomial basis functions.

In meshless methods, the discrete equations can be formulated using the interpolation functions applied to the strong or weak form formulation; for those meshless methods using the weak form of Galerkin, the discrete equations can be obtained by applying to the differential equation governing the physic phenomenon the weighted residual method of Galerkin. Thus, the system of equations can be established and arranged in a local nodal matrix and then assembled into a global system of matrix equations. Then, to obtain the displacement field, it can be used a linear solution algorithm, such as the Gauss elimination method.

3.2 Radial Point Interpolation Method formulation

3.2.1 Influence-Domains and Nodal Connectivity

Generally, in meshless methods the nodal connectivity is obtained by the overlap of the influence-domain of each node. That is the case for the RPIM. These influence-domains are based on the nodal discretization and the definition of the background integration mesh, and can have a fixed or flexible size - Figure 3.3. The size or shape variation of these influence-domains along the problem's domain affects the performance and the final solution of the meshless method; literature recommends that all the influence-domains contain approximately the same number of nodes, which, regardless of the meshless technique, should be a number between 9 and 16 for 2D problems or a number between 27 and 70 for 3D problems [6], [37], [38]. Since the number of nodes inside the influence-domain does not depend on the density of the nodal discretizations, once selected, the value is valid for all domain discretizations within the same analysis.

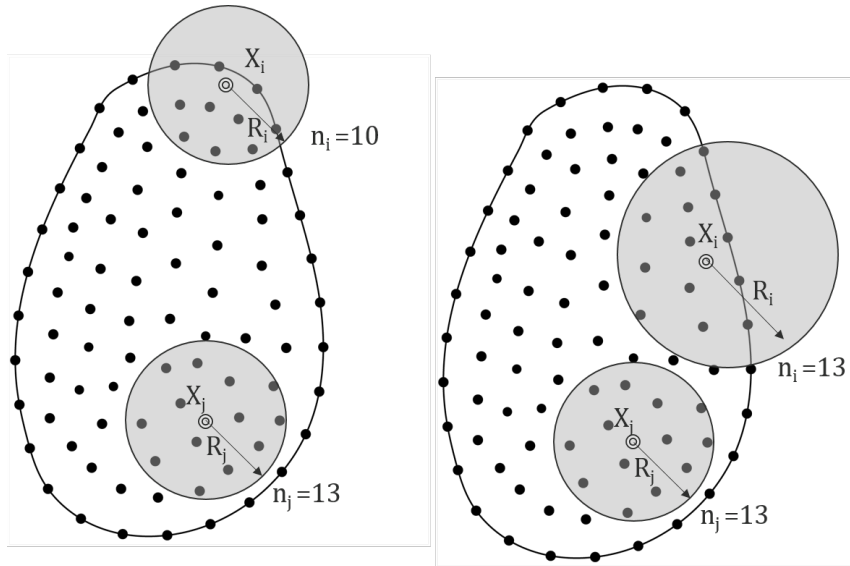


FIGURE 3.3: Fixed size influence-domain (left); Variable size influence-domain (right) [6].

3.2.2 Numerical Integration

In the RPIM, the differential equations of the Galerkin weak form are integrated using the Gauss-Legendre quadrature. In order to do this, a background mesh must be created first. This background mesh can be composed of the cells created by connecting the nodes discretizing the problem domain.

The cells composing the background mesh can be, for instance, triangular or quadrilateral, as seen in Figure 3.4 (a) and (b). Then, inside each cell, the integration points, also called Gauss Points, are distributed - Figure 3.5.

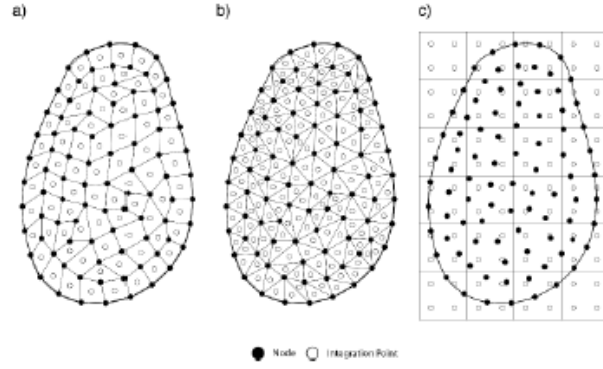


FIGURE 3.4: a) Quadrilateral cell background; b) Triangular cell background; c) Quadrilateral grid background [6].

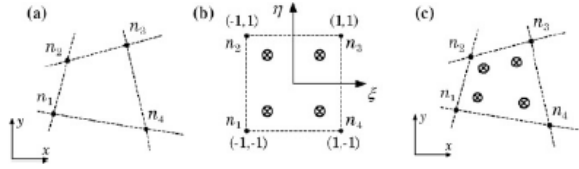


FIGURE 3.5: a) Initial quadrilateral; b) Transformation of the previous quadrilateral into an isoparametric square and application of the 2x2 quadrature point rule; c) Return to the initial quadrilateral shape. Adapted from [6].

The Gauss-Legendre integration scheme can be represented by the following equation:

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(\xi, \eta) d\xi d\eta \simeq \sum_{p=1}^{n_p} \sum_{q=1}^{n_q} f(\xi_p, \eta_q) W_p W_q \quad (3.2)$$

with $f(\xi, \eta)$ being the integrand function, n_p e n_q the number of integration points along each natural coordinate ξ e η , respectively; ξ_p e η_q the natural coordinates of point p , and W_p e W_q the corresponding weights. Table 3.1 and Table 3.2 contain the weights of the isoparametric integration points for quadrilateral (1 point, 2x2 points and 9 points) and triangular (1 point, 3 points and 4 points) elements background meshes. An example is represented in Figure 3.5 for the 4 node quadrilateral.

TABLE 3.1: Integration points and correspondent weights for different quadrilateral isoparametric cells [3].

Gauss-Legendre Point	ξ	η	Weight
I	0	0	4
I	$-\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
II	$+\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	1
III	$-\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	1
IV	$+\frac{1}{\sqrt{3}}$	$+\frac{1}{\sqrt{3}}$	1
I	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
II	0	$-\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
III	$+\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
IV	$-\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
V	0	0	$\frac{25}{81}$
VI	$+\sqrt{\frac{3}{5}}$	0	$\frac{40}{81}$
VII	$-\sqrt{\frac{3}{5}}$	$+\sqrt{\frac{3}{5}}$	$\frac{25}{81}$
VIII	0	$+\sqrt{\frac{3}{5}}$	$\frac{40}{81}$
IX	$+\sqrt{\frac{3}{5}}$	$+\sqrt{\frac{3}{5}}$	$\frac{25}{81}$

TABLE 3.2: Integration points and correspondent weights for different triangular isoparametric cells [3].

Gauss-Legendre Point	ξ	η	Weight
I	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$
I	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
II	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
III	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$
I	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{27}{96}$
II	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{25}{96}$
III	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{25}{96}$
IV	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{25}{96}$

After the distribution of the integration points in each isoparametric cell, the Cartesian coordinates for the points are obtained using the following interpolation:

$$\mathbf{x} = \sum_{i=1}^n N_i(\xi, \eta) x_i \quad ; \quad \mathbf{y} = \sum_{i=1}^n N_i(\xi, \eta) y_i \quad (3.3)$$

Where $N_i(\xi, \eta)$ is the isoparametric shape function, n is the number of nodes defining the cell, and x_i and y_i are the Cartesian coordinates of the node i .

For quadrilaterals, 4 shape functions can be defined:

$$\begin{aligned} N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\ N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\ N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\ N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta) \end{aligned} \quad (3.4)$$

While for triangles,

$$\begin{aligned} N_1 &= 1 - \eta - \xi \\ N_2 &= \eta \\ N_3 &= \xi \end{aligned} \quad (3.5)$$

Then, the integration weight is obtained by multiplying the isoparametric weight of the integration point with the determinant of the Jacobian matrix, defined by:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (3.6)$$

Thus, the differential equation integration can be calculated in the integration point, x_i , as follows:

$$I = \int_{-1}^{+1} \int_{-1}^{+1} f(x_i) \det(\mathbf{J}) \, d\xi \, d\eta = \sum_{p=1}^{n_p} \sum_{q=1}^{n_q} f(x_i) W_p W_q \det(\mathbf{J}) \quad (3.7)$$

3.2.3 Interpolation functions

Since the RPIM and the NNRPIM use the same the same interpolation functions, the formulation for said functions will be explained in the present section and not in following section for the NNRPIM. Consider a function, $u(x)$, defined in a domain ω , discretized by a set of N nodes. It is assumed that only the nodes that are inside the influence-domain of the interest point x_I are considered for the function $u(x)$. For the RPIM, said function passes through all nodes using a radial basis function (RBF). Considering the value of the function at the interest point, $\mathbf{u}(x_I)$, one can define:

$$u(x_I) = \sum_{i=1}^n R_i(x_I) \mathbf{a}_i(x_I) + \sum_{j=1}^n p_j(x_I) b_j(x_I) = \mathbf{R}^T(x_I) \mathbf{a}(x_I) + \mathbf{p}^T(x_I) \mathbf{b}^T(x_I) \quad (3.8)$$

where n is the number of nodes within the influence-domain of the interest point, x_I , $R_i(x_I)$ is the radial basis function (RBF) and $a(x_I)$ and $b_j(x_I)$ are non-constant coefficients of $R_i(x_I)$ and $p_j(x_I)$, respectively, where $p_j(x_I)$ represents the monomials of the polynomial basis (m being the basis monomial number).

These vectors are defined by:

$$\begin{aligned}\mathbf{R}^T(x_I) &= \{R_1(x_I), R_2(x_I), \dots, R_n(x_I)\} \\ \mathbf{p}^T(x_I) &= \{p_1(x_I), p_2(x_I), \dots, p_m(x_I)\} \\ \mathbf{a}^T(x_I) &= \{a_1(x_I), a_2(x_I), \dots, a_n(x_I)\} \\ \mathbf{b}^T(x_I) &= \{b_1(x_I), b_2(x_I), \dots, b_m(x_I)\}\end{aligned}\tag{3.9}$$

Then, equation (3.8) can be rewritten as:

$$u(x_I) = \{\mathbf{R}^T(x_I), \mathbf{p}^T(x_I)\} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}\tag{3.10}$$

In order to obtain a more stable function, the number of monomial m of the polynomial basis should be $m < n$. The purpose of adding the polynomials into the basis function is to ensure the consistency of the RPI functions, since that adding to the RBF a linear polynomial allows to ensure C1 consistency and helps the RPI pass the standard patch test.

Nevertheless, Belinha et. al [7] determined that that the polynomial basis function is not necessary for the RPI to pass the standard patch test. This allows to reduce the computational cost and results in shape functions dependent solely on the RBFs.

A Multiquadratic Radial Basis Function (MQ-RBF), proposed initially by Hardy [39], is

$$R(r_{ij}) = (r_{ij}^2 + c^2)^p\tag{3.11}$$

with c and p being the shape parameters that must to be optimized in order to maximize the method's performance.

In earlier works on the RPI [40], [41], optimal values for the shape functions have been proposed, such as $c = 1,42$ and $p = 1,03$. However, these were then found to be inaccurate and somewhat poor in stability and precision [7], [42]. One of the most relevant problems of meshless methods using RBFs is that the variation of the shape parameters affects significantly the performance of the RBFs, and there are no standard or precise theoretical methods to determine the optimal values of these parameters. Belinha et al. [6] performed a optimization study for the shape parameters and found that the values $c = 0,0001$ and $p = 0,9999$ are more suitable, and therefore, are the ones that will be used throughout this work.

The Euclidean distance between the interest point x_I and a node x_i , belonging to the influence-domain (for the RPIM) or influence-cell (for the NNRPIM), can be defined, for the 2D case,

$$r_{ij} = \sqrt{(x_I - x_i)^2 + (y_I - y_i)^2}\tag{3.12}$$

and for the 3D case,

$$r_{ij} = \sqrt{(x_I - x_i)^2 + (y_I - y_i)^2 + (z_I - z_i)^2}\tag{3.13}$$

And the polynomial basis functions, for the 2D case,

$$\mathbf{p}^T = \{1, x, y, x^2, y^2, \dots\} \quad (3.14)$$

and for the 3D case,

$$\mathbf{p}^T = \{1, x, y, z, x^2, xy, y^2, yz, z^2, \dots\} \quad (3.15)$$

The sequence of terms presented in equations (3.14) and (3.15) depend on the chosen number m . The polynomial functions for both 2D and 3D are represented in Table 3.3 and Table 3.4, respectively.

TABLE 3.3: Polynomial basis functions - 2D.

Basis	Polynomial Function
Null (m=0)	$\mathbf{p}^T(x) = \{0\}$
Constant (m=1)	$\mathbf{p}^T(x) = \{1\}$
Linear (m=3)	$\mathbf{p}^T(x) = \{1, x, y\}$
Quadratic (m=6)	$\mathbf{p}^T(x) = \{1, x, y, x^2, xy, y^2\}$

TABLE 3.4: Polynomial basis functions - 3D.

Basis	Polynomial Function
Constant (m=1)	$\mathbf{p}^T(x) = \{1\}$
Linear (m=4)	$\mathbf{p}^T(x) = \{1, x, y\}$
Quadratic (m=10)	$\mathbf{p}^T(x) = \{1, x, y, z, x^2, xy, y^2, yz, z^2, zx\}$

For the entirety of this work, a constant polynomial basis was applied, in order to reduce computational costs and increase the NNRPIM's accuracy. The polynomial basis has to satisfy an extra requirement in order to obtain a unique solution, given by:

$$\sum_{i=1}^n p_j(x_i) a_i(x_i) = 0 \quad (3.16)$$

with $j = \{1, 2, \dots, m\}$.

Therefore, combining equations (3.14) and (3.16), we can obtain:

$$\begin{Bmatrix} \mathbf{u}_s \\ 0 \end{Bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{p}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mathbf{G} \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} \quad (3.17)$$

in which $\mathbf{u}_s$ is given by,

$$\mathbf{u}_s = \{u_1, u_2, \dots, u_n\}^T \quad (3.18)$$

with the matrix $\mathbf{R}$, of size $[n \times n]$, containing all interest points of the RBFs,

$$\mathbf{R} = \begin{bmatrix} R(r_{11}) & R(r_{12}) & \cdots & R(r_{1n}) \\ R(r_{21}) & R(r_{22}) & \cdots & R(r_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ R(r_{n1}) & R(r_{n2}) & \cdots & R(r_{nn}) \end{bmatrix} \quad (3.19)$$

and the matrix $\mathbf{p}$, of size $[n \times m]$,

$$\mathbf{p} = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix} \quad (3.20)$$

The $\mathbf{G}$ matrix is symmetric, since the distance between the interest points and the nodes that belong to a certain influence-domain are independent of directional terms. Solving the system of equations in (3.21),

$$\begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \mathbf{G}^{-1} \begin{Bmatrix} \mathbf{u}_s \\ 0 \end{Bmatrix} \quad (3.21)$$

And then, substituting equation (3.21) into equation (3.8), it is possible to obtain:

$$u(x_I) = \{ \mathbf{R}^T(x_I), \mathbf{p}^T(x_I) \} \mathbf{G}^{-1} \begin{Bmatrix} \mathbf{u}_s \\ 0 \end{Bmatrix} = \phi(x_I) \mathbf{u}_s \quad (3.22)$$

where $\phi(x_I)$ is the interpolation function of the interest point x_I , given by

$$\phi(x_I) = \{ R_{\xi}^T(x_I), p_{\xi}^T(x_I) \} G^{-1} \quad (3.23)$$

with the partial derivatives of the MQ-RBF, in order of the same variable ξ , being:

$$R_{\xi}(r_{ij}) = 2p(r_{ij}^2 + c^2)^{p-1}(\xi_j - \xi_i) \quad (3.24)$$

As the RPI functions are interpolation functions that possess the Kronecker delta property, they pass through all nodes, which makes the imposition of the essential and natural boundary conditions easier [6]. Also, they satisfy the partition of unity, that is,

$$\sum_{i=1}^n \phi_i(x_i) = 1 \quad (3.25)$$

If the interpolation function $\phi_i(x_i)$ possesses reproducing properties and the constant of the polynomial basis is included, then,

$$\sum_{i=1}^n \phi_i(x_i) x_i = x \quad (3.26)$$

on the condition that the first order monomial is included in the polynomial basis.

3.3 Natural Neighbour Radial Point Interpolation Method formulation

As previously mentioned in this chapter, while the RPIM uses influence-domains in order to impose the nodal connectivity, the NNRPIM uses mathematical concepts such as Voronoï diagrams and Delaunay tessellations in order to construct the influence-cells - the fundamental structure for nodal connectivity and the background integration mesh.

Again, the only required information for the application of the NNRPIM is the nodal distribution of the problem's domain, making it a truly "meshless" method [6].

3.3.1 Natural neighbours and Voronoï Diagrams

The concept of natural neighbours, which is used in the NNRPIM to impose the nodal connectivity, combined with Voronoï diagrams for the determination of the discretized domain, was first presented by Sibson [43].

On the other hand, a Voronoï diagram consists of a set of Voronoï cells, in which every cell has a single node associated with it, and contains all of the other nodes in the plane whose distance to said node is less than or equal to its distance to any other point in the problem's domain [44]. For the sake of simplicity, only 2D procedures will be presented, although this can be applied to any given $\mathbb{R}^d$ space.

Consider the problem domain, Ω , bounded by a physical boundary, Γ , with a random discretization of nodes $N = \{n_0, n_1, \dots, n_n\} \in \mathbb{R}^2$, as the one represented in Figure 3.6:

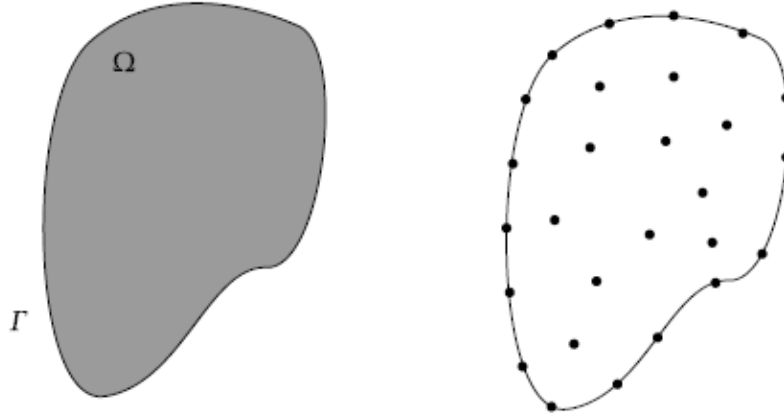


FIGURE 3.6: Problem domain discretized with randomly distributed nodes [45].

As shown in Figure 3.7, the Voronoï diagram of N results of the partition of the domain in sub-regions V_I , closed and convex, while each sub-region V_I is associated to node I , n_I , in such a way that any point contained in V_I is closer to n_I than any other node n_J , where,

$$n_J \in N \wedge J \neq I \quad (3.27)$$

Thus, V_I consists of the geometric place where all points are closer to n_I than any other node, with the sub-regions V_k being the Voronoï cells - which form the Voronoï diagram for $k = \{1, 2, \dots, N\}$. Mathematically, this concept can be expressed as follows:

$$V_I = \{x \in \mathbb{R}^2 : E_n(x, x_I) < E_n(x, x_n), \forall J \neq I\} \quad (3.28)$$

with $E_n(x, x_I)$ being the Euclidian distance between an interest point, x , and a node x_I , and $E_n(x, x_J)$ the same, but instead regarding a node outside the cell ($J \neq I$).

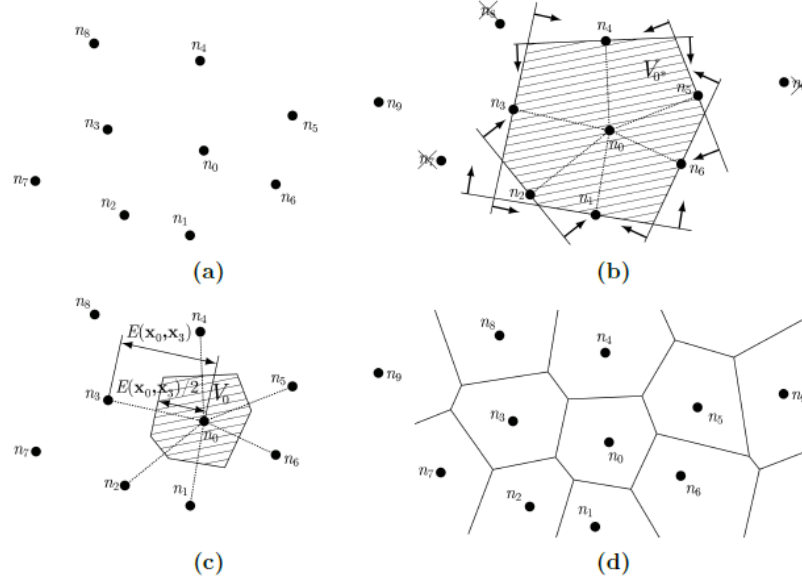


FIGURE 3.7: (a) Initial nodal set of potential neighbour nodes. (b) Construction of the cell containing only neighbour nodes. (c) Voronoi cell. (d) Voronoi diagram [45].

The construction of the sub-region V_I is represented in Figure 3.7. First, for an interest node n_0 , a set of potential neighbour nodes is identified - Figure 3.7 (a). A set of dashed lines, starting from the interest node and ending at potential neighbour nodes are drawn. Simultaneously, straight and full lines, perpendicular to the dashed lines, are drawn as well - each of these lines containing a neighbour node. The neighbour nodes are, then, obtained by the intersection of domains whose limits are defined by the lines which intersect the nodes J and are normal to the dashed lines.

A provisional Voronoi cell, V_0^* , is created - Figure 3.7 (b) - and all of the remaining nodes, located outside of this cell, are discarded; only the nodes inside the cell are considered the neighbour nodes of node n_0 .

The final Voronoi cell, V_0 , is obtained - Figure 3.7 (c) - by reducing the size of the provisional cell, V_0^* . The reduction is made by defining the distance between node n_0 and the neighbour nodes as half of the Euclidean distance between node n_0 and the neighbour node in question. Therefore, the final cell, V_0 , is the homothetic form of V_0^* .

The same process is carried for each node in the problem domain, resulting in a total Voronoi cell discretization of the geometry, as represented in Figure 3.7 (d).

3.3.2 Influence-Cells and Nodal Connectivity

As previously mentioned for the case of the RPIM, the size and shape of the influence domains will, in fact, affect the performance and accuracy of the method. Likewise, for the NNRPIM, the nodal connectivity is imposed by the overlapping of influence cells.

In Figure 3.8, two different types of influence-cells are represented, and the difference between them is very intuitive: the first degree influence-cell is solely composed by the first natural neighbours of interest point x_I , while the second degree influence-cell, contains as well the natural neighbours of the interest points' natural neighbours.

As such, the second degree influence-cell represents a larger area and, consequently, has a higher nodal connectivity than the first degree influence-cell. Using higher degree influence-cells permits to achieve results with greater accuracy. Therefore, throughout this work, only second-degree influence-cells will be considered.

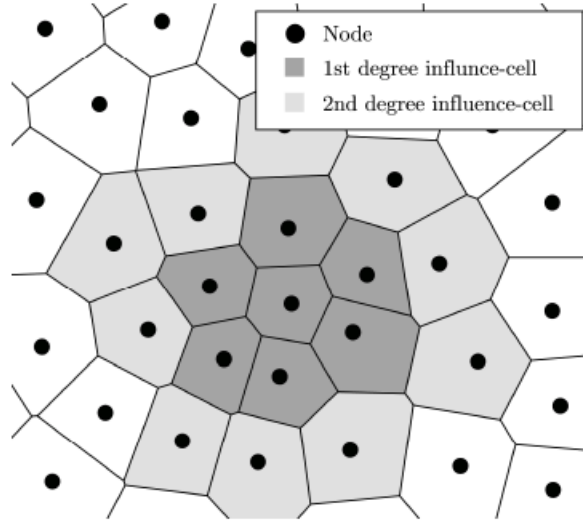


FIGURE 3.8: First and second degree influence-cell [45].

3.3.3 Numerical Integration

Once the Voronoï diagram is fully defined, it is necessary to create an integration mesh, in order to numerically integrate the differential equation that govern the physical problem being studied. This is made possible by using a particular geometric construction - called the Delaunay tessellation. This geometric construction is the geometrical dual of the Voronoï diagram, and it is constructed by connecting the nodes whose Voronoï cells have common boundaries - see Figure 3.9 - a Delaunay edge exists between two nodes in the plane if and only if their Voronoï cells share a common edge [6].

Belinha et. al. [7] proposed an integration scheme based on the Voronoï diagram and the Delaunay tessellation. The 2D integration scheme is considered below, for the sake of simplicity.

The Delaunay tessellation allows the original Voronoï cell area, A^{V_j} , of a certain node, x_j , to be divided in a number of K sub-areas (quadrilaterals or triangles), $A_i^{V_j}$, so that:

$$A^{V_j} = \sum_{i=1}^K A_i^{V_j} \quad (3.29)$$

For a regular mesh, the middle points of the Delaunay edges are coincident with the edge intersection points - in which case, the constructed sub-areas are triangular.

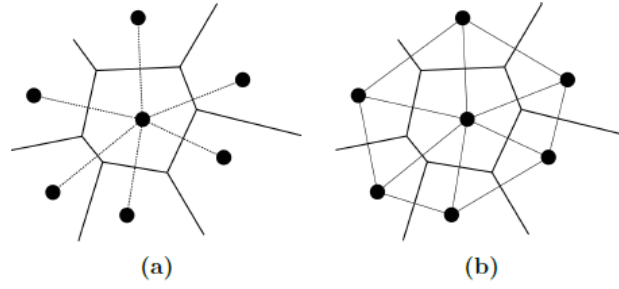


FIGURE 3.9: (a) Initial Voronoi diagram; (b) Delaunay tessellation [45].

Finally, in order to obtain the integration mesh, integration points are distributed inside each sub-area, $A_i^{V_j}$, following the Gauss-Legendre quadrature rule, for each Voronoi cell discretizing the problem domain - Figure 3.10 (b). This is an "Order 0" integration scheme and consists of calculating the Cartesian coordinates of each integration point and performing the integration using the area of the sub-cell as the weight of each integration point, which is equivalent with the 1x1 integration point Gauss-Legendre scheme for triangle and quadrilateral shapes, already explained in previous sections in this Chapter. By using this integration scheme, it is possible to obtain a numerical integration mesh dependent only on the nodal distribution of the problem domain.

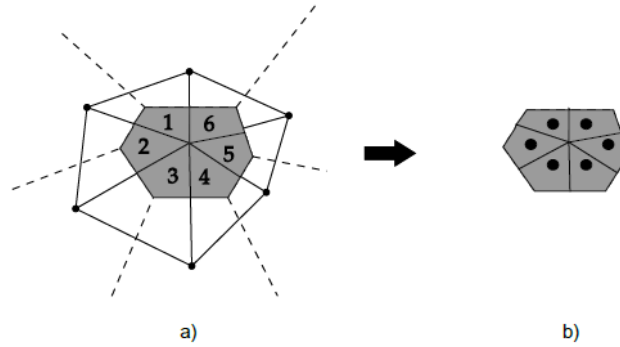


FIGURE 3.10: (a) Sub-cell obtained through the overlapping of the Voronoi diagram and the Delaunay tessellation; (b) Quadrature points following the Gauss-Legendre integration scheme. Adapted from [45].

There are other integration schemes which divide each sub-cell into smaller quadrilateral sub-cells, in order to add more integration points to the original sub-cells. Nevertheless, adding more integration points represents an increase of the computational cost, while it does not necessarily increase the accuracy of the solution significantly.

3.4 Solid Mechanics Fundamentals

Throughout the next sections, the fundamentals of Solid Mechanics which constitute the foundation for the numerical structural analysis carried in this work are presented. Firstly, general concepts of strain and stress fields are approached with the equilibrium equations being presented. Subsequently, the strong and weak form formulation is described, and the discrete system of equations, based on the Principle of Virtual Work, is deduced, using the previously presented shape functions as trial functions.

Since non-linear and linear mechanics share a common ground in some topics such as kinematics, stress and equilibrium equations, a review on these subjects is crucial to understanding of the numerical analysis performed in this work.

3.4.1 Strain Field, Stress Tensor and Equilibrium Equations

For a given 3D problem, the Cauchy stress tensor is defined as:

$$\Lambda = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (3.30)$$

The Cauchy tensor, Λ , represents the stresses present in a body for a certain stress state.

The stress tensor can also be written in the the Voigt notation, which is particularly useful for numerical implementation. In this case,

$$\sigma = \{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{zz} \quad \sigma_{xy} \quad \sigma_{yz} \quad \sigma_{zx} \} \quad (3.31)$$

The stress tensor is obtained by multiplying the constitutive matrix, c , by the strain tensor, σ , giving:

$$\sigma = \{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \gamma_{xy} \quad \gamma_{yz} \quad \gamma_{zx} \} \quad (3.32)$$

with the constitutive matrix, c , being:

$$c = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \quad (3.33)$$

with E being the Young's modulus, ν being the Poisson's coefficient, and G the shear modulus, which is given by:

$$G = \frac{E}{2(1+\nu)} \quad (3.34)$$

The strain components can be calculated from the displacement field, which is composed of the displacements u , v and w , along the Cartesian coordinate system axis x , y and z , respectively.

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
\end{aligned} \tag{3.35}$$

For an infinitesimal element, the equilibrium equations can be established as

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + F_z &= 0
\end{aligned} \tag{3.36}$$

where F_x , F_y and F_z are the applied body forces, along the three Cartesian axis.

3.4.2 Galerkin's strong and weak form

The differential equations that govern a physical system must obey the equilibrium equations. Thus, the strong form consists of the governing equations as well as the boundary conditions for the physical system. This means that the aforesaid equations must be satisfied at any given point of the domain [3], [46].

Due to the difficulty to solve these equations whenever the physical system is complex, due to its form or boundary conditions, the weak form is usually preferred - instead of solving the system of differential equations that govern the system, the weak form allows to obtain a discretized system of equations, leading to the approximate solution of the problem.

At every integration point, the weak form is established, which means that the precision and accuracy of the obtained solution is dependent on the number of nodes in which the problem's domain is discretized. For that reason, a higher nodal density tends to return results with higher accuracy. In addition to that, the imposition of the boundary conditions is much easier, since they can be applied directly at the nodes.

Galerkin Weak Form

Both the RPIM and NNRPIM use the Galerkin weak form formulation to convert the differential equations into a discrete equation system, in order to obtain approximated solutions to the strong form formulation. For this purpose, all of the admissible displacements must satisfy the following requirements:

- All of the displacements are compatible with the problem domain;
- The displacement history must satisfy the constraints at the initial and final times.

In order to obtain the Galerkin weak form, the Hamilton's principle is applied, which can be stated as follows: "Of all possible time histories of displacement states that satisfy the compatibility equations and the constraints or the kinematic boundary conditions and that also satisfy the conditions at initial and final times (t_1 and t_2), the history corresponding to the actual solution makes the Lagrangian functional a minimum." [47], [46].

$$L = T - U + W_f \quad (3.37)$$

with T being the kinetic energy, U being the strain energy and W_f the work performed by external forces. The Lagrangian functional, L , contains all physical information concerning the system and the forces acting on it. Once the Lagrangian is defined, we may establish:

$$\int_{t_1}^{t_2} [\delta T - \delta U + \delta W_f] dt = 0 \quad (3.38)$$

which, by substituting into equation (3.37), gives:

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (3.39)$$

For a generic solid, such as the one represented in Figure 3.11, where the domain, Ω , is bounded by a boundary Γ , one can define the surface on which the external forces, $\bar{t}$, are applied, the natural boundary, Γ_t , and the essential boundary, Γ_u where the displacements are constrained. Besides these, a body force, b , is considered.

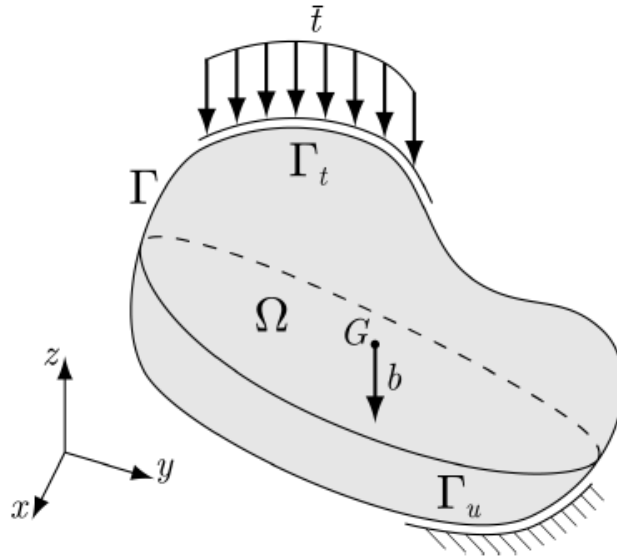


FIGURE 3.11: Continuous solid subjected to different kind of forces [45].

The energy terms of equation (3.37) can be defined as:

$$T = \frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} d\Omega \quad (3.40)$$

$$U = \frac{1}{2} \int_{\Omega} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} d\Omega \quad (3.41)$$

$$W_f = \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma \quad (3.42)$$

with Ω being the solid volume, $\dot{\mathbf{u}}$ the velocity, ρ the solid mass density, ε the strain vector, σ the stress, $\mathbf{u}$ the displacement and $\mathbf{b}$ the body forces.

One can, then, rewrite the Lagrangian functional, L , as

$$L = \frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} d\Omega - \frac{1}{2} \int_{\Omega} \varepsilon^T \sigma d\Omega + W_f = \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma \quad (3.43)$$

Applying the Hamilton's principle,

$$\delta \int_{t_1}^{t_2} \left[\frac{1}{2} \int_{\Omega} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} d\Omega - \frac{1}{2} \int_{\Omega} \varepsilon^T \sigma d\Omega + \int_{\Omega} \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \mathbf{u}^T \bar{\mathbf{t}} d\Gamma \right] dt = 0 \quad (3.44)$$

For any given static problem, the kinetic energy term can be discarded. Also, by moving the variation operator δ inside each integral, we obtain:

$$\int_{t_1}^{t_2} \left[-\frac{1}{2} \int_{\Omega} \delta(\varepsilon^T \sigma) d\Omega + \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma \right] dt = 0 \quad (3.45)$$

The first term can be rewritten as:

$$\delta(\varepsilon^T \sigma) = \delta \varepsilon^T \sigma + \varepsilon^T \delta \sigma \quad (3.46)$$

And since these terms are scalar, they can be transposed:

$$\varepsilon^T \delta \sigma = (\varepsilon^T \delta \sigma)^T = \delta \sigma^T \varepsilon \quad (3.47)$$

Considering that the constitutive matrix is symmetrical, $\mathbf{c}^T = \mathbf{c}$,

$$\delta \sigma^T \varepsilon = \delta(\mathbf{c} \varepsilon)^T \varepsilon = \delta \varepsilon^T \mathbf{c}^T \varepsilon = \delta \varepsilon^T \mathbf{c} \varepsilon = \delta \varepsilon^T \sigma \quad (3.48)$$

Introducing equation (3.48) into equation (3.49),

$$\delta(\varepsilon^T \sigma) = 2\delta \varepsilon^T \sigma \quad (3.49)$$

Then, substituting this result into equation (3.45),

$$\int_{t_1}^{t_2} \left[-\int_{\Omega} \delta(\varepsilon^T \sigma) d\Omega + \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma \right] dt = 0 \quad (3.50)$$

In order for equation (3.50) to be satisfied for all possible values of $\mathbf{u}$, for any given initial and final times, t_1 and t_2 , the integrand must be equal to zero. Moreover, considering that the strain field can be expressed as a multiplication of a differential operator, L , by the displacement field, $\mathbf{u}$, $\varepsilon = L \cdot \mathbf{u}$, we can obtain the Galerkin weak form:

$$\int_{\Omega} \delta(L \mathbf{u})^T \mathbf{c}(L \mathbf{u}) d\Omega - \int_{\Omega} \delta \mathbf{u}^T \mathbf{b} d\Omega - \int_{\Gamma_t} \delta \mathbf{u}^T \bar{\mathbf{t}} d\Gamma = 0 \quad (3.51)$$

3.5 Discrete system of equations

The discrete system of equations needed for the RPIM and NNRPIM can be obtained using the Galerkin weak form, which was established in equation (3.59), combined with the interpolation functions presented earlier in this Chapter.

For a solid in a static equilibrium, the Principle of Virtual Work dictates that the total virtual work done by the body's internal stresses and applied external forces is zero, for a set of infinitesimal virtual displacements from equilibrium. The Galerkin weak form, combined with this principle, allows for the establishment of the discrete system of equations considered in meshless methods.

If we recall equation (3.1),

$$\mathbf{u}(x_I) = \sum_i^n \phi_i(x_I) \mathbf{u}(x_i) \quad (3.52)$$

we can establish that the virtual displacements of the nodes within the influence cell (for the NNRPIM) or influence domain (for the RPIM) of an interest point x_I , will cause a virtual displacement of the integration point, which can be interpolated considering:

$$\delta \mathbf{u}(x_I) = \sum_{i=1}^n \phi_i(x_I) \delta \mathbf{u}(x_i) \quad (3.53)$$

Combining this result with the Galerkin weak form,

$$\begin{aligned} & - \int_{\Omega} \left(\sum_{i=1}^n \phi_i(x_I) \delta \mathbf{u}(x_i) \right)^T \mathbf{L}^T \mathbf{c} \mathbf{L} \left(\sum_{j=1}^n \phi_j(x_j) \delta \mathbf{u}(x_j) \right) d\Omega + \\ & + \int_{\Omega} \left(\sum_{i=1}^n \phi_i(x_I) \delta \mathbf{u}(x_i) \right)^T \mathbf{b} d\Omega - \\ & - \int_{\Gamma_t} \left(\sum_{i=1}^n \phi_i(x_I) \delta \mathbf{u}(x_i) \right)^T \bar{\mathbf{t}} d\Gamma = 0 \end{aligned} \quad (3.54)$$

The equation presented above can also be expressed in a matrix form; for the 3D case,

$$\phi(x) = \begin{bmatrix} \phi_1(x) & 0 & 0 & \phi_2(x) & 0 & 0 & \dots & \phi_n(x) & 0 & 0 \\ 0 & \phi_1(x) & 0 & 0 & \phi_2(x) & 0 & \dots & 0 & \phi_n(x) & 0 \\ 0 & 0 & \phi_1(x) & 0 & 0 & \phi_2(x) & \dots & 0 & 0 & \phi_n(x) \end{bmatrix} \quad (3.55)$$

The general differential operator, $\mathbf{L}$, for a 3D problem is:

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad (3.56)$$

The deformability matrix, $\mathbf{B}$, can be defined as:

$$\mathbf{B} = \mathbf{L} \cdot \phi(x) \quad (3.57)$$

which, when introduced in (3.54), gives:

$$-\delta \mathbf{U}^T [\mathbf{K} \cdot \mathbf{U} - \mathbf{F}] = 0 \quad (3.58)$$

with $\mathbf{U} = \mathbf{u}$ and $\mathbf{F} = \mathbf{f}_b + \mathbf{f}_f$. Solving the previous equation, the linear system of equations is obtained:

$$\mathbf{K} \cdot \mathbf{U} = \mathbf{F} \quad (3.59)$$

3.5.1 Boundary Conditions

Since the interpolation functions used in the RPIM and NNRPIM formulations possess the Kronecker delta property, expressed by,

$$\phi(x_i) = \delta_{ij} \quad (3.60)$$

where

$$\begin{aligned} \delta_{ij} &= 1, & \text{if } i &= j \\ \delta_{ij} &= 0, & \text{if } i &\neq j \end{aligned} \quad (3.61)$$

Thus, the boundary conditions can be applied directly to the stiffness matrix, in equation (3.59). These boundary conditions may refer to either essential boundary conditions (constrained displacements) or natural boundary conditions (related to the applied forces).

Chapter 4

Plasticity

A good understanding of the elastoplastic deformation of materials used in structural applications, such as metals and polymers, including the constitutive description of these materials and the analysis of structures undergoing plastic deformation is of high interest. In this study, in addition to the linear elastic analysis performed for the conventional and re-entrant honeycomb structures, an elastoplastic analysis was performed using FEMAS. Therefore, throughout the following sections, some insight is given on the elastoplasticity topic, focusing on the isotropic behaviour of materials.

4.1 Elastoplasticity

While in the elastic domain a material which deforms under stress returns to its original shape when the stress is removed, in the plastic domain the deformation of a material implies non-reversible, permanent changes of shape in response to the applied forces[48]. In this study, only elastoplastic materials are considered - these are materials which, initially, show an elastic behavior, but after yielding, become plastic; and by removing the loading, the material becomes elastic again. Thus, the mathematical theory of plasticity gives a theoretical description of the relationship between stress and strain for a material which exhibits an elastoplastic response [49].

Three main types of elastoplastic material behaviours can be considered - Figure 4.1.

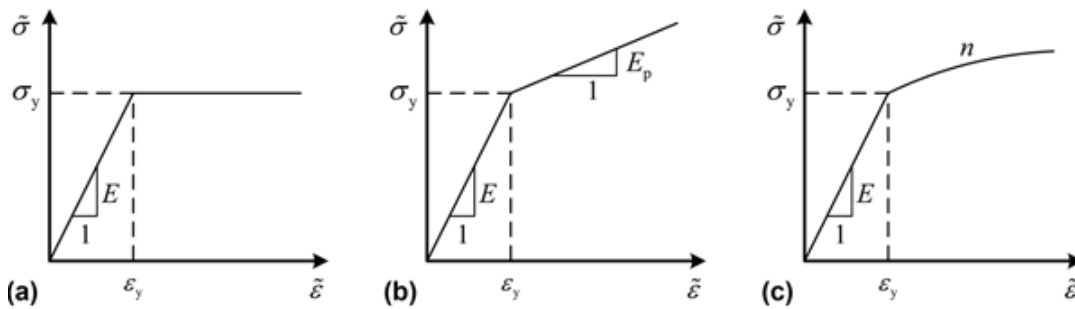


FIGURE 4.1: Elastoplastic materials' behaviours: (a) elastic-perfectly plastic model; (b) bilinear elastoplastic model; (c) non-linear elastoplastic model [48].

For the elastic-perfectly plastic model (a), the elastic part is linear, and characterized by a slope which represents the Young's Modulus, E , and thus, $\alpha_1 = \tan^{-1}(E)$. However, after reaching the yield stress, σ_y , the material maintains the stress level, while the deformation (plastic only) increases.

In case (b), for the bilinear elastoplastic model, the first slope, α_1 characterizes the elastic part; once the yield stress is reached, a new slope begins, defined by

$\alpha_2 = \tan^{-1}(E_T)$, representing the plastic part, with E_T being the tangent modulus. Finally, there is case (c), which is the non-linear elastoplastic model: this one is very similar to the previous one, case (b), except that after reaching the yield stress, σ_Y , the considered stress relation is no longer linear [50].

By assuming small (infinitesimal) deformations, the elastic and plastic strains can be decomposed using a basic rule of plasticity, usually called additive decomposition of the strain tensor, which is represented by

$$\varepsilon = \varepsilon_e + \varepsilon_p \quad (4.1)$$

where ε_e represents the elastic strain component and ε_p the plastic strain component. These are represented for the bilinear case, with some more detail, in Figure 4.2.

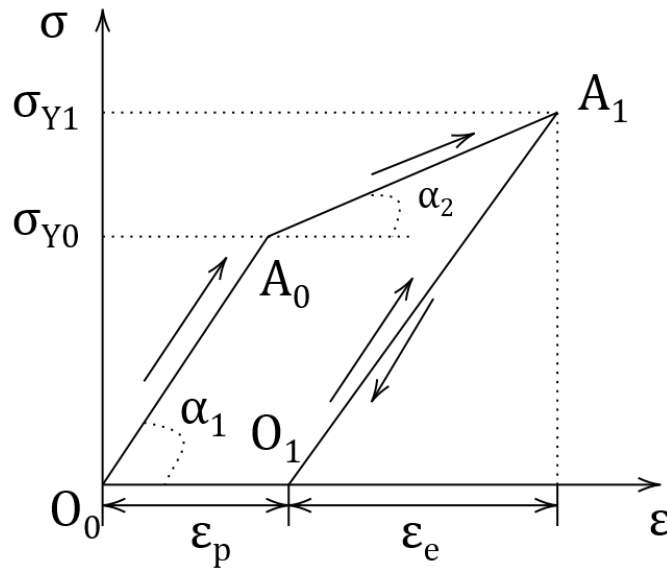


FIGURE 4.2: Bilinear elastoplastic model. Adapted from [51].

From Figure 4.2 it becomes clear that if the load is removed in point A1, then, the elastic part of the strain is recovered, but the plastic part is not. Otherwise, point O₀ would be achieved, instead of point O₁. This occurs because plastic deformation is associated with the deformation energy and, as such, it is an irreversible process. In addition to that, this makes the plastic deformation dependent on the load history.

