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The Venture Capitalist Investment Decision: A Dynamic Real Options Approach

Ricardo Jorge Maia Ferreira

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Professor Paulo Pereira, PhD

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Biographical Note

Ricardo Jorge Maia Ferreira is 25 years old. Born in Chamusca, his youth was dedicated to canoeing, where life-changing adventures shape most personality traits that characterize him nowadays.

Has completed the Bachelor of Economics at School of Economics and Management of University of Porto (FEP) prior to joining the Master in Finance at the very same institution.

Ricardo is currently a Corporate Finance professional at KPMG. Among previous working experiences highlights the roles of M&A and Corporate Finance Summer Trainee at BPI and Venture Capital Summer Trainee at Seedstars.

Academically, finishing the Master in Finance is nothing but the beginning of the journey.

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Seedstars has changed the way I see the world, the way I see myself in it, how it is possible to make a difference and "impact people's lives through technology and entrepreneurship". I am really thankful as it invaluably contributed to my increasing passion for Venture Capital, Entrepreneurship and Social Impact.

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Abstract

The model herein presented analyses the investment decision taken by a Venture Capitalist (VC) who has an opportunity to invest in a growth option hold by a start-up firm for which the Entrepreneur who owns it cannot afford to fully finance the growth option alone.

Since the VC has to account for the exit of such an investment when it first decide to invest, the model uses a rational similar to the one implemented by Lukas and Welling (2012) where the VC determines an optimal premium, determined as function of the synergies possessed by the buyer, when determining the optimal exit trigger. The model on the entry decision, besides the exit option, accounts for two different settings regarding the growth prospects that both the VC and Entrepreneur have. There must be present homogeneous beliefs where both VC and Entrepreneur agree on the same growth prospects for the expansion project, or heterogeneous beliefs where both players agree on different growth prospects. This model is implemented using a framework similar to the one Tavares-Gärtner et al (2018a) developed. Moreover, this entry-exit option hold by the VC – which so far considered a perpetual time horizon –, is extended using the contribution of Pereira and Rodrigues (2014) therefore implementing a time restriction so the investment cannot last longer than the investment maturity a VC is typically subject to.

The model determines, among other things, the optimal entry and exit triggers, the post-money ownership the VC must request when investing in the venture opportunity, as well as the optimal premium to be requested when exiting the investment – as function of the later buyer synergies. Also, it allows to compute the exit multiple a VC can expect when considering the investment and analyse its sensitivity to – among other variables – the investment volatility and maturity.

Keywords: Investment Decisions, Venture Capital, Entrepreneurial Finance, Entrepreneurship, Start-ups, Real Options, Growth Options

JEL-Codes: G11, G24, G31, G34, L26, M13

Sumário

O presente trabalho analisa a decisão de investimento tomada por um investidor de Capital de Risco (VC) quando este é confrontado com uma oportunidade de investimento de um projecto de expansão de uma *start-up*, uma vez que o empreendedor que detém a *start-up* não dispõe da totalidade dos fundos necessários para investir sem recorrer a capitais alheios.

Uma vez que o VC tem de conceptualizar de imediato a decisão posterior de desinvestimento, é utilizado uma adaptação do modelo de Lukas e Welling (2012) onde é determinado um prémio óptimo que o VC deve exigir ao seu comprador, em função das sinergias que se sabem existentes por parte do comprador. No modelo que considera simultaneamente a entrada e saída do VC do capital da *start-up* são consideradas duas abordagens distintas, com expectativas homogéneas e heterogéneas no que respeita às convicções do VC e do empreendedor quanto ao potencial de crescimento da *start-up* no que respeita à performance do projecto de expansão. Este modelo baseia-se no trabalho de Tavares-Gärtner et al (2018a). O modelo descrito na sua concepção inicial apresenta uma duração perpétua, mas sendo o típico VC sujeito a uma restrição temporal nos seus investimentos, é implementada uma extensão ao modelo através do contributo de Pereira e Rodrigues (2014), através da qual a exposição do VC à *start-up* tem uma maturidade máxima.

O modelo permite determinar, entre outros, momentos óptimos para investir e desinvestir, a proporção óptima que o VC deve exigir quando entra no capital da *start-up*, ou o prémio óptimo que o comprador deve pagar no momento da saída do VC, em função das sinergias existentes. O modelo permite também analisar os múltiplos de saída esperados com as transacções, e analisá-los em função da sensibilidade a variáveis como a volatilidade ou a maturidade do investimento.

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1. Introduction

"If I have seen further it is by standing on the shoulders of Giants."

Isaac Newton, 1675

As discussed by Feld and Mendelson (2016), one of the very first ventures being financed through Venture Capital is as old as 1957, where American Research and Development Corporation (AR&D) invested $70,000 in Digital Equipment Corporation (DEC), which went public in 1968 resulting in a return of over 500 times the invested amount, with a stake worth more than $355 million. As the authors point out, at the time, the term sheet determining the terms in which the deal would happen would be as simple as to mention the amount to be invested, and the respective equity ownership to be required. Nowadays, term sheets may be many pages long discussing every little detail and account for every imaginable and unimaginable situation that might happen. As for the industry growing relevance, CB Insights and PwC (2018) in its *Venture Capital Funding Report 2017* report 11,042 deals worth over $164,4 Billion for the year of 2017, proving how hot and sexy Venture Capital is becoming. Moreover, corporates are looking into it as well, with the KPMG (2018) reporting an increase in corporate Venture Capital investment, as a percentage of the total investments, of over 18.7% in the fourth quarter of 2017, proving that even large corporations have a growing appetite for Venture Capital investments.

The present master dissertation aims to breakdown the Venture Capitalist's investment decision considering an investment in a growth option that a certain start-up firm has, to be financed through the Entrepreneur's own funds and the remaining to be financed by a Venture Capital fund. It considers the decision of the Venture Capitalist (VC) of investing and exiting the investment afterwards, through a real options approach.

The VC decision to invest considers the impact of the expected investment exit, aiming to determine the optimal entry timing and the optimal ownership to be required considering a set of variables, and similar or different growth prospects both the VC and Entrepreneur have regarding the growth option – in a setting of, respectively, homogeneous or heterogeneous beliefs, using a framework similar to the one created by Tavares-Gärtner et. al. (2018a). The investment disposal considers a setting similar to the one presented by Lukas and Welling (2012) whereas the share of the synergies possessed by the buyer is the main driver to achieve the optimal exit trigger. The model is extended by implementing a time restriction to the base case model – which has a perpetual approach –, so that the model can fit better the industry reality with investment maturities typically short. This time restriction is put in place through the contribution of Pereira and Rodrigues (2014).

The model developed in this dissertation allow, among other things, to determine the optimal timing to enter the investment, the optimal post-money ownership the VC must request in order to invest, the optimal share of synergies to require within the exit of the investment, the optimal exit trigger, to assess the impact of the investment maturity within the mentioned triggers, to determine the expected exit multiple, and assess how sensible it is to variables such as the volatility or the investment maturity.

This dissertation unfolds as follows: in chapter 2 is made a synthesis of the literature related to Venture Capital, real options and real options applied to Venture Capital. Chapter 3 explains the base case model and has a numerical approach to it. The model is extended in chapter 4 to account for the time restriction VCs face and has also a numerical example to clearly observe its impact and expected results. Chapter 5 has some future research ideas whose purpose is to make the model fit even better the industry reality, and chapter 6 concludes.

2. Literature Review

2.1. Venture Capital

Context and Players

The Venture Capital industry is one of such great complexity, whereby four main players interact so that each of them can prosper. A simple explanation of what happens in the Venture Capital industry may be illustrated as Zider (1998, p.7), in the following diagram:

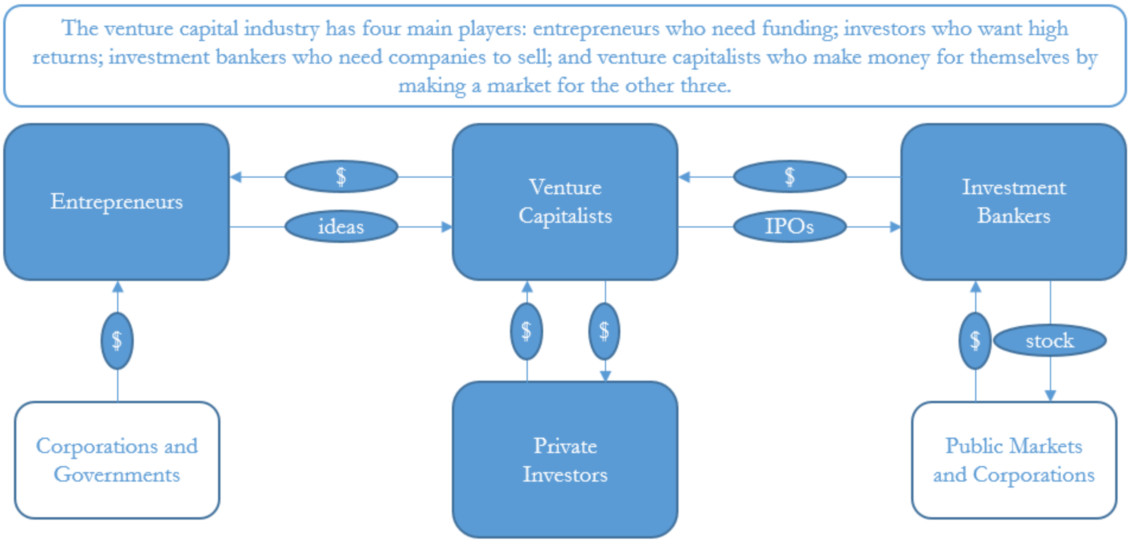

Figure 1 – Venture Capital Business Environment, as in Zider (1998, p.7).

Sahlman (1990, p.473) describes the Venture Capital industry as one well adapted to environments characterized by uncertainty and information asymmetries between agents, of "professionally managed pool of capital that is invested in equity-linked securities of private ventures at various stages in their development". The same author, Sahlman (1990), enumerates as the main stages of VC investments the following: i) seed investments, ii) start-ups , iii) first stage – early development, iv) second stage – expansion, v) third stage – profitable but cash poor, vi) fourth stage – rapid growth toward liquidity point, vii) bridge stage – mezzanine investment, viii) liquidity stage – cash-out or exit. Worth mention, as described by Feld and Mendelson (2016), that different start-up financing rounds are often regarded as series, and defined by a letter, being the first round referred to as *Series A*, the second as *Series B*, and so on. Not that often, numbers may be attached to the letter of the series (e.g. *Series C-2*, or *Series D-1*), which might be introduced in order to limit the extension of the use of letters. Most recently introduced, and representing an exception to the letter nomenclature, the *series seed financing* has been considered as a financing round that precedes the *Series A*. As one of their main characteristic, Venture Capitalists are actively involved managing their ventures, sometimes becoming members of the board of directors and having attached some economic rights in addition to their ownership rights. The primarily organization form is the limited partnership, with Venture Capitalists as general partners and the outside investors as limited partners. A Venture Capital partnership intermediates, through contracts among investors and entrepreneurs, the flow of funds to ventures.

As stated by Sahlman (1990, p.474) "the contracts share certain characteristics, such as i) staging the commitment of capital and preserving the option to abandon, ii) using compensation systems directly linked to value creation, and iii) preserving ways to force management to distribute investment proceeds. These elements of the contracts address fundamental problems, such as i) the sorting problem – in sourcing both the best VCs and entrepreneurs, ii) the agency problem, and iii) the high operating-cost problem."

Also addressed by Sahlman (1990), governance systems used by Venture Capital organizations and traditional corporations are very different. The environment is highly uncertain about payoffs on each investment and a high degree of information asymmetry between investors and agents. To deal with this matter, procedures and contracts have evolved to mechanisms such as the inclusion of staging the commitment of capital, in attach compensation to the value created, and the use of mechanisms to somehow pressure agents to distribute capital and profits.

VC Risk, Returns and Other Outcomes

Giat and Hackman (2007) used a dynamic model to associate entrepreneurs' optimism to projects performance and, consequentially, higher VC returns. In an association of Behaviour Economics, Real Options, and Venture Capital relations with entrepreneurs, the authors derived a qualitative assessment of the influence of optimism in entrepreneurs with the economic value generated, the contract structures signed between VCs and Start-ups, the duration of the relationship, and even how this characteristic might reduce agency costs of risk-sharing. A considered to be "Entrepreneurs Optimism Premium" explains the discrepancy of VC's discount rates of over 40% and the expected returns of VC projects which usually yields around 15%.

Cochrane (2001) measured metrics as the mean, standard deviation, alpha and beta of Venture Capital investments, correcting them for selection bias. The author found out that the second, third, and fourth rounds of financing account for less risk, measured by progressively lower volatility. The betas of successive rounds declined from around 1 in the first round to around zero in the fourth.

As for possible reasons behind the difference of risk and return among private equities and publicly traded stocks, even when holding similar betas or belong to the same industry, share the small size and financial structure (e.g. book/market ratio), Cochrane (2001) points out the following:

- Liquidity – the lack of liquidity within private companies may imply a higher premium to compensate investors.
- Poor diversification – since typically each stake in a private company is a large one, it becomes harder to diversify, therefore accounting for a higher risk.
- Information and monitoring – compensation due to the capacity of Venture Capitalists to perform a monitoring role in the firm, by having a sit on the board of directors and sometimes the right to hire or fire managers.

Kortum and Lerner (2001) examine the impact of VC on technological innovation. The authors estimate that Venture Capital accounts for 8% of Industrial Innovation in the decade ending in 1992 in the US, and assuming a constant effectiveness of venture funding, by 1998 Venture Capital has accounted for 14% of US innovative activity.

The institutional framework

As stated by Feld and Mendelson (2016), there are three main entities within a VC fund structure: i) the management company, typically owned by the most senior partners, which is the entity that employs all the staff and pays for all the fund expenses for a regular operation. Also, while there might be the setup of various funds with a predetermined maturity, the management company can hold itself forever. ii) The limited partnership, which is a vehicle through which investors (the limited partners) interact with the general partners (VCs) financing them. A management company can hold several funds, through the setup of several limited partnerships, with different focus for each of the funds within the same management company. The final entity, iii) the general partnership, is the legal structure behind a general partner or several general partners. There can be several general partnerships within a management company, but a single limited partnership can only be attached to a single general partnership.

Feld and Mendelson (2016) explain how a typical VC fund is raised, whose process and contractual definition is known as the Limited Partnership Agreement. This is an agreement signed with the limited partners – that can assume a huge variety of personas, from high net worth individuals to other funds, funds of funds, governmental institutions and so on –, and define the amount and conditions through which the general partners can make the investments on behalf of the limited partners.

It is often considered that general partners risk only their credibility and reputation within the VC industry, but Sahlman (1990) shows that the contribution of general partners to the funds raised is usually, at least, of 1% of the total amount. This partially proves i) the commitment of the general partners to the fund, and assure ii) favourable tax treatment to the general partners as well.

The role played by time restrictions

The commitment period, as mentioned by Feld and Mendelson (2016) – which may be addressed as the investment period –, usually of a 5 years length, is the time period available for the VCs to source, screen, assess and invest in new start-ups for the fund. Once this period is over, no further investments in new ventures can be made on behalf of the limited partners of the fund or the fund itself. However, the fund can continue investing in existing ventures started to be financed during the commitment period. The authors mention this particularity of funds to be the justification for new funds within the same management company to be set up every three to five years, since fresh investment is needed to keep VCs active as investors.

As for the total length of time that a fund can remain active, the concept applied is the investment term. While new investments can only happen through the commitment period, follow-on investments can still be made until the end of the investment term. Feld and Mendelson (2016) argue that a typical fund has a 10 year investment term with either two to three one-year extension option or one two-year extension option. Although these results in a twelve to thirteen year length, for very early stage investment funds this maturity may be very short and the authors mention that this type of investment funds can last up to seventeen years.

