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BUCKLING ANALYSIS OF END RESTRAINED
IMPERFECT TUBULAR BEAM COLUMNS
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IMPERFECT TUBULAR BEAM COLUMNS

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BUCKLING ANALYSIS OF END RESTRAINED
IMPERFECT TUBULAR BEAM COLUMNS

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Dissertation

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ABSTRACT

The Influence Coefficient Method is currently available for the buckling analysis of elasto-plastic, end-restrained, space tubular beam-columns. A consistent formulation of the effects of initial imperfections, eccentricities of the loading, applied end moments, material nonlinearities and end-restraints, is presented in the context of small deflection theory. The derivation and calibration of beam-column end-restraint systems is reported. Experimental performance of complex end restraint joint systems is investigated to determine the characteristic calibration curves of spatial end-restraint. In addition the results of a sequence of experimental buckling tests, conducted for obtaining empirical data on end-restrained tubular beam-columns, are described. In this dissertation the elasto-plastic buckling analysis is improved by using a modified Crisfield's constrained displacement length method. Several examples are included to demonstrate the usage of the numerical procedure, and the results are compared with those of the experiments. A parametric study of end-restraint effects on imperfect tubular beam-columns is presented. In addition, effective length factors are derived. Evidence is given on the range of numerical values of effective length factors to be used in the design of end restrained spatial tubular beam-columns.
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CHAPTER 1
INTRODUCTION

As the purpose of design in engineering is optimization, structural systems throughout the years have evolved towards high strength, slender and light structures. This trend leads not only to the immediate savings of material, but essentially to the indirect savings in fabrication, construction and service life of the structures.

In rigid frame construction, columns may carry transverse loads which tend to produce bending about both axes. When using plastic design methods, wind bracing in at least one plane is an essential practice. Therefore, bracing constitutes an important element of structural design and plastic design demands adequate bracing.

Structural tubing is often preferred to other steel column sections when resistance to torsion is required and when a smooth, closed section is esthetically desirable. In addition, structural tubing may often be the economical choice for compression members subjected to moderate loads.

Manufactured tubes are considered to be any tubes produced by piercing, forming and welding, cupping or extruding, in a plant devoted specifically to the production of tubes. They are distinguished from the tubes fabricated from plates by riveting,
bolting or welding, in a structural fabrication shop. Long large
diameter fabricated tubes require circumferential joints as well as
longitudinal joints. Such circumferential joints, which are not
present in manufactured tubes, produce a discontinuity in the tube.
Therefore, fabricated tubes, in general, may be expected to have
more severe imperfections than manufactured tubes, and consequently
a lower buckling strength. In addition to differences in the
magnitude of imperfections, there are also differences among manu-
factured tubes of different types.

Although circular tubes are widely used in structural
members susceptible of buckling, most structural specifications
[e.g., 47, 77, 80] do not give complete design information on the
buckling of such tubes.

Inelastic behavior of tubes has been studied in relation to
deepwater pipelaying and offshore jacket designs. One of the
design requirements for fixed steel offshore structures is to avoid
collapse under an extreme earthquake that might occur at the site.
Economy makes it necessary to take advantage of the inelastic
energy absorption capacity of the structure, and to assure that it
maintains its integrity through inelastic deformation. This requires
reliable prediction of column response in the plastic range and a
basic knowledge of how real individual tubular columns behave.

Some structures reveal high structural sensitivity to local
failure and damage. Furthermore, the behavior of bracing members
acting as beam-columns is quite complex, because it is influenced
by both buckling and yielding. It is likely that most of the
buckling problems encountered by the engineer designing such columns, will involve some degree of plasticity. Then, column stability requirements improperly used may cause an increase of the design effective length of the column and ultimately column and/or structure failure.

1.1 Research Needs on the Stability of Beam-Columns

Experimental and analytical evidence demonstrates that material properties, eccentricities of loading, residual stresses, initial imperfections and end restraints, are major determinants of beam-column performance. Numerical studies have shown that initial crookedness and residual stresses can significantly reduce the strength of hinged-end columns. However, for columns with partially restrained ends, the effect of those two factors is not clearly understood.

Chen [16] attempted to define this subject, describing the current research effort by the Structural Stability Research Council (SSRC), and emphasizing the considerable importance to clarify the influence of end-restraints on the strength of columns.

Bjorhovde [9] presented an organized survey of research needs in the various areas related to the stability of metal structures. Specifically under the general heading of tubular columns it was admitted that many of the current design rules for tubular members were originally developed for other shapes, and verification of this practice is needed.
The investigation of the effect of end restraint on realistic building frame members received additional impetus with the formation by the SSRC in 1979, of its Task Group 23 entitled "Effect of End Restraint on Initially Crooked Columns". In connection with the widespread tendency to base column design on the strength of initially crooked columns, it is of considerable importance to clarify the influence of end restraints on such columns. This is particularly important because most research on column members has been directed toward the study of columns with hinged end conditions.

However, so far very little attention has been given to the spatial behavior and strength of end restrained steel tubular beam-columns, whose usage as main columns or as bracing systems in the building industry—especially for fixed steel offshore platforms—can be improved in the viewpoints of structural integrity, safety and economy. Moreover, postbuckling analysis of such members has not yet been successfully conducted.

Considerable interest has recently been generated in the inelastic dynamic analysis of structures in response to rare, severe earthquakes. For structures which are detailed to maintain their integrity through inelastic deformation, earthquake motions beyond their elastic capacity may result in permanent distortion and localized damage—but not collapse. This design philosophy is particularly pertinent in pile-supported, tubular steel offshore drilling and production platforms. One of the industry consensus standards, API RP 2A, describes ductility requirements in excess of the elastic strength level design criteria.
Various approaches to inelastic analysis have been applied to the problem. A common feature of these analytical efforts is that they rely to a considerable extent on empirical descriptions of tubular member behavior in the inelastic range. Given this need, and the rudimentary understanding of the problem as it has developed within the last few years, it is not surprising that there has been a sudden increase in such research and testing.

1.2 Scope and Objective of this Study

The primary purpose of the dissertation is to establish an analytical procedure to examine the maximum strength and behavior of end-restrained imperfect tubular spatial beam-columns, under no-sway conditions, within the requirements of engineering accuracy. Elasto-plastic buckling defining ultimate strength, will be analyzed under static loading for columns of various slenderness ratios.

So far, only hinged end conditions have been studied including both effects of initial imperfection and residual stresses. The available specifications [78] ensure that column initial imperfections are bounded by allowable values. Also the effects of residual stresses have been thoroughly investigated [19,87]. However, the effect of initial geometric imperfection and of end restraints on strength and behavior of crooked columns, is expected to be much more considerable in the region of intermediate to high slenderness ratio of practical columns. This has not yet been systematically studied in depth. This dissertation therefore investigates the effects of end restraints on the buckling strength and behavior of imperfect tubular beam-
columns. Moreover, new experimental and theoretical evidence of effective length factors to be used in the designs involving such structural members is derived.

The present research extends the work of Ross [60], on behavior and strength of fabricated tubular pinned-end columns, by including in the buckling analysis the effects of complex spatial end-restraint functions. In particular, the Influence Coefficient Method (ICM) is utilized in the present study.

In order to establish the validity of the analytical model, experimental data is obtained conducting small-scale tests as a part of this investigation. Moreover, the postbuckling behavior is of interest for the load carrying capacity after buckling. Ductility ratios appropriate for tubular members of the type used in offshore structures need to be ascertained. Therefore, postbuckling behavior up to practically acceptable deformations is observed, which develops a needed empirical data base. Although small-scale testing precludes examination of multi-directional residual stresses, as present in prototype fabricated tubular steel columns, the primary emphasis herein is on the effects of end restraints.

In Chapter 2, a literature survey of elasto-plastic beam-column analysis is presented. The in-plane plastic buckling of columns is first briefly reviewed in a historical perspective. Then, the elasto-plastic biaxial bending of beam-columns is outlined in a general context. Thereafter, a synopsis of the state of the art on tubular beam-columns is described with direct reference on the
research reported herein. Finally, the assumptions for the nonlinear buckling analysis of elasto-plastic beam-columns are briefly summarized.

In Chapter 3, the construction and calibration of two complex beam-column end-restraint systems is described in detail. Test results from this calibration program are used to formulate descriptive mathematical equations for the end-restraint moment functions.

In Chapter 4, the experimental program on the behavior and strength of end restrained, steel circular tubes as spatial beam-columns is presented. The experimental results, constituting empirical data with adequate accuracy, are discussed.

In Chapter 5, detailed attention is focused on the Influence Coefficient Method for the elasto-plastic buckling analysis of end restrained tubular beam-columns. The Newton-Raphson iterative solution technique is used with efficient convergence criteria. A consistent derivation of the gradient matrix for the iterative procedure is presented. The method is able to model material inelasticity, eccentricities of loading, applied end moments, residual stresses, initial imperfections and end-restraints.

In Chapter 6, solution methods to solve systems of nonlinear equations in structural engineering are briefly reviewed, for both the prebuckling and postbuckling analysis. From the existing methods, an adaptation of the constrained displacement length method is used in the context of the Influence Coefficient Method. It is a modified Crisfield's method usually used in nonlinear finite element analysis.
In Chapter 7, several numerical examples are given to demonstrate the validity and performance of the developed analytical procedure for elasto-plastic buckling analysis of end restrained tubular beam-columns. The experimental results outlined in Chapter 4, are compared with the theoretical results obtained by the Influence Coefficient Method. A parametric study of the effects of end-restraint and initial crookedness in the buckling strength and behavior is also presented. In addition, the magnitude of effective length factors to be used in the design of tubular structural members is ascertained.

Finally, in Chapter 8 a summary of the research is given, emphasizing conclusions and suggestions for further research.
CHAPTER 2

REVIEW OF ELASTO-PLASTIC BEAM-COLUMN ANALYSIS

2.1 Introduction

Technologic improvements in the last four decades have led to continuous interest in the behavior and design of beam-columns beyond the elastic limit. Spurred initially by demands of the aerospace industries, these efforts were continued and improved by attempts to apply the theories to bridges and high-rise building designs.

The failure of a beam-column is always the result of inelastic stability which involves the combination of one or more of the following phenomena: yielding and deformation in the plane of bending (in plane behavior) or in space (biaxial bending), lateral flexural torsional buckling and local buckling.

For tubular circular beam-columns, lateral torsional buckling is not a problem. Also, according to the Guide to the Stability Design Criteria for Metal Structures [39], true column type (primary) buckling (elastic or inelastic) of tubular column members may occur with no local buckling (shell buckling, elastic or inelastic) as long as

\[ \frac{D}{t} < \frac{3300}{\sigma_y} \]  \hspace{1cm} (2.1)

where \( \frac{D}{t} \) is the diameter to thickness ratio of the member, and \( \sigma_y \) is the yield stress in ksi. The \( \frac{D}{t} \) ratios of the columns analyzed

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in this dissertation are well below this limit and consequently there will be no interaction between column type and local buckling. Thus, general instability patterns like the flattening (ovaling) in bending (Brazier effect), inward bulge in bending, or propagating buckle [38], will not be included herein since they are outside the scope of this investigation.

2.2 In-Plane Plastic Buckling

The purpose of this section is to provide historical background and continuity for the next section on biaxial bending. By the end of the 19th century, works of Considere [23] and Engesser [31] laid the foundation for determining the carrying capacity of inelastic geometrically imperfect columns. Based on two different sequences of loading, two major plastic buckling theories evolved: the tangent modulus theory and the reduced modulus theory.

The tangent modulus load [83] is the largest load that a column can support without strain reversal. It only depends on the shape of the stress strain curve for the material of the column, and gives a good prediction of a real column situation.

The reduced modulus load [53] is based on recognizing an elastic process in the unloading fibers of a buckling column. It depends on the shape of the stress strain curve and on the shape of the column cross section, and gives an upper limit for the load carrying capacity of real columns. The reduced modulus theory has been generally considered to be the true theoretical solution for perfect columns until Shanley raised the point that the column may
bend simultaneously with increasing axial load [69], and subsequently presented the load deflection solution of a simplified column model [70]. For this 'Shanley model' the column starts to buckle at the tangent modulus load, after which the load carrying capacity increases approaching asymptotically the reduced modulus load.

However, real columns have disturbing indefinite factors such as initial imperfections, eccentricities of the compressive loads and residual stresses due to processing and fabrication. Thus, strain reversal (unloading) in the column section is inevitable. The increasing effect of this unloading on the load carrying capacity is counteracted (at least partially) by the decreasing effect of initial imperfections. The maximum load is therefore usually slightly above the tangent modulus load. Theoretically the load carrying capacity of a centrally loaded column lies between the tangent modulus load and the reduced modulus load.

A procedure for determining the inelastic buckling of geometrically imperfect columns eccentrically loaded advanced by von Karman, was thoroughly explained by Bleich [10]. Basic assumptions are:

(i) Displacements are small compared with cross section dimensions.

(ii) Plane sections remain plane and normal to the center line after bending.

(iii) The relationship between stress and strain in any longitudinal fibers follows the elasto-plastic mechanical properties of the material.
(iv) The plane of bending is a plane of symmetry of the column. Then a compatible deformed column shape corresponding to a given axial force is determined for a given imperfection. By making a large number of such equilibrium solutions for different bent configurations an entire path of equilibrium for an initially assumed imperfection can be established. The maximum load of this path of equilibrium is the ultimate load capacity of the column.

2.3 Elasto-Plastic Biaxial Bending of Beam-Columns

In a three-dimensional building framework, beam-columns are subjected to biaxial bending with end loadings $P$, $M_x$, and $M_y$ (as shown in Figure 1). Since plastic action is load path dependent, statically equivalent load conditions will cause different plastic behavior of two equal columns, unless $P$, $M_x$, and $M_y$ increase proportionally (radial loading). It is known that biaxial bending causes rotation (twisting) of the cross sections, in addition to lateral displacements. This can significantly reduce the strength of such columns, even in the elastic range, especially if their cross sections have small torsional rigidity.

However, the closed section of tubular columns has high torsional rigidity so that lateral torsional buckling is not a problem. Due to the axisymmetry of the cross sections, warping moment and warping deformations do not exist in circular tubular members. So, the perfect cylindrical tube is structurally the optimum section for a column, free to buckle in any direction; the radius of
Figure 1. Tubular Beam-Column with Unequal End Moments
gyration is constant in all directions and the material is used most efficiently.

Exact elasto-plastic solutions of biaxially loaded beam-columns are difficult to obtain. Among the difficulties are:

(i) Dependence of plastic behavior on the load path.
(ii) Elasto-plastic boundary (yield surface) is moving.
(iii) Stress-strain relations for loading and unloading are different.

To overcome the above difficulties step-by-step solutions, following the load history, are required. Also some simplifying assumptions are used concerning compatibility (geometry, kinematics), equilibrium (static or dynamic), and the stress strain relationship. A fundamental assumption about strain distribution is that plane cross sections remain plane after loading. This enables the stress distribution over the cross section for a given stress strain curve to be obtained once the curvatures at each station along the beam-column are known.

Approximate formulation of equilibrium equations is made in terms of generalized stress and strain, and the force boundary conditions are expressed in terms of the stress resultants and displacements. Materials with sharply defined yield strength, may be idealized as elastic-perfectly plastic. However, in materials where sharp yield point seldom occurs, there is a more or less rapid decrease in the elastic modulus as the stress is increased above the proportional limit. Those may be idealized either as bilinear (Osgood [54,55]) or as an elastic-perfectly plastic, not neglecting
work hardening but averaging its effect over the range of plastic flow. Effective modulus of elasticity also reduces under biaxial stress situations, when Tresca yield criteria is used.

The effect of lateral deflection on the geometry has to be considered for long or intermediate length beam-columns, since instability arises from the magnification of the primary moments by the axial load acting on the deformed beam-column. Biaxially loaded beam columns exhibit the nonbifurcation type of buckling instability in which the deflection increases until a maximum load is reached, beyond which static equilibrium can only be sustained by decreasing the load.

To obtain solutions of the differential equations governing the equilibrium of isolated members or small structures in the nonlinear elasto-plastic range, numerical methods have been commonly used. These include:

(i) Numerical integration methods

(ii) Finite difference methods

However, for practical purposes, the differential equation can still be simplified by introducing additional assumptions. These include:

(i) Establishing equilibrium at a number of stations along the length of the beam-column.

(ii) Assuming the displacements of beam-columns to be given by known simple functions.
These approximate approaches are found quite accurate for many cases, and through them practical analysis methods have been developed [18], namely the Influence Coefficient Method (Chapter 5).

In this method, the beam-column is treated not as a continuous member, but as a member discretized at several stations. Instead of equilibrium differential equations, there are interstational relationships. An influence coefficient matrix of the system of stations can be developed, and the deflections are obtained solving iteratively the resulting system of equations starting from initially assumed values.

The Finite Element Method (FEM) has also been applied to flexural torsional buckling and to three dimensional nonlinear analysis of biaxially loaded thin walled columns [18]. Solutions are obtained through systematic use of the matrix stiffness method. Large structures and complex beam-column problems may be solved by this method. Although the FEM is a powerful and versatile tool for a wide range of problems in structural mechanics, it requires relatively large amounts of computer memory and time even for most efficient computer codes. Hence, it will not be used under the present investigation.

2.4 Literature Survey on Tubular Beam-Columns

A comprehensive investigation described by Ellis [30], set forth a theory for predicting the collapse loads of circular tubular columns (with no residual stresses) when subjected to end loads of equal and unequal end eccentricities. A buckling criteria previously
developed by Horne [36] was used. Ellis conducted large scale experiments to support the analysis, the results of which showed good agreement with the theory. From a comparison with the elastic theory it appeared that the plastic theory would allow considerable economy in the design of circular tubular columns.

This excellent earlier contribution prompted the research on tubular structural members for the 1960's and 1970's. Following are brief commentaries on the major tubular beam-column information reported in the literature.

A series of three tests on square tubular sections were conducted by Dwyer and Galambos [29]. These columns had different slenderness ratios and axial load ratios, chosen in such a way to produce beam type behavior, beam-column and predominantly column behavior in separate tests. The objective of these tests was to study the plastic behavior of tubular beam-columns. The effects of residual stresses were not considered in the evaluation of the results. Theoretical moment-axial load-curvature curves were derived for the tubular section, and were proven to be quite different from those of wide flange sections. The dissimilarity between the two curves occurs when yielding is initiated, and is due to clearly different distributions of area about the centroidal axis. These tests showed that the tubular section is appreciably stronger than a similar wide flange section.

Schilling [68] reported on structural design information on the buckling strength of circular steel tubes. Column buckling (and local buckling) under axial compression, bending, torsion,
transverse shear, and combinations of these loadings, were investigated. It was found that the column buckling strength of circular tubes can be obtained by the Engesser or tangent modulus formula, for tubes with round-house stress-strain curves or with yield point stress-strain curves. To include the effect of residual stresses for yield point tubes, the tangent modulus was obtained from a short (stub) column test. The effect of manufacturing and fabrication methods on the magnitude of geometric imperfections in the tubes, and hence on buckling strength, was examined.

Snyder and Lee [76,45] analyzed the inelastic behavior of tubular cantilever beam-columns. The interaction curves were derived analytically by means of a numerical integration technique, as used by Ellis [30]. Then, using the techniques of Hauck and Lee [35,44], the moment-thrust-curvature curves were approximated by semi-empirical formulas to facilitate the design of such columns under arbitrary boundary conditions and end loads.

To evaluate the strength of structural columns made from hollow rectangular tubes it is necessary to know both the magnitude and profile of the residual stress distribution throughout the component. Sherman [72] provided a method of determining the residual stress pattern in a tubular member. Owing to the forming and cooling procedures in manufacturing, a double pattern of residual stresses is induced. One component is the variation of average stress around the cross section and the other component is the variation of stress through the wall thickness. A slicing
technique, using mechanical measuring equipment, was applied to rectangular structural tubing with satisfactory accuracy.

A review of the research done in the 1960's on the inelastic behavior of isolated beam-columns in biaxial bending, was published in 1968 by Chen and Santathadaporn [21]. During the 1970's a considerable amount of research on the space behavior of elasto-plastic tubular beam-columns was conducted.

Pillai and Ellis [57] showed that the interaction equation for biaxially bent beam-columns recommended by design specifications, like the Guide to Design Criteria for Metal Compression Members (2nd ed., 1966) of the Column Research Council (CRC), yielded inconsistent predictions of the ultimate strength of hollow tubular section columns. They proposed a modified equation for such sections. To verify the accuracy of this, the results of small scale tests on 21 hollow square box columns, reported by Marshall and Ellis [48], were compared with the predicted strengths using both the original CRC interaction equation and the proposed modified equation. Satisfactory predictions were obtained using the modified equation.

A simple method for obtaining the exact interaction relationships of doubly symmetric sections under combined axial force and biaxial bending moments was presented by Chen and Atsuta [17]. The exactness of the solution was checked by upper and lower bound limit analyses. A lower bound solution was obtained integrating, over the cross section, a state of stress satisfying the yield condition and the equilibrium conditions. An upper bound solution was obtained by equating internal rate of energy dissipation to
rate of external work. For illustration, interaction curves for a wide-flange section, a box section and a circular section were developed using the method.

Sherman [73] conducted a series of tests on 10-3/4 in. (270mm) tubes, with six different D/t ratios, to determine the plastic rotation capacity of tubular beams. Cantilever and fixed-end beams, with moment gradient (shears) and restraints against ovalizing at the end, were compared with the behavior of the constant moment region of simple beams. Members with D/t of 35 or less could develop a plastic moment and sustain sufficient rotation to fully redistribute the moments in fixed ended beams, with the equivalent of a uniform load. Under these conditions, ovalization at the hinge zone was slight and buckles did not occur. Tubes with D/t greater than about 50 did not have sufficient plastic hinge rotation capacity to develop the classical ultimate strength.