Furthermore, in order to fully describe the elastoplastic material behavior, besides the elastic stress-strain relations, three requirements must be fulfilled: [6], [50]

- A yield criterion, based on a function which defines the moment at which the yielding can occur, taking into account the values of the stresses;
- A flow rule, which relates the plastic strain increments to the current stresses and the stress increments outside the yield function;
- A hardening rule, which specifies how the yield function is modified during the plastic flow.

Thus, before the stress state reaches the yield function, here defined as σ_Y , the Hooke's law is still valid. However, for stress states after that point (point A₀) the

linear Hooke's law is no longer valid, and a flow rule is needed to define the stress state of the problem. Moreover, a hardening rule must be defined as well, because due to the plastic strain component, the yield surface begins to evolve. The aforementioned concepts will be treated in the following sections.

4.1.1 Yield Criterion

The characterization of the yield for a given material is idealized by a yield surface, which splits the space of stresses in an elastic and a plastic domain, as shown in Figure 4.3.

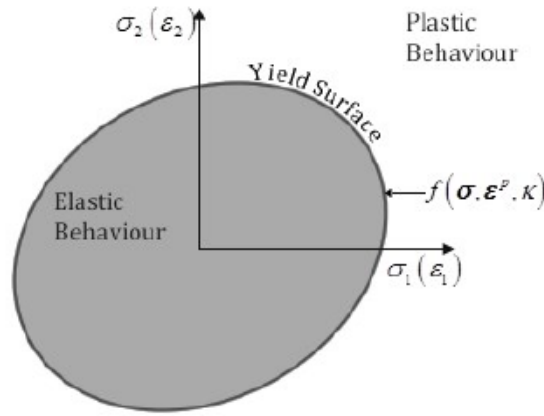


FIGURE 4.3: Representation of the Yield Surface [52].

The yield occurs only if the stress satisfies the general yield criterion, usually stated in the following form:

$$F(\sigma, \varepsilon_p, k) = f(\sigma, \varepsilon_p, k) - \sigma_Y(k) = 0 \quad (4.2)$$

where $f(\sigma, \varepsilon_p, k)$ is the yield function, which is dependent of the stress state, σ , the plastic strain, ε_p , and a hardening parameter, k . The yield stress associated to the material is represented by $\sigma_Y(k)$.

For an isotropic material, whose yield stress is independent of the hardening parameter, k , the previous equation can be simplified as:

$$F(\sigma) = f(\sigma) - \sigma_Y(k) = 0 \quad (4.3)$$

The yield criterion used in this work for isotropic materials is the von Mises yield criterion, which is commonly applied to metals. There are some advantages on the computational implementation of the von Mises yield criterion when compared to the Tresca yield criterion, since the second one presents some edges in the surface, which would lead to some discontinuities - Figure 4.4.

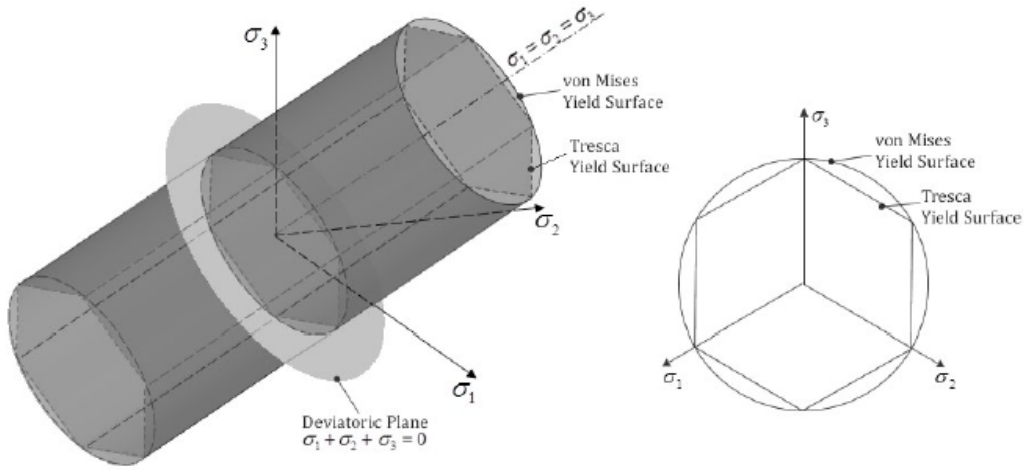


FIGURE 4.4: Comparison between the Von Mises and the Tresca yield surfaces [52].

Thus, following the von Mises yield criterion, it is assumed that the yielding occurs when the second deviatoric stress invariant, I_2 , reaches a critical value, given by:

$$I_2 = \left(\frac{\sigma_Y}{\sqrt{3}} \right)^2 \quad (4.4)$$

The von Mises yield criterion can also be expressed in terms of the effective stress $\bar{\sigma}$, from which we can obtain

$$\bar{\sigma}^2 = \bar{\sigma}_Y^2 = 3I_2 \quad (4.5)$$

and defining I_2 in terms of the stress tensor components, we obtain:

$$\bar{\sigma} = \sqrt{\frac{1}{2}[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2)]} \quad (4.6)$$

Applying to equation (4.3) the aforementioned considerations, the yield criterion for isotropic materials is expressed by the following equation:

$$F(\sigma) = \sqrt{\frac{1}{2}[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2)]} - \sigma_Y(k) = 0 \quad (4.7)$$

4.1.2 Plastic flow

As the yield function, f , reaches a value equal to the yield stress of the material, σ_Y , the material is in the threshold of the elastic domain and starts the transition towards the beginning of the plastic behaviour. Once the material is in the plastic domain, its behaviour is conditioned by the variation of the value of the yield function, f with respect to the stress state, σ , as the following equation shows,

$$df = \left(\frac{df}{d\sigma} \right)^T d\sigma \quad (4.8)$$

where $df/d\sigma$ is the f gradient, and therefore, an orthogonal vector to the yield surface for a considered stress variation $d\sigma$, which moves tangentially to the surface, being orthogonal to the f gradient. Thus, $df/d\sigma$ is normal to the yield function, which invokes orthogonality. The satisfaction of the orthogonality rule is essential to prove its uniqueness [53].

Furthermore, by applying the Prandtl-Reuss flow rule to the yield function, the equation (4.9) is obtained, with $d\lambda$ being the plastic strain-rate multiplier. The plastic flow is schematically represented in Figure 4.5.

$$d\epsilon_p = d\lambda \frac{df}{d\sigma} \quad (4.9)$$

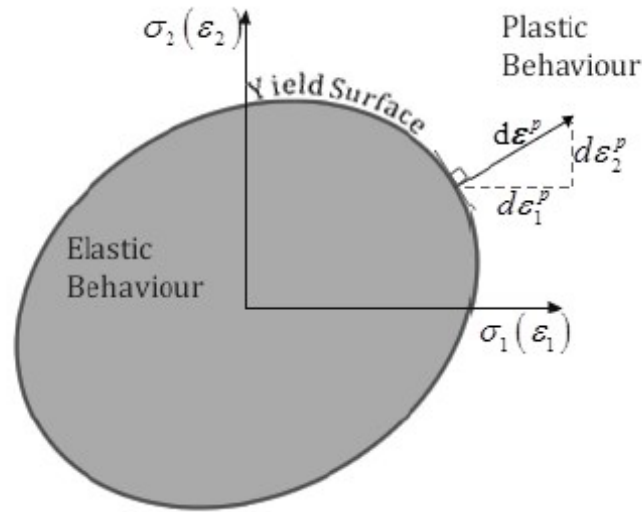


FIGURE 4.5: Flow rule (orthogonality principle) [52].

The following conclusions can be withdrawn from the analysis of Figure 4.5:

- $d\lambda < 0$: Elastic unloading situation. Stress state returns to the inside of the yield surface (softening). The Hooke's law is still valid.
- $d\lambda = 0$: The stress state is on the yield surface. In a material without hardening, σ_Y is independent of the parameter k , which means that we are in presence of a perfectly plastic model, as represented in Figure 4.1. If the material has hardening, then, this situation corresponds to the beginning of the plasticity.
- $d\lambda > 0$: The stress state is on a constantly expanding yield surface. That is the case of a material subjected to hardening ($df/d\sigma \neq 0$). The stress point remains on the expanding yield surface.

4.1.3 Hardening rule

The hardening rule describes how the yield surface changes as the result of plastic deformation. Due to hardening, the elastic domain changes its size, position or both. The elastic-perfectly plastic model (represented in Figure 4.1 (a)) is an exception, since the elastic domain remains constant. As a consequence, the stress state can move along the yield surface, but the surface itself remains unchanged. For all other cases, the hardening rule specifies how the yield function is modified during

plastic flow, usually determined by one of the three hardening rules presented in Figure 4.6 [6], [54], [55].

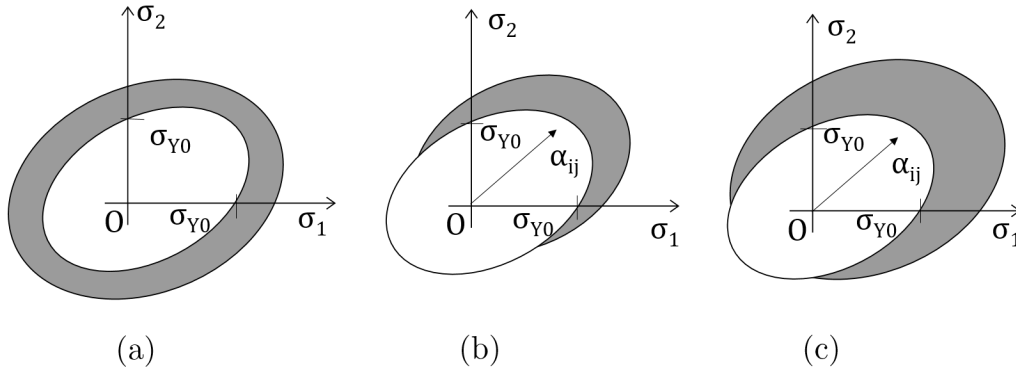


FIGURE 4.6: Hardening Rule: (a) Isotropic Hardening; (b) Kinematic Hardening; (c) Independent Hardening.

The Isotropic Hardening Rule (Figure 4.6 (a)) is the simplest, regarding both the formulation and its implementation. It assumes that the evolution of the yield surface during plastic flow is an expansion of the initial surface, with the same origin; in other words, in this case, the yield surface expands uniformly during plastic flow, only changing its size. The isotropic hardening rule does not consider the Bauschinger effect, since an increase of the yield function has the same effect on tension and compression [6], [55].

In order to understand the Bauschinger Effect, we shall consider a simple example of an uniaxial test, with plastic deformation and stress reversal, as shown in Figure 4.7.

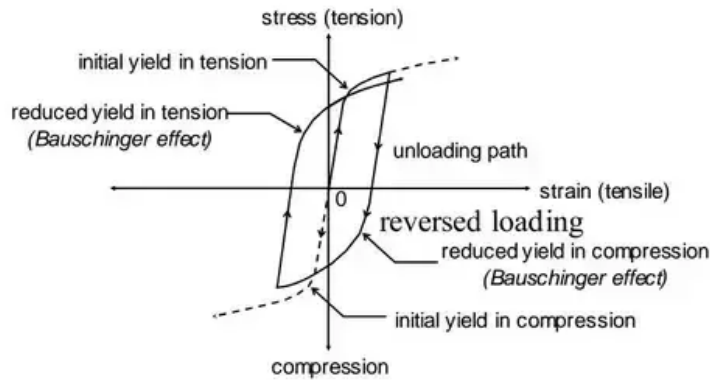


FIGURE 4.7: Stress-strain curve with Bauschinger effect for uniaxial strain hardening [56].

At the start of the loading phase, there is no initial stress or strain in the origin of the diagram, and a tensile load increases in a linear way. As long as the stress is below the yield stress, σ_Y , the material is in the pure elastic range, and Hooke's law, $\sigma = E \cdot \varepsilon$, is valid. Once the yield stress is reached, a plastic deformation occurs and the slope of the diagram is changed. While the load increases, the plastic deformation and strain continue to increase. Once the maximum point is reached, the load is reversed. The unloading is completely elastic and a compressive stress

is developed as the load path crosses the strain axis. However, note that, during the subsequent plastic deformations in the compressive regime, this deformation occurs, for the kinematic hardening, at a stress level which is lower than the initial yield stress. [57]

This phenomenon is known as the Bauschinger effect [57], which is typically produced in metallic materials and requires plastic pre-straining with subsequent load reversal. It acts a kind of directional anisotropy created by the plastic deformation; it is, therefore, extremely important for the design of structural components that may undergo such kind of cyclic tension-compression loading [58].

In the case of the Kinematic Hardening Rule (Figure 4.6 (b)), the evolution of the yield surface consists of the translation of the surface. Then, the size of the yield surface remains constant, due to the unchanged elastic part during hardening. Nevertheless, the centre of the yield surface changes its position (new centre defined by α_{ij}). This kinematic hardening rule considers the Bauschinger effect [6], [55], [57].

The last rule is the Independent Hardening Rule (Figure 4.6 (c)). In this case, the yield surface not only suffers an expansion, but also a translation and/or a rotation. The material is assumed to be hardened independently in tension and in compression [6], [55].

For this work, the isotropic hardening rule, represented in Figure 4.6, was the one considered.

4.2 Nonlinear solution algorithms

The elastoplastic study requires a nonlinear analysis of structures - as such, there is no direct methodology capable of solving the set of non-linear equations; the use of incremental methods is mandatory in order to obtain the solution. By using incremental methods, we intend to achieve an approximate solution of the problem through the application of successive load increments, such as the one schematically represented in Figure 4.8 (a), where the displacement, u , for each load stage, δf , can be obtained with equation (4.12) [54].

$$\delta u = \left(\frac{df}{du} \right)^{-1} \delta f = K_r^{-1} \Delta f \quad (4.10)$$

However, as the example in Figure 4.8 (a) shows, small errors will accumulate throughout the several load stages, due to the lack of equilibrium and, therefore, inevitably, the solution will drift away from the true equilibrium curve. As it would be expected, the more nonlinear the solution is ($df/du \rightarrow 0$), the more deteriorated the obtained results will be. To overcome this problem, an iterative solution, such as the Newton-Raphson method, shall be used. In its simplest form, the Newton-Raphson method only provides the displacement, u , of a fixed load, f , as represented in Figure 4.8 (b), being the iterative changes, j , defined by:

$$\partial u^j = \left(\frac{dq^j}{du} \right)^{-1} q^j(u^j) \quad (4.11)$$

and the successive displacements given by

$$\partial u^j = u^{j-1} + \partial u^{j-1} \quad (4.12)$$

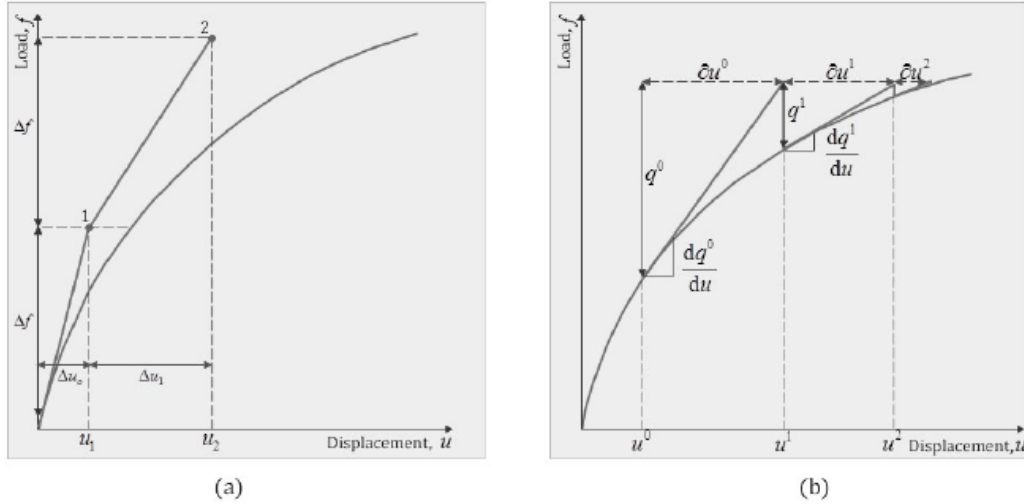


FIGURE 4.8: (a) Incremental (Euler) solution scheme; (b) Newton Raphson method [52].

Furthermore, the iterative Newton-Raphson method can be combined with the incremental method - which we shall call KT-All (Figure 4.9 (a)) - thus, allowing to obtain the problem's solution and the reduction of the lack of equilibrium presented on the incremental process.

As shown in Figure 4.9 (a), the incremental method acts as a predictor, providing the starting solution u_i^0 for the iterative procedure. In Figure 4.9 (a), u_i^j represents the displacement of iteration j for a load increment i . Note that the use of the predictor (incremental method) allows for an improvement of the convergence of the iterative procedure, which, consequently, leads to a reduction of the number of iterations.

For this non-linear solution algorithm, the stiffness matrix, K_{ij} , and its inverse, K_{ij}^{-1} , are calculated at the beginning of each iteration of each of increment. Therefore, for every single increment, the algorithm calculates the stiffness matrix, considering the initial conditions and the previous increments; while this might make the process slow and costly, the number of iterations needed for the convergence of the solution is reduced, as the stiffness matrix is updated regularly. Nevertheless, some variants of this method were progressively formulated, which we will briefly refer to next.

One of these variants is a combination of incremental predictors with the modified Newton-Raphson iterations - KT-1 - where only one tangent stiffness is considered for each stage of the incremental load (generally, the stiffness matrix is calculated for the first iteration of each increment), as represented in Figure 4.9 (b). The fact that the tangent stiffness is not calculated in each of the iterations usually allows for greater savings of time and resources; on the other hand, without the stiffness matrix being updated every iteration, the KT-1 algorithm needs more iterations for each increment to achieve the solution - which tends to translate into a slower convergence of the algorithm.

Moreover, the KT-1 method can be even further simplified, if rather than calculating the stiffness matrix at the beginning of each increment, the same tangent stiffness for each and every incremental and iterative process is considered. In that case, the considered tangent stiffness matrix would be the one from the iteration of the first increment. This method is called the initial stress method combined with an incremental solution - commonly, KT-0 - and is schematically represented in Figure 4.9 (c).

Bearing all of the three algorithms in mind,

- The KT-0 algorithm's advantage would be the time saved by only calculating the stiffness matrix, at the beginning of the very first increment, for the very first iteration. However, the number of iterations needed for the solution to converge ends up being much higher than the other algorithms considered.
- Between the KT-1 and KT-0 methods, the KT-1 usually leads to an inferior number of iterations; notwithstanding, it is necessary to devote more time to the calculation of the tangent stiffness matrix and its inverse.
- The KT-ALL will take more time calculating the stiffness matrix and its inverse at each and every increment and iteration, but will require far less iterations than both the KT-1 and KT-0. The pros and cons of each algorithm shall be weighted for each case.

In this work, only one of the previous referred variants was applied - the Combined incremental and Newton-Raphson method (KT-All), presented in Figure 4.9 (a) for a one dimensional problem.

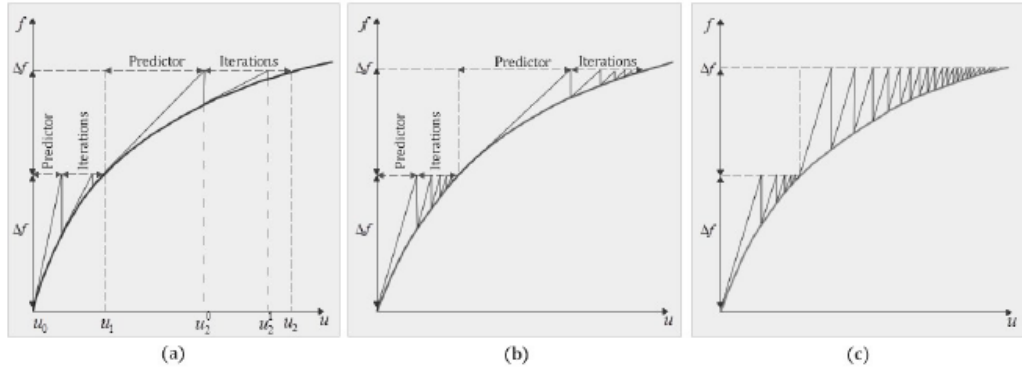


FIGURE 4.9: (a) Combined incremental and Newton Raphson method (KT-All); (b) Combination of incremental predictors with modified Newton-Raphson iterations (KT-1); (c) Initial stress method combined with an incremental solution (KT-0) [52].

4.2.1 Computational/Mathematical Plasticity

Throughout this section, the elastoplastic concepts defined previously in Chapter 4 are now presented in a matrix form.

The von Mises yield criterion, from equation (4.2), is,

$$F(\sigma, \varepsilon_p, k) = f(\sigma, \varepsilon_p, k) - \sigma_Y(k) = 0 \quad (4.13)$$

from which, by differentiating, we obtain

$$dF = \left(\frac{\partial f}{\partial \sigma} \right)^T d\sigma - \left(\frac{\partial \sigma_Y}{\partial k} \right) dk = 0 \quad (4.14)$$

or, as an alternative,

$$dF = a^T d\sigma - A d\lambda = 0 \quad (4.15)$$

where a is the flow vector ($a = \partial f / \partial \sigma$), normal to the yield surface, $d\lambda$ is the plastic strain-rate multiplier, and A is given by the following expression:

$$A = \frac{1}{d\lambda} \frac{\partial \sigma_Y}{\partial k} dk \quad (4.16)$$

The Prandtl-Reuss flow rule for the generic yield criterion, previously defined in equation (4.9), can now be rewritten, as presented in equation (4.17), for the von Mises yield criterion. The effective stress, $\bar{\sigma}$, was also defined in the previous section, in equation (4.6).

$$d\varepsilon_p = d\lambda a = d\lambda \begin{Bmatrix} \partial f / \partial \sigma_{xx} \\ \partial f / \partial \sigma_{yy} \\ \partial f / \partial \sigma_{zz} \\ \partial f / \partial \sigma_{xy} \\ \partial f / \partial \sigma_{yz} \\ \partial f / \partial \sigma_{zx} \end{Bmatrix} = \frac{d\lambda}{2\bar{\sigma}} \begin{Bmatrix} 2\sigma_{xx} - \sigma_{yy} - \sigma_{zz} \\ 2\sigma_{yy} - \sigma_{xx} - \sigma_{zz} \\ 2\sigma_{zz} - \sigma_{xx} - \sigma_{yy} \\ 6\sigma_{xy} \\ 6\sigma_{yz} \\ 6\sigma_{zx} \end{Bmatrix} \quad (4.17)$$

Stress changes are related with strain changes by the following relation:

$$d\sigma = c d\varepsilon_e \quad (4.18)$$

Taking into account the additive decomposition of the strain tensor presented in equation (4.1) and the plastic flow in equation (4.9), it is possible to rewrite equation (4.18) in the following form:

$$d\sigma = c d\varepsilon_e - d\lambda c\mathbf{a} \quad (4.19)$$

In order to define the plastic strain-rate multiplier, $d\lambda$, equation (4.21) is applied to equation (4.15), from which we obtain:

$$d\lambda = \frac{\mathbf{a}^T c d\varepsilon}{\mathbf{a}^T c \cdot \mathbf{a} + A} \quad (4.20)$$

The substitution of equation (4.20) in equation (4.15) gives,

$$d\sigma = c d\varepsilon_e - \frac{\mathbf{a}^T c d\varepsilon}{\mathbf{a}^T c \cdot \mathbf{a} + A} \cdot c \cdot \mathbf{a} = \left(c - \frac{c \cdot \mathbf{a} \cdot \mathbf{a}^T c}{\mathbf{a}^T c \cdot \mathbf{a} + A} \right) d\varepsilon = c^{ep} d\varepsilon \quad (4.21)$$

where c^{ep} is the elastoplastic tangential modular matrix.

At this point, A is the only parameter whose determination is still needed; it was defined previously in equation (4.16), where it becomes clear that A is dependent on the considered hardening rule.

In order to determine this parameter, the isotropic work hardening concept must be applied. Then, from equation (4.13), the yield stress becomes a function of the plastic work, $\sigma_Y(k = W_p)$ where the plastic work, W_p is given by equation (4.22):

$$k = W_p = \int \sigma_Y d\varepsilon_p = \int \sigma^T d\varepsilon_p = \int d\lambda \sigma^T \mathbf{a} \quad (4.22)$$

Thus, the plastic work rate, dk is given by:

$$dk = dW_p = \sigma_Y d\varepsilon_p = \sigma^T d\varepsilon_p = d\lambda \sigma^T \mathbf{a} \quad (4.23)$$

Replacing equation (4.23) into equation (4.16), the hardening constant can be written as:

$$A = \sigma^T \mathbf{a} \frac{\partial \sigma_Y}{\partial k} \quad (4.24)$$

Defining the explicit relation between the uniaxial yield stress (σ_Y) and the hardening parameter, k , the value of A would be totally defined. This relation can be written in the form presented in equation (4.2.1) where the variables are related to the plastic strain ε_p .

$$\frac{\partial \sigma_Y}{\partial k} = \frac{\partial \sigma_Y}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial k} \quad (4.25)$$

Applying equation (4.23) to equation , we obtain:

$$\frac{\partial \sigma_Y}{\partial k} = \frac{\partial \sigma_Y}{\partial \varepsilon_p} \frac{1}{\sigma_Y} \quad (4.26)$$

The relation between σ_Y and ε_p of equation (4.26) can be taken from the uniaxial stress/plastic strain relationship. Then, from Figure 4.2, the aforementioned relation can be defined as:

$$\frac{\partial \sigma_Y}{\partial \varepsilon_p} = \frac{d\sigma}{d\varepsilon - d\varepsilon_e} = \frac{1}{\frac{d\varepsilon}{d\sigma} - \frac{d\varepsilon_e}{d\sigma}} = \frac{1}{\frac{1}{E_T} - 1E} = \frac{E_T}{1 - \frac{E_T}{E}} = H' \quad (4.27)$$

Substituting equation (4.27) in equation (4.26),

$$\frac{\partial \sigma_Y}{\partial k} = \frac{H'}{\sigma_Y} \quad (4.28)$$

And then, the hardening constant, A , is defined as:

$$A = \frac{H' \sigma^T a}{\sigma_Y} \quad (4.29)$$

If the yield function is an homogeneous function of order one, the Euler's theorem shows that:

$$\frac{\partial f^T a}{\partial \sigma_Y} \sigma = a^T \sigma = \sigma_Y \quad (4.30)$$

Hence, $A = H'$, as concluded in equation (4.27).

4.2.2 Stress Returning Algorithm

As previously mentioned, the main purpose of the use of incremental-iterative methods is to avoid the accumulation of errors during the solving of the non-linear equations that govern the elastoplastic problems. Therefore, it is necessary to ensure that the obtained stresses remain under the yield surface or, at least, very close to it. Otherwise, errors would accumulate and the computed load will generally be over predicted.

To overcome this issue, some procedures - which we call stress returning algorithms - can be applied when the stresses are outside the yield surface [54]. The procedure adopted, in this work, for such cases, is the backward-Euler scheme - represented in Figure 4.10.

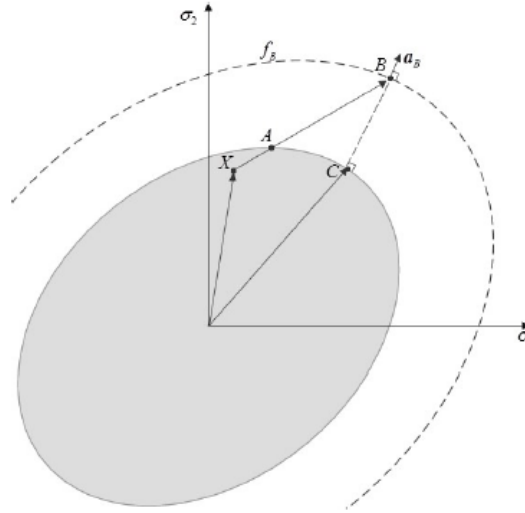


FIGURE 4.10: Backward-Euler scheme [52].

In this method, which acts at the Gauss point level to solve the nonlinear equations, there is no need to know the intersection point of the incremental load with the yield surface.

Thereby, after the incremental load application, and for each Gauss point, there is a verification of whether the obtained stress state is inside or outside the yield

surface. In order to achieve that, the yield criterion presented in equation (4.31) is verified, where $f(\sigma)$ is the yield surface, defined in previous sections, and σ_Y^* is the updated yield stress, defined in equation (4.32). In equation (4.32), ε_{i-1}^p is the accumulated effective plastic strain from the previous increment, $(i - 1)$. Then, if the Gauss point in question has not entered in plasticity yet, the second term of equation (4.32) will be null, and so, the updated yield stress equals the reference yield stress.

$$f(\sigma) \leq \sigma_Y^* \quad (4.31)$$

$$\sigma_Y^* = \sigma_Y + [E_T \cdot \varepsilon_{i-1}^p] \quad (4.32)$$

This stress returning algorithm is called every time that equation (4.31) is not verified. In that case, the present point is point B (of Figure 4.10), and this state must be pushed into point C, on the yield surface. In order to do that, the implemented algorithm starts with a predictor. This predictor simulates that point B is under a yield function f_B , and thus, avoids the necessity of actually computing the intersection point A. The flow vector, a , is calculated for point B, and the yield function for this point B is given by $f_B = \bar{\sigma}_B - \sigma_Y^*$. Afterwards, the plastic multiplier, d is calculated, by equation (4.33), with H' being defined in equation (4.27):

$$d\lambda = \frac{f_B}{\mathbf{a}_B^T \cdot \mathbf{c} \cdot \mathbf{a}_B + H'} \quad (4.33)$$

Thus, it is possible to calculate an estimate of the stress in point C, as:

$$\sigma_C = \sigma_B - d\lambda \cdot \mathbf{c} \cdot \mathbf{a}_B \quad (4.34)$$

Note that this is only an estimation, since the correct procedure to achieve the stress in point C would be using the flow vector of that same point, using equation (4.35):

$$\sigma_C = \sigma_B - d\lambda \cdot \mathbf{c} \cdot \mathbf{a}_C \quad (4.35)$$

Nevertheless, the flow vector, $\mathbf{C}$, cannot be computed only with the data from points X and B. Thus, an iterative procedure using the data from point B must be used in order to approximate point B to point C. In each of these iterations, the effective plastic strain is updated, in order to apply equation (4.32) and repeat all the referred process. At the end, point B would be very close to point C - with "very close" being associated with a given tolerance, that is considered acceptable for the proximity. The whole process involves several iterations until this tolerance is reached and the next Gauss point is analyzed.

4.2.3 The KT-All Algorithm - Mathematical Formulation

Since the KT-All will be the main non-linear solution algorithm to be used in this work, some details on its mathematical formulation are provided.

In order to initiate the analysis, one must introduce some data, such as the problem dimensions, the type of nodal mesh that is intended to obtain, the material's properties, the value and type of the maximum load to apply, the essential boundary conditions and, at last, the number of increments, iterations and the value of the tolerance.

For all the numerical methods used, the pre-processing must occur before the algorithm initializes. Therefore, for either FEM, RPIM or NNRPIM, the creation of

the nodal mesh, integration mesh and the shape functions is performed beforehand. While for the FEM, a specific type of element is chosen, for both RPIM and NNRPIM, certain parameters must be specified - as explained previously in Chapter 3.

Taking into account the material's properties and the connectivity between nodes, the initial stiffness matrix is determined. Afterwards, the natural and essential boundary conditions are applied, and the first stiffness matrix, K_{11} , is obtained. Note that, for this algorithm, the stiffness matrix will be updated at the start of each iteration of each increment. As such, in terms of nomenclature, K_{ij} refers to the stiffness matrix of increment i and iteration j .

Then, the incremental load process is started - the incremental load, f_i , applied is given by equation (4.36), where F is the maximum load and inc is the considered number of increments.

$$\mathbf{f}_i = \frac{F}{inc} \quad (4.36)$$

Once f_i is defined, it is possible to determine the displacement field by using equation (4.37) and then, the stress field, using equation (4.38).

$$\mathbf{u}_{ij} = \mathbf{K}_{ij}^{-1} \mathbf{f}_i \quad (4.37)$$

$$\sigma_{ij} = \mathbf{c} \cdot \mathbf{B} \cdot \mathbf{u}_{ij} \quad (4.38)$$

Then, the stress on each Gauss point is obtained, to verify if it is outside of the yield surface - in which case, it has entered the plastic domain. Thus, in order to do that, the stress returning algorithm referred in the previous section is applied. At the end of this algorithm, the updated stress vector of the increment, $\delta\sigma$, is obtained. The stress vector of the iteration can be obtained by equation (4.39), where $\sigma_i(j-1)$ is the stress vector of the previous iteration:

$$\sigma_{ij} = \sigma_{i(j-1)} + \delta\sigma_{ij} \quad (4.39)$$

Furthermore, if the present iteration of the present increment on a specific Gauss point has passed beyond the yield surface, the residual forces are calculated,

$$\mathbf{f}_i^{res} = \mathbf{f}_i - \int_{\Omega} \mathbf{B}^T \delta\sigma d\Omega \quad (4.40)$$

After calculating the residual force, it is necessary to evaluate the significance of its value. This tolerance verification serves, as well, as a stop criterion for the algorithm. If the residual forces are under the given tolerance, the iteration stops and the next increment is applied. If not, the next iteration within the same increment begins.

After the application of all the load increments, the load-displacement curves can be obtained, as well as the stress and displacements fields.

Chapter 5

Mechanics of honeycombs and foams

The mechanical response of cellular solids can be modeled by one of three ways: analyzing a unit cell using classical mechanics; using a dimensional analysis without specifying the exact cell geometry (which requires experiments to determine the constants related to the cell geometry), or using numerical methods, such as the finite element method or meshless methods. Although this work will be focused on the application of numerical methods, in this chapter, some of the analytical expressions obtained throughout the years are reviewed. These allow for a better understanding and prediction of the numerically and experimentally analyzed structures, and can be used as a reference or starting point. The application of interest of these geometries - the sandwich structures with foam cores - is then briefly analyzed.

5.1 Honeycombs

Studying the mechanics of honeycombs is important for two main reasons: the first one is that they are, in fact, used in a variety of applications, from cheap doors to advanced aerospace components; the second one is that the results obtained shed light on the mechanics of much more complex three-dimensional foams. While analyzing foams is a difficult task, honeycombs, due to their regular geometry, are much easier to understand. Their deformations can be analyzed more or less exactly to give equations describing their properties, and the results of the analysis can be verified by experiments on tailor-made models.

In this section, the analysis will be focused solely on the uniaxial loading of elastic-plastic solids. More information on elastomeric materials, brittle materials and biaxial loading can be found, for example, in [10].

A simple representation of a hexagonal honeycomb structure can be found in Figure 5.1. The general picture of the deformation mechanism is the following:

If a honeycomb is compressed in-plane, the cell walls at first bend, giving linear elastic deformation. Beyond a critical strain the cells collapse by elastic buckling, plastic yielding, creep or brittle fracture, depending on the nature of the cell wall material. Cell collapse ends once the opposing cell walls begin to touch each other; as the cells close up the structure densifies and its stiffness increases rapidly. In tension the cell walls at first bend (as in compression) but elastic buckling is not possible. If the cell wall material yields plastically, the honeycomb itself shows extensive plasticity; if the cells are brittle, it fractures. Thus, the in-plane stiffnesses and strengths (X_1 — X_2 plane) are the lowest.

On loading out-of-plane, the cell walls suffer extension or compression. The out-of-plane stiffnesses and strengths (those in the X_3 direction) are, therefore, much

larger, as they require the axial extension or compression of the cell walls. The out-of-plane analysis gives the additional stiffnesses which are needed for the design of honeycomb cores in sandwich structures [10].

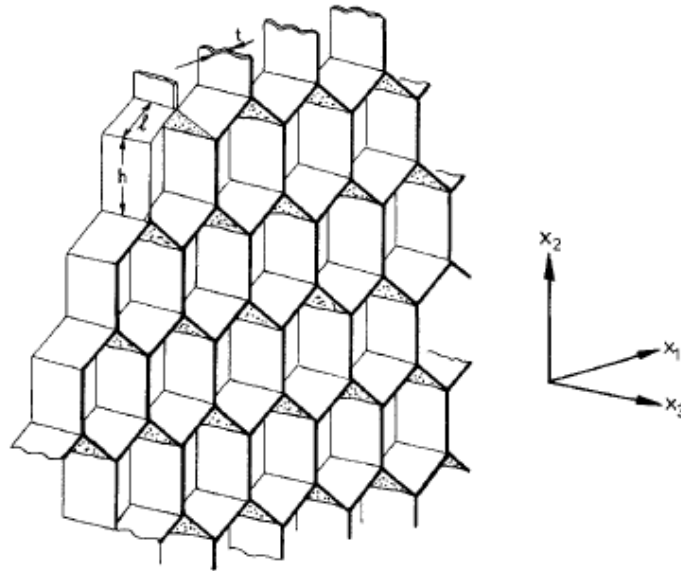


FIGURE 5.1: A honeycomb with hexagonal cells. The in-plane properties are those relating to loads applied in the X_1 - X_2 plane. Responses to loads applied to the faces normal to X_3 are referred to as the out-of-plane properties. [10].

5.1.1 In-plane deformation

Figure 5.2 shows compressive and tensile stress-strain curves for a honeycomb made of an elastic-plastic material.

In **compression**, there are three regimes: a linear-elastic regime, followed by a plateau of roughly constant stress, leading into a final regime of steeply rising stress. Each regime is associated with a *mechanism of deformation*. On first loading, the cell walls bend, giving linear elasticity (provided, of course, that the cell wall material is itself linear-elastic). But when a critical stress is reached the cells begin to collapse, by the formation of plastic hinges at the section of maximum moment in the bent members. Eventually, at high strains, the cells collapse sufficiently that opposing cell walls touch (or their broken fragments pack together) and further deformation compresses the cell wall material itself. This gives the final, steeply rising portion of the stress-strain curve, labelled *densification*.

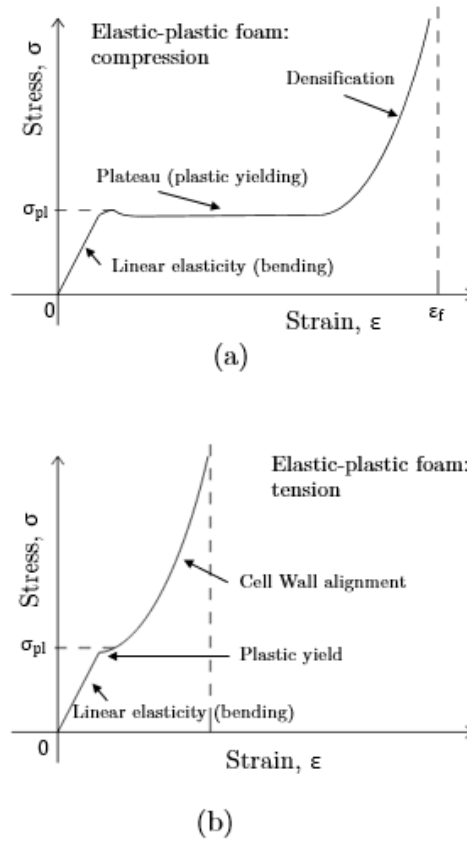


FIGURE 5.2: Compressive and tensile stress-strain curves for an elastic-plastic honeycomb (in-plane deformation). Adapted from [10].

An increase in the relative density of a honeycomb increases the relative thickness of the cell walls. Then, the resistance to cell wall bending and cell collapse goes up, giving a higher modulus and plateau stress; and the cell walls touch sooner, reducing the strain at which densification begins. The mechanisms for **compressive deformation** of honeycombs, and shows how the stress-strain curve changes with increasing relative density or the relation t/l is presented in Figure 5.3.

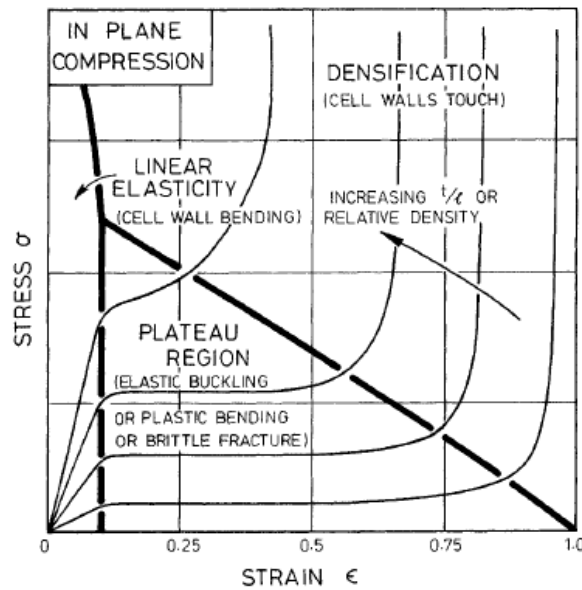


FIGURE 5.3: Schematic diagram for a honeycomb loaded in compression in the X_1 — X_2 plane, showing the linear-elastic, collapse and densification regimes, and the way the stress-strain curve changes with t/l [10].

In **tension**, plastic honeycombs (in particular) behave in a similar way as they do in compression: initially, the cell walls bend, giving linear-elastic deformation with the same slope (and so the same modulus) as in compression. Then, plastic hinges form, allowing large deformations at a nearly constant *plateau* stress; however, the geometry change introduces a difference, which usually represents pushing the tensile curve above the compressive one.

5.1.2 Out-of-plane deformation

Honeycombs are much stiffer and stronger when loaded along the cell axis - the X_3 direction in Figure 5.1. The same is true for honeycombs loaded in out-of-plane shear (as they are in sandwich panels loaded in bending). In this case, the initial linear-elastic deformation involves significant axial or shear deformations of the cell walls themselves.

In **compression**, the linear-elastic regime is truncated by plastic buckling and final failure is by tearing or crushing, giving a stress-strain curve similar to the one shown in Figure 5.4. In **tension**, the honeycomb yields plastically. The stress-strain curves for honeycombs with a range of relative densities again form a family as shown in Figure 5.5.

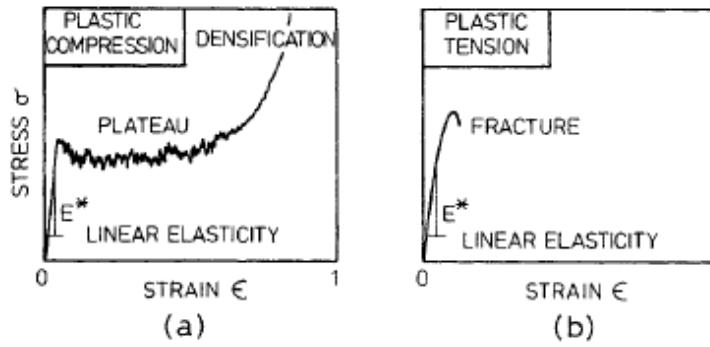


FIGURE 5.4: Stress-strain curves for the axial (X3) loading of a honeycomb, in (a) compression and (b) tension [10].

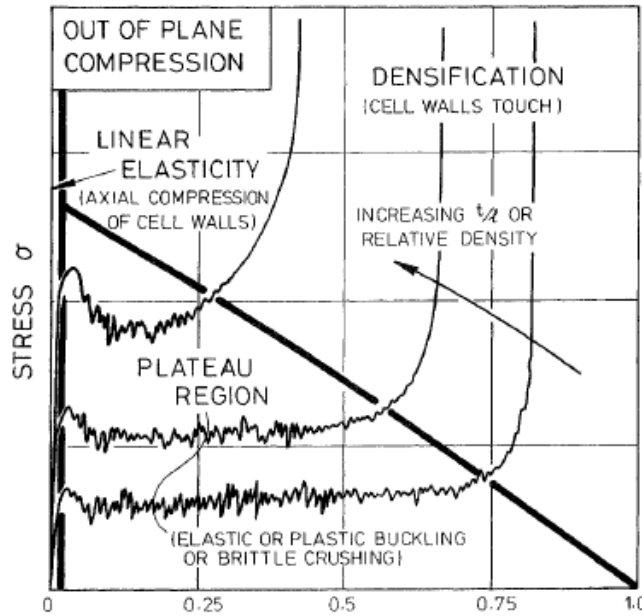


FIGURE 5.5: A schematic diagram for honeycombs loaded in the axial (X3) direction, showing regimes of linear elasticity, collapse and densification, and the way in which the stress-strain curve changes with t/l [10].