The way VCs can still invest in existing ventures after the commitment period is over and prior to the expiration of the investment term is through reserves. Feld and Mendelson (2016, p.126) describe it as the "theoretical future amount of the fund to invest in follow-on rounds" defined at the moment of the first investment. These reserves lock a predetermined amount whose destination will only be the venture it was designed for, therefore reducing the amount of the fund available for new ventures.

Some interesting covenants within VC financing contracts – the term sheet

Feld and Mendelson (2016, p.49) define economics and control as the most important things VCs do look to when negotiating an investment in a start-up firm. In their own words, "*Economics* refers to the return the investor will ultimately get in a liquidation event, usually either a sale of the company or an initial public offering (IPO), and the terms that have direct impact on this return", while "*Control* refers to the mechanism that allows the investors to affirmatively exercise control over the business or to veto certain decisions the company can make".

Regarding economics, the same authors mention (Feld and Mendelson (2016, p.52)) that "it's a mistake to focus only on the valuation when considering the economics of a deal". This is an obvious hot topic – the dark magic versus random science of both pricing and valuation of a start-up – which, besides its subjectiveness, is ruled out by long and complex spreadsheets and numbers that might find justified and defensible assumptions in it making it somehow more objective. But again, this is not always the most difficult battlefield VCs and entrepreneurs face in a financing negotiation. According to the authors, a complete approach to the economics of a start-up firm financing deal must include, not being limited to, the pricing, liquidation preferences, important covenants such as the pay-to-play or how to treat the ownership of the stakeholders using, for example, the vesting concept, how to think and negotiate the employee pool, or account for antidilution clauses.

For the control issues, the biggest portion of negotiation time is spent on issues that regard the board of directors, protective provisions defined by Feld and Mendelson (2016, p. 75) as "veto rights that investors have on certain actions by the company" on both the VC and key employees side, drag along rights, and conversion.

Compensation schemes for VCs - How Venture Capitalists make money

To keep its operations running, Feld and Mendelson (2016) show how VCs rule their activity by claiming, mainly, three types of income: i) management fees, ii) reimbursement for expenses and iii) carried interest. Let aside the reimbursement expenses, that only represent the repayment – by start-ups who were invested in by VCs – of costs incurred to meet start-ups responsibilities like board, clients or suppliers meetings, management fees and carried interest are, respectively, the oxygen and true motivation that feed VCs activities.

The management fee is defined as a percentage, ranging from 1.5% to 2.5%, of the total amount raised by the fund, which is claimed annually, but paid through shorter periods of time. Since funds typically last for about 10 years, the amount of management fees can reach up to a total of 15% to 25% of the committed capital, and this amount is used to pay for all the costs a fund might have, from salaries to rents or travels. Worth to mention that, as previously stated, a management company might possess many funds, each of them receiving management fees to feed the management company costs. Interestingly, and once that the management fees decrease the capital amount available to invest in new ventures which is the true purpose of a VC fund, the funds might, in cases where some profitable investment exits occurs earlier in the fund life span, reinvest the amount of management fees in other ventures, thus leveraging the total investment capital.

Where does the true motivation for setting up a VC fund lies? Carried interest. Although the management fees might boost interesting salaries for partners and staffs and a high quality breathable oxygen for a VC fund to be kept on running, the share of profits returned to the limited partners are the carrot to which all general partners and fund related staffs run to, when performing their day to day tasks. The carried interest is a predetermined proportion of the final profits that limited partners give up on to the general partners to motivate and stimulate their best use of knowledge and expertise. This carried interest is usually of 20%, and might in some cases be defined above a certain predetermined hurdle rate.

Entrepreneurial financing decision

The investment decision taken by VCs is very hard and often taken with a conjunction of highly subjective criteria. Feld and Mendelson (2016) enumerate some of the most important ones VCs take into account when considering a certain investment. These criteria includes, but are not limited to, the stage of the start-up, competition with other financing sources, previous experience of the team – specially its founders –, the natural entry point of the fund, "numbers, numbers, numbers" – meaning the metrics VC typically use to asses and value the start-ups to invest –, and the macroeconomic environment.

The literature keeps growing, and although many metrics can be found, both financially driven and business driven, such as, for example, the positivity of unit economics, there are many other concerns that Venture Capitalists take into account that determines the investment decisions. Those concerns, as stated by Visagie (2011), regards primarily the quality of the team, market drivers, competitive hedge of the product or service, the capacity for the project to be scalable, the use of the so-called MVP – minimum viable product –, described as the commercial proof of concept also hugely highlighted in the Lean Startup from Eric Ries (2011), and VC specific factors.

The following table sums up the evidence collected by Visagie (2011, p.31) for the use of these investment criteria:

Criteria	Evidence for Use
Management Team	Means the entrepreneurial team and their characteristics.
	Tyebjee (1981), Goslin (1986), Hisrich (1990), Hutt (1985).
Market Drivers	Means the size of the market/industry, the market need, the access to market and target market. Associated with the market, are competition considerations including barriers to entry.
	Tyebjee (1984), MacMillan (1985), Hutt (1985), Hisrich (1990), Kahn (1987), Muzyka et al (1996).
Unique, Disruptive Product	Means uniqueness of the product, attributes of the product or profit margins.
	Tyebjee (1984), Hutt (1985), Kahn (1987), Hisrich (1990).
Scalable Business Model	Could possibly form part of the "Market Drivers" criterion, but were included separately to ascertain whether certain VCs consider potential scale more important relative to the other criteria.
	Tyebjee (1984), MacMillan (1985).
Commercial Proof of Concept	Means the development of a product to the point of a functioning prototype that has potential to generate profit. This criterion was included as a stand-alone criterion to ascertain its importance relative to the other criteria, and to ascertain whether early or late stage investors would view the importance of this criterion differently.
	MacMillan (1985).
VC Specific Factors	Means the factors specific to the VC such as the fund portfolio, fund phase or timeframe within which a return is required in order to fit in with the time horizon of the fund.
	Fulghieri and Sevilir (2009), Petty (2009).

Table 1 – Investment Criteria for Venture Capitalists, as in Visagie (2011, p.31).

Is there a methodology able to solve the investment decision problem VCs face?

So far it becomes clear that VCs i) has to profit a lot for their credibility to grow and be able to raise capital for further funds, ii) has to profit as much as possible on ventures exits so their carried interest – after eventual hurdle rates –, can be highly rewarding, iii) have a time restriction preventing them to wait for the right moment to exit from some ventures.

Putting together the necessity to maximize profits on a time constraint setting needs some audacity to be successfully accomplished. Is there a methodology able to solve the investment decision problem faced by VCs?

Indeed, among the myriad of possibilities to tackle the problem under sight, there is a methodology that clearly stands out against any other. Real Options.

2.2. Real Options

Financial Options – The beginning

Funny story to know that the solution for such a breakthrough in finance as the pricing of financial options was built upon an already mainstream mathematical rational of the ancient science of physics. Immortalized by Merton (1973), the work of Black and Scholes (1973) in the paper "*The Pricing of Options and Corporate Liabilities"* could not be more revolutionary for the field of finance, and the best of that methodology application were yet to be unleashed.

Real Options are born

Not long after, Myers (1977) has established the beginning of the history of Real Options, in its work entitled *"Determinants of Corporate Borrowing"*. In the paper, the author explores how companies growth opportunities can be seen as call options, and therefore, when dealing with highly uncertain projects, the value of the future option should be added to their Net Present Value.

Not many people are able to fully understand the mathematical rational behind financial options but yet their application is now fully accepted, opening doors for the same destiny for real options.

Circumstances under which Real Options are applied

Dixit and Pindyck (1995), address the shortcomings of the orthodox theory, and explain why capital budgeting problems should be addressed in an options point of view. They share the options frameworks upon which the rational can be based on, and give a series of industry examples, highly detailed in their book *Investment Under Uncertainty* (Dixit and Pindyck (1994)).

It shows, among many other things, that the NPV rule takes a certain project investment decision as a now or never execution problem, giving up the possibility of waiting so more decisive information can arrive and increase or decrease dramatically the value of the project. Given that, the NPV rule should account for the value of the option, and the investment decision should take a different form: the GPV should exceed the investment cost, plus the value option to invest in the project at a later time with more complete and useful information.

The same authors in the very same paper address the conditions under which capital budgeting is facing an option. Investment decisions must be taken under uncertainty, projects execution must be flexible, and investment costs should be, at a certain extent, irreversible.

Uncertainty

Whenever there is no uncertainty, real options does not fit strategic decisions. If there is no uncertainty, the prediction of future cash flows is perfectly accurate, all information is immediately available, and in such a world, valuing options makes no sense, since there will be no event to trigger, given that the value of a project is perfectly predictable. Once that there is no such world, uncertainty is of huge importance when considering the use of Real Options.

Flexibility

When there is no flexibility for intake a project, or not, delay it or not, invest or not, the decision will always be a "now or never" situation where does not makes sense to determine the value of an option.

Irreversibility

Cost irreversibility, at least at a certain extent, must be present in order for Real Options to be considered as a methodology. If irreversibility is not present, delay an investment, for example, never makes sense once that all the investment is fully recoverable if or when something goes wrong when undertaking a project. Roche (2015), for example, addresses the implications of irreversibility in investments, specially its impact regarding projects financed by debt, emphasizing the value of waiting to invest when the cost of external funds is endogenously determined.

2.3. Real Options applied to Venture Capital

The applications of Real Options to the field of Venture Capital (VC) are still reduced, although the range of studies already covers a wide range of important issues. These issues include matters such as valuation, risk, uncertainty, financing needs, earn outs or convertible notes, timings for entrepreneurs to look for investment, moral hazard, managerial replacement, among many others.

As Venture Capital entails a large spectre of company stages and purposes – ranging from early to later stages, for which financing may be used to, for example, prototype ideas, or to growth focus –, many different characteristics might have the chance of being modelled through a Real Options perspective.

Imai, Y (2017) modelled the relation of a second-round equity investor and a convertible note holder, and how the value of the option to convert is impacted by the investor's belief on the ability of the entrepreneur to raise funds and increase the start-up value in each of the further capital rounds.

Regarding staged financing, Koçkesen and Ozerturk (2002) showed that the highest benefit of this financing timing is related to the possibility of investing in a certain start-up in a further round of investment with better conditions than VC competitors by having a competitive advantage related to the lack of information that outsiders might have. The paper also reveals that adverse selection resulting from asymmetric information represents an exit barrier for entrepreneurs with good prospects and creates an endogenous lock-in.

Meng (2008) developed a model based on a duopoly patent race showing that patent races cause over investment, value-dissipation, a higher CAPM beta, and a higher return volatility, in excess of 100% sometimes, compared to a joint monopoly. As for the high level of return volatility, is considered to happen due to technological risks.

Tavares-Gärtner, Pereira and Brandão (2018b) introduced a taxonomy of contingent payment mechanisms based on the maturity and amount of investment, through which analysed the decision of an entrepreneur – facing a wealth constraint – looking for an external equity provider to back a growth opportunity. The authors concluded that the choice of the optimum mechanism depends on exogenous variables as liquidity preferences or constraints, timing requirements, post-deal integration or overall deal terms.

Maya (2004) created an approach named Creative Destruction - Real Options Approach to valuing start-ups when only technological uncertainty is present, when a start-up is in a context of a Creative Destruction process. This, as mentioned by Schumpeter (1942), happens when new consumer goods, new methods of production or transportation, new markets appear to disrupt the economic structure destroying the previous one, in a cyclical way. In this approach the value of the firm results in the sum of the project value without flexibility, and adding the value of the real options the project offers to the firm, which should account for the option to invest conditional on the discovery of a new product considered as a "drastic innovation". The approach explains the high prices investors may be willing to pay for certain growth stocks, proving that overpricing may not be the case, but is instead a recognition of the large growth potential due to innovation.

Siller-Pagaza and Otalora (2008) explains that when managers are entrepreneurs contributing with intangible assets to firm – be the expertise, networking or other mean –, the moral hazard when seeking outside equity depends on the value of real options and the percentage he / she receives of free cash flows. The greater the value of the options, the greater the percentage he / she must earn from dividends, to decrease the moral hazard possibility.

Leshchinskii and Brisley (2006) studied, in a two-period framework that allow stage financing, how the information available from potential investors determines an entrepreneur choice of financing from a pool of potential investors that includes business angels, Venture Capitalists and traditional atomistic investors. The decision of being funded depends on the additional value that investors abilities might bring to solve some problems that can occur along the projects duration and by the actions they can take, such as replacing the manager or cutting the investment. The results show that the entrepreneurs choose either angel or Venture Capital financing with a cost-benefit analysis on the resolution of possible uncertainties, meaning that more value is created than its cost.

The list goes on, and the challenges of Venture Capital being tackled through the Real Options methodology keeps increasing. As a short conclusion on the literature review, the foundations of both Real Options and Venture Capital were addressed, and finally some examples were given on existing frameworks that apply Real Options to Venture Capital. Interestingly, few are the examples of Real Options approaches to Venture Capital from the Venture Capitalist – the investor – point of view, and it is precisely where the present work will be focused on.

3. The Model

The model comprises the investment and posterior exit of a Venture Capitalist in an established entrepreneurial firm, in a dynamic real options approach. This entrepreneurial firm is assumed to be owned by a single Entrepreneur, and hold a growth opportunity, θ $(\theta > 1)$, defined by an expansion of its value, V_t, given an investment totalling K.

The limited to no access to debt attached to both the start-up firm and the Entrepreneur pushes the growth opportunity to be financed through the limited financial sources of the Entrepreneur, $K_E(0 < K_E < K)$, together with the Venture Capitalist's (VC) funds, $K - K_E$, in a jointly backed equity round. Transaction costs associated to the Venture Capitalist initial investment are considered to be included in $K - K_E$, and those of the Entrepreneur in K_E.

The Venture Capitalist is assumed to not have any funding constraints neither burden any additional opportunity cost from other potential investments in other ventures, or equivalently that the current investment is the best available one. Also, the capital increase (either by the Entrepreneur or Venture Capitalist) is made at no premium or discount.

Post-equity round firm ownership held by the Venture Capitalist is of $0 < Q_{VC} < 1$, while the new Entrepreneur ownership will be of $0 < Q_E < 1$ and $Q_{VC} = 1 - Q_E$.

After investing $K - K_E$ to possess $Q_{VC}\,\theta\,V_t$ of the start-up firm value, the Venture Capitalist main concern is to dispose the investment at a considerable profit within a predetermined timing. This time restriction will be addressed latter on.

From the moment the deal took effectiveness, the Venture Capitalist is considered to be the seller – S – who owns $Q_{VC}\,\theta\,V$ of the start-up firm value, the target. For another player B (either another Venture Capital fund, a Private Equity fund or another company) – the buyer – the same target has a higher value of $(1+\gamma)\,Q_{VC}\,\theta\,V_t$, $(\gamma > 0)$. Those synergies, represented by γ, are assumed to be well known by the industry due to Venture Capitalists' experience on past deals used as benchmarks or even deep specific sector knowledge, since γ can be interpreted as a sector specificity.

The seller is only willing to dispose the investment if a certain proportion of the synergies possessed by the buyer, γ, is shared, and so the transaction will be settled at a premium $\emptyset$, $(\gamma > \emptyset > 0)$. It is important to notice that the premium being discussed, $\emptyset$, is not about the value added by the VC to the start-up, but a premium built upon that added value already incorporated in $\theta\, V_t$. Thus, the premium, $\emptyset$, is a proportion of the buyer synergies, γ, that is simultaneously the minimum proportion of synergies that the VC is willing to accept in order to sell the start-up stake, and the maximum one the bidder is willing to give up on to buy the start-up.