A significant achievement presented by Wagner, Mueller, and Erzurumlu [87] described a mathematical model used in the determination of ultimate capacities of tubular beam-columns, providing a basis for the development of interaction diagrams. A general purpose beam-column computer program was developed, generating the moment-thrust-curvature data for the member cross section and predicting the ultimate strength of circular tubes subjected to selected patterns of combined axial force and flexure. The computer model also provided the option of incorporating longitudinal residual stress patterns and nonlinear configuration of stress-strain relationships. By the use of the mathematical model, a comparison
was made between interaction diagrams developed with and without residual stresses, showing that the ultimate strength of circular tubes subjected to axial compression and bending is decreased by the inclusion of residual stresses. This effect is more noticeable for the higher values of the ratio $P/P_y$ between the axial compressive load $P$ and the axial load $P_y$ causing complete yielding of the cross section. The analytical results were also compared with the experimental data obtained by Ellis [30], and reasonable agreement was observed between the predicted and measured values of ultimate capacities. However, there was an apparent need for additional experimental data on the residual stress distribution of circular tubes.

Chen and Ross [20] reported on the behavior, strength and failure mode of fabricated tubular steel columns of medium slenderness ratio (range between 39 to 83) and diameter-to-thickness ratios of 48 and 70, with essentially pin-ended conditions. Stub column tests gave a stress-strain curve showing the effect of residual stresses. The method of sectioning and the hole drilling technique were used to measure the longitudinal and circumferential residual stresses. Their combined effect on the fabricated "can" is considerable, as detailed elsewhere [19], completely justifying their inclusion in any theoretical analysis of column behavior. The maximum strength of these columns was compared with the CRC column curve and good agreement was observed.

Sherman, Erzurumlu, and Mueller [74] investigated the influence of residual stresses and the degree of end restraint on
the interaction curves of welded tubular beam-columns. They concluded that their effect is significant in all portions of the interaction curves. The AISC interaction equation for ultimate conditions was compared with the analytical results and good correlation was obtained. Finally, the effect of external hydrostatic pressure on strength was also considered. The additional axial force in the tubular member affects the moment-curvature-thrust behavior. However it normally does not influence column stability [39].

An analytical model predicting theoretically the behavior of fabricated tubular columns, with essentially pin-ended conditions, was described by Toma and Chen [84]. The computer model included the effects of residual stresses, geometric imperfections, end eccentricities and material nonhomogeneity. Extensive comparisons were made with the results of 10 tests on actual large diameter long columns obtained by Chen and Ross [20], demonstrating the validity of the model. Using this computer model, theoretical column strength curves for a fabricated tubular column were obtained. Also, design equations for the column strength curve and for the interaction curve for fabricated tubular columns were developed.

Korol [42] conducted a series of 11 tests on circular hollow tubular steel beams, comparing the experimental results of critical strains with inelastic bending and axial compression theories of buckling [2,41]. Theoretical predictions were higher than the data values. Due to its simplicity, the buckling strain for the axially compressed cylinder was used as a useful estimation for the circular tube subjected to bending. For ductile materials with given D/t
ratios, the buckling strain is inversely proportional to yield stress raised to a power exponent lying between 0.5 and 1.0. That exponent tends to unity for steels with nearly flat yield plateaus (elastic-plastic case) and high D/t, while for a high tangent modulus and small D/t it tends towards zero.

The modelling of tubular members for the inelastic earthquake response analyses of offshore platforms was improved by Anagnostopoulos [1], by selecting plastic-hinge properties taking into account the spread of yielding. Average post-yield stiffness and idealized yield moment ratios were derived as functions of the rotational ductility factor. Finally, a typical platform was analyzed and the results obtained from an iterative scheme were compared to those from the first analysis.

Ross, Chen, and Tall [61] used the Newton-Raphson iterative technique as an economic and accurate numerical technique to model the strength and behavior of axially loaded, fabricated, tubular, steel columns, having highly variable section properties along their length. Consideration was given to longitudinal residual stresses and initial profiles of out-of-straightness. A new family of column buckling curves was derived, showing the dependence of the ultimate column load on the circumferential location of the longitudinal welds. The experimental data reported by Chen and Ross [20], confirmed the fact that weld staggering during fabrication produces columns with buckling strength between the upper and lower bounds theoretically possible for those columns.
Large displacement analysis of elastic-plastic bending problems were studied by Saleeb and Chen [64]. A numerical procedure based on the finite segment method [18], treated the pipe as an assemblage of cans. The moment-curvature relation of the tubular cross section was approximated by simple mathematical expressions. Material and geometrical nonlinearities were included using a modified incremental tangent stiffness approach [18,65]. Four numerical check problems were solved by a computer program developing the aforementioned formulation and good agreement was obtained with the available exact solutions.

Very recently some results of research on end restrained columns have been published. Chapuis and Galambos [14] compared the strengths of restrained and pinned crooked aluminum columns. It was pointed out that the amount of restraint necessary to compensate the effect of initial imperfections is strongly dependent on the slenderness ratio of the member.

Sugimoto and Chen [82] conducted a parametric study of the behavior and strength of axially loaded wide flange steel columns, with small end restraints. The beam-column analysis included the effects of residual stresses, geometric imperfections and small end restraints, using the approximate deflection method [18] and bilinear fitting for test data of moment-rotation relationships of the beam-to-column connections. The validity of the computer model developed was demonstrated by comparison with the results of tests on actual columns.
A theoretical method taking into account the influence of residual stresses, geometric imperfections and end-restraints was described by Vinnakota [85] for analyzing the planar strength of I-section beam-columns. A computer program using the finite difference technique [18] was developed, the validity of which was verified by comparison with available experimental data.

2.5 Summary of Assumptions for the Buckling Analysis of Tubular Beam-Columns

This section briefly reviews characteristics of circular tubular sections and the assumptions used in this dissertation for the nonlinear elasto-plastic buckling analysis of tubular beam-columns. These can be outlined as follows:

(i) Elastic-perfectly plastic material.

(ii) The material is homogeneous and isotropic in both elastic and plastic states.

(iii) The yield stresses in tension and compression are equal.

(iv) The deflections and slopes are small.

(v) Plane sections remain plane during bending.

(vi) The effect of shear stresses on yielding is neglected.

(vii) Column type of buckling (no significant distortion of the cross sections).

(viii) No flexural torsional buckling.

(ix) End loaded beam-column.

Based on the above mentioned assumptions, the Influence Coefficient Method will be thoroughly derived in Chapter 5.
CHAPTER 3
CALIBRATION OF A BEAM-COLUMN END RESTRAINT SYSTEM

3.1 Introduction

The degree of end restraint offered to a column is known to affect its buckling and postbuckling behavior greatly in both the elastic and plastic regions of column stability. In an effort to better understand the effect of end restraint on the behavior of tubular columns, an experimental and analytical program was initiated in 1981 at the Civil Engineering Department of The University of Akron, Akron, Ohio. A system of variable end restraints, capable of modelling the end restraints offered to beam-columns under biaxial bending, was developed for experimental testing of such columns. The construction and calibration of these restraint systems is described in this chapter, along with the development of mathematical expressions to model the restraint behavior. The experiments were conducted on an electro-hydraulic universal testing machine (CGS Lawrence) with a load capacity of ± 10,000 lbs.

3.2 Description of the End-Restraint Systems

Two complex end-restraint "joint" systems were screwed to the end plates of the testing machine (Figure 2), into which tubular columns can be fitted. Each of these end-restraint systems consists
essentially of two perpendicular steel rods (taken as the x and y axes) connected to a rigid central bearing through which axial compressive load can be applied (Figure 3). A diagram of a restraint "joint" is shown in Figure 4 emphasizing its constituents and a legend is added for completeness.

The rods are 3/8 inch round, solid steel bars with a modulus of elasticity of 29,000 ksi (200 GPa). Their effective restraint length can be varied in the range 3.9—7.9 inches (10–20 cm) by an adjustable collar which also provides an axially variable contact surface for the specimen.

The rotations of the restraint system through the two perpendicular directions of the restraint bars are recorded throughout experiments. This is accomplished by two rotational potentiometer differential transducers (KOU's) installed on the rigid rotation shafts of the end-restraint system, as shown in Figures 2 and 3. Trans-tek Inc. RV07's (Series 600) were used; their choice seemed justifiable because of their characteristic infinite resolution, no reactive torque, and high dependability.
essentially of two perpendicular steel rods (taken as the x and y axes) connected to a rigid central bearing through which axial compressive load can be applied (Figure 3). A diagram of a restraint "joint" is shown in Figure 4 emphasizing its constituents and a legend is added for completeness.

The rods are 3/8 inch round, solid steel bars with a modulus of elasticity of 29,000 ksi (200 GPa). Their effective restraint length can be varied in the range 3.9-7.9 inches (10-20 cm) by an adjustable end clamp. This assures that under the anticipated maximum axial load to be applied to the end-restraints and column specimens (less than 2000 lbs), the restraint bars behave elastically. These bars are clamped at their outer ends and their continuity with the central bearing and adjacent collar (insertion section) is obtained by driven spring pins.

Any rotation of the central bearing on its machine-plate contact area, is restrained by the semi-rigid collar and in torsion by the solid bars. Thus the end-restraints are "complex end-restraint joint systems", and not just as trivial torsion bar systems.

The rotations of the restraint system through the two perpendicular directions of the restraint bars are recorded throughout experiments. This is accomplished by two rotational variable differential transducers (RVDT's) installed on the rigid rotation shafts of the end-restraint system, as shown in Figures 2 and 3. Herein, Trans-Tek Inc. RVDT's (Series 600) were used; their choice seemed justifiable because of their characteristic infinite resolution, no reactive torque, and high dependability.
Figure 3. Detailed View of an End-Restraint System
Axial Load $P$

Column Specimen

Torsion Bars

Figure 4. Diagram of an End-Restraint System
3.3 Experimental Calibration Program

This section describes the calibration procedure used to enable the estimation of the x and y restraint moments ($R_{xB}$, $R_{yB}$, $R_{xT}$, and $R_{yT}$) of the bottom and top end-restraint systems as the bottom and top central bearings rotate in space.

To accomplish this a rigid lever arm is fixed by a bullet-like rigid end fixture to the central bearing of the restraint system being tested at the bottom machine plate. Through it, variable moments can be applied by suspending variable weights from the extremity of a hanger bar (Figure 5). A calibration tubular column was used, fixed at the top and restrained at the bottom by the end-restraint system being tested. To ensure continuity between the column and the central bearings, the end fixtures were machined with a slip fit to the column ends.

3.3.1 Development of the End-Restraint Systems

The end-restraint systems, allowing variable rotational stiffnesses, were designed to function as elastic restraints under testing conditions. The estimation of the end-restraint functions by means of experimental set-ups as described, requires that two characteristics be satisfied for a useful, repeatable use of the testing data:

(i) Uncoupling between restraint bars of the end-restraint systems, and

(ii) Frictionless bearings.
The former assures that each direction can have an independently variable restraint. The latter reflects the ideal independence of experimental results from axial load applied to the column specimen.

With the eccentric restraint system, the decoupling between restraint and load by loading and unloading the lever arm, portion of the 80° wedge, and the HOD's consistently residual load was normal to the end-restraint retention plate in a squeezing and unlatching of the lever arms 111° apart. Pursued a slightly different direct geometric decomposition of the sensitive and negative $x,y$ restrained strain in the spring pin correction planes of the restraint system to either. This was detected when the lever arm was loaded consecutively in two symmetrically opposite positions. This problem was significantly reduced, although not totally eliminated, by driving specially acorned tapered spring pins into the joint pin holes. The minimized incursions still incorporates total angular rotations between positive and negative $x,y$ directions having the measured values indicated at Table 1 for the bottom and top end-restraint systems.

Figure 5. Application of Moments During Calibration Testing
The former assures that each direction can have an independently, variable restraint. The latter reflects the ideal independence of experimental results from axial load applied to the column specimens.

With no applied load on the restraint system, the uncoupling between restraint bars was verified by loading and unloading the lever arm, positioned on its 0° and 90° orientation. The RVDT's consistently monitored electrical signals proportional to the end-restraint rotations.

In a complementary manner systematic loading and unloading of the lever arm on first quadrant stations oriented 15° apart permitted a check of the assumption of independent trigonometric decomposition of actions (applied moments) and responses (measured x and y rotations).

These procedures were repeated sequentially for all quadrants, thereby completing a full spectrum of positive and negative x,y restraint characteristics. For the initial end-restraint systems tested, there was some looseness of the spring pins connecting pieces of the restraint systems together. This was detected when the lever arm was loaded consecutively in two diametrically opposite positions. This problem was significantly reduced, although not totally eliminated, by driving specially machined tapered spring pins into the joint pin holes. The minimized looseness still incorporates total slop rotations between positive and negative x,y directions having the measured values indicated at Table 1 for the bottom and top end-restraint systems.
TABLE 1. Slope Rotations of the End-Restraint Systems

<table>
<thead>
<tr>
<th>End-Restraint System</th>
<th>X-Slope Rotation (degrees)</th>
<th>Y-Slope Rotation (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom</td>
<td>1.10</td>
<td>0.25</td>
</tr>
<tr>
<td>Top</td>
<td>0.50</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Friction is impossible to ignore in column experiments, since it is always present to some degree. Nevertheless, with proper procedures its effect can be minimized, diminishing the differences between calibration results at no axial load and under compressive axial load.

As indications of acceptability of the developed restraints, the following tests were deemed essential:

(i) Repeatability of data for the same loadings
(ii) Unloading return
(iii) Consistent data variation trends for different loadings

In the originally constructed sliding support (Figure 6), a considerable central bearing contact area was a source of significant friction-related disturbances. At the optimized rolling support (Figure 7), a small contact area minimizes frictional influence.

Rosette strain gages were also installed on each restraint bar of both restraint systems. They were used to ascertain the behavior of the end-restraints, either as torsion shafts or as complex end-restraint joint systems. The end-restraint systems were shown by this means to act as complex end-restraint joint systems. Therefore, the x and y monitored rotations are just like
Figure 6. Restraint Joint Sliding Support
Column Specimen

Figure 7. Restraint Joint Rolling Support
control measures of the restraint offered by each end-restraint system, and complex torsion and/or bending actions are involved in the composite system. The results were consistent, repeatable, and predictable, giving useful information once the systems were calibrated.

3.3.2 Calibration Experiments

The end-restraint system to be calibrated was installed at the bottom machine plate and the balanced lever arm was fixed to its central bearing at a 45° orientation. This allowed the restraint functions of both perpendicular directions of the restraint system to be determined simultaneously. The restraint bars were locked in a position corresponding to an intermediate effective restraint length of 5.9 inches (15 cm). The other end-restraint system was installed at the top machine plate and its restraint bars were tightly locked in the position corresponding to the smallest effective restraint length. By this procedure, the top restraint system behaved as a fixed end during calibration of the other end. (This was demonstrated experimentally.)

Increasing end moments through the xz and yz planes were applied by hanging progressively increasing weights at the extremity of the lever arm. The maximum applied moment was 107 lb.in (12.1 N.m).

The calibration of the bottom and top end-restraint systems is divided in three basic phases for each x and y direction as follows:
(i) Phase 1 - Restraint system alone resisting the applied moments.

(ii) Phase 2 - Restraint system and calibration column resisting the applied moments with no axial load applied to the column.

(iii) Phase 3 - Restraint system and calibration column resisting the applied moments under applied compressive axial load.

To ensure repeatability four cycles of this three phase sequence were performed.

During Phase 1 readings of the x and y restraint rotations were taken at each incremental step of moment application. During this phase the applied moments were resisted solely by the restraint moments.

Then, a tubular steel calibration column was installed between the end-restraint systems in such a way that the central bearing end fixtures had a push fit connection to the column ends. During this stage (Phase 2) no compressive load was induced in the column specimen by the loading machine plates.

The moments applied to the bottom complex joint through the rigid central bearing were resisted elastically in both x and y directions by the end-restraint and by bending of the calibration column.

The mathematical model describing each in-plane elastic behavior is shown in Figure 8, for no axial load. It is a discontinuous column analogue formed by an elastic column of length
Figure 8. Mathematical Column Model
L and a rigid stub of length "a" through which moment M is applied. It is fixed at the top end and is restrained at the bottom end by an elastic spring whose restraint function R is as yet unknown. The elastic calibration column was a steel tube 22.2 inches (56.5 cm) long with an outside diameter of 0.5 inches, thickness of 0.09 inches, and a modulus of elasticity of 29,000 ksi. The rigid stubs model the rigid part of the central bearing, lever arm and stub-like end fixture that connected the column to the plates of the loading machine. Their lengths for the bottom and top end-restraint systems were respectively 1.89 inches (4.8 cm) and 2.01 inches (5.1 cm).

During Phase 2 readings of the x and y bottom rotations were taken at each incremental moment step. These bottom rotations were the rotations of both the restraint spring and the bottom column end.

For Phase 3 the testing machine was preset in load-control mode. The experimental setup was as for Phase 2, but now the column was subjected to compressive axial loads up to 1200 lbs. The in-plane behavior can be described by the mathematical model shown in Figure 8. During each load step of Phase 3 readings of the x and y bottom rotations were taken as the applied moment was gradually applied and then taken off.

Each experiment was repeated four times, loading and unloading on diametrically perpendicular axes. This allowed a comprehensive evaluation of the x and y restraint characteristics in both positive and negative directions.
The end restraint systems were interchanged and the other joint system was calibrated following the sequential calibration program just described.

3.4 Calibration Tests Results

For the x and y directions of each end-restraint system, each of the three phases of the calibration experiments provided 20 observed pairs of data on two variables: applied moment $M$ and end rotation $\theta$. The analysis of data followed herein refers to the positive and negative x and y end rotations (loading lever arm at 45° and 225° orientation) of the bottom and top end-restraint systems. Complete data for both restraints in all rotation directions was obtained during calibration.

The pairs of measurements $(M, \theta)$ provided by each phase exhibited a nonlinear relationship. Using appropriate rectification rules [67], best fit curves were obtained for functions of the type (a) $y = mx^n$, and (b) $y = bx + cx^2$. The least squares method of approximation theory [26] was used in fitting equations of types (a) and (b) to each set of 20 data pairs. The correlation coefficient associated with curve fittings by functions of the type (a) was higher than the one corresponding to type (b) functions. This indicated that greater confidence could be achieved using type (a) approximations. It was found that correlation between $x$ and $y$ exists at the 0.1% significant level and better, for all sets of data pairs, when type (a) fitting functions are used. The unknown
restraint functions $R_{xB}$, $R_{yB}$, $R_{xT}$, and $R_{yT}$ were also determined as approximation functions of the type $y = mx^n$.

At Phase 1 the applied moments $M$ in each perpendicular direction are equal to the restraint moments $R_{xB}$, $R_{yB}$, $R_{xT}$ and $R_{yT}$. The restraint functions $R_{xB}$, $R_{yB}$, $R_{xT}$ and $R_{yT}$ for $P = 0$ were found to be:

- $R_{xB} = 74.4 \left( \theta_{xB} \right)^{0.824} \text{ (lb. in)} = 8.41 \left( \theta_{xB} \right)^{0.824} \text{ (N.m)}$ \hspace{1cm} (3.1)
- $R_{yB} = 73.4 \left( \theta_{yB} \right)^{0.823} \text{ (lb. in)} = 8.29 \left( \theta_{yB} \right)^{0.823} \text{ (N.m)}$ \hspace{1cm} (3.2)
- $R_{xT} = 88.6 \left( \theta_{xT} \right)^{0.613} \text{ (lb. in)} = 10.0 \left( \theta_{xT} \right)^{0.613} \text{ (N.m)}$ \hspace{1cm} (3.3)
- $R_{yT} = 68.7 \left( \theta_{yT} \right)^{0.865} \text{ (lb. in)} = 7.76 \left( \theta_{yT} \right)^{0.865} \text{ (N.m)}$ \hspace{1cm} (3.4)

for positive end rotations, and

- $R_{xB} = 72.6 \left( \theta_{xB} \right)^{0.828} \text{ (lb. in)} = 8.21 \left( \theta_{xB} \right)^{0.828} \text{ (N.m)}$ \hspace{1cm} (3.5)
- $R_{yB} = 76.7 \left( \theta_{yB} \right)^{0.817} \text{ (lb. in)} = 8.66 \left( \theta_{yB} \right)^{0.817} \text{ (N.m)}$ \hspace{1cm} (3.6)
- $R_{xT} = 79.8 \left( \theta_{xT} \right)^{0.705} \text{ (lb. in)} = 9.02 \left( \theta_{xT} \right)^{0.705} \text{ (N.m)}$ \hspace{1cm} (3.7)
- $R_{yT} = 67.4 \left( \theta_{yT} \right)^{0.883} \text{ (lb. in)} = 7.61 \left( \theta_{yT} \right)^{0.883} \text{ (N.m)}$ \hspace{1cm} (3.8)

for negative end rotations, where the bottom and top end rotations $\theta_{xB}$, $\theta_{yB}$, $\theta_{xT}$ and $\theta_{yT}$ are expressed in degrees.