5.1.3 In-plane loading

In this work, it is intended to analyse its response to loads applied in the X1—X2 plane. If the hexagon is regular (that is, the sides are equal and the angles are all 120°) and the cell walls are all of the same thickness, then the in-plane properties are isotropic: they do not depend on direction. Such a structure has two independent elastic material properties (a Young's modulus E^* and a shear modulus G^* , for instance) and a single value of the plateau stress, σ^* . But when the hexagon is irregular or the cell walls in one direction are thicker than those in the other directions, the properties are anisotropic and a complete description of the in-plane properties requires four material properties (E_1^* , E_2^* , G_{12}^* and ν_{12}^* , and two values for the plateau stress (σ_1^* and σ_2^*).

For a general honeycomb, for which h and l differ and for arbitrary cell wall angle, θ , the isotropic results are easily obtained. We assume that the honeycomb has a low relative density, ρ^*/ρ_s , so that t/l is small; simple geometry gives the relation between the two as:

$$\frac{\rho^*}{\rho_s} = \frac{t/l(h/l + 2)}{2\cos\theta(h/l + \sin\theta)} \quad (5.1)$$

being the variables defined in Figure 5.6. We also assume that deformations are sufficiently small that changes in geometry can be neglected.

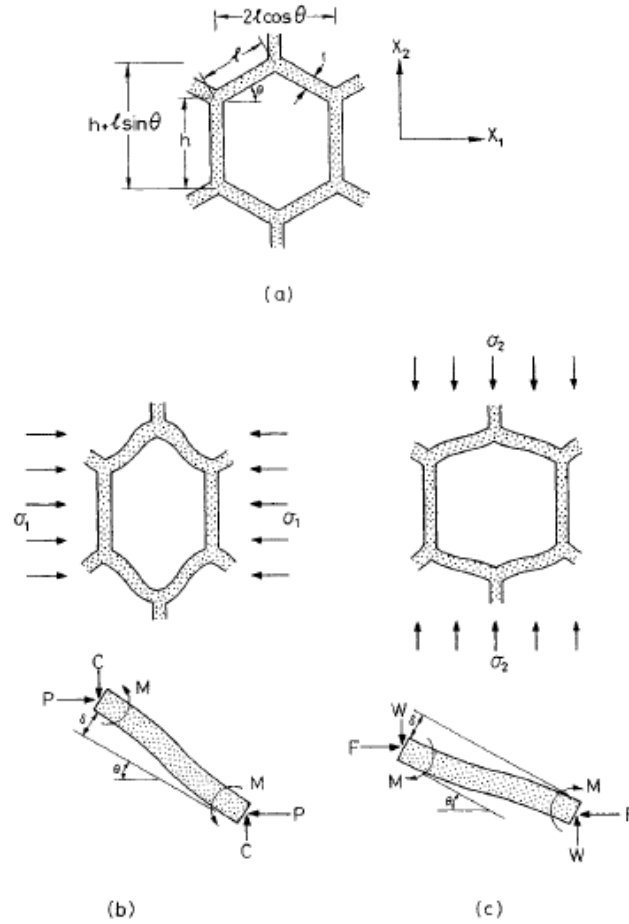


FIGURE 5.6: Cell deformation by cell wall bending, giving linear elastic extension or compression of the honeycomb: (a) the undeformed honeycomb; (b) and (c) the bending caused by loads in the X_1 and X_2 directions [10].

Linear-elastic deformation

When a honeycomb, loaded in the X_1 or the X_2 direction, deforms in a linear elastic way, the cell walls bend; the response is conveniently described by five material properties: two Young's material properties E_1^* , E_2^* , a shear modulus G_{12}^* and two Poisson's ratios, ν_{12}^* and ν_{21}^* . But the five are not independent. There is a reciprocal relation,

$$E_1^* \nu_{21}^* = E_2^* \nu_{12}^* \quad (5.2)$$

for which the explanation can be found on [10], which reduces the number of independent material properties to four. They are the same in tension as in compression, and are calculated as follows.

The two Young's material properties are calculated by the method illustrated in 5.6. A stress σ_1 parallel to X1 causes one set of cell walls - those of length l - to bend; one cell wall is shown in detail beneath the cell. Equilibrium requires that the component of force C parallel to X2 be zero. The moment M tending to bend the cell wall (which we treat as a beam of length, l , thickness, t , depth b , and Young's modulus, E_s) is:

$$M = \frac{Pl \sin \theta}{2} \quad (5.3)$$

where

$$P = \sigma_1(h + l \sin \theta)b \quad (5.4)$$

And, from standard beam theory, the wall deflects by:

$$\delta = \frac{Pl^3 \sin \theta}{12E_s I} \quad (5.5)$$

where I is the second moment of inertia of the cell wall ($I = bt^3/12$) for a wall of uniform thickness t). Of this, a component $\delta \sin \theta$ is parallel to the X1 axis, giving a strain:

$$\epsilon_1 = \frac{\delta \sin \theta}{l \cos \theta} = \frac{\sigma_1(h + l \sin \theta)bl^2 \sin^2 \theta}{12E_s I \cos \theta} \quad (5.6)$$

And therefore, the Young's modulus **parallel to X1** is just $E_1^* = \sigma_1 / \epsilon_1$, giving:

$$\frac{E_1^*}{E_s} = \left(\frac{t}{l}\right)^3 \frac{\cos \theta}{(h/l + \sin \theta) \sin^2 \theta} \quad (5.7)$$

Loading in the X2 direction is shown in Figure 5.6(c); the forces acting on the cell wall of length, l , and depth, b , are shown in the bottom part. The Young's Modulus for loading in the X2 direction and the shear modulus for loading in the X1-X2 direction can be found in a similar way [10]. These are given by:

$$\frac{E_2^*}{E_s} = \left(\frac{t}{l}\right)^3 \frac{(h/l + \sin \theta)}{\cos^3 \theta} \quad (5.8)$$

For regular hexagons with walls of uniform thickness, both Young's material properties, E_1^* and E_2^* reduce to the same value:

$$\frac{E_1^*}{E_s} = \frac{E_2^*}{E_s} = 2.3 \left(\frac{t}{l}\right)^3 \quad (5.9)$$

And the shear modulus,

$$\frac{G_{12}^*}{E_s} = \left(\frac{t}{l}\right)^3 \frac{(h/l + \sin \theta)}{(h/l)^2(1 + 2h/l) \cos \theta} \quad (5.10)$$

Which reduces, for regular, uniform hexagons, to:

$$\frac{G_{12}^*}{E_s} = 0.57 \left(\frac{t}{l} \right)^3 = \frac{1}{4} \frac{E^*}{E_s} \quad (5.11)$$

Which correctly obeys the well known relation $G = E/(2(1 + \nu))$ for isotropic solids.

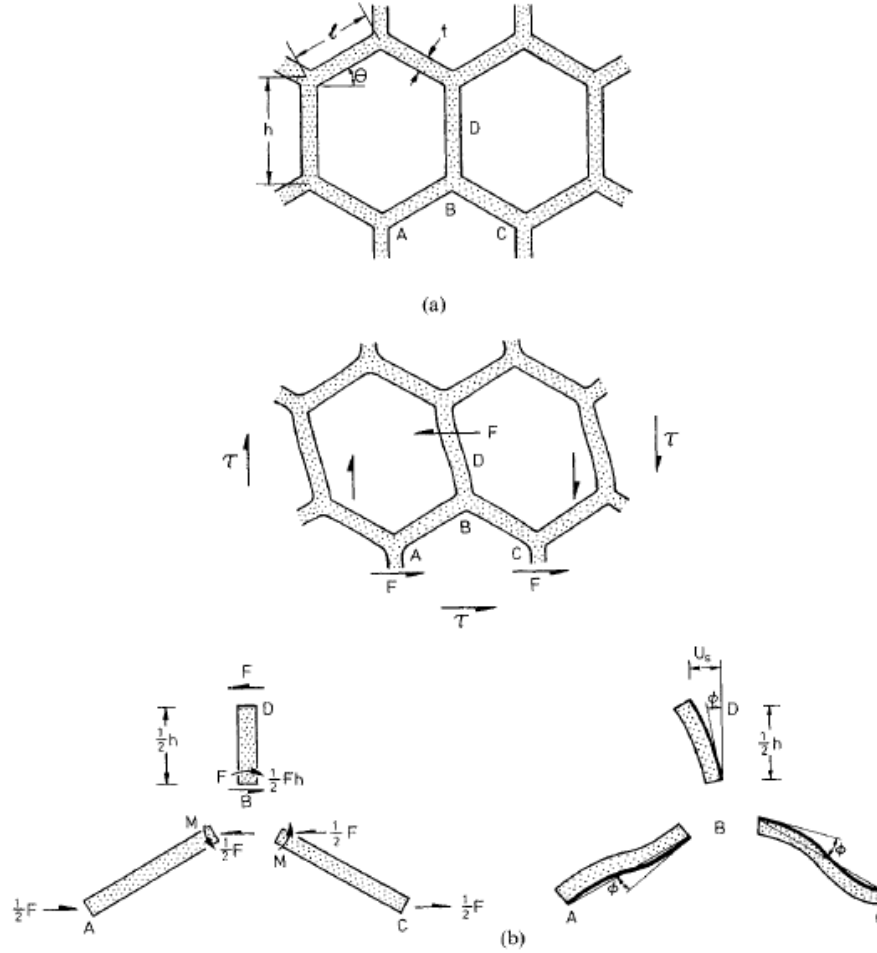


FIGURE 5.7: Cell deformation by cell wall bending and rotation, giving linear elastic shear of the honeycomb: (a) the undeformed honeycomb; (b) the loads, moments, displacements and rotations caused by a shear stress [10].

Then, the Poisson's ratios are calculated by taking the negative ratio of the strains normal to, and parallel to, the loading direction. We find for **loading in the X1 direction**:

$$\nu_{12}^* = -\frac{\varepsilon_2}{\varepsilon_1} = \frac{\cos^2 \theta}{(h/l + \sin \theta) \sin \theta} \quad (5.12)$$

which, for regular hexagons, gives $\nu_{12}^* = 1$. For **loading in the X2 direction**:

$$\nu_{21}^* = -\frac{\varepsilon_1}{\varepsilon_2} = \frac{(h/l + \sin \theta) \sin \theta}{\cos^2 \theta} \quad (5.13)$$

again, giving $\nu_{21}^* = 1$ for regular hexagons. It is interesting to note that when θ is negative (that is, the cells are inverted), the Poisson's ratio is negative, implying that a compressive stress in one direction produces a contraction in the normal plane-direction, instead of the usual expansion. When that is the case, we say that the material exhibits an auxetic behaviour. [9]. An example of this is presented in Figure 5.8.

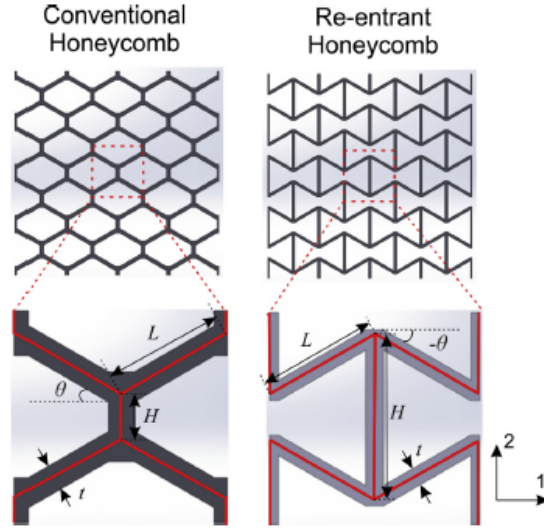


FIGURE 5.8: Example of a conventional honeycomb vs. a honeycomb with auxetic behaviour [9].

Plastic collapse

Honeycombs made of elastic-plastic solids collapse plastically when the bending moment in the cell walls reaches the fully plastic moment. This gives a stress-strain curve with a plateau both in compression and in tension at the plastic collapse stress, σ_{pl}^* . Consider the loading in the X1 direction first, Figure 5.9. An upper bound on the plastic collapse stress is given by equating the work done by the force:

$$P = \sigma_1(h + l\sin\theta)b \quad (5.14)$$

During a plastic rotation ϕ of the four plastic hinges, A, B, C and D, to the plastic work done at the hinges giving:

$$4M_p\phi \geq 2\sigma_1b(h + l\sin\theta)\phi l\sin\theta \quad (5.15)$$

where σ_{ys} is the yield stress of the cell-wall material. It follows that:

$$\frac{(\sigma_{pl}^*)_1}{\sigma_{ys}} = \left(\frac{t}{l}\right)^2 \frac{1}{2(h/l + \sin\theta)\sin\theta} \quad (5.16)$$

On the other hand, a lower bound is given by equating the maximum moment in the beam, M_p . The maximum moment is:

$$(M_{max})_1 = \frac{1}{2}\sigma_1(h + l\sin\theta)bl\sin\theta \quad (5.17)$$

from which:

$$\frac{(\sigma_{pl}^*)_1}{\sigma_{ys}} = \left(\frac{t}{l}\right)^2 \frac{1}{2(h/l + \sin\theta)\sin\theta} \quad (5.18)$$

The lower and upper bounds are identical, and thus define the exact solution to the problem. For regular uniform hexagons, it reduces to:

$$\frac{(\sigma_{pl}^*)_1}{\sigma_{ys}} = \frac{2}{3} \left(\frac{t}{l}\right)^2 \quad (5.19)$$

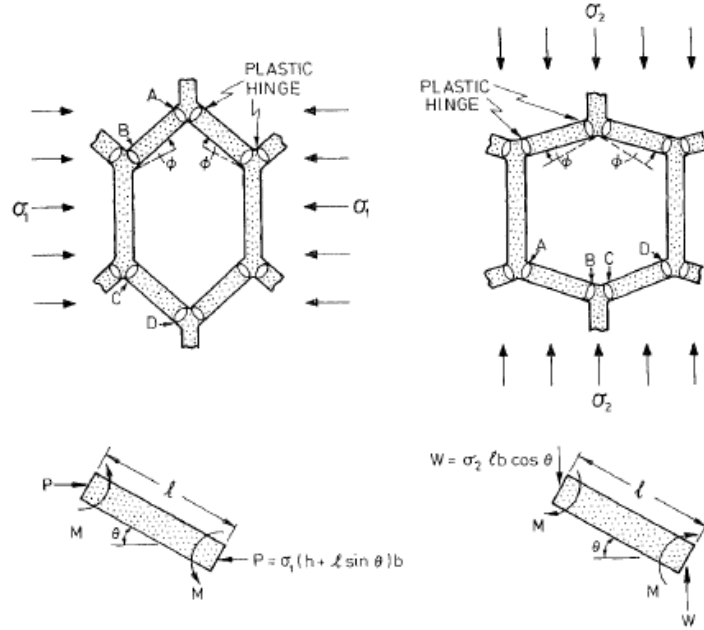


FIGURE 5.9: Cell deformation by plastic collapse: the rotations, forces and moments for loading in the X1 and X2 directions [10].

Plastic collapse in the X2 direction is treated in the same way. An upper bound is given by equating the plastic work done in a postulated compatible deformation to the work done by the stress (see Figure 5.9). A lower bound is given by equating the maximum moment in the beam:

$$(M_{max})_2 = \frac{\sigma_2 l^2 b \cos 2\theta}{2} \quad (5.20)$$

to its fully plastic moment. As before, the two results are identical, and thus equal to the exact equation:

$$\frac{(\sigma_{pl}^*)_2}{\sigma_{ys}} = \left(\frac{t}{l}\right)^2 \frac{1}{2\cos^2\theta} \quad (5.21)$$

which, for regular hexagons ($\theta = 30^\circ$) reduces to equation (5.19), that is, regular hexagons have the same plastic collapse in the X1 and X2 directions.

Plastic shear is dealt with in a similar way. If a honeycomb is loaded in simple shear with a shear stress τ_{12} acting in the X1—X2 plane, plastic hinges form in the vertical walls. The shear force on each vertical wall is $F = 2\tau_{12}bl\cos\theta$. If the moment $Fh/2$ exerted by this force exceeds the fully plastic moment of Equation (5.17), the honeycomb shears plastically, giving a lower bound for the in-plane shear strength of:

$$\frac{(\tau_{pl}^*)_{12}}{\sigma_{ys}} = \frac{1}{4} \left(\frac{t}{l} \right)^2 \frac{1}{h/l \cos \theta} \quad (5.22)$$

which, for regular hexagons, reduces to:

$$\frac{(\tau_{pl}^*)_{12}}{\sigma_{ys}} = \frac{1}{2\sqrt{3}} \left(\frac{t}{l} \right)^2 \quad (5.23)$$

The upper bound for the in-plane shear strength is found by equating the plastic work done by the force F on the vertical wall with the plastic work done by the plastic moment at the hinge. The result is the same as the lower bound, which therefore gives the exact solution.

Under some conditions elastic buckling can precede plastic collapse. For this to happen, the density must be lower than a critical value which is found when the elastic collapse stress, $(\sigma_{el}^*)_2$ equals the plastic collapse stress, $(\sigma_{pl}^*)_2$, giving:

$$\left(\frac{t}{l} \right)_{crit} = \frac{12}{n^2 \pi^2 \cos \theta} \left(\frac{h}{l} \right)^2 \frac{\sigma_{ys}}{E_s} \quad (5.24)$$

Giving, for regular hexagons,

$$\left(\frac{t}{l} \right)_{crit} = 3 \frac{\sigma_{ys}}{E_s} \quad (5.25)$$

For many metals, σ_{ys}/E_s is of order 10^{-3} ; for these, elastic buckling precedes plastic collapse only when relative density $(2t/\sqrt{3}l)$ is very low - less than 0.3%. However, many polymers yield at a stress of around $E_s/100$ and then, elastic collapse occurs first for all densities below about 3%.

5.1.4 Out-of-plane loading

Honeycombs are often used as cores in sandwich panels. One must note that the function of the honeycomb core is to carry normal and shear loads in planes containing the axis of the hexagonal prisms - the X_3 direction in Figure 5.10. When loaded in this direction, the cell walls are extended or compressed (rather than bent) and the material properties, for hexagonal honeycombs, are much larger than those calculated in the previous section, for the in-plane loading. The plastic collapse strengths, too, are larger, and for the same reason: that axial, as well as bending, deformations are involved. Throughout this section, we assume a low density, so that $t \ll l$, and that all the walls have the same thickness, t ; the extension to walls of differing thicknesses is straightforward.

Linear-elastic deformation

Five additional material properties are needed to describe out-of-plane deformation, giving a total of **nine** for a complete description of the honeycomb.

One is obvious: Young's modulus E_3^* for normal loading in the X_3 direction simply reflects the solid modulus, E_s , scaled by the area of the load-bearing section:

$$\frac{E_3^*}{E_s} = \frac{h/l + 2}{2(h/l + \sin\theta)\cos\theta} \frac{t}{l} = \rho^*/\rho_s \approx \frac{t}{l} \quad (5.26)$$

Two more are straightforward: the two Poisson's ratios ν_{31}^* and ν_{32}^* are simply equal to those for the solid itself, that is:

$$\nu_{31}^* = \nu_{32}^* = \nu_s \quad (5.27)$$

The Poisson's ratios ν_{13}^* and ν_{23}^* are then found from the reciprocal relations (explained in detail in [10]):

$$\nu_{13}^* = \frac{E_1^*}{E_3^*} \nu_s \approx 0, \quad \nu_{23}^* = \frac{E_2^*}{E_3^*} \nu_s \approx 0. \quad (5.28)$$

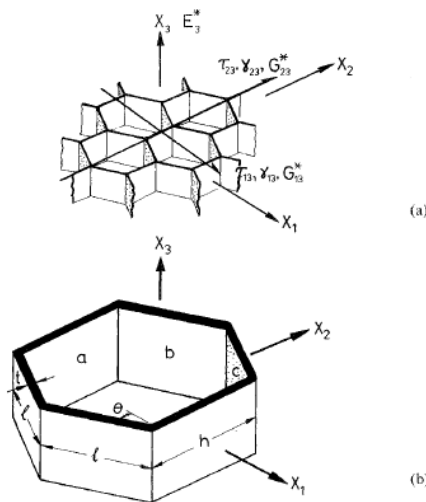


FIGURE 5.10: (a) A honeycomb carrying loads on the faces normal to X_3 ; (b) One cell, showing the walls a, b and c [10].

The shear material properties, however, are more complicated, because of the fact that the stress distribution in a sheared honeycomb is not so simple; each cell face suffers a non-uniform deformation because of the constraints imposed on it by its neighbours, and the initially plane honeycomb may not remain plane. Exact calculations are possible by using numerical methods. But upper and lower bounds for the two shear material properties can be formulated relatively easily by using a simplification of the method used by Kelsey et al., in 1958, which is thoroughly explained in [10]. The results obtained are, for loading in the X1 direction,

$$\frac{G_{13}^*}{G_s} = \frac{\cos\theta}{(h/l + \sin\theta)} \left(\frac{t}{l} \right) \quad (5.29)$$

and for loading in the X2 direction,

$$\frac{G_{23}^*}{G_s} = \frac{(h/l + \sin\theta)}{(1 + 2h/l)\cos\theta} \left(\frac{t}{l} \right) \quad (5.30)$$

For regular hexagons, both equations (5.29) and (5.30) reduce to:

$$\frac{G_{23}^*}{G_s} = 0.577 \left(\frac{t}{l} \right) \quad (5.31)$$

Which confirms that regular hexagons are isotropic in the X1-X2 plane. It is important to note that in calculating the upper and lower bounds, it was assumed that the shear stresses are uniform within the cell walls.

Plastic collapse

If the net section stress in the plane normal to X3 (roughly $\sigma_3(l/t)$), exceeds the yield strength σ_{ys} of the cell-wall material, then the cell walls will yield axially. This defines an upper limit for the plastic collapse strength of the honeycombs:

$$\frac{(\sigma_{pl}^*)_3}{\sigma_{ys}} = \frac{(h/l + 2)}{2\cos\theta(h/l + \sin\theta)} \frac{t}{l} = \frac{\rho^*}{\rho_s} \quad (5.32)$$

This equation properly describes the axial strength in tension; however, in compression, this limit is seldom reached because plastic buckling occurs first. The axial collapse of hexagonal honeycombs by plastic buckling has been treated, for example, by Wierzbicki, in 1983 [10]. The cells fold progressively with a wavelength λ , which is often roughly equal to the cell side length l . Figure 5.11(a) shows a two-dimensional schematic of the buckling, but this conceals the problem of describing the displacement where the cell faces meet. Wierzbicki shows that a geometry of collapse is possible, which involves reduced extension or compression of the cell wall - it is achieved largely by bending. An isolated wall can collapse, as shown in Figure 5.11(b). He gives an estimate of the plastic buckling stress as:

$$\frac{(\sigma_{pl}^*)_3}{\sigma_{ys}} \approx \frac{\pi}{4} \frac{(h/l + 2)}{(h/l + \sin\theta)\cos\theta} \left(\frac{t}{l} \right)^2 \quad (5.33)$$

which, for regular hexagons with a uniform thickness t , reduces to

$$\frac{(\sigma_{pl}^*)_3}{\sigma_{ys}} = 5.6 \left(\frac{t}{l} \right)^{5/3} \quad (5.34)$$

or, for honeycombs in which two of the six cell walls are double:

$$\frac{(\sigma_{pl}^*)^3}{\sigma_{ys}} = 6.6 \left(\frac{t}{l} \right)^{5/3} \quad (5.35)$$

The stress for collapse by plastic buckling given by this equation is less than that for simple plastic compression, equation (5.32) when $t/l < 0.1$, so for all but rather dense honeycombs, the compressive collapse mode is by plastic buckling.

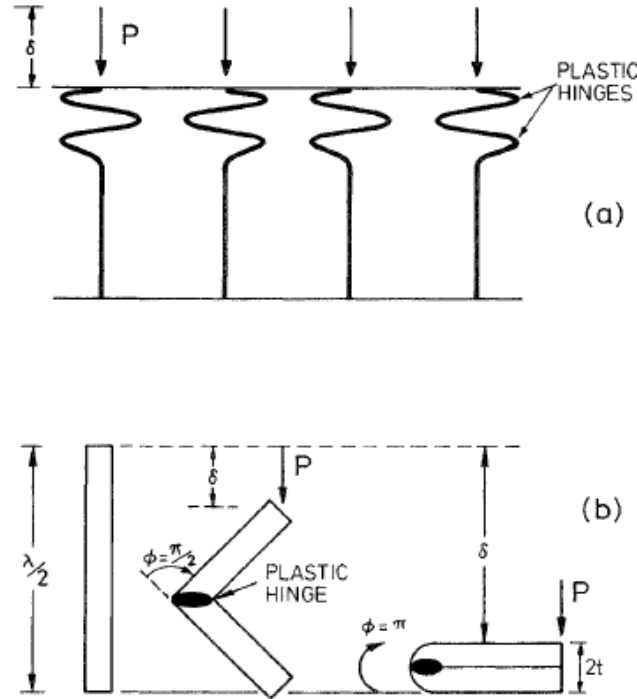


FIGURE 5.11: (a) Schematic of the plastic buckling of a honeycomb, loaded in the X3 direction: the walls fold in a periodic way; (b) the work done by the load P on one wall is dissipated in the plastic hinge [10].

As a conclusion for the analysis of honeycombs in uniaxial loading: honeycombs deform by a number of different mechanisms. When loaded in-plane, the cell walls at first bend, giving linear elasticity which can persist to strains as large as 10%. This is possible because the honeycomb is like a spring: the geometry allows a large distortion of the structure with only small strains in its members (the strain in the cell wall is less, by a factor of about t/l , than that of the honeycomb as a whole). When loaded out-of-plane, the cell walls are extended, compressed or sheared, and the structure is much stiffer. Simple mechanics gives the full set of 9 elastic constants required to describe a general orthotropic honeycomb. For regular honeycombs the number is reduced to five.

The complications start when linear elasticity breaks down. If it is made of a plastic material, the cell walls yield - by plastic bending for compressive and tensile in-plane loading and by plastic buckling or axial yield for compressive or tensile loading out-of-plane. This gives a plateau in the stress-strain curve, and the deformation associated with it is permanent. The in-plane strength, in every instance, is lower than that for out-of-plane loading because bending predominates in the first case, while axial deformation predominates in the second.

5.2 Foams

The efficient use of foams requires a detailed understanding of their mechanical behaviour. This section will be focused only on open cell foams and which are made of elastic-plastic materials. Thus, the deformation mechanisms, as well as their mechanical properties in both compression and tension for the linear elasticity and plastic collapse domains are analyzed.

5.2.1 Deformation mechanisms in foams

The salient structural features of a foam are its relative density, ρ^*/ρ_s , the degree to which the cells are open or closed, and their shape anisotropy ratios R_{12} and R_{13} . The crucial cell-wall properties for an elastic-plastic material are the solid density, ρ_s , the Young's modulus, E_s , the yield strength, σ_{ys} . Throughout this chapter, symbols with the subscript 's' refer to the properties of the solid, while those with a subscripted asterisk '*' refer to the foam properties.

Figure 5.12 shows a schematic **compressive** stress-strain curve for an elastic-plastic foam. Like those for honeycombs, they show linear elasticity at low stresses, followed by a long collapse *plateau*, truncated by a regime of densification, in which the stress rises steeply.

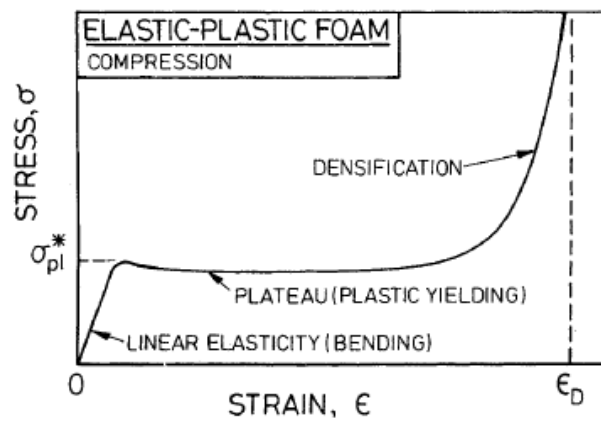


FIGURE 5.12: Schematic compressive stress-strain curves for foams, showing the three regimes of linear elasticity, collapse and densification of an elastic-plastic foam [10].

Linear elasticity is controlled by cell wall bending and, if the cells are closed, by cell face stretching; Young's modulus, E^* , is the initial slope of the stress-strain curve. When loading is compressive, the plateau is associated with collapse of the cells, by the formation of plastic hinges in a foam which yields (such as a metal or some polymers). When the cells have almost completely collapsed, opposing cell walls touch and further strain compresses the solid itself, giving the final region of rapidly increasing stress.

The example presented in Figure 5.13 shows that increasing the relative density of the foam increases Young's modulus, raises the plateau stress and reduces the strain at which densification starts.

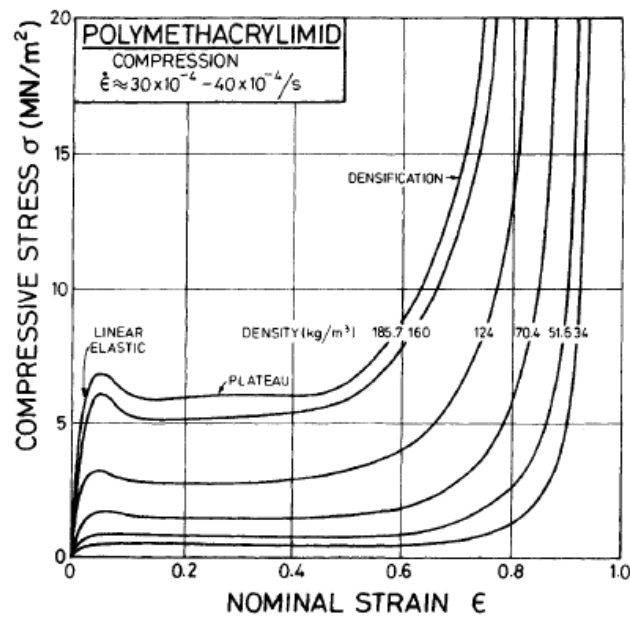


FIGURE 5.13: Compressive stress-strain curve for a polymethacrylimid foam [10].

The **tensile** response of an elastic-plastic foam is shown schematically in Figure 5.14. Again, it resembles that for honeycombs. The initial linear elasticity is caused by cell wall bending.

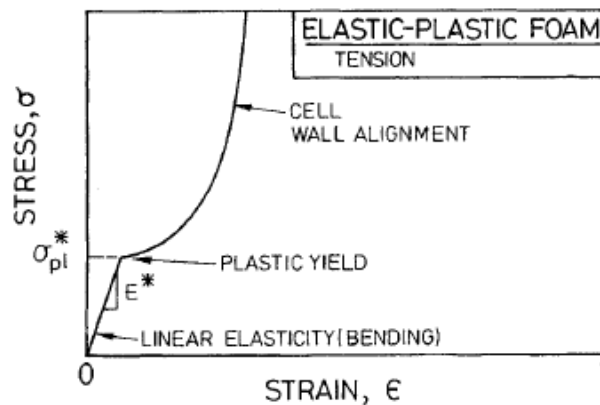


FIGURE 5.14: Schematic tensile stress-strain curves for foams, showing the regimes for an elastic-plastic foam [10].

The cell walls in the plastic foam rotate towards the tensile axis (by plastic bending), giving a yield point followed by a rising stress-strain curve, which is ultimately truncated by fracture. Some foams (rigid polymer foams are examples) are plastic in compression but brittle in tension. This is because of the stress-concentrating effect of a crack, which can cause cell wall failure and fast fracture when loaded in tension, but which is less damaging in compression.

5.2.2 Mechanical properties of open-cell foams: compression

Linear Elasticity

The linear elastic behaviour of a foam is characterized by a set of material properties. Two material properties are needed to describe an isotropic foam; usually they are chosen from the Young's modulus, E^* , the shear modulus, G^* , the bulk modulus, K^* , and the Poisson's ratio, ν^* . More are required for one that is not isotropic: five if the structure is axisymmetric; nine if it is orthotropic. Here we give expressions for E^* , G^* , K^* and ν^* , for an isotropic foam in terms of the cell-wall modulus, E_s , and the relative density, ρ^*/ρ_s of the foam itself.

The mechanism of linear elasticity depends on whether the cells are open or closed. At low relative densities, open-cell foams deform primarily by cell-wall bending. As the relative density increases ($\rho^*/\rho_s > 0.1$) the contribution of simple extension or compression of the cell walls becomes more significant. Each of the mechanisms which contribute to the linear-elastic response of foams is shown schematically in Figure 5.15.

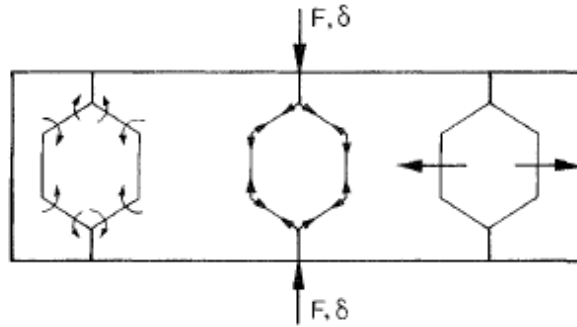


FIGURE 5.15: The mechanisms of deformation in open-cell foams: cell-wall bending + cell-wall axial deformation + fluid flow between cells [10].

At the simplest level, an open-cell foam can be modelled as a cubic array of members of length l and square cross-section of side t (Figure 5.16). Adjoining cells are staggered so that their members meet at their midpoints. The cell-shapes in real foams are, of course, more complex than that of the figure. But if they deform and fail by the same mechanisms, their properties can be understood using dimensional arguments which omit all constants arising from the specific cell geometry. The relative density of the cell, ρ^*/ρ_s , and the second moment of area of a member, I , are related to the dimensions t and l by:

$$\frac{\rho^*}{\rho_s} \propto \left(\frac{t}{l}\right)^2 \quad \text{and} \quad I \propto t^4 \quad (5.36)$$

Young's modulus for the foam is calculated from the linear-elastic deflection of a beam of length l loaded at its midpoint by a load F . Standard beam theory gives this deflection, δ , as proportional to $Fl^3/E_s I$, where E_s is the Young's modulus for the material of the beam. When a uniaxial stress is applied to the foam, so that each cell edge transmits a force F (Figure 5.17), the edges bend and the linear-elastic deflection of the structure as a whole is proportional to $Fl^3/E_s I$. The force, F , is related to the remote compressive stress, σ , by $F \propto \sigma l^2$ and the strain ϵ is related to

the displacement, δ , by $\epsilon \propto \delta/l$. It follows that Young's modulus for the foam is given by:

$$E^* = \frac{\sigma}{\epsilon} = \frac{C_1 E_s I}{l^4} \quad (5.37)$$

From which, for open cells, gives:

$$\frac{E^*}{E_s} = C_1 \frac{\rho^*}{\rho_s} \quad (5.38)$$

where C_1 includes all of the geometric constraints of proportionality. Any equiaxed cell shape leads to this result; the only difference lies in the constant C_1 . Data obtained by experimentation show that $C_1 \approx 1$.

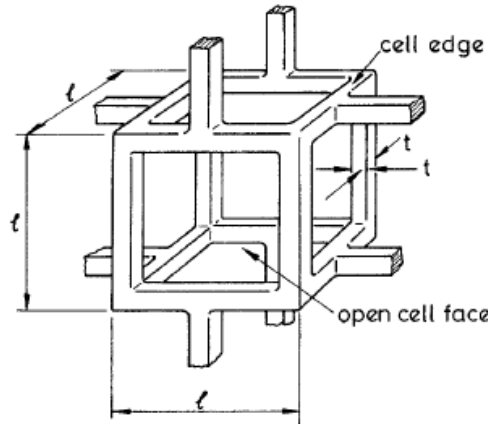


FIGURE 5.16: A cubic model for an open-cell foam showing the edge length, l , and the edge thickness, t [10].

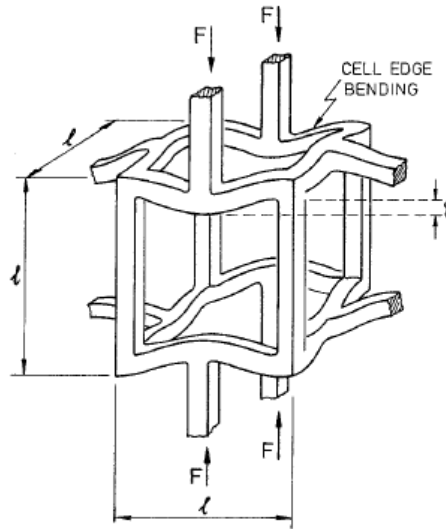


FIGURE 5.17: Cell edge bending during linear elastic deformation [10].

The shear modulus is calculated in a similar way. If a shear stress, τ , is applied to a foam, the cell members again respond by bending. Since the bending deflection, δ , is proportional to $Fl^3/E_s I$, and the overall stress, τ , and strain, γ , are proportional to Fl^2 and δ/l , respectively, it is obtained,

$$G^* = \frac{\tau}{\gamma} = \frac{C_2 E_s I}{l^4} \quad (5.39)$$

from which, for equiaxed shape,

$$\frac{G^*}{E_s} = C_2 \left(\frac{\rho^*}{\rho_s} \right)^2 \quad (5.40)$$

Data found in literature suggests that $C_2 \approx 3/8$.

Poisson's ratio, ν^* , is the negative ratio of the lateral to the axial strain. Since both are proportional to a bending deflection per cell length, their ratio is a constant, and for this reason, Poisson's ratio is solely a function of cell geometry and is independent of density. For a material which is linear-elastic and isotropic,

$$G = \frac{E}{2(1 + \nu)} \quad (5.41)$$

from which we obtain:

$$\nu^* = \frac{C_1}{2C_2} - 1 = C_3 \quad (5.42)$$

By comparison with data for open-cell foams, taking $C_3 = 1/3$ led to good fitting.

One must note that the analysis for E^* , G^* and ν^* contains a number of approximations. The way in which density is calculated double-counts the cell corners, for instance; and the axial and shear displacements of the cell walls have been neglected because they are usually small compared with the bending displacements. These approximations have been examined by Gibson and Ashby (1982), who showed that the errors they introduce at large values of ρ^*/ρ_s fortuitously cancel. We have also assumed that the cellular material can be treated as a continuum; that is, that the cell size is small relative to the size of a test specimen or a component. Lakes, in 1983, shows that classical elasticity, with its continuum assumption, fails to describe the elastic behaviour of foam specimens with a ratio of specimen diameter to cell size of less than 20.

Plastic collapse

Foams made from materials which have a plastic yield point (rigid polymers, or metals, for instance) **collapse plastically** when loaded beyond the linear-elastic regime. Plastic collapse, like elastic buckling, gives a long horizontal plateau to the stress-strain curve, though the strain is no longer recoverable. The failure is localized in a band transverse to the loading direction which propagates throughout the foam with increasing strain. By analogy with ductile honeycombs, it is expected that this localization is associated with a local maximum in the stress-strain curve of a unit cell. The long stress plateau is exploited in foams, for example, in crash protection and energy-absorbing systems.

Plastic collapse in an open-cell foam occurs when the moment exerted on the cell walls exceeds the fully plastic moment creating plastic hinges, like those shown in Figure 5.18.

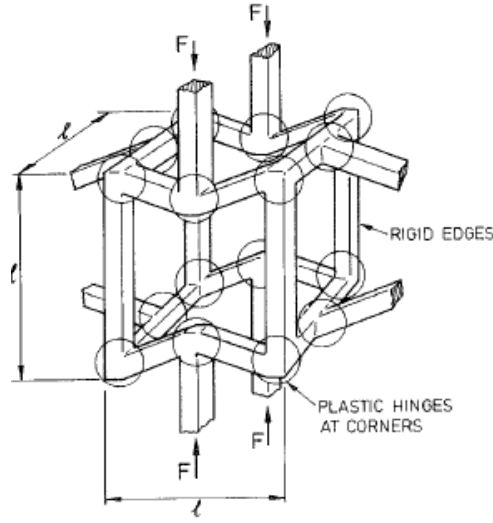


FIGURE 5.18: The formation of plastic hinges in an open-cell foam [10].

Plastic collapse occurs when the moment exerted by the force F exceeds the fully plastic moment of the cell edges. For a beam with a square section of side t , this moment is:

$$M_p = \frac{1}{4} \sigma_{ys} t^3 \quad (5.43)$$

where σ_{ys} is the yield strength of the cell wall material. If the force F has a component normal to the cell edge (of length l), the maximum bending moment is proportional to Fl . The stress on the foam, as before, is proportional to F/l^2 . Combining the results gives the plastic-collapse strength of the foam:

$$\sigma_{pl}^* \propto \frac{M_p}{l^3} \quad (5.44)$$

Using the relation $\rho^*/\rho_s \propto (t/l)^2$ for open cells, we obtain:

$$\frac{\sigma_{pl}^*}{\sigma_{ys}} = C_5 \left(\frac{\rho^*}{\rho_s} \right)^{3/2} \quad (5.45)$$

where C_5 contains all the constants of proportionality.

Two limits on this behaviour are worth noting: At large relative densities ($\rho^*/\rho_s > 0.3$) the beam-bending concept breaks down. The cell walls are now so short yet squat that they yield axially (in compression or tension) before they bend. Above this density the material is better thought of as a solid with holes in it, not as a foam. At the other extreme - that of very low relative density - it is possible for elastic collapse to precede plastic collapse, even though the subsequent large-strain behaviour (and thus the plateau stress) may be determined by plasticity. This will happen when

$$\sigma_{el}^* < \sigma_{pl}^* \quad (5.46)$$

that is (using $C_5 = 0.3$, as it was a good fit for data), when:

$$0.05 E_s \left(\frac{\rho^*}{\rho_s} \right)^2 < 0.3 \sigma_{ys} \left(\frac{\rho^*}{\rho_s} \right)^{3/2} \quad (5.47)$$

or

$$\frac{\rho^*}{\rho_s} < 36 \left(\frac{\sigma_{ys}}{E_s} \right)^2 \quad (5.48)$$

For elastomers, this criterion is always satisfied. For 'rigid' polymers, $\sigma_{ys}/E_s \approx 1/30$, with the result that, for densities below about 0.04, elastic collapse precedes (and thus triggers) plastic collapse. For metals, $\sigma_{ys}/E_s \approx 1/300$, so that plastic collapse dominates at all densities.

5.2.3 Mechanical properties of foams: tension

Structural foams, and foam cores in sandwich structures, are subjected to tension as well as compression. In most cases the tensile behaviour is described by a simple adaptation of results developed in the last section.

Linear elasticity

The small strain linear-elastic modulus of a foam in tension is the same as that in compression. The material properties of open-cell foams are determined by cell-edge bending.

Plastic collapse

A plastic foam yields in tension by the same mechanism, and at essentially the same stress, as in compression (Equation (5.45)). The post-yield behaviour, however, is different. In compression, the rotation of the cell walls causes the bending moment to remain constant or even increase a little, giving a long, horizontal plateau. In tension, the rotation (Figure 5.19) reduces the bending moment, so that after a strain of about 1/3 the cell walls are substantially aligned with the tensile axis; then, further strain requires the plastic extension of the cell walls themselves. The stress σ^* rises over this strain interval from the value σ_{pl}^* , calculated earlier (Equation (5.45)) to the value:

$$\frac{\sigma_A^*}{\sigma_{ys}} = \frac{\rho^*}{\rho_s} \quad (5.49)$$

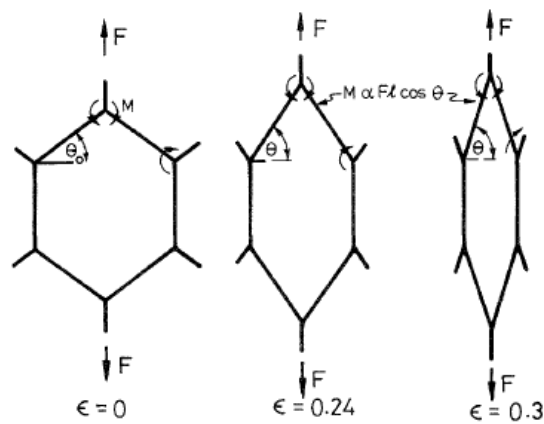


FIGURE 5.19: Alignment of the cell edges during tensile loading. The limiting strain is about 0.3 [10].

5.2.4 Stretch-Dominated structures

The foams analyzed so far respond to load by cell-edge bending. Not all cellular structures show this bending-dominated behaviour. To understand this, it is required the *Maxwell stability criterion*: the condition that a two-dimensional pin-jointed truss (meaning a structure that is hinged at its corners) made up of b struts and j frictionless joints is both statically and kinematically determinate (meaning that it is rigid and does not fold up when loaded) is:

$$M = b - 2j + 3 = 0 \quad (5.50)$$

In three dimensions, the equivalent is:

$$M = b - 3j + 6 = 0 \quad (5.51)$$

If $M < 0$, the frame is a *mechanism*. It has no stiffness or strength; it collapses if loaded. If its joints are locked (Figure 5.20(a), preventing rotation (as they are in a cellular lattice) the bars of the frame bend when the structure is loaded (Figure 5.20(b)). If, instead, $M > 0$, the frame ceases to be a mechanism (Figure 5.20(c) - if it is loaded, its members carry tension or compression (even when pin-jointed) and it becomes a *stretch-dominated* structure. Locking the joints now makes little difference because slender structures are much stiffer when stretched than when bent. There is an underlying principle here: stretch-dominated structures have high structural efficiency; bending dominated structures have low.