By selling the target from S to B sunk transaction costs of C arises either for S and B, in a proportion of ε and $(1-\varepsilon)$ respectively, with $\varepsilon \in [0,1]$.

It is assumed that the value of the start-up firm follows a geometric Brownian motion:

$$dV_t = \alpha\, V_t\, dt + \sigma\, V_t\, dz, \qquad where\ V_0 > 0,$$

with σ^2 as the volatility of the start-up value, α as its growth rate – of the start-up – and dz as an increment of a Wiener process with zero mean and variance equal to dt.

It is also assumed that all agents are risk neutral and that the riskless interest rate $r, (r \geq \alpha)$ controls for the time-value of money. Accordingly, α is a risk neutral drift.

In order to achieve the *entry-exit* option value two different timings will be considered: (i) the *entry-option* – an option to invest –, determining the optimal ownership that the VC has to request considering a set of variables, and (ii) the *exit-option*, negotiating the optimal premium to exit the investment in the case of an offer contemplating the right premium.

For the determination of the optimal ownership to be required, the rational in which the model will be built upon is the contribution of Tavares-Gärtner *et. al.* (2018a), approaching the optimal ownership through both homogeneous and heterogeneous beliefs on the growth prospects.

The determination of the optimal premium upon which the start-up must be disposed by the Venture Capitalist in the *exit-option* (even if the predetermined investment maturity is not reached) will be done through the contribution of Lukas and Welling (2012). Firstly in a setting that comprises a perpetual option and therefore excluding any consideration of the investment maturity, and secondly considering the time restriction of the investment maturity through the use of the contribution of Pereira and Rodrigues (2014).

Since the *entry-option* value is dependent on the *exit-option* value, a backward procedure must be used, and the *exit-option* has to be the first to be determined.

3.1. VC Exit Option

The option value is determined through an adjusted model of Lukas and Welling (2012), conceptualized and applied from the seller perspective.

Once the target is sold, the seller (Venture Capitalist) gets the sales price $(1 + \emptyset)\ Q_{VC}\ \theta\ V_t$, $(\emptyset > 0)$, has to pay the transaction costs of εC and transfer the target, of value $Q_{VC}\ \theta\ V_t$, to the buyer. It does not incur a loss, if $\emptyset \geq \frac{\varepsilon C}{Q_{VC}\ \theta\ V_t}$.

By buying the start-up stake, the buyer gets $(1 + \gamma)\ Q_{VC}\ \theta\ V_t$ and in return has to pay the sales price $(1 + \emptyset)\ Q_{VC}\ \theta\ V_t$, and the transaction cost $(1 - \varepsilon)$C. It does not incur a loss if $\gamma \geq \emptyset + \frac{(1-\varepsilon)C}{Q_{VC}\ \theta\ V_t}$.

Consequently, a sale of the target from S to B will create a surplus if and only if $\emptyset\ Q_{VC}\ \theta\ V_t > C$.

The surplus is $\gamma\ Q_{VC}\ \theta\ V_t - C$, and its partitioning has to be negotiated by the choice of $\emptyset$.

Therefore, at time t_0, the seller is requiring $\emptyset > 0$ to the buyer, which can accept the offer or reject it. The buyer does not has to decide immediately at t_0 if it accepts or rejects the offer, having the possibility of postpone the decision.

It is assumed that there is no possibility for further rounds of negotiation or counteroffers. Hence, accepting the offer leads to an acquisition of the target. As mentioned by Lukas and Welling (2012), the absence of counteroffers might be the result of the assumption that the first mover hold the dominant bargaining power thus being able to avoid further negotiations.

Generalizing, the seller receives upon closing the deal $\emptyset\ Q_{VC}\ \theta\ V_t - \varepsilon C$ while the buyer receives $\gamma\ Q_{VC}\ \theta\ V_t - (1 - \varepsilon)C$.

Time is continuous, i.e. $t \in (t_0, \infty)$, and the seller sets $\emptyset \in (0, \infty)$ and after it, at every moment in time, the buyer decides whether to $\{$accept, wait$\}$.

The process has a Markovian Perfect Nash Equilibrium path to determine the optimal decision for each parties. Particularly, the seller places the bid defining optimally $\emptyset$ in stage one, and the buyer, conditional on the required premium, $\emptyset$, will choose a threshold value $Q_{VC}\ \theta\ {V_t}^*(\emptyset)$ in stage two at which the offer will be accepted.

This corresponds to an optimal timing decision with $t^* = \min\{t \geq t_0 | V_t > V_t{}^*\}$. Hence, this degree of managerial flexibility can be interpreted as a real option.

The exercising of the option manifests itself on accepting the offer. At this stage, it is assumed that this flexibility is not limited to a fixed maturity, and therefore the possibility to accept the offer is a perpetual real option.

Consequently, the value of the option to acquire the target – the start-up – held by the buyer is the solution of the following maximization problem in stage two:

$$F\big(V_{Buyer}(t)\big) = \max_t E\left[\Big((\gamma - \emptyset)\ Q_{VC}\ \theta\ V_{Buyer}(t) - (1-\varepsilon)C\Big)\ e^{-rt}\right] \qquad (1)$$

Where $E[\cdot]$ denotes the expectation operator.

Solving the previous equation yields[1]:

$$F\big(V_{Buyer}\big) = \Big((\gamma - \emptyset)\ Q_{VC}\ \theta\ V^*{}_{Buyer} - (1-\varepsilon)C\Big)\left(\frac{V}{V^*{}_{Buyer}}\right)^{\beta} \qquad (2)$$

With $\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1$, and

$$V^*{}_{Buyer} = \frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\ \theta\ (\gamma - \emptyset)} \qquad (3)$$

In contrast, the seller will choose $\emptyset$ in stage one such that it maximizes:

$$F(V_{VC-seller}(t)) = \max_t E\left[\big(\emptyset\ Q_{VC}\ \theta\ V^*{}_{Buyer} - \varepsilon C\big)e^{-rt^*}\right] \qquad (4)$$

Solving the previous equation yields[2]:

$$F(V_{VC-seller}) =$$

$$\left(\emptyset\ Q_{VC}\ \theta\ \left(\frac{\beta}{\beta-1} \times \frac{(1-\varepsilon)C}{Q_{VC}\ \theta\ (\gamma-\emptyset)}\right) - \varepsilon C\right)\left(\frac{V_{VC-seller}}{\left(\frac{\beta}{\beta-1} \times \frac{(1-\varepsilon)C}{Q_{VC}\ \theta\ (\gamma-\emptyset)}\right)}\right)^{\beta} \qquad (5)$$

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[1] Please refer to Appendix 1 for the solution through the use of contingent claims.
[2] Please refer to Appendix 2 for the solution through the use of contingent claims.

Or equivalently:

$$F(V_{VC-seller}) = \left(\left(\frac{\beta}{\beta-1}\times\frac{(1-\varepsilon)C\,\emptyset}{(\gamma-\emptyset)}\right)-\varepsilon C\right)\left(\frac{V_{VC-seller}}{\left(\frac{\beta}{\beta-1}\times\frac{(1-\varepsilon)C}{Q_{VC}\,\theta\,(\gamma-\emptyset)}\right)}\right)^{\beta} \qquad (6)$$

Proposition 1 *The optimal required premium for the seller, similar to Lukas and Welling (2012), equals:*

$$\emptyset^* = \frac{\gamma(1+(\beta-2)\varepsilon)}{\beta-\varepsilon} \qquad (7)$$

Proposition 2 *The optimal timing threshold* ${V^*}_{Buyer}$, *similar to Lukas and Welling (2012), becomes:*

$${V^*}_{Buyer}(\emptyset^*) = \frac{\beta}{(\beta-1)^2}\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta} \qquad (8)$$

The triggers determined by Propositions 1 and 2 define the optimal premium and timing for the transaction to happen, and are crucial pieces for the entry decision to be taken by the Venture Capitalist, whose *entry-option* value is to be determined next.

3.2. VC Entry-Exit Option

As previously mentioned, the *entry-option* aim is to optimally define the ownership the Venture Capitalist should require considering an investment amount, $K-K_E$, and the growth prospects, θ, on the start-up value regarding that investment amount. Since there could be, or not, an equal understanding on that growth prospect among the Entrepreneur and the Venture Capitalist, the *entry-option* is calculated accounting for both scenarios – of homogeneous beliefs and then of heterogeneous beliefs –, using the contribution of Tavares-Gärtner et. al. (2018a).

3.2.1. Homogeneous Beliefs

This setting assumes that both the Entrepreneur and Venture Capitalist have homogeneous beliefs regarding the start-up growth prospects, θ $(\theta > 1)$, whereby the players individually assess their decision whether to invest or not in the growth opportunity, and jointly determine the ownership each will optimally take by undertaking the investment.

Since both players held an option to wait for the optimal ownership trigger, and so flexibility, uncertainty and cost irreversibility – at least at certain extent – are in place, the problem deserves a real options approach in order to be glamorously solved.

Given the homogeneous beliefs in place, with a growth prospect θ, $(\theta > 1)$, agreed by both players, the optimal ownership for the Venture Capitalist, who has to immediately account for its next step of an exit of that venture – due to its business nature –, considering the investment $K - K_E$, can be seen as follows:

$$F(V_{VC-hom}(t)) = Q_{VC}\,\theta\,V + Exit\ Option_{VC} - (K - K_E) \qquad (9)$$

Consequently[3], the Venture Capitalist's option to enter in the venture is given by

$$F\big(V_{VC-hom}(t)\big) = \max_t E\left[\left(\left(Q_{VC}\,\theta\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}\right)\right) - (K - K_E)\right)e^{-rt}\right] \qquad (10)$$

Solving the previous equation yields[4] the following proposition.

Proposition 3 *The optimal timing to entry in the start-up capital in the homogeneous beliefs setting is:*

$$V^*{}_{VC-hom} = \frac{\beta}{\beta-1}\times\frac{(K-K_E)}{Q_{VC}\,\theta} \qquad (11)$$

The value function for the Entrepreneur, considering $Q_E = 1 - Q_{VC}$, comes as follows:

$$F(V_{E-hom}(t)) = \max_t E[((1-Q_{VC})\,\theta\,V - V - K_E)e^{-rt}] \qquad (12)$$

Solving the previous equation yields[5]:

$$V^*{}_{E-hom} = \frac{\beta}{\beta-1}\times\frac{K_E}{\big((1-Q_{VC})\,\theta - 1\big)} \qquad (13)$$

[3] The simplified version of the VC Exit Option is as follows: $F(V_{VC-seller}(\emptyset^*, t)) = \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}$

[4] Please refer to Appendix 3 for the solution through the use of contingent claims.

[5] Please refer to Appendix 4 for the solution through the use of contingent claims.

Aligning the optimum investment timing through the optimal ownership yields the following proposition.

Proposition 4 *Venture Capitalist's post-money optimum ownership in the homogeneous beliefs setting:*

$$Q^{*}{}_{VC} = \frac{(K - K_E)(\theta - 1)}{\theta\, K} \qquad (14)$$

Consequently, the optimal Entrepreneur's post-money ownership is:

$$Q^{*}{}_{E} = \frac{K + K_E\,(\theta - 1)}{\theta\, K} \qquad (15)$$

Once the exit and entry triggers are well defined, as above, it is possible to extrapolate an exit multiple, and assess its behaviour and to which variables the multiple is sensible the most. This multiple is the result of the exit trigger, as per defined in Eq. (8), divided by the entry trigger, as per defined in Eq. (11).

Proposition 5 *The exit multiple implied in the homogeneous beliefs setting is:*

$$Exit\ multiple = \frac{\beta}{(\beta - 1)} \frac{\left(1 - \frac{\varepsilon}{\beta}\right) C}{(K - K_E)\,\gamma} \qquad (16)$$

3.2.2. Heterogeneous Beliefs

The heterogeneous beliefs approach relax the assumption that the agents have the same homogeneous expectations about the growth opportunity. It differs from the previous one in a sense that exists different growth prospects θ, $(\theta > 1)$, attached to the Entrepreneur θ_E, $(\theta_E > 1)$, and Venture Capitalist θ_{VC}, $(\theta_{VC} > 1)$, for the start-up growth option value considering the very same investment amount of, respectively, K_E and $K - K_E$.

As Tavares-Gärtner et. al. (2018a) pointed out, it is important to emphasize the assumption that both the Entrepreneur and Venture Capitalist truthfully share their own beliefs on the growth prospects with each other, thus the Entrepreneur believes and shares his/her growth prospects for the start-up, θ_E, and so does the Venture Capitalist regarding his/her growth prospects, θ_{VC}.

Given so, with a growth prospect θ_{VC}, $(\theta_{VC} > 1)$, disclosed by the Venture Capitalist, the optimal ownership for the latter, who ,again, has to immediately account for its next step of an exit of that venture – due to its business nature –, considering the investment $K - K_E$, can be seen as follows:

$$F\big(V_{VC-het}(t)\big) =$$

$$\max_t E\left[\left(\left(Q_{VC}\,\theta_{VC}\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}\right)\right) - (K-K_E)\right)e^{-rt}\right]$$

(17)

Solving the previous equation yields[6] the following proposition:

Proposition 6 *The optimal timing to entry in the start-up capital in the heterogeneous beliefs setting is:*

$$V^*{}_{VC-het} = \frac{\beta}{\beta-1}\times\frac{(K-K_E)}{Q_{VC}\,\theta_{VC}} \qquad (18)$$

The value function for the Entrepreneur, considering $Q_E = 1 - Q_{VC}$, comes as follows:

$$F(V_{E-het}(t)) = \max_t E\big[\big((1-Q_{VC})\,\theta_E\,V - V - K_E\big)e^{-rt}\big] \qquad (19)$$

Solving the previous equation yields[7]:

$$V^*{}_{E-het} = \frac{\beta}{\beta-1}\times\frac{K_E}{\big((1-Q_{VC})\theta_E - 1\big)} \qquad (20)$$

Aligning the optimum investment timing through optimal ownership results in the following proposition:

Proposition 7 *Venture Capitalist's post-money optimum ownership in the heterogeneous beliefs setting:*

$$Q^*{}_{VC} = \frac{(K-K_E)(\theta_E-1)}{\theta_E\,K + K_E(\theta_{VC}-\theta_E)} \qquad (21)$$

Consequently, the optimal Entrepreneur's post-money ownership is:

$$Q^*{}_{E} = \frac{K + K_E\,(\theta_{VC}-1)}{\theta_E\,K + K_E(\theta_{VC}-\theta_E)} \qquad (22)$$

[6] Please refer to Appendix 5 for the solution through the use of contingent claims.
[7] Please refer to Appendix 6 for the solution through the use of contingent claims.

Again, it is possible to extrapolate the exit multiple implicit in the heterogeneous beliefs setting, but the indicator is precisely the same as in Proposition 5, Eq. (16). This happens due to the fact that the indicator is not sensitive to changes in the growth prospects.

Interestingly, for both the homogeneous and heterogeneous settings, neither the value triggers nor optimal ownerships in the *entry-exit options* are affected by the *exit option*. This result makes the present model yielding very similar triggers as the ones derived by Tavares-Gärtner et. al. (2018a) whose differences lies on the model methodology since their model use cash-flows other than the start-up value as the stochastic variable to assess.

3.3. Numerical Example

A numerical example is provided in order to clearly visualize how variables relate to each other and their impact on triggers, how both models behave and which differences can be found among them, as well as the most relevant considerations on the models.