In Phase 2, the applied moments are equal to the restraint moments found in Phase 1 plus the moments required for column bending $M_{cb}$. Using column analogy method [50], the rotational stiffness
coefficient for the bottom end of the mathematical column model in Figure 8 is found to be

\[ K_{\theta\theta} = \frac{4EI}{L} \left( 1 + \frac{3a}{L} \left( 1 + \frac{a}{L} \right) \right) \]  \hspace{1cm} (3.9)

in which EI is the bending stiffness. The second term inside the parenthesis accounts for the stiffening effect of the rigid end stub where \( a \) is the length of the stub. Therefore, the column bending moment of the idealized model for \( P = 0 \) is:

\[ M_{cb} \left( \frac{EI}{L}, \frac{a}{L}, \theta \right) = \frac{4EI}{L} \left( 1 + \frac{3a}{L} \left( 1 + \frac{a}{L} \right) \right) \theta \]  \hspace{1cm} (3.10)

Theoretically, the Phase 2 equilibrium equation of the idealized model can be expressed as \( M - R = M_{cb} \), where \( R \) is the restraint function for \( P = 0 \). However, actual columns always have initial curvature as well as eccentricity of the loading. Moreover, the end fixtures cannot assure full continuity of the column-stub connection section when no axial load is applied. In addition, the complete fixity at the top end of the idealized model is not fully provided in the experimental system. During Phase 2 and 3 the RVDT's monitored maximum top end rotations of 0.04° and 0.34° respectively in the x and y directions. The mentioned difficulties are sources for new redistributions of moments throughout the calibration column. The ideal column modeled theoretically does therefore not really exist. Therefore, the equilibrium of the actual calibration column for \( P = 0 \) is considered to be given for each x and y direction in the form:

\[ M - R = (M_{cb})_{\text{experimental}} = a (M_{cb})_{\text{model}} \]  \hspace{1cm} (3.11)
in which $\alpha$ is an experimental function of the end rotation $\Theta$, correcting the column bending moments of the idealized model to their actual values.

Substituting equation (3.10) into (3.11) and rearranging terms, the correction function $\alpha$ is expressed as:

$$
\alpha(\Theta) = \frac{M - R}{\frac{4EI}{L} \left( 1 + 3 \frac{a}{L} \frac{1}{1 + \frac{a}{L}} \right) \Theta}
$$

(3.12)

where $R$ is given by equations (3.1) through (3.8) and the $x$ and $y$ applied moments $M$ are related to the $x$ or $y$ rotations by functions of the type $y = mx^n$.

For the bottom end-restraint system, the average value of $\alpha$ in the range of restraint rotations from $0^\circ$ to $1^\circ$ was found to be approximately 0.20 and 0.30 for the $x$ and $y$ directions respectively. For the top end-restraint system, the average value of $\alpha$ in the range of restraint rotations from $0^\circ$ to $1^\circ$ was found to be approximately 0.35 and 0.15 for the $x$ and $y$ directions respectively.

The calibration of the restraint systems with applied axial load was complicated by the evaluation of the column bending moment capacity for different values of the applied compressive axial load. Consider an ideal model of an elastic beam-column as shown in Figure 8. Solving the beam-column boundary value problem [12] of the idealized model, the resulting equilibrium equation can be shown to be:

$$
M = R + \frac{EI}{L} \left\{ \frac{1}{\tan \left( \frac{\beta L}{2} \right)} \left[ 1 - \frac{\beta L}{\tan (\beta L)} + \frac{a}{L} \left( 1 + \frac{a}{L} (\beta L)^2 \right) \right] \Theta \right\}
$$

(3.13)
in which \( \beta = \sqrt{P/(EI)} \) and the second term at the right hand side of equation (3.13) refers to the fraction of the applied moments that are resisted by bending of the beam-column. Thus, the column bending moments are now identified as:

\[
M_{cb} \left( \frac{EI}{L}, \frac{a}{L}, \beta L, \theta \right) = \frac{EI}{L} f_{cb} \theta \tag{3.14}
\]

in which \( f_{cb} \) is a column bending function given by:

\[
f_{cb} = \frac{1}{-1 + \frac{\tan (\beta L/2)}{\tan (\beta L) + \frac{a}{L} (1 + \frac{a}{L}) (\beta L)^2}} \tag{3.15}
\]

The column bending functions of the beam-column idealized model associated with the calibration of the bottom and top end-restraint systems are illustrated in Figure 9. It is a non-dimensional plot of \( f_{cb} \) versus \( \beta L \) depicting the decrease of bending moment capacity with increasing values of compressive axial load.

As the length "a" of the rigid end stub tends to zero, the column bending function \( f_{cb} \) tends to the stability function \( S \), discussed by Livesley and Chandler [46]. Moreover, the rotational stiffness coefficient \( K_{\theta\theta} \) expressed by equation (3.9) is the limit of the product \( \frac{EI}{L} f_{cb} \) as the axial compressive load \( P \) tends to zero (i.e., \( BL \to 0 \)). Thus, \( \frac{EI}{L} f_{cb} \) is the stiffness of the beam-column model.

Recalling the considerations that led to the formulation of equation (3.11), the Phase 3 equilibrium equations for each \( x \) and \( y \) direction of the experimental calibration system are considered to be expressed as:

\[
R = M - \alpha (M_{cb})_{\text{model}} = M - \alpha \frac{EI}{L} f_{cb} \theta \tag{3.16}
\]
CALIBRATION OF END-RESTRAINT SYSTEMS

Figure 9. Column Bending Function (Bottom and Top Restraint)
where \( \alpha \) is the experimental column bending correction function defined by equation (3.12) for each \( x \) and \( y \) direction, and the \( x \) or \( y \) applied moments \( M \) are related to the \( x \) or \( y \) restraint rotations \( \Theta \) by functions of the type \( y = mx^n \).

The restraint function formulation given by equation (3.16) enables the evaluation of points \((R, \Theta)\) of the restraint moment curves for each value of the applied compressive axial load \( P \). For both \( x \) and \( y \) directions the pairs \((R, \Theta)\) are approximated by functions of the type \( y = mx^n \), using the method of least squares [26]. Here again correlation between \( x \) and \( y \) exists at the 0.1% significant level and better for all the approximations. The restraint moment functions \( R_{xB}, R_{yB}, R_{xT}, \) and \( R_{yT} \) are plotted in Figures 10 through 17 for \( P \) values of 0 (Phase 1), 500, 600, 800, 1000, 1200 lbs. In a similar manner Tables 2 through 9 give the corresponding \( m \) and \( n \) parameters of the approximations \( R = m\Theta^n \).

Typically, the \((R, \Theta)\) relationship of Figures 10 through 17 together with the resulting parameters \( m \) and \( n \) shown in Tables 2 through 9 for different values of the applied compressive axial load, characterize the family of end-restraint moment curves associated with all rotation directions of the bottom and top end-restraint systems.

3.5 Use of the Calibration Results

This section explains how to use the available experimentally derived end-restraint functions under other loadings and end-restraint characteristics.
Figure 10. Bottom Restraint Moment Function (for positive x rotation)
Figure 11. Bottom Restraint Moment Function (for positive y rotation)
Figure 12. Top Restraint Moment Function (for positive x rotations)
Figure 13. Top Restraint Moment Function
(for positive y rotations)
Figure 14. Bottom Restraint Moment Function
(for negative x rotations)
Figure 15. Bottom Restraint Moment Function (for negative y rotations)
Figure 16. Top Restraint Moment Function (for negative x rotations)
Figure 17. Top Restraint Moment Function (for negative y rotations)
TABLE 2. \( m \) and \( n \) Parameters in \( R_{xB} = m \ (\theta_{xB})^n \), for positive rotations \( \theta_{xB} \\

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load P (1)(^a)</th>
<th>( m ) (2)(^b)</th>
<th>( n ) (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>74.4</td>
<td>0.824</td>
</tr>
<tr>
<td>(0)</td>
<td>(8.41)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>124.</td>
<td>0.550</td>
</tr>
<tr>
<td>(2225)</td>
<td>(14.0)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>159.</td>
<td>0.654</td>
</tr>
<tr>
<td>(2670)</td>
<td>(18.0)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>172.</td>
<td>0.682</td>
</tr>
<tr>
<td>(3560)</td>
<td>(19.4)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>166.</td>
<td>0.673</td>
</tr>
<tr>
<td>(4450)</td>
<td>(18.7)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>161.</td>
<td>0.663</td>
</tr>
<tr>
<td>(5340)</td>
<td>(18.1)</td>
<td></td>
</tr>
</tbody>
</table>

\( a \) Units are pounds (Newtons)  
\( b \) Units are pounds \times \text{inch} (\text{Newton} \times \text{meter})
**TABLE 3.**  \( m \) and \( n \) Parameters in \( R_{yB} = m (\theta_{yB})^n \),
for positive rotations \( \theta_{yB} 

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load P (1)(^{a})</th>
<th>( m ) (2)(^{b})</th>
<th>( n ) (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>73.4</td>
<td>0.823</td>
</tr>
<tr>
<td>(0)</td>
<td>(8.29)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>89.2</td>
<td>0.610</td>
</tr>
<tr>
<td>(2225)</td>
<td>(10.1)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>95.2</td>
<td>0.584</td>
</tr>
<tr>
<td>(2670)</td>
<td>(10.8)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>110.</td>
<td>0.623</td>
</tr>
<tr>
<td>(3560)</td>
<td>(12.5)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>121.</td>
<td>0.679</td>
</tr>
<tr>
<td>(4450)</td>
<td>(13.7)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>120.</td>
<td>0.676</td>
</tr>
<tr>
<td>(5340)</td>
<td>(13.5)</td>
<td></td>
</tr>
</tbody>
</table>

\(^{a}\) Units are pounds (Newtons)

\(^{b}\) Units are pounds x inch (Newton x meter)
TABLE 4. m and n Parameters in $R_{xt} = m \theta_x^n$, for positive rotations $\theta_{xt}$

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load P (1)(^a)</th>
<th>m (^b)</th>
<th>n (^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>88.6</td>
<td>0.613</td>
</tr>
<tr>
<td>(0)</td>
<td>(10.0)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>162.</td>
<td>0.587</td>
</tr>
<tr>
<td>(2225)</td>
<td>(18.3)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>206.</td>
<td>0.688</td>
</tr>
<tr>
<td>(2670)</td>
<td>(23.3)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>262.</td>
<td>0.779</td>
</tr>
<tr>
<td>(3560)</td>
<td>(29.6)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>257.</td>
<td>0.775</td>
</tr>
<tr>
<td>(4450)</td>
<td>(29.0)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>271.</td>
<td>0.817</td>
</tr>
<tr>
<td>(5340)</td>
<td>(30.6)</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) Units are pounds (Newtons)

\(^b\) Units are pounds x inch (Newton x meter)
TABLE 5.  $m$ and $n$ Parameters in $\mathbf{R}_{yT} = m (\theta_{yT})^n$, for positive rotations $\theta_{yT}$

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load $P$ (1)⁰</th>
<th>$m$ (2)⁰</th>
<th>$n$ (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>68.7</td>
<td>0.865</td>
</tr>
<tr>
<td>(0)</td>
<td>(7.76)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>130.</td>
<td>0.490</td>
</tr>
<tr>
<td>(2225)</td>
<td>(14.6)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>161.</td>
<td>0.568</td>
</tr>
<tr>
<td>(2670)</td>
<td>(18.2)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>197.</td>
<td>0.640</td>
</tr>
<tr>
<td>(3560)</td>
<td>(22.3)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>213.</td>
<td>0.672</td>
</tr>
<tr>
<td>(4450)</td>
<td>(24.0)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>207.</td>
<td>0.651</td>
</tr>
<tr>
<td>(5340)</td>
<td>(23.4)</td>
<td></td>
</tr>
</tbody>
</table>

a Units are pounds (Newtons)

b Units are pounds x inch (Newton x meter)
TABLE 6. m and n Parameters in $R_{xB} = m (\theta_{xB})^n$, for negative rotations $\theta_{xB}$

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load P (1)</th>
<th>m (2)</th>
<th>n (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>72.6</td>
<td>0.828</td>
</tr>
<tr>
<td>(0)</td>
<td>(8.21)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>190.</td>
<td>0.804</td>
</tr>
<tr>
<td>(2225)</td>
<td>(21.5)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>195.</td>
<td>0.799</td>
</tr>
<tr>
<td>(2670)</td>
<td>(22.0)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>218.</td>
<td>0.841</td>
</tr>
<tr>
<td>(3560)</td>
<td>(24.6)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>220.</td>
<td>0.850</td>
</tr>
<tr>
<td>(4450)</td>
<td>(24.8)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>226.</td>
<td>0.880</td>
</tr>
<tr>
<td>(5340)</td>
<td>(25.5)</td>
<td></td>
</tr>
</tbody>
</table>

a Units are pounds (Newtons)
b Units are pounds x inch (Newton x meter)
<table>
<thead>
<tr>
<th>Applied Compressive Axial Load ( P ) (1)(^a)</th>
<th>( m ) (2)(^b)</th>
<th>( n ) (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>76.7</td>
<td>0.817</td>
</tr>
<tr>
<td>(0)</td>
<td>(8.66)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>152.</td>
<td>0.648</td>
</tr>
<tr>
<td>(2225)</td>
<td>(17.1)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>155.</td>
<td>0.660</td>
</tr>
<tr>
<td>(2670)</td>
<td>(17.5)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>170.</td>
<td>0.718</td>
</tr>
<tr>
<td>(3560)</td>
<td>(19.2)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>161.</td>
<td>0.681</td>
</tr>
<tr>
<td>(4450)</td>
<td>(18.2)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>166.</td>
<td>0.733</td>
</tr>
<tr>
<td>(5340)</td>
<td>(18.7)</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) Units are pounds (Newtons)

\(^b\) Units are pounds x inch (Newton x meter)
TABLE 8. \( m \) and \( n \) Parameters in \( R_{\theta} = m (\theta_x)^n \), for negative rotations \( \theta_x \)

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load ( P ) ((1)^a)</th>
<th>( m ) ((2)^b)</th>
<th>( n ) ((3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>79.8</td>
<td>0.705</td>
</tr>
<tr>
<td>(0)</td>
<td>(9.02)</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>111.</td>
<td>0.498</td>
</tr>
<tr>
<td>(2225)</td>
<td>(12.5)</td>
<td></td>
</tr>
<tr>
<td>600</td>
<td>141.</td>
<td>0.580</td>
</tr>
<tr>
<td>(2670)</td>
<td>(16.0)</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>190.</td>
<td>0.709</td>
</tr>
<tr>
<td>(3560)</td>
<td>(21.5)</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>211.</td>
<td>0.759</td>
</tr>
<tr>
<td>(4450)</td>
<td>(23.8)</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>219.</td>
<td>0.801</td>
</tr>
<tr>
<td>(5340)</td>
<td>(24.8)</td>
<td></td>
</tr>
</tbody>
</table>

\(a\) Units are pounds (Newtons)

\(b\) Units are pounds x inch (Newton x meter)
TABLE 9. \( m \) and \( n \) Parameters in \( R_{yT} = m (\theta_{yT})^n \), for negative rotations \( \theta_{yT} \\

<table>
<thead>
<tr>
<th>Applied Compressive Axial Load P ( P ) (1)(^a)</th>
<th>( m ) (2)(^b)</th>
<th>( n ) (3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ( (0) )</td>
<td>67.4</td>
<td>0.083</td>
</tr>
<tr>
<td>500 ( (2225) )</td>
<td>166.</td>
<td>0.586</td>
</tr>
<tr>
<td>600 ( (2670) )</td>
<td>193.</td>
<td>0.646</td>
</tr>
<tr>
<td>800 ( (3560) )</td>
<td>209.</td>
<td>0.639</td>
</tr>
<tr>
<td>1000 ( (4450) )</td>
<td>224.</td>
<td>0.661</td>
</tr>
<tr>
<td>1200 ( (5340) )</td>
<td>228.</td>
<td>0.673</td>
</tr>
</tbody>
</table>

a Units are pounds (Newtons)

b Units are pounds x inch (Newton x meter)
A study of Figures 10 through 17 and Tables 2 through 9 reveals that the restraint curves corresponding to P values of 800, 1000 and 1200 lbs lie in a narrow band. This suggests that in each case a single curve could be used predicting the restraint moments for \( P \geq 800 \) lbs. Based on the data of Tables 2 through 9, the m and n parameters of such a single curve were derived for all the rotation directions of the bottom and top end-restraint systems. For an applied axial load less than 800 lbs, the end-restraint functions (i.e., m and n) appear to be a function of the axial load \( P \). The relationships \( m(P) \) and \( n(P) \) describing this "seating effect" were approximated by semi-empirical polynomials. The results of these approximations are outlined below.

For positive rotations \( \theta_{x_B} \), \( m = 166 \) lb. in (18.8 N.m) and \( n = 0.673 \), for \( P \geq 800 \) lbs. For \( P < 800 \) lbs:

\[
m(P) = 73.8 + 0.125 P - 6.90 \times 10^{-6} P^2 \text{ (lb. in)} \quad (3.17)
\]

\[
n(P) = 0.821 - 9.48 \times 10^{-4} P + 9.73 \times 10^{-7} P^2 \quad (3.18)
\]

with \( P \) in lbs, or:

\[
m(P) = 8.34 + 3.17 \times 10^{-3} P - 3.94 \times 10^{-6} P^2 \text{ (N.m)} \quad (3.19)
\]

\[
n(P) = 0.821 - 2.13 \times 10^{-4} P + 4.79 \times 10^{-8} P^2 \quad (3.20)
\]

with \( P \) in Newtons.

For positive rotations \( \theta_{y_B} \), \( m = 117 \) lb. in (13.2 N.m) and \( n = 0.659 \), for \( P \geq 800 \) lbs. For \( P < 800 \) lbs:
\[ m(P) = 72.9 - 9.03 \times 10^{-2} \ P + 1.86 \times 10^{-4} \ P^2 \text{ (lb. in)} \] (3.21)

\[ n(P) = 0.819 - 1.59 \times 10^{-3} \ P + 1.78 \times 10^{-6} \ P^2 \] (3.22)

with \( P \) in lbs, or:

\[ m(P) = 8.24 - 2.29 \times 10^{-3} \ P + 1.06 \times 10^{-6} \ P^2 \text{ (N.m)} \] (3.23)

\[ n(P) = 0.819 - 3.58 \times 10^{-4} \ P + 8.96 \times 10^{-8} \ P^2 \] (3.24)

with \( P \) in Newtons.

For positive rotations \( \theta_x \), \( m = 263 \text{ lb. in} \) (29.7 N.m) and \( n = 0.790 \), for \( P \geq 800 \text{ lbs} \). For \( P < 800 \text{ lbs} \):

\[ m(P) = 88.1 + 6.23 \times 10^{-2} \ P + 1.99 \times 10^{-4} \ P^2 \text{ (lb. in)} \] (3.25)

\[ n(P) = 0.611 - 3.91 \times 10^{-4} \ P + 7.80 \times 10^{-7} \ P^2 \] (3.26)

with \( P \) in lbs, or:

\[ m(P) = 9.96 + 1.58 \times 10^{-3} \ P + 1.14 \times 10^{-6} \ P^2 \text{ (N.m)} \] (3.27)

\[ n(P) = 0.611 - 8.80 \times 10^{-5} \ P + 3.94 \times 10^{-8} \ P^2 \] (3.28)

with \( P \) in Newtons.

For positive rotations \( \theta_y \), \( m = 206 \text{ lb. in} \) (23.2 N.m) and \( n = 0.654 \), for \( P \geq 800 \text{ lbs} \). For \( P < 800 \text{ lbs} \):

\[ m(P) = 68.4 + 6.09 \times 10^{-2} \ P + 1.41 \times 10^{-4} \ P^2 \text{ (lb. in)} \] (3.29)

\[ n(P) = 0.863 - 1.43 \times 10^{-3} \ P + 1.48 \times 10^{-6} \ P^2 \] (3.30)

with \( P \) in lbs, or:
\[ m(P) = 7.73 + 1.55 \times 10^{-3} P + 8.02 \times 10^{-7} P^2 \text{ (N.m)} \]  
\[ n(P) = 0.863 - 3.22 \times 10^{-4} P + 7.45 \times 10^{-8} P^2 \]  
with \( P \) in Newtons.

For negative rotations \( \theta_{xB} \), \( m = 221 \text{ lb. in} \) (25.0 N.m) and \( n = 0.857 \), for \( P \geq 800 \text{ lbs.} \). For \( P < 800 \text{ lbs.} \):

\[ m(P) = 72.9 + 0.298 P - 1.42 \times 10^{-4} P^2 \text{ (lb. in)} \]  
\[ n(P) = 0.829 - 2.28 \times 10^{-4} P + 3.25 \times 10^{-7} P^2 \]  
with \( P \) in lbs., or:

\[ m(P) = 8.24 + 7.56 \times 10^{-3} P - 8.12 \times 10^{-7} P^2 \text{ (N.m)} \]  
\[ n(P) = 0.829 - 5.12 \times 10^{-5} P + 1.64 \times 10^{-8} P^2 \]  
with \( P \) in Newtons.

For negative rotations \( \theta_{yB} \), \( m = 165 \text{ lb. in} \) (18.7 N.m) and \( n = 0.711 \), for \( P \geq 800 \text{ lbs.} \). For \( P < 800 \text{ lbs.} \):

\[ m(P) = 76.8 + 0.206 P - 1.20 \times 10^{-4} P^2 \text{ (lb. in)} \]  
\[ n(P) = 0.817 - 6.71 \times 10^{-4} P + 6.74 \times 10^{-7} P^2 \]  
with \( P \) in lbs., or:

\[ m(P) = 8.68 + 5.23 \times 10^{-3} P - 6.85 \times 10^{-7} P^2 \text{ (N.m)} \]  
\[ n(P) = 0.817 - 1.51 \times 10^{-4} P + 3.40 \times 10^{-8} P^2 \]  
with \( P \) in Newtons.
For negative rotations $\theta_{xI}$, $m = 206$ lb. in (23.3 N.m) and $n = 0.756$, for $P \geq 800$ lbs. For $P < 800$ lbs.:

\[
m(P) = 79.7 - 8.61 \times 10^{-2} P + 3.06 \times 10^{-4} P^2 \quad \text{(lb. in)} \quad (3.41)
\]

\[
n(P) = 0.704 - 1.15 \times 10^{-3} P + 1.52 \times 10^{-6} P^2 \quad (3.42)
\]

with $P$ in lbs., or:

\[
m(P) = 9.00 - 2.19 \times 10^{-3} P + 1.75 \times 10^{-6} P^2 \quad \text{(N.m)} \quad (3.43)
\]

\[
n(P) = 0.704 - 2.57 \times 10^{-4} P + 7.67 \times 10^{-8} P^2 \quad (3.44)
\]

with $P$ in Newtons.