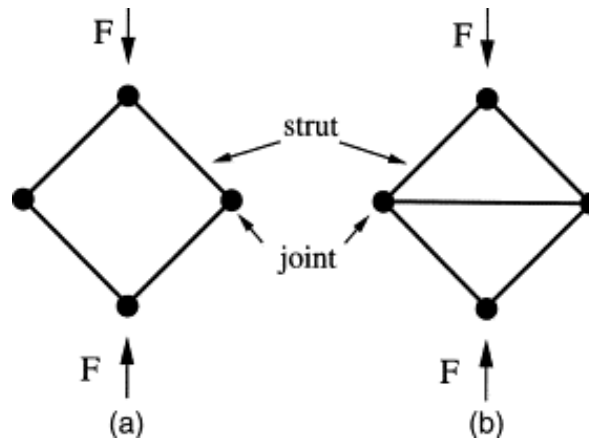


FIGURE 5.20: The frame (a) deforms by cell-edge bending when it is loaded. The triangulated frame at (b) is stiff when loaded because the transverse bar carries tension: its behaviour is stretch-dominated.

Adapted from [13].

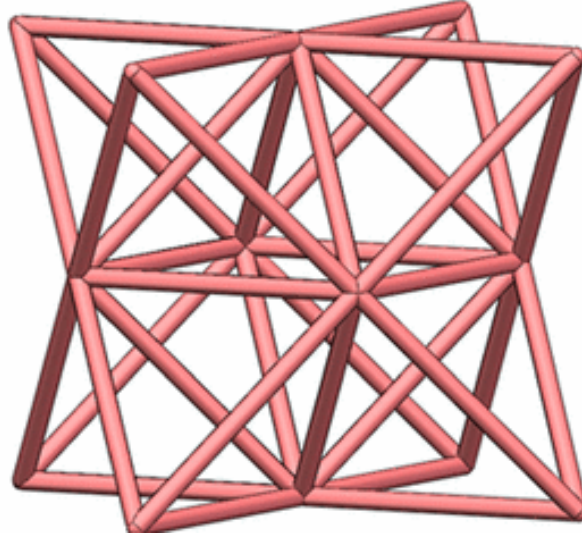


FIGURE 5.21: A stretch-dominated unit cell. Adapted from [13].

These criteria give a basis for the design of efficient micro-truss structures. For the cellular structures of a typical honeycomb or a generic cubic open-cell foam, $M < 0$, and bending dominates. For the structure shown in Figure 5.21, however, $M > 0$, and it behaves as an almost isotropic, stretch-dominated structure. On average, one third of its bars carry tension, when the structure is loaded in a simple tension, regardless of the loading direction. Thus,

$$\frac{E^*}{E_s} = \frac{1}{3} \frac{\rho^*}{\rho_s} \quad (5.52)$$

and its collapse stress is:

$$\frac{\sigma^*}{\sigma_{ys}} = \frac{1}{3} \frac{\rho^*}{\rho_s} \quad (5.53)$$

This is an upper bound, since it assumes that the struts yield in tension or compression when the structure is loaded. If the struts are slender, they may buckle before they yield. Then, the "strength", like that of a buckling foam, is:

$$\frac{\sigma_{el}^*}{E_s} = 0.2 \left(\frac{\rho^*}{\rho_s} \right)^2 \quad (5.54)$$

These results are summarized in Figure 5.22, in which the modulus E^* is plotted against the density of the foam, ρ^* . Stretch-dominated, prismatic microstructures have material properties that scale as ρ^*/ρ_s (slope 1); bending-dominated, foam-like microstructures have material properties that scale as $(\rho^*/\rho_s)^2$ (slope 2). Given that the density can be varied through a wide range, this allows great scope for material design. A good example of an application of stretch-dominated structures, using both octahedral and octet stretching-dominated structures, fabricated by fused deposition modeling (FDM), is presented in [59].

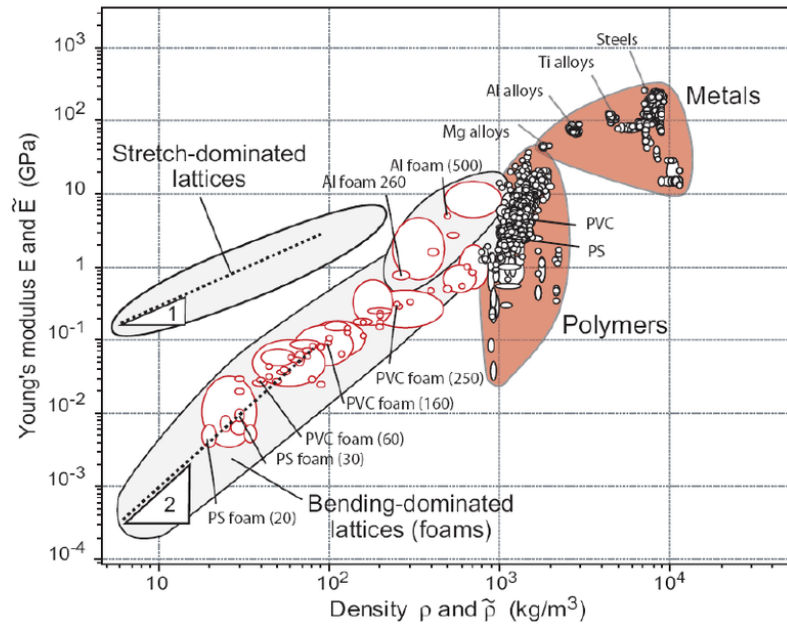


FIGURE 5.22: The contrast between bend- and stretch-dominated behaviour, shown here for aluminum foams and lattices. Real aluminum foams and lattices have properties that occupy the shaded regions [60].

5.3 Application of honeycombs and foams in sandwich structures

The mechanical behaviour of a sandwich structure depends on the properties of the face and core materials and on its geometry. In most applications, the panel must have some required minimum stiffness, it must not fail under some maximum service loading and it must be as light as possible. Its design can be formulated as an optimization problem: the goal is the panel with minimum weight which meets the constraints on stiffness and strength. The optimization can be carried out with respect to the core and skin thicknesses, with respect to the core and skin materials and (since foams have continuously variable density) with respect to the core density. [30], [31]

5.3.1 The stiffness of sandwich structures and its optimization

Consider a sandwich beam under a load P in three-point bending - Figure 5.23. The span is l , the width, b , the core thickness, c , and the face thickness, t .

The central deflection is δ . The density and Young's modulus of the faces are ρ_f and E_f , respectively. Those of the core are ρ_c^* and E_c^* . The shear modulus of the core is G_c^* . Note that the subscript ' f ' refers to the faces, the subscript ' c ' to the foamed core and the subscript ' s ' to the solid from which the core is made.

The core material properties vary continuously with the foam density, ρ_c^* ; at their simplest, they are given by:

$$E_c^* = C_1 E_s \left(\frac{\rho_c^*}{\rho_s} \right)^2 \quad (5.55)$$

and

$$G_c^* = C_2 E_s \left(\frac{\rho_c^*}{\rho_s} \right)^2 \quad (5.56)$$

where $C_1 (\approx 1)$ and $C_2 (\approx 0.4)$ are constants of proportionality.

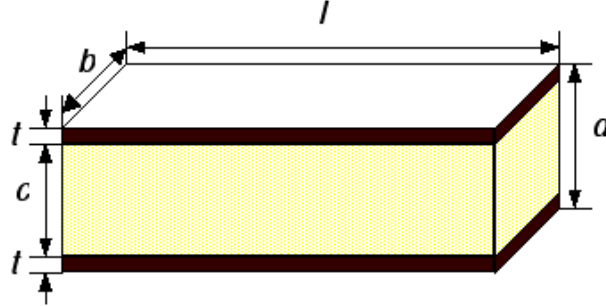


FIGURE 5.23: Schematic of a sandwich beam. Adapted from [13].

The simplest case to consider is that of point loading in 3 point-bending, though the analysis is identical for other forms of bending load, with the exception of the different values for the geometrical constants. The faces of the sandwich beam largely carry the normal stresses, while the core carries nearly all the shear stresses. The deflection of the beam is the sum of its bending and shear deflections, calculated from the equivalent flexural rigidity, $(EI)_{eq}$, and the equivalent shear rigidity, $(AG)_{eq}$, respectively. The equivalent flexural rigidity for a rectangular cross-section is:

$$(EI)_{eq} = \frac{E_f b t^3}{6} + \frac{E_c^* b c^3}{12} + \frac{E_f b t d^2}{2} \quad (5.57)$$

where $d = c + t$. The first and second terms describe the bending stiffnesses of the faces and of the core about their centroids: together they give the stiffness of the beam if the faces were not bonded to the core. In optimal sandwich design, both are small compared to the third term, which describes the bending stiffness of the faces about the centroid of the beam itself. In practice, the faces are always much stiffer and thinner than the core, so that $d \approx c$. Then, to a good approximation (assuming that $E_f \gg E_s^*$),

$$(EI)_{eq} = \frac{E_f b t c^2}{2} \quad (5.58)$$

The equivalent shear rigidity is, then:

$$(AG)_{eq} = \frac{G_c^* b d^2}{c} \approx b c G_c^* \quad (5.59)$$

The deflection δ is the sum of the bending and shear components:

$$\delta = \delta_b + \delta_s = \frac{P l^3}{B_1 (EI)_{eq}} + \frac{P l}{B_2 (AG)_{eq}} \quad (5.60)$$

where B_1 and B_2 are constants which depend on the geometry of loading [10].

5.4 The strength of sandwich structures

The obvious attraction of sandwich structures is that they are light and stiff. But stiffness alone is not enough. The beam or panel must also have strength: it must carry the design loads without failing. At least five different failure modes (described below) are possible; a given sandwich will fail by the one which occurs at the lowest load. And as the geometry and loading change the failure mode can change, too. Then, it is not enough to design against one mode; all must be considered, and the dominant mode - the one which determines failure - identified and evaluated.

The failure modes of a sandwich beam or panel are shown in Figure 5.24. It may fail by the yielding or fracture of the faces; the compression face may 'wrinkle' - a local buckling of the skin into the core or it may 'dimple' - a local buckling of the compression face on a honeycomb core, with a wavelength equal to twice the cell size. The core, too, can fail, usually in shear, though compressive or tensile failure is also possible. Then there is the bond between the face and core: it can fail; and since resin adhesives are usually brittle, debonding is by brittle fracture - however, here it is assumed that this bond is sufficiently strong that delamination is avoided. Finally, there is the possibility of indentation of the faces and core at the loading point; but because this can be suppressed by distributing the load over an area about equal to the core section, it is easily avoided.

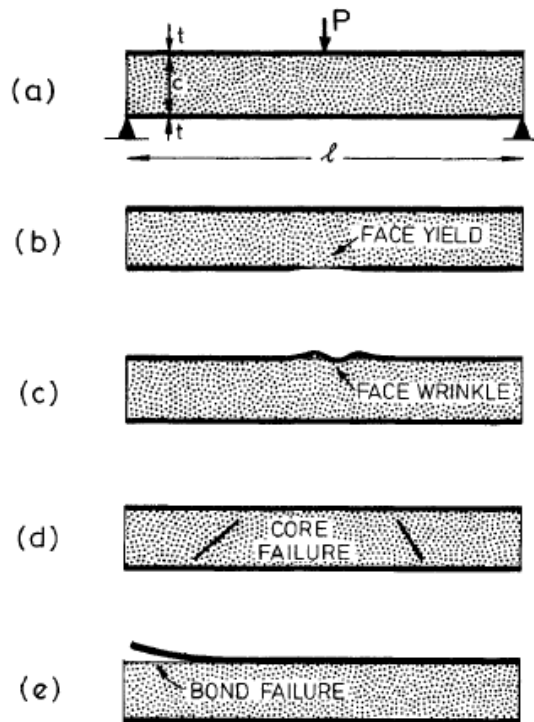


FIGURE 5.24: Failure modes for sandwich panels: (a) the loading geometry, (b) face yielding, (c) face wrinkling, (d) core failure, (e) decohesion of the interfacial bond [10].

The analysis of these failure modes requires expressions for the normal stresses, σ_f and σ_c , and the shear stresses, τ_f and τ_c , acting in the faces and the core. The maximum normal stresses are related to the applied moment by:

$$\sigma_f = \frac{MyE_f}{(EI)_{eq}} = \frac{M}{btc} \quad (5.61)$$

$$\sigma_c = \frac{MyE_c^*}{(EI)_{eq}} = \frac{M}{btc} \frac{E_c^*}{E_f} \quad (5.62)$$

where M is the applied moment, y is the distance from the neutral axis, and $(EI)_{eq}$ is given by Equation (5.58). The shear stress varies through the face and core in a parabolic fashion; but if the faces are much stiffer and thinner than the core (as, in practice, they always are) the shear stress can be assumed as linear through the face and constant through the core at the value - Figure 5.25.

$$\tau_c = \frac{Q}{bc} \quad (5.63)$$

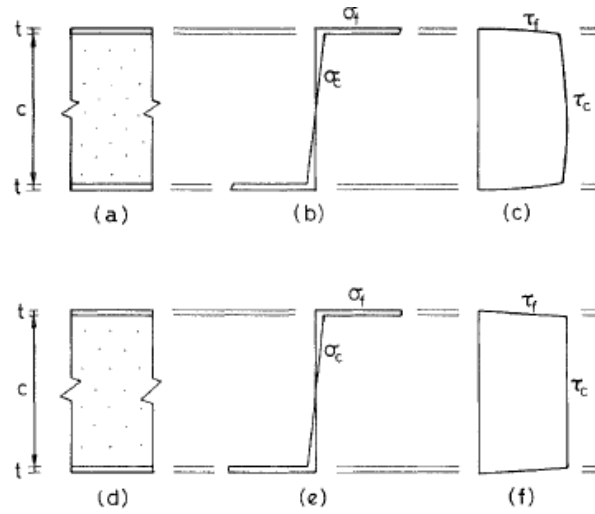


FIGURE 5.25: Stress distributions in a rectangular sandwich beam: (a) beam cross-section, (b) normal stress distribution, (c) shear stress distribution, (d) beam cross-section, (e) approximate normal stress distribution, (f) approximate shear stress distribution [10].

The stress is maximum when $y = (c/2) + t \approx c/2$ for thin faces, relative to the core, and when $M = M_{max}$. The maximum moment can be related to the load P , and the span of the beam, l , by $M = Pl/B_3$. Values of B_3 for common loading configurations can be found in Table 5.1.

Face yielding occurs when the normal stress in the face equals the strength of the face material, σ_{yf} , or:

$$\sigma_f = \frac{Pl}{B_3 btc} = \sigma_{yf} \quad (5.64)$$

Face wrinkling occurs when the normal stress in the compressive face of the beam reaches the local instability stress, that is, when the compressive stress in the face is:

$$\sigma_f = \frac{Pl}{B_3 btc} = \frac{3E_f^{1/3} E_c^{*2/3}}{(12(3 - \nu_c^*)^2 (1 + \nu_c^*)^2)^{1/3}} \quad (5.65)$$

for a Poisson's ratio of the core of 0.33, and using the relation in (5.58), we obtain:

$$\sigma_f = 0.57 E_f^{1/3} E_s^{2/3} \left(\frac{\rho_c^*}{\rho_s} \right)^{4/3} \quad (5.66)$$

Finally, **core failure** occurs when the shear stress in the core equals the shear strength of the core, $\tau_c = \tau_c^*$:

$$\frac{P}{B_4 b c} = C_{11} \left(\frac{\rho_c^*}{\rho_s} \right)^{3/2} \sigma_{ys} \quad (5.67)$$

where C_{11} is a constant of proportionality for the foam shear strength (assuming failure by plastic yielding), and B_4 is a constant relating the maximum shear force in the beam to the applied load (see Table 5.1).

TABLE 5.1: Constants B for bending and failure of sandwich beams

Mode of Loading	B1	B2	B3	B4
Cantilever, end load (P)	3	1	1	1
Cantilever, uniformly distributed load (P/l)	8	2	2	1
Three Point Bend, central load (P)	48	4	4	2
Three point Bend, uniformly distributed load (P/l)	384/5	8	8	2
Ends built in, central load (P)	192	4	8	2
Ends built in, uniformly distributed load (P/l)	384	8	12	2

Finally, interesting works focused on the failure modes of sandwich structures include [61], [16] and [62], just to name a few.

Chapter 6

Additive Manufacturing

While some manufacturing processes might allow for the production of foam cores used in sandwich structures, the most recently developed and of growing interest is Additive manufacturing. In this short chapter, a review on the FDM technique and its application to polymeric foams, in particular, for sandwich structures is provided.

6.1 An introduction to Additive manufacturing

Additive Manufacturing (AM) has been defined as the process of joining materials to make parts from a 3-dimensional model data one layer at a time [63]. Typically, the process starts with the drawing of a three-dimensional model using a computer-aided design - CAD - software. The model is then saved as an *.stl* file format. Usually, 3D printer manufacturers provide their own software to slice the model in the *.stl* file into individual layers. The sliced file will then be sent to the Additive Manufacturing device, or, as commonly called, the printer. The printer will then progressively print one 2D layer on top of the other, and thereby forming a three-dimensional object - the whole process flow is summarized in Figure 6.1 [63], [14].

Throughout the last years, the notable progress in the additive manufacturing technologies has enabled the design of complex materials with location-specific architecture, which, allied with the continuous advances in computational power and computational methods, enables the optimization of an architected material to meet all desired objectives. [8], [9], [2].

The technology has demonstrated the ability to fabricate architectures that are impossible to fabricate through conventional processing techniques. Although quality and reliability of additive manufacturing processes are currently still debatable, a better process monitoring should eventually enable extremely well controlled qualities, low failure rates, and a rapid certification of additive manufactured architected materials. Some fabrication approaches for architected materials, in decreasing maturity, are, for example, Fused deposition modeling, Selective laser melting, Electron Beam melting, Polyjet printing, Stereolithography, Binder printing, Electrophoretic deposition and Direct ink writing, just to name a few [2], [64].

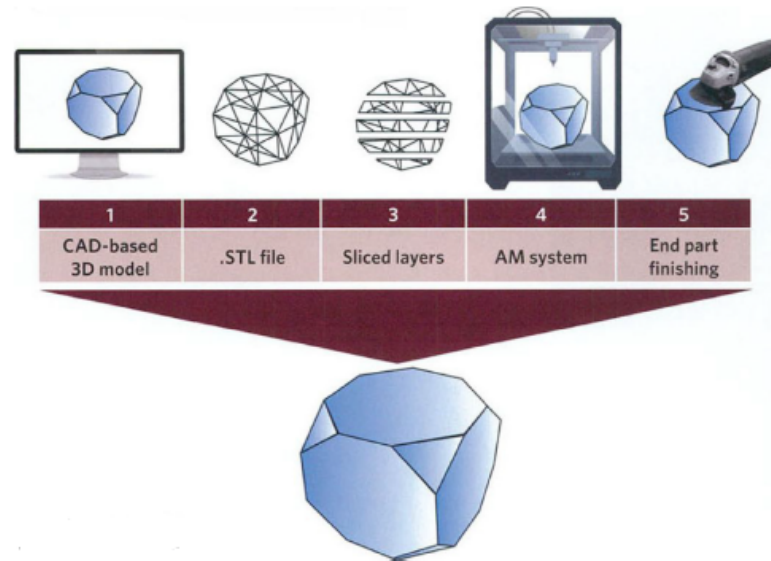


FIGURE 6.1: The Additive Manufacturing Process Flow [63].

In the context of polymeric architected cellular materials, the first additive manufacturing methods to mature were the SLA, stereolithography, in which layers of liquid photopolymer resin are cured with UV light, and FDM, fused deposition modeling, in which thermoplastic polymers are extruded on a small scale. Support structures are usually required for overhanging features (Figure 6.2).

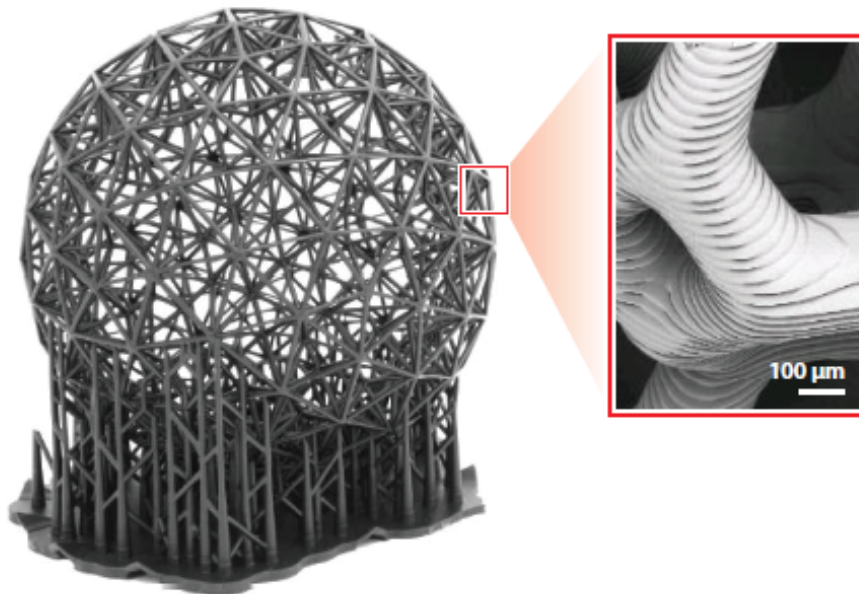


FIGURE 6.2: Architected polymer structure 3D printed via stereolithography. The inset shows the steps in the surface caused by the layer-by-layer printing process [2].

Scott Crump patented the Fused Deposition Modeling (FDM) in 1989 [65]. FDM-based 3D printers are presently the most popular consumer-level 3D printers for printing polymer composites that is based on extrusion additive manufacturing (AM)

systems. Among other extrusion-based techniques, FDM is a material-melting technique which uses a spool of thermoplastic filament such as PC, ABS and PLA with varying diameters to be melted and extruded through a heated nozzle [66]. As shown in Figure 6.3, the extruded semi-liquid polymer that is actually printed onto the build platform solidifies instantly, translating the sliced layers of digital data into an actual printed object [63].

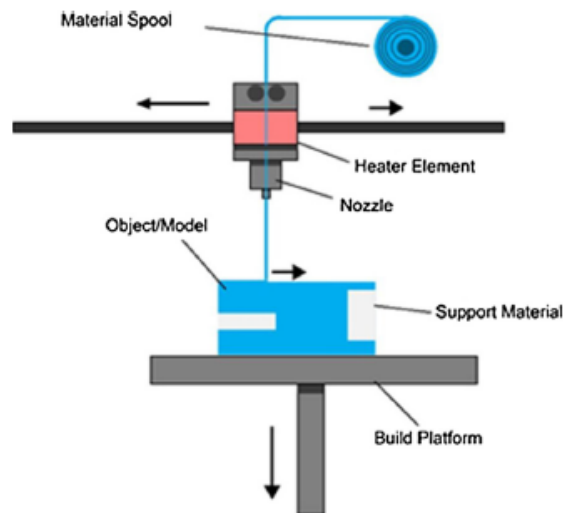


FIGURE 6.3: Typical FDM setup [63].

PLA, or polylactic acid, is a type of thermoplastic polymer whose industrial and general use has increased throughout the years due to its environmental friendliness and biocompatibility, and many desktop consumer printer models now exclusively use PLA [27]. As a consequence, a careful examination should fall on the user-controllable aspects of the FDM process and the resultant effect on PLA parts [67], [27], [28].

This examination of the dependencies on controllable printer parameters can lead to optimized components from each of the source materials. Print orientation and infill density alone have been shown to alter compressive, tensile, flexural, and impact strengths by an order of magnitude in some cases [67].

Examples of applications of PLA or other polymers using the FDM 3D printing technique include [14], [15], [16], [66], [68] and [59].

Although 3D printing of polymer composites has undergone significant developments in recent years, it is still not widely accepted by most industries. The limitations of this technology that shall to be overcome reside mostly on the: [32], [17]

- **Material:** The wide application of 3D printing is severely limited by printable materials. Currently, only thermoplastic polymer with low glass transition temperature and suitable melting viscosity, powder formed materials and a few photo polymers could be used in 3D printing. However, these limited materials could not meet the variety requirements of industry application and thus the diversity of materials must increase. Synthesis of matrix materials with special properties, discover of new reinforcement and discover of suitable mixing composition are critical to increase the versatility of composites printing technology. Sustainable materials are also promising to be developed to reduce material cost and environmental impact.

- **Performance:** Although reinforcement helps to improve the performance of polymer composites, compared with polymer composites manufactured by traditional molding methods, most of the printed composites still have low mechanical strength and are not able to meet the functional requirement. Therefore, additional post-treatment steps involving infiltration or consolidation have been used to improve the performance of printed products. However, these steps further increase the cost and processing time. Moreover, the repeatability and consistency of manufactured parts cannot be guaranteed by printing, thus, approaches to ensure the uniform properties of printed parts need to be investigated in depth.
- **Machine:** Most printing processes are time-consuming now, and it is difficult to fabricate parts that have large volume. These inhibit their industrial interest. New printing techniques based on scalable and fast processing of materials should be developed. Another area of growth centers on the need for feedback systems. If an error occurs during printing now, the process needs to be suspended, which causes the waste of time and materials. Feedback systems should be included in the printer to have a response to the process change. Additional progress for 3D printers is to increase the printing resolution without extending printing time or sacrificing geometry complexity of products.

6.2 Additive Manufacturing for sandwich structures

The same engineering and architectural principles that have been used to increase the mechanical efficiency of these structures can be applied at the material scale. At the same time, additive manufacturing offers the opportunity to tailor the architecture to location-specific requirements. By varying cellular architecture, material properties can be altered throughout a part in such a way that the target application is maximally enabled. Powerful computational tools to optimize structures for mechanical efficiency allow the most structurally efficient and lightweight architecture for a complex load case to be computationally determined, and with the rapid progress in additive manufacturing, the limits on what can be made are progressively vanishing. Additional advances must still be made to enhance the scale, productivity, reliability, and palette of available materials for broad commercialization [2].

Using additive manufacturing techniques for the cores of sandwich structures offers several advantages over conventional honeycomb or foam manufacturing, including (a) the ability to form net shape cores, (b) the ability to create cores with complex curvature, and (c) the possibility of designing cores with graded cellular architectures. [69]

As the complexity of lightweight structures continues to grow with ever-increasing structural, aerodynamic, and packaging requirements, performance and weight benefits may be achieved using complex-shaped sandwich cores with spatially graded properties. By designing a core that has a location-specific density, which is matched to the local loads, substantial weight savings can be obtained over conventional cores, which have an uniform density and whose properties must be selected in order to match the highest local load. A problem specific to honeycombs is that moisture is easily trapped in the closed cells, which increases weight and shifts the

center of gravity, an effect that can be mitigated by open-celled cores. Lastly, additively manufacturing cores, or coating 3D-printed core templates, widens the available base material palette. [2]. Examples of works using 3D printed sandwich panels with honeycomb cellular core structures include [70], [71] and [72].

Chapter 7

Numerical and Experimental Analysis

Once the foundation concepts necessary for this study were gathered, namely the formulation of the RPIM and NNRPIM, the solid mechanics fundamentals, the continuum mechanics of plasticity, and the general behaviour for a honeycomb made of an elastic-plastic material, the numerical analysis was carried. At first, a benchmark example was studied in order to assess the convergence of the solutions using the three referred methods. Then, an analysis using the linear elastic regime was considered, for both conventional and re-entrant honeycombs, for both in-plane directions, x and y , and the phenomenological curves for the linear elastic regimen were obtained, for 7 different relative densities of the two types of honeycombs. Afterwards, the elastoplastic regime was studied, considering the same 7 relative densities, using the FEM and the KT-All non-linear algorithm; simultaneously, samples were prepared through a FDM 3D printing process, and tested in compression, as well as modelled and analysed in FEMAS using an homogenization process, in order to establish a comparison between the simulations and the obtained experimental results. The procedures, results and respective discussions are presented throughout this Chapter.

7.1 Preliminary study on the convergence of the used methods - a benchmark example

In order to verify and compare the convergence and efficiency of the numerical methods used in this work, presented previously in Chapter 3, namely the Finite Element Method (FEM), the Radial Point Interpolation Method (RPIM) and the Natural Neighbour Radial Point Interpolation Method (NNRPIM), a convergence test for both 2D and 3D examples was carried. Figures 7.1 and 7.2 are schematic representations of the thick cantilever beam under a distributed load on the free end - CD line for the 2D case, and the CDEF plane for the 3D case.

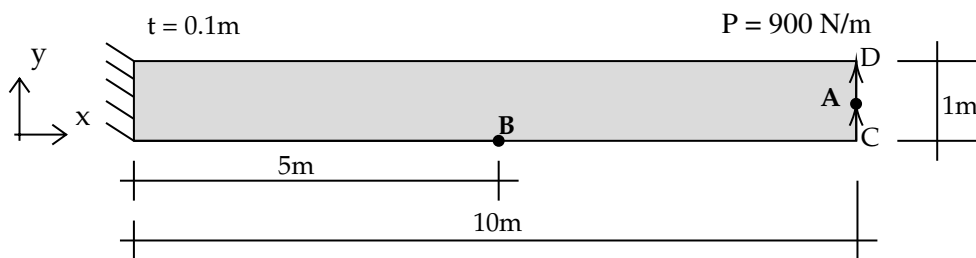


FIGURE 7.1: 2D cantilever beam example. Adapted from [3].

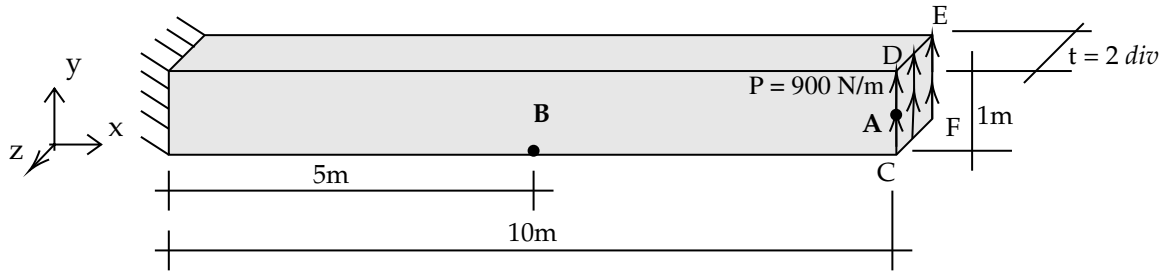


FIGURE 7.2: 3D cantilever beam example. Adapted from [3].

The mechanical properties are given below:

- $E = 2 \times 10^8$ [Pa]
- $\nu = 0.2$
- $P = 900$ [N/m] for the 2D case, and $P = 900/A$ [N/m²], with A being the cross section area
- Thickness, $t = 0.1$ m for the 2D case and $t = 2 \times div$ with *div* being the inter-nodal distance.

The 2D and 3D models as well as their meshes were created in FEMAP - some examples for 2D and 3D meshes are presented in Figures 7.3 and 7.4. For each case, (a), (b) and (c) represent the most sparse mesh, the second most sparse mesh and the most refined mesh, respectively. **In every case, 3 node-triangular elements were used.** Note that, for the 3D model, the thickness is progressively smaller, and 2 elements were used in every case along this thickness.

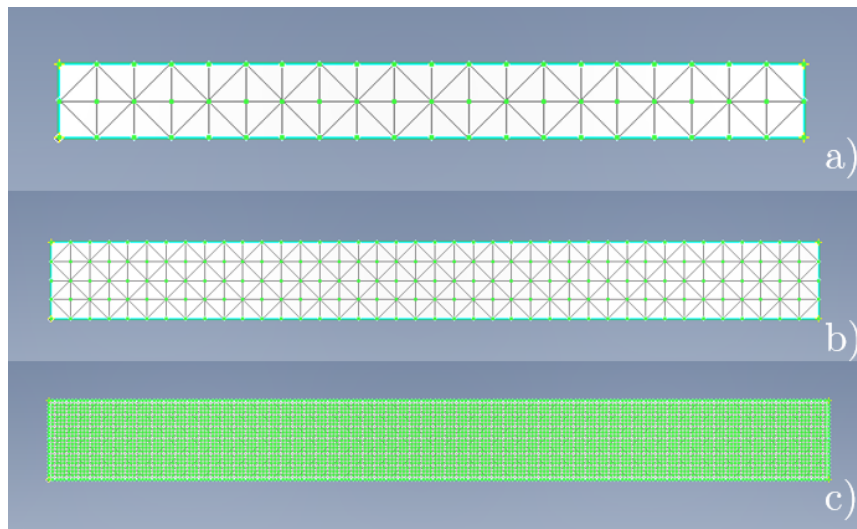


FIGURE 7.3: 2D meshed models: (a) 2x20 mesh; (b) 4x40 mesh; (c) 20x200 mesh

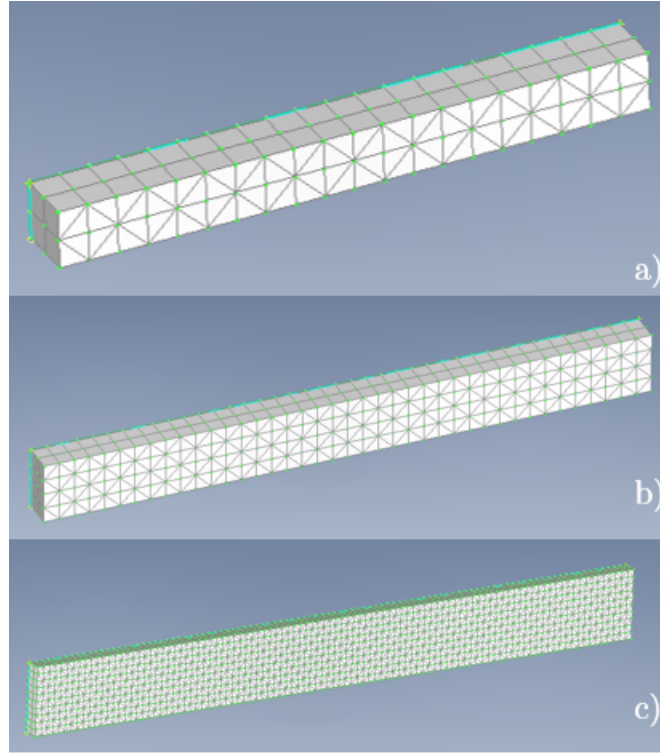


FIGURE 7.4: 3D meshed models: (a) 2x20x2 mesh; (b) 4x40x2 mesh; (c) 10x100x2 mesh

The points used as a reference for the solutions were found in literature in [3]. The normalized exact solution (for $t = 1$) for the vertical displacement of point A is $v_A = 0,018m$ and the normalized exact solution (for $t = 1$) for the normal stress component of point B is $\sigma_{xx}^B = 27$ kPa. The models were imported to FEMAS and the calculations were made using a standard Finite Element Method using the previously created mesh, and for the meshless methods, the following parameters - Table 7.1 - were considered to lead to more accurate results, as previously stated in Chapter 3.

TABLE 7.1: Parameters used for the RPIM and NNRPIM tests.

	RPIM	NNRPIM
Type of mesh	Regular	Regular
Nodal Connectivity	Fixed radial influence-domain	Second degree influence-cell
Shape parameter, C	0,0001	0,0001
Shape parameter, P	0,9999	0,9999
Polynomial	Constant	Constant
Integration Points	1	1

The results obtained for both v_A and σ_{xx}^B in the 2D and 3D models using the three different methods are presented in Figures 7.5, 7.6, 7.7 and 7.8.

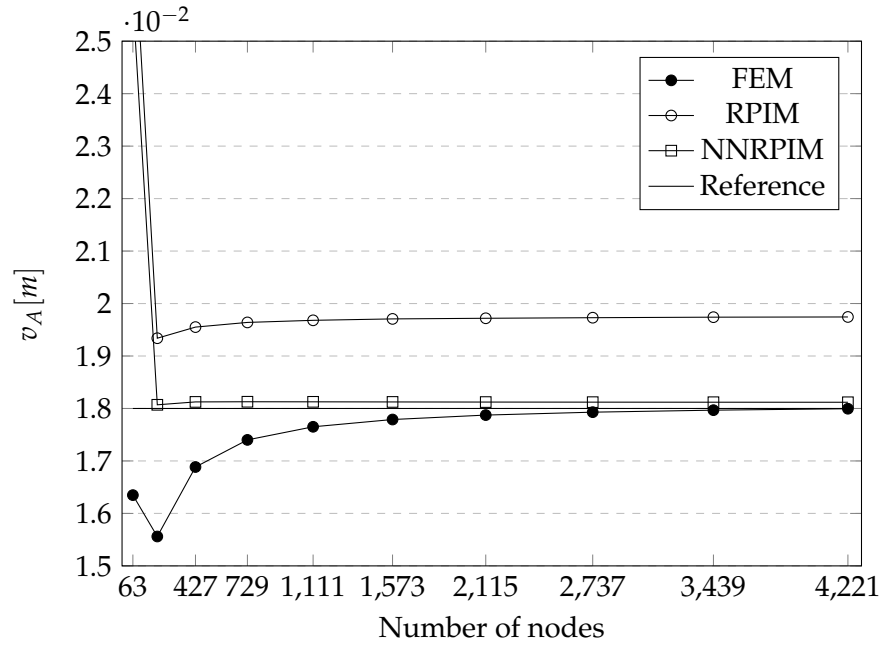


FIGURE 7.5: Results for the vertical displacement in point A, v_A , for the 2D model

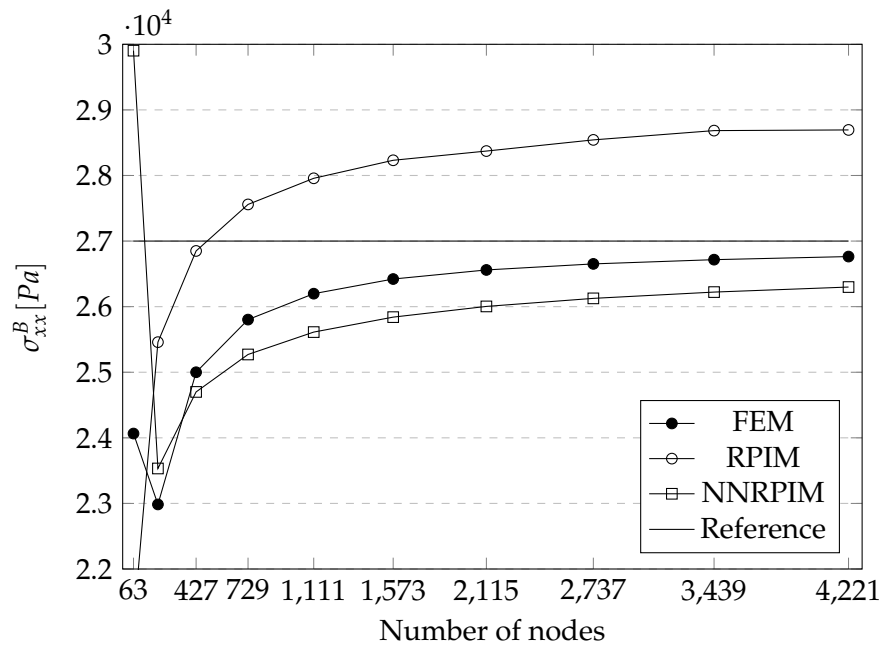


FIGURE 7.6: Results for the normal stress component of point B, σ_{xx}^B , for the 2D model

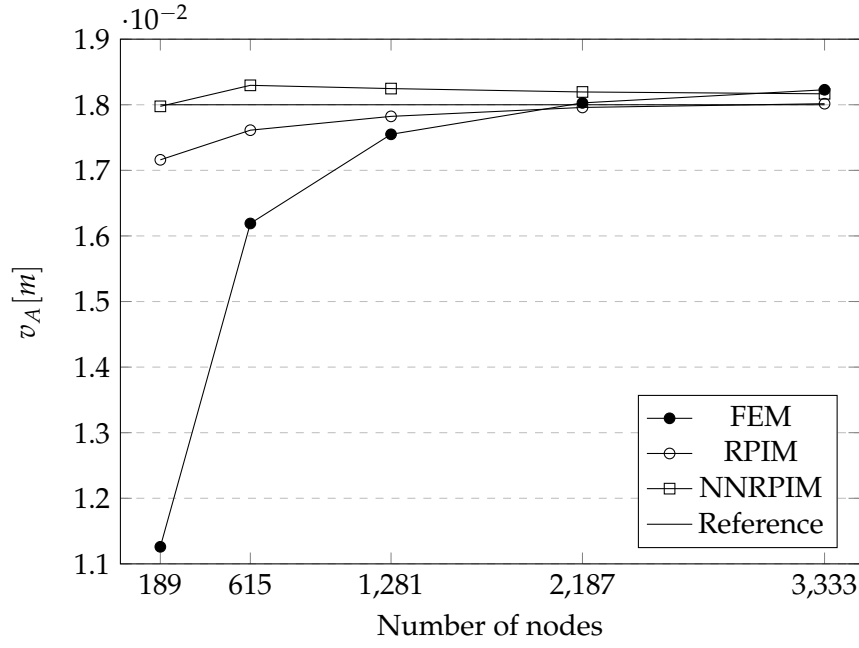


FIGURE 7.7: Results for the vertical displacement in point A, v_A , for the 3D model

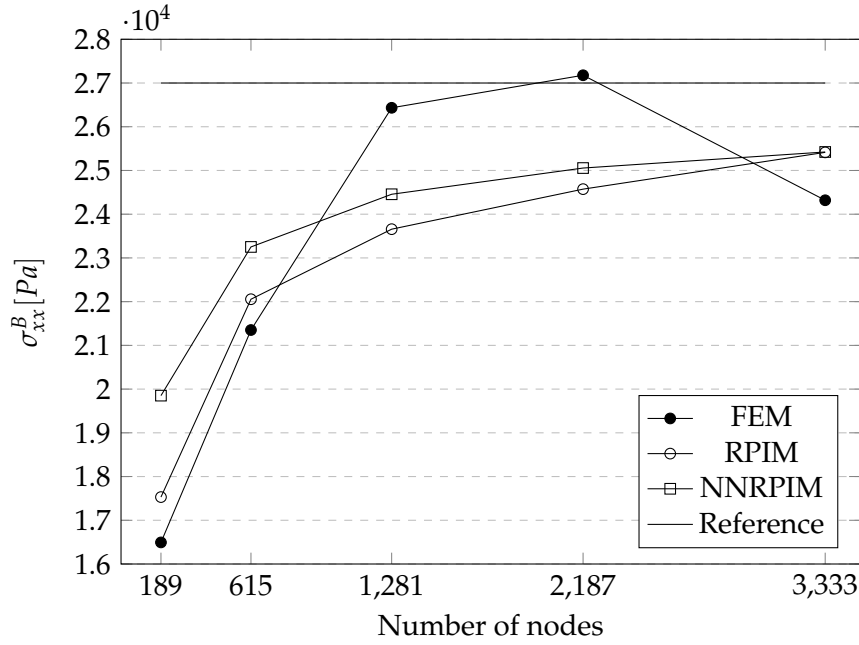


FIGURE 7.8: Results for the normal stress component of point B, σ_{xx}^B , for the 3D model

The results for every number of nodes (and mesh used) for the three methods are presented in Tables 7.2 and 7.3. Tables 7.4 and 7.5 highlight the difference between the results obtained and exact solutions for each case. This difference can be determined by the following expression:

$$\delta = \frac{X_{method} - X_{ref}}{X_{ref}} \quad (7.1)$$

In this case, this relation will be applied to the vertical displacement in point A, v_A . The results are shown in Tables 7.6 - for the 2D model - and 7.7, for the 3D model.

TABLE 7.2: Vertical displacement for point A - 2D model [m].

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
63	2x20	1,6349E-02	2,7473E-02	2,5806E-02	1,80E-02
205	4x40	1,5560E-02	1,9339E-02	1,8071E-02	1,80E-02
427	6x60	1,6884E-02	1,9551E-02	1,8125E-02	1,80E-02
729	8x80	1,7401E-02	1,9640E-02	1,8129E-02	1,80E-02
1111	10x100	1,7651E-02	1,9680E-02	1,8127E-02	1,80E-02
1573	12x120	1,7789E-02	1,9707E-02	1,8124E-02	1,80E-02
2115	14x140	1,7874E-02	1,9719E-02	1,8122E-02	1,80E-02
2737	16x160	1,7930E-02	1,9730E-02	1,8121E-02	1,80E-02
3439	18x18	1,7968E-02	1,9740E-02	1,8119E-02	1,80E-02
4221	20x200	1,7995E-02	1,9744E-02	1,8119E-02	1,80E-02

TABLE 7.3: Normal stress component of point B - 2D model [Pa].