Following Tavares-Gärtner et. al. (2018a) rational, that mention Meza and Southey (1996), Manove and Padilla (1999), Koellinger et al. (2007), Hmieleski and Baron (2009) or Landier and Thesmar (2009) on the perspective that Entrepreneurs' growth prospects for a certain investment are more optimistic than the ones of the Venture Capitalists, and so $\theta_E > \theta_{VC}$. The volatility measure, σ, also follows Tavares-Gärtner et. al. (2018a), being based on Liu and Yang (2015), whereas the riskless interest rate meets FED and ECB long term inflation targets. Returns of the start-up is regarded as more conservative, $\delta = 5\%$, and as known $\alpha = r - \delta$. The base case parameters are shown in Table 2.

Key numerical assumptions		
Variable	Value	Description
γ	15%	Synergies of the Buyer
ε	50%	Proportion of exit transaction costs supported by the VC
C	80	Transaction costs of the exit
K	800	Total investment needed to finance the growth opportunity
K_E	250	Entrepreneur's own funds
α	-3%	Growth rate of the start-up
r	2%	Riskless interest rate
σ	25%	Volatility
θ	2	Growth prospects - homogeneous beliefs setting
θ_{VC}	2	Growth prospects of the VC - heterogeneous beliefs setting
θ_E	3	Growth prospects of the Entrepreneur - heterogeneous beliefs setting

Table 2 – Key numerical assumptions, perpetual model.

The first variable to be stressed is the volatility impact on the entry trigger for both the VC and Entrepreneur – below in Figure 2 –, which allow to have a clear perspective on the fact that both the Venture Capitalist and the Entrepreneur are aligned in what concerns the start-up value at the VC's entry point. Also, and since the volatility is the parameter being stressed, it also becomes clear the fact that these triggers increase exponentially with the increase of volatility, $\frac{\partial V^*{}_{VC-hom}}{\partial \sigma} > 0$, $\frac{\partial V^*{}_{E-hom}}{\partial \sigma} > 0$, $\frac{\partial V^*{}_{VC-het}}{\partial \sigma} > 0$ and $\frac{\partial V^*{}_{E-het}}{\partial \sigma} > 0$.

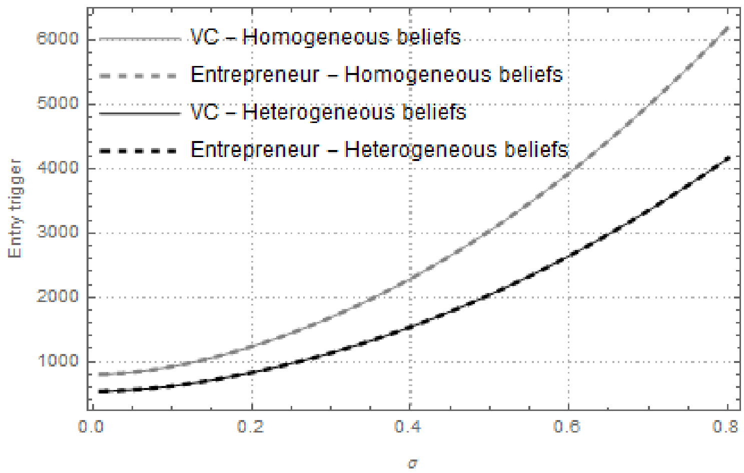

Figure 2 – Computing the optimal VC entry trigger, $V^*{}_{VC-hom}$ and $V^*{}_{VC-het}$, and the optimal Entrepreneur entry trigger, $V^*{}_{E-hom}$ and $V^*{}_{E-het}$, in both an homogeneous and heterogeneous setting, for different volatility, σ, levels. Keeping all the assumptions taken in Table 2, except for the stressed ones.

The homogeneous beliefs setting represents a higher entry trigger of the start-up both for the VC and the Entrepreneur against the setting of heterogeneous beliefs trigger, which given the assumption that the Entrepreneur is the most optimistic on the growth prospects makes absolute sense. These triggers difference also increases with the increase of volatility, which is accurate from an economic meaning point of view.

An important aspect to be interpreted using the previous assumptions, in Table 2, is the behaviour of the optimal premium the VC must request in order to sell the start-up stake it has acquired. It is observable in Figure 3 that a straight line with a slope of 0.64 derived from Proposition 1, $\frac{(1+(\beta-2)\varepsilon)}{\beta-\varepsilon}$, yields an optimal premium of 9.65% when the synergies, γ, are of 15%. This means that the VC, for the given assumptions, will always bear 64.43% of the total synergies the buyer is expecting.

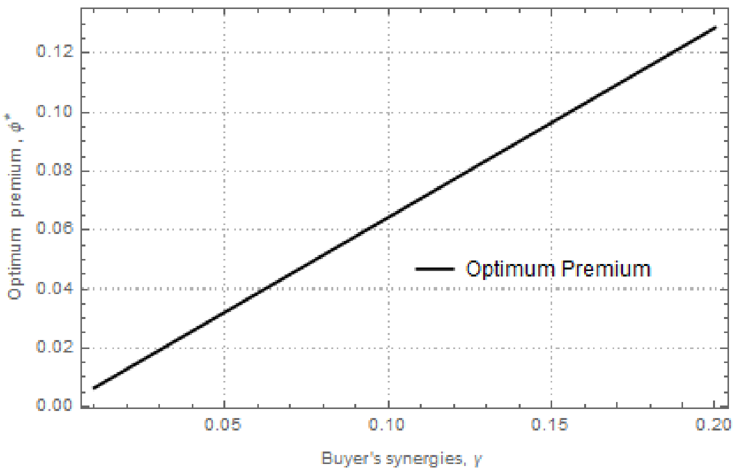

Figure 3 – Computing the optimal premium, $\emptyset$, the VC is willing to accept in order to sell the start-up stake, for different values of the synergies, γ, possessed by the Buyer. Keeping all the assumptions taken in Table 2, except for the stressed ones.

When observing the behaviour of the exit triggers, $V^*{}_{Buyer}(\emptyset^*)$, in Figure 4, for different values of the growth prospects for homogeneous and heterogeneous beliefs[8], on different values for θ, θ_{VC} and θ_E, homogeneous and heterogeneous beliefs cross each other lines for an exit trigger, V^*, of 1960.6, when $\theta_{VC} = 2$ and of 980. 3 when $\theta_E = 3$. Interestingly, Heterogeneous beliefs curves do never cross each other for any combination of values of θ_{VC} and θ_E. As observed, $\frac{\partial V^*{}_{Buyer}(\emptyset^*)}{\partial\, \theta} < 0$, $\frac{\partial V^*{}_{Buyer}(\emptyset^*)}{\partial\, \theta_{VC}} < 0$ and $\frac{\partial V^*{}_{Buyer}(\emptyset^*)}{\partial\, \theta_E} < 0$, meaning that the bigger the growth prospects – either for the VC or the Entrepreneur, since VC and Entrepreneurs are always aligned on these triggers –, the sooner the VC will be willing to dispose the start-up stake.

[8] The expression for the exit trigger considering heterogeneous beliefs is similar to the one of the homogeneous beliefs expressed in Eq. (8), being the growth prospects the one of the VC:
$V^*{}_{Buyer}(\emptyset^*)_{heterogeneous\ beliefs} = \frac{\beta}{(\beta-1)^2}\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}$

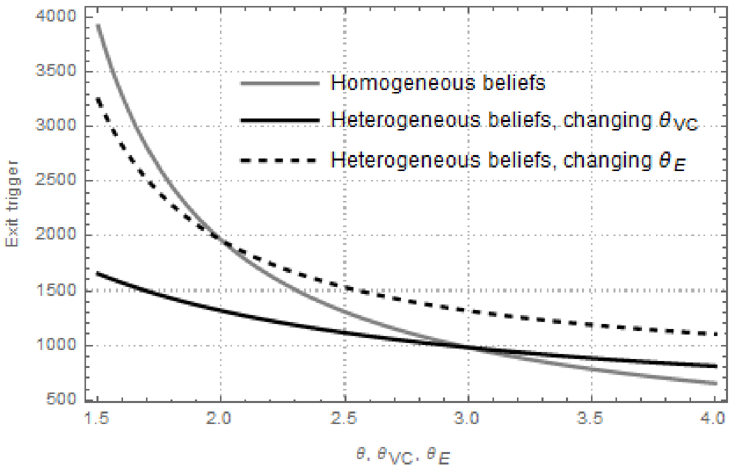

Figure 4 – Computing the optimal VC exit trigger, $V^*_{Buyer}(\emptyset^*)$, for different growth prospects, θ, θ_{VC} and θ_E, applied to both homogeneous and heterogeneous beliefs. Keeping all the assumptions taken in Table 2, except for the stressed ones.

The entry triggers, V^*_{VC-hom} and V^*_{VC-het}, in Figure 5, do not behave much differently than the exit triggers, $V^*_{Buyer}(\emptyset^*)$, for different growth prospects, θ, θ_{VC} and θ_E, for either the homogeneous and heterogeneous beliefs. Besides the fact that, obviously, the values upon which the VC is willing to invest are lower than those of the exit for the same levels of growth prospects, everything else is interestingly similar. Homogeneous and heterogeneous beliefs curves cross each other lines for an entry trigger, V^*, of 1442.5, when $\theta_E = 2$ and of 721.3 when $\theta_{VC} = 3$, and there is no combination of values of θ_{VC} and θ_E for which the heterogeneous beliefs curves cross each other. Again, the derivatives of the entry triggers in regard to growth prospects is negative, $\frac{\partial V^*_{VC-hom}}{\partial \theta} < 0$, $\frac{\partial V^*_{VC-het}}{\partial \theta_{VC}} < 0$ and $\frac{\partial V^*_{VC-het}}{\partial \theta_E} < 0$, meaning that the bigger the growth prospects the sooner the VC will be willing to invest in the start-up.

Considering the assumptions taken, it is interesting to note that the homogeneous beliefs setting yields the highest trigger when $\theta < 2$ compared to any of the heterogeneous settings. It has the lowest one when $\theta > 3$, and stands in between the heterogeneous belief curve that considers changes in θ_E – in the upper side –, and the heterogeneous belief curve that considers changes in θ_{VC} – in the lower side for $2 > \theta > 3$.

These results are clearly influenced by the initial growth prospects chosen as the base case. Since the derivatives of the value function for the VC in regard to the growth prospects are negative, $\frac{\partial V^*_{VC-hom}}{\partial \theta} < 0$, $\frac{\partial V^*_{VC-het}}{\partial \theta_{VC}} < 0$ and $\frac{\partial V^*_{VC-het}}{\partial \theta_{E}} < 0$, it was already shown that the lower the growth prospect, the higher are both the entry and exit triggers. Nevertheless, when stressing the growth prospects assumptions for the different models – homogeneous beliefs, heterogeneous beliefs changing θ_E and heterogeneous beliefs changing θ_{VC} –, the remaining growth prospects assumptions are kept equal.

Given this, when having $\theta < 2$, $\theta_{VC} < 2$ and $\theta_E < 2$ for, respectively, the homogeneous beliefs, heterogeneous beliefs changing θ_{VC} and heterogeneous beliefs changing θ_E settings, the assumption is always $\theta_E = 3$ in the case of heterogeneous beliefs changing θ_{VC}, and $\theta_{VC} = 2$ in the case of heterogeneous beliefs changing θ_E. This combinations of $\theta_E = 3$ for $\theta_{VC} < 2$ and $\theta_{VC} = 2$ for $\theta_{VC} < 2$ results in a combined growth prospects which are more optimistic, as a all, than the one of the homogeneous beliefs, and so the higest trigger will always come from the homogeneous beliefs setting. The opposite rational applies for $\theta > 3$, $\theta_{VC} > 3$ and $\theta_E > 3$.

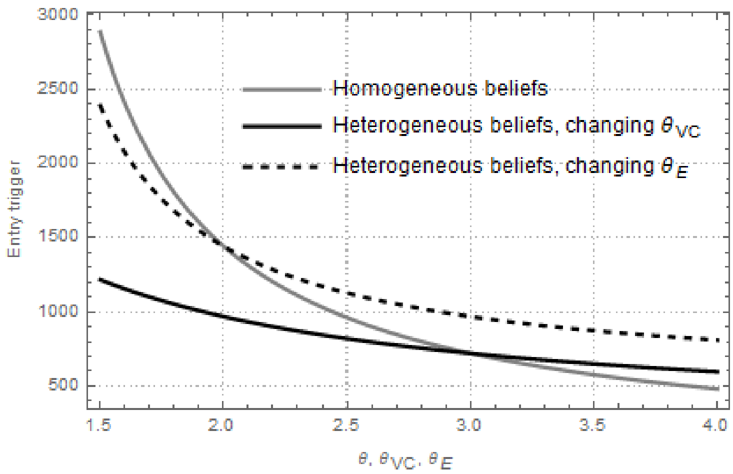

Figure 5 – Computing the optimal VC entry trigger, V^*_{VC-hom} and V^*_{VC-het}, for different growth prospects, θ, θ_{VC} and θ_E, applied to both homogeneous and heterogeneous beliefs. Keeping all the assumptions taken in Table 2, except for the stressed ones.

In order to compare the VC triggers – both exit triggers and entry triggers –, which, when divided, yields the expected exit multiple of the Venture Capitalist, Figure 6 displays their behaviour for different growth prospects in the homogeneous and heterogeneous settings.

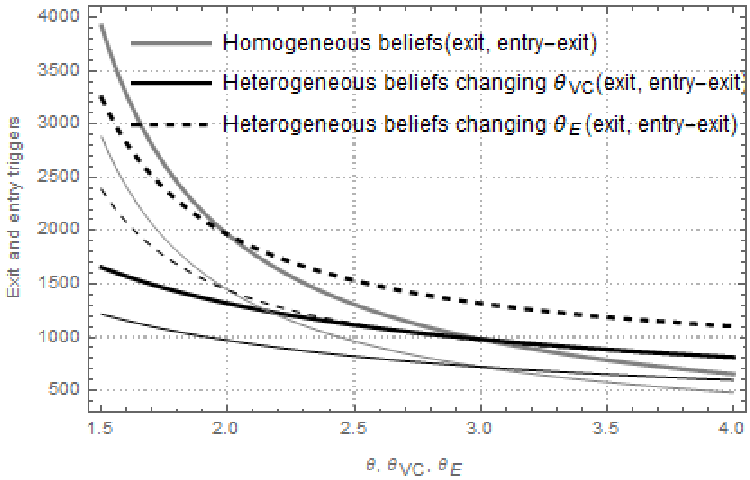

Figure 6 – Comparing the optimal VC exit trigger, $V^*_{Buyer}(\emptyset^*)$, with the optimal VC entry trigger, V^*_{VC-hom} and V^*_{VC-het}, for different growth prospects, θ, θ_{VC} and θ_E, applied to both homogeneous and heterogeneous beliefs. Keeping all the assumptions taken in Table 2, except for the stressed ones.

Although the previous figure is not crystal clear regarding each setting considered, the exit triggers for every setting is the upper curve in each curve style, meaning that, for example, the dashed grey upper line regards the VC exit trigger in the heterogeneous beliefs setting, for different growth prospects of the Entrepreneur, while the lower dashed grey line regards the VC entry trigger also in the heterogeneous beliefs setting for different growth prospects of the Entrepreneur.

Taking a better look at Figure 6, it is easy to extrapolate that the exit multiple is static, or close to it, and Figure 7 proves it.