For negative rotations $\theta_{yI}$, $m = 220$ lb. in (24.8 N.m) and $n = 0.658$, for $P \geq 800$ lbs. For $P < 800$ lbs.:

\[
m(P) = 67.1 + 0.227 P - 4.27 \times 10^{-5} P^2 \quad \text{(lb. in)} \quad (3.45)
\]

\[
n(P) = 0.881 - 9.84 \times 10^{-4} P + 8.94 \times 10^{-7} P^2 \quad (3.46)
\]

with $P$ in lbs., or:

\[
m(P) = 7.58 + 5.76 \times 10^{-3} P - 2.43 \times 10^{-7} P^2 \quad \text{(N.m)} \quad (3.47)
\]

\[
n(P) = 0.881 - 2.21 \times 10^{-4} P + 4.51 \times 10^{-8} P^2 \quad (3.48)
\]

with $P$ in Newtons.

The similar behavior found for the $x$ and $y$ directions of both end-restraint systems gives further evidence that the behavior of the end-restraints was largely friction independent at higher axial loads. Thus, experimental calibration results indicated that
each direction of each end-restraint system resists the rotations by restraint moment functions of the type:

\[ R = m(P) \theta^{n(P)} \]  

(3.49)

where \( m(P) \) and \( n(P) \) are given either by expression similar to (3.17) through (3.48) for \( 0 \leq P < 800 \text{ lbs} \), or have constant values for \( P \geq 800 \text{ lbs} \).

This refers to a specific calibration system with restraint bar reference lengths of 5.9 inches (15 cm). For end-restraint systems with restraint bar length different from the reference length, it was expected that the restraint moments would vary with restraint bar lengths. With complete generality the restraint moment functions can be approximated by:

\[ R = m(P) \left( \frac{\theta}{s} \right)^{n(P)} \]  

(3.50)

where \( s \) is the ratio between actual restraint bar length and reference length. This conclusion was verified experimentally for the bottom and top end restraint systems using new restraint bar lengths of 6.85 inches (17.4 cm) in both directions. At the rotation of one degree expression (3.50) predicted the bottom and top restraint moments with a maximum error of 7.5% and 4% respectively.
CHAPTER 4

TUBULAR BEAM-COLUMN EXPERIMENTAL INVESTIGATION

4.1 Introduction

This chapter outlines the experimental program undertaken as part of this investigation. The purpose of the program was to provide reliable results on the buckling and initial postbuckling behavior of end-restrained tubular beam-columns.

Included among preliminary tests were those to determine the mechanical and cross-section properties, as well as those measuring unavoidable initial crookedness of the columns. Four steel tubular columns, 24 inches long, were tested in axial compression. The columns were restrained on both ends by a complex end-restraint system. Preliminary tests on the characteristic properties of the end-restraint systems have been reported in detail (Chapter 3, [62]).

This chapter reports the results of an experimental study on the behavior of end-restrained, steel circular tubes, as spatial beam-columns, in a form useful for subsequent theoretical comparison. The study was prompted by the necessity of obtaining information on the strength and deflection characteristics of end-restrained tubular beam-columns. All the experiments as part of this investigation, were conducted in the Structural Engineering Laboratory of the University of Akron.

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The shape of the stress-strain curve and the magnitude of unavoidable geometric imperfections, such as out-of-roundness, affect the buckling strength of circular tubes. Therefore, a brief review is presented of the shape of the stress-strain curve and the degree of imperfections that can be expected in tubes of different materials produced by different methods.

The tubes examined herein have two different types of stress-strain curves: yield point and round-house. In the yield point curve, stress is linearly proportional to strain up to the yield point (elastic region) and thereafter is nearly constant over a large range of strain (plastic region). A round-house curve deviates from a linear relationship at stresses below the yield strength and does not exhibit a region in which stress remains constant over a large range of strains. The following classifications are intended only to group tubes with similar buckling properties and are not necessarily commercial or specification classifications.

4.2 Tensile Coupons

Structural steels, such as structural carbon steel, high strength low-alloy steel, and construction alloy steel, are widely used in tubes. All these steels have yield point stress-strain curves, but a full sized tube of any of these steels may have a round-house stress-strain curve after manufacture, as shown by stub column tests. Three types of manufactured structural-steel tubes may be considered. Seamless tubes have essentially yield point stress-strain curves, although cooling residual stresses cause a
slight rounding between the straight elastic and plastic parts of
the curve. But welded tubes and cold expanded or cold worked tubes
may have round-house stress-strain curves as the result of the
Bauschinger effect and residual stresses resulting from the manu-
facturing process.

Stainless steel tubes and aluminum alloy tubes have round-
house stress-strain curves regardless of the production method,
because the metal itself has a round-house curve. However, the
production method may affect the exact shape of the curves.

In the testing program, four circular tubular column
specimens were tested. They were cut from a manufactured structural
carbon steel seamless tube, with an average outside diameter of 1/2
inch and an average thickness of 0.09 inches.

Prior to the experiments with columns, tensile coupon tests
were made on short (10 in.) tubes of the same tubing to determine
the average stress-strain characteristics of the column when
instability effects are excluded. This information was also used
for the determination of the tangent modulus curves, needed for
the theoretical prediction of the critical load of the axially com-
pressed end restrained columns.

In accordance with the practice outlined in the supplement
II of the ASTM standard A370-77 [79], the tension test specimens,
cut from the tubing, were of full size tubular sections. To permit
the testing machine jaws to grip the specimens properly without
crushing, snug-fitting metal plugs were inserted in the ends of
the tubular tension specimens. These plugs did not extend into the
part of the specimen on which the elongation is measured (gage length zone). Dial gages were used for the measurement of strains within a 2 inch (50 mm) gage length [79]. Care was taken in proper alignment of the specimens. The load was applied axially in increments at a rate not exceeding 0.05 in/min, and readings of load and strain were taken at each increment.

A total of two tension tests were performed to determine a numerical average of the properties of the structural carbon steel seamless tubing used. Figure 18 shows the experimental stress-strain curve derived. The modulus of elasticity $E$ and the failure elongation were respectively $29 \times 10^6$ psi ($2 \times 10^{11}$ N/m$^2$) and 14.5%. Because the concepts of elastic limit and yield point are so useful in design, numerous artificial elastic limits and yield points have been suggested for use with materials for which these properties are ill-defined. A more rational approach is to use the proportional limit and yield point of an idealized similar material which really has these properties, and which differs as little as possible in behavior from the real material. Such approach, recently summarized by Donnel [28], is used herein. By plotting stress versus the slope of the stress-strain diagram, the resulting stress-slope curve of the actual material can be closely approximated by two straight lines, which constitute the stress-slope diagram of a similar ideal material: a vertical line at the slope $E$ and a diagonal line intersecting the actual material diagram at the slopes $E/4$ and $3E/4$. To this corresponds the stress-strain diagram of the similar material,
Figure 18. Experimental and Approximate Stress-Strain Curves
whose proportional limit $\sigma_p$ and yield point $\sigma_y$ --artificial proportional limit and yield point of the real material--are defined as:

$$\sigma_p = 1.5 \sigma_{3E/4} - 0.5 \sigma_{E/4}$$

(4.1)

$$\sigma_y = 1.5 \sigma_{E/4} - 0.5 \sigma_{3E/4}$$

(4.2)

where $\sigma_{E/4}$ and $\sigma_{3E/4}$ are the stresses at which the slopes of the stress-strain diagram of the real material are $E/4$ and $3E/4$ respectively.

With these definitions, the artificial proportional limit and yield point are 16875 psi ($1.16 \times 10^8$ N/m$^2$) and 64375 psi ($4.44 \times 10^8$ N/m$^2$), respectively. Figure 18 compares diagramatically the approximate and experimental curves. It can be observed that the approximate curve of an idealized similar material, closely depicts elastic-perfect plastic behavior.

4.3 Initial Imperfections

Beam-columns strength and behavior can be significantly influenced by at least two types of initial, geometric, manufacturing or fabricating imperfections: out-of-roundness and out-of-straightness. In recent works [20,60,84] considerable attention was given to the measurements of the initial crookedness in fabricated specimens. It was concluded that the effect of out-of-roundness on the column performance is negligibly small, since in general there is less than 1% difference between two perpendicular diameters at all positions along the column length. In fact, the API specifications [78] allow a maximum tolerance of 0.1% for
out-of-roundness fabrication imperfections. At all the stations of all the specimens, the out-of-roundness measurements confirmed the fact that out-of-roundness manufacturing imperfections are negligible. A consistent outside diameter of 0.5 in. (1.27 cm) was measured through two perpendicular planes. Therefore, considerable effort was devoted to the measurement of out-of-straightness imperfections, which may be a critical parameter on the columns performance. According to the same specifications [78] the maximum out-of-straightness amplitude specified for a prototype fabricated tubular column is 0.1%. Herein, some of the measurement procedures developed for the fabricated specimens were followed.

Four column specimens, 24 inches long (61 cm), were cut from the seamless tubing. Prior to the out-of-straightness measurements the columns were rolled on a flat surface to establish approximately for each specimen a plane of maximum out-of-straightness. Then, with this as reference, four longitudinal lines 90 degrees apart were established on each specimen using an alignment jig, defining two perpendicular diametrical planes. In general, it was possible to establish one of the diametrical planes close to the reference maximum out-of-straightness plane. Also, five circumferential lines were established on each specimen at stations located at 1/4, 3/8, 1/2, 5/8 and 3/4 of the column length. At the intersection of longitudinal and circumferential lines, out-of-roundness and out-of-straightness measurements were made. These line intersections were also useful for measuring column deflections during testing.
To obtain for each specimen the magnitude of the out-of-straightness, an alignment jig and a dial gage with a 1/1000 in. precision were used. The profiles of out-of-straightness along each set of two diametrically opposite longitudinal lines were obtained, from which average profiles of out-of-straightness through the two perpendicular diametrical planes were established. Typical out-of-straightness patterns are shown in Figures 19, 20, 21 and 22, for the column specimens nos. 1, 2, 3 and 4. Table 10 gives the magnitude of out-of-straightness and also the form of the out-of-straightness pattern, as observed prior to testing.

**TABLE 10. Maximum Column Out-of-Straightness**

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Plane A-C</th>
<th>Plane B-D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Inches (mm)</td>
<td>Form of Curvature</td>
</tr>
<tr>
<td>1</td>
<td>0.0033 (0.085)</td>
<td>Single</td>
</tr>
<tr>
<td>2</td>
<td>0.0016 (0.041)</td>
<td>Double</td>
</tr>
<tr>
<td>3</td>
<td>0.014 (0.36)</td>
<td>Single</td>
</tr>
<tr>
<td>4</td>
<td>0.0043 (0.11)</td>
<td>Single</td>
</tr>
</tbody>
</table>

An analysis of this table clearly shows the out-of-straightness amplitudes to be very small, the maximum being about 0.05%. However, the numerical values of out-of-straightness associated with the patterns shown in Figures 19 through 22, are crucial input data for computer numerical modeling of the beam-column performance.
Figure 20. Specimen 2  Average Out-of-Straightness as Measured Prior to Testing
Figure 21. Specimen 3 Average Out-of-Straightness as Measured Prior to Testing
Figure 22. Specimen 4 Average Out-of-Straightness as Measured Prior to Testing
4.4 Experimental Program

The four beam-column experiments were conducted on an electro-hydraulic universal testing machine. Each column specimen was fitted between the bottom and top complex end-restraint systems, described in Chapter 3. The restraint systems were each screwed to the end plates of the testing machine. The axial compressive load, P, was applied through the rigid central bearing of the end-restraint fixtures. A view of the general test apparatus is shown in the photograph of Figure 23.

The testing machine was operated in displacement control mode, which permitted the postbuckling behavior of the column specimens to be monitored. Curves of load P versus overall column shortening, were drawn automatically on an X-Y plotter connected to the control panel of the testing machine. The applied axial load was also measured by a digital voltmeter connected to the load cell.

In the process of observing the buckling and postbuckling behavior of a beam-column, it is necessary to measure lateral deflections as well as end rotations of the end fixtures. These measurements were taken in the same perpendicular longitudinal planes used for the out-of-straightness measurements. For each column specimen, the two longitudinal planes were made coincident with the XX and YY planes defined by the torsion bars of the end-restraint systems (Figure 24), using a plumb-line device.

It was decided to measure lateral deflections at midheight, where beam-column deflections were likely to be largest. However, these measurements posed some problems. The direction of lateral
Figure 23. General View of a Column under Testing
deflection was unpredictable, and in general varied during loading. Also, deflection measurements were taken at a point on a curved contour. These difficulties were overcome by using deflection-measuring transducers, attached to a support on the testing machine at midheight of the column, on which a horizontal string could be attached. Figure 25 shows this arrangement.

To ensure that only one deflection in only one direction was measured, the string was held in place 0.18 to 0.20 m above the center of the column being tested. This produced a negligible affect on the transducer readings. The transducers were required to record the deflections for the other directions.

The results of the rotations of the end-restraint system are presented in a differential form associated with the development of the end-restraint system already described in Chapter 3. To ensure a conditioning of the end-restraint system similar to the calibration point, a measured predetention axial load was applied to the columns followed by subsequent unloading. By this procedure, any initial looseness in the system was minimized. This proved to be successful since only one direction of the ten end-restraint system had measured rotations that seemed to have some looseness for low values of loading.

Figure 24. Detailed View of a Column under Testing
deflection was unpredictable, and in general varied during loading. Also, deflection measurements were taken at a point on a curved contour. These difficulties were overcome by using deflection-measuring transducers, attached to a support on the testing machine at midheight along the column specimen length, from which a horizontal string could be stretched to (and around) the column specimen. Figure 25 shows a photograph of this measuring system.

To ensure that each transducer would measure deflections in only one direction, the string length used was 7 to 8 in. (0.18 to 0.20 m), which is considered long with respect to the cross section dimensions of the column specimen. Any movement of the column perpendicular to the string produced a negligible effect on the transducer reading. A separate calibration of the transducers was required. One transducer-string apparatus was used to measure deflections, at midheight, in each of two perpendicular directions.

The measurements of the central bearing head rotations of the end-restraint systems, were done with rotational variable differential transducers (RVDT's). The difficulties associated with the development of the end-restraint systems were already described in Chapter 3. To ensure a conditioning of the end-restraint systems similar to the calibration phase, a small predetermined axial load was applied to the columns followed by subsequent unloading. By this procedure, any initial looseness in the system was minimized. This proved to be successful since only one direction of the top end-restraint system had measured rotations that seemed to have some looseness for low values of loading.
Values of end eccentricities were essential for the theoretical analysis, in which the column is treated as a member under biaxial bending and axial load. Although care was taken in the alignment of each test specimen, the ideal geometric condition in which the center of each end of the specimen is aligned with the center of the bearing block at that end, is difficult to obtain in practice. For each beam-column experiment, the best possible alignment was obtained and the remaining end eccentricities were determined.

Table gives the measured eccentricities between the center of tubes and the bearing block. It seems that the reasons for these small eccentricities may be associated with the loading and the handling of the specimens. The table below shows the eccentricities:

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Y Eccentricity (inches)</th>
<th>X Eccentricity (inches)</th>
<th>Z Eccentricity (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.12</td>
<td>0.15</td>
<td>-0.01</td>
</tr>
<tr>
<td>2</td>
<td>-0.12</td>
<td>0.12</td>
<td>-0.01</td>
</tr>
<tr>
<td>3</td>
<td>-0.12</td>
<td>0.08</td>
<td>-0.04</td>
</tr>
<tr>
<td>4</td>
<td>-0.12</td>
<td>0.19</td>
<td>-0.09</td>
</tr>
</tbody>
</table>

Figure 25. Deflection-Measuring Transducer

The experimental program tested four circular tubular beam-columns with the same length and radius of gyration, but different end-restraint characteristics and initial crookednesses. The
Values of end eccentricities are essential for the theoretical analysis, in which the column is treated as a member under biaxial bending and axial load. Although care was taken in the alignment of each test specimen, the ideal geometric condition in which the center of each end of the specimen is aligned with the center of the bearing block at that end, is difficult to obtain in practice. For each beam-column experiment, the best possible alignment was obtained and the remaining end eccentricities were determined. Table 11 gives the measured end eccentricities between the center of tubes and the center of end blocks. It seems that the reason for these eccentricities lies in some inherent misalignment between the loading machine heads. Notwithstanding this, experiments showed the end eccentricities to be quite small.

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Bottom Head</th>
<th>Top Head</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X-Direction</td>
<td>Y-Direction</td>
</tr>
<tr>
<td>1</td>
<td>-0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>2</td>
<td>-0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>-0.12</td>
<td>0.08</td>
</tr>
<tr>
<td>4</td>
<td>-0.12</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The experimental program tested four circular tubular beam-columns with the same length and radius of gyration, but different end-restraint characteristics and initial crookednesses. The
combinations of end-restraint properties were chosen in such a way to cover the range of end restraint characteristics evaluated in Chapter 3. The end-restraint moment functions offered by the end-restraint fixtures, depend on the restraint bar lengths used in the x and y directions of the complex bottom and top restraint systems. Table 12 summarizes the ratios s between actual restraint bar lengths, in each coordinate direction of each end-restraint system, and the reference length of 5.9 in. (15 cm). These ratios are used in determining the end-restraint moment functions by expressions analogous to (3.50).

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Bottom Restraint</th>
<th>Top Restraint</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X-Direction</td>
<td>Y-Direction</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2</td>
<td>1.333</td>
<td>1.333</td>
</tr>
<tr>
<td>3</td>
<td>1.333</td>
<td>0.667</td>
</tr>
<tr>
<td>4</td>
<td>0.667</td>
<td>0.667</td>
</tr>
</tbody>
</table>

TABLE 12. End Restraints Properties (s ratio)

With the loading machine in displacement control mode, the procedure for each test was as follows:

(i) The axial load (displacement) was applied in increments. The size of the load increment varied, depending how close to the expected buckling load was the applied load.

(ii) After each load (displacement) increment, sufficient time was allowed for the system to come to a complete rest, attaining
equilibrium before the measurements of column behavior were taken. In this way the effects of strain rate were eliminated and the measurements of column behavior (axial load, lateral deflections and end rotations) represented a static condition. Because of the need to monitor postbuckling beam-column behavior in the inelastic range, increments of deflection rather than increments of load were used.

(iii) Loading was usually continued beyond the buckling load until the column was so deformed (Figure 26) that the decrease in column strength could be monitored.

A considerable portion of postbuckling phase was observed for each specimen. An additional criteria for limiting the observed postbuckling phase, was based on allowable limits of elastic usefulness of the end-restraint systems. In order to prevent damage to the complex joint end-restraint systems they could be stressed only in the elastic range of material behavior.

4.5 Test Results and Discussion

The results of experiments are here individually presented in graph and tabular form. In order to compare the predicted theoretical values and the experimental values obtained, graphs are drawn with the data furnished from each column specimen. For each specimen, the graphs drawn display the displacement path at midheight, the bottom and top end rotation paths, and the static axial load versus total midheight displacement. The set of figures, Figures 27 through 30 (for specimen 1), Figures 31 through 34 (for specimen 2),
Figure 26. Deformed Column in Postbuckling Phase
Figure 27. Midheight Displacement Path (for Specimen 1)
Figure 28. Bottom Rotation Path (for Specimen 1)
Figure 29. Top Rotation Path (for Specimen 1)
Figure 30. Load-Lateral Displacement at Midheight Data (for Specimen 1)
Figure 31. Midheight Displacement Path (for Specimen 2)
Figure 32. Bottom Rotation Path (for Specimen 2)
Figure 33. Top Rotation Path (for Specimen 2)
LOAD-DISPLACEMENT DATA

Figure 34. Load-Lateral Displacement at Midheight Data (for Specimen 2)
Figures 35 through 38 (for specimen 3), Figures 39 through 42 (for specimen 4), show buckling and postbuckling behavior and strength of the four end-restrained beam-columns tested.

A characteristic of all the specimens was that significant lateral movement was observed at 70%-80% of the recorded maximum static column axial load (buckling load). Furthermore, buckling was approached gradually, followed by a zone of relatively large lateral deflections--postbuckling phase--at which the sustained load was 90%-97% of the buckling load. This shows that the beam-column had a relatively high postbuckling load capacity, because the elastic end-restraint systems provided part of the observed postbuckling strength. The transducer-string apparatus monitored the postbuckling midheight deflections, which in previous experimental research had been difficult to measure. Considerable elastic straightening of the specimens was observed as the applied load was released after the tests.

The low D/t ratio of the column specimens (D/t = 5.55) ensures the buckling instability is characterized by general yielding of the material at the critical section, without significant cross section distortion. Therefore, the observed buckling modes were of the general inelastic instability type (Figure 26), rather than interactive instability which is characterized by high cross section distortions at a localized section. In fact, out-of-roundness measurements at all stations of all the buckled specimens did not reveal any measurable cross section ovalization.
Figure 35. Midheight Displacement Path (for Specimen 3)
Figure 36. Bottom Rotation Path (Specimen 3)
Figure 37. Top Rotation Path (for Specimen 3)
Figure 38. Load-Lateral Displacement at Midheight Data
(for Specimen 3)
Figure 39. Midheight Displacement Path (for Specimen 4)
Figure 40. Bottom Rotation Path (for Specimen 4)
Figure 41. Top Rotation Path (for Specimen 4)
Figure 42. Load-Lateral Displacement at Midheight Data (for Specimen 4)
Tables 13 and 14 illustrate typical postbuckling behavior observed. They contain a brief description of the direction of buckling, total midheight deflections and end rotations of the beam-columns, at the onset of buckling and at the end of each beam-column experiment. The maximum lateral deflections measured at midheight of each specimen are approximately of the order of 1.0%–2.3% of the column length. At these deflections, the remaining axial load-carrying capacity of the beam-columns is of the order of 90% of the buckling load (Table 15). For each column the critical direction of buckling changed slightly from the onset of buckling until the end of the experiment, except for specimen 3 for which a secondary critical buckling direction was observed.