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
63	2x20	2,4065E+05	3,1144E+05	2,9903E+05	2,70E+05
205	4x40	2,2985E+05	2,5459E+05	2,3533E+05	2,70E+05
427	6x60	2,5000E+05	2,6849E+05	2,4700E+05	2,70E+05
729	8x80	2,5803E+05	2,7559E+05	2,5271E+05	2,70E+05
1111	10x100	2,6198E+05	2,7957E+05	2,5612E+05	2,70E+05
1573	12x120	2,6421E+05	2,8232E+05	2,5840E+05	2,70E+05
2115	14x140	2,6560E+05	2,8372E+05	2,6004E+05	2,70E+05
2737	16x160	2,6652E+05	2,8543E+05	2,6126E+05	2,70E+05
3439	18x18	2,6717E+05	2,8684E+05	2,6222E+05	2,70E+05
4221	20x200	2,6764E+05	2,8695E+05	2,6299E+05	2,70E+05

TABLE 7.4: Vertical displacement for point A - 3D model [m].

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
189	2x20x2	1,1260E-02	1,7160E-02	1,7976E-02	1,80E-02
615	4x40x2	1,6190E-02	1,7612E-02	1,8296E-02	1,80E-02
1281	6x60x2	1,7549E-02	1,7823E-02	1,8247E-02	1,80E-02
2187	8x80x2	1,8029E-02	1,7959E-02	1,8195E-02	1,80E-02
3333	10x100x2	1,8228E-02	1,8016E-02	1,8167E-02	1,80E-02

TABLE 7.5: Normal stress component of point B - 3D model [Pa].

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
189	2x20x2	1,6494E+04	1,7530E+04	1,9851E+04	2,70E+04
615	4x40x2	2,1351E+04	2,2055E+04	2,3251E+04	2,70E+04
1281	6x60x2	2,6433E+04	2,3657E+04	2,4456E+04	2,70E+04
2187	8x80x2	2,7178E+04	2,4573E+04	2,5056E+04	2,70E+04
3333	10x100x2	2,4319E+04	2,5414E+04	2,5423E+04	2,70E+04

TABLE 7.6: Difference between the vertical displacement obtained for point A and the reference solution - 2D model.

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
63	2x20	-1,6512E-03	9,4729E-03	7,8056E-03	1,80E-02
205	4x40	-2,4398E-03	1,3389E-03	7,0619E-05	1,80E-02
427	6x60	-1,1155E-03	1,5509E-03	1,2535E-04	1,80E-02
729	8x80	-5,9882E-04	1,6401E-03	1,2895E-04	1,80E-02
1111	10x100	-3,4914E-04	1,6801E-03	1,2675E-04	1,80E-02
1573	12x120	-2,1052E-04	1,7075E-03	1,2426E-04	1,80E-02
2115	14x140	-1,2588E-04	1,7187E-03	1,2223E-04	1,80E-02
2737	16x160	-7,0484E-05	1,7305E-03	1,2066E-04	1,80E-02
3439	18x180	-3,2293E-05	1,7404E-03	1,1947E-04	1,80E-02
4221	20x200	-4,8540E-06	1,7435E-03	1,1855E-04	1,80E-02

TABLE 7.7: Difference between the vertical displacement obtained for point A and the reference solution - 3D model.

Number of nodes	Mesh	FEM	RPIM	NNRPIM	Reference
189	2x20x2	-3,7444E-01	-3,7444E-01	-1,3333E-03	1,8000E-02
615	4x40x2	-1,0056E-01	-1,0056E-01	1,6461E-02	1,8000E-02
1281	6x60x2	-2,5039E-02	-2,5039E-02	1,3700E-02	1,8000E-02
2187	8x80x2	1,5833E-03	1,5833E-03	1,0833E-02	1,8000E-02
3333	10x100x2	1,2689E-02	1,2689E-02	9,2667E-03	1,8000E-02

In addition to the numeric values, the distribution of the output values, such as the stresses found in the model, are often presented in the form of color maps. Therefore, since the normal stress component was used as a reference, the color maps for this output obtained with FEMAS, for the most sparse and most refined mesh, for each of the three different methods, are provided - Figure 7.9 and Figure 7.10 for the 2D model and Figures 7.11 and 7.12 for the 3D model.

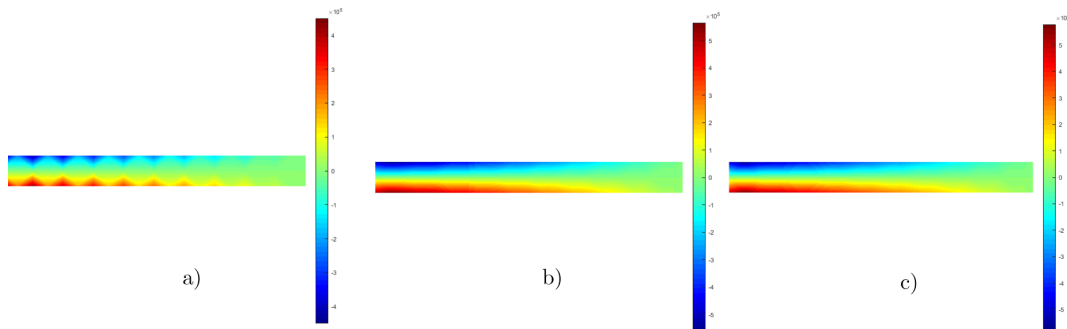


FIGURE 7.9: Color maps obtained for $\sigma_{xx}[Pa]$ using the 2x20 mesh (2D model) - (a) FEM; (b) RPIM; (c) NNRPIM

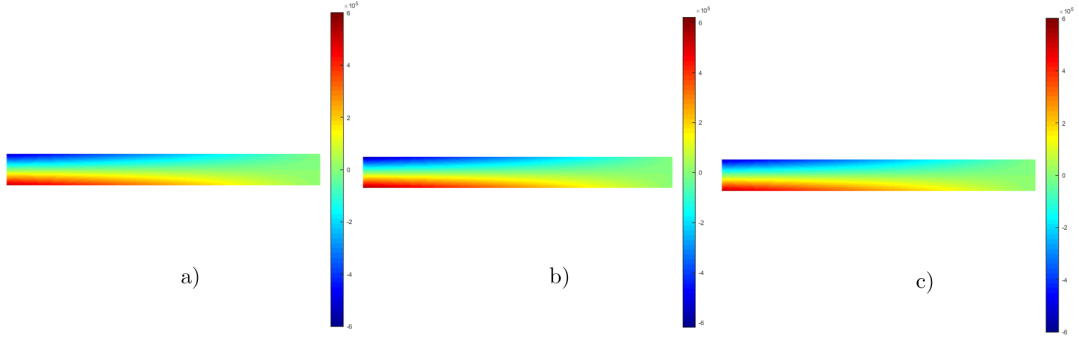


FIGURE 7.10: Color maps obtained for $\sigma_{xx}[Pa]$ using the 20x200 mesh (2D model) - (a) FEM; (b) RPM; (c) NNRPIM

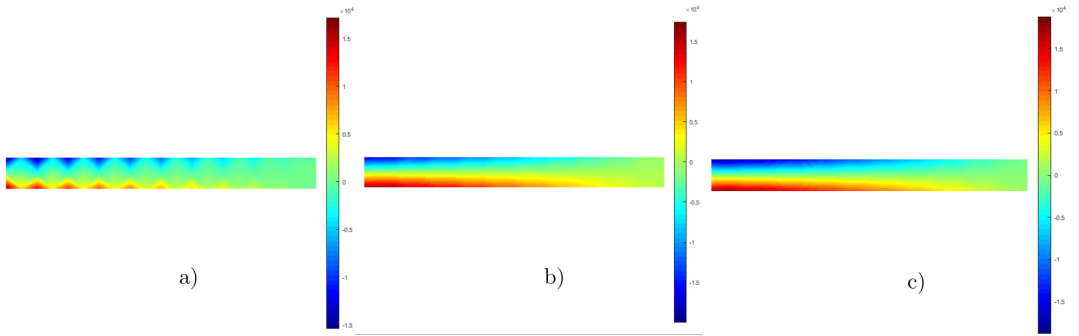


FIGURE 7.11: Color maps obtained for $\sigma_{xx}[Pa]$ using the 2x20x2 mesh - (a) FEM; (b) RPM; (c) NNRPIM

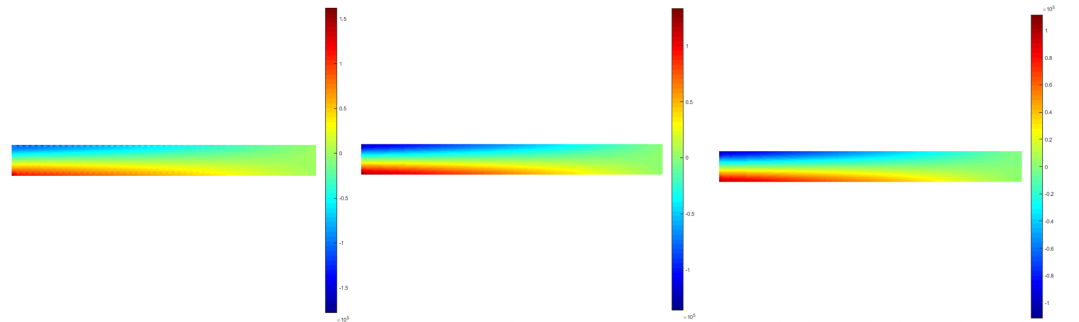


FIGURE 7.12: Color maps obtained for $\sigma_{xx}[Pa]$ using the 10x100x2 mesh - (a) FEM; (b) RPM; (c) NNRPIM

The convergence of the distinct methods is verified by the notorious increasing accuracy obtained as the number of nodes is raised. For the 2D model, the FEM lead to the lowest differences between the exact solution and the simulation, thus, allowing for the best accuracy between the three different methods. For the 3D model, it was the NNRPIM which led to the lowest differences between the exact solution and the simulation. The evolution of the color maps shows that the general distribution obtained from the models with sparse meshes is identical to the ones on the most refined meshes; however, the transitions between zones with lower stress concentration zones, shown in red in the figures becomes progressively smoother as the meshes used are more refined. Furthermore, the accuracy of the numeric values obtained for the vertical displacement (in point A) and normal stress component (in point B), used as a reference, clearly improved as the meshes became more refined,

at the cost of longer simulation periods. Bearing all of this in mind, this study allows to understand the potential, efficiency and accuracy of the presented methods - FEM, RPIM and NNRPIM - and this knowledge shall be extended and transposed for simulations using different models and more intricate geometries.

7.2 Numerical analysis of honeycomb structures

In order to study the effect of the relative density, ρ^*/ρ_s , which translates into the thickness of the trabecular walls of the core material of the honeycomb, as well as the effect of the chosen geometry, different geometries were modelled, considering a repeatable pattern of each geometry, as represented in Figures 7.13 and 7.14.

As previously described in Chapter 2, the Homogenization process will be used for the numerical analysis. This process allows us to derive the representative properties of the studied structures through direct numerical simulation of unit cell (microscale) models; the resulting homogenized properties are then used assuming uniform distribution of the cellular structure to compute the macroscale structure [8].

Thus, the conventional honeycomb and the re-entrant honeycomb geometries were firstly designed as 2D structures. The models were designed using the *Solid-Works* software, then meshed in *FEMAP*, and finally imported to *FEMAS*, where the tests for all three FEM, RPIM and NNRPIM were carried.

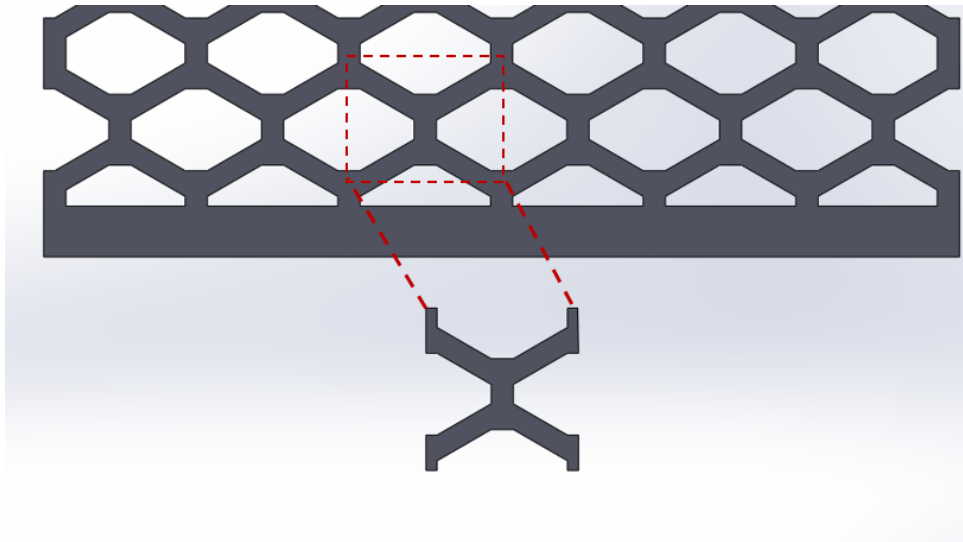


FIGURE 7.13: Repeating pattern of the conventional honeycomb core

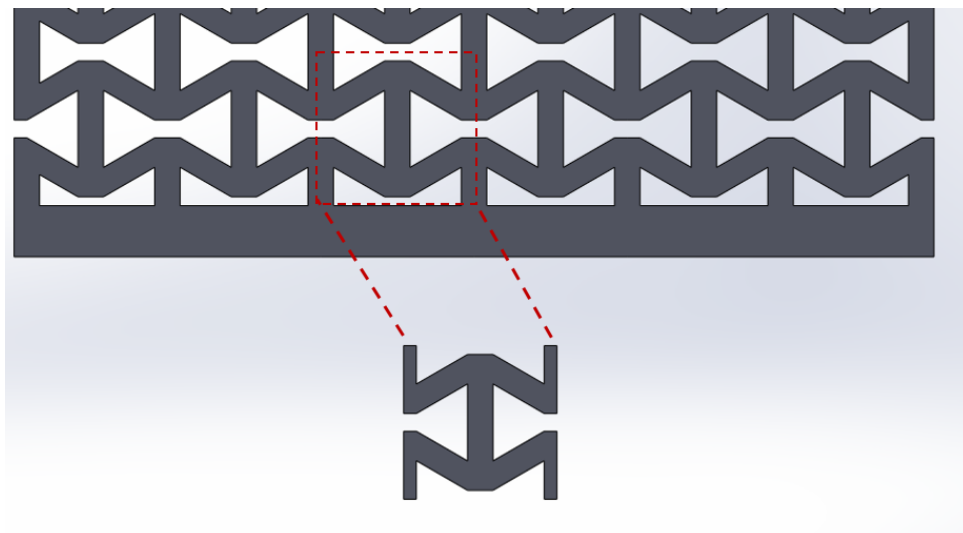


FIGURE 7.14: Repeating pattern of the re-entrant honeycomb core.

7.2.1 Structural design

The geometric features of the proposed sandwich core design of conventional honeycombs and re-entrant honeycombs must be firstly characterized.

These geometries are fully defined in Figure 7.15(a) and (b), respectively.

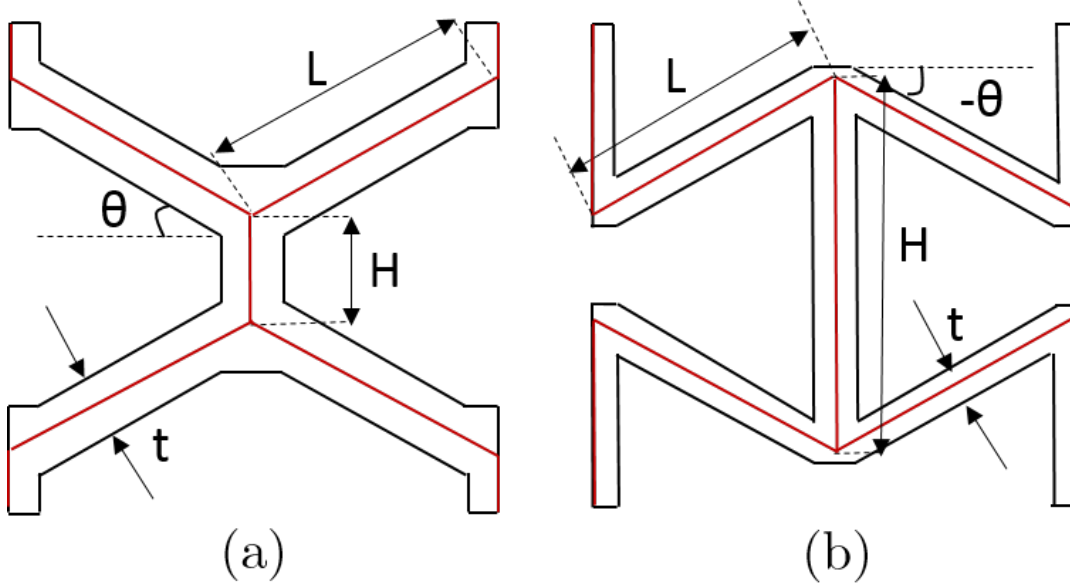


FIGURE 7.15: Necessary dimensions to fully define the geometry of the (a) conventional honeycomb; and (b) re-entrant honeycomb.

The geometries defined above were modelled considering a total of 7 different relative densities, ranging from 0.2 (20%) to 0.5 (50%), for increments of 0.05 (5%).

May we recall the expression obtained in Chapter 5 for the relative density of the honeycomb:

$$\frac{\rho^*}{\rho_s} = \frac{t/l(h/l + 2)}{2\cos\theta(h/l + \sin\theta)} \quad (7.2)$$

which applies for both conventional and re-entrant honeycombs [10]. Therefore, considering the dimensions $A \times A = 10 \times 10$ mm for the unit cell in both cases, the needed parameters for each model were obtained.

For the conventional honeycomb, by fixing $A = 10$ mm and $\theta = 30^\circ$, we can obtain H and L :

$$\theta = \tan^{-1} \left(\frac{1}{2} \frac{(A - 2H)}{A/2} \right) \quad (7.3)$$

which gives, for every conventional honeycomb, $H = 2.113$ mm.

Then, in order to obtain L ,

$$A = \frac{H}{2} + L\sin(\theta) + H + L\sin(\theta) + \frac{H}{2} \quad (7.4)$$

which gives, for every conventional honeycomb, $L = 5.7735$ mm. These parameters, along with the thickness, t , defined for each geometry by the relation expressed in equation (7.2), are enough to design the full geometry. The same applies to the re-entrant honeycomb model, for which $H = 7.8867$ mm and $L = 5.7735$ mm were

obtained. The parameters for each model are presented in Table 7.8 and Table 7.9, for the conventional honeycomb and re-entrant honeycomb models, respectively.

TABLE 7.8: Parameters for all conventional honeycomb models used in the linear-elastic analysis.

ρ^*/ρ_s	θ (°)	H [mm]	L [mm]	t [mm]
20%	30	2,113	5,7735	0,732
25%	30	2,113	5,7735	0,915
30%	30	2,113	5,7735	1,098
35%	30	2,113	5,7735	1,281
40%	30	2,113	5,7735	1,464
45%	30	2,113	5,7735	1,647
50%	30	2,113	5,7735	1,830

TABLE 7.9: Parameters for all re-entrant honeycomb models used in the linear-elastic analysis.

ρ^*/ρ_s	θ (°)	H [mm]	L [mm]	t [mm]
20%	30	7,88675	5,7735	1,109
25%	30	7,88675	5,7735	1,386
30%	30	7,88675	5,7735	1,663
35%	30	7,88675	5,7735	1,940
40%	30	7,88675	5,7735	2,217
45%	30	7,88675	5,7735	2,495
50%	30	7,88675	5,7735	2,772

Once modelled in *SolidWorks*, the geometries were meshed using a reference of 10 elements for each unit of length, resulting models containing up to 9000 nodes. Examples of the meshes created for each conventional and re-entrant honeycomb are represented in Figures 7.16 and 7.17.

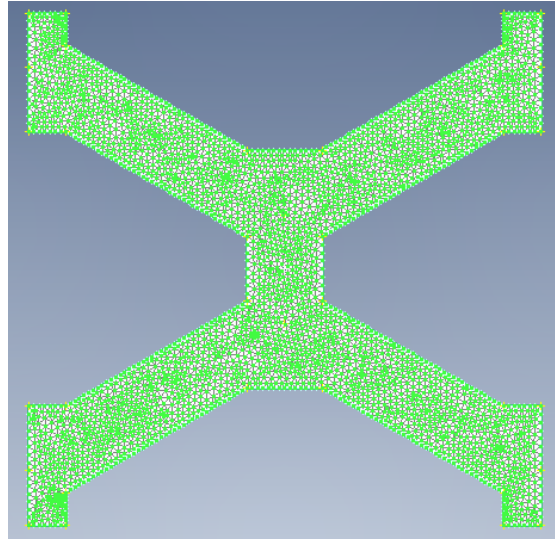


FIGURE 7.16: Meshed model of the conventional honeycomb

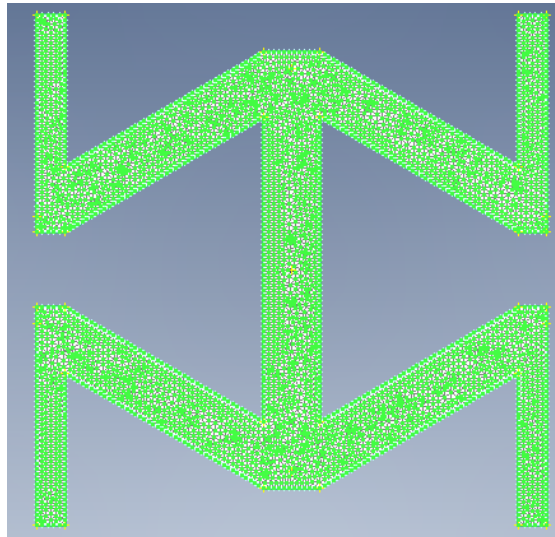


FIGURE 7.17: Meshed model of the re-entrant honeycomb

7.2.2 Linear elastic analysis

Afterwards, these meshed models were exported to *FEMAS*, where the appropriate essential and natural boundary conditions were considered - Figures 7.18 and 7.19.

Both honeycombs were assumed confined in two distinct load cases. First, the honeycombs were tested in the Oy direction, Figure 7.18, in which the nodes belonging to the left and a right vertical boundaries were assumed as constrained in the Ox direction and the nodes belonging to the $y=0$ mm plane were assumed as constrained in the Oy direction. Furthermore, in the top arms of the honeycomb ($y=5$ mm), it was assumed a downward vertical distributed load.

Afterwards, the honeycombs were tested in the Ox direction, Figure 7.19, in which the nodes belonging to the horizontal top and bottom boundaries ($y=0$ and $y=5$ mm) were assumed as constrained in the Oy direction and the nodes belonging to the left vertical boundary ($x=0$ mm) were assumed as constrained in the Ox direction. Additionally, on the right boundary of the honeycomb ($x=5$ mm), a right-to-left horizontal distributed load was assumed.

The mechanical properties considered for the linear-elastic material were $E = 3500\text{MPa}$ and $\nu = 0,36$.

Therefore, for the linear elastic analysis, the models of both conventional and re-entrant honeycombs were tested both in the Oy direction - Figure 7.18 - and the Ox direction - 7.19 - in *FEMAS*, using the 3 numerical methods (FEM, RPIM and NNRPIM) which translated into a total of 84 numerical tests.

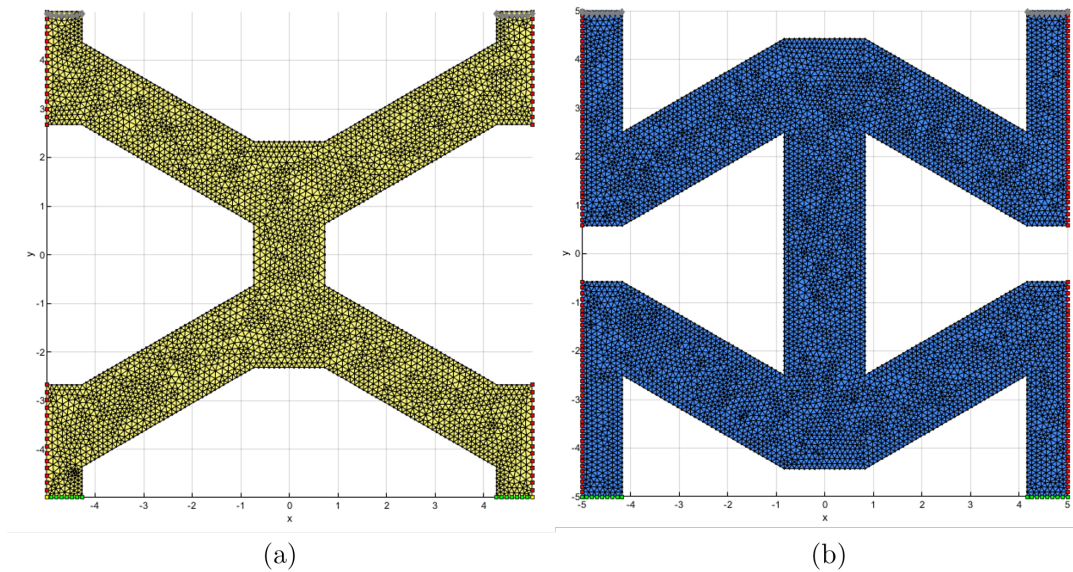


FIGURE 7.18: Boundary conditions for the tests in the y direction - (a) conventional honeycomb; (b) re-entrant honeycomb.

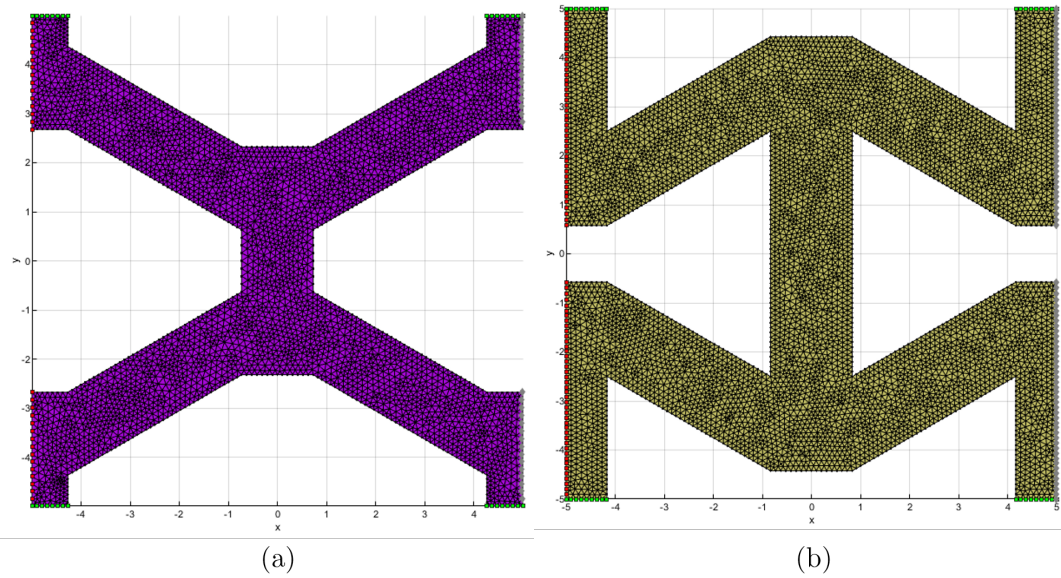


FIGURE 7.19: Boundary conditions for the tests in the x direction - (a) conventional honeycomb; (b) re-entrant honeycomb.

Once all the tests were carried, the obtained results can be represented, for example, in color maps. Just to show a few, the von Mises stress obtained for the conventional honeycomb with a 40% relative density in the Oy , direction and the same in the x direction, both obtained by the FEM, are represented in Figures 7.20 and 7.21; the von Mises stress obtained for the re-entrant honeycomb with a 30% relative density in the y direction, and the same in the x direction, both obtained by the FEM, is represented in Figures 7.22 and 7.23. The remaining color maps for the von Mises stress obtained in the linear-elastic simulations are present in Appendix A.

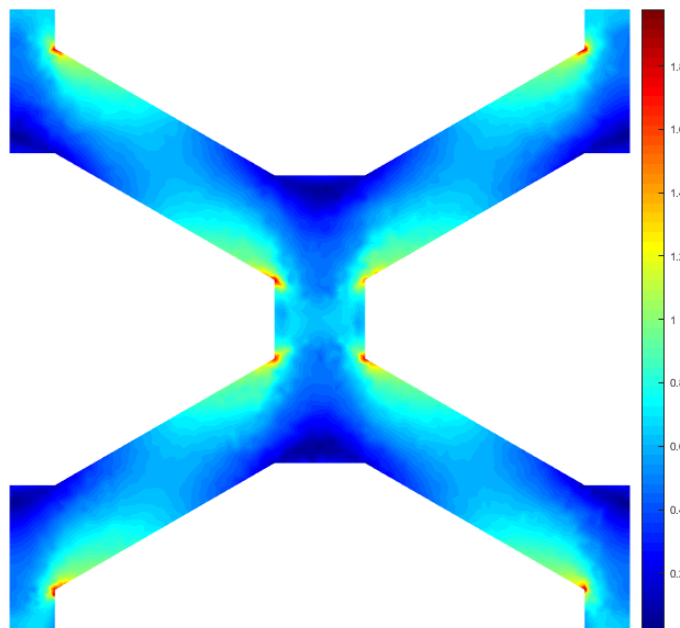


FIGURE 7.20: von Mises stress [MPa] for the conventional honeycomb (relative density 40%), y direction, obtained with the FEM.

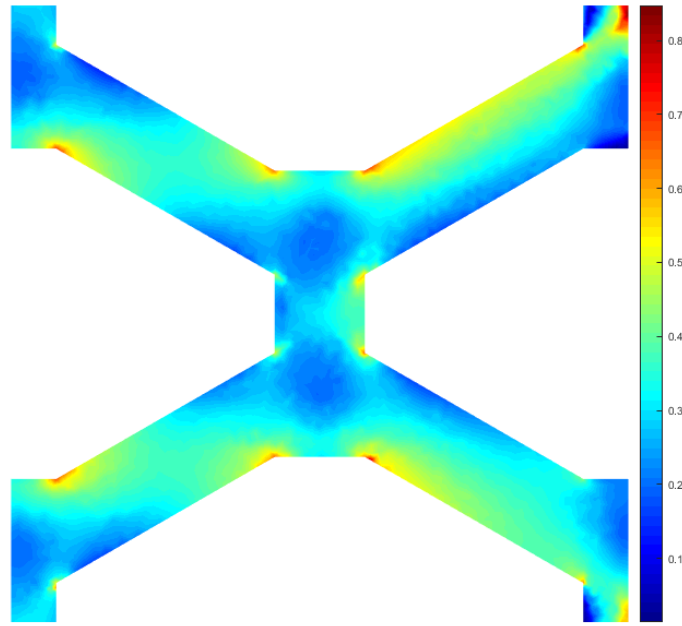


FIGURE 7.21: von Mises stress [MPa] for the conventional honeycomb (relative density 40%), x direction, obtained with the FEM.

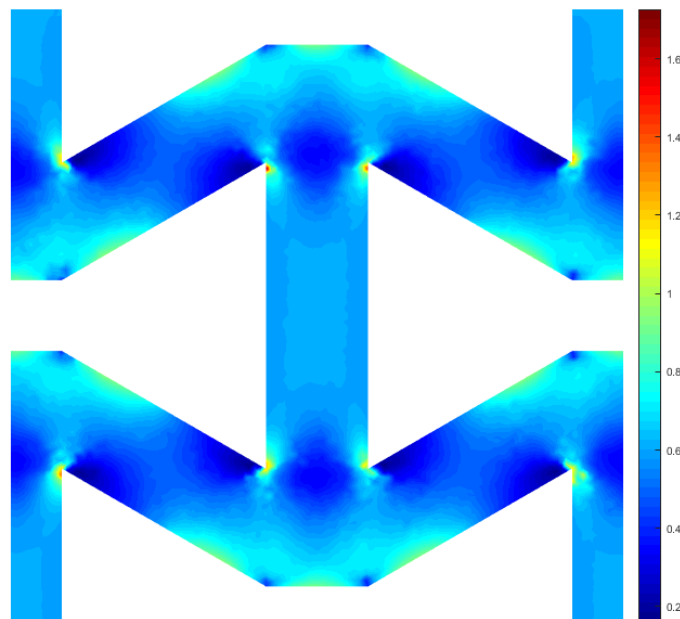


FIGURE 7.22: von Mises stress [MPa] for the re-entrant honeycomb (relative density 30%), y direction, obtained with the FEM.

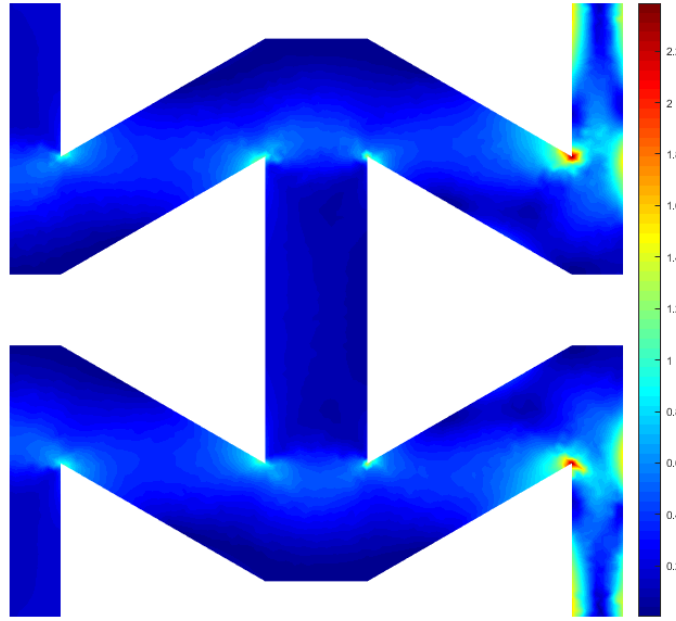


FIGURE 7.23: von Mises stress [MPa] for the re-entrant honeycomb (relative density 30%), x direction, obtained with the FEM.

Furthermore, in order to define the stiffness and maximum force supported by the geometry in both the Ox and Oy directions, the maximum deflection of the geometry and the maximum von Mises stress in each model were considered. For the stiffness along either the x or the y direction, the average displacement found in the points where the forces were applied, in that direction, was calculated. Then, considering the respective force for each case, the directional global stiffness can be obtained through the following a simple static-equilibrium expression:

$$K.u = f \quad (7.5)$$

with K being the stiffness along either x or y , u being the average of the displacements of the points where the forces were applied obtained, in the respective direction, and f the total force applied to the model.

The results obtained for both the y and x directions are represented, respectively, in Figures 7.24 and 7.25, for the conventional honeycomb, and Figures 7.28 and 7.29, for the re-entrant honeycomb.

In addition to that, the maximum force supported by each model was calculated considering the maximum von Mises stress obtained in each case, using a simple correspondence: for the applied force of 1 N in each model, the maximum von Mises stress can be found as an output. Thus, for the yielding stress of a given material, the maximum force which can be applied before the material starts to have a plastic behaviour, instead of a linear-elastic behaviour, can be obtained:

$$\frac{F}{\sigma_{max}^{model}} = \frac{F_{max}}{\sigma_{ced}} \quad (7.6)$$

with F being the applied force ($F = 1\text{ N}$), σ_{max}^{model} the maximum von Mises stress obtained in the model, σ_{ced} the normal yield stress of the considered material, and F_{max} the maximum force supported by the structure. Thus,

$$F_{max} = F \times \frac{\sigma_{ced}}{\sigma_{max}^{model}} \quad (7.7)$$

For each value of the wall thickness, this maximum force was represented for both the y and x directions, respectively, in Figures 7.26 and 7.27, for the conventional honeycomb, and Figures 7.30 and 7.31, for the re-entrant honeycomb.

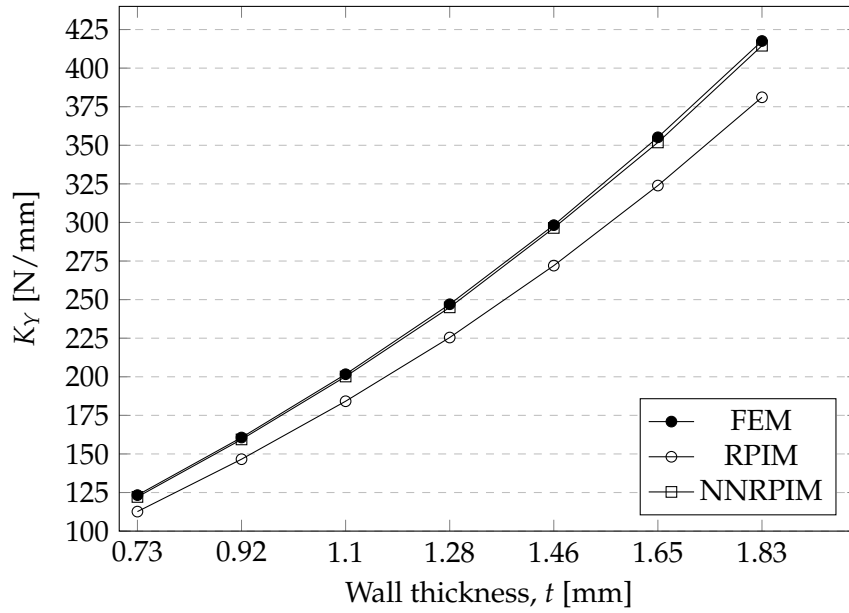


FIGURE 7.24: Stiffness obtained for the conventional honeycomb in the Y direction, K_Y , for each value of wall thickness, t .

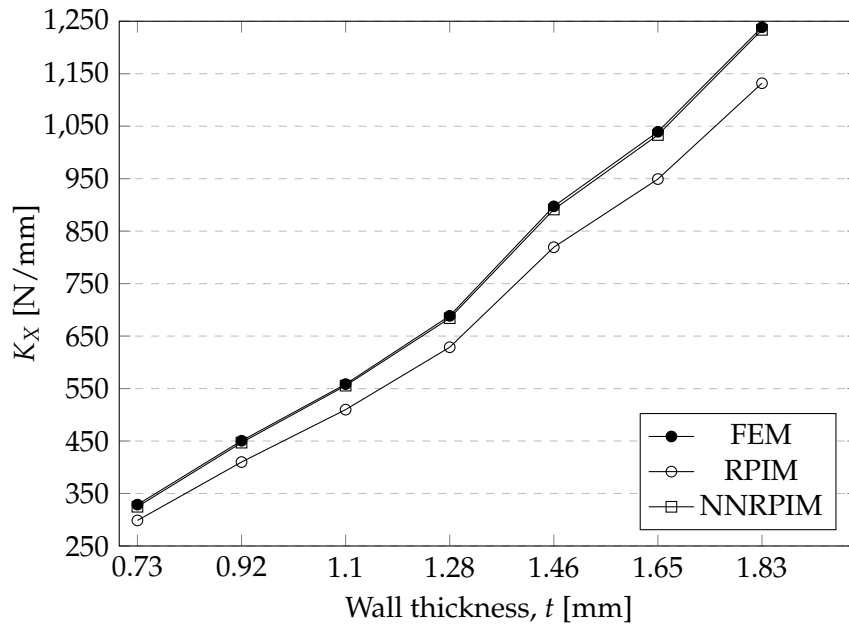


FIGURE 7.25: Stiffness obtained for the conventional honeycomb in the X direction, K_X , for each value of wall thickness, t .

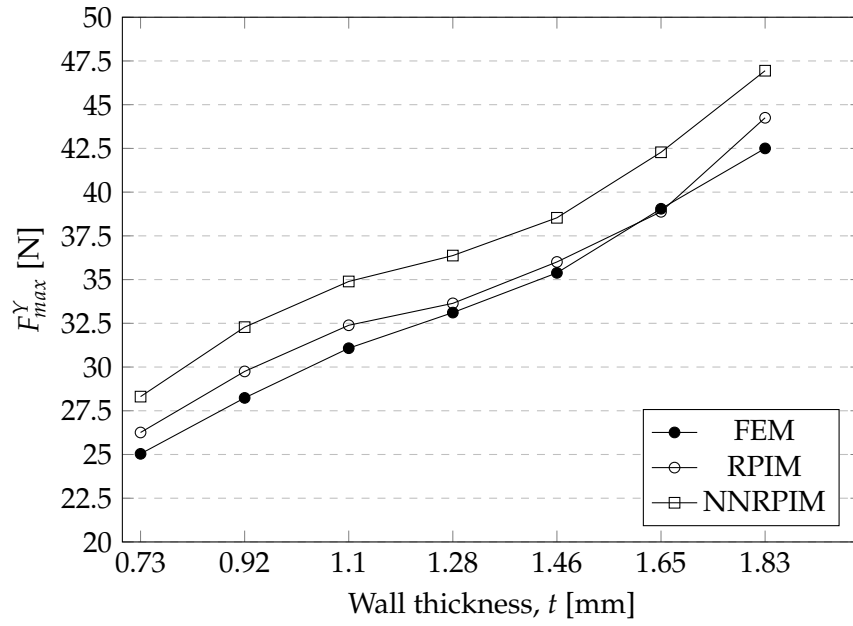


FIGURE 7.26: Maximum force supported by the conventional honeycomb in the Y direction, F_{max}^Y for each value of wall thickness, t .

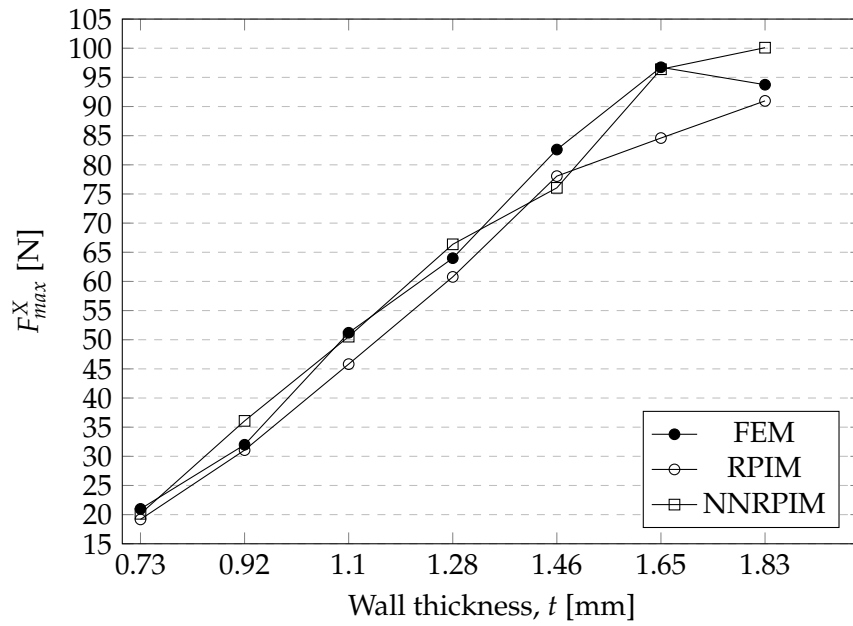


FIGURE 7.27: Maximum force supported by the conventional honeycomb in the X direction, F_{max}^X , for each value of wall thickness, t .

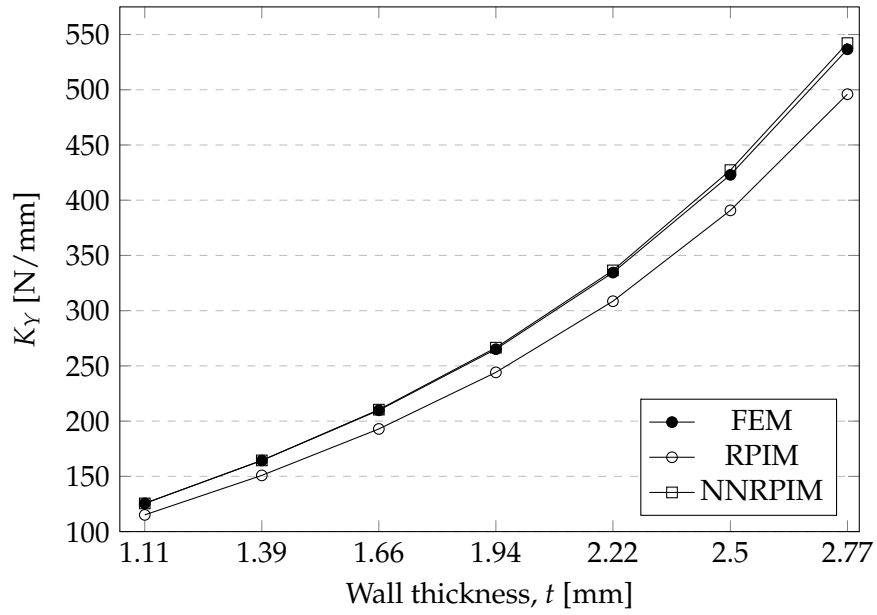


FIGURE 7.28: Stiffness obtained for the re-entrant honeycomb in the Y direction, K_Y , for each value of wall thickness, t .