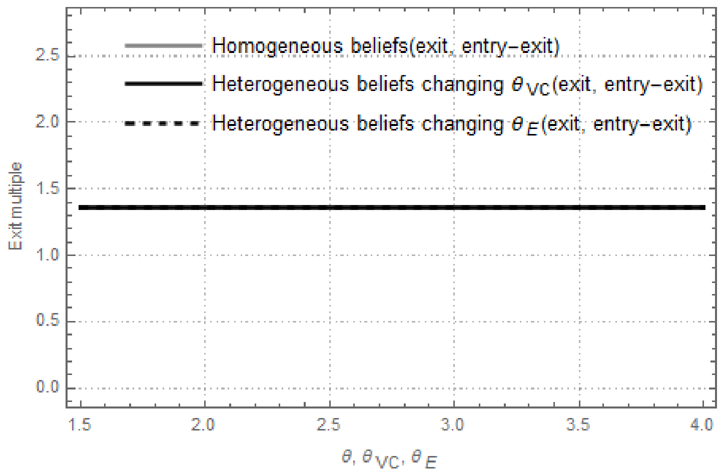

Figure 7 – Computing the implied exit multiple considering the optimal VC triggers for both entering, $V^*{}_{VC-hom}$ and $V^*{}_{VC-het}$, and exiting, $V^*{}_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for changes in the growth prospects, θ, θ_{VC} and θ_E. Keeping all the assumptions taken in Table 2, except for the stressed ones.

The exit multiple for the VC does not has any sensibility to the growth prospects, θ, θ_{VC} and θ_E, in either homogeneous or heterogeneous beliefs, either considering the Entrepreneur or the Venture Capitalist as the most optimist regarding their personal view on growth prospects. For the considered assumptions, the multiple is always of 1.36, representing an upside for the VC of over 35% on the exit.

The picture is quite different when considering the variables that incorporate the beta, β[9], especially the volatility, σ, and the growth rate of the start-up, α, as shown below.

[9] $\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1.$

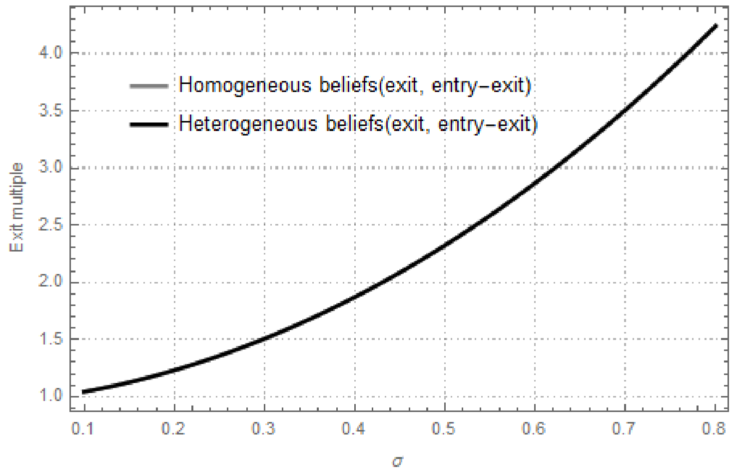

Figure 8 – Computing the implied exit multiple considering the optimal VC triggers for both entering, ${V^*}_{VC-hom}$ and ${V^*}_{VC-het}$, and exiting, ${V^*}_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, levels. Keeping all the assumptions taken in Table 2, except for the stressed ones.

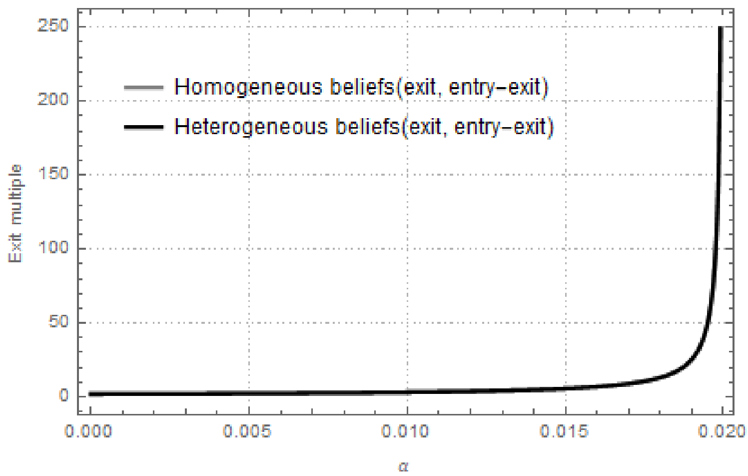

Figure 9 – Computing the implied exit multiple considering the optimal VC triggers for both entering, ${V^*}_{VC-hom}$ and ${V^*}_{VC-het}$, and exiting, ${V^*}_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different growth rates of the start-up, α, levels. Keeping all the assumptions taken in Table 2, except for the stressed ones.

Figures 8 and 9 proves how sensitive the exit multiple is for the volatility, σ, and growth rate of the start-up, α, levels. The behaviour of those variables have an equal sign repercussion on the exit multiple, although with different intensities. The multiple grows exponentially as the volatility increases, $\frac{\partial\ Exit\ multiple}{\partial\ \sigma} > 0$, and the same happens with the increase of the start-up growth rate, $\frac{\partial\ Exit\ multiple}{\partial\ \alpha} > 0$.

Figure 10 allows to observe both behaviours simultaneously with an exit multiple exponentially bigger for higher measures for volatility, σ, and growth rate of the start-up, α.

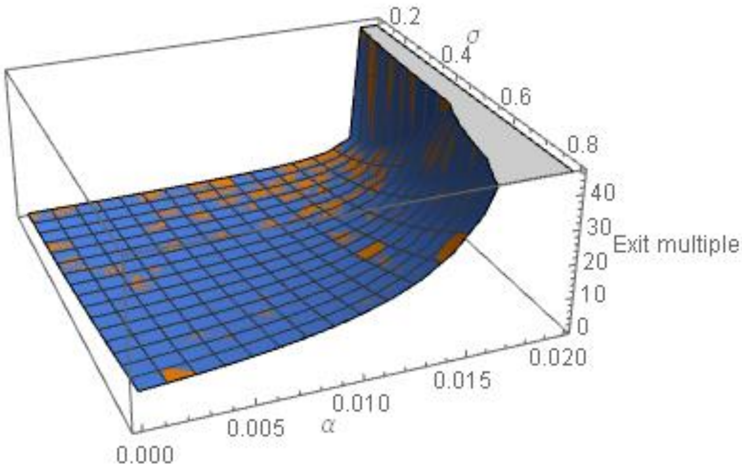

Figure 10 – Computing the implied exit multiple considering the optimal VC triggers for both entering, V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, and growth rates of the start-up, α, levels. Keeping all the assumptions taken in Table 2, except for the stressed ones.

As for the behaviour of the ownership triggers for both homogeneous and heterogeneous beliefs, considering different growth prospects, θ, θ_{VC} and θ_E, the results can be seen in the Figure 11 below.

Considering the assumptions of Table 2, only stressing the growth prospects, homogeneous and heterogeneous beliefs cross each other lines for an optimal ownership, Q^*_{VC}, of 34.38%, when $\theta_E = 2$ and of 45.83% when $\theta_{VC} = 3$.

Consistent with its economic intuition, the optimal ownership required by the VC for the same investment cost decreases as its growth prospects increases against the Entrepreneurs', $\frac{\partial\ V^*_{VC-het}}{\partial\ \theta_{VC}} < 0$. The opposite happens as the Entrepreneurs' growth prospects increases against the ones of the VC, $\frac{\partial\ V^*_{VC-het}}{\partial\ \theta_E} > 0$, and within the homogeneous beliefs setting, $\frac{\partial\ V^*_{VC-hom}}{\partial\ \theta} > 0$.

Unlike the value triggers, the optimal post-money VC ownership in the heterogeneous setting has a crossing point, for $\theta_E = \theta_{VC} = 2.7$, which results in a Q^*_{VC} of 47.3%.

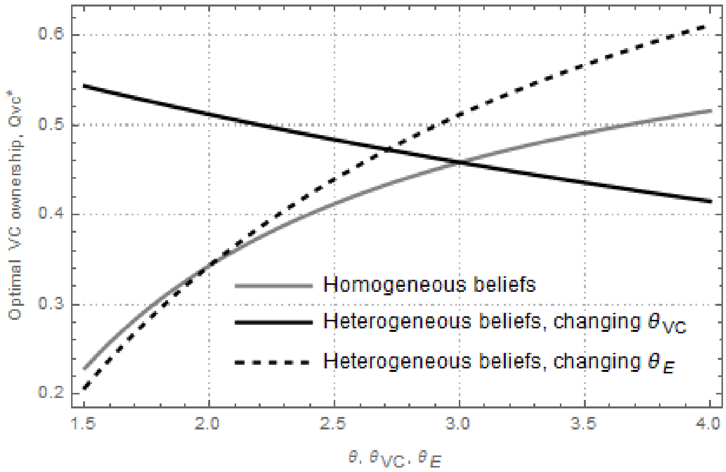

Figure 11 – Computing the VC post-money optimum ownership, Q^*_{VC}, for different growth prospects, θ, θ_{VC} and θ_E, applied to both homogeneous and heterogeneous beliefs. Keeping all the assumptions taken in Table 2, except for the stressed ones.

4. Model Extension - Implementing a Time Restriction

Since VC funds typically last for a predetermined maturity – averaging 10 years, as stated by Feld and Mendelson (2016) –, and the above models considers a perpetual *entry-exit option*, a time constrain must be put in place in order to fully adapt the model to reality.

In order to do so, the time restriction is incorporated through the contribution of Pereira and Rodrigues (2014), considering a short position in a Forward Start Option (FSO) conceptualized upon the one used for certain-lived monopolies under preemption.

Considering a limited maturity project available for T years, this is equivalent to consider a short position in a Forward Start Call Option on the project, i.e. an option that ceases to be available after T, or as soon as the trigger is achieved.

Under the risk neutral expectation the value of the option is:

$$FSO(V(t)) = E[V_{VC-seller}(t)]e^{-r(T-t)} \qquad (23)$$

Where $FSO(V(t))$ is the present value of the option to invest in a limited maturity project and $V_{VC-seller}(t)$ is the value of the seller option to invest at time T. At that moment, the state variable $V(t)$ at T can be either below or above the trigger ${V^*}_{Buyer}$ $(\emptyset^*)$ given in Eq. (8). For the latter case, it will sell the start-up stake in exchange of the present value of future cash flows, plus the premium, $\emptyset$, which is similar to an European call option with maturity T exercised if $\left(V_{Buyer}(\emptyset, t) > {V^*}_{Buyer}(\emptyset^*, T)\right)$.

However, if the firm does not sell at time T, the option expires losing its value.

Proposition 8[10] *The value of the Forward Start Option considering homogeneous beliefs is given by:*

$$FSO(V_{VC-seller}(t)) =$$

$$\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta} N\left(-d3(V_{VC-seller})\right) \qquad (24)$$

[10] Please refer to Appendix 7 for the proofs of the Proposition 8.

Where $N(\cdot)$ is the cumulative normal integral and

$$d3(V_{VC-seller}) = d1(V_{VC-seller}) + (\beta_1 - 1)\sigma\sqrt{T} \qquad (25)$$

where[11]:

$$d1(V_{VC-seller}) = \frac{\ln\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2} \times \frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \qquad (26)$$

The probability distribution $N\left(-d3(V_{VC-seller})\right)$ captures the value of exercising the option to sell the start-up in a later stage (after T) if the trigger $V^*{}_{Buyer}(\emptyset^*)$ is not reached at $\left(V_{Buyer}(\emptyset, t) < V^*{}_{Buyer}(\emptyset^*, T)\right)$.

It is extremely important to notice that, for the purposes of the theoretical challenge embraced, the rational must be inverted, and a short position on the Forward Start Option should be considered in order to limit the Venture Capitalist exposure to a certain investment in a certain start-up for the predetermined maturity of the investment.

4.1. Time Restricted Entry-Exit Option, considering Homogeneous Beliefs

The current setting includes the purchase of a stake in a given start-up to be entered at time t_0, with a planned exit to happen at T, or prior to that if the right premium, $\emptyset$, is payed. For this to happen, it is necessary to combine the option to exit, built upon Lukas and Welling (2012), with a short position in the Forward Start Call Option, adapted from Pereira and Rodrigues (2014), together with model that determines the optimal ownership structure of the investment considering homogeneous and heterogeneous beliefs adapted from Tavares-Gärtner *et. al.* (2018a).

[11] For further details please refer to Pereira and Rodrigues (2014).

At this stage, the combination of both models yield the following equation[12]:

$$F_{hom}(V_{VC-hom}(t)) = Q_{VC}\,\theta\,V + Exit\ Option_{VC} - (K - K_E) - FSO(V(t)) \qquad (27)$$

After rearranging, the equation becomes:

$$F_{hom}(V_{VC-hom}(t)) =$$

$$Q_{VC}\,\theta\,V + \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta} N\big(d3(V_{VC-seller})\big) - (K - K_E),$$

$$for\ t < T \qquad (28)$$

For the entire rational to be complete, it is still needed to account for the value of the option once the time matures without being exercised, when the option becomes worthless. Thus, the Venture Capitalist is pushed to dispose the start-up at T at its market value. When this happens, a net loss arise from the transaction costs, of εC, since the Venture Capitalist has no other options available.

At T, it is assumed that the bargaining power belongs entirely to the buyer and therefore the seller does not benefit from any of the synergies that might exist. Even though, this assumption can and should be relaxed through, for example, a setting where, at T there is the share of synergies but in a smaller proportion, $\emptyset^{*}{}_{t>T} \ll \emptyset^{*}$, thus being one of the suggestions for further research.

As described by Nielsen (1992), the risk-adjusted probability that the option will be exercised is captured by $N\big(d2(V_{VC-seller})\big)$, similar to what happens within the Black and Scholes formula. Given that, by multiplying the present value of the transaction costs by $N\big(-d2(V_{VC-seller})\big)$ the function[13] will capture the probability of the option to expire worthless, thus generating only the net loss whose amount are the transaction costs.

[12] Again, the simplified version of the VC Exit Option is: $F(V_{VC-seller}(\emptyset^{*},t)) = \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}$

[13] The present value of the transaction cost are of $E[\varepsilon C]e^{-rT} = \varepsilon C\ e^{-rT}\ N\big(-d2(V_{VC-seller})\big)$

Hence, considering this scenario of reaching T without selling at the premium $(1+\emptyset)$, and accounting for the probability of this to happens, comes:

Proposition 9 *The optimal timing to entry in the start-up capital in the time restricted homogeneous beliefs setting is given by:*

$$F_{hom}(V_{VC-hom}(t)) =$$

$$Q_{VC}\,\theta\,V + \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta} N\big(d3(V_{VC-seller})\big) - (K-K_E)$$

$$-\varepsilon C\, e^{-rT}\, N\big(-d2(V_{VC-seller})\big) \qquad (29)$$

where:

$$d2(V_{VC-seller}) = d1(V_{VC-seller}) - \sigma\sqrt{T} \qquad (30)$$

The first term of Eq. (29) represents the value the VC gets, $Q_{VC}\,\theta\,V$, in exchange for $(K-K_E)$. The second term represents the time restricted *exit option* of the Venture Capitalist, exercised if $V_{Buyer}\big(\emptyset,(T-t)\big) > V^{*}{}_{Buyer}(\emptyset^{*},(T-t))$. The third term represents the initial investment of the Venture Capitalist, of $(K-K_E)$. The last one accounts for the transaction costs of the Venture Capitalist, which is the only cash-outflow in the case it does not completes the transaction prior to T. This happens since the option lose its value and the VC is forced to dispose the start-up at its market value, V, that represents an inflow, which is offset by the loss of the ownership – representing an outflow of V –, thus having a net loss attached to the transaction due to the fact that has to bear the transaction costs.