### TABLE 13. Buckling Direction and Midheight Deflection, at Buckling and at End of Tests

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Direction of buckling with respect to XX plane</th>
<th>Total Midheight Deflection</th>
<th>( \frac{\delta_m}{\delta_b} )</th>
<th>( \frac{\delta_m}{L} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>At buckling</td>
<td>At end of test</td>
<td>At buckling</td>
<td>At end of test</td>
</tr>
<tr>
<td>1</td>
<td>-42°</td>
<td>-35°</td>
<td>0.185 (4.70)</td>
<td>0.374 (9.51)</td>
</tr>
<tr>
<td>2</td>
<td>-46°</td>
<td>-40°</td>
<td>0.193 (4.90)</td>
<td>0.538 (13.7)</td>
</tr>
<tr>
<td>3</td>
<td>-41°</td>
<td>-29°</td>
<td>0.159 (4.03)</td>
<td>0.373 (9.47)</td>
</tr>
<tr>
<td>4</td>
<td>-31°</td>
<td>-29°</td>
<td>0.173 (4.40)</td>
<td>0.258 (6.55)</td>
</tr>
</tbody>
</table>
TABLE 14. End Rotations of Columns, at Buckling and at End of Tests

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Bottom End Rotation $\theta_B$ degrees</th>
<th>Top End Rotation $\theta_T$ degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\theta_{B,b}$ At buckling</td>
<td>$\theta_{B,m}$ At end of test</td>
</tr>
<tr>
<td>1</td>
<td>1.17</td>
<td>1.94</td>
</tr>
<tr>
<td>2</td>
<td>0.99</td>
<td>2.84</td>
</tr>
<tr>
<td>3</td>
<td>0.92</td>
<td>2.10</td>
</tr>
<tr>
<td>4</td>
<td>0.87</td>
<td>1.28</td>
</tr>
</tbody>
</table>

|                 | $\theta_{T,b}$ At buckling             | $\theta_{T,m}$ At end of test     |
| 1               | 0.89                                   | 1.68                              |
| 2               | 0.75                                   | 2.68                              |
| 3               | 0.56                                   | 1.77                              |
| 4               | 0.63                                   | 1.09                              |

TABLE 15. Maximum Column Loads

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Static Column Loads $P_s$ lbs (Newtons)</th>
<th>Dynamic-to-Static Column Load Ratio at Buckling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P_{s,b}$ At buckling</td>
<td>$P_{s,m}$ At end of test</td>
</tr>
<tr>
<td>1</td>
<td>1260 (5607)</td>
<td>1221 (5433)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1179 (5247)</td>
<td>1108 (4931)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1454 (6470)</td>
<td>1292 (5749)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1333 (5932)</td>
<td>1300 (5785)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A summary of the buckling loads and the maximum measured loads (static) for the four beam-columns tested, is given in Table 15. The distinction between "static" axial buckling load, $P_s$, and "dynamic"
axial buckling load, $P_d$, as obtained during testing, has already been established [39,60]. It was found that the difference between static and dynamic loads is usually noticeable only as the axial load approaches the column buckling load, beyond which the applied load on the column tends to drop to its static value. Table 15 shows additionally the dynamic-to-static column load ratio at buckling, which varied within the range $P_d/P_s = 1.02$ to $1.04$. 
CHAPTER 5
ELASTO-PLASTIC TUBULAR BEAM-COLUMN BUCKLING ANALYSIS

5.1 Introduction

This chapter outlines the theory used in this dissertation for the analysis of the space behavior and strength of long tubular beam-columns. The theoretical formulation can be applied to both manufactured and fabricated tubes since it has the ability of modeling various parameters influencing column space behavior and strength, namely: material inelasticity, initial imperfections, eccentricities in the axial compressive force, residual stresses and end restraints. The results of this analytical method will be compared in Chapter 7 with the available experimental data base outlined in Chapter 4.

During the pre-buckling stage and until the buckling load, the Influence Coefficient Method is used. In this efficient numerical procedure, the displacements are first assumed at each point along the length of the beam-column, from which the corresponding curvatures, strains, stresses and internal moments are calculated. Using these internal moments and the applied loads, new values of displacements are calculated from the equilibrium conditions. In order to ascertain the convergence of the solution, modified values of displacement are calculated through an influence coefficient matrix, using a Newton-Raphson iterative technique.
Virdi and Dowling [86] first used the Influence Coefficient Method with equally spaced stations to obtain the ultimate strength of a pin-ended composite beam-column. Although in their analysis all the twisting effects were ignored, the convergence was achieved extremely rapidly because of the high twisting rigidity of the solid cross sections. The method is applicable to many biaxially loaded beam-columns of solid cross section, as well as manufactured and fabricated thin-walled steel tubular sections. Ross [60] also used this method to study the strength and behavior of fabricated tubular steel columns, idealized as an assemblage of short "cans" of highly variable lengths.

The elasto-plastic material behavior was already defined in Chapter 4, together with the initial out-of-straightness and eccentricities of loading of the column specimens. In Chapter 3 the general end restraint characteristics were derived (force boundary functions). In the following analysis of the buckling space behavior and strength it is assumed that the moment-axial load-curvature relationships (M-P-φ curves) of tubular columns sections are known—or may be established numerically—for each stage of loading.

For in-plane behavior the nonlinear moment-curvature relation, for a given axial load, is computed by an incremental iterative procedure. The cross section is idealized into a number of small area elements in each of which the normal stress is assumed uniform. Then, the axial load and bending moment are calculated from a proposed plane strain profile across the cross section. If they agree closely with the applied loading to the section, the corresponding
Curvature is taken as correct. Otherwise the plane strain profile is modified and the procedure repeated. Therefore, the complete moment-curvature relation corresponding to a given axial load can be derived. Ellis [30] studied the in-plane plastic behavior of annular columns (with no residual stresses) loaded unsymmetrically by unequal end moments. Closed-form expressions for the moment, the axial load and the curvature were obtained in parametric form, from which the moment-curvature curve for a given axial load can be determined by solving the two simultaneous transcendental equations expressing the moment and the axial load. More recently, Wagner, Mueller and Erzurumlu [87] presented an open-form solution technique to determine the M-P-Φ data of tubular beam-column steel sections. As developed, their model permits the inclusion of nonlinear configurations of stress-strain relationships and longitudinal residual stress patterns.

However, to obtain the moment-curvature curves for given axial loads requires a time-consuming set of calculations. The computational time can be reduced significantly, if closed-form moment-axial load-curvature expressions are developed to approximate the actual behavior of the cross sections. Using the formulation of reference [15], simple analytical expressions to fit the real moment-axial load-curvature curves of tubular cross section to a high degree of approximation, can be derived. The results, recently presented by Saleeb and Chen [64], compare the actual exact M-P-Φ curves with the approximate M-P-Φ curves. The fact that actual and approximate curves are almost identical proves the high degree of accuracy of the approximation.
For space behavior, the computation of the moment-axial load-curvature relationships is further complicated not only by dependence on the history of loading but also by two-dimensional residual stress distribution patterns. No closed-form expressions and/or approximations are possible in this case.

The M-P-\( \phi \) curves for space behavior define the biaxial moments \( M_x, M_y \), in terms of the biaxial curvatures \( \phi_x, \phi_y \) and the axial load \( P \), by

\[
M_x = M_x (\phi_x, \phi_y, P) \quad (5.1)
\]

\[
M_y = M_y (\phi_x, \phi_y, P) \quad (5.2)
\]

These relations are obtained by performing a nonlinear analysis of a column section of unit height by means of the tangent stiffness method [18,65].

Basically, it is assumed that the plane sections remain plane after bending and that the stress-strain relations of the material are elastic-perfectly plastic. Then, the elements of the tangent stiffness matrix and the internal generalized stress resultants (axial load \( P \), biaxial moments \( M_x, M_y \)) of a plastic section are evaluated numerically, by dividing the cross section into finite area elements as shown in Figure 43. The strains and stresses in each element are computed as the average value at its centroid, at every stage of deformation and loading. The loading history, applied in an incremental manner to a unit height column section, is as follows:
Figure 43. Division of a Cross Section into Elemental Areas
(i) The required axial load is applied, and the equilibrium iterations are performed.

(ii) With the axial load $P$ held constant, a new set of equilibrium iterations are performed for the applied biaxial moment $M_x$.

(iii) With both axial load $P$ and biaxial moment $M_x$ held constant, the moment-curvature relationship $(M_y, \phi_y)$ for a gradually increasing biaxial moment $M_y$, is developed in an incremental manner.

The processing and fabrication of steel tubular beam-columns, as used in multistory and offshore oil drilling structures, induces significant circumferential and longitudinal residual stresses. Ross [60] derived the $M$-$P$-$\phi$ curves for a fabricated steel tubular section with various combinations of applied loads and inclusion of residual stresses in two perpendicular directions. Typical nondimensional results have shown that residual stresses significantly affect the knee portion of the $M$-$P$-$\phi$ curves, without altering the ultimate moment plateaux (plastic limit moment).

A fabricated tubular steel column has highly discontinuous section properties along its length. Not only each individual "can" of the column assemblage has variable length and/or different location of the longitudinal weld (longitudinal welds staggering), but also the steel in different "cans" may be of different yield strengths. Therefore, it is appropriate to develop a numerical procedure allowing for different distances between successive stations along the column length. At each station independent section properties may be
considered. These include material behavior characteristics (such as, directional location of longitudinal weld and yield strength) and random out-of-straightness profile measured in two perpendicular directions along the column length.

The present research extends the previous work of Ross, Chen and Tal [61] by including the effects of end restraints in two perpendicular directions, which have not been adequately considered in the available references [18, 61].

5.2 Mathematical Formulation of the Influence Coefficient Method

Consider a beam-column of length \( L \), subjected to an axial compressive load \( P \) and biaxial end moments \( M_{xB}, M_{yB} \) at the bottom end \( B \) and \( M_{xT}, M_{yT} \) at the top end \( T \) (Figure 44). Denote by \( e_{xB}, e_{yB} \) and \( e_{xT}, e_{yT} \) the \( x \) and \( y \) end eccentricities of the axial load at the bottom and top ends, respectively. The space beam-column is restrained at the ends, in the two perpendicular \( x,y \) directions, by nonlinear springs whose moment-rotation relationships are assumed known.

Let \( u_o \) and \( v_o \) denote the initial lateral deflections (out-of-straightness) of the centroid of the beam-column cross sections in an unloaded state. If \( u \) and \( v \) are assumed net lateral deflections of the centroid of the cross sections, the total curvatures of the column in the \( yz \) and \( xz \) planes are respectively

\[
\phi_x = -\frac{v'' + v'}{[1 + (v'_o + v')^2]^{3/2}} \quad (5.3)
\]

\[
\phi_y = \frac{u'' + u'}{[1 + (u'_o + u')^2]^{3/2}} \quad (5.4)
\]
Figure 44. Notation for Beam-Column Analysis
Under the assumption of small deflection theory \(((u'_0 - u'), (v'_0 + v')) \ll 1\). The total curvatures may be written as \(\Phi_x = -(v''_0 + v'')\) and \(\Phi_y = u''_0 + u''\) from which the initial curvatures of the unloaded column \(\phi_{ox}, \phi_{oy}\) and the net curvatures \(\phi_x, \phi_y\) (respectively in the yz and xz planes) may be written as

\[
\phi_{ox} = -\frac{\partial^2 v_0}{\partial z^2} \tag{5.5}
\]

\[
\phi_{oy} = \frac{\partial^2 u_0}{\partial z^2} \tag{5.6}
\]

\[
\phi_x = -\frac{\partial^2 v}{\partial z^2} \tag{5.7}
\]

\[
\phi_y = \frac{\partial^2 u}{\partial z^2} \tag{5.8}
\]

Assigning a discrete number of stations \(i\) to \(n+1\) from bottom end \(B\) to top end \(T\), with varying interstation distances \(h_i\) as defined in Figure 45, the differential expressions can be computed by finite difference operators. At every station \(i\) along the column length, the equilibrium equations can be expressed by

\[
M_{xi} = P (v_{oi} + v_i) + M_{xB} + (M_{xT} - M_{xB}) \frac{z_i}{L} \tag{5.9}
\]

\[
M_{yi} = -P (u_{oi} + u_i) + M_{yB} + (M_{yT} - M_{yB}) \frac{z_i}{L} \tag{5.10}
\]

where

\[
z_i = \sum_{s=1}^{i-1} h_s \tag{5.11}
\]
Figure 45. Defining Terms in yz Plane for Curvature Derivation
Solving equations (5.9) and (5.10) for the net deflections \( u_i \) and \( v_i \) at the station \( i \) gives

\[
\begin{align*}
    u_i &= -\frac{1}{P} \left[ M_{yi} - (1 - \frac{z_i}{L}) M_{yB} - \frac{z_i}{L} M_{yT} \right] - u_{oi} \quad (5.12) \\
    v_i &= \frac{1}{P} \left[ M_{xi} - (1 - \frac{z_i}{L}) M_{xB} - \frac{z_i}{L} M_{xT} \right] - v_{oi} \quad (5.13)
\end{align*}
\]

The internal moments \( M_{xi}, M_{yi} \) are calculated from the moment-axial load-curvature relations (Equations (5.1) and (5.2)) by

\[
\begin{align*}
    M_{xi} &= M_x (\phi_{xi}, \phi_{yi}, P) \quad (5.14) \\
    M_{yi} &= M_y (\phi_{xi}, \phi_{yi}, P) \quad (5.15)
\end{align*}
\]

in which, referring to Figure 45, the biaxial curvatures \( \phi_{xi}, \phi_{yi} \) are calculated from the net deflections \( v, u \) by finite difference operators as

\[
\begin{align*}
    \phi_{xi} = \left( -\frac{\partial^2 v}{\partial z^2} \right)_i &= \frac{-2}{h_i(h_{i-1} + h_i)} \left[ v_{i+1} - (1 + \frac{h_i}{h_{i-1}}) v_i + \frac{h_i}{h_{i-1}} v_{i-1} \right] \\
    \phi_{yi} = \left( \frac{\partial^2 u}{\partial z^2} \right)_i &= \frac{2}{h_i(h_{i-1} + h_i)} \left[ u_{i+1} - (1 + \frac{h_i}{h_{i-1}}) u_i + \frac{h_i}{h_{i-1}} u_{i-1} \right] \\
    & \quad (5.16)
\end{align*}
\]

In practice, however, it is not necessary to obtain the whole \( M-P-\phi \) curve to obtain the internal generalized stress resultants \( M_{xi}, M_{yi} \). In fact, with an axial strain value computed from the applied axial load and with the curvatures computed from equations (5.16) and (5.17), it is possible to sum their effects over elemental areas of the cross section and compute directly the stress resultants \( M_{xi}, M_{yi} \) at each station.
Equilibrium at the ends of the beam-column expresses the biaxial end moments $M_{xB}', M_{yB}', M_{xT}, M_{yT}$ in terms of the externally applied end moments $\overline{M}_{xB}', \overline{M}_{yB}', \overline{M}_{xT}, \overline{M}_{yT}$ and the end restraint moments $R_{xB}', R_{yB}', R_{xT}, R_{yT}$. In accordance with the sign convention outlined in Figure 44, these equilibrium relations are

\begin{align}
M_{xB} &= \overline{M}_{xB} - R_{xB} \\
M_{yB} &= \overline{M}_{yB} + R_{yB} \\
M_{xT} &= \overline{M}_{xT} - R_{xT} \\
M_{yT} &= \overline{M}_{yT} + R_{yT}
\end{align}

in which the externally applied end moments (here produced by the end eccentricities of the axial load) are

\begin{align}
\overline{M}_{xB} &= -P \, e_{yB} \\
\overline{M}_{yB} &= P \, e_{xB} \\
\overline{M}_{xT} &= -P \, e_{yT} \\
\overline{M}_{yT} &= P \, e_{xT}
\end{align}

The nonlinear end restraint moment functions are expressed in terms of the corresponding net end rotations $\theta_{xB}', \theta_{xT}, \theta_{yB}', \theta_{yT}$ in the manner derived already in Chapter 3. Referring to Figure 46 and using the same sign convention for end rotations and end moments, the net end rotations are calculated by finite difference operators as
Figure 46. Slope Conditions at Bottom End B in the yz Plane.
\[ \theta_{xB} = \left( \frac{\partial v}{\partial z} \right)_B = \frac{-1}{h_2(1 + \frac{h_2}{h_1})} \left[ v_3 - (1 + \frac{h_2}{h_1})^2 v_2 \right] \] (5.26)

\[ \theta_{xT} = \left( -\frac{\partial v}{\partial z} \right)_T = \frac{-1}{h_{n-1}(1 + \frac{h_{n-1}}{h_n})} \left[ v_{n-1} - (1 + \frac{h_{n-1}}{h_n})^2 v_n \right] \] (5.27)

\[ \theta_{yB} = \left( -\frac{\partial u}{\partial z} \right)_B = \frac{1}{h_2(1 + \frac{h_2}{h_1})} \left[ u_3 - (1 + \frac{h_2}{h_1})^2 u_2 \right] \] (5.28)

\[ \theta_{yT} = \left( \frac{\partial u}{\partial z} \right)_T = \frac{1}{h_{n-1}(1 + \frac{h_{n-1}}{h_n})} \left[ u_{n-1} - (1 + \frac{h_{n-1}}{h_n})^2 u_n \right] \] (5.29)

Establishing the equilibrium of all the stations along the column length, results in the column equilibrium equations

\[ u_i = U_i (u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1}) \] (5.30)

\[ v_i = V_i (u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1}) \] (5.31)

for \( i = 1, 2, \ldots, n+1 \), in which \( U_i \) and \( V_i \) are known functions of the assumed displacements \( u \) and \( v \) at all the stations, mapping the \( 2(n+1) \)-dimensional space, \( \mathbb{R}^{2(n+1)} \), into the real line \( R \). The above relations clearly define fixed point iterations for functions of several variables [13]. Using vector notation to represent the deflections at all stations, these relations can be written for simplicity as

\[ \{w\} = \{F \{w\}\} = \{U \{w\}\} = \{V \{w\}\} \] (5.32)

in which \( \{w\} = (u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_{n+1})^T \), and \( F \) is a function mapping \( \mathbb{R}^{2(n+1)} \) into \( \mathbb{R}^{2(n+1)} \) by
\{ F(w) \} = (U_1(w), U_2(w), \ldots U_{n+1}(w), V_1(w), V_2(w), \ldots V_{n+1}(w))^T \quad (5.33)

Alternatively, \{ u \} = (u_1, u_2, \ldots u_{n+1})^T, \{ v \} = (v_1, v_2, \ldots v_{n+1})^T, and the functions U, V map \mathbb{R}^{2(n+1)} into \mathbb{R}^{n+1}.

The displacement boundary conditions ensure that the deflections at both ends of the column are always zero (u_1 = v_1 = u_{n+1} = v_{n+1} = 0). Therefore, the equilibrium equations of station 1 and n+1 are deleted.

The functional fixed point iteration defined by equation (5.32) may be efficiently solved by the application of the Newton-Raphson iterative technique for nonlinear systems of equations. If \{ w^k \} is a vector of approximate deflections after k iterations, an improved solution \{ w^{k+1} \}, after k+1 iterations, is calculated by

\{ w^{k+1} \} = \{ w^k \} - [I - J]^{-1} \{ w^k - \bar{w} \} \quad (5.34)

in which \{ \bar{w} \} = \{ F((w^k)) \}, [I] is the unit matrix of rank (n-1) and [J] is a Jacobian matrix whose terms, at each iteration K, are defined as

\[ J_{ij} = \frac{\partial F_i}{\partial w_j} \quad (5.35) \]

The iterative technique outlined above is repeated until for a particular iteration k

\[ \frac{||\{ w^k \} - \{ \bar{w} \} ||_2}{||\{ w^k \} ||_2} < \epsilon_D \quad (5.36) \]

where ||.||_2 is a Euclidean norm and \epsilon_D is a convergence tolerance for displacements whose value usually is of order 10^{-2} to 10^{-6},
depending on the desired accuracy and available computer equipment [5]. Details of derivation of the iterative recurrence relation (equation (5.34)) are given in the Appendix.