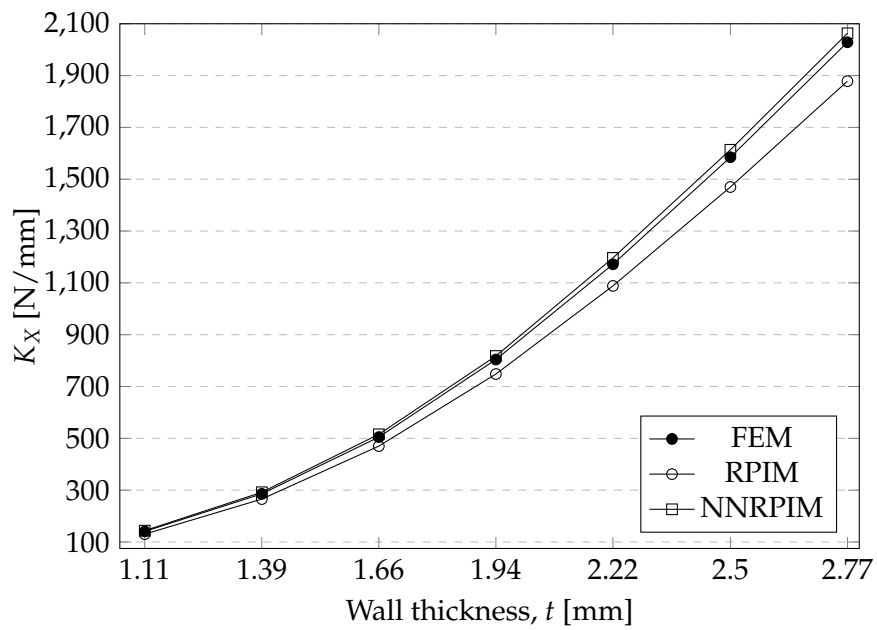


FIGURE 7.29: Stiffness obtained for the re-entrant honeycomb in the X direction, K_X , for each value of wall thickness, t .

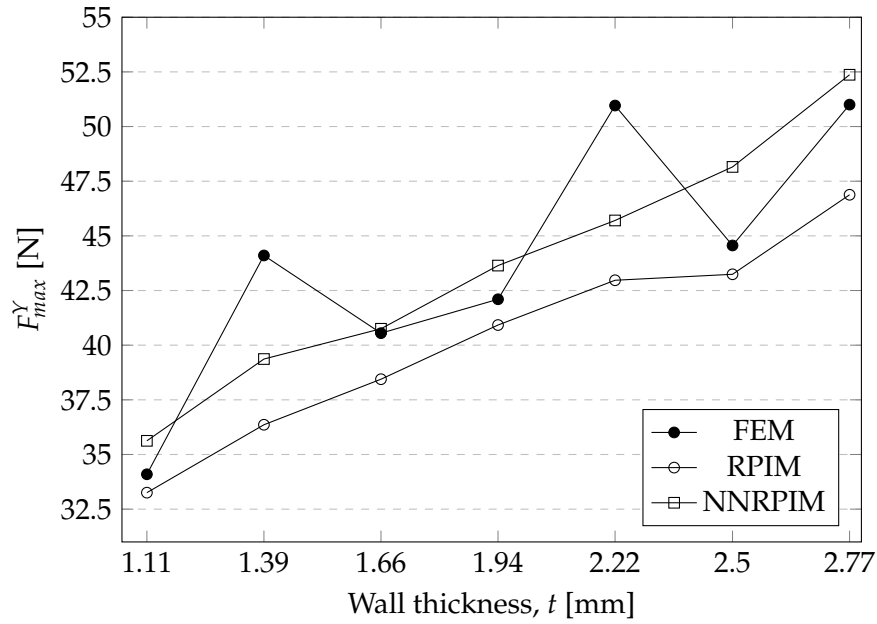


FIGURE 7.30: Maximum force supported by the re-entrant honeycomb in the Y direction, F_{max}^Y for each value of wall thickness, t .

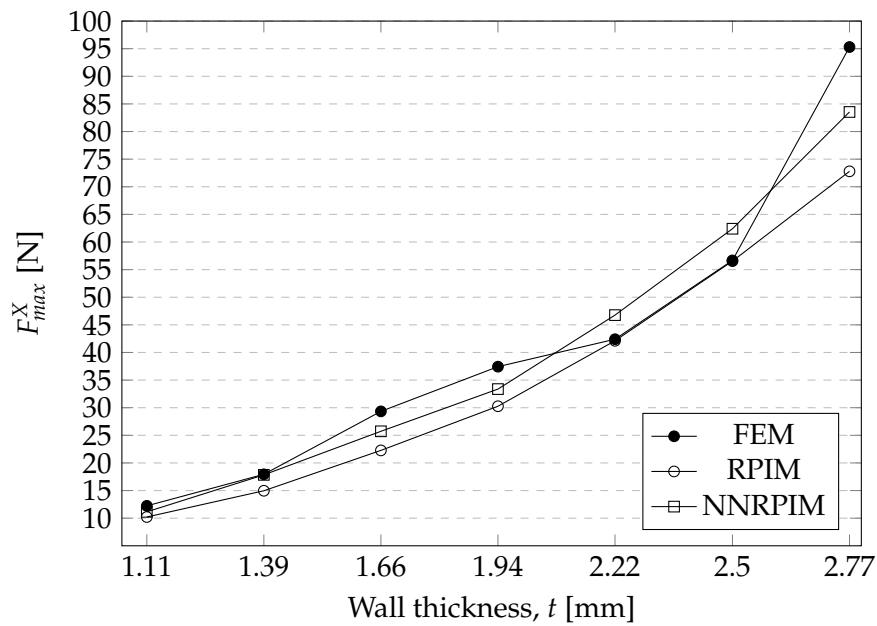


FIGURE 7.31: Maximum force supported by the re-entrant honeycomb in the X direction, F_{max}^X , for each value of wall thickness, t .

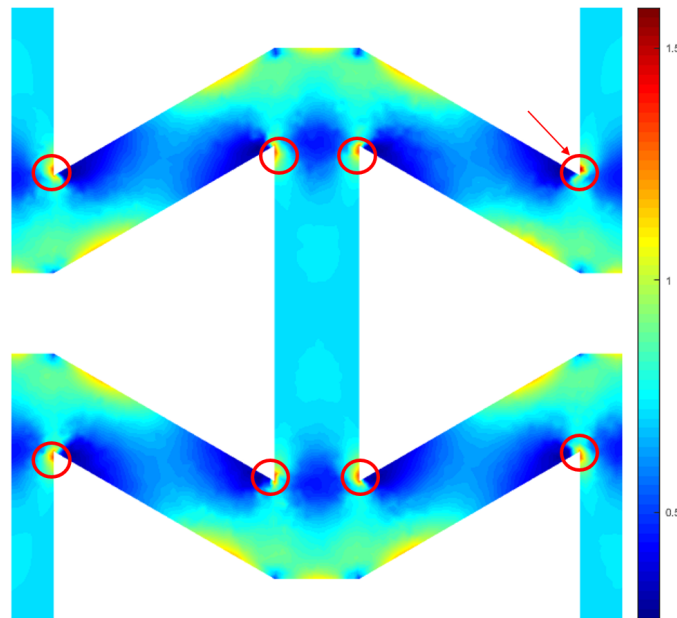
The results on the obtained stiffness, for both conventional and re-entrant honeycombs, in both x and y directions can be adjusted by polynomial functions, thus defining phenomenological curves in the domain of the considered thicknesses: in these expressions, K represents the stiffness in N/mm and t the wall thickness - Table 7.10.

TABLE 7.10: Adjusted polynomial functions to the obtained phenomenological curves.

Method	HC, y direction	HC, x direction	RHC, y direction	RHC, x direction
FEM	$K = 77,045t^2 + 69,66t + 31,662$	$K = 254,47t^2 + 176,65t + 65,098$	$K = 95,712t^2 - 129,85t + 156,33$	$K = 402,98t^2 - 413,37t + 92,648$
RPIM	$K = 70,537t^2 + 62,813t + 29,447$	$K = 231,3t^2 + 166,08t + 54,879$	$K = 89,021t^2 - 121,52t + 144,49$	$K = 368,1t^2 - 362,74t + 69,272$
NNRPIM	$K = 75,857t^2 + 70,723t + 30,396$	$K = 254,16t^2 + 175,31t + 62,788$	$K = 97,938t^2 - 135,04t + 159,31$	$K = 407,11t^2 - 409,67t + 85,953$

Note that, for the maximum force supported by the re-entrant honeycomb in the x direction, there were two densities (25% and 40 %) for which the behaviour of the model did not follow the expected progression; however, this was only found for the FEM, while for the RPIM and NNRPIM the obtained values were visibly more coherent and progressed in a smoother fashion. This may be explained by the presence of a denser mesh nearby these points of stress concentration when compared to the rest of the structure, which, for the FEM in particular, led to the unusual occurrence of a higher von Mises stress. These stress concentration zones are highlighted in Figures 7.32 and 7.33.

Nevertheless, the meshless methods, which instead use the concepts of influence domains (RPIM) and influence cells (NNRPIM) did not lead to such results, and returned a smoother and more coherent evolution instead.

FIGURE 7.32: Detail of the stress concentration zones for the re-entrant model (relative density 25%), tested in the y direction, obtained with the FEM.

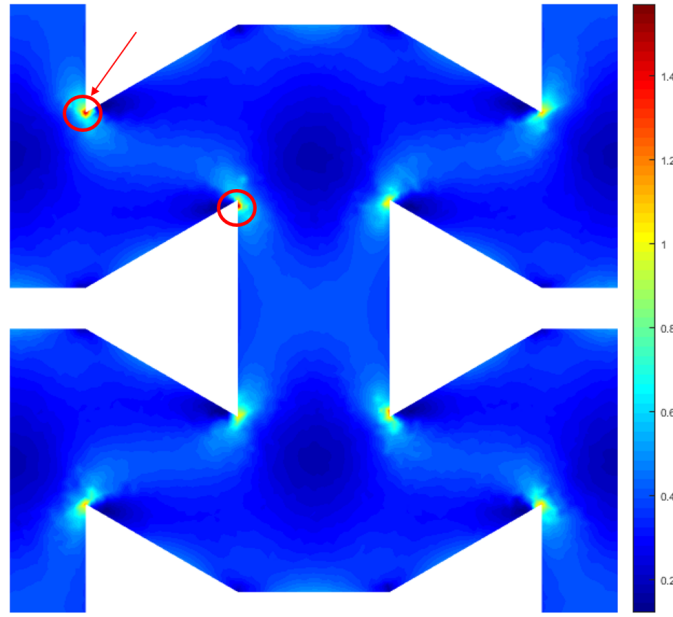


FIGURE 7.33: Detail of the stress concentration zones for the re-entrant model (relative density 40%), tested in the y direction, obtained with the FEM.

7.2.3 Elastoplastic analysis

The linear-elastic analysis presented earlier in this chapter was extended by an elastoplastic one, which allows to establish a direct comparison with the experimental results, to be obtained and described afterwards. Thus, all 7 relative densities for both conventional and re-entrant honeycombs were again considered. The analysis was carried using the FEM, and considering the KT-All non-linear algorithm, described in detail in Chapter 4, in order to reduce the higher processing cost associated with the elastoplastic analysis.

For this part of the study, the mechanical properties considered for the constitutive material were obtained from the adjusting of experimental stress-strain curves with an elastoplastic bi-linear behaviour. These curves were obtained from the mechanical testing in compression of specimens produced by 3D printing, in "Instituto Superior Técnico" of the University of Lisbon, within the scope of other Master's thesis [73], [74]. These had 10x10x4 mm dimensions and were tested at a speed of 5mm/min, following the International Standard ISO 604. In the context of those studies, they have been used to study the influence of the printing orientation on the mechanical properties of the specimens.

Note that these specimens were produced with a 100% infill of the same PLA material used in the experimental analysis of this work (which will be described in the subsequent sections), and therefore it will be used to obtain the mechanical properties of the extruded PLA to be considered in the elastoplastic analysis - namely the Young's Modulus, E , the tangent modulus, E_T , and the yield stress in compression, σ_Y . The chosen value for the Poisson's coefficient, ν , was the same as the one used in previous tests, found in literature to be $\nu \approx 0,36$.

The obtained stress-strain curves for the aforementioned specimens are represented in Figure 7.34.

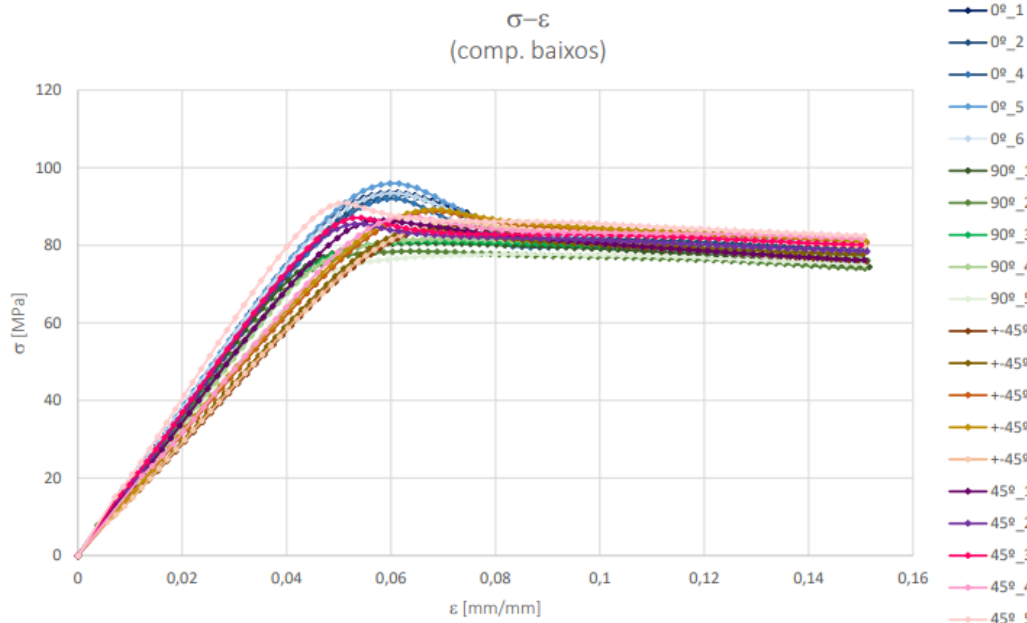


FIGURE 7.34: Obtained stress-strain curves for 3D printed specimens tested in compression, produced with PLA.

This group of curves was narrowed to include only the ones which were printed in alternated 45 °angles - which was the considered option for the experimental analysis in this work, and therefore, will give a more accurate representation of the properties of the specimens produced for this work - Figure 7.35.

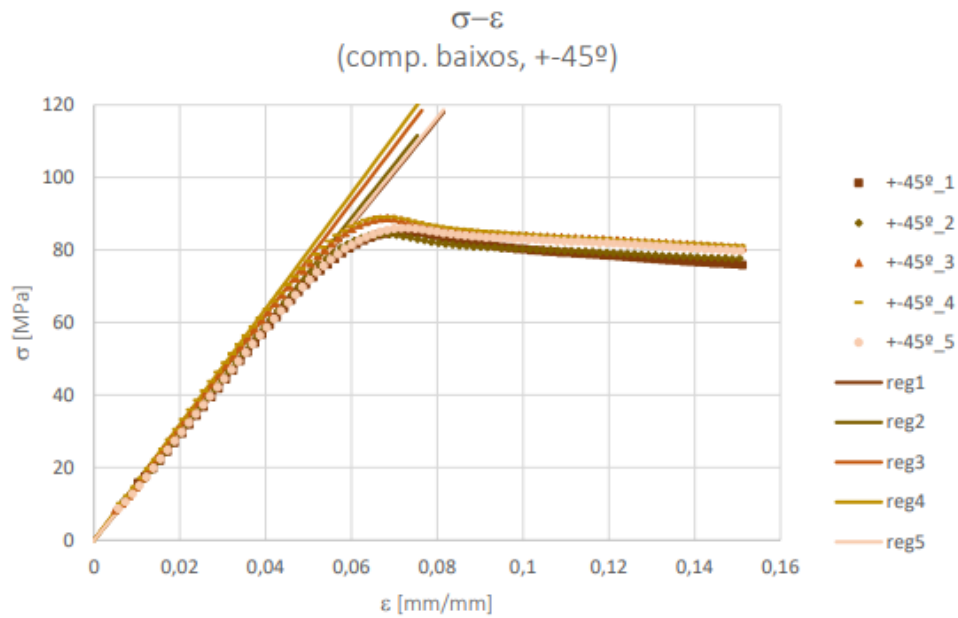


FIGURE 7.35: Obtained stress-strain curves for the 3D printed specimens printed in alternated 45 °angles.

From these curves, the following mechanical properties for the extruded PLA were obtained:

- $E = 3025,05 \text{ MPa}$

- $E_T = 1008,35 \text{ MPa}$
- $\sigma_Y = 89,18 \text{ MPa}$

Once the constitutive material's properties were defined, the models which were previously designed for the linear-elastic regimen were then tested on the elastoplastic domain. In order to perform this elastoplastic analysis and obtain the needed stress-strain curves for the PLA-made honeycomb unit cell, three parameters must be defined: the tolerance, the number of increments and the ratio. As previously described in Chapter 4, the tolerance acts as the stop criteria for each increment. This tolerance was defined to be 0,01 for every test. The number of increments will define the smoothness and accuracy of the obtained curve: the more increments are considered, the more information is gathered and the more accurate the obtained result is, at the cost of a longer simulation time. For the two densities used in the experimental analysis, the respective elastoplastic numerical analysis was performed using 10 increments. For the remaining models, only 5 increments were considered. Finally, the ratio is, in fact, a ratio between the expected extension of the curve in the vertical axis and the linear elastic part of the curve; therefore, it defines how much of the elastoplastic part is represented. Since this parameter is not known beforehand, the process becomes iterative: generally, the ratio is estimated to be similar of the one found in a stress-strain curve of the original material; therefore, for the analysis of PLA honeycombs, the ratio was estimated to be around 1.5 at the start, and tended to increase as the density (and therefore, the stiffness) of the honeycomb structure increased. If the chosen ratio is too low, a significant part of the elastoplastic domain may not even be obtained in the analysis, and therefore, the whole process must be repeated.

In every way similar to the linear-elastic analysis, the geometry, mechanical properties and both essential and natural boundary conditions are defined in FEMAS, and the analysis is started.

In order to demonstrate the evolution of the plastic regions in the honeycomb structure, the color maps representing the equivalent plastic strain for the re-entrant honeycomb model with a relative density of 30% are shown here, for 6 of the 10 considered increments - Figures 7.36, 7.37 and 7.38.

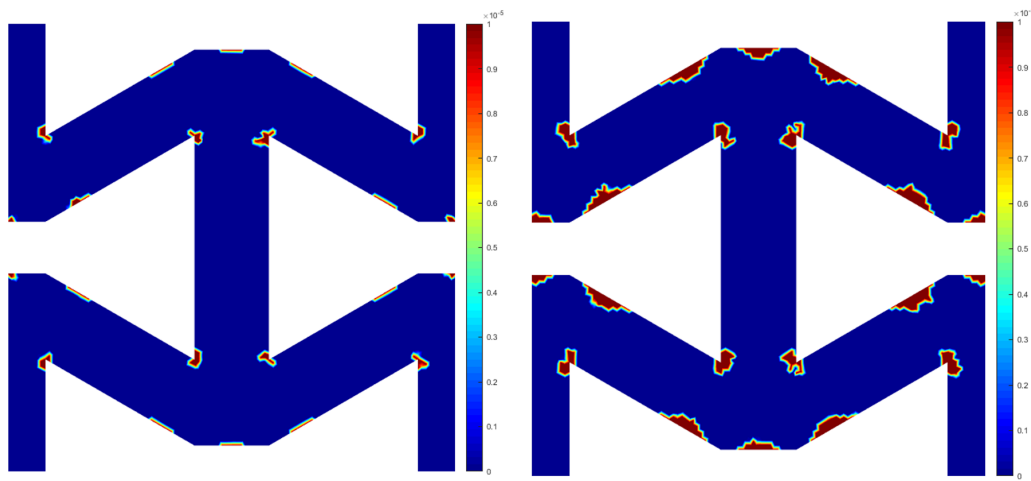


FIGURE 7.36: Progression of the equivalent plastic strain in the elastoplastic domain in the lower increments - re-entrant honeycomb with 30% relative density.

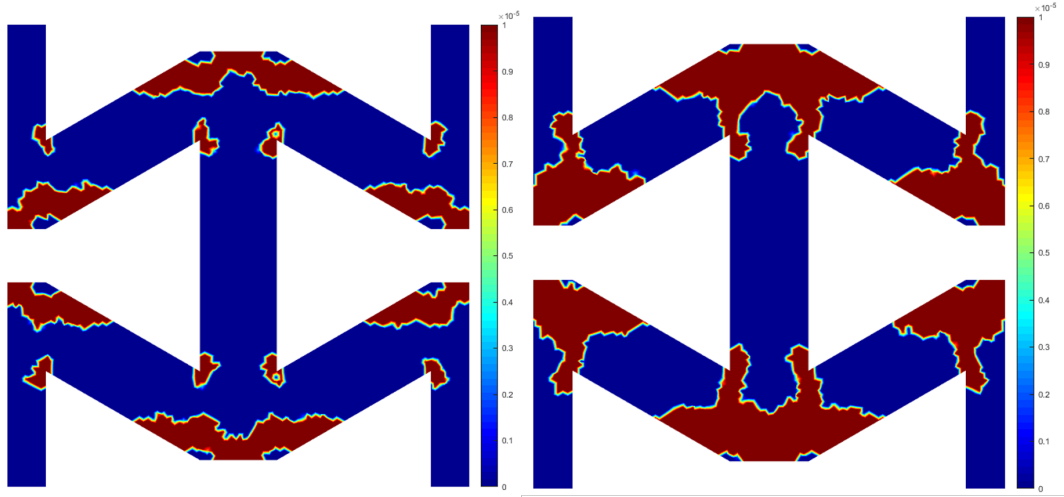


FIGURE 7.37: Progression of the equivalent plastic strain in the elasto-plastic domain in the intermediate increments - re-entrant honeycomb with 30% relative density.

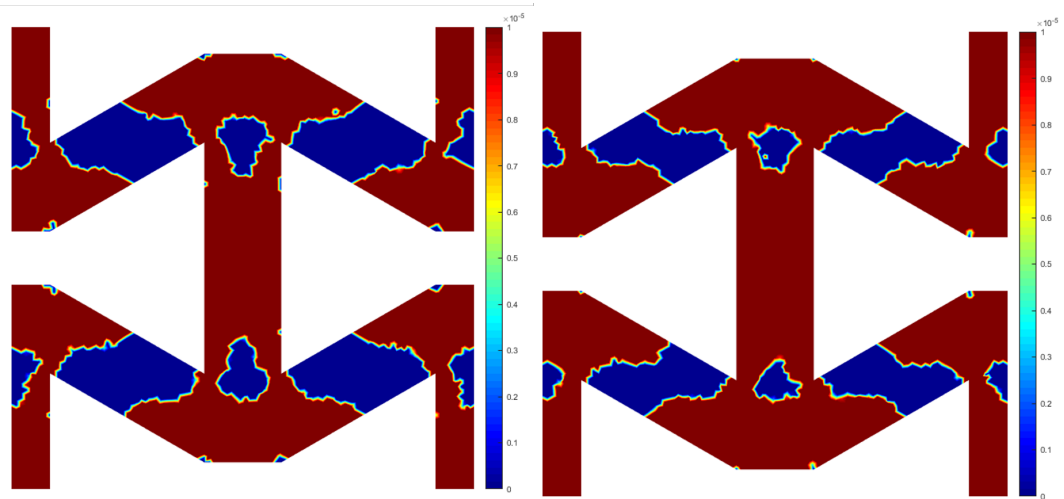


FIGURE 7.38: Progression of the equivalent plastic strain in the elasto-plastic domain in the higher increments - re-entrant honeycomb with 30% relative density.

Some interesting conclusions can be retained from these results: firstly, the evolution of the plastic regions starts precisely at the stress concentration regions obtained in the linear-elastic domain - recall Figure 7.22 - since these are the places where the von Mises stress is higher at the moment the plastification of the structure begins. Secondly, with a more careful inspection, one can notice the perfect symmetry of the plastic regions when imagining the compound structure of several honeycombs - in Figures 7.39 and 7.40, this fact becomes even more evident: zones A, B and C are clearly repeated patterns in the unit cell, which will then be multiplied to create the full sandwich structure. The fact that the unit cells are periodically repeated not only allows for an easier manufacturing and quality control, but also for an easier prediction of the behaviour of the structures using these cores.

Another interesting fact is that the obtained regions which first enter the elasto-plastic regimen are perfectly coherent with the example shown in Chapter 5, in Figure 5.18, which shows the formation of plastic hinges in an open-cell cubic foam.

Curiously enough, when compressed, the open cell cubic unit cell totally resembles the re-entrant honeycomb design.

These same conclusions can be retained for the conventional honeycomb geometry as well - Figure 7.40 shows the detailed view of the regions which notably enter the elastoplastic domain, in which the symmetry is even more evident.

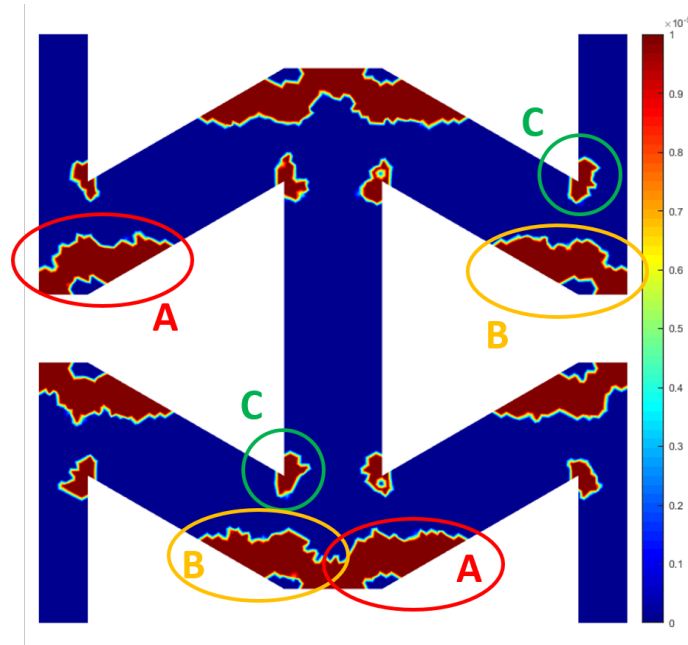


FIGURE 7.39: Details on the progression of the equivalent plastic strain in the elastoplastic domain of the re-entrant honeycomb unit cell.

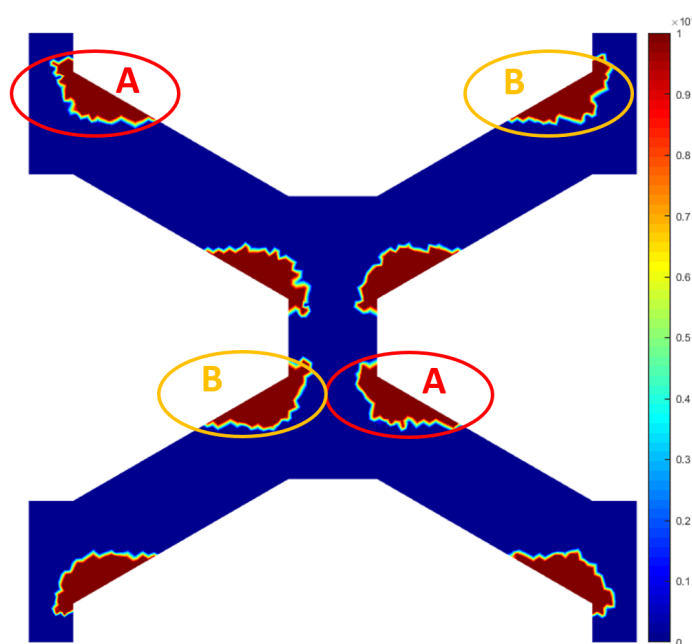


FIGURE 7.40: Details on the progression of the equivalent plastic strain in the elastoplastic domain of the conventional honeycomb unit cell.

In addition to the color maps, the obtained results were represented in stress-strain curves, in order to obtain the respective Young's Modulus, E , tangent modulus, E_T , and the normal yield stress in compression, σ_Y , of these honeycomb unit cells with the extruded PLA as constitutive material. For the two densities considered in the experimental analysis, the force-displacement curves were first obtained, for a point contained in the top plane of the honeycomb, that is, a point where the force is directly applied and the displacement is maximum.

However, in order to adjust the curve to a bi-linear model and obtain the mechanical properties, one must use a stress-strain curve. Therefore, the apparent stress can be obtained by dividing the applied force by the considered area, $\sigma = F/A_t$, with A_t being $10 \times 0,1\text{mm}$ in both cases ($0,1\text{mm}$ thickness), and the apparent strain is obtained by dividing the displacement by the height of the model, $\varepsilon = \delta/A$, with A being the height defined in Figure 7.15.

Bearing this in mind, the apparent/approximated stress-strain curves for all 7 densities of the conventional honeycomb unit cells and the re-entrant honeycomb unit cells were obtained. As an example, for the conventional honeycomb with 40% relative density and the re-entrant honeycomb with 30% relative density, the stress curves are represented in Figures 7.42 and 7.41.

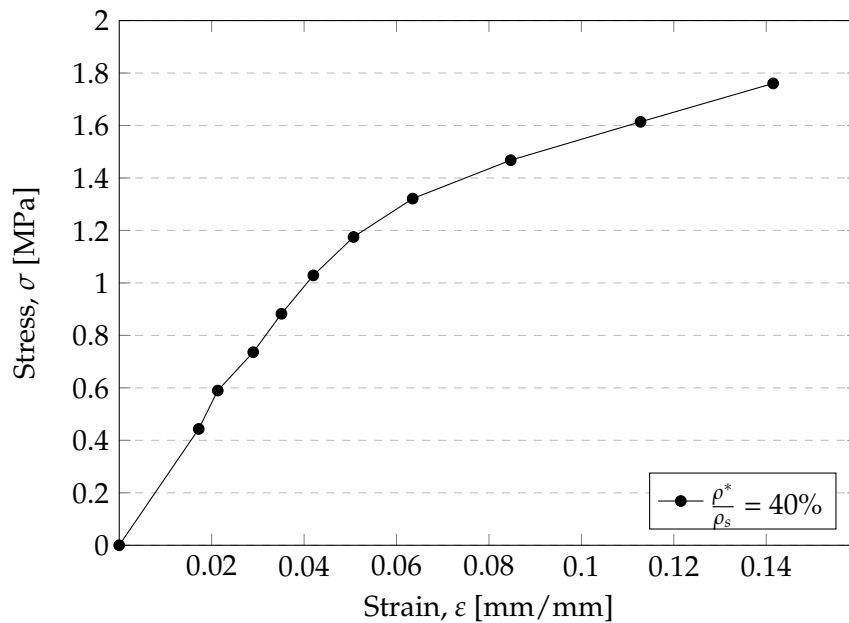


FIGURE 7.41: Stress-strain curve for the conventional honeycomb unit cell with a relative density of 40%.

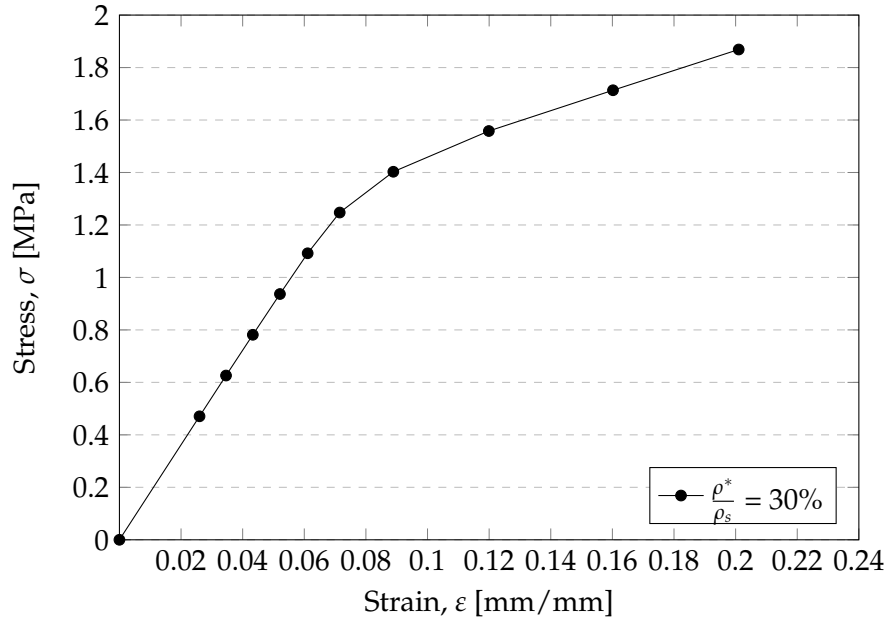


FIGURE 7.42: Stress-strain curve for the re-entrant honeycomb unit cell with a relative density of 30%.

From the curves represented in Figures 7.41 and 7.42, it is possible to estimate the following mechanical properties, for the extruded PLA conventional honeycombs with a 40% relative density:

- $E = 25,7188 \text{ MPa}$
- $E_T = 6,4485 \text{ MPa}$
- $\sigma_Y = 1,0287 \text{ MPa}$

and for the extruded PLA re-entrant honeycombs with a 30% relative density:

- $E = 18,1320 \text{ MPa}$
- $E_T = 4,1577 \text{ MPa}$
- $\sigma_Y = 1,4028 \text{ MPa}$

The remaining curves obtained from the elastoplastic analysis are present in Appendix B.

Nevertheless, all of the obtained values for Young's Modulus, tangent modulus and yield stress, for every relative density, are summarized in tables 7.11 and 7.12.

TABLE 7.11: Obtained mechanical properties for the conventional honeycomb unit cells from the elastoplastic analysis.

ρ^*/ρ_s	E [MPa]	E_T [MPa]	σ_Y [MPa]
20%	10,7581	2,3021	0,6016
25%	13,9640	2,7477	0,7926
30%	14,7329	4,2353	0,8384
35%	12,5148	3,5681	1,0798
40%	25,7188	6,4485	1,0287
45%	30,7470	6,0987	1,2439
50%	21,3692	7,8001	1,2559

TABLE 7.12: Obtained mechanical properties for the re-entrant honeycomb unit cells from the elastoplastic analysis.

ρ^*/ρ_s	E [MPa]	E_T [MPa]	σ_Y [MPa]
20%	10,8532	2,3132	0,9132
25%	14,2102	3,5452	1,0705
30%	18,1320	4,1577	1,4028
35%	22,8282	5,9940	1,6100
40%	28,9236	6,8947	1,9702
45%	36,5768	24,4686	2,2268
50%	46,4493	33,3125	2,5648

These equivalent properties are the foundation of the homogenization process. As previously referred, this technique allows to model the real specimens, without actually considering the number of honeycombs which constitute the core of the specimen - in fact, it is quite cumbersome to model all the geometrical details of the unit cell at the macroscopic scale. Nevertheless, in this work, this fully detailed design has been done since otherwise the additive manufacturing of these geometries would not be possible.

However, if a numerical analysis of these fully detailed designs was carried, the resulting numerical models would be constituted by an impractical number of nodes, thus requiring a considerable computational effort (and an unmanageable simulation cost) without the actual guarantee of accurate results; instead, one can make use of an homogenization procedure which allows to replace, at the macroscopic scale, the true complex geometry of the unit cell (also referred in literature as a Representative Volume Element - RVE) by an equivalent homogeneous medium whose mechanical response is described by a set of equivalent material properties.

Furthermore, new models considering two different constitutive materials - the

extruded PLA, and the "honeycomb-shaped" PLA - were designed and tested. Notice that the considered values for the mechanical properties of the honeycomb geometries are adjusted from the elastoplastic analysis results. Therefore, a cumulative error must be considered, at the exchange of rather faster and practical simulations of the models which represent the real specimens, to be described in detail throughout next section.

7.3 Experimental analysis

After modelling the geometries of the conventional and re-entrant honeycombs, with different relative densities and using three numerical methods, namely the FEM, the RPIM and the NNRPIM, in order to characterize the geometries both in the elastic-plastic and the elastoplastic domain, specimens were manufactured with 3D printing and mechanically tested as described in the subsequent sections.

Throughout this study, 7 relative densities of the honeycombs were considered for the numerical analysis: 20%, 25%, 30%, 35%, 40%, 45% and 50%. From these, two densities were selected for the experimental analysis - 30% and 40% - using as the main criteria the optimization of the obtained thickness of the trabecular walls for the additive manufacturing of the specimens - these particular choices and all of the challenges associated with the manufacturing of the specimens are now discussed. The obtained results and the comparison with the numerical analysis of the respective specimens' models are presented as well.

7.3.1 Sample preparation

For every specimen, a face sheet of 1 mm was produced in the same material as the core; the chosen constitutive material for the specimens was a Premium PLA by *FormFutura*. This Premium PLA is a "high-end general purpose Poly-Lactic Acid type of 3D printer filament, which has an excellent thermal stability and a slightly faster crystallisation process of the 3D printed layers and is often used for large scale prototyping on industrial-sized 3D printers" [75].

Afterwards, 4 different types of cube specimens were created: one filled with a conventional honeycomb design, with two face sheets, one at the bottom, one at the top - Figure 7.43, and its equivalent with the re-entrant honeycomb design - Figure 7.44; and one filled with a conventional honeycomb design, but containing a 1 mm face sheet in every direction (giving it the appearance of a full, solid cube) - Figure 7.45 - as well as its equivalent using the re-entrant honeycomb design as the core geometry - Figure 7.46.

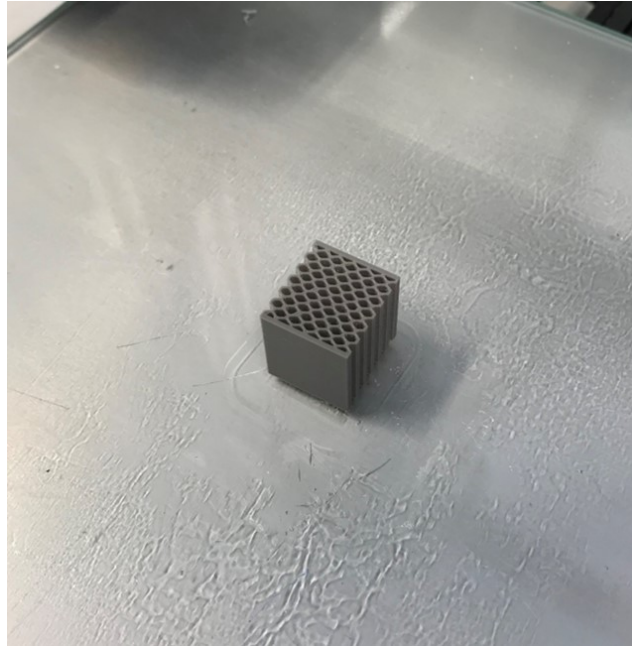


FIGURE 7.43: Cube specimen with a conventional honeycomb core and face sheet at the top and bottom.

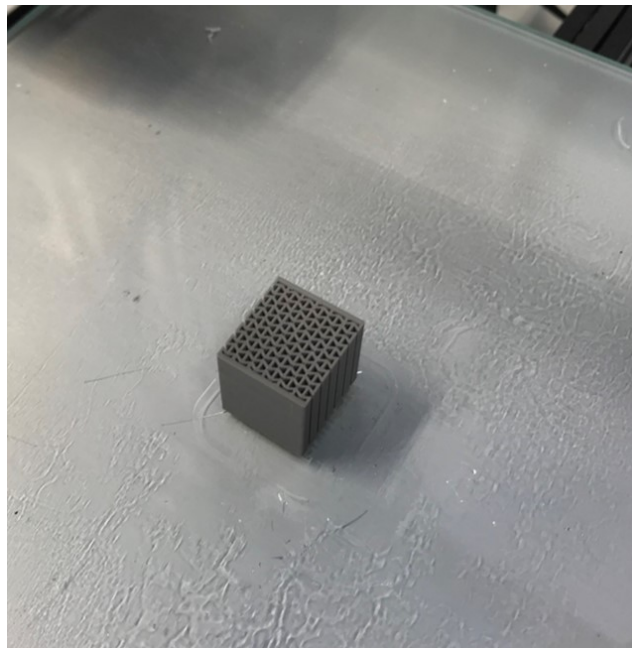


FIGURE 7.44: Cube specimen with a re-entrant honeycomb core and face sheet at the top and bottom.

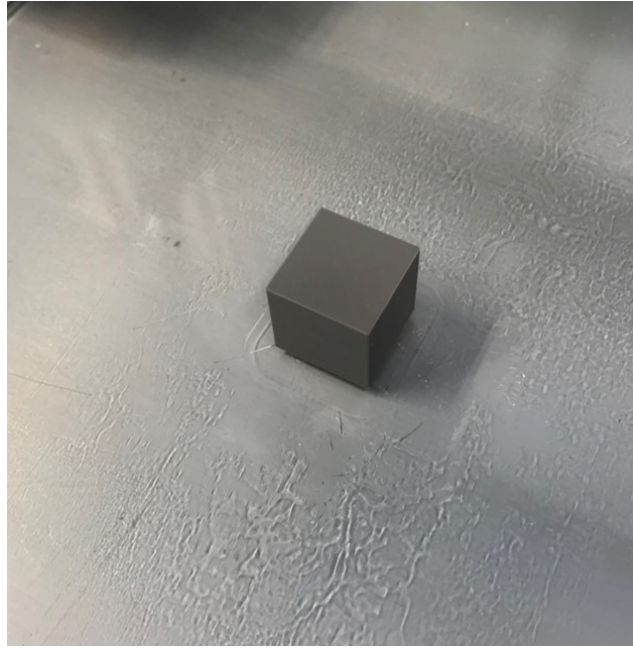


FIGURE 7.45: Cube specimen with a conventional honeycomb core and face sheet in every direction.

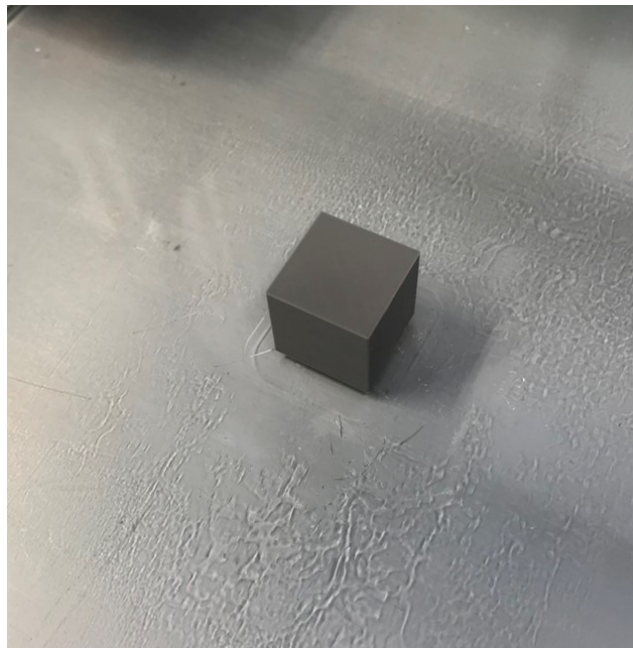


FIGURE 7.46: Cube specimen with a re-entrant honeycomb core and face sheet in every direction.

A noteworthy mention to be made in the context of the design of these specimens is concerned with the restrictions imposed by the manufacturing process. Therefore, finding the optimized dimensions of the models to be tested constitutes an interesting challenge, since these models should have maximum dimensions allowed by the 3D printer (in this case, 220x220x250 mm), but compatible with the norm to be used in the mechanical tests; and a minimum thickness of the walls defined by its nozzle - in this case, 0.2 mm -, while still using an ideal number of honeycombs in the core of the model (the higher the number of honeycombs, the thinner are the walls of

each unit). A low number of honeycomb unit cells may lead to the occurrence of local effects such as stress concentration or premature failure due to eventual printing errors, and lower the accuracy of the homogenization process from the numerical analysis.

For this study, each specimen contains 6x6 honeycombs, which are then extruded and produced in the positions shown in the figures presented above. Therefore, considering a 20x20x20 mm specimen and a face sheet of 1 mm at the bottom and top ends, each honeycomb unit cell will be 3x3 mm.

Considering the resulting wall thickness in every case, the two densities were 40% for the conventional honeycombs and 30% for the re-entrant honeycombs. Recall that the wall thickness had corresponding thicknesses which were approximately multiples of a number between 0.22 and 0.24 mm (the minimum thickness produced by the 0.2 mm nozzle).