Here, the value function for the Entrepreneur keeps unchanged, since the time restriction only apply for the VC's decision to exit the investment:

$$F_{hom}(V_{E-hom}(t)) = (1-Q_{VC})\,\theta\,V - V - K_E \qquad (31)$$

4.2. Time Restricted Entry-Exit Option, considering Heterogeneous Beliefs

For the heterogeneous beliefs model to be also time restricted, the only changes are attached to the expectation on the value added by the investment on the expansion. The Entrepreneur and VC believe, respectively, on an increase in the start-up value of θ_E and θ_{VC}.

The combination of Lukas and Welling (2012), Pereira and Rodrigues (2014) and Tavares-Gärtner *et. al.* (2018a) models yield the following equation[14]:

$$F_{het}(V_{VC-het}(t)) = Q_{VC}\,\theta\,V + Exit\ Option_{VC} - (K - K_E) - FSO(V(t)) \qquad (32)$$

After rearranging, the equation becomes:

$$F_{het}(V_{VC-het}(t)) =$$

$$Q_{VC}\,\theta_{VC}\,V + \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta} N\big(d3(V_{VC-seller})\big) - (K - K_E),$$

$$for\ t < T \qquad (33)$$

Again, for the value of the option considering the possibility of the maturity to expire without the option being exercised, when the Venture Capitalist is pushed to dispose the start-up after T at its market value, only transaction costs, of εC, arise.

Proposition 10 *The optimal timing to entry in the start-up capital in the time restricted heterogeneous beliefs setting is given by:*

$$F_{het}(V_{VC-het}(t)) =$$

$$Q_{VC}\,\theta_{VC}\,V + \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta} N\big(d3(V_{VC-seller})\big) - (K - K_E)$$

$$-\varepsilon C\,e^{-rT}\,N\left(-d2\big({V^*}_{Buyer}(\emptyset^*)\big)\right) \qquad (34)$$

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[14] The VC Exit Option considering heterogeneous beliefs is: $F(V_{VC-seller}(\emptyset^*, t)) = \frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}$

The FSO accounting for heterogeneous beliefs is: $\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta} N\big(-d3(V_{VC-seller})\big)$

As for the value function for the Entrepreneur, it keeps unchanged since the time restriction only applies to the VC's decision to exit the investment:

$$F_{het}(V_{E-het}(t)) = (1 - Q_{VC})\,\theta_E\,V - V - K_E \qquad (35)$$

4.3. Numerical Example

Both models – considering homogeneous and heterogeneous beliefs –, when time restricted, do not yield an analytical result, and therefore their outputs can only be shown and proved through a numerical approach. Given so, the same assumptions used in Table 2 are replicated below, extended by the assumption of the investment maturity, T, which was considered to be of 7 years. This maturity was chosen due to the fact that it is an intermediate value standing above the 5 years of the typical commitment period as per described in Feld and Mendelson (2016) and below the 10 years of the typical investment term also pointed in Feld and Mendelson (2016) and already explained in the literature review.

Key numerical assumptions		
Variable	Value	Description
γ	15%	Synergies of the Buyer
ε	50%	Proportion of exit transaction costs supported by the VC
C	80	Transaction costs of the exit
K	800	Total investment needed to finance the growth opportunity
K_E	250	Entrepreneur's own funds
α	-3%	Growth rate of the start-up
r	2%	Riskless interest rate
σ	25%	Volatility
θ	2	Growth prospects - homogeneous beliefs setting
θ_{VC}	2	Growth prospects of the VC - heterogeneous beliefs setting
θ_E	3	Growth prospects of the Entrepreneur - heterogeneous beliefs setting
T	7	Investment maturity

Table 3 – Key numerical assumptions, time-restricted model.

The numerical example below compares the perpetual and time-restricted models, in order to assess the impact time has in the Venture Capitalist's decisions. The point keeps being the alignment of Entrepreneur and VC, determining the entry-exit value triggers as well as the optimum VC ownership – which can be approached in two different ways –, and the behaviour of the exit multiple stressing the investment maturity.

Figure 12 shows the behaviour of the time-restricted entry-exit trigger against the perpetual one. The time-restricted entry-exit triggers have the particularity of being calculated keeping constant the ownership required by the VC, to be considered the static one, for the considered invest cost – and this is the one resulting from the perpetual model, respectively equations (14) and (21) for homogeneous and heterogeneous beliefs.

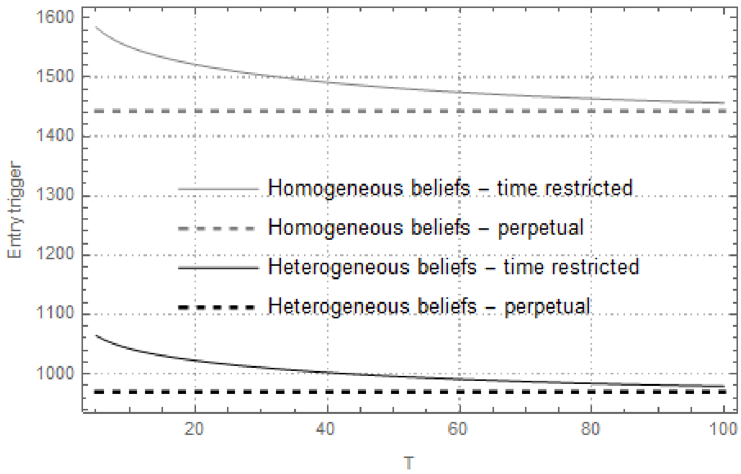

Figure 12 – Computing the optimal VC entry trigger, V^*_{VC-hom} and V^*_{VC-het}, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time-restricted models, for different maturities, T, considering the static VC post-money optimum ownership, Q^*_{VC}, within the time-restricted model. Keeping all the assumptions taken in Table 3, except for the stressed ones.

As intuitively expected, keeping static the ownership required by the VC increases the value trigger in the time-restricted setting when compared to the perpetual model. The shorter the investment maturity considered the more pronounced the difference among the triggers becomes. On the other hand, both models become closer from each other while the investment maturity increases, which is consistent with the economic intuition behind it.

The functions explicit in Figure 13 has a different approach, dynamically determining the optimal VC ownership that align both the perpetual and time-restricted models entry-exit value triggers for any given investment maturity within the time-restricted approach.

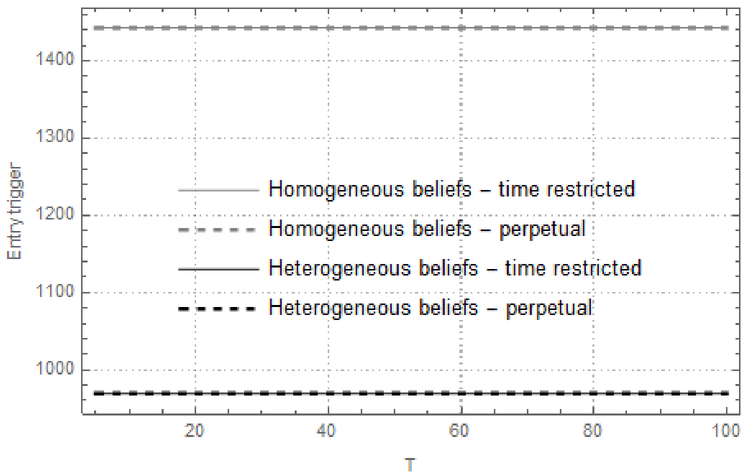

Figure 13 – Computing the optimal VC entry trigger, ${V^*}_{VC-hom}$ and ${V^*}_{VC-het}$, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time-restricted models, for different maturities, T, considering a dynamic approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in Table 3, except for the stressed ones.

Those optimal VC ownerships, determined using the dynamic approach to the time-restricted model allowing both models to be align for any investment maturity considered, are shown in Figure 14.

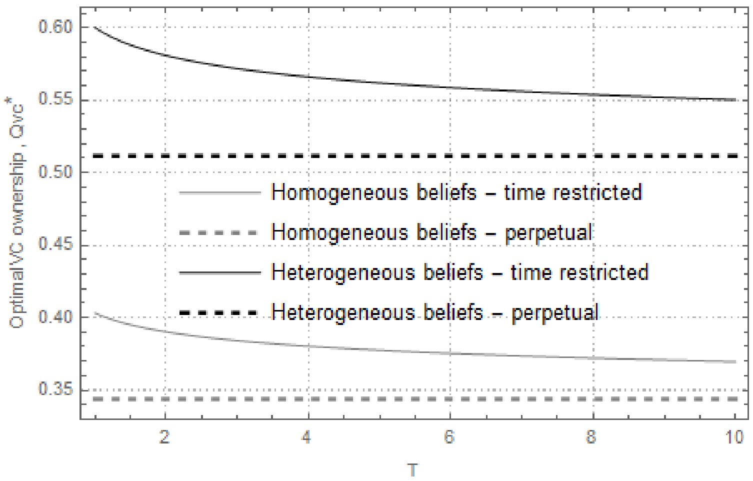

Figure 14 – Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual (static) and time-restricted (dynamic) models, for different maturities, T. Keeping all the assumptions taken in Table 3, except for the stressed ones.

The economic intuition behind this results are quite similar to the one reached within the analysis to Figure 12 whereas it is clear that for an investment which is time-restricted, either the entry-exit value trigger increases while the maturity shortens for a static VC growth prospect ownership, or the entry-exit value trigger might be kept constant and aligned with the perpetual model, but as time shortens the optimal ownership has to increase.

Figure 15 shows only the convergence of the time-restricted model to the perpetual one, using a maturity up until 100 years, which, although unrealistic, proves how impactful a time restriction might be in such an industry as Venture Capital.

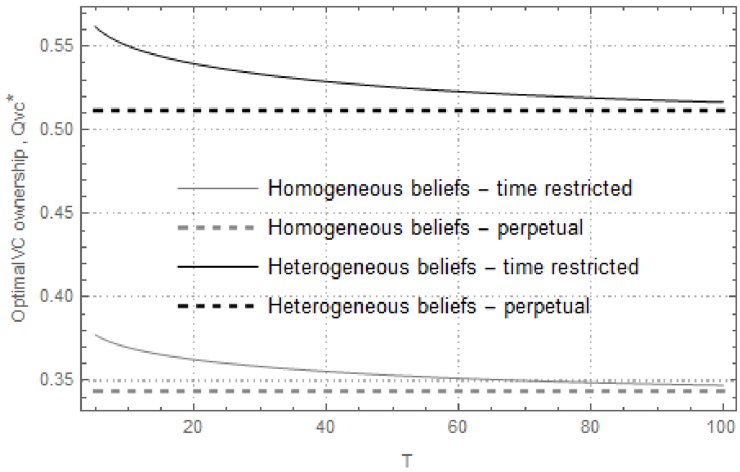

Figure 15 – Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual (static) and time-restricted (dynamic) models, for different and higher maturities, T, than those of Figure 14. Keeping all the assumptions taken in Table 3, except for the stressed ones.

As for the impact of different growth prospects, θ, θ_{VC} and θ_E, on the entry-exit value triggers for both homogeneous and heterogeneous beliefs, Figure 16 shows that each of the lines cross each other for an entry-exit trigger, V^*, of 1442.5, when $\theta_E = 2$ and of 721.3 when $\theta_{VC} = 3$. The present results, as Figure 13 helps to determine, are the same as if no time-restriction was in place, since both triggers are aligned through the increase in the required optimal ownership.

Keeping constant the optimal ownership determined in the perpetual model, the entry-exit trigger, V^*, would become 1568.0, when $\theta_E = 2$ and 784.0 when $\theta_{VC} = 3$, representing an increase of respectively 125.5 or 8,7% and 62.7 or 8.7% for $\theta_E = 2$ and $\theta_{VC} = 3$.

With the same behaviour as when the perpetual model applies, the derivatives of the value function for the VC in regard to every growth prospects are negative, $\frac{\partial V^*_{VC-hom}}{\partial \theta} < 0$, $\frac{\partial V^*_{VC-het}}{\partial \theta_{VC}} < 0$ and $\frac{\partial V^*_{VC-het}}{\partial \theta_E} < 0$, on the right economic intuition that the smaller the growth prospects, the bigger opportunity cost, and therefore the higher the entry trigger, since the VC's upside potential is reduced.

Again, the chart behaviour is strongly influenced by the base case assumptions on the growth prospects and that is why, as explained on the analysis of Figure 4, the crossing points are $\theta_E = 2$ and $\theta_{VC} = 3$, and the homogeneous beliefs curve yields a higher trigger when compared to any of the heterogeneous beliefs setting when $\theta < 2$, $\theta_{VC} < 2$ and $\theta_E < 2$, and a lower trigger when θ>3, $\theta_{VC} > 3$ and $\theta_E > 3$.

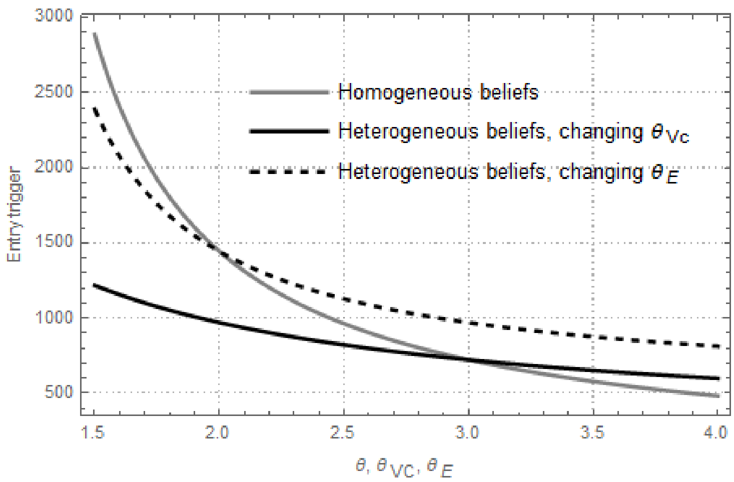

Figure 16 – Computing the optimal VC entry trigger, V^*_{VC-hom} and V^*_{VC-het}, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time-restricted models, for different growth prospects, θ, θ_{VC} and θ_E, considering a dynamic approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in Table 3, except for the stressed ones.

Figure 17 presents the behaviour of the ownership triggers for both homogeneous and heterogeneous beliefs, considering different growth prospects, θ, θ_{VC} and θ_E, for the investment maturity of 7 years.

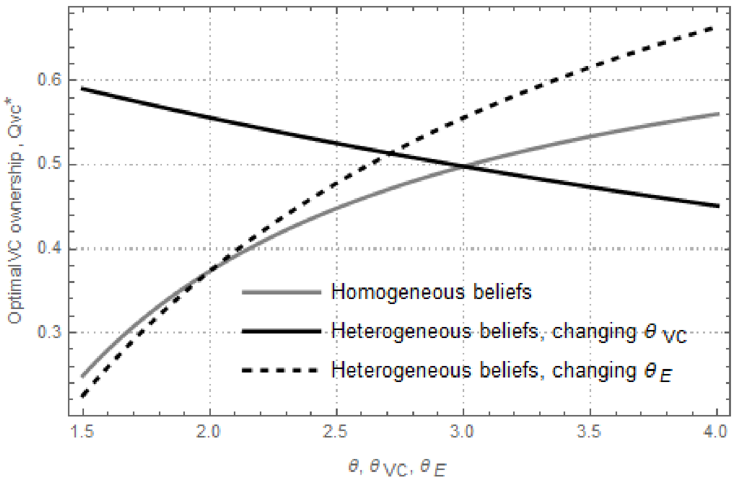

Figure 17 – Computing the VC post-money optimum ownership, in a setting of homogeneous and heterogeneous beliefs, applied to both the perpetual and time-restricted models, for different growth prospects, θ, θ_{VC} and θ_E. Keeping all the assumptions taken in Table 3, except for the stressed ones.