5.3 Explicit Form of the Jacobian Matrix

Referring to equation (5.32), the Jacobian Matrix \([J]\) may be partitioned accordingly as

\[
[J] = \begin{bmatrix}
\frac{\partial U}{\partial u} & \frac{\partial U}{\partial v} \\
\frac{\partial V}{\partial u} & \frac{\partial V}{\partial v}
\end{bmatrix}
\] (5.37)

in which each submatrix is calculated for each iteration \(k\) of the iterative procedure. (The superscript \(k\) has been omitted for clarity.) Substituting the right hand side of equation (5.12) and (5.13) into equations (5.30) and (5.31), permits expressing the elements of the Jacobian matrix explicitly. These are

\[
\frac{\partial U_i}{\partial u_j} = -\frac{1}{\rho} \left[ \frac{\partial M_i y_i}{\partial u_j} - \left(1 - \frac{z_i}{L}\right) \frac{\partial M_i y_B}{\partial u_j} - \frac{z_i}{L} \frac{\partial M_i y_I}{\partial u_j} \right]
\] (5.38)

\[
\frac{\partial U_i}{\partial v_j} = -\frac{1}{\rho} \left[ \frac{\partial M_i y_i}{\partial v_j} - \left(1 - \frac{z_i}{L}\right) \frac{\partial M_i y_B}{\partial v_j} - \frac{z_i}{L} \frac{\partial M_i y_I}{\partial v_j} \right]
\] (5.39)

\[
\frac{\partial V_i}{\partial u_j} = \frac{1}{\rho} \left[ \frac{\partial M_i x_i}{\partial u_j} - \left(1 - \frac{z_i}{L}\right) \frac{\partial M_i x_B}{\partial u_j} - \frac{z_i}{L} \frac{\partial M_i x_I}{\partial u_j} \right]
\] (5.40)

\[
\frac{\partial V_i}{\partial v_j} = \frac{1}{\rho} \left[ \frac{\partial M_i x_i}{\partial v_j} - \left(1 - \frac{z_i}{L}\right) \frac{\partial M_i x_B}{\partial v_j} - \frac{z_i}{L} \frac{\partial M_i x_I}{\partial v_j} \right]
\] (5.41)
The partial derivatives on the right hand side of equation (5.38) through (5.41) can be systematically obtained using equations (5.16) through (5.21), and (5.26) through (5.29). With respect to equation (5.38)

$$\frac{\partial M_{yi}}{\partial u_j} = \frac{\partial M_{yi}}{\partial \phi_{yi}} \frac{\partial \phi_{yi}}{\partial u_j}$$

$$= \begin{cases} \frac{2}{h_i(h_{i-1} + h_i)} & , (j = i + 1) \\ -\frac{2}{h_{i-1} h_i} & , (j = i) \\ \frac{2}{h_{i-1}(h_{i-1} + h_i)} & , (j = i-1) \\ 0 & \text{otherwise} \end{cases}$$

(5.42)

$$\frac{\partial M_{yB}}{\partial u_j} = \frac{\partial M_{yB}}{\partial \theta_{yB}} \frac{\partial \theta_{yB}}{\partial u_j}$$

$$= \begin{cases} -\frac{(h_1 + h_2)}{h_1 h_2} & , (j = 2) \\ \frac{h_1}{h_2(h_1 + h_2)} & , (j = 3) \\ 0 & , (j > 3) \end{cases}$$

(5.43)
\[
\frac{\partial M_{yi}}{\partial u_j} = \frac{\partial M_{yi}}{\partial \theta_{yi}} \frac{\partial \theta_{yi}}{\partial u_j}
\]

\[
\left\{
\begin{array}{l}
-\frac{h_{n-1} + h_n}{h_{n-1} h_n}, (j = n) \\
\frac{h_n}{h_{n-1}(h_{n-1} + h_n)}, (j = n-1) \\
0, (j < n-1)
\end{array}
\right.
\]

Therefore,

\[
-\frac{2}{h_{i-1} + h_i} \frac{\partial M_{yi}}{\partial \theta_{yi}} = h_{i-1} \frac{\partial M_{yi}}{\partial u_{i-1}} = h_i \frac{\partial M_{yi}}{\partial u_{i+1}} = -\frac{h_{i-1} h_i}{h_{i-1} + h_i} \frac{\partial M_{yi}}{\partial u_i}
\]

(5.45)

\[
\frac{h_1}{h_2(h_1 + h_2)} \frac{\partial R_{yB}}{\partial \theta_{yB}} = \frac{\partial M_{yB}}{\partial u_3} = \frac{-h_1^2}{(h_1 + h_2)^2} \frac{\partial M_{yB}}{\partial u_2}
\]

(5.46)

\[
\frac{h_n}{h_{n-1}(h_{n-1} + h_n)} \frac{\partial R_{yi}}{\partial \theta_{yi}} = \frac{\partial M_{yi}}{\partial u_{n-1}} = \frac{-h_n^2}{(h_{n-1} + h_n)^2} \frac{\partial M_{yi}}{\partial u_n}
\]

(5.47)

With respect to equation (5.41)

\[
\frac{\partial M_{xi}}{\partial v_j} = \frac{\partial M_{xi}}{\partial \phi_{xi}} \frac{\partial \phi_{xi}}{\partial v_j}
\]

\[
\left\{
\begin{array}{l}
-\frac{2}{h_i(h_{i-1} + h_i)}, (j = i+1) \\
\frac{2}{h_{i-1} h_i}, (j = 1) \\
-\frac{2}{h_{i-1}(h_{i-1} + h_i)}, (j = i-1) \\
0, \text{ otherwise}
\end{array}
\right.
\]

(5.48)
\[
\frac{\partial M_{xB}}{\partial \theta_{xB}} = \frac{\partial M_{xB}}{\partial \theta_{xB}} \frac{\partial \theta_{xB}}{\partial v_j} \\
= - \frac{\partial R_{xB}}{\partial \theta_{xB}} \begin{cases} 
\frac{h_1 + h_2}{h_1 h_2} & , (j = 2) \\
-\frac{h_1}{h_2(h_1 + h_2)} & , (j = 3) \\
0 & , (j > 3) 
\end{cases} 
\] 

\[
\frac{\partial M_{xT}}{\partial \theta_{xT}} = \frac{\partial M_{xT}}{\partial \theta_{xT}} \frac{\partial \theta_{xT}}{\partial v_j} \\
= - \frac{\partial R_{xT}}{\partial \theta_{xT}} \begin{cases} 
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & , (j = n) \\
-\frac{h_n}{h_{n-1}(h_{n-1} + h_n)} & , (j = n-1) \\
0 & , (j < n-1) 
\end{cases} 
\] 

Therefore,

\[
-2 \frac{h_i}{h_{i-1} + h_i} \frac{\partial M_{x_i}}{\partial \phi_{x_i}} = h_{i-1} \frac{\partial M_{x_i}}{\partial v_{i-1}} = h_i \frac{\partial M_{x_i}}{\partial v_{i+1}} = -\frac{h_{i-1} h_i}{h_{i-1} + h_i} \frac{\partial M_{x_i}}{\partial v_i} 
\] 

\[
\frac{h_1}{h_2(h_1 + h_2)} \frac{\partial R_{xB}}{\partial \phi_{xB}} = \frac{\partial M_{xB}}{\partial v_3} = \frac{-h_1^2}{(h_1 + h_2)^2} \frac{\partial M_{xB}}{\partial v_2} 
\] 

\[
\frac{h_n}{h_{n-1}(h_{n-1} + h_n)} \frac{\partial R_{xT}}{\partial \theta_{xT}} = \frac{\partial M_{xT}}{\partial v_{n-1}} = \frac{-h_n^2}{(h_{n-1} + h_n)^2} \frac{\partial M_{xT}}{\partial v_n} 
\]
In a similar manner, for equation (5.39) and (5.40)

\[
\frac{-2 AM_{yi}}{h_{i-1} + h_i} \phi_{xi} = h_{i-1} \frac{AM_{yi}}{\theta v_{i-1}} = \frac{AM_{yi}}{\theta v_{i+1}} = \frac{-h_{i-1} h_i}{h_{i-1} + h_i} \frac{AM_{yi}}{\theta v_i}
\]  

(5.54)

\[
\frac{2 AM_{xi}}{h_{i-1} + h_i} \phi_{yi} = h_{i-1} \frac{AM_{xi}}{\theta u_{i-1}} = h_i \frac{AM_{xi}}{\theta u_{i+1}} = \frac{-h_{i-1} h_i}{h_{i-1} + h_i} \frac{AM_{xi}}{\theta u_i}
\]  

(5.55)

and \(\frac{AM_y B}{\theta v_j} = \frac{AM_y T}{\theta v_j} = \frac{AM_x B}{\theta u_j} = \frac{AM_x T}{\theta u_j} = 0\) since the end rotations \(\theta_{yB}, \theta_{yT}\) are not dependent on the \(v_j\), and \(\theta_{xB}, \theta_{xT}\) are not dependent on the \(u_j\).

In the previous expressions, the cross section bending rigidities \(\frac{AM_{yi}}{\phi_{yi}}, \frac{AM_{xi}}{\phi_{xi}}\) are evaluated from the M-P-\(\phi\) relations equation (5.14) and (5.15), or alternatively can be evaluated numerically by computing the moment changes induced by small changes in the \(u\) and \(v\) deflections at the station \(i\), respectively (i.e., small changes in the \(y\) and \(x\) curvatures \(\phi_{yi}, \phi_{xi}\) at the station \(i\), respectively). The end restraint tangent stiffnesses

\[-\frac{\theta yB}{\theta yB}, -\frac{\theta yT}{\theta yT}, -\frac{\theta xB}{\theta xB}, -\frac{\theta xT}{\theta xT}\]

can be explicitly derived, for each level of rotation, from the nonlinear end restraint moment functions obtained in Chapter 3.
Therefore, the submatrices of equation (5.37) are

\[ \frac{\partial U}{\partial u} = -\frac{1}{P} \begin{bmatrix} \frac{\partial M_y}{\partial \phi y_2} & (0) & [DFI] + \\
\frac{\partial M_y}{\partial \phi y_1} & \frac{\partial M_y}{\partial \phi y_1} & \\
(0) & \frac{\partial M_y}{\partial \phi y_n} & \end{bmatrix} \]

\[ + \frac{1}{P} [DIETA] \begin{bmatrix} \frac{\partial R}{\partial y_B} \\
\frac{\partial R}{\partial y_B} \\
(0) \\
\frac{\partial R}{\partial y_I} \\
\frac{\partial R}{\partial y_I} \\
(0) \\
\end{bmatrix} \]

(5.56)

\[ \frac{\partial U}{\partial v} = \frac{1}{P} \begin{bmatrix} \frac{\partial M_y}{\partial \phi x_2} & (0) & [DFI] \\
\frac{\partial M_y}{\partial \phi x_1} & \\
(0) & \frac{\partial M_y}{\partial \phi x_n} \end{bmatrix} \]

(5.57)

\[ \frac{\partial V}{\partial u'} = \frac{1}{P} \begin{bmatrix} \frac{\partial M_x}{\partial \phi y_2} & [DFI] \\
\frac{\partial M_x}{\partial \phi y_1} \\
\frac{\partial M_x}{\partial \phi y_n} \end{bmatrix} \]

(5.58)
\[
\frac{\partial V}{\partial v} = -\frac{1}{P} \begin{bmatrix}
\frac{\partial M_{x2}}{\partial \phi_{x2}} \\
\frac{\partial M_{x1}}{\partial \phi_{x1}} \\
\frac{\partial M_{xn}}{\partial \phi_{xn}}
\end{bmatrix} [DFI] + \\
+ \frac{1}{P} [DTETA] \begin{bmatrix}
\frac{\partial R_{xB}}{\partial \theta_{xB}} \\
\frac{\partial R_{xB}}{\partial \theta_{xB}} \\
\frac{\partial R_{xT}}{\partial \theta_{xT}}
\end{bmatrix} (0)
\]

in which [DFI] and [DTETA] are the \((n-1) \times (n-1)\) matrices expressed by equations (5.60) and (5.61).
\[ [DFI] = \]

\[
\begin{bmatrix}
\frac{-2}{h_1 h_2} & \frac{2}{h_2 (h_1 + h_2)} \\
\frac{2}{h_2 (h_2 + h_3)} & \frac{-2}{h_2 h_3} & \frac{2}{h_3 (h_2 + h_3)} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{2}{h_1 (h_1 + h_1)} & \frac{-2}{h_1 h_1} & \frac{2}{h_1 (h_1 + h_1)} \\
\frac{2}{h_{n-2} (h_{n-2} + h_{n-1})} & \frac{-2}{h_{n-2} h_{n-1}} & \frac{2}{h_{n-1} (h_{n-2} + h_{n-1})} \\
\frac{2}{h_{n-1} (h_{n-1} + h_n)} & \frac{-2}{h_{n-1} h_n} & \end{bmatrix}
\]
\[
\begin{align*}
\begin{array}{c|c|c}
\hline 
\text{Row 1} & \text{Row 2} & \text{Row 3} \\
\hline 
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-2} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\hline 
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c|c|c}
\hline 
\text{Row 4} & \text{Row 5} & \text{Row 6} \\
\hline 
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-2} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\hline 
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c|c|c}
\hline 
\text{Row 7} & \text{Row 8} & \text{Row 9} \\
\hline 
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-2} h_s}{h_1 (1 - L)} \\
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-1} h_s}{h_1 (1 - L)} \\
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-1} h_s}{h_1 (1 - L)} \\
\hline 
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c|c|c}
\hline 
\text{Row 10} & \text{Row 11} & \text{Row 12} \\
\hline 
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-2} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\hline 
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c|c|c}
\hline 
\text{Row 13} & \text{Row 14} & \text{Row 15} \\
\hline 
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-2} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\hline 
\end{array}
\end{align*}
\]

\[
[DIEA] =
\begin{align*}
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-2} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{h_{n-1} + h_n}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-2} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{h_{n-1} (h_{n-1} + h_n)}{h_{n-1} h_n} & \frac{\sum_{s=1}^{n-1} h_s}{h_{n-1} h_n} \\
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-2} h_s}{h_1 (1 - L)} \\
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-1} h_s}{h_1 (1 - L)} \\
\frac{h_n}{h_1 (1 - L)} & \frac{h_n}{h_1 (1 - L)} & \frac{\sum_{s=1}^{n-1} h_s}{h_1 (1 - L)} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-2} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\frac{h_1 + h_2}{h_1} & \frac{h_1 + h_2}{h_1} & \frac{\sum_{s=1}^{n-1} h_s}{h_1} \\
\end{align*}
\]
CHAPTER 6

IMPROVED COMPUTATION OF THE BUCKLING CONDITION

6.1 Introduction

During the last decade a considerable amount of research effort has been put into the development of numerical techniques for the analysis of nonlinear structural systems. These numerical methods have been applied to structural mechanics problems of large deformations and stability, elasto-plasticity, creep, etc. The applications of the Finite Element Method to nonlinear problems have proliferated, resulting in the development of general purpose nonlinear programs such as MARC, NONSAP, NFAP and ADINA.

Nevertheless, nonlinear programs require the user to have some understanding of the theories and assumptions behind the algorithms involved. Besides, there are also many nonlinear problems whose solution algorithm may require adaptation of a particular numerical technique used in a particular nonlinear program. Thus each numerical technique appears to be appropriate for a limited range of nonlinear problems. Therefore, it has been suggested by Bergan and Soreide [7] that general nonlinear programs should include several alternative numerical techniques for the solution of nonlinear systems, and also allow for combinations of these.
In a similar manner, in this investigation an attempt is made to extend the application of the nonlinear analysis described in Chapter 5. The present chapter begins by introducing some of the various solution methods for solving nonlinear structural problems. The most important solution techniques were derived in the context of the Finite Element Method spatial discretization of the variational or differential formulations of the nonlinear problems [51,89].

Using such techniques as a basis, this chapter thereafter introduces a possible extension of the Influence Coefficient Method, presented in Chapter 5 for the buckling analysis of tubular beam-columns. It is a scheme that provides the experienced user with the possibility of obtaining efficient, reliable and improved solutions to the buckling condition of end-restrained tubular beam-columns.

6.2 Description of Solution Methods for Nonlinear Analysis

In general structural mechanics problems may be formulated mathematically in several different ways. The numerical solution technique of a nonlinear structural problem is dictated by the form of the set of nonlinear simultaneous equations. It is possible to classify the available solution methods in three major categories: minimization techniques, iterative techniques and incremental techniques.

In the first method, the solution is obtained by seeking stationary values of a potential functional (objective function) expressed in terms of generalized loads and displacements using variational principles of nonlinear mechanics [88]. Direct numerical
search techniques of unconstrained minimization [32] use the objective function directly while gradient methods, like the straightforward method of steepest descent [32], use the gradient of the objective function. There are, however, some problems with applications of the minimization techniques to nonlinear structural problems. In fact, certain types of nonlinear problems do not allow potential function formulation. That is the case of problems involving path-dependent material nonlinearities or deformation dependent loads (e.g., non-conservative follower forces), such as Beck's problem [75]. Moreover, structural ill-conditioning often precludes using both methods because of slow convergence.

The second method for solution of nonlinear structural problems expresses directly the equilibrium between internal and external forces. These equilibrium equations can be established either by equating the variation of the total potential energy function to zero, by the virtual work principle, or simply by expressing the equilibrium balance. The internal generalized forces are often expressed in terms of a secant stiffness matrix dependent on the current configuration and, for non-conservative systems, on the load history. The equilibrium equations are usually solved for an updated displacement set, \( \{ w^{k+1} \} \), using a Newton-Raphson iterative scheme [52] applied to a current displacement set \( \{ w^k \} \). The general form of this method, for cycle \( k \), is

\[
\{ w^{k+1} \} = \{ w^k \} - \gamma^k \left( \text{\frac{\partial R}{\partial w}} \right)^{-1} \{ R(w^k) \} \quad (6.1)
\]

Here, \( \{ R(w^k) \} \) is a vector of residuals (unbalances), \( \left( \text{\frac{\partial R}{\partial w}} \right) \) is the
at every iteration ensuring quadratic convergence. Nevertheless this method may become expensive to apply due to the computation time taken in inverting the gradient matrix. Therefore variations of the full Newton-Raphson method are often used. In the modified Newton-Raphson iterative scheme the same gradient matrix is used throughout each load step. To improve the low convergence rate inherent in this, some acceleration techniques have been developed. Jennings [37] used a modification of the Aitken's \( \delta^2 \) process. More recently Crisfield [24] used a scheme in which the iterative displacement change is a scalar multiple of the previous iterative change plus another scalar multiple of the usual unaccelerated change. The quasi-Newton-Raphson iterative scheme [27] constitutes another class of iterative methods known also as matrix updating procedures. In these methods the gradient matrix is updated at the end of each iteration cycle by a correction matrix. The simplest of these is the one reported by Broyden [11], in which the improved approximation to the gradient matrix is obtained by adding to the previous approximation a correction matrix of rank one. But the most effective quasi-Newton-Raphson method for finite element analysis in the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method applied by Matthies and Strang [49], in which the correction matrix is of rank two.

The third method for the solution of nonlinear structural problems expresses the incremental form of the equilibrium equations. These are derived either by an incremental form of the virtual work principle or by a series expansion of the potential energy function about an equilibrium state. The internal generalized forces are
expressed in terms of an incremental tangent stiffness matrix. The external load is applied in sufficiently small increments for the increment of deformation to be linear. At the end of each load increment the incremental stiffness matrix is updated and a new load increment is applied. In this way, the complete nonlinear solution is generated as a sequence of piecewise linear steps, each of which is solved by integration as an initial value problem.

A more detailed comparative study of the usage, capabilities and limitations of these three methods can be seen in references [6], [34] and [81]. The use of appropriate numerical integration schemes of the incremental equations, complemented by the use of appropriate Newton-Raphson iterative schemes for the equilibrium iterations within each load step, constitutes the most efficient approach for solving large systems of nonlinear equations by the Finite Element Method [51].

However, when these techniques alone are applied to post-limit analysis they do not converge satisfactorily, and may diverge. This occurs because of the singularity of the tangent stiffness matrix at the limit point (buckling) and also because of the multiple stable solutions available near the instability zone. To overcome these difficulties, four categories of methods have been developed: prescribed displacement method, fictitious spring method, current stiffness parameter method, and constrained displacement length method.

In the prescribed displacement method [3,33] the singularity of the tangent stiffness matrix is avoided by selecting certain displacement components to have specific assigned values. Although a numerical handling of the resulting equations has been developed to
preserve the symmetric banded character of the stiffness matrix, the method has limitations. It is difficult to select the appropriate displacement variable and it cannot be applied in cases where the load-displacement curve has a dynamic-snap under displacement control.

In the fictitious spring method suggested by Sharifi and Popov [71], the positive definiteness of the tangent stiffness matrix is preserved by introducing fictitious springs at the loaded degrees of freedom. The method is valid when only one spring is used. However, when several springs are added this method does not have any criteria for the selection of the appropriate springs.

The current stiffness parameter [8] was used by Bergan et al. [4,6,8] to predict the position of the local singularities of a load-displacement curve. Equilibrium iterations are suppressed in the neighborhood of the limit points and the sign of the incremental load vector is reversed according to the sign of the determinant of the tangent stiffness matrix (or alternatively, the sign of the current stiffness parameter). Because of the suppression of the equilibrium corrections, very small load increments are required near the limit points. This causes a local displacement drift from the true equilibrium path.

In the constrained displacement length method, the length of the incremental displacement is constrained throughout an iterative scheme until equilibrium is reached. In a recent research, Sawamiphakdi [66] concluded that the constrained displacement length method offers the best alternative for the postbuckling analysis of shells by the Finite Element Method.
The Influence Coefficient Method for the analysis of end-restrained beam-columns may be grouped with others under the iterative technique category. Clearly, the unaccelerated scheme of equation (6.1) is formally identical to the iterative relation (5.34), the vector of unbalances standing for \( \{ R \} = \{ w \} - \{ F(\{ w \}) \} \). Moreover, in the equilibrium equations (5.32) the internal generalized forces (biaxial internal moments) are computed through the \((M-P-\Theta)\) curves. Thus they are not expressed in terms of a coefficient matrix, as is the case in a tangent stiffness matrix. None of the first three methods of post-limit analysis can thus be formally applied in the context of the Influence Coefficient Method, since their formulation relies mainly in modifications or special handling of a coefficient matrix expressing the internal forces. However, the constrained displacement length method was applied by the author to improve the estimation of the buckling state. In the following section, further details on the method are given, in the context of its use with the Influence Coefficient Method.