Therefore, the respective models' internal walls can be printed using 2 perimeters on the printing settings, allowing for an optimized and generally faster printing process. Thus, these were the chosen relative densities for the 3D printed specimens.

All the dimensions required to define the honeycomb geometries are presented in tables 7.13 and 7.14, using the nomenclature previously defined at the start of the honeycomb structural design section, in Figure 7.15.

TABLE 7.13: Parameters for all the conventional honeycomb models used in the elastoplastic analysis.

ρ^* / ρ_s	θ (°)	H [mm]	L [mm]	t [mm]
20%	30	0,6339	1,7321	0,220
25%	30	0,6339	1,7321	0,275
30%	30	0,6339	1,7321	0,329
35%	30	0,6339	1,7321	0,384
40%	30	0,6339	1,7321	0,439
45%	30	0,6339	1,7321	0,494
50%	30	0,6339	1,7321	0,549

TABLE 7.14: Parameters for all the re-entrant honeycomb models used in the elastoplastic analysis.

ρ^*/ρ_s	θ (°)	H [mm]	L [mm]	t [mm]
20%	30	2,3660	1,7321	0,333
25%	30	2,3660	1,7321	0,416
30%	30	2,3660	1,7321	0,499
35%	30	2,3660	1,7321	0,582
40%	30	2,3660	1,7321	0,665
45%	30	2,3660	1,7321	0,748
50%	30	2,3660	1,7321	0,832

Once every detail of the geometries was defined, the specimens were printed using the FDM technique and a *Crealty Ender 3* 3D printer. The main specifications provided by the manufacturer are presented in Table 7.15.

TABLE 7.15: Specifications for the Crealty Ender 3 3D printer

Modeling Technology	FDM (Fused Deposition Modeling)
Printer type	Material Extrusion
Year	2018
Filament Diameter	1.75mm
Printable materials	ABS, PLA
Print size millimeters (xyz)	220 x 220 x 250 mm
Layer height	0.1 - 0.4mm
Max nozzle temperature:	255 °C
Max heated bed temperature	100 °C

The software which was used to read the *.stl* files, select every printing parameter and estimate each printing time was *Slic3r Prusa Edition*. Despite the possibility of creating a specimen in this software with a honeycomb core and a face sheet of our choice, a simple yet determinant problem arose: when choosing a different relative density for the core, it was not possible to define, in an independent way, the number of honeycombs in the specimen and the thickness of the walls of each honeycomb - Figures 7.47 and 7.48.

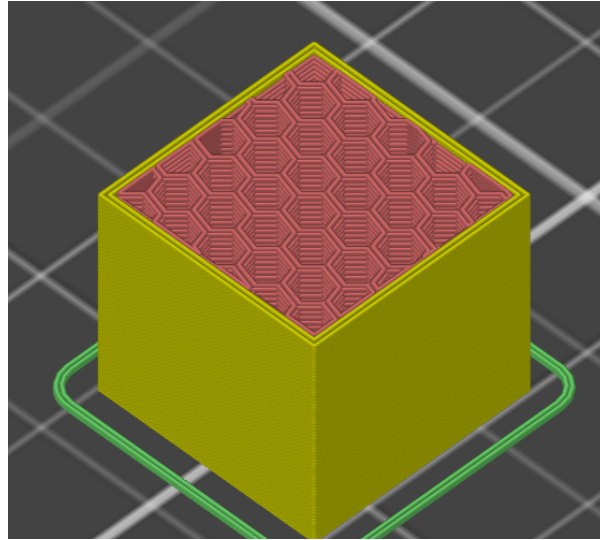


FIGURE 7.47: Available design of the honeycomb infill using the default Slic3r options - 25% relative density.

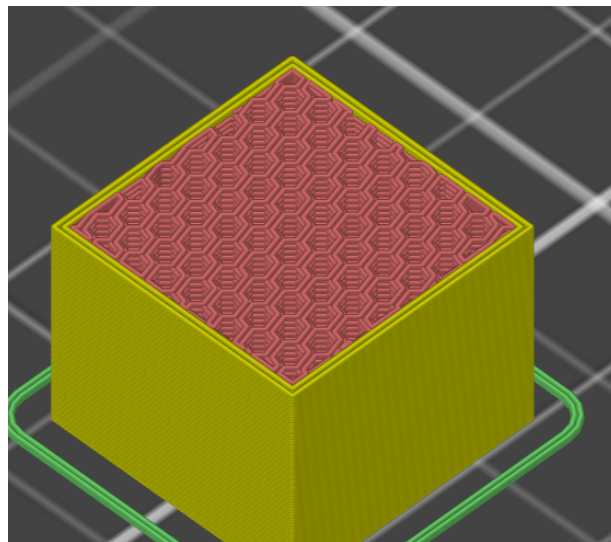


FIGURE 7.48: Available design of the honeycomb infill using the default Slic3r options - 40% relative density.

Therefore, each and every core geometry and face sheet was designed in *SolidWorks*, for both the conventional and re-entrant honeycomb-filled specimens and subsequently exported to *Slic3r* - Figures 7.49 and 7.50.

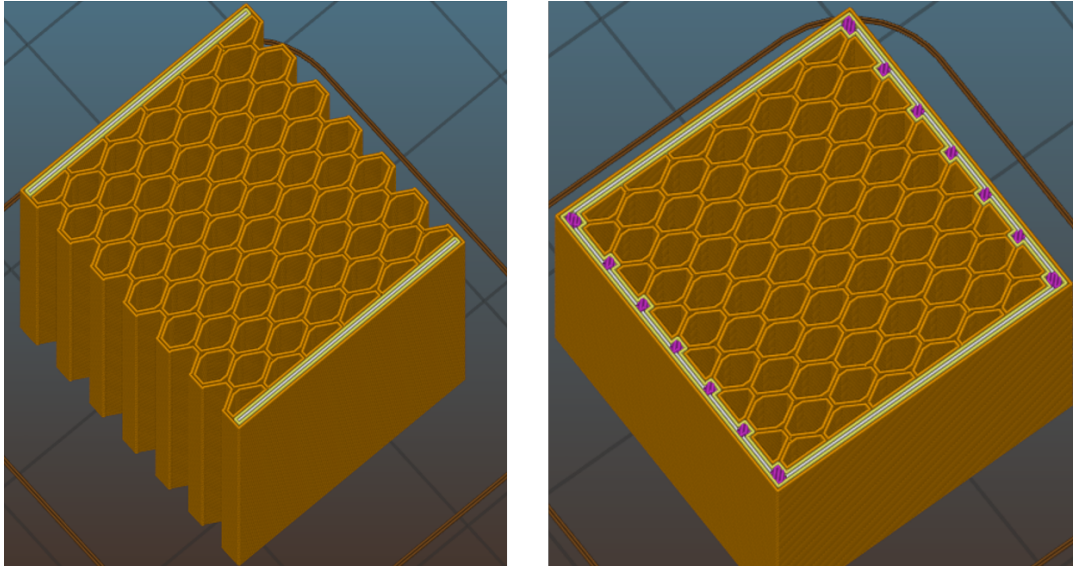


FIGURE 7.49: Designed models of the conventional honeycomb specimens.

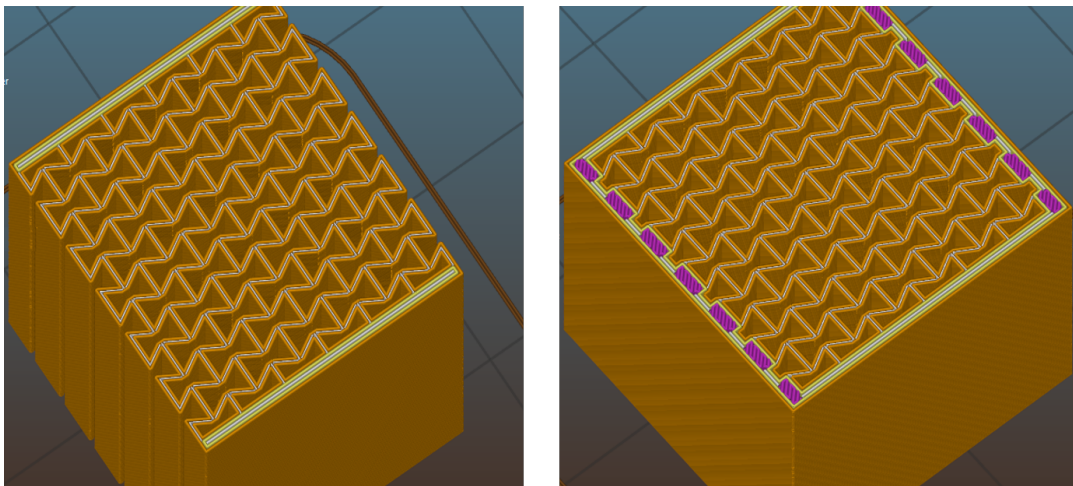


FIGURE 7.50: Designed models of the re-entrant honeycomb specimens.

Finally, the parameters used for the 3D printing of every specimen are described in Table 7.16. The printing process of each specimen ranged from 5 to 8 hours.

TABLE 7.16: Parameters used for the printing of the specimens.

Layer height	0.1 mm
Number of perimeters	2
Extruder's temperature	200 °C
Bed's temperature	60 °C
Filament diameter	1.75 mm
Type	Premium PLA (<i>Formfutura</i>)
Nozzle diameter	0,2 mm

Considering the anisotropic nature of 3D printing, due to the layer-by-layer fabrication process, many factors such as the layer orientation, layer height and nozzle thickness were found to influence the mechanical properties of the material in literature [32], [17], [63]; therefore, all the specimens were printed along the same orientation, using alternated 45 °angles, a consistent layer height of 0,1 mm and 2 perimeters, and the PLA filament was always kept in the same conditions. In addition to that, 3 specimens of each type were manufactured, in order to guarantee a representative sample of specimens - as recommended by the International Standard used for the mechanical testing.

7.3.2 Mechanical testing

The compression tests for the cube specimens and the three-point bending tests for the beam specimens were performed in the Mechanical Engineering Department of FEUP, using a MTS 810 Material Test System and the International Standard ISO 844:2014 - "Rigid cellular plastics - Determination of compression properties" [76].

The parameters used for each test are given in Table 7.17:

TABLE 7.17: Parameters used for testing of the specimens.

Compression rate	2 mm/min
Sampling frequency	10 Hz
Load Cell	MTS, 10kN

The specimens were tested as represented in Figures 7.51 and 7.52. Note that the specimens with the face sheet covering every face are not represented here as they had the appearance of a full solid cube. These are, however, shown in the subsequent figures.



FIGURE 7.51: Mechanical testing of the specimens with the conventional honeycomb core geometry and only top and bottom face sheets, in compression.

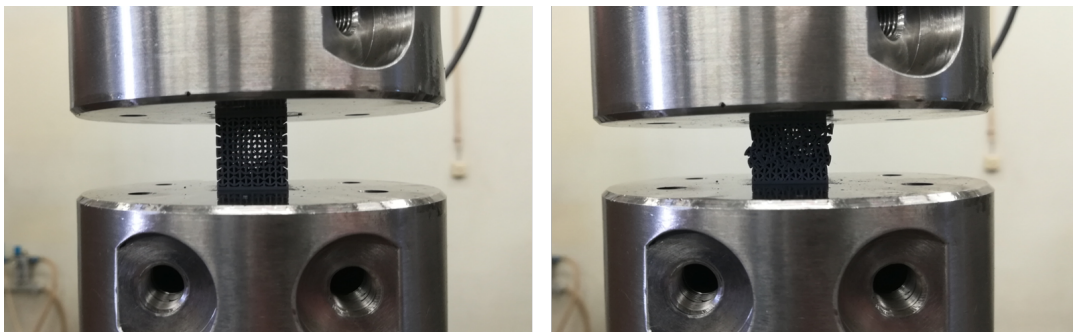
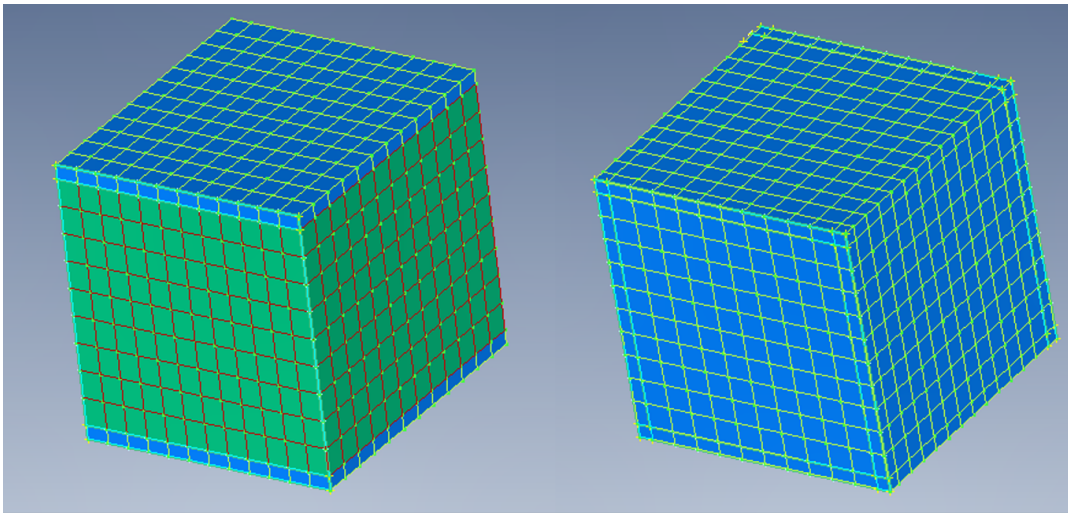


FIGURE 7.52: Mechanical testing of the specimens with the re-entrant honeycomb core geometry and only top and bottom face sheets, in compression.

7.3.3 Obtained experimental results and comparison with the corresponding models constructed for the numerical analysis

In order to establish the direct comparison with the tested specimens, new models using the previously mentioned homogenization technique were developed. These were first designed and meshed in FEMAP - Figure 7.53(a) and (b), and then exported into FEMAS for the elastoplastic numerical analysis, in which the materials' mechanical properties and the boundary conditions were defined - Figures 7.54(a) and (b) - and the analysis was performed. Note that only two models are shown here, because the only difference between the numerical analysis using conventional honeycombs and re-entrant honeycombs stems from the different mechanical properties for each one, obtained on the previous elastoplastic analysis of the respective unit cells.



(a)

(b)

FIGURE 7.53: Models for the specimens designed in FEMAP - (a) specimen with a face sheet at the top and bottom; (b) specimen with a face sheet in every direction.

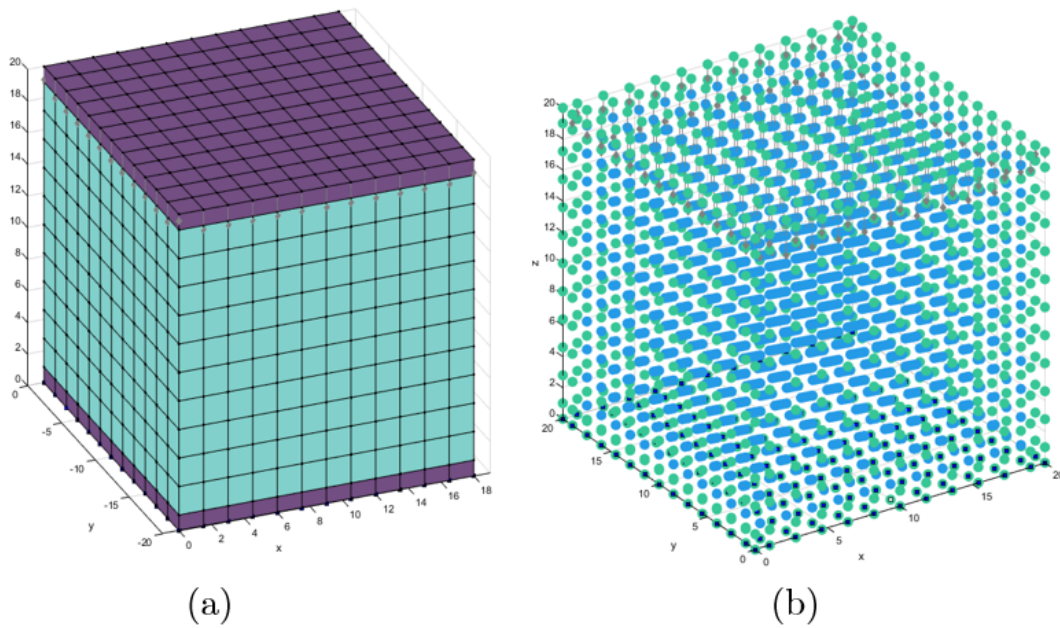


FIGURE 7.54: Essential and natural boundary conditions applied in the model, in FEMAS - (a) specimen with a face sheet at the top and bottom; (b) specimen with a face sheet in every direction.

The force-displacement curves which are obtained from this analysis can be directly compared with the ones obtained from the mechanical testing of the real specimens.

The 3D printed, deformed specimens are shown in Figure 7.55 (a) and (b) for the conventional honeycombs cores and Figure 7.56 (a) and (b) for the re-entrant honeycombs cores.

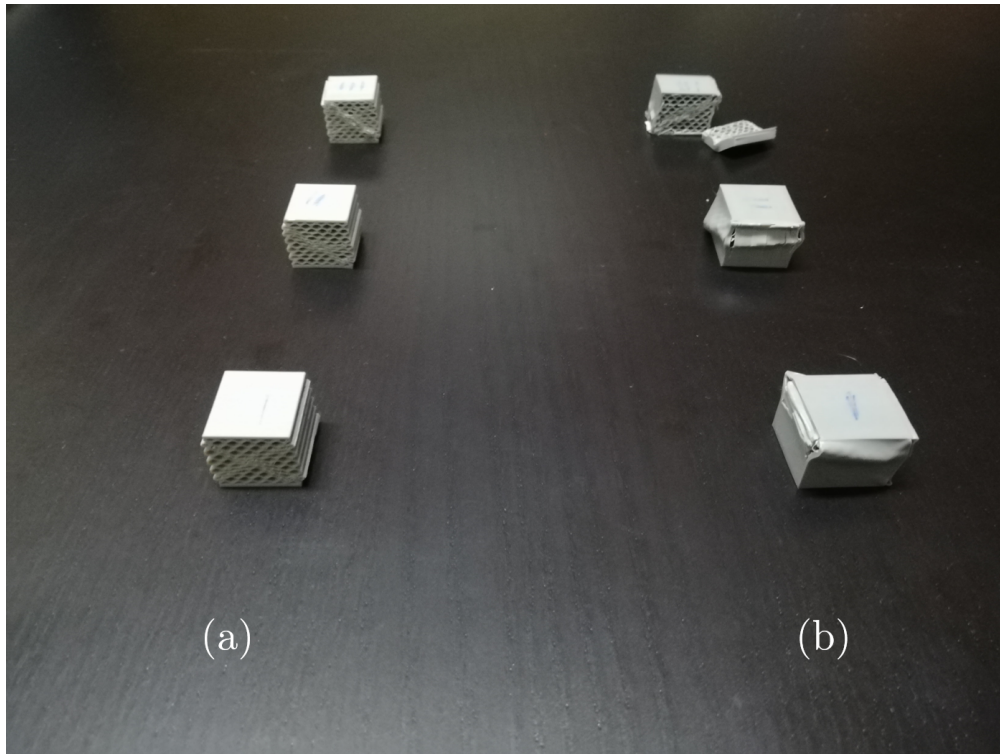


FIGURE 7.55: Deformed specimens with the conventional honeycomb core geometry and (a) face sheets at top and bottom; (b) a face sheet in every direction.

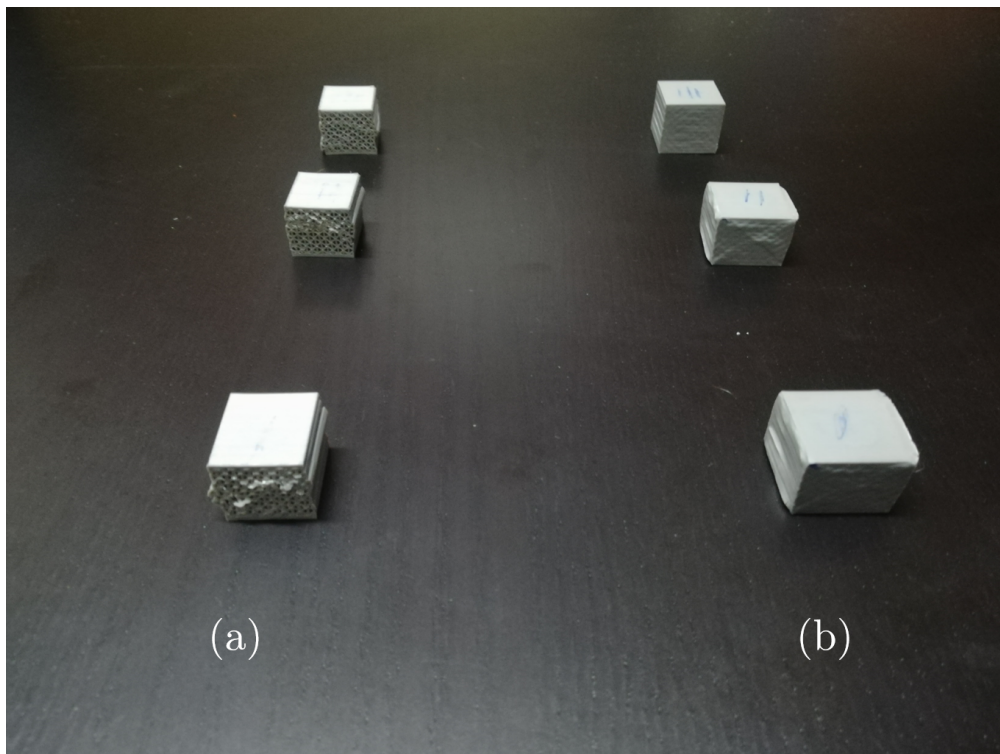


FIGURE 7.56: Deformed specimens with the re-entrant honeycomb core geometry and (a) face sheets at top and bottom; (b) a face sheet in every direction.

There are some important aspects to retain solely from the visual inspection of the deformed specimens: for the specimens with a conventional honeycomb core, there is a clear tendency on the honeycombs to deform and fracture in a 45° angle on the lateral face, positioned at 90° to the loading face; the failure is, therefore, localized in a band transverse to the loading direction, which propagates throughout the core with the increasing strain. This localization is associated with a local maximum in the stress-strain curve of the honeycomb's unit cell.

This phenomenon might occur due to the printing directions used in the manufacturing of the specimens, which was a combined and alternated 45° deposition in every layer. It is, however, coherent with the progressive formation of plastic hinges represented in Figure 7.40.

In terms of the failure modes, explained previously towards the end of Chapter 5 and represented in 5.24 of said chapter, the specimens with conventional honeycombs and a face sheet at the top and bottom - Figure 7.55 (a) - tend to fail, precisely, by core failure. The 1 mm face sheet does not yield or wrinkle, and the evolution towards the plastic domain is rather smooth. However, for the conventional honeycomb specimens and a face sheet in every direction, different failure modes were observed: while the core failed as well - which is perfectly observable in the third specimen of row (b) in Figure 7.55 - the lateral faces have notably wrinkled, highlighting the expansion of the conventional honeycombs in a transverse direction to the loading, and in the last case, caused the total delamination of one of the lateral walls; however, the presence of these lateral walls translated into a significantly higher stiffness for the whole structure, which can be verified in the force-displacement curves represented below.

Figures 7.57, 7.58, 7.59 and 7.60 show the comparison of the experimental data and the numerical analysis for each case. In these figures, the letters HSC, HCC, RHCS and RCHCC represent, respectively, the specimens with a conventional honeycomb core and two face sheets (top and bottom), conventional honeycomb core and six face sheets (every direction), re-entrant honeycomb core and two face sheets (top and bottom) and re-entrant honeycomb core and six face sheets (every direction). For every case, three specimens were produced as recommended in the ISO standard used in this work for the testing of specimens.

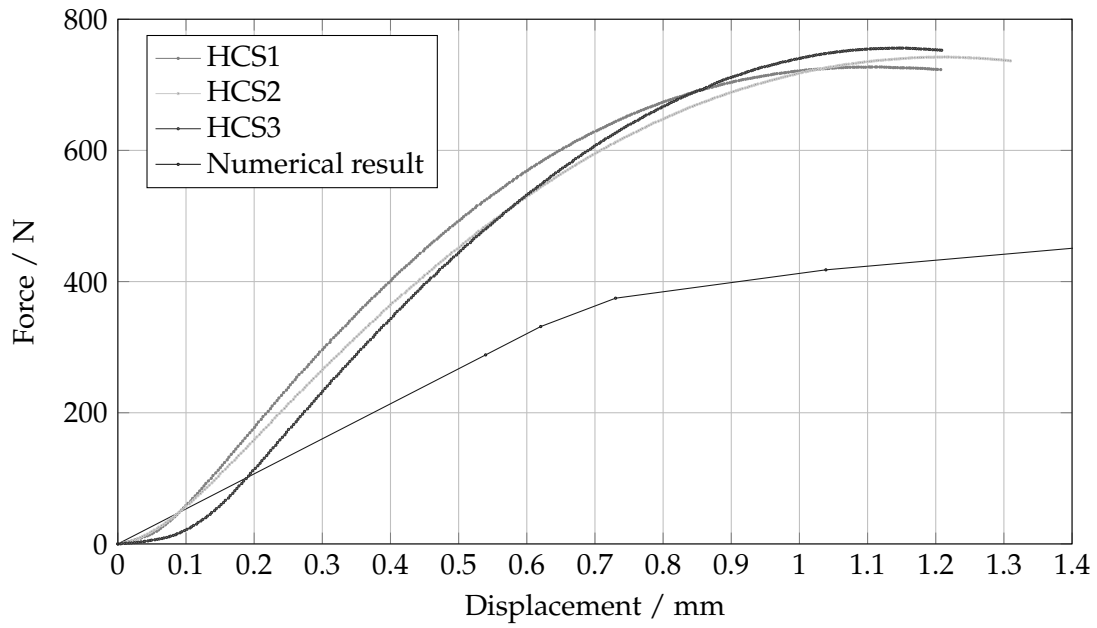


FIGURE 7.57: Comparison of the obtained Force vs. Displacement curves between the experimental and the numerical analysis - Conventional honeycomb cores with top and bottom face sheet.

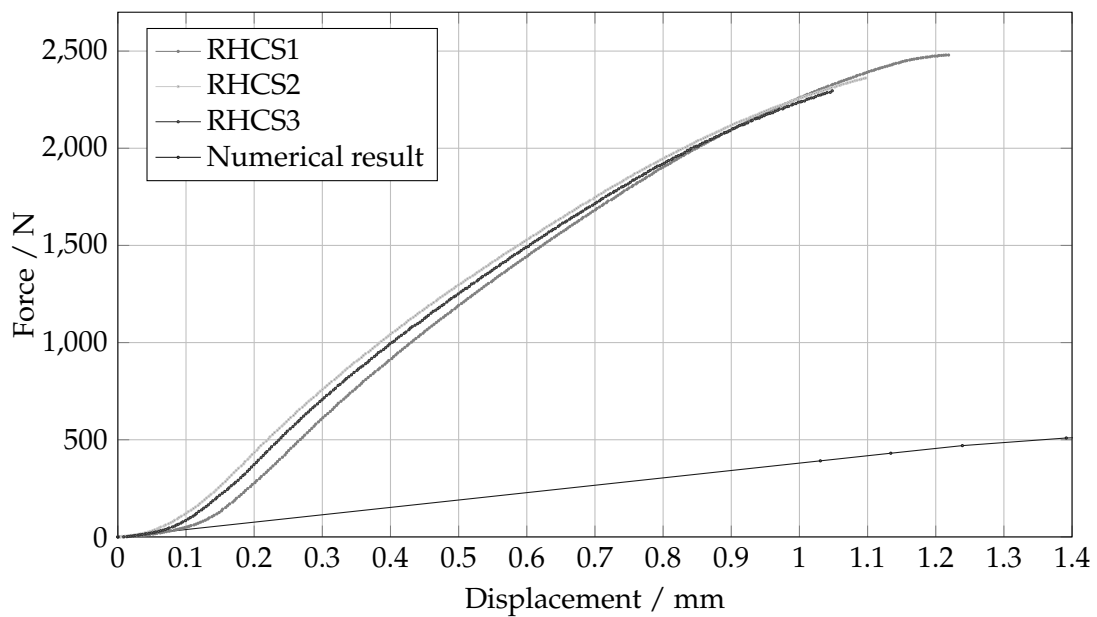


FIGURE 7.58: Comparison of the obtained Force vs. Displacement curves between the experimental and the numerical analysis - Re-entrant honeycomb cores with top and bottom face sheet.

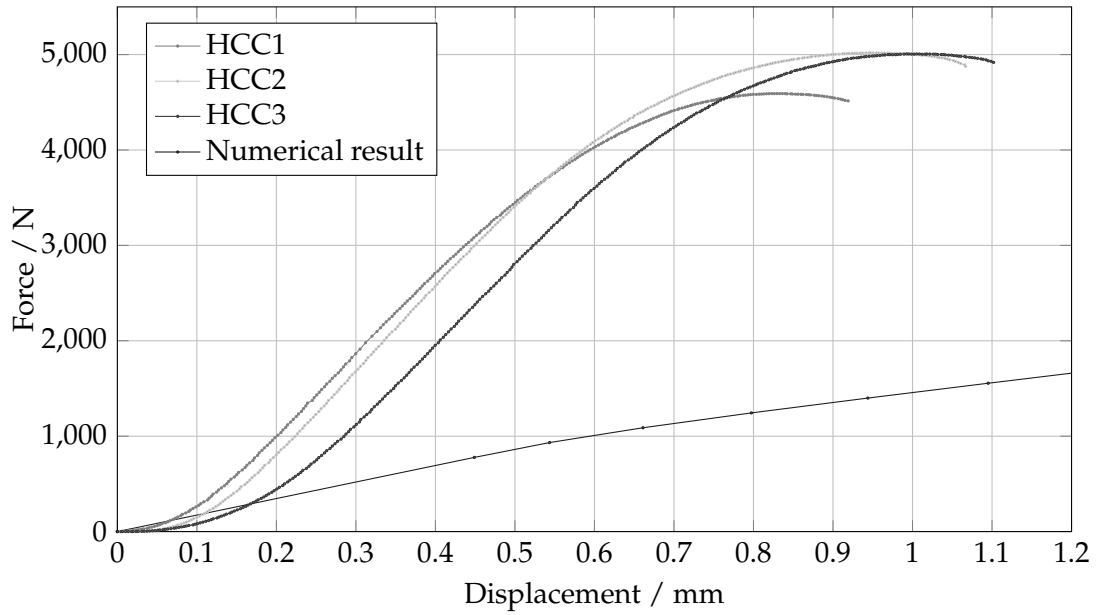


FIGURE 7.59: Comparison of the obtained Force vs. Displacement curves between the experimental and the numerical analysis - Conventional honeycomb cores with a face sheet in every direction.

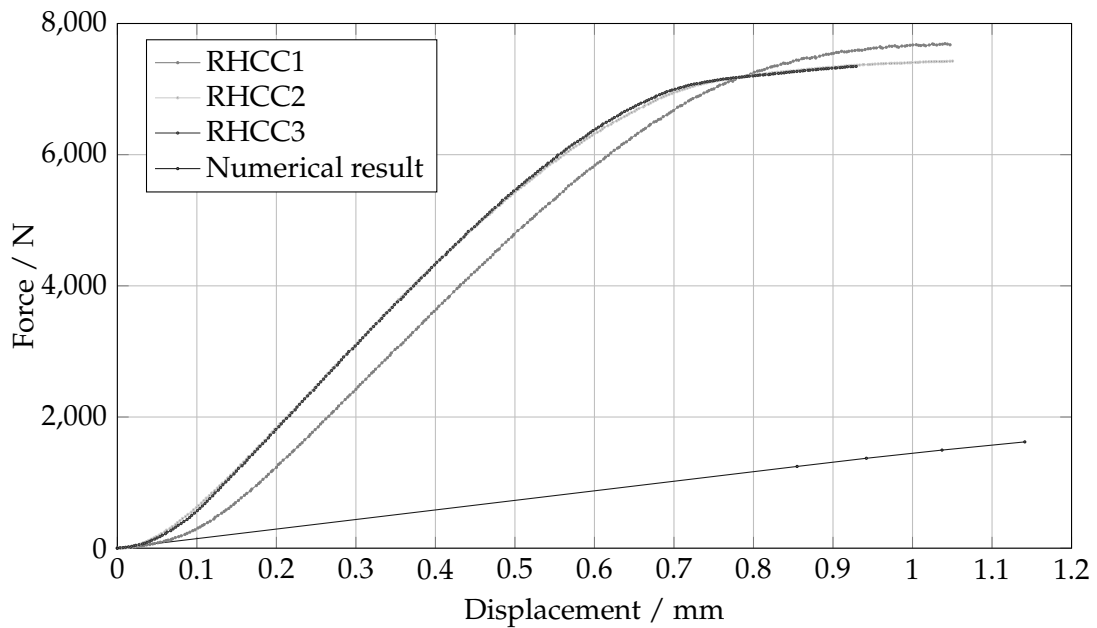


FIGURE 7.60: Comparison of the obtained Force vs. Displacement curves between the experimental and the numerical analysis - Re-entrant honeycomb cores with a face sheet in every direction.

As for specimens with the re-entrant honeycomb core, these have presented a higher yield stress, σ_Y in spite of a lower Young's Modulus, E , when compared to the conventional honeycombs - which is coherent and therefore could be expected from the stress-strain curves obtained in the numerical analysis for the respective unit cells.

Some significant differences between the numerical simulations and the experimental data were found, and these are a result of a wide range of factors, from

the factors associated with the numerical methods to the ones associated with the fabrication process.

An important factor is that these numerical elastoplastic analyses considered for the conventional and re-entrant honeycombs, the extruded PLA as a constitutive material, whose properties were obtained, in turn, from the 3D printing and testing of PLA specimens (which were not the ones shown in this work). The ones used and tested in this work are the ones shown in Figures 7.55 and 7.56, which were manufactured for the specific validation of this numerical analysis).

Therefore, all of the eventual errors in the manufacturing and testing of these and the other previous specimens, as well as the adjusted properties taken from the stress-strain curves which were used as a reference, will influence not only the considered properties for the extruded PLA, but also for the obtained homogenized properties of the honeycombs which used this same extruded PLA as a constitutive material.

Moreover, small errors in the adjustment of both the stress-strain curves of the mechanical testing of extruded PLA specimens and the stress-strain curves of the elastoplastic analysis of conventional and re-entrant honeycombs will then have an effect on the final properties of the specimens' models, which are directly compared to the obtained force-displacement curves of the specimens which were designed, manufactured by 3D printing and tested for the purpose of this work.

On the other hand, the fidelity of the 3D printing process will also affect the materials properties: the resolution and accuracy of the printing process can affect the volume fraction and distribution of the printed materials, and some specific aspects resulting from this manufacturing process are not taken into consideration in the numerical model, such as the:

- Printing direction(s);
- Anisotropy due to the layer-by-layer fabrication process;
- Porosity at the material's level;
- Imperfections introduced during the 3D printing, namely on the first layer deposition;
- Contraction of the material in the deposition process, influencing its mechanical properties.

Most of these aspects depend on a large number of factors associated with the additive manufacturing process in general, or the fused deposition modeling - FDM - in particular. Some of these include:

- The temperature at which every specimen is collected from the base;
- The temperature variation from the center to the periphery of the base;
- The room temperature;
- The calibration of the printing process for the specific material and the variation in tolerances after a number of printed specimens;
- The various printing parameters influencing the printing quality, such as the layer height, nozzle diameter, number of perimeters, etc.;

- The handling of the material in each step of the process, as well as the conditions in which it is kept (these are usually kept in hermetic packages)
- Any previous defects present in the material filament.

Therefore, despite the differences found between the experimental data and the numerical results, this work is a step towards the characterization of both conventional and re-entrant honeycombs. As previously mentioned throughout this text, characterizing the behaviour of honeycomb structures is not an easy task; only some analytical solutions are established, and an alternative efficient way to analyse them is by modeling them and using numerical simulations.

The fact that, in this work, the numerical models did not consider the information that the materials are not isotropic and the mechanical properties are significantly dependent on the printing orientation, may be enough to induce such differences between the predictions and reality. The investigation on honeycombs and its mechanical properties must be continued.

Chapter 8

Conclusions and Future work

Due to the nature of this thesis, which included both the development of numerical models to be tested in a numerical analysis and the 3D printing of specimens to be experimentally tested, many limitations and obstacles were found throughout time, which ultimately led to a progressive optimization of each step of the process: at first, the formulation of the meshless methods - RPIM and NNRPIM - was understood, as well as the mechanics and behaviour of honeycombs and foams made of elastic-plastic materials; then, simple benchmark models were created, and a better capability to interpret and analyze the obtained results was gained.

Furthermore, this knowledge was applied to 2 different geometries: conventional honeycombs and re-entrant honeycombs, for which numerical models were created from the ground up - first, the design on SolidWorks, using literature as a reference for defining the geometries, then, FEMAP for meshing the parts, and finally FEMAS, for the structural analysis. These models were created for 7 different relative densities, and tested in both X and Y directions (in-plane) with three different numerical methods - FEM, RPIM and NNRPIM - for the linear elastic analysis. Therefore, phenomenological laws were established for both x and y directions, for both conventional and re-entrant honeycombs.

This linear-elastic analysis was followed by an elastoplastic one, which allowed for a more interesting and precise understanding of the results, as well as the comparison with the experimental results to be obtained afterwards. Thus, all 7 relative densities for both conventional and re-entrant honeycombs were considered once again, but due to the higher processing times, the analysis was carried using only the FEM, and considering the KT-All non-linear algorithm.

Simultaneously, 3D printed models were created in order to validate the models used in numerical analysis, for two different densities, and 2 different type of models for each - cubes with a covering layer in each x , y and z direction, and sandwich structures in their original concept: two thin layers at the top and bottom sections, separated by a honeycomb core. Due to limitations in the printing process and software, these detailed models were all designed, fully defined and optimized for the printing process in SolidWorks and Slic3r.

Despite the previous study of the 3D printing of polymers and the FDM process itself, the connection between the computational analysis and the real 3D printed parts constituted an interesting challenge to overcome: models to be tested should have maximum dimensions allowed by the 3D printer, but compatible with the norm to be used in the mechanical tests; and a minimum thickness defined by its nozzle, while using a number of honeycombs in the core of the model preventing the occurrence of local effects (the higher the number of honeycombs, the thinner are the walls of each unit), such as stress concentration or premature failure due to eventual printing errors. All these factors must be considered in order to obtain the models

in a reasonable period of time - considering that a single model can take some hours up to a day to print, depending on many factors combined.

Moreover, the fact that the majority of these steps possess an iterative nature usually leads to an experience gained through a trial/error process. In this way, simulation times ranged from a few minutes to some hours, whose minimum threshold was imposed by the available processing capabilities and the meshes' quality, and many different models to be 3D printed were created before an optimal point between print quality and printing time was finally found for the two studied geometries.

As future works, it would be interesting to study the behaviour of these structures in bending, using the same principles in terms of the development of numerical models and 3D printed parts. In particular, studying the effect of using materials for the face sheets different than the core material may be one of the more intuitive options. In addition to that, the study on re-entrant honeycombs may be continued, as they curiously led to better mechanical responses overall when compared to the conventional honeycombs, which are the established state-of-the-art geometries for common sandwich structures. Literature indicates that the re-entrant honeycombs are auxetic geometries - with a negative Poisson's coefficient - and while it implies a more in-depth experimental analysis, this parameter can be studied. These re-entrant honeycombs are associated as well with a higher energy absorption potential, which is an interesting feature for many applications. Besides the honeycombs, many other geometries can be studied; as mentioned at the start of the introductory chapter, nature has designed so many complex geometries that we may never be able to replicate; however, nature may serve as an inspiration for the solutions we find.

The advances of numerical methods, such as the meshless methods applied here, found in parallel with the progresses found in manufacturing techniques such as 3D printing, allow us to create structures which are progressively more efficient, with architectures which can be tailored in order to optimize the structure for every application. In fact, the opportunity to use meshless methods in this work not only made it unique, but allowed to perform more tests, compare more results, and obtain better representations of the studied physical phenomenon.

One must keep in mind that there are, however, still many challenges associated with both numerical methods and 3D printing techniques which must be overcome with time. In particular, the study of geometries such as honeycombs and foams is highly interesting. Although preliminary, this work represents a step towards the characterization of both conventional and re-entrant honeycombs. It would be interesting and highly relevant to simulate the same honeycomb geometries considering more advanced non-linear analysis, assuming large deformation and buckling effects, as well as including self-contact.

Moreover, the whole process carried in this work allowed to connect the knowledge for both the numerical and the experimental analysis, and thus obtain a better understanding of the difficulties that must be overcome in the project and development of sandwich structures with foam cores, for lightweight structural applications. The findings presented here provide new insights into the development of sandwich composite structures with unique mechanical properties for a wide range of mechanical and structural applications.

To conclude, all the aforementioned factors must be considered in an engineering context - fueled by the need to find the optimal solutions within the available options, while striving to understand, predict and find even better solutions, the mechanical engineer may contribute to the progress of the science and engineering fields, in a world of growing knowledge and sustainability.