Homogeneous and heterogeneous beliefs cross each other lines for an optimal ownership, ${Q^*}_{VC}$, of 37.4%, when $\theta_E = 2$ and of 49.8% when $\theta_{VC} = 3$. These results, when compared with the perpetual model numerical analysis, given an optimal ownership, ${Q^*}_{VC}$, of 34.4%, when $\theta_E = 2$ and of 45.8% when $\theta_{VC} = 3$, represents an increase of the required VC ownership of 3.0 p.p., when $\theta_E = 2$ and of 4.0 p.p. when $\theta_{VC} = 3$.

Heterogeneous beliefs curves cross each other for $\theta_E = \theta_{VC} = 2.7$, at a ${Q^*}_{VC}$ of 51.4%, which is the exact same combination of growth prospects where heterogeneous lines cross each other within the perpetual model, but for a ${Q^*}_{VC}$ of 47.3%, representing an increase of 4.1 p.p.

As economically intuitive – and already explained within the analysis of the perpetual model –, the optimal ownership required by the VC – for the same investment cost – decreases as its growth prospects increases against the Entrepreneurs', $\frac{\partial\ {Q^*}_{VC-het}}{\partial\ \theta_{VC}} < 0$. The opposite happens as the Entrepreneurs' growth prospects increases against the ones of the VC, $\frac{\partial\ {Q^*}_{VC-het}}{\partial\ \theta_E} > 0$, and within the homogeneous beliefs setting, $\frac{\partial\ {Q^*}_{VC-hom}}{\partial\ \theta} > 0$.

For the determination of the exit multiple for the VC in the time restricted setting, the exit perpetual option – because it does not rely on time to be determined – has to be divided by the time restricted *entry-exit option*. Since the time restricted *entry-exit option* can be determined holding static ${Q^*}_{VC}$ thus adjusting V^*, or through a dynamic ${Q^*}_{VC}$ for which V^* yields the same result as in the perpetual setting, both scenarios are considered in Figure 18.

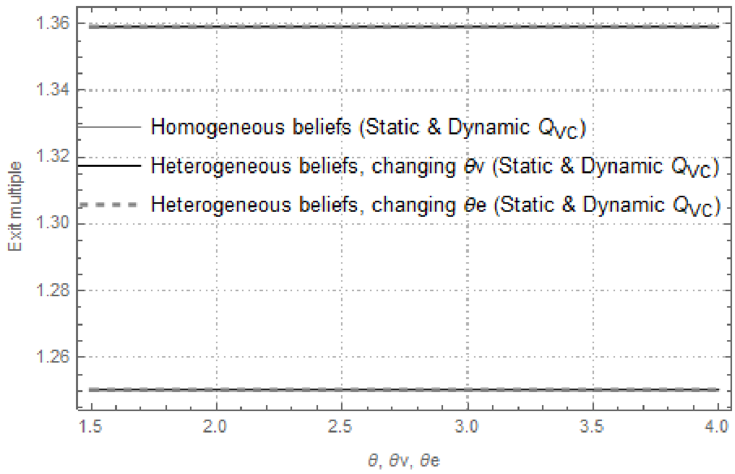

Figure 18 – Computing the implied exit multiple considering the optimal VC triggers for both entering, ${V^*}_{VC-hom}$ and ${V^*}_{VC-het}$, and exiting, ${V^*}_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for changes in the growth prospects, θ, θ_{VC} and θ_E, considering a dynamic approach for the optimal ownership structure within the time-restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in Table 3, except for the stressed ones.

The multiple, as in the perpetual setting, does not change for different growth prospects neither for homogeneous nor heterogeneous beliefs considering different growth prospects for either the Entrepreneur or Venture Capitalist. However, the case is different when considering a static or dynamic approach for ${Q^*}_{VC}$. As intuitively extrapolated, since the dynamic ${Q^*}_{VC}$ is designed for V^* to match the perpetual results, the multiple resulting from this approach yields the precise same multiple as the perpetual model, of 1,36. As for the multiple arising from the static ${Q^*}_{VC}$ – the one resulting from the perpetual model – the multiple for an investment maturity of 7 years is reduced to 1,2504.

Knowing from the analysis performed to the perpetual model that volatility is the parameter to which the multiple is sensible the most, Figure 19 shows how sensitive to volatility both multiples are – considering a static and dynamic approach for ${Q^*}_{VC}$.

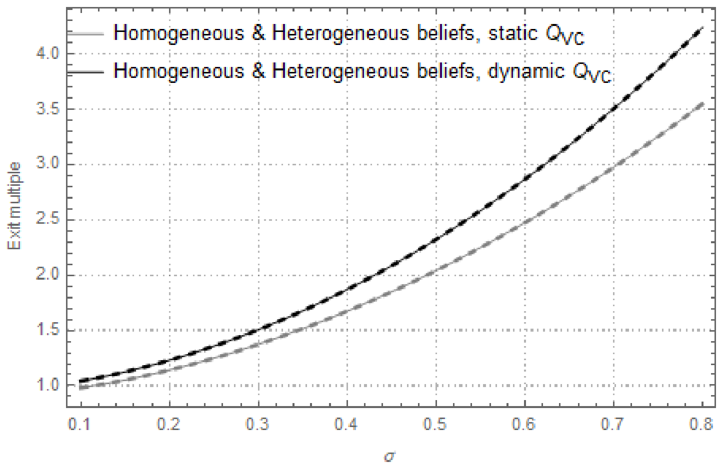

Figure 19 – Computing the implied exit multiple considering the optimal VC triggers for both entering, V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, levels, considering a dynamic approach for the optimal ownership structure within the time-restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in Table 3, except for the stressed ones.

From Figure 19 it is possible to already have a hint on what behaviour the multiple will have when stressing the investment maturity. The dynamic approach to Q^*_{VC} is the one yielding the biggest multiple for every level of volatility – which is the same as the perpetual model, once again –, and the gap for the static Q^*_{VC} gets bigger as the levels of volatilities increase.

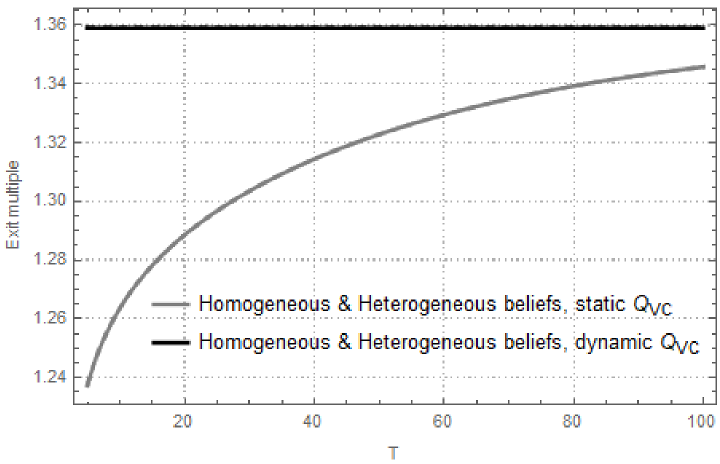

Figure 20 – Computing the implied exit multiple considering the optimal VC triggers for both entering, V^*_{VC-hom} and V^*_{VC-het}, and exiting, $V^*_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different investment maturities, T, levels, considering a dynamic approach for the optimal ownership structure within the time-restricted model in the upper line and a static approach for the optimal ownership structure within the time-restricted model in the lower line. Keeping all the assumptions taken in Table 3, except for the stressed ones.

As Figure 19 left behind the curtain, the exit multiple yielded when used the dynamic approach to ${Q^*}_{VC}$ is the same as the perpetual model thus being static for every investment maturity. This happens due to the fact that ${Q^*}_{VC}$ is the variable that changes in order for V^* to be kept at the perpetual model level.

The exit multiple computed using the perpetual model ${Q^*}_{VC}$ – the static one –, yields, for increasing values of the investment maturity, results which are closer from the multiple calculated in the perpetual model.

The results shows that the shorter the investment maturity, the smaller the exit multiple. This is consistent with its economic intuition, since the entry triggers are higher for lower investment maturities, thus reducing the exit multiple since the exit trigger is hold constant due to its independency from the investment maturity – being driven by the synergies of the buyer, and the share of those synergies with the Venture Capitalist.

Figure 21 allows to observe both behaviours simultaneously with an exit multiple exponentially bigger for higher measures for volatility, σ, and lower for smaller investment maturities in a converging path to the perpetual model results keeping everything else constant.

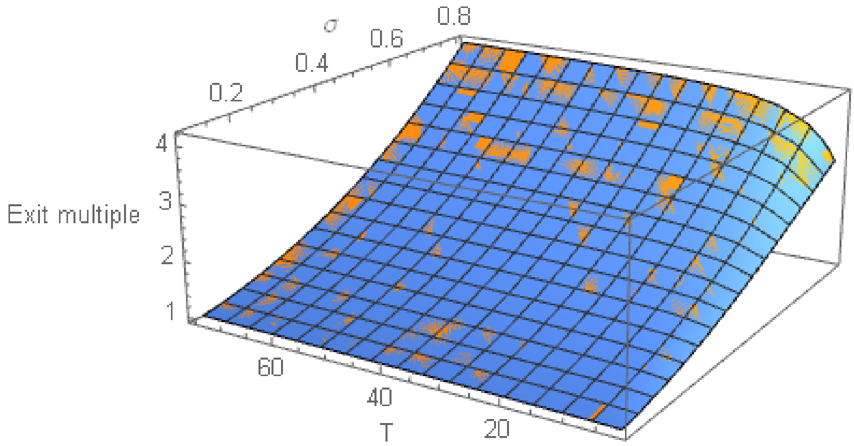

Figure 21 – Computing the implied exit multiple considering the optimal VC triggers for both entering, ${V^*}_{VC-hom}$ and ${V^*}_{VC-het}$, and exiting, ${V^*}_{Buyer}(\emptyset^*)$, the investment, applied to both homogeneous and heterogeneous beliefs, for different volatility, σ, and investment maturity, T, levels, considering a static approach for the optimal ownership structure within the time-restricted model. Keeping all the assumptions taken in Table 3, except for the stressed ones.

5. Future Research

The current thesis is focused on analysing the overall Venture Capitalist investment decision, from the first decision to invest until the investment exit, considering an optimal outcome on the exit deal, through the share of the buyer synergies, but also extended in a setting that allow the model to fit reality the best possible way through the implementation of, perhaps, the most important variable – other than risk – which is time. Nevertheless, many other important considerations should be implemented in order to achieve a practical and fully applicable model to today VC specificities. Some of those specificities include:

- Relax the assumption about the perpetual entry time, thus applying a time-restriction to the decision to invest would bring meaningful conclusions to the table, since the current setting considers an infinite time to wait for the optimum timing to invest. Venture Capital firms face severe competition to find the best start-ups to invest as sooner as possible, to maximize their upside potential, so time plays a very important role even to enter.
- The case for ${\emptyset^*}_{t>T} \ll \emptyset^*$. The time restriction in the present model is too strict regarding the timing to exit, considering a now or never approach to sell the start-up at the optimal premium – if reached before the investment maturity –, or lose the option value. A refreshed setup whereas after the investment maturity the share of synergies assume a lower premium for the seller is a plausible hypothesis. The time restriction VC face are legally extendable for shorter periods, as stated by Feld and Mendelson (2016), giving plausibility for this idea.
- Consider a "peak behaviour" of the buyer when the maturity is closer to the end. For $t \geq T$, or when t is really closer to T, many games can be played in order to bargain the most on purchase and selling negotiations. One plausible extension is to consider that the closer the VC is to the investment maturity, the bigger is the buyer bargaining power on the purchasing negotiation and consequently the smaller is the share of the total synergies.
- Account for further financing rounds within the investment maturity, accounting for the dilution effect is one of the most likely events both VCs and Entrepreneurs face, being an exceptional upgrade to consider in the model as well. This is something quite usual during the lifetime of start-up, whereas many financing rounds take place until the company either goes public through an IPO, or acquired by another player.

6. Conclusions

The present work envisage the entire investment decision of a Venture Capitalist when considering to invest in a star-up firm. Since VCs face high risk when investing, and the maturity of this type of investment are typically short, it is important to conceptualize, sooner than latter, which exit multiple to require, determining the optimal amount to pay and optimal ownership, so then the start-up can be disposed for the optimal share of the synergies a given buyer is expected to have. This implemented setting allowed for some interesting conclusions on the economic intuitions behind the decisions a VC usually takes.

Firstly, it is important to mention that although the model assumes one only Venture Capitalist investing in a seed round of a certain star-up, this assumption, even if relaxed, does not impact the results since the typical behaviour when more than a Venture Capitalist is to invest is to recognize one as the lead investor, which centralize the decisions and negotiation process thus smoothing the investment decision.

Curiously, the entry decision the Venture Capitalist takes when assessing a start-up is not affected by its exit in a perpetual setting. The economic intuition attached to this conclusion is that without a time restriction to enter in the venture the Venture Capitalist can wait until the optimum timing for the exit to arrive and it does not matter how far in time that optimum will arrive. This means that the perpetual model yield results very similar to the ones achieved by Tavares-Gärtner et. al. (2018a), whose only differences lie on the modelling methodology used since the stochastic variable used by the authors was the start-up cash flows whereas the present model considers the start-up value.

The previous results change a lot when the time restriction is put in place, where the exit decision does impact the entrance level. Generically, the intuition would be an increasing in the entry trigger as shorter the investment maturity becomes, but this only happens keeping constant the optimal post-money ownership. For the model also considers the possibility of keeping the entry triggers unchanged thus increasing the required optimal post-money ownership for decreasing investment maturities, which is consistent with its economic intuition.

When stressing the growth prospects either for the perpetual and for the time restricted models, the intuition is both a decreasing entry triggers and exit triggers for increasing growth prospects and the other way around. For the optimal VC post-money ownership, it increases for higher growth prospects within the homogeneous beliefs setting, and also for increasing growth prospects of the Entrepreneur against a static VC's growth prospects. On the other hand, for increasing VC's growth prospects against static Entrepreneur's growth prospects – typically less common –, the VC post-money ownership decreases.

As for the exit multiple analysis, the variable to which this indicator is most sensible to is the volatility, with a positive correlation. A statement which is true in both the perpetual and time restricted models. The exit multiple increases exponentially with higher levels of volatility, and that is also truth for the growth rates of the start-up. An interesting result comes with the time restriction setting, since the multiple, as economically intuitive, decreases with shorter maturities, which can be explained by a decrease in the opportunity cost due to an expected shorter investment exposure, and also due to the fact that the entry trigger is also bigger therefore reducing the upside potential since the exit trigger is hold constant.

The current model, although producing very interesting results and confirming economic intuitions and dynamics associated to the Venture Capitalist's investment decision, has to be continuously upgraded in order to increase its fit to reality and in order to be fully regarded as a hands on tool for investment professionals, where the future research topics are considered useful tips in this regard.