6.3 The Constrained Displacement Length Method

This method was originally introduced by Riks [59] as a way of overcoming the local extremum points in the solution of systems of nonlinear equations. It involves solving a length constraint equation, fixing the length of the incremental load step in the load-deflection space, in combination with the equilibrium equations. Based on numerical experience, Crisfield [25] implemented Riks approach applied in conjunction with the modified Newton-Raphson iterative scheme. An
adaptation of Crisfield's procedure to be used in the context of the Influence Coefficient Method was developed by the author, and can be outlined as follows.

The vector \( \{ w_L \} \) is defined as the last converged displacement configuration of the tubular beam-column in equilibrium with the axial load \( P_L \). A forward stable solution near the instability zone, on the onset of divergence, satisfies the following equilibrium equation

\[
\{ w_L + \Delta w \} = \{ F \{ w_L + \Delta w \} \}
\]  

(6.2)

This can alternatively be written as

\[
\{ g(\lambda) \} = \{ w_L \} + \lambda \{ \Delta w \} - \{ F \{ w_L \} + \lambda \{ \Delta w \} \} = 0
\]  

(6.3)

where \( \{ \Delta w \} \) is an incremental displacement vector and \( \lambda \) is a scaling parameter.

Using a technique similar to the one used by Batoz and Dhatt [3] for displacement control, the unbalanced vector of residuals can be expressed as

\[
\{ g(\lambda + \delta \lambda) \} = \{ g(\lambda) \} + \delta \lambda \{ \Delta w \}
\]  

(6.4)

During the equilibrium iterations, the incremental displacement vector after \( k \) iterations is

\[
\{ \Delta w^{k+1} (\lambda_{k+1}) \} = \{ \Delta w (\lambda_k) \} + \{ \Delta^k (\lambda_{k+1}) \}
\]  

(6.5)

where the iterative change \( \{ \Delta^k (\lambda_{k+1}) \} \), expressed by the product of the gradient matrix for the iteration and the residuals, is
\[
\{\Delta^k(\lambda_{k+1})\} = -[H(\{w_L\})]^{-1}\{g(\lambda_k + \delta\lambda_k)\}
\]

\[
= \{\Delta^k(\lambda_k)\} + \delta\lambda_k \{\Delta^o(\lambda_o)\}
\]  \hspace{1cm} (6.6)

and \[
[H] = [I] - [J]
\]  \hspace{1cm} (6.7)

\[
\{\Delta^k(\lambda_k)\} = -[H(\{w_L\})]^{-1}\{g(\lambda_k)\}
\]  \hspace{1cm} (6.8)

\[
\{\Delta^o(\lambda_o)\} = -[H(\{w_L\})]^{-1}\{g(\lambda_o)\}
\]  \hspace{1cm} (6.9)

For each equilibrium iteration \(k\), the incremental displacement length of the subsequent iteration is constrained by the length of the previous iteration. Therefore the constraint equation is stated as

\[
\{\Delta w^{k+1}(\lambda_{k+1})\}^T\{\Delta w^{k+1}(\lambda_{k+1})\} = \{\Delta w^k(\lambda_k)\}^T\{\Delta w^k(\lambda_k)\}
\]  \hspace{1cm} (6.10)

The scaling parameter \(\lambda_k\) starts with the unity value at the beginning of the equilibrium iterations \((\lambda_o = 1)\) and is updated throughout the constrained iterations by

\[
\lambda_{k+1} = \lambda_k + \delta\lambda_k
\]  \hspace{1cm} (6.11)

Substituting equations (6.5) and (6.6) into (6.10), gives the following quadratic equation for the change \(\delta\lambda_k\) in the scaling parameter

\[
a_1 \delta\lambda_k^2 + a_2 \delta\lambda_k + a_3 = 0
\]  \hspace{1cm} (6.12)

where

\[
a_1 = [\Delta^o(\lambda_o)]^T\{\Delta^o(\lambda_o)\}
\]  \hspace{1cm} (6.13)

\[
a_2 = 2\{[\Delta w^k(\lambda_k)] + [\Delta^k(\lambda_k)]\}^T\{\Delta^o(\lambda_o)\}
\]  \hspace{1cm} (6.14)
\[ a_3 = \{\{\Delta w^k \lambda_k\} + \{\Delta^k \lambda_k\}\}^T \{\{\Delta w^k \lambda_k\} + \{\Delta^k (\lambda_k)\}\} - \{\Delta w^k \lambda_k\}^T \{\Delta w^k (\lambda_k)\} \] (6.15)

From the two roots \(\delta \lambda_{k1}\) and \(\delta \lambda_{k2}\) of equation (6.12), the appropriate value of \(\delta \lambda_k\) is the one which gives a positive angle between the incremental displacement vectors \(\{\Delta w^k (\lambda_k)\}\) and \(\{\Delta w^{k+1} (\lambda_{k+1})\}\), thus precluding doubling back on the original solution path. The two angles \(\theta_1\) and \(\theta_2\) are expressed by the following scalar products:

\[ \theta_1 = \{\Delta w^{k+1} (\lambda_k + \delta \lambda_{k1})\}^T \{\Delta w^k (\lambda_k)\} \] (6.16)

\[ \theta_2 = \{\Delta w^{k+1} (\lambda_k + \delta \lambda_{k2})\}^T \{\Delta w^k (\lambda_k)\} \] (6.17)

If both angles are positive, the appropriate root is that closest to the linear solution \((-a_3/a_2)\).

Convergence of the equilibrium iterations is achieved when equation (5.36) and/or

\[ \delta \lambda \lesssim e_\lambda \] (6.18)

are fulfilled, where \(e_\lambda\) is a convergence tolerance on the increments of the scaling parameter.

Conceptually, the intersection of the equilibrium iterations hyperplanes with the constraint hypersurface is used as a guess for the unknown function of the solution path.
6.4 Attempts at Initial Postbuckling Analysis with the Influence Coefficient Method

The elasto-plastic analysis of end-restrained tubular beam-columns was achieved, as outlined in Chapter 5, and improved, as outlined in section 6.3. The possibility of analyzing the initial postbuckling behavior by means of the ICM was then investigated. Several unsuccessful approaches were attempted, the most important of which are outlined herein.

Up to the limit point, most analytical schemes (including the ICM) use a load increment approach characteristic of a load control process. When analyzing relatively simple structures, it is tempting to try to avoid the singularity of the limit point by using some sort of displacement control. Such displacement control would simulate the physical testing procedure of a testing machine, in which the loading can be controlled and static equilibrium positions are obtained at successive lower loads, under increasing deformations. Two simple kinds of displacement control were used: initial forward guess of the displacement vector, with and without unloading.

In the case of forward guess of displacement vector without unloading, the use of the ICM iterative relation (5.34), with or without constraints, gives a converged displacement vector in agreement with the one of the last available solution to the buckling state. If such forward guess is applied together with unloading, the iterative solution lies on the original load deformation path.

To interpret the reason for this backward travelling to the original solution path, one must have further insight into the form of the iterative relation (4.34). As mentioned earlier, the usage of
the first three methods of postbuckling analysis in the context of
the ICM is precluded by the lack of a coefficient matrix (such as a
tangent stiffness matrix. When such a coefficient matrix exists the
gradient matrix for the iterative solution is the coefficient matrix,
and therefore the singularities of the load deformation path are
accompanied by singularities on the gradient matrix. However, the
iterative relation (5.34) cannot be applied successfully to equilibrium
solutions beyond the limit buckling point, because the non-positive
definiteness of the gradient matrix [H] precludes such usage. In fact
the gradient matrix for the iteration (equation 6.7) is not singular
at buckling, and furthermore the Jacobian matrix [J] (equation 5.37)
is nonsymmetric. It is therefore misleading to consider that the
Newton-Raphson iterative scheme applied in the context of the ICM,
uses the concept of vanishing stiffness as the criteria for limit axial
load [61]. Rather, the ICM criteria for ultimate axial load is full
yielding of a station and/or divergence. No stiffness concept is
involved. While it is true that the stiffness of the beam-column
vanishes at buckling, such reasoning is useful only in a stiffness
approach formulation of the tubular beam-column buckling analysis.

Therefore the remaining alternative for an initial postbuck-
ling analysis would be to use the equilibrium equations (5.32),
defining a functional fixed point iteration of recurrence relation

$$\{w^{k+1}\} = \{F(\{w^k\})\}$$

(6.19)

The method of successive substitutions is often unsuited to
solve systems of nonlinear equations. In fact the direct use of
equation (6.19) with both simple forms of displacement control gave a diverging sequence of displacement vectors.

A further attempt was made to handle equation (6.19) in the form

$$\{w^{k+1}\} = \{w^k\} - [I] \{\{w^k\} - F(\{w^k\})\}$$

(6.20)

The simplest quasi Newton-Raphson method available (Broyden's method [11]) was applied in an attempt to correct the gradient matrix of the iterative scheme (6.20) and force a forward convergence. For the kth iteration, the gradient matrix $[B^k]$ would be expressed by

$$[B^k] = [B^{k-1}] - \frac{([B^{k-1}](q) - (p)) \{p\}^T [B^{k-1}]}{\{p\}^T [B^{k-1}](q)}$$

(6.21)

where $\{p\} = \{w^k\} - \{w^{k-1}\}$

(6.22)

$$\{q\} = \{\{w^k\} - F(\{w^k\})\} - \{\{w^{k-1}\} - F(\{w^{k-1}\})\}$$

(6.23)

$$[B^0] = [I]$$

(6.24)

This scheme resulted also in a diverging sequence of displacement vectors, but at a slower rate than the one using equation (6.19). The author therefore concluded that the highly nonlinear functions on the right-hand-side of equation (6.19) made this equation insoluble by a functional fixed point iteration scheme.

These same procedures were also applied unsuccessfully to tubular beam-columns without end-restraint. Therefore the end restraints could not be viewed as a source of divergence of the numerical schemes.
In view of the above discussion, it is the author's opinion that the ICM cannot be applied to the initial post limit analysis of beam-columns. The main reason is associated with the fact that the ICM is not a stiffness formulation of the equilibrium balance. Nevertheless, the ICM constitutes an effective method for the elasto-plastic buckling analysis of end-restrained tubular beam-columns. The end-restraint effects are formulated directly in a Jacobian matrix. The solution of the nonlinear equilibrium equations can be obtained efficiently up to the limit buckling point.
CHAPTER 7

NUMERICAL RESULTS

7.1 Introduction

The analytical procedure discussed in Chapters 5 and 6 has been implemented into a nonlinear buckling analysis program. The computer program has been developed to investigate the influence of end-restraints on the strength and behavior of tubular beam-columns in three dimensions. The main features of the computer program used, include the following:

(i) It traces full load-deflection behavior up to the limit buckling load.

(ii) It permits any initial out-of-straightness to be assumed in two perpendicular planes.

(iii) It allows spread of yield throughout the member, using Tresca's yield criterion and associated flow rule.

(iv) It permits any pattern of residual stresses to be assumed, using a straight-line approximation suggested by Chen and Ross [20].

(v) It permits inclusion of any appropriate restraint moment-rotation function.

In this chapter, the validity of the program is verified by comparison with experimental data reported by Chen and Ross [20].
for fabricated tubular columns. Thereafter, the experimental data obtained by the author for end-restrained tubular beam-columns (as outlined in Chapter 4) is compared with the analytical predictions.

A parametric study of the influence of semi-rigid end-restraints on the strength and behavior of imperfect tubular beam-columns is conducted, for the determination of effective length factors.

7.2 Verification of the Need to Include End-Restraint in Beam-Column Modeling

The Lehigh data [20] corresponding to column specimens numbers 2, 3 and 7 has been used to check the computer program. All specimens have an outside diameter of 15 inches and thickness of 0.3125 inches. The lengths of specimens 2, 3 and 7 are 18, 25 and 36 feet respectively, giving nominal slenderness ratios of 42, 60 and 83 respectively. The three fabricated columns were modeled with essentially pinned end conditions. The circumferential and longitudinal residual stresses patterns were approximated by straight lines. Measurements of column out-of-straightness showed the columns possessed single, double and triple initial curvatures in two perpendicular planes. Tables 16, 17 and 18 compare the analytical and experimental results of the buckling load, midheight deflections at buckling and buckling directions, for these Lehigh tests.

Agreement between experimental and calculated results is not very good, with the calculated maximum load underestimating the experimental failure load by up to 32%. The difference between
TABLE 16. Comparison of Buckling Loads, for Lehigh Tests

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Load Kips (kilonewtons)</th>
<th>% Error Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>2</td>
<td>566.6 (2521)</td>
<td>560 (2492)</td>
</tr>
<tr>
<td>3</td>
<td>450.8 (2006)</td>
<td>510 (2270)</td>
</tr>
<tr>
<td>7</td>
<td>375.0 (1669)</td>
<td>497 (2212)</td>
</tr>
</tbody>
</table>

TABLE 17. Comparison of Total Buckling Deflections at Midheight, for Lehigh Tests

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Deflection in (mm)</th>
<th>% Error Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>2</td>
<td>1.307 (33.2)</td>
<td>0.485 (12.3)</td>
</tr>
<tr>
<td>3</td>
<td>2.044 (51.9)</td>
<td>0.253 (6.43)</td>
</tr>
<tr>
<td>7</td>
<td>1.573 (40.0)</td>
<td>0.599 (15.2)</td>
</tr>
</tbody>
</table>

analytical and experimental results is thought to be due to the likelihood that the end restraints—which were assumed to be pinned—do in fact possess rotational stiffness.

It is therefore pertinent to model and study the effects of end-restraints on buckling strength and behavior, as mentioned in the objective of this dissertation.
TABLE 18. Comparison of Buckling Directions, for Lehigh Tests

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Direction with respect to XX plane</th>
<th>Difference in Buckling Angle, Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>2</td>
<td>146°</td>
<td>167°</td>
</tr>
<tr>
<td>3</td>
<td>146°</td>
<td>110°</td>
</tr>
<tr>
<td>7</td>
<td>4°</td>
<td>13°</td>
</tr>
</tbody>
</table>

7.3 Comparison of Theoretical and Observed Experimental Results

The results of calibration experiments on the bottom and top end-restraint systems obtained in Chapter 3 were input into the program for modelling of the experimental results in Chapter 4. These, together with the initial imperfections and eccentricities associated with the four column experiments conducted by the author, constitute relevant initial data for the theoretical buckling analysis of the four end-restrained tubular beam-columns. The numerical results outlined herein, were obtained using a displacement tolerance of $10^{-3}$ on the displacement norm equation (5.36).

The comparison of theoretical and observed experimental results is presented herein in graphical and tubular form. The set of figures, Figures 47 through 50 (for Specimen 1), Figures 51 through 54 (for Specimen 2), Figures 55 through 58 (for Specimen 3), Figure 59 through 62 (for Specimen 4), compare the theoretical (continuous line) and experimental (dotted data) buckling behavior
Figure 47. Comparison of Midheight Displacement Path (for Specimen 1)
Figure 48. Comparison of Bottom Rotation Path (for Specimen 1)
TOP ROTATION PATH

Figure 49. Comparison of Top Rotation Path (for Specimen 1)
Figure 50. Comparison of Load-Lateral Displacement at Midheight Curves (for Specimen 1)
Figure 51. Comparison of Midheight Displacement Path (for Specimen 2)
Figure 52. Comparison of Bottom Rotation Path (for Specimen 2)
Figure 53. Comparison of Top Rotation Path (for Specimen 2)
Figure 54. Comparison of Load-Lateral Displacement at Midheight Curves (for Specimen 2)
Figure 55. Comparison of Midheight Displacement Path (for Specimen 3)
Figure 56. Comparison of Bottom Rotation Path (for Specimen 3)
Figure 57. Comparison of Top Rotation Path (for Specimen 3)
Figure 58. Comparison of Load-Lateral Displacement at Midheight Curves (for Specimen 3)
MIDHEIGHT DISPLACEMENT PATH

Figure 59. Comparison of Midheight Displacement Path (for Specimen 4).
Figure 60. Comparison of Bottom Rotation Path (for Specimen 4)
Figure 61. Comparison of Top Rotation Path (for Specimen 4)
LOAD-DISPLACEMENT DATA

Figure 62. Comparison of Load-Lateral Displacement at Midheight Curves (for Specimen 4)
and strength of the four end-restrained beam-columns tested. In addition, Figures 50, 54, 58 and 62 also show the theoretical load-deflection behavior of pinned beam-columns, with the same initial imperfections and eccentricities of the four restrained columns. In each case the experimental data continues into the post-buckling range while the analytical data does not.

A striking feature of this graphical comparison is its consistency. The paths of the theoretical predictions follow closely the experimental data, except for the initial zones of the top rotation path. The comparison of results relative to the top rotation path indicate a possible malfunction of one of the top end-restraint system RVDT's.

The agreement between experimental and analytical results is further enhanced by the comparison of results at buckling as summarized in Tables 19 through 21. The ultimate loads and the total buckling deflections at midheight are predicted with relative errors less than 3.5% and 3%, respectively. The buckling directions are predicted with relative errors less than 5%, except for Specimen 4 for which the relative error is 16%.

Furthermore, examination of the load-deflection curves for pinned and restrained beam-columns leads to the conclusion that inclusion of the end-restraint systems studied in this dissertation, produces:

(i) An increase in the maximum strength of the order of 65%-100%.
### TABLE 19. Comparison of Buckling Loads

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Load (lbs)</th>
<th>Buckling Load (Newtons)</th>
<th>% Error Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1224 (5536)</td>
<td>1260 (5607)</td>
<td>1.29</td>
</tr>
<tr>
<td>2</td>
<td>1150 (5118)</td>
<td>1179 (5247)</td>
<td>2.52</td>
</tr>
<tr>
<td>3</td>
<td>1408 (6266)</td>
<td>1454 (6470)</td>
<td>3.27</td>
</tr>
<tr>
<td>4</td>
<td>1293 (5754)</td>
<td>1333 (5932)</td>
<td>3.09</td>
</tr>
</tbody>
</table>

### TABLE 20. Comparison of Total Buckling Deflections at Midheight

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Deflection (in)</th>
<th>% Error Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>1</td>
<td>0.180 (4.57)</td>
<td>0.185 (4.70)</td>
</tr>
<tr>
<td>2</td>
<td>0.189 (4.80)</td>
<td>0.193 (4.90)</td>
</tr>
<tr>
<td>3</td>
<td>0.159 (4.03)</td>
<td>0.159 (4.03)</td>
</tr>
<tr>
<td>4</td>
<td>0.171 (4.35)</td>
<td>0.173 (4.40)</td>
</tr>
</tbody>
</table>
TABLE 21. Comparison of Buckling Directions

<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>Buckling Direction with respect to XX plane</th>
<th>% Error Experiment vs. Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theory</td>
<td>Experiment</td>
</tr>
<tr>
<td>1</td>
<td>-43°</td>
<td>-42°</td>
</tr>
<tr>
<td>2</td>
<td>-48°</td>
<td>-46°</td>
</tr>
<tr>
<td>3</td>
<td>-43°</td>
<td>-41°</td>
</tr>
<tr>
<td>4</td>
<td>-37°</td>
<td>-31°</td>
</tr>
</tbody>
</table>

(ii) A decrease in the total buckling deflection at midheight of the order of 63%-70%.

These conclusions are proven valid for long columns (the nominal slenderness ratio of the experimental columns tested is 188).

7.4 Evaluation of Effective Length Factors

The concept of effective column length is widely used in design as a method of allowing for different degrees of end restraints. Current design manuals \[58,63,77]\ provide the user with theoretical effective length factors, K, for a few end conditions. In particular, the American Institute of Steel Construction (AISC) specifications \[77]\ give theoretical K-values for three idealized end conditions with sidesway inhibited, their values ranging from 0.5 to 1.0. In addition, design values recommended by the SSRC are suggested for use when these end conditions are approximate. These recommended values range from 0.65 to 1.0. It is also possible to use a nomograph when framing members are known.
In the context of fixed offshore structures, both design practices [50,63] give effective length factors to be used in design, unless different values are justified by rational analyses including joint fixity. The Det Norske Veritas design rules [63] recommend the use of $K$-values of 1.0 and 0.8 respectively for chords and bracing members of fixed platforms. The American Petroleum Institute recommended practice (API RP 2A) [50] suggests a set of $K$-values ranging from 0.7 to 1.0, to be used depending on the particular situation of the tubular structural member on the fixed platform.

Justification and/or improvement of current design practices is desirable. In view of these inadequacies, it seemed appropriate to evaluate the effective length factors of the beam-columns tested.

In the approach used herein, the effective length is defined as that length which gives on the critical Euler load (for pinned ends) the same load carrying capacity as the failure load for the actual restrained column. The effective length factor, $K$, is then approximately obtained as:

\[
K = \sqrt{\frac{P_{b,o}}{P_b}}
\]  

(7.1)

where $P_{b,o}$ is the maximum column load of a similarly imperfect pinned column, and $P_b$ is the maximum column load of the actual end restrained column. Thus, the model of each column was repeated without end restraints to obtain $P_{b,o}$.

Therefore, with the results showed in Figures 50, 54, 58 and 62, the effective length factors, $K$, of the four tubular beam-columns tested are tabulated in Table 22.
<table>
<thead>
<tr>
<th>Specimen Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>K</td>
<td>0.750</td>
<td>0.780</td>
<td>0.707</td>
<td>0.735</td>
</tr>
</tbody>
</table>

Although the four tested columns have different initial imperfections and eccentricities of loading, the range of calculated K-values, 0.70-0.80, is found to be consistent. Denoting by $K_i$ the effective length factor associated with specimen $i$, the following comparisons can be made:

(i) Specimen 2 is less restrained than Specimen 1. Accordingly, $K_2 > K_1$.

(ii) Specimen 3 and 4 are more restrained than Specimen 2. Accordingly, $K_3 < K_2$ and $K_4 < K_2$.