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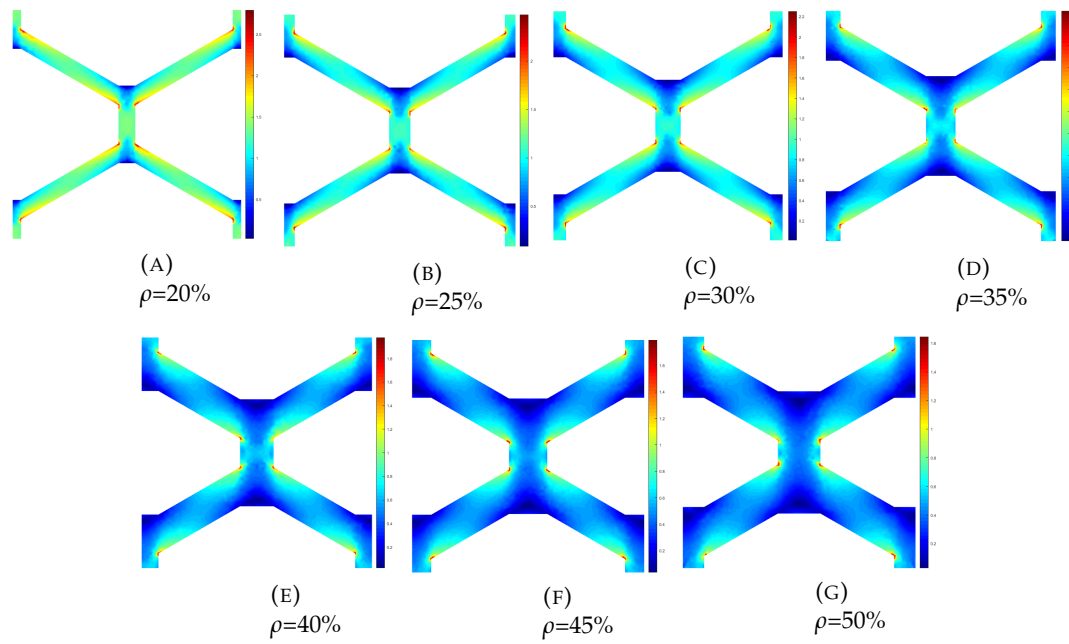
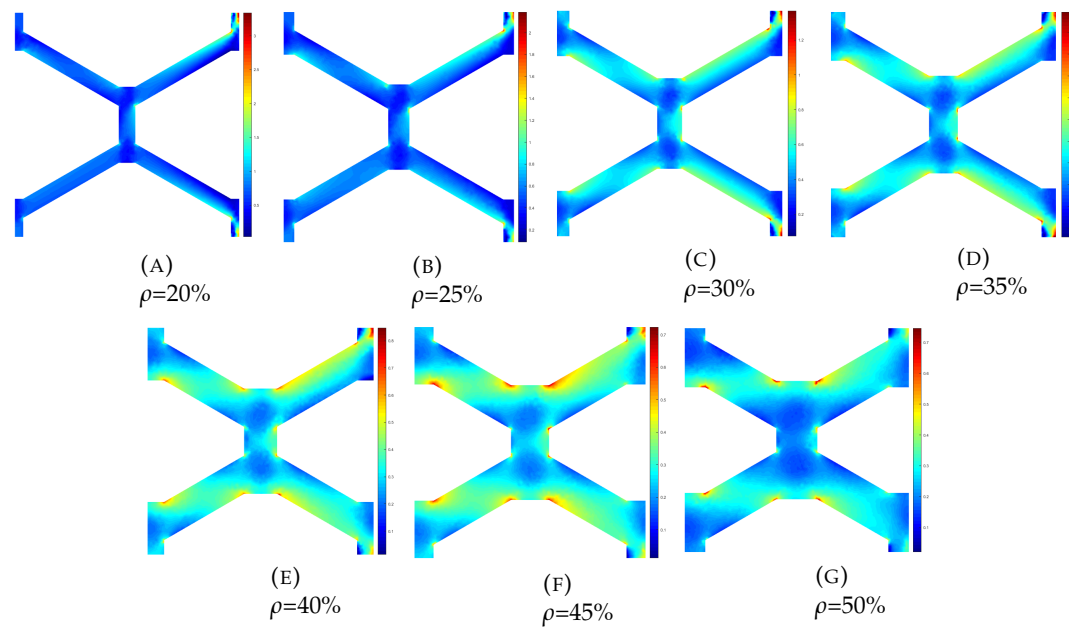
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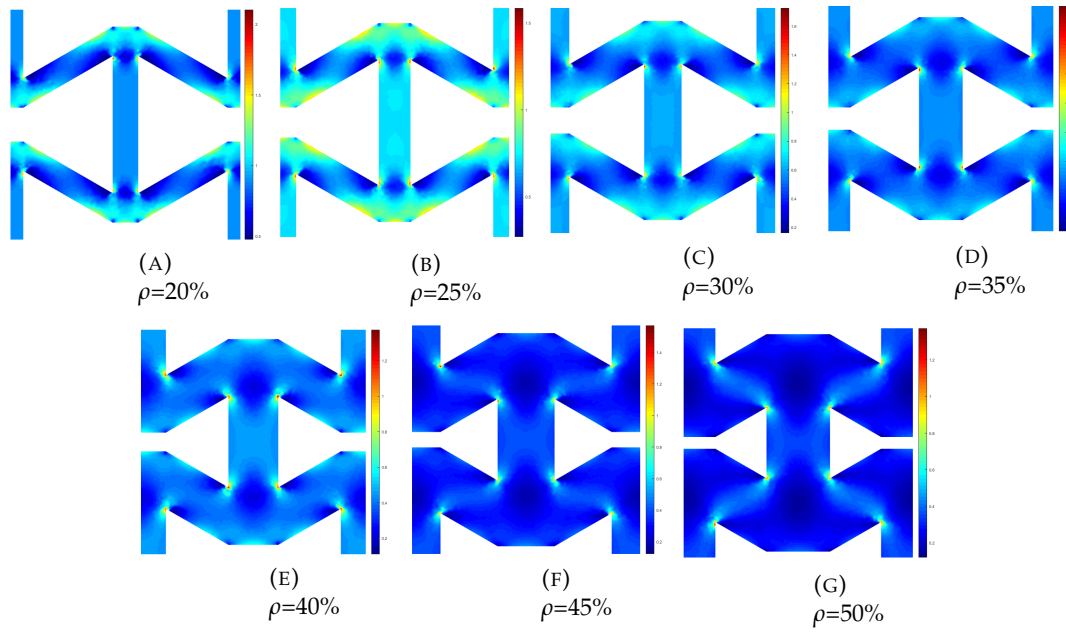
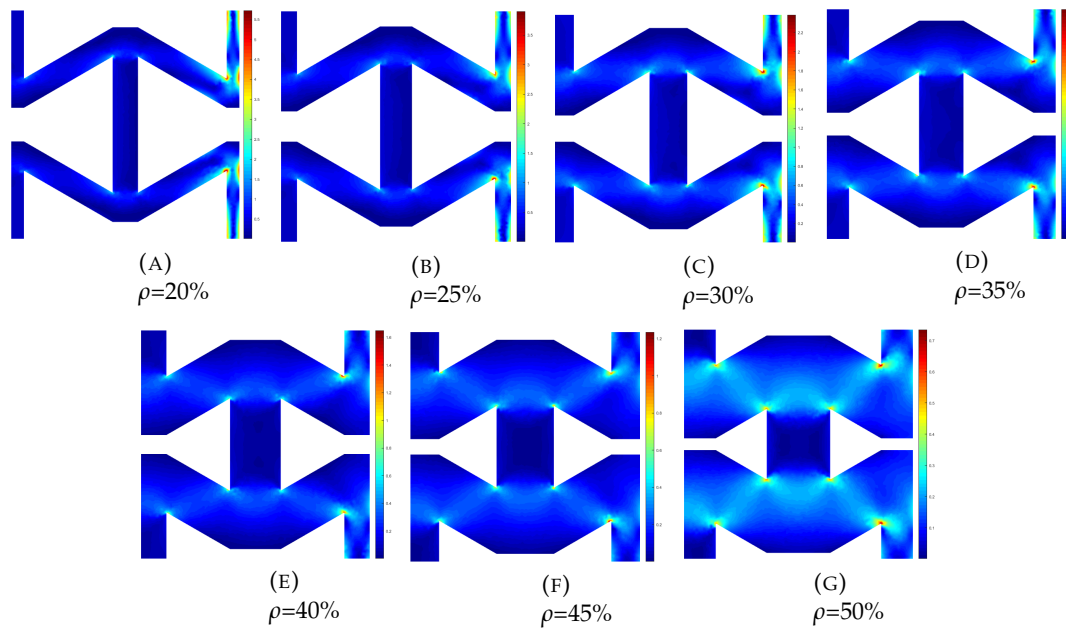
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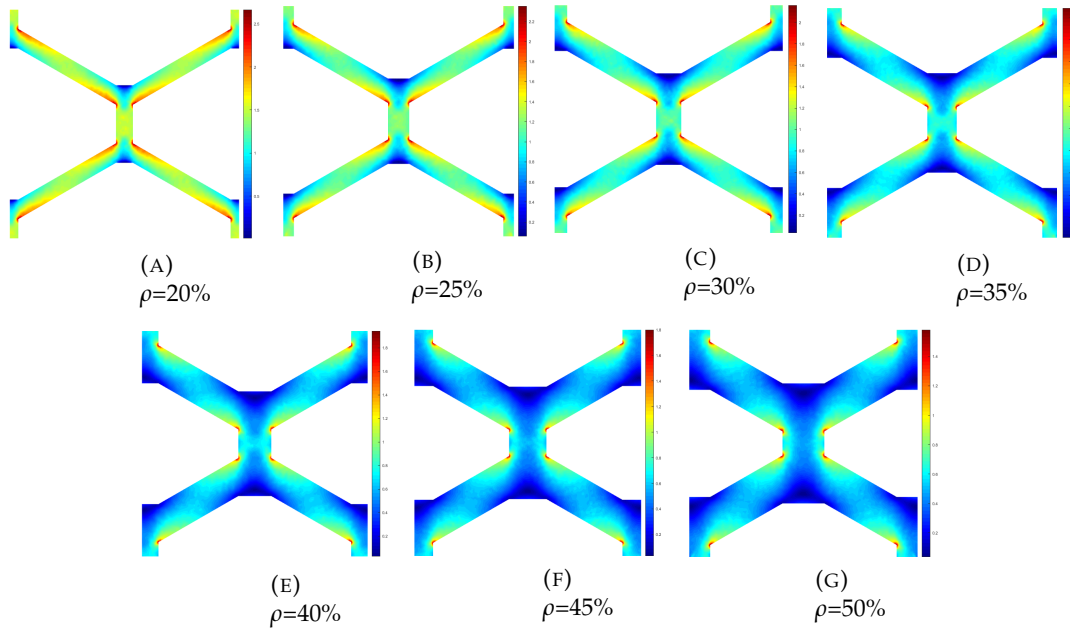
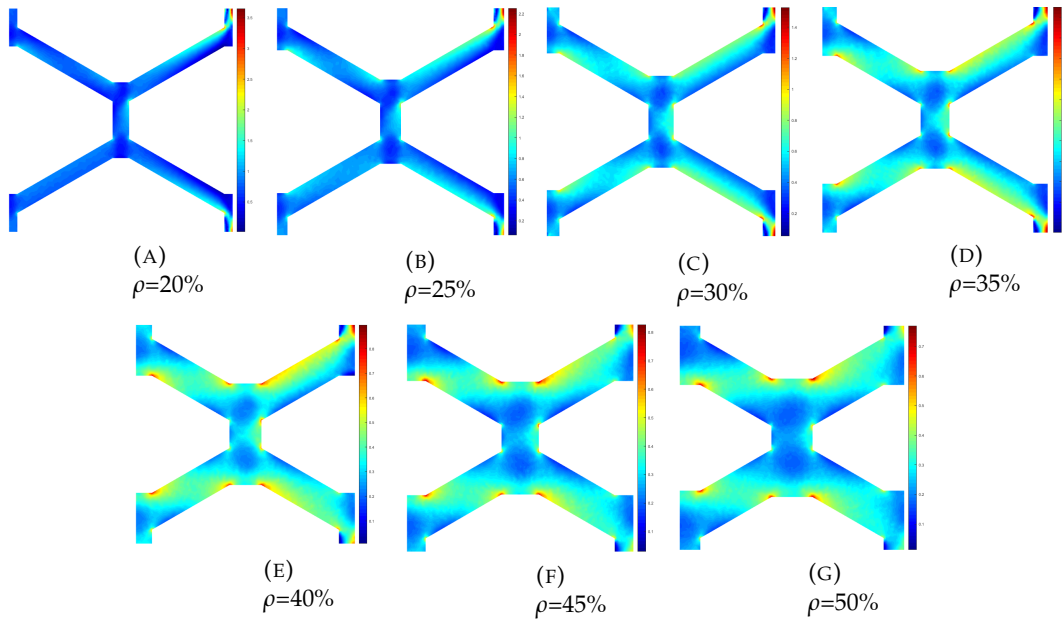
Appendix A

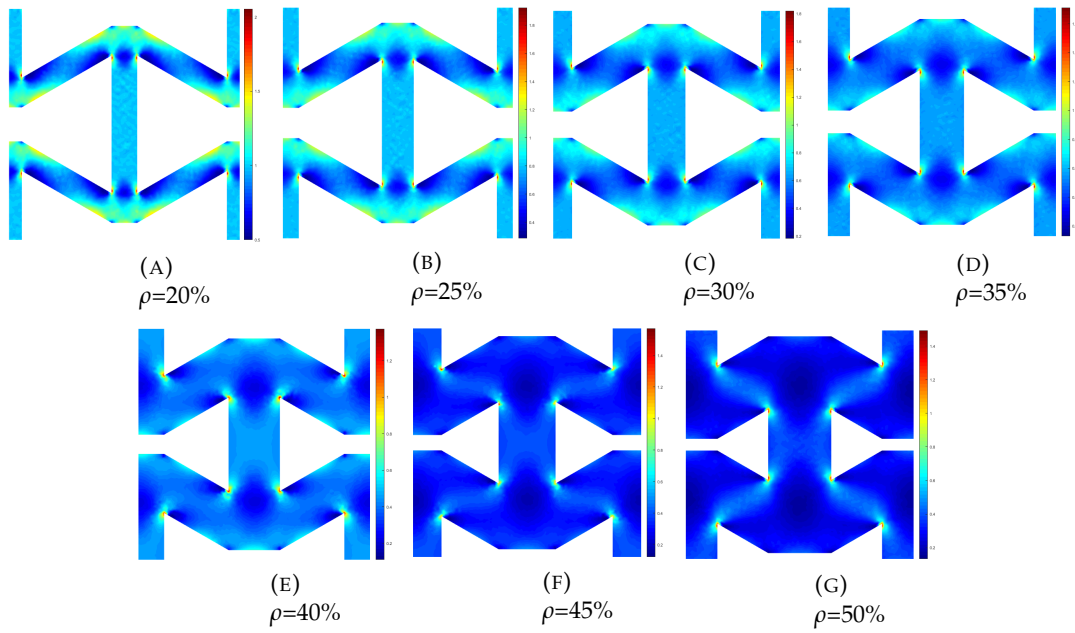
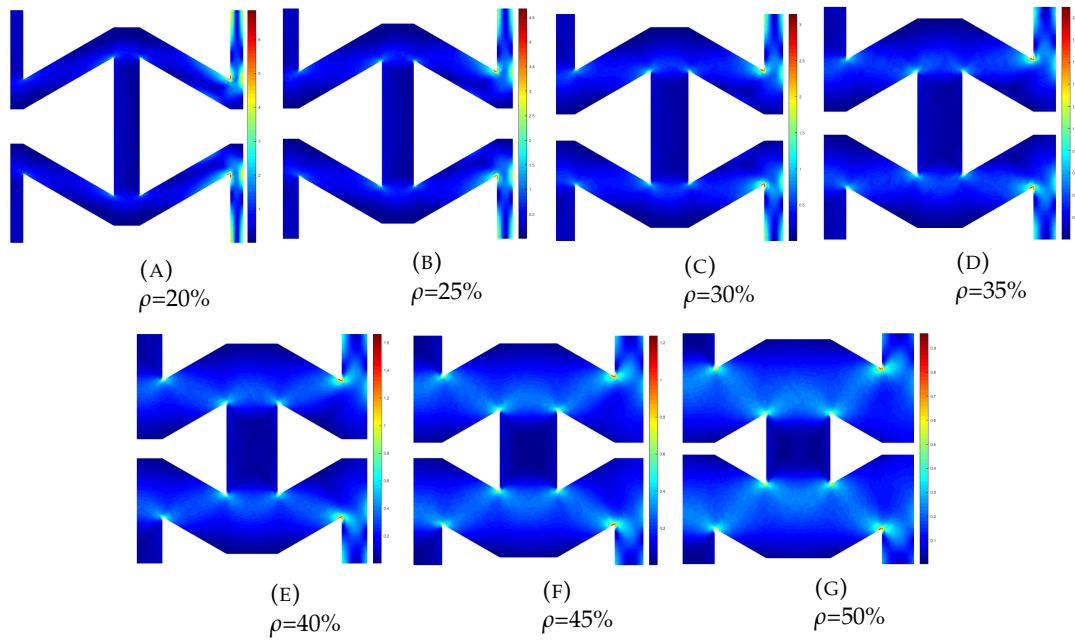
Linear Elastic Results

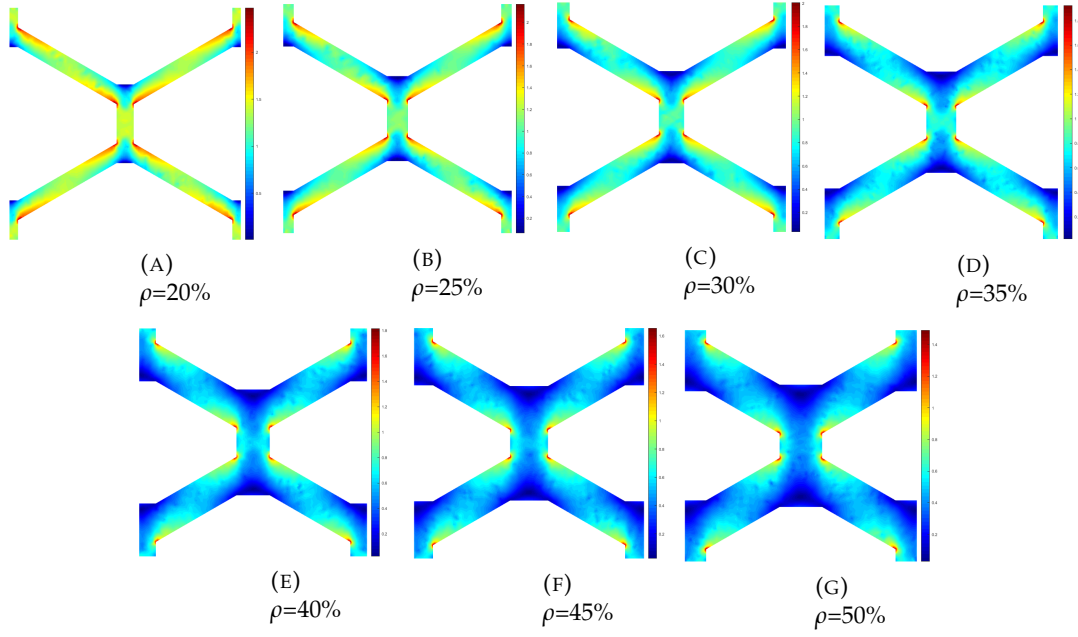
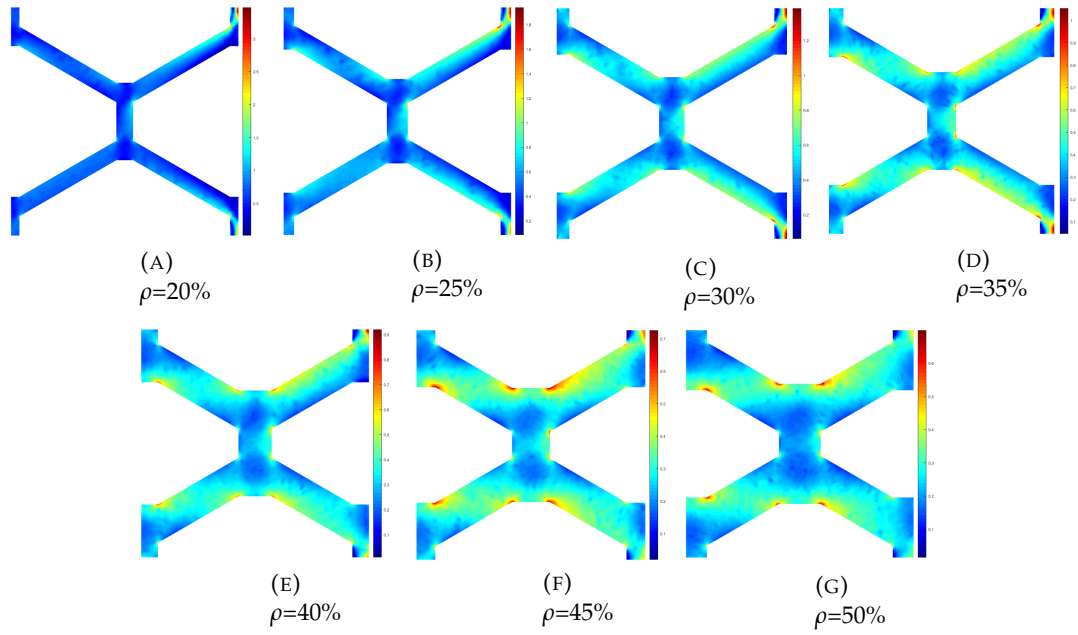
Throughout the numerical analysis on the conventional and re-entrant honeycombs, the obtained color maps for the von Mises stress in the linear-elastic analysis were gathered. These are all presented here, for each of the 7 considered values of the relative density (abbreviated here as ρ) - 20%, 25%, 30%, 35%, 40%, 45% and 50% - for all of the three methods, in this order - FEM, RPIM and NNRPIM - as well as in the two in-plane directions - y and x .

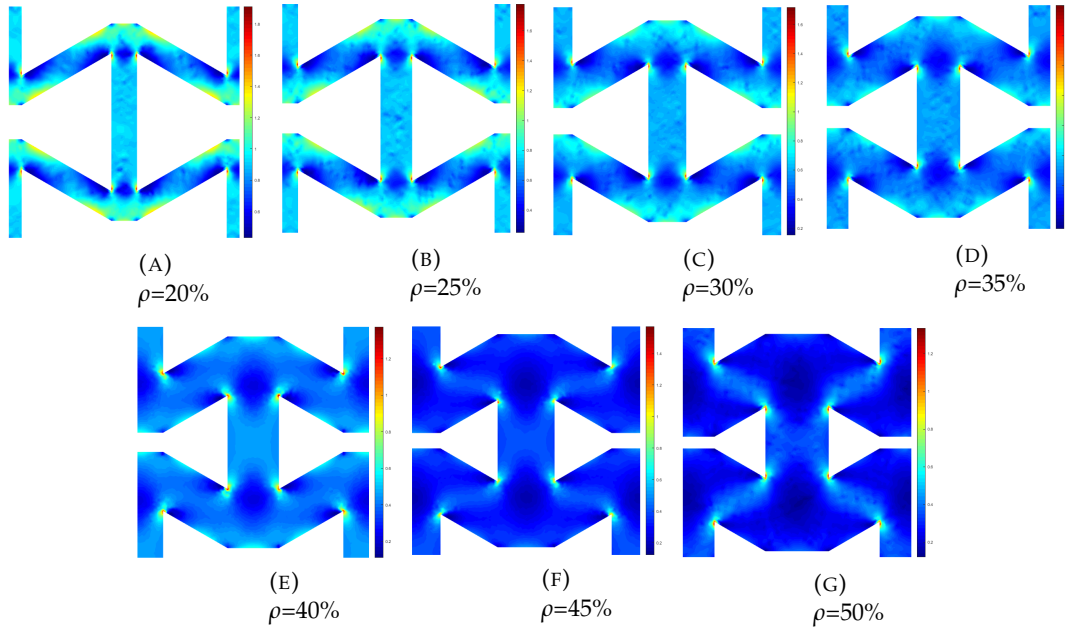
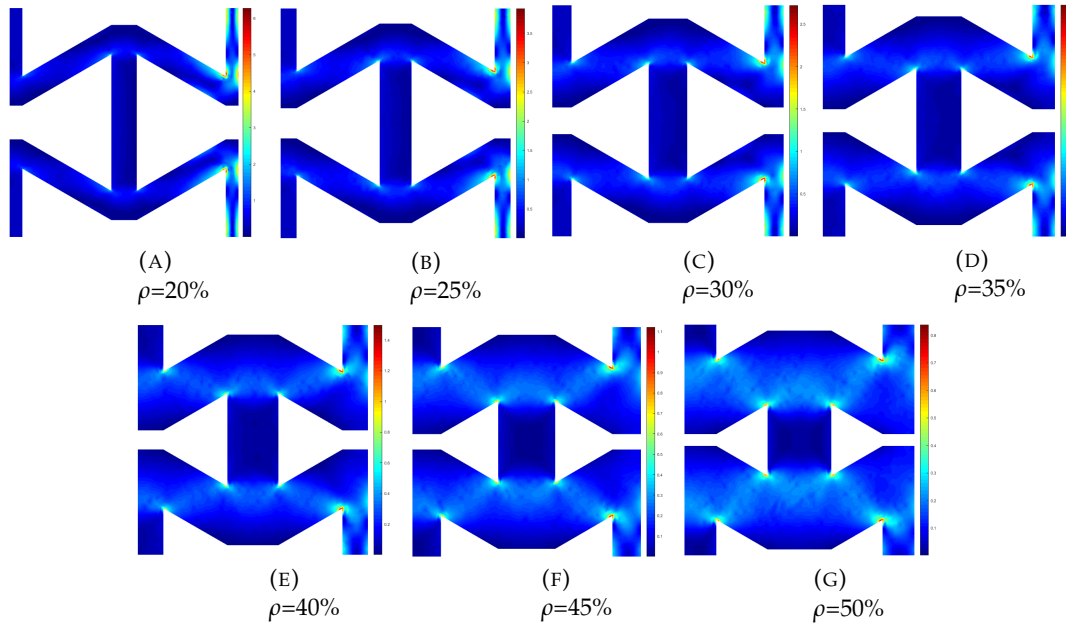
FIGURE A.1: Conventional honeycombs - FEM - y directionFIGURE A.2: Conventional honeycombs - FEM - x direction

FIGURE A.3: Re-entrant honeycombs - FEM - y directionFIGURE A.4: Re-entrant honeycombs - FEM - x direction

FIGURE A.5: Conventional honeycombs - RPIM - y directionFIGURE A.6: Conventional honeycombs - RPIM - x direction

FIGURE A.7: Re-entrant honeycombs - RPIM - y directionFIGURE A.8: Re-entrant honeycombs - RPIM - x direction

FIGURE A.9: Conventional honeycombs - NNRPIM - y directionFIGURE A.10: Conventional honeycombs - NNRPIM - x direction

FIGURE A.11: Re-entrant honeycombs - NNRPIM - y directionFIGURE A.12: Re-entrant honeycombs - NNRPIM - x direction

Appendix B

Elastoplastic numerical results

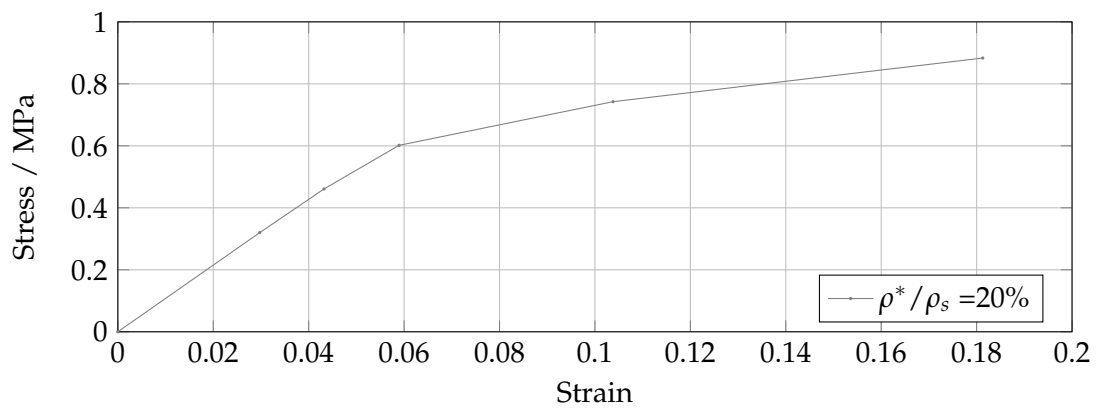


FIGURE B.1: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 20%.

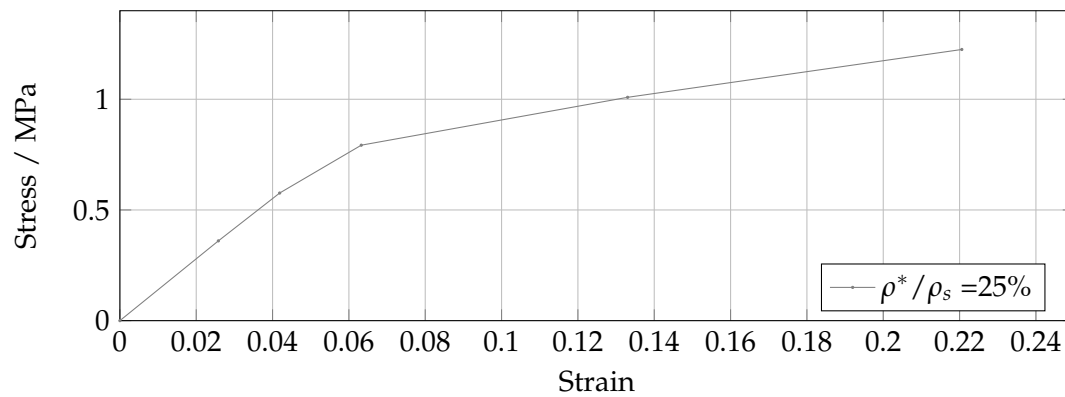


FIGURE B.2: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 25%.

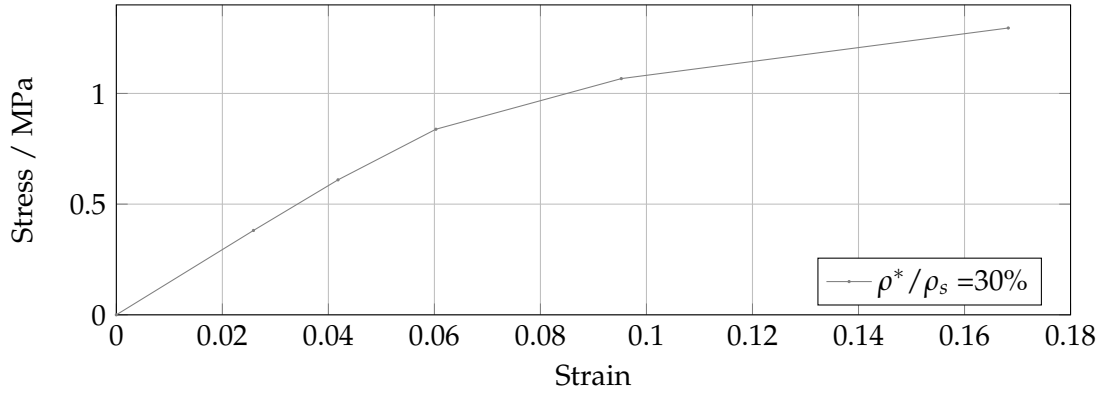


FIGURE B.3: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 30%.

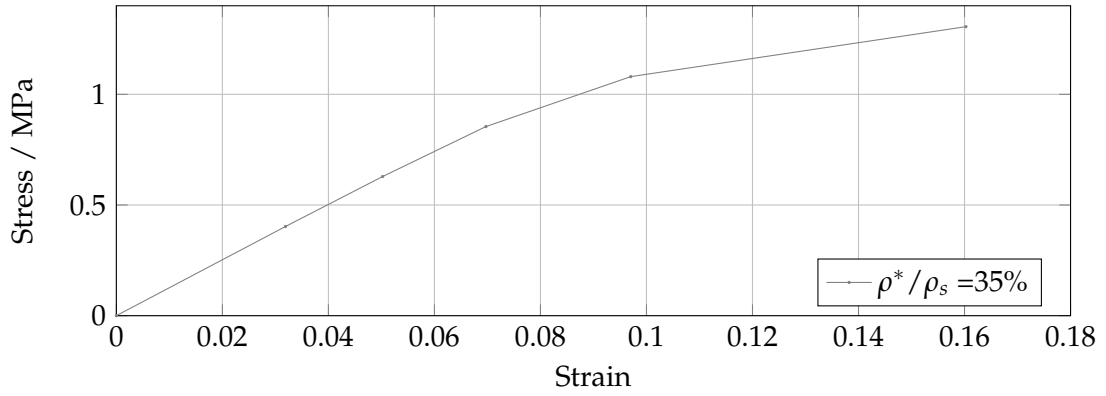


FIGURE B.4: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 35%.

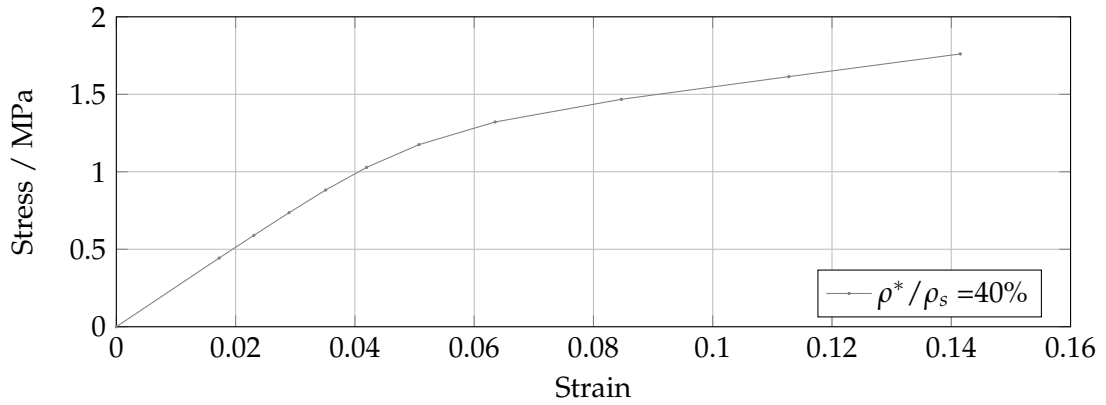


FIGURE B.5: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 40%.

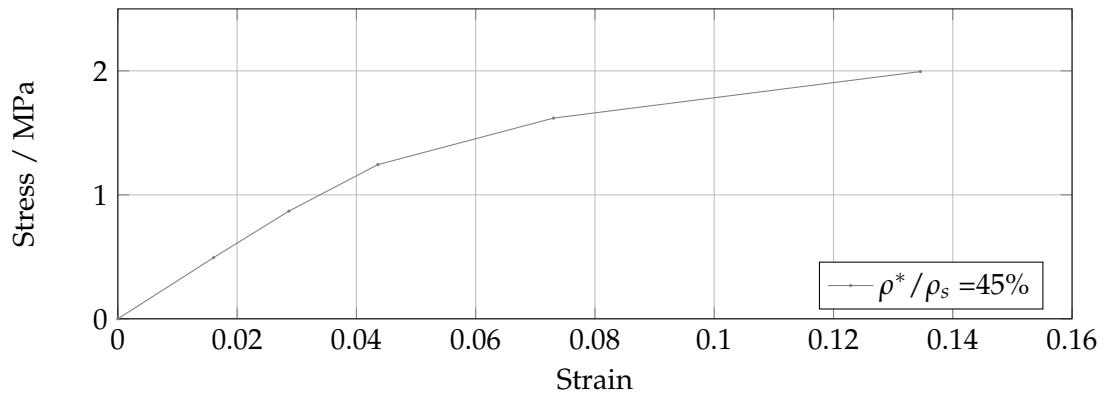


FIGURE B.6: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 45%.

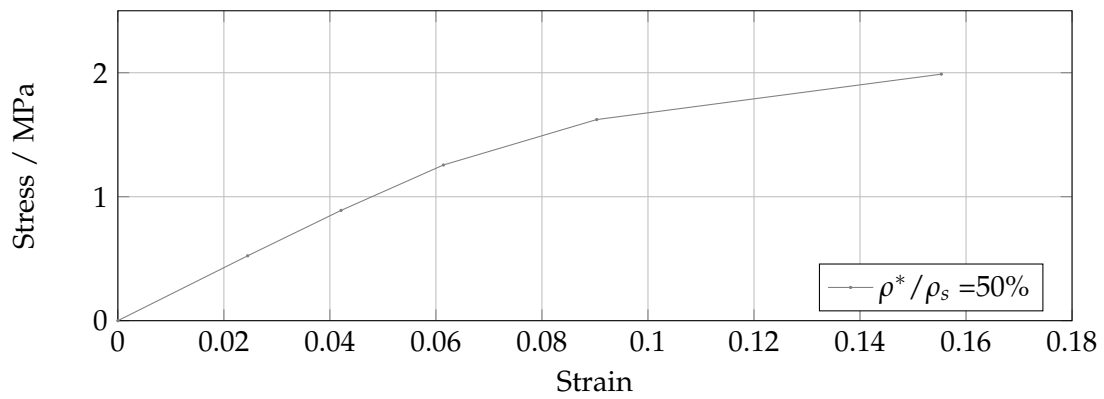


FIGURE B.7: Stress-Strain curve for the elastoplastic analysis of the conventional honeycomb unit cell with relative density 50%.

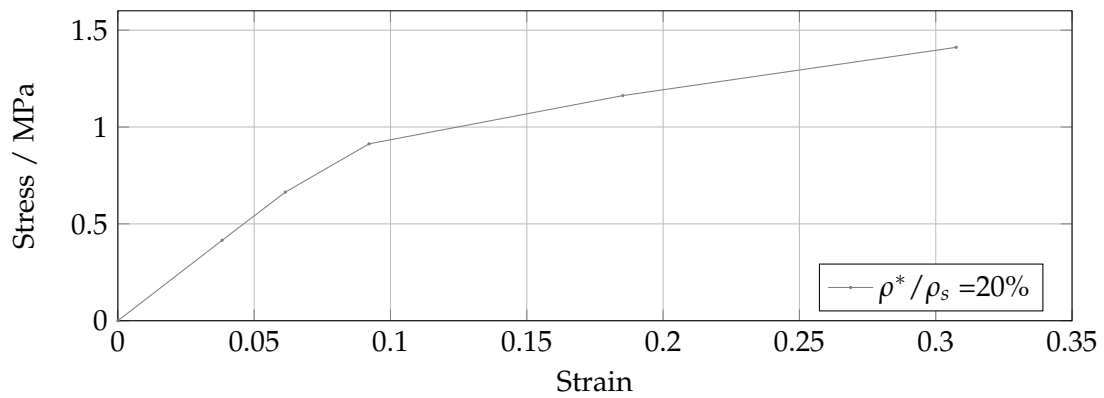


FIGURE B.8: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 20%.

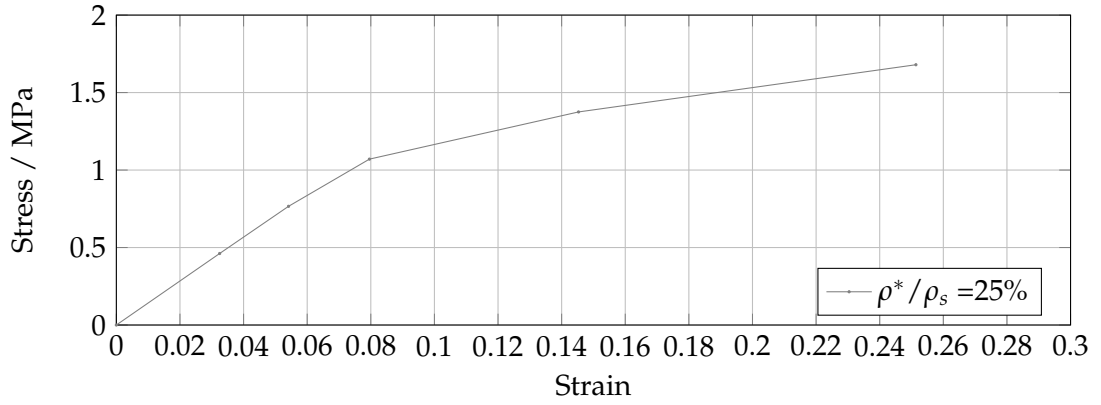


FIGURE B.9: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 25%.

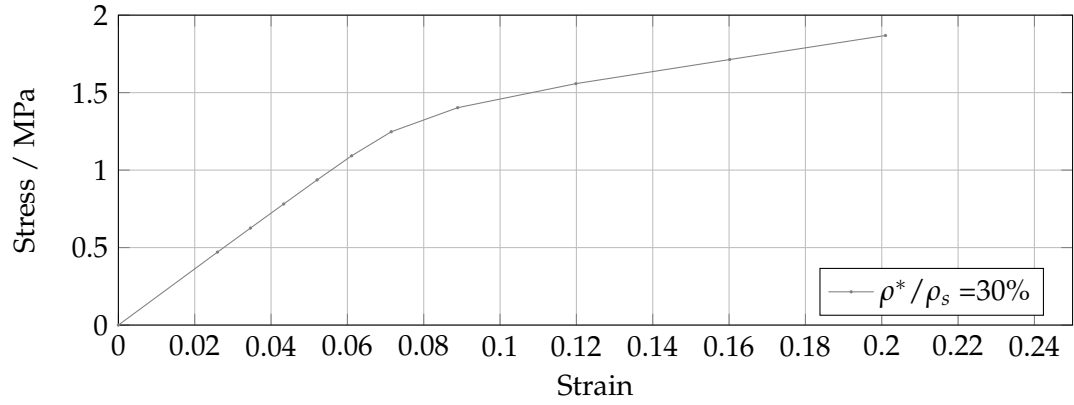


FIGURE B.10: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 30%.

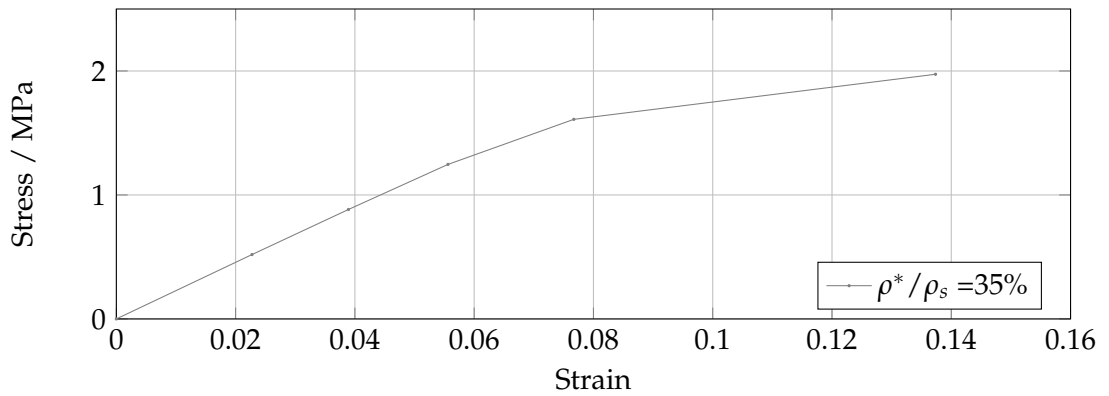


FIGURE B.11: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 35%.

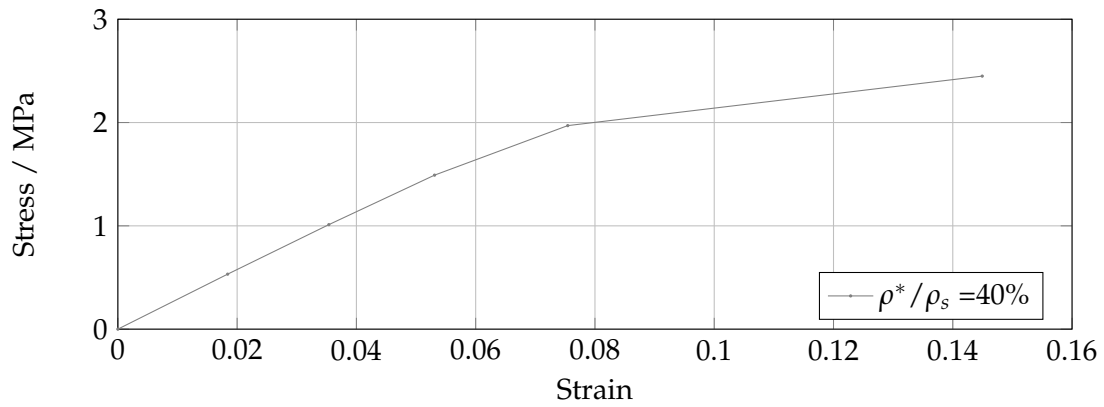


FIGURE B.12: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 40%.

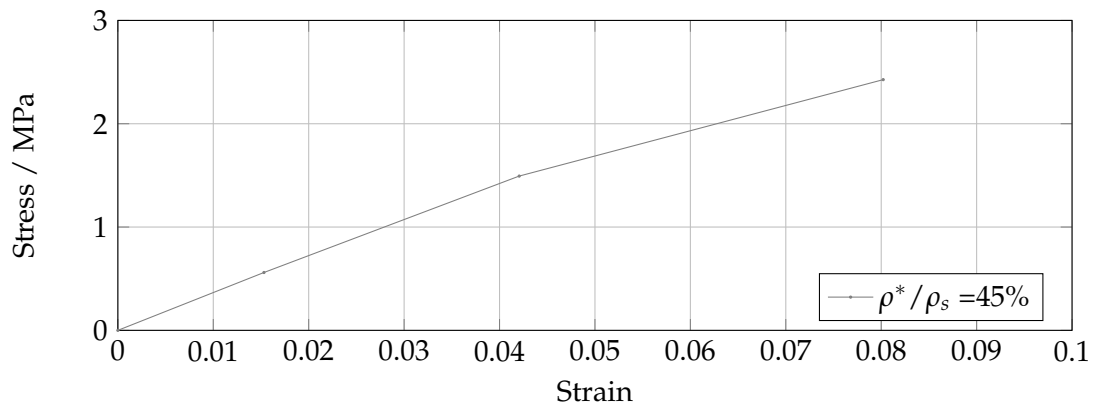


FIGURE B.13: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 45%.

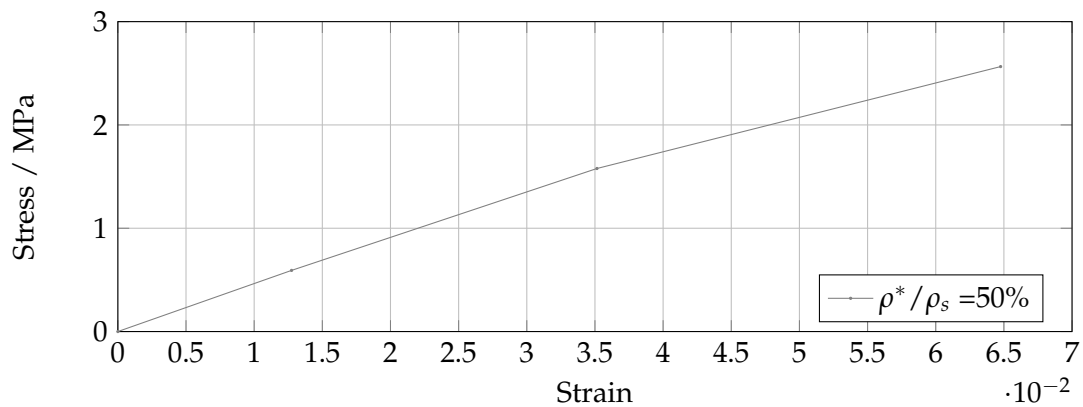


FIGURE B.14: Stress-Strain curve for the elastoplastic analysis of the re-entrant honeycomb unit cell with relative density 50%.