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Appendices

Appendix 1

The following expression, Eq. (1), can also be solved through the contingent-claim approach:

$$F\left(V_{Buyer}\right) = \max_{t} E\left[\left((\gamma - \emptyset)\, Q_{VC}\, \theta\, V_{Buyer}(t) - (1-\varepsilon)C\right) e^{-rt}\right] \qquad (A1.1)$$

Based on Dixit and Pindyck (1994), where the value of the option held by the buyer to invest in the start-up owned by the VC, must satisfy the following ordinary differential equation (ODE)[15]:

$$\frac{1}{2}\sigma^2 V^2 F''(V) - (r-\delta)VF'(V) - rF(V) = 0 \qquad (A1.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F\left(V_{Buyer}\right) = A_1{}^{\beta 1} + A_2{}^{\beta 2} \qquad (A1.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta 1$ and $\beta 2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \qquad (A1.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A1.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A1.6)$$

[15] In rigorous terms this ODE is valid for perpetual options, where there is no time decay, and so the value of the option to invest in not a function of time. However, we can say that, for long-lived options, the passing time (dt) only slightly approaches the option to its maturity, and so it can be assumed that ∂F (V,t) ∂t ≈ 0. For this reason, and for convenience, the impact of time on the value of long-lived investment opportunity can be, in this context, ignored

In addition, $F(V_{Buyer})$must satisfy the following boundary conditions:

$$\lim_{V_{Buyer} \to 0} F(V_{Buyer}) = 0 \qquad (A1.7)$$

$$\lim_{V_{Buyer} \to V^*_{Buyer}} F(V_{Buyer}) = (\gamma - \emptyset)\, Q_{VC}\, \theta\, V_{Buyer} - (1 - \varepsilon)C \qquad (A1.8)$$

$$\lim_{V_{Buyer} \to V^*_{Buyer}} F'(V_B) = (\gamma - \emptyset)\, Q_{VC}\, \theta \qquad (A1.9)$$

Condition $F(0) = 0$ arises from the observation that if V_{Buyer} goes to zero, it will stay at zero (this is an implication of the stochastic process for V_{Buyer}). Therefore, the option will be of no value when $V = 0$. The other two conditions come from consideration of optimal investment. ${V_{Buyer}}^*$ is the price at which it is optimal to invest, then $F({V_{Buyer}}^*) = (\gamma - \emptyset)\, Q_{VC}\, \theta\, {V_{Buyer}}^*(t) - (1 - \varepsilon)C$ is the value-matching condition: it just says that upon investing, the firm receives a net payoff $(\gamma - \emptyset) Q_{VC}\, \theta\, {V_{Buyer}}^*(t) - (1 - \varepsilon)C$. Finally, condition $F'({V_B}^*) = (\gamma - \emptyset)\, Q_{VC}\, \theta$ is the "smooth-pasting" condition. If $F(V_{Buyer})$ were not continuous and smooth at the critical exercise point,${V_{Buyer}}^*$, one could do better by exercising at a different point.

To find $F(V_{Buyer})$, we must solve equation $\frac{1}{2}\sigma^2 V^2 F''(V) - (r - \delta)VF'(V) - rF(V) = 0$ subject to the boundary conditions. In order to respect the first boundary condition, and realizing that $\lim_{V \to 0} {A_2}^{\beta 2} = \infty$ (because $\beta 2 < 0$), A_2 must be set equal to zero and the solution must take the form: $F(V) = A_1 V^{\beta 1}$. Where A_1 is a constant that is yet to be determined, and $\beta 1$> 1 is a known constant whose value depends on the parameters δ, σ and r of the differential equation.

$$F(V_{Buyer}) = \begin{cases} \left((\gamma - \emptyset)\, Q_{VC}\, \theta\, V^*_{Buyer} - (1 - \varepsilon)C\right)\left(\dfrac{V}{{V_{Buyer}}^*}\right)^{\beta}, for\ V_{Buyer} < V^*_{Buyer} \\ (\gamma - \emptyset) Q_{VC}\, \theta\, V_{Buyer}(t) - (1 - \varepsilon)C\,, \qquad for\ V_{Buyer} \geq V^*_{Buyer} \end{cases} \qquad (A1.10)$$

Appendix 2

The following expression, Eq. (4), can also be solved through the contingent-claim approach:

$$F(V_{VC-seller}) =$$

$$\max_t E\left[(\emptyset\ Q_{VC}\ \theta\ \ V^*{}_B - \varepsilon C)e^{-rt^*}\right],\ \ V^*{}_{Buyer} = \frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\ \theta\ (\gamma - \emptyset)} \qquad (A2.1)$$

Based on Dixit and Pindyck (1994), where the value of the option held by the Venture Capitalist (seller) to dispose the start-up, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 V^2 F''(V) - (r-\delta)VF'(V) - rF(V) = 0 \qquad (A2.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F(V_{VC-seller}) = A_1{}^{\beta 1} + A_2{}^{\beta 2} \qquad (A2.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta 1$ and $\beta 2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \qquad (A2.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A2.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A2.6)$$

In addition, $F(V_{VC-seller})$ must satisfy the following boundary conditions:

$$\lim_{VC-seller \to 0} F(V_{VC-seller}) = 0 \qquad (A2.7)$$

$$\lim_{V_{VC-seller} \to V^*{}_{VC-seller}} F(V_{VC-seller}) = \emptyset\, Q_{VC}\, \theta \left(\frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\, \theta\, (\gamma - \emptyset)} \right) - \varepsilon C$$

$$(A2.8)$$

$$\lim_{V_{VC-seller} \to V^*{}_{VC-seller}} F'(V_{VC-seller}) = \left. \frac{\partial F(V_{VC-seller})}{\partial V_{VC-seller}} \right|_{V_{VC-seller} = V^*{}_{VC-seller}} \qquad (A2.9)$$

Following the standard procedures, the function $F(V_{VC-seller})$ comes as follows:

$$F(V_{VC-seller}) =$$

$$\begin{cases} \left(\emptyset\, Q_{VC}\, \theta \left(\frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\, \theta\, (\gamma - \emptyset)} \right) - \varepsilon C \right) \left(\frac{V_{VC-seller}}{\left(\frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\, \theta\, (\gamma - \emptyset)} \right)} \right)^{\beta}, \\ \qquad for\ V_{VC-seller} < V^*{}_{VC-seller} \\ \emptyset\, Q_{VC}\, \theta \left(\frac{\beta}{\beta - 1} \times \frac{(1-\varepsilon)C}{Q_{VC}\, \theta\, (\gamma - \emptyset)} \right) - \varepsilon C, \\ \qquad for\ V_{VC-seller} \geq V^*{}_{VC-seller} \end{cases} \qquad (A2.10)$$

Appendix 3

The following expression, Eq. (10), can also be solved through the contingent-claim approach:

$$F(V_{VC-hom}) = Q_{VC}\,\theta\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}\right) - (K-K_E)$$

$$(A3.1)$$

Based on Dixit and Pindyck (1994), where the value of the entry-option held by the Venture Capitalist to support the growth prospect in a certain proportion of the total investment amount and in an homogeneous beliefs setting, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2V^2F''(V) - (r-\delta)VF'(V) - rF(V) = 0 \qquad (A3.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F(V_{VC-hom}) = A_1{}^{\beta 1} + A_2{}^{\beta 2} \qquad (A3.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta 1$ and $\beta 2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \qquad (A3.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A3.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A3.6)$$

In addition, $F(V_{VC})$ must satisfy the following boundary conditions:

$$\lim_{V_{VC-hom}\to 0} F(V_{VC-hom}) = 0 \qquad (A3.7)$$

$$\lim_{V_{VC-hom}\to V^*_{VC-hom}} F(V_{VC}) = Q_{VC}\,\theta\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}\right) - (K-K_E) \qquad (A3.8)$$

$$\lim_{V_{VC-hom}\to V^*_{VC-hom}} F'(V_{VC-hom}) = \left.\frac{\partial F(V_{VC-hom})}{\partial V_{VC-hom}}\right|_{V_{VC-hom}=V^*_{VC-hom}} \qquad (A3.9)$$

Following the standard procedures, the function $F(V_{VC-hom})$ comes as follows:

$$F(V_{VC-hom}) = \begin{cases} Q_{VC}\,\theta\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}\right) - (K-K_E)\left(\frac{V}{V^*_{VC-hom}}\right)^{\beta}, \\ \qquad for\ V_{VC-hom} < V^*_{VC-hom} \\ Q_{VC}\,\theta\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}\right) - (K-K_E), \\ \qquad for\ V_{VC-hom} \geq V^*_{VC-hom} \end{cases} \qquad (A3.10)$$

Appendix 4

The following expression, Eq. (12), can also be solved through the contingent-claim approach:

$$F(V_{E-hom}) = (1 - Q_{VC})\,\theta\,V - V - K_E \qquad (A4.1)$$

Based on Dixit and Pindyck (1994), where the value of the option held by the entrepreneur to invest in the growth prospect in an homogeneous beliefs setting, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 V^2 F''(V) - (r - \delta)VF'(V) - rF(V) = 0 \qquad (A4.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F(V_{E-hom}) = {A_1}^{\beta 1} + {A_2}^{\beta 2} \qquad (A4.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta 1$ and $\beta 2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r \qquad (A4.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A4.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} - \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A4.6)$$

In addition, $F(V_{E-hom})$ must satisfy the following boundary conditions:

$$\lim_{V_{E-hom}\to 0} F(V_{E-hom}) = 0 \qquad (A4.7)$$

$$\lim_{V_{E-hom}\to {V^*}_{E-hom}} F(V_{E-hom}) = (1 - Q_{VC})\,\theta\, V - V - K_E \qquad (A4.8)$$

$$\lim_{V_{E-hom}\to {V^*}_{E-hom}} F'(V_{E-hom}) = \left.\frac{\partial F(V_{E-hom})}{\partial V_{E-hom}}\right|_{V_{E-hom}={V^*}_{E-hom}} \qquad (A4.9)$$

Following the standard procedures, the function $F(V_{E-hom})$ comes as follows:

$$F(V_{E-hom}) =$$

$$\begin{cases} (1 - Q_{VC})\,\theta\, V - V - K_E \left(\dfrac{V}{{V^*}_{E-hom}}\right)^{\beta}, for\ V_{E-hom} < {V^*}_{E-hom} \\ (1 - Q_{VC})\,\theta\, V - V - K_E, \qquad for\ V_{E-hom} \geq {V^*}_{E-hom} \end{cases} \qquad (A4.10)$$

Appendix 5

The following expression, Eq. (17), can also be solved through the contingent-claim approach:

$$F(V_{VC-het}) = Q_{VC}\,\theta_{VC}\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}\right) - (K-K_E)$$

$$(A5.1)$$

Based on Dixit and Pindyck (1994), where the value of the entry-option held by the Venture Capitalist to support the growth prospect in a certain proportion of the total investment amount and in an heterogeneous beliefs setting, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2V^2F''(V) - (r-\delta)VF'(V) - rF(V) = 0 \qquad (A5.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F(V_{VC-het}) = A_1{}^{\beta1} + A_2{}^{\beta2} \qquad (A5.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta1$ and $\beta2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r \qquad (A5.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A5.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A5.6)$$

In addition, $F(V_{VC-het})$ must satisfy the following boundary conditions:

$$\lim_{V_{VC-het}\to 0} F(V_{VC-het}) = 0 \qquad (A5.7)$$

$$\lim_{V_{VC-het}\to V^*{}_{VC-het}} F(V_{VC-het}) = Q_{VC}\,\theta_{VC}\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}\right) - (K-K_E) \qquad (A5.8)$$

$$\lim_{V_{VC-het}\to V^*{}_{VC-het}} F'(V_{VC-het}) = \left.\frac{\partial F(V_{VC-het})}{\partial V_{VC-het}}\right|_{V_{VC-het}=V^*{}_{VC-het}} \qquad (A5.9)$$

Following the standard procedures, the function $F(V_{VC-het})$ comes as follows:

$$F(V_{VC-het}) = \begin{cases} Q_{VC}\,\theta_{VC}\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}\right) - (K-K_E)\left(\frac{V}{V^*{}_{VC-het}}\right)^{\beta}, & for\ V_{VC-het} < V^*{}_{VC-het} \\ Q_{VC}\,\theta_{VC}\,V + \left(\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta_{VC}}\right)}\right)^{\beta}\right) - (K-K_E), & for\ V_{VC-het} \geq V^*{}_{VC-het} \end{cases} \qquad (A5.10)$$

Appendix 6

The following expression, Eq. (19), can also be solved through the contingent-claim approach:

$$F(V_{E-het}) = (1 - Q_{VC})\,\theta_E\, V - V - K_E \qquad (A6.1)$$

Based on Dixit and Pindyck (1994), where the value of the option held by the entrepreneur to invest in the growth prospect in an heterogeneous beliefs setting, must satisfy the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 V^2 F''(V) - (r - \delta)VF'(V) - rF(V) = 0 \qquad (A6.2)$$

Subject to some conditions that must be imposed, in order to obtain the appropriate solution. The general solution for this ODE is well known and takes the form:

$$F(V_{E-het}) = {A_1}^{\beta 1} + {A_2}^{\beta 2} \qquad (A6.3)$$

where A_1 and A_2 are arbitrary constants that need to be determined, and $\beta 1$ and $\beta 2$ are the roots of the fundamental quadratic:

$$Q(\beta) = \frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r \qquad (A6.4)$$

i.e.:

$$\beta_1 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A6.5)$$

$$\beta_2 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} - \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \qquad (A6.6)$$

In addition, $F(V_{E-het})$ must satisfy the following boundary conditions:

$$\lim_{V_{E-het}\to 0} F(V_{E-het}) = 0 \qquad (A6.7)$$

$$\lim_{V_{E-het}\to V^*{}_{E-het}} F(V_{E-het}) = (1-Q_{VC})\,\theta_E\,V - V - K_E \qquad (A6.8)$$

$$\lim_{V_{E-het}\to V^*{}_{E-het}} F'(V_{E-het}) = \left.\frac{\partial F(V_{E-het})}{\partial V_{E-het}}\right|_{V_{E-het}=V^*{}_{E-het}} \qquad (A6.9)$$

Following the standard procedures, the function $F(V_{E-het})$ comes as follows:

$$F(V_{E-het}) = \begin{cases} (1-Q_{VC})\,\theta_E\,V - V - K_E\left(\dfrac{V}{V^*{}_{E-het}}\right)^{\beta}, \\ \qquad for\ V_{E-het} < V^*{}_{E-het} \\ (1-Q_{VC})\,\theta_E\,V - V - K_E, \\ \qquad for\ V_{E-het} \geq V^*{}_{E-het} \end{cases} \qquad (A6.10)$$

Appendix 7 – Proof of Proposition 8

In order to derive the value of the Forward Start Option it is needed to discount the expected risk-neutral value of the seller:

$$FSO(V(t)) = E[V_{VC-seller}]e^{-r(T-t)} \qquad (A7.1)$$

Where $FSO(V(t))$ is the present value of the option to invest in a limited maturity project and $V_{VC-seller}(t)$ is the value of the seller's option to invest at time T

This Forward Start Option follows the rational of Pereira and Rodrigues (2014), inspired in Shackleton and Wojakowski (2007, p.3849–3850), being the $FSO(V(t))$ an asset-or-nothing call option on $\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta}$, with trigger $V^*{}_{Buyer}(\emptyset^*)$ and maturity T:

$$E[V_{VC-seller}]e^{-r(T-t)}\ 1_{V(t)\geq V^*{}_{Buyer}(\emptyset^*)} =$$

$$\frac{(\beta-\varepsilon)C}{(\beta-1)^2}\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)^{\beta} N\big(d3(V_{VC-seller})\big) \qquad (A7.2)$$

Where:

$1_{condition}$ equals 1 if the condition is met, and 0 otherwise

$$d1(V_{VC-seller}) = \frac{\ln\left(\frac{V}{\left(\frac{\beta}{(\beta-1)^2}\times\frac{(\beta-\varepsilon)C}{Q_{VC}\,\gamma\,\theta}\right)}\right)+\left(\alpha+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \qquad (A7.3)$$

$$d3(V_{VC-seller}) = d1(V_{VC-seller}) + (\beta_1 - 1)\sigma\sqrt{T} \qquad (A7.4)$$

For further details, please refer to Pereira and Rodrigues (2014).