(iii) Specimen 4 is of the fixed-pinned type, for which the SSRC recommended K-value, when ideal conditions are approximated, is 0.8 [77]. The realistic rational analysis conducted herein lead to $K_4 < 0.8$, because the bottom end is considerably more restrained than an "imperfect" pinned condition.

Thus, in view of the above results, it can be concluded that for long columns with a variety of end-restraint conditions similar to those modelled herein, the expected effective length factors lie in the range 0.7-0.8. The design practice of using a K-value of 0.8 seems, therefore, justifiable for long columns because it is
a conservative upperbound. But this depends on the extent of end restraints, in a manner as yet undetermined.

7.5 Parametric Study of Semirigid End-Restrains

In theory, end-restraint has a very simple effect: it merely changes the effective column length. In practice, however, the accurate determination of the effects of end-restraint for any column, except a pin-ended column, is a difficult problem.

There are three types of construction permitted under the provisions of Section 1.2 of the Specifications for the Design, Fabrication and Erection of Structural Steel for Buildings [77] issued by the AISC. Type 1, designated as rigid-frame (continuous frame); Type 2, designated as simple-frame (unrestrained, free-ended); and Type 3, designated as semirigid framing (partially restrained). Type 3 construction assumes that the connections between beams and girders possess an intermediate moment capacity between the rigidity of Type 1 and the flexibility of Type 2.

When conditions preclude use of bracing or shear walls, moment resisting frames have to be used to resist lateral forces. In such cases, the framing may be rigid or semirigid. Rigid framing, providing complete continuity in a frame, maintains the original angles between beams and columns. Semirigid framing employs partially restrained girder-column connections with known, dependable moment capacity. The semirigid construction offers greater potential for lower cost.
Current design methods of steel frames are based on the assumption that the end-connections are either pinned or provide full rotational continuity between adjacent members. Tests have indicated that it is beneficial to the design to take advantage of the partial fixity [83]. Experimental investigations of joint behavior have shown that the type of connections which are normally assumed pinned do possess significant rotational stiffness, thus behaving as flexible shear connections transferring some moment. Likewise, some degree of flexibility often exists in nominally rigid connections. Therefore, it is appropriate to group the three types of construction under the heading of semirigid construction.

However, Type 3 construction is allowed only upon evidence that the joint connections to be used will provide a predictable proportion of full end-restraint. This calls for reliable descriptions of the flexural behavior of joint connections, which is best depicted by the moment-rotation curves (M - θ connection curves) as shown graphically in Figure 63. Historically, research work has been concentrated on connections between wide flange members, deriving the connection curves either experimentally [56] or analytically [43], and applying these, through an analytical procedure, to the buckling analysis.

Although the stability of frames and the effective lengths of members within frames with semirigid connections have been considered recently [40], most of the research is based upon experimental data on the in-plane flexural behavior of connections. The
Figure 63. Types of Moment-Rotation Connection Curves
three dimensional behavior of structural members with semirigid connection has received little attention.

The load-deflection and stability behavior of tubular beam-columns with semirigid connections is unknown, and investigation is required to take advantage of any possible economy due to real end-restraint conditions. Although the behavior of a tubular semirigid connection is complex and its $M-\theta$ curve is expected to be nonlinear over the complete loading range, no data appears to be available on the three dimensional connection behavior in all degrees of freedom.

Therefore, the parametric study of semirigid end-restraints undertaken herein, in the context of tubular beam-columns, assumes linear restraint moment-rotation relationship. Table 23 outlines the four end-restraint cases to be analyzed in the parametric study, where $EI$ in the bending stiffness of the tubular member and $L$ is the member length. These four cases also resemble the four end-restraint experiments conducted by the author (as outlined in Chapter 4), except that the connection curves are now linear relationships. Case 1 models imperfect rigid connections. Case 2 models imperfect pinned connections. Case 4 models imperfect pinned-fixed boundary conditions. Case 3 models are an intermediate restraint circumstance between cases 2 and 4. However, it should be mentioned that once real $M-\theta$ curves describing real three-dimensional behavior of tubular connections are available, their inclusion in a parametric study can be readily achieved.
<table>
<thead>
<tr>
<th>End Restraint Case Number</th>
<th>Bottom Restraint</th>
<th>Top Restraint</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X-Direction</td>
<td>Y-Direction</td>
</tr>
<tr>
<td>1</td>
<td>2 (EI/L)</td>
<td>2 (EI/L)</td>
</tr>
<tr>
<td>2</td>
<td>½ (EI/L)</td>
<td>½ (EI/L)</td>
</tr>
<tr>
<td>3</td>
<td>½ (EI/L)</td>
<td>2 (EI/L)</td>
</tr>
<tr>
<td>4</td>
<td>2 (EI/L)</td>
<td>2 (EI/L)</td>
</tr>
</tbody>
</table>

The process of structural optimization of a tubular member frame using semirigid joints, may lead to designs which are liable to exhibit buckling instabilities. An optimum structure is by its very nature imperfection sensitive, therefore inevitable manufacturing imperfections will erode the benefits of a nominally optimum design.

Four end-restrained tubular columns were chosen, without applied biaxial moments, end eccentricities or residual stresses, for an in-depth study of the effects of end-restraints on initially crooked columns. The columns selected differed only in length, and the different lengths considered enable the range of intermediate to high slenderness ratio to be studied. Each column was divided into ten segments with variable interstation lengths. The yield strength of the material used was assumed to be constant along each column. All the sections were allowed to perform inelastically, thus modelling the spread of yield more adequately. A half-period sinusoidal initial out-of-straightness pattern was specified for each column in each direction. The maximum out-of-straightness at
midheight were taken to be 0.001 x L—the usual fabrication tolerance [78]—and 0.0005 x L, respectively.

The numerical results of the parametric study of each column with both crookedness profiles are shown in Tables 24 through 27, for each end-restraint case. The strengths of the corresponding pinned columns are shown in Table 28. Here, $P_y$ is the axial load causing complete yielding of the cross section. The lateral movement at midheight is calculated at buckling. The $K$-values are obtained in accordance with section 7.4. In addition, the outputs also permitted to analyze the spread of yield throughout the columns.
<table>
<thead>
<tr>
<th>Initial out-of-straightness amplitude</th>
<th>Nominal slenderness ratio</th>
<th>$\frac{P_b}{P_y}$</th>
<th>Lateral movement at midheight/column length (%)</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001 L</td>
<td>52.0</td>
<td>0.999</td>
<td>1.38</td>
<td>0.924</td>
</tr>
<tr>
<td>0.001 L</td>
<td>74.3</td>
<td>0.834</td>
<td>0.62</td>
<td>0.857</td>
</tr>
<tr>
<td>0.001 L</td>
<td>96.6</td>
<td>0.604</td>
<td>0.85</td>
<td>0.811</td>
</tr>
<tr>
<td>0.001 L</td>
<td>118.9</td>
<td>0.423</td>
<td>1.07</td>
<td>0.788</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>52.0</td>
<td>0.999</td>
<td>0.59</td>
<td>0.955</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>74.3</td>
<td>0.889</td>
<td>0.20</td>
<td>0.868</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>96.6</td>
<td>0.673</td>
<td>0.43</td>
<td>0.774</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>118.9</td>
<td>0.463</td>
<td>0.52</td>
<td>0.757</td>
</tr>
<tr>
<td>Initial out-of-straightness amplitude</td>
<td>Nominal slenderness ratio</td>
<td>$\frac{P_b}{P_y}$</td>
<td>Lateral movement at midheight/ column length (%)</td>
<td>K</td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>---------------------------</td>
<td>-------------------</td>
<td>-----------------------------------------------</td>
<td>----</td>
</tr>
<tr>
<td>0.001 L</td>
<td>52.0</td>
<td>0.943</td>
<td>1.03</td>
<td>0.951</td>
</tr>
<tr>
<td>0.001 L</td>
<td>74.3</td>
<td>0.675</td>
<td>0.51</td>
<td>0.952</td>
</tr>
<tr>
<td>0.001 L</td>
<td>96.6</td>
<td>0.460</td>
<td>0.89</td>
<td>0.922</td>
</tr>
<tr>
<td>0.001 L</td>
<td>118.9</td>
<td>0.316</td>
<td>0.86</td>
<td>0.911</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>52.0</td>
<td>0.976</td>
<td>0.45</td>
<td>0.966</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>74.3</td>
<td>0.748</td>
<td>0.33</td>
<td>0.946</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>96.6</td>
<td>0.490</td>
<td>0.48</td>
<td>0.908</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>118.9</td>
<td>0.325</td>
<td>0.62</td>
<td>0.903</td>
</tr>
<tr>
<td>Initial out-of-straighness amplitude</td>
<td>Nominal slenderness ratio</td>
<td>$\frac{P_b}{P_y}$</td>
<td>Lateral movement at midheight/column length (%)</td>
<td>K</td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>--------------------------</td>
<td>-------------------</td>
<td>-----------------------------------------------</td>
<td>---</td>
</tr>
<tr>
<td>0.001 L</td>
<td>52.0</td>
<td>0.976</td>
<td>0.93</td>
<td>0.935</td>
</tr>
<tr>
<td>0.001 L</td>
<td>74.3</td>
<td>0.726</td>
<td>0.36</td>
<td>0.918</td>
</tr>
<tr>
<td>0.001 L</td>
<td>96.6</td>
<td>0.498</td>
<td>0.79</td>
<td>0.893</td>
</tr>
<tr>
<td>0.001 L</td>
<td>118.9</td>
<td>0.344</td>
<td>1.20</td>
<td>0.874</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>52.0</td>
<td>0.984</td>
<td>1.61</td>
<td>0.962</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>74.3</td>
<td>0.791</td>
<td>0.26</td>
<td>0.920</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>96.6</td>
<td>0.527</td>
<td>0.70</td>
<td>0.875</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>118.9</td>
<td>0.353</td>
<td>0.89</td>
<td>0.867</td>
</tr>
<tr>
<td>Initial out-of-straightness amplitude</td>
<td>Nominal slenderness ratio</td>
<td>( \frac{P_b}{P_y} )</td>
<td>Lateral movement at midheight/ column length (%)</td>
<td>K</td>
</tr>
<tr>
<td>-------------------------------------</td>
<td>--------------------------</td>
<td>-------------------</td>
<td>-----------------------------------------------</td>
<td>----</td>
</tr>
<tr>
<td>0.001 L</td>
<td>52.0</td>
<td>0.992</td>
<td>1.91</td>
<td>0.928</td>
</tr>
<tr>
<td>0.001 L</td>
<td>74.3</td>
<td>0.767</td>
<td>0.36</td>
<td>0.894</td>
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<tr>
<td>0.001 L</td>
<td>96.6</td>
<td>0.545</td>
<td>0.63</td>
<td>0.854</td>
</tr>
<tr>
<td>0.001 L</td>
<td>118.9</td>
<td>0.366</td>
<td>0.97</td>
<td>0.846</td>
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<tr>
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<td>52.0</td>
<td>0.996</td>
<td>1.67</td>
<td>0.956</td>
</tr>
<tr>
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<td>0.821</td>
<td>0.27</td>
<td>0.902</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>96.6</td>
<td>0.553</td>
<td>0.57</td>
<td>0.854</td>
</tr>
<tr>
<td>0.0005 L</td>
<td>118.9</td>
<td>0.366</td>
<td>0.38</td>
<td>0.851</td>
</tr>
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</table>
### TABLE 28. Parametric Study for Pinned Case

<table>
<thead>
<tr>
<th>Initial out-of-straightness amplitude</th>
<th>Nominal slenderness ratio</th>
<th>( \frac{P_{b,o}}{P_y} )</th>
<th>Lateral movement at midheight/column length (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001 L</td>
<td>52.0</td>
<td>0.854</td>
<td>0.23</td>
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<tr>
<td>0.001 L</td>
<td>74.3</td>
<td>0.612</td>
<td>0.54</td>
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<td>96.6</td>
<td>0.397</td>
<td>0.75</td>
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<tr>
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<td>118.9</td>
<td>0.262</td>
<td>0.76</td>
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<tr>
<td>0.0005 L</td>
<td>52.0</td>
<td>0.911</td>
<td>0.11</td>
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<tr>
<td>0.0005 L</td>
<td>74.3</td>
<td>0.669</td>
<td>0.41</td>
</tr>
<tr>
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<td>96.6</td>
<td>0.404</td>
<td>0.41</td>
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<tr>
<td>0.0005 L</td>
<td>118.9</td>
<td>0.265</td>
<td>0.40</td>
</tr>
</tbody>
</table>
7.6 Discussion of Results

A comparison of the consistent results of Tables 24 through 28, permits the following conclusions:

(i) The effective length factor is more dependent on slenderness ratio than on out-of-straightness.

(ii) For a particular crookedness and end-restraint, the effective length factor decreases with increasing slenderness.

(iii) For a particular column with corresponding crookedness and slenderness ratios, less restrained cases have higher K-values, as was expected.

(iv) In view of the fact mentioned in (iii), the four end-restraint cases modelling different 'imperfect' connections can be scaled in decreasing degree of overall connection stiffness, respectively as: case 1, case 4, case 3, case 2. Therefore, not only the values of end-restraint are important but also their locations.

(v) Columns with stiffer connections are more prone to variations of slenderness ratios. This emphasizes the importance of defining properly the force boundary conditions in column design, in all the ranges of slenderness.

(vi) For shorter columns (with slenderness ratios less than 74), the end-restraints increase the buckling load more for columns of high initial out-of-straightness than for straighter columns.

(vii) For longer columns (with slenderness ratios bigger than 96.6), the end-restraints increase the buckling load more
for columns of small initial out-of-straightness than for those of large out-of-straightness.

(viii) The effect of end-restraints on K-values is more important in the region of high slenderness ratio, than in the region of intermediate and short slenderness ratio. This is due to the fact that for short and intermediate slenderness ratio columns, the buckling and plasticity phenomena interfere causing the overall column buckling to be affected by the (local) spread of yield.

(ix) For any particular end-restraint case and initial crookedness studied, the increase of columns strength relative to the strength of pinned columns, is more pronounced for higher slenderness ratios (longer columns).

At this stage it seemed pertinent to quantify the gain of strength that might be achieved in each of the four cases of 'imperfect' connections, in relation to the strength of pinned columns. The values given herein refer to columns with the usual out-of-straightness tolerance and with nominal slenderness ratios in the range 52-118.9. For columns restrained by imperfect fixed connections (case 1), imperfect pinned-fixed connection (case 4), end-restraint case 3, and imperfect pinned connections (case 2), the increase in column load carrying capacity lies respectively in the ranges 17.0%-61.5%, 16.2%-39.7%, 14.3%-31.3%, and 10.4%-20.6%.

(x) In the range of slenderness studied, there is a tendency
for an overall increase of buckling deflections at mid-
height, for all the cases of initial crookedness and of
end-restraint, in relation to the buckling deflections at
midheight of pinned end columns. This increase in
deflections is not explicitly due to the end-restraints
effect, but is a consequence of the increase in column load
carrying capacity. In fact, for a bigger sustained column
load there is more spread of yield throughout the column,
which in turn induces a magnification of deflections as
the column becomes plastic.

(xi) The increase in deflections mentioned in (x) tends to be
higher for smaller slenderness ratios (shorter columns).
This is due to the fact that shorter columns have higher
$P_s/P_y$ ratios.

Moreover, the analysis of the spread of yield showed that
for pinned columns failure occurred almost immediately after first
yield in the case of longer (slender) members, or after a limited
amount of yielding for shorter (stocky) members. For restrained
columns, the spread of yield is more gradual and substantial, and
not necessarily symmetric (because of varying end-restraints).

7.7 Design Recommendations

The parametric study undertaken represents an analysis of
imperfect tubular beam-columns restrained by ideal semirigid connec-
tions, from which expected levels of column strength, expected degrees
of lateral movement and expected ranges of effective length factors
have been derived for columns with intermediate to high slenderness ratios. In view of these facts, it is the author's opinion that the following recommendations are pertinent in real designs involving tubular beam-columns:

(i) Properly selected semirigid connections may improve economy in tubular column designs, by the material savings associated with the reduction of as much as 25% in effective length factors.

(ii) In designs of shorter columns with nominal slenderness ratios less than 52, little is expected to be gained by considering end rotation restraint.

(iii) For all likely combinations of slenderness ratios in the range 52-180 (the upper bound includes experimental study) and end-restraint, the expected range of K-values is 0.7-1.0.

(iv) Potentially greater effects of end-restraint can be more fully utilized for long (slender) columns because of reduced significance of column yield in such columns.

(v) Conservatism should be used in selecting a K-value, because an excess of actual restraint (over the assumed value) can do good, but a lack of restraint (under the assumed value) can do much harm.

In view of this fact, the alignment charts for effective length of columns in continuous frames (with sidesway prevented) cannot be conservatively used in the design of space tubular beam-columns, in the range of
slenderness studied. The K-values obtained by the alignment charts are considerably lower than the K-values obtained in this dissertation, with slenderness ratios varying in the range 52-188. As an example, for the imperfect fixed connections (case 1) the expected range of K-values is 0.75-0.96, while the alignment charts give a constant value of 0.61. For the imperfect pinned connection (case 2) the expected range of K-values is 0.78-0.95, while the alignment charts give a constant value of 0.77. A basic assumption of the governing equations for the nomographs has been violated, namely elastic behavior.

As one's knowledge increases with discovery and recognition of more of the facts affecting the behavior of tubular beam-columns and proper account is put in design analysis, the latitude of errors due to uncertainties decreases. Accordingly, the potential properties of the material can be more fully utilized with confidence. All savings of steel would be absolutely genuine savings.
CHAPTER 8
SUMMARY

8.1 Summary and Conclusions

The construction and calibration of a beam-column end-restraint system has been described. Test results from this calibration program were used to formulate descriptive mathematical equations for the end-restraint moment functions. In addition, an empirical database on the buckling and postbuckling behavior and strength of end-restrained tubular beam-columns was obtained, by conducting a series of buckling tests as part of this investigation.

The Influence Coefficient Method has been consistently derived for the inelastic buckling analysis of end-restrained, crooked, space tubular beam-columns. In this efficient method the end-restraint characteristics, for each level of loading, were formulated directly into a Jacobian matrix.

Different solution methods of nonlinear structural problems were described. The Newton-Raphson iterative algorithm has been used in the context of the Influence Coefficient Method, to establish convergent solutions up to the limit buckling load. The elasto-plastic buckling analysis was improved by obtaining more efficient and reliable solutions of the buckling state, using the modified Newton-Raphson method in conjunction with the constrained
displacement length method. Also it was found that the Influence Coefficient Method cannot be applied to the initial postbuckling analysis of tubular columns, because it is not a stiffness formulation of the equilibrium condition.

A number of beam-column problems were analyzed to check the validity of the analytical procedure and results indicated very good performance of the Influence Coefficient Method in handling end-restraints, initial imperfections, eccentricities of loading, spread of yield and residual stresses. A parametric study of the effects of semirigid end-restraints and initial crookedness in the buckling strength and behavior of tubular columns was also conducted. From this effective length factors were derived, showing evidence on the expected range of effective length factors for columns with intermediate to high slenderness ratios. It was concluded that the reduction in effective length factors suggests the possibility of improved economy in tubular column designs, if semirigid end-restraints are properly accounted for. Moreover, it was shown that efficiently designed tubular members cannot be made proportionally stronger by increasing the amount of end-restraint. In effect, the end-restraint causes a more marked increase in buckling strength for long slender columns, than for shorter ones. Possible reductions of as much as 25% in the effective length factor of restrained beam-columns, have been estimated by this investigation.
8.2 Suggestions for Further Research

This final section advances a number of areas in which future work would be advisable. A few brief comments on each of these topics are mentioned. Further research on tubular beam-columns may be extended in the following directions:

(i) Experimental and theoretical information on three dimensional behavior of real tubular connections is needed, defining the moment rotation connection curves. Care must be taken to ensure that correct connection rotation values are measured in the experiments. The modelling of the theoretical connection behavior by bilinear curves might be improved by polynomial, exponential or spline curve fitting techniques.

Once the real tubular connection behavior has been modelled, a simple realistic relationship between the effective length factor and the slenderness ratio may be established, for different magnitudes of initial out-of-straightness.

(ii) A parametric study, similar to the one undertaken herein, coupled by full scale tubular column tests should be performed, to estimate the combined effects of the three determinant factors influencing column maximum strength, namely: end-restraint, magnitude of the initial imperfections, and magnitude and distribution of residual stresses. An analytical procedure such as the one described in this dissertation, should then be compared
with the prototype tubular steel columns predictions, thus enabling a more accurate assessment of column strength and performance, when end-restraint characteristics are known.

(iii) The combined effect of large eccentricities and end-restraint is an attractive subject for further research. End-restraint and end-eccentricities together with initial imperfections and residual stresses can all be included in an isolated column parametric study. End-restraint in an eccentric column reduces the effective eccentricity. In the elastic case the reduction increases as the column load is increased. In the inelastic case it might be expected that this effect should be even more pronounced.

As a result of such parametric study, the relationship between effective length factor and slenderness ratios might be further developed and refined for practical use. Only after this can studies of columns in simple frames with sidesway be followed.

(iv) Application of multiple column curves to tubular beam-columns is needed, to ascertain the strengthening effect of end-restraint as compared to the weakening effect of initial crookedness. This, however, is dependent on realistic estimation of the tubular joints connection curves.

(v) A Finite Element Method formulation (or other stiffness method formulation) of biaxially loaded columns may be
used as an alternative method to the Influence Coefficient Method used in this study. Although possibly less cost effective, such formulation would have the ability to produce a complete load-deflection curve including the postbuckling unloading portion. Moreover, a formulation by finite elements could predict significant deformation and stresses along the tubular beam-column accurately, and even account for axial variation of ovalization.
REFERENCES


[44] Lee, S.L., & Hauck, G.F. Buckling of steel columns